MINIMUM DUST ABUNDANCES FOR PLANETESIMAL FORMATION VIA SECULAR GRAVITATIONAL INSTABILITIES

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ABSTRACT

We estimate minimum dust abundances required for secular gravitational instability (SGI) to operate at the midplane dust layer of protoplanetary disks. For SGI to be a viable process, the growth time of the instability $T_{\text{grow}}$ must be shorter than the radial drift time of the dust $T_{\text{drift}}$. The growth time depends on the turbulent diffusion parameter $\alpha$ because the modes with short wavelengths are stabilized by turbulent diffusion. Assuming that turbulence is excited via the Kelvin–Helmholtz or streaming instabilities in the dust layer, and that its strength is controlled by the energy supply rate from dust accretion, we estimate the diffusion parameter and the growth time of the instability. The condition $T_{\text{grow}} < T_{\text{drift}}$ requires that the dust abundance must be greater than a critical abundance $Z_{\text{min}}$, which is a function of the Toomre parameter $Q_g$ and the aspect ratio $h_g/r$ of the gas disk. For a wide range of parameter space, the required dust abundance is less than 0.1. A slight increase in dust abundance opens a possible route for the dust to directly collapse to planetesimals.

Key words: planets and satellites: formation – protoplanetary disks

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1. INTRODUCTION

Formation processes of planetesimals have not been well understood in planet formation theory. The gravitational instability of the dust layer at the midplane of a protoplanetary disk has been proposed as a possible route to planetesimal formation. The classical scenario of gravitational instability requires high densities of dust layers to surpass the Roche limit (Goldreich & Ward 1973; Sekiya 1983). Dust sedimentation only in the vertical direction hardly achieves such a high density (Sekiya 1998). The dust layer becomes turbulent via Kelvin–Helmholtz (KH) or streaming instabilities when its dust density exceeds the gas density, which is much less than the Roche density, and further dust settling is significantly suppressed (Chiang & Youdin 2010 for review). Radial drift of the dust caused by gas drag may provide a possible route for further accumulation of the dust. When the dissipative effects of gas drag are included, the dust layer is secularly gravitationally unstable to the modes of dust accumulation in the radial direction (Ward 2000; Youdin 2005a, 2005b). This secular gravitational instability (SGI) occurs in any dust layers even if their densities are small (i.e., there is no criterion on their Toomre’s $Q$ values for SGI; Youdin 2011; Shariff & Cuzzi 2011; Michikoshi et al. 2012; these papers are referred to hereafter as Y11, SC11, and MKI12, respectively). However, the growth rate of SGI is greatly suppressed by gas drag, particularly for small particles, and could be too slow to have any effect on planetesimal formation. Y11, SC11, and MKI12 argue that the growth timescale must be shorter than the lifetime of the dust drift toward the star owing to gas drag. The wavelength of the most unstable mode is determined by the balance between the radial accumulation of dust particles due to self-gravity and their diffusion due to gas turbulence. Only perturbations with wavelengths sufficiently long are unstable. For stronger turbulence, wavelengths of the unstable modes, as well as growth timescales, are longer. Y11 and SC11 have calculated the growth timescales for various values of the turbulent diffusion parameter $\alpha$ and have derived an upper limit on $\alpha$ required for the growth timescale to be shorter than the drift timescale. Their approach is general and can be applied for any disk turbulence, provided that the value of $\alpha$ is known.

In this paper, we focus on disk turbulence induced in the dust layer, assuming that the gas disk itself is a globally laminar flow. In such a disk, the concentration of dust particles at its midplane triggers turbulence. The velocity difference between the dust and gas induces the KH or streaming instabilities and causes turbulence in the dust layer (Chiang & Youdin 2010). The ultimate energy source for turbulence is the accretion energy of the dust drifting toward the star, regardless of the kind of instability that occurs in the dust layer. Thus, a simple energetics can be applied to estimate the turbulence strength, provided that the dust accretion velocity is known. Takeuchi et al. (2012, hereafter T12) calculate the energy supply rate to turbulence, using the classical formulae on the drift velocity of the dust (Nakagawa et al. 1986; Weidenschilling 2003; Youdin & Chiang 2004), and then estimate the turbulence strength or the value of $\alpha$. T12 have shown that, from the comparison between the estimated value of $\alpha$ and the recent results of numerical simulations of turbulence by Johansen et al. (2006) and Bai & Stone (2010), the turbulence strength excited via KH and/or streaming instabilities can be estimated from the dust accretion rate.

While Y11, SC11, and MKI12 have derived the growth time of SGI as a function of $\alpha$ and other disk parameters (Equation (5) of Y11), T12 have obtained $\alpha$ values of turbulence in the dust layer (Equation (5) below). Combining these results gives an expression of the growth time without the unknown parameter $\alpha$, as shown in Section 2.1. Consequently, in Section 2.3, the criterion for SGI to operate is obtained as a condition on the dust abundance $Z$ in the disk. For SGI to operate faster than the dust drift timescale, the dust abundance must be greater than a specific critical value $Z_{\text{min}}$, which is a function of the Toomre $Q$ value of the gas disk and the deviation fraction $\eta$ of the gas velocity from the Keplerian velocity. The $Z_{\text{min}}$ value derived in this paper gives the minimum dust abundance required for SGI because we consider only turbulence induced in the dust layer. If turbulence were caused by another mechanism...
such as magneto-rotational instability (MRI) of the gas disk, $\alpha$ would be larger than the value we adopted, and consequently the required value of $Z_{\text{mid}}$ would increase. Hence, $Z_{\text{mid}}$ derived in this paper is considered as the minimum value required for SGI in realistic disks.

2. CONDITION FOR SECULAR GRAVITATIONAL INSTABILITY

We consider a protoplanetary disk initially in a laminar flow state. Sedimentation of dust particles to the midplane of the disk induces turbulence via KH or streaming instabilities caused by velocity differences between the dust and gas. A (quasi-)steady dust layer forms when turbulent diffusion matches dust settling in which steady state turbulence is maintained. We neglect intermittency of turbulence. If an extra source of turbulence is present such as global turbulence of the gas disk via MRI, the turbulence would be stronger than that excited only by the dust-gas velocity difference. Thus, in this paper, we consider the minimum strength of turbulence.

The dust particles are characterized by their stopping time $t_s$, which is the timescale of damping the velocity difference from the gas. We define the non-dimensional stopping time, $T_s = t_s/\Omega_K$, normalized by the Keplerian frequency $\Omega_K$. In this study, we focus on the dynamics of relatively small particles such that $T_s \lesssim 1$. Particles with $T_s \sim 1$ experience high-speed collisions and the fastest radial drift, which likely hinder particle growth. SGI of the dust layer is a possible route for directly forming large bodies ($\sim$ planetesimals) via planetesimal formation.

2.1. Timescale of Secular Gravitational Instability

In his Equation (51), Y11 shows that the growth time of SGI for small particles with $T_s \lesssim 1$, normalized by the Keplerian time $\Omega_K^{-1}$, is

$$T_{\text{grow}} \approx \frac{\alpha Q_g^2}{Z^2 T_s^2}. \quad (1)$$

Here $\alpha$ is the turbulent diffusion parameter (see below), $Q_g$ is the Toomre stability parameter of the gas disk,

$$Q_g = \frac{c_s \Omega_K}{\pi G \Sigma_g}, \quad (2)$$

where $c_s$ is the sound speed of the gas, $G$ is the gravitational constant, and $\Sigma_g$ is the surface density of the gas disk. The disk “metallicity” $Z$ is the ratio of the surface densities between the dust and gas, $Z = \Sigma_d/\Sigma_g$. The growth time depends on the turbulent diffusion parameter $\alpha$ through its stabilizing effect for perturbations with short wavelengths.

Turbulence in the dust layer is induced by velocity differences between the dust and gas. If the drag force between the gas and dust were not effective, the dust would orbit with the Keplerian velocity $v_K$, while the gas would orbit with a sub-Keplerian velocity $v_g = (1 - \eta)v_K$, where

$$\eta = -2(\rho_g r\Omega_K)^{-1} \partial P/\partial r \sim (c_s/v_K)^2 \sim 10^{-3} - 10^{-2},$$

$\rho_g$ is the gas density, and $P$ is the gas pressure. In the calculation of $\alpha$, a slightly different definition of $\eta$ is useful (see T12 or the Appendix in this paper). We introduce $\tilde{\eta}$ defined by

$$\tilde{\eta} = \left(\frac{v_K}{c_g}\right)^2 \eta^2 = C_\eta \eta \quad (4)$$

where $C_\eta$ is a factor of the order of unity, which depends on the radial profile of the gas disk. In this paper, we adopt $C_\eta = 1$ (i.e., $\tilde{\eta} = \eta$) for simplicity, except in Section 3 where specific disk models are considered.

T12 show that the dust particles accrete toward the star due to gas drag and supply energy to turbulence via KH and/or streaming instabilities. Dust accretion is either caused by the gas drag acting on individual particles or by turbulent drag acting on the surface of the dust layer. The former (individual drag) is effective if the dust-to-gas ratio at the disk midplane, $f_{\text{mid}} = \rho_d/\rho_g|_{z=0}$, is less than unity, and the latter (collective drag) is effective if $f_{\text{mid}} \gtrsim 1$. The Appendix briefly summarizes dust accretion due to both drag types (see T12 for detailed discussions). The turbulence strength or the parameter $\alpha$ is determined by the energy supply rate from dust accretion toward the star. The approximate expression of $\alpha$ for $T_s \lesssim 1$ particles is given by

$$\alpha \approx [(C_1 C_{\text{eff}} \tilde{\eta} Z)^{-1} + (C_2 C_{\text{eff}} \tilde{\eta} Z^{-1})^{-2}]^{-1}, \quad (5)$$

where $C_{\text{eff}} = 0.19$ is the energy supply efficiency (see T12), and $C_1 = 1.0$ and $C_2 = 1.6$ are the numerical factors. This expression connects the approximate formulae of $\alpha$ for the two limiting regimes of $f_{\text{mid}} \ll 1$ and $f_{\text{mid}} \gg 1$ (Equations (A2) and (A3)). Note that the condition $f_{\text{mid}} \ll 1$ (or $f_{\text{mid}} \gg 1$) corresponds to the condition $Z \ll (C_{\text{eff}} \tilde{\eta})^{1/2}$ (or $Z \gg (C_{\text{eff}} \tilde{\eta})^{1/2}$; see Equation (A5)). The numerical factors $C_1$ and $C_2$ are adjusted to make an appropriate fit to the overall behavior of the numerical result. A comparison of this approximate expression to the numerically calculated $\alpha$ is shown in the Appendix.

Substituting Equation (5) into Equation (1) gives

$$T_{\text{grow}} \approx \frac{Q_g^2}{Z^2 T_s^2} [(C_1 C_{\text{eff}} \tilde{\eta} Z)^{-2} + (C_2 C_{\text{eff}} \tilde{\eta} Z^{-1})^{-2}]^{-1}, \quad (6)$$

showing that $T_{\text{grow}}$ is a function of the disk parameters $(\tilde{\eta}, Z, Q_g)$ and is inversely proportional to $T_s$.

2.2. Timescale of Radial Drift

According to T12, the non-dimensional timescale of the radial drift of dust particles due to gas drag is estimated as

$$T_{\text{drift}} \approx \frac{f_{\text{mid}} + 1}{2 \tilde{\eta} T_s}, \quad (7)$$

where the midplane dust-to-gas ratio $f_{\text{mid}}$ is approximately given by

$$f_{\text{mid}} \approx \left(\frac{Z^2}{C_1 C_{\text{eff}} \tilde{\eta}}\right)^{1/2} + \left(\frac{Z^2}{C_2 C_{\text{eff}} \tilde{\eta}}\right). \quad (8)$$

In the above estimate, both individual and collective drags are considered. Derivation of the above expressions is described in the Appendix. The timescale of the radial drift is a function of $(\tilde{\eta}, Z)$ and is inversely proportional to $T_s$. 

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2.3. Comparison of Timescales

2.3.1. Minimum Dust Abundances

Y11, SC11, and MKI12 show that dust layers are always unstable and subject to SGI, but the growth rate for particles tightly coupled to the gas is strongly suppressed by gas drag. The growth timescale can be larger than the other timescales, in which the dust layer evolves significantly by other processes such as the dispersal of the gas disk and the radial drift of the dust to the star. The condition for SGI to be a relevant process for planetesimal formation dictates that its growth timescale must be shorter than the other evolution timescales. Y11 shows that the radial drift imposes the most stringent condition for a wide range of disk parameters. We discuss the condition for $T_{\text{grow}} < T_{\text{drift}}$. Because both $T_{\text{grow}}$ and $T_{\text{drift}}$ are inversely proportional to $T_*$, the ratio $T_{\text{grow}}/T_{\text{drift}}$ is independent of $T_*$; that is, the condition $T_{\text{grow}} < T_{\text{drift}}$ is not affected by particle size. We consider dependence of $T_{\text{grow}}/T_{\text{drift}}$ on the parameters $(\eta, Z, Q_g)$.

First, we examine the limiting cases of $Z \ll (C_{\text{eff}}\eta)^{1/2}$ and $Z \gg (C_{\text{eff}}\eta)^{1/2}$ ($f_{\text{mid}} < 1$ and $f_{\text{mid}} > 1$). In both limits,

$$\frac{T_{\text{grow}}}{T_{\text{drift}}} \approx \begin{cases} 2C_{\eta}^{-1}(C_{\text{eff}}\eta)^{3/2}\eta^{5/3}Q_g^2Z^{-4/3} & \text{for } Z \ll (C_{\text{eff}}\eta)^{1/2} \\ 2C_{\eta}^{-1}(C_2C_{\text{eff}}\eta)^{2/3}\eta^{2}Q_g^2Z^{-6} & \text{for } Z \gg (C_{\text{eff}}\eta)^{1/2} \end{cases}$$

For $Z \ll (C_{\text{eff}}\eta)^{1/2}$, $T_{\text{grow}}/T_{\text{drift}}$ is proportional to $Z^{-4/3}$, while for $Z \gg (C_{\text{eff}}\eta)^{1/2}$ it rapidly decreases as $T_{\text{grow}}/T_{\text{drift}} \propto Z^{-6}$. In Figure 1, $T_{\text{grow}}/T_{\text{drift}}$ is plotted against $Z$ for various values of $\eta$ and $Q_g$. For $Z \gg (C_{\text{eff}}\eta)^{1/2} \sim 10^{-2}$, it is evident that $T_{\text{grow}}/T_{\text{drift}}$ rapidly decreases. For sufficiently large values of $Z$, $T_{\text{grow}}/T_{\text{drift}}$ is less than unity, indicating that SGI operates faster than the radial drift of the dust. From Figure 1, the minimum dust abundance required for SGI, $Z_{\text{min}}$, is determined as $Z$ satisfying $T_{\text{grow}}/T_{\text{drift}} = 1$. The minimum dust abundance $Z_{\text{min}}$ ranges between $10^{-2}$ and $10^{-1}$ for parameters $(\eta, Q_g)$ adopted in Figure 1.

In Figure 2(a), the minimum dust abundance $Z_{\text{min}}$ required for $T_{\text{grow}} < T_{\text{drift}}$ is plotted as contours on the $\tilde{\eta}$-$Q_g$ plane. The locus of $Z_{\text{min}} = (C_{\text{eff}}\tilde{\eta})^{1/2}$ is represented as a dashed line on the $\tilde{\eta}$-$Q_g$ plane. On this line, the midplane dust-to-gas ratio $f_{\text{mid}}$ becomes close to unity when $Z = Z_{\text{min}}$. Substitution of $Z = (C_{\text{eff}}\tilde{\eta})^{1/2}$ into Equation (9) shows that the condition $T_{\text{grow}}/T_{\text{drift}} = 1$ (i.e., $Z = Z_{\text{min}}$) becomes $Q_g \approx \eta^{-1/2}$. Thus, the dashed line in Figure 2(a) is linear with a slope of $-1/2$. Above this line, $Z_{\text{min}} > (C_{\text{eff}}\tilde{\eta})^{1/2}$ (and $f_{\text{mid}} > 1$ for $Z = Z_{\text{min}}$); below it, $Z_{\text{min}} < (C_{\text{eff}}\tilde{\eta})^{1/2}$ (and $f_{\text{mid}} < 1$). The dependence of $Z_{\text{min}}$ on $(\tilde{\eta}, Q_g)$ is expressed as

$$Z_{\text{min}} \approx \begin{cases} (8C_{\eta}^{-3}C_{\text{eff}}^2\eta^7Q_g^6)^{1/4} & \text{for } Q_g \ll \tilde{\eta}^{-1/2} \\ (2C_{\eta}^{-5}C_{\text{eff}}^2\eta^3Q_g^2)^{1/6} & \text{for } Q_g \gg \tilde{\eta}^{-1/2} \end{cases} \quad (10)$$
Larger dust abundances are required for larger $\eta$ (i.e., hotter gas disks) and larger $Q_g$ (i.e., less massive gas disks). In standard models for protoplanetary disks, $\eta = 10^{-3} - 10^{-2}$ and $Q_g = 10 - 1000$ (see Figure 6 below). Thus, if the dust abundance $Z$ is larger than 0.1, such dusty disks operate SGI. If the gas disk is so cold that $\eta$ is as small as $10^{-3}$ and if it is so massive that $Q_g$ is as small as 10, then even a standard value $Z = 0.01$ is sufficiently large to operate SGI.

2.3.2. Dependence of $Z_{\text{min}}$ on Some Model Parameters

The growth time $T_{\text{grow}}$ is just a time for density perturbations to increase $e$-fold. Because many $e$-folds are needed for a significant density increase, actual condition for SGI would be $RT_{\text{grow}} < T_{\text{drift}}$, where $R > 1$ is the required number of $e$-folding. From Equation (9) the minimum dust abundance $Z_{\text{min}}$ scales as $Z_{\text{min}} \propto R^{3/4}$ for $Q_g \ll (R\eta)^{-1/2}$ and $Z_{\text{min}} \propto R^{1/2}$ for $Q_g \gg (R\eta)^{-1/2}$. Figure 2(b) shows $Z_{\text{min}}$ derived for the condition $10T_{\text{grow}} < T_{\text{drift}}$. For less massive gas disks ($Q_g \gg (R\eta)^{-1/2}$, above the dashed line in Figure 2), $Z_{\text{min}}$ depends only weakly on $R$. For massive gas disks ($Q_g \ll (R\eta)^{-1/2}$, required $Z_{\text{min}}$ is higher for larger $R$ ($Z_{\text{min}} \propto R^{3/4}$), but would not be too high (i.e., $Z_{\text{min}} \lesssim 0.1$ even for $R = 10$; see the region below the dashed line in Figure 2(b)). In the following discussions, we use the condition $T_{\text{grow}} < T_{\text{drift}}$.

In this paper, it is assumed that about 20% of the dust accretion energy is used for turbulence excitation ($C_{\text{eff}} = 0.19$). This value is determined by comparison with the numerical simulations of turbulence excited via KH instability by Johansen et al. (2006). Because their simulation was two dimensional, the realistic value of $C_{\text{eff}}$ could be different. For example, three-dimensional simulations by Lee et al. (2010) show that the critical Richardson number for KH instability increases with $Z$, implying that $C_{\text{eff}}$ also increases with $Z$ (see discussion in Section 5.2.1 of T12).

As shown in Equation (10), $Z_{\text{min}} \propto C_{\text{eff}}^{1/2}$. For the maximum efficiency ($C_{\text{eff}} = 1$), $Z_{\text{min}}$ would be about two times larger than our estimate.

2.4. Conditions on $T_{\text{grow}}$, $\alpha$, and $\lambda$

In the previous subsection, the condition for $T_{\text{grow}} < T_{\text{drift}}$ was discussed. Next, we discuss other constraints required for SGI operation. First, we consider the condition for the growth time to be less than the disk lifetime. The growth time multiplied by the stopping time, $T_{\text{grow}}T_s$, is plotted on the $\eta\bar{Q}_g$ plane in Figure 3. The values of $T_{\text{grow}}(\eta, Q_g)$ are calculated using the minimum dust abundance $Z = Z_{\text{min}}(\eta, Q_g)$, the values of which differ according to $(\eta, Q_g)$ as shown in Figure 2. For larger values of $Z$, $T_{\text{grow}}$ is shorter ($T_{\text{grow}} \propto Z^{-3/4}$ for $Z < (C_{\text{eff}}\eta)^{1/2}$ and $T_{\text{grow}} \propto Z^{-1}$ for $Z > (C_{\text{eff}}\eta)^{1/2}$). Because $T_{\text{grow}}$ is inversely proportional to $T_s$, the contours are labeled by $\log(T_{\text{grow}}T_s)$. Thus, the growth timescale is obtained by dividing the value in Figure 3 by $T_s$. For $\eta > 10^{-3}$ and $Q_g < 10^2$, $T_{\text{grow}}$ is less than $10^3 T_s^{-1} \Omega^{-1}$. Thus at 1 AU, the dust layer composed of $T_s \gtrsim 10^{-3}$ particles (size $a \gtrsim 1$ mm) with the dust abundance $Z_{\text{min}}$ operates SGI within a disk lifetime of $\sim 1$ Myr.

Figure 4 shows the diffusion parameter $\alpha$ due to turbulence induced in the dust layer with the abundance $Z_{\text{min}}$. Because $\alpha$ is proportional to $T_s$, the contours are labeled by $\log(\alpha/T_s)$. Thus, the $\alpha$ value is obtained by multiplying the value in Figure 4 by $T_s$. For $\eta > 10^{-3}$ and $Q_g < 10^2$, $\alpha$ is larger than $10^{-4} T_s$, except for very small $Q_g \lesssim 10$ and $\eta \lesssim 3 \times 10^{-3}$. If the disk is turbulent due to other mechanisms than that originating from the dust layer and if turbulent diffusion is stronger than that shown in Figure 4, then the required value for $Z$ would be higher. Even in the dead zone where MRI is not active, the gas could have turbulent motion, causing diffusion of the dust (Fleming & Stone 2003). Okuzumi & Hirose (2011) show that lower values of the diffusion coefficient in the dead zone are realized for a wider dead zone or weaker vertical magnetic fields. For example, if the plasma $\beta$ (the ratio of gas pressure to magnetic pressure) is greater than $3 \times 10^9$, the diffusion coefficient due to MRI turbulence is less than $10^{-5}$ (see model X1b of Okuzumi & Hirose 2011). In such a weakly magnetized disk and for $T_s > 0.1$ particles, turbulence originating in the dust layer is stronger than MRI turbulence in the dead zone.

The wavelength of the most unstable mode must be smaller than the disk radius. The wavelength $\lambda$ is given in Equation (56) of Y11. The ratio of $\lambda$ to half of the disk radius $r$ is

$$\frac{2\lambda}{r} \approx \frac{4\pi \alpha \bar{Q}_g \bar{\eta}^{1/2}}{C_s Z T_s}$$

and is plotted in Figure 5 for disks with the minimum dust abundances $Z_{\text{min}}$. Note that $2\lambda/r$ does not depend on $T_s$ because
$\lambda$ is proportional to $T$, Figure 5 shows that $\lambda$ is smaller than $r/2$ if $\eta$ is less than $3 \times 10^{-2}$, although it is still comparable to $r/2$ even for $\eta$ as small as $3 \times 10^{-3}$. In the analysis by Y11, SC11, and MK12, the gas disk is assumed to behave as a stationary background, and its velocity profiles are not affected by the gravity of the dust layer. However, the response of the gas to perturbations with smaller wavelengths than those shown in Figure 2 are required. The wavelength varies as $\lambda \propto Z^{-1}$ for $Z < (C_{eq}\eta)^{1/2}$ and $\lambda \propto Z^{-3}$ for $Z > (C_{eq}\eta)^{1/2}$. In the latter regime ($Z > (C_{eq}\eta)^{1/2}$), an increase by a factor of two in $Z$ results in an order of magnitude decrease in $\lambda$, while in the former regime, reduction in the wavelength requires a large increase in $Z$.

3. MINIMUM DUST ABUNDANCES FOR VARIOUS DISK MODELS

In the previous section, minimum dust abundances for SGI were obtained for given parameters ($\eta$, $Q_g$). In this section, we consider several disk models. We adopt power-law profiles for density and temperature distributions of the gas disk, such that

$$\Sigma_g = \Sigma_0 r_P^{-p}$$

$$T = T_0 r_AU^{-q}$$

where $r_AU$ is the distance from the star measured in AU. In such a disk, $Q_g \propto r_AU^{(2p - q - 3)/2}$ and $\eta \propto r_AU^{-q}$. We consider two cases of hot and cold gas disks, $T_0 = 300$ K and 150 K, while we fix the power-law index $q = 1/2$. For the density profile, $p = 1.0$ and 1.5 are adopted and the mass of the gas disk inside 100 AU is varied within $M_{disk} = 10^{-2}$–$10^{-1} M_\odot$. Figure 6 shows the variability in the values of $\eta$ and $Q_g$ between the models and with $r$. In this figure, models with $M_{disk} = 3 \times 10^{-2} M_\odot$ are shown. For various values of $M_{disk}$, $Q_g$ scales as $Q_g \propto M_{disk}$.

Figure 7 shows the required minimum dust abundances $Z_{\min}$ against $r_AU$ for various disk models. For the steep density profile ($p = 1.5$; blue dashed lines), $Z_{\min} \propto r_AU^{1/4}$. (In the regime of $Q_g \ll \eta^{-1/2}$ of Equation (10), $Z_{\min} \propto \eta^{5/4} Q_g^{1/2} \propto r_AU^{1/4}$; in the $Q_g \gg \eta^{-1/2}$ regime, $Z_{\min} \propto \eta^{3/2} Q_g^{1/4} \propto r_AU^{1/4}$.) SGI is more viable at the inner part of the disk. For $p = 1.0$ (red solid lines), $Z_{\min} \propto r_AU^{1/2}$ only weakly depends on $r$ (in the $Q_g \gg \eta^{-1/2}$ regime). The outer part of the disk with $M_{disk} = 10^{-1} M_\odot$
and \( p = -1 \) is in the \( Q_s \ll \tilde{\eta}^{-1/2} \) regime, and \( Z_{\text{min}} \) decreases as \( Z_{\text{min}} \propto r^{-1/2} \). Even in the hot gas disks (\( T_0 = 300 \) K) with a mass as small as \( M_{\text{disk}} = 10^{-2} M_\odot \), the required dust abundances for SGI are less than 0.1. If the temperature at the disk midplane is as cold as \( T_0 \approx 150 \) K, as expected in passive disks that are heated only by stellar radiation (Chiang & Goldreich 1997; Tanaka et al. 2005), SGI operates for \( Z_{\text{min}} \approx 0.05 \) at 10 AU of disks with \( M_{\text{disk}} = 10^{-2} M_\odot \).

4. DISCUSSION

Analysis in this paper assumes that turbulence is induced in the dust layer. If the dust-to-gas ratio at the midplane \( Z \) is larger than unity (\( Z > \tilde{\eta} \)), turbulence weakens with increasing \( Z \) as \( \alpha \propto Z^{-2} \) (Equation (5)). The growth timescale of SGI rapidly decreases as \( T_{\text{grow}} \propto Z^{-4} \) (Equation (6)). A slight increase in \( Z \) makes SGI viable. Y11 also estimated the minimum dust abundance \( Z_{\eta} \) required for SGI (Equation (69) of Y11), assuming that the turbulent diffusion parameter of a dense dust layer (\( f_{\text{mid}} > 1 \)) was \( \alpha_{\eta} \propto T_{\text{eff}} \tilde{\eta} \) (Equation (68) of Y11), independent of \( Z \). This estimate for \( \alpha_{\eta} \) adopted by Y11 corresponds to the maximum value of \( \alpha(Z) \) at \( Z \approx \tilde{\eta}^{1/2} \) in our estimate (Equation (5)). If \( \alpha \) had a constant value \( \alpha_{\eta} \) for all \( Z \), the condition \( T_{\text{grow}} < T_{\text{drift}} \) would be

\[
Z > Z_{\eta} \approx \tilde{\eta}^{5/6} Q_s^{2/3},
\]

where we used \( f_{\text{mid}} = Z h_d/h_g \) and \( h_d/h_g \approx \tilde{\eta}^{1/2} \). Note that this Equation (14) differs from Equation (69) of Y11 because we use Equation (7), in which \( T_{\text{drift}} \propto f_{\text{mid}}^{1/2} \) while Y11 assumes \( T_{\text{drift}} \propto f_{\text{mid}}^{1/2} \) (see discussion following Equation (A6) below). As discussed in T12, accumulation of the dust increases the inertia of the dust layer and accelerates the inward drift of the dust, resulting in weaker turbulence. Considering \( \alpha \propto Z^{-2} \), the condition \( T_{\text{grow}} < T_{\text{drift}} \) becomes

\[
Z > Z_{\text{min}} \approx \tilde{\eta}^{2/3} Q_s^{1/3},
\]

as shown in Equation (10). We ignore numerical coefficients such as \( C_{\text{eff}} \) and \( C_{\eta} \). These two conditions (Equations (14) and (15)) coincide when \( f_{\text{mid}} = 1 \) (i.e., \( Z_{\eta} = Z_{\text{min}} \approx \tilde{\eta}^{1/2} \)) or when \( Q_s = Q_{s,0}(\tilde{\eta}) \propto \tilde{\eta}^{-1/2} \) (i.e., on the dashed line in Figures 2–5). If the disk’s \( Q_s \) is greater than \( Q_{s,0} \), then the latter condition (Equation (15)) permits smaller dust abundances for SGI by a factor of \( Z_{\text{min}}/Z_{\eta} = (Q_{s,0}/Q_s)^{-1/3} \). For example, at 1 AU on a disk with \( M_{\text{disk}} = 0.03 M_\odot \), \( \Sigma_g \propto r^{-1} \), and \( T_0 = 300 \) K, Equation (15) gives \( Z_{\text{min}} \approx 0.11 \), which is smaller by a factor of two than the value \( Z_{\eta} = 0.23 \) predicted by Equation (14). Considering the \( Z \) dependence of \( \alpha \), the required minimum value \( Z_{\text{min}} \) for SGI as compared with that derived by Y11.

5. SUMMARY

We analyze the condition for SGI occurrence in the dust layer. The growth timescale of the instability must be shorter than the radial drift timescale of the dust. The growth timescale decreases as turbulence weakens. The necessary condition for SGI is obtained by considering turbulence induced in the dust layer. Using the turbulence strength estimated from the energy supply rate from the accreting dust, the minimum dust abundances for SGI, \( Z_{\text{min}} \), are derived as a function of the Toomre \( Q_s \) parameter of the gas disk and the deviation fraction \( \eta \) of the gas velocity from the Keplerian value. If the dust particles are small and their stopping time is less than the Keplerian time (\( T_s < 1 \)), \( Z_{\text{min}} \) is independent of \( T_s \). For disks with \( Q_s \sim 10^2 \) and \( \eta \sim 10^{-2} \), SGI occurs if \( Z \gtrsim 0.1 \). The required \( Z \) decreases with decreasing \( Q_s \) and \( \eta \), becoming as small as 0.01 for \( Q_s \sim 10 \) and \( \eta \sim 10^{-3} \). Such an increase in \( Z \) from the solar abundance is expected to occur through several processes, including the radial drift of the dust and dispersal of the gas from the disk (Youdin & Shu 2002; Takeuchi & Lin 2002; Takeuchi et al. 2005). Therefore, SGI provides a possible route for small particles to directly collapse to planetesimals in an initially laminar disk. If the gas disk were globally turbulent via MRI, for example, the required \( Z \) would be larger than \( Z_{\text{min}} \) estimated in this paper.

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APPENDIX

APPROXIMATE EXPRESSIONS OF DIFFUSION PARAMETER AND RADIAL DRIFT RATE

In this Appendix, we evaluate an approximate expression of the turbulent diffusion parameter \( \alpha \) for small particle limit (\( T_s \ll 1 \)) according to T12. We consider turbulence excited via KH or streaming instabilities of the dust layer. The strength of the turbulence is controlled by the dust accretion rate. The dust particles accrete toward the star either due to the gas drag on individual particles (Nakagawa et al. 1986) or collective drag exerted on the entire dust layer (Weidenschilling 2003; Youdin & Chiang 2004). Individual drag dominates if the dust-to-gas ratio in the dust layer is less than unity (\( f_{\text{mid}} \lesssim 1 \)), while collective drag is important if \( f_{\text{mid}} \gtrsim 1 \).

First, we consider individual drag (for \( f_{\text{mid}} \lesssim 1 \)). Using the formula derived by Nakagawa et al. (1986; or Equation (9) of T12, hereafter Equation (T9)), the accretion velocity of small dust particles (\( T_s \ll 1 \)) is

\[
\dot{v}_{d,r} = -2T_s \eta v_K.
\]

To satisfy angular momentum conservation, the gas moves outward with the velocity \( v_{g,r} = -\dot{v}_{d,r} \rho_d/\rho_g \), where \( \rho_d (\rho_g) \) is the dust (gas) density. The effective gravities (including the pressure gradient force) acting on the dust and gas are \( g_d = -r \Omega_K^2 \) and \( g_g = -(1-2\eta)\Omega_K^2 \), respectively. The net work done by these effective gravities on a unit surface area of the dust layer is calculated as \( \Delta E_{\text{drag}} \sim \int \rho_d g_d v_{d,r} + \rho_d g_g v_{g,r} dz \sim \eta^3 \tilde{\eta}^{1/2} Q_s T_s \Sigma_d \) (Equation (T12)). The energy dissipation due to turbulence is estimated as \( \Delta E_{\text{turb}} \sim \Sigma \nu \Omega_K^2 \), where \( \Sigma \) is the surface density of the dust layer, and \( \nu \) and \( \Omega_K \) are the typical values for the velocity and turnover time of the largest eddies, respectively. Adopting the so-called \( \alpha \) prescription, we evaluate \( \nu \sim \tilde{\alpha} \) and \( \nu \Omega_K \sim \tilde{\alpha} K \) (Cuzzi et al. 2001). For \( f_{\text{mid}} \lesssim 1 \), the column density of the dust layer is dominated by the gas, and thus \( \Sigma \sim \Sigma_d \). Here, \( \theta_d/h_g \) is the ratio of the scale heights of the dust layer and gas disk, and is given by \( h_d/h_g \approx \sqrt{\tilde{\alpha} T_s} \) (Equation (T4)). Then, the energy dissipation rate is \( \Delta E_{\text{turb}} \sim \tilde{\alpha}^3 / T_s h_g^2 \Omega_K^2 \Sigma_d \). Equating the energy supply and dissipation rates of turbulence, with an efficiency parameter
for the energy supply, $\Delta E_{\text{turb}} \sim C_{\text{eff}} \Delta E_{\text{drag}}$, gives

$$\alpha \sim (C_{\text{eff}} \eta Z)^{2/3} T_s.$$  \hfill (A2)

The collective drag force in the $\theta$-direction on a unit surface area is estimated by the plate drag approximation for $f_{\text{mid}} \gtrsim 1$ as $P_{\theta z} \sim \rho_\theta \nu \partial \nu_{\theta z}/\partial z \sim -\rho_\theta \nu \eta v_k/\hbar_d$, where $\nu$ is the turbulent viscosity, and the velocity shear in the vertical direction $\partial \nu_{\theta z}/\partial z$ is estimated as $\eta v_k/\hbar_d$. This drag causes accretion of the dust layer and an outward motion of the upper gas layer. The dust layer loses energy $\Omega_k^2 P_{\theta z}$, and the upper gas layer gains energy $(1 - \eta)\Omega_k^2 P_{\theta z}$, where the work done by gas pressure is taken into account. The net energy liberation rate is $\Delta E_{\text{vis}} \sim -\eta \Omega_k^2 P_{\theta z} \sim \eta^2 v_k^2 \Omega_k T_s \Sigma_d/f_{\text{mid}}$ (Equation (T18)), where we used $\nu \sim h_d^3 T_s \Omega_k$ and $h_d = h_d \Sigma_d/(\Sigma_k f_{\text{mid}})$ (Equations (T16) and (T6)). The energy dissipation rate is $\Delta E_{\text{turb}} \sim \Sigma_{\text{layer}} \eta^2 v_k^2/\tau_{\text{eddy}} \sim \alpha \eta v_k^2 \Omega_k \Sigma_d$, where $\Sigma_{\text{layer}} \sim \Sigma_d$ is dominated by the dust. Equating $\Delta E_{\text{turb}} \sim C_{\text{eff}} \Delta E_{\text{vis}}$ gives

$$\alpha \sim (C_{\text{eff}} \eta Z)^{-1} T_s.$$  \hfill (A3)

where we use $f_{\text{mid}} \approx Z \sqrt{T_s/\alpha}$ (Equation (T6)). Expressions (A2) and (A3) are the same as Equation (T28), except for numerical factors and the adoption of a slightly steeper power-law index ($\delta = 1$ in Equation (A3) instead of $\delta = 0.94$ in Equation (T28)) to simplify analytical expressions. Connecting these expressions gives

$$\alpha \approx [(C_1 C_{\text{eff}} \eta Z)^{-1} + (C_2 C_{\text{eff}} \eta Z)^{-1}]^{-1} T_s,$$  \hfill (A4)

where the numerical factors $C_1 = 1.0$ and $C_2 = 1.6$ are set to fit the numerical result of T12. Figure 8(a), the approximate expression (A4) is compared to the numerical result of T12. The approximation overestimates $\alpha$ around $Z = \sqrt{C_{\text{eff}} \eta}$ and underestimates it for large $Z$. However, the error is less than 2 for $10^{-1} < Z < 10^{-1}$, and the overall behavior appears to be acceptable.

The midplane dust-to-gas ratio $f_{\text{mid}} = Z h_d/\hbar_d \approx Z \sqrt{T_s/\alpha}$ is, from Equations (A2) and (A3), $f_{\text{mid}} \sim [Z^2/(C_{\text{eff}} \eta)]^{1/3}$ for $f_{\text{mid}} \lesssim 1$, and $f_{\text{mid}} \sim Z^2/(C_{\text{eff}} \eta)$ for $f_{\text{mid}} \gtrsim 1$. Connecting these two expressions gives an approximate formula for $f_{\text{mid}}$ as

$$f_{\text{mid}} \approx \left(\frac{Z^2}{C_1 C_{\text{eff}} \eta}\right)^{1/9} + \left(\frac{Z^2}{C_2 C_{\text{eff}} \eta}\right)^{1/9}. \hfill (A5)$$

From Equation (A5), it is seen that $f_{\text{mid}} \approx 1$ when $Z \approx (C_{\text{eff}} \eta)^{1/2}$.

The drift velocity of the dust in the limit of $f_{\text{mid}} \ll 1$ and $T_s \ll 1$ is given in Equation (A1). In the limit of $f_{\text{mid}} \gg 1$, $v_{d,r} \sim \rho_{d,v} \nu_k/\theta_{d,r}$ (Equation (T51)). Connecting these limits, we estimate

$$v_{d,r} \approx \frac{2T_s \eta v_k}{f_{\text{mid}} + 1}, \hfill (A6)$$

which gives the dust drift timescale given in Equation (7). This approximate expression is compared with the numerical calculation by T12 in Figure 8(b), which shows that the above expression provides a good approximation. Note that $v_{d,r} \propto f_{\text{mid}}$ for $f_{\text{mid}} \gg 1$. Equation (2.11) of Nakagawa et al. (1986) shows that, due to individual drag, the radial velocity of the dust particles at the midplane is proportional to $f_{\text{mid}}^{-2}$. However, the average
Takeuchi, T., & Lin, D. N. C. 2002, ApJ, 581, 1344
Takeuchi, T., Muto, T., Okuzumi, S., Ishitsu, N., & Ida, S. 2012, ApJ, 744, 101 (T12)
Tanaka, H., Himeno, Y., & Ida, S. 2005, ApJ, 625, 414
Ward, W. R. 2000, in Origin of the Earth and Moon, ed. R. M. Canup & K. Righter (Tucson, AZ: Univ. Arizona Press), 75
Youdin, A. N. 2005a, arXiv:astro-ph/0508659
Youdin, A. N. 2005b, arXiv:astro-ph/0508662
Youdin, A. N. 2011, ApJ, 731, 99 (Y11)
Youdin, A. N., & Chiang, E. I. 2004, ApJ, 601, 1109
Youdin, A. N., & Shu, F. H. 2002, ApJ, 580, 494
Weidenschilling, S. J. 2003, Icarus, 165, 438
Youdin, A. N. 2005a, arXiv:astro-ph/0508659
Youdin, A. N. 2005b, arXiv:astro-ph/0508662
Youdin, A. N. 2011, ApJ, 731, 99 (Y11)
Youdin, A. N., & Chiang, E. I. 2004, ApJ, 601, 1109
Youdin, A. N., & Shu, F. H. 2002, ApJ, 580, 494