Large Deviations for Backward Stochastic Differential Equations Driven by G-Brownian Motion

Ibrahim Dakaou · Abdoulaye Soumana Hima

Received: 27 October 2018 / Revised: 14 March 2020 / Published online: 24 March 2020 © Springer Science+Business Media, LLC, part of Springer Nature 2020

Abstract

In this paper, we consider forward–backward stochastic differential equation driven by G-Brownian motion (G-FBSDEs in short) with small parameter $\varepsilon > 0$. We study the asymptotic behavior of the solution of the backward equation and establish a large deviation principle for the corresponding process.

Keywords Large deviations · G-stochastic differential equation · Backward SDEs · Contraction principle

Mathematics Subject Classification (2010) 60F10 · 60H10 · 60H30

1 Introduction

The large deviation principle (LDP in short) characterizes the limiting behavior, as $\varepsilon \to 0$, of family of probability measures $\{\mu_\varepsilon\}_{\varepsilon > 0}$ in terms of a rate function. Several authors have considered large deviations and obtained different types of applications mainly to mathematical physics. General references on large deviations are: [1,3,15].

Let $X^{x,x,\varepsilon}$ be the diffusion process that is the unique solution of the following stochastic differential equation (SDE in short)

$$X^{x,x,\varepsilon}_t = x + \int_s^t \beta(X^{x,x,\varepsilon}_r)dr + \sqrt{\varepsilon} \int_s^t \sigma(X^{x,x,\varepsilon}_r)dW_r, \ 0 \leq s \leq t \leq T$$

(1.1)
where \( \beta \) is a Lipschitz function defined on \( \mathbb{R}^d \) with values in \( \mathbb{R}^d \), \( \sigma \) is a Lipschitz function defined on \( \mathbb{R}^d \) with values in \( \mathbb{R}^{d \times k} \), and \( W \) is a standard Brownian motion in \( \mathbb{R}^k \) defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The existence and uniqueness of the strong solution \( X_{s,x}^{\varepsilon} \) of (1.1) is standard. Thanks to the work of Freidlin and Wentzell [5], the sequence \( (X_{s,x}^{\varepsilon})_{\varepsilon > 0} \) converges in probability, as \( \varepsilon \) goes to 0, to \( (\varphi_{s,t})_{s \leq t \leq T} \) solution of the following deterministic equation

\[
\varphi_{s,t}^{x,x} = x + \int_{s}^{t} \beta(\varphi_{r}^{x,x})dr, \quad 0 \leq s \leq t \leq T
\]

and satisfies a large deviation principle (LDP in short).

Rainero [13] extended this result to the case of backward stochastic differential equations (BSDEs in short) and Essaky [4] and N’zi and Dakaou [11] to reflected BSDEs.

Gao and Jiang [7] extended the work of Freidlin and Wentzell [5] to stochastic differential equations driven by \( G \)-Brownian motion (\( G \)-SDEs in short). The authors considered the following \( G \)-SDE: for every \( 0 \leq t \leq T \),

\[
X_t^{x,x} = x + \int_{0}^{t} b^\varepsilon(X_r^{x,x})dr + \varepsilon \int_{0}^{t} h^\varepsilon(X_r^{x,x})d\langle B, B \rangle_r/\varepsilon + \varepsilon \int_{0}^{t} \sigma^\varepsilon(X_r^{x,x})dB_r/\varepsilon
\]

and use discrete time approximation to establish LDP for \( G \)-SDEs.

The aim of this paper is to establish LDP for \( G \)-BSDEs. More precisely, we consider the following forward–backward stochastic differential equation driven by \( G \)-Brownian motion: for every \( 0 \leq s \leq t \leq T \),

\[
\begin{aligned}
X_s^{x,x} &= x + \int_{s}^{t} b(X_r^{x,x})dr + \varepsilon \int_{s}^{t} h(X_r^{x,x})d\langle B, B \rangle_r + \varepsilon \int_{s}^{t} \sigma(X_r^{x,x})dB_r \\
Y_s^{x,x} &= \Phi(X_T^{x,x}) + \int_{s}^{t} f(r, X_r^{x,x}, Y_r^{x,x}, Z_r^{x,x})dr - \int_{s}^{t} Z_r^{x,x}dB_r \\
&\quad + \int_{s}^{t} g(r, X_r^{x,x}, Y_r^{x,x}, Z_r^{x,x})d\langle B, B \rangle_r - (K_T^{x,x} - K_s^{x,x})
\end{aligned}
\]

We study the asymptotic behavior of the solution of the backward equation and establish a LDP for the corresponding process.

The remaining part of the paper is organized as follows. In Sect. 2, we present some preliminaries that are useful in this paper. Section 3 is devoted to the large deviations for stochastic differential equations driven by \( G \)-Brownian motion obtained by Gao and Jiang [7]. The large deviations for backward stochastic differential equations driven by \( G \)-Brownian motion are given in Sect. 4.

## 2 Preliminaries

We review some basic notions and results about \( G \)-expectation, \( G \)-Brownian motion and \( G \)-stochastic integrals (see, [8,12], for more details).
Let $\Omega$ be a complete separable metric space, and let $\mathcal{H}$ be a linear space of real-valued functions defined on $\Omega$ satisfying: if $X_i \in \mathcal{H}$, $i = 1, \ldots, n$, then

$$\varphi(X_1, \ldots, X_n) \in \mathcal{H}, \quad \forall \varphi \in \mathcal{C}_i, \text{Lip}(\mathbb{R}^n),$$

where $\mathcal{C}_i, \text{Lip}(\mathbb{R}^n)$ is the space of real continuous functions defined on $\mathbb{R}^n$ such that for some $C > 0$ and $k \in \mathbb{N}$ depending on $\varphi$,

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \quad \forall x, y \in \mathbb{R}^n.$$

**Definition 1** (Sublinear expectation space). A sublinear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

1. Monotonicity: if $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;
2. Constant preservation: $\mathbb{E}[c] = c$;
3. Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$;
4. Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, for all $\lambda \geq 0$.

$(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space.

**Definition 2** (Independence). Fix the sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. A random variable $Y \in \mathcal{H}$ is said to be independent of $(X_1, X_2, \ldots, X_n)$, $X_i \in \mathcal{H}$, if for all $\varphi \in \mathcal{C}_i, \text{Lip}(\mathbb{R}^{n+1})$,

$$\mathbb{E}[\varphi(X_1, X_2, \ldots, X_n, Y)] = \mathbb{E}\left[\mathbb{E}[\varphi(x_1, x_2, \ldots, x_n, Y)] \bigg| (x_1, x_2, \ldots, x_n) = (X_1, X_2, \ldots, X_n)\right].$$

Now we introduce the definition of $G$-normal distribution.

**Definition 3** ($G$-normal distribution). A random variable $X \in \mathcal{H}$ is called $G$-normally distributed, noted by $X \sim \mathcal{N}(0, [\sigma^2, \overline{\sigma}^2])$, $0 \leq \sigma^2 \leq \overline{\sigma}^2$, if for any function $\varphi \in \mathcal{C}_i, \text{Lip}(\mathbb{R})$, the function $u$ defined by $u(t, x) := \mathbb{E}[\varphi(x + \sqrt{t}X)]$, $(t, x) \in [0, \infty) \times \mathbb{R}$, is a viscosity solution of the following $G$-heat equation:

$$\partial_t u - G\left(D_x^2 u\right) = 0, \quad u(0, x) = \varphi(x),$$

where

$$G(a) := \frac{1}{2}(\sigma^2 a^+ - \overline{\sigma}^2 a^-).$$

In multi-dimensional case, the function $G(\cdot) : \mathbb{S}_d \rightarrow \mathbb{R}$ is defined by

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}(\gamma \gamma^T A),$$

where $\mathbb{S}_d$ denotes the space of $d \times d$ symmetric matrices and $\Gamma$ is a given nonempty, bounded and closed subset of $\mathbb{R}^{d \times d}$ which is the space of all $d \times d$ matrices.
Throughout this paper, we consider only the non-degenerate case, i.e., $\sigma^2 > 0$.

Let $\Omega := C([0, \infty))$, which equipped with the raw filtration $\mathcal{F}$ generated by the canonical process $(B_t)_{t \geq 0}$, i.e., $B_t(\omega) = \omega_t$, for $(t, \omega) \in [0, \infty) \times \Omega$. Let us consider the function spaces defined by

$$\text{Lip}(\Omega_T) := \left\{ \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}) : n \geq 1, \right. \left. 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T, \varphi \in C_{l,\text{Lip}}(\mathbb{R}^n) \right\},$$

for $T > 0$,

$$\text{Lip}(\Omega) := \bigcup_{n=1}^{\infty} \text{Lip}(\Omega_n).$$

**Definition 4** (G-Brownian motion and G-expectation). On the sublinear expectation space $(\Omega, \text{Lip}(\Omega), \hat{E})$, the canonical process $(B_t)_{t \geq 0}$ is called a G-Brownian motion if the following properties are verified:

1. $B_0 = 0$
2. For each $t, s \geq 0$, the increment $B_{t+s} - B_t \sim \mathcal{N}(0, [s\sigma^2, s\sigma^2])$ and is independent from $(B_{t_1}, \ldots, B_{t_n})$, for $0 \leq t_1 \leq \cdots \leq t_n \leq t$.

Moreover, the sublinear expectation $\hat{E}$ is called G-expectation.

**Remark 1** For each $\lambda > 0$, $\left(\sqrt{\lambda}B_t/\lambda\right)_{t \geq 0}$ is also a G-Brownian motion. This is the scaling property of G-Brownian motion, which is the same as that of the classical Brownian motion.

**Definition 5** (Conditional G-expectation). For the random variable $\xi \in \text{Lip}(\Omega_T)$ of the following form:

$$\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}), \quad \varphi \in C_{l,\text{Lip}}(\mathbb{R}^n),$$

the conditional G-expectation $\hat{E}_{t_i}[\cdot]$, $i = 1, \ldots, n$, is defined as follows

$$\hat{E}_{t_i}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})] = \tilde{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}),$$

where

$$\tilde{\varphi}(x_1, \ldots, x_i) = \hat{E}\left[\varphi\left(x_1, \ldots, x_i, B_{t_{i+1}} - B_{t_i}, \ldots, B_{t_n} - B_{t_{n-1}}\right)\right].$$

If $t \in (t_i, t_{i+1})$, then the conditional G-expectation $\hat{E}_t[\xi]$ could be defined by reformulating $\xi$ as

$$\xi = \tilde{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_t - B_{t_i}, B_{t_{i+1}} - B_t, \ldots, B_{t_n} - B_{t_{n-1}}), \quad \tilde{\varphi} \in C_{l,\text{Lip}}(\mathbb{R}^{n+1}).$$
For $\xi \in \text{Lip}(\Omega_T)$ and $p \geq 1$, we consider the norm $\|\xi\|_{L^p_G} := \left(\mathbb{E}\left[|\xi|^p\right]\right)^{1/p}$. Denote by $L^p_G(\Omega_T)$ the Banach completion of Lip$(\Omega_T)$ under $\|\cdot\|_{L^p_G}$. It is easy to check that the conditional $G$-expectation $\mathbb{E}_t[\cdot] : \text{Lip}(\Omega_T) \rightarrow \text{Lip}(\Omega_t)$ is a continuous mapping and thus can be extended to $\mathbb{E}_t[\cdot] : L^p_G(\Omega_T) \rightarrow L^p_G(\Omega_t)$.

**Definition 6** (G-martingale). A process $M = (M_t)_{t \in [0,T]}$ with $M_t \in L^1_G(\Omega_t)$, $0 \leq t \leq T$, is called a G-martingale if for all $0 \leq s \leq t \leq T$, we have

$$\mathbb{E}_s[M_t] = M_s.$$ 

The process $M = (M_t)_{t \in [0,T]}$ is called symmetric G-martingale if $-M$ is also a G-martingale.

**Theorem 1** (Representation theorem of $G$-expectation, see [2,10]). There exists a weakly compact set $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$, the set of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that

$$\hat{\mathbb{E}}[\xi] := \sup_{P \in \mathcal{P}} E_P[\xi] \quad \text{for all} \quad \xi \in L^1_G(\Omega_T).$$

$\mathcal{P}$ is called a set that represents $\hat{\mathbb{E}}$.

Let $\mathcal{P}$ be a weakly compact set that represents $\hat{\mathbb{E}}$. For this $\mathcal{P}$, we define the capacity of a measurable set $A$ by

$$\hat{C}(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega_T).$$

A set $A \in \mathcal{B}(\Omega_T)$ is a polar if $\hat{C}(A) = 0$. A property holds quasi-surely (q.s.) if it is true outside a polar set.

An important feature of the $G$-expectation framework is that the quadratic variation $\langle B \rangle$ of the $G$-Brownian motion is no longer a deterministic process, which is given by

$$\langle B \rangle_t := \lim_{\delta(\pi_t^N) \rightarrow 0} \sum_{j=0}^{N-1} (B_{i,j+1}^N - B_{i,j}^N)^2,$$

where $\pi_t^N = \{t_0, t_1, \ldots, t_N\}$, $N = 1, 2, \ldots$, are refining partitions of $[0, t]$. By [12], for all $s, t \geq 0$, $\langle B \rangle_{t+s} - \langle B \rangle_t \in [s\sigma_1^2, s\sigma_2^2]$, q.s.

Let $M^0_G(0, T)$ be the collection of processes in the following form: for a given partition $\pi_T^N := \{t_0, t_1, \ldots, t_N\}$ of $[0, T]$,

$$\eta_t(\omega) := \sum_{j=0}^{N-1} \xi_j(\omega)1_{[t_j, t_{j+1})}(t), \quad (2.1)$$

where $\xi_i \in \text{Lip}(\Omega_i)$, for all $i = 0, 1, \ldots, N - 1$. For $p \geq 1$ and $\eta \in M^0_G(0, T)$, let

$$\|\eta\|_{H^p_G} := \left(\mathbb{E}\left[\left(\int_0^T |\eta_s|^2ds\right)^{p/2}\right]\right)^{1/p}, \quad \|\eta\|_{M^p_G} := \left(\mathbb{E}\left[\int_0^T |\eta_s|^pds\right]\right)^{1/p}$$

and denote
by $H^p_G(0,T)$, $M^p_G(0,T)$ the completions of $M^0_G(0,T)$ under the norms $\| \cdot \|_{H^p_G}$, $\| \cdot \|_{M^p_G}$, respectively.

Let $\mathcal{S}^0_G(0,T) := \{ h(t, B_{1,t}, B_{2,t}, \ldots, B_{t_{n-1},t}) : 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T, \, h \in \mathcal{C}_b, \text{Lip}(\mathbb{R}^{n+1}) \}$, where $\mathcal{C}_b, \text{Lip}(\mathbb{R}^{n+1})$ is the collection of all bounded and Lipschitz functions on $\mathbb{R}^{n+1}$. For $p \geq 1$ and $\eta \in \mathcal{S}^0_G(0,T)$, we set $\eta \| \eta \|_G := \left( \mathbb{E} \left[ \sup_{t \in [0,T]} | \eta_t |^p \right] \right)^{1/p}$. We denote by $\mathcal{S}^p_G(0,T)$ the completion of $\mathcal{S}^0_G(0,T)$ under the norm $\| \cdot \|_G$.

**Definition 7** For $\eta \in M^0_G(0,T)$ of the form (2.1), the Itô integral with respect to $G$-Brownian motion is defined by the linear mapping $\mathcal{I} : M^0_G(0,T) \to L^2_G(\Omega_T)$,

$$\mathcal{I}(\eta) := \int_0^T \eta_t dB_t = \sum_{k=0}^{N-1} \xi_k (B_{k+1} - B_k),$$

which can be continuously extended to $\mathcal{I} : H^1_G(0,T) \to L^2_G(\Omega_T)$. On the other hand, the stochastic integral with respect to $(B_t)_{t \geq 0}$ is defined by the linear mapping $\mathcal{Q} : M^0_G(0,T) \to L^1_G(\Omega_T)$,

$$\mathcal{Q}(\eta) := \int_0^T \eta_t dB_t = \sum_{k=0}^{N-1} \xi_k (B_{k+1} - B_k),$$

which can be continuously extended to $\mathcal{Q} : H^1_G(0,T) \to L^1_G(\Omega_T)$.

**Lemma 1** (BDG-type inequality, see Theorem 2.1 in [6]). Let $p \geq 2$, $\eta \in H^p_G(0,T)$ and $0 \leq s \leq t \leq T$. Then,

$$c_p \sigma^p \mathbb{E} \left[ \left( \int_0^T | \eta_s |^2 ds \right)^{p/2} \right] \leq \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^t \eta_r dB_r \right|^p \right] \leq C_p \sigma^p \mathbb{E} \left[ \left( \int_0^T | \eta_s |^2 ds \right)^{p/2} \right],$$

where $0 < c_p < C_p < \infty$ are constants independent of $\eta$, $\sigma$ and $\overline{\sigma}$.

For $\xi \in \text{Lip}(\Omega_T)$, let

$$\mathcal{Q}(\xi) := \mathbb{E} \left( \sup_{t \in [0,T]} \mathbb{E}_t [\xi] \right).$$

$\mathcal{Q}$ is called the $G$-evaluation.

For $p \geq 1$ and $\xi \in \text{Lip}(\Omega_T)$, define

$$\| \xi \|_{p,\mathcal{Q}} := \left( \mathcal{Q}(\| \xi \|^p) \right)^{1/p}$$
and denote by $L_p^E(\Omega_T)$ the completion of $\text{Lip}(\Omega_T)$ under the norm $\| \cdot \|_{p,E}$.

The following estimate will be used in this paper.

**Theorem 2** (see [14]). For any $\alpha \geq 1$ and $\delta > 0$, we have $L^{\alpha+\delta}_G(\Omega_T) \subset L^\alpha_E(\Omega_T)$. More precisely, for any $1 < \gamma < \beta := (\alpha + \delta)/\alpha$, $\gamma \leq 2$ and for all $\xi \in \text{Lip}(\Omega_T)$, we have

$$\hat{E}\left[ \sup_{t \in [0,T]} \hat{E}_t[|\xi|^\alpha] \right] \leq C \left\{ \left( \hat{E}[|\xi|^{\alpha+\delta}] \right)^{\alpha/(\alpha+\delta)} + \hat{E}[|\xi|^{\alpha+\delta}]^{1/\gamma} \right\},$$

where

$$C = \frac{\gamma}{\gamma - 1} \left( 1 + 14 \sum_{i=1}^\infty i^{-\beta/\gamma} \right).$$

**Remark 2** By $\frac{\alpha}{\alpha+\delta} < \frac{1}{\gamma} < 1$, we have

$$\hat{E}\left[ \sup_{t \in [0,T]} \hat{E}_t[|\xi|^\alpha] \right] \leq 2C \left\{ \left( \hat{E}[|\xi|^{\alpha+\delta}] \right)^{\alpha/(\alpha+\delta)} + \hat{E}[|\xi|^{\alpha+\delta}] \right\}.$$ 

Set

$$C_1 = 2 \inf \left\{ \frac{\gamma}{\gamma - 1} \left( 1 + 14 \sum_{i=1}^\infty i^{-\beta/\gamma} \right) : 1 < \gamma < \beta, \gamma \leq 2 \right\},$$

then

$$\hat{E}\left[ \sup_{t \in [0,T]} \hat{E}_t[|\xi|^\alpha] \right] \leq C_1 \left\{ \left( \hat{E}[|\xi|^{\alpha+\delta}] \right)^{\alpha/(\alpha+\delta)} + \hat{E}[|\xi|^{\alpha+\delta}] \right\}, \quad (2.2)$$

where $C_1$ is a constant only depending on $\alpha$ and $\delta$.

### 3 Large Deviations for $G$-SDEs

In this section, we present the large deviations for $G$-SDEs obtained by Gao and Jiang [7]. The authors use discrete time approximation to obtain their results.

First, we recall the following notations on large deviations under a sublinear expectation.

Let $(\chi,d)$ be a Polish space. Let $(U^\varepsilon, \varepsilon > 0)$ be a family of measurable maps from $\Omega$ into $(\chi,d)$ and let $\delta(\varepsilon)$, $\varepsilon > 0$ be a positive function satisfying $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

A nonnegative function $I$ on $\chi$ is called to be (good) rate function if $\{ x : I(x) \leq \alpha \}$ (its level set) is (compact) closed for all $0 \leq \alpha < \infty$.

The family $\{ \hat{C}(U^\varepsilon \in \cdot \} \}_{\varepsilon > 0}$ is said to satisfy large deviation principle with speed $\delta(\varepsilon)$ and with rate function $I$ if for any measurable closed subset $\mathcal{F} \subset \chi$,

$$\limsup_{\varepsilon \to 0} \delta(\varepsilon) \log \hat{C}(U^\varepsilon \in \mathcal{F}) \leq - \inf_{x \in \mathcal{F}} I(x),$$

and for any measurable open subset $\mathcal{O} \subset \chi$,

$$\liminf_{\varepsilon \to 0} \delta(\varepsilon) \log \hat{C}(U^\varepsilon \in \mathcal{O}) \geq - \inf_{x \in \mathcal{O}} I(x).$$
In [7], for any $\varepsilon > 0$, the authors considered the following random perturbation SDEs driven by $d$-dimensional $G$-Brownian motion $B$

$$X_t^{x,\varepsilon} = x + \int_0^t b^{\varepsilon}(X_r^{x,\varepsilon})dr + \varepsilon \int_0^t h^{\varepsilon}(X_r^{x,\varepsilon})d\langle B, B \rangle_r + \varepsilon \int_0^t \sigma^{\varepsilon}(X_r^{x,\varepsilon})dB_r$$

where $(B, B)$ is treated as a $d \times d$-dimensional vector,

$$b^{\varepsilon} = (b_1^{\varepsilon}, \ldots, b_n^{\varepsilon})^T : \mathbb{R}^n \rightarrow \mathbb{R}^n, \sigma^{\varepsilon} = (\sigma_{i,j}^{\varepsilon}) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$$

and $h^{\varepsilon} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d^2}$.

Consider the following conditions:

(1) $b^{\varepsilon}$, $\sigma^{\varepsilon}$ and $h^{\varepsilon}$ are uniformly bounded;
(2) $b^{\varepsilon}$, $\sigma^{\varepsilon}$ and $h^{\varepsilon}$ are uniformly Lipschitz continuous;
(3) $b^{\varepsilon}$, $\sigma^{\varepsilon}$ and $h^{\varepsilon}$ converge uniformly to $b := b^0$, $\sigma := \sigma^0$ and $h := h^0$, respectively.

Let $\mathcal{C}([0, T], \mathbb{R}^n)$ be the space of $\mathbb{R}^n$-valued continuous functions $\varphi$ on $[0, T]$ and $\mathcal{C}_0([0, T], \mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued continuous functions $\tilde{\varphi}$ on $[0, T]$ with $\tilde{\varphi}_0 = 0$.

Define

$$\mathcal{H}^d := \{ \varphi \in \mathcal{C}_0([0, T], \mathbb{R}^d) : \varphi \text{ is absolutely continuous and} \}

$$

$$\| \varphi \|_{\mathcal{H}^d}^2 := \int_0^T |\varphi'(r)|^2 dr < +\infty \},

$$

$$\mathcal{A} := \{ \eta = \int_0^t \eta'(r) dr; \ \eta' : [0, T] \rightarrow \mathbb{R}^{d \times d} \text{ Borel measurable and} \}

$$

$$\eta'(t) \in \Sigma \text{ for all } t \in [0, T] \}.$$

We recall the following result of a joint large deviation principle for $G$-Brownian motion and its quadratic variation process.

**Theorem 3** (see p. 2225 in [7]). \{ $\mathcal{C} \left( (\varepsilon B_t^{\varepsilon}, \varepsilon \langle B \rangle_t^{\varepsilon}) |_{t \in [0, T]} \in \cdot \right) \}_{\varepsilon > 0}$ satisfies large deviation principle with speed $\varepsilon$ and with rate function

$$J(\varphi, \eta) = \begin{cases} 
\frac{1}{2} \int_0^T (\varphi'(r), (\eta'(r))^{-1} \varphi'(r)) dr, & \text{if } (\varphi, \eta) \in \mathcal{H}^d \times \mathcal{A}, \\
+\infty, & \text{otherwise.}
\end{cases}$$

For any $(\varphi, \eta) \in \mathcal{H}^d \times \mathcal{A}$, let $\Psi(\varphi, \eta) \in \mathcal{C}([0, T], \mathbb{R}^n)$ be the unique solution of the following ordinary differential equation (ODE in short)

$$\Psi(\varphi, \eta)(t) = x + \int_0^t b(\Psi(\varphi, \eta)(r))dr + \varepsilon \int_0^t \sigma(\Psi(\varphi, \eta)(r))\varphi'(r)dr 

$$

$$+ \int_0^t h(\Psi(\varphi, \eta)(r))\eta'(r)dr.$$
Theorem 4 (see p. 2233 in [7]). Let (H1), (H2) and (H3) hold. Then for any closed subset \( F \) and any open subset \( O \) in \((C_0([0, T], \mathbb{R}^d), \| \cdot \|) \times (C_0([0, T], \mathbb{R}^d), \| \cdot \|) \times (C_0([0, T], \mathbb{R}^n), \| \cdot \|)\),
\[
\lim_{\varepsilon \to 0} \sup \varepsilon \log \mathcal{C} \left( (\varepsilon B_{t/\varepsilon}, \varepsilon (B)_{t/\varepsilon}, X^{t,\varepsilon} - x) \mid t \in [0, T] \right) \in F \right) \leq - \inf_{(\phi, \eta, \psi) \in F} \mathcal{I}(\phi, \eta, \psi),
\]
and
\[
\lim_{\varepsilon \to 0} \inf \varepsilon \log \mathcal{C} \left( (\varepsilon B_{t/\varepsilon}, \varepsilon (B)_{t/\varepsilon}, X^{t,\varepsilon} - x) \mid t \in [0, T] \right) \in O \right) \geq - \inf_{(\phi, \eta, \psi) \in O} \mathcal{I}(\phi, \eta, \psi),
\]
where
\[
\mathcal{I}(\phi, \eta, \psi) = \left\{ J(\phi, \eta), \text{ if } (\phi, \eta) \in \mathbb{R}^d \times \mathbb{A}, x + \psi = \Psi(\phi, \eta) \right\} + \infty, \text{ otherwise.}
\]

For \( 0 \leq \alpha < 1 \) given and \( n \geq 1 \), for each \( \psi \in C_0([0, T], \mathbb{R}^n) \), set
\[
\| \psi \|_{\alpha} := \sup_{s, t \in [0, T]} \frac{\| s - t \|^{\alpha}}{\| s - t \|^{\alpha}}
\]
and
\[
C_0^\alpha([0, T], \mathbb{R}^n)
:= \left\{ \psi \in C_0([0, T], \mathbb{R}^n) : \lim_{\delta \to 0} \sup_{|s - t| < \delta} \frac{\| s - t \|^{\alpha}}{\| s - t \|^{\alpha}} = 0, \| \psi \|_{\alpha} < \infty \right\}.
\]

Theorem 5 (see p. 2227 in [7]). Let \( 0 \leq \alpha < 1/2 \) and let (H1), (H2) and (H3) hold. Then for any closed subset \( F \) and any open subset \( O \) in \((C_0^\alpha([0, T], \mathbb{R}^n), \| \cdot \|_{\alpha})\),
\[
\lim_{\varepsilon \to 0} \sup \varepsilon \log \mathcal{C} \left( (X_{t}^{x, \varepsilon} - x) \mid t \in [0, T] \right) \in F \right) \leq - \inf_{\psi \in F} I(\psi),
\]
and
\[
\lim_{\varepsilon \to 0} \inf \varepsilon \log \mathcal{C} \left( (X_{t}^{x, \varepsilon} - x) \mid t \in [0, T] \right) \in O \right) \geq - \inf_{\psi \in O} I(\psi),
\]
where
\[
I(\psi) = \inf \left\{ J(\phi, \eta) : \psi = \Psi(\phi, \eta) - x \right\}.
\]

We immediately have the following result which will be used in the following section.

Corollary 1 Let (H1), (H2) and (H3) hold. Then for any closed subset \( F \) and any open subset \( O \) in \( C_0([0, T], \mathbb{R}^n) \),
\[
\lim_{\varepsilon \to 0} \sup \varepsilon \log \mathcal{C} \left( (X_{t}^{x, \varepsilon} - x) \mid t \in [0, T] \right) \in F \right) \leq - \inf_{\widehat{\psi} \in F} A(\widehat{\psi}),
\]
and
\[
\liminf_{\varepsilon \to 0} \varepsilon \log \tilde{C} \left( (X_t^{x,\varepsilon} - x) \ | _{t \in [0,T]} \in \Theta \right) \geq - \inf_{\tilde{\varphi} \in \Theta} \Lambda(\tilde{\varphi}),
\]
where
\[
\Lambda(\tilde{\varphi}) = \inf \left\{ J(\phi, \eta) : x + \tilde{\varphi} = \Psi(\phi, \eta) \right\}
\]

In the following section, we consider the following G-SDE: for every \( s \leq t \leq T \), \( x \in \mathbb{R}^n \),
\[
X_t^{s,x,\varepsilon} = x + \int_s^t b(X_r^{s,x,\varepsilon})dr + \varepsilon \int_s^t h(X_r^{s,x,\varepsilon})dB_r + \varepsilon \int_s^t \sigma(X_r^{s,x,\varepsilon})dB_r,
\]
where \( b, \sigma \) and \( h \) are bounded. In order to use the large deviation principle obtained by Gao and Jiang [7], we will transform the G-SDE (3.1) in the following form:
\[
\tilde{X}_t^{s,x,\varepsilon} = x + \int_s^t b^{\varepsilon}(\tilde{X}_r^{s,x,\varepsilon})dr + \varepsilon \int_s^t h^{\varepsilon}(\tilde{X}_r^{s,x,\varepsilon})d\tilde{B}_r + \varepsilon \int_s^t \sigma^{\varepsilon}(\tilde{X}_r^{s,x,\varepsilon})d\tilde{B}_r,
\]
where \( \tilde{B}_t := \frac{1}{\sqrt{\varepsilon}}B_t, b^{\varepsilon} := \varepsilon b, h^{\varepsilon} := \varepsilon h \) and \( \sigma^{\varepsilon} := \sqrt{\varepsilon} \sigma \).

4 Large Deviations for G-BSDEs

Hu et al. [8] obtained the existence, uniqueness and a priori estimates of the following backward stochastic differential equation driven by G-Brownian motion
\[
Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T g(r, Y_r, Z_r)d\langle B \rangle_r - \int_t^T Z_r dB_r - (K_T - K_t),
\]
where \( K \) is a decreasing G-martingale, under standard Lipschitz conditions on \( f(r, y, z), g(r, y, z) \) in \( (y, z) \) and the integrability condition on \( \xi \). The unique solution of the BSDE (4.1) is the triple \((Y, Z, K)\). The solution of an SDE is one process, say \( X \). The solution of a “traditional” BSDE is a pair \((Y, Z)\), the solution of a BSDE driven by a G-Brownian motion is a triplet.

To establish large deviation principle for G-BSDEs, we consider the following forward–backward stochastic differential equation driven by G-Brownian motion (we use Einstein convention): for every \( s \leq t \leq T \), \( x \in \mathbb{R}^n \),
\[
dX_t^{x,\varepsilon} = b(X_t^{x,\varepsilon})dt + \varepsilon h_{ij}(X_t^{x,\varepsilon})dB^j + \varepsilon \sigma_{ij}(X_t^{x,\varepsilon})dB_t^j, X_s^{x,\varepsilon} = x,
\]
\[
Y_t^{s,x,\varepsilon} = \Phi(X_T^{s,x,\varepsilon}) + \int_t^T f(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon})dr
+ \int_t^T g_{ij}(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon})dB^i_r - \int_t^T Z_r^{s,x,\varepsilon}dB_r - (K_T^{s,x,\varepsilon} - K_s^{s,x,\varepsilon}),
\]
where

\[ b, h_{ij}, \sigma_j : \mathbb{R}^n \to \mathbb{R}^n; \ \Phi : \mathbb{R}^n \to \mathbb{R}; \ f, g_{ij} : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \]

are deterministic functions and satisfy the following assumptions:

(A0) \( b, \sigma \) and \( h \) are bounded, i.e., there exists a constant \( L > 0 \) such that

\[
\sup_{x \in \mathbb{R}^n} \max \left\{ |b(x)|, \|\sigma(x)\|_{HS}, \|h(x)\|_{HS} \right\} \leq L,
\]

where \( \|A\|_{HS} := \sqrt{\sum_{ij} a_{ij}^2} \) is the Hilbert-Schmidt norm of a matrix \( A = (a_{ij}) \).

(A1) \( h_{ij} = h_{ji} \) and \( g_{ij} = g_{ji} \) for \( 1 \leq i, j \leq d \);

(A2) \( f \) and \( g_{ij} \) are continuous in \( t \);

(A3) There exist a positive integer \( m \) and a constant \( L > 0 \) such that

\[
|b(x) - b(x')| + \sum_{i,j=1}^d |h_{ij}(x) - h_{ij}(x')| \\
+ \sum_{j=1}^d |\sigma_j(x) - \sigma_j(x')| \leq L|x - x'|,
\]

\[
|\Phi(x) - \Phi(x')| \leq L(1 + |x|^m + |x'|^m)|x - x'|,
\]

\[
|f(t, x, y, z) - f(t, x', y', z')| + \sum_{i,j=1}^d |g_{ij}(t, x, y, z) - g_{ij}(t, x', y', z')| \\
\leq L \left( 1 + |x|^m + |x'|^m \right)|x - x'| + |y - y'| + |z - z'|.
\]

It follows from [8,12] that, under the assumptions (A0)–(A3), the G-BSDE (4.2) has a unique solution \( \{(Y^{s,x,e}_t, Z^{s,x,e}_t, K^{s,x,e}_t) : s \leq t \leq T \} \). Moreover, for any \( \alpha > 1 \), we have \( Y^{s,x,e} \in \mathcal{S}^\alpha_G(0, T), Z^{s,x,e} \in H^\alpha_G(0, T) \) and \( K^{s,x,e}_T \) is a decreasing G-martingale with \( K^{s,x,e}_s = 0 \) and \( K^{s,x,e}_T \in L^\alpha_G(\Omega_T) \).

We consider the following deterministic system: for every \( s \leq t \leq T, x \in \mathbb{R}^n \),

\[
d\psi^{s,x}_t = b(\psi^{s,x}_t)dt, \ \psi^{s,x}_s = x,
\]

\[
\psi^{s,x}_t = \Phi(\psi^{s,x}_T) + \int_t^T f(r, \psi^{s,x}_r, \psi^{s,x}_r, 0) dr \\
+ 2 \int_t^T G(g(r, \psi^{s,x}_r, \psi^{s,x}_r, 0)) dr. \tag{4.3}
\]

Lemma 2 Let (A0), (A1) and (A3) hold. Then

1. Let \( p \geq 2 \). For any \( \varepsilon \in (0, 1] \), there exists a constant \( C_p > 0 \), independent of \( \varepsilon \), such that
\[
\mathbb{E}\left(\sup_{s \leq t \leq T} |X_{t}^{s,x,\varepsilon} - \varphi_{t}^{s,x}|^p\right) \leq C_{p}\varepsilon^p.
\]  (4.4)

2. Moreover, \(\{\tilde{C}\left(\left((X_{t}^{s,x,\varepsilon} - x)\right)_{t \in [s,T]}\right)\}_{\varepsilon > 0}\) satisfies a large deviation principle with speed \(\varepsilon\) and with rate function

\[
\Lambda(\tilde{\varphi}) = \inf \left\{ J(\phi, \eta) : x + \tilde{\varphi} = \tilde{\Psi}(\phi, \eta) \right\},
\]

where \(\tilde{\Psi}(\phi, \eta) \in \mathcal{C}([s,T], \mathbb{R}^n)\) be the unique solution of the following ODE

\[
\tilde{\Psi}(\phi, \eta)(t) = x + \int_{s}^{t} b(\tilde{\Psi}(\phi, \eta)(r))dr.
\]

**Proof**

1. Let \(u \in [s,T]\), we have

\[
X_{u}^{s,x,\varepsilon} - \varphi_{u}^{s,x} = \int_{s}^{u} \left( b(X_{r}^{s,x,\varepsilon}) - b(\varphi_{r}^{s,x}) \right) dr + \varepsilon \int_{s}^{u} h(X_{r}^{s,x,\varepsilon})d\langle B, B \rangle_{r}
\]

\[
+ \varepsilon \int_{s}^{u} \sigma(X_{r}^{s,x,\varepsilon})dB_r.
\]

Then, there exists a constant \(C_{p} > 0\),

\[
|X_{u}^{s,x,\varepsilon} - \varphi_{u}^{s,x}|^p \leq C_{p}\left\{ \int_{s}^{u} |b(X_{r}^{s,x,\varepsilon}) - b(\varphi_{r}^{s,x})|^p dr
\]

\[
+ \varepsilon^p \int_{s}^{u} \|h(X_{r}^{s,x,\varepsilon})\|^p dr
\]

\[
+ \varepsilon^2 \int_{s}^{u} \sigma(X_{r}^{s,x,\varepsilon})d|B_r|\}
\]

\[
\leq C_{p}\left\{ \int_{s}^{u} |X_{r}^{s,x,\varepsilon} - \varphi_{r}^{s,x}|^p dr + \varepsilon^p
\]

\[
+ \varepsilon^p \int_{s}^{u} \sigma(X_{r}^{s,x,\varepsilon})d|B_r|\}
\]

For \(t \in [s,T]\),

\[
\sup_{s \leq u \leq t} |X_{u}^{s,x,\varepsilon} - \varphi_{u}^{s,x}|^p \leq C_{p}\left\{ \sup_{s \leq u \leq t} \int_{s}^{u} |X_{r}^{s,x,\varepsilon} - \varphi_{r}^{s,x}|^p dr + \varepsilon^p
\]

\[
+ \varepsilon^p \sup_{s \leq u \leq t} \int_{s}^{u} \sigma(X_{r}^{s,x,\varepsilon})d|B_r|\}
\]

\[
\leq C_{p}\left\{ \int_{s}^{t} \sup_{s \leq u \leq t} |X_{u}^{s,x,\varepsilon} - \varphi_{u}^{s,x}|^p dr + \varepsilon^p
\]

\[
+ \varepsilon^p \sup_{s \leq u \leq t} \int_{s}^{u} \sigma(X_{r}^{s,x,\varepsilon})d|B_r|\}
\]

\[
\square \text{ Springer}
\]

\[
\text{Springer}
\]
So taking the $G$-expectation, it follows from the BDG inequality that
\[
\hat{E}\left[ \sup_{s \leq u \leq t} |X_u^{s,x,\varepsilon} - \varphi_u^{s,x}|^p \right] \leq C_p \varepsilon^p + C_p \int_s^t \hat{E}\left[ \sup_{s \leq u \leq r} |X_u^{s,x,\varepsilon} - \varphi_u^{s,x}|^p \right] dr.
\]
Therefore, by Gronwall’s inequality,
\[
\hat{E}\left( \sup_{s \leq u \leq T} |X_u^{s,x,\varepsilon} - \varphi_u^{s,x}|^p \right) \leq C_p \varepsilon^p.
\]

2. Set $\tilde{B}_t = \frac{1}{\sqrt{\varepsilon}} B_{t/\varepsilon}$. Thanks to Remark 1, $\tilde{B}$ is a $G$-Brownian motion. Then, we have $B_t = \sqrt{\varepsilon} \tilde{B}_{t/\varepsilon}$, $(B, B)_t = \varepsilon (\tilde{B}, \tilde{B})_{t/\varepsilon}$. Therefore, by the uniqueness of the solution of the $G$-SDEs, it is easy to check that $\{X_t^{s,x,\varepsilon} : s \leq t \leq T\}$ is the solution of the following $G$-SDE:
\[
\tilde{X}_t^{s,x,\varepsilon} = x + \int_s^t b^\varepsilon(\tilde{X}_r^{s,x,\varepsilon}) d\tilde{r} + \varepsilon \int_s^t h^\varepsilon(\tilde{X}_r^{s,x,\varepsilon}) d(\tilde{B}_r, \tilde{B})_r + \varepsilon \int_s^t \sigma^\varepsilon(\tilde{X}_r^{s,x,\varepsilon}) d\tilde{B}_r,
\]
where $b^\varepsilon$, $h^\varepsilon$ and $\sigma^\varepsilon$ have already been defined at the end of Sect. 3. Therefore, in view of assumption (A0), the proof follows by virtue of Corollary 1.

Proposition 1 Let $p \geq 2$. For any $\varepsilon \in (0, 1)$, we have
\[
\hat{E}\left[ \sup_{s \leq t \leq T} |X_t^{s,x,\varepsilon}|^p \right] \leq C (1 + |x|^p),
\]
where the constant $C$ depends on $L$, $G$, $p$, $n$ and $T$.

Proof By Proposition 4.1 in [9], there exists a constant $C > 0$ such that
\[
\hat{E}_s\left[ \sup_{s \leq t \leq T} \left| X_t^{s,x,\varepsilon} - x \right|^p \right] \leq C \left( 1 + |x|^p \right).
\]
Then
\[
\hat{E}\left[ \sup_{s \leq t \leq T} \left| X_t^{s,x,\varepsilon} - x \right|^p \right] \leq C \left( 1 + |x|^p \right),
\]
which implies the desired result.

Theorem 6 Let (A0)–(A3) hold. For any $\varepsilon \in (0, 1)$, there exists a constant $C > 0$, independent of $\varepsilon$, such that
\[
\hat{E}\left( \sup_{s \leq t \leq T} \left| Y_t^{s,x,\varepsilon} - \psi_t^{s,x} \right|^2 \right) \leq C \varepsilon^2.
\]
Proof We consider the following $G$-BSDE: for every $s \leq t \leq T$, $x \in \mathbb{R}^n$,

$$Y^x_t = \Phi(\varphi^x_T) + \int_t^T f(r, \varphi^x_r, Y^x_r, Z^x_r)dr$$

$$+ \int_t^T g_{ij}(r, \varphi^x_r, Y^x_r, Z^x_r)dB^i_r - \int_t^T Z^x_r dB_r - (K^x_T - K^x_t).$$

(4.6)

Let $M^{s,x}$ be the following decreasing $G$-martingale:

$$M^{s,x}_t := \int_s^t g_{ij}(r, \varphi^x_r, \psi^x_r, 0)d(B^i_r, B^j_r) - 2\int_s^t G(r, \varphi^x_r, \psi^x_r, 0) dr.$$

Thanks to Eq. (4.3) and the uniqueness of the solution of the $G$-BSDEs, it is easy to check that $\{(\psi^x_t, 0, M^{t,x}_t) : s \leq t \leq T\}$ is the solution of the $G$-BSDE (4.6).

So, by Proposition 2.16 in [9], there exists a constant $C > 0$ such that

$$\mathbb{E}\left[ \sup_{s \leq t \leq T} |Y^{s,x,e}_t - \psi^{s,x}_t|^2 \right] \leq C \left\{ \mathbb{E}\left[ \sup_{t \in [s, T]} \mathbb{E}_t[|\Phi(X^{s,x,e}_T) - \Phi(\psi^{s,x}_T)|^2] \right] \right. \right.$$

$$\left. + \left( \mathbb{E}\left[ \sup_{t \in [s, T]} \mathbb{E}_t\left[ \left( \int_s^T \tilde{h}_r dr \right)^4 \right] \right] \right)^{1/2} \right.$$\n
$$\left. + \mathbb{E}\left[ \sup_{t \in [s, T]} \mathbb{E}_t\left[ \left( \int_s^T \tilde{h}_r dr \right)^4 \right] \right] \right\},$$

where

$$\tilde{h}_t = |f(r, X^{s,x,e}_r, \psi^{s,x}_r, 0) - f(r, \varphi^{s,x}_r, \psi^{s,x}_r, 0)|$$

$$+ \sum_{i,j=1}^d |g_{ij}(r, X^{s,x,e}_r, \psi^{s,x}_r, 0) - g_{ij}(r, \varphi^{s,x}_r, \psi^{s,x}_r, 0)|.$$
Then, by Hölder’s inequality, (4.4) in Lemma 2 and (4.5) in Proposition 1, we can get

\[
D_1 = \mathbb{E}\left[ \sup_{t \in [s, T]} (1 + |X_t^s, \epsilon|^m + |\varphi_T^s|^m) |X_t^s, \epsilon - \varphi_T^s|^2 \right],
\]

\[
D_2 = \mathbb{E}\left[ \sup_{t \in [s, T]} \left( \int_s^T (1 + |X_r^s, \epsilon|^m + |\varphi_r^s|^m) |X_r^s, \epsilon - \varphi_r^s|^4 |dr\right)^4 \right].
\]

By Theorem 2 and (2.2) in Remark 2, for any \( \delta_1 > 0 \), we get

\[
D_1 \leq C_1 \left\{ \mathbb{E}\left[ (1 + |X_T^s, \epsilon|^m + |\varphi_T^s|^m)^{2+\delta_1} |X_T^s, \epsilon - \varphi_T^s|^{2+\delta_1} \right] \right\}^{\frac{2}{2+\delta_1}}
+ \mathbb{E}\left[ (1 + |X_T^s, \epsilon|^m + |\varphi_T^s|^m)^{2+\delta_1} |X_T^s, \epsilon - \varphi_T^s|^{2+\delta_1} \right]
= C_1 \left\{ D_{1,1} + D_{1,2} \right\}.
\]

Similarly, for any \( \delta_2 > 0 \), we get

\[
D_2 \leq C_2 \left\{ \mathbb{E}\left[ \left( \int_s^T (1 + |X_r^s, \epsilon|^m + |\varphi_r^s|^m) |X_r^s, \epsilon - \varphi_r^s|^4 |dr\right)^{4+\delta_2} \right] \right\}^{\frac{4}{4+\delta_2}}
+ \mathbb{E}\left[ \left( \int_s^T (1 + |X_r^s, \epsilon|^m + |\varphi_r^s|^m) |X_r^s, \epsilon - \varphi_r^s|^4 |dr\right)^{4+\delta_2} \right]
= C_2 \left\{ D_{2,1} + D_{2,2} \right\}.
\]

Then, by Hölder’s inequality, (4.4) in Lemma 2 and (4.5) in Proposition 1, we can get

\[
D_{1,2} \leq C \left\{ \mathbb{E}\left[ (1 + |X_T^s, \epsilon|^m + |\varphi_T^s|^m)^{4+2\delta_1} \right] \right\}^{1/2}
\times \left\{ \mathbb{E}\left[ |X_T^s, \epsilon - \varphi_T^s|^{4+2\delta_1} \right] \right\}^{1/2}
\leq C \varepsilon^{2+\delta_1}.
\]

Thus

\[
D_1 \leq C \left( \varepsilon^2 + \varepsilon^{2+\delta_1} \right). \quad (4.8)
\]

Furthermore

\[
\mathbb{E}\left[ \left( \int_s^T (1 + |X_r^s, \epsilon|^m + |\varphi_r^s|^m) |X_r^s, \epsilon - \varphi_r^s|^4 |dr\right)^{4+\delta_2} \right]
\leq C \mathbb{E}\left[ \left( 1 + \sup_{r \in [s, T]} |X_r^s, \epsilon|^m + \sup_{r \in [s, T]} |\varphi_r^s|^m \right)^{4+\delta_2} \left( \sup_{r \in [s, T]} |X_r^s, \epsilon - \varphi_r^s|^4 \right)^{4+\delta_2} \right]
\leq C \left\{ \mathbb{E}\left[ \left( 1 + \sup_{r \in [s, T]} |X_r^s, \epsilon|^m + \sup_{r \in [s, T]} |\varphi_r^s|^m \right)^{8+2\delta_2} \right] \right\}^{1/2}
\times \left\{ \mathbb{E}\left[ \left( \sup_{r \in [s, T]} |X_r^s, \epsilon - \varphi_r^s|^{8+2\delta_2} \right) \right] \right\}^{1/2}
\]
Therefore \( D_2 \leq C^2 (\varepsilon^4 + \varepsilon^4 + \delta_2) \). (4.9)

So, by virtue of (4.7), (4.8) and (4.9), we have

\[
\mathbb{E} \left[ \sup_{s \leq t \leq T} |Y_t^{s,x,\varepsilon} - \psi_t^{s,x}|^2 \right] \leq C \varepsilon^2 \left( 1 + \varepsilon^{\delta_1/2} + 1 + \varepsilon^{\delta_2/2} + \varepsilon^2 + \varepsilon^{2+\delta_2} \right),
\]

which leads to the end of the proof. \( \square \)

We have an immediate consequence of Theorem 6.

**Corollary 2** For any \( \varepsilon \in (0, 1] \) and all \( x \) in a compact subset of \( \mathbb{R}^n \), there exists a constant \( C > 0 \), independent of \( s, x \) and \( \varepsilon \), such that

\[
\mathbb{E} \left( \sup_{s \leq t \leq T} |Y_t^{s,x,\varepsilon} - \psi_t^{s,x}|^2 \right) \leq C \varepsilon^2.
\]

**Theorem 7** Let (A0)–(A3) hold. For any \( \varepsilon \in (0, 1] \), there exists a constant \( C > 0 \), independent of \( \varepsilon \), such that

\[
\mathbb{E} \left[ \int_s^T |Z_r^{s,x,\varepsilon}|^2 \, dr \right] + \mathbb{E} \left( \sup_{s \leq t \leq T} |K_t^{s,x,\varepsilon} - M_t^{s,x}|^2 \right) \leq C \varepsilon^2,
\]

where \( M_t^{s,x} \) is the following decreasing \( G \)-martingale:

\[
M_t^{s,x} = \int_s^t g_{ij}(r, \varphi_r^{s,x}, \psi_r^{s,x}, 0) \, d\langle B^i, B^j \rangle_r - 2 \int_s^t G \left( g(r, \varphi_r^{s,x}, \psi_r^{s,x}, 0) \right) \, dr.
\]

**Proof** Applying Itô’s formula to \( |Y_t^{s,x,\varepsilon} - \psi_t^{s,x}|^2 \), we have

\[
|Y_t^{s,x,\varepsilon} - \psi_t^{s,x}|^2 + \int_s^T |Z_r^{s,x,\varepsilon}|^2 \, d\langle B \rangle_r
\]

\[
= \left| \Phi(X_T^{s,x,\varepsilon}) - \Phi(\psi_T^{s,x}) \right|^2
\]

\[
+ 2 \int_s^T (Y_r^{s,x,\varepsilon} - \psi_r^{s,x}) \left( f(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon}) - f(r, \varphi_r^{s,x}, \psi_r^{s,x}, 0) \right) \, dr
\]

\[
+ 2 \int_s^T (Y_r^{s,x,\varepsilon} - \psi_r^{s,x}) \left( g(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon}) - g(r, \varphi_r^{s,x}, \psi_r^{s,x}, 0) \right) \, d\langle B \rangle_r
\]

\[
- 2 \int_s^T (Y_r^{s,x,\varepsilon} - \psi_r^{s,x}) Z_r^{s,x,\varepsilon} \, dB_r
\]

\[
- 2 \int_s^T (Y_r^{s,x,\varepsilon} - \psi_r^{s,x}) \, d(K_r^{s,x,\varepsilon} - M_r^{s,x}).
\]
Therefore, in view of assumption (A3), we have 
\[
\int_s^T |Z_r^{s,x,\varepsilon}|^2 d(B)_r
\leq |\Phi(X_T^{s,x,\varepsilon}) - \Phi(\psi_{s,x}^{s,x})|^2 \\
+ 2L(1 + d^2 \sigma^2) \int_s^T (1 + |X_r^{s,x,\varepsilon}|^m + |\varphi_r^{s,x,\varepsilon}|^m) |Y_r^{s,x,\varepsilon} - \psi_r^{s,x,\varepsilon}| |X_r^{s,x,\varepsilon} - \varphi_r^{s,x,\varepsilon}| dr \\
+ 2L(1 + d^2 \sigma^2) \int_s^T |Y_r^{s,x,\varepsilon} - \psi_r^{s,x,\varepsilon}|^2 dr \\
+ 2L(1 + d^2 \sigma^2) \int_s^T |Y_r^{s,x,\varepsilon} - \psi_r^{s,x,\varepsilon}| |Z_r^{s,x,\varepsilon}| dr \\
+ 2 \left| \int_s^T (Y_r^{s,x,\varepsilon} - \psi_r^{s,x,\varepsilon}) Z_r^{s,x,\varepsilon} dB_r \right| \\
+ 2C \sup_{s \leq r \leq T} |Y_r^{s,x,\varepsilon} - \psi_r^{s,x,\varepsilon}| \sup_{s \leq r \leq T} |K_r^{s,x,\varepsilon} - M_r^{s,x,\varepsilon}|.
\]

On the other hand,
\[
(K_t^{s,x,\varepsilon} - M_t^{s,x,\varepsilon}) = (Y_t^{s,x,\varepsilon} - \psi_t^{s,x,\varepsilon}) - (Y_s^{s,x,\varepsilon} - \psi_s^{s,x,\varepsilon}) \\
+ \int_s^t \tilde{f}_r dr + \int_s^t \tilde{g}_r d(B)_r - \int_s^t Z_r^{s,x,\varepsilon} dB_r.
\]

where
\[
\tilde{f}_r = |f(r, X_r^{s,x,\varepsilon}, \psi_r^{s,x,\varepsilon}, 0) - f(r, \varphi_r^{s,x,\varepsilon}, \psi_r^{s,x,\varepsilon}, 0)| \\
\tilde{g}_r = \sum_{i,j=1}^d |g_{ij}(r, X_r^{s,x,\varepsilon}, \psi_r^{s,x,\varepsilon}, 0) - g_{ij}(r, \varphi_r^{s,x,\varepsilon}, \psi_r^{s,x,\varepsilon}, 0)|.
\]

Thus
\[
|K_t^{s,x,\varepsilon} - M_t^{s,x,\varepsilon}| \leq \left\{ 2 \sup_{s \leq r \leq T} |Y_r^{s,x,\varepsilon} - \psi_r^{s,x,\varepsilon}| + \int_s^t \tilde{f}_r dr \\
+ \int_s^t \tilde{g}_r d(B)_r + \left| \int_s^t Z_r^{s,x,\varepsilon} dB_r \right| \right\}. \tag{4.10}
\]

Then
\[
\int_s^T |Z_r^{s,x,\varepsilon}|^2 d(B)_r \\
\leq C_1 \sup_{s \leq r \leq T} |Y_r^{s,x,\varepsilon} - \psi_r^{s,x,\varepsilon}|^2 \\
+ C_2 \int_s^T (1 + |X_r^{s,x,\varepsilon}|^m + |\varphi_r^{s,x,\varepsilon}|^m) |Y_r^{s,x,\varepsilon} - \psi_r^{s,x,\varepsilon}| |X_r^{s,x,\varepsilon} - \varphi_r^{s,x,\varepsilon}| dr
\]
Then, taking \( \lambda \)

Therefore, by the same arguments as above, we get

The proof is complete.

As a consequence of Theorems 6 and 7, we get

\[
\hat{\mathbb{E}} \left[ \sup_{s \leq t \leq T} |K_t^{s,x,e} - M_t^{s,x}|^2 \right] \leq C \varepsilon^2.
\]

Theorem 3

As a consequence of Theorems 6 and 7, we get

\[
\hat{\mathbb{E}} \left[ \sup_{s \leq t \leq T} |Y_t^{s,x,e} - \psi_t^{s,x}|^2 + \int_s^T |Z_r^{s,x,e}|^2 \,dr + \sup_{s \leq t \leq T} |K_t^{s,x,e} - M_t^{s,x}|^2 \right] \leq C \varepsilon^2,
\]

where \( C \) is a positive constant and then the solution \( \{(Y_t^{s,x,e}, Z_t^{s,x,e}, K_t^{s,x,e}): s \leq t \leq T\} \) of the G-BSDE (4.2) converges to \( \{(|\psi_t^{s,x}, 0, M_t^{s,x}): s \leq t \leq T\} \) where \( \psi_t^{s,x} \) is the solution of the following backward ODE:

\[
\psi_t^{s,x} = \Phi(\varphi_T^{s,x}) + \int_t^T f(r, \varphi_r^{s,x}, \psi_r^{s,x}, 0) \,dr + 2 \int_t^T g(r, \varphi_r^{s,x}, \psi_r^{s,x}, 0) \,dr.
\]
and $M^{s,x}$ is the following decreasing $G$-martingale:

$$M^{s,x}_t = \int_s^t g_{ij}(r, \varphi^{s,x}_r, \psi^{s,x}_r, 0) d\langle B^i, B^j \rangle_r - 2 \int_s^t G(g(r, \varphi^{s,x}_r, \psi^{s,x}_r, 0)) dr.$$ 

We recall a very important result in large deviation theory, used to transfer a LDP from one space to another.

**Lemma 3** (Contraction principle). Let $\{\mu_\varepsilon\}_{\varepsilon > 0}$ be a family of probability measures that satisfies the large deviation principle with a good rate function $\Lambda$ on a Hausdorff topological space $\chi$, and for $\varepsilon \in (0, 1]$, let $f_\varepsilon : \chi \to \Upsilon$ be continuous functions, with $(\Upsilon, d)$ a metric space. Assume that there exists a measurable map $f : \chi \to \Upsilon$ such that for any compact set $K \subset \chi$,

$$\limsup_{\varepsilon \to 0} \sup_{x \in K} d(f_\varepsilon(x), f(x)) = 0. \quad (4.11)$$

Suppose further that $\{\mu_\varepsilon\}_{\varepsilon > 0}$ is exponentially tight. Then the family of probability measures $\{\mu_\varepsilon \circ f_\varepsilon^{-1}\}_{\varepsilon > 0}$ satisfies the LDP in $\Upsilon$ with the good rate function

$$\Pi(y) = \inf \left\{ \Lambda(x) : x \in \chi, y = f(x) \right\}.$$ 

**Proof** First, observe that the condition (4.11) implies that for any compact set $K \subset \chi$, the function $f$ is continuous on $K \subset \chi$ (consequently that $f$ is continuous everywhere).

Since $\{\mu_\varepsilon\}_{\varepsilon > 0}$ is exponentially tight, for every $\alpha < \infty$, there exists a compact set $K_{\alpha} \subset \chi$ such that

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(K_{\alpha}^c) < -\alpha.$$ 

For every $\delta > 0$, set

$$\Gamma_{\varepsilon, \delta} = \{ x \in \chi : d(f_\varepsilon(x), f(x)) > \delta \}.$$ 

We have

$$\mu_\varepsilon(\Gamma_{\varepsilon, \delta}) \leq \mu_\varepsilon(\Gamma_{\varepsilon, \delta} \cap K_{\alpha}) + \mu_\varepsilon(K_{\alpha}^c).$$

Given $\delta > 0$, the first term on the right is zero for $\varepsilon$ small enough, so that

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(\Gamma_{\varepsilon, \delta}) \leq \limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(K_{\alpha}^c) < -\alpha$$

and letting $\alpha \to \infty$, we obtain

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(\Gamma_{\varepsilon, \delta}) = -\infty.$$ 

Therefore, the lemma follows from Corollary 4.2.21 p. 133 in Dembo and Zeitouni [1].

$\square$ Springer
Now consider
\[ u^\varepsilon(t, x) = Y_t^{t,x,\varepsilon}, \quad (t, x) \in [0, T] \times \mathbb{R}^n. \] (4.12)

In [9] it is shown that \( u^\varepsilon \) is a viscosity solution of the following nonlinear partial differential equation (PDE in short):
\[
\begin{aligned}
\partial_t u^\varepsilon + \mathcal{L}^\varepsilon \left( D_x^2 u^\varepsilon, D_x u^\varepsilon, u^\varepsilon, x, t \right) &= 0, \\
u^\varepsilon(T, x) &= \Phi(x),
\end{aligned}
\]
where
\[
\mathcal{L}^\varepsilon \left( D_x^2 u^\varepsilon, D_x u^\varepsilon, u^\varepsilon, x, t \right) = G \left( H \left( D_x^2 u^\varepsilon, D_x u^\varepsilon, u^\varepsilon, x, t \right) \right) + \langle b(x), D_x u^\varepsilon \rangle \\
&\quad + f \left( t, x, u^\varepsilon, \langle \varepsilon \sigma_1(x), D_x u^\varepsilon \rangle, \ldots, \langle \varepsilon \sigma_d(x), D_x u^\varepsilon \rangle \right),
\]
and
\[
H_{ij} \left( D_x^2 u^\varepsilon, D_x u^\varepsilon, u^\varepsilon, x, t \right) = \langle D_x^2 u^\varepsilon \varepsilon \sigma_i(x), \varepsilon \sigma_j(x) \rangle + 2 \langle D_x u^\varepsilon, \varepsilon h_{ij}(x) \rangle \\
&\quad + 2g_{ij} \left( t, x, u^\varepsilon, \langle \varepsilon \sigma_1(x), D_x u^\varepsilon \rangle, \ldots, \langle \varepsilon \sigma_d(x), D_x u^\varepsilon \rangle \right)
\]

We define the following
\[ u^0(t, x) = \psi_t^{t,x}, \quad (t, x) \in [0, T] \times \mathbb{R}^n. \] (4.13)

**Proposition 2** For any \( \varepsilon > 0 \) and all \( x \in \mathbb{R}^n \),
\[ Y_s^{s,x,\varepsilon} = u^\varepsilon(t, X_t^{s,x,\varepsilon}), \quad \forall t \in [s, T]. \]

**Proof** Using the Markov property of the \( G \)-SDE and the uniqueness of the solution \( Y^{s,x,\varepsilon} \) of the \( G \)-BSDE (4.2) to show that
\[ Y_r^{s,x,\varepsilon} = Y_r^{t, X_t^{s,x,\varepsilon}}, \quad s \leq t \leq r \leq T. \]

Taking \( r = t \), we deduce that \( Y_t^{s,x,\varepsilon} = u^\varepsilon(t, X_t^{s,x,\varepsilon}) \), which leads to the end of the proof. \( \square \)

Let \( \mathcal{C}_{0,s}([s, T], \mathbb{R}^n) \) be the space of \( \mathbb{R}^n \)-valued continuous functions \( \bar{\varphi} \) on \([s, T] \) with \( \bar{\varphi}_s = 0 \).

Let \( s \in [0, T] \) and \( \varepsilon \geq 0 \). We define the mapping \( F^\varepsilon : \mathcal{C}_{0,s}([s, T], \mathbb{R}^n) \rightarrow \mathcal{C}([s, T], \mathbb{R}^n) \) by
\[ F^\varepsilon(\bar{\varphi}) = \left[ t \mapsto u^\varepsilon(t, x + \bar{\varphi}_t) \right] \quad s \leq t \leq T, \quad \bar{\varphi} \in \mathcal{C}_{0,s}([s, T], \mathbb{R}^n). \] (4.14)

where \( u^\varepsilon \) is given by (4.12) and \( u^0 \) by (4.13).
By virtue of (4.14) and Proposition 2, for any $\varepsilon > 0$ and all $x \in \mathbb{R}^n$, we have

$$Y^{s,x,\varepsilon} = F^\varepsilon (X^{s,x,\varepsilon} - x).$$

We have the following result of large deviations

**Theorem 8** Let (A0)–(A3) hold. Then for any closed subset $\mathcal{F}$ and any open subset $\mathcal{O}$ in $\mathcal{C}([s, T], \mathbb{R}^n)$,

$$\limsup_{\varepsilon \to 0} \varepsilon \log \hat{C} \left( Y^{s,x,\varepsilon} \in \mathcal{F} \right) \leq - \inf_{\psi \in \mathcal{F}} \Pi(\psi),$$

and

$$\liminf_{\varepsilon \to 0} \varepsilon \log \hat{C} \left( Y^{s,x,\varepsilon} \in \mathcal{O} \right) \geq - \inf_{\psi \in \mathcal{O}} \Pi(\psi),$$

where

$$\Pi(\psi) = \inf \left\{ A(\tilde{\psi}) : \psi_t = F^0(\tilde{\psi})(t) = u^0(t, x + \tilde{\psi}_t), t \in [s, T], \tilde{\psi} \in \mathcal{C}_{0,s}([s, T], \mathbb{R}^n) \right\}.$$

**Proof** Since the family $\left\{ \hat{C} \left( (X^t_t, x, \varepsilon) \mid t \in [s, T] \varepsilon > 0 \right) \right\}$ is exponentially tight (see Lemma 3.4 p. 2235 in Gao and Jiang [7]), by virtue of Lemma 3 (contraction principle) and Lemma 2, we just need to prove that $F^\varepsilon$, $\varepsilon > 0$ are continuous and $\{F^\varepsilon\}_{\varepsilon > 0}$ converges uniformly to $F^0$ on every compact subset of $\mathcal{C}_{0,s}([s, T], \mathbb{R}^n)$, as $\varepsilon \to 0$.

**Continuity of $F^\varepsilon$**:

Let $\varepsilon > 0$ and $\tilde{\varphi} \in \mathcal{C}_{0,s}([s, T], \mathbb{R}^n)$. Let $(\tilde{\varphi}^m)_m$ be a sequence in $\mathcal{C}_{0,s}([s, T], \mathbb{R}^n)$ which converges to $\tilde{\varphi}$ under the uniform norm.

We set $\varphi^m = x + \tilde{\varphi}^m$, $\varphi = x + \tilde{\varphi}$. So, $(\varphi^m)_m$ is a sequence in $\mathcal{C}([s, T], \mathbb{R}^n)$ which converges to $\varphi$ under the uniform norm. Fix $\varepsilon > 0$. Since $\|\varphi^m - \varphi\|_\infty \to 0$, there exists $M > 0$ such that,

$$\|\varphi^m\|_\infty \leq M, \|\varphi\|_\infty \leq M. \quad (4.15)$$

Since $u^\varepsilon$ is a continuous function in $[0, T] \times \mathbb{R}^n$, it follows that $u^\varepsilon$ is uniformly continuous in $[s, T] \times B(0, M)$ where $B(0, M)$ is the closed ball centered at the origin with radius $M$ in $\mathbb{R}^n$. Therefore, there exists $\eta > 0$ such that for $r_1, r_2 \in [s, T]$ and $z_1, z_2 \in B(0, M)$, $|r_1 - r_2| < \eta$ and $|z_1 - z_2| < \eta$, we have

$$|u^\varepsilon(r_1, z_1) - u^\varepsilon(r_2, z_2)| \leq \varepsilon.$$

Since there exists $m_0$ such that $\forall m \geq m_0, \|\varphi^m - \varphi\|_\infty \leq \eta$, in view of (4.15), for any $r \in [s, T]$ and for all $m \geq m_0$, we have

$$\varphi^m_r, \varphi_r \in B(0, M) \quad \text{and} \quad |u^\varepsilon(r, \varphi^m_r) - u^\varepsilon(r, \varphi_r)| \leq \varepsilon.$$

Thus

$$|u^\varepsilon(r, x + \tilde{\varphi}^m_r) - u^\varepsilon(r, x + \tilde{\varphi}_r)| \leq \varepsilon.$$

So we conclude that $F^\varepsilon(\tilde{\varphi}^m) \to F^\varepsilon(\tilde{\varphi})$, which proves the continuity of $F^\varepsilon$ at $\tilde{\varphi}$.

**Uniform convergence of $F^\varepsilon$**:
Let $\mathcal{K}$ be a compact subset of $C_0([s, T], \mathbb{R}^n)$ and let
\[ \mathcal{L} = \{ \varphi_r : \tilde{\varphi} \in \mathcal{K}, \varphi = x + \tilde{\varphi}, r \in [s, T] \}. \]

Obviously, $\mathcal{L}$ is a compact subset of $\mathbb{R}^n$. Thanks to Corollary 2, there exists a positive constant $C$ such that
\[
\sup_{\tilde{\varphi} \in \mathcal{K}} \| F^\varepsilon(\tilde{\varphi}) - F^0(\tilde{\varphi}) \|_\infty^2 = \sup_{\tilde{\varphi} \in \mathcal{K}} \sup_{r \in [s, T]} | u^\varepsilon(r, x + \tilde{\varphi}_r) - u^0(r, x + \tilde{\varphi}_r) |^2 \\
= \sup_{\tilde{\varphi} \in \mathcal{K}} \sup_{r \in [s, T]} | u^\varepsilon(r, \varphi_r) - u^0(r, \varphi_r) |^2 \\
= \sup_{\tilde{\varphi} \in \mathcal{K}} \sup_{r \in [s, T]} | Y_{r, x, \varepsilon} - \psi_{r, x} |^2 \\
\leq \sup_{x \in \mathcal{L}} \sup_{r \in [s, T]} | Y_{r, x, \varepsilon} - \psi_{r, x} |^2 \\
\leq C \varepsilon^2.
\]

Therefore the uniform convergence of the mapping $F^\varepsilon$ toward $F^0$ follows. \qed

**Acknowledgements** The authors would like to thank the anonymous referee and the AE for their helpful comments and suggestions that greatly improved the paper.

**References**

1. Dembo, A., Zeitouni, O.: Large Deviations Techniques and Applications, 2nd edn. Springer, Berlin (1998)
2. Denis, L., Hu, M., Peng, S.: Function spaces and capacity related to a sublinear expectation: application to $G$-brownian motion paths. Potential Anal. 34, 139–161 (2011)
3. Deuschel, J.D., Stroock, D.W.: Large Deviations. Academic Press Inc., Boston (1989)
4. Essaky, E.H.: Large deviation principle for a backward stochastic differential equation with subdifferential operator. Comptes Rendus Math. 346, 75–78 (2008)
5. Freidlin, M.I., Wentzell, A.D.: Random Perturbations of Dynamical Systems. Springer, Berlin (1984)
6. Gao, F.: Pathwise properties and homeomorphic flows for stochastic differential equations driven by $G$-brownian motion. Stoch. Process. Appl. 119, 3356–3382 (2009)
7. Gao, F., Jiang, H.: Large deviations for stochastic differential equations driven by $G$-brownian motion. Stoch. Process. Appl. 120, 2212–2240 (2010)
8. Hu, M., Ji, S., Peng, S., Song, Y.: Backward stochastic differential equations driven by $G$-brownian motion. Stoch. Process. Appl. 124, 759–784 (2014)
9. Hu, M., Ji, S., Peng, S., Song, Y.: Comparison theorem, Feynman–Kac formula and Girsanov transformation for BSDEs driven by $G$-brownian motion. Stoch. Process. Appl. 124, 1170–1195 (2014)
10. Hu, M., Peng, S.: On representation theorem of $G$-expectations and paths of $G$-Brownian motion. Acta Mathe. Appl. Sin. Engl. Ser. 25, 539–546 (2009)
11. N’zi, M., Dakaou, I.: Large deviation for multivalued backward stochastic differential equations. Random Oper. Stoch. Equ. 22(2), 119–127 (2014)
12. Peng, S.: Nonlinear expectations and stochastic calculus under uncertainty. arXiv:1002.4546v1 [math.PR] (2010)
13. Rainero, S.: Un principe de grandes déviations pour une équation différentielle stochastique progressive rétrograde. C. R. Acad. Sci. Paris 343, 141–144 (2006)
14. Song, Y.: Some properties on $G$-evaluation and its applications to $G$-martingale decomposition. Sci. China Math. 54(2), 287–300 (2011)
15. Varadhan, S.R.S.: Large Deviations and Applications. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (1984)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.