Entangled Subspaces and Quantum Symmetries

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Abstract

Entanglement is defined for each vector subspace of the tensor product of two finite-dimensional Hilbert spaces, by applying the notion of operator entanglement to the projection operator onto that subspace. The operator Schmidt decomposition of the projection operator defines a string of Schmidt coefficients for each subspace, and this string is assumed to characterize its entanglement, so that a first subspace is more entangled than a second, if the Schmidt string of the second majorizes the Schmidt string of the first. The idea is applied to the antisymmetric and symmetric tensor products of a finite-dimensional Hilbert space with itself, and also to the tensor product of an angular momentum $j$ with a spin $1/2$. When adapted to the subspaces of states of the nonrelativistic hydrogen atom with definite total angular momentum (orbital plus spin), within the space of bound states with a given total energy, this leads to a complete ordering of those subspaces by their Schmidt strings.

1 Introduction

If quantum entanglement [1] is to be regarded as a physical resource [2], then it seems sensible to consider the entanglement not only of individual states,
but also of collections of states of a given composite quantum system. On the other hand, it is not clear how to combine measures of entanglement of individual states in such a collection, because of the possibility of superposing given states to form new ones. Since the natural organizational unit for any collection of states is the vector (sub)space spanned by those states, we are led to the problem of quantifying the degree of entanglement inherent in a given vector subspace of the whole state space of a quantum system. Examples of vector subspaces of interest might be the space of states with a given total energy, or the space with a given total angular momentum. More generally, a state subspace might be labelled by the eigenvalues of any incomplete set of commuting observables. A situation that arises often is one where the state space is associated with a tensor product representation of some symmetry group or algebra, and the subspaces of interest carry irreducible subrepresentations of that algebra or group. Again, the example of angular momentum springs to mind; we may be interested in a subspace of states carrying a definite total angular momentum, for a quantum system made up of several subsystems, each contributing angular momentum to the total. In such cases, the problem of quantifying the entanglement of individual irreducible subspaces is seen to have an essentially group-theoretical character; any measure of entanglement of such a subspace must surely involve such group-theoretical constructs as the Clebsch-Gordan coefficients of the corresponding group or algebra. Conversely, considerations of the entanglement of such irreducible subspaces seems likely to throw interesting new light on familiar group-theoretical reduction problems.

In what follows, we consider only bipartite systems, and vector subspaces $V$ of a complex, finite-dimensional state space $H = H_1 \otimes H_2$, where the two factor spaces have dimensions $d_1$ and $d_2$ respectively, and are equipped with the usual scalar products. Extensions to the multipartite case seem likely to face the same sort of difficulties as entanglement measures for state vectors of multipartite systems.

**Example 1:** As more specific motivation, consider the following 3-dimensional vector subspaces of $H$, in the case where $H_1 \equiv H_2$ has orthonormal basis $\{e_1, e_2, e_3, \ldots e_N\}$, $N \geq 3$:

(A) $V_A$ is spanned by the orthonormal vectors $(e_1 \otimes e_2 - e_2 \otimes e_1)/\sqrt{2},$
\( (e_2 \otimes e_3 - e_3 \otimes e_2)/\sqrt{2}, \ (e_3 \otimes e_1 - e_1 \otimes e_3)/\sqrt{2}. \)

(S) $V_S$ is spanned by the orthonormal vectors $(e_1 \otimes e_2 + e_2 \otimes e_1)/\sqrt{2},$
\( (e_1 \otimes e_1 - e_2 \otimes e_2)/\sqrt{2}, \ (e_1 \otimes e_1 + e_2 \otimes e_2)/\sqrt{2}. \)

The bases of $V_A$ and $V_S$ so defined consist of three maximally entangled
vectors in each case. However, it is possible to find another basis in \( V_S \) with only one entangled vector, namely the set \( \{ e_1 \otimes e_2 + e_2 \otimes e_1 \} / \sqrt{2}, e_1 \otimes e_1, e_2 \otimes e_2 \} \), whereas every choice of basis in \( V_A \) consists entirely of maximally entangled vectors. It is intuitive that \( V_A \) is ‘more entangled’ than \( V_S \), and we seek to quantify such differences.

2 Operator entanglement and subspaces

In general, any vector subspace \( V \leq \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \), of dimension \( 1 \leq d \leq d_1 d_2 \), is characterized by a corresponding hermitian projection operator \( \hat{P} \),

\[
\hat{P} V = V, \quad \hat{P}^\dagger = \hat{P} = \hat{P}^2, \quad \text{Tr}(\hat{P}) = d,
\]

and we suggest that measures of entanglement of the operator \( \hat{P} \) provide suitable measures of entanglement of \( V \).

Measures of operator entanglement have been considered previously in other contexts [3, 4]. The central idea is to consider each linear operator (matrix) \( \hat{A} \) as an ‘operator vector’ \( |A\rangle \rangle \) in the \( (d_1 d_2)^2 \)-dimensional ‘operator vector space’ \( E_{\mathcal{H}} \) of all linear operators on \( \mathcal{H} \), with Hilbert-Schmidt scalar product

\[
\langle \langle A, B \rangle \rangle = \text{Tr}(\hat{A}^\dagger \hat{B}).
\]

Similarly, for \( r = 1, 2 \), linear operators on \( \mathcal{H}_r \) can be considered as operator vectors in the \( d_r^2 \)-dimensional operator vector space \( E_{\mathcal{H}_r} \). If \( \hat{A} = \hat{B} \otimes \hat{C} \), then \( |A\rangle \rangle = |B\rangle \rangle \otimes |C\rangle \rangle \), and \( \hat{A} \) is unentangled. Otherwise, \( \hat{A} \) is entangled.

The operator vector

\[
|\hat{P}\rangle \rangle = \frac{1}{\sqrt{d}} |P\rangle \rangle
\]

corresponding to the projector \( \hat{P} \) divided by \( \sqrt{d} \), is a unit operator vector in \( E_{\mathcal{H}} \), according to (1) and (2). We can define measures of entanglement of this unit operator vector in \( E_{\mathcal{H}} \), just as we define measures of entanglement of unit vectors in \( \mathcal{H} \). To this end, we note firstly that \( |\hat{P}\rangle \rangle \) will have a Schmidt decomposition,

\[
|\hat{P}\rangle \rangle = \sqrt{p_1} |E_1\rangle \otimes |F_1\rangle + \sqrt{p_2} |E_2\rangle \otimes |F_2\rangle + \ldots \sqrt{p_K} |E_K\rangle \otimes |F_K\rangle,
\]

where

\[
K \leq \bar{K} = \min\{d_1^2, d_2^2\},
\]

\[
p_1 \geq p_2 \geq \ldots \geq p_K > 0, \quad p_1 + p_2 + \ldots + p_K = 1,
\]

(5)
while $|E_1\rangle$, $|E_2\rangle$, $\ldots$, $|E_K\rangle$ are orthonormal operator vectors in $E_{\mathcal{H}_1}$, and $|F_1\rangle$, $|F_2\rangle$, $\ldots$, $|E_K\rangle$ are orthonormal operator vectors in $E_{\mathcal{H}_2}$. If we introduce the superoperator [5] density matrix, of dimension $(d_1 d_2)^2 \times (d_1 d_2)^2$,

$$\hat{R} = |\hat{P}\rangle\langle\hat{P}|,$$

and then define the reduced superoperator density matrices $\hat{R}^{(1)}$ and $\hat{R}^{(2)}$ by tracing over the second (respectively, the first) vector subspace of $E_{\mathcal{H}}$, then $|E_1\rangle$, $|E_2\rangle$, $\ldots$, $|E_K\rangle$ are eigen operator vectors of $\hat{R}^{(1)}$, and $|F_1\rangle$, $|F_2\rangle$, $\ldots$, $|E_K\rangle$ are eigen operator vectors of $\hat{R}^{(2)}$, in each case with eigenvalues $p_1$, $p_2$, $\ldots$, $p_K$. Furthermore, the unentangled unit operator vector closest to $|\hat{P}\rangle\rangle$ — in the sense of the norm defined by the scalar product [2] — is $|E_1\rangle\otimes|F_1\rangle\rangle$, and its distance from $|\hat{P}\rangle\rangle$ is

$$\mathcal{E}_D(|\hat{P}\rangle\rangle) = \left(\langle\langle\hat{P}| - \langle\langle E_1| \otimes \langle\langle F_1|\rangle\langle\hat{P}| - |E_1\rangle\otimes|F_1\rangle\rangle\right)^{1/2} = \sqrt{2(1 - \sqrt{p_1})}.$$  

(7)

This distance provides a partial measure of the entanglement of $|\hat{P}\rangle\rangle$, and of $\mathcal{V}$, and we shall also write it as $\mathcal{E}_D(\mathcal{V})$.

The entanglement of $\mathcal{V}$ is fully characterized by its corresponding $K$-dimensional ‘Schmidt string’

$$\mathcal{S}(\mathcal{V}) = (p_1, p_2, \ldots, p_K, p_{K+1} = 0, p_{K+2} = 0, \ldots, p_K = 0).$$

(8)

Various partial measures of entanglement can be defined in terms of the Schmidt string, including $\mathcal{E}_D$ as above. Thus the ‘information’ measure of entanglement of $|\hat{P}\rangle\rangle$, and hence of $\mathcal{V}$, is

$$\mathcal{E}_I(|\hat{P}\rangle\rangle) = \mathcal{E}_I(\mathcal{V}) = -\text{Tr}(\hat{R}^{(1)} \log_2(\hat{R}^{(1)})) = -\text{Tr}(\hat{R}^{(2)} \log_2(\hat{R}^{(2)})) = -\sum_{\alpha=1}^{K} p_\alpha \log_2(p_\alpha),$$

(9)

while the ‘trace’ measure of entanglement is

$$\mathcal{E}_T(|\hat{P}\rangle\rangle) = \mathcal{E}_T(\mathcal{V}) = 1 - \text{Tr}(\hat{R}^{(1)} \hat{R}^{(1)}) = 1 - \text{Tr}(\hat{R}^{(2)} \hat{R}^{(2)}) = 1 - \sum_{\alpha=1}^{K} p_\alpha^2.$$  

(10)
A better indicator of entanglement is provided with the help of the notion of majorization \[6, 7, 8\]. Thus we may say that \( V \leq H \) is more entangled than \( W \leq H \), with Schmidt string \( S(W) = (q_1, q_2, \ldots) \), if \( p_1 \leq q_1 \) AND \( p_1 + p_2 \leq q_1 + q_2 \) AND \( \ldots \), that is to say, if \( S(V) \) is majorized by \( S(W) \), which we write as \( S(V) \prec S(W) \). When \( S(V) \prec S(W) \), it can be shown \[7\] that \( E_D(V) \geq E_D(W) \), \( E_I(V) \geq E_I(W) \) and \( E_T(V) \geq E_T(W) \). But when neither of \( S(V) \) and \( S(W) \) majorizes the other, some of these inequalities and not others may be reversed. In that situation, it is best to say only that \( V \) and \( W \) are differently entangled.

The preceding two paragraphs merely paraphrase for operator (or subspace) entanglement what is well-known for state entanglement. Many statements that hold true for the tensor product space of states \( H \), go over to the tensor product space of operators \( E_H \), without the need for new proofs. In what follows, we give some properties of subspace entanglement as defined, and then some examples of naturally arising entangled subspaces and their Schmidt strings.

Property 1: If in the situation described above, \( \mathcal{H}_1 \) is embedded as a subspace in a larger space \( \mathcal{H}_1' \), and \( \mathcal{H}_2 \) is embedded as a subspace in a larger space \( \mathcal{H}_2' \), so that
\[
\mathcal{V} \leq \mathcal{H}_1 \otimes \mathcal{H}_2 \leq \mathcal{H}_1' \otimes \mathcal{H}_2' , \quad \mathcal{H}_1 \leq \mathcal{H}_1' , \quad \mathcal{H}_2 \leq \mathcal{H}_2' ,
\]
then \( S(\mathcal{V}) \), when \( \mathcal{V} \) is regarded as a subspace of \( \mathcal{H}_1' \otimes \mathcal{H}_2' \), differs from \( S(\mathcal{V}) \) when \( \mathcal{V} \) is regarded as a subspace of \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), only by the addition of the appropriate number of zeros on the right-hand end. In this sense, our notion of entanglement of a subspace is stable against embeddings.

Property 2: If \( \mathcal{V} \) has the form \( \mathcal{V}_1 \otimes \mathcal{V}_2 \), where \( \mathcal{V}_1 \leq \mathcal{H}_1 \) and \( \mathcal{V}_2 \leq \mathcal{H}_2 \), then \( |\bar{P}\rangle = |\bar{P}_1\rangle \otimes |\bar{P}_2\rangle \) is unentangled, and so also is \( \mathcal{V} \), according to our definition. In this case, the Schmidt string has the form
\[
S(\mathcal{V}) = (1, 0, 0, \ldots, 0).
\]

In particular, \( \mathcal{H} \) itself is unentangled, and \( S(\mathcal{H}) \) has the form \[12\].

Property 3: If \( \mathcal{V} \) is 1-dimensional, spanned by the unit vector \( |v\rangle \) say, with Schmidt decomposition
\[
|v\rangle = \sum_{\alpha=1}^{k} \sqrt{p_\alpha} |e_\alpha\rangle \otimes |f_\alpha\rangle ,
\]

then
\[\hat{P} = |v\rangle \langle v| = \sum_{\alpha, \beta=1}^{k} \sqrt{p_\alpha p_\beta} |e_\alpha\rangle \langle e_\beta| \otimes |f_\alpha\rangle \langle f_\beta|. \tag{14}\]

This defines the Schmidt decomposition of \(|\hat{P}\rangle\rangle\), whose Schmidt string then has as components the \(p_\alpha p_\beta\), with a suitable ordering. From this it is easily deduced that the Schmidt string of \(|v\rangle\) majorizes the Schmidt string of \(|u\rangle\) if and only if the Schmidt string of the subspace spanned by \(|v\rangle\) majorizes the Schmidt string of the subspace spanned by \(|u\rangle\). This guarantees that our notion of entanglement of 1-dimensional subspaces is consistent with that for state vectors. In particular, it is also easily seen that

\[E_I(V) = -\sum_{\alpha, \beta=1}^{k} p_\alpha p_\beta \log_2 (p_\alpha p_\beta) \]

\[= -2 \sum_{\alpha=1}^{k} p_\alpha \log_2 (p_\alpha) = 2E_I(|v\rangle). \tag{15}\]

Thus the information measures of entanglement of a 1-dimensional subspace and of any unit vector within that subspace differ only by the constant factor 2.

3 Antisymmetric and symmetric subspaces

Example 2: As a generalization of (A) in Example 1 above, consider the ‘antisymmetric tensor product’ space \(V_A \leq \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2\), where \(\mathcal{H}_1 \cong \mathcal{H}_2\) has an orthonormal basis \(\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}\). An orthonormal basis for \(V_A\) is provided by the \(n(n-1)/2\) vectors

\[|e_{kl}\rangle = (|e_k\rangle \otimes |e_l\rangle - |e_l\rangle \otimes |e_k\rangle)/\sqrt{2}, \quad k < l, \quad k, l \in \{1, 2, \ldots, n\}, \tag{16}\]

and the projector onto \(V_A\) is then

\[\hat{P}_A = \sum_{k<l=1}^{n} |e_{kl}\rangle \langle e_{kl}| \]

\[= \frac{1}{2} \sum_{k<l=1}^{n} \left(|e_k\rangle \langle e_k| \otimes |e_l\rangle \langle e_l| + |e_l\rangle \langle e_l| \otimes |e_k\rangle \langle e_k| \right. \]

\[\left. - |e_k\rangle \langle e_k| \otimes |e_l\rangle \langle e_l| - |e_l\rangle \langle e_l| \otimes |e_k\rangle \langle e_k| \right). \tag{17}\]
Labelling the unit operator vectors in $E_{\mathcal{H}_1}$ and $E_{\mathcal{H}_2}$ as

$$
|1\rangle = |e_1\rangle\langle e_1|, \quad |2\rangle = |e_2\rangle\langle e_2|, \ldots, \quad |n\rangle = |e_n\rangle\langle e_n|,
$$

$$
|n+1\rangle = |e_1\rangle\langle e_2|, \quad |n+2\rangle = |e_2\rangle\langle e_1|, \ldots, \quad |3n-2\rangle = |e_n\rangle\langle e_1|,
$$

$$
|3n-1\rangle = |e_2\rangle\langle e_3|, \quad |3n\rangle = |e_3\rangle\langle e_2|, \ldots, \quad |5n-6\rangle = |e_n\rangle\langle e_2|,
$$

$$
\ldots, \quad |n^2\rangle = |e_n\rangle\langle e_{n-1}|,
$$

we then have from (17), the unit operator vector

$$
|\tilde{P}\rangle = \frac{1}{\sqrt{2n(n-1)}} \sum_{r,s=1}^{n^2} A_{rs} |r\rangle \otimes |s\rangle
$$

(19)

where the $n^2 \times n^2$ matrix $A$ with matrix elements $A_{rs}$ takes the form

$$
A = B \oplus C \oplus C \oplus \ldots \oplus C.
$$

(20)

Here $B$ is $n \times n$, with all diagonal elements equal to 0, and all nondiagonal elements equal to 1. Each of the $n(n-1)/2$ copies of $C$ is $2 \times 2$, with diagonal elements equal to 0, and off-diagonal elements equal to $-1$.

It follows from (19) that the matrix elements of $\hat{R}^{(1)}$ in this case are just those of $AA\dagger$, whose eigenvalues are easily calculated from (20) to be

$$
S(V_A) = \frac{1}{2n(n-1)} \left( (n-1)^2, 1, 1, \ldots, 1 \right),
$$

(21)

where the 1 appears $n^2 - 1$ times. Then (21) is the Schmidt string for $V_A$. It follows that

$$
\mathcal{E}_D(V_A) = \sqrt{2(1 - \sqrt{(n-1)/(2n)})},
$$

$$
\mathcal{E}_I(V_A) = \log_2 \left( 2n(n-1)^{1/n} \right), \quad \mathcal{E}_T(V_A) = (n+1)(3n-4)/[4n(n-1)].
$$

(22)

Note that, as $n \to \infty$, $\mathcal{E}_D(V_A)$ and $\mathcal{E}_T(V_A)$ tend to constants, whereas $\mathcal{E}_I(V_A) \sim \log_2(n)$.

**Example 3:** Consider again the space $\mathcal{H}$ as in Example 2, and let $V_S$ denote the ‘symmetric tensor product’ space of dimension $n(n+1)/2$, with orthonormal basis $|e_{kl}\rangle$, $k \leq l = 1, 2, \ldots, n$, where

$$
|e_{kl}\rangle = (|e_k\rangle \otimes |e_l\rangle + |e_l\rangle \otimes |e_k\rangle)/\sqrt{2}, \quad k < l,
$$

$$
|e_{kk}\rangle = |e_k\rangle \otimes |e_k\rangle.
$$

(23)
A similar calculation to that for the antisymmetric case shows that the Schmidt string for $V_S$ is

$$S(V_S) = \frac{1}{2n(n+1)}((n+1)^2, 1, 1, \ldots, 1), \quad (24)$$

where again the 1 appears $n^2 - 1$ times. Then

$$E_D(V_S) = \sqrt{2(1 - \sqrt{(n+1)/(2n)})}, \quad E_I(V_S) = \log_2\left(\frac{2n}{(n+1)^{1/n}}\right), \quad E_T(V_S) = \frac{(n-1)(3n+4)}{4n(n+1)}. \quad (25)$$

The asymptotic behaviour of these quantities as $n \to \infty$ is similar to that in the antisymmetric case of Example 2.

We note from (21) and (24) that, in Example 1, the 3-dimensional antisymmetric and symmetric subspaces of the $N^2$-dimensional space $H$ have $N^2$-dimensional Schmidt strings

$$S(V_A) = \frac{1}{12}(4, 1, 1, 1, 1, 0, 0, \ldots, 0)$$
$$S(V_S) = \frac{1}{12}(9, 1, 1, 1, 0, 0, \ldots, 0), \quad (26)$$

so that $S(V_A) \prec S(V_S)$, consistent with our intuition that $V_A$ is more entangled than $V_S$.

### 4 Coupled angular momenta

**Example 4**: Consider the coupling of two angular momenta, $(\hat{J}_1, \hat{J}_2, \hat{J}_3)$ with spin $j$, and $(\hat{S}_1, \hat{S}_2, \hat{S}_3)$ with spin 1/2. In this case, the full state space is

$$H = \mathcal{H}_j \otimes \mathcal{H}_{1/2} = \mathcal{V}_{j+1/2} \oplus \mathcal{V}_{j-1/2}, \quad (27)$$

where the space $\mathcal{H}_k$, for $k = j$ or 1/2, or $\mathcal{V}_k$, for $k = j + 1/2$ or $j - 1/2$, has dimension $(2k + 1)$, and carries the corresponding irreducible representation of the spin Lie algebra $su(2)$. We are interested in the entanglement of $\mathcal{V}_{j\pm1/2}$, regarded as a subspace of $H$.

Let $|m\rangle$, $m = j, j - 1, \ldots, -j$ denote the usual orthonormal basis of eigenstates of $\hat{J}_3$ in $\mathcal{H}_j$, and let $|+\rangle$ and $|-\rangle$ denote the usual orthonormal basis of eigenstates of $\hat{S}_3$ in $\mathcal{H}_{1/2}$. Then let

$$|1\rangle = |+\rangle|+\rangle, \quad |2\rangle = |-\rangle|-\rangle, \quad |3\rangle = |+\rangle|-\rangle, \quad |4\rangle = |-\rangle|+\rangle, \quad (28)$$
defining an orthonormal basis of operator vectors in $E_{1/2}$, and let
\[ |m, n\rangle = |m\rangle \langle n|, \quad m, n = j, j - 1, \ldots, -j, \quad (29) \]
defining an orthonormal basis of operator vectors in $E_j$. Then $\hat{S}_{\pm} = \hat{S}_1 \mp i\hat{S}_2$ and $\hat{S}_3$, regarded as operator vectors in $E_{1/2}$, take the form
\[ |S_+\rangle = |3\rangle, \quad |S_-\rangle = |4\rangle, \quad |S_3\rangle = \frac{1}{2}(|1\rangle - |2\rangle). \quad (30) \]
while $\hat{J}_{\pm}$ and $\hat{J}_3$, regarded as operator vectors in $E_j$, take the form
\[ |J_+\rangle = \sqrt{(1)(2j)}|j, j - 1\rangle + \sqrt{(2)(2j - 1)}|j - 1, j - 2\rangle + \ldots \]
\[ \ldots + \sqrt{(2j)(1)}| - j + 1, -j\rangle \]
\[ |J_-\rangle = \sqrt{(1)(2j)}|j - 1, j\rangle + \sqrt{(2)(2j - 1)}|j - 2, j - 1\rangle + \ldots \]
\[ \ldots + \sqrt{(2j)(1)}| - j, -j + 1\rangle \]
\[ |J_3\rangle = j|j, j\rangle + (j - 1)|j - 1, j - 1\rangle + \ldots + (-j)| - j, -j\rangle. \quad (31) \]
Recall [9] that the $\mathcal{H}$ operator $\hat{X}$, defined by
\[ \hat{X} = \hat{J}_+ \otimes \hat{S}_- + \hat{J}_- \otimes \hat{S}_+ + 2\hat{J}_3 \otimes \hat{S}_3, \quad (32) \]
takes the eigenvalue $j$ on the subspace $\mathcal{V}_{j+1/2}$ and the eigenvalue $-(j + 1)$ on the subspace $\mathcal{V}_{j-1/2}$. It follows that the projector from $\mathcal{H}$ onto $\mathcal{V}_{j\pm 1/2}$ is given by
\[ \hat{P}_{j\pm 1/2} = \pm \frac{1}{2j + 1} \left( \hat{X} + \frac{1}{2} \hat{I} \pm (j + \frac{1}{2}) \hat{I} \right), \quad (33) \]
where $\hat{I}$ denotes the unit operator on $\mathcal{H}$.
Consider firstly the projector $\hat{P}_{j+1/2}$. From (31), (32), (33) and (34), we see that this operator, regarded as an operator vector on $E_{\mathcal{H}}$, and normalized to a unit operator vector, takes the form
\[ |\hat{P}_{j+1/2}\rangle = \frac{1}{\sqrt{2(j + 1)}} \frac{1}{2j + 1} \left\{ |J_+\rangle \otimes |S_-\rangle + |J_-\rangle \otimes |S_+\rangle \right. \]
\[ + 2|J_3\rangle \otimes |S_3\rangle \]
\[ + (j + 1)(|j, j\rangle + |j - 1, j - 1\rangle + \ldots + | - j, -j\rangle) \otimes (|1\rangle + |2\rangle) \right\}. \quad (34) \]
Here the terms on the last line represent the operator vector corresponding to the operator \((j + 1)\hat{I}\). Expression (34) has the general form

\[
\left| \tilde{P}_{j+1/2} \right\rangle = \sum_{(m,n)=(-j,-j)}^{(j,j)} 4 \sum_{\alpha=1}^{(m,n)} A_{(m,n),\alpha} |m, n\rangle \otimes |\alpha\rangle ,
\]

and we wish to calculate the eigenvalues of the reduced superoperator density matrix

\[
\hat{R}^{(2)} = \sum_{(m,n)=(-j,-j)}^{(j,j)} 4 \sum_{\alpha,\beta=1}^{(m,n)} \left\{ A_{(m,n),\alpha} A_{(m,n),\beta}^* \right\} |\alpha\rangle \langle \beta| ,
\]

or, what is the same thing, the eigenvalues of the 4×4 matrix \(Q\) with elements

\[
Q_{\alpha\beta} = \sum_{(m,n)=(-j,-j)}^{(j,j)} \left\{ A_{(m,n),\alpha} A_{(m,n),\beta}^* \right\} .
\]

The only nonzero elements are, from (31) and (34),

\[
Q_{11} = Q_{22} = \frac{1}{2(j+1)} \frac{1}{(2j+1)^2} \left( (2j+1)^2 + (2j)^2 + \ldots + (1)^2 \right)
= \frac{4j + 3}{6(2j + 1)} ,
\]

\[
Q_{33} = Q_{44} = \frac{1}{2(j+1)} \frac{1}{(2j + 1)^2} \left( (1)(2j) + (2)(2j-1) + \ldots + (2j)(1) \right)
= \frac{j}{3(2j + 1)} ,
\]

and

\[
Q_{12} = Q_{21} = \frac{1}{2(j+1)} \frac{1}{(2j + 1)^2} \left( (2j+1)(1) + (2j)(2) + \ldots + (1)(2j + 1) \right)
= \frac{2j + 3}{6(2j + 1)} ,
\]

and the eigenvalues of \(Q\) are now easily calculated to be \((j + 1)/(2j + 1)\) (multiplicity 1) and \(j/(6j + 3)\) (multiplicity 3). The Schmidt string of \(V_{j+1/2}\) is therefore

\[
S(V_{j+1/2}) = \frac{1}{2j + 1} \left( j + 1, \frac{j}{3}, \frac{j}{3}, \frac{j}{3} \right) .
\]
We then have as scalar partial measures of the entanglement of this subspace,

\[ E_D(V_{j+1/2}) = \sqrt{2(1 - \sqrt{(j + 1)/(2j + 1)})}, \]
\[ E_I(V_{j+1/2}) = -\log_2 \left( \frac{\left(\frac{\sqrt{2}}{3}\right)^{(j+1)/(2j+1)} (j + 1)^{(j+1)/(2j+1)}}{2j + 1} \right), \]
\[ E_T(V_{j+1/2}) = 2j(4j + 3)/[3(2j + 1)^2]. \] (42)

We see that as \( j \to \infty \), all these quantities approach constant values. This is a consequence of the fact that the Schmidt string (41) approaches the constant value
\[ S_0 = \left( \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right), \] (43)
and can perhaps be understood as follows: as \( j \) gets large, the number of states in \( V_{j+1/2} \) with larger and larger positive or negative eigenvalue of \( \hat{J}_3 \otimes \hat{I}_{1/2} + \hat{J}_j \otimes \hat{S}_3 \) increases, and these states have smaller and smaller entanglement, with the entanglement reaching zero for the highest and lowest weight states.

A similar calculation shows that the Schmidt string of \( V_{j-1/2} \) is
\[ S(V_{j-1/2}) = \frac{1}{2j + 1} \left( j, \frac{j + 1}{3}, \frac{j + 1}{3}, \frac{j + 1}{3} \right). \] (44)
In this case,
\[ E_D(V_{j-1/2}) = \sqrt{2(1 - \sqrt{j/(2j + 1)})}, \]
\[ E_I(V_{j-1/2}) = -\log_2 \left( \frac{\left(\frac{\sqrt{2}}{3}\right)^{(j+1)/(2j+1)} j^{(j+1)/(2j+1)}}{2j + 1} \right), \]
\[ E_T(V_{j-1/2}) = 2(j + 1)(4j + 3)/[3(2j + 1)^2]. \] (45)
As \( j \to \infty \), these quantities approach the same constant values as in the previous case.

Note that \( S(V_{j+1/2}) \geq S(V_{j-1/2}) \), so that \( V_{j-1/2} \) is more entangled than \( V_{j+1/2} \).

5 Application: Electron spin and H-atom

Example 5: Consider the space \( H^{(n)} \) of bound states of the nonrelativistic hydrogen atom with principal quantum number \( n \), where \( n \) is a positive
integer. This space is \( n^2 \)-dimensional, with the structure
\[
\mathcal{H}^{(n)} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_{n-1},
\] 
(46)
where \( \mathcal{H}_l \) is \((2l + 1)\)-dimensional, corresponding to the orbital angular momentum content \( l = 0, 1, \ldots, n-1 \). Allowing for the spin of the electron, we have as the relevant state space including spin,
\[
\mathcal{H} = \mathcal{H}^{(n)} \otimes \mathcal{H}_{1/2} = (\mathcal{H}_0 \otimes \mathcal{H}_{1/2}) \oplus (\mathcal{H}_1 \otimes \mathcal{H}_{1/2}) \oplus \ldots \oplus (\mathcal{H}_{n-1} \otimes \mathcal{H}_{1/2}) \\
= (\mathcal{V}_{1/2}) \oplus \left( \mathcal{W}_{1/2} \oplus \mathcal{V}_{3/2} \right) \oplus \left( \mathcal{W}_{3/2} \oplus \mathcal{V}_{5/2} \right) \oplus \\
\ldots \oplus \left( \mathcal{W}_{n-3/2} \oplus \mathcal{V}_{n-1/2} \right).
\] 
(47)
From the results of Section 4, we see that the Schmidt strings corresponding to these subspaces are
\[
S(\mathcal{V}_k) = \frac{1}{4k} \left( \frac{2k+1}{3}, \frac{2k-1}{3}, \frac{2k-1}{3}, \frac{2k-1}{3} \right) \\
\text{for } k = \frac{1}{2}, \frac{3}{2}, \ldots, n-\frac{1}{2}, \text{ and}
\]
\[
S(\mathcal{W}_k) = \frac{1}{4(k+1)} \left( \frac{2k+1}{3}, \frac{2k+3}{3}, \frac{2k+3}{3}, \frac{2k+3}{3} \right) \\
\text{for } k = \frac{1}{2}, \frac{3}{2}, \ldots, n-\frac{3}{2}.
\] 
(48)
Now we see a remarkable ordering of these subspaces by their spin-orbit entanglement. From least entangled to most entangled, as indicated by their Schmidt strings, we have:
\[ S(\mathcal{V}_{1/2}) = \left(1, 0, 0, 0\right) \succ S(\mathcal{V}_{3/2}) \succ \ldots \]

\[ \ldots \succ S(\mathcal{V}_{n-1/2}) = \frac{1}{4n-2} \left(2n, \frac{2n-2}{3}, \frac{2n-2}{3}, \frac{2n-2}{3}\right) \succ \ldots \]

\[ \succ S_0 = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) \succ \]

\[ S(\mathcal{V}_{n-3/2}) = \frac{1}{4n-2} \left(2n-2, \frac{2n}{3}, \frac{2n}{3}, \frac{2n}{3}\right) \succ \ldots \]

\[ \ldots \succ S(\mathcal{V}_{3/2}) \succ S(\mathcal{V}_{1/2}) = \left(\frac{1}{3}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}\right). \]

(49)

Here the limiting Schmidt string \( S_0 \) as in (43), is approached from above by\( S(\mathcal{V}_{n-1/2}) \), and from below by \( S(\mathcal{V}_{n-3/2}) \), as \( n \to \infty \).

In this example, it seems that the notion of subspace entanglement has to be interpreted as a kind of mean entanglement per basis vector, rather than a total entanglement. Thus the 2-dimensional subspace \( \mathcal{V}_{1/2} \), according to this notion, has a greater entanglement than, say, \( \mathcal{V}_{n-1/2} \), although the latter subspace may be of much greater dimension, containing many entangled states.

6 Concluding remarks

The notion of subspace entanglement that has been introduced has some interesting features. It is desirable in future work to try and establish that it does indeed make good sense in the context of applications to physics, perhaps along the lines that have been explored [10] in the case of state entanglement. To do that, it may well be necessary to relate more closely than we have done here, the entanglement of individual basis vectors in a subspace, with the notion of subspace entanglement.

Are there other ways to measure subspace entanglement? Another way might be to consider an arbitrary orthonormal basis of the subspace, and to consider the system to be in a mixed state of those basis states, each with probability \( 1/d \), where \( d \) is the subspace dimension. Then we could associate
the entanglement of that mixed state with the entanglement of the subspace, using existing measures of entanglement of mixed states \cite{11, 12, 13, 14}. The density operator for the mixed state in this case is simply a multiple (by $1/d$) of the projection operator onto the subspace, so we would then be considering in a different way, the entanglement associated with a projection operator. It should be interesting to explore the connections between these two notions of subspace entanglement.

We might also consider the possibility that the subspace of states associated with a given quantum system is itself uncertain. In that situation it would seem appropriate to consider the extension of the superoperator density matrix $\hat{\rho}$ to the mixed case, with probabilities $p_1, p_2, \ldots, p_N$ associated with different subspaces $V_1, V_2, \ldots, V_N$. Then we would need to extend existing notions of entanglement of mixed states to this new situation.

We have considered above the coupling of an angular momentum $j$ with a spin $1/2$. There is a challenge to calculate the entanglement of irreducible subspaces with definite total angular momentum, in the case when two arbitrary angular momenta are coupled together. When this is done, it should be possible to see how the entanglement is related to the values of Clebsch-Gordan coefficients, in particular. This ‘reduction entanglement’ problem has an obvious extension to representations of other groups and algebras, and seems to open up a new aspect of the tensor product reduction problem in general, including cases involving infinite-dimensional representations, and cases involving tensor products of more than two representations.

The concept of entanglement of a vector subspace of a tensor product space seems clearly to be of mathematical interest. It is less clear what may be its importance for physics, but we hope that the examples above are suggestive of important applications.

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