Scalarization of horizonless reflecting stars: neutral scalar fields non-minimally coupled to Maxwell fields

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Abstract

We analyze condensation behaviors of neutral scalar fields outside horizonless reflecting stars in the Einstein-Maxwell-scalar gravity. It was known that minimally coupled neutral scalar fields cannot exist outside horizonless reflecting stars. In this work, we consider non-minimal couplings between scalar fields and Maxwell fields, which is included to aim to trigger formations of scalar hairs. We analytically demonstrate that there is no hair theorem for small coupling parameters below a bound. For large coupling parameters above the bound, we numerically obtain regular scalar hairy configurations supported by horizonless reflecting stars.

PACS numbers: 11.25.Tq, 04.70.Bw, 74.20.-z

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I. INTRODUCTION

The recent observation of gravitational waves may provide a way to test the nature of astrophysical black holes [1–3]. In classical general relativity, one famous property of black holes is no hair theorem, which states that a nontrivial static scalar field cannot exist in the exterior region of asymptotically flat black holes, see references [4]-[11] and reviews [12, 13]. However, some candidate quantum-gravity models suggested that, due to quantum effects [14–18], the classical absorbing horizon should be replaced by a reflecting surface [19–25]. Interestingly, no hair behaviors also appear for such horizonless reflecting stars [26]-[38]. In particular, even for static scalar fields non-minimally coupled to the Ricci curvature, no hair theorem still holds in backgrounds of black holes and horizonless reflecting stars [39–44].

Intriguingly, for static scalar fields non-minimally coupled to the Gauss-Bonnet invariant, scalar field hairs can exist outside asymptotically flat black holes [45–48]. Spinning hairy black holes were also numerically obtained in the scalar-Gauss-Bonnet theory [49]. In addition, analytical formula of the scalar-Gauss-Bonnet coupling parameter was explored in [50]. These scalarization models are constructed by introducing an additional term $f(\psi)R_{GB}^2$, where $f(\psi)$ is a function of the scalar field $\psi$ and $R_{GB}^2$ is the Gauss-Bonnet invariant. In the scalar-Gauss-Bonnet gravity, under scalar perturbations, the bald black hole is thermodynamically unstable and it may evolve into a hairy black hole [46, 47]. This intriguing mechanism of hair formations is usually called spontaneous scalarization, which was found long ago for neutron stars in the context of scalar-tensor theories [51]. At present, lots of spontaneous scalarization models were constructed in the background of black holes [52–60].

As mentioned above, some candidate quantum-gravity models suggested that quantum effects may prevent the formation of horizons and a reflecting wall may lay above the would-be horizon position [14–21]. Interestingly, it was found that neutral scalar field hairs cannot exist outside such horizonless reflecting stars (even the star is charged) [26]. When considering scalar-Gauss-Bonnet couplings, we showed that the coupling can lead to the formation of neutral scalar field hairs in the background of horizonless reflecting stars [61]. In fact, black hole spontaneous scalarization is a very universal property, which also can be induced by another type of non-minimal couplings between scalar fields and Maxwell fields [62–67]. As a further step, it is very interesting to examine whether scalar-Maxwell couplings can trigger condensations of neutral scalar fields outside horizonless reflecting stars.
This work is organized as follows. We start by introducing a model with a neutral scalar field coupled to the Maxwell field in the charged horizonless reflecting star spacetime. For small coupling parameters, the neutral scalar field cannot exist. In contrast, for large coupling parameters, we get numerical solutions of scalar hairy horizonless reflecting stars. Main conclusions are presented in the last section.

II. INVESTIGATIONS ON THE COUPLING PARAMETER BETWEEN SCALAR FIELDS AND MAXWELL FIELDS

We take the Lagrange density with scalar fields non-minimally coupled to Maxwell fields in the asymptotically flat background. It is defined by the following expression

\[ L = R - \nabla^\nu \nabla_\nu \Psi - \mu^2 \psi^2 + f(\Psi) I. \]  

(1)

Here \( R \) is the scalar curvature. \( \Psi \) is the static neutral scalar field with mass \( \mu \). \( f(\Psi) \) is a function coupled to \( I = F_{\rho\sigma} F^{\rho\sigma} \). In the linearized regime, there is \( I = -\frac{Q^2}{r^4} \) and the general coupling function can be expressed as \( f(\Psi) = 1 - \alpha \Psi^2 \), where \( \alpha \) is the model parameter describing coupling strength. In the limit of \( \alpha \to 0 \), it returns to the usual Einstein-Maxwell-scalar gravity.

The scalar field differential equation is

\[ \nabla^\nu \nabla_\nu \Psi - \mu^2 \Psi + \frac{f' \psi I^2}{2} = 0. \]  

(2)

The charged static spherically symmetric background is

\[ ds^2 = -N(r) dt^2 + \frac{dr^2}{N(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]  

(3)

In the weak-field limit, the metric function \( N(r) \) is

\[ N(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \]  

(4)

with \( M \) and \( Q \) representing the star mass and star charge respectively. We point out that this background metric is valid on the condition \( \alpha \psi^2 \ll 1 \).

We take the scalar field decomposition

\[ \Psi(r, \theta, \phi) = \sum_{lm} e^{im\phi} S_{lm}(\theta) R_{lm}(r). \]  

(5)

For simplicity, we label \( R_{lm}(r) \) as \( \psi(r) \). With relations (2), (3), (5) and \( I = -\frac{Q^2}{r^4} \), we derive the ordinary differential equation

\[ \psi'' + \left( \frac{2}{r} + \frac{N'}{N} \right) \psi' + \left( \frac{\alpha Q^2}{r^2 N} - \frac{l(l + 1)}{r^2 N} - \frac{\mu^2}{N} \right) \psi = 0. \]  

(6)
Here \( l \) is the spherical harmonic index and \( l(l+1) \) is the characteristic eigenvalue of the angular scalar eigenfunction \( S_{lm}(\theta) \).

We label \( r_s \) as the radial coordinate of the star surface. Since we focus on the compact star without a horizon, the star surface is outside the gravitational radius, which can be expressed as \( r_s > M + \sqrt{M^2 - Q^2} \).

At the star surface, we take scalar reflecting surface boundary conditions \( \psi(r_s) = 0 \). In the far region, the physical massive static scalar fields asymptotically behave as \( \psi(r \to \infty) \sim \frac{1}{r} e^{-\mu r} \). So the scalar field satisfies bound-state conditions

\[
\psi(r_s) = 0, \quad \psi(\infty) = 0. \tag{7}
\]

According to boundary conditions (7), one concludes that the function \( \psi(r) \) must possess (at least) one extremum point \( r = r_{\text{peak}} \) between the star surface \( r = r_s \) and spatial infinity. It can be a positive maximum extremum point or a negative minimum extremum point. With the symmetry \( \psi \to -\psi \) of equation (6), without loss of generality, we can only study the case of positive maximum extremum points. Then the scalar field around the extremum point is characterized by

\[
\psi(r_{\text{peak}}) > 0, \quad \psi'(r_{\text{peak}}) = 0, \quad \psi''(r_{\text{peak}}) \leq 0. \tag{8}
\]

(6) and (8) yield the relation

\[
\frac{\alpha Q^2}{r_{\text{peak}}^4 N} - \frac{l(l+1)}{r_{\text{peak}}^2 N} - \frac{\mu^2}{N} \geq 0 \quad \text{for} \quad r = r_{\text{peak}}. \tag{9}
\]

Since we concentrate on horizonless stars, the extremum point is outside the gravitational radius satisfying

\[
N(r_{\text{peak}}) = 1 - \frac{2M}{r_{\text{peak}}} + \frac{Q^2}{r_{\text{peak}}^2} > 0. \tag{10}
\]

With (9) and (10), we get the relation

\[
\frac{\alpha Q^2}{r_{\text{peak}}^4} - \frac{l(l+1)}{r_{\text{peak}}^2} - \mu^2 \geq 0. \tag{11}
\]

According to the inequality (11), we deduce a bound on the coupling parameter

\[
\alpha \geq \frac{\mu^2 r_{\text{peak}}^4 + l(l+1)r_{\text{peak}}^2}{Q^2} = \frac{\mu^2 r_s^4 + l(l+1)r_s^2}{Q^2} > \frac{\mu^2(M + \sqrt{M^2 - Q^2})^4 + l(l+1)(M + \sqrt{M^2 - Q^2})^2}{Q^2}. \tag{12}
\]

If compact reflecting stars are surrounded with static neutral scalar hairs, the parameter \( \alpha \) should be above the bound (12). In other words, we obtain a no hair theorem for small coupling parameters

\[
\alpha \leq \frac{\mu^2(M + \sqrt{M^2 - Q^2})^4 + l(l+1)(M + \sqrt{M^2 - Q^2})^2}{Q^2}. \tag{13}
\]
It implies that neutral massive static exterior scalar fields usually cannot exist in cases of large field mass, large spherical harmonic index, large star mass or small star charge. In particular, for $\alpha = 0$, the no hair condition (13) always holds, which means that charged stars cannot support minimally coupled neutral scalar hairs. In the following, we numerically show that scalar hairs can be induced by large non-minimal coupling parameters satisfying (12).

III. NEUTRAL SCALAR FIELD HAIRS NON-MINIMALLY COUPLED TO MAXWELL FIELDS

We numerically solve the equation (6) together with boundary conditions (7). Besides parameters $r_s, M, Q, l$ and $\alpha$, we also need initial values of $\psi(r_s)$ and $\psi'(r_s)$ to integrate the equation. The reflecting condition of (7) gives the value $\psi(r_s) = 0$. According to the symmetry $\psi \rightarrow k\psi$ of equation (6), we firstly set $\psi'(r_s) = 1$ without loss of generality. Since the equation (6) also satisfies the symmetry $r \rightarrow \gamma r, \mu \rightarrow \mu/\gamma, M \rightarrow \gamma M, Q \rightarrow \gamma Q$, we use dimensionless parameters $\mu r_s, \mu M, \mu Q, l$ and $\alpha$ to describe the system. For given values of $\mu r_s, \mu M, \mu Q$ and $l$, using standard shooting methods, we search for the proper $\alpha$ with the vanishing condition $\psi(\infty) = 0$. The equation of motion of the scalar field is linear with respect to $\psi$ and the solution is scale free. After getting numerical solutions, we modify the boundary condition $\psi'(r_s) = 1$ so that $\alpha\psi^2 \ll 1$ holds. In this work, we take very small $\alpha\psi^2$ satisfying $\alpha\psi^2 < 10^{-7}$.

In the case of $\mu r_s = 2.7, \mu M = 1.5, \mu Q = 1.0$ and $l = 0$, we choose various $\alpha$ to try to get the physical solution with $\psi(\infty) = 0$. As shown by red curves in Fig. 1, if we choose $\alpha = 322$, the solution diverges quickly to be $\infty$. For green curves in Fig. 1, if we choose a little larger value $\alpha = 323$, then the solution decreases to be $-\infty$ in the larger $r$ region. It turns out that $\alpha = 322$ and $\alpha = 323$ are not related to the physical scalar field solution with decaying behaviors at infinity.

![FIG. 1: (Color online) We show behaviors of $\psi(r)$ in cases of $\mu r_s = 2.7, \mu M = 1.5, \mu Q = 1.0, l = 0$ and different values of $\alpha$. The red line corresponds to $\alpha = 322$ and the green line is with $\alpha = 323$.](image)
In fact, general mathematical solutions of equation (6) behave as $\psi \approx A \cdot \frac{1}{r_1 e^{-\mu r}} + B \cdot \frac{1}{r_1 e^{\mu r}}$ with $r \to \infty$. The red line of Fig. 1 corresponds to $B > 0$ and the green line of Fig. 1 represents the case of $B < 0$. As the value $B$ should change continuously with $\alpha$, indicating the existence of a critical $\alpha$ corresponding to $B = 0$. For this critical $\alpha$, physical scalar fields asymptotically decay as $\psi \propto \frac{1}{r} e^{-\mu r}$ at infinity. With $\mu r_s = 2.7$, $\mu M = 1.5$, $\mu Q = 1.0$ and $l = 0$, we numerically obtain a discrete value $\alpha \approx 322.083016$, which corresponds to the solution satisfying $\psi(\infty) = 0$. We plot the physical solution with blue curves in Fig. 2, which asymptotically approaches zero in the far region. For higher modes $l \geq 1$, we showed physical solutions with discrete $\alpha$ in Fig. 3. Similarly, in other cases of black holes, scalar hairy configurations are also characterized by discrete coupling parameters in the linearized regime.

![FIG. 2: (Color online) We show the function $\psi(r)$ in the case of $\mu r_s = 2.7$, $\mu M = 1.5$, $\mu Q = 1.0$, $l = 0$ and $\alpha = 322.083016$.](image1)

![FIG. 3: (Color online) We Plot the function $\psi(r)$ in the case of $\mu r_s = 2.7$, $\mu M = 1.5$ and $\mu Q = 1.0$. The left panel corresponds to $l = 1$ and $\alpha = 347.375787$. The right panel is with $l = 2$ and $\alpha = 397.388732$. The two physical solutions with different $l$ behave very similarly to each other.](image2)

Now we study how parameters $\mu r_s$, $\mu M$, $\mu Q$ and $l$ can affect the discrete coupling parameter $\alpha$, which corresponds to the decaying scalar field. In Table I, for $\mu M = 1.5$, $\mu Q = 1.0$ and fixed $l$, we show effects of $\mu r_s$ on discrete $\alpha$. It can be seen that a larger radius $\mu r_s$ corresponds to a larger discrete $\alpha$. With $\mu r_s = 2.7$, $\mu Q = 1$ and fixed $l$, according to data in Table II, the discrete $\alpha$ decreases as we choose a larger star mass.
In Table III, we see that the discrete $\alpha$ decreases as a function of the star charge $\mu Q$. Results in Table III also implies that $\alpha$ becomes smaller when we choose a small $\sqrt{\frac{Q^2}{\mu^2} - \frac{\mu^2}{\mu M}} = \frac{\mu Q}{\mu M}$, which is qualitatively the same as cases of black holes expressed by analytical formula (17) in [64]. From data in Tables I, II and III, we see that larger spherical harmonic index $l$ leads to a larger discrete coupling parameter $\alpha$.

### TABLE I: The parameter $\alpha(l)$ with $\mu M = 1.5, \mu Q = 1.0$ and various $\mu r_s$

| $\mu r_s$ | 2.62 | 2.66 | 2.70 | 2.74 | 2.78 |
|------------|------|------|------|------|------|
| $\alpha(l = 0)$ | 258.102979 | 298.304665 | 322.083016 | 343.522473 | 364.215671 |
| $\alpha(l = 1)$ | 280.963138 | 322.681432 | 347.375787 | 369.638470 | 391.118639 |
| $\alpha(l = 2)$ | 326.150845 | 370.872697 | 397.388732 | 421.291276 | 444.340681 |

### TABLE II: The parameter $\alpha(l)$ with $\mu r_s = 2.7, \mu Q = 1.0$ and various $\mu M$

| $\mu M$ | 1.48 | 1.49 | 1.50 | 1.51 | 1.52 |
|----------|------|------|------|------|------|
| $\alpha(l = 0)$ | 329.865858 | 326.322158 | 322.083016 | 316.754944 | 309.433789 |
| $\alpha(l = 1)$ | 355.357693 | 351.725147 | 347.375787 | 341.903046 | 334.372292 |
| $\alpha(l = 2)$ | 405.760364 | 401.953940 | 397.388732 | 391.632426 | 383.690319 |

### TABLE III: The parameter $\alpha(l)$ with $\mu r_s = 2.7, \mu M = 1.5$ and various $\mu Q$

| $\mu Q$ | 0.92 | 0.96 | 1.00 | 1.04 | 1.08 |
|----------|------|------|------|------|------|
| $\alpha(l = 0)$ | 351.506957 | 339.345528 | 322.083016 | 304.458791 | 287.540951 |
| $\alpha(l = 1)$ | 380.541786 | 366.508043 | 347.375787 | 328.019191 | 309.520394 |
| $\alpha(l = 2)$ | 437.965671 | 420.222598 | 397.388732 | 374.603527 | 352.976403 |

### IV. CONCLUSIONS

We investigated formations of neutral scalar field hairs outside asymptotically flat spherical horizonless reflecting stars. We considered scalar fields non-minimally coupled to Maxwell fields. We showed that the coupling parameter plays an important role in scalar condensations. We analytically got a bound on the coupling parameter expressed in the form $\alpha \leq \frac{\mu^2(M + \sqrt{M^2 - Q^2})^l + (l+1)(M + \sqrt{M^2 - Q^2})^2}{\sqrt{Q^2}}$, where $\alpha$ is the coupling parameter, $\mu$ is the scalar field mass, $M$ is the star mass, $Q$ is the star charge and $l$ is the spherical harmonic index. For $\alpha$ below this bound, no neutral scalar hair theorem holds. In contrast, for large $\alpha$ above this bound, with shooting methods, we obtained regular scalar hairy configurations with a horizonless reflecting
star in the center. We also examined effects of star radii, star mass, star charge and spherical harmonic index on condensations of neutral scalar fields.

Acknowledgments

This work was supported by the Shandong Provincial Natural Science Foundation of China under Grant No. ZR2018QA008. This work was also supported by a grant from Qufu Normal University of China under Grant No. xkjje201906.

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