Gravitational potential of a point mass in a brane world

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Abstract

In brane world models, combining the extra dimensional field modes with the standard four dimensional ones yields interesting physical consequences that have been proved from high energy physics to cosmology. Even some low energy phenomena have been considered along these lines to set bounds on the brane model parameters. In this work we extend to the gravitational realm a previous result which gave finite electromagnetic and scalar potentials and self energies for a source looking pointlike to an observer sitting in a 4D Minkowski subspace of the single brane of a Randall-Sundrum spacetime including compact dimensions. We calculate here the gravitational field for the same type of source by solving the linearized Einstein equations. Remarkably, it turns out to be also non singular. Moreover, we use gravitational experimental results of the Cavendish type and the Parameterized Post Newtonian (PPN) coefficients, to look for admissible values of the brane model parameters. The anti de Sitter radius hereby obtained is concordant with previous results based on Lamb shift in hydrogen. However, the resulting PPN parameters lie outside the acceptable value domain.

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I. INTRODUCTION

We are close to celebrate 100 years of the birth of General Relativity (GR), one of the most beautiful and spectacular theories in physics ever conceived. GR is beautiful because at the time, it introduced deep and unexpected physical concepts that allowed to understand the gravitational field and its relation with the geometry of the space-time. It is spectacular because, despite its age, the GR equations of motion have remained immutable in form and they describe with great accuracy most of the observable gravitational physics. Over the years GR has passed most of the experimental tests concerning the theory, however it is well known that there exist some phenomena escaping an accurate description within the framework of GR and the Standard Model of particle physics such as dark matter and dark energy. In order to develop a consistent theory that could describe these kind of phenomena, physicists have tried to modify either GR or Quantum Mechanics and consider possible extensions of the Standard Model. Indeed, the effort to place limits on possible deviations from the standard formulations of such theories continue nowadays.

Since its inception there have been many attempts to modify GR with different purposes. Soon after its conception there were notable proposals with the idea to extend it and incorporate it in a larger unified theory. A relevant example for the purpose of this work is the higher dimensional theory introduced by Kaluza [1] and refined by Klein [2]. More recently and with the aim to solve the hierarchy problem, the ideas of large extra dimensions [3–5] and brane worlds were introduced [6–10]. Of course in the literature there are many other attempts to modify GR (see e.g. [11] and references therein) and currently people continue exploring the physical consequences predicted from them all and most important confronting them with experimental data. This work follows the same strategy, we will explore a particular characteristic of the gravitational field, specifically the behavior of the gravitational potential generated by a point like source in the so called RSIIp model which modifies GR by including extra dimensions, and we will confront it with experimental data available today.

The RSIIp model is an extension of the 5D Randall-Sundrum (RS) model with one brane (RSII) extended by $p$ compact extra dimensions (RSIIp) [12–14], its construction was motivated for the need to improve the localization properties of matter fields within the standard RS model. Specifically in the RSII model there exists a problem to localize spin 1
fields on the brane and a way out to this problem can be achieved by extending the model
with $p$ compact dimensions \cite{15,16}. Thus the RSII$p$ setup contains all the nice features of
the RS model and additionally has the advantage to localize every kind of field on the brane.
These higher dimensional models have the property to modify gravity in the low scale length
regime and a huge amount of physical phenomena have been studied over the years, ranging
from particle physics (see e.g. \cite{17,18} and references therein) to cosmology (see e.g. \cite{19,20}
and references therein). Moreover, recently it was shown that an electric source lying in
the single brane of a RSII$p$ spacetime which looks pointlike to an observer sitting in usual
3D space, produces a static potential which is nonsingular at the 3D point position \cite{21,22}
and, furthermore, it matches Coulomb potential outside a small neighborhood. Amusingly,
coping with classical singularities goes back to the non-linear proposal made by Born and
Infeld \cite{23}. In regard to the divergences in field theory, over the years there have been many
attempts to formulate a theory that avoids the problem, or at least that could improve, for
instance, the high energy behavior of GR. Among them we have for instance: String Theory
(see e.g. \cite{24} and references therein), non commutative theories (see e.g. \cite{25} and references
therein) and the recent attempt made by Horava \cite{26} of a modified UV theory of gravity.

In this work we extend the analysis of \cite{21} to the case of the gravitational field. As we
will show the classical potential due to an effective 4D punctual source becomes regular at
the position of the source in analogous way to the scalar and gauge cases. To complement
our study we compare the consequences of this feature with some experimental observations,
in particular we have chosen to compare the predictions of the model with the experimental
data of a Cavendish type experiment which imposes a bound to the anti de Sitter (adS)
radius of the bulk adS metric. We also obtain the Parameterized Post Newtonian (PPN)
coefficients of the resulting effective theory.

The paper is organized as follows. In section II we describe briefly the RSII$p$ scenarios.
In section III we discuss the linearized Einstein equations in the low energy regime for a
massive particle with the topology of $T^p$ torus, but which is seen as punctual by an observer
in our 4D world. In section IV we obtain the metric perturbations and in section V we
discuss a Cavendish type experiment and we give the PPN coefficients. We give a short
discussion of our results in section VI.
II. RANDALL-SUNDRUM II\_p SCENARIOS

The way in which the Randall-Sundrum II\_p (RSII\_p) scenarios arise has been discussed several times in the literature (see e.g. [12–16]), so here we just give a short summary including the most important features of the model. The RSII\_p setups consist of a \((3 + p)\)-brane with \(p\) compact dimensions and positive tension \(\sigma\), embedded in a \((5 + p)\) spacetime whose metrics are two patches of anti-de Sitter (AdS\(_{5+p}\)) having curvature radius \(\epsilon\) (for convenience in some equations we will use instead of the radius, its inverse: \(\kappa \equiv \epsilon^{-1}\)). The model arise from considering the \((5 + p)\D\) Einstein action with bulk cosmological constant \(\Lambda\) and the action of a \((3 + p)\)-brane

\[
S = \frac{1}{16\pi G_{5+p}} \int d^4x \, dy \prod_{i=1}^{p} R_i d\theta_i \sqrt{|g^{(5+p)}|} \left( R^{(5+p)} - 2\Lambda \right) + S_{brane},
\]

which leads to the Einstein equations of motion

\[
R_{MN} - \frac{1}{2} R g^{(5+p)}_{MN} + \Lambda g^{(5+p)}_{MN} = 8\pi G_{5+p} T_{MN}.
\]

In these equations we use the following notation for the \(5+p\) coordinates: \(X^M \equiv (x^\mu, \theta_i, y)\), where \(\mu = 0, 1, 2, 3\), and \(i = 1, \ldots, p\). The four coordinates \(x^\mu\) denote to the coordinates that mimic our universe, the \(\theta_i\)'s \(\in [0, 2\pi]\) denote to the \(p\) compact coordinates and the \(R_i\)'s signal the sizes of the corresponding compact dimensions. Finally \(y\) denotes the non-compact extra dimension. The superscript in the determinant \(g^{(5+p)}\) emphasizes the fact that the metric is \((5 + p)\D\). \(G_{5+p}\) is the Newton constant in \((5 + p)\D\) and the energy-momentum tensor

\[
T_{MN} \equiv \frac{2}{\sqrt{|g^{(5+p)}|}} \frac{\delta S}{\delta g^{MN}},
\]

corresponds to the one produced by the brane.

With this setup and appropriate fine-tuning between the brane tension \(\sigma\) and the bulk cosmological constant \(\Lambda\), which are related to \(\kappa\) as follows

\[
\sigma = \frac{2(3 + p)}{8\pi G_{5+p}} \kappa, \quad \Lambda = -\frac{(3 + p)(4 + p)}{16\pi G_{5+p}} \kappa^2 = -\frac{(4 + p)\sigma}{4} \kappa,
\]

there exists a solution to the \((5+p)\D\) Einstein equations with metric

\[
ds_{5+p}^2 = e^{-2\kappa y} \left[ \eta_{\mu\nu} dx^\mu dx^\nu - \sum_{i=1}^{p} R_i^2 d\theta_i^2 \right] - dy^2.
\]

Here \(\eta_{\mu\nu}\) is the 4D Minkowski metric and without loss of generality it was assumed that the brane is at the position \(y = 0\). At \(y =\)constant we have 4D flat hypersurfaces extended by \(p\) compact extra dimensions.
The interest in these setups comes from its property of localizing on the brane: scalar, gauge and gravity fields due to the gravity produced by the brane itself. We emphasize that this property is valid whenever there are $p$ extra compact dimensions \[12, 13\]. In the limiting case $p = 0$, the model localizes scalar and gravity fields but not gauge fields. A short discussion about the consistency of both the KK and the RS compactifications, as well as a discussion of the moduli fixing mechanisms and stability of the setup can be found for instance in \[21\]. In the literature there are already different analysis of low energy physics effects in these setups such as the electric charge conservation \[12\], the Casimir effect between two conductor hyperplates \[27–30\], the Liennard-Wiechert potentials, the Hydrogen Lamb shift \[22\] and perturbations to the ground state of the Helium atom \[31\] among others.

III. LOW ENERGY LINEARIZED EINSTEIN EQUATIONS

In this section we determine the linearized Einstein equations for the perturbations produced by a static source. In analogy with the scalar and gauge fields cases discussed in \[21\], we consider a source with the topology of a $p$-dimensional torus sitting on the $(3 + p)$ brane, which is seen as a punctual mass from the perspective of an observer living in the usual 3D low-energy observable part of the brane. In order to solve the equations, we follow closely the technique used in \[32\] where authors studied highly energetic particles that leave the 4D brane and propagate into the bulk of the 5D RSII model. The main difference of the physical situation discussed here respect to the ones previously reported in the literature \[32–34\], is the inclusion of the $p$ extra compact dimensions.

A. Linearized Einstein equations

Our starting point are the $(5 + p)$D Einstein equations \[2\]. Taking the trace of these equations and replacing the value of $R$, we obtain the convenient equivalent form

$$R_{MN} = 8\pi G_{5+p} \left(T_{MN} - \frac{1}{3+p} T g_{MN} \right) + \frac{2}{3+p} \Lambda g_{MN}. \quad (5)$$

In general the linearized Einstein equations that result from considering metric perturbations $h_{MN}$ to a known metric solution $g_{MN}$

$$ds^2 = g_{MN}dx^Mdx^N + h_{MN}dx^Mdx^N, \quad (6)$$
and energy-momentum tensor perturbations $\delta T_{MN}$, to the equations of motion (5) are given by

$$\delta R_{MN} = 8\pi G_{5+p} \left[ \delta T_{MN} - \frac{1}{3+p} (h_{MNT} + g_{MN}\delta T) \right] + \frac{2}{3+p} \Lambda h_{MN},$$

where (see for instance [35])

$$\delta R_{MN} = -\frac{1}{2} \left[ \nabla_M \nabla_N \hat{h} + \nabla^A \nabla_A h_{MN} - \nabla^A \nabla_M h_{NA} - \nabla^A \nabla_N h_{MA} \right],$$

and $\hat{h} \equiv g^{MN} h_{MN}$. Following [32] we will work in Gaussian Normal (GN) coordinates. In such a frame one has

$$h_{yy} = h_{y\bar{M}} = 0,$$

where the coordinates $X^{\bar{M}}$ label the coordinates of the 4D flat brane and the compact dimensions: $X^{\bar{M}} \equiv \{x^\mu, R_i \theta_i\}$. Accordingly the linearized theory is described by the metric

$$ds^2 = a^2(y)\eta_{\bar{M}\bar{N}} dx^{\bar{M}} dx^{\bar{N}} + h_{\bar{M}\bar{N}} dx^{\bar{M}} dx^{\bar{N}} - dy^2,$$

where $\eta_{\bar{M}\bar{N}} = \text{diag}(1,-1,\ldots,-1)$ is a $(4+p)$D flat metric. We have also introduced the shorthand notation $a(y) \equiv e^{-k|y|}$. It is clear that for the metric of the RSII\textsubscript{p} setup $\hat{h}$ is given simply by $\hat{h} = a^{-2}\eta_{\bar{M}\bar{N}} h_{\bar{M}\bar{N}} \equiv a^{-2}h$.

As for the perturbation of the energy-momentum tensor, we shall consider a static source at the position $y_0$ with the topology of a $p$D torus, i.e., we consider that the massive object is located a distance $y_0$ away from the brane. From these considerations it is clear that during the computation, the perturbed energy-momentum tensor resides entirely on the bulk and is given by

$$\delta T^{MN} = \frac{m^{(5+p)}}{\sqrt{|g^{(5+p)}|}} \frac{dx^M}{ds} \frac{dx^N}{ds} \delta^3(\vec{x} - \vec{x}_0) \delta(y - y_0),$$

where $\frac{dx^M}{ds} = (1, \vec{0})$. A technicality of our calculation is that if $y_0 > 0$ in (11), it means we are considering an energy-momentum tensor residing to the right of the brane, however the RSII\textsubscript{p} model owns the symmetry $z \rightarrow -z$. Then although we will work entirely only to the right of the brane it should be understood that matter is symmetric with respect to the brane and therefore there exists another source located at position $-y_0$. The two symmetrical located sources together with the fact that we are considering only symmetric perturbations to the metric justify the way in which the computation is done [32]. Because we are interested in the gravitational potential produced by a source placed on the brane, so after computing the solution to the linearized equations we will consider the limit $y_0 \rightarrow 0,$
and the perturbations will appear as generated by a source of mass \( M = 2m^{(5+p)} \) on the brane.

It is clear that for an energy-momentum tensor on the bulk, the second term on the right hand of the equation (7) vanishes and the third term becomes: \( \delta T = a^{-2} \eta^{MN} \delta T_{MN} \equiv a^{-2} \delta T^0_0 \). Under these considerations the non vanishing linearized Einstein equations on the bulk are

\[
\delta R_{yy} = 8\pi G_{5+p} \frac{1}{3 + p} \delta T^0_0,
\]

\[
\delta R_{\bar{M}\bar{N}} - \frac{2\Lambda}{3 + p} h_{\bar{M}\bar{N}} = 8\pi G_{p+5} \left[ \delta T_{\bar{M}\bar{N}} - \frac{1}{3 + p} \eta_{\bar{M}\bar{N}} \delta T^0_0 \right],
\]

where the variation of the Ricci tensor (8), can be explicitly written as

\[
\delta R_{yy} = -\partial_y \left[ \frac{\partial_y h}{2a^2} \right],
\]

\[
\delta R_{\bar{M}\bar{N}} = \frac{1}{2} \partial^2_y h_{\bar{M}\bar{N}} - \frac{p}{2} \kappa \partial_y h_{\bar{M}\bar{N}} + 2\kappa^2 h_{\bar{M}\bar{N}} - \left( \kappa^2 h + \frac{\kappa^2}{2} \partial_y h \right) \eta_{\bar{M}\bar{N}}
\]

\[
+ \frac{1}{2a^2} \left( \partial^L \partial_{\bar{M}} h_{\bar{N}L} + \partial^L \partial_{\bar{N}} h_{\bar{M}L} - \partial^L \partial_{\bar{L}} h_{\bar{M}\bar{N}} - \partial_{\bar{N}} \partial_{\bar{M}} h \right).
\]

Notice that the role of the \( p \) compact extra dimensions at the level of the variation of the Ricci tensor is given by the second term in the right hand side of the equation (15). In the case \( p = 0 \), we recover the expression of the variation of the Ricci tensor for the standard RS model [32, 36].

**B. The perturbation in modes**

In order to solve the linearized Einstein equations, we start solving equation (13) by inserting (15) into it

\[
\frac{1}{2} h''_{\bar{M}\bar{N}} - \frac{p}{2} \kappa h'_{\bar{M}\bar{N}} + \frac{1}{2a^2} \left( \partial^L \partial_{\bar{M}} h_{\bar{N}L} + \partial^L \partial_{\bar{N}} h_{\bar{M}L} - \partial^L \partial_{\bar{L}} h_{\bar{M}\bar{N}} - h_{,\bar{M}\bar{N}} \right) + 2\kappa^2 h_{\bar{M}\bar{N}}
\]

\[-(4 + p)\kappa^2 h_{\bar{M}\bar{N}} = 8\pi G_{p+5} \left[ \delta T_{\bar{M}\bar{N}} - \frac{1}{3 + p} \eta_{\bar{M}\bar{N}} \delta T^0_0 \right] + \left( \kappa^2 h + \frac{\kappa^2}{2} h' \right) \eta_{\bar{M}\bar{N}},
\]

where the prime denotes the derivative with respect to the \( y \) coordinate. At this point it is convenient to introduce a consideration about the modes spectrum of the metric perturbations into the equation, dictated by the geometry of the setup. Formally we write down the metric perturbation in a Fourier series expansion due to the compact coordinates

\[
h_{\bar{M}\bar{N}}(x, \theta_i, y) = \prod_{k=1}^{p} \frac{1}{\sqrt{2\pi R_k}} \sum_{\vec{n}} (h_{\bar{M}\bar{N}}(x, y))_{(\vec{n})} e^{i\vec{n} \vec{\theta}},
\]
where \( \vec{n} \) denotes the collection of \( p \) different indexes \( \vec{n} = (n_1, n_2, \ldots, n_p) \) taking values in \( \mathbb{Z} \), \( \vec{\theta} \) is a \( p \) dimensional vector whose components are the \( p \) compact coordinates \( \theta_k : \vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_p) \) and \( \sum_{\vec{n}} \) is the collection of \( p \) sums \( \sum_{\vec{n}} = \sum_{n_1 = -\infty}^{\infty} \cdots \sum_{n_p = -\infty}^{\infty} \). The functions \( e^{i\vec{n} \cdot \vec{\theta}} \) correspond to the basis of the Fourier decomposition along the compact directions. It is well known that toroidal dimensional reductions a la Kaluza-Klein lead to consistent lower dimensional theories (see e.g., [37] and references therein) which although do not come with a mechanism to fix the radii of the \( T^p \) torus, invoking agreement with phenomenology at enough low energies, in particular agreement with the value of the electron charge, it is possible to set a bound to the radius of the order Planck length [2]. In the following we shall consider a low energy approximation so that we truncate the massive KK modes of the compact dimensions but keeping those that correspond to the noncompact dimension (so far encoded in the \( y \) dependence of \( h_{MN} \)), meaning that we assume the scale energy of the former is much smaller than that of the latter. Under these considerations we are performing the dimensional reduction on the \( T^p \) torus or equivalently we are keeping only the zero mode of the Fourier expansion, i.e.

\[
h_{\vec{M}\vec{N}}(x, \theta_i, y) \approx (h_{\vec{M}\vec{N}}(x, y))_0.
\]

\( \xi \)From here onwards we replace in equations (12) and (13), the whole metric perturbation by its zero mode.

Under this consideration the laplacian operator simplifies to: \( \partial^\vec{L} \partial_{\vec{L}} = \Box + \partial^\theta_i \partial_{\theta_i} = \Box \), and equation (16) can be rewritten as

\[
\frac{1}{2} h''_{\vec{M}\vec{N}} - \frac{p}{2} \kappa h'_{\vec{M}\vec{N}} + \frac{1}{2a^2} \left( \partial^\vec{L} \partial_{\vec{L}} h_{\vec{N}\vec{L}} + \partial^\vec{L} \partial_{\vec{L}} h_{\vec{M}\vec{L}} - \Box h_{\vec{M}\vec{N}} - h_{,\vec{M}\vec{N}} \right) + 2\kappa^2 h_{\vec{M}\vec{N}} = - (4 + p)\kappa^2 h_{\vec{M}\vec{N}} = 8\pi G_{p+5} \left[ \delta T_{\vec{M}\vec{N}} - \frac{1}{3 + p} \eta_{\vec{M}\vec{N}} \delta T_0^0 \right] + \left( \kappa^2 h + \frac{\kappa}{2} h' \right) \eta_{\vec{M}\vec{N}}. \tag{19} \]

Introducing the shorthand definition

\[
\xi_{\vec{M}} = h^\vec{L}_{\vec{M},\vec{L}} - \frac{1}{2} h_{,\vec{M}}, \tag{20} \]

equation (19) takes the form

\[
\frac{1}{2} h''_{\vec{M}\vec{N}} - \frac{p}{2} \kappa h'_{\vec{M}\vec{N}} - \frac{1}{2a^2} \Box h_{\vec{M}\vec{N}} - (2 + p)\kappa^2 h_{\vec{M}\vec{N}} = 8\pi G_{p+5} \left[ \delta T_{\vec{M}\vec{N}} - \frac{1}{3 + p} \eta_{\vec{M}\vec{N}} \delta T_0^0 \right] + \left( \kappa^2 h + \frac{\kappa}{2} h' \right) \eta_{\vec{M}\vec{N}} - \frac{1}{2a^2} \left( \xi_{\vec{M},\vec{N}} + \xi_{\vec{N},\vec{M}} \right). \tag{21} \]
We can consider the following gauge transformation
\[ h_{\bar{M}\bar{N}} = \tilde{h}_{\bar{M}\bar{N}} + u_{\bar{M},\bar{N}} + u_{\bar{N},\bar{M}}, \]  
(22)
where \( u_{\mu} \) satisfies
\[ u''_{\bar{M}} - p\kappa u'_{\bar{M}} - 2(2 + p)\kappa^2 u_{\bar{M}} - \frac{1}{a^2} \Box u_{\bar{M}} = -\frac{1}{a^2} \xi_{\bar{M}}. \]  
(23)
It follows then that \( \bar{h}_{\bar{M}\bar{N}} \) should satisfy
\[ \frac{1}{2} \bar{h}_{\bar{M}\bar{N}}'' - \frac{p}{2} \kappa \bar{h}_{\bar{M}\bar{N}}' - \frac{1}{2a^2} \Box \bar{h}_{\bar{M}\bar{N}} - (2 + p)\kappa^2 \bar{h}_{\bar{M}\bar{N}} = 8\pi G_{p+5} \left[ \delta T_{\bar{M}\bar{N}} - \frac{1}{3 + p} \eta_{\bar{M}\bar{N}} \delta T_{0}^0 \right] + \left( \kappa^2 + \frac{\kappa^2}{2} \right) \eta_{\bar{M}\bar{N}}. \]  
(24)
The strategy to solve this equation is the following. We can think the right hand side of the equation \( (24) \) as an effective energy-momentum tensor \( T^{\text{eff}}_{\bar{M}\bar{N}} \), in such a way that
\[ 8\pi G_{p+5} \left[ \delta T_{\bar{M}\bar{N}} - \frac{1}{3 + p} \eta_{\bar{M}\bar{N}} \delta T_{0}^0 \right] + \left( \kappa^2 + \frac{\kappa^2}{2} \right) \eta_{\bar{M}\bar{N}} = 8\pi G_{p+5} T^{\text{eff}}_{\bar{M}\bar{N}}. \]  
(25)
Therefore the equation \( (24) \) takes the form
\[ \frac{1}{2} \bar{h}_{\bar{M}\bar{N}}'' - \frac{p}{2} \kappa \bar{h}_{\bar{M}\bar{N}}' - \frac{1}{2a^2} \Box \bar{h}_{\bar{M}\bar{N}} - (2 + p)\kappa^2 \bar{h}_{\bar{M}\bar{N}} = 8\pi G_{p+5} T^{\text{eff}}_{\bar{M}\bar{N}}. \]  
(26)
Solving this equation requires to know the solutions to the homogeneous equations, once we have these solutions we can compute the Green function and with it solving the inhomogeneous equation \( (26) \). It is also convenient at this point to expand \( \bar{h}_{\bar{M}\bar{N}}(x, y) \) in terms of the functions \( \psi_m(y) \), which correspond to the modes structure of the metric perturbations due to the non-compact dimension \( y \)
\[ (h_{\bar{M}\bar{N}}(x, y))_{\bar{0}} = \left( \int (h_{\bar{M}\bar{N}}(x))_m \psi_m(y) \, dm \right)_{(\bar{0})}. \]  
(27)
Plugging this ansatze in the left hand side of equation \( (26) \), allows us to perform a separation of variables in the differential operator. Introducing the separation constant \( m \) lead us to have an equation for \( \psi_m(y) \) of the following form
\[ \left( \partial_y^2 - p\kappa \partial_y - 2(2 + p)\kappa^2 + \frac{m^2}{a^2} \right) \psi_m(y) = 0. \]  
(28)
This equation can be rewritten as a Bessel equation. In order to do that we perform the variable change \( \xi(y) = \epsilon a^{-1}(y) \), and we introduce the rescaled function \( \tilde{\psi}(\xi) = \xi^{p/2} \psi(\xi) \), obtaining
\[ \left[ \partial_{\xi}^2 + \frac{1}{\xi} \partial_{\xi} + m^2 - \frac{\alpha^2}{\xi^2} \right] \tilde{\psi} = 0, \]  
(29)
where the constant $\alpha \equiv 2 + \frac{p}{2}$, contains the information about the number of extra compact dimensions.

For the massless mode ($m = 0$) the solution is

$$\tilde{\psi}_0(\xi) = A_1 \xi^\alpha + A_2 \xi^{-\alpha} \Rightarrow \psi_0(\xi) = a_1 \xi^{p+2} + a_2 \xi^{-2},$$

(30)

where $a_i$ are integration constants. We take $a_1 = 0$ in order to have a normalizable solution, which is explicitly given by

$$\psi_0(y) = \sqrt{ \left( 1 + \frac{p}{2} \right) \kappa } e^{-2\kappa y}. $$

(31)

For the massive modes ($m > 0$) we obtain

$$\psi_m(y) = e^{\frac{m}{2}y} \sqrt{\frac{m}{2\kappa}} \left[ a_m J_\alpha \left( \frac{m}{\kappa} e^{\kappa y} \right) + b_m N_\alpha \left( \frac{m}{\kappa} e^{\kappa y} \right) \right],$$

(32)

where the constants $a_m$ and $b_m$ are given by

$$a_m = -\frac{A_m}{\sqrt{1 + A_m^2}}, \quad b_m = \frac{1}{\sqrt{1 + A_m^2}},$$

(33)

with

$$A_m = \frac{N_{\alpha-1} \left( \frac{m}{\kappa} \right) - 2 \kappa N_\alpha \left( \frac{m}{\kappa} \right)}{J_{\alpha-1} \left( \frac{m}{\kappa} \right) - 2 \kappa J_\alpha \left( \frac{m}{\kappa} \right)}.$$  

(34)

In order to simplify further this expression it is convenient to take the approximation of light modes $m \ll \kappa^{-1}$, this is plausible because these are the modes contributing the most to the potential. In this approximation

$$A_m = \frac{\Gamma(\alpha - 1) \Gamma(\alpha)}{\pi} \left( \frac{m}{2\kappa} \right)^{2-2\alpha},$$

(35)

and therefore the coefficients $a_m$ and $b_m$ are given by

$$a_m = -1, \quad b_m = \frac{\pi}{\Gamma(\alpha - 1) \Gamma(\alpha)} \left( \frac{m}{2\kappa} \right)^{2\alpha - 2}. $$

(36)

Plugging these coefficients into equation (32) and considering the same light modes approximation in the Bessel and Neumann functions we get

$$\psi_m(0) = \sqrt{\frac{m}{2\kappa} \frac{1}{\Gamma(\alpha - 1)} \left( \frac{m}{2\kappa} \right)^{\alpha - 2}},$$

(37)

$$\psi_m(y) = -e^{\frac{\kappa y}{2}} \sqrt{\frac{m}{2\kappa} J_\alpha \left( \frac{m}{\kappa} e^{\kappa y} \right)}. $$

(38)

Notice we are computing the massive modes in two different points of the $y$ coordinate because with these functions we are constructing the two points Green function.
C. The Green function

With the eigenfunctions $\psi_m(y)$ it is straightforward to construct the Green function
\[
G_R(x, x', y = 0, y') = -\frac{\psi_0(0)\psi_0(y')}{4\pi r} - \int_0^\infty dm\psi_m(0)\psi_m(y')\frac{e^{-mr}}{4\pi r} \tag{39}
\]
\[
= -\frac{1}{4\pi r} \left(1 + \frac{p}{2}\right) \frac{1}{\kappa \xi^2} + \frac{\xi^{\frac{p}{2}}}{\Gamma(\alpha - 1)2^{\alpha - 1}\kappa^{1-\alpha+\frac{p}{2}}\sqrt{\pi}} \int_0^\infty dm\, m^\alpha J_{\alpha}(m\xi) \frac{e^{-mr}}{m}. \tag{40}
\]

Explicit evaluation of this function depends on the parity of the number $p$ of compact dimensions.

1. $p$ odd:

For this case we have that $\alpha$ takes semi-integer values and the Green function is
\[
G_R(x, x', y = 0, y') = \frac{1}{4\pi r} \sqrt{\frac{2}{\pi}} \frac{(-1)^{\alpha - \frac{1}{2}} \xi^{\alpha + \frac{p}{2}}}{\Gamma(\alpha - 1)2^{\alpha - 1}\kappa^{1-\alpha+\frac{p}{2}}} \left( \frac{d}{\xi d\xi} \right)^{\alpha - \frac{1}{2}} \left[ \frac{\pi}{2\xi} - \frac{\arctan \left( \frac{r}{\xi} \right)}{\xi} \right] - \frac{1}{4\pi r} \left(1 + \frac{p}{2}\right) \frac{1}{\kappa \xi^2}. \tag{41}
\]

Using the relation
\[
\left( \frac{d}{\xi d\xi} \right)^{\alpha - \frac{1}{2}} \left[ \frac{1}{\xi} \right] = \frac{(-1)^{\alpha - \frac{1}{2}} [2 (\alpha - 1)]!(\alpha - 1)!}{2^{\alpha - \frac{3}{2}} 2^{-2\alpha + 2} \sqrt{\pi} \xi^{2\alpha} \Gamma(\alpha - 1)^2 (2 (\alpha - 1))!} = \frac{(-1)^{\alpha - \frac{1}{2}} (\alpha - 1)!}{2^{-\alpha + \frac{1}{2}} \sqrt{\pi} \xi^{2\alpha}}, \tag{42}
\]
the Green function can be written as
\[
G_R(x, x', y = 0, y') = -\frac{1}{4\pi r} \sqrt{\frac{2}{\pi}} \frac{(-1)^{\alpha - \frac{1}{2}} \xi^{\alpha + \frac{p}{2}}}{\Gamma(\alpha - 1)2^{\alpha - 1}\kappa^{1-\alpha+\frac{p}{2}}} \left( \frac{d}{\xi d\xi} \right)^{\alpha - \frac{1}{2}} \left[ \frac{\arctan \left( \frac{r}{\xi} \right)}{\xi} \right]. \tag{43}
\]

The derivative can be evaluated, recalling the relation
\[
\frac{d}{\xi d\xi} f(\xi) = 2 \frac{d}{d\beta} f \left( \sqrt{\xi^2 + \beta} \right) \Big|_{\beta = 0}, \tag{44}
\]
which leads to the final form of the Green function
\[
G_R(x, x', y = 0, y') = -\frac{1}{4\pi r} \sqrt{\frac{2}{\pi}} \frac{(-1)^{\alpha - \frac{1}{2}} \xi^{\alpha + \frac{p}{2}}}{\Gamma(\alpha - 1)2^{\alpha - 1}\kappa^{1-\alpha+\frac{p}{2}}} \nonumber
\]
\[
\left[ -\Gamma \left( \alpha + \frac{1}{2} \right) \frac{r(-1)^{\alpha - \frac{1}{2}}}{2\alpha (r^2 + \xi^2)^{\alpha + \frac{1}{2}}} F \left( 1, \alpha + \frac{1}{2}; \alpha + 1; \frac{\xi^2}{r^2 + \xi^2} \right) \right] + \frac{(1)^{\alpha - \frac{1}{2}} \Gamma(\alpha)}{\sqrt{\pi} \xi^{2\alpha}} \arcsin \left( \frac{\xi}{\sqrt{r^2 + \xi^2}} \right) + \frac{(1)^{\alpha - \frac{1}{2}} \Gamma(\alpha) \arctan \left( \frac{r}{\xi} \right)}{\sqrt{\pi} \xi^{2\alpha}}. \tag{45}
\]
2. \( p \) even:

For even \( p \), \( \alpha \) takes integer values and the Green function is

\[
G_R(x, x', y = 0, y') = -\frac{1}{4\pi r} \left( 1 + \frac{p}{2} \right) \frac{1}{\kappa \xi^2} + \\
\frac{1}{4\pi r} \frac{(-1)^{\alpha} \xi^{\alpha + \frac{d}{2}}}{\Gamma(\alpha - 1)2^{\alpha - 1}\kappa^{1 - \alpha + \frac{d}{2}}} \left( \frac{d}{\xi d\xi} \right)^{\alpha - 1} \left[ \frac{1}{\xi^2} - \frac{r}{\xi^2 \sqrt{r^2 + \xi^2}} \right].
\]

In a similar way as the former case, we obtain finally the Green function for even compact dimensions

\[
G_R(x, x', y = 0, y') = \frac{1}{4\pi r} \frac{(-1)^{\alpha} \xi^{\alpha + \frac{d}{2}}}{\Gamma(\alpha - 1)2^{\alpha - 1}\kappa} \left( \frac{d}{\xi d\xi} \right)^{\alpha - 1} \left[ \frac{r}{\xi^2 \sqrt{r^2 + \xi^2}} \right].
\]

IV. SOLUTIONS

We are now in position to compute the solutions to the linearized Einstein equations in the low energy regime. The order in which the solutions are obtained is the following. We start solving the equation \((12)\) where the Ricci tensor is given by equation \((14)\). This happen because we have to know the expression for the combination: \(\kappa^2 h(x', y') + \frac{\kappa}{2} \partial_y h(x', y')\), in order to solve for the perturbations \(\bar{h}_{MN}\) of the equations \((13)\).

A. Solution of the \(yy\) equation

We start integrating twice equation \((12)\)

\[
- \partial_y \left[ \frac{\partial_y h}{2a^2} \right] = 8\pi G_{5+p} \frac{1}{3 + p} \delta T_0^0.
\]

After the first integral we directly get

\[
h' = -2a^2 \frac{8\pi G_{5+p}}{3 + p} \int_y^\infty dy \delta T_0^0 + 2a^2 C(x),
\]

and after the second integral we obtain

\[
h = -\int_y^\infty dy \left[ 2a^2 \frac{8\pi G_{5+p}}{3 + p} \int_y^z dy' \delta T_0^0(y') - 2a^2 C(x) \right] + D(x),
\]

here \(C(x)\) y \(D(x)\) are functions to be determined. From the explicit form of \(a(y)\) we can evaluate in a straightforward way, the second term of the integral in the equation above

\[
\int_y^\infty dy \frac{a^2(y)}{\kappa},
\]

(50)
whereas for the first term we use an integration by parts
\[ \int_{\gamma}^\infty dy y^2 a^2 \int_{\gamma}^\infty dz \delta T^0_0(z) = \frac{a^2}{\kappa} \int_{\gamma}^\infty dy \delta T^0_0 - \int_{\gamma}^\infty dy a^2 \frac{\delta T^0_0}{\kappa}, \] obtaining that \( h(y) \) is of the form
\[ h(y) = -\frac{8\pi G_{5+p}}{(3+p)\kappa} \left[ a^2 \int_{\gamma}^\infty dy \delta T^0_0 - \int_{\gamma}^\infty dy a^2 \delta T^0_0 \right] + \frac{a^2}{\kappa} C(x) + D(x). \] (52)
So far we have only considered the perturbation in the bulk. The role played by the brane in the solution appears through the junction conditions
\[ K_{\bar{M}\bar{N}} = -\frac{8\pi G_{5+p}}{2} \left( S_{\bar{M}\bar{N}} - \frac{1}{3+\eta_{\bar{M}\bar{N}}} a^2 S \right), \] (53)
which constitute a connection between the metric perturbations living in the bulk and the matter perturbations confined to the brane \( S_{\bar{M}\bar{N}} \) [38]. In a GN coordinate system, the extrinsic curvature is given by the simple expression
\[ K_{\bar{M}\bar{N}} = \frac{1}{2} \partial_y \left( a^2 \eta_{\bar{M}\bar{N}} + h_{\bar{M}\bar{N}} \right), \] (54)
whereas the energy-momentum tensor on the brane is given by
\[ S_{\bar{M}\bar{N}} = -\sigma \left( a^2 \eta_{\bar{M}\bar{N}} + h_{\bar{M}\bar{N}} \right) + \delta T_{\bar{M}\bar{N}}. \] (55)
In equation (53) we are using the definition \( S \equiv a^{-2} \eta^{\bar{M}\bar{N}} S_{\bar{M}\bar{N}} \). Plugging in the expressions (54) and (55) in the equation (53) and considering the energy momentum tensor (11) and the relation between the brane tension and the adS radius (3), we obtain after taking the trace of the junction condition that
\[ \partial_y h + 2\kappa h = \frac{8\pi G_{5+p}}{3+p} \delta T \bigg|_{y=0} = \frac{4\pi G_{5+p} \kappa}{3+p} \frac{m_{(5+p)}}{a^{2+p}(y')} \delta(y - y_0) \delta^3(\vec{x} - \vec{x}_0) \bigg|_{y=0}. \] (56)
This means that the points \( y = 0 \) and \( y_0 \) never coincide and therefore \( \delta(y' - y_0) \) is null. Substitution of expression (52) into Eq. (56), allows us to find the expression for the function \( D(x) \) which is given by the equation
\[ 2\kappa D(x) + 16\pi G_{5+p} \int_0^\infty a^2(y') \delta T(y') dy' = 0. \] (57)
Once we know the value of \( D(x) \), we can evaluate the combination of \( h \) and \( h' \) that appears in the definition (25) of \( T_{\bar{M}\bar{N}}^{\text{eff}} \)
\[ \kappa^2 h(x', y') + \frac{\kappa}{2} \partial_{y'} h(x', y') = -8\pi G_{5+p} \kappa \int_0^{y'} a^2(z) \delta T(z) dz = -\frac{8\pi G_{5+p}}{a^{2+p}(y_0)} \theta(y' - y_0) \delta^3(\vec{x}' - \vec{x}_0). \] (58)
B. The $\bar{h}_{00}$ component

Once we have the solution of the equation (12) and as a consequence the expression for the combination (58), we can compute the expressions for the metric perturbations. Using the Green function of the subsection (III C) we have that

$$\bar{h}_{00}(r, y = 0) = 8\pi G_{5+p} \int d^3 x' \int dy' G(\bar{x}, y = 0; x', y') \left[ \left( \delta T_{00}(x', y') - \frac{1}{3 + p} \eta_{00} \delta T^0_0(x', y') \right) \right]$$

$$+ \frac{1}{8\pi G_{5+p}} \left( \kappa^2 h(x', y') + \frac{\kappa}{2} \partial_y h(x', y') \right) \eta_{00},$$

(59)

where according to the energy-momentum tensor (11)

$$\delta T_{00}(x', y') - \frac{1}{3 + p} \eta_{00} \delta T^0_0(x', y') = \frac{2 + p}{3 + p} m^{(5+p)} a^{2+p}(y_0) \delta(y' - y_0) \delta^3(\bar{x}' - \bar{x}_0).$$

(60)

Plugging in expressions (58) and (60) in (59) we obtain

$$\bar{h}_{00}(r, y = 0) = 8\pi G_{5+p} \frac{2 + p}{3 + p} \frac{m^{(5+p)}}{a^{2+p}(y_0)} G_R(x, x' = x_0, y = 0, y' = y_0)$$

$$- \frac{8\pi G_{5+p} \kappa m^{(5+p)}}{a^{2+p}(y_0)} \int dy' G_R(x, x' = x_0, y = 0, y') \theta(y' - y_0).$$

(61)

As we have discussed, the explicit form of the Green function depends on the parity of the number of compact extra dimensions $p$, and therefore this also happen for the metric component $\bar{h}_{00}$

1. $p$ odd

In the case in which $p$ is odd, we use the Green function (44) obtaining

$$\bar{h}_{00}(r, y = 0) = -8\pi G_{5+p} \frac{2 + p}{3 + p} \frac{m^{(5+p)}}{a^{2+p}(y_0)} \frac{1}{4\pi r} \sqrt{\frac{\kappa}{\pi}} \frac{1}{\Gamma(\alpha - 1) 2^{\alpha - 1}} \left( \frac{d}{d\xi} \right)^{\alpha - \frac{1}{2}} \left[ \arctan \left( \frac{\xi}{\alpha} \right) \right] \bigg|_{\xi = \xi_0}$$

$$+ \frac{8\pi G_{5+p} \kappa m^{(5+p)}}{a^{2+p}(y_0)} \int \frac{d\xi}{k \xi} \frac{1}{4\pi r} \sqrt{\frac{\kappa}{\pi}} \frac{1}{\Gamma(\alpha - 1) 2^{\alpha - 1}} \left( \frac{d}{d\xi} \right)^{\alpha - \frac{1}{2}} \arctan \left( \frac{\xi}{\alpha} \right) \theta(\xi - \xi_0).$$

(62)

• Example: $p=1$
In particular, if we take the value \( p = 1 \), we have
\[
G_R(x, x', y = 0, y')^{(1)} = -\frac{1}{4\pi r} \frac{\epsilon \xi^3}{\pi} \left( \frac{d}{\xi d \xi} \right)^2 \left[ \frac{\text{arctan} \left( \frac{\xi}{\xi} \right)}{\xi} \right]
\]
\[
= -\frac{1}{4\pi r} \frac{\epsilon}{\pi} \left[ \frac{5r}{\xi^3 (1 + \frac{r^2}{\xi^2})} - \frac{2r^3}{\xi^5 (1 + \frac{r^2}{\xi^2})^2} + \frac{3}{\xi^2} \text{arctan} \left( \frac{r}{\xi} \right) \right], \quad (63)
\]
and \( \bar{h}_{00} \) is given by
\[
\bar{h}_{00} = \frac{6\pi G_6 m^{(6)}}{a^2(y_0)} \frac{\epsilon}{4\pi^2 r} \left[ \frac{5r}{\epsilon \xi_0^3 (1 + \frac{r^2}{\xi_0^2})} - \frac{2r^3}{\epsilon \xi_0^5 (1 + \frac{r^2}{\xi_0^2})^2} + \frac{3}{\epsilon \xi_0^2} \text{arctan} \left( \frac{r}{\xi_0} \right) \right]
\]
\[
- \frac{8\pi G_6 m^{(6)}}{a^2(y_0)} \frac{\epsilon}{4\pi^2 r} \left[ -\frac{3}{2} \frac{\xi}{\xi^2} \text{arctan} \left( \frac{\xi}{r} \right) - \frac{3}{2} \frac{1}{r \xi} - \frac{1}{r^2} \text{arctan} \left( \frac{\xi}{r} \right) + \frac{1}{r \xi (1 + \frac{r^2}{\xi^2})} \right] \bigg|_{\xi = \xi_0} \quad (64)
\]
Taking the limit when \( y_0 \to 0, \xi_0 = \epsilon \), we obtain for this component
\[
\bar{h}_{00} = -\frac{3G_6 m^{(6)}}{2\pi^2} \left[ \frac{5}{\epsilon^2 (1 + \frac{r^2}{\epsilon^2})} - \frac{2r^2}{\epsilon^4 (1 + \frac{r^2}{\epsilon^2})^2} + \frac{3}{r \epsilon} \text{arctan} \left( \frac{r}{\epsilon} \right) \right]
\]
\[
+ \frac{2G_6 m^{(6)}}{\pi^2} \left[ \frac{3}{2} \frac{\arctan \left( \frac{r}{\epsilon} \right)}{r \epsilon} + \frac{3}{2} \frac{1}{r^2} + \frac{\epsilon}{2} \arctan \left( \frac{r}{\epsilon} \right) - \frac{1}{r^2 (1 + \frac{r^2}{\epsilon^2})} - \frac{1}{4 \epsilon r^3} \right]. \quad (65)
\]
It is illustrative to calculate the short and the long distance limits
\[
\bar{h}_0 = -\frac{G_6 m^{(6)}}{\pi^2} \left[ \frac{20}{3 \epsilon^2} - \frac{44}{5 \epsilon^4} r^2 + \ldots \right], \quad r \to 0, \quad (66)
\]
\[
\bar{h}_0 = -\frac{G_6 m^{(6)}}{\pi^2} \left[ \frac{3 \pi}{4 \epsilon r} + \frac{\pi}{2 r^3} + \ldots \right] \sim -\frac{2G_N m}{r} \left( 1 + \frac{2\epsilon^2}{3 r^2} \right), \quad r \to \infty, \quad (67)
\]
where we have defined the effective 4D Newton constant in terms of the 6D one as
\[
G_N = \frac{3G^{(6)}}{8\pi \epsilon}. \quad (68)
\]

2. **Example: \( p = 2 \)**

For the even case we give as an example the value \( p = 2 \). In this case the Green function is
\[
G_R(x, x', y = 0, y')^{(2)} = -\frac{1}{4\pi r} \frac{\epsilon \xi^4}{4} \left( \frac{d}{\xi d \xi} \right)^2 \left[ \frac{r}{\xi^2 \sqrt{r^2 + \xi^2}} \right]
\]
\[
= -\frac{1}{4\pi} \left[ \frac{8}{\xi^2 \sqrt{r^2 + \xi^2}} + \frac{4}{(r^2 + \xi^2)^{3/2}} + \frac{3\xi^2}{(r^2 + \xi^2)^{5/2}} \right], \quad (69)
\]
and the potential is given by

\[ h_{00}^{(2)} = -\frac{4 \pi G_7 m_7}{5 \alpha^3(y_0)} \left[ \frac{8}{\xi^2 \sqrt{r^2 + \xi^2}} + \frac{4}{(r^2 + \xi^2)^{3/2}} + \frac{3 \xi^2}{(r^2 + \xi^2)^{3/2}} \right] \bigg|_{\xi = \xi_0} + \frac{8 \pi G_7 m_7}{\epsilon \alpha^3(y_0)} \int_{\xi_0}^{\infty} \frac{e \epsilon d \xi}{\xi} \left[ \frac{8}{\xi^2 \sqrt{r^2 + \xi^2}} + \frac{4}{(r^2 + \xi^2)^{3/2}} + \frac{3 \xi^2}{(r^2 + \xi^2)^{3/2}} \right] \theta(\xi - \xi_0). \tag{70} \]

Evaluating the integral we finally have

\[ h_{00}^{(2)} = -\frac{2 G_7 m_7}{5} \left[ \frac{8}{\epsilon \sqrt{r^2 + \epsilon^2}} + \frac{4 \epsilon}{(r^2 + \epsilon^2)^{3/2}} + \frac{3 \epsilon^3}{(r^2 + \epsilon^2)^{3/2}} \right] + \frac{G_7 m_7}{2} \frac{4 r^2 + 5 \epsilon^2}{\epsilon (\epsilon^2 + r^2)^{3/2}}. \tag{71} \]

The short and long distance limits for this case are

\[ h_{00}^{(2)} = -G_7 m_7 \left[ \frac{7}{2 \epsilon^2} - \frac{21}{4 \epsilon^4} r^2 + \ldots \right], \quad r \to 0, \tag{72} \]
\[ h_{00}^{(2)} = -G_7 m_7 \left[ \frac{6}{5 \epsilon} r + \frac{1}{2 r^3} + \ldots \right] \sim -\frac{2 G_N m}{r} \left( 1 + \frac{5 \epsilon^2}{12 r^2} \right), \quad r \to \infty, \tag{73} \]

where the 4D Newton constant is

\[ G_N = \frac{3 G_7}{5 \epsilon}. \tag{74} \]

C. The \( h_{ij} \) components

For the spatial components of the induced metric on the brane we proceed as before. In this case the Green function of the subsection \( \text{III C} \) reads

\[ h_{ij}(r, y = 0) = 8 \pi G_{5+p} \int d^4 x' \int dy' G(\bar{x}, y = 0; x', y') \left[ \left( \delta T_{ij}(x', y') - \frac{1}{3 + p} \eta_{ij} \delta T^0_0(x', y') \right) \right. \]
\[ \left. + \frac{1}{8 \pi G_{5+p}} \left( \kappa^2 h(x', y') + \frac{\kappa}{2} \partial_y h(x', y') \right) \right] \eta_{ij}, \tag{75} \]

where this time, according to \( \text{III} \)

\[ \delta T_{ij}(x', y') - \frac{1}{3 + p} \eta_{ij} \delta T(x', y') = -\frac{\eta_{ij}}{3 + p} \frac{m^{(5+p)}}{a^{1+p}(y')} \delta(y' - y_0) \delta^3(\bar{x} - \bar{x}_0). \tag{76} \]

Thus in this case we have in general that

\[ h_{ij}(r, y = 0) = -8 \pi G_{5+p} \frac{\eta_{ij}}{3 + p} \frac{m^{(5+p)}}{a^{2+p}(y_0)} G_R(x, x' = x_0, y = 0, y' = y_0) \]
\[ - \frac{8 \pi G_{5+p} \kappa m^{(5+p)}}{a^{2+p}(y_0)} \eta_{ij} \int dy' G_R(x, x' = x_0, y = 0, y') \theta(y' - y_0). \tag{77} \]

Again the computations have to be worked out in two separate cases depending on the parity of the number of extra compact dimensions.
1. \( p \) odd

This case correspond to have integer values of the parameter \( \alpha \), so the expression of the components \( \bar{h}_{ij} \) is given by

\[
\bar{h}_{ij}(r, y = 0) = 8\pi G_5 + \frac{p}{3} a^{5+p}(y_0) \frac{\eta_{ij}}{4\pi r} \sqrt{\frac{2}{\pi}} \frac{(-1)^{\alpha-\frac{1}{2}} \epsilon \xi^{\alpha+\frac{1}{2}} (\frac{d}{\xi d\xi})^{\alpha-\frac{1}{2}}}{\xi^{\alpha-1}} \left[ \arctan \left( \frac{\xi}{\xi_0} \right) \right] \Bigg|_{\xi = \xi_0} \\
+ 8\pi G_5 + \frac{p}{3} \kappa a^{5+p}(y_0) \int dy' \frac{1}{4\pi r} \sqrt{\frac{2}{\pi}} \frac{(-1)^{\alpha-\frac{1}{2}} \epsilon \xi^{\alpha+\frac{1}{2}} (\frac{d}{\xi d\xi})^{\alpha-\frac{1}{2}}}{\xi^{\alpha-1}} \left[ \arctan \left( \frac{\xi}{\xi} \right) \right] \theta(y' - y_0).
\]

- **Example \( p = 1 \)**

Evaluating the Green function for this case lead us to the expression

\[
G_R(x, x', y = 0, y')^{(1)} = -\frac{1}{4\pi r} \frac{\epsilon \xi^{3}}{\pi} \left( \frac{\frac{d}{\xi d\xi}}{\xi} \right)^2 \left[ \arctan \left( \frac{\xi}{\xi_0} \right) \right] \\
= -\frac{1}{4\pi r} \frac{\epsilon}{\pi} \left[ \frac{5r}{\xi^3 (1 + \frac{r^2}{\xi^2})} - \frac{2r^3}{\xi^5 (1 + \frac{r^2}{\xi^2})} + \frac{3}{\xi^2} \arctan \left( \frac{r}{\xi} \right) \right], \quad (78)
\]

hence \( \bar{h}_{ij} \) is after taking the limit \( y_0 \to 0 \)

\[
\bar{h}_{ij} = \frac{G_6 m^{(6)} \eta_{ij}}{2\pi^2} \left[ \frac{5}{\epsilon^2 (1 + \frac{r^2}{\xi^2})} - \frac{2r^2}{\epsilon^4 (1 + \frac{r^2}{\xi^2})^2} + \frac{3}{r} \arctan \left( \frac{r}{\epsilon} \right) \right] \\
+ \frac{2G_6 m^{(6)} \eta_{ij}}{\pi^2} \left[ \frac{3}{2} \frac{\arctan \left( \frac{r}{\xi} \right)}{r} + \frac{3}{2r^2} + \frac{\arctan \left( \frac{\xi}{r} \right)}{r^3} - \frac{1}{r^2 (1 + \frac{r^2}{\xi^2})} - \frac{1}{\pi \epsilon} \right]. \quad (79)
\]

For astrophysical applications is convenient to calculate the long distances limit

\[
\bar{h}_{ij} = -\frac{G_6 m^{(6)} \eta_{ij}}{\pi^2} \left[ -\frac{9\pi}{4} \frac{1}{4r} + \frac{\pi}{2r^3} + \ldots \right] \sim -\frac{G_6 m}{r} \left( -3 + \frac{2\epsilon}{3} \right) \eta_{ij}, \quad r \to \infty, \quad (80)
\]

where the Newton constant is the same as in equation (68).

2. \( p = 2 \)

In this case \( \alpha \) is a semi-integer number and the Green function is given by

\[
G_R(x, x', y = 0, y')^{(2)} = -\frac{1}{4\pi r} \frac{\epsilon \xi^4}{4} \left( \frac{\frac{d}{\xi d\xi}}{\xi^2 \sqrt{r^2 + \xi^2}} \right)^2 \left[ \frac{r}{\xi^2 \sqrt{r^2 + \xi^2}} \right] \\
= -\frac{1}{4\pi} \frac{\epsilon}{4} \left[ \frac{8}{\xi^2 \sqrt{r^2 + \xi^2}} + \frac{4}{(r^2 + \xi^2)^{\frac{3}{2}}} + \frac{3\xi^2}{(r^2 + \xi^2)^{\frac{5}{2}}} \right]. \quad (81)
\]
thus the potential is written as

\[
\bar{h}^{(2)}_{ij} = \frac{G_7 m^{(7)} \epsilon_{\eta ij}}{10 a^3(y_0)} \left[ \frac{8}{\xi^2 \sqrt{r^2 + \xi^2}} + \frac{4}{(r^2 + \xi^2)^{3/2}} + \frac{3 \xi^2}{(r^2 + \xi^2)^{5/2}} \right] |_{\xi = \xi_0}
\]

+ \frac{G_7 m^{(7)} \epsilon_{\eta ij}}{2a^3(y_0)} \left[ \frac{-4r^2 + 5\xi^2}{\xi^2 (\xi^2 + r^2)^{3/2}} \right] |_{\xi = \xi_0}.
\]  

(82)

Evaluating the limits explicitly we have

\[
\bar{h}^{(2)}_{ij} = \frac{G_7 m^{(7)} \epsilon_{\eta ij}}{10 a^3(y_0)} \left[ \frac{8}{\epsilon \sqrt{r^2 + \epsilon^2}} + \frac{4 \epsilon}{(r^2 + \epsilon^2)^{3/2}} + \frac{3 \epsilon^3}{(r^2 + \epsilon^2)^{5/2}} \right]
\]

+ \frac{G_7 m^{(7)} \epsilon_{\eta ij}}{2} \frac{4r^2 + 5\epsilon^2}{\epsilon (\epsilon^2 + r^2)^{3/2}}.
\]  

(83)

Taking the large distances limit \((r \rightarrow \infty)\) we obtain

\[
\bar{h}^{(2)}_{ij} = -G_7 m^{(7)} \epsilon_{\eta ij} \left[ \frac{14}{5\epsilon} \frac{1}{r} + \frac{\epsilon}{2r^3} + \ldots \right] \sim -\frac{2G_N m}{r} \left( -\frac{7}{3} + \frac{5\epsilon^2}{12} \right) \eta_{ij},
\]  

(84)

where \(G_N\) is given by (74).

V. EXPERIMENTAL TESTS

In this section we consider two gravitational experiments in order to set bounds to the parameters of the model. First we look at a Cavendish type experiment. As a second test we compare the perturbed induced metric of the model with the generic PPN metric generated by a static non rotating compact object.

A. Cavendish type test

In the context of the 5D Randall-Sundrum model, in \[39\] authors obtained the relative force corrections to the Newton’s gravitational force between two massive spheres. The analysis was performed computing both the exact (considering the whole Kaluza-Klein massive tower) and the approximated gravitational potential (long distances limit) and comparing them in order to find out where the application of the approximate solution is appropriate. For their analysis they used the long distances limit of the potential generated by a massive particle (of mass \(m\)) in the RS model, which is of the form

\[
\varphi(r) \approx -\frac{mG_N}{r} \left( 1 + \frac{\alpha}{r^2} \right), \quad \alpha = l^2/2,
\]  

(85)
where $l$ is proportional to the anti de Sitter radius. This potential leads to the following gravitational force between two massive spheres

$$F(r) = \frac{G_N m_1 m_2}{r^2} (1 + \delta_F),$$

(86)

with

$$\delta_F = -\frac{9\alpha}{8R^3 R'} \left\{ \ln \left( \frac{r^2 - (R' + R)^2}{r^2 - (R' - R)^2} \right) \left[ -\frac{1}{4} r^4 + \frac{1}{2} r^2 \left( R'^2 + R^2 \right) - \frac{1}{4} \left( R'^2 - R^2 \right)^2 \right] 
- r^2 R'R + R^3 R + R'R^3 \right\}.\tag{87}$$

Here $R$ and $R'$ are the radii of the spheres. Experimental data to verify this expression of the force is obtained from the Moscow Cavendish-type experiment [40], where one of the spheres was made of platinum with a radius $R \approx 0.087$ cm and mass $m_1 = 59.25 \times 10^{-3}$ gr., whereas the second sphere was made of tungsten with a radius $R' \approx 0.206$ cm and mass $m_2 = 706 \times 10^{-3}$ gr. The center of the spheres were separated by a distance of $r = 0.3773$ cm.

To obtain a bound on $l$, it is necessary to use an accurate value of Newton’s gravitational constant. The values given by CODATA in 2010 [41] are

$$\frac{G_N}{10^{-11} \text{kgs}^2} m^3 = 6.674215 \pm 0.000092 \quad \text{and} \quad 6.674252 \pm 0.000124,$$

(88)

here the relative error $\Delta G_N/G_N$ shows the agreement of the measurements of the gravitational constant with the $r^{-2}$ experiments [39], i.e., the relation $|\Delta G_N/G_N| = \delta_F$ gives the upper limit for $\delta_F$, in order to not detect experimental deviations from the Newton’s law. In the 5D RS model this implies that $l \leq 9.067 \mu m$ and $l \leq 10.527 \mu m$. A second approach using the complete solution gives $l \leq 9.070 \mu m$ and $l \leq 10.531 \mu m$. For practical use we can take $l \leq 10 \mu m$, which combined with the expressions (67) and (73) that we have obtained for the potentials in the brane produces a bound to the adS radius

$$l^2 = \frac{4 \epsilon^2}{3} \Rightarrow \epsilon = \sqrt{\frac{3}{4} l} \approx 0.86 l = 8.6 \mu m, \quad \text{for } p=1,$$

(89)

$$l^2 = \frac{5 \epsilon^2}{6} \Rightarrow \epsilon = \sqrt{\frac{6}{5} l} \approx 1.09 l = 10.9 \mu m, \quad \text{for } p=2.$$

(90)

These bounds are not in conflict with other previously reported in the literature, nevertheless the ones obtained here are weaker than for instance, the ones obtained by the Lamb shift, which gives bounds of the order $\epsilon \sim 10^{-14} m$ for $p = 1$ and $\epsilon \sim 10^{-13} m$ for $p = 2$ [22].
B. The four dimensional effective metric on the brane

Here we want to obtain the effective metric on the brane and look at the Newtonian and the parametrized post-Newtonian (PPN) limits in order to set some bounds on the parameters of the theory. The PPN limit of metric theories of gravity contains 10 real valued parameters and to every metric theory of gravitation corresponds a set of values of the PPN parameters. The observational values of the parameters have been measured in the Solar System as also in binary neutron stars [42, 43].

The corresponding PPN metric in “standard” spherical coordinates for a non rotating object is (Will 2006)

\[ ds^2_{PPN} = \left[ 1 - \frac{2G_N m}{\rho} + \frac{2G_N^2 m^2 (\beta - \gamma)}{\rho^2} + \ldots \right] dt^2 - \left[ 1 + \frac{2G_N m \gamma}{\rho} + \ldots \right] d\rho^2 - \rho^2 d\Omega. \]  

(91)

For this case only the \( \beta \) and \( \gamma \) parameters appear. The \( \gamma \) parameter measures how much space curvature \( g_{ij} \) is produced by unit rest mass, while \( \beta \) measures how much nonlinearity is there in the superposition law for gravity \( g_{00} \). These two parameters are involved in the astrophysical effects of the perihelion shift and light deflection as follows [44]:

\[ \delta_{\text{prec}} = \frac{1}{3} (2 + 2 \beta - \gamma) \left( \frac{6 \pi G_N m}{c^2 a (1 - e^2)} \right), \]  

(92)

here \( a \) is the orbit’s semi-major axis and \( e \) is the eccentricity.

\[ \delta_{\text{def}} = \frac{1 + \gamma}{2} \frac{4G_N m}{c^2 b}, \]  

(93)

in this case, \( b \) is the impact parameter of the light ray.

The four dimensional effective metric on the brane is given by

\[ ds^2 = (1 + h_{00}) dt^2 + (-\delta_{ij} + h_{ij}) dx^i dx^j, \]  

(94)

For the cases of one and two extra compact dimensions \( (p=1,2) \), taking into account the results [67], [73], [80] and [84] the metric is, to the lowest order that is needed here,

\[ ds^2 = \left[ 1 - \frac{2G_N m}{r} \left( 1 + \frac{k_p}{r^2} \right) \right] dt^2 + \left[ -1 + \frac{2G_N m}{r} l_p \right] \delta_{ij} dx^i dx^j, \]  

(95)

with

\[ k_1 = \frac{2 \varepsilon_1}{3}, \quad l_1 = 3; \quad k_2 = \frac{5 \varepsilon_2}{12}, \quad l_2 = \frac{7}{3}. \]  

(96)
In spherical coordinates we have

\[ ds^2 = \left[ 1 - \frac{2G_N m}{r} \left( 1 + \frac{k_p}{r^2} \right) \right] dt^2 + \left[ -1 - 2l_p \frac{G_N m}{r} \right] \left[ dr^2 + r^2(\theta^2 + \sin^2(\theta)d\phi^2) \right]. \] (97)

This metric is given in the isotropic form and we want it in the “standard” form, which is the one adopted for the calculation of the PPN form of the metric of a static non rotating compact object. In order to obtain that form for our metric we take the coordinate transformation

\[ \rho = r \left( 1 + \frac{l_p G_N m}{r} + \ldots \right). \] (98)

The metric in the new coordinates is

\[ ds^2 = \left[ 1 - \frac{2G_N m}{\rho} - \frac{2l_p G_N^2 m^2}{\rho^2} + \ldots \right] dt^2 - \left[ 1 + \frac{2l_p G_N m}{\rho} + \ldots \right] d\rho^2 - \rho^2 d\Omega. \] (99)

The corresponding PPN metric is (Will 2006)

\[ ds^2_{PPN} = \left[ 1 - \frac{2G_N m}{\rho} + \frac{2G_N^2 m^2(\beta - \gamma)}{\rho^2} + \ldots \right] dt^2 - \left[ 1 + \frac{2G_N m \gamma}{\rho} + \ldots \right] d\rho^2 - \rho^2 d\Omega. \] (100)

We notice by comparing the metrics that we do have the Newtonian limit. The values of the PPN coefficients \( \beta \) and \( \gamma \) for this theory are

\[ \beta = 0; \ \gamma = l_p. \] (101)

At the order of approximation considered here, the quantity \( k_p \) does not appear, implying that the astrophysical tests do not impose a constraint on the anti de Sitter length. The obtained values for the PPN parameters for this theory, in the cases where we have one or two extra compact dimensions, disagree with the observed values, since they are very close to one (the values for general relativity). We cannot tell if taking more compact dimensions will ameliorate the problem.

VI. DISCUSSION

The perspective on known phenomena changes in light of models of spacetime that include extra dimensions. In particular, brane world models have provided new possibilities in high energy physics and cosmology to try to solve some problems like the hierarchy [6,7] or dark matter/energy problems [19,20]. However little attention has been devoted to low energy physical effects which may shed light in regard to the viability of such higher dimensional
scenarios by making reference to known experimental data including the Casimir effect, Lamb shift and others [21, 22, 29, 30]. In fact even there some unexpected results may emerge as it is the case of non singular field configurations like the reported here and in previous works [21, 22].

In the present work we studied the gravitational potential produced by a source which looks pointlike to a 4D observer sitting in the single brane of an extended Randall-Sundrum-II scenario. Such source extends along the $p$ compact extra dimensions of the single brane thus forming a $T^p$ torus touching our usual 3D space at one point. A linear approximation for the hyper dimensional Einstein equations appropriate for such models was used. We obtained a gravitational potential which is non singular at the position of the source in 4D. In line with our motivation we also calculated the gravitational force between two spheres in order to compare it with experimental data. This sets a bound for the adS radius of the order $10\mu m$ which is consistent with previous more stringent electromagnetic results based on Lamb shift in hydrogen [22]. On the other hand we obtained the PPN parameters for the field configuration corresponding to the point like source. The Newtonian limit is correctly contained in our results and this was proved explicitly for $p = 1, 2$ extra compact dimensions. However the PPN values obtained for the parameters of the RSII$p$ model are out of range of the experimental data. This is not a problem as far as we do consider our brane model RSII$p$ as a test scenario rather than a realistic proposal to describe our world.

Future work along the lines we have developed here include the following. The gravitational radiation reaction problem may be reanalyzed in a setting similar to the one presented here. This may help to further understand the role of its specific features that allow to solve the divergent character of the standard 4D case. In particular it would be of interest to pinpoint what are the elements relevant for the resolution of the divergence in connection with the source, namely, whether is it linked to its topology, extension, codimension or something else. Further divergences in field theory may acquire a different form in brane worlds and we think they deserve some effort. This may be the case for instance for quantum field theory in a brane world background.
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