The essence of nonclassicality: more effect than cause

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Abstract Nonclassical properties of correlations—like unpredictability, no-cloning and uncertainty—are known to follow from two assumptions: nonlocality and no-signaling. For two-input-two-output correlations, we derive these properties from a single, unified assumption: namely, the excess of the communication cost over the signaling in the correlation. This is relevant to quantum temporal correlations, resources to simulate quantum correlations and extensions of quantum mechanics. We generalize in the context of such correlations the nonclassicality result for nonlocal-nonsignaling correlations (Masanes, Acin and Gisin, 2006) and the uncertainty bound on nonlocality (Oppenheim and Wehner, 2010), when the no-signaling condition is relaxed. An analogy of nonclassicality with Gödel incompleteness is suggested, motivated by the expectation that quantum unpredictability is somehow comparable with metamathematical undecidability. This line of research could shed light on why randomness may be inevitable in Nature.

Keywords Bell’s theorem · nonclassicality · signaling

1 Introduction

It is known that bi-partite correlations in nonclassical theories, such as quantum mechanics or the world of PR boxes [1], possess common properties absent in classical probability theory, and these properties can be traced to two basic assumptions: no-signaling and nonlocality [2]. By “nonclassical”, we mean features like unpredictability [3] of outcomes of measurement on pure states, uncertainty of conjugate pairs of observables, monogamy of nonlocal correlations [4], no-cloning [5], etc. By ‘pure states’, we mean the extreme points of the polytope of the state space.

In this work, we unify these two assumptions into a single weaker assumption: namely, the existence of a gap \( \eta \) by which the signal in bipartite correlation \( \mathbf{P} \) falls short of the communication cost of \( \mathbf{P} \). We describe correlation \( \mathbf{P} \) as the vector \( P_{AB|ab} \), where \( a \) and \( b \) are Alice’s and Bob’s input measurements respectively, and \( A \) and \( B \) are their measurement outcomes. We show that the condition \( \eta > 0 \) suffices to generate nonclassical properties [7], even if we relax the condition of no-signaling. We interpret this gap, called the signal deficit, as an indication of nonclassicality and also suggest its analogy with metamathematical incompleteness. Conventionally, almost all research on quantum correlations has been in the no-signaling scenario, and here we point to a direction to go beyond this. In particular, our results will be relevant to the study of temporal correlations [8], resources required to simulate quantum mechanics [9], and extensions of QM.

2 Correlation inequality and signaling

The correlations \( \mathbf{P} = P_{AB|ab} \) we consider here will be restricted to measurement outcomes \( A, B = \pm 1 \) with measurement inputs \( a, b = 0, 1 \). \( \mathbf{P} \) fails to admit a deterministic local hidden variable (LHV) model [10]:

\[
P_{AB|ab} = \int \rho(\lambda) P(A|a, \lambda) P(B|b, \lambda) d\lambda
\]

if it violates the condition

\[
\Lambda(\mathbf{P}) \equiv |E(0,0) + E(0,1) - E(1,0) - E(1,1)| \leq 2
\]
where \( E(a, b) = \sum_{A,B} AB \times P_{AB|ab} \) indicates the average outcome upon measuring two given input observables, and \( P_{AB|ab} \) satisfies the positivity and normalization conditions. By checking the non-violation of each of the CHSH inequalities obtained by permuting settings and outcomes (which here effectively gives three other inequalities, with the minus sign in Ineq. \( ^{(2)} \) displaced), we would know that \( P \) admits a LHV model.

If \((a = 0, a = 1)\) and \((b = 0, b = 1)\) refer to spatially separated measurements on two different particles, then \( P \) must be non-signaling and Eq. \( ^{(1)} \) corresponds to the condition of local-realism and Ineq. \( ^{(2)} \) is the the Clauser-Horne-Shimony-Holt (CHSH) inequality \( ^{(11)} \), a Bell-type inequality \( ^{(12)} \). By contrast, if \((a = 0, a = 1)\) and \((b = 0, b = 1)\) refer to temporally separated measurements on the same particle, then Eq. \( ^{(1)} \) corresponds to the condition of noninvasive-realism and Ineq. \( ^{(2)} \) is the the Leggett-Garg (LG) inequality \( ^{(13)} \) in its two-time variant \( ^{(13)} \). Unlike in the spatial case, relativity does not forbid \( P \) from being signaling in the temporal case. Since Eq. \( ^{(1)} \) implies the no-signaling condition

\[
\sum_B P_{AB|00} = \sum_B P_{AB|01}; \quad \sum_A P_{AB|00} = \sum_A P_{AB|10},
\]

the presence of signal in \( P \) by itself guarantees violation of noninvasive-realism. The situation with contextuality inequalities is similar to Bell-type inequalities, in that they must be non-signaling \( ^{(15)} \).

As our results below will apply to any of these three kinds of inequalities, it will be convenient to use a uniform terminology and refer to Ineq. \( ^{(2)} \) as a correlation inequality and to Eq. \( ^{(1)} \) as the separability condition.

3 The signaling polytope and communication cost

It will be convenient to characterize a bi-partite correlation \( P \) in terms of two parameters, the signaling and communication cost. One way to define the signal from Alice to Bob is by

\[
s_{A\rightarrow B}(P) = \max_b |P(B = 0|0, b) - P(B = 0|1, b)|,
\]

and similarly for the signal from Bob to Alice. The signal \( s(P) \) is the maximum of \( s_{A\rightarrow B} \) and \( s_{B\rightarrow A} \).

The average communication cost \( C \) is the minimum number of bits that Alice must send to Bob in a classical simulation of \( P \). In general, \( C \) must convey something to Bob about both Alice’s settings and outcomes \( ^{(10)} \), but assuming freewill of both players and allowing for outcome information to be part of pre-shared information, \( C \) only carries her settings information, \( a \).

The bi-partite 2-input-2-output possibly signaling correlation \( P \), which has 4 possible inputs and 4 possible outputs, is a list of \( 4 \times 4 \) numbers. Taking into consideration the \( 4 \) probability conservation conditions for each input, but not the \( 4 \) independent no-signaling conditions \( ^{(3)} \), there are \( 12 \) free parameters, which is the dimension \( D_S \) of the “signaling polytope” \( S \). The \( 4 \) possible outputs on each input entails that there are \( 4^4 = 256 \) deterministic \( P \), or deterministic boxes \( d \), that are the extreme points of \( S \). This is appropriate for a classical simulation of \( P \) \( ^{(17)} \).

Of the \( d \)’s, sixteen are deterministic 0-bit boxes (for which \( C = 0 \)), and are the extreme points of the local polytope \( L \), and the remaining 240 are deterministic boxes requiring 1 bit (in the case of 1-way signaling) or 2 bits (in case of 2-way signaling) for their simulation \( ^{(18)} \). The familiar no-signaling polytope \( N \) is a subset of \( S \), and exists in an \( 8 \)-dimensional space. It has 24 pure states, 16 of which are the pure points of \( L \), while the remaining 8 of which are the PR boxes \( ^{(11)} \).

A (convex) polytope can be defined in terms of its vertices or facet inequalities. It turns out that Ineq. \( ^{(2)} \) is a facet inequality for the local polytope \( ^{(17)} \). Eight of these local deterministic boxes are:

| \( ab \) | \( d_{10} \) | \( d_{15} \) | \( d_{60} \) | \( d_{65} \) | \( d_{80} \) | \( d_{85} \) | \( d_{90} \) | \( d_{95} \) |
|---|---|---|---|---|---|---|---|---|
| 00 | 00 | 00 | 01 | 11 | 00 | 10 | 10 | 10 |
| 01 | 00 | 00 | 00 | 10 | 01 | 11 | 11 | 11 |
| 10 | 00 | 10 | 01 | 01 | 10 | 10 | 01 | 11 |
| 11 | 00 | 10 | 00 | 00 | 11 | 11 | 01 | 11 |


\[
(5)
\]

for which \( \Lambda = +2 \) in Ineq. \( ^{(2)} \), while eight 1-way-signaling \( d \)’s are:

| \( ab \) | \( d_{11} \) | \( d_{15} \) | \( d_{62} \) | \( d_{64} \) | \( d_{67} \) | \( d_{71} \) | \( d_{75} \) |
|---|---|---|---|---|---|---|---|
| 00 | 00 | 11 | 00 | 11 | 00 | 11 | 00 |
| 01 | 00 | 11 | 00 | 11 | 00 | 00 | 11 |
| 10 | 01 | 01 | 10 | 10 | 10 | 01 | 10 |
| 11 | 00 | 00 | 11 | 11 | 00 | 00 | 11 |


\[
(6)
\]

for which \( \Lambda = +4 \) in Ineq. \( ^{(2)} \). For spatial quantum correlations, the decomposition in terms of the deterministic boxes of Eqs. \( ^{(3)} \) and \( ^{(5)} \) is optimal to determine \( C \) \( ^{(17,19)} \).

As the number of these different deterministic boxes exceeds the dimension \( D_S \), in general, there will be multiple decompositions of \( P \):

\[
P \equiv P_{AB|ab} = \sum_{\lambda=0}^{7} q_{\lambda} d_{AB|ab}^{\lambda} + \sum_{\lambda=0}^{7} q_{\lambda} d_{AB|ab}^{\lambda*}.
\]

\[
(7)
\]
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Of these, the optimal decomposition is one for which the quantity
\[ C(P) = \sum_{\lambda=0}^{7} q_{\lambda_i} \]  
\[ \text{(8)} \]
must be minimum, and quantifies the average communication cost.

Now we note that \( C \) for a mixture of deterministic correlations is simply the average of the communication cost of the constituent deterministic boxes. For example
\[ C(pd^{b_1} +qd^{a_1}) = p + q. \]  
\[ \text{(9)} \]
whereas the signaling \( s \) is a convex function. This is because the pattern of signaling, namely Bob’s input to observe Alice’s signal or Bob’s output for his same input, will not be the same between two 1-bit boxes. For example, in the deterministic 1-bit boxes of Table 11 to receive Alice’s signal, in the case of \( pd^b \), Bob chooses \( b = 0 \), while for \( dq^1 \), he chooses \( b = 1 \).

One finds that
\[ s(pd^b +qd^{a_1}) = \max (p,q) \leq p + q. \]
\[ s(pd^b +qd^{a_1}) = |p - q| \leq p + q. \]  
\[ \text{(10)} \]
It may be verified that this bound holds for any pair of 1-bit strategies \[ [6] \], and thus we have:
\[ s(P) \leq \sum_{\lambda} q_{\lambda_i}. \]  
\[ \text{(11)} \]
Combining Eqs (8) and (11) we have:
\[ s(P) \leq C(P). \]  
\[ \text{(12)} \]
A more general proof of Ineq. (12) for \( P \) with arbitrary number of inputs and outputs, using an entropic argument, is presented elsewhere [20].

4 Geometric nonclassicality

Now suppose a pair of 1-bit strategies \[ [6] \], e.g., \( (d^0_1 \text{ and } d^{a_1}) \), occurs together in the optimal decomposition \[ [7] \], then necessarily \( s(P) < C(P) \) because of the different geometric properties of \( s \) and \( C \), as discussed above. Thus, mixing any such pair of strategies results in the gap \( \eta(P) = C(P) - s(P) \) being larger than 0. The converse is also true, i.e., if \( \eta(P) \) is larger than 0, then there are two or more distinct 1-bit strategies occurring with non-zero probability in the optimal decomposition \[ [7] \]. To see this, note that in Eq. (7) only if a single 1-bit strategy occurs is it the case that \( s(P) = C(P) \).

In other words, if \( \eta > 0 \), then there is non-separable unpredictability—i.e., mixing of more than one 1-bit strategies of the type \[ [9] \] with non-zero probability. Note that mixing 0-bit strategies like \[ [5] \] cannot lead to \( \eta > 0 \). Without distinguishing between the non-separable and local contributions, we may quantify local unpredictability by:
\[ I(P) \equiv \max \min_{a,b} \{P_{o(a,b)} + 1 - P_{o(a,b)}\}, \]  
\[ \text{(13)} \]
where \( o \) is \( A \) or \( B \). Here \( I(P) \) is so called because it is obtained by considering the unpredictability observed by Alice and Bob locally, and then taking the larger of them. If \( P \) corresponds to a pure state, then \( I(P) > 0 \) says that there is fundamental unpredictability in the theory.

Whereas the results so far are quite general, still what we call classical or otherwise is a matter of (aesthetic) choice. We indicate three criteria based on unpredictability:

C0. According to the most stringent definition of classicality, classical pure states necessarily give predictable outcomes. Correspondingly, we get the weakest interpretation of nonclassicality, namely, any theory with pure state having non-vanishing \( I(P) \). Local quantum mechanics is “weakly nonclassical”, because of randomness in the outcomes of measuring states that are not eigenstates of the measured observable. Further, since \( \eta(P) > 0 \) implies \( I(P) > 0 \), thus signal deficit states (i.e., those with \( s < C \)) are nonclassical.

C1. A weaker definition of classicality, whereby classical pure states are any states whose measurement outcome statistics can be simulated by local deterministic non-contextual hidden variables. Local quantum mechanics then is nonclassical by this criterion, as proven by the Kochen-Specker theorem [21].

C2. An still weaker definition of classicality, defined as characterizing a theory where measurement outcomes can be simulated by local indeterministic non-contextual hidden variables. Local quantum mechanics is classical by this criterion. We infer this by noting that there is an indeterministic but measurement non-contextual model for local quantum mechanics—the Beltrametti-Bugajski model [22]. Correspondingly one has the most stringent interpretation of nonclassicality. Even so, a theory that allows \( \eta > 0 \) is “strongly nonclassical”, because even allowing for outcome indeterminism, a local hidden variable theory cannot model states with \( \eta > 0 \).

It is worth noting that the above considerations are purely formal, and do not depend on the physical interpretation of the correlations as being spatial (the measurements of Alice and Bob are on two distinct particles) or temporal (the measurements of Alice and Bob
are sequentially on the same particle). In the former case, Alice and Bob refer to observations “here” and “there”, while in the temporal case, they refer to observations “now” and “later”. There is one caveat, though: if the correlation \( P \) is signaling \( (s(P) > 0) \), then the chronology of measurements must be consistent with the signal, with the signal sender’s measurement preceding the receiver’s. (Otherwise, the free will of the sender gets restricted.)

This perspective allows us to study the nonclassicality of spatial and temporal correlations on the same footing. Both the violation of Bell’s inequality and the violation of the Leggett-Garg inequality are strongly nonclassical, by criterion (C2), since the “here-there” correlations relevant to the former inequality, as well as the “now-later” correlations relevant to the latter inequality, contain non-separable unpredictability.

If Alice and Bob hold \( d \)-level systems, then the largest signal possible by transmitting such a system is \( \log(d) \).

In the context of temporal correlations, dimension witnesses are Bell-type inequalities for successive preparation and measurement on \( d \)-level quantum systems, whose violation requires a communication cost \( C > \log(d) \). Therefore, irrespective of the signal level, a correlation that violates a dimension witness \( \eta^* \) is nonclassical according to the stronger criterion (C2). We can in fact define a ‘super-strong’ criterion of nonclassicality (say, C3), according to which the necessary condition for nonclassicality is that \( \eta^* > 0 \), where \( \eta^* \equiv C - \log(d) \) (cf. Ref. [24]). However, such a criterion would be applicable only to temporal correlations, and not to spatial correlations, and would thus be unsuitable to our present purpose of identifying the basic nonclassical elements of correlations, that would be indifferent to the spatio-temporal status of the correlations.

While \( \eta > 0 \) implies nonclassicality, the converse is not true, since one can still have weak non-classicality according to the C0 criterion. We may think of C2-nonclassicality as being Bell-certified, while some C0-nonclassicality can be simulated using local randomness. A weakly-but-not-strongly nonclassical theory would be one that is locally nonclassical, in the sense that no-cloning holds good in each local sector. Consider a toy theory \( \Omega_0 \) in which the local correlations like (5) and 1-bit correlations like (6) are pure states. Geometrically, the pure-state decomposition for mixed states in this theory will not be unique, even for local states. To see this, we note that:

\[
\frac{1}{2}(d^{10} \otimes d^{20}) = \frac{1}{2} (d^{10} + d^{20}).
\]

Multiplicpity of decomposition implies that the state space is not a simplex, and contains a no-cloning theorem, making it nonclassical in that sense [25].

Finally, a classical theory is one where pure states lack fundamental unpredictability, and thus is classical even by the strongest criterion C0. An example of a classical theory would be one with 0-bit correlation states like those given by Eq. (3), and with 1-bit correlation states like those given by Eq. (6). But in contrast to \( \Omega_0 \), these correlations are no longer ‘boxes’ but instead are strategies for classical simulation of local or nonlocal correlations. Thus the state space is a 255-dimensional simplex, whose vertices are all the 256 two-party two-input-two-output deterministic strategies like those listed in Eqs. (5) and (6). Further, as the state space is simplex, this classical theory lacks no-cloning. By contrast, the theory \( \Omega_0 \) is a non-simplex polytope in \( 4 \cdot 4 \cdot 4 = 12 \) dimensions.

5 Signalizing undermines nonclassicality

The preceding Section shows that the gap between \( C \) and \( s \) guarantees unpredictability. This qualitative observation can be made quantitative, and can be extended to other nonclassical properties besides unpredictability [7][20], like no-cloning, uncertainty, monogamy, etc., that are consequences of the assumption of nonlocality \( (C > 0) \) and no-signaling \( (s = 0) \) [2]. In the case of each property, one can show that the property persists, though diminishingly, when the signal level is raised at constant \( C \).

5.1 Fundamental unpredictability

Here we illustrate the idea quantitatively for the fundamental unpredictability and uncertainty in an operational theory \( T \). Fundamental unpredictability is the quantity in Eq. (13) maximized over all pure states \( \psi \) in \( T 

\[
I(T) = \sup_{\psi} I(P_\psi),
\]

where \( P_\psi \) is the correlation generated by making measurements on \( \psi \).

For state \( \psi \) characterized by communication cost \( C \), there exists a complementarity between the signaling and local randomness in any resource (PR boxes, classical communication, signaling boxes, etc.) that simulates \( \psi \) [9]:

\[
s + 2I \geq C,
\]

from which it follows by direct substitution that

\[
I \geq \frac{\eta}{2}
\]
The result in Ref. [2], that fundamental unpredictability (referred to as “intrinsic randomness” in that reference) is a consequence of no-signaling and nonlocality, is generalized in Ineq. (17) in the context of two-input-two-output correlations, by relaxing the no-signaling condition.

Ineq. (17) can be interpreted as showing that unpredictability that cannot be modeled using separable resources (i.e., “Bell-certified randomness”) is weakened by signaling, in the sense that for fixed degree of non-separability as quantified by $C$, an increase in $s$ reduces the lower bound on $I$. From this viewpoint, signaling can be said to undermine strong (C2) nonclassicality. One can still have weak (C0) nonclassicality, i.e., unpredictability originating from the separable component of the correlations.

Suppose we are given two correlations, $P_a$ and $P_s$, both violating Ineq. [2] at the same level, but with the former correlation being non-signaling and the latter signaling. Ineq. (17) does not imply that the signaling correlation is less classical. This is because the degree of violation of a correlation inequality only lower-bounds $C$. It may be the case that $C(P_s) \geq C(P_a) + s$, so that $\eta(P_s) \geq \eta(P_a)$, meaning that the signaling correlation can be more C2-nonclassical.

5.2 Uncertainty

Uncertainty measures the incompatibility of two observables, say $a = 0$ and $a = 1$. We will find it convenient to use the concept of uncertainty directly related to unpredictability, though any other definition (entropic, standard deviation or fine-grained) would do as well. On any one side, the uncertainty on input may be quantified as:

\[
\Delta_A^b = \max_{a} \min_{b} P_A[a,b]
\]

\[
\Delta_B^b = \max_{a} \min_{B} P_B[a,b],
\]

which is unpredictability [15], but without the maximization over the local input. Uncertainty exists on Alice’s side if

\[
\mathcal{U}_A \equiv \Delta_A^0 + \Delta_A^1 > 0,
\]

and similarly for Bob. Inequality [19] is an uncertainty relation because it says that there is no state such that both measurements $a = 0$ and $a = 1$ are simultaneously predictable.

Now consider a pure state $\psi$ in an operational theory, which is given by a mixture of the 0-bit strategies $d^{0_{1}}$ and $d^{0_{2}}$. One finds $\mathcal{U}_A = 0$. Now let us suppose a non-vanishing gap $\eta$. This comes from a mixture of $d^{\eta_{1}}$ strategies, which for simplicity, we confine to those that signal from Alice to Bob, i.e., the first four strategies of $d^{0_{1}}$, whose equations are given by:

\[
\begin{array}{c|c|c}
\text{d} & A = 0 & B = a \cdot (b + 1) \\
\text{d} & A = 1 & B = a \cdot (b + 1) + 1 \\
\text{d} & A = a & B = a \cdot b \\
\text{d} & A = a + 1 & B = a \cdot b + 1 \\
\end{array}
\]

where ‘+’ indicates mod-2 addition. These four strategies are obtained by imposing the locality condition (column 2 of the Table in Eq. (20) on the CHSH condition $x + y = a \cdot b + 1$, which would violate Ineq. (2) to its algebraic maximum. Consider a decomposition [7] whose 1-bit part mixes only two of these, say $d^{0_{1}}$ and $d^{2_{1}}$ with probabilities $p_0$ and $p_2$.

Wlog, let $p_0 \leq p_2$. We find $\Delta_A^0 = 0$ while $\Delta_A^1 = p_0$, and so, using definition [19], $U_A = p_0$. On the other hand, $s \geq p_2 - p_0 = C - 2p_0$, so that:

\[
s + 2U_A \geq C,
\]

which is analogous to the result [16] for unpredictability. In general, by mixing two distinct 1-bit boxes with probabilities $p_{\min}$ and $p_{\max}$ with $p_{\min} < p_{\max}$, we get $U \geq p_{\min}$ and $s \geq p_{\max} - p_{\min}$. Thus we have $s + 2U \geq p_{\max} + p_{\min} = C$. The result (21) can be shown to hold for arbitrary mixtures of strategies separable and non-separable strategies [5] and [6], and for uncertainty of Bob, too [20].

Rewriting Ineq. (21), we have:

\[
\mathcal{U}_A \geq \frac{\eta}{2}.
\]

This generalizes from nonsignaling to (possibly) signaling correlations, in the context of two-input-two-output correlations, the result of [2] that uncertainty is a consequence of no-signaling and nonlocality. As with Ineq. (17), Ineq. (22) can be interpreted as showing that Bell-certified uncertainty is weakened by signaling. One can still have uncertainty originating from the separable component of the correlations. In quantum mechanics, this would be related to quantum discord, which arises from local non-commutativity, even for separable states [20].

Our result (21) also generalizes, for the two-input-two-output case, the Oppenheim-Wehner theorem that uncertainty bounds nonlocality [27], which, in our approach, would be

\[
\mathcal{U}_A \geq \frac{C}{2},
\]

obtained from Ineq. (22) by setting $s := 0$. Ineq. (21) can be interpreted as asserting that uncertainty and signaling jointly upper-bound nonlocality. In other words,
the nonlocality (as quantified by $C$ rather than by the probability to win a CHSH game or by some other measure) can violate the Oppenheim-Wehner uncertainty bound in the form (23) for the case of signaling correlations. In the absence of signaling, uncertainty by itself determines nonlocality, and Ineq. (22) reduces to the result of (21).

It is worth noting our uncertainty bound on nonlocality (22), in contrast to Ref. (21), does not invoke steering. Instead, (22) can be seen simply as a consequence of the complementarity between uncertainty and signaling in resources required to simulate nonseparable unpredictability, analogous to the complementarity between local randomness and signaling to simulate singlet statistics (20). The reason is essentially that Ref. (21) employs the approach of fine-grained uncertainty based on a “random access coding” accessed by Bob conditioned on Alice’s input $a$ and output $A$, which makes the uncertainty ensemble-dependent, whereas our method quantifies uncertainty unilaterally, as a “conjugate unpredictability” rather than by fine-graining. This contrast can be illustrated for a PR box (11), which is maximally nonlocal ($C = 1$). In the approach of (21), a PR box lacks uncertainty, and the nonlocality comes purely from perfect steering. However, according to the present approach, from Ineq. (22), setting $\eta = 1$, we find $U = \frac{1}{\tau}$, i.e., the nonlocality constrains the uncertainty to be maximal. For classical systems, $U$ and $\eta$ identically vanish.

A subtlety worth noting here is that only in the case of spatial correlations does $U$ truly represent uncertainty. In the temporal case, $U$ must be interpreted as a mix of measurement uncertainty and measurement disturbance (20).

6 Metamathetical incompleteness

We now mention briefly the similarity of our main result to a theorem in metamathematics, the study of mathematics using mathematical tools. A metatheory $\mathcal{T}_1$ is a theory to hold a discourse about another theory—a base theory $\mathcal{T}_0$, which is either an axiomatic theory, such as propositional logic and predicate logic, or an axiomatization of an ‘informal’ theory, $\mathcal{T}$, such as say arithmetic. An ‘axiomatization’ of $\mathcal{T}$ is a reduction of $\mathcal{T}$ to a collection of axioms and rules of inference. A meta-theory $\mathcal{T}_1$ identifies proofs and theorems in $\mathcal{T}_0$.

A celebrated metamathematical result is the incompleteness theorem due to Gödel (30), according to which any axiomatization $A$ of arithmetic is, if consistent, then incomplete, in the sense of there being undecidable propositions, in particular true propositions expressible in $A$ but not provable within $A$. Key to Gödel’s result is that arithmetic is sufficiently rich to embed self-referential propositions about arithmetic in arithmetic statements. An existential proof of his result is that the set of true propositions in $A$ has the cardinality of the continuum, i.e., it is uncountably large, whilst the number of proofs is only countably infinite. Worded differently, there are more truths than proofs.

In this Section, an analogy is drawn between a hidden-variable model of an operational theory in an ontological framework, and a metatheory of a (base) theory. The idea is that the manner in which an ontic model ‘explains’ the workings of an operational theory in some sense parallels the way a metatheory relates to a base theory, and that quantum nonclassicality and unpredictability are somehow analogous to metamathematical incompleteness and undecidability.

Operational theories are those whose variables can be accessed by physical experimentalists in a real or toy world. Spekkens’ toy models (31) generalized probability theories due to Barrett (32), or quantum mechanics are examples. Hidden-variables models may involve features, like hidden variables or ontic states, which are experimentally inaccessible.

Exploring this analogy between hidden-variable models and metatheories, while still admittedly very sketchy, in our opinion offers potential clarity in the way we think about physical theories, by permitting an ‘outside view’. It may eventually elucidate why nonclassicality is somehow natural to physics, and may shed light on the question: Is nonclassicality in physics somehow inevitable, like intrinsic properties of numbers, such as undecidability and uncomputability in arithmetic?

True propositions $\tau$ of arithmetic correspond to the concept of correct, deterministic predictions in an operational theory, which, in the present case, are predictions of the outcomes ($A, B$) obtained by Alice and Bob by measuring their respective particles. For our purpose, a proof is an explanation (in a theory or metatheory), whose conclusion is the true proposition that is being proved. A protocol implementing decomposition (7) is a proof of observed correlations in the ontic theory, because it can deterministically predict Alice’s and Bob’s outcomes. In particular, each deterministic strategy (6) corresponds to a proof for correlations that give rise to strong nonclassicality.

Thus an ontological analog of metamathematical completeness is Einstein-Podolsky-Rosen (EPR) completeness: an operational theory is complete if one can make deterministic predictions about measurements on properties, without disturbing those properties (33). Therefore metamathematical incompleteness, i.e., the existence of one or more undecidable propositions, is comparable
with (operational) unpredictability. The analog of syntactic consistency is empirical consistency: an operational theory is empirically consistent if its predictions on measurement outcomes are consistent with actual experiments in the lab.

EPR showed that quantum mechanics is incomplete, because any completion of quantum mechanics would be empirically inconsistent. Their argument can be extended to any operational theory with C2-nonclassicality. With the above identifications, the argument for the EPR-incompleteness of C2-nonclassical operational theories is seen to be analogous to G"odel-incompleteness of arithmetic. When a correlation is strongly nonclassical, i.e., $\eta > 0$, we saw that there is non-separable unpredictability, so that an argument for EPR-incompleteness follows, which is analogous to G"odel incompleteness. The elements of this parallelism between hidden-variable models and meta-theories is tabulated in Table 6.

Since ultimately we are dealing with an operational theory that is physical, the above meta-theoretic considerations require a physical interpretation. Accordingly, true propositions are physically interpreted as empirical effects, which are observed in a laboratory by Alice and Bob. A proof, as a theoretical explanation for that effect, can be interpreted as the formal cause for the effect. For example, the deflection of the needle of a compass is an observed effect, of which the theoretical construct called the ‘magnetic field of Earth’ is the cause. Unpredictability, which signifies that there are more true propositions than proofs for them, thus seems to a laboratory observer equipped with an operational theory as an instance of more effects than causes.

The association with G"odel’s incompleteness can also be cast in an information theoretic version, due to Chaitin [34]. By using a self-referential argument called Berry’s paradox as his point of departure, rather than the modified liar’s paradox used by G"odel, Chaitin showed that a theorem (true proposition) cannot be proven from a theory, whose information content is smaller than that of the theorem. Here by information content is meant Kolmogorov or algorithmic complexity $K$ of the theory, which is the minimum number of bits (in some encoding) required to describe the axioms and rules of inference in the theory. More precisely, Chaitin showed that to a sufficiently complex axiomatic system $S$, we may associate a number $K_0$, such that there is no string $\sigma$ in $S$, such that the truth of $K(\sigma) \geq K_0$ can be proven in $S$.

Our concept of strong nonclassicality implies that there are propositions that can be proven (in the sense mentioned above) in the HV system, but by virtue of unpredictability, are not provable in the operational theory. From the perspective of algorithmic complexity, nonclassicality arises because the information content of a HV model is greater than that of the operational theory. The latter partially or fully lacks information about hidden variables, of which full information exists in the HV model. Thus strong nonclassicality can, in this approach, be quantified by the algorithmic complexity of the minimal hidden variables required to complete an operational theory.

7 Conclusions

Nonclassical properties like intrinsic randomness, no-cloning and uncertainty, which are known to be consequences of the twin assumptions of nonlocality and no-signaling, are shown to subsist even when no-signaling is relaxed, in the context of two-input-two-output correlations, provided there is a nonvanishing signal deficit, $\eta$, which is the excess of the communication cost over the signaling in the correlation. This result, which can also be generalized to higher dimensions, is shown to imply the presence of non-separable unpredictability. This forms our criterion of strong nonclassicality which is independent of whether the correlation is spatial (between two geographically separated particle) or temporal (between two events on the same particle). Weaker versions of nonclassicality were also indicated.

In particular we show that signal diminishes strong nonclassicality in the sense that the lower bound on the nonclassical properties like unpredictability and uncertainty reduces with increasing signal at fixed communication cost. This generalizes to the case of nonvanishing signal the existence of unpredictability and uncertainty etc proven for the nonlocal-nonsignaling correlations [2] and the uncertainty bound on nonlocality [27].

An analogy of strong nonclassicality with the metamathematical concept of G"odel incompleteness is suggested, motivated by the expectation that quantum unpredictability is somehow analogous to metamathematical undecidability. This line of research can potentially shed light on why randomness is inevitable in quantum mechanics.

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Table 1  Suggested correspondence between nonclassicality in an ontological model and Gödel incompleteness

| **Ontological model feature** | **Meta-mathematical analog** |
|--------------------------------|-------------------------------|
| Operational theory            | Base theory                   |
| Ontological theory            | Meta-theory                   |
| EPR-completeness              | Metamathematical completeness |
| Empirical consistency         | Syntactic consistency         |
| Nonseparable unpredictability | Undecidability                |
| Strongly (C2) nonclassical theories | Sufficiently complex theories like arithmetic |
| Classical theory              | Metamathematically Complete and consistent theory |
| Joint outcome of Alice and Bob in a given event $e$ | True proposition $p$ |
| Signal $s$ for event $e$ deterministically demonstrating nonlocal correlation | Provability of $p$ |

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