MHD free convective heat and mass transfer flow of dissipative Casson fluid with variable viscosity and thermal conductivity effects

Amos S. Idowu, Mojeed T. Akolade, Jos U. Abubakar and Bidemi O. Falodun

Department of Mathematics, University of Ilorin, Ilorin, Nigeria

1. Introduction

Generally, alloy structures like liquid metals, electrolytes and plasma demonstrate higher thermal conductivity, this is ascribed to its effectiveness in converting energy from a heat source into liquid and high possession of electrical conductivity. This application can be found in electronics, chemical industries, MHD pumps, power generation, among others [1]. Thus the dynamics of an electrically conducting fluid known as magneto fluid dynamics (i.e. the idea that produces its magnetic field from the caused electrical current, solely as a result of fluid movement into a magnetic field) is therefore essential [2]. To this, the attention of various researchers drawn into the MHD modelling in different physical properties was analysed, among which Mahanthesh et al. [3] presented their study on Cu-H2O nanofluid, Nasir et al. [4] on a Carbon nanotube, Bilal et al. [5] on Carreau fluid, Idowu and Falodun [2] on Water’s B viscoelastic fluid, Arthur et al. [6], Ajayi et al. [7] and Malik et al. [8] on Casson fluid accordingly. Combined MHD and heat transfer effects past a moving plate was studied by Ellahi et al. [9]. Recently, Kumar and Srinivas [10] highlighted the effects of unsteady MHD and thermal radiation past an inclined stretching sheet. They noticed a rise in fluid velocity with a higher magnitude of solutal and local Grashof number.

Recent attention on MHD provides more emphasis on its usefulness in industries, engineering, medical processes among others. Then, Asha and Sunitha [11] present a study on MHD peristaltic blood flow of Eyring–Powell nanofluid through an irregular channel, Abubakar and Adeoye [12] on MHD porous tapered stenosed artery subject to radiation effect, Idowu and Falodun [13] on Soret–Dufour, thermophoresis and heat and mass transfer of Casson nanofluid over an inclined plate, Khan et al. [14] on peristaltic flow of MHD nanofluids in an asymmetric channel with different nanoparticle shapes. While El-Masry et al. [15] employed the variation iteration techniques on the flow of Eyring–Powell fluid to seek the solution of MHD variation on free convective heat transfer analysis across the wavy channel. These studies highlighted that the magnetic field effect gives rise to an opposing force which produces a reduction in the frictional force of MHD conducting fluid. However, Abubakar and Adeoye [12] thus concluded that the MHD effect gains a maximum momentum in the converging region relative to the diverging region of the stenosed artery.

An investigation over a stretching flat sheet is dated back to Crane [16]. For its enormous usages in industries and engineering operations (textile industry, cooling of papers, glass, to mention few) [17], several studies past a stretching/shrinking sheet were called for, among which Rao et al. [18] experimented heat transfer and slip effects over a stretching surface. Similar studies include, results on Casson fluid [19–21], MHD flow with power-law velocity in [22], Maxwell fluid in [23] and Nanofluid containing gyrotactic microorganisms by...
Mehran et al. [17]. Double stratifications and viscous dissipation effects were presented in [24]. And recently, Hussanan [25] presented the effects of a non-linear stretching sheet and Newtonian heating on MHD Casson fluid. While [26] presents its analysis on the isothermal sphere in MHD Casson nanofluid. And the effects of thermophysical properties on Casson nanofluid were discussed by Gbadeyan et al. [27], where variable properties were found to raise the velocity, thus, decreases energy and volume fraction respectively.

The physics of stratification in geophysical flows is paramount since it negates ambient heating from gaining access into the fluid region, thereby, minimize concentration and energy distributions [28–30]. To this, Omowaye and Animasaun [31] presented the differences in thermal properties over a melting sheet of an upper-converted Maxwell fluid. They deduced that the flow distributions declined to a rise in fluid stratification. In assuming the physical nature of problems, considering the thermophysical properties is essential. To this, a viscous variability of reacting flow over a heated plate was examined by Makinde et al. [32]. They found that for a temperature-dependent viscosity, skin friction decreases while the Nusselt number is appreciated. Animasaun et al. [20] examined the thermophysical properties on Casson fluid with stratification effects. While Agunbiade and Dada [33] reported an increase in resultant fluid velocity due to the rise in dissipation function and some other fluid properties. Ellahi et al. [34] studied the variable thermal properties on Kerosene-Al₂O₃ nano liquid and its application to the cooling system. While Zaib et al. [35] stated that viscous dissipation enhances heat absorption at the surface.

Marin [36] highlighted the contradiction between the theory of relativity and Fourier’s law of heat conduction. The study is essential since each material exhibits a distinct time necessary for heat transfer. In an attempt to address these contradictions, a series of the modified laws of Fourier (Paradox of Heat Conduction (PHC)) were suggested, among which the Maxwell–Cattaneo model was found suitable [37]. However, the modified law is found useful in heat convection analysis. As a result, Malik et al. [38], Ali and Sandeep [39], Mustafa [40], Hayat et al. [41] and Arthur et al. [42] consider the modified model and observed an enhancement on heat transfer rate to rise in material time. Recently, Prasad [43] considered the theory of Cattaneo–Christov model on Williamson nanofluid. The theory was found to decelerate both concentration and energy profiles. Nandeppanavar and Shakunthala [44] analysed the modified theory on MHD carbon nanofluid past a stretching sheet. Zhai et al. [45] assumed an effective thermal conductivity on the modified model in a micropolar Casson ferrofluid. While Ijaz and Ayub [46] employed the modified heat flux in the analysis of dual diffusion and stratification with activation energy effects on waters-B viscoelastic fluid.

Among numerical techniques, Chebyshev Spectral Collocation Method (CSCM) based on spectral technique was found satisfactorily good for obtaining an approximate solution due to its potentials in rapid convergence approximations and ability to handle highly non-linear systems of equations [47], alongside its efficiency in the field of Computational Fluid Dynamics (CFD) [48]. Hence, CSCM is a strong numerical technique used in solving PDEs and ODEs systems [49]. Thus Ehrenstein and Peyret [50] solved the unsteady two-dimensional Naiver–Stokes equations by applying CSCM. Elbarbary [51] utilized a new spectral differentiation matrix to reduce the round-off error. And recently, Babatin [52] utilized this method in the modelling of an unsteady stretching surface. While in an attempt to solve the fractional-order advection–dispersion equation, Malawi et al. [53] stated that this numerical scheme performed better than the existing schemes.

From the literature, the effects of stratifications and applied magnetic field over different flow assumptions past the stretching sheet had been carried out. While, to the best knowledge of authors, no studies reported yet on chemically reacting MHD free convective heat and mass transfer flow of dissipative Casson fluid with variable viscosities and thermal conductivity effects, using Cattaneo–Christov flux phenomena, thus this research work.

2. Mathematical formulation

Figure 1 described the physical coordinate and model system of free convective, MHD dissipative Casson fluid encompassed two-dimensional, incompressible, laminar, and modified flux model of viscous flow past a variable stretching sheet. Thus the following flow assumptions were made:

- the usual Boussinesq approximation hold, the stretching velocity is $u = U_w(x + b)^n$ such that $n \neq 1$, this is illustrated with the profile $y = A(x + b)^{(1−n)/2}$. $n$ denote the power index, $b$ a positive constant, $U_w$ is the reference velocity and $A$ is a constant assumed minimal;
- flow field is MHD conducting with magnetic field strength $B(x)$. While magnetic Reynolds number is considered so small compared to that of applied magnetic field and thus neglected;
- Cattaneo–Christov heat flux model was assumed over Fourier’s law,
- the fluid variability are taking to be a linear function of temperature, $\mu(T) = \mu(\frac{1}{T} + j_1(T_w - T))$ and $\kappa(T) = \kappa(\frac{1}{T} + j_2(T - T_{\infty}))$ where $j_1$ and $j_2$ are temperature-dependent viscous parameter and temperature-dependent thermal conductivity parameter respectively,
Casson fluid

\[ \tau_y = \begin{cases} 
\left( \mu_b + \frac{P_b}{\sqrt{2\pi}} \right) 2\epsilon_y & \text{if } \pi > \pi_c \\
\left( \mu_b + \frac{P_y}{\sqrt{2\pi}} \right) 2\epsilon_y & \text{if } \pi < \pi_c 
\end{cases} \]

was considered as the base fluid,

- variations in temperature and concentration (stratifications) were assumed and,

- space-dependent heat generation/absorption viscous dissipation were therefore considered.

Considering the above assumptions, the governing equations; continuity, momentum, energy, and concentration were given as follows:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1} \]

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\mu_b(T)}{\rho} \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho} \left( 1 + \frac{1}{\beta} \right) \frac{\partial u \partial \mu(T)}{\partial y} - \frac{\sigma \beta^2}{\rho} u, \tag{2} \]

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + \left[ \frac{\partial u}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial T}{\partial y} + u \frac{\partial T}{\partial y} + v \frac{\partial^2 T}{\partial y^2} \right] + h_i \left[ \frac{\partial u}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial T}{\partial y} + 2uv \frac{\partial^2 T}{\partial y \partial x} + u \frac{\partial^2 T}{\partial x^2} \right] = \frac{1}{\rho C_p} \frac{\partial}{\partial y} \left( k(T) \frac{\partial T}{\partial y} \right) + \left( 1 + \frac{1}{\beta} \right) \frac{\mu_b(T)}{\rho C_p} \left( \frac{\partial u}{\partial y} \right)^2 + \frac{Q_0(T - T_\infty)}{\rho C_p}, \tag{3} \]

The physical coordinate system of the model.

\[ \frac{\partial C}{\partial x} + \frac{\partial C}{\partial y} = D_p \frac{\partial^2 C}{\partial y^2} - K(C - C_\infty). \tag{4} \]

The boundary conditions are:

\[ u = U_w(x + b)^n, \quad v = 0, \quad T = T_w, \quad C = C_w \text{ at } y = \infty, \tag{5} \]

\[ u \rightarrow 0, \quad T \rightarrow T_\infty, \quad C \rightarrow C_\infty \text{ as } y \rightarrow \infty. \]

Following the work presented in [31], the double stratification of thermal and solutal are given accordingly,

\[ T_w - T_0 = z_1(x + b)^{(1-n)/2}, \tag{6} \]

\[ T_\infty - T_0 = z_2(x + b)^{(1-n)/2}, \]

\[ C_w - C_0 = z_3(x + b)^{(1-n)/2}, \tag{6} \]

\[ C_\infty - C_0 = z_4(x + b)^{(1-n)/2}. \]

Hence, these relations hold:

\[ j_1(T_w - T_0) = j_1z_1(x + b)^{(1-n)/2}, \]

\[ j_1(T_\infty - T_0) = j_1z_2(x + b)^{(1-n)/2}. \tag{7} \]

Thus produces double differences in temperature: first by stratification that take place for all \( x \) at a point \( y = A(x + b)^{(1-n)/2} \) and the other for all \( x \) as \( y \rightarrow \infty \). To this, temperature-dependent thermal conductivity (\( \gamma \)) and plastic dynamic viscosity (\( \delta \)) is defined as \( \gamma = j_2(T_\infty - T_0) \) and \( \delta = j_1(T_w - T_0) \) respectively along with the relation \( \gamma \varphi = j_2(T_w - T_0) \) and \( \delta \varphi = j_1(T_w - T_0) \). The non-dimensional stratification parameters are produced by taking the ratio of the two term in (7) (Thermal and Solutal) \( \varphi = T_w - T_0/T_\infty - T_0 = z_1/z_2 \) and \( \varphi = C_w - C_0/C_\infty - C_0 = z_3/z_4 \) respectively. Transforming the governing PDEs in (1)–(4) together with (5) into an ODEs, using similarity transformations.
below as presented in [7, 8].

\[
\begin{align*}
\psi &= \left(\frac{2v\mathcal{U}_w(x + b)^n(1 + n)}{n + 1}\right)^\frac{1}{2} f^n(\eta),
\eta &= \left(\frac{(n + 1)\mathcal{U}_w(x + b)^n(1 + n)}{2v}\right)^\frac{1}{2} y,
\theta(\eta) &= \frac{T - T_\infty}{T_w - T_\infty},
\phi(\eta) &= \frac{C - C_\infty}{C_w - C_\infty}.
\end{align*}
\]

Equation (1) is satisfied, invoking (8), the governing equations (2)–(4) along with (5) are reduced to;

\[
\begin{align*}
&\left(1 + \frac{1}{\beta}\right)\left[Q_1 f^{\prime\prime\prime}(\eta) - \delta \theta^\prime(\eta) f^{\prime\prime}(\eta)\right] + f(\eta)f^{\prime\prime}(\eta) - \left(\frac{2}{n + 1}\right) f(\eta) [n f(\eta) + M] = 0, \quad (9) \\
&Q_2 \theta^{\prime\prime\prime}(\eta) + \gamma (\theta^\prime(\eta))^2 - Pr\left(\frac{1 - n}{n + 1}\right) \theta(\eta)f(\eta) - Pr\varphi \left(\frac{1 - n}{n + 1}\right) f^{\prime}(\eta)
\end{align*}
\]

\[
\begin{align*}
&\left(\frac{n + 1}{2}\right) f(\eta) \theta^\prime(\eta)f(\eta) - \left(\frac{n - 1}{2(n + 1)}\right) f(\eta)^2 \theta^\prime(\eta) - \varphi \left(\frac{n - 1}{2(n + 1)}\right) f^{\prime}(\eta)^2 \\
&- Pr h_0 \left[\frac{n - 1}{2}\right] f(\eta) \theta^\prime(\eta)f(\eta) + \varphi \left(\frac{n - 1}{2}\right) f(\eta)f^{\prime\prime}(\eta) + \varphi \left(\frac{n - 1}{2}\right) f^{\prime}(\eta)^2 \eta^\prime(\eta)
\end{align*}
\]

\[
\begin{align*}
&+ Pr \theta^\prime(\eta)f(\eta) + EcPr Q_1 \left(1 + \frac{1}{\beta}\right) [f^{\prime\prime}(\eta)]^2 + Pr H_0 \left(\frac{n - 1}{n + 1}\right) \theta(\eta) = 0, \quad (10)
\end{align*}
\]

\[
\begin{align*}
\phi^{\prime\prime}(\eta) + Scf(\eta)\phi^{\prime}(\eta)
+ \left(\frac{n - 1}{n + 1}\right) \left[Scf(\eta)\phi(\eta) + Sc\varphi f(\eta)\right]
- Sc\lambda \left(\frac{2}{n + 1}\right) \phi(\eta) = 0.
\end{align*}
\]

where \(Q_1 = [i_1 + \delta - \delta \theta(\eta) - \delta \phi] \) and \(Q_2 = [i_2 + \gamma \theta(\eta)]\). Since the sheet is stretching, we choose \(\eta = A(n + 1)\mathcal{U}_w(2x)^{\frac{1}{2}}\) as the least similarity variable. Then, Equation (5) becomes

\[
\begin{align*}
&f(\eta) = \chi \left(\frac{1 - n}{1 + n}\right), \quad f^{\prime}(\eta) = 1, \quad \theta(\eta) = 1, \\
&\phi(\eta) = 1 \quad \text{at} \ \eta = \chi, \\
&f^{\prime}(\eta) \rightarrow 0, \quad \theta(\eta) \rightarrow 0, \quad \phi(\eta) \rightarrow 0 \quad \text{as} \ \eta \rightarrow \infty.
\end{align*}
\]

where \(\chi = A(n + 1)\mathcal{U}_w(2x)^{\frac{1}{2}}\) represent the wall surface. Equations (9)–(11) with (12) are in the domain \([\chi, \infty)\). Dimensionalizing into \([0, \infty)\) we set \(g(\tau) = g(\eta - \chi) = f(\eta), h(\tau) = h(\eta - \chi) = \theta(\eta)\) and \(\Phi(\tau) = \Phi(\eta)\) then we have

\[
\begin{align*}
&\left(1 + \frac{1}{\beta}\right) [Q_3 g^{\prime\prime\prime}(\tau) - \delta h^\prime(\tau) g^{\prime\prime}(\tau)] \\
&+ g(\tau) g^{\prime\prime}(\tau) - \left(\frac{2}{n + 1}\right) g(\tau) \left[ng(\tau) + M\right] = 0,
\end{align*}
\]

\[
\begin{align*}
&Q_4 h^{\prime\prime}(\tau) + \gamma (h^\prime(\tau))^2 - Pr \left(\frac{1 - n}{n + 1}\right) h(\tau) g^{\prime\prime}(\tau)
\end{align*}
\]

\[
\begin{align*}
&- Pr\varphi \left(\frac{1 - n}{n + 1}\right) g^{\prime}(\tau)
\end{align*}
\]

\[
\begin{align*}
&\left(\frac{n - 1}{2}\right) \left[Sc^{\prime\prime}(\tau) \Phi(\tau) + Sc\varphi g^{\prime}(\tau)\right]
\end{align*}
\]

\[
\begin{align*}
&- Sc\lambda \left(\frac{2}{n + 1}\right) \Phi(\tau) = 0.
\end{align*}
\]

and Equation (12) gives

\[
\begin{align*}
g(\tau) = \chi \left(\frac{1 - n}{1 + n}\right), \quad g^{\prime}(\tau) = 1, \quad h(\tau) = 1, \\
\Phi(\tau) = 1 \quad \text{at} \ \tau = 0, \\
g^{\prime}(\tau) \rightarrow 0, \quad h(\tau) \rightarrow 0, \quad \Phi(\tau) \rightarrow 0 \quad \text{as} \ \tau \rightarrow \infty.
\end{align*}
\]

\[
\begin{align*}
Q_1 = [i_1 + \delta - \delta \theta(\tau) - \delta \phi] \quad \text{and} \quad Q_2 = [i_2 + \gamma \theta(\eta)].
\end{align*}
\]

\[
\begin{align*}
\Phi(\tau) = 1 \quad &\text{at} \ \tau = 0, \\
g^{\prime}(\tau) \rightarrow 0, \quad h(\tau) \rightarrow 0, \quad \Phi(\tau) \rightarrow 0 \quad &\text{as} \ \tau \rightarrow \infty.
\end{align*}
\]

where \(Q_1 = [i_1 + \delta - \delta \theta(\tau) - \delta \phi]\) and \(Q_2 = [i_2 + \gamma \theta(\tau)]\), \(M = \sigma B_0^2 / \rho \mathcal{U}_w(x + b)^{n - 1}, P_\tau = C_p \mu / \kappa, h_0 = h \mathcal{U}_w(x + b)^{n - 1}, \delta = j_1(T - T_0), \gamma = j_2(T - T_0), H_0 = Q_0(x + b)^{1 - n}/\rho C_p, Sc = \nu / D_p, \lambda = K_\epsilon / \rho C_p \mathcal{U}_w(x + b)^{n - 1}, \psi = z_1 / z_2, q = z_3 / z_4, \chi = A(n + 1)\mathcal{U}_w(2x)^{\frac{1}{2}}\), \(E_c = U_w^2(x + b)^{2n} / C_p z_3(x + b)^{1 - n}/2\), and \(i_1, i_2\) are taken to be one.

3. Method of solution

To solve the formulated problem, the Chebyshev Spectral Collocation Method (CSCM) was employed. Due to its global nature (i.e. it approximate value, at any point in space depends on the entire domain) and its ability
to handle linear/nonlinear, PDEs/ODEs systems of equations [54]. Define the $n$th-order Chebyshev polynomial denoted by $T_i(x)$; $i \geq 0$ as

$$T_i(x) = \cos(N \cos^{-1} x); \quad -1 \leq x \leq 1,$$  \hspace{0.5cm} (17)

and the recursive formula is given by $T_{i+1} = 2xT_i(x) - T_{i-1}(x); \quad i \geq 1$. To apply CSCM, the flow domain $[0, \infty)$ is approximated to $[0, L]$. L is the scaling parameter, the choice of L determines the convergence of the result at infinity. And initial trial would be taken to guide the right choice L by observing the convergence. Thus the domain $[0, L]$ is transformed to the domain $[-1, 1]$ by applying the mapping in Equation (18).

$$\xi = \frac{2x}{L} - 1, \quad \xi \in [-1, +1]. \hspace{0.5cm} (18)$$

Thus, it is assumed that the unknown functions, $g(\tau)$, $h(\tau)$ and $\Phi(\tau)$ are approximated by the sum of basis function $T_k(\xi)$.

$$g(\tau) \approx g_N(\tau) = \sum_{k=0}^{N} a_k T_k(\tau),$$

$$h(\tau) \approx h_N(\tau) = \sum_{k=0}^{N} b_k T_k(\tau),$$

$$\Phi(\tau) \approx \Phi_N(\tau) = \sum_{k=0}^{N} c_k T_k(\tau) \hspace{0.5cm} (19)$$

Using the basis function in Equation (17) where $a_k$, $b_k$ and $c_k$ are unknown to be determined, in order to compute the residue, substituting Equation (19) into the governing equations (13)–(15) the non-zero residue is obtained. For the purpose of minimizing error, residues are equated to zero at $N$th collocation points. Chebyshev collocation points used is defined by Kayvan et al. [54].

$$r_j = \cos\left(\frac{\pi j}{N}\right), \quad j = 0, \ldots, N. \hspace{0.5cm} (20)$$

Owing to this, a system of $3N + 3$ algebraic equations with $3N + 3$ unknown coefficient expansion $a_k$, $b_k$ and $c_k$ were obtained. Newton iteration method Finlayson [55] was imposed on the system of derived equations. The constants $a_k$, $b_k$, and $c_k$ are obtained using MATHEMATICA and substituted into Equation (19) to get the required approximate solution for the flow distributions.

### 4. Results and discussion

The following pertinent parameters were used $Pr = 0.72$, $\beta = 0.2$, $M = 1$, $Ec = \delta = \gamma = 0.5$, $Sc = 0.62$, $n = H_0 = \lambda = h_0 = \vartheta = \varphi = \chi = 0.1$. These values are kept constant throughout the study unless otherwise stated in the respective tables and figures. The analysis shows the influence of the flow parameters on momentum $g'(\tau)$, temperature $h(\tau)$ and concentration $\Phi(\tau)$ boundary layers.

To authenticate the method used, we compared the obtained results with the existing results in the literature. Table 1 presents the skin friction result obtained by Nadeema et al. [56], where Adomian Decomposition Method (ADM) along with Pade approximation was employed. And the heat transfer coefficient of the work presented by Pramanik [57] for various values of Prandtl number, and the results show a good agreement with the existing literature. The simulation presented in Figures 2 and 3 highlighted the effectiveness of the method used. And was carried out by analysing graphically the velocity and temperature profile obtained using the Chebyshev Spectral Collocation Method (CSCM) and Galerkin Weighted Residual Method (GWRM).

Table 3 shows the effects of some flow parameters on the flow model. Bearing in mind, other pertinent parameters remain the same as stated earlier. It was observed that an increase in the parameters $\beta$, $n$ and $\gamma$ increases the skin friction and decreases the energy as well as mass gradient close to the wall. Where $\beta$ tends to attain

![Figure 2. Comparison of the velocity profiles obtained using CSCM and GWRM.](image-url)
the maximum value of temperature gradient within the range of 0.4–0.6 and declined afterwards. Meanwhile, the addition of $M$, $h_0$ and $\delta$ decreases the Sherwood number, Nusselt number and skin friction accordingly. Also, a rise in $\chi$ and $Pr$ presented an increase in the temperature and concentration while it decreases the skin friction.

Figure 4 presents the effect of thermal relaxation time ($h_0$) on energy distribution, it found decelerating the temperature field. Due to thermal relaxation enhancement in temperature, the material particle (Casson fluid) needed extra time for heat transfer to its surrounding particles. However, a higher $Pr$ is expected to bring down energy and velocity profiles due to the heat generated. $Pr$ the ratio of mass diffusion to thermal diffusion enhances the thinning thermal boundary layer while it reduces thermal diffusion. Thus found decelerating the temperature penetration depth as shown in Figure 5.

The effect of variable plastic dynamic viscosity ($\delta$) on flow distributions is presented in Figures 6 and 7. The rise in $\delta$ reduces the fluid velocity near the stretching surface $r \leq 2$ but increases as it moves far away. Hence, it gave rise to the heat transfer rate and produces no significant effect on mass distribution across the flow field. Physically, this effect described the flow resistance.

The effect of variable plastic dynamic viscosity ($\delta$) on skin friction is presented in Table 3. It is observed that skin friction decreases as the plastic viscosity increases.

### Table 2

Comparison, skin friction coefficient $[g''(0)]$ and heat transfer coefficient $[-h'(0)]$ for different values of $n$ and $\chi$ when $Pr = 0.7$, $\beta \to \infty$ and $Ec = \delta = \gamma = Sc = h_0 = \lambda = h_0 = \varrho = \varphi = 0$.

| $\beta$ | $[g''(0)]$ | $[-h'(0)]$ | $\Phi(0)$ | $[g''(0)]$ | $[-h'(0)]$ | $\Phi(0)$ |
|-------|-----------|-----------|---------|-----------|-----------|---------|
| 0.1   | -5.59247486  | 0.1770193178  | 0.957091728  | 0.1   | -4.06291010  | 0.2302894562  | 0.918161778  |
| 0.5   | -2.38591590  | 0.233680834  | 0.938556713  | 0.1 | -0.64665031  | 0.2301594690  | 0.918120515  |
| 1     | -2.29938561  | 0.2065882874  | 0.816728263  | 0.3 | -0.7437670  | 0.23036910  | 0.91803718  |
| $M$   | -2.15939884  | 0.1874742000  | 0.904832189  | 0.5 | -0.8223560  | 0.229846146  | 0.917954250  |
| 0     | -2.15939884  | 0.1874742000  | 0.904832189  | 0.5 | -0.8223560  | 0.229846146  | 0.917954250  |
| 1     | -2.29938561  | 0.2065882874  | 0.816728263  | 0.3 | -0.7437670  | 0.23036910  | 0.91803718  |
| 2     | -2.03966685  | 0.1874742000  | 0.904832189  | 0.5 | -0.8223560  | 0.229846146  | 0.917954250  |
| 3     | -2.29938561  | 0.2065882874  | 0.816728263  | 0.3 | -0.7437670  | 0.23036910  | 0.91803718  |

### Table 3

Computed effects of $\beta$, $M$, $\delta$, $\gamma$, $n$, $h_0$, $Pr$ and $\chi$ on skin friction $[\Gamma g''(0)]$, heat transfer $-h'(0)$ and mass transfer coefficients $-\Phi(0)$, where $\Gamma = (1 + \beta)$.

| Values | $[\Gamma g''(0)]$ | $[-h'(0)]$ | $-\Phi(0)$ | Values | $[\Gamma g''(0)]$ | $[-h'(0)]$ | $-\Phi(0)$ |
|--------|----------------|-----------|---------|--------|----------------|-----------|---------|
| $\beta$ | 0.1 | -5.59247486  | 0.1770193178  | 0.957091728  | 0.1 | -4.06291010  | 0.2302894562  | 0.918161778  |
| 0.5 | -2.38591590  | 0.233680834  | 0.938556713  | 0.1 | -0.64665031  | 0.2301594690  | 0.918120515  |
| 1 | -2.29938561  | 0.2065882874  | 0.816728263  | 0.3 | -0.7437670  | 0.23036910  | 0.91803718  |
| $M$ | 0 | -2.15939884  | 0.1874742000  | 0.904832189  | 0.5 | -0.8223560  | 0.229846146  | 0.917954250  |
| 1 | -2.29938561  | 0.2065882874  | 0.816728263  | 0.3 | -0.7437670  | 0.23036910  | 0.91803718  |
| 2 | -2.03966685  | 0.1874742000  | 0.904832189  | 0.5 | -0.8223560  | 0.229846146  | 0.917954250  |
| 3 | -2.29938561  | 0.2065882874  | 0.816728263  | 0.3 | -0.7437670  | 0.23036910  | 0.91803718  |

Figure 3. Comparison of the temperature profiles obtained using CSCM and GWRM.

Figure 4. Effect of $h_0$ on temperature profile.

Figure 5. Effect of $Pr$ on temperature profile.
to fluids which are helpful in many applications, such as lubricants that perform well under variable temperature conditions.

Figure 8 displayed the influence of thermal conductivity ($\gamma$) known as random movement of molecular motion across the temperature gradient on energy distribution. This influenced the thermal boundary layer to generate heat and aided the temperature profile. Interestingly, high thermal conductivity helps in heat conduction which is used in heat sink applications. Meanwhile, rise in magnetic parameter ($M$) decreases $g'(\tau)$ profile, this allowed the perpendicular magnetic field to create an electromagnetic force corresponding to the resistance force that acts as a retarding force, hence, provides resistance to the momentum boundary layer as displayed in Figure 9. Consequently, an increase in $M$ produces a significant rise in the temperature field as shown in Figure 10.

The variation of Casson parameter ($\beta$) increases both the energy and concentration profile across the flow field while it decelerates the flow field (see Figures 11–13). Physically, a larger and smaller value of $\beta$ parameter corresponds to Newtonian and non-Newtonian fluids respectively (i.e. $\beta$ reduces the yield stress). However, the result obtained in Figure 12 contradicts the stated physical analogy which is due to the temperature injected that affects the fluid temperature thus behave contrarily. Moreover, this behaviour of Casson fluid with variable viscosity was also observed by Ajayi et al. [7] and other related literature.

Figures 14–16 analysed the variation of wall thickness parameter ($\chi$) on flow, energy and mass transfer respectively. Literarily, materials with variable thickness exhibit a lower magnitude towards the flow distributions for free convective flow. Thus an increase in $\chi$ results to a decrease in boundary layer thickness, velocity and other flow distributions. Meanwhile, a rise in velocity power index ($n$) gives rise to the stretching velocity which produces distortion in the fluid region, thereby gave rise to $g'(\tau)$ and $h(\tau)$ distributions as shown in Figures 17 and 18 respectively. Practically, a
rise in $n$ generates more force in the same direction, hence, appreciate the flow distributions.

Figure 19 displays the variations in physical quantities Eckert number ($Ec$). $Ec$ gives rise to the temperature profile due to heat energy stored in the liquid as a result of frictional heating, physically, resulting in a reduction in the velocity distribution. In Figure 20, behaviour of $H_0$ on temperature distribution for both heat generation ($H_0 < 0$) and absorption ($H_0 > 0$) result to higher magnitude of heat, thus enhanced the fluid temperature. And Figure 21 accounts for the magnitude of $\lambda$ which decelerates the fluid concentration.
5. Conclusion

A study on the boundary layer flow of MHD dissipative Casson fluid has been investigated. The fluid is assumed to encompass variable viscosity, variable thermal conductivity along with thermal and solutal stratifications. The effects of a stretching sheet with a variable thickness on free convective heat and mass transfer were examined. While discovering heat relocation, the phenomena employed was the Cattaneo–Christov heat flux model. The governing nonlinear coupled ordinary differential equations were solved using the Spectral Method based on Chebyshev Collocation (CSCM) and the following conclusions were drawn.

(i) Existence of thermal relaxation time and Prandtl number brings down the temperature distribution accordingly

(ii) Introduction of magnetic field effect reduces the velocity field across the flow region while it appreciates the temperature field

(iii) Higher magnitude of Casson parameter results to the reduction in velocity and concentration fields while it increases temperature field

(iv) Variable viscosity parameter decreases the fluid velocity near the wall and gives rise to both velocity and temperature distributions as it approaches the free stream region. While temperature-dependent thermal conductivity appreciates the temperature distribution through the entire boundary region

(v) Wall thickness parameter ($\chi$) slows down the flow distributions (velocity, energy and concentration), while these distributions were appreciated on varying the velocity power index ($n$)

(vi) Heat generation/absorption increases the velocity distribution, and a hike in Eckert number appreciates the temperature profiles accordingly.

The effect of thermal relaxation on material time in this study helps in model decision making. In particular, Casson fluid is considered the most preferable model for rheological data in comparison to viscoelastic fluid models, which found useful in the model of blood oxygenators and haemodialysers. It is, however, applicable in the area of magnetic drug targeting, where the blood flow can be represented by Casson fluid while the magnetic nanoparticles are considered as drug carriers. External MHD field will be employed to control the drug to reach the target, and thermal relaxation time will be utilized in determining the nature of the material used. This study can as well address the minimization of heat in an electrical voice coil and engine coolant by employing higher thermal conductivity (radiators) which leads to a reduction in aerodynamic drag thus reduces the consumption of the radiators.
**Nomenclature**

| Symbol | Description |
|--------|-------------|
| $\chi$ | Wall thickness parameter (−) |
| $T_{\infty}$ | Free stream temperature (K) |
| $T$ | Fluid temperature (K) |
| $C_p$ | Specific heat ($\frac{J}{kg \cdot K}$) |
| $S_c$ | Schmidt number (−) |
| $\phi$ | Dimensional concentration ($\frac{mol}{L}$) |
| $\rho$ | Fluid density ($\frac{kg}{m^3}$) |
| $B_0$ | Magnetic field parameter (tesla) |
| $\varepsilon$ | Solutal stratification (−) |
| $M$ | Dimensionless magnetic field parameter (−) |
| $\sigma^*$ | Stefan–Boltzmann constant ($\frac{W}{m^2 \cdot K}$) |
| $n$ | Velocity power index parameter (−) |
| $\tau_w$ | Skin-friction coefficient (−) |
| $g'(r)$ | Dimensionless fluid velocity (−) |
| $\nu$ | Kinematic viscosity ($\frac{m^2}{s}$) |
| $P_y$ | Yield stress of the fluid (−) |
| $\Phi(r)$ | Dimensionless concentration (−) |
| $L$ | Scaling parameter (−) |
| $K_r$ | Chemical reaction ($\frac{mol}{L \cdot s}$) |
| $\beta$ | Casson parameter (−) |
| $u, v$ | Velocity components along x and y direction ($\frac{m}{s}$) |
| $\pi$ | Product of the component deformation rate (−) |
| $T_w$ | Temperature at the wall (K) |
| $\lambda$ | Chemical reaction parameter (−) |
| $\delta$ | Dimensionless variable viscosity (−) |
| $\beta_g$ | Soret–Dufour parameter (−) |
| $h_0$ | Dimensionless heat generation/absorption (−) |
| $\kappa$ | Thermal diffusivity (−) |
| $C$ | Fluid concentration (mol) |
| $D_p$ | Mass diffusivity term (−) |
| $\theta$ | Dimensional temperature (K) |
| $\sigma$ | Fluid electrical conductivity ($\frac{s}{m}$) |
| $\mu_0$ | Dynamical viscosity ($\frac{kg}{m \cdot s}$) |
| $\mu_b$ | Plastic dynamic viscosity ($\frac{kg}{m \cdot s}$) |
| $\psi$ | Dimensionless thermal stratification (−) |

**Acknowledgements**

The authors are sincerely thankful to the University of Ilorin, Nigeria, for providing an accommodating environment for the completion of this research interest.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

**ORCID**

Mojeed T. Akolade [http://orcid.org/0000-0002-6876-7203]
Jos U. Abubakar [http://orcid.org/0000-0002-2079-5731]
Bidemi O. Falodun [http://orcid.org/0000-0003-1020-1677]

**References**

[1] Miner A, Ghoshal U. Cooling of high-power-density microdevices using liquid metals coolants. Appl Phys Lett. 2004;85(3):506–508. doi:10.1063/1.1772862.
[2] Idowu AS, Falodun BO. Soret–Dufour effects on MHD heat and mass transfer of Walter’s-B viscoelastic fluid over a semi-infinite vertical plate: spectral relaxation analysis. J Taibah Univ Sci. 2019;13(1):49–62.
[3] Mahanthesh BJ, Gireesha B, Gorlac RSR. Heat and mass transfer effects on the mixed convective flow of chemically reacting nanofluid past a moving/stationary vertical plate. Alex Eng J. 2016;55(1):569–581.
[4] Nasir S, Shah Z, Islam S, et al. Radiative flow of magnetohydrodynamics single-walled carbon nanotube over a convectively heated stretchable rotating disk with velocity slip effect. Adv Mech Eng. 2019;11(3):1–11.
[5] Bilal S, Shaqfatullah , Alshomrani AS, et al. Analysis of Carreau fluid in the presence of thermal stratification and magnetic field effect. Saha Inst Nucl Phys. 2018;10:118–125.
[6] Arthur EM, Seini IY, Bortteir LB. Analysis of Casson fluid flow over a vertical porous surface with chemical reaction in the presence of magnetic field. J Appl Math Phys. 2015;3:713–723. doi:10.4236/jamp.2015.36085.
[7] Ajayi TM, Omowaye AJ, Animasaun IL. Effects of viscous dissipation and double stratification on MHD Casson fluid flow over a surface with variable thickness: boundary layer analysis. Int J Eng Res Afr. 2017;28:73–89. doi:10.4028/www.scientific.net/IERA.28.73.
[8] Malik MY, Khan M, Salahuddin T, et al. Variable viscosity and MHD flow in Casson fluid with Cattaneo-Christov heat flux model, using Keller box method. J Eng Sci Tech. 2016;19:19-1985–1992.
[9] Elahi K, Alamri SZ, Basit A, et al. Effects of MHD and slip on heat transfer boundary layer flow over a moving plate based on specific entropy generation. J Taibah Univ Sci. 2018;12(4):476–482. doi:10.1080/16583655.2018.1483795.
[10] Kumar B, Srinivas S. Unsteady hydromagnetic flow of Eyring–Powell nanofluid over an inclined permeable stretching sheet with Joule heating and thermal radiation. J Comput Mech. 2020;6(2):259–270. doi:10.22055/JACM.2019.29520.1608.
[11] Asha SK, Sunitha G. Effect of joule heating and MHD on peristaltic blood flow of Eyring–Powell nanofluid in a non-uniform channel. J Taibah Univ Sci. 2019;13(1):155–168. doi:10.1080/16583655.2018.1549530.
[12] Abubakar JU, Adeoye AD. Effects of radiative heat and magnetic field on blood flow in an inclined tapered stenosed porous artery. J Taibah Univ Sci. 2020;14(1):77–86. doi:10.1080/16583655.2019.1701397.
[13] Idowu AS, Falodun BO. Effects of thermophoresis, Soret–Dufour on heat and mass transfer flow of magnetohydrodynamics non-Newtonian nanofluid over an inclined plate. Arab J Basic Appl Sci. 2020;27(1):149–165. doi:10.1080/25765299.2020.1746017.

[14] Khan LA, Raza M, Mir N, et al. Effects of different shapes of nanoparticles on peristaltic flow of MHD nanofluids filled in an asymmetric channel. J Therm Anal Calorim. 2020;140:879–890. doi:10.1007/s10973-019-08348-9.

[15] El-Masyr YAS, Abd Elmaboud Y, Abdel-Sattar MA. The impacts of varying magnetic field and free convection heat transfer on an Eyring–Powell fluid flow with peristalsis: VIM solution. J Taibah Univ Sci. 2020;14(1):19–30. doi:10.15838/jtnms.2019.1698277.

[16] Crane LJ. Flow past a stretching plate. J Appl Math Phys. 1970;2:645–647.

[17] Mehryan SAM, Kashkooli FM, Soltani M, et al. Fluid flow and heat transfer analysis of a nanofluid containing motile gyrotactic Micro-Organisms passing a nonlinear stretching vertical sheet in the presence of a Non-Uniform magnetic field; numerical approach. PLoS One. 2016;11(6):e0157598. doi:10.1371/journal.pone.0157598.

[18] Rao AS, Prasad V, Nagendra N, et al. Numerical modeling of Non-Similar mixed convection heat transfer over a stretching surface with slip conditions. World J Mech. 2015;5:117–128. doi:10.4236/wjm.2015.56013.

[19] Chiam TC. Hydromagnetic flow over a surface stretching with a power-law velocity. Int J Eng Sci. 1995;33:429–435.

[20] Animasaju IL, Fagbade AI. Casson fluid flow with variable thermo-physical property along exponentially stretching sheet with suction and exponentially decaying internal heat generation using the homotopic analysis method. J Niger Math Soc. 2016;35(1):1–17. doi:10.1016/j.jnms.2015.02.001.

[21] Khan NA, Khan S. Dual solution of Casson fluid over a porous medium: exact solutions with extra boundary condition. Acta Univ Cibi Tech Ser. 2016;68(1):35–49. doi:10.1515/aucts-2016-0008.

[22] Ahmed J, Begum A, Shahzad A, et al. MHD axisymmetric flow of power-law fluid over an unsteady stretching sheet with convective boundary conditions. Results Phys. 2016;6:973–981.

[23] Singh V, Agarwal S. MHD flow and heat transfer for Maxwell fluid over an exponentially stretching sheet with variable thermal conductivity in porous medium. Therm Sci. 2014;18(2):5599–5615.

[24] Ajayi TM, Omowaye AJ, Animasaun IL. Effects of viscous dissipation and double stratification on MHD Casson fluid flow over a surface with variable thickness: boundary layer analysis. Int J Eng Res Afr. 2017;28:73–89. doi:10.4028/www.scientific.net/JERA.28.73.

[25] Hussanan A, Ahmad SAA, Zaiheh MA. Double stratification effects on boundary layer over a stretching cylinder with chemical reaction and heat generation. J Phys: Conf Ser. 2017;890:012019. doi:10.1088/1742-6596/890/1/012019.

[26] Sebnem E. Effects of thermal stratification and mixing on reservoir water quality. Limnology. 2008;9:135–142.

[27] Noor FHMS, Ahmad SAA, Zaiheh MA. Double stratification effects on heat and mass transfer in unsteady MHD nanofluid flow over a flat surface. Asia Pac J Comput Eng. 2017;4(2):1. doi:10.1186/s40540-017-0021.

[28] Mustaku WN, Makinde OD. Double stratification effects on heat and mass transfer over a stretching sheet subjected to thermal stratification. J Appl Fluid Mech. 2016;9(4):1777–1790. doi:10.18869/acadpub.jafm.68.235.24939.

[29] Makinde OD, Khan WA, Gulham JR. MHD variable viscosity reacting flow over a convectively heated plate in a porous medium with thermophoresis and radiative heat transfer. Int J Heat Mass Trans. 2016;93:595–604.

[30] Agunbiade SA, Dada MS. Effects of viscous dissipation on convective rotationally chemically reacting Rivlin-Ericksen flow past a porous vertical plate. J Taibah Univ Sci. 2019;13(1):402–413. doi:10.1080/16583665.2019.1582149.

[31] Ellahi R, Zeeshan A, Shehzad N, et al. Structural impact of kerosene-Al2O3 nanoliquid on MHD poiseuille flow with variable thermal conductivity; application of cooling process. J Mol Liq. 2018;264:607–615. doi:10.1016/j.molliq.2018.05.010.

[32] Prasad KV, Vaidya H, Vajravelu K. Analytical study of variable viscosity reacting flow over a melting surface situated in hot environment subject to thermal stratification. J Appl Fluid Mech. 2019;14(3):377–388. doi:10.1080/25765299.2019.1698277.

[33] Malik MY, Khan M, Salahuddin T, et al. Variable viscosity and MHD flow in Casson fluid with Cattaneo–Christov heat flux model: using Keller box method. Eng Sci Technol Int J. 2016;19:1985–1992.

[34] Marin E. Does Fourier’s law of heat conduction contradict the theory of relativity. Latin Am J Phys Educ. 2011;5(2):402–405.

[35] Hayat T, Khan MI, Farooq M, et al. Impact of Cattaneo–Christov heat flux model for Williamson–Eyring–Powell fluid over an exponentially permeable shrinking sheet with viscous dissipation. Hindawi. 2016; 8: ID 6968371. doi:10.1155/2016/6968371.

[36] Marin E. Does Fourier’s law of heat conduction contradict the theory of relativity. Latin Am J Phys Educ. 2011;5(2):402–405.

[37] Christov CI. On frame indifferent formulation of the Maxwell–Cattaneo model of finite-speed heat conduction. Mech Res Comm. 2009;36:481–486.

[38] Mustafa M. Cattaneo–Christov heat flux model for rotationally chemically reacting flow of Maxwell–Cattaneo model of finite-speed heat conduction. J Heat Mass Transfer. 2016;99:702–710. doi:10.1016/j.jhmt.2015.07.014.

[39] Ali ME, Sandeep N. Cattaneo–Christov model for radiative heat transfer of magnetohydrodynamic Casson-ferrofluid: A numerical study. Results Phys. 2017;7:21–30.

[40] Malik MY, Khan M, Salahuddin T, et al. Variable viscosity and MHD flow in Casson fluid with Cattaneo–Christov heat flux model: using Keller box method. Eng Sci Technol Int J. 2006;19:1985–1992.

[41] Ali ME, Sandeep N. Cattaneo–Christov model for radiative heat transfer of magnetohydrodynamic Casson-ferrofluid: A numerical study. Results Phys. 2017;7:21–30.

[42] Mustafa M. Cattaneo–Christov heat flux model for rotating flow and heat transfer of upper-convected Maxwell fluid. AIP Adv. 2015;5:047109.

[43] Hayat T, Khan MI, Farooq M, et al. Impact of Cattaneo–Christov heat flux model in flow of variable thermal conductivity fluid over a variable thickened surface. Int J Heat Mass Transfer. 2016;99:702–710. doi:10.1016/j.ijheatmasstransfer.2016.09.034.

[44] Arthur E, Seini I, Borteir L. Analysis of Casson fluid flow over a vertical porous surface with chemical reaction in the presence of magnetic field. J Appl Math Phys. 2015;3:713–723. doi:10.4236/jamp.2015.36085.

[45] Prasad KV, Vaidya H, Vajravelu K. Analytical study of Cattaneo–Christov heat flux model for Williamson–Casson fluid flow over a slender elastic sheet with variable thickness. J Nanofluids. 2018;7(3):583–594. doi:10.1166/jnon.2018.1475.
[44] Nandeppanavar MM, Shakunthala S. Impact of Cattaneo–Christov heat flux on magnetohydrodynamic flow and heat transfer of Carbon nanofluid due to stretching sheet. J Nanofluids. 2019;8(4):746–755. doi:10.1166/jon.2019.1629.

[45] Zhah Z, Dawar A, Khan I, et al. Cattaneo–Cristov model for electrical magnetic micropolar Casson ferrofluid over a stretching/shrinking sheet using effective thermal conductivity model. Case Stud Therm Eng. 2018;13:100352.

[46] Ijaz M, Ayub M. Activation energy and dual stratification effects for Walter-B fluid in view of Cattaneo–Christov double diffusion. Helin. 2019;5.

[47] Fox L, Parker IB. Chebyshev polynomials in numerical analysis. Oxford: Clarendon Press; 1968.

[48] Canuto C, Quarteroni A, Hussaini MY, et al. Spectral methods in fluid dynamics. New York, NY: Springer; 1988. (Springer Series in Computational Physics).

[49] Khater AH, Temsah RS, Hassan MM. A Chebyshev spectral collocation method for solving burgers-type equations. J Comput Appl Math. 2008;222:333–350.

[50] Ehrenstein U, Peyret R. A Chebyshev spectral collocation method for the Navier–Stokes equations with application to double-diffusive convection. Int J Numer Meth Fl. 1989;9:427–452.

[51] Elbarbary EME. Chebyshev finite difference method for the solution of boundary-layer equations. Appl Math Comput. 2005;160:487–498.

[52] Babatin MM. Numerical treatment for the flow of Casson fluid and heat transfer model over an unsteady stretching surface in the presence of internal heat generation/absorption and thermal radiation. Appl Appl Math. 2018;13(2):854–862.

[53] Mallawi F. Application of a legendre collocation method to the space time variable fractional-order advection dispersion equation. J Taibah Univ Sci. 2019;13(1):324–330. doi:10.1080/16583655.2019.1576265.

[54] Kayvan S, Hadi H, Seyed-Mohammad T. Stagnation-Point flow of Upper-Convected Maxwell fluids. Int J Nonlin Mech. 2006;41:1242–1247.

[55] Finlayson BA. The method of weighted residuals and variational principles. New York: Academic Press; 1972.

[56] Nadeema S, Haqa RU, Lee C. MHD flow of a Casson fluid over an exponentially shrinking sheet. Sci Iran. 2012;19:1550–1553. doi:10.1016/j.sci.2012.10.021.

[57] Pramanik S. Casson fluid flow and heat transfer past an exponentially porous stretching surface in presence of thermal radiation. Ain Shams Eng J. 2014;5:205–212. doi:10.1016/j.asej.2013.05.003.