ABUNDANT NOVEL SOLUTIONS OF THE CONFORMABLE LAKSHMANAN-PORSEZIAN-DANIEL MODEL

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Abstract. In this paper, three images of nonlinearity to the fractional Lakshmanan Porsezian Daniel model in birefringent fibers are investigated. The new bright, periodic wave and singular optical soliton solutions are constructed via the \((m+G'/G)\) expansion method, which are applicable to the dynamics within the optical fibers. All solutions are novel compared with solutions obtained via different methods. All solutions verify the conformable Lakshmanan-Porsezian-Daniel model and also, for the existence the constraint conditions are utilized. Moreover, 2D and 3D for all solutions are plotted to more understand its physical characteristics.

1. Introduction. Progress in the formation of solitons and its application in differential equations has been noticeable. Analyses, discussion, and modeling of solitary energy propagating on behalf of a chain of other biological molecules have pulled forward interesting. The physical phenomena of nonlinear partial differential equations (NLPDEs) can connect to a lot of areas of sciences, for example, plasma physics, optical fibers, nonlinear optics, fluid mechanics, chemistry, biology, geochemistry, and engineering sciences [32].

Scientists were used and improved many methods to obtain exact and numerical solution of (NLPDEs), such as sine-Gordon expansion method [3,24,36], a functional variable method [29], the degenerate Darboux transformation [44], the extended sinh-Gordon expansion method [18,19,51], the Lie symmetries along with \((G'/G)\)-expansion method [39], the inverse mapping method [21], the Riccati-Bernoulli sub-ODE method [55], the extended trial equation method [11,46], the modified simple equation method [37,41], the couple of integration schemes [13], the undetermined coefficients method [53], the modified auxiliary expansion method [26], the Riccati differential equation method [42,56], the simple equation method [20], Lie group

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approach [40], the tan(ϕ(ξ)/2)-expansion method [30, 45, 47], the Hirota bilinear method [43], the decomposition-Sumudu-like-integral-transform method [59], the Jacobi elliptic function method [25], the modified exponential function method and the extended sinh-Gordon method [27], the modified exp(−φ(ξ))-expansion method [17, 28], the tanh function method [38], the shooting method [1, 2, 33, 35, 63], the Haar wavelet method [48], the homotopy perturbation method [57, 62], the finite forward difference method [52, 61], the Adomian decomposition method [34], the Adams-Bashforth-Moulton method [9], homotopy perturbation Sumudu transform method [8] and the improved Adams-Bashforth algorithm [7,10,49].

The fractional Lakshmanan-Porsezian-Daniel (LPD) model [58] is written as:

\[ iD^\alpha_t u + aD^2_x u + bD^\alpha_x D^\alpha_t u + cF \left( |u|^2 \right) u = \sigma D^4_x u + \vartheta D^2_x u^* + \rho |D^3_x u|^2 u + \gamma |u|^2 D^2_x u + \varpi u^2 D^2_x u^* + \delta |u|^4 u \]  

(1)

In Eq. (1) \( u(x,t) \) depict the complex-valued wave function. The first expression on the left-hand side depicts the temporal evolution of the optical pulse, also the coefficients \( a, b \) represent the group-speed dispersion (GDV) and the spatial-temporal dispersion (STD), respectively. A real-valued algebraic function represents the nonlinearity that \( F \left( |u|^2 \right) u : C \rightarrow C \), in more detail, this function \( F \left( |u|^2 \right) u \) is \( p \)-times continuously differentiable, so that

\[ F \left( |u|^2 \right) u \in \bigcup_{m, n=1}^{\infty} CP \left( (-s,s) \times (-r,r) : R^2 \right) \]  

(2)

On the right-hand side of Eq. (1), represents the coefficient of fourth-order dispersion and symbolizes two-photon absorption. The coefficients of \( \vartheta, \rho, \gamma \) and \( \varpi \) depicts the perturbation terms include nonlinear forms of dispersion.

In the field of nonlinear fiber optics, optical solitons are one of the most important research fields. In this instance, the LPD model, which drives the dynamics of pulse transmission with optical fibers has become quite popular lately. Different methods are utilized to retrieve the optical soliton solutions of the LPD model. In Ref. [5], the Riccati equation method has been applied to extract dark, singular and bright-singular combo soliton solutions of Eq. (1) include Kerr law nonlinearity. In Ref. [50], the sine-Gordon expansion method has been used to retrieve analytical solutions of the LPD model includes three nonlinearity laws. The extended Jacobi’s elliptic function expansion method has been applied on LPD model and periodic solutions, singular solitons, dark solitons were obtained [22]. The extended trial function method has been utilized on LPD model includes Kerr and power laws to retrieve bright solitons, dark solitons and periodic solitary waves [12]. The singular and dark soliton solutions to the model have been constructed by using the modified simple equation method [16]. In Ref. [15], singular, bright and dark soliton solutions via the trial equation method have been obtained. The undetermined coefficients method [54], the semi-inverse variational principle [4], a multipliers method [14], the modified extended direct algebraic method [31] and exp (−ϕ(ξ))-expansion method [6] have been used to reveal soliton solutions of the LPD model. The modified simple equation method has been implemented to build dark and singular soliton solutions, but the method fails to gain a bright soliton solution for the model [23].
Moreover, in Ref. [60], fractional LPD model that contains M-derivative is studied via the Jacobi elliptic function anz"at"z method. In this manuscript, we use \( \left( m + \frac{C'}{\alpha} \right) \) expansion method on the conformable LPD model in birefringent fibers include three images of nonlinearity to obtain novel optical soliton solutions in terms of bright-singular soliton, singular soliton, bright soliton and periodic wave solutions.

2. The conformable fractional derivative. The conformable derivative has the following definitions, properties and theorems:

**Definition 2.1.** Suppose that \( f(t) \) be a conformable fractional derivative of order \( \alpha \) and defines as \( f : (0, \infty) \rightarrow \mathbb{R} \) then
\[
D_\alpha^t f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad \forall \ t > 0, \ 0 < \alpha \leq 1.
\]

**Definition 2.2.** Suppose that \( f(t) \) be a function defined on \( (a, t] \) and \( \alpha \in \mathbb{R} \), then, the \( \alpha \)-fractional integral of the function \( f(t) \) can be stated as
\[
iD_\alpha^a f(t) = \frac{\alpha}{\gamma} \int_a^t f(\xi) d\xi, \quad \text{where} \ a \geq 0 \ \text{and} \ t \geq a,
\]
Providing that the Riemann improper integral exists.

**Theorem 2.3.** Suppose that \( f(t) \) and \( g(t) \) be \( \alpha \)-conformable differentiable at a point \( t > 0 \), such that \( \alpha \in (0, 1] \), then
1. \( D_\alpha^t (a f(t) + b g(t)) = a D_\alpha^t f(t) + b D_\alpha^t g(t) \), \quad \text{for all} \ a, b \in \mathbb{R}.
2. \( D_\alpha^t (t^\mu) = \mu t^{\mu-\alpha}, \quad \text{for all} \ \mu \in \mathbb{R} \).
3. \( D_\alpha^t (f(t) g(t)) = g(t) D_\alpha^t (f(t)) + f(t) D_\alpha^t (g(t)) \).
4. \( D_\alpha^t \left( \frac{f(t)}{g(t)} \right) = \frac{g(t) D_\alpha^t (f(t)) - f(t) D_\alpha^t (g(t))}{g(t)^2} \).

In addition, if the function \( f(t) \) is a differentiable function, then
\[
D_\alpha^t (f(t)) = t^{1-\alpha} \frac{df(t)}{dt}.
\]
The chain rule for conformable fractional derivatives is set out in the following theorem [64].

**Theorem 2.4.** Suppose that \( f(0, \infty) \rightarrow R \) be both a \( \alpha \)-conformable differentiable function and classic differentiable function. Assume that \( g(t) \) be a classic differentiable function defined in the range of \( f(t) \), then
\[
D_\alpha^t (f g)(t) = t^{1-\alpha} g(t)^{\alpha-1} g_t(t) D_\alpha^t (f(t))_{t=g(t)}.
\]

3. General form of \( \left( m + \frac{C'}{\alpha} \right) \) method. The mainly modified steps of this technique can be taken as follows:

**Step 1.** Assuming a NLPDE as follows:
\[
(D_\alpha^\beta u, D_\alpha^\alpha u, D_\alpha D_\alpha^\beta u, D_\alpha^2 u, \ldots) = 0,
\]
and utilizing the traveling wave transformation,
\[
u(x, y, t) = U(\xi), \ \xi = \frac{x^\beta}{\beta} - \nu \frac{t^\alpha}{\alpha}.
\]
Inserting Eq. (4) to Eq. (3) yields:
\[
N(U, U', U'' , \ldots) = 0.
\]
Step 2. Take trial equation of solution for Eq. (4) as following:

\[ U(\xi) = \sum_{i=-n}^{n} a_i (m + F)^i = a_{-n} (m + F)^{-n} + \ldots + a_1 (m + F) + \ldots + a_n (m + F)^n, \]

where \( a_n, n = 0, 1, \ldots, n \) and \( m \) are nonzero constants. According to the principles of balance, we find the value of \( n \). In this manuscript, we define a function \( F \) as:

\[ F = \frac{G'(\xi)}{G(\xi)}, \]

where \( G(\xi) \) satisfy

\[ G'' + (\lambda + 2m) G' + \mu G = 0. \]

Step 3. Putting the Eq. (6) to Eq. (5) and using (7), then collect all terms with the same order of the \((m + F)^n\), we get the system of algebraic equations for \( \nu, a_n, n = 0, 1, \ldots, n, \lambda \) and \( \mu \).

Step 4. As a result, solving the obtained system and substitute \( \nu, a_n, n = 0, 1, \ldots, n \) and the obtained solution of Eq. (6) to Eq. (5), we get the analytic solutions of Eq. (3).

4. Application on \((m + G')\) method. To solve Eq. (1), via the scheme mentioned above, we define the transformation as

\[ u(x,t) = U(\xi)e^{i\theta}, \quad \xi = \frac{\alpha}{\beta}x - \nu \frac{t}{\alpha}, \quad \theta = -\frac{\kappa}{\beta}x + \omega \frac{t}{\alpha} + \epsilon. \]

In the above equation, \( \theta(x,t) \) symbolize the phase component of the soliton, \( \kappa \) represent the soliton frequency, while \( \omega \) denoting the wave number, \( \epsilon \) symbolize the phase constant and \( \nu \) symbolize the velocity of the soliton. Substituting wave transformation into Eq. (1) and then splitting the outcomes equations into real and imaginary parts. We can write the real part

\[ \sigma U''' - (6\kappa^2\sigma - b\nu + a) U'' - (b\kappa \omega - \kappa^4 \sigma - a\kappa^2 - \omega) U - cFU^2U' + \sigma U^5 \]

\[ - \kappa^2 (\varpi - \beta + \gamma + \vartheta) U + (\vartheta + \beta) U(U')^2 + (\varpi + \gamma) U^2U'' = 0, \]

and the imaginary part can be written as

\[ (b\kappa \nu - \nu + b\omega - 2a\kappa - 4\kappa^3 \sigma) U' + 2\kappa(\gamma + \vartheta - \varpi) U^2U' + 4\kappa \sigma U''' = 0. \]

In real and complex parts, if we set the coefficients of the linearly independent functions to zero, we obtain:

\[ \vartheta + \rho = 0, \quad \varpi + \gamma = 0, \quad \sigma = 0, \quad \gamma + \vartheta - \varpi = 0, \]

and therefore, the velocity of the soliton can be rewritten as

\[ \nu = \frac{b\omega - 2a\kappa}{1 - b\kappa}, \quad b\kappa \neq 1. \]

Hence, the real part Eq. (9) can be rewritten as

\[ (a - b\nu) U'' + (b\kappa \omega - a\kappa^2 - \omega) U + (cF - 4\gamma \kappa^2) U^3 - \delta U^5 = 0. \]
4.1. Kerr law. Suppose that \( F(u) = u \), then Eq. (1) can be rewrite

\[
iD_t^\alpha u + aD_x^\beta u + bD_x^\alpha D_t^\gamma u + c \left( |u|^2 \right) u = \sigma D_x^{4\beta} u + \vartheta (D_x^{\beta} u)^2 u^* + \rho |D_x^\beta u|^2 u + \gamma |u|^2 D_x^{2\beta} u + \omega u^2 D_x^{2\beta} u^* + \delta |u|^4 u
\]  \tag{12}

so, Eq. (11) becomes

\[
(a - bv) U'' + (bk\omega - a\kappa^2 - \omega) U + (c - 4\gamma\kappa^2) U^3 - \delta U^5 = 0.
\]  \tag{13}

Defining \( U = V^{1/2} \), Eq. (13) yields

\[
(a - bv) \left( 2VV'' - (V')^2 + 4(bk\omega - a\kappa^2 - \omega) V^2 + 4(c - 4\gamma\kappa^2) V^3 - 4\delta V^4 \right) = 0.
\]  \tag{14}

Using the balance method between and , we get . Therefore, Eq. (6) becomes

\[
U(\xi) = \sum_{i=-1}^1 a_i(m + F)^i = a_{-1}(m + F)^{-1} + m a_0 + a_1 (m + F).
\]  \tag{15}

By taking every summation of the coefficients of the polynomial identities of the same power to be zero, one can conclude the following optical soliton solutions. We notice that the value of all square roots is defined more than zero and \( \Delta = (2m + \lambda)^2 - 4\mu \).

**Case 1.** When \( a_{-1} = \frac{\sqrt[3]{3(m\lambda + m\lambda) - \mu}}{2\sqrt{3}}, a_0 = \frac{\sqrt[3]{3\lambda\sqrt{a - bv}}}{2\sqrt{3}}, a_1 = \frac{\sqrt[3]{3\sqrt{a - bv}}}{2\sqrt{3}}, \)

\( c = 4\gamma\kappa^2 \mp \frac{2\sqrt{3}\lambda\sqrt{a - bv}}{\sqrt{3}}, \omega = -\frac{a((2m + \lambda)^2 - a(k^2 + \mu)) + b((2m + \lambda)^2 - 4\mu)^2}{-4 + 4bk\omega} \), we conclude the following solution sets:

**Set 1.** In case \( \Delta > 0 \), we have a hyperbolic function solution:

\[
u(x, t) = \frac{3^{1/4}}{2\sqrt{2}} e^{i\theta} \left( \frac{(A_1^2 - A_2^2)}{\sqrt{3}} \left( A_2 \cosh \left( \frac{\sqrt{3}\xi}{2} \right) + A_1 \sinh \left( \frac{\sqrt{3}\xi}{2} \right) \right) \right)^{1/4} \left( -A_1 \sqrt{\Delta} + A_2 \lambda \right) \cosh \left( \frac{\sqrt{3}\xi}{2} \right) + \left( -A_2 \sqrt{\Delta} + A_1 \lambda \right) \sinh \left( \frac{\sqrt{3}\xi}{2} \right),
\]  \tag{16}

where \( A_1 \) & \( A_2 \) are numbers, \( \xi = \frac{\beta}{\sigma} - \nu \frac{\alpha}{\tau} \) and \( \theta = -\kappa \frac{\beta}{\sigma} + \omega \frac{\alpha}{\tau} + \epsilon \).

**Set 2.** In case \( \Delta < 0 \), we have trigonometric function solutions

\[
u(x, t) = \frac{3^{1/4}}{2\sqrt{2}} e^{i\theta} \left( \frac{(A_1^2 + A_2^2)}{\sqrt{3}} \left( 4m^2 + 4m\lambda + \lambda^2 - 4\mu \right) \sqrt{a - bv} \right)^{1/4} \left( -A_1 \sqrt{\Delta} + A_2 \lambda \right) \cos \left( \frac{\sqrt{3}\xi}{2} \right) + \left( A_2 \sqrt{\Delta} + A_1 \lambda \right) \sin \left( \frac{\sqrt{3}\xi}{2} \right),
\]  \tag{17}

where \( A_1 \) & \( A_2 \) are numbers, \( \xi = \frac{\beta}{\sigma} - \nu \frac{\alpha}{\tau} \) and \( \theta = -\kappa \frac{\beta}{\sigma} + \omega \frac{\alpha}{\tau} + \epsilon \).
Figure 1. 3D graphic of Eq. (16) when $\alpha = 0.9, A_1 = 3, A_2 = 2, a = 0.4, b = 0.1, \delta = 0.2, \nu = 0.2, \kappa = 0.1, \epsilon = 0.4, \beta = 0.9, \lambda = 1, m = 1, \mu = -1$ and $t = -2$ for 2D.

Figure 2. 3D figure of Eq. (17), when $\alpha = 0.8, A_1 = 3, A_2 = 2, a = 0.4, b = 0.1, \delta = 0.2, \nu = -0.2, \kappa = 0.1, \epsilon = 0.4, \beta = 0.8, \lambda = -1, m = 1, \mu = 1$ and $t = 2$ for 2D.

Set 3. In case $\Delta = 0$, we have rational function solutions as:

$$u(x,t) = \sqrt{3} e^{i \left( \frac{\alpha \beta \sqrt{a^2 - b^2}}{\sqrt{a^2 - b^2}} \right)} \frac{\sqrt{A_2^2 \alpha^2 \beta^2 \sqrt{a - b} \nu}}{\sqrt{\sqrt{5} (A_1^2 \alpha^2 \beta^2 + A_1 A_2 \alpha \beta (2x^3 \alpha + \Gamma) + A_2^2 (x^2 \beta \alpha^2 + x^3 \alpha \Gamma + t^2 \beta \nu \Gamma))}}$$

(18)
where $A_1$ & $A_2$ are numbers and $\Gamma = \beta (\alpha - 2t^\alpha \nu)$.

**Figure 3.** 3D figure of Eq. (18) when $\alpha = 0.5$, $A_1 = -3$, $A_2 = 0.2$, $a = 0.4$, $b = 0.1$, $\delta = 0.2$, $\nu = -0.2$, $\kappa = 0.5$, $\epsilon = 0.1$, $\beta = 0.5$, $\lambda = -2$, $m = 2$, $\mu = 1$ and $t = 2$ for 2D.

**Case 2.** When $a_{-1} = \frac{\sqrt{3}(m(m+\lambda)-\mu)\sqrt{a-b\nu}}{2\sqrt{\delta}}$, $a_0 = \frac{i\sqrt{3}m(m+\lambda)-3\mu\sqrt{a-b\nu}}{\sqrt{3}}$, $a_1 = -\frac{\sqrt{3}\sqrt{a-b\nu}}{2\sqrt{\delta}}$, $\omega = \frac{(a-b\nu)(20m^2+4\nu^2+20m\lambda-\lambda^2+12\lambda\sqrt{m(m+\lambda)-\mu}-20\mu)}{-4+4\nu\varepsilon}$, and $c = \frac{4\gamma\kappa^2}{\sqrt{3}}$, we conclude the following solution sets:

**Set 1.** In case $\Delta > 0$, we have hyperbolic function solutions:

$$u(x,t) = \frac{\sqrt{3}e^{i\theta}}{\sqrt{\delta\sqrt{2}}} - \frac{2(m(m+\lambda)-\mu)}{A_1\sqrt{\Delta} + A_2\lambda} \left( A_2 \cosh \left( \frac{\sqrt{\Delta}}{2} \right) + A_1 \sinh \left( \frac{\sqrt{\Delta}}{2} \right) \right) + \left( -A_1\sqrt{\Delta} + A_2\lambda \right) \cosh \left( \frac{\sqrt{\Delta}}{2} \right) + \left( -A_2\sqrt{\Delta} + A_1\lambda \right) \sinh \left( \frac{\sqrt{\Delta}}{2} \right) + \frac{2 \left( A_2 \cosh \left( \frac{\sqrt{\Delta}}{2} \right) + A_1 \sinh \left( \frac{\sqrt{\Delta}}{2} \right) \right) - m - 2i\sqrt{m(m+\lambda)-\mu}}{A_1(2m+\lambda)} \cosh \left( \frac{\sqrt{\Delta}}{2} \right) + \left( -A_2\sqrt{\Delta} + A_1\lambda \right) \sinh \left( \frac{\sqrt{\Delta}}{2} \right) \right)$$ (19)

where $A_1$ & $A_2$ are numbers, $\xi = \frac{x^\beta}{\sqrt{\nu}} - \nu \frac{t^\alpha}{\alpha}$ and $\theta = -\kappa \frac{x^\beta}{\sqrt{\nu}} + \omega \frac{t^\alpha}{\alpha} + \epsilon$. 
Figure 4. 3D graphic of Eq. (19) when $\alpha = 0.5, A_1 = -3, A_2 = 1, a = 0.3, b = -2, \delta = 0.2, \nu = 0.2, \kappa = 0.1, \epsilon = 2, \beta = 0.5, \lambda = 1, m = 1, \mu = -1$ and $t = -2$ for 2D.

Set 2. In case $\Delta < 0$, we have trigonometric function solutions

$$u(x,t) = \frac{\sqrt{3}(a - b\nu)e^{i\theta}}{\sqrt{\delta \sqrt{2}}}$$

$$- \frac{2(m^2 + m\lambda - \mu)(A_2 \cos \left(\frac{\sqrt{-\Delta} + \xi}{2}\right) + A_1 \sin \left(\frac{\sqrt{-\Delta} + \xi}{2}\right))}{(-A_1\sqrt{-\Delta} + A_2\lambda) \cos \left(\frac{\sqrt{-\Delta} + \xi}{2}\right) + (A_2\sqrt{-\Delta} + A_1\lambda) \sin \left(\frac{\sqrt{-\Delta} + \xi}{2}\right)} +$$

$$\left(\frac{-A_1\sqrt{-\Delta} + A_2(2m + \lambda)}{A_2\sqrt{-\Delta} + A_1(2m + \lambda)}\right) \cos \left(\frac{\sqrt{-\Delta} + \xi}{2}\right) + \left(\frac{-A_1\sqrt{-\Delta} + A_2(2m + \lambda)}{A_2\sqrt{-\Delta} + A_1(2m + \lambda)}\right) \sin \left(\frac{\sqrt{-\Delta} + \xi}{2}\right)$$

$$\sqrt{-m^2 + 2m\lambda - \mu, \epsilon.}$$

where $A_1$ & $A_2$ are numbers, $\xi = \frac{x^\beta}{\beta} - \nu \frac{t^\alpha}{\alpha}$ and $\theta = -\kappa \frac{\xi^\beta}{\beta} + \omega \frac{t^\alpha}{\alpha} + \epsilon$.

Set 3. In case $\Delta = 0$, we have fractional function solutions as:

$$u(x,t) = \frac{\sqrt{3}(a - b\nu)e^{i\theta}}{\sqrt{\delta \sqrt{2}}}$$

$$\left(\frac{-m^2 + m\lambda - \mu + \sqrt{\mu} - \frac{A_2}{A_1 + A_2\xi}}{(m + \lambda - \mu)(A_1 + A_2\xi)}\right) \sqrt{A_2 + A_1 (m - \sqrt{\mu}) + A_2 (m - \sqrt{\mu}) \xi}, \epsilon.$$

where $A_1$ & $A_2$ are numbers, $\xi = \frac{x^\beta}{\beta} - \nu \frac{t^\alpha}{\alpha}$ and $\theta = -\kappa \frac{\xi^\beta}{\beta} + \omega \frac{t^\alpha}{\alpha} + \epsilon$.
Figure 5. 3D figure of Eq. (20) when $\alpha = 0.5, A_1 = 3, A_2 = 1, a = 0.3, b = 0.2, \delta = 0.2, \nu = -0.2, \kappa = 0.1, \epsilon = 2, \beta = 0.5, \lambda = -1, m = 1, \mu = 1$ and $t = 2$ for 2D.

Figure 6. 3D figure of Eq. (21) when $\alpha = 0.5, A_1 = 3, A_2 = 1, a = 0.3, b = 0.2, \delta = 0.2, \nu = -0.2, \kappa = 0.1, \epsilon = 2, \beta = 0.5, \lambda = -2, m = 2, \mu = 1$ and $t = 2$ for 2D.

4.2. Parabolic law. Suppose that $F(u) = c_1 u + c_2 u^2$, then Eq. (1) can be rewritten

$$iD_t^\alpha u + aD_x^{2\beta} u + bD_x^\beta D_t^\alpha u + \left(c_1 |u|^2 + c_2 |u|^4\right) u = \sigma D_x^{4\beta} u + \vartheta (D_x^\beta u)^2 u^* + \rho |D_x^\beta u|^{2} u + \gamma |u|^2 D_x^{2\beta} u + \omega u^2 D_x^{2\beta} u^* + \delta |u|^4 u,$$

(22)
so, Eq. (11) becomes

\[(a - bv)U'' + (b\omega - a\kappa^2 - \omega)U + (c_1 - 4\gamma\kappa^2)U^3 + (c_2 - \delta)U^5 = 0.\]  \hspace{1cm} (23)

Defining \(U = V^{1/2}\), Eq. (23) yields

\[(a - bv)\left(-V'' + 2VV''\right) + 4\left(b\omega - a\kappa^2 - \omega\right)V^2 + 4\left(c_1 - 4\gamma\kappa^2\right)V^3\]

\[+ 4(c_2 - \delta)V^4 = 0.\]  \hspace{1cm} (24)

Using the balance method between \(V''\) and \(V^4\), we get \(n = 1\). Therefore, Eq. (6) becomes

\[U(\xi) = \sum_{i=-1}^{1} a_i(m + F)^i = a_{-1}(m + F)^{-1} + ma_0 + a_1(m + F).\]  \hspace{1cm} (25)

By taking every summation of the coefficients of the polynomial identities of the same power to be zero, we conclude the following soliton solutions. We notice that the value of all square roots is more than zero and \(\Delta = (2m + \lambda)^2 - 4\mu\).

**Case 3.** When \(a_{-1} = -(m(m + \lambda) - \mu)\), \(a_0 = \lambda a_1\), \(c_1 = 4\gamma\kappa^2 + \frac{\lambda(a-bv)}{a_1}\), \(c_2 = \delta - \frac{(a-bv)}{m\lambda}\), \(\omega = \frac{-a((2m + \lambda)^2 - 4(\kappa^2 + \mu)) + b((2m + \lambda)^2 - 4\mu)\nu}{4 + 4\kappa}\), we conclude the following solution sets:

**Set 1.** In case \(\Delta > 0\), we have a hyperbolic function solutions:

\[u(x, t) = \frac{1}{2}e^{i\theta} \sqrt{\frac{(A_1^2 - A_2^2)(2m + \lambda)^2 - 4\mu a_1}{A_2 \cosh \left(\frac{\sqrt{\Delta}}{2}\right) + A_1 \sinh \left(\frac{\sqrt{\Delta}}{2}\right)}} \]

\[\times \frac{1}{\sqrt{-A_1\sqrt{\Delta} + A_2\lambda} \cosh \left(\frac{\sqrt{\Delta}}{2}\right) + (-A_2\sqrt{\Delta} + A_1\lambda) \sinh \left(\frac{\sqrt{\Delta}}{2}\right)},\]

where \(A_1\) & \(A_2\) are numbers, \(\xi = \frac{a^{\alpha}}{a} - \nu \frac{\lambda}{\nu}\) and \(\theta = -\kappa \frac{a^{\beta}}{a} + \omega \frac{\kappa}{\omega} + \epsilon\).

**Set 2.** In case \(\Delta < 0\), we have a trigonometric function solutions

\[u(x, t) = \frac{e^{i\theta}}{2} \sqrt{\frac{(A_1^2 + A_2^2)(4m^2 + 4m\lambda + \lambda^2 - 4\mu) a_1}{A_2 \cos \left(\frac{\sqrt{-\Delta}}{2}\right) + A_1 \sin \left(\frac{\sqrt{-\Delta}}{2}\right)}} \]

\[\times \frac{1}{\sqrt{(A_2\lambda - A_1\sqrt{-\Delta}) \cos \left(\frac{\sqrt{-\Delta}}{2}\right) + (A_1\lambda + A_2\sqrt{-\Delta}) \sin \left(\frac{\sqrt{-\Delta}}{2}\right)}},\]

where \(A_1\) & \(A_2\) are numbers, \(\xi = \frac{a^{\alpha}}{a} - \nu \frac{\lambda}{\nu}\) and \(\theta = -\kappa \frac{a^{\beta}}{a} + \omega \frac{\kappa}{\omega} + \epsilon\).

**Set 3.** In case \(\Delta = 0\), we have a fractional function solutions as:

\[u(x, t) = \sqrt{A_1}e^{i\theta} \sqrt{\frac{m + \lambda + \frac{A_2 - A_1\sqrt{\mu} - A_2\sqrt{\mu}\xi}{A_1 + A_2\xi}}{A_2 + A_1m - A_1\sqrt{\mu} + A_2m\xi - A_2\sqrt{\mu}\xi}},\]  \hspace{1cm} (28)

where \(A_1\) & \(A_2\) are numbers, \(\xi = \frac{a^{\alpha}}{a} - \nu \frac{\lambda}{\nu}\) and \(\theta = -\kappa \frac{a^{\beta}}{a} + \omega \frac{\kappa}{\omega} + \epsilon\).
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Figure 7. 3D figure of Eq. (26) when $\alpha = 0.9, A_1 = 1, A_2 = 3, a = 0.3, b = 0.2, \delta = 0.2, \nu = 0.1, \kappa = 2, \epsilon = 0.2, a_1 = 2, \beta = 0.9, \lambda = 1, m = 1, \mu = -1$ and $t = -2$ for 2D.

Figure 8. 3D figure of Eq. (27) when $\alpha = 0.9, A_1 = 1, A_2 = 3, a = 0.3, b = 0.2, \delta = 0.2, \nu = -0.1, \kappa = 2, \epsilon = 0.2, a_1 = -2, \beta = 0.9, \lambda = -1, m = 1, \mu = 1$ and $t = 2$ for 2D.

Case 4. When $a_{-1} = -\frac{\lambda((m+\lambda)-\mu)(a-b\nu)}{c_1-4\gamma\kappa^2}, a_0 = \frac{\lambda^2(a-b\nu)}{c_1-4\gamma\kappa^2}, a_1 = \frac{\lambda(a-b\nu)}{c_1-4\gamma\kappa^2}, c_2 = \delta \frac{3(c_1-4\gamma\kappa^2)^2}{4\lambda^2(a-b\nu)}$, $\omega = \frac{-a((2m+\lambda)^2-4(\kappa^2+\mu))+b((2m+\lambda)^2-4\mu)\nu}{-4+4bc}$, we conclude the following solution sets:

Set 1. In case $\Delta > 0$, we have hyperbolic function solutions:

$$ u(x,t) = \frac{e^{i\theta}}{\sqrt{2}} \left( \frac{(A_1^2 - A_2^2) \lambda (2m + \lambda)^2 - 4\mu (a - b\nu)}{(c_1 - 4\gamma\kappa^2) \left( A_2 \cosh \left( \frac{\sqrt{\Delta} \xi}{2} \right) + A_1 \sinh \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right)} \right) $$
Figure 9. 3D figure of Eq. (28) when $\alpha = 0.5, A_1 = 3, A_2 = 2, a = 2, b = 0.1, \delta = 0.2, \nu = 0.2, \kappa = 1, \epsilon = 0.2, a_1 = 2, \beta = 0.5, \lambda = -2, m = 2, \mu = 1$ and $t = 2$ for 2D.

\[
\frac{1}{\sqrt{\left((A_2 \lambda - A_1 \sqrt{\Delta}) \cosh \left(\frac{\sqrt{\Delta} \xi}{2}\right) + (A_1 \lambda - A_2 \sqrt{\Delta}) \sinh \left(\frac{\sqrt{\Delta} \xi}{2}\right)\right)}}
\]  \hspace{1cm} (29)

where $A_1$ & $A_2$ are numbers, $\xi = \frac{\pi^\beta}{\beta} - \nu \frac{\omega}{\alpha}$ and $\theta = -\kappa \frac{\pi^\beta}{\beta} + \omega \frac{\mu}{\alpha} + \epsilon$.

Figure 10. 3D figure of Eq. (29) when $\alpha = 0.5, A_1 = 2, A_2 = 3, a = 0.2, b = 0.2, \delta = 0.2, \nu = -0.2, \kappa = 1, \epsilon = 2, c_1 = 1, \beta = 0.5, \gamma = -1, \lambda = 1, m = 1, \mu = -1$ and $t = 2$ for 2D.
Set 2. In case $\Delta < 0$, we have a trigonometric function solution

$$u(x,t) = \frac{e^{i\theta}}{2} \sqrt{\left( \frac{(A_1^2 + A_2^2) \lambda (4m^2 + 4m\lambda + \lambda^2 - 4\mu) (a - b\nu)}{(c_1 - 4\gamma\kappa^2)} \left( A_2 \cos \left( \frac{\sqrt{-\Delta} \xi}{2} \right) + A_1 \sin \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right) \right)} \sqrt{\left( (-A_1\sqrt{-\Delta} + A_2\lambda) \cos \left( \frac{\sqrt{-\Delta} \xi}{2} \right) + (A_2\sqrt{-\Delta} + A_1\lambda) \sin \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right)}.$$  

where $A_1$ & $A_2$ are numbers, $\xi = \frac{x^\alpha}{\beta} - \nu \frac{t^\alpha}{\alpha}$ and $\theta = -\kappa \frac{x^\beta}{\beta} + \omega \frac{t^\alpha}{\alpha} + \epsilon$.

![Figure 11. 3D figure of Eq. (30) when $\alpha = 0.9, A_1 = 2, A_2 = 3, a = 3, b = -2, \delta = 0.2, \nu = 0.1, \kappa = 2, \epsilon = 0.2, c_1 = 1, \beta = 0.9, \gamma = 1, \lambda = -1, m = 1, \mu = 1$ and $t = 2$ for 2D.](image)

Set 3. In case $\Delta = 0$, we have fractional function solutions as:

$$u(x,t) = \frac{\lambda (a - b\nu)}{c_1 - 4\gamma\kappa^2} e^{i\theta} \sqrt{\frac{m + \lambda - \sqrt{\mu}}{A_1 + A_2 \xi} - \frac{(m (m + \lambda) - \mu) (A_1 + A_2 \xi)}{A_2 + A_1 (m - \sqrt{\mu}) + A_2 (m - \sqrt{\mu}) \xi}},$$

where $A_1$ & $A_2$ are numbers, $\xi = \frac{x^\alpha}{\beta} - \nu \frac{t^\alpha}{\alpha}$ and $\theta = -\kappa \frac{x^\beta}{\beta} + \omega \frac{t^\alpha}{\alpha} + \epsilon$.

4.3. Anti-cubic law. Suppose that $F(s) = c_1 \frac{s}{s^2} + c_2 s + c_3 s^2$, then Eq. (1) can be rewritten

$$iD_t^\alpha u + aD_x^\beta u + bD_x^\alpha D_t^\alpha u + \left( c_1 |u|^{-4} + c_2 |u|^2 + c_3 |u|^4 \right) u = \sigma D_x^\beta u + \nu \left( D_x^\alpha u \right)^2 u^* + \gamma |u|^2 D_x^2 u + \varepsilon |u|^2 D_x^\beta u + \delta |u|^4 u,$$

so, Eq. (11) becomes

$$(a - b\nu) U'' + (b\kappa \omega - a\kappa^2 - \omega) U + c_1 U^{-3} + (c_2 - 4\gamma\kappa^2) U^3 + (c_3 - \delta) U^5 = 0.$$  

(33)
Figure 12. 3D figure of Eq. (31) when $\alpha = 0.9, A_1 = -3, A_2 = 2, a = 2, b = 0.1, \delta = 0.2, \nu = 0.2, \kappa = 1, \epsilon = 0.2, c_1 = 1, \beta = 0.9, \gamma = -2, \lambda = -2, m = 2, \mu = 1$ and $t = 2$ for 2D.

Defining $U = V^{\frac{1}{2}}$, Eq. (33) yields

$$
(a - b\nu) \left(-\left(\frac{V'}{V}\right)^2 + 2VV''\right) + 4 \left(bk\omega - a\kappa^2 - \omega\right) V^2 + 4c_1 + 4 \left(c_2 - 4\gamma\kappa\right) V^3 + 4 \left(c_3 - \delta\right) V^4 = 0.
$$  (34)

Using the balance method between $VV''$ and $V^4$, we get $n = 1$. Therefore, Eq. (6) becomes

$$
U(\xi) = \sum_{i=-1}^{1} a_i (m + F)^i = a_{-1} (m + F)^{-1} + m a_0 + a_1 (m + F). 
$$  (35)

By taking every summation of the coefficients of the polynomial identities of the same power to be zero, we conclude the following soliton solutions. We notice that the value of all square roots is defined more than zero and $\Delta = (2m + \lambda)^2 - 4\mu$.

**Case 5.** When $a_{-1} = \frac{(m(m+\lambda) - \mu)\sqrt{-3a + 3b\nu}}{2\sqrt{c_3 - \delta}}, a_1 = -\frac{\sqrt{-3a + 3b\nu}}{2\sqrt{c_3 - \delta}}, c_2 = \frac{2}{3} (6\gamma\kappa^2 - \sqrt{3} - \delta) \left(a - b\nu\right) + (c_3 - \delta) a_0^2,$

$$
\omega = \frac{-a \left((2m + \lambda)^2 - 4\left(\kappa^2 + \mu\right)\right) + b \left((2m + \lambda)^2 - 4\mu\right) \nu}{-4 + 4b\kappa} + \frac{4\sqrt{3}c_3 - \delta\lambda\sqrt{-a + b\nu a_0 + 8 (c_3 - \delta) a_0^2}}{-4 + 4b\kappa},
$$

$$
c_1 = \frac{-\left(3(m + \lambda) - \mu\right) (a - b\nu) + (-c_3 + \delta) a_0^2}{12(c_3 - \delta)^{3/2}} \times \left(3\sqrt{c_3 - \delta}\lambda^2 (a - b\nu) - 4 (c_3 - \delta) a_0 \left(\lambda\sqrt{-3a + 3b\nu + \sqrt{c_3 - \delta} a_0}\right)\right),
$$
we conclude the following solution sets:

**Set 1.** In case $\Delta > 0$, we have hyperbolic function solutions:

$$u(x, t) = 4\sqrt{-3a + 3b\nu e^{i\theta}}$$

$$\frac{\sqrt{c_3 - \delta} \left(S_2 \cosh \left(\frac{\sqrt{-\Delta} \xi}{2}\right) + S_1 \sinh \left(\frac{\sqrt{-\Delta} \xi}{2}\right)\right) + a_0}{\sqrt{c_3 - \delta} \left(A_2 \cosh \left(\frac{\sqrt{-\Delta} \xi}{2}\right) + A_1 \sinh \left(\frac{\sqrt{-\Delta} \xi}{2}\right)\right)}$$

where $A_1$, $A_2$ are numbers, $S_1 = -A_1\sqrt{-\Delta} + A_2\lambda$, $S_2 = -A_2\sqrt{-\Delta} + A_1\lambda$, $\xi = \frac{x^\alpha}{\beta} - \nu \frac{\omega^\alpha}{\alpha}$ and $\theta = -\kappa \frac{x^\alpha}{\beta} + \omega \frac{\omega^\alpha}{\alpha} + \epsilon$.

![3D figure of Eq. (36) when $\alpha = 1/2, A_1 = 0.2, A_2 = 0.3, a = -0.2, b = 1, \delta = -0.2, \nu = 0.2, \kappa = 0.1, \epsilon = 0.2, c_1 = 2, \gamma = 0.1, a_0 = 0.1, c_3 = 0.1, \beta = 1/2, \lambda = 1, m = 1, \mu = -1$ and $t = -2$ for 2D.](image)

**Set 2.** In case $\Delta < 0$, we have trigonometric function solutions

$$u(x, t) = 4\sqrt{-3a + 3b\nu e^{i\theta}}$$

$$\frac{\sqrt{c_3 - \delta} \left(S_2 \cos \left(\frac{\sqrt{-\Delta} \xi}{2}\right) + S_4 \sin \left(\frac{\sqrt{-\Delta} \xi}{2}\right)\right) + a_0}{\sqrt{c_3 - \delta} \left(A_2 \cos \left(\frac{\sqrt{-\Delta} \xi}{2}\right) + A_1 \sin \left(\frac{\sqrt{-\Delta} \xi}{2}\right)\right)}$$

where $A_1$, $A_2$ are numbers, $S_3 = -A_1\sqrt{-\Delta} + A_2\lambda$, $S_4 = -A_2\sqrt{-\Delta} + A_1\lambda$, $\xi = \frac{x^\alpha}{\beta} - \nu \frac{\omega^\alpha}{\alpha}$ and $\theta = -\kappa \frac{x^\alpha}{\beta} + \omega \frac{\omega^\alpha}{\alpha} + \epsilon$. 
Figure 14. 3D figure of Eq. (37) when $\alpha = 0.5, A_1 = 1, A_2 = 3, a = -0.2, b = 2, \delta = -0.2, \nu = 0.2, \kappa = 1, \epsilon = 2, c_1 = 1, \gamma = 1, a_0 = 5, c_3 = 1, \beta = 0.5, \lambda = -1, m = 1, \mu = 1$ and $t = -2$ for 2D.

Set 3. In case $\Delta = 0$, we have fractional function solutions as:

\[
\mathfrak{u}(x,t) = e^{i\theta} \left( \frac{(m(m + \lambda) - \mu) \sqrt{-3a + 3b\nu (A_1 + A_2 \xi)}}{2\sqrt{c_3 - \delta \left( A_2 + A_1 (m - \sqrt{\mu}) + A_2 (m - \sqrt{\mu}) \xi \right)}} - \frac{\sqrt{-\alpha - 3b\nu \left( A_2 + A_1 (m - \sqrt{\mu}) + A_2 (m - \sqrt{\mu}) \xi \right)}}{2\sqrt{c_3 - \delta \left( A_1 + A_2 \xi \right)}} + a_0, \right)
\]

where $A_1 \& A_2$ are numbers, $\xi = \frac{\alpha}{\beta} - \nu \frac{\omega}{\alpha}$ and $\theta = -\kappa \frac{\epsilon}{\beta} + \omega \frac{a_1}{\alpha} + \epsilon$.

Case 6. When $\nu = -(m(m + \lambda) - \mu) a_1, c_3 = \delta - \frac{3(a - b\nu)}{4a_1^2},$

\[
a_0 = \frac{\lambda a_1}{2} \pm \left( \frac{(b\nu - a)}{2}\right) \left( \frac{a(8m^2 + 8m\lambda - \lambda^2 - 8(\kappa^2 + \mu)) + b(-8m^2 - 8m\lambda + \lambda^2 + 8\mu) \nu + 8(-1 + b\kappa)\omega}{2\sqrt{3(a - b\nu)}} \right) a_1^2.
\]

\[
c_1 = \left( \frac{a_1^2 \alpha^2}{144(a - b\nu)} - \frac{2a_1^2 b\nu}{144(a - b\nu)} + \frac{a_1^2 \beta^2 \nu^2}{144(a - b\nu)} \right) \left( -80m^4 - 160m^3\lambda + (-4\kappa^2 + \lambda^2)^2 - 8(8\kappa^2 + 7\lambda^2)\mu \right)
\]

\[
-80\mu^2 + 8m^2 \left( 8\kappa^2 - 3\lambda^2 + 20\mu \right) + 8m\lambda \left( 8\kappa^2 + 7\lambda^2 + 20\mu \right) - \left( \frac{8\omega(-1 + b\kappa)a_1^2}{144(a - b\nu)} + \frac{8b(-1 + b\kappa)a_1^2 \nu \omega}{144(a - b\nu)} \right) \left( \frac{8m^2 + 4\kappa^2}{8m\lambda - \lambda^2 - 8\mu} \right) + \frac{16(-1 + b\kappa)^2 \omega^2 a_1^2}{144(a - b\nu)} \pm \lambda (m(m + \lambda) - \mu) a_1 \times
\]
where $\Delta$. In case $\Delta$ we conclude the following solution sets:

$$x_{\beta, t} > \alpha$$

$$\alpha = 1, \beta = 0.5, \lambda = -2, m = 2, \mu = 1 \text{ and } t = 2 \text{ for 2D.}$$

$$\sqrt{(a-b\nu) \left( \frac{8m^2 + 8m\lambda - \lambda^2}{8a} - \frac{8a}{8a} \right)} \frac{a^2_1}{2\sqrt{3}},$$

$$c_2 = 4\gamma\kappa^2 \pm \sqrt{\frac{-(a-b\nu) a^2_1 \left( \frac{(a-b\nu) (8m^2 + 8m\lambda - \lambda^2)}{-8a (\kappa^2 + \mu)} + 8b\nu\mu + 8(-1 + b\kappa) \omega} \right)} \frac{a^2_1}{\sqrt{3a^2_1}},$$

we conclude the following solution sets:

**Set 1.** In case $\Delta > 0$, we have hyperbolic function solutions:

$$u(x, t) = e^{i\theta} \left[ \lambda a_1 + \frac{(K_1 \cosh \frac{\sqrt{\xi}}{2} + K_2 \sinh \frac{\sqrt{\xi}}{2}) a_1}{2 \left( A_2 \cosh \frac{\sqrt{\xi}}{2} + A_1 \sinh \frac{\sqrt{\xi}}{2} \right)} + \frac{2(m (m + \lambda) - \mu) \left( A_2 \cosh \frac{\sqrt{\xi}}{2} + A_1 \sinh \frac{\sqrt{\xi}}{2} \right)}{-K_1 \cosh \frac{\sqrt{\xi}}{2} - K_2 \sinh \frac{\sqrt{\xi}}{2}} \right],$$

where $A_1$ & $A_2$ are numbers, $K_1 = A_1 \sqrt{\Delta} - A_2 \lambda, K_2 = A_2 \sqrt{\Delta} - A_1 \lambda, \xi = \frac{x^\beta}{r} - \nu \frac{c}{\alpha}$ and $\theta = -\kappa \frac{x^\beta}{r} + \omega \frac{c}{\alpha} + \epsilon.$
Figure 16. 3D figure of Eq. (39) when $\alpha = 0.5, A_1 = 0.2, A_2 = 0.3, b = -0.2, \delta = 0.2, \nu = 0.2, \kappa = 1, \epsilon = 2, c_1 = 0.2, \gamma = 1, c_3 = 1, \beta = 0.5, a_1 = 1, \omega = 1, a = 0.2, \lambda = 1, m = 1, \mu = -1$ and $t = -2$ for 2D.

**Set 2.** In case $\Delta < 0$, we have trigonometric function solutions

$$u(x,t) = e^{i\theta} \frac{2 \left( m^2 + m\lambda - \mu \right) \left( A_2 \cos \left( \frac{\sqrt{-\Delta}}{2} \right) + A_1 \sin \left( \frac{\sqrt{-\Delta}}{2} \right) \right) a_1}{\lambda a_1 - \frac{2 \left( K_3 \cos \left( \frac{\sqrt{-\Delta}}{2} \right) + K_4 \sin \left( \frac{\sqrt{-\Delta}}{2} \right) \right) a_1}{2 \left( A_2 \cos \left( \frac{\sqrt{-\Delta}}{2} \right) + A_1 \sin \left( \frac{\sqrt{-\Delta}}{2} \right) \right) + \lambda a_1} + \frac{\left( a - b\nu \right) \left( 8m^2 + 8m\lambda - \lambda^2 - 8\mu \right)}{2\sqrt{3} (a - b\nu)} \left( \frac{-8\kappa^2 + 8 \left( 1 + b\kappa \right) \omega}{-8\kappa^2 + 8 \left( -1 + b\kappa \right) \omega} \right) a_1^2}$$

(40)

where $A_1$ & $A_2$ are numbers, $K_3 = A_2 \lambda - A_1 \sqrt{-\Delta}$, $K_4 = A_2 \sqrt{-\Delta} + A_1 \lambda$, $\xi = \frac{x^\beta}{\beta} - \nu \frac{t^\alpha}{\alpha}$ and $\theta = -\kappa \frac{x^\beta}{\beta} + \omega \frac{t^\alpha}{\alpha} + \epsilon$. **Set 3.** In case $\Delta = 0$, we have fractional function solutions as:

$$u(x,t) = e^{i\theta} \frac{\lambda a_1}{2 \lambda a_1 - \frac{\left( A_2 + A_1 \left( m - \sqrt{\mu} \right) + A_2 \left( m - \sqrt{\mu} \right) \right) \xi}{\lambda a_1 + A_2 \xi} \left( A_2 + A_1 \left( m - \sqrt{\mu} \right) + A_2 \left( m - \sqrt{\mu} \right) \right) a_1}{\lambda a_1 - \frac{\left( a - b\nu \right) \left( 8m^2 + 8m\lambda - \lambda^2 - 8\mu \right)}{2\sqrt{3} (a - b\nu)} \left( \frac{-8\kappa^2 + 8 \left( 1 + b\kappa \right) \omega}{-8\kappa^2 + 8 \left( -1 + b\kappa \right) \omega} \right) a_1^2}$$

(41)

where $A_1$ & $A_2$ are numbers, $\xi = \frac{x^\beta}{\beta} - \nu \frac{t^\alpha}{\alpha}$ and $\theta = -\kappa \frac{x^\beta}{\beta} + \omega \frac{t^\alpha}{\alpha} + \epsilon.$
Figure 17. 3D figure of Eq. (40) when $\alpha = 0.9, A_1 = 0.4, A_2 = 2, b = -0.2, \delta = 0.2, \nu = 0.2, \kappa = 1, \epsilon = 2, c_1 = 1, \gamma = 1, c_3 = 1, \beta = 0.9, a_1 = 0.1, \omega = 1, a = 0.2, \lambda = -1, m = 1, \mu = 1$ and $t = 2$ for 2D.

Figure 18. 3D figure of Eq. (41) when $\alpha = 0.9, A_1 = 0.4, A_2 = 2, b = -0.2, \delta = 0.2, \nu = 0.2, \kappa = 1, \epsilon = 2, c_1 = 1, \gamma = 1, c_3 = 1, \beta = 0.9, a_1 = 0.1, \omega = 1, a = 0.2, \lambda = -2, m = 2, \mu = 1$ and $t = 2$ for 2D.
5. Conclusion. In this article, some new exact bright, periodic wave, bright-singular and singular optical soliton solutions of fractional Lakshmanan-Porsezian-Daniel model are constructed via the \((m + \frac{G'}{G})\) expansion method. We investigated three images of nonlinearity named the parabolic law, Kerr law and anti-cubic law nonlinearity. By using this method with computer-based symbolic computation, we construct broad classes such as hyperbolic, trigonometric and fractional solutions of nonlinear differential equations that arise in applied physics. Comparing our gained solutions with the other solutions obtained in Refs. [4–6, 12, 14–16, 22, 23, 31, 50, 54, 60], we conclude that our obtained soliton solutions are novel. Our resultant may appreciate in the telecommunication area and photonics study.

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