Microwave Conductivity of a High Purity $d$-wave Superconductor

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The cusp-like behavior of the microwave conductivity observed in clean ortho-II $\text{YB}_2\text{Cu}_3\text{O}_{6.50}$ at low temperature and low frequency is shown to be related directly to a linear in frequency dependence of the impurity scattering rate with a negligibly small value at zero frequency. In the weak scattering limit, the conductivity decreases linearly with the frequency. In the vortex state, assuming a random (Gaussian) distribution of vortices, we show that the magnetic field profoundly alters the impurity scattering rate, which now acquires a finite zero frequency value. As a consequence, we predict a Drude-like line shape in the microwave conductivity at low frequency.

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At low temperature elastic impurity scattering is expected to dominate over its inelastic counterpart in high-$T_c$ cuprates. In much of the early literature on impurity scattering in $d$-wave superconductors, only the unitary (strong) and Born (weak) limit were studied. Later, it was recognized that neither of these two limits apply and an intermediate impurity potential $V_i$ is realistic for the high purity samples which are now available. In particular, Schachinger and Carbotte found that a finite value of $c = 1/[\pi N(0)V_i]$ with $N(0)$ the density of states in the normal state, is required to qualitatively fit the microwave data of ortho-I $\text{YB}_2\text{Cu}_3\text{O}_{6.99}$. Similarly, intermediate scattering was necessary to explain the thermal conductivity of another high purity ortho-I sample. Additional evidence for a finite value of the scattering strength is the zero temperature residual absorption observed in ortho-II $\text{YB}_2\text{Cu}_3\text{O}_{6.50}$ while at the same time the penetration depth is linear in $T$ and the microwave conductivity at low frequency exhibits a cusp.

We consider the microwave conductivity of an ortho-II sample with and without a magnetic field ($H$). Our theoretical results with $c = 0.4$ semi-quantitatively explain the experimental data in zero field. We do not attempt an exact fit to the data, instead we explore the origin of the cusp behavior of the microwave conductivity at low frequency and changes resulting from the application of a magnetic field. We find that the cusp arises because the impurity scattering rate is the form $\gamma(\omega) \simeq \gamma_0 + \alpha \omega$ for small $\omega$ with $\gamma_0$ negligibly small. It is the linear increase of $\gamma(\omega)$ that is responsible for the cusp behavior. In the vortex state, the impurity scattering is profoundly changed, and is no longer linear. Moreover, its zero frequency value is increased significantly. Consequently, we predict that the cusp disappears and a Drude-like behavior results.

Roughly speaking, the magnetic field has two independent effects on the conductivity. It increases the number of quasiparticles, and so the conductivity. The other is its effect on the impurity scattering. This is the more interesting and significant aspect for understanding quasiparticle transport in $d$-wave superconductors. The formalism to account for field effects on physical quantities such as thermodynamic and transport properties is well established. The basic approach uses a semiclassical approximation in which the essential effect of the field is embodied via the Doppler shift due to the circulating supercurrent ($v_s$) around the vortex cores. This approximation is valid when $H_{c1} \lesssim H \ll H_{c2}$. Another approximation valid for the low $T$ microwave conductivity is that the nodal quasiparticles dominantly determine the transport properties.

Due to the inhomogeneous nature of the vortex state, the impurity scattering $\gamma(\omega, \mathbf{r})$ depends on local position ($\mathbf{r}$) in space. Since the microwave conductivity is significantly affected by the impurity scattering, it is quite crucial to determine the scattering rate self-consistently. The self-consistency comes through the T-matrix approximation in which the local $2 \times 2$ Green’s function $G(k, i\omega - v_s(r) \cdot k)$ appears with $\omega = \omega - \Sigma_i(\mathbf{r}, \mathbf{r})$ and $\gamma(\omega, \mathbf{r}) = -\text{Im}\Sigma_{i,ret}(\omega, \mathbf{r})$. The local impurity self-energy is determined by

$$\Sigma_{i,ret}(\omega, \mathbf{r}) = -\frac{\Gamma G_0(\omega, \mathbf{r})}{\epsilon^2 - G_0^2(\omega, \mathbf{r})},$$

where $G_0(\omega, \mathbf{r}) = [2\pi N(0)]^{-1} \sum_k \text{Tr}[G_{ret}(k, i\omega - v_s(\mathbf{r}) \cdot \mathbf{k})]$ and $\Gamma$ is proportional to the impurity concentration. Since the nodal quasiparticles dominantly contribute to transport at low $T$, the Doppler shift effectively depends only on momentum along the nodes; namely, $\epsilon_i = v_s(\mathbf{r}) \cdot \mathbf{k}_{ni}$, where $i$ is a node index. Then,

$$G_0(\omega, \epsilon_1, \epsilon_2) \simeq \frac{1}{2\pi} \sum_{i=1}^{4} \frac{\zeta_i}{4\Delta_0} \ln \left( \frac{\zeta_i}{4\epsilon_i} \right)$$

where $\zeta_i = \omega - \epsilon_i$ with $\epsilon_3 = -\epsilon_1$ and $\epsilon_4 = -\epsilon_2$. Now $\omega = \omega - \Sigma_i(\omega, \epsilon_1, \epsilon_2)$ and $\gamma(\omega, \epsilon_1, \epsilon_2) = -\text{Im}\Sigma_i(\omega, \epsilon_1, \epsilon_2)$. Consequently, only two nodes (say, node 1 and node 2) need to be considered independently in the self-consistent calculation for $\Sigma_{i,ret}(\omega, \epsilon_1, \epsilon_2)$. Once the
self-energy is determined, the local microwave conductivity \( \sigma_{xx}(\Omega, T, \epsilon_1, \epsilon_2) \) can be evaluated. However, it is still necessary to average the conductivity over space to compare with experimental data:

\[
\sigma_{xx}(\Omega, T, H) = \int d\epsilon_1 d\epsilon_2 \mathcal{L}(\epsilon_1, \epsilon_2) \sigma_{xx}(\Omega, T, \epsilon_1, \epsilon_2),
\]

where \( \mathcal{L}(\epsilon_1, \epsilon_2) \) is the vortex distribution function taking into account the two nodes. As in the standard formalism, the optical conductivity is given by the spectral function corresponding to the \( 2 \times 2 \) Green’s function of the vortex state. Later, we will present the explicit form of the microwave conductivity. However, from the theoretical point of view, it is practically a formidable task. Here we reduce it to a simpler, if approximate, form valid for low \( T \) and for a wide range of the magnetic field.

We begin with a discussion of the zero field case. Microwave data has been obtained at various \( T \) by Turner et al. for high purity samples of ortho-I YB\(_2\)Cu\(_3\)O\(_{6.99}\) and ortho-II YB\(_2\)Cu\(_3\)O\(_{6.50}\). Here we consider only the ortho-II sample. For \( H = 0 \), the microwave conductivity (\( \sigma_{xx} \equiv \sigma \)) is given by

\[
\sigma(\Omega, T) = \frac{e^2}{2\pi^2} \left( \frac{v_f}{v_g} \right) \int d\omega \left[ \frac{f(\omega) - f(\omega + \Omega)}{\Omega} \right] A_\sigma(\Omega, \omega),
\]

where \( A_\sigma(\Omega, \omega) = 2(a + b + c)/d \) with

\[
a = \gamma \gamma \Omega \left[ \tilde{\omega}_\Omega^2 - \omega^2 + \gamma^2 + \gamma^2 \right] \ln \left[ \frac{\omega^2 + \gamma^2}{\tilde{\omega}_\Omega^2 + \gamma^2} \right],
\]

\[
b = 2\gamma \left[ \tilde{\omega}_\Omega (\gamma^2 + \gamma^2) - 2\tilde{\omega}_\Omega \gamma + \tilde{\omega}_\Omega (\tilde{\omega}_\Omega - \omega)^2 \right] \tan^{-1} \left( \frac{\tilde{\omega}_\Omega}{\gamma} \right),
\]

\[
c = 2\gamma \left[ \tilde{\omega}_\Omega (\gamma^2 + \gamma^2) - 2\tilde{\omega}_\Omega \gamma + \tilde{\omega}_\Omega (\tilde{\omega}_\Omega - \omega)^2 \right] \tan^{-1} \left( \frac{\tilde{\omega}_\Omega}{\gamma} \right),
\]

\[
d = \left[ (\tilde{\omega}_\Omega - \tilde{\omega})^2 + (\gamma - \gamma)^2 \right] \left[ (\tilde{\omega}_\Omega - \tilde{\omega})^2 + (\gamma + \gamma)^2 \right],
\]

where \( \tilde{\omega} = \omega - \text{Re} [\Sigma_{i,\text{ret}}(\omega)] \) while \( \omega_\Omega = \omega + \Omega - \text{Re} [\Sigma_{i,\text{ret}}(\omega + \Omega)] \). The impurity scattering rate \( \gamma(\omega) \) is given by the imaginary part of the self-energy, \(-\text{Im}\Sigma_{i,\text{ret}}(\omega)\) and \( \gamma_\Omega \) denotes \( \gamma(\omega + \Omega) \). The self-consistent equation determining \( \Sigma_{i,\text{ret}} \) is now \( \Sigma_{i}(\tilde{\omega}) = \frac{G_0(\tilde{\omega})}{\tilde{\omega} - G_0(\tilde{\omega})} \), where \( G_0(\omega) \approx \frac{2}{\pi} \bar{\omega} \ln \frac{\omega}{4\Delta_0} \) with \( \bar{\omega} = \omega - \Sigma_i(\tilde{\omega}) \) and \( \gamma(\omega) = -\text{Im}\Sigma_i(\tilde{\omega}) \).

An extensive search of parameter space for \( c \) and \( \Gamma \) has yielded a reasonable fit to the zero field microwave data of the ortho-II sample at \( T = 1.3, 2.7, 4.3, \) and \( 6.7\, \text{K} \), which is shown in Fig. 1. The parameters are \( c = 0.4 \) and \( \Gamma/\Delta_0 = 0.003 \). The solid curves are the results of our calculations while the symbols are for the experimental data of Ref. [11]. For the comparison we assumed \( \sigma_{00} \approx 2.5 \times 10^5 \text{\Omega}^{-1}\text{m}^{-1} \). The self-consistently obtained \( \gamma(\omega) \) is shown in the inset of Fig. 1. The curve peaks around \( \omega/\Delta_0 = 0.2 \) and has maximum height \( \gamma(\omega)/\Delta_0 \approx 0.008 \). However, at low temperature only the very small \( \omega \) region of \( \gamma(\omega) \) plays a significant role in the calculation of \( \sigma(\Omega, T) \). In this region, we can approximate \( \gamma(\omega) \approx \gamma_{00} + \omega \alpha \) with a value of \( \gamma_{00} \) which is practically zero, and \( \alpha \approx 0.021 \). As we will see, what is crucial for a cusp to appear in the microwave conductivity at low frequency is that \( \gamma_{00} \ll T \) at the temperature considered and a linear increase of \( \gamma(\omega) \) at small \( \omega \). Moreover, at low frequency the conductivity is determined mainly by \( \alpha \) and not by \( \gamma_{00} \). This implies that the cusp-like behavior in highly pure samples is robust.

In the weak impurity scattering regime (\( \gamma(\omega) < T \)), we can simplify the equation for the microwave conductivity significantly and obtain:

\[
\frac{\sigma(\Omega, T)}{\sigma_{00}} \approx \int d\omega \left( \frac{\partial f}{\partial \omega} \right) |\omega| \pi \omega \frac{2\gamma(\omega)}{\Omega^2 + 4\gamma^2(\omega)}. \tag{5}
\]

If we use the above mentioned analytic form of \( \gamma(\omega) \) in Eq. (5), we obtain remarkably good agreement with the full numerical calculation based on Eq. (4). It is clear that the complicated behavior of the microwave conductivity observed simply reflects the \( \omega \) dependence of the underlying impurity scattering rate \( \gamma(\omega) \). Consequently, instead of making use of the self-consistent calculation for \( \gamma(\omega) \), one could simply use a fitting procedure based on Eq. (5) to get the impurity scattering rate which gives the best fit to the experimental data.

When \( \gamma_{00} < \Omega \ll T \), we can further simplify Eq. (5). The analytic expression we obtained, which is valid at small \( \Omega(< T) \), is

\[
\frac{\sigma}{\sigma_{00}} \approx \frac{\pi}{2\alpha} \left[ 1 - \frac{\pi}{8\alpha T} \Omega \right]. \tag{6}
\]

This equation is one of our important results and explains the observed cusp-like behavior. As the dashed line in Fig. 1 shows for \( T = 1.3\, \text{K} \), the dc conductivity is determined only by the slope \( \alpha \), and also we see from Eq. (6), it is independent of \( \gamma_{00} \). Further, the value of the dc conductivity is independent of \( T \). The full numerical calculation shows a slight \( T \) dependence particularly at \( T = 6.7\, \text{K} \). This can be traced to the fact that the approximate expression for \( \gamma(\omega) \) is no longer completely valid. Eq. (6) shows that at low frequency, the microwave conductivity decreases linearly with \( \Omega \), and the slope of the
The microwave conductivity $\sigma(\Omega, \omega, T, H)$ as a function of frequency is also fundamentally altered in the presence of a magnetic field. The physical fact that the field has two effects on the conductivity is important to find an alternative, if approximate, way to evaluate the low temperature microwave conductivity. To get the formal expression for $\sigma(\Omega, T, H)$, which can be compared with experimental data. To get the formal expression for $\sigma(\Omega, T, \varepsilon_1, \varepsilon_2)$, we need to determine $\gamma(\omega, \varepsilon_1, \varepsilon_2)$ to evaluate $\sigma(\Omega, T, \varepsilon_1, \varepsilon_2)$. Then an averaging procedure is necessary to obtain $\sigma(\Omega, T, H)$, where $i$ is again the node index. For the node $i$, $\omega \rightarrow \omega - Re [\Sigma_{i,rel}(\omega, \varepsilon_1, \varepsilon_2)] - \varepsilon_i$ and the corresponding $\gamma(\omega) \rightarrow \gamma(\omega, \varepsilon_1, \varepsilon_2)$. Obviously, such a procedure requires a huge numerical calculation. Because of this it is important to find an alternative, if approximate, way to evaluate $\sigma(\Omega, T, H)$. We base our simplification on the physical fact that the field has two effects on the conductivity. The first is to create quasiparticles. The second is to alter the impurity scattering itself, and this will significantly change the shape of the microwave conductivity as a function of frequency. In our alternative way, we treat the two effects separately. First we deal with the field effect on the impurity self-energy to obtain $\Sigma_{i,rel}(\omega, H)$. Then, we take into account the increase in the number of quasiparticles. This procedure simplifies calculations of the microwave conductivity considerably. It reduces the contributions from the quasiparticles in the two distinct nodal areas to one from a single average node. From the theoretical point of view, this means that the space-averaged Green’s function $G(k, i\omega, H)$ is used in the T-matrix approximation to obtain

$$\Sigma(\omega, H) = \frac{\Gamma G_0(\omega, H)}{i^2 - G_0^2(\omega, H)},$$

where $G_0(\omega, H) \approx \frac{1}{\hbar} \int de \mathcal{P}(\epsilon) \frac{1}{\Delta_0} \ln \frac{\Delta_0}{4\sqrt{\Delta_0}}$ where $z = i\omega - \epsilon$ and $\mathcal{P}(\epsilon) = \int de' \mathcal{L}(\epsilon, \epsilon')$.

Within this approximation, the only change needed in Eq. (4) is to replace $A_\sigma(\Omega, \omega)$ with $A_\sigma(\Omega, \omega, \epsilon, H)$ and the averaging procedure requires the single node distribution of vortices $\mathcal{P}(\epsilon)$ as already indicated in the self-consistent calculation for the impurity self-energy. We have tested the single-node approximation involving $A_\sigma(\Omega, \omega, \epsilon, H)$ with the two-node method using $A_{\sigma,i}(\Omega, \omega, \epsilon_1, \epsilon_2)$, on the dc conductivity at zero temperature, for which only $\gamma(0, \epsilon_1, \epsilon_2)$ or $\gamma(0, H)$ enters the calculation. We found that these two methods give almost identical results for $0 < E_H/\Delta_0 \leq 0.1$ when we consider a random (Gaussian) distribution of vortices [15]. Therefore, as a first approximation, we use the simple procedure of Eq. (7) to evaluate the low temperature microwave conductivity.

In Fig. 2, we show our results for $\gamma(\omega, H)$ as a function of $\omega$ obtained self-consistently for various values of the magnetic energy $E_H/\Delta_0 = 0.02$, 0.04, and 0.08. We estimate $E_H \approx 20\sqrt{\mu K T^{-1/2}}$ with the magnetic field...
in units of Tesla. The same values of the parameters $c$ and $\Gamma$ are used as before. For $E_H/\Delta_0 = 0$, the result for $\gamma(\omega, H)$ is the same as shown in the inset of Fig. 1. As one can see, the field has a drastic effect on the low frequency region of the impurity scattering, and this is the region that is mainly sampled in the low $T$ microwave conductivity. The two most important changes in $\gamma(\omega, H)$ are i) $\gamma(0, H)$ is no longer negligible and ii) for low $\omega$, the increase in $\gamma(\omega, H)$ is now quadratic: $\gamma(\omega, H)/\Delta_0 \simeq (\gamma(0, H)/\Delta_0 + \beta (\omega/\Delta_0)^2$, where $\beta \simeq 0.6, 0.4,$ and $0.26$ for $E_H/\Delta_0 = 0.02, 0.04,$ and $0.08$, respectively. The results for the microwave conductivity $\sigma(\Omega, T, H)$ are shown in Fig. 3 up to $20\text{GHz}$ at $T = 1.3K$. The black solid curve ($E_H/\Delta_0 = 0$) is repeated from Fig. 1 for comparison. The other curves correspond to $E_H/\Delta_0 = 0.02$ (red), $0.04$ (blue), and $0.08$ (green), respectively. The magnetic field removes the cusp at low frequency. This is due to the drastic change in impurity scattering in the vortex state as we will show below.

Even in the vortex state, we can still simplify the equation for the microwave conductivity in the weak scattering limit if $\gamma(w, H) < T$ for the important range of $\omega$. We obtain

$$\sigma / \sigma_{00} \simeq \int d\omega \left( -\frac{\partial f \pi N(\omega, H)}{\partial \omega} \right) \frac{2\gamma(\omega, H)}{\Omega^2 + 4\gamma^2(\omega, H)},$$

where the effective density of states is $N(\omega, H) = \int d\epsilon \, \rho(\epsilon) |\omega - \epsilon|$. Eq. (8) clearly shows both the effect of the magnetic field on the density of states and on the impurity scattering. The symbols in Fig. 3 are based on Eq. (8) with the approximate quadratic expression for $\gamma(\omega, H)$ quoted. It is clear that the microwave conductivity is well represented by the weak impurity scattering limit even in the vortex state. Moreover, from the comparison with Eq. (8), we can see immediately the field effect in Eq. (8). It represents the appropriate generalization of Eq. (8), and is another of our important results. For the random vortex distribution, the effective density of states becomes $N(\omega, H) = (E_H/\sqrt{\pi}) e^{-\omega^2/E_H^2} + \omega \text{erf}(\omega/E_H)$. In the field dominated regime, we can further simplify Eq. (8) utilizing the fact that $\gamma(\omega, H) \approx \gamma(0, H)$ and $N(\omega, H) \approx E_H/\sqrt{\pi}$ at very low $\omega \ll E_H$. Now it is possible to approximate the microwave conductivity as:

$$\sigma(\Omega, T, H) / \sigma_{00} \simeq \frac{\sqrt{\pi}E_H}{2\gamma(0, H)} \left[ 1 - \frac{\Omega^2}{4\gamma^2(0, H)} \right].$$

This equation shows that the microwave conductivity now has no cusp and a Drude-like behavior appears in the vortex state which can be traced to the characteristic change in the impurity scattering which now has a finite $\gamma(0, H)$ value.

In conclusion, we have considered the low $T$ microwave conductivity of a high purity $d$-wave superconductor with and without a magnetic field. We find that the cusp-like behavior in the microwave conductivity arises because the impurity scattering rate is negligibly small at zero frequency and increases linearly at low frequency: $\gamma(\omega) \simeq \gamma_0 + \alpha \omega$. The low frequency microwave conductivity does not depend on $\gamma_0$ but the slope $\alpha$ plays a significant role. The steepness of the cusp is inversely proportional to $T$. In the vortex state, the impurity scattering is drastically changed, and its zero frequency value is no longer negligible. Moreover, it increases quadratically with frequency. We predict that this characteristic change in the scattering rate eliminates the cusp in the microwave conductivity. Drude-like behavior appears instead. We hope our prediction inspires an experiment of the microwave conductivity in the vortex state.

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