GENERALIZED COMPLETE INTERSECTIONS WITH LINEAR RESOLUTIONS

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Abstract. We determine the simplicial complexes $\Delta$ whose Stanley-Reisner ideals $I_{\Delta}$ have the following property: for all $n \geq 1$ the powers $I_{\Delta}^n$ have linear resolutions and finite length local cohomologies.

Keywords: FLC, generalized Cohen-Macaulay monomial ideal, linear resolution, local cohomology

MSC Primary: 13F55, Secondary: 13D02, 13D45

Introduction

Let $S = K[X_1, \ldots, X_n]$ be a standard graded polynomial ring over a field $K$ and $m = (X_1, \ldots, X_n)$. Recall that a graded ideal $I \subset S$ is called generalized Cohen-Macaulay or simply FLC (finite local cohomology) when the local cohomology $H^i_m(S/I)$ has finite length for all $i < \dim S/I$. We find many examples of such ideals in algebraic geometry: the defining ideals of Cohen-Macaulay projective schemes over the field are all FLC ideals. However, as far as the authors are concerned, we do not know very much about FLC monomial ideals. The second author gave a combinatorial characterization of FLC monomial ideals \cite{8} as an extension of the well known case of squarefree monomial ideals, i.e., Buchsbaum Stanley-Reisner ideals. On the other hand, the notion of generalized complete intersections (gCI) has been introduced in \cite{4}. A gCI $I \subset S$ is a squarefree monomial ideal such that the Stanley-Reisner ring $S/I$ is complete intersection over the punctured spectrum. All the powers $I^n$ ($n \geq 1$) of a gCI $I \subset S$ are FLC, and a combinatorical characterization of gCI has been given.

The purpose of this paper is to give a combinatorial characterization of the simplicial complexes $\Delta$ whose Stanley-Reisner ideals $I_{\Delta}$ are gCI and all the powers $I_{\Delta}^n$ ($n \geq 1$) have linear resolutions. We will show that $I_{\Delta}$ is gCI and $I_{\Delta}^n$ has a linear resolution for all $n \geq 1$ if and only if $\Delta$ is as follows: $\Delta$ is a finite set of points, the disjoint union of paths or, if $\dim \Delta \geq 2$, then $\Delta$ is a pure simplicial complex and is the disjoint union of facets $H$ and pairs $(F, G)$ of facets such that $|F \setminus G| = |G \setminus F| = 1$ (Theorem 2.3). We also show that if the Buchsbaum Stanley-Reisner ring $S/I_{\Delta}$ has minimal multiplicity in the sense of Goto \cite{3}, then any power $I_{\Delta}^\ell$, $\ell \geq 1$ is FLC and has a linear resolution (Corollary 4). Finally, we give a computation of part of local cohomologies of gCI with linear resolutions (Proposition 4.3).
1. Generalized Complete Intersection

In this section, we recall some definitions and already known results, which will be used in the next section.

A Stanley-Reisner ring $K[\Delta] = S/I_\Delta$ is called \textit{generalized complete intersection} (gCI) if $K[\Delta]_P$ is a complete intersection for every prime ideal $P(\neq m)$ and $\Delta$ is a pure simplicial complex. For a simplicial complex $\Delta$, we will always assume that $\{i\} \in \Delta$ for all $i \in [n]$.

**Theorem 1.1** (cf. Th. 2.5 [4]). Let $\Delta$ be a simplicial complex on the vertex set $[n] = \{1, \ldots, n\}$. Then the following conditions are equivalent:

(i) $K[\Delta]$ is a gCI;

(ii) $S/I_\Delta^{+1}$ has FLC for an arbitrary integer $\ell \geq 0$.

If one of these conditions holds, $K[\Delta]$ is Buchsbaum.

A special case of gCI is complete intersection. In order to exclude this uninteresting case, we use the notion of core of simplicial complex. Let $\Delta$ be a simplicial complex on the vertex set $[n]$. For $F \in \Delta$, we define $\text{st}_\Delta(F) = \{G \in \Delta \mid G \cup F \in \Delta\}$. We also define $\text{core}[n] = \{i \in [n] \mid \text{st}_\Delta(\{i\}) \neq \Delta\}$. Then the core of $\Delta$ is defined by $\text{core}\Delta = \{F \cap \text{core}[n] \mid F \in \Delta\}$. A gCI is a complete intersection if and only if core$\Delta \neq \Delta$.

A combinatorial characterization of gCI is given as follows:

**Theorem 1.2** (cf. Th. 3.16 [4]). Let $K[\Delta]$ be a Stanley-Reisner ring with $\Delta = \text{core}\Delta$ (i.e., $K[\Delta]$ is not a complete intersection). Let $G(I_\Delta) = \{u_1, \ldots, u_\ell\}$ be the minimal set of generators of $I_\Delta$ and $F_\Delta = \{\text{supp}(u_j) \mid j = 1, \ldots, \ell\}$, where $\text{supp}(u) = \{j \in [n] \mid X_j \text{ divides } u\}$. Then $K[\Delta]$ is a gCI if and only if the following conditions hold:

(i) $\Delta$ is pure.

(ii) for every $U \in F_\Delta$ with $|U| \geq 3$, there exists a non-empty subset $C(U)$ of $[n]$ such that

(a) $C(U) \cap U = \emptyset$,

(b) for every $i \in C(U)$, we have $E_{ij} := \{i, j\} \in F_\Delta$ for all $j \in U$.

Moreover if $U \cap T \neq \emptyset$ for $T \in F_\Delta$, then $T = E_{ij}$ for some $i, j$,

(c) for every $k \in C(U) \cup U$, we have $\{i, k\} \in F_\Delta$ for all $i \in C(U)$.

(iii) Any two elements $i, j \in [n]$ are linked with a path $P = \{i_k, i_{k+1}\} \mid k = 1, \ldots, r\}$, with edges $\{i_k, i_{k+1}\} \in F_\Delta$ for $k = 1, \ldots, r$ such that $i = i_1$ and $j = i_{r+1}$,

(iv) If there exists a length 4 path $P = \{i_p, i_{p+1}\} \in F_\Delta \mid p = 1, 2, 3, 4\}$ (with $i_1 \neq i_5$), then there must be an edge $\{i_1, i_q\} \in F_\Delta$ with $q = 3, 4$ or 5.

Recall that a graded ideal $I \subset S$ is said to have a linear resolution if all entries in the matrices representing the differentials in a graded minimal $S$-free resolution of $I$ are linear. It is an interesting question which ideal $I$ has the property that all the powers $I^n$ ($n \geq 2$) have a linear resolution.

An immediate corollary to Th. 1.2 is
Corollary 1. Let $I_\Delta \subset S$ is a gCI with $\text{core}\Delta = \Delta$. If $I_\Delta$ has a linear resolution, $I_\Delta$ is generated in degree 2. Namely, a gCI may have only a 2-linear resolution.

Proof. If $I_\Delta$ has a $q$-linear resolution with $q \geq 3$. Then $\deg u_i = q(\geq 3)$ for $i = 1, \ldots, \ell$. But the condition (ii) of Th. 1.2 implies the existence of a degree 2 element in $G(I_\Delta)$, a contradiction. $\Box$

For 2-linear resolution, we have the following result by Herzog-Hibi-Zheng.

Theorem 1.3 (Th. 3.2 [5]). Let $I$ be a monomial ideal generated in degree 2. Then the following conditions are equivalent:

(i) $I$ has a linear resolution.
(ii) $I^\ell$ has a linear resolution for every $\ell \geq 1$.

In particular, when we consider Stanley-Reisner ideals, a monomial ideal $I$ generated in degree 2 can be described in terms of edge graph. Namely, for a finite graph $G$ we define the edge ideal $I_G = (X_iX_j \mid \{i, j\} \text{is a edge of } G)$. Any Stanley-Reisner ideal generated in degree 2 is the edge ideal of a graph, which we call the edge graph. Notice that when $I_\Delta$ is generated in degree 2, $F_\Delta$ in Th. 1.2 can be identified with (the edge set of) the edge graph $G_\Delta$ corresponding to the edge ideal $I_\Delta$.

For linear resolutions of such ideals, we have

Theorem 1.4 (Fröberg [2]). Let $G$ be a graph. Then the edge ideal $I_G$ has a linear resolution if and only if the complementary graph $G'$ is chordal.

Recall that the complemental graph $G'$ of $G$ is the graph whose vertex set is the same as that of $G$ and whose edges are the non-edges of $G$. A graph $G$ is called chordal if each cycle of length > 3 has a chord. Notice that the complemental graph $G'$ in Th. 1.4 is exactly the 1-skeleton $\Delta_1$ of the simplicial complex $\Delta$ corresponding to the edge ideal $I_G$.

As an immediate consequence, we have

Corollary 2. Let $K[\Delta]$ be a Stanley-Reisner ring with $\Delta = \text{core}\Delta$. Then $K[\Delta]$ is a gCI and $I_\Delta^\ell$ has a linear resolution for every $\ell \geq 1$ if and only if the following conditions hold:

(i) $\Delta$ is pure and $I_\Delta$ is generated in degree 2.
(ii) The 1-skeleton $\Delta_1$ is chordal.
(iii) Any two elements $i, j \in [n]$ are linked with a path $P = \{i_k, i_{k+1} \mid k = 1, \ldots, r\}$, with edges $\{i_k, i_{k+1}\} \in G_\Delta$ for $k = 1, \ldots, r$ such that $i = i_1$ and $j = i_{r+1}$.
(iv) If there exists a length 4 path $P = \{i_p, i_{p+1}\} \in G_\Delta \mid p = 1, 2, 3, 4\}$ (with $i_1 \neq i_5$), then there must be an edge $\{i_1, i_q\} \in G_\Delta$ with $q = 3, 4$ or 5.

In the next section, we will give a precise description of the simplicial complexes satisfying the conditions in Cor. 2.
2. Generalized complete intersections with 2-linear resolutions

In this section, we will give a precise description of the simplicial complexes \( \Delta \) such that the Stanley-Reisner rings \( K[\Delta] \) are gCI and every power \( I_\Delta^\ell \) has a linear resolution. As we showed in the previous section, this means gCI with a 2-linear resolution.

For a graph \( G \) over the vertex set \( V \), we define \( \text{Simp}(G) \) to be the set of all subsets \( F \) of \( V \) such that \( F \) is the vertex set of a subgraph \( H \) of \( G \) isomorphic to a complete graph. For a simplicial complex \( \Delta \), \( \text{Simp}(\Delta) \), where \( \Delta_1 \) denotes the 1-skeleton, is the simplicial complex obtained by filling all the simplicial cycles in \( \Delta \).

**Proposition 2.1** (cf. Prop. 6.1.25 [10]). Let \( I_\Delta \) be a Stanley-Reisner ideal generated. Then \( I_\Delta \) is generated in degree 2 if and only if \( \Delta = \text{Simp}(\Delta_1) \).

**Proposition 2.2.** Let \( \Delta \) be a pure simplicial complex on the vertex set \([n]\) with \( \Delta = \text{core}\Delta \). Assume that \( K[\Delta] \) is a gCI and \( I_\Delta \) has a 2-linear resolution. Then, for any two distinct facets \( F \) and \( H \) such that \( F \cap H \neq \emptyset \), there exist a unique element \( \{i_1, i_2\} \in G_\Delta \), i.e., \( X_{i_1}X_{i_2} \in I_\Delta \), with \( i_1 \in F \setminus H \) and \( i_2 \in H \setminus F \).

**Proof.** Assume that there exists a pair of distinct facets \( F \) and \( H \) with \( F \cap H \neq \emptyset \) such that no edge \( \{i_1, i_2\} \), where \( i_1 \in F \setminus H \) and \( i_2 \in H \setminus F \), is in \( G_\Delta \). Then the complete graph \( K \) over the vertex set \( F \cup H \) is contained in the 1-skeleton \( \Delta_1 \) of \( \Delta \). But since \( I_\Delta \) is generated in degree 2, we know that \( F \cup H \in \Delta \) by Prop. 2.1 which contradicts the assumption that \( F \) and \( H \) are facets. Thus we have proved the existence of the pair \( \{i_1, i_2\} \in G_\Delta \) with the required property.

Now we show the uniqueness of the pair. Let \( F \) and \( H \) be distinct facets with \( F \cap H \neq \emptyset \). We may assume \( |F \setminus H| = |H \setminus F| \geq 2 \) and there exists a pair \( \{i_1, i_2\} \in G_\Delta \) with \( i_1 \in F \setminus H \) and \( i_2 \in H \setminus F \).

First of all, suppose there exists \( j_2 \neq i_2 \in H \setminus F \) such that \( \{i_1, j_2\} \in G_\Delta \). Take any \( i \in F \cap H \). Then, since \( F = \text{core}\Delta \), there exists \( j \in [n] \setminus (F \cup H) \) such that \( \{i, j\} \in G_\Delta \). By Cor. 2(iii), there exists a path in \( G_\Delta \) connecting \( j \) and \( i_2 \). Furthermore, by using Cor. 2(iv), we can take the path to be an edge \( \{j, i_2\} \in G_\Delta \). Consequently, we obtain the length 4 path \( \{i, j\}, \{j, i_2\}, \{i_2, i_1\}, \{i_1, j_2\}\) in \( G_\Delta \), so that by Cor. 2 at least one of \( \{i, i_2\}, \{i, i_1\}, \{i, j_2\} \in G_\Delta \). But since \( \{i, i_2\}, \{i, j_2\} \in H \) and \( \{i, i_1\} \in F \), this cannot happen. Thus such \( j_2 \) does not exist.

Next suppose that there exist \( j_1 \in F \setminus H \) and \( j_2 \in H \setminus F \) such that \( j_1 \neq i_1 \), \( j_2 \neq i_2 \) and \( \{j_1, j_2\} \in G_\Delta \). By the similar discussion as above, we know that \( \{i_1, j_2\}, \{i_2, j_1\} \notin G_\Delta \). Thus \( \Delta_1 \) contains the length 4 cycle \( \{i_1, j_2\}, \{j_2, i_2\}, \{i_2, j_1\}, \{j_1, i_1\}\) without any chord. But, since \( I_\Delta \) has a 2-linear resolution, this contradicts Fröberg’s condition Th. 1.3.

**Corollary 3.** Let \( \Delta \) be as in Prop. 2.2. Then, for any distinct facets \( F \) and \( H \) such that \( F \cap H \neq \emptyset \), we have \(|F \setminus H| = |H \setminus F| = 1\).
Theorem 2.3. Let $\Delta$ be a pure simplicial complex with $\text{core}\Delta = \Delta$ and $\text{dim}\Delta = 0$. Then $I_\Delta$ is generated in degree 2 and has FLC and a linear resolution for all $\ell \geq 1$. Now assume that $I_\Delta$ is generated in degree 2.

Consider any distinct facets $F$ and $G$ with $F \cap G \neq \emptyset$ and assume that there exists the third facet $H$ with $H \cap (F \cup G) \neq \emptyset$. By Cor. 1 we know that we must have $C := F \cap G = G \cap H = F \cap H \neq \emptyset$ and $\{i_1\} = F \setminus C$, $\{i_2\} = G \setminus C$ and $\{i_3\} = H \setminus C$ for some distinct $i_1$, $i_2$ and $i_3$. Let $i \in C$ be arbitrary. There, then exists $j \notin F \cup G \cup H$ since $\text{core}\Delta = \Delta$. Now, as in the proof of Prop. 2.1, we obtain the length 4 path $\{i, j\}, \{j, i_2\}, \{i_2, i_3\}, \{i_3, i_1\}$ in the edge graph $G_\Delta$, for which $\{i, i_1\}, \{i, i_2\}, \{i, i_3\} \notin G_\Delta$ since each of them is in a facet. This contradicts Cor. 2 (iv). Thus such a facet $H$ does not exist. This implies that $\Delta$ is as stated above.

Corollary 4. Let $\Delta$ be a pure simplicial complex with $\text{core}\Delta = \Delta$. Then the power of the Stanley-Reisner ideal $I_\Delta^\ell$ has FLC and a linear resolution for all $\ell \geq 1$, if and only if $\Delta$ is as in Th. 2.3.
3. **Stanley-Reisner ring with minimal multiplicity**

Let \( A = K[A_1] \) be a homogeneous Buchsbaum \( K \)-algebra of dimension \( d \) with the unique homogeneous maximal ideal \( m = A_+ \). Then \( A \) is called a *Buchsbaum ring with minimal multiplicity* \( \mathbb{K} \) if \( e(A) = 1 + \sum_{i=1}^{d-1} (-1)^{i-1} l_A(H^n_m(A)) \), where \( e(A) \) denotes the multiplicity of \( A \) with regard to \( m \), \( H^n_m(A) \) is the \( i \)th local cohomology and \( \ell_A(M) \) denotes the length of the \( A \)-module.

On the other hand, a Stanley-Reisner ideal \( I_\Delta \subset S \) generated in the same degree \( \delta \) is called *matroidal* of degree \( \delta \) if \( \mathcal{B} := \{ \text{supp}(u) \mid u \in G(I_\Delta) \} \) forms a matroid base of degree \( \delta \). Namely, \( |B| = \delta \) for all \( B \in \mathcal{B} \), and for all \( B_1, B_2 \in \mathcal{B} \) and all \( i \in B_1 \setminus B_2 \), there exists \( j \in B_2 \setminus B_1 \) such that \( (B_1 \setminus \{i\}) \cup \{j\} \in \mathcal{B} \). In particular, for a Stanley-Reisner ideal \( I_\Delta \) generated in degree 2 is matroidal if the edge graph \( G \) has the following property: for any disjoint edges \( \{i, j\} \) and \( \{p, q\} \) of the edge graph \( G_\Delta \), each vertex \( \{k\} \), \( k = i, j \) is linked with at least one of the vertices of the edge \( \{p, q\} \), and vice versa. Matroidal ideals have linear resolutions. See \[5\] for the detail of this fact.

We note that, for a matroidal ideal \( I_\Delta \), the corresponding simplicial complex is not always pure. For example,

\[
I_\Delta = (X_1X_3, X_1X_4, X_2X_3, X_2X_4, X_3X_5, X_4X_5),
\]

is matroidal, and the simplicial complex \( \Delta \) is spanned by the facets \( \{1, 2, 5\} \) and \( \{3, 4\} \).

Now we show the following.

**Proposition 3.1.** Let \( K[\Delta] \) be a Buchsbaum Stanley-Reisner ring with \( \dim K[\Delta] = d + 1 \) and \( \text{core} \Delta = \Delta \). Then the followings are equivalent:

(i) \( K[\Delta] \) has minimal multiplicity.

(ii) \( \Delta \) is the disjoint union of \( d \)-simplexes.

(iii) \( I_\Delta \) is matroidal of degree 2.

**Proof.** The equivalence of (i) and (ii) is shown in Example 3.1 [9]. Assume that \( \Delta = \langle H_1, \ldots, H_r \rangle \) (disjoint union of \( d \)-simplexes). Then \( I_\Delta = (X_iX_j \mid i \in H_p, j \in H_q \) for some \( 1 \leq p < q \leq r \), so that we easily know that this ideal is matroidal. Now we have only to show (iii) to (ii).

\( \Delta \) is pure since \( K[\Delta] \) is Buchsbaum. Assume that \( I_\Delta \) is matroidal of degree 2 but \( \Delta \) is not the disjoint union of \( d \)-simplexes. Then there exist facets \( F \) and \( G \) such that \( F \cap G \neq \emptyset \). Choose any \( i \in F \cap G \). Since \( \text{core} \Delta = \Delta \), there exists \( j \notin F \cup G \) such that \( \{i, j\} \in G_\Delta \).

On the other hand, by Th. 1.2 we easily know that \( I_\Delta \) is a gCI. Thus there exists \( \{i_1, i_2\} \in G_\Delta \) with \( i_1 \in F \setminus G \) and \( i_2 \in G \setminus F \) by Prop 2.2.

Since \( \{i_1, i_2\}, \{i_1, i_2\} \notin G_\Delta \), the existence of two disjoint edges \( \{i, j\} \) and \( \{i_1, i_2\} \) contradicts the assumption that \( I_\Delta \) is matroidal. \( \square \)

**Corollary 5.** Let \( K[\Delta] \) be a Buchsbaum Stanley-Reisner ring with minimal multiplicity. Then \( I_\Delta^\ell \) has FLC and a linear resolution for all \( \ell \geq 1 \).
4. LOCAL COHOMOLOGIES OF GENERALIZED COMPLETE INTERSECTION WITH LINEAR RESOLUTIONS

In this section, we consider local cohomologies of the generalized complete intersection with linear resolutions. For a positively graded $K$-algebra $R$ with the graded maximal ideal $m = \bigoplus_{n>0} R_n$, we denote by $H^i_m(R)$ the $i$th local cohomology module with regard to $m$. Since we consider monomial ideals, the local cohomology modules have the $\mathbb{Z}^n$-grading. In the following, we will denote by $[H^i_m(R)]_a$, where $a \in \mathbb{Z}$ or $a \in \mathbb{Z}^n$, the $a$-th graded component of the module.

It is well known that, for Stanley-Reisner ideals, FLC and Buchsbaum are equivalent notions and for a Buchsbaum Stanley-Reisner ring $K[\Delta]$, we have $[H^i_m(K[\Delta])]_j = 0$ for all $i < \dim K[\Delta]$ and for all $j \neq 0$. For monomial ideals, if $I \subset S$ is FLC then $[H^i_m(S/I)]_a = 0$ for all $i < \dim S/I$ and $j < 0$ or $\sum_{k=1}^n \rho_k - n < j$ where $\rho_k$ is the maximal exponent of the variable $X_k$ in the minimal set $G(I)$ of monomial generators. In particular, we have

**Proposition 4.1** (Prop. 1 [8]). For a monomial ideal $I \subset S$, the local cohomology $H^i_m(S/I)$, $i \neq \dim S/I$, has finite length if and only if $H^i_m(S/I)_a = 0$ for all $a \in \mathbb{Z}^n$ such that $a_j < 0$ for some $1 \leq j \leq n$.

See [8] for the results on FLC monomial ideals.

We will now compute $[H^i_m(S/I_\ell \Delta)]_0$ for all $i < \dim S/I_\ell \Delta = \dim S/I_\Delta$ and all $\ell \geq 1$, for gCI $I_\Delta$ with a linear resolution.

We first recall a few results. The local cohomologies of a monomial ideal $I$ and its radical $\sqrt{I}$ can be compared by

**Proposition 4.2** (cf. Cor. 2.3 [7]). Let $I \subset S$ be a monomial ideal. Then we have the following isomorphisms of $K$-vector spaces

$$[H^i_m(S/I)]_a \cong [H^i_m(S/\sqrt{I})]_a$$

for all $a \in \mathbb{Z}^n$ with $a_i \leq 0$ for all $1 \leq i \leq n$.

For the local cohomologies of Stanley-Reisner ideal, we recall Hochster’s formula:

**Theorem 4.3** (cf. Th. 5.3.8 [1]). The Hilbert series of $K[\Delta]$ with respect to the $\mathbb{Z}^n$-grading is given by

$$\text{Hilb}(H^i_m(K[\Delta]), t) = \sum_{F \in \Delta} \dim_K \tilde{H}_{i-\abs{F}-1}(\text{lk} F; K) \prod_{j \in F} \frac{t_j^{-1}}{1-t_j^{-1}}.$$ 

Now we show the following

**Proposition 4.4.** Let $I_\Delta \subset S$ be a gCI with a linear resolution. Then we have the following isomorphism as $K$-vector spaces

$$[H^i_m(S/I_\ell \Delta)]_0 \cong \begin{cases} 
K^{a-1} & \text{if } i = 1 \\
0 & \text{if } i \neq 1, d
\end{cases}$$
where \( d = \dim S/I_\Delta = \dim \Delta + 1 \) and \( \alpha \) is the number of connected components of \( \Delta \).

**Proof.** By Prop. 4.1 and Prop. 4.2 we have 
\[
[H_m^i(S/I_\Delta^\ell)]_0 = [H_m^i(S/I_\Delta^\ell)]_0 = [H_m^i(S/\sqrt{I_\Delta^\ell})]_0 = [H_m^i(S/I_\Delta^\ell)]_0 \quad \text{for all } i < d \text{ and } \ell \geq 1,
\]
where \( 0 = (0, \ldots, 0) \in \mathbb{Z}^n \).

Now, since \( I_\Delta \subset S \) is a gCI with a 2-linear resolution, we compute 
\[
[H_m^i(S/I_\Delta^\ell)]_0 = 0 \quad \text{for } i \neq 1 \text{ and } i < d
\]
using Th. 4.3 (see Prop. 1.1(3) in [9]).

Now we consider the case of \( i = 1 \). Since \( \Delta \) is as in Th. 2.3, \( \text{lk}(F) \) \( (F \in \Delta) \)
and \( H_{\cdot} := \tilde{H}_{\cdot - |F|-1}^i(\text{lk}(F); K) = \tilde{H}_{|F|}^i(\text{lk}(F); K) \) are as follows:

(i) \( \text{lk}(F) = \Delta \) if \( F = \emptyset \). Then \( H \cong K^{\alpha-1} \), where \( \alpha \) is the number of connected components in \( \Delta \).

(ii) \( \text{lk}(F) = \{\emptyset\} \), if \( F \) is an isolated facet. Then \( H = 0 \).

(iii) \( \text{lk}(F) \) is a simplex, if \( F \) is one of the following;

(a) \( F \) is a non-facet of an isolated facet,

(b) for a pair \( \langle F', G' \rangle \) of intersecting facets we have \( F \subset F' \) and also \( F' \setminus G' \subset F \).

Then \( H = 0 \).

(iv) \( \text{lk}(F) \) is two points, if \( F \) is the intersection of two facets \( F' \) and \( G' \) such that \( F' \cap G' = F \) and \( |F' \setminus G'| = |G' \setminus F'| = 1 \). Then, \( H = 0 \).

(v) \( \text{lk}(F) \) is a union of two simplexes \( \langle F', G' \rangle \), where \( F' \cup F \) and \( G' \cup F \) are facets. Then, \( H = 0 \).

Then we obtain the desired result by Th. 4.3. \( \square \)

**References**

[1] W. Bruns and T. Hibi, Stanley-Reisner rings with pure resolutions, Communications in Algebra 23-4 (1995), 1201-1217. MR1317395 (96h:13059)

[2] R. Föberg, On Stanley-Reisner rings, in: Topics in algebra, Banach Center Publications, 26 part 2, (1990), 57-70

[3] S. Goto, On the associated graded rings of parameter ideals in Buchsbaum rings, J. Algebra 85 (1983), 490-534. MR0725097 (85e:13021)

[4] S. Goto and Y. Takayama, Stanley-Reisner ideals whose powers have finite length cohomologies, Proc. Amer. Math. Soc. (to appear)

[5] J. Herzog, T. Hibi and X. Zheng, Monomial ideals whose powers have a linear resolution, Math. Scand. 95 (2004), no. 1, 23-32.

[6] J. Herzog and Y. Takayama, Resolutions by mapping cones, Homology Homotopy Appl. 4 (2002), no.2, part 2, 277–294.

[7] J. Herzog, Y. Takayama and N. Terai, On the radical of a monomial ideal. Arch. Math. (Basel) 85 (2005), no. 5, 397–408.

[8] Y. Takayama, Combinatorial characterizations of generalized Cohen-Macaulay monomial ideals, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 48(96) (2005), no. 3, 327–344.

[9] N. Terai and K. Yoshida, Buchsbaum Stanley-Reisner rings with minimal multiplicity, Proc. Amer. Math. Soc. 134 (2006), no. 1, 55–65

[10] R. H. Villarreal, MONOMIAL ALGEBRAS, Monographs and Textbooks in Pure and Applied Mathematics, 238. Marcel Dekker, Inc., New York, 2001.

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