Gap-Townes Solitons and Delocalizing Transitions of Multidimensional Bose–Einstein Condensates in Optical Lattices

Mario Salerno, F. Kh. Abdullaev, and B.B. Baizakov

Abstract We show the existence of gap-Townes solitons for the multidimensional Gross–Pitaevskii equation with attractive interactions and in two- and three-dimensional optical lattices. In absence of the periodic potential the solution reduces to the known Townes solitons of the multi-dimensional nonlinear Schrödinger equation, sharing with these the property of being unstable against small norm (number of atoms) variations. We show that in presence of the optical lattice the solution separates stable localized solutions (gap-solitons) from decaying ones, characterizing the delocalizing transition occurring in the multidimensional case. The link between these higher dimensional solutions and the ones of one dimensional nonlinear Schrödinger equation with higher order nonlinearities is also discussed.

Keywords Matter-waves · Gap-Townes soliton · Collapse · Delocalizing transition

1 Introduction

One interesting phenomenon occurring in ultracold atomic gases trapped in periodic potentials is the possibility to localize matter in states which can stay for a long time due to an interplay between nonlinearity, dispersion and periodicity. Such states (also called gap-solitons) have been observed in Bose–Einstein condensates (BEC) and in arrays of nonlinear optical waveguides [1–4]. For attractive atomic interactions in BEC and in absence of a periodic potential, stable localized solutions are possible only in a one-dimensional (1D) setting since in two (2D) and three (3D) dimensions the phenomenon of collapse appears [5]. More precisely, one observes that when the number of atoms exceeds a critical threshold, the solution collapses in a finite time (blow-up) while for number of atoms below the critical threshold

M. Salerno (✉)
Dipartimento di Fisica “E.R. Caianiello” and Consorzio Nazionale Interuniversitario per le Scienze Fisiche della Materia (CNISM), Università di Salerno, I-84081 Baronissi (SA), Italy
e-mail: salerno@sa.infn.it

F.Kh. Abdullaev and B.B. Baizakov
Physical-Technical Institute of the Uzbek Academy of Sciences, 100084, Tashkent, Uzbekistan
there is an irreversible decay of the state into background radiation. For the higher dimensional nonlinear Schrödinger (NLS) equation, however, it is known that there exists an unstable localized solution, the so called Townes soliton [6], which separates decaying solutions from collapsing ones. Townes soliton, however, exists only for a single value of the number of atoms, being unstable against fluctuations around it (for slightly overcritical or undercritical number of atoms the solution collapses or decays, respectively). The situation is drastically changed in presence of an optical lattice (OL). To this regard, it has been shown that stable 2D and 3D solitons can exist in OLs both in BEC and nonlinear optics contexts [7–11]. Moreover, it is known that while the periodic potential can only marginally shift the critical value for collapse, it can substantially move the delocalizing transition curve, thereby increasing the soliton existence range in parameter space from a single point to a whole interval [12]. The typical situation with 2D and 3D BEC solitons in OLs is therefore the following: in the parameter space the stable localized solutions are confined from above by the collapse curve and from below by the delocalizing transition curve, thus, in contrast with the one dimensional case where there are no limits for the existence of localized states, strict limitations for soliton existence appear in multidimensional cases. From this point of view it is clear that for possible experimental observation of multidimensional BEC solitons the parameter design becomes very important. Since the collapse curve is only marginally affected by the periodic potential, to enlarge existence ranges of solitons it is of interest to give a full characterization of the delocalizing curve in parameter space.

The aim of this paper is just devoted to this, i.e. we characterize 2D and 3D delocalizing curves of gap-solitons in terms of an unstable solution of the multi-dimensional Gross–Pitaevskii equation (GPE), which we call gap-Townes soliton. This solution can be viewed as a separatrix (it separates gap soliton states from extended (Bloch) states) and reduces to the known Townes soliton when the strength of the OL goes to zero. Similar solutions were found also for the 1D NLS equation with higher order nonlinearities in [13], where they were called gap-Townes solitons, and in [14] where they were termed Townes solitons. Conditions for the occurrence of the delocalizing transition phenomenon of one-dimensional localized modes of several nonlinear continuous periodic and discrete systems of the nonlinear Schrödinger type were also recently discussed in [15]. For the periodic multidimensional GPE the delocalizing curve has been characterized in [12] as the critical threshold for the existence of one bound state in an effective potential. The characterization given here, however, is more general since it is valid also for 1D NLS with higher order nonlinearities. To this regard we remark that in absence of confining potential the 2D and 3D GPE behaves similarly to the 1D NLS with quintic and septic nonlinearities, respectively. The interplay between dimensionality and nonlinearity has been used to investigate collapse in lower dimensional NLS on the basis of pure dimensional arguments. In particular, the critical condition for collapse has been characterized as \( D(n-1) - 4 = 0 \), where \( n \) is the order of the nonlinearity in the equation and \( D \) is the dimensionality of the system [16]. In the following we take advantage of this interplay to construct approximate gap-Townes soliton solutions of the GPE with multidimensional separable OLs, in terms of products of exact gap-Townes solutions of the 1D NLS with higher nonlinearities. Remarkably, we find