The phase diagram of approximation rates for deep neural networks

Dmitry Yarotsky
Skolkovo Institute of Science and Technology
d.yarotsky@skoltech.ru

Anton Zhevnerchuk
Skolkovo Institute of Science and Technology
Anton.Zhevnerchuk@skoltech.ru

Abstract

We explore the phase diagram of approximation rates for deep neural networks. The phase diagram describes theoretically optimal accuracy-complexity relations and their qualitative properties. Our contribution is three-fold. First, we generalize the existing result on the existence of deep discontinuous phase in ReLU networks to functional classes of arbitrary positive smoothness, and identify the boundary between the feasible and infeasible rates. Second, we demonstrate that standard fully-connected architectures of a fixed width independent of smoothness can adapt to smoothness and achieve almost optimal rates. Finally, we discuss how the phase diagram can change in the case of non-ReLU activation functions. In particular, we prove that using both sine and ReLU activations theoretically leads to very fast, nearly exponential approximation rates, thanks to the emerging capability of the network to implement efficient lookup operations.

1 Introduction

The topic of expressiveness of deep neural networks has received much attention in recent years. One of the fundamental questions in this area is the complexity of networks required to approximate classes of functions of given smoothness. Given a class $F$ of maps from the $d$-dimensional cube $[0,1]^d$ to $\mathbb{R}$, we want to identify network architectures of minimal complexity sufficient to approximate all $f \in F$ with given accuracy. In this paper we focus on the classical setting in which the sets $F$ are Sobolev- or Hölder balls, approximation is with respect to the uniform norm $\|\cdot\|_\infty$, and the complexity of the network is measured by the number of weights $W$. In this case we expect a power law relation between accuracy and complexity:

$$\|f - \tilde{f}_W\|_\infty = O(W^{-p}), \quad \forall f \in F,$$

(1)

where $\tilde{f}_W$ is an approximation of $f$ by a network with $W$ weights, and $p$ is an $F$-dependent constant that we will call an approximation rate.

Remarkably, this argument extends to non-linear weight assignment and reconstruction maps under the assumption that the weight assignment is continuous. More precisely, it was proved in [2] that, under this assumption, $p$ in Eq. (1) cannot be larger than $r/d$.

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An even more important set of ideas is related to estimates of Vapnik-Chervonenkis dimensions of deep neural networks. The concept of expressiveness in terms of VC-dimension (based on finite set shattering) is weaker than expressiveness in terms of uniform approximation, but upper bounds on the VC-dimension directly imply upper bounds on feasible approximation rates. In particular, VC-dimension of networks with piecewise-polynomial activations are $O(W^2)$ ([3]), which implies that $p$ cannot be larger than $\frac{2r}{d}$ – note the additional factor 2 coming from the power 2 in the VC bound. We refer to the book [4] for a detailed exposition of this and related results.

Returning to approximations with networks, the above arguments suggest that the rate $p$ in Eq.(1) can be up to $\frac{r}{d}$ assuming the continuity of the weight assignment, and up to $\frac{2r}{d}$ without assuming the continuity, but assuming a piecewise-polynomial activation function such as ReLU. We then face the constructive problem of showing that these rates can indeed be fulfilled by a network computation. One standard general strategy of proving the rate $p = \frac{r}{d}$ is based on polynomial approximations of $f$ (in particular, via the Taylor expansion). A survey of early results along this line for networks with a single hidden layer and suitable activation functions can be found in [5]. An interesting aspect of piecewise-linear approximations such as ReLU is that the rate $p = \frac{r}{d}$ cannot be achieved with shallow networks, but can be achieved with deeper networks implementing approximate multiplication and polynomials ([6],[7],[8],[9]).

It was shown in [10] that ReLU networks can also achieve rates $p$ beyond $\frac{r}{d}$. The result of [10] is stated in terms of the modulus of continuity of $f$; when restricted to Hölder functions with constant $r \leq 1$, it implies that on such functions ReLU networks can provide rates $p$ in the interval $(\frac{r}{d}, \frac{2r}{d})$, in agreement with the mentioned upper bound $\frac{2r}{d}$. The construction is quite different from the case $p = \frac{r}{d}$ and has a “coding theory” rather than “analytic” flavor. The central idea is to divide the domain $[0,1]^d$ into suitable patches and encode an approximation to $f$ in each patch by a single network weight using a binary-type representation. Then, the network computes the approximation $f(x)$ by finding the relevant weight and decoding it using the bit extraction technique of [11]. In agreement with continuous approximation theory and existing VC bounds, the construction inherently requires discontinuous weight assignment (as a consequence of coding finitely many values) and network depth (necessary for the bit extraction part). In this sense, at least in the case of $r \leq 1$ one can distinguish two qualitatively different “approximation phases”: the shallow continuous one corresponding to $p = \frac{r}{d}$ (and lower values), and the deep discontinuous one corresponding to $p \in (\frac{r}{d}, \frac{2r}{d})$. It was shown in [8] [12] that the shallow rate $p = \frac{r}{d}$, but not faster rates, can be achieved if the network weights are discretized with the precision of $O(\log(1/\epsilon))$ bits, where $\epsilon$ is the approximation accuracy.

**Contribution of this paper.** The developments described above leave many questions open. One immediate question is whether and how the deep discontinuous approximation phase generalizes to higher values of smoothness ($r > 1$). Another natural question is how much the network architectures providing the maximal rate $p = \frac{2r}{d}$ depend on the smoothness class. Yet another question is how sensitive the phase diagram is with respect to changing ReLU to other activation functions. In the present paper we resolve some of these questions. Our contribution is three-fold:

- In Section 3 we prove that the approximation phase diagram indeed generalizes to arbitrary $r > 0$, with the deep discontinuous phase occupying the region $\frac{r}{d} < p \leq \frac{2r}{d}$.

- In Section 4 we prove that the standard fully-connected architecture of a sufficiently large constant width $H$ only depending on the dimension $d$, say $H = 2d + 10$, can serve for implementing approximations that are asymptotically almost optimal (with rate $p = \frac{2r}{d}$), up to a logarithmic correction. This can be described as a phenomenon of “universal adaptivity to smoothness” exhibited by such architectures.

- In Section 5 we discuss how the ReLU phase diagram can change if ReLU is replaced or supplemented by other activation functions. In particular, we show that the ReLU-infeasible region is fully feasible if the network is allowed to include the sine function in addition to ReLU. Our key observation leading to this result is that the networks containing both ReLU and sin can implement lookup operations more efficient than the sequential lookup provided by the bit extraction technique of [11].
2 Preliminaries

Smooth functions. The paper revolves about what we informally describe as “functions of smoothness $r$”, for any $r > 0$. It is convenient to precisely define them as follows. If $r$ is integer, we consider the standard Sobolev space $W^{r, \infty}([0, 1]^d)$ with the norm
\[
\|f\|_{W^{r, \infty}([0, 1]^d)} = \max_{k: |k| \leq r} \sup_{x \in [0, 1]^d} |D^k f(x)|.
\]
Here $D^k f$ denotes the (weak) partial derivative of $f$. For an $f \in W^{r, \infty}([0, 1]^d)$, the derivatives $D^k f$ of order $|k| < r$ exist in the strong sense and are continuous. The derivatives $D^k f$ of order $|k| = r - 1$ are Lipschitz, and $\max_{k: |k| = r} \sup_{x \in [0, 1]^d} |D^k f(x)|$ can be upper- and lower-bounded in terms of the Lipschitz constants of these derivatives.

In the case of non-integer $r$, we consider Hölder spaces that provide a natural interpolation between the above Sobolev spaces. For any integer $k \geq 0$ and $0 < \alpha \leq 1$, we define the Hölder space $C^{k, \alpha}([0, 1]^d)$ as a subspace of $k$ times continuously differentiable functions having a finite norm
\[
\|f\|_{C^{k, \alpha}([0, 1]^d)} = \max \left\{ \|f\|_{W^{k, \infty}([0, 1]^d)}, \sup_{x, y \in [0, 1]^d, x \neq y} \frac{|D^k f(x) - D^k f(y)|}{\|x - y\|^\alpha} \right\}.
\]
At $\alpha = 1$ the norm in $C^{k,1}([0, 1]^d)$ is equivalent to the norm in $W^{k+1, \infty}([0, 1]^d)$. Given a non-integer $r$, we define “$r$-smooth functions” as those belonging to $C^{\lfloor r \rfloor, r - \lfloor r \rfloor}([0, 1]^d)$, where $\lfloor \cdot \rfloor$ is the floor function. We choose the sets $F$ appearing in Eq. (1) to be unit balls in the Sobolev spaces $W^{k, \infty}([0, 1]^d)$ for integer $r > 0$ or in the Hölder spaces $C^{\lfloor r \rfloor, r - \lfloor r \rfloor}([0, 1]^d)$ for non-integer $r$; we denote these balls by $F_{r,d}$.

Neural networks. We consider conventional feedforward neural networks in which each hidden unit performs a computation of the form $\sigma(\sum_{k=1}^R w_k z_k + h)$, where $z_k$ are input signals coming from some of the previous units, and $w_k$ and $h$ are the weights associated with this unit. In addition to input units and hidden units, the network is assumed to have a single output unit performing a computation similar to that of hidden units, but without the activation function. In Sections 3 and 4 we assume that the activation function is ReLU: $\sigma(x) = \max(0, x)$.

In the general results of Sections 3 and 5, we do not make any special connectivity assumptions about the architecture. On the other hand, in Section 4 we consider a particular family of architectures in which the hidden units are divided into a sequence of layers, with each layer having a constant number of units. Two units are connected if and only if they belong to neighboring layers. The input units are connected to the units of the first hidden layer and only to them; the output unit is connected to the units of the last hidden layer, and only to them. We refer to this as a standard deep fully-connected architecture of constant width.

Approximations. In the accuracy–complexity relation (1), we assume that approximations $\tilde{f}_W$ are obtained by assigning $f$-dependent weights to a network architecture $\eta_W$ common to all $f \in F$. In particular, this allows us to speak of the weight assignment map $G_W : f \mapsto w_f \in \mathbb{R}^W$ associated with a particular architecture $\eta_W$. We say that the weight assignment is continuous if this map is continuous with respect to the topology of uniform norm $\| \cdot \|_\infty$ on $F$. We will be interested in considering different approximation rates $p$, and we interpret Eq. (1) in a precise way by saying that a rate $p$ can be achieved if
\[
\inf_{\eta_W, G_W} \sup_{f \in F} \|f - \tilde{f}_{\eta_W, G_W}\|_\infty \leq c_{F, p} W^{-p},
\]
where $\tilde{f}_{\eta_W, G_W}$ denotes the approximation obtained by the weight assignment $G_W$ in the architecture $\eta_W$. Here and in the sequel we generally denote by $c_{a,b,\ldots}$ various positive constants possibly dependent on $a, b, \ldots$ (typically on smoothness $r$ and dimension $d$). Throughout the paper, we will treat $r$ and $d$ as fixed parameters in the asymptotic accuracy-complexity relations.

3 The phase diagram of ReLU networks

Our first main result is the phase diagram of approximation rates for ReLU networks, shown in Fig. 1.
\begin{align*}
f(x) &= x^2 \\
x \sim W^{pd/r-1}, \\
\text{Deep NN with} \quad L = c_{r,d} W^{pd/r-1}, \text{discontinuous weight assignment}
\end{align*}

Figure 1: The phase diagram of approximation rates for ReLU networks.

to $\frac{r}{d} < p \leq \frac{2r}{d}$, and the infeasible region corresponds to $p > \frac{2r}{d}$. Our main new contribution is the exact location of the deep discontinuous phase for all $r > 0$. The precise meaning of the diagram is explained by the following series of theorems (partly established in earlier works).

**Theorem 3.1** (The shallow continuous phase). The approximation rate $p = \frac{r}{d}$ in Eq. (2) can be achieved by ReLU networks having $L \leq c_{r,d} \log W$ layers, and with a continuous weights assignment. This result was proved in [6] in a slightly weaker form, for integer $r$ and with error $O(W^{-r/d} \log^{r/d} W)$ instead of $O(W^{-r/d})$. The proof is based on ReLU approximations of local Taylor expansions of $f$. The extension to non-integer $r$ is immediate thanks to our definition of general $r$-smoothness in terms of Hölder spaces. The logarithmic factor $\log^{r/d} W$ can be removed by observing that the computation of the approximate Taylor polynomial can be isolated from determining its coefficients and hence only needs to be implemented once in the network rather than for each local patch as in [6] (see Remark A.1; the idea of isolation of operations common to all patches is developed much further in the proof Theorem 3.3 below, and is applicable in the special case $p = \frac{r}{d}$).

**Theorem 3.2** (Feasibility of rates $p > \frac{r}{d}$).

1. Approximation rates $p > \frac{2r}{d}$ are infeasible for networks with piecewise-polynomial activation function and, in particular, ReLU networks;
2. Approximation rates $p \in \left(\frac{r}{d}, \frac{2r}{d}\right]$ cannot be achieved with continuous weights assignment;
3. If an approximation rate $p \in \left(\frac{r}{d}, \frac{2r}{d}\right]$ is achieved with ReLU networks, then the number of layers $L$ in $\eta_W$ must satisfy $L \geq c_{p,r,d} W^{pd/r-1} / \log W$ for some $c_{p,r,d} > 0$.

These statements follow from existing results on continuous nonlinear approximation ([2] for statement 2) and from upper bounds on VC-dimensions of neural networks ([3] for statement 1 and [13] for statement 3), see [6, Theorem 1] for a derivation. The extensions to arbitrary $r$ are straightforward.

The main new result in this section is the existence of approximations with $p \in \left(\frac{r}{d}, \frac{2r}{d}\right]$:

**Theorem 3.3** (The deep discontinuous phase). For any $r > 0$, any rate $p \in \left(\frac{r}{d}, \frac{2r}{d}\right]$ can be achieved with deep ReLU networks with $L \leq c_{r,d} W^{pd/r-1}$ layers.

This result was proved in [10] in the case $r \leq 1$. We generalize this to arbitrary $r$ by combining the coding-based approach of [10] with Taylor expansions. The technical details are given in Section A, but we explain now the main ideas.

**Sketch of proof.** We use two length scales for the approximation: the coarser one $\frac{1}{N}$ and the finer one $\frac{1}{M}$, with $M \gg N$. We start by partitioning the cube $[0,1]^d$ into $\sim N^d$ patches (particularly, simplexes)
of linear size \( \sim \frac{1}{N} \), and then sub-partitioning them into patches of linear size \( \sim \frac{1}{M} \). In each of the finer \( M \)-patches \( \Delta_M \), we approximate the function \( f \in F_{r,d} \) by a Taylor polynomial \( P_{\Delta_M} \) of degree \( \lceil r \rceil - 1 \). Then, from the standard Taylor remainder bound, we have \( |f(x) - P_{\Delta_M}(x)| = O(M^{-r}) \) on \( \Delta_M \). This shows that \( \epsilon \) is the required approximation accuracy, we should choose \( M \sim \epsilon^{-1/r} \).

Now, if we tried to simply save the Taylor coefficients for each \( M \)-patch in the weights of the network, we would need at least \( \sim M^d \), i.e. \( \sim \epsilon^{-d/r} \), weights in total. This corresponds to the classical rate \( p = \frac{d}{r} \). In order to save on the number of weights and achieve higher rates, we collect Taylor coefficients of all \( M \)-patches lying in one \( N \)-patch and encode them in a single encoding weight associated with this \( N \)-patch. Given \( p > \frac{d}{r} \), we choose \( N \sim \epsilon^{-1/(pd)} \), so that in total we create \( \sim \epsilon^{-1/p} \) encoding weights, each containing information about \( \sim (M/N)^d \), i.e. \( \sim \epsilon^{-(d/r-1)/p} \), Taylor coefficients. The number of encoding weights then matches the desired complexity \( W \sim \epsilon^{-1/p} \).

To encode the Taylor coefficients we actually need to discretize them first. Note that to reconstruct the Taylor approximation in an \( M \)-patch with accuracy \( \epsilon \), we need to know the Taylor coefficients of order \( k \) with precision \( \sim M^{-(r-k)} \). We implement an efficient sequential encoding/decoding procedure for the approximate Taylor coefficients of orders \( k < \lceil r \rceil \) for all \( M \)-patches lying in the given \( N \)-patch \( \Delta_N \). Specifically, choose some sequence \( (\Delta_M)_t \) of the \( M \)-patches in \( \Delta_N \) so that neighboring elements of the sequence correspond to neighboring patches. Then, the order-\( k \) Taylor coefficients at \( (\Delta_M)_t \) can be determined with precision \( \sim M^{-(r-k)} \) from the respective and higher order coefficients at \( (\Delta_M)_t \) using \( O(1) \) predefined discrete values. This allows us to encode all the approximate Taylor coefficients in all the \( M \)-patches of \( \Delta_N \) by a single \( O((M/N)^d) \)-bit number.

To reconstruct the approximate Taylor polynomial for a particular input \( x \in \Delta_M \subset \Delta_N \), we sequentially reconstruct all the coefficients for the sequence \( (\Delta_M)_t \), and, among them, select the coefficients at the patch \( (\Delta_M)_t(x) = \Delta_M \). The sequential reconstruction can be done by a deep subnetwork with the help of the bit extraction technique [11]. The depth of this subnetwork is proportional to the number of \( M \)-patches in \( \Delta_N \), i.e. \( \sim (M/N)^d \), which is \( \sim \epsilon^{-(d/r-1)/p} \) according to our definitions of \( N \) and \( M \). If \( p \leq \frac{2d}{r} \), then \( d/r - \frac{1}{p} \leq \frac{1}{p} \) and hence this depth is smaller or comparable to the number of encoding weights, \( \epsilon^{-1/p} \). However, if \( p > \frac{2d}{r} \), then the depth is asymptotically larger than the number of encoding weights, so the total number of weights is dominated by the depth of the decoding subnetwork, which is \( \sim \epsilon^{-d/(2r)} \), and the approximation becomes less efficient than at \( p = \frac{2d}{r} \). This explains why \( p = \frac{2d}{r} \) is the boundary of the feasible region.

Once the (approximate) Taylor coefficients at \( \Delta_M \ni x \) are determined, an approximate Taylor polynomial \( \tilde{P}_{\Delta_M}(x) \) can be computed by a ReLU subnetwork implementing efficient approximate multiplications [6]. □

4 Fixed-width networks and their universal adaptivity to smoothness

The network architectures constructed in the proof of Theorem 3.3 to provide the faster rates \( p \in (\frac{d}{r}, \frac{2d}{r}] \) are relatively complex and \( r \)-dependent. We can ask if such rates can be supported by some simple conventional architectures. It turns out that we can achieve nearly optimal rates with standard fully-connected architectures of sufficiently large constant widths only depending on \( d \):

**Theorem 4.1.** Let \( \eta_W \) be standard fully-connected ReLU architectures of width \( 2d + 10 \) with \( W \) weights. Then

\[
\inf_{G_W} \sup_{f \in F_{r,d}} \| f - \tilde{f}_{\eta_W,G_W} \|_\infty \leq c_{r,d} W^{-2r/d} \log^{2r/d} W. \tag{3}
\]

The rate in Eq. (3) differs from the optimal rate with \( p = \frac{2d}{r} \) only by the logarithmic factor \( \log^{2r/d} W \).

An interesting result proved in [14, 15] (see also [16] for a related result for ResNets) states that standard fully-connected ReLU architectures of a fixed width \( H \) can approximate any \( d \)-variate continuous function if and only if \( H \geq d + 1 \). Theorem 4.1 shows that with slightly larger widths, such networks can not only adapt to any function, but also adapt to its smoothness. The results of [14, 15] also show that Theorem 4.1 cannot hold with \( d \)-independent widths.
In the case $r \leq 1$, it was proved in [10] that standard networks of width $2d + 10$ allow to achieve the highest feasible rate $p = \frac{2}{r}$. Details of the proof of Theorem 4.1 are given in Section 5; we explain now the main idea.

**Sketch of proof.** The proof is similar to the proof of Theorem 3.3 but requires a different implementation of the reconstruction of $f(x)$ from encoded Taylor coefficients. The network constructed in Theorem 3.3 traverses $M$-knots of an $N$-patch and computes Taylor coefficients at the new $M$-knot by updating the coefficients at the previous $M$-knot. This computation can be arranged within a fixed-width network, but its width depends on $r$, since we need to store the coefficients from the previous step, and the number of these coefficients grows with $r$ (see [10] for the constant-width fully-connected implementation in the case of $r \leq 1$, in which the Taylor expansion degenerates into the 0-order approximation).

To implement the approximation using an $r$-independent network width, we can decode the Taylor coefficients afresh at each traversed $M$-knot, instead of updating them. This is slightly less efficient and leads to the additional logarithmic factor in Eq. (3), as can be seen in the following way. First, since we need to reconstruct the Taylor coefficients of degree $k$ with precision $O(M^{-(r-k)})$, we need to store $\sim \log M$ bits for each coefficient in the encoding weight. Since $M \sim \varepsilon^{-1/r}$, this means a $\sim \log(1/\varepsilon)$-fold increase in the depth of the decoding subnetwork. Moreover, an approximate Taylor polynomial must be computed separately for each $M$-patch. Multiplications can be implemented with accuracy $\varepsilon$ by a fixed-width ReLU network of depth $\sim (\log(1/\varepsilon))$ (see [6]). Computation of an approximate polynomial of the components of the input vector $x$ can be arranged as a chain of additions and multiplications in a network of constant width independent of the degree of the polynomial – assuming the coefficients of the polynomial are decoded from the encoding weight and supplied as they become required. This shows that we can achieve accuracy $\varepsilon$ with a network of constant width independent of $r$ at the cost of taking the larger depth $\sim \varepsilon^{-d/(2r)} \log(1/\varepsilon)$ (instead of simply $\sim \varepsilon^{-d/(2r)}$ as in Theorem 3.3). Since $W$ is proportional to the depth, we get $W \sim \varepsilon^{-d/(2r)} \log(1/\varepsilon)$. By inverting this relation, we obtain Eq. (3). □

## 5 Non-piecewise-polynomial activation functions

We discuss now how much the ReLU phase diagram depends on the activation function. We note first that it is well-known that some exotic activation functions allow to achieve much higher rates than those discussed in the previous sections. For example, a result of [17] based on the Kolmogorov Superposition Theorem ([11] p. 553) shows the existence of a strictly increasing analytic activation function $\sigma$ such that any $f \in C([0,1]^d)$ can be approximated with arbitrary accuracy by a three-layer $\sigma$-network with only $9d + 3$ units.

On the other hand, note that statement 1 of Theorem 3.3 holds not only for ReLU, but for any piecewise-polynomial activation functions, so that the region $p > \frac{2}{d}$ remains infeasible for any such activation. Also, since all piecewise-linear activation functions are essentially equivalent (see e.g. [6], Proposition 1), the phase diagram for any piecewise-linear activation is the same as for ReLU.

A remarkable class of functions that can be seen as a far-reaching generalization of polynomials are the Pfaffian functions [18]. Level sets of these functions admit bounds on the number of their connected components that are similar to analogous bounds for algebraic sets, and this is a key property in establishing upper bounds on VC dimensions of networks. In particular, it was proved in [19] that the VC-dimension of networks with the standard sigmoid activation function $\sigma(x) = 1/(1 + e^{-x})$ is upper-bounded by $O(W^2 k^2)$, where $k$ is the number of computation units (see also [4], Theorem 8.13]). Since $k \leq W$, the bound $O(W^2 k^2)$ implies the slightly weaker bound $O(W^4)$. Then, by mimicking the proof of statement 1 of Theorem 3.3 and replacing there the bound $O(W^2)$ for piecewise-polynomial activation by the bound $O(W^4)$ for the standard sigmoid activation, we find that the approximation rates $p > \frac{W^2}{d}$ are infeasible for networks with the standard sigmoid activation function. It appears that there remains a significant gap between the upper and lower VC dimension bounds for networks with $\sigma(x) = 1/(1 + e^{-x})$ (see a discussion in [4], Chapter 8)). Likewise, we do not know if the approximation rates up to $p = \frac{W^2}{d}$ are indeed feasible with this $\sigma$.

We note, at the same time, that the network expressiveness in terms of covering numbers can be upper bounded for any Lipschitz activation function if the network weights are bounded, see [4], Theorem 14.5]. Assuming moderately growing weights, this implies $p \leq \frac{2r}{d}$ (see Section 6).
Our main result in this section is the proof that the ReLU-infeasible sector \( p > \frac{2r}{d} \) becomes fully feasible if we allow some hidden units of the network to have the \( \sin \) activation function and make no restriction on the weights. Moreover, the approximation rate becomes exponential in a power of \( W \):

**Theorem 5.1.** Let \( \eta_W \) be feed-forward networks with \( W \) weights containing both ReLU and \( \sin \) activation functions. Then, for any \( r > 0 \) and \( d \in \mathbb{N} \),

\[
\inf_{\eta_W} \sup_{f \in F_{r,d}} \left\| f - \tilde{f}_{\eta_W, G_W} \right\|_{\infty} \leq \exp \left( -c_{r,d}W^{1/2} \right) \tag{4}
\]

with some \( r, d \)-dependent constant \( c_{r,d} > 0 \).

On the one hand, this result is not very surprising since \( \sin \) has level sets with infinitely many connected components. It is well-known, for example, that the family of classifiers \( \{ \text{sgn}(\sin(a \cdot)) \} : \mathbb{R} \to \{0, 1\} \}_{a \in \mathbb{R}} \), where \( \text{sgn}(x) = 1_{x>0} \), has an infinite VC-dimension. On the other hand, note that our network can be considered as a generalization of the Fourier series expansion \( f(x) \sim \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n \cdot x} \), which can be viewed as a neural network with one hidden layer, the \( \sin \) activation function, and predefined weights in the first layer. Standard convergence bounds for Fourier series (see e.g. [20]) correspond to the shallow continuous rate \( p = \frac{1}{d} \), in agreement with the fact that the conventional assignment of Fourier coefficients is linear in \( f \). Thus, adding depth and the ReLU activation to the Fourier expansion makes it substantially more expressive.

In any case, it is interesting to pinpoint the particular constructive mechanism that leads to the very fast approximation rates of Theorem [5.1]. Our proof is based on the observation that networks including both ReLU and \( \sin \) can implement an **efficient, dichotomy-based lookup**. We sketch the main idea of the proof; see details in Section [D].

**Sketch of proof.** Recall the concepts of coarser partition on the scale \( \frac{1}{2^d} \) and the finer partition on the scale \( \frac{1}{2^d} \) used in the proofs of Theorems [3.3] and [4.1]. In those theorems, both \( N \) and \( M \) were \( \sim W^d \) with some constant powers \( a \). In contrast, we choose now \( N = 1 \), and we’ll set \( M \) to grow much faster (roughly exponentially) with \( W \): this will be possible thanks to the much more efficient decoding available with the \( \sin \) activation.

Specifically, note first that we can implement an almost perfect approximation of the parity function \( \theta : x \mapsto (-1)^{\lfloor x \rfloor} \) using a constant size networks, by computing \( a \sin(\pi x) \) with a large \( a \) and then thresholding the result at 1 and -1 using ReLU operations (the approximation only fails in small neighborhoods of the integer points). If the cube \([0, 1]^d\) is partitioned into cubic \( M \)-patches, we can apply rescaled versions of \( \theta \) coordinate-wise to create a binary dictionary of these patches. Specifically, we can construct a network of size \( \sim d \log_2 M \) that maps a given \( x \in [0, 1]^d \) to a size-\( K \) binary sequence encoding the place of the patch \( \Delta_M \ni x \) in the cube \([0, 1]^d\), with \( K \sim d \log_2 M \). We call this network the **patch-encoder**.

Given a function \( f \in F_{r,d} \), we approximate it by a function \( \tilde{f} \) which is constant in each \( M \)-patch. Suppose for simplicity and without loss of generality that the smoothness \( r \leq 1 \), then this approximation has accuracy \( \epsilon \sim M^{-r} \). Let \( \tilde{f}_{\Delta_M} \) be the value that the approximation returns on the patch \( \Delta_M \). It is sufficient to define \( \tilde{f}_{\Delta_M} \) with precision \( \sim M^{-r} \). Consider the binary expansion of \( f_{\Delta_M} \) that provides this precision: \( \tilde{f}_{\Delta_M} = -1 + \sum_{k=0}^{R} \tilde{f}_{\Delta_M,k} 2^{-k} \), where \( R \sim r \log_2 M \) and \( \tilde{f}_{\Delta_M,k} \in \{0, 1\} \). Suppose that for each \( k \) we can construct a network that maps each patch \( \Delta_M \) to the corresponding bit \( \tilde{f}_{\Delta_M,k} \). Summing these **patch-classifiers** with coefficients \( 2^{-k} \), we then reconstruct the full approximation \( \tilde{f} \).

We have thus reduced the task to efficiently implementing an arbitrary binary classifier on the \( M \)-partition of \([0, 1]^d\). The patch-encoder constructed above efficiently encodes each \( M \)-patch by a binary \( K \)-bit sequence. We can then think of the classifier as an assignment \( A : \{0, 1\}^K \to \{0, 1\} \) that must be implemented by our network. We show below that this can be done by a size-\( O(K) \) network, with the assignment encoded in a single weight \( w_A \). The full number of network weights (including the patch-encoder and the patch-classifiers on all \( R \) scales) can then be bounded by \( W = O(K R) \), i.e. \( W = O(rd \log_2^2 M) \). The relations \( \epsilon \sim M^{-r} \) and \( W \sim rd \log_2^2 M \) then yield \( \epsilon \sim 2^{-c W^{1/2}} \) (with \( c' \sim \sqrt{r/d} \)), as claimed in Eq. [4].
To make these arguments fully rigorous, we need to handle the issue of our approximation to the parity function \( \theta \) becoming invalid near the boundaries of the patches. This is done in Section 10 using partitions of unity; the resulting complications do not affect the asymptotic.

We explain now how an arbitrary assignment \( A : \{0,1\}^K \rightarrow \{0,1\} \) can be implemented by a network of size \( O(K) \) with a single encoding weight \( w_A \). Let us define two sequences, \( a_k \) and \( l_k \):

\[
I_1 = 0.1, \quad a_1 = 10, \quad l_k = \frac{1}{a_k}, \quad a_k = \frac{4}{l_{k-1}}.
\]

Consider iterations \( g_1 \circ g_2 \circ \ldots \circ g_K(w) \), in which each \( g_k \) can be either the identity function \( g_k(w) = w \), or \( g_k(w) = \sin(a_kw) \), with some initial value \( w \). For each \( z \in \{0,1\}^K \), let us define \( H_K,w_s(z) \) as the \( \text{sgn} \) of the value obtained by substituting the respective functions:

\[
H_K,w_s(z) = \text{sgn} \circ \begin{cases} 
\text{Id}, & z_1 = 0, \\
\sin(a_1), & z_1 = 1
\end{cases} \circ \begin{cases} 
\text{Id}, & z_2 = 0, \\
\sin(a_2), & z_2 = 1
\end{cases} \circ \ldots \circ \begin{cases} 
\text{Id}, & z_K = 0, \\
\sin(a_K), & z_K = 1
\end{cases} (w)
\]

**Lemma 5.1.** For any assignment \( A : \{0,1\}^K \rightarrow \{0,1\} \) there exists \( w_A \in \mathbb{R} \) such that \( H_K,w_A(z) = A(z) \) for all \( z \in \{0,1\}^K \).

**Proof.** Proof by induction on \( K \), but of a slightly sharper statement: the desired \( w_A \) not only exist, but still (at least) an interval \( I_K \subset [-1,1] \) of length \( l_K \).

The base \( K = 1 \) is immediate. Suppose we have proved the statement for \( K - 1 \). Given an assignment \( A : \{0,1\}^K \rightarrow \{0,1\} \), consider it as a pair of assignments \( A_0 : \{0,1\}^{K-1} \rightarrow \{0,1\}, A_1 : \{0,1\}^{K-1} \rightarrow \{0,1\} \). By the hypothesis, we can find two intervals \( I_{K-1}^{(0)} \) and \( I_{K-1}^{(1)} \) of length \( l_{K-1} \) such that \( H_{K-1,w_0}(z) = A_0(z) \) and \( H_{K-1,w_1}(z) = A_1(z) \) for all \( w_0 \in I_{K-1}^{(0)}, w_1 \in I_{K-1}^{(1)} \) and \( z \in \{0,1\}^{K-1} \). Consider the set

\[
I = \{ w \in \mathbb{R} : w \in I_{K-1}^{(0)} \text{ and } \sin(a_Kw) \in I_{K-1}^{(1)} \}.
\]

Then for any \( w \in I \), we have the desired property \( H_{K,w}(z) = A(z), \forall z \in \{0,1\}^K \). We need to show now that \( I \) contains an interval of length \( l_K \). This follows from Eq. (5) since \( |\frac{d}{dw} \sin(a_Kw)| \leq a_K \) and since \( \sin(a_Kw) \) has the period \( \frac{2\pi}{a_K} \) twice as small as the length \( l_{K-1} \) of \( I_{K-1}^{(1)} \).

This lemma shows that the network can implement any classifier \( A \) if the network can somehow branch into applying either \( \text{Id} \) or \( \sin(a_k \cdot) \) depending on the signal bit \( b \in \{0,1\} \) that is output by the patch-encoder subnetwork. This branching can be easily implemented by forming the linear combination \( (1 - b)x + b \sin(a_k x) \), and also noting that a product of any \( x \in \{0,1\} \) and \( y \in \{0,1\} \) admits the ReLU implementation \( xy = \max(0, y + x - 1) \).

We remark that the construction in Lemma 5.1 can be interpreted as an efficient lookup if we think of the assignment \( A \) as a binary sequence of size \( S = 2^K \). In each of the network steps we divide the sequence in half, ultimately locating the desired bit in \( K \sim \log_2 S \) steps. We can compare this with the less efficient bit extraction procedure of [14] (for which it is however sufficient to only have the ReLU activation in the network). In this latter procedure, the bits are extracted from the encoding weight one-by-one, and so the lookup requires \( \sim S \) steps.

Our results highlight a tradeoff between complexity of the architecture and complexity of network weights: optimization of the number of weights forces the weights to represent information in intricate ways. While we have not treated the topic of weight precision in this paper (cf. [21, 8, 12]), the arguments of Section 5 show that approximation with accuracy \( \epsilon \) requires the encoding weights to contain \( \sim (M/N)^d \), i.e., \( \sim \epsilon^{-d/r-1/p} \) bits. For ReLU/\sin networks of Section 5 the required precision of the encoding weight \( w_A \) can be estimated from the lengths \( l_k \) of the intervals \( I_k \) considered in Lemma 5.1. Using Eq. (5), we find that \( \log l_k \sim -2K \). Since we used \( K \sim \frac{d}{r} \log_2 (1/\epsilon) \), this means that \( w_A \) should contain about \( \sim \epsilon^{-d/r} \) bits. These estimates agree with the observation that the information required to specify a function \( f \in F_{r,d} \) with accuracy \( \epsilon \) is \( \sim \epsilon^{-d/r} \) bits [22], and this information is uniformly distributed over the encoding weights of the network (\( \sim \epsilon^{-1/p} \) weights in ReLU networks or \( R = O(\log(1/\epsilon)) \) weights in ReLU/\sin networks).
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A Theorem 3.3: proof details

We follow the paper [10] where Theorem 3.3 was proved for $r \leq 1$, and generalize it to arbitrary $r > 0$ using the strategy explained in Section 3. Given $p \in \left( \frac{1}{2}, \frac{2}{r} \right)$ we show that it is possible to construct a network architecture with $W$ weights and $L = O(W^{pd/r-1})$ layers which approximates every $f \in F_{r,d}$ with error $O(W^{-p})$. In Remark A.1 we deal with the case $p = \frac{2}{r}$.

We start by describing the space partition and related constructions. Then we give an overview of the network structure. Finally, we describe in more detail the network computation of the Taylor approximations, which is the main novel element of Theorem 3.3.

A.1 Space partitions

For an integer $N \geq 1$ we denote by $\mathcal{P}_N$ a standard triangulation of $\mathbb{R}^d$ into simplexes:

$$\Delta_{N,n,\rho} = \left\{ x \in \mathbb{R}^d : 0 \leq x_{\rho(1)} - \frac{n_{\rho(1)}}{N} \leq \cdots \leq x_{\rho(d)} - \frac{n_{\rho(d)}}{N} \right\},$$

where $n \in \mathbb{Z}^d$ and $\rho$ is a permutation of $d$ elements. The vertices of these simplices are the points of the grid $(\mathbb{Z}/N)^d$. We call the set of all the vertices the $N$-grid and a particular vertex an $N$-knot. For an $N$-knot we call the union of simplexes it belongs to an $N$-patch. We denote a set of all $N$-knots $K_N$.

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be the “spike” function defined as the continuous piecewise linear function such that:

1. $\phi$ is linear on every simplex from the triangulation $\mathcal{P}_1$;
2. $\phi(0) = 1$, $\phi(n) = 0$ for all other $n \in \mathbb{Z}^d$.

The function $\phi(x)$ can be computed by a feed-forward ReLU network with $O(d^2)$ weights (see [10], Section 4.2) for details). We treat $d$ as a constant, so we can say that $\phi(x)$ can be computed by a network with a constant number of weights. Note that for integer $N$ and $n \in \mathbb{Z}^d \cap [0,N]^d$, the function $\phi(Nx - n)$ is a continuous piecewise linear function which is linear in each simplex from $\mathcal{P}_N$, is equal to 1 at $x = \frac{n}{N}$, and vanishes at all other $N$-knots of $(\mathbb{Z}/N)^d$.

It is convenient to keep in mind two following simple propositions:

**Proposition A.1.** Suppose we have $K$ $N$-knots $\frac{n_1}{N}, \ldots, \frac{n_K}{N}$, $n_i \in \mathbb{Z}^d$ and corresponding numbers $\ell_1, \ldots, \ell_K$. Then the function

$$g(x) = \sum_{k=1}^{K} \ell_k \phi(Nx - n_k)$$

has the following properties:

1. $g(x)$ is linear on each simplex from $\mathcal{P}_N$;
2. $g\left(\frac{n_k}{N}\right) = \ell_k$ for $k = 1, \ldots, N$. For other $N$-knots $\frac{n_k}{N}$, $h$ is zero: $h\left(\frac{n_k}{N}\right) = 0$;
3. $g(x)$ can be computed exactly by a network with $O(K)$ weights and $O(1)$ layers.

**Proposition A.2.** Suppose we have $K$ $N$-knots $\frac{n_1}{N}, \ldots, \frac{n_K}{N}$, $n_i \in \mathbb{Z}^d$ and corresponding numbers $s_1, \ldots, s_K$. Suppose also that $N$-patches associated with $\frac{n_1}{N}, \ldots, \frac{n_K}{N}$ are disjoint. Then there exists function $h(x)$ with the following properties:

1. $h(x)$ is linear on each simplex from $\mathcal{P}_N$;
2. For $k = 1, \ldots, N$, $h(x) = s_k$ at an $N$-patch associated with $\frac{n_k}{N}$;
3. $h(x)$ can be computed exactly by a network with $O(K)$ weights and $O(1)$ layers.

**Proof.** Follows directly from Prop. A.1. We assign value $s_k$ to all $N$-knots in $N$-patch associated with $\frac{n_k}{N}$ and apply Prop. A.1. Since $N$-patches of interest are disjoint, each $N$-knot has at most one assigned value. \hfill \Box
A.2 The filtering subgrids

Given the total number of weights \( W \), we set \( N = W^{1/d} \). We will assume without loss of generality that \( N \) is integer. We consider triangulation \( \mathcal{P}_N \) of \([0,1]^d\) on length scale \( \frac{1}{N} \).

It is convenient to split the \( N \)-grid into \( 3^d \) disjoint subgrids with the \( 3 \times 3 \) grid spacing:

\[
N_q = \{ \frac{n}{N} : n \in (q + (3Z)^d) \cap [0,N]^d \}, \quad q \in \{0,1,2\}^d.
\]

Clearly, each subgrid contains \( O(N^d) \) knots. Note that \( N \)-patches associated with \( N \)-knots in \( N_q \) are disjoint. It means, in particular, that any point \( x \in [0,1]^d \) lies in at most one such \( N \)-patch. It also means that Prop. A.2 is applicable to \( N_q \). We will use this observation in subsection A.3 for constructing an efficient approximation in a neighbourhood of \( N_q \) for a single \( q \). We call the union of these \( N \)-patches a domain of \( N_q \).

We compute the full approximation \( \tilde{f} \) as a sum

\[
\tilde{f}(x) = \sum_{q \in \{0,1,2\}^d} \tilde{w}_q(x) \tilde{f}_q(x). \tag{6}
\]

Function \( \tilde{f}_q(x) \) computes \( f(x) \) with error \( O(W^{-p}) \) for every \( x \) in the domain of \( N_q \). For \( x \) out of the domain of \( N_q \) it computes some garbage value. We describe \( \tilde{f}_q(x) \) in subsection A.3. The final approximation \( f(x) \) is a weighted sum of \( \tilde{f}_q(x) \) with weights \( \tilde{w}_q(x) \). We choose such functions \( \tilde{w}_q(x) \), that \( \tilde{w}_q(x) \) vanishes outside the domain of \( N_q \) and

\[
\sum_{q \in \{0,1,2\}^d} \tilde{w}_q(x) \equiv 1.
\]

It follows that \( \tilde{f}(x) \) is a weighted sum (with weights with the sum 1) of terms approximating \( f(x) \) with error \( O(W^{-p}) \). Consequently, \( \tilde{f}(x) \) approximates \( f(x) \) with error \( O(W^{-p}) \).

Function \( \tilde{w}_q(x) \) is given by applying Prop. A.1 to \( N \)-knots from \( N_q \) with all values \( \ell_1, \ell_2, \ldots, \ell_{|N_q|} \) equals to 1. Clearly, \( \tilde{w}_q(x) \) vanishes outside the domain of \( N_q \). Sum \( \sum_{q \in \{0,1,2\}^d} \tilde{w}_q(x) \) is linear on each simplex from \( \mathcal{P}_N \) and equals to 1 at all \( N \)-knots, because each \( N \)-knot belongs to exactly one set \( N_q \). Consequently, this sum equals to 1 for every \( x \in [0,1]^d \). It follows from Prop. A.1 that network implementing \( \tilde{w}_q(x) \) has \( O(N^d) = O(W) \) weights and \( O(1) \) layers.

Multiplication \( \tilde{w}_q(x) \tilde{f}_q(x) \) is implemented approximately, with error \( O(W^{-p}) \), by network given by \cite{6} Proposition 3 and requires \( O(\log W) \) additional weights.

A.3 The approximation for a subgrid

Here we describe how we construct \( \tilde{f}_q(x) \) for a single \( q \in \{0,1,2\}^d \). Remind that \( \tilde{f}_q(x) \) computes accurate approximation for \( f(x) \) only on the domain of \( N_q \).

For any \( N \)-knot \( \frac{n}{N} \) in \( N_q \) we consider a cube with center at \( \frac{n}{N} \) and edge \( \frac{2}{N} \):

\[
\left\{ x \in \mathbb{R}^d : \max_{1 \leq i \leq d} |x_i - \frac{n_i}{N}| \leq \frac{1}{N} \right\}.
\]

We call such cube an \emph{N-cube} and denote it by \( C_n \). Note that \( C_n = \frac{n}{N} + C_0 \).

Remind that the domain of \( N_q \) consists of \( |N_q| \) disjoint \( N \)-patches associated with \( N \)-knots from \( N_q \). Each \( x \) from the domain of \( N_q \) belongs to exactly one such \( N \)-patch. We call this patch an \emph{N-patch for} \( x \) and associated \( N \)-knot an \emph{N-knot for} \( x \). Let us denote an \( N \)-knot for \( x \) by \( \frac{n_q(x)}{N} \).

We set \( M = W^{p/r} \). Note that \( M^{-r} = W^{-p} \) and, therefore, we need to construct an approximation of error \( O(M^{-r}) \). We will assume without loss of generality that \( M \) is integer and \( M \) is divisible by \( N \). Then \( \mathcal{P}_M \) is a subpartition of \( \mathcal{P}_N \). We define \( M \)-knot and \( M \)-patch similarly to \( N \)-knot and \( N \)-patch. We denote a set of all \( M \)-knots by \( K_M \). Note that there are \( O((M/N)^d) \) \( M \)-knots in each \( N \)-patch and \( N \)-cube. See Fig. 2 for an illustration of all described constructions.
Suppose that \( x \) lies in an \( M \)-patch associated with an \( M \)-knot \( \frac{m}{M} \). Consider a Taylor polynomial \( P_{m/M}(x) \) at \( \frac{m}{M} \) of order \( \lceil r \rceil - 1 \). Standard bounds for the remainder of Taylor polynomial imply that it approximates \( f(x) \) with error \( O(M^{-r}) \) uniformly for \( f \in F_{r,d} \). Taylor polynomial at \( \frac{m}{M} \) (and actually any polynomial) can be implemented with error \( O(M^{-r}) \) by a network with \( O(\log M) \) weights and layers. We refer reader to [6, Proposition 3] and a proof of [6, Theorem 1] for details.

We can approximate \( f(x) \) with error \( O(M^{-r}) \) with a weighted sum of Taylor polynomials \( P_{m/M}(x) \) at all \( M \)-knots:

\[
\tilde{f}(x) = \sum_{\frac{m}{M} \in K_M} \phi \left( \frac{M}{x} - m \right) P_{m/M}(x). \tag{7}
\]

Note that \( \phi \left( \frac{M}{x} - m \right) \) vanishes outside an \( M \)-patch associated with \( \frac{m}{M} \) and

\[
\sum_{\frac{m}{M} \in K_M} \phi \left( \frac{M}{x} - m \right) = 1.
\]

There are \( M^d \) terms in (7) and calculating single term requires \( O(\log M) \) weights. So, the total number of weights needed to implement (7) is \( O(M^d \log M) = O(W^{pd/r} \log W) \). It is clearly infeasible for \( p > \frac{r}{d} \). For \( p = \frac{r}{d} \) it leads to approximation error \( O(W^{-r/d} \log^{r/d} W) \) and makes a statement of [6, Theorem 1]. Note that in this construction Taylor coefficients at \( M \)-knots are the weights of network.

Note that terms of (7) are nonzero only for \( M \)-knots in an \( N \)-cube for \( x \). Suppose that \( x \) lies in the domain of \( N_q \), and, therefore, has well defined \( N \)-knot \( \frac{n_q(x)}{N} \). For such \( x \) we can write

\[
\tilde{f}_q(x) = \sum_{\frac{m}{M} \in K_M \cap C_{n_q(x)}} \phi \left( \frac{M}{x} - m \right) P_{m/M}(x)
\]

\[
= \sum_{\frac{m}{M} \in K_M \cap C_0} \phi \left( M \left( x - \frac{n_q(x)}{N} \right) - m \right) P_{m/M+n_q(x)/N}(x). \tag{8}
\]
There are only $(M/N)^d = W^{pd}/r^{d-1}$ terms in $f(x)$. Therefore, if we know $\frac{n_q(x)}{N}$ and Taylor coefficients for $\frac{m}{M} + \frac{n_q(x)}{N}$, then $f_q(x)$ can be implemented with error $O(M^{-r})$ by a network with $O((M/N)^d \log M) = O(W^{pd}/r^{d-1} \log W)$ weights.

For $x$ in the domain of $N_q$, it holds that $\tilde{f}_q(x) = \tilde{f}(x)$. It follows that $\tilde{f}_q(x)$ indeed approximates $f(x)$ with error $O(M^{-r}) = O(W^{-p})$ on the domain of $N_q$.

If $x$ lies in the domain of $N_q$, then we can compute a single coordinate of $\frac{n_q(x)}{N}$ with a network given by Proposition $\textit{A.2}$. We need to take $\frac{n_q}{N} \in N_q$ and set $s_i$ to be a corresponding coordinate of $n$. We compute $\frac{n_q(x)}{N}$ by applying this observation to all coordinates. Constructed network has $O([|N_q|] = O(N^d) = O(W)$ weights and $O(1)$ layers.

In subsection $\textit{A.4}$ we show, that (approximated) Taylor coefficients for $(M/N)^d M$-knots $\frac{m}{M} + \frac{n_q(x)}{N}$, $\frac{m}{M} \in K_M \cap C_0$ can be computed by a network with $O((M/N)^d)$ weights and layers from $e_{r,d} \leq 2(d + 1)^{-1}$ $n$-dependent values. We call this values encoding weights for $n$.

In subsection $\textit{A.4}$ we describe how we construct encoding weights for a particular function $f$ and an $N$-knot $\frac{m}{M}$. We show that using approximated Taylor coefficients computed from encoding weights instead of real ones leads to error bounded by $O(M^{-r}) = O(W^{-p})$. For $x$ in the domain of $N_q$ we can calculate encoding weights for $n_q(x)$ by a network given by Proposition $\textit{A.2}$.

Let us finalize a structure of network computing $\tilde{f}_q(x)$. For $x$ in the domain of $N_q$ it

1. Computes $n_q(x)$ and encoding weights for $n_q(x)$. This step is implemented by applying Proposition $\textit{A.2}$ and requires $O(N^d) = O(W)$ weights and $O(1)$ layers;

2. Given encoding weights for $n_q(x)$, computes (approximated) Taylor coefficients for all $M$-knots $\frac{m}{M} + \frac{n_q(x)}{N}$, $\frac{m}{M} \in K_M \cap C_0$. This step requires a network with $O((M/N)^d) = O(W^{pd}/r^{d-1})$ weights and layers and described in subsection $\textit{A.4}$.

3. Given (approximated) Taylor coefficients achieved at the previous step, computes an approximation for $P_{m/M+n_q(x)/N}$ for all $M$-knots $\frac{m}{M} + \frac{n_q(x)}{N}$, $\frac{m}{M} \in K_M \cap C_0$. The approximation with error $O(M^{-r}) = O(W^{-p})$ for a single $P_{m/M+n_q(x)/N}$ can be implemented by a network with $O(\log M) = O(\log W)$ weights and layers. Total number of weights needed at this step is, therefore, $O(|K_M \cap C_0| \log M) = O((M/N)^d \log M) = O(W^{pd}/r^{d-1} \log W)$. Computation for different $M$-knots can be done in parallel, so the total number of layers is still $O(\log W)$;

4. Given $n_q(x)$, computed at first step, computes $\phi \left( \frac{M}{x - \frac{n_q(x)}{N}} - m \right)$ for all $M$-knots $\frac{m}{M} + \frac{n_q(x)}{N}$, $\frac{m}{M} \in K_M \cap C_0$. It requires $O(|K_M \cap C_0|) = O((M/N)^d) = O(W^{pd}/r^{d-1})$ weights and $O(1)$ layers;

5. Combines outputs of steps 3 and 4 in the final approximation with $f(x)$. Multiplication with accuracy $O(M^{-r})$ can be implemented by a network with $O(\log M)$ weights and layers, so this step requires $O(|K_M \cap C_0| \log M) = O((M/N)^d \log M) = O(W^{pd}/r^{d-1} \log W)$ weights and $O(\log M) = O(\log W)$ layers.

Clearly we can pass forward values achieved at early steps without increasing an asymptotic for needed number of weights and layers.

If we sum up the total number of weights needed at each step, we obtain $O(W + W^{pd}/r^{d-1} \log W)$.

For $\frac{5}{2} < p < \frac{3d}{d-1}$ it is equivalent to $O(W)$ and matches the desired approximation rate. For $p = \frac{3d}{2d-1}$ it is equivalent to $O(W \log W)$ and leads to the desired approximation rate up to a logarithmic factor.

We show how to deal with it in subsection $\textit{A.5}$.

The total number of needed layers is $O(W^{pd}/r^{d-1})$ and matches the desired.
A.4 Encoding and decoding Taylor coefficients

It is known that $\sim \epsilon^{-d/r}$ bits are needed to specify a function $f \in F_{r,d}$ with accuracy $\epsilon$ [22]. It follows from the bounds for Kolmogorov $\epsilon$-entropy of $F_{r,d}$ derived in [22, § 4]. Here we describe how this specification can be implemented by a neural network.

First we introduce some notation. Suppose we have an $M$-knot $\frac{m}{M}$. Taylor expansion $P_{m/M}(x)$ of $f(x)$ at $\frac{m}{M}$ is given by

$$P_{m/M}(x) = \sum_{k:|k|\leq |r|-1} \frac{D^k f \left( \frac{m}{M} \right)}{k!} \left( x - \frac{m}{M} \right)^k.$$ 

We use usual convention $k! = \prod_{i=1}^d k_i$ and $(x - \frac{m}{M})^k = \prod_{i=1}^d (x_i - \frac{m_i}{M})^{k_i}$. We denote

$$a_{m,k} = D^k f \left( \frac{m}{M} \right).$$

We denote an approximated Taylor coefficients to be defined further in this section by $\hat{a}_{m,k}$. Corresponding approximated Taylor expansion is given by

$$\hat{P}_{m/M}(x) = \sum_{k:|k|\leq |r|-1} \frac{\hat{a}_{m,k}}{k!} \left( x - \frac{m}{M} \right)^k.$$ 

For any $x$ in the $M$-patch associated with $\frac{m}{M}$

$$|f(x) - P_{m/M}(x)| \leq c_{r,d}M^{-r},$$

for all $f \in F_{r,d}$ and some constant $c_{r,d}$, which does not depend on $M$ and $m$.

We first show how we construct encoding weights associated with an $N$-knot $\frac{m}{M}$. Our construction is quite similar to one from the proof of [22] Theorem XIV, where bounds for Kolmogorov $\epsilon$-entropy of $F_{r,d}$ were derived. Then we discuss how approximated Taylor coefficients at $M$-knots in the $N$-cube $C_n$ are computed from encoding weights by a network.

Our goal is to construct such approximated Taylor coefficients $\hat{a}_{m,k}$, that for any $x$ in the $M$-patch associated with $\frac{m}{M}$ holds $|\hat{P}_{m/M}(x) - P_{m/M}(x)| \leq c_{r,d}M^{-r}$ for some $M$-independent constant $c_{r,d}$. The following proposition states sufficient condition on such $\hat{a}_{m,k}$.

**Proposition A.3.** Suppose that

$$|a_{m,k} - \hat{a}_{m,k}| \leq \frac{M^{k^r}}{r!} \quad \forall k: |k| \leq |r| - 1. \quad (9)$$

Then for any $x$ in an $M$-patch associated with $\frac{m}{M}$

$$|\hat{P}_{m/M}(x) - P_{m/M}(x)| \leq (d+1)^{|r|-1}M^{-r}.$$ 

**Proof.**

$$|\hat{P}_{m/M}(x) - P_{m/M}(x)| \leq \sum_{k:|k|\leq |r|-1} \frac{1}{k!} |\hat{a}_{m,k} - a_{m,k}| \left| \left( x - \frac{m}{M} \right)^k \right|$$

$$\leq \sum_{k:|k|\leq |r|-1} M^{|k|^r} M^{-|k|}$$

$$\leq (d+1)^{|r|-1}M^{-r}.$$ 

Suppose that two $M$-knots $\frac{m_1}{M}$ and $\frac{m_2}{M}$ are adjacent and we have $\hat{a}_{m_1,k}, |k| \leq |r| - 1$ satisfying (9). Another convenient proposition we use further shows how to construct an accurate approximation for Taylor coefficients at $\frac{m_2}{M}$. 

14
Proposition A.4. Suppose that two $M$-knots $\frac{m_1}{M}$ and $\frac{m_2}{M}$ are adjacent. Suppose that approximated Taylor coefficients $\tilde{a}_{m_1,k}$ for $|k| \leq |r| - 1$ at $\frac{m_1}{M}$ satisfy (9). Then we can find such $c_{k,k}$ and $\tilde{a}_{m_2,k}$ for $|k|, |\tilde{k}| \leq |r| - 1$, that

1. For all $k : |k| \leq |r| - 1$
   \[ \tilde{a}_{m_2,k} = \sum_{\tilde{k} : |\tilde{k}| \leq |r| - 1} c_{k,\tilde{k}} \cdot a_{m_1,\tilde{k}} ; \]

2. For all $k : |k| \leq |r| - 1$
   \[ |a_{m_2,k} - \tilde{a}_{m_2,k}| < 4M^{\lceil |k| - r \rceil} ; \]

3. Coefficients $c_{k,k}$ depend only on the relative position of $\frac{m_1}{M}$ and $\frac{m_2}{M}$.

Proof. Remind that $M$-knots $\frac{m_1}{M}$ and $\frac{m_2}{M}$ are adjacent. Let us consider first component of $m_1$ and $m_2$ independently and assume without loss of generality that $m_1 = (m_1, \bar{m})$ and $m_2 = (m_1 + 1, \bar{m})$.

Standard bounds for a remainder of Taylor series partial sum imply, that for any $k = (k_1, \ldots, k_d)$ and $f \in F_{r,d}$

\[ \left| D^{(k_1, \ldots, k_d)} f \left( \frac{M_2}{M} \right) - \sum_{n=0}^{[r] - 1 - |k|} \frac{D^{(k_1 + n, \ldots, k_d)} f \left( \frac{m_1}{M} \right)}{n!} \cdot \frac{1}{M^n} \right| \leq M^{\lceil |k| - r \rceil} . \]

In our notation

\[ \left| a_{m_2,(k_1, \ldots, k_d)} - \sum_{n=0}^{[r] - 1 - |k|} \frac{a_{m_1,(k_1 + n, \ldots, k_d)}}{n!} \cdot \frac{1}{M^n} \right| \leq M^{\lceil |k| - r \rceil} . \] (11)

From the proposition that coefficients $\tilde{a}_{m_1,k}$ satisfy (9) it follows that

\[ \left| \sum_{n=0}^{[r] - 1 - |k|} \frac{(a_{m_1,(k_1 + n, \ldots, k_d)} - \tilde{a}_{m_1,(k_1 + n, \ldots, k_d)})}{n!} \cdot \frac{1}{M^n} \right| \leq \sum_{n=0}^{[r] - 1 - |k|} \frac{M^{\lceil |k| + n - r \rceil}}{n!} \cdot \frac{1}{M^n} \]

\[ = M^{\lceil |k| - r \rceil} \sum_{n=0}^{[r] - 1 - |k|} \frac{1}{n!} \]

\[ < eM^{\lceil |k| - r \rceil} < 3M^{\lceil |k| - r \rceil} . \] (12)

Combining (11) and (12) we obtain

\[ \left| a_{m_2,(k_1, \ldots, k_d)} - \sum_{n=0}^{[r] - 1 - |k|} \frac{a_{m_1,(k_1 + n, \ldots, k_d)}}{n!} \cdot \frac{1}{M^n} \right| < 4M^{\lceil |k| - r \rceil} . \]

It follows that if for each $k = (k_1, \ldots, k_d)$ we set

\[ \tilde{a}_{m_2,(k_1, \ldots, k_d)} = \sum_{n=0}^{[r] - 1 - |k|} \frac{a_{m_1,(k_1 + n, \ldots, k_d)}}{n!} \cdot \frac{1}{M^n} , \]

then $\tilde{a}_{m_2,k}$ satisfy (10). It remains to note that coefficients in (13) depend only on the relative position of $\frac{m_1}{M}$ and $\frac{m_2}{M}$, but not on $f \in F_{r,d}$, values $\tilde{a}_{m_1,k}$ or $M$-knots $\frac{m_1}{M}$ and $\frac{m_2}{M}$ themselves. \( \square \)

Now we are ready to describe how we find $\tilde{a}_{m,k}$ for all $M$-knots $\frac{m}{M}$ from a given $N$-cube $C_n$. We enumerate $M$-knots lying in $C_n$ with numbers $i = 1, \ldots, (2M/N + 1)^d$ and denote them $\frac{m_n}{M}$. We inductively construct $\tilde{a}_{m_{n+1},k}$ satisfying (9) for all $M$-knots $\frac{m_{n+1}}{M}$. We choose such an enumeration, that two consequent $M$-knots are adjacent.
We set \( \hat{a}_{m_n,k} \) first compute \( \hat{\tilde{a}} \) the total number of weights and layers is \( O \). Then coefficients \( \hat{a}_{m_n,k} \) we first apply Prop. A.4 to get \( \tilde{a}_{m_n,k} \) and \( \hat{\tilde{a}}_{m_n,k} \) satisfying (9). Given \( \hat{a}_{m_n,k} \), we first apply Prop. A.4 to get \( \hat{a}_{m_{n+1},k} \) satisfying (10). This step is illustrated by the brown dashed arrow and brown cross is \( \hat{a}_{m_{n+1},k} \). Then we choose such \( B_{n,k,t} \in \{-3\ldots,3\} \), that \( \hat{a}_{m_{n+1},k} = \hat{a}_{m_{n+1},k} \) and \( M|k|\)\( B_{n,k,t} \) satisfy (9).

We set \( \hat{a}_{m_n,k} = a_{m_n,k} \). Such \( \hat{a}_{m_n,k} \) clearly satisfy (9). Suppose that we have constructed \( \hat{a}_{m_n,k} \) satisfying (9). Since \( M \)-knots \( \frac{m_n}{M} \) and \( \frac{m_{n+1}}{M} \) are adjacent, we can apply Prop. A.4 to get \( \hat{a}_{m_{n+1},k}, |k| \leq |r| - 1 \) satisfying (10). It follows that there exist such integers \( B_{n,k,t} \), that \( |B_{n,k,t}| \leq 3 \) and

\[
\left| \hat{a}_{m_{n+1},k} - \hat{a}_{m_{n+1},k} - M|k|\)\( B_{n,k,t} \right| \leq M|k| - r.
\]

We set

\[
\hat{a}_{m_{n+1},k} = \hat{a}_{m_{n+1},k} + M|k|\)\( B_{n,k,t} \).
\]

Then coefficients \( \hat{a}_{m_{n+1},k} \) satisfy (9) as desired. See Fig. 3 for an illustration of algorithm of determining \( \hat{a}_{m_{n+1},k} \).

For a single \( k \) we encode \( (2M/N + 1)^d \) values \( B_{n,k,t} \) by a single base-7 number \( b_{n,k} \)

\[
b_{n,k} = \sum_{t=1}^{(2M/N+1)^d} 7^{-t} (B_{n,k,t} + 3)
\]

Numbers \( b_{n,k} \) and \( \hat{a}_{m_n,k} = a_{m_n,k} \) are encoding weights for \( n \). There are \( c_r,d \leq 2(d+1)^{|r|} - 1 \) encoding weights.

Now we describe how a network reconstruct all \( \hat{a}_{m_n,k} \) from encoding weights. Numbers \( B_{n,k,t} \) can be reconstructed from \( b_{n,k} \) by a ReLU network with \( O((M/N)^d) \) weights and layers. We refer to [10 5.2.2], where similar reconstruction is described for ternary numbers. Given \( \hat{a}_{m_n,k} \) and \( B_{n,k,t} \), we first compute \( \hat{a}_{m_{n+1},k} \) with (13) and then we compute \( \hat{a}_{m_{n+1},k} \) with (14). We need \( O(1) \) weights and layers at each step, so the total number of needed weights and layers is \( O((M/N)^d) \).

For given \( x \in [0,1]^d \) and \( q \in \{0, 1, 2\}^d \) we obtain encoding weights for \( n_q(x) \) by applying Prop. A.2. Note that Prop. A.4 implies that coefficients in (13) depend only on the relative position of \( M \)-knots.
We split all weights \( \hat{\varphi}_{m_{n,k}} \) to compute such \( \tilde{f}_q(x) \) and can exchange information. We reserve \( d \) channels for passing forward the scalar components of \( m_{n,k} \). Following \[10\], we think of the width-\( N,M \) increase in the size of the network. Accordingly, we define parameters \( N, M \) in terms of the required accuracy \( \epsilon \) rather than the number of weights: specifically, we set \( M = \epsilon^{-1/r} \) and \( N = \epsilon^{-1/(2r)} \).

We split all \( M \)-knots lying in \( N \)-cube \( C_0 \) into a disjoint union of \( 3^d \) sets

\[
M_s = \left\{ \frac{m}{M} : m \in (s + (3Z)^d) \cap C_0, \quad s \in \{0,1,2\}^d \right\}.
\]

\( M \)-patches associated with \( M \)-knots in \( M_s \) are disjoint. We call their union the domain of \( M_s \). If \( x - \frac{n_q(x)}{N} \) lies in the domain of \( M_s \), there is exactly one such \( \frac{n_q(x)}{M} \), that \( x - \frac{n_q(x)}{N} \) lies in the \( M \)-patch associated with \( \frac{m}{M} \). We can rewrite \( \tilde{f}_q(x) \) as

\[
\tilde{f}_q(x) = \sum_{s \in \{0,1,2\}^d} \left[ \tilde{f}_{q,s}(x) \sum_{m \in \frac{n_q(x)}{M} + M_s} \phi \left( M \left( x - \frac{n_q(x)}{N} \right) - m \right) \right]. \tag{15}
\]

Here \( \tilde{f}_{q,s}(x) \) is a function, which calculates \( P_{\frac{n_q(x)}{M} + n_q/N}(x) \) if \( x - \frac{n_q(x)}{N} \) lies in the domain of \( M_s \), and some garbage value otherwise. We also require that \( \tilde{f}_{q,s}(x) \) computes an approximation for a Taylor series partial sum only once. The total number of partial sums computed by network implementing \( \tilde{f}_q(x) \) in form \( \tilde{f}_q(x) \) is therefore reduced to \( 3^d \). The total number of weights needed to implement \( \tilde{f}_q(x) \) reduces from \( O(W \log W) \) to \( O(W) \).

To compute such \( \tilde{f}_{q,s}(x) \) we only need to determine approximated Taylor coefficients for \( \frac{n_q(x)}{M} + \frac{n_q(x)}{N} \) among all coefficients. For each \( \frac{m}{M} \in M_s \) we construct function \( \tilde{\varphi}_{s,m}(x) \), which equals to 1 in the \( M \)-patch associated with \( \frac{m}{M} \) and vanishes in other patches of the domain of \( M_s \). Knowing values \( \tilde{\varphi}_{s,m}(x - \frac{n_q(x)}{N}) \) we clearly can get Taylor coefficients for \( \frac{n_q(x)}{M} + \frac{n_q(x)}{N} \) from all Taylor coefficients computed by network.

**Remark A.1.** Similar reasoning can be applied to the case \( p = \frac{r}{2} \). In this case we do not consider an \( M \)-grid at all, but we still can split \( N \)-grid into \( 3^d \) disjoint sets and compute approximated Taylor sum once for each set. In this case weight assignment map is continuous and even linear on \( f \).

**B. Theorem 4.1: proof details**

We follow the network construction used in the proof of Theorem 3.3 and described in Subsections A.2, A.3. We want to show that this construction can be realized within a ReLU network of width \( 2d + 10 \). As explained in Section 4, we slightly modify the construction, so that we don’t update the Taylor coefficients at new \( M \)-patches, but rather compute them afresh. This will give a slight increase in the size of the network. Accordingly, we define parameters \( N, M \) in terms of the required accuracy \( \epsilon \) rather than the number of weights: specifically, we set \( M = \epsilon^{-1/r} \) and \( N = \epsilon^{-1/(2r)} \).

Following \[10\], we think of the width-\( (2d+10) \) network as \( 2d+10 \) “channels” that are interconnected and can exchange information. We reserve \( d \) channels for passing forward the scalar components of...
We return now to the computation of \( \widetilde{f}(x) \). The other channels are used for intermediate computations.

The first step in computing the approximation of \( \widetilde{f}(x) \) is the finite decomposition of \( \overline{f} \) over \( q \)-subgrids. The decomposition can be implemented in the width-\((2d+10)\) network in the serial fashion, so we only need to consider computation of a single term \( \overline{w}_q(x) \overline{f}_q(x) \).

The weight \( \overline{w}_q(x) \) is just a linear combination of \( O(N^d) \) functions \( \phi(Nx - n) \), and \( \phi \) can be computed by a constant-size chain of linear and ReLU operations (see [10] Section 4.2). Thus, \( \overline{w}_q(x) \) can be computed by a subnetwork using just 2 channels and depth \( O(\epsilon^{-d/2r}) \). On the other hand, we will show below that \( \overline{f}_q(x) \) can be computed by a subnetwork using \( d+8 \) channels and depth \( O(\epsilon^{-d/(2r)} \log(1/\epsilon)) \). We can then pass the values \( \overline{w}_q(x) \) and \( \overline{f}_q(x) \) to the third subnetwork computing an \( O(\epsilon) \)-approximation to the product \( \overline{w}_q(x) \overline{f}_q(x) \). This approximate product can be computed by a width-4 subnetwork of depth \( O(\log(1/\epsilon)) \) (see [6] Proposition 3). Thus the total computation of the term \( \overline{w}_q(x) \overline{f}_q(x) \), and hence of the whole approximation \( \widetilde{f}(x) \) can be done with necessary accuracy \( \epsilon \) within the width-\((2d+10)\) network of depth \( L = O(\epsilon^{-d/(2r)} \log(1/\epsilon)) \). By inverting this relation, we get \( \epsilon = O(L^{-2d/\log(2d/L)}) \), as desired.

We return now to the computation of \( \overline{f}_q(x) \). It is based on the expansion [8] and can be performed as described later in that subsection. We examine now individual steps and how they can be implemented in our fixed-depth network.

1. The \( N \)-knot positions \( n_q(x) \) associated with \( x \) are computed using a linear combination of \( O((M/N)^d) \) functions of the form \( \phi(Nx - n_k) \). This computation can be performed in a subnetwork of width 2 and depth \( O(\epsilon^{-d/2r}) \). We reserve \( d \) channels to pass forward the scalar components of \( n_q(x) \). Additionally, we reserve one channel for passing forward the encoding weight corresponding to this \( n_q(x) \). The encoding weight gets transformed as it passes along the network and bits get decoded from it. Additional 3 channels are sufficient for bit decoding (see [10] for a description of the decoding procedure).

2. We traverse the \( O((M/N)^d) \) \( M \)-knots of the \( N \)-patch corresponding to \( n_q \) and decode from the encoding weight the Taylor coefficients of degree up to \( |r| - 1 \) at these knots. It is sufficient to know these coefficients with precision \( O(\epsilon^r) \), so each Taylor coefficient can be encoded by \( K_{\max} = O(\log(1/\epsilon)) \) bits \( \{b_k\}_{k=0}^{K_{\max}} \), and reconstructed by accumulating the linear combination \( \sum_{k=0}^{K_{\max}} 2^{-k} b_k \). Thus, the total required number of bits in the encoding weight is \( O(\epsilon^{-d/(2r)} \log(1/\epsilon)) \). Also, all the necessary coefficients can be reconstructed using \( O(\epsilon^{-d/(2r)} \log(1/\epsilon)) \) layers of width 4.

3. At each \( M \)-knot \( m/M + n_q(x)/N \) in the \( N \)-patch, we compute the respective Taylor polynomial \( P_{m/M+n_q(x)/N}(x) = \sum_{k:|k| \leq |r|-1} a_k (x-(m/M+n_q(x)/N))^k \). The values of \( x \) and \( n_q(x) \) are provided from the reserved channels, and \( m \) is defined in the network weights. We don’t need to know all the coefficients at once, since the polynomial can be computed serially, one monomial at a time, and one multiplication after another. To ensure accuracy \( \epsilon \), each multiplication requires depth \( O(\log(1/\epsilon)) \) and width 4. The total polynomial can then be accumulated using a subnetwork of depth \( O(\log(1/\epsilon)) \) and width 5.

4. Computation of the values \( \phi(M(x - \frac{n_q(x)}{N}) - m) \) can be performed in 2 channels using \( O(\epsilon^{-d/(2r)}) \) layers in total.

5. Once the factors are computed, each product \( \phi(M(x - \frac{n_q(x)}{N}) - m) P_{m/M+n_q(x)/N}(x) \) can be computed with accuracy \( O(\epsilon) \) in a subnetwork of width 4 and with \( O(\log(1/\epsilon)) \) layers, which gives \( O(\epsilon^{-d/(2r)} \log(1/\epsilon)) \) layers in total.

Summarizing, we see that the computation of \( \overline{f}_q(x) \) can be implemented with accuracy \( O(\epsilon) \) in a subnetwork occupying \( d+8 \) channels and spanning \( O(\epsilon^{-d/(2r)} \log(1/\epsilon)) \) layers, as claimed.
C Expressiveness of networks with Lipschitz activation functions and slowly growing weights

In this section we clarify why, as mentioned in Section 5, under mild assumptions on the growth of network weights, networks with any bounded Lipschitz activation function (in particular, the standard sigmoid \( \sigma(x) = 1/(1 + e^{-x}) \)) can only achieve the approximation rates \( p \leq \frac{2c}{d} \). This follows from existing upper bounds on the covering numbers for such networks, in particular [4, Theorem 14.5].

Specifically, suppose that \( \sigma \) is an activation function such that \( |\sigma(x)| \leq b \) and \( |\sigma(x) - \sigma(y)| \leq a \) for all \( x, y \in \mathbb{R} \). Suppose that there is a constant \( V > 1/a \) such that for any weight vector \( w \) associated with a particular neuron, its \( l^1 \)-norm \( ||w||_1 \) is bounded by \( V \). Assume that the network has \( L \geq 2 \) layers, with connections only between adjacent layers, and has \( W \) weights. Let \( F \) denote the family of functions on \([0,1]^d \) implementable by feedforward networks under these constraints.

For any finite subset \( S \subset [0,1]^d \) consider the restriction \( F|_S \) as a subset of \( \mathbb{R}^{|S|} \) equipped with the uniform norm \( \| \cdot \|_\infty \). We define the covering number \( N_\infty(\epsilon, F, S) \) as the smallest number of \( \epsilon \)-balls in \( \mathbb{R}^{|S|} \) covering the set \( F|_S \). Then, for any integer \( m \geq 0 \), we define the covering number \( N_\infty(\epsilon, F, m) = \max_{S \subset [0,1]^d, |S|=m} N_\infty(\epsilon, F, S) \). We then have the following bound.

**Theorem C.1** (Theorem 14.5 of [4]). \( N_\infty(\epsilon, F, m) \leq \left( \frac{4ecbhW(aV)^L}{\epsilon(aV-1)} \right)^W \).

To obtain the desired bound on approximaton rates for Hölder balls \( F_{r,d} \), we can now lower-bound \( N_\infty(\epsilon, F, m) \) using the \( \epsilon \)-capacity of Hölder balls. Specifically, observe that the Hölder ball \( F_{r,d} \) contains a set \( \Phi_\epsilon \) of at least \( M_\epsilon = \frac{2c\epsilon d}{r} \) functions separated by \( \| \cdot \|_\infty \)-distance \( 4\epsilon \) with some constant \( c_{r,d} > 0 \). These functions can be constructed by a standard argument in which we choose \( r \) elements of \( \Phi_\epsilon \), and then place a properly rescaled spike function with the sign + or – at each point of the grid. The functions of \( \Phi_\epsilon \) are mutually \( 4\epsilon \)-separated when restricted to the grid \( S_r \). If our family \( F \) of network-implementable functions can \( \epsilon \)-approximate any function from the balls \( F_{r,d} \), then any \( \epsilon \)-net for \( F|_S \) is a \( 2\epsilon \)-net for \( \Phi_\epsilon|_S \), and thus must contain at least \( M_\epsilon \) elements. Hence, \( M_\epsilon \leq N_\infty(\epsilon, F_1, S_r) \leq N_\infty(\epsilon, F, c_{r,d}(\epsilon e)^{-d/r}) \), i.e.

\[
C_{r,d}(\epsilon e)^{-d/r} \leq W \log_2 \left( \frac{4ec_{r,d}(\epsilon e)^{-d/r}bW(aV)^L}{\epsilon(aV-1)} \right).
\]

Assuming that \( 1/\epsilon, W, L, V \) grow while the other parameters are held constant, this bound implies that

\[
\epsilon \geq c_{r,d,a,b}(WL)^{-r/d} \ln^{-r/d} V
\]

with some \( c_{r,d,a,b} > 0 \).

Now suppose that \( V \) is a function of \( W \), i.e. the magnitude of the weights is allowed to depend on the network size. Suppose that the network achieves the approximation rate \( p \), i.e.

\[
\epsilon \leq C_{r,d,a,b}W^{-p}.
\]

Since \( L \leq W \), comparing Eq. (16) with Eq. (17), we then find that

\[
\ln V \geq c'_{r,d,a,b} W^{pd/r} - 2.
\]

Thus, the rates \( p > \frac{2c}{d} \) require \( V \) to very rapidly grow with \( W \). This observation agrees with the main result of Section 5 – Theorem 5.1 – describing approximation with arbitrary rates \( p \) by networks with the sigmoid function. In the proof of this theorem, the network weights are defined with the help of rapidly growing constants \( a_K \) given in Eq. (5). In particular, we have \( \log a_K \sim 2K \) with \( K \sim W^{1/2} \), which agrees with the lower bound (18).

D Theorem 5.1: proof details

Examining the sketch of proof given in Section 5 we see that the only significant gap in the given argument is the treatment of boundaries of the patches. Namely, recall that we use approximations to the parity function \( \vartheta(x) = (-1)^{[x]} \). The approximations can be defined by a finite expression in terms of linear, ReLU and \( \sin \) operations:

\[
\tilde{\vartheta}_a(x) = \min(1, \max(-1, a \sin(\pi x))).
\]
By taking \(a\) large, we can make \(\tilde{\theta}_a\) to equal \(\theta\) outside some small neighborhood of \(\mathbb{Z}\). Now, recall that we choose patches \(\Delta_M\) as cubes \([\frac{m_1}{M}, \frac{m_1+1}{M}] \times [\frac{m_2}{M}, \frac{m_2+1}{M}] \times \cdots \times [\frac{m_d}{M}, \frac{m_d+1}{M}]\). Assume without loss of generality that \(M = 2^U\) with some integer \(U\). The patch-encoding functions \(g_{u,k} : \mathbb{R} \rightarrow \mathbb{R}\) map the cubes \(\Delta_M\) to the values \(\pm 1\) everywhere except near the boundaries of these cubes. If we could slightly “shrink” the cubes \(\Delta_M\) so that they were disjoint, we could adjust \(a\) in \(\tilde{\theta}_a\) so that the functions \(g_{u,k}\) were perfectly equal to \(\pm 1\) on the whole cubes. The remaining construction of patch-classifying networks in Section 5 then becomes fully functional and yields the desired asymptotic relation (4).

Thus, we need to show how to reduce the problem to the case of disjoint patches. This can be done by using suitable filtering functions, similarly to the proofs of Theorems 3.3 and 4.1. Fix some \(a_0 > 1\) and consider the functions \(\Psi_0, \Psi_1 : \mathbb{R} \rightarrow [0, 1]\) defined by

\[
\Psi_0(x) = \frac{1}{2}(1 + \tilde{\theta}_a(2Mx)), \quad \Psi_1 = 1 - \Psi_0.
\]

The functions \(\Psi_0\) and \(\Psi_1\) form a two-element partition of unity. Furthermore, since \(a_0 > 1\), there is \(\delta > 0\) such that

\[
\Psi_0(x) = 0 \text{ for } x \in \left(\frac{3}{4M} - \delta, \frac{3}{4M} + \delta\right) + \mathbb{Z}/M, \quad (19)
\]

\[
\Psi_1(x) = 0 \text{ for } x \in \left(\frac{1}{4M} - \delta, \frac{1}{4M} + \delta\right) + \mathbb{Z}/M. \quad (20)
\]

Taking the product of the partitions of unity over the \(d\) coordinates, we can write for \(\tilde{f} : [0, 1]^d \rightarrow \mathbb{R}\):

\[
\tilde{f} = \sum_{q \in \{0, 1\}^d} \left(\prod_{s=1}^d \Psi_{q_s}\right) \tilde{f}.
\]

Thanks to Eqs. (19), (20), for each \(q \in \{0, 1\}^d\), the filtering function \(\prod_{s=1}^d \Psi_{q_s}\) vanishes in \([0, 1]^d\) outside an \(\frac{1}{2M}\)-grid of disjoint cubic patches, exactly as desired. We can then look for the approximation \(\tilde{f}\) in the form

\[
\tilde{f} = \sum_{q \in \{0, 1\}^d} \left(\prod_{s=1}^d \Psi_{q_s}\right) \tilde{f}_q,
\]

where \(\tilde{f}_q\) has the required values only on the patches \([0, 1]^d\setminus\text{supp}(\prod_{s=1}^d \Psi_{q_s})\) and can be constructed as described in Section 5.

Having implemented these approximations \(\tilde{f}_q\), the final approximation is obtained by implementing approximate products with the filters \(\Psi_{q_s}\) and performing summation over \(q \in \{0, 1\}^d\). As shown in [6, Proposition 3], multiplication with accuracy \(\epsilon\) requires a ReLU subnetwork with \(O(\log(1/\epsilon))\) connections. This is asymptotically negligible compared to our bound for the total complexity of the patch-classifiers (which is \(O(\log^2(1/\epsilon))\)).