Abstract: Reduze is a computer program for reducing Feynman integrals to master integrals employing a variant of Laporta’s reduction algorithm. This article describes version 2 of the program. New features include the distributed reduction of single topologies on multiple processor cores. The parallel reduction of different topologies is supported via a modular, load balancing job system. Fast graph and matroid based algorithms allow for the identification of equivalent topologies and integrals.
1 Introduction

In perturbative quantum field theory, the traditional method to compute cross sections and distributions for a physical process involves generating tree and loop amplitudes via Feynman diagrams and interfering them. Simplifications of the expressions are performed at the analytical level. Here an essential part is the reduction of the typically dimensionally regularized loop integrals to a small number of standard integrals. This step can be performed at the amplitude level for tensor integrals or, after contraction of Lorentz indices, at the level of interferences for scalar integrals. Considering the case of scalar integrals,
integration by parts (IBP) identities \cite{2,3} and Lorentz invariance (LI) identities \cite{4} may be used for a systematic reduction to a set of independent integrals, called master integrals. The standard reduction algorithm by Laporta \cite{5} defines an ordering for Feynman integrals, generates identities and solves the resulting system of linear equations. Alternative methods to exploit IBP and LI identities for reductions have been proposed \cite{6–9}, see also \cite{10,11} and references therein. Public implementations of different reduction algorithms are available with the computer programs AIR \cite{12}, FIRE \cite{13} and the first version of Reduze \cite{14}.

This article presents the new public reduction program Reduze 2. It is written in C++ and represents a major rewrite and extension of its predecessor Reduze. In the following, the name Reduze refers to the new version presented here.

In Reduze, integrals are indexed by integral families (“auxiliary topologies”) and sectors (“topologies”) therein. For the reduction, the program implements a fully distributed variant of Laporta’s algorithm using the Message Passing Interface (MPI). In this way, not only different sectors can be reduced in parallel, but also the integrals of a single sector can be reduced in a distributed computation.

The program allows to utilize multiple integral families within a calculation. Special emphasis has been placed on finding relations between sectors of the same or different integral families and employing them to eliminate integrals. Besides a straightforward combinatorial matcher, the program implements graph and matroid theory based algorithms to compute such relations, taking into account possible crossings of external momenta. Similar to the program DIANA \cite{15}, Reduze may be used to shift loop momenta of Feynman diagrams generated by a program like QGRAF \cite{16} or FeynArts \cite{17} to match sectors of integral families.

Other features include the generation of differential equations for Feynman integrals and the computation of (bare) amplitude interferences up to master integrals, starting from Feynman diagrams generated by QGRAF. For storing intermediate results of a reduction, optionally, the transitional open source database Berkeley DB \cite{18} can be used.

For the normalization of algebraic coefficients in the identities, one can choose between GiNaC \cite{19} and Fermat \cite{20}. Reduction identities and other results can be exported to FORM \cite{21}, Mathematica \cite{22}, and Maple \cite{23} format. Configuration and job files use the YAML format \cite{24} and are parsed with the yaml-cpp parser \cite{25}.

Reduze 2 was used to calculate the two-loop leading color corrections to heavy-quark pair production in the gluon fusion channel \cite{26}. Last but not least, Reduze is published as open source under the GNU General Public License (GPL) v3 and has no mandatory dependencies on proprietary software.

2 Basic concepts and notations

2.1 Integral families, sectors, and integrals

A propagator $P$ is defined as the expression $1/(q^2 - a)$ where $q$ is a four-momentum and $a$ is constant. The momentum $q$ of a propagator (defined up to a minus sign) is a linear combination of loop momenta $k_i$ and external momenta $p_i$, and $q^2$ is a scalar product in Minkowski space with the metric convention $g = \text{diag}(1,-1,-1,-1)$. In Reduze, also
generalized propagators \(1/(ql - m^2)\) with the scalar product of two different momenta \(q\) and \(l\) are available to support more general irreducible numerators.

An \(l\)-loop integral family (or “auxiliary topology”) \(F\) is an ordered set \(\{P_1, \ldots, P_n\}\) of propagators \(P_i, i = 1, \ldots, n\), which is minimal and complete in the sense, that any scalar product of a loop momentum \(k_i\) with a loop momentum \(k_j\) or an external momentum \(p_j\) can be uniquely expressed as a linear combination of inverse propagators and kinematic invariants. Denoting the number of independent external momenta by \(m\), an integral family must contain exactly \(l(l + 1)/2 + lm\) propagators, where the first term counts the scalar products between loop momenta only and the second term the products involving both loop and external momenta. A new feature of \texttt{Reduze}\ is its ability to handle several integral families simultaneously.

A selection of \(t\) propagators of an integral family defines a sector of this family. Assuming a sector has the propagators \(P_{j_1}, \ldots, P_{j_t}\) with \(\{j_1, \ldots, j_t\} \subset \{1, \ldots, n\}\), then its identification number is defined as

\[
ID = \sum_{k=1}^{t} 2^{j_k - 1}.
\]  

(2.1)

There are in general \(\binom{n}{t}\) different \(t\)-propagator sectors and \(\sum_{t=0}^{n} \binom{n}{t} = 2^n\) sectors are contained in an integral family. Their identification numbers fulfill \(0 \leq ID \leq 2^n - 1\).

A sector whose propagators form a subset of the propagators of another sector of the same integral family is a subsector of the other sector.

The purpose of an integral family is to index scalar loop integrals. To every \(t\)-propagator sector with propagators \(P_{j_1}, \ldots, P_{j_t}\) belongs a infinite set of \(d\)-dimensionally regularized \(l\)-loop integrals \([\ldots]\) which all share the same propagators. These integrals have the generic form

\[
I = \int d^d k_1 \ldots \int d^d k_l P_{j_1}^{r_1} \ldots P_{j_t}^{r_t} P_{j_{t+1}}^{-s_1} \ldots P_{j_n}^{-s_{n-t}}
\]  

(2.2)

with integer exponents \(r_i \geq 1\) and \(s_i \geq 0\). In \texttt{Reduze} such an integral is represented by

\[
I(F, t, ID, r, s, \{v_1, \ldots, v_n\})
\]  

(2.3)

where \(F\) denotes the integral family, \(r = \sum_{i=1}^{t} r_i \geq t\), \(s = \sum_{i=1}^{n-t} s_i \geq 0\) and \(v_i\) is the exponent of propagator \(P_i\). Positive \(v_i\) denote powers of regular propagators (non-trivial denominator), negative \(v_i\) denote powers of inverse propagators (non-trivial numerator), and zero means absence of a propagator. The numbers \(t, r, s\) as well as the identification number \(ID\) of the sector, to which the integral belongs, can be calculated from the vector \(v\).

Consider a \(t\)-propagator sector of a \(n\)-propagator integral family. The number of integrals that one can build for certain values of \(r\) and \(s\) is given by \(N(n, t, r, s) = \binom{r-1}{t-1} \binom{s + n - t - 1}{n - t - 1}\). The two binomial factors count all possible ways to arrange the exponents of the propagators in the denominator and numerator, respectively.

The integral with \(r = t\) and \(s = 0\) of some sector is called corner integral of this sector.
2.2 Integration by parts (IBP) identities

In dimensional regularization [1] the integral over a total derivative is zero. Let $I'$ be the integrand of an integral of the form (2.2). Then, working out the differentiation in

$$ \int d^d k_i \frac{\partial}{\partial k_i^\mu} [q^\mu I'(p_1, \ldots, p_m, k_1, \ldots, k_l)] = 0 $$

(2.4)

leads to the integration by parts (IBP) identities [2, 3]. The momentum $q$ is an arbitrary loop or external momentum. The index $\mu$ is summed over but the index $i$ is not. If there are $l$ loop momenta and $m$ independent external momenta one can therefore build $l(l+m)$ equations from one integral, the seed integral.

2.3 Lorentz invariance (LI) identities

One can also use the Lorentz invariance of the integrals [4]. Taking an integral $I(p_1, \ldots, p_m)$ the following equation holds

$$ \sum_{n=1}^m \left( p_{n}^{\nu} \frac{\partial}{\partial p_{n\mu}} - p_{n}^{\mu} \frac{\partial}{\partial p_{n\nu}} \right) I(p_1, \ldots, p_m) = 0. $$

(2.5)

The derivatives can be shifted directly to the integrand of the integral $I$. This equation can be contracted with all possible antisymmetric combinations of the external momenta, e.g. $p_{1\mu}p_{2\nu} - p_{1\nu}p_{2\mu}$, which leads to $m(m-1)/2$ equations where $m$ denotes the number of independent external momenta. As it was shown in [10] the LIs do not give new linearly independent equations in addition to the IBPs. However, they can accelerate the convergence in a reduction, since in general an LI identity generated from one seed integral cannot be reproduced with the IBP identities generated from the same seed integral alone. Reduze offers the possibility to use the LIs.

2.4 Zero sectors

It is possible that a whole sector is zero which means that all integrals belonging to this sector are zero. A sector of an $l$-loop integral family is trivially zero if it does not allow for a selection of $l$ propagator momenta which are independent with respect to the $l$ loop momenta. The graph based methods in Reduze, see section 3.1, will automatically detect these cases. As a second method, a sector is set to zero if the reduced IBP identities generated from the seed integrals of this sector with $r = t$ and $s = 0, 1$ show that its corner integral is zero.

2.5 Sector relations

Given a scalar loop integral as well as one or several integral families, suppose it is possible to map the integral to a linear combination of indexed integrals of type (2.3). In general, such a map is not unique. Ambiguities may arise if sectors from different integral families have the same set of propagators or if a transformation of loop and external momenta in
(2.2) leads to a different linear combination of type (2.3). For the corner integral of some sector $S$ written in the form (2.2), consider the transformation of integration variables

$$k_i \rightarrow \sum_{j=1}^{l} M_{ij} k_j + \sum_{j=1}^{m} N_{ij} p_j$$

(2.6)

with $|\det M| = 1$. If the new integrand factors can be identified with propagators of a sector $S'$, the “shift” transformation (2.6) defines a sector relation between $S$ and $S'$. In this case, any integral in the sector $S$ can be expressed as a linear combination of integrals in the sector $S'$ and subsectors of $S'$. If $S$ and $S'$ are different, such a relation can be used to eliminate one of the two sectors completely. The case where $S$ and $S'$ are identical is discussed in section 2.6.

Reduce is able to automatically detect sector relations or handle relations supplied by the user. Sector relations will be used to eliminate redundant sectors or integrals, usually at the earliest possible stage. As a particularly useful special case, a shift (2.6) might map each propagator of an integral family to another propagator of the same integral family. This leads to a one–to–one mapping between integrals as well as sectors of the integral family. Such a relation can be entered for the full integral family via permutations of propagators and allows for particularly efficient removal of redundancies.

As a generalization of the above described concepts, Reduce also allows for crossings of external momenta in addition to (2.6). If the involved crossing leaves the kinematic invariants unchanged, the corresponding relations can be directly exploited for relations between sectors as described before. In the general case, Reduce also handles relations between sectors of integral families defined with crossed kinematics.

### 2.6 Sector symmetries

Special shifts of the loop momenta as in (2.6) which transform a sector to itself are called sector symmetries. These shifts are also allowed to contain a permutation of the external momenta as long as it does not change the kinematic invariants. Sector symmetries may be used to express integrals in terms of other integrals in the same sector and its subsectors. These relations may provide information complementary to the IBP and LI identities and can be used in the reduction to find a minimal number of master integrals. Reduce is capable of automatically determining sector symmetries or handling user supplied rules, and offers to exploit them for reductions.

### 3 Graph and matroid based algorithms for sectors

#### 3.1 Physical sectors

A physical sector is a sector whose propagators correspond to edges in a graph such that momentum is conserved. The construction of a graph from a sector, i.e. from the momenta of a set of propagators with $l$-loop momenta, can be done by choosing $l$ propagators which have independent momenta with respect to the loop momenta and identifying them as edges in a graph with both ends glued together in a single root vertex. External edges
are also attached to this root vertex with one of their ends. Subsequently, for each of
the remaining propagators, a vertex of the graph (first the root vertex) is cleaved into two
vertices, and a new edge is inserted between these vertices, such that the edge’s momentum
(determined by momentum conservation) exactly matches the propagator’s momentum.

With this procedure \texttt{Reduze} automatically constructs graphs for sectors where this is
possible and thus identifies physical sectors. The possibility of having graph representations
for sectors gives access to fast algorithms for identifying isomorphic graphs and finding
sector relations and sector symmetries.

3.2 Sector relations

If graphs constructed for two different physical sectors are isomorphic, a shift of the form
(2.6) between the two sets of loop momenta can be derived by identifying the edges of
the two graphs together with their oriented momentum flow labeling. \texttt{Reduze} offers the
possibility to find relations between physical sectors by this strategy, allowing also crossings
of external legs for the graph isomorphism but restricting to cases with $|\det M| = 1$. For
the graph of each physical sector a standard form of its adjacencies, a \textit{canonical label}, is
computed with the algorithm [27]. We distinguish different masses by replacing massive
edges with a chain of several edges, where the length of such a chain labels a mass uniquely.

Graph isomorphism takes into account the ambiguities in labeling the \textit{nodes} of a graph.
While isomorphic graphs can be described by the same propagators (possibly with a cross-
ing of external momenta), the inverse is not true. Consider for instance the two vacuum
graphs in figure 1. The two graphs are non-isomorphic but can be described by the same
propagators, i.e. by the same sector. Here, a more appropriate object to consider is not
the graph of the sector, but the associated matroid. Matroids are based on the notion
of a set of linearly independent sets and may be considered as generalizations of graphs.
For a graph an associated graph matroid (or cycle matroid) can be defined via its edges.
Definitions and fundamental properties are given in the review [28] and references therein.
Essential for us is the following chain of statements. The relevant properties of a vacuum
graph where all edges share a common mass is encoded in the first Symanzik polynomial
($U$ polynomial). For brevity of the argument let us furthermore restrict to biconnected
graphs. The generalization to arbitrary vacuum components with different masses is rather
straightforward. An immediate combinatorial approach to isomorphisms of the Symanzik
polynomials, which is not restricted to vacuum graphs, was presented in [29]. Here, we
note that the first Symanzik polynomials of two graphs are equal up to a permutation.
of edge variables exactly if their matroids are isomorphic, see [28] and references therein. Two matroids of biconnected vacuum graphs are isomorphic exactly if the two graphs are isomorphic up to a series of twists, a statement known as Whitney’s theorem [30]. A twist operation starts by breaking a graph into two graphs such that identification of separation pairs of nodes in both graphs restore the original graph. As the second step of the twist, the separation pairs are identified with flipped orientation. In figure 1 the graphs are related by a twist around the left-most and right-most nodes and thus have the same matroid.

These statements can be turned into an algorithm to detect shifts (2.6) between vacuum sectors, which is sketched in the following and implemented in 

Reduze. We extend the graph isomorphism based shift detection by modifying the generated graphs with twists such that their canonical labels are minimized. A graph of a physical sector is decomposed into biconnected components with the algorithm [31]. Possible separation pairs of biconnected components are identified via a decomposition into triconnected components. We implemented the algorithm [32, 33] for this purpose. In order to generate at least one representative for each graph isomorphism class, it is necessary to perform twists around separation pairs as specified by virtual edges as well as twists which correspond to permutations of edges within polygon components of the decomposition. While twisting we track the edges including their orientations to be able to identify propagator momenta and different masses. Graphs with external legs are handled by intermediating joining their external nodes into one node and restricting to those twists, which keep the external legs joined into one node with their original orientation.

Alternatively, 

Reduze also offers a procedure to find all sector relations, which tries to identify sets of propagators in a straightforward approach based on linear algebra and combinatorics. While this procedure finds all shifts between arbitrary sectors, including sectors not corresponding to a graph, it is usually much less efficient than the graph based methods.

3.3 Sector symmetries

Sector symmetries are shifts of the form (2.6) with map a sector onto itself. Additional permutations of the external momenta are permitted as long as the kinematic invariants are unchanged. Different sector symmetries for physical sectors can be found by the underlying symmetries of the graph as vertex permutations from the automorphism group and permutations of multi-edges. The automorphism group of a graph consists of all permutations of the vertices which leave the canonical label of the graph unchanged. These transformations are calculated in 

Reduze with the algorithm [27] and are used to derive the mapping of the edges between pairs of vertices and the associated shift of the loop momenta. In the case where there is more than one edge between two vertices also permutations of the edges with the same mass are considered.

Alternatively, a complete set of sector symmetries can be calculated by 

Reduze using a combinatorial propagator matching approach.
I_5 + c_{14}I_4 + c_{13}I_3 = 0
I_5 + c_{24}I_4 + c_{22}I_2 = 0
I_5 + c_{33}I_3 + c_{32}I_2 = 0
I_3 + c_{42}I_2 = 0
I_3 + c_{51}I_1 = 0
I_2 + c_{61}I_1 = 0

**Figure 2.** Left: Three blocks of equations for loop integrals $I_i$ with coefficients $c_{ij}$ depending on kinematic invariants and the space–time dimension. Each block (shaded rectangle) contains equations for the same leading integral (bold face $I_i$). Right: communication topology for MPI processes involved in a distributed reduction.

### 3.4 Matching of diagrams to sectors

As discussed above, the assignment of momenta to propagators of Feynman diagrams is ambiguous. If diagrams are generated with a program like QGRAF \[16\], typically the assigned loop momenta have to be shifted in order to index the involved loop integrals via integral families. Reduce can compute these shifts for diagram files generated with QGRAF. Reduce can also handle permutations of external momenta and find matchings of diagrams to crossed sectors.

### 4 Distributed reduction algorithm

#### 4.1 Load balanced system solving

Reduce computes reductions for Feynman integrals by generating identities for a range of seed integrals and reducing this linear system of equations. The seed integrals are usually chosen for ranges in the propagator exponent sums $r$ and $s$. In a typical application, reductions for integrals from several sectors are needed. Moreover, a full reduction of a specific sector requires in general also the reduction of subsector integrals. Reduce proceeds bottom-up: the reduction of a sector is started only after all subsector results are available. Sectors which are no subsectors of each other can be reduced independently such that these tasks are easily parallelized. This kind of parallelization is available in the first version of Reduce via a shell script, which launches programs for different sectors. In version 2 of Reduce, two levels of parallelization are implemented via the message–passing–interface (MPI) standard. The reduction of a sector becomes a job and several independent such reduction jobs can be executed in parallel. The job system is described in more detail in the next section. On top of this first level of parallelization, this version of Reduce implements a distributed reduction algorithm for the reduction of a single sector. Since in this case the parallelization is less obvious we give some details about our implementation in the following subsection.

In Reduce, a total ordering is defined for indexed integrals of the type (2.3). The ordering defines integrals of a sector to be more complicated than integrals of its subsectors.
In the following, terms like “leading” or “lower” integral refer to this ordering. To choose specific integrals as master integrals the user may adjust the ordering, possibly at a later stage. In order to reduce integrals of a given sector, IBP and LI identities are generated for a specified range of seed integrals. This results in a sparse homogeneous linear system of equations for the indexed integrals where the coefficients are rational functions of the kinematic invariants and the space–time dimension. The equations are sorted into blocks of equations with the same leading integral, see left panel of figure 2.

The blocks are reduced bottom up, starting from the block with the lowest leading integral. For each block, integrals are reduced, i.e. replaced by linear combinations of lower integrals, according to the results from lower blocks and subsectors (“back substitution”) if possible. If a block contains several equations, one is kept and used to eliminate the leading integral from all other equations in the block (“forward elimination”), which are subsequently solved for the new leading integral. The coefficients of the integrals are normalized such that zeros are detected and numerator and denominator are coprime. This requires multivariate polynomial greatest–common–divisor (GCD) computations which typically present the most time consuming part of the full calculation. The result of this reduction of a block is one equation for the block and possibly further equations with lower leading integrals to be inserted into lower blocks. The next block to be selected is the lowest block which contains more than one equation or involves integrals which can be reduced.

Reduze offers both, a purely serial reduction for one sector as well as a distributed execution. In the serial version, the above steps are performed in deterministic order on one core. The distributed version employs a star topology of MPI processes with one manager and one or more workers, see right panel of figure 2. The workers perform the actual reduction steps for a block, while the manager keeps track of the complete system of equations and balances the work between the workers. More specifically, an idle worker contacts the manager to request work. The manager looks up the next block to process and sends its equations together with equations needed for back substitutions to the worker. The worker reduces the block and sends the result to the manager.

Our motivation for this distributed algorithm is as follows. Experiments show that in typical applications the time needed for the reduction of one block can easily differ by more than 6 orders of magnitude. Moreover, the exact order of the individual reduction steps significantly determines the execution time for the full reduction and a bottom-up order typically shows the best performance. Both issues are directly addressed by the dynamical load balancing mechanism presented above, at least for a not–too–large number of worker processes. How well this works in practice is quantified in the following subsection.

Reduze allows to choose between GiNaC [19] and Fermat [20] for the GCD computation needed to normalize coefficients. During reduction, the equations are stored either in RAM or optionally in a transactional database as implemented by the open source Berkeley DB [18]. With transactions turned on, an aborted reduction of a single sector may be resumed at a later time; recovering completed reductions for sectors is available in any case. Reduze supports different ways to split a calculation into several runs, this is described in the tutorial provided with the Reduze distribution. Reductions for crossed integrals are automatically be obtained via reduction results from its uncrossed counterpart in order to
save computation time and disk space.

4.2 Performance results for single sectors

Performance results for the reduction of sectors for two-loop contributions to heavy-quark pair production are shown in figure 3. These sectors have \( t = 4 \) or 5, respectively, and reductions were computed for integrals with \( r = t \ldots 7 \) and \( s = 0 \ldots 3 \) or 4, respectively. We used a computer with 48 CPU cores operating at 2.1 GHz and started \texttt{Reduze} with \( n_{\text{workers}} + 2 \) MPI processes (one job center process and one manager process should be overbooked for a small number of available cores).

In general, we observe that the scaling with the number of processes is problem specific and depends on the configuration of \texttt{Reduze}, such as the chosen computer algebra system. The upper two curves in the figure show an example with a good scaling for up to 22 workers. Indeed, we find examples where the scaling is good up to 48 workers. We think this good scaling behavior is noteworthy, given the fact that it describes the distributed computation of a not too loosely coupled system. As expected, we observe that beyond some number of worker processes the run time decreases less and less with additional workers and finally increases for even larger number of workers. Contributions to this behavior is expected from serial parts of the code, communication overhead, but potentially also from a “less ideal” order of evaluation when solving the system of equations with a larger number of workers. The lowest curve was obtained for an example of a reduction of a comparably simple system of equations, where the onset of such a behavior is clearly visible. It is also not difficult to find examples with worse scaling, where a minimal runtime is obtained for only a few worker processes.

Using \texttt{Fermat} instead of \texttt{GiNaC} for the GCD computations can easily result in a speed-
up by an order of magnitude, see the two blue curves for sector B in the figure. For the displayed Fermat benchmarks, the system to be reduced was kept in RAM, while for the GiNaC benchmarks a database was used. Our tests indicate that for examples of this type, the performance penalty for using a database is considerably less than the differences due to the two algebra systems considered here. Nevertheless, also this option is relevant for optimal performance, especially for a larger number of workers. Typically, if the coefficients in the equations become more involved, the reduction benefits considerably from a larger amount of workers compared to cases with compact coefficients. The performance in a real application example, involving the reduction of several sectors, is discussed at the end of the next section.

5 Job system

5.1 Load balanced execution of jobs

In Reduze, a job represents a sequence of computations which can be performed once its dependencies, specified via the presence of input files, are fulfilled. Most jobs in Reduze are serial jobs: they are executed on a single core but possibly in parallel to other jobs. Parallel execution of different jobs represents the top layer parallelization mechanism of Reduze which is automatically available for any job type added to Reduze. Reduction of identities is an example for a distributed job. Such a job can be executed by several processes in parallel: one process, the manager, is responsible for the full execution of the job, other processes, the workers, help for some time with the execution. In order to employ this second layer parallelization, a dedicated distributed algorithm needs to be implemented for the job. For each run of Reduze, the user specifies a sequence of such jobs, which are inserted into a job queue. A job may generate additional auxiliary jobs automatically. The job queue is responsible for resolving the dependencies between the jobs and determining the next job to be executed.
If Reduce is started with several MPI processes, one process will act as a job center and dynamically balance work between the other processes, the clients of the job center, see figure 4. The job center schedules jobs using the job queue and assigns them to clients. An idle client contacts the job center and requests work. The job center responds by assigning a job to the client, either to be executed as a manager or as a worker. The client changes its role accordingly. As a worker, it contacts the responsible manager of the job and helps with the job execution. As a manager, it can either execute the job by itself or register as an employer at the job center to request workers.

In order to optimize the efficiency of the parallelization, the job center periodically collects performance measurements from the employers and estimates an optimal distribution of workers based on it. According to this estimate, workers will effectively be reassigned to other employers by requesting release of workers from “inefficient” employers and assigning idle customer to “efficient” employers. The basic idea is to avoid low efficiencies due to overloaded managers by assigning workers preferably to managers which idle a high percentage of their CPU time.

5.2 Performance results for multiple sectors

Performance results for the reduction of a selection of sectors are shown in figure 5. These sectors are encountered in the calculation of two-loop corrections to heavy-quark pair production. Reductions are calculated for integrals of the depicted sectors and all of their subsectors, where $r = t \ldots 7$ and $s = 0 \ldots 4$. The benchmarks for the solid curve were performed on a computer with 48 CPU cores operating at 2.1 GHz using Fermat and keeping all equations in RAM. The measurements for the dashed curve were obtained using GiNaC.
the “Schrödinger” cluster of the University of Zurich with 2.8 GHz cores; these runs were configured to use GiNaC and a database.

The figure shows for this realistic application example that the calculation benefits considerably from up to 96 cores, if available. Let us stress again, that the scaling behavior is problem specific and may be worse for other types of applications. In the present example, the run–time decrease due to additional cores is quite close to an optimal $1/n_{\text{cores}}$ behavior for smaller numbers of cores.

6 Other features

6.1 Differential equations for master integrals

One method to solve master integrals consists of deriving differential equations by taking derivatives in the kinematic invariants, replacing new integrals with the reduction results and solving the differential equations by integration. In particular for sectors with several independent integrals, a change of basis for the master integrals may be required. Reduze offers the possibility to derive differential equations for Feynman integrals and reduce these equations for some user choice of master integrals.

6.2 Interference terms

Starting from diagrams generated by QGRAF, Reduze can compute scalar interferences of (bare) QED or QCD amplitudes in dimensional regularization. This includes insertion of user–defined Feynman rules, contraction of Lorentz vector indices, performing Dirac traces, and evaluating color structures. In case of an interference of a tree-level diagram with a diagram that could be matched to a sector of an integral family (section 3.4) also the occurring integrals, which belong to the matched sector and its subsectors, are indexed by the corresponding integral family. In a further step, these integrals can be reduced to master integrals and substituted correspondingly in the interference terms. Each computation of an interference of two diagrams is treated as a job, and when MPI is used, these jobs can be performed in parallel. Specific examples are distributed with the Reduze package.

7 Usage

The package Reduze can be downloaded from the web page http://projects.hepforge.org/reduze. The distribution contains a tutorial with detailed description of installation and usage as well as several example files.

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