r-MODES IN THE OCEAN OF A MAGNETIC NEUTRON STAR

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ABSTRACT

We study the dynamics of r-modes in the ocean of a magnetic neutron star. We model the star’s ocean with a spherical rotating thin shell and assume that the magnetic field symmetry axis is not aligned to the shell’s spin axis. In the magnetohydrodynamic approximation, we calculate the frequency of $l = m$ r-modes in the shell of an incompressible fluid. Different r-modes with $l$ and $l \pm 2$ are coupled by the inclined magnetic field. Kinematical secular effects for the motion of a fluid element in the shell undergoing the $l = m = 2$ r-mode are studied. The magnetic-corrected drift velocity of a given fluid element undergoing the $l = m$ r-mode oscillations is obtained. The magnetic field increases the magnitude of the fluid drift produced by the r-mode oscillations. The drift velocity is strongly modulated by the inclined magnetic field. We show that the magnetic field is distorted by the high- $l$ magnetic r-modes more strongly than by the low- $l$ modes. Furthermore, because of the shear produced by the r-mode drift velocity, the high- $l$ modes in the ocean fluid will damp faster than the low- $l$ ones.

Subject headings: stars: magnetic fields — stars: neutron — stars: oscillations — stars: rotation

1. INTRODUCTION

In a series of papers, Bildsten & Cutler (1995) and Bildsten, Ustimirsky, & Cutler (1996) showed that neutron stars accreting at high accretion rates, $\gtrsim 10^{-9} M_\odot$ yr$^{-1}$, are covered with massive oceans with densities $\gtrsim 10^{9}$ g cm$^{-3}$. Both theoretical consideration of the nature of burning at these rates (Fusiki & Lamb 1987; Bildsten 1993, 1995) and observational features of type I X-ray bursts (van der Klis 1995) reveal that the star burns the accreting hydrogen and helium directly to iron-group elements via the type I X-ray bursts, which are crystallized by the larger Coulomb force at much lower densities. Furthermore, they conjectured that the waves in these oceans might modulate the outgoing X-ray flux at frequencies comparable to what is observed.

In this paper, motivated by the conjectures of Bildsten et al., we study the evolution of r-modes in the ocean of magnetic neutron stars. The r-modes are analogous to Rossby waves in the Earth’s oceans, and are driven by the Coriolis force in the rotating stars. Their motions are predominantly toroidal, and their oscillation frequencies are proportional to the rotation rate of the star, $\Omega$. Since even the small magnitude of the Coriolis force drives r-modes in a rotating fluid, they should be considered in the dynamics of the fluid. We model the star’s ocean with a thin rotating shell of incompressible inviscid fluid at the radius $R$, which is sandwiched between two hard spheres. The role of the spheres is to make sure that the fluid motion is restricted to a two-dimensional spherical surface.

The paper is organized as follows. In § 2, we set up the magnetohydrodynamics equations for a spherical thin shell in the background of a uniform magnetic field in which the magnetic field symmetry axis is not aligned with shell’s spin axis. We find the magnetic coupling coefficients and then the corrected eigenfrequencies and eigenvectors for the $l = m$ r-mode. In § 3 we consider the kinematical secular drift of r-modes, introduced by Rezzolla et al. (2000, 2001a, 2001b), in a magnetic ocean fluid. We show that in addition to the magnetic field strength causing the drift magnitude to increase, the inclination of the magnetic field axis modulates the r-mode trajectories. The latter was not considered in previous studies. Section 4 is devoted to further discussion.

2. r-MODES IN MAGNETIC ROTATING SHELL

Historically, the study of oscillations of fluids in a rotating shell in the presence of a magnetic field was begun about 50 years ago by geophysicists. Pioneering works by Hide (1966) and Stewartson (1967) are concerned with magnetic modes of the Earth’s core that can be attributed to the observed secular changes in the main geomagnetic field at the Earth’s surface. It is well known that the Earth’s magnetic field at the surface drifts slowly westward, with a period of the order of 1000 yr. They assumed an unrealistic magnetic field configuration characterized by a toroidal field of constant magnitude on the shell, and found that the magnetic-corrected toroidal motions might cause the field drift at the surface of the Earth.

In this paper we investigate the MHD perturbations of a uniformly rotating thin spherical shell of incompressible fluid of radius $R$, endowed with a uniform magnetic field (a more realistic configuration) given by

$$B = B_\rho(\cos \theta e_\rho - \sin \theta e_\theta)$$

on the shell. Since the shell is sandwiched between two spherical hard covers, the motion of the fluid is restricted to a two-dimensional spherical surface. Furthermore, we assume that the symmetry axis of the field makes an angle $\beta$ to the shell’s rotation axis. We work in an ideal MHD framework, so that the field lines are frozen into the fluid, and the magnetic field rotates at the same rate as the shell.
For a uniformly rotating shell of incompressible fluid with angular frequency $\Omega$ around the $z$-axis, the fluid velocity perturbation field in the corotating frame is exactly determined by

$$
\delta v = \alpha \xi_{lm},
$$
(2a)

$$
\xi_{lm}(\theta, \phi, t) = \frac{U_{lm}(R)}{R} \left[ -\frac{\partial_r Y_{lm}(\theta, \phi)}{\sin \theta} e_\theta + \frac{\partial_\theta Y_{lm}(\theta, \phi)}{\sin \theta} e_\phi \right] e^{-i \omega mt},
$$
(2b)

for any arbitrary function $U_{lm}(R)$ and with dispersion relation

$$
\omega_{lm} = -\frac{2m\Omega}{(l+1)}.
$$
(2c)

Here $\alpha$ is the amplitude of the mode. Equation (2) is the exact solution of the fluid equations of motion in fully nonlinear regime; see Levin & Ushomirsky (2001) and references therein for more detail.

Recently, Morsink & Rezania (2002) have studied the dynamics of a rotating star in the presence of a magnetic field where its symmetric axis is not aligned to the star’s spin axis. For the $l = m$ $r$-modes of an incompressible rotating fluid in the background of uniform magnetic field, they obtained a shift proportional to the ratio of the magnetic energy to the rotational energy of the star in the $r$-mode eigenfrequencies only.

Following Morsink & Rezania (2002), we consider that the magnetic force produced by the fluid’s perturbations of the rotating shell drives the motion of fluid elements. As they have shown, the eigenvalue equation for eigenfrequencies and eigenvectors of the normal modes of a fluid in the presence of a magnetic field will be corrected by

$$
\omega_A \left( c_A - \frac{1}{2} \sum_D \kappa_{AD} e_D \right) = \sigma_A c_A,
$$
(3)

where the $c_A$ and $\sigma_A$ are the magnetic-corrected eigenvector and eigenfrequency in the rotating frame, respectively. Here $c_A$ is the energy of the mode in the rotating frame. The magnetic coupling coefficients, $\kappa_{AD}$, can be calculated on the surface of sphere as

$$
\kappa_{AD} = -\frac{1}{4\pi \mathcal{M}} \int \delta B_A^* \delta B_D d\Omega,
$$
(4a)

$$
\delta B_A = \mathbf{V} \times (\xi_A \times \mathbf{B}),
$$
(4b)

where $\mathcal{M} = B_0^2/4\pi$ is the magnetic field energy density.\(^1\)

Note that in the above calculations we expand any magnetic perturbations in terms of the modes of rotating star, $\xi_A$, as

$$
\xi = \sum_A c_A \xi_A e^{-i \omega t}.
$$
(5)

Here uppercase Latin subscripts are used to label solutions and corresponds to the unique set of quantum numbers that describe the solutions.

\(^1\) Equation (1) leads to $\mathbf{V} \times \mathbf{B} = 0$. As a result, the term containing $\mathbf{J}$ vanishes. The term containing $\xi \times \mathbf{B}$ does not vanish, but its contribution is smaller by $\Omega^2 R^2/c^2$ for $r$-modes, and we neglect it in equation (4).

To calculate the magnetic coupling we use the method described by Morsink & Rezania (2002). They used the spin-weighted formalism in which (1) the vectors are expressed in the orthonormal but complex basis $\{e_+, e_-\}$ that is related to the usual orthonormal basis $\{e_\theta, e_\phi\}$ by

$$
e_{\pm} = \frac{1}{\sqrt{2}} (e_\theta \pm e_\phi);
$$
(6)

and (2) all functions are expanded by spin-weighted spherical harmonics, $s Y_{lm}(\theta, \phi)$, that are defined by (Campbell 1971)

$$
s Y_{lm}(\theta, \phi) = \sqrt{\frac{(2l+1)}{4\pi}} d_{-m}^l(\theta) e^{im\phi},
$$
(7)

where the $d_{-m}^l(\theta)$ are the matrix representations for rotations through an angle $\theta$ discussed in detail by Edmonds (1974). When $s = 0$, the spin-weighted spherical harmonics reduce to the regular spherical harmonics. The orthogonality relations are

$$
\int d \Omega s Y^*_m Y_{lm} = \delta_{lm} \delta_{mm}.
$$
(8)

We use the convention that $s Y_{lm}(\theta, \phi) = (-1)^{s+m} Y_{l-m}(\theta, \phi)$. A uniform magnetic field tilted by an angle $\beta$ from the shell’s spin axis can be written in spin-weight formalism as

$$
\mathbf{B}(\theta, \phi) = \sum_{m=-l}^{l} \left[ b_0^l(a) Y_{lm}(\theta, \phi) e_0 + b_+^l(a) Y_{lm}(\theta, \phi) e_+ + b_-^l(a) Y_{lm}(\theta, \phi) e_- \right],
$$
(9)

where the functions $b_\pm^l(a)$ are given by

$$
b_\pm^l = \mp b_0^l = \sqrt{\frac{4\pi}{3}} d_{0\pm}^l(\beta),
$$
(10)

and the $d_{0\pm}^l$ are given by

$$
d_{10}^l(\beta) = \frac{1}{\sqrt{2}} \sin \beta, \quad d_{00}^l(\beta) = \cos \beta,
$$
(11)

To rewrite equation (2) in spin-weighted formalism, at first we normalize $\xi$ such that the energy density of the modes in the corotating system is $\varepsilon_\mathcal{F} = \mathcal{F}$, where $\mathcal{F}$ is rotational kinetic energy density of the shell. Therefore, the $r$-modes of a nonmagnetic incompressible thin shell have the form

$$
\omega_{lm\mathcal{F}} = -\frac{2m\Omega}{l(l+1)},
$$
(12a)

$$
U_{lm\mathcal{F}} = R^2 \sqrt{\frac{4l(l+1)}{l(l+1) + 1}},
$$
(12b)

$$
\varepsilon_{lm\mathcal{F}} = \mathcal{F} = \frac{1}{2} \rho R^2 \Omega^2.
$$
(12c)

The spin-weighted decomposition of the $A$th $r$-mode is

$$
\xi_A(x) = f_+^A Y_{lm\mathcal{F}} e_+ + f_-^A Y_{lm\mathcal{F}} e_-,
$$
(13)
where the functions $f^A_\lambda(r)$ are given by (Schenk et al. 2002)

$$f^A_\lambda = f^A = \frac{R}{2\sqrt{2}} \frac{l(l+1)}{m_A} .$$

Different components of the perturbed magnetic field, $\delta B_{A\lambda}$, due to the $r$-modes in the thin shell, can be found in spin-weighted formalism as

$$\delta B_{A\lambda}(\theta, \phi) = \sum_{\lambda, \mu} \delta B_{A\lambda}^{\lambda\mu} Y_{\lambda\mu}^* ,$$

$$\delta B_{A\lambda}^{\lambda\mu} = \sum_{m=-1}^{+1} C(l_A, 1, \lambda) \left( \begin{array}{c} \lambda & l_A & 1 \\ \mu & m_A & \bar{m} \end{array} \right) \frac{1}{a} b_0^{m} f_\lambda^{A}(1 + (-1)^{\lambda+l_A}) ,$$

$$\delta B_{A+}^{\lambda\mu} = \frac{1}{a} \left[ \frac{l_A(l_A+1)}{2} \left( \begin{array}{c} \lambda & l_A & 1 \\ -1 & 0 & 1 \end{array} \right) + \frac{l_A(l_A+1)(l_A+2)}{2} \left( \begin{array}{c} \lambda & l_A & 1 \\ -1 & 2 & -1 \end{array} \right) \right] ,$$

$$\delta B_{A-}^{\lambda\mu} = \sum_{m=-1}^{+1} C(l_A, 1, \lambda) \left( \begin{array}{c} \lambda & l_A & 1 \\ \mu & m_A & \bar{m} \end{array} \right) \frac{1}{a} b_0^{m} f_\lambda^{A}(1 + (-1)^{\lambda+l_A}) ,$$

where

$$\left( \begin{array}{ccc} I & k & \lambda \\ -s & -t & \sigma \end{array} \right)$$

is a Wigner 3-j symbol (Edmonds 1974), and the constant $C(l_A, I, \lambda)$ is defined by

$$C(l_A, I, \lambda) = ( -1 )^{l_A+\lambda+I} \sqrt{\frac{2l_A+1)(2I+1)(2\lambda+1)}{4\pi} .$$

In equation (15a), $\lambda$ takes on values $l_A$, $l_A \pm 1$, although equation (15b) will survive only for $\lambda = l_A$. Since the spin-weighted spherical harmonics obey the orthogonality relation (8), integrals over all angles of quantities quadratic in $\delta B$ have the simple form

$$\kappa_{A\lambda} = \int \delta B_{A\lambda} \cdot \delta B_{D\lambda} d\Omega = \sum_{\lambda, \mu} \left[ \delta B_{A00}^{\lambda\mu} \delta B_{D00}^{\lambda\mu} + \delta B_{A+}^{\lambda\mu} \delta B_{D+}^{\lambda\mu} + \delta B_{A-}^{\lambda\mu} \delta B_{D-}^{\lambda\mu} \right] ,$$

$$= \sum_{\lambda, \mu} \left\{ \delta B_{A00}^{\lambda\mu} \delta B_{D00}^{\lambda\mu} \delta B_{A+}^{\lambda\mu} \delta B_{D+}^{\lambda\mu} + \delta B_{A-}^{\lambda\mu} \delta B_{D-}^{\lambda\mu} \right\} \times [1 + (-1)^{l_A+l_D}] ,$$

where $\delta_{AD}$ is the Kronecker delta function. The last step is due to the symmetry property (eq. [15c]). It then follows that the magnetic coupling coefficients between $r$-modes is zero unless both modes have the same parity. Since the triangle inequalities $l_A - 1 \leq \lambda \leq l_A + 1$ and $l_D - 1 \leq \lambda \leq l_D + 1$ must be satisfied, only modes satisfying $l_D = l_A$, $l_D \geq 2$ have nonzero magnetic coupling. Evaluating the terms appearing in equations (15b) and (15d) for $l_A = m_A$, the individual terms for the allowed values of $\lambda$ are

$$\delta B_{A0}^{l_A, 0} = (-1)^{l_A+1} \sqrt{\frac{2l_A+1}{4}(l_A+1)} \times \sum_{m=-1}^{+1} d_m^{l_A} \left( \frac{l_A}{(l_A+m)} \frac{l_A}{l_A} \right) ,$$

$$\delta B_{A+}^{l_A+1, 0} = (-1)^{l_A+1} \sqrt{\frac{2l_A+1(l_A+1)}{8l_A}} \times \sum_{m=-1}^{+1} d_m^{l_A} \left( \frac{l_A+1}{(l_A+m)} \frac{l_A}{l_A} \right) ,$$

$$\delta B_{A-}^{l_A-1, 0} = (-1)^{l_A+1} \sqrt{\frac{l_A-1(l_A+1)}{8l_A}} \times \sum_{m=-1}^{+1} d_m^{l_A} \left( \frac{l_A-1}{(l_A+m)} \frac{l_A}{l_A} \right) .$$

The magnetic coupling coefficients between different $r$-modes can now be computed. A straightforward calculation yields the self-coupling term, $\kappa_{A\lambda}$, as

$$\kappa_{A\lambda} = - \frac{l_A+1}{4(2l_A+3)} \left[ l_A(l_A+1)(l_A+2) + 3 \right]$$

$$+ \frac{1}{2} \left[ l_A(l_A+1)(2l_A^2 + 2l_A - 3) \right] \sin^2 \beta .$$

The off-diagonal terms, $\kappa_{A+2\lambda}$ and $\kappa_{A-2\lambda}$, will be

$$\kappa_{A+2\lambda} = \frac{1}{8} \sqrt{\frac{l_A(l_A+3)}{(2l_A+3)(2l_A+5)}} \times l_A(l_A+1)(l_A+2) \sin^2 \beta ,$$

$$\kappa_{A-2\lambda} = \frac{1}{8} \sqrt{\frac{l_A(l_A-2)(l_A+1)}{(2l_A-1)(2l_A+1)}} \times (l_A-2)(l_A+1)^2 \sin^2 \beta .$$

Therefore, the equation of motion for mode expansion coefficients of the magnetically modified $A\lambda$ $r$-modes reduces to

$$\sigma_{AC} = \omega_A \left[ \frac{\mu}{2\pi} |\kappa_{A\lambda}| c_{A+2\lambda} + \left( 1 + \frac{\mu}{2\pi} |\kappa_{A\lambda}| c_{A-2\lambda} \right) \right] .$$

The final equations (19)–(21) give the magnetically corrected eigenfrequencies $\sigma_A$ and eigenvectors $c_A$ for the $r$-modes. To compute these eigenfrequencies and eigen-
vectors one should solve the $N$-dimensional linear algebraic system (eq. [21]), where $N$ is the number of modes. Therefore, for a given $N = 1, 2, \ldots$, the $N \times N$ matrix should be diagonalized. Since it is not possible to solve an infinite-dimensional system in practice, we truncate the infinite matrix to a finite matrix and large enough $N$. Furthermore, we note that for the lower inclination angles, the magnetic coupling coefficients $\kappa_{AD}$ increase by $\frac{1}{l_A}$, while for the larger inclination angles by $\frac{1}{l_A^2}$. Since we are working in the linear regime, we must keep $(\mathcal{M}/2\mathcal{F})\kappa_{AD} < 1$. The ratio of magnetic energy density to rotational energy density in the covered ocean at the surface of star is

$$\frac{\mathcal{M}}{2\mathcal{F}} = 8 \times 10^{-3} \left( \frac{B}{10^{13} \text{ G}} \right)^2 \left( \frac{10^9 \text{ g cm}^{-3}}{\rho} \right) \times \left( \frac{10\text{ km}}{R} \right)^2 \left( \frac{10^3 \text{ Hz}}{\Omega} \right)^2,$$  

where $\rho$ is the ocean’s density. In Tables 1 and 2 we present the eigenfrequencies of $l_A = m_A = 1, 2, \ldots, 6$ r-modes. In Table 1 we assume that the inclination angle between the field axis and shell’s spin axis is zero, $\beta = 0$, and calculate the eigenfrequencies $\sigma_A$ for different ratios of the magnetic energy density to the rotational energy density, $\mathcal{M}/2\mathcal{F}$, by increasing the magnetic field strength; see equation (22). In Table 2 we compute the eigenfrequencies by increasing the inclination angle for a specific energy ratio $\mathcal{M}/2\mathcal{F} = 10^{-2}$. As a result, the eigenfrequency of the mode is increased by increasing both magnetic field energy density and inclination angle.

In the next section we consider the kinematical secular effects of r-modes in the oceanic fluid.

### Table 1

| $l_A$ | $l_A = 1$ | $l_A = 2$ | $l_A = 3$ |
|-------|-----------|-----------|-----------|
| $0.0$ | $-1.0090$ | $-0.68595$ | $-0.53500$ |
| $\sigma/6$ | $-1.0000$ | $-0.68874$ | $-0.54775$ |
| $\pi/4$ | $-1.0068$ | $-0.69038$ | $-0.55827$ |
| $\pi/3$ | $-1.0056$ | $-0.69155$ | $-0.56837$ |
| $\pi/2$ | $-1.0043$ | $-0.69250$ | $-0.57834$ |

Note.—The ratio of magnetic energy to the rotational energy is $\mathcal{M}/2\mathcal{F} = 10^{-2}$.

### Table 2

| $\beta$ | $l_A = 1$ | $l_A = 2$ | $l_A = 3$ |
|---------|-----------|-----------|-----------|
| $0.0$ | $-1.0090$ | $-0.68595$ | $-0.53500$ |
| $\sigma/6$ | $-1.0000$ | $-0.68874$ | $-0.54775$ |
| $\pi/4$ | $-1.0068$ | $-0.69038$ | $-0.55827$ |
| $\pi/3$ | $-1.0056$ | $-0.69155$ | $-0.56837$ |
| $\pi/2$ | $-1.0043$ | $-0.69250$ | $-0.57834$ |

Note.—The angle between magnetic field axis and shell’s spin axis is $\beta = 0$. The energy in the magnetic field is $\mathcal{M} = B_0^2/4\pi$, and the rotational energy is $\mathcal{F} = \rho \omega^2 \Omega^2/2$.

### 3. r-MODE DRIFT BY THE MAGNETIC FIELD

In a series of papers, Rezzolla et al. (2000, 2001a, 2001b) showed the existence of a kinematic secular velocity field of r-mode oscillations that interacts with the background magnetic field of the star. Although this interaction is nonlinear and should be considered in the nonlinear regime, one can find some second-order quantities from a linear result (Rezzolla et al. 2000). As an example, the second-order secular toroidal drift that appears on isobaric surfaces can be obtained through the linear velocity field.

To investigate this secular effect in the fluid motion undergoing the magnetic-corrected r-mode oscillations in the ocean fluid, we consider the real part of the magnetic-corrected fluid velocity of an $l_A = m_A$ r-mode, equation (2),

$$\theta(t) = -\sum_A \alpha Q(l_A)c_A \sigma_A (\sin \theta)^{l_A-1} \cos(l_A \phi - 2\pi \sigma_A t),$$

$$(23a)$$

$$(23b)$$

$$Q(l_A) = \left( -\frac{1}{2} \right) \frac{1}{l_A^4} \frac{1}{l_A^2} \sqrt{\frac{\pi l_A (l_A + 1)(2l_A + 1)!}{4}}.$$

Here $\sigma_A$ and $c_A$ are the magnetic-corrected eigenfrequency and eigenvector, respectively, and the time, $t$, is in unit of the star’s period, $P$.

In Figures 1, 2, and 3 we plot the result of numerical integration of equation (23) in the shell’s northern hemisphere for the magnetic-corrected $l_A = m_A = 2$ r-mode with amplitude $\alpha = 0.1$. Note that we assumed that at $t = 0$, only the $l_A = m_A = 2$ r-mode is present in the fluid, and all calculations are done in the corotating frame.

In the case of zero inclination angle, $\beta = 0$, our results for all mode’s latitudes are in good agreement with the results reported by Rezzolla et al. (2000, 2001a). In Figure 1 we plot $\theta(t)$ versus $\phi(t)$ from $t = 0$ to $7.45 P$ (almost 5 cycles of the mode), with initial values $\phi(0) = 0, \theta(0) = \pi/2$, $l_A = m_A = 2$, zero inclination angle $\beta = 0$, and two different values of $\mathcal{M}/2\mathcal{F} = 0$ and $10^{-1}$. The aligned field causes the drift in the $\phi$-direction (produced by the $r$-mode) to increase. The corrected expression$^2$ for the drift velocity of a given fluid element in the background of the uniform

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$^2$ Following Rezzolla et al. (2000, 2001a), equation (24) is obtained by expanding equations (23) in powers of $\alpha$, averaging over a gyration, and retaining only the lowest order nonvanishing term.
magnetic field will be

\[
\mathbf{v}_d = \sum_A K_A(\theta) \alpha(t) \sigma_A(t) \mathbf{e}_\phi ,
\]

(24a)

\[
K_A(\theta) = Q^2(l_A) c_A^2 \sin^2\theta (\sin^2\theta - 2(l_A - 1) \cos^2\theta) .
\]

(24b)

As a result, because of the nonzero inclination angle, the drift velocity of a fluid element undergoing the \( l_A \) r-mode oscillation is affected by all \( l_A \pm 2 \) r-mode oscillations. Equation (24) reduces to one obtained by Rezzolla et al. (2000, 2001a) for the nonmagnetic \( l_A = m_A = 2 \) r-mode, i.e.,

\[
\mathbf{v}_d = K_2(\theta) \alpha(t) \mathbf{w}_2(t) \mathbf{e}_\phi .
\]

Therefore, the total displacement in \( \phi \) from the onset of the oscillation at \( t_0 \) to time \( t \) is

\[
\Delta \phi(\theta, t) = \sum_A K_A(\theta) \int_{t_0}^{t} \alpha^2(t') \sigma_A(t') dt' .
\]

(25)

The change in \( \phi \) due to the magnetic field is given in Tables 3 and 4. As a result, \( \Delta \phi = \phi_{\text{mag}} - \phi_{\text{nonmag}} \) is increased by increasing the magnetic energy density. Furthermore, the high-\( l_A \) magnetic r-modes drift more than the low-\( l_A \) ones.

As shown in Figures 2 and 3, the r-mode trajectories in both the \( \theta \)- and \( \phi \)-directions are strongly modulated by the inclined magnetic field. In Figure 2, from left to right, we plotted time evolution of r-mode displacements in the \( \theta \)-direction, the \( \phi \)-direction, and the projected (\( \theta \), \( \phi \)) r-mode trajectory, respectively. All panels correspond to \( \mathcal{M}/2 \mathcal{F} = 10^{-2} \) and are plotted with initial conditions \( \phi(0) = 0, \theta(0) = \pi/2 \). The inclination angle is \( \beta = 0 \) for the top, \( \beta = \pi/6 \) for the middle, and \( \beta = \pi/2 \) for the bottom panels. The larger the field’s angle, the stronger the modulation. The first and second columns of Figure 2 show how \( \theta \)- and \( \phi \)-displacements of the initial \( l_A = m_A = 2 \) r-mode are affected by the inclined field, respectively. In the third column we show the motion of the fluid element in the northern hemisphere of the shell undergoing the \( l_A = m_A = 2 \) r-mode oscillations. The projected trajectories \( \theta(t) \sin \theta(t) \cos \phi(t) \) and \( \phi(t) \sin \theta(t) \cos \phi(t) \) show how the inclined magnetic field changes the motions that will be detected by a co-rotating observer in the rotation equator of the shell.

It is interesting to note that the results show that the modulations are more sensitive to the field inclination angle than to the magnetic field strength. This can be understood from equation (23). A nondiagonal matrix of the corrected eigenvectors \( c_A \) causes modulations in both \( \theta \) and \( \phi \) trajectories. For any nonzero value of \( \mathcal{M}/2 \mathcal{F} \neq 0 \), no matter how small, and \( \beta \neq 0 \), the eigenvectors matrix is not diagonal. Furthermore, the magnitude of the corrected eigenvector \( c_A \) changes significantly by varying \( \beta \), while it is almost constant for different (and nonzero) values of \( \mathcal{M}/2 \mathcal{F} \). This fact is shown in Figure 4. In the left panel, we plot the element \( c_2^2(2, 2) \) as a function of \( \beta \) for a fixed value of \( \mathcal{M}/2 \mathcal{F} \approx 10^{-10} \). As shown, by increasing \( \beta \) the value of \( c_2^2(2, 2) \) drops by \( \sim 0.02 \), while the change in the element \( c_2^2(2, 2) \) due to the \( \mathcal{M}/2 \mathcal{F} \) for a fixed value \( \beta = \pi/2 \) is \( \approx 10^{-8} \), which is shown in the right panel of Figure 4. Therefore, no matter how small the value of the energy ratio, the fluid motions are strongly modulated by a magnetic field making a large inclination angle to the star’s symmetry axis.

Furthermore, the larger the fluid element’s latitude, the stronger the modulation. This feature is shown in Figure 3 which is plotted for different initial conditions \( \phi(0) = 0, \theta(0) = \pi/6 \) for both \( \beta = 0 \) (top panels) and \( \beta = \pi/2 \) (bottom panels). Comparing both bottom panels in Figures 2 and 3 reveals that the magnetic field (having the same energy density and inclination angle) has less effect on the lower latitude modes.

Finally, we note that \( \Delta \phi \) is also increased by increasing the inclination angle; see the third column of Figures 2 and 3. Therefore, the motion of a fluid element of the shell...
undergoing the magnetic r-mode oscillation drifts more than nonmagnetic r-mode oscillation due to both field energy density and field inclination angle. These properties can be understood by considering that both $D/D_{30}$ and $v_d$ are proportional to the corrected eigenfrequency and eigenvector; see equations (24) and (25). The corrected eigenvector changes by changing the inclination angle, while the corrected eigenfrequency increases by increasing the field energy density.

The toroidal fluid motions produced by the r-mode are perpendicular to the magnetic field. They distort the field and increase the field energy density. As shown by Rezzolla et al. (2000), if the energy required by the mode to distort the magnetic field is greater than the mode energy during an r-mode oscillation, the distortion of the magnetic field that is proportional to the $v_d$ will prevent the oscillation from occurring. To make an estimate of at which field strength this might get to be important, we calculate the ratio of the change in the magnetic energy density of the shell, $\delta E_m = \int (\delta B^2/4\pi) d\Omega$, to the mode energy density during one oscillation as

$$\frac{\delta E_m}{\epsilon_A} \approx \frac{M}{2\pi |\kappa_{AA}|},$$

$$\approx 10 \left( \frac{B}{10^{15} \text{ G}} \right)^2 \left( \frac{10^9 \text{ g cm}^{-3}}{\rho} \right) \left( \frac{10 \text{ km}}{R} \right)^2 \times \left( \frac{10^3 \text{ Hz}}{\Omega} \right)^2 \left\{ \begin{array}{ll} \frac{l_3}{l_3} & \text{for small } \beta, \\ \frac{l_3}{l_3} & \text{for large } \beta. \end{array} \right.$$

Thus, $\delta E_m > \epsilon_A$ if $B > B_{\text{critical}} \approx 3 \times 10^{14} \rho_9^{1/2} R_{10} \Omega_3 l_3^2$, where $\eta = -3/2$ and $-2$ for small and large inclination angles, respectively. Here $\rho_9 = \rho/10^9 \text{ g cm}^{-3}$, $R_{10} = R/10 \text{ km}$, and $\Omega_3 = \Omega/10^3 \text{ s}^{-1}$. Hence, the smaller field strength would prevent the high-$l_A$ r-mode oscillations. Furthermore, equation (26) shows that for the higher field inclination angle the distortion effect will be stronger.

In addition, the increase in the energy of the magnetic field produced by the r-mode drift reduces the mode energy

Fig. 2.—Motion of a fiducial fluid element in the northern hemisphere of a shell undergoing $l_A = m_A = 2$ r-mode with amplitude $\alpha = 0.1$. The projected trajectories are plotted for 20 periods of the star (almost 13 cycles of the mode), with initial values $\phi(0) = 0$, $\theta(0) = \pi/2$, and $\dot{\phi}/2\pi = 10^{-2}$. Both $\theta$ and $\phi$ are in radians, and $t$ is in units of star’s period, $P$. Top panels are plotted for $\beta = 0$, middle panels for $\beta = \pi/6$, and bottom panels for $\beta = \pi/2$. 


and causes damping. The rate at which the magnetic field extracts energy density from the mode can be estimated by

$$\frac{1}{4\pi} \int B_\phi (dB_\phi/dt) d\Omega \approx \sum_A K_A(\theta) \gamma^2 \sigma_A B_\phi B_\phi.$$ \hspace{1cm} (27)$$

The azimuthal field $B_\phi$ is generated by the $r$-mode drift velocity $v_d$ from the background poloidal field. The change in $B_\phi$ can be calculated approximately for the magnetic $r$-modes from the time $t_0$ that the oscillation begins to time $t$ by $\Delta B_\phi(t) \approx \sum_A \int_{t_0}^{t} K_A(\theta) \gamma^2 \sigma_A(i) B_\phi dt$ (Rezzolla et al. 2000). Equation (27) shows that the energy loss from the mode increases when either the field energy density or the inclination angle is increased. Furthermore, since the high-$l_A$ magnetic $r$-modes drift more than the low-$l_A$ ones, they will damp faster than the low-$l_A$ modes.

4. DISCUSSION

In this paper, by assuming that neutron stars in the low-mass X-ray binaries (LMXBs) are covered by degenerate massive oceans, we have studied the interaction of the

[Figures 3 and 4 are shown here, depicting graphs and data related to the study's findings.]
inclined magnetic field with \( r \)-mode oscillations of an incompressible ocean fluid in a thin-shell approximation. Our analysis shows that in the case of large inclination angle, even for weak magnetic field energy density in comparison with the star’s rotational energy density, \( l \) and \( l/\sqrt{C_6} \)-modes are coupled significantly. The kinematical secular drift produced by the \( r \)-mode is studied in the background of the inclined magnetic field. The results show that the magnetic field tends to increase the fluid drift in the \( \phi \)-direction. As a result, for the same time period, the fluids in the presence of the magnetic field undergo larger drift than fluids with no background magnetic field.

Since the magnetic field is distorted by the \( r \)-mode, we showed that the field distortion by the high-\( l \) magnetic \( r \)-mode is stronger than the low-\( l \) modes. Thus, a smaller critical field is needed to prevent the high-\( l \) \( r \)-mode oscillation from occurring. Furthermore, the shear produced by the drift velocity of the magnetic \( r \)-mode will damp high-\( l \) modes faster than the low-\( l \) ones. Note that in the present paper, we assume that the \( r \)-mode is stable, so we do not deal with its unstable growth in the magnetic star, as has been considered by Rezzolla et al. (2000).

Observational and phenomenological studies of more than 20 LMXBs reveal that these accreting neutron stars are a source of millisecond oscillations, burst oscillations, and quasi-periodic oscillations in the Hz–kHz range. Different authors proposed different possible mechanisms to explain the observations in LMXBs, but the regularity of these oscillations in many sources led the investigators to consider radial or nonradial oscillations in a coronal accretion flow near the Eddington limit; see the review by van der Klis (2000). By assuming neutron stars in LMXBs covered by an ocean fluid, Bildsten & Cutler (1995) and Bildsten et al. (1996) showed that both low and high radial orders of ocean \( g \)-modes might be responsible for observed low-frequency (\( \sim \)Hz) oscillations in these stars. Recently, Heyl (2001) discussed different possible oscillations (the \( g \)-modes, the Kelvin modes, and the buoyant \( r \)-modes) that may excite in the neutron-star ocean during a type I burst, and concluded that the buoyant \( r \)-modes may meet the observations of low-frequency oscillations in type I X-ray bursts quite well.

![Fig. 5. Time evolution of \( \theta(t) \)–\( \theta(0) \) displacement of the \( l_1 = m_4 = 2 \) \( r \)-mode, from \( t = 0 \) to \( t = 100P \), for different initial mode amplitudes \( \alpha \). Top left: \( \alpha = 0.1 \); top right: \( \alpha = 0.5 \); bottom left: \( \alpha = 0.8 \); bottom right: \( \alpha = 0.9 \). All panels correspond to \( \mu/2\pi = 10^{-2} \), \( \beta = \pi/2 \) and are plotted with initial conditions \( \phi(0) = 0 \) and \( \theta(0) = \pi/2 \).](image)
Figures 2 and 3 show how the inclined magnetic field modulates $l_A = m_A = 2$ $r$-modes. The associated envelope frequency, $f_{\text{env}}$, decreases as $f_{\text{env}} \sim \nu_s / 10 - \nu_s / 16$ when the inclination angle increases from $\beta \sim \pi / 6$ to $\pi / 2$. Here $\nu_s = \Omega / 2\pi$ is the star’s spin frequency. The frequencies of these new modes are comparable to those observed oscillations in the LMXBs in low frequencies. For the star’s spin frequency $\nu_s \approx 300$ Hz, the wave envelope frequency is $f_{\text{env}} \approx 20$–$30$ Hz. Different frequencies can be obtained by different initial $l_A = m_A$ $r$-modes. Furthermore, the amplitude of the modulated oscillations is increased by increasing the inclination angle and/or mode’s altitude.

Finally, we note that during the evolution of the $r$-mode instability, one would expect the growth of the amplitude of the new modes. This feature can be understood from equation (23), which shows that both $\theta$ and $\phi$ trajectories of the $r$-mode are proportional to the mode’s amplitude, $\alpha$. Furthermore, changes in the star’s spin frequency in this period lead to changes in the frequency of modulated modes as well. In Figure 5 we plot the time evolution of the $\theta(t)$-displacement of the $l_A = m_A = 2$ $r$-mode, from $t = 0$ to $100P$, for different initial mode’s amplitudes $\alpha = 0.1$, $0.5$ (top panels) and $\alpha = 0.8$, $0.9$ (bottom panels). All panels correspond to $\dot{\mathcal{H}} / 2\mathcal{F} = 10^{-2}$, $\beta = \pi / 2$ and are plotted with initial conditions $\phi(0) = 0$ and $\theta(0) = \pi / 2$. By increasing $\alpha$, both the amplitude and the period of the modulated oscillations are increased. Furthermore, for $\alpha$ close to the saturation limit, $\alpha_{\text{sat}} = 1.0$, the behavior of the modulated oscillations changes drastically. Therefore, to analyze the problem carefully, one needs to solve both the nonlinear equation (23) and the equations governing the $r$-mode evolution together. This is currently under investigation (V. Rezania 2002, in preparation).

Although there is still no firm observational evidence for the low magnetic field strength in the LMXBs, these objects are believed to be neutron stars with weak magnetic fields ($B \lesssim 10^9$ G), high accretion rates ($\dot{M} \gtrsim 10^{-9} M_\odot$ yr$^{-1}$), and milliseconds periods (van der Klis 2000). Our calculations show that the existence of an inclined (even weak) magnetic field modulates the normal modes of an ocean fluid to a new mode with frequency $f_{\text{env}}$. Therefore, by observing normal modes in the neutron star’s ocean, one would expect to observe these new modes.

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