Gaugeon formalism for the two-form gauge fields

Masataka Aochi

Asahikawa Jitsugyo High School, Asahikawa, Hokkaido 071-8138, Japan

Ryusuke Endo*

Department of Physics, Yamagata University, Yamagata, 990-8560, Japan

Hikaru Miura

Yamagata Meisei High School, Yamagata 990-2322, Japan

Abstract

We present a BRST symmetric gaugeon formalism for the two-form gauge fields. A set of vector gaugeon fields is introduced as a quantum gauge freedom. One of the gaugeon fields satisfies a higher derivative field equation; this property is necessary to change the gauge-fixing parameter of the two-form gauge field. A naive Lagrangian for the vector gaugeon fields is itself invariant under a gauge transformation for the vector gaugeon field. The Lagrangian of our theory includes the gauge-fixing terms for the gaugeon fields and corresponding Faddeev–Popov ghosts terms.

* endo@sci.kj.yamagata-u.ac.jp
I. INTRODUCTION

The standard formalism of canonically quantized gauge theories [1–5] does not consider quantum-level gauge transformations. There is no quantum gauge freedom, since the quantum theory is defined only after the gauge fixing. Within the broader framework of Yokoyama’s gaugeon formalism [6], we can consider quantum gauge transformations as q-number gauge transformations. In this formalism, quantum gauge freedom is provided by an extra field, called a gaugeon field. The gaugeon formalism has been developed so far for various gauge fields, such as, Abelian gauge fields [6–11], non-Abelian gauge fields [12–19], Higgs models [20, 21], chiral gauge theories [22], Schwinger’s model [23], spin-3/2 gauge fields [24], string theories [25, 26], and gravitational fields [27, 28].

Recently, gaugeon formalisms for the Abelian two-form gauge fields are considered by Upadhyay and Panigrahi [29] (in the framework of the “very special relativity” [30]), and by Dwivedi [31]. They introduced a vector gaugeon field which would play a role of the quantum gauge freedom of the two-form gauge field. The vector gaugeon field itself has a property of gauge fields. It has a gauge invariance. In fact, the Lagrangians given in Refs.[29, 31] are invariant under the gauge transformation of the vector gaugeon field. So, we should fix the gauge before quantizing the vector gaugeon field. However, the authors of Refs.[29, 31] did not fix the gauge. Thus, their vector gaugeon field was not quantized. Namely, their theories are incomplete as a gaugeon formalism for the two-form gauge fields; they do not permit the quantum level gauge transformation, which is an essential ingredient of the gaugeon formalism.

The aim of this paper is quantizing the vector gaugeon field and obtaining a correct gaugeon theory for the two-form gauge field.

This paper is organized as follows. In sect. 2, we first review the standard formalism for the covariantly quantized two-form gauge field. Then, we show that the vector gaugeon field must be a massless dipole field, that is, its propagator have a term proportional to $1/(p^2)^2$. In sect. 3, we covariantly fix the gauge of the massless dipole vector field and quantize the system. In section 4, incorporating the massless dipole vector field as the gaugeon field, we present a correct gaugeon theory of the two-form gauge field. Section 5 is devoted to summary and comments.
II. STANDARD FORMALISM

A Faddeev-Popov quantization of the antisymmetric tensor gauge field (the two-form gauge field) [32, 33] was first performed by Townsend [34]. He revealed that the Faddeev-Popov (FP) ghosts themselves have gauge invariance and thus the ghosts for ghosts are necessary. His theory, however, violates unitarity because of inappropriate ghost contents. To ensure the unitarity, counting of ghosts should have been improved. The correct mode-counting was given by Kimura [35] and Siegel [36]. In the BRST quantization scheme [2–4], Kimura [35] has introduced a correct number of FP ghosts and auxiliary multiplier fields which form an off-shell nilpotent BRST symmetry. The unitarity of the theory is assured by Kugo-Ojima’s mechanism of BRST quartets [4, 5]. Kimura also gave canonically quantized theories of the antisymmetric tensor gauge fields of third rank [37] and of arbitrary rank [38]. In the path integral formalism, Siegel [36] gave the precise ghost counting by a careful application of the ’t Hooft averaging to the arbitrary rank antisymmetric tensor gauge fields. In this section, we review Kimura’s theory as a standard formalism.

The classical (gauge-unfixed) Lagrangian of a two-form gauge field $B_{\mu\nu}$ is given by

$$L_0 = \frac{1}{12} F_{\lambda\mu\nu} F_{\lambda\mu\nu},$$

(2.1)

where the third-rank antisymmetric tensor $F_{\lambda\mu\nu}$ is the field strength of $B_{\mu\nu}$ defined by

$$F_{\lambda\mu\nu} = \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}.$$  

(2.2)

The tensor $F_{\lambda\mu\nu}$ and thus the Lagrangian (2.1) are invariant under the gauge transformation

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu,$$

(2.3)

where $\Lambda_\mu$ is an arbitrary vector field. Thus, to obtain a quantized theory, we need gauge-fixing and appropriate ghosts and auxiliary fields. Note that the second term on the right hand side of (2.3) is invariant under a “gauge transformation” $\Lambda_\mu \rightarrow \Lambda_\mu + \partial_\mu \Lambda$ with an arbitrary scalar function $\Lambda$. This is the origin why we need ghosts for ghosts in the quantized theory of the antisymmetric tensor gauge theories.

---

1 Kimura’s Lagrangians were also given [39–41] by Bonora-Tonin’s superspace method [42] of the BRST symmetry.
2 See also Ref. [43, 44].
The quantum Lagrangian given by Kimura [35] is

\[ \mathcal{L}_K = \mathcal{L}_0 - \partial^\mu B^\nu B_{\mu\nu} - \frac{\alpha}{2} B^\mu B_\mu + B^\mu \phi_\mu \eta + \partial^\mu \phi_\mu \partial_\mu \phi \\
- \frac{i}{2} (\partial^\mu c_\nu - \partial^\nu c_\mu) (\partial_\rho c_\mu - \partial_\mu c_\rho) + ic_\mu \partial_\mu d + i \partial^\mu d_\mu + i \beta d_\nu d, \tag{2.4} \]

where \( \alpha \) and \( \beta \) are real parameters, \( B_\mu \) is (partly) a multiplier field imposing a gauge condition \( \partial^\mu B_{\mu\nu} = \alpha B_\nu + \cdots \) on \( B_{\mu\nu} \) as a field equation, \( c_\mu \) and \( c^\nu_\mu \) are FP ghosts, and scalar fields \( \phi, \phi^*, d, d^* \) and \( \eta \) play the roles of ghosts for ghosts or multiplier fields. One may expect these roles by observing the following BRST transformations under which Kimura’s Lagrangian (2.4) is invariant: \(^3\)

\[
\begin{align*}
\delta_B B_{\mu\nu} &= \partial_\mu c_\nu - \partial_\nu c_\mu, \\
\delta_B c_\mu &= -i \partial_\mu \phi, \\
\delta_B c^\nu_\mu &= i B_\mu, \\
\delta_B \phi^* &= d^*, \\
\delta_B \phi &= d, \\
\delta_B B_\mu &= \delta_B d^* = \delta_B d = \delta_B \phi = 0.
\end{align*} \tag{2.5} \]

These BRST transformations satisfy the off-shell nilpotency \( \delta_B^2 = 0 \). The corresponding BRST charge \( Q_{B(K)} \) can be written as

\[
Q_{B(K)} = \int \left[ B^\lambda \overset{\rightarrow}{\partial}_0 c_\lambda + d^* \overset{\rightarrow}{\partial}_0 \phi + (1 - \beta) B_0 d \right] d^{D-1}x, \tag{2.6} \]

where we consider in \( D \)-dimensional space-time, and \( \overset{\rightarrow}{\partial}_0 = \overset{\rightarrow}{\partial}_0 - \overset{\leftarrow}{\partial}_0 \). This charge is also nilpotent: \( Q_{B(K)}^2 = 0 \). Figure 1 shows the field contents and their BRST transformations. With these field contents, Kugo–Ojima’s quartet mechanism [4, 5] works and all the unphysical modes are removed by Kugo–Ojima’s physical subsidiary condition,

\[
Q_{B(K)}|_{\text{phys}} = 0. \tag{2.7} \]

Especially, the fields \( \eta \) and \( d \) are necessary in correct mode-counting; without these fields the longitudinal modes of \( B_\mu \) and \( c^\nu_\mu \) could not form a BRST quartet.

The field equations for the zero-ghost-number fields derived from (2.4) are

\[
\begin{align*}
\partial^\lambda F_{\lambda\mu\nu} + \partial_\mu B_\nu - \partial_\nu B_\mu &= 0, \tag{2.8} \\
\partial^\lambda B_\lambda &= \alpha B_\mu + \partial_\mu \eta, \tag{2.9} \\
\partial^\lambda B_\lambda &= 0. \tag{2.10}
\end{align*} \]

\(^3\) The Lagrangian (2.4) is also invariant under the anti-BRST transformation [39–41].
FIG. 1. Field contents and BRST transformations of Kimura’s theory. The arrows represent the directions of the BRST transformations. The fields of odd ghost numbers are fermionic, while those of even ghosts numbers bosonic.

from which we also have

\[ \Box B_\mu = \Box \eta = 0. \] (2.11)

We regard the equation (2.9) as the Lorenz-like gauge condition for the gauge field \( B_{\mu\nu} \) and \( \alpha \) as a gauge-fixing parameter. Now we consider a possibility to change the gauge-fixing parameter \( \alpha \) by an appropriate \( q \)-number gauge transformation, which would be given by

\[ B_{\mu\nu} \to \hat{B}_{\mu\nu} = B_{\mu\nu} + \tau (\partial_\mu Y_\nu - \partial_\nu Y_\mu), \] (2.12)

where the vector field \( Y_\mu \) is a would-be gaugeon field and \( \tau \) is a real parameter. One possibility is that \( Y_\mu \) satisfies

\[ \partial^\mu (\partial_\mu Y_\nu - \partial_\nu Y_\mu) = B_\nu, \] (2.13)

so that the gauge condition (2.9) transforms under (2.12) as

\[ \partial^\mu \hat{B}_{\mu\nu} - (\alpha + \tau) B_\nu + \partial_\nu \eta = 0. \] (2.14)

Thus the gauge-fixing parameter changes from \( \alpha \) to \( \alpha + \tau \). From (2.13) together with (2.10) we presume the field equation for the gaugeon field \( Y_\mu \) to be

\[ \Box \partial^\mu (\partial_\mu Y_\nu - \partial_\nu Y_\mu) = 0, \] (2.15)

which suggests that the gaugeon for the two-form \( B_{\mu\nu} \) would be a massless dipole field.
III. QUANTUM THEORY OF A MASSLESS DIPOLE VECTOR FIELD

A. classical theory

Here we consider the quantization of the massless dipole vector field $Y_\mu$, whose classical equation is given by (2.15). To avoid a higher derivative Lagrangian we imitate the Froissart model [45] describing a dipole scalar field. Simply generalizing the Froissart model to our case, we adopt
\[
\mathcal{L}_{vF0} = - \frac{1}{2} \left( \partial^\mu Y^\nu - \partial^\nu Y^\mu \right) \left( \partial_\mu Y_\nu - \partial_\nu Y_\mu \right) - \frac{\varepsilon}{2} Y_\mu^\mu Y^{*}_\mu
\] (3.1)
as a starting Lagrangian, where $\varepsilon$ is a sign factor $\varepsilon = \pm 1$, and $Y^{*}_\mu$ is an auxiliary vector field. We call this model a massless vector-Froissart model.\(^4\) The field equations derived from (3.1) are
\[
\partial^\mu \left( \partial_\mu Y_\nu - \partial_\nu Y_\mu \right) - \varepsilon Y^{*}_\nu = 0, \tag{3.2}
\]
\[
\partial^\mu \left( \partial_\mu Y^{*}_\nu - \partial_\nu Y^{*}_\mu \right) = 0, \tag{3.3}
\]
from which we also have
\[
\partial^\nu Y^{*}_\nu = 0, \tag{3.4}
\]
\[
\Box Y^{*}_\nu = 0. \tag{3.5}
\]
From (3.2) and (3.5) we obtain the desired equation for $Y_\mu$:
\[
\Box \partial^\mu \left( \partial_\mu Y_\nu - \partial_\nu Y_\mu \right) = 0. \tag{3.6}
\]

B. gauge fixing

To quantize the Lagrangian (3.1) we need appropriate gauge-fixing terms since the Lagrangian is invariant under the gauge transformation
\[
Y_\mu \rightarrow Y_\mu + \partial_\mu \Lambda,
\] (3.7)
where $\Lambda$ is an arbitrary scalar function. Our gauge fixed Lagrangian is
\[
\mathcal{L}_{vF0+GF} = \mathcal{L}_{vF0} + Y^{*}_\mu \partial_\mu Y + \partial^\mu Y^{*}_\mu Y_\mu + \beta^\mu Y_\mu Y, \tag{3.8}
\]
\(^4\) A brief report of the quantization of this model was given by one of the authors (M. A.) [46].
where \(Y_*\) and \(Y\) are scalar multiplier fields and \(\beta'\) is a gauge-fixing parameter. The field equations derived from (3.8) are

\[
\partial^\mu (\partial_\mu Y_\nu - \partial_\nu Y_\mu) - \varepsilon Y_{\nu\nu} + \partial_\nu Y = 0,
\]

(3.9)

\[
\partial^\mu (\partial_\mu Y_* - \partial_\mu Y_{*\mu}) + \partial_\nu Y_\mu = 0,
\]

(3.10)

\[
\partial^\mu Y_\mu = \beta' Y,
\]

(3.11)

\[
\partial^\mu Y_{*\mu} = \beta' Y_*,
\]

(3.12)

which lead to higher derivative field equations for \(Y_\mu\),

\[
\Box^2 Y_\nu + \left( \frac{1}{\beta'^2} - 1 \right) \Box \partial_\nu \partial^\mu Y_\mu = 0.
\]

(3.13)

The higher derivative of the field equations suggests higher pole propagators. In fact, we have

\[
\langle Y_\mu Y_\nu \rangle \sim \frac{\varepsilon (p^2)}{(p^2)^2} \left[ g_{\mu\nu} + (\beta'^2 - 1) \frac{p_{\mu} p_{\nu}}{p^2} \right].
\]

(3.14)

C. BRST symmetry

Because of the higher derivative field equations, the Fock space of the quantum theory derived from (3.8) is not positive definite. We must remove these unphysical modes from the theory. This was done for the scalar Froissart model by introducing BRST symmetry [19, 47]. We imitate here again the Froissart model with BRST symmetry.

We introduce vector FP ghosts \(K_\mu\) and \(K_{*\mu}\), together with scalar FP ghosts \(K\) and \(K_*\), and define our Lagrangian by

\[
\mathcal{L}_{VF} = -\frac{1}{2} (\partial^\mu Y_\nu - \partial^\nu Y_\mu)(\partial_\mu Y_\nu - \partial_\nu Y_\mu) - \frac{\varepsilon}{2} Y_*^{\mu\nu} Y_{*\mu} - \partial^\mu Y_* Y_\mu + Y_*^{\mu} \partial_\mu Y + \beta' Y_* Y
\]

\[
- \frac{i}{2} (\partial^\mu K_* - \partial^\nu K_{*\nu})(\partial_\mu K_\nu - \partial_\nu K_\mu) + i \partial^\mu K_* K_\mu + i K_*^{\mu} \partial_\mu K + i \beta' K_* K.
\]

(3.15)

Note that the first term on the second line is invariant under the “gauge transformations” \(K_\mu \rightarrow K_\mu + \partial_\mu \theta\) and \(K_{*\mu} \rightarrow K_{*\mu} + \partial_\mu \theta_*\) where \(\theta\) and \(\theta_*\) are arbitrary Grassmann odd functions. The remaining terms on the second line are the gauge-fixing terms for the gauge freedom; \(K_*\) and \(K\) play the role of the multiplier fields. These gauge-fixing terms are necessary for the vector FP ghosts to have propagators.
The arrows represent the directions of the BRST transformations.

The Lagrangian (3.15) is invariant under the BRST transformations,

\[ \delta_B Y_\mu = K_\mu, \quad \delta_B K_\mu = 0, \quad \delta_B K_{*\mu} = iY_{*\mu}, \quad \delta_B Y_{*\mu} = 0, \]
\[ \delta_B Y = K, \quad \delta_B K = 0, \quad \delta_B K_* = iY_*, \quad \delta_B Y_* = 0, \]  

which clearly satisfy the off-shell nilpotency \( \delta_B^2 = 0 \). The BRST invariance of (3.15) is easily confirmed when we rewrite (3.15) as

\[ \mathcal{L}_vF = i\delta_B \left[ \partial^\mu K_*^\nu \left( \partial_\nu Y_\mu - \partial_\mu Y_\nu \right) + \frac{\varepsilon}{2} K_*^\mu Y_{*\mu} - \partial^\mu K_* Y_\mu - K_*^\mu \partial_\mu Y - \beta' K_* Y \right] \]  

The corresponding BRST charge can be expressed by

\[ Q_{B(vF)} = \int \left[ Y_{*\mu} \partial_0 K_\mu + (1 - \beta')(Y_{*0} K - Y_0 K) \right] d^{D-1}x. \]

Figure 2 shows the field contents and their BRST transformations of the quantized vector-Froissart model. All of the unphysical modes are removed by Kugo–Ojima’s quartet mechanism; any physical states satisfying \( Q_{B(vF)}|_{\text{phys}} = 0 \) are zero-normed states.

**IV. GAUGEON FORMALISM**

**A. Lagrangian and field equations**

Combining the Lagrangians of Kimura’s theory (2.4) and the massless vector-Froissart model (3.15), we present the Lagrangian of the gaugeon formalism for the two-form gauge
field $B_{\mu\nu}$:

$$
\mathcal{L} = \mathcal{L}_K(\alpha = 0, \beta) + \mathcal{L}_\text{VF}(\beta' = \beta) + \frac{\varepsilon}{2} Y^\mu_\ast Y_{\ast\mu} - \frac{\varepsilon}{2} (Y^\mu_\ast + aB^\mu)(Y_{\ast\mu} + aB_\mu)$$

$$= \frac{1}{12} F^{\lambda\mu} F_{\lambda\mu} - \partial^\mu B^\nu B_{\mu\nu} + B^\mu \partial_\mu \eta + \partial^\mu \phi \partial_\mu \phi$$

$$- \frac{\varepsilon}{2} (Y^\mu_\ast + aB^\mu)(Y_{\ast\mu} + aB_\mu)$$

$$- \partial^\mu Y^\nu_\ast (\partial_\mu Y_\nu - \partial_\nu Y_\mu) + Y^\mu_\ast \partial_\mu Y + \partial^\mu Y_\ast Y_\mu + \beta Y_\ast Y$$

$$- i \partial^\mu c_\ast^\nu (\partial_\mu c_\nu - \partial_\nu c_\mu) + i c_\ast^\nu \partial_\nu d + i \partial^\mu d c_\mu + i\beta d_\nu d$$

$$- i \partial^\mu K^\nu_\ast (\partial_\mu K_\nu - \partial_\nu K_\mu) + i K^\mu_\ast \partial_\mu K + i\beta d K$$

(4.1)

where $a$ is a real parameter. The third and fourth terms of the right-hand-side of (4.1) show that the term $- (\varepsilon/2) Y^\mu_\ast Y_{\ast\mu}$ in $\mathcal{L}_\text{VF}$ has been replaced by $- (\varepsilon/2) (Y^\mu_\ast + aB^\mu)(Y_{\ast\mu} + aB_\mu)$. As seen later, the gauge-fixing parameter $\alpha$ of Kimura’s theory (2.4) can be identified through the parameter $a$ as

$$\alpha = \varepsilon a^2.$$  

(4.3)

The field equations derived from (4.2) are

$$\partial^\lambda F_{\lambda\mu\nu} + \partial_\mu B_\nu - \partial_\nu B_\mu = 0,$$

$$\partial^\mu B_{\mu\nu} + \partial_\nu \eta - \varepsilon a (Y_{\ast\nu} + aB_\nu) = 0,$$

$$\partial^\mu B_\mu = \Box \phi = \Box \phi_\ast = 0,$$

$$\partial^\mu (\partial_\mu Y_\nu - \partial_\nu Y_\mu) - \varepsilon (Y_{\ast\nu} + aB_\nu) + \partial_\nu Y = 0, \quad \partial^\mu Y_\mu = \beta Y$$

$$\partial^\mu (\partial_\mu Y_{\ast\nu} - \partial_\nu Y_{\ast\mu}) + \partial_\nu Y_\ast = 0, \quad \partial^\mu Y_{\ast\mu} = \beta Y_\ast,$$

(4.4)

for bosonic fields, and

$$\partial^\mu (\partial_\mu c_\nu - \partial_\nu c_\mu) + \partial_\nu d = 0, \quad \partial^\mu c_\mu = \beta d,$$

$$\partial^\mu (\partial_\mu c_{\ast\nu} - \partial_\nu c_{\ast\mu}) + \partial_\nu d_{\ast} = 0, \quad \partial^\mu c_{\ast\mu} = \beta d_{\ast},$$

$$\partial^\mu (\partial_\mu K_\nu - \partial_\nu K_\mu) + \partial_\nu K = 0, \quad \partial^\mu K_\mu = \beta K,$$

$$\partial^\mu (\partial_\mu K_{\ast\nu} - \partial_\nu K_{\ast\mu}) + \partial_\nu K_{\ast} = 0, \quad \partial^\mu K_{\ast\mu} = \beta K_{\ast},$$

(4.5)

for fermionic fields. We emphasize here that we have chosen the gauge-fixing parameter $\beta'$ of the vector-Froissart fields $Y_\mu$ and $Y_{\ast\mu}$ as $\beta' = \beta$. As a result, four pairs of FP ghosts $(c_\mu, d), (c_{\ast\mu}, d_{\ast}), (K_\mu, K)$ and $(K_{\ast\mu}, K_{\ast})$, as well as $(Y_{\ast\mu}, Y_\ast)$, satisfy the same field equations.
FIG. 3. Field contents and BRST transformations of the gaugeon formalism. The arrows show the BRST transformations. The fields in the parentheses represent corresponding fields of Refs. [29, 31]; there are no counterparts of our fields \( Y, Y^*, K, \) and \( K^* \). Instead, two BRST-singlet fields \( Z \) and \( Z^* \) were introduced as ghosts for ghosts in [29, 31].

B. BRST symmetry

The Lagrangian (4.2) is invariant under the BRST transformations which are defined by

\[
\begin{align*}
\delta_B B_{\mu\nu} &= \partial_\mu c_\nu - \partial_\nu c_\mu, \\
\delta_B c_\mu &= -i\partial_\mu \phi, \\
\delta_B c_{*\mu} &= iB_\mu, \\
\delta_B \phi &= d_*, \\
\delta_B \eta &= d, \\
\delta_B B_\mu &= \delta_B d_* = \delta_B d = \delta_B \phi = 0.
\end{align*}
\]

(4.6)

for the fields of the standard formalism sector and

\[
\begin{align*}
\delta_B Y_\mu &= K_\mu, \\
\delta_B K_\mu &= 0, \\
\delta_B K_{*\mu} &= iY_{*\mu}, \\
\delta_B Y_{*\mu} &= 0, \\
\delta_B Y &= K, \\
\delta_B K &= 0, \\
\delta_B K_* &= iY_*, \\
\delta_B Y_* &= 0.
\end{align*}
\]

(4.7)

for the fields of gaugeon sector. Field contents and their BRST transformations are shown in Figure 3. (Fields introduced in Refs. [29, 31] are also shown in the figure for comparison.) Because of the off-shell nilpotency \( \delta_B^2 = 0 \), the BRST invariance of the Lagrangian is easily
understood when we rewrite (4.2) as

\[
\mathcal{L} = \frac{1}{12} F^{\lambda \mu \nu} F_{\lambda \mu \nu} + i \delta_B \left[ \partial^\mu c_\nu^\alpha (\partial_\nu B_\mu - \partial_\mu B_\nu) - c^\mu_\nu \partial_\mu \eta + \partial^\mu \phi_\nu c_\mu - \beta \delta_2 \frac{\varepsilon}{2} (K^\mu_\nu + ac^\mu_\nu) (Y^\nu_\mu + aB_\mu) + \partial^\mu K^\nu_\nu (\partial_\nu Y_\mu - \partial_\mu Y_\nu) - K^\mu_\nu \partial_\mu Y + \partial^\mu K^\mu_\nu - \beta K^\nu Y \right].
\]

(4.8)

The corresponding BRST charge \( Q_B \) can be written as

\[
Q_B = \int \left[ B^\lambda \overleftarrow{\partial_0} c_\lambda + d_\star \overleftarrow{\partial_0} \phi + (1 - \beta) B_0 d \right. \\
\left. + Y^\lambda_\star \overleftarrow{\partial_0} K_\lambda + (1 - \beta) (Y_0 K - Y_\star K_0) \right] d^{D-1}x.
\]

(4.9)

With the help of the charge, we can define the physical subspace of the Fock space by

\[
\mathcal{V}_{\text{phys}} = \ker Q_B = \{ |\Phi \rangle; Q_B |\Phi \rangle = 0 \}.
\]

(4.10)

C. \( q \)-number gauge transformations

The Lagrangian (4.2) permits the \( q \)-number gauge transformation where we vary the gauge-fixing parameter \( a \). Under the field redefinitions

\[
\hat{B}_{\mu \nu} = B_{\mu \nu} + \tau (\partial_\mu Y_\nu - \partial_\nu Y_\mu), \quad \hat{c}_\mu = \hat{c}_\mu + \tau K_\mu,
\]
\[
\hat{Y}_{\star \mu} = Y_{\star \mu} - \tau B_\mu, \quad \hat{K}_{\star \mu} = K_{\star \mu} - \tau c_{\star \mu},
\]
\[
\hat{B}_\mu = B_\mu, \quad \hat{Y}_\mu = Y_\mu, \quad \hat{c}_{\star \mu} = c_{\star \mu}, \quad \hat{K}_\mu = K_\mu,
\]
\[
\hat{\eta} = \eta + \tau Y, \quad \hat{d} = d + \tau K,
\]
\[
\hat{Y}_\star = Y_\star, \quad \hat{K}_\star = K_\star - \tau d_\star,
\]
\[
\hat{Y} = Y, \quad \hat{d}_\star = d_\star, \quad \hat{K} = K, \quad \hat{\phi} = \phi, \quad \hat{\phi}_\star = \phi_\star
\]

(4.11)

with \( \tau \) being a real parameter, the Lagrangian (4.2) becomes

\[
\mathcal{L}(\Phi_A; a, \beta) = \mathcal{L}(\hat{\Phi}_A; \hat{a}, \beta),
\]

(4.12)

where \( \Phi_A \) collectively represents all fields and \( \hat{a} \) is defined by

\[
\hat{a} = a + \tau.
\]

(4.13)
The form invariance (4.12) concludes that the field equations transform gauge covariantly under the $q$-number gauge transformations (4.11): $\hat{\Phi}_A$ satisfies the same field equation as $\Phi_A$ if the parameter $a$ replaced by $\hat{a}$.

It should be noted that the $q$-number gauge transformations (4.11) commute with the BRST transformations (4.6) and (4.7). As a result, the BRST charge is invariant under the $q$-number gauge transformations:

$$\hat{Q}_B = Q_B. \quad (4.14)$$

The physical subspace $V_{\text{phys}}$ is, therefore, also invariant under the $q$-number gauge transformation:

$$\hat{V}_{\text{phys}} = V_{\text{phys}}. \quad (4.15)$$

D. gauge structure of the Fock space

In addition to the BRST charge (4.9), the Lagrangian (4.2) has several conserved BRST-like charges. We focus on here the following three charges:

$$Q_{B(K)} = \int \left[ B^\mu \partial_0 c_\mu + d_\nu \partial_0 \phi + (1 - \beta)B_0 d \right] d^{D-1}x, \quad (4.16)$$

$$Q_{B(\nu F)} = \int \left[ Y^\mu \partial_0 K_\mu + (1 - \beta')(Y_0 K - Y_0 K_0) \right] d^{D-1}x, \quad (4.17)$$

$$\tilde{Q}_B = \int \left[ B^\mu \partial_0 K_\mu + (1 - \beta')B_0 K \right] d^{D-1}x, \quad (4.18)$$

which are nilpotent and anticommuting with each other:

$$Q_{B(K)}^2 = Q_{B(\nu F)}^2 = Q_B^2 = 0,$$

$$\{Q_{B(K)}, Q_{B(\nu F)}\} = \{Q_{B(\nu F)}, \tilde{Q}_B\} = \{\tilde{Q}_B, Q_{B(K)}\} = 0. \quad (4.19)$$

The $Q_{B(K)}$ generates the BRST transformation (4.6) acting only on the fields of the standard formalism sector, while $Q_{B(\nu F)}$ generates the transformation (4.7) acting only on the fields of the gaugeon sector. The total BRST charge (4.9) can be expressed as

$$Q_B = Q_{B(K)} + Q_{B(\nu F)}. \quad (4.20)$$

The charge $\tilde{Q}_B$ (4.18) generates the transformation $\tilde{\delta}_B$:

$$\tilde{\delta}_B B_{\mu\nu} = \partial_\mu K_\nu - \partial_\nu K_\mu, \quad \tilde{\delta}_B K_{\mu\nu} = iB_{\mu\nu}, \quad \tilde{\delta}_B \eta = K,$$

$$\tilde{\delta}_B (\text{other fields}) = 0. \quad (4.21)$$
In (4.10) we have defined the physical subspace \( V_{\text{phys}} \) using the charge \( Q_B \). Instead, we may consider another subspace by

\[
V^{(a)}_{\text{phys}} = \ker Q_B(K) \cap \ker Q_B(vF) = \{ \langle \Phi \rangle; Q_B(K)|\Phi\rangle = Q_B(vF)|\Phi\rangle = 0 \}. \tag{4.22}
\]

The condition \( Q_B(K)|\Phi\rangle = 0 \) removes the unphysical modes included in the standard formalism sector, while the condition \( Q_B(vF)|\Phi\rangle = 0 \) removes the modes of the gaugeon sector. As is easily seen, the space \( V^{(a)}_{\text{phys}} \) is a subspace of \( V_{\text{phys}} \):

\[
V^{(a)}_{\text{phys}} \subset V_{\text{phys}}. \tag{4.23}
\]

We have attached the index \( (a) \) to \( V^{(a)}_{\text{phys}} \) to emphasize that its definition depends on the gauge-fixing parameter \( a \). In fact, under the \( q \)-number gauge transformation (4.11), the BRST charges \( Q_B(K) \) and \( Q_B(vF) \) transform as

\[
\hat{Q}_B(K) = Q_B(K) + \tau \tilde{Q}_B, \quad \hat{Q}_B(vF) = Q_B(vF) - \tau \tilde{Q}_B, \tag{4.24}
\]

while their sum \( Q_B \) remains invariant.

Let us define a subspace \( \mathcal{V}^{(a)} \) of the total Fock space by

\[
\mathcal{V}^{(a)} = \ker Q_B(vF). \tag{4.25}
\]

This space can be identified with the total Fock space of the standard formalism in the \( \alpha = \varepsilon a^2 \) gauge. We can understand this by rewrite the Lagrangian (4.2) as,

\[
\mathcal{L} = \mathcal{L}_K(\alpha = \varepsilon a^2) + i\{Q_B, \Theta\}, \tag{4.26}
\]

where \( \Theta \) being given by

\[
\Theta = \varepsilon \frac{K^\mu Y_{*\mu} + 2aB_\mu}{\partial^\mu K^\nu (\partial_\mu Y_\nu - \partial_\nu Y_\mu)} - K^\mu \partial_\mu Y + \partial^\mu K_\mu Y - \beta K_\mu Y. \tag{4.27}
\]

The first term of (4.26) corresponds to the Lagrangian of the standard formalism (2.4). The second term becomes null-operator in the subspace \( \mathcal{V}^{(a)} \). Namely, we can ignore the second term of (4.26) in \( \mathcal{V}^{(a)} \).

We emphasize that the same arguments hold if we start from the \( q \)-number transformed charges (4.24) rather than \( Q_B(K) \) and \( Q_B(vF) \). We define the subspaces \( \mathcal{V}^{(a+\tau)} \) and \( \mathcal{V}^{(a+\tau)}_{\text{phys}} \) by

\[
\mathcal{V}^{(a+\tau)} = \ker \hat{Q}_B(vF), \quad \mathcal{V}^{(a+\tau)}_{\text{phys}} = \ker \hat{Q}_B(K) \cap \ker \hat{Q}_B(vF). \tag{4.28}
\]
The space $\mathcal{V}^{(a+\tau)}$ can be identified with the Fock space of the standard formalism in the $\alpha = \varepsilon(a + \tau)^2$ gauge, and $\mathcal{V}_{\text{phys}}^{(a+\tau)}$ corresponds to its physical subspace. Thus various Fock spaces of the standard formalism in different gauges are embedded in the single Fock space of the present theory.

V. SUMMARY AND COMMENTS

We have presented the BRST symmetric gaugeon formalism for the two-form gauge theory. For this purpose, we have covariantly quantized the massless vector-Froissart model (a dipole vector gauge theory), as vector gaugeon fields of our theory. Since this model has gauge invariance at the classical level, we have first considered gauge-fixing for the model; the necessity of the gauge-fixing for the gaugeon fields was overlooked in the previous literature [29, 31]. Using three kinds of BRST charges as well as the total BRST charge, we have shown that our total Fock space contains the subspaces which are identified with the Fock spaces of the standard formalism in various gauges.

In the following, we add some comments.

A. Type II theory

In the theory presented in the last section, we can change the value of the gauge-fixing parameter $a$ by the $q$-number gauge transformation (4.11). The gauge-fixing parameter $\alpha$ of the standard formalism (2.4) is identified with $\alpha = \varepsilon a^2$ ($\varepsilon = \pm 1$). This means that we cannot change the sign of the parameter of the standard parameter $\alpha$ by the $q$-number transformation. The situation is analogous to Type I gaugeon theory for QED [7]. There are two types of gaugeon theories, Type I and Type II. The gauge-fixing parameter $a$ can be shifted as $\hat{a} = a + \tau$ by the $q$-number gauge transformation in both theories. The standard gauge-fixing parameter $\alpha$ is expressed as $\alpha = \varepsilon a^2$ in Type I theory, and $\alpha = a$ in Type II theory; the sign of $\alpha$ can be changed in Type II theory. We comment here that Type II theory can also be formulated for the two-form gauge fields.

We consider the Lagrangian, rather than (4.1),

$$L_{\text{II}} = L_K(\alpha = a; \beta') + L_{\nu F}(\beta' = \beta) + \frac{\varepsilon}{2} Y^\mu Y_{\ast \mu} - \frac{1}{2} Y^\mu B_\mu. \quad (5.1)$$
Under the $q$-number transformation (4.11), this Lagrangian is also form invariant:

$$\mathcal{L}_{II}(\Phi_A; a, \beta) = \mathcal{L}_{II}(\hat{\Phi}_A; \hat{a}, \beta)$$

(5.2)

with $\hat{a} = a + \tau$. The standard gauge-fixing parameter $\alpha$ can be identified with

$$\alpha = a$$

(5.3)

in the present case, thus we can change the parameter $\alpha$ quite freely without any limitation for the sign of $\alpha$.

The Lagrangian (5.1) is also invariant under all of the transformations corresponding to the BRST charges (4.9), (4.16), (4.17), and (4.18). Thus, the similar arguments to those in the last section on the gauge structure of the Fock space are also available. For example, $\mathcal{V}^{(a)} = \ker Q_{B(F)} \left[ \mathcal{V}^{(a+\tau)} = \ker \hat{Q}_{B(F)} \right]$ is identified with the Fock space of the standard formalism in $\alpha = a \ [\alpha = a + \tau]$ gauge.

**B. gaugeons for gaugeons**

The standard formalism (2.4) has two gauge-fixing parameters $\alpha$ and $\beta$. As seen in the last section (and in the last subsection), the value of the parameter $\alpha$ can be changed by the $q$-number gauge transformation (4.11), while the value of $\beta$ cannot. One might attempt to find a $q$-number transformation which can change the value of the parameter $\beta$, the gauge-fixing parameter for the FP ghosts $c_\mu$ and $c^*_\mu$. Let us consider this possibility here.

To introduce the $q$-number gauge transformation for the vector FP ghosts, we would need ghost-number $\pm 1$ gaugeon fields (gaugeons for ghosts) and their FP ghosts (ghosts for gaugeons for ghosts); the FP ghosts have $\pm 2$ ghost numbers and thus might be identified with the fields $Z$ and $Z^*$ introduced in Refs.[29, 31]. Furthermore, remembering that $\beta (= \beta')$ is a gauge-fixing parameter also for the gaugeon fields $Y_\mu$ and $Y^*_\mu$, we would need zero-ghost-number gaugeons (gaugeons for gaugeons) too, and their FP ghosts (ghosts for gaugeons for gaugeons). An early attempt of this program is seen in Ref. [48].

[1] N. Nakanishi, Prog. Theor. Phys. Suppl.51, 1 (1972).

[2] T. Kugo and I. Ojima, Phys. Lett. B 73, 459 (1978).
[3] T. Kugo and I. Ojima, Prog. Theor. Phys. 60, 1869 (1978).
[4] T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl. 66, 1 (1979).
[5] T. Kugo, Quantum Theory of Gauge Field I, II (Baifukan, Tokyo, 1989), [in Japanese].
[6] K. Yokoyama, Prog. Theor. Phys. 51, 1956 (1974).
[7] K. Yokoyama and R. Kubo, Prog. Theor. Phys. 52, 290 (1974).
[8] K. Izawa, Prog. Theor. Phys. 88, 759 (1992).
[9] M. Koseki, M. Sato, and R. Endo, Prog. Theor. Phys. 90, 1111 (1993).
[10] R. Endo, Prog. Theor. Phys. 90, 1121 (1993).
[11] T. Saito, R. Endo, and H. Miura, Prog. Theor. Exp. Phys. 2016, 023B02 (2016).
[12] K. Yokoyama, Prog. Theor. Phys. 59, 1699 (1978).
[13] K. Yokoyama, M. Takeda, and M. Monda, Prog. Theor. Phys. 60, 927 (1978).
[14] K. Yokoyama, Prog. Theor. Phys. 60, 1167 (1978).
[15] K. Yokoyama, Phys. Lett. B 79, 79 (1978).
[16] K. Yokoyama, M. Takeda, and M. Monda, Prog. Theor. Phys. 64, 1412 (1980).
[17] M. Abe, The Symmetries of the Gauge-Covariant Canonical Formalism of Non-Abelian Gauge Theories, Master Thesis, Kyoto University, 1985.
[18] M. Koseki, M. Sato, and R. Endo, Bull. of Yamagata Univ., Nat. Sci. 14, 15 (1996) (doi:10.15022/00003947) [Errata: R. Endo, Bull. of Yamagata Univ., Nat. Sci. 18, 33 (2017) (doi:10.15022/00004109)].
[19] S. Sakoda, Prog. Theor. Phys. 117, 745 (2007).
[20] K. Yokoyama and R. Kubo, Prog. Theor. Phys. 54, 848 (1975).
[21] H. Miura and R. Endo, Prog. Theor. Phys. 117, 695 (2007).
[22] K. Yokoyama, S. Yamagami, and R. Kubo, Prog. Theor. Phys. 54, 1532 (1975).
[23] Y. Nakawaki, Prog. Theor. Phys. 98, 1193 (1997).
[24] R. Endo and M. Koseki, Prog. Theor. Phys. 103, 685 (2000).
[25] M. Faizal, Commun. Theor. Phys. 57, 637 (2012).
[26] M. Faizal, Mod. Phys. Lett. A27, 1250147 (2012).
[27] S. Upadhyay, Ann. Phys. 344, 290 (2014).
[28] S. Upadhyay, Eur. Phys. J. C 74 2737(2014).
[29] S. Upadhyay and P. K. Panigrahi, Nucl. Phys. B 915, 168 (2017).
[30] A. G. Cohen and S. L. Glashow, Phys. Rev. Lett. 97, 021601 (2006).
[31] M. K. Dwivedi, Int. J. Theor. Phys. (2017). doi:10.1007/s10773-017-3302-1.

[32] K. Hayashi, Phys. Lett. B 44, 497 (1973).

[33] M. Kalb and P. Ramond, Phys. Rev. D 9, 2273 (1974).

[34] P. K. Townsend, Phys. Lett. B 88, 97 (1979).

[35] T. Kimura, Prog. Theor. Phys. 64, 357 (1980).

[36] W. Siegel, Phys. Lett. B 93, 170 (1980).

[37] T. Kimura, J. Phys. A: Math. Gen. 13, L353 (1980).

[38] T. Kimura, Prog. Theor. Phys. 65, 338 (1981).

[39] Marchetti and Tonin, Il Nuovo Cimento A63, 459 (1981).

[40] S. Kawasaki and T. Kimura, Prog. Theor. Phys. 70, 1436 (1983).

[41] J. Thierry-Mieg and L. Baulieu, Nucl. Phys. B 228, 259 (1983).

[42] L. Bonora and M. Tonin, Phys. Lett. B 98, 48 (1981).

[43] Yu. N. Obukhov, Theor. Math. Phys. 50, 229 (1982).

[44] Yu. N. Obukhov, Phys. Lett. B 109, 195 (1982).

[45] M. Froissart, Nuovo Cim. Suppl. 14, 197 (1959).

[46] M. Aochi, Soryushiron Kenkyu 105, E25 (2004), [in Japanese].

[47] T. Kashiwa, Prog. Theor. Phys. 70, 1124 (1983).

[48] M. Aochi, Gaugeon Formalism for the Second-Rank Antisymmetric Tensor Gauge Field, [in Japanese], Master Thesis, Yamagata University, 2004.