Does the fivebrane have a nonclassical BV-structure?

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Abstract

The fivebrane in M-theory comes equipped with a higher order gauge field which should have a formulation in terms of a 2-gerbe on the fivebrane. One can pose the question if the BV-quantization scheme for such a higher order gauge theory should differ from the usual BV-algebra structure. We give an algebraic argument that this should, indeed, be the case and a fourth order equation should appear as Master equation, in this case. We also discover a second order term in this equation which seems to indicate that deformation theory (i.e. solving the Master equation) in this case involves a nonlinear algebraic theory which goes beyond complexes and cohomology.

1 Introduction

The fivebrane in M-theory comes equipped with a higher order gauge field which - due to the coupling to the 4-form G-flux of eleven dimensional supergravity - should be given by a 2-gerbe on the fivebrane. In the case of usual gauge theories, given by principal bundles on manifolds, the BV-quantization scheme can, roughly speaking, be seen as considering the complex and cohomology - with values in the space of functions on connections - of the Lie algebra of the gauge group. It is therefore of no surprise that the general
structure of BRST-cohomology resembles Lie algebra cohomology (or also group cohomology): One has a differential, a Gerstenhaber bracket (called the BV-antibracket), and a Maurer-Cartan equation (called the Master equation, in this context). One can now pose the question if this general structure might be different for the quantization of the higher gauge theory on the fivebrane. Of course, quantization of the degrees of freedom of the fivebrane is a deep and completely unsolved issue. We do certainly not claim to present any solution to this important problem, here. What we do is studying, only, a deformation question for a general algebraic structure which should be implied by a 2-gerbe formulation of the higher gauge field. Then we calculate the analogue of the Maurer-Cartan equation for the deformation theory of this algebraic structure. We find a fourth order equation which might be seen as a hint that a new BV-structure could emerge in the M5-brane case. We also find a new second order term which seems to indicate that the deformation theory linked to this higher order Master equation is not determined by cohomology alone but involves a nonlinear algebraic level beyond complexes and cohomology.

In section 2, we study the algebraic deformation problem in the context of monoidal bicategories. In section 3, we show that the general algebraic structure which we find in section 2 is stable under passing to the more general setting of the deformation theory of enriched categories. Section 4 contains some concluding remarks.

For a different approach to the deformation theory of monoidal bicategories - which is, especially, structurally not stable under the generalization to the setting of enriched categories and considers a slightly different “gauge freedom” for monoidal bicategories - we would like to refer to [Elg 2002, Elg 2003].

## 2 Deformation theory of monoidal bicategories

Suppose an M5-brane with world-volume $X$ together with its higher order gauge theory, in the form of a 2-gerbe on $X$, is given. Whatever the details of a precise formulation of this 2-gerbe might be, can one say something about the general algebraic structure which should appear for its local gauge transformations? Since it seems to be clear that the 2-gerbe should be related to a kind of principal bundle with bicategories as fibers, one has to expect that the
gauge symmetry is given by a monoidal bicategory (see [GPS] for the technical definition) instead of a group. Now, as indicated in the introduction, it is basically the fact that we have a gauge group for a usual gauge theory with principal bundle which is responsible for the structure of a BV-algebra appearing for the BV-quantization scheme, i.e. already the deformation theory of groups leads one to the correct general algebraic structure for the BV-quantization scheme. It is therefore tempting to speculate that quantization of the higher gauge theory on the M5-brane should - concerning the general algebraic structure - be related to the deformation problem for monoidal bicategories. We will, in this paper, study the question if BV-quantization of a 2-gerbe gauge theory might involve a nonclassical Master equation from precisely this point of view.

What are the relevant structures which one can deform, in this case? A bicategory consists of objects, 1-morphisms, and 2-morphisms and two types of compositions. For a monoidal bicategory there is, in addition, a third product structure given by $\otimes$ as a functor of bicategories. In a bicategory not only the associativity of $\otimes$ can be weak (as in a monoidal category) but also associativity of the composition of 1-morphisms needs only to hold up to 2-isomorphisms. So, we have two types of associators involved. In addition, the exchange rule for $\otimes$ with a product $\bullet$

$$(a \otimes b) \bullet (c \otimes d) = (a \bullet c) \otimes (b \bullet d)$$

needs only to hold in the weak form for the exchange of $\otimes$ with the composition of 1-morphisms (for the details of the weak rules, see, again [GPS]). So, we have a new kind of “weak exchange morphism” appearing in addition to the two associators for monoidal bicategories.

Suppose for a moment that all structures would be strict, i.e. we would have strict associativity for all three products and the strict exchange rule $\otimes$ would hold for all pairs of products. The deformation theory of a strict monoidal category was studied in [GPS] and it was discovered there that the Maurer-Cartan equation is replaced by a system of three coupled differential equations, consisting of two Maurer-Cartan equations (for the associativity of the two products) plus a constraint (for the exchange rule). The structure of the deformation complex receives an additional ingredient beyond the Gerstenhaber bracket which resembles a curvature tensor on the complex. So, one gets a kind of nonlinear complex for the deformation theory.

It is straightforward to see that this structure generalizes to the case of a strict monoidal bicategory (also called a monoidal 2-category in the
literature). One gets a coupled system of three Maurer-Cartan equations plus constraints for all the exchange rules, in this case. The general structure of the deformation complex remains very much of the same type as found for strict monoidal categories. For monoidal categories, one can exclude the case of a nontrivial associator for $\otimes$ and the corresponding deformation theory for the associator because it can always be gauged away by Mac Lane’s coherence theorem, i.e. the deformation theory of an associator should contribute trivially to the general deformation theory of a monoidal category. In physical terms one has learned to see the existence of a coherence theorem for the weak structure which allows to gauge transform to the strict one as saying that the additional data of the weak structure arise from BRST-exact terms, only (compare e.g. to the $A_\infty$-structures appearing in topological string theory).

For the case of monoidal bicategories it is known that a coherence theorem in this sense does not exist but one has only equivalence of a general monoidal bicategory to a so called semistrict monoidal bicategory (instead of equivalence to a strict monoidal bicategory, see GPS). Roughly speaking, this means that we can always gauge away the two associators involved in the definition of a monoidal bicategory but we can, in general, not gauge away the “weak exchange morphism”. It turns out that it suffices to have a “weak exchange morphism” for $\Pi$ with $b$ and $c$ identities which is what is called a cubic functor in the literature (see GPS). We will write this down in more detail, below.

So, for monoidal bicategories we have to include the deformation theory of the cubic functor, in addition to the deformation theory of a strict monoidal bicategory because this should lead to non-BRST-exact contributions which should therefore be relevant to the structure on observables. Since the deformation theory of a strict monoidal bicategory would basically mean redoing GS, we exclude this case, here, and restrict to the cubic functor, alone. So, the deformation problem which we consider is the problem of deforming a semistrict monoidal bicategory into another semistrict monoidal bicategory such that $\otimes$ and all compositions of morphism remain fixed and the cubic functor is deformed into a new one.

**Remark 1** Monoidal bicategories can be seen as tricategories with one object. So, the deformation problem considered, here, is precisely the deformation problem which was suggested in Sch to be linked to the study of the
universality question of noncommutative field theories (passing to the case of more than one object does not make a difference for the deformation theory). A semistrict tricategory is sometimes also called a Gray-category (category enriched over the monoidal category Gray of 2-categories with the Gray-tensor product - see [GPS] - which induces the nontrivial cubic functors). So, a semistrict monoidal bicategory is a Gray-category with one object.

Let us now write down the data of a cubic functor in a little bit more detail (for the full technical definition, see [GPS] or the nice review included in [Lau]). Let

\[ f : A \to A' \]

and

\[ g : B \to B' \]

be 1-morphisms in the semistrict monoidal bicategory \( \mathcal{C} \). Then there should always be given a 2-isomorphism \( K_{f,g} \)

\[ K_{f,g} : (f \otimes 1_{B'}) (1_A \otimes g) \to (1_{A'} \otimes g) (f \otimes 1_B) \]

where juxtaposition means composition of 1-morphisms. The whole family \( K_{f,g} \) gives the central part of the data of a cubic functor. From the conditions imposed on \( K_{f,g} \), we will keep only one for the deformation theory. The reason is that Hochschild or also Hopf algebra cohomology shows that only the constraints on product like structures - like associativity or coassociativity - have to be imposed while e.g. constraints on unital elements come along, automatically. We will, heuristically, proceed as if this would also be true for the cubic functor structure. But the analogous theorems are an open problem for future work, in this case.

Denoting horizontal composition of 2-morphisms by \( \circ \) and vertical composition by juxtaposition, the constraint reads as

\[
\begin{align*}
(K_{f',g'} K_{f,g}) \circ (K_{f,g}) & = (K_{f,g} \circ K_{f',g'}) (K_{f',g'} \circ K_{f,g}) \\
& = K_{f',g'}
\end{align*}
\]

We will now assume that \( \mathcal{C} \) is a \( \mathbb{C} \)- or \( \mathbb{R} \)-linear, semistrict monoidal bicategory, i.e. all morphism spaces are \( (\mathbb{C} \text{- or } \mathbb{R} \text{-}) \) vector spaces and both compositions of morphisms, as well, as \( \otimes \) are assumed to be bilinear. We proceed in
the same way, then, as e.g. in the case of the usual deformation theory of an associative algebra. For simplicity, we will always neglect the fact that actually one passes to a category of formal power series with coefficients in \( \mathcal{C} \) in the deformation theory.

We replace the cubic functor \( K_{f,g} \) by a new cubic functor \( \widetilde{K}_{f,g} \) with

\[
\widetilde{K}_{f,g} = K_{f,g} + \Psi_{f,g}
\]

(3)

where \( \Psi_{f,g} \) is a collection of 2-morphisms in \( \mathcal{C} \) (observe that we assume both compositions in \( \mathcal{C} \), as well, as \( \otimes \) to be fixed in the process of the deformation). Next, we want to calculate the analogue of the Maurer-Cartan equation for \( \Psi_{f,g} \).

By inserting (3) into (2) written for \( \widetilde{K}_{f,g} \), we get in 0-th order the original equation (2) for \( K_{f,g} \). In first order, we get for the left hand side

\[
(K_{f',g'} K_{f',g}) \circ (K_{f,g} \Psi_{f,g}) + (K_{f',g'} K_{f',g}) \circ (\Psi_{f,g} K_{f,g})
\]

\[
+ (K_{f',g'} \Psi_{f',g}) \circ (K_{f,g} K_{f,g}) + (\Psi_{f',g'} K_{f,g}) \circ (K_{f,g} K_{f,g})
\]

and for the right hand side

\[
(K_{f',g'} \circ K_{f,g}) (K_{f',g} \circ \Psi_{f,g}) + (K_{f',g'} \circ K_{f,g}) (\Psi_{f,g} \circ K_{f,g})
\]

\[
+ (K_{f',g'} \circ \Psi_{f,g}) (K_{f',g} \circ K_{f,g}) + (\Psi_{f,g} \circ K_{f',g}) (K_{f',g} \circ K_{f,g})
\]

where both have to be equal to

\( \Psi_{f',g'} \)

For the second order terms, we calculate for the left hand side

\[
(K_{f',g'} K_{f',g}) \circ (\Psi_{f,g} \Psi_{f,g}) + (K_{f',g'} \Psi_{f,g}) \circ (K_{f,g} \Psi_{f,g})
\]

\[
+ (K_{f',g'} \Psi_{f',g}) \circ (\Psi_{f,g} K_{f,g}) + (\Psi_{f',g'} K_{f,g}) \circ (K_{f,g} \Psi_{f,g})
\]

\[
+ (\Psi_{f',g'} K_{f',g}) \circ (\Psi_{f,g} K_{f,g}) + (\Psi_{f',g'} \Psi_{f',g}) \circ (K_{f,g} K_{f,g})
\]

and for the right hand side

\[
(K_{f',g'} \circ K_{f,g}) (\Psi_{f',g} \circ \Psi_{f,g}) + (K_{f',g'} \circ \Psi_{f,g}) (K_{f',g} \circ \Psi_{f,g})
\]

\[
+ (K_{f',g'} \circ \Psi_{f,g}) (K_{f',g} \circ K_{f,g}) + (\Psi_{f',g'} \circ K_{f,g}) (K_{f',g} \circ K_{f,g})
\]

\[
+ (\Psi_{f',g'} \circ K_{f,g}) (\Psi_{f,g} \circ K_{f,g}) + (\Psi_{f',g'} \circ \Psi_{f,g}) (K_{f',g} \circ K_{f,g})
\]
As third order terms, we get

\[(K_{f',g'} \Psi_{f',g'}) \circ (\Psi_{f,g} \Psi_{f,g}) + (\Psi_{f',g'} K_{f',g'}) \circ (\Psi_{f,g} \Psi_{f,g})
+ (\Psi_{f',g'} \Psi_{f',g}) \circ (K_{f,g} \Psi_{f,g}) + (\Psi_{f',g'} \Psi_{f',g}) \circ (\Psi_{f,g} \Psi_{f,g})\]

respectively

\[(K_{f',g'} \circ \Psi_{f',g'}) (\Psi_{f',g'} \circ \Psi_{f,g}) + (\Psi_{f',g'} K_{f',g'}) (\Psi_{f',g'} \circ \Psi_{f,g})
+ (\Psi_{f',g'} \circ \Psi_{f',g}) (K_{f',g'} \circ \Psi_{f,g}) + (\Psi_{f',g'} \circ \Psi_{f',g}) (\Psi_{f',g'} \circ K_{f,g})\]

Finally, the fourth order terms are

\[(\Psi_{f',g'} \Psi_{f',g'}) \circ (\Psi_{f,g} \Psi_{f,g})\]

for the left hand side and

\[(\Psi_{f',g'} \circ \Psi_{f',g'}) (\Psi_{f,g} \circ \Psi_{f,g})\]

for the right hand side, respectively.

Let us now discuss the structural properties of these terms. Consider $K_{\ldots}$ as a map of two variables from 1-morphisms in $\mathcal{C}$ to 2-morphisms in $\mathcal{C}$. Since $K_{f,g}$ measures the failure of the square shaped diagram, defined by the 1-morphisms $f$ and $g$ as given above, to commute. We can now assemble four such diagrams as two times two into a larger square shaped diagram, as required for equation (2). Obviously, the failure of this larger diagram to commute is then measured by a map of four variables from 1-morphisms in $\mathcal{C}$ to 2-morphisms in $\mathcal{C}$. Equation (2) states that in the case of $K_{f,g}$ no new family of maps arises in this way but the four variable map is independent of the two ways to compose in the big two times two diagram and is given by $K_{f'g',gg'}$. For a general map of two variables from 1-morphisms in $\mathcal{C}$ to 2-morphisms in $\mathcal{C}$ this need not be true and for such a general map $\Phi_{f,g}$, we define $d_K \Phi_{f,g}$ as the four variable map given by

\[
d_K \Phi_{f,g} = (K_{f',g'} K_{f',g}) \circ (K_{f,g} \Phi_{f,g}) + (K_{f',g'} K_{f',g}) \circ (\Phi_{f,g} K_{f,g})
+ (K_{f',g'} \Phi_{f,g}) \circ (K_{f,g} K_{f,g}) + (\Phi_{f,g} \Phi_{f,g}) \circ (K_{f,g} K_{f,g})
- (K_{f',g'} \circ K_{f,g}) (K_{f',g} \circ \Phi_{f,g}) - (K_{f',g'} \circ K_{f,g}) (\Phi_{f,g} \circ K_{f,g})
- (K_{f',g'} \circ \Phi_{f,g}) (K_{f',g} \circ K_{f,g}) - (\Phi_{f',g'} \circ \Phi_{f,g}) (K_{f',g} \circ K_{f,g})\]
Here, the notation $d_K$ indicates the dependence on the cubic functor $K$ (understood as the whole family $K_{f,g}$). The requirement

$$d_K \Phi_{f,g} = 0$$

means, then, that, at least in first order correction, new contributions to the deficiency of the big square diagram to commute - besides the ones deriving from the deficiency in commutativity of the four small squares - vanish. This is precisely what is satisfied in a linear approximation by $\Psi_{f,g}$.

Starting from the square shaped diagram and then building the big two times two diagram out of this, one can proceed in this way. In the next step, one constructs a “huge diagram”, consisting of two times two big diagrams. Obviously, the deficiency of the huge diagram to commute is measured by a function of eight variables. We can proceed iteratively, then. Starting from a given diagram and its deficiency function, we can pass to the next larger diagram and calculate in first order the additional contribution to the noncommutativity of the larger diagram, besides the contribution arising form the deficiency function of the four smaller diagrams. In the same way as above, we define a differential $d_K$ for the deficiency function of the smaller diagram, in this way. One has a coherence property in this setting: Building larger and larger two times two diagrams completely reduces to iterating the first step which leads from the square shaped diagram to the big square shaped diagram. It follows from this coherence property that

$$d^2_K = 0 \quad (4)$$

We complete the construction of the deformation complex by including as 0-cochains the real or complex numbers (depending on whether $\mathcal{C}$ is $\mathbb{R}$- or $\mathbb{C}$-linear). We interpret a number $\lambda$ as giving the trivial deficiency function $K_{\lambda}$ which is constantly the identity, rescaled (remember that we are in a linear setting) by the factor $\lambda$. The cocycle condition

$$d_K \lambda = 0$$

is trivial, then. Using (4), we can pass from the deformation complex to cohomology. Observe that the resulting cohomology theory has two exotic features:

- It is a strange exponential cohomology: While e.g. in Hochschild cohomology of associative algebras the $n$-cochains are $n$-variable maps, here, the $n$-cochains are maps of $2^n$ variables.
As a result of this exponential behavior, the first order deformations of a cubic functor live in first cohomology while in Hochschild cohomology the first order deformations live in second cohomology.

Observe that we can not expect for an analogue of the result of [Kon] to hold for the deformation theory of a cubic functor: Since we have a fourth order Master equation, we can not expect deformations to be determined by first order terms but we expect that we also have to prescribe a second and a third order term to determine a deformation. Hence, we can not expect that obstructions to extending beyond first order deformations are of a purely cohomological nature. We will next try to understand the higher order terms of the Master equation in more detail.

Let us start with the fourth order terms. Define the difference of the left and the right hand side of (2) as an operator $J(K)$, i.e. (2) can be rephrased as

$$J(K) = 0$$

We can extend this operator to any two variable map $\Phi$ by just replacing $K$ by $\Phi$ in the definition. The fourth order terms are just of the form $J(\Psi)$, then. The operator $J$ is independent of the given cubic functor $K$. This is analogous to Hochschild cohomology where the highest order terms appearing in the Maurer-Cartan equation

$$d_m a + a \circ_G a = d_m a + \frac{1}{2} [a, a]_G = 0$$

are given by the Gerstenhaber product $\circ_G$ or its odd commutator in the form of the Gerstenhaber bracket $[,]_G$ and are also independent of the given algebra product $m$ one wants to deform. Also, associativity of the original product $m$ is expressed as

$$[m, m]_G = 0$$

in the Hochschild setting which is analogous to (6). The Maurer-Cartan equation (6) says that the deficiency of $a$ to satisfy (7) is measured by the differential $d_m$ applied to $a$. The Master equation for the deformation of a cubic functor says that the deficiency of $\Psi$ to satisfy (3) is measured by the differential $d_K$ applied to $\Psi$, plus the second and third order terms.

The third order terms can be concisely summarized as

$$d_\Psi K$$
where we define $d_Ψ$ by the formula resulting from the definition of $d_K$ after replacing $K$ by $Ψ$. Since $Ψ$ is not a cubic functor, we have to expect that in general

$$d_Ψ^2 \neq 0$$

This means that in the same way as $J(Ψ)$ normally does not vanish, in general there is a deficiency to $d_Ψ$ being a true differential. So, besides the symmetry between the 0-th and the highest order term under exchanging $K$ with $Ψ$ - which is also known from the Maurer-Cartan equation of Hochschild cohomology - we have discovered another symmetry of this type for the Master equation for the deformation theory of a cubic functor, as a symmetry between the first and third order terms.

Finally, we summarize the second order terms as

$$Ψ ⊡_K Ψ$$

where $⊡_K$ can be extended in the obvious way to a product $Φ_1 ⊡_K Φ_2$ on general two variable maps $Φ_1$ and $Φ_2$. The Master equation for the deformation theory of a cubic functor can then be written as

$$d_KΨ + Ψ ⊡_K Ψ + d_Ψ K + J(Ψ) = 0 \quad (8)$$

While the Maurer-Cartan equation says that the deficiency of $a$ to satisfy (7) is given by $d_m a$, we can interpret equation (8) also as follows: Besides the cubic functor $K$, we have to consider its cohomology given by cocycles with

$$d_KΨ = 0 \quad (9)$$

In general

$$J(Ψ) \neq 0$$

and

$$d_Ψ K \neq 0$$

but the deficiency that at least the weaker condition

$$d_Ψ K + J(Ψ) = 0$$

would be satisfied is measured by the second order term $Ψ ⊡_K Ψ$ if $Ψ$ is a cocycle, i.e. (9) holds. We would like to stress that the product $⊡_K$ is not of the type of the Gerstenhaber product because $⊡_K$ depends on $K$, much
like the differential $d_K$ is dependent on $K$. So, besides the cohomological first order part and a kind of deformed (because of the deficiency of $d_\Psi$ to square to zero) cohomology given by $d_\Psi$, we have a new structural ingredient in the Master equation $\square$ in the form of the product $\square_K$. So, to determine a solution of the Master equation, a new nonlinear algebraic level beyond complexes and cohomology is needed.

In the next section, we will study the deformation problem of a cubic functor from a slightly different perspective. We will see that similar structural properties for the Master equation arise.

## 3 Deformation theory of enriched categories

As we mentioned in Remark 1, semistrict monoidal bicategories and, more generally, semistrict tricategories can also be understood as categories enriched over the monoidal category $\text{Gray}$ of 2-categories with the $\text{Gray}$ tensor product (see [GPS], see [Kel] for background on enriched categories). The structure of a cubic functor is induced from the $\text{Gray}$ tensor product. If one uses the usual cartesian product as tensor product, instead, one gets strict tricategories, i.e. trivial cubic functors, only. So, we can understand the deformation problem of a cubic functor also as being related to the deformation problem for the tensor product of the monoidal category over which the enrichment is taken. We will explain this in a little bit more detail, now.

Let $\mathcal{V}$ be a monoidal category. A category $\mathcal{C}$ enriched over $\mathcal{V}$, also called a $\mathcal{V}$-category, is given by the following data:

- A class $\text{Obj}(\mathcal{C})$ of objects of $\mathcal{C}$.
- For any pair of objects $a, b$ of $\mathcal{C}$, we have an object $\text{Hom}(a, b)$ in $\mathcal{V}$.
- For objects $a, b, c$ of $\mathcal{C}$, there is a composition morphism $f_{a,b,c}$ in $\mathcal{V}$

$$f_{a,b,c} : \text{Hom}(a, b) \otimes \text{Hom}(b, c) \to \text{Hom}(a, c)$$

which is supposed to be associative. Here, $\otimes$ is the tensor product of $\mathcal{V}$.
In addition, one has straightforward requirements on existence of units which we do not explicitly give, here, since we will not need them in the sequel (see e.g. [Kel] for the full details).

Deforming $\otimes$ - to pass e.g. from the case of a trivial to a nontrivial cubic functor - one automatically has to include deformations of the function system $f_{a,b,c}$ since the domain of definition of $f_{a,b,c}$ is, in general, changed under deformations of the tensor product. We will consider an even more general situation, here, where we allow for deformations of the composition of morphisms in $\mathcal{V}$ to become deformed, too. So, the deformation problem we consider for enriched categories is, in the case of Gray-categories, even more general than the one discussed in the previous section. The deformation problem for an enriched category is then the problem to deform $f_{a,b,c}$, $\otimes$ and the composition $\bullet$ of $\mathcal{V}$ at the same time, i.e we deform $\mathcal{C}$ into a new enriched category $\tilde{\mathcal{C}}$ where $\tilde{\mathcal{C}}$ is enriched over the monoidal category $\tilde{\mathcal{V}}$, resulting from the deformation of $\otimes$ and $\bullet$.

We assume, now, that $\mathcal{C}$ and $\mathcal{V}$ are $\mathbb{R}$- or $\mathbb{C}$-linear, completely analogous to the corresponding assumption in the previous section. Let

$$\tilde{f} = f + \alpha$$

where we write $f$ for the whole function system $f_{a,b,c}$,

$$\tilde{\otimes} = \otimes + \beta$$

and

$$\tilde{\bullet} = \bullet + \gamma$$

The first requirement in the deformation theory is then that $\tilde{\bullet}$ and $\tilde{\otimes}$ constitute a monoidal category, again. This results in the deformation theory described in [GS]. In addition, we have the requirement that $\tilde{f}$ has to be associative. Since the associativity constraint on $\tilde{f}$ also involves $\tilde{\otimes}$ and $\tilde{\bullet}$, this does not result in the usual Maurer-Cartan equation for the deformation theory of an associative product, in this case. We will now study the equation which generalizes the Maurer-Cartan equation, in this case (we will call this the Master equation for the deformation theory of an enriched category but remember that this Master equation has to be coupled to the coupled system of three differential equations of [GS] as a fourth equation, to arrive
at the full deformation theory of an enriched category). We will find a fourth order equation, again.

The associativity constraint for $\tilde{f}$ is

$$\tilde{f} \bullet (\tilde{f} \otimes 1) = \tilde{f} \bullet (1 \otimes \tilde{f}) \quad (10)$$

where we have suppressed indices of $\tilde{f}$ (actually (10) refers to four objects $a, b, c, d$ of $\mathcal{C}$, of course). In the same way, we have not expelled the index for the unit object 1. Inserting $\tilde{f}$, $\otimes$, and $\bullet$ as given above into (10), we get in 0-th order the associativity constraint

$$f \bullet (f \otimes 1) = f \bullet (1 \otimes f)$$

for $f$. Rewriting (10) as

$$\tilde{f} \bullet (\tilde{f} \otimes 1) - \tilde{f} \bullet (1 \otimes \tilde{f}) = 0$$

we get for the first order terms

$$f \bullet (\alpha \otimes 1) - f \bullet (1 \otimes \alpha) + f \bullet \beta (f, 1) - f \bullet \beta (1, f)$$

$$+ \alpha \bullet (f \otimes 1) - \alpha \bullet (1 \otimes f) + \gamma (f, f \otimes 1) - \gamma (f, 1 \otimes f)$$

For the second order terms, we have

$$f \bullet \beta (\alpha, 1) - f \bullet \beta (1, \alpha) + \alpha \bullet (\alpha \otimes 1) - \alpha \bullet (1 \otimes \alpha)$$

$$+ \alpha \bullet \beta (f, 1) - \alpha \bullet \beta (1, f) + \gamma (f, f \otimes 1) - \gamma (f, 1 \otimes f)$$

and the third order terms are given by

$$\alpha \bullet \beta (\alpha, 1) - \alpha \bullet \beta (1, \alpha) + \gamma (f, \beta (\alpha, 1)) - \gamma (f, \beta (1, \alpha))$$

$$+ \gamma (\alpha, \alpha \otimes 1) - \gamma (\alpha, 1 \otimes \alpha) + \gamma (\alpha, \beta (f, 1)) - \gamma (\alpha, \beta (1, f))$$

Finally, the fourth order terms result in

$$\gamma (\alpha, \beta (\alpha, 1)) - \gamma (\alpha, \beta (1, \alpha))$$

Let us comment, again, on the structural properties of these terms. We summarize the first order terms as

$$d_{\tilde{f}, \otimes, \bullet} (\alpha, \beta, \gamma)$$
where the notation indicates the dependence of $d_{f, \otimes, \bullet}$ on $f, \otimes$ and $\bullet$. As in the previous section, there is a symmetry between the first and third order terms and we can formally write the third order terms as

$$d_{\alpha, \beta, \gamma}(f, \otimes, \bullet)$$

Also, there is a symmetry, again, between the 0-th and the fourth order terms: Define for a function system $\chi$, or more detailed $\chi_{a,b,c}$, of two variables $\chi_{a,b,c}$:

$$
\text{Hom}(a, b) \otimes \text{lin} \text{Hom}(b, c) \rightarrow \text{Hom}(a, c)
$$

(where the notation $\otimes_{\text{lin}}$ indicates that the left hand side agrees with $\text{Hom}(a, b) \otimes \text{Hom}(b, c)$ as taken in $\mathcal{C}$ only as a linear space over $\mathbb{R}$ or $\mathbb{C}$; remember in this context that the linear structures of $\mathcal{C}$ and $\mathcal{V}$ are supposed to stay fixed in the process of deformation), and corresponding function systems $\varsigma$ and $\rho$, generalizing $\otimes$ and $\bullet$, the operator $G$ as

$$G(\chi, \varsigma, \rho) = \rho(\chi, \varsigma(\chi, 1)) - \rho(\chi, \varsigma(1, \chi))$$

Then the associativity constraint on $f$ is equivalent to

$$G(f, \otimes, \bullet) = 0$$

and we can summarize the fourth order terms as

$$G(\alpha, \beta, \gamma)$$

Again, $G$ is independent of $f, \otimes$ and $\bullet$ and the Master equation gives a deficiency of $\alpha, \beta, \gamma$ to satisfy \ref{eq:11}. As in the previous section, the second order terms can be written as a product depending on $f, \otimes$ and $\bullet$ which we denote as $\square_{f, \otimes, \bullet}$, i.e. the second order terms are given by

$$(\alpha, \beta, \gamma) \square_{f, \otimes, \bullet}(\alpha, \beta, \gamma)$$

So, the whole Master equation for the deformation theory of an enriched category can be written as

$$d_{f, \otimes, \bullet}(\alpha, \beta, \gamma) + (\alpha, \beta, \gamma) \square_{f, \otimes, \bullet}(\alpha, \beta, \gamma) + d_{\alpha, \beta, \gamma}(f, \otimes, \bullet) + G(\alpha, \beta, \gamma) = 0 \tag{12}$$

The discussion of the remaining structural properties of \ref{eq:12} is completely analogous, then, to the discussion for the Master equation of the deformation
theory of a cubic functor in the previous section and we will, therefore, not repeat it, here. Especially, we expect that, as a consequence of the coherence property for the associativity constraint (10) (i.e. iterated brackets with four and more variables can be rebracketed automatically as a consequence of (10)),

$$d_{f,\otimes} = 0$$

holds.

In conclusion, the deformation theory of an enriched category is described by the Master equation (12), which is very similar to the Master equation of the previous section, but with an additional coupling to the deformation theory of a strict monoidal category as given in [GS].

The viewpoint starting from enriched categories for the deformation theory, as taken in this section, might also be of interest in physics besides the setting of semistrict monoidal bicategories. E.g. also the differential graded categories and $A_\infty$-categories appearing in topological string theory can be understood as enriched categories. The deformation theory of enriched categories, as presented here, would considerably generalize the Hochschild complex for these structures and one can pose the question what the even larger moduli spaces that would arise this way should mean for topological string theory.

4 Conclusion

We have studied the deformation theory of a semistrict monoidal bicategory in this paper and found structural properties and a Master equation which considerably differ from the usual structure of the BV-quantization scheme. We found a similar result when studying the deformation problem not as the deformation problem for the cubic functor of a semistrict monoidal bicategory but in the more general setting of enriched categories.

For function algebras on quantum groups, taking the case where the deformation parameter $q$ is a root of unity, several authors have undertaken to construct a monoidal bicategory from the data (see e.g. [CF], [KS]). Especially, this means that the deformation theory developed, here, should be applicable in the setting of the $SU(2)$-WZW model. We plan to study this application in a separate paper.
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