NEW GAUSSIAN RIESZ TRANSFORMS ON VARIABLE LEBESGUE SPACES

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Abstract. We give sufficient conditions on the exponent $p: \mathbb{R}^d \to [1, \infty)$ for the boundedness of the non-centered Gaussian maximal function on variable Lebesgue spaces $L^{p(\cdot)}(\gamma_d)$, as well as of the new higher order Riesz transforms associated with the Ornstein–Uhlenbeck semigroup, which are the natural extensions of the supplementary first order Gaussian Riesz transforms defined by A. Nowak and K. Stempak in [25].

1. Introduction

Gaussian harmonic analysis on $\mathbb{R}^d$ is represented by a differential operator called Ornstein–Uhlenbeck which is defined as

$$\mathcal{L} := -\frac{1}{2} \Delta + x \cdot \nabla,$$

where $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and $\nabla = \left( \frac{\partial}{\partial x_i} \right)_{i=1}^{d}$ is the classical gradient. This operator is factored, on each variable, into two derivatives as follows. Indeed, by naming $\delta_i = \frac{i}{\sqrt{2}} \frac{\partial}{\partial x_i}$ and $\delta_i^* = -\frac{i}{\sqrt{2}} e^{-|x|^2} \frac{\partial}{\partial x_i} \left( e^{-|x|^2} \right)$, the formal adjoint of $\delta_i$ with respect to the $d$-dimensional non-standard Gaussian measure $d\gamma_d(x) = \frac{e^{-|x|^2}}{\pi^{d/2}} \, dx$, we have the differential operator $\mathcal{L}_i = \delta_i^* \delta_i$.

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and

\[ \mathcal{L} = \sum_{i=1}^{d} \mathcal{L}_i. \]

Let us remark that \( \mathcal{L} \) is an unbounded non-negative symmetric operator on \( L^2(\gamma_d) \). Besides, there is a dense linear subspace of this space where \( \mathcal{L} \) turns out to be a self-adjoint operator (see [17]). It has a discrete spectrum \( \sigma(\mathcal{L}) = \{0, 1, 2, 3, \ldots\} =: \mathbb{N}_0 \) which are all eigenvalues of \( \mathcal{L} \) and its eigenfunctions are the \( d \)-dimensional Hermite polynomials (see, for instance, [8] for the one-dimensional case, and [28] for higher dimensions).

Following the notation of [25] we have \( d \) more differential operators which are associated with the \( \delta_i^* \)-derivatives. For \( i = 1, \ldots, d \) let us define

\[ M_i := \mathcal{L} + [\delta_i, \delta_i^*]I, \]

where \( I \) is the identity operator and \( [\delta_i, \delta_i^*] = \delta_i \delta_i^* - \delta_i^* \delta_i \) is the commutator corresponding to the \( i \)-th derivatives. In this case, \( [\delta_i, \delta_i^*] = 1 \) for all \( i \). Thus \( M_1 = M_2 = \cdots = M_d =: \tilde{\mathcal{L}} = \mathcal{L} + I \). This differential operator has also a discrete spectrum, being \( \sigma(\tilde{\mathcal{L}}) = \mathbb{N} \).

Associated with these two differential operators there exist two diffusion semigroups, i.e. \( T^t = e^{-\mathcal{L}t} \) and \( \tilde{T}^t = e^{-\tilde{\mathcal{L}}t} \), which are defined through the spectral decomposition of \( \mathcal{L} \) and \( \tilde{\mathcal{L}} \) on \( L^2(\gamma_d) \), respectively.

In order to define the Gaussian Riesz transforms, in this article we are going to consider two different transforms. We need the fractional integrals, say, for \( \beta > 0 \),

\[ I_\beta = \mathcal{L}^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} T^t dt, \quad \tilde{I}_\beta = \tilde{\mathcal{L}}^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} \tilde{T}^t dt. \]

We set \( I_0 = \tilde{I}_0 = I \).

Now we define the two \( i \)-th Gaussian Riesz transforms (see [25]). For \( i = 1, \ldots, d \), let

\[ R_i f(x) = \delta_i I_{\frac{x}{2}} f(x), \quad R_i^* f(x) = \delta_i^* \tilde{T}_{\frac{x}{2}} f(x). \]

Like in classical harmonic analysis, these Gaussian Riesz transforms verify the equation

\[ \sum_{i=1}^{d} R_i^* R_i = I \]

on \( L^2(\gamma_d) \), the space of square integrable functions with zero average. That is, they decompose the identity.
Now, we want to define higher order Gaussian Riesz transforms that retain this property. Let us introduce some notation. For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d \), we define \( |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d \), \( \delta^\alpha = \delta_1^{\alpha_1} \delta_2^{\alpha_2} \cdots \delta_d^{\alpha_d} \), and similarly \( \delta_*^\alpha = (\delta_1^*)^{\alpha_1} (\delta_2^*)^{\alpha_2} \cdots (\delta_d^*)^{\alpha_d} \), where we set \( \delta_i^0 = (\delta_i^*)^0 = I \) for \( i = 1, \ldots, d \). Thus, we are ready to define the higher order Gaussian Riesz transforms as follows:

\[
R_\alpha f(x) = \delta^\alpha I_{a_1/2} I_{a_2/2} \cdots I_{a_d/2} f(x),
\]
\[
R_*^\alpha f(x) = \delta_*^\alpha \overline{I}_{a_1/2} \overline{I}_{a_2/2} \cdots \overline{I}_{a_d/2} f(x).
\]

Taking into account that \( I_{\beta I_{\varepsilon}} = I_{\beta + \varepsilon} \) and \( \overline{I}_{\beta I_{\varepsilon}} = \overline{I}_{\beta + \varepsilon} \), we can rewrite the Gaussian Riesz transforms as

\[
R_\alpha f(x) = \delta^\alpha I_{|\alpha|/2} f(x), \quad R_*^\alpha f(x) = \delta_*^\alpha \overline{I}_{|\alpha|/2} f(x).
\]

Let us remark that \( R_{e_i} = R_i \) and \( R_{e_i}^* = R_i^* \), where \( e_i \) is the \( i \)-th canonical unit vector of \( \mathbb{N}_0^d \). Let us also note that W. Urbina-Romero in [28] (see also [23]) has defined alternative higher order Gaussian Riesz transforms \( \overline{R}_\beta \) but he does not recover the suitable supplementary first order Gaussian Riesz transforms \( R_i^* \) given by A. Nowak and K. Stempak [25] when \( \beta = e_i \).

We will refer to \( R_\alpha \) as the “old” Gaussian Riesz transform, and to \( R_*^\alpha \) as the “new” Gaussian Riesz transform. The reason why we are considering the word “new” added to the higher order Gaussian Riesz transforms is because they were firstly used in [2], in order to distinguish them from the existed first ones which were extensively studied before. The difference between them is in the choice of the derivatives in which the Ornstein–Uhlenbeck differential operator is factored out.

The operator \( R_\alpha \) turns out to be bounded on \( L^p(\gamma_d) \), for \( 1 < p < \infty \), with constant independent of dimension (see [22], [18], [14]). For the first order Gaussian Riesz transforms \( R_i^* \), \( i = 1, \ldots, d \), \( L^p(\gamma_d) \)-dimension-free estimates were obtained in [12] and [29], for \( 1 < p < \infty \). By means of Meyer’s multiplier theorem, the “new” higher order Riesz transforms are also bounded on \( L^p(\gamma_d) \), as can be proved similarly to [28, Corollary 9.14], with constant independent of dimension. According to [3], the Euclidean space \( \mathbb{R}^d \) can be extended to an infinite-dimensional real Hausdorff locally convex space \( X \), where one can introduce a standard Gaussian measure \( \gamma \). In this context we can define the analogous diffusion semigroup \( T^t \) whose infinitesimal generator is the Ornstein–Uhlenbeck operator \( \mathcal{L} = \text{div}_\gamma D_H \), being \( D_H \) the gradient on a Cameron–Martin space \( H \). Taking into account these derivatives and the potentials associated with the corresponding Sobolev spaces, one can define singular integrals on this context and the boundedness of them on \( L^p(\gamma) \) can be obtained from their boundedness on \( L^p(\gamma_d) \), with constants independent of dimension.
In [7], we have proved that each $R_\alpha$ is bounded on variable Lebesgue spaces with respect to the non-standard Gaussian measure. Here, we cannot obtain a constant independent of dimension. It is an open problem to find a technique similar to the Littlewood-Paley one that gives a boundedness independent of dimension. Inspired by [7], the main aim of the present article is to show the following boundedness property.

**Theorem 1.1.** The new Gaussian Riesz transforms $R^*_\alpha$ are bounded on $L^p(\gamma_d)$ provided that $p^- > 1$ and $p \in LH_0(\mathbb{R}^d) \cap \mathcal{P}^\infty(\mathbb{R}^d)$.

For the definitions and notations involved in the theorem above, see Section 2.

In order to get a proof of Theorem 1.1, in Section 3 we will introduce a more general operator which contains these Riesz transforms and show its boundedness, following closely the ideas of our previous article [7]. Indeed, Theorem 1.1 will be a particular case of Theorem 4.2.

On the other hand, in Section 5, we consider the variable $L^p(\cdot)$-boundedness of the non-centered Gaussian Hardy–Littlewood maximal function $M_{\gamma_d}$. Moreover, we prove a more general theorem dealing with such a boundedness for a maximal operator associated to a measure $\mu$ defined on a metric space, finding a geometric condition on the measure $\mu$ over the balls similar to L. Diening’s condition (2.3) for the case of the Lebesgue measure. We prove that the very same properties on the exponents required in Theorem 1.1 are also sufficient for its boundedness on $L^p(\cdot)(\gamma_d)$. An equivalent condition to $\mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d)$ is also established.

2. Preliminaries

We now give some definitions about variable Lebesgue spaces on a measure space.

Given a $\sigma$-finite measure $\mu$ over $\mathbb{R}^d$, we shall denote with $\mathcal{P}(\mathbb{R}^d, \mu)$ the set of *exponents*, that is, the set of $\mu$-measurable and bounded functions $p: \mathbb{R}^d \to [1, \infty)$. When $\mu$ is the Lebesgue measure, we simply write $\mathcal{P}(\mathbb{R}^d)$. For a $\mu$-measurable set $E$, we write

$$p_E^- = \text{ess inf}_{x \in E} p(x), \quad p_E^+ = \text{ess sup}_{x \in E} p(x),$$

and, for the whole space, we denote $p^- = p^-_{\mathbb{R}^d}$ and $p^+ = p^+_{\mathbb{R}^d}$.

Given $p \in \mathcal{P}(\mathbb{R}^d, \mu)$, we say that a $\mu$-measurable function $f$ belongs to $L^p(\cdot)(\mathbb{R}^d, \mu)$ if, for some $\lambda > 0$,

$$\int_{\mathbb{R}^d} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} d\mu(x) < \infty.$$
The natural norm for these spaces is the Luxemburg norm, defined by
\[ \|f\|_{p(\cdot),\mu} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, d\mu(x) \leq 1 \right\}, \]
which recovers the classical norm \( \|f\|_{p,\mu} = \left( \int_{\mathbb{R}^d} |f(x)|^p \, d\mu(x) \right)^{1/p} \) whenever \( p(x) \equiv p, \, 1 \leq p < \infty \). It is also well-known that \( (L^p(\mathbb{R}^d,\mu),\|\cdot\|_{p(\cdot),\mu}) \) is a Banach function space ([10, Theorem 3.2.7]). When \( \mu \) is the classical Lebesgue measure, we simply write \( L^p(\mathbb{R}^d) \) and then norms \( \|\cdot\|_p \).

Associated with each exponent \( p \in \mathcal{P}(\mathbb{R}^d,\mu) \), we have another exponent \( p' \in \mathcal{P}(\mathbb{R}^d,\mu) \), which is the generalization to the variable context of Hölder’s conjugate exponent given by
\[ \frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{for all} \quad x \in \mathbb{R}^d. \]
As expected, a generalization of Hölder’s inequality holds for variable exponents ([10, Lemma 3.2.20]). Given a measure \( \mu \) as above, for every pair of functions \( f \in L^{p(\cdot)}(\mathbb{R}^d,\mu) \) and \( g \in L^{p'(\cdot)}(\mathbb{R}^d,\mu) \),
\[ \int_{\mathbb{R}^d} |f(x)g(x)| \, d\mu(x) \leq 2\|f\|_{p(\cdot),\mu}\|g\|_{p'(\cdot),\mu}. \]

Another important property is the norm conjugate formula: for any \( \mu \)-measurable function \( f \), the following inequalities
\[ \frac{1}{2} \|f\|_{p(\cdot),\mu} \leq \sup_{\|g\|_{p'(\cdot),\mu} \leq 1} \int_{\mathbb{R}^d} |f(x)g(x)| \, d\mu(x) \leq 2\|f\|_{p(\cdot),\mu}. \]
hold ([10, Corollary 3.2.14]). For more information about \( L^{p(\cdot)} \) spaces, see, for instance, [6] or [10].

The measure we shall be dealing with is the non-standard Gaussian measure \( \gamma_d \), which is a finite, non-doubling and upper Ahlfors \( d \)-regular measure on \( \mathbb{R}^d \). From now on, \( \mu = \gamma_d \).

The exponents we will consider are not arbitrary, but we may allow them to have some continuity properties. The following conditions on the exponent arise related with the boundedness of the Hardy–Littlewood maximal function \( M_{H-L} \) on \( L^{p(\cdot)}(\mathbb{R}^d) \) (see, for example, [4] or [9]).

**Definition 2.1.** Let \( p \in \mathcal{P}(\mathbb{R}^d) \).

(1) We will say that \( p \in LH_0(\mathbb{R}^d) \) if there exists \( C_{\log}(p) > 0 \) such that, for any pair \( x, y \in \mathbb{R}^d \) with \( 0 < |x - y| < 1/2 \),
\[ |p(x) - p(y)| \leq \frac{C_{\log}(p)}{-\log(|x - y|)}. \]
We will say that \( p \in LH_{\infty}(\mathbb{R}^d) \) if there exist constants \( C_\infty > 0 \) and \( p_\infty \geq 1 \) for which

\[
|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}, \quad \text{for all } x \in \mathbb{R}^d.
\]

We will say \( p \in LH(\mathbb{R}^d) \) when \( p \in LH_0(\mathbb{R}^d) \cap LH_{\infty}(\mathbb{R}^d) \).

Conditions (2.1) and (2.2) are usually called the local log-Hölder condition and the decay log-Hölder condition, respectively. When \( p \in LH(\mathbb{R}^d) \), we simply say that it is log-Hölder continuous. It is well-known that whenever \( 1 < p^- \leq p^+ < \infty \), \( p \in LH(\mathbb{R}^d) \) is a sufficient condition for the continuity of the Hardy–Littlewood maximal operator \( M_{H-L} \) on variable Lebesgue spaces (see, for instance, [4]). However, it is not a necessary condition although it was proved in [6, Examples 4.1 and 4.43] that both \( LH_0(\mathbb{R}^d) \) and \( LH_{\infty}(\mathbb{R}^d) \) are the sharpest possible pointwise conditions on \( p \). The authors in [10] gave a necessary and sufficient condition for the \( L^{p(\cdot)} \)-boundedness of \( M_{H-L} \), but it is not easy to work with from the practical point of view. In this article, we do not expect to characterize the exponents, but to give sufficient easy-to-check conditions for them in order to obtain the boundedness properties for the operators in study.

Regarding the local log-Hölder condition (2.1), L. Diening gave a geometric characterization of it (see [9]) in order to prove the boundedness of \( M_{H-L} \) on bounded subsets of \( \mathbb{R}^d \) or in the whole Euclidean space assuming \( p \) is constant outside of a fixed ball.

**Lemma 2.2** [9]. Given \( p \in P(\mathbb{R}^d) \), \( p \in LH_0(\mathbb{R}^d) \) if and only if there exists a positive constant \( C \) such that

\[
|B|^{p^+_B - p^-_B} \geq C,
\]

for every ball \( B \).

An analogous property can also be obtained when dealing with the boundedness of \( M_\mu \), the non-centered maximal function associated with the measure \( \mu \). We will consider it in Section 5.

Whilst Diening’s geometric condition can be applied to control the behavior of \( f \) when it is large, condition (2.2) happens to be useful when \( f \) is small. This is evidenced in the following lemma, which establishes that we can change a variable exponent \( p \) for its limit \( p_\infty \), and vice versa, adding an integrable error (for a proof, see for instance [6, Lemma 3.26]). Previous results of this kind were given in [4,5], closely related with the boundedness of the Hardy–Littlewood maximal operator in the Euclidean setting.

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Lemma 2.3 [6]. Let \( p \in LH_\infty(\mathbb{R}^d) \) with \( 1 < p^- \leq p^+ < \infty \). Then, there exists a constant \( C \), depending on \( d \) and \( C_\infty \), such that for any set \( E \) and any function \( G \) with \( 0 \leq G(y) \leq 1 \) for \( y \in E \),

\[
\int_E G(y)^{p(y)} \, dy \leq C \int_E G(y)^{p_\infty} \, dy + \int_E (e + |y|)^{-dp^-} \, dy,
\]

\[
\int_E G(y)^{p_\infty} \, dy \leq C \int_E G(y)^{p(y)} \, dy + \int_E (e + |y|)^{-dp^-} \, dy.
\]

We will now recall the class of exponents introduced in our previous article [7], which is related with the boundedness of the “old” Riesz transforms.

Definition 2.4. Given \( p \in P(\mathbb{R}^d, \gamma_d) \), we will say that \( p \in P_\infty(\mathbb{R}^d) \) if there exist constants \( C_{\gamma_d} > 0 \) and \( p_\infty \geq 1 \) such that

\[
|p(x) - p_\infty| \leq \frac{C_{\gamma_d}}{|x|^2}, \quad \text{for all } x \in \mathbb{R}^d \setminus \{(0, \ldots, 0)\}.
\]

As it was observed in [7, Remark 2.4], if \( p \in P_\infty(\mathbb{R}^d) \), then \( p \in LH_\infty(\mathbb{R}^d) \), and, if \( p^- > 1 \), also \( p' \in P_\infty(\mathbb{R}^d) \) with \( (p')_\infty = (p_\infty)' = p'_\infty < \infty \). Since, in this case, \( p_\infty = \lim_{|x| \to \infty} p(x) \), we have \( p_\infty > 1 \) whenever \( p^- > 1 \).

From now on, we will use the following notation: given two functions \( f \) and \( g \), by \( \lesssim \) and \( \gtrsim \) we will mean that there exists a positive constant \( c \) such that \( f \leq cg \) and \( cf \geq g \), respectively. When both inequalities hold, i.e., \( f \lesssim g \gtrsim f \), we will write it as \( f \approx g \).

As it is usual in the Gaussian context, we consider the “local” and “global” parts of several operators, in order to analyze their properties. For this partition, we may recall the definition of the hyperbolic ball

\[
B(x) := \{ y \in \mathbb{R}^d : |y - x| \leq d(1 \wedge 1/|x|) \}, \quad x \in \mathbb{R}^d,
\]

where \( \alpha \wedge \beta = \min\{\alpha, \beta\} \), \( \alpha, \beta \in \mathbb{R} \). Given a sublinear operator \( S \), we say that \( S(f \chi_{B(\cdot)}) \) is the local part and \( S(f \chi_{B^c(\cdot)}) \), being \( B^c(x) := \mathbb{R}^d \setminus B(x) \), is the global part.

3. The “new” higher order Gaussian Riesz transforms on variable Lebesgue spaces

We have that the “old” higher order Gaussian Riesz transforms \( R_\alpha f \) can be written as an integral operator with a kernel \( K_\alpha \)

\[
R_\alpha f(x) = \text{p.v.} \int_{\mathbb{R}^d} K_\alpha(x, y) f(y) \, dy,
\]

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with
\[
K_\alpha(x, y) = C_\alpha \int_0^1 r^{\alpha|\alpha|-1} \left( \frac{-\log r}{1 - r^2} \right)^{\frac{\alpha|\alpha|-2}{2}} H_\alpha \left( \frac{y - rx}{\sqrt{1 - r^2}} \right) \frac{e^{-\frac{|y-\sqrt{1-t}y|^2}{1-r^2}}}{(1 - r^2)^{\frac{d}{2}+1}} \, dr.
\]

On the other hand, the “new” higher order Gaussian Riesz transforms are given by
\[
R_\alpha^* f(x) = \text{p.v.} \int_{\mathbb{R}^d} \overline{K}_\alpha(x, y) f(y) \, dy,
\]
where
\[
\overline{K}_\alpha(x, y) = C_\alpha \int_0^1 \left( \frac{-\log r}{1 - r^2} \right)^{\frac{\alpha|\alpha|-2}{2}} H_\alpha \left( \frac{x - ry}{\sqrt{1 - r^2}} \right) \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1 - r^2)^{\frac{d}{2}+1}} \, dr \, e^{|x|^2 - |y|^2}
\]
and \( \alpha \) is a multi-index in \( \mathbb{N}_0^d \setminus \{(0, \ldots, 0)\} \).

As we said in the introduction, in the spirit of [7] and [26], we will introduce a larger class of singular integrals, containing \( R_\alpha^* \), and prove their boundedness on \( L^p(\gamma_d) \).

Let \( F \in C^1(\mathbb{R}^d) \) be a function which is orthogonal with respect to the Gaussian measure, i.e.
\[
\int_{\mathbb{R}^d} F(x) \, d\gamma_d(x) = 0,
\]
and for every \( \varepsilon > 0 \) there exists some constant \( C_\varepsilon > 0 \) such that for all \( x \in \mathbb{R}^d \)
(i) \( |F(x)| \leq C_\varepsilon \, e^{\varepsilon|x|^2} \),
(ii) \( |\nabla F(x)| \leq C_\varepsilon \, e^{\varepsilon|x|^2} \).

We define the singular integral operator
\[
\overline{R}_F f(x) = \text{p.v.} \int_{\mathbb{R}^d} \overline{K}_F(x, y) f(y) \, dy,
\]
with kernel
\[
\overline{K}_F(x, y) = \int_0^1 \left( \frac{-\log r}{1 - r^2} \right)^{\frac{m-2}{2}} F \left( \frac{x - ry}{\sqrt{1 - r^2}} \right) \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1 - r^2)^{\frac{d}{2}+1}} \, dr \, e^{|x|^2 - |y|^2}.
\]
If we make the change of variables \( t = 1 - r^2 \) in the integral defining the above kernel we obtain
\[
\overline{K}_F(x, y) = \int_0^1 \psi_m(t) \, F \left( \frac{x - \sqrt{1-t}y}{\sqrt{t}} \right) e^{-\frac{|x-\sqrt{1-t}y|^2}{t}} \left( \frac{d}{t^\frac{d}{2}+1} \right) \frac{dt}{\sqrt{1-t}} \, e^{|x|^2 - |y|^2},
\]

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with
\[ \psi_m(t) = \left( \frac{\log \frac{1}{1-t}}{t} \right)^{\frac{m-2}{2}} = \left( -\frac{\log(1-t)}{t} \right)^{\frac{m-2}{2}} 2^{-\frac{m-2}{2}}. \]

If we set \( F(x) = C_\alpha H_\alpha(x) \), then \( \overline{R}_F = R_\alpha^* \), with \( m = |\alpha| = \alpha_1 + \cdots + \alpha_d \).

The singular integral will be splitting into the local and global parts as follows:
\[ \overline{R}_F f(x) = \overline{R}_F(f \chi_B(x))(x) + \overline{R}_F(f \chi_{B^c}(x))(x) =: Lf(x) + Gf(x). \]

### 3.1. The local part.

In this section, we shall prove that

**Lemma 3.1.** For every \( x \in \mathbb{R}^d \),
\[
Lf(x) \lesssim \sum_{B \in \mathcal{F}} (|T_F(f \chi_B)(x)| + M_{H-L}(f \chi_B)(x)) \chi_B(x),
\]
where \( T_F \) is a singular integral operator, \( M_{H-L} \) is the non-centered Hardy–Littlewood maximal function, and \( \mathcal{F} = \{ B \} \) and \( \mathcal{F} = \{ B \} \) are the families of balls given by [7, Lemma 3.1].

We are going to look at the kernel written as
\[
K_F(x, y) = \int_0^1 \psi_m(t) \frac{F\left( x - \sqrt{1-t} y \right)}{\sqrt{t}} e^{-\frac{|y - \sqrt{1-t}x|^2}{t}} \frac{1}{t^{d+1}} \, dt
\]

where, as before, \( \psi_m(t) = \left( -\frac{\log(1-t)}{t} \right)^{\frac{m-2}{2}} 2^{-\frac{m-2}{2}} \), and \( \phi_m(t) = \frac{\psi_m(t)}{\sqrt{1-t}} \). We remark that \( \lim_{t \to 0^+} \phi_m(t) = 2^{-\frac{m-2}{2}} \), so we can define \( \phi_m(0) = 2^{-\frac{m-2}{2}} \). Besides,
\[
|\phi_m(t) - \phi_m(0)| \leq C \frac{t}{1-t}.
\]

Then
\[
K_F(x, y) = \left( \int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^1 \right) \phi_m(t) F\left( x - \sqrt{1-t} y \right) e^{-\frac{|y - \sqrt{1-t}x|^2}{t}} \frac{1}{t^{d+1}} \, dt
\]

\[ := K_F^1(x, y) + K_F^2(x, y). \]

Let
\[
u(t) := \frac{|y - \sqrt{1-t}x|^2}{t},
\]

\[ (3.2) \]

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and

\[ \overline{w}(t) := \frac{|x - \sqrt{1 - ty}|^2}{t}. \]  

Then \( \overline{w}(t) = u(t) + |x|^2 - |y|^2 \). On \( B(x) \) we have \( u(t) \geq \frac{|x-y|^2}{t} - 2d \) and \( ||x|^2 - |y|^2| \leq 4d \). Thus

\[ |K_F^2(x, y)| \lesssim \int_1^1 \frac{(- \log(1 - t))^{m-2}}{\sqrt{1-t}} e^{\varepsilon \overline{w}(t)} e^{-u(t)} \frac{dt}{t^{d/2}} \]

\[ = \int_1^1 \frac{(- \log(1 - t))^{m-2}}{\sqrt{1-t}} e^{\varepsilon (|x|^2 - |y|^2)} e^{-u(t)} \frac{dt}{t^{d/2}} \]

\[ \lesssim \int_0^1 \frac{(- \log(1 - t))^{m-2}}{\sqrt{1-t}} e^{-u(t)} \frac{dt}{t^{d/2}} \]

\[ \lesssim \left( \int_0^\infty s^{m-2} e^{-s} ds \right) \sup_{t>0} \omega \sqrt{t} (x - y), \]

where we have applied the change of variables \( s = - \log(1 - t) \), and considered \( \omega_t(z) := t^{-d} e^{-(1-\varepsilon)|z|^2/t^2} \).

Now

\[ \overline{K_F^1}(x, y) = \phi_m(0) \int_0^1 \left( \frac{x - \sqrt{1 - ty}}{\sqrt{t}} \right) e^{-u(t)} \frac{dt}{t^{d/2} + 1} \]

\[ - \phi_m(0) \int_{1/2}^1 F \left( \frac{x - \sqrt{1 - ty}}{\sqrt{t}} \right) e^{-u(t)} \frac{dt}{t^{d/2} + 1} \]

\[ + \int_{1/2}^1 (\phi_m(t) - \phi_m(0)) F \left( \frac{x - \sqrt{1 - ty}}{\sqrt{t}} \right) e^{-u(t)} \frac{dt}{t^{d/2} + 1} \]

\[ =: \tilde{K}(x, y) + \overline{K}_{F,1}(x, y) + \overline{K}_{F,2}(x, y). \]

Let us observe that, for \( j = 1, 2 \), we can proceed as before to obtain

\[ |\overline{K}_{F,j}(x, y)| \lesssim e^{\varepsilon (|x|^2 - |y|^2)} \int_0^1 e^{-u(t)} \frac{dt}{t^{d/2}} \lesssim \sup_{t>0} \omega \sqrt{t} (x - y). \]

Following [26], if we define, for \( x \neq 0 \),

\[ K_F(x) = \int_0^\infty F \left( \frac{x}{\sqrt{t}} \right) e^{-\frac{|x|^2}{t}} \frac{dt}{t^{d/2} + 1} = \frac{\Omega(x')}{|x|^d}, \]

\[ Analysis\ Mathemathica\ 48,\ 2022 \]
with \( \Omega(x') = 2 \int_0^\infty F(sx') s^{d-1} e^{-s^2} \, ds \) and \( x' = \frac{x}{|x|} \), we have

\[
\int_{S^{d-1}} \Omega(x') \, d\sigma(x') = 2\pi^{\frac{d}{2}} \int_{\mathbb{R}^d} F(x) \, d\gamma_d(x) = 0,
\]

i.e., \( \mathcal{K}_F(x) \) is a homogeneous kernel of degree \(-d\), and therefore

\[
(3.5) \quad \mathcal{T}_F(f)(x) = \text{p.v.} \, \mathcal{K}_F * f(x)
\]
is a singular integral operator with homogeneous kernel, an example of a singular integral of Calderón–Zygmund type.

Now we write

\[
\overline{K}_F^1(x, y) = \mathcal{K}_F(x - y) + \overline{K}_F^1(x, y) + \overline{K}_F^1(x, y) + \overline{K}_F^1(x, y),
\]

with \( \overline{K}_F^1(x, y) = \tilde{K}(x, y) - \mathcal{K}_F(x - y) \).

Let us recall that for every \( x \in B \in \mathcal{F}, B(x) \subset \mathcal{B} \). Hence, looking at \([26, p. 506]\) (see also \([7, (3.4) \text{ and } (3.8)\]) and taking into account that on \( B(x), |x| \approx |y| \), we have

\[
\int_{B(x)} \overline{K}_F^1(x, y)|f(y)| \, dy \lesssim \left( 1 + |x|^{\frac{1}{2}} \right) \int_{\mathbb{R}^d} \frac{1}{|x - y|^{d-1/2}} |f(y)| \chi_B(y) \, dy 
\lesssim M_{H-L}(f \chi_B)(x),
\]

\[
\int_{B(x)} \overline{K}_F^1(x, y)|f(y)| \, dy \lesssim \int_{\mathbb{R}^d} \sup_{t>0} \omega \sqrt{t} (x - y)|f(y)| \chi_B(y) \, dy 
\lesssim M_{H-L}(f \chi_B)(x) \quad \text{for } j = 1, 2,
\]

and also, from \((3.4)\),

\[
\int_{B(x)} \overline{K}_F^2(x, y)|f(y)| \, dy \lesssim \int_{\mathbb{R}^d} \sup_{t>0} \omega \sqrt{t} (x - y)|f(y)| \chi_B(y) \, dy \lesssim M_{H-L}(f \chi_B)(x).
\]

Finally, for \( x \in B \),

\[
\text{p.v.} \int_{B(x)} \mathcal{K}_F(x - y) f(y) \, dy = \mathcal{T}_F(f \chi_B)(x) - \int_{B \setminus B(x)} \mathcal{K}_F(x - y) f(y) \, dy,
\]

and

\[
\left| \int_{B \setminus B(x)} \mathcal{K}_F(x - y) f(y) \, dy \right| \lesssim M_{H-L}(f \chi_B)(x).
\]

This ends the estimates for the local part of the operator and \((3.1)\) yields.

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3.2. The global part. The aim of this section is to show that

**Lemma 3.2.** For every \( x \in \mathbb{R}^d \), and \( 0 < \varepsilon < \frac{1}{\rho_{\infty}} \wedge \frac{1}{d} \),

\[
Gf(x) \lesssim e^{\varepsilon |x|^2} \left( \int_{\mathbb{R}^d} |f(y)|^{p^*} \gamma_d(dy) \right)^{1/p^*} + e^{\varepsilon |x|^2} \int_{B(x)^c} P(x,y) |f(y)| e^{-\frac{|y|^2}{p(y)}} dy
\]

where \( P(x,y) = |x+y|^d e^{-\alpha_{\infty} |x-y||x+y|} \) and \( \alpha_{\infty} = \frac{1-\varepsilon}{2} - \frac{1}{\rho_{\infty}} - \frac{1+\varepsilon}{2} \).

Let us notice that the global part of these new Gaussian Riesz transforms is strictly larger than the global part of the old ones, but still we get the right estimates on this kernel such that the boundedness of this part also holds.

In order to study the global part of the new higher order Gaussian Riesz transforms, we follow the ideas of [26]. To that end, we recall some notation and results from that article (see also [20] or [21]).

For \( x, y \in \mathbb{R}^d \), we set

\[
a = a(x,y) := |x|^2 + |y|^2 \quad \text{and} \quad b = b(x,y) := 2 \langle x,y \rangle.
\]

On the complement of \( B(x) \), we know that \( a > d/2 \) and, whenever \( b > 0 \), we also have \( \sqrt{a^2 - b^2} = |x+y||x-y| > d \).

Recall the definitions of \( u \) and \( \overline{u} \) given in (3.2) and (3.3). Hence, we can write \( u(t) = \frac{a}{t} - \sqrt{\frac{a^2 - b^2}{a + \sqrt{a^2 - b^2}}} b - |y|^2 \). Both \( u \) and \( \overline{u} \) have a minimum and it is attained at \( t_0 \), given by

\[
t_0 = \begin{cases} 
2 \frac{\sqrt{a^2 - b^2}}{a + \sqrt{a^2 - b^2}} & \text{if } b > 0 \\
1 & \text{if } b \leq 0.
\end{cases}
\]

The minimum value is

\[
u_0 := u(t_0) = \begin{cases} 
\frac{|y|^2 - |x|^2 + |x+y||x-y|}{2} & \text{if } b > 0 \\
|y|^2 & \text{if } b \leq 0.
\end{cases}
\]

Then we have

\[
e^{-u(t)} \frac{1}{t^{d/2}} \leq C e^{-u_0} \frac{1}{t_0^{d/2}} \quad \text{and} \quad e^{-\overline{u}(t)} \frac{1}{t^{d/2}} \leq C e^{-(u_0 + |x|^2 - |y|^2)} \frac{1}{t_0^{d/2}}.
\]

Moreover, the following result holds.

**Lemma 3.3.** Let us consider the kernel \( \overline{K}_F(x,y) \) in the global part, that is, for \( y \in B^c(x) \). The following statements hold:

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(i) If \( b \leq 0 \), for each \( 0 < \varepsilon < 1 \), there exists \( C_\varepsilon > 0 \) such that
\[
|\mathcal{K}_F(x, y)| \leq C_\varepsilon e^{\varepsilon |x|^2 - |y|^2};
\]

(ii) If \( b > 0 \), for each \( 0 < \varepsilon < 1/d \) there exists \( C_\varepsilon > 0 \) such that
\[
|\mathcal{K}_F(x, y)| \leq C_\varepsilon \frac{e^{-(1-\varepsilon)u_0}}{t_0^{d/2}} e^{\varepsilon (|x|^2 - |y|^2)},
\]
where \( t_0 \) and \( u_0 \) are as in (3.7) and (3.8), respectively.

**Proof.** If \( b \leq 0 \), then
\[
\frac{a}{t} - |x|^2 \leq u(t) = \frac{a}{t} - \frac{\sqrt{1-t}}{t} b - |x|^2 \leq \frac{2a}{t},
\]

\[
\mathcal{K}_F(x, y) = \left( \int_0^{1/2} + \int_{1/2}^1 \right) \psi_m(t) F\left( \frac{x - \sqrt{1-t}y}{\sqrt{t}} \right) \frac{e^{-u(t)}}{t^{d/2}} \frac{dt}{\sqrt{1-t}} e^{\varepsilon |x|^2 - |y|^2}
=: I + II.
\]

If \( 0 < m \leq 2 \), \( \psi_m \) is bounded on \([0, 1]\) and
\[
(3.9) \quad |\mathcal{K}_F(x, y)| \lesssim \int_0^1 \frac{e^{-(1-\varepsilon)u(t)}}{t^{d/2+1}} \frac{dt}{\sqrt{1-t}} e^{\varepsilon (|x|^2 - |y|^2)}.
\]

From [26, p. 500], \( |\mathcal{K}_F(x, y)| \) is bounded by \( e^{\varepsilon |x|^2 - |y|^2} \).

On the other hand, if \( m > 2 \), taking into account that \( \psi_m(t) \) is bounded on \([0, 1/2]\), we have
\[
|I| \lesssim \int_0^{1/2} \frac{e^{-(1-\varepsilon)u(t)}}{t^{d/2+1}} \frac{dt}{\sqrt{1-t}} e^{\varepsilon |x|^2 - |y|^2} \lesssim \int_0^1 \frac{e^{-(1-\varepsilon)u(t)}}{t^{d/2+1}} \frac{dt}{\sqrt{1-t}} e^{\varepsilon (|x|^2 - |y|^2)},
\]
and
\[
|II| \lesssim \int_{1/2}^1 (- \log(1-t))^{m-2} e^{-(1-\varepsilon)u(t)} \frac{dt}{\sqrt{1-t}} e^{\varepsilon |x|^2 - |y|^2}
= \int_{1/2}^1 (- \log(1-t))^{m-2} e^{-(1-\varepsilon)u(t)} \frac{dt}{\sqrt{1-t}} e^{\varepsilon |x|^2 - |y|^2}.
\]

As before, \(|I| \lesssim e^{\varepsilon |x|^2 - |y|^2} \). On the other hand, from [26] again with the change of variables \( s = \frac{a}{t} - a \) we get that
\[
|II| \lesssim e^{-(1-\varepsilon)|y|^2} \frac{1}{\sqrt{a}} \int_0^a \left( \log \left( 1 + \frac{a}{s} \right) \right)^{m-2} e^{-(1-\varepsilon)s} \frac{ds}{\sqrt{s}} e^{\varepsilon (|x|^2 - |y|^2)}
\]
which, in turn, by the change of variables $w = \log(1 + \frac{u}{s})$, can be bounded as

$$|II| \lesssim \frac{1}{\sqrt{a}} \int_{\log 2}^{\infty} w^{m-2} \frac{a}{(1 - e^{-w})^2} e^{-\frac{w}{2}(1 - e^{-w}) \frac{1}{2}} \, dw \, e^{\varepsilon |x|^2 - |y|^2} \lesssim \int_{0}^{\infty} w^{m-2} e^{-\frac{w}{2}} \, dw \, e^{\varepsilon |x|^2 - |y|^2} \leq C e^{\varepsilon |x|^2 - |y|^2}.$$

Now, we assume $b > 0$. If $0 < m \leq 2$, we repeat the estimate of (3.9). For $m > 2$,

$$|K_F(x, y)| \lesssim \int_{0}^{1} \psi_m(t) \frac{e^{-(1-\varepsilon)\pi(t)}}{t^\frac{d}{2} + 1} \frac{dt}{\sqrt{1 - t}} \, e^{\varepsilon |x|^2 - |y|^2} = (\int_{0}^{1} + \int_{1}^{1}) \psi_m(t) \frac{e^{-(1-\varepsilon)u(t)}}{t^\frac{d}{2} + 1} \frac{dt}{\sqrt{1 - t}} \, e^{\varepsilon |x|^2 - |y|^2} = I + II.$$

To estimate $I$ we use that $\psi_m$ is bounded and the estimates in [26, p. 500] to get

$$I \lesssim \frac{e^{-(1-\varepsilon)u_0}}{t^d}_0 e^{\varepsilon |x|^2 - |y|^2}.$$

For the other term, we have

$$II \lesssim \int_{0}^{1} (- \log(1 - t))^{m-2} e^{-(1-\varepsilon)u(t)} \frac{dt}{\sqrt{1 - t}} \, e^{\varepsilon |x|^2 - |y|^2} \lesssim \int_{0}^{1} (- \log(1 - t))^{m-2} \frac{dt}{\sqrt{1 - t}} \, e^{-(1-\varepsilon)u_0} e^{\varepsilon |x|^2 - |y|^2}.$$

$$\lesssim \int_{0}^{\infty} w^{m-2} \frac{e^{-(1-\varepsilon)u_0}}{t^d}_0 \frac{dt}{\sqrt{1 - t}} \, e^{\varepsilon |x|^2 - |y|^2} \leq C e^{\varepsilon |x|^2 - |y|^2}. \quad \square$$

Now, we are in position to prove (3.6). From Lemma 3.3(i),

$$\int_{B^c(x) \cap \{b \leq 0\}} |K_F(x, y)||f(y)| \, dy \lesssim e^{\varepsilon |x|^2} \int_{\mathbb{R}^d} |f(y)| \, d\gamma_d(y) \lesssim e^{\varepsilon |x|^2} \left( \int_{\mathbb{R}^d} |f(y)|^p \, d\gamma_d(y) \right)^{1/p^\times}.$$

On the other hand, from Lemma 3.3(ii)

$$\int_{B^c(x) \cap \{b > 0\}} |K_F(x, y)||f(y)| \, dy \lesssim \int_{B^c(x)} \frac{e^{-(1-\varepsilon)u_0}}{t^d}_0 \frac{e^{\varepsilon |x|^2 - |y|^2}}{t^d}_0 \, e^{\varepsilon |x|^2 - |y|^2} |f(y)| \, dy.$$
\[
\int_{B^c(x)} e^{-(1-\varepsilon)u_0} e^{\frac{|y|^2}{p(y)}} e^{\frac{|x|^2}{p(x)}} e^\varepsilon(|x|^2-|y|^2) |f(y)| e^{-\frac{|y|^2}{p(y)}} dy.
\]

Since \( p \in \mathcal{P}_\gamma^\infty(\mathbb{R}^d) \), from [7] we know that
\[
\frac{e^{-(1-\varepsilon)u_0} e^{\frac{|y|^2}{p(y)}} e^{\frac{|x|^2}{p(x)}}}{t_0^{d/2}} \lesssim \frac{e^{(|y|^2-|x|^2)(\frac{1}{p_\infty} - \frac{1-\varepsilon}{2})} e^{-\frac{1-\varepsilon}{2}|x+y||x-y|}}{t_0^{d/2}} \lesssim |x + y|^d e^{(|y|^2-|x|^2)(\frac{1}{p_\infty} - \frac{1-\varepsilon}{2})} e^{-\frac{1-\varepsilon}{2}|x+y||x-y|}.
\]

Then, applying this to (3.10) and taking into account that \(||y|^2-|x|^2| \leq |x + y||x - y|\), we obtain
\[
\int_{B^c(x) \cap \{b>0\}} |K_F(x,y)||f(y)| dy \lesssim e^{\frac{|x|^2}{p(x)}} \int_{B^c(x)} |x+y|^d e^{(|y|^2-|x|^2)(\frac{1}{p_\infty} - \frac{1-\varepsilon}{2})} e^{-\frac{1-\varepsilon}{2}|x+y||x-y|} |f(y)| e^{-\frac{|y|^2}{p(y)}} dy \lesssim e^{\frac{|x|^2}{p(x)}} \int_{B^c(x)} |x+y|^d e^{-\alpha_\infty |x+y||x-y|} |f(y)| e^{-\frac{|y|^2}{p(y)}} dy.
\]

Finally, we may choose \( \varepsilon \) in such a way that \( \alpha_\infty = \frac{1-\varepsilon}{2} - \frac{1}{p_\infty} - \frac{1+\varepsilon}{2} > 0 \); for example we can take \( 0 < \varepsilon < \frac{1}{p_\infty} \wedge \frac{1}{d} \).

4. Proof of main results

In order to prove the \( L^{p(\cdot)} \)-boundedness of \( R_F \) in the Gaussian context, we will use the continuity properties of Calderón–Zygmund singular integrals on the Lebesgue setting.

It is known that \( p \in LH(\mathbb{R}^d) \) is sufficient for the boundedness on \( L^{p(\cdot)}(\mathbb{R}^d) \) of singular integral operators (see [6, Theorem 5.39]). Here, it will be enough to consider singular integral operators with homogeneous kernels. That is, operators of the form
\[
Tf(x) = \lim_{\varepsilon \to 0} \int_{\{|y| \geq \varepsilon\}} \frac{\Omega(y')}{|y|^d} f(x - y) dy,
\]
for \( f \in \mathcal{S} \) (the class of Schwartz functions), where \( \Omega \) is defined on the unit sphere \( S^{d-1} \), is integrable with zero average and \( y' = y/|y| \). This kind of operators, and a wider class of singular integrals, are bounded on \( L^{p(\cdot)}(\mathbb{R}^d) \) (see [11,15,16]). Moreover, the next result is valid on the variable setting.
Theorem 4.1 [4,6]. Let $p \in LH(R^d)$ with $1 < p^- \leq p^+ < \infty$. Then, the Hardy–Littlewood maximal operator $M_{H-L}$ and singular integrals with homogeneous kernels of the form (4.1) are bounded on $L^{p(\cdot)}(R^d)$.

We can now prove our main result.

Theorem 4.2. Let $p \in LH_0(R^d) \cap P_{\gamma_d}^\infty(R^d)$, with $p^- > 1$. Then, there exists a positive constant $C$ such that

$$\|RFf\|_{p(\cdot),\gamma_d} \leq C\|f\|_{p(\cdot),\gamma_d}$$

for every $f \in L^{p(\cdot)}(\gamma_d)$.

Proof. Since for every $x \in R^d$, $|RFf(x)| \lesssim Lf(x) + Gf(x)$, the proof of Theorem 4.2 will follow from the boundedness of $L$ and $G$ on $L^{p(\cdot)}(\gamma_d)$.

The boundedness of $L$ follows the same lines as the proof of [7, Theorem 3.3], by means of (3.1) and applying Theorem 4.1 since the operator $T_F$ given in (3.5) falls in its scope.

In order to prove that

$$\|Gf\|_{p(\cdot),\gamma_d} \lesssim \|f\|_{p(\cdot),\gamma_d},$$

we will carry out the same steps we used in [7] for the “old” Gaussian Riesz transforms, with a few changes. Let $f \in L^{p(\cdot)}(\gamma_d)$ such that $\|f\|_{p(\cdot),\gamma_d} = 1$. We will prove that $\int_{R^d} (Gf(x))^{p(x)} d\gamma_d(x) \lesssim 1$, then by homogeneity the general boundedness will yield.

It is easy to prove that

$$\int_{R^d} e^{\varepsilon p(x)|x|^2} \left( \int_{R^d} |f(y)|^{p^-} d\gamma_d(y) \right)^{p(x)/p^-} d\gamma_d(x) \lesssim 1.$$

Indeed, since $p \in P_{\gamma_d}^\infty(R^d)$, $p(x) \leq p_\infty + \frac{C_{\gamma_d}}{|x|^2}$, and thus

$$e^{\varepsilon p(x)|x|^2} \leq e^{\varepsilon p_\infty|x|^2} e^\varepsilon C_{\gamma_d}.$$

Also

$$\int_{R^d} |f(y)|^{p^-} d\gamma_d(y) \leq 1 + \int_{|f| > 1} |f(y)|^{p^-} d\gamma_d(y) \leq 1 + \int_{R^d} |f(y)|^{p(y)} d\gamma_d(y) \leq 2,$$

so we have

$$\int_{R^d} e^{\varepsilon p(x)|x|^2} \left( \int_{R^d} |f(y)|^{p^-} d\gamma_d(y) \right)^{p(x)/p^-} d\gamma_d(x) \lesssim \int_{R^d} e^{-(1-\varepsilon p_\infty)|x|^2} 2^{p(x)} dx \lesssim 2^{p^+} \int_{R^d} e^{-(1-\varepsilon p_\infty)|x|^2} dx.$$
and the last integral is finite if we take $0 < \varepsilon < \frac{1}{p_\infty}$. Thus, we obtain $(4.2)$ by choosing $0 < \varepsilon < \min\{\frac{1}{p_\infty}, \frac{1}{d}\}$.

On the other hand, it can be proved (see [7]) that

$$D := \sup_{x \in \mathbb{R}^d} \int_{B^c(x)} P(x, y)|f(y)| e^{-\frac{|y|^2}{p(y)}} dy < \infty.$$  

Then, if we set $g(y) := |f(y)| e^{-\frac{|y|^2}{p(y)}} = g_1(y) + g_2(y)$ with $g_1 = g\chi_{\{g > 1\}}$, as it was done in [7], taking into account that $0 \leq \frac{1}{D} \int_{B^c(x)} P(x, y)g_1(y) dy \leq 1$, using Lemma 2.3 conveniently, and realizing that both

$$0 \leq \frac{1}{D} \int_{B^c(x)} P(x, y)g_2(y) dy \leq 1 \quad \text{and} \quad 0 \leq g_2(y) \leq 1,$$

we have

$$\int_{\mathbb{R}^d} |x|^2 \left( \int_{B^c(x)} P(x, y)g(y) dy \right)^{p(x)} d\gamma_d(x) \lesssim \int_{\mathbb{R}^d} \left( \frac{1}{D} \int_{B^c(x)} P(x, y)g_1(y) dy \right)^{p(x)} dx + \int_{\mathbb{R}^d} \left( \frac{1}{D} \int_{B^c(x)} P(x, y)g_2(y) dy \right)^{p(x)} dx \lesssim \int_{\mathbb{R}^d} \left( \left( \int_{B^c(x)} P(x, y)g_1(y) dy \right)^{p^-} + \left( \int_{B^c(x)} P(x, y)g_2(y) dy \right)^{p_\infty} \right) dx + 1 \lesssim \int_{\mathbb{R}^d} (g_1(y)^{p^-} + g_2(y)^{p_\infty}) dy + 1 \lesssim \int_{\mathbb{R}^d} |f(y)|^{p(y)} d\gamma_d(y) + \int_{\mathbb{R}^d} g_2(y)^{p(y)} dy + 1 \lesssim 1.$$

Thus, $\|Gf\|_{p(\cdot), \gamma_d} \lesssim 1$. This ends the proof of Theorem 4.2. \(\square\)

5. The non-centered Gaussian maximal function

Now let us introduce here the non-centered maximal function associated to the non-standard Gaussian measure, i.e.,

$$\mathcal{M}_{\gamma_d} f(x) = \sup_{B \ni x} \frac{1}{\gamma_d(B)} \int_B |f(y)| d\gamma_d(y),$$

where the supremum is taken over every ball $B$ of $\mathbb{R}^d$ containing $x$. 

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It is known that this maximal function is a bounded operator on $L^p(\gamma_d)$ for $1 < p \leq \infty$ (see [13]) and it is not weak type $(1,1)$ (see [27]).

Under certain conditions on the exponent $p(\cdot)$, we are going to prove the boundedness of $M_{\gamma_d}$ on $L^{p(\cdot)}(\gamma_d)$ whenever $1 < p^- \leq p^+ < \infty$.

As a matter of fact we will prove a result that contains this one where we extend the space $\mathbb{R}^d$ to a metric space $X$ in which a positive $\sigma$-finite measure $\mu$ is defined such that $0 < \mu(B) < \infty$ for all ball $B$ in $X$. So the non-centered maximal function associated to $\mu$ is

$$M_\mu f(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y),$$

where the supremum is taken over every ball $B$ of $X$ containing $x$.

In [1, Theorem 1.7] they gave a proof of this result for the centered maximal function

$$M^c_\mu f(x) = \sup_{r > 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| \, d\mu(y).$$

In their technique of proof, they used, among other things, that this centered maximal function is weak-type $(1,1)$ which in the case of the non-centered maximal function this statement need not be true (see [27]).

For our purposes, $M_\mu$ will be defined pointwise for $\mu$-a.e. $x \in X$. Since we are dealing with averages over balls, and with pointwise estimates for them (see Proposition 5.9) we are going to assume, when calculating the maximal function, that the supremum over all balls will coincide with the supremum over a collection of countable balls that cover all of $X$. That is, there exists a countable family of balls $\mathcal{F}$ such that $\bigcup \mathcal{F} = X$ and

$$M_\mu f(x) = \sup_{B \in \mathcal{F}, B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y).$$

From now on, we are going to consider just those balls belonging to $\mathcal{F}$, without mentioning it. Let us remark that this is true for the case we are concerned, that is, when $\mu = \gamma_d$.

We give some notation. We will denote by $\mathcal{P}(X,d,\mu)$ the set of bounded exponents over the metric space $(X,d)$ with respect to a positive $\sigma$-finite measure $\mu$. It can be proved that the variable Lebesgue space $L^{p(\cdot)}(\mu)$ is a Banach space (see [19, Lemma 3.1]). We will still denote, as before, by $\varrho_{p(\cdot),\mu}$ and $\|\cdot\|_{p(\cdot),\mu}$ the modular and the norm on $L^{p(\cdot)}(\mu)$, respectively.

**Definition 5.1.** Let $(X,d)$ be a metric space in which a positive $\sigma$-finite measure $\mu$ is defined such that for every ball $B$ in $X$, $0 < \mu(B) < +\infty$. We
say that a \(\mu\)-measurable function \(p: X \to [1, +\infty)\) belongs to \(\mathcal{P}_\mu(X)\) if there exists a constant \(c_\mu\) with \(0 < c_\mu < 1\) such that

\[
\mu(B)^{p^+_\mu - p^-_\mu} \geq c_\mu,
\]

for all ball \(B\) in \(X\).

The relationship between this measure \(\mu\) and the exponent function \(p\) expressed in (5.1) says a lot about the behaviour of \(p\) both locally and its decay at infinity if we are dealing with an unbounded space \(X\). For the case of the non-standard Gaussian measure \(\gamma_d\), we will get necessary and sufficient continuity conditions which \(p\) must meet in order to hold inequality (5.1) true.

In this context we will prove the following theorem.

**Theorem 5.2.** Let \(\mu\) be a positive \(\sigma\)-finite measure on \(X\) a metric space such that \(0 < \mu(B) < \infty\) for every ball \(B\). Let \(p \in \mathcal{P}_\mu(X)\) with \(p^- > 1\).

If \(\mu(X) = \infty\), we also assume that there exists \(p_\infty \in [1, \infty)\) such that

\[1 \in L^{s(\cdot)}(\mu), \text{ where } \frac{1}{s(x)} = \left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right|.
\]

Then, if \(M_\mu\) is bounded on \(L^{p_\infty}(\mu)\), we have

\[
\|M_\mu f\|_{p(\cdot)} \leq K\|f\|_{p(\cdot)}
\]

for every \(f \in L^{p(\cdot)}(\mu)\).

**Remark 5.3.** In the Euclidean setting, the assumption \(1 \in L^{s(\cdot)}(\mathbb{R}^d)\) is nothing but Nekvinda's integral condition on the exponent \(p\) (see [24]). This property is strictly weaker than \(p \in LH_\infty(\mathbb{R}^d)\) (see for example [6, Proposition 4.9]) but also sufficient, together with the local log-Hölder condition \(LH_0(\mathbb{R}^d)\), for the boundedness of \(M_{H-L}\) on \(L^{p(\cdot)}(\mathbb{R}^d)\) as proved by Nekvinda in the mentioned article [24].

**Remark 5.4.** The proof can be done as in [1]. But in their proof the authors obtain the result for the centered maximal function. Indeed, they use that this maximal function is weak-type \((1, 1)\) which for the non-centered one this claim need not be true as aforementioned, see [27]. However, we extend this result, following closely their proof, to the non-centered maximal function.

We should also note that in the lemmas and theorems their proof is based on, the phrase “\(\mu\)-almost everywhere” should be included since the exponent function may not necessarily be continuous under the given conditions.

We recall some auxiliary results given in [1] for the sake of completeness, and we state them taking into account the previous remark.

Next lemma corresponds to [1, Lemma A1]. Here, we establish the precise constant \(\beta \in (0, 1)\) for our case, given that \(p^+ < \infty\). Indeed, \(\beta = c_\mu\).
LEMMA 5.5. Let $p \in \mathcal{P}_\mu(X)$. Then

$$
\left( c_\mu \left( \frac{\lambda}{\mu(B)} \right)^{\frac{1}{p'}} \right)^{p(x)} \leq \frac{\lambda}{\mu(B)},
$$

for every $\lambda \in [0, 1]$, for $\mu$-a.e. $x \in B$ and for each ball $B$ in $X$.

The above condition yields the following estimate, that will lead to a pointwise inequality for $\mathcal{M}_\mu$.

LEMMA 5.6. Let $p \in \mathcal{P}_\mu(X)$ be given and define $q : X \times X \to [1, +\infty]$ as follows:

$$
\frac{1}{q(x, y)} = \max \left\{ \frac{1}{p(x)} - \frac{1}{p(y)}, 0 \right\}.
$$

Then, for every $\gamma \in (0, 1)$, there exists $\delta \in (0, 1)$ such that

$$
(5.2) \quad \left( \delta \int_B |f(y)| d\mu(y) \right)^{p(x)} \leq \int_B |f(y)|^{p(y)} d\mu(y) + \int_B \gamma^{q(x,y)} d\mu(y),
$$

for every ball $B$ in $X$, $\mu$-a.e. $x \in B$, and $f \in L^{p(\cdot)}(\mu)$ with $\|f\|_{p(\cdot), \mu} \leq \frac{1}{2}$. Here, we set $\gamma^\infty = 0$.

PROOF. Taking into account the embedding result given in [10, Theorem 3.3.11], for $f \in L^{p(\cdot)}(\mu)$ there exist $f_0 := \max\{|f| - 1, 0\} \in L^{\frac{p(\cdot)}{p^-}}(\mu)$ and $f_1 := \min\{|f|, 1\} \in L^{\infty}(\mu)$ such that $|f| = f_0 + f_1$ and their norms verify $\|f_0\|_{p(\cdot)/p^-, \mu} + \|f_1\|_{\infty, \mu} \leq 2\|f\|_{p(\cdot), \mu} \leq 1$, since we shall assume $\|f\|_{p(\cdot), \mu} \leq \frac{1}{2}$.

Let $B \subset X$ be a ball and $E_B \subset B$ with $\mu(E_B) = 0$ such that for any $x \in B \setminus E_B$, $1 \leq p(x) < +\infty$. Fix such an $x$.

Let $\beta \in (0, 1)$ be the constant obtained in Lemma 5.5. We can also assume $\beta \leq \gamma$. Now we will call $g$ to either $f_0$ or $f_1$, and split it into the sum of three functions:

$$
g_1(y) = g(y) \chi_{\{z \in B : |g(z)| > 1\}}(y),
g_2(y) = g(y) \chi_{\{z \in B : |g(z)| \leq 1, p(z) \leq p(x)\}}(y),
g_3(y) = g(y) \chi_{\{z \in B : |g(z)| \leq 1, p(z) > p(x)\}}(y).
$$

Let us remark that $g \chi_B = g_1 + g_2 + g_3$. Let us also observe that $(f_1)_1 \equiv 0$.

By the convexity of $t \mapsto t^{p(x)}$,

$$
\left( \frac{\beta}{3} \int_B g(y) d\mu(y) \right)^{p(x)} \leq \frac{1}{3} \sum_{j=1}^{3} \left( \beta \int_B g_j(y) d\mu(y) \right)^{p(x)} =: \frac{1}{3}(I_1 + I_2 + I_3).
$$
Let us prove that
\[ I_j \leq \int_B g(y)^{p(y)} \, d\mu(y), \quad j = 1, 2 \]
and
\[ I_3 \leq \int_B g(y)^{p(y)} \, d\mu(y) + \int_B \gamma^{q(x,y)} \, d\mu(y). \]

By applying Hölder’s inequality and taking into account that \( t \mapsto t^{p(x)} \) is a non-decreasing function, we get
\[ I_1 \leq \left( \beta \left( \int_B g_1(y)^{p(y)} \, d\mu(y) \right)^{\frac{1}{p_B}} \right)^{p(x)}. \]

Since \( g_1 = 0 \) or \( g_1 > 1 \) and \( p_B \leq p(y) \) \( \mu \)-a.e. \( y \in B \), we have \( g_1^{p_B}(y) \leq g_1^{p(y)}(y) \) \( \mu \)-a.e. \( y \in B \). Then
\[ I_1 \leq \left( \beta \left( \int_B g_1(y)^{p(y)} \, d\mu(y) \right)^{\frac{1}{p_B}} \right)^{p(x)}. \]

Since \( (f_1)_1 = 0 \) we have \( I_1 = 0 \) for \( g = f_1 \). On the other hand, since \( \|f\|^p_{p(\cdot),\mu} \leq \frac{1}{2} \) then \( \int_B (f_0)_1^{p(y)} \, d\mu(y) \leq 1 \). So by applying Lemma 5.5 with \( \lambda = \int_B g_1^{p(y)} \, d\mu(y), 0 \leq \lambda \leq 1 \), we get
\[ I_1 \leq \int_B g_1(y)^{p(y)} \, d\mu(y) \leq \int_B g(y)^{p(y)} \, d\mu(y). \]

The Jensen inequality implies that
\[ I_2 \leq \int_B (\beta |g_2(y)|)^{p(x)} \, d\mu(y). \]

Since \( \beta |g_2(y)| \leq |g_2(y)| \leq 1 \) and \( t^{p(x)} \leq t^{p(y)} \) for all \( t \in [0, 1] \) when \( p(y) \leq p(x) \), we obtain that
\[ I_2 \leq \int_B (|g_2(y)|)^{p(y)} \, d\mu(y) \leq \int_B (|g(y)|)^{p(y)} \, d\mu(y) \leq \int_B |g(y)|^{p(y)} \, d\mu(y). \]

Finally, for \( I_3 \), by the Jensen inequality we get
\[ I_3 \leq \int_B (\beta |g(y)|)^{p(x)} \chi_{\{|y\in B:|g(y)| \leq 1, p(y) > p(x)\}} \, d\mu(y). \]
Now, Young’s inequality (see e.g. [10, Lemma 3.2.15]), the definition of \( q(x,y) \) and the inequality \( \beta \leq \gamma \) give that

\[
I_3 \leq \int_B \left( \left( \frac{\beta |g(y)|}{\gamma} \right)^p(y) + \gamma^q(x,y) \right) \chi\{y \in B : |g(y)| \leq 1, p(y) > p(x)\} \, d\mu(y)
\]

\[
\leq \int_B |g(y)|^p(y) \, d\mu(y) + \int_B \gamma^q(x,y) \, d\mu(y).
\]

This proves inequality (5.2) for \( f_0 \) and \( f_1 \). To get that inequality for \( f \), taking into account that \( t \mapsto t^{p(x)} \) is a convex function, we argue as follows:

\[
\left( \frac{\beta}{6} \int_B |f(y)| \, d\mu(y) \right)^{p(x)} \leq \frac{1}{2} \left[ \left( \frac{\beta}{3} \int_B f_0(y) \, d\mu(y) \right)^{p(x)} + \left( \frac{\beta}{3} \int_B f_1(y) \, d\mu(y) \right)^{p(x)} \right].
\]

By applying the lemma for \( f_0 \) and \( f_1 \) and taking into account that \( f_j \leq |f| \) for \( j = 0, 1 \), we prove this lemma for \( f \) as well by choosing \( \delta = \frac{\beta}{6} \). \( \square \)

The following lemma is immediate, and will be used in the proof of Theorem 5.2 for the case \( \mu(X) = \infty \).

**Lemma 5.7.** Let \( q \) be the exponent defined in Lemma 5.6 and define a new exponent \( s : X \to [1, +\infty] \) by

\[
\frac{1}{s(x)} = \left| \frac{1}{p(x)} - \frac{1}{p_\infty} \right|,
\]

for some constant \( p_\infty \in [1, \infty) \). Then

\[
t^{q(x,y)} \leq t^{\frac{s(x)}{2}} + t^{\frac{s(y)}{2}}
\]

for every \( t \in [0,1] \).

By combining Lemmas 5.6 and 5.7, the following result can be deduced.

**Theorem 5.8.** Let \( p \in \mathcal{P}_\mu(X) \). Then for every \( \gamma \in (0,1) \) there exists \( \delta \in (0,1) \) such that

\[
\left( \delta \int_B |f(y)| \, d\mu(y) \right)^{p(x)} \leq \int_B |f(y)|^p(y) \, d\mu(y) + \int_B \left( \gamma^\frac{s(x)}{2} + \gamma^\frac{s(y)}{2} \right) \, d\mu(y),
\]

for every ball \( B \) in \( X \), \( \mu \)-a.e. \( x \in B \), \( f \in L^{p(\cdot)}(\mu) \) with \( \|f\|_{p(\cdot),\mu} \leq \frac{1}{2} \), being \( s(\cdot) \) as in Lemma 5.7.

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Proposition 5.9. Let \( p \in P_\mu(X) \). Then for every \( \gamma \in (0, 1) \) there exists \( \delta \in (0, 1) \) such that

\[
(5.3) \quad (\delta M_\mu f(x))^{p(x)} \leq M_\mu(|f|^{p(\cdot)}(x) + 2M_\mu(\gamma^{\frac{1}{\gamma}}))(x),
\]

for all \( f \in L^{p(\cdot)}(\mu) \) with \( \|f\|_{p(\cdot),\mu} \leq \frac{1}{2} \), \( \mu \)-a.e. \( x \in X \), being \( s(\cdot) \) as in Lemma 5.7.

Proof. It is immediate from Theorem 5.8 taking the supremum over all \( \gamma \in (0, 1) \) \( \cap \) \( P(X) \). By applying inequality (5.3) from Proposition 5.9 for \( \mu \) we get

\[
(\delta M_\mu f(x))^{p(x)} = (|f|^{q(\cdot)}(x))^{p(x)} \leq (\lambda M_\mu(|f|^{q(\cdot)}(x)))^{p(x)} + (M_\mu(\gamma^{\frac{1}{\gamma}}))(x)^{p(x)}.
\]

Integrating over \( X \) yields

\[
\varrho_{p(\cdot),\mu}^{(\delta M_\mu f)} \lesssim \|M_\mu(|f|^{q(\cdot)})\|_{p^{-,\mu}}^{p^{-}} + \|M_\mu(\gamma^{\frac{1}{\gamma}})\|_{p^{-,\mu}}^{p^{-}}.
\]

If \( \mu(X) < +\infty \), we use that the maximal \( M_\mu \) is bounded on both \( L^{p^{-}}(\mu) \) and \( L^{\infty}(\mu) \) taking into account that \( |f|^{q(\cdot)} \in L^{p^{-}}(\mu) \) and \( \gamma^{\frac{1}{\gamma}} \in L^{\infty}(\mu) \) for every \( \gamma \in (0, 1) \) with \( \|\gamma^{\frac{1}{\gamma}}\|_{\infty,\mu} \leq 1 \). Thus

\[
\varrho_{p(\cdot),\mu}^{(\delta M_\mu f)} \lesssim \|f|^{q(\cdot)}\|_{p^{-,\mu}}^{p^{-}} + \mu(X) \lesssim 1 + \mu(X) < +\infty.
\]

If \( \mu(X) = +\infty \), since \( 1 \in L^{s(\cdot)}(\mu) \) then there exists \( \lambda > 1 \) such that

\[
\int_X \left(\frac{1}{\lambda}\right)^{s(y)} d\mu(y) < +\infty.
\]

By taking \( \gamma = \lambda^{-2} \in (0, 1) \) we have that \( \gamma^{\frac{1}{\gamma}} \in L^{1}(\mu) \cap L^{\infty}(\mu) \) and hence \( \gamma^{\frac{1}{\gamma}} \in L^{p^{-}}(\mu) \). And since \( M_\mu \) is bounded on \( L^{p^{-}}(\mu) \) we have

\[
\varrho_{p(\cdot),\mu}^{(\delta M_\mu f)} \lesssim \|f|^{q(\cdot)}\|_{p^{-,\mu}}^{p^{-}} + \|\gamma^{\frac{1}{\gamma}}\|_{p^{-,\mu}}^{p^{-}} \lesssim 1 + \|\gamma^{\frac{1}{\gamma}}\|_{p^{-,\mu}}^{p^{-}} < +\infty.
\]

And with this we end the proof of this theorem. \( \square \)
We are now interested in giving sufficient pointwise conditions on $p(\cdot)$ such that $p \in \mathcal{P}_{\gamma_d}(\mathbb{R}^d)$ holds.

Condition $p \in \mathcal{P}_\mu(X)$ is a generalization of Diening’s geometric condition (2.3) when $\mu$ is the Lebesgue measure and $(X, d) = (\mathbb{R}^d, |\cdot|)$. However, it is not necessarily true that this is equivalent to the local log-Hölder condition $LH_0(\mathbb{R}^d)$ for every measure, see [19, Lemma 3.6].

From now on, for a given ball $B$ of radius $r_B > 0$, we denote by $q_B$ the point in the closure of $B$ whose distance to the origin is minimal, i.e., $q_B \in B$ and $|q_B| = \text{dist}(0, B)$.

The next lemma is technical and although it can be found as a partial result in the proof of [13, Lemma 1] we are including it here for the sake of completeness.

**Lemma 5.10 [13].** Let $B$ be a ball of $\mathbb{R}^d$ of radius $r_B > 0$, and let $q_B$ as defined before. If $|q_B| \geq 1$ and $r_B \geq 1/|q_B|$, then
\[
\gamma_d(B) \geq C e^{-\frac{|q_B|^2}{|q_B|^2}} \left( 1 \wedge \left( \frac{r_B}{|q_B|^2} \right)^{\frac{d-1}{2}} \right),
\]
where $C$ does not depend on $B$.

**Proof.** Consider the hyperplane orthogonal to $q_B$ whose distance from the origin is $|q_B| + t$, with $\frac{1}{2|q_B|} < t < \frac{1}{|q_B|}$. Its intersection with $B$ is a $(d-1)$-dimensional ball whose radius is at least $C \sqrt{r_B t} \geq \tilde{C} \sqrt{r_B/|q_B|}$. Integrating the Gaussian density first along this $(d-1)$-dimensional ball and then in $t$, we get
\[
\gamma_d(B) \geq \int_{1/(2|q_B|)}^{1/|q_B|} e^{-(t+|q_B|)^2} \int_{|v|<\tilde{C} \sqrt{r_B/|q_B|}} e^{-|v|^2} dv dt
\]
where $v$ is a $(d-1)$-dimensional variable. The inner integral here is at least $C \left( 1 \wedge \left( \frac{r_B}{|q_B|^2} \right)^{(d-1)/2} \right)$, and $e^{-(t+|q_B|)^2} \geq Ce^{-|q_B|^2}$ for these $t$. Therefore
\[
\gamma_d(B) \geq C e^{-\frac{|q_B|^2}{|q_B|^2}} \left( 1 \wedge \left( \frac{r_B}{|q_B|^2} \right)^{\frac{d-1}{2}} \right). \quad \square
\]

Now we are in position to give sufficient conditions for the validity of $p \in \mathcal{P}_{\gamma_d}(\mathbb{R}^d)$ and, consequently, for the boundedness of $\mathcal{M}_{\gamma_d}$.

**Lemma 5.11.** Let $p \in LH_0(\mathbb{R}^d)$ be given and assume that there exists a constant $C_{\gamma_d}$ such that
\[
(5.4) \quad p_B^+ - p_B^- \leq \frac{C_{\gamma_d}}{|q_B|^2}
\]
for every ball $B$. Then $p \in \mathcal{P}_{\gamma_d}(\mathbb{R}^d)$.

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Proof. Let $B$ be a ball of center $c_B$ and radius $r_B > 0$. Then, there exists $0 < c < 1$ such that

$$
\begin{align*}
\gamma_d(B) &\geq c e^{-|q_B|^2} |B| & \text{if } r_B \leq 1 \wedge \frac{1}{|q_B|} \\
\gamma_d(B) &\geq c & \text{if } |q_B| < 1 \text{ and } r_B > 1 \\
\gamma_d(B) &\geq c e^{-(d+1)|q_B|^2} & \text{if } |q_B| \geq 1 \text{ and } r_B > \frac{1}{|q_B|}.
\end{align*}
$$

(5.5)

Indeed, if $r_B \leq 1 \wedge \frac{1}{|q_B|}$ and taking into account that $|y| < |q_B| + 2r_B$ for $y \in B$, then

$$
\gamma_d(B) = \frac{1}{\pi^{d/2}} \int_B e^{-|y|^2} dy \geq \frac{1}{\pi^{d/2}} e^{-\frac{1}{2}(|q_B|^2 + 2r_B)^2} |B| \geq \frac{e^{-8}}{\pi^{d/2}} e^{-|q_B|^2} |B|.
$$

If we are in the case $|q_B| < 1$ and $r_B > 1$, first assume that $|c_B| \leq 1$. Then $B(c_B, 1) \subset B = B(c_B, r_B)$ and, thus, $\gamma_d(B) \geq \gamma_d(B(c_B, 1))$. In this case, we also have $|y| \leq 2$ for every $y \in B(c_B, 1)$, hence $\gamma_d(B) \geq e^{-4(\omega_d/\pi^{d/2})}$. On the other hand, if $|c_B| > 1$, we have $B(q_B + \frac{1}{|c_B|} c_B, 1) \subset B$, and for every $y \in B(q_B + \frac{1}{|c_B|} c_B, 1)$, $|y| \leq 3$. Hence, $\gamma_d(B) \geq e^{-9(\omega_d/\pi^{d/2})}$.

Now let us consider the case $|q_B| \geq 1$ and $r_B > \frac{1}{|q_B|}$. Then, from Lemma 5.10, we know that

$$
\gamma_d(B) \geq c \frac{e^{-|q_B|^2}}{|q_B|} \left(1 \wedge \left(\frac{r_B}{|q_B|}\right)^{\frac{d-1}{2}}\right) \geq c \frac{e^{-|q_B|^2}}{|q_B|} \left(1 \wedge \left(\frac{1}{|q_B|^2}\right)^{\frac{d-1}{2}}\right)
$$

$$
= c \frac{e^{-|q_B|^2}}{|q_B|^d} = c e^{-(|q_B|^2 + d \log |q_B|)} \geq c e^{-(d+1)|q_B|^2}.
$$

This finishes the proof of (5.5).

Now, taking into account these estimates and condition (5.4), it can be easily seen that there exists a constant $c$ independent of $B$ such that

$$
\gamma_d(B)^{p_B^{\infty} - p_B} \geq c
$$

for every ball $B$ in $\mathbb{R}^d$. The assumption $p \in LH_0(\mathbb{R}^d)$ is needed to estimate the case $r_B \leq 1 \wedge \frac{1}{|q_B|}$. □

Condition (5.4) is actually equivalent to $\mathcal{P}^{\infty}_{\gamma_d}$, the condition considered at previous sections for the Gaussian Riesz transforms. We have the following lemma.

**Lemma 5.12.** Let $p \in \mathcal{P}(\mathbb{R}^d, \gamma_d)$. The following conditions are equivalent.

(i) $p$ verifies (5.4);

(ii) $p$ verifies (5.4) for every half-space $\mathbb{R}^d_+$.

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(ii) \( p \in \mathcal{P}_{\gamma_d}^\infty \), that is, for some constant \( p_\infty \in [1, \infty) \), there exists \( C_{\gamma_d} > 0 \) such that

\[
|p(x) - p_\infty| \leq \frac{C_{\gamma_d}}{|x|^2} \quad \text{for all } x \in \mathbb{R}^d \setminus \{(0, \ldots, 0)\};
\]

(iii) \( p \) satisfies the inequality

\[
|p(x) - p(y)| \leq \frac{C_{\gamma_d}}{|x|^2} \quad \text{for all } |y| \geq |x|,
\]

for some \( \widehat{C}_{\gamma_d} > 0 \).

**Proof.** It is easy to see that condition \( p \in \mathcal{P}_{\gamma_d}^\infty \) is equivalent to the condition (5.7) where \( p_\infty \) happens to be the limit of \( p(x) \) as \( |x| \to \infty \), uniformly in all directions.

Keeping in mind this fact, we prove (5.4) \( \Rightarrow \) (5.7) and (5.6) \( \Rightarrow \) (5.4).

Assume (5.4) holds. Take \( |x| > \sqrt{2} \) (for \( |x| \leq \sqrt{2} \) the proof is immediate).

Let us define \( q_x = (|x| - \frac{1}{|x|}) \frac{x}{|x|} \) and the hyperspace

\[
H_0 = \{ z \in \mathbb{R}^d : x \cdot (z - x) \geq 0 \}.
\]

For \( y \in H_0 \) we have \( x \cdot y \geq |x|^2 \) and it is easy to check that \( |y| \geq |x| \). Now we can choose a ball \( B \) with \( q_B = q_x \) and \( x, y \in B \). Indeed, \( B = B(c_B, r_B) \) is chosen in such a way that its center is \( c_B = \lambda x \) for some \( \lambda > 1 \) and its radius

\[
r_B = |q_x - c_B| = (\lambda - 1)|x| + \frac{1}{|x|}.
\]

It is immediate that \( |x - c_B| < r_B \). The parameter \( \lambda \) will be chosen greater than 1 and depending on \( x \) and \( y \) subject to the condition \( |y - c_B| < r_B \). That is, \( \lambda > \frac{2 + |y|^2 - |x|^2 + \frac{1}{|x|^2}}{2(1 + x \cdot y - |x|^2)} \). Then, taking into account (5.4) we have

\[
|p(y) - p(x)| \leq p^+_B - p^-_B \leq \frac{4C_{\gamma_d}}{|q_x|^2} \leq \frac{4C_{\gamma_d}}{|x|^2},
\]

since \( |q_x| \geq \frac{|x|}{2} \). Thus, \( |p(y) - p(x)| \leq \frac{4C_{\gamma_d}}{|x|^2} \), for every \( y \in H_0 \).

Now let us fix an angle \( \theta \in (-\pi, \pi) \) and consider \( \rho_\theta \) a rotation of an angle \( \theta \) about the origin. Let us call \( q_\theta = \rho_\theta q_x \) and \( x_\theta = \rho_\theta x \) and define

\[
H_\theta = \{ z \in \mathbb{R}^d : x_\theta \cdot (z - x_\theta) \geq 0 \}.
\]

Since the module of a vector in \( \mathbb{R}^d \) is invariant under rotations, we have \( |q_\theta| = |q_x| \geq \frac{|x|}{2} \) and \( |x_\theta| = |x| \). Now we apply the same procedure as before and we get that

\[
|p(y) - p(x_\theta)| \leq \frac{4C_{\gamma_d}}{|x|^2}
\]

for every \( y \in H_\theta \).
Let us remark that \( H_\theta \cap H_0 \neq \emptyset \) if and only if \(-\pi < \theta < \pi\).

For \( y \in \mathbb{R}^d \) such that \(|y| \geq |x|\), let \( \theta \) be the angle between \( x \) and \( y \) such that \(|\theta| < \pi\), then \( y \in H_\theta \). Let \( z \in H_\theta \cap H_0 \), then

\[
|p(y) - p(x)| \leq |p(y) - p(x_\theta)| + |p(x_\theta) - p(z)| + |p(z) - p(x)| \leq \frac{12C_{\gamma_d}}{|x|^2}.
\]

We have proved (5.7) for \( y \notin (B(0, |x|) \cup \{\alpha x : \alpha \leq -1\}) \). Since we have \( \{\alpha x : \alpha \leq -1\} \subset H_\pi \) then for \( 0 < \theta < \pi \), we have \( H_\pi \cap H_\theta \neq \emptyset \) and proceeding as before we get the estimate \((16C_{\gamma_d})/|x|^2\) for \( y \in \{\alpha x : \alpha \leq -1\}\). This ends the proof of (5.7).

Now, let us assume (5.6) holds. For \( B \) a ball with center at \( c_B \) and radius \( r_B \) such that \( q_B \neq 0 \), and \( x \in B \), we know that \(|p(x) - p_\infty| \leq \frac{C_d}{|q_B|} \), and then

\[
p_\infty - \frac{C_{\gamma_d}}{|q_B|} \leq p_B^- \leq p_B^+ \leq p_\infty + \frac{C_d}{|q_B|}.
\]

From this we easily get (5.4).

□

Combining the previous lemmas, we deduce the following fact.

**Corollary 5.13.** Let \( p \in LH_0(\mathbb{R}^d) \cap P_{\gamma_d}^\infty(\mathbb{R}^d) \). Then \( p \in P_{\gamma_d}(\mathbb{R}^d) \).

Since \( M_{\gamma_d} \) is bounded on \( L^{p^-}(\gamma_d) \) and \( \gamma_d(\mathbb{R}^d) < \infty \), we can apply Corollary 5.13 and Theorem 5.2 in order to get the continuity of \( M_{\gamma_d} \) on variable Lebesgue spaces with respect to \( \gamma_d \) under the same sufficient conditions obtained for the boundedness of the Gaussian Riesz transforms.

**Theorem 5.14.** Let \( p \in LH_0(\mathbb{R}^d) \cap P_{\gamma_d}^\infty(\mathbb{R}^d) \) with \( p^- > 1 \). Then, there exists a constant \( K > 0 \) such that

\[
\|M_{\gamma_d}f\|_{p(\cdot),\gamma_d} \leq K \|f\|_{p(\cdot),\gamma_d}
\]

for every \( f \in L^{p(\cdot)}(\gamma_d) \).

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**References**

[1] T. Adamowicz, P. Harjulehto, and P. Hästö, Maximal operator in variable exponent Lebesgue spaces on unbounded quasimetric measure spaces, *Math. Scand.*, 116 (2015), 5–22.

[2] H. Aimar, L. Forzani, and R. Scotto, On Riesz transforms and maximal functions in the context of Gaussian harmonic analysis, *Trans. Amer. Math. Soc.*, 359 (2007), 2137–2154.

[3] V. I. Bogachev, Ornstein–Uhlenbeck operators and semigroups, *Russian Math. Surveys*, 73 (2018), 191–260.

[4] D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer, The maximal function on variable \( L^p \) spaces, *Ann. Acad. Sci. Fenn. Math.*, 28 (2003), 223–238.
[5] E. Dalmazzo and R. Scotto, Corrections to: “The maximal function on variable $L^p$ spaces” Ann. Acad. Sci. Fenn. Math., 29 (2004), pp. 247–249.

[6] D. Cruz-Uribe and A. Fiorenza, Variable Lebesgue Spaces, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer (Heidelberg, 2013).

[7] E. Dalmazzo and R. Scotto, Riesz transforms on variable Lebesgue spaces with Gaussian measure, Integral Transforms Spec. Funct., 28 (2017), 403–420.

[8] E. B. Davies, Heat Kernels and Spectral Theory, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press (Cambridge, 1990).

[9] L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(x)}$, Math. Inequal. Appl., 7 (2004), 245–253.

[10] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, Lebesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Mathematics, vol. 2017, Springer (Heidelberg, 2011).

[11] J. Duoandikoetxea, Fourier Analysis, Graduate Studies in Mathematics, vol. 29, American Mathematical Society (Providence, RI, 2001).

[12] L. Forzani, E. Sasso, and R. Scotto, $L^p$ boundedness of Riesz transforms for orthogonal polynomials in a general context, Studia Math., 231 (2015), 45–71.

[13] L. Forzani, R. Scotto, P. Sjögren, and W. Urbina, On the $L^p$ boundedness of the non-centered Gaussian Hardy–Littlewood maximal function, Proc. Amer. Math. Soc., 130 (2002), 73–79.

[14] L. Forzani, R. Scotto, and W. Urbina, A simple proof of the $L^p$ continuity of the higher order Riesz transforms with respect to the Gaussian measure $\gamma_d$, in: Séminaire de Probabilités, XXXV, Lecture Notes in Math., vol. 1755, Springer, (Berlin, 2001), pp. 162–166.

[15] J. García-Cuerva and J. L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co. (Amsterdam, 1985).

[16] L. Grafakos, Classical Fourier Analysis, 3rd ed., Graduate Texts in Mathematics, vol. 249, Springer (New York, 2014).

[17] A. Grigor’yan, Heat Kernel and Analysis on Manifolds, AMS/IP Studies in Advanced Mathematics, vol. 47, American Mathematical Society (Providence, RI), International Press (Boston, MA, 2009).

[18] C. E. Gutiérrez, C. Segovia, and J. L. Torrea, On higher Riesz transforms for Gaussian measures, J. Fourier Anal. Appl., 2 (1996), 583–596.

[19] P. Harjulehto, P. Hästö, and M. Pere, Variable exponent Lebesgue spaces on metric spaces: the Hardy–Littlewood maximal operator, Real Anal. Exchange, 30 (2004/05), 87–103.

[20] T. Menárguez, S. Pérez, and F. Soria, Pointwise and norm estimates for operators associated with the Ornstein–Uhlenbeck semigroup, C. R. Acad. Sci. Paris Sér. I Math., 326 (1998), 25–30.

[21] T. Menárguez, S. Pérez, and F. Soria, The Mehler maximal function: a geometric proof of the weak type 1, J. London Math. Soc. (2), 61 (2000), 846–856.

[22] P. A. Meyer, Transformations de Riesz pour les lois gaussiennes, in: Seminar on Probability, XVIII, Lecture Notes in Math., vol. 1059, Springer (Berlin, 1984), pp. 179–193.

[23] E. Navas, E. Pineda, and W. O. Urbina, The boundedness of general alternative singular integrals with respect to the Gaussian measure, J. Stoch. Anal., 1 (2020), Article 14, 26 pp.

[24] A. Nekvinda, Hardy–Littlewood maximal operator on $L^{p(x)}(\mathbb{R})$, Math. Inequal. Appl., 7 (2004), 255–265.

[25] A. Nowak and K. Stempak, $L^2$-theory of Riesz transforms for orthogonal expansions, J. Fourier Anal. Appl., 12 (2006), 675–711.
[26] S. Pérez, The local part and the strong type for operators related to the Gaussian measure, *J. Geom. Anal.*, **11** (2001), 491–507.

[27] P. Sjögren, A remark on the maximal function for measures in \( \mathbb{R}_n \), *Amer. J. Math.*, **105** (1983), 1231–1233.

[28] W. Urbina-Romero, *Gaussian Harmonic Analysis*, Springer Monographs in Mathematics, Springer (Cham, 2019).

[29] B. Wróbel, Dimension-free \( L^p \) estimates for vectors of Riesz transforms associated with orthogonal expansions, *Anal. PDE*, **11** (2018), 745–773.