The Fundamental Limits of Measurable Q Factor

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Abstract—This paper describes an analytical evaluation of the radiation Q factor of any separable system in which the vector potential is known. The proposed method uses a potential definition of active and reactive power, implicitly avoiding infinite entire space integration and extraction of radiation energy. Furthermore, the proposed method uses solely measurable quantities and is thus not influenced by commonly used, but ambiguous, separation into electric and magnetic energies. As a result, all the used quantities are finite, and the calculated quality factors are always non-negative smooth functions of frequency.

The theory is presented on the canonical example of the currents flowing on a spherical shell. The fundamental limit, known as the Chu formula, is reformulated analytically without any approximations. These measurable limits are presented for both spherical TM mode and for TE mode in closed forms, including both internal and external energies. The asymptotic behaviour for electrically small and large radiators is discussed, and the minima of the Q factors as well as the minima of the total stored energies are introduced. The proposed analytical method is verified by numerical calculations, including utilization of the theory of characteristic modes.

Index Terms—Antenna theory, electromagnetic theory, electrically small antennas, spherical antennas, Q factor.

I. INTRODUCTION

The radiation Q factor is recognized as one of the most significant parameters of the radiating system, especially if the electrical dimensions are small. The definition of radiation Q and its evaluation for antennas has long been discussed in the literature, see e.g. [1] and references therein. However, there are many as yet unresolved issues associated with the radiation Q factor and the available bandwidth. One of these problems, the exact definition and physical interpretation of the fundamental limits for electrically small antennas (ESA), is solved in this paper.

The classical work of Chu [2] considers a sphere of radius $a$ that encloses an ESA. The normalized radial wave impedance for the dominant spherical TM mode is expressed as a continued fraction equivalent to a ladder network with particular R, L, C elements. In this way, the fundamental limits can be found. However, Chu did not include the internal energy of the sphere, and thus the limit is overly optimistic. The Chu method is restricted to the spherical modes only. Later, Wheeler [3] reduced the basic radiators, dipole and loop, to the circuit elements and derived practically oriented limits. Expansion to the spherical harmonics was also used by Harrington [4] to evaluate the electric and magnetic energy for each mode. The method, however, operates with separated – and thus ambiguous – energies and the problematic entire space integration. The same approach was presented by Collin and Rothschild [5] for spherical and cylindrical modes. McLean [6] verified the Chu formula. He obtained the same results, but his approach is based on the field radiated by the Herzyan dipole. Thiele, Detweiler and Penno [7] used the “far-field method”, based on separating of the far-field pattern into its visible and invisible parts [8]. That [9] used the ladder network to reformulate the fundamental limit by including the energies inside the sphere, and he thus improved the limits given by Chu. The same assumption was made by Hansen and Collin [10], but the calculation was performed over the E- and H-fields. Hansen, Kim and Breinbjerg [11] generalized the results of Hansen and Collin for any spherical TM and TE mode and for a sphere filled with an isotropic medium. The limitations of the dual mode case were studied by Fante [12] and recently by Kim [13].

The most recent approaches to the Q factor calculation utilized the source current distribution. There are obvious benefits: the resultant functionals are of bilinear forms, the calculation is very effective and it is possible to use any current distribution that is available thanks to modern EM simulators or that could even be user-defined. This opens new possibilities in optimization [14] and modal decomposition [15]. The excellent work by Vandenbosch [16] is inspired by the pioneering research of Geiy [17], and directly uses Maxwell equations and the source currents. The same theory has been generalized in the time domain [18]. However, some non-observable terms [19] are neglected, and the technique suffers from the fact that the separation of the electric and magnetic energies is non-unique. Another approach by Gustafsson, Sohl and Kristensson [20] utilized static polarizability. Gustafsson and Jonsson [21] also postulated the uncertainty in Vandenbosch’s definition of Q. Unfortunately, their contribution opens a new question about the coordinate dependent term which is strictly nonphysical.

Some attempts have also been made to obtain the fundamental limits by utilizing of the Q factor defined by the sources. The fundamental limitations were investigated by Vandenbosch and Volski [22], but the method is encumbered with the difficulties mentioned above, and thus the results are provided only for a small radiator. Very interesting work has been done by Seshadri [23], closely related with [24], where the complex power of the spherical modes is already known analytically.

Together with the theoretical achievements, many scientists have sought for an antenna prototype that achieves the given limits, see e.g. [25], [26]. The folded multi-arm spherical helix antenna designed by Best [27] achieved roughly 1.5 times the Chu limit and almost exactly the limit predicted by Hansen.
and Collin. An attempt to reach the Chu limit was undertaken by Kim and Breinbjerg [28], using a magnetic-coated PEC core.

In connection with realistic antennas, the question arises, whether the classical Q limits, based on knowledge of unmeasurable field energies, should not be replaced by limits imposed on a “measurable-Q”. In fact, such a radiation Q factor (based on the antenna’s input impedance) has been proposed by Yaghjian and Best [29] and has recently been revisited by ourselves [30].

This paper attempts to redefine the fundamental limits in the context of the measurable Q factor previously derived in [30], which is based on differentiating the complex power, expressed by electromagnetic potentials rather than fields. In this way, the issues with divergent integrals [16] are automatically eliminated, since subtraction of the far-field energy is inherently present. The complex power differentiation is free of any ambiguous energy separations which give rise to nonphysical quantities like coordinate dependent terms [21] or non-unique or negative energies.

The merits of the proposed theory are presented on an example of spherical modes, which have been in the spotlight in recent decades for their ability to establish a general lower bound of Q. It is important to stress that thanks to the proposed theory, the whole process is completely analytical, without any approximations or numerical calculations. The final expressions, presented in the closed form, are easy to work with and are compatible with all previous observations. Furthermore, our methodology can establish the Q limits not only for the spherical coordinate system, but for any system in which the vector wave equation is separable [31] and thus the vector potential is analytically known. This gives a possibility of practical Q limits tailored for a particular antenna design.

The paper is organized as follows. The definition of measurable Q is briefly recapitulated in Section II. The complex power and all necessary power and energy terms of the dominant spherical TM and TE modes are presented in Section III and Section IV. Section V presents the measurable and thus practically available fundamental limits of ESA that are represented by the dominant spherical modes, further denoted as TM$_{10}$ and TE$_{10}$. Section VI depicts the asymptotic behaviour of measurable Q factors, and also the values where the important quantities reach the minimum.

II. DEFINITION OF MEASURABLE Q

The exact derivation of the measurable Q factor [29] in terms of sources is provided in [30], including the related discussion and numerical verification, and reads

$$Q^{(4)}_Z = \left| Q^{(4)}_R + jQ^{(4)}_X \right|$$

$$= \frac{ka}{2(P_m - P_e)} \left| \frac{\partial}{\partial ka} \left( (P_m - P_e) + j\omega (W_m - W_e) \right) \right|,$$

(1)

where $j = \sqrt{-1}$, $\omega$ is the angular frequency of the time harmonic field [32] under the convention $\mathcal{F}(t) = \sqrt{2} R \{ \mathcal{F}(\omega) e^{j\omega t} \}$, where $\mathcal{F}$ is any time-harmonic quantity, $k = \omega/c_0$ is the wavenumber, $c_0$ is the speed of light, $a$ is the smallest radius of a sphere circumscribing all the sources, $P_m - P_e$ is the total radiated power, $\omega (W_m - W_e)$ is the total reactive power, and the total input current at the antenna’s port is normalized to $I_0 = 1A$. Considering an arbitrary source current distribution $\mathbf{J}$ and charge density $\rho$ inside $\Omega$ source region, and $A$ and $\varphi$ as the vector and scalar potential [33], the separated Q factors in [1] are

$$Q^{(4)}_R = \frac{P_m + P_e + P_{rad} + P_{\omega}}{2(P_m - P_e)},$$

and

$$Q^{(4)}_X = \frac{\omega(W_m + W_e + W_{rad} + W_{\omega})}{2(P_m - P_e)},$$

(3)

where the particular terms are expressed as

$$W_m - j\frac{P_m}{\omega} = \int_{\Omega} \mathbf{A} \cdot \mathbf{J}^* \, d\mathbf{r},$$

$$W_e - j\frac{P_e}{\omega} = \int_{\Omega} \varphi \rho^* \, d\mathbf{r},$$

$$W_{rad} - j\frac{P_{rad}}{\omega} = -jk \left( k^2 \mathcal{L}_{rad} (\mathbf{J}, \mathbf{J}) - \mathcal{L}_{rad} (\nabla \cdot \mathbf{J}, \nabla \cdot \mathbf{J}) \right),$$

$$W_{\omega} - j\frac{P_{\omega}}{\omega} = k^2 \mathcal{L}_{\omega} (\mathbf{J}, \mathbf{J}) - \mathcal{L}_{\omega} (\nabla \cdot \mathbf{J}, \nabla \cdot \mathbf{J}),$$

(4c)

with

$$\mathcal{L}_{rad} (\mathbf{U}, \mathbf{V}) = \frac{1}{4\pi \epsilon_0 \omega^2} \int_{\Omega} \int_{\Omega'} \mathbf{U}(\mathbf{r}) \cdot \mathbf{V}^* (\mathbf{r}') e^{-jkR} \, d\mathbf{r} \, d\mathbf{r}',$$

(5a)

$$\mathcal{L}_{\omega} (\mathbf{U}, \mathbf{V}) = \frac{1}{4\pi \epsilon_0 \omega^2} \int_{\Omega} \int_{\Omega} \partial \mathbf{U}(\mathbf{r}) \cdot \mathbf{V}^* (\mathbf{r}') e^{-jkR} \frac{\partial}{\partial \omega} \frac{1}{R} \, d\mathbf{r} \, d\mathbf{r}',$$

(5b)

in which $R = |\mathbf{r} - \mathbf{r}'|$ is the Euclidean distance, $\epsilon$ is the vacuum permittivity, and $*$ denotes complex conjugation.

The $Q^{(4)}_{X}$ in (3) and $Q^{(4)}_{Z}$ in [1] are untuned Q factors [1] and thus they will be denoted as $Q^{(4)}_{X}(\text{un.})$ and $Q^{(4)}_{Z}(\text{un.})$ in the rest of the paper. One can, however, tune the antenna to its resonance at angular frequency $\omega_0$ by a reactive lumped element. Then, the tuned Q factors will be denoted as $Q^{(4)}_{X}(\text{tuned})$ and $Q^{(4)}_{Z}(\text{tuned})$, and they can be evaluated as

$$Q^{(4)}_{X}(\text{tuned}) = 2\omega_0 \max \left\{ W_m, W_e \right\} + \omega_0 \left( W_{rad} + W_{\omega} \right).$$

(6)

Note that tuning by purely reactive elements leaves the $Q^{(4)}_R$ factor unchanged. For a physical interpretation and further details see [30].

III. COMPLEX POWER AND Q OF THE TM$_{10}$ MODE

Let us consider the TM$_{10}$ mode, which is described by the current density

$$\mathbf{J} = \frac{\sin (\varphi)}{2\pi a} \delta (r - a) \, \vartheta_0$$

(7)
flowing on a spherical shell of radius \( a \) situated in a vacuum, where \( \delta \) is the Dirac delta and \( \vartheta_0 \) is the unit vector codirectional with \( \vartheta \), see Fig. 1. The current density (7) is normalized so that the current flowing through the \( x \times y \) plane is \( I_0 = 1 \text{A} \). The corresponding charge density is

\[
\rho = \frac{\cos (\vartheta)}{\omega \pi a^2} \delta (r - a). \tag{8}
\]

The vector and scalar potentials of the TM\(_{10}\) mode are (see appendix A)

\[
A_{\vartheta} = -\frac{\gamma}{2\pi ka} \sin (\vartheta) \left( 2h_1^{(2)} (ka) j_1 (ka) + \left( h_1^{(2)} (ka) - ka h_0^{(2)} (ka) \right) (j_1 (ka) - ka j_0 (ka)) \right) \tag{9}
\]

and

\[
\varphi = \frac{\omega \mu}{\pi k} j_1^{(2)} (ka) j_1 (ka) \cos (\vartheta), \tag{10}
\]

where \( j_n \) and \( h_n^{(2)} \) are the spherical Bessel and Hankel functions of the \( n \)th order \([34]\). Substituting the potentials into (4a) and (4b) leads to

\[
P_m = \frac{4}{6\pi} Z_0 \left( 2j_1^2 (ka) + (j_1 (ka) - ka j_0 (ka))^2 \right), \tag{11a}
\]

\[
P_e = \frac{4}{3\pi} Z_0 j_1^2 (ka), \tag{11b}
\]

\[
\omega W_m = -\frac{4}{6\pi} Z_0 \left( 2y_1 (ka) j_1 (ka) + (y_1 (ka) - ka y_0 (ka)) (j_1 (ka) - ka j_0 (ka)) \right), \tag{11c}
\]

\[
\omega W_e = -\frac{4}{3\pi} Z_0 y_1 (ka) j_1 (ka), \tag{11d}
\]

where \( Z_0 = \sqrt{\mu/\varepsilon} \) is the free space impedance. Note here that the distribution (7) does not vary with the frequency, \( \partial J (\vartheta) / \partial \omega = 0 \), and thus from (4d) we have

\[
P_\omega = \omega W_\omega = 0. \tag{12}
\]

Finally, by comparing (1) with (2) and (3), and using (12), we can deduce that

\[
P_{\text{rad}} = k a \partial \left( P_m + P_e \right) / \partial k a - (P_m + P_e), \tag{13a}
\]

\[
\omega W_{\text{rad}} = k a \partial \left( W_m - W_e \right) / \partial k a - (W_m + W_e). \tag{13b}
\]

The above expressions have been simplified in Mathematica \([35]\) and numerically evaluated in Matlab \([36]\), and the results are depicted in Fig. 2.

We now turn to a brief discussion of untuned Q factors of the TM\(_{10}\) mode, which are depicted in Fig. 3. First of all, both \( Q_R^{(4)} \) and \( Q_X^{(4)} \text{(un.)} \) can be negative. This means that the standalone \( Q_X^{(4)} \text{(un.)} \) cannot be regarded as a radiation Q factor. The overall \( Q_{\text{Z}}^{(4)} \text{(un.)} \) is always non-negative thanks to the absolute value in (1). It reaches zero at \( ka = \sqrt{2} \), and the first maximum is located at \( ka \approx 2.7437 \), where \( W_m - W_e = 0 \). Interestingly, the results in Fig. 3 can be verified by implementing the characteristic modes \([37]\). The characteristic current distribution of the TM\(_{10}\) mode is the same as (7), except for a constant which ensures the unitary radiated power \([38]\). The inherence of the frequency dependent constant is eliminated by setting \( P_\omega = \omega W_\omega = 0 \).
in postprocessing. Thus, after discretization of the sphere [39], preparation of the impedance matrix and decomposition in Matlab [40], we arrive at the modal current which was used for calculating (4). The modal Q factor is compared with \(Q_Z^{(4)}(\text{un.})\) in Fig. 3. Checking the green circular markers, one can observe that the results are in excellent agreement with the analytical calculation.

IV. COMPLEX POWER AND Q OF THE TE10 MODE

The procedure from the previous section can be used for the TE10 mode as well. In that case, the current density is

\[
J = \frac{\sin (\theta)}{2\alpha} \delta (r - a) \varphi_0, \tag{14}
\]

where \(\varphi_0\) is the unit vector codirectional with \(\varphi\), see Fig. 1. The current density is normalized so that the current flowing through the \(z \times (x > 0)\) half-plane is \(I_0 = 1\)A. The corresponding charge density vanishes, \(\rho = 0\), so

\[
\varphi = 0, \tag{15a}
\]

\[
P_m = 0, \tag{15b}
\]

\[
\omega W_m = 0. \tag{15c}
\]

Furthermore, as the current is frequency independent, (2) is still valid. The vector potential is again found by the method described in appendix A and is equal to

\[
A_\phi = -\frac{j\epsilon}{2} \sin (\theta) ka j_1 (ka) h_1^{(2)} (ka), \tag{16}
\]

which leads to

\[
P_m = \frac{2\pi}{3} Z_0 (ka)^2 j_1^2 (ka), \tag{17a}
\]

\[
\omega W_m = -\frac{2\pi}{3} Z_0 (ka)^2 j_1 (ka) y_1 (ka). \tag{17b}
\]

All non-zero terms related to the TE10 mode are depicted in Fig. 4. Their behaviour is very similar to the TM10 case. An interesting point is located near \(ka = 4.5\), where \(P_m = \omega W_m = P_{\text{rad}} = 0\) and thus only \(\omega W_{\text{rad}}\) has a non-zero value.

The untuned Q factors of the TE10 mode are depicted in Fig. 5. Both \(Q^{(4)}_R\) and \(Q^{(4)}_X(\text{un.})\) can be negative, both \(Q^{(4)}_R\) and \(Q^{(4)}_X(\text{un.})\) are smooth functions of frequency, except from the jumps of \(Q^{(4)}_R\). The measurable \(Q_Z^{(4)}(\text{un.})\) is smooth as well, positive everywhere, and has the first minimum 1.1098 at \(ka \approx 2.3323\). As in the case of the TM10 mode, the results were verified by integrating the characteristic current, see the green circular markers in Fig. 5.

V. THE LIMITATIONS FOR ESA AND ASYMPTOTIC BEHAVIOUR OF THE Q FACTORS FOR THE TM10 MODE AND FOR THE TE10 MODE

In this section, we will discuss the tuned Q factor for the spherical TM10 and TE10 modes, including their analytical expressions. The results are compared with the limits already found by Chu [2], and by Hansen and Collin [10] for the ESA region where \(ka < 0.5\).

According to Chu, the fundamental limit for the TM10 mode is

\[
Q_{\text{Chu}} = \frac{1}{(ka)^2} + \frac{1}{ka}, \tag{18}
\]

This limit has been rectified by Hansen and Collin for the TM10 mode as

\[
Q_{\text{Hansen}} = \frac{1.4959}{(ka)^3} + \frac{0.7133}{ka}, \tag{19}
\]

and approximately for the TE10 mode as

\[
Q_{\text{Hansen}} \approx 3Q_{\text{Chu}}. \tag{20}
\]

These limits assume that the ESA is tuned to the resonance by the ideal and lossless lumped element [29].

The above mentioned Q factors are tuned. Therefore, the concept of the tuned and measurable Q factor [6] has to be utilized [30] to obtain the fundamental limits analytically. The measurable Q factor of the TM10 mode for \(ka \in (0, 2\pi)\) is depicted in Fig. 6 where the ESA region is highlighted. The agreement between \(Q^{(4)}_Z(\text{tuned})\) and \(Q_{\text{Hansen}}\) is excellent in the region where \(Q_{\text{Hansen}}\) is valid (\(ka < 0.5\)). The \(Q^{(4)}_Z(\text{tuned})\) reaches its minimum 0.4851 at \(ka \approx 1.4756\). At this point, the value of \(Q^{(4)}_Z(\text{tuned})\) is below \(Q_{\text{Chu}}\). However,
Fig. 6. Comparison of the limits of radiation Q: measurable $Q_Z^{(4)}$ (tuned), the Chu limit $Q_{Chu}$ [18] and the Hansen and Collin limit $Q_{Hansen}$ for the TM$_{10}$ mode [19]. The electrically small antenna regime is highlighted. The minimum value of $Q_Z^{(4)}$ (tuned) $\approx 0.4851$ occurs at $ka \approx 1.4756$.

Fig. 7. Comparison of the limits of radiation Q: the measurable $Q_Z^{(4)}$ (tuned), the Chu limit $Q_{Chu}$ [18] and the Hansen and Collin limit $Q_{Hansen}$ for the TE$_{10}$ mode [19]. The electrically small antenna regime is highlighted. The minimum value of $Q_Z^{(4)}$ (tuned) $\approx 0.9494$ occurs $ka \approx 2.2923$.

the $Q_{Chu}$ limit is not valid here, and then we postulate a new minimum Q of the TM$_{10}$ mode.

The situation is similar in the case of the tuned Q factor of the TE$_{10}$ mode, see Fig. 7. The $Q_Z^{(4)}$ (tuned) reaches its minimum 0.9494 at $ka \approx 2.2923$. It is obvious that the Chu limit is very optimistic and cannot be practically achieved due to the fact that Chu excluded the reactive power inside the sphere. This is particularly true in the case of the TE$_{10}$ mode, which stores a great deal of the reactive power in the near-field.

It is often of practical interest to define a power series expansion of the exact formulas, which simplifies the final expressions. We expand the $Q_Z^{(4)}$ and $Q_X^{(4)}$ (tuned) terms separately to point out the differences between the TM$_{10}$ and the TE$_{10}$ cases. Asymptotic behaviour of the TM$_{10}$ mode for $ka < 1$ is

$$Q_X^{(4)}(tuned) = \frac{3}{2}(ka)^3 + \frac{3}{5}(ka) - 0.5807ka + \mathcal{O}(ka)^3,$$

$$Q_Z^{(4)}(tuned) \approx Q_X^{(4)}(tuned),$$

and the series expansion of the TE$_{10}$ mode for $ka < 1$ is

$$Q_R^{(4)} = 2 - \frac{(ka)^2}{5} + \mathcal{O}(ka)^4,$$

$$Q_X^{(4)}(tuned) = \frac{3}{(ka)^3} + \frac{3}{ka} - \frac{174ka}{175} + \mathcal{O}(ka)^3,$$

$$Q_Z^{(4)}(tuned) \approx Q_X^{(4)}(tuned),$$

where only the terms up to order of $(ka)^2$ are explicitly shown. Note the very good correspondence between (21b), (22b) and (18), (19). For both modes, the overall Q factor $Q_Z^{(4)}(tuned)$ is almost identical to $Q_X^{(4)}(tuned)$ for $ka < 1$, since $Q_Z^{(4)}(tuned) \gg Q_R^{(4)}$, see Fig. 3 and Fig. 5. The first two terms in (21b) and (22b) are equal to the Q factors found numerically in [16].

It should be noted that the TM$_{10}$ mode for $ka \rightarrow 0$ behaves like the Hertzian dipole with $\omega W_e \rightarrow \infty$ [41]. Conversely, the TE$_{10}$ mode for $ka \rightarrow 0$ behaves like a small loop with $W_m = L_{TE}$, where $L_{TE}$ is the self-inductance of the mode [42].

VI. ON THE TOTAL STORED ENERGY OF THE TM$_{10}$ MODE AND THE TE$_{10}$ MODE

We have enough information to go back and recall the classical definition of the radiation Q factor, which is inspired by [5]

$$Q = \frac{\omega \tilde{W}}{P_o},$$

where $P_o = P_m - P_r$ is the total radiated power and $\tilde{W}$ is the time-averaged stored energy. Seeking for the proper meaning of (23), we compare definitions (1) and (23) and directly obtain

$$\tilde{W} = \left| P_m + P_e + P_{rad} + P_\omega + j\omega (W_m + W_e + W_{rad} + W_\omega) \right|_2,$$

To verify (24), we investigate the behaviour of the numerator and the denominator of (23) for the TM$_{10}$ and the TE$_{10}$ modes, see Fig. 8 and Fig. 9. The important observation is that the total stored energy $\tilde{W}$ is constant for $ka \gg 1$. The product of $\omega \tilde{W}$ correlates with the physical nature of the TM$_{10}$ and the TE$_{10}$ modes: while $\omega W_m$ for the TM$_{10}$ mode starts from positive infinity (and reaches zero at $ka = \sqrt{2}$), the $\omega W_m$ for the TE$_{10}$ mode starts from zero.

The asymptotic behaviour for $ka \gg 1$ can be characterized by $W \approx 40a/c_0$ for the TM$_{10}$ mode and by $W = 395a/c_0$ for the TE$_{10}$ mode, see Fig. 8 and Fig. 9. Considering the excellent agreement between all presented results, it is evident what $P$- and $W$-terms are essential for the proper Q definition and how the total stored energy $\tilde{W}$ is established. The important extrema, from both the scientific
and the practical point of view are depicted in Table I. The presented results indirectly verify the hypothesis that the radiated energy travels in the radial direction at the speed of light from the antenna sphere to the far-field region, since they are in good accordance with the separation methods [5], [6], in which the speed of light is assumed for outgoing energy. Thus, the alternative concepts [43] seem not to be valid.

VII. CONCLUSION

The potential theory has been employed to obtain the measurable radiation $Q$ of important spherical current distributions, particularly of the fundamental TM and TE modes. It has been shown that the presented “source” approach is effective, leading to unique and finite energy terms with the far-field extraction implicitly included. For the presented cases of the spherical coordinate system, radiation $Q$ was obtained in closed form for any $ka$. The fundamental lower limit of radiation $Q$ of electrically small antennas was then obtained by series expansion of these expressions for small $ka$. Excellent agreement with the previous work of Thal and Hansen was observed.

Furthermore, a comparison with the classical $Q$ definition showed that the total stored energy was formed not only by reactive power terms, but also by radiated power and by two qualitatively new terms related to the frequency variation of the source distributions and to the radiation fields. These new terms were shown to be essential for a concept of measurable $Q$ that correctly represents the physical reality.

Although the proposed approach has been presented on spherical modes, it is not restricted to them, and can also easily be extended to other separable coordinate systems. In this respect, the elliptic coordinates may be of considerable interest, as they can closely match the shape of many realistic antennas. The lower bounds of measurable $Q$ obtained in this way would then represent practically oriented limits for antenna designers. Another challenge is to investigate the measurable $Q$ of the dual mode case.

APPENDIX A

VECTOR AND SCALAR POTENTIALS OF THE TM$_{10}$ AND TE$_{10}$ MODE

The vector and scalar potentials are found by the expansion method of appendix [9]. For the particular case of the TM$_{10}$ and the TE$_{10}$ modes, we obtain the corresponding vector and scalar potentials regular for $r = [0, \infty)$ by using (36a)–(36c) with $a = z_0$, $\psi_0 = z_0(kr)$ for $M$-, $N$-terms and with $\psi_10 = z_1(kr) \cos(\vartheta)$ for $L$-terms, where $z_n(x)$ is a spherical Bessel function of order $n$ and where we will use $z_n(x) = j_n(x)$ for $r < a$ and $z_n = h_n^{(2)}(x)$ for $r > a$. The resulting vector wave functions read

$$ M_{10} = \varphi_{10} k z_1(kr) \sin(\vartheta), \quad (25a) $$

$$ N_{10} = r_0 - \frac{1}{r} \frac{1}{r} (z_1(kr) - kr z_0(kr)) \cos(\vartheta), \quad (25b) $$

$$ L_{10} = r_0 - \frac{1}{r} \frac{1}{r} (z_1(kr) - kr z_2(kr)) \cos(\vartheta) - \vartheta_0 \frac{1}{r} z_1(kr) \sin(\vartheta). \quad (25c) $$

The vector potential of the TM$_{10}$ mode will be expressed as a linear combination of (25b) and (25c) because of the non-vanishing charge density and the need for the $L_{10}$-term. The vector potential of the TE$_{10}$ mode will be expressed in terms of (25a) only, since there is no charge density and thus no scalar potential.

| $W_{m} - W_{e}$ | $\min_{ka} \{ Q_{Z}^{(4)}(\text{tuned}) \}$ | $\min_{ka} \{ \omega W \}$ |
|-----------------|-----------------------------|-----------------------------|
| 0.2.7437        | 0.4851                     | 0.0                      |
| 0.2.3323        | 1.4756                     | 0.9494                    |
| 0.2.7983        | 2.2923                     | 2.3323                    |

TABLE I

COMPARISON OF SOME IMPORTANT PARAMETERS RELATED TO THE LIMITS OF THE RADIATING $Q$ FACTOR. THE VALUES WERE FOUND FROM THE ANALYTICAL EQUATIONS PROCESSED IN MATHEMATICA.
According to the above, in order to find the vector and the scalar potential of the TM\(_{10}\) mode, we choose

\[
\begin{align*}
A &= C_1 N_{10} + D_1 L_{10}, \\
\varphi &= -j\omega D_1 \psi_{10}
\end{align*}
\]

\(r < a\) \quad (26a)

\[
\begin{align*}
A &= C_2 N_{10} + D_2 L_{10}, \\
\varphi &= -j\omega D_2 \psi_{10}
\end{align*}
\]

\(r > a\) \quad (26b)

where \(C, D\) are constants to be determined. The \(C_{1,2}\) can be determined from the boundary conditions on the current shell at \(r = a\), i.e. by continuity of the tangential electric field \(n_0 \times (E_1 - E_2) = 0\) and discontinuity of the tangential magnetic field \(n_0 \times (H_1 - H_2) = K\), where \(K\) is the surface current density and where the normal \(n_0\) points to the region \(1, [33]\). The boundary conditions lead to

\[
\begin{align*}
C_1 &= -\frac{j \mu}{2\pi k} \left( h_1^{(2)}(ka) - ka h_0^{(2)}(ka) \right), \\
C_2 &= -\frac{j \mu}{2\pi k} (j_1(ka) - ka j_0(ka)).
\end{align*}
\]

\(27a\) and \(27b\).

For the unknown constants \(D_{1,2}\) in \(26a\) and \(26b\), the only condition that needs to be satisfied is the wave equation for the scalar potential in the Lorentz gauge

\[
\nabla^2 \varphi + k^2 \varphi = \frac{\rho}{\epsilon}, \quad (28)
\]

Choosing then the scalar potential being continuous at \(r = a\), \(28\) dictates

\[
\frac{\partial \varphi}{\partial r} \bigg|_{r=a^+} - \frac{\partial \varphi}{\partial r} \bigg|_{r=a^-} = -j \frac{\cos \vartheta}{\omega \epsilon a^2}, \quad (29)
\]

which leads to

\[
\begin{align*}
D_1 &= \frac{j \mu}{k \pi} h_1^{(2)}(ka), \\
D_2 &= \frac{j \mu}{k \pi} j_1(ka).
\end{align*}
\]

\(30a\) and \(30b\).

Putting all together we have for \(r < a\)

\[
\begin{align*}
A_\theta &= -\frac{j \mu}{2\pi kr} \sin \vartheta \left( 2 h_1^{(2)}(ka) j_1(kr) \\
&+ \left( h_1^{(2)}(ka) - ka h_0^{(2)}(ka) \right) \left( j_1(kr) - kr j_0(kr) \right) \right) \\
\varphi &= \frac{\omega \mu}{k} h_1^{(2)}(ka) j_1(kr) \cos \vartheta,
\end{align*}
\]

\(31a\) and \(31b\).

and for \(r > a\)

\[
\begin{align*}
A_\theta &= -\frac{j \mu}{2\pi kr} \sin \vartheta \left( 2 h_1^{(2)}(kr) j_1(ka) \\
&+ \left( h_1^{(2)}(kr) - kr h_0^{(2)}(kr) \right) \left( j_1(ka) - ka j_0(ka) \right) \right) \\
\varphi &= \frac{\omega \mu}{k} h_1^{(2)}(kr) j_1(ka) \cos \vartheta,
\end{align*}
\]

\(32a\) and \(32b\).

where the first terms in the vector potential come from \(L_{10}\) and the second terms come from \(N_{10}\).

The derivation of the scalar and vector potential of the TE\(_{10}\) mode is analogous to the above, and results in

\[
\begin{align*}
A_\varphi &= -\frac{j \mu}{2} \sin \vartheta \left( ka j_1(ka) h_1^{(2)}(ka) \\
&- \varphi = 0,
\end{align*}
\]

\(33a\) and \(33b\).

for \(r < a\), and

\[
\begin{align*}
A_\varphi &= -\frac{j \mu}{2} \sin \vartheta \left( ka j_1(ka) h_1^{(2)}(kr) \\
&= \frac{\rho}{\epsilon}, \quad (34a)
\end{align*}
\]

\(34b\) for \(r < a\).

**APPENDIX B**

**EXPANSION OF THE VECTOR WAVE EQUATION IN SEPARABLE SYSTEMS**

In this appendix, we recall the expansion of the vector wave equation and point out some aspects important for a consistent definition of \(Q\). This approach leads to the analytical calculation of the \(Q\) factor for the separable systems.

According to \([44]\), the general solution of \(\nabla^2 A + k^2 A = 0\) can be written as

\[
A = \sum_n (\alpha_n M_n + \beta_n N_n + \gamma_n L_n), \quad (35)
\]

where

\[
\begin{align*}
M_n &= \nabla \times (av_n), \\
N_n &= \frac{1}{k} \nabla \times M_n, \\
L_n &= \nabla \psi_n,
\end{align*}
\]

\(36a\) \(36b\) and \(36c\).

\(a\) is a constant vector, and scalar function \(\psi_n\) satisfies

\[
\nabla^2 \psi_n + k^2 \psi_n = 0. \quad (37)
\]

The conventional notation from \([45]\) is used for clarity of the paper.

Taking now the vector field \(A\) as a magnetic vector potential in the Lorentz gauge, one can verify that the scalar potential is

\[
\varphi = -\frac{1}{j \omega \mu} \nabla \cdot A = -j \omega \sum_n \gamma_n \psi_n, \quad (38)
\]

where \(\mu\) is the permeability of the vacuum, and that the field quantities read

\[
\begin{align*}
E &= -\frac{j \omega}{k^2} (\nabla \varphi \cdot A + k^2 A) = -j \omega \sum_n (\alpha_n M_n + \beta_n N_n), \\
H &= \frac{1}{\mu} \nabla \times A = k \sum_n (\alpha_n N_n + \beta_n M_n),
\end{align*}
\]

\(39a\) \(39b\).

where we used the fact that \(\nabla \times N_m = k M_n\).

It is worth noting that any measurable quantity is independent of \(L_m\)-terms (which is equivalent to gauge invariance).

Particularly, if a volume is chosen so that it contains all the sources and if the vector potential \(35\) is divided as

\[
A = A^{M,N} + A^L,
\]

with \(A^{M,N}\) belonging to \(M_n\)-, \(N_n\)-terms and \(A^L\) belonging to \(L_n\)-terms, then, one can easily realize that

\[
\int \Omega (A^L \cdot J^* - \varphi \rho^*) \, dr = 0, \quad (40)
\]

and thus that only the \(M_n\)-, \(N_n\)-terms participate in the definition of the complex power \([35]\).

\[
\int \Omega (A \cdot J^* - \varphi \rho^*) \, dr = \int \Omega (A^{M,N} \cdot J^*) \, dr. \quad (41)
\]
ACKNOWLEDGEMENT

The authors would like to thank Dr. Zdenek Hradecky for his encouragement and unremitting support.

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