CHARACTERIZATION OF AFFINE AUTOMORPHISMS AND ORTHO-ORDER AUTOMORPHISMS OF QUANTUM PROBABILISTIC MAPS

ZHAOFANG BAI AND SHUANPING DU*

Abstract. In quantum mechanics, often it is important for the representation of quantum system to study the structure-preserving bijective maps of the quantum system. Such maps are also called isomorphisms or automorphisms. In this note, using the Uhlhorn-type of Wigner’s theorem, we characterize all affine automorphisms and ortho-order automorphisms of quantum probabilistic maps.

1. Introduction

In quantum physics, of particular importance for the representation of physical system and symmetries are structure-preserving bijective maps of the system. Such maps are also called isomorphisms or automorphisms. Automorphisms or isomorphisms are frequently amenable to mathematical formulation and can be exploited to simplify many physical problems. By now, they have been extensively studied in different quantum systems, and systematic theories have been achieved [10]. Recently, most deepest results in this field have been obtained by Lajos Monlar in a series of articles [13, 14, 15, 16, 17, 18]. And an overview of recent results can be found in [5, 19].

Let us now fix the notations and set the problem in mathematical terms. Let $H$ be a separable Hilbert space with dimension at least 3 and inner product $\langle ., . \rangle$. Let $B_1(H)$ be the complex Banach space of the trace class operators on $H$, with trace $tr(T)$ and trace norm $\|T\|_1 = tr(|T|), |T| = \sqrt{T^*T}, T \in B_1(H)$. The self-adjoint part of $B_1(H)$ is denoted by $B_{1r}(H)$ which is a real Banach space. By $B_{1r}^+(H)$ we denote the positive cone of $B_{1r}(H)$. As usual, the unit ball of $B_{1r}^+(H)$ is denoted by $S_1(H) = \{T \in B_{1r}^+(H) : tr(T) = \|T\|_1 \leq 1\}$, the surface of $S_1(H)$ by $V = \{T \in B_{1r}^+(H) : tr(T) = 1\}$. With reference to the quantum physical applications, $B_{1r}(H)$ is called state space, the elements of $B_{1r}^+(H)$ and $V$ are called density operators and states, respectively (see [4, 10]). Naturally, $S_1(H)$ can be equipped with several algebraic operations. Clearly, $S_1(H)$ is a convex set, so one can consider the convex
combinations on it. Furthermore, one can define a partial addition on it. Namely, if \( T, S \in S_1(H) \) and \( T + S \in S_1(H) \), then one can set \( T \oplus S = T + S \). Moreover, as for a multiplicative operation on \( S_1(H) \), note that in general, \( T, S \in S_1(H) \) does not imply that \( TS \in S_1(H) \). However, we all the time have \( TST \in S_1(H) \), since \( TST \in B_{tr}^+(H) \) and \( tr(TST) = \|TST\|_1 \leq \|T\|_1\|S\|_1\|T\|_1 \leq 1 \). This multiplication is a nonassociative operation and sometimes called Jordan triple product also appears in infinite dimensional holomorphy as well as in connection with the geometrical properties of \( C^* \)-algebras. Finally, there is a natural partial order \( \leq \) on \( S_1(H) \) which is induced by the usual order between selfadjoint operators on \( H \). So, for any \( T, S \in S_1(H) \) we write \( T \leq S \) if and only if \( <Tx, x> \leq <Sx, x> \) holds for every \( x \in H \). Physically, the most interesting order may be spectral order (see [20]). The detailed definition is as follows. Let \( T, S \in S_1(H) \) and consider their spectral measures \( E_T, E_S \) defined on the Borel subsets of \( \mathbb{R} \). We write

\[
T \preceq S \text{ if and only if } E_T(-\infty, t] \geq E_S(-\infty, t] \quad (t \in \mathbb{R}).
\]

The spectral order has a natural interpretation in quantum mechanics. In fact, the spectral projection \( E_T(-\infty, t] \) represents the probability that a measurement of \( T \) detects its value in the interval \( (-\infty, t] \). Hence for \( T, S \in S_1(H) \) the relation \( T \preceq S \) means for every \( t \in [0, 1] \) we have \( E_T(-\infty, t] \geq E_S(-\infty, t] \) in each state of the system, i.e., the corresponding distribution functions are pointwise ordered.

Because of the importance of \( S_1(H) \), it is a natural problem to study the automorphisms of the mentioned structures. The aim of this paper is to contribute to these investigations. In [2], the automorphisms of \( S_1(H) \) with the partial addition and Jordan triple product were characterized. In this paper, we are aimed to characterize the affine automorphisms and ortho-order automorphisms of \( S_1(H) \). The core of the proof is to reduce the problem to using the Uhlhorn-type of Wigner’s theorem (see [22]).

Now, let us give the concrete definitions of affine automorphism and ortho-order automorphism. A bijective map \( \Phi : S_1(H) \to S_1(H) \) is an affine automorphism if

\[
\Phi(\lambda T + (1 - \lambda)S) = \lambda \Phi(T) + (1 - \lambda)\Phi(S) \quad \text{for all } T, S \in S_1(H), 0 \leq \lambda \leq 1.
\]

A bijective map \( \Phi : S_1(H) \to S_1(H) \) is called an ortho-order automorphism if

(i) \( TS = 0 \Leftrightarrow \Phi(T)\Phi(S) = 0 \) for all \( T, S \in S_1(H) \),

(ii) \( T \preceq S \Leftrightarrow \Phi(T) \preceq \Phi(S) \) for all \( T, S \in S_1(H) \).

Here, it is worth mentioning that the affine automorphism has an intimate relationship with the so-called operation of \( B_1(H) \) (see [6, 9]), which is a fundamental notion in quantum theory. Recall that an operation \( \Phi \) is a completely positive linear mapping on \( B_1(H) \) such that \( 0 \leq tr(\Phi(T)) \leq 1 \) for every \( T \in V \). An operation represents a
probabilistic state transformation. Namely, if $\Phi$ is applied on an input state $T$, then the state transformation $T \rightarrow \Phi(T)$ occurs with the probability $\text{tr}(\Phi(T))$, in which case the output state is $\Phi(T)$. By the Kraus representation theorem (see [3, 9]), $\Phi$ is an operation if and only if there exists a countable set of bounded linear operators $\{A_k\}$ such that $\sum_k A_k^* A_k \leq I$ and $\Phi(T) = \sum_k A_k T A_k^*$ holds for all $T \in \mathcal{B}_1(H)$. This is very important in describing dynamics, measurements, quantum channels, quantum interactions, quantum error, correcting codes, etc [21]. Since operation $\Phi$ is linear and $0 \leq \text{tr}(\Phi(T)) \leq 1$ for every $T \in V$, it is evident such $\Phi$ maps $S_1(H)$ into $S_1(H)$ and possesses the affine condition mentioned in the definition of affine automorphism. Thus operations on $\mathcal{B}_1(H)$ can be reduced to maps on $S_1(H)$. Furthermore, if the reduction of operation on $S_1(H)$ is bijective, from our Theorem 2.1, an explicit description can be given even without completely positive assumption.

2. Affine automorphisms on $S_1(H)$

In this section, we present a structure theorem of affine automorphisms on $S_1(H)$. The following are the main results of this section.

**Theorem 2.1.** If $\Phi : S_1(H) \rightarrow S_1(H)$ is an affine automorphism, then there exists an either unitary or antiunitary operator $U$ on $H$ such that $\Phi(T) = UTU^*$ for all $T \in S_1(H)$.

**Corollary 2.2.** If $\dim H < +\infty$, $\Phi$ is a $\|\cdot\|_1$-isometric automorphism of $S_1(H)$ (which is a bijection of $S_1(H)$ and satisfies $\|T - S\|_1 = \|\Phi(T) - \Phi(S)\|_1$ for all $T, S \in S_1(H)$), then there exists an either unitary or antiunitary operator $U$ on $H$ such that $\Phi(T) = UTU^*$ for all $T \in S_1(H)$.

We remark that, in the above two results, the bijectivity assumption is indispensable to obtain a nice form of $\Phi$. To show it, an example originating from the Kraus representation theorem will be given after the proof of theorem 2.1 and corollary 2.2.

Before the proof of Theorem 2.1, let us recall the general structure of density operators (see for instance [1]). For $T \in \mathcal{B}_1^+(H)$, there exists an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of $H$ and numbers $\lambda_n > 0$ such that

$$T = \sum_{n=1}^{+\infty} \lambda_n P_n,$$

or

$$Tx = \sum_{n=1}^{+\infty} \lambda_n < x, e_n > e_n, \forall x \in H \quad \text{and} \quad 0 < \text{tr}(T) = \sum_{n=1}^{+\infty} \lambda_n < +\infty,$$

where $P_n$ is the one dimensional projection onto the eigenspace spanned by the eigenvector $e_n$. Let $\mathcal{P}_1(H)$ stand for the set of all one dimensional projections on $H$. With
reference to the quantum physical applications, the elements of $\mathcal{P}_1(H)$ are called pure states.

Now, we are in a position to prove our first theorem.

**Proof of Theorem 2.1.** we will finish the proof by checking 3 claims.

**Claim 1.** $\Phi$ is continuous in the trace norm.

To see the continuity of $\Phi$, consider the affine transformation $\Psi : S_1(H) \hookrightarrow B_{1r}(H)$ defined by $\Psi(T) = \Phi(T) - \Phi(0)$ for every $T \in S_1(H)$. It is easy to see that $\Psi$ is injective. In the following, we will prove $\Psi$ has a unique linear extension from $S_1(H)$ to $B_{1r}(H)$. Since $\Psi$ is affine and $\Psi(0) = 0$, for each $\lambda \in [0, 1]$ and every $T \in S_1(H)$, $\Psi(\lambda T) = \lambda \Psi(T)$. For $T \in B_{1r}^+(H)$, a natural extension of $\Psi$ from $S_1(H)$ to $B_{1r}^+(H)$ is to define

$$\tilde{\Psi}(T) = \|T\|_1 \Psi\left(\frac{T}{\|T\|_1}\right).$$

Then for any $\lambda \geq 0$, one gets $\tilde{\Psi}(\lambda T) = \lambda \tilde{\Psi}(T)$, which is the positive homogeneity. For $T, S \in B_{1r}^+(H)$, suppose $\tilde{\Psi}(T) = \tilde{\Psi}(S)$, without loss of generality, assume further $\|T\|_1 \leq \|S\|_1$. Then $\tilde{\Psi}\left(\frac{T}{\|S\|_1}\right) = \tilde{\Psi}\left(\frac{S}{\|S\|_1}\right)$, $\frac{T}{\|S\|_1}, \frac{S}{\|S\|_1} \in S_1(H)$. By the injectivity of $\Psi$, $T = S$ and thus $\tilde{\Psi}$ is injective. For $T_1, T_2 \in B_{1r}^+(H)$, we can rewrite $T_1 + T_2$ in the form

$$T_1 + T_2 = \left(\frac{\|T_1\|_1}{\|T_1\|_1 + \|T_2\|_1}\right)T_1 + \left(\frac{\|T_2\|_1}{\|T_1\|_1 + \|T_2\|_1}\right)T_2.$$

The positive homogeneity of $\tilde{\Psi}$ and the affine property of $\Psi$ yield the additivity of $\tilde{\Psi}$, that is, $\tilde{\Psi}(T_1 + T_2) = \tilde{\Psi}(T_1) + \tilde{\Psi}(T_2)$.

Next for $T \in B_{1r}(H)$, write $T = T^+ - T^-$, where $T^+ = \frac{1}{2}(|T| + T), T^- = \frac{1}{2}(|T| - T), |T| = (T^*T)^{\frac{1}{2}}$. Let

$$\tilde{\Psi}(T) = \tilde{\Psi}(T^+) - \tilde{\Psi}(T^-).$$

Also if $T = T_1 - T_2$ for some other $T_1, T_2 \in B_{1r}^+(H)$, then $T^+ + T_2 = T^- + T_1$, by the additivity of $\tilde{\Psi}$, $\tilde{\Psi}(T^+) - \tilde{\Psi}(T^-) = \tilde{\Psi}(T_1) - \tilde{\Psi}(T_2)$, which shows $\tilde{\Psi}$ is well defined. Furthermore, for $T \in B_{1r}(H)$, it is easy to see $\tilde{\Psi}(-T) = -\tilde{\Psi}(T)$, combining the homogeneity of $\tilde{\Psi}$ over non-negative real number, we know $\tilde{\Psi}$ is linear. Assume $\tilde{\Psi}(T) = 0$, from the definition of $\tilde{\Psi}$, $\tilde{\Psi}(T^+) = \tilde{\Psi}(T^-)$, i.e., $\tilde{\Psi}(T^+) = \tilde{\Psi}(T^-)$. Now, the injectivity of $\tilde{\Psi}$ implies $T^+ = T^-$, so $T = 0$ and $\tilde{\Psi}$ is injective. If $\Gamma : B_{1r}(H) \rightarrow B_{1r}(H)$ is another
linear map which extends $\Psi$, then for any $T \in B_{1r}(H)$,
\[
\Gamma(T) = \Gamma(T^+ - T^-) = \Gamma(T^+) - \Gamma(T^-) = \|T^+\|_1 \Gamma\left(\frac{T^+}{\|T^+\|_1}\right) - \|T^-\|_1 \Gamma\left(\frac{T^-}{\|T^-\|_1}\right)
\]
\[
= \|T^+\|_1 \Psi\left(\frac{T^+}{\|T^+\|_1}\right) - \|T^-\|_1 \Psi\left(\frac{T^-}{\|T^-\|_1}\right)
\]
\[
= \widehat{\Psi}(T^+) - \widehat{\Psi}(T^-) = \widehat{\Psi}(T).
\]
This shows the extension is unique, as desired.

Now, $\widehat{\Psi} : B_{1r}(H) \to B_{1r}(H)$ is a linear injection. We assert that $\widehat{\Psi}$ is continuous in the trace norm $\|\cdot\|_1$. For any $T \in S_1(H)$, clearly, $\|\widehat{\Psi}(T)\|_1 = \|\Phi(T) - \Phi(0)\|_1 \leq 2$. For arbitrary $T \in B_{1r}(H)$, $\|T\|_1 \leq 1$, it is easy to see $T^+ \in S_1(H), T^- \in S_1(H)$. Thus $\|\widehat{\Psi}(T)\|_1 = \|\widehat{\Psi}(T^+) - \widehat{\Psi}(T^-)\|_1 \leq \|\widehat{\Psi}(T^+)\|_1 + \|\widehat{\Psi}(T^-)\|_1 \leq 4$. It follows that $\widehat{\Psi}$ is bounded on the unit ball of $B_{1r}(H)$, hence $\widehat{\Psi}$ is continuous. Note that $\Psi$ is the restriction of $\widehat{\Psi}$ on $S_1(H)$, therefore $\Psi$ is continuous and so $\Phi$ is continuous on $S_1(H)$, as desired.

**Claim 2.** $\Phi(0) = 0$, $\Phi(P_1(H)) = P_1(H)$.

Clearly, $\Phi$ preserves the extreme points of $S_1(H)$ which are exactly the one dimensional projections and zero in $S_1(H)$ (see [2, Lemma 1]). Since $\Phi^{-1}$ has the same properties as $\Phi$, $\Phi(P_1(H) \cup \{0\}) = P_1(H) \cup \{0\}$. We claim that $\Phi(0) = 0$. Assume on the contrary, $\Phi(0) \neq 0$, then there exists one dimensional projection $P$ such that $\Phi(P) = 0$. Note that for $P_1, P_2 \in P_1(H)$, $\|P_1 - P_2\|_1 = 2\sqrt{1 - tr(P_1P_2)}$. Therefore we can choose a sequence $\{P_n\}_{n=1}^\infty$ in $P_1(H)$ such that $\|P_n - P\|_1 \to 0 (n \to +\infty)$. By Claim 1 and the property of $\Phi$, $\|\Phi(P_n) - \Phi(P)\|_1 = 1 \to 0 (n \to +\infty)$, a contradiction. This tells us $\Phi(0) = 0$ and so $\Phi(P_1(H)) = P_1(H)$.

**Claim 3.** There exists an either unitary or antiunitary operator $U$ on $H$ such that $\Phi(T) = U T U^*$ for all $T \in S_1(H)$.

From Claim 2, $\Phi = \Psi$, so we denote $\widehat{\Psi} = \widehat{\Phi}$. From the proof of Claim 1, one can easily get $\widehat{\Phi}$ is also surjective. Thus $\Phi$ has a unique positive linear bijective extension from $S_1(H)$ to $B_{1r}(H)$. In addition, Since $\Phi^{-1}$ has the same properties as $\Phi$, a direct computation shows $\widehat{\Phi}^{-1} = \widehat{\Phi}^{-1}$.

In the following, we will prove $\widehat{\Phi}$ is trace norm preserving. Firstly, it will be shown that $\widehat{\Phi}$ is trace preserving, i.e. $tr(T) = tr(\Phi(T))$ for every $T \in B_{1r}(H)$. Assume $T \in B_{1r}^+(H)$, and $T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n$, where $\{P_i\}_{i=1}^n$ are mutually orthogonal one dimensional projections, $\lambda_h > 0, i = 1, 2 \cdots, n$. Then
\[
\widehat{\Phi}(T) = \lambda_1 \widehat{\Phi}(P_1) + \lambda_2 \widehat{\Phi}(P_2) + \cdots + \lambda_n \widehat{\Phi}(P_n)
\]
\[
= \lambda_1 \Phi(P_1) + \lambda_2 \Phi(P_2) + \cdots + \lambda_n \Phi(P_n).
\]
By Claim 2, we can obtain $tr(T) = \Sigma_{i=1}^{n} \lambda_i = tr(\hat{\Phi}(T))$. For any $T \in \mathcal{B}_{1r}^+(H)$, by the spectral theorem of positive operators, there exists monotone increasing sequence $\{T_n = \Sigma_{i=1}^{n} \lambda_i P_i\}_{n=1}^{\infty}$ such that $\|T_n - T\|_1 = tr(T - T_n) = tr(T) - tr(T_n) \to 0 (n \to \infty)$, where $\{P_i\}_{i=1}^{n}$ are mutually orthogonal one dimensional projections, $\lambda_i > 0, i = 1, 2, \cdots, n$. Since $\hat{\Phi}$ is positive preserving and continuous, $\{\hat{\Phi}(T_n)\}_{n=1}^{\infty}$ is monotone increasing and $\|\hat{\Phi}(T_n) - \hat{\Phi}(T)\|_1 = tr(\hat{\Phi}(T)) - tr(\hat{\Phi}(T_n)) \to 0 (n \to \infty)$. Note that $tr(\hat{\Phi}(T_n)) = tr(T_n)$, so for every $T \in \mathcal{B}_{1r}^+(H), tr(T) = tr(\hat{\Phi}(T))$. For any $T \in \mathcal{B}_{1r}(H)$,

$$tr(\hat{\Phi}(T)) = tr(\hat{\Phi}(T^+)) - tr(\hat{\Phi}(T^-)) = tr(T^+) - tr(T^-) = tr(T),$$

So $\hat{\Phi} : \mathcal{B}_{1r}(H) \to \mathcal{B}_{1r}(H)$ is positive and trace preserving.

Next, we will show that $\hat{\Phi}$ preserves the trace norm. In fact, for any $T \in \mathcal{B}_{1r}(H)$, we have

$$\|\hat{\Phi}(T)\|_1 = \|\hat{\Phi}(T^+ - T^-)\|_1 = \|\hat{\Phi}(T^+) - \hat{\Phi}(T^-)\|_1$$

$$\leq \|\hat{\Phi}(T^+)\|_1 + \|\hat{\Phi}(T^-)\|_1 = tr(\hat{\Phi}(T^+)) + tr(\hat{\Phi}(T^-))$$

$$= tr(T^+) + tr(T^{-}) = tr(T^+ + T^-) = tr(|T|) = \|T\|_1.$$ 

So $\hat{\Phi} : \mathcal{B}_{1r}(H) \to \mathcal{B}_{1r}(H)$ is contractive, i.e., for $T \in \mathcal{B}_{1r}(H), \|\hat{\Phi}(T)\|_1 \leq \|T\|_1$. Since $\hat{\Phi}^{-1}$ has the same properties as $\hat{\Phi}$, we have $\|\hat{\Phi}(T)\|_1 \geq \|T\|_1$ and thus $\|\hat{\Phi}(T)\|_1 = \|T\|_1$, that is $\hat{\Phi}$ is a $\|\cdot\|_1$-isometry of $\mathcal{B}_{1r}(H)$.

Note that, for $P, Q \in \mathcal{P}_1(H), \|P - Q\|_1 = 2\sqrt{1 - tr(PQ)}$. Thus $PQ = 0 \iff \|P - Q\|_1 = 2$. Since $\hat{\Phi}$ is trace norm preserving, we have $PQ = 0 \iff \hat{\Phi}(P)\hat{\Phi}(Q) = 0$. By Claim 2, $\hat{\Phi}|_{\mathcal{P}_1(H)} : \mathcal{P}_1(H) \to \mathcal{P}_1(H)$ is a bijection with the property $PQ = 0 \iff \hat{\Phi}(P)\hat{\Phi}(Q) = 0, P, Q \in \mathcal{P}_1(H)$. Using the well-known Uhlhorn-type of Wigner’s theorem (see [22]), we have $\hat{\Phi}(P) = UPU^*(P \in \mathcal{P}_1(H))$ for some unitary or antiunitary operator $U$ on $H$. By the spectral theorem of selfadjoint operators and the continuity of $\hat{\Phi}$, for all $T \in \mathcal{B}_{1r}(H), \hat{\Phi}(T) = UU^*$, therefore $\hat{\Phi}(T) = UU^*$ for all $T \in \mathcal{S}_1(H)$, as desired.

Based on the Theorem 2.1, we can prove Corollary 2.2.

**Proof of Corollary 2.2.** Firstly, we recall a nice result of Mankiewicz, namely [11] Theorem 5] which states that if we have a bijective isometry between convex set in normed linear space with nonempty interiors, then this isometry can be uniquely extended to a bijective affine isometry between the whole space. Clearly, in the finite dimensional case, the convex set $S_1(H)$ has nonempty interiors in the normed linear space of $\mathcal{B}_{1r}(H)$ (In fact, the interior of $S_1(H)$ consists of all invertible positive operators). Consequently, applying the result of Mankiewicz, we know that $\hat{\Phi}$ is automatically affine. Combing this with Theorem 1, we get the desired.

**Remark 2.3.** Now, in order to illustrate that the bijective assumption is indispensable in theorem 2.1 and corollary 2.2. we give an example, the idea is come
from the Kraus representation theorem (see [9]): Suppose that $H$ is a complex separable infinite dimensional Hilbert space such that $H$ can be presented as a direct sum of mutually orthogonal closed subspaces, $H = (\oplus_{k=1}^{N} H_k) \oplus H_0$, $N \in \mathbb{N}$, $\dim H_k = \dim H$, $k = 1, 2, \cdots, N$. Let $U_k : H \to H_k$ be unitary or antiunitary operators, $\lambda_1, \lambda_2, \cdots, \lambda_N \in (0, 1), \sum_{k=1}^{N} \lambda_k = 1$. Let $\Phi(T) = \sum_{k=1}^{N} \lambda_k U_k T U_k^*$, $\forall T \in S_1(H)$. Then $tr(\Phi(T)) = \sum_{k=1}^{N} \lambda_k tr(U_k T U_k^*) = tr(T)$, the last equality being due to $U_k U_k^* = I$.

This implies that $\Phi$ is indeed a mapping which maps $S_1(H)$ into $S_1(H)$. Furthermore, it is easy to see that $\Phi(\lambda T + (1 - \lambda)S) = \lambda \Phi(T) + (1 - \lambda)\Phi(S)$ for all $T, S \in S_1(H), 0 \leq \lambda \leq 1$. Finally, all $\Phi_k : T \to U_k T U_k^*$ are isometric and for all $T \in S_1(H), |\Phi_k(T)||\Phi_l(T)| (i.e., |\Phi_k(T)||\Phi_l(T)| = 0)$ if $k \neq l$. Thus

$$\|\Phi(T) - \Phi(S)\|_1 = \|\sum_{k=1}^{N} \lambda_k U_k(T - S) U_k^*\|_1 = \sum_{k=1}^{N} \lambda_k(\|T - S\|_1) = \|T - S\|_1$$

for all $T, S \in S_1(H)$. But, in general, such $\Phi$ is not a bijection and does not have a nice form as theorem 2.1 and corollary 2.2.

3. Ortho-order automorphisms on $S_1(H)$

The purpose of this section is to characterize the ortho-order automorphisms of $S_1(H)$, that is, the bijective map $\Phi$ preserves the spectral order in both directions and preserves orthogonality in both directions. The following is the main result.

**Theorem 3.1.** If $\Phi : S_1(H) \to S_1(H)$ is an ortho-order automorphism, then there exists an either unitary or antiunitary operator $U$ on $H$, a strictly increasing continuous bijection $f : [0, 1] \mapsto [0, 1]$ such that $\Phi(T) = U f(T) U^*$ for all $T \in S_1(H)$, where $f(T)$ is obtained from the continuous function calculus.

Before embarking on the proof of Theorem 3.1, we need some terminologies and facts about spectral order.

First of all, by a resolution of identity we mean a function from $\mathbb{R}$ into the lattice $(\mathcal{P}(H), \leq)$ of all projections on $H$ which is increasing, right-continuous, for all small real numbers it takes the value 0, while for large enough real numbers it takes value $I$ (the identity operator of $H$). It is well-known that there is a one-to-one correspondence between the compactly supported spectral measures on the Borel sets of $\mathbb{R}$ and the resolutions of the identity (see [8, Page 360]). In fact, every resolution of the identity is the form $t \mapsto E(-\infty, t]$. If $T \in S_1(H)$, the resolution of the identity corresponding to $E_T$ is called the spectral resolution of $T$. Next, the spectral order implies the usual order: if $T, S \in S_1(H)$ and $T \preceq S$, then $T \leq S$. Furthermore, $T \preceq S$ if and only if $T^n \preceq S^n$ for every $n \in \mathbb{N}$. For commuting $T, S \in S_1(H)$, by the spectral theorem of positive operators, it is easy to see $T \preceq S$ if and only if $T \leq S$. Finally, for $T, S \in S_1(H)$,
the supremum of the set \( \{T, S\} \) in this structure denoted by \( T \vee S \) exists. Similarly, the infimum of the set \( \{T, S\} \) denoted by \( T \wedge S \) also exists. For details, one can see [20].

After these preparations, we turn to the proof of Theorem 3.1.

**Proof of Theorem 3.1.** The proof is divided into 3 claims.

**Claim 1.** \( \Phi \) preserves the rank of operators.

Let \( \Phi \) be an ortho-order automorphism of \( S_1(H) \). Since \( 0 = \wedge S_1(H) \), it follows that \( \Phi(0) = 0 \). For \( T \in S_1(H) \), we denote by \( \{T\}^\perp = \{S \in S_1(H) : TS = 0\} \), i.e., the set of all elements of \( S_1(H) \) which are orthogonal to \( T \). By the spectral theorem of positive operators, it is easy to see that \( T \in S_1(H) \) is of rank \( n \) if and only if \( \{T\}^\perp \) contains \( n \) pairwise orthogonal nonzero elements but it does not contain more. As \( \Phi \) preserves orthogonality in both directions, it is now clear that \( \Phi \) preserves the rank of operators.

**Claim 2.** There exists a strictly increasing continuous bijection \( f : [0, 1] \rightarrow [0, 1] \) such that \( \Phi(\lambda P) = f(\lambda) \Phi(P) \) for all \( P \in P_1(H) \).

By Claim 1, \( \Phi \) preserves the rank of operators. In particular, \( \Phi \) preserves the rank one elements of \( S_1(H) \). Since \( \Phi^{-1} \) has the same properties as \( \Phi \), we have \( \Phi \) preserves the rank one elements in both directions, i.e., \( T \in S_1(H) \) is rank one if and only if \( \Phi(T) \) is rank one. Note that the rank one projections are exactly the maximal elements of the set of all rank one elements in \( S_1(H) \). This implies \( \Phi \) preserves rank one projections in both directions, i.e., \( \Phi(P_1(H)) = P_1(H) \).

Let \( P \) be a rank one projection. For \( \lambda \in [0, 1] \), then \( \lambda P \preceq P \) and so we have \( \Phi(\lambda P) \preceq \Phi(P) \). This implies that there is a scalar \( f_P(\lambda) \in [0, 1] \) such that

\[
\Phi(\lambda P) = f_P(\lambda) \Phi(P).
\]

It follows from the properties of \( \Phi \) that \( f_P \) is a strictly increasing continuous bijection of \([0, 1] \). Now, \( \Phi(0) = 0 \) together with \( \Phi \) preserves rank one projection implies \( f_P(0) = 0 \), \( f_P(1) = 1 \).

In the following, we will prove that \( f_P \) does not depend on \( P \).

Let \( E, F \), \( E \neq F \), be rank one projections and \( 0 < \lambda \leq \mu \leq 1 \). Computing the spectral resolution of \( \lambda E \) and \( \mu F \), we have

\[
E_{\lambda E}(-\infty, t] = \begin{cases} 
0 & t < 0; \\
I - E & 0 \leq t < \lambda; \\
I & \lambda \leq t,
\end{cases}
\]

\[
E_{\mu F}(-\infty, t] = \begin{cases} 
0 & t < 0; \\
I - F & 0 \leq t < \mu; \\
I & \mu \leq t.
\end{cases}
\]
From [20], we know that \( E_{\lambda E \lor \mu F}(-\infty, t] = E_{\lambda E}(-\infty, t] \land E_{\mu F}(-\infty, t] \), and so

\[
E_{\lambda E \lor \mu F} = \begin{cases} 
0 & t < 0; \\
(I - E) \land (I - F) & 0 \leq t < \lambda; \\
I - F & \lambda \leq t < \mu; \\
I & \mu \leq t.
\end{cases}
\]

Note that \((I - E) \land (I - F) = I - E \lor F\), thus we have

\[
\lambda E \lor \mu F = \lambda(E \lor F - F) + \mu F.
\]

This tells us that the nonzero eigenvalues of the operator \( \lambda E \lor \mu F \) are \( \lambda \) and \( \mu \).

Let \( R \) be a rank two projection of \( H \) and pick \( \lambda \leq \frac{1}{2} \), then \( \lambda R \in S_1(H) \). Since \( \Phi \) preserves the rank of operators, we have \( \Phi(\lambda R) \) is a rank two operator and hence it can be written in the form

\[
\Phi(\lambda R) = \alpha P' + \beta Q'
\]

with mutually orthogonal rank one projection \( P', Q' \) and \( 0 < \alpha \leq \beta < 1 \). Pick any two different rank one subprojections \( P, Q \) of \( R \). Then we compute

\[
\Phi(\lambda R) = \Phi(\lambda P \lor \lambda Q) = \Phi(\lambda P) \lor \Phi(\lambda Q) = f_P(\lambda)\Phi(P) \lor f_Q(\lambda)\Phi(Q).
\]

It follows that

\[
\{\alpha, \beta\} = \{f_P(\lambda), f_Q(\lambda)\}.
\]

We claim that \( \alpha = \beta \). Suppose on the contrary that \( \alpha \neq \beta \). Without loss of generality, we may assume that \( f_P(\lambda) = \alpha \) and \( f_Q(\lambda) = \beta \). Pick a third rank one subprojection \( R_0 \) of \( R \) which is different from \( P \) and \( Q \). Then repeating the above argument for the pair \( P, R_0 \), we have \( f_{R_0}(\lambda) = \beta \). Similarly, for the pair \( R_0, Q \), we have \( f_{R_0}(\lambda) = \alpha \). This contradiction yields that \( \alpha = \beta \) and so \( f_P(\lambda) = f_Q(\lambda) \) for all \( \lambda \in [0, \frac{1}{2}] \). Thus we can denote \( f_P(\lambda) = f(\lambda) \) for all \( \lambda \in [0, \frac{1}{2}] \).

The remained is to prove \( f_P(\lambda) \) does not depend on \( P \) for every \( \lambda \in (\frac{1}{2}, 1) \). Let \( Q \) be a rank one projection such that \( PQ = 0 \). We compute

\[
\Phi(\lambda P + (1 - \lambda)Q) = \Phi(\lambda P \lor (1 - \lambda)Q)) = \Phi(\lambda P) \lor \Phi((1 - \lambda)Q).
\]

Since \( \Phi(\lambda P) \) and \( \Phi((1 - \lambda)Q) \) are orthogonal, It follows that

\[
\Phi(\lambda P + (1 - \lambda)Q) = f_P(\lambda)\Phi(P) + f(1 - \lambda)\Phi(Q).
\]

For \( T \in S_1(H) \), by the spectral theorem of positive operator, \( tr(T) = 1 \) if and only if there does not exist \( S \in S_1(H) \) such that \( T \preceq S \). By the properties of \( \Phi \) and \( \Phi^{-1} \), we have \( \Phi(V) = V \), recall that \( V \) is the surface of \( S_1(H) \). Combing this with \( \Phi(\lambda P + (1 - \lambda)Q) = f_P(\lambda)\Phi(P) + f(1 - \lambda)\Phi(Q) \), we can obtain \( f_P(\lambda) + f(1 - \lambda) = 1 \).
Clearly, $f_P(\lambda) = 1 - f(1 - \lambda)$ and so $f_P(\lambda)$ does not depend on $P$ for every $\lambda \in (\frac{1}{2}, 1)$. This completes the proof of this claim.

Claim 3. There exists an either unitary or antiunitary operator $U$ on $H$, a strictly increasing continuous bijection $f : [0, 1] \mapsto [0, 1]$ such that $\Phi(T) = Uf(T)U^*$ for all $T \in S_1(H)$, where $f(T)$ is obtained from the continuous function calculus.

Now $\Phi : \mathcal{P}_1(H) \to \mathcal{P}_1(H)$ is a bijection and preserves orthogonality in both directions. By the Uhlhorn-type of Wigner's theorem (see [22]), there exists a unitary or antiunitary operator $U$ on $H$ such that $\Phi(P) = UPU^*$ for all $P \in \mathcal{P}_1(H)$.

For $T_n = \lambda_1 P_1 + \lambda_1 P_2 + \cdots + \lambda_n P_n$, $\lambda_i \in (0, 1]$ ($i = 1, 2, \cdots, n$), $\sum_{i=1}^{n} \lambda_i \in (0, 1]$, $P_i P_j = 0 (i \neq j, i, j = 1, 2, \cdots n)$. Then

$$\Phi(T) = \Phi(\lambda_1 P_1 + \lambda_1 P_2 + \cdots + \lambda_n P_n)$$
$$= \Phi(\lambda_1 P_1 \vee \lambda_1 P_2 \vee \cdots \vee \lambda_n P_n)$$
$$= f(\lambda_1)UP_1U^* + \cdots + f(\lambda_n)UP_nU^*$$
$$= Uf(T)U^*,$$

where $f(T)$ is obtained from the continuous function calculus. For every $T \in S_1(H)$, by the spectral theorem of positive operators, there exists a monotonically increasing sequence $\{T_n\}_{n=1}^{+\infty}$ of $S_1(H)$ such that $T = \vee_{n=1}^{+\infty} T_n$. Since $\Phi$ preserves the spectral order of operators in both directions, it follows that $\Phi(T) = Uf(T)U^*$ for every $T \in S_1(H)$.

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(Zhaofang Bai) School of Mathematical Sciences, Xiamen University, Xiamen, 361000, P. R. China.

(Shuanping Du) School of Mathematical Sciences, Xiamen University, Xiamen, 361000, P. R. China.

E-mail address, Shuanping Du: shuanpingdu@yahoo.com