Relation between the 4d superconformal index and the $S^3$ partition function

Yosuke Imamura$^1$

$^1$ Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan

Abstract

A relation between the 4d superconformal index and the $S^3$ partition function is studied with focus on the 4d and 3d actions used in localization. In the case of vanishing Chern-Simons levels and round $S^3$ we explicitly show that the 3d action is obtained from the 4d action by dimensional reduction up to terms which do not affect the exact results. By combining this fact and a recent proposal concerning a squashing of $S^3$ and $SU(2)$ Wilson line, we obtain a formula which gives the partition function depending on the Weyl weight of chiral multiplets, real mass parameters, FI parameters, and a squashing parameter as a limit of the index of a parent 4d theory.

$^*$E-mail: imamura@phys.titech.ac.jp
1 Introduction

Recent years, exactly calculable quantities in gauge theories play important roles in study of gauge theories themselves and their relation to string/M theory. In this paper we discuss a relation between two of such quantities.

One is the $S^3$ partition function\cite{1,2,3}. It is used to confirm dualities among 3d theories\cite{4,5,6,7} and predictions of AdS$_4$/CFT$_3$\cite{8,9,10,11,12}. Furthermore, this function provides a simple way to determine the R-charge at IR fixed points\cite{2}. The partition function is evaluated exactly by localization. We choose a nilpotent supercharge $Q$ and deform the action by $Q$-exact terms.

The partition function is given by

$$Z = \int D\Phi \exp \left( -S_0^{(3d)} - u \int_{S^3} \sqrt{g} L^{(3d)} d^3 x \right),$$

(1)

$S_0^{(3d)}$ is the original action of the 3d theory and the second term in the exponent is the $Q$ exact action. This path integral does not depend on $u$, and is evaluated exactly in the weak coupling limit $u \to \infty$.

The other exactly calculable quantity we consider is the $N=1$ superconformal index for 4d theories\cite{13,14}. The index is defined by

$$I(t, x, h_i) = \text{tr} \left[ (-1)^F q^{\frac{3}{2} R - 2J_L t^R + 2J_R x} \prod_i h_i^{F_i} \right],$$

(2)

where $D$ (the dilatation), $R$ (the R-charge), $J_L$ and $J_R$ ($SU(2)_L \times SU(2)_R$ spins), are Cartan generators of the $N=1$ superconformal algebra $PSU(2, 2|1)$, and $F_i$ are Cartan generators of the flavor symmetry. Only operators saturating the BPS bound

$$D - \frac{3}{2} R - 2J_L \geq 0$$

(3)

contribute to the index, and (2) is independent of the variable $q$. This quantity is exactly calculable, and is conveniently used as a tool to check Seiberg-duality\cite{15,16,17,18} and AdS$_5$/CFT$_4$\cite{13,19,20,21,22}.

One way to compute the index is to use localization. We choose a nilpotent supercharge $Q$ and deform the action by $Q$-exact terms. The index can be expressed in the path integral form

$$I(t, x, h_i) = \int D\Phi \exp \left( -S_0^{(4d)} - u \int_{S^3 \times S^1} \sqrt{g} L^{(4d)} d^4 x \right),$$

(4)

where $S_0^{(4d)}$ is the original action of the theory defined in $S^3 \times S^1$, and the second term in the exponent is the $Q$-exact deformation action. The chemical potentials are introduced as non-trivial Wilson lines around $S^1$. Let $r$ and $\beta r$ be the $S^3$ radius and the $S^1$ period, respectively. The ratio $\beta$ is related to the parameter $q$ by $q = e^{-\beta}$. In the case of the index, the deformation term does not necessarily have to be $Q$-exact because the index does not depend on
continuous coupling constants; even so, we adopt a $Q$-exact deformation action in this paper for the reason which will become clear shortly.

The similarity between (1) and (4) strongly suggests that there exists some relation between the index and the partition function. If we consider 4d and 3d theories with the same gauge group $G$ and the same matter contents, we naturally expect that the partition function is obtained by taking a small $S^1$ limit of the index. Such a relation was recently studied in [23, 24].

In [23] it is shown for particular examples of gauge theories that a relation between the 3d partition function and the 4d index follows from certain mathematical properties of special functions appearing in the index and partition function. A similar relation is also studied in [24], and a limiting procedure which reduces the superconformal index of 4d $\mathcal{N} = 2$ theories to the $S^3$ partition function of corresponding 3d $\mathcal{N} = 4$ theories is proposed. In these works, only the final expressions for the partition function and the index are studied, and physical origin of the relation is not so obvious. The purpose of this paper is to extend the relation obtained in [23, 24] to general 3d $\mathcal{N} = 2$ and 4d $\mathcal{N} = 1$ theories, and to establish the relation at more fundamental level by comparing 3d and 4d actions. For this purpose, it is convenient to use as similar deformation terms as possible in two computations. We use $Q$-exact deformation terms in both cases with closely related supercharges $Q$ in 3d and 4d theories.

In both (1) and (4), the deformation terms dominate the actions in the weak coupling limit $u \to \infty$, and only few terms in the original actions are relevant to the partition function and the index. Let $S_{\text{rel}}^{(3d)}$ and $S_{\text{rel}}^{(4d)}$ be the relevant terms including the deformation terms. $S_{\text{rel}}^{(3d)}$ consists of (supersymmetric completion of) Chern-Simons and FI terms in the original action $S_{(3d)}^{(3d)}$ and the $Q$-exact terms

$$ S_{\text{rel}}^{(3d)} = S_{\text{CS}}^{(3d)} + S_{\text{FI}}^{(3d)} + u \int_{S^3} \sqrt{g} L^{(3d)} d^3 x, \quad (5) $$

while $S_{\text{rel}}^{(4d)}$ consists of the (supersymmetric completion of) FI terms and the deformation terms

$$ S_{\text{rel}}^{(4d)} = S_{\text{FI}}^{(4d)} + u \int_{S^4 \times S^1} \sqrt{g} L^{(4d)} d^4 x. \quad (6) $$

We consider 3d and 4d theories with the same gauge group $G$ and chiral multiplets $\Phi_I$ belonging to the same $G$-representations $R_I$. We assume that the Weyl weight $\Delta_I$ of each chiral multiplet is the same in 3d and 4d. We explicitly show for a 3d theory without Chern-Simons terms on round $S^3$ that $S_{\text{rel}}^{(3d)}$ is obtained by dimensional reduction of $S_{\text{rel}}^{(4d)}$ provided that an appropriate Wilson line is turned on.

\footnotetext[1]{The deformation terms are not invariant under the dilatation, and the dilatation is broken in the deformed theories. For this reason, the parameters $\Delta_I$ in the deformed theories should be regarded not as the Weyl weights but as parameters appearing in the $Q$ transformation laws for chiral multiplets. The absence of the dilatation symmetry in the deformed theories does not cause any problem because we need only the fermionic symmetry $Q$ for the computation of the exact results.}
The symmetry associated with the Wilson line may not be a symmetry of the original action $S_{0}^{(4d)}$, but is a symmetry of $S_{\text{rel}}^{(4d)}$. (The symmetry may be anomalous. We discuss the treatment of anomalous symmetries at the end of §7.) $S_{\text{rel}}^{(4d)}$ has the symmetry rotating each chiral multiplet independently. For each chiral multiplet $\Phi_{I}$ we define the charge $F_{I}$ rotating only $\Phi_{I}$ and the corresponding chemical potential $h_{I}$. By comparing $S_{\text{rel}}^{(3d)}$ and $S_{\text{rel}}^{(4d)}$, we obtain a formula which gives the partition function $Z$ as a small radius limit of the index $I(t, x, h_{I})$. We further generalize the relation by using the recently proposed connection between squashing parameter $s$ of $S^{3}[25]$ and $SU(2)R$ Wilson line. The most general formula we propose in this paper is

$$Z = \lim_{q \to 1} I(t = q, x = q^{s}, h_{I} = q^{-ir\mu_{I}+\frac{1}{3}\Delta_{I}})\big|_{\zeta^{(4d)}} = \frac{1}{\zeta^{(3d)}}$$

where $\mu_{I}$ are real mass parameters, and $\zeta^{(4d)}$ and $\zeta^{(3d)}$ are 4d and 3d FI parameters, respectively. Unfortunately, when Chern-Simons levels $k_{a}$ of 3d theory are non-vanishing, we could not reproduce the $S^{3}$ partition function from the index due to the difficulty in obtaining Chern-Simons terms by dimensional reduction.

The paper is organized as follows. After explaining our notation for spinors in the next section, we summarize the superconformal algebra and the supersymmetry transformation laws in §3 and §4. Exact computations of the $S^{3}$ partition function and the 4d superconformal index are briefly reviewed in §5 and §6, respectively. In §7 we compare the 3d and 4d actions, and find the relation between the partition function and the index in the case of $\mu_{I} = k_{a} = \zeta^{(3d)} = s = 0$. Generalization to non-vanishing parameters is discussed in §8. Conclusions are presented in §9.

2 Notation for spinors

Because we consider both 3d and 4d theories, we use notation for spinors such that the expression of 3d and 4d theories look as similar as possible.

For 3d spacetime, we use coordinates $x^{m}$ ($m = 1, 2, 3$). Although we can define Majorana spinors in 3d Minkowski spacetime, all spinors we use are complex spinors. For a complex spinor $\psi$, we denote its Majorana conjugate by $\bar{\psi}$. In Euclidean spacetime $\psi$ and $\bar{\psi}$ should be treated as independent spinors.

For 4d spacetime, we use coordinates $x^{\mu}$ ($\mu = 1, 2, 3, 4$). When we consider $S^{3} \times S^{1}$ background, we use $x^{m}$ for $S^{3}$ and $x^{4}$ for $S^{1}$. The 4d Dirac matrices are expressed in terms of the 3d Dirac’s matrices by

$$\gamma^{m} = \left( \begin{array}{cc} 0 & \gamma^{m} \\ \gamma^{m} & 0 \end{array} \right) \quad m = 1, 2, 3, \quad \gamma^{4} = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right).$$

We use the same symbol $\gamma^{m}$ for 3d and 4d Dirac’s matrices. The charge conjugation and the chirality in 4d are

$$C_{ab} = \left( \begin{array}{cc} \epsilon_{ab} & 0 \\ 0 & \epsilon_{ab} \end{array} \right), \quad \gamma^{5} = \left( \begin{array}{cc} 1_{2} & 0 \\ 0 & -1_{2} \end{array} \right).$$
We call the upper (lower) half of a four-component spinor left-handed (right-handed). Namely, a left-handed (right-handed) spinor has positive (negative) chirality. 3d and 4d completely anti-symmetric tensors $\epsilon_{mnp}$ and $\epsilon_{\mu\nu\rho\sigma}$ are defined by
\[\gamma_{mnp} = i\epsilon_{mnp}1_{2}, \quad \gamma^{5}\gamma_{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma}1_{4}.\] (10)
We raise and lower spinor indices by the relation $\psi_{a} = \psi^{b}\epsilon_{ba}$. Spinor indices are contracted by the NW-SE rule. For example, for spinors $\psi$ and $\chi$, $\psi\chi = \psi^{a}\chi_{a} = \psi^{a}\chi_{b}\epsilon_{ba}$.

In 4d we use two-component representation. We use a symbol without and with bar for a left-handed and right-handed spinor, respectively. For example, when we use symbol $\psi$ and $\bar{\psi}$ as 4d two-component spinors, their four-component representations are
\[\left(\begin{array}{c}
\psi \\
0
\end{array}\right), \quad \left(\begin{array}{c}
0 \\
\bar{\psi}
\end{array}\right).\] (11)
Note that $\bar{\psi}$ is not the Dirac’s conjugate of $\psi$. We will never use Dirac’s conjugate in this paper.

We use indices $\mu, \nu, \ldots$ not only in 4d but also in 3d. In that case we assume that all fields do not depend on $x^{4}$, and the 4-th component of a gauge field $A_{\mu}$ is regarded as a Hermitian scalar field $\sigma$. For example, if the gauge covariant derivative is given by $D_{\mu} = \partial_{\mu} - iA_{\mu}$, the fermion kinetic term $-\tilde{(\psi\gamma^{m}D_{m}\psi)}$ represents in 3d the sum of two terms $-\tilde{(\psi\gamma^{m}D_{m}\psi)}$ and $-(\psi\sigma\psi)$.

3 Superconformal algebra

Before considering actions and transformation laws, let us compare the 4d $\mathcal{N} = 1$ superconformal algebra and 3d $\mathcal{N} = 2$ superconformal algebra.

The 4d algebra contains the generators
\[M_{\mu\nu}, \ P_{\mu}, \ K_{\mu}, \ D, \ R, \ Q, \ \overline{Q}, \ S, \ \overline{S}, \] (12)
while the 3d algebra contains the same generators with vector indices $\mu$ and $\nu$ running over 1, 2, 3 only. For later use we define Cartan generators of the rotation groups,
\[M_{12} = iJ_{3} \ (3d), \quad M_{12} = i(J_{L} + J_{R}), \quad M_{34} = i(J_{L} - J_{R}) \ (4d).\] (13)
Almost all (anti-)commutation relations are the same in 3d and 4d.

\[[M_{\mu
u},M_{\rho\sigma}] = \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho},\]
\[[M_{\mu\nu},P_\rho] = \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu, \quad [M_{\mu\nu},K_\rho] = \eta_{\mu\rho}K_\nu - \eta_{\nu\rho}P_\mu,\]
\[[D,P_\mu] = P_\mu, \quad [D,K_\mu] = -K_\mu, \quad [P_\mu,K_\nu] = -2M_{\mu\nu} + 2\eta_{\mu\nu}D,\]
\[[M_{\mu\nu},S] = \frac{1}{2}\gamma_{\mu\nu}S, \quad (S = Q, \overline{Q}, S, \overline{S}),\]
\[[R,Q] = -Q, \quad [R,\overline{Q}] = \overline{Q}, \quad [R,S] = S, \quad [R,\overline{S}] = -\overline{S},\]
\[[D,Q] = \frac{1}{2}Q, \quad [D,\overline{Q}] = \frac{1}{2}\overline{Q}, \quad [D,S] = \frac{1}{2}S, \quad [D,\overline{S}] = -\frac{1}{2}\overline{S},\]
\[[S,P_\mu] = \gamma_\mu S, \quad [\overline{S},P_\mu] = \gamma_\mu Q, \quad [Q,K_\mu] = \gamma_\mu S, \quad [\overline{Q},K_\mu] = \gamma_\mu S,\]
\[\{Q_\alpha,\overline{Q}_\beta\} = 2(\gamma^{\mu})_{ab}P_\mu, \quad \{S_\alpha,\overline{S}_\beta\} = 2(\gamma^{\mu})_{ab}K_\mu.\] (14)

Differences between 3d and 4d arise only in \{S,Q\} and \{\overline{S},Q\}. In the 3d algebra, they are

\[\{S_\alpha,\overline{Q}_\beta\} = (\gamma^{mn})_{ab}M_{mn} + 2\epsilon_{ab}D + 2\epsilon_{ab}R,\]
\[\{\overline{S}_\alpha,\overline{Q}_\beta\} = (\gamma^{mn})_{ab}M_{mn} + 2\epsilon_{ab}D - 2\epsilon_{ab}R,\] (15)

while in the 4d algebra, the coefficients of the R-charge terms are different.

\[\{S_\alpha,\overline{Q}_\beta\} = (\gamma^{\mu\nu})_{ab}M_{\mu\nu} + 2\epsilon_{ab}D + 3\epsilon_{ab}R,\]
\[\{\overline{S}_\alpha,\overline{Q}_\beta\} = (\gamma^{\mu\nu})_{ab}M_{\mu\nu} + 2\epsilon_{ab}D - 3\epsilon_{ab}R.\] (16)

In radial quantization, the dilatation \(D\) is regarded as Hamiltonian, and \(\overline{Q}^\dagger\) and \(\overline{S}_a\) are treated to be Hermitian conjugate to each other. From (15) and (16) we can derive BPS bounds. In particular, the bound obtained from \{\overline{S}_1,\overline{Q}\} is important in the following computations. In 3d, it is

\[\{\overline{S}_1,\overline{Q}\} = 2D - 2R - 2J_3 \geq 0.\] (17)

In 4d, we obtain the bound with different coefficients

\[\{\overline{S}_1,\overline{Q}\} = 2D - 3R - 4J_L \geq 0.\] (18)

### 4 Supersymmetry transformations

Because the Poincare subalgebra in (14) generated by \(M_{\mu\nu}, P_\mu, Q\) and \(\overline{Q}\) is the same in 3d and 4d, (up to the absence of \(P_4\) and \(M_{4\alpha}\) in 3d, \(Q\) and \(\overline{Q}\)-transformation laws in the flat background take the same form in 3d and 4d.
For a vector multiplet \((A_\mu, \lambda, \bar{\lambda}, D)\), the \(Q\) and \(\bar{Q}\)-transformations are
\[
\delta^0 A_\mu = i(\epsilon \gamma_\mu \lambda) - i(\tau \gamma_\mu \lambda),
\]
\[
\delta^0 \lambda = \frac{i}{2} \gamma^\mu \epsilon F_{\mu \nu} + D \epsilon,
\]
\[
\delta^0 \bar{\lambda} = - \frac{i}{2} \gamma^\mu \tau F_{\mu \nu} + D \tau,
\]
\[
\delta^0 D = - (\epsilon \gamma^\mu D_\mu \lambda) - (\tau \gamma^\mu D_\mu \lambda).
\]  
(19)

We use the symbol \(\delta^0\) rather than \(\delta\) to emphasize that these are rules for the flat background. When we regard these as rules for 3d theory, all fields are assumed to be independent of \(x^4\), and \(A_4\) should be regarded as a Hermitian scalar field \(\sigma\). For a chiral multiplet \((\phi, \psi, F)\), the transformation laws are
\[
\delta^0 \phi = \sqrt{2} (\epsilon \psi),
\]
\[
\delta^0 \phi^\dagger = \sqrt{2} (\bar{\epsilon} \bar{\psi}),
\]
\[
\delta^0 \psi = - \sqrt{2} \gamma^\mu \tau D_\mu \phi + \sqrt{2} \epsilon F,
\]
\[
\delta^0 \bar{\psi} = - \sqrt{2} \gamma^\mu \epsilon D_\mu \phi^\dagger + \sqrt{2} \tau F^\dagger,
\]
\[
\delta^0 F = - \sqrt{2} (\tau \gamma^\mu D_\mu \psi) - 2 (\bar{\tau} \bar{\lambda}) \phi,
\]
\[
\delta^0 F^\dagger = - \sqrt{2} (\epsilon \gamma^\mu D_\mu \bar{\psi}) - 2 \phi^\dagger (\epsilon \lambda).
\]  
(20)

We can construct supersymmetry transformation laws for an arbitrary conformally flat background from (19) and (20) by Weyl-covariantization. By a Weyl transformation
\[
e^a_\mu = e^{\alpha} e^{\alpha}_\mu,
\]  
(21)
a field \(\varphi\) with Weyl weight \(\Delta \varphi\) is transformed by
\[
\varphi = e^{\Delta \varphi} \varphi'.
\]  
(22)

Even if a field \(\varphi\) has definite Weyl weight, its derivative is not transformed covariantly as (22) and terms containing \(\partial \mu \alpha\) arise. There are such non-covariant terms in the transformation laws (19) and (20). To extend them to a general conformally flat background, we should covariantize them with respect to Weyl transformation by adding terms containing derivatives of parameters \(\epsilon\) and \(\tau\). \(\delta^0 \lambda\) in 3d and \(\delta^0 \psi\) contain terms proportional to \((D_\mu \varphi) \gamma^\mu \tau\) with \(\varphi = \sigma\) and \(\phi\), respectively. \(\delta^0 \bar{\lambda}\) in 3d and \(\delta^0 \bar{\psi}\) also contain similar scalar derivative terms. In \(d\)-dimensional spacetime, we can covariantize terms of this form by the replacement
\[
(D_\mu \varphi) \gamma^\mu \tau \rightarrow (D_\mu \varphi) \gamma^\mu \tau + \frac{2 \Delta \varphi}{d} \varphi \gamma^\mu D_\mu \tau,
\]
\[
(D_\mu \varphi) \gamma^\mu \epsilon \rightarrow (D_\mu \varphi) \gamma^\mu \epsilon + \frac{2 \Delta \varphi}{d} \varphi \gamma^\mu D_\mu \epsilon.
\]  
(23)
The fermion derivative terms in $\delta^0 D \delta^0 F$, and $\delta^0 F^\dagger$ are covariantized by the replacement

\[
(\bar{\epsilon} \gamma^\mu D_\mu \chi) \to (\bar{\epsilon} \gamma^\mu D_\mu \chi) + \frac{2\Delta_\chi + 1 - d}{d} (D_\mu \bar{\epsilon} \gamma^\mu \chi),
\]

\[
(\epsilon \gamma^\mu D_\mu \bar{\chi}) \to (\epsilon \gamma^\mu D_\mu \bar{\chi}) + \frac{2\Delta_\chi + 1 - d}{d} (D_\mu \epsilon \gamma^\mu \bar{\chi}).
\]

We can easily confirm that (23) and (24) are transformed covariantly by the Weyl transformation (21) and (22) as fields with weight $\Delta_\phi + 1/2$ and $\Delta_\chi + 1/2$, respectively.

5 \textbf{S}^3 \textbf{ partition function}

In this section we briefly review the computation of the $\text{S}^3$ partition function. We here only consider the case with $\mu_l = \zeta_A = k_a = s = 0$.

Both a 3d $\mathcal{N} = 2$ theory and a 4d $\mathcal{N} = 1$ theory have eight supercharges. Four of them ($Q$ and $\overline{S}$) correspond to the parameter $\epsilon$ and the other four ($\overline{Q}$ and $S$) to $\bar{\tau}$. When we use localization, we choose a nilpotent supercharge $Q$, and add $Q$-exact terms to the action. Because we should use a linear combination of $\overline{Q}$ and $S$ for computation of the index (2), we consider only transformations by $\bar{\tau}$ in the following.

On a conformally flat 3d background the parameter $\bar{\tau}$ must satisfy the Killing equation

\[
D_m \bar{\tau} = \gamma_m \kappa,
\]

where $\kappa$ is an arbitrary spinor. Corresponding to four supercharges $\overline{Q}_a$ and $S_a$, there are four linearly independent solutions to (25). In the case of $\text{S}^3$, two of them are right-invariant, and belong to the (2,$1\overline{1}$) representation of the $SO(4) = SU(2)_L \times SU(2)_R$ isometry group. Let us denote spinors with $J_L = +1/2$ and $J_L = -1/2$ by $\tau_1$ and $\tau_2$, respectively. We adopt $\delta(\tau_1)$ as $Q$. Both $\tau_1$ and $\tau_2$ satisfy

\[
D_m \tau = - \frac{i}{2} \gamma_m \tau, \tag{26}
\]

and $(\tau_1 \tau_2)$ is constant on $\text{S}^3$. The other two solutions of (25), which we will not use in this paper, are left-invariant, and satisfy a similar equation to (26) with opposite sign on the right hand side. In the following the parameter $\tau$ is always assumed to satisfy (26).

The $\delta(\tau)$ transformation laws for fields on $\text{S}^3$ are obtained from (19) and (20) by the Weyl-covariantization. The vector multiplet transformation laws are

\[
\begin{align*}
\delta(\tau) A_m &= -i(\overline{\tau} \gamma_m \lambda), & \delta(\tau) \sigma &= (\overline{\tau} \lambda), & \delta(\tau) \lambda &= 0, \\
\delta(\tau) \bar{\tau} &= - \frac{i}{2} \gamma^{mn} \tau F_{mn} - \gamma^m \tau D_m \sigma - D \bar{\tau} + \frac{i}{r} \bar{\tau} \sigma, \\
\delta(\tau) D &= -(\overline{\tau} \gamma^m D_m \lambda) - (\overline{\tau} [\sigma, \lambda]) - \frac{i}{2r} (\lambda \bar{\tau}). \tag{27}
\end{align*}
\]
The chiral multiplet transformation laws are
\[
\delta(\tau)\phi^\dagger = \sqrt{2}(\tau\overline{\psi}), \quad \delta(\tau)\phi = 0, \quad \delta(\tau)\overline{\psi} = \sqrt{2}F^\dagger, \quad \delta(\tau)F^\dagger = 0,
\]
\[
\delta(\tau)\psi = -\sqrt{2}\gamma^m\tau D_m\phi + \sqrt{2}\tau\sigma\phi + \frac{\sqrt{2}i}{r}\Delta_d\tau\phi,
\]
\[
\delta(\tau)F = -\sqrt{2}\tau\gamma^mD_m\psi - \sqrt{2}\sigma(\tau\psi) - 2(\overline{\tau\Delta})\phi - \frac{\sqrt{2}i}{r}\left(\Delta_d - \frac{1}{2}\right)(\tau\psi), \quad (28)
\]
where \(\Delta_d\) is the Weyl weight of the chiral multiplet, which is defined as the Weyl weight of the dynamical scalar component field.

There is an ambiguity in the choice of the \(Q\)-exact deformation Lagrangian density \(L\). We adopt the following one obtained by applying \(\delta(\tau_1)\) and \(\delta(\tau_2)\) to an anti-chiral operator,
\[
(\tau_1\tau_2)L = \delta(\tau_1)\delta(\tau_2)\left(-\frac{1}{4}\text{tr}(\lambda\lambda) - \frac{1}{2} \sum_I \phi_I^\dagger F_I\right), \quad (29)
\]
where \(\text{tr}\) represents a gauge invariant positive definite inner product. \(29\) can be used both in 3d and 4d. In 3d, by using the 3d transformation laws \(27\) and \(28\), we obtain
\[
L^{(3d)} = \text{tr}\left[\frac{1}{4}F_{mn}F^{mn} + \frac{1}{2}D_m\sigma D^m\sigma - \frac{1}{2}\epsilon^{mnp}F_{mn}D_p\sigma + \frac{1}{2}\left(\frac{1}{r}\sigma - iD\right)^2 \right.
\]
\[
- (\lambda\gamma^mD_m\lambda) - (\overline{\lambda}[\sigma, \lambda]) + \frac{i}{2r}(\overline{\lambda}\lambda)
\]
\[
+ \sum_I \left[-\phi_I^\dagger D_mD^m\phi_I + \phi_I^\dagger\sigma\phi_I + \phi_I^\dagger D\phi_I
\right.
\]
\[
- \frac{i(1 - 2\Delta_I)}{r}\phi_I^\dagger\sigma\phi_I - \frac{\Delta_I(\Delta_I - 2)}{r^2}\phi_I^\dagger\phi_I - F_I^\dagger F_I
\]
\[
- (\overline{\psi}_I\gamma^mD_m\psi_I) - (\overline{\psi}_I\sigma\psi_I) - \frac{i(\Delta_I - \frac{1}{2})}{r}(\overline{\psi}_I\psi_I) - \sqrt{2}\phi_I^\dagger(\lambda\psi_I) - \sqrt{2}(\overline{\psi}_I\overline{\lambda}\phi_I).
\]
\[
(30)
\]
(We use notation that in Euclidean signature the Hermitian conjugate of the auxiliary fields \(D\) and \(F_I\) are \(-D\) and \(-F_I^\dagger\), respectively.) In the large \(u\) limit, we can perform the path integral \(1\), and obtain the matrix model integral\(1\)[1][2][3]
\[
Z = \int_{\text{rank } G} d\sigma J^{(3d)}(\sigma)Z^{\text{vector}}(\sigma) \prod_I Z^{\text{chiral}}(\sigma). \quad (31)
\]
Integration variable \(\sigma\) in \(31\) is an element of the Cartan subalgebra of the gauge group \(G\). The Jacobian factor \(J^{(3d)}(\sigma)\) is
\[
J^{(3d)}(\sigma) = \prod_{\alpha \in \Delta} \pi\alpha(r\sigma). \quad (32)
\]
$Z_{\text{vector}}(\sigma)$ and $Z_{\Phi I}^{\text{chiral}}(\sigma)$ are 1-loop partition function of vector and chiral multiplets. They are given by

$$Z_{\text{vector}}(\sigma) = \prod_{\alpha \in \Delta} \frac{\sinh(\pi \alpha(r\sigma))}{\pi \alpha(r\sigma)},$$

$$Z_{\Phi I}^{\text{chiral}}(\sigma) = \prod_{\rho \in R_I} \prod_{k=1}^{\infty} \frac{(k + 1 - \Delta_I - i\rho(r\sigma))^k}{(k - 1 + \Delta_I + i\rho(r\sigma))^k}. \quad (33)$$

### 6 Superconformal index

Let us consider a 4d $\mathcal{N} = 1$ theory in $S^3 \times S^1$. The background is conformally flat, and the parameter $\tau$ must satisfy the Killing equation

$$D_\mu \tau = \gamma_\mu \kappa. \quad (34)$$

To relate 3d spinor $\tau(x^m)$ satisfying (26) and 4d spinor $\tau(x^\mu)$, we take the anzats

$$\tau(x^\mu) = f(x^4) \bar{\tau}(x^m). \quad (35)$$

From (26) the 4d spinor $\tau(x^\mu)$ satisfies

$$D_\mu \tau(x^\mu) = \frac{1}{2r} \gamma_\mu \gamma_4 \tau(x^m) \quad (36)$$

for $\mu = 1, 2, 3$. For $\tau$ to be a Killing spinor in 4d, this must hold for $\mu = 4$, too. This determines the function $f(x^4)$ up to normalization as

$$f(x^4) = e^{\frac{4}{r}}. \quad (37)$$

Corresponding to the Killing spinors $\bar{\tau}_1(x^m)$ and $\bar{\tau}_2(x^m)$ in 3d, we define two Killing spinors in 4d, which are denote by the same symbols $\bar{\tau}_1$ and $\bar{\tau}_2$. We adopt $\delta(\bar{\tau}_1(x^\mu))$ as $\mathcal{Q}$ in the same way as in 3d.

We want to compute a quantity in the form

$$I = \text{tr}[-1] \mathcal{O} q^D], \quad (38)$$

where $\mathcal{O}$ is an operator constructed from the Cartan generators of the superconformal and flavor symmetries. The most general form of $\mathcal{O}$ is

$$\mathcal{O} = y^{-\frac{3}{2}R-2J_L + R + 2J_L} x^{2J_R} \prod_i h_i^x \mathcal{F}_i. \quad (39)$$

This is equivalent to imposing the boundary condition

$$\Phi(x^m, x^4) = \mathcal{O} \Phi(x^m, x^4 + \beta r), \quad (40)$$

on an arbitrary field $\Phi$. For localization to be applicable the supercharge $\mathcal{Q}$ must commute with the operator $\mathcal{O}$. Equivalently, the Killing spinor $\bar{\tau}_1$ must
satisfy the boundary condition (40). This requires \( y = q \), and in this case \( \mathbf{3} \) becomes the index \( \mathbf{2} \).

The 4d supersymmetry transformation laws are obtained from (19) and (20) by using (23), (24), and (36). The transformation laws for a vector multiplet are

\[
\begin{align*}
\delta(\tau) A_\mu &= -i(\tau \gamma_\mu \lambda), \\
\delta(\tau) \bar{\lambda} &= -\frac{i}{2} \gamma^{\mu\nu} \tau F_{\mu\nu} + D\tau, \\
\delta(\tau) D &= -(\bar{\tau} \gamma^\mu D_\mu \lambda).
\end{align*}
\]

The chiral multiplet transformation laws are

\[
\begin{align*}
\delta(\tau) \phi^\dagger &= \sqrt{2} (\bar{\tau} \psi), \\
\delta(\tau) \phi &= 0, \\
\delta(\tau) \bar{\psi} &= \sqrt{2} \tau F^\dagger, \\
\delta(\tau) F^\dagger &= 0, \\
\delta(\tau) D &= -\sqrt{2} \gamma^{\mu \nu} D_\mu \phi - \frac{\sqrt{2} \Delta \lambda}{r} \phi^\dagger (\bar{\tau} \psi^4).
\end{align*}
\]

The 4d deformation Lagrangian density \( \mathcal{L}^{(4d)} \) is given by (29) with the 4d transformation laws (41) and (42).

\[
\begin{align*}
\mathcal{L}^{(4d)} &= \text{tr} \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} F_{\mu \nu} F_{\rho \sigma} - \frac{1}{2} D^2 - (\bar{\lambda} \gamma^\mu D_\mu \lambda) \right] \\
&+ \sum_I \left[ - F_I^{\dagger} F_I - \phi_I^\dagger D_\mu \phi_I + \phi_I^\dagger D \phi_I = \frac{\Delta_I^2 - 2 \Delta_I}{r^2} \phi_I^\dagger \phi_I - \frac{2(\Delta - 1)}{r} \phi_I^\dagger D_4 \phi_I \\
&- (\bar{\psi}_I \gamma^\mu D_\mu \psi_I) - \frac{\Delta_I}{r} (\bar{\psi}_I \gamma^4 \psi_I) - \sqrt{2} \phi_I^\dagger (\lambda \psi_I) - \sqrt{2} \bar{\psi}_I (\bar{\lambda} \phi_I). \right]
\end{align*}
\]

Note that this action contains only the anti-self-dual part of \( F_{\mu\nu} \), and we need to change the coefficient of the topological term \( \propto \text{tr}(F \wedge F) \) to localize the path integral to flat connections. This is possible because the index does not depend on the coefficient of this term as well as other coupling constants consistent with the symmetry of the system. This Lagrangian density is essentially the same as what is derived in [14]. The index can be computed exactly by performing the path integral (4) in the large \( u \) limit. The result is (15)

\[
I(t, x, h_i) = \int_{\text{rank} G} dA_4 J^{(4d)}(A_4) \text{Pexp} f(q^{tr} A_4, t, x, h_i). \tag{44}
\]

The \( A_4 \) integral is taken over the maximal torus of the gauge group \( G \). Pexp is the plethystic exponential

\[
\text{Pexp} f(g, t, x, h_i) = \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} f(g^m, t^m, x^m, h_i^m) \right). \tag{45}
\]
\(J^{(4d)}(A_4)\) is the Jacobian factor associated with the gauge fixing,
\[
J^{(4d)}(A_4) = \prod_{\alpha \in \Delta} \sin( \pi \beta \alpha (r A_4)) \over \beta. \tag{46}
\]
f\((g, t, x, h_i)\) is the letter index. The contribution of vector multiplets is
\[
f^{\text{vector}}(q^{ir A_4}, t, x, h_i) = \sum_{\alpha \in G} q^{i \alpha (r A_4)} \left( \sum_{l=0}^{\infty} \sum_{k=-l/2}^{l/2} t^l x^{2k} - \sum_{l=1}^{\infty} \sum_{k=-l/2}^{l/2} t^l x^{-2k} \right).
\tag{47}
\]

The contribution of a chiral multiplet \(\Phi_I\) belonging to a gauge representation \(R_I\) is
\[
f^{\text{chiral}}(q^{ir A_4}, t, x, h_i) = \sum_{\rho \in R_I} \sum_{l=0}^{\infty} \sum_{k=-l/2}^{l/2} q^{i \rho (r A_4)} t^l x^{2k} \prod_i h_i^{F_i(\Phi_I)} - q^{-i \rho (r A_4)} t^l x^{-2k} \prod_i h_i^{-F_i(\Phi_I)} \left(1 - tx(1 - tx^{-1}) \right).
\tag{48}
\]

7 Comparison of the deformation actions

In order to relate the \(S^3\) partition function and the index, let us compare the Lagrangian densities \(L^{(3d)}\) in (30) and \(L^{(4d)}\) in (43). They look similar, but not the same. The difference is partially absorbed by shifting the auxiliary \(D\)-field.
\[
D^{(4d)} = D^{(3d)} + \frac{i}{r} \sigma. \tag{49}
\]

Even after this shift the actions are still different. If we assume there are no non-trivial background Wilson lines around \(S^1\) and the covariant derivative \(D_4\) reduces to \(-i A_4 = -i \sigma\) in dimensional reduction, the difference is
\[
L^{(3d)} - L^{(4d)} = \frac{1}{2r} \left[ (\bar{\chi} \gamma^4 \chi) - \sum_I (\bar{\psi}_I \gamma^4 \psi_I) \right]. \tag{50}
\]

This difference can be removed by introducing a suitable Wilson line if the theory has the symmetry \(R_0\) with the charge assignments
\[
R_0(\lambda) = +1, \quad R_0(\psi_I) = -1, \quad R_0(A_\mu) = R_0(\phi) = 0. \tag{51}
\]
We weakly gauge this symmetry and introduce the gauge field $V_\mu$ for this symmetry. If we turn on the Wilson line

$$(V_4) = -\frac{i}{2r},$$

the difference is canceled by the terms arising from the 4d fermion kinetic terms in (43). This is equivalent to the insertion of the operator

$$O = q^{-\pi R_0}$$

in (38), and thus we expect that the partition function $Z$ is given by

$$Z = \lim_{q \to 1} \text{tr}[-(-1)^F q^{-\pi R_0} q^D].$$

If the 4d parent theory has non-vanishing superpotential, the symmetry $R_0$ is in general broken. However, the superpotential does not affect the index. The relevant part of the deformed action $S^{(4d)}_{\text{rel}}$ has the large symmetry rotating chiral multiplets independently. Let $F_I$ denote the generator rotating only a chiral multiplet $\Phi_I$ by charge 1. Correspondingly, we introduce chemical potentials $h_I$. The symmetry $R_0$ is related to $R$, the $R$-symmetry in the superconformal algebra, by

$$R_0 = R - \frac{2}{3} \sum_I \Delta_I F_I,$$

and we can express (54) as a special limit of the index,

$$Z = \lim_{q \to 1} I(t = q, x = 1, h_I = q^{\frac{2}{3} \Delta_I}).$$

It is easily checked that this relation indeed holds for (31) and (44) as follows. Because the radius of the maximal torus $T_{\text{rank} G}$ is inversely proportional to the $S^1$ period $\beta r$, it becomes $R_{\text{rank} G}$ in the limit $\beta \to 0$. We also see that the Jacobian factor (46) reduces to (32) when $\beta \to 0$. We obtain

$$\lim_{q \to 1} \int_{T_{\text{rank} G}} dA_4 J^{(4d)}(A_4) = \int_{R_{\text{rank} G}} d\sigma J^{(3d)}(A_4).$$

For the letter indices, we first express the plethystic exponential of (47) and (48) as the infinite products,

$$\text{Pexp}_{\text{vector}}(q^{irA_4}, t, x, h_I) = \prod_{\alpha \in G} \prod_{l=0}^{\infty} \prod_{k=-l/2}^{l/2} \frac{1 - q^{i\alpha(rA_4)t} x^{-2k}}{1 - q^{i\alpha(rA_4)t+2x^{2k}}},$$

$$\text{Pexp}_{\Phi_I} \Phi_I(q^{irA_4}, t, x, h_I) = \prod_{\rho \in R_I} \prod_{l=0}^{\infty} \prod_{k=-l/2}^{l/2} \frac{1 - q^{-i\rho(rA_4)t} x^{-\frac{2}{3} \Delta_I + 2x^{2k} h_I F_I}}{1 - q^{i\rho(rA_4)t+\frac{2}{3} \Delta_I - 2x^{2k} h_I F_I}}.$$
Once we obtain these infinite products, it is straightforward to confirm the following relations.

\[
\lim_{q \to 1} \text{Pexp}_f \mathbf{A}_4(q, q, 1, q^4) = Z_{\text{vector}}(A_4),
\]

\[
\lim_{q \to 1} \text{Pexp}_f \mathbf{A}_4(q, q, 1, q^4) = Z_{\text{chiral}}(A_4).
\] (59)

Combining (57) and (59), we obtain the relation (56).

Before ending this section, let us argue the anomaly associated with the inserted operator \( O \). We consider the quantity

\[
I(t, x, h) = \text{tr}[(-1)^F q^X q^D],
\] (60)

where we denote the inserted operator \( O \) by \( q^X \). As we mentioned in §1 the symmetry generated by \( X \) may be anomalous, and then the quantity (60) is not well defined. This can be regarded as inconsistency in the \( S^1 \) compactification. If \( X \) is anomalous, the rotation by \( q^X \) does not keep the effective action \( \Gamma \) invariant but changes it by

\[
\Gamma \to \Gamma' = \Gamma + \int_{S^1 \times S^1} \frac{\beta}{8\pi^2} \text{tr}_F (XF \wedge F)
\] (61)

where \( \text{tr}_F \) is the trace over Weyl fermions of positive chirality, which contribute to the anomaly.

This change of the effective action obstacles the compactification \( x^4 + \beta r \sim x^4 \). We can remove this obstruction by adding the following term to the tree-level action.

\[
S' = - \int_{S^1 \times S^1} \frac{x^4}{8\pi^2 r} \text{tr}_F (XF \wedge F)
\]

\[
= \int_{S^1} dx^4 \int_{S^3/\pi^2 r} \text{tr}_F \left[ X \left( A \wedge F - \frac{2i}{3} A \wedge A \wedge A \right) \right].
\] (62)

Due to the \( x^4 \) dependence of the \( \theta \) angle, the change of \( S' \) under the shift \( x^4 \to x^4 + \beta r \) cancels the anomalous change (61). With the inclusion of the term (62) in the action, we can consistently compactify the \( x^4 \) direction with the twist by \( O = q^X \). When \( X \) is anomalous, we define the quantity (60) by the path integral (2) with the action improved by (62).

Let us consider whether it is possible to extend the additional term \( S' \) in a supersymmetric way. (62) is a three-dimensional Chern-Simons term except that fields depend on the fourth coordinate \( x^4 \) along \( S^1 \). If all fields were \( x^4 \)-independent, we could actually construct the supersymmetric completion

\[
S'_{\text{SUSY}} = \int_{S^1 \times S^3} d^4x \frac{\sqrt{g}}{8\pi^2 r} \text{tr}_F \left[ X \left\{ \frac{i}{2} e^{mnp} \left( A_m \partial_n A_p - \frac{2i}{3} A_m A_n A_p \right) + (\lambda \phi) - D^{(4d)} A_4 + \frac{i}{r} A_4^2 \right\} \right].
\] (63)
For fields depending on $x^4$, however, this action is not supersymmetry invariant. We have non-vanishing supersymmetry transformation of the action

$$\delta S_{\text{SUSY}} = \int_{S^4 \times S^3} d^4x \frac{\sqrt{g}}{8\pi^2 r} \text{tr}_F [X(\lambda \gamma^\mu \gamma^\nu) \partial_4 A_{\mu}],$$

(64)

It is even worse that (63) is not even gauge invariant due to terms containing $A_4$. Unfortunately, we could not remedy these defects in (63), and we use the non-supersymmetric term (62) to turn on non-trivial Wilson lines for anomalous symmetries. In the large $u$ limit, the term (62) is irrelevant, and $I(t, x, h_I)$ is still given by the formula (44). However, the absence of the supersymmetry spoils the $u$-independence of the path integrals, and we can no longer regard $I(t, x, h_I)$ computed by the formula (44) as the index of the original theory. In the small $S^1$ limit $\beta \to 0$, the term (62) vanishes and the relation (56) still holds.

8 Generalization

Up to here we have been assuming that parameters of the 3d theory, $\mu_I$, $\zeta_A$, $k_a$, and $s$ all vanish. Let us consider how we can obtain partition function for a theory with these parameters turned on.

If the 3d theory has a flavor $U(1)$ symmetry, we can introduce a real mass proportional to the flavor charge for each chiral multiplet. We here focus only on the relevant part $S^{(3d)}_{\text{rel}}$, and we can introduce real mass $\mu_I$ for each chiral multiplet $\Phi_I$ by weakly gauging $F_I$ and turning on the scalar component $\sigma_I$ of the corresponding vector multiplet $(\sigma_I, A_{I,m}, \lambda_I, \overline{\lambda}_I, D_I)$. (If some of $F_I$ are anomalous, we need to introduce the term (62) in the definition of the index.) Note that we should turn on the auxiliary field $D_I$, too, to preserve the supersymmetry (27).

$$\langle \sigma_I \rangle = \mu_I, \quad \langle D_{I}^{(3d)} \rangle = -\frac{i}{r} \mu_I, \quad \langle A_{I,m} \rangle = \langle \lambda_I \rangle = \langle \overline{\lambda}_I \rangle = 0.$$  (65)

From the viewpoint of 4d theory, this is realized by turning on the Wilson line for the flavor symmetry $F_I$,

$$\langle A_{I,4} \rangle = \mu_I, \quad \langle D_{I}^{(4d)} \rangle = \langle A_{I,m} \rangle = \langle \lambda_I \rangle = \langle \overline{\lambda}_I \rangle = 0.$$  (66)

This is equivalent to the insertion of the operator

$$q^{-i r \sum_I \mu_I F_I}$$

(67)

in (54).

The next parameter we consider is a squashing parameter $s$. The partition function of a theory on squashed $S^3$ is investigated in [25], and it is found that the partition function is changed when both the isometries $SU(2)_L$ and $SU(2)_R$ are broken to $U(1)$. It is proposed recently in [24] that the partition function depending on the squashing parameter is reproduced from the index by turning
on $SU(2)_R$ Wilson line in the case of 4d $\mathcal{N} = 2$ theories. We consider the insertion of the operator
\[ q^{2sJ_R}, \]  
(68)
in a general 4d $\mathcal{N} = 1$ theory.

By inserting (67) and (68) into (54), we obtain
\[ Z = \lim_{q \to 1} \text{tr} \left[ (-1)^F q^{-\frac{1}{2} R_0} q^{-ir\mu_I I} q^{2sJ_R} q^D \right] \]
\[ = \lim_{q \to 1} I(t = q, x = q^s, h_I = q^{1 - ir\mu_I + \frac{1}{2} \Delta_I}). \]  
(69)

This is the relation (7) with vanishing FI parameters. Let us confirm that (69) reproduces the partition function of a 3d theory with non-vanishing real masses and squashing parameter. From the infinite product representation (68) we easily obtain
\[ \lim_{q \to 1} \text{Pexp} f^{\text{chiral}}(q^{irA_4}, q, q^s, q^{ir\mu_I + \frac{1}{2} \Delta_I}) \]
\[ = \prod_{\rho \in R} \prod_{m,n \geq 0} \frac{(m(1+s) + n(1-s) - \Delta_I + 2 - i\rho(rA_4) - ir\mu_I)}{m(1-s) + n(1+s) + \Delta_I + i\rho(rA_4) + ir\mu_I}, \]
\[ \lim_{q \to 1} \text{Pexp} f^{\text{vector}}(q^{irA_4}, q, q^s, q^{ir\mu_I + \frac{1}{2} \Delta_I}) \]
\[ = \prod_{\alpha \in G} \prod_{m,n \geq 0, (m,n) \neq (0,0)} \frac{(m(1-s) + n(1+s) + i\alpha(rA_4))}{\prod_{m,n \geq 0} (m(1+s) + n(1-s) + 2 + i\alpha(rA_4))}. \]  
(70)

These are consistent with known results. When $\mu_I = 0$, these agree with the results in [25] by the identification of parameters
\[ \frac{\ell}{\tilde{\ell}} = \frac{1 + s}{1 - s}. \]  
(71)

The $\mu_I$ dependence of (70) is consistent with the holomorphic dependence of the partition function on $\Delta_I + ir\mu_I$ [2].

One may think that this result is inconsistent with the result in [25] because the expression (69) for the partition function does not break the $SU(2)_L$ symmetry. Ref [22] shows that an $SU(2) \times U(1)$ invariant squashing does not change the partition function. The reason for these different results is as follows. The squashing considered in [25] is a left-invariant squashing which preserves $SU(2)_L$ isometry, and a Wilson line is turned on so that a half of left-invariant Killing spinors is preserved. There is in fact another essentially inequivalent possibility. We can realize a left-invariant squashing with right-invariant Killing spinors by taking a different graviphoton background from [25]. In the above we use right-invariant Killing spinors, and the $SU(2)_R$ Wilson line (68) preserves $SU(2)_L$ isometry. This is a different situation from [25].

In our case, the squashed metric is obtained from the 4d background metric
corresponding to the insertion \((68)\)

\[
ds^2 = r^2 \left[ (\mu_1)^2 + (\mu_2)^2 + (\mu_3 + isdx^4)^2 + (dx^4)^2 \right]
= r^2 \left[ (\mu_1)^2 + (\mu_2)^2 + \frac{1}{1-s^2}(\mu_3)^2 + (1-s^2) \left( dx^4 + \frac{is}{1-s^2}\mu_3 \right)^2 \right], \quad (72)
\]

where \(\mu^a\) are left-invariant one-forms used in \([25]\). We can read off the squashed metric of the base manifold,

\[
ds^2 = r^2 \left[ (\mu_1)^2 + (\mu_2)^2 + \frac{1}{1-s^2}(\mu_3)^2 \right]. \quad (73)
\]

It is interesting problem to confirm directly in 3d that the partition function for this squashed manifold with right-invariant Killing spinors agree with \((70)\).

As the last extension, let us introduce FI parameters. Let \((A_{A,m}, \sigma_A, \lambda_A, \bar{\lambda}_A, D_A^{(3d)})\) be \(U(1)\) vector multiplets for which we want to turn on the FI parameters. If the 3d original action contains the supersymmetry completion of FI terms

\[
S_{\text{FI}}^{(3d)} = - \sum_A \zeta_A^{(3d)} \int_{S^3} \sqrt{g} \left( D_A^{(3d)} - \frac{i}{r} \sigma_A \right) d^3x, \quad (74)
\]

the additional factor

\[
\exp \left( -4\pi i r^2 \sum_A \zeta_A^{(3d)} \sigma_A \right) \quad (75)
\]

should be included in the integrand in \((31)\). \(S_{\text{FI}}^{(3d)}\) in \((74)\) is obtained by dimensional reduction of 4d FI terms. Note that the 4d FI term must be accompanied by smeared Wilson line to preserve the supersymmetry,

\[
S_{\text{FI}}^{(4d)} = - \sum_A r_A^{(4d)} \int_{S^3 \times S^1} \sqrt{g} \left( D_A^{(4d)} - \frac{2i}{r} A_{A,4} \right) d^4x. \quad (76)
\]

Due to the coupling to the gauge fields, the 4d FI parameters must be quantized, and thus the index can depend on them. If we keep the relation \(\beta r \zeta_A^{(4d)} = \zeta_A^{(3d)}\) in the small radius limit, we reproduce \((74)\) from \((76)\) and the factor corresponding to \((75)\) arises in the index formula \((44)\). Taking account of this relation, we obtain the most general relation \((7)\).

Finally we comment on Chern-Simons terms. The supersymmetric completion of Chern-Simons term is

\[
S_{\text{CS}}^{(3d)} = \int_{S^3} \sqrt{g} \text{tr'} \left[ \frac{i}{2} \epsilon^{mnp} \left( A_m \partial_n A_p - \frac{2i}{3} A_m A_n A_p \right) + (\lambda \bar{\lambda}) - D\bar{\sigma} \right] d^3x, \quad (77)
\]

where \(\text{tr'}\) is a gauge invariant inner product containing Chern-Simons levels. If these terms exist in the original action in \((1)\), the extra factor

\[
e^{-2\pi^2 i \text{tr'} (r^2 \sigma^2)} \quad (78)
\]

arises in the integrand in \((31)\). Unfortunately, we cannot reproduce this contribution from the index due to the difficulty in constructing 4d action which gives Chern-Simons terms through dimensional reduction.
9 Conclusions

In this paper we investigated a relation between 3d and 4d actions used for computation of two exactly calculable quantities, the $S^3$ partition function and the 4d superconformal index.

When the 3d theory does not have Chern-Simons terms, the relevant part of the action, which affects the $S^3$ partition function, consists of $Q$-exact deformation terms and the supersymmetric completion of FI terms. In the case of round $S^3$, we showed that this relevant part of the 3d action is obtained by dimensional reduction from the corresponding terms in 4d action used for the computation of the 4d superconformal index. From this fact, we obtained a relation which gives the $S^3$ partition function as a small radius limit of the 4d superconformal index suitably generalized so that we can introduce chemical potentials to anomalous symmetries.

To obtain the most general relation (7), we used a connection between a squashing of $S^3$ and $SU(2)_R$ Wilson line. Although the squashing we considered in this paper, the left-invariant squashing with right-invariant Killing spinors, is different from squashings studied in [25], our result agree with the partition function for the $U(1) \times U(1)$ symmetric squashed $S^3$ derived in [25].

For 3d theory with Chern-Simons terms, we could not give a 4d action reproducing the $S^3$ partition function.

Acknowledgments

I would like to thank Daisuke Yokoyama and Shuichi Yokoyama for daily discussions. I would also like to thank Kazuo Hosomichi and Yu Nakayama for valuable comments.

References

[1] A. Kapustin, B. Willett and I. Yaakov, “Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter,” JHEP 1003, 089 (2010) [arXiv:0909.4559 [hep-th]].

[2] D. L. Jafferis, “The Exact Superconformal R-Symmetry Extremizes $Z$,” [arXiv:1012.3210 [hep-th]].

[3] N. Hama, K. Hosomichi and S. Lee, “Notes on SUSY Gauge Theories on Three-Sphere,” [arXiv:1012.3512 [hep-th]].

[4] A. Kapustin, B. Willett and I. Yaakov, “Nonperturbative Tests of Three-Dimensional Dualities,” JHEP 1010, 013 (2010) [arXiv:1003.5694 [hep-th]].

[5] D. Jafferis, X. Yin, “A Duality Appetizer,” [arXiv:1103.5700 [hep-th]].

[6] A. Kapustin, “Seiberg-like duality in three dimensions for orthogonal gauge groups,” [arXiv:1104.0466 [hep-th]].
[7] B. Willett and I. Yaakov, “N=2 Dualities and Z Extremization in Three Dimensions,” arXiv:1104.0487 [hep-th].

[8] N. Drukker, M. Marino and P. Putrov, “From weak to strong coupling in ABJM theory,” arXiv:1007.3837 [hep-th].

[9] C. P. Herzog, I. R. Klebanov, S. S. Pufu and T. Tesileanu, “Multi-Matrix Models and Tri-Sasaki Einstein Spaces,” Phys. Rev. D 83, 046001 (2011) arXiv:1011.5487 [hep-th].

[10] D. Martelli and J. Sparks, “The large N limit of quiver matrix models and Sasaki-Einstein manifolds,” arXiv:1102.5289 [hep-th].

[11] S. Cheon, H. Kim and N. Kim, “Calculating the partition function of N=2 Gauge theories on $S^3$ and AdS/CFT correspondence,” arXiv:1102.5565 [hep-th].

[12] D. L. Jafferis, I. R. Klebanov, S. S. Pufu, B. R. Safdi, “Towards the F-Theorem: N=2 Field Theories on the Three-Sphere,” arXiv:1103.1181 [hep-th].

[13] J. Kinney, J. M. Maldacena, S. Minwalla and S. Raju, “An Index for 4 dimensional super conformal theories,” Commun. Math. Phys. 275, 209 (2007) arXiv:hep-th/0510251.

[14] C. Romelsberger, “Counting chiral primaries in $N = 1$, $d=4$ superconformal field theories,” Nucl. Phys. B 747, 329 (2006) arXiv:hep-th/0510060.

[15] C. Romelsberger, “Calculating the Superconformal Index and Seiberg Duality,” arXiv:0707.3702 [hep-th].

[16] F. A. Dolan, H. Osborn, “Applications of the Superconformal Index for Protected Operators and q-Hypergeometric Identities to N=1 Dual Theories,” Nucl. Phys. B818, 137-178 (2009). arXiv:0801.4947 [hep-th].

[17] V. P. Spiridonov, G. S. Vartanov, “Superconformal indices for $N = 1$ theories with multiple duals,” Nucl. Phys. B824, 192-216 (2010). arXiv:0811.1909 [hep-th].

[18] V. P. Spiridonov, G. S. Vartanov, “Elliptic hypergeometry of supersymmetric dualities,” arXiv:0910.5944 [hep-th].

[19] Y. Nakayama, “Index for orbifold quiver gauge theories,” Phys. Lett. B 636, 132 (2006) arXiv:hep-th/0512280.

[20] Y. Nakayama, “Index for supergravity on AdS(5) x T***(1,1) and conifold gauge theory,” Nucl. Phys. B 755, 295 (2006) arXiv:hep-th/0602284.

[21] S. Benvenuti, B. Feng, A. Hanany and Y. H. He, “Counting BPS operators in gauge theories: Quivers, syzygies and plethystics,” JHEP 0711, 050 (2007) arXiv:hep-th/0608050.
[22] A. Gadde, L. Rastelli, S. S. Razamat and W. Yan, “On the Superconformal Index of N=1 IR Fixed Points: A Holographic Check,” arXiv:1011.5278 [hep-th].

[23] F. A. H. Dolan, V. P. Spiridonov, G. S. Vartanov, “From 4d superconformal indices to 3d partition functions,” arXiv:1104.1787 [hep-th].

[24] A. Gadde and W. Yan, “Reducing the 4d Index to the $S^3$ Partition Function,” arXiv:1104.2592 [hep-th].

[25] N. Hama, K. Hosomichi, S. Lee, “SUSY Gauge Theories on Squashed Three-Spheres,” arXiv:1102.4716 [hep-th].