NONCOMMUTATIVE RESOLUTION, F-BLOWUPS AND D-MODULES

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Abstract. We explain the isomorphism between the $G$-Hilbert scheme and the F-blowup from the noncommutative viewpoint after Van den Bergh. In doing this, we immediately and naturally arrive at the notion of $D$-modules. We also find, as a byproduct, a canonical way to construct a noncommutative resolution at least for a few classes of singularities in positive characteristic.

1. Introduction

The starting point of this work is the isomorphism between the $G$-Hilbert scheme and the F-blowup found in [24]. The $G$-Hilbert scheme, introduced by Ito and Nakamura [11], is associated to a smooth $G$-variety $M$ with $G$ a finite group, while the $e$-th F-blowup introduced by the second author [24] is associated to the $e$ times iteration of the Frobenius morphism $F : X \to X$ of a singular variety in positive characteristic. Both are the moduli spaces of certain 0-dimensional subschemes.

From now on, we work over an algebraically closed field $k$ of characteristic $p > 0$. Then under some condition, the mentioned isomorphism connects the $G$-Hilbert scheme, $\text{Hilb}^G(M)$, of a $G$-variety $M$ and the $e$-th F-blowup, $\text{FB}_e(X)$, of the quotient variety $X = M/G$.

$$\text{Hilb}^G(M) \cong \text{FB}_e(X)$$

Our motivation is the following:

Problem 1.1. Understand a mechanism behind this phenomenon!

The $G$-Hilbert scheme and the F-blowup fit into similar diagrams:

$$\begin{array}{ccc}
\text{Univ. fam.} & \longrightarrow & M \\
\downarrow & & \downarrow \\
\text{Hilb}^G(M) & \longrightarrow & M/G
\end{array}$$
In each diagram, the right vertical arrow is a finite and dominant morphism, the left is finite and flat, the horizontal ones are projective and birational. Moreover in each diagram, the bottom one is the universal birational flattening of the right. A difference between the two diagrams is that the vertical arrows in the first are $G$-covers, in particular, separable, while the ones in the second are purely inseparable. Bridgeland, King and Reid [4] proved that in the first diagram, under some condition, the Fourier-Mukai transform associated to the universal family gives the equivalence

\[
D(Coh(\text{Hilb}^G(M))) \cong D(Coh^G(M)).
\]

Here $\text{Coh}(-)$ ($\text{Coh}^G(-)$) denotes the category of coherent ($G$-)sheaves and $D(-)$ denotes the bounded derived category. It is natural to ask:

**Problem 1.2.** Does a similar result hold for the second diagram?

We will address Problems 1.1 and 1.2 in terms of the noncommutative resolution due to Van den Bergh [21, 22]. Now let us recall his observation. For simplicity, suppose $M = \mathbb{A}^d_k$ and $G \subset SL_d(k)$. Let $S$ and $R$ be the coordinate rings of $M$ and $X = M/G$. Then the endomorphism ring $A := \text{End}_R(S)$ is a noncommutative crepant resolution. Namely $A$ is regular in the sense that it has finite global dimension and satisfies the condition corresponding to the crepancy. Now the $G$-Hilbert scheme is identified with some moduli space $W$ of $A$-modules and a coherent $G$-sheaf on $M$ is identified with an $A$-module. Thus (1) translates into

\[
D(Coh(W)) \cong D(A\text{-mod})
\]

and the Galois group $G$ disappears from view. The equivalence in this form can fit to the situation of F-blowup.

Consider an affine scheme $X = \text{Spec } R$ over $k$. For $q = p^e$, the $e$-th Frobenius morphism of $X$ is identified with the morphism $X \to X_e := \text{Spec } R^q$ defined by the inclusion map $R^q \hookrightarrow R$. We suppose that it is a finite morphism. The relevant noncommutative ring is

\[
D_{R,e} := \text{End}_{R^q}(R).
\]

Here particularly interesting is that $D_{R,e}$ is a ring of differential operators on $R$ and $\bigcup_e D_{R,e}$ is the ring of all differential operators on $R$. The following is our answer to Problems 1.1 and 1.2.
Theorem 1.3. Let \( M = \mathbb{A}_k^d = \text{Spec } S, \ G \subset \text{GL}_d(k) \) a small finite subgroup and \( X := M/G = \text{Spec } R \). Then for sufficiently large \( e \), we have an equivalence of abelian categories
\[
\text{End}_R(S)-\text{mod} \cong D_{R,e}-\text{mod}.
\]
Hence \( D_{R,e} \) has global dimension \( d \). Moreover \( \text{Hilb}^G(M) \) and \( \text{FB}_e(X) \) are the moduli spaces corresponds to each other via this equivalence, hence clearly isomorphic to each other.

The point is that \( R \) as well as \( S \) and \( S^q \) consists of the so-called modules of covariants as an \( R^q \)-module. This was shown by Smith and Van den Bergh [20] except for the fact that every module of covariants appears in \( R \) (if \( q \) is sufficiently large). Then the last fact follows from Bryant’s theorem [5] in the representation theory (for details, see \( \S 2.2 \)).

Now the equivalence (2) is directly translated to the F-blowup situation. In view of these results, we may say that \( D_{R,e} \) is a noncommutative counterpart of F-blowup.

Remark 1.4. It is natural that differential operators appear. For, in the Galois theory for purely inseparable extensions, derivations play a role of automorphisms in the Galois theory of normal extensions (see [12]).

We will also see that the F-blowup of an F-pure variety can be expressed as the moduli space of \( D_{R,e} \)-modules explicitly.

Since \( D_{R,e} \) is defined for arbitrary \( k \)-algebra \( R \), we can ask:

Problem 1.5. When is \( D_{R,e} \) a noncommutative (crepant) resolution?

We see that for at least three different classes of singularities, the answer is affirmative:

Theorem 1.6. Suppose that \( R \) is a complete local ring and has one of the following singularity type:

1. 1-dimensional analytically irreducible singularity
2. tame quotient singularity
3. simple singularity of type \( A_1 \) (odd characteristic)

Then for sufficiently large \( e \), \( D_{R,e} \) is a noncommutative resolution, that is, of finite global dimension. (However it is not crepant in general. See Section 6.)

The paper is organized as follows. Sections 2 and 3 are devoted to the case of tame quotient singularities, which is the core of the paper. Here we prove the mentioned equivalence \( \text{End}_R(S)-\text{mod} \cong D_{R,e}-\text{mod} \) and derive the correspondence of moduli spaces. In Section 4, we express an F-blowup of an F-pure variety as the moduli space of \( D_{R,e} \)-modules explicitly. In Sections 5 and 6, we treat the 1-dimensional analytically irreducible singularity and the simple singularity of type \( A_1 \) respectively.
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Convention and notation. Throughout the paper, we work over an algebraically closed field $k$ of characteristic $p > 0$. We always denote by $q$ the $e$-th power of $p$ for some $e \in \mathbb{Z}_{>0}$. For a commutative $k$-algebra $R$, we write $R^q := \{f^q \mid f \in R\} \subset R$ and $D_{R,e} := \text{End}_R(R^q)$. For the affine scheme $X = \text{Spec} R$, we write $X_e := \text{Spec} R^q$. We will use the symbol, gldim, to denote the global dimension respectively. For a ring $A$, we denote by $A\text{-mod}$ the category of left $A$-modules.

2. Tame quotient singularities

2.1. Functors between module categories. In this subsection, we collect a few results on functors between module categories, which will be needed below. These are probably well-known.

Fix a commutative ring $R$. In the following, $L, M, N$ denote $R$-modules. For convenience, we regard $L$ as a $(\text{End}_R(L), R)$-bimodule, and the same for $M, N$. Then for instance, $\text{Hom}_R(L, M)$ is a $(\text{End}_R(M), \text{End}_R(L))$-bimodule.

Proposition 2.1. Suppose that $L$ is a direct summand of $M^{\oplus r}$ for some $r \in \mathbb{Z}_{>0}$. Then the natural morphism of $(\text{End}_R(M), \text{End}_R(L))$-bimodules

$$(3) \quad \text{Hom}_R(M, N) \otimes_{\text{End}_R(M)} \text{Hom}_R(L, M) \to \text{Hom}_R(L, N), \quad f \otimes g \mapsto g \circ f$$

is an isomorphism. Hence the composition of the functors

$$\text{Hom}_R(L, M) \otimes_{\text{End}_R(L)} - : \text{End}_R(L)\text{-mod} \to \text{End}_R(M)\text{-mod}$$

and

$$\text{Hom}_R(M, N) \otimes_{\text{End}_R(M)} - : \text{End}_R(M)\text{-mod} \to \text{End}_R(N)\text{-mod}$$

is canonically isomorphic to

$$\text{Hom}_R(L, N) \otimes_{\text{End}_R(L)} - : \text{End}_R(L)\text{-mod} \to \text{End}_R(N)\text{-mod}.$$ 

Proof. By assumption, $\text{Hom}_R(L, M)$ (resp. $\text{Hom}_R(L, N)$) is a direct summand of $\text{End}_R(M)^{\oplus r}$ (resp. $\text{Hom}_R(M, N)^{\oplus r}$). So (3) is a direct summand of the isomorphism

$$\text{Hom}_R(M, N) \otimes_{\text{End}_R(M)} \text{End}_R(M)^{\oplus r} \to \text{Hom}_R(M, N)^{\oplus r}.$$ 

It follows that (3) is also an isomorphism. \qed

The following is a direct consequence of the proposition.
Corollary 2.2. [cf. [19, Corollary 2.5.8]] Suppose that for some positive integers \( r, s, M \) is a direct summand of \( N^{\oplus s} \) and \( N \) is a direct summand of \( M^{\oplus r} \). Then the functors
\[
\text{Hom}_R(M, N) \otimes_{\text{End}_R(M)} - : \text{End}_R(M)\text{-mod} \to \text{End}_R(N)\text{-mod}
\]
\[
\text{Hom}_R(N, M) \otimes_{\text{End}_R(N)} - : \text{End}_R(N)\text{-mod} \to \text{End}_R(M)\text{-mod}
\]
are inverses to each other. In particular \( \text{End}_R(M) \) and \( \text{End}_R(N) \) are Morita equivalent.

Proposition 2.3. [cf. [1 §19, Ex. 4]] Suppose that \( M \) is a direct summand of \( L \), and \( e \in \text{End}_R(L) \) denotes the projection \( L \to M \subset L \). Then for any left \( \text{End}_R(L) \)-module \( N \), \( eN \) is an \( \text{End}_R(M) \)-module and isomorphic to \( \text{Hom}_R(L, M) \otimes_{\text{End}_R(L)} N \).

Proof. Since the functors \( N \mapsto eN \) and \( N \mapsto \text{Hom}_R(L, M) \otimes_{\text{End}_R(L)} N \) are exact, it suffices to show the proposition in the case \( N = \text{End}_R(L) \), which we can see as follows. Write \( L = M \oplus M' \). Then
\[
\text{End}_R(L) = \begin{pmatrix}
\text{End}_R(M) & \text{Hom}_R(M', M) \\
\text{Hom}_R(M, M') & \text{End}_R(M')
\end{pmatrix}
\]
and \( e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).

Hence
\[
e \text{End}_R(L) = \begin{pmatrix}
\text{End}_R(M) & \text{Hom}_R(M', M) \\
0 & 0
\end{pmatrix} = \text{Hom}_R(L, M).
\]

\[
\square
\]

2.2. Modules of covariants and Frobenius maps. Let \( V \) be a \( d \)-dimensional \( k \)-vector space and \( G \subset GL(V) \) a finite subgroup. We assume the tameness condition that \( p \) does not divide \( |G| \). Let \( S \) be the symmetric algebra \( S^*V \) with the natural \( G \)-action. Set \( R := S^G \), the invariant ring.

For a finite dimensional \( G \)-representation \( U, R(U) := (S \otimes_k U)^G \) is a finitely generated \( R \)-module, called a module of covariants (over \( R \)). Let \( U_1, \ldots, U_l \) be the complete set of irreducible representations. Then every module of covariants is the direct sum of copies of \( R(U_i) \)'s. We say that \( R(U) \) is full if it contains every \( R(U_i) \) as a direct summand. Since \( S \cong (k[G] \otimes_k S)^G \), \( S \) is a full module of covariants. Similarly \( S^q \) is a full module of covariants over \( R^q \).

Lemma 2.4. Let \( m := S^{>0}V \subset S \) be the homogeneous maximal ideal and \( m^{[q]} \) its \( q \)-th Frobenius power, that is, the ideal of \( S \) generated by \( f^q, f \in m \).

1. For sufficiently large \( q \), the quotient \( G \)-representation \( S/m^{[q]} \) of \( S \) contains all irreducible \( G \)-representations as direct summands.

2. In addition if \( G \) is abelian, then the preceding assertion holds for \( q \geq |G| \).
Proof. (1) It follows from Bryant’s theorem [5] that for large $l$, the set of polynomials of degree at most $l$, $S^{\leq l}V \subset S$, contains all irreducible representations. Then if $q \gg l$, since $m[q] \subset S^{>l}V$, the natural map $S^{\leq l}V \to S/m[q]$ is injective, and the assertion follows.

(2) There is a decomposition of $V$ into 1-dimensional representations,

$$V = V_1 \oplus \cdots \oplus V_d.$$ 

Again from Bryant’s theorem, every irreducible representation is of the form $V_1^{\otimes n_1} \otimes \cdots \otimes V_d^{\otimes n_d}$ for some $n = (n_1, \ldots, n_d)$, $0 \leq n_i < \sharp G$. Now it is easy to see the assertion.

\[\square\]

**Proposition 2.5.** Suppose that $q$ is large enough as in the preceding lemma. (In particular, if $G$ is abelian, $q \geq \sharp G$ is enough.) Then $R$ is a full module of covariants over $R^q$.

*Proof.* Note that the proposition is a direct consequence of [20, Proposition 3.2.1] and the preceding lemma. Indeed since $S$ is isomorphic to $S/m[q] \otimes_k S^q$, $R$ is isomorphic to $(S/m[q] \otimes_k S^q)^G$. Therefore the proposition follows from the lemma.

\[\square\]

**Remark 2.6.** If in the non-abelian case, we had an effective estimation on how large $q$ is enough in Lemma 2.4, then we would have one in Proposition 2.5 too. In characteristic zero, the assertion of Lemma 2.4 is valid under the condition $q \geq \sharp G$ even in the non-abelian case (see [18, Prop. II.1.3 and Cor. II.3.4]).

**Corollary 2.7.** If $q$ is large enough as above, the functors

$$\Phi := \text{Hom}_{R^q}(S^q, R) \otimes_{\text{End}_{R^q}(S^q)} - : \text{End}_{R^q}(S^q)\text{-mod} \to D_{R,e}\text{-mod}$$

and

$$\text{Hom}_{R^q}(R, S^q) \otimes_{D_{R,e}} - : D_{R,e}\text{-mod} \to \text{End}_{R^q}(S^q)\text{-mod}$$

are equivalences which are inverses to each other.

*Proof.* This follows from Corollary 2.2 and Proposition 2.5.

\[\square\]

**Corollary 2.8.** If $q$ is large enough as above, then $\text{gldim } D_{R,e} = d$. Furthermore $D_{R,e}$ is a Cohen-Macaulay $R^q$-module. The same is true for the completion of $R$ with respect to the maximal ideal $m \cap R$.

*Proof.* We may and will suppose that $d \geq 2$ and that $G$ is small, that is, has no reflection. Then $\text{End}_{R^q}(S^q)$ is isomorphic to the skew group ring $S^q[G]$ (see [2]). It is well-known that $S^q[G]$ has global dimension $d$. Being Morita equivalent to $S^q[G]$, $D_{R,e}$ also has global dimension $d$. 

\[\square\]
We easily see that $S^q[G]$ is a Cohen-Macaulay $R^q$-module. Since
\[
S^q[G] \cong \text{End}_{R^q}(S^q) \cong \bigoplus_{i,j} \text{Hom}_{R^q}(R^q(U_i), R^q(U_j))^{a_{ij}}, \quad a_{ij} > 0,
\]
\[
D_{R,e} \cong \bigoplus_{i,j} \text{Hom}_{R^q}(R^q(U_i), R^q(U_j))^{b_{ij}}, \quad b_{ij} > 0,
\]
$D_{R,e}$ is also Cohen-Macaulay. The last assertion is obvious. \(\square\)

**Remark 2.9.** The Cohen-Macaulayness is the condition corresponding the crep-ancy in Van den Bergh’s sense (see [21, Lemma 4.2]). Note however that he defined the noncommutative crepant resolution only for Gorenstein singulari-
ties.

## 3. Moduli spaces of stable objects

We keep the notation of the preceding section. Suppose that $q$ is sufficiently large.

### 3.1. Stability.

For $i = 1, \ldots, l$, define functors
\[
\Psi_{1,i} := \text{Hom}_{R^q}(S^q, R^q(U_i)) \otimes_{\text{End}_{R^q}(S^q)} - : \text{End}_{R^q}(S^q)-\text{mod} \to \text{End}_{R^q}(R^q(U_i))-\text{mod}
\]
\[
\Psi_{2,i} : \text{Hom}_{R^q}(R, R^q(U_i)) \otimes_{D_{R,e}} - : D_{R,e}-\text{mod} \to \text{End}_{R^q}(R^q(U_i))-\text{mod}.
\]

From Proposition [2.1], for each $i$, $\Psi_{1,i}$ and $\Psi_{2,i}$ are compatible with the equivalence $\Phi$.

**Definition 3.1.** For a (closed) point $x$ of $X_e := \text{Spec } R^q$ and a left $\text{End}_{R^q}(S^q) \otimes_{R^q} k(x)$-module $V$, we define the dimension vector $\text{dim } V := (\dim_k \Psi_{1,i}(V))_i \in \mathbb{Z}^l$.

Let $\lambda \in \mathbb{Z}^l$. We say that $V$ is $\lambda$-(semi)stable if $(\lambda, \text{dim } V) = 0$ and for any proper submodule $W$ of $V$, we have $(\lambda, \text{dim } W) > 0$ (\geq 0). Here $(\cdot, \cdot)$ is the standard inner product. We similarly define the dimension vector and the (semi)stability for $D_{R,e} \otimes_{R^q} k(x)$-modules with $\Psi_{2,i}$ instead of $\Psi_{1,i}$.

Given $\alpha \in \mathbb{Z}^l_{\geq 0}$, we say that $\lambda$ is generic with respect to dimension vector $\alpha$ if every $\lambda$-semistable $V$ with $\text{dim } V = \alpha$ is $\lambda$-stable.

Our stability corresponds to Craw-Ishii’s one [6] as follows.

**Definition 3.2.** A $G$-constellation on $\text{Spec } S^q(\cong A_k^d)$ is an $S^q[G]$-module $M$ which is isomorphic to $k[G]$ as a $k[G]$-module. A $G$-cluster is a $G$-constellation which is a quotient of $S^q$. Let $R(G) := \bigoplus_i \mathbb{Z}[U_i]$ be the representation ring and $\theta : R(G) \to \mathbb{Z}$ a map of abelian groups such that $\theta(k[G]) = 0$. A $G$-constellation $M$ is $\theta$-stable (resp. $\theta$-semistable) if for every proper $S[G]$-submodule $L \subset M$, $\theta(L) > 0$ (resp. $\geq 0$).
For such $\theta$, put $\lambda_\theta := (\theta(U^*_i), \ldots, \theta(U^*_i))$. Here $U^*_i$ is the dual representation of $U_i$. A $G$-constellation $M$ naturally becomes an $\text{End}_{R^e}(S^q) \otimes_{R^e} k(x)$-module for some and unique $x \in X_e$. From the following lemma, $\theta(M) = (\lambda_\theta, \text{dim } M)$. Hence the $\theta$-(semi)stability of the $G$-constellation in the sense of Craw-Ishii coincides with the $\lambda_\theta$-(semi)stability of the $\text{End}_{R^e}(S^q) \otimes_{R^e} k(x)$-module.

**Lemma 3.3.** Let $N$ be an $\text{End}_{R^e}(S^q)$-module and $N = \bigoplus_{i=1}^l N_{U_i}$ the isotypic decomposition of $N$ as a $G$-representation. (That is, for each $i$, $N_{U_i}$ is a direct sum of copies of $U_i$.) Then for each $i$, we have a natural isomorphism

$$\text{Hom}_{R^e}(S^q, R^i(U_i)) \otimes_{\text{End}_{R^e}(S^q)} N \cong (N_{U_i} \otimes U_i)^G.$$  

In particular, if $N_{U_i} = (U^*_i)^{\oplus r}$ ($r < \infty$), then $\text{dim} \Psi_{1,i}(N) = r$.

**Proof.** We first see

$$\text{Hom}_{R^e}(S^q, R^i(U_i)) = \text{Hom}_{R^e}(S^q, (S^q \otimes U_i)^G)$$

$$\cong (\text{Hom}_{R^e}(S^q, S^q) \otimes U_i)^G$$

$$\cong (\text{End}_{R^e}(S^q) U_i \otimes U_i)^G.$$  

Thus the first assertion holds when $N$ is a free module. We can prove the general case by using a free presentation of $N$.

The first assertion implies the second. \hfill \square

Our stability also corresponds to Van den Bergh’s one [21] as follows. Write $S^q = \bigoplus_i R^i(U_i)^{\oplus r_i}$ and $e_i : S^q \to R^i(U_i)^{\oplus r_i} \subset R$ the projections. Then the $e_i \in \text{End}_{R^e}(S^q)$ are pairwise orthogonal idempotents with $\sum_i e_i = 1$. From Proposition 2.3 for an $\text{End}_{R^e}(S^q)$-module $N$, we have

$$\Psi_{1,i}(N)^{\oplus r_i} \cong \text{Hom}_{R^e}(S^q, R^i(U_i)^{\oplus r_i}) \otimes_{\text{End}_{R^e}(S^q)} N \cong e_i N.$$  

Hence if write $r = (r_1, \ldots, r_l)$, the dimension vector $\text{dim}^1 N$ of $N$ with respect to the $e_i$ in the sense of Van den Bergh is equal to $r \cdot \text{dim } N$ (componentwise multiplication). Consequently, for $\lambda \in \mathbb{Z}^l$, putting $\lambda' := (\lambda_1/r_1, \ldots, \lambda_l/r_l)$, we have $(\lambda', \text{dim}^1 N) = (\lambda, \text{dim } N)$. It follows that our stability with respect to $\lambda$ corresponds to Van den Bergh’s one with respect to the $e_i$ and $\lambda'$. Similarly for $D_{R,e}$ in place of $\text{End}_{R^e}(S^q)$.

**Remark 3.4.** The stabilities of Craw-Ishii and Van den Bergh are both derived from King’s [13] and hence must a priori correspond to each other. We just described their stabilities with functors and confirmed their compatibility with the equivalence of module categories.

### 3.2. Moduli space.

Choose $\alpha \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathbb{Z}^l$ which is generic with respect to $\alpha$. Let $A$ be either $\text{End}_{R^e}(S^q)$ or $D_{R,e}$. From [21], there exists the fine moduli space $W$ of $\lambda$-stable $A$-modules with dimension vector $\alpha$. This is a projective $X_e$-scheme, say with the structure morphism $\pi : W \to X_e$. Each
point \( x \in W \) represents an isomorphism class of \( \lambda \)-stable \( A \otimes_{R^\ell} k(\pi(x)) \)-modules with dimension vector \( \alpha \).

When \( A = \text{End}_{R^\ell}(S^q) \cong S^q[G] \) and when \( \alpha = \dim k[G] \) and \( \lambda = \lambda_\theta \) as above, then \( W \) is the moduli space of \( \theta \)-stable \( G \)-constellations. If \( \theta(U_i) > 0 \) for all nontrivial representations \( U_i \), then \( W \) is the moduli space of the \( G \)-clusters (see [6, page 266] and also [10]), that is, the \( G \)-Hilbert scheme.

If we replace \( \text{End}_{R^\ell}(S^q) \) with \( D_{R,e} \) and keep \( \alpha \) and \( \lambda \) unchanged, then the resulting moduli space \( W' \) is canonically isomorphic to \( W \). For \( W \) and \( W' \) are the moduli spaces of stable objects of two equivalent abelian categories respectively with respect to stability conditions corresponding to each other. Thus the canonical isomorphism \( W \to W' \) is nothing but the restriction of the equivalence \( \Phi \) to objects belonging to \( W \).

### 3.3. The maps from the \( G \)-Hilbert scheme to F-blowups.

From Proposition [21] the equivalence \( \Phi : \text{End}_{R^\ell}(S^q)\text{-mod} \to D_{R,e}\text{-mod} \) factors into two functors

\[
\Phi_1 := \text{Hom}_{R^\ell}(S^q, S) \otimes_{\text{End}_{R^\ell}(S^q)} - : \text{End}_{R^\ell}(S^q)\text{-mod} \to \text{End}_{R^\ell}(S)\text{-mod}
\]

\[
\Phi_2 := \text{Hom}_{R^\ell}(S, R) \otimes_{\text{End}_{R^\ell}(S)} - : \text{End}_{R^\ell}(S)\text{-mod} \to D_{R,e}\text{-mod}.
\]

For a left \( \text{End}_{R^\ell}(S^q)\)-module \( M \), we have isomorphisms of \( S \)-modules,

\[
\Phi_1(M) \cong (\text{End}_{R^\ell}(S^q) \otimes_{S^q} S) \otimes_{\text{End}_{R^\ell}(S^q)} M \cong S \otimes_{S^q} M.
\]

Thus if we forget the \( \text{End}_{R^\ell}(S) \)-module structure and remember only the \( S \)-module structure, then \( \Phi_1 \) is just the pull-back by the Frobenius morphism \( \text{Spec } S \to \text{Spec } S^q \), which is flat of rank \( q^d \) since \( \text{Spec } S \) is smooth.

On the other hand, for a left \( \text{End}_{R^\ell}(S) \)-module \( N \), the group \( G \), regarded as a subset of \( \text{End}_{R^\ell}(S) \), acts on \( N \) and we have a natural isomorphism of \( R \)-modules

\[
\Phi_2(N) \cong N^G.
\]

Indeed the isomorphism is obvious if \( N \) is free. In the general case, we can show this, considering a free presentation of \( N \). In summary, as \( R \)-modules, \( \Phi(M) \cong (S \otimes_{S^q} M)^G \).

If \( \theta : R(G) \to \mathbb{Z} \) is such that \( \theta(U_i) > 0 \) for every nontrivial irreducible representation \( U_i \), then as mentioned above, the corresponding moduli space \( W \) of \( G \)-constellation is the \( G \)-Hilbert scheme. Now the isomorphism \( \Phi : W \to W' \) restricted to the irreducible component of \( W \) dominating \( X_e \) coincides with the isomorphism of the \( G \)-Hilbert scheme to the F-blowup constructed in [21].

In particular, the \( e \)-th F-blowup of \( \text{Spec } R \) is an irreducible component of the moduli space \( W \) of stable \( D_{R,e} \)-modules for the stability \( \lambda_\theta \).

Summarizing, we have:

**Theorem 3.5.** The \( e \)-th F-blowup of \( X = \text{Spec } R \) is (an irreducible component of) the moduli space of certain \( D_{R,e} \)-modules which corresponds to the \( G \)-Hilbert scheme.
scheme via the equivalence $\Phi$. Hence the isomorphism between the $G$-Hilbert scheme and the $F$-blowup is just the restriction of the equivalence.

3.4. Fourier-Mukai transform. Applying [21, Theorem 6.3.1] to our situation, we obtain:

**Corollary 3.6.** (We still suppose that $e$ is sufficiently large.) Let $Y \to X_e$ be the $e$-th $F$-blowup of $X$. Suppose that $\dim Y \times_X Y \leq d+1$ and that $G \subset SL_d(k)$. Then $Y$ is a crepant resolution and we have the equivalence

$$D(\text{Coh}(Y)) \cong D(D_{R,e}\text{-mod})$$

defined as the Fourier-Mukai transform associated to the universal family of $D_{R,e}$-modules over $Y$.

4. F-blowups of F-pure singularities

Let $R$ be a commutative finitely generated domain over $k$ of dimension $d$. Suppose that $R$ is $F$-pure, that is, the inclusion map $R^p \hookrightarrow R$ splits (as an $R^p$-homomorphism). Then for any $q = p^f$, $R^q \hookrightarrow R$ splits. So we write $R = R^q \oplus M$ with $M \subset R$ an $R^q$-submodule. Let $e_1, e_2 \in \text{End}_{R^q}(R)$ be the projections $e_1 : R \to R^q \subset R$ and $e_2 : R \to M \subset R$ respectively. Then we can consider dimension vectors and the stability with respect to $e_1$ and $e_2$ in Van den Bergh’s sense.

**Proposition 4.1.** Put $\alpha := (1, q^d - 1)$ and $\lambda := (1 - q^d, 1)$. Then the $e$-th $F$-blowup of $X$ is canonically isomorphic to the unique irreducible component dominating $X_e$ of the moduli scheme $W$ of $\lambda$-stable $D_{R,e}$-modules with dimension vector $\alpha$.

**Proof.** First note that $\lambda$ is generic with respect to $\alpha$, since for any $0 < \alpha' < \alpha$, $(\alpha', \lambda) \neq 0$. The proof here is a modification of Craw-Ishii’s argument [6, page 266]. If $x \in X_e$ is a smooth point, then $D_{R,e} \otimes_{R^q} k(x) \cong M_{q^d \times q^d}(k)$ and there exists one and only one $D_{R,e} \otimes k(x)$-module of $k$-dimension $q^d$ modulo isomorphisms. A canonical representative of it is $R \otimes_{R^q} k(x)$ with the canonical $D_{R,e} \otimes k(x)$-module structure. It has dimension vector $(1, q^d - 1)$ with respect to the idempotents $e_1, e_2$. It follows that $W$ has a unique irreducible component dominating $X_e$.

If $x \in X_e$ is an arbitrary point, then any $D_{R,e} \otimes k(x)$-module $M$ of dimension vector $\alpha$ which is a quotient of $R \otimes_{R^q} k(x)$ is $\lambda$-stable. Indeed if $M' \subset M$ is a submodule with $e_1 M' \neq 0$, then we must have $M' = M$. Hence for every nonzero proper submodule $L \subset M$, we have $\dim L = (0, r)$, $r > 0$ and $(\lambda, \dim L) = r > 0$. Thus $M$ is $\lambda$-stable. It follows that the $e$-th $F$-blowup is a subscheme of $W$, which completes the proof. □

**Remark 4.2.** The $F$-blowup of $F$-pure singularities has another nice property. Namely the sequence of $F$-blowups satisfies the monotonicity [23].
5. 1-DIMENSIONAL ANALYTICALLY IRREDUCIBLE SINGULARITIES

Let $R$ be a 1-dimensional complete integral domain over $k$ with the normalization $S := k[[x]]$.

**Lemma 5.1.** For any $q$, we have ring isomorphisms

$\text{End}_{R^q}(S^q) \cong \text{End}_{S^q}(S^q) \cong S^q$.

**Proof.** The second isomorphism is trivial. For the first one, we first see

$\text{End}_{R^q}(S^q) \supset \text{End}_{S^q}(S^q)$.

Take $\phi \in \text{End}_{R^q}(S^q)$. For any $s \in S^q$, there exists $n \in q\mathbb{Z}_{\geq 0}$ with $x^n, x^n s \in R^q$. Then

$x^n \phi(s) = \phi(x^n s) = x^n s \phi(1)$.

Hence $\phi(s) = s \phi(1)$ and $\phi \in \text{End}_{S^q}(S^q)$. Thus we have proved the lemma. \qed

**Theorem 5.2.** Suppose that $x^q \in R$. (This holds for sufficiently large $q$.) Then we have the ring isomorphism

$D_{R,e} \cong M_{q \times q}(S^q)$.

**Proof.** By assumption, $S^q \subset R$. Being a torsion-free $S^q$-module of rank $q$, $R$ is isomorphic to $(S^q)^{\oplus q}$ as an $S^q$-module and as an $R^q$-module too. It follows that

$D_{R,e} \cong \text{End}_{R^q}((S^q)^{\oplus q}) \cong M_{q \times q}(\text{End}_{R^q}(S^q)) \cong M_{q \times q}(S^q)$.

\qed

**Corollary 5.3.** For sufficiently large $e$, $\text{glidim } D_{R,e} = 1$. Furthermore $D_{R,e}$ is a Cohen-Macaulay $R^q$-module.

**Proof.** It is well-known that $M_{q \times q}(S^q)$ is Morita equivalent to $S$. Hence $D_{R,e}$ has global dimension 1.

It is clear that $D_{R,e} \cong M_{q \times q}(S^q)$ is a Cohen-Macaulay $R^q$-module. \qed

**Problem 5.4.** For large $e$, the $e$-th $F$-blowup of a curve separates all the analytic branches. [24]. Is there a categorical counterpart of separation of branches? And can we generalize the result to arbitrary 1-dimensional singularity?

See the next section for the node $R = k[[x, y]]/(x^2 + y^2)$ in odd characteristic.

6. THE SIMPLE SINGULARITY OF TYPE $A_1$

A hypersurface singularity $R = k[[x_0, \ldots, x_d]]/(f)$, which is necessarily a reduced ring, is called simple, if it is of finite Cohen-Macaulay type, that is, it has only finitely many indecomposable MCM (maximal Cohen-Macaulay) modules up to isomorphisms. See [9] for the classification of simple singularities in positive characteristic.
For such \( R \), an MCM \( R \)-module \( M \) is called a \textit{representation generator} if \( M \) contains every indecomposable MCM module as a direct summand. The following theorem of Leuschke [17] provides a useful sufficient condition for an endomorphism ring to have finite global dimension.

**Theorem 6.1.** For \( R \) as above, if \( M \) is a representation generator, then
\[
\text{gldim } \text{End}_R(M) \leq \max\{2, d\}.
\]
Moreover if \( d \geq 2 \), then the equality holds.

Now suppose that \( k \) has odd characteristic and \( R \) is the \( d \)-dimensional simple singularity of type \( A_1 \):
\[
R = k[[x_0, \ldots, x_d]]/(x_0^2 + x_1^2 + \cdots + x_d^2).
\]
Then we see as follows that for \( e > 1 \),
\[
\text{gldim } D_{R,e} \leq \max\{2, d\}
\]
with the equality in the case \( d \geq 2 \).

**Proof.** Since the monomial \( x_0^{p-1}x_1^{p-1} \) with nonzero coefficient \( \binom{p-1}{(p-1)/2} \) appears in \( (x_0^2 + x_1^2 + \cdots + x_d^2)^{p-1} \), we have \( (x_0^2 + x_1^2 + \cdots + x_d^2)^{p-1} \notin (x_0^p, \ldots, x_d^p) \). From Fedder’s criterion [8, Proposition 2.1], \( R \) is F-pure, that is, for any \( q \), \( R^q \hookrightarrow R \) splits as an \( R^q \)-module homomorphism.

If \( d \) is even, then \( R \) has only two indecomposable MCM modules, one of which is the trivial one. To see this, it is enough to check the case where \( d = 2 \), thanks to the Knörrer periodicity [15]. In this case, it was proved by Auslander [3]. The same is true for \( R^q \). From Kunz [16], \( R \) is not a free \( R^q \)-module. So \( R \) must be a representation generator as an \( R^q \)-module and the assertion follows.

If \( d = 1 \), \( R \) has three indecomposable MCM modules, \( R \), \( R/(x_0 + x_1) \) and \( R/(x_0 - x_1) \). The two nontrivial MCM modules are interchanged by a coordinate change. In fact, these three are the only indecomposable MCM modules (see Dieterich and Wiedemann [7] and Kiyek and Steinke [14]). Then the same holds for \( R^q \). Again from the Knörrer periodicity, the same holds for arbitrary odd \( d \). Since \( R \) is not a free \( R^q \)-module, \( R \) contains one of the two nontrivial indecomposable MCM modules as a direct summand. But from the symmetry, it must contain the other too and hence is a representation generator. The assertion follows.

If \( d \geq 3 \), then from a result of Quarles [19, Corollary 3.5.5], the above \( D_{R,e} \) is not Cohen-Macaulay. Namely it is not crepant in the sense of Van den Bergh [21].

**Problem 6.2.** How about simple singularities of other types?
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