ON GENERALISED PALEY GRAPHS AND THEIR AUTOMORPHISM GROUPS

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1. Introduction

The generalised Paley graphs are, as their name suggests, a generalisation of the Paley graphs, first defined by Paley in 1933 (see [15]). They arise as the relation graphs of symmetric cyclotomic association schemes. However, their automorphism groups may be much larger than the groups of the corresponding schemes. We determine the parameters for which the graphs are connected, or equivalently, the schemes are primitive. Also we prove that generalised Paley graphs are sometimes isomorphic to Hamming graphs and consequently have large automorphism groups, and we determine precisely the parameters for this to occur. We prove that in the connected, non-Hamming case, the automorphism group of a generalised Paley graph is a primitive group of affine type, and we find sufficient conditions under which the group is equal to the one-dimensional affine group of the associated cyclotomic association scheme. The results have been applied in [11] to distinguish between cyclotomic schemes and similar twisted versions of these schemes, in the context of homogeneous factorisations of complete graphs.

Let $\mathbb{F}_q$ be a finite field with $q$ elements such that $q \equiv 1 \pmod{4}$. Let $\omega$ be a primitive element in $\mathbb{F}_q$ and $S$ the set of nonzero squares in $\mathbb{F}_q$, so $S = \{\omega^2, \omega^4, \ldots, \omega^{q-1} = 1\} = -S$. The Paley graph, denoted by $\text{Paley}(q)$, is the graph with vertex set $\mathbb{F}_q$ and edges all pairs $\{x, y\}$ such that $x - y \in S$. The class of Paley graphs is one of the two infinite families of self-complementary arc-transitive graphs characterised by W. Peisert in [16]. Moreover, Paley graphs are also examples of distance-transitive graphs, of strongly regular graphs, and of conference graphs, see [8, Section 10.3]. The automorphism group $\text{Aut}(\text{Paley}(q))$ of $\text{Paley}(q)$ is of index 2 in the affine group $\text{AGL}(1, q)$ and each permutation in $\text{AGL}(1, q) \setminus \text{Aut}(\text{Paley}(q))$ interchanges $\text{Paley}(q)$ and its complementary graph. The generalised Paley graphs are defined similarly.

**Definition 1.1. (Generalised Paley Graph)** Let $\mathbb{F}_q$ be a finite field of order $q$, and let $k$ be a divisor of $q - 1$ such that $k \geq 2$, and if $q$ is odd then $\frac{q-1}{k}$ is even. Let $S$ be the subgroup of order $\frac{q-1}{k}$ of the multiplicative group $\mathbb{F}_q^*$. Then the generalised Paley graph $\text{GPaley}(q, \frac{q-1}{k})$ of $\mathbb{F}_q$ is the graph with vertex set $\mathbb{F}_q$ and edges all pairs $\{x, y\}$ such that $x - y \in S$.

The generalised Paley graphs $\text{GPaley}(q, \frac{q-1}{k})$ are the relation graphs of the symmetric cyclotomic association scheme $\text{Cyc}(q,k)$ defined in Subsection [11]. They also arise as the factors of cyclotomic homogeneous factorisations of complete graphs [11, Theorem 1.1]. Our main theorem determines precise

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conditions under which \( \text{Cyc}(q, k) \) is primitive (defined in Subsection 1.1), and shows how the automorphism group of a generalised Paley graph depends heavily on the parameters \( k \) and \( q \).

**Theorem 1.2.** Let \( V = \mathbb{F}_q \), where \( q = p^R \) with \( p \) a prime, and let \( k \mid (q - 1) \) such that \( k > 1 \) and either \( q \) is even or \( \frac{q - 1}{k} \) is even. Let \( \text{Cyc}(q, k) \) be a \( k \)-class symmetric cyclotomic scheme with relation graphs \( \Gamma_1, \ldots, \Gamma_k \), and let \( \Gamma = \text{GPaley}(q, \frac{q - 1}{k}) \). Then \( \Gamma_i \cong \Gamma \) for each \( i \) and:

1. \( \text{Cyc}(q, k) \) is primitive if and only if \( k \) is not a multiple of \( \frac{q - 1}{p^R - 1} \) for any proper divisor \( a \) of \( R \).
2. \( \Gamma \) is a Hamming graph if and only if \( k = \frac{a(q - 1)}{R(q - 1)} \) for some proper divisor \( a \) of \( R \).
3. If \( \Gamma \) is connected and is not a Hamming graph, then \( \text{Aut}(\Gamma) \) is a primitive subgroup of \( \text{AGL}(R, p) \) containing the group of translations \( \mathbb{Z}_p^R \).
4. If \( k \) divides \( p - 1 \), then \( \text{Aut}(\Gamma) = \text{Aut}(\text{Cyc}(q, k)) < \text{AGL}(1, q) \).

Commentary on the significance and consequences of this result is given in Subsection 1.2. In particular, Hamming graphs and their automorphism groups are discussed there. The theorem may be contrasted with McConnel’s Theorem [14], proved in 1963, that the automorphism group of \( \text{Cyc}(q, k) \) (the intersection of the automorphism groups of its relation graphs) is always a subgroup of \( \text{AGL}(1, q) \).

1.1. **Cyclotomic association schemes and cyclotomic factorisations.** Here we describe briefly the relationship between generalised Paley graphs, symmetric cyclotomic association schemes, and cyclotomic homogeneous factorisations of complete graphs. Symmetric association schemes are defined as follows.

**Definition 1.3.** [4, p. 43] A symmetric \( k \)-class association scheme is a pair \((V, R)\) such that

1. \( R = \{R_0, R_1, \ldots, R_k\} \) is a partition of \( V \times V \);
2. \( R_0 = \{(x, x) \mid x \in V\} \);
3. \( R_i = R_i^T \) (that is, \((x, y) \in R_i \Rightarrow (y, x) \in R_i\)) for all \( i \in \{0, 1, \ldots, k\} \);
4. there are constants \( p_{ij}^0 \) (called the intersection numbers of the scheme) such that for any pair \((x, y) \in R_i, (x, z) \in R_i \) and \((z, y) \in R_j \), equals \( p_{ij}^0 \).

For each \( i \geq 1 \), the class \( R_i \) corresponds to the undirected graph \( \Gamma_i = (V, E_i) \), where \( E_i = \{(x, y) \mid (x, y) \in R_i\} \) (see for example [7, Chapter 12]). These graphs are called the relation graphs of the scheme and are not in general isomorphic. The scheme is said to be primitive if each of the \( \Gamma_i \) is connected, and otherwise it is called imprimitive. The automorphism group \( \text{Aut}(V, R) \) is the largest subgroup of \( \text{Sym}(V) \) that, in its natural action on \( V \times V \), fixes each of the relations \( R_1, \ldots, R_k \) setwise, that is to say, \( \text{Aut}(V, R) = \bigcap_{i=1}^{k} \text{Aut}(\Gamma_i) \). The edge sets of the \( \Gamma_i \) form a partition \( E = \{E_1, \ldots, E_k\} \) of the edge set of the complete graph \( K_n \), where \(|V| = n\), and hence the relation graphs form a factorisation of \( K_n \).

If the subgroup of \( \text{Sym}(V) \) fixing \( R \) setwise permutes transitively the set \( \{\Gamma_1, \ldots, \Gamma_k\} \) and if \( \text{Aut}(V, R) \) is transitive on \( V \), then the factorisation is called homogeneous. In particular, in this case the relation graphs \( \Gamma_i \) are pairwise isomorphic.
Moreover, in Theorem 1.2, a more detailed version of this result, we prove that, if \( \text{GPaley}(q, k) \) is indeed equal to the automorphism group of \( \text{Cyc}(q, k) \), then its automorphism group is a primitive group of affine type. Although this gives a lot of information about the group, it does not determine it completely. In a special case, that is relevant to our work on homogeneous factorisations in [11], we were able to show that the automorphism group of \( \text{GPaley}(q, k) \) is indeed equal to the automorphism group of \( \text{Cyc}(q, k) \) (and no larger), see Theorem 1.2 (4).

We make additional detailed comments about Theorem 1.2 and its consequences in Remark 1.4.

1.2. Commentary on Theorem 1.2: automorphism groups of graphs and schemes. In 1963, McConnel [14] proved that the full automorphism group of \( \text{Cyc}(q, k) \) is a subgroup of \( \text{AGL}(1, q) \). However, the automorphism groups of the relation graphs of \( \text{Cyc}(q, k) \), that is, of the generalised Paley graphs \( \text{GPaley}(q, \frac{q-1}{k}) \), may be much larger. This paper initiates a study of these automorphism groups for various ranges of values of the parameters \( q \) and \( k \).

The ‘easiest’ way for the automorphism group of \( \text{GPaley}(q, \frac{q-1}{k}) \) to be larger is if the graph is not connected. Since all the relation graphs of \( \text{Cyc}(q, k) \) are isomorphic to \( \text{GPaley}(q, \frac{q-1}{k}) \), it follows that \( \text{Cyc}(q, k) \) is primitive if and only if \( \text{GPaley}(q, \frac{q-1}{k}) \) is connected. We determine in Theorem 1.2 (1) the precise parameter values for \( \text{GPaley}(q, \frac{q-1}{k}) \) to be connected, or equivalently, for \( \text{Cyc}(q, k) \) to be primitive. Moreover in Theorem 2.2, a more detailed version of this result, we prove that, if \( \text{GPaley}(q, \frac{q-1}{k}) \) is disconnected, then its connected components are generalised Paley graphs over a proper subfield and generate a cyclotomic scheme over this subfield.

Suppose now that \( \text{GPaley}(q, \frac{q-1}{k}) \) is connected. It is possible for \( \text{GPaley}(q, \frac{q-1}{k}) \) to be a Hamming graph, for certain \( q \) and \( k \), and hence have a large automorphism group. For positive integers \( a > 1, b > 1 \), the Hamming graph \( H(a, b) \) has as vertices all \( b \)-tuples with entries from a set \( \Delta \) of size \( a \). Two vertices are adjacent in \( H(a, b) \) if and only if the two \( b \)-tuples differ in exactly one entry. The full automorphism group of \( H(a, b) \) is the wreath product \( S_a \wr S_b \) in its product action on \( \Delta^k \), see [4 Theorem 9.2.1]. Section 2 contains details about the product action. We determine in Theorem 1.2 (2) the precise parameter values for \( \text{GPaley}(q, \frac{q-1}{k}) \) to be a Hamming graph.

Moreover we prove, in Theorem 1.2 (3), that if \( \text{GPaley}(q, \frac{q-1}{k}) \) is connected and not a Hamming graph, then its automorphism group is a primitive group of affine type. Although this gives a lot of information about the group, it does not determine it completely. In a special case, that is relevant to our work on homogeneous factorisations in [11], we were able to show that the automorphism group of \( \text{GPaley}(q, \frac{q-1}{k}) \) is indeed equal to the automorphism group of \( \text{Cyc}(q, k) \) (and no larger), see Theorem 1.2 (4).

Let \( V = \mathbb{F}_q \), let \( k \mid (q - 1) \) such that \( k > 1 \) and either \( q \) is even or \( \frac{q-1}{k} \) is even, and let \( S(k) = \langle \omega^k \rangle \subseteq \mathbb{F}_q^* \) (the multiplicative group of \( \mathbb{F}_q \)). Then the \( k \)-class symmetric cyclotomic scheme \( \text{Cyc}(q, k) = (V, \mathcal{R}) \), has \( \mathcal{R}_i = \{(x, y) \mid y - x \in S(k)\omega^i\} \) for \( 1 \leq i \leq k \). Note that condition (3) of Definition 1.3 holds since \(-1 \in S(k)\). Also, the relation graph \( \Gamma_k \) is the generalised Paley graph of \( \mathbb{F}_q \) relative to \( S(k) \), and in particular if \( k = 2 \), then \( \Gamma_2 \) is the Paley graph of \( \mathbb{F}_q \) (see [4 p. 66]).

Moreover, the affine group \( \text{AGL}(1, q) \) fixes \( \mathcal{R} \) setwise and permutes the relation graphs transitively, and \( \text{Aut}(\text{Cyc}(q, k)) = \cap_{i=1}^k \text{Aut}(\Gamma_i) \) contains the group of translations which is transitive on \( V \). Thus the relation graphs for \( \text{Cyc}(q, k) \) form a homogeneous factorisation of \( K_q \) called a cyclotomic factorisation. In particular all of the relation graphs are isomorphic to \( \text{GPaley}(q, \frac{q-1}{k}) \), and \( \text{Aut}(\text{GPaley}(q, \frac{q-1}{k})) \) contains the subgroup of \( \text{AGL}(1, q) \) of order \( q(q - 1)/k \) acting arc-transitively.
Remark 1.4. (a) The graph $G_{\text{Paley}}(q, \frac{q^2-1}{k})$ is a Cayley graph for the translation subgroup $T$ of $A\Gamma L(1,q)$, see Subsection 2.2. Theorem 1.2 (3) proves that, provided $G_{\text{Paley}}(q, \frac{q^2-1}{k})$ is connected and not a Hamming graph, then it is a normal Cayley graph, that is, the translation subgroup $T$ of automorphisms is normal in the full automorphism group.

(b) The condition $k \mid (p - 1)$ holds in particular if $q = p$, that is to say, if $R = 1$. In this case, Theorem 1.2 (4) follows from an old result of W. Burnside about primitive permutation groups of prime degree, see [20, 11.7].

(c) The proof of Theorem 1.2 (3) uses results from [17, 18], and that of Theorem 1.2 (4) depends heavily on results in [9]. As the results used from these papers rely on the classification of the simple groups, these two parts of Theorem 1.2 also rely on that classification.

(d) Apart from the possibilities that $G_{\text{Paley}}(q, \frac{q^2-1}{k})$ may be disconnected or isomorphic to a Hamming graph, which are dealt with in Theorem 1.2 (1) and (2), there are other cases where $\text{Aut}(G_{\text{Paley}}(q, \frac{q^2-1}{k}))$ is not a one-dimensional affine group. In Example 1.6, we give an explicit example. Thus despite our identifying the disconnected and Hamming cases precisely in Theorem 1.2, there remain some mysteries to be solved concerning generalised Paley graphs: Problem 1.5 below is still largely open.

(e) Our interest in generalised Paley graphs arose from our study of homogeneous factorisations of complete graphs (see Subsection 1.1). These factorisations were introduced in [12] as a generalisation of vertex-transitive self-complementary graphs. Our study in [11] gave a classification of arc-transitive homogeneous factorisations of complete graphs. In addition to the cyclotomic factorisations, we discovered a new family of examples that generalise an infinite family of vertex-transitive self-complementary graphs constructed and characterised by Peisert [16]. They may be viewed as a twisted version of cyclotomic factorisations. Using Theorem 1.2 (4), we proved in [11] that factor graphs in these two homogeneous factorisations, with the same parameters $q$ and $k$, were non-isomorphic; we showed that their automorphism groups had non-isomorphic intersections with $A\Gamma L(1,q)$.

Problem 1.5. Determine the precise conditions on $k$ and $p^R$ under which the conclusion of Theorem 1.2 (4) holds.

Example 1.6. Take $R = 4$, $p = 3$ and $k = 4$, so that $\frac{p^R - 1}{k} = 20$, and let $\Gamma = G_{\text{Paley}}(81, 20)$. Then $k \neq a\frac{p^R - 1}{k}$ for any proper divisor $a$ of $R$. Hence, by Theorem 1.2, $\Gamma$ is connected and not a Hamming graph, and $\text{Aut}(\Gamma)$ is a primitive subgroup of $A\Gamma L(4, 3)$. Using MAGMA [3], we computed $\text{Aut}(\Gamma)$. Its order is $|\text{Aut}(\Gamma)| = 233280$, greater than $|A\Gamma L(1, 81)| = 25920$. Thus $\text{Aut}(\Gamma)$ is not contained in the one-dimensional affine group. A further check using MAGMA showed that a point-stabiliser $A_0$ of $A := \text{Aut}(\Gamma)$, which has order 2880, contains a normal subgroup $B$ isomorphic to $A_6$, and $A_0/B \cong D_8$, so $\text{Aut}(\Gamma) = Z_3^4 \rtimes (A_6 \cdot D_8)$.

In Section 2, we introduce some terminology and definitions needed for subsequent results, and we prove Theorem 1.2 (1). We prove Theorem 1.2 (2), (3) and (4) in Sections 3, 4 and 5 respectively.
2. Preliminaries and Proof of Theorem [1,2] (1)

2.1. Cayley graphs. All graphs considered are finite, undirected and without loops or multiple edges. Thus a graph \( \Gamma = (V, E) \) consists of a vertex set \( V \) and a subset \( E \) of unordered pairs from \( V \), called the edge set. An arc is an ordered pair \((u, v)\) where \( \{u, v\} \) is an edge. The generalised Paley graphs belong to a larger class of graphs called the Cayley graphs, defined as follows.

**Definition 2.1. (Cayley Graph)** For a group \( K \) and a nonempty subset \( H \) of \( K \) such that \( 1_K \notin H \) and \( H = H^{-1} = \{h^{-1} | h \in H\} \), the *Cayley graph* \( \Gamma = \text{Cay}(K, H) \) of \( K \) relative to \( H \) is the graph with vertex set \( K \) such that \( \{x, y\} \) is an edge if and only if \( xy^{-1} \in H \).

A Cayley graph \( \Gamma = \text{Cay}(K, H) \) is connected if and only if \( \langle H \rangle = K \). Furthermore, if \( \Gamma = \text{Cay}(K, H) \) is disconnected, then each connected component is isomorphic to \( \text{Cay}(\langle H \rangle, H) \) and the number of connected components equals \( \frac{|K|}{|\langle H \rangle|} \). A permutation group \( G \) on \( V \) is semiregular if the only element fixing a point in \( V \) is the identity element of \( G \); and \( G \) is regular on \( V \) if it is both semiregular and transitive. In Definition 2.1, the group \( K \) acts regularly on vertices by \( y : x \mapsto xy \), for \( x, y \in K \). Conversely, see [2, Lemma 16.3], a graph \( \Gamma \) is isomorphic to a Cayley graph for some group if and only if \( \text{Aut}(\Gamma) \) has a subgroup which is regular on vertices.

2.2. Generalised Paley graphs as Cayley graphs. In follows from Definition 1.1 that

\[
\text{GPaley}(q, \frac{q-1}{k}) = \text{Cay}(V, S),
\]

where \( V \) is the additive group of the field \( \mathbb{F}_q \) and \( S \) is the unique subgroup of order \( \frac{q-1}{k} \) of the multiplicative group \( \mathbb{F}_q^* \). Let \( \omega \) be a primitive element of \( \mathbb{F}_q \). Then \( S = \langle \omega^k \rangle \). Thus \( \text{GPaley}(q, \frac{q-1}{k}) \) admits the additive group of \( \mathbb{F}_q \) acting regularly by \( t_y : x \mapsto x + y \) (for \( x, y \in \mathbb{F}_q \)) as a subgroup of automorphisms. To distinguish this subgroup from the vertex set \( V \), we denote it by \( T = \{ t_y | y \in \mathbb{F}_q \} \), and call it the translation group of \( A\Gamma L(1,q) \).

Now \( T \cong \mathbb{Z}_p^k \), where \( q = p^k \) with \( p \) prime, and \( T \) is the unique minimal normal subgroup of \( A\Gamma L(1,q) \). Let \( \tilde{\omega} \) denote the scalar multiplication map \( \tilde{\omega} : x \mapsto x \omega \) (for all \( x \in \mathbb{F}_q \)) corresponding to the primitive element \( \omega \), and let \( \alpha \) denote the Frobenius automorphism of \( \mathbb{F}_q \), that is, \( \alpha : x \mapsto x^p \). Then \( A\Gamma L(1,q) = T \rtimes \langle \tilde{\omega}, \alpha \rangle \), and the one-dimensional general semilinear group \( \Gamma L(1,q) = \langle \tilde{\omega}, \alpha \rangle \). Now both \( \tilde{\omega}^k \) and \( \alpha \) fix \( 0 \) and fix \( S \) setwise, and hence \( T \rtimes \langle W, \alpha \rangle \) is a subgroup of automorphisms of \( \text{GPaley}(q, \frac{q-1}{k}) \), where \( W := \langle \tilde{\omega}^k \rangle \). In fact, \( T \rtimes W \) is arc-transitive, and \( W \) is transitive on \( S = \{1, \omega^k, \omega^{2k}, \ldots, \omega^\left((q-1)/k\right)k^{-k}\} = 1^W \). (For a permutation group \( K \) on \( V \) and a point \( v \in V \) we denote by \( v^K \) the \( K \)-orbit \( \{ v^x | x \in K \} \) containing \( v \).)

2.3. Hamming graphs and Cayley graphs. Let \( H \) be a group, \( b \) a positive integer and \( K \) be a subgroup of the symmetric group \( S_b \). Then the *wreath product* \( H \wr K \) is the semidirect product \( H^b \rtimes K \) where elements of \( K \) act on \( H^b \) by permuting the “entries” of elements of \( H^b \), that is, \((h_1, h_2, \ldots, h_b)^{k^{-1}} = (h_{1^k}, h_{2^k}, \ldots, h_{b^k}) \) for all \((h_1, h_2, \ldots, h_b) \in H^b \) and \( k \in K \). Now suppose \( H \leq \text{Sym}(\Delta) \). Then the *product*
action of $H \triangleright K$ on $\Delta^b$ is defined as follows. Elements of $H^b$ act coordinate-wise on $\Delta^b$ and elements of $K$ permute the coordinates: for $(h_1, \ldots, h_b) \in H^b$, $k \in K$, and $(\delta_1, \ldots, \delta_b) \in \Delta^b$,
\[
(\delta_1, \ldots, \delta_b)^{(h_1, \ldots, h_b)} = (\delta_1^{h_1}, \ldots, \delta_b^{h_b})
\]
\[
(\delta_1, \ldots, \delta_b)^{k^{-1}} = (\delta_1^k, \ldots, \delta_b^k).
\]

If $A$ is a regular subgroup of $S_a$ then $A^b < S_a \wr S_b$ and $A^b$ acts regularly on the vertices of $H(a, b)$. Thus (see Subsection 2.1) $H(a, b)$ is a Cayley graph. If $a$ is a prime power $q$, then $A$ can be identified with the additive group of a finite field $\mathbb{F}_q$, and the vertex set of $H(a, b)$ can be identified with $\mathbb{F}_q^b$.

2.4. Primitive permutation groups. Let $G$ be a transitive permutation group acting on a finite set $V$. A nonempty subset $\Delta \subseteq V$ is called a block for $G$ if for every $g \in G$, either $\Delta \cap \Delta^g = \emptyset$ or $\Delta = \Delta^g$. A block $\Delta$ is said to be trivial if $|\Delta| = 1$ or $\Delta = V$. Otherwise, $\Delta$ is called nontrivial. We say that the group $G$ is primitive if the only blocks for $G$ are the trivial ones.

The possible structures of finite primitive permutation groups up to permutational isomorphism are described by the O’Nan-Scott Theorem (for example see [5] or [13]). Here, we will briefly describe the three types of finite primitive permutation groups relevant to this paper (we refer readers to [5, 13] for further details about the remaining types).

A finite primitive permutation group $G$ on $V$ is of type $HA$ (holomorph of an abelian group) if $G = T \rtimes G_0$ is a subgroup of an affine group $AGL(R, p)$ on $V$, where $T \cong \mathbb{Z}_p^R$ is the (regular) group of translations (and we may identify $V$ with $\mathbb{Z}_p^R$) and $G_0$ is an irreducible subgroup of $GL(R, p)$. We often say that a primitive group of this type is of affine type. A primitive permutation group $G$ is of type $AS$ if $G$ is an almost simple group, that is $N \leq G \leq \text{Aut}(N)$ where $N$ is a finite nonabelian simple group. Such a group can equivalently be defined as a primitive group $G$ having a unique minimal normal subgroup $N$ which is nonabelian and simple. Finally, a primitive permutation group $G$ on $V$ is of type $PA$ (product action) if $V = \Delta^b$ and $N^b \leq G \leq H \rtimes S_b \leq \text{Sym}(\Delta) \rtimes S_b$ in its product action, where $H$ is a primitive permutation group on $\Delta$ of type $AS$ with simple normal subgroup $N$.

2.5. Proof of Theorem 1.2 (1). The following result Theorem 2.2 relates the connectedness of $\Gamma$ with the action of $W$ on $V$, and Theorem 1.2 (1) follows immediately from it.

Theorem 2.2. Let $\Gamma = \text{GPaley}(q, \frac{q-1}{k}) = \text{Cay}(V, S)$, where $V = \mathbb{F}_q$ and $S = \langle \omega^k \rangle$, with $k$ a divisor of $q - 1$ such that $k \geq 2$ and either $q$ or $\frac{q-1}{k}$ is even. Let $q = p^R$ with $p$ prime, and let $\text{Cyc}(q, k) = (V, \mathcal{R})$.

(1) The following are equivalent:

(i) $\Gamma$ is connected,

(ii) $\text{Cyc}(q, k)$ is primitive,

(iii) $(\omega^k)$ acts irreducibly on $V$,

(iv) $k$ is not a multiple of $\frac{q-1}{p^a-1}$ for any proper divisor $a$ of $R$. 

We note that in part (2), $k'$ may equal 1, and if this happens we still use the notation $\text{GPaley}(p^n, \frac{p^n-1}{k'})$ for $\Gamma_0$ even though in this case $\Gamma_0 = \text{Cay}(\mathbb{F}_{p^n}, S) \cong K_{p^n}$, the complete graph on $p^n$ vertices.

Proof. (1) The equivalence of parts (i) and (ii) follows from our discussion in Subsection 1.1. Let $U$ be the $\mathbb{F}_p$-span of $S$, that is, $U = \{ \sum_{s \in S} \lambda_s \omega^s \mid \lambda_s \in \mathbb{F}_p \}$. Since $W = \langle \omega^k \rangle$ leaves $S$ invariant, it also leaves invariant the $\mathbb{F}_p$-span $U$ of $S$. Also we note that $\Gamma$ is connected if and only if $U = V$ (see Definition 2.1).

Suppose $W$ acts irreducibly on $V$. Then as $U$ is $W$-invariant and nonzero, $U = V$ and hence $\Gamma$ is connected. Conversely suppose $\Gamma$ is connected. Then $S$ is an $\mathbb{F}_p$-spanning set for $V$ (that is $U = V$). Now $S$ is the orbit $1W = \langle \omega^k \rangle$ (note that $W$ acts by field multiplication). Also, for each $\omega^i \in V^*$, $\hat{\omega}^i$ maps $S$ to $S\omega^i$, and as $\hat{\omega}^i \in \text{GL}(1, p^R)$, it follows that $S\omega^i$ is also an $\mathbb{F}_p$-spanning set for $V$. However $S\omega^i$ is the $W$-orbit containing $\omega^i$. Hence every $W$-orbit in $V^*$ is a spanning set for $V$, and so $W$ is irreducible on $V$. Thus (i) and (iii) are equivalent.

Suppose that $k$ is a multiple of $\frac{p^n-1}{p^R-1}$ for some proper divisor $a$ of $R$. Then $S$ is a subgroup of the multiplicative group of the proper subfield $\mathbb{F}_{p^a}$ of $\mathbb{F}_q$. Thus $U \subset \mathbb{F}_{p^a}$ and so $\Gamma$ is disconnected. Therefore condition (iv) implies condition (i). The reverse implication will follow from (2).

(2) Suppose $\Gamma$ is disconnected. Let $U$ be the vertex set of the connected component of $\Gamma$ containing $1 \in \mathbb{F}_{p^R}$. Then $U$ is the $\mathbb{F}_p$-span of $S$. It follows that all the connected components of $\Gamma$ are isomorphic to $\text{Cay}(U, S)$. We claim that $U$ is a subfield of $V = \mathbb{F}_{p^n}$.

Since $U$ is $W$-invariant, $U\omega^i = U$ for each $\omega^i \in W$, and hence $U\omega^i = U$ for each $\omega^i \in S$. Thus $U$ is closed under multiplication by elements of $S$. Now each element of $U$ is of the form $\sum_{\omega^i \in S} \lambda_i \omega^i$ for some $\lambda_i \in \mathbb{F}_p$, and (by regarding $\lambda_i$ as an integer in the range $0 \leq \lambda_i \leq p - 1$) each $\lambda_i\omega^i$ is equal to the sum $\omega^i + \cdots + \omega^i$ ($\lambda_i$-times). Thus each element of $U$ is a sum of a finite number of elements of $S$. Since $U$ is closed under addition and under multiplication by elements of $S$, it follows that $U$ is closed under multiplication. Thus $U$ is a subring of $V$. Also $U$ contains the identity 1 of $V$ (since $1 \in S$). Let $u \in U \setminus \{0\}$. Then since $V$ is finite, $u^i = u^{i+j}$ for some $i \geq 1$ and $j \geq 1$, and hence $u^{-1} = u^{i-1} \in U$. Thus $U$ is a subfield of $V = \mathbb{F}_{p^n}$ as claimed.

Hence $|U| = p^a$ for some proper divisor $a$ of $R$. Also, since $S \subseteq U^*$, it follows that $|S| = \frac{p^a - 1}{k'}$ divides $p^a - 1$. Let $k' = \frac{(p^a - 1)k}{p^R-1}$. Then $|S| = \frac{p^a - 1}{k'} = \frac{p^a - 1}{k}$, and by definition of a generalised Paley graph, we have $\text{Cay}(U, S) = \text{GPaley}(p^a, \frac{p^a-1}{k'})$ (though perhaps $k' = 1$). Since there are $p^R-a$ connected components in $\Gamma$, it follows that $\text{Aut}(\Gamma) = \text{Aut}(\text{GPaley}(p^a, \frac{p^a-1}{k'})) \leq S_{p^{R-a}}$. \qed
From Theorem 2.2 if \( \Gamma = \text{GPaley}(q, \frac{q-1}{k}) \) is disconnected, then the connected components are generalised Paley graphs for subfields. In the rest of the paper we will assume that \( \Gamma = \text{GPaley}(q, \frac{q-1}{k}) \) is connected.

3. Proof of Theorem 1.2 (4)

Suppose first that \( \Gamma = \text{GPaley}(p^R, \frac{p^R-1}{k}) \cong H(p^a, b) \), where \( R = ab \) with \( b > 1 \). The valency of \( \Gamma \) is \( \frac{b}{k} = b(p^a - 1) \), so \( k = \frac{a(p^R - 1)}{R(p^a - 1)} \) as required. Conversely suppose that \( k = \frac{a(p^R - 1)}{R(p^a - 1)} \) where \( R = ab \) and \( 1 < a < R \). Then since \( \Gamma = \text{GPaley}(p^R, \frac{p^R-1}{k}) \) is connected, the \( \mathbb{F}_p \)-span of \( S = \langle \omega^k \rangle \) equals \( V \), that is, \( \{ \sum_{\lambda \in \mathbb{F}_p} \lambda \omega^{ik} | \lambda \in \mathbb{F}_p \} = V \). Let \( U \) be the \( \mathbb{F}_p \)-span of the set \( X := \{1, \omega^k, \omega^{2k}, \ldots, \omega^{(b-1)k}\} \).

We claim that \( U = V \).

Now \( \mathbb{F}_p^a, X \subseteq V = \mathbb{F}_p \), and hence \( U \subseteq V \). Moreover, since \( k = \frac{a(p^R - 1)}{R(p^a - 1)} \), the set \( S = \langle \omega^k \rangle \) has order \( \frac{b}{k} = \frac{a(p^R - 1)}{R(p^a - 1)} \), and also \( \omega^{bk} \) has order \( p^a - 1 \) \( \Rightarrow \langle \omega^{bk} \rangle = \mathbb{F}_p^a \). Now suppose \( v \neq 0 \) and \( v \in V \). Then \( v = \sum \lambda_i \omega^{ik} \) where \( \lambda_i \in \mathbb{F}_p \) (not all zero) and the sum is over all \( \omega^{ik} \in S \). Let \( i = bx_i + r_i \) where \( 0 \leq r_i < b \). Then \( \omega^{ik} = \omega^{(bx_i + r_i)k} = \omega^{bx_i k} \cdot \omega^{r_i k} \) and we have

\[
v = \sum \lambda_i \omega^{ik} = \sum \lambda_i \omega^{bx_i k} \cdot \omega^{r_i k}.
\]

Since \( \mathbb{F}_p^a = \langle \omega^{bk} \rangle \), it follows that \( \omega^{bx_i k} \in \mathbb{F}_p^b \). Thus \( \lambda_i \omega^{bx_i k} \in \mathbb{F}_p^b \) and so \( v \in U \). Thus \( U = V \) as claimed.

From now on we shall regard \( V \) as a vector space over \( \mathbb{F}_p \). We have shown that \( V \) is spanned by the set \( X = \{1, \omega^k, \omega^{2k}, \ldots, \omega^{(b-1)k}\} \), and as \( \dim_{\mathbb{F}_p^a}(\mathbb{F}_p^b) = b \), it follows that \( X \) is an \( \mathbb{F}_p^a \)-basis for \( V \). Define \( \Theta : V \rightarrow \mathbb{F}_p^b \) as follows. For \( u = \sum_{j=0}^{b-1} \mu_j \omega^{jk} \) with \( \mu_j \in \mathbb{F}_p \), let \( \Theta(u) = (\mu_0, \mu_1, \ldots, \mu_{b-1}) \in \mathbb{F}_p^b \). Since \( X \) is an \( \mathbb{F}_p^a \)-basis for \( V \), \( \Theta \) is a bijection.

Next, we determine the image of the connecting set \( S \subset V \) under \( \Theta \). As we observed above, \( |S| = |\langle \omega^k \rangle| = b(p^a - 1) \). Thus each element of \( S \) can be expressed uniquely as \( \omega^{ik} \) for some \( i \) such that \( 0 \leq i < b \).

As before, we write \( i = bx_i + r_i \) where \( 0 \leq r_i \leq b - 1 \). Thus \( \omega^{ik} = \omega^{bx_i k} \cdot \omega^{r_i k} \). Now \( \omega^{bx_i k} \in \mathbb{F}_p^a = \langle \omega^{bk} \rangle \), and so

\[
\Theta(\omega^{ik}) = \Theta(\omega^{bx_i k} \cdot \omega^{r_i k}) = (0, \ldots, 0, \omega^{bx_i k}, 0, \ldots, 0).
\]

Observe that \( x_i \) can be any integer satisfying \( 0 \leq x_i \leq p^a - 2 \), and therefore \( \omega^{bx_i k} \) takes on each of the values in \( \mathbb{F}_p^b \). Moreover each of these values occurs exactly once in each of the positions \( r_i \), for \( 0 \leq r_i \leq b - 1 \). Thus \( \Theta(S) \) is the set of all elements of \( \mathbb{F}_p^b \) with exactly one component non-zero, that is, the set of “weight-one” vectors.

Now \( \Theta \) determines an isomorphism from \( \Gamma \) to the Cayley graph for \( \mathbb{F}_p^b \) with connecting set \( \Theta(S) \). In this Cayley graph, two \( b \)-tuples \( u, v \in \mathbb{F}_p^b \) are adjacent if and only if \( u - v \in \Theta(S) \), that is, if and only if \( u - v \) has exactly one non-zero component. Thus \( \Gamma = \text{GPaley}(p^R, \frac{p^R-1}{k}) \) is mapped under the isomorphism \( \Theta \) to the Hamming graph \( H(p^a, b) \) where \( b = \frac{R}{a} \). \( \square \)
4. Proof of Theorem 1.2 (3)

Recall that by Theorem 2.2 (1), if $\Gamma = \text{GPaley}(q, \frac{q-1}{k})$ is connected, then $W = \langle \mathbb{Z}^k \rangle$ acts irreducibly on $V$. It follows that the group $G = T \rtimes W$ is a vertex-primitive subgroup of $\text{Aut}(\Gamma)$ of affine type. Thus $\text{Aut}(\Gamma)$ is a primitive permutation group on $V$ containing $G$. We will use results from [17, 18] concerning such groups.

Proof of Theorem 1.2 (3). Suppose that $\Gamma = \text{GPaley}(p^a, \frac{p^a-1}{k}) = \text{Cay}(V, S)$ is connected and not a Hamming graph. Let $G = T \rtimes W$ where $T \cong \mathbb{Z}^p$ and $W = \langle \mathbb{Z}^k \rangle$, as in Subsection 2.2. By Theorem 2.2 (2), $k \neq \frac{a(p^a-1)}{R(p^a-1)}$ for any $a \mid R$ with $1 \leq a < R$. Also, by Subsection 2.2 and Theorem 2.2 G $\leq X := \text{Aut}(\Gamma)$, and $W$ is irreducible on $V$, so $G$ is a primitive subgroup of $\text{AGL}(R, p)$. Suppose, for a contradiction, that $X$ is not contained in $\text{AGL}(R, p)$. Since $k \geq 2$, $\Gamma$ is not a complete graph and so $X \neq S_{p^a}$ or $A_{p^a}$. By [17, Proposition 5.1], it follows that $X$ is primitive of type $PA$. Thus (see Subsection 2.3) $R = ab$ with $b \geq 2$, $V = \Delta^b$ where $|\Delta| = p^a$, and $N^b \leq X \leq H \leq S_b$ with $H$ primitive on $\Delta$ of type $AS$ with simple normal subgroup $N$. Moreover from [17, Proposition 5.1] and [18, Proposition 2.1], either $N = A_{p^a}$ or $N$ and $p^a$ are as listed in [17, Table 2] (denoted as $L_1$ in [17]). In all cases $N$ acts 2-transitively on $\Delta$, and since $N$ is nonabelian simple, it follows that $|\Delta| = p^a \geq 5$.

We will prove that $\frac{p^a-1}{k} = b(p^a - 1)$, contradicting the assumption on $k$, and therefore proving the theorem. Let $\gamma \in \Delta$ and consider the point $u = (\gamma, \ldots, \gamma) \in \Delta^b = V$. Since $N^b$ is transitive on $V$, $X = N^b X_u$, where $X_u$ is the stabiliser of $u$ in $X$. Now $X_u$ contains $(N^b)_u = (N_u)^b$, and $N_u$ is transitive on $\Delta - \{\gamma\}$. If $v \neq u$, then $v := (\delta_1, \ldots, \delta_b)$ with, say, $\ell$ entries different from $\gamma$, where $1 \leq \ell \leq b$, and the length of the $(N^b)_{\ell}$-orbit containing $v$ is $(p^a - 1)^{\ell}$.

Since $X$ is a primitive subgroup of $\text{Sym}(\Delta)/S_b$, $X$ projects to a transitive subgroup of $S_b$ (for instance, see [5, Theorem 4.5]). Moreover, since $X = N^b X_u$, it follows that $X_u$ also projects to a transitive subgroup of $S_b$. Thus the $\ell$-subset of subscripts $i$ such that $\delta_i \neq \gamma$ has $n_\ell$ distinct images under $X_u$, where $n_\ell \geq b/\ell$. It follows that the length of the $X_u$-orbit containing $v$ is at least $n_\ell \cdot (p^a - 1)^{\ell} \geq \frac{b}{\ell} \cdot (p^a - 1)^{\ell}$. Suppose now that the point $v$ has been chosen to lie in $\Gamma(u)$ (the set of all vertices in $\Gamma$ adjacent to $u$) so that $v^{N^b} = \Gamma(u)$ has size $\frac{p^a-1}{k}$. We therefore have

\begin{align}
(1) \quad \frac{p^R - 1}{k} &\geq \frac{b}{\ell} \cdot (p^a - 1)^{\ell}.
\end{align}

On the other hand, $X_u$ contains $G_u = W = \langle \mathbb{Z}^k \rangle$, and all $W$-orbits in $V \setminus \{u\}$ have length $\frac{p^R - 1}{k}$. It follows that all orbits of $X_u$ in $V \setminus \{u\}$ have length a multiple of $\frac{p^R - 1}{k}$. Now there exists an orbit of $X_u$ in $V \setminus \{u\}$ of length $b(p^a - 1)$ (the set of $b$-tuples with exactly one entry different from $\gamma$), and so

\begin{align}
(2) \quad b(p^a - 1) &\geq \frac{p^R - 1}{k}.
\end{align}

Combining inequalities (1) and (2), we obtain

\begin{align}
(3) \quad \ell &\geq (p^a - 1)^{\ell-1}.
\end{align}
Since $p^a \geq 5$, the inequality (3) holds if and only if $\ell = 1$, and hence $\frac{p^R - 1}{R} = b(p^a - 1)$ as claimed. This implies that $k = \frac{p^R - 1}{a(p^a - 1)} = \frac{b(a^b - 1)}{a(p^a - 1)}$, which is a contradiction. Thus $X$ is a primitive subgroup of $\text{AGL}(R, p)$. 

5. The case where $k \mid (p - 1)$: Proof of Theorem 1.2 (4)

Let $\Gamma = \text{GPaley}(q, \frac{q^R}{2}) = \text{Cay}(V, S)$, where $q = p^R$ and $V$, $S$ are as defined in Definition 1.1 and suppose that $k$ divides $p - 1$. Let $A := \text{Aut}(\Gamma)$. Recall from Section 1 that $A$ contains $X := T \rtimes (W, \alpha)$ as an arc-transitive subgroup, where $W = \langle \hat{\omega} \rangle$. We will prove that $A = X$.

If $k = 2$, then $\Gamma = \text{GPaley}(p R, \frac{p^R - 1}{2})$ is a Paley graph and, see for example [16], $A = X$. Thus we may assume that $k \geq 3$. Then, since $p - 1 \geq k \geq 3$, we have $p \geq 5$. Suppose that $R = ab$ with $b > 1$. Then $\frac{a^b - 1}{b(p^a - 1)} = \frac{a^b - 1}{b(p^a - 1)} > \frac{b^R - 1}{b(p^a - 1)} > \frac{a^b - 1}{b(p^a - 1)}$. However, this contradicts the assumption $k \mid (p - 1)$. Thus it follows from Theorem 1.2 (3) that $A = T \rtimes A_0 \leq \text{AGL}(R, p)$, where $(W, \alpha) \leq A_0 \leq \text{GL}(R, p)$. Note that $A_0$ preserves $S \subset V$, and hence $A_0$ does not contain $\text{SL}(R, p)$.

We identify $V = \mathbb{F}_p^R$ with an $R$-dimensional vector space over the prime field $\mathbb{F}_p$. Let $a$ be minimal such that $a \geq 1$, $a \mid R$ and $A_0$ preserves on $V$ the structure of an $a$-dimensional vector space over a field of order $q_0 = p^R/a$. Then $A_0 \leq \Gamma L(a, q_0)$ acting on $V = \mathbb{F}_{q_0}^a$. Let $Z := \langle \hat{\omega} \rangle^{(q^R - 1)/(q_0^R - 1)} = Z(\text{GL}(a, q_0)) \cong \mathbb{Z}_{q_0^R - 1}$. 

Lemma 5.1. If $a \leq 2$ then $A = X$.

Proof. Suppose first that $a = 1$. Then $A_0 \leq \Gamma L(1, p^R) = \langle \hat{\omega}, \alpha \rangle$. Since $\langle \hat{\omega} \rangle$ is regular on $V^*$, and since $A_0$ leaves $S = \langle \omega^k \rangle$ invariant, it follows that $A_0 \cap \langle \hat{\omega} \rangle = \langle \hat{\omega} \rangle^k$. Hence $A_0 = W$ and $A = X$.

Suppose now that $a = 2$. Consider the canonical homomorphism $\varphi : \text{GL}(2, q_0) \rightarrow \text{PGL}(2, q_0)$, and for $H \leq \text{GL}(2, q_0)$ let $\overline{H} := \varphi(H) = HZ/Z$. Now $\overline{A_0} \nsubseteq \text{PSL}(2, q_0)$ since $A_0 \nsubseteq \text{SL}(2, q_0)$. Also $\overline{W} = WZ/Z$, and $\overline{WZ} = \langle \hat{\omega}^k, \hat{\omega}^{q_0 + 1} \rangle$ has order $q_0^2 - 1$ if $k$ is odd and $\frac{q_0^2 - 1}{2}$ if $k$ is even. Hence

$$\overline{\langle W, \alpha \rangle} \cong \begin{cases} D_{2(q_0^2 + 1)} & \text{if } k \text{ is odd} \\ D_{q_0^2 + 1} & \text{if } k \text{ is even} \end{cases}$$

It follows from the classification of the subgroups of $\text{PGL}(2, q_0)$ and $\text{PSL}(2, q_0)$ (see [19], p. 417) that either $\overline{A_0} \leq \langle \hat{\omega}, \alpha \rangle \cong D_{2(q_0^2 + 1)}$ or $\overline{A_0} \in \{ A_4, S_4, A_5 \}$. In the former case, $A_0 \leq \langle \hat{\omega}, \alpha \rangle = \Gamma L(1, q)$ and $a = 1$, which is a contradiction. In the latter case, since $\overline{A_0} \geq \mathbb{Z}_{(q_0^2 + 1)/2}$ and $p \geq 5$, it follows that $q_0 = p = 5$ or 7. Moreover since $\overline{A_0} \geq D_{p + 1}$ and $\overline{A_0} \nsubseteq \text{PSL}(2, p)$, it follows that $\overline{A_0} = S_4$ and in both cases $\overline{A_0}$ is transitive on 1-spaces. Thus $S$ consists of, say, $s$ points from each 1-space and $|S| = (p + 1)s$. Since $|S| = |V^*|/k$, we have $k = |V^*|/(p + 1)s) = (p - 1)$ / $s$. Also, since $\Gamma$ is an undirected Cayley
applying the results of [9], the dimension $a$ and $q$ have multiplicative order $R$ (see also Tables 2 - 5) with order divisible by the primitive prime divisor $r$ (or see [9]), there is a prime divisor of $pR - 1$. Thus $p$ has multiplicative order $R$ modulo $r$, and in particular $R$ divides $r - 1$. Thus $r = Rs + 1$ for some $s \geq 1$. Such a prime $r$ is called a primitive prime divisor of $pR - 1$.

Let $A_1 := A_0 \cap \text{GL}(a, q_0)$. Since $W \subseteq \text{GL}(a, q_0)$, it follows that $W \subseteq A_1$, so $r$ divides $|A_1|$ and $A_1$ is irreducible. By the minimality of $a$, $A_1$ is not contained in a proper `extension field subgroup' of $\text{GL}(a, q_0)$. Also, since $k \mid (p - 1)$, the order of $W$ is divisible by $\frac{p^{R-1}}{p-1}$, and it follows that $W$, and hence also $A_1$, cannot be realised over a proper subfield of $\mathbb{F}_{q_0}$. By [9] [Main Theorem] (noting that the groups in [9] Examples 2.2, 2.3, 2.4] do not have all of these properties), either

(A) $A_1$ belongs to one of the families of Examples 2.1 or 2.5 in [9], or

(B) $A_1$ is nearly simple, that is, $L \leq A_1/(A_1 \cap Z) \leq \text{Aut}(L)$ for some nonabelian simple group $L$, with $L$ as in one of Examples 2.6 - 2.9 in [9].

Lemma 5.2. The group $A_1$ satisfies condition (B).

Proof. Suppose that $A_1$ is a subgroup of $\text{GL}(a, q_0)$ in [9] Example 2.1]. Then $A_1$ is a classical group containing $Y = \text{SL}(a, q_0)$, or (for $a$ even) $\text{Sp}(a, q_0)$ or $\Omega^\pm(a, q_0)$, or (if $aq_0$ is odd) $\Omega(a, q_0)$, or (if $q_0$ is a square) $\text{SU}(a, q_0)$. Since $A_1$ is not transitive on $V^*$, $A_1$ cannot contain $\text{SL}(a, q_0)$ or $\text{Sp}(a, q_0)$. Also $A_1$ contains the irreducible element $\tilde{\omega}^{p-1}$ of order $\frac{p^{R-1}}{p-1}$, whereas (see [11]) for the remaining groups $Y$, $|\tilde{\omega}| \leq (q_0^{a/2} + 1)(q_0 - 1)$.

Next suppose that $A_1$ is as in [9] Example 2.5]. Then $a = 2^m$, $r = a + 1$, and $A_1$ is contained in $Z \circ (S \cdot M_0)$ where $S$ and $M_0$ are as listed in Table 1. Also, since $r \geq R + 1$, it follows that $a = R$, and so $r = R + 1 = 2^{m+1} \geq 5$ and $q_0 = p$. Elements in $S$ and $M_0$ have orders at most 4 and $2^{2m-1} - 1 = R^2 - 1$ respectively (see [11] Proof of Lemma 2]), and $S \cap Z \cong \mathbb{Z}_2$. Thus elements of $A_1$ have order at most $2(R^2 - 1)(p - 1)$, and as $A_1$ contains $\tilde{\omega}^{p-1}$, of order $\frac{p^{R-1}}{p-1}$ we have $\frac{p^{R-1}}{p-1} < 2(R^2 - 1)(p - 1)$. Since $p \geq 5$ and $R \geq 4$,

$$5^{R-2} - 1 \leq p^{R-2} - 1 < \frac{p^{R-1}}{p-1} < 2(R^2 - 1).$$

Since $R \geq 4$, this implies that $(R, p) = (4, 5)$. However, $r = R + 1 = 5$ does not divide $5^4 - 1$, contradicting the definition of $r$. □

Thus case (B) holds, and we need to consider the possibilities for $A_1$ from Examples 2.6 - 2.9 in [9] (see also Tables 2 - 5] with order divisible by the primitive prime divisor $r$ of $pR - 1$, where $R = ab, a \geq 3$, $q_0 = p^b \geq 5^b$. Here, $L \leq A_1/(A_1 \cap Z) \leq \text{Aut}(L)$ for some non-abelian simple group $L$. Note that in applying the results of [9], the dimension $a$ is equal to $d = e$ in [9] and $b$ is the parameter $a$ in [9]. Also,
most of the examples in Tables 2 - 5 have additional conditions for $q_0$, $p$, $r$ or $a$. We will not mention them here, unless they are necessary for our calculations.

\begin{table}
\centering
\begin{tabular}{cccc}
\hline
$S$ & $M_0$ & $p$ \\
\hline
$4 \circ 2^{1+2m} = Z_4 \circ D_8 \circ \cdots \circ D_8$ & $\text{Sp}(2m, 2)$ & $p \equiv 1 \mod 4$ \\
$2^{1+2m}_- = D_8 \circ \cdots \circ D_8 \circ Q_8$ & $\text{O}^-(2m, 2)$ & \\
$2^{1+2m}_+ = D_8 \circ \cdots \circ D_8$ & $\text{O}^+(2m, 2)$ & \\
\hline
\end{tabular}
\caption{Table 1.}
\end{table}

\begin{table}
\centering
\begin{tabular}{llllll}
\hline
Example 2.6 (b) of [9] \\
n & 7 & 6 & 5 & 5 & 7 & 7 \\
a & 4 & 4 & 2 & 4 & 3 & 6 \\
r & 5 & 5 & 5 & 7 & 7 & 7 \\
b & 1 & 1 & 2 & 1 & 2 & 1 \\
\hline
\end{tabular}
\caption{Table 2.}
\end{table}

\begin{table}
\centering
\begin{tabular}{llllllll}
\hline
Example 2.7 of [9] \\
$L$ & $M_{11}$ & $M_{12}$ & $M_{22}$ & $M_{23}$ & $J_2$ & $J_3$ & $\text{Ru}$ & $\text{Suz}$ \\
\hline
$a$ & 10 & 10 & 10 & 22 & 6 & 18 & 28 & 12 \\
r & 11 & 11 & 11 & 23 & 7 & 19 & 29 & 13 \\
\hline
\end{tabular}
\caption{Table 3.}
\end{table}

\begin{table}
\centering
\begin{tabular}{llllll}
\hline
Example 2.9 of [9, Table 7] \\
$L$ & $G_2(4)$ & $\text{PSU}(4, 2)$ & $\text{PSU}(4, 3)$ & $\text{PSL}(3, 4)$ \\
\hline
$a$ & 12 & 4 & 6 & 6 \\
r & 13 & 5 & 7 & 7 \\
\hline
\end{tabular}
\caption{Table 4.}
\end{table}

Lemma 5.3. The group $A_1$ does not satisfy condition (B).

Proof. Recall that $r \geq R + 1 \geq a + 1 \geq 4$ and $p \geq 5$. Also $\varpi^{p-1} \in A_1$ has order $\frac{p^a - 1}{p-1}$, a multiple of $r$. Let $\max(A_1)$ denote the maximum order of an element of $A_1$ having order a multiple of $r$. An easy
calculation shows that

\[
\max(A_1) \geq \frac{p^R - 1}{p - 1} = \begin{cases} 
2(p^2 - 1)(R + 1) & \text{if either } R \geq 5, p \geq 5, \text{ or } R = 4, p \geq 11 \\
(p^2 - 1)(R + 1) & \text{if } R = 4, p = 7 \\
2(p - 1)(R + 1) & \text{if } R \geq 4, p \geq 5.
\end{cases}
\]

**CASE [9] Example 2.6** Suppose first that \(A_n \leq A_1 \leq S_n \times Z\) with \(n = a + 1\) or \(a + 2\), and \(r = a + 1\), so \(R = a, q_0 = p\). Then any element of \(S_n\) of order a multiple of \(r\) is an \(r\)-cycle, and therefore \(\omega^{p^R - 1}\) has order at most \(r(p - 1)\). Hence \((R + 1)(p - 1) = r(p - 1) \geq \frac{n}{p - 1}\), contradicting (3). Thus \(L = A_n\) with \(n, a, r, b\) as in one of the columns of Table 2 (see [9] Tables 2, 3 and 4). In all cases \(r = ab + 1 = R + 1\) and \(r \geq n - \delta\), where \(\delta = 1\) except for column 1 where \(\delta = 2\). Thus an element of \(S_n\) of order a multiple of \(r\) has order at most \(\delta r\). Hence an element of \(A_1\) of order a multiple of \(r\) has order at most \(\delta r(q_0 - 1) = \delta(p^b - 1)(R + 1)\). By (4), column 3 of Table 2 holds but with \(r = p = 5\), contradicting the definition of \(r\).

**CASE [9] Example 2.7**, see Table 5 which contains the examples from [9] Table 5] for which \(r\) is a primitive prime divisor of \(p^R - 1\) In all cases \(q = p\) and \(r = R + 1\). An element of \(\text{Aut}(L)\) of order a multiple of \(r\) has order at most \(2r = 2(R + 1)\) (see the Atlas [9]), so \(\max(A_1) \leq 2(p - 1)(R + 1)\), contradicting (3).

**CASE [9] Example 2.8**. These examples are listed in [9] Table 6 and the only ones for which \(r\) is a primitive prime divisor of \(p^R - 1\) are \(L = G_2(q_0)\) with \((a, p) = (6, 2)\) and \(L = S_2(q_0)\) with \((a, p) = (4, 2)\). However these are not examples for us since \(p \geq 5\).

**CASE [9] Example 2.9**, see Tables 3 and 5 which contain the examples from [9] Table 7 and 8 respectively for which \(r\) is a primitive prime divisor of \(p^R - 1\) and \(p \geq 5\). We deal with Table 4 first. Here \(R = a\) and \(r = R + 1\). An element of \(\text{Aut}(L)\) of order a multiple of \(r\) has order at most \(\delta r\), where \(\delta\) is 1, 2, 4 and 3 for the columns of Table 4 respectively, (see the Atlas [9]). Thus \(\max(A_1) \leq \delta(p - 1)(R + 1)\), contradicting (1) in all four cases.

Now we turn to the examples in Table 5. In all cases \(\gcd(s, p) = 1\), and we have \(s = s_0^c\) for some prime \(s_0 \neq p\) and \(c \geq 1\). Following the notation used in [1], we let \(m(K)\) denote the maximum of the orders of the elements of a finite group \(K\). Then by (1), \(m(A_1) \geq \max(A_1) \geq \frac{\delta^c - 1}{p - 1} \geq \frac{s_0^c - 1}{q_0 - 1} > q_0^{c-1} \).
Moreover, in all cases, \( m(A_1) \leq (q_0 - 1)m(\text{Aut}(L)) \), so

\[
(5) \quad m(\text{Aut}(L)) > q_0^{a-2}.
\]

In column 1 of Table 5, \( a \geq s^2 + s \geq 6 \), and \( m(\text{Aut}(L)) \leq m(\text{GL}(n, s)) \cdot 2c = 2c(s^n - 1) = 2c(s - 1)(a + 1) < s(s - 1)(a + 1) < a(a + 1) \) so by (5), \( q_0^{a-2} < a(a + 1) \), which is a contradiction since \( a \geq 6 \), \( q_0 \geq 5 \).

In column 2 of Table 5, \( a \geq s^2 - s \geq c \), and \( m(\text{Aut}(L)) \leq m(\text{GL}(n, s^2)) \cdot 2c = 2c(s^{2n} - 1) < 2c(s^n + 1)^2 = 2c(s + 1)^2(a + 1)^2 \leq 2a(a + 1)^4 \). By (5), \( q_0^{a-2} < 2a(a + 1)^4 \) and, since \( q_0 \geq 5 \), this implies that \( a \leq 9 \). Also, as \( r = \frac{a + 1}{2} = a + 1 \) is prime, \( n \) is odd and \( 3 \leq a \leq 9 \), it follows that \( (r, a, s, n) = (7, 6, 3, 3) \) and \( L = \text{PSU}(3, 3) \). By the Atlas [6], \( m(\text{Aut}(\text{PSU}(3, 3))) = 12 \) and we have a contradiction to (5).

In column 3 of Table 5, \( a \geq s + 1 \geq c \), and \( m(\text{Aut}(L)) \leq m(\text{GL}(2n, s)) \cdot 2c = 2c(s^{2n} - 1) = 2c \cdot 2a(2a + 2) < 8a^2(a + 1) \). By (5), \( q_0^{a-2} < 8a^2(a + 1) \) and, since \( q_0 \geq 5 \), this implies that \( a \leq 6 \). Also, as \( r = \frac{a + 1}{2} = a + 1 \) is prime, \( n \geq 2 \) and \( 3 \leq a \leq 6 \), it follows that \( (r, a, s, n) = (5, 4, 3, 2) \) and \( L = \text{PSp}(4, 3) \). By the Atlas [6], \( m(\text{Aut}(\text{PSp}(4, 3))) = 12 \) and we have a contradiction to (5).

In columns 4 and 5 of Table 5, \( L = \text{PSL}(2, s) \) with \( s \geq 7 \), \( s \neq p \) and \( a \geq \frac{s+1}{2} \geq 3 \). Then \( m(\text{Aut}(L)) = s + 1 \) (see [1]) and hence (5) yields \( q_0^{a-2} < s + 1 \leq 2a + 2 \). Since \( q \geq 5 \) and \( a \geq \frac{s+1}{2} \geq 3 \), this implies that \( a = 3 = \frac{s+1}{2} \) and \( q_0 \leq 7 \). Thus \( L = \text{PSL}(2, 7) \) and \( p \neq s = 7 \). So \( q_0 = p = 5 \) (since \( p \geq 5 \)) and \( a = R = 3 \). However the only primitive divisor of \( p^R - 1 = 5^3 - 1 \) is 31 whereas by Table 5 \( r \leq s = 7 \). □

It follows from the discussion and results of this section that Theorem 1.2 (4) is proved.

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