Signatures of nonclassical effects in optical tomograms

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Abstract

Several nonclassical effects displayed by wave packets governed by nonlinear Hamiltonians can be identified and assessed directly from tomograms without attempting to reconstruct the Wigner function or the density matrix explicitly. We have demonstrated this for both single-mode and bipartite systems. We have shown that a wide spectrum of effects such as the revival phenomena, quadrature squeezing, and Hong–Mandel and Hillery type higher-order squeezing in a generic single-mode system and the double-well Bose–Einstein condensate (BEC) can be obtained from appropriate tomograms in a straightforward manner. We have examined the manner in which decoherence affects the nature of the state of a generic single-mode system at specific instants during temporal evolution. We have investigated entropic squeezing of the subsystem state of a bipartite system as it evolves in time, solely from tomograms. The procedures that we have demonstrated can be readily adapted to multimode systems. Further, for the double-well BEC we have identified an indicator of entanglement between subsystems that can be obtained directly from the tomogram. This mirrors the qualitative behavior of the subsystem von Neumann entropy and the subsystem linear entropy.

Keywords: tomograms, nonclassical effects, revival phenomena, squeezing and higher-order squeezing, entropic squeezing, quantum entanglement

(Some figures may appear in colour only in the online journal)

1. Introduction

Atom optics provides an ideal framework for investigating nonclassical effects such as revivals of quantum wave packets and squeezing of quantum states. Both these effects have been examined theoretically and investigated experimentally in a variety of physical systems [1, 2], particularly in the context of the interaction of the radiation field with atomic media. An initial wave packet $|\psi(0)\rangle$ governed by a nonlinear Hamiltonian is said to revive fully at an instant $T_{\text{rev}}$ during its dynamical evolution if the wave packet $|\psi(T_{\text{rev}})\rangle$ differs from $|\psi(0)\rangle$ only by an overall phase. This happens due to very specific quantum interference between the basis states that comprise the wave packet. In certain systems revivals periodically occur at integer multiples of $T_{\text{rev}}$. Under certain circumstances, $m$-subpacket fractional revivals ($m$: positive integer) of the wave packet can occur at specific instants between two successive revivals [3]. At these instants the initial wave packet becomes $m$ superposed copies of itself each of amplitude less than that of the initial state. The system governed by the Kerr Hamiltonian $\hbar \chi a^2 a^\dagger$ is an example where periodic revivals and fractional revivals occur. Here, $a^\dagger$ and $a$ are the photon creation and destruction operators satisfying the commutation relation $[a, a^\dagger] = 1$, and $\chi$ is the third-order nonlinear susceptibility of the medium.

It is an interesting fact that squeezing occurs in this system in the neighborhood of revivals and two-subpacket fractional revivals of the field wavefunction. Squeezing is intimately related to quantum noise reduction in the measurement of either of two noncommuting observables. Detection of squeezed states of light [4] provided early proof of this nonclassical effect.

Bipartite systems with Hamiltonians in which operators corresponding to both subsystems appear explicitly, provide an ideal framework for examining a wider spectrum of nonclassical effects such as quantum entanglement. The possibility of occurrence of revivals in this case is very sensitive to the extent of entanglement and the ratio of the strengths of the nonlinearity and coupling between the subsystems.
To estimate the extent of nonclassical effects manifested by a state at any instant during temporal evolution would involve the entire machinery of state reconstruction at that instant. The first step in this program is to identify a finite set of appropriate observables for the physical system considered, which can be measured in experiments. For single-mode systems such an optimal set are the matrix elements \(\langle X_\theta, \theta|p|X_\theta, \theta \rangle\) (also referred to as the tomogram \(w(X_\theta, \theta)\)) [5]. Here, \(p\) is the density operator at the instant considered and \(|X_\theta, \theta\rangle\) are the eigenbasis of the set of rotated quadrature operators \(X_\theta = \frac{1}{\sqrt{2}}(a^\dagger e^{i\theta} + ae^{-i\theta})\) \((0 \leq \theta \leq \pi)\). This concept can be extended in a straightforward manner to multimode systems comprising \(n\) subsystems. This is done by introducing tomogram variables \((X_\theta, \theta_i)\) \((i\) takes values 1, 2, \(\ldots n\)), and defining quadrature operators corresponding to each of these pairs. The tomogram corresponding to the full system is now a function of all the \(2n\) tomogram variables. Homodyne measurements of the quorum of rotated quadrature operators are made on an ensemble of identical copies of the system and this quadrature histogram (the tomogram) is used for state reconstruction. This is a somewhat cumbersome procedure involving maximum likelihood estimates which can be carried out only approximately in general and are therefore error-prone.

It would therefore be very useful to ‘read-off’ as much information about a state directly from the tomogram itself. In particular, identifying signatures of nonclassical effects through simple manipulations of the relevant tomograms alone becomes an interesting and important exercise. Investigations carried out in this regard employ the ‘inverse procedure’ of starting with a known state and obtaining the tomogram with a purpose to understand how tomographic patterns carry signatures of nonclassical effects. Earlier reports in the literature of immediate relevance to us demonstrate how tomogram patterns carry signatures of (1) fractional revivals of a wave packet governed by the Kerr Hamiltonian [6] and (2) entanglement at the output port of a quantum beamsplitter [7].

However more detailed qualitative and quantitative analysis needs to be undertaken on other single-mode and multipartite physical systems of experimental interest before clear signatures of different nonclassical effects are identified from the tomograms alone. For instance, (i) if the Kerr Hamiltonian is modified to have an additional term proportional to \(a^3 a^\dagger\), the delicate balance between the coefficients of the Kerr term and this additional term can lead to super-revivals, i.e., revivals in a system with more than one time scale [8, 9]. Super-revivals have been experimentally detected in systems of alkali atoms subject to an external field (see for instance, [10]). Further, such nonlinear Hamiltonians are important in the context of understanding the behavior of nonlinear optical systems where higher order susceptibilities are sought to be enhanced using high intensity lasers and properties of atomic coherence [11]. The important role played by higher-order nonlinearities in the context of multiphoton processes and engineering of nonclassical states is extensively discussed in [12]. Investigations on higher-order nonlinear susceptibilities are specially important in making new materials (see for instance, [13]). These systems differ both in the qualitative aspects and in the instants of occurrence of revivals (or near-revivals) from the simpler case governed by the Kerr Hamiltonian and straightforward correlations between tomograms and fractional revivals are in general not expected. The effects of decoherence also needs to be examined in such systems, (ii) Squeezing and higher-order squeezing properties have not been investigated by merely exploiting tomograms. These include both Hillery and Hong–Mandel type squeezing. The former involves squeezing in the expectation values of \((a^\dagger + a^\dagger)\) \((g\) a positive integer), and the latter in expectation values of powers of \((a + a^\dagger)\). (iii) A third aspect concerns estimating the information entropy and the extent of entropic squeezing of the state of a subsystem of a bipartite system directly from the tomogram. The subsystem information entropy \(S(\theta_i)\) of the \(i\)th subsystem of a full system is the quantum analog of the classical Shannon entropy. This is of the form \(S = \int_{-\infty}^{\infty} dX_\theta \log w_i\).

In a bipartite system the quantity \(w_i = w(X_\theta, \theta_i)\) for instance, is obtained by fixing \(\theta_i\) and integrating over \(X_\theta\). For any fixed value of \(\theta_i\), interesting properties of the entropy have been discussed [14]. Corresponding to the \(x\)-quadrature for instance, i.e., for \(\theta_i = 0\), the entropy \(S(x)\) in a state satisfies \(S(x) \leq (1/2)[1 + \log \pi + \log(2(\Delta x)^2)]\), where \(\Delta x^2\) is the variance in \(x\) in that state. Further, it exhibits entropic squeezing if \(S(x) < (1/2)(1 + \log \pi)\). (iv) Another exercise of importance would be to identify an indicator of entanglement between subsystems during temporal evolution, directly from the tomogram.

In this paper we have examined all the four aspects mentioned above. The single-mode system considered is the radiation field governed by a Hamiltonian with both the Kerr term and the cubic nonlinearity \(a^3 a^\dagger\). The bipartite system is a double-well Bose–Einstein condensate (BEC) [15–17] on which recently several experiments on nonclassical effects have been performed. The plan of the paper is as follows: In the next section we briefly describe the salient features of tomograms which are of direct relevance to our calculations. In section 2, we also examine the tomograms of the single-mode system for signatures of revivals and fractional revivals, amplitude squeezing and higher-order squeezing without indulging in state-reconstruction procedures. Since these effects are sensitive to the precise initial state considered, we have carried out our investigations with initial states \(|\alpha\rangle\) \((\alpha \in \mathbb{C})\) that exhibit ideal coherence and with states that depart from coherence in a quantifiable manner. The latter are 1-photon-added coherent states \(|\alpha, 1\rangle\) (PACSs) and can be obtained from \(|\alpha\rangle\) by applying \(a\) on it and normalizing the state. This state has been identified experimentally [18] using quantum state tomography and is therefore an ideal candidate for our purpose. We also consider the effects of dissipation due to amplitude decay and phase damping in the single-mode system. In section 3, we extend our investigations to a double-well BEC with interacting atomic species that have Kerr nonlinearities. We have identified and assessed the manner in which the revival phenomena and two-mode amplitude squeezing properties are manifested for initial states which are factored products of the states of the individual subsystems.
These are combinations of $|\alpha\rangle$ and $|\alpha, 1\rangle$ (the latter quantifying departure from macroscopic coherence of the initial condensate). (The operators and number states refer in this case to the atomic species.) Further, from the tomograms alone (a) we have determined the extent of entropic squeezing of the condensate in one of the wells and (b) identified an indicator of entanglement between the two subsystems of the condensate, as it evolves in time. We conclude with brief comments on the results of our investigations.

2. Single-mode system: a tomographic approach

2.1. Properties of tomograms: a brief review

We summarize below certain basic properties of tomograms which are useful in examining nonclassical effects. To be able to reconstruct a quantum state we need a set of operators (quorums) whose statistics gives us tomographically complete information about the state. For optical tomography of a single-mode radiation field (i.e., a single system) the set of rotated quadrature operators [19, 20] given by

$$X_\theta = \frac{1}{\sqrt{2}} (a e^{i\theta} + a^* e^{-i\theta}),$$

with $\theta$ ranging from 0 to $\pi$, constitutes a quorum of observables that carry complete information about the state. The tomogram [19, 21]

$$w(X_\theta, \theta) = \langle X_\theta, \theta | \rho | X_\theta, \theta \rangle.$$  

(2)

Here [22]

$$|X_\theta, \theta\rangle = \frac{1}{\pi^{1/4}} \exp \left( -\frac{X^2_\theta}{2} - \frac{1}{2} e^{2i\theta} a^\dagger a^2 + \sqrt{2} e^{i\theta} X_\theta a^\dagger \right) |0\rangle.$$  

(3)

Essentially the tomogram is a collection of probability distributions corresponding to the quadrature operators and for every $\theta$ it satisfies

$$\int_{-\infty}^{\infty} dX \, w(X_\theta, \theta) = 1.$$  

(4)

Thus a tomogram is simply the pattern $w(X_\theta, \theta)$ shown with $X$ on the $x$-axis and $\theta$ on the $y$-axis.

These ideas can be extended in a straightforward manner to multimode systems. We will use this generalization in section 3 where we examine a bipartite system. For this purpose tomographic variables $(X_\theta, \theta_i)$ for the $i$th subsystem of the multipartite system are introduced and quadrature operators corresponding to each of these pairs are defined. Correspondingly, we have

$$\int_{-\infty}^{\infty} dX_\theta \int_{-\infty}^{\infty} dX_{\theta_1} \cdots \int_{-\infty}^{\infty} dX_{\theta_i} \cdots w(X_\theta, \theta_1, X_{\theta_2}, \theta_2, \ldots) = 1.$$  

(5)

(Here, $\theta_i$ are constants and the above equation holds for all values of $\theta_i$.)

The tomogram $w(X_\theta, \theta)$ of a single-mode system can be seen to have the symmetry property

$$w(X_\theta, \theta + \pi) = w(X_\theta, \theta),$$

(6)

and hence information for $0 \leq \theta < \pi$ is sufficient in principle, to reconstruct the state. However we choose the full range $0 \leq \theta < 2\pi$ to help visualize various features of the tomogram better. A convenient expression for $w(X_\theta, \theta)$ has been derived in terms of Hermite polynomials by realizing that

$$e^{i\theta} a^\dagger a + a^* e^{-i\theta} = X_\theta.$$  

(7)

This expression follows from the Baker–Campbell–Hausdorff identity. Hence the eigenvectors of $X_\theta$, satisfy,

$$|X_\theta, \theta\rangle = e^{i\theta} a^\dagger a |\Theta\rangle,$$

(8)

where $|\Theta\rangle$ is the eigenstate of the position operator. It has been shown [23] that as a consequence of (2), (8), and $|X| = (e^{-X^2/2} H_0(X)) / (\pi^{1/4} \sqrt{n!2^n})$, the tomogram of a normalized pure state $|\psi\rangle$, which can be expanded in the photon number basis as $\sum_n c_n |n\rangle$, is given by

$$w(X_\theta, \theta) = e^{-X^2} \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} c_n e^{-i\theta} |H_0(X)\rangle^2.$$  

(9)

where $H_0(X_\theta)$ are Hermite polynomials. We will use this expression as the time-dependence is reflected only in the coefficients $c_n$ of the number basis thus facilitating numerical computations.

2.2. Revivals and fractional revivals

In this section we investigate the manner in which full and fractional revivals are mirrored in tomograms as a single-mode system with Hamiltonian $\hat{H} = (\hbar \chi a^\dagger a^2 + \hbar \chi_2 a^3 a^\dagger)$ evolves in time. In what follows we set $\hbar = 1$ for convenience. It has been shown earlier [6] that if the field is governed by the Kerr Hamiltonian $\hat{H} = \hbar a^\dagger a^2$ the tomogram is composed of distinct strands at instants of revivals and fractional revivals in contrast to a blurred pattern at other generic instants during temporal evolution of the field. The authors use the method of ‘strand-counting’ in the tomogram to study revival patterns, and they infer that for the Kerr system, the number of strands in a tomogram is equal to the number of subpackets at instants of fractional revivals. A limitation in this method is that individual strands in the tomogram corresponding to a subpacket fractional revival will not be distinct for $|\Theta\rangle$, and the latter quantifying interference effects. However, to understand the broad features of revival phenomena it suffices to employ this procedure without resorting to state-reconstruction methods. We employ the same procedure and first consider a system with effective Hamiltonian $\hat{H}' = \chi a^\dagger a^3 = \chi N (N - 1)(N - 2)$, where $N = a^\dagger a$ and $\chi$ is a constant. We then move on to consider the full Hamiltonian $\hat{H}$. We corroborate our numerical findings with analytical explanations for the revival patterns that we observe.
In the system with Hamiltonian $H'$ it can be easily seen that an initial CS or PACS revives fully at instants $T_{\text{rev}} = \pi / \chi$. Hence we examine tomogram patterns at $T_{\text{rev}}$ and at fractional revival times, i.e., at instants

$$t = \frac{\pi}{l \chi} = \frac{T_{\text{rev}}}{l},$$

where $l$ is a positive integer. The tomograms of an initial CS corresponding to specific fractional revivals are shown in figures 1(a)–(i). We observe that at both $t = 0$ (equivalently $T_{\text{rev}}$) and $T_{\text{rev}} / 3$ the tomograms look similar. This is in sharp contrast to the case of the system defined by the Kerr Hamiltonian alone, where at $T_{\text{rev}} / 3$, the tomogram has three strands. We outline the reason for this difference later in this section. Again at instants $T_{\text{rev}} / 2$ and $T_{\text{rev}} / 6$ the tomograms are similar and have four strands each, in contrast to what is reported in [6], where at $T_{\text{rev}} / 2$ the tomogram has two strands and at $T_{\text{rev}} / 6$ it has six strands.

The new features in our system follow from the properties of the unitary time evolution operator corresponding to this system. It is known for instance that for the simpler system with Hamiltonian $a^2 \hat{a}^2$ the number of subpackets $p$ of a wave packet at an instant of fractional revival $T_{\text{rev}} / p$ is a consequence of the periodicity of the unitary time evolution operator which can be Fourier decomposed at that instant in the form

$$U(T_{\text{rev}} / p) = \sum_{m=0}^{p-1} f_m \exp \left( -\frac{2\pi i m N}{p} \right),$$

where $f_m$ is a Fourier coefficient. As a consequence, an initial state $|\alpha\rangle$ evolves to a superposition of $p$ coherent states at that instant [24].

While the system at hand is more complicated, the full revival at $T_{\text{rev}} / 3$ is a simple consequence of the fact that $n(n-1)(n-2)/3$ is even for $n \in \mathbb{N}$. Here, $N[n] = n[n]$, with $|n\rangle$ denoting the photon number basis. Hence corresponding to an initial state $|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$, the state at instant $T_{\text{rev}} / 3$ is

$$|\psi(T_{\text{rev}} / 3)\rangle = U(T_{\text{rev}} / 3)|\psi(0)\rangle = \sum_{n=0}^{\infty} e^{-i\pi(n-1)(n-2)/3} c_n |n\rangle = |\psi(0)\rangle.$$
An analysis of the properties of the time evolution operator would, in principle, explain the appearance of a specific number of strands in the tomogram at different instants $T_{\text{rev}}/l$. However, accounting for the number of strands in a tomogram is not always straightforward in this case, in contrast to the Kerr system.

We are now in a position to investigate tomogram patterns for the full Hamiltonian

$$H = \left( \chi_1 a^\dagger a^2 + \chi_2 a^\dagger a^3 \right).$$

In this case,

$$T_{\text{rev}} = \pi \text{LCM} \left( \frac{1}{\lambda_1}, \frac{1}{\lambda_2} \right).$$

As before, we proceed to examine tomogram patterns at different instants of time. If the ratio $\chi_1/\chi_2$ is irrational, revivals are absent and the generic tomogram at any instant is blurred. This is illustrated in figure 2 for an initial CS with $\alpha = \sqrt{10} \exp(i\pi/4)$ at $t = \pi/\lambda_2$.

For rational $\chi_1/\chi_2$, revivals and fractional revivals are seen. Fractional revivals occur at instants $T_{\text{rev}}/l$ as before, but the corresponding tomogram patterns are sensitive to the ratio $\chi_1/\chi_2$. We expect that for a given $l$ the tomogram will have $l$ strands as a consequence of the Kerr term in $H$. While this is one possibility, the effect of the cubic term in $H$ allows for the possibility of other tomogram patterns. We illustrate this for an initial CS with Hamiltonian $H$ and $\alpha = \sqrt{10} \exp(i\pi/4)$ in figures 3, 4, and 5. At $t = T_{\text{rev}}/2$, apart from the two-strand tomogram for $\chi_1 = 1$ and $\chi_2 = 2.048 \times 10^{-7}$, one of the other possibilities is a four-strand tomogram for $\chi_1 = 1$ and $\chi_2 = 1.024 \times 10^{-7}$ (figure 3). Similarly at $t = T_{\text{rev}}/3$, the tomogram has three strands for $\chi_1 = 1$ and $\chi_2 = 2.048 \times 10^{-7}$ and a single strand similar to the tomogram of a CS for $\chi_1 = 1$ and $\chi_2 = (4/3) \times 10^{-7}$ (figure 4).

At $t = T_{\text{rev}}/4$ three specimen tomograms which are distinctly different from each other are shown in figures 5(a)–(c) where $\chi_1 = 1$ and $\chi_2 = 2.048 \times 10^{-7}$, $1.024 \times 10^{-7}$ and $4.096 \times 10^{-7}$ respectively.

These features can be explained on a case by case basis as before by examining the periodicity properties of the unitary time evolution operator at appropriate instants. It is however evident that the simple inference that an $l$-subpacket fractional revival is associated with an $l$-strand tomogram alone does not hold when more than one time scale is involved in the Hamiltonian, and there can be several tomograms possible at a given instant depending on the interplay between different time scales in the system.

Thus distinct signatures of higher-order nonlinearities are captured in tomograms corresponding to specific instants of fractional revivals. Whereas a Kerr Hamiltonian allows only for a tomogram with $n$ strands at the instant $T_{\text{rev}}/n$, even inclusion of cubic nonlinearities in the Hamiltonian allows for more possibilities in tomogram patterns at that instant. This observation could be of considerable use in understanding the role of higher-order susceptibilities through experiments. Practical difficulties arise because the susceptibility parameter is numerically small for the Kerr medium. (In our discussions we have scaled $\chi_1$ to unity.) Typically, the loss rate $\kappa$ of the field is such that damping occurs much faster than $T_{\text{rev}}$. It is therefore considerably difficult to sustain field collapses and revivals till $T_{\text{rev}}$ in experiments. However a forerunner to such studies [25] has been successfully carried out recently by engineering an artificial Kerr medium with sufficiently high susceptibility in which collapses and revivals of a coherent state has been observed. It should therefore be possible in future to capture the tomographic signatures reported above, particularly because they can be seen even at the instant $T_{\text{rev}}/4$.

We now examine the role played by interaction of the single-mode system with an external environment and the manner in which the purity of the system gets affected. For illustrative purposes, we consider the state at the instant $T_{\text{rev}}/2$ corresponding to an initial CS under the Hamiltonian (12). This is allowed to interact with the environment. The procedure used is similar to that employed in the case of the Kerr Hamiltonian examined in [6] so as to facilitate comparison of the two cases.

We consider two models of decoherence, namely, amplitude decay and phase damping of the state. First, dissipation is modeled using the master equation for amplitude decay

$$\frac{d\rho}{dt} = -\Gamma (a' a \rho - 2a' \rho a + \rho a' a).$$

Here, $\Gamma$ is the rate of loss of photons and $t$ the time parameter is reckoned from the instant $T_{\text{rev}}/2$.

The solution to this master equation is [26]

$$\rho(\tau) = \sum_{n,n'=0}^{\infty} \rho_{n,n'}(\tau) |n\rangle |n\rangle^\dagger,$$

where,

$$\rho_{n,n'}(\tau) = e^{-\Gamma (n+n') \tau} \sum_{r=0}^{\infty} \binom{n+r}{r} \binom{n'+r}{r} \left(1 - e^{-2\Gamma \tau}\right)^{n+n'-r} \rho_{n+n'-r,n=n'+r}(\tau = 0).$$
Figure 3. Tomograms of an initial CS at \( t = \frac{T_{rev}}{2} \) with \( \alpha = \sqrt{10} e^{i\pi/4} \) and \( \chi_1 = 1 \) for (a) \( \chi_2 = 2.048 \times 10^{-7} \) and (b) \( \chi_2 = 1.024 \times 10^{-7} \).

Figure 4. Tomograms of an initial CS at \( t = \frac{T_{rev}}{3} \) with \( \alpha = \sqrt{10} e^{i\pi/4} \) and \( \chi_1 = 1 \) for (a) \( \chi_2 = 2.048 \times 10^{-7} \) and (b) \( \chi_2 = (4/3) \times 10^{-7} \).

Figure 5. Tomograms of an initial CS at \( t = \frac{T_{rev}}{4} \) with \( \alpha = \sqrt{10} e^{i\pi/4} \) and \( \chi_1 = 1 \) for (a) \( \chi_2 = 2.048 \times 10^{-7} \), (b) \( \chi_2 = 1.024 \times 10^{-7} \), and (c) \( \chi_2 = 4.096 \times 10^{-7} \).

Figure 6. Tomograms of an initial CS at \( T_{rev}/2 \) subject to amplitude decay for \( \Gamma \tau = 1 \) and \( \chi_1 = 1 \) and (a) \( \chi_2 = 2.048 \times 10^{-7} \), (b) \( \chi_2 = 1.024 \times 10^{-7} \), (c) \( \text{Tr}(\rho^2) \) versus \( \Gamma \tau \) for \( \chi_1 = 1 \) and \( \chi_2 = 2.048 \times 10^{-7} \) (green dashed curve) and \( \chi_2 = 1.024 \times 10^{-7} \) (solid red curve).
As is to be expected from (15) and (16) as \( \Gamma \tau \to \infty \) the state evolves to \( |0\rangle \langle 0| \). The purity of the state, \( \text{Tr}(\rho^2) \) initially decreases from 1 (corresponding to the initial pure state) and subsequently increases back to 1 when \( \Gamma \tau \approx 4.5 \) (figure 6(c)). Depending on the ratio \( \chi_1/\chi_2 \) we see different extents of loss of purity of the state. The solid red (dashed green) curve corresponds to \( \chi_1 = 1 \) and \( \chi_2 = 1.024 \times 10^{-7} \) (\( \chi_2 = 2.048 \times 10^{-7} \)). Further, new aspects of decoherence in the system considered by us arise as a consequence of the possibility of more than one distinctly different tomograms at \( T_{\text{rev}}/2 \), depending on the ratio \( \chi_1/\chi_2 \). Figures 6(a) and (b) mirror the effect of decoherence at \( \Gamma \tau = 1 \) for two such ratios.

We also consider dissipation through phase damping of the state. This is modeled using the master equation [26],

\[
\frac{d\rho}{d\tau} = -\Gamma_p(N^2\rho - 2N\rho N + \rho N^2),
\]

(17)

Here, \( \Gamma_p \) is the rate of decoherence. The solution to this master equation also can be expressed in the form given in (15), with [26]

\[
\rho_{n,n'}(\tau) = e^{-\Gamma_p(n-n')^2} \rho_{n,n'}(\tau = 0).
\]

As \( \Gamma_p\tau \to \infty \), it is evident that the off-diagonal terms of the density matrix vanish while the diagonal terms of \( \rho_{n,n'}(\tau = 0) \) remain unchanged due to phase damping. In contrast to amplitude damping, the state does not go back to a pure state as \( \Gamma_p\tau \to \infty \) but remains a mixed state. Further, the differences between tomograms with different strand structures (consequent to different ratios of \( \chi_1/\chi_2 \)) disappear faster than in the case of amplitude damping. Figures 7(a) and (b) show the effect of phase damping at \( \Gamma_p\tau = 0.1 \) for two such ratios. However, for \( \Gamma_p\tau = 1 \) the differences are barely visible.

2.3. Squeezing and higher-order squeezing

We now proceed to examine the squeezing and higher-order squeezing properties of the system with Hamiltonian \( H \). Once again, our aim is to identify and assess these nonclassical effects directly from the tomogram without attempting to reconstruct the state of the system at any instant of time. The extent of quadrature squeezing at a given instant is essentially determined by the numerical value of the variance of the quadrature observables. The state of the system is said to be squeezed in \( x \) if the variance \( \langle (\Delta x)^2 \rangle = \langle (x - \langle x \rangle)^2 \rangle \) is less than the variance of \( x \) in a CS \( |\alpha\rangle \) which is \( 1/2 \) in dimensionless units. Generalization of this definition to include higher-order squeezing allows for two possibilities. Hong–Mandel squeezing of order 4 in \( x \) requires that \( \langle (x - \langle x \rangle)^2 \rangle \) for the given state is less than the corresponding expectation value for the CS which is \( 3/4 \) in dimensionless units. We will estimate the extent of Hong–Mandel squeezing by calculating the central moments of the probability distribution corresponding to the appropriate quadrature. For example, if we wish to determine the extent of second-order Hong–Mandel squeezing in the \( x \) quadrature, we simply calculate the fourth central moment of a horizontal cut of the tomogram at \( \theta = 0 \). Note that for \( q = 1 \), Hong–Mandel squeezing is identical to quadrature squeezing.

Hillery type squeezing of order \( q \) corresponds to squeezing in either of the pair of operators \( Z_q = (a^q + a^{q\dagger})/\sqrt{2} \) and \( Z_2 = (a^q - a^{q\dagger})/\sqrt{2}i \), where \( q = 2, 3, ... \). A useful quantity \( D_q \) of \( q \)th-power squeezing in \( Z_q \) for instance, is defined [27] in terms of the commutator \( [a^q, a^{q\dagger}] = F_q(N) \) as

\[
D_q = \frac{\langle (\Delta Z_q)^2 \rangle - \langle F_q(N) \rangle}{\langle F_q(N) \rangle},
\]

(18)

where \( \langle (\Delta Z)^2 \rangle \) is the variance in \( Z_1 \). A similar definition holds for \( q \)th power squeezing in \( Z_2 \). We note that \( F_q(N) \) is a polynomial function of order \( q - 1 \) in \( N \). A state is \( q \)th-power squeezed if \( -1 \leq D_q < 0 \). It is clear that \( Z_1 \) and \( Z_2 \) cannot be obtained in a straightforward manner from the tomogram as they involve terms with products of powers of different rotated quadratures and hence cannot be assigned probability distributions directly from a set of tomograms.

However an illustrative treatment of the problem of expressing the expectation value of a product of moments of creation and destruction operators in terms of the tomogram \( w \) and Hermite polynomials [28] leads to the result (see appendix A for details)

\[
\langle a^k a^l \rangle = C_{kl} \sum_{m=0}^{k+l} \exp \left( -i(k - l) \left( \frac{mm}{k + l + 1} \right) \right) \times \int_{-\infty}^{\infty} dX_0 w \left( X_0, \frac{mm}{k + l + 1} \right) H_{k+l}(X_0),
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Figure 8. Hong–Mandel squeezing as a function of scaled \( t/T_{\text{rev}} \) for initial states (a) \(|\alpha\rangle\) and (b) \(|\alpha, 1\rangle\) with \( \alpha = 1 \). The solid (dashed) line corresponds to the Kerr (respectively cubic) Hamiltonian. The horizontal line at 0.75 denotes the value below which the state is squeezed.

\( C_l = \frac{k!l!}{(k + l + 1)!\sqrt{2^{k+l}}} \)  

(20)

This form is readily amenable to numerical computations. We therefore need to consider \((k + l + 1)\) values of the tomogram variable \( \theta \) in order to calculate a moment of order \((k + l)\). In a single tomogram, this amounts to using \((k + l + 1)\) probability distributions \( w(X_\theta) \) corresponding to these chosen values of \( \theta \), in order to calculate the squeezing parameter \( D_\nu \) from the tomogram without resorting to detailed state reconstruction. As the system evolves in time, the extent of squeezing at various instants are determined from the instantaneous tomograms in this manner.

For the squeezed vacuum, \(|\alpha\rangle\) and \(|\alpha, 1\rangle\) we have verified that the variance and hence the Hong–Mandel (equivalently Hillery type squeezing) properties inferred from tomograms are in excellent agreement with corresponding results obtained analytically by calculating the variance from the state. We have also computed \((\Delta x)^2\) (equivalently the higher-order Hong–Mandel squeezing parameter) directly from the tomogram for initial states \(|\alpha\rangle\) and \(|\alpha, 1\rangle\) evolving under the Kerr Hamiltonian and the cubic Hamiltonian \(a^\dagger a^3\), setting \( \alpha = 1 \) in both cases without loss of generality (figures 8(a) and (b)). From these figures it is evident that independent of the precise nature of the initial field state \(\langle (\Delta x)^2 \rangle\) oscillates more rapidly in the case of the cubic Hamiltonian compared to the Kerr Hamiltonian. Earlier these squeezing properties have been investigated for an initial CS and PACS evolving under a Kerr Hamiltonian alone by calculating \(\langle (\Delta x)^2 \rangle\) explicitly for the state at different times [29]. Our results from the tomograms for the Kerr Hamiltonian are in excellent agreement with these.

We have also examined the manner in which the squeezing properties depend on the numerical value of \( \alpha \), directly from the relevant tomograms. For real \( \alpha \) in the range 0 to \( \sqrt{3} \) we have plotted \( D_1 \), the quantifier of the extent of squeezing versus \( \nu = |\alpha|^2 \) at the instant \( T_{\text{rev}}/2 \) for an initial CS evolving under both the Kerr and cubic Hamiltonians (figure 9(a)). It is evident that the state is squeezed for a larger range of values of \( \nu \) in the case of the Kerr Hamiltonian than for the cubic Hamiltonian. Further the extent of squeezing as measured by the numerical value of \( D_1 \) is more for a given \( \nu \) in the former case as compared to the latter.

For an initial state \(|\alpha, 1\rangle\) we have computed the extent of higher-order Hillery type squeezing from the tomograms over the range \( 0 \leq \nu \leq 3 \), at \( T_{\text{rev}}/2 \) for both Hamiltonians. While \( D_2 \) is not negative for any \( \nu \) in this range for the cubic Hamiltonian, it becomes negative for \( \nu \geq 0.8 \) approximately

Figure 9. (a) \( D_1 \) for initial \(|\alpha\rangle\), (b) \( D_2 \) for initial \(|\alpha, 1\rangle\), and (c) \( \langle (\Delta x)^4 \rangle \) for initial \(|\alpha, 1\rangle\) versus \( \nu \) for real \( \alpha \) at instant \( T_{\text{rev}}/2 \). The solid (dashed) line corresponds to the Kerr (respectively cubic) Hamiltonian. The horizontal line at 0 in (a) and (b) (3/4 in (c)) denotes the value below which the state is squeezed.
for the Kerr Hamiltonian (figure 9(b)). In contrast \((\Delta x)^4\) is never less than \(3/4\) in both cases over this range of values of \(\nu\) (figure 9(c)).

3. The double-well BEC: a tomographic approach

3.1. The revival phenomena

We now proceed to examine nonclassical effects in a BEC condensed in a double-well potential. The effective Hamiltonian of this bipartite system is \[H_{\text{bec}} = \omega_0 N_{\text{tot}} + \omega_1 (a^\dagger a - b^\dagger b) + U_{ab} N_{\text{tot}}^2 - \lambda (a^\dagger b + a b^\dagger).\] (21)\(a, a^\dagger\) and \((b, b^\dagger)\) are the boson annihilation and creation operators corresponding to the two subsystems \(A\) and \(B\) which comprise the atomic species condensed in the two wells. They satisfy \([a, a^\dagger] = 1, [b, b^\dagger] = 1\) with all other commutators vanishing. Here, \(N_{\text{tot}} = (a^\dagger a + b^\dagger b)\) and \(U_{ab}\) is the strength of the nonlinearity. It is convenient to define the effective interaction strength by the parameter \(\lambda = \sqrt{\omega_1^2 + \lambda^2}\). It is easy to see that \([H_{\text{bec}}, N_{\text{tot}}] = 0\).

Before we examine revivals and squeezing phenomena in this system by investigating relevant tomograms, we recall that we now have two quadrature operators to consider, one for each subsystem. The tomogram is therefore denoted by \(w(X_{ab}, \theta_1; X_{ab}, \theta_2)\) with subscripts 1 and 2 corresponding to subsystems \(A\) and \(B\) respectively.

Analogous to the single-mode example we now consider a pure state \(|\psi\rangle\) expanded in the Fock bases \(|m\rangle\}, \{|n\}\) corresponding to subsystems \(A\) and \(B\) respectively as \(|\psi\rangle = \sum_{m,n=0}^{\infty} c_{mn}|m; n\rangle\), where \(|m; n\rangle\) is a short-hand notation for \((|m\rangle \otimes |n\rangle)\) and \(c_{mn}\) are the expansion coefficients.

It is straightforward to extend the procedure for expressing the optical tomogram in the single-mode example in terms of Hermite polynomials to a generic bipartite system whose subsystems are infinite-dimensional. We can see that analogous to (9), in this case we have \[w(X_{ab}, \theta_1; X_{ab}, \theta_2) = \frac{\exp(-X_{ab}^2 - X_{ab}^2)}{\pi} \times \left| \sum_{m,n=0}^{\infty} c_{mn} e^{-i(m\theta_1+n\theta_2)} H_m(X_{ab}) H_n(X_{ab}) \right|^2.\] (22)

The tomograms corresponding to the two subsystems (reduced tomograms) are given by \[w_1(X_{ab}, \theta_1) = \int_{-\infty}^{\infty} w(X_{ab}, \theta_1; X_{ab}, \theta_2) dX_{ab},\] (23) for any fixed value of \(\theta_2\) and \[w_2(X_{ab}, \theta_2) = \int_{-\infty}^{\infty} w(X_{ab}, \theta_1; X_{ab}, \theta_2) dX_{ab},\] (24) for any fixed value of \(\theta_1\).

We denote by \(|\alpha_a\rangle\) (respectively \(|\alpha_b\rangle\)) the CS formed from the condensate corresponding to subsystem \(A\) (respectively \(B\)) and by \(|\alpha_{a, 1}\rangle\) (respectively \(|\alpha_{b, 1}\rangle\)) a 1-boson added CS corresponding to subsystem \(A\) (respectively \(B\)). The initial states considered by us are factored product states of the form \(|\alpha_a\rangle \otimes |\alpha_b\rangle\) (denoted by \(|\psi_{00}\rangle\), \(|\alpha_{a, 1}\rangle \otimes |\alpha_{b, 1}\rangle\) (denoted by \(|\psi_{11}\rangle\)) and \(|\alpha_{a, 1}\rangle \otimes |\alpha_{b}\rangle\) (denoted by \(|\psi_{01}\rangle\)).

The state at a subsequent time \(t\) is entangled in general and corresponding to the initial state \(|\psi_{00}\rangle\) we have \[|\psi_{00}(t)\rangle = \exp(-n/2) \sum_{p,q=0} \frac{\alpha(t)^p \beta(t)^q}{\sqrt{p!q!}} \exp(-i(t(p + q) + U_{ab}(p + q)^2))|p, q\rangle.\] (25)

Here,

\[\alpha(t) = \alpha_a \cos(\lambda t) + i \frac{\sin(\lambda t)}{\lambda} (\lambda \alpha_b - \omega_1 \alpha_a),\] (26)

\[\beta(t) = \alpha_b \cos(\lambda t) + i \frac{\sin(\lambda t)}{\lambda} (\lambda \alpha_a + \omega_1 \alpha_b),\] (27)

and \(n = (|\alpha_a|^2 + |\alpha_b|^2)\). \[|\psi_{00}(t)\rangle = \frac{1}{d_{10}} (a^\dagger \lambda_1 \cos(\lambda_1 t) + i \lambda_2 \sin(\lambda_1 t)) \times \exp(-iU_{ab}(N + 1))|\psi_{00}(t)\rangle,\] (28)

and

\[|\psi_{11}(t)\rangle = \frac{1}{d_{11}} (2\omega_1^2 a^\dagger b^\dagger + \omega_1 \lambda (a^\dagger^2 - b^\dagger^2) + \cos(2\lambda_1 t)(2\lambda_1 \lambda a^\dagger a^\dagger b^\dagger - \omega_1 \lambda (a^\dagger^2 - b^\dagger^2)) + \sin(2\lambda_1 t)(\lambda a^\dagger a^\dagger b^\dagger + b^\dagger^2)) \times \exp(-4iU_{ab}(N + 1))|\psi_{00}(t)\rangle.\] (29)

Here \(d_{10} = \lambda_1 \exp(i\omega_0 t) \sqrt{1 + |\alpha_b|^2}\) and \(d_{11} = 2\lambda_1 \exp(2i\omega_0 t) \sqrt{1 + |\alpha_a|^2} \sqrt{1 + |\alpha_b|^2}\).

We have obtained a form for the density matrix which is very useful in numerical computations. The salient steps in the calculation of the density matrix \(\rho_{m_1 m_2}\) corresponding to the factored product state \(|\alpha_{a, m_1}\rangle \otimes |\alpha_{b, m_2}\rangle\) for any positive integer value of \(m_1\) and \(m_2\) are outlined in appendix B. There we show that \(\rho_{m_1 m_2}\) can be expressed in terms of an operator \(M_{m_1 m_2}(t)\) and \(|\psi_{00}(t)\rangle\) as

\[\rho_{m_1 m_2}(t) = M_{m_1 m_2}(t)|\psi_{00}(t)\rangle \langle \psi_{00}(t)| M_{m_1 m_2}^\dagger(t).\] (30)

Defining \(p_{\text{max}} = (k + m_2 - l)\) and \(q_{\text{max}} = (l + m_1 - k)\) we
have

\[ M_{m_1,m_2}(t) = \frac{1}{\kappa} \sum_{k=0}^{p_{\max}} \sum_{p=0}^{q_{\max}} (-1)^{k-p} \binom{m_1}{k} \binom{m_2}{l} \times \left( \frac{p_{\max}}{p} \right) q_{\max} \exp(-i\lambda t(2k-l) + m_2 - m_3) \times (\cos(\gamma/2))^{k+l+p+q} (\sin(\gamma/2))^{2(m+m_2-(k+l+p+q))} \times a^{(p_{\max}+m_2-q)} b^{(q_{\max}+m_3-p)} \exp(-i\omega t(m_1+m_2)) \times \exp(-iU_{ab}(m_1+m_2)(2N_{tot} + m_1 + m_2)). \]  

(31)

Here \( \kappa = \sqrt{m_1!L_{m_1}(-|\alpha|^2)m_2!L_{m_2}(-|\bar{\alpha}|^2)} \) and \( \gamma = \cos^{-1}((\omega_1/\lambda_1)) \). \( L_m \) are the Laguerre polynomials which appear in the normalization of \( |\alpha, m\rangle \). This expression for the general density matrix can be easily seen to reduce to the forms needed in our case where the states are \( |\psi_{00}(t)\rangle, |\psi_{11}(t)\rangle \) and \( |\psi_{10}(t)\rangle \).

We now proceed to examine the revival phenomena by studying appropriate tomograms in this case. A straightforward calculation reveals that full and fractional revivals occur provided \( \omega_0 = mU_{ab}, \lambda_0 = mU_{ab}, m, m' \in \mathbb{Z}, \) and \((m+m')\) is odd. Fractional revivals occur at fractions of the revival period \( T_{rev} \) (which is equal to \( T_{tot}/2 \)). This follows (similar to the single-mode case) from the periodicity property of \( \exp(-iU_{ab}N_{tot}) \). For instance, we can easily show that an initial state \( |\psi_{00}\rangle \) evolves at an instant \( \pi/(sU_{ab}) \) (s: even integer) to

\[ |\psi_{00}(\pi/sU_{ab})\rangle = \sum_{j=0}^{s-1} a_j |\alpha(\pi/sU_{ab})e^{-i(m+2j)/s}\rangle \otimes |\beta(\pi/sU_{ab})e^{-i(m+2j)/s}\rangle, \]  

(32)

If \( s \) is an odd integer,

\[ |\psi_{00}(\pi/sU_{ab})\rangle = \sum_{j=0}^{s-1} b_j |\alpha(\pi/sU_{ab})e^{-i(m+2j+1)/s}\rangle \otimes |\beta(\pi/sU_{ab})e^{-i(m+2j+1)/s}\rangle. \]  

(33)

Note that \( \alpha(\pi/sU_{ab}) \) and \( \beta(\pi/sU_{ab}) \) are obtained from (26) and (27).

We recall that in the single-mode system the occurrence of full and fractional revivals are related to the presence of distinct strands in the tomogram. In this bipartite system however, the tomogram is a four-dimensional hypersurface. Hence we need to consider appropriate sections to identify and examine nonclassical effects. The two-dimensional section \( (X_{01} - X_{02}) \) obtained by setting \( \theta_1 \) and \( \theta_2 \) constant is a natural choice for investigating not only revival phenomena but also squeezing properties. In contrast to the single-mode case where strands appear in the tomograms these sections are characterized by ‘blobs’ at instants of fractional revivals. The number of blobs gives the number of subpackets in the wave packet.

Although \( |\alpha\rangle \) is expanded as an infinite superposition of photon number states, in practice, a numerical computation can be carried out using only a large but finite sum of these basis states. An alternative is to use truncated coherent state (TCS) [31] instead of the standard CS. The latter are defined as

\[ |\alpha_{\text{TCS}}\rangle = \frac{1}{\sqrt{N_{\text{max}}}} \sum_{n=0}^{N_{\text{max}}} \sum_{p=0}^{N_{\text{max}}} \alpha_p^n |p\rangle, \]  

(34)

where \( N_{\text{max}} \) is a sufficiently large but finite integer. We have worked with an initial state \( |\alpha_{\text{TCS}}\rangle \otimes |\alpha_{\text{TCS}}\rangle \) instead of \( |\psi_{00}\rangle \) and verified that as this state evolves under \( H_{\text{tot}} \) the revival phenomena and squeezing properties mimic that of \( |\psi_{00}\rangle \) remarkably well. Hence we have used initial states \( |\psi_{00}\rangle, |\psi_{11}\rangle \) and \( |\psi_{10}\rangle \) for our numerical computations. We set \( \theta_1 = \theta_2 = 0, \omega_0 = 10, \omega_1 = 3, \lambda = 4, U_{ab} = 1, \) and \( \alpha_0 = \alpha_0 = \sqrt{10} \) in figures 10(a)-(d) where we present tomograms corresponding to different fractional revivals for the initial state \( |\psi_{00}\rangle \). At instants \( T_{rev}/4, T_{rev}/3 \) and \( T_{rev} \) (figures 10(a), (b) and (d) respectively) we see 4, 3 and a single blob in the tomogram along with interference patterns as expected for this choice of values of \( \lambda_0 \) and \( \omega_0 \) for they satisfy the conditions \( \omega_0 = mU_{ab}, \lambda_0 = mU_{ab}, m, m' \in \mathbb{Z}, \) and \((m+m')\) is odd, necessary for the revival phenomena to occur. However in figure 10(c) corresponding to the instant \( T_{rev}/2 \) blobs are absent and we merely see interference patterns. This is primarily due to the specific choice of real values of \( \alpha_0 \) and \( \alpha_0 \) as explained above. At the instant \( T_{rev}/2 \) it follows from (25) that the state of the system can be expanded in terms of superpositions of factored products of CS corresponding to \( A \) and \( B \) as

\[ |\psi_{00}(T_{rev}/2)\rangle = \frac{(1-i)}{2} |\psi_{11}(T_{rev}/2)\rangle \otimes |\psi_{10}(T_{rev}/2)\rangle \]  

(35)

It is now straightforward to see why the interference patterns alone appear in the tomogram, as a simple calculation gives

\[ |\psi_{00}(X_{10}, X_{20})\rangle = |X_{10}, 0; X_{20}, 0|\psi_{00}(T_{rev}/2)\rangle |^2 = \frac{1}{\pi} e^{-\gamma^2 + \bar{\gamma}^2} (1 - \sin(4X_{10} + 7X_{20}/\sqrt{5})). \]  

(36)

Here \( X_{10} \) denotes \( X_{\theta_1} \) for \( \theta_1 = 0 \) and \( X_{20} \) denotes \( X_{\theta_2} \) for \( \theta_2 = 0 \). In contrast, it can be seen that if \( \alpha_0 \) and \( \alpha_0 \) were chosen to be generic complex numbers two blobs together with the interference pattern would be seen to appear at \( T_{rev}/2 \) also.

Similar results hold in the case of initial states \( |\psi_{11}\rangle \) and \( |\psi_{10}\rangle \).

3.2. Squeezing and higher-order squeezing of the condensate

The extent of Hong–Mandel squeezing is simply obtained as in the single-mode example, by calculating the central moments of the probability distribution corresponding to the quadrature. We examine two-mode squeezing by evaluating appropriate moments of the quadrature variable \( \eta = (a + a^\dagger + b + b^\dagger)/2\sqrt{2} \). These are obtained from the \( \theta_1 = \theta_2 = 0 \) section of the tomogram as the system evolves in time. These moments have also been obtained by explicit calculation of the relevant expectation values of \( \eta \) in the state of the system at different times. In both cases the initial state
considered is $|\psi_0\rangle$. In figures 11(a)–(d), the variance and 2$q$-order moments for $q = 1, 2, 3$ and 4 obtained both from the states and directly from the tomogram are plotted as a function of time. It is evident that they are in excellent agreement with each other at all instants. The horizontal line in each figure denotes the value below which the state is squeezed ($1/4, 3/16, 15/64$, and $105/256$ for $q = 1, 2, 3, 4$ respectively). For all values of $q$ considered, the state is squeezed (higher-order squeezed) in the neighborhood of revivals, and the actual magnitude at various instants is considerably sensitive to the value of $q$ as expected. We have also verified that the extent of this Hong–Mandel squeezing depends on the magnitude of $\alpha_a$ and $\alpha_b$ at various instants of time.

For numerical computation of the Hillery type higher-order squeezing recall that in the single-mode case we used the expression [28] for moments of the creation and destruction operators in terms of the tomogram and Hermite...
polynomials given by
\[
\langle a^{\dag^k}a^m \rangle = C_{kl} \sum_{m=0}^{k+1} \exp \left( -i(k-l) \frac{m\pi}{k+l+1} \right) \times \int_{-\infty}^{\infty} dX_0 \left( X_0, l \frac{m\pi}{k+l+1} \right) H_{k+l}(X_0),
\]
where
\[
C_{kl} = \frac{k!}{(k+l+1)!} \sqrt{2^{k+l+1}}.
\]
A straightforward extension to the two-mode system gives us the required expression
\[
\langle a^{\dag^k}a_t^{\dag^m}b^p b^q \rangle = C_{klm} \sum_{p=0}^{k+1} \sum_{q=0}^{l+1} \exp \left( -i(k-l)\theta_{1p} \right) \times \exp \left( -i(m-n)\theta_{2q} \right) \int_{-\infty}^{\infty} dX_0 \int_{-\infty}^{\infty} dX_2 \times w(X_0, \theta_{1p}, X_2, \theta_{2q}) H_{k+l}(X_0) H_{m+n}(X_2).
\]
Here
\[
C_{klm} = \frac{k!m!n!}{(k+l+1)!(m+n+1)!} \sqrt{2^{k+l+m+n}},
\]
\[
\theta_{1p} = \frac{p\pi}{k+l+1}, \quad \text{and} \quad \theta_{2q} = \frac{q\pi}{m+n+1}.
\]
Note that \((k+l+1)\) gives the number of two-dimensional slices of the tomogram that are required to calculate \(\langle a^{\dag^k}a_t^{\dag^m}b^p b^q \rangle\).

In figures 12(a)–(d) \(D_q(t)\) obtained both from the states and directly from the tomogram for an initial state \(\{|\alpha_0\rangle\}\) are compared and seem to be in excellent agreement. It is clear from the figures that for higher values of \(q\) there are more instants of time when higher-order squeezing occurs as expected from the fact that more cross terms involving the creation and destruction operators arise with increase in \(q\). We have also verified that both the Hong–Mandel and Hillery type squeezing and higher-order squeezing parameters obtained from tomograms and from expectation values of appropriate operators for the initial states \(|\psi_{10}\rangle\) and \(|\psi_{11}\rangle\) were equal to each other at all instants between \(t = 0\) and \(T_{rev}\).

### 3.3. Subsystem entropies from tomograms

This bipartite system provides an ideal framework for investigating a subsystem’s nonclassical properties from the tomogram. We are primarily interested in computing the quantum information entropy and entropic squeezing properties of the subsystem as it evolves in time. Further, we extract an indicator which mimics the extent of entanglement between the two condensates trapped in the double well. (Without loss of generality we have considered subsystem \(A\) by setting \(\theta_2 = 0\), and integrating over the full range of \(X_0\) to obtain \(w(X_0, \theta_1)\).)

In order to examine entropic squeezing we have further set \(\theta_1 = 0\) and investigated the manner in which \(S_0\) (the information entropy of subsystem \(A\)) given by
\[- \int_{-\infty}^{\infty} dX_0 \, w(X_0) \log w(X_0)\]
varies as the system evolves in time. (Here \(w(X_0)\) denotes \(w(X_0, \theta_1)\).) This information entropy has been plotted in figures 13(a)–(c) for initial states \(|\psi_{00}\rangle\), \(|\psi_{10}\rangle\) and \(|\psi_{11}\rangle\) respectively. The horizontal line in these figures denotes the numerical value below which entropic squeezing occurs in the quadrature considered. We see that entropic squeezing occurs close to \(t = 0\) and \(T_{rev}\). At other instants the entropy is significantly higher in the case of initial states \(|\psi_{10}\rangle\) and \(|\psi_{11}\rangle\) compared to initial ideal coherence. This feature is very prominent close to \(T_{rev}/2\). Further, comparing (b) and (c) it is clear that \(S_0\) is larger at all instants if both subsystems depart from coherence initially, compared to the case where one of them displays initial coherence.

Figures 14(a), (b) and (d) show the variation of \(S_0\) corresponding to subsystem \(A\) with \(|\alpha_0|^2\) and \(|\alpha_0|^4\), at \(T_{rev}/2\), for initial states \(|\psi_{00}\rangle\), \(|\psi_{10}\rangle\) and \(|\psi_{11}\rangle\) respectively. Figure 14(c) corresponds to the entropy of subsystem \(B\) for an initial state \(|\psi_{10}\rangle\). This facilitates comparison of the features in figures 14(b) and (c) where for the same asymmetric initial state the two subsystems examined are different. It is evident that \(S_0\) corresponding to \(A\) is not squeezed while that corresponding to \(B\) exhibits squeezing for some values of \(\alpha_0\) and \(\alpha_0^4\). The role played by the asymmetry in the initial states of the two subsystems is thus clearly brought out in these figures.
It is also clear from figures 14(a)–(d) that entropic squeezing is more if the initial states of the subsystems are coherent.

We now proceed to investigate if an indicator of entanglement can be obtained from the tomogram alone. A standard quantifier of entanglement is the subsystem von Neumann entropy (SVNE) defined as

$$S_{i} = -\sum_{k} \lambda_{k} \log \lambda_{k}$$

where $\{\lambda_{k}\}$ are the eigenvalues of the density matrix $\rho$. Quantum mutual information $I(\rho_{AB})$ for a bipartite system with density matrix $\rho_{AB}$ and reduced density matrices $\rho_{i}(i = A, B)$ is defined as [32],

$$I(\rho_{AB}) = \sigma(\rho_{A}) + \sigma(\rho_{B}) - \sigma(\rho_{AB}).$$

Computation of SVNE involves $\text{Tr}(\rho \log \rho) = \int_{0}^{\infty} dX \int_{0}^{\infty} dX' \langle X|\rho|X'\rangle \langle X'|\log(\rho)|X\rangle$ where $|X\rangle$ are eigenvectors of the position operator. Elements of the form $\langle X|\rho|X\rangle$ are needed for computing other standard measures of entanglement as well. In our approach, we are limited by the fact that a generic tomogram is given by $|\psi_{Q}\rangle$, and therefore, for instance for $\theta = 0$, it does not carry direct information on $\langle X|\rho|X\rangle$. A natural question that arises is whether we can exploit the tomographic properties of a quorum (obtained by choosing several values of $\theta$) to at least identify an indicator of entanglement which mimics the standard measures of entanglement.
In what follows we outline our procedure for obtaining an indicator which is similar in form to quantum mutual information in terms of quantities which are accessible from the tomogram. However, it is to be noted that it does not satisfy the requirements of an entanglement measure [32, 33].

We recall that $w_1$ is independent of the value of $\theta_2$ and hence the information entropy
\[ S(\theta_1) = -\int_{-\infty}^{\infty} d\theta_1 w_1 \log w_1, \]
is also independent of the value of $\theta_2$. A similar statement holds for $S(\theta_2)$. The notation in [32, 34] is analogous in form to quantum mutual information (40).

We can now define, following the notation in $S(\theta_1 : \theta_2) = S(\theta_1) + S(\theta_2) - S(\theta_1, \theta_2)$,

which is analogous in form to quantum mutual information (40). Here,
\[ S(\theta_1, \theta_2) = -\int_{-\infty}^{\infty} d\theta_1 \int_{-\infty}^{\infty} d\theta_2 \log w(X_{\theta_1}, \theta_1; X_{\theta_2}, \theta_2). \]

$S(\theta_1 : \theta_2)$ is averaged over $\theta_1$ and $\theta_2$ (the quorum) to get $S(A : B)$ which is seen to closely mimic SVNE (figure 15) and quantum mutual information (figure 16). We have numerically verified that with as few as five values of $\theta_i$ ($i = 1, 2$), which are equally spaced over the interval $[0, \pi]$, and hence 25 different pairs $(\theta_1, \theta_2)$, the temporal behavior of $S(A : B)$ (scaled by 11) effectively captures the qualitative behavior of quantum mutual information $I(\rho_{AB})$. We have compared $S(A : B)$ with SVNE and also the subsystem linear entropy which is defined as, $\delta_A = 1 - \text{Tr}(\rho_A^2) = 1 - \sum_k \lambda_k^2$. We see that the indicator mirrors the temporal behavior of both the entanglement measures as well. This is evident from figures 15(a)–(c) and 16 where we have set $\alpha_a = \alpha_b = 1$ and considered the three initial states $|\psi_{00}\rangle$, $|\psi_{10}\rangle$ and $|\psi_{11}\rangle$. We emphasize that while we could quantify the extent of squeezing, higher-order squeezing and entropic squeezing from tomograms alone, $S(A : B)$ is merely an indicator and does not satisfy the requirements of a measure of entanglement.

In this paper we have established how tomograms can be exploited to identify and characterize a variety of nonclassical effects such as the wave packet revival phenomena and squeezing and higher-order squeezing in both single-mode and bipartite systems. While a simple relation has been shown to exist between the number of strands in tomogram patterns and the nature of fractional revivals when a single-mode radiation field propagates in a Kerr medium [6] our investigations reveal that this no longer holds even in single-mode systems which display super-revivals during temporal evolution. The role played by decoherence due to amplitude decay and phase damping of the state has also been discussed. We have also analyzed the revival phenomena in bipartite systems such as the double-well BEC evolving in time, solely from tomograms. We have obtained the extent of squeezing, higher-order Hong–Mandel and Hillery type squeezing for both systems from tomograms. We have also investigated entropic squeezing and the extent of bipartite entanglement in detail from the tomograms alone in the case of the double-well BEC. For pure states, we have identified an indicator which can be obtained from the tomograms, that mimics the quantum mutual information at all instants of time. In the single-mode example we have considered initial states which are either CS or PACS and in the bipartite system the initial states are factored products of CS and states which marginally depart from macroscopic coherence.

We have also undertaken similar investigations on the Jaynes–Cummings model of field-atom interactions. This helps in identifying how tomograms of systems with a few
We now consider the special case when 

Using \( F_{kl} \) and \( (k - s)F[l - s] \), it follows that the expansion of \( F \) above is valid.

We outline the procedure [28] for obtaining normal-ordered moments for infinite dimensional single-mode systems from optical tomograms. We expand any normal-ordered operator \( F \) in the form

\[
F = \sum_{k,l,m,n=0}^\infty F_{kl}a^k a^\dagger_l.
\] (A.1)

Here the coefficients

\[
F_{kl} = \sum_{s=0}^{[k,l]} \frac{(-1)^s}{s! \sqrt{(k - s)!(l - s)!}} \langle k - s\mid F\mid l - s \rangle,
\] (A.2)

and \( [k, l] \) is used to denote min(\( k, l \)). (This can be verified by first obtaining \( \langle p\mid F\mid q \rangle \) using (A.1) and then substituting it in (A.2) to get

\[
F_{kl} = \sum_{s=0}^{[k,l]} \sum_{u=0}^{[k-l,s]} \frac{(-1)^s}{s! u!} F_{k-s\mid u\mid l-s-u}.
\] (A.3)

Now replacing \( u \) with \( u' = s + u \), changing the order of the sums, and using

\[
\sum_{s=0}^{[k,l]} \frac{(-1)^s}{s! \sqrt{(k - s)!(l - s)!}} = \delta_{k,l},
\] (A.4)

it follows that the expansion of \( F \) above is valid.

We write the projection operator

\[
|k\rangle\langle l| = (k!)^{-1/2} \sum_{u=0}^{[k,l]} \frac{(-1)^s}{u!} a^u a^\dagger_{l-u}
\] (A.5)

Using \( F = \sum_{k,l=0}^\infty \langle k\mid F\rangle \langle l| \) and (A.5), we get

\[
F = \sum_{k,l=0}^\infty A_{k,l} Tr(a^k a^\dagger_l).
\] (A.6)

Here

\[
A_{k,l} = \sum_{s=0}^{[k,l]} \frac{(-1)^s}{s! \sqrt{(k - s)!(l - s)!}} \langle k - s\mid k \rangle.
\] (A.7)

We now consider the special case when \( F \) is the density operator \( \rho \). Using (A.6) and (A.7),

\[
\langle X_\theta, \theta\mid m \rangle \langle n\mid X_\theta, \theta \rangle = \frac{e^{-x^2} e^{-i(m-n)\theta}}{\sqrt{\pi} \sqrt{m!n!} \sqrt{2m+n}} H_m(X_\theta) H_n(X_\theta),
\] (A.8)

and the following property of the Hermite polynomials

\[
H_{k+l}(X_\theta) = \sum_{s=0}^{[k,l]} (-1)^s k! l! H_k(X_\theta) H_l(X_\theta) \frac{(k-s)!(l-s)!}{s!}.
\] (A.9)

Using the orthonormality property of the Hermite polynomials together with the expression

\[
\sum_{u=0}^{n} \exp(2\pi ius/(n + 1)) = (n + 1) \delta_{u0},
\] (A.11)

in (A.10) gives

\[
\langle a^{k+l} \rangle = Tr(a^k a^\dagger_l \rho)
\]

\[
= C_{kl} \sum_{m=0}^{k+l} \exp \left( -i(k-l) \left( \frac{m \pi}{k+l+1} \right) \right) \int_{-\infty}^{\infty} dX_\theta w(X_\theta, \frac{m \pi}{k+l+1}) H_{k+l}(X_\theta).
\] (A.12)

where

\[
C_{kl} = \frac{k! l!}{(k+l+1)! \sqrt{2^{k+l}}}.
\] (A.13)

This procedure can be extended in a straightforward manner to multimode systems.

### Appendix B

#### B.1. Numerical computation of the two-mode density matrix

We outline the essential steps in computing the density matrix \( \rho_{m_1 m_2}(t) \) of the double-well BEC, for initial states \( |\psi_{m_1 m_2}\rangle = |\alpha_1, m_1 \rangle \otimes |\alpha_2, m_2 \rangle \) with Hamiltonian \( H_{\text{BEC}} \).

The procedure for obtaining \( \rho_{00}(t) \) the time-evolved density matrix corresponding to the initial state \( |\alpha_1 \rangle \otimes |\alpha_2 \rangle \) is outlined in [30]. We obtain \( \rho_{m_1 m_2}(t) \) from \( \rho_{00}(t) \) through appropriate transformations. We first write

\[
|\psi_{m_1 m_2}\rangle(t) = M_{m_1 m_2}(t)|\psi_{00}\rangle(t),
\] (B.1)

where

\[
M_{m_1 m_2}(t) = \frac{1}{\kappa} \exp(-iH_{\text{BEC}} t) a^m b^{m^*} \exp(iH_{\text{BEC}} t),
\] (B.2)

and \( \kappa \) is given in terms of Laguerre polynomials as \( \sqrt{m_1 L_{m_1}(-|\alpha_1|^2) m_2 L_{m_2}(-|\alpha_2|^2)} \). In order to recast \( M_{m_1 m_2}(t) \) in a simpler form we introduce the operator
This expression can now be simplified using the following identities which can be obtained in a straightforward manner by using the Baker–Hausdorff lemma.

\[
V' a' V = a' \cos(\gamma/2) + b' \sin(\gamma/2),
\]

\[
V' b' V = b' \cos(\gamma/2) - a' \sin(\gamma/2),
\]

\[
V' a' V = a' \cos(\gamma/2) - b' \sin(\gamma/2),
\]

\[
V' b' V = b' \cos(\gamma/2) + a' \sin(\gamma/2),
\]

\[
\exp(-i\lambda(t)(a' - b' b))a' \rho b' \exp(i\lambda(t)(a' - b' b)) = a' \rho b' \exp(-i(p - q)\lambda(t)),
\]

\[
\exp(-i\omega(t)N_{\text{tot}})a' \rho b' \exp(i\omega(t)N_{\text{tot}}) = a' \rho b' \exp(-i(p + q)\omega(t)),
\]

\[
\exp(-iU_{ab}N_{\text{tot}}^2)a' \rho b' \exp(iU_{ab}N_{\text{tot}}^2) = a' \rho b' \exp(-iU_{ab}(p + q)2N_{\text{tot}} + p + q)).
\]

Further, using binomial expansions for the two commuting operators \(a'\) and \(b'\) and defining \(p_{\text{max}} = (k + m_2 - l)\) and \(q_{\text{max}} = (l + m_1 - k)\), we arrive at the following simplified expression for the density matrix \(M_{m_1 m_2}(t)\).

\[
M_{m_1 m_2}(t) = \frac{1}{\kappa} \sum_{k=0}^{m_1} \sum_{l=0}^{m_2} \sum_{j=0}^{p_{\text{max}}} \sum_{p=0}^{q_{\text{max}}} (-1)^{k-j-p} \binom{m_1}{k} \binom{m_2}{l} \binom{p_{\text{max}}}{p} \binom{q_{\text{max}}}{q} 
\]

\[
\exp(-i\lambda(t)(2(k - l) + m_2 - m_1))
\]

\[
\cos(\gamma/2)^{(k+1+p+q)} \sin(\gamma/2)^{(2(m_1+m_2)-(k+1+p+q))}
\]

\[
a'(1-\omega(t)m_2 + \omega(t)m_1)\exp(-i\omega(t)(m_1 + m_2))
\]

\[
\times \exp(-iU_{ab}(m_1 + m_2)(2N_{\text{tot}} + m_1 + m_2)).
\]

The density matrix can now be expressed in terms of \(M_{m_1 m_2}(t)\) and \(\rho_{00}(t)\) as

\[
\rho_{m_1 m_2}(t) = M_{m_1 m_2}(t) \rho_{00}(t) M_{m_1 m_2}^\dagger(t).
\]