Structured FRW universe leads to acceleration: a non-perturbative approach

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(Dated: March 20, 2022)

We propose a model universe in the matter dominated phase described by a FRW background with local inhomogeneities, like our local patch, grown out of the primordial fluctuations. Our sub-horizon local patch consisting of different structures is approximated as an inhomogeneous cosmic fluid described by a LTB metric embedded in a background FRW universe, in which the observer could be located anywhere. Within the exact general relativistic formulation, the junction conditions for the only possible matching without a thin shell at the boundary, neglected so far in the literature, constrains the model in such a way that the luminosity distance-red shift relation mimics a FRW universe with dark energy. Therefore, the dimming of SNIa is accounted for in such a structured FRW universe. We have also calculated the exact general relativistic backreaction term and shown how it influences the global Hubble parameter and the effective density of the cosmic fluid By using an exact formulation of the general relativistic dynamics of structures in a homogeneous universe, the claim is therefore stressed that the backreaction of cosmological perturbations leads to an apparent dimming of the cosmological distances.

PACS numbers: 98.80.cq, 95.35.+d, 4.62.+v

I. INTRODUCTION

The Copernican turn in cosmology, which happened after the identification of two different population of Cepheids in the mid fifties of the last century, was a milestone for the acceptance of homogeneity of the universe and the FRW metric as the metric of our universe. Although the inhomogeneity of the local structure of the universe had been observed, but all the observational data had led to the acceptance of the homogeneity at scales larger than some hundred mega parsecs, and its generalization to all of the universe. The introduction of a homogeneous cosmic fluid describing the matter content of the universe was the natural theoretical formulation of this observational finding. So far there has been excellent agreement between theory and observation within the limit of observational precision. That is why FRW universe had been accepted as the model universe on which the interpretation of all the observational data is based, albeit many observations in the last decade show us explicitly the inhomogeneity of the cosmic structures on the cosmological scales of our surrounding.

The precision cosmology is, however, already so far developed that we can not ignore any more the local inhomogeneity of the universe, and the theoretical concept of the homogeneous cosmic fluid, i.e. the basic concept of the FRW models, has to be modified. On the other hand, the well established successes of standard FRW cosmology can not be abandoned so easily.

The deviation from the standard homogeneous cosmic fluid in the matter dominated phase of the universe in our proximity should be reflected in the data from cosmological objects. Now, recent observational data on SNIa imply a larger distance to supernovae than predicted by the conventional FRW universe. Different factors, such as evolution effects, dust absorption, gravitational lensing, or dynamics of the universe, may have led to this dimming of SNIa’s. Detailed studies show that all these factors have negligible effects, except the dynamics, which had led to the "term acceleration of the universe" as the sole model-independent interpretation of the data. Of course, the familiar interpretation is within the context of the standard FRW cosmology, as all the theoretical ingredients, such as cosmological constant and all the conceivable equation of states derivable from a scalar field been developed before. Hence, the concept of dark energy, in addition to the baryonic and dark matter content of the universe, for the interpretation of the dimming of SNIa within the FRW universe is still the prevailing hypotheses. Since then, papers related to dark energy are increasing in torrent, many different models have been developed, and many terms related to it like quintessence, k-essence, spin-essence, phantom, mirage, and so on have been coined in the last years, many of them violating basic physics intuition. Leaving aside modified gravity theories in 4
Incorporating the large scale structures of the universe into a model universe has led different authors to look at the inhomogeneous models and its consequences for the cosmological parameters. Even long before the new SNIa data, the possibility of distinguishing observationally the homogeneous from inhomogeneous past light-cone had been investigated by Partovi and Mashhoon [18]. Just after the release of the new data on SNIa, Celerier [19] published an interesting paper questioning the dark energy interpretation of the acceleration of the universe. She showed, using a Lemaitre-Tolman-Bondi (LTB) inhomogeneous solution of the Einstein equations [20] and the corresponding luminosity distance relation in it, that large scale inhomogeneities may mimic a cosmological constant, say dark energy, in a homogeneous universe. Tomita [21, 22], in an attempt to explain the acceleration, uses an inhomogeneous model universe consisting of an underdense FRW region, a void of about 200 Mpc extend, immersed in the FRW bulk, and saw hints of acceleration due to the underdensity of the void, without answering the question of the effect of the thin shell supporting energy and momentum needed at the boundary of the bulk. Giovannini [23], along the same line as the authors of [16, 17], assuming an arbitrary inhomogeneous metric, shows that- within the limit of some approximations- the deceleration parameter of a matter dominated universe is always positive. The no-go theorem adopted in these references is repeated in other papers dealing with a LTB inhomogeneous universe. Giovannini then concludes that the claimed acceleration must be the result of extrapolation of a specific solution in a regime where both the perturbative expansion breaks down and the constraints are violated. Wiltshire [24], generalizing Tomita’s model, assumes an inhomogeneous underdense model within a bulk FRW universe, without going into the dynamics of matching of two different solutions of the Einstein equations and finds ‘promising’ consequences. Moffat [25] also assumes an inhomogeneous LTB void within a FRW universe where the matching is placed at about $z = 20$, not going into the detail of the dynamics of Einstein equation for such a matching. The result is again the possibility of explaining the acceleration without any need of a dark energy. A1nes et. al. [26], assuming again a model of an inhomogeneous void immersed smoothly in a FRW universe, go a step further considering different models according to the distribution of the density in the void and looking at their cosmological consequences such as luminosity distance relation, and the position of the first CMB peak. They too, ignoring the details of the dynamics of the Einstein equations for such a matching, come to the conclusion that in some of the models proposed it maybe possible to explain the acceleration without any use of a dark energy. Bolejko [27], using again a void embedded in a FRW universe, and assuming 6 different models for the density distribution, comes to the result that there is no realistic model which could explain the observed dimming of supernovae without a cosmological constant, not noting the impossibility of such a junction. Models of an inhomogeneous bubble embedded in a FRW have the advantage of admitting, in principle, an exact approach, remedying the main shortcoming of the perturbative back-action approach as has been mentioned by many authors. But, as mentioned before, all the papers published so far neglect one important point: is it at all possible, within general relativity, to have an underdense spherical bubble embedded in a bulk FRW universe? A negative answer would catapult all these analyses into the range of approximative approaches with the result that all the criticism published so far on the back-reaction approaches will apply to these models too. We have analyzed this question in detail some years ago [28]. It has been shown there explicitly that the matching of an underdense LTB bubble to a FRW universe is not possible, except for the case of having a thin shell supporting energy and momentum on the boundary to the FRW background. Apart from lacking astrophysical indications of such a spherical thin mass condensation around us at about $z = 0.46$, i.e. the boundary of transition from the accelerating to decelerating epochs of the universe [29], we do not know of any other theoretical indication for it from the large scale studies of the universe. On the contrary, an overdense region surrounded by a void as the result of the evolution of the primordial perturbation is the most expected one. The case of the great wall being considered as a massive thin shell around a putative local void [30], even if it turns out to be observationally viable, is just a perturbation within the cosmic fluid of our local patch we are going to consider which extends up to about $z = 0.46$. Here we report on a realistic exact GR model of the universe, consisting of inhomogeneous patches embedded smoothly in a FRW background without any thin shell to be required. We suggest minimal changes to the FRW universe incorporating the inhomogeneity in our cosmic neighborhood. The theoretical description of the real inhomogeneity of the structures in our proximity is modelled again by a cosmic fluid which, contrary to the FRW case, is inhomogeneous. We will have to constrain the infinite number of degrees of freedom of such a cosmic fluid
model to the smallest possible number to match the observations, otherwise the complexity of the model makes it useless for observational cosmology. Therefore, our main task is to provide a more realistic model universe capable of matching the precision cosmology of today and its future developments. Much of the work done so far within the FRW models have to be repeated now to incorporate the inhomogeneity of the local patches and provide a new theoretical framework for interpretation of the observational data. The structured FRW (SFRW) model we are proposing is to be considered as a first step in that direction.

The local patches grown out of the primordial perturbations and their backreactions to the homogeneous background are modelled exactly as a truncated flat LTB manifold embedded in a FRW universe from which a sphere of the same extent as the LTB patch is removed. It turns out that as a result of the junction conditions the mean density of any such inhomogeneous patch, with over- and under-dense regions, has to be equal to the density of the FRW bulk. Therefore, the Copernican principle is in no way violated and we are led to a structured universe where the local patches are distributed homogeneously in the bulk and having the same mass as a local FRW patch would have, accounting for all the structures we see grown out of the primordial perturbation within a FRW universe. The analysis of the luminosity distance relation in our structured FRW model shows explicitly a dimming of objects within a patch relative to what it would be inferred from a standard FRW universe. The so-called 'bang time function' \[19, 27, 30\], which is an integration function in LTB bubble models, is interpreted very naturally as the time of nucleation of mass condensation in a patch and its behavior is fixed through the junction conditions at the transition epoch.

II. THE STRUCTURED FRW (SFRW) MODEL

The assumed homogeneity of the universe is at the scales greater than some hundred mega parsecs. In regions below that we have different structures showing the inhomogeneity in the smaller scales. Up to now, we have always interpreted the astrophysical data, at any scale whatsoever, on the basis of the assumption that the matter content of the universe is best modelled through the homogeneous cosmic fluid, which is achieved over some large smoothing scale\[2\], and always tacitly assumed that the fine grained details are smoothed out and ignored. The simplicity of FRW universe, reducing the infinite degrees of freedom of the real universe to just one scale factor, has been the compelling reason for all the data interpretations so far.

Now, let us go a step back in the smoothing process and make our model more realistic to see if any substantial differences in the interpretation of data may result. We remove a spherical patch, resembling our local neighborhood in the universe up to about \(z = 0.5\)\[29\], out of the FRW universe model and replace it by a simple inhomogeneous spherical mass distribution. The simplest way of modeling our local patch is to use a LTB flat metric, without any cosmological parameter. Our local patch is embedded in a flat FRW universe. This does not contradict the cosmological principle, nor is it a reaction to Copernican turn, as the universe is full of different patches like ours distributed homogeneously in the background FRW due to the existence of primordial density fluctuations. Therefore, the result of interpretation of the observed large scale data maybe the same in any other patch within the FRW universe. As our local patch is the consequence of a primordial perturbation, or mass condensation, within a FRW background, it must be matched to a FRW metric smoothly. Otherwise, we have either to change our gravity theory, abandon the exactness of our calculation and accept the perturbative nature of it, or assume a thin mass condensation at the boundary of our local patch to the FRW and looking for a mass deficit or surplus in the patch in comparison to the mass density of the background universe. None of these alternatives is desired or observed. Having this model in mind, we look for the exact dynamic of such a model. Note that we are defining in principle a FRW universe having structured patches within it distributed homogeneously and isotropically, although each patch is inhomogeneous. Our SFRW model could be considered as a generalization of the idea of the Swiss cheese model, in which the subhorizon inhomogeneous patches are distributed homogeneously and we are living somewhere in one of the patches. The model is exact in the sense of being an exact solution of the Einstein equations.
A. Dynamics of a patch within the structured FRW universe

Our local inhomogeneous matter dominated patch is modelled as an inhomogeneous spherically symmetric manifold of comoving radius \( r_b = L \) with an arbitrary density profile glued to a homogeneous pressure-free FRW background from which a sphere of matter of the same radius is removed. Our calculation is based on an exact general relativistic formulation of gluing manifolds. This may be considered as a generalization of the work done by Olson and Silk\cite{31} within the Newtonian dynamics where there is no need to be cautious about the matching conditions. According to a theorem in general relativity, there is no solution of Einstein equations representing a time-dependent fluid sphere with finite radius having an equation of state in the form \( \rho = \rho(p) \)\cite{32}. Inhomogeneous dust fluid defined by \( p = 0 \) representing the matter dominated phase of our local patch of the universe does not violate this theorem, contrary to the radiation dominated phase defined by the equation of state \( \rho = 3p \). Therefore, we may continue with our model, and take a finite matter dominated patch of the universe represented by an inhomogeneous dust cosmic fluid obeying \( p = 0 \).

Our spherical inhomogeneous patch containing dust matter is represented by a LTB metric embedded in a pressure-free FRW background universe with the uniform density \( \rho_0 \). We choose the LTB metric to be written in the synchronous comoving coordinates in the form\cite{28}:

\[
 ds^2 = -dt^2 + \frac{R^2}{1 + 2E(r)}dr^2 + R^2(r, t)(d\theta^2 + \sin^2 \theta d\phi^2).
\] (1)

The overdot and prime will thereafter denote partial differentiation with respect to \( t \) and \( r \), respectively, and \( E(r) \) is an arbitrary real function such that \( E(r) > -\frac{1}{2} \). Then the corresponding Einstein equations turn out to be

\[
 \dot{R}^2(r, t) = 2E(r) + \frac{2M(r)}{R},
\] (2)

\[
 4\pi \rho(r, t) = \frac{M'(r)}{R^2 R'}.
\] (3)

The density \( \rho(r, t) \) is in general an arbitrary function of \( r \) and \( t \), and the integration time-independent function \( M(r) \) is defined by

\[
 M(r) = 4\pi \int_0^{R(r, t)} \rho(r, t)R^2dR = \frac{4\pi}{3} \overline{\rho}(r, t)R^3,
\] (4)

where \( \overline{\rho} \), as a function of \( r \) and \( t \), is the average density up to the radius \( R(r, t) \). Furthermore, in order to avoid shell crossing of dust matter during their radial motion, we must have \( R'(r, t) > 0 \). Solutions to the above equations show that an overdense spherical inhomogeneity with \( E(r) < 0 \) within \( R \) evolves just like a closed universe, namely it reaches to a maximum radius at a certain time, then the expansion ceases and undergoes a gravitational collapse so that a bound object forms in such a way. In other words, \( E(r) \) plays the role of the curvature scalar \( k \) in the FRW universe.

For the sake of simplicity and comparison to the astrophysical parameters, we take the solution of the dynamical equation (2) which corresponds to \( E(r) = 0 \), the so-called flat or parabolic case. The solution can be written in the form\cite{19,27,30}:

\[
 R(r, t) = \left( \frac{9M(r)}{2} \right)^{\frac{1}{3}}(t - t_n(r))^\frac{2}{3},
\] (5)

where \( t_n(r) \) is an arbitrary function of \( r \) appearing as an integration 'constant'. This arbitrary function has puzzled different authors who give it the name of 'bang time function' corresponding to the big bang singularity\cite{19,27,30}. It has, however, a simple astrophysical meaning within our structured FRW universe. As \( R(r, t) \) is playing the role of radius of our local patch, the time \( t = t_n \), leading to \( R = 0 \), means the time of onset of the mass condensation or nucleation within the homogeneous cosmic fluid. That is why we prefer to use the subscript \( n \) for it indicating the time of nucleation. In the next section we will see its crucial role in the luminosity distance relation and the impact of the junction conditions on its running.

The metric\cite{11} can also be written in a form similar to the Robertson-Walker metric. The definition

\[
 a(t, r) = \frac{R(t, r)}{r}, \quad k(r) = -\frac{2E(r)}{r^2}
\]
brings the metric into the form

\[ ds^2 = -dt^2 + a^2 \left[ \left(1 + \frac{a'}{a} \right)^2 \frac{dr^2}{1 - k(r)r^2} + r^2 d\Omega^2 \right]. \]

For a homogeneous universe, \( a \) and \( k \) don’t depend on \( r \) and we get the familiar Robertson-Walker metric. In our SFRW universe, the metric outside the inhomogeneous patch, is Robertson-Walker again.

The corresponding field equations and the solution for the parabolic case \( E(r) = 0 \) can be written in the following familiar form:

\[ \left( \frac{a}{a} \right) = 1 \frac{\rho_c(r)}{3 a^3} - k a^2, \]

where we have introduced \( \rho_c(r) = \frac{\dot{M}(r)}{r^5} \). These are very similar to the familiar Friedmann equations, except for the \( r \)-dependence of the different quantities. The solution (5) for the parabolic case can now be written in the form:

\[ a(r) = \left( \frac{3}{4} \rho_c(r) \right)^{\frac{1}{2}} (t - t_n(r))^{\frac{3}{2}}. \]

Now, let us denote by \( \Sigma \) the (2+1)-dimensional timelike boundary of the two distinct spherically symmetric regions glued together. We will show that the gluing a LTB patch to the background FRW is in general not possible except for \( \Sigma \) being a singular hypersurface carrying energy and momentum. To this end we write down the appropriate Israel junction equation on \( \Sigma \) [28]. It reads:

\[ \epsilon_{\text{in}} \sqrt{1 + \left( \frac{dR}{d\tau} \right)^2} - \frac{8\pi \rho_b}{3} R^2 - \epsilon_{\text{out}} \sqrt{1 + \left( \frac{dR}{d\tau} \right)^2} - \frac{8\pi \rho_b}{3} R^2 \stackrel{\Sigma}{=} 4\pi \sigma R, \]

where \( \Sigma \) means that all functions on both sides of the equality are evaluated on \( \Sigma \), \( \tau \) is the proper time of the comoving observer on \( \Sigma \), \( \sigma \) is the surface energy density of the boundary \( \Sigma \), \( \rho_b \) is the density at the boundary being just a function of time and equal to the density of the background FRW universe, and \( \rho_b \) is the LTB mean density defined by the Eq. (4) evaluated at the boundary \( \Sigma \), i.e. the mean density of the local patch. The sign functions are fixed according to the convention \( \epsilon_{\text{in}}(\epsilon_{\text{out}}) = +1 \) for \( R \) increasing in the outward normal direction to \( \Sigma \), while \( \epsilon_{\text{in}}(\epsilon_{\text{out}}) = -1 \) for decreasing \( R \). For the case we are considering with flat FRW metric, it can be shown that [28, 33]

\[ \epsilon_{\text{out}} = \text{sgn} \left( 1 + v_b H_b R_b \right), \]

where \( H_b \) is the Hubble parameter of the bulk, and \( v_b \) is the radial peculiar velocity of \( \Sigma \) relative to the bulk.

**B. Constraints from the junction**

Now, without going into the detail discussion (see [28]), we may easily infer from the junction equation (10) that, in general, the matching is only possible for \( \sigma \neq 0 \), i.e. if a thin layer is formed on the boundary of the mass condensation where our local patch joins the background FRW. This is a mathematical possibility not observed yet, so we are going to discard it. The only exception is the case where \( \epsilon_{\text{in}} = \epsilon_{\text{out}} \) and

\[ \rho_b := \mathcal{P}_b = \rho_b, \]

We, therefore, are left with the only case imposed by the dynamics of Einstein equations in which the mean density of our local patch is exactly equal to the density of the background FRW universe: a desired exact dynamical result reflecting the validity of the cosmological principle at large, contrary to the concerns of many authors assuming an underdense LTB region [18, 21, 27]. This fact can be seen as a concrete example of the integral constraint in perturbing an energy-momentum tensor seen first by Traschen [34]. Each nucleated patch within the FRW universe have the same average mass density as the bulk. Being distributed statistically, the patches does not have to destroy the homogeneity of the bulk. The total mass in a local patch, being equal to the background density times the volume of the patch, is distributed individually due to its self-gravity, leading to overdense structures and voids to compensate it. Assuming again the matter inside each patch to be smoothed out in the form of an inhomogeneous cosmic fluid, we expect it to be overdense at the center decreasing smoothly to an underdense compensation region, a void, up to the point of matching to the background. We, therefore, have to expect voids around us, as it is indicated in different
observations. Other cases is, however, possible depending on the functional form of $t_n$, being only constrained by the mean density. However, more general cases are conceivable, such as elliptic and hyperbolic cases in which $E(r) \neq 0$, even if the background is a flat FRW, which are outside the scope of this paper.

The equality of both sign functions is also an astrophysically trivial result. We know already from the technology of gluing manifolds that the sign functions are, for static metrics, related to the topology of the matching. In the case of non-static metrics, like FRW and LTB, the interpretation is more complicated. Fortunately we are left with only one relatively trivial choice $\epsilon_{\text{out}} = \epsilon_{\text{in}} = +1$. In fact, the case $-1$ is also possible, but it can easily be seen that it is isometric to the case where both sign functions are positive. Within our model of an expanding FRW background from which a matter sphere is removed and replaced by a part of a LTB metric everything is topologically simple and is translated in the mathematical language as the positivity of both sign functions. This, again, is a desired astrophysical result coming out of the dynamics of Einstein equations. It remains to check the condition (7), which can now be written in the form

$$(1 + v_b H_0 R_b) > 0,$$

where all quantities are to be taken at the boundary $\Sigma$. For a sub-horizon local patch, as it is assumed in our model, the inequality is valid for all values of the peculiar velocity $v_b$.

We are, therefore, left with a structured FRW universe for which the mean density of each local patch is equal to the FRW bulk density. The density distribution within a patch must be such that the overdensity of structures are compensated by voids. Of course, for the actual mass distribution, taking into account the fine structure of the patch including the substructures, we have to rely on the overall observations and the matter power spectrum.

### III. LUMINOSITY DISTANCE-RED SHIFT RELATION AND ITS ASTROPHYSICAL CONSEQUENCES

The luminosity distance in an LTB universe has been considered in many papers and also in a perturbed FRW universe. We will follow the paper as it is most suited to our purpose of comparing to observational data.

#### A. Luminosity distance for small $z$ values

The luminosity distance $d_L$ is assumed to be an explicit function of the red shift $z$, remembering its implicit dependence on the parameters of the model. According to the recent observations we expect the dimming of the cosmological objects at values of $z \approx 0.5 < 1$. Besides this, just to check the possibility of explaining the dimming of cosmological objects as a consequence of the local inhomogeneities, we restrict our calculation in this paper to small $z$ values. Therefore, we may legitimately use the Taylor expansion around $z = 0$:

$$d_L(z) = d_1 z + d_2 z^2 + d_3 z^3 + O(z^4).$$

In the FRW universe the coefficients of the expansion are given by

$$d_1 = \frac{1}{H_0},$$

$$d_2 = \frac{1}{4H_0} (2 - \Omega_m + 2\Omega_\Lambda),$$

$$d_3 = \frac{1}{8H_0} (-2\Omega_m - 4\Omega_\Lambda - 4\Omega_m\Omega_\Lambda + \Omega_m^2 + 4\Omega_\Lambda^2),$$

where $H_0$ is the present time Hubble parameter and $\Omega_m$ and $\Omega_\Lambda$ are the familiar mass density- and cosmological constant-density parameters. A straightforward calculation along the familiar line in FRW universe yields the coefficients of expansion in Eq.(10)
for a LTB flat model\[16]:

\[
d_1 = \frac{1}{H_0},
\]

\[
d_2 = \frac{1}{4H_0} \left( 1 - 6 \frac{t_n'}{(\rho_c)^{\frac{1}{2}}t^\frac{3}{2}} \right),
\]

\[
d_3 = \frac{1}{8H_0} \left( -1 + 4 \frac{t_n'}{(\rho_c)^{\frac{1}{2}}t^\frac{3}{2}} + 7 \frac{t_n''}{(\rho_c)^{\frac{3}{2}}t^\frac{5}{2}} - 10 \frac{t_n'}{(\rho_c)^{\frac{3}{2}}t^\frac{5}{2}} \right).
\]

Here we have introduced the new coordinate \( r \) such that \( M(r) = \frac{1}{6}\pi\rho_0 r^3 \) with constant \( \rho_0 \) in contrast to \( \rho_c(r) \) defined in Eq.(7), indicating the comoving density. Obviously all functions are evaluated at the present time of the observer. The Hubble parameter is defined as\[15, 43\]

\[H_0 = \frac{1}{d_1} = \left( \frac{\dot{R}}{R} \right)_0.\]

The similarity of \( d_1 \) in both the FRW and LTB case should not obscure the fact that in the case of FRW the Hubble function \( H \) is homogeneous and independent of the space coordinates. In our structured FRW model, we have to take into account the radial dependence of the Hubble function, in addition to its time dependence, reflected also in the peculiar velocity at the boundary of our patch. Hence, in fitting the observational data to the structured FRW model one has to use a local value for the Hubble parameter, and differentiate it from its mean global value. It can be seen from the coefficients of the expansion of the luminosity distance in powers of the redshift \( z \) that the conventional FRW universe may mimic the coarse-grained FRW without the dark energy term. In fact, a comparison with the corresponding FRW coefficients shows that the luminosity distance coefficients of the structured FRW goes over to those of the FRW if one sets

\[
\Omega_m = 1 + 5 \frac{t_n'}{(\rho_c)^{\frac{1}{2}}t^\frac{3}{2}} + 29 \frac{t_n''}{(6\pi\rho_c)^{\frac{3}{2}}t^\frac{5}{2}} + \frac{5}{2} \frac{t_n'}{(6\pi\rho_c)^{\frac{3}{2}}t^\frac{5}{2}},
\]

\[
\Omega_{\Lambda} = -\frac{1}{2} \frac{t_n'}{(\rho_c)^{\frac{1}{2}}t^\frac{3}{2}} + 29 \frac{t_n''}{8(\rho_c)^{\frac{3}{2}}t^\frac{5}{2}} + \frac{5}{4} \frac{t_n'}{(\rho_c)^{\frac{3}{2}}t^\frac{5}{2}}.
\]

These correspondence equations make a comparison with the observational data easier, as everyone is accustomed to the FRW jargon. Let us take as an example the data of\[4\] in the form

\[0.8\Omega_m - 0.6\Omega_{\Lambda} = -0.2 \pm 0.1.\]

The standard interpretation of this result in a FRW universe is that there is a dark energy \( \Omega_{\Lambda} > 0 \). Substituting from equations (19 and 20) into (21) we obtain for the corresponding interpretation in a structured FRW universe

\[4.3 \frac{t_n'}{(\rho_c)^{\frac{1}{2}}t^\frac{3}{2}} + 3.625 \frac{t_n''}{(\rho_c)^{\frac{3}{2}}t^\frac{5}{2}} + 1.25 \frac{t_n'}{(\rho_c)^{\frac{3}{2}}t^\frac{5}{2}} = -1 \pm 0.1.\]

We may also take the result of the first year of the 5-year Supernova Legacy Survey (SNLS)\[46\]. According to this survey we have

\[\Omega_m - \Omega_{\Lambda} = -0.49 \pm 0.12.\]

Substituting from the Eqs. (19 and 20) leads to

\[5.5 \frac{t_n'}{(\rho_c)^{\frac{1}{2}}t^\frac{3}{2}} + 3.625 \frac{t_n''}{(\rho_c)^{\frac{3}{2}}t^\frac{5}{2}} + 1.25 \frac{t_n'}{(\rho_c)^{\frac{3}{2}}t^\frac{5}{2}} = -1.49 \pm 0.12.\]
B. The off-center observer and the nucleation time

According to the cosmological principle we may be located anywhere off the center of our local patch. Therefore, our luminosity distance-redshift relation should be based on an off-center position of the observer and the corresponding past null geodesics. The off-center geometry of the light cone for LTB metric may be found in [45, 47]. Let us take the simplest case of a radial off-center observation, where the observer is located at the point \( P \) defined by the fixed coordinate \( r = r_P \), and the source is located such that the center of the patch, the observer, and the source are aligned in the \( \theta = \pi/2 \) plane. In this case, the luminosity distance is given with a relation similar of the case of a central observer except for the functions \( R \) and its derivatives which are to be taken at the point \( r_P \) (equations (35, 36) in the reference [47] with the angle \( \psi = 0 \)). Therefore, the expansion by \( z \) is now at \( r = r_P \), and we have to look at the behavior of \( t_n \) and its derivatives at this point.

Let us now look at the behavior of the nucleation time \( t_n' \). From the equation (5) we have

\[
t - t_n = R^\frac{3}{2}(\frac{9M}{2})^{-\frac{1}{2}}.
\]

Differentiating with respect to \( r \) yields

\[
t_n' = -9\pi R^\frac{1}{2}(\frac{9M}{2})^{-\frac{1}{2}}(\bar{\rho} - \rho),
\]

Now, we have seen that \( \dot{R} > 0 \), to avoid shell crossing. Therefore, the sign of \( t_n' \) at any point is determined by the difference between the mean density up to the coordinate value \( r \) to the density at \( r \). Depending on our position within the patch, we may have \( \bar{\rho} - \rho > 0 \) or \( < 0 \). Therefore, \( t_n' \) may be either negative or positive. To avoid singularity at \( r = 0 \) we assume \( \bar{\rho} = \rho \) at the center which leads to \( t_n'(r = 0) = 0 \). At our position within the local patch defined now by \( z = 0 \), however, we may assume \( t_n'(r = r_P) < 0 \), i.e. the mean density of structures up to our position, \( r = r_P \), is larger than the density at our position, which means we are in an underdense region. This is the value we have to plug in the above equations to compare the luminosity distance to the observational data.

As it is well known [3, 4, 19] to interpret the SNIa data we need to take into account at least up to the \( d_L \) term in the expression for the luminosity distance. The \( d_L \) consists of terms proportional to the derivatives of \( t_n \) up the the second order. Now, the influence of the \( d_L \) term in the Eqs. (26, 28) can be seen in all the three terms on the left hand side of these equations. In a first approximation, we will now neglect the effect of the second derivative of \( t_n \). From the fact that for a homogeneous metric \( \dot{R}(r,t) = a(t)r \), we can easily see that

\[
\alpha := \frac{t_n'}{(6\pi r_P)^{\frac{3}{2}}} \sim \frac{dt_n}{d(ar)}.
\]

Therefore, the term \( \alpha \) is approximately equal to the running of \( t_n \) with respect to the physical-, or even luminosity-, distance. Now, we can write the Eq. (28) as a second order equation in \( \alpha \), which has the acceptable solution \( \alpha = -0.35 \) in accordance with the assumption of our position to be in an underdense region. Ignoring the term proportional to the second derivative in the Eq. (24) we obtain \( \Omega_m = 0.14 \) and \( \Omega_\Lambda = 0.63 \). Note that in the evaluation of the observational data (23) the mean Hubble value, and not the local \( H_0 \), which is needed for a LTB comparison, is used. These are rough approximations which show explicitly the effect of inhomogeneity. Depending on the actual mass power spectrum, the second derivative may also be negative, which will increase the term corresponding to the dark energy. Note that

\[
\alpha = -\frac{1}{6}(1 - \Omega_m + 2\Omega_\Lambda),
\]

easily derived from Eqs. (23, 24), gives an exact relation between \( \Omega_m \), \( \Omega_\Lambda \), and the first derivative of \( t_n \), independent of the second derivative of \( t_n \) present in both \( \Omega_m \) and \( \Omega_\Lambda \). If we assume the constraint \( \Omega_m + \Omega_\Lambda = 1 \), and take the value of \( \alpha = -0.35 \), we arrive at \( \Omega_m = 0.3 \) and \( \Omega_\Lambda = 0.7 \), which fits well to the results reported in [47] processing the data from their first sample of 75 low red shift and 43 high red shift SNIa.

We see, therefore, that our toy model for a realistic structured FRW universe, taking into account the impact of the inhomogeneities in our local patch, leads to a dimming of distant objects without the use of a dark energy.

IV. EXACT BACKREACTION OF OUR LOCAL INHOMOGENEOUS LTB PATCH

The traditional way of doing cosmology is to take the average of the matter distribution in the universe and write down the Einstein equations for it, adding some symmetry requirement. One then solves the equations \( G_{\mu\nu} = \langle T_{\mu\nu} \rangle \),
assuming homogeneity and isotropy of the mass distribution as the underlying symmetry. Note that in doing this we are taking the average of the energy momentum tensor at a constant time in the comoving coordinates, otherwise we would not come along with a homogeneously distributed matter content of the universe. As far as the precision of the observations allow, we may go ahead with this simplification. The more exact equation, however, is \( \langle G_{\mu\nu} \rangle = \langle T_{\mu\nu} \rangle \). Calling the difference \( G_{\mu\nu} - \langle G_{\mu\nu} \rangle = Q_{\mu\nu} \), one may write the correct equation as \( G_{\mu\nu} = \langle T_{\mu\nu} \rangle + Q_{\mu\nu} \). The backreaction term \( Q \) has so far been neglected in cosmology because of its smallness. Now that measuring \( Q \) is within the range of observational capabilities we have to take it into account. Of course, the averaging process is neither trivial nor unambiguous, but let us see what is the effect of a volume averaging in a comoving coordinates as it is done in the case of FRW model universe.

### A. Volume averaging in the local patch

We intend to average the inhomogeneities within our patch to get again a homogeneous patch within the FRW background and look for differences between this smoothed out SFRW and the original FRW. The difference caused by the backreaction is not vanishing and may have observational effects. To this end we will use the averaging formalism, developed mainly by Thomas Buchert[49, 50, 51, 52], which can easily be adapted to our LTB patch, having the same background and look for differences between this smoothed out SF RW and the original FRW. The difference caused by the case of FRW model universe.

The averaged scale factor is defined using the volume of our patch \( V_D \) and its time derivative:

\[
\theta_D \equiv \langle \theta \rangle = \frac{\dot{V}}{V} = 3 \frac{\dot{a}_D}{a_D} = 3H_D.
\]

where \( \theta \) being equal to the minus of the trace of the second fundamental form of the hypersurface \( t = \text{const.} \), is now a function of \( r \) and \( t \). The right hand side trivially vanishes for a FRW universe because of the homogeneity. This fact has far-reaching consequences for observational cosmology in our non-homogeneous neighborhood. The variation of the Hubble function with respect to the red-shift is not so simple any more as in the simple case of FRW universe. This affects a lot of observational data processing which so far has been done assuming homogeneity of the universe. Depending on the smoothing width \( \Delta z \), the bins, and the matter power spectrum there may be large effects due to the non-commutativity of the averaging process[53].

The averaged scale factor is defined using the volume of our patch \( D \) by \( a_D \equiv V(t)_D \). Now it can be shown that[49, 54]

\[
\frac{\ddot{a}}{a} - \langle \frac{\ddot{a}}{a} \rangle = \frac{\ddot{a}_D}{a_D} - \langle \frac{\ddot{a}_D}{a_D} \rangle,
\]

where we have used the notation \( \dot{a}_D \equiv \frac{d}{dt}a_D \), and denoted the average Hubble function as \( H_D \). Averaging over the local patch means we are taking it as an effective FRW patch. Therefore all the derived quantities should be based on the average value \( a_D \). This is why we take the above definition for the mean Hubble parameter and not \( \langle \frac{\ddot{a}}{a} \rangle \), which is different from \( \frac{\ddot{a}_D}{a_D} \). A similar difference holds for the second derivative of \( a \):

\[
\langle \frac{\ddot{a}}{a} \rangle \neq \langle \ddot{a} \rangle \neq \frac{\ddot{a}_D}{a_D}.
\]

Therefore, the definition of the averaged deceleration parameter is not without ambiguity, specially because there is no nice relation like (9) for the deceleration parameter. To choose the most appropriate definition, we make recourse to the fact that in the averaging process we are taking our patch to be homogeneous and FRW-like. Therefore, in averaging the redshift as a function \( a \), we always encounter \( a_D \) and its time derivatives \( \dot{a}_D \) and \( \ddot{a}_D \). This justifies the above definition of the mean Hubble parameter and motivates us to make the following definition for the deceleration parameter:

\[
q_D = -\frac{\ddot{a}_D a_D}{\dot{a}_D^2} = \frac{\ddot{a}_D}{a_D} \frac{1}{H_D^2}.
\]
as was done in the literature so far \cite{49,52}. Now, we are ready to take the average of the Einstein equations in our local patch to see how the mean field equations will look like and what are the differences to the simple FRW field equations. Buchert’s backreaction term is defined by \cite{49,52}

\[
Q = \langle \sigma^2 \rangle - \frac{1}{3} \langle (\theta - \langle \theta \rangle)^2 \rangle \tag{36}
\]

\[
\sigma^2 \langle \theta^2 \rangle - \frac{1}{3} \langle \theta^2 \rangle \theta_D, \tag{37}
\]

where \( \sigma \) is the shear scalar and \( \theta \) is the expansion. Although \( \theta_D \) and \( H_D \) are proportional, \( \langle \theta^2 \rangle \) and \( \langle H^2 \rangle \) are not. Hence, the relations (30, 37) can not be written in terms of \( H \), as was done in \cite{53}. The averages of the Einstein equations using the Hamiltonian constraint and the Raychaudhuri equation, taking into account the subtleties of the observation just mentioned, is then written in the following form \cite{49,52}:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3} (\rho_b + \Lambda + Q) \tag{38}
\]

\[
\frac{\ddot{a}}{a} = -\frac{1}{6} (\rho_b - 2\Lambda + 4Q), \tag{39}
\]

where we have set \( \langle \rho \rangle = \rho_b \), the density of the background FRW universe, as a result of the junction conditions reflected in the eq. (11), and added the cosmological term for completeness. Note that in the so-called Friedmann equation (38) the averaged Hubble parameter enters instead of the global background one \( H_b \). The effect of the backreaction within the local patch is realized as an effective extra perfect fluid having the density \( \rho_Q = \frac{Q}{4\pi G} \), and pressure \( p_Q \), and the equation of state

\[
\rho_Q = p_Q. \tag{40}
\]

A positive \( Q \) would lead to an increased Hubble parameter relative to the background \( H_b \). In fact we have \( H^2_b = H^2_D - \frac{Q}{4} \). Therefore, the averaged Hubble parameter measured in our subhorizon local patch is bigger than the background global one.

### B. Explicit value of \( Q \) and its interpretation

The value of \( Q \) is determined by the balance between the mean values of the shear and the term related to the mean values of the Hubble parameter and the expansion scalar in a complex manner depending on the running of the density and the nucleation time. Given this complex behavior of the backreaction term, let us study it for the simplest case of the nucleation time satisfying the necessary conditions in our neighborhood discussed in the last section. We then approximate \( t_n \) in the following way:

\[
t_n = t_0 - \frac{\tau}{L^2} r^2, \tag{41}
\]

where \( L = r_b \) is the comoving radius of the patch. For \( \tau > 0 \) the above expansion satisfies all the necessary conditions to be fullfilled by \( t_n \) at the center of the patch and in our observational vicinity. We obtain the following expression for the backreaction:

\[
Q = \frac{(-5.8 t + 1.4 \tau) + (2\tau)^{-1/2} t^{3/2} [10.1 \arctan(\sqrt{\tau t^{-1}}) - 1.7 \arctan 1.5(\sqrt{\tau t^{-1}})]}{4\tau(t + \tau)^2} \tag{42}
\]

At the onset of nucleation, i.e. \( t - t_0 \ll \tau \), the effects of backreaction is negligible. However, for the late time \( t - t_0 \gg \tau \) we obtain \( Q \approx 0.1 \frac{1}{\tau} \). This is to be compared with \( \frac{1}{\tau} \) behavior of the matter density. This suggest that the dimming of the SNIa distances we have seen in the last section must be due to the late time increase of \( Q \) relative to the mass density and its effect on the background Hubble parameter.

### V. CONCLUSION

The precision cosmology is already so far developed that we can not ignore any more the effect of the local inhomogeneities on the global cosmology. On the other hand, the well established successes of standard FRW cosmology can
not be abandoned so easily. The structured FRW model (SFRW) we are proposing is just a step further towards a more realistic model universe, and is in accordance with the cosmological principle. In fact it could be considered as an exact Swiss Cheese model, in which the cheese is in the holes: a FRW model in which there are local subhorizon inhomogeneous patches embedded homogeneously as an exact solution of the Einstein equations. Each local patch is approximated, therefore, by an inhomogeneous cosmic fluid represented by a LTB metric up to a radius $r_b$ embedded in a background homogeneous FRW universe. Each local inhomogeneous sphere is then glued to a FRW homogeneous universe from which a sphere of the same radius is removed. The dynamics of the Einstein equations leave only one possibility for such a matching which is astrophysically appealing: The patches consist of overdense and underdense compensating regions such that the mean density in each patch is equal to the background FRW density. Taking into account this junction condition, the luminosity distance-redshift relation shows a dimming of astrophysical objects relative to what may be inferred from a simple FRW universe. We have analyzed the luminosity distance from a cosmic object to an on-center or off-center observer in such a SFRW universe and shown that a dimming of cosmic objects, which could mimic a dark energy, is the result of the inhomogeneity.

We are used to average out the inhomogeneities within the universe and take a global FRW metric to represent it. Within the proposed SFRW universe we may also average out the inhomogeneities. The process of averaging is adopted to the familiar understanding that the local inhomogeneities in the universe at any time $t$ should be smoothed out. To that end we have used the volume averaging of the Einstein equations developed so far to see the effect of the backreaction within a homogeneous scenario. The consequence is a backreaction term that maybe interpreted in terms of a new effective energy momentum tensor, although it is just a geometric term modifying the Friedmann equations. Using this interpretation, one could say that in addition to the mean cosmic matter fluid, we have a backreaction fluid with a density and pressure which has a very peculiar behavior. The back reaction density behaves as $1/t$ at late times and leads to a reduced background Hubble parameter.

Our simple model of a structured FRW universe shows how important it is in the era of precision cosmology to go beyond the model of a simple homogeneous fine-grained cosmic fluid and replace it by a scenario with coarse-grained local patches. The fact that the value of the mean local Hubble parameter, is not equal to the global one, and the collective peculiar velocity of the objects reflected in that of cosmic fluid are just some of the changes in the terminology of the universe models we have to incorporate in the interpretation of the astrophysical data.

VI. ACKNOWLEDGMENTS

It is a pleasure to thank members of the cosmology group at Sharif University of Technology, specially S. Khaksarynia and S. Rahvar, the members of the cosmology group at McGill University, specially Robert Brandenberger, and E. W. Kolb for discussions about the main idea of this work. My special thanks to McGill Physics department and Robert Brandenberger for the hospitality.

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