Phenomenological model of propagation of the elastic waves in a fluid-saturated porous solid with non-zero boundary slip velocity

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Abstract

It is known that a boundary slip velocity starts to play important role when the length scale over which the fluid velocity changes approaches the slip length, i.e. when the fluid is highly confined, for example, fluid flow through porous rock or blood vessel capillaries. Zhu & Granick [Phys. Rev. Lett. 87, 096105 (2001)] have recently experimentally established existence of a boundary slip in a Newtonian liquid. They reported typical values of the slip length of the order of few micro-meters. In this light, the effect of introduction of the boundary slip into the theory of propagation of elastic waves in a fluid-saturated porous medium formulated by Biot [J. Acoust. Soc. Am., 28, 179-191 (1956)] is investigated. Namely, the effect of introduction of boundary slip upon the function $F(\kappa)$ that measures the deviation from Poiseuille flow friction as a function of frequency parameter $\kappa$ is studied. By postulating phenomenological dependence of the slip velocity upon frequency, notable deviations in the domain of intermediate frequencies, in the behavior of $F(\kappa)$ are introduced with the incorporation of the boundary slip into the model. It is known that $F(\kappa)$ crucially enters Biot’s equations, which describe dynamics of fluid-saturated porous solid. Thus, consequences of the non-zero boundary slip by calculating the phase velocities and attenuation coefficients of the both rotational and dilatational waves with the variation of frequency are investigated. The new model should allow to fit the experimental seismic data in circumstances when Biot’s theory fails, as the introduction of phenomenological dependence of the slip velocity upon frequency, which is based on robust physical arguments, adds an additional degree of freedom to the model. If fact, it predicts higher than the Biot’s theory values of attenuation coefficients of the both rotational and dilatational waves in the intermediate frequency domain, which is in qualitative agreement with the experimental data. Therefore, the introduction of the boundary slip yields three-fold benefits: (A) Better agreement of theory with an experimental data since the parametric space of the model is larger (includes effects of boundary slip); (B) Possibility to identify types of porous medium and physical situations where boundary slip is important; (C) Constrain model parameters that are related to the boundary slip.

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I. INTRODUCTION

It has been a common practice in the fluid dynamics to assume that when a fluid flows over an interface with a solid, the fluid molecules adjacent to the solid have zero relative velocity with the respect to solid. So far, this widely used assumption, known as "no-slip boundary condition", has been successfully applied to the theoretical modeling of almost all macroscopic experiments. As relevantly noticed by Craig et al. [1], the success of this assumption does not reflect its accuracy, but rather insensitivity of the experiment to a partial-slip boundary condition. It is known that the boundary slip becomes important only when the length scale over which the fluid velocity changes approaches the slip length, that is the distance behind the interface at which the fluid velocity extrapolates to zero, i.e. when the fluid becomes highly confined, e.g., blood flow through capillaries or fluid flow through natural porous rock. Recently, authors of Refs. [1, 2] presented a convincing experimental evidence of a boundary slip in a Newtonian liquid. They performed direct measurements of the hydrodynamic drainage forces, which show a clear evidence of boundary slip. Also, they found that the boundary slip is a function of the fluid viscosity and the shear rate. These results have important implications for the blood dynamics in capillaries, the permeability of porous media, and lubrication of nano-machines. For example, results of Craig et al. suggest that red blood cells squeeze through narrow capillary walls more easily and induce less shear stress on capillary walls due to the boundary slip. Also, in oil production industry, the residual oil is difficult to produce due to its naturally low mobility. Thus, the enhanced oil recovery operations are used to increase production. It has been experimentally proven and theoretically validated that there is a substantial increase in the net fluid flow through a porous medium if the latter is treated with elastic waves [2, 4, 5]. We may conjecture that the elastic waves via the pore wall vibration cause boundary slip of the residual oil droplets, which likewise red blood cells, squeeze through pores with less resistance, effectively increasing permeability of the porous medium.

A quantitative theory of propagation of elastic waves in a fluid-saturated porous solid has been formulated in the classic paper by Biot [7]. After its appearance, this theory has seen numerous modifications and generalizations. One of the major findings of Biot’s work was that there was a breakdown in Poiseuille flow above a certain characteristic frequency specific to this fluid-saturated porous material. Biot theoretically studied this breakdown
by considering the flow of a viscous fluid in a tube with longitudinally oscillating walls under an oscillatory pressure gradient. Biot’s theory can be used to describe interaction of fluid-saturated porous solid with sound for a classic Newtonian fluid assuming no-slip boundary condition at the pore walls holds. However, in the light of recent experimental results of Ref. [2], revision of the classic theory is needed in order to investigate novelties bought about by the boundary slip.

Biot’s theory has been a successful tool for interpreting the experimental data for decades, but there are circumstances when it fails. Gist [8] performed ultrasonic velocity and attenuation measurements in sandstones with a variety of saturating fluids and compared the data with the predictions of the Biot’s theory. *Velocity data show systematic deviations from Biot theory as a function of pore fluid viscosity. Ultrasonic attenuation is much larger than the Biot prediction* and for several sandstones is nearly constant for a three decade variation in viscosity. This behavior contrasts with synthetic porous media such as sintered glass beads, where Biot theory provides accurate predictions of both velocity and attenuation. The similar claim has been also made by earlier experimental studies e.g. Ref. [9]. Even Biot himself acknowledged [10] that the model of a purely elastic solid matrix saturated with a viscous fluid is rather idealistic and there are rare occasions where it gives satisfactory agreement with the experiment. Thus, he generalized theory applying the correspondence principle, and incorporated the solid and fluid attenuation, and dissipation due to non-connected pores. In this work Biot introduced the viscodynamic operator, which is a sum of the viscosity and inertial terms, and it accounts for the dynamic properties of the fluid motions in pores valid for the both low and high frequencies. The theory can model other, more complex, mechanisms such as thermoelastic dissipation, which produce a relaxation spectrum, or local flow mechanisms as the ’squirt’ flow [11], [12]. A time domain formulation based on memory variables has been also successfully introduced [13], [14]. In order to narrow down the source of discrepancy between Biot theory and experiment, Gist [8] tried to modify natural sandstone by curing a residual saturation of epoxy in the pore space, filling small pores and micro-cracks. This altered rock has a significantly reduced attenuation, demonstrating the dominance of small pores in controlling ultrasonic attenuation. Then he suggested that two non-Biot viscous attenuation mechanisms are needed: local flow in micro-cracks along grain contacts, and attenuation from pore-wall surface roughness. The model model based on these two mechanisms seemed to succeed in improvement of fitting the data. However,
this by no means does not exclude other possibilities, and all plausible effects that could explain the discrepancy should be explored. At low frequencies, the fluid flow is of Poiseuille type and the inertia effects are obviously negligible in comparison with the effects of the viscosity. At high frequencies, however, these effects are confined to a thin boundary layer in the vicinity of the pore walls and the inertia forces are dominant. Therefore, we conjecture that in such situations, apart from other yet unknown effects, non-zero boundary slip effect maybe responsible for the deviations between the theory and experiment. This justifies our aim to formulate a model that would account for the boundary slip.

In the section II we formulate theoretical basis of our model and in section III we present our numerical results. In the section IV we conclude with the discussion of the results.

II. THE MODEL

In our model we study a Newtonian fluid flowing in a cylindrical tube, which mimics a natural pore, whose walls are oscillating longitudinally and the fluid is subject to an oscillatory pressure gradient. We give analytical solutions of the problem in the frequency domain.

The governing equation of the problem is the linearized momentum equation of an incompressive fluid

$$\rho \frac{\partial \vec{v}}{\partial t} = -\nabla p + \mu \nabla^2 \vec{v}. \quad (1)$$

Here, $\vec{v}$, $p$, $\rho$ denote velocity, pressure and mass density of the fluid, while $\mu$ is the viscosity coefficient.

Now, let $u$ be a velocity of the wall of the tube which oscillates in time as $e^{-i\omega t}$. The flow of fluid in a cylindrical tube with longitudinally oscillating walls can be described by a single component of the velocity, namely, its $z$-component $v_z$ ($z$ axis is along the centerline of the tube). We use cylindrical coordinate system ($r, \phi, z$) in treatment of the problem. We introduce the relative velocity $U_1$ as $U_1 = v_z - u$. Thus, assuming that all physical quantities vary in time as $e^{-i\omega t}$, we arrive at the following master equation for $U_1$

$$\nabla^2 U_1 + \frac{i\omega}{\nu} U_1 = -\frac{X}{\nu}. \quad (2)$$

Here, we have introduced the following notations:

$$\rho X = -(\nabla p + \rho \frac{\partial u}{\partial t}),$$
which is a sum of the applied pressure gradient and force exerted on the fluid from the oscillating wall of the tube and, \( \nu \), which is \( \nu = \mu / \rho \).

The solution of Eq.(2) can be found to be

\[
U_1(r) = -\frac{X}{i\omega} + CJ_0(\beta r),
\]

where \( J_0 \) is the Bessel function and \( \beta = \sqrt{i\omega / \nu} \).

Assuming that the slip velocity is \( U_1(a) = U_s \) at the wall of the tube, where \( a \) is its radius, we obtain

\[
U_1(r) = -\frac{X}{i\omega} \left[ 1 - (1 + U_s) \frac{J_0(\beta r)}{J_0(\beta a)} \right].
\]

(3)

Here,

\[
U_s \equiv U_s i\omega = U_s \frac{\nu}{a^2} (\beta a)^2.
\]

Defining the cross-section averaged velocity as

\[
\bar{U}_1 = \frac{2}{a^2} \int_0^a U_1(r) rdr,
\]

we obtain

\[
\bar{U}_1 = -\frac{X a^2}{\nu} \left[ \frac{1}{(\beta a)^2} \left( 1 - \frac{2(1 + U_s) J_1(\beta a)}{(\beta a) J_0(\beta a)} \right) \right].
\]

(4)

Following work of Biot we calculate the stress at the wall \( \tau \),

\[
\tau = -\mu \left( \frac{\partial U_1(r)}{\partial r} \right)_{r=a} = \frac{\mu \beta X}{i\omega} \left( 1 + U_s \right) \frac{J_1(\beta a)}{J_0(\beta a)}.
\]

(5)

The total friction force is \( 2\pi a \tau \). Following Biot we calculate the ratio of total friction force to the average velocity, i.e.

\[
\frac{2\pi a \mu}{\bar{U}_1} = -2\pi \mu (\beta a) \left( 1 + U_s \right) \frac{J_1(\beta a)}{J_0(\beta a)} \times \left[ 1 - \frac{2 \left( 1 + U_s \right) J_1(\beta a)}{(\beta a) J_0(\beta a)} \right]^{-1}.
\]

(6)

Simple analysis reveals that (assuming \( U_s \to 0 \) as \( \omega \to 0 \), see discussion below)

\[
\lim_{\omega \to 0} \frac{2\pi a \tau}{\bar{U}_1} = 8\pi \mu,
\]

which corresponds to the limiting case of Poiseuille flow. Following Biot, we also introduce a function \( F(\kappa) \) with \( \kappa \) being frequency parameter, \( \kappa = a \sqrt{\omega / \nu} \), in the following manner

\[
\frac{2\pi a \tau}{\bar{U}_1} = 8\pi \mu F(\kappa),
\]

(6)
thus,

\[ F(\kappa) = -\frac{1}{4} \kappa \sqrt{i} \left(1 + \bar{U}_s\right) \frac{J_1(\kappa \sqrt{i})}{J_0(\kappa \sqrt{i})} \times \left[1 - \left(1 + \bar{U}_s\right) \frac{2J_1(\kappa \sqrt{i})}{\kappa \sqrt{i} J_0(\kappa \sqrt{i})}\right]^{-1}. \] (7)

Note, that \( F(\kappa) \) measures the deviation from Poiseuille flow friction as a function of frequency parameter \( \kappa \). The Biot’s expression for \( F(\kappa) \) in the no boundary slip regime can be easily recovered from Eq.(7) by putting \( \bar{U}_s \to 0 \) for all \( \kappa \)'s.

So far, we did not specify \( \bar{U}_s \), however there are certain physical constraints it should satisfy:

(A) Authors of Ref. [1] demonstrated that the slip length (which, in fact, is proportional to the slip velocity) is a function of the approach rate and they showed why several previous careful measurement of confined liquids have not observed evidence for boundary slip. Under the low approach rates employed in previous measurements slip length is zero and no-slip boundary condition is applicable. Experiments reported in Ref. [1] were performed when half-sphere approached a plane with liquid placed in-between at different approach rates. However, we should clearly realize what term ”approach rate” means in the context of Biot’s theory: Biot investigated fluid flow in the cylindrical tube whose walls are harmonically oscillating in the longitudinal direction as \( x(t) = Ae^{-i\omega t} \), therefore if similar to Ref.[1] experiment would be done for the oscillation tube, the ”approach rate” would be the amplitude of \( \dot{x}(t) \), i.e. \(-i\omega A\). Thus, when \( \omega \to 0 \), \( \bar{U}_s \) should also tend to zero.

(B) At high frequencies the viscous effects are negligible compared to the inertial effects. Thus, fluid at high frequencies behaves as an ideal (non-viscous) pore fluid, which allows for an arbitrary slip. Therefore, when \( \omega \to \infty \), \( \bar{U}_s \) should tend to zero. When we examine Fig. 6 from Ref.[1] and Fig. 3 (bottom panel) from Ref.[2] closely we should understand that the authors measured ascending part where slip length increases with the increase of the approach (flow) rate. Since, the viscous effects should be negligible compared to the inertial effects for large approach rates, there must be also a descending part – clearly slip length (or the same as slip velocity) cannot grow infinitely as approach rate increases. Viscous effects should give in to the inertial effects and fluid should behave an ideal fluid allowing for an arbitrary slip.

Based upon the above two physical arguments we conclude that \( \bar{U}_s \) should be a smooth function which tends to zero when \( \omega \) (or the same as \( \kappa \)) tends to both zero and \( \infty \). Therefore,
we postulate following *phenomenological* expression for $\bar{U}_s$

$$Re[\bar{U}_s(\kappa)] = Im[\bar{U}_s(\kappa)] = \xi \frac{B \kappa^4}{(A + \kappa^2)^5}$$

This function has a maximum at $\kappa = \sqrt{2A/3}$. By fixing $B$ at $5^5A^3/(4 \times 3^3)$ we force it to achieve its maximum equal to unity at $\kappa = \kappa_*$ (when $\xi = 1$). We introduced $\xi$ (usually $0 \leq \xi \leq 1$) as a sort of ”weight” of the boundary slip effect into the solution. Putting, $\xi = 0$ this way would easily allow us to recover no-slip case, while the increase of $\xi$ we would be able to trace effect of non-zero slip onto the all physical quantities. We plot this function in Fig. 1 for $A = 25$ and $\xi = 1$.

It is worthwhile to mention that, at first glance, it seems that non-zero boundary slip, which appears at ”high approach rates”, should be attributed to the non-linear effects. However, thorough interpretation of the experimental results of Ref. [1], in the context of the oscillatory tube (see points (A) and (B) above) allows us to conclude that the non-zero boundary slip is also incorporated into the linear Biot’s theory.

Next, we plot both $F_r(\kappa) = Re[F(\kappa)]$ and $F_i(\kappa) = -Im[F(\kappa)]$ in Fig. 2 for the three cases: when there is no boundary slip ($\xi = 0$), and $\xi = 0.05$, 0.1. We gather from the plot that the $\xi = 0$ is identical to Fig. 4 from Ref.[7] as it should be. However, we also notice a noticeable difference from the classic case when $\xi$ is non-zero in the intermediate frequencies domain. Of course, according to our definition of the phenomenological form of $\bar{U}_s(\kappa)$, even for non-zero $\xi$ (non-zero boundary slip), when $\kappa \rightarrow 0$ and $\kappa \rightarrow \infty$, asymptotically $F(\kappa)$ behaves as classic Biot’s solution [7], i.e.

$$\lim_{\kappa \rightarrow 0} F_r(\kappa) = 1, \quad (8)$$

and

$$\lim_{\kappa \rightarrow 0} F_i(\kappa) = 0, \quad (9)$$

$$\lim_{\kappa \rightarrow \infty} F(\kappa) = \frac{\kappa}{4} \sqrt{i} = \frac{\kappa}{4} \left(1 + \frac{i}{\sqrt{2}}\right). \quad (10)$$

Since in our phenomenological model we allow for the deviations in the intermediate frequency domain, it is easy to foresee that these will have an impact on all the predictions of the Biot’s theory precisely in that frequency range. Namely, all observable quantities predicted by the Biot’s theory, such as phase velocities and attenuation coefficients of the
both rotational and dilatational waves will be affected by the introduction of boundary slip into the model in the intermediate frequency range.

Biot [7] showed that the general equations which govern propagation of rotational and dilatational high-frequency waves in a fluid-saturated porous medium are the same as in the low-frequency range provided the viscosity is replaced by its effective value as a function of frequency. In practice, it means replacing the resistance coefficient $b$ by $bF(\kappa)$.

The equations describing dynamics of the rotational waves are

$$\frac{\partial^2}{\partial t^2}(\rho_{11}\vec{\omega} + \rho_{12}\vec{\Omega}) + bF(\kappa)\frac{\partial}{\partial t}(\vec{\omega} - \vec{\Omega}) = N\nabla^2\vec{\omega},$$

$$\frac{\partial^2}{\partial t^2}(\rho_{12}\vec{\omega} + \rho_{22}\vec{\Omega}) - bF(\kappa)\frac{\partial}{\partial t}(\vec{\omega} - \vec{\Omega}) = 0,$$

where, $\rho_{11}, \rho_{12}$ and $\rho_{22}$ are mass density parameters for the solid and fluid and their inertia coupling; $\vec{\omega} = \text{curl} \vec{u}$ and $\vec{\Omega} = \text{curl} \vec{U}$ describe rotations of solid and fluid with $\vec{u}$ and $\vec{U}$ being their displacement vectors, while the rigidity of the solid is represented by the modulus $N$.

Substitution of a plane rotational wave of the form

$$\omega = C_1e^{i(lx+\chi t)}, \quad \Omega = C_2e^{i(lx+\chi t)},$$

into Eqs.(11) and (12) allows us to obtain a characteristic equation

$$\frac{NL^2}{\rho a^2} = E_r - iE_i,$$

where $l$ is wavenumber, $\chi = 2\pi f$ is wave cyclic frequency, $\rho = \rho_{11} + 2\rho_{12} + \rho_{22}$ is the mass density of the bulk material and $a$ is a pore radius.

The real and imaginary parts of Eq.(14) can be written as

$$E_r = \frac{(\gamma_{11}\gamma_{22} - \gamma_{12}^2)(\gamma_{22} + \epsilon_2) + \gamma_{22}\epsilon_2 + \epsilon_1^2 + \epsilon_2^2}{(\gamma_{22} + \epsilon_2)^2 + \epsilon_1^2},$$

and

$$E_i = \frac{\epsilon_1(\gamma_{12} + \gamma_{22})^2}{(\gamma_{22} + \epsilon_2)^2 + \epsilon_1^2},$$

where $\gamma_{ij} = \rho_{ij}/\rho$, $\epsilon_1 = (\gamma_{12} + \gamma_{22})(f_c/f) F_r(\kappa) = (\gamma_{12} + \gamma_{22})(f_c/f) F_r(\delta\sqrt{f/f_c})$, $\epsilon_2 = (\gamma_{12} + \gamma_{22})(f_c/f) F_i(\kappa) = (\gamma_{12} + \gamma_{22})(f_c/f) F_i(\delta\sqrt{f/f_c})$. The function $F(\kappa)$ was written here more conveniently as a function of frequency $f$, i.e. $F(\kappa) = F(\delta\sqrt{f/f_c})$, where $\delta$ is a factor dependent on pore geometry. For the hollow cylinder-like pores, $\delta = \sqrt{8}$ and we use this value throughout the paper. $f_c$ is the critical frequency above which the Poiseuille flow

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breaks down, and it equals \( b/(2\pi \rho_2) \). Here \( \rho_2 \) denotes the product of porosity and fluid mass density.

In order to obtain phase velocity and attenuation coefficient of the rotational waves, we put \( l = \text{Re}[l] + i\text{Im}[l] \). Thus, the phase velocity is then \( v_r = \chi/|\text{Re}[l]| \). Introducing a reference velocity as \( V_r = \sqrt{N/\rho} \), we obtain the dimensionless phase velocity as

\[
\frac{v_r}{V_r} = \frac{\sqrt{2}}{\sqrt{E_i^2 + E_r^2 + E_r}}^{1/2}.
\]  

(17)

To obtain the attenuation coefficient of the rotational waves, we introduce a reference length, \( L_r \), defined as \( L_r = V_r/(2\pi f_c) \). The length \( x_a \) represents the distance over which the rotational wave amplitude is attenuated by a factor of \( 1/e \). Therefore we can construct the dimensionless attenuation coefficient as \( L_r/x_a \),

\[
\frac{L_r}{x_a} = \frac{f}{f_c} \left[ \frac{\sqrt{E_i^2 + E_r^2 - E_r}}{\sqrt{2}} \right]^{1/2}.
\]  

(18)

The equations describing dynamics of the dilatational waves are [7]

\[
\nabla^2(Pe + Q\epsilon) = \frac{\partial^2}{\partial t^2}(\rho_{11}e + \rho_{12}\epsilon) + bF(\kappa) \frac{\partial}{\partial t}(e - \epsilon),
\]  

(19)

\[
\nabla^2(Q\epsilon + R\epsilon) = \frac{\partial^2}{\partial t^2}(\rho_{12}e + \rho_{22}\epsilon) - bF(\kappa) \frac{\partial}{\partial t}(e - \epsilon),
\]  

(20)

where, \( P, Q \) and \( R \) are the elastic coefficients, \( e = \text{div} \, \vec{u} \) and \( \epsilon = \text{div} \, \vec{U} \) are the divergence of solid and fluid displacements. Again, substitution of a plane dilatational wave of the form

\[
e = C_1 e^{i(lx + \chi t)}, \quad \epsilon = C_2 e^{i(lx + \chi t)},
\]  

(21)

into Eqs.(19) and (20) allows us to obtain a characteristic equation

\[
(z - z_1)(z - z_2) + iM(z - 1) = 0,
\]  

(22)

where \( z = l^2 V_c^2/\chi^2 \), \( V_c^2 = (P + R + 2Q)/\rho \) represents the velocity of a dilatational wave when the relative motion between fluid and solid is absent, \( z_{1,2} = V_c^2/V_{1,2}^2 \) with \( V_{1,2} \) being the velocities of the purely elastic waves with subscripts 1,2 referring to the two roots of Eq.(22), and finally \( M = (\epsilon_1 + i\epsilon_2)/(\sigma_{11}\sigma_{22} - \sigma_{12}^2) \) with \( \sigma_{11} = P/(P + R + 2Q) \), \( \sigma_{22} = R/(P + R + 2Q) \) and \( \sigma_{12} = Q/(P + R + 2Q) \).
Eq.(22) has two complex roots $z_I$ and $z_{II}$. Phase velocities of the two kinds of dilatational waves can be defined as
\[
\frac{v_I}{V_c} = \frac{1}{Re[\sqrt{z_I}]}, \quad \frac{v_{II}}{V_c} = \frac{1}{Re[\sqrt{z_{II}}]},
\]
while the corresponding attenuation coefficients can be also introduced as
\[
\frac{L_c}{x_I} = Im[\sqrt{z_I}] \frac{f}{f_c}, \quad \frac{L_c}{x_{II}} = Im[\sqrt{z_{II}}] \frac{f}{f_c}.
\]

III. NUMERICAL RESULTS

In order to investigate the novelties brought about into classical Biot’s theory of propagation of elastic waves in porous medium [7] by the inclusion of boundary slip, we have studied the full parameter space of the problem.

In all forthcoming results, we calculate phase velocities and attenuation coefficients for the case 1 from Table I taken from Ref. [7], which is $\sigma_{11} = 0.610, \sigma_{22} = 0.305, \sigma_{12} = 0.043, \gamma_{11} = 0.500, \gamma_{22} = 0.500, \gamma_{12} = 0, z_1 = 0.812, \text{and } z_2 = 1.674$.

We calculated normalized phase velocity of the plane rotational waves, $v_r/V_r$, and the attenuation coefficient $L_r/x_a$ using our more general expression for $F(\kappa)$ (which takes into account non-zero boundary slip) given by Eq.(7).

In Fig. 3 we plot phase velocity $v_r/V_r$ as a function of frequency for the three cases: the solid curve corresponds to $\xi = 0$ (no boundary slip), while long-dashed and short-dashed curves correspond to $\xi = 1.0$ and $\xi = 1.5$ respectively (we have used these large values in order to emphasize the effect of the boundary slip). Note that the $\xi = 0$ case perfectly reproduces the curve 1 in Fig. 5 from Ref.[7]. For $\xi = 1.0$ and 1.5 we notice a deviation from the classic behavior in the form of a decrease of phase velocity in the domain of intermediate frequencies.

Fig. 4 shows the attenuation coefficient $L_r/x_a$ of the rotational wave as a function of frequency for the three values of $\xi$: the solid curve corresponds to $\xi = 0$ (no boundary slip), while long-dashed and short-dashed curves represent the cases $\xi = 1.0$ and $\xi = 1.5$ respectively. Note that $\xi = 0$ case coincides with curve 1 in Fig. 6 from Ref.[7]. We observe that, in the intermediate frequency range, with increase of boundary slip our model yields substantially higher values of the attenuation coefficient than in the classic no-slip case, which is in qualitative agreement with the experimental data [8], [9].
We also calculated normalized phase velocities of the plane dilatational waves, \(v_I/V_c\) and \(v_{II}/V_c\), and the attenuation coefficients \(L_c/x_I\) and \(L_c/x_{II}\) using our more general expression for \(F(\kappa)\) given by Eq. (7).

In Figs. 5 and 6 are similar to Fig. 3 except that now we plot phase velocities \(v_I/V_c\) and \(v_{II}/V_c\) as a function of frequency for \(\xi = 0, 0.05, 0.1\). Note that the solid curves on the both graphs reproduce curves 1 in Figs. 11 and 12 from Ref. [7]. We gather from the graph that introduction of the non-zero boundary slip leads to decrease of \(v_I/V_c\), while \(v_{II}/V_c\) increases in the domain of intermediate frequencies as boundary slip, \(\xi\), increases.

In Figs. 7 and 8 we plot the attenuation coefficients \(L_c/x_I\) and \(L_c/x_{II}\) in a similar manner as in Fig. 4, but now for \(\xi = 0, 0.05, 0.1\). Again we observe that the \(\xi = 0\) case reproduces curves 1 in Figs. 13 and 14 from Ref. [7]. We gather from Figs. 7 and 8 that introduction of the non-zero boundary slip yields higher than in the no boundary slip case values of the attenuation coefficients for both types of the dilatational waves in the domain of intermediate frequencies, which, again, is in qualitative agreement with the experimental results [8], [9].

It is worthwhile to note that for no-zero \(\xi\) the asymptotic behavior of the elastic waves in the \(f/f_c \to \infty\) as well as \(f/f_c \to 0\) limit is identical to the classic behavior established by Biot [7]. This is a consequence of the assumptions of our phenomenological model, which is based on robust physical arguments.

IV. DISCUSSION

In this paper we have studied the effect of introduction of the boundary slip in the theory of propagation of elastic waves in a fluid-saturated porous solid originally formulated by Biot [7]. Biot’s theory does not account for boundary slip effect, however, the boundary slip becomes important when the length scale over which the fluid velocity changes approaches the slip length, i.e. when the fluid is highly confined, for instance, fluid flow through porous rock or blood vessel capillaries. In the light of recent convincing experimental evidence of a boundary slip in a Newtonian liquid [3], it is necessary to take into account this effect into the Biot’s theory where appropriate. We have studied the effect of introduction of boundary slip upon the function \(F(\kappa)\) that measures the deviation from Poiseuille flow friction as a function of frequency parameter \(\kappa\). This function crucially enters Biot’s equations which describe dynamics of fluid-saturated porous solid. Therefore, a revision of Biot’s theory
was needed in order to incorporate boundary slip effect into the all measurable predictions of this theory such as phase velocities and attenuation coefficients of the both rotational and dilatational waves. We have performed such analysis, and in summary, we found that the introduction of the non-zero boundary slip into the Biot’s theory of propagation of the elastic waves in a fluid-saturated porous solid results in

- the decrease, as compared to the no-slip limiting case, of the phase velocities of both rotational waves \( v_r/V_r \) and dilatational wave of the first kind \( v_1/V_c \) in the domain of intermediate frequencies. On contrary, the phase velocity of the dilatational wave of the second kind \( v_{II}/V_c \) experiences an increase as compared to the no-slip limiting case in the domain of intermediate frequencies.

- in the domain of intermediate frequencies the attenuation coefficients of both the rotational \( L_r/x_a \) and dilatational waves \( L_c/x_I \) and \( L_c/x_{II} \) are increased as compared to the no-slip limiting case as the boundary slip increases, which is in qualitative agreement with the experimental data [8], [9].

- behavior of all physical quantities which describe the elastic waves in the asymptotic limits of both small and large frequencies is not affected by the introduction of the non-zero boundary slip. The deviation occurs only in the domain of intermediate frequencies, as prescribed by our phenomenological model.

The investigation of properties of elastic waves is important for a number of applications. The knowledge of phase velocities and attenuation coefficients of elastic waves is necessary, for example, to guide the oil-field exploration applications, acoustic stimulation of oil producing fields (in order to increase the amount of recovered residual oil), and the acoustic clean up of contaminated aquifers [3, 4, 5, 6]. Therefore, our results would be useful for various applications in oil production as well as in ecology.

From the recent experimental results of Ref.[2] we gathered that there are physical situations were the no-slip boundary condition becomes invalid. We have formulated a phenomenological model of elastic waves in the fluid-saturated porous medium based on Biot’s linear theory and certain physically justified assumptions on the variation of boundary slip velocity with frequency, \( U_s(\kappa) \). Since, there are no experimental measurements of \( U_s(\kappa) \), for a cylindrical tube, on which "Biot-like" theory relies, there is a certain freedom of choice,
which could be used to obtain a better fit of experimental data with the theory \textit{in cases where classic Biot’s theory fails to do so}. If fact, our model predicts \textit{higher than the Biot’s theory values of attenuation coefficients of the both rotational and dilatational waves} in the intermediate frequency domain, which is in qualitative agreement with the experimental data [3], [4]. Therefore, the introduction of the boundary slip yields three-fold benefits:

- Better agreement of theory with an experimental data, since, the parametric space of the model is larger (includes effects of boundary slip).

- Possibility to identify types of porous medium and physical situations where boundary slip is important.

- Constrain model parameters that are related to the boundary slip.

We would like to close this paper with the following two remarks: First, the reported slip length varies from tens of nano-meters [1] to a few microns [2], depending on a type of a liquid and differences in the experimental set up. The latter experimental work clearly demonstrates the applicability of our model to the elastic wave phenomena in a usual porous media found in Nature. Second, in this work we \textit{postulated} a phenomenological expression for the boundary slip velocity based on some physically sound arguments (in fact, similar, successful, approach has been used by the authors of Ref.[15] in the context of solar physics). However, perhaps it may be possible to derive an expression based on more general ”first principles”, which should be a subject of a future work.

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Figure Captions

Fig. 1: Behavior of $Re[U_s(\kappa)] = Im[U_s(\kappa)]$ as function of frequency parameter, $\kappa$, for $\xi = 1$, $A = 25$.

Fig. 2: Behavior of $F_r(\kappa)$ (thick curves) $F_i(\kappa)$ (thin curves) as function of frequency parameter, $\kappa$, according to Eq.(7). Solid curves correspond to the case when $\xi = 0$ (no boundary slip), while long-dashed and short-dashed curves correspond to $\xi = 0.05$ and $\xi = 0.1$ respectively.

Fig. 3: Behavior of dimensionless, normalized phase velocity of the rotational wave, $v_r/V_r$, as a function of frequency. Solid curve corresponds to the case when $\xi = 0$ (no boundary slip), while long-dashed and short-dashed curves correspond to $\xi = 1.0$ and $\xi = 1.5$ respectively.

Fig. 4: Behavior of dimensionless, normalized attenuation coefficient of the rotational wave, $L_r/x_a$, as a function of frequency. Solid curve corresponds to the case when $\xi = 0$ (no boundary slip), while long-dashed and short-dashed curves correspond to $\xi = 1.0$ and $\xi = 1.5$ respectively.

Fig. 5: Behavior of dimensionless, normalized phase velocity of the dilatational wave, $v_l/V_c$, as a function of frequency. Solid curve corresponds to the case when $\xi = 0$ (no boundary slip), while long-dashed and short-dashed curves correspond to $\xi = 0.05$ and $\xi = 0.1$ respectively.

Fig. 6: The same as in Fig. 5 except for the curves are for $v_{II}/V_c$.

Fig. 7: Behavior of dimensionless, normalized attenuation coefficient of the dilatational wave, $L_c/x_I$, as a function of frequency. Solid curve corresponds to the case when $\xi = 0$ (no boundary slip), while long-dashed and short-dashed curves correspond to $\xi = 0.05$ and $\xi = 0.1$ respectively.

Fig. 8: The same as in Fig. 7 except for the curves are for $L_c/x_{II}$.
