Manifolds with Pointwise Ricci Pinched Curvature

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Abstract In this paper, we proved a compactness result about Riemannian manifolds with an arbitrary pointwisely pinched Ricci curvature tensor.

1. Introduction

Let $M^n$ be an $n$-dimensional complete Riemannian manifold with $n \geq 3$. One of the basic problems is under which condition on its curvature the Riemannian manifold is compact. The classical Bonnet-Myers’ theorem states that a complete Riemannian manifold with positive lower bound for its Ricci curvature is compact.

In [11], Hamilton proved that:

\begin{center}
Any convex hypersurface with dimension $\geq 3$ in Euclidean space with second fundamental form $h_{ij} \geq \delta \cdot \frac{\text{tr}(h)}{n}$ must be compact.
\end{center}

In [5], Chen-Zhu proved an intrinsic analogue of the Hamilton’s result by using the Ricci flow which was introduced by Hamilton in 1982. They proved that:
If $M^n$ is a complete $n$-dimensional ($n \geq 4$) Riemannian manifold with positive and bounded scalar curvature and satisfies the following pointwisely pinching condition

$$|W|^2 + |V|^2 \leq \delta_n (1 - \varepsilon)^2 |U|^2,$$

for $\varepsilon > 0$, $\delta_4 = \frac{1}{5}$, $\delta_5 = \frac{1}{10}$ and $\delta_n = \frac{2}{(n-2)(n+1)}$, ($n \geq 6$), where $W, V, U$ denote the Weyl conformal curvature tensor, traceless Ricci part and the scalar curvature part of the curvature operator respectively. Then $M^n$ is compact.

For the 3-dimensional case, they weaken the curvature operator pinching condition to an arbitrary Ricci curvature pinching condition:

Let $M$ be a complete 3-dimensional Riemannian manifold with bounded and nonnegative sectional curvature. If $M$ satisfies the positive Ricci pinching condition:

$$R_{ij} \geq \varepsilon \cdot \text{scal} \cdot g_{ij} > 0$$

for some $\varepsilon > 0$. Then $M$ must be compact.

Recently, by the Ricci flow and the new invariant cone construction introduced by Böhm-Wilking [1], Ni-Wu [13] proved the following compactness result in terms of curvature operator:

If $M^n$ is a complete $n$-dimensional ($n \geq 3$) Riemannian manifold with bounded curvature and satisfies

$$Rm \geq \delta U > 0$$

for $\delta > 0$, where $Rm, U$ denote the curvature operator and its scalar curvature part. Then $M^n$ must be compact.

Naturally, from the above results, one expects that: any complete Riemannian manifold with dimension $\geq 3$ and has positive Ricci pinched curvature must be compact. This is already true in 3-dimensional case by the result in Chen-Zhu [5]. In this paper, by using the Yamabe flow, we give an affirmative answer in the class of locally conformally flat manifolds. Our main result is the following:

**Theorem 1.1** Let $n \geq 3$. Suppose $M^n$ is a smooth complete locally conformally flat $n$-dimensional manifold with bounded and positive scalar curvature. Suppose $M^n$ has nonnegative sectional curvature and satisfies the following Ricci curvature pinching condition

$$R_{ij} \geq \varepsilon \cdot \text{scal} \cdot g_{ij}$$

(1.1)
for some $\varepsilon > 0$. Then $M^n$ is compact.

We briefly describe the proof of the theorem. Our proof of Theorem 1.1 depends on the Yamabe flow and the limit solution of Yamabe flow. Suppose there exists such a noncompact Riemannian manifold satisfying the Ricci pinching condition (1.1), we evolve it by the Yamabe flow. By the short-time existence result [6] and the Ricci pinching condition, we can obtain a long-time existence result. In section 2, we will study the asymptotic behaviors of the solution to the Yamabe flow. Finally in section 3, we will complete the proof of the main theorem by using the results obtained in section 2.

2. The Asymptotic Behaviors of the Yamabe Flow

In the geometric flows, in order to know the initial manifold well, we usually need to study the asymptotic behaviors of the solution of the flow. In this section, we study the asymptotic behaviors of the Yamabe flow. First we recall the Li-Yau-Hamilton inequality of Chow [8] on locally conformally flat manifolds.

**Theorem 2.1** (Chow [8]) Suppose $(M^n, g_{ij})$ is a smooth $n$-dimensional $(n \geq 3)$ complete locally conformally flat manifold with bounded and nonnegative Ricci curvature. Let $R(x, t)$ be the scalar curvature of the solution of the Yamabe flow with $g_{ij}$ as initial metric. Then we have

$$\frac{\partial R}{\partial t} + \langle \nabla R, X \rangle + \frac{1}{2(n-1)} R_{ij} X^i X^j + \frac{R}{t} \geq 0$$

for any vector $X$ on $M$.

In his paper [8], Chow proved the above theorem for compact locally conformally flat manifolds with positive Ricci curvature. However, by a perturbation argument as in [9], it is clear that the Li-Yau-Hamilton inequality actually holds for complete locally conformally flat manifolds with nonnegative Ricci curvature.

**Lemma 2.2** Let $g_{ij}(t)$ be a locally conformally flat complete solution to the Yamabe flow for $t > 0$ which has bounded and positive Ricci curvature. If the Harnack quantity

$$Z = \frac{\partial R}{\partial t} + \langle \nabla R, X \rangle + \frac{1}{2(n-1)} R_{ij} X^i X^j + \frac{R}{t}$$
is positive for all $X \in T_{x_0}M^n$ at some point $x = x_0$ and $t = t_0 > 0$, then it is positive for all $X \in T_xM^n$ at every point $x \in M^n$ for any $t > t_0$.

**Proof.** By the calculation in [8], we know

$$\left(\frac{\partial}{\partial t} - (n-1)\triangle\right)Z \geq (R - \frac{2}{t})Z \geq -\frac{2}{t}Z.$$  \hspace{1cm} (2.1)

Since $Z$ is positive for all $X \in T_{x_0}M^n$ at $t = t_0 > 0$, we can find a nonnegative function $F$ on $M^n$ with support in a neighborhood of $x_0$ so that $F(x_0) > 0$ and $Z \geq \frac{F}{t}$ for all $X$ everywhere at $t = t_0$. Let $F$ evolve by the heat equation

$$\frac{\partial F}{\partial t} = (n-1)\triangle F.$$  \hspace{1cm} (2.2)

It then follows the usual strong maximum principle that $F > 0$ everywhere for any $t > t_0$. We only need to prove that

$$Z \geq \frac{F}{t^2}, \quad \text{for all } t \geq t_0.$$  

By (2.1) and (2.2) we know

$$\left(\frac{\partial}{\partial t} - (n-1)\triangle\right)(Z - \frac{F}{t^2}) \geq -\frac{2}{t}(Z - \frac{F}{t^2}),$$

for $t \geq t_0$. By the maximum principle we get $Z \geq \frac{F}{t^2}$.

This completes the proof of the Lemma 2.2.

Before we give the main result of this section, we first recall some definitions for the classification of the asymptotic behaviors of the solution of the Yamabe flow as $t \to +\infty$.

**Definition 2.3** (i) A complete solution to the Yamabe flow is called a Type I limit solution if the solution has nonnegative Ricci curvature and exists for $-\infty < t < \Omega$ for some constant $\Omega$ with $0 < \Omega < +\infty$ and $R \leq \frac{\Omega}{\Omega - t}$ everywhere with equality somewhere at $t = 0$.

(ii) A complete solution to the Yamabe flow is called a Type II limit solution if the solution has nonnegative Ricci curvature and exists for $-\infty < t < +\infty$ and $R \leq 1$ everywhere with equality somewhere at $t = 0$.
A complete solution to the Yamabe flow is called a Type III limit solution if the solution has nonnegative Ricci curvature and exists for \(-A < t < +\infty\) for some constant \(A\) with \(0 < A < +\infty\) and \(R \leq \frac{A}{A+t}\) everywhere with equality somewhere at \(t = 0\).

**Definition 2.4**

(i) We call a solution to the Yamabe flow a steady soliton, if it satisfies
\[
Rg_{ij} = g_{jk} \nabla_i X^k,
\]
where \(X^i\) is a vector field on the manifold.

(ii) We call a solution to the Yamabe flow a shrinking soliton, if it satisfies
\[
(R - \lambda)g_{ij} = g_{jk} \nabla_i X^k,
\]
where \(X^i\) is a vector field on the manifold and \(\lambda\) is a positive constant.

(iii) We call a solution to the Yamabe flow an expanding soliton, if it satisfies
\[
(R + \lambda)g_{ij} = g_{jk} \nabla_i X^k,
\]
where \(X^i\) is a vector field on the manifold and \(\lambda\) is a positive constant.

Moreover, if the vector field \(X\) is the gradient of some function \(f\), then we will call the corresponding soliton a steady, shrinking, expanding gradient soliton respectively.

We now follow Hamilton [10] and Chen-Zhu [5] (or also Cao [2]) to give a classification for Type II and Type III limit solutions.

**Theorem 2.5** Let \(M^n\) be a smooth \(n\)-dimensional locally conformally flat and simply connected Riemannian manifold. Then:

(i) any Type II limit solution with positive Ricci curvature to the Yamabe flow on \(M^n\) is necessarily a homothetically steady gradient soliton;

(ii) any Type III limit solution with positive Ricci curvature to the Yamabe flow on \(M^n\) is necessarily a homothetically expanding gradient soliton.

**Proof.** The following arguments are adapted from Hamilton [10] and Chen-Zhu [5] (or also Cao [2]), where the classification for the limit solutions of the Ricci flow were given. We only give the complete proof of (ii), since the proof of (i) is
similar and easier. At the end of the proof we point the difference between (i) and (ii), and then it is easy to see that the rest of the arguments are the same.

By the definition of the Type III limit solution, after a shift of the time variable, we may assume the Type III limit solution \( g_{ij}(t) \) is defined for \( 0 < t < +\infty \) with uniformly bounded curvature and positive Ricci curvature where \( tR \) assumes its maximum in space-time.

Suppose \( tR \) assumes its maximum at a point \((x_0, t_0)\) in space-time, then \( t_0 > 0 \) and the Harnack quantity

\[
Z = \frac{\partial R}{\partial t} + \langle \nabla R, X \rangle + \frac{1}{2(n-1)} R_{ij} X^i X^j + \frac{R}{t}, \tag{2.3}
\]

vanishes in the direction \( X = 0 \) at \((x_0, t_0)\). By Lemma 2.2 we know that at any earlier time \( t < t_0 \) and at every point \( x \in M^n \), there is a vector \( X \in T_x M^n \) such that \( Z = 0 \).

By the first variation of \( Z \) in \( X \)

\[
\nabla_i R + \frac{1}{n-1} R_{ij} X^j = 0, \tag{2.4}
\]

which implies that such a null vector \( X \) is unique at each point and varies smoothly in space-time.

Combining (2.3) and (2.4) we obtain that

\[
\frac{\partial R}{\partial t} + \frac{R}{t} + \frac{1}{2} \nabla_i R \cdot X^i = 0. \tag{2.5}
\]

By (2.4) and (2.5) and a direct computation, we have

\[
\begin{align*}
X^i (\frac{\partial}{\partial t} - (n-1)\Delta)(\nabla_i R) &+ \frac{1}{2(n-1)} X^i X^j (\frac{\partial}{\partial t} - (n-1)\Delta)R_{ij} \\
- \nabla_k R_{ij} \nabla_k X^j X^i &- (n-1) \nabla_k \nabla_i R \cdot \nabla_k X^i \\
+ (\frac{\partial}{\partial t} - (n-1)\Delta)(\frac{\partial R}{\partial t} + \frac{R}{t}) & = 0, \tag{2.6}
\end{align*}
\]

\[
\begin{align*}
(\frac{\partial}{\partial t} - (n-1)\Delta)(\nabla_i R) & = \nabla_i [(\frac{\partial}{\partial t} - (n-1)\Delta)R] - (n-1)R_{il} \nabla_l R \\
& = \nabla_i (R^2) - (n-1)R_{il} \nabla_l R, \tag{2.7}
\end{align*}
\]
\( \left( \frac{\partial}{\partial t} - (n-1)\Delta \right) (\frac{\partial R}{\partial t} + \frac{R}{t}) = 3(n-1)R\Delta R + \frac{1}{2}(n-1)(2-n)|\nabla R|^2 \)  
\[ (2.8) \]

\[ + 2R^2 + \frac{R^2}{t} - \frac{R}{t^2}, \]

\( \left( \frac{\partial}{\partial t} - (n-1)\Delta \right)R_{ij} = \frac{1}{n-2}B_{ij}, \)  
\[ (2.9) \]

where \( B_{ij} = (n-1)|Ric|^2g_{ij} + nRR_{ij} - n(n-1)R^2g_{ij} \). The combination of \( (2.6)-(2.9) \) gives

\[-R(R + \frac{1}{t})^2 + \frac{1}{2(n-1)(n-2)}B_{ij}X^iX^j - \frac{1}{2(n-1)}RR_{ij}X^iX^j \]
\[ + \frac{n}{2(n-1)}R_{il}R_{lj}X^iX^j + R_{ij}\nabla_kX^i\nabla_kX^j = 0. \]  
\[ (2.10) \]

On the other hand, by \( (2.4) \) we have

\[ \nabla_k \nabla_iR = -\frac{1}{n-1}(X^j \cdot \nabla_kR_{ij} + R_{ij} \cdot \nabla_kX^j), \]  
\[ (2.11) \]

and then by taking trace and using the evolution equation of the scalar curvature,

\[ R_{ij}((R + \frac{1}{t})g_{ij} - \nabla_iX^j) = 0. \]  
\[ (2.12) \]

Hence it follows from \( (2.10) \) and \( (2.12) \) that:

\[ R_{ij}(\nabla_kX^i - (R + \frac{1}{t})g_{ik})(\nabla_kX^j - (R + \frac{1}{t})g_{jk}) + A_{ij}X^iX^j = 0, \]  
\[ (2.13) \]

where \( A_{ij} = \frac{1}{2(n-1)(n-2)}B_{ij} + \frac{1}{2(n-1)}(nR_{il}R_{lj} - RR_{ij}). \)

In local coordinate \( \{x^i\} \) where \( g_{ij} = \delta_{ij} \) and the Ricci tensor is diagonal, i.e., \( Ric = diag(\lambda_1, \lambda_2, \cdots, \lambda_n) \), with \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \), and \( e_i, (1 \leq i \leq n) \) is the direction corresponding to the eigenvalue \( \lambda_i \) of the Ricci tensor, we have

\[ \sum_i \lambda_i(\nabla_kX^i - (R + \frac{1}{t})g_{ik})^2 + A_{ij}X^iX^j = 0 \]

and

\[ A_{ij} = diag(\nu_1, \nu_2, \cdots, \nu_n), \]

where

\[ \nu_i = \frac{1}{2(n-1)(n-2)} \sum_{k,l \neq i, k > l} (\lambda_k - \lambda_l)^2 \geq 0. \]
So
\[ \nabla_j X^i = (R + \frac{1}{t})g_{ij}, \quad \text{and} \quad A_{ij}X^iX^j = 0. \]
Thus \( \nabla_j X^i \) is symmetric and by the simply connectedness of \( M^n \), there exists a function \( f \) such that
\[ \nabla_i X^j = \nabla_i \nabla_j f. \]
Hence
\[ (R + \frac{1}{t})g_{ij} = \nabla_i \nabla_j f. \]
This means that \( g_{ij}(t) \) is a homothetically expanding gradient soliton.

So we have proved that if the solution exists on \( 0 < t < +\infty \), and the Harnack quantity
\[ Z = \frac{\partial R}{\partial t} + \langle \nabla R, X \rangle + \frac{1}{2(n-1)} R_{ij}X^iX^j + \frac{R}{t} \]
vanishes, then it must be an expanding gradient soliton. If we have a solution on \( \alpha < t < +\infty \), we can replace \( t \) by \( t - \alpha \) in the Harnack quantity. Then if \( \alpha \to -\infty \), the expression \( \frac{1}{t-\alpha} \to 0 \) and disappears. So the Harnack quantity becomes
\[ Z = \frac{\partial R}{\partial t} + \langle \nabla R, X \rangle + \frac{1}{2(n-1)} R_{ij}X^iX^j. \]
Then the rest of the arguments for the proof of (i) follows.

Hence we complete our proof of Theorem 2.5.

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In order to prove our Theorem 1.1, we need to get more information about the limit solutions of the Yamabe flow under our assumptions. So we give two Propositions which are necessary in our proof in the following section. We first deal with the case of the Type III limit solutions.

**Proposition 2.6** There exists no noncompact locally conformally flat Type III limit solution of the Yamabe flow which satisfies the Ricci pinching condition:

\[ R_{ij} \geq \varepsilon \cdot \text{scal} \cdot g_{ij} > 0, \]

for some \( \varepsilon > 0 \).
Proof. We argue by contradiction. Suppose there is a noncompact locally conformally flat Type III limit solution $g_{ij}(t)$ on $M$ which satisfies the above Ricci pinching condition. By Theorem 2.5, we know that the solution must be a homothetically expanding gradient soliton. This means that for any fixed time $t = t_0$, we have:

$$(R + \rho)g_{ij} = \nabla_i \nabla_j f$$

for some positive constant $\rho$ and some function $f$ on $M$.

Differentiating the equation (2.14) and switching the order of differentiations and then taking trace, we have

$$-(n - 1)\nabla_i R = R_{ij} \nabla_j f.$$  \hspace{1cm} (2.15)

Fix the time $t = t_0$ and consider a long shortest geodesic $\gamma(s)$, $0 \leq s \leq \bar{s}$. Let $x_0 = \gamma(0)$ and $X(s) = \dot{\gamma}(s)$. Following by the same arguments as in the proof of Lemma 1.2 of Perelman [14] (or see the proof of Lemma 6.4.1 of [3] for the details) and using the Ricci pinching condition, we can obtain that

$$|\frac{df}{ds} - \rho s| \leq \text{const.}$$  \hspace{1cm} (2.16)

and

$$|f - \frac{1}{2}\rho s^2| \leq \text{const} \cdot (s + 1)$$  \hspace{1cm} (2.17)

for $s$ large enough. From (2.16) and (2.17) we obtain that

$$|\nabla f|^2(x) \geq c\rho f(x) \geq \frac{c}{2}\rho^2 s^2 = \frac{c}{2}\rho^2 d^2(x, x_0)$$

for some constant $c > 0$. Then by the same argument as in Theorem I in [5], we can obtain a contradiction!

Hence we complete the proof of Proposition 2.6.

For the case of Type II limit solution of the Yamabe flow, we have the following result:

**Proposition 2.7** Suppose $(M^n, g_{ij}(t))$ is an $n$-dimensional ($n \geq 3$) complete noncompact locally conformally flat steady gradient soliton with bounded and positive
Ricci curvature. Assume the scalar curvature assumes its maximum at a point \( p \in M \), then the asymptotic scalar curvature ratio is infinite, i.e.,

\[
A = \limsup_{s \to +\infty} Rs^2 = +\infty
\]

where \( s \) is the distance to the point \( p \).

**Proof.** We argue by contradiction. Suppose \( R \leq \frac{C}{s^2} \), for some constant \( C > 0 \). By the equation of steady gradient soliton, we have

\[
R g_{ij} = \nabla_i \nabla_j f,
\]

for some smooth function \( f \) on \( M \).

Consider the integral curve \( \gamma(s), 0 \leq s \leq \bar{s} \), of \( \nabla f \) with \( \gamma(0) = p \) and \( X(s) = \dot{\gamma}(s) \). We first claim that \( M \) is diffeomorphic to \( \mathbb{R}^n \). Indeed, by differentiating the equation (2.18) and switching the order of differentiations and then taking trace, we have

\[
-(n - 1)\nabla_i R = R_{ij} \nabla_j f.
\]

By the positivity of the Ricci curvature, we have

\[
(n - 1)\nabla_X R + CR \nabla_X f \geq 0,
\]

for some positive constant \( C \) depends only on \( n \). This is equivalent to

\[
\nabla_X ((n - 1) \log R + Cf) \geq 0.
\]

That is the function \( (n - 1) \log R + Cf \) is nondecreasing along \( \gamma(s) \).

But by the assumption

\[
R \leq \frac{C}{s^2},
\]

we have

\[
\log R \to -\infty \quad \text{as} \quad s \to +\infty.
\]

So \( f(\gamma(s)) \to +\infty \) as \( s \to +\infty \). That is \( f \) is a exhaustion function on \( M \). By (2.18) we know that \( f \) is a strictly convex function, so any two level sets of \( f \) are diffeomorphic via the gradient curves of \( f \). Combining these and \( f \) is a exhaustion function, we know that \( M \) is diffeomorphic to \( \mathbb{R}^n \). So we have proved the claim. (We can have another proof by using the main result of [4].)
Next, we follow the argument of Hamilton [12] to prove that we can take a limit on $M - \{p\}$ of $g_{ij}(x, t)$ as $t \to -\infty$ and the limit is flat.

By (2.18) we have
$$\nabla_X \nabla_X f = R.$$ Integrating it we obtain
$$X(f(\gamma(s))) - X(f(\gamma(0))) = \int_0^s Rds \geq C_0 > 0$$ for some constant $C_0 > 0$. So we have $|\nabla f| \geq C_0 > 0$. Then we can evolve the function $f$ backward with time along the gradient of $f$. When we go backward in time, this is equivalent to following outwards along the gradient of $f$, and the speed $|\nabla f| \geq C_0 > 0$. So we have
$$\frac{s}{|t|} \geq C_0 \quad \text{as} \quad |t| \quad \text{large.}$$ Then
$$R \leq \frac{C}{s^2} \leq \frac{C}{C_0^2 |t|^2} \quad \text{as} \quad |t| \quad \text{large.} \quad (2.20)$$

By the equation of the Yamabe flow, we obtain
$$0 \geq \frac{\partial}{\partial t} g_{ij} = -R g_{ij} \geq -\frac{C}{C_0^2 |t|^2} g_{ij}.$$ Then by the same argument as in [12], we can take a limit on $M - \{p\}$ of $g_{ij}(x, t)$ as $t \to -\infty$ and the limit is flat.

Since $M$ is diffeomorphic to $R^n$, we know that $M - \{p\}$ is diffeomorphic to $S^{n-1} \times R$, but for $n \geq 3$, there exists no flat metric on it. So we obtain a contradiction.

Hence we complete the proof of the Proposition 2.7.

3. The Proof of the Main Theorem

Proof of the Main Theorem 1.1. We will argue by contradiction to prove our Theorem. Let $M^n$ be a noncompact conformally flat manifold with nonnegative
sectional curvature. Suppose $M^n$ has positive and bounded scalar curvature and satisfies the Ricci pinching condition:

$$R_{ij} \geq \varepsilon \cdot \text{scal} \cdot g_{ij}$$

for some $\varepsilon > 0$. We evolve the metric by the Yamabe flow:

$$\frac{\partial g_{ij}(x,t)}{\partial t} = -Rg_{ij}(x,t),$$

$$g_{ij}(x,0) = g_{ij}(x),$$

Then by Theorem 2.3 in [6], we know that the equation has a smooth solution on a maximal time interval $[0, T)$ with $T > 0$ such that either $T = +\infty$ or the evolving metric contracts to a point at a finite time $T$.

Moreover, for locally conformally flat manifolds, we have

$$R_{ijkl} = \frac{1}{n-2} (R_{ik}g_{jl} + R_{jli}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}) - \frac{R}{(n-1)(n-2)} (g_{ik}g_{jl} - g_{il}g_{jk}).$$

Then by direct computation, we have the following evolution equation:

$$\frac{\partial}{\partial t} R_{ijkl} = (n-1) \Delta R_{ijkl} - R \cdot R_{ijkl} + \frac{n-1}{(n-2)^2} [(R_{imkn}R_{mn} - R_{ik}^2)g_{jl} + (R_{jmln}R_{mn} - R_{jl}^2)g_{ik} - (R_{jmkn}R_{mn} - R_{jk}^2)g_{il} - (R_{imln}R_{mn} - R_{il}^2)g_{jk}]$$

$$= (n-1) \Delta R_{ijkl} - R \cdot R_{ijkl} + \frac{1}{(n-2)^2} (B_{ik}g_{jl} + B_{jl}g_{ik} - B_{il}g_{jk} - B_{jk}g_{il}),$$

where $B_{ij} = (n-1)|Ric|^2 g_{ij} + nRR_{ij} - n(n-1)R_{ij}^2 - R^2 g_{ij}$. In a moving frame, we have:

$$\frac{\partial}{\partial t} R_{abcd} = (n-1) \Delta R_{abcd} - R \cdot R_{abcd} + \frac{1}{(n-2)^2} (B_{ac}g_{bd} + B_{bd}g_{ac} - B_{ad}g_{bc} - B_{be}g_{ad})$$

$$+ \frac{R}{n-2} \cdot (R_{ac}g_{bd} + R_{bd}g_{ac} - R_{ad}g_{bc} - R_{be}g_{ad}).$$

At a point where $g_{ab} = \delta_{ab}$ and the Ricci tensor is diagonal:

$$Ric = \text{diag}(\lambda_1, \lambda_2, \cdots , \lambda_n),$$

with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, we also have $B_{ab}$ is diagonal and the sectional curvature

$$R_{abab} = \frac{1}{n-2} (\lambda_a + \lambda_b) - \frac{R}{(n-1)(n-2)}.$$
If at some point, the sectional curvature $R_{1212} = 0$, then $\lambda_1 + \lambda_2 = \frac{R}{n-1}$. Hence if $n \geq 4$, we have:

$$\frac{1}{(n-2)^2} (B_{aa} + B_{bb}) + \frac{R}{n-2} (\lambda_a + \lambda_b)$$

$$= \frac{1}{(n-2)^2} [2(n-1)|\text{Ric}|^2 + nR(\lambda_a + \lambda_b) - n(n-1)(\lambda_a^2 + \lambda_b^2) - 2R^2] + \frac{R^2}{n(n-2)(n-1)}$$

$$\geq \frac{1}{(n-2)^2} [\frac{2(n-1)}{n} R^2 + \frac{nR^2}{n-1} - n(n-1)\frac{R^2}{(n-1)^2} - 2R^2] + \frac{R^2}{n(n-2)(n-1)}$$

$$= \frac{n^2 - 4n + 2}{n(n-1)(n-2)^2} R^2$$

$$> 0,$$

if $n = 3$, by direct calculation, we have:

$$\frac{1}{(n-2)^2} (B_{11} + B_{22}) + \frac{R}{n-2} (\lambda_1 + \lambda_2)$$

$$= B_{11} + B_{22} + \frac{1}{2} R^2$$

$$= 4|\text{Ric}|^2 + 3R(\lambda_1 + \lambda_2) - 6(\lambda_1^2 + \lambda_2^2) - 2R^2 + \frac{1}{2} R^2$$

$$= 4(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - 6(\lambda_1^2 + \lambda_2^2)$$

$$= 4\lambda_3^2 - 2(\lambda_1^2 + \lambda_2^2)$$

$$= R^2 - 2(\lambda_1^2 + \lambda_2^2)$$

$$= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1 \lambda_2 + 2(\lambda_1 + \lambda_2)\lambda_3 - 2(\lambda_1^2 + \lambda_2^2)$$

$$= (\lambda_3^2 - \lambda_1^2) + (\lambda_3^2 - \lambda_2^2) + 2\lambda_1 \lambda_2$$

$$> 0.$$

So we obtain that the nonnegative sectional curvature is preserved under the Yamabe flow.

Next we claim that under our assumption, the solution $g_{ij}(t)$ has a long-time existence. Otherwise, using the same argument as in Theorem 1.2 in [8], we know that the Ricci pinching condition is preserved under the Yamabe flow. Then by a scaling argument as in Ricci flow, we can take a limit to obtain a noncompact solution to the Yamabe flow with constant positive Ricci curvature, which is a contradiction with Bonnet-Myers’ Theorem. So we have the long-time existence result.
By a standard rescaling argument similarly as in Ricci flow, we know that there exists a sequence of dilations of the solution which converges to a noncompact limit solution, which we also denote by $g_{ij}(t)$, of Type II or Type III with positive scalar curvature and it still satisfies the Ricci pinching condition.

Now we consider its universal covering space, then we also have a solution on its universal cover which is of Type II or Type III. So in the following we consider the limit solution is defined on its universal cover.

If the limit solution is of Type III, then by Theorem 2.5, we know that it is a homothetically expanding gradient soliton, but from Proposition 2.6, we know that there exists no such limit solution of Type III satisfies the Ricci pinching condition. So the limit must be of Type II.

Suppose the limit solution is of Type II, then by Theorem 2.5, we know that it is a homothetically steady gradient soliton. From Proposition 2.7, we also know that

$$\limsup_{s \to +\infty} Rs^2 = +\infty,$$

where $s$ is the distance function from the point $p$ where the scalar curvature $R$ assumes its maximum. Then by the result of Hamilton [12], we can take a sequence of points $x_k$ divergent to infinity and a sequence of $r_k$, such that $r_k^2 R(x_k) \to +\infty$ and

$$\frac{d(p,x_k)}{r_k} \to +\infty$$

and

$$R(x) \leq 2R(x_k)$$

for all points $x \in B(x_k,r_k)$. Then by a same argument as in Ricci flow, we obtain that $(M, R(x_k)g_{ij}, x_k)$ converge to a limit manifold $(\tilde{M}, \tilde{g}_{ij}, \tilde{x})$ with nonnegative sectional curvature. By Proposition 2.3 in [7], we know that the limit manifold will split a line. Since the Ricci pinching condition is preserved under dilations, we conclude that the limit must be also satisfies the Ricci pinching condition. And this is a contradiction.

Therefore the proof of the main theorem 1.1 is completed.
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