LINEAR RECURSIONS FOR INTEGER POINT TRANSFORMS

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Abstract. We consider the integer point transform \( \sigma_P(x) = \sum_{m \in P \cap \mathbb{Z}^n} x^m \in \mathbb{Z}[x_1^\pm 1, \ldots, x_n^\pm 1] \) of a polytope \( P \subset \mathbb{R}^n \). We show that if \( P \) is a lattice polytope then for any polytope \( Q \) the sequence \( \{\sigma_k P + Q(x)\}_{k \geq 0} \) satisfies a multivariate linear recursion that only depends on the vertices of \( P \). We recover Brion’s Theorem and by applying our results to Schur polynomials we disprove a conjecture of Alexandersson (2014).

1. Introduction

A polytope is the convex hull of finitely many points in \( \mathbb{R}^n \). A polytope is a lattice polytope if all its vertices lie in the integer lattice \( \mathbb{Z}^n \). The integer point transform of a polytope \( P \) is defined by

\[ \sigma_P(x) = \sum_{m \in P \cap \mathbb{Z}^n} x^m \in \mathbb{Z}[x_1^\pm 1, \ldots, x_n^\pm 1], \]

where \( x^m \) denotes \( x_1^{m_1} \cdots x_n^{m_n} \) for all \( m \in \mathbb{Z}^n \). In this note we study sequences \( \{\sigma_k P(x)\}_{k \geq 0} \) of integer point transforms of integer dilates of polytopes \( P \) and relatives. We prove the following linear recursion.

**Theorem 1.1.** Let \( Q \) be a polytope in \( \mathbb{R}^n \) and let \( P \) be a lattice polytope with vertex set \( V(P) = \{v_1, \ldots, v_r\} \). Then the sequence \( \{\sigma_k P + Q(x)\}_{k \geq 0} \) satisfies the linear recursion

\[ \sigma_{(k+r)P+Q}(x) = \sum_{\emptyset \neq I \subseteq [r]} (-1)^{1+|I|} x^{\sum_{i \in I} v_i} \sigma_{(k+r-|I|)P+Q}(x) \]

with characteristic polynomial

\[ \chi_{P;Q}(X) := \prod_{v \in V(P)} (X - x^v). \]

If \( Q \) is a lattice polytope, then \( \chi_{P;Q} \) is minimal.

In particular, the recursion only depends on the vertices of \( P \). This improves results by Alexandersson [2] where it was assumed that \( P \) has the integer decomposition property and \( Q = \{0\} \).

Employing classical results from valuation theory, in Section 2 we first prove a recursion for indicator functions of dilated polytopes. Then, in Section 3, we apply these results to integer point transforms and prove Theorem 1.1. We recover Brion’s Theorem in Section 4 and by applying our results to Schur polynomials we disprove a conjecture of Alexandersson [1] in Section 5.

2. Characteristic functions and valuations

In this section we prove a linear recursion for indicator functions of integer dilates of a polytope \( P \). Let \( P \) denote the set of polytopes in \( \mathbb{R}^n \) and let \( G \) be an abelian group. A valuation is a map \( \varphi: P \to G \) such that \( \varphi(\emptyset) = 0 \) and

\[ \varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q), \]
for all $P, Q \in \mathcal{P}$ such that also $P \cup Q \in \mathcal{P}$. The volume, the number of lattice points inside a polytope and the integer point transform are examples of valuations. It was shown by Volland [17] that every valuation satisfies the inclusion-exclusion property. That is, for polytopes $P, P_1, \ldots, P_r$

$$\varphi(P) = \sum_{\emptyset \neq I \subseteq [r]} (-1)^{|I|+1} \varphi(P_I),$$

where $P_I := \bigcap_{i \in I} P_i$. Stronger even, it follows from a result of Groemer [8], that if $\sum \alpha_i 1_{P_i} = 0$ for polytopes $P_1, \ldots, P_m$ and some $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}$ then $\sum \alpha_i \varphi(P_i) = 0$ where $1_P$ denotes the indicator function for every polytope $P$. A function of the form $\sum \alpha_i 1_{P_i}$ is called a polytopal simple function. By Groemer’s result, every valuation uniquely defines a homomorphism from the abelian group of polytopal simple functions to $G$, that is, every polytope can be identified with its indicator function. For valuation on lattice polytopes this was proved by McMullen [10]. It is well-known that for every affine linear map $T : \mathbb{R}^n \to \mathbb{R}^m$

$$1_P \mapsto 1_{T(P)},$$

defines a valuation. Moreover, for fixed $Q \in \mathcal{P}$, $1_P \mapsto 1_{P+Q}$ defines a valuation (see, e.g., [14]) where $P+Q = \{p+q : p \in P, q \in Q\}$ is the Minkowski sum. The family of all polytopal simple functions forms an algebra where the multiplicative structure is given by the Minkowski sum of polytopes: $1_P \star 1_Q := 1_{P+Q}$ for all polytopes $P$ and $Q$. The following result is a special case of [12, Theorem 1].

**Theorem 2.1** ([12]). Let $P$ be a polytope and $\{v_1, \ldots, v_r\} = V(P)$ the set of vertices of $P$. Then

$$1_P - 1_{v_1} \star \cdots \star (1_P - 1_{v_r}) = 0.$$  

As a consequence we obtain the following recursion on indicator functions.

**Theorem 2.2.** Let $P$ be a polytope in $\mathbb{R}^n$ and $\{v_1, \ldots, v_r\} = V(P)$ the vertex set of $P$. Then

$$1_{(k+r)P} = \sum_{\emptyset \neq I \subseteq [r]} (-1)^{|I|} 1_{Q^k_I}$$

for all $k \geq 0$ where $Q^k_I = (k+r-|I|)P + \sum_{i \in I} v_i$.

**Proof.** The proof follows from Theorem 2.1 by expanding equation (2) and multiplying both sides with $1_{kP}$. \qed

We also give a more elementary proof of Theorem 2.2 following an approach of Sam [13, Proof of Theorem 4].

**2nd proof of Theorem 2.2.** We first assume that $P$ is a simplex. After applying an affine transformation we may furthermore assume that $P$ is the $(d-1)$-dimensional standard simplex in $\mathbb{R}^d$ spanned by the unit vectors $e_1, \ldots, e_d$. Its $(k+d)$-th dilate is given by

$$(k+d)\Delta_{d-1} = \{x \in \mathbb{R}^d : x_1 + \cdots + x_d = d+k, x_1, \ldots, x_d \geq 0\}$$

For all $I \subseteq [d]$, let

$$P_I = (k+d)\Delta_{d-1} \cap \{x \in \mathbb{R}^d : x_i \geq 1 \text{ for all } i \in I\}.$$  

Then $P_I = \bigcap_{i \notin I} P_{\{i\}}$ for all $\emptyset \neq I \subseteq [r]$. As in [13] we observe that $(k+d)\Delta_{d-1} = P_\emptyset = \bigcup_{i \in [d]} P_{\{i\}}$ for all $k \geq 0$. Therefore, by inclusion-exclusion,

$$1_{(k+d)\Delta_{d-1}} = \sum_{\emptyset \neq I \subseteq [d]} (-1)^{|I|} 1_{P_I}$$

and we finish the proof of this case by observing that $P_I = (k+d-|I|)\Delta_{d-1} + \sum_{i \notin I} e_i$.

For the general case, we recall that every polytope is an affine linear projection of a standard simplex and, thus, the claim follows from equation (1). \qed
By the discussion above, Theorem 2.2 is equivalent to the following.

**Theorem 2.3.** Let $P$ be a polytope in $\mathbb{R}^n$ and $\{v_1, \ldots, v_r\} = V(P)$ the vertex set of $P$, and $\varphi: P \to G$ a valuation. Then

$$
\varphi((k+r)P) = \sum_{\varnothing \neq I \subseteq [r]} (-1)^{|I|} \varphi(Q^k_I)
$$

for all $k \geq 0$ where $Q^k_I = (k+r-|I|)P + \sum_{i \in I} v_i$.

3. A multivariate recursion

A sequence $a = \{a_k\}_{k \geq 0}$ of elements in $\mathbb{C}(x_1, \ldots, x_n)$ satisfies a **linear recursion** of order $d \geq 1$ if there are $c_1, \ldots, c_d \in \mathbb{C}(x_1, \ldots, x_n)$, $c_d \neq 0$, such that

$$a_k = \sum_{j=1}^{d} c_j a_{k-j}
$$

for all $k \geq d$. The corresponding **characteristic polynomial** $\chi_e$ is defined as $X^d - \sum_{j=1}^{d} c_j X^{d-j} \in \mathbb{C}(x_1, \ldots, x_n)[X]$. The polynomial $\chi_e$ is called **minimal** if for every vector $c' = (c'_1, \ldots, c'_d)$ corresponding to a linear recursion of $a$ we have $\chi_e|\chi_{c'}$. Since $\mathbb{C}(x_1, \ldots, x_n)[X]$ is a principal ideal domain a uniquely determined minimal polynomial exists.

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $r = |V(P)|$ be the number of vertices of $P$. Since the maps $P \to P+Q$ and also $P \to \sigma_P(x)$ define valuations, by Theorem 2.3

$$
\sigma_{(k+r)P+Q}(x) = \sum_{\varnothing \neq I \subseteq [r]} (-1)^{|I|} \sigma_{(k+r-|I|)P+\sum_{i \in I} v_i+Q}(x)
$$

where the last equation follows by observing that $\sigma_{P+v}(x) = x^v \sigma_P(x)$ for all $v \in \mathbb{Z}^n$. We observe that $\chi_{P,Q}$ is the characteristic polynomial of this linear recursion.

Now let $Q$ be a lattice polytope and suppose that $\chi_{P,Q}$ is not minimal. Then, for some vertex $u$ of $P$, $\{\sigma_{kP+Q}(x)\}_{k \geq 0}$ satisfies a linear recursion with characteristic polynomial $\prod_{v \in V(P) \setminus \{u\}} (X - x^v)$. That is

$$\sigma_{(k+r)P+Q}(x) + \sum_{j=1}^{\left| V(P) \right| - 1} (-1)^je_j(\{x^v : v \in V(P) \setminus \{u\}\})\sigma_{(k+r-j)P+Q}(x) = 0$$

where $e_j$ denotes the $j$-th elementary symmetric polynomial in $|V(P)| - 1$ variables. Now let $v$ be a vertex of $Q$ such that $u + v$ is a vertex of $P + Q$. Then $(k+r)u + v$ is a vertex of $(k+r)P + Q$ and thus $x^{(k+r)u+v}$ appears as a summand in $\sigma_{(k+r)P+Q}(x)$. However, it does not appear in $e_j(\{x^v : v \in V(P) \setminus \{u\}\})\sigma_{(k+r-j)P+Q}(x)$ for any $1 \leq j \leq |V(P)| - 1$. To see that, it suffices to argue that $(k+r)u + v$ is not contained in $(k+r-j)P + Q + \sum_{i \in I} v_i$ for any choice of $v_1, \ldots, v_j \in V(P) \setminus \{u\}$. For that, let $\ell : \mathbb{R}^n \to \mathbb{R}$ be a linear functional such that $\ell(u) > \ell(p)$ for all $p \neq u$ in $P$ and $\ell(v) > \ell(q)$ for all $q \neq v$ in $Q$. Then $\ell((k+r-j)p + q + \sum v_i) = (k+r-j)\ell(p) + \ell(q) + j(\ell(v) < (k+r-j)\ell(u) + \ell(v) + j\ell(u) = \ell((k+r)u + v)$ for all $p \in P$ and $q \in Q$. The conclusion follows.

Every linear map $f : \mathbb{R}^n \to \mathbb{R}^l$ with the property that $f(\mathbb{Z}^n) \subseteq \mathbb{Z}^l$ induces an algebra homomorphism

$$\tilde{f} : \mathbb{C}[x_1^\pm 1, \ldots, x_n^\pm 1] \to \mathbb{C}[x_1^\pm 1, \ldots, x_l^\pm 1]$$

$x^m \mapsto x^{\tilde{f}(m)}$
As a consequence of Theorem 1.1 we therefore obtain the following.

**Proposition 3.1.** Let $Q$ be a polytope in $\mathbb{R}^n$ and $P$ be a lattice polytope, and let $f : \mathbb{R}^n \to \mathbb{R}^d$ a linear map such that $f(\mathbb{Z}^n) \subseteq \mathbb{Z}^d$. Then $\{\bar{f}(\sigma_{kP+Q}(x))\}_{k \geq 0}$ satisfies a linear recursion with characteristic polynomial

$$
\chi_{fP,Q}(X) := \prod_{v \in V(P)} (X - x^{f(v)}).
$$

The following two examples show that the minimality of a characteristic polynomial is not necessarily preserved under affine transformations or taking Minkowski sums.

**Example 3.2.** If $Q$ in Theorem 1.1 is not a lattice polytope then $\chi_{P,Q}$ is not necessarily minimal. A counterexample is given by the lattice segment $P = [0, 1]$ and the point $Q = \{(0.5, 0.5)\}$ in $\mathbb{R}^2$. In that case $\sigma_{kP+Q} \equiv 0$ is constant.

**Example 3.3** (Ehrhart polynomials). For $f : \mathbb{R}^n \to \mathbb{R}_0$ and $f \equiv 0$ we obtain $\bar{f}(\sigma_{kP}(x)) = |kP \cap \mathbb{Z}^n|$ and thus recover the Ehrhart function counting lattice points in integer dilates of $P$. If $P$ is a lattice polytope then this function is known to agree with a polynomial of degree $\dim P$ as was demonstrated in [6]. Therefore the order of the minimal polynomial of the sequence is $\dim P$ as was demonstrated in [13] and is thus in general smaller than $|V(P)|$.

These examples motivate the following question.

**Question 1.** What are necessary and sufficient conditions on $Q$ and on $f$ that guarantee that $\chi_{fP,Q}$ is minimal?

### 4. Brion’s Theorem

In this section we provide a proof of Brion’s Theorem using the recursion given in Theorem 1.1. For a polytope $P \subseteq \mathbb{R}^n$ and a vertex $v$ of $P$ the tangent cone $K_v$ is defined as $\{v + w : v + \epsilon w \in P$ for $0 < \epsilon \ll 1\}$. If the polytope $P$ has rational edge directions, in particular, if it is a lattice polytope, then the integer point transform of $K_v$ is a rational function.

**Theorem 4.1** (Brion’s Theorem [5]). Let $P$ be a lattice polytope. Then

$$
\sigma_P(x) = \sum_{v \in V(P)} \sigma_{K_v}(x)
$$

as rational functions.

The following is an immediate consequence of [4, Lemma 13.5.]

**Lemma 4.2.** [4] Let $u_1, \ldots, u_k \in \mathbb{Z}^n$ such that the cone $K := \text{cone}(u_1, \ldots, u_k)$ generated by $u_1, \ldots, u_k$ is pointed. Then

$$
\sigma_K(x) = \sum_{m \in K \cap \mathbb{Z}^n} x^m
$$

is a rational function and converges absolutely for all $x$ in $\{x \in \mathbb{C}^n : |x^m| < 1$ for $i = 1, \ldots, k\}$.

A further ingredient for our proof of Brion’s Theorem is the following well-known result (see, e.g., [15, Chapter 5]).

**Lemma 4.3.** Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of elements of a field $K$ that satisfy a linear recursion of order $d$ with characteristic polynomial

$$
\prod_{i=1}^{d}(X - r_i).
$$

If all roots $r_1, \ldots, r_d$ are distinct then there are $\alpha_1, \ldots, \alpha_d \in K$ such that

$$
a_n = \sum_{i=1}^{d} \alpha_i r_i^n
$$
for all $n \in \mathbb{N}$.

Proof of Theorem 4.1. By Theorem 1.1 and Lemma 4.3 there are $c_v \in \mathbb{C}(x_1, \ldots, x_n)$ for all $v \in V(P)$ such

$$\sigma_k P(x) = \sum_{v \in V(P)} c_v x^v$$

for all $k \geq 0$. Our goal is to show that $c_w \cdot x^w = \sigma_{K_0} (x)$ as rational functions for all $w \in V(P)$, or, equivalently, that $c_w$ equals the integer point transform of the tangent cone $K_0$ of the vertex 0 of the translated polytope $P - w$. Equation (5) is equivalent to $\sigma_{k(P-w)}(x) = \sum_{v \in V(P)} c_v x^{k(v-w)}$.

As $k$ goes to infinity $\sigma_{k(P-w)}(x)$ converges absolutely to $\sigma_{K_0} (x)$ on $W_{K_0} = \{ x \in \mathbb{C}^n : |x|^w < 1 \}$ for all $v \in V(P) \setminus \{ w \}$ by Lemma 4.2. On the other hand, $\sum_{v \in V(P)} c_v x^{k(v-w)}$ converges to $c_w$. Thus $\sigma_{K_0} (x)$ and $c_w$ coincide on $W_{K_0}$ and are therefore the same as rational functions. \hfill \Box

5. Schur polynomials

In this section we apply our results to Schur polynomials.

A partition is a vector $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n)$ of weakly decreasing nonnegative integers. The number of strictly positive entries $\lambda_i$ is called the length of $\lambda$. A partition $\mu$ is smaller than a partition $\lambda$ with respect to the inclusion order if $\mu_i \leq \lambda_i$ for all $i$. The partition $\mu$ is smaller than a partition $\lambda$ with respect to the domination order, denoted $\lambda \geq \mu$, if $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n \mu_i$ and $\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i$ for all $k$. A skew Young diagram of shape $\lambda/\mu$ is an axis-parallel arrangement of unit squares in the plane centered at the coordinates $\{(i,j) \in \mathbb{Z}^2 : \mu_i < j \leq \lambda_i\}$. A semi-standard Young tableau is a Young diagram together with a filling of the boxes with natural numbers such that the numbers are strictly increasing in each column and weakly increasing in each row. Let $T_{\lambda/\mu}$ denote the set of semi-standard Young tableaux filled with numbers in $[n] = \{1, 2, \ldots, n\}$. For every $T$ in $T_{\lambda/\mu}$ let $w(T)$ be the vector $t = (t_1, \ldots, t_n)$ where $t_i$ is the number of boxes filled with $i$. The vector $w(T)$ is called the weight of $T$. The Kostka coefficient $K_{\lambda/\mu, \nu}$ equals the number of tableaux of shape $\lambda/\mu$ with weight $\mu$. In particular, $K_{\lambda/\mu, \nu} > 0$ if and only if there is $T \in T_{\lambda/\mu}$ with $w(T) = \nu$. The skew Schur polynomial of shape $\lambda/\mu$ is defined as

$$s_{\lambda/\mu}(x) = \sum_{T \in T_{\lambda/\mu}} x^{w(T)} \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

In [1] Alexandersson proved the following recursion for Schur polynomials.

Theorem 5.1 ([1, Theorem 1]). Let $n$ be a natural number and let $\nu, \lambda, \mu, \nu$ be partitions of length at most $n$ such that $\lambda \supseteq \mu$ and $k \lambda \supseteq \nu + k \mu$ for some positive integer $k$. Then there is a natural number $r$ such that the sequence $\{s_{\lambda+\nu/\lambda+\mu}(x)\}_{i=r}^\infty$ satisfies a linear recursion with characteristic polynomial

$$\chi(X) = \prod_{T \in T_{\lambda/\mu}} (X - x^{w(T)}).$$

Furthermore, in [1] the following conjecture concerning the minimal polynomial was stated. For every vector $w$ let $\overline{w}$ denote the vector obtained from $w$ by rearranging its coordinates in non-increasing order.

Conjecture 5.2 ([1, Conjecture 25]). Let $\nu, \lambda, \mu, \nu$ be as in Theorem 5.1 and let

$$W = \{ w \in \mathbb{N}^n : K_{\lambda/\mu, w} > 0 \text{ and } \overline{w} \supseteq \overline{\lambda - \mu} \}.$$

Then, for sufficiently large $r$, $\{s_{\lambda+\nu/\lambda+\mu}(x)\}_{i=r}^\infty$ satisfies a linear recursion with minimal polynomial

$$\chi(X) = \prod_{w \in W} (X - x^w).$$
Let Example 5.4 shows that also the polynomial given in Conjecture 5.2 is not minimal in general, thus refuting the characteristic polynomial given in Theorem 5.1 is in general not minimal. The next example since typically there are more lattice points in and its corresponding Gelfand-Tsetlin pattern for all $l\geq r$. Then there is an integer $r \gg 0$ such that $s_{\kappa + i\lambda/\mu}(x)$ equals $\sum_{k=1}^{n}(x_{i,k} - x_{i+1,k})$ for all $1 \leq i \leq n$. Further details may be found in [16]. It follows that

$$s_{\lambda/\mu}(x) = \sum_{p} x^{w(p)} ,$$

where $p$ is over all lattice points in $\mathbb{GL}_{\lambda/\mu}$.

As a corollary of Theorem 1.1 we obtain the following.

**Corollary 5.3.** Let $n$ be a natural number and let $\kappa, \lambda, \mu, \nu$ be partitions of length at most $n$ such that $\lambda \supseteq \mu$ and $\kappa + k\lambda \supseteq \nu + k\mu$ for some positive integer $k$. Let $V$ be the set of vertices of $\mathbb{GL}_{\lambda/\mu}$. Then there is an integer $r \gg 0$ such that $(s_{\kappa + i\lambda/\mu}(x))_{i=1}^{\infty}$ satisfies a linear recursion with characteristic polynomial

$$\chi(X) = \prod_{v \in V} (X - x^{w(v)}) .$$

**Proof.** Let $f = (\lambda, \mu)$ and $g = (\kappa, \nu)$. Then there is an $r \gg 0$ such that if $f_i < f_j$ then $rf_i + g_i < rf_j + g_j$ for all $i \neq j$. In particular, one can find a permutation $\sigma \in S_{2n}$ such that

$$f_{\sigma(1)} \leq f_{\sigma(2)} \leq \cdots \leq f_{\sigma(2n)} \text{ and } rf_{\sigma(1)} + g_{\sigma(1)} \leq \cdots \leq rf_{\sigma(2n)} + g_{\sigma(2n)} .$$

Then, by Theorem [7, Theorem 2.10],

$$\mathbb{GL}_{\kappa + i\lambda/\mu} = \mathbb{GL}_{\kappa + i\lambda/\mu} + (l - r)\mathbb{GL}_{\lambda/\mu}$$

for all $l \geq r$. The claim now follows from Proposition 3.1 since the weight function $w$ is linear. \qed

Since typically there are more lattice points in $\mathbb{GL}_{\lambda/\mu}$ than vertices, Corollary 5.3 shows that the characteristic polynomial given in Theorem 5.1 is in general not minimal. The next example shows that also the polynomial given in Conjecture 5.2 is not minimal in general, thus refuting it.

**Example 5.4.** Let $n = 3$, $\lambda = (5, 3, 1)$ and $\mu = (3, 0, 0)$. Consider the skew Young tableau $T$ and its corresponding Gelfand-Tsetlin pattern $p$ depicted in Figure 2. Then

$$w(T) = w(p) = (4, 2, 0) \succeq (3, 2, 1) = \lambda - \mu .$$
From the face structure studied in [9, 11] it follows that the coordinates of any vertex of $\text{GL}_{\lambda/\mu}$ are in the set \{0, 1, 3, 5\}. Let $x = \{x_{i,j}\}$ be a Gelfand-Tsetlin pattern that is a vertex of $\text{GL}_{\lambda/\mu}$. Then $x_{4,1}, x_{4,2}$, and $x_{3,1}$ are 0. Furthermore, $x_{2,1} \in \{0, 1\}$. If $x_{2,1} = 0$, then the sum of entries of the first row of $x$ is odd and the sum of entries of the second is even, therefore $w(x)_1$ is odd and $w(x) \neq (4, 2, 0)$. On the other hand, if $x_{2,1} = 1$, then $x_{3,2} \in \{1, 3\}$ and in that case $w(x)_2$ is odd and again $w(x) \neq (4, 2, 0)$. In summary, $(4, 2, 0) \in W$ is not the weight of a vertex of $\text{GL}_{\lambda/\mu}$ and therefore

$$
\prod_{w \in W} (X - x^w) \not\equiv \prod_{v \in V} (X - x^{w(v)}).
$$

Therefore, by Corollary 5.3, $\prod_{w \in W} (X - x^w)$ cannot be the minimal polynomial.

![Skew Young tableau](image)

**Figure 2.** The skew Young tableau $T$ and its corresponding Gelfand-Tsetlin pattern $p$.

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