Indistinguishability and the external correlation of mixtures

K. A. Kirkpatrick

New Mexico Highlands University, Las Vegas, New Mexico 87701

Experimental evidence, the heuristics of indistinguishability, and its logical inconsistency with quantum formalism all argue against the existence of a quantum mixture uncorrelated with the exterior, that is, argue for the postulate “The state of a system uncorrelated with its exterior is pure.” This is shown to be equivalent with “The state of a system describable in terms of indistinguishable pure states is pure,” and with “The state of the universe is pure”; further, it yields a quantitative expression of the traditional relation of welcher Weg information to partial coherence. It is concluded that all mixtures are “improper,” the trace-reduction of a composite system’s pure state.

1. INTRODUCTION

“The concept of interfering alternatives is fundamental to all of quantum mechanics” (Feynman and Hibbs, 1965). The relation of their interference to the indistinguishability of these alternatives is equally fundamental: “the loss of coherence [interference] in measurements on quantum states can always be traced to correlations between . . . the measuring apparatus and the system” (Scully et al., 1989). These distinguishing correlations are traditionally called welcher Weg (“which path”) information; the lack of welcher Weg correlations implies indistinguishability and coherence. Coherence, in quantum mechanics, appears as the purity of the state descriptor, and incoherence as a mixed state descriptor.

In classical probability, a preparation state is pure if it is sharp (dispersionless) in at least one variable. The mixed state, or mixture, is a convex combination of pure states, and has dispersion in all variables.

The mixture was introduced ad hoc into quantum mechanics in direct analogy with the classical mixture, as the mixing of pure state preparations — John von Neumann (1955): “if we do not even know what state is actually present — for example, when several states \( \phi_1, \phi_2, \ldots \) with the respective probabilities \( w_1, w_2, \ldots \) constitute the description . . .” we have a mixture, represented by the statistical operator \( \rho = \sum_s w_s \left| \phi_s \right\rangle \left\langle \phi_s \right| \); Bernard d’Espagnat (1995): “An ensemble obtained by combining all the elements of several [pure state] subensembles is a mixture . . .”; Asher Peres (1993): “[A] procedure in which we prepare various pure states \( u_\alpha \) with respective probabilities \( p_\alpha \) leads to a mixture. However, each of these statements ignores the issue of distinguishability: in quantum mechanics, the indistinguishable mixing of pure states results in a pure state, not a mixture.

But alongside this ad hoc introduction, the mixture arose deductively out of the quantum formalism: With the simple requirement that the statistics of a proposition not be changed by its conjunction with the trivial proposition in another system, von Neumann (1955) proved that \( \rho^S \), the statistical descriptor of a subsystem \( S \) of a joint system \( S \oplus M \) in the state \( \rho^{SM} \), is uniquely given by the partial trace \( \rho^S = \text{Tr}_M \{ \rho^{SM} \} \), which is a mixture if there is a distinguishing correlation between the variables of \( S \) and \( M \).

It is the purpose of the present paper to establish the converse (\( \text{Tr}_M \{ \rho^{SM} \} \) is not a mixture if there is no such distinguishing correlation), thereby rejecting any distinction between the mixture introduced ad hoc and the mixture representing a subsystem of a joint system: If a system’s state is a mixture, that system is necessarily correlated with another, external, system. There are no uncorrelated mixtures in quantum mechanics; mixtures that

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1 E-mail: kirkpatrick@nmhu.edu
2 A quantal state is pure iff the state operator is a projector.
3 As shown by example (Kirkpatrick, 2003a), even in classical probability the pure state needn’t be sharp in all variables.
are assigned \textit{ad hoc} to the mixing of preparations are in reality mixtures which arise from correlation with the exterior — absent such correlation the \textit{ad hoc} assignment is incorrect, and the state resulting from the mixing is pure.

Sec. 2 presents the necessary fundamentals of probability and correlation (both in the classical and the quanl setting); it also presents (perhaps for the first time) probabilistic definitions of indistinguishability independent of quantum mechanics. Sec. 3 presents a micro-review of the experimental basis for the relation between indistinguishability and coherence as stated by the well-known distinguishibility heuristics (stated here as \textbf{IP} and \textbf{DP} for combining preparation states indistinguishably or distinguishably, respectively). Sec. 4 establishes that a mixture uncorrelated with its exterior is an anomaly in quantum mechanics, which strongly suggests the non-existence of uncorrelated mixtures: \textit{A system which is uncorrelated with its exterior is in a pure state}; we denote this statement \textbf{HP}. (This is the converse of the well known fact that a system in a pure state is uncorrelated with its exterior; cf. Thm. 2.)

In Sec. 5, we show that \textbf{HP} is equivalent with a heuristic regarding the indistinguishability of the states describing (supporting the state operator of) a system. The implications of \textbf{HP} for the case of intermediate distinguishability are discussed in Sec. 6. And finally, in Sec. 7 we distinguish true mixtures from the mixture-like mathematical expression used to estimate the state resulting from an uncertain preparation.

2. \textbf{FUNDAMENTALS}

Throughout this work, probability is understood statistically: that an event has a given probability implies well-known statistical statements regarding the frequency of occurrence of that event.

When dealing with a probabilistic physical system, it is necessary to distinguish the \textit{probability state} (p-state) and the \textit{value state} (v-state): The p-state carries the probabilities of all possible events (it is the card-count in a deck of cards, the density matrix of a quantum mechanical system). The events themselves are described by the occurrence values of the variables of the systems; this description is given by the v-state.

We will consider physical systems which have several discrete-valued variables. The probability that a variable \( P \) of a system \( S \) takes on the value \( p_j \) is denoted by \( \Pr_{\sigma^S}(p_j) \), in which \( \sigma^S \) is the \textit{probability state} (p-state) determined by the preparation of \( S \). (The value proposition \( F = p_j \) will generally be denoted simply by the value \( p_j \).) A set of value propositions \( \{p_j \} \) is \textit{disjoint} iff \( \Pr_{\sigma^S}(p_j \land p_j') = \delta_{jj'} \) for all \( \sigma^S \) (i.e., all preparations), and is \textit{complete} iff \( \Pr_{\sigma^S}(\bigvee p_j) = 1 \) for all \( \sigma^S \). If the set is disjoint, complete\( \sigma^p \)-ness may be written \( \sum_j \Pr_{\sigma^S}(p_j) = 1 \). The value propositions of a single variable are complete\( \sigma^p \) and disjoint. We may also consider sets of value propositions which are not necessarily disjoint (that is, they are not the values of a single variable); we will generally denote such by Greek rather than Roman letters: \( \{ \phi_j \} \).

A system \( S \oplus M \) may be considered the \textit{composite} of the two systems \( S \) and \( M \) iff every p-state \( \sigma^S \land M \) of \( S \oplus M \) implies unique p-states \( \sigma^S \) and \( \sigma^M \) of \( S \) and \( M \), respectively, such that \( \Pr_{\sigma^S}(q) = \Pr_{\sigma^S \land M}(q \land T) \) for every value proposition \( q \) of \( S \) (with \( T \) the trivial proposition of \( M \)) — and similarly with \( S \) and \( M \) reversed. The probability of the occurrence of a value \( p \) in system \( S \) given the occurrence of the value \( q \) in system \( M \) is the \textit{conditional probability} \( \Pr_{\sigma^S \land M}(p \mid a) = \Pr_{\sigma^S \land M}(p \land a) / \Pr_{\sigma^M}(a) \).

In quantum mechanics a system \( S \) is described in a Hilbert space \( \mathcal{H}^S \). A value proposition \( q \) of the system \( S \) is represented by the normalized vector \( | q \rangle \in \mathcal{H}^S \); vectors representing disjoint value propositions are orthogonal. A p-state \( \sigma^S \) corresponds to a Hermitian statistical operator \( \hat{\rho}^S \) with \( \text{Tr}\{\hat{\rho}^S\} = 1 \), in terms of which \( \Pr_{\sigma^S}(\phi) = \text{Tr}\{\hat{\rho}^S | \phi \rangle \langle \phi |\} \). (If

\footnote{We distinguish completeness in the probability sense, \textit{complete}\( \sigma^p \), from completeness in the vector-space sense, \textit{complete}\( \sigma^v \).}
$\rho^S$ is a projector, $\rho^S = |\Psi^S\rangle\langle\Psi^S|$, and $|\Psi^S\rangle$ also represents $\sigma^S$. A disjoint set $\{p_j\}$ which is complete with respect to $\sigma^S$ corresponds to an orthogonal set $\{|p_j\rangle\}$ which spans (is complete on) the support of $\rho^S$. $\{|p_j\rangle\}$ may always be extended to an orthonormal basis of $H^S$. The joint system $S \otimes M$ is described in the product space $H^S \otimes H^M$. The conjunction of value propositions $\phi$ of $S$ and $\eta$ of $M$ is represented by the direct product of their projectors: $|\phi\eta\rangle\langle\phi\eta|$. (As a notational shorthand, we denote the direct product $|\phi\rangle\langle\eta|$ as $|\phi\eta\rangle$.) Given a joint system’s statistical operator $\rho^{SM}$, the statistical operator of a subsystem $S$ is the partial trace over the other system(s): $\rho^{S} = Tr_M \{\rho^{SM}\}$.

For the “entangled” p-state $\sigma^{AB}$ represented by $|\Psi^{AB}\rangle = \sum_s \psi_s |\alpha_s\beta_s\rangle$, we have the conditional probability expression

$$Pr_{\sigma^{AB}}(a | b) = \frac{1}{N(b)} \sum_s \psi_s \langle b | \beta_s | \alpha_s \rangle$$

(1)

(where $N(b) > 0$ normalizes $|\Psi(b)\rangle$). If the $\{|\beta_j\rangle\}$ are orthonormal, then

$$Pr_{\sigma^{AB}}(a | \beta_k) = |\langle a | \alpha_k \rangle|^2$$

(2)

### 2.1. Correlation and hermeticity

The value propositions $\phi$ of $S$ and $\eta$ of $M$ are uncorrelated in the p-state $\sigma^{SM}$ of $S \oplus M$ iff $Pr_{\sigma^{SM}}(\phi \land \eta) = Pr_{\sigma^S}(\phi) Pr_{\sigma^M}(\eta)$. Systems which are completely uncorrelated with their exteriors are an important part of quantum mechanics; because there is no common term for this condition, I introduce the definitions

**Definition 1.** A value of $S$ is **hermetic** iff it is uncorrelated with any value of any system exterior to $S$. A system is hermetic iff each of its values is hermetic.

**Definition 2.** The **hermetic environment** of a system $S$ is the smallest hermetic system which includes $S$, less $S$ itself.

(A system’s hermetic environment always exists — in the worst case, it would consist of all other systems in the universe.)

Several quantal results regarding hermeticity follow:

**Theorem 1.** The quantal system $S$ is hermetic iff $\rho^{SM} = \rho^S \otimes \rho^M$ for every system $M$.

**Proof:** Sufficiency is obvious. Necessity: Hermeticity of $S$ implies $Pr_{\sigma^{SM}}(\phi \land \eta) = Pr_{\sigma^S}(\phi) Pr_{\sigma^M}(\eta)$ for every value $\phi$ of $S$ and every value $\eta$ of every system $M$; in quantum mechanics, this is $Tr_{SM} \{\rho^{SM} | \phi\rangle\langle\phi| \otimes |\eta\rangle\langle\eta|\} = Tr_S \{\rho^S | \phi\rangle\langle\phi|\} Tr_M \{\rho^M | \eta\rangle\langle\eta|\}$ for every $|\phi\rangle \in H^S$ and every $|\eta\rangle \in H^M$; $Tr_S \{A\} Tr_M \{B\} = Tr_{SM} \{A \otimes B\}$. □

**Theorem 2.** A quantal system whose probability state is pure is hermetic.

**Proof:** The state operator of a composite system factors if one of the systems has a pure p-state (cf., e.g., Ballentine (1998), p. 219); apply Thm. 1. □

**Theorem 3.** If the p-state of a composite quantal system is pure, and one of its subsystems has no correlations with the rest of the composite system, then the p-state of that subsystem is pure.

**Proof:** Assume the state operator of the subsystem is not pure. Then it may be written as a convex sum of orthogonal projectors, the eigenstates of some observable of that subsystem. Because the composite system is pure, the Schrödinger-HJW Theorem (Schrödinger (1936); see also Kirkpatrick (2003b)) applies: those eigenstates are correlated with some observable of the remainder of the composite system, contradicting the assumption. □
2.2. Indistinguishability

That several values of a system \( S \) are indistinguishable means that nothing in the world external to \( S \) reflects which value has occurred: the statistics of every value of the exterior of \( S \) must be independent of the various indistinguishable values of \( S \). Thus,

**Definition 3.** The values \( \{ \phi_j \mid j \in I \} \), a subset of the values of a system \( S \), are **indistinguishable** iff \( \Pr_{\sigma_{S,M}}(\eta \mid \phi_j) \) is independent of \( j \in I \), for every value \( \eta \) of every system \( M \) exterior to \( S \).

**Theorem 4.** All values of a hermetic system are indistinguishable.

**Proof:** Take \( S \) to be any hermetic system; each value \( \phi_j \) is hermetic, hence uncorrelated with the exterior of \( S \). Thus, for each \( j \), \( \Pr_{\sigma_{S,M}}(\eta \mid \phi_j) = \Pr_{\sigma_M}(\eta) \) for every \( \eta \) of every exterior \( M \): the \( \{ \phi_j \} \) are indistinguishable. \( \square \)

**Theorem 5.** If a complete \( p \) set of disjoint states are indistinguishable, they are hermetic.

**Proof:** \( \{ p_j \} \) disjoint, complete \( p \): \( \sum_t \Pr_{\sigma_S}(p_t) = 1 \), \( \sum_t \Pr_{\sigma_{S,M}}(p_t \land \eta) = \Pr_{\sigma_M}(\eta) \). If the \( \{ p_j \} \) are indistinguishable, then \( \Pr_{\sigma_{S,M}}(\eta \mid p_j) = f(\eta) \) for all \( j \), hence \( \Pr_{\sigma_{S,M}}(p_j \land \eta) = \Pr(p_j)f(\eta) \); sum over \( j \) to find \( f(\eta) = \Pr_{\sigma_M}(\eta) \), hence \( p_j \) is uncorrelated with \( \eta \). \( \square \)

2.3. Full distinguishability

If several values of a system are fully distinguishable, there must be a dependable external sign of which value has occurred; given the external sign, the statistics of the system must be compatible with the corresponding occurring value. Thus,

**Definition 4.** The values \( \{ \phi_k \mid k \in D \} \) of a system \( S \) are **fully distinguishable** iff there exists, exterior to \( S \), a system \( M \) which has a set of disjoint values \( \{ b_k \mid k \in D \} \) for which \( \Pr_{\sigma_{S,M}}(q \mid b_k) = \Pr_{\sigma_S}(q) \) for all \( k \in D \), for every value \( q \) of \( S \) (where \( \sigma_{S,M} \) is the \( p \)-state of the joint system \( S \oplus M \)).

(Only if the \( \{ \phi_j \} \) are disjoint and complete \( p \) may we define their full distinguishability by the more obvious \( \Pr_{\sigma_{S,M}}(\phi_j \mid b_k) = \delta_{jk} \).)

The appendix contains several lemmas regarding distinguishability and indistinguishability in quantum systems.

3. INDISTINGUISHABILITY AND PREPARATION

Classically, the combining ("mixing") of distinct pure-state preparations necessarily results in a mixture. In quantum mechanics, however, such mixing may result in a pure \( p \)-state rather than a mixture, as we show in Sec. 3.1.

It is often claimed that the mixing of independent preparations must necessarily result in a mixture due to the indeterminacy of the phase of the several prepared pure \( p \)-states. But this conflates a reduction of the visibility of coherence with an actual loss of coherence, as we show in Sec. 3.2.

In Sec. 3.3 we present the distinguishability heuristics \( \text{IP} \) and \( \text{DP} \).

3.1. Evidence for the coherent mixing of preparations

A system is prepared randomly by one or another of several sources; each source prepares the system in a distinct pure \( p \)-state. The intensities of the sources are sufficiently low that never more than one system is available at a time. If the sources are aligned in such
a way that it is impossible in principle to determine from which source an occurrence of the system arose — hence impossible in principle to determine which pure p-state the system was prepared in — then this mixing of preparations yields a p-state which is pure, a superposition of the several source preparation states, rather than a mixture.

This particularly “quantal” behavior — the indistinguishability, the lack of welcher Weg information, of alternative pure-state preparations leading to coherent superposition — has been recognized from the very earliest days of quantum mechanics. Over the years it has been given quite direct experimental demonstration, particularly by Leonard Mandel’s group (below). The theoretical and experimental expressions of Marlan Scully’s quantum eraser (Scully and Druhl, 1982; Scully et al., 1991; Kim et al., 2000) fully exercise the connection between the presence or absence of welcher Weg information and the absence or presence of coherent superposition.

Pfleegor and Mandel (1967) demonstrated single-photon coherence with a pair of independent equal-frequency lasers so arranged that, in their words, “the localization of a photon at the detector makes it intrinsically uncertain from which of the two sources it came.” The photon which has appeared at the detector is the result of mixing two pure-state preparations (the emission of a photon from each laser), and the observed interference shows that this mixing is at least partially coherent.

The 1991 experiment of Zou, Wang, and Mandel (Wang et al., 1991; Zou et al., 1991) provides a clear demonstration of the influence of welcher Weg information. Photons from a pair of independent but phase-coherent sources (signal photons from a pair of coherently-pumped downconversion crystals) travel on variable-differential-length paths to a detector; it is extremely unlikely that photons from both sources are in the interferometer simultaneously. The correlated idler photons leave the apparatus on a common path; a variable-transmission filter causes the source of the idler photons, and hence of the corresponding signal photons, to range from indistinguishable to fully distinguishable. When indistinguishable, the signal photons are maximally coherent; as the distinguishability increases, the coherence of the signal photons decreases, so that, for full distinguishability, the signal photons are incoherently mixed. It is significant to note that the idler photons are not “detected” in any way — they, if not absorbed in the transmission filter, pass out of the apparatus and travel through space until they collide with whatever arbitrary matter might be on their path — and that the signal photons are not physically affected in any way which happens to the idlers, in particular not by changes in the transmissivity of the filter. Whether these independently prepared signal photons combine incoherently or coherently depends only on the existence or the nonexistence of their distant, distinct correlates.

3.2. Invisible coherence

Consider two sources producing the system $S$ in the p-state $e^{i\theta_j} |\psi_j \rangle$, with probabilities $w_j$ respectively; $j = 1, 2$. If these are indistinguishable productions — if there is no correlation of these two preparations with the exterior — then the p-state of the produced particle is necessarily $\sqrt{w_1} e^{i\theta_1} |\psi_1 \rangle + \sqrt{w_2} e^{i\theta_2} |\psi_2 \rangle$, a pure state, not a mixture. The coherence of this pure state may be seen in the interference in the probability of passage through a $|x\rangle$-filter, $w_1 |\langle \psi_1 |x\rangle|^2 + w_2 |\langle \psi_2 |x\rangle|^2 + \sqrt{w_1 w_2} |\langle x |\psi_1 \rangle \langle \psi_2 |x\rangle| \cos(\theta_2 - \theta_1 + \phi)$; the cosine term is the interference. However, if the phase difference fluctuates widely, the time average of this interference term vanishes: This pure, but time-dependent, p-state behaves as the mixture $w_1 |\psi_1 \rangle + w_2 |\psi_2 \rangle$ for all practical purposes (FAPP). We see here the truth at the core of the claim that the emerging ensemble is a mixture — it is a “mixture” FAPP, though pure in actuality.

Fluctuation of the phase difference is a matter of the experimental situation; if experimental technique results in $\theta_2 - \theta_1$ being constant during the observation, interference will

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4 Only the phase difference $\theta_2 - \theta_1$ is physically meaningful.
be visible and the coherence of the p-state will be apparent. Pfleegor and Mandel repeatedly “looked quickly,” each look taken over such a short time period that the phase difference was constant; Wang, Zou, and Mandel time-stabilized the phase difference by coherent pumping. (A related discussion of this matter is found in Englert et al. (1999).)

On the other hand, not only may a FAPP pseudo-mixture arise from time-fluctuating phases in an indistinguishable mixing of preparations, but a true mixture may well arise out of sloppy technique — an interferometer on a wobbly table, say. In such case, the position of the table is correlated with the phases of the photons, and directly averaging over the phases, obtaining a mixture, is equivalent with tracing out the entangled environment (the table). (See Stern et al. (1990) for an interesting treatment of this general question.) In any discussion of fundamentals, we must take care to distinguish pure states which only seem to be mixtures from true mixtures.

3.3. Distinguishable and indistinguishable preparations

If we prepare a system by randomly mixing several pure states, and do no more, then (in the interaction picture) the resulting p-state must be describable in terms of those pure states:

Restriction on mixing-preparation. The p-state of a system prepared by randomly mixing a set of alternative pure states \( \{ \phi_j \mid j \in D \} \) is supported by those pure states:

\[
\rho = \sum_{j \in D} w_j |\phi_j\rangle \langle \phi_j|.
\]

(That the support of the p-state is spanned by the several pure preparations is implicitly assumed in all discussions of the fundamentals of quantum mixtures.)

Perhaps the clearest statement — certainly the most consistent use — of the principles regarding welcher Weg distinguishability and mixing preparations is found in Feynman et al. (1965). The experiments reviewed in Sec. 3.1 strongly support this statement, which may be expressed in terms of preparation states (rather than Feynman’s processes) as the two heuristics

Heuristic for Indistinguishable Preparations (IP). The p-state of a system prepared by randomly mixing a set of indistinguishable alternative pure states \( \{ \phi_j \mid j \in D \} \), each with probability \( w_j \), is pure, \( \rho = \sum_{j \in D} w_j |\phi_j\rangle \langle \phi_j| \) (with \( |\psi_j|^2 = w_j \)).

Heuristic for Distinguishable Preparations (DP). The p-state of a system prepared by randomly mixing a set of fully distinguishable alternative pure states \( \{ \phi_j \mid j \in D \} \), each with probability \( w_j \), is a mixture, \( \rho = \sum_{j \in D} w_j |\phi_j\rangle \langle \phi_j| \).

Though the Heuristic for Indistinguishable Preparations must be considered a fundamental part of quantum mechanics, it cannot be derived from the usual Hilbert-space formalization of quantum mechanics. However, rather than postulating IP itself, we show in the next section that IP strongly suggests another condition, HP, which itself implies IP and is more suitable for statement as a postulate in a Hilbert-space formalism.

4. THE HERMETIC MIXTURE HAS NO PLACE IN QUANTUM MECHANICS

If the p-state of a hermetic system (cf. Def. 1) is not pure, we call it a hermetic mixture. In this section we critically consider the place of hermetic mixtures in quantum mechanics.

4.1. The hermetic mixture cannot be created by mixing pure states

Suppose no hermetic mixtures already exist — could we create one by mixing pure states? The following theorem answers, No — the resulting p-state would be pure or it would be a non-hermetic mixture:

Theorem 6. Assuming IP and given the prior absence of hermetic mixtures in the environment, it is impossible to construct a hermetic mixture by mixing pure-state preparations.
Proof: Let the preparation of the system $S$ vary randomly among a set of distinct possible output p-states $\{ |\alpha_j\rangle \}$. The corresponding p-states of the environment must be pure: By hypothesis, there are no hermetic mixtures, while if the p-state of this “environment” were a correlated mixture, then it must be only a subsystem of the actual environment — whose p-state therefore must be pure. Thus, when $S$ is prepared in the p-state $|\alpha_j\rangle$, the p-state of its environment $E$ is a pure state $|\eta_j\rangle$, so the p-state of $S \oplus E$ is $|\alpha_j\eta_j\rangle$. Now if these p-states $\{ |\alpha_j\eta_j\rangle \}$ were not indistinguishable, there necessarily would be a system $X$ so that for each production the p-state of $S \oplus E \oplus X$ would be $|\alpha_j\eta_j\gamma_j\rangle$, the $\{ \gamma_j \}$ not collinear (Lemma A1); but clearly then the environment of $S$ is truly $E \oplus X$, contrary to assumption, so in fact the p-states $\{ |\alpha_j\eta_j\rangle \}$ must be indistinguishable. Therefore, according to IP, the p-state of $S \oplus E$ must be the pure state $|\Psi^{SE}\rangle = \sum \gamma_t |\alpha_t\eta_t\rangle$ (for some set $\{ \gamma_t \}$). If the $\{ |\eta_j\rangle \}$ are all collinear, the state of $S$ is the pure state $|\Psi^S\rangle = \sum \gamma_t |\alpha_t\rangle$. If the $\{ |\eta_j\rangle \}$ are not all collinear, then, utilizing the Schmidt decomposition of a pure state of a composite system, we obtain $|\Psi^{SE}\rangle = \sum \psi_j |p_t\alpha_t\rangle$, with both the $\{ |p_t\rangle \}$ and the $\{ |a_j\rangle \}$ orthonormal, and with more than one $\psi_j$ non-vanishing. Then the p-state of $S$ is $\rho^S = \sum \psi_j^2 |p_t\rangle \langle p_t|$ — a non-hermetic mixture.

This is exhaustive: the hermetic mixture cannot arise from any such construction. $\square$

### 4.2. Quantum mechanics provides no formalism regarding the mixing of hermetic mixtures

Suppose, if it were possible to prepare a system as a hermetic mixture, that we were to mix several such preparations randomly — for example, suppose $S$ were prepared in the states $\rho^S_1$ and $\rho^S_2$ with the respective probabilities $w_1$ and $w_2$ — what would the resulting state be?

The intuitively obvious answer is, of course, $\rho^S = w_1 \rho^S_1 + w_2 \rho^S_2$. The equally obvious question arises immediately: How do we know this? And the only answer I can find is: this is what we would expect of mixtures in classical probability — perhaps not the most convincing approach to take in quantal matters.

So let’s look at this more carefully, using the specific example of an atomic-Young double slit apparatus, with the p-states $|p_1\rangle$ and $|p_2\rangle$ representing passage through each slit (again with probabilities $w_1$ and $w_2$). Suppose that the p-state of the environment is a mixture, $\rho^E$, uncorrelated with slit passage (so the slit passages are indistinguishable); then the p-state of $S \oplus E$ is, for each slit passage, the direct product of the pure state of $S$ with the mixed state of $M$: $|p_j\rangle \langle p_j| \otimes \rho^M$. Following the “intuitively obvious” rule, the p-state of $S \oplus M$ for the double-slit process is $\rho^{SM} = w_1 |p_1\rangle \langle p_1| \otimes \rho^E + w_2 |p_2\rangle \langle p_2| \otimes \rho^M$; thus $\rho^S = \text{Tr}_M \{ \rho^{SM} \} = w_1 |p_1\rangle \langle p_1| + w_2 |p_2\rangle \langle p_2|$ — an incoherent mixture lacking any double-slit interference! The intuitively obvious rule leads to contradiction with standard quantum mechanics. And it’s rather obvious that there’s no un-intuitive rule that’s going to save the situation: there’s not enough information in the specification of the problem to obtain $|\Psi^S\rangle = \sqrt{w_1} |p_1\rangle + \sqrt{w_2} |p_2\rangle$.

The situation is quite different if $\rho^M$ is a correlated mixture: Suppose the system $M \oplus E$ is in the pure state $|\Psi^{ME}_j\rangle$ when $S$ passes through slit $j$, with $\text{Tr}_E \{ |\Psi^{ME}_j\rangle \langle \Psi^{ME}_j| \} = \rho^M$. Then (with $|\alpha_j|^2 = w_j$), $|\Psi^{SME}_j\rangle = \alpha_1 |p_1\rangle |\Psi^{ME}_j\rangle + \alpha_2 |p_2\rangle |\Psi^{ME}_j\rangle$, and so

$$\rho^{SM} = (w_1 |p_1\rangle \langle p_1| + w_2 |p_2\rangle \langle p_2|) \otimes \rho^M + (\alpha_1 \alpha_2^* |p_1\rangle \langle p_2| \otimes \text{Tr}_E \{ |\Psi^{ME}_1\rangle \langle \Psi^{ME}_1| \} + \text{h. c.}),$$

the second term being what is missing after application of the “intuitively obvious” rule. The p-state of the system is thus

$$\rho^S = w_1 |p_1\rangle \langle p_1| + w_2 |p_2\rangle \langle p_2| + (\alpha_1 \alpha_2^* |\Psi^{ME}_1\rangle \langle \Psi^{ME}_1| |p_1\rangle \langle p_2| + \text{c. c.});$$

if the $|\Psi^{ME}_j\rangle$ are orthogonal (so the slit passages are distinguishable), $\rho^S$ is an incoherent mixture, while if they are collinear (so the slit passages are indistinguishable), $\rho^S$ is the projector of the pure state $\alpha_1 |p_1\rangle + \alpha_2 |p_2\rangle$. 

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4.3. Thus spake Ockham: The hermetic mixture does not exist

The hermetic mixture (a) cannot be distinguished phenomenologically from the correlated mixture, other than by establishing a negative — the absence, in the particular case, of an ancillary correlate; (b) is treated only partially and inconsistently by the standard formalism of quantum mechanics; (c) cannot be created by known quantal processes; and (d) plays only a metaphysical role in the discussion of unobservable aspects of “subensemble” membership. That is to say, neither reason nor evidence — only a kind of superstitious intuition — supports the physical existence of the hermetic mixture. Scientific conservatism, as expressed by Ockham, requires that the hermetic mixture be excluded by postulating that all physically existent mixtures are correlated with the exterior. This, our central result, may be stated more simply as

Postulate of Hermetic Purity (HP). The p-state of a hermetic quantal system is pure.

The following sections will explore the implications of this postulate, but several results follow immediately:

Theorem 7. \( \rho^{SM} = \rho^S \otimes \rho^M \) for every system \( M \) exterior to \( S \) iff \( \rho^S \) is a 1-projector.

Proof: Necessity: Thm. 1 and HP. Sufficiency: Thms. 1 and 2. \( \square \)

The concept of “quantum state of the universe” is controversial, particularly from a positivist viewpoint; it is interesting, though, that

Theorem 8. The purity of the p-state of the universe is equivalent with the Postulate of Hermetic Purity.

Proof: Assume HP; the universe is necessarily hermetic, hence pure. Conversely, assume the p-state of the universe to be pure; taking the composite system in Thm. 3 to be the universe, that theorem becomes HP. \( \square \)

5. DISTINGUISHABILITY HEURISTICS IN TERMS OF STATE DESCRIPTIONS

A set of states which span the support of the statistical operator are sufficient to describe the state; we formalize this as

Definition 5. The p-state of \( S \) is described by the states \( \{ | \varphi_j \rangle \in H^S | j \in D \} \) iff \( \rho^S = \sum_{\nu' \in D} w_{\nu'} | \varphi_{\nu'} \rangle \langle \varphi_{\nu'} | \) (where \( w_{\nu'} = w_{\nu' *} \) and \( \sum_{\nu' \in D} w_{\nu'} \langle \varphi_{\nu'} | \varphi_{\nu} \rangle = 1 \)).

The Restriction stated in Sec. 3.3 requires that the mixing of pure states must result in a p-state described by these pure states. Further, hermeticity of a system implies indistinguishability of its values (Thm. 4), and hermeticity of a system implies purity of the p-state (HP). This suggests a strengthening of the indistinguishability heuristics from preparations to state descriptors, and suggests that a system described by a set of indistinguishable p-states must have a pure state — that is, this suggests the

Heuristic for Indistinguishable Descriptors (ID). The p-state of a system described by (supported by) a set of indistinguishable alternative pure states is pure.

ID establishes that if any set of state descriptors is indistinguishable, then all sets are; in fact, as we show next, ID is equivalent to HP. (Although HP is proposed as a postulate, the following discussion treats it as merely a proposition whose logical relation with ID, also taken as a proposition, is to be explored.)

Theorem 9. The Postulate of Hermetic Purity is equivalent to the Heuristic for Indistinguishable Descriptors (HP \( \iff \) ID).

Proof:

ID \( \Rightarrow \) HP: Assume ID. Take \( S \) to be any hermetic system, its p-state described by a set of vectors \( \{ | \varphi_j \rangle \in H^S \} \); because \( S \) is hermetic, the \( \{ \varphi_j \} \) are indistinguishable (Thm. 4). ID requires the \( \{ | \varphi_j \rangle \} \) be linearly combined to get the state vector of \( S \), a pure state: HP.
\textbf{HP} \Rightarrow \textbf{ID}: Assume \textbf{HP}. Consider a system \( S \) described by a set of indistinguishable pure states \( \{ \phi_j \mid j \in \mathcal{I} \} \); these correspond to the set \( \{ |\phi_j\rangle \in \mathcal{H}^S \} \). Let \( E \) be the hermetic environment of \( S \). By \textbf{HP}, the p-state of \( S \oplus E, |\Psi^{SE}\rangle \), is pure; it can be expanded in terms of direct products of the \( \{ |\phi_j\rangle \} \) and any orthonormal set \( \{ |b_j\rangle \in \mathcal{H}^E \} \):

\[
|\Psi^{SE}\rangle = \sum_{s \in \mathcal{L}, t} \gamma_{st} |\phi_s b_t\rangle.
\]

\(|\Psi^{SE}\rangle\) may be rewritten in terms of a linearly independent subset of \( \{ |\phi_j\rangle \} \): Let \( \mathcal{L} \subset \mathcal{I} \) be the index set of a maximal linearly independent subset of \( \{ |\phi_j\rangle \} \), so \( |\phi_j\rangle = \sum_{p \in \mathcal{L}} \alpha_{js} |\phi_s\rangle \); for \( j \in \mathcal{L}, \alpha_{jk} = \delta_{jk} \). Then

\[
|\Psi^{SE}\rangle = \sum_{s \in \mathcal{L}, t} \hat{\gamma}_{st} |\phi_s b_t\rangle, \text{ where } \hat{\gamma}_{st} = \sum_{p \in \mathcal{L}} \gamma_{pt} \alpha_{ps}, s \in \mathcal{L}.
\]

This may be written in the correlated form

\[
|\Psi^{SE}\rangle = \sum_{s \in \mathcal{L}} \mu_s |\phi_s \lambda_s\rangle, \text{ with } \mu_j |\lambda_j\rangle = \sum_{t} \hat{\gamma}_{jt} |b_t\rangle.
\]

But, by Lemma A1(b), the \( \{ |\lambda_j\rangle \} \) must be collinear: \( |\lambda_j\rangle = e^{i\theta_j} |\Lambda\rangle \) for all \( j \). Thus the p-state of \( S \) is pure, the linear combination of the indistinguishable p-states:

\[
|\Psi^S\rangle = \sum_{s \in \mathcal{L}} e^{i\theta_s} \mu_s |\phi_s\rangle = \sum_{s \in \mathcal{L}} \psi_s |\phi_s\rangle
\]

(the \( \{ \psi_j \} \) not uniquely determined): \textbf{ID}. \( \square \)

In parallel with \textbf{ID}, we state the

**Heuristic for Distinguishable Descriptors (DD).** The p-state of a system described by (supported by) a set of fully distinguishable alternative pure states \( \{ \phi_j \mid j \in \mathcal{D} \} \) is a mixture, \( \sum_{t \in \mathcal{D}} w_t |\phi_t\rangle |\phi_t\rangle \).

**Theorem 10.** The Postulate of Hermetic Purity implies the Heuristic for Distinguishable Descriptors (\textbf{HP} \Rightarrow \textbf{DD}).

**Proof:** Assume \textbf{HP}, and a system \( S \) with a complete set of fully distinguishable values \( \{ |\phi_j\rangle \in \mathcal{H}^S \} \). The hermetic environment of \( S \) is \( E \); by \textbf{HP}, the p-state of \( S \oplus E \) is pure, \( |\Psi^{SE}\rangle \). Using the Schmidt decomposition, \( |\Psi^{SE}\rangle = \sum_s \mu_s |p_s a_s\rangle \), with the \( \{ |p_j\rangle \in \mathcal{H}^S \} \) and the \( \{ |a_j\rangle \in \mathcal{H}^E \} \) orthonormal. Expanding \( |p_s\rangle = \sum_t \gamma_{st} |\phi_t\rangle \), we obtain

\[
|\Psi^{SE}\rangle = \sum_t \mu_t |\phi_t \lambda_t\rangle, \text{ with } \mu_t |\lambda_t\rangle = \sum_s \gamma_{st} \psi_s |a_s\rangle.
\]

By Lemma A2, \( \rho^S = \sum_t w_t |\phi_t\rangle |\phi_t\rangle \): \textbf{DD}. \( \square \)

If the \( \{ |\phi_j\rangle \} \) are not linearly independent, Lemma A3(b) does not apply, and the \( \{ |\lambda_j\rangle \} \) may be fully distinguishable even though the associated p-states of the exterior (the \( \{ |\lambda_j\rangle \} \)) are not orthogonal.

**Theorem 11.** The Heuristic for Indistinguishable Descriptors implies the Heuristic for Indistinguishable Preparations; the Heuristic for Distinguishable Descriptors implies the Heuristic for Distinguishable Preparations (\textbf{ID} \Rightarrow \textbf{IP}; \textbf{DD} \Rightarrow \textbf{DP}).

**Proof:** The p-state of a system prepared by randomly mixing a set of indistinguishable alternative pure states is supported by those indistinguishable states, hence by \textbf{ID} is pure; the p-state of a system prepared by randomly mixing a set of fully distinguishable alternative pure states is supported by those distinguishable states, hence by \textbf{DD} is mixed. \( \square \)

Thus, postulating \textbf{HP} recovers the traditional distinguishability heuristics \textbf{IP} and \textbf{DP}.
6. THE GENERAL SITUATION OF PARTIAL DISTINGUISHABILITY

The categories fully distinguishable and indistinguishable, although disjoint, are not complete: the intermediate case of partial distinguishability is not dealt with by the traditional welcher Weg heuristics. Let us see what follows from HP:

A system $S$ is described in terms of the linearly independent $\{|\phi_j\rangle \in \mathcal{H}^S\}$. Let $E$ be the hermetic environment of $S$, so $S \oplus E$ is hermetic; by HP, the p-state of $S \oplus E$ is pure, $|\Psi^{SE}\rangle$. We may write this in terms of the $\{|\phi_j\rangle\}$ and any orthonormal basis $\{|b_j\rangle \in \mathcal{H}^E\}$:

$$|\Psi^{SE}\rangle = \sum_{s \gamma_s} \gamma_s |\phi_s b_s\rangle;$$

defining the $\{|\lambda_j\rangle\}$ by $\mu_s |\lambda_s\rangle = \sum_{s \gamma_s} \gamma_s |\phi_s b_s\rangle$, we obtain the expression

$$|\Psi^{SE}\rangle = \sum_s \mu_s |\phi_s \lambda_s\rangle,$$

and $\rho^{SE} = \sum_{ss'} \mu_s \mu^*_{s'} \langle \lambda_{s'} | \lambda_s \rangle |\phi_s\rangle \langle \phi_{s'} |$.

Since, by Lemma A1, indistinguishability of the $\{\phi_j\}$ requires the collinearity of the $\{|\lambda_j\rangle\}$ while, by Lemma A3, full distinguishability of the $\{\phi_j\}$ requires the orthonormality of the $\{|\lambda_j\rangle\}$, a set $\{|\lambda_j\rangle\}$ which is neither orthonormal nor collinear is necessary and sufficient to the intermediate case of partial distinguishability of the $\{\phi_j\}$. Thus Eq. (10) yields the entire range of possibilities for the $\{\phi_j\}$ between indistinguishability and full distinguishability; only the extremes are accounted for by the distinguishability heuristics.

An empirical failure of Eq. (10) would not point unambiguously to an error in the heuristics ID and DD: in contrast, such failure would directly falsify HP. Because of this greater degree of falsifiability, HP is scientifically stronger than these heuristics.

A particularly interesting situation of partial distinguishability is that in which the $\{|\lambda_j\rangle\}$ are, pairwise, either orthonormal or collinear — as for an incomplete ideal measurement. In this case, the $\{\phi_j\}$ will divide into a number of subsets of indistinguishable values, the subsets fully distinguishable from one another. Take, for example, a three-slit atomic Young apparatus with an ideal passage detector at slit 1, where the $\{\phi_j\}$ represent the passage through the slits and the $\{|\lambda_j\rangle\}$ represent the passage detector: The activation of the detector is $|d_1\rangle$ and its non-activation is $|d_0\rangle$, with $\langle d_1 | d_0 \rangle = 0$; the assumption of ideality of the passage detector gives us $|\lambda_1\rangle = |d_1\rangle$ and $|\lambda_2\rangle = |\lambda_3\rangle = |d_0\rangle$. Detecting the atoms at the screen yields the reduced-visibility interference

$$\rho^S = |\mu_1|^2 |\phi_1\rangle \langle \phi_1 | + (|\mu_2|^2 |\phi_2\rangle + |\mu_3|^2 |\phi_3\rangle) (|\mu_2|^2 |\phi_2\rangle + |\mu_3|^2 |\phi_3\rangle) (|\mu_2|^2 |\phi_2\rangle + |\mu_3|^2 |\phi_3\rangle)$$

(11) due to partial distinguishability. We see (using Eq. (2)) that selecting out the atoms at the screen which arrived in anti-coincidence with the passage detector results in full-visibility interference between $\phi_2$ and $\phi_3$ due to their indistinguishability. (Imaginative extension of the welcher Weg heuristics may yield this result; such extensions, however, are ad hoc and limited to the case at hand.)

7. THE “IGNORANCE MIXTURE” IS NOT NECESSARILY A MIXTURE

The preparation determines the p-state, from which (taking into account the further action of “the whole experimental arrangement”) all probability predictions arise. That preparation may be a random mixing of several pure-state sub-preparations $\{\phi_j\}$ each with probability $w_j$ (e.g., a polarizer whose orientation varies a little due to random rotational vibration of the optical bench). Let us specify the case (call it A) of fully distinguishable mixing; the resulting p-state, as we’ve discussed at length, is the mixture $\rho = \sum_s w_s |\phi_s\rangle \langle \phi_s |$.

There is another situation (call it B), easily conflated with (A): the preparation is one of several possibilities, $\{\phi_j\}$; we don’t know which one, but we can assign a probability (subjective or objective, depending on circumstances) $w_j$ to the possibilities (e.g., a rigidly mounted polarizer whose orientation is set with limited accuracy). In this case, we must guess at the p-state — and our best initial estimate of the preparation is the mixture-like expression $\rho_{est} = \sum_s w_s |\phi_s\rangle \langle \phi_s |$.

Observations on any number of systems produced as per (A) will be statistically consistent with the preparation $\rho = \sum_s w_s |\phi_s\rangle \langle \phi_s |$; in contrast, after the observation of even a few
systems produced as per (B), it is likely that the results will be more consistent with a different expression \( \rho_{\text{est}} = \sum_s w_s' |\phi_s\rangle\langle\phi_s| \), with the \( \{w'_s\} \) more “peaked”; with more observations, this would be expected to converge with confidence to a pure state \( |\phi_J\rangle\langle\phi_J| \) for some (presently unknown) \( J \).

Situation (B) is a true situation of ignorance: we simply do not know which preparation \( \{\phi_j\} \) was used, and, as we observe the systems, the estimate \( \rho_{\text{est}} \) “collapses” to a better estimate \( \rho_{\text{est}} \), reducing our ignorance. Ignorance is quite irrelevant to situation (A); perhaps we know, at each occurrence, in which \( \phi_j \) the system was prepared, perhaps not — nonetheless the statistics of the occurrences continue to be described by \( \rho = \sum_s w_s |\phi_s\rangle\langle\phi_s| \).

\( \rho \) is the statistical descriptor of a mixture which arises (in quantum mechanics) from correlation with the exterior; \( \rho_{\text{est}} \) is merely an estimator of the true (pure and hermetic) p-state. The use of a mixture-like estimator \( \rho_{\text{est}} \) does not contradict the conclusion of this paper that all true mixtures in quantum mechanics are accompanied by external correlation.

8. SUMMARY AND CONCLUSION

The Heuristics of Indistinguishable and Distinguishable Preparations (IP and DP) have been accepted as a fundamental part of quantum mechanics from its earliest days. They, and the intermediate case of partially distinguishable preparations, are directly and strongly supported by experiment. We have shown that IP implies that the hermetic mixture cannot be prepared, in the absence of neighboring hermetic mixtures, by mixing pure-state preparations; further, we have seen that logical anomalies would arise within the quantum formalism if hermetic mixtures were to exist. These results point to the non-existence of the hermetic mixture. This result, which we have called HP, is stronger than the traditional distinguishability heuristics (IP and DP), which are expressed in terms of mixed preparations; HP is in fact equivalent with the stronger heuristics ID and DD, which are expressed in terms of state descriptors (basis vectors).

We conclude that the statement If \( S \) is hermetic then \( \rho^S \) is pure (HP) is a necessary postulate of quantum mechanics, the correct formalization of the relation between welcher Weg information and coherence. It follows that any quantum mixture is the trace-reduction of a composite system’s pure state — an “improper” (d’Espagnat, 1995) mixture — and that every expression of a mixture in the form of a convex sum of projectors implies a correlation of those projector-states to an ancillary variable in another system.

Finally, a comment of possible interest to an area of current research: when calculating entanglement in a bipartite mixture, it is always physically legitimate to treat the problem as tripartite and pure (the third system unknown, but physically existent).

APPENDIX

The lemmas of this appendix depend on definitions 3 and 4 (of indistinguishability and full distinguishability, respectively).

In each, the p-state \( \sigma^{SM} \) of \( S \oplus M \) is the pure state \( \Psi^{SM} \).

Lemma A1. \( \Psi^{SM} = \sum_i \mu_i |\phi_i\rangle \).

(a) If the \( \{\phi_j\} \) are collinear, the \( \{\phi_j\} \) are indistinguishable.
(b) If the \( \{\phi_j\} \) are linearly independent and indistinguishable, the \( \{|\lambda_j\}\} \) are collinear.

Proof:
(a) Write the collinear \( \{|\lambda_j\}\} |\lambda_j\rangle = e^{i\theta_j} |\Lambda\rangle \) for all \( j \); then \( \Psi^{SM} = |\Psi^S\rangle \otimes |\Lambda\rangle \), with \( |\Psi^S\rangle = \sum_i \mu_i e^{i\theta_i} |\phi_i\rangle \). Because the p-state of \( S \) is pure, it is hermetic (cf. Thm. 2); by Thm. 4 the p-states \( \{\phi_j\} \) are indistinguishable.
(b) Using Eq. (1), the expression, in quantum terms, of the indistinguishability of the \( \{\phi_j\} \) in the p-state \( \Psi^{SM} \) becomes

\[
|\langle\eta|\chi_j\rangle|^2 \text{ is independent of } j \text{ for all } |\eta\rangle, \text{ where } |\chi_j\rangle = \frac{1}{N_j} \sum_i \mu_i |\phi_j\rangle |\phi_i\rangle |\lambda_i\rangle.
\]

(A1)
Thus the \( \{ |\chi_j\rangle \} \) is collinear: for all \( j \), \( |\chi_j\rangle = e^{i\theta_j} |X\rangle \).

Define \( \mathcal{Z} \equiv \{ |x\rangle \in \mathcal{H}^M \mid \langle x | X \rangle = 0 \} \). Then, from Eq. (A1),

\[
\langle \phi_j \| \left( \sum_t \mu_t \langle x | \lambda_t \| \phi_t \rangle \right) = 0, \quad \text{for all } j, \quad \text{for all } |x\rangle \in \mathcal{Z}. \quad (A2)
\]

The vector in parentheses lies in the subspace spanned by the \( \{ |\phi_j\rangle \} \) and is orthogonal to all of them; it must therefore vanish. Since the \( \{ |\phi_j\rangle \} \) are linearly independent, that vanishing requires \( \langle x | \lambda_j \rangle = 0 \) for all \( j \), for all \( |x\rangle \in \mathcal{Z} \); thus each \( \lambda_j \) is collinear with \( X \). \( \square \)

**Lemma A2.** \( |\Psi^{S,M}\rangle = \sum_t \mu_t |\phi_t \lambda_t\rangle \).

If the \( \{ \phi_j \} \) are fully distinguishable, then there must exist an orthonormal set \( \{ |b_j\rangle \in \mathcal{H}^M \} \) such that \( |\Psi^{S,M}\rangle = \sum \psi_t |\phi_t b_t\rangle \); if, further, the \( \{ |\phi_j\rangle \} \) are linearly independent, then \( |\lambda_j\rangle = |b_j\rangle \) and \( \mu_j = \psi_j \).

**Proof:** The \( \{ \phi_j \} \) are fully distinguishable, thus by definition there must exist a complete, disjoint set \( \{ b_j \} \) of \( \mathcal{M} \) in terms of which \( \Pr_{\Psi^{S,M}}(q \mid b_j) = \Pr_{\phi_j}(q \mid b_j) \) for all values \( q \) of \( \mathcal{S} \).

Using Eq. (1) with the complete, orthonormal set \( \{ |b_j\rangle \} \) corresponding to the \( \{ b_j \} \), we have the quantum expression

\[
\Pr_{\Psi^{S,M}}(q \mid b_j) = |\langle q | \Psi_j\rangle|^2, \quad \text{with } |\Psi_j\rangle = \frac{1}{N_j} \sum \mu_t |b_j \langle \lambda_t\|\phi_t\rangle \rangle. \quad (A3)
\]

Of course, \( \Pr_{\phi_j}(q) = |\langle q | \phi_j\rangle|^2 \). Full distinguishability then requires that, for all \( |q\rangle \), \( |\langle q | \Psi_j\rangle|^2 = |\langle q | \phi_j\rangle|^2 \); then \( |\Psi_j\rangle = e^{i\theta_j} |\phi_j\rangle \), and we have \( N_j e^{i\theta_j} |\phi_j\rangle = \sum \mu_t |b_j \langle \lambda_t\|\phi_t\rangle \rangle \).

Multiply this by \( \otimes |b_j\rangle \) and sum on \( j \). The \( \{ |b_j\rangle \} \) are complete and orthonormal, so \( \sum_j |b_j\rangle \langle b_j| = \mathbb{I} \) and thus \( \sum_j \sum_t \mu_t |b_j \langle \lambda_t \| \phi_t\rangle \rangle \otimes |b_j\rangle = \sum_t \mu_t |\phi_t \rangle \otimes |\lambda_t\rangle \).

Thus

\[
\sum_j N_j e^{i\theta_j} |\phi_j b_j\rangle = \sum_t \mu_t |\phi_t \lambda_t\rangle; \quad (A4)
\]

set \( \psi_j = N_j e^{i\theta_j} \). If the \( \{ |\phi_j\rangle \} \) are linearly independent, \( \psi_j |b_j\rangle = \mu_j |\lambda_j\rangle \). \( \square \)

**Lemma A3.** \( |\Psi^{S,M}\rangle = \sum_t \mu_t |\phi_t \lambda_t\rangle \).

(a) If the \( \{ |\lambda_j\rangle \} \) are orthonormal, the \( \{ \phi_j \} \) are fully distinguishable.

(b) If the linearly independent \( \{ |\phi_j\rangle \} \) are fully distinguishable, the \( \{ |\lambda_j\rangle \} \) are orthonormal.

**Proof:**

(a) By Eq. (2), \( \Pr_{\Psi^{S,M}}(q \mid \lambda_j) = \Pr_{\phi_j}(q \mid \lambda_j) \).

(b) Apply Lemma A2 to the linearly independent \( \{ \phi_j \} \). \( \square \)

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