HECKE-BOCHNER IDENTITY AND EIGENFUNCTIONS ASSOCIATED TO GELFAND PAIRS ON THE HEISENBERG GROUP

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Abstract. Let $\mathbb{H}^n$ be the $(2n+1)$-dimensional Heisenberg group, and let $K$ be a compact subgroup of $U(n)$, such that $(K, \mathbb{H}^n)$ is a Gelfand pair. Also assume that the $K$-action on $\mathbb{C}^n$ is polar. We prove a Hecke-Bochner identity associated to the Gelfand pair $(K, \mathbb{H}^n)$. For the special case $K = U(n)$, this was proved by Geller [6], giving a formula for the Weyl transform of a function $f$ of the type $f = Pg$, where $g$ is a radial function, and $P$ a bigraded solid $U(n)$-harmonic polynomial. Using our general Hecke-Bochner identity we also characterize (under some conditions) joint eigenfunctions of all differential operators on $\mathbb{H}^n$ that are invariant under the action of $K$ and the left action of $\mathbb{H}^n$.

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1. INTRODUCTION

This paper is concerned with two fundamental problems in Harmonic analysis on the Heisenberg group, $\mathbb{H}^n$. The first one is the Hecke-Bochner identity and the second one is a characterization of joint eigenfunctions for a certain family of invariant differential operators on $\mathbb{H}^n$. We first briefly recall the known results in this direction.

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The Hecke-Bochner identity on $\mathbb{R}^n$ states that (see [15], Theorem-3.10, page-158) the Fourier transform of a function $f = Pg$, where $P$ is a homogeneous solid $SO(n)$-harmonic polynomial (of degree $k$ say) and $g$ is radial, is given by $\hat{P}g = Ph$, where $h$ is a radial function given by

$$h(r) = i^{-k} \int_{s=0}^{\infty} g(s) \frac{J_{\frac{n}{2}+k-1}(rs)}{(rs)^{\frac{n}{2}+k-1}} s^{n+k-1} ds,$$

where $J_{\frac{n}{2}+k-1}$ is the Bessel’s function of order $\frac{n}{2} + k - 1$. Secondly, any eigenfunction $\varphi$ of $\triangle$, the Laplacian on $\mathbb{R}^n$, with eigenvalue $-\lambda^2$ is given by the integral representation

$$\varphi(x) = \int_{S^{n-1}} e^{i\lambda x \cdot \omega} dT(\omega),$$

where $T$ is a certain analytic functional. See Helgason ([8], Theorem 2.1, page-5) for $n = 2$ and Hashizume et al [7] for general case. Both these results can be interpreted in terms of harmonic analysis on the Gelfand pair $(\mathbb{R}^n \rtimes SO(n), SO(n))$. Note that a solid homogeneous harmonic polynomial of degree $k$ is an element which transforms according to a class one representation of $SO(n)$. Next, the Laplacian $\triangle$ is the generator of $\mathbb{R}^n \rtimes SO(n)$ invariant differential operators on $\mathbb{R}^n$. This point of views have a natural generalization to other homogeneous spaces.

In the context of Riemannian symmetric spaces $X = G/K$, Helgason ([11], Corollary 7.4) characterized all $K$-finite joint eigenfunctions for $D(G/K)$. The characterization of arbitrary joint eigenfunctions for $D(G/K)$ was done by Helgason ([10], Chapter IV, Corollary 1.6) when rank$X = 1$ and by Kashiwara et al [12] in the general case. A Hecke-Bochner type identity was established, when $X$ is of rank one, by Bray [3]. For general case see [9], Chapter-III, Corollary 5.5.

In this paper, we consider these two questions on the Heisenberg group associated to the Gelfand pair $(K, \mathbb{H}^n)$, where $K \subset U(n)$ and the $K$-action on $\mathbb{C}^n$ is polar. We prove a Hecke-Bochner type identity (Theorem 7.4), giving a formulae for the Weyl transform of a function which transforms according to a class one representation of $K$. We will see that the formulae involves generalized $K$-spherical functions, as in the case of Euclidean spaces and Riemannian symmetric spaces. For the special
case $K = U(n)$ this was already proved by Geller ([6], Theorem 4.2). Let $\mathcal{L}_K(h_n)$ be the algebra of all differential operators on $\mathbb{H}^n$ that are invariant under the action of $K$ and the left action of $\mathbb{H}^n$. Any joint eigenfunction of all $D \in \mathcal{L}_K(h_n)$ has to be of the form $f(z, t) = e^{i\lambda t} g(z)$ for some complex number $\lambda$. Following the viewpoint of Thangavelu in [16], under the assumptions that $\lambda$ is non-zero real and $e^{-((|\lambda| - \epsilon)|z|^2)}|g(z)| \in L^p(\mathbb{C}^n)$ for some $\epsilon > 0$ and $1 \leq p \leq \infty$, we characterize all $K$-finite joint eigenfunctions $f(z, t)$ of all $D \in \mathcal{L}_K(h_n)$, in terms of the representations of the Heisenberg group (Theorem 8.3). We extend this result for arbitrary (with the same growth condition) joint eigenfunctions, when $\dim V^M_\delta = 1$ for all class one representations $\delta$ of $K$; here $M$ is the stabilizer of a $K$-regular point, $V_\delta$ is the (finite dimensional) Hilbert space where the representation $\delta$ is realized and $V^M_\delta$ is the space of $M$-fixed vectors in $V_\delta$. This can be put in a different form, giving an integral representation of eigenfunctions, which for $K = U(n)$ is precisely Theorem 4.1 in [16]. We also obtain a different integral representation with an explicit kernel.

The plan of the paper is as follows. In section 2., we recall the definition of polar action of $K \subset SO(n)$ on $\mathbb{R}^n$, develop a system of polar coordinates and state some results about polar actions. In section 3., we show that the Kostant-Rallis Theorem holds for polar actions i.e each $K$-harmonic polynomial is determined by its values on a regular $K$-orbit. We also discuss the class one representations of $K$ realized on the space of $K$-harmonic polynomials and on the space of their restriction to a regular $K$-orbit. In section 4., for a class one representation $\delta$ of $K$, we consider $\delta$ type Hom($V_\delta, V_\delta$)-valued functions $G$ i.e $G : \mathbb{R}^n \rightarrow$Hom($V_\delta, V_\delta$) such that $G(k \cdot x) = \delta(k)G(x)$. We show that such a $G$ can be written in a special form, which, for the case $K = SO(n)$, is equivalent to considering a function of type $Pg$, where $P$ is a solid homogeneous $SO(n)$-harmonic polynomial of certain degree and $g$ is radial. In section 5., we mainly recall some basic facts related to the Heisenberg group, its representations and Weyl transform. We also state some results about Gelfand pairs and bounded $K$-spherical functions from [2]. Section 6., deals with the Weyl transform of $K$-invariant functions. In section 7., we prove the
main results of this paper. We start with defining generalized $K$-spherical functions, prove a Hecke-Bochner type identity for the Weyl transform. Using this we prove the uniqueness (upto a right multiplication by a constant matrix) of generalized $K$-spherical functions. We also give a formulae of generalized $K$-spherical function in terms of the representations of Heisenberg group. This formulae together with the uniqueness of generalized $K$-spherical functions will imply characterizations of $K$-finite joint eigenfunctions (with the usual growth condition) of all $D \in \mathcal{L}_K(h_n)$, which we present in section 8. Section 9. deals with square integrable (modulo the center) joint eigenfunctions. In the final section, we discuss the special case when $\dim V_\delta^M = 1$ for all class one representations $\delta$ of $K$.

2. Polar actions and coordinates

In this section we recall polar actions and develop a system of polar coordinates on the spaces upon they act. References for this section are Conlon [4], Dadok [5] and Lander [13]. Let $K$ be a compact connected subgroup of $SO(n)$ which acts naturally on $\mathbb{R}^n$. Let $\mathfrak{k}$ be the Lie algebra of $K$. We denote the inner product on $\mathbb{R}^n$ by $(\cdot, \cdot)$. Let $N_x := \{k \cdot x : k \in K\}$ be the $K$-orbit through $x$, and $K_x := \{k : k \cdot x = x\}$ be the isotropy subgroup of $x$, hence $N_x \cong K/K_x$. A $K$-orbit of maximal dimension is called a regular orbit, and any point on a regular orbit is called a regular point. A $K$-orbit through a point $x$ is called a principal orbit if $K_x$ is a subgroup of a conjugate of any other isotropy subgroup. Clearly any principal orbit is also a regular orbit. The action of $K$ on $\mathbb{R}^n$ is called polar action if there is a linear subspace $T$ of $\mathbb{R}^n$ which meets every $K$-orbit and is orthogonal to the $K$-orbit at every point i.e $(\mathfrak{k} \cdot x, T) = 0$ for all $x \in T$. Such a linear subspace $T$ is called a $K$-transversal domain. This is precisely the condition $(A)$ in the introduction of [4]. Then $\dim (T) = \dim (\mathbb{R}^n) - \dim.$ of a regular orbit ([4], Proposition 1.1). Therefore if we take a regular point $x \in T$ then clearly $A_x = T$ where $A_x = \{y \in \mathbb{R}^n : (y, \mathfrak{k} \cdot x) = 0\}$. Consequently $A_x$ meets all the orbits orthogonally. Hence the above definition of polar action is equivalent to that of Dadok [5]. Also, for polar action any orbit of maximal
dimension is principal ([4], Proposition 2.2). Therefore, regular orbits and principal orbits are equivalent for polar action.

From now on we always assume that $K$ is a compact connected subgroup of $SO(n)$ whose action on $\mathbb{R}^n$ is polar. We state some results from Conlon [4] and derive some easy consequences. Since regular orbits and principal orbits are same, we only use the word “regular orbit” instead of using both. As mentioned above we have,

**Proposition 2.1.** (Conlon [4], Proposition 1.1) Let $N \subset \mathbb{R}^n$ be a $K$-orbit of maximal dimension. Then $\dim(N) = \dim(\mathbb{R}^n) - \dim(T)$.

**Theorem 2.2.** (Conlon [4], Theorem II) Let $T \subset \mathbb{R}^n$ be a $K$-transversal domain. Then there is a finite collection $P_1, P_2, \ldots, P_r$ of hyperplanes in $T$, together with positive integers $m(i), i = 1, 2, \cdots, r$, such that for each $x \in T$,

$$\dim(N_x) = \dim(\mathbb{R}^n) - \dim(T) - \sum_{i \in I_x} m(i),$$

where $I_x = \{i : x \in P_i\}$.

**Definition 2.3.** Each $P_i$ as above is called a singular variety of multiplicity $m(i)$, and each connected component of $T \setminus \cup P_i$ a Weyl domain in $T$. The Weyl group $W = W(K,T)$ is the group of transformations of $T$ consisting of those $k \in K$ such that $k \cdot T = T$.

**Theorem 2.4.** (Conlon [4], Theorem III) If $T$ is a $K$-transversal domain, then the orthogonal reflection of $T$ in each singular variety $P_i$ exists, $W$ is a finite group generated by all such reflections, and $W$ permutes simply transitively the set of Weyl domains in $T$. If $x \in T$ lies on no singular variety, then $W$ permutes simply transitively the set $N_x \cap T$.

Fix a Weyl domain $T^+$ in $T$. As an easy consequence of the above three results we get the following corollary.

**Corollary 2.5.** All the points of $T^+$ are regular, and each regular $K$-orbit intersects $T^+$ exactly at one point.
Lemma 2.6. If $x \in T$ is regular then $K_x = K_T$, where $K_T := \{ k \in K : k \cdot q = q, \ \forall q \in T \}$ is the stabilizer of $T$.

**Proof.** See the proof of Proposition 1.1 in Conlon [4]. □

Let $M = K_T$ be as defined in the above lemma. Define the “polar coordinate mapping”

$$
\phi : T^+ \times K/M \longrightarrow \mathbb{R}^n \ \text{by} \ \phi(r, kM) = k \cdot r.
$$

Clearly $\phi$ is well defined and by Corollary 2.5, its image is precisely the set of all regular points. If $k_1 \cdot r_1 = k_2 \cdot r_2$ for $k_1, k_2 \in K$ and $r_1, r_2 \in T^+$, then $K$-orbits through $r_1$ and $r_2$ are same. By Corollary 2.5, $r_1 = r_2$ (say). Consequently $k_1^{-1}k_2$ fixes $r$ and hence belongs to $K_T$ by Lemma 2.6. Therefore $(r_1, k_1M) = (r_2, k_2M)$. So, we have proved the following proposition.

**Proposition 2.7.** The polar coordinate mapping $\phi$, defined above, is a bijection of $T^+ \times K/M$ onto the set of regular points in $\mathbb{R}^n$ whose complement has measure zero.

For a regular point $x$, if $x = \phi(r, kM)$ for $r \in T^+$ and $k \in K$ then we simply write $x = (r, kM)$ and call this the polar coordinates of $x$. It is clear from the definition of $\phi$, that $k_1 \cdot (r, k_2M) = (r, k_1k_2M)$, $r \in T^+$ and $k_1, k_2 \in K$.

**Remark 2.8.** Let $K = SO(n)$. Consider the $K$-regular point $e_1 = (1, 0, 0, \ldots, 0) \in \mathbb{R}^n$. A $K$-transversal domain $T$ can be chosen to be $T = A_{e_1} = \{(x, 0, 0, \ldots, 0) \in \mathbb{R}^n : x \in \mathbb{R} \}$, and $T^+ = \{(r, 0, 0, \ldots, 0) \in \mathbb{R}^n : r > 0 \}$, which can be identified with $(0, \infty)$. If $M$ is the stabilizer of $e_1$, via the map $kM \rightarrow k \cdot e_1$, we have the identification $K/M = K \cdot e_1 = S^{n-1}$. Therefore, by the above proposition, it follows that each regular point $x \in \mathbb{R}^n$ can be written uniquely as $x = (r, \omega) = r \omega$, $r > 0, \omega \in S^{n-1}$, which gives the usual polar coordinate system on $\mathbb{R}^n$.

We conclude this section by relating polar actions and symmetric space actions, due to Dadok [5]. First we define a symmetric space action.
Definition 2.9. The action of a connected subgroup $G$ of $SO(n)$ with Lie algebra $\mathfrak{g}$ on $\mathbb{R}^n$ is called a symmetric space action if there is a real semisimple Lie algebra $\mathfrak{u}$ with Cartan decomposition $\mathfrak{u} = \mathfrak{k}' + \mathfrak{p}$, a Lie algebra isomorphism $A : \mathfrak{g} \to \mathfrak{k}'$, and a real vector space isomorphism $L : \mathbb{R}^n \to \mathfrak{p}$ such that $L(X \cdot y) = [A(X), L(y)]$ for all $X \in \mathfrak{g}$ and $y \in \mathbb{R}^n$. Here $[,]$ denotes the Lie algebra bracket on $\mathfrak{g}$.

Remark 2.10. Let the action of $G$ be a symmetric space action. If $U$ is a connected Lie group with Lie algebra $\mathfrak{u}$, and $K'$ is a connected subgroup of $U$ with Lie algebra $\mathfrak{k}'$, then the action of $G$ on $\mathbb{R}^n$ is isomorphic to that of $\text{Ad}(K')$ on $\mathfrak{p}$, i.e. if we identify $\mathbb{R}^n$ and $\mathfrak{p}$ via the map $L$, then $G$-orbits and $\text{Ad}(K')$-orbits coincide.

The relation between a polar and a symmetric space action is provided by the following proposition.

Proposition 2.11. (Dadok [5], Proposition 6) Let $K$ be a connected, compact subgroup of $SO(n)$ whose action on $\mathbb{R}^n$ is polar. Then there exists a connected subgroup $G$ of $SO(n)$ whose action on $\mathbb{R}^n$ is a symmetric space action and whose orbits coincide with those of $K$.

3. $K$-Harmonic Polynomials

Throughout this section we assume that $K$ is a connected compact subgroup of $SO(n)$ whose action on $\mathbb{R}^n$ is polar, $T$ a $K$-transversal domain, and $M = K_T$, the centralizer of $T$. Let $S$ denote the space of polynomials on $\mathbb{R}^n$, $I \subset S$ the set of $K$-invariants in $S$ and $I_+$ the set of polynomials in $I$ without the constant term. Let $H \subset S$ denote the set of $K$-harmonic polynomials, that is, polynomials annihilated by the constant coefficient differential operators on $\mathbb{R}^n$ defined by elements in $I_+$. For more details about $K$-harmonic polynomials see Helgason [8], Chapter III. The following result is proved there (Theorem 1.1).

Theorem 3.1. $S=IH$, that is, each polynomial $p$ on $\mathbb{R}^n$ has the form $p = \sum k i_k h_k$ where $i_k$ is $K$-invariant and $h_k$ is $K$-harmonic.
Since the action of $K$ is polar by Proposition \[2.11\], there is a connected subgroup $G$ of $SO(n)$ whose action on $\mathbb{R}^n$ is a symmetric space action and whose orbits coincide with those of $K$. Let $L$, $K'$ and $p$ are as in Definition \[2.9\]. Therefore, by Remark \[2.10\], if we identify $\mathbb{R}^n$ and $p$ via the map $L$, then $K$ and $\text{Ad}(K')$ orbits coincide. Hence $I$ and $I_+$ are same for both actions and consequently so is $H$. So, for polar actions we have the following version of Kostant-Rallis Theorem (see Helgason [9], Chapter III, Theorem 2.4).

**Theorem 3.2.** Each $K$-harmonic polynomial is determined by its values on a regular $K$-orbit.

Now we briefly describe the class one representations of $K$ realized on the space $H$ and on the space of their restriction to a regular $K$-orbit. This is similar to the symmetric space theory (see Helgason [9], page-236,237 and 298,299; [8], page-533). For $x \in T$ regular, consider the embedding $K/M = N_x \subset \mathbb{R}^n$ via the map $kM \rightarrow k \cdot x$. Then as is well known (Helgason [8], Exercise A1 (iv), page-73) each $K$-finite function on $K/M$ is the restriction of a polynomial $p \in S$ which by Theorem [3.1] can be taken to be harmonic. Thus, by Theorem \[3.2\] we see that the restriction mapping $h \rightarrow h \mid_{N_x}$ is a bijection of $H$ onto the space of $K$-finite functions in $E(K/M)$ (the space of smooth functions on $K/M$). Let $\tilde{K}_M$ be the set of all inequivalent unitary irreducible representation of $K$ having $M$ fixed vector. If $\delta \in \tilde{K}_M$, let $H_\delta$ (respectively $E_\delta(K/M)$) denote the space of $K$-finite functions in $H$ (respectively $E(K/M)$) of type $\delta$. Then the restriction mapping maps $H_\delta$ onto $E_\delta(K/M)$. Let $V_\delta$ be the (finite dimensional) Hilbert space on which $\delta$ is realized and let $V_\delta^M \subset V_\delta$ be the space of $M$-fixed vectors. Let $v_1, v_2, \cdots, v_{d(\delta)}$ be an orthonormal basis of $V_\delta$ such that $v_1, v_2, \cdots, v_{l(\delta)}$ span $V_\delta^M$. Then the functions
\[kM \rightarrow \langle v_j, \delta(k)v_i \rangle \mid 1 \leq j \leq d(\delta), 1 \leq i \leq l(\delta)\]
form a basis of $E_\delta(K/M)$ (Theorem 3.5, chapter V, Helgason [8]), and
\[E_\delta(K/M) = \bigoplus_{i=1}^{l(\delta)} E_{\delta,i}(K/M),\]
(3.1)
where \( \mathcal{E}_{\delta,i}(K/M) \) is the space of functions

\[
F_{v,i}(K/M) = \langle v, \delta(k)v_i \rangle, \ v \in V_{\delta}.
\]

The map \( v \rightarrow F_{v,i} \) is an isomorphism of \( V_{\delta} \) onto \( \mathcal{E}_{\delta,i}(K/M) \) commuting with the action of \( K \). Consequently \( H_{\delta} \) decomposes into \( l(\delta) \) copies of \( \delta \). Thus we write

\[
H_{\delta} = \bigoplus_{i=1}^{l(\delta)} H_{\delta,i}, \tag{3.2}
\]

where the action of \( K \) on each \( H_{\delta,i} \) is equivalent to \( \delta \) (by decomposing \( H_{\delta} \) first into homogeneous components we can assume that the \( H_{\delta,i} \) consists of homogeneous polynomials of degree say \( d_i(\delta) \)), and the vector space \( F_{\delta} = \text{Hom}_K(V_{\delta}, H_{\delta}) \) of linear maps \( \eta \) of \( V_{\delta} \) into \( H_{\delta} \) satisfying

\[
\eta(\delta(k)v) = k \cdot (\eta(v)) \ k \in K, \ v \in V_{\delta} \tag{3.3}
\]

has dimension \( l(\delta) \).

**Remark 3.3.** Let \( K = SO(n) \). Let \( e_1, T, T^+; M \) be as in the Remark 2.8. Also we have the identification \( K/M = S^{n-1} \). In this special case, note that, the space \( H \) consists of all polynomials \( P \) such that \( \Delta P = 0 \), where \( \Delta = \Sigma \partial^2 / \partial x_i^2 \) is the usual Laplacian on \( \mathbb{R}^n \). Let \( \mathcal{H}_m \) denotes the space of all \( m \)th degree homogeneous polynomials in \( H \) and \( S_m \) denotes the space of restrictions of elements of \( \mathcal{H}_m \) to \( S^{n-1} \). The elements of \( \mathcal{H}_m \) are called solid harmonics of degree \( m \), and those of \( S_m \) are called spherical harmonics of degree \( m \). The \( K \)-action on \( K/M = S^{n-1} \) defines a unitary representation on \( L^2(S^{2n-1}) \). Clearly each \( S_m \) is a \( K \)-invariant subspace. Let \( \delta_m \) denotes the restriction of \( \delta \) to \( S_m \). In fact these describe all inequivalent, irreducible, unitary representations in \( \hat{K}_M \). Note that according to our general notation, \( H_{\delta_m} = \mathcal{H}_m, \mathcal{E}_{\delta_m}(K/M) = S_m, \) and \( l(\delta_m) = \dim V_{\delta_m}^{M} = 1 \). Let \( v^m \) be the unique (upto constant multiple) unit \( M \)-fixed vector in \( V_{\delta_m} \). Then the one dimensional vector space \( F_{\delta_m} = \text{Hom}(V_{\delta_m}, \mathcal{H}_m) \) is spanned by the linear map \( \eta_{\delta_m} : V_{\delta_m} \rightarrow \mathcal{H}_m \), where for \( v \in V_{\delta_m}, \eta_{\delta_m}(v) \) is the unique element in \( \mathcal{H}_m \) whose restriction to \( S^{n-1} \) is \( Y_v(kM) := \langle v, \delta_m(k)v^m \rangle \in S_m \) i.e \( \eta_{\delta_m}(v)(x) = |x|^m Y_v(x/|x|) \).
4. K-Type functions in matrix form

We assume that $K$ is a connected, compact subgroup of $SO(n)$ whose action on $\mathbb{R}^n$ is polar. We use all the notation from the previous two sections. For two finite dimensional vector spaces $V$ and $W$ denote the space of all linear maps from $V$ into $W$, by $\text{Hom}(V, W)$. For two positive integers $p$ and $q$ denote the space of all $p \times q$ matrices with complex entries by $\mathcal{M}_{p \times q}$. If $A$ is a set and $f : A \rightarrow \mathcal{M}_{p \times q}$ a function, then we define $f_{ij} : A \rightarrow \mathbb{R}^n$ by $f_{ij}(a) = (i, j)$th entry of $f(a)$, for $a \in A$. For $\delta \in \hat{K}_M$ define $X^\delta(\mathbb{R}^n)$ to be the set of all functions $F : \mathbb{R}^n \rightarrow \text{Hom}(V^M_\delta, V_\delta)$ satisfying the condition

$$F(k \cdot x) = \delta(k)F(x) \quad \forall \; x \in \mathbb{R}^n, \; k \in K,$$

and $Y^\delta(\mathbb{R}^n)$ to be the set of all functions $G : \mathbb{R}^n \rightarrow \text{Hom}(V_\delta, V_\delta)$ satisfying the conditions

$$G(k \cdot x) = \delta(k)G(x), \quad G(x)\delta(m) = G(x) \quad \forall \; x \in \mathbb{R}^n, \; k \in K, \; m \in M.$$

Here the multiplications are the compositions of linear maps. Proposition 3.1 below says that the sets $X^\delta(\mathbb{R}^n)$ and $Y^\delta(\mathbb{R}^n)$ can be identified. Also, define $\mathcal{E}^\delta(\mathbb{R}^n)$ to be the space of all smooth functions in $X^\delta(\mathbb{R}^n)$. Choose an orthonormal ordered basis $b = \{v_1, v_2, \ldots, v_{d(\delta)}\}$ for $V_\delta$, so that $b^M = \{v_1, v_2, \ldots, v_{l(\delta)}\}$ form an ordered basis for $V^M_\delta$. Identify $\delta$ with its matrix representation with respect to the basis $b$. Then we can identify $X^\delta(\mathbb{R}^n)$ with the space of all functions

$$F : \mathbb{R}^n \rightarrow \mathcal{M}_{d(\delta) \times l(\delta)}$$

satisfying (4.1) (but now, the multiplications are simply matrix multiplications), via the matrix representation with respect to bases $b$ for $V_\delta$ and $b^M$ for $V^M_\delta$. Similarly identify $Y^\delta(\mathbb{R}^n)$ and $\mathcal{E}^\delta(\mathbb{R}^n)$ with their corresponding matrix representations with
respect to bases $b$ and $b^M$. Through out this paper, we use these identifications with respect to the basis $b$ and $b^M$. Define
\[
Y^\delta : K/M \longrightarrow \mathcal{M}_{d(\delta) \times l(\delta)}
\]
by
\[
Y^\delta_{ij}(kM) = \delta_{ij}(k) = \langle \delta(k)v_j, v_i \rangle, \ 1 \leq i \leq d(\delta), 1 \leq j \leq l(\delta).
\]
If $\check{\delta}$ denote the contragredient representation, choose $V_\check{\delta} = V_\delta^\ast$ (the dual vector space of $V_\delta$) with inner product $\langle \cdot, \cdot \rangle$ defined by $\langle v^\ast, w^\ast \rangle = \langle w, v \rangle$, $v, w \in V_\delta$. Take the orthonormal ordered basis $b^\ast$ of $V_\check{\delta} = \mathcal{V}_\check{\delta}$ to be the dual basis $\{v^\ast_1, v^\ast_2, \ldots, v^\ast_{d(\delta)}\}$. Then $b^{\ast M} = \{v^\ast_1, v^\ast_2, \ldots, v^\ast_{l(\check{\delta})}\}$ will be a basis for $V_\check{\delta}^M$. Identify $\check{\delta}$ with its matrix representation with respect to the basis $b^\ast$. Then $\check{\delta}_{ij}(k) = \delta_{ij}(k)$. Therefore $\{Y^\delta_{ij}(kM) : 1 \leq i \leq d(\delta), 1 \leq j \leq l(\delta)\}$ form a basis for $E_\check{\delta}(K/M)$. For more details about contragredient representation see Helgason \[8\], page-393,533. Now, take an ordered basis $e = \{\eta_1, \eta_2, \ldots, \eta_{l(\check{\delta})}\}$ for $F_\check{\delta} = \text{Hom}_K(V_\check{\delta}, H_\check{\delta})$. Define
\[
P^\delta : \mathbb{R}^n \longrightarrow \mathcal{M}_{d(\delta) \times l(\delta)}
\]
by
\[
P^\delta_{ij}(x) = \eta_j(v^\ast_i)(x), \ 1 \leq i \leq d(\delta), 1 \leq j \leq l(\delta).
\]
Since $\check{\eta}_j(kv^\ast_i) = k \cdot (\check{\eta}_j(v^\ast_i))$, using the fact that $\check{\delta}_{ij}(k) = \delta_{ij}(k)$ and $\check{\delta}$ is unitary, one can show that $P^\delta(k \cdot x) = \delta(k)P^\delta(x)$. Hence $P^\delta \in \mathcal{E}^\delta(\mathbb{R}^n)$. Define
\[
\Upsilon^\delta : \mathbb{R}^n \longrightarrow \mathcal{M}_{l(\check{\delta}) \times l(\check{\delta})}
\]
by
\[
\Upsilon^\delta(x) = [P^\delta(x)]^\ast[P^\delta(x)].
\] (4.3)
Here $\ast$ denotes the matrix adjoint. Clearly $\Upsilon^\delta$ is $K$-invariant.

**Remark 4.1.** Let $K = SO(n)$. We describe $Y^\delta$, $P^\delta$ and $\Upsilon^\delta$ in this special case. Let $e_1, T, T^+$, $M$ be as in the Remark 2.8 and $\mathcal{H}_m, S_m, \delta_m, v^m, \eta^m$, be as in Remark 3.3. Choose an ordered orthonormal basis $\{v_1, v_2, \ldots, v_{d(m)}\}$ for $V_{\delta_m}$, such that $\{v_1 = v^m\}$
is the orthonormal basis for $V_{\delta_m}^M$. Then \( \{ Y_{i1}^{\delta_m}(kM) = \langle \delta_m(k)v_1, v_i \rangle : 1 \leq i \leq d(m) \} \) forms an orthogonal basis for \( E_{\delta_m}^v(K/M) = S_m = S_m \), and \( \sum_{i=1}^{d(\delta)} |Y_{i1}^{\delta_m}(kM)|^2 = 1 \).

Take \( \{ \eta_i \} \) as a basis for \( F_{\delta_m}^v \). Then, by Remark 3.3, for \( x = (r, kM) = (r, \omega) \),

\[
P_{i1}^{\delta_m}(x) = \eta_i^v(v_i^*)(x) = r^m \langle v_i^*, \delta_m(k)v_i^* \rangle = r^m \langle \delta_m(k)v_1, v_i \rangle = r^m Y_{i1}^{\delta_m}(kM) = |x|^m Y_{i1}^{\delta_m}(\omega)
\]

i.e. \( P_{i1}^{\delta_m} \) is the unique element in \( H_m \) whose restriction to \( S^{n-1} \) is \( Y_{ij}^{\delta_m} \).

From the above discussion we can prove the following: Take \( P_{i}^{m} \in H_m \), and \( Y_{i}^{m} \in S_m \) to be their restrictions to \( S^{n-1} \) so that \( \{ Y_{i}^{m} : i = 1, 2, \ldots, d(m) \} \) forms an orthonormal basis for \( S_m \). Then it is possible to choose orthonormal ordered bases \( b = \{ v_1, v_2, \ldots, v_d(m) \} \) for \( V_{\delta_m} \) and \( b^M = \{ v_1 \} \) for \( V_{\delta_m}^M \), so that, with respect to these bases, \( Y_{\delta_m} : S^{n-1} \rightarrow M_{d(m) \times 1} \) is given by

\[
Y_{\delta_m}(\omega) = \sqrt{\frac{|S^{n-1}|}{d(m)}} \begin{bmatrix} Y_{1}^{m}(\omega), Y_{2}^{m}(\omega), \ldots, Y_{d(m)}^{m}(\omega) \end{bmatrix}^t, \omega \in S^{n-1}.
\] (4.4)

We can choose a basis \( e \) for \( F_{\delta_m} \) so that, \( P_{\delta_m} : \mathbb{R}^n \rightarrow M_{d(m) \times 1} \) is given by

\[
P_{\delta_m}(x) = \sqrt{\frac{|S^{n-1}|}{d(m)}} \begin{bmatrix} P_{1}^{m}(x), P_{2}^{m}(x), \ldots, P_{d(m)}^{m}(x) \end{bmatrix}^t, x \in \mathbb{R}^n.
\] (4.5)

In particular,

\[
P_{\delta_m}(x) = |x|^m Y_{\delta_m}(x/|x|).
\] (4.6)

Also, we have

\[
\Upsilon_{\delta_m}(x) = \frac{|S^{n-1}|}{d(m)} \sum_{i=1}^{d(m)} P_{i}^{m}(x) P_{i}^{m}(x) = \frac{|S^{n-1}|}{d(m)} |x|^{2m} \sum_{i=1}^{d(m)} |Y_{i}^{m}(\omega)|^2 = |x|^{2m}.
\]

**Proposition 4.2.** Each \( G \in Y^\delta(\mathbb{R}^n) \) is determined by its restriction on \( V_{\delta}^M \).

**Proof.** If \( G \in Y^\delta(\mathbb{R}^n) \), then \( G \) is identified with its \( (d(\delta) \times d(\delta)) \) matrix with respect to the fixed basis \( b \). Hence it is enough to show that all the entries in last \( (d(\delta) - l(\delta)) \) columns of \( G \) are zero. Since \( G(x)\delta(m) = G(x) \) for all \( m \in M \), equating the matrix entries on both sides we get, for \( 1 \leq i, j \leq d(\delta) \),

\[
G_{ij}(x) = \sum_{p=1}^{d(\delta)} G_{ip}(x)\delta_{pj}(m), \forall m \in M.
\] (4.7)
Since \( v_j \in (V_\delta^M)\perp \) for \( j \geq l(\delta) \), \( \int_M \delta(m)v_jdm = 0 \) if \( j \geq l(\delta) \). So,

\[
\int_M \delta_{p_j}(m)dm = \int_M \langle \delta(m)v_j, v_p \rangle dm = 0, \quad 1 \leq p \leq d(\delta), \quad j \geq l(\delta). \tag{4.8}
\]

Therefore for \( j \geq l(\delta) \), integrating both side of \([4.7]\) over \( M \) we get the desired result. \( \square \)

**Lemma 4.3.** Suppose \( F \) is in \( \mathcal{E}^\delta(\mathbb{R}^n) \). Then there is a unique function \( G_0 : T^+ \rightarrow \mathcal{M}_{l(\delta)\times l(\delta)} \) such that for all regular points \( x = (r, kM) \),

\[
F(x) = Y^\delta(kM)G_0(r).
\]

**Proof.** First note that the uniqueness follows from the fact that \( Y^\delta(kM) \) has a left inverse namely \( [Y^\delta(kM)]^\ast \). Since \( F(\sigma \cdot x) = \delta(\sigma)F(x) \) for all \( \sigma \in K \), we can write (for \( x = (r, kM) \) regular)

\[
F(x) = \int_K \delta(\sigma)^{-1}F(\sigma \cdot x)d\sigma
\]

\[
= \int_K \delta(\sigma)^{-1}F(\sigma \cdot (r, kM))d\sigma
\]

\[
= \int_K \delta(\sigma)^{-1}F(r, \sigma kM)d\sigma
\]

\[
= \int_K \delta(\sigma k^{-1})^{-1}F(r, \sigma M)d\sigma
\]

\[
= \delta(k) \int_K \delta(\sigma)^{-1}F(r, \sigma M)d\sigma
\]

\[
= \delta(k) \int_K \int_M \delta(\sigma m)^{-1}F(r, \sigma M)d\sigma dm = \delta(k)G'_0(r),
\]

where

\[
G'_0(r) = \int_K \int_M \delta(\sigma m)^{-1}F(r, \sigma M)d\sigma dm.
\]

Now,

\[
\delta_{ij}(\sigma m) = \sum_{p=1}^{d(\delta)} \delta_{ip}(\sigma)\delta_{pj}(m).
\]

Integrating both sides over \( M \) and using \([4.8]\) we get (for each \( \sigma \in K \)),

\[
\int_M \delta_{ij}(\sigma m)dm = 0, \quad 1 \leq i \leq d(\delta), \quad j \geq l(\delta).
\]
Since \( \delta(\sigma m)^{-1} = \delta(\overline{\sigma m})^t \), all the entries in last \( (d(\delta) - l(\delta)) \) rows of the matrix \( \int_M \delta(\sigma m)^{-1} dm \) are zero for all \( \sigma \in K \), and consequently so is for the \( d(\delta) \times l(\delta) \) matrix 

\[
G'_0(r) = \int_K \int_M \delta(\sigma m)^{-1} F(r, \sigma M) d\sigma dm
\]

(note that \( F \) is a \( (d(\delta) \times l(\delta)) \) matrix). Therefore, if we define \( \delta(k) G'_0(r) = Y^\delta(kM) G_0(r) \), since first \( l(\delta) \) columns in the matrix \( \delta(k) \) are precisely the columns in \( Y^\delta(kM) \). Hence the proof.

\[ \square \]

**Corollary 4.4.** Let \( F \in \mathcal{E}^\delta(\mathbb{R}^n) \). Then the \( j \)th column of \( F \) is determined by \( F_{1j} \).

In particular \( F \) is determined by its first row.

**Proof.** Let \( F \in \mathcal{E}^\delta(\mathbb{R}^n) \) be such that all the entries in first row are identically zero. We have to show that \( F \equiv 0 \). Let \( G_0 \) be as in the previous lemma. We have

\[
\sum_{p=1}^{l(\delta)} Y^\delta_{1p}(kM)(G_0)_{pj}(r) = 0 \text{ for all regular points } (r, kM).
\]

Since \( Y^\delta_{1p} \)'s are linearly independent, for each \( r \in T^+ \), \( (G_0)_{pj}(r) = 0 \), \( p = 1, 2, \cdots l(\delta) \); and consequently the \( j \)th column of \( F \) is zero on the set of regular points. The set of regular points being dense in \( \mathbb{R}^n \), we are done.

\[ \square \]

**Remark 4.5.** Using the above arguments, one can show the following : For \( F \in \mathcal{E}^\delta(\mathbb{R}^n) \), let \( V_F \) denote the finite dimensional vector space spanned by \( F_{ij} \)'s, and \( V_F^i \) denote the space spanned by the entries of \( i \)th row in \( F \). Let \( m(\delta) \) be the number of linearly independent columns in \( F \). Also assume that first \( m(\delta) \) columns are linearly independent. Then \( \{F_{ij} : j = 1, 2, \cdots, m(\delta)\} \) form a basis for \( V_F^i \); and \( \{F_{ij} : i = 1, 2, \cdots d(\delta); j = 1, 2, \cdots, m(\delta)\} \) form a basis for \( V_F \). In particular, \( \dim V_F = m(\delta) l(\delta) \).
Lemma 4.6. There is a unique function $J_\delta : T^+ \longrightarrow \mathcal{M}_{l(\delta) \times l(\delta)}$ such that for all regular point $x = (r, kM)$ in $\mathbb{R}^n$,

$$P^\delta(x) = Y^\delta(kM)J_\delta(r). \quad (4.9)$$

Also for each $r \in T^+$, $J^\delta(r)$ is invertible, and consequently for all regular point $x = (r, kM)$

$$Y^\delta(kM) = P^\delta(x)[J_\delta(r)]^{-1}. \quad (4.10)$$

Proof. First part follows from Lemma 4.3 since $P^\delta \in \mathcal{E}_\delta(\mathbb{R}^n)$. Now, if possible, let for some $r_0$ in $T^+$, $J_\delta(r_0)$ be not invertible i.e $\det(J_\delta(r_0)) = 0$ (where det stands for the determinant). This implies that the columns of $J_\delta(r_0)$ namely $[J_{1j}(r_0), J_{2j}(r_0), \ldots, J_{l(\delta)j}(r_0)]^t$, $1 \leq j \leq l(\delta)$ are linearly dependent as vectors in $\mathbb{C}^{l(\delta)}$. Equating the entries of first row in (4.9) for $x = (r_0, kM)$ we get ($1 \leq j \leq l(\delta)$),

$$P^\delta_{1j}(x) = Y^\delta_{11}(kM)J_{1j}(r_0) + Y^\delta_{12}(kM)J_{2j}(r_0) + \cdots + Y^\delta_{1l(\delta)}(kM)J_{l(\delta)j}(r_0).$$

Therefore $P^\delta_{1j}$ is linearly dependent when restricted to the orbit through $r_0$ and hence by Kostant-Rallis Theorem (Theorem 3.2) $P^\delta_{1j}$ are linearly dependent which is a contradiction. Therefore $J^\delta(r)$ is invertible for all $r \in T^+$.

Remark 4.7. (i) Let $x = (r, kM)$ be a regular point. Since $Y^\delta(kM)$ has a left inverse, and $J_\delta(r)$ is invertible $P^\delta(x)$ also has a left inverse.

(ii) The function $J_\delta$ is related to $\Upsilon_\delta$ (see (4.3)) by

$$[J_\delta(r)]^*[J_\delta(r)] = \Upsilon_\delta(r), \quad \forall \ r \in T^+. \quad (4.11)$$

(iii) Let $K = SO(n)$. Let $Y^\delta_m$, $P^\delta_m$ be as in (4.4) and (4.5). Then we have seen that

$$P^\delta_m(x) = r^mY^\delta_m(\omega), \quad x = r\omega, \quad r > 0, \omega \in S^{n-1}.$$

Therefore $J^\delta_m : T^+ = (0, \infty) \rightarrow \mathcal{M}_{1 \times 1}$ is given by $J^\delta_m(r) = r^m$.

The next proposition follows by using (4.10) in Lemma 4.3 and by (i) in the previous remark.
Proposition 4.8. Suppose $F$ is in $\mathcal{E}^\delta(\mathbb{R}^n)$. Then there is a unique (on the set of regular points) $K$-invariant function $G : \mathbb{R}^n \to \mathcal{M}_{l(\delta) \times l(\delta)}$ such that for all regular points $x$ (hence for almost every $x$),

$$F(x) = P^\delta(x)G(x).$$

Throughout this paper we use the following convention: when we say that a matrix-valued function is a polynomial we mean that each entry of the function is a polynomial.

Corollary 4.9. Let $F \in \mathcal{E}^\delta(\mathbb{R}^n)$ be a polynomial. Then there is a unique $K$-invariant polynomial $G : \mathbb{R}^n \to \mathcal{M}_{l(\delta) \times l(\delta)}$ such that for all $x \in \mathbb{R}^n$,

$$F(x) = P^\delta(x)G(x).$$

Proof. Since the set of regular points is dense in $\mathbb{R}^n$, uniqueness follows from Remark 4.7 (i). Now let $G$ be as in the previous proposition. It is enough to show that each entry of $G$ is equal to a polynomial on the set of regular points. Consider $F_{11}$. For all regular points $x$ we have

$$F_{11}(x) = \sum_{p=1}^{l(\delta)} P^\delta_{1p}(x)G_{p1}(x). \tag{4.12}$$

By Theorem 3.1 we have

$$S^\gamma_\delta = IH^\gamma_\delta,$$

where $S^\gamma_\delta \subset S$ denotes the space of all polynomials of type $^\gamma_\delta$. Clearly $F_{11} \in S^\gamma_\delta$. Therefore there exists $K$-invariant polynomials $I_{ij}$ such that for all $x \in \mathbb{R}^n$

$$F_{11}(x) = \sum_{i=1}^{d(\delta)} \sum_{j=1}^{l(\delta)} I_{ij}(x)P^\delta_{ij}(x). \tag{4.13}$$

Since $P^\delta_{ij}$s are linearly independent, by Kostant-Rallis Theorem (Theorem 3.2) so are their restrictions to any regular orbit. Comparing equations (4.12) and (4.13), restricted to a orbit passing through a regular point $x$, we get $G_{p1}(x) = I_{p1}(x)$ for all $p = 1, 2, \cdots l(\delta)$. Similar proof works for other entries of $G$. \qed
5. HEISENBERG GROUP: REPRESENTATIONS, WEYL TRANSFORM, SPHERICAL FUNCTIONS

The Heisenberg group $H^n$ is the Lie group with underlying manifold $\mathbb{C}^n \times \mathbb{R}$ and group operation

$$(z,t)(z',t') = (z + z', t + t' + \text{Im}(z \cdot \bar{z}')).$$

For the following see Geller [6]. For real non-zero $\lambda$, let $H_\lambda = \{u \text{ holomorphic on } \mathbb{C}^n : \int_{\mathbb{C}^n} |u(w)|^2 d\bar{w}^\lambda = ||u||^2 < \infty\}$, where the measure $d\bar{w}^\lambda$ is given by

$$d\bar{w}^\lambda = (2|\lambda|/\pi)^n e^{-2|\lambda||w|^2} dwd\bar{w}.$$

The space $H_\lambda$ is a Hilbert space and an orthonormal basis is given by $\{u_\lambda^\nu : \nu \in \mathbb{Z}_+^n\}$, where $\mathbb{Z}_+^n$ is the set of non-negative $n$-tuple, and

$$u_\lambda^\nu(w) = [(2|\lambda|)^{1/2}w]^{\nu}(\nu!)^{1/2}.$$

(Here $\nu! = \Pi_{j=1}^n \nu_j!$ and $w^\nu = \Pi_{j=1}^n w_j^{\nu_j}$.) Let $\mathcal{O}(H^\lambda)$ denote the set of all linear operators in $H^\lambda$ whose domain of definition contains $\mathcal{P}(\mathbb{C}^n)$, the space of holomorphic polynomial on $\mathbb{C}^n$. For $\lambda > 0$, define $\overline{W}_j^\lambda$, $W_j^\lambda \in \mathcal{O}(H^\lambda)$ as follows: if $P \in \mathcal{P}(\mathbb{C}^n)$,

$$\overline{W}_j^\lambda P(w) = 2|\lambda|w_j P(w) \text{ and } W_j^\lambda P(w) = \frac{\partial P}{\partial w_j}(w),$$

while if $\lambda < 0$ the situation is reversed (In Geller [6], the notation $W_j^+, W_j^-$ are used for $\overline{W}_j^\lambda$, $W_j^\lambda$ respectively). We have the commutation relations

$$[\overline{W}_j^\lambda, W_k^\lambda] = -2\delta_{jk}\lambda I, \; [W_j^\lambda, W_k^\lambda] = 0, \; [\overline{W}_j^\lambda, \overline{W}_k^\lambda] = 0,$$

where $I$ denote the identity operator. Let $\overline{W}^\lambda = (\overline{W}_1^\lambda, \overline{W}_1^\lambda, \ldots, \overline{W}_n^\lambda)$;

$W^\lambda = (W_1^\lambda, W_1^\lambda, \ldots W_n^\lambda)$; and for $z \in \mathbb{C}^n$, let $z \cdot \overline{W}^\lambda$, $z \cdot W^\lambda$ denote the operators $z_1\overline{W}_1^\lambda + z_2\overline{W}_2^\lambda + \cdots + z_n\overline{W}_n^\lambda$ and $z_1W_1^\lambda + z_2W_2^\lambda + \cdots + z_nW_n^\lambda$ respectively. Then $i(-z \cdot \overline{W}^\lambda + \bar{z} \cdot W^\lambda)$ being self-adjoint,

$$V_z^\lambda = \exp(-z \cdot \overline{W}^\lambda + \bar{z} \cdot W^\lambda)$$
extends to a unitary operator on \( \mathcal{H}^\lambda \) which satisfy

\[
V_z^\lambda V_w^\lambda = \exp(2i\lambda \Im(z \cdot \bar{w})) V_{z+w}^\lambda,
\]

and has an explicit formulae given as follows: if \( u \in \mathcal{H}_\lambda \),

\[
(V_z^\lambda u)(w) = u(w + \bar{z}) \exp[-2\lambda(w \cdot z + |z|^2/2)] \text{ for } \lambda > 0 \\
= u(w - z) \exp[2\lambda(-w \cdot \bar{z} + |z|^2/2)] \text{ for } \lambda < 0.
\]

In view of (5.2) we have a representation \( \Pi^\lambda \) of \( \mathbb{H}^n \) on \( \mathcal{H}_\lambda \), given by \( \Pi^\lambda(z,t) = e^{i\lambda t} V_z^\lambda \).

Explicitly \( \Pi^\lambda \) is given as follows: if \( u \in \mathcal{H}_\lambda \),

\[
(\Pi^\lambda(z,t)u)(w) = u(w + \bar{z}) \exp[-2\lambda(w \cdot z + |z|^2/2)] e^{i\lambda t} \text{ for } \lambda > 0 \\
= u(w - z) \exp[2\lambda(-w \cdot \bar{z} + |z|^2/2)] e^{i\lambda t} \text{ for } \lambda < 0.
\]

In fact, these are all the unitary irreducible representations of \( \mathbb{H}^n \) which are non-trivial on the center. Note that \( \Pi^\lambda(z,0) = V_z^\lambda \). We will write \( \Pi^\lambda(z) \) instead of \( \Pi^\lambda(z,0) \). Since \( V_z^\lambda \) is unitary, we can define a map \( G^\lambda : \mathcal{S}(\mathbb{C}^n) \rightarrow \mathcal{O}(\mathcal{H}_\lambda) \) by

\[
G^\lambda f = \int_{\mathbb{C}^n} f(z) V_z^\lambda dzd\bar{z} = \int_{\mathbb{C}^n} f(z) \Pi^\lambda(z) dzd\bar{z}.
\]

The operator \( G^\lambda f \) is called the Weyl transform of \( f \). Let \( \mathcal{S}_2(\mathcal{H}_\lambda) \) stand for the Hilbert space of Hilbert-Schmidt operators on \( \mathcal{H}_\lambda \) with the inner product \( \langle T,S \rangle = \text{tr}(TS^*) \).

Let \( ||.||_{\text{HS}} \) denote the Hilbert-Schmidt norm. Now we state the Plancherel theorem for Weyl transform.

**Theorem 5.1.** (Geller [6], Theorem 1.2) If \( f \in \mathcal{S}(\mathbb{C}^n) \), then \( G^\lambda f \in \mathcal{S}_2(\mathcal{H}_\lambda) \) and

\[
||f||^2_2 = \pi^{-n}(2|\lambda|)^n ||G^\lambda f||^2_{\text{HS}}.
\]

The map \( G^\lambda \) may then be extended as a constant multiple of a unitary map from \( L^2(\mathbb{C}^n) \) onto \( \mathcal{S}_2(\mathcal{H}_\lambda) \). A polarization of the above formula gives

\[
\langle f,g \rangle = \pi^{-n}(2|\lambda|)^n \langle G^\lambda f, G^\lambda g \rangle,
\]

where \( f,g \in L^2(\mathbb{C}^n) \).
For $f, g \in L^2(\mathbb{C}^n)$, define the twisted convolution
\[
f \times^\lambda g(z) = \int_{\mathbb{C}^n} f(z - w)g(w)e^{2i\lambda \text{Im}(z \cdot \bar{w})}dw.
\]

Then it is well-known that, $f \times^\lambda g \in L^2(\mathbb{C}^n)$ and
\[
\mathcal{G}^\lambda(f \times^\lambda g) = \mathcal{G}^\lambda(f)\mathcal{G}^\lambda(g).
\]

Next, we extend this definition to a suitable subset of $\mathcal{S}'(\mathbb{C}^n)$, the space of tempered distributions on $\mathbb{C}^n$ (see Geller [6], page 624-625). We say that $T \in \mathcal{S}'(\mathbb{C}^n)$ is Weyl transformable if there exist $R \in \mathcal{O}(\mathcal{H}^\lambda)$ such that
\[
T(f) = \pi^{-n}(2|\lambda|)^n \sum_{\nu \in \mathbb{Z}_+^n} \left( R_{u_\nu}^\lambda, (\mathcal{G}^\lambda f)_{u_\nu}^\lambda \right) \forall f \in \mathcal{S}(\mathbb{C}^n),
\]
where the series converges absolutely. It is shown in [6] that if such an $R$ exists then it is unique. In this case we call $R$ to be the Weyl transform of $T$ and write $\mathcal{G}^\lambda(T) = R$. It is clear from the polarization of Plancherel Theorem (Theorem 5.1) that this definition agrees with the previous definition of Weyl transform if $T$ is given by an $L^2$-function. In the course of proving the uniqueness of $R$, Geller proved that, if we fix a $\gamma \in \mathbb{Z}_+^n$, then for each $\alpha, \beta \in \mathbb{Z}_+^n$ there exist $f_{\alpha\beta} \in \mathcal{S}(\mathbb{C}^n)$ such that $\mathcal{G}^\lambda(f_{\alpha\beta})_{u_\alpha}^\lambda = \delta_{\alpha\gamma}u_\alpha^\lambda$. Taking $\beta = \alpha$, in particular we have the following: Fix $\gamma \in \mathbb{Z}_+^n$. Then for each $\alpha \in \mathbb{Z}_+^n$, there exists $f_\alpha \in \mathcal{S}(\mathbb{C}^n)$ such that $\mathcal{G}^\lambda(f_\alpha)_{u_\alpha}^\lambda = \delta_{\alpha\gamma}u_\alpha^\lambda$. From this fact, the next proposition follows easily.

**Proposition 5.2.** Let $\{T_j\}$ be a sequence of tempered distributions which converge to a tempered distribution $T$ in the topology of $\mathcal{S}'(\mathbb{C}^n)$, i.e $T_j(f) \to T(f)$ for all $f \in \mathcal{S}(\mathbb{C}^n)$. Assume that all $T_j$’s and $T$ are Weyl transformable. Then for any $u, v \in \mathcal{P}(\mathbb{C}^n)$, $\langle \mathcal{G}^\lambda(T_j)u, v \rangle \to \langle \mathcal{G}^\lambda(T)u, v \rangle$.

Define $\mathcal{F} : \mathcal{S}(\mathbb{C}^n) \to \mathcal{S}(\mathbb{C}^n)$ by
\[
(\mathcal{F}f)(\zeta) = \int_{\mathbb{C}^n} \exp(-z \cdot \bar{\zeta} + \bar{z} \cdot \zeta)f(z)dzd\bar{z}.
\]
This is a modification of the usual Euclidean Fourier transform $\mathcal{F}$, the relation being that $(\mathcal{F}'f)(\zeta) = (\mathcal{F}f)(-2i\zeta)$. So we can extend $\mathcal{F}'$ as a continuous, linear, one-to-one mapping of $\mathcal{S}'(\mathbb{R}^n)$ onto $\mathcal{S}'(\mathbb{C}^n)$. Let $T \in \mathcal{S}'(\mathbb{C}^n)$ be such that $\mathcal{F}'^{-1}T$ is Weyl transformable. Then we define the Weyl correspondence $\mathcal{W}^\lambda$ of $T$ by

$$\mathcal{W}^\lambda(T) = \mathcal{S}^\lambda(\mathcal{F}'^{-1}T).$$

On $\mathbb{H}^n$, the differential operators

$$T = \frac{\partial}{\partial t}, \quad Z_j = \frac{\partial}{\partial z_j} - i\bar{z}_j \frac{\partial}{\partial t}, \quad \bar{Z}_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}$$

are the left invariant vector fields corresponding to the one parameter family of subgroups $\Gamma_0 = \{(0,s) : s \in \mathbb{R}\}$, $\Gamma_j = \{se_j, 0 : s \in \mathbb{R}\}$ and $\Gamma_{\bar{j}} = \{s\bar{e}_j, 0 : s \in \mathbb{R}\}$ respectively, where $\{e_1, e_2, \cdots, e_n\}$ be the usual basis for $\mathbb{C}^n$. In [6], page-651, the notation differ slightly. Geller uses $\bar{Z}_j$ for our operator $Z_j$ (and $Z_{\bar{j}}$ for $\bar{Z}_j$). These form a basis for $L(\mathfrak{h}_n)$, the set of all left invariant differential operators on $\mathbb{H}^n$. Here $\mathfrak{h}_n$ is the Lie algebra of $\mathbb{H}^n$. For each $D \in L(\mathfrak{h}_n)$, let $D^\lambda$ denote the operator on $\mathbb{C}^n$ obtained by replacing each copy of $\partial/\partial t$ in $D$ by $-i\lambda$. Define

$$L^\lambda(\mathbb{C}^n) = \{D^\lambda : D \in L(\mathfrak{h}_n)\}, \text{ and } \mathcal{R}^\lambda(\mathbb{C}^n) = \{D^{-\lambda} : D \in L(\mathfrak{h}_n)\}.$$

Then

$$L_j^\lambda = \frac{\partial}{\partial z_j} - \lambda z_j, \quad \bar{L}_j^\lambda = \frac{\partial}{\partial z_j} + \lambda \bar{z}_j$$

form a basis for $L^\lambda(\mathbb{C}^n)$, and

$$R_j^\lambda = \frac{\partial}{\partial z_j} + \lambda z_j, \quad \bar{R}_j^\lambda = \frac{\partial}{\partial z_j} - \lambda \bar{z}_j$$

form a basis for $\mathcal{R}^\lambda(\mathbb{C}^n)$. In [6], page-619, these are denoted by $\tilde{Z}_j^\lambda$, $\tilde{Z}_{\bar{j}}^\lambda$, $\tilde{Z}^R_j$, $\tilde{Z}^R_{\bar{j}}$ respectively. Note that the action of $Z_j$ and $\bar{Z}_j$ on a function of the form $e^{-i\lambda t}f(z)$ are given by

$$Z_j(e^{-i\lambda t}f) = e^{-i\lambda t}L_j^\lambda(f), \quad \bar{Z}_j(e^{-i\lambda t}f) = e^{-i\lambda t}\bar{L}_j^\lambda(f).$$

We also have the commutation relations

$$[L_j^\lambda, L_k^\lambda] = -2\delta_{jk}\lambda I, \quad [L_j^\lambda, \bar{L}_k^\lambda] = 0, \quad [L_j^\lambda, \bar{L}_k^\lambda] = 0. \quad (5.3)$$
\([\mathcal{R}_j^\lambda, \mathcal{R}_k^\lambda] = 2\delta_{jk}\lambda I, \ [\mathcal{R}_j^\lambda, \mathcal{R}_k^\lambda] = 0, \ [\mathcal{R}_j^\lambda, \mathcal{R}_k^\lambda] = 0. \) (5.4)

The following proposition tells how the operators \(L_j^\lambda, L_j^\lambda, R_j^\lambda, R_j^\lambda\) behave under \(G^\lambda\). The proof can be found in [6], page 624-625.

**Proposition 5.3.** If \(T \in \mathcal{S}'(\mathbb{C}^n)\) is Weyl transformable, then so are \(L_j^\lambda T, L_j^\lambda T, R_j^\lambda T, R_j^\lambda T\). Also

\[G^\lambda(L_j^\lambda T) = -G^\lambda(T)W_j^\lambda, \quad G^\lambda(L_j^\lambda T) = G^\lambda(T)W_j^\lambda;\]

\[G^\lambda(R_j^\lambda T) = -W_j^\lambda G^\lambda(T), \quad G^\lambda(R_j^\lambda T) = W_j^\lambda G^\lambda(T).\]

Let \(\delta_0\) denote the Dirac delta distribution at origin. Since \(G^\lambda(\delta_0)\) is the identity operator, from the above proposition we get the following corollary. We write \(G^\lambda(D)\) for \(G^\lambda(D\delta_0)\), if \(D \in \mathcal{L}^\lambda(\mathbb{C}^n)\) or \(\mathcal{R}^\lambda(\mathbb{C}^n)\).

**Corollary 5.4.** Let \(T\) be a Weyl transformable tempered distribution. Then \(G^\lambda(DT) = G^\lambda(T)G^\lambda(D)\) if \(D \in \mathcal{L}^\lambda(\mathbb{C}^n); \) and \(G^\lambda(DT) = G^\lambda(D)G^\lambda(T)\) if \(D \in \mathcal{R}^\lambda(\mathbb{C}^n)\).

Let \(\mathcal{P}(\mathbb{C}^n_R)\) denote the space of all polynomials on the underlying real vector space \(\mathbb{C}^n_R\) of \(\mathbb{C}^n\). Clearly \(\mathbb{C}^n_R\) can be identified with \(\mathbb{R}^{2n}\). In other words the elements of \(\mathcal{P}(\mathbb{C}^n_R)\) are polynomials in \(z\) and \(\bar{z}\) with complex coefficients. From now on we use the following convention: when we write "polynomial", we mean a polynomial in \(z\) and \(\bar{z}\), i.e. an element in \(\mathcal{P}(\mathbb{C}^n_R)\), and elements of \(\mathcal{P}(\mathbb{C}^n)\) are called "holomorphic polynomials" i.e. polynomials in \(z\) only. For a monomial \(p(\zeta) = \zeta^\rho\zeta^\gamma\) (\(\rho, \gamma\) multi-indices), we set

\[\theta^\lambda_1(p) = (\mathcal{R}^\lambda)^\gamma(-\mathcal{R}^\lambda)^\rho, \quad \theta^\lambda_2(p) = (-\mathcal{R}^\lambda)^\rho(\mathcal{R}^\lambda)^\gamma\]

and

\[\tau^\lambda_1(p) = (\mathcal{W}^\lambda)^\gamma(\mathcal{W}^\lambda)^\rho, \quad \tau^\lambda_2(p) = (\mathcal{W}^\lambda)^\rho(\mathcal{W}^\lambda)^\gamma.\]

In the above \((\mathcal{R}^\lambda)^\gamma = (\mathcal{R}_1^\lambda)^{\gamma_1} \cdots (\mathcal{R}_n^\lambda)^{\gamma_n}\) (order does not matter because of commutation relations (5.4)), where \(\gamma = (\gamma_1, \cdots, \gamma_n)\). The other expressions are similarly
defined. Define
\[ \theta^\lambda(p) = \frac{1}{2}(\theta^\lambda_1(p) + \theta^\lambda_2(p)), \quad \tau^\lambda(p) = \frac{1}{2}(\tau^\lambda_1(p) + \tau^\lambda_2(p)). \]

We extend them to all polynomials by linearity. Note that by Proposition 5.3,
\[ G^\lambda_1(\theta^\lambda_1(p)) = \tau^\lambda_1(p), \quad G^\lambda_2(\theta^\lambda_2(p)) = \tau^\lambda_2(p), \quad G^\lambda_1(\theta^\lambda_1(p)) = \tau^\lambda(p), \]
for any polynomial \( p \).

**Proposition 5.5.** (Geller [6], Proposition 2.1 (a), 2.7)

(a) If \( p \) is a polynomial, then \( F^{-1}p \) is Weyl transformable and hence \( W^\lambda(p) \) is well defined.

(b) If \( p \) is a U(n)-harmonic polynomial, then \( W^\lambda(p) = \tau^\lambda_1(p) = \tau^\lambda_2(p) = \tau^\lambda(p) \).

**Remark 5.6.** In fact one can prove that for any polynomial \( p \), \( W^\lambda(p) = \tau^\lambda(p) \). Since we will be dealing with only harmonic polynomials we don’t need this general result.

We conclude this section with a short discussion about Gelfand pairs and \( K \)-spherical functions on \( \mathbb{H}^n \). For details see Benson et al. [2]. Let \( K \) be a compact Lie subgroup of \( \text{Aut}(\mathbb{H}^n) \), the group automorphisms of \( \mathbb{H}^n \). Each \( k \in U(n) \), the group of \( n \times n \) unitary matrices on \( \mathbb{C}^n \), gives rise to an automorphism of \( \mathbb{H}^n \), via \( k \cdot (z, t) = (k \cdot z, t) \). So we can consider \( U(n) \) as a subgroup of \( \text{Aut}(\mathbb{H}^n) \). In fact \( U(n) \) is a maximal connected, compact subgroup of \( \text{Aut}(\mathbb{H}^n) \), and thus any connected, compact subgroup of \( \text{Aut}(\mathbb{H}^n) \) is the conjugate of a subgroup \( K \) of \( U(n) \). The pair \( (K, \mathbb{H}^n) \) is called a Gelfand pair if \( L^1_K(\mathbb{H}^n) \), the convolution subalgebra of \( K \)-invariant \( L^1 \) functions on \( \mathbb{H}^n \), is commutative. Since conjugates of \( K \) form Gelfand pairs with \( \mathbb{H}^n \) if and only if \( K \) does, and produce the same joint eigenfunctions for all \( D \in \mathcal{L}_K(\mathfrak{h}_n) \) (the set of all differential operators on \( \mathbb{H}^n \) that are invariant under the action of \( K \) and the left action of \( \mathbb{H}^n \)), which is our main interest in this paper, we will always assume that we are dealing with a connected, compact subgroup \( K \) of \( U(n) \). The \( K \)-action on \( \mathbb{C}^n \) gives rise to a natural action on a function \( f \) on \( \mathbb{C}^n \).
given by $k \cdot f(z) = f(k^{-1} \cdot z)$. Under this action we have the decomposition of $\mathcal{P}(\mathbb{C}^n)$ into $K$-irreducible subspaces as

$$\mathcal{P}(\mathbb{C}^n) = \bigoplus_{\alpha \in \Lambda} V_\alpha \text{ (algebraic direct sum)}.$$  

Here $\Lambda$ denotes a countably infinite index set. Since $\mathcal{P}_m(\mathbb{C}^n)$, the space of homogeneous holomorphic polynomials of degree $m$, is invariant under the $K$-action (as $K \subset U(n)$), we can take each $V_\alpha$ to be contained in some $\mathcal{P}_m(\mathbb{C}^n)$. Define the unitary representation $U^\lambda$ of $K$ on the Hilbert space $\mathcal{H}^\lambda$ as follows: if $u \in \mathcal{H}^\lambda$,

$$U^\lambda(k)u = \begin{cases} \bar{k} \cdot u & \text{if } \lambda > 0 \\ k \cdot u & \text{if } \lambda < 0 \end{cases}.$$  

Since $(K, \mathbb{H}^n)$ is a Gelfand pair, $U^\lambda$ is multiplicity free (see Benson et al. [2], Theorem 1.7). $\mathcal{P}(\mathbb{C}^n)$ being dense in $\mathcal{H}^\lambda$, we get the same decomposition of $\mathcal{H}^\lambda$ into $U^\lambda$-irreducible subspaces:

$$\mathcal{H}^\lambda = \bigoplus_{\alpha \in \Lambda} V_\alpha \text{ (orthogonal Hilbert space decomposition)}.$$  

Choose a basis $\{e_{\alpha \nu}^\lambda : \nu = 1, 2, \cdots d(\alpha)\}$ for each $V_\alpha$ so that $\{e_{\alpha \nu}^\lambda : \alpha \in \Lambda, \nu = 1, 2, \cdots d(\alpha)\}$ is an orthonormal basis for $\mathcal{H}^\lambda$. We will use this basis in the later sections. The behaviour of $K$-action on a function under Weyl transform is given by the following proposition.

**Proposition 5.7.** (Geller [6], Proposition 1.3)

(a) $\Pi^\lambda(k \cdot z) = (U^\lambda(k))^{-1} \Pi^\lambda(z)(U^\lambda(k))$.

(b) If $f \in L^2(\mathbb{C}^n)$, $\mathcal{G}^\lambda(k \cdot f) = (U^\lambda(k)) \mathcal{G}^\lambda f (U^\lambda(k))^{-1}$.

(c) For any polynomial $p$, $\mathcal{W}^\lambda(k \cdot p) = (U^\lambda(k)) \mathcal{W}^\lambda(p) (U^\lambda(k))^{-1}$.

In fact, (c) is not proved in [6]. But using the definition of Weyl transform of a tempered distribution, one can show that (b) is true for any Weyl transformable tempered distribution. Since Euclidean Fourier transform commutes with the action of $K$, (c) follows.
A smooth $K$-invariant function $\phi : \mathbb{H}^n \rightarrow \mathbb{C}$ is called $K$-spherical if $\phi(0, 0) = 1$ and $\phi$ is a joint eigenfunction for all $D \in \mathcal{L}_K(\mathfrak{h}_n)$. In [2], the authors describe all bounded $K$-spherical functions, their forms, and the corresponding eigenvalues. We summarise these in the following theorem. Assume that $dk$ is the normalized Haar measure on $K$.

**Theorem 5.8.** There are two distinct classes of bounded $K$-spherical functions.

(a) The first type is parametrized by $(\lambda, \alpha) \in \mathbb{R}^* \times \Lambda$ ($\mathbb{R}^*$ denotes the set of all non-zero real numbers), and given by

$$\phi^\lambda_\alpha(z, t) = \int_{K} \langle \Pi^\lambda (k \cdot (z, t)) \rangle v, v \rangle dk,$$

for any unit vector $v \in V_\alpha$. Each $\phi^\lambda_\alpha$ has the form

$$\phi^\lambda_\alpha(z, t) = e^{i\lambda t} q^\lambda(z)e^{-|\lambda||z|^2},$$

where $q^\lambda(z)$ is a polynomial. The corresponding eigenvalue $\tilde{\mu}^\lambda_\alpha$s are distinct, and can be obtained from the equation (for any non-zero $v \in V_\alpha$),

$$\Pi^\lambda(D)v = \tilde{\mu}^\lambda_\alpha(D)v \forall D \in \mathcal{L}_K(\mathfrak{h}_n).$$

(b) The second type is parametrized by $\mathbb{C}^n/K$, the space of $K$-orbits in $\mathbb{C}^n$. For $\omega \in \mathbb{C}^n$ we write $\eta_\omega$, for the associated $K$-spherical function. One has $\eta_\omega = \eta_{\omega'}$, if $K \cdot \omega = K \cdot \omega'$. $\eta_\omega(z, t)$ is independent of $t$, and is given by

$$\eta_\omega(z, t) = \int_{K} e^{i\text{Re}(\omega \cdot k)z} dk.$$

Let $\mathcal{L}^\lambda_K(\mathbb{C}^n) = \{ D^\lambda : D \in \mathcal{L}_K(\mathfrak{h}_n) \}$. Clearly $\mathcal{L}^\lambda_K(\mathbb{C}^n) \subset \mathcal{L}^\lambda(\mathbb{C}^n)$. Define

$$\psi^\lambda_\alpha(z) = \frac{1}{\kappa^\lambda_\alpha} \int_{K} \langle \Pi^\lambda (k \cdot z) \rangle v, v \rangle dk,$$

where $v$ is any unit vector in $V_\alpha$, and $\kappa^\lambda_\alpha$ is the square of $L^2$ norm of $\int_{K} \langle \Pi^\lambda (k \cdot z) \rangle v, v \rangle dk$. The functions $\psi^\lambda_\alpha$ are real valued and $\psi^\lambda_\alpha = \psi^{-\lambda}_\alpha$ (see [2], Remark, page-428). Therefore we can write $\phi^{-\lambda}_\alpha(z, t) = \kappa^\lambda_\alpha e^{-i\lambda t} \psi^\lambda_\alpha$. Then with the property $||\psi^\lambda_\alpha||^2 = \frac{1}{\kappa^\lambda_\alpha}$, $\psi^\lambda_\alpha(z)$ is the unique (upto constant multiple) bounded joint eigenfunction for all $D^\lambda \in \mathcal{L}^\lambda_K(\mathbb{C}^n)$ with the eigenvalue $\mu^\lambda_\alpha$, where $\mu^\lambda_\alpha(D^\lambda) = \tilde{\mu}^{-\lambda}_\alpha(D)$ for all $D \in \mathcal{L}_K(\mathfrak{h}_n)$. 
Remark 5.9. Equation (5.6) can be restated in terms of Weyl transform as (for any non-zero $v \in V_\alpha$)

$$\mathcal{S}_\lambda(D)v = \mu_\alpha^\lambda(D)v \ \forall D \in \mathcal{L}_K^\lambda(\mathbb{C}^n).$$

Remark 5.10. Let $K = U(n)$. $\mathcal{L}_K^\lambda(\mathbb{C}^n)$ is generated by the special Hermite operator

$$\mathcal{L}_\lambda := \sum_{j=1}^n L_j^\lambda \overline{L}_j^\lambda + \sum_{j=1}^n \frac{\partial}{\partial \overline{z}_j} \frac{\partial}{\partial z_j} - \lambda |z|^2 + \sum_{j=1}^n \left( \overline{z}_j \frac{\partial}{\partial \overline{z}_j} - z_j \frac{\partial}{\partial z_j} \right).$$

The decomposition of $\mathcal{P}(\mathbb{C}^n)$ into $K$-invariant subspaces is given by $\mathcal{P}(\mathbb{C}^n) = \bigoplus_{k \in \mathbb{Z}^+} \mathcal{P}_k(\mathbb{C}^n)$. Recall that $\mathcal{P}_k(\mathbb{C}^n)$ is the space of all homogeneous holomorphic polynomials on $\mathbb{C}^n$ of degree $k$. The bounded $K$-spherical functions are parametrized by $\mathbb{Z}^+$, the set of non negative integers. The corresponding $\psi^\lambda_k$’s are given by

$$\psi^\lambda_k(z) = \pi^{-n}(2|\lambda|)^n L_k^{n-1}(2|\lambda||z|^2) e^{-|\lambda||z|^2},$$

where $L_k^{n-1}$ is the Laguerre polynomial of type $n - 1$, and the corresponding eigenvalues are given by $\mu^\lambda_k(\mathcal{L}_\lambda) = -2|\lambda|(2k + n)$. It is easy to see that (or by Corollary 2.3 in [2]), $\psi^\lambda_k = \sum_{V_\alpha \subset \mathcal{P}_k(\mathbb{C}^n)} \psi_\alpha^\lambda$.

6. Weyl transform of $K$-invariant functions

Throughout this section we assume that $(K, \mathbb{H}^n)$ is a Gelfand pair. Let $\lambda \in \mathbb{R}^*$ be fixed.

**Proposition 6.1.** Let $T \in \mathcal{S}'(\mathbb{C}^n)$ be $K$-invariant and Weyl transformable. Then $\mathcal{S}_\lambda T$ is a constant multiple of the identity operator on each $V_\alpha$.

**Proof.** For simplicity of notation we suppress the superscript $\lambda$ from the notation introduced in the previous section. Since $T$ is $K$-invariant, there exists a sequence $\{f_j\}$ of smooth, compactly supported, $K$-invariant functions on $\mathbb{C}^n$ such that $f_j$ converge to $T$ in the topology of $\mathcal{S}'(\mathbb{C}^n)$. By Proposition 5.7 (b), each $\mathcal{S} f_j$ commutes with all $U(k)$. Since the representation $U$ of $K$ on the various $V_\alpha$’s are irreducible and inequivalent, $\mathcal{S} f_j$ preserves each $V_\alpha$. Thus, by Schur’s Lemma, $\mathcal{S} f_j$ is constant on each $V_\alpha$. Hence by Proposition 5.2 we are done. \qed
Since the Euclidean Fourier transform commutes with the action of \( K \), an easy consequence of the above proposition and Proposition 4.5 (a) is the following corollary.

**Corollary 6.2.** Weyl correspondence of a \( K \)-invariant polynomial is constant on each \( V_\alpha \).

**Proposition 6.3.** \( \mathcal{G}(\psi^\lambda) = \mathcal{P}_\alpha \), where \( \mathcal{P}_\alpha \) denotes the projection operator onto \( V_\alpha \).

**Proof.** As usual we suppress the superscript \( \lambda \). By the previous proposition, \( \mathcal{G}(\psi_\alpha) \) is constant on each \( V_\beta \), say \( c_\beta I \). Let \( v \in V_\beta \) be non-zero. Then

\[
\mathcal{G}(\psi_\alpha)v = c_\beta v.
\]

Let \( D \in \mathcal{L}_K(\mathbb{C}^n) \). By Corollary 5.4 we have

\[
\mathcal{G}(\psi_\alpha)\mathcal{G}(D)v = \mathcal{G}(D\psi_\alpha)v = \mu_\alpha(D)\mathcal{G}(\psi_\alpha)v = c_\beta \mu_\alpha(D)v.
\]

Again, by Remark 5.9

\[
\mathcal{G}(\psi_\alpha)\mathcal{G}(D)v = \mu_\beta(D)\mathcal{G}(\psi_\alpha)v = c_\beta \mu_\beta(D)v.
\]

Therefore we have

\[
c_\beta \mu_\alpha(D)v = c_\beta \mu_\beta(D)v.
\]

This is true for all \( D \in \mathcal{L}_K(\mathbb{C}^n) \). Since \( \mu_\beta \neq \mu_\alpha \) for \( \beta \neq \alpha \), we get \( c_\beta = 0 \). Therefore \( \mathcal{G}(\psi_\alpha) \) is zero on \( V_\beta \), if \( \beta \neq \alpha \). Now take a unit vector \( u \) from \( V_\alpha \). Then

\[
c_\alpha = \langle \mathcal{G}(\psi_\alpha)u, u \rangle = \int_{\mathbb{C}^n} \psi_\alpha(z)\langle \Pi(z)u, u \rangle dz = \int_K \int_{\mathbb{C}^n} \psi_\alpha(z)\langle \Pi(k\cdot z)u, u \rangle dzdk.
\]

Since \( \Pi \) is unitary, \( |\langle \Pi(k\cdot z)u, u \rangle| \leq 1 \). Therefore we can use Fubini’s theorem to interchange the integrals and the fact that \( \psi_\alpha \) are real valued to get

\[
c_\alpha = \kappa_\alpha \int_{\mathbb{C}^n} \psi_\alpha(z)\psi_\alpha(z)dz = \kappa_\alpha ||\psi_\alpha||^2 = 1.
\]

Hence the proof. \( \Box \)
Proposition 6.4. Let \( f \in L^2(\mathbb{C}^n) \). Then \( f = \sum_{\alpha \in \Lambda} f \times \lambda_{\alpha} \), where the series converges in \( L^2(\mathbb{C}^n) \).

Proof. Since the index set \( \Lambda \) is countable, we can identify \( \Lambda \) with the set of natural numbers \( \mathbb{N} \). For \( j \in \mathbb{N} \),

\[
G(f)|_j = G(f)P_j = G(f)G(\psi_j) = G(f \times \psi_j).
\]

Therefore,

\[
\left\| G(f) - G\left( \sum_{j=1}^{N} f \times \psi_j \right) \right\|_{HS}^2 = \sum_{j>N} \sum_{\nu=1}^{d(j)} \left\| G(f)e_{j\nu} \right\|_2^2 \rightarrow 0
\]
as \( N \rightarrow \infty \), since \( G(f) \) is a Hilbert-Schmidt operator. Hence by the Plancherel theorem (Theorem 5.1) we are done.

The above proposition was also proved in [14].

7. generalized spherical functions and Weyl transform of K-type functions

From now on we assume that \( (K, \mathbb{H}^n) \) is a Gelfand pair, where \( K \) is a connected, compact subgroup of \( U(n) \), whose action on \( \mathbb{C}^n \) is polar. More precisely, if we identify \( U(n) \) as a subgroup of \( SO(2n) \) and \( \mathbb{C}^n \) with \( \mathbb{R}^{2n} \), then the action of \( K \subset SO(2n) \) on \( \mathbb{R}^{2n} \) is polar, so that we can use all the results about polar actions from the first three sections. Our main aims are to find all generalized \( K \)-spherical functions (Theorem 7.14) and give a formulae for Weyl transform of a function \( F \in \mathcal{F}^\delta(\mathbb{C}^n) \) (Theorem 7.4). Here \( \mathcal{F}^\delta(\mathbb{C}^n) := \{ F \in \mathcal{E}^\delta(\mathbb{C}^n) : F_{ij} \in \mathcal{F}(\mathbb{C}^n) \} \). Theorem 7.4 can be thought of as a generalization of the Theorem 4.2 in [6], which is a Hecke-Bochner type identity. To prove his theorem, Geller introduced certain Hilbert spaces of linear operators which turned out to be analogous to \( L^2(S^{2n-1}) \) and showed that the Weyl correspondence of \( U(n) \)-harmonic polynomials are dense in these Hilbert spaces. We show that a similar result holds for any \( K \) (Proposition 7.8), and use this to prove our theorems.
For two positive integers $p$ and $q$, let $R_{p \times q}^\lambda(C^n)$ denote the set of all $p \times q$ matrices whose entries belong to $R^\lambda(C^n)$; $O_{p \times q}(H^\lambda)$ denote the set of all $p \times q$ matrices whose entries belong to $O(H^\lambda)$; and $H_{p \times q}^\lambda$ denote the same whose entries belong to $H^\lambda$. For $T \in O_{p \times q}(H^\lambda)$, define its action as follows: if $u \in P(C^n)$, the $(i,j)$th entry of $Tu$ is equal to $T_{ij}u$. Let $H_{p \times q}^\lambda(C^n)$ denote the set of all $p \times q$ matrices with entries from $H_{p \times q}^\lambda$. For $g \in H_{p \times q}^\lambda(C^n)$ and $F \in H_{p \times q}^\lambda(C^n)$, we define the following whenever they make sense. For a differential operator $D$ on $C^n$, define $DF : C^n \to M_{p \times q}$, by $(DF)_{ij}(z) = DF_{ij}(z)$. If $D \in R_{p \times q}^\lambda(C^n)$, define $Dg : C^n \to M_{p \times q}$, by $(Dg)_{ij}(z) = D_{ij}g(z)$. Define $G^\lambda F, W^\lambda F \in O_{p \times q}(H^\lambda)$; $P^\delta \in O(d(\delta) \times (d(\delta))(H^\lambda)$; and $P^\delta \in R_{p \times q}^\lambda(C^n)$, $T^{-1}P^\delta \in T_{d(\delta) \times (d(\delta))}(C^n)$ similarly. For $S \in O(H^\lambda_\alpha)$, $T \in O_{p \times q}(H^\lambda)$, define $TS \in O_{p \times q}(H^\lambda)$ by $(TS)_{ij} = T_{ij} \circ S$. Similarly define $ST \in O_{p \times q}(H^\lambda)$. For a $r \times p$ constant matrix $C$, define $C T \in O_{r \times q}(H^\lambda)$, by $(C T)_{ij} = \Sigma_{k=1}^p C_{ik} T_{kj}$.

If $f$ is a joint eigendistribution of all $D \in L^\lambda_K(C^n)$, then $K$ being a subgroup of $U(n)$, it is also a joint eigendistribution of all $D \in L^\lambda_{U(n)}(C^n)$. But $L^\lambda_{U(n)}(C^n)$ is generated by the special Hermite operator $L^\lambda$, which is elliptic (see Remark 5.10). So we can assume that $f$ is smooth. Therefore, we will consider only smooth joint eigenfunctions for $L^\lambda_K(C^n)$.

Definition 7.1. A function $\Psi \in E^\delta(C^n)$ is said to be a generalized $K$-spherical function of type $\delta$ corresponding to $\mu^\lambda_\alpha$, if it is a joint eigenfunction for all $D \in L^\lambda_K(C^n)$ with eigenvalue $\mu^\lambda_\alpha$, i.e $D \Psi = \mu^\lambda_\alpha(D) \Psi$ for all $D \in L^\lambda_K(C^n)$.

By Proposition 5.5 (b) it follows that, for any $K$-harmonic (hence $U(n)$-harmonic) polynomial $p$, $\theta^\lambda_1(p) = \theta^\lambda_2(p)$. Hence $\theta^\lambda(P^\delta) = \theta^\lambda_1(P^\delta) = \theta^\lambda_2(P^\delta)$. Define $\Psi^\delta_\alpha \in E^\delta(C^n)$, by

$$\Psi^\delta_\alpha = \theta^\lambda(P^\delta) \psi^\lambda_\alpha.$$

Proposition 7.2. $\Psi^\delta_\alpha$ is a generalized $K$-spherical function of type $\delta$ corresponding to $\mu^\lambda_\alpha$. 

Proof. We suppress the superscript $\lambda$. Since $\theta(P^\delta) \in R_{d(\delta) \times l(\delta)}(\mathbb{C}^n)$, by Corollary 5.4

$$G(\Psi_\alpha^\delta) = G(\theta(P^\delta))G(\psi_\alpha) = \tau(P^\delta)G(\psi_\alpha) = W(P^\delta)G(\psi_\alpha).$$

Therefore, by Proposition 5.7, for $k \in K$,

$$G(k^{-1} \cdot \Psi_\alpha^\delta) = U(k)^{-1}G(\Psi_\alpha^\delta)U(k) = U(k)^{-1}W(P^\delta)G(\psi_\alpha)U(k) = \delta(k)W(P^\delta)G(\psi_\alpha) = \delta(k)G(\Psi_\alpha^\delta).$$

So we get $\Psi_\alpha^\delta(k \cdot z) = \delta(k)\Psi_\alpha^\delta(z)$, and hence $\Psi_\alpha^\delta \in E^\delta(\mathbb{C}^n)$. Again, $D\Psi_\alpha^\delta = \mu_\alpha(D)\Psi_\alpha^\delta$ for all $D \in L_K(\mathbb{C}^n)$, because

$$G(D\Psi_\alpha^\delta) = G(\Psi_\alpha^\delta)G(D) = G(\theta(P^\delta))G(\psi_\alpha)G(D) = G(\theta(P^\delta))G(D\psi_\alpha) = \mu_\alpha(D)G(\Psi_\alpha^\delta).$$

Therefore $\Psi_\alpha^\delta$ is a generalized $K$-spherical function of type $\delta$ corresponding to $\mu_\alpha$. \qed

Note that $\Psi_\alpha^{\delta,\lambda}(z)$ is equal to $e^{-|\lambda||z|^2}$ times a polynomial in $E^\delta(\mathbb{C}^n)$, and hence by Corollary 4.9, there is a unique $K$-invariant polynomial $L_\alpha^{\delta,\lambda} : \mathbb{C}^n \to M_{l(\delta) \times l(\delta)}$ such that

$$\Psi_\alpha^{\delta,\lambda}(z) = P^\delta(z)L_\alpha^{\delta,\lambda}(z)e^{-|\lambda||z|^2}. \quad (7.1)$$

Define the $l(\delta) \times l(\delta)$ constant matrix $A_\alpha^{\delta,\lambda}$ by

$$A_\alpha^{\delta,\lambda} = \int_{\mathbb{C}^n} [\Psi_\alpha^{\delta,\lambda}(z)]^*\Psi_\alpha^{\delta,\lambda}(z)dz = \int_{\mathbb{C}^n} [L_\alpha^{\delta,\lambda}(z)]^*\gamma^\delta(z)L_\alpha^{\delta,\lambda}(z)e^{-2|\lambda||z|^2}dz.$$

Clearly $A_\alpha^{\delta,\lambda}$ is positive definite. Let $\alpha(\delta)$ denote the number of linearly independent columns in $\Psi_\alpha^{\delta,\lambda}$. Let $C_i$ denote the $i$th column. Choose $C_{l(1)}, C_{l(2)}, \cdots, C_{l(\alpha(\delta))}$ linearly independent and $l(1) < l(2), \cdots < l(\alpha(\delta))$. Let the remaining columns be $C_{m(1)}, C_{m(2)}, \cdots, C_{m(l(\delta) - \alpha(\delta))}$, with $m(1) < m(2) < \cdots < m(l(\delta) - \alpha(\delta))$. Let

$$I_1 = \{l(1), l(2), \cdots, l(\alpha(\delta))\}.$$
\[ I_2 = \{ m(1), m(2), \ldots, m(l(\delta) - \alpha(\delta)) \}. \]

Then \( I_1 \) and \( I_2 \) are disjoint, and

\[ I_1 \cup I_2 = \{ 1, 2, \ldots, l(\delta) \}. \]

Let \( \tilde{\Psi}_{\alpha}^{\delta,\lambda} \) be the \( d(\delta) \times \alpha(\delta) \) matrix whose \( r \)th column is \( C_{l(r)} \), where \( l(r) \in I_1 \); and \( \tilde{L}_{\alpha}^{\delta,\lambda} \) be the \( l(\delta) \times \alpha(\delta) \) matrix whose \( r \)th column is \( C_{l(\delta)} \), where \( l(r) \in I_1 \). Then clearly we have

\[ \tilde{\Psi}_{\alpha}^{\delta,\lambda}(z) = P^{\delta}(z)\tilde{L}_{\alpha}^{\delta,\lambda}(z)e^{-|\lambda||z|^2}. \]

Define

\[ \tilde{A}_{\alpha}^{\delta,\lambda} = \int_{C^n} [\tilde{\Psi}_{\alpha}^{\delta,\lambda}(z)]^*\tilde{\Psi}_{\alpha}^{\delta,\lambda}(z)dz = \int_{C^n} [\tilde{L}_{\alpha}^{\delta,\lambda}(z)]^*\Upsilon_{\delta}(z)\tilde{L}_{\alpha}^{\delta,\lambda}(z)e^{-2|\lambda||z|^2}dz. \]

Note that \( \tilde{A}_{\alpha}^{\delta,\lambda} \) is precisely the \( \alpha(\delta) \times \alpha(\delta) \) matrix, obtained by deleting the \( m(r) \)th rows and columns from \( A_{\alpha}^{\delta,\lambda} \), where \( m(r) \in I_2 \).

**Lemma 7.3.** If \( \alpha(\delta) > 0 \), \( \tilde{A}_{\alpha}^{\delta,\lambda} \) is invertible.

**Proof.** As usual we suppress the superscript \( \lambda \). If \( \tilde{A}_{\alpha}^{\delta} \) is not invertible, then there exist a non-zero vector \( \xi \in C^{\alpha(\delta)} \), such that \( \langle (\tilde{A}_{\alpha}^{\delta})\xi, \xi \rangle = 0 \) (here \( \langle , \rangle \) denotes the usual hermitian inner product on \( C^{\alpha(\delta)} \)). Since

\[ \tilde{A}_{\alpha}^{\delta} = \int_{C^n} [\tilde{\Psi}_{\alpha}^{\delta}(z)]^*\tilde{\Psi}_{\alpha}^{\delta}(z)dz, \]

we get \( \tilde{\Psi}_{\alpha}^{\delta}(z)\xi = 0 \) for all \( z \), implying that the columns of \( \tilde{\Psi}_{\alpha}^{\delta} \) are linearly dependent, which is a contradiction. Hence the proof. \( \square \)

Before we state one of our main theorems we introduce some more notation. For \( F \in \mathcal{E}(C^n) \), define \( l(\delta) \times l(\delta) \) matrix \( C_{\alpha}^{\delta,\lambda}(F) \) as follows : Let \( \tilde{C}_{\alpha}^{\delta,\lambda}(F) \) denotes the \( \alpha(\delta) \times l(\delta) \) matrix whose \( r \)th row is the \( l(r) \)th row of \( C_{\alpha}^{\delta,\lambda} \), where \( l(r) \in I_1 \); and
\( \tilde{C}^{\delta,\lambda}_\alpha(F) \) denotes the \((l(\delta) - \alpha(\delta)) \times l(\delta)\) matrix whose \(r\)th row is the \(m(r)\)th row of \(C^{\delta,\lambda}_\alpha\), where \(m(r) \in I_2\).

\[
\tilde{C}^{\delta,\lambda}_\alpha(F) = \left( \tilde{A}^{\delta,\lambda}_\alpha \right)^{-1} \int_{\mathbb{C}^n} [\tilde{\Psi}^{\delta,\lambda}(z)]^* F(z) dz = \left( \tilde{A}^{\delta,\lambda}_\alpha \right)^{-1} \int_{\mathbb{C}^n} [\tilde{L}^{\delta,\lambda}(z)]^* \Upsilon_\delta(z) G(z) e^{-|\lambda||z|^2} dz,
\]

whenever the integrals exist.

**Theorem 7.4.** (Hecke-Bochner identity) Suppose \( F = P^\delta G \in \mathcal{P}^\delta(\mathbb{C}^n) \), where \( G \) is \( K \)-invariant. Then \( \mathcal{G}^\lambda(F) = \mathcal{W}^\lambda(P^\delta) S \), where \( S \in \mathcal{O}_{l(\delta) \times l(\delta)}(\mathcal{H}^\lambda) \) whose action on each \( V_\alpha \) is the \( l(\delta) \times l(\delta) \) constant matrix \( C^{\delta,\lambda}_\alpha(F) \); equivalently if \( F = P^\delta G \in \mathcal{P}^\delta(\mathbb{C}^n) \), where \( G \) is \( K \)-invariant, then \( F \times^\lambda \psi_\alpha^\lambda = \Psi_\alpha^\lambda C^{\delta,\lambda}_\alpha(F) \), where \( C^{\delta,\lambda}_\alpha(F) \) is defined by (7.2).

Before proving this general theorem let us consider the special case \( K = U(n) \), and describe \( \Psi_\alpha^\lambda, \tilde{\Psi}_\alpha^\lambda, L_\alpha^\lambda, \ldots \). We also show that, in this special case, the above theorem is precisely Theorem 4.2 in [6]. We put these in the following remark.

**Remark 7.5.** For this remark \( K \) always stands for \( U(n) \). Let \( M \) be the stabilizer of the \( K \)-regular point \( e_1 = (1, 0, \cdots, 0) \in \mathbb{C}^n \); then \( M \) can be identified with \( U(n-1) \). Via the map \( kM \to k \cdot e_1 \), we have the identification \( K/M = K \cdot e_1 = S^{2n-1} \).

Note that, in this special case, the space \( H \) consists of all polynomials \( P \) such that \( \Delta_{\mathbb{C}^n} P = 0 \), where \( \Delta_{\mathbb{C}^n} = \sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_j} \), for each pair of non-negative integers \((p, q)\), let \( P_{pq} \) be the space of all polynomials \( P \) in \( z \) and \( \bar{z} \) of the form

\[
P(z) = \sum_{|\alpha|=p, |\beta|=q} c_{\alpha\beta} z^\alpha \bar{z}^\beta.
\]

Let \( \mathcal{H}_{pq} = H \cap P_{pq} \), and \( S_{pq} \) denote the space of restrictions of elements of \( \mathcal{H}_{pq} \) to \( S^{2n-1} \). The elements of \( \mathcal{H}_{pq} \) are called bigraded solid harmonics of degree \((p, q)\), and those of \( S_{pq} \) are called bigraded spherical harmonics of degree \((p, q)\). The \( K \)-action on \( S^{2n-1} \) defines an unitary representation on \( L^2(S^{2n-1}) \). Clearly each \( S_{pq} \)
is a $K$-invariant subspace. Let $\delta_{pq}$ denotes the restriction of this representation to $S_{pq}$. It is well known that these describe all inequivalent, irreducible, unitary representations in $\hat{K}_M$. Note that, according to our general notation (in Section 3), $H_{\delta_{pq}} = \mathcal{H}_{pq}, E_{\delta_{pq}}(K/M) = S_{pq}$ and $l(\delta_{pq}) = \dim V_{\delta_{pq}}^M = 1$.

Using the similar arguments as in Remark 4.1 we can prove the following: first note that, in this case, $\hat{\psi}_{\delta_{pq}}$ is equivalent to $\delta_{qp}$. Take $P_{pq}^i \in \mathcal{H}_{pq}$, and $Y_{pq}^i$ to be their restrictions to $S_{pq}$. Then it is possible to choose orthonormal ordered basis $b = \{v_1, v_2, \cdots, v_{d(p,q)}\}$ for $V_{\delta_{pq}}$, and $b_M = \{v_1\}$ for $V_{\delta_{pq}}^M$, and a basis $e$ for $F_{\delta_{pq}} = \text{Hom}(V_{\delta_{pq}}, \mathcal{H}_{pq})$, so that, with respect to these bases, $Y_{\delta_{pq}}: S_{2n-1} \rightarrow M_{d(p,q) \times 1}$ and $P_{\delta_{pq}}: \mathbb{C}^n \rightarrow M_{d(p,q) \times 1}$ are given by

\[
Y_{\delta_{pq}}(\omega) = \sqrt{\frac{S_{2n-1}}{d(p,q)}} \begin{bmatrix} Y_{pq}^1(\omega), Y_{pq}^2(\omega), \cdots, Y_{pq}^{d(p,q)}(\omega) \end{bmatrix}^t, \omega \in S_{2n-1},
\]

\[
P_{\delta_{pq}}(z) = \sqrt{\frac{S_{2n-1}}{d(p,q)}} \begin{bmatrix} P_{pq}^1(z), P_{pq}^2(z), \cdots, P_{pq}^{d(p,q)}(z) \end{bmatrix}^t, z \in \mathbb{C}^n.
\]

In particular,

\[
P_{\delta_{pq}}(z) = |z|^{p+q} Y_{\delta_{pq}}(z/|z|).
\]

Also, we have

\[
\Upsilon_{\delta_{pq}}(z) = |z|^{2(p+q)}.
\]

(i) Recall the $U(n)$-spherical functions $\psi_k^\lambda$’s from Remark 5.10. The corresponding generalized $K$-spherical functions are given by $\Psi_k^{\delta_{pq},\lambda} = \theta^\lambda(P^{\delta_{pq}})\psi_k^\lambda$. Note that, $\Psi_k^{\delta_{pq},\lambda}: \mathbb{C}^n \rightarrow M_{d(p,q) \times 1}$, $L_k^{\delta_{pq},\lambda}: \mathbb{C}^n \rightarrow M_{1 \times 1}$, and $A_k^{\delta_{pq},\lambda}$ is a $1 \times 1$ matrix. Let $L_k^\gamma$ denotes the $k$ th degree Laguerre polynomial of type $\gamma$. We will show the following
Since $\theta > 0$, therefore, to prove (7.3) it is enough to show that

$$L_{k-p}^\lambda (z) = \begin{cases} (-1)^q \pi^{-n} (2|\lambda|)^{n+p+q} L_{k-p}^{n+p+q-1} (2|\lambda||z|^2), & \text{if } p \leq k \\ 0, & \text{if } p > k, \end{cases}$$

(7.3)

$$\Psi_k^\lambda (z) = \begin{cases} (-1)^q \pi^{-n} (2|\lambda|)^{n+p+q} \psi_k^\lambda (z) L_{k-p}^{n+p+q-1} (2|\lambda||z|^2)e^{-|\lambda||z|^2}, & \text{if } p \leq k \\ 0, & \text{if } p > k, \end{cases}$$

(7.4)

$$A_k^\lambda = \begin{cases} \pi^{-n} (2|\lambda|)^{n+p+q} \frac{\Gamma(k+n+q)}{\Gamma(n)\Gamma(k-p+1)}, & \text{if } p \leq k \\ 0, & \text{if } p > k; \end{cases}$$

(7.5)

consequently $\Psi_k^\lambda = \Psi_k^\lambda$, $\tilde{L}_k^\lambda = L_k^\lambda$ and $A_k^\lambda = A_k^\lambda$ if $p \leq k$. When $\lambda < 0$, the role of $p$ and $q$ will be interchanged in the above formulae. We give a proof assuming $\lambda > 0$. The proof for $\lambda < 0$ will be similar. Since

$$\Psi_k^\lambda = \theta^\lambda (\psi_k^\lambda) = \psi_k^\lambda (z) L_{k-p}^\lambda (z) e^{-|\lambda||z|^2},$$

and $z_1^p z_2^q \in H_{pq}$, it follows that

$$\theta^\lambda (z_1^p z_2^q)^\lambda = z_1^p z_2^q L_{k-p}^\lambda (z) e^{-|\lambda||z|^2}.$$

Therefore, to prove (7.3) it is enough to show that

$$\theta^\lambda (z_1^p z_2^q)^\lambda = \begin{cases} (-1)^q \pi^{-n} (2|\lambda|)^{n+p+q} z_1^p z_2^q L_{k-p}^{n+p+q-1} (2|\lambda||z|^2)e^{-|\lambda||z|^2}, & \text{if } p \leq k \\ 0, & \text{if } p > k. \end{cases}$$

(7.6)

Since $\theta^\lambda (\tilde{z}_2) = \frac{\partial}{\partial z_2} - \lambda \tilde{z}_2$, and

$$\frac{\partial}{\partial z_2} \left[ L_{k-p}^{n+p+q-1} (2|\lambda||z|^2) \right] = 2 \lambda|z|_2 \left[ L_{k-p}^{n+p+q-1}' (2|\lambda||z|^2) \right],$$

an easy calculation shows that

$$\theta^\lambda (\tilde{z}_2) \psi_k^\lambda (z) = \pi^{-n} (-1) (2|\lambda|)^{n+1} \tilde{z}_2 \left[ (L_{k-p}^{n-1})' - L_{k-p}^{n-1} \right] (2|\lambda||z|^2)e^{-|\lambda||z|^2}.$$

Using the well-known relations

$$(L_k^\alpha)' = -L_{k-1}^{\alpha+1}, \quad L_k^{\alpha+1} = L_{k-1}^{\alpha+1} + L_k^\alpha,$$
we get
\[ \theta^\lambda(z_2)\psi^\lambda_k(z) = \pi^{-n}(-1)(2|\lambda|)^{n+1}z_2L^{n+1-1}e^{-|\lambda||z|^2}. \]

Since \( \theta^\lambda(z_2^{m+1}) = (R^\lambda_2)^{m+1} = R^\lambda_2\theta^\lambda(z_2^m) \), by an induction argument we can prove that
\[ \theta^\lambda(z_2^q)\psi^\lambda_k(z) = \pi^{-n}(-1)^q(2|\lambda|)^{n+1}z_2^qL^{n+1-1}e^{-|\lambda||z|^2}, \]
for all non negative integer \( q \). Now fix a \( q \). Then again by induction on \( p \) and using a similar argument we can prove that
\[ \theta^\lambda(z_1^p z_2^q)\psi^\lambda_k(z) = \pi^{-n}(-1)^q(2|\lambda|)^{n+p+q}z_1^p z_2^qL^{n+p+q-1}e^{-|\lambda||z|^2}, \]
whenever \( p \leq k \). In particular we get for any fixed \( q \), \( \theta^\lambda(z_1^k z_2^q)\psi^\lambda_k(z) \) is equal to a constant times \( z_1^k z_2^q e^{-|\lambda||z|^2} \). Since \( \theta^\lambda(z_1^{k+1} z_2^q) = (-R^\lambda_1)\theta^\lambda(z_1^k z_2^q) \) and
\[ R^\lambda_1(z_1^k z_2^q e^{-|\lambda||z|^2}) = (\partial/\partial \bar{z}_1 + \lambda z_1)(z_1^k z_2^q e^{-|\lambda||z|^2}) = 0 \text{ (as} \lambda > 0), \]
we get \( \theta(z_1^{k+1} z_2^q)^\lambda \psi^\lambda(z) = 0 \), and consequently \( \theta^\lambda(z_1^p z_2^q)\psi^\lambda_k(z) = 0 \) for all \( p > k \). This finishes the proof of (7.3). (7.4) follows immediately from (7.3). \( A^\lambda_{k+1} \) has the formulae
\[ A^\lambda_{k+1} = \int_{\mathbb{C}^n} [L^\lambda_{k+1}(z)]^* Y^\lambda_{k+1}(z)L^\lambda_{k+1}(z)e^{-2|\lambda||z|^2}dz. \]
Therefore, (7.5) follows by (7.3) and the fact that
\[ \int_0^{\infty} [L^\gamma_k(r)]^2 e^{-r}r^\gamma dr = \frac{\Gamma(k + \gamma + 1)}{\Gamma(k + 1)}. \]

(ii) From the above discussion it is immediate that, for the special case \( K = U(n) \), Theorem 7.4 can be restated as follows (which is precisely Theorem 4.2 in [6]) : Suppose \( pg \in \mathcal{S}(\mathbb{C}^n) \), where \( P \in \mathcal{H}_{pq} \) and \( g \) is a radial function. For \( \lambda > 0 \), \( \mathcal{S}^\lambda(Pg) = W^\lambda(P)S, S \in \Theta(\mathcal{H}^\lambda), \) whose action on each \( \mathcal{P}_k(\mathbb{C}^n) \) is a constant \( c_k \), where \( c_k = 0 \) if \( p > k \), and for \( p \leq k \), it is given by
\[ c_k = (-1)^q \frac{\Gamma(n)\Gamma(k - p + 1)}{\Gamma(k + n + q)} \int_{\mathbb{C}^n} g(z)L^{n+p+q-1}_k(z)z^{2(p+q)} e^{-|\lambda||z|^2}. \]
when \( \lambda < 0 \), the role of \( p \) and \( q \) will be interchanged in the definition of \( c_k \).
To prove Theorem 7.4, we need several steps. For a finite dimensional subspace \( V \) of \( \mathcal{H}^\lambda \), let \( \mathcal{O}(V) \) stand for the vector space of all bounded linear operators \( R : V \rightarrow \mathcal{H}^\lambda \). Define an inner product \( \langle \cdot, \cdot \rangle^\lambda \) on \( \mathcal{O}(V) \) by

\[
\langle R, R' \rangle^\lambda = \sum_{j=1}^d \langle Rv_j, R'v_j \rangle,
\]

for an orthonormal basis \( \{v_1, v_2, \cdots v_d\} \). Clearly the definition is independent of the orthonormal basis. One can see that the norm defined by the above inner product is equivalent to the operator norm. Since \( \mathcal{O}(V) \) is a Banach space with respect to the operator norm, we conclude that \( \mathcal{O}(V) \) is a Hilbert space with the above inner product. If \( V \subset \mathcal{P}(\mathbb{C}^n) \) we can view \( \mathcal{O}(\mathcal{H}^\lambda) \) as a subset of \( \mathcal{O}(V) \) by restricting the elements in \( \mathcal{O}(\mathcal{H}^\lambda) \) to \( V \). If \( V = V_\alpha \), we denote the inner product by \( \langle \cdot, \cdot \rangle^\lambda_\alpha \). In this case, by Schur’s orthogonality relation we have another formula for \( \langle \cdot, \cdot \rangle^\lambda_\alpha \), given by

\[
\langle R, R' \rangle^\lambda_\alpha = \int_K \langle R(k \cdot v), R'(k \cdot v) \rangle dk,
\]

for any unit vector \( v \in V_\alpha \).

**Lemma 7.6.**  
(a) Suppose \( f \in \mathcal{S}(\mathbb{C}^n) \), and \( \delta \in \widehat{K}_M \). Then

\[
\langle G^\lambda(f), W^\lambda(P^\delta) \rangle^\lambda_\alpha = \pi^n (2|\lambda|)^{-n} \langle f, \Psi^\lambda_\alpha \rangle.
\]

Here the equality is entry wise.

(b) For two \( K \)-harmonic polynomials \( p, q \)

\[
\langle W^\lambda(p), W^\lambda(q) \rangle^\lambda_\alpha = \pi^n (2|\lambda|)^{-n} \langle \theta^\lambda(p) \psi^\lambda_\alpha, \theta^\lambda(q) \psi^\lambda_\alpha \rangle.
\]

**Proof.** The proof is similar to that of Lemma 2.1 in [16].
By Corollary 5.4
\[ \mathcal{G}(\theta(P^\delta)) \mathcal{G}(\psi_\alpha) = \mathcal{G}(\theta(P^\delta)\psi_\alpha) = \mathcal{G}(\Psi^\delta_\alpha). \]

Hence by the Plancherel Theorem (Theorem 5.1), (a) follows. The proof of (b) is similar. \(\Box\)

Lemma 7.7. \(\{\mathcal{W}^\lambda(p) : p \in \mathcal{P}(\mathbb{C}^n)\} \) is dense in \(\mathcal{O}(V_\alpha)\).

Proof. It is shown in \(\text{[6]}\) (See Proposition 2.10 (b)) that \(\{\mathcal{W}(p) : p \in \mathcal{P}(\mathbb{C}^n)\} \) is dense in \(\mathcal{O}(\mathcal{P}_m(\mathbb{C}^n))\). Since \(V_\alpha\) is contained in some \(\mathcal{P}_m(\mathbb{C}^n)\), we can extend any operator \(T \in \mathcal{O}(V_\alpha)\) to \(T' \in \mathcal{O}(\mathcal{P}_m(\mathbb{C}^n))\) by defining \(T'\) to be zero on the complement of \(V_\alpha\) in \(\mathcal{P}_m(\mathbb{C}^n)\). From this, it is easy to see that, \(\{\mathcal{W}(p) : p \in \mathcal{P}(\mathbb{C}^n)\} \) is dense in \(\mathcal{O}(V_\alpha)\). \(\Box\)

Let \(H^i_\delta\) be the subspace of \(H^\delta\), spanned by the entries of \(i\)th row in \(P^\delta\). Using Schur’s orthogonality it can be shown that, if \(p \in H^i_\delta, q \in H^{i'}_{\delta'}\) with \((\delta, i) \neq (\delta', i')\), then
\[
\int_K p(k \cdot z)\overline{q(k \cdot z)}\,dk = 0. \tag{7.7}
\]

Proposition 7.8. \(\mathcal{O}(V_\alpha) = \bigoplus_{\delta \in \hat{K}_M} \bigoplus_{i=1}^{d(\delta)} \mathcal{W}^\lambda(H^i_\delta)|_{V_\alpha} \) (orthogonal Hilbert space decomposition).

Proof. In Lemma 7.6 (b), if we take \(p = P^\delta_{ij}\) and \(q = P^{\delta'}_{i'j'}\), then \(\theta(p)\psi_\alpha = (i, j)\)th entry of \(\theta(P^\delta)\psi_\alpha = (i, j)\)th entry of \(\Psi^\delta_\alpha\), and similarly \(\theta(q)\psi_\alpha = (i', j')\)th entry of \(\Psi^{\delta'}_\alpha\). Note that \(\Psi^{\delta'}_\alpha\) has the form \(\text{(7.1)}\). Therefore if \((\delta, i) \neq (\delta', i')\), then by \(\text{(7.7)}\) we have,
\[
\int_K [\theta(p)\psi_\alpha](k \cdot z)[\overline{\theta(q)\psi_\alpha}](k \cdot z)\,dk = 0 \forall z.
\]
Integrating both sides over \(\mathbb{C}^n\), and then making a change of variable, namely \(z \to k^{-1} \cdot z\), we get \(\langle \theta(p)\psi_\alpha, \theta(q)\psi_\alpha \rangle = 0\) if \((\delta, i) \neq (\delta', i')\). Hence the orthogonality is proved.

By Lemma 7.7 \(\{\mathcal{W}(p) : p \in \mathcal{P}(\mathbb{C}^n)\} \) is dense in \(\mathcal{O}(V_\alpha)\). Therefore to complete the proof of the proposition it is enough to show that for any \(\delta \in \hat{K}_M\),
\[
\mathcal{W}(IH^i_\delta)|_{V_\alpha} = \mathcal{W}(H^i_\delta)|_{V_\alpha}. \tag{7.8}
\]
For a polynomial $p$, let $p(D)$ denote the constant coefficient differential operator obtained by replacing $z_j$ by $-\partial/\partial \bar{z}_j$ and $\bar{z}_j$ by $\partial/\partial z_j$. Then it is an easy consequence of the Euclidean Fourier transform that $\mathcal{F}^{-1}p = p(D)\delta_0$. Also, we can write

$$\theta(p) = p(D) + \varepsilon(p),$$

where $\varepsilon(p)$ is a polynomial coefficient differential operator of order strictly less than the degree of $p$. Let $P_m$ be the space of all polynomials in $z, \bar{z}$ whose degree is less than or equal to $m$. We prove (7.8) by showing that

$$W(IH_\delta \cap P_m)|_{V_\alpha} \subset W(H_\delta)|_{V_\alpha},$$

for all non negative integers $m$. We do it by induction on $m$. Since $W(1) = \mathcal{G}(\delta_0) = \text{identity operator}$, (7.9) is true for $m = 0$. Now suppose (7.9) is true for $m = k$. It is enough to show that $W(p)|_{V_\alpha} \in W(H_\delta)|_{V_\alpha}$, for any polynomial $p$ of the type $p = jh$, where $j \in I, h \in H_\delta$ and degree $p = (k + 1)$.

$$W(p) = \mathcal{G}(\mathcal{F}^{-1}p) = \mathcal{G}(h(D)j(D)\delta_0)$$

$$= \mathcal{G}(\theta(h)j(D)\delta_0) - \mathcal{G}(\varepsilon(h)j(D)\delta_0)$$

$$= W(h)W(j) - W(\mathcal{F}(\varepsilon(h)j(D)\delta_0)).$$

By Corollary [6.2], $W(j)$ is a scalar on $V_\alpha$. Hence

$$[W(h)W(j)]|_{V_\alpha} \in W(H_\delta)|_{V_\alpha}.$$

Now note that $\varepsilon(h)j(D)\delta_0$ is a distribution supported at the origin, whose order is less than or equal to $k$. Therefore $\mathcal{F}(\varepsilon(h)j(D)\delta_0)$ is a polynomial of degree at most $k$. Again from (7.10) we have

$$\mathcal{F}^{-1}p = \theta(h)j(D)\delta_0 - \varepsilon(h)j(D)\delta_0,$$

which implies that

$$\mathcal{F}(\varepsilon(h)j(D)\delta_0) \in S_\delta = IH_\delta.$$
Therefore, by the induction hypothesis,

\[ \mathcal{W}(\mathcal{F}'(\varepsilon(h)j(D)\delta_0)) \big|_{V_\alpha} \in \mathcal{W}(H_\delta) \big|_{V_\alpha}. \]

So ultimately we get that

\[ \mathcal{W}(p) \big|_{V_\alpha} \in \mathcal{W}(H_\delta) \big|_{V_\alpha}, \]

as desired. Hence the proof of the proposition is complete. \( \square \)

**Proof. (Proof of Theorem 7.4)** By Lemma 7.6 (a),

\[ \langle G(f), \mathcal{W}(P_{\delta'}) \rangle_\alpha = \pi^n(2|\lambda|)^{-n} \langle f, \Psi_{\delta'}^\alpha \rangle, \]

for \( f \in \mathcal{F}(\mathbb{C}^n) \) and \( \delta' \in \hat{K}_M \). Again, \( \Psi_{\delta'}^\alpha \) has the form (7.1). Therefore if we take

\[ f(z) = F_{ij}(z) = \sum_{k=1}^{l(\delta)} P_{\delta}^i(z)G_{kj}(z), \]

then by (7.7), we get

\[ \langle G(F_{ij}), \mathcal{W}(P_{\delta''}) \rangle_\alpha = 0, \]

if \( (\delta', i') \neq (\delta, i) \). By Proposition 7.8 in particular for \( i = 1 \), we get

\[ \mathcal{G}(F_{1j}) \big|_{V_\alpha} \in \mathcal{W}(H_\delta^1) \big|_{V_\alpha}, \]

for all \( j = 1, 2, \ldots, l(\delta) \); and consequently there are constants \( c_{kj} \) such that

\[ \mathcal{G}(F_{1j}) \big|_{V_\alpha} = \sum_{k=1}^{l(\delta)} c_{kj} \mathcal{W}(P_{1k}) \big|_{V_\alpha}, \quad (7.11) \]

which implies

\[ F_{1j} \times \psi_\alpha = \sum_{k=1}^{l(\delta)} c_{kj} \theta(P_{1k}) \psi_\alpha \]

\[ = (1, j)\text{th entry of } [\theta(P_{\delta}) \psi_\alpha] C_\alpha \]

\[ = (1, j)\text{th entry of } \Psi_{\delta}^\alpha C_\alpha, \]

where \( C_\alpha \) is the \( l(\delta) \times l(\delta) \) constant matrix whose \( (k, j) \)th entry is \( c_{kj} \). Therefore by Corollary 4.4, \( F \times \psi_\alpha = \Psi_{\delta}^\alpha C_\alpha \). This is true for all \( \alpha \in \Lambda \). Hence we get \( \mathcal{G}(F) = \mathcal{W}(P_\delta)S \), if we define the \( l(\delta) \times l(\delta) \) linear operator \( S \) by \( S \big|_{V_\alpha} = C_\alpha \).
Since $\Pi(z)^* = \Pi(-z)$, a direct calculation shows that $\mathcal{G}(\bar{f}) = \mathcal{G}(f^-)^*$, for all $f \in \mathcal{S}(\mathbb{C}^n)$, where $f^-(z) = f(-z)$. Also, note that $(L_j f^-) = (-L_j) f^-$ and $(\bar{L}_j f) = (-\bar{L}_j) f^-$. Therefore if $p, q$ be two $K$-harmonic polynomials then

$$
\mathcal{G}(\theta(p)\psi_\alpha \times \overline{\theta(q)\psi_\beta}) = \mathcal{G}(\theta(p)\psi_\alpha)\mathcal{G}(\theta(q^-)\psi_\beta)^* \\
= W(p)\mathcal{G}(\psi_\alpha)[W(q^-)\mathcal{G}(\psi_\beta)]^* \\
= \tau(p)\mathcal{G}(\psi_\alpha)\mathcal{G}(\psi_\beta)\tau(q^-)^* \\
= 0,
$$

if $\beta \neq \alpha$. The last two equality holds on the domain of $\tau(q^-)^*$. Since $W_j$ and $\bar{W}_j$ are adjoint to each other we see that $\tau(q^-)^* = \tau(\bar{q}^-)$ whose domain contains $\mathcal{P}(\mathbb{C}^n)$. Hence we get $\theta(p)\psi_\alpha \times \overline{\theta(q)\psi_\beta} = 0$. In particular $\int_{\mathbb{C}^n} \overline{\theta(q)\psi_\beta(z)} \theta(p)\psi_\alpha(z)dz = 0$ if $\beta \neq \alpha$. Applying this, we have for $\beta \neq \alpha$,

$$
\int_{\mathbb{C}^n} [\Psi_\beta^\delta(z)]^* \Psi_\alpha^\delta(z)dz = 0. \tag{7.12}
$$

Since (by Proposition 6.4)

$$
F = \sum_{\beta \in \Lambda} F \times \psi_\beta = \sum_{\beta \in \Lambda} \Psi_\beta^\delta C_\beta,
$$

we get

$$
A_\alpha^\delta C_\alpha = \int_{\mathbb{C}^n} [\Psi_\alpha^\delta(z)]^* F(z)dz \\
= \int_{\mathbb{C}^n} [L_\alpha^\delta(z)]^* \Psi_\delta(z)G(z)e^{-|\lambda||z|^2}dz. \tag{7.13}
$$

Now we show that it is possible to chose $C_\alpha = C_\alpha^{\delta,\lambda}(F)$, as defined in (7.2). Without loss of generality assume that $l(r) = r$, i.e. first $\alpha(\delta)$ columns in $\Psi_\alpha^\delta$ are linearly independent. By Remark 4.5 \{$\Psi_{\alpha_1 j}^\delta = \theta(P_{1j}^\delta)\psi_\alpha : j = 1, 2, \cdots \alpha(\delta)$\} form a basis for $V_{\Psi_\delta}^1 = \text{span}\{\Psi_{\alpha_1 j}^\delta = \theta(P_{1j}^\delta)\psi_\alpha : j = 1, 2, \cdots l(\delta)\}$; which is equivalent to saying that \{$W(P_{1j}^\delta)|_{V_\alpha} : j = 1, 2, \cdots \alpha(\delta)$\} form a basis for $W(H_\delta^1)|_{V_\alpha}$. Therefore in (7.11) we can take $c_{kj} = 0$ for $k > \alpha(\delta)$. Consequently from (7.13) we get

$$
\tilde{A}_\alpha^\delta \tilde{C}_\alpha = \int_{\mathbb{C}^n} [\tilde{L}_\alpha^\delta(z)]^* \Psi_\delta(z)G(z)e^{-|\lambda||z|^2}dz,
$$
where $\tilde{C}_\alpha$ denotes the $\alpha(\delta) \times l(\delta)$ matrix whose rows are precisely the first $\alpha(\delta)$ rows of $C_\alpha$. But by Lemma 7.3, $\tilde{A}_{\alpha}^{\delta,\lambda}$ is invertible. Therefore we can write

$$\tilde{C}_\alpha = (\tilde{A}_{\alpha}^{\delta,\lambda})^{-1} \int_{\mathbb{C}^n} [\tilde{T}_\alpha^\delta(z)]^* \tilde{Y}_\delta(z) G(z) e^{-|\lambda||z|^2} dz.$$ 

Hence by the definition of $C_{\alpha}^{\delta,\lambda}(F)$, $C_\alpha = C_{\alpha}^{\delta,\lambda}(F)$ as desired. \[\square\]

Now we extend Theorem 7.4 to a larger class of functions. Let

$$\mathcal{E}^\lambda(\mathbb{C}^n) = \{ f \in \mathcal{E}(\mathbb{C}^n) : e^{-|\lambda||z|^2} |f(z)| \in L^p(\mathbb{C}^n), \text{ for some } \epsilon > 0, 1 \leq p \leq \infty \},$$

and for $\delta \in \hat{K}_M$,

$$\mathcal{E}^{\delta,\lambda}(\mathbb{C}^n) = \{ F \in \mathcal{E}^\delta(\mathbb{C}^n) : \text{each } F_{ij} \in \mathcal{E}^\lambda(\mathbb{C}^n) \}.$$ 

Since $\psi_\alpha^\lambda(z)$ is equal to $e^{-|\lambda||z|^2}$ times a polynomial, clearly (by Holder’s inequality)

$$f \times^\lambda \psi_\alpha^\lambda(z) = \int_{\mathbb{C}^n} f(z-w) \psi_\alpha^\lambda(w) e^{2i\lambda \text{Re}(z \cdot w)} dw$$

is well defined, whenever $f \in \mathcal{E}^\lambda(\mathbb{C}^n)$. For $\epsilon > 0$ and $z \in \mathbb{C}^n$, define

$$\tau_z^\epsilon \psi_\alpha^\lambda(w) = e^{(|\lambda| - \epsilon)|w|^2} [\psi_\alpha^\lambda(z-w) e^{-2i\lambda \text{Im}(z \cdot w)}],$$

which clearly belongs to $\mathcal{S}(\mathbb{C}^n)$. Note that if $f \in \mathcal{E}^\lambda(\mathbb{C}^n)$, then for some $\epsilon > 0$, we can think of $e^{-|\lambda| |z|^2} f(z)$ as a tempered distribution and then clearly

$$f \times^\lambda \psi_\alpha^\lambda(z) = e^{-|\lambda| |z|^2} f(\tau_z^\epsilon \psi_\alpha^\lambda).$$

**Lemma 7.9.** Let $f$ be a distribution on $\mathbb{C}^n$, such that for some $\epsilon > 0$, $e^{-|\lambda| |z|^2} f$ is a tempered distribution. Let $D$ be a polynomial coefficient differential operator on $\mathbb{C}^n$. Then

(a) $e^{-|\lambda| |z|^2} D f$ is also a tempered distribution.

(b) Let $f_j \in \mathcal{S}(\mathbb{C}^n)$ be such that $e^{-|\lambda| |z|^2} f_j \rightarrow e^{-|\lambda| |z|^2} f$ in $\mathcal{S}'(\mathbb{C}^n)$. Then $e^{-|\lambda| |z|^2} D f_j \rightarrow e^{-|\lambda| |z|^2} D f$ in $\mathcal{S}'(\mathbb{C}^n)$. Consequently for each $z \in \mathbb{C}^n$,

$$D f_j \times^\lambda \psi_\alpha^\lambda \rightarrow e^{-|\lambda| |z|^2} D f(\tau_z^\epsilon \psi_\alpha^\lambda).$$

(c) In particular, if $f \in \mathcal{E}^\lambda(\mathbb{C}^n)$, $f_j \times^\lambda \psi_\alpha^\lambda(z) \rightarrow f \times^\lambda \psi_\alpha^\lambda(z)$ for each $z \in \mathbb{C}^n$. 


Proof. When the action of $D$ on $f$ is multiplication by a polynomial, clearly (a) and (b) are true. Note that

$$e^{-|\lambda|\epsilon|z|^2} \frac{\partial f}{\partial z_j} = \frac{\partial}{\partial z_j} (e^{-|\lambda|\epsilon|z|^2} f) + (|\lambda| - \epsilon) z_j f,$$

which immediately proves (a), as well as (b) when $D = \partial/\partial z_j$. General case follows by an induction argument. Assertion (c) is immediate from (b). □

Theorem 7.10. Suppose $F = P^\delta G \in \mathcal{E}^{\delta,\lambda}(\mathbb{C}^n)$, $G$ is $K$-invariant. Then $F \times^\lambda \psi^\lambda_\alpha = \Psi^\delta,\lambda_\alpha (F)$, where $C^\delta,\lambda_\alpha (F)$ is defined by (7.2).

Proof. Each entry of $F$ belongs to $\mathcal{E}^\lambda(\mathbb{C}^n)$. Take $F_j \in \mathcal{S}(\mathbb{C}^n)$ such that $e^{-|\lambda|\epsilon|z|^2} F_j \to e^{-|\lambda|\epsilon|z|^2} F$ entry wise in $\mathcal{S}'(\mathbb{C}^n)$. For each $F_j$ we can apply Theorem 7.4 to get

$$F_j \times^\lambda \psi_\alpha(z) = \Psi^\delta_\alpha(z) C^\delta,\lambda_\alpha (F_j),$$

(7.14)

where $C^\delta,\lambda_\alpha (F_j)$ is defined by equation (7.2). A similar argument used in the proof of the previous lemma shows that $\lim_{j \to \infty} C^\delta,\lambda_\alpha (F_j) = C^\delta,\lambda_\alpha (F)$. On the other hand, by (c) of the previous lemma, for each $z \in \mathbb{C}^n$,

$$\lim_{j \to \infty} [F_j \times^\lambda \psi_\alpha(z)] = F \times^\lambda \psi_\alpha(z).$$

Hence for each $z \in \mathbb{C}^n$, taking limit, as $j \to \infty$, in (7.14), the proof follows. □

Now we proceed to prove the uniqueness (upto right multiplication by a constant matrix) of generalized $K$-spherical function when it belongs to $\mathcal{E}^{\delta,\lambda}(\mathbb{C}^n)$.

Lemma 7.11. Let $f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ be a joint eigenfunction for all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue $\mu^\lambda_\alpha$. Then $f \times^\lambda \psi_\beta = 0$ if $\beta \neq \alpha$.

Proof. By definition of $\mathcal{E}^\delta(\mathbb{C}^n)$, $e^{-|\lambda|\epsilon|z|^2} f$ is a tempered distribution for some $\epsilon > 0$. Take $f_j \in \mathcal{S}(\mathbb{C}^n)$ such that $e^{-|\lambda|\epsilon|z|^2} f_j \to e^{-|\lambda|\epsilon|z|^2} f$ in $\mathcal{S}'(\mathbb{C}^n)$. Let $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$. Since $Df = \mu_\alpha(D)f$, by Lemma 7.9 (b),

$$\lim_{j \to \infty} Df_j \times^\lambda \psi_\beta(z) = e^{-|\lambda|\epsilon|z|^2} \mu_\alpha(D) f (\tau^\epsilon \psi_\beta) = \mu_\alpha(D) f \times^\lambda \psi_\beta.$$

(7.15)
Now we will show that \( Df_j \times \psi_\beta = \mu_\beta(D)f_j \times \psi_\beta \) for all \( j \). By Proposition 5.3 it follows that \( \text{range}[\mathcal{S}(D)\mathcal{P}_\beta] \subset \mathcal{P}_N \) for some natural number \( N \). Here \( \mathcal{P}_N \) denotes the space of all holomorphic polynomials of degree less than or equal to \( N \). Hence

\[ \mathcal{S}(D)\mathcal{P}_\beta = \left( \sum_{\gamma \in \mathcal{P}_N} \mathcal{P}_\gamma \right) \mathcal{S}(D)\mathcal{P}_\beta. \]

Enlarging \( \mathcal{P}_N \) if necessary we may assume that \( V_\beta \subset \mathcal{P}_N \). Therefore we have

\[ \mathcal{S}(Df_j \times \psi_\beta) = \mathcal{S}(Df_j)\mathcal{P}_\beta = \mathcal{S}(f_j)\mathcal{S}(D)\mathcal{P}_\beta = \mathcal{S}(f_j) \left( \sum_{\gamma \in \mathcal{P}_N} \mathcal{P}_\gamma \right) \mathcal{S}(D)\mathcal{P}_\beta. \]

But,

\[ \mathcal{P}_\gamma \mathcal{S}(D) = \mathcal{S}(\psi_\gamma)\mathcal{S}(D) = \mathcal{S}(D\psi_\gamma) = \mu_\gamma(D)\mathcal{P}_\gamma. \]

Hence

\[ \mathcal{S}(Df_j \times \psi_\beta) = \mu_\beta(D)\mathcal{S}(f_j)\mathcal{P}_\beta = \mu_\beta(D)\mathcal{S}(f_j \times \psi_\beta). \]

Therefore \( Df_j \times \psi_\beta(z) = \mu_\beta(D)f_j \times \psi_\beta(z) \) for all \( z \in \mathbb{C}^n \). Now taking limit as \( j \to \infty \) and using (7.15), (7.16) we get \( \mu_\alpha(D)f \times \psi_\beta(z) = \mu_\beta(D)f \times \psi_\beta(z) \). This is true for all \( D \in \mathcal{L}_K^\lambda(\mathbb{C}^n) \). Since \( \mu_\beta \neq \mu_\alpha \) for \( \beta \neq \alpha \), we get \( f \times \psi_\beta(z) = 0 \) if \( \beta \neq \alpha \). Hence the proof.

\[ \square \]

**Lemma 7.12.** Let \( \mathcal{L}^\lambda \) be the special Hermite operator and \( \psi_k^\lambda \)'s be the \( U(n) \)-spherical functions (see Remark 5.10). Let \( f \in \mathcal{E}^\lambda(\mathbb{C}^n) \) be an eigenfunction of \( \mathcal{L}^\lambda \) with eigenvalue \(-2|\lambda|(2k + n)\). Then \( f = f \times^\lambda \psi_k^\lambda \).

**Proof.** For this proof let \( K = U(n) \) and \( M \) be the subgroup of \( U(n) \) that fixes the coordinate vector \( e_1 = (1,0,\cdots,0) \) in \( \mathbb{C}^n \). For \( \delta \in \widehat{K}_M \), let \( \chi_\delta(k) = tr(\delta(k)). \)

Define \( f_\delta(z) = \int_K f(k^{-1} \cdot z)\chi_\delta(k)dk \). Clearly each \( f_\delta \) is an eigenfunction of \( \mathcal{L} \) with eigenvalue \(-2|\lambda|(2k + n)\). Applying the previous lemma for \( K = U(n) \) to each \( f_\delta \) we get \( f_\delta \times \psi_m = 0 \) if \( m \neq k \). Again Proposition 4.5 \[16\], in particular, implies that each \( f_\delta \in \mathcal{S}(\mathbb{C}^n) \). Hence by Proposition 6.4, \( f_\delta = \sum_{m \in \mathbb{N}} f_\delta \times \psi_m \). Consequently we get \( f_\delta = f_\delta \times \psi_k \) for all \( \delta \in \widehat{K}_M \). Since \( \psi_k \) is radial, an easy calculation shows
that \((f \times \psi_k)_\delta = f_\delta \times \psi_k\). Therefore \(f_\delta = (f \times \psi_k)_\delta\) for all \(\delta \in \hat{K}_M\). But for any smooth function \(g\) it is well known that \(g(z) = \sum_{\delta \in \hat{K}_M} g_\delta(z)\), where the right hand side converges uniformly over compact set. Hence we conclude that \(f = f \times \psi_k\). □

**Proposition 7.13.** Let \(f \in E^\lambda(\mathbb{C}^n)\) be a joint eigenfunction for all \(D \in L^\lambda_K(\mathbb{C}^n)\) with eigenvalue \(\mu^\lambda_\alpha\). Then \(f = f \times^\lambda \psi^\lambda_\alpha\).

**Proof.** \(V_\alpha \subset P_k(\mathbb{C}^n)\) for some \(k \in \mathbb{N}\). Then \(f\) is an eigenfunction of \(L_\delta\) with eigenvalue \(-(2k + n)|\lambda|\). Therefore by Lemma 7.12 \(f = f \times \psi_k\). Since \(\psi_k = \sum_{\beta \subset V_k(\mathbb{C}^n)} \psi_\beta\), we get \(f = \sum_{\beta \subset V_k(\mathbb{C}^n)} f_\beta \times \psi_\beta\). But then by Lemma 7.11 \(f = f \times \psi_\alpha\). □

As an immediate consequence of Theorem 7.10 and Proposition 7.13 we get the following Theorem.

**Theorem 7.14.** If \(\Psi \in E^{\delta,\lambda}(\mathbb{C}^n)\) is a generalized \(K\)-spherical function of type \(\delta\) corresponding to the eigenvalue \(\mu^\lambda_\alpha\), then \(\Psi = \Psi^{\delta,\lambda}_\alpha C\), where \(C = C^{\delta,\lambda}_\alpha(\Psi)\) as defined by (7.2).

We conclude this section by giving another formulae for \(\Psi^{\delta,\lambda}_\alpha\) which will be used in the next section. Define

\[
\Phi^{\delta,\lambda}_\alpha(z) = \langle \Pi^\lambda(z), W^\lambda(P^\delta) \rangle^\lambda_\alpha = \sum_{\nu=1}^{d(\alpha)} \langle \Pi^\lambda(z)e^\lambda_\alpha, W^\lambda(P^\delta)e^\lambda_\alpha \rangle
\]

**Proposition 7.15.** \(\Psi^{\delta,\lambda}_\alpha = \pi^{-n}(2|\lambda|)^n \Phi^{\delta,\lambda}_\alpha\). Consequently \(\theta^\lambda(p) \psi^\lambda_\alpha = \langle \Pi^\lambda(z), W^\lambda(p) \rangle^\lambda_\alpha\) whenever \(p \in H_\delta\).

**Proof.** Note that, on the one hand a direct calculation shows

\[
\langle G^\lambda(f), W^\lambda(P^\delta) \rangle^\lambda_\alpha = \langle f, \Phi^{\delta,\lambda}_\alpha \rangle,
\]

and on the other hand, by Lemma 7.6(a), we have

\[
\langle G^\lambda(f), W^\lambda(P^\delta) \rangle^\lambda_\alpha = \pi^n(2|\lambda|)^{-n} \langle f, \Psi^{\delta,\lambda}_\alpha \rangle,
\]

for all \(f \in \mathcal{S}(\mathbb{C}^n)\). Hence \(\Psi^{\delta,\lambda}_\alpha = \pi^{-n}(2|\lambda|)^n \Phi^{\delta,\lambda}_\alpha\). This also can be proved directly using the inversion formulae for Weyl transform. □
8. $K$-finite eigenfunctions

Following the viewpoint of Thangavelu in [16] (see Theorem 3.3 there), we obtain a representation for $K$-finite joint eigenfunctions in $\mathcal{E}^\lambda(\mathbb{C}^n)$.

**Theorem 8.1.** Let $f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ be a $K$-finite joint eigenfunction for all $D \in \mathcal{L}^\lambda_K(\mathbb{C}^n)$ with eigenvalue $\mu^\lambda_\alpha$. Then $f(z) = \langle \Pi^\lambda(z), \mathcal{W}^\lambda(P) \rangle^\lambda_\alpha$ for some $K$-harmonic polynomial $P$.

To prove the theorem we first prove the following lemma, which is an easy consequence of Theorem 7.14 and Proposition 7.15.

**Lemma 8.2.** Suppose $F : \mathbb{C}^n \rightarrow M_{d(\delta) \times d(\delta)}$ is a smooth, square integrable joint eigenfunction for all $D \in \mathcal{L}^\lambda_K(\mathbb{C}^n)$ with eigenvalue $\mu^\lambda_\alpha$. Also assume $F(k \cdot z) = \delta(k)F(z)$, for some $\delta \in \hat{K}_M$. Then, there exists a $l(\delta) \times d(\delta)$ constant matrix $C$ such that $F = \Phi^\lambda_\alpha C$.

**Proof.** We suppress the superscript $\lambda$. For each $j \in \{1, 2, \ldots, d(\delta)\}$, define $F^j : \mathbb{C}^n \rightarrow M_{d(\delta)}$ to be the matrix whose first column is precisely the $j$th column of $F(z)$ and else are zero. Then clearly each $F^j$ is square integrable generalized $K$-spherical function. Hence, by Theorem 7.14 and Proposition 7.15 it follows that there exist $l(\delta) \times l(\delta)$ constant matrix $C^j$ such that $F^j = \Phi^\delta_\alpha C^j$. Equating the entries in first column we get

$$F_{ij} = F^j_{i1} = \sum_{k=1}^{l(\delta)} (\Phi^\delta_\alpha)_{ik} C^j_{k1}, \quad 1 \leq i, j \leq d(\delta);$$

which in matrix form can be written as $F = \Phi^\delta_\alpha C$, where $C$ is the $l(\delta) \times d(\delta)$ constant matrix given by $C_{kj} = C^j_{k1}$. Hence the proof.

Let $\hat{K}$ denote the set of all inequivalent unitary irreducible representations of $K$. For $\delta \in \hat{K}$, let $\chi_\delta(k) = \text{tr}[\delta(k)]$. 


Proof. (Proof of Theorem 8.1) Since \( f \) is \( K \)-finite, by Lemma 1.7, Chapter IV of [8], there is a finite subset \( \hat{K}(f) \) of \( \hat{K} \) such that

\[
f(z) = \sum_{\delta \in \hat{K}(f)} d(\delta) \chi_{\delta} \ast f(z) := \sum_{\delta \in \hat{K}(f)} d(\delta) \int_{K} \chi_{\delta}(k) f(k^{-1} \cdot z) dk = \sum_{\delta \in \hat{K}(f)} d(\delta) \text{tr}(f^{\delta}),
\]

where

\[
f^{\delta}(z) = \int_{K} f(k^{-1} \cdot z) \delta(k) \, dk.
\]

Clearly \( f^{\delta} \in \mathcal{E}^{\lambda}(\mathbb{C}^{n}) \). Since any \( D \in \mathcal{L}_{K}(\mathbb{C}^{n}) \) commutes with the action of \( K \), clearly each \( f^{\delta} \) is also a joint eigenfunction for all \( D \in \mathcal{L}_{K}(\mathbb{C}^{n}) \) with eigenvalue \( \mu_{\alpha} \). Also note that \( f^{\delta}(k \cdot z) = \delta(k)f^{\delta}(z) \). Now, for \( z = (r, kM) \) and \( m \in M \),

\[
f^{\delta}(z) = f^{\delta}(r, kmM) = f^{\delta}(km \cdot (r, M)) = \delta(k)\delta(m)f^{\delta}(r, M).
\]

Therefore if \( \delta \not\in \hat{K}_{M} \), integrating both side of the above equation over \( M \), we get \( f^{\delta}(z) = 0 \). So assume that \( \delta \in \hat{K}_{M} \). But then by the previous lemma, each \( f^{\delta}_{ij} \) can be written as \( f^{\delta}_{ij}(z) = \langle \Pi^{\lambda}(z), \mathcal{W}^{\lambda}(\tilde{P}^{\delta}_{ij}) \rangle_{\alpha} \) for some \( \tilde{P}^{\delta}_{ij} \in H_{\delta} \). Hence the proof follows. \( \square \)

Let \( f(z, t) \) be a joint eigenfunction for all \( D \in \mathcal{L}_{K}(\mathfrak{h}_{n}) \) with eigenvalue \( \tilde{\mu}_{\alpha}^{\lambda} \). Since

\[
\frac{\partial}{\partial t}(\phi_{\alpha}^{\lambda}) = i\lambda \phi_{\alpha}, \quad \tilde{\mu}_{\alpha}^{\lambda}(\partial/\partial t) = i\lambda, \quad f \text{ has the form } f(z, t) = e^{i\lambda t}g(z).
\]

Clearly \( g \) is a joint eigenfunction for all \( D \in \mathcal{L}_{K}^{\lambda}(\mathbb{C}^{n}) \) with eigenvalue \( \mu_{\alpha}^{-\lambda} \). Therefore, Theorem 8.1 implies the following theorem on the Heisenberg group.

**Theorem 8.3.** Let \( f \) be a \( K \)-finite joint eigenfunction for all \( D \in \mathcal{L}_{K}(\mathfrak{h}_{n}) \) with eigenvalue \( \tilde{\mu}_{\alpha}^{\lambda} \) such that \( f(z, 0) \in \mathcal{E}^{\lambda}(\mathbb{C}^{n}) \). Then \( f(z, t) = \langle \Pi^{\lambda}(z, t), \mathcal{W}^{\lambda}(P) \rangle_{\alpha}^{\lambda} \) for some \( K \)-harmonic polynomial \( P \).

The following proposition says that \( \mu_{\alpha}^{\lambda} \)'s are the only possible eigenvalues for joint eigenfunctions of all \( D \in \mathcal{L}_{K}^{\lambda}(\mathbb{C}^{n}) \), which belong to \( \mathcal{E}^{\lambda}(\mathbb{C}^{n}) \). Hence Theorem 8.1 actually describes all \( K \)-finite joint eigenfunctions of all \( D \in \mathcal{L}_{K}^{\lambda}(\mathbb{C}^{n}) \), which belong to \( \mathcal{E}^{\lambda}(\mathbb{C}^{n}) \). Consequently, Theorem 8.3 actually describes all \( K \)-finite joint eigenfunctions \( f(z, t) \) of all \( D \in \mathcal{L}(\mathfrak{h}_{n}) \) with eigenvalue \( \tilde{\mu} \), such that \( \tilde{\mu}(\partial/\partial t) \) is a non zero real number and \( f(z, 0) \in \mathcal{E}^{\lambda}(\mathbb{C}^{n}) \).
Proposition 8.4. Let \( f \in \mathcal{E}^\lambda(\mathbb{C}^n) \) be a joint eigenfunction of all \( D \in \mathcal{L}_K^\lambda(\mathbb{C}^n) \) with eigenvalue \( \mu \). Then \( \mu = \mu_\alpha^\lambda \) for some \( \alpha \in \Lambda \).

Proof. From Remark 7.5, recall \( S_{pq} \), the space of bigraded spherical harmonics of degree \((p, q)\). Take an orthonormal basis \( \{ Y_{pq}^j(\omega) : j = 1, 2, \cdot \cdot \cdot, d(p, q) \} \) for \( S_{pq} \), so that \( \{ Y_{pq}^j(\omega) : j = 1, 2, \cdot \cdot \cdot, d(p, q) ; p + q = k ; k = 0, 1, \cdot \cdot \cdot \infty \} \) form a basis for \( L^2(S^{2n-1}) \).

Therefore for each \( r > 0 \), \( f(r\omega) \) has the bigraded spherical harmonic expansion

\[
f(r\omega) = \sum_{m=0}^{\infty} \sum_{p+q=m} d(p, q) \sum_{j=1}^{d(p, q)} f_{pq}^j(r) Y_{pq}^j(\omega), \quad \omega \in S^{2n-1},
\]

where

\[
f_{pq}^j(r) = \int_{S^{2n-1}} f(r\omega) Y_{pq}^j(\omega) d\omega, \quad r > 0.
\]

Since \( f \) is smooth, clearly \( f_{pq}^j(r) \) is bounded at zero. Now let \( \mu(\mathcal{L}) = -2|\lambda|(2a + n), \quad a \in \mathbb{C} \). i.e \( f \) is an eigenfunction of \( \mathcal{L} \) with eigenvalue \(-2|\lambda|(2a + n)\). Then it can be shown that (see the proof of Proposition 4.5 [16]), each \( f_{pq}^j(r) Y_{pq}^j(\omega) \) is also an eigenfunction of \( \mathcal{L} \) with eigenvalue \(-2|\lambda|(2a + n)\) i.e

\[
\mathcal{L}[f_{pq}^j(r) Y_{pq}^j(\omega)] = -2|\lambda|(2a + n)[f_{pq}^j(r) Y_{pq}^j(\omega)].
\]

Writing \( \mathcal{L} \) in polar coordinate, using the fact that \( Y_{pq}^j \) is an eigenfunction of the spherical Laplacian on \( S^{2n-1} \), and then making a change of variable

\[
f_{pq}^j(r) = r^{p+q} u(2|\lambda|r^2) e^{-|\lambda|r^2},
\]

we get (for details see the proof of Proposition 4.4 [16]) that \( u \) satisfies the following confluent hypergeometric equation

\[
tu''(t) + (d - t)u'(t) - (p - a)u(t) = 0, \quad (8.1)
\]

where \( d = n + p + q \). The equation (8.1) has two linearly independent solutions \( u_1 \) and \( u_2 \), with the following asymptotic behaviour (see [11], page-145):

(i) If \((p - a) \neq 0, -1, -2, \cdot \cdot \cdot \),

\[
u_1(t) \sim \frac{(d - 1)!}{\Gamma(p - a)} e^t t^{p-a-d}, \quad u_2(t) \sim t^{-(p-a)} \text{ as } t \to +\infty
\]
$u_1(t) \sim 1, \ u_2(t) \sim \left\{ \begin{array}{ll}
\frac{-\log t}{\Gamma(p-a)} & \text{if } d = 1 \\
c \frac{\Gamma(d-a)}{t^{d-1}} & \text{if } d \geq 2
\end{array} \right.,$ as $t \to 0^+$,

where $c$ is a non zero constant.

(ii) If $(p-a)$ is a non positive integer,

$u_1(t) = L^{d-1}_{a-p}(t), \ u_2(t) \sim e^{t(-t)^{a-p-d}}$ as $t \to +\infty$.

Therefore, under the conditions on $f$, the only possibility is $(p-a)$ is a non positive integer and consequently

$f^j_{pq}(r) = r^{p+q} L^{n+p+q-1}_{a-p}(2|\lambda|r^2) e^{-|\lambda|r^2}$.

So there exists non-positive integer $k$ such that $a = k$. Hence $f$ is a eigenfunction of $L$ with eigenvalue $-2|\lambda|(2k + n)$. Therefore by Lemma 7.12 $f = f \times \psi_k = \sum_{V_\beta \subset P_k(\mathbb{C}^n)} f \times \psi_\beta$. Since $f$ is non zero, there exists $\alpha \in \Lambda$ with $V_\alpha \subset P_k(\mathbb{C}^n)$ such that $f \times \psi_\alpha \neq 0$. Now let $D \in \mathcal{L}_K(\mathbb{C}^n)$. Then

$Df = \sum_{V_\beta \subset P_k(\mathbb{C}^n)} D[f \times \psi_\beta] = \sum_{V_\beta \subset P_k(\mathbb{C}^n)} f \times D\psi_\beta = \sum_{V_\beta \subset P_k(\mathbb{C}^n)} \mu_\beta(D) f \times \psi_\beta.$

Again

$Df = \mu(D)f = \sum_{V_\beta \subset P_k(\mathbb{C}^n)} \mu(D) f \times \psi_\beta.$

So we get

$\sum_{V_\beta \subset P_k(\mathbb{C}^n)} [\mu(D) - \mu_\beta(D)] f \times \psi_\beta = 0.$

Hence

$\sum_{V_\beta \subset P_k(\mathbb{C}^n)} [\mu(D) - \mu_\beta(D)] f \times \psi_\beta \times \psi_\alpha = 0,$

which implies that $[\mu(D) - \mu_\alpha(D)] f \times \psi_\alpha = 0$. Since $f \times \psi_\alpha \neq 0$, we get $\mu(D) = \mu_\alpha(D)$. But $D \in \mathcal{L}_K(\mathbb{C}^n)$ is arbitrary. Hence $\mu = \mu_\alpha$. \hfill \Box

Remark 8.5. Theorem 8.1 holds true even if we assume that $f$ is a distribution such that $e^{-|\lambda|-\epsilon)|\cdot|^2} f$ defines a tempered distribution for some $\epsilon > 0$. 


9. SQUARE INTEGRABLE EIGENFUNCTIONS

In this section we prove the following theorem characterizing square integrable joint eigenfunctions of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$. This is analogous to Theorem 3.3 in [16].

**Theorem 9.1.** The square integrable joint eigenfunctions of all the operators $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue $\mu_\alpha$ are precisely $f(z) = \langle \Pi^\lambda(z), S \rangle_\alpha$, where $S \in \mathcal{O}(V_\alpha)$. Moreover $\|f\|^2_2 = \pi^n(2|\lambda|)^{-n}\|S\|^2_\alpha$.

**Proof.** Let $f \in L^2(\mathbb{C}^n)$ be a joint eigenfunction of all $D \in \mathcal{L}_K(\mathbb{C}^n)$ with eigenvalue $\mu_\alpha$. We have

$$f = \sum_{\delta \in \hat{K}_M} d(\delta)\chi_\delta \ast f = \sum_{\delta \in \hat{K}_M} d(\delta)\text{tr}(f^{\delta}),$$

where the series converges in $L^2(\mathbb{C}^n)$. Clearly each $f^{\delta} : \mathbb{C}^n \to M_{d(\delta) \times d(\delta)}$ is a joint eigenfunction of all $D \in \mathcal{L}_K(\mathbb{C}^n)$ with eigenvalue $\mu_\alpha$. Also $f^{\delta}(k \cdot z) = \delta(k)f^{\delta}(z)$. Therefore by Lemma 8.2, there is a $(l(\delta) \times d(\delta))$ constant matrix $C_\delta$ such that

$$d(\delta)f^{\delta} = \Psi_\alpha^\delta C_\delta = \pi^{-n}(2|\lambda|)^n\langle \Pi(z), W(P^{\delta}) \rangle_\alpha C_\delta.$$

Hence

$$f(z) = \pi^{-n}(2|\lambda|)^n \sum_{\delta \in \hat{K}_M} \text{tr}\left[\langle \Pi(z), W(P^{\delta}) \rangle_\alpha C_\delta\right], \quad (9.1)$$

and

$$\|f\|^2_2 = \sum_{\delta \in \hat{K}_M} \left\|\text{tr}[\Psi_\alpha^\delta C_\delta]\right\|^2_2 = \sum_{\delta \in \hat{K}_M} \left\|\text{tr}[\theta(P^{\delta})\psi_\alpha C_\delta]\right\|^2_2 \quad = \quad \pi^{-n}(2|\lambda|)^n \sum_{\delta \in \hat{K}_M} \left\|\text{tr}[W(P^{\delta})C_\delta]\right\|^2_\alpha,$$

where the last equality follows from Lemma 7.6 (b). Therefore

$$S := \pi^{-n}(2|\lambda|)^n \sum_{\delta \in \hat{K}_M} \text{tr}[W(P^{\delta})|_{V_\alpha}C_\delta]$$

defines an element in $\mathcal{O}(V_\alpha)$, and consequently from (9.1) we get $f(z) = \langle \Pi(z), S \rangle_\alpha$.

Conversely let $f(z) = \langle \Pi(z), S \rangle_\alpha$ for some $S \in \mathcal{O}(V_\alpha)$. Let $\hat{K}(\alpha) = \{ \delta \in \hat{K}_M : W(H_\delta)|_{V_\alpha} \neq \{0\} \}$. For each $\delta \in \hat{K}(\alpha)$, choose $p^{\delta}_j \in H_\delta$, $j = 1, 2, \ldots n_\alpha(\delta)$ so that
\{W(p_j^\delta)|_{V_\alpha} : j = 1, 2, \cdots, n_\alpha(\delta)\} forms an orthonormal basis for \(W(H_\delta)|_{V_\alpha}\). Hence by Proposition 7.8 \(\{W(p_j^\delta)|_{V_\alpha} : j = 1, 2, \cdots, n_\alpha(\delta); \delta \in \hat{K}(\alpha)\}\) is an orthonormal basis for \(O(V_\alpha)\). Therefore we can write for each \(z \in \mathbb{C}^n\),

\[
f(z) = \langle \Pi(z), S \rangle_\alpha = \sum_{\delta \in \hat{K}(\alpha)} \sum_{j=1}^{n_\alpha(\delta)} \langle \Pi(z), W(p_j^\delta) \rangle_\alpha \langle W(p_j^\delta), S \rangle_\alpha
\]

\[= \pi^n(2|\lambda|)^{-n} \sum_{\delta \in \hat{K}(\alpha)} \sum_{j=1}^{n_\alpha(\delta)} [\theta(p_j^\delta)\psi_\alpha](z) \langle W(p_j^\delta), S \rangle_\alpha \tag{9.2}
\]

by Proposition 7.15. But

\[
\sum_{\delta \in \hat{K}(\alpha)} \sum_{j=1}^{n_\alpha(\delta)} |\langle W(p_j^\delta), S \rangle_\alpha|^2 = ||S||^2_\alpha \leq \infty,
\]

and by Lemma 7.6 (b),

\[
\langle \theta(p_j^\delta)\psi_\alpha, \theta(p_{j'}^\delta)\psi_\alpha \rangle = \begin{cases} 0 & \text{if } (\delta, j) \neq (\delta', j') \\ \pi^{-n}(2|\lambda|)^{n} & \text{if } (\delta, j) = (\delta', j') \end{cases}.
\]

Therefore it follows that the series for \(f\) defined by equation (9.2) converges in \(L^2(\mathbb{C}^n)\). In particular \(f \in L^2(\mathbb{C}^n)\). Since any \(D \in \mathcal{L}_K(\mathbb{C}^n)\) is a polynomial coefficient differential operator we have

\[Df(z) = \pi^n(2|\lambda|)^{-n} \sum_{\delta \in \hat{K}(\alpha)} \sum_{j=1}^{n_\alpha(\delta)} D[\theta(p_j^\delta)\psi_\alpha](z) \langle W(p_j^\delta), S \rangle_\alpha
\]

in the distribution sense. But \(D[\theta(p_j^\delta)\psi_\alpha] = \mu_\alpha(D)[\theta(p_j^\delta)\psi_\alpha]\). Therefore we can conclude that \(Df = \mu_\alpha(D)f\). Hence \(f\) is a joint eigenfunction of all \(D \in \mathcal{L}_K(\mathbb{C}^n)\) with eigenvalue \(\mu_\alpha\). Also note that \(||f||^2_2 = \pi^n(2|\lambda|)^{-n}||S||^2_\alpha\). Thus the proof is complete. \(\square\)

10. INTEGRAL REPRESENTATIONS OF EIGENFUNCTIONS WHEN \(\text{dim} V_\delta^M = 1\)

As usual let \((K, \mathbb{H}^n) (K \subset U(n))\) be a Gelfand pair such that the \(K\)-action on \(\mathbb{C}^n\) is polar. In this section we consider the special case when \(\text{dim} V_\delta^M = 1\) for all \(\delta \in \hat{K}_M\), and (under the usual growth condition) characterize any joint eigenfunction
of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$. We show that for this special case it is enough to consider subgroups of the type $K = U(n_1) \times U(n_2) \times \cdots \times U(n_m)$, $n_1 + n_2 + \cdots + n_m = n$. Then the generalized $K$-spherical functions are given in terms of certain Laguerre polynomials. We use the well-known asymptotic behaviour of Laguerre polynomials to characterize joint eigenfunctions. We give two such characterizations. The first one is a direct generalization of Theorem 8.1 and Theorem 9.1. We will see that this is actually analogous to Theorem 4.1 in [16], which gives an integral representation of eigenfunctions. Though the ideas behind the proof are similar to that in [16], we give the details here, since we will be dealing with $K = U(n_1) \times U(n_2)$ instead of $K = U(n)$. The second one gives a different integral representation of eigenfunctions with an explicit kernel.

**Lemma 10.1.** Suppose $\dim V_\delta^M = 1$ for all $\delta \in \hat{K}_M$. Also assume that the decomposition of $\mathcal{P}_1(\mathbb{C}^n)$ into $K$-irreducible subspaces is as follows : $\mathcal{P}_1(\mathbb{C}^n) = \bigoplus_{j=1}^m V_j$, where $V_1 = \text{span}\{z, z_1, \ldots, z_n\}$, $V_2 = \text{span}\{z_{n_1+1}, z_{n_1+2}, \ldots, z_{n_1+n_2}\}, \ldots; n_1 + n_2 + \cdots + n_m = n$. Then $\mathcal{P}(\mathbb{C}_K^n) = \mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)^K$, where $K_0 = U(n_1) \times U(n_2) \cdots \times U(n_m)$.

**Proof.** For simplicity of the proof we take $m = 2$. Let $\{v_1, v_2, \ldots, v_{d(\alpha)}\}$ be an orthonormal (in $\mathfrak{H}^\lambda$ for $\lambda = \frac{1}{2}$) basis for $V_{\alpha}$ and $p_{\alpha} = \sum_{i=1}^{d(\alpha)} v_i \bar{v}_i$. Then $\{p_{\alpha}\}_{\alpha \in \Lambda}$ is a vector space basis for $\mathcal{P}(\mathbb{C}_K^n)$ (see Proposition 3.9 in [2]). In particular, any $K$-invariant second degree homogeneous polynomial has to be a linear combination of $\gamma_1$ and $\gamma_2$, where $\gamma_1(z) = |z_1|^2 + |z_2|^2 + \cdots |z_n|^2$ and $\gamma_2(z) = |z_{n_1+1}|^2 + |z_{n_1+2}|^2 + \cdots |z_n|^2$. Since $U(n_1) \times U(n_2)$ invariant elements in $\mathcal{P}(\mathbb{C}_K^n)$ are generated by $\gamma_1$ and $\gamma_2$, to prove the theorem it is enough to show that any $K$-invariant homogeneous polynomial can be written as a polynomial in $\gamma_1(z)$ and $\gamma_2(z)$. We prove this by induction on the degree of $K$-invariant homogeneous polynomials. Note that degree of a $K$-invariant homogeneous polynomial is always even. Denote the representation of $K$ on $V_j$ by $\delta_j$. Since $\ell(\delta_j) = \dim V_j^M = 1$, $H_{\delta_j} = V_j$ (see [3,2]). Now if $V_{\alpha} \subset \mathcal{P}_2(\mathbb{C}^n)$ then degree of $p_{\alpha}$ is 4. Since $\mathcal{F}(\frac{\partial p_{\alpha}}{\partial z_1})$ is equal to $z_1$ times a $K$-invariant distribution, which transform according to representation $\delta_1$ and $K$-action commutes with $\mathcal{F}$,
the same is true for \( \frac{\partial p_\alpha}{\partial \bar{z}_1} \). Hence \( \frac{\partial p_\alpha}{\partial \bar{z}_1} \in IH_{\delta_1} = IV_1 \), where \( I = \mathcal{P}(\mathbb{C}^n)^K \). But \( \frac{\partial p_\alpha}{\partial \bar{z}_1} \) being a third degree homogeneous polynomial and any \( K \)-invariant second degree homogeneous polynomial being a linear combination of \( \gamma_1 \) and \( \gamma_2 \), \( \frac{\partial p_\alpha}{\partial \bar{z}_1} \) has to be of the following form:

\[
\frac{\partial p_\alpha}{\partial \bar{z}_1}(z) = \sum_{j=1}^{n_1} z_j[a_j \gamma_1(z) + b_j \gamma_2(z)]. \tag{10.1}
\]

Similarly, as \( \frac{\partial p_\alpha}{\partial \bar{z}_{n_1+1}} \) is a third degree homogeneous polynomial which belongs to \( IH_{\delta_2} = IV_2 \), it has the following form:

\[
\frac{\partial p_\alpha}{\partial \bar{z}_{n_1+1}}(z) = \sum_{j=1}^{n_2} z_{n_1+j}[a'_j \gamma_1(z) + b'_j \gamma_2(z)]. \tag{10.2}
\]

From the above two equations we get

\[
\frac{\partial^2 p_\alpha}{\partial \bar{z}_{n_1+1} \partial \bar{z}_1} = \sum_{j=1}^{n_1} b_j z_j z_{n_1+1} = \sum_{j=1}^{n_2} a'_j z_{n_1+j} z_1,
\]

which implies that \( b_j = 0 \) if \( j \neq 1 \). So (10.1) becomes

\[
\frac{\partial p_\alpha}{\partial \bar{z}_1}(z) = z_1[a_1 \gamma_1(z) + b_1 \gamma_2(z)] + \sum_{j=2}^{n_1} z_j a_j \gamma_1(z). \tag{10.3}
\]

Now let

\[
v_i(z) = z_1(c_1 z_1 + c_2 z_2 + \cdots c_n z_n) + q_1(z), \quad i = 1, 2, \cdots d(\alpha),
\]

where \( q_1(z) \) is a second degree homogeneous holomorphic polynomial in \( z_2, z_3, \cdots z_n \).

Then

\[
\frac{\partial p_\alpha}{\partial \bar{z}_1}(z) = \sum_{i=1}^{d(\alpha)} [z_1(c_1 z_1 + c_2 z_2 + \cdots c_n z_n) + q_1(z)][2\bar{c}_1 \bar{z}_1 + \bar{c}_2 \bar{z}_2 + \cdots \bar{c}_n \bar{z}_n]. \tag{10.4}
\]

Equating the coefficient of \( z_2 z_1 \bar{z}_1 \) from the right hand sides of (10.3) and (10.4) we get \( a_2 = 2\Sigma c_{i2} \bar{c}_{i1} \). Again equating the coefficients of \( z_1^2 \bar{z}_2 \) from the right hand sides of (10.3) and (10.4) we get \( \Sigma c_{1i} \bar{c}_{i2} = 0 \). Hence \( a_2 = 0 \). similarly \( a_3 = a_4 = \cdots = a_{n_1} = 0 \). So from (10.3) we get

\[
\frac{\partial p_\alpha}{\partial \bar{z}_1}(z) = z_1[a_1 \gamma_1(z) + b_1 \gamma_2(z)].
\]
Similarly we get
\[ \frac{\partial p_\alpha}{\partial \bar{z}_j}(z) = z_j[a_j\gamma_1(z) + b_j\gamma_2(z)], j = 1, 2, \cdots n. \]

Hence we can write \( p_\alpha \) as
\[ p_\alpha(z) = \begin{cases} 
  a_j\gamma_1(z)^2 + b_j\gamma_1(z)\gamma_2(z) + r_j(z) & \text{if } j = 1, 2, \cdots, n_1 \\
  c_j\gamma_1(z)\gamma_2(z) + d_j\gamma_2(z)^2 + r_j(z) & \text{if } j = n_1 + 1, n_1 + 2, \cdots, n_1 + n_2 = n. 
\end{cases} \]

Here \( r_j(z) \) is a fourth degree homogeneous polynomial in \( z, \bar{z} \), which is independent of \( \bar{z}_j \). For \( j = 1, 2 \), equating the coefficients of \( |z_1|^2|z_2|^2 \), we get \( a_1 = a_2 \). Similarly we can show that all \( a_j \)'s are same; and all \( d'_j \)'s are same. Again for \( j = i \) and \( j = n_1 + k \) (\( i = 1, 2, \cdots, n_1; k = 1, 2, \cdots, n_2 \)), equating the coefficients of \( |z_i|^2|z_{n_1+k}|^2 \), we get \( b_i = c_{n_1+k} \). Hence we can write
\[ p_\alpha(z) = a_1\gamma_1(z)^2 + b_1\gamma_1(z)\gamma_2(z) + d_1\gamma_2(z)^2 + \tilde{r}_j(z), \forall j = 1, 2, \cdots, n, \]
where \( \tilde{r}_j(z) \) is a fourth degree homogeneous polynomial in \( z, \bar{z} \), which is independent of \( \bar{z}_j \). Therefore
\[ p_\alpha(z) = a_1\gamma_1(z)^2 + b_1\gamma_1(z)\gamma_2(z) + d_1\gamma_2(z)^2 + r(z), \]
where \( r(z)(= \tilde{r}_1(z) = \tilde{r}_2(z) = \cdots = \tilde{r}_n(z)) \) is a fourth degree homogeneous polynomial in \( z \) only. But since \( p_\alpha \) has the form \( p_\alpha = \Sigma_{i=1}^{d(\alpha)} v_i \bar{v}_i \), it follows that \( r(z) \equiv 0 \).

Hence
\[ p_\alpha(z) = a_1\gamma_1(z)^2 + b_1\gamma_1(z)\gamma_2(z) + d_1\gamma_2(z)^2. \]

So we have proved that if \( V_\alpha \subset \mathcal{P}_2(\mathbb{C}^n) \), then \( p_\alpha \) can be written as a polynomial in \( \gamma_1 \) and \( \gamma_2 \). Hence, it follows that any \( K \)-invariant, 4th degree, homogeneous polynomial can be written as a polynomial in \( \gamma_1 \) and \( \gamma_2 \). Now, let any \( K \)-invariant homogeneous polynomial of degree \( 2N \) can be written as a polynomial in \( \gamma_1 \) and \( \gamma_2 \). We have to show that any \( K \)-invariant homogeneous polynomial of degree \( 2(N+1) \) can be written as a polynomial in \( \gamma_1 \) and \( \gamma_2 \). But for this, it is enough to show the following: If \( V_\alpha \subset \mathcal{P}_{N+1}(\mathbb{C}^n) \), then \( p_\alpha \) can be written as a polynomial in \( \gamma_1 \) and \( \gamma_2 \). So fix a \( \alpha \in \Lambda \), such that \( V_\alpha \subset \mathcal{P}_{N+1}(\mathbb{C}^n) \). Then \( \frac{\partial p_\alpha}{\partial \gamma_1} \) is a \((2N+1)\)th
degree homogeneous polynomial which belongs to $IV_1$. Therefore by the induction hypothesis, \( \frac{\partial p_\alpha}{\partial z_1} (z) \) has the following form

\[
\frac{\partial p_\alpha}{\partial z_1} (z) = \sum_{j=1}^{n_1} z_j \left[ \sum_{l=0}^{N} d_j^l (\gamma_1(z))^{N-l} (\gamma_2(z))^l \right].
\]

Similarly, \( \frac{\partial p_\alpha}{\partial \bar{z}_{n_1+1}} \) has the following form

\[
\frac{\partial p_\alpha}{\partial \bar{z}_{n_1+1}} (z) = \sum_{j=1}^{n_2} z_{n_1+j} \left[ \sum_{l=0}^{N} b_j^l (\gamma_1(z))^{N-l} (\gamma_2(z))^l \right].
\]

Now using similar arguments as before, it is possible to show that \( p_\alpha \) is generated by \( \gamma_1 \) and \( \gamma_2 \). Hence the proof. \( \square \)

**Proposition 10.2.** Suppose \( \dim V^M_\delta = 1 \) for all \( \delta \in \hat{K}_M \). Then there exist \( J \in U(n) \), and positive integers \( n_1, n_2, \ldots, n_m \) with \( n_1 + n_2 + \cdots + n_m = n \), such that \( \mathcal{P}(\mathbb{C}^n)_K = \mathcal{P}(\mathbb{C}_R^n)^K_0 \), where \( K_0 = J \left[ U(n_1) \times U(n_2) \times \cdots \times U(n_m) \right] J^{-1} \).

**Proof.** Let \( \mathcal{P}_1(\mathbb{C}^n) = \bigoplus_{l=1}^{m} V_l \) be the decomposition of \( \mathcal{P}_1(\mathbb{C}^n) \) into \( K \)-irreducible subspaces. \( V_l \)'s are pairwise orthogonal in \( \mathcal{H}^2 \). Let \( \dim V_l = n_l \) so that \( n_1 + n_2 + \cdots + n_m = n \). Let \( u_i(z) = z_i \). Choose an orthonormal (in \( \mathcal{H}^2 \)) basis \( \{ v_i(z) = \Sigma_{j=1}^{n} c_{ij} u_j : i = 1, 2, \ldots, n \} \) for \( \mathcal{P}_1(\mathbb{C}^n) \) such that \( \{ v_1, v_2, \ldots, v_{n_1} \}, \{ v_{n_1+1}, v_{n_1+2}, \ldots, v_{n_1+n_2} \}, \ldots \) form a basis for \( V_1, V_2, \ldots \) respectively. Since \( \{ u_1(z), u_2(z), \ldots, u_n(z) \} \) is an orthonormal set in \( \mathcal{H}^2 \), we get \( \Sigma_{j=1}^{n} |c_{ij}|^2 = 1 \); and \( \Sigma_{j=1}^{n} c_{ij} \overline{c_{ij}} = 0 \) if \( i \neq i' \). Hence the matrix \( J := (\bar{c}_{ij})_{n \times n} \) is unitary. Therefore \( J^{-1} = J^* = (c_{ji})_{n \times n} \), which implies that \( (J \cdot u_i)(z) = u_i(J^{-1} \cdot z) = v_i(z) \) or \( J \cdot u_i = v_i \). Consequently the decomposition of \( \mathcal{P}_1(\mathbb{C}^n) \) into \( J^{-1} K J \)-irreducible subspaces is given by \( \mathcal{P}_1(\mathbb{C}^n) = \bigoplus_{l=1}^{m} V'_l \), where \( V'_1 = \text{span}\{ u_1, u_2, \ldots, u_{n_1} \} \), \( V'_2 = \text{span}\{ u_{n_1+1}, u_{n_1+2}, \ldots, u_{n_1+n_2} \} \). Next, \( M = K_{z_0} \) for some \( K \)-regular point \( z_0 \). Then clearly \( J^{-1} \cdot z_0 \) is a \( J^{-1} K J \)-regular point, and \( J^{-1} M J = [J^{-1} K J] J^{-1} z_0 \). Let \( K' = J^{-1} K J \) and \( M' = J^{-1} M J \). For each \( \delta \in \hat{K}_M \) define the irreducible unitary representation \( \delta' \) of \( K' \) on \( V_\delta' = V_\delta \) by \( \delta'(J^{-1} k J) = \delta(k) \) for all \( k \in K \). Then it is easy to see that the map \( \delta \rightarrow \delta' \) is a bijection from \( \hat{K}_M \) onto \( \hat{K}'_{M'} \), and \( \dim V^{M'}_\delta = 1 \) for all \( \delta' \in \hat{K}'_{M'} \). Therefore by
the previous lemma $\mathcal{P}(\mathbb{C}^n_{\mathbb{R}})^{K'} = \mathcal{P}(\mathbb{C}^n_{\mathbb{R}})^{K''}$, where $K' = U(n_1) \times U(n_2) \cdots \times U(n_m)$. Hence the proof.

If $\dim V_{\delta}^{M} = 1$ for all $\delta \in \hat{K}_M$, the above proposition says that with respect to a suitable coordinate system on $\mathbb{C}^n$, the $K$-invariant polynomials are same as that of $U(n_1) \times U(n_2) \times \cdots \times U(n_m)$, $n_1 + n_2 + \cdots + n_m = n$. Since $\mathcal{P}(\mathbb{C}^n_{\mathbb{R}})^K$ determines $\mathcal{L}_K(h_n)$ (see [2], section-3), and hence $\mathcal{L}^\lambda_K(\mathbb{C}^n)$, to find joint eigenfunctions of all $D \in \mathcal{L}^\lambda_K(\mathbb{C}^n)$ for this special case, it is enough to consider the groups $U(n_1) \times U(n_2) \times \cdots \times U(n_m)$, $n_1 + n_2 + \cdots + n_m = n$. For simplicity of notation, here we only deal with the particular case : $m = 2$.

So, from now on $K$ always stands for $U(n_1) \times U(n_2)$, and $M$ the stabilizer of the $K$-regular point $e = (1, 0, \cdots , 1, 0, \cdots , 0) \in \mathbb{C}^n$, where the second 1 is at the $(n_1 + 1)$th position. Via the map $kM \to k \cdot e$, we have the identification

$$K/M = K \cdot e = \{z \in \mathbb{C}^n : \sum_{j=1}^{n_1} |z_j|^2 = 1, \sum_{j=n_1+1}^{n} |z_j|^2 = 1\}.$$

If we identify $\mathbb{C}^n$ with $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ by the map $z \to ((z_1, z_2, \cdots , z_{n_1}), (z_{n_1+1}, z_{n_1+2}, \cdots , z_n))$, then $K/M = S^{2n_1-1} \times S^{2n_2-1}$, where $S^{2n_1-1}$ is the unit sphere in $\mathbb{C}^{n_1}$ and $S^{2n_2-1}$ is the unit sphere in $\mathbb{C}^{n_2}$. Now we explicitly describe the spaces $H_\delta$ and $\mathcal{E}_\delta(K/M) = \mathcal{E}_\delta(S^{2n_1-1} \times S^{2n_2-1})$. Since $I_+$, the set of polynomials in $I = \mathcal{P}(\mathbb{C}^n_{\mathbb{R}})^K$ without constant term, is generated by $\Sigma_{j=1}^{n_1} |z_j|^2, \Sigma_{j=n_1+1}^{n} |z_j|^2$, the set of $K$-harmonic polynomials is given by

$$H = \{P \in \mathcal{P}(\mathbb{C}^n_{\mathbb{R}}) : \Delta_1 P = 0, \Delta_2 P = 0\},$$

where

$$\Delta_1 = \sum_{j=1}^{n_1} \frac{\partial^2}{\partial z_j \partial \overline{z}_j}, \quad \Delta_2 = \sum_{j=n_1+1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_j}.$$ 

For $z \in \mathbb{C}^n$, let $z^1 = (z_1, z_2, \cdots z_{n_1}) \in \mathbb{C}^{n_1}$, $z^2 = (z_{n_1+1}, z_{n_1+2}, \cdots , z_n)$. Let $i = 1$ or 2. For each pair of positive integer $(p, q)$, we define $\mathcal{P}_{pq}^i$ to be the subspace of $\mathcal{P}(\mathbb{C}^n_{\mathbb{R}})$ consisting of all polynomials of the form

$$P(z^i) = \sum_{|\alpha_i|=p} \sum_{|\beta_i|=q} (z^i)^{\alpha_i} (\overline{z}^i)^{\beta_i}.$$
Here $\alpha_i$ and $\beta_i$ are multi-indices of non negative integers of length $n_i$. We let

$$H^i = \{ P \in \mathcal{P}(\mathbb{C}_R^n) : \Delta_i P = 0 \}; \quad H^i_{pq} = \{ P \in \mathcal{P}^i_{pq} : \Delta_i P = 0 \}.$$

We have the identification $\mathcal{P}(\mathbb{C}_R^n) = \mathcal{P}(\mathbb{C}_R^n) \otimes \mathcal{P}(\mathbb{C}_R^n)$, $H = H^1 \otimes H^2$, and consequently $H$ has the algebraic direct sum decomposition:

$$H = \bigoplus_{p_1,q_1,p_2,q_2 \in \mathbb{Z}^+} H^1_{p_1,q_1} \otimes H^2_{p_2,q_2}.$$

Here $\mathbb{Z}^+$ denotes the set of non negative integers. Also note that each $P \in \mathcal{P}^1_{p_1,q_1} \otimes \mathcal{P}^2_{p_2,q_2}$ satisfy the homogeneity condition

$$P(\lambda_1 z^1, \lambda_2 z^2) = \lambda_1^{p_1} \lambda_2^{p_2} \overline{\lambda_1}^{q_1} \overline{\lambda_2}^{q_2} P(z)$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}$. Let $\mathcal{E}_{pq}^i$ stand for the restrictions of members of $H^i_{pq}$ to $S^{2n_i-1}$. The relation between $P \in H^i_{pq}$ and its restriction $Y^i_{pq}$ is given by $P^i_{pq}(z^i) = |z^i|^{p+q} Y^i_{pq}(|z^i|)$, if $z^i = r^i \omega^i$, $r^i > 0$, $\omega^i \in S^{2n_i-1}$. The natural action of $U(n_i)$ defines a unitary representation, $\delta^i_{pq}$ on each of these spaces $\mathcal{E}_{pq}^i$, considered as a Hilbert subspace of $L^2(S^{2n_i-1})$. Clearly the restriction of $H^1_{p_1,q_1} \otimes H^2_{p_2,q_2}$ to $S^{2n_1-1} \times S^{2n_2-1}$ is given by $\mathcal{E}_{p_1,q_1}^1 \otimes \mathcal{E}_{p_2,q_2}^2$. If we consider this as a Hilbert subspace of $L^2(S^{2n_1-1} \times S^{2n_2-1})$, then the natural action of $K$ on each of these spaces $\mathcal{E}_{p_1,q_1}^1 \otimes \mathcal{E}_{p_2,q_2}^2$ defines a unitary representation which is same as $\delta_{p_1,q_1}^1 \otimes \delta_{p_2,q_2}^2$. Now for each fixed $i \in \{1, 2\}$, we have the following well known facts about the class one representations of $U(n_i)$ (see [17], page: 64-69): The representations $\delta^i_{pq}$ of $U(n_i)$ on $\mathcal{E}_{pq}^i$ are irreducible. $\delta^i_{pq}$ and $\delta^i_{p',q'}$ are unitarily equivalent if and only if $(p, q) = (p', q')$. Let $M_i \subset U(n_i)$ be the stabilizer of $(1, 0, \cdots, 0) \in \mathbb{C}^{n_i}$, so that $M = M_1 \times M_2$. Any $\delta \in \hat{U(n_i)}_{M_i}$ is equivalent to some $\delta^i_{pq}$. $L^2(S^{2n_i-1})$ has the orthogonal Hilbert space decomposition:

$$L^2(S^{2n_i-1}) = \bigoplus_{p,q \in \mathbb{Z}^+} \mathcal{E}_{pq}^i.$$ From these facts we can prove the following proposition.

**Proposition 10.3. (a)** The representations $\delta_{p_1,q_1}^1 \otimes \delta_{p_2,q_2}^2$ of $K$ on $\mathcal{E}_{p_1,q_1}^1 \otimes \mathcal{E}_{p_2,q_2}^2$ are irreducible. $\delta_{p_1,q_1}^1 \otimes \delta_{p_2,q_2}^2$ and $\delta_{p_1,q_1'}^1 \otimes \delta_{p_2,q_2'}^2$ are unitarily equivalent if and only if $(p_1, q_1, p_2, q_2) = (p_1', q_1', p_2', q_2')$. Moreover any $\delta \in \hat{K}_M$ is equivalent to some $\delta_{p_1,q_1}^1 \otimes \delta_{p_2,q_2}^2$. 
(b) We have the orthogonal Hilbert space decomposition of \( L^2(S^{2n_1-1} \times S^{2n_2-1}) \):

\[
L^2(S^{2n_1-1} \times S^{2n_2-1}) = \bigoplus_{p_1,q_1,p_2,q_2 \in \mathbb{Z}^+} E_{p_1q_1}^1 \otimes E_{p_2q_2}^2.
\]

By the above proposition, the decomposition of \( \mathcal{P}(\mathbb{C}^n) \) into \( K \)-irreducible subspaces is given by

\[
\mathcal{P}(\mathbb{C}^n) = \bigoplus_{m_1,m_2 \in \mathbb{Z}^+} V_{m_1m_2},
\]

where

\[
V_{m_1m_2} = \text{span}\{(z^1)^{\alpha_1}(z^2)^{\alpha_2} : |\alpha_1| = m_1, |\alpha_2| = m_2\}.
\]

Denote the corresponding bounded \( K \)-spherical functions by \( \phi^\lambda_{m_1m_2}(z, t) = e^{i\lambda t} \psi^\lambda_{m_1m_2}(z) \). Then \( \psi^\lambda_{m_1m_2}(z) \) is a joint eigenfunction of all \( D \in \mathcal{L}_K^\lambda(\mathbb{C}^n) \) with eigenvalue, say \( \mu^\lambda_{m_1m_2} \).

Note that here \( \mathcal{L}_K^\lambda(\mathbb{C}^n) \) is generated by

\[
\mathcal{L}_1^\lambda := \sum_{j=1}^{n_1} L_j^\lambda T_j^\lambda + L_j^\lambda L_j^\lambda, \quad \text{and} \quad \mathcal{L}_2^\lambda := \sum_{j=n_1+1}^{n_2} L_j^\lambda T_j^\lambda + L_j^\lambda L_j^\lambda.
\]

Let \( L_k^\alpha \) be the \( k \)th degree Laguerre polynomial of type \( \alpha \). For any \( \nu \in \mathbb{N} \), and any \( \zeta \in \mathbb{C}^\nu \), define

\[
\varphi^\alpha_{k,\lambda}(\zeta) = L_k^\alpha(2|\lambda| ||\zeta||^2) e^{-|\lambda ||\zeta||^2}.
\]

**Proposition 10.4.** \( \mu^\lambda_{m_1m_2}(\mathcal{L}_i^\lambda) = -2|\lambda|(2m_i + n_i), \ i = 1, 2. \psi^\lambda_{m_1m_2} \) has the following formulae in terms of Laguerre polynomials:

\[
\psi^\lambda_{m_1m_2}(z) = \pi^{-n}(2|\lambda|)^n \prod_{i=1}^{2} \varphi^{m_i-1}_{m_i,\lambda}(z^i).
\]

**Proof.** As usual we drop the superscript \( \lambda \). Take \( z_1^{m_1}z_{n_1+1}^{m_2} \in V_{m_1m_2} \). By Remark 5.9

\[
\mathcal{G}(\mathcal{L}_1)[z_1^{m_1}z_{n_1+1}^{m_2}] = \mu_{m_1m_2}(\mathcal{L}_1)[z_1^{m_1}z_{n_1+1}^{m_2}],
\]

which, by Proposition 5.3, reduces to

\[
\left( -\sum_{j=1}^{n_1} W_j W_j + W_j \mathbf{W}_j \right)[z_1^{m_1}z_{n_1+1}^{m_2}] = \mu_{m_1m_2}(\mathcal{L}_1)[z_1^{m_1}z_{n_1+1}^{m_2}].
\]

Using the definition of \( W_j \) and \( \mathbf{W}_j \), an easy calculation shows that \( \mu_{m_1m_2}(\mathcal{L}_1) = -2|\lambda|(2m_1 + n_1) \). Similarly \( \mu_{m_1m_2}(\mathcal{L}_2) = -2|\lambda|(2m_2 + n_2) \). Since \( \varphi^{m_i-1}_{m_i,\lambda}(z^i) \) is an
eigenfunction of $\mathcal{L}_i$ with eigenvalue $-2|\lambda|(2m_i + n_i)$, $\prod_{i=1}^2 \varphi_{m_i,\lambda}^{n_i-1}(z^i)$ is a joint eigenfunction of $\mathcal{L}_K(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1m_2}$. Hence $\psi_{m_1m_2}(z) = c\prod_{i=1}^2 \varphi_{m_i,\lambda}^{n_i-1}(z^i)$, for some constant $c$. To calculate the constant $c$, first note that by Proposition 6.4, $\psi_{m_1m_2}(z) = \psi_{m_1m_2} \times \psi_{m_1m_2}(z)$. In particular, putting $z = 0$, we get

$$cL_{m_1}^{n_1-1}(0)L_{m_2}^{n_2-1}(0) = c^2 \prod_{i=1}^2 \int_{\mathbb{C}^n_i} \left[ L_{m_i}^{n_i-1}(2|\lambda||z^i|^2) \right]^2 e^{-2|\lambda||z^i|^2} dz^i.$$

Using the well-known facts

$$L_k^\alpha(0) = \frac{\Gamma(k + \alpha + 1)}{\Gamma(k + 1)\Gamma(\alpha + 1)}, \quad \text{and} \quad \int_0^\infty [L_k^\alpha(r)]^2 e^{-r} r^{\alpha} dr = \frac{\Gamma(k + \alpha + 1)}{\Gamma(k + 1)},$$

we can deduce that $c = \pi^{-n}(2|\lambda|)^n$. Hence the proof.

\[\square\]

From now on we always assume that $\lambda > 0$ and state our results only for $\lambda > 0$. The corresponding results for $\lambda < 0$ can be obtained by interchanging the role of $p_i$ and $q_i$.

**Proposition 10.5.** Let $P \in H_{p_1q_1}^1 \otimes H_{p_2q_2}^2$. Then

$$\theta^\lambda(P)\psi_{m_1m_2}^\lambda(z) = \pi^{-n}(2|\lambda|)^nP(z)\prod_{i=1}^2 (-1)^{q_i}(2|\lambda|)^{p_i+q_i}\varphi_{m_i-p_i,\lambda}^{n_i}(z^i)$$

if $p_i \leq m_i$ for all $i = 1, 2$; otherwise $\theta^\lambda(P)\psi_{m_1m_2}^\lambda(z) = 0$.

**Proof.** Since $\delta_{p_1q_1} \otimes \delta_{p_2q_2}$ has a unique (upto a constant multiple) $M$-fixed vector, by Corollary 4.4 it is enough to prove the proposition for $P(z) = z_1^{p_1}z_2^{q_1}z_{n_1+1}^{p_2}z_{n_2+2}^{q_2}$, which clearly belongs to $H_{p_1q_1}^1 \otimes H_{p_2q_2}^2$. Since $\theta(P) = T_{p_1q_1} R_{n_1+1}^{q_2}(-R_1)^{p_1}(R_{n_1+1})^{q_2}$, it follows that

$$\theta(P)\psi_{m_1m_2} = \left[ \theta(z_1^{p_1}z_2^{q_1})\psi_{m_1}(z^1) \right] \left[ \theta(z_{n_1+1}^{p_2}z_{n_2+2}^{q_2})\psi_{m_2}(z^2) \right],$$

where

$$\psi_{m_i}(z^i) = \pi^{-n_i}(2|\lambda|)^{n_i}\varphi_{m_i,\lambda}^{n_i-1}(z^i), \quad i = 1, 2.$$

Hence the proof follows by (7.6). \[\square\]
Corollary 10.6. (a) Let \( P g \in E^\lambda(\mathbb{C}^n) \) be a joint eigenfunction of all \( D \in L^\lambda_R(\mathbb{C}^n) \) with eigenvalue \( \mu^\lambda_{m_1m_2} \), where \( g \) is \( K \)-invariant and \( P \in H^1_{p_1q_1} \otimes H^2_{p_2q_2} \). Then there is a constant \( c_{p_1q_1p_2q_2} \) such that

\[
P(z)g(z) = c_{p_1q_1p_2q_2} P(z) \prod_{i=1}^{2} (-1)^{q_i} (2|\lambda|)^{p_i+q_i} \varphi^{n_i+p_i+q_i-1}_{m_i-p_i,\lambda}(z^i),
\]

when \( p_i \leq m_i \) for all \( i = 1, 2 \); otherwise \( P g = 0 \).

(b) Let \( P \in H^1_{p_1q_1} \otimes H^2_{p_2q_2} \). Then

\[
\langle \Pi^\lambda(z), W^\lambda(P) \rangle_{m_1m_2} = P(z) \prod_{i=1}^{2} (-1)^{q_i} (2|\lambda|)^{p_i+q_i} \varphi^{n_i+p_i+q_i-1}_{m_i-p_i,\lambda}(z^i),
\]

when \( p_i \leq m_i \) for all \( i = 1, 2 \); otherwise \( \langle \Pi^\lambda(z), W^\lambda(P) \rangle_{m_1m_2} = 0 \).

Proof. Since \( \delta^1_{p_1q_1} \otimes \delta^2_{p_2q_2} \) has unique (up to a constant multiple) \( M \)-fixed vector, (a) follows from Theorem 7.14 and Proposition 10.5 (b) follows from Proposition 7.15 and Proposition 10.5.

\[\square\]

Lemma 10.7. Let \( \{Y^j_{p_1q_1p_2q_2} : j = 1, 2, \ldots, d(p_1, q_1, p_2, q_2)\} \) be an orthonormal basis for \( E^1_{p_1q_1} \otimes E^2_{p_2q_2} \) so that \( \{Y^j_{p_1q_1p_2q_2} : j = 1, 2, \ldots, d(p_1, q_1, p_2, q_2) ; p_1, q_1, p_2, q_2 \in \mathbb{Z}^+\} \) forms an orthonormal basis for \( L^2(S^{2n_1-1} \times S^{2n_2-1}) \). Let \( Y^j_{p_1q_1p_2q_2} \) be the restriction of \( \tilde{Y}^j_{p_1q_1p_2q_2} \in H^1_{p_1q_1} \otimes H^2_{p_2q_2} \) i.e

\[
\tilde{Y}^j_{p_1q_1p_2q_2}(z) = \tilde{Y}^j_{p_1q_1p_2q_2}(z^1, z^2) = \begin{pmatrix} 1 \\ p_1+q_1 \\ r_2 \end{pmatrix} Y^j_{p_1q_1p_2q_2}(|\lambda^1, \lambda^2),
\]

where \( z^i = r_i \omega^i, \omega^i \in S^{2n_i-1} \). Define

\[
P^j_{p_1q_1p_2q_2}(z) = \prod_{i=1}^{2} \Gamma(\lambda_i)(2|\lambda|)^{-(p_i+q_i)} \frac{\Gamma(m_i-p_i+1)}{\Gamma(m_i+n_i+q_i)} \tilde{Y}^j_{p_1q_1p_2q_2}(z).
\]

Then

\[
\{P^j_{p_1q_1p_2q_2}(z) : j = 1, 2, \ldots, d(p_1, q_1, p_2, q_2) ; p_i \leq m_i, q_i \in \mathbb{Z}^+ ; i = 1, 2\}
\]

forms an orthonormal basis for \( \mathcal{O}^\lambda(V_{m_1m_2}) \).
Proof. Taking \( p = q = P^j_{p_1,q_1,p_2,q_2} \) in Lemma \( \ref{7.6} \) (b), we get
\[
\left\| \mathcal{W}(P^j_{p_1,q_1,p_2,q_2}) \right\|_{m_1m_2} = \sqrt{\pi^n(2|\lambda|)^{-n}\left\| \theta(P)\psi_{m_1m_2} \right\|_2}.
\]
Therefore by Proposition \( \ref{10.5} \), \( \mathcal{W}(P^j_{p_1,q_1,p_2,q_2}) = 0 \), unless \( p_1 \leq m_1, p_2 \leq m_2 \); and if \( p_1 \leq m_1, p_2 \leq m_2 \), writing the right hand side of the above equation in polar coordinates and using the formulae
\[
\int_0^\infty [I^\alpha_k(r)]^2 e^{-r}r^\alpha dr = \frac{\Gamma(k + \alpha + 1)}{\Gamma(k + 1)},
\]
we can deduce that \( \left\| \mathcal{W}(P^j_{p_1,q_1,p_2,q_2}) \right\|_{m_1m_2} = 1 \). But then, since each \( \delta^1_{p_1,q_1} \otimes \delta^2_{p_2,q_2} \) has a unique (upto a constant multiple) \( M \)-fixed vector, the proof follows from Proposition \( \ref{7.8} \).

\[ \square \]

Lemma 10.8. Let \( \tilde{P}^j_{p_1,q_1,p_2,q_2} \) and \( P^j_{p_1,q_1,p_2,q_2} \) are as in the previous lemma. If \( f \) is a joint eigenfunction of all \( D \in \mathcal{L}_K^\lambda(\mathbb{C}^n) \) with eigenvalue \( \mu^\lambda_{m_1m_2} \) satisfying \( \chi_\delta \ast f \in \mathcal{E}_K^\lambda(\mathbb{C}^n) \) for each \( \delta \in K_M \), then there exist constants \( a^j_{p_1,q_1,p_2,q_2} \) such that
\[
f(z) = \sum_{p_1,q_1,p_2,q_2 \in \mathbb{Z}^+} \sum_{j=1}^{d(p_1,q_1,p_2,q_2)} a^j_{p_1,q_1,p_2,q_2} \langle \Pi^\lambda(z), \mathcal{W}^\lambda(P^j_{p_1,q_1,p_2,q_2}) \rangle_{m_1m_2}^{\lambda}, \tag{10.6}
\]
where the series converges uniformly over compact subsets of \( \mathbb{C}^n \). \( a^j_{p_1,q_1,p_2,q_2} \)'s satisfy the following :
\[
\sum_{p_1,q_1,p_2,q_2 \in \mathbb{Z}^+} \sum_{p_1 \leq m_1, p_2 \leq m_2}^{d(p_1,q_1,p_2,q_2)} \left| a^j_{p_1,q_1,p_2,q_2} \right|^2 \prod_{i=1}^2 \frac{k_{q_i}^{q_i}}{\Gamma(n_i + p_i + q_i)} < \infty, \quad \forall k_1, k_2 \in \mathbb{N}. \tag{10.7}
\]

Proof. By Proposition \( \ref{10.3} \) (b), we have the expansion, for fixed \( r_1, r_2 > 0 \),
\[
f(z) = f(r_1 \omega^1, r_2 \omega^2) = \sum_{p_1,q_1,p_2,q_2 \in \mathbb{Z}^+} \sum_{p_1 \leq m_1, p_2 \leq m_2}^{d(p_1,q_1,p_2,q_2)} f^j_{p_1,q_1,p_2,q_2}(r_1, r_2) Y^j_{p_1,q_1,p_2,q_2}(\omega^1, \omega^2), \tag{10.8}
\]
where the right hand side converges in \( L^2(S^{2n_1-1} \times S^{2n_2-1}) \). Here
\[
f^j_{p_1,q_1,p_2,q_2}(r_1, r_2) = \int_{S^{2n_1-1}} \int_{S^{2n_2-1}} f(r_1 \omega^1, r_2 \omega^2) Y^j_{p_1,q_1,p_2,q_2}(\omega^1, \omega^2) d\omega^1 d\omega^2.
\]
By a representation theoretic argument it can be shown that (see the proof of Proposition 4.5 in [16]) if \( \delta = \delta_{p_1,q_1} \otimes \delta_{p_2,q_2} \), \( z = (r_1 \omega^1, r_2 \omega^2) \),

\[
f^j_{p_1,q_1,p_2,q_2}(r_1, r_2)Y^j_{p_1,q_1,p_2,q_2}(\omega^1, \omega^2) = d(\delta) \int_K f(k \cdot z)(\delta(k^{-1})Y^j_{p_1,q_1,p_2,q_2} \cdot Y^j_{p_1,q_1,p_2,q_2})dk.
\]

Hence we can conclude that each \( f^j_{p_1,q_1,p_2,q_2}(r_1, r_2)Y^j_{p_1,q_1,p_2,q_2}(\omega^1, \omega^2) \in \mathcal{E}^\lambda(\mathbb{C}^n) \) is a joint eigenfunction of all \( D \in \mathcal{L}^\lambda_K(\mathbb{C}^n) \) with eigenvalue \( \mu^\lambda_{m_1m_2} \). But then by Corollary 10.6 (a), it follows that (for \( z = (z^1, z^2) = (r_1 \omega^1, r_2 \omega^2) \))

\[
f^j_{p_1,q_1,p_2,q_2}(r_1, r_2)Y^j_{p_1,q_1,p_2,q_2}(\omega^1, \omega^2) = a^j_{p_1,q_1,p_2,q_2} P^j_{p_1,q_1,p_2,q_2}(z) \prod_{i=1}^2 (-1)^{q_i} (2|\lambda|)^{p_i+q_i} (\varphi^m_{q_{m_i+p_i}})^{y_i} (z^i),
\]

for some constant \( a^j_{p_1,q_1,p_2,q_2} \). Hence (10.6) follows from Corollary 10.6 (b). Since \( Y^j_{p_1,q_1,p_2,q_2} \) and \( P^j_{p_1,q_1,p_2,q_2} \) are related by (10.5), from the above equation we get

\[
f^j_{p_1,q_1,p_2,q_2}(r_1, r_2) = a^j_{p_1,q_1,p_2,q_2} b^j_{p_1,q_1,p_2,q_2} \prod_{i=1}^2 r_i^{p_i+q_i} L^{n_i+p_i+q_i-1}_{m_i-p_i}(2|\lambda| r_i^2) e^{-|\lambda| r_i^2},
\]

where

\[
b^j_{p_1,q_1,p_2,q_2} = \prod_{i=1}^2 (-1)^{q_i} (2|\lambda|)^{p_i+q_i} \sqrt{\Gamma(n_i)(2|\lambda|)^{-(p_i+q_i)} \Gamma(m_i-p_i+1)} \Gamma(m_i+n_i+q_i).\]

Now fix \( r_1, r_2 > 0 \). Since

\[
\sum_{p_1,q_1,p_2,q_2 \in \mathbb{Z}^+} |f^j_{p_1,q_1,p_2,q_2}(r_1, r_2)|^2 = \left\| f(r_1 \omega^1, r_2 \omega^2) \right\|_{L^2(S^{2n_1-1} \times S^{2n_2-1})}^2 < \infty;
\]

\[
\sum_{p_1,q_1,p_2,q_2 \in \mathbb{Z}^+} \left[ a^j_{p_1,q_1,p_2,q_2} \prod_{i=1}^2 (2|\lambda| r_i^2)^{q_i} \right] \prod_{i=1}^2 \frac{\Gamma(n_i+p_i+q_i)}{\Gamma(m_i+n_i+q_i)} \left( L^{n_i+p_i+q_i-1}_{m_i-p_i}(2|\lambda| r_i^2) \right)^2 < \infty.
\]

Therefore to prove (10.7), it is enough to show that for large \( q_1, q_2 \),

\[
\prod_{i=1}^2 \frac{\Gamma(n_i+p_i+q_i)}{\Gamma(m_i+n_i+q_i)} \left( L^{n_i+p_i+q_i-1}_{m_i-p_i}(2|\lambda| r_i^2) \right)^2 > c \quad (10.9)
\]

for all \( p_i \leq m_i \). Now if \( \alpha + 1 > 2kt \), then

\[
|L^\alpha_k(t)| \geq \frac{1}{2} \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1)}.
\]
and hence
\[
L_{m_i-p_i}^{n_i+p_i+q_i-1}(2|\lambda|r^2_i) \geq \frac{1}{2} \frac{\Gamma(m_i + q_i + n_i)}{\Gamma(m_i - p_i + 1) \Gamma(n_i + p_i + q_i)} \geq \frac{1}{2} \frac{\Gamma(m_i + q_i + n_i)}{\Gamma(m_i + 1) \Gamma(n_i + p_i + q_i)}.
\]

Therefore for all \( q_i > 2m_i(2|\lambda|r^2_i) - n_i \) and \( p_i \leq m_i \), we have
\[
(L_{m_i-p_i}^{n_i+p_i+q_i-1}(2|\lambda|r^2_i))^2 \geq \frac{1}{4} \frac{\Gamma(m_i + q_i + n_i)}{\Gamma(m_i + 1)^2 \Gamma(n_i + p_i + q_i)},
\]
which implies (10.9). Hence the proof is complete. \( \square \)

Following [16], for each positive integer \( k \), we define \( B_k \) to be the subspace of operators \( S \in C(V_{m_1m_2}) \) for which
\[
\sum_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \atop p_1 \leq m_1, p_2 \leq m_2} ||\mathcal{P}_{W^\lambda(H^1_{p_1q_1} \otimes H^2_{p_2q_2})} S||_{m_1m_2}^2 \prod_{i=1}^2 \frac{\Gamma(n_i + p_i + q_i)}{k^{q_i}} < \infty,
\]
where \( \mathcal{P}_{W^\lambda(H^1_{p_1q_1} \otimes H^2_{p_2q_2})} \) is the projection on \( W^\lambda(H^1_{p_1q_1} \otimes H^2_{p_2q_2}) \), that is
\[
\mathcal{P}_{W^\lambda(H^1_{p_1q_1} \otimes H^2_{p_2q_2})} S = \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} \langle S, W^\lambda(P^j_{p_1q_1p_2q_2}) \rangle_{m_1m_2}^\lambda W^\lambda(P^j_{p_1q_1p_2q_2}).
\]
Then \( B_k \) becomes a Hilbert space if we define the inner product as
\[
\langle S_1, S_2 \rangle_{B_k} = \sum_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \atop p_1 \leq m_1, p_2 \leq m_2} \langle \mathcal{P}_{W^\lambda(H^1_{p_1q_1} \otimes H^2_{p_2q_2})} S_1, \mathcal{P}_{W^\lambda(H^1_{p_1q_1} \otimes H^2_{p_2q_2})} S_2 \rangle_{m_1m_2}^\lambda \prod_{i=1}^2 \frac{\Gamma(n_i + p_i + q_i)}{k^{q_i}}.
\]
Note that for each \( k \in \mathbb{N} \), \( B_k \subset B_{k+1} \) and the inclusion \( B_k \hookrightarrow B_{k+1} \) is continuous.

We define \( B = \cup_{k \in \mathbb{N}} B_k \) and equip it with the inductive limit topology.

**Lemma 10.9.** For each fixed \( z \in \mathbb{C} \), \( \Pi^\lambda(z) \in B \).

**Proof.** Fix \( z \in \mathbb{C}^{n} \).

\[
||\mathcal{P}_{W(H^1_{p_1q_1} \otimes H^2_{p_2q_2})} \Pi(z)||_{m_1m_2} = \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} ||\Pi(z), W(P^j_{p_1q_1p_2q_2})||_{m_1m_2}^2
\]
\[
= \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} ||P^j_{p_1q_1p_2q_2}(z) \prod_{i=1}^2 (-1)^{q_i} (2|\lambda|)^{p_i+q_i} \mathcal{P}_{m_i-p_i, \lambda}^{n_i+p_i+q_i-1}(z)||_{m_1m_2}^2.
\]
Lemma 10.10. Let $B^*$ be the dual of $B$. If $v \in B^*$ then

$$||v||^2 := \sum_{p_1,q_1,p_2,q_2 \in \mathbb{Z}^+ \atop p_1 \leq m_1, p_2 \leq m_2} d(p_1,q_1,p_2,q_2) \sum_{j=1}^2 |v(W^\lambda(P^j_{p_1,q_1,p_2,q_2}))|^2 \prod_{i=1}^2 \frac{k^{q_i}}{\Gamma(n_i + p_i + q_i)} < \infty, \quad (10.10)$$

for all $k \in \mathbb{N}$. Conversely if the constants $a^j_{p_1,q_1,p_2,q_2}$'s satisfy

$$\sum_{p_1,q_1,p_2,q_2 \in \mathbb{Z}^+ \atop p_1 \leq m_1, p_2 \leq m_2} d(p_1,q_1,p_2,q_2) \sum_{j=1}^2 |a^j_{p_1,q_1,p_2,q_2}|^2 \prod_{i=1}^2 \frac{k^{q_i}}{\Gamma(n_i + p_i + q_i)} < \infty \quad (10.11)$$

for all $k \in \mathbb{N}$, then there is a unique $v \in B^*$ such that $v(W^\lambda(P^j_{p_1,q_1,p_2,q_2})) = a^j_{p_1,q_1,p_2,q_2}$. 

Since $|Y^j_{p_1,q_1,p_2,q_2}(\omega^1, \omega^2)| \leq c_1 \prod_{i=1}^2 (p_i + q_i)^{n_i-1}$ and

$$|\varphi_{m_i - p_i, \lambda}^{n_i + p_i + q_i}(z^j)| \leq c_2 \frac{\Gamma(m_i + n_i + q_i)}{\Gamma(m_i - p_i + 1)\Gamma(n_i + p_i + q_i)},$$

we get

$$||P_{\mathcal{W}(H_{p_1,q_1} \otimes H_{p_2,q_2})}P(z)||^2_{m_1m_2} \leq c_3 \prod_{i=1}^2 (p_i + q_i)^{2n_i - 2} \frac{\Gamma(m_i + n_i + q_i)(2|\lambda|_i^2)^{p_i + q_i}}{\Gamma(m_i - p_i + 1)(\Gamma(n_i + p_i + q_i))^2} \leq c_3 \prod_{i=1}^2 (p_i + q_i)^{2n_i} \frac{(m_i + n_i + q_i)^{m_i - p_i}(2|\lambda|_i^2)^{p_i + q_i}}{\Gamma(m_i - p_i + 1)\Gamma(n_i + p_i + q_i)} \leq c_3 \prod_{i=1}^2 \frac{q_i^{2n_i + m_i}(2|\lambda|_i^2)^{q_i}}{\Gamma(n_i + p_i + q_i)},$$

for all $q_i \in \mathbb{Z}^+$, $p_i \leq m_i$. Therefore $\Pi(z) \in \mathcal{B}_k$. Hence the proof. \hfill \Box
Proof. Since the topology on $\mathcal{B}$ is the inductive limit topology, $v \in \mathcal{B}^*$ if and only if $v \in \mathcal{B}_k^*$ for all $k$. Fix a $k$. Then as $\mathcal{B}_k$ is a Hilbert space, there exists $S_k \in \mathcal{B}_k$ such that $v(S) = \langle S, S_k \rangle_{\mathcal{B}_k}$ for all $S \in \mathcal{B}_k$. Taking $S = \mathcal{W}(P_{p_1,q_1,p_2,q_2}^j)$, we get

$$
v(\mathcal{W}(P_{p_1,q_1,p_2,q_2}^j)) = \langle \mathcal{W}(P_{p_1,q_1,p_2,q_2}^j), S_k \rangle_{\mathcal{B}_k} = \langle \mathcal{W}(P_{p_1,q_1,p_2,q_2}^j), S_k \rangle_{m_1m_2} \prod_{i=1}^2 \frac{\Gamma(n_i + p_i + q_i)}{k^{q_i}}.
$$

Since $S_k \in \mathcal{B}_k$,

$$
\sum_{p_1,q_1,p_2,q_2 \in \mathbb{Z}^+} \sum_{p_1 \leq m_1, p_2 \leq m_2} d(p_1,q_1,p_2,q_2) \langle S_k, \mathcal{W}(P_{p_1,q_1,p_2,q_2}^j) \rangle_{m_1m_2}^2 \prod_{i=1}^2 \frac{\Gamma(n_i + p_i + q_i)}{k^{q_i}} < \infty.
$$

Hence (10.10) follows. Conversely, let the constants $a_{p_1,q_1,p_2,q_2}^j$ satisfy (10.11). Then we can define an operator $S_k \in \mathcal{B}_k$ by

$$
\langle \mathcal{W}(P_{p_1,q_1,p_2,q_2}^j), S_k \rangle_{m_1m_2} = a_{p_1,q_1,p_2,q_2}^j \prod_{i=1}^2 \frac{k^{q_i}}{\Gamma(n_i + p_i + q_i)}.
$$

For each $k \in \mathbb{N}$, define $v_k \in \mathcal{B}_k^*$, by $v_k(S) = \langle S, S_k \rangle_{\mathcal{B}_k}$ for all $S \in \mathcal{B}_k$. Note that

$$
v_k(S) = \sum_{p_1,q_1,p_2,q_2 \in \mathbb{Z}^+} \sum_{p_1 \leq m_1, p_2 \leq m_2} d(p_1,q_1,p_2,q_2) a_{p_1,q_1,p_2,q_2}^j \langle S, \mathcal{W}(P_{p_1,q_1,p_2,q_2}^j) \rangle_{m_1m_2} S \in \mathcal{B}_k.
$$

Therefore for any $S \in \mathcal{B}$, if we define $v(S)$ to be equal to the right hand side of the above equation then $v \mid \mathcal{B}_k = v_k \in \mathcal{B}_k^*$. Hence $v \in \mathcal{B}^*$. Also note that $v(\mathcal{W}(P_{p_1,q_1,p_2,q_2}^j)) = a_{p_1,q_1,p_2,q_2}^j$. Uniqueness of $v$ follows from the fact that

$$\left\{ \prod_{i=1}^2 \frac{k^{q_i}}{\Gamma(n_i + p_i + q_i)} \mathcal{W}(P_{p_1,q_1,p_2,q_2}^j) : j = 1, 2, \ldots, d(p_1,q_1,p_2,q_2); p_i \leq m_i, q_i \in \mathbb{Z}^+ \right\}
$$

forms an orthonormal basis for $\mathcal{B}_k$. Hence the proof is complete. \hfill $\square$

Theorem 10.11. Let $f$ be a joint eigenfunction of all $D \in \mathcal{L}^{\lambda}_{K}(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1m_2}^{\lambda}$ such that $\lambda_{\delta} \ast f \in \mathcal{E}^{\lambda}(\mathbb{C}^n)$ for all $\delta \in \tilde{K}_M$. Then $f(z) = v(\Pi^{\lambda}(z))$ for a unique $v \in \mathcal{B}^*$. Conversely, if $f(z) = v(\Pi^{\lambda}(z))$ for some $v \in \mathcal{B}^*$, then $f$ is a joint eigenfunction of all $D \in \mathcal{L}^{\lambda}_{K}(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1m_2}^{\lambda}$ and $\lambda_{\delta} \ast f \in \mathcal{E}^{\lambda}(\mathbb{C}^n)$ for all $\delta \in \tilde{K}_M$. 
Proof. Let \( v \in \mathcal{B}^* \) and \( f(z) = v(\Pi(z)) \). We claim that
\[
f(z) = \sum_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} v(W(P^{j}_{p_1, q_1, p_2, q_2}))(\Pi(z), W(P^{j}_{p_1, q_1, p_2, q_2}))_{m_1, m_2},
\]
where the right hand side converges absolutely and uniformly over every compact subset of \( \mathbb{C}^n \). To prove the claim fix \( r_i > 0 \). Then the proof of Lemma 10.9 shows that there exist \( k \in \mathbb{N} \) (depending on \( r_i \)) such that \( \Pi(z) \in \mathcal{B}_k \) and \( ||\Pi(z)||_{\mathcal{B}_k} < c \) for all \( z \in \mathbb{C}^n \) with \( |z^i| \leq r_i \). Since \( \Pi(z) \in \mathcal{B}_k \), it follows that
\[
\sum_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} \langle \Pi(z), W(P^{j}_{p_1, q_1, p_2, q_2}) \rangle_{m_1, m_2} W(P^{j}_{p_1, q_1, p_2, q_2})
\]
converges to \( \Pi(z) \) in the Hilbert space \( \mathcal{B}_k \). Since \( v \in \mathcal{B}^*_k \), we get
\[
v(\Pi(z)) = \sum_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} v(W(P^{j}_{p_1, q_1, p_2, q_2}))(\Pi(z), W(P^{j}_{p_1, q_1, p_2, q_2}))_{m_1, m_2} \tag{10.12}
\]
Multiply \( v(W_{m_1, m_2}(P^{j}_{p_1, q_1, p_2, q_2})) \) by \( \Pi_{i=1}^{2} k^{|n_i + p_i + q_i|} \langle \Pi(z), W(P^{j}_{p_1, q_1, p_2, q_2}) \rangle_{m_1, m_2} \)
by \( \Pi_{i=1}^{2} k^{-q_i} \Gamma(n_i + p_i + q_i) \) and then use the Cauchy-Schwarz inequality to get
\[
\sum_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} |v(W(P^{j}_{p_1, q_1, p_2, q_2}))(\Pi(z), W(P^{j}_{p_1, q_1, p_2, q_2}))_{m_1, m_2}|
\]
\[
\leq ||v||_k ||\Pi(z)||_{\mathcal{B}_k} \leq c ||v||_k
\]
for all \( z \in \mathbb{C}^n \) such that \( |z^i| \leq r_i \). Since \( r_i > 0 \) was arbitrary, the claim follows. In particular \( f \) is a smooth function. Since any \( D \in \mathcal{L}_K(\mathbb{C}^n) \) is a polynomial coefficient differential operator we have
\[
Df(z) = \sum_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} v(W(P^{j}_{p_1, q_1, p_2, q_2}))(\Pi(z), W(P^{j}_{p_1, q_1, p_2, q_2}))_{m_1, m_2} D \left[ \langle \Pi(z), W(P^{j}_{p_1, q_1, p_2, q_2}) \rangle_{m_1, m_2} \right]
\]
in the distribution sense. But
\[
D \left[ \langle \Pi(z), W(P^{j}_{p_1, q_1, p_2, q_2}) \rangle_{m_1, m_2} \right] = \mu_{m_1, m_2} (D) \langle \Pi(z), W(P^{j}_{p_1, q_1, p_2, q_2}) \rangle_{m_1, m_2}.
\]
Therefore we can conclude that $Df = \mu_{m_1m_2}(D)f$. Hence $f$ is a joint eigenfunction of all $D \in \mathcal{L}_K(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1m_2}$. Now, if $\delta = \delta_{p_1q_1} \otimes \delta_{p_2q_2}$, equation (10.12) implies that, $\chi_\delta \ast f = 0$ if $p_1 > m_1$ or $p_2 > m_2$; and when $p_i \leq m_i$ for $i = 1, 2$,

$$\chi_\delta \ast f = \frac{1}{d(p_1, q_1, p_2, q_2)} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} v\left(\mathcal{W}(P^j_{p_1q_1p_2q_2})\right)\langle \Pi(z), \mathcal{W}(P^j_{p_1q_1p_2q_2}) \rangle_{m_1m_2}.$$ 

But, by Proposition 7.15, $\langle \Pi(z), \mathcal{W}(P^j_{p_1q_1p_2q_2}) \rangle_{m_1m_2} = \theta(P^j_{p_1q_1p_2q_2})\psi_{m_1m_2}$ which clearly equals to $e^{-|\lambda||z|^2}$ times a polynomial. Hence it follows that $\chi_\delta \ast f \in \mathcal{E}^\lambda(\mathbb{C}^n)$.

Conversely let $f$ be a joint eigenfunction of all $D \in \mathcal{L}_K(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1m_2}$ such that $\chi_\delta \ast f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ for each $\delta \in \widehat{K}_M$. By Lemma 10.8 there exist constants $a^j_{p_1q_1p_2q_2}$ such that

$$f(z) = \sum_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+, p_1 \leq m_1, p_2 \leq m_2} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} a^j_{p_1q_1p_2q_2} \langle \Pi(z), \mathcal{W}(P^j_{p_1q_1p_2q_2}) \rangle_{m_1m_2},$$

and $a^j_{p_1q_1p_2q_2}$’s satisfy the following:

$$\sum_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+, p_1 \leq m_1, p_2 \leq m_2} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} |a^j_{p_1q_1p_2q_2}|^2 \prod_{i=1}^2 \frac{k_{q_i}}{\Gamma(n_i + p_i + q_i)} < \infty, \ \forall k \in \mathbb{N}.$$ 

Then by the previous lemma there exists $v \in \mathcal{B}^*$ such that $v\left(\mathcal{W}(P^j_{p_1q_1p_2q_2})\right) = a^j_{p_1q_1p_2q_2}$, and consequently by (10.12), $f(z) = v(\Pi(z))$.

Now we prove the uniqueness of $v$ which will complete the proof of the theorem. So let $v \in \mathcal{B}^*$ and $v(\Pi(z)) = 0$ for all $z \in \mathbb{C}^n$. We must prove that $v = 0$. It is enough to show that $v\left(\mathcal{W}(P^j_{p_1q_1p_2q_2})\right) = 0$ for all $j = 1, 2, \cdots, d(p_1, q_1, p_2, q_2); p_i \leq m_i, q_i \in \mathbb{Z}^+$. But this follows, since (10.12) and Corollary 10.6 (b) imply that for each fixed $r_1, r_2 > 0$,

$$\langle v(\Pi(r_1\cdot, r_2\cdot)), Y^j_{p_1q_1p_2q_2} \rangle_{L^2(S^{2n_1-1} \times S^{2n_2-1})}$$

$$= b^j_{p_1q_1p_2q_2} v\left(\mathcal{W}(P^j_{p_1q_1p_2q_2})\right) \prod_{i=1}^2 r_i^{p_i+q_i} L^{n_i+p_i+q_i-1}(2|\lambda|r_i^2) e^{-|\lambda|r_i^2}$$

for some non zero constants $b^j_{p_1q_1p_2q_2}$.
We have already mentioned that the above characterization is analogous to the viewpoint of Thangavelu [16] (see Theorem 4.1 there). Now we make this analogy clear by showing that the above theorem can be reformulated (Theorem 10.12 below), which is similar to Theorem 4.1 in [16]. Consider

\[ L^2_{m_1,m_2}(S^{2n_1-1} \times S^{2n_2-1}) := \text{span}\{Y^j_{p_1q_1p_2q_2} : j = 1, 2, \ldots, d(p_1, q_1, p_2, q_2); p_i \leq m_i, q_i \in \mathbb{Z}^+\} \]

as Hilbert subspace of \( L^2(S^{2n_1-1} \times S^{2n_2-1}) \). Then the map

\[ J : \mathcal{O}^\lambda(V_{m_1m_2}) \rightarrow L^2_{m_1,m_2}(S^{2n_1-1} \times S^{2n_2-1}) \]

defined by

\[ J(W^\lambda_{p_1q_1p_2q_2}) = Y^j_{p_1q_1p_2q_2} \]

is an Hilbert space isomorphism. Note that \( J(B_k) \) is the subspace of all functions \( \phi \) in \( L^2_{m_1,m_2}(S^{2n_1-1} \times S^{2n_2-1}) \) such that

\[ \sum_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \atop p_1 \leq m_1, p_2 \leq m_2} \left| \phi_{\delta_{p_1q_2} \otimes \delta_{p_2q_2}}(\omega) \right|^2 \prod_{i=1}^2 \frac{\Gamma(n_i + p_i + q_i)}{k_{n_i}} < \infty, \]

where, for \( \omega \in S^{2n_1-1} \times S^{2n_2-1} \),

\[ \phi_{\delta_{p_1q_2} \otimes \delta_{p_2q_2}}(\omega) := d(p_1, q_1, p_2, q_2)[\chi_{\delta_{p_1q_1} \otimes \delta_{p_2q_2}} * \phi](\omega) = d(p_1, q_1, p_2, q_2) \int K \chi_{\delta_{p_1q_1} \otimes \delta_{p_2q_2}}(k) \phi(k^{-1} \cdot \omega) dk \]

\[ = \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} \langle \phi, Y^j_{p_1q_1p_2q_2} \rangle Y^j_{p_1q_1p_2q_2}(\omega). \]

Each \( J(B_k) \) becomes a Hilbert space with the inner product

\[ \langle \phi_1, \phi_2 \rangle_{J(B_k)} = \langle J^{-1} \phi_1, J^{-1} \phi_2 \rangle_{B_k}. \]

Explicitly

\[ \langle \phi_1, \phi_2 \rangle_{J(B_k)} = \sum_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \atop p_1 \leq m_1, p_2 \leq m_2} \left( \langle \phi_1 \rangle_{\delta_{p_1q_2} \otimes \delta_{p_2q_2}}, \langle \phi_2 \rangle_{\delta_{p_1q_2} \otimes \delta_{p_2q_2}} \right) \prod_{i=1}^2 \frac{\Gamma(n_i + p_i + q_i)}{k_{n_i}}. \]
Consider $\mathcal{I}(\mathcal{B}) = \bigcup_{k \in \mathbb{N}} \mathcal{I}(\mathcal{B}_k)$ and equip this space with the inductive limit topology. Let

$$\mathcal{P}_{m_1m_2}^\lambda(z,\omega) = \sum_{p_1, q_1, p_2, q_2 \in \mathbb{Z}_+} \sum_{p_1 \leq m_1, p_2 \leq m_2} \langle \Pi^\lambda(z), \mathcal{W}^\lambda(P_{p_1q_1p_2q_2}) \rangle_{m_1m_2}^\lambda Y^j_{p_1q_1p_2q_2}(\omega), \quad (10.13)$$

$\omega \in S^{2n_1-1} \times S^{2n_2-1}$. It is easy to see that $\mathcal{I}(\Pi^\lambda(z)) = \mathcal{P}_{m_1m_2}^\lambda(z,\cdot)$. Then one can show that Theorem 10.11 is equivalent to the following theorem:

**Theorem 10.12.** Let $f$ be a joint eigenfunction of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1m_2}^\lambda$ such that $\chi_{\delta} \ast f \in \mathcal{E}(\mathbb{C}^n)$ for all $\delta \in \hat{K}_M$. Then

$$f(z) = \int_{S^{2n_1-1} \times S^{2n_2-1}} \mathcal{P}_{m_1m_2}^\lambda(z,\omega)d\nu(\omega),$$

for a unique $\nu \in \mathcal{B}^*$. Conversely, if

$$f(z) = \int_{S^{2n_1-1} \times S^{2n_2-1}} \mathcal{P}_{m_1m_2}^\lambda(z,\omega)d\nu(\omega),$$

for some $\nu \in \mathcal{B}^*$, then $f$ is a joint eigenfunction of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1m_2}^\lambda$ and $\chi_{\delta} \ast f \in \mathcal{E}(\mathbb{C}^n)$ for all $\delta \in \hat{K}_M$.

The above theorem gives an integral representation of joint eigenfunctions, where the kernel $\mathcal{P}_{m_1m_2}^\lambda(z,\omega)$ is given by the series in (10.13). Now we shall give another integral representation, where the kernel can be given explicitly. Fix $a_1, a_2 > 0$ so that

$$L_{m_1-p_1}^{n_1+p_1+q_i-1}(2|\lambda|a_i^2) \neq 0$$

for all $p_i \leq m_i, q_i \in \mathbb{Z}_+; i = 1, 2$. Define

$$\Omega_{m_1m_2}^\lambda(z,\omega) = e^{-2i\lambda \text{Im}(z^i(a_1^i\omega_1^i, a_2^i\omega_2^i))} \psi_{m_1m_2}^\lambda(z - (a_1^i\omega_1^i, a_2^i\omega_2^i)) \cdot \pi^{-n}(2|\lambda|)^n \prod_{i=1}^2 e^{2i\lambda a_i \text{Im}(z^i\omega_i^i)} \mathcal{C}_{m_i, \lambda}^{n_i-1}(z^i - a_i\omega_i^i),$$

where $z = (z^1, z^2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ and $\omega = (\omega_1, \omega_2) \in S^{2n_1-1} \times S^{2n_2-1}$. For each positive integer $k$, define $\mathcal{A}_k$ to be the subspace of functions $\phi$ in $L_{m_1m_2}^2(S^{2n_1-1} \times S^{2n_2-1})$ for
which
\[
\sum_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+} \left| \delta_{p_1 q_2} \otimes \delta_{p_2 q_2} \right|^2 \prod_{i=1}^2 \left[ \frac{\Gamma(n_i + p_i + q_i)}{k^{n_i}} \right]^2 < \infty.
\]

Each $A_k$ becomes a Hilbert space with the following inner product:
\[
\langle \phi_1, \phi_2 \rangle_{A_k} = \sum_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+} \langle (\phi_1)_{p_1 q_2} \otimes (\phi_2)_{p_2 q_2} \rangle \prod_{i=1}^2 \left[ \frac{\Gamma(n_i + p_i + q_i)}{k^{n_i}} \right]^2.
\]

We take $A = \bigcup_{k \in \mathbb{N}} A_k$ and equip it with the inductive limit topology. Let $A^*$ be the dual of $A$ with respect to this topology. Then we have the following integral representation of joint eigenfunctions of all $D \in \mathcal{L}^\lambda_K(\mathbb{C}^n)$.

**Theorem 10.13.** Let $f$ be a joint eigenfunction of all $D \in \mathcal{L}^\lambda_K(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1 m_2}^\lambda$ such that $\chi_\delta * f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ for all $\delta \in \tilde{K}_M$. Then

\[
f(z) = \int_{S^{2n_1-1} \times S^{2n_2-1}} Q_{m_1 m_2}^\lambda(z, \omega) d\nu(\omega),
\]

for a unique $\nu \in A^*$. Conversely, if

\[
f(z) = \int_{S^{2n_1-1} \times S^{2n_2-1}} Q_{m_1 m_2}^\lambda(z, \omega) d\nu(\omega),
\]

for some $\nu \in A^*$, then $f$ is a joint eigenfunction of all $D \in \mathcal{L}^\lambda_K(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1 m_2}^\lambda$ and $\chi_\delta * f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ for all $\delta \in \tilde{K}_M$.

**Proof.** The theorem can be proved using arguments similar to the proof of Theorem 10.11 once we have the following claim:

\[
\int_{S^{2n_1-1} \times S^{2n_2-1}} Q_{m_1 m_2}^\lambda(z, \omega) Y_{p_1 q_1 p_2 q_2}^\lambda(\omega) d\omega = c_{p_1 q_1 p_2 q_2} (P_{p_1 q_1 p_2 q_2}^\lambda)'(z) \prod_{i=1}^2 \varphi_{m_i - p_i + q_i - 1}(z_i),
\]

where $c_{p_1 q_1 p_2 q_2} = 0$ if either $p_1 > m_1$ or $p_2 > m_2$, and for $p_i \leq m_i$, it is given by

\[
c_{p_1 q_1 p_2 q_2} = \pi^{-n} (2|\lambda|)^n \prod_{i=1}^2 (2|\lambda|)^{p_i + q_i} \frac{\Gamma(n_i) \Gamma(m_i - p_i + 1)}{\Gamma(m_i + n_i + q_i)} \frac{a_i^{(p_i + q_i)}}{a_i^{2m_i - 1}} L_{m_i - p_i}^{n_i + p_i + q_i - 1}(2|\lambda| a_i^2) e^{-|\lambda| a_i^2}.
\]
To prove the claim, first note that we can write

\[
\int_{S^{2n_1-1} \times S^{2n_2-1}} Q^\lambda_{m_1m_2}(z,\omega) Y^j_{p_1q_1p_2q_2}(\omega) d\omega
\]

\[
= \left[ \prod_{i=1}^{2} \frac{1}{a_i^{2n_i+p_i+q_i-1}} \right] (P^j_{p_1q_1p_2q_2})' d\mu_{a_1,a_2} \times^\lambda \psi^\lambda_{m_1m_2}(z)
\]

\[
= \left[ \prod_{i=1}^{2} (2|\lambda|)^{(p_i+q_i)} \Gamma(m_i-p_i+1) \Gamma(m_i+n_i+q_i) \left( a_i^{2n_i+p_i+q_i-1} \right) \right]^{-1}
\]

\[
\times \left[ P^j_{p_1q_1p_2q_2} d\mu_{a_1,a_2} \times^\lambda \psi^\lambda_{m_1m_2}(z) \right],
\]

where \(d\mu_{a_1,a_2}\) is the surface measure on \(a_1S^{2n_1-1} \times a_2S^{2n_2-1}\). But then the claim follows, if we can prove the following lemma.

Lemma 10.14. Let \(a_1, a_2 > 0\) and \(d\mu_{a_1,a_2}\) be the surface measure on \(a_1S^{2n_1-1} \times a_2S^{2n_2-1}\). Let \(P \in H^1_{p_1q_1} \otimes H^2_{p_2q_2}\). Then

\[
Pd\mu_{a_1,a_2} \times^\lambda \psi^\lambda_{m_1m_2} = b_{p_1q_1p_2q_2} \pi^{-n} (2|\lambda|)^nP(z) \prod_{i=1}^{2} (-1)^{q_i} (2|\lambda|)^{p_i+q_i} \psi^\lambda_{m_1m_2}(z),
\]

\[
b_{p_1q_1p_2q_2} = \prod_{i=1}^{2} (-1)^{q_i} \frac{\Gamma(n_i)}{\Gamma(m_i+n_i+q_i)} \left( a_i^{2n_i+p_i+q_i} \right) \left( L^{m_i-p_i} \right) \left( 2|\lambda|a_i^2 \right)^{-1},
\]

if \(p_i \leq m_i\) for all \(i = 1, 2\); otherwise \(Pd\mu_{a_1,a_2} \times^\lambda \psi^\lambda_{m_1m_2} = 0\).

Proof. Let \(\delta = \delta_1^{p_1q_1} \otimes \delta_2^{p_2q_2}\). Take \(P^j_{p_1q_1p_2q_2} : j = 1, 2, \ldots, d(p_1, q_1, p_2, q_2)\), as Lemma 10.7 so that \(\{P^j_{p_1q_1p_2q_2} : j = 1, 2, \ldots, d(p_1, q_1, p_2, q_2)\}\) forms a basis for \(H^\delta\). Also we have \(\|W(P^j_{p_1q_1p_2q_2})\|^2_{m_1m_2} = 1\) if \(p_i \leq m_i, i = 1, 2\). We can choose suitable bases \(b\) for \(V^\delta\) and \(e\) for \(F^\delta = \text{Hom}_K(V^\delta, H^\delta)\) so that with respect to these bases \(P^\delta : \mathbb{C}^n \to M_d(\delta)\times 1\) can be given as follows : \(P^\delta_j = P^j_{p_1q_1p_2q_2}\). Since \(\Psi^\delta_{m_1m_2} = \theta(P^\delta)\psi^\lambda_{m_1m_2}\), by Proposition 10.5, we can say that, \(\Psi^\delta_{m_1m_2} = \tilde{\Psi}^\delta_{m_1m_2}\) if \(p_i \leq m_i\) for all \(i = 1, 2\);
otherwise $\Psi_{m_1m_2}^\delta = 0$. Now let $p_i \leq m_i$ for $i = 1, 2$. Then
\[ \tilde{A}_{m_1m_2}^\delta = A_{m_1m_2}^\delta = \int_{\mathbb{C}^n} [\Psi_{m_1m_2}^\delta(z)]^* [\Psi_{m_1m_2}^\delta(z)] dz \]
\[ = \sum_{j=1}^{d(p_1,q_1,p_2,q_2)} ||\theta(P_{p_1,q_1,p_2,q_2}^j)\psi_{m_1m_2}||^2_2 \]
\[ = \pi^{-n}(2|\lambda|)^n d(p_1,q_1,p_2,q_2), \text{ by Lemma 7.6 (b)}, \]
and
\[ \tilde{L}_{m_1m_2}^\delta(z) = \tilde{L}_{m_1m_2}^\lambda(z) = \pi^{-n}(2|\lambda|)^n \prod_{i=1}^{2} (-1)^{q_i} (2|\lambda|)^{p_i+q_i} L_{m_i-p_i}^{n_i+p_i+q_i-1}(2|\lambda| z_i^2). \]
Also we have
\[ \Upsilon_{\delta}(z) = [P_{\delta}(z)]^* [P_{\delta}^*(z)] \]
\[ = \sum_{j=1}^{d(p_1,q_1,p_2,q_2)} ||P_{p_1,q_1,p_2,q_2}^j(z)||^2 \]
\[ = \prod_{i=1}^{2} (2|\lambda|)^{-(p_i+q_i)} \frac{\Gamma(n_i)\Gamma(m_i-p_i+1)}{\Gamma(m_i+n_i+q_i)} r_i^{2(p_i+q_i)} \sum_{j=1}^{d(p_1,q_1,p_2,q_2)} ||Y_{p_1,q_1,p_2,q_2}^j(\omega)||^2. \]
Therefore from Theorem 7.10 we can show that, for $p_i \leq m_i$, $i = 1, 2$,
\[ P_{\delta}^\lambda d\mu_{a_1,a_2}^\lambda \psi_{m_1m_2}^\lambda = b_{p_1,q_1,p_2,q_2} \Psi_{m_1m_2}^\delta, \]
where $b_{p_1,q_1,p_2,q_2}$ is given by
\[ b_{p_1,q_1,p_2,q_2} = \prod_{i=1}^{2} (-1)^{q_i} \frac{\Gamma(n_i)\Gamma(m_i-p_i+1)}{\Gamma(m_i+n_i+q_i)} a_i^{2(p_i+q_i)} L_{m_i-p_i}^{n_i+p_i+q_i-1}(2|\lambda| a_i^2) e^{-|\lambda| a_i^2}. \]
Hence the proof follows. \[ \square \]

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HEIKE-BOCHNER IDENTITY AND EIGENFUNCTIONS

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