ON BELTRAMI EQUATIONS WITH INVERSE CONDITIONS AND HYDRODYNAMIC NORMALIZATION

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Abstract. We consider problems concerning the existence of solutions of the Beltrami equations and their convergence in the complex plane. We are mainly interested in the case when these solutions satisfy the so-called hydrodynamic normalization condition in the neighborhood of infinity. Under some conditions on dilatations of inverse mappings, we have established the existence of such solutions in the class of continuous Sobolev mappings. We have also obtained results on the locally uniform limit of a sequence of such solutions.

1. Introduction

In our recent papers, some problems related to the existence of solutions of Beltrami equations were considered (see [25,26]). Note that, as a rule, we were talking about solutions satisfying the normalization \( f(0) = 0 \) and \( f(1) = 1 \). In this manuscript, we will consider some other normalization. On this occasion, we mention papers [10] and [5,6]. The main distinguishing feature of the article is the consideration of the Beltrami equations with not one but two complex characteristics.

Note that problems of this type may be solved by the moduli method for families of paths. In fact, the solutions of the Beltrami equations are mappings that distort the modulus mentioned above according to certain rules, and this is precisely the key to solving problems. The modulus of families of paths, as a tool for research, is discussed in more detail in the second section. Let us point out numerous publications on mapping theory

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related to the modulus of families of paths, cf. [3,4], [10], [15,16], [21,22] and [24].

Let $D$ be a domain in $\mathbb{C}$. In what follows, a mapping $f: D \to \mathbb{C}$ is assumed to be sense-preserving, moreover, we assume that $f$ has partial derivatives almost everywhere. Put $f_z = (f_x + if_y)/2$ and $f_{\bar{z}} = (f_x - if_y)/2$. Denote

$$D = \{ z \in \mathbb{C} : |z| < 1 \}.$$ 

Let $\mu: D \to \mathbb{D}$ and $\nu: D \to \mathbb{D}$ be Lebesgue measurable functions. We set

$$K_{\mu,\nu}(z) = \frac{1 + |\mu(z)| + |\nu(z)|}{1 - |\mu(z)| - |\nu(z)|}.$$ 

The complex dilatation of $f$ at $z \in D$ is defined as follows: $\mu(z) = \mu_f(z) = f_{\overline{z}}/f_z$ for $f_z \neq 0$ and $\mu(z) = 0$ otherwise. The maximal dilatation of $f$ at $z$ is the following function:

$$K_\mu(z) = K_{\mu_f}(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}.$$ 

Note that the Jacobian of $f$ at $z \in D$ may be calculated according to the relation

$$J(z, f) = |f_z|^2 - |f_{\overline{z}}|^2.$$ 

Since we assume that the map $f$ is sense preserving, the Jacobian of this map is positive at all points where $f$ is differentiable. By direct calculations, we may show that

$$K_{\mu_f}(z) = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|}$$

whenever partial derivatives of $f$ exist at $z \in D$ and, in addition, $J(z, f) \neq 0$. Set $\| f'(z) \| = |f_z| + |f_{\overline{z}}|$. Recall that a homeomorphism $f: D \to \mathbb{C}$ is said to be quasiconformal if $f \in W^{1,2}_{\text{loc}}(D)$ and, in addition, $\| f'(z) \|^2 \leq K \cdot |J(z, f)|$ for some constant $K \geq 1$ almost everywhere.

A Beltrami equation with two characteristics is a differential equation of the form

$$(3) \quad f_{\overline{z}} = \mu(z) \cdot f_z + \nu(z) \cdot \overline{f_z},$$

where $\mu = \mu(z)$ and $\nu = \nu(z)$ are given measurable functions with $|\mu(z)| < 1$ and $|\nu(z)| < 1$ a.a. Let $\mu: D \to \mathbb{D}$ and $\nu: D \to \mathbb{D}$ be functions such that the
a solution to the usual Beltrami equation

In the contrary case, we put

\[ \mu_n(z) = \begin{cases} 
\mu(z), & z \in \mathbb{C}, \ K_{\mu,\nu}(z) \leq n, \\
0, & \text{otherwise in } \mathbb{C},
\end{cases} \]

and

\[ \nu_n(z) = \begin{cases} 
\nu(z), & z \in \mathbb{C}, \ K_{\mu,\nu}(z) \leq n, \\
0, & \text{otherwise in } \mathbb{C}.
\end{cases} \]

Let \( f_n: \mathbb{C} \to \mathbb{C} \) be a homeomorphic solution of the equation

\[(f_n)_z = \mu_n(z) \cdot (f_n)_z + \nu_n(z) \cdot \overline{(f_n)_z} \]

(it exists by \([10, \text{Theorem 9.2}]\)). Set \( g_n(z) := f_n^{-1}(z) \). Observe that \( f_n \) is conformal at the neighborhood of the infinity, so there is a continuous extension \( f_n: \overline{\mathbb{C}} \to \mathbb{C} \). Thus \( f_n(\mathbb{C}) = \mathbb{C} \) and \( f_n(\infty) = \infty \). Note that, \( g_n: \mathbb{C} \to \mathbb{C} \) is quasiconformal, in particular, \( g_n \) is almost everywhere differentiable in \( \mathbb{C} \).

By \([10, \text{Proposition 2.1}]\), \( f_n(z) = a_n z + b_n + o(1) \) as \( z \to \infty \), where \( a_n, b_n \in \mathbb{C} \) and \( a_n \neq 0 \). We may consider that \( a_n = 1 \) and \( b_n = 0 \) for any \( n \in \mathbb{N} \).

In the contrary case, we put \( \tilde{f}_n(z) := \frac{1}{a_n} (f_n(z) - b_n) \), while \( \tilde{f}_n \) satisfies the equation \( \tilde{f}_z = \mu(z) \cdot \tilde{f}_z + \nu(z) \cdot \overline{\tilde{f}_z} \) in \( \mathbb{C} \), as well.

Note that such a function \( \mu \) is unique. Indeed, the same function is a solution to the usual Beltrami equation \( f_z = \mu_n^*(z) \cdot f_z \), where \( \mu_n^*(z) = \mu_n(z) + \nu_n(z) \cdot \overline{f_z} \), and, by the triangle inequality, \( |\mu_n^*(z)| \leq |\mu_n(z)| + |\nu_n(z)| \leq \frac{n+1}{n+1} < 1 \). Thus, \( f \) is unique by \([1, \text{Theorem 20.4.15}]\).

Let \( K_{\mu_n}(w) \) be a maximal dilatation of \( g_n \), namely,

\[ K_{\mu_n}(w) = \frac{|(g_n)_w|^2 - |(g_n)\overline{w}|^2}{|(g_n)_w| - |(g_n)\overline{w}|)^2}. \]

We also define the inner dilatation of \( g_n \) of the order \( p \) at a point \( w \) by the relation

\[ K_{I,p}(w, g_n) = \frac{|(g_n)_w|^2 - |(g_n)\overline{w}|^2}{|(g_n)_w| - |(g_n)\overline{w}|)^2}. \]

The following statement holds.

**Theorem 1.** Let \( D \) be a domain in \( \mathbb{C} \) such that \( \overline{D} \) is a compact set in \( \mathbb{C} \), let \( \mu: \mathbb{C} \to \mathbb{D} \) and \( \nu: \mathbb{C} \to \mathbb{D} \) be Lebesgue measurable functions vanishing outside \( D \) such that the relation \( |\mu(z)| + |\nu(z)| < 1 \) holds for almost any \( z \in D \).
In addition, let \( \mu_n, \nu_n, f_n \) and \( g_n \) as above, \( n = 1, 2, \ldots \). Let \( Q : \mathbb{C} \to [1, \infty] \) be a Lebesgue measurable function. Assume that the following conditions hold:

1) for each \( 0 < r_1 < r_2 < 1 \) and \( y_0 \in \mathbb{C} \) there is a set \( E \subset [r_1, r_2] \) of positive linear Lebesgue measure such that the function \( Q \) is integrable over the circles \( S(y_0, r) \) for any \( r \in E \);

2) there exists a number \( 1 < p \leq 2 \) such that, for any bonded domain \( G \subset \mathbb{C} \) there exists a constant \( M = M_G > 0 \) such that

\[
\int_G K_{I,p}(w, g_n) \, dm(w) \leq M
\]

for all \( n = 1, 2, \ldots \), where \( K_{I,p}(w, g_n) \) is defined in (7);

3) the inequality

\[
K_{\mu_n}(w) \leq Q(w)
\]

holds for a.e. \( w \in \mathbb{C} \), where \( K_{\mu_n} \) is defined in (6). Then the equation (3) has a continuous \( W^{1,p}_{\text{loc}}(\mathbb{C}) \)-solution \( f \) in \( \mathbb{C} \) such that \( f(z) = z + \varepsilon(z) \), where \( \varepsilon(z) \to 0 \) as \( z \to \infty \).

**Corollary 2.** In particular, the conclusion of Theorem 1 holds if, in this theorem, we abandon condition 1), accept condition 3), and replace condition 2) with the requirement \( Q \in L^1_{\text{loc}}(\mathbb{C}) \). If \( G \) is some bounded domain in \( \mathbb{C} \) and \( K \) is a compactum in \( G \), then there is some domain \( G' \subset \mathbb{C} \) and a function \( Q' \) equal \( Q \) in \( G' \) and vanishing outside \( G' \) such that \( Q' \) is integrable in \( \mathbb{C} \) and the relation

\[
|f(x) - f(y)| \leq \frac{C}{\log^{1/2} \left( 1 + \frac{r_0}{2|x-y|} \right)}
\]

holds for any \( x, y \in K \), where \( C = C(K, \|Q'\|_1, G) > 0 \) is some constant depending only on \( K, G \) and \( \|Q'\|_1, \|Q'\|_1 \) denotes \( L^1 \)-norm of \( Q' \) in \( \mathbb{C} \), and \( r_0 = d(K, \partial G) \).

2. Proof of the main results

Given sets \( E \) and \( F \) and a domain \( D \) in \( \mathbb{R}^n = \mathbb{R}^n \cup \{ \infty \} \), we denote by \( \Gamma(E, F, D) \) the family of all paths \( \gamma : [0, 1] \to \mathbb{R}^n \) joining \( E \) and \( F \) in \( D \), that is, \( \gamma(0) \in E, \gamma(1) \in F \) and \( \gamma(t) \in D \) for all \( t \in (0, 1) \). Everywhere below, unless otherwise stated, the boundary and the closure of a set are understood in the sense of the extended Euclidean space \( \mathbb{R}^n \). Let \( x_0 \in \overline{D}, x_0 \neq \infty \),

\[
B(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| < r \}, \quad \mathbb{B}^n = B(0, 1)
\]
Let $f: D \to \mathbb{R}^n$, $n \geq 2$, and let $Q: \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function such that $Q(y) = 0$ for $y \in \mathbb{R}^n \setminus f(D)$. Let $A = A(y_0, r_1, r_2)$ and let $\Gamma_f(y_0, r_1, r_2)$ denotes the family of all paths $\gamma: [a, b] \to D$ such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$, i.e., $f(\gamma(a)) \in S(y_0, r_1)$, $f(\gamma(b)) \in S(y_0, r_2)$, and $f(\gamma(t)) \in A(y_0, r_1, r_2)$ for any $a < t < b$. We say that $f$ satisfies the inverse Poletsky inequality at $y_0 \in f(D)$ if the relation

$$M(\Gamma_f(y_0, r_1, r_2)) \leq \int_A Q(y) \cdot \eta^n(|y - y_0|) \, dm(y)$$

holds for any $0 < r_1 < r_2 < r_0 := \sup_{y \in D'} |y - y_0|$ and any Lebesgue measurable function $\eta: (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) \, dr \geq 1.$$  

For domains $D, D' \subset \mathbb{R}^n$, $n \geq 2$, and a Lebesgue measurable function $Q: \mathbb{R}^n \to [0, \infty]$ equal to zero outside the domain $D'$, we define by $\mathcal{R}_Q(D, D')$ the family of all open discrete mappings $f: D \to D'$ such that relation (12) holds for each point $y_0 \in D'$. Note that the definition of class $\mathcal{R}_Q(D, D')$ does not require that the domain $D$ be mapped onto the domain $D'$ surjectively under the mapping $f \in \mathcal{R}_Q(D, D')$. In what follows, $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure. The following statement holds (see [26, Theorem 1.1]).

**Proposition 3.** Let $D$ and $D'$ be domains in $\mathbb{R}^n$, $n \geq 2$, and let $D'$ be a bounded domain. Suppose that, for each point $y_0 \in D'$ and for every $0 < r_1 < r_2 < r_0 := \sup_{y \in D'} |y - y_0|$, there is a set $E \subset [r_1, r_2]$ of a positive linear Lebesgue measure such that the function $Q$ is integrable with respect to $\mathcal{H}^{n-1}$ over the spheres $S(y_0, r)$ for every $r \in E$. Then the family of mappings $\mathcal{R}_Q(D, D')$ is equicontinuous at each point $x_0 \in D$.

**Remark 4.** For domains $D, D' \subset \mathbb{R}^n$, $n \geq 2$, and a Lebesgue measurable function $Q: \mathbb{R}^n \to [0, \infty]$ equal to zero outside the domain $D'$, we define by $\mathcal{R}_Q(D, D')$ the family of all open discrete mappings $f: D \to D'$ such that relation

$$M(\Gamma_f(y_0, r_1, r_2)) \leq \frac{\omega_{n-1}}{T^{n-1}},$$

holds for each point $y_0 \in D'$ with

$$I = \int_{r_1}^{r_2} \frac{dt}{t q_y(t)}, \quad \omega_q(y_0, r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{S(y_0, r)} Q(y) \, d\mathcal{H}^{n-1}(y)$$

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\(\omega_{n-1}\) is the area of the unit sphere \(S^{n-1}\) in \(\mathbb{R}^n\), and any \(0 < r_1 < r_2 < r_0 := \sup_{y \in D'} |y - y_0|\). Suppose that, for each point \(y_0 \in D'\) and for every \(0 < r_1 < r_2 < r_0 := \sup_{y \in D'} |y - y_0|\) there is a set \(E \subset [r_1, r_2]\) of a positive linear Lebesgue measure such that the function \(Q\) is integrable with respect to \(\mathcal{H}^{n-1}\) over the spheres \(S(y_0, r)\) for every \(r \in E\). Then the family of mappings \(\mathcal{R}_Q(D, D')\) is equicontinuous at each point \(x_0 \in D\).

The proof of this assertion repeats almost exactly the proof of [26, Theorem 1.1] and is therefore omitted. Note that the function \(Q\) in the above statement can be extended outside the domain \(D'\) arbitrarily, and not necessarily by zero.

For the case of mappings of a domain \(D\) onto itself, the following lemma is proved in [7, Lemma 2], cf. [10, Theorem 9.1] and [26, Lemma 5.1]. We need to formulate it in a somewhat more general case, when the mappings do not, in general, map the domain \(D\) onto itself. We note that the proof of this assertion is quite similar to the proof of the above particular case, however, we present it in full in the text.

**Lemma 5.** Let \(1 < p \leq 2\), and let \(f_n, n = 1, 2, \ldots\), be a sequence of sense-preserving homeomorphisms of a domain \(D\) into \(\mathbb{C}\) which belong to the class \(W_{loc}^{1,2}(D)\) and satisfying the equation

\[
(14) \quad \overline{\partial f}_n = \partial f_n \mu_n(z) + \overline{\partial f}_n \nu_n(z),
\]

where \(\partial f = f_z\), \(\overline{\partial f} = f_{\overline{z}}\) and \(\mu_n\) and \(\nu_n\) are Lebesgue measurable functions satisfying the relation \(|\nu_n(z)| + |\mu_n(z)| < 1\) almost everywhere. Assume that \(f_n\) converges to a mapping \(f: D \to \mathbb{C}\) locally uniformly as \(n \to \infty\), and \(\mu_n(z)\) and \(\nu_n(z)\) converge to \(\mu(z)\) and \(\nu(z)\) as \(n \to \infty\) a.e. Assume that the inverse mappings \(g_n := f_n^{-1}\) belong to \(W_{loc}^{1,2}(f_n(D))\), while

\[
\int_{f_n(D)} K_{1,p}(w, g_n) \, dm(w) \leq M
\]

for some \(M > 0\) and any \(n = 1, 2, \ldots\). Then \(f \in W_{loc}^{1,p}(D)\) and, in addition, \(\mu\) and \(\nu\) are complex characteristics of the mapping \(f\), in other words, \(\overline{\partial f} = \partial f \mu(z) + \overline{\partial f} \nu(z)\) for almost any \(z \in D\).

**Proof.** We will generally follow the scheme outlined in [10, Theorem 9.1], cf. [26, Lemma 5.1]. Let \(C\) be arbitrary compactum in \(D\). Since \(g_n = f_n^{-1} \in W_{loc}^{1,2}\), by the assumption, \(g_n\) have \(N\)-property of Luzin, see e.g. [14, Corollary B]. Then \(J(z, f_n)\) almost everywhere is not equal to zero, see, for example, [17, Theorem 1]. Moreover, the change of variables formula under the integral holds, see [8, Theorem 3.2.5]. In this case, we obtain that

\[
(15) \quad \int_C \|f_n'(z)\|^p \, dm(z) = \int_C \frac{\|f_n'(z)\|^p}{J(z, f_n)} \cdot J(z, f_n) \, dm(z)
\]
\[ \int_{f_n(C)} K_{I,p}(w,g_n) \, dm(w) \leq M < \infty. \]

It follows from (15) that \( f \in W^{1,p}_{\text{loc}} \) and, besides that, \( \partial f_n \) and \( \overline{\partial f_n} \) converge weakly in \( L^1_{\text{loc}}(D) \) to \( \partial f \) and \( \overline{\partial f} \), respectively (see [18, Lemma III.3.5]; cf. [22, Lemma 2.1]).

It remains to show that \( f \) is a solution of the equation \( f_z = \mu(z) \cdot f_z + \nu(z) \cdot \overline{f_z} \). Set

\[ \zeta(z) = \overline{\partial f(z)} - \mu(z) \partial f(z) - \nu \partial f(z) \]

and let us to show that \( \zeta(z) = 0 \) almost everywhere. Let \( B \) be arbitrary disk belonging to \( D \) with its closure. By the triangle inequality

\[ \left| \int_B \zeta(z) \, dm(z) \right| \leq I_1(n) + I_2(n) + I_3(n), \]

where

\[ I_1(n) = \left| \int_B (\overline{\partial f(z)} - \overline{\partial f_n(z)}) \, dm(z) \right|, \]

\[ I_2(n) = \left| \int_B (\mu(z) \partial f(z) - \mu_n(z) \partial f_n(z)) \, dm(z) \right| \]

and

\[ I_3(n) = \left| \int_B (\nu(z) \overline{\partial f(z)} - \nu_n(z) \overline{\partial f_n(z)}) \, dm(z) \right|. \]

Due to the mentioned above, \( I_1(n) \to 0 \) as \( n \to \infty \). It remains to deal with the expressions \( I_2(n) \) and \( I_3(n) \). To do this, note that, by the triangle inequality, \( I_2(n) \leq I'_2(n) + I''_2(n) \), where

\[ I'_2(n) = \left| \int_B \mu(z)(\partial f(z) - \partial f_n(z)) \, dm(z) \right| \]

and

\[ I''_2(n) = \left| \int_B (\mu(z) - \mu_n(z)) \partial f_n(z) \, dm(z) \right|. \]

Due to the weak convergence of \( \partial f_n \to \partial f \) in \( L^1_{\text{loc}}(D) \) as \( n \to \infty \), we obtain that \( I'_2(n) \to 0 \) as \( n \to \infty \), because \( \mu \in L^\infty(D) \). Moreover, since the above mapping \( \partial f \) is integrable in degree \( p > 1 \), the integral \( \int_E |\partial f(z)| \, dm(z) \) is
absolute continuous with respect to $E$. Besides that, since $\partial f_n \to \partial f$ weakly in $L^1_{\text{loc}}(D)$, given $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

\begin{equation}
\int_E |\partial f_n(z)| \, dm(z) \\
\leq \int_E |\partial f_n(z) - \partial f(z)| \, dm(z) + \int_E |\partial f(z)| \, dm(z) < \varepsilon,
\end{equation}

whenever $m(E) < \delta$, $E \subset B$, and numbers $n$ are sufficiently large.

Finally, by Egorov’s theorem (see [23, Theorem III.6.12]), for any $\delta > 0$ there is a set $S \subset B$ such that $m(B \setminus S) < \delta$ and $\mu_n(z) \to \mu(z)$ uniformly in $S$. Then $|\mu_n(z) - \mu(z)| < \varepsilon$ for all $n \geq n_0$, some $n_0 = n_0(\varepsilon)$ and all $z \in S$. Due to the conditions (20), (15) and by Hölder’s inequality, we obtain that

\begin{equation}
I_2''(n) \leq \varepsilon \int_S |\partial f_n(z)| \, dm(z) + 2 \int_{B \setminus S} |\partial f_n(z)| \, dm(z)
< \varepsilon \left\{ \left( \int_{f_n(D)} K_{I,p}(w, g_n) \, dm(w) \right)^{1/p} \cdot (m(B))^{(p-1)/p} + 2 \right\}
\leq \varepsilon \cdot \left\{ M^{1/p} \cdot (m(B))^{(p-1)/p} + 2 \right\}
\end{equation}

for the same $n \geq n_0$. Thus, $I_2''(n) \to 0$ as $n \to \infty$, therefore $I_2(n) \to 0$ as $n \to \infty$. The fact that

\begin{equation}
I_3(n) \to 0
\end{equation}

as $n \to \infty$ may be proved similarly. Thus, by (17), (18), (19), (21) and (22) it follows that $\int_B \zeta(z) \, dm(z) = 0$ for any disks $B$, compactly embedded in $D$. Based on the Lebesgue theorem on differentiation of an indefinite integral (see [23, IV(6.3)]), it follows that $\zeta(z) = 0$ almost everywhere in $D$. The lemma is proved. □

**Proof of Theorem 1.** Let $f_n$, $n = 1, 2, \ldots$, be a mapping from the condition of the theorem. Let us to prove that $\{f_n\}_{n=1}^{\infty}$ forms a normal family of mappings. Fix an arbitrary compact set $C \subset \mathbb{C}$. Since $\overline{D}$ is a compactum in $\mathbb{C}$, there is a domain $G \subset \mathbb{C}$ with a compact closure in $\mathbb{C}$ such that $C \cup \overline{D} \subset G$.

We put $\tilde{f}_n = \frac{1}{f_n(1/z)}$. Since $f_n(z) = z + o(1)$ as $z \to \infty$, $\lim_{z \to \infty} f_n(z) = \infty$. Set $\tilde{f}_n(0) = 0$. Since $f_n(z) = z + o(1)$ as $z \to \infty$, there is a neighborhood $U$ of the origin and a function $\varepsilon : U \to \mathbb{C}$ such that $f_n(1/z) = 1/z + \varepsilon(1/z)$, where $z \in U$ and $\varepsilon(1/z) \to 0$ as $z \to 0$. Thus,

\[\frac{\tilde{f}_n(\Delta z) - \tilde{f}_n(0)}{\Delta z} = \frac{1}{\Delta z} \cdot \frac{1}{1/(\Delta z) + \varepsilon(1/\Delta z)} = \frac{1}{1 + (\Delta z) \cdot \varepsilon(1/\Delta z)} \to 1\]
as $\Delta z \to 0$. This proves that $\tilde{f}_n'(0) = 1$ exists. Since $\mu(z)$ vanishes outside $G$, the mapping $f_n$ is conformal in some neighborhood $V := \mathbb{C} \setminus B(0, 1/r_0)$ of the infinity, and the number $1/r_0$ depends only on $G$, and $G \subset B(0, 1/r_0)$. In this case, the mapping $\tilde{f}_n = \frac{1}{f_n(z)}$ is conformal in $B(0, r_0)$. In addition, the mapping $F_n(z) := \frac{1}{r_0} \cdot \tilde{f}_n(r_0z)$ is a homeomorphism of the unit disk such that $F_n(0) = 0$ and $F_n'(1) = 1$. By Koebe’s theorem on $1/4$ (see, e.g., [2, Theorem 1.3]) $F_n(\mathbb{D}) \supset B(0, 1/4)$. Then

$$\tilde{f}_n(B(0, r_0)) \supset B(0, r_0/4).$$

By (23)

$$\frac{1}{f_n}(\mathbb{C} \setminus B(0, 1/r_0)) \supset B(0, r_0/4).$$

Taking into account formula (24), we show that

$$f_n(\mathbb{C} \setminus B(0, 1/r_0)) \supset \mathbb{C} \setminus B(0, 4/r_0).$$

Indeed, let $y \in \mathbb{C} \setminus B(0, 4/r_0)$. Now, $\frac{1}{y} \in B(0, r_0/4)$. By (24), $\frac{1}{y} = (1/f_n)(x)$, $x \in \mathbb{C} \setminus B(0, 1/r_0)$. Thus, $y = f_n(x)$, $x \in \mathbb{C} \setminus B(0, 1/r_0)$, which proves (25).

Since $f_n$ is a homeomorphism in $\mathbb{C}$, by (25) we obtain that

$$f_n(B(0, 1/r_0)) \subset B(0, 4/r_0).$$

On the other hand, since $f_n$ are $n$-quasiconformal, the mappings $g_n = f_n^{-1}$ are also quasiconformal; in particular, they belong to $W^{1,2}_{loc}(B(0, 4/r_0))$. By [15, Theorem 6.10], (9) and (26) we obtain that

$$M(g_n(\Gamma)) \leq \int_{f_n(B(0, 1/r_0))} K_{\mu_{g_n}}(w) \cdot \rho_*^2(w) \, dm(w)$$

$$\leq \int_{B(0, 4/r_0)} Q(w) \cdot \rho_*^2(w) \, dm(w)$$

for any $n \in \mathbb{N}$, any path family $\Gamma$ in $f_n(B(0, 1/r_0))$, and each function $\rho_* \in \text{adm}\Gamma$. It follows from (27) that

$$M(g_n(\Gamma)) \leq \int_{B(0, 4/r_0)} Q(w) \cdot \rho_*^2(w) \, dm(w)$$

for the same functions $\rho$. By Proposition 3 the family $\{f_n\}_{n=1}^\infty$ is equicontinuous in $B(0, 1/r_0)$. Thus, by the Arzelà–Ascoli theorem $f_n$ is a normal family of mappings (see e.g. [27, Theorem 20.4]), in other words, there is a subsequence $f_{n_k}$ of $f_n$, converging locally uniformly in $\mathbb{C}$ to some map...
Let us to prove that the limit mapping $f$ satisfies the condition $f(z) = z + o(1)$ as $z \to \infty$. Note that the family of mappings

$$F_{n_k}(z) := \frac{1}{r_0} \cdot \frac{1}{f_{n_k}(\frac{1}{r_0 z})}$$

is compact in the unit disk (see, e.g., [2, Theorem 1.10]). Without loss of generality, we may consider that $F_{n_k}$ converges locally uniformly in $\mathbb{D}$. Now $F(z) = \frac{1}{r_0} \cdot \frac{1}{f(\frac{1}{r_0 z})}$ belongs to the class $S$, consisting of conformal mappings $F$ of the unit disk that satisfy the conditions $F(0) = 0$, $F'(0) = 1$. Then the expansions of functions $F$ and $F_{n_k}$ in a Taylor series at the origin have the form

$$F_{n_k}(z) = z + c_k z^2 + \varepsilon_k(z), \quad k = 1, 2, \ldots,$$

(28)  

$$F(z) = z + c_0 z^2 + \varepsilon_0(z),$$

(29)

where $\varepsilon_k(z)$ and $\varepsilon_0(z)$ tend to zero as $z \to 0$. It follows from (28) and (29) that

$$f_{n_k}(t) = \frac{r_0 t^2}{r_0 t + c_k + \varepsilon_k(\frac{1}{r_0 t})}, \quad f_{n_k}(t) - t = -\frac{c_k + \varepsilon_k(\frac{1}{r_0 t})}{r_0 + \frac{c_k}{t} + \frac{\varepsilon_k(\frac{1}{r_0 t})}{t}}, \quad k = 1, 2, \ldots,$$

(30)

$$f(t) = \frac{r_0 t^2}{r_0 t + c_0 + \varepsilon_0(\frac{1}{r_0 t})}, \quad f(t) - t = -\frac{c_0 + \varepsilon_0(\frac{1}{r_0 t})}{r_0 + \frac{c_0}{t} + \frac{\varepsilon_0(\frac{1}{r_0 t})}{t}}.$$

(31)

In particular, passing to the limit in (30) as $t \to \infty$, we obtain that $f_{n_k}(t) - t \to -\frac{c_k}{r_0}$. Since $f_{n_k}(t) - t \to 0$ as $t \to \infty$, we obtain that $c_k = 0$. By the Weierstrass theorem on the convergence of the coefficients of the Taylor series (see, e.g., [9, Theorem 1.1.1]) we obtain that $c_k = 0 \to c_0$ as $k \to \infty$. Thus, $c_0 = 0$ in (31), in other words, the mapping $f$ also has a hydrodynamic normalization: $f(z) = z + o(1)$ as $z \to \infty$. Theorem is proved. □

**Proof of Corollary 2.** Let $G$ and $K$ be the same as in the condition of the corollary. Repeating the proof of Theorem 1, we observe that $g_n$
satisfy the relation

\[ M(g_n(\Gamma)) \leq \int_{f_n(B(0,1/r_0))} K_{\mu_{g_n}}(w) \cdot \rho^2_* (w) \, dm(w) \leq \int_{C} Q'(w) \cdot \rho^2_* (w) \, dm(w), \]

where

\[ Q'(w) = \begin{cases} Q(w), & w \in B(0,4/r_0), \\ 0, & \text{otherwise}, \end{cases} \]

where the function \( Q' \) is integrable in \( \mathbb{C} \). Now, by [26, Theorem 4.1] the relation

\[ |f_n(x) - f_n(y)| \leq \frac{C}{\log^{1/2} \left( 1 + \frac{r_0}{2|x-y|} \right)} \]

holds for any \( x, y \in K \) and any compact set \( K \) in \( G \), where

\[ C = C(K, \| Q' \|_1, G) > 0 \]

is some constant depending only on \( K, G \) and \( \| Q' \|_1, \| Q' \|_1 \) denotes \( L^1 \)-norm of \( Q' \) in \( \mathbb{C} \), and \( r_0 = d(K, \partial G) \). Passing here to the limit as \( n \to \infty \), we obtain the required relation (10). □

3. A convergence theorem

Recall that a mapping \( f: D \to \mathbb{R}^n, \ n \geq 2, \) is called a mapping with a finite distortion, if \( f \in W^{1,1}_{loc}(D) \) and there is a function \( K: D \to [1, \infty) \) such that \( \| f'(x) \|^n \leq K(x) \cdot |J(x,f)| \) for almost all \( x \in D \).

Given a function \( Q: \mathbb{C} \to \mathbb{C} \), numbers \( 1 < p \leq 2, \ 0 < M < \infty \) and a domain \( G \) such that \( \overline{G} \) is a compactum in \( \mathbb{C} \), denote by \( \mathcal{F}_{Q,p,M}(G) \) a family of all \( W^{1,1}_{loc} \)-homeomorphisms \( f: \mathbb{C} \to \mathbb{C} \) with a finite distortion which are conformal outside \( G \) and are solutions of the equation (3) in \( \mathbb{C} \) such that

1) \( f(z) = z + \varepsilon_f(z) \), where \( \varepsilon_f(z) \to 0 \) as \( z \to \infty \);
2) there exists a number \( 1 < p \leq 2 \) such that, for any bonded domain \( G \subset \mathbb{C} \) there exists a constant \( M = M_G > 0 \) such that

\[ \int_{G} K_{I,p}(w,g) \, dm(w) \leq M \] (32)

for all \( n = 1, 2, \ldots, \) where \( g := f^{-1} \) and \( K_{I,p}(w,g) \) is defined in (7);
3) the inequality

\[ K_{\mu_g}(w) \leq Q(w) \] (33)

holds for \( g := f^{-1} \) and for a.e. \( w \in \mathbb{C} \), where \( K_{\mu_g} \) is defined in (6).

The following statement holds.
THEOREM 6. Assume that, for each $0 < r_1 < r_2 < 1$ and $y_0 \in \mathbb{C}$ there is a set $E \subset [r_1, r_2]$ of positive linear Lebesgue measure such that the function $Q$ is integrable over the circles $S(y_0, r)$ for any $r \in E$. Then $\mathfrak{F}_{Q,p,M}(G)$ is a normal family. If $f_n \in \mathfrak{F}_{Q,p,M}(G)$, $n = 1, 2, \ldots$, $f_n \to f$ as $n \to \infty$ locally uniformly, and $\mu_n \to \mu$ and $\nu_n \to \nu$ as $n \to \infty$, then $f$ satisfies the equation (3). In this case, $f(z) = z + \varepsilon(z)$, $\varepsilon(z) \to 0$ as $z \to \infty$, as well.

COROLLARY 7. In particular, the statement of Theorem 6 holds, if instead of the conditions on the function $Q$, specified in this theorem, we require that $Q \in L^1_{\text{loc}}$.

PROOF OF THEOREM 6. Note that the proof follows the same principle as the proof of the previous Theorem 1. Due to this, we restrict us the sketch of the proof.

Firstly, we prove that $\mathfrak{F}_{Q,p,M}(G)$ forms a normal family of mappings. Fix an arbitrary domain $G \subset \mathbb{C}$ with a compact closure. Now, arguing similarly to the proof of the relation (26), we obtain that the relation

$$f(B(0, 1/r_0)) \subset B(0, 4/r_0)$$

holds for any $f \in \mathfrak{F}_{Q,p,M}(G)$.

Observe that, if $f$ has a finite distortion, then $g := f^{-1}$ is of finite distortion, as well (see [11, Theorem 1.2]). Then by [13, Lemma 3.1 and Proposition 2.1]

$$M(g(\Sigma_{r_1, r_2})) \geq \int_{r_1}^{r_2} dr \frac{dr}{\|Q\|_1(r)},$$

where $\Sigma_{r_1, r_2}$ denotes the family of all circles $S(y_0, r) \cap f(G)$, $r \in (r_1, r_2)$, and

$$\|Q\|_1(r) = \int_{S(y_0, r) \cap f(G)} Q(y) d\mathcal{H}^1(y).$$

Applying the Ziemer and Hesse theorems, see [28, Theorem 3.13] and [12, Theorem 5.5], we obtain that

$$M\left(g(\Gamma(S(y_0, r_1), S(y_0, r_2), f(G)))\right) \leq \frac{2\pi}{\int_{r_1}^{r_2} \frac{dr}{rq_{y_0}(r)}},$$

or, in other form,

$$M\left(\Gamma_f(y_0, r_1, r_2)\right) \leq \frac{2\pi}{\int_{r_1}^{r_2} \frac{dr}{rq_{y_0}(r)}}.$$

Now, by Remark 4 it follows that $\mathfrak{F}_{Q,p,M}(G)$ is equicontinuous in $G$. Finally, by the Arzelà–Ascoli theorem $\mathfrak{F}_{Q,p,M}(G)$ is a normal family of mappings (see e.g. [27, Theorem 20.4]).
Assume now that \( f_n \in \mathfrak{S}_Q p, M(G) \), \( n = 1, 2, \ldots \), is a sequence converging locally uniformly in \( \mathbb{C} \) to some map \( f: \mathbb{C} \to \mathbb{C} \). Note also that \( \mu_n(z) \to \mu(z) \) as \( n \to \infty \) and \( \nu_n(z) \to \nu(z) \) as \( n \to \infty \) for almost all \( z \in D \), because \( |\mu(z)| < 1 \) a.e. and, therefore, \( K_\mu(z) \) in (2) is finite for almost all \( z \in D \). Then by (8) and Lemma 5 the map \( f \) belongs to the class \( W^{1,p}_{\text{loc}}(D) \) and, in addition, \( f \) is a solution of (3).

Finally, the limit mapping \( f \) satisfies the condition \( f(z) = z + o(1) \) as \( z \to \infty \). The proof of this fact is similar to the last part of the proof of Theorem 1. \( \square \)

Let us consider the following example, see [26, Example 3].

**Example 8.** Let \( p = 2 \), let \( q \geq 1 \) be an arbitrary number and let \( 0 < \alpha < 2/q \). As usual, we use the notation \( z = re^{i\theta}, r \geq 0, \theta \in [0, 2\pi) \). Put

\[
\mu(z) = \begin{cases} 
  e^{2i\theta \frac{2r-\alpha(2r-1)}{2r+\alpha(2r-1)}}, & 1/2 < |z| < 1, \\
  0, & |z| \leq 1/2.
\end{cases}
\]

Using the ratio

\[
\mu_f(z) = \frac{\partial f}{\partial \bar{z}} = e^{2i\theta} \frac{r f_r + i f_\theta}{r f_r - i f_\theta},
\]

see in [16, (11.129)], we obtain that the mapping

\[
f(z) = \begin{cases} 
  \frac{z}{|z|} (2|z| - 1)^{1/\alpha}, & 1/2 < |z| < 1, \\
  0, & |z| \leq 1/2
\end{cases}
\]

is a solution of the equation \( f_\mu = \mu(z) \cdot f_z \), where \( \mu \) is defined by (35). Note that for \( \mu \) in (35), the corresponding maximal dilatation \( K_\mu \) is the function

\[
K_\mu(z) = \begin{cases} 
  \frac{2|z|}{\alpha(2|z|-1)}, & 1/2 < |z| < 1, \\
  1, & |z| \leq 1/2
\end{cases}.
\]

Let \( k > 1/\alpha \). Observe that \( K_\mu(z) \leq k \) for \( |z| \geq 1/2 \cdot \frac{k\alpha}{k\alpha - 1} \) and \( K_\mu(z) > k \) otherwise. As above, we set

\[
\mu_k(z) = \begin{cases} 
  \mu(z), & K_\mu(z) \leq k, \\
  0, & K_\mu(z) > k.
\end{cases}
\]

Observe that the mappings

\[
f_k(z) = \begin{cases} 
  \frac{z}{|z|} (2|z| - 1)^{1/\alpha}, & \frac{1}{2} \cdot \frac{k\alpha}{k\alpha - 1} < |z| < 1, \\
  \frac{z}{1/2 \cdot \frac{k\alpha}{k\alpha - 1}}, & \left( \frac{1}{k\alpha - 1} \right)^{1/\alpha}, |z| \leq \frac{1}{2} \cdot \frac{k\alpha}{k\alpha - 1}.
\end{cases}
\]
are homeomorphic solutions of the equation $f_\varepsilon = \mu_k(z) \cdot f_z$. Besides that, the inverse mappings $g_k(y) = f_k^{-1}(y)$ are calculated by the relations

\begin{equation}
(38) \quad g_k(y) = \begin{cases}
\frac{y(|y|^{\alpha}+1)}{2|y|}, & \left( \frac{k\alpha}{k\alpha-1} - 1 \right)^{1/\alpha} < |y| < 1, \\
\frac{y}{2^{k\alpha-1}} \left( \frac{k\alpha}{k\alpha-1} - 1 \right)^{1/\alpha}, & \frac{1}{2} \cdot \frac{k\alpha}{k\alpha-1} < |y| < 1, \\
\left( \frac{k\alpha}{k\alpha-1} - 1 \right)^{1/\alpha}, & |y| \leq \frac{1}{2} \cdot \frac{k\alpha}{k\alpha-1}.
\end{cases}
\end{equation}

It follows from (37) that

\begin{equation}
(39) \quad K_{\mu_k}(z) = \begin{cases}
\frac{4|z|}{2\alpha(2|z|-1)}, & \frac{1}{2} \cdot \frac{k\alpha}{k\alpha-1} < |z| < 1, \\
1, & |z| \leq \frac{1}{2} \cdot \frac{k\alpha}{k\alpha-1}.
\end{cases}
\end{equation}

We should check that relation (8) holds for some function $Q$ that is integrable in $\mathbb{D}$. For this purpose, we substitute the maps $g_k$ from (38) into the maximal dilatation $K_{\mu_k}$ defined by the equality (39). Then

\begin{equation}
K_{\mu_{g_k}}(y) = \begin{cases}
\frac{|y|^{\alpha}+1}{\alpha|y|^\alpha}, & \left( \frac{k\alpha}{k\alpha-1} - 1 \right)^{1/\alpha} < |y| < 1, \\
1, & |y| \leq \left( \frac{k\alpha}{k\alpha-1} - 1 \right)^{1/\alpha}.
\end{cases}
\end{equation}

Note that $K_{\mu_{g_k}}(y) \leq Q(y) := \frac{|y|^{\alpha}+1}{\alpha|y|^\alpha}$ for all $y \in \mathbb{D}$. Moreover, the function $Q$ is integrable in $\mathbb{D}$ even in the degree $q$, and not only in the degree 1 (see the arguments used in considering [16, Proposition 6.3]). We extend each of the mappings $f_k$ identically to the whole plane, and set $Q(y) \equiv 1$ for $y \not\in \mathbb{C}$. It follows from the considerations mentioned above that, all mappings $f_k$, $k = 1, 2, \ldots$, belong to some class $\mathcal{F}(G)$ with $Q$ mentioned above. Note that the family $f_k$, $k = 1, 2, \ldots$, is normal, but not compact, since the limit mapping $f$ of this sequence is not a homeomorphism. In order for the compactness of the class to be satisfied, other conditions on the characteristics of mappings are necessary, which will be considered below.

Let $D$ be a domain in $\mathbb{C}$. Suppose that a function $\varphi: D \to \mathbb{R}$ is locally integrable in some neighborhood of a point $z_0 \in D$. We say that $\varphi$ has a finite mean oscillation at $z_0 \in D$, and we write $\varphi \in FMO(z_0)$, if the relation

\[ \limsup_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \int_{B(z_0, \varepsilon)} |\varphi(z) - \varphi_\varepsilon| \, dm(z) < \infty \]

holds, where $\varphi_\varepsilon = \frac{1}{\pi \varepsilon^2} \int_{B(z_0, \varepsilon)} \varphi(z) \, dm(z)$ (see, e.g., [21, Section 2]). We say that a function $\varphi$ has a finite mean oscillation in $D$, and we write $\varphi \in FMO(D)$, if $\varphi \in FMO(z_0)$ for any $z_0 \in D$.

Given a function $Q: \mathbb{C} \to \mathbb{C}$ and a domain $G$ with a compact closure in $\mathbb{C}$, denote by $\mathfrak{R}_Q(G)$ a family of all $W^{1,1}_{loc}$-homeomorphisms $f: \mathbb{C} \to \mathbb{C}$.
with a finite distortion which are conformal outside $G$ and are solutions of the equation (3) in $\mathbb{C}$ such that

1) $f(z) = z + \varepsilon f(z)$, where $\varepsilon f(z) \to 0$ as $z \to \infty$;
2) the inequality

$$K_{\mu_g}(w) \leq Q(w)$$

holds for $g := f^{-1}$ and for a.e. $w \in \mathbb{C}$, where $K_{\mu_g}$ is defined in (6).

**Theorem 9.** If 1) $Q \in FMO(\mathbb{C})$, or 2) $Q \in L^1_{\text{loc}}(\mathbb{C})$ and, in addition,

$$\int_0^{\delta(w_0)} \frac{dt}{tq_{w_0}(t)} = \infty$$

for any $w_0 \in \mathbb{C}$ and some $\delta(w_0) > 0$,

$$q_{w_0}(r) = \frac{1}{2\pi} \int_0^{2\pi} Q(w_0 + re^{i\theta}) \, d\theta.$$  \hspace{1cm} (41)

Then the class $\mathcal{R}_Q(G)$ is compact.

**Proof.** Since the normality of $\mathcal{R}_Q(G)$ follows by Theorem 6, we need to prove the closeness of $\mathcal{R}_Q(G)$. Let $f_n \in \mathcal{R}_Q(G)$, $n = 1, 2, \ldots$, and let $f_n \to f$ as $n \to \infty$. Since $f_n(z) = z + o(1)$ as $z \to \infty$, denoting by $g_n := f_n^{-1}$ and $f_n(z) = w$ we obtain that $z = g_n(w) = w - o(1) = w - \varepsilon(z(w)) = w + \varepsilon_1(w)$, where $\varepsilon_1(w) \to 0$ as $w \to \infty$. Arguing similarly to the proof of Theorem 1, we obtain that $g_n(B(0, 1/r_0)) \subset B(0, 4/r_0)$ for any $n \in \mathbb{N}$ and any sufficiently small $r_0 > 0$. Then the sequence $g_n := f_n^{-1}$ forms an equicontinuous family of mappings, as well (see [20, Theorems 6.1 and 6.5]). Therefore, by the Arzelà–Ascoli theorem $g_k$ is a normal family (see e.g. [27, Theorem 20.4]), in other words, there is a subsequence $g_{nk}$ of $g_n$ converging locally uniformly in $\mathbb{C}$ to some map $g: \mathbb{C} \to \mathbb{C}$. Arguing as in the proof of Theorem 1, we obtain that $g(z)$ has a hydrodynamical normalization at the infinity. Then, by virtue of [19, Theorems 4.1, 4.2] the mapping $g$ is a homeomorphism in $\mathbb{C}$ and $g: \mathbb{C} \to \mathbb{C}$. Moreover, $g(\mathbb{C}) = \mathbb{C}$ because $g$ is a homeomorphism and the point $\infty$ is isolated boundary point of $g$ (see, e.g., [16, Theorem 6.2]). Thus, by [19, Lemma 3.1], we also have that $g_{nk}^{-1} \to f = g^{-1}$ as $k \to \infty$ locally uniformly in $\mathbb{C}$. Observe that, since $f_{nk}$ has a finite distortion, then $g_{nk} := f_{nk}^{-1}$ is of finite distortion, as well (see [11, Theorem 1.2]). Now, $K_{\mu_g}(w) \leq Q(w)$ for almost any $w \in \mathbb{C}$ by [19, Theorem 16.1]. \hfill \Box

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