New type of chimera and mutual synchronization of spatiotemporal structures in two coupled ensembles of nonlocally interacting chaotic maps

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We study numerically the dynamics of a network made of two coupled one-dimensional ensembles of discrete-time systems. The first ensemble is represented by a ring of nonlocally coupled Henon maps, and the second one - by a ring of nonlocally coupled Lozi maps. We find that the network of coupled ensembles can realize all the spatio-temporal structures which are observed both in the Henon map ensemble and in the Lozi map ensemble when uncoupled. Moreover, we reveal a new type of spatiotemporal structure, a solitary state chimera, in the considered network. We also establish and describe the effect of mutual synchronization of various complex spatiotemporal patterns in the system of two coupled ensembles of Henon and Lozi maps.

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I. INTRODUCTION

Recently studying the formation and evolution of various spatiotemporal patterns in ensembles or networks of coupled oscillators has become one of the most rapidly developing and highly attractive research topics in the nonlinear science and its applications. This exclusive interest is especially related to the discovery of a novel type of spatiotemporal structure – a chimera state. A lot of attention is paid to the dynamics of coupled ensembles of identical elements with various coupling topologies, but of particular interest are coupled ensembles with different types of network elements. In the latter case, the enrichment of regimes as well as the synchronization of spatiotemporal patterns is expected to be observed. In the present paper we analyze the spatiotemporal dynamics of a network made of two coupled rings of Henon and Lozi maps with nonlocal coupling. Our numerical studies have shown that this network can demonstrate both the spatiotemporal regimes, which are observed in separate rings, and a new type of chimera structure, called a solitary state chimera. We have also established the possibility of realizing the mutual synchronization of various complex spatiotemporal structures in the network of two coupled rings. The identity of synchronous patterns is confirmed by calculating the cross-correlation coefficient. The existence of a finite region of synchronization in the coupling parameters plane of the considered system is shown for an exemplary synchronous structure.

1. INTRODUCTION

Studying the dynamics of complex ensembles of coupled nonlinear oscillators has been the subject of intensive research for many years. At present, a special attention in this research direction is targeted to the analysis of formation and evolution of a novel type of spatiotemporal patterns, a so-called chimera state. The chimera state represents the coexistence of clearly identified clusters of oscillators with asynchronous (incoherent) and synchronous (coherent) dynamics. The efforts of many researchers are aimed at searching for new types of chimera structures, at describing their dynamical and statistical characteristics, and at the ability to give their justified classification. Up to now, several types of chimeras have been revealed, for example, phase and amplitude chimeras, coherence resonance chimera, double-well chimera, nonstationary chimeras of switching type, and others. Chimera structures have been found in ensembles of coupled oscillators of a very different nature with regular and chaotic dynamics. Besides, chimeras can be realized and observed both numerically and experimentally. It is important to note that the majority of works is devoted to the study of one- or two-dimensional networks of identical oscillators with different coupling topology between them. It appears to be very interesting to explore the dynamics of coupled ensembles in the case when the individual ensembles consist of elements of different types. In such more complicated networks one can expect the complex cooperative dynamics and the appearance of new interesting incoherence (chimera) states. Moreover, one can naturally formulate and study the problem of synchronization of various spatiotemporal structures in networks of coupled ensembles.

In the present paper we explore the spatiotemporal dynamics of a network which consists of two coupled one-dimensional ensembles of discrete-time systems. The first
ensemble is represented by a ring of nonlocally coupled chaotic Henon maps, and the second one – by a ring of
nonlocally coupled chaotic Lozi maps. It is important to mention that the first network elements possess non-
hyperbolic chaotic attractors, while the second network elements are characterized by hyperbolic chaotic attrac-
tors. In the papers\cite{16,24,25} it has been shown that the one-dimensional ensemble of nonlocally coupled Henon
maps can realize phase, amplitude and nonstationary
solitary state structures and no chimeras appear. In this
connection we are especially interested in studying various
types of spatiotemporal structures and their possible
mutual synchronization in a network of two coupled one-
dimensional networks of the types indicated above.

II. MATHEMATICAL MODEL OF THE SYSTEM
UNDER STUDY

In our research we explore the network of two interacting
one-dimensional rings of Henon and Lozi maps.
When uncoupled, each of the ensembles is characterized
by nonlocal coupling between their individual elements.
These two rings are coupled in the way as shown in Fig. 1.
Each ith element of the first ensemble is coupled with the
ith element of the second one with the coupling coefficient
γ. The network of two coupled ensembles is described by
the following equations:

\[
\begin{align*}
    x_i^{t+1} &= f_i^t + \frac{\sigma_1}{2P} \sum_{j=i-P}^{i+P} [f_j^t - f_i^t] + \gamma F_i^t, \\
    y_i^{t+1} &= \beta x_i^t, \\
    u_i^{t+1} &= g_i^t + \frac{\sigma_2}{2R} \sum_{j=i-R}^{i+R} [g_j^t - g_i^t] + \gamma G_i^t, \\
    v_i^{t+1} &= \beta u_i^t.
\end{align*}
\]

Here \(i = 1, 2, \ldots, N\) is the element number in each ring,
\(N = 1000\) is the total number of elements in each ensem-
ble, \(\sigma_1\) and \(\sigma_2\) are the nonlocal coupling strengths in the
Henon and Lozi map rings, respectively. \(P\) and \(R\) denote
the number of couplings of the ith element with its near-
est neighbors from each side in the Henon and Lozi map
ensembles, respectively, and \(\gamma\) is the coupling coefficient
between two elements from different ensembles.

The first pair of equations in the system (1) describes
the ring of nonlocally coupled Henon maps, where \(f_i^t = f(x_i^t, y_i^t) = 1 - \alpha (x_i^t)^2 + y_i^t\). The number of neighbors \(P\)
and the coupling strength \(\sigma_1\) are set in such a way that,
without coupling between the two rings, the Henon map
demonstrates a chimera state.

The second pair of equations in (1) determines the ring
of nonlocally coupled Lozi maps, where \(g_i^t = g(u_i^t, v_i^t) = 1 - \alpha |u_i^t| + v_i^t\). The number of neighbors \(R\) and the
 coupling strength \(\sigma_2\) are chosen so that the regime of solitary
states is realized in the Lozi map ensemble when the two
rings are uncoupled. The control parameters of both net-
works’ elements are fixed as \(\alpha = 1.4\) and \(\beta = 0.3\), that
corresponds to the chaotic behavior in all the individual
elements of the considered system.

Functions \(F_i^t\) and \(G_i^t\) in the network (1) are responsible
for the coupling type between the two ensembles. In our
numerical simulation we deal with dissipative and inertial
coupling types. We have for the dissipative coupling

\[
\begin{align*}
    F_i^t &= \tilde{F}_i^t(x_i^t, y_i^t, u_i^t, v_i^t) = g(u_i^t, v_i^t) - f(x_i^t, y_i^t), \\
    G_i^t &= \tilde{G}_i^t(x_i^t, y_i^t, u_i^t, v_i^t) = f(x_i^t, y_i^t) - g(u_i^t, v_i^t),
\end{align*}
\]

and for the inertial coupling

\[
\begin{align*}
    F_i^t &= \tilde{F}_i^t(x_i^t, y_i^t, u_i^t, v_i^t) = u_i^t - x_i^t, \\
    G_i^t &= \tilde{G}_i^t(x_i^t, y_i^t, u_i^t, v_i^t) = x_i^t - u_i^t.
\end{align*}
\]

These coupling type definitions correspond to the inter-
action character of coupled rings. The dissipative cou-
pling tends to make equal the instantaneous amplitudes
of coupled elements, while the inertial coupling possesses
the ability to keep (retain) the memory about the previ-
ous state of the system.

III. NUMERICAL RESULTS

As can be seen from (1), we have two uncoupled en-
sembles of Henon and Lozi maps when \(\gamma = 0\). In this
case, the Henon map network can demonstrate two types
of chimera structures – phase and amplitude chimeras.
They can be observed, for example, with \(\sigma_1 = 0.32\) and
\(P = 320\), as was shown in the papers\cite{16,24,25}.
These structures are exemplified in Fig. 2(a). The phase chimera
represents the coexistence of a coherence cluster (100 \( \leq i \leq 400\))
and incoherence clusters of elements (1 \( \leq i \leq 100\)
and 400 \( \leq i \leq 550\)) with periodic dynamics and irregular
phases. Consequently, the cross-correlation coefficient
\(R_{1,i}\) for the phase chimera is equal to unity in modulus
but changes irregularly its sign \((R_{1,i} = \pm 1)\) inside the
incoherence cluster. The spatial cross-correlation coefficient
\(R_{1,i}\) is estimated for the elements of the spatiotemporal
structures realized in one of the considered ensembles, as
was described in the paper\cite{31}:

\[
R_{1,i} = \frac{\langle x_i(t) x_{i}(t) \rangle}{\sqrt{\langle x_i^2(t) \rangle \langle x_{i}^2(t) \rangle}}, \quad i = 2, \ldots, N,
\]
Here, $\hat{x} = x(t) - \langle x(t) \rangle$. The amplitudes of oscillations $x(t)$ in Fig. 4 correspond to the values of the $x$ (or $u$) coordinate of one of the ensembles.

The elements belonging to the amplitude chimera (700 $\leq i \leq$ 820 in Fig. 2(a)) oscillate strongly chaotically and are uncorrelated to each other. Besides, this chaotic regime is characterized by temporal intermittency and the lifetime of the amplitude chimera is finite.

As was described in the papers, the ring of nonlocally coupled Lozi maps demonstrates the regime of solitary states. This spatiotemporal structure is displayed in Fig. 2(b) for $\sigma = 0.225$ and $R = 190$.

A. Spatiotemporal structures in the network of coupled ensembles

We now consider the dynamics of the network of two coupled ensembles (1) with the dissipative coupling (2) for finite values of $\gamma > 0$. Instantaneous profiles (snapshots) for the states in both rings are shown in Fig. 3 for $\gamma = 0.115$. Our numerical calculations have indicated that if the two rings are dissipatively coupled, each of them can demonstrate phase and amplitude chimeras and solitary states. With this, for instance, the Lozi map ensemble can exhibit the regime which cannot be observed without coupling with the Henon map ring. As can be seen from Fig. 3(b), the amplitude (800 $\leq i \leq$ 980) and phase (210 $\leq i \leq$ 230) and (490 $\leq i \leq$ 570) chimeras are realized in the Lozi map ring.

In a similar manner, solitary state structures can be realized in the Henon map ensemble, while they cannot be observed in this ensemble without coupling with the Lozi map ring. The regime of solitary states in the Henon map ensemble is exemplified in Fig. 4(a) for the dissipative coupling (2) with the Lozi map ensemble. As follows from this figure, the solitary states are realized within the interval of elements 1 $\leq i \leq$ 280, and the amplitude chimera occupies the elements 570 $\leq i \leq$ 640. The corresponding snapshot of the dynamics of the Lozi map ensemble is shown in Fig. 4(b).

It has been shown in the paper that the lifetime of the amplitude chimera in the ensemble of nonlocally coupled Henon map is finite and can be indefinitely increased by the influence of noise. If the amplitude chimera regime is established in the system (1) of coupled ensembles of Henon and Lozi maps (Fig. 3), then its lifetime can be controlled by varying the coupling coefficient $\gamma$ between the ensembles. Our calculations have shown that the dependence of the lifetime on $\gamma$ is essentially nonlinear and may vary over a wide range, including regimes when the lifetime grows infinitely. This effect is achieved in the autonomous system (1) without external influences.

In the case of inertial coupling (3) in (1) and for certain values of the coupling parameters, the coupled ensembles of Henon and Lozi maps can also demonstrate new spatiotemporal patterns which cannot be realized when they are uncoupled ($\gamma = 0$). For example, the Henon map ring can exhibit the regime of traveling waves, which is not observed without coupling with the Lozi map ensemble.

B. Solitary state chimera

Now we would like to pay attention to one more spatiotemporal structure which is found in the coupled rings and can be referred to as a solitary state chimera. This new structure is observed in the Henon map ring (Fig. 3(a)), while the solitary state regime is detected in the Lozi map ensemble (Fig. 5(b)). The newly found spatiotemporal mode can be classified as a chimera state, since this incoherence cluster, which includes the net-
work elements with a different behavior (as compared with the other elements), is strongly localized in the space. The main differences between the new structure and the phase chimera consist in the facts that the elements from the solitary state chimera represent solitary states and demonstrate asynchronous chaotic oscillations. Moreover, the coherence domain corresponds to complete chaotic synchronization, while we deal with periodic or weakly chaotic behavior in the case of phase chimera.

Our finding can be confirmed quantitatively by estimating the cross-correlation coefficient \( R_{1,i} \) for the new chimera structure. The calculation results are presented in Fig. 5(c,d) for the Henon map ensemble and the Lozi map ensemble, respectively. As can be seen from Fig. 5(c), all the values of \( R_{1,i} \) do not achieve unity in absolute value (\( |R_{1,i}| < 1 \)) and this corroborates the fact that all the elements in the solitary state chimera behave chaotically. In the case of phase chimera the values of \( R_{1,i} \) are irregularly switched (alternated) between two levels \( \pm 1 \).

The novel type of chimera state (Fig. 5(a)) has been revealed for the inertial coupling in the network for (a,c) solitary state chimera in the Henon map ensemble, and (b,d) solitary states in the Lozi map ensemble. The solitary state chimera forms the cluster with elements \( 490 \leq i \leq 590 \) in Fig. 5(a). Parameters: \( \sigma_1 = 0.34, \sigma_2 = 0.205, P = 320, R = 193, \) and \( \gamma = 0.02 \).

FIG. 5. Snapshots of the \( x_i \) and \( u_i \) states and cross-correlation coefficient \( R_{1,i} \) in the network for the inertial coupling for (a,c) solitary state chimera in the Henon map ensemble, and (b,d) solitary states in the Lozi map ensemble. The solitary state chimera forms the cluster with elements \( 490 \leq i \leq 590 \) in Fig. 5(a). Parameters: \( \sigma_1 = 0.34, \sigma_2 = 0.205, P = 320, R = 193, \) and \( \gamma = 0.02 \).

FIG. 6. Snapshot of the variables \( x_i \) (a) and the cross-correlation coefficient \( R_{1,i} \) (b) for the dissipative coupling in the network for the solitary state chimera in the ensemble of Henon maps. This chimera consists of the elements \( 1 < i < 100 \) and \( 750 < i < 1000 \). Parameters: \( \sigma_1 = 0.39, \sigma_2 = 0.1, P = 320, R = 190, \) and \( \gamma = 0.110 \).

FIG. 7. Synchronous spatiotemporal structures in the network for the dissipative coupling: (a) amplitude and phase chimera, and (b) phase chimera at \( \sigma_1 = 0.32, \gamma = 0.375, \) and (c) complex spatiotemporal structure at \( \sigma_1 = 0.18, \gamma = 0.375 \). Snapshots of the dynamics of the coupled Henon maps and of the coupled Lozi maps are displayed in the left and middle columns, respectively, and the mutual correlation coefficient \( R_i \) is pictured in the right column. Other parameters: \( P = 320, R = 190, \sigma_2 = 0.15 \).

C. Mutual synchronization of spatiotemporal structures

As our numerical studies have shown, the coupled ensembles can demonstrate another interesting and important phenomenon which is very typical in coupled systems. Varying the coupling strength \( \gamma \) between the rings often leads to the identity of spatiotemporal structures which are observed in the ensemble of Henon maps and the ensemble of Lozi maps. We mean here a possible mutual synchronization of the spatiotemporal dynamics of the structures in these networks. Let us examine this issue in more detail.

Figure 7 illustrates typical examples of different identical structures in the system of coupled ensembles for the dissipative coupling. Snapshots for the dynamics of the ensemble of Henon maps and of the ensemble of...
Lozi maps are shown in the left and middle columns, respectively. In order to argue that indeed we deal with the effect of mutual synchronization of spatiotemporal structures, we need to justify that the following two conditions are fulfilled: (i) the identity of oscillatory processes in the corresponding oscillators of synchronous structures is quantified, and (ii) the synchronization effect is realized in a finite region of the control parameters of the network (1).

The identity of the oscillatory processes in the coupled ensembles of Henon and Lozi maps is diagnosed by calculating the cross-correlation coefficient \( R_i \):

\[
R_i = \frac{\langle \tilde{x}_i(t) \tilde{u}_i(t) \rangle}{\sqrt{\langle \tilde{x}_i^2(t) \rangle \langle \tilde{u}_i^2(t) \rangle}}.
\]

Unlike \( R_{1,2} \), the value of \( R_i \) characterizes the mutual correlation between the \( i \)th elements of the ensembles of Henon and Lozi maps. The calculation results for \( R_i \) for the exemplary structures are presented in Fig. 7 in the right column. As can be seen from the plots, the values of \( R_i \) are very close to unity, i.e., 0.99 \( \leq R_i \leq 1.0 \), in all the three cases. This finding indicates that the oscillations of the elements in the coupled ensembles of Henon and Lozi maps are practically identical for the spatiotemporal structures shown in Fig. 7.

We note that the first two modes (Fig. 7(a,b)) correspond to the effect of mutual synchronization of chimera structures, while the latter (Fig. 7(c)) illustrates the mutual synchronization of a more complex spatiotemporal structure. As is known, the equality to unity characterizes the effect of complete synchronization, which is possible only in the case when interacting oscillators are identical. In our case the oscillators are different (non-identical) and the fact that \( R_i \) is close to unity testifies to nearly identical processes in the relevant oscillators.

In order to justify that the synchronization effect is observed in a finite range of parameters’ values of the network (1), we construct the synchronization region in the parameter plane of the system under study. Figure 8 shows the region of synchronization in the \((\gamma, \sigma_1)\) parameter plane for the complex spatiotemporal structure shown in Fig. 7(a). As can be seen, the phenomenon of mutual synchronization is implemented in a rather wide range of changes in the coupling parameters. Similar results have also been obtained for the region of synchronization in the plane of the other coupling parameters.

IV. CONCLUSIONS

We have studied numerically the spatiotemporal dynamics of the network made of two coupled one-dimensional ensembles of nonlocally coupled chaotic Henon and Lozi maps. Our simulations have shown that for both dissipative and inertial couplings between the elements in the coupled ensembles (1) \((\gamma > 0)\), all the spatiotemporal structures that occur in the individual ensembles can be realized, e.g., phase and amplitude chimeras, traveling waves and solitary states. The amplitude chimera regime in the network (1), as well as in the individual ensemble of coupled Henon maps, is nonstationary and demonstrates irregular temporal switchings between amplitude and phase chimeras. The lifetime of the amplitude chimera in (1) is, as a rule, finite and, as our studies have demonstrated, it can be controlled by varying the coupling coefficient \( \gamma > 0 \) in fairly wide ranges.

A novel type of chimera state called a solitary state chimera has been revealed in the network (1) for both the dissipative and inertial couplings between the ensembles of coupled Henon and Lozi maps. This chimera state is characterized by the coexistence of an incoherence cluster with uncorrelated chaotic oscillations of the cluster elements and a coherence cluster with synchronous chaotic oscillations.

In conclusion, we have established the effect of mutual synchronization of various spatiotemporal structures in the coupled ensembles (1). The synchrony of oscillations in the elements of the considered structures has been verified by calculating the cross-correlation coefficient \( R_i \) (5), whose values in the synchronization mode are close to unity. Additionally, we have constructed the region of synchronization for a selected spatiotemporal structure in the plane of coupling parameters of the network under study and this result also corroborates the fact that the effect of synchronization is realized in the multilayer system (1).

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