COMPUTATIONAL PROBLEMS ON SOME SYMBOL ALGEBRAS OF PRIME DEGREE

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Abstract. Let $p$ be an odd prime, let $K = \mathbb{Q}(\epsilon)$ where $\epsilon$ is a primitive cubic root of unity, and let $L$ be the Kummer field $\mathbb{Q}(\epsilon, \sqrt[3]{\alpha})$. In this paper we obtain a characterization of the splitting behavior of the symbol algebras $(\frac{\alpha}{p}, K, \epsilon)$ and $(\frac{\alpha^h}{p}, K, \epsilon)$, where $h_p$ is the order in the class group $Cl(L)$ of a prime ideal of $\mathcal{O}_L$ which divides $p\mathcal{O}_L$.

1. Introduction

Let $n \geq 3$ be an arbitrary positive integer, and let $F$ be a field with $\text{char}(F) \nmid n$. Let $\xi$ be a primitive $n$-th root of unity in $F$. If $a, b \in F \setminus \{0\}$, the algebra $A$ over $F$ generated by two elements $x$ and $y$ satisfying

$$x^n = a, y^n = b, yx = \xi xy$$

is called a symbol algebra and it is denoted by $(\frac{a, b}{F, \xi})$. When $n = 2$ we obtain the well known generalized quaternion algebra over the field $F$, and indeed a symbol algebra is a natural generalization of a quaternion algebra. Quaternion algebras and symbol algebras are central simple algebras of dimension $n^2$ over the base field $F$. The results about quaternion algebras and symbol algebras have strong connections with number theory (especially with the ramification theory in algebraic number fields). Different criteria are known for a quaternion algebra or a symbol algebra to split \cite{7, 9, 15}. Explicit conditions for a quaternion algebra over the field of rationals numbers to be split or else a division algebra were studied in \cite{4}. In the paper \cite{13} we investigated the splitting behavior of quaternion algebras over quadratic fields, and of specific symbol algebras over cyclotomic fields. In \cite{14} we found a sufficient condition for a quaternion algebra over a quadratic field to split. Next, in \cite{2} we gave necessary and sufficient conditions for a quaternion algebra $H(\alpha, m)$ to split over a quadratic field $K$, and in \cite{3} we obtained a complete characterization of division quaternion algebras $H(p, q)$, where $p, q$ are prime integers, over the composite $K$ of $n$ quadratic number fields.

In this paper we study some symbol algebras of prime degree over very specific cyclotomic fields. In Section 2 we recall some useful results about

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symbol algebras, cyclotomic fields and Kummer fields which we will use later. In Section 3 we find conditions for a symbol algebra of prime degree over a cyclotomic field to split. Our results have been computationally validated by using the computer algebra packages MAGMA.

2. Some basic results

We recall here the decomposition behavior of a prime integer resp. in the ring of integers of a cyclotomic field and of a Kummer field. We will use these facts later to prove our results.

**Proposition 2.1.** \[8\] Let \( l \geq 3 \) be an integer, \( \xi \) be a primitive root of the unity of order \( l \). Let \( K = \mathbb{Q}(\xi) \) and let \( \mathcal{O}_K \) be the ring of integers of the cyclotomic field \( K \). Then \( \mathcal{O}_K = \mathbb{Z}[\xi] \).

**Theorem 2.2.** \[8\] Let \( l \geq 3 \) be an integer, and let \( \xi \) be a primitive root of the unity of order \( l \). If \( p \) is a prime number which does not divide \( l \) and \( f \) is the smallest positive integer such that \( p^f \equiv 1 \mod l \), then we have \( p\mathbb{Z}[\xi] = P_1P_2\ldots P_r \), where \( r = \varphi(l)/f \), where \( \varphi \) is the Euler’s function and \( P_j, j = 1, \ldots, r \) are different prime ideals in the ring \( \mathbb{Z}[\xi] \).

**Theorem 2.3.** \[11\] Let \( \xi \) be a primitive root of the unity of order \( l \), where \( l \) is a prime number, and let \( A \) be the ring of integers of the Kummer field \( \mathbb{Q}(\xi, \sqrt[l]{\mu}) \). A prime ideal \( P \) of \( \mathbb{Z}[\xi] \) decomposes in \( A \) as follows:

- It is equal to the \( l \)-power of a prime ideal of \( A \), if the \( l \)-power characteristic \( \left( \frac{\mu}{P} \right)_l = 0 \);
- It is a prime ideal of \( A \), if \( \left( \frac{\mu}{P} \right)_l \) is a root of order \( l \) of unity, different from 1;
- It is equal to the product of \( l \) different prime ideals of \( A \), if \( \left( \frac{\mu}{P} \right)_l = 1 \).

Let’s recall now some results about central simple algebras. Let \( A \) be a central simple algebra over a field \( K \). Then the dimension \( n \) of \( A \) over \( K \) is a square; its positive square root is called the degree of the algebra \( A \). If the equations \( ax = b \), \( ya = b \) have unique solutions for all \( a, b \in A \), with \( a \neq 0 \), then the algebra \( A \) is called a division algebra. If \( A \) is a finite-dimensional algebra, then \( A \) is a division algebra if and only if \( A \) has no zero divisors \((x \neq 0, y \neq 0 \Rightarrow xy \neq 0)\).

Let \( L/K \) be a field extension and let \( A \) be a central simple algebra over \( K \). We recall that:

- \( A \) is called split by \( K \) if \( A \) is isomorphic to a full matrix algebra over \( K \);
- \( A \) is called split by \( L \), and \( L \) is called a splitting field for \( A \), if \( A \otimes_K L \) is a full matrix algebra over \( L \).

The following splitting criteria for symbol algebras is known:

**Theorem 2.4.** \[7\] Let \( K \) be a field which contains a primitive \( n \)-th root of unity \( \xi \), and let \( a, b \in K^* \). Then the following statements are equivalent:
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- The cyclic algebra \( A = \left( \frac{a,b}{K,\zeta} \right) \) is split.
- The element \( b \) is a norm from the extension \( K \subseteq K(\sqrt[3]{a}) \).

For symbol algebras of degree prime it is true the following:

**Remark 2.5.** [10] Let \( K \) be a field with \( \text{char} K \neq 2 \) and let \( \alpha, \beta \in K \setminus \{0\} \). Let \( n \) be a positive integer, \( n \geq 3 \) and let \( \zeta \) be a primitive \( n \)-th root of unity. Let \( K \) be a field such that \( \zeta \in K, \alpha, \beta \in K^* \). If \( n \) is prime, then the symbol algebra \( \left( \frac{\alpha, \beta}{K,\zeta} \right) \) is either split or a division algebra.

**Lemma 2.6.** [5] Let \( n \) be a positive integer, \( n \geq 2 \) and let \( \zeta \) be a primitive root of unity of order \( n \). Let \( < \zeta > = \mu_n \subseteq K \). Let \( Br(K) \) be the Brauer group of the field \( K \). Then, the assignment \( (\alpha, \beta) \mapsto \left( \frac{\alpha, \beta}{K,\zeta} \right) \) induces a \( \mathbb{Z} \)-bilinear map \( K^*/(K^*)^n \times K^*/(K^*)^n \mapsto_n Br(K) \).

3. Symbol algebras which split over specific cyclotomic fields

In this section we study the symbol algebras \( \left( \frac{\alpha, p}{K,\zeta} \right) \) when \( \zeta \) is a primitive root of unity of prime order \( q \) and \( L \) is the Kummer field \( L = \mathbb{Q}(\zeta, \sqrt[3]{q}) \), for some particular values of \( c \). We start with a small remark about such algebras:

**Remark 3.1.** Let \( n \) be a positive integer, \( n \geq 3 \), and let \( \zeta \) be a primitive root of order \( n \) of the unity and let \( K = \mathbb{Q}(\zeta) \). Let \( \alpha \in K^* \) and let \( L \) be the Kummer field \( K(\sqrt[n]{\alpha}) \). Then, the symbol algebras \( A = \left( \frac{\alpha, 1}{K,\zeta} \right) \) splits.

**Proof.** By Lemma 2.6 the symbol algebra \( A = \left( \frac{\alpha, 1}{K,\zeta} \right) \) lies in the same class of the algebra \( \left( \frac{\alpha, 1}{K,\zeta} \right) \) in the Brauer group of \( K \). But \( \left( \frac{\alpha, 1}{K,\zeta} \right) \) splits by Theorem 2.4, since 1 is always a norm. \( \square \)

In the paper [6] the authors obtained some results about the symbol algebras of the form \( \left( \frac{\alpha, p^h L}{K,\zeta} \right) \). In that paper there is a small mistake, which we fix in the next proposition and corollary.

**Proposition 3.2.** [6] Prop. 4.1] Let \( \epsilon \) be a primitive cubic root of unity and let \( K = \mathbb{Q}(\epsilon) \). Let \( \alpha \in K^* \) be a cubic residue modulo \( p \) with \( p \neq 3 \) a prime integer. Let \( h_L \) be the class number of the Kummer field \( L = K(\sqrt[3]{\alpha}) \). Then, there exists a unit \( u \in U(\mathbb{Z}[\epsilon]) \) such that the symbol algebra \( A = \left( \frac{\alpha, \epsilon p^h L}{K,\zeta} \right) \) splits.

**Corollary 3.3.** [6] Cor. 4.2] Let \( q \) be an odd prime integer, let \( \xi \) be a primitive root of unity of order \( q \), and let \( K = \mathbb{Q}(\xi) \). Let \( p \neq q \) be a prime integer and let \( \alpha \in K^* \) be a \( q \) power residue modulo \( p \). Let \( h_L \) be the class number of the Kummer field \( L = K(\sqrt[q]{\alpha}) \). Then, there exists a unit \( u \in \)
\( U(\mathbb{Z}[\xi]) \) such that the symbol algebra \( A = \left( \frac{\alpha \cdot \eta^h L}{K, \xi} \right) \) splits.

In the next two propositions we show how to generalize the previous results, by replacing \( h_L \) with some divisors \( h \) of \( h_L \).

**Proposition 3.4.** Let \( \epsilon \) be a primitive cubic root of unity and let \( K = \mathbb{Q}(\epsilon) \). Let \( \alpha \in K^* \) be a cubic residue modulo \( p \) with \( p \neq 3 \) a prime integer. Let \( h_L \) be the class number of the Kummer field \( L = K(\sqrt[3]{\alpha}) \). Let \( h_p \) be the order of the class of a prime ideal in \( \mathcal{O}_L \), which divides \( p\mathcal{O}_L \), in the class group \( Cl(L) \). Then, there exists a unit \( u \in U(\mathbb{Z}[\epsilon]) \) such that the symbol algebra \( A = \left( \frac{\alpha \cdot \eta^h L}{K, \xi} \right) \) splits.

**Proof.** It is known that \( \mathcal{O}_K = \mathbb{Z}[\epsilon] \) is a principal ideal domain. We split the proof into two cases:

- \( p \equiv 2 \pmod{3} \).
  - From Theorem 2.4 it follows that \( p \) remains prime in the ring \( \mathcal{O}_K \).
  - Since the cubic residual symbol \( \left( \frac{\alpha}{p} \right)_3 \) is equal to 1, from Theorem 2.3 it follows that we have the following decomposition of the ideal \( p\mathcal{O}_L \) as a product of prime ideals in \( \mathcal{O}_L \):
    \[ p\mathcal{O}_L = P_1 P_2 P_3, \]
  - Let \( h_p \) be the order of the ideal class of \( P_1 \) in the group \( Cl(L) \).
  - Since the ideals \( P_1, P_2, P_3 \) are conjugate under the action of the Galois group \( \text{Gal}(L/K) \), it follows that the ideals classes of \( P_1, P_2, P_3 \) have the same order \( h_p \) in the group \( Cl(L) \).
  - Now \( (p\mathcal{O}_L)^{h_p} = P_1^{h_p} P_2^{h_p} P_3^{h_p} = (\beta \mathcal{O}_L)(\sigma(\beta) \mathcal{O}_L)(\sigma^2(\beta) \mathcal{O}_L) \) for some \( \beta \in \mathcal{O}_L \), where \( \sigma \) is a generator of \( \text{Gal}(L/K) \). Hence \( P^{h_p} \mathcal{O}_L = N_{L/K}(\beta) \mathcal{O}_L \).
  - This means that \( P^{h_p} \) and \( N_{L/K}(\beta) \) differ by a unit of \( \mathcal{O}_L \), but, since \( P^{h_p} \) is an element of \( \mathcal{O}_L \) with \( N_{L/K}(\beta) \in K \), they really differ by a unit of \( \mathcal{O}_K \). So, there exists a unit \( u \in U(\mathbb{Z}[\epsilon]) \) such that \( u \cdot P^{h_p} = N_{L/K}(\beta) \).
    - According to Theorem 2.4 the symbol algebra \( A = \left( \frac{\alpha \cdot \eta^h L}{K, \xi} \right) \) splits.

- \( p \equiv 1 \pmod{3} \).
  - According to Theorem 2.2 we have \( p\mathcal{O}_K = p_1 \mathcal{O}_K \cdot p_2 \mathcal{O}_K \), where \( p_1, p_2 \) are prime elements in \( \mathcal{O}_K \). Since \( \alpha \) is a cubic residue modulo \( p \), it follows that \( \alpha \) is a cubic residue modulo \( p_1 \) and \( \alpha \) is a cubic residue modulo \( p_2 \). From Theorem 2.3 we get:
    \[ p\mathcal{O}_L = p_1 \mathcal{O}_L p_2 \mathcal{O}_L = P_{11} P_{12} P_{13} P_{21} P_{22} P_{23}, \]
  - where \( P_{1j} \) and \( P_{2j} \) are prime ideals in \( \mathcal{O}_L \). These ideals are conjugate under the action of the Galois group, so their classes have the same order \( h_p \). Hence
    \[ (p\mathcal{O}_L)^{h_p} = (P_{11} P_{21})^{h_p} (P_{12} P_{22})^{h_p} (P_{13} P_{23})^{h_p}, \]
so there is an element \( \gamma \in \mathcal{O}_L \) such that \( p^h \mathcal{O}_L = N_{L/K}(\gamma) \mathcal{O}_L \). It follows that there exists a unit \( u \in U(\mathbb{Z}[\xi]) \) such that \( u - p^h = N_{L/K}(\gamma) \).

From Theorem 2.4 it follows that the symbol algebras \( A = \left( \frac{\alpha, u\cdot p^h L}{K, \xi} \right) \) splits.

\[ \Box \]

By using the package MAGMA \([12]\) we found some examples of split symbol algebras which satisfy the hypotheses of the previous theorems.

- Let \( K = \mathbb{Q}(\xi) \), where \( \xi^3 = 1, \xi \neq 1 \); the class number of Kummer field \( L = \mathbb{Q}(\xi, \sqrt{43}) \) is 48. The ideal \( 23 \mathcal{O}_L \) decomposes into the product of three prime ideals of \( L \), \( P_1, P_2, P_3 \) in the notations of our example. We denote with \( [I] \) be the class of the ideal \( I \) in the class group of \( L \). We have \( \text{ord} ([P_3]) = \text{ord} ([P_1]) = \text{ord} ([P_2]) = 12 \). The norm equation \( 23^{12} = N_{L/\mathbb{Q}(\xi)}(a) \) has solutions, but the norm equations \( 23^2 = N_{L/\mathbb{Q}(\xi)}(a), 23 = N_{L/\mathbb{Q}(\xi)}(a) \) does not have any. This example agrees with the assertion of Proposition 3.4.

- Let \( K = \mathbb{Q}(\xi) \), where \( \xi^3 = 1, \xi \neq 1 \); the class number of Kummer field \( L = \mathbb{Q}(\xi, \sqrt{43}) \) is 48. The ideal \( 11 \mathcal{O}_L \) decomposes into the product of three prime ideals of \( L \), \( P_1, P_2, P_2 \). We have \( \text{ord} ([P_3]) = \text{ord} ([P_1]) = \text{ord} ([P_2]) = 2 \). The norm equation \( 11^2 = N_{L/\mathbb{Q}(\xi)}(a) \) has solutions. This example agrees again with the assertion of Proposition 3.4.

The same argument used above leads to the following result:

**Proposition 3.5.** Let \( q \) be an odd prime positive integer, \( q \geq 5 \), and \( \xi \) be a primitive root of order \( q \) of unity and let the cyclotomic field \( K = \mathbb{Q}(\xi) \). Let \( \alpha \in K^* \), \( p \) be a prime rational integers, \( p \neq q \) and let \( L = K(\sqrt[3]{\alpha}) \) be the Kummer field such that \( \alpha \) is a \( q \) power residue modulo \( p \). Let \( \mathcal{O}_L \) be the ring of integers of the field \( L \). Let \( \text{Cl}(L) \) be the ideal class group of the ring \( \mathcal{O}_L \), let \( h_L \) be the class number of \( L \) and let \( h_p \) the order of a class of a prime ideal in \( \mathcal{O}_L \), which divides \( p \mathcal{O}_L \), in the group \( \text{Cl}(L) \). Then, there exists a unit \( u \in U(\mathbb{Z}[\xi]) \) such that the symbol algebra \( A = \left( \frac{\alpha, u\cdot p^h L}{K, \xi} \right) \) splits.

The proof of the next proposition follows very closely the proof of the first case of Proposition 3.4.

**Proposition 3.6.** Let \( q \) be an odd prime positive integer and \( \xi \) be a primitive root of order \( q \) of unity and let the cyclotomic field \( K = \mathbb{Q}(\xi) \). Let \( \alpha \in K^* \) and let \( L = K(\sqrt[3]{\alpha}) \) be the Kummer field. Let \( \pi \in \mathbb{Z}[\xi] \) be a prime element in the ring \( \mathbb{Z}[\xi] \) such that \( \alpha \) is a \( q \) power residue modulo \( \pi \). Let \( \mathcal{O}_L \) be the ring of integers of the field \( L \). Let \( \text{Cl}(L) \) be the ideal class group of the ring \( \mathcal{O}_L \), let \( h_L \) be the class number of \( L \) and let \( h_\pi \) be the order of a class of a prime ideal in \( \mathcal{O}_L \), which divides \( \pi \mathcal{O}_L \) in the group \( \text{Cl}(L) \). Then, there exists a unit \( u \in U(\mathbb{Z}[\xi]) \) such that the symbol algebra \( A = \left( \frac{\alpha, u\cdot \pi^h L}{K, \xi} \right) \) splits.
In Proposition 3.4 we obtained a sufficient condition for a symbol algebra \( \left( \frac{\alpha, u \cdot p}{K, \epsilon} \right) \) to split over the cyclotomic field \( K = \mathbb{Q}(\epsilon) \), where \( \epsilon \) is a primitive cubic root of unity. Next, let’s ask ourselves if this condition is also necessary for the symbol algebra \( \left( \frac{\alpha, u \cdot p}{K, \epsilon} \right) \) to split. The answer is affirmative, according to the following result.

**Proposition 3.7.** Let \( \epsilon \) be a primitive root of order 3 of unity and let \( K = \mathbb{Q}(\epsilon) \) be the cyclotomic field. Let \( \alpha \in K^* \), \( p \) be a prime rational integers, \( p \neq 3 \) and let \( L = K(\sqrt[3]{\alpha}) \) be the Kummer field. Let \( \mathcal{O}_L \) be the ring of integers of the field \( L \). Let \( \mathcal{Cl}(L) \) be the ideal class group of the ring \( \mathcal{O}_L \), let \( h_L \) be the class number of \( L \) and let \( h_p \) the order of a class of a prime ideal in \( \mathcal{O}_L \), which divides \( p \mathcal{O}_L \), in the group \( \mathcal{Cl}(L) \). Then, there exists a unit \( u \in U(\mathbb{Z}[\epsilon]) \) such that the symbol algebra \( A = \left( \frac{\alpha, u \cdot p}{K, \epsilon} \right) \) splits if and only if \( \alpha \) is a cubic residue modulo \( p \).

**Proof.** We have proved the sufficiency in Proposition 3.4, hence all we need to do is to prove the necessity. Let’s assume that there exist a unit \( u \in U(\mathbb{Z}[\epsilon]) \) such that the symbol algebra \( A = \left( \frac{\alpha, u \cdot p}{K, \epsilon} \right) \) splits, where \( \alpha \) is not a cubic residue modulo \( p \) - this will lead to a contradiction. Clearly \( p \equiv 1 \pmod{3} \) and \( \alpha \equiv \frac{p^4}{\epsilon^3} \not\equiv 1 \pmod{3} \). Since \( \mathcal{O}_K = \mathbb{Z}[\epsilon] \) is a principal ideal domain, from Theorem 2.2 and Theorem 2.3 we get

\[
(3.1) \quad p \mathcal{O}_K = p_1 \mathcal{O}_K \cdot p_2 \mathcal{O}_K,
\]

where \( p_1, p_2 \) are prime elements from \( \mathcal{O}_K \) which remain primes in \( \mathcal{O}_L \), so

\[
(3.2) \quad p \mathcal{O}_L = P_1 P_2,
\]

where \( P_1 = p_1 \mathcal{O}_L, P_2 = p_2 \mathcal{O}_L \). Since \( P_1 \) and \( P_2 \) are principal ideals of \( \mathcal{O}_L \), it follows that \( h_p = 1 \), so

\[
(3.3) \quad (p \mathcal{O}_L)^{h_p} = P_1 P_2,
\]

Now \( p_1 \mathcal{O}_K \) and \( p_2 \mathcal{O}_K \) are conjugate under the action of the Galois group \( \text{Gal}(K/\mathbb{Q}) \). However, the right side of the equality (3.3) is not a product of all the ideals which are conjugate under the action of the Galois group \( \text{Gal}(L/K) \), hence it can not be the norm of an ideal in the field extension \( L/K \). *A fortiori* the ideal \( P_1 P_2 \) which is a principal ideal of \( \mathcal{O}_L \) can not be generated by the norm of an element of \( L \). According to Theorem 2.4, the symbol algebra \( A = \left( \frac{\alpha, u \cdot p}{K, \epsilon} \right) \) does not split, contradicting our hypothesis. Therefore \( \alpha \) must be a cubic residue modulo \( p \). \[\square\]

In the paper [13] we obtained the following result:

**Proposition 3.8.** [13] Thm. 3.7 Let \( p \) and \( q \) be prime positive integers such that \( p \equiv 1 \pmod{q} \), let \( \xi \) be a primitive root of order \( q \) of unity and let \( K = \mathbb{Q}(\xi) \). Then there is an integer \( \alpha \) not divisible by \( p \) whose residue class mod \( p \) does not belong to \( \left( \mathbb{F}_p^* \right)^q \), and for every such an \( \alpha \), we have:
• the algebra $A \otimes_{K} Q_{p}$ is a division algebra over $Q_{p}$, where $A$ is the symbol algebra $A = \left( \frac{\alpha \cdot p}{K, \xi} \right)$;

• the symbol algebra $A$ is a division algebra over $K$.

We consider now a Kummer field with class number 1, for example $L = Q(\epsilon, \sqrt[3]{5})$, where $\epsilon^3 = 1$, $\xi \neq 1$. We consider the prime integers 17 and 19. Now, 5 is a cubic residue modulo 17, but 5 is not a cubic residue modulo 19. By using again the computer algebra system MAGMA, we get that the norm equation $17 = N_{L/Q(\epsilon)}(a)$ has solutions, but the norm equation $N_{L/Q(\epsilon)}(a) = 19$ does not have a solution.

From Proposition 3.2 and Proposition 3.8 we obtain in a very particular situation, i.e. when $L$ is a Kummer field of class number 1, a necessary and sufficient condition for a symbol algebra to split over the third cyclotomic field.

**Proposition 3.9.** Let $\epsilon$ be a primitive root of order 3 of unity and let $K = Q(\epsilon)$ be the cyclotomic field. Let $\alpha \in K^{*}$, $p$ a prime rational integer, $p \neq 3$ and let $L = K(\sqrt[3]{\alpha})$ be a Kummer field with $h_{L} = 1$. Then, there exists a unit $u \in U(\mathbb{Z}[\epsilon])$ such that the symbol algebra $A = \left( \frac{\alpha \cdot u \cdot p}{K, \epsilon} \right)$ splits if and only if $\alpha$ is a cubic residue modulo $p$.

**Proof.** In order to prove the necessity, we note that, according to Remark 2.5 with $u = 1$, the symbol algebra $A = \left( \frac{\alpha \cdot p}{K, \epsilon} \right)$ splits if and only if $A$ is not division algebra over $K$. From Proposition 3.8 it follows that $\alpha$ is a cubic residue modulo $p$ or $p$ is not congruent to 1 modulo 3. But, if $p$ is not congruent to 1 modulo 3 and $p \neq 3$, it follows that $p\equiv 2 \pmod{3}$ and this implies that $\left( \frac{\alpha}{p} \right)_3$ is equal to 1. Hence, from our previous results, we obtain that $\alpha$ is a cubic residue modulo $p$.

In order to prove the sufficiency, we note that, if $\alpha$ is a cubic residue modulo $p$, by applying Proposition 3.2 with $h_{L} = 1$ it follows that there exists a unit $u \in U(\mathbb{Z}[\epsilon])$ such that the symbol algebra $A = \left( \frac{\alpha \cdot u \cdot p}{K, \epsilon} \right)$ splits. □

In the future we will try to generalize the results contained in Proposition 3.7 and Proposition 3.9 to Kummer fields $L = Q(\xi, \sqrt[3]{\alpha})$, where $l \geq 5$ is a prime integer and $\xi$ is a primitive root of order $l$ of the unity.

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