Odd-periodic Grover Walks

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Abstract
The Grover walk is one of the most well-studied quantum walks on graphs. In this paper, we investigate its periodicity to reveal the relationship between the quantum walk and the underlying graph, focusing particularly on the characterization of graphs exhibiting a periodic Grover walk. Graphs having a periodic Grover walk with periods of 2, 3, 4, and 5 have previously been characterized. It is expected that graphs exhibiting a periodic Grover walk with odd period correspond to cycles with odd length. We address that problem and are able to perfectly characterize the class of graphs exhibiting an odd-periodic Grover walk by using a combinatorial method.

Keywords Grover walk · Periodicity · Characteristic polynomial · Matching

Mathematics Subject Classification 05C50 · 05C81 · 81Q99

1 Introduction
A quantum walk is a quantum analog of a random walk [2, 10], and it has been applied to several research fields, including graph theory, quantum physics, and computer science. Details can be found in [8, 18, 20, 25, 26]. Quantum walks on graphs are currently in the limelight because of their ability to show us specific characteristics that are not seen in classical random walks, and these specific characteristics are the focuses of this paper. Specifically, we address periodicity of a discrete-time quantum walk. If there exists a positive integer $k$ such that the time evolution $U$ satisfies $U^k = I$, where $I$ is the identity operator, then the quantum walk is periodic. In other words, any initial state $\varphi_0$ returns to itself after $k$ applications of the time evolution. If there exists such an integer, then the minimum one is called the period.

One of motivation to study the periodicity is to see discrete analogy of the perfect state transfer [6, 9]. If a quantum walk is periodic, then any state returns to itself with
finite time so the perfect state transfer often occurs in every point. It is applied to the field of quantum cryptography [19]. From the viewpoint of quantum searching, the periodicity is important. In quantum searching, existence of a state becoming asymptotically periodic by application of a time evolution operator gives speed-up [4, 23]. Our study is to study the case where any state becomes exactly periodic. Hence, characterizing graph in which quantum walk becomes periodic sometimes helps to solve such a searching problem.

Nowadays, research on the periodicity of discrete-time quantum walks is a topic of much interest, especially its ability to characterize graphs exhibiting a periodic quantum walk. Konno et al. [15] considered the periodicity of the Hadamard walk in cycles and was able to determine conditions that have periodic Hadamard walks. Additionally, Higuchi et al. [11] discussed periodicities of discrete-time quantum walks in several graphs with high symmetry, including complete graphs, complete bipartite graphs, strongly regular graphs, and cycles. Moreover, Saito [22] formulated the Fourier walk on cycles and addressed the periodicity of it. Kubota et al. [17] proposed the staggered walk on a generalized line graph induced by a Hoffman graph and addressed its periodicity.

In this paper, we study the periodicity of the Grover walk, which is strongly related to the structure of the underlying graph. We call a graph exhibiting a periodic Grover walk a periodic graph. The Grover walk is only defined by the underlying graph, and it strongly reflects the graph structure. If a graph is periodic, then its structure is sometimes restricted. For example, elements of the graph structure, such as the degree and the diameter, are restricted if the Grover walk on the graph is periodic. Such restriction is seen in [16, 28]. Thus, the Grover walk is a suitable model for revealing the relation between graph structure and its induced quantum walk. Yoshie [27] studied its periodicity and found classes of graphs exhibiting a periodic Grover walk with a given period. The results for periodic graphs are summarized in the following table, where $P_n$ and $C_n$ are the path graph on $n$ vertices and the cycle graph on $n$ vertices, respectively. The researcher in [27] studied the characterization of periodic graphs with odd period. Using a relatively easy calculation, they demonstrated that every cycle graph with odd length induces a periodic Grover walk of odd period, where the length of the cycle is the number of the vertices. Hence, being a cycle graph with odd length is a sufficient condition for inducing an odd-periodic Grover walk. However, demonstrating its necessity is not so easy. It is thought that cycle graphs with odd length are the only ones that admit an odd-periodic Grover walk (i.e., necessity holds). We address this problem in this paper and show that the statement is true. Specifically, we

| Period | Graph |
|--------|-------|
| 2      | $P_2$ |
| 3      | $C_3$ |
| 4      | Every complete bipartite graph |
| 5      | $C_5$ |
| Over 6 | Still open |

Table 1 List of periodic graphs

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show that an odd-periodic graph with period $k$ is the only cycle graph with length $k$ for an odd $k$. This is formally stated in the following theorem.

**Theorem 1.1** Let $k$ be an odd integer. A graph $G$ is periodic with period $k$ if and only if $G = C_k$.

Therefore, we completely characterize periodic graphs with an odd period. This statement implies that the period of almost every periodic graph is even.

The remainder of this paper is organized as follows. In Sect. 2, we give definitions of graphs and the Grover walk and introduce the periodicity. Additionally, we give the transition matrix of the graphs, which plays an important role in this paper. Using them, we prepare spectral tools: eigenvalues and the coefficient of the characteristic polynomial of the transition matrix, for example. After that, we state the main theorem of the paper. In Sect. 3, we prove our main theorem. Using the tools we prepared, we control the inner structure of graphs and bring them closer to a cycle with odd length. Section 4 is devoted to the summary of our work and to discussion of our future research goals.

2 Preliminaries

2.1 Graph

In this paper, we only treat simple, connected, and finite graphs. Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. If two vertices $u$ and $v$ are adjacent by an edge, then we denote this as $u \sim v$. For $u \in V$, we define $\deg u := |\{v \mid v \sim u\}|$. For $uv \in E$, we denote the arc from $u$ to $v$ as $(u, v)$. Moreover, the origin and the terminus of $e = (u, v)$ are denoted by $o(e)$ and $t(e)$, respectively. We denote by $\bar{e}$ the inverse arc of $e$. Define $\mathcal{A} := \{(u, v), (v, u) \mid uv \in E\}$. In this paper, we denote $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{C}$ and $\mathbb{Z}$ by the sets of natural, rational, complex numbers and integers, respectively.

A graph containing no odd cycle as a subgraph is called a bipartite graph. Furthermore, an **odd unicycle graph** is a non-bipartite graph satisfying $|V| = |E|$. In other words, an odd unicycle graph is a graph containing exactly one odd cycle as a subgraph (examples are seen in Fig. 1). The length of the cycle is called the **girth**. For $k \in \mathbb{N}$, we define $\mathcal{O}(k)$ to be the class of odd unicycle graphs of girth $k$. Note that the cycle graph $C_k$ is a member of $\mathcal{O}(k)$. Additionally, a graph containing no cycle as a subgraph is called a **tree**. Furthermore, the set of disjoint edges of $G$ is called a **matching**. If the number of edges of a matching is $k$, then it is called a **$k$-matching**. Throughout this paper, we denote the $n \times n$ identity matrix as $I_n$. 
2.2 Grover walk and its periodicity

In this section, we define the time evolution operator of the Grover walk and consider its periodicity. For a graph $G$, let us define $U = \{ U_{e,f} \}$ on $\mathbb{C}^{|A|}$ by

$$U_{e,f} = \begin{cases} 2/\deg t(f), & \text{if } t(f) = o(e), e \neq \bar{f} \\ 2/\deg t(f) - 1, & \text{if } e = \bar{f} \\ 0, & \text{otherwise.} \end{cases}$$

(1)

Let $\varphi_0 \in \mathbb{C}^{|A|}$ be the initial state. Then the state at time $k$, $\varphi_k$, is given by $\varphi_k = U^k \varphi_0$. If there exists $k \in \mathbb{N}$ such that $U^k = I_{|A|}$,

then the Grover walk is periodic. If the Grover walk on a graph $G$ is periodic, then the minimum integer satisfying the condition in (2) is called the period. If the period is specified as $k$, then we call the graph a $k$-periodic graph, but if the period of the Grover walk induced by a periodic graph $G$ is odd (resp. even), then we call $G$ an odd-(resp. even-) periodic graph.

2.3 Transition matrix and its characteristic polynomial

Let us define the transition matrix $T = \{ T_{u,v} \}$ on $\mathbb{C}^V$ induced by a graph $G$ as

$$T_{u,v} = \begin{cases} 1/\deg u, & \text{if } u \sim v \\ 0, & \text{otherwise}, \end{cases}$$

(3)

which is a weighted adjacency matrix expressing the isotropic random walk on $G$. Note that

$$T f(v) = \sum_{u \sim v} \frac{1}{\deg v} f(u)$$
for $f \in \mathbb{C}^{[V]}$, where $f(u)$ is the entry of $f$ corresponding to $u$. Let $\text{Spec}(\cdot)$ be the set of eigenvalues. We can now present a spectral mapping theorem for the Grover walk.

**Theorem 2.1** (Higuchi and Segawa [12]) Let $U$ be defined as above. Then the set of eigenvalues of $U$ can be expressed as

$$\text{Spec}(U) = \{e^{\pm i \cos^{-1}(\text{Spec}(T))}\} \cup \{1\} \cup \{-1\}.$$ 

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $U$. If it holds that there exists $k_i \in \mathbb{N}$ such that

$$\lambda_i^{k_i} = 1$$

and

$$\lambda_i^j \neq 1, \ j < k_i$$

for $1 \leq i \leq n$, then $G$ is periodic and the period is given by

$$\text{lcm}(k_1, \ldots, k_n),$$

where $\text{lcm}(\cdot, \ldots, \cdot)$ is the least common multiple. If $G$ is $k$-periodic, then it follows from Theorem 2.1 that $\lambda^k = 1$ for any $\lambda \in \text{Spec}(U)$. From the part of the spectrum given by $\{e^{\pm i \cos^{-1}(\text{Spec}(T))}\}$, we obtain the following.

**Corollary 2.2** Let $T$ be defined as above. A graph $G$ is periodic if and only if

$$\cos^{-1}(\text{Spec}(T)) \subset \pi \mathbb{Q}.$$ 

We now introduce another tool for investigating the periodicity. Let $n = |V|$, and let $\rho_j$ be the coefficient of the characteristic polynomial of $T$ for $0 \leq j \leq n$, that is,

$$\det (xI_n - T) = \sum_{j=0}^{n} \rho_j x^j.$$ 

**Lemma 2.3** (Yoshie [28]) If $G$ is periodic, then

$$2^j \rho_{n-j} \in \mathbb{Z}$$

for $0 \leq j \leq n$.

Recall the definition of the determinant. Let $Y = xI_n - T$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then it holds that

$$\det Y = \sum_{\sigma} \text{sgn}(\sigma) Y_{v_1, \sigma(v_1)} Y_{v_2, \sigma(v_2)} \cdots Y_{v_n, \sigma(v_n)},$$

$	ext{sgn}$ Springer
where $\sigma$ runs over the permutations on $V(G)$, and $\text{sgn}(\cdot)$ is the signature. Additionally, we denote by $|\sigma|$ the length of $\sigma$, and a permutation with length 2 is called a *transposition*. We call permutations like

$$\sigma = \begin{pmatrix} v_1 & v_2 & \ldots & v_r \\ v_2 & v_3 & \ldots & v_1 \end{pmatrix}$$

and

$$\sigma^{-1} = \begin{pmatrix} v_1 & v_2 & \ldots & v_r \\ v_r & v_1 & \ldots & v_{r-1} \end{pmatrix}$$

cyclic permutations with length $r$. The permutation $\sigma$ maps $v_i$ to $v_{i+1}$ for $1 \leq i \leq r-1$ and $v_r$ to $v_1$. Note that

$$Y_{v_i, \sigma(v_i)} = \begin{cases} x, & \text{if } v_i = \sigma(v_i) \\ -\frac{1}{\deg v_i}, & \text{if } v_i \sim \sigma(v_i) \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the permutations contributing to the coefficient of $x^{n-j}$ in (6) are expressed by a product of cyclic permutations $\sigma_1, \sigma_2, \ldots, \sigma_k$ so that $|\sigma_1| + |\sigma_2| + \cdots + |\sigma_k| = j$. Hence, a permutation like

$$\sigma = \begin{pmatrix} v_1^{(1)} & v_2^{(1)} & \ldots & v_{j_1}^{(1)} \\ v_2^{(1)} & v_3^{(1)} & \ldots & v_1^{(1)} \\ & v_2^{(2)} & v_3^{(2)} & \ldots & v_{j_2}^{(2)} \\ & & v_2^{(k)} & v_3^{(k)} & \ldots & v_{j_k}^{(k)} \end{pmatrix}$$

with $\{v_1^{(i)}, v_2^{(i)}, \ldots, v_{j_i}^{(i)}\} \cap \{v_1^{(l)}, v_2^{(l)}, \ldots, v_{j_l}^{(l)}\} = \phi$ for $i \neq l$ and $v_1^{(i)} \sim v_2^{(i)} \sim \cdots \sim v_{j_i}^{(i)} \sim v_1^{(i)}$, and $j_1 + j_2 + \cdots + j_k = j$ contributes the term

$$\text{sgn}(\sigma) \prod_{i=1}^{k} \prod_{s=1}^{j_i} \left( -\frac{1}{\deg v_s^{(i)}} \right)$$

(7)

to the coefficient of $x^{n-j}$. Thus, $\rho_{n-j}$ is obtained by summing (7) over all permutation with length $j$. A cyclic permutation and a transposition correspond to a cycle and an edge in the underlying graph. Thus, computing the coefficient of the characteristic polynomial is equivalent to finding a combination of these cycles and matchings.

### 2.3.1 Example

Let us compute the coefficients of the characteristic polynomial of $T$ of $G = C_4$ with $V(G) = \{v_1, v_2, v_3, v_4\}$. With a proper labeling of the vertices, we have
\[ T = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}. \]

Put

\[ \det (xI_d - T) = \sum_{j=0}^{4} \rho_j x^j. \]

It clearly holds that \( \rho_4 = 1 \) and \( \rho_3 = \text{Tr}(T) = 0 \). Let us consider \( \rho_2 \). This coefficient is contributed by transpositions on two adjacent vertices. Such permutations are

\[ (v_1 v_2), (v_2 v_3), (v_3 v_4), (v_4 v_1) \]

and we draw the edge corresponding to a transposition which switches the endpoint vertices of the edge by a thick line in Fig. 2. Since the signature of a transposition is \(-1\) and the degree of the vertices are 2, it follows form (7) that such a combination of transpositions gives

\[ (-1) \cdot \left(\frac{-1}{2}\right) \cdot \left(\frac{-1}{2}\right) \times 4 = -1, \]

which coincides with \( \rho_2 \). Here, there is no cycles with length 3 in \( G \) and it implies that there is no cyclic permutations on three vertices which are mutually adjacent. Thus, \( \rho_1 = 0 \). Finally, we compute \( \rho_0 \). This is contributed by permutations on four vertices in \( G \). Such permutations are cyclic ones on all the vertices, the inverses of these cyclic permutations, and the combinations of two transpositions on a pair of distinct two vertices. Then the permutations are

\[ (v_1 v_2 v_3 v_4), (v_1 v_2 v_3 v_4) \]

and

\[ (v_1 v_2), (v_3 v_4), (v_1 v_4), (v_2 v_3). \]

Such permutations are seen in Fig. 3. The signatures of the cyclic permutations are \(-1\). and these permutations give

\[ (-1) \cdot \left(\frac{-1}{2}\right) \cdot \left(\frac{-1}{2}\right) \cdot \left(\frac{-1}{2}\right) \cdot \left(\frac{-1}{2}\right) \times 2 = -\frac{1}{8}. \]
The signatures of the combinations of two transpositions are $+1$ and these permutations give

$$(+1) \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{1}{2}\right) \times 2 = \frac{1}{8}. $$

Thus, it follows from (7) that $\rho_0 = -\frac{1}{8} + \frac{1}{8} = 0$.

### 3 Proof of Theorem 1.1

#### 3.1 Outline of the proof

In this section, we prove Theorem 1.1 which is the main result in this paper. We will now give an outline of the proof. We firstly introduce a graph which is not odd-periodic seen in Fig. 5. Let the class of the periodic graphs and the odd unicycle graphs be $P$ and $O$, respectively. We divide $P$ into $P_o$: odd-periodic graphs, and $P_e$: even-periodic graphs. Then $P = P_o \cup P_e$ and $P_o \cap P_e = \phi$. Theorem 3.2 says a known fact that $P_o \subset O$. Next, we divide $O$ into the class of the odd cycles, say $O_1$, and the one of the odd unicycle graphs satisfying the condition (ii) in Theorem 3.3, say $O_2$. Then $O = O_1 \cup O_2$ and $O_1 \cap O_2 = \phi$. Our result is equivalent to $P_o \subset O_1$. To this end, we show that

$$P \cap O_2 \subset P_e. \hspace{1cm} (8)$$

In the next part, we give Theorems 3.5, 3.8 and 3.9 which give the proof of (8). These theorems mention conditions of graphs in $P \cap O_2$. It follows form the conditions that such graphs are not members of $P_o$. Indeed, we obtain that the graphs in $P \cap O_2$ can only be the ones in Fig. 5. To demonstrate this main theorem, we prove Theorems 3.5 and 3.8 using Lemmas 3.4, 3.6, and 3.7 (the outline of the proof is seen in Fig. 4).
3.2 Non-odd-periodic graph

Here, we present a non-odd-periodic graph that plays an important role. Let us introduce a Chebyshev polynomial. We inductively define a sequence of polynomials \( \{U_n(x)\}_{n=0}^{\infty} \) through the following process: \( U_0(x) = 1, \) \( U_1(x) = 2x, \) and

\[
U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \tag{9}
\]

for \( n \geq 1. \) This is the Chebyshev polynomial of the second kind. The following is a well-known property of the Chebyshev polynomial:

\[
U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}.
\]

It is convenient to set \( U_{-1}(x) = -1. \) More detailed properties of the Chebyshev polynomials can be found in Rivlin [21].

We construct a graph \( G \) by connecting an odd cycle \( C_k \) and two path graphs \( P_r \) with the same length. Now, we identify a vertex in the cycle \( C_k \) and the endpoints of two paths \( P_r \) as a single vertex \( u. \) Let \( V(G) = \{u, v_1, \ldots, v_{k-1}, s_1, \ldots, s_{r-1}, w_1, \ldots, w_{r-1}\} \) and

\[
E(G) = \{v_{i-1}v_i \mid 1 \leq i \leq k\} \cup \{s_{i-1}s_i \mid 1 \leq i \leq r - 1\} \cup \{w_{i-1}w_i \mid 1 \leq i \leq r - 1\},
\]

where \( v_0 = v_k = s_0 = w_0 = u \) (see Fig. 5).

**Theorem 3.1** The graph illustrated in Fig. 5 is not an odd-periodic graph.

**Proof** For \( 1 \leq l \leq r - 1, \) set \( \lambda_l = \cos \frac{2l-1}{2(r-1)} \pi. \) Define \( f_l \in \mathbb{C}^{|V|} \) as

\[
\begin{align*}
    f_l(s_j) &= U_{j-1}(\lambda_l), \\
    f_l(w_j) &= -U_{j-1}(\lambda_l), \\
    f_l(v_i) &= 0
\end{align*}
\]
for $1 \leq i \leq k$ and $1 \leq j \leq r - 1$. We show that $f_i$ is an eigenvector of $T$ associated with $\lambda_j$. First, it clearly holds that $T f_i(v_i) = \lambda_i f_i(v_i)$ for $1 \leq i \leq k - 1$. Next, we have

$$T f_i(u) = \frac{1}{4}(f_i(v_1) + f_i(v_{k-1}) + f_i(s_1) + f_i(w_1)) = \frac{1}{4}(0 + 0 + U_0(\lambda_i) - U_0(\lambda_i)) = 0 = \lambda_i f_i(u).$$

Moreover, it holds that

$$T f_i(s_j) = \frac{1}{2}(f_i(s_{j+1}) + f_i(s_{j-1})) = \frac{1}{2}(U_j(\lambda_i) + U_{j-2}(\lambda_i)) = \frac{1}{2}(2\lambda_i U_{j-1}(\lambda_i) - U_{j-2}(\lambda_i) + U_{j-2}(\lambda_i)) = \lambda_i U_{j-1}(\lambda_i) = \lambda_i f_i(s_j)$$

for $1 \leq j \leq r - 2$. Setting $j = r - 1$, we have

$$T f_i(s_{r-1}) = \frac{1}{1} f_i(s_{r-2}) = U_{r-3}(\lambda_i)$$
Similarly, it holds that $T f_l(w_j) = \lambda_l f_l(w_j)$ for $1 \leq j \leq r - 1$. Thus, it holds that $T f_l = \lambda_l f_l$, so $f_l$ is an eigenvector associated with $\lambda_l$. Then

$$\left\{ \cos \frac{2l - 1}{2(r - 1)} \pi \mid 1 \leq l \leq r - 1 \right\} \subset \text{Spec}(T).$$

Note that $\cos \frac{2l - 1}{2(r - 1)} \pi$ is the real part of a $4(r - 1)$-th root of unity. By (4), the period is a multiple of $4(r - 1)$ if $G$ is periodic. Therefore, $G$ is not odd-periodic.

\[ \top \top \]

### 3.3 Necessary condition for graphs to be odd-periodic

A necessary condition for graphs to be odd-periodic can be found in [27].

**Theorem 3.2** (Yoshie [27]) If $G$ is odd-periodic, then $G$ is an odd unicycle graph.

We want to obtain a stronger condition for necessity. Suppose that a graph $G$ is odd-periodic. It follows from Theorem 3.2 that $G$ is an odd unicycle graph. In other words, $G \in \mathcal{O}(k)$ for an odd integer $k \geq 3$. Define $u_0, u_1, \ldots, u_{k-1}$ as the vertices of the unique cycle in $G$. This is the unique combination of vertices satisfying $u_0 \sim u_1 \sim \cdots \sim u_{k-1} \sim u_0$. First, we prove the following statement.

**Theorem 3.3** If $G \in \mathcal{O}(k)$ for an odd $k$ is periodic, then one of the following holds:

(i) $\deg u_0 = \deg u_1 = \cdots = \deg u_{k-1} = 2$

(ii) there exists $0 \leq i \leq k - 1$ such that $\deg u_i = 4$ and $\deg u_l = 2$ for $l \neq i$.

**Proof** We begin by analyzing the coefficient of the characteristic polynomial of the transition matrix $T$. Let

$$\det (x I_n - T) = \sum_{j=0}^{n} \rho_j x^j,$$

where $n = |V|$. Then the only permutations that contribute $\rho_{n-j}$ are ones like

$$\sigma = \begin{pmatrix} u_0 & u_1 & \cdots & u_{k-1} \\ u_1 & u_2 & \cdots & u_0 \end{pmatrix}$$

and $\sigma^{-1}$ since $k$ is odd and $\{u_0, u_1, \ldots, u_{k-1}\}$ is the unique set of vertices satisfying $u_0 \sim u_1 \sim \cdots \sim u_{k-1} \sim u_0$. Note that $\text{sgn}(\sigma) = \text{sgn}(\sigma^{-1}) = 1$ as $k$ is odd. By (7), we have

$$\rho_{n-k} = \text{sgn}(\sigma) \prod_{i=0}^{k-1} \left( -\frac{1}{\deg u_i} \right) + \text{sgn}(\sigma^{-1}) \prod_{i=0}^{k-1} \left( -\frac{1}{\deg u_i} \right) = -2 \prod_{i=0}^{k-1} \frac{1}{\deg u_i}$$

\[ \text{Springer} \]
For $G$ to be periodic, it must hold that $2^k \rho_{n-k} \in \mathbb{Z}$ by Lemma 2.3. Then

$$-2^{k+1} \prod_{i=0}^{k-1} \frac{1}{\deg u_i} \in \mathbb{Z}.$$ 

Since $\deg u_i \geq 2$, we have that either $\deg u_i = 2$ for $0 \leq i \leq k-1$ or there uniquely exists $0 \leq j \leq k-1$ such that $\deg u_j = 4$ and $\deg u_l = 2$ for $l \neq j$. Thus we complete the proof. \hfill \Box

If an odd unicycle graph $G$ satisfies (i) in Theorem 3.3, then $G$ is nothing but $C_k$, which is known to be a $k$-periodic graph [27]. From now on, we only treat odd unicycle graphs satisfying condition (ii). Our aim is to show that periodic odd unicycle graphs satisfying (ii) are the only ones illustrated in Fig. 5. However, these graphs are not odd-periodic.

### 3.4 Key statements

From now on, let $G$ be an odd-periodic graph in $\mathcal{O}(k)$ satisfying the condition (ii) in Theorem 3.3 and $n$ be the number of the vertices of $G$. Thus, $G$ is a graph as is seen in Fig. 6. Let the vertices of the unique cycle $C_k$ be $\{u, u_1, \ldots, u_{k-1}\}$, where $u$ is the unique vertex of degree 4. Hence, there exist two vertices $v_1, w_1 \notin V(C_k)$ such that $v_1 \sim u$ and $w_1 \sim u$. Since $G$ is a unicycle graph, the subgraph induced by $V(G) \setminus V(C_k)$, say $S$, does not contain cycles. We decompose $S$ into two subgraphs $S_1$ and $S_2$ and denote some edges by $f_1 = uu_1$, $f_2 = uu_{k-1}$, $g_1 = uv_1$, and $h_1 = uw_1$ (see Fig. 6). Let $E_u = \{f_1, f_2, g_1, h_1\}$ and define $G'$ as the subgraph of $G$ given by

$$V(G') = V(C_k) \cup \{v_1, w_1\}$$
$E(G') = E(C_k) \cup \{g_1, h_1\}$. 

Furthermore, for $e = xy \in E(G)$, we define a map $M : E(G) \to \mathbb{Q}$ by

$$M(e) = \frac{1}{\deg x \deg y}.$$ 

We also consider the coefficient of the characteristic polynomial of $T$. Throughout this paper, we denote by $\sum_{e_1, \ldots, e_l \in E(S)}$ the summation over the combinations of $l$ edges in $S$ that form an $l$-matching in $E(S)$ so $e_1, \ldots, e_l$ are disjoint. Let

$$K_{2l} = \sum_{e_1, \ldots, e_l \in E(S)} \prod_{i=1}^l M(e_i).$$

If there is no such matching in $E(S)$, we define $K_{2l} = 0$.

For example, let us compute $K_4$ of the graph illustrated in Fig. 7. We name some edges as are seen in the figure. The value $K_4$ is obtained by combinations of two disjoint edges in $E(S)$. Such combinations are $(e_1, e_4)$, $(e_1, e_5)$, $(e_2, e_5)$, $(e_3, e_4)$, $(e_3, e_5)$ and $(e_4, e_5)$. Since $M(e_1) = M(e_3) = \frac{1}{4}, M(e_4) = M(e_5) = \frac{1}{2}$ and $M(e_2) = \frac{1}{8}$, we have

$$K_4 = M(e_1)M(e_4) + M(e_1)M(e_5) + M(e_2)M(e_5) + M(e_3)M(e_4) + M(e_3)M(e_5) + M(e_4)M(e_5) = \frac{13}{16}.$$
Lemma 3.4 Let $\rho_{n-k-2t}$ be defined as above for $t \in \mathbb{N}$ with $k + 2t \leq n$. If $2^{k+2t}\rho_{n-k-2t} \in \mathbb{Z}$, then

$$2^{2t}K_{2t} \in \mathbb{Z}.$$  

\textbf{Proof} The permutations contributing $\rho_{n-k-2t}$ are the ones with length $k + 2t$, which are decomposed into cyclic permutations or transpositions. Since $k + 2t$ is odd, the permutations are expressed by the product of a cyclic permutation on $V(C_k)$ and $t$ disjoint transpositions in $V(S)$, which correspond to a $t$-matching in $E(S)$. The cyclic permutations are

$$\sigma = \begin{pmatrix} u & u_1 & \ldots & u_{k-1} \\ u_1 & u_2 & \ldots & u \end{pmatrix}$$

and

$$\sigma^{-1} = \begin{pmatrix} u & u_1 & \ldots & u_{k-1} \\ u_{k-1} & u & \ldots & u_{k-2} \end{pmatrix}.$$  

Then we have

$$\text{sgn}(\sigma) \prod_{i=0}^{k-1} \left(-\frac{1}{\deg u_i}\right) = \text{sgn}(\sigma^{-1}) \prod_{i=0}^{k-1} \left(-\frac{1}{\deg u_i}\right) = -\frac{1}{2^{k+1}}, \quad (10)$$

where $u_0 = u$. Let $\{x_1y_1, x_2y_2, \ldots, x_ty_t\} \subset E(S)$ be a $t$-matching in $E(S)$ and $e_i = x_iy_i$. The transposition on $\{x_i, y_i\}$ is

$$\tau = \begin{pmatrix} x_i & y_i \\ y_i & x_i \end{pmatrix}$$

and

$$\text{sgn}(\tau) \left(-\frac{1}{\deg x_i}\right) \left(-\frac{1}{\deg y_i}\right) = -M(e_i).$$

Then all the $t$-matchings in $E(S)$ yield

$$\sum_{e_1, e_2, \ldots, e_t \in E(S)} (-1)^t \prod_{i=1}^{t} M(e_i). \quad (11)$$

Inserting (10) and (11) into (7), we have

$$\rho_{n-k-2t} = 2 \left(- \prod_{i=0}^{k-1} \frac{1}{\deg u_i}\right) \cdot \left( \sum_{e_1, e_2, \ldots, e_t \in E(S)} (-1)^t \prod_{i=1}^{t} M(e_i) \right)$$
\[ (-1)^{t+1} \frac{1}{2^k} \sum_{e_1, e_2, \ldots, e_t \in E(S)} \prod_{i=1}^{t} M(e_i). \]

The condition \( 2^{k+2t} \rho_{n-2t} \in \mathbb{Z} \) implies
\[
2^{k+2t} \left( (-1)^{t+1} \frac{1}{2^k} \sum_{e_1, e_2, \ldots, e_t \in E(S)} \prod_{i=1}^{t} M(e_i) \right) = (-1)^{t+1} 2^{2t} \sum_{e_1, e_2, \ldots, e_t \in E(S)} \prod_{i=1}^{t} M(e_i) = (-1)^{t+1} 2^{2t} K_{2t} \in \mathbb{Z},
\]
which completes the proof. \( \square \)

**Theorem 3.5** Let \( v_1 \) and \( w_1 \) be defined as above. If \( G \) is periodic, then it holds that \( \deg v_1 = \deg w_1 = 1 \) or 2.

**Proof** Suppose that a graph \( G \) illustrated in Fig. 6 is periodic. Then it follows from Lemma 2.3 that \( 2^{k+2} \rho_{n-2} \in \mathbb{Z} \). By Lemma 3.4, this condition is reduced to \( 2^2 K_2 \in \mathbb{Z} \). It holds that
\[
K_2 = \sum_{e \in E(G)} M(e) = \sum_{e \in E(G)} M(e) - \sum_{e \in E(G')} M(e).
\]

Note that \( \sum_{e \in E(G)} M(e) = -\rho_{n-2} \) by (7). Then we have
\[
K_2 = \sum_{e \in E(G)} M(e) - \sum_{e \in E(G')} M(e) = -\rho_{n-2} - \frac{1}{4} (k - 2) - \frac{1}{4} \deg v_1 - \frac{1}{4} \deg w_1.
\]

The condition \( 2^2 K_2 \in \mathbb{Z} \) gives rise to
\[
-2^2 \rho_{n-2} - (k - 2) - 1 - \left( \frac{1}{\deg v_1} + \frac{1}{\deg w_1} \right) \in \mathbb{Z}.
\]

Since \( G \) is periodic, it holds that \( 2^2 \rho_{n-2} \in \mathbb{Z} \) by Lemma 2.3. Thus, it necessarily holds that
\[
\left( \frac{1}{\deg v_1} + \frac{1}{\deg w_1} \right) \in \mathbb{Z}.
\]

This implies that \( \deg v_1 = \deg w_1 = 1 \) or 2. \( \square \)
If $\deg v_1 = \deg w_1 = 1$, then $G$ is a member of the graphs seen in Fig. 5, which are not odd-periodic by Theorem 3.1. Thus, we suppose $\deg v_1 = \deg w_1 = 2$. Here, we recursively define the vertices $v_t \in V(S_1)$ and $w_t \in V(S_2)$ for $t \in \mathbb{N}$ as follows: If $\deg v_1 = \cdots = \deg w_{t-1} = 2$, and $\deg v_1 = \cdots = \deg w_{t-1} = 2$, then we define

$$v_t := \{v \in V(S_1) \mid d(u, v) = t\} \quad (12)$$
$$w_t := \{w \in V(S_2) \mid d(u, w) = t\}. \quad (13)$$

In this situation, the vertices are uniquely determined (see Fig. 8). For $1 \leq i \leq t$, we set edges $g_i$ and $h_i$ using $g_i = v_{i-1}v_i$ and $h_i = w_{i-1}w_i$, respectively. We define

$$K_{2i}^g(2, \ldots, r) := \sum_{e_1, \ldots, e_i \in E(S) \setminus \{g_2, \ldots, g_r\}} \prod_{l=1}^{i} M(e_l),$$
$$K_{2i}^h(2, \ldots, r) := \sum_{e_1, \ldots, e_i \in E(S) \setminus \{h_2, \ldots, h_r\}} \prod_{l=1}^{i} M(e_l),$$

and

$$L_{2i}(2, \ldots, r) := K_{2i}^g(2, \ldots, r) + K_{2i}^h(2, \ldots, r)$$
for $1 \leq r \leq t - 1$. It is convenient to set $K^g(2, \ldots, r) = K^h(2, \ldots, r) = 1$ and $L(2, \ldots, r) = 2$.

For example, we compute $K^g(2, 3)$ of the graph illustrated in Fig. 9. The value $K^g(2, 3)$ is given by combinations of two disjoint edges in $E(S) \setminus \{g_2, g_3\}$. Such combinations are $(e_1, h_2)$, $(e_1, h_3)$, $(e_1, e_4)$, $(e_2, h_2)$, $(e_2, h_3)$, $(e_2, h_4)$, $(e_3, h_2)$, $(e_3, h_3)$, $(e_3, e_4)$ and $(h_2, e_4)$. Since $M(e_1) = M(e_2) = M(e_3) = M(h_2) = M(h_3) = \frac{1}{4}$ and $M(e_4) = \frac{1}{2}$, we have

$$K^g(2, 3) = M(e_1)M(h_2) + M(e_1)M(h_3) + M(e_1)M(e_4)$$
$$+ M(e_2)M(h_2) + M(e_2)M(h_3) + M(e_2)M(e_4)$$
$$+ M(e_3)M(h_2) + M(e_3)M(h_3) + M(e_3)M(e_4) + M(h_2)M(e_4)$$
$$= \frac{7}{8}.$$
\[ L_{2i}(2, \ldots, r) = 2K_{2i} - \sum_{j=2}^{r} \left( \frac{1}{4} L_{2(i-1)}(2, \ldots, j+1) \right). \]

**Proof** Recall that \( K_{2i}^{s}(2, \ldots, r) = \sum_{e_1, \ldots, e_i \in E(S) \setminus \{s_2, \ldots, s_r\}} \prod_{l=1}^{i} M(e_l) \) for \( s \in \{g, h\} \). Define \( \sum_{e_1, \ldots, e_i} \) as the summation over the combinations of edges in \( E(S) \setminus \{s_2, \ldots, s_{j+1}\} \), which form an \( i \)-matching containing \( s_j \). Then it holds that

\[
K_{2i}^{s}(2, \ldots, r) = \sum_{e_1, \ldots, e_i \in E(S) \setminus \{s_2, \ldots, s_r\}} \prod_{l=1}^{i} M(e_l) \\
= \sum_{e_1, \ldots, e_i \in E(S)} \prod_{l=1}^{i} M(e_l) - \sum_{j=2}^{r} \sum_{e_1, \ldots, e_i} \prod_{l=1}^{i} M(e_l) \\
= K_{2i} - \sum_{j=2}^{r} \sum_{e_1, \ldots, e_i} \prod_{l=1}^{i} M(e_l).
\]

Here, we have

\[
\sum_{e_1, \ldots, e_i} \prod_{l=1}^{i} M(e_l) = M(s_j) \sum_{e_1, \ldots, e_{i-1} \in E(S) \setminus \{s_2, \ldots, s_{j+1}\}} \prod_{l=1}^{i-1} M(e_l) \\
= \frac{1}{4} K_{2(i-1)}^{s}(2, \ldots, j+1).
\]

Therefore, it holds that

\[
K_{2i}^{s}(2, \ldots, r) = K_{2i} - \sum_{j=2}^{r} \left( \frac{1}{4} K_{2(i-1)}^{s}(2, \ldots, j+1) \right)
\]

and

\[
L_{2i}(2, \ldots, r) = K_{2i}^{g}(2, \ldots, r) + K_{2i}^{h}(2, \ldots, r) \\
= 2K_{2i} - \sum_{j=2}^{r} \left( \frac{1}{4} L_{2(i-1)}(2, \ldots, j+1) \right).
\]

\( \square \)

**Lemma 3.7** Suppose \( \deg v_1 = \deg w_1 = 2 \) and let \( K_{2t} \) be defined as above for \( 1 \leq t \leq \lfloor \frac{n}{2} \rfloor \). It holds that

\[
K_{2t} = \rho_{n-2t} - \sum_{j=0}^{t-1} (A_j^{(t)} + B_j^{(t)} + C_j^{(t)}),
\]

\( \square \) Springer
where

\[ A_j^{(t)} = \left( \frac{1}{2} \right)^{2(t-j)} K_{2j} a_j^{(t)}, \]
\[ B_j^{(t)} = \left( \frac{1}{2} \right)^{2(t-j)} K_{2j} b_j^{(t)}, \]
\[ C_j^{(t)} = \left( \frac{1}{2} \right)^{(2(t-j)+1)} L_{2j} c_j^{(t)}, \]

for some constants \( a_j^{(t)}, b_j^{(t)}, c_j^{(t)} \in \mathbb{Z} \).

**Proof** Recall that

\[ K_{2t} = \sum_{e_1, \ldots, e_t \in E(S)} \prod_{l=1}^{t} M(e_l). \]

Define \( \sum'_{e_1, \ldots, e_t} \) to be the summation over the combinations of edges in \( E(G) \) forming an \( t \)-matching containing at least one edge in \( E(G') \). Then we have

\[
K_{2t} = \sum_{e_1, \ldots, e_t \in E(S)} \prod_{l=1}^{t} M(e_l) = \sum_{e_1, \ldots, e_t \in E(G)} \prod_{l=1}^{t} M(e_l) - \sum'_{e_1, \ldots, e_t} \prod_{l=1}^{t} M(e_l) = \rho_{n-2t} - \sum'_{e_1, \ldots, e_t} \prod_{l=1}^{t} M(e_l).
\]

Let \( X_t = \sum'_{e_1, \ldots, e_t} \prod_{l=1}^{t} M(e_l) \). The value \( X_t \) is constructed by \( t \)-matchings containing at least one edge in \( E(G') \). Suppose that there is a \( t \)-matching with \( j \) edges in \( E(S) \) and \((t-j)\) edges in \( E(G') \) for \( 0 \leq j \leq t - 1 \). The candidates for such a \((t-j)\)-matching are separated into the three cases (see Fig. 6 again):

(A- \( j \)) : It has no edges from \( E_u \),
(B- \( j \)) : It has \( f_1 \) or \( f_2 \),
(C- \( j \)) : It has \( g_1 \) or \( h_1 \).

In the case of (A- \( j \)), the \( t \)-matching is constructed using \((t-j)\) edges in \( E(G') \setminus E_u \) and \( j \) edges in \( E(S) \), which yield

\[
\sum_{e_1, \ldots, e_{t-j} \in E(G') \setminus E_u} \prod_{i=1}^{t-j} M(e_i).
\]
and

$$\sum_{e_1, \ldots, e_j \in E(S)} \prod_{l=1}^{j} M(e_l) = K_{2j},$$

respectively. Denote the product of them as $A_{j}^{(t)}$. The value $A_{j}^{(t)}$ is a term contributing to $X_t$, which is constructed from $t$-matchings in $E(S)$ whose $(t-j)$ edges satisfy (A-$j$). For $e \in E(G') \setminus E_u$, it holds that $M(e) = \frac{1}{4}$. For $e_1, \ldots, e_{t-j} \in E(G') \setminus E_u$, we thus have

$$\prod_{i=1}^{t-j} M(e_i) = \left(\frac{1}{2}\right)^{2(t-j)}.$$ 

Note that this value does not depend on the choice of the $(t-j)$-matching in $E(G') \setminus E_u$. Denote the number of these $(t-j)$-matchings as $a_{j}^{(t)}$. Thus, we have

$$A_{j}^{(t)} = \left(\frac{1}{2}\right)^{2(t-j)} K_{2j} a_{j}^{(t)}.$$ 

We next consider the case of (B-$j$). Here, the value

$$M(f_1) \sum_{e_1, \ldots, e_{t-j-1} \in E(G') \setminus E_u} \prod_{i=1}^{t-j-1} M(e_i) \sum_{e_1, \ldots, e_j \in E(S)} \prod_{l=1}^{j} M(e_l) = \frac{1}{8} \sum_{e_1, \ldots, e_{t-j-1} \in E(G') \setminus E_u} \prod_{i=1}^{t-j-1} M(e_i) K_{2j}$$

is constructed from $t$-matchings with $f_1$ and $(t-j-1)$ edges in $E(G') \setminus E_u$ and $j$ edges in $E(S)$. Similarly, $t$-matchings for (B-$j$) with $f_2$ yield the same value. Let

$$B_{j}^{(t)} = \frac{1}{4} \sum_{e_1, \ldots, e_{t-j-1} \in E(G') \setminus E_u} \prod_{i=1}^{t-j-1} M(e_i) K_{2j}$$

for $0 \leq j \leq t-2$ and $B_{t-1}^{(t)} = \frac{1}{4} K_{2(t-1)}$. This is a term contributing to $X_t$, which is constructed by $t$-matchings whose $(t-j)$ edges satisfy (B-$j$). It similarly holds that $M(e) = \frac{1}{4}$ for $e \in E(G') \setminus E_u$, and we have

$$\prod_{i=1}^{t-j-1} M(e_i) = \left(\frac{1}{2}\right)^{2(t-j-1)}.$$
for \(e_1, \ldots, e_{t-j-1} \in E(G') \setminus E_u\). This does not depend on the choice of \((t-j-1)\)-matchings. Denoting the number of these \((t-j-1)\)-matchings as \(b_j^{(t)}\), we have

\[
B_j^{(t)} = \frac{1}{4} \left( \frac{1}{2} \right)^{2(t-j-1)} K_{2j} b_j^{(t)} = \left( \frac{1}{2} \right)^{2(t-j)} K_{2j} b_j^{(t)}.
\]

Finally, we consider the case of \((C-j)\). A \(t\)-matching for this case is constructed using \(g_1\), \((t-j-1)\) edges in \(E(G') \setminus E_u\), and \(j\) edges in \(E(S) \setminus \{g_2\}\). Such a matching yields

\[
M(g_1) \sum_{e_1, \ldots, e_{t-j-1} \in E(G') \setminus E_u} \prod_{i=1}^{t-j-1} M(e_i) \sum_{e_1, \ldots, e_j \in E(S) \setminus \{g_2\}} \prod_{l=1}^{j} M(e_l)
= \frac{1}{8} \sum_{e_1, \ldots, e_{t-j-1} \in E(G') \setminus E_u} \left( \frac{1}{4} \right)^{t-j-1} K_{2j}^g (2)
= \sum_{e_1, \ldots, e_{t-j-1} \in E(G') \setminus E_u} \left( \frac{1}{2} \right)^{2(t-j)+1} K_{2j}^g (2).
\]

A \(t\)-matching in this case with \(h_1\) similarly yields

\[
\sum_{e_1, \ldots, e_{t-j-1} \in E(G') \setminus E_u} \left( \frac{1}{2} \right)^{2(t-j)+1} K_{2j}^h (2).
\]

Let

\[
C_j^{(t)} = \sum_{e_1, \ldots, e_{t-j-1} \in E(G') \setminus E_u} \left( \frac{1}{2} \right)^{2(t-j)+1} K_{2j}^g (2) + \sum_{e_1, \ldots, e_{t-j-1} \in E(G') \setminus E_u} \left( \frac{1}{2} \right)^{2(t-j)+1} K_{2j}^h (2)
= \sum_{e_1, \ldots, e_{t-j-1} \in E(G') \setminus E_u} \left( \frac{1}{2} \right)^{2(t-j)+1} L_{2j} (2)
\]

for \(0 \leq j \leq t-2\) and \(C_{t-1}^{(t)} = \frac{1}{8} L_{2j} (2)\). The value in this summation does not depend on the choice of \((t-j-1)\)-matching as was the case above. Denoting the number of these \((t-j-1)\)-matchings as \(c_j^{(t)}\), we have

\[
C_j^{(t)} = \left( \frac{1}{2} \right)^{2(t-j)+1} L_{2j} (2) c_j^{(t)},
\]

which is the final term contributing to \(X_t\).
Therefore, we have

\[ X_t = \sum_{j=0}^{t-1} (A_j^{(t)} + B_j^{(t)} + C_j^{(t)}) \]

and

\[ K_{2t} = \rho_{n-2t} - X_t \]
\[ = \rho_{n-2t} - \sum_{j=0}^{t-1} (A_j^{(t)} + B_j^{(t)} + C_j^{(t)}). \]

\[ \square \]

Now, we give key statements needed to prove our main result.

**Theorem 3.8** Suppose that \( G \) is periodic, \( \deg v_1 = \cdots = \deg v_t = 2 \), and \( \deg w_1 = \cdots = \deg w_t = 2 \). For \( 1 \leq i \leq t \), we have

\[ 2^{2i} K_{2i} \in \mathbb{Z}, \] \hspace{1cm} (14)
\[ 2^{2(i-1)-1} L_{2(i-1)}(2) \in \mathbb{Z}, \] \hspace{1cm} (15)
\[ 2^{2(i-j)-1} L_{2(i-j)}(2, \ldots, j + 1) \in \mathbb{Z}, \] \hspace{1cm} (16)

**Proof** We demonstrate this using induction on \( i \). The case of \( t \leq 2 \) is clear, so we turn our attention to \( t > 2 \). Suppose that (14), (15), and (16) hold for \( 1 \leq i \leq t - 1 \). We first prove that (16) holds for \( i = t \) and every \( j \) with \( 2 \leq j \leq t - 1 \). We begin with the case of \( i = t \) and \( j = t - 1 \). Here, we have

\[ K_2^g(2, \ldots, t) = \sum_{e \in E(S \setminus \{g_2, \ldots, g_t\})} M(e) \]
\[ = K_2 - \sum_{j=2}^{t} M(g_j) \]
\[ = K_2 - \frac{1}{4}(t - 1). \]

Similarly, we have \( K_2^h(2, \ldots, t) = K_2 - \frac{1}{4}(t - 1) \). Hence, it holds that

\[ L_2(2, \ldots, t) = K_2^g(2, \ldots, t) + K_2^h(2, \ldots, t) = 2K_2 - \frac{1}{2}(t - 1), \]

and this gives rise to

\[ 2L_2(2, \ldots, t) = 2^2 K_2 - (t - 1) \in \mathbb{Z} \]
by the assumption in (14). Then (16) holds for \( i = t \) and \( j = t - 1 \). Using \( i = t \) and 
\( j = t - 2 \) in (16), we have

\[
L_4(2, \ldots, t - 1) = 2K_4 - \frac{1}{4} \sum_{j=2}^{t-1} L_2(2, \ldots, j + 1)
\]

by Lemma 3.6. Then

\[
2^3L_4(2, \ldots, t - 1) = 2^4K_4 - 2 \sum_{j=2}^{t-1} L_2(2, \ldots, j + 1).
\]

Now, it holds that \( 2^4K_4 \in \mathbb{Z} \) and \( 2L_2(2, \ldots, j + 1) \in \mathbb{Z} \) for \( 2 \leq j \leq t - 2 \) by the assumption. Additionally, it holds that \( 2L_2(2, \ldots, t) \in \mathbb{Z} \) by the previous argument, and this gives rise to \( 2^3L_4(2, \ldots, t - 1) \in \mathbb{Z} \). Taking \( i = t \) and \( j = t - k \) for \( 3 \leq k \leq t - 2 \), we can show inductively that

\[
2^{2k-1}L_{2k}(2, \ldots, t - k + 1) \in \mathbb{Z}.
\]

Hence, we have

\[
2^{2(t-j)-1}L_{2(t-j)}(2, \ldots, j + 1) \in \mathbb{Z}
\]

for \( 2 \leq j \leq t - 1 \). Thus, (16) holds for \( i = t \) and \( 2 \leq j \leq t - 1 \).

We next show that (15) holds for \( i = t \). By Lemma 3.6, we have

\[
L_{2(t-1)}(2) = 2K_{2i} - \frac{1}{4} L_{2(t-2)}(2, 3).
\]

Therefore, it holds that

\[
2^{2(t-1)-1}L_{2(t-1)}(2) = 2^{2(t-1)}K_{2(t-1)} - 2^{2(t-2)-1}L_{2(t-2)}(2, 3).
\]

It follows from the assumptions for the case of \( i = t - 1 \) in (14) and (16) that 
\( 2^{2(t-1)}K_{2(t-1)} \in \mathbb{Z} \) and \( 2^{2(t-2)-1}L_{2(t-2)}(2, 3) \in \mathbb{Z} \). Thus, (15) holds for \( i = t \).

Finally, we show that (14) holds for \( i = t \). By Lemma 3.7, it holds that

\[
K_{2t} = \rho_{n-2t} - \sum_{j=0}^{t-1} (A_j^{(t)} + B_j^{(t)} + C_j^{(t)}),
\]

where \( A_j^{(t)} \), \( B_j^{(t)} \), and \( C_j^{(t)} \) are defined as in the statement in Lemma 3.7. Now, it holds that \( 2^{2t} \rho_{n-2t} \in \mathbb{Z} \) by Lemma 2.3. Additionally, we have

\[
2^{2t} A_j^{(t)} = 2^{2j} K_{2j} a_j^{(t)}
\]
for $0 \leq j \leq t - 1$. By the assumption, we have $2^{2j}K_{2j} \in \mathbb{Z}$ for $0 \leq j \leq t - 1$. Moreover, it follows from the assumption and the above argument that $2^{2j-1}L_{2j}(2) \in \mathbb{Z}$ for $0 \leq j \leq t - 1$. Hence, we have

$$2^{2t} \sum_{j=0}^{t-1} (A_j^{(t)} + B_j^{(t)} + C_j^{(t)}), \in \mathbb{Z}$$

and this gives rise to

$$2^{2t} K_{2t} \in \mathbb{Z}.$$ 

Therefore, we have completed the proof.

**Theorem 3.9** Suppose that $G$ is periodic, $\deg v_1 = \cdots = \deg v_t = 2$, and $\deg w_1 = \cdots = \deg w_t = 2$. We have

$$\deg v_{t+1} = \deg w_{t+1} = 1 \text{ or } 2.$$ 

**Proof** Recall that the numbers $n$ and $k$ are the number of vertices of $G$ and the length of the unique cycle in $G$, respectively. Since $G$ is periodic, it holds that $2^{k+2(t+1)}\rho_{n-k-2(t+1)} \in \mathbb{Z}$, and it follows from Lemma 3.4 that $2^{2(t+1)}K_{2(t+1)} \in \mathbb{Z}$. By Lemma 3.7, we have

$$K_{2(t+1)} = \rho_{n-2(t+1)} - \sum_{j=0}^{t} (A_j^{(t+1)} + B_j^{(t+1)} + C_j^{(t+1)}).$$

Recall that $2^{2(t+1)}\rho_{n-2(t+1)} \in \mathbb{Z}$. As is seen in the proof of Theorem 3.8, we have

$$2^{2(t+1)} A_j^{(t+1)}, 2^{2(t+1)} B_j^{(t+1)} \in \mathbb{Z}$$

for $0 \leq j \leq t$. Thus, the condition of $2^{2(t+1)}K_{2(t+1)} \in \mathbb{Z}$ is reduced to

$$2^{2(t+1)} \sum_{j=1}^{t} C_j^{(t+1)} \in \mathbb{Z}. \quad (17)$$

Now, recall that

$$C_j^{(t+1)} = \left(\frac{1}{2}\right)^{2(t+1-j)+1} L_{2j}(2)c_j^{(t+1)},$$

$\square$ Springer
where $c_j^{(t+1)}$ is the number of $(t + 1 - j)$-matchings in $E(G') \cup \{g_1, h_1\}$ with $g_1$, which coincides with those with $h_1$. Thus, we have

$$2^{2(t+1)}c_j^{(t+1)} = 2^{2j-1}L_{2j}(2)c_j^{(t+1)}.$$  

By Theorem 3.8 (15), it holds that

$$2^{2j-1}L_{2j}(2) \in \mathbb{Z}$$  

for $1 \leq j \leq t - 1$. The condition in (17) then implies

$$2^{2t-1}L_{2t}(2)c_t^{(t+1)} \in \mathbb{Z}.$$  

Note that $c_t^{(t+1)} = 1$ since the number of 1-matchings in $E(G') \cup \{g_1, h_1\}$ with $g_1$ is one. It suffices to consider

$$2^{2t-1}L_{2t}(2) \in \mathbb{Z}. \quad (18)$$  

By Lemma 3.6, we have

$$L_{2t}(2) = 2K_{2t} - \frac{1}{4}L_{2(t-1)}(2, 3)$$  

and

$$2^{2t-1}L_{2t}(2) = 2^{2t}K_{2t} - 2^{2(t-1)-1}L_{2(t-1)}(2, 3).$$  

By Theorem 3.8 (14), we have

$$2^{2t}K_{2t} \in \mathbb{Z},$$  

and the condition in (18) is reduced to

$$2^{2(t-1)-1}L_{2(t-1)}(2, 3) \in \mathbb{Z}. \quad (19)$$  

By Lemma 3.6, we have

$$L_{2(t-1)}(2, 3) = 2K_{2(t-1)} - \frac{1}{4}L_{2(t-2)}(2, 3) - \frac{1}{4}L_{2(t-2)}(2, 3, 4).$$  

Then

$$2^{2(t-1)-1}L_{2(t-1)}(2, 3) = 2^{2(t-1)}K_{2(t-1)} - 2^{2(t-2)-1}L_{2(t-2)}(2, 3) - 2^{2(t-2)-1}L_{2(t-2)}(2, 3, 4).$$
Since $2^{2(t-1)} K_{2(t-1)} \in \mathbb{Z}$ and $2^{2(t-2)} L_{2(t-2)} (2, 3) \in \mathbb{Z}$ by Theorem 3.8 (14) and (16), the condition in (19) is reduced to

$$2^{2(t-2)} L_{2(t-2)} (2, 3, 4) \in \mathbb{Z}. \tag{20}$$

Similarly, the condition in (20) implies

$$2^{2(t-3)} L_{2(t-3)} (2, 3, 4, 5) \in \mathbb{Z}. \tag{21}$$

Repeating this process, we obtain the condition

$$2^{2(t+1-j)} L_{2(t+1-j)} (2, \ldots, j + 1) \in \mathbb{Z},$$

which is reduced to

$$2^{2(t-j)} L_{2(t-j)} (2, \ldots, j + 2) \in \mathbb{Z} \tag{21}$$

for $4 \leq j \leq t - 1$. Setting $j = t - 1$ in (21), we obtain

$$2L_2 (2, \ldots, t + 1) \in \mathbb{Z}. \tag{22}$$

On the other hand, we have

$$L_2 (2, \ldots, t + 1) = K_2^g (2, \ldots, t + 1) + K_2^h (2, \ldots, t + 1)$$

$$= \sum_{e \in E(S) \setminus \{g_2, \ldots, g_{t+1}\}} M(e) + \sum_{e \in E(S) \setminus \{h_2, \ldots, h_{t+1}\}} M(e).$$

Now,\[
\sum_{e \in E(S) \setminus \{s_2, \ldots, s_{t+1}\}} M(e) = \sum_{e \in E(S)} M(e) - \sum_{j=2}^{t+1} M(s_j) = K_2 - \frac{1}{4} (t - 1) - \frac{1}{2 \deg v_{t+1}}
\]

for $s \in \{g, h\}$. So, we have

$$L_2 (2, \ldots, t + 1) = 2K_2 - \frac{1}{2} (t - 1) - \frac{1}{2} \left( \frac{1}{\deg v_{t+1}} + \frac{1}{\deg w_{t+1}} \right).$$

Hence, the condition in (22) gives
\[
\frac{1}{\deg v_{t+1}} + \frac{1}{\deg w_{t+1}} \in \mathbb{Z},
\]
which implies that \( \deg v_{t+1} = \deg w_{t+1} = 1 \) or \( 2 \). Thus, the proof is completed. □

### 3.5 Proof of Theorem 1.1

We will now finally prove Theorem 1.1. Let \( G \) be an odd-periodic graph. Then it follows from Theorem 3.2 that \( G \) is an odd unicycle graph, that is, \( G \) is a member of \( \mathcal{O}(k) \) for an odd \( k \in \mathbb{N} \). Denote the vertices of the unique cycle in \( G \) as \( \{ u_0, u_1, \ldots, u_{k-1} \} \). By Theorem 3.3, one of the following holds:

(i) \( \deg u_0 = \deg u_1 = \cdots = \deg u_{k-1} = 2 \),
(ii) there exists \( 0 \leq i \leq k - 1 \) such that \( \deg u_i = 4 \) and \( \deg u_l = 2 \) for \( l \neq i \).

If (i) holds, then \( G \) is nothing but an odd cycle. Suppose that (ii) holds and let \( \deg u_0 = 4 \) and \( \deg u_i = 2 \) for \( 1 \leq i \leq k - 1 \). Thus, \( G \) is the kind of graph seen in Fig. 6. Let two vertices \( v_1 \) and \( w_1 \) be defined as in this figure. In this case, it holds that \( \deg v_1 = \deg w_1 = 2 \) by Theorem 3.5. Then we define two vertices \( v_2 \) and \( w_2 \) as in (12) and (13), respectively. Combining Theorems 3.8 and 3.9, we have \( \deg v_2 = \deg w_2 = 1 \) or \( 2 \). If \( \deg v_2 = \deg w_2 = 2 \), then we define two vertices \( v_3 \) and \( w_3 \) as in (12) and (13), respectively. Repeating this process, we have that if \( \deg v_1 = \deg v_2 = \cdots = \deg v_t = \deg w_1 = \deg w_2 = \cdots = \deg w_t = 2 \) for some \( t \in \mathbb{N} \), then \( \deg v_{t+1} = \deg w_{t+1} = 1 \) or \( 2 \). Since \( G \) is a finite graph, there exists \( t \in \mathbb{N} \) such that \( \deg v_1 = \deg v_2 = \cdots = \deg v_t = \deg w_1 = \deg w_2 = \cdots = \deg w_t = 2 \) and \( \deg v_{t+1} = \deg w_{t+1} = 1 \). This graph is nothing but the one illustrated in Fig. 5. However, it is not odd periodic due to Theorem 3.1. It follows then that odd-periodic odd unicycle graphs do not satisfy condition (ii). Therefore, the odd-periodic graph is only an odd cycle, and the proof is completed.

### 4 Summary and discussion

In this paper, we tried to extract the graphs on which the induced Grover walk is odd-periodic, the odd-periodic graphs. It was previously observed that the odd-periodic graphs are included in a family of odd unicycle graphs (see Theorem 3.2). However, there are conditions that periodic odd unicycle graphs should obey (see Theorems 3.3, 3.5, 3.8, and 3.9). These conditions are derived from the characteristic polynomial of the transition matrix defined in Sect. 2.3. Combining them, we see that the periodic odd unicycle graph illustrated in Fig. 5 is not an odd cycle, but the graph is not odd periodic. Therefore, the periodic odd unicycle graph whose period is odd is an odd cycle. In other words, the odd-periodic graph is only an odd cycle. It is generally hard to characterize the graphs from the information of spectrum. This paper succeeded in such an inverse problem. Thus, we believe that this work gives a contribution to the field of spectral graph theory.

This result strongly depends on the combinatorial method used to compute the coefficients of the characteristic polynomial. We are currently focusing on counting.
combinations of cycles and matchings in the underlying graph to obtain the coefficients. Our ultimate goal is to control graph structure by utilizing properties of quantum walks. We believe that the work detailed in this paper helps us reach that goal.

By applying the above procedure, we can perfectly characterize the odd-periodic graphs. How to characterize the even-periodic graphs is still an open question, and it is one of our next research directions.

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Data Availability The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The author declares no conflict of interest associated with manuscript.

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