\[ E_0/L = \frac{1}{\beta} \]

Graph showing the relationship between \( E_0/L \) and \( 1/L \) for different values of \( \beta \):

- \( \beta = 1.2 \)
- \( \beta = 1.5 \)
- \( \beta = 1.8 \)
- \( \beta = 2.0 \)
- \( \beta = 2.3 \)
- \( \beta = 2.6 \)
- \( \beta = 2.9 \)
- \( \beta = 3.5 \)
\[
\beta = 1.2 \\
\beta = 1.5 \\
\beta = 1.8 \\
\beta = 2 \\
\beta = 2.3 \\
\beta = 2.6 \\
\beta = 2.9 \\
\beta = 3.5
\]
$\beta = 1.2$
$\beta = 1.5$
$\beta = 1.8$
$\beta = 2.0$
$\beta = 2.3$
$\beta = 2.6$
$\beta = 2.9$
$\beta = 3.5$
Conformal field theory in Tomonaga-Luttinger model with $1/r^\beta$ long-range interaction

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Abstract

I attempt to construct $U(1)$ conformal field theory (CFT) in the Tomonaga-Luttinger (TL) liquid with $1/r^\beta$ long-range interaction (LRI). Treating the long-range forward scattering as a perturbation and applying CFT to it, I derive the finite size scalings which depend on the power of the LRI. The obtained finite size scalings give the nontrivial behaviours when $\beta$ is odd and is close to 2. I find the consistency between the analytical arguments and numerical results in the finite size scaling of energy.

1 Introduction

Electron systems have attracted our much attention in the low energy physics. As the dimension of the electron systems decrease, the charge screening effects become less important. In spite of these facts, models with short-range interaction have been adopted in many researches of one dimensional electron systems. The recent advanced technology makes it possible to fabricate quasi-one-dimension systems. Actually in low temperature the effect of Coulomb force has been observed in GaAs quantum wires [1], quasi-one-dimensional conductors [2, 3, 4] and 1D Carbon nanotubes [5, 6, 7].

The systems with $1/r$ Coulomb repulsive forward scattering was investigated on the long distance properties by bosonization techniques [8]. The charge correlation function decays with the distance as $\exp(-\text{const.}(\ln x)^{1/2})$ more slowly than any power law. The momentum distribution function and the density of state does not show the simple power law singular behaviour. The logarithmic behaviours appear in the power [9]. These mean that the system is driven to

*corresponding phone num.:011-81-948-22-0711
the Wigner crystal which is quite different from the ordinary TL liquid. The investigation for the interaction $1/r^{1-\epsilon}$ through the path integral approach [10] reconfirms the slower decaying of the single particle Green function for $\epsilon = 0$ and leads the faster decay for $0 \leq \epsilon (\ll 1)$ than any power type.

The numerical calculation in the electron system with the Coulomb interaction shows that the larger range of the interaction causes the insulator (charge density wave) to metal (metallic Wigner crystal) transition [11]. In the spinless fermion system, the convergence of the Luttinger parameters exhibits the quasi-metallic behaviour different from the simple TL one [12].

As I will discuss below, the forward scattering is irrelevant for $\beta > 1$. As an instance of the effect of the long-range Umklapp scattering, it was reported that the $1/r^2$ interaction makes the system gapless to gapful through the generalized Kosterlitz-Thouless transition [13].

In this paper I discuss CFT in the system with LRI. The basic assumptions of CFT are symmetries of translation, rotation, scale and special conformal transformation. Besides them I assume the short-range interaction in the CFT. Hence it is a subtle problem whether the CFT can describe the system with LRI.

Of LRIs, up to now, the solvable models with $1/r^2$ interaction were discussed [14] [15] [16] [17]. With the Bethe ansatz, the conformal anomaly and the conformal dimensions were calculated and the system proved to be described by $c = 1$ CFT. In fact the central charge from the specific heat agrees with $c = 1$. On the other hand, the ground state energy is affected by the LRI and the periodic nature. The effective central charge deviates from $c = 1$.

In general, the CFT for LRI which breaks the locality, has been left as unsettled problem. It is significant to clarify the validity of the CFT to the systems with LRI. I investigate the tight-binding model with $1/r^\beta$ interaction as one of such problems. The low energy effective model consists of TL liquid, the long-range forward scattering and the long-range spatially oscillating Umklapp scattering. Extending arguments appearing in Ref. [18] to the TL liquid with the long-range forward scattering, I derive the finite size scalings. In the tight-binding model with $1/r^\beta$ interaction, I calculate numerically the size dependences of energy and the coefficients of $1/L^y$. And I see numerically the relations between the velocity, susceptibility and Drude weight, which CFT requires.

2 Field theoretical approach

I consider the following tight-binding Hamiltonian of the interacting spinless Fermions:

$$
H = -\sum_{j}^{L} (c_{j}^\dagger c_{j+1} + \text{h.c}) + \frac{g}{2} \sum_{i \neq j}^{L} (\rho_{i} - 1/2)V(i-j)(\rho_{j} - 1/2),
$$

\[1\]

I define the "effective central charge" by $c' = \frac{\pi v}{\beta}$ in the finite size scaling of the ground state energy $E_g = aL - \frac{b}{L}$. I use the word "effective central charge" in this sense.
where the operator $c_j$ ($c_j^\dagger$) annihilates (creates) the spinless Fermion in the site $j$ and $\rho_j = c_j^\dagger c_j$ is the density operator. In order to treat this model under the periodic boundary condition, I define the chord distance between the sites $i$ and $j$: $r_{i,j} = (\frac{L}{\pi} \sin \frac{\pi(i-j)}{L})$ where $L$ is the site number. Using this, I express the LRI as $V(i-j) = (\frac{1}{\pi} \sin \frac{\pi(i-j)}{L})^\beta$.

By the bosonization technique, I obtain the effective action of the Hamiltonian (1) for the arbitrary filling:

$$S = \int d\tau dx \frac{1}{2\pi K} (\nabla \phi)^2 + g \int d\tau dx x' \partial_x \phi(x, \tau) V(x-x') \partial_{x'} \phi(x', \tau)$$

$$+ \ g' \int d\tau dx x' \cos(2k_F x + \sqrt{2} \phi(x, \tau)) V(x-x') \cos(2k_F x' + \sqrt{2} \phi(x', \tau)),$$

where $V(x) = \frac{1}{|x|^\beta}$, $K$ is the TL parameter and $k_F$ is the Fermi wave number. And $g'$ is the coupling constant proportional to $g$. The first term of (2) is the TL liquid and the second term is the long-range forward scattering. The last term is the spatially oscillating Umklapp process which includes $2\sqrt{2}\phi$ which comes from the interaction between the neighbour sites.

Schulz analyzed the effects of the Coulomb forward scattering by the bosonization technique in the electron system [8]. He discussed the quasi-Wigner crystal of electrons due to the Coulomb forward scattering. Here I focus on the effects of the $1/\nu^\beta$ forward scattering in the spinless Fermions system. I treat the action:

$$S = \int d\tau dx \frac{1}{2\pi K} (\nabla \phi)^2 + g \int d\tau dx x' \partial_x \phi(x, \tau) V(|x-x'|) \partial_{x'} \phi(x', \tau)$$

for any filling $k_F$. To investigate in the Fourier space, I choose the form $V(x) = \frac{1}{(x^2 + \alpha^2)^{\beta/2}}$, where $\alpha$ is the ultra-violet cut-off. In the wave number space, the action (3) is expressed as

$$S = \int dq dw \frac{2\pi}{K} (q^2 + w^2) + gq^2 V(q) |\phi(q, w)|^2,$$

where $V(q)$ is the Fourier transformation of $V(x)$:

$$V(q) = \frac{2\sqrt{\pi}}{\Gamma(\beta/2)2^{\beta/2-1/2}} (\alpha q)^{\beta/2-1/2} K_{\beta/2-1/2}(\alpha q).$$

Here $K_{\nu}(x)$ is the modified Bessel function of $\nu$th order and $\Gamma(x)$ is the gamma function. From this, the dispersion relation is

$$w^2 = q^2 \{1 + \frac{gK}{2\pi} V(q)\}.$$

The long wavelength behaviors of $V(q)$ are given by

$$V(q) \sim \begin{cases} A + B(q^2)^\beta + C(q^2)^{\beta-1} + \cdots & \beta > 0 \text{ and } \beta \neq \text{odd} \\ A + B \ln q\alpha + \cdots & \beta = 1 \\ A + B(q^2)^\beta \ln q\alpha + C(q^2)^2 + \cdots & \beta = 3 \\ A + B(q^2)^2 + C(q^2)^4 \ln q\alpha + D(q^2)^4 + \cdots & \beta = 5 \\ \cdots, \end{cases}$$

where $\CD$.
where $A, B, C$ and $D$ are the functions of $\beta$. For the case where $\beta > 0$ and $\neq$ odd, the coefficient

$$B = B(\beta), C = C(\beta)$$

is given by

$$B(\beta) = \frac{\pi^{3/2}}{4} \frac{1}{2^{\beta/2-1/2} \Gamma(\frac{5-\beta}{2}) \Gamma(\beta/2) \sin \left(\frac{\beta-1}{2}\pi\right)}$$

$$C(\beta) = \frac{-\pi^{3/2}}{2^{\beta-1} \Gamma(\frac{\beta+1}{2}) \Gamma(\beta/2) \sin \left(\frac{\beta-1}{2}\pi\right)}.$$ (8)

From the eqs. (6) and (7), I see that $(q\alpha)^{\beta-1}$ and $\ln q\alpha$ terms for $0 < \beta \leq 1$ affect the linear dispersion essentially. Especially for $\beta = 1$, there is the analysis by Schulz, where the charge density correlation function is calculated [8]. According to it, in the present spinless case, the LRI drives the ground state from the TL liquid to the quasi-Wigner crystal as $\beta \to 1+$. The slowest decaying part of the density correlation function is given by

$$\langle \rho(x)\rho(0) \rangle \sim \cos(2k_F x) \exp(-c\sqrt{\log x}),$$ (9)

where $c$ is a function of $K$, which exhibits slower spatial decay than the power decay of TL liquid.

Then I see the effects of the long-range forward scattering in the standpoints of the renormalization of $g$. The renormalization group eqs. of $g, v$ and $K$ are simply derived for the long wave-length (see Appendixes.). From the renormalization eqs., the $g$ terms are relevant for $\beta < 1$, marginal for $\beta = 1$ and irrelevant for $\beta > 1$. Thus it is expected that the system becomes the quasi-Wigner crystal caused by the forward scattering for $\beta \leq 1$ and the system becomes the TL liquid when $\beta > 1$. I see that

$$\Phi(x) \equiv \int dx' \partial_x \phi(x, \tau)V(x-x')\partial_{x'} \phi(x', \tau)$$ (10)

has the scaling dimension $x_g = \beta + 1$ for $1 < \beta < 3$ and $4$ for $\beta > 3$. As the weak logarithmic corrections appear for $\beta=$odd, I here distinguish $\Phi(x)$ for $\beta=3$ from the scaling functions. I also find the consistency on these scaling dimensions by CFT. By using the first order perturbation, I can know the effects of the long-range forward scattering. Based on CFT, the finite size scalings of energies for no perturbations are given [13, 20, 21] by

$$\Delta E_n = \frac{2\pi v x_n}{L}$$

$$E_g = e_g L - \frac{\pi v c}{6L},$$ (11)

where $x_n$ is the scaling dimension of the primary field denoted by $n$, $v$ is the sound velocity and $c$ is the central charge. Considering the LRI, I can extract the corrections to these energy size scalings(see Appendixes.):

$$\Delta E_n = \frac{2\pi v x_n}{L}(1 + \frac{g(0) \text{ const.}}{x_n L^{\beta-1}} + O(1/L^2))$$

$$E_g = (e_g + g(0) \text{ const.}) L - \frac{\pi v c}{6L}(c + g(0) \text{ const.} + \frac{g(0) \text{ const.}}{L^{\beta-1}} + O(1/L^2)),$$ (12)
where $\beta(>1)$ is not odd. And the constants are the functions of $\beta$. Note that for $\beta = \text{odd}$ cases, the logarithmic corrections appear. They correspond to the integer points of the modified Bessel function, which appear in the long-wave behaviours (7). I can reproduce these anomalies for $\beta = \text{odd}$ by the CFT. Moreover from CFT I can show that there are the anomalies in the general excitations and the ground state energy. The details are shown in Appendixes. The $O(1/L^2)$ terms come from the irrelevant field $L_{-2} \bar{L}_{-2} \mathbf{1}$ and the long-range $g$ term. The first eq. of (12) means that the long-range forward scattering $\Phi(x)$ has the scaling dimension $x_g = \beta + 1$ for $1 < \beta < 3$ and $4$ for $\beta > 3$ effectively. These respective scaling dimensions are consistent with the estimation from the renormalization group eqs. of $g$, that I mentioned above (see Appendixes).

The energy finite size scalings (12) mean that the LRI has the higher order influences than $1/L$ to the excitation energy and the LRI affects the $1/L$ term in the finite size scaling of the ground state energy. Here I note that it becomes difficult to calculate the central charge from finite size scalings (11) unless the effects of the LRI to $O(1/L)$ terms are known.

It is notable to compare the eqs. (12) with the case where the perturbations are of short-range type. Ludwig and Cardy calculated the contributions of the short-range perturbation [18]. The results for the irrelevant perturbation, $-g \sum_r \phi(r)$, which has the scaling dimensions $x > 2$ are

$$\Delta E_n = \frac{2\pi v x_n}{L}(1 + \frac{g(0)}{x_n} C_{nn} q(\frac{2\pi}{L})^{x-2})$$

$$E_g = (e_g + g(0)\text{const.})L - \frac{\pi v}{6L}(c + g(0)^2 \text{const.} \frac{L}{2x-4} + O(1/L^{3x-6})), \quad (13)$$

where the $O(g)$ terms do not appear in the ground state scaling because we set $\langle \phi \rangle = 0$ for the short-range interaction. These results mean that the $x > 2$ irrelevant field has influences of the higher order to the finite size scalings (11). And their result contains parts not so simple. There are the special points of scaling dimension $x = 1, 3, 5$, and $x = 2$ which is related to the appearance of logarithmic corrections.

To the contrary, I see $\langle \Phi \rangle \neq 0$ in the long-range case, where $\Phi$ is defined in eq. (10). The LRI gives the $O(1/L)$ intrinsic influence to the finite size scaling of the ground state energy, as appearing in the scalings (12), even if the LRI is irrelevant, that is, $x_g > 2$.

### 3 Numerical calculations

Through the Jordan-Wigner transformation, I transform the model (1) to $S = 1/2$ spin Hamiltonian for the numerical calculations:

$$H = -\sum_j (S_j^+ S_{j+1}^- + \text{h.c}) + \frac{g}{2} \sum_{i \neq j} S_i^z V(|i-j|) S_j^z. \quad (14)$$
I impose the periodic boundary condition $S_{L+1} = S_1$ to this model. Using the Lanczos algorithm I perform the numerical calculations for the Hamiltonian (14).

I have found analytically the corrections to the energy scalings (11) caused by the long-range forward scattering. If the oscillating Umklapp process term of (2) is irrelevant and does not disturb the energy scalings, the finite size corrections due to the forward scattering are expected to appear in the excited state energies and the ground state energy. I attempt to detect the contribution of the forward scattering.

I numerically calculate the size dependences of the excitation energy $\Delta E(m = 1/L)$ and the ground state energy $E_g(m = 0)$, $E_g(m = 1/L)$ for $g = 0.5$. Here I define the magnetization $m \equiv \sum_j S_j^z / L$ which is the conserved quantity. Fitting the one particle excitation energy as $L\Delta E(m = 1/L) = a + bL^c + dL^2$, I show the power $c$ versus the powers $\beta$ in Fig. 1. I see the power $c$ agrees with theoretical predictions: $\beta - 1$ except for $\beta = 2$. I shall discuss the $\beta = 2$ case later.

Fitting the ground state energy per site as $E_g/L = a + bL^\beta + cL^2$, I plot the powers $d$ versus $\beta$ in Fig. 2. I see that the power $d$ do not show agreements with theoretical predictions $\beta + 1$ in $E_g(m = 0)/L$. These disagreements may be caused by the oscillating Umklapp process which becomes relevant at only $m = 0$ filling. On the contrary, the oscillating Umklapp process is irrelevant at $m \neq 0$. Actually, in Fig. 2 I see that the power $d$ show agreements with theoretical predictions $\beta + 1$. I shall discuss the $\beta = 2$ case later.
Figure 2: The numerically calculated powers $d$ in the ground state energies $E_g(m = 1/L)/L$ and $E_g(m = 0)/L$ are shown versus $\beta$ for $g = 0.5$. Here I use the scaling form: $E_g/L = a + \frac{b}{L^2} + \frac{c}{L^4}$, where $a, b, c$ and $d$ are determined numerically. If the LRI is not present, the energy finite size scaling must take the form: $E_g/L = A + \frac{B}{L^2} + \frac{C}{L^4}$, where $A, B$ and $C$ are constant values.
predictions $\beta + 1$ in $E_g(m = 1/L)/L$ except for $\beta = 2$.

As I stated above, for $\beta = 2$, the power $c$ in the excitation energy $L\Delta E(m = 1/L) = a + \frac{b}{L^2} + \frac{d}{L^3}$ apparently shows disagreement with theoretical value $\beta - 1$ and likewise for $\beta = 2$, the power $d$ in the ground state energy $E_g(m = 1/L)/L = a + \frac{b}{L^2} + \frac{c}{L^3}$ apparently shows disagreement with theoretical value $\beta + 1$. I investigate the reason for these disagreements.

In Fig. 3 I show the numerically obtained coefficient of $1/L^d$ in the size scalings $E_g(m = 0)/L, E_g(m = 1/L)/L = a + \frac{b}{L^2} + \frac{c}{L^3}$ and the numerically obtained coefficient of $1/L^c$ in the size scaling $L\Delta E(m = 1/L) = a + \frac{b}{L^2} + \frac{d}{L^3}$. I observe that the coefficient of $1/L^c$ in $L\Delta E(m = 1/L)$ and the coefficient of $1/L^d$ in $E_g(m = 1/L)/L$ become small around $\beta = 2$. So for $\beta = 2$, $1/L^2$ dependence appears rather than $1/L$ in $L\Delta E(m = 1/L)$ (see Fig. 1). Likely for $\beta = 2$, $1/L^4$ dependence appears rather than $1/L^3$ in $E_g(m = 1/L)/L$ (see Fig. 2). I observe that the coefficient of $1/L^d$ in $E_g(m = 0)/L$ show the different behaviour from those in $E_g(m = 1/L)/L$ in Fig. 3. This difference may come from the spatially oscillating Umklapp process that opens the gap at $m = 0$ and disturbs the finite size scaling.

I can obtain $A(\beta)$ in the scalings (40) and (45) by evaluating the integrals. The results are
shown in Fig. 4 (a) and (b). The analytical $A(\beta)$ in the scalings (40) and $A(\beta, s)$ in the scalings (45) for $s = 0$ fit with the points in Fig. 3 well. The curve only for $s = 0$ in Fig. 4(b) shows the good fitting. This point shall be discussed later. These reveal that the present numerical calculation of the tight-binding model agrees with the CFT analysis of the long-range forward scattering.

![Figure 4](image-url)

Figure 4: (a) $A(\beta)$, the coefficient of $1/L^{\beta}$, in the eq. (40) is shown. I see that $A(\beta)$ has zero point close to $\beta = 2$. This curve coincides with the results from the numerical calculation in the tight-binding model shown in Fig. 3 (b) $A(\beta)$, the coefficient of $1/L^{\beta}$, in the eq. (45) is shown for some $s$. Analytically only $s = 0$ is meaningful for particle excitations. $A(\beta)$ for $s = 0$ has zero point close to $\beta = 2$. This coincides with the results from the numerical calculation in the tight-binding model shown in Fig. 3.

Next I survey whether the long-range tight-binding model satisfies the necessary condition of CFT. The operator $\cos \sqrt{2} \phi$ has the scaling dimensions $K/2$ and the operator $e^{\pm i \sqrt{2} \theta}$ has $1/2K$ in the regime of the TL liquid. The two quantities $2K/v$ and $vK/2$ are the compressibility and the Drude weight respectively in the regime of the TL liquid. If $c = 1$ CFT is valid to the tight-binding model with the LRI, the two quantities are related to the two excitations with the
symmetries \( q = \pi, m = 0 \) and \( q = \pi, m = 1/L \) respectively:
\[
2K/v = 1/(L\Delta E(m = 1/L, q = \pi)) \equiv \chi
\]
\[
vK/2 = L\Delta E(q = \pi) \equiv D.
\] (15)

I show the numerically calculated quantities \( \chi \) and \( D \) in Fig. 5 and 6 where I use the sizes \( L = 16, 18 \) and 20 and extrapolate the data. For \( g < 0 \), \( \chi \) (which is the susceptibility, irrespective of the CFT arguments) exhibits the rapid increase which suggests the phase separation. In spin variables’ language for (1), this phase separation is nothing but the ferromagnetic phase. Hence for the larger \( \beta \) the point of the phase separation approaches to \(-1\). For \( g > 0 \) I see the weak tendency that the quantity \( \chi \) becomes smaller as \( \beta \) is smaller for \( g \) less than about 1. I find that the quantity \( D \) of \( g > 0 \) become larger as \( \beta \) approaches to \( \beta = 1 \).

In Fig. 7 I plot the velocity versus the strength \( g \) for the various powers \( \beta \), where the velocity is defined by
\[
v = \frac{L}{2\pi}\Delta E(q = 2\pi/L).
\] (16)

I see that the velocities are finite values for \( \beta > 1 \), as is expected. There are the points where the velocities are zero, implying the phase separation.

In Fig. 8 I plot the quantity \( \frac{D}{\chi^2} \) versus the strength \( g \) for the various powers \( \beta \). If the present system is described by \( c = 1 \) CFT, this quantity is 1 from eqs. (15). I find the regions
Figure 6: The extrapolated $vK (= 2D)$ is plotted versus the strength $g$. I use the scaling form $L\Delta E = a + \frac{b}{L^c}$, where $a,b$ and $c$ are determined numerically.

Figure 7: The extrapolated spin wave velocity $v$ is plotted versus the strength $g$. I use the scaling form $L\Delta E = a + \frac{b}{L^c}$, where $a,b$ and $c$ are determined numerically.
where $\frac{D}{\chi v^2} = 1$ in Fig. 8. The regions become wider as $\beta$ approaches to 1 for $g > 0$. For larger $\beta$, the normalization breaks owing to the generations of mass.

4 Discussion

I have investigated the system with the $1/r^\beta$ interaction by applying CFT to it and by the numerical calculation. At first I have analyzed TL liquid with the $1/r^\beta$ forward scattering by utilizing the CFT and I have found that the $1/r^\beta$ forward scattering works as higher order corrections in the excitation energy, whereas the effective central charge in the scaling of the ground state energy depends on the interaction and it deviates from $c = 1$. The deviation are like the solvable $1/r^2$ models \[14, 15, 16, 17\]. Next I have numerically calculated the ground state energy and excitations energies in the tight-binding model with $1/r^\beta$ interaction, which is expected to include the above $1/r^\beta$ forward scattering in the low energy. The numerical results are in accordance with the analysis with CFT of the long-range forward scattering. Furthermore I have numerically checked the normalization $\frac{D}{\chi v^2} = 1$, which is the necessary condition for $c = 1$ CFT.

For $\beta \approx 2$, the coefficient $A(\beta)$ in the ground state energy vanishes. This seems to correspond to the exact solution for $\beta = 2$ \[17\] which states that the finite size scaling of ground state has no higher order term than $1/L$. The coefficient $D(\beta)$ of $1/L^3$ in eq. (42) does not vanish for $\beta = 2$. However the present argument is the first order perturbation theory. With higher order
treatments, I may clarify this. In any case, with consistency in many points I could construct CFT in the system with non-local interaction.

The numerical calculations in the tight-binding model support the finite size scalings (40) and (45). In one particle excitation energy \( L\Delta E(m = 1/L) \), the coefficients of \( 1/L^\beta \) fit with \( s = 0 \) case in Fig. 4. The coefficients from the long-range forward scattering are related with \( \langle \varphi \rangle \) and (45). In one particle excitation energy \( e \), I may clarify this. In any case, with consistency in many points I could construct the size effect of the compressibility: \( \text{forward scattering enhances the metallic character. For fairy large interaction the long-range effect of the Drude weight (proportional to the charge stiffness) is now given by} \)

\[
\Delta E(z) = 1/4K (z - z')^2 : e^{i\sqrt{2}z}\theta(z,z') : +\text{reg.}
\]

\[
\tilde{T}(z) : e^{i\sqrt{2}z}\theta(z,z') : = 1/4K (z - z')^2 : e^{i\sqrt{2}z}\theta(z,z') : +\text{reg.,}
\]

where I define \( T(z) = -\frac{2}{K}(\partial\varphi(z))^2 \), \( \tilde{T}(z) = -\frac{2}{K}(\tilde{\partial}\varphi(z))^2 \) and \( \theta(z,z) \equiv \frac{1}{L}(\varphi(z) - \varphi(z)) \). From the first and the second eqs., I see \( C_{\alpha 0} = -i\sqrt{2}/4 \), \( C_{\alpha 1} = i\sqrt{2}/4 \) for \( \alpha = 1 \) and \( C_{\alpha 0} = C_{\alpha 1} = 0 \) otherwise, where \( 0(0) \) and \( 1 \) denote \( \partial\varphi(z) \) \( (\tilde{\partial}\varphi(z)) \) and \( e^{i\sqrt{2}z}\theta(z,z) : \). From the third and the fourth eqs., I see : \( e^{i\sqrt{2}z}\theta(z,z) : \) have the conformal dimension (1/4K, 1/4K) and spin 0. As \( i(\partial\varphi(z) - \tilde{\partial}\varphi(z))/2 \) is associated with \( \partial_\sigma(\sigma) \) for \( z = \exp(2\pi i) \), I obtain

\[
\langle \alpha|\partial_\sigma(\sigma)|1 \rangle = \begin{cases} 
\frac{2\pi}{T} i(C_{110} - C_{110})/2 = \frac{\sqrt{2}}{4} \frac{2\pi}{L} \text{ for } \alpha = 1 \\
0 \text{ otherwise,}
\end{cases}
\]

which means that only \( s = 0 \) is relevant for the particle excitation and the last eq. in (13) has no cosine term.

I would discuss the size effects for \( \beta = 1 \). As seen in the eqs. (29) and (45), the velocity shows the weak divergence for the size and the Luttinger parameter vanishes gradually for increasing size. This is consistent with the numerical tendency (see Figs. 8 and 9 in Ref. [12]). The size effect of the Drude weight (proportional to the charge stiffness) is now given by \( vk/2 \sim \text{const.} \) as the logarithmic contributions cancel. The numerical data (see Fig. 7 in Ref. [12]) shows the metallic behaviour at small and intermediate magnitude interaction strength (larger than the CDW transition point \( V = 2 \) by the short range interaction). I think that the long-range forward scattering enhances the metallic character. For fairy large interaction the long-range Umklapp scattering becomes relevant and the charge stiffness is suppressed. Finally I would add the size effect of the compressibility:

\[
L\Delta E(\rho = 1/2 + 1/L) \equiv 1/\chi = O(\ln L) \to \infty, \tag{19}
\]
which comes from the results (29) by the RG analysis and the CFT arguments. The compressibility \( \chi \) goes to 0 weakly for increasing size.

To summarize, within the perturbation theory I have constructed CFT in the TL liquid with \( 1/r^\beta \) long-range forward scattering. I have found that the interaction gives the nontrivial behaviour for \( \beta = \text{odd} \) and \( \beta \approx 2 \). I have numerically checked the finite size scalings obtained from CFT in the tight-binding model with \( 1/r^\beta \) LRI. Our analysis and numerical calculations exhibit consistency with each other.

Appendixes

1. Renormalization group equation

\[ 0 < \beta < 1 \text{ or } 1 < \beta < 3 \]

I derive the renormalization group equations heuristically. Let us start from the action (4):

\[
S = \sum_w \sum_{q=-\Lambda}^{\Lambda} \frac{2\pi}{K} (q^2 + w^2) |\phi(q,w)|^2 + g \sum_w \sum_{q=-\Lambda}^{\Lambda} q^2 V(q)|\phi(q,w)|^2
\]

\[
= \sum_w \{ \sum_{q=-\Lambda/b}^{\Lambda/b} + \sum_{q=-\Lambda}^{\Lambda} \} + g \sum_w \{ \sum_{q=-\Lambda/b}^{\Lambda/b} + \sum_{q=-\Lambda}^{\Lambda} \}.
\]

(20)

The partition function is

\[
Z = \int D\phi_{\text{slow}} D\phi_{\text{fast}} \exp(-S^0_{\text{slow}} - S^0_{\text{fast}} - S^g_{\text{slow}} - S^g_{\text{fast}}).
\]

(21)

Thus I can integrate out \( S_{\text{fast}} \) (\(|q| > \Lambda/b \) component) simply and obtain

\[
Z = \int D\phi_{\text{slow}} \exp(-S^0_{\text{slow}} - S^g_{\text{slow}}).
\]

(22)

The remaining procedure of the renormalization is the scale transformation

\[ q \to q/b, \ w \to w/b \text{ and } \phi \to \phi b^2, \]

(23)

where I choose the dynamical exponent 1. The results are

\[
S^0_{\text{slow}} \to S^0
\]

\[
S^g_{\text{slow}} \to g \sum_w \sum_{q=-\Lambda}^{\Lambda} q^2 V(q/b)|\phi(q,w)|^2
\]

\[
\to gb^{1-\beta} \sum_w \sum_{q=-\Lambda}^{\Lambda} q^2 V(q)|\phi(q,w)|^2,
\]

(24)
where I use \( V(q) - A \sim q^{\beta-1} \) from the behaviours (7). Hence I obtain the renormalization group eq.

\[
\frac{dg(b)}{db} = (1 - \beta) \frac{g(b)}{b}.
\] (25)

Substituting \( l = \ln b \) into this, I obtain renormalization group eqs.

\[
\frac{dg}{dl} = (1 - \beta) g, \\
\frac{d}{dl} \left( \frac{v}{K} \right) = 0, \\
\frac{d}{dl} \left( \frac{1}{vK} \right) = 0.
\] (26)

The TL parameter \( K \) is not renormalized but it shifts due to the constant \( A \).

\( \beta = 1 \)

The dispersion relation of the Coulomb interaction includes the marginal part \( w \sim q \) and \( w \sim q \sqrt{\ln q} \) as well as \( \beta > 1 \) case. Integrating out the fast moving part, I obtain the effective action of the slow part

\[
S_{\text{slow}} = \sum_{w} \sum_{q=\Lambda/b}^{\Lambda/b} \frac{2\pi}{K} (vq^2 + w^2/v)|\phi(q, w)|^2 + g \sum_{w} \sum_{q=\Lambda/b}^{\Lambda/b} q^2 V(q)|\phi(q, w)|^2,
\] (27)

where I dare to leave the velocity in the Gaussian part. Note that I need not the renormalization of the velocity in the case \( \beta > 1 \). After the scale transformation, I obtain the eqs.

\[
\frac{dg}{dl} = 0, \\
\frac{d}{dl} \left( \frac{v}{K} \right) = \frac{gA}{2\pi}, \\
\frac{d}{dl} \left( \frac{1}{vK} \right) = 0,
\] (28)

where \( A \) is the constant appearing in (7). I see that \( K \) and the velocity \( v \) is renormalized instead of the no renormalization of \( g \). The forward scattering become relevant through \( K, v \) and drive the system away from the TL fixed point. Note that this result holds irrespective of any filling \( k_F \). From these eqs., the size dependences of \( v \) and \( K \) are given by

\[
v(b) \sim \sqrt{\ln L}, \\
K(b) \sim 1/\sqrt{\ln L}.
\] (29)

The velocity diverges weakly for long distances, which is consistent with the estimations of \( v = \frac{dw}{dq} \) from the behaviours (7).
\[ \beta = 3 \]

I use \( V(q) = A + Bq^2 \ln q + Cq^2 + \cdots \) in the behaviours (7). The g term of (20) is

\[ \sum \sum_{w, q = -\lambda/b}^{\lambda/b} q^2(g_1(0)q^2 \ln q + g_2(0)q^2), \quad (30) \]

where the couplings \( g_1(0) \) and \( g_2(0) \) are defined by \( gB \) and \( gC \) respectively. For the scale transformation (23), the g term is changed to

\[ \sum \sum_{w, q = -\lambda}^{\lambda} q^2[g_1(0)q^2 \ln q/b^2 + (g_2(0)/b^2 + g_1(0)q^2 \ln q/b^2)], \quad (31) \]

Thus I obtain

\[ g_1(b) = \frac{1}{b^2} g_1(0) \]
\[ g_2(b) = g_1(0) \frac{1}{b^2} \ln \frac{1}{b} + g_2(0) \frac{1}{b^2}. \quad (32) \]

By \( l = \ln b \), I write this as

\[ \frac{dg_1(l)}{dl} = -2g_1(l) \]
\[ \frac{dg_2(l)}{dl} = -2g_2(l) - g_1(l). \quad (33) \]

\[ \beta > 3 \]

This case is same as eq. (25) putting \( \beta = 3 \).

2. CFT in the TL liquid with LRI

The Hamiltonian in the finite strip from the action (3) is

\[ H = H_{\text{TL}} + g \int_D d\sigma_1 d\sigma_2 \partial_\sigma_1 \phi(\sigma_1) \partial_\sigma_2 \phi(\sigma_2) V(\mid \sigma_1 - \sigma_2 \mid) \theta(\mid \sigma_1 - \sigma_2 \mid - \alpha_0), \quad (34) \]

where \( H_{\text{TL}} \) is TL liquid and \( D \) means the region \( D = \{ \mid \sigma_1 - \sigma_2 \mid \leq L, -L/2 \leq \sigma_1, \sigma_2 \leq L/2 \} \).

I introduce the step function \( \theta(x) \) to avoid the ultra violet divergences which come from \( V(x) \) and the operator product expansion of \( \partial_\sigma \phi(\sigma) \). For the small perturbation \( g \) the ground state energy \( E_g \) varies as

\[ E'_g - E_g = g \int_D d\sigma_1 d\sigma_2 V(\mid \sigma_1 - \sigma_2 \mid) < 0|\partial_\sigma_1 \phi(\sigma_1) \partial_\sigma_2 \phi(\sigma_2)|0 > \theta(\mid \sigma_1 - \sigma_2 \mid - \alpha_0) \]
\[ = - \frac{g}{4} \int_D d\sigma_1 d\sigma_2 V(\mid \sigma_1 - \sigma_2 \mid) < 0|\partial_\sigma_1 \phi(\sigma_1) \partial_\sigma_2 \phi(\sigma_2)|0 > \mid_{\tau_1 = \tau_2 = 0} \theta(\mid \sigma_1 - \sigma_2 \mid - \alpha_0), \quad (35) \]
where I introduce the coordinates \( w = \tau + i\sigma \) \((-L/2 \leq \sigma \leq L/2, -\infty < \tau < \infty\) and \( |0\rangle \) is the ground state of \( H_{\text{TL}} \). From the characters of the Gaussian part (TL liquid part) I can separate as \( \phi(\sigma, \tau) = \varphi(w) + \tilde{\varphi}(\bar{w}) \) and derive \( < 0|\partial_{w_1} \tilde{\varphi}(\bar{w}_1)\partial_{w_2} \varphi(w_2)|0 >= 0 \). The content of the brackets is modified as follows:

\[
< 0|\partial_{w_1} \varphi(w_1)\partial_{w_2} \varphi(w_2)|0 > + < 0|\partial_{\bar{w}_1} \tilde{\varphi}(\bar{w}_1)\partial_{\bar{w}_2} \tilde{\varphi}(\bar{w}_2)|0 > \big|_{\tau_1=\tau_2=0} = \frac{K}{4} \left( \frac{2\pi}{L} \right)^{2\Delta} \frac{1}{z_1 (1 - \frac{2\Delta}{z_1})^2} + \left( \frac{2\pi}{L} \right)^{2\Delta} \frac{1}{z_1 (1 - \frac{2\Delta}{z_1})^2} \big|_{\tau_1=\tau_2=0}
\]

where I transform the correlation function \( < \partial_{z_1} \tilde{\varphi}(z_1)\partial_{z_2} \tilde{\varphi}(z_2) > = -\frac{K}{4(z_1-z_2)^2} \) in \( \infty \times \infty \) plane to that in the strip \( w \) through \( z = \exp \frac{2\pi w}{L} \). At present case \( \partial_w \tilde{\varphi}(w) \) (\( \partial_w \tilde{\varphi}(\bar{w}) \)) have the spin \( s = 1(-1) \) and conformal dimension \( \Delta = 1(\bar{\Delta} = 1) \). Hence I obtain

\[
E'_g - E_g = \frac{gK\pi^2}{4} \left( \frac{\pi}{L} \right)^{\beta} \int_{-1/2}^{1/2} dx' \frac{1}{(\sin \pi |x'|)^{3\beta} \sin^2 \pi x} \theta(|x'| - \alpha_0)
\]

(36)

where I impose the periodic boundary condition and use the interaction potential \( V(x) = 1/(\frac{L}{\pi} \sin(\frac{\pi}{L}))^\beta \). Putting \( \epsilon = \alpha_0/L \) for convenience, I give the differential of the integral part:

\[
\frac{\partial}{\partial \epsilon} \int_{-1/2}^{1/2} dx' \frac{1}{(\sin \pi |x'|)^{3\beta} \sin^2 \pi x} \theta(|x'| - \epsilon) = -\frac{2}{(\sin \pi |\epsilon|)^{3\beta} \sin^2 \pi \epsilon}
\]

(38)

After integrating the Taylor expansion about \( \epsilon \) of this quantity, I obtain

\[
\int_{-1/2}^{1/2} dx' \frac{1}{(\sin \pi |x'|)^{3\beta} \sin^2 \pi x} \theta(|x'| - \epsilon) = \text{const.} + \frac{2}{\pi} \left[ (\pi \epsilon)^{-\beta-1} + \frac{\beta + 2}{6(\beta - 1)} (\pi \epsilon)^{-\beta + 1}
\right.
\]

\[
+ \frac{1}{\beta - 3} \left\{ \frac{1}{120} (\beta + 2) - \frac{1}{72} (\beta + 1)(\beta + 2) \right\} (\pi \epsilon)^{-\beta + 3}
\]

\[
+ O((\pi \epsilon)^{-\beta + 5})
\]

(39)

where \( \beta \neq \text{odd} \). Therefore I can write the corrections in the form:

\[
E'_g - E_g = \frac{gK}{2} \left[ \frac{A(\beta)}{L^\beta} + B(\beta) L + \frac{C(\beta)}{L^3} + \frac{D(\beta)}{L^5} + O(\frac{1}{L^7}) \right].
\]

(40)

Here \( B(\beta) \), \( C(\beta) \) and \( D(\beta) \) are given by

\[
B(\beta) = \frac{\alpha_0^{-\beta-1}}{1 + \beta}
\]

\[
C(\beta) = \frac{\pi^2(2 + \beta)}{6(\beta - 1)} \alpha_0^{-\beta + 1}
\]

\[
D(\beta) = \frac{\pi^4}{\beta - 3} \left[ \frac{1}{120} (\beta + 2) - \frac{1}{72} (\beta + 1)(\beta + 2) \right] \alpha_0^{3-\beta}.
\]

(41)

I can obtain \( A(\beta) \) by evaluating the above integral numerically. The result is shown in Fig. 4(a).
For $\beta=\text{odd}$, there exists the logarithmic correction instead of the eq. (11). The results for respective $\beta$ are

$$E'_g - E_g = \begin{cases} 
  g\left[A_1 + BL + C_1 \ln \frac{1}{L} + D_1 \frac{1}{L^3} + O(\frac{1}{L^5})\right] & \text{for } \beta = 1 \\
  g\left[A_3 + BL + C_3 + D_3 \ln \frac{1}{L} + O(\frac{1}{L^3})\right] & \text{for } \beta = 3 \\
  g\left[A_5 + BL + C_5 + D_5 \ln \frac{1}{L} + O(\frac{1}{L^3})\right] & \text{for } \beta = 5 \\
  \cdots 
\end{cases} \quad (42)$$

The C terms in eqs. (10) and (42) contribute to deviation of the central charge. The present LRI inevitably contains the contribution from the short range interaction: $\delta(x)$. The C term does not come from the short range types of interactions because the vacuum expected value $\langle (\partial_x \phi)^2 \rangle$ vanishes. Thus C term is intrinsic in the present system with the LRI under periodic boundary condition. Because the velocity is not renormalized as I have seen from the renormalization group eqs., the C term contributes to the deviations of the effective central charge from the ground state energy.

Next I derive the corrections for the energy of the excited state:

$$E'_n - E_n = g \int_D d\sigma_1 d\sigma_2 V(|\sigma_1 - \sigma_2|) < n|\partial_{\sigma_1} \phi(\sigma_1) \partial_{\sigma_2} \phi(\sigma_2)| n > \theta(|\sigma_1 - \sigma_2| - \alpha_0)$$

$$\quad = g \int_D d\sigma_1 d\sigma_2 V(|\sigma_1 - \sigma_2|) \sum_{\alpha} < n|\partial_{\sigma_1} \phi(\sigma_1)| \alpha > < \alpha| \partial_{\sigma_2} \phi(\sigma_2)| n > \theta(|\sigma_1 - \sigma_2| - \alpha_0)$$

$$\quad = 4g \sum_{\alpha} C_{nj\alpha} C_{n'j\alpha} \left(\frac{2\pi}{L}\right)^2 \int_0^{1/2} dy \frac{1}{(\sin \pi |y|)^{\beta}} \cos 2\pi (s_n - s_{\alpha}) y \theta(|y| - \frac{\alpha_0}{L}), \quad (43)$$

where I use the results by Cardy [22]:

$$< n|\phi(\sigma)| \alpha > = C_{nj\alpha} \left(\frac{2\pi}{L}\right)^{\beta} e^{\frac{2\pi i (s_n - s_{\alpha})^2}{L}}. \quad (44)$$

Here $j$ means $\partial \phi$ and $|n>$ is the excited state of $H_{TL}$. I can derive the size dependence of eq. (43) from likewise treatments as the ground state. After taking the derivative about $1/L$, I expand about $1/L$. Integrating them, I obtain

$$E'_n - E_n = \begin{cases} 
  16\pi^2 g \sum_{\alpha} C_{nj\alpha} C_{n'j\alpha} \left\{ \frac{A(s_n - s_{\alpha}, \beta)}{L^3} + \frac{B(\beta)}{L} + \frac{C(s_n - s_{\alpha})}{L^2} + \frac{D(s_n - s_{\alpha})}{L} + O(\frac{1}{L}) \right\} & \beta \neq \text{odd} \\
  16\pi^2 g \sum_{\alpha} C_{nj\alpha} C_{n'j\alpha} \left\{ \frac{A(s_n - s_{\alpha})}{L^3} + B(1) \ln \frac{1}{L} + C(s_n - s_{\alpha}) \frac{1}{L^2} + D(s_n - s_{\alpha}) \frac{1}{L} + O(\frac{1}{L^3}) \right\} & \beta = 1 \\
  16\pi^2 g \sum_{\alpha} C_{nj\alpha} C_{n'j\alpha} \left\{ \frac{A(s_n - s_{\alpha})}{L^3} + B(1) + C(s_n - s_{\alpha}) \ln \frac{5}{L} + D(s_n - s_{\alpha}) \frac{1}{L^2} + O(\frac{1}{L^3}) \right\} & \beta = 3 \\
  16\pi^2 g \sum_{\alpha} C_{nj\alpha} C_{n'j\alpha} \left\{ \frac{A(s_n - s_{\alpha})}{L^3} + B(1) + C(s_n - s_{\alpha}) \ln \frac{5}{L} + D(s_n - s_{\alpha}) \frac{1}{L^2} + O(\frac{1}{L^3}) \right\} & \beta = 5 \\
  \cdots \quad (45)
\end{cases}$$
where $B$ are the constant independent of $s_n, s_\alpha$. Here for $\beta \neq$ odd, $B(\beta), C(s_n - s_\alpha, \beta)$ and $D(s_n - s_\alpha, \beta)$ are given by

\[
B(\beta) = \frac{1}{(\alpha_0 \pi)^{\beta-1}\pi(\beta - 1)} \\
C(s_n - s_\alpha, \beta) = \frac{1}{(\alpha_0 \pi)^{\beta-3} \beta^3 - 2(s_n - s_\alpha)^2} \frac{1}{\pi(\beta - 3)} \\
D(s_n - s_\alpha, \beta) = \frac{1}{(\alpha_0 \pi)^{\beta-5}} \left[ -\frac{(s_n - s_\alpha)^2}{3} + \frac{1}{180} \right] + \frac{\beta^2}{72} + \frac{2(s_n - s_\alpha)^4}{3} \frac{1}{\pi(\beta - 5)}. \tag{46}
\]

It is not straightforward to determine $A(s_n - s_\alpha, \beta)$ generally. However about one particle excitation($s_n = 0$), I can obtain $A(\beta, s_\alpha)$, which is shown for some $s_\alpha$ in Fig. 4(b). Actually further consideration about the operator product expansion leads $s_n = s_\alpha = 0$ (see section 4.).

I refer to the $O(1/L)$ dependences. These are due to the fact that the LRI includes the short range type interaction. Actually I can derive the same form

\[
\frac{g}{L} \sum_\alpha C_{n\alpha}C_{\alpha n} \tag{47}
\]

as an ordinary finite scaling by replacing as $V(|x|)\theta(|x| - \alpha_0) \rightarrow \delta(x)$. As $(\partial \phi)^2$ is a part of the TL liquid, the $O(1/L)$ term can be erased under subtracting such the contributions first. Thus the $O(1/L)$ term is not intrinsic.

Summarizing the discussions in this appendix, I can prove that the Hamiltonian (34) is described by $c = 1$ CFT for $\beta > 1$ in the excitation energy. However the effective central charge from the ground state depends on the interaction and deviates from 1. I find the nontrivial behaviors when $\beta = \text{odd}$, which corresponds to the integer points of the modified Bessel function as appearing in the behaviours (7).

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Conformal field theory in Tomonaga-Luttinger model with $1/r^\beta$
long-range interaction

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March 22, 2022

Abstract

I attempt to construct $U(1)$ conformal field theory (CFT) in the Tomonaga-Luttinger (TL) liquid with $1/r^\beta$ long-range interaction (LRI). Treating the long-range forward scattering as a perturbation and applying CFT to it, I derive the finite size scalings which depend on the power of the LRI. The obtained finite size scalings give the nontrivial behaviours when $\beta$ is odd and is close to 2. I find the consistency between the analytical arguments and numerical results in the finite size scaling of energy.

1 Introduction

Electron systems have attracted our much attention in the low energy physics. As the dimension of the electron systems decrease, the charge screening effects become less important. In spite of these facts, models with short-range interaction have been adopted in many researches of one dimensional electron systems. The recent advanced technology makes it possible to fabricate quasi-one-dimension systems. Actually in low temperature the effect of Coulomb force has been observed in GaAs quantum wires [1], quasi-one-dimensional conductors [2, 3, 4] and 1D Carbon nanotubes [5, 6, 7].

The systems with $1/r$ Coulomb repulsive forward scattering was investigated on the long distance properties by bosonization techniques [8]. The charge correlation function decays with the distance as $\exp(-\text{const.}(\ln x)^{1/2})$ more slowly than any power law. The momentum distribution function and the density of state does not show the simple power law singular behaviour. The logarithmic behaviours appear in the power [9]. These mean that the system is driven to
the Wigner crystal which is quite different from the ordinary TL liquid. The investigation for the interaction \(1/r^{1-\epsilon}\) through the path integral approach [10] reconfirms the slower decaying of the single particle Green function for \(\epsilon = 0\) and leads the faster decay for \(0 \leq \epsilon(\ll 1)\) than any power type.

The numerical calculation in the electron system with the Coulomb interaction shows that the larger range of the interaction causes the insulator (charge density wave) to metal (metallic Wigner crystal) transition [11]. In the spinless fermion system, the convergence of the Luttinger parameters exhibits the quasi-metallic behaviour different from the simple TL one [12].

As I will discuss below, the forward scattering is irrelevant for \(\beta > 1\). As an instance of the effect of the long-range Umklapp scattering, it was reported that the \(1/r^2\) interaction makes the system gapless to gapful through the generalized Kosterlitz-Thouless transition [13].

In this paper I discuss CFT in the system with LRI. The basic assumptions of CFT are symmetries of translation, rotation, scale and special conformal transformation. Besides them I assume the short-range interaction in the CFT. Hence it is a subtle problem whether the CFT can describe the system with LRI.

Of LRIs, up to now, the solvable models with \(1/r^2\) interaction were discussed [14, 15, 16, 17]. With the Bethe ansatz, the conformal anomaly and the conformal dimensions were calculated and the system proved to be described by \(c = 1\) CFT. In fact the central charge from the specific heat agrees with \(c = 1\). On the other hand, the ground state energy is affected by the LRI and the periodic nature. The effective central charge deviates from \(c = 1\).

In general, the CFT for LRI which breaks the locality, has been left as unsettled problem. It is significant to clarify the validity of the CFT to the systems with LRI. I investigate the tight-binding model with \(1/r^\beta\) interaction as one of such problems. The low energy effective model consists of TL liquid, the long-range forward scattering and the long-range spatially oscillating Umklapp scattering. Extending arguments appearing in Ref. [18] to the TL liquid with the long-range forward scattering, I derive the finite size scalings. In the tight-binding model with \(1/r^\beta\) interaction, I calculate numerically the size dependences of energy and the coefficients of \(1/L^y\). And I see numerically the relations between the velocity, susceptibility and Drude weight, which CFT requires.

2 Field theoretical approach

I consider the following tight-binding Hamiltonian of the interacting spinless Fermions:

\[
H = -\sum_j (c_j^\dagger c_{j+1} + \text{h.c}) + \frac{g}{2} \sum_{i \neq j} (\rho_i - 1/2) V(i - j)(\rho_j - 1/2),
\]

I define the “effective central charge” by \(c = \frac{6\pi}{\beta} \) in the finite size scaling of the ground state energy \(E_g = aL - \frac{b}{L}\). I use the word “effective central charge” in this sense.
where the operator \( c_j (c_j^\dag) \) annihilates (creates) the spinless Fermion in the site \( j \) and \( \rho_j = c_j^\dag c_j \) is the density operator. In order to treat this model under the periodic boundary condition, I define the chord distance between the sites \( i \) and \( j \): \( r_{i,j} = (\frac{L}{\pi} \sin \frac{\pi(i-j)}{L}) \) where \( L \) is the site number. Using this, I express the LRI as \( V(i-j) = \frac{1}{(\frac{L}{\pi} \sin \frac{\pi(i-j)}{L})^{\beta}} \).

By the bosonization technique, I obtain the effective action of the Hamiltonian (1) for the arbitrary filling:

\[
S = \int d\tau dx \frac{1}{2\pi K} (\nabla \phi)^2 + g \int d\tau dx dx' \partial_x \phi(x, \tau) V(x-x') \partial_x \phi(x', \tau)
+ g' \int d\tau dx dx' \cos(2k_F x + \sqrt{2} \phi(x, \tau)) V(x-x') \cos(2k_F x' + \sqrt{2} \phi(x', \tau)),
\]

where \( V(x) = \frac{1}{|x|^\beta} \), \( K \) is the TL parameter and \( k_F \) is the Fermi wave number. And \( g' \) is the coupling constant proportional to \( g \). The first term of (2) is the TL liquid and the second term is the long-range forward scattering. The last term is the spatially oscillating Umklapp process which includes \( \cos 2\sqrt{2} \phi \) which comes from the interaction between the neighbour sites.

Schulz analyzed the effects of the Coulomb forward scattering by the bosonization technique in the electron system [8]. He discussed the quasi-Wigner crystal of electrons due to the Coulomb forward scattering. Here I focus on the effects of the \( 1/\mu^\beta \) forward scattering in the spinless Fermions system. I treat the action :

\[
S = \int d\tau dx \frac{1}{2\pi K} (\nabla \phi)^2 + g \int d\tau dx dx' \partial_x \phi(x, \tau) V(|x-x'|) \partial_x \phi(x', \tau)
\]

for any filling \( k_F \). To investigate in the Fourier space, I choose the form \( V(x) = \frac{1}{(x^2+\alpha^2)^{\beta/2}} \), where \( \alpha \) is the ultra-violet cut-off. In the wave number space, the action (3) is expressed as

\[
S = \int dq dw \left\{ \frac{2\pi}{K} q^2 + w^2 \right\} + gq^2 V(q) |\phi(q, w)|^2,
\]

where \( V(q) \) is the Fourier transformation of \( V(x) \):

\[
V(q) = \frac{2\sqrt{\pi}}{\Gamma(\beta/2) 2^{\beta/2-1/2}} (\alpha q)^{\beta/2-1/2} K_{\beta/2-1/2}(\alpha q).
\]

Here \( K_\nu(x) \) is the modified Bessel function of \( \nu \)th order and \( \Gamma(x) \) is the gamma function. From this, the dispersion relation is

\[
w^2 = q^2 \{ 1 + \frac{gK}{2\pi} V(q) \}.
\]

The long wavelength behaviors of \( V(q) \) are given by

\[
V(q) \sim \begin{cases} 
A + B(q^\alpha)^2 + C(q^\alpha)^{\beta-1} + \cdots & \beta > 0 \text{ and } \beta \neq \text{ odd} \\
A + B \ln q^\alpha + \cdots & \beta = 1 \\
A + B(q^\alpha)^2 \ln q^\alpha + C(q^\alpha)^2 + \cdots & \beta = 3 \\
A + B(q^\alpha)^2 + C(q^\alpha)^4 \ln(q^\alpha) + D(q^\alpha)^4 + \cdots & \beta = 5 \\
\cdots,
\end{cases}
\]
where $A, B, C$ and $D$ are the functions of $\beta$. For the case where $\beta > 0$ and $\neq$ odd, the coefficient $B = B(\beta), C = C(\beta)$ is given by

$$B(\beta) = \frac{\pi^{3/2}}{4} \frac{1}{2^{\beta/2-1/2} \Gamma(\frac{5-\beta}{2}) \Gamma(\beta/2) \sin \left(\frac{\beta-1}{2}\right)}$$

$$C(\beta) = -\frac{\pi^{3/2}}{4} \frac{1}{1\cdot 2^{\beta/2-1/2} \Gamma(\frac{1+\beta}{2}) \Gamma(\beta/2) \sin \left(\frac{\beta-1}{2}\right)}.$$  

(8)

From the eqs. (6) and (7), I see that $(q\alpha)^{\beta-1}$ and $\ln q\alpha$ terms for $0 < \beta \leq 1$ affect the linear dispersion essentially. Especially for $\beta = 1$, there is the analysis by Schulz, where the charge density correlation function is calculated [8]. According to it, in the present spinless case, the LRI drives the ground state from the TL liquid to the quasi-Wigner crystal as $\beta \to 1+$. The slowest decaying part of the density correlation function is given by

$$< \rho(x)\rho(0) > \sim \cos(2k_Fx)\exp(-c\sqrt{\log x}),$$

(9)

where $c$ is a function of K, which exhibits slower spatial decay than the power decay of TL liquid.

Then I see the effects of the long-range forward scattering in the standpoints of the renormalization of $g$. The renormalization group eqs. of $g, v$ and $K$ are simply derived for the long wave-length (see Appendixes.). From the renormalization eqs., the $g$ terms are relevant for $\beta < 1$, marginal for $\beta = 1$ and irrelevant for $\beta > 1$. Thus it is expected that the system becomes the quasi-Wigner crystal caused by the forward scattering for $\beta \leq 1$ and the system becomes the TL liquid when $\beta > 1$. I see that

$$\Phi(x) \equiv \int dx^\prime \partial_x \phi(x, \tau)V(x - x^\prime)\partial_{x^\prime} \phi(x^\prime, \tau)$$

(10)

has the scaling dimension $x_g = \beta + 1$ for $1 < \beta < 3$ and $4$ for $\beta > 3$. As the weak logarithmic corrections appear for $\beta =$ odd, I here distinguish $\Phi(x)$ for $\beta = 3$ from the scaling functions. I also find the consistency on these scaling dimensions by CFT. By using the first order perturbation, I can know the effects of the long-range forward scattering. Based on CFT, the finite size scalings of energies for no perturbations are given [19, 20] by

$$\Delta E_n = \frac{2\pi v x_n}{L}$$

$$E_g = e_g L - \frac{\pi v c}{6L},$$

(11)

where $x_n$ is the scaling dimension of the primary field denoted by $n$, $v$ is the sound velocity and $c$ is the central charge. Considering the LRI, I can extract the corrections to these energy size scalings(see Appendixes.):

$$\Delta E_n = \frac{2\pi v x_n}{L}(1 + \frac{g(0)\text{const.}}{x_n L^{\beta-1}} + O(1/L^2))$$

$$E_g = (e_g + g(0)\text{const.})L - \frac{\pi v}{6L}(c + g(0)\text{const.}) + g(0)\frac{\text{const.}}{L^{\beta-1}} + O(1/L^2)),$$

(12)
where $\beta(>1)$ is not odd. And the constants are the functions of $\beta$. Note that for $\beta = \text{odd}$ cases, the logarithmic corrections appear. They correspond to the integer points of the modified Bessel function, which appear in the long-wave behaviours \cite{7}. I can reproduce these anomalies for $\beta = \text{odd}$ by the CFT. Moreover from CFT I can show that there are the anomalies in the general excitations and the ground state energy. The details are shown in Appendixes. The $O(1/L^2)$ terms come from the irrelevant field $L_{-2}L_{-2}$ and the long-range $g$ term. The first eq. of \cite{12} means that the long-range forward scattering $\Phi(x)$ has the scaling dimension $x_g = \beta + 1$ for $1 < \beta < 3$ and $4$ for $\beta > 3$ effectively. These respective scaling dimensions are consistent with the estimation from the renormalization group eqs. of $g$, that I mentioned above (see Appendixes.).

The energy finite size scalings \cite{12} mean that the LRI has the higher order influences than $1/L$ to the excitation energy and the LRI affects the $1/L$ term in the finite size scaling of the ground state energy. Here I note that it becomes difficult to calculate the central charge from finite size scalings \cite{11} unless the effects of the LRI to $O(1/L)$ terms are known.

It is notable to compare the eqs. \cite{12} with the case where the perturbations are of short-range type. Ludwig and Cardy calculated the contributions of the short-range perturbation \cite{18}. The results for the irrelevant perturbation, $-g \sum_r \phi(r)$, which has the scaling dimensions $x > 2$ are

$$
\Delta E_n = \frac{2\pi v x_n}{L} (1 + \frac{g(0)}{x_n} C_{nng}(\frac{2\pi}{L})^2 - 2)
$$

$$
E_g = (e_g + g(0)\text{const.}) L - \frac{\pi v}{6L} (c + g(0)\text{const.}) L^{2x-4} + O(1/L^{3x-6}),
$$

where the $O(g)$ terms do not appear in the ground state scaling because we set $\langle \phi \rangle = 0$ for the short-range interaction. These results mean that the $x > 2$ irrelevant field has influences of the higher order to the finite size scalings \cite{11}. And their result contains parts not so simple. There are the special points of scaling dimension $x = 1, 3, 5$, and $x = 2$ which is related to the appearance of logarithmic corrections.

To the contrary, I see $\langle \Phi \rangle \neq 0$ in the long-range case, where $\Phi$ is defined in eq. \cite{10}. The LRI gives the $O(1/L)$ intrinsic influence to the finite size scaling of the ground state energy, as appearing in the scalings \cite{12}, even if the LRI is irrelevant, that is, $x_g > 2$.

3 Numerical calculations

Through the Jordan-Wigner transformation, I transform the model \cite{1} to $S = 1/2$ spin Hamiltonian for the numerical calculations:

$$
H = -\sum_j (S_j^+ S_{j+1}^- + \text{h.c.}) + \frac{g}{2} \sum_{i \neq j} S_i^z V(|i-j|) S_j^z.
$$
I impose the periodic boundary condition $S_{L+1} = S_1$ to this model. Using the Lanczos algorithm I perform the numerical calculations for the Hamiltonian (14).

I have found analytically the corrections to the energy scalings (11) caused by the long-range forward scattering. If the oscillating Umklapp process term of (2) is irrelevant and does not disturb the energy scalings, the finite size corrections due to the forward scattering are expected to appear in the excited state energies and the ground state energy. I attempt to detect the contribution of the forward scattering.

I numerically calculate the size dependences of the excitation energy $\Delta E(m = 1/L)$ and the ground state energy $E_g(m = 0)$, $E_g(m = 1/L)$ for $g = 0.5$. Here I define the magnetization $m \equiv \sum_j S_j^z / L$ which is the conserved quantity. Fitting the one particle excitation energy as $L\Delta E(m = 1/L) = a + \frac{b}{L^c} + \frac{d}{L^2}$, I show the power $c$ versus the powers $\beta$ in Fig. 1. I see the power $c$ agrees with theoretical predictions: $\beta - 1$ except for $\beta = 2$. I shall discuss the $\beta = 2$ case later. Fitting the ground state energy per site as $E_g/L = a + \frac{b}{L^d} + \frac{c}{L^d}$, I plot the powers $d$ versus $\beta$ in Fig. 2. I see that the power $d$ do not show agreements with theoretical predictions $\beta + 1$ in $E_g(m = 0)/L$. These disagreements may be caused by the oscillating Umklapp process which becomes relevant at only $m = 0$ filling. On the contrary, the oscillating Umklapp process is irrelevant at $m \neq 0$. Actually, in Fig. 2 I see that the power $d$ show agreements with theoretical
Figure 2: The numerically calculated powers $d$ in the ground state energies $E_g(m = 1/L)/L$ and $E_g(m = 0)/L$ are shown versus $\beta$ for $g = 0.5$. Here I use the scaling form: $E_g/L = a + \frac{b}{L^d} + \frac{c}{L^4}$, where $a, b, c$ and $d$ are determined numerically. If the LRI is not present, the energy finite size scaling must take the form: $E_g/L = A + \frac{B}{L^d} + \frac{C}{L^4}$, where $A, B$ and $C$ are constant values.
predictions $\beta + 1$ in $E_g(m = 1/L)/L$ except for $\beta = 2$.

As I stated above, for $\beta = 2$, the power $c$ in the excitation energy $L\Delta E(m = 1/L) = a + \frac{b}{L^\beta} + \frac{d}{L^\gamma}$ apparently shows disagreement with theoretical value $\beta - 1$ and likewise for $\beta = 2$, the power $d$ in the ground state energy $E_g(m = 1/L)/L = a + \frac{b}{L^\beta} + \frac{c}{L^\gamma}$ apparently shows disagreement with theoretical value $\beta + 1$. I investigate the reason for these disagreements.

In Fig. 3 I show the numerically obtained coefficient of $1/L^d$ in the size scalings $E_g(m = 0)/L, E_g(m = 1/L)/L = a + \frac{b}{L^\beta} + \frac{c}{L^\gamma}$ and the numerically obtained coefficient of $1/L^c$ in the size scaling $L\Delta E(m = 1/L) = a + \frac{b}{L^\beta} + \frac{d}{L^\gamma}$. I observe that the coefficient of $1/L^c$ in $L\Delta E(m = 1/L)$ and the coefficient of $1/L^d$ in $E_g(m = 1/L)/L$ become small around $\beta = 2$. So for $\beta = 2$, $1/L^2$ dependence appears rather than $1/L$ in $L\Delta E(m = 1/L)$ (see Fig. 11). Likely for $\beta = 2$, $1/L^4$ dependence appears rather than $1/L^3$ in $E_g(m = 1/L)/L$ (see Fig. 21). I observe that the coefficient of $1/L^d$ in $E_g(m = 0)/L$ show the different behaviour from that in $E_g(m = 1/L)/L$ in Fig. 3. This difference may come from the spatially oscillating Umklapp process that opens the gap at $m = 0$ and disturbs the finite size scaling.

Figure 3: I show the numerically obtained coefficients of $1/L^d$ in the size scalings $E_g(m = 0)/L, E_g(m = 1/L)/L = a + \frac{b}{L^\beta} + \frac{c}{L^\gamma}$ and the numerically obtained coefficient of $1/L^c$ in the size scaling $L\Delta E(m = 1/L) = a + \frac{b}{L^\beta} + \frac{d}{L^\gamma}$. I observe that the coefficient of $1/L^d$ in $E_g(m = 1/L)/L$ and the coefficient of $1/L^c$ in $L\Delta E(m = 1/L)$ become small around $\beta = 2$. The coefficients of $1/L^d$ in $E_g(m = 0)/L$ show the different behaviour from that in $E_g(m = 1/L)/L$. This difference may be caused by the spatially oscillating Umklapp process term.

I can obtain $A(\beta)$ in the scalings (40) and (45) by evaluating the integrals. The results are
shown in Fig. 4 (a) and (b). The analytical $A(\beta)$ in the scalings (40) and $A(\beta, s)$ in the scalings (45) for $s = 0$ fit with the points in Fig. 3 well. The curve only for $s = 0$ in Fig. 4(b) shows the good fitting. This point shall be discussed later. These reveal that the present numerical calculation of the tight-binding model agrees with the CFT analysis of the long-range forward scattering.

![Figure 4](image)

Figure 4: (a) $A(\beta)$, the coefficient of $1/L^\beta$, in the eq. (40) is shown. I see that $A(\beta)$ has zero point close to $\beta = 2$. This curve coincides with the results from the numerical calculation in the tight-binding model shown in Fig. 3 (b) $A(\beta)$, the coefficient of $1/L^\beta$, in the eq. (45) is shown for some $s$. Analytically only $s = 0$ is meaningful for particle excitations. $A(\beta)$ for $s = 0$ has zero point close to $\beta = 2$. This coincides with the results from the numerical calculation in the tight-binding model shown in Fig. 3.

Next I survey whether the long-range tight-binding model satisfies the necessary condition of CFT. The operator $\cos \sqrt{2} \phi$ has the scaling dimensions $K/2$ and the operator $e^{\pm i \sqrt{2} \theta}$ has $1/2K$ in the regime of the TL liquid. The two quantities $2K/v$ and $vK/2$ are the compressibility and the Drude weight respectively in the regime of the TL liquid. If $c = 1$ CFT is valid to the tight-binding model with the LRI, the two quantities are related to the two excitations with the
symmetries $q = \pi, m = 0$ and $q = \pi, m = 1/L$ respectively:

\[
2K/v = 1/(L\Delta E(m = 1/L, q = \pi)) \equiv \chi \\
vK/2 = L\Delta E(q = \pi) \equiv D.
\] (15)

I show the numerically calculated quantities $\chi$ and $D$ in Fig. 5 and 6 where I use the sizes $L = 16, 18$ and $20$ and extrapolate the data. For $g < 0$, $\chi$ (which is the susceptibility, irrespective of the CFT arguments) exhibits the rapid increase which suggests the phase separation. In spin variables’ language for (1), this phase separation is nothing but the ferromagnetic phase. Hence for the larger $\beta$ the point of the phase separation approaches to $-1$. For $g > 0$ I see the weak tendency that the the quantity $\chi$ becomes smaller as $\beta$ is smaller for $g$ less than about 1. I find that the the quantity $D$ of $g > 0$ become larger as $\beta$ approaches to $\beta = 1$.

In Fig. 7 I plot the velocity versus the strength $g$ for the various powers $\beta$, where the velocity is defined by

\[
v = \frac{L}{2\pi} \Delta E(q = 2\pi/L).
\] (16)

I see that the velocities are finite values for $\beta > 1$, as is expected. There are the points where the velocities are zero, implying the phase separation.

In Fig. 8 I plot the quantity $\frac{D}{\chi^2}$ versus the strength $g$ for the various powers $\beta$. If the present system is described by $c = 1$ CFT, this quantity is 1 from eqs. (15). I find the regions
Figure 6: The extrapolated $vK(= 2D)$ is plotted versus the strength $g$. I use the scaling form $L\Delta E = a + \frac{b}{L^c}$, where $a,b$ and $c$ are determined numerically.

Figure 7: The extrapolated spin wave velocity $v$ is plotted versus the strength $g$. I use the scaling form $L\Delta E = a + \frac{b}{L^c}$, where $a,b$ and $c$ are determined numerically.
where $\frac{D}{\chi v^2} = 1$ in Fig. 8. The regions become wider as $\beta$ approaches to 1 for $g > 0$. For larger $g$, the normalization breaks owing to the generations of mass.

### 4 Discussion

I have investigated the system with the $1/r^\beta$ interaction by applying CFT to it and by the numerical calculation. At first I have analyzed TL liquid with the $1/r^\beta$ forward scattering by utilizing the CFT and I have found that the $1/r^\beta$ forward scattering works as higher order corrections in the excitation energy, whereas the effective central charge in the scaling of the ground state energy depends on the interaction and it deviates from $c = 1$. The deviation are like the solvable $1/r^2$ models [14, 15, 16, 17]. Next I have numerically calculated the ground state energy and excitations energies in the tight-binding model with $1/r^\beta$ interaction, which is expected to include the above $1/r^\beta$ forward scattering in the low energy. The numerical results are in accordance with the analysis with CFT of the long-range forward scattering. Furthermore I have numerically checked the normalization $\frac{D}{\chi v^2} = 1$, which is the necessary condition for $c = 1$ CFT.

For $\beta \approx 2$, the coefficient $A(\beta)$ in the ground state energy vanishes. This seems to correspond to the exact solution for $\beta = 2$ [17] which states that the finite size scaling of ground state has no higher order term than $1/L$. The coefficient $D(\beta)$ of $1/L^3$ in eq. (42) does not vanish for $\beta = 2$. However the present argument is the first order perturbation theory. With higher order
treatments, I may clarify this. In any case, with consistency in many points I could construct CFT in the system with non-local interaction.

The numerical calculations in the tight-binding model support the finite size scalings (40) and (45). In one particle excitation energy \( L \Delta E (m = 1/L) \), the coefficients of \( 1/L^\beta \) fit with \( s = 0 \) case in Fig. 4. The coefficients from the long-range forward scattering are related with \( \langle \varphi \rangle \) and \( (45) \). In any case, with consistency in many points I could construct CFT in the system with non-local interaction.

I can prove that only \( s = 0 \) case is relevant for the particle excitation. Using \( \langle \varphi(z)\varphi(z') \rangle = -\frac{K}{\pi} \ln(z - z') \) and \( \langle \tilde{\varphi}(\tilde{z})\tilde{\varphi}(\tilde{z}') \rangle = -\frac{K}{\pi} \ln(\tilde{z} - \tilde{z}') \), I confirm the operator product expansions:

\[
\begin{align*}
\partial \varphi(z) : e^{i\sqrt{2}\theta(z,z')}: &= -\frac{i\sqrt{2}}{4} \frac{1}{z-z'} : e^{i\sqrt{2}\theta(z',\tilde{z}')} : + \text{reg.} \\
\tilde{\partial} \tilde{\varphi}(\tilde{z}) : e^{i\sqrt{2}\theta(z',\tilde{z}')}: &= \frac{i\sqrt{2}}{4} \frac{1}{\tilde{z}-\tilde{z}'} : e^{i\sqrt{2}\theta(z',\tilde{z}')} : + \text{reg.} \\
T(z) : e^{i\sqrt{2}\theta(z',\tilde{z}')}: &= \frac{1}{4K} \frac{1}{(z-z')^2} : e^{i\sqrt{2}\theta(z',\tilde{z}')}: + \frac{i\sqrt{2}}{K \frac{1}{z-z'}} : \partial \varphi(z) ( e^{i\sqrt{2}\theta(z',\tilde{z}')}. ) : + \text{reg.} \\
\tilde{T} (\tilde{z}) : e^{i\sqrt{2}\theta(z',\tilde{z}')}: &= \frac{1}{4K} \frac{1}{(\tilde{z}-\tilde{z}')^2} : e^{i\sqrt{2}\theta(z',\tilde{z}')}: - \frac{i\sqrt{2}}{K \frac{1}{\tilde{z}-\tilde{z}'}} : \tilde{\partial} \tilde{\varphi}(\tilde{z}) ( e^{i\sqrt{2}\theta(z',\tilde{z}')}. ) : + \text{reg.},
\end{align*}
\]

where I define \( T(z) \equiv -\frac{2}{K} (\partial \varphi(z))^2 \), \( \tilde{T}(\tilde{z}) \equiv -\frac{2}{K} (\tilde{\partial} \tilde{\varphi}(\tilde{z}))^2 \), and \( \theta(z,\tilde{z}) \equiv \frac{1}{K} (\varphi(z) - \tilde{\varphi}(\tilde{z})) \). From the first and the second eqs., I see \( C_{\alpha 0} = -i\sqrt{2}/4, C_{\alpha 10} = i\sqrt{2}/4 \) for \( \alpha = 1 \) and \( C_{\alpha 10} = C_{\alpha 0} = 0 \) otherwise, where \( 0(0) \) and \( 1 \) denote \( \partial \varphi(z) ( \tilde{\partial} \tilde{\varphi}(\tilde{z}) ) \) and \( : e^{i\sqrt{2}\theta(z,\tilde{z})}. \). From the third and the fourth eqs., I see : \( e^{i\sqrt{2}\theta(z,\tilde{z})} \) have the conformal dimension \( (1/4K, 1/4K) \) and spin 0. As \( i(\partial \varphi(z) - \tilde{\partial} \tilde{\varphi}(\tilde{z}))/2 \) is associated with \( \partial \sigma(\sigma) \) for \( z = \exp(\frac{2\pi i}{L}) \), I obtain

\[
\langle \alpha | \partial \sigma(\sigma) | 1 \rangle = \begin{cases} 
\frac{2\pi}{L} \frac{i(C_{110} - C_{110})}{2} = \frac{\sqrt{2}2\pi}{L} & \text{for } \alpha = 1 \\
0 & \text{otherwise},
\end{cases}
\]

which means that only \( s = 0 \) is relevant for the particle excitation and the last eq. in (13) has no cosine term.

I would discuss the size effects for \( \beta = 1 \). As seen in the eqs. (29) and (45), the velocity shows the weak divergence for the size and the Luttinger parameter vanishes gradually for increasing size. This is consistent with the numerical tendency (see Figs. 8 and 9 in Ref. [12]). The size effect of the Drude weight (proportional to the charge stiffness) is now given by \( v k/2 \sim \text{const.} \) as the logarithmic contributions cancel. The numerical data (see Fig. 7 in Ref. [12]) shows the metallic behaviour at small and intermediate magnitude interaction strength (larger than the CDW transition point \( V = 2 \) by the short range interaction). I think that the long-range forward scattering enhances the metallic character. For fairy large interaction the long-range Umklapp scattering becomes relevant and the charge stiffness is suppressed. Finally I would add the size effect of the compressibility:

\[
L \Delta E (\rho = 1/2 + 1/L) \equiv 1/\chi = O(\ln L) \to \infty, \tag{19}
\]
which comes from the results \([29]\) by the RG analysis and the CFT arguments. The compressibility \(\chi\) goes to 0 weakly for increasing size.

To summarize, within the perturbation theory I have constructed CFT in the TL liquid with \(1/\beta\) long-range forward scattering. I have found that the interaction gives the nontrivial behaviour for \(\beta = \text{odd}\) and \(\beta \approx 2\). I have numerically checked the finite size scalings obtained from CFT in the tight-binding model with \(1/\beta\) LRI. Our analysis and numerical calculations exhibit consistency with each other.

Appendixes

1. Renormalization group equation

\(0 < \beta < 1\) or \(1 < \beta < 3\)

I derive the renormalization group equations heuristically. Let us start from the action \([4]\):

\[
S = \sum_w \sum_{q=-\Lambda}^{\Lambda} \frac{2\pi}{K} (q^2 + w^2)|\phi(q, w)|^2 + g \sum_w \sum_{q=-\Lambda}^{\Lambda} q^2 V(q)|\phi(q, w)|^2 \\
= \sum_w \left\{ \sum_{q=-\Lambda/b}^{\Lambda/b} - \sum_{q=-\Lambda}^{\Lambda} \right\} + g \sum_w \left\{ \sum_{q=-\Lambda/b}^{\Lambda/b} - \sum_{q=-\Lambda}^{\Lambda/b} \right\}.
\]

(20)

The partition function is

\[
Z = \int \mathcal{D}\phi_{\text{slow}} \mathcal{D}\phi_{\text{fast}} \exp(-S_{\text{slow}}^0 - S_{\text{fast}}^0 - S_{\text{slow}}^g - S_{\text{fast}}^g).
\]

(21)

Thus I can integrate out \(S_{\text{fast}}\) (\(|q| > \Lambda/b\) component) simply and obtain

\[
Z = \int \mathcal{D}\phi_{\text{slow}} \exp(-S_{\text{slow}}^0 - S_{\text{slow}}^g).
\]

(22)

The remaining procedure of the renormalization is the scale transformation

\[
q \rightarrow q/b, \ w \rightarrow w/b \ \text{and} \ \phi \rightarrow \phi b^2,
\]

(23)

where I choose the dynamical exponent 1. The results are

\[
S_{\text{slow}}^0 \rightarrow S^0 \\
S_{\text{slow}}^g \rightarrow g \sum_w \sum_{q=-\Lambda}^{\Lambda} q^2 V(q/b)|\phi(q, w)|^2 \\
\rightarrow gb^{1-\beta} \sum_w \sum_{q=-\Lambda}^{\Lambda} q^2 V(q)|\phi(q, w)|^2,
\]

(24)
where I use \( V(q) - A \sim q^{\beta - 1} \) from the behaviours \( \text{(7)} \). Hence I obtain the renormalization group eq.

\[
\frac{dg(b)}{db} = (1 - \beta) \frac{g(b)}{b}.
\]  

(25)

Substituting \( l = \ln b \) into this, I obtain renormalization group eqs.

\[
\begin{align*}
\frac{dg}{dl} & = (1 - \beta) g \\
\frac{d}{dl} \left( \frac{v}{K} \right) & = 0 \\
\frac{d}{dl} \left( \frac{1}{vK} \right) & = 0.
\end{align*}
\]

(26)

The TL parameter \( K \) is not renormalized but it shifts due to the constant \( A \).

\( \beta = 1 \)

The dispersion relation of the Coulomb interaction includes the marginal part \( w \sim q \) and \( w \sim q \sqrt{\ln q} \) as well as \( \beta > 1 \) case. Integrating out the fast moving part, I obtain the effective action of the slow part

\[
S_{\text{slow}} = \sum_w \sum_{q=-\Lambda/b}^{\Lambda/b} 2\pi K \left( vq^2 + w^2/v \right) |\phi(q, w)|^2 + g \sum_w \sum_{q=-\Lambda/b}^{\Lambda/b} q^2 V(q) |\phi(q, w)|^2,
\]

(27)

where I dare to leave the velocity in the Gaussian part. Note that I need not the renormalization of the velocity in the case \( \beta > 1 \). After the scale transformation, I obtain the eqs.

\[
\begin{align*}
\frac{dg}{dl} & = 0 \\
\frac{d}{dl} \left( \frac{v}{K} \right) & = \frac{gA}{2\pi} \\
\frac{d}{dl} \left( \frac{1}{vK} \right) & = 0,
\end{align*}
\]

(28)

where \( A \) is the constant appearing in \( \text{(7)} \). I see that \( K \) and the velocity \( v \) is renormalized instead of the no renormalization of \( g \). The forward scattering become relevant through \( K, v \) and drive the system away from the TL fixed point. Note that this result holds irrespective of any filling \( k_F \). From these eqs., the size dependences of \( v \) and \( K \) are given by

\[
\begin{align*}
v(b) & \sim \sqrt{\ln L} \\
K(b) & \sim 1/\sqrt{\ln L}.
\end{align*}
\]

(29)

The velocity diverges weakly for long distances, which is consistent with the estimations of \( v = \frac{dv}{dq} \) from the behaviours \( \text{(7)} \).
\[ \beta = 3 \]

I use \( V(q) = A + Bq^2 \ln q + Cq^2 + \cdots \) in the behaviours (1). The \( g \) term of (20) is

\[
\sum_w \sum_{q=-\lambda/b}^{\lambda/b} q^2 (g_1(0)q^2 \ln q + g_2(0)q^2),
\]

where the couplings \( g_1(0) \) and \( g_2(0) \) are defined by \( g_B \) and \( g_C \) respectively. For the scale transformation (23), the \( g \) term is changed to

\[
\sum_w \sum_{q=-\lambda}^{\lambda} q^2 [g_1(0)q^2 \ln q/b^2 + (g_2(0)/b^2 + g_1(0)q^2 \ln q/b^2)].
\]

Thus I obtain

\[
g_1(b) = \frac{1}{b^2} g_1(0), \quad g_2(b) = g_1(0) \frac{1}{b^2} \ln \frac{1}{b} + g_2(0) \frac{1}{b^2}. \tag{32}
\]

By \( l = \ln b \), I write this as

\[
\frac{dg_1(l)}{dl} = -2g_1(l), \quad \frac{dg_2(l)}{dl} = -2g_2(l) - g_1(l). \tag{33}
\]

\[ \beta > 3 \]

This case is same as eq. (25) putting \( \beta = 3 \).

2. CFT in the TL liquid with LRI

The Hamiltonian in the finite strip from the action (3) is

\[
H = H_{\text{TL}} + g \int d\sigma_1 d\sigma_2 \partial_\sigma \phi(\sigma_1) \partial_\sigma \phi(\sigma_2) V(|\sigma_1 - \sigma_2|) \theta(|\sigma_1 - \sigma_2| - \alpha_0), \tag{34}
\]

where \( H_{\text{TL}} \) is TL liquid and \( D \) means the region \( D = \{ |\sigma_1 - \sigma_2| \leq L, -L/2 \leq \sigma_1, \sigma_2 \leq L/2 \} \). I introduce the step function \( \theta(x) \) to avoid the ultra violet divergences which come from \( V(x) \) and the operator product expansion of \( \partial_\sigma \phi(\sigma) \). For the small perturbation \( g \) the ground state energy \( E_g \) varies as

\[
E'_g - E_g = g \int d\sigma_1 d\sigma_2 V(|\sigma_1 - \sigma_2|) \langle 0 | \partial_\sigma_1 \phi(\sigma_1) \partial_\sigma_2 \phi(\sigma_2) | 0 \rangle > \theta(|\sigma_1 - \sigma_2| - \alpha_0)
\]

\[
= -\frac{g}{4} \int d\sigma_1 d\sigma_2 V(|\sigma_1 - \sigma_2|) \langle 0 | \partial_{w_1} \varphi(w_1) \partial_{w_2} \varphi(w_2) | 0 \rangle > \tau_1 = \tau_2 = 0 \theta(|\sigma_1 - \sigma_2| - \alpha_0), \tag{35}
\]
where I introduce the coordinates $w = \tau + i\sigma$ ($-L/2 \leq \sigma \leq L/2$, $-\infty < \tau < \infty$) and $|0>$ is the ground state of $H_{TL}$. From the characters of the Gaussian part (TL liquid part) I can separate as $\phi(\sigma, \tau) = \varphi(w) + \bar{\varphi}(\bar{w})$ and derive $<0|\partial_{w_1}\bar{\varphi}(\bar{w}_1)\partial_{w_2}\varphi(w_2)|0> = 0$. The content of the brackets is modified as follows:

$$<0|\partial_{w_1}\varphi(w_1)\partial_{w_2}\varphi(w_2)|0> + <0|\partial_{\bar{w}_1}\bar{\varphi}(\bar{w}_1)\partial_{\bar{w}_2}\bar{\varphi}(\bar{w}_2)|0>_{\tau_1=\tau_2=0}$$

$$= \frac{K}{4}[(\frac{2\pi}{L})^{2\Delta} \frac{1}{z_1(1-\frac{z_1}{z_2})^2} + (\frac{2\pi}{L})^{2\Delta} \frac{1}{z_1(1-\frac{z_2}{z_1})^2}]_{\tau_1=\tau_2=0}$$

$$= -\frac{K}{4}(\frac{2\pi}{L})^2 \frac{1}{2\sin^2 \pi|\sigma_1-\sigma_2|},$$

(36)

where I transform the correlation function $<\partial_{z_1}\bar{\varphi}(z_1)\partial_{z_2}\bar{\varphi}(z_2)> = \frac{K}{4(z_1-z_2)^2}$ in $\infty \times \infty$ plane to that in the strip $w$ through $z = \exp\frac{2\pi w}{L}$. At present case $\partial_{w}\bar{\varphi}(w)$ ($\partial_{\bar{w}}\bar{\varphi}(\bar{w})$) have the spin $s = 1(-1)$ and conformal dimension $\Delta = 1(\Delta = 1)$. Hence I obtain

$$E'_{g} - E_{g} = \frac{gK\pi^2}{4}(\frac{\pi}{L})^3 \int_{-1/2}^{1/2} dx' \frac{1}{(\sin \pi|x'|)^{\beta} \sin^2 \pi x} \theta(|x'| - \frac{\alpha_0}{L}),$$

(37)

where I impose the periodic boundary condition and use the interaction potential $V(x) = 1/(\frac{L}{\pi} \sin(\frac{\pi x}{L}))^\beta$. Putting $\epsilon = \alpha_0/L$ for convenience, I give the differential of the integral part:

$$\frac{\partial}{\partial \epsilon} \int_{-1/2}^{1/2} dx' \frac{1}{(\sin \pi|x'|)^{\beta} \sin^2 \pi x} \theta(|x'| - \epsilon) = -\frac{2}{(\sin \pi|\epsilon|)^{\beta} \sin^2 \pi \epsilon}.$$  

(38)

After integrating the Taylor expansion about $\epsilon$ of this quantity, I obtain

$$\int_{-1/2}^{1/2} dx' \frac{1}{(\sin \pi|x'|)^{\beta} \sin^2 \pi x} \theta(|x'| - \epsilon) = \text{const.} + \frac{2}{\pi} \left( \frac{1}{\beta+1} + \frac{\beta+2}{6(\beta-1)} (\pi\epsilon)^{-\beta+1} \right. 

+ \frac{1}{\beta-3} \left\{ \frac{1}{120} (\beta+2) - \frac{1}{72} (\beta+1)(\beta+2) \right\} (\pi\epsilon)^{-\beta+3}

+ O((\pi\epsilon)^{-\beta+5})],$$

(39)

where $\beta \neq \text{odd}$. Therefore I can write the corrections in the form:

$$E'_{g} - E_{g} = \frac{gK}{2} \left( \frac{A(\beta)}{L^\beta} + B(\beta)L + C(\beta) \frac{\beta}{L^\beta} + D(\beta) \frac{1}{L^\beta} \right).$$

(40)

Here $B(\beta)$, $C(\beta)$ and $D(\beta)$ are given by

$$B(\beta) = \frac{\alpha_0}{1+\beta},$$

$$C(\beta) = \frac{\pi^2(2+\beta)}{6(\beta-1)} \frac{\alpha_0}{1+\beta},$$

$$D(\beta) = \frac{\pi^4}{\beta-3} \left( \frac{1}{120} (\beta+2) - \frac{1}{72} (\beta+1)(\beta+2) \right) \alpha_0^{3-\beta}.$$  

(41)

I can obtain $A(\beta)$ by evaluating the above integral numerically. The result is shown in Fig. 4(a).
For $\beta = \text{odd}$, there exists the logarithmic correction instead of the eq. (11). The results for respective $\beta$ are

$$E'_g - E_g = \begin{cases} 
\frac{g[\frac{A}{L^2} + BL + \frac{C}{L} \ln \frac{1}{L} + D \frac{1}{L^2} + O(\frac{1}{L^3})]}{\beta = 1} \\
\frac{g[\frac{A}{L^2} + BL + \frac{C}{L} + D \frac{1}{L^2} \ln \frac{1}{L} + O(\frac{1}{L^3})]}{\beta = 3} \\
\frac{g[\frac{A}{L^2} + BL + \frac{C}{L} + D \frac{1}{L^2} + E \frac{1}{L^2} \ln \frac{1}{L} + O(\frac{1}{L^3})]}{\beta = 5}
\end{cases}$$

(42)

The C terms in eqs. (10) and (42) contribute to deviation of the central charge. The present LRI inevitably contains the contribution from the short range interaction: $\delta(x)$. The C term does not come from the short range types of interactions because the vacuum expected value $\langle (\partial_x \phi)^2 \rangle$ vanishes. Thus C term is intrinsic in the present system with the LRI under periodic boundary condition. Because the velocity is not renormalized as I have seen from the renormalization group eqs., the C term contributes to the deviations of the effective central charge from the ground state energy.

Next I derive the corrections for the energy of the excited state:

$$E'_n - E_n = g \int_D d\sigma_1 d\sigma_2 V(|\sigma_1 - \sigma_2|) < n|\partial_{\sigma_1} \phi(\sigma_1) \partial_{\sigma_2} \phi(\sigma_2)|n > \theta(|\sigma_1 - \sigma_2| - \alpha_0)$$

$$= g \int_D d\sigma_1 d\sigma_2 V(|\sigma_1 - \sigma_2|) \sum_{\alpha} < n|\partial_{\sigma_1} \phi(\sigma_1)||\alpha > < \alpha|\partial_{\sigma_2} \phi(\sigma_2)|n > \theta(|\sigma_1 - \sigma_2| - \alpha_0)$$

$$= 4g \sum_{\alpha} C_{n \alpha} C_{\alpha n} \frac{(2\pi)^2}{L^2} \int_0^{1/2} dy \frac{1}{(\sin \pi |y|)^\beta} \cos 2\pi (s_n - s_\alpha) y \theta(|y| - \frac{\alpha_0}{L}),$$

(43)

where I use the results by Cardy [22]:

$$< n|\phi(\sigma)|\alpha > = C_{n \alpha} (\frac{2\pi}{L})^{2\beta} e^{\frac{2\pi i (s_n - s_\alpha) \alpha}{L}}.$$ 

(44)

Here $j$ means $\partial \phi$ and $|n>$ is the excited state of $H_{TL}$. I can derive the size dependence of eq. (13) from likewise treatments as the ground state. After taking the derivative about $1/L$, I expand about $1/L$. Integrating them, I obtain

$$E'_n - E_n$$

$$= \begin{cases} 
16\pi^2 g \sum_{\alpha} C_{n \alpha} C_{\alpha n} \left\{ \frac{A(s_n - s_\alpha, \beta)}{L^2} + B(\beta) + C(s_n - s_\alpha) \right\} \frac{1}{L^2} + O\left(\frac{1}{L^3}\right) \beta \neq \text{odd} \\
16\pi^2 g \sum_{\alpha} C_{n \alpha} C_{\alpha n} \left\{ \frac{A(s_n - s_\alpha, \beta)}{L^2} \right\} + B(\beta) + C(s_n - s_\alpha) \frac{1}{L^2} + D(s_n - s_\alpha) \frac{1}{L^2} + O\left(\frac{1}{L^3}\right) \beta = 1 \\
16\pi^2 g \sum_{\alpha} C_{n \alpha} C_{\alpha n} \left\{ \frac{A(s_n - s_\alpha)}{L^2} \right\} + B(\beta) + C(s_n - s_\alpha) \frac{1}{L^2} + D(s_n - s_\alpha) \frac{1}{L^2} + E \frac{1}{L^2} + O\left(\frac{1}{L^3}\right) \beta = 3 \\
16\pi^2 g \sum_{\alpha} C_{n \alpha} C_{\alpha n} \left\{ \frac{A(s_n - s_\alpha)}{L^2} \right\} + B(\beta) + C(s_n - s_\alpha) \frac{1}{L^2} + D(s_n - s_\alpha) \frac{1}{L^2} + E \frac{1}{L^2} + F \frac{1}{L^2} + O\left(\frac{1}{L^3}\right) \beta = 5 \\
\cdots
\end{cases}$$

(45)
where $B$ are the constant independent of $s_n, s_\alpha$. Here for $\beta \neq$ odd, $B(\beta), C(s_n-s_\alpha, \beta)$ and $D(s_n-s_\alpha, \beta)$ are given by

$$
B(\beta) = \frac{1}{(\alpha_0\pi)^{\beta-1}} \pi(\beta-1)
$$

$$
C(s_n-s_\alpha, \beta) = \frac{1}{(\alpha_0\pi)^{\beta-3}} \left[ \frac{\beta}{6} - 2(s_n-s_\alpha)^2 \right] \frac{1}{\pi(\beta-3)}
$$

$$
D(s_n-s_\alpha, \beta) = \frac{1}{(\alpha_0\pi)^{\beta-5}} \left[ -\frac{(s_n-s_\alpha)^2}{3} + \frac{1}{180} \right] \beta + \frac{2(s_n-s_\alpha)^4}{72} \frac{1}{\pi(\beta-5)}.
$$

(46)

It is not straightforward to determine $A(s_n-s_\alpha, \beta)$ generally. However about one particle excitation ($s_n = 0$), I can obtain $A(\beta, s_\alpha)$, which is shown for some $s_\alpha$ in Fig. 4(b). Actually further consideration about the operator product expansion leads $s_n = s_\alpha = 0$ (see section 4.).

I refer to the $O(1/L)$ dependences. These are due to the fact that the LRI includes the short range type interaction. Actually I can derive the same form

$$
\frac{g}{L} \sum_\alpha C_{n\alpha} C_{\alpha jn}
$$

as an ordinary finite scaling by replacing as $V(|x|)\theta(|x|-\alpha_0) \to \delta(x)$. As $(\partial \phi)^2$ is a part of the TL liquid, the $O(1/L)$ term can be erased under subtracting such the contributions first. Thus the $O(1/L)$ term is not intrinsic.

Summarizing the discussions in this appendix, I can prove that the Hamiltonian $\mathcal{H}$ is described by $c = 1$ CFT for $\beta > 1$ in the excitation energy. However the effective central charge from the ground state depends on the interaction and deviates from 1. I find the nontrivial behaviors when $\beta = \text{odd}$, which corresponds to the integer points of the modified Bessel function as appearing in the behaviors (7).

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