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A resolution of the $K(2)$-local sphere

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Abstract

At the prime $p = 3$, we write the spectrum $L_{K(2)}S^0$ as the inverse limit of a short tower of fibrations where the fibers are (suspensions of) explicit homotopy fixed point spectra $E^n_F$ with $F$ a finite subgroup of the Morava stabilizer group.

Fix a prime $p$ and let $K(n)_*$ denote the $n$-th Morava $K$-theory at $p$. The purpose of this paper is to write down an organizing principle for the homotopy type of the $K(n)$-localization $L_{K(n)}S^0$ of the stable sphere. This is the spectrum that will, in some sense, capture the $v_n$-periodic part of the stable homotopy groups of spheres. We focus on the case $n = 2$ and $p = 3$ because this is at the edge of our current knowledge. The homotopy type and homotopy groups of $L_{K(1)}S^0$ are classically well-understood at all primes and intimately connected with the image of the $J$-homomorphism; if $n = 2$ and $p > 3$, the Adams-Novikov spectral sequence (of which more below) calculating $\pi_\ast L_{K(2)}S^0$ collapses and cannot have extensions; hence, the problem becomes algebraic, although not easy. Compare [18].

It should be noticed immediately that there has been a great deal of calculation with the homotopy groups of closely related spectra, most notably by Shimomura and his coauthors. (See, for example, [19] and [20].) One aim of this paper is to provide some general conceptual framework for further advances.

The $K(n)$-local category of spectra is governed by the $K(n)$-version of the homology theory built from the Lubin-Tate or Morava theory $E_n$. This is a ring spectrum with coefficient ring

$$(E_n)_* = W(F_\mathbb{F}_p^n)[[u_1, \ldots, u_{n-1}]] [u^\pm 1]$$

with the power series ring over the Witt vectors in degree 0 and the degree of $u$ equal to $-2$. $(E_n)_*$ is a complex oriented theory and the formal group law over $(E_n)_*$ can be taken as the universal deformation of

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the Honda formal group law $\Gamma_n$ of height $n$ over the field $\mathbb{F}_p^n$ with $p^n$ elements; $\Gamma_n$ is the usual formal group law associated to $K(n)$. By the Hopkins-Miller theorem ([17]) $E_n$ supports an action by the group

$$G_n = \text{Aut}(\Gamma_n) \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p).$$

The group $\text{Aut}(\Gamma_n)$ of automorphism of the formal group law $\Gamma_n$ is also known as the Morava stabilizer group and will be denoted $\mathbb{S}_n$. There is an Adams-Novikov spectral sequence

$$E_2^{s,t} := H^{s,t}(\mathbb{S}_n, (E_n)_t)^{\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)} \Rightarrow \pi_{s+t} L_{K(n)} S^0.$$ 

(See [6] for a basic description.) The group $G_n$ is a profinite group and it acts continuously on $(E_n)_t$. The cohomology here is continuous cohomology. We note that by [4] $L_{K(n)} S^0$ can be identified with the homotopy fixed point spectrum $E_n^{h\mathbb{S}_n}$ and the Adams-Novikov spectral sequence can be interpreted as a homotopy fixed point spectral sequence.

The qualitative behaviour of this spectral sequence depends very much on qualitative cohomological properties of the group $\mathbb{S}_n$, in particular on its cohomological dimension. This in turn depends very much on $n$ and $p$.

If $p - 1$ does not divide $n$ (for example, if $n < p - 1$) then the $p$-Sylow subgroup of $\mathbb{S}_n$ is of cohomological dimension $n^2$. Furthermore, if $n^2 < 2(p - 1) - 1$ (for example, if $n = 2$ and $p > 3$) then this spectral sequence is sparse enough so that there can be no differentials or extensions.

However, if $n$ divides $p - 1$, then the cohomological dimension of $\mathbb{S}_n$ is infinite and the Adams-Novikov spectral sequence has a more complicated behaviour. The reason for infinite cohomological dimension is the existence of elements of order $p$ in $\mathbb{S}_n$. However, in this case at least the virtual cohomological dimension remains finite, in other words there are finite index subgroups with finite cohomological dimension. In terms of resolutions of the trivial module $\mathbb{Z}_p$ this means, that while there are no projective resolutions of the trivial $\mathbb{S}_n$-module $\mathbb{Z}_p$ of finite length, one might still hope that there exist “resolutions” of $\mathbb{Z}_p$ of finite length in which the individual modules are direct sums of modules which are permutation modules of the form $\mathbb{Z}_p[[G_2/F]]$ where $F$ is a finite subgroup of $\mathbb{G}_n$. Note that in the case of a discrete group which acts properly and cellulary on a finite dimensional contractible spaces $X$ such a “resolution” is provided by the cellular complex of $X$.

This phenomenon is already visible for $n = 1$ in which case $G_1 = \mathbb{S}_1$ can be identified with $\mathbb{Z}_p^\times$, the units in the $p$-adics integers. Thus $G_1 \cong \mathbb{Z}_p \times C_{p-1}$ if $p$ is odd while $G_1 \cong \mathbb{Z}_2 \times C_2$ if $p = 2$. In both cases there is a short exact sequence

$$0 \rightarrow \mathbb{Z}_p[[G_1/F]] \rightarrow \mathbb{Z}_p[[G_1/F]] \rightarrow \mathbb{Z}_p \rightarrow 0$$

of continuous $G_1$-modules (where $F$ is the maximal finite subgroup of $G_1$). If $p$ is odd this sequence is a projective resolution of the trivial module while for $p = 2$ it is only a resolution by permutation modules. These resolutions are the algebraic analogues of the fibrations (see [6])

$$(0.1) \quad L_{K(1)} S^0 \cong E_1^{hC_1} \rightarrow E_1^{hF} \rightarrow E_1^{hF}.$$ 

We note that $p$-adic complex $K$-theory $K\mathbb{Z}_p$ is in fact a model for $E_1$, the homotopy fixed points $E_1^{hC_2}$ can be identified with 2-adic real $K$-theory $KO\mathbb{Z}_2$ if $p = 2$ and $E_1^{hC_2}$ is the Adams summand of $K\mathbb{Z}_p$ if $p$ is odd, so that the fibration of (0.1) agrees indeed with that of [6].

In this paper we produce a resolution of the trivial module $\mathbb{Z}_p$ by (direct summands of) permutation modules in case $n = 2$ and $p = 3$ and we use it to build $L_{K(2)} S^0$ as the top of a finite tower of fibrations where the fibers are (suspensions of) spectra of the form $E_2^{hF}$ where $F \subseteq G_2$ is a finite subgroup.
In fact, if $n = 2$ and $p = 3$, only two subgroups appear. The first is a subgroup $G_{24} \subseteq G_2$; this is a finite subgroup of order 24 containing a normal cyclic subgroup $C_3$ with quotient $G_{24}/C_3$ isomorphic to the quaternion group $Q_8$ of order 8. The other group is the semidihedral group $SD_{16}$ of order 16. The two spectra we will see, then, are $E_2^{bG_{24}}$ and $E_2^{bSD_{16}}$.

The discussion of these and related subgroups of $G_2$ occurs in section 1 (see Remark 1.1, 1.2). The homotopy groups of these spectra are known. We will review the calculation in section 3.

Our main result can be stated as follows (see Theorem 5.4 et Theorem 5.5).

**0.1 Theorem.** There is a sequence of maps between spectra

$$L_{K(2)}S^0 \to E_2^{bG_{24}} \to \Sigma^8 E_2^{bSD_{16}} \vee E_2^{bG_{24}} \to \Sigma^8 E_2^{bSD_{16}} \vee \Sigma^{40} E_2^{bSD_{16}} \to \Sigma^{48} E_2^{bG_{24}} \to \Sigma^{48} E_2^{bG_{24}}$$

with the property that the composite of any two successive maps is zero and all possible Toda brackets are zero modulo indeterminacy.

Because the Toda brackets vanish, this “resolution” can be refined to a tower of spectra with $L_{K(2)}S^0$ at the top. The precise result is given in Theorem 5.6. There are many curious features of this resolution, of which we note here only two. First, this is not an Adams resolution for $E_2$, as the spectra $E_2^F$ are not $E_2$-injective, at least if 3 divides the order of $F$. Second, there is a certain superficial duality to the resolution which should somehow be explained by the fact that $S_3$ is virtual Poincaré duality group, but we do not know how to make this thought precise.

Our method is by brute force. The hard work is really in section 4, where we use the calculations of [10] in an essential way to produce the short resolution of the trivial $G_2$-module $Z_3$ by (summands of) permutation modules over the form $Z_3[[G_2/F]]$ where $F$ is finite. In section 2, we calculate the homotopy type of the function spectra $F(E^{hK_1}, E^{hK_2})$ if $K_1$ is a closed and $K_2$ a finite subgroup of $G_n$; this will allow us to make the Toda bracket calculations. Here the work of [4] is crucial. These calculations also explain the role of the suspension by 48 which is really a homotopy theoretic phenomenon while the other suspensions can be explained in terms of the algebraic resolution constructed in section 4.

## 1 Lubin-Tate Theory and the Morava stabilizer group

The purpose of this section is to give a summary of what we will need about deformations of formal group laws over perfect fields. The primary point of this section is to establish notation and to run through some of the standard algebra needed to come to terms with the $K(n)$-local stable homotopy category.

Fix a perfect field $k$ of characteristic $p$ and a formal group law $\Gamma$ over $k$. A deformation of $\Gamma$ to a complete local ring $A$ (with maximal ideal $m$) is a pair $(G, i)$ where $G$ is a formal group law over $A$, $i: k \to A/m$ is a morphism of fields and one requires $i_* \Gamma = \pi_* G$, where $\pi: A \to A/m$ is the quotient map. Two such deformations $(G, i)$ and $(H, j)$ are $*$-isomorphic if there is an isomorphism $f: G \to H$ of formal group laws which reduces to the identity modulo $m$. Write $\text{Def}_+(A)$ for the set of $*$-isomorphism classes of deformations of $\Gamma$ over $A$.

A common abuse of notation is to write $G$ for the deformation $(G, i)$; $i$ is to be understood from the context.
Now suppose the height of $\Gamma$ is finite. Then the theorem of Lubin and Tate [13] says that the functor $A \mapsto \text{Def}_{\Gamma}(A)$ is representable. Indeed let

\begin{equation}
E(\Gamma, k) = W(k)[[u_1, \ldots, u_{n-1}]]
\end{equation}

where $W(k)$ denotes the Witt vectors on $k$ and $n$ is the height of $\Gamma$. This is a complete local ring with maximal ideal $m = (p, u_1, \ldots, u_{n-1})$ and there is a canonical isomorphism $q : k \cong E(\Gamma, k)/m$. Then Lubin and Tate prove there is a deformation $(G, q)$ of $\Gamma$ over $E(\Gamma, k)$ so that the natural map

\begin{equation}
\text{Hom}_n(E(\Gamma, k), A) \rightarrow \text{Def}_{\Gamma}(A)
\end{equation}

sending a continuous map $f : E(\Gamma, k) \rightarrow A$ to $(f_*G, f_*q)$ (where $f$ is the map on residue fields induced by $f$) is an isomorphism. Continuous maps here are very simple: they are the local maps; that is, we need only require that $f(m)$ be contained in the maximal ideal of $A$. Furthermore, if two deformations are $*$-isomorphic, then the $*$-isomorphism between them is unique.

We'd like to now turn the assignment $(\Gamma, k) \mapsto E(\Gamma, k)$ into a functor. For this we introduce the category $\mathcal{FG}_n$ of height $n$ formal group laws over perfect fields. The objects are pairs $(\Gamma, k)$ where $\Gamma$ is of height $n$. A morphism

$$(f, j) : (\Gamma_1, k_1) \rightarrow (\Gamma_2, k_2)$$

is a homomorphism of fields $j : k_1 \rightarrow k_2$ and an isomorphism of formal group laws $f : j_*\Gamma_1 \rightarrow \Gamma_2$.

Let $(f, j)$ be such a morphism and let $G_1$ and $G_2$ be the fixed universal deformations over $E(\Gamma_1, k)$ and $E(\Gamma_2, k)$ respectively. If $\bar{f} \in E(\Gamma_2, k_2)[[x]]$ is any lift of $f \in k_2[[x]]$, then we can define a formal group law $H$ over $E(\Gamma_2, k_2)$ by requiring that $\bar{f} : H \rightarrow G_2$ is an isomorphism. Then the pair $(H, f)$ is a deformation of $\Gamma_1$, hence we get a homomorphism $E(\Gamma_1, k_1) \rightarrow E(\Gamma_2, k_2)$ classifying the $*$-isomorphism class of $H$ -- which, one easily checks, is independent of the lift $\bar{f}$. Thus if $\text{Rings}_c$ is the category of complete local rings and local homomorphisms, we get a functor

$$E(\cdot, \cdot) : \mathcal{FG}_n \rightarrow \text{Rings}_c.$$ 

In particular, note that any morphism in $\mathcal{FG}_n$ from a pair $(\Gamma, k)$ to itself is an isomorphism. This is the “big” Morava stabilizer group of the formal group law; it contains the subgroup of elements $(f, id_k)$. This formal group law and hence also its automorphism group is determined up to isomorphism by the height of $\Gamma$ if $k$ is separably closed.

Specifically, let $\Gamma$ be the Honda formal group law over $\mathbb{F}_{p^n}$; thus the $p$-series of $\Gamma$ is

$$[p](x) = x^{p^n}.$$ 

From this formula it immediately follows that any automorphism $f : \Gamma \rightarrow \Gamma$ over any finite extension field of $\mathbb{F}_{p^n}$ actually has coefficients in $\mathbb{F}_{p^n}$; thus we obtain no new isomorphisms by making such extensions. Let $S_n$ be the group of automorphisms of this $\Gamma$ over $\mathbb{F}_{p^n}$; this is the classical Morava stabilizer group. If we let $G_n$ be the group of automorphisms of $(\Gamma, \mathbb{F}_{p^n})$ in $\mathcal{FG}_n$ (the big Morava stabilizer group of $\Gamma$), then one easily sees that

$$G_n \cong S_n \times \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p).$$

Of course, $G_n$ acts on $E(\Gamma, \mathbb{F}_{p^n})$. Also, we note that the Honda formal group law is defined over $\mathbb{F}_p$, although it won’t get its full group of automorphisms until changing base to $\mathbb{F}_{p^n}$. 

4
Next we put in the gradings. This requires a paragraph of introduction. For any commutative ring $R$, the morphism $R[[x]] \to R$ of rings sending $x$ to $0$ makes $R$ into a $R[[x]]$-module. Let $\text{Der}_R(R[[x]], R)$ denote the $R$-module of continuous $R$-derivations; that is, continuous $R$-module homomorphisms

$$\partial : R[[x]] \longrightarrow R$$

so that

$$\partial(f(x)g(x)) = \partial(f(x))g(0) + f(0)\partial(g(x)).$$

If $\partial$ is any derivation, write $\partial(x) = u$; then, if $f(x) = \sum a_i x^i$,

$$\partial(f(x)) = a_1 \partial(x) = a_1 u.$$

Thus $\partial$ is determined by $u$, and we write $\partial = \partial_u$. We then have that $\text{Der}_R(R[[x]], R)$ is a free $R$-module of rank one, generated by any derivation $\partial_u$ so that $u$ is a unit in $R$. In the language of schemes, $\partial_u$ is a generator for the tangent space at $0$ of the formal scheme $A^1_R$ over $\text{Spec}(R)$.

Now consider pairs $(F, u)$ where $F$ is a formal group law over $R$ and $u$ is a unit in $R$. Thus $F$ defines a smooth one dimensional commutative formal group scheme over $\text{Spec}(R)$ and $\partial_u$ is a chosen generator for the tangent space at $0$. A morphism of pairs

$$f : (F, u) \longrightarrow (G, v)$$

is an isomorphism of formal group laws $f : F \to G$ so that

$$u = f'(0)v.$$ 

Note that if $f(x) \in R[[x]]$ is a homomorphism of formal group laws from $F$ to $G$, and $\partial$ is a derivation at $0$, then $(f^*\partial)(x) = f'(0)\partial(x)$. In the context of deformations, we may require that $f$ be a $\ast$-isomorphism.

This suggests the following definition: let $\Gamma$ be a formal group law of height $n$ over a perfect field $k$ of characteristic $p$, and let $A$ be a complete local ring. Define $\text{Def}_\Gamma(A)_\ast$ to be equivalence classes of pairs $((G, i), u)$ where $(G, i)$ is a deformation of $\Gamma$ to $A$ and $u$ is a unit in $A$. The equivalence relation is given by $\ast$-isomorphisms transforming the unit as in the last paragraph. We now have that there is a natural isomorphism

$$\text{Hom}_e(E(\Gamma, k)[u^\pm 1], A) \cong \text{Def}_\Gamma(A)_\ast.$$ 

We impose a grading by giving an action of the multiplicative group scheme $\mathbb{G}_m$ on the scheme $\text{Def}_\Gamma(\cdot)_\ast$ (on the right) and thus on $E(\Gamma, k)[u^\pm 1]$ (on the left); if $v \in A^\times$ is a unit and $(G, u)$ represents an equivalence class in $\text{Def}_\Gamma(A)_\ast$ define a new element in $\text{Def}_\Gamma(A)_\ast$ by $(G, v^{-1}u)$. In the induced grading on $E(\Gamma, k)[u^\pm 1]$, one has $E(\Gamma, k)$ in degree $0$ and $u$ in degree $-2$.

This grading is essentially forced by topological considerations. See the remarks before Theorem 20 of [21] for an explanation. In particular, it is explained there why $u$ is in degree $-2$ rather than 2.

The rest of the section will be devoted to what we need about the Morava stabilizer group. The group $S_n$ is the group of units in the endomorphism ring $\mathcal{O}_n$ of the Honda formal group law of height $n$. The ring $\mathcal{O}_n$ can be described as follows (See [10] or [15]). One adjoins a non-commuting element $S$ to the Witt vectors $\mathbb{W} = W(F_{p^n})$ subject to the conditions that

$$Sa = \phi(a)S \quad \text{and} \quad S^n = p$$

5
where \( a \in \mathbb{W} \) and \( \phi : \mathbb{W} \to \mathbb{W} \) is the Frobenius. (In terms of power series, \( S \) corresponds to the endomorphism of the formal group law given by \( f(x) = x^p \).) This algebra \( \mathcal{O}_n \) is a free \( \mathbb{W} \)-module of rank \( n \) with generators \( 1, S, \ldots, S^{n-1} \) and is equipped with a valuation \( \nu \) extending the standard valuation of \( \mathbb{W} \); since we assume that \( \nu(p) = 1 \), we have \( \nu(S) = 1/n \). Define a filtration on \( S_n \) by

\[
F_k S_n = \{ x \in S_n \mid \nu(x - 1) \geq k \}.
\]

Note that \( k \) is a fraction of the form \( a/n \) with \( a = 0, 1, 2, \ldots \). We have

\[
F_0 S_n / F_1 / n S_n \cong \mathbb{F}_{p^n},
\]

\[
F_a / n S_n / F(a+1) / n S_n \cong \mathbb{F}_{p^n}, \quad a \geq 1
\]

and

\[
S_n \cong \lim_a \frac{S_n}{F_n / S_n}.
\]

If we define \( S_n = F_1 / n S_n \), then \( S_n \) is the \( p \)-Sylow subgroup of the profinite group \( S_n \). Note that the Teichmüller elements \( \mathbb{F}_{p^n}^\times \subset \mathbb{W}^\times \subseteq \mathcal{O}_n^\times \) define a splitting of the projection \( S_n \to \mathbb{F}_{p^n}^\times \) and, hence, \( S_n \) is the semi-direct product of \( \mathbb{F}_{p^n}^\times \) and the \( p \)-Sylow subgroup.

The action of the Galois group \( \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \) on \( \mathcal{O}_n \) is the obvious one; the Galois group is generated by the Frobenius \( \phi \) and

\[
\phi(a_0 + a_1 S + \cdots + a_{n-1} S^{n-1}) = \phi(a_0) + \phi(a_1)S + \cdots + \phi(a_{n-1})S^{n-1}.
\]

We are, in this paper, concerned mostly with the case \( n = 2 \) and \( p = 3 \). In this case, every element of \( S_2 \) can be written as a sum

\[
a + b S, \quad a, b \in W(\mathbb{F}_9) = \mathbb{W}
\]

with \( a \not\equiv 0 \mod 3 \). The elements of \( S_2 \) are of the form \( a + b S \) with \( a \equiv 1 \mod 3 \).

The following subgroups of \( S_2 \) will be of particular interest to us. The first two are choices of maximal finite subgroups. \(^1 \) The last one (see Remark 1.3) is a closed subgroup which is, in some sense, complementary to the center.

1.1. Choose an eighth root of unity \( \omega \in \mathbb{F}_9 \). We will write \( \omega \) for the corresponding element in \( \mathbb{W} \) and \( S_2 \). The element

\[
s = -\frac{1}{2}(1 + \omega S)
\]

is of order 3; furthermore,

\[
\omega^2 s \omega^6 = s^2.
\]

Hence the elements \( s \) and \( \omega^2 \) generate a subgroup of order 12 in \( S_2 \) which we label \( G_{12} \). As a group, it is abstractly isomorphic to the unique non-trivial semi-direct product

\[
\mathbb{Z}/3 \mathbb{Z} \rtimes \mathbb{Z}/4 \mathbb{Z}.
\]

Any other subgroup of order 12 in \( S_2 \) is conjugate to \( G_{12} \). In the sequel, when discussing various representations, we will write the element \( \omega^2 \) in \( G_{12} \) as \( t \).

\(^1 \) The first author would like to thank Haynes Miller for several lengthy and informative discussions about finite subgroups of the Morava stabilizer group.
We note that the subgroup $G_{12} \subseteq S_2$ is a normal subgroup of a group $G_{24}$ of the larger group $G_2$. Indeed, there is a diagram of short exact sequences of groups

$$
1 \longrightarrow G_{12} \longrightarrow G_{24} \longrightarrow \text{Gal}(\mathbb{F}_9/\mathbb{F}_3) \longrightarrow 1
$$

Since the action of the Galois group on $S_2$ does not preserve any choice of $G_{12}$, this is not transparent. In fact, while the lower sequence is split the upper sequence is not. More concretely we let

$$
\psi = \omega \phi \in S_2 \times \text{Gal}(\mathbb{F}_9/\mathbb{F}_3) = G_2
$$

where $\omega$ is our chosen 8th root of unity and $\phi$ is the generator of the Galois group. Then if $s$ and $t$ are the elements of order 3 and 4 in $G_{12}$ chosen above, we easily calculate that $\psi s = s \psi$, $t \psi = \psi t^3$ and $\psi^2 = t^2$. Thus the subgroup of $G_2$ generated by $G_{12}$ and $G_{24}$ has order 24, as required. Note that the 2-Sylow subgroup of $G_{24}$ is the quaternion group $Q_8$ of order 8 generated by $t$ and $\psi$ and that indeed

$$
1 \longrightarrow G_{12} \longrightarrow G_{24} \longrightarrow \text{Gal}(\mathbb{F}_9/\mathbb{F}_3) \longrightarrow 1
$$

is not split.

1.2. The second subgroup is the subgroup $SD_{16}$ generated by $\omega$ and $\phi$. This is the semidirect product

$$
\mathbb{F}_9^* \rtimes \mathbb{Z}/2
$$

and is also known as the semidihedral group of order 16.

1.3. For the third subgroup, note that the evident right action of $S_n$ on $\mathcal{O}_n$ defines a group homomorphism $S_n \rightarrow \text{GL}(\mathcal{W})$. The determinant homomorphism $S_n \rightarrow \mathcal{W}^*$ extends to a homomorphism

$$
\mathbb{G}_n \rightarrow \mathcal{W}^* \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)
$$

For example, if $n = 2$, this map sends $(a + bS, \phi^e)$, $e \in \{0, 1\}$, to

$$(a \phi(a) - pb\phi(b), \phi^e)$$

where $\phi$ is the Frobenius. It is simple to check (for all $n$) that the image of this homomorphism lands in

$$
\mathbb{Z}_p^* \times \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \subseteq \mathcal{W}^* \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)
$$

If we identify the quotient of $\mathbb{Z}_p^*$ by its subgroup $C_{n-1}$ of elements of finite order with $\mathbb{Z}_p$, we get a “reduced determinant” homomorphism

$$
\mathbb{G}_n \rightarrow \mathbb{Z}_p
$$

Let $\mathbb{G}_1$ be the kernel of this map. One also easily checks that the center of $\mathbb{G}_n$ is $\mathbb{Z}_p^* \subseteq \mathcal{W}^* \subseteq S_n$ and that the composite

$$
\mathbb{Z}_p^* \rightarrow \mathbb{G}_n \rightarrow \mathbb{Z}_p^*
$$

sends $a$ to $a^n$. Thus, if $p$ doesn’t divide $n$, we have

$$
\mathbb{G}_n \cong \mathbb{Z}_p \times \mathbb{G}_1^n.
$$

Note that any finite subgroup of $\mathbb{G}_n$ is a subgroup of $\mathbb{G}_1^n$. 

7
2 The $K(n)$-local category and the Lubin-Tate theories $E_n$

The purpose of this section is to collect together the information we need about the $K(n)$-local category and the role of the functor $(E_n)_\ast(-)$ in governing this category. But attention! $(E_n)_\ast X$ is not the homology of $X$ defined by the spectrum $E_n$, but a completion thereof: see Definition 2.1 below.

Most of the information in this section is collected from [2], [3], and [11].

Fix a prime $p$ and let $K(n)$, $1 \leq n < \infty$, denote the $n$-th Morava $K$-theory spectrum. Then $K(n)_\ast \cong F_p[u\pm 1]$ where the degree of $u$ is $2(p^n - 1)$. This is a complex oriented theory and the formal group law over $K(n)_\ast$ is of height $n$. As is customary, we specify that the formal group law over $K(n)_\ast$ is the graded variant of the Honda formal group law; thus, the $p$-series is

$$[p](x) = v_n x^n.$$ 

Following Hovey and Strickland, we will write $K_n$ for the category of $K(n)$-local spectra. We will write $L_{K(n)}$ for the localization functor from spectra to $K_n$.

Next let $K_n$ be the extension of $K(n)$ with $(K_n)_\ast \cong F_p[u\pm 1]$ with the degree of $u = -2$. The inclusion $K(n)_\ast \subseteq (K_n)_\ast$ sends $v_n$ to $u^{-2(p^n - 1)}$. There is a natural isomorphism of homology theories

$$(K_n)_\ast \otimes_{K(n)_\ast} (K(n)_\ast) X \cong (K_n)_\ast X$$

and $(K_n)_\ast \to (K_n)_\ast$ is a faithfully flat extension; thus the two theories have the same local categories and weakly equivalent localization functors.

If we write $F$ for the graded formal group law over $K(n)_\ast$ we can extend $F$ to a formal group law over $(K_n)_\ast$ and define a formal group law $\Gamma$ over $F_p = (K_n)_0$ by

$$x + y = \Gamma(x, y) = u^{-1}F(u x, u y) = u^{-1}(ux + uy).$$

Then $F$ is chosen so that $\Gamma$ is the Honda formal group law.

We note that -- as in [3] -- there is a choice of the universal deformation $G$ of $\Gamma$ such that the $p$-series of the associated graded formal group law $G_0$ over $E(\Gamma, F_p)[u\pm 1]$ satisfies

$$[p](x) = v_0 x + G_0 v_1 x^p + G_0 v_2 x^{p^2} + G_0 \cdots$$

with $v_0 = p$ and

$$v_k = \begin{cases} 
    u^{1-p^k} u_k & 0 < k \leq n; \\
    u^{1-p^n} & k = n; \\
    0 & k > n.
\end{cases}$$

This shows that the functor $X \mapsto (E_n)_\ast \otimes_{BP_\ast} BP_\ast X$ (where $(E_n)_\ast$ is considered a $BP_\ast$-module via the evident ring homomorphism) is a homology theory which is represented by a spectrum $E_n$ with coefficients

$$\pi_\ast(E_n) \cong E(\Gamma, F_p)[u\pm 1] \cong \mathbb{W}[u_1, \ldots, u_{n-1}][u^{\pm 1}] .$$

The inclusion of the subring $E(n)_\ast = \mathbb{Z}(p)[v_1, \ldots, v_n, v_{n+1}]$ into $(E_n)_\ast$ is again faithfully flat; thus, these two theories have the same local categories. We write $L_n$ for the category of $E(n)$-local spectra and $L_n$ for the localization functor from spectra to $L_n$. 

8
The reader will have noticed that we have avoided using the expression \((E_n)_* X\); we now explain what we mean by this. The \(K(n)\)-local category \(\mathcal{K}_n\) has internal smash products and (arbitrary) wedges given by
\[
X \wedge_{\mathcal{K}_n} Y = L_{K(n)}(X \wedge Y)
\]
and
\[
\bigvee_{\mathcal{K}_n} X_n = L_{K(n)}\left(\bigvee_{\mathcal{K}_n} X_n\right).
\]

In making such definitions, we assume we are working in some suitable model category of spectra, and that we are taking the smash product between cofibrant spectra; that is, we are working with derived smash product. The issues here are troublesome, but well understood, and we will not dwell on these points. See \([7]\) or \([5]\). If we work in our suitable categories of spectra the functor \(Y \mapsto X \wedge_{\mathcal{K}_n} Y\) has a right adjoint \(Z \mapsto F(X, Z)\).

We define a version of \((E_n)_* (\cdot)\) intrinsic to \(\mathcal{K}_n\) as follows.

**2.1 Definition.** Let \(X\) be a spectrum. Then we define \((E_n)_* X\) by the equation
\[
(E_n)_* X = \pi_*(L_{K(n)}(E_n, X)).
\]

We remark immediately that \((E_n)_* (\cdot)\) is not a homology theory in the usual sense; for example, it will not send wedges to sums of abelian groups. However, it is tractable, as we now explain. First note that \(E_n\) itself is \(K(n)\)-local; indeed, Lemma 5.2 of \([11]\) demonstrates that \(E_n\) is a finite wedge of spectra of the form \(L_{K(n)} E(n)\). Therefore if \(X\) is a finite CW spectrum, then \(E_n \wedge X\) is already in \(\mathcal{K}_n\), so

\[
(E_n)_* X = \pi_*(E_n \wedge X).
\]

Let \(I = (i_0, \ldots, i_{n-1})\) be a sequence of positive integers and let
\[
m^I = (p^{i_0}, u_1^{i_1}, \ldots, u_n^{i_n}) \subseteq m \subseteq (E_n)_*
\]
where \(m = (p, u_1, \ldots, u_n)\) is the maximal ideal in \((E_n)_*\). These form a system of ideals in \((E_n)_*\) and produce a filtered diagram of rings \(\{(E_n)_*/m^I\}\); furthermore
\[
(E_n)_* = \varinjlim I \frac{(E_n)_*}{m^I}.
\]
There is a cofinal diagram \(\{(E_n)_*/m^J\}\) which can be realized as a diagram of spectra in the following sense: using nilpotence technology, one can produce a diagram of finite spectra \(\{M_J\}\) and an isomorphism
\[
\{(E_n)_* M_J\} \cong \{(E_n)_*/m^J\}
\]
as diagrams. See §4 of \([11]\). Here \((E_n)_* M_J = \pi_* E_n \wedge M_J = \pi_* L_{K(n)}(E_n \wedge M_J)\). The importance of this diagram is that (see \([11]\), Proposition 7.10) for each spectrum \(X\)
\[
L_{K(n)} X \cong \varinjlim J M_J \wedge L_n X.
\]
This has the following consequence, immediate from Definition 2.1: there is a short exact sequence
\[
0 \to \varinjlim (E_n)_{k+1} (X \wedge M_J) \to (E_n)_k X \to \varinjlim (E_n)_k (X \wedge M_J) \to 0.
\]
This suggests $(E_n)_*X$ is closely related to some completion of $\pi_*(E_n \wedge X)$ and this is nearly the case. The details are spelled out in §8 of [11], but we won’t need the full generality there. In fact, all of the spectra we consider here will satisfy the hypotheses of Proposition 2.2 below.

If $M$ is a $(E_n)_*$-module, let $M^\wedge_m$ denote the completion of $M$ with respect to the maximal ideal of $(E_n)_*$. A module of the form

\[
\bigoplus_{\alpha} \Sigma^k_{\alpha} (E_n)_* \mapsto_m
\]

will be called pro-free.

2.2 Proposition. If $X$ is a spectrum so that $K(n)_*X$ is concentrated in even degrees, then

\[
(E_n)_*X \cong \pi_*(E_n \wedge X) \mapsto_m
\]

and $(E_n)_*X$ is pro-free as an $(E_n)_*$-module.

See Proposition 8.4 of [11].

As with anything like a flat homology theory, the object $(E_n)_*X$ is a comodule over some sort of Hopf algebroid of co-operations; it is our next project to describe this structure. In particular, this brings us to the role of the Morava stabilizer group. We begin by identifying $(E_n)_*E_n$.

Let $\mathbb{G}_n$ be the (big) Morava stabilizer group of $\Gamma$, the Honda formal group law of height $n$ over $\mathbb{F}_{p^n}$. For the purposes of this paper, a Morava module is a complete $(E_n)_*$-module $M$ equipped with a continuous $\mathbb{G}_n$-action subject to the following compatibility condition: if $g \in \mathbb{G}_n$, $a \in (E_n)_*$ and $x \in M$, then

\[
g(ax) = g(a)g(x).
\]

For example, if $X$ is any spectrum with $K(n)_*X$ concentrated in even degrees, then $(E_n)_*X$ is a complete $(E_n)_*$-module (by Proposition 2.2) and the action $\mathbb{G}_n$ on $E_n$ defines a continuous action of $\mathbb{G}_n$ on $(E_n)_*X$. This is a prototypical Morava module.

Now let $M$ be a continuous $(E_n)_*$-module and let

\[
\text{Hom}^\ell(\mathbb{G}_n, M)
\]

be the abelian group of continuous maps from $\mathbb{G}_n$ to $M$. If $M$ is complete, then

\[
\text{Hom}^\ell(\mathbb{G}_n, M) \cong \lim \text{colim}_k \text{map}(\mathbb{G}_n/U_k, M/m^k M)
\]

where $U_k$ runs over any system of open subgroups of $\mathbb{G}_n$ with $\bigcap_k U_k = \{e\}$. To give $\text{Hom}^\ell(\mathbb{G}_n, M)$ a structure of a continous $(E_n)_*$-module let $\phi: \mathbb{G}_n \to M$ be continuous and $a \in (E_n)_*$. We define $a\phi$ by the formula

\[
(a\phi)(g) = g(a)\phi(g).
\]

There also is a standard continuous action of $\mathbb{G}_n$ on $\text{Hom}^\ell(\mathbb{G}_n, M)$: if $g \in \mathbb{G}_n$ and $\phi: \mathbb{G}_n \to M$ is continuous, then one defines $g\phi: \mathbb{G}_n \to M$ by the formula

\[
(g\phi)(g') = \phi(g'g).
\]

With this action, and the action of $(E_n)_*$, defined in (2.5), the formula of (2.3) holds. If $M$ is complete, then (2.4) shows that $\text{Hom}^\ell(\mathbb{G}_n, M)$ is complete. We note that the functor $M \mapsto \text{Hom}^\ell(\mathbb{G}_n, M)$ from complete modules to Morava modules is right adjoint to the forgetful functor.
For example, if $X$ is a spectrum such that $(E_n)_* X$ is $(E_n)_*$-complete, the $\mathbb{G}_n$-action on $(E_n)_* X$ is encoded by the map

$$(E_n)_* X \to \text{Hom}^c(\mathbb{G}_n, (E_n)_* X)$$

adjoint to the identity.

The next result says that this is essentially all the structure that $(E_n)_* X$ supports. For any spectrum $X$, $\mathbb{G}_n$ acts on

$$(E_n)_*(E_n \wedge X) = \pi_* L_{K(n)}(E_n \wedge E_n \wedge X)$$

by operating in the left factor of $E_n$. The multiplication $E_n \wedge E_n \to E_n$ defines a morphism of $(E_n)_*$-modules

$$(E_n)_*(E_n \wedge X) \to (E_n)_* X$$

and by composing we obtain a map

$$\phi : (E_n)_*(E_n \wedge X) \to \text{Hom}^c(\mathbb{G}_n, (E_n)_*(E_n \wedge X)) \to \text{Hom}^c(\mathbb{G}_n, (E_n)_* X).$$

If $(E_n)_* X$ is complete, this is a morphism of Morava modules.

We now record:

**2.3 Proposition.** For any cellular spectrum $X$ with $(K_n)_* X$ concentrated in even degrees the morphism

$$\phi : (E_n)_*(E_n \wedge X) \to \text{Hom}^c(\mathbb{G}_n, (E_n)_* X)$$

is an isomorphism of Morava modules.

**Proof.** See [4] and [21] for the case $X = S^0$. The general case follows in the usual manner. First, it’s true for finite spectra by a five lemma argument. For this one needs to know that the functor

$$M \mapsto \text{Hom}^c(\mathbb{G}_n, M)$$

is exact on finitely generated $m$-complete modules. This follows from Equation (2.4). Then one argues the general case, by noting first that by taking colimits over finite cellular subspectra

$$\phi : (E_n)_*(E_n \wedge M \wedge X) \to \text{Hom}^c(\mathbb{G}_n, (E_n)_*(M \wedge X))$$

is an isomorphism for any $J$ and any $X$. (At this point one uses that $E_n \wedge M \wedge X$ is already $K(n)$-local for any $X$ and therefore $L_{K(n)}$ commutes with the homotopy colimits in question!). Finally one takes limits with respect to $J$. Here the hypothesis on $X$ is used to assure that

$$(E_n)_*(E_n \wedge X) \cong \lim (E_n)_*(E_n \wedge M \wedge X).$$

We next turn to the results of Devinatz and Hopkins ([4]) on homotopy fixed point spectra. Let $\mathcal{O}_{\mathbb{G}_n}^\text{op}$ be the orbit category of $\mathbb{G}_n$. Thus an object in $\mathcal{O}_{\mathbb{G}_n}$ is an orbit $\mathbb{G}_n/F$ where $F$ is a closed subgroup and the morphisms are continuous $\mathbb{G}_n$-maps. Then Devinatz and Hopkins have defined a functor

$$\mathcal{O}_{\mathbb{G}_n}^\text{op} \to \mathcal{K}$$
sending $\mathbb{G}_n / F$ to a $K(n)$-local spectrum $E_n^{hF}$. If $F$ is finite, then $E_n^{hF}$ is the usual homotopy fixed point spectrum defined by the action of $F \subseteq \mathbb{G}_n$. By the results of [4], the morphism $\phi$ of Proposition 2.3 restricts to an isomorphism (for any closed $F$)

$$\pi_* L_{K(n)}(E_n^{hF} \wedge E_n) \overset{\approx}{\longrightarrow} \text{Hom}^c(\mathbb{G}_n / F, (E_n)_*)$$

This, of course, computes $(E_n)_* E_n^{hF}$ to even, with some care, as a Morava module. Indeed, we give the $(E_n)_*$-module $\text{Hom}^c(\mathbb{G}_n / F, (E_n)_*)$ the “diagonal” $\mathbb{G}_n$ action defined as follows: if $g \in \mathbb{G}_n$ and $\phi : \mathbb{G}_n / F \to (E_n)_*$ a continous map, set

$$(g\phi)(x) = g\phi(g^{-1}x).$$

Then the isomorphism of 2.6 and the switch map give an isomorphism of Morava modules

$$(E_n)_* E_n^{hF} \cong \text{Hom}^c(\mathbb{G}_n / F, (E_n)_*)$$

where the target has the diagonal action.

Now let $U \subseteq \mathbb{G}_n$ be any open subgroup and consider the universal coefficient spectral sequence

$$E_2^{*,*} \cong \text{Ext}^*_*((E_n)_*(E_n^{hU}), (E_n)_*) \Longrightarrow \pi_* s(F(E_n^{hU}, E_n)) \cong E_2^{*,*}$$

relating the homology theory $(E_n)_*$ as defined in Definition 2.1 and the cohomology theory $(E_n)^*$ represented by the spectrum $E_n$. By the isomorphism of (2.7) $(E_n)_*(E_n^{hU}) \cong \text{Hom}^c(\mathbb{G}_n / U, (E_n)_*)$ is $(E_n)_*$-projective and therefore the edge homomorphism

$$\pi_* F((E_n)^{hU}, E_n) \to \text{Hom}_{(E_n)_*}(\text{map}^c(\mathbb{G}_n / U, (E_n)_*), (E_n)_*) \cong (E_n)_*[\mathbb{G}_n / U]$$

is an isomorphism (where for a finite set $S$ we denote the free $(E_n)_*$-module with bases $S$ by $(E_n)_*[S]$). This allows us to identify the homotopy type of the function spectrum $F((E_n)^{hU}, E_n)$ as follows: by [4] the finite group $\mathbb{G}_n / U$ acts on the homotopy fixed point spectrum $(E_n)^{hU}$ and the composition of this action with the multiplication on $E_n$ gives us a map

$$E_n \wedge (\mathbb{G}_n / U)_+ \wedge (E_n)^{hU} \to E_n$$

whose adjoint

$$E_n \wedge (\mathbb{G}_n / U)_+ \to F((E_n)^{hU}, E_n)$$

is an equivalence. More generally, if $X = \lim_n X_n$ is a profinite set, and $E$ is a spectrum we define

$$E[[X]] = \text{holim}_n E \wedge (X_n)_+.$$ 

Note that if $K$ is a group, $X$ a continuous $K$-set and $E$ an $K$-spectrum, then $E[[X]]$ is a $K$-spectrum via the diagonal action.

2.4 Proposition. Let $K_1$ be a closed subgroup and $K_2$ a finite subgroup of $\mathbb{G}_n$. Then there is a natural equivalence

$$E_n[[\mathbb{G}_n / K_1]]^{hK_2} \simeq F(E_n^{hK_1}, E_n^{hK_2}).$$
Furthermore, there is a natural decomposition

\[ E_n[[G_n/K_1]]^{hK_2} \cong \prod_{K_2 \subseteq G_n/K_1} E_n^{hK_x} \]

where \( K_x = K_2 \cap xK_1x^{-1} \) is the isotropy subgroup of the coset \( xK_1 \) and \( K_2 \backslash G_n/K_1 \) is the (profinite) set of double cosets.

Proof. If \( K_1 \) is a closed subgroup and if \( U_k \) is a decreasing sequence of subgroups with \( \bigcap U_k = K_1 \) then by construction of the homotopy fixed point spectra there is an equivalence

\[ E_n^{hK_2} \cong \text{hocolim}_k E_n^{hU_k} \).

If \( K_2 = \{1\} \) then the first statement follows simply from (2.9) by passing to homotopy colimits (which get turned into homotopy limits on the level of function spectra). The case of a general finite subgroup \( K_2 \) then follows by taking homotopy fixed points with respect to \( K_2 \).

To prove the second statement, write \( G_n/K_1 = \lim_k X_k \) where \( X_k \) are finite left \( G_n \)-sets. Then one easily checks that

\[ E_n[X_k]^{hK_2} \cong \prod_x E_n^{hK_x} \]

where \( x \) runs over orbit representatives for the \( K_2 \)-action on \( X_k \). The result now follows by taking homotopy inverse limits.

\[ \Box \]

2.5 Remark. By careful examination of the constructions of [4], we could probably make sense of, and prove, a weak equivalence

\[ E_n[[G_n/K_1]]^{hK_2} \cong F(E_n^{hK_1},E_n^{hK_2}) \]

with \( K_2 \) closed but not necessarily finite. The second equation of Proposition 2.4 would not hold in general, however. For example, the calculations of [21] show

\[ F(E_n,E_n^{hS^0}) \cong F(E_n,L_{K(n)}S^0) \cong \Sigma^{-n^2} E_n \].

We will be interested in the \( E_n \)-Hurewicz homomorphism

\[ \pi_0 F(E_n^{hK_1},E_n^{hK_2}) \to \text{Hom}(E_n)_*,E_n((E_n)_*E_n^{hK_1},(E_n)_*E_n^{hK_2}) \]

where \( \text{Hom}(E_n)_*,E_n \) denotes morphisms in the category of Morava modules. Let

\[ (E_n)_*[G_n] = \lim_k (E_n)_{*[G_n/U_k]} \]

denote the completed group ring and give this the structure of a Morava module by setting

\[ g(\sum a_i x_i) = \sum g(a_i) g(x_i). \]
2.6 Proposition. Let $K_1$ and $K_2$ be closed subgroups of $\mathbb{G}_n$ and suppose that $K_2$ is finite. Then there is an isomorphism

$$
\left((E_n)_*\left[[\mathbb{G}_n/K_1]\right]\right)^{K_2} \cong \text{Hom}(E_n)_*(E_n)^{E_{nK_1}}, (E_n)_*(E_n)^{E_{nK_2})}
$$

such that the following diagram commutes

$$
\begin{array}{ccc}
\pi_* E_n[[\mathbb{G}_n/K_1]]^{hK_2} & \cong & \left((E_n)_*\left[[\mathbb{G}_n/K_1]\right]\right)^{K_2} \\
& \Downarrow \cong & \Downarrow \\
\pi_* F(E_{nK_1}, E_{nK_2}) & \longrightarrow & \text{Hom}(E_n)_*(E_n)^{E_{nK_1}}, (E_n)_*(E_n)^{E_{nK_2})
\end{array}
$$

where the top horizontal map is the edge homomorphism in the homotopy fixed point spectral sequence, the left hand vertical map is the isomorphism of Proposition 2.4 and the bottom horizontal map is the $E_n$-Hurewicz homomorphism.

Proof. If $K_2$ is the trivial group then all other three maps in the diagram are isomorphisms; the top horizontal map for trivial reasons and the left hand vertical map by the equivalence of Proposition 2.4. For the bottom horizontal map we use Proposition 2.3 (in case $X = S^0$) and the adjunction isomorphism

$$
\text{Hom}(E_n)_*(M, \text{Hom}^\ast((\mathbb{G}_n, (E_n)_*))) \cong \text{Hom}(E_n)_*(M, (E_n)_*)
$$

(see the discussion after Proposition 2.2) together with the following easy observation: the $E_n$-Hurewicz homomorphism followed by the adjunction isomorphism agrees with the edge homomorphism of the universal coefficient spectral sequence and this edge homomorphism is again an isomorphism in our case (to see this pass to the limit in (2.8)). This establishes the isomorphism and the commutativity of the diagram in case $K_2 = \{1\}$.

The case of a general $K_2$ follows now by passing to $K_2$-invariants. \hfill \Box

3 The homotopy groups of $E_2^{hF}$ at $p = 3$

To construct our tower we are going to need some information about $\pi_* E_2^{hF}$ for various finite subgroups of the stabilizer group $\mathbb{G}_2$. Much of what we say here can be recovered from various places in the literature (for example, [9], [14], or [8]) and the point of view and proofs expressed are certainly those of Mike Hopkins. What we add here to the discussion in [8] is that we pay careful attention to the Galois group. In particular we treat the case of the finite group $G_{24}$.

Recall that we are working at the prime 3. We will write $E$ for $E_2$, so that we may write $E_\ast$ for $(E_2)_\ast$.

In Remark 1.1 we defined a subgroup

$$
G_{24} \subseteq \mathbb{G}_2 = \mathbb{S}_2 \rtimes \text{Gal} (\mathbb{F}_9/\mathbb{F}_3)
$$

generated by elements $s$, $t$ and $\psi$ of orders 3, 4 and 4 respectively. The cyclic subgroup $C_3$ generated by $s$ is normal, and the subgroup $Q_8$ generated by $t$ and $\psi$ is the quaternion group of order 8.

The first results are algebraic in nature; they give a nice presentation of $E_\ast$ as a $G_{24}$-algebra. First we define an action of $G_{24}$ on $\mathbb{W} = W(\mathbb{F}_9)$ by the formulas:

$$
s(a) = a \quad t(a) = \omega^2 a \quad \psi(a) = \omega \phi(a)
$$
where $\phi$ is the Frobenius. Note the action factors through $\mathbb{G}_{24}/C_3 \cong Q_8$. Restricted to the subgroup $G_{12} = \mathbb{S}_2 \cap G_{24}$ this action is $\mathbb{W}$-linear, but over $G_{24}$ it is simply linear over $\mathbb{Z}_3$. Let $\chi$ denote the resulting $G_{24}$-representation and $\chi'$ its restriction to $Q_8$.

This representation is a module over a twisted version of the group ring $\mathbb{W}[G_{24}]$. The projection

$$G_{24} \longrightarrow \text{Gal}(\mathbb{F}_9/\mathbb{F}_3)$$

defines an action of $G_{24}$ on $\mathbb{W}$ and we use this action to twist the multiplication in $\mathbb{W}[G_{24}]$. We should really write $\mathbb{W}\rho[G_{24}]$ for this twisted group ring, but we forebear, so as to not clutter notation. Note that $\mathbb{W}[Q_8]$ has a similar twisting, but $\mathbb{W}[G_{12}]$ is the ordinary group ring.

Define a $G_{24}$-module $\rho$ by the short exact sequence

$$(3.1) \quad 0 \rightarrow \chi \rightarrow \mathbb{W}[G_{24}] \otimes_{\mathbb{W}[Q_8]} \chi' \rightarrow \rho \rightarrow 0$$

where the first map takes a generator $e$ of $\chi$ to

$$(1 + s + s^2)e \in \mathbb{W}[G_{24}] \otimes_{\mathbb{W}[Q_8]} \chi'.$$

3.1 Lemma. There is is a morphism of $G_{24}$-modules

$$\rho \rightarrow E_2$$

so that the induced map

$$\mathbb{F}_9 \otimes_{\mathbb{W}} \rho \rightarrow E_0/(3, u_1^3) \otimes_{E_0} E_2$$

is an isomorphism. Furthermore, this isomorphism sends the generator $e$ of $\rho$ to an invertible element in $E_\ast$.

Proof. We need to know a bit about the action of $\mathbb{G}_2$ on $\mathbb{E}_\ast$. The relevant formulas have been worked out by Devinatz and Hopkins. Let $m \subseteq E_0$ be the maximal ideal and $a + bS \in \mathbb{S}_2$. Then Proposition 3.3 and Lemma 4.9 of [3] together imply that, modulo $m^2E_{-2}$

$$(3.2) \quad (a + bS)u = au + \phi(b)uu_1$$

$$(3.3) \quad (a + bS)uu_1 = 3bu + \phi(a)uu_1 .$$

In some cases we can be more specific. For example, if $\alpha \in \mathbb{F}_9^\times \subseteq W(\mathbb{F}_9)^\times \subseteq \mathbb{G}_2$, then the induced map of rings

$$\alpha_\ast : E_\ast \rightarrow E_\ast$$

is the $\mathbb{W}$-algebra map defined by the formulas

$$(3.4) \quad \alpha_\ast(u) = \alpha u \quad \text{and} \quad \alpha_\ast(uu_1) = \alpha^3uu_1 .$$

Finally, since the Honda formal group is defined over $\mathbb{F}_9$ the action of the Frobenius on $E_\ast = \mathbb{W}(\mathbb{F}_9)[[u_1]][u \pm 1]$ is simply extended from the action on $\mathbb{W}(\mathbb{F}_9)$. Thus we have

$$(3.5) \quad \psi(a) = \omega \phi(a)$$

for all $a \in E_2$. 
The formulas (3.2) and (3.5) imply that $E_0 / (3, u_1^2) \otimes_{E_0} E_{-2}$ is isomorphic to $E_0 \otimes_W \rho$ as a $G_{24}$-module and, further, that we can choose as a generator the residue class of $u$. In [8] (following [14], who learned from Hopkins) we found a class $y \in E_{-2}$ so that

$$y \equiv \omega u \mod (3, u_1) \ .$$

and so that

$$(1 + s + s^2)y = 0 \ .$$

This element might not yet have the correct invariance property with respect to $\psi$; to correct this, we average and set

$$x = \frac{1}{8} (x + \omega^{-2} t_*(y) + \omega^{-4}(t^2)_*(y) + \omega^{-6}(t^3)_*(y) + \omega^{-1}(\psi^2)_*(y) + \omega^{-7}(\psi t)_*(y) + \omega^{-5}(\psi^3 t^2)_*(y) + \omega^{-3}(\psi^3 t^2)_*(y)).$$

We can now send the generator of $\rho$ to $x$. Note also that the formulas (3.2) and (3.5) imply that $x \equiv \omega u$ modulo $(3, u_1^2)$.

We now make a construction. The morphism of $G_{24}$-modules constructed in this last lemma defines a morphism of $W$-algebras

$$S(\rho) = S_W(\rho) \longrightarrow E_*$$

sending the generator $e$ of $\rho$ to an invertible element in $E_2$. The symmetric algebra is over $W$ and the map is a map of $W$-algebras. The group $G_{24}$ acts through $\mathbb{Z}_3$-algebra maps, and the subgroup $G_{12}$ acts through $W$-algebra maps. If $a \in W$ is a multiple of the unit, then $\psi(a) = \phi(a)$.

Let

$$N = \prod_{g \in G_{12}} ge \in S(\rho);$$

then $N$ is invariant by $G_{12}$ and $\psi(N) = -N$ so that the get a morphism of graded $G_{24}$-algebras

$$S(\rho)[N^{-1}] \longrightarrow E_*$$

(where the grading on the target is determined by putting $\rho$ in degree $-2$). Inverting $N$ inverts $e$, but in an invariant manner. This map is not yet an isomorphism, but it is an inclusion onto a dense subring. The following result is elementary (cf. Proposition 2 of [8]):

**3.2 Lemma.** Let $I = S(\rho)(N^{-1}) \cap m$. Then completion at the ideal $I$ defines an isomorphism of $G_{24}$-algebras

$$S(\rho)[N^{-1}]_I \cong E_*.$$

Thus the input for the calculation of the $E_2$-term $H^*(G_{24}, E_*)$ of the homotopy fixed point spectral sequence associated to $E_2^{hG_{24}}$ will be discrete. Indeed, let $A = S(\rho)[N^{-1}]$. Then the essential calculation is that of $H^*(G_{24}, A)$. For this we begin with the following. For any finite group $G$ and any $G$ module $M$, let

$$\text{tr}_G : M \longrightarrow M^G = H^0(G, M)$$

be the transfer: $\text{tr}(x) = \sum_{g \in G} gx$. In the following result, an element listed as being in bidegree $(s, t)$ is in $H^s(G, A_t)$.

If $e \in \rho$ is the generator, define $d \in A$ to be the multiplicative norm with respect to the cyclic group generated by $s$: $d = s^2(e)s(e)e$. By construction $d$ is invariant with respect to the subgroup generated by $s$. 

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3.3 Lemma. Let $C_3 \subseteq G_{12}$ be the normal subgroup of order three. Then there is an exact sequence

$$A \xrightarrow{\text{tr}} H^*(C_3, A) \rightarrow F_0[a, b, d^{\pm 1}] / (a^2) \rightarrow 0$$

where $a$ has bidegree $(1, -2)$, $b$ has bidegree $(2, 0)$ and $d$ has bidegree $(0, -6)$. Furthermore the action of $t$ and $\psi$ is described by the formulas

$$t(a) = -\omega^2 a \quad t(b) = -b \quad t(d) = \omega^6 d$$

and

$$\psi(a) = \omega a \quad \psi(b) = b \quad \psi(d) = \omega^3 d .$$

Proof. This is the same argument as in Lemma 3 of [8], although here we keep track of the Frobenius.

Let $F$ be the $G_{24}$-module $W[G_{24}] \otimes W[Q] \chi$; thus Lemma 3.1 gives a short exact sequence of $G_{24}$-modules

$$(3.8) \quad 0 \rightarrow S(F) \otimes \chi \rightarrow S(F) \rightarrow S(\rho) \rightarrow 0 .$$

In the first term, we set the degree of $\chi$ to be $-2$ in order to make this an exact sequence of graded modules. We use the resulting long exact sequence for computations. We may choose $W$-generators of $F$ labelled $x_1$, $x_2$, and $x_3$ so that if $s$ is the chosen element of order 3 in $G_{24}$, then $s(x_1) = x_2$ and $s(x_2) = x_3$. Furthermore, we can choose $x_1$ so that it maps to the generator $e$ of $\rho$ and is invariant under the action of the Frobenius. Then we have

$$S(F) = W[x_1, x_2, x_3]$$

with the $x_i$ in degree $-2$. Under the action of $C_3$ the orbit of a monomial in $W[x_1, x_2, x_3]$ has three elements unless that monomial is a power of $\sigma_3 = x_1 x_2 x_3$ — which, of course, maps to $d$. Thus, we have a short exact sequence

$$S(F) \xrightarrow{\text{tr}} H^*(C_3, S(F)) \rightarrow F_0[b, d] \rightarrow 0$$

where $b$ has bidegree $(0, 2)$ and $d$ has bidegree $(0, -6)$. Here $b \in H^2(\mathbb{Z}/3, \mathbb{Z}_3) \subseteq H^2(\mathbb{Z}/3\mathbb{Z}, W)$ is a generator and $W \subseteq S(F)$ is the submodule generated by the algebra unit. Note that the action of $t$ is described by

$$t(d) = \omega^6 d \quad \text{and} \quad t(b) = -b .$$

The last is because the element $t$ acts non-trivially on the subgroup $C_3 \subseteq G_{24}$ and hence on $H^2(C_3, W)$. Similarly, since the action of the Frobenius on $d$ is trivial and $x$ acts trivially on $C_3$, we have

$$\psi(d) = \omega^3 d \quad \text{and} \quad \psi(b) = b .$$

The short exact sequence $(3.8)$ and the fact that $H^1(C_3, S(F)) = 0$ now imply that there is an exact sequence

$$S(\rho) \xrightarrow{\text{tr}} H^*(C_3, S(\rho)) \rightarrow F_0[a, b, d] / (a^2) \rightarrow 0 .$$

The element $a$ maps to

$$b \in H^2(C_3, S_0(F) \otimes \chi) = H^2(\mathbb{Z}/3\mathbb{Z}, \chi)$$

under the boundary map (which is an isomorphism)

$$H^1(C_3, \rho) = H^1(C_3, S_1(\rho)) \rightarrow H^2(C_3, \chi) ;$$

thus $a$ has bidegree $(1, -2)$ and the actions of $t$ and $\psi$ are twisted by $\chi$:

$$t(a) = -\omega^2 a = \omega^6 a \quad \text{and} \quad \psi(a) = \omega a .$$

\qed

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We next write down the invariants $E^0_F$ for the various finite subgroups $F$ of $G_{24}$. To do this, we work up from the symmetric algebra $S(\rho)$, and we use the presentation of the symmetric algebra as given in the exact sequence (3.8). Recall that we have written $S(F) = \mathbb{W}[x_1, x_2, x_3]$ where the normal subgroup of order three in $G_{24}$ cyclically permutes the $x_i$. This action by the cyclic group extends in an obvious way to an action of the symmetric group $\Sigma_3$ on three letters; thus we have an inclusion of algebras

$$\mathbb{W}[\sigma_1, \sigma_2, \sigma_3] = \mathbb{W}[x_1, x_2, x_3]^\Sigma_3 \subseteq S(F)^{C_3}.$$  

There is at least one other obvious element invariant under the action of $C_3$: set

$$\epsilon = x_1^2x_2 + x_2^2x_3 + x_3^2x_1 - x_2^3x_1 - x_1^2x_3 - x_3^2x_2.$$  

This might be called the “anti-symmetrization” (with respect to $\Sigma_3$) of $x_1^2x_2$.

**3.4 Lemma.** There is an isomorphism

$$\mathbb{W}[\sigma_1, \sigma_2, \sigma_3, \epsilon]/(\epsilon^2 - f) \cong S(F)^{C_3}$$

where $f$ is determined by the relation

$$\epsilon^2 = -27\sigma_3^2 - 4\sigma_2^3 - 4\sigma_3\sigma_1^3 + 18\sigma_1\sigma_2\sigma_3 + \sigma_1^2\sigma_2^2.$$  

Furthermore, the actions of $t$ and $\psi$ are given by

$$t(\sigma_1) = \omega^2\sigma_1 \quad t(\sigma_2) = -\sigma_2 \quad t(\sigma_3) = \omega^6\sigma_3 \quad t(\epsilon) = \omega^2\epsilon$$

and

$$\psi(\sigma_1) = \omega\sigma_1 \quad \psi(\sigma_2) = \omega^2\sigma_2 \quad \psi(\sigma_3) = \omega^3\sigma_3 \quad \psi(\epsilon) = \omega^3\epsilon.$$  

**Proof.** Except for the action of $\psi$, this is Lemma 4 of [8]. The actions of $t$ and $\psi$ are a matter of a simple computation.  

This immediately leads to the following result.

**3.5 Proposition.** There is an isomorphism

$$\mathbb{W}[\sigma_2, \sigma_3, \epsilon]/(\epsilon^2 - g) \cong S(\rho)^{C_3}$$

where $g$ is determined by the relation

$$\epsilon^2 = -27\sigma_3^2 - 4\sigma_2^3$$

with the actions of $t$ and $x$ as given above in Lemma 3.4. Under this isomorphism $\sigma_3$ maps to $d$.

**Proof.** This follows immediately from Lemma 3.4, the short exact sequence (3.8), and the fact (see the proof of Lemma 3.3) that $H^1(C_3, S(F)) = 0$. Together these imply that

$$S(\rho)^{C_3} \cong S(F)^{C_3}/(\sigma_1).$$  

\[\Box\]
The next step is to invert the element \( N \) of (3.7). This element is the image of \( \sigma_3^4 \); thus, we are effectively inverting the element \( d = \sigma_3 \in S(\rho)^{C_2} \). We begin with the observation that if we divide
\[
\epsilon^2 = -27\sigma_3^2 - 4\sigma_3^3
\]
by \( \sigma_3^6 \) we obtain the relation
\[
\left( \frac{\epsilon}{\sigma_3^2} \right)^2 + 4\left( \frac{\sigma_2}{\sigma_3} \right)^3 = -\frac{27}{\sigma_3^2}.
\]
Thus if we set
\[
(3.10)
\]
\[
c_4 = -\frac{\omega^2\sigma_2}{\sigma_3^2}, \quad c_6 = \frac{\omega^3\epsilon}{2\sigma_3^3}, \quad \Delta = -\frac{\omega^6}{4\sigma_5^3} = \frac{\omega^2}{4\sigma_3^2}
\]
then we get the expected relation \(^2\)
\[
c_4^2 - c_6^3 = 27\Delta.
\]
Furthermore, \( c_4, c_6, \) and \( \Delta \) are all invariant under the action of the entire group \( G_{24} \). (Indeed, the powers of \( \omega \) are introduced so that this happens.)

To describe the group cohomology, we define elements
\[
\alpha = \frac{\omega a}{d} \in H^1(C_5, (S(\rho)[N^{-1}])_4)
\]
and
\[
\beta = \frac{\omega^3 b}{d^2} \in H^2(C_5, (S(\rho)[N^{-1}])_{12})
\]
These elements are fixed by \( t \) and \( \psi \) and, for degree reasons, acted on trivially by \( c_4 \) and \( c_6 \). The following is now easy.

3.6 Proposition. 1.) The inclusion
\[
\mathbb{Z}_3[c_4, c_6, \Delta^{\pm 1}]/(c_4^2 - c_3^3 = 27\Delta) \to S(\rho)[N^{-1}]^{G_{24}}
\]
is an isomorphism.

2.) There is an exact sequence
\[
S(\rho)[N^{-1}] \xrightarrow{\text{tr}} H^*(G_{24}, S(\rho)[N^{-1}]) \to F_3[\alpha, \beta, \Delta^{\pm 1}]/(\alpha^2) \to 0
\]
and \( c_4 \) and \( c_6 \) act trivially on \( \alpha \) and \( \beta \).

Then a completion argument, as in Theorem 6 of [8] or [14] implies the next result.

3.7 Theorem. 1.) There is an isomorphism
\[
(E_*)^{G_{24}} \cong \mathbb{Z}_3[[c_4^3\Delta^{-1}]]/(c_4^2 - c_3^3 = 27\Delta).
\]

2.) There is an exact sequence
\[
E_* \xrightarrow{\text{tr}} H^*(G_{24}, E_*) \to F_3[\alpha, \beta, \Delta^{\pm 1}]/(\alpha^2) \to 0
\]
and \( c_4 \) and \( c_6 \) act trivially on \( \alpha \) and \( \beta \).

\(^2\)This is the relation appearing in theory of modular forms [1], except here 2 is invertible so we can replace 1728 by 27. There is some discussion of the connection in [9]. The relation could be arrived at more naturally by choosing, as our basic formal group law, one arising from the theory of elliptic curves, rather than the Honda formal group law.
3.8 Remark. The same kind of reasoning can be used to obtain the group cohomologies \( H^*(F, E_*) \) for other finite subgroups of \( G_{24} \). First define an element
\[
\delta = \sigma_3^{-1} \in S(\rho)[N^{-1}].
\]
Then \( \Delta = (\omega^2/4)^{\delta^2} \); thus \( -\Delta \) has a square root:
\[
(-\Delta)^{1/2} = \frac{\omega^3}{2} \delta^2.
\]
The elements \( t \) and \( \psi \) of \( G_{24} \) act on \( \delta \) by the formulas
\[
t(\delta) = \omega^2 \delta \quad \text{and} \quad \psi(\delta) = \omega^5 \delta.
\]
The element \( (-\Delta)^{1/2} \) is invariant under the action of \( t^2 \) and \( \psi \) (whereas the evident square root of \( \Delta \) is not fixed by \( \psi \)).

Let \( C_{12} \) be the cyclic subgroup of order 12 in \( G_{24} \) generated by \( s \) and \( \psi \). This subgroup has a cyclic subgroup \( C_6 \) of order 6 generated by \( s \) and \( t^2 = \psi^2 \). We have
\[
(E_*)^{C_6} \cong \mathbb{W}[c_4, c_6, \Delta^{-1}]/(c_6^3 - c_4^3 = 27\Delta)
\]
\[
(E_*)^{C_{12}} \cong \mathbb{Z}_6[c_4, c_6, (\Delta)^{\pm 1}/(c_6^3 - c_4^3 = 27\Delta)]
\]
\[
(E_*)^{C_6} \cong \mathbb{W} \otimes_{\mathbb{Z}_6} (E_*)^{C_{12}}
\]
\[
(E_*)^{G_{12}} \cong \mathbb{W}[c_4, c_6, \Delta, (\Delta)^{\pm 1}/(c_6^3 - c_4^3 = 27\Delta)] \cong \mathbb{W} \otimes_{\mathbb{Z}_6} (E_*)^{G_{24}}.
\]
Furthermore, for all these groups, the analogue of Theorem 3.7.2 holds. For example, there are exact sequences
\[
E_* \xrightarrow{tr} H^*(C_3, E_*) \rightarrow F_3[\alpha, \beta, \delta, \Delta^{-1}/(\Delta^2)] \rightarrow 0
\]
\[
E_* \xrightarrow{tr} H^*(C_{12}, E_*) \rightarrow F_3[\alpha, \beta, (\Delta)^{\pm 1}/(\Delta^2)] \rightarrow 0
\]
and \( c_4 \) and \( c_6 \) act trivially on \( \alpha \) and \( \beta \).

These results allow one to completely write down the various homotopy fixed point spectral sequences for computing \( \pi_*E^{hG} \) for the various finite groups in question. The differentials in the spectral sequence follow from Toda’s classical results and the following easy observation: every element in the image of the transfer is a permanent cycle. We record:

3.9 Lemma. In the spectral sequence
\[
H^*(G_{24}, E_*) \Longrightarrow \pi_*E^{hG_{24}}
\]
the only non-trivial differentials are \( d_5 \) and \( d_9 \). They are determined by
\[
d_5(\Delta) = a_1 \alpha \beta^2 \quad \text{and} \quad d_9(\alpha \Delta^2) = a_2 \beta^5
\]
where \( a_1 \) and \( a_2 \) are units in \( F_3 \).
Proof. These are a consequence of Toda’s famous differential (see [23]) and nilpotence. See Proposition 7 of [8] or, again, [14]. There it is done for $G_{12}$ rather than $G_{24}$, but because we are working at the prime 3, this is sufficient.

This immediately calculates the differentials in the other spectral sequences; for example, if one wants fixed points with respect to the $C_3$-action, we have, up to units,

$$d_5(\delta) = \delta^{-3} \alpha \beta^2 \quad \text{and} \quad d_6(\alpha \delta^2) = \delta^{-6} \beta^5.$$

Here is the main homotopy theoretic calculation of this section.

3.10 Theorem. In the spectral sequence

$$H^*(C_3, E_\ast) \Longrightarrow \pi_\ast E^{hC_3}$$

we have an inclusion of subrings

$$E_\infty^{0, \ast} \cong \mathbb{W}[\langle c_4^3 \Delta^{-1} \rangle | c_4, c_6, c_4 \delta^\pm 1, c_6 \delta^\pm 1, 3 \delta^\pm 1, 3 \delta^\pm 3 \rangle (c_4^3 - c_6^2 = 27 \Delta) \subseteq E_2^{0, \ast}.$$

In positive filtration $E_\infty$ is additively generated by the elements $\alpha, \delta \alpha, \alpha \beta, \delta \alpha \beta, \beta^j, 1 \leq j \leq 4$ and all multiples of these elements by $\delta^\pm 3$. Furthermore, these elements are of order 3 and are annihilated by $c_4, c_6, c_4 \delta^\pm 1, c_6 \delta^\pm 1$ and $3 \delta^\pm 1$.

For the case of the cyclic group $C_6$ of order 6 generated by $s$ and $t^2$, one note that $t^2(\delta) = -\delta$ and the spectral sequence can now be read off Theorem 3.10. This also determines the case of $C_{12}$, the group generated by $s$ and $\psi$. We leave the details to the reader but state the result in case of $G_{24}$.

3.11 Theorem. In the spectral sequence

$$H^*(G_{24}, E_\ast) \Longrightarrow \pi_\ast E^{hG_{24}}$$

we have an inclusion of subrings

$$E_\infty^{0, \ast} \cong \mathbb{Z}[\langle c_4^3 \Delta^{-1} \rangle | c_4, c_6, c_4 \Delta^\pm 1, c_6 \Delta^\pm 1, 3 \Delta^\pm 1, 3 \Delta^\pm 3 \rangle (c_4^3 - c_6^2 = 27 \Delta) \subseteq E_2^{0, \ast}.$$

In positive filtration $E_\infty$ is additively generated by the elements $\alpha, \Delta \alpha, \alpha \beta, \Delta \alpha \beta, \beta^j, 1 \leq j \leq 4$ and all multiples of these elements by $\Delta^\pm 3$. Furthermore, these elements are of order 3 and are annihilated by $c_4, c_6, c_4 \Delta^\pm 1, c_6 \Delta^\pm 1$ and $3 \Delta^\pm 1$.

3.12 Remark. 1.) Note that $E^{hC_3}$ is periodic of period 18 and that $\delta^3$ detects a periodicity class. The spectra $E^{hC_6}$ and $E^{hC_{12}}$ are periodic with period 36 and $(-\Delta)^{3/2}$ detects the periodicity generator. Finally $E^{hG_{12}}$ and $E^{hG_{24}}$ are periodic with period 72 and $\Delta^3$ detects the periodicity generator.

2.) By contrast, $E_\ast E^{hG_{24}}$ is of period 24. To see this, note that the isomorphism of (2.7) supplies an isomorphism of Morava modules

$$E_\ast E^{hG_{24}} \cong \text{Hom}_G(G/G_{24}, E_\ast)$$

with $G_2$ acting diagonally. Since $\Delta$ is $G_{24}$-invariant post-multiplication by $\Delta$ induces an automorphism which gives period at least 24. On the other hand the image of the coset of the unit element in $G_2$ must be contained in the $G_{24}$-invariants $E^{G_{24}}$. Together this gives that the period is equal to 24.
3.) If $F$ has order prime to 3, then $\pi_*(E^{hF}) = (E_*)^F$ is easy to calculate. For example, using (3.4) and (3.5) we obtain
$$\pi_*(E^{hS_{D_{16}}}) = \mathbb{Z}_3[[u_1^4]][u^\pm 8, u^6 u_1, u^4 u_1^2, u^2 u_1^3]] \subseteq E_*$$
and
$$\pi_*(E^{hQ_8}) = \mathbb{Z}_3[[\omega^2 u_1^2]][\omega^2 u_1^\pm 2, \omega^2 u_1^4]] \subseteq E_* .$$
Note that $E^{hS_{D_{16}}}$ has periodicity of order 16 and $E^{hQ_8}$ has periodicity of order 8.

We finish this section by listing exactly the computational results we will use in building the tower in section 5.

**3.13 Corollary.** 1.) Let $F \subseteq \mathbb{G}_2$ be any finite subgroup. Then the edge homomorphism
$$\pi_0 E_2^{hF} \longrightarrow (E_2)^F$$
is an isomorphism.

2.) Let $F \subseteq \mathbb{G}_2$ be any finite subgroup. Then
$$\pi_{24} E_2^{hF} \longrightarrow (E_2)^{F_{24}}$$
is an injection.

*Proof. If $F$ has order prime to 3, both of these statements are clear. If 3 divides the order of $F$, then the 3-Sylow subgroup of $F$ is conjugate to $C_3$; hence $\pi_* E_2^{hF}$ is a retract of $\pi_* E_2^{hC_3}$ and the result follows from Theorem 3.10. □

**3.14 Corollary.** Let $F \subseteq \mathbb{G}_2$ be a finite subgroup. If the order of $F$ is prime to 3, then
$$\pi_1 E_2^{hF} = 0 .$$
For all finite subgroups $F$,
$$\pi_{25} E_2^{hF} = 0 .$$

*Proof. Again apply Theorem 3.10. □

**3.15 Corollary.** Let $F \subseteq \mathbb{G}_2$ be any finite subgroup containing the central element $\omega^4 = -1$. Then
$$\pi_{26} E_2^{hF} = 0 .$$

*Proof. Equation 3.4 implies that $(E_*)^F$ will be concentrated in degrees congruent to 0 mod 4. Combining this observation with Theorem 3.10 proves the result. □
4 The algebraic resolution

We continue to work at the prime 3. In 1.3 we wrote down a splitting of the group $G_2$ as $G_2 \times \mathbb{Z}_3$. In this section we will construct our resolution of the trivial $G_2^1$-module $\mathbb{Z}_3$.

Recall that we have selected a maximal finite subgroup $G_{24} \subseteq G_2$, it is generated by an element $s$ of order three, $t = \omega^2$, and $\psi = \omega \phi$ where $\omega$ is a primitive eighth root of unity and $\phi$ is the Frobenius. Let $C_3$ be the normal subgroup of order 3 in $G_{24}$ and let $Q_8$ be the subgroup of order 8 generated by $t$ and $\psi$. The group $G_2^1$ also contains a copy of the semidihedral group $SD_{16}$ generated by $\omega$ and $\psi$, namely

$$SD_{16} = \mathbb{F}_9 \rtimes \text{Gal}(\mathbb{F}_9/\mathbb{F}_3).$$

The group $Q_8$ is a subgroup of index 2. Let $\chi$ be the sign representation (over $\mathbb{Z}_3$) of $SD_{16}/Q_8$; we regard $\chi$ as representation of $SD_{16}$ using the quotient map. (Note that this is not the same $\chi$ as in section 3!). In this section, the continuous $G_2^1$-module

$$\chi \uparrow_{SD_{16}}^{G_2^1} \overset{\text{def}}{=} \mathbb{Z}_3[[G_2^1]] \otimes \mathbb{Z}_3[SD_{16}] \chi$$

will play an important role. If $\tau$ is the trivial representation of $SD_{16}$, there is an isomorphism of $SD_{16}/Q_8$-modules

$$\mathbb{Z}_3[SD_{16}/Q_8] \cong \chi \oplus \tau. \tag{4.2}$$

Thus, we have that $\chi \uparrow_{SD_{16}}^{G_2^1}$ is a direct summand of the permutation module $\mathbb{Z}_3[[G_2^1]/Q_8]]$.

The following is the main algebraic result of the paper. It will require the entire section to prove.

**4.1 Theorem.** There is an exact sequence of $G_2^1$-modules

$$0 \to \mathbb{Z}_3[[G_2^1/G_{24}]] \to \chi \uparrow_{SD_{16}}^{G_2^1} \to \chi \uparrow_{SD_{16}}^{G_2^1} \to \mathbb{Z}_3[[G_2^1/G_{24}]] \to \mathbb{Z}_3 \to 0. \tag{4.1}$$

Some salient features of this “resolution” (we will use this word even though not all of the modules are projective) are that each module is a summand of a permutation module and that each module is free over $K$, where $K \subseteq G_2^1$ is a subgroup so that we can decompose the 3-Sylow subgroup $S_3^1$ as $K \times C_3$. Important features of $K$ include that it is a torsion-free 3-adic Poincaré duality group of dimension 3. (See before Lemma 4.10 for more on $K$.)

Since $G_2 \cong G_2^1 \times \mathbb{Z}_3$, we may tensor the resolution of Theorem 4.1 with the standard resolution for $\mathbb{Z}_3$ to get the following as an immediate corollary.

**4.2 Corollary.** There is an exact sequence of $G_2$-modules

$$0 \to \mathbb{Z}_3[[G_2/G_{24}]] \to \mathbb{Z}_3[[G_2/G_{24}]] \oplus \mathbb{Z}_3[[G_2/G_{24}]] \oplus \mathbb{Z}_3[[G_2/G_{24}]] \to \mathbb{Z}_3[[G_2/G_{24}]] \to \mathbb{Z}_3 \to 0. \tag{4.2}$$

As input for our calculation we will use $H^q(S_2^1, \mathbb{F}_3) = H^q(S_2^1)$, as calculated by the second author in [10]. This is an effective starting point because of the following lemma and the fact that

$$H^q(S_2^1, \mathbb{F}_3) = \text{Ext}_{\mathbb{Z}_3[[S_2^1]]}^q(\mathbb{Z}_3, \mathbb{F}_3) \cong \text{Tor}_{\mathbb{Z}_3}^q(\mathbb{Z}_3, \mathbb{F}_3)^*.$$ 

Here ($-)^*$ means $F_p$-linear dual.
A profinite group $G$ is called finitely generated if there is a finite set of elements $X \subseteq G$ so that the subgroup generated by $X$ is dense. This is true of all the groups in this paper. If $G$ is a $p$-profinite group and $I \subseteq \mathbb{Z}_p[[G]]$ is the kernel of the augmentation $\mathbb{Z}_p[[G]] \to F_p$, then

$$\mathbb{Z}_p[[G]] \cong \lim_{\to} \mathbb{Z}_p[[G]] / I^n.$$ 

A $\mathbb{Z}_p[[G]]$-module will be called complete if it is $I$-adically complete.

### 4.3 Lemma

Let $G$ be a finitely generated $p$-profinite group and $f : M \to N$ a morphism of complete $\mathbb{Z}_p[[G]]$-modules. If

$$F_p \otimes f : F_p \otimes_{\mathbb{Z}_p[[G]]} M \to F_p \otimes_{\mathbb{Z}_p[[G]]} N$$

is surjective, then $f$ is surjective. If

$$\text{Tor}(F_p, f) : \text{Tor}^p_{\mathbb{Z}_p[[G]]}(F_p, M) \to \text{Tor}^p_{\mathbb{Z}_p[[G]]}(F_p, N)$$

is an isomorphism for $q = 0$ and onto for $q = 1$, then $f$ is an isomorphism.

**Proof.** This is an avatar of Nakayama's Lemma. To see this, suppose $K$ is some complete $\mathbb{Z}_p[[G]]$-module so that $F_p \otimes_{\mathbb{Z}_p[[G]]} K = 0$. Then an inductive argument shows

$$\mathbb{Z}_p[[G]] / I^n \otimes_{\mathbb{Z}_p[[G]]} K = 0$$

for all $n$; hence $K = 0$. This is the form of Nakayama's lemma we need. The result is then proved using the long exact sequence of Tor groups: the weaker hypothesis implies that the cokernel of $f$ is trivial; the stronger hypothesis then implies that the kernel of $f$ is trivial. □

We next turn to the details about $H^*(S^2_3; F_3)$ from [10]. (See Theorem 4.3 of that paper.) We will omit the coefficients $F_3$ in order to simplify our notation. The key point here is that the cohomology of the group $S^2_3$ is detected on the centralizers of the cyclic subgroups of order 3. There are two conjugacy classes of such subgroups of order 3 in $S^2_3$; namely, $C_3$ and $\omega C_3 \omega^{-1}$. The element $s = s_1$ is our chosen generator for $C_3$; thus we choose as our generator for $\omega C_3 \omega^{-1}$ the element $s_2 = \omega s \omega^{-1}$. The Frobenious $\phi$ also conjugates $C_3$ to $\omega C_3 \omega^{-1}$ and a short calculation shows that

$$\phi(s_1) = \phi(s) = s_2^2.$$ 

The centralizer $C(C_3)$ in $S^2_3$ is isomorphic to $C_3 \times Z_3$ and $\omega \phi$ commutes with $C(C_3)$ (see [8]). In particular, for every $x \in C(C_3)$ we have

$$\omega x \omega^{-1} = \phi(x)^{-1} \in C(\omega C_3 \omega^{-1}) .$$ 

Note that $C(\omega C_3 \omega^{-1}) = \omega C(C_3) \omega^{-1}$. Write $E(X)$ for the exterior algebra on a set $X$. Then

$$H^*(C(C_3)) \cong F_3[y_1] \otimes E(x_1, a_1)$$

and

$$H^*(C(\omega C_3 \omega^{-1})) \cong F_3[y_2] \otimes E(x_2, a_2) .$$

We know that $C_4$ (which is generated by $\omega^2$) acts on

$$H^*(C(C_3)) \cong F_3[y_1] \otimes E(x_1, a_1)$$

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sending all three generators to their negative. This action extends to an action of $SD_{16}$ on the product

$$H^*(C(C_3)) \times H^*(C(\omega C_3 \omega^{-1}))$$

as follows. By equation 4.3 and 4.4 the action of the generators $\omega$ and $\phi$ of $SD_{16}$ is given by

(4.5) $\omega(x_1) = x_2, \quad \omega(y_1) = y_2, \quad \omega(a_1) = a_2, \quad \phi(x_1) = -x_2, \quad \phi(y_1) = -y_2, \quad \phi(a_1) = -a_2.$

(4.6) $\omega(x_2) = -x_1, \quad \omega(y_2) = -y_1, \quad \omega(a_2) = -a_1, \quad \phi(x_2) = -x_1, \quad \phi(y_2) = -y_1, \quad \phi(a_2) = -a_1.$

4.4 Theorem. [10][11.] The inclusions $\omega^i C(C_3) \omega^{-i} \to S^1_2$, $i = 0, 1$ induce an $SD_{16}$-equivariant homomorphism

$$H^*(S^1_2) \to \prod_{i=1}^2 F_3[y_i] \otimes E(x_i, a_i)$$

which is an injection onto the subalgebra generated by $x_1, x_2, y_1, y_2, x_1a_1 - x_2a_2, y_1a_1$ and $y_1a_2$.

2.) In particular, $H^*(S^1_2)$ is free as a module over $F_3[y_1 + y_2]$ on generators $1, x_1, x_2, y_1, x_1a_1 - x_2a_2, y_1a_1, y_2a_2$, and $y_1x_1a_1$.

We will produce the resolution of Theorem 4.1 from this data and by splicing together the short exact sequences of Lemma 4.5, 4.6, and 4.7 below. Most of the work will be spent in identifying the last module; this is done in Theorem 4.9.

In the following computations, we will write

$$\text{Ext}(M) = \text{Ext}^*_Z[[S^1_2]](M, F_3).$$

This graded vector space is a module over

$$H^*(S^1_2) = \text{Ext}^*_{Z_3[[S^1_2]]}(Z_3, F_3) = \text{Ext}(Z_3),$$

and, hence is also a module over the sub-polynomial algebra of $H^*(S^1_2)$ generated by $y_1 + y_2$. If $M$ is actually a continuous $Z_3[[G^2_3]]$-module, then $\text{Ext}(M)$ has an action by $SD_{16}$ which extends the action by $H^*(S^1_2)$ in the obvious way; if $\alpha \in SD_{16}, a \in H^*(S^1_2)$ and $x \in \text{Ext}(M)$, then

$$\alpha(ax) = a(\alpha(a)) \alpha(x).$$

We can write this another way. Let’s define $\text{Ext}(Z_3) \circ F_3[SD_{16}]$ to be the algebra constructed by taking

$$\text{Ext}(Z_3) \otimes F_3[SD_{16}]$$

with twisted product

$$(a \otimes \alpha)(b \otimes \beta) = a\alpha(b) \otimes \alpha\beta.$$ 

The above remarks imply that if $M$ is a $Z_3[[G^2_3]]$-module, then $\text{Ext}(M)$ is an $\text{Ext}(Z_3) \circ F_3[SD_{16}]$-module.

This structure behaves well with respect to long exact sequences. If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence of $Z_3[[G^2_3]]$-modules we get a long exact sequence in $\text{Ext}$ which is a long exact sequence of $\text{Ext}(Z_3) \circ F_3[SD_{16}]$-modules. As a matter of notation, if $x \in \text{Ext}(Z_3)$ we will write $\overline{x} \in \text{Ext}(M)$ if $x$ is the image of $\overline{x}$ under some unambiguous and injective sequence of boundary homomorphisms in the long exact sequences.

Write $z$ for $x_1a_1 - x_2a_2 \in H^2(S^1_2)$. 

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4.5 Lemma. There is a short exact sequence of $\mathbb{Z}_3[[G_2]]$-modules

$$0 \to N_1 \to \mathbb{Z}_3[[G_2^1/G_{24}]] \xrightarrow{\epsilon} \mathbb{Z}_3$$

where the map $\epsilon$ is the augmentation. The cohomology group $\text{Ext}(N_1)$ is a module over $\mathbb{F}_3[y_1 + y_2]$ on generators $a$, $\bar{a}$, $\bar{y}_1a_1$, $\bar{y}_2a_2$, and $\bar{y}_1x_1a_1$ of degrees $0$, $1$, $2$, and $3$ respectively. The last four generators are free and $(y_1 + y_2)a = 0$. The action of $SD_{16}$ is determined by the action on $\text{Ext}(\mathbb{Z}_3)$ and the facts that

$$\omega_*(a) = -a = \phi_*(a).$$

Proof. As a continuous $S^1_2$-module, there is an isomorphism

$$\mathbb{Z}_3[[G_2^1/G_{24}]] \cong \mathbb{Z}_3[[S_2^1/C_3]] \oplus \mathbb{Z}_3[[S_2^1/\omega C_3 \omega^{-1}]].$$

Hence there is an isomorphism

$$\text{Ext}(\mathbb{Z}_3[[G_2^1/G_{24}]]) \cong H^*(C_3, \mathbb{F}_3) \times H^*(\omega C_3 \omega^{-1}, \mathbb{F}_3)$$

and the map $\text{Ext}(\mathbb{Z}_3) \to \text{Ext}(\mathbb{Z}_3[[G_2^1/G_{24}]]$) corresponds via this isomorphism to the restriction map. The result now follows from Theorem 4.4.

Recall that $\chi$ is the rank one (over $\mathbb{Z}_3$) representation of $SD_{16}$ obtained by pulling back the sign representation along the quotient map $\varepsilon : SD_{16} \to SD_{16}/Q_8 \cong \mathbb{Z}/2$.

4.6 Lemma. There is a short exact sequence of $\mathbb{Z}_3[[G_2^1]]$-modules

$$0 \to N_2 \to \chi \downarrow_{SD_{16}} \to N_1 \to 0.$$ 

The cohomology module $\text{Ext}(N_2)$ is a freely generated module over $\mathbb{F}_3[y_1 + y_2]$ on generators $\bar{a}$, $\bar{y}_1a_1$, $\bar{y}_2a_2$, and $\bar{y}_1x_1a_1$ of degrees $0$, $1$, $2$, and $2$ respectively. The action of $\omega$ is determined by the action on $\text{Ext}(\mathbb{Z}_3)$.

Proof. By the previous result the $SD_{16}$-module $\mathbb{F}_3 \otimes \mathbb{Z}_3[[S_2^1]]$, $N_1$ is one dimensional over $\mathbb{F}_3$ generated by (the class of) $a$ and the action is given by the sign representation along $\varepsilon$. Lift $a$ to an element $y \in N_1$. Then $SD_{16}$ may not act correctly on $y$, but we can average $y$ to obtain an element $x$ on which $SD_{16}$ acts correctly and which reduces to the same element in $\mathbb{F}_3 \otimes \mathbb{Z}_3[[S_2^1]] N_1$; indeed,

$$x = \frac{1}{16} \sum_{\alpha \in SD_{16}} \varepsilon(\alpha)^{-1} \alpha_*(y).$$

This defines the morphism

$$\chi \downarrow_{SD_{16}} \to N_1.$$ 

Lemma 4.3 now implies that this map is surjective and we obtain the exact sequence we need. For the calculation of $\text{Ext}(N_2)$ note that we have an isomorphism of $S^1_2$-modules

$$\chi \downarrow_{SD_{16}} \cong \mathbb{Z}_3[[S_2^1]].$$

The result now follows from the previous lemma and the long exact sequence. 

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4.7 Lemma. There is a short exact sequence of $\mathbb{Z}_3[[G_2]]$-modules
\[
0 \to N_3 \to \chi \downarrow_{SD_{16}} \to N_2 \to 0
\]
where $\text{Ext}(N_3)$ is a free module over $F_3[y_1 + y_2]$ on generators $y_1a_1, y_2a_2, y_1x_1a_1,$ and $y_2x_2a_2$ of degree 0, 0, 1 and 1 respectively. In fact, the iterated boundary homomorphisms
\[
\text{Ext}^\ast(N_3) \to \text{Ext}^\ast + 3(Z_3) = H^\ast + 3(S_2^1, F_3)
\]
define an injection onto an $\text{Ext}(Z_3) \odot F_3[SD_{16}]$-submodule isomorphic to $\text{Ext}(Z_p[[G_2^1/G_2^2]])$.

Proof. The $SD_{16}$-module $F_3 \otimes_{Z_3} N_2$ is $F_3 \otimes_{Z_3} \chi$ generated by $\tilde{x}$. As in the proof the last lemma, we can now form a surjective map
\[
\chi \downarrow_{SD_{16}} \to N_2 ,
\]
and this map defines $N_2$. The calculation of $\text{Ext}(N_3)$ follows from the long exact sequence.

To make use of this last result we prove a lemma.

4.8 Lemma. Let $A = \text{Ext}(Z_3) \odot F_3[SD_{16}]$ and $M = \text{Ext}(Z_3[[G_2^1/G_2^2]])$, regarded as an $A$-module. Then $M$ is a simple $A$-module; in fact,
\[
\text{End}_A(M) \cong F_3 .
\]

Proof. Let $e_1$ and $e_2$ in
\[
\text{Ext}(Z_3[[G_2^1/G_2^2]]) \cong H^\ast(C_3, F_3) \times H^\ast(\omega C_3 \omega^{-1}, F_3)
\]
be the evident two generators in degree 0. If $f : M \to M$ is any $A$-module endomorphism, we may write
\[
f(e_1) = ae_1 + be_2
\]
where $a, b \in F_3$. Then (using the notation of Theorem 4.4) we have
\[
0 = f(y_2e_1) = by_2e_2 .
\]
Since $y_2e_2 \neq 0$, we have $b = 0$. Also, since $\omega(e_1) = e_2$, we have $f(e_2) = ae_2$. Finally, since every homogeneous element of $M$ is of the from $x_1y_1e_1 + x_2y_2e_2$, we have $f = \text{Id}_M$. \hfill \Box

This means that in order to prove the following result, we need only produce a map $f : N_3 \to Z_3[[G_2^1/G_2^2]]$ of $G_2$-modules which induces a non-zero map on Ext groups.

4.9 Theorem. There is an isomorphism of $G^1_2$-modules
\[
N_3 \cong Z_3[[G_2^1/G_2^2]] .
\]

This requires a certain amount of preliminaries, and some further lemmas. We are looking for a diagram (see diagram (4.11) below) which will build and detect the desired map.

The first ingredient of our calculation is a spectral sequence. Let us write
\[
0 \to C_3 \to C_2 \to C_1 \to C_0 \to Z_3 \to 0
\]

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for the resolution obtained by splicing together the short exact sequences of Lemma 4.5, 4.6, and 4.7:

$$0 \to N_3 \to \chi \to \chi \to Z_3[[G_2]]_3 \to Z_3 \to 0.$$ 

By extending the resolution $C \to Z_3$ to a bicomplex of projective $Z_3[[G_2]]$-modules, we get, for any $Z_3[[G_2]]$-module $M$ and any closed subgroup $H \subseteq G_2$, a first quadrant cohomology spectral sequence

$$E_1^{p,q} = \text{Ext}^p_{Z_3[H]}(C_q, M) \implies H^{p+q}(H, M).$$

In particular, because $E_1^{0,q} = 0$ for $q > 3$, there is an edge homomorphism

$$\text{Hom}_{Z_3[H]}(N_3, M) = \text{Hom}_{Z_3[H]}(C_3, M) \to H^3(H, M).$$

Dually, there are homology spectral sequences

$$E_1^{p,q} = \text{Tor}^p_{Z_3[H]}(M, C_q) \implies H_{p+q}(H, M)$$

with an edge homomorphism

$$H_3(H, M) \to M \otimes_{Z_3[H]} N_3.$$

That said, we remark that the important ingredient here is that $G_2$ contains a subgroup $K$ which is a Poincaré duality group of dimension three and which has good cohomological properties. The reader is referred to [22] for a modern discussion of a duality theory in the cohomology of profinite groups.

To define $K$, we use the filtration on the 3-Sylow subgroup $S_3^1 = F_1/\mathbb{Z}_3$ described in the first section. There is a projection

$$S_3^1 \to F_1/\mathbb{Z}_3 \cong C_3.$$ 

We follow this by the map $F_9 \to F_9/F_3 \cong C_3$ to define a group homomorphism $S_3^1 \to C_3$; then, we define $K \subseteq S_3^1$ to be the kernel. The chosen subgroup $C_3 \subseteq S_3^1$ of order 3 provides a splitting of $S_3^1 \to C_3$; hence $S_3^1$ can be written as a semi-direct product $K \ltimes C_3$. Note that every element of order three in $S_3^1$ maps to a non-zero element in $C_3$ so that $K$ is torsionfree.

From [10], we know a good deal about $K$, some of which is recorded in the following lemma. Let $j : K \to S_3^1$ denote the inclusion.

**4.10 Lemma.** The group $K$ is a 3-adic Poincaré duality group of dimension 3, and if $[K] \in H_3(K, \mathbb{Z}_3)$ is a choice of fundamental class, then

$$j_*[K] \in H_3(S_3^1, \mathbb{Z}_3)$$

is a non-zero $SD_{16}$-invariant generator of infinite order. Furthermore, under the edge homomorphism (4.10) the element $j_*[K]$ maps to a non-zero $SD_{16}$-invariant element in $F_3 \otimes_{Z_3}[S_3^1] \otimes_{\mathbb{Z}_3} N_3$.

**Proof.** The fact that $K$ is a Poincaré duality group is discussed in [10]; this discussion is an implementation of the theory of Lazard [12]. We must now address the statements about $j_*[K]$. For this, we first compute with cohomology, and we use the results and notation of Theorem 4.4.

It is known (see Proposition 4.3 and 4.4 of [10]), that the Lyndon-Serre-Hochschild spectral sequence

$$H^p(C_3, H^q(K, F_3)) \implies H^{p+q}(S_3^1, F_3)$$

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collapses and that $H^0(C_3, H^q(K, F_3))$ is one dimensional for $0 \leq q \leq 3$. Since the composites

$$H^1(C_3, F_3) \rightarrow H^1(S^1_2, F_3) \rightarrow H^1(\omega^1 C_3 w^{-i}, F_3)$$

of the inflation with the restriction maps are isomorphisms for $i = 1, 2$, the image of the generator of $H^1(C_3, F_3)$ is some linear combination $a x_1 + b x_2$ with both $a \neq 0$ and $b \neq 0$. This implies that $j^*(x, y_i) = 0$ for $i = 1, 2$; for example

$$a j^*(x_1, y_1) = j^*(y_1 (a x_1 + b x_2)) = 0.$$ 

But since $j^* : H^3(S^1_2, F_3) \rightarrow H^3(K, F_3)$ is onto and $H^3(S^1_2, F_3)$ is generated by $x_1 y_1$ and $a_i y_i$ for $i = 1, 2$ it is impossible that $j^*(a_i y_i)$ is trivial for both $i = 1, 2$. Because $K$ is a Poincaré duality group of dimension 3 we also know that the Bockstein $\beta : H^3(K, F_3) \rightarrow H^3(K, F_3)$ is zero; hence

$$j^*(a_1 y_1 - a_2 y_2) = j^*(\beta(x_1 a_1 - x_2 a_2)) = 0$$

and therefore

$$j^*(a_1 y_1) = j^*(a_2 y_2) \neq 0.$$ 

This shows that $H^3(S^1_2, F_3) \rightarrow H^3(K, F_3)$ is onto and factors through the $SD_{16}$-coinvariants, or dually $H_3(K, F_3) \rightarrow H_3(S^1_2, F_3)$ is an injection and lands in the $SD_{16}$-invariants. Furthermore, $H_3(K, F_3)$ even maps to the kernel of the Bockstein $\beta : H_3(S^1_2, F_3) \rightarrow H_3(S^1_2, F_3)$ and the induced map

$$H_3(K, F_3) \rightarrow \frac{\text{Ker} \beta : H_3(S^1_2, F_3) \rightarrow H_2(S^1_2, F_3)}{\text{Im} \beta : H_4(S^1_2, F_3) \rightarrow H_2(S^1_2, F_3)} \cong H_3(S^1_2; Z_3) \otimes_{Z_3} F_3$$

is an isomorphism, yielding that $j_*[K]$ is a generator of infinite order.

To prove the last statement we proceed as follows. Consider the spectral sequence of (4.9)

$$\text{Tor}_{Z_3[[K]]}^q(F_3, C_q) \Rightarrow H_{p+q}(K, F_3).$$

We know that $C_0 = Z_3[[G_2/G_{24}]]$ is a free $Z_3[[K]]$-module of rank 2 and $C_1 = C_2 = \chi^{|S^1_{16}|}$ are free $Z_3[[K]]$-modules of rank 3. Since the cohomological dimension of $K$ is 3 we see that $N_3$ is projective as a $Z_3[[K]]$-module and because the Euler characteristic of $K$ is zero, we obtain

$$\text{Tor}_{Z_3[[K]]}^q(F_3, N_3) \cong \begin{cases} F_3 \oplus F_3; & q = 0 \\ 0; & q > 0. \end{cases}$$

Thus we can use Lemma 4.3 to say that $N_3$ is a free $Z_3[[K]]$-module of rank 2. Since $F_3 \otimes_{Z_3[[K]]} N_3 \rightarrow F_3 \otimes_{Z_3[[S^1_{2}]]} N_3$ is a surjective morphism of vector spaces of the same dimension (see Lemma 4.7), it must be an isomorphism. Furthermore,

$$H_3(K, F_3) \rightarrow F_3 \otimes_{Z_3[[K]]} N_3$$

is an injection. The result now follows from the diagram

$$\begin{array}{ccc}
H_3(K, F_3) & \rightarrow & F_3 \otimes_{Z_3[[K]]} N_3 \\
\downarrow j_* & & \downarrow \psi \\
H_3(S^1_2, F_3) & \rightarrow & F_3 \otimes_{Z_3[[S^1_{2}]]} N_3
\end{array}$$

$\square$
We will use cap product with the elements $[K]$ and $j_*[K]$ to construct a commutative diagram for detecting maps $N_3 \to \mathbb{Z}_3[[G_2/G_{24}]]$. In the form we use the cap product, it has a particularly simple expression. Let $G$ be a profinite group and $M$ a continuous $\mathbb{Z}_p[[G]]$-module. If $a \in H^n(G, M)$ and $x \in H_n(G, \mathbb{Z}_p)$ we may define $a \cap x \in H_0(G, M)$ as follows: choose a projective resolution $Q_\bullet \to \mathbb{Z}_p$ and represent $a$ and $x$ by a cocycle $\alpha : Q_n \to M$ and a cycle $y \in \mathbb{Z}_p \otimes_{\mathbb{Z}_p[[G]]} Q_n$ respectively. Then $\alpha$ descends to a map

$$\overline{\alpha} : \mathbb{Z}_p \otimes_{\mathbb{Z}_p[[G]]} Q_n \to \mathbb{Z}_p \otimes_{\mathbb{Z}_p[[G]]} M$$

and $a \cap x$ is represented by $\overline{\alpha}(y)$. It is a simple matter to check that this is well-defined; in particular, if $y = \partial z$ is a boundary, then $\alpha(y) = 0$ because $\alpha$ is a cocycle. The usual naturality statements apply, which we record in a lemma. Note that part (2) is a special case of part (1) (with $K = G$).

4.11 Lemma. 1.) If $j : K \to G$ is a continuous homomorphism of profinite groups, and $x \in H_n(K, \mathbb{Z}_p)$ and $a \in H^n(G, M)$, then

$$j_*(j^*a \cap x) = a \cap j_*(x).$$

2.) Suppose $K \subseteq G$ is the inclusion of a normal subgroup and $M$ is a $G$-module. Then $G/K$ acts on $H^*(K, M)$ and $H_*(K, \mathbb{Z}_p)$ and for $g \in G/K$, $a \in H^n(K, M)$, and $x \in H^n(K, \mathbb{Z}_p)$

$$g_*(g^*a \cap x) = a \cap g_*(x).$$

Here is our main diagram. Let $i : K \to G_2^1$ be the inclusion.

\[
\begin{array}{ccc}
\text{Hom}_{\mathbb{Z}_3^2}[N_3, \mathbb{Z}_3[[G_2/\mathbb{Z}_3]]] & \xrightarrow{\text{edge}} & H^3(G_2^1, \mathbb{Z}_3[[G_2/G_{24}]]) \\
\text{Hom}_{S_4^5}(N_3, \mathbb{Z}_3[[G_2/\mathbb{Z}_3]]) & \xrightarrow{\cap_i[K]} & H_0(G_2^1, \mathbb{Z}_3[[G_2/G_{24}]]) \\
\text{Hom}_{S_4^5}(F_3, \mathbb{Z}_3[[G_2/\mathbb{Z}_3]]) & \xrightarrow{\text{ev}} & H_0(G_2^1, F_3[[G_2/G_{24}]])
\end{array}
\]

We now annotate this diagram. The maps labelled ev are defined by evaluating a homomorphism at the image of $j_*[K]$ under the edge homomorphism

$$H_3(S_2^1, \mathbb{Z}_3) \to \mathbb{Z}_3 \otimes_{\mathbb{Z}_3[[S_2]]} N_3$$

resp.

$$H_3(S_2^1, \mathbb{Z}_3) \to \mathbb{Z}_3 \otimes_{\mathbb{Z}_3[[S_2]]} N_3 \to F_3 \otimes_{\mathbb{Z}_3[[S_2]]} N_3$$

of (4.10). By Lemma 4.10, this is an $SD_{16}$-invariant element and hence we get an element in

$$(\mathbb{Z}_3 \otimes_{\mathbb{Z}_3[[S_2]]} \mathbb{Z}_3[[G_2/G_{24}]])^{SD_{16}}.$$  

We now take the image of that element under the projection map from the invariants to the coinvariants (which in our case is an isomorphism because the order of $SD_{16}$ is prime to 3)

$$(\mathbb{Z}_3 \otimes_{\mathbb{Z}_3[[S_2]]} \mathbb{Z}_3[[G_2/G_{24}]])^{SD_{16}} \xrightarrow{\text{cap}} H_0(G_2^1, \mathbb{Z}_3[[G_2/G_{24}]]) .$$

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Similar remarks apply to $F_3$-coefficients.

The diagram commutes, by the definition of cap product. Theorem 4.9 now follows from Lemma 4.3, Lemma 4.7, Lemma 4.8 and the final two Lemmas 4.12 and 4.13; in fact, once we have proved the lemmas below, diagram (4.11) will then show that we can choose a morphism of continuous $G_2$-modules

$$f : N_3 \longrightarrow Z_3[[G_2/G_{24}]]$$

so that

$$F_3 \otimes f : F_3 \otimes_{Z_3[[S_3]]} N_3 \longrightarrow F_3 \otimes_{Z_3[[S_3]]} Z_3[[G_2/G_{24}]]$$

is non-zero. Then Lemmas 4.7, 4.8 and 4.3 imply that $f$ is an isomorphism.

Let $i : K \to G_2$ denote the inclusion.

\textbf{4.12 Lemma.} The homomorphism

$$\cap i_* [K] : H^3(G_2^1, Z_3[[G_2/G_{24}]] \to H_0(G_2^1, Z_3[[G_2/G_{24}]]$$

is an isomorphism.

\textit{Proof.} Recall that we have denoted the inclusion $K \to S_2$ by $j$. We begin by demonstrating that

$$\cap j_* [K] : H^3(S_2^1, Z_3[[G_2/G_{24}]] \to H_0(S_2^1, Z_3[[G_2/G_{24}]]$$

is an isomorphism. Since $[K] \in H_3(K, Z_3)$ is invariant under the action of $C_3$, Lemma 4.11 supplies a commutative diagram

$$\begin{array}{ccc} H^3(S_2^1, Z_3[[G_2/G_{24}]] & \xrightarrow{j^*} & H^3(K, Z_3[[G_2/G_{24}]]^C_3 \\
\cap j_* [K] & & \cap i_* [K] \\
H_0(S_2^1, Z_3[[G_2/G_{24}]] & \xrightarrow{j_*} & H_0(K, Z_3[[G_2/G_{24}]]^C_3 \end{array}$$

The morphism $\cap [K]$ is an isomorphism by Poincaré duality. As $S_2$-modules, we have

$$Z_3[[G_2/G_{24}]] \cong Z_3[[S_2/G_{24}]] \oplus Z_3[[S_2/G_{24}]] - \omega$$

on generators $\epsilon G_{24}$ and $\omega G_{24}$ in $G_2^1/G_{24}$; hence, as $K$-modules

$$Z_3[[G_2/G_{24}]] \cong Z_3[[K]] \oplus Z_3[[K]]$$

and $C_3$ acts trivially on $H_0$. Thus $j_*$ is an isomorphism. In addition, since $H^q(K, Z_3[[G_2/G_{24}]] = 0$ if $q \neq 3$, the Lyndon-Serre-Hochschild spectral sequence shows that $j^*$ is an isomorphism.

To finish the proof, we continue in the same manner. Let $r : S_2^1 \to G_2^1$ be the inclusion, so that $i = rj : K \to G_2^1$. Lemma 4.10 and 4.11 supply a diagram

$$\begin{array}{ccc} H^3(G_2^1, Z_3[[G_2/G_{24}]] & \xrightarrow{r^*} & H^3(S_2^1, Z_3[[G_2/G_{24}]]^SD_3 \\
\cap i_* [K] & & \cap j_* [K] \\
H_0(G_2^1, Z_3[[G_2/G_{24}]] & \xrightarrow{r_*} & H_0(S_2^1, Z_3[[G_2/G_{24}]]^SD_3 \end{array}$$

and we have just shown that $\cap j_* [K]$ is an isomorphism. The map $r_*$ sends invariants to coinvariants and, since the order of $SD_3$ is prime to 3, is an isomorphism. Again, because the order of $SD_3$ is prime to 3 the spectral sequence of the extension $S_2^1 \to G_2^1 \to SD_3$ collapses at $E_2$ and therefore the map $r^*$ is an isomorphism. This completes the proof.
4.13 Lemma. The edge homomorphism
\[ \text{Hom}_{G_2}(N_3, Z_3[[G_2^1/G_2]]) \to H^3(G_2^1, Z_3[[G_2^1/G_2]]) \]
is surjective.

Proof. We examine the spectral sequence of (4.7):
\[ E_1^{p,q} \cong \text{Ext}^p_{Z_3[[G_2]]}(C_q, Z_3[[G_2^1/G_2]]) \implies H^{p+q}(G_2^1, Z_3[[G_2^1/G_2]]) \]
We need only show that
\[ \text{Ext}^p_{Z_3[[G_2]]}(C_q, Z_3[[G_2^1/G_2]]) = 0 \]
for \( p + q = 3 \) and \( q < 3 \). If \( q = 1 \) or 2, then \( C_q = \chi \mid_{SD_{16}} \). Now \( \chi \mid_{SD_{16}} \) is projective as a \( Z_3[SD_{16}] \)-module and therefore \( \text{Ext}^p_{Z_3[[G_2]]}(C_q, Z_3[[G_2^1/G_2]]) \) is trivial for \( p > 0 \).
If \( q = 0 \), then \( C_0 = Z_3[[G_2^1/G_2]] \) and
\[ \text{Ext}^3_{Z_3}(C_0, Z_3[[G_2^1/G_2]]) = H^3(G_2, Z_3[[G_2^1/G_2]]) \cong H^3(C_3, Z_3[[G_2^1/G_2]])^Q \cdot \]
As a \( C_3 \)-module, we have an isomorphism
\[ Z_3[[G_2^1/G_2]] \cong \prod_{C_2 \times G_2} Z_3[C_3/C_x] \]
where \( C_2 \times G_2 \) runs over the double cosets of \( G_2^1/G_2 \) of the left action of \( C_3 \) and \( C_x \) is the stabilizer subgroup of \( xG_2 \). Note that \( C_x \) is either trivial or all of \( C_3 \). Thus \( H^3(C_3, Z_3[[C_3/C_x]]) = 0 \) for either of the two cases and
\[ \text{Ext}^3_{Z_3}(C_0, Z_3[[G_2^1/G_2]]) \subseteq \prod_{C_2 \times G_2} H^3(C_3, Z_3[[C_3/C_x]]) = 0 \cdot \]

5 The tower

In this section we write down the five stage tower whose homotopy inverse limit is \( L_{K(2)}S^0 = E_2^{hG_2} \) and the four stage tower whose homotopy inverse limit is \( E_2^{hG_0} \). We will write \( E = E_2 \) and recall that we have fixed the prime 3.

To state our results, we will need a new spectrum. Let \( \chi \) be the representation of the subgroup \( SD_{16} \subseteq G_2 \) that appeared in (4.1) and let \( \epsilon \chi \) be an idempotent in the group ring \( Z_3[SD_{16}] \) that picks up \( \chi \). The action of \( SD_{16} \) on \( E \) gives us a spectrum \( E^\chi \) which is the telescope associated to this idempotent: \( E^\chi := \epsilon \chi E \).

Then we have an isomorphism of Morava modules
\[ E_\ast E^\chi \cong \text{Hom}_{Z_3[SD_{16}]}(\chi, E_\ast E_\ast) \cong \text{Hom}_{Z_3[SD_{16}]}(\chi, \text{Hom}^c(G_2, E_\ast)) \]
\[ \cong \text{Hom}_{Z_3[G_2]}(\chi \mid_{SD_{16}}, \text{Hom}^c(G_2, E_\ast)) \cong \text{Hom}^c(\chi \mid_{SD_{16}}, E_\ast) \cdot \]

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The Hom groups here are groups of continuous homomorphisms and $\text{Hom}^\sim(G_2, E_\ast)$ is a Morava module via the diagonal $G_2$-action, and a $\mathbb{Z}_2[SD_{16}]$-module via the translation action on $G_2$.

It is clear from (4.2) that $E^\chi$ is a direct summand of $E^hQ^4$ and a module spectrum over $E^hSD_{16}$. In fact, it is easy to check that this homotopy is free of rank 1 as a $\pi_\ast(E^hSD_{16})$-module on a generator $\omega^2u^4 \in \pi_5(E^hQ^4) \subset \pi_\ast(E)$: this generator determines a map of module spectra from $\Sigma^8E^hSD_{16}$ to $E^\chi$ which is an equivalence. From now on we will use this equivalence to replace $E^\chi$ by $\Sigma^8E^hSD_{16}$. We note that $E^\chi$ is periodic with period 16.

5.1 Lemma. There is an exact sequence of Morava modules

$$0 \to E_\ast \to E_\ast E^{hG_{24}} \to E_\ast \Sigma^8E^{hSD_{16}} \oplus E_\ast E^{hG_{24}} \to E_\ast \Sigma^8E^{hSD_{16}} \oplus E_\ast \Sigma^4E^{hSD_{16}} \to E_\ast \Sigma^4E^{hSD_{16}} \oplus E_\ast \Sigma^4E^{hG_{24}} \to E_\ast \Sigma^4E^{hG_{24}} \to 0.$$ 

Proof. Take the exact sequence of continuous $G(2)$-modules of Corollary 4.2 and apply the functor $\text{Hom}^\sim_{G_2}(\ast, E_\ast)$. Then use the isomorphism $E_\ast \Sigma^8E^{hSD_{16}} = \text{Hom}^\sim_{G_2}(\chi, E_\ast)$ above and the isomorphisms $E_\ast E^{hF} \cong \text{Hom}^\sim(G_2/F, E_\ast)$ supplied by (2.7) to get an exact sequence of Morava modules

$$0 \to E_\ast \to E_\ast E^{hG_{24}} \to E_\ast \Sigma^8E^{hSD_{16}} \oplus E_\ast E^{hG_{24}} \to E_\ast \Sigma^8E^{hSD_{16}} \oplus E_\ast \Sigma^8E^{hSD_{16}} \to E_\ast \Sigma^8E^{hSD_{16}} \oplus E_\ast E^{hG_{24}} \to E_\ast E^{hG_{24}} \to 0.$$ 

Finally, we use that $\Sigma^8E^{hSD_{16}} \cong \Sigma^4E^{hSD_{16}}$ because $E^{hSD_{16}}$ is periodic of period 16 and $E_\ast E^{hG_{24}} \cong E_\ast \Sigma^4E^{hG_{24}}$ as Morava modules (see Remark 3.12.2).

5.2 Remark. In the previous lemma, replacing $\Sigma^8E^{hSD_{16}}$ by $\Sigma^4E^{hSD_{16}}$ is merely aesthetic: it emphasizes some sort of duality. However, $E^{hG_{24}}$ and $\Sigma^4E^{hG_{24}}$ are different spectra, even though $E_\ast E^{hG_{24}} \cong E_\ast \Sigma^4E^{hG_{24}}$. This substitution is essential to the solution to the Toda bracket problem which arises in Theorem 5.5.

In the same way, one can immediately prove

5.3 Lemma. There is an exact sequence of Morava modules

$$0 \to E_\ast E^{hG_{24}} \to E_\ast E^{hG_{24}} \to E_\ast \Sigma^8E^{hSD_{16}} \to E_\ast \Sigma^4E^{hSD_{16}} \to E_\ast \Sigma^4E^{hG_{24}} \to 0.$$ 

5.4 Theorem. The exact sequence of Morava modules

$$0 \to E_\ast \to E_\ast E^{hG_{24}} \to E_\ast \Sigma^8E^{hSD_{16}} \oplus E_\ast E^{hG_{24}} \to E_\ast \Sigma^8E^{hSD_{16}} \oplus E_\ast \Sigma^4E^{hSD_{16}} \to E_\ast \Sigma^4E^{hSD_{16}} \oplus E_\ast \Sigma^4E^{hG_{24}} \to E_\ast \Sigma^4E^{hG_{24}} \to 0.$$ 

can be realized in the homotopy category of $K(2)$-local spectra by a sequence of maps

$L_{K(2)}S^0 \to E^{hG_{24}} \to \Sigma^8E^{hSD_{16}} \lor E^{hG_{24}} \to \Sigma^8E^{hSD_{16}} \lor \Sigma^4E^{hSD_{16}} \to \Sigma^4E^{hSD_{16}} \lor \Sigma^4E^{hG_{24}} \to \Sigma^4E^{hG_{24}}$ so that the composite of any two successive maps is null homotopic.
Proof. The map $L_{K(2)}S^0 \to E^{hG_{24}}$ is the unit map of the ring spectrum $E^{hG_{24}}$. To produce the other maps, we use the diagram of Proposition 2.6. It is enough to show that the $E_*$-Hurewicz homomorphism

$$\pi_0 F(X, Y) \to \text{Hom}_{E_*E}(E_* X, E_* Y)$$

is onto when $X$ and $Y$ belong to the set $\{\Sigma^8 E^{hSD_{16}}, E^{hG_{24}}\}$. (Notice that the other suspensions cancel out nicely, since $E^{hSD_{16}}$ is 16-periodic.) Since $\Sigma^8 E^{hSD_{16}}$ is a retract of $E^{hQ_8}$, it is sufficient to show that

$$\pi_0 F(E^{hK_1}, E^{hK_2}) \to \text{Hom}_{E_*E}(E_* E^{hK_1}, E_* E^{hK_2})$$

is onto for $K_1$ and $K_2$ in the set $\{Q_8, G_{24}\}$. Using Proposition 2.4, we see that it is enough to show that

$$\pi_0 E^{hK} \to (E_0)^K$$

is surjective whenever $K \subseteq K_2 \cap xK_1 x^{-1}$. But now we can apply Corollary 3.13.

To show the successive compositions are zero, we proceed similarly, again using Proposition 2.6, but now we have to show that various Hurewicz maps are injective. In this case, the suspensions do not cancel out, and we must show

$$\pi_0 F(E^{hK_1}, \Sigma^{48k} E^{hK_2}) \to \text{Hom}_{E_*E}(E_* E^{hK_1}, E_* \Sigma^{48k} E^{hK_2})$$

is injective for $k = 0$ and 1, at least for $K_1$ of the form $G_2$, $G_{24}$, or $Q_8$ and $K_2$ of the form $G_{24}$ or $Q_8$. Since all the spectra in involved are 72-periodic, it is sufficient to show

$$\pi_{24k} E^{hK} \to (E_{24k})^K$$

is injective for $k = 0$ and 1 and $K \subseteq K_2 \cap xK_1 x^{-1}$. The result follows from Corollary 3.13.1 for the case $k = 0$ and Corollary 3.13.2 in the case $k = 1$. Note that in the latter case, this map is not an isomorphism.

The following result will let us build the tower.

5.5 Theorem. In the sequence of spectra

$$L_{K(2)}S^0 \to E^{hG_{24}} \to \Sigma^8 E^{hSD_{16}} \vee E^{hG_{24}} \to \Sigma^8 E^{hSD_{16}} \vee \Sigma^{40} E^{hSD_{16}} \to \Sigma^{40} E^{hSD_{16}} \vee \Sigma^{48} E^{hG_{24}} \to \Sigma^{48} E^{hG_{24}}$$

all the possible Toda brackets are zero modulo their indeterminacy.

Proof. There are three possible three-fold Toda brackets, two possible four-fold Toda bracket and one possible five-fold Toda bracket. All but the last lie in zero groups.

Because $\Sigma^8 E^{hSD_{16}}$ is a summand of $E^{hQ_8}$, the three possible three-fold Toda brackets lie in

$$\pi_1 F(E^{hG_2}, E^{hQ_8} \vee \Sigma^{32} E^{hQ_8}), \quad \pi_1 F(E^{hG_{24}} \vee \Sigma^{32} E^{hQ_8} \vee \Sigma^{48} E^{hG_{24}}), \quad \pi_1 F(E^{hQ_8} \vee \Sigma^{32} E^{hQ_8} \vee \Sigma^{48} E^{hG_{24}})$$

which are all zero by Proposition 2.4 and Corollary 3.14. The most interesting calculation is the middle of these three and the most interesting part of that calculation is

$$\pi_1 F(E^{hG_{24}} \vee \Sigma^{48} E^{hG_{24}}) \cong \prod_{G_{24} \cap \Omega_2 / G_{24}} \pi_{23} E^{hF_x}.$$ 

This is zero by Corollary 3.14; however, notice that without the suspension by 48 this group is non-zero.
The two possible four-fold Toda brackets lie in
\[ \pi_2 F(E^{hG_{24}}, \Sigma^{32} E^{hG_{24}} \lor \Sigma^{48} E^{hG_{24}}) \quad \text{and} \quad \pi_2 F(E^{hG_{24}}, \Sigma^{48} E^{hG_{24}}) \]
We claim these are also the zero groups. All of the calculations here present some interest. For example, consider
\[ \pi_2 F(E^{hG_{24}}, \Sigma^{48} E^{hG_{24}}) \cong \prod_{G_{24} \setminus G_2/24} \pi_{26} E^{hK_x}. \]
where \( K_x = xG_{24}x^{-1} \cap G_{24} \). Since the element \(-1 = \omega^4 \in G_{24}\) is in the center of \( G_2 \), it is in \( K_x \) for every \( x \) and the result follows from Corollary 3.15.
Finally, the five-fold Toda bracket lies in
\[ \pi_3 F(E^{hG_{24}}, \Sigma^{48} E^{hG_{24}}) \cong \pi_{27} E^{hG_{24}} \cong \mathbb{Z}/3. \]
Thus, we do not have the zero group; however, we claim that the map \( E^{hG_{24}} \to E^{hG_{12}} \) at the beginning of our sequence supplies a surjective homomorphism
\[ \pi_2 F(E^{hG_{24}}, \Sigma^{48} E^{hG_{24}}) \to \pi_2 F(E^{hG_{24}}, \Sigma^{48} E^{hG_{24}}). \]
This implies that the indeterminacy of the five-fold Toda bracket is the whole group, completing the proof.
To prove this claim, note the \( E^{hG_{24}} \to E^{hG_{24}} \) is the inclusion of the homotopy fixed points by a larger subgroup into a smaller one. Thus Proposition 2.4 yields a diagram
\[
\begin{array}{c}
F(E^{hG_{24}}, E^{hG_{24}}) \xrightarrow{=} F(E^{hG_{24}}, E^{hG_{24}}) \\
\cong \quad \cong \\
E[[G_2/24]]^{hG_{24}} \xrightarrow{=} E[[G_2/24]]^{hG_{24}} \xrightarrow{=} E^{hG_{24}}
\end{array}
\]
and the contribution of the coset \( eG_{24} \) in \( E[[G_2/24]]^{hG_{24}} \) shows that the horizontal map is a split surjection of spectra. \( \square \)

The following result is now an immediate consequence of Theorems 5.4 and 5.5:

**5.6 Theorem.** There is a tower of fibrations in the \( K(n) \)-local category

\[
\begin{array}{cccccc}
L_K(2)S^0 & \xrightarrow{X_3} & X_2 & \xrightarrow{X_1} & E^{hG_{24}} \\
\Sigma^{44} E^{hG_{24}} & \lor & \Sigma^{45} E^{hG_{24}} & \lor & \Sigma^{37} E^{hS_{D_{16}}} & \lor & \Sigma^{36} E^{hS_{D_{16}}} & \lor & \Sigma^{38} E^{hS_{D_{16}}} & \lor & \Sigma^{39} E^{hS_{D_{16}}} & \lor & \Sigma^{-1} E^{hG_{24}}
\end{array}
\]

Using Lemma 5.3 and the very same program, we may produce the following result. The only difference will be that the Toda brackets will all lie in zero groups.

**5.7 Theorem.** There is a tower of fibrations in the \( K(n) \)-local category

\[
\begin{array}{cccccc}
E^{hG_{24}} & \xrightarrow{X_2} & X_1 & \xrightarrow{E^{hG_{24}}} \\
\Sigma^{45} E^{hG_{24}} & \lor & \Sigma^{38} E^{hS_{D_{16}}} & \lor & \Sigma^{37} E^{hS_{D_{16}}}
\end{array}
\]

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