Spectral Analysis
of the Zeta and $L$-Functions

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1. Aim of talk
We will show an instance of applications of the theory of automorphic representations to a genuinely traditional problem in the theory of the zeta and allied functions. We shall restrict ourselves to very basic issues and results, because of the purpose of the workshop.

2. The original problem
Bounding the zeta-function $\zeta(s)$, $s = \sigma + it$, especially on the critical line $\sigma = \frac{1}{2}$, is one of the central problems in analytic number theory, as it is indispensable in the theory of the distribution of prime numbers and more centrally in investigating the distribution of complex zeros of the zeta-function. This is the same as treating non-trivially the zeta-sum

$$\sum_{N < n \leq 2N} n^it,$$

where $N, t > 0$ are large and independent of each other. In other words, a massive cancellation among the waves $\{n^it\}$ is expected and to be detected.

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Talk at the workshop ‘$L$-functions of automorphic forms and related problems’ March 2012, Tokyo University.
3. Divide and conquer†
The first significant contribution was achieved by H. Weyl (1921). It was greatly improved by J.G. van der Corput (1921–37) using a tool from harmonic analysis (Poisson’s sum formula).

Simplifying the story, we chop up the zeta-sum into pieces, and apply Cauchy’s inequality, getting the expression

\[ \sum_{n \in \mathbb{N}} w(n) \left| \sum_{0 < m \leq M} (m + n)^i t \right|^2, \]

with an appropriate test function \( w \). We then expand the squares, take the sum over \( n \) inside, and appeal to Poisson’s sum formula in handling non-diagonal parts.‡

The resulting non-trivial bounds were exploited in the theory of the zero-free region and the zero-density concerning the zeta-function, and yielded eventually G. Hoheisel’s detection (1930) of prime numbers in short intervals, without any hypothesis like Riemann’s. It was the dawn of the modern theory of the zeta-function and the distribution of prime numbers.

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† Appellation by M.N. Huxley.
‡ Thus follows the famous subconvexity bound of van der Corput: \( \zeta(\frac{1}{2} + it) \ll t^{1/6} \log t, \ t \geq 2. \)
4. **The most fundamental problem in ANT**

This is to discuss the sum

\[ \sum_{n} f(n). \]

5. **Lifting to higher dimension**

This is essentially due to F.V. Atkinson (1949) and can be regarded as a generalisation of Weyl’s idea. Thus, we square the last sum and consider instead the two dimensional sum

\[ \sum_{m} \sum_{n} f(m, n), \]

where \( f \) has a new specification (the same convention applies to later steps). Then, we classify the summands according to the deviation from the symmetric axis \( m = n \). The sum

\[ \sum_{m} \sum_{n} f(n, n + m) \]

emerges.

6. **A surprise**

Then, we apply Poisson’s sum formula to the last inner sum, assuming \( f \) is appropriate to do so. Often a better result comes out. Well, it is better than what can be achieved by applying Poisson’s sum formula to the original double sum. This mysterious effect of *divide and conquer* was observed by van der Corput in an explicit way.
But, why is such a mystery possible? We are not claiming any full excavation of this under-sea *hoard* but only try to reveal a group structure and indicate that an ocean of orthogonal waves embraces the mystery.

### 7. Seeking axis of symmetry

It is reasonable to surmise that the axis $m = n$ in the original double sum is *not* in general the central line of symmetry. Also, we note that the application of Poisson’s sum formula is the same as looking for hidden orthogonality. As much as, but not too much, orthogonality are to be exploited. The situation reminds us of the circle method of Hardy–Littlewood–Ramanujan: To seek an *optimal* dissection.

### 8. Further lifting

Being in the NT community, we try the line $km = ln$ with an arbitrary pair of non-zero integers $k, l$; that is, turning the line $m = n$ to $m = (l/k)n$ in order to achieve the best focusing. The Farey sequence is visible here, in an analogous context as in the circle method. Since we do adjusting inside a view finder, weights are to be attached to each pair $k, l$ in order to see a particular set of points $\{l/k\}$ better than the rest so that the control is kept in our hands.
We are thus lead to a division of general quadruple sums:

\[
\sum_{k,l,m,n} f(k, l, m, n) = \left\{ \sum_{km=ln} + \sum_{km>ln} + \sum_{km<ln} \right\} f(k, l, m, n)
\]

9. Rendering with matrices
It occurred to us, well more than 2 decades ago†, that the last identity could be better expressed in terms of matrices:

With \( M = \begin{pmatrix} k & l \\ n & m \end{pmatrix} \in M_2(\mathbb{Z}) \),

\[
\sum_{k,l,m,n} f(k, l, m, n) = \sum_M f(M) = \left\{ \sum_{\text{det} M=0} + \sum_{\text{det} M>0} + \sum_{\text{det} M<0} \right\} f(k, l, m, n).
\]

10. Hecke and \( \text{SL}_2 \)
We call the first sum on the right the \textit{Ramanujan term} with a good reason. Perhaps, \textit{Rankin–Selberg} is to be attached as well. The second and the third sums can be associated with E. Hecke; perhaps, with \textit{Hurwitz–Mordell} also.

† Proc. Amalfi Conference 1989, ed. E. Bombieri et al, Salerno Univ., 1992, pp. 325–344.
Thus we have

$$\sum_{\det M > 0} f(M) = \sum_{n > 0} (T(n)F)(1),$$

$$F(g) = \sum_{M \in \text{SL}(2, \mathbb{Z})} f(Mg), \quad g \in \text{SL}(2, \mathbb{R}),$$

where \(\{T(n)\}\) are Hecke operators associated with the full modular group.

11. Spectral decomposition
Assuming \(f\) be sufficiently smooth and of rapid decay, we may apply, to the Poincaré series \(F(g)\), the spectral decomposition of the Hilbert space \(L^2(\Gamma \backslash G)\) or rather that of automorphic representations occurring there. Here \(G = \text{SL}(2, \mathbb{R})\) and \(\Gamma = \text{SL}(2, \mathbb{Z})\). But a more convenient choice is

\(G = \text{PSL}(2, \mathbb{R}), \quad \Gamma = \text{PSL}(2, \mathbb{Z}),\)

as we may assume that \(f\) is even. We shall keep this convention hereafter.

12. Upshot, in fact an intermezzo
Applying the above consideration to the integral transform

$$Z_2(\zeta^2, g) = \int_{-\infty}^{\infty} |\zeta(\frac{1}{2} + it)|^4 g(t)dt,$$
of a nice weight \( g \), we are able to achieve a complete and explicit spectral decomposition in terms of the spectral resolution of the Casimir operator acting over \( L^2(\Gamma \backslash G) \):

\[
\mathcal{Z}_2(\zeta^2, g) = \mathcal{M}(g) + \sum_V |\varrho_V(1)|^2 H_V^3(\frac{1}{2}) \Theta(\nu_V, g)
\]

\[
+ \frac{1}{4\pi i} \int_{(0)} \frac{|\zeta(\frac{1}{2} + \nu)|^6}{|\zeta(1 + \nu)|^2} \Theta(\nu, g) d\nu.
\]

There will be no need to explain the notation in detail. We shall be a little more precise later.

This was achieved in a series of our works (1989–1997)\(^\dagger\). Initially we used the spectral theory of sums of Kloosterman sums due to R.W. Bruggeman (1978) and N.V. Kuznetsoy (1977–81). However, later their results were dispensed with in our joint work\(^\ddagger\) with Bruggeman.

The formula gave rise to a variety of new facts on the zeta-function which had been unattainable before. Main applications were done in our joint works with A. Ivić concerning the plain fourth power mean

\[
\int_{-T}^T |\zeta(\frac{1}{2} + it)|^4 dt,
\]

to which a part of Ivić’s talk at this workshop is devoted.

\(^\dagger\) Cambridge Tracts in Math., 127.

\(^\ddagger\) Crelle 579 (2005), 75–114
13. Real issues

The spectral decomposition for $Z_2(\zeta^2, g)$ is only the top tip of a great iceberg. It is beautiful but is just the very beginning of a story to be unfolded in the future.

There are at least two obvious directions in which we should go deeper:

$$Z_k(\zeta^2, g) = \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + it\right)\right|^{2k} g(t) \, dt,$$

and

$$Z_k(L_V, g) = \int_{-\infty}^{\infty} \left| L_V\left(\frac{1}{2} + it\right)\right|^k g(t) \, dt,$$

where, as is to be made precise later, $L_V(s)$ is the $L$-

function associated with an irreducible cuspidal automorphic representation occurring in $L^2(\Gamma\backslash G)$.

We shall discuss only $Z_2(L_V, g)$, since both for any integral $k \geq 3$ belong presently to a terra incognita; thus here is a challenging problem, whose resolution will yield fundamental changes throughout ANT. On the other hand, $Z_2(L_V, g)$ is a natural extension of $Z_2(\zeta^2, g)$. ‘Natural’, because $\zeta^2$ corresponds to the Eisenstein series associated with $\Gamma$, which is a kind of automorphic form. However, it is by no means a ready-made extension, since the above scheme developed for $Z_2(\zeta^2, g)$ does not yield any incision, especially when dealing with $V$ in the unitary principal series, i.e., the irreducible subspace generated by repeated applications of Maass derivatives to a particular real-analytic cusp-form on the hyperbolic upper half-plane.
14. An arithmetic–analytic issue

In the Weyl–van der Corput setting, the function $f$ needs to be ‘smooth’ actually. But in general $f$ is an arithmetic function, and what is essential is to have a sufficiently detailed asymptotical result on the shifted convolution

$$\sum_{n \in \mathbb{N}} f_1(n)f_2(n + m)W(n/m), \quad m > 0,$$

with a test function $W$. Thus:

*The Poisson sum formula is to be replaced by something more arithmetic.*

When both $f_j$ are divisor functions, this is called an additive divisor problem/sum

$$\sum_{n \in \mathbb{N}} d(n)d(n + m)W(n/m),$$

to which corresponds the mean value $\mathcal{Z}_2(\zeta^2, g)$. This sum looks undisguised, but it is in fact a deep problem which has an essential relation with the Ramanujan conjecture on the size of Hecke eigenvalues. We need to appeal either to the spectral theory of sums of Kloosterman sums or to a careful construction of Poincaré series. Here the key-point is that the divisor function has an inner-structure $\sum_{a \mid n} 1$
which can be effectively and readily exploited.†

Then, what will happen, if \( f_j(n) \) are Fourier coefficients of cusp forms? There does not seem to exist any corresponding inner-structure. This problem was posed by A. Selberg (1965), and was only recently resolved by ourselves‡ appealing to the Kirillov model of automorphic representations occurring in \( L^2(\Gamma \backslash G) \). We are going to indicate salient points of our idea.

14. Normalisation

We have the Iwasawa co-ordinate system

\[
G = NAK \ni g = n[x]a[y]k[\theta],
\]

with \( n[x] = \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} \), \( a[y] = \begin{bmatrix} \sqrt{y} \\ 1/\sqrt{y} \end{bmatrix} \), \( k[\theta] = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \).

The Casimir operator is

\[
\Omega = -y^2(\partial_x^2 + \partial_y^2) + y\partial_x\partial_\theta.
\]

The (orthonormal) spectral decomposition is rendered as

\[
L^2(\Gamma \backslash G) = \mathbb{C} \cdot 1 \oplus \mathbb{C}L^2(\Gamma \backslash G) \oplus eL^2(\Gamma \backslash G)
\]

\[
0L^2(\Gamma \backslash G) = \oplus V, \quad V = \bigoplus \mathbb{C}\lambda_V^{(\ell)}.
\]

† Our result on this is one of the essential implements to prove a uniform subconvexity bound for \( L_V(s) \), which is a wide extension of van der Corput’s exponent \( \frac{1}{6} \) for \( \zeta \). See M. Jutila and Y.M: Acta Math., 195 (2005), 61–115.

‡ Proc. Japan Acad., 80A (2004), 28–33.
The subspace $eL^2(\Gamma \backslash G)$ is generated by integrals of Eisenstein series, the details of which is omitted. The set $\{\lambda^{(\ell)}_V\}$ is a complete orthonormal system of the cuspidal subspace $^0L^2(\Gamma \backslash G)$, whose irreducible subspaces are $\{V\}$; and

$$\Omega\lambda^{(\ell)}_V = \left(\frac{1}{4} + \nu^2_V\right)\lambda^{(\ell)}_V, \quad \partial_\theta \lambda^{(\ell)}_V = 2i\ell \lambda^{(\ell)}_V.$$ 

Also we have the Fourier expansion

$$\lambda^{(\ell)}_V(g) = \left|\pi^{-2\nu_V} \frac{\Gamma(|\ell| + \nu_V + \frac{1}{2})}{\Gamma(|\ell| - \nu_V + \frac{1}{2})}\right|^{1/2} \times \sum_{n \neq 0} \frac{\varrho_V(n)}{\sqrt{|n|}} A_{\text{sgn}(n)}^{\ell}(a[|n|]g, \nu_V).$$

Here $\phi_\ell(g, \nu) = y^{1/2+\nu} \exp(2i\ell \theta)$, and the Jacquet transform

$$A^\delta \phi(g) = \int_{-\infty}^\infty e(-\delta \xi) \phi(wn[\xi]g) d\xi,$$

$$e(\xi) = \exp(2\pi i \xi), \quad \delta = \pm, \quad w = k\left[\frac{1}{2} \pi\right].$$

This normalisation allows us to regard the Fourier coefficients of cusp forms as being inherent to each $V$ but not the respective forms.†

With this, we define

$$L_V(s) = \sum_{n=1}^\infty \varrho_V(n)n^{-s}, \quad \text{Re } s > 1.$$ 

† See our exposition: arXiv:1112.4226v1[math.NT]
The integral $Z_2(L_V, g)$ is expressible in terms of a generalised additive divisor problem/sum or the shifted convolution sum

$$\sum_{n=1}^{\infty} \varphi_V(n) \varphi_V(n+m) W(n/m).$$

The main problem is how to generate this sum by means of the automorphic forms $\{\lambda^{(\ell)}_V\}$.

15. Discrete series
If $V$ belongs to the discrete series, then $\nu_V = k - \frac{1}{2}$ with a $k \in \mathbb{N}$, and $A^\delta \phi_\ell(g, \nu_V)$ is essentially the exponential function. Since the exponential function is a kind of additive character, and we are done: In fact we may use the trivial but very basic identity

$$\int_0^1 \exp(2\pi i n x) \exp(2\pi i (n + m) x) dx = \delta_{m,0}.$$

With this we may pick up $\{\varphi_V(n) \varphi_V(n+m)\}$ by considering the Fourier expansion for $|\lambda^{(0)}_V|^2$.

16. Unitary principal series
If $V$ belongs to the unitary principal series, then $\nu_V \in i\mathbb{R}$, and $A^\delta \phi_\ell(g, \nu_V)$ is essentially a Whittaker function which is a generalisation of the $K$-Bessel function (the Kelvin function) and does not admit any property like an additive character.
Then, is there any element in $V$ whose generic Fourier coefficient is a product of $\varrho_V(n)$ and a function that admits an additive property similar to the exponential function?

That is, we ask naively: *Is there an element in $V$ whose outward appearance comes very close to the discrete series?*

17. Kirillov map/model

Our idea is to use the Kirillov model to find such an automorphic form inside $V$. By the Kirillov map we mean the correspondence†:

$$
\mathcal{K} : \lambda_V^{(\ell)} \mapsto A^{\text{sgn}(u)} \phi_\ell(a[|u|], \nu_V), \quad u \in \mathbb{R}^\times.
$$

This extends linearly to the whole of $V$ and becomes a unitary and surjective map between $V$ and $L^2(\mathbb{R}^\times, d^\times/\pi)$, with $d^\times u = du/|u|$. The right action $r$ of $G$ over the space $V$ is realised faithfully in $L^2(\mathbb{R}^\times, d^\times/\pi)$:

$$
r_h \lambda_V^{(\ell)} \mapsto A^{\text{sgn}(u)} (r_h \phi_\ell)(a[|u|], \nu_V), \quad h \in G.
$$

18. Inverse map

With this, we pick up a function $\omega \in L^2(\mathbb{R}^\times, d^\times/\pi)$ which admits the necessary additive property, and exploit $\mathcal{K}^{-1}\omega$. For instance, we put

$$
\omega(u) = \begin{cases} 
  u^{\alpha+1/2} \exp(-2\pi u) & u > 0, \\
  0 & u \leq 0,
\end{cases}
$$

† loc. cit.
with a sufficiently large $\alpha$ to gain a rapid decay on the boundary. Then the function $(\mathcal{K}^{-1}\omega)(g)$ is in the space $V$, and we have the expansion

$$(\mathcal{K}^{-1}\omega)(n[x]a[y]) = y^{\alpha+1/2} \sum_{n>0} \varrho_V(n)n^{\alpha} \exp(2\pi in(x + iy)).$$

This is similar to holomorphic cusp forms (i.e., elements in the discrete series). Then we consider the function

$$|(\mathcal{K}^{-1}\omega)(g)|^2 \in L^2(\Gamma\backslash G),$$

whose Fourier expansion involves $\{\varrho_V(n)\varrho_V(n + m)\}$ in a nice way. The spectral decomposition of $|(\mathcal{K}^{-1}h)(g)|^2$ can be done explicitly by using again the Kirillov map. We are essentially done. As a matter of fact, we need to be a little more careful in choosing the seed function $\omega$, but this point appears to be immaterial in our present discussion.

We have thus established the complete spectral expansion†

$$\mathcal{Z}_2(L_V, g) = \mathcal{M}(g; V) + \text{Re} \left\{ \sum_U \varrho_U(1)H_U \left(\frac{1}{2}\right) \Theta(\nu_U, g; V) \right. \right.$$\n
$$+ \left. \frac{1}{4\pi i} \int_{(0)} \frac{|\zeta(\frac{1}{2} + \nu)|^2}{|\zeta(1 + 2\nu)|^2} L_V \otimes V \left(\frac{1}{2} + \nu\right) \Theta(\nu, g; V) d\nu \right\}.$$

† Proc. Japan Acad., 83A (2007), 73–78.
This is an exact extension of our spectral expansion for the fourth moment of the Riemann zeta-function. Here $V$ is a particular irreducible representation while $U$ runs over all cuspidal irreducible representations. It should be stressed again that this result is beyond the reach of the spectral theory of sums of Kloosterman sums. It belongs genuinely to the theory of automorphic representations, that is, it appears to us that without representation theory of Lie groups the result would be very hard to achieve, if not impossible.

As an application, we are currently developing a detailed quantitative analysis of the plain mean value

$$\int_{-T}^{T} |L_V(\frac{1}{2} + it)|^2 dt.$$  

We have already obtained assertions analogous to those on the fourth moment of the zeta-function. Our effort is now focused to their uniformisation with respect to $V$.

Our way of using the Kirillov model has been exploited recently by V. Blomer and G. Harcos† and by others in investigating the problem of uniformly bounding various shifted convolution sums. Also it is worth noting that our idea can be extended to higher dimensional situation in a straightforward way, though the analysis becomes extremely complicated.

† arXiv: 0703246v1[math.NT]