Existence and stability of standing waves for nonlinear Schrödinger systems involving the fractional Laplacian

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Abstract

In the present paper we consider the coupled system of nonlinear Schrödinger equations with the fractional Laplacian

\[
\begin{aligned}
(-\Delta)^\alpha u_1 &= \lambda_1 u_1 + f_1(u_1) + \partial_1 F(u_1, u_2) \quad \text{in } \mathbb{R}^N, \\
(-\Delta)^\alpha u_2 &= \lambda_2 u_2 + f_2(u_2) + \partial_2 F(u_1, u_2) \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]

where \(u_1, u_2 : \mathbb{R}^N \rightarrow \mathbb{C}, N \geq 2, \) and \(0 < \alpha < 1.\) By studying an appropriate family of constrained minimization problems, we obtain the existence of solutions in \(H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)\) satisfying

\[
\int_{\mathbb{R}^N} |u_1|^2 \, dx = \sigma_1 \quad \text{and} \quad \int_{\mathbb{R}^N} |u_2|^2 \, dx = \sigma_2
\]

for given \(\sigma_j > 0.\) The numbers \(\lambda_1\) and \(\lambda_2\) in the system appear as Lagrange multiplier. The method is based on the concentration compactness arguments, but introduces a new way to verify some of the properties of the variational problem that are required in order for the concentration compactness method to work. We consider the case when \(f_j(s) = \mu_j |s|^{p_j-2} s\) and \(F(s, t) = \beta |s|^{r_1} |t|^{r_2}\) with \(\mu_j > 0, \beta > 0,\) and the values \(r_i > 1, 2 < p_j, r_1 + r_2 < 2 + \frac{4 \alpha}{N}.\) The method also enables us to prove the stability result of standing wave solutions associated with the set of global minimizers.

1 Introduction

The present study is concerned with the coupled nonlinear fractional Schrödinger system of the form

\[
\begin{aligned}
i \partial_t \Psi_1 + (-\Delta)^\alpha \Psi_1 &= f_1(\Psi_1) + \partial_1 F(\Psi_1, \Psi_2), \quad x \in \mathbb{R}^N, \quad t > 0, \\
i \partial_t \Psi_2 + (-\Delta)^\alpha \Psi_2 &= f_2(\Psi_2) + \partial_2 F(\Psi_1, \Psi_2), \quad x \in \mathbb{R}^N, \quad t > 0,
\end{aligned}
\]
where $\Psi_1, \Psi_2 : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}$, $N \geq 2, \alpha \in (0,1)$ is the fractional parameter, the function $f_j \in C(\mathbb{R}^N \times \mathbb{C})$ satisfies $f_j(x, ze^{i\theta}) = e^{i\theta} f_j(z)$, and $F \in C(\mathbb{R}^N \times \mathbb{C}^2)$ satisfies
\[ \partial_j F(x, z_1 e^{i\theta_1}, z_2 e^{i\theta_2}) = e^{i\theta_j} \partial_j F(z_1, z_2), \quad j = 1, 2. \]
For any parameter $\alpha \in (0,1)$, the fractional Laplacian $(-\Delta)^\alpha$ is defined via Fourier transform as
\[ (-\Delta)^\alpha u(\xi) = |\xi|^{2\alpha} \hat{u}(\xi), \quad u \in S(\mathbb{R}^N), \]
where $S(\mathbb{R}^N)$ denotes the Schwartz space of rapidly decaying $C^\infty(\mathbb{R}^N)$ functions.

Nonlinear fractional Schrödinger equation was introduced by N. Laskin in a series of papers \cite{8, 9, 10} by generalizing the Feynman path integral over Brownian-like paths to Lévy-like quantum paths. In other words, if the Feynman path integral over Brownian trajectories allows one to reproduce the well known NLS equation, then the path integral over Lévy trajectories leads one to a space-fractional Schrödinger equation (see \cite{10}). The models involving the fractional Laplacian arise in the description of a wide variety of phenomena in the applied sciences such as finance, plasma physics, obstacle problems, semipermeable membrane, anomalous diffusion, to name a few. For instance, the reader may consult \cite{16} for a rigorous derivation of fractional NLS type equations starting from a family of models for charge transport in biopolymers like the deoxyribonucleic acid (DNA) and \cite{17, 18} for the derivation of many fractional differential equations asymptotically from Lévy random walk models. Research in the fractional Schrödinger equations have recently begun to receive more attention and the literature for this research area is still expanding and rather young in the mathematics realm.

In this paper we study existence and stability of standing wave solutions of the system (1.1). A standing wave for the system (1.1) is a solution of the form $\Psi_j(t,x) = e^{i\lambda_j t} u_j(x)$ for some numbers $\lambda_1, \lambda_2 \in \mathbb{R}$. Plugging the standing wave ansatz into (1.1), one sees that the functions $u_1, u_2 : \mathbb{R}^N \to \mathbb{C}$ satisfy the following coupled system of time independent equations
\[
\begin{cases}
(-\Delta)^\alpha u_1 = \lambda_1 u_1 + f_1(u_1) + \partial_1 F(u_1, u_2) \quad \text{in } \mathbb{R}^N, \\
(-\Delta)^\alpha u_2 = \lambda_2 u_2 + f_2(u_2) + \partial_2 F(u_1, u_2) \quad \text{in } \mathbb{R}^N.
\end{cases}
\]
The study of standing wave solutions is of particular interest in physics. The question of existence of solutions has been well studied in the literature for the standard nonlinear Schrödinger type equations and their coupled versions. The approach to stability theory taken here will be the variational approach. The orbital stability of standing waves for the standard nonlinear Schrödinger equations was proved by Cazenave and Lions \cite{5} by using the concentration compactness principle \cite{11, 12} to characterize these special waves as minimizers of certain variational problems. There
are also many results concerning the stability of standing waves for coupled systems of nonlinear Schrödinger equations (for example, see \cite{2, 4, 6, 13}). Concerning the fractional Schrödinger equations, Guo and Huang \cite{7} have recently proved the stability of standing waves by using Cazenave and Lions’ concentration compactness arguments. In the present paper we prove results for coupled fractional Schrödinger systems which are in the same spirit of the results of the above cited papers. To our knowledge, none has addressed the questions of existence and stability of solutions with prescribed $L^2$-norms for coupled systems of fractional Schrödinger equations.

In the study of nonlinear systems such as (1.3), it is important to choose proper function space. In this work, we consider (1.3) in the energy space $H^s$ with the norm $\| \cdot \|_{H^s}$, with prescribed knowledge, none has addressed the questions of existence and stability of solutions of nonlinear Schrödinger equations (for example, see \cite{2, 4, 6, 13}). Concerning the fractional order Sobolev space $H^s$ denote by $\| \cdot \|_{H^s}$ the norm in the space $L^p(\mathbb{R}^N)$. The function spaces appearing in this paper will, unless otherwise stated, all have complex-valued functions.

Following closely the strategies of \cite{1, 2, 3, 4}, we first address the existence question of standing wave profiles $(u_1, u_2)$ in $H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} |u_1|^2 \, dx = \tau_1 \text{ and } \int_{\mathbb{R}^N} |u_2|^2 \, dx = \tau_2$$

(1.4)

for given $\tau_1 > 0$ and $\tau_2 > 0$. In literature these special solutions are also called $L^2$ normalized (or simply normalized) solutions. To avoid technicalities, we only consider the case when $f_1(s) = \mu_1 |s|^{p_1-2}s$, $f_2(s) = \mu_2 |s|^{p_2-2}s$, and $F(s,t) = \beta |s|^{r_1}|t|^{r_2}$ and assume throughout that the following assumptions hold:

$$N \geq 2, \ 0 < \alpha < 1, \mu_j > 0, \beta > 0, r_i > 1, \text{ and } 2 < p_1, p_2, r_1 + r_2 < 2 + \frac{4\alpha}{N}. \quad (1.5)$$

A standard way often used to describe standing wave solutions is as critical points of constrained variational problems, in which the functional being minimized and the constraint functionals are conserved quantities. For purposes of investigating the stability of standing waves, however, one needs to characterize them not just as critical points, but as absolute minimizers of constrained variational problems. In this paper, we consider, for any $\tau = (\tau_1, \tau_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, the problem of finding minimizers of the energy

$$E(u_1, u_2) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |(-\Delta)^{\alpha/2} u_1|^2 + |(-\Delta)^{\alpha/2} u_2|^2 \right) \, dx$$

$$- \int_{\mathbb{R}^N} \left( \frac{\mu_1}{p_1} |u_1|^{p_1} + \frac{\mu_2}{p_2} |u_2|^{p_2} + F(u_1, u_2) \right) \, dx$$

(1.6)
over the set $\mathcal{S}_r = S_{\tau_1} \times S_{\tau_2}$, where $S_r = \{ u \in H^\alpha(\mathbb{R}^N) : \| u \|_{L^2(\mathbb{R}^N)} = \sqrt{r} \}$ for any $r > 0$. If there exists a minimizer $(u_1, u_2)$ for the problem $(E, \mathcal{S}_r)$, then it is a solution of (1.3) and the function $(\Psi_1, \Psi_2)$ defined by $\Psi_j(t, x) = e^{i\lambda_j t}u_j(x)$ is a standing wave solution of (1.1). The numbers $\lambda_1$ and $\lambda_2$ are the Lagrange multipliers associated to the stationary point $(u_1, u_2)$ on $\mathcal{S}_r$. The stability of the set of minimizers follows by a standard principle since both the energy $E(u_1, u_2)$ and the constraint functionals $\int_{\mathbb{R}^N} |u|^2\, dx$ are conserved along the flow of (1.1). To prove the precompactness of an energy-minimizing sequence via the method of concentration compactness, we establish certain strict inequalities of the function involving the infimum of the problem $(E, \mathcal{S}_r)$ as a function of two parameters $\tau_1$ and $\tau_2$.

Our first main theorem addresses the existence of minimizers of $(E, \mathcal{S}_r)$.

**Theorem 1.1.** Suppose that the assumptions (1.5) hold. Then

(i) for any given $\tau \in \mathbb{R}^+ \times \mathbb{R}^+$, there exists a nonempty subset $V(\tau)$ of $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ consisting of minimizers of the problem $(E, \mathcal{S}_r)$.

(ii) any minimizer $(u_1, u_2) \in V(\tau)$ satisfies (1.3), for some $\lambda \in \mathbb{R}^+ \times \mathbb{R}^+$, and the function $(\Psi_1, \Psi_2)$ defined by $\Psi_j(t, x) = e^{i\lambda_j t}u_j(x)$ is a standing wave solution of (1.1) satisfying

$$\| \Psi_1 \|_{L^2(\mathbb{R}^N)} = \sqrt{\tau_1} \quad \text{and} \quad \| \Psi_2 \|_{L^2(\mathbb{R}^N)} = \sqrt{\tau_2}.$$ 

(iii) if $\{ (u_1^n, u_2^n) \}_{n \geq 1}$ is an energy-minimizing sequence for the problem $(E, \mathcal{S}_r)$, then there exists a sequence of points $\{ y_k \} \subset \mathbb{R}^N$ and a subsequence $\{ (u_1^{n_k}, u_2^{n_k}) \}_{k \geq 1}$ such that $(u_1^{n_k}(\cdot + y_k), u_2^{n_k}(\cdot + y_k)) \to (u_1, u_2)$ strongly in $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$, as $k \to \infty$, where $(u_1, u_2)$ is some minimizer for $(E, \mathcal{S}_r)$. Moreover, $(u_1^n, u_2^n) \to V(\tau)$ in the following sense

$$\lim_{n \to \infty} \inf_{(u_1, u_2) \in V(\tau)} \| (u_1^n, u_2^n) - (u_1, u_2) \|_\alpha = 0.$$ 

(iv) the family of sets $V(\tau)$ is mutually disjoint in the following sense

$$\forall \tau, \sigma \in \mathbb{R}^+ \times \mathbb{R}^+, \tau \neq \sigma \Rightarrow V(\tau) \cap V(\sigma) = \emptyset.$$ 

The following is our orbital stability result of solutions, which is a direct consequence of the result of relative compactness.

**Theorem 1.2.** For any $\tau \in \mathbb{R}^+ \times \mathbb{R}^+$, the set of standing wave profiles $V(\tau)$ is stable, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that if the initial condition $\Psi_0 = (\Psi_1, 0, \Psi_2, 0) \in H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ satisfies

$$\inf_{u \in V(\tau)} \| \Psi_0 - u \|_\alpha < \delta,$$ 

then there exists a sequence of points $\{ y_k \} \subset \mathbb{R}^N$ and a subsequence $\{ (\Psi_1^{n_k}, \Psi_2^{n_k}) \}_{k \geq 1}$ such that $(\Psi_1^{n_k}(\cdot + y_k), \Psi_2^{n_k}(\cdot + y_k)) \to (\Psi_1, \Psi_2)$ strongly in $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$, as $k \to \infty$, where $(\Psi_1, \Psi_2)$ is some minimizer for $(E, \mathcal{S}_r)$. Moreover, $(\Psi_1^{n_k}, \Psi_2^{n_k}) \to V(\tau)$ in the following sense

$$\lim_{k \to \infty} \inf_{(u_1, u_2) \in V(\tau)} \| (\Psi_1^{n_k}, \Psi_2^{n_k}) - (u_1, u_2) \|_\alpha = 0.$$
then any solution $\Psi(t) = (\Psi_1(t), \Psi_2(t))$ of the system (1.1) emanating from $\Psi_0$ satisfies

$$\sup_{t \geq 0} \inf_{u \in V(t)} \|\Psi(t) - u\|_\alpha < \varepsilon.$$  

Theorem 1.2 must however be understood in a qualified sense because we lack a suitable global well-posedness theory for the initial value problem.

The paper is organized as follows. In Section 2 we provide some well-known results about fractional Sobolev spaces and prove some preliminary lemmas which play the most important roles in this paper. Section 3 proves the existence theorem and the stability result is proved in Section 4. Throughout this paper, the same letter $C$ might be used to denote various positive constants which may take different values within the same string of inequalities.

## 2 Preliminary lemmas

For the reader’s convenience, we first provide some basic properties of the fractional order Sobolev spaces $H^\alpha(\mathbb{R}^N)$ which will be used throughout the paper. The first lemma provides an alternative way of defining the fractional Sobolev space $H^\alpha(\mathbb{R}^N)$.

**Lemma 2.1.** Let $0 < \alpha < 1$ and $N \geq 1$. Then $u \in H^\alpha(\mathbb{R}^N)$ if and only if

$$u \in L^2(\mathbb{R}^N) \text{ and } (x, y) \mapsto \frac{|u(x) - u(y)|}{|x - y|^{2+\alpha}} \in L^2(\mathbb{R}^{2N}, dxdy).$$

Moreover, the norm $\|\cdot\|_{H^\alpha(\mathbb{R}^N)}$ in the space $H^\alpha(\mathbb{R}^N)$ is equivalent to

$$\|u\|_{H^\alpha(\mathbb{R}^N)} = \left(\|u\|^2_{L^2(\mathbb{R}^N)} + [u]^2_{H^\alpha(\mathbb{R}^N)}\right)^{1/2},$$

where the quantity $[u]^2_{H^\alpha(\mathbb{R}^N)}$ (so-called Gagliardo seminorm of $u$) is given by

$$[u]^2_{H^\alpha(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} \, dxdy.$$

The proof of the following alternative definition of the fractional Laplacian operator can be found, for example, in [19].

**Lemma 2.2.** For any $0 < \alpha < 1$, the fractional Laplacian operator $(-\Delta)^\alpha : \mathcal{S}(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ as defined in (1.2) can be expressed as

$$(-\Delta)^\alpha u(x) = C(N, \alpha) \, P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} \, dy = C(N, \alpha) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} \, dy,$$
where \( P.V. \) denotes abbreviation for the Cauchy principal value of the singular integral and \( C(N, \alpha) > 0 \) is some normalization constant given by

\[
C(N, \alpha) = \left( \int_{\mathbb{R}^N} \frac{1 - \cos x_1}{|x|^{N+2\alpha}} \, dx \right)^{-1} = \frac{\Gamma((2\alpha + N)/2)}{|\Gamma(-\alpha)|} 2^{2\alpha-1}\pi^{-N/2}.
\]

This above integral representation shows the nonlocal feature of the fractional Laplacian \((-\Delta)^\alpha u\) and can be used to define the operator for more general functions, for example, \( u \in C^2(\mathbb{R}^N) \). The next lemma provides the relationship between the fractional Laplacian \((-\Delta)^\alpha\) and the fractional Sobolev space \( H^\alpha(\mathbb{R}^N) \) (for a proof, see Propositions 4.2 and 4.4 of [19]).

**Lemma 2.3.** Let \( 0 < \alpha < 1 \). Then for all \( u \in H^\alpha(\mathbb{R}^N) \),

\[
[u]_{H^\alpha(\mathbb{R}^N)}^2 = \left( \frac{C_{N,\alpha}}{2} \right)^{-1} \int_{\mathbb{R}^N} |\xi|^{2\alpha} |\hat{u}(\xi)|^2 \, d\xi = \left( \frac{C_{N,\alpha}}{2} \right)^{-1} \|(-\Delta)^{\alpha/2} u\|_{L^2(\mathbb{R}^N)}^2.
\]

The following lemma is the Sobolev-type inequality for the fractional order Sobolev spaces. An elementary proof of this is given in Theorem 6.5 of [19].

**Lemma 2.4.** Let \( 0 < \alpha < 1 \) be such that \( N > 2\alpha \). Then there exists a positive constant \( C = C(N, \alpha) \) such that

\[
\|u\|_{L^p(\mathbb{R}^N)} \leq C \, [u]_{H^\alpha(\mathbb{R}^N)}^2, \quad \forall u \in H^\alpha(\mathbb{R}^N),
\]

where \( 2^*_\alpha \equiv 2^*_\alpha(N, \alpha) = \frac{2N}{N-2\alpha} \) is the fractional critical exponent and \([u]_{H^\alpha(\mathbb{R}^N)}^2\) is the Gagliardo seminorm of \( u \) as defined in Lemma 2.7.

Consequently, the space \( H^\alpha(\mathbb{R}^N) \) is continuously embedded into \( L^q(\mathbb{R}^N) \) for any \( q \in [2, 2^*_\alpha] \) and compactly embedded into \( L^q_{loc}(\mathbb{R}^N) \) for any \( q \in [2, 2^*_\alpha) \).

The next lemma is the fractional Gagliardo-Nirenberg inequality.

**Lemma 2.5.** Let \( 1 \leq p < \infty, 0 < \alpha < 1, \) and \( N > 2\alpha \). Then for any \( u \in H^\alpha(\mathbb{R}^N) \),

\[
\|u\|_{L^p(\mathbb{R}^N)} \leq C \, [u]_{H^\alpha(\mathbb{R}^N)}^\lambda \|u\|_{L^q(\mathbb{R}^N)}^{1-\lambda},
\]

where \( q \geq 1, \lambda \in [0, 1], C = C_{N,\alpha,\lambda} \) is a positive constant, and \( \lambda \) satisfies

\[
\frac{N}{p} = \frac{\lambda(N - 2\alpha)}{2} + \frac{N(1 - \lambda)}{q}.
\]
Proof. This is a consequence of the Hölder inequality and the Sobolev-type inequality (Lemma 2.4). The case \( p = 1 \) is clear. For \( p > 1 \), using the Hölder inequality, one has
\[
\|u\|_{L^p(\mathbb{R}^N)} \leq \|u\|_{L^2(\mathbb{R}^N)}^{1/2} \|u\|_{L^q(\mathbb{R}^N)}^{1/2}, \quad \text{where} \quad \frac{1}{p} = \frac{1}{2} \frac{1}{q} + \frac{\lambda}{2}.
\]
Now using the Sobolev inequality (Lemma 2.4), we obtain that
\[
\|u\|_{L^p(\mathbb{R}^N)} \leq (C_{N,\alpha})^{\lambda/2} [u]_{H^{\alpha}(\mathbb{R}^N)}^{\lambda} \|u\|_{L^q(\mathbb{R}^N)}^{1-\lambda},
\]
where \( C_{N,\alpha} \) is the same constant as in Lemma 2.3. The desired inequality follows by taking \( C = (C_{N,\alpha})^{\lambda/2} \) in the last inequality. \( \square \)

We now turn our attention to proving some properties of the variational problem \((E, S_r)\) and its minimizing sequences, which will be used in the proof of the existence theorem. Throughout the rest of this paper, we shall use the following notation
\[
e_i(u) = \frac{1}{2} \|D^\alpha u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu_i}{p_i} \|u\|_{L^{p_i}(\mathbb{R}^N)}^{p_i} \quad \text{for} \quad i = 1, 2, \quad (2.1)
\]
where \( D^\alpha = (-\Delta)^{\alpha/2} \). We denote by \( \gamma_1 = \mu_1/p_1 \) and \( \gamma_2 = \mu_2/p_2 \) the constants appearing in (2.1). For any \( \beta = (\beta_1, \beta_2) \), we define the function \( E_\beta \) by
\[
E_\beta = \inf \{ E(u_1, u_2) : (u_1, u_2) \in S_\beta \}. \quad (2.2)
\]
We first prove that \( E_\tau \) is finite and negative.

**Lemma 2.6.** For any \( \tau \in \mathbb{R}^+ \times \mathbb{R}^+ \), one has \(-\infty < E_\tau < 0\).

**Proof.** The proof that \( E_\tau < 0 \) follows a standard scaling argument. Indeed, for a given \((u_1, u_2) \in S_\tau\), define \( u_1^\lambda = \lambda^{1/2} u_1(\lambda^{1/2} x) \) and \( u_2^\lambda = \lambda^{1/2} u_2(\lambda^{1/2} x) \) for any \( \lambda > 0 \). Then, we have that \((u_1^\lambda, u_2^\lambda) \in S_\tau\) as well. Since the fractional Laplacian \( D^\alpha \) behaves like differentiation of order \( \alpha \), i.e., \( D^\alpha h_\theta(x) = \theta^\alpha D^\alpha h(\theta x) \) for any \( \theta > 0 \) and \( h_\theta(x) = h(\theta x), \ x \in \mathbb{R}^N \), we obtain that
\[
E(u_1^\lambda, u_2^\lambda) \leq \frac{\lambda^{2\alpha/N}}{2} \int_{\mathbb{R}^N} (|D^\alpha u_1|^2 + |D^\alpha u_2|^2) \ dx - \lambda \theta \int_{\mathbb{R}^N} F(u_1, u_2) \ dx
\]
where \( \theta = (r_1 + r_2 - 1) < \frac{2\alpha}{N} \) by \( r_1 + r_2 < 2 + \frac{\alpha}{N} \). Thus one can take \( \lambda > 0 \) sufficiently small such that \( E(u_1^\lambda, u_2^\lambda) < 0 \) and consequently \( E_\tau < E(u_1^\lambda, u_2^\lambda) < 0 \).

Next, making use of the Hölder inequality and the Sobolev inequality, we obtain
\[
\left( \int_{\mathbb{R}^N} |u_1|^{p_1} \ dx \right)^{1/p_1} \leq \left( \int_{\mathbb{R}^N} |u_1|^2 \ dx \right)^{\lambda/(1-\lambda)} \left( \int_{\mathbb{R}^N} |u_1|^{2\lambda} \ dx \right)^{(1-\lambda)/2} 
\leq C \left( \int_{\mathbb{R}^N} |u_1|^2 \ dx \right)^{(1-\lambda)/2} \|u_1\|_{H^{\alpha}(\mathbb{R}^N)}^\lambda, \quad (2.3)
\]
where $\lambda = N(p_1 - 2)/2p_1\alpha$. We now use the Young inequality to obtain
\[
\int_{\mathbb{R}^N} |u_1|^{p_1} \, dx \leq \varepsilon \|u_1\|_{H^\alpha(\mathbb{R}^N)}^2 + C\varepsilon \left( \int_{\mathbb{R}^N} |u_1|^2 \, dx \right)^\mu, \quad \mu = \frac{(1 - \lambda)p_1}{2 - \lambda p_1},
\]
for sufficiently small $\varepsilon > 0$ and $C\varepsilon$ depends on $\varepsilon$ but not on $u_1$. Similar inequality holds for $\int_{\mathbb{R}^N} |u_2|^{p_2} \, dx$. Then, for any $(u_1, u_2) \in S_r$, one obtains
\[
E(u_1, u_2) \geq \frac{1}{2} \|u_1\|_{H^\alpha(\mathbb{R}^N)}^2 + \frac{1}{2} \|u_2\|_{H^\alpha(\mathbb{R}^N)}^2 - C \left( \|u_1\|_{L^{p_1}(\mathbb{R}^N)} + \|u_2\|_{L^{p_2}(\mathbb{R}^N)} \right) - \tau_3 \geq \frac{1 - \varepsilon}{2} \left( \|u_1\|_{H^\alpha(\mathbb{R}^N)}^2 + \|u_2\|_{H^\alpha(\mathbb{R}^N)}^2 \right) - C,
\]
where $\tau_3 = (\tau_1 + \tau_2)/2$ and $C = C(N, p_1, p_2, \alpha, \varepsilon, \tau)$ are positive constants. Taking $2\varepsilon < 1$, this last inequality shows that $E_r > -\infty$ holds for all $N < \infty$. \quad \Box

In the next lemma, we prove that any energy-minimizing sequence for the problem $(E, S_r)$ must be bounded in $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ and collect some of their special properties.

**Lemma 2.7.** Suppose $\{(u_1^n, u_2^n)\}_{n \geq 1}$ be any sequence of functions in $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ such that
\[
\lim_{n \to \infty} \|u_i^n\|_{L^2(\mathbb{R}^N)} = \sqrt{\tau_i} \quad \text{and} \quad \lim_{n \to \infty} E(u_1^n, u_2^n) = E_r.
\]
Then the following assertions hold:

(i) there exists $B > 0$ such that $\|u_1^n\|_{H^\alpha(\mathbb{R}^N)} + \|u_2^n\|_{H^\alpha(\mathbb{R}^N)} \leq B$ for all $n$.

(ii) there exists $\delta_i > 0$ and $N_{\delta_i}$ such that $\|u_i^n\|_{L^{p_i}(\mathbb{R}^N)} \geq \delta_i$ for all $n \geq N_{\delta_i}$.

(iii) for any $\lambda > 1$ and for all sufficiently large $n$, the energy $e_i(u)$ satisfies the following scaling property
\[
e_i(\lambda u_i^n) < \lambda^2 e_i(u_i^n) \quad \text{for } i = 1, 2.
\]

**Proof.** For any minimizing sequence $\{(u_1^n, u_2^n)\}_{n \geq 1}$, it follows from (2.4) that
\[
\int_{\mathbb{R}^N} |u_1^n|^{p_1} \, dx \leq C \|u_1^n\|_{H^\alpha(\mathbb{R}^N)}^{N(p_1-2)/2\alpha} \leq C \|(u_1^n, u_2^n)\|_{H^\alpha(\mathbb{R}^N)}^{N(p_1-2)/2\alpha},
\]
where $C = C(p_1, N, \tau_1, \alpha)$. Similar estimate holds for $\int_{\mathbb{R}^N} |u_2^n|^{p_2} \, dx$. Using the Hölder inequality and the estimates for $\int_{\mathbb{R}^N} |u_1^n|^{p_1} \, dx$ and $\int_{\mathbb{R}^N} |u_2^n|^{p_2} \, dx$, we also have
\[
\int_{\mathbb{R}^N} |u_1^n|^{r_1} |u_2^n|^{r_2} \, dx \leq \left( \int_{\mathbb{R}^N} |u_1^n|^{r_1} \right)^{1/q} \left( \int_{\mathbb{R}^N} |u_2^n|^{r_2} \right)^{1/q'} \leq C \|(u_1^n, u_2^n)\|_{H^\alpha(\mathbb{R}^N)}^{\mu_1 + \mu_2},
\]
where the exponents $\mu_1$ and $\mu_2$ are given by $\mu_1 = N(r_1 q - 2)/2q\alpha$ and $\mu_2 = N(r_2 q' - 2)/2q'\alpha$ with $1/q + 1/q' = 1$ and $C = C(N, p_1, p_2, r_1, r_2, \tau, q, \alpha)$. Now put $F_n = (u_1^n, u_2^n)$ and observe that

$$\frac{1}{2}\|F_n\|^2_{H^\alpha(\mathbb{R}^N)} = E(F_n) + \gamma_2\|u_1^n\|^p_{L^p(\mathbb{R}^N)} + \gamma_2\|u_2^n\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N} F(F_n) \, dx + \tau_3.$$

Since the sequence $\{E(u_1^n, u_2^n)\}_{n \geq 1}$ is bounded, we obtain that

$$\frac{1}{2}\|F_n\|^2_{H^\alpha(\mathbb{R}^N)} \leq C \left( \|F_n\|^N_{H^{\alpha/2}(\mathbb{R}^{N,p_1-2})} + \|F_n\|^N_{H^{\alpha/2}(\mathbb{R}^{N,p_2-2})} + \|F_n\|^\mu_{\mu_1+\mu_2} \right).$$

Since the exponent $N(p_1-2)/2\alpha$ belongs to $(0,2)$ for any $p_i \in (2,2+4\alpha/N)$ and $\mu_1 + \mu_2 < 2$, it follows that the sequence $\{(u_1^n, u_2^n)\}_{n \geq 1}$ is bounded in $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$.

To prove that $L^p$-norms are bounded away from zero for large $n$, we argue by contradiction. If no such a positive number $\delta_1$ exists, then $\liminf_{n \to \infty} \int_{\mathbb{R}^N} |u_1^n|^p_\mathbb{R}^N \, dx = 0$. Consequently, $\int_{\mathbb{R}^N} F(u_1^n, u_2^n) \, dx \to 0$ as $n \to \infty$, and we have that

$$E_\tau = \lim_{n \to \infty} E(u_1^n, u_2^n) \geq \liminf_{n \to \infty} e_2(u_2^n). \quad (2.7)$$

On the other hand, select $\phi \geq 0$ with $\|\phi\|^2_{L^2(\mathbb{R}^N)} = \tau_1$ and for any number $\theta > 0$, put $u(x) = \theta^{1/2}\phi(\theta^{1/N}x)$. Then, for all $n \in \mathbb{N}$, one obtains that

$$E_\tau \leq e_2(u_2^n) + \frac{1}{2}\theta^{2\alpha/N}\|(-\Delta)^\alpha/2\phi\|^2_{L^2(\mathbb{R}^N)} - \theta^{(p_1-2)/2}\gamma_1\|\phi\|^p_{L^p(\mathbb{R}^N)}. \quad (2.8)$$

Now by selecting $\theta$ sufficiently small, one can obtain

$$\frac{1}{2}\theta^{2\alpha/N}\|(-\Delta)^\alpha/2\phi\|^2_{L^2(\mathbb{R}^N)} - \theta^{(p_1-2)/2}\gamma_1\|\phi\|^p_{L^p(\mathbb{R}^N)} < 0.$$

It then follows from (2.8) that $E_\tau < \liminf_{n \to \infty} e_2(u_2^n)$, this contradicts the inequality obtained above in (2.7). The proof that $\|u_2^n\|^p_{L^p(\mathbb{R}^N)} \geq \delta_2$ follows the same argument.

To prove statement (iii), let $\lambda > 1$. By Lemma 2.7, since the $L^p$-norms are bounded away from zero for all sufficiently large $n$, one has that

$$e_1(\lambda f_1^n) = \lambda^2 e_1(f_1^n) + \left(\lambda^2 - \gamma_1\lambda^p_1\right)\|f_1^n\|^p_{L^p(\mathbb{R}^N)}$$

$$\leq \lambda^2 e_1(f_1^n) + \left(\lambda^2 - \gamma_1\lambda^p_1\right)\delta_2 < \lambda^2 e_1(f_1^n).$$

The proof of the scaling property $e_2(\lambda f_2^n) < \lambda^2 e_1(f_2^n)$ is similar. \hfill \square
Lemma 2.8. Let \( N \geq 2 \) and \( 2^*_\alpha = 2N/(N - 2\alpha) \). Assume \( \{w_n\}_{n \geq 1} \) be a bounded sequence of functions in \( H^\alpha(\mathbb{R}^N) \). If there is some \( R > 0 \) such that

\[
\lim_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{y + B_R(0)} |w_n(x)|^2 \, dx \right) = 0,
\]

then one has \( \lim_{n \to \infty} \|w_n\|_{L^q(\mathbb{R}^N)} = 0 \) for any \( 2 < q < 2^*_\alpha \).

**Proof.** This is a version of Lemma I.1 of P. L. Lions [12]. We provide a proof here for the sake of completeness. Denote

\[
\epsilon_n = \sup_{y \in \mathbb{R}^N} \int_{y + B_R(0)} |w_n|^2 \, dx,
\]

so that \( \lim_{n \to \infty} \epsilon_n = 0 \). Let \( 2 < q < 2^*_\alpha = 2N/(N - 2\alpha) \). For every point \( y \in \mathbb{R}^N \) and any number \( R > 0 \), using the Hölder inequality, for every \( n \) we obtain that

\[
\|w_n\|_{L^q(y + B_R(0))} \leq \|w_n\|_{L^2(y + B_R(0))} \|w_n\|_{L^{2^*_\alpha}(y + B_R(0))}^{1+\lambda} \|
\]

where \( \lambda \) satisfies \( \frac{\lambda}{2} + \frac{(1+\lambda)q}{2s} = 1 \). Taking \( \lambda q = s \), it then follows from the Sobolev inequality that

\[
\int_{y + B_R(0)} |w_n|^q \, dx \leq C \epsilon_n^s \|w_n\|_{L^{2^*_\alpha}(y + B_R(0))} \|w_n\|_{H^\alpha}^s \leq C \epsilon_n^s \|w_n\|_{L^{2^*_\alpha}(y + B_R(0))},
\]

where \( s = 2/q \). Now cover the \( n \)-space \( \mathbb{R}^N \) by \( n \)-balls of radius \( R \) in such a way that each \( x \in \mathbb{R}^N \) lies at most \( N + 1 \) of these \( n \)-balls, then by summing the inequality \( (2.10) \) over all \( n \)-balls in the covering and making another use of the Sobolev inequality, we obtain that

\[
\int_{\mathbb{R}^N} |w_n|^q \, dx \leq (N + 1) C \epsilon_n^s \|w_n\|_{L^{2^*_\alpha}(\mathbb{R}^N)} \leq C \epsilon_n^s,
\]

which gives the desired result. \( \Box \)

We require the following result of [7] concerning the existence of minimizers of the energy functional \( e_i(f) \) associated with the standard nonlinear fractional Schrödinger equations.

**Lemma 2.9.** Suppose \( N \geq 2, 0 < \alpha < 1, \) and \( 2 < p_1, p_2 < 2 + \frac{4\alpha}{N} \). Let \( i \in \{1, 2\} \) and let the functional \( e_i : H^\alpha(\mathbb{R}^N) \to \mathbb{C} \) be as defined in \( (2.1) \). Then, for any \( \tau_i > 0 \), if a sequence \( \{u^n_i\}_{n \geq 1} \) in \( H^\alpha(\mathbb{R}^N) \) be such that \( \|u^n_i\|_{L^2(\mathbb{R}^N)} \to \sqrt{\tau_i} \) as \( n \to \infty \) and

\[
\lim_{n \to \infty} e_i(u^n_i) = \inf \{e_i(u) : u \in H^\alpha(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^2 \, dx = \tau_i\},
\]
then the sequence \( \{u^n_i\}_{n \geq 1} \) is compact in \( H^\alpha(\mathbb{R}^N) \) up to spatial translations and the extraction of subsequence. The limit function \( \psi_i \) satisfies

\[
(-\Delta)^\alpha Q = \omega Q + \gamma |Q|^{p-2}Q, \quad \text{for some } \omega \in \mathbb{R}.
\]

In the next few lemmas, we closely follow techniques of [2, 3, 4] to prove strict inequalities involving the minimization problem \((E, M_\sigma)\) as function of constraint variables. These inequalities will play a key role later to excluding the possibility of dichotomy for an energy-minimizing sequence while applying the concentration compactness principle.

Lemma 2.10. For any \( \sigma, \tau \in \mathbb{R}^+ \times \mathbb{R}^+ \), one has \( E_{\sigma+\tau} < E_{\sigma} + E_{\tau} \).

Proof. For any \( \sigma, \tau \in \mathbb{R}^+ \times \mathbb{R}^+ \), let \( \{(u^n_{1,i}, u^n_{2,i})\}_{n \geq 1} \) be any sequence of functions in \( H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N) \) satisfying the conditions

\[
\lim_{n \to \infty} \|u^n_{j,1}\|_{L^2(\mathbb{R}^N)} = \sqrt{\sigma_j}, \quad \lim_{n \to \infty} \|u^n_{j,2}\|_{L^2(\mathbb{R}^N)} = \sqrt{\tau_j} \quad \text{for } j = 1, 2,
\]

\[
\lim_{n \to \infty} E(u^n_{1,1}, u^n_{1,2}) = E_{\sigma}, \quad \text{and} \quad \lim_{n \to \infty} E(u^n_{1,2}, u^n_{2,2}) = E_{\tau}.
\]

By passing to suitable subsequences, one may assume that the following values exists

\[
A_1 = \frac{1}{\sigma_1} \lim_{n \to \infty} \left( e_1(u^n_{1,1}) - \int_{\mathbb{R}^N} F(u^n_{1,1}, u^n_{2,1}) \, dx \right), \quad B_1 = \frac{1}{\sigma_2} \lim_{n \to \infty} e_2(u^n_{2,1}),
\]

\[
A_2 = \frac{1}{\tau_1} \lim_{n \to \infty} \left( e_1(u^n_{1,2}) - \int_{\mathbb{R}^N} F(u^n_{1,2}, u^n_{2,2}) \, dx \right), \quad B_2 = \frac{1}{\tau_2} \lim_{n \to \infty} e_2(u^n_{2,2}).
\]

We first consider the case that \( A_1 < A_2 \). Without loss of generality, we may assume that \( u^n_{1,i} \) and \( u^n_{2,i} \) are non-negative. By a density argument, we may also suppose that \( u^n_{1,i} \) and \( u^n_{2,i} \) have compact support. We denote \( \tilde{u}^n_{2,1}(\cdot) = u^n_{2,1}(\cdot - b_{2,1} \kappa) \), where \( \kappa \) is some unit vector in \( \mathbb{R}^N \). Choose \( b_{2,1} \) such that the supports of \( \tilde{u}^n_{2,1} \) and \( u^n_{2,2} \) are disjoint and let \( u^n_2 = \tilde{u}^n_{2,1} + u^n_{2,2} \) in \( \mathbb{R}^N \). Then, we have that \( \lim_{n \to \infty} \|u^n_2\|_{L^2(\mathbb{R}^N)} = \sqrt{\sigma_2 + \tau_2} \). Let us denote \( q_{1,1} = 1 + \frac{\sigma_1}{\tau_1} \) and \( q_{2,2} = 1 + \frac{\tau_2}{\sigma_2} \). Then it is obvious that

\[
E_{\sigma+\tau} \leq \lim_{n \to \infty} E((q_{1,1})^{1/2}u^n_{1,1}, u^n_2). \tag{2.11}
\]

Introducing the notation \( J(u, v) = e_1(u) - \int_{\mathbb{R}^N} F(u, v) \, dx \) for \( u, v \in H^\alpha(\mathbb{R}^N) \). We now make use of the fact that \( q_{1,1} > 1 \) to obtain

\[
\lim_{n \to \infty} J((q_{1,1})^{1/2}u^n_{1,1}, u^n_2) \leq \lim_{n \to \infty} J((q_{1,1})^{1/2}u^n_{1,1}, \tilde{u}^n_{2,1})
\]

\[
= q_{1,1} \lim_{n \to \infty} \int_{\mathbb{R}^N} \left( \frac{1}{2} |D^\alpha u^n_{1,1}|^2 - q_{1,1}\gamma_1 |u^n_{1,1}|^{p_1} - F(u^n_{1,1}, \tilde{u}^n_{2,1}) \right) \, dx \tag{2.12}
\]

\[
\leq q_{1,1} \lim_{n \to \infty} J(u^n_{1,1}, \tilde{u}^n_{2,1}) = q_{1,1} \sigma_1 A_1 = \sigma_1 A_1 + \tau_1 A_2 - \delta,
\]
where $\delta = \tau_1 (A_2 - A_1)$. Since $A_1 < A_2$, it is obvious that $\delta > 0$. Applying (2.12) into (2.11), one can easily deduce that
\[
E_{\sigma + \tau} \leq \lim_{n \to \infty} \left( E\left( u_{1,1}^n, u_{2,1}^n \right) + E\left( u_{1,2}^n, u_{2,2}^n \right) \right) - \delta < E_\sigma + E_\tau.
\]

The proof in the case $A_1 > A_2$ follows the same argument except that we swap the indices and so will not be repeated here. Next, suppose that $A_1 = A_2$ and $B_1 \leq B_2$. Invoking Lemma 2.7(iii), there exists $\delta > 0$ such that
\[
E_{\sigma + \tau} \leq E\left( (q_{1,1})^{1/2} u_{1,1}^n, (q_{2,2})^{1/2} u_{2,1}^n \right)
\leq q_{2,2} e_2 (u_{2,1}^n) + J\left( (q_{1,1})^{1/2} u_{1,1}^n, (q_{2,2})^{1/2} f_{2,1}^n \right) - \delta
\leq q_{2,2} e_2 (u_{2,1}^n) + q_1 J\left( u_{1,1}^n, u_{2,1}^n \right) - \delta
\leq E\left( u_{1,1}^n, u_{2,1}^n \right) + \frac{\tau_2}{\sigma_2} e_2 (u_{2,1}^n) + \frac{\tau_1}{\sigma_1} J\left( u_{1,1}^n, u_{2,1}^n \right) - \delta.
\]

Passing the limit as $n \to \infty$ on both sides of the preceding inequality and making use of the facts $A_1 = A_2$ and $B_1 \leq B_2$, one obtains that
\[
E_{\sigma + \tau} \leq E_\sigma + \frac{\tau_2}{\sigma_2} (\sigma_2 B_1) + \frac{\tau_1}{\sigma_1} (\sigma_1 A_1) - \delta
\leq E_\sigma + \tau_2 B_2 + \tau_1 A_2 - \delta < E_\sigma + E_\tau.
\]

The proof in the case $A_1 = A_2$ and $B_1 \geq B_2$ follows a similar argument. \qed

**Lemma 2.11.** For any $\tau \in \mathbb{R}^+ \times \mathbb{R}^+$ and $\sigma = (0, \sigma_2)$ with $\sigma_2 > 0$, one has
\[
E_{\sigma + \tau} < E_\sigma + E_\tau. \tag{2.13}
\]

**Proof.** Suppose $i \in \{1, 2\}$ and let $\{(u_{1,i}^n, u_{2,i}^n)\}_{n \geq 1}$ be sequence of functions in $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ satisfying the conditions
\[
\lim_{n \to \infty} \|u_{1,i}^n\|_{L^2(\mathbb{R}^N)} = 0, \quad \lim_{n \to \infty} \|u_{2,i}^n\|_{L^2(\mathbb{R}^N)} = \sqrt{\sigma_2}, \quad \lim_{n \to \infty} E\left( u_{1,i}^n, u_{2,i}^n \right) = E_\sigma,
\]
\[
\lim_{n \to \infty} \|u_{j,i}^n\|_{L^2(\mathbb{R}^N)} = \sqrt{\tau_j} \text{ for } j = 1, 2, \text{ and } \lim_{n \to \infty} E\left( u_{1,i}^n, u_{2,i}^n \right) = E_\tau.
\]

We look for sequence of functions $(u_1^n, u_2^n)$ in $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ satisfying $\|u_1^n\|_{L^2(\mathbb{R}^N)} \to \sqrt{\tau_1}$, $\|u_2^n\|_{L^2(\mathbb{R}^N)} \to \sqrt{\sigma_2 + \tau_2}$, $E(u_1^n, u_2^n) \to E_{\sigma + \tau}$ as $n \to \infty$, and such that (2.13) holds. As before, one can pass to a subsequence if necessary and consider the values
\[
D_1 = \frac{1}{\sigma_2} \lim_{n \to \infty} \left( e_2 (u_{2,1}^n) - \int_{\mathbb{R}^N} F\left( u_{1,2}^n, u_{2,1}^n \right) \, dx \right) \quad \text{and}
\]
\[
D_2 = \frac{1}{\tau_2} \lim_{n \to \infty} \left( e_2 (u_{2,2}^n) - \int_{\mathbb{R}^N} F\left( u_{1,2}^n, u_{2,2}^n \right) \, dx \right).
\]
Assume first that $D_1 < D_2$. With $q_{2,2}$ defined as above, it follows that
\[
E_{\sigma + \tau} \leq E (u_{1,2}^n, (q_{2,2})^{1/2} u_{2,1}^n) = e_1 (u_{1,2}^n) \\
+ q_{2,2} \int_{\mathbb{R}^N} \left( \frac{1}{2} |D^n u_{2,1}^n|^2 - q_{2,2} \gamma_2 |u_{2,1}^n|^{p_2} - q_{2,2} F (u_{1,2}^n, u_{2,1}^n) \right) \, dx \\
\leq e_1 (u_{1,2}^n) + q_{2,2} e_2 (u_{2,1}^n) - q_{2,2} \int_{\mathbb{R}^N} F (u_{1,2}^n, f_{2,1}^n) \, dx.
\]

Put $\delta = \tau_2 (D_2 - D_1)$. Since $D_1 < D_2$, we have that $\delta > 0$. Passing the limit as $n \to \infty$ in the preceding inequality and using the definition of $q_{2,2}$, we obtain that
\[
E_{\sigma + \tau} \leq \lim_{n \to \infty} E (u_{1,1}^n, u_{2,1}^n) + \lim_{n \to \infty} e_1 (u_{1,2}^n) + \frac{\tau_2}{\sigma_2} (\sigma_2 D_1) \\
= E_\sigma + \lim_{n \to \infty} e_1 (u_{1,2}^n) + \tau_2 D_2 - \delta \\
= E_\sigma + \lim_{n \to \infty} E (u_{1,2}^n, u_{2,2}^n) - \delta < E_\sigma + E_{\tau}.
\]

Next consider the case that $D_1 > D_2$. With $p_{2,2} = 1 + \frac{\sigma_2}{\tau_2}$, we have that
\[
E_{\sigma + \tau} \leq E (u_{1,2}^n, (p_{2,2})^{1/2} u_{2,2}^n) = e_1 (u_{1,2}^n) \\
+ p_{2,2} \int_{\mathbb{R}^N} \left( \frac{1}{2} |D^n u_{2,2}^n|^2 - p_{2,2} \gamma_2 |u_{2,2}^n|^{p_2} - p_{2,2} F (u_{1,2}^n, u_{2,2}^n) \right) \, dx \\
\leq e_1 (u_{1,2}^n) + p_{2,2} e_2 (u_{2,2}^n) - p_{2,2} \int_{\mathbb{R}^N} F (u_{1,2}^n, u_{2,2}^n) \, dx.
\]

Put $\delta = \sigma_2 (D_1 - D_2)$. Then $\delta > 0$. Passing the limit as $n \to \infty$ in the preceding inequality and using the definition of $p_{2,2}$, it follows that
\[
E_{\sigma + \tau} \leq \lim_{n \to \infty} E (u_{1,2}^n, u_{2,2}^n) + \frac{\sigma_2}{\tau_2} (\tau_2 D_2) = E_{\tau} + \sigma_2 D_1 - \delta \\
\leq E_{\tau} + \lim_{n \to \infty} E (u_{1,1}^n, u_{2,1}^n) - \delta < E_{\tau} + E_{\sigma}.
\]

Finally, suppose that $D_1 = D_2$. Let $q_{2,2}$ be as defined in the proof of Lemma 2.10 and $u_2^n = (q_{2,2})^{1/2} u_{2,1}^n$. Then, using Lemma 2.7(iii), one can find $\delta > 0$ such that
\[
E_{\sigma + \tau} \leq E (u_{1,2}^n, u_2^n) \leq e_1 (u_{1,2}^n) + q_{2,2} e_2 (u_{2,1}^n) - q_{2,2} \int_{\mathbb{R}^N} F (u_{1,2}^n, u_{2,1}^n) \, dx - \delta.
\]

Since $q_{2,2} = 1 + \frac{\sigma_2}{\tau_2}$, one can pass limit as $n \to \infty$ on both sides of the last inequality and make use of the fact $D_1 = D_2$ to obtain
\[
E_{\sigma + \tau} \leq E_\sigma + \lim_{n \to \infty} e_1 (u_{1,2}^n) + \frac{\tau_2}{\sigma_2} (\sigma_2 D_1) - \delta \\
= E_\sigma + \lim_{n \to \infty} e_1 (u_{1,2}^n) + \tau_2 D_2 - \delta \\
\leq E_\sigma + \lim_{n \to \infty} E (u_{1,2}^n, u_{2,2}^n) - \delta < E_\sigma + E_{\tau}.
\]
This completes the proof of the inequality (2.13) in all three possible cases according to values of $D_1$ and $D_2$.  

One can follow the same argument as in the proof of (2.13) to prove the following version of subadditivity inequality.

**Lemma 2.12.** For any $\sigma \in \mathbb{R}^+ \times \mathbb{R}^+$ and $\tau \in \mathbb{R}^+ \times \{0\}$, one has $E_{\sigma + \tau} < E_{\sigma} + E_{\tau}$.

We are now able to establish the following version of the subadditivity condition.

**Lemma 2.13.** For all $\sigma, \tau \in (\mathbb{R}^+ \times \mathbb{R}^+) \cup \{0\}$ with $\sigma + \tau = \beta \in \mathbb{R}^+ \times \mathbb{R}^+$ and $\sigma, \tau \neq \{0\}$, one has

$$E_{\beta} < E_{\sigma} + E_{\tau}. \quad (2.14)$$

**Proof.** To prove (2.14), we consider four separate cases: (i) $\sigma, \tau \in \mathbb{R}^+ \times \mathbb{R}^+$; (ii) $\tau \in \mathbb{R}^+ \times \mathbb{R}^+$ and $\sigma \in \{0\} \times \mathbb{R}^+$; (iii) $\sigma \in \mathbb{R}^+ \times \mathbb{R}^+$ and $\tau \in \mathbb{R}^+ \times \{0\}$; and (iv) $\sigma \in \{0\} \times \mathbb{R}^+$ and $\tau \in \mathbb{R}^+ \times \{0\}$. All other remaining cases can be reduced to one of cases above by swapping the indices. In view of lemmas 2.10, 2.11, and 2.12, it only remains to prove (2.14) in the case when $\sigma \in \{0\} \times \mathbb{R}^+$ and $\tau \in \mathbb{R}^+ \times \{0\}$.

Assuming $\tau_2 = \sigma_2$ in Lemma 2.9 let $\psi_{\tau_1}$ and $\psi_{\sigma_2}$ be minimizers of $e_1(u)$ and $e_2(u)$ over $S_{\tau_1}$ and $S_{\sigma_2}$, respectively. Then it is obvious that $\int_{\mathbb{R}^N} F(\psi_{\tau_1}, \psi_{\sigma_2}) \, dx > 0$ and (2.14) holds. □

**Lemma 2.14.** For any $\sigma \in \mathbb{R}^+ \times \mathbb{R}^+$, let $\{(u_1^n, u_2^n)\}_{n \geq 1}$ be a minimizing sequence for the problem $(E, S_{\sigma})$. Define $Q_n : [0, \infty) \to [0, \sigma_1 + \sigma_2]$ by

$$Q_n(t) = \sup_{y \in \mathbb{R}} \int_{y + B_t(0)} (|u_1^n(x)|^2 + |u_2^n(x)|^2) \, dx, \quad n \geq 1, \ t > 0. \quad (2.15)$$

Then the following assertions hold:

(i) Every sequence of non-decreasing functions $(Q_n)$ has a subsequence converging pointwise to a non-decreasing function $Q : [0, \infty) \to [0, \sigma_1 + \sigma_2]$.

(ii) If $\gamma$ is defined by $\gamma = \lim_{t \to \infty} Q(t)$, then there exists an ordered pair $\tau = (\tau_1, \tau_2) \in [0, \sigma_1] \times [0, \sigma_2]$ such that $\gamma = \tau_1 + \tau_2$ and

$$E_{\sigma} \geq E_{\tau} + E_{\sigma - \tau}. \quad (2.16)$$

**Proof.** Each function $Q_n$ is non-decreasing on $[0, \infty)$ and by Helly’s selection theorem $Q(t) = \lim_{t \to \infty} Q_n(t)$ (perhaps passing to a subsequence) is a non-decreasing function on $[0, \infty)$. It is easy to check that $0 \leq \gamma \leq \sigma_1 + \sigma_2$. To prove statement (ii), let $\epsilon > 0$ be an arbitrary. We continue to assume that the subsequence associated with $\gamma$ is the whole sequence. It follows from the definition of $\gamma$ that there exists $t_\epsilon > 0$ and $N_\epsilon \in \mathbb{N}$ such that for every $t \geq t_\epsilon$ and $n \geq N_\epsilon$, one has $\gamma - \epsilon < Q(t) \leq Q(2t) < \gamma$. 

$$\int_{y + B_{2t}(0)} (|u_1^n(x)|^2 + |u_2^n(x)|^2) \, dx \leq \int_{y + B_t(0)} (|u_1^n(x)|^2 + |u_2^n(x)|^2) \, dx + \epsilon \quad (2.17)$$

By the definition of $B_{2t}(0)$, we have $\int_{y + B_{2t}(0)} (|u_1^n(x)|^2 + |u_2^n(x)|^2) \, dx \leq \int_{\mathbb{R}^N} (|u_1^n(x)|^2 + |u_2^n(x)|^2) \, dx$. Hence, we get

$$\int_{y + B_t(0)} (|u_1^n(x)|^2 + |u_2^n(x)|^2) \, dx \leq \int_{\mathbb{R}^N} (|u_1^n(x)|^2 + |u_2^n(x)|^2) \, dx + \epsilon.$$
and $\gamma - \epsilon < Q_n(t) \leq Q_n(2t) \leq \gamma + \epsilon$. Thus, by the definition of the concentration functions $Q_n$, for every $n \geq N_\epsilon$ there exists a sequence of points $\{y_n\} \subset \mathbb{R}^N$ such that

$$
\int_{y_n + B_{2i}(0)} \rho_n(x) \, dx > \gamma - \epsilon \quad \text{and} \quad \int_{y_n + B_{2i}(0)} \rho_n(x) \, dx < \gamma + \epsilon,
$$

where $\rho_n = |u_1^n|^2 + |u_2^n|^2$. Now, for any $\eta > 0$, we set $\rho_\eta(x) = \rho(x/\eta)$ and $\sigma_\eta(x) = \sigma(x/\eta)$, where $\rho \in C_0^\infty(B_2(0))$ be such that $\rho \equiv 1$ on $B_1(0)$ and $\sigma \in C^\infty(\mathbb{R}^N)$ be such that $\rho^2 + \sigma^2 \equiv 1$ on $\mathbb{R}^N$. Next, define the functions

$$
\begin{align*}
(u_1^{1,n}(x), u_1^{2,n}(x)) &= \rho_i(x - y_k) (u_1^n(x), u_2^n(x)), \ x \in \mathbb{R}^N, \\
(u_2^{1,n}(x), u_2^{2,n}(x)) &= \sigma_i(x - y_k) (u_1^n(x), u_2^n(x)), \ x \in \mathbb{R}^N.
\end{align*}
$$

Then, one can pass to a subsequence to find the numbers $\tau_1 \in [0, \sigma_1]$ and $\tau_2 \in [0, \sigma_2]$ such that $\|f_1^{i,n}\|_{L^2(\mathbb{R}^N)} \to \sqrt{\tau_i}$ for $i = 1, 2$, whence it also follows immediately that $\|f_2^{i,n}\|_{L^2(\mathbb{R}^N)} \to \sqrt{\sigma_i - \tau_i}$ for $i = 1, 2$. Now taking into account of these and making use of the inequalities (2.17), it is easy to check that $|\tau_1 + \tau_2 - \gamma| < \epsilon$. Suppose for now that the following holds:

$$
E(u_1^{1,n}, u_1^{2,n}) + E(u_2^{1,n}, u_2^{2,n}) \leq E(u_1^n, u_2^n) + C\epsilon, \ \forall n.
$$

To prove the inequality (2.16), since for any given $\epsilon > 0$, each of the terms in both sides of (2.18) is bounded independently of $n$, thus up to a subsequence, one may assume that $E(u_1^{1,n}, u_1^{2,n}) \to \Lambda_1$ and $E(u_2^{1,n}, u_2^{2,n}) \to \Lambda_2$. In turn, it follows that

$$
\Lambda_1 + \Lambda_2 \leq E_\sigma + C\epsilon.
$$

Since the number $\epsilon$ can be chosen arbitrarily small and $t$ can be taken arbitrarily large, taking account into the results obtained in the preceding paragraphs, one sees that for every $m \in \mathbb{N}$, one can find sequences of functions $(u_{1,m}^{1,n}, u_{1,m}^{2,n})$ and $(u_{2,m}^{1,n}, u_{2,m}^{2,n})$ in $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ such that $\|u_{1,m}^{i,n}\|_{L^2(\mathbb{R}^N)} \to \sqrt{\tau_i(m)}$, $\|u_{2,m}^{i,n}\|_{L^2(\mathbb{R}^N)} \to \sqrt{\sigma_i - \tau_i(m)}$, and $E(u_{1,m}^{i,n}, u_{2,m}^{i,n}) = \Lambda_i(m)$ for $i = 1, 2$, where $\tau_i(m) \in [0, \sigma_1]$, $\tau_2(m) \in [0, \sigma_2]$, $|\tau_1(m) + \tau_2(m) - \gamma| \leq \epsilon$, and $\Lambda_1(m) + \Lambda_2(m) \leq E_\sigma + \frac{1}{m}$.

Passing to a subsequence, we can further suppose that $\tau_1(m) \to \tau_1 \in [0, \sigma_1]$, $\tau_2(m) \to \tau_2 \in [0, \sigma_2]$, $\Lambda_1(m) \to \Lambda_1$, and $\Lambda_2(m) \to \Lambda_2$. Moreover, by redefining the sequences $(u_{1,n}^{1,n}, u_{1,n}^{2,n})$ and $(u_{2,n}^{1,n}, u_{2,n}^{2,n})$ to be diagonal subsequences $(u_{1,n}^{1,n}, u_{1,n}^{2,n}) = (u_{1,1,n}^{1,n}, u_{1,1,n}^{2,n})$ and $(u_{2,n}^{1,n}, u_{2,n}^{2,n}) = (u_{2,1,n}^{1,n}, u_{2,1,n}^{2,n})$, one can assume that $\|u_{1,n}^{i,n}\|_{L^2(\mathbb{R}^N)} \to \sqrt{\tau_i}$, $\|u_{2,n}^{i,n}\|_{L^2(\mathbb{R}^N)} \to \sqrt{\sigma_i - \tau_i}$, and $E(u_{1,n}^{i,n}, u_{2,n}^{i,n}) \to \Lambda_i$ for $i = 1, 2$. Now, letting the limit as $m \to \infty$ in
the first inequality of (2.19), one obtains \( \gamma = \tau_1 + \tau_2 \). The condition (2.16) follows from the second inequality in (2.19) provided one can show that

\[
\Lambda_1 \geq E_\tau \quad \text{and} \quad \Lambda_2 \geq E_{\sigma - \tau}.
\]  

(2.20)

To see the first inequality in (2.20), assume first that both \( \tau_1 \) and \( \tau_2 \) are positive and define

\[
\beta_1^n = \frac{(\tau_1)^{1/2}}{\|u_1^{1,n}\|_{L^2(\mathbb{R}^N)}} \quad \text{and} \quad \beta_2^n = \frac{(\tau_2)^{1/2}}{\|u_2^{2,n}\|_{L^2(\mathbb{R}^N)}}.
\]

Then, it is obvious that \( E(\beta_1^n u_1^{1,n}, \beta_2^n u_2^{2,n}) \geq E_\tau \). Since scaling factors all tend to 1 as \( n \to \infty \), we have that \( E(\beta_1^n u_1^{1,n}, \beta_2^n u_2^{2,n}) \to \Lambda_1 \) and hence, the first inequality in (2.20) follows. Next, suppose that one of \( \tau_1 \) or \( \tau_2 \) is zero, say \( \tau_1 = 0 \). Then, it follows from the Sobolev interpolation inequality that \( \int_{\mathbb{R}^N} |u_1^{1,n}|^{\tau_1} |u_2^{2,n}|^{\tau_2} \, dx \to 0 \) as \( n \to \infty \) and hence, we obtain that

\[
\Lambda_1 = \lim_{n \to \infty} E(u_1^{1,n}, u_2^{2,n}) = \lim_{n \to \infty} \left( e_2(\beta_1^n) + \int_{\mathbb{R}^N} |D^n u_1^{1,n}|^2 \, dx \right) \geq E_{\tau_2}.
\]

This concludes the proof of the first inequality in (2.20). The proof of second inequality in (2.20) uses the same argument with \( \sigma_1 - \tau_1 \) and \( \sigma_2 - \tau_2 \) enjoying the roles of \( \tau_1 \) and \( \tau_2 \), respectively.

To complete the proof of lemma, it only remains to prove the condition (2.18). We will make use of the following commutator estimates result.

**Lemma 2.15.** If \( 0 < \alpha < 1 \) and \( f, g \in S(\mathbb{R}^N) \), then

\[
\| [D^n, f] g \|_{L^2(\mathbb{R}^N)} \leq C \left( \|\nabla f\|_{L^p(\mathbb{R}^N)} \|D^{\alpha-1} g\|_{L^q(\mathbb{R}^N)} + \|D^\alpha f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)} \right),
\]

where \([X,Y] = XY - YX\) is the commutator, \( q_1, q_2 \in [2, \infty) \), and

\[
\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{2}.
\]

This lemma is proved in [7] by combining the commutator estimates established by Kato and Ponce in (Lemma X1 of [14]) with a version of Kenig, Ponce, and Vega’s result in (Lemma 2.10 of [15]).

We now show that the condition (2.18) holds. For ease of notation, let us write the shifted functions \( \rho_t(x - y_k) \) and \( \sigma_t(x - y_k) \) simply as \( \rho_t \) and \( \sigma_t \), respectively. By Lemma 2.15 with \( f = \rho_t \) and \( g = u_1^{1,n} \), we estimate

\[
\| [D^n, \rho_t] u_1^{1,n} \|_{L^2(\mathbb{R}^N)} = \| D^n (\rho_t u_1^{1,n}) - \rho_t D^n u_1^{1,n} \|_{L^2(\mathbb{R}^N)} \leq C \left( \|\nabla \rho_t\|_{L^p(\mathbb{R}^N)} \|D^{\alpha-1} u_1^{1,n}\|_{L^q(\mathbb{R}^N)} + \|D^\alpha \rho_t\|_{L^p(\mathbb{R}^N)} \|u_1^{1,n}\|_{L^q(\mathbb{R}^N)} \right).
\]
Taking \( p_1 = \infty, q_1 = 2; p_2 = 2 + \frac{N}{\alpha}, \) and \( q_2 = 2 + \frac{4\alpha}{N} \) so that \( \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{2}, \) and using the Sobolev inequality, we obtain that
\[
\| [D^\alpha, \rho_t] u_1^{1,n} \|_{L^2(\mathbb{R}^N)} \leq C \| \nabla \rho_t \|_\infty \| D^{\alpha-1} u_1^{1,n} \|_{L^2(\mathbb{R}^N)} + C \frac{1}{t^\mu} \| D^\alpha \rho \|_{L^{p_2}(\mathbb{R}^N)} \| u_1^{1,n} \|_{H^\alpha(\mathbb{R}^N)},
\]
where \( \mu = 2\alpha^2/(2\alpha + N). \) By the definition of \( \rho_t, \) we have \( \| \nabla \rho_t \|_\infty = \| \nabla \rho \|_\infty / t. \) It immediately follows from this last inequality that
\[
\int_{\mathbb{R}^N} |D^\alpha(\rho_t u_1^{1,n})|^2 dx \leq \int_{\mathbb{R}^N} (\rho_t)^2 |D^\alpha u_1^{1,n}|^2 dx + C \varepsilon. \tag{2.21}
\]
holds for all sufficiently large \( t. \) Using the estimate for \( \int_{\mathbb{R}^N} |D^\alpha(\rho_t u_1^{1,n})|^2 dx \) obtained in (2.21) and a similar estimate for \( \int_{\mathbb{R}^N} |D^\alpha(\rho_t u_2^{1,n})|^2 dx, \) we can write
\[
E(u_1^{1,n}, u_1^{2,n}) \leq \int_{\mathbb{R}^N} \rho_t^2 \left( |D^\alpha u_1^{1,n}|^2 + |D^\alpha u_2^{1,n}|^2 - \gamma_1 |u_1^{1,n}|^{p_1} - \gamma_2 |u_2^{1,n}|^{p_2} - F(u_1^{1,n}, u_2^{1,n}) \right) dx + \int_{\mathbb{R}^N} \left( \gamma_1 U_{\rho}^{p_1} |u_1^{1,n}|^{p_1} + \gamma_2 U_{\rho}^{p_2} |u_2^{1,n}|^{p_2} + U_{\rho}^{r_1+r_2} F(u_1^{1,n}, u_2^{1,n}) \right) dx + C \varepsilon,
\]
where \( U_r^x = \rho_t^2 - |\rho_t|^r. \) A similar estimate holds for the quantity \( E(u_2^{1,n}, u_2^{2,n}). \) Since \( \rho_t \) and \( \sigma_t \) satisfy \( \rho_t^2(x) + \sigma_t^2(x) \equiv 1 \) for \( x \in \mathbb{R}^N, \) we obtain that
\[
E(u_1^{1,n}, u_1^{2,n}) + E(u_2^{1,n}, u_2^{2,n}) \leq E(u_1^{1,n}, u_2^{1,n}) + \gamma_1 \int_{\mathbb{R}^N} U_{\rho}^{p_1} |u_1^{1,n}|^{p_1} dx + \gamma_2 \int_{\mathbb{R}^N} U_{\rho}^{p_2} |u_2^{1,n}|^{p_2} dx + \int_{\mathbb{R}^N} U_{\rho}^{r_1+r_2} F(u_1^{1,n}, u_2^{1,n}) dx + C \varepsilon, \tag{2.22}
\]
where \( U_r \) is given by \( U_r = U_{\rho}^r + U_{\sigma}^r. \) Now denote \( D = B(y_k, 2r) - B(y_k, r). \) Then, using the inequalities we obtained in (2.17), it immediately follows that
\[
\int_{\mathbb{R}^N} U_{\rho}^{p_1} |u_1^{i,n}|^{p_1} dx \leq 4 \int_{D} |u_1^{i,n}|^{p_1} dx \leq C \varepsilon, \quad i = 1, 2,
\]
\[
\int_{\mathbb{R}^N} U_{\rho}^{r_1+r_2} F(u_1^{1,n}, u_2^{1,n}) dx \leq 4 \beta \|u_1^{1,n}\|_{L^2}^{r_1} \int_{D} |u_2^{1,n}|^{r_2} dx \leq C \varepsilon,
\]
where the letter \( C \) represents various positive constants independent of \( t \) and \( n. \) Taking into account all these inequalities, (2.18) follows from (2.22). \( \square \)
3 Proof of existence result

Armed with all preliminaries lemmas, we are now able to prove the existence theorem. To begin, consider any energy-minimizing sequence \( \{(u_1^n, u_2^n)\}_{n \geq 1} \) for \( E_{\gamma} \). Define \( Q_n \) and \( \gamma \) as in Lemma 2.14. Denoting the subsequence associated to \( \gamma \) again by \( \{(u_1^n, u_2^n)\}_{n \geq 1} \), suppose for now that \( \gamma = \sigma_1 + \sigma_2 \) (called the case of compactness).

Then for every \( k \in \mathbb{N} \) there exists \( t_k \in \mathbb{R} \) and points \( y_n \in \mathbb{R}^N \) such that for all sufficiently large \( n \),

\[
\|u_1^n\|_{L^2(y_n + B_{t_k}(0))}^2 + \|u_2^n\|_{L^2(y_n + B_{t_k}(0))}^2 > (\sigma_1 + \sigma_2) - \frac{1}{k}.
\]

(3.1)

For ease of notation, let us denote the shifted sequences \( w_1^n(x) = u_1^n(x + y_n) \) and \( w_2^n(x) = u_2^n(x + y_n) \) for \( x \in \mathbb{R}^N \). Then using (3.1), for every \( k \in \mathbb{N} \), we have that

\[
\|w_1^n\|_{L^2(B_{t_k}(0))}^2 + \|w_2^n\|_{L^2(B_{t_k}(0))}^2 > (\sigma_1 + \sigma_2) - \frac{1}{k}.
\]

(3.2)

Since the translated sequence \( \{(w_1^n, w_2^n)\} \) is bounded uniformly in the space \( H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N) \), so by taking a subsequence if necessary, one may assume that \( (w_1^n, w_2^n) \rightharpoonup (u_1, u_2) \) in \( H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N) \). Invoking the Fatou’s lemma as \( n \to \infty \), we have that \( \|u_1^n\|_{L^2(\mathbb{R}^N)}^2 + \|u_2^n\|_{L^2(\mathbb{R}^N)}^2 \leq \sigma_1 + \sigma_2 \). For each \( k \in \mathbb{N} \), the inclusion of \( H^\alpha(B_{t_k}(0)) \) into \( L^2(B_{t_k}(0)) \) is compact, so up to a subsequence, we may assume that \( (w_1^n, w_2^n) \to (u_1, u_2) \) strongly in \( L^2(B_{t_k}(0)) \). Moreover, using a standard diagonalization technique, one may assume that a single subsequence of the translated sequence \( \{(w_1^n, w_2^n)\}_{n \geq 1} \) has been chosen which enjoys this property for every \( k \). Now, by passing the limit as \( n \to \infty \) on both sides of (3.2), we obtain that

\[
\|u_1\|_{L^2(\mathbb{R}^N)}^2 + \|u_2\|_{L^2(\mathbb{R}^N)}^2 \geq \|u_1^n\|_{L^2(B_{t_k}(0))}^2 + \|u_2^n\|_{L^2(B_{t_k}(0))}^2 \geq (\sigma_1 + \sigma_2) - \frac{1}{k}.
\]

Since \( \|u_1^n\|_{L^2(\mathbb{R}^N)}^2 + \|u_2^n\|_{L^2(\mathbb{R}^N)}^2 \leq \sigma_1 + \sigma_2 \) and \( k \in \mathbb{N} \) was arbitrary, this last inequality in turn implies that \( \|u_1\|_{L^2(\mathbb{R}^N)}^2 + \|u_2\|_{L^2(\mathbb{R}^N)}^2 = \sigma_1 + \sigma_2 \). Thus \( (w_1^n, w_2^n) \to (u_1, u_2) \) strongly in \( L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \). Now, using the interpolation inequality for \( L^p \)-norms and the Sobolev inequality, we have that

\[
\|w_1^n - u_1\|_{L^p(\mathbb{R}^N)} \leq \|w_1^n - u_1\|_{L^2(\mathbb{R}^N)}^{\frac{1}{p_1}} \|w_1^n - u_1\|_{L^2(\mathbb{R}^N)}^{\frac{1}{p_2}} \leq C \|w_1^n - u_1\|_{H^\alpha(\mathbb{R}^N)} \|w_1^n - u_1\|_{H^\alpha(\mathbb{R}^N)} \leq C \|w_1^n - u_1\|_{L^2(\mathbb{R}^N)},
\]

(3.3)

where \( \lambda \) satisfies \( \frac{1}{p_1} = \frac{\lambda}{2} + \frac{1-\lambda}{2}. \) Making use of the fact that \( w_1^n \to u_1 \) strongly in \( L^2(\mathbb{R}^N) \) and the estimate (3.3), we obtain that \( \|w_1^n\|_{L^p(\mathbb{R}^N)} \to \|u_1\|_{L^p(\mathbb{R}^N)} \) as
$n \to \infty$. Similarly, we have that $\|w_2^n\|_{L^p(R^N)} \to \|u_2\|_{L^p(R^N)}$ as $n \to \infty$. We also have
\[
\int_{R^N} |w_1^n|^r_1 |w_2^n|^r_2 \, dx \to \int_{R^N} |u_1|^r_1 |u_2|^r_2 \, dx
\]
(3.4) as $n \to \infty$. By Lemma 2.7, since the translated sequence \{$(w_1^n, w_2^n)$\}$_{n \geq 1}$ is bounded in $H^\alpha(R^N) \times H^\alpha(R^N)$, (3.4) can be proved by writing
\[
\int_{R^N} (|w_1^n|^r_1 |w_2^n|^r_2 - |u_1|^r_1 |u_2|^r_2) \, dx \leq \int_{R^N} |w_1^n|^r_1 (|w_2^n|^r_2 - |u_2|^r_2) \, dx
\]
+ \int_{R^N} |u_2|^r_2 (|w_1^n|^r_1 - |u_1|^r_1) \, dx.

and estimating separately the limiting behavior of the integrals on the right-hand side as $n \to \infty$. Now, invoking the Fatou’s lemma once again, we obtain that
\[
E_\sigma = \lim_{n \to \infty} E(w_1^n, w_2^n) \geq E(u_1, u_2),
\]
whence $E(u_1, u_2) = E_\sigma$. Thus $(u_1, u_2)$ is a minimizer for $E_\sigma$. Finally, since equality in (3.5) in fact implies that
\[
\lim_{n \to \infty} \left( \|D^\alpha w_1^n\|^2_{L^2(R^N)} + \|D^\alpha w_2^n\|^2_{L^2(R^N)} \right) = \|D^\alpha u_1\|^2_{L^2(R^N)} + \|D^\alpha u_2\|^2_{L^2(R^N)},
\]
and therefore, $(w_1^n, w_2^n) \to (u_1, u_2)$ strongly in $H^\alpha(R^N) \times H^\alpha(R^N)$. Thus, in order to complete the proof of existence theorem, one needs to show that $\gamma = \sigma_1 + \sigma_2$ is the only possibility. To see this, we claim that the following holds for any energy-minimizing sequence:

(i) $\gamma > 0$
(ii) $\gamma \notin (0, \sigma_1 + \sigma_2)$.

The possibility $\gamma = 0$ is called the case of vanishing and the possibility $\gamma \in (0, \sigma_1 + \sigma_2)$ is called the dichotomy. To prove $\gamma > 0$, we argue by contradiction. If $\gamma = 0$ and \{$(u_1^n, u_2^n)$\}$_{n \geq 1}$ is the subsequence associated with $\gamma$, then for all $R > 0$, one has
\[
\lim_{n \to \infty} \left( \sup_{y \in R^N} \int_{y + B_R(0)} (|u_1^n(x)|^2 + |u_2^n(x)|^2) \, dx \right) = 0.
\]
By the part (i) of Lemma 2.14, the sequences \{|$u_1^n$|\} and \{|$u_2^n$|\} are both bounded in the space $H^\alpha(R^N)$. Since $2 < p_1, p_2 < 2 + \frac{4N}{N - 2\alpha}$, Lemma 2.8 proves that $\|u_1^n\|_{L^{p_1}(R^N)} \to 0$ and $\|u_2^n\|_{L^{p_2}(R^N)} \to 0$ as $n \to \infty$. Since $2 < 2r_1, 2r_2 < \frac{2N}{N - 2\alpha}$, using the Hölder inequality and another use of Lemma 2.8, it also follows that
\[
\int_{R^N} |f_1^n|^{r_1} |f_2^n|^{r_2} \, dx \leq \left( \int_{R^N} |f_1^n|^{2r_1} \, dx \right)^{1/2} \left( \int_{R^N} |f_2^n|^{2r_2} \, dx \right)^{1/2} \to 0
\]
as \( n \to \infty \). Taking account into these convergence properties, we obtain that the infimum of the energy satisfies
\[
E_\sigma = \lim_{n \to \infty} E(f_1^n, f_2^n) \geq \liminf_{n \to \infty} \int_{\mathbb{R}^N} \frac{1}{2} \left( |D^\alpha f_1^n|^2 + |D^\alpha f_2^n|^2 \right) \, dx \geq 0,
\]
which contradicts the Lemma 2.16 and hence, \( \gamma > 0 \).

Next we show that dichotomy is also not an option for an energy-minimizing sequence. Suppose that \( \gamma \in (0, \sigma_1 + \sigma_2) \) holds. Let \( \tau = (\tau_1, \tau_2) \) be the same ordered pair that was found in the part (iii) of Lemma 2.14 and define \( \rho \) by \( \rho = \sigma - \tau \). Then, we have that \( \tau + \rho = \beta \in \mathbb{R}^+ \times \mathbb{R}^+ \) and \( \tau, \rho \neq \{0\} \). Consequently, Lemma 2.13 obtains that \( E_\beta < E_\tau + E_\rho \). On the other hand, the part (iii) of Lemma 2.14 implies that \( E_\sigma \geq E_\tau + E_{\sigma - \tau} \), which is same as \( E_\beta \geq E_\tau + E_\rho \), a contradiction. Therefore, \( \gamma \in (0, \sigma_1 + \sigma_2) \) cannot occur here. This completes the proof of Theorem 3.

The proof of part (iv) of the existence theorem follows a similar argument as in Theorem 1.2(v) of [3] and we do not repeat here.

4 Proof of stability result

The stability result can be proved by a classical argument, which we repeat for completeness. If the claim were not true, then there would exist a number \( \varepsilon > 0 \), a sequence \( \{(\Psi_1^n(0), \Psi_2^n(0))\}_{n \geq 1} \subset H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N) \), and \( t_n \geq 0 \) such that for all \( n \),
\[
\inf_{(u_1, u_2) \in V(\tau)} \| (\Psi_1^n(0), \Psi_2^n(0)) - (u_1, u_2) \|_\alpha < \frac{1}{n},
\]
and
\[
\inf_{(u_1, u_2) \in V(\tau)} \| (\Psi_1^n(t_n), \Psi_2^n(t_n)) - (u_1, u_2) \|_\alpha \geq \varepsilon,
\]
where \( (\Psi_1^n(t_n), \Psi_2^n(t_n)) \) is the solution of (1.1) emanating from \( (\Psi_1^n(0), \Psi_2^n(0)) \). Since \( (u_1, u_2) \in V(\tau) \), one has that
\[
E(\Psi_1^n(0), \Psi_2^n(0)) \to E_\tau = E(u_1, u_2), \quad \| \Psi_j^n(0) \|_{L^2(\mathbb{R}^N)} \to \sqrt{\tau_j}, \quad j = 1, 2,
\]
as \( n \to \infty \). Since the energy \( E(f_1, f_2) \) and \( \int_{\mathbb{R}^N} |f_j|^2 \, dx \) both are conserved functionals, therefore the sequence of functions
\[
\{(\Psi_1^n(t_n), \Psi_2^n(t_n))\}_{n \geq 1} \subset H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)
\]
is a minimizing sequence for the problem \( (E, \mathcal{S}_\tau) \). By the existence theorem, this sequence must be relatively compact in \( H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N) \) up to a translation.
Hence, there exists a subsequence \( \{(\Psi_{nk}(t_{nk}), \Psi_{nk}(t_{nk}))\}_{k \geq 1} \), a sequence of points \( x(t_{nk}) \in \mathbb{R}^N \), and an element \((u_1, u_2) \in V(\tau)\) such that

\[
\inf_{(u_1, u_2) \in V(\tau)} \| (\Psi_{1k}(t_{nk}, \cdot + x(t_{nk})), \Psi_{2k}(t_{nk}, \cdot + x(t_{nk}))) - (u_1, u_2) \|_\alpha \to 0
\]

as \( k \to \infty \), which is a contradiction, and hence the set \( V(\tau) \) is stable. \( \square \)

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