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On the generation of the coefficient field of a newform by a single Hecke eigenvalue

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1. Statement of the results

The principal result of this paper is the following theorem. Its corollaries below completely resolve the question of the density of the set of primes \( p \) such that the \( p \)-th coefficient of \( f \) generates a given field.

**Theorem 1.1.** Let \( f \) be a newform (i.e., a new normalized cuspidal Hecke eigenform) of weight \( k \geq 2 \), level \( N \) and Dirichlet character \( \chi \) which does not have complex multiplication (CM, see [5, p. 48]). Let \( E_f = \mathbb{Q}(a_n(f) : (n,N) = 1) \) be the field of coefficients of \( f \) and \( F_f = \mathbb{Q}\left(\frac{a_n(f)^2}{\chi(n)} : (n,N) = 1\right) \).

The set \( \{ \text{prime} : \mathbb{Q}\left(\frac{a_p(f)^2}{\chi(p)}\right) = F_f \} \) has density 1.
A twist of $f$ by a Dirichlet character $\epsilon$ is said to be inner if there exists a (necessarily unique) field automorphism $\sigma_\epsilon : E_f \to E_f$ such that

\begin{equation}
    a_p(f \otimes \epsilon) = a_p(f)\epsilon(p) = \sigma_\epsilon(a_p(f))
\end{equation}

for almost all primes $p$. For a discussion of inner twists we refer the reader to [5, §3] and [6, §3]. Here we give several statements that will be needed for the sequel. The $\sigma_\epsilon$ belonging to the inner twists of $f$ form an abelian subgroup $\Gamma$ of the automorphism group of $E_f$. The field $F_f$ is the subfield of $E_f$ fixed by $\Gamma$. It is well-known that the coefficient field $E_f$ is either a CM field or totally real. In the former case, the formula

\begin{equation}
    a_p(f) = \chi(p) - 1 a_p(f)
\end{equation}

which is easily derived from the behaviour of the Hecke operators under the Petersson scalar product, shows that $f$ has a nontrivial inner twist by $\chi - 1$ with $\sigma_{\chi - 1}$ being complex conjugation. If $N$ is square free, $k = 2$ and the Dirichlet character $\chi$ of $f$ is the trivial character, then there are no nontrivial inner twists of $f$.

**Lemma 1.1.** The field $F_f$ is totally real and $\mathbb{Q}(a_p(f))$ contains $a_p(f)^2$.  

**Proof.** Equation 1.2 gives $\frac{a_p(f)^2}{\chi(p)} = a_p(f)a_p(f)$, whence $F_f$ is totally real. Since every subfield of a CM field is preserved by complex conjugation, $\mathbb{Q}(a_p(f))$ contains $a_p(f)$, thus it also contains $\frac{a_p(f)^2}{\chi(p)}$. \hfill $\square$

We immediately obtain the following two results.

**Corollary 1.1.** Let $f$ and $E_f$ be as in Theorem 1.1. If $f$ does not have any nontrivial inner twists (e.g. if $k = 2$, $N$ is square free and $\chi$ is trivial), then the set

\begin{equation}
    \{p \text{ prime} : \mathbb{Q}(a_p(f)) = E_f\}
\end{equation}

has density 1.

**Corollary 1.2.** Let $f$ and $F_f$ be as in Theorem 1.1. The set

\begin{equation}
    \{p \text{ prime} : F_f \subseteq \mathbb{Q}(a_p(f))\}
\end{equation}

has density 1.

To any subgroup $H$ of $\Gamma$, we associate a number field $K_H$ as follows. Consider the inner twists as characters of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and let $\epsilon_1, \ldots, \epsilon_r$ be the inner twists such that $H = \{\sigma_{\epsilon_1}, \ldots, \sigma_{\epsilon_r}\}$. Let $K_H$ be the minimal number field on which all $\epsilon_i$ for $1 \leq i \leq r$ are trivial, i.e. the field such that its absolute Galois group is the kernel of the map

\begin{equation}
    \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\epsilon_1 \cdots \epsilon_r} \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times.
\end{equation}

We use this field to express the density of the set of primes $p$ such that $a_p(f)$ is contained in a given subfield of the coefficient field.
Corollary 1.3. Let \( f, E_f \) and \( F_f \) be as in Theorem 1.1. Let \( L \) be any subfield of \( E_f \). Let \( M_L \) be the set
\[
\{ \text{prime } p : a_p(f) \in L \}.
\]
(a) If \( L \) does not contain \( F_f \), then \( M_L \) has density 0.
(b) If \( L \) contains \( F_f \), then \( L = E_f^H \) for some subgroup \( H \subseteq \Gamma \) and \( M_L \) has density \( \frac{1}{[K_H : Q]} \).

Proof. Suppose first that \( L \) does not contain \( F_f \). Then \( a_p(f) \in L \) implies that \( F_f \) is not a subfield of \( Q(a_p(f)) \). Thus by Corollary 1.2, \( M_L \) is a subset of a set of density 0 and is consequently itself of density 0. We now assume that \( L = E_f^H \). Then we have
\[
M_L = \{ \text{prime } p : \sigma(a_p(f)) = a_p(f) \forall \sigma \in H \}
= \{ \text{prime } p : a_p(f) \epsilon_i(p) = a_p(f) \forall i \in \{1, \ldots, r\} \}.
\]
Since the set of \( p \) with \( a_p(f) = 0 \) has density 0 (see for instance [7], p. 174), the density of \( M_L \) is equal to the density of
\[
\{ \text{prime } p : \epsilon_i(p) = 1 \forall i \in \{1, \ldots, r\} \} = \{ \text{prime } p : p \text{ splits completely in } K_H \},
\]
yielding the claimed formula. \( \square \)

A complete answer as to the density of the set of \( p \) such that \( a_p(f) \) generates a given field \( L \subseteq E_f \) is given by the following immediate result.

Corollary 1.4. Let \( f, E_f \) and \( F_f \) be as in Theorem 1.1. Let \( L \) be \( E_f^H \) with \( H \) some subgroup of \( \Gamma \). The density of the set
\[
\{ \text{prime } p : Q(a_p(f)) = L \}.
\]
is equal to the density of the set
\[
\{ \text{prime } p : \epsilon_i(p) = 1 \forall i \in \{1, \ldots, r\} \text{ and } \epsilon_j(p) \neq 1 \forall j \in \{r + 1, \ldots, s\} \},
\]
where the \( \epsilon_j \) for \( j \in \{r + 1, \ldots, s\} \) are the inner twists of \( f \) that belong to elements of \( \Gamma - H \).

This corollary means that the above density is completely determined by the inner twists of \( f \). We illustrate this by giving two examples. In weight 2 there is a newform on \( \Gamma_0(63) \) with coefficient field \( Q(\sqrt{3}) \). It has an inner twist by the Legendre symbol \( p \mapsto (\frac{p}{3}) \). Consequently, the field \( F_f \) is \( Q \) and the set of \( p \) such that \( a_p(f) \in Q \) has density \( \frac{1}{2} \).

For the next example we consider the newform of weight 2 on \( \Gamma_0(512) \) whose coefficient field has degree 4 over \( Q \). More precisely, the coefficient field \( E_f \) is \( Q(\sqrt{2}, \sqrt{3}) \) and \( F_f = Q \). Hence, \( \Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{1, \sigma_1, \sigma_2, \sigma_3\} \). There are thus nontrivial inner twists \( \epsilon_1, \epsilon_2 \) and \( \epsilon_3 \), all of which are quadratic, as their values must be contained in the totally real
field $E_f$. As $\sigma_1\sigma_2 = \sigma_3$, it follows that $\epsilon_1(p)\epsilon_2(p) = \epsilon_3(p)$. This equation already excludes the possibility that all $\epsilon_i(p) \neq 1$, whence there is not a single $p$ such that $a_p(f)$ generates $E_f$. Furthermore, the set of $p$ such that $a_p$ generates the quadratic field $E_f^{(\sigma_1)}$ is equal to the density of \(\{p \text{ prime} : \epsilon_1(p) = 1 \text{ and } \epsilon_2(p) \neq 1\}\), which is $\frac{1}{4}$. Similar arguments apply to the other two quadratic fields. The set of $p$ such that $a_p \in \mathbb{Q}$ also has density $\frac{1}{4}$.

In the literature there are related but weaker results concerning Corollary 1.1, which are situated in the context of Maeda’s conjecture, i.e., they concern the case of level 1 and assume that the space $S_k(1)$ of cusp forms of weight $k$ and level 1 consists of a single Galois orbit of newforms (see, e.g., [4] and [1]). We now show how Corollary 1.1 extends the principal results of these two papers.

Let $f$ be a newform of level $N$, weight $k \geq 2$ and trivial Dirichlet character $\chi = 1$ which neither has CM nor nontrivial inner twists. This is for instance true when $N = 1$. Let $\mathbb{T}$ be the $\mathbb{Q}$-algebra generated by all $T_n$ with $n \geq 1$ inside $\text{End}(S_k(N,1))$ and let $\mathfrak{P}$ be the kernel of the $\mathbb{Q}$-algebra homomorphism $\mathbb{T} \xrightarrow{T_n \mapsto a_n(f)} E_f$. As $f$ is new, the map $\mathbb{T}/\mathfrak{P} \xrightarrow{T_n \mapsto a_n(f)} E_f$ is a ring isomorphism with $\mathbb{T}/\mathfrak{P}$ the localization of $\mathbb{T}$ at $\mathfrak{P}$. Non canonically $\mathbb{T}/\mathfrak{P}$ is also isomorphic as a $\mathbb{T}/\mathfrak{P}$-module (equivalently as an $E_f$-vector space) to its $\mathbb{Q}$-linear dual, which can be identified with the localization at $\mathfrak{P}$ of the $\mathbb{Q}$-vector space $S_k(N,1; \mathbb{Q})$ of cusp forms in $S_k(N,1)$ with $q$-expansion in $\mathbb{Q}[[q]]$. Hence, $\mathbb{Q}(a_p(f)) = E_f$ precisely means that the characteristic polynomial $P_p \in \mathbb{Q}[X]$ of $T_p$ acting on the localization at $\mathfrak{P}$ of $S_k(N,1; \mathbb{Q})$ is irreducible. Corollary 1.1 hence shows that the set of primes $p$ such that $P_p$ is irreducible has density 1.

This extends Theorem 1 of [4] and Theorem 1.1 of [1]. Both theorems restrict to the case $N = 1$ and assume that there is a unique Galois orbit of newforms, i.e., a unique $\mathfrak{P}$, so that no localization is needed. Theorem 1 of [4] says that

$$\# \{ p < X \text{ prime} : P_p \text{ is irreducible in } \mathbb{Q}[X] \} \gg \frac{X}{\log X}$$

and Theorem 1.1 of [1] states that there is $\delta > 0$ such that

$$\# \{ p < X \text{ prime} : P_p \text{ is reducible in } \mathbb{Q}[X] \} \ll \frac{X}{(\log X)^{1+\delta}}.$$  

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2. Group theoretic input

Lemma 2.1. Let $q$ be a prime power and $\epsilon$ a generator of the cyclic group $\mathbb{F}_q^\times$.
(a) The conjugacy classes $c$ in $\text{GL}_2(\mathbb{F}_q)$ have the following four kinds of representatives:
\[
S_a = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad T_a = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}, \quad U_{a,b} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad V_{x,y} = \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}
\]
where $a \neq b$, and $y \neq 0$.
(b) The number of elements in each of these conjugacy classes are: $1$, $q^2 - 1$, $q^2 + q$, and $q^2 - q$, respectively.

Proof. See Fulton-Harris [3], page 68. $\square$

We use the notation $[g]_G$ for the conjugacy class of $g$ in $G$.

Proposition 2.1. Let $q$ be a prime power and $r$ a positive integer. Let further $R \subseteq \widetilde{R} \subseteq \mathbb{F}_q^\times$ be subgroups. Put $\sqrt{\widetilde{R}} = \{ s \in \mathbb{F}_q^\times : s^2 \in \widetilde{R} \}$. Set $H = \{ g \in \text{GL}_2(\mathbb{F}_q) : \det(g) \in R \}$ and let $G \subseteq \{ g \in \text{GL}_2(\mathbb{F}_q^r) : \det(g) \in \widetilde{R} \}$ be any subgroup such that $H$ is a normal subgroup of $G$. Then the following statements hold.
(a) The group $G/(G \cap \mathbb{F}_q^\times)$ (with $\mathbb{F}_q^\times$ identified with scalar matrices) is either equal to $\text{PSL}_2(\mathbb{F}_q)$ or to $\text{PGL}_2(\mathbb{F}_q)$. More precisely, if we let $\{ s_1, \ldots, s_n \}$ be a system of representatives for $\sqrt{\widetilde{R}}/R$, then for all $g \in G$ there is $i$ such that $g \begin{pmatrix} s_{i}^{-1} & 0 \\ 0 & s_{i}^{-1} \end{pmatrix} \in G \cap \text{GL}_2(\mathbb{F}_q)$ and $\begin{pmatrix} s_{i} & 0 \\ 0 & s_{i} \end{pmatrix} \in G$.
(b) Let $g \in G$ such that $g \begin{pmatrix} s_{i}^{-1} & 0 \\ 0 & s_{i}^{-1} \end{pmatrix} \in G \cap \text{GL}_2(\mathbb{F}_q)$ and $\begin{pmatrix} s_{i} & 0 \\ 0 & s_{i} \end{pmatrix} \in G$. Then
\[
[g]_G = [g \begin{pmatrix} s_{i}^{-1} & 0 \\ 0 & s_{i}^{-1} \end{pmatrix}]_{G \cap \text{GL}_2(\mathbb{F}_q)} \begin{pmatrix} s_{i} & 0 \\ 0 & s_{i} \end{pmatrix}.
\]
(c) Let $P(X) = X^2 - aX + b \in \mathbb{F}_q[X]$ be a polynomial. Then the inequality
\[
\sum_C |C| \leq 2|\widetilde{R}/R|(q^2 + q)
\]
holds, where the sum runs over the conjugacy classes $C$ of $G$ with characteristic polynomial equal to $P(X)$. 
Proof. (a) The classification of the finite subgroups of $\text{PGL}_2(F_q)$ yields that the group $G/(G \cap \mathbb{F}_q^\times)$ is either $\text{PGL}_2(F_q^u)$ or $\text{PSL}_2(F_q^u)$ for some $u \mid r$. This, however, can only occur with $u = 1$, as $\text{PSL}_2(F_q^u)$ is simple. The rest is only a reformulation.

(b) This follows from (a), since scalar matrices are central.

(c) From (b) we get the inclusion

$$\bigcup C \subseteq \bigcup_{i=1}^n D \left( \begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix} \right),$$

where $C$ runs over the conjugacy classes of $G$ with characteristic polynomial equal to $P(X)$ and $D$ runs over the conjugacy classes of $G \cap \text{GL}_2(F_q)$ with characteristic polynomial equal to $X^2 - a s_i^{-1} X + b s_i^{-2}$ (such a conjugacy class is empty if the polynomial is not in $F_q[X]$). The group $G \cap \text{GL}_2(F_q)$ is normal in $\text{GL}_2(F_q)$, as it contains $\text{SL}_2(F_q)$. Hence, any conjugacy class of $\text{GL}_2(F_q)$ either has an empty intersection with $G \cap \text{GL}_2(F_q)$ or is a disjoint union of conjugacy classes of $G \cap \text{GL}_2(F_q)$. Consequently, by Lemma 2.1, the disjoint union $\bigcup D$ is equal to one of

(i) $[U_{a,b}]_{\text{GL}_2(F_q)} \left( \begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix} \right),$  

(ii) $[V_{x,y}]_{\text{GL}_2(F_q)} \left( \begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix} \right)$ or  

(iii) $[S_a]_{\text{GL}_2(F_q)} \left( \begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix} \right) \sqcup [T_a]_{\text{GL}_2(F_q)} \left( \begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix} \right).$

Still by Lemma 2.1, the first set contains $q^2 + q$, the second set $q^2 - q$ and the third one $q^2$ elements. Hence, the set $\bigcup C$ contains at most $2|\tilde{R}/R|(q^2 + q)$ elements. □

3. Proof

The proof of Theorem 1.1 relies on the following important theorem by Ribet, which, roughly speaking, says that the image of the mod $\ell$ Galois representation attached to a fixed newform is as big as it can be for almost all primes $\ell$.

**Theorem 3.1 (Ribet).** Let $f$ be a Hecke eigenform of weight $k \geq 2$, level $N$ and Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. Suppose that $f$ does not have CM. Let $E_f$ and $F_f$ be as in Theorem 1.1 and denote by $\mathcal{O}_{E_f}$ and $\mathcal{O}_{F_f}$ the corresponding rings of integers. For almost all prime numbers $\ell$ the following statement holds:

Let $\mathcal{L}$ be a prime ideal of $\mathcal{O}_{E_f}$ dividing $\ell$. Put $\mathcal{L} = \mathcal{L} \cap \mathcal{O}_{F_f}$ and $\mathcal{O}_{F_f}/\mathcal{L} \cong \mathbb{F}$. Consider the residual Galois representation

$$\bar{\rho}_{f,\mathcal{L}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathcal{O}_{E_f}/\mathcal{L})$$
attached to \( f \). Then the image \( \bar{\rho}_{f,\tilde{L}}(\text{Gal}(\overline{\mathbb{Q}}/K_\Gamma)) \) is conjugate to
\[
\{ g \in \text{GL}_2(\mathbb{F}) : \det(g) \in \mathbb{F}_\ell^{\times(k-1)} \},
\]
where \( K_\Gamma \) is the field defined in Section 1.

Proof. It suffices to take Ribet [6, Thm. 3.1] mod \( \tilde{L} \).

\[ \square \]

Theorem 3.2. Let \( f \) be a non-CM newform of weight \( k \geq 2 \), level \( N \) and Dirichlet character \( \chi \). Let \( F_f \) be as in Theorem 1.1 and let \( L \subset F_f \) be any proper subfield. Then the set
\[
\left\{ p \text{ prime} : \frac{a_p(f)^2}{\chi(p)} \in L \right\}
\]
has density zero.

Proof. Let \( L \subset F_f \) be a proper subfield and \( \mathcal{O}_L \) its integer ring. We define the set
\[
S := \{ \mathcal{L} \subset \mathcal{O}_{F_f} : \text{prime ideal} : \mathcal{O}_{F_f}/\mathcal{O}_L/(L \cap \mathcal{L}) \geq 2 \}.
\]
Notice that this set is infinite. For, if it were finite, then all but finitely many primes would split completely in the extension \( F_f/L \), which is not the case by Chebotarev’s density theorem.

Let \( \mathcal{L} \in S \) be any prime, \( \ell \) its residue characteristic and \( \tilde{\mathcal{L}} \) a prime of \( \mathcal{O}_{E_\ell} \) lying over \( \mathcal{L} \). Put \( \mathbb{F}_q = \mathcal{O}_{L}/(L \cap \mathcal{L}) \), \( \mathbb{F}_{q^r} = \mathcal{O}_{F_f}/\mathcal{L} \) and \( \mathbb{F}_{q^{rs}} = \mathcal{O}_{E_\ell}/\tilde{\mathcal{L}} \). We have \( r \geq 2 \). Let \( W \) be the subgroup of \( \mathbb{F}_q^{\times(rs)} \) consisting of the values of \( \chi \) modulo \( \tilde{\mathcal{L}} \); its size \( |W| \) is less than or equal to \(|(\mathbb{Z}/N\mathbb{Z})^{\times}|\).

Let \( R = \mathbb{F}_\ell^{\times(k-1)} \) be the subgroup of \((k-1)\)st powers of elements in the multiplicative group \( \mathbb{F}_\ell^{\times} \) and let \( \tilde{R} = \langle R, W \rangle \subset \mathbb{F}_{q^{rs}}^{\times} \). The size of \( \tilde{R} \) is less than or equal to \(|R| \cdot |W|\). Let \( H = \{ g \in \text{GL}_2(\mathbb{F}_{q^r}) : \det(g) \in \tilde{R} \} \) and \( G = \text{Gal}(\overline{\mathbb{Q}}_{\ker \bar{\rho}_{f,\tilde{L}}}/\mathbb{Q}) \). By Galois theory, \( G \) can be identified with the image of the residual representation \( \bar{\rho}_{f,\tilde{L}} \); and we shall make this identification from now on. By Theorem 3.1 we have the inclusion of groups
\[
H \subseteq G \subseteq \{ g \in \text{GL}_2(\mathbb{F}_{q^{rs}}) : \det(g) \in \tilde{R} \}
\]
with \( H \) being normal in \( G \).

If \( C \) is a conjugacy class of \( G \), by Chebotarev’s density theorem the density of
\[
\{ p \text{ prime} : [\bar{\rho}_{f,\tilde{L}}(\text{Frob}_p)]_G = C \}
\]
equals \(|C|/|G|\). We consider the set
\[
M_\mathcal{L} := \bigcup_C \{ p \text{ prime} : [\bar{\rho}_{f,\tilde{L}}(\text{Frob}_p)]_G = C \} \supseteq \left\{ p \text{ prime} : \frac{a_p(f)^2}{\chi(p)} \in \mathbb{F}_q \right\},
\]
where the reduction modulo $L$ of an element $x \in \mathcal{O}_{F_f}$ is denoted by $\bar{x}$ and $C$ runs over the conjugacy classes of $G$ with characteristic polynomials equal to some $X^2 - aX + b \in \mathbb{F}_{q^r}[X]$ such that
\[
a^2 \in \{ t \in \mathbb{F}_{q^r} : \exists u \in \mathbb{F}_q \exists w \in W : t = uw \}
\]
and automatically $b \in \bar{R}$. The set $M_L$ has the density $\delta(M_L) = \sum_C |C| |G|^{-1}$ with $C$ as before. There are at most $2q|W|^2 \cdot |R|$ such polynomials. We are now precisely in the situation to apply Prop. 2.1, Part (c), which yields the inequality
\[
\delta(M_L) \leq 4|W|^3 q(q^{2r} + q^r) = O\left(\frac{1}{q^{r-1}}\right) \leq O\left(\frac{1}{q^r}\right),
\]
where for the denominator we used $|G| \geq |H| = |R| \cdot |\text{SL}_2(\mathbb{F}_{q^r})|$. Since $q$ is unbounded for $L \in S$, the intersection $M := \bigcap_{L \in S} M_L$ is a set having a density and this density is 0. The inclusion
\[
\left\{ p \text{ prime} : \frac{a_p(f)^2}{\chi(p)} \in L \right\} \subseteq M
\]
finishes the proof.

Proof of Theorem 1.1. It suffices to apply Theorem 3.2 to each of the finitely many subextension of $F_f$. □

4. Reducibility of Hecke polynomials: questions

Motivated by a conjecture of Maeda, there has been some speculation that for every integer $k$ and prime number $p$, the characteristic polynomial of $T_p$ acting on $S_k(1)$ is irreducible. See, for example, [2], which verifies this for all $k < 2000$ and $p < 2000$. The most general such speculation might be the following question: if $f$ is a non-CM newform of level $N \geq 1$ and weight $k \geq 2$ such that some $a_p(f)$ generates the field $E_f = \mathbb{Q}(a_n(f) : n \geq 1)$, do all but finitely many prime-indexed Fourier coefficients $a_p(f)$ generate $E_f$? The answer in general is no. An example is given by the newform in level 63 and weight 2 that has an inner twist by $(\cdot 3)$. Also for non-CM newforms of weight 2 without nontrivial inner twists such that $[E_f : \mathbb{Q}] = 2$, we think that the answer is likely no.

Let $f \in S_k(\Gamma_0(N))$ be a newform of weight $k$ and level $N$. The degree of $f$ is the degree of the field $E_f$, and we say that $f$ is a reducible newform if $a_p(f)$ does not generate $E_f$ for infinitely many primes $p$.

For each even weight $k \leq 12$ and degree $d = 2, 3, 4$, we used [8] to find newforms $f$ of weight $k$ and degree $d$. For each of these forms, we computed the reducible primes $p < 1000$, i.e., the primes such $a_p(f)$ does not generate $E_f$. The result of this computation is given in Table 1. Table 2
contains the number of reducible primes $p < 10000$ for the first 20 newforms of degree 2 and weight 2. This data inspires the following question.

**Question 4.1.** If $f \in S_2(\Gamma_0(N))$ is a newform of degree 2, is $f$ necessarily reducible? That is, are there infinitely many primes $p$ such that $a_p(f) \in \mathbb{Z}$?

Tables 4–6 contain additional data about the first few newforms of given degree and weight, which may suggest other similar questions. In particular, Table 3 contains data for all primes up to $10^6$ for the first degree 2 form $f$ with $L(f, 1) \neq 0$, and for the first degree 2 form $g$ with $L(g, 1) = 0$. We find that there are 386 primes $< 10^6$ with $a_p(f) \in \mathbb{Z}$ and 309 with $a_p(g) \in \mathbb{Z}$.

**Question 4.2.** If $f \in S_2(\Gamma_0(N))$ is a newform of degree 2, can the asymptotic behaviour of the function

$$N(x) := \# \{ p \text{ prime} : p < x, a_p(f) \in \mathbb{Z} \}$$

be described as a function of $x$?

The authors intend to investigate these questions in a subsequent paper.

### Table 1. Counting Reducible Characteristic Polynomials

| $k$ | $d$ | $N$ | reducible $p < 1000$ |
|-----|-----|-----|---------------------|
| 2   | 2   | 23  | 13, 19, 23, 29, 43, 109, 223, 229, 271, 463, 673, 677, 883, 991 |
| 2   | 3   | 41  | 17, 41 |
| 2   | 4   | 47  | 47 |
| 4   | 2   | 11  | 11 |
| 4   | 3   | 17  | 17 |
| 4   | 4   | 23  | 23 |
| 6   | 2   | 7   | 7 |
| 6   | 3   | 11  | 11 |
| 6   | 4   | 17  | 17 |
| 8   | 2   | 5   | 5 |
| 8   | 3   | 17  | 17 |
| 8   | 4   | 11  | 11 |
| 10  | 2   | 5   | 5 |
| 10  | 3   | 7   | 7 |
| 10  | 4   | 13  | 13 |
| 12  | 2   | 5   | 5 |
| 12  | 3   | 7   | 7 |
| 12  | 4   | 21  | 3, 7 |
Table 2. First 20 Newforms of Degree 2 and Weight 2

| $k$ | $d$ | $N$ | #\{reducible $p < 10000$\} |
|-----|-----|-----|-----------------------------|
| 2   | 2   | 23  | 47                          |
| 2   | 2   | 29  | 42                          |
| 2   | 2   | 31  | 78                          |
| 2   | 2   | 35  | 48                          |
| 2   | 2   | 39  | 71                          |
| 2   | 2   | 43  | 43                          |
| 2   | 2   | 51  | 64                          |
| 2   | 2   | 55  | 95                          |
| 2   | 2   | 62  | 77                          |
| 2   | 2   | 63  | 622 (inner twist by $\left(\frac{3}{p}\right)$) |
| 2   | 2   | 65  | 43                          |
| 2   | 2   | 65  | 90                          |
| 2   | 2   | 67  | 51                          |
| 2   | 2   | 67  | 19                          |
| 2   | 2   | 68  | 53                          |
| 2   | 2   | 69  | 47                          |
| 2   | 2   | 73  | 43                          |
| 2   | 2   | 73  | 55                          |
| 2   | 2   | 74  | 52                          |
| 2   | 2   | 74  | 21                          |

Table 3. Newforms 23a and 67b: values of $\psi(x) = \#\{\text{reducible } p < x \cdot 10^6\}$

| $k$ | $d$ | $N$ | $r_{\text{ran}}$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 2   | 2   | 23  | 0                | 127 | 180 | 210 | 243 | 277 | 308 | 331 | 345 | 360 | 386 |
| 2   | 2   | 67  | 1                | 111 | 159 | 195 | 218 | 240 | 257 | 276 | 288 | 301 | 309 |

Table 4. First 5 Newforms of Degrees 3, 4 and Weight 2

| $k$ | $d$ | $N$ | #\{reducible $p < 10000$\} |
|-----|-----|-----|-----------------------------|
| 2   | 3   | 41  | 17, 41                      |
| 2   | 3   | 53  | 13, 53                      |
| 2   | 3   | 61  | 61, 2087                    |
| 2   | 3   | 71  | 23, 31, 71, 479, 647, 1013, 3181 |
| 2   | 3   | 71  | 13, 71, 509, 3613          |

| $k$ | $d$ | $N$ | #\{reducible $p < 10000$\} |
|-----|-----|-----|-----------------------------|
| 2   | 4   | 47  | 47                          |
| 2   | 4   | 95  | 5, 19                       |
| 2   | 4   | 97  | 97                          |
| 2   | 4   | 109 | 109, 4513                   |
| 2   | 4   | 111 | 3, 37                       |
### Table 5. First 5 Newforms of Degrees 2, 3 and Weight 4

| $k$ | $d$ | $N$ | reducible $p < 1000$ |
|-----|-----|-----|---------------------|
| 4   | 2   | 11  | 11                  |
| 4   | 2   | 13  | 13                  |
| 4   | 2   | 21  | 3, 7                |
| 4   | 2   | 27  | 3, 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, 103, 109, 127, 139, 151, 157, 163, 181, 193, 199, 211, 223, 229, 241, 271, 277, 283, 307, 313, 331, 337, 349, 367, 373, 379, 397, 409, 421, 433, 439, 457, 463, 487, 499, 523, 541, 547, 571, 577, 601, 607, 613, 619, 631, 643, 661, 673, 691, 709, 727, 733, 739, 751, 757, 769, 787, 811, 823, 829, 853, 859, 877, 883, 907, 919, 937, 967, 991, 997 (has inner twists) |
| 4   | 2   | 29  | 29                  |

### Table 6. Newforms on $\Gamma_0(389)$ of Weight 2

| $k$ | $d$ | $N$ | reducible $p < 10000$ |
|-----|-----|-----|-----------------------|
| 2   | 1   | 389 | none (degree 1 polynomials are all irreducible) |
| 2   | 2   | 389 | 5, 11, 59, 97, 157, 173, 223, 389, 653, 739, 859, 947, 1033, 1283, 1549, 1667, 2207, 2417, 2909, 3121, 4337, 5431, 5647, 5689, 5879, 6151, 6323, 6373, 6607, 6763, 7583, 7589, 8363, 9013, 9371, 9767 |
| 2   | 3   | 389 | 7, 13, 389, 503, 1303, 1429, 1877, 5443 |
| 2   | 6   | 389 | 19, 389 |
| 2   | 20  | 389 | 389 |
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