Direct multistep method for solving delay differential equation with boundary conditions

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Abstract. This study will consider the implementation of direct multistep method for solving Delay Differential Equations (DDEs) with boundary conditions. The approach of this method is solving DDEs directly using the proposed method without reducing to the system of first order. To execute the direct multistep method to solve Boundary Value Problems (BVPs), we apply the shooting technique by using Newton’s method to compute the guessing value. Some numerical examples are presented to show that the proposed method are capable for solving DDEs with boundary conditions.

1. Introduction
Delay differential equations (DDEs) have been used as mathematical model in engineering and science fields. Time delay exists in real life phenomenas such as in transport delay (driver reaction time), biological processes (cell division time) and pharmaceutical kinetics (body reaction’s to carbon dioxide) [1]. This study is focus on second order DDEs of pantograph delay. The general form is:

\[ y'' = f(x, y(x), y(qx), y'(qx)), x \in [a, b] \] (1)

with the boundary conditions:

\[ y(a) = \alpha, y(b) = \beta \] (2)

where \( q \) is a positive constant, \( 0 < q < 1 \). The difference between ordinary differential equations (ODEs) and DDEs is that the ODEs approximate the solutions at the current time while DDEs approximate at the pass time. Since 1970s, there were numerous studies have been done to solve DDEs with boundary conditions. However, not much studies have been done in recent years. Thus, this become a motivation in this paper. The first study was done in 1971 by Nevers and Schmitt [2] where the DDEs have been solved using Euler’s method by implementing shooting technique for BVPs. Bellen and Zennaro [3] solved these problems by using collocation procedure with polynomial and piecewise polynomial approximation while Agarwal and Chow [4] used finite difference method. Bakke and Jackiewicz [5] solved using the Richardson extrapolation while Qu and Agarwal [6] used subdivision approach. The previous studies on second order DDEs of pantograph delay with boundary conditions was done by Bica [7] by using fixed point technique,
the trapezoidal quadrature rule, and the cubic spline interpolation procedure. Then, Wazwaz et al. [8] have solved analytically the pantograph delay using Adomian decomposition method and the variational iteration method. From these literature reviews, it can be observed that multistep block methods have never been applied in solving directly second order DDEs of pantograph delay with boundary conditions. Thus, it become a motivation of this study.

2. Derivation of method

Figure 1. The two point block explicit order 4 method

The interval \([a, b]\) will be divided into a series of block as in Figure 1. In this study, two point block method is proposed where \(y_{i+1}\) and \(y_{i+2}\) are solved simultaneously at points \(x_{i+1}\) and \(x_{i+2}\) respectively for each block. The back values chosen for each block of order four are in diagonally manner which is at the first point \(y_{i+1}\), the back values are \(\{x_i, x_{i-1}, x_{i-2}, x_{i-3}\}\) and at the second point \(y_{i+2}\), the back values are \(\{x_{i+1}, x_i, x_{i-1}, x_{i-2}\}\). The first and second point will be integrated once and twice over the interval \([x_i, x_{i+1}]\) and \([x_i, x_{i+2}]\) respectively.

First point:
Integrate once:
\[
\int_{x_i}^{x_{i+1}} y''(x)dx = \int_{x_i}^{x_{i+1}} f(x, y(x), y'(x))dx
\]
\[
y'(x_{i+1}) = y'(x_i) + \int_{x_i}^{x_{i+1}} f(x, y(x), y'(x))dx
\]  (3)

Integrate twice:
\[
\int_{x_i}^{x_{i+1}} \int_{x_i}^{x} y''(x)dx dx = \int_{x_i}^{x_{i+1}} \int_{x_i}^{x} f(x, y(x), y'(x))dx dx
\]
by using integration by parts for the right hand side and rearrange the equation, we then get:
\[
y(x_{i+1}) = y(x_i) + hy'(x_i) + \int_{x_i}^{x_{i+1}} [xf(x, y(x), y'(x))]^{x_{i+1}}_{x_i} dx - \int_{x_i}^{x_{i+1}} xf(x, y(x), y'(x))dx
\]
then, after the substitution of limits:
\[
y(x_{i+1}) = y(x_i) + hy'(x_i) + \int_{x_i}^{x_{i+1}} x_{i+1}f(x, y(x), y'(x))dx - \int_{x_i}^{x_{i+1}} x_{i}f(x, y(x), y'(x))dx
\]
\[
- \int_{x_i}^{x_{i+1}} xf(x, y(x), y'(x))dx
\]
\[
y(x_{i+1}) = y(x_i) + hy'(x_i) + \int_{x_i}^{x_{i+1}} (x_{i+1} - x)f(x, y(x), y'(x))dx
\]  (4)
where \( h = x_{i+1} - x_i \).

Second point:
Integrate once:
\[
\int_{x_i}^{x_{i+2}} y''(x) dx = \int_{x_i}^{x_{i+2}} f(x, y(x), y'(x)) dx
\]
\[
y'(x_{i+2}) = y'(x_i) + \int_{x_i}^{x_{i+2}} f(x, y(x), y'(x)) dx
\]
\[
(5)
\]
Integrate twice:
\[
\int_{x_i}^{x_{i+2}} \int_{x_i}^{x} y''(x) dx dx = \int_{x_i}^{x_{i+2}} \int_{x_i}^{x} f(x, y(x), y'(x)) dx dx
\]
by using integration by parts for the right hand side and rearrange the equation, we then get:
\[
y(x_{i+2}) = y(x_i) + 2h y'(x_i) + \int_{x_i}^{x} \left[ x f(x, y(x), y'(x)) \right]^{x_{i+2}}_{x_i} dx - \int_{x_i}^{x_{i+2}} x f(x, y(x), y'(x)) dx
\]
then, after the substitution of limits:
\[
y(x_{i+2}) = y(x_i) + 2h y'(x_i) + \int_{x_i}^{x} x_{i+2} f(x, y(x), y'(x)) dx - \int_{x_i}^{x_{i+2}} x_i f(x, y(x), y'(x)) dx
\]
\[
y(x_{i+2}) = y(x_i) + 2h y'(x_i) + \int_{x_i}^{x_{i+2}} (x_{i+2} - x) f(x, y(x), y'(x)) dx
\]
\[
(6)
\]
where \( h = x_{i+1} - x_i \).

By the definition of Lagrange polynomial:
\[
L_{q,j}(x) = \prod_{n=0, n \neq j}^{q} \frac{x - x_{i-n}}{x_{i-j} - x_{i-n}}.
\]
where \( L_{q,j}(x_{i-n}) = \begin{cases} 1, & n = j \\ 0, & n \neq j \end{cases} \) for \( n = 1, 2, ..., q \).

Thus, \( f(x, y(x), y'(x)) \) can be approximated by:
\[
P_q(x) = \sum_{j=0}^{q} L_{q,j}(x) f_{i-j}
\]
where \( P_q(x) \) is the polynomial of degree at most \( q \).

By choosing \( x = x_{i+2} + sh \) and \( dx = hds \), then, the limit is changed from -2 to -1 for the first point and -2 to 0 for the second point. Thus, the two point block explicit method of order four
(2PBEM4) is obtained as below:

\[
y_{i+1} = y_i + \frac{h}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3})
\]

\[
y_i + 1 = y_i + hy_i + \frac{h^2}{360}(323f_i - 264f_{i-1} + 159f_{i-2} - 38f_{i-3})
\]

\[
y_i + 2 = y_i + \frac{h}{3}(8f_i + 5f_{i-1} - 4i_{i-2} - f_{i-3})
\]

\[
y_i + 2 = y_i + 2hy_i + \frac{h^2}{45}(62f_i + 24f_i + 6f_{i-1} - 2f_{i-2})
\]

(7)

3. Implementation of method

3.1. Delay differential equation

The idea of solving DDEs of pantograph delay is by taking previous solutions at delay arguments which are \( y(qx) \) and \( y'(qx) \) that already being computed previously or else use the Lagrange interpolation of order 5 to approximate the values of \( y(qx) \) and \( y'(qx) \) if there are no previous solutions computed.

3.2. Boundary value problems

Since the problems are of type BVPs thus we need to solve the BVPs by using shooting technique. The BVP is changed to initial value problem (IVP) and since there is only one initial value given at the boundary, thus we have to guess the other initial value \( y'(a) \) by using Newton’s method.

Suppose that (1) can be written in the form of such that the solution depends on both \( x \) and the parameter \( t \) as the following:

\[
y''(x,t) = f(x,y(x,t),y(qx,qt),y'(qx,qt))
\]

with initial conditions: \( y(a,t) = \alpha, y'(a,t) = t_1 \)

The parameters \( t = t_k \) is chosen such that

\[
\lim_{k \to \infty} y(b,t_k) - \beta = 0
\]

The first initial guessing, \( t_1 \) obtained by using:

\[
t_1 = \frac{\beta - \alpha}{b - a}
\]

Then, (8) will be differentiated by using partial derivative with respect to \( t \) and by assuming

\[
z(x,t) = \frac{\delta y}{\delta t}(x,t)
\]

thus, we will get

\[
z''(x,t) = \frac{\delta f}{\delta y}(x,y,y')z(x,t) + \frac{\delta f}{\delta y'}(x,y,y')z'(x,t)
\]

with initial conditions:

\[
z(a,t) = 0, z'(a,t) = 1
\]

Hence, the two IVPs, (8) and (9) will be solved simultaneously to obtain the two approximate solutions \( y(x) \) and \( z(x) \) in order to use the Newton’s method to approximate the next guessing value of \( t_k \). The formula for Newton’s method involved is:

\[
t_k = t_{k-1} - \frac{y(b,t_{k-1}) - \beta}{z(b,t_{k-1})}
\]

The process is repeated until the absolute error, \( |y(b,t_{k-1}) - \beta| \leq TOL \) where \( TOL \) is the tolerance that is chosen.
4. Analysis of method

4.1. Order of method

The linear multistep block method can be written in the following form [9]:

\[
\sum_{j=0}^{k} \alpha_j y_{i+j} = h \sum_{j=0}^{k} \beta_j y'_{i+j} + h^2 \sum_{j=0}^{k} \gamma_j f_{i+j}. \tag{10}
\]

The proposed method, two point block method for the second order DDEs (1) can be portrayed by the matrix difference equation as

\[
\alpha Y_N = h \beta Y'_N + h^2 \gamma F_N
\]

where \( Y_N = [ y_{i-3} \ y_{i-2} \ y_{i-1} \ y_i \ y_{i+1} \ y_{i+2} ]^T \), \( Y'_N = [ y'_{i-3} \ y'_{i-2} \ y'_{i-1} \ y'_i \ y'_{i+1} \ y'_{i+2} ]^T \) and \( F_N = [ f_{i-3} \ f_{i-2} \ f_{i-1} \ f_i \ f_{i+1} \ f_{i+2} ]^T \). The linear difference operator \( L \) is described as in Lambert [10]:

\[
L[y(x); h] = \sum_{j=0}^{k} [a_j y(x + jh) + b_j y'[(x + jh) - h^2 \gamma y''(x + jh)] \tag{11}
\]

Then, \( y(x + jh) \) and its derivatives \( y'[(x + jh), y''[(x + jh)] \) is expanded about \( x \) by using Taylor series. Thus, after collecting terms of (11) will yields as following:

\[
L[y(x); h] = C_0 y(x) + C_1 h y^{(1)}(x) + \ldots + C_r h^r y^{(r)}(x) + \ldots,
\]

where coefficients \( C_r \) are in terms of constants given as

\[
C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_s,
\]

\[
C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + s\alpha_s - (\beta_0 + \beta_1 + \beta_2 + \ldots + \beta_s),
\]

\[
C_2 = \frac{1}{2!} (\alpha_1 + 2^2 \alpha_2 + 3^2 \alpha_3 + \ldots + s^2 \alpha_s) - (\beta_1 + 2\beta_2 + 3\beta_3 + \ldots + s\beta_s) - (\gamma_0 + \gamma_1 + \gamma_2 + \ldots + s\gamma_s),
\]

\[
C_r = \frac{1}{r!} (\alpha_1 + 2^r \alpha_2 + 3^r \alpha_3 + \ldots + s^r \alpha_s) - \frac{1}{(r-1)!} (\beta_1 + 2^{r-1} \beta_2 + 3^{r-1} \beta_3 + \ldots + k^{r-1} \beta_s)
\]

\[- \frac{1}{(r-2)!} (\gamma_1 + 2^{r-2} \gamma_2 + 3^{r-2} \gamma_3 + \ldots + s^{r-2} \gamma_s),
\]

\[r = 3, 4, 5, \ldots \tag{12}
\]

where \( s \) is the column vector chosen in the matrix form.

Thus, the 2PBM4 formula (7) can be written in the following matrix difference form when the column vector, \( s = 5 \)

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
y_{i-3} \\
y_{i-2} \\
y_{i-1} \\
y_i \\
y_{i+1} \\
y_{i+2}
\end{bmatrix}
= h
\begin{bmatrix}
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y'_{i-3} \\
y'_{i-2} \\
y'_{i-1} \\
y'_i \\
y'_{i+1} \\
y'_{i+2}
\end{bmatrix}
\]
Thus, the method is said of order $p$ if $C_0 = C_1 = C_2 = ... = C_{p+1} = 0$ and $C_{p+2} \neq 0$. The value of $C_{p+2}$ is the error constant of the method. Since $C_6 \neq 0$ therefore $p = 4$ makes the proposed method, 2PBEM4 is of order four.

4.2. Consistency of method

The linear multistep method is said to be consistent if it has order $p \geq 1$ [10]. Since it has been proved that 2PBEM4 is of order four thus $p = 4 \geq 1$, hence 2PBEM4 is consistent.

4.3. Stability of method

Zero stability concerns the stability of the system only in the limit as $h \to 0$.

The linear multistep method (10) is said to be zero stable if the roots $R_j$ of the first characteristic polynomial $\rho(R)$ identified as $\rho(R) = \det[\sum_{i=0}^{k} A_{(i)} R^{k-i}] = 0$ will satisfy $|R_j| \leq 1$.

The formula for 2PBEM4 in (7) can written in the following matrix form:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y_{i+1}^j \\
y_{i+2}^j \\
y_{i+3}^j \\
y_{i+4}^j
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y_{i-1}^j \\
y_{i}^j \\
y_{i+1}^j \\
y_{i+2}^j
\end{bmatrix}
+ h^2 \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y_{i-1}^j \\
y_{i}^j \\
y_{i+1}^j \\
y_{i+2}^j
\end{bmatrix}
\]

Thus, the first characteristic polynomial of 2PBEM4 is

\[
\rho(R) = \det[\sum_{i=0}^{k} A_{(i)} R^{k-i}] = \det[A_0 R - A_1] = 0
\]
also using the Lipschitz condition will yields:

\[ R^2(R - 1)^2 = 0 \]

and

\[ R = 0, 0, 1, 1 \]

where \( A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \) and \( A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \).

Since \( |R_j| \leq 1 \), the proposed method is said to be zero stable.

### 4.4. Convergence analysis

Definition 1: \( \lim y(x_i) = Y(x_i) \) is the convergence condition for the approximates where \( y(x_i) \) is the approximate solution and \( Y(x_i) \) is the exact solution.

Consider the exact solution as follows:

\[
Y_{i+1} = Y_i + \frac{h}{24}(55F_i - 59F_{i-1} + 37F_{i-2} - 9F_{i-3}) + \frac{251}{720}h^5Y^{(5)}(\xi_i)
\]

\[
y_{i+1} = y_i + hY'_i + \frac{h^2}{360}(323F_i - 264F_{i-1} + 159F_{i-2} - 38F_{i-3}) + \frac{3}{32}h^5Y^{(5)}(\xi_i)
\]

\[
y'_{i+2} = y'_i + \frac{h}{3}(8F_{i+1} - 5F_i + 4F_{i-1} - F_{i-2}) + \frac{29}{90}h^5Y^{(5)}(\xi_i)
\]

\[
y_{i+2} = y_i + 2hY'_i + \frac{h^2}{45}(62F_{i+1} + 24F_i + 6F_{i-1} - 2F_{i-2}) + \frac{1}{18}h^5Y^{(5)}(\xi_i)
\]

Consider the approximate solution as (7) and the Lipschitz condition is as below:

\[
|F(x_i, Y_i, Y'_i) - f(x_i, y_i, y'_i)| \leq L \max \{|Y_i - y_i|, |Y'_i - y'_i|\}
\]

where \( L > 0 \). Hence, by subtracting the exact solution (14) and approximate solution (7) and also using the Lipschitz condition will yields:

\[
|w_i| \geq \left( \frac{24}{24 + 55hL} \right)|w_{i+1}| + \left( \frac{4 - 4hL}{24 + 55hL} - 1 \right)|w_{i+2}| + \left( \frac{24 + 18hL}{24 + 55hL} - 1 \right)|w_{i-1}|
\]

\[
+ \left( \frac{24 + 46hL}{24 + 55hL} - 1 \right)|w_{i-3} - \frac{251}{30(24 + 55hL)}h^5Y^{(5)}(\xi_i)
\]

\[
|d_i| \geq \left( \frac{360}{360 + 323h^2L} \right)|d_{i+1}| - \left( \frac{360 + 59h^2L}{360 + 323h^2L} - 1 \right)|d_{i-1}|
\]

\[
+ \left( \frac{360 + 164h^2L}{360 + 323h^2L} - 1 \right)|d_{i-2} + \left( \frac{360 + 285h^2L}{360 + 323h^2L} - 1 \right)|d_{i-3} - \frac{135}{4(360 + 323h^2L)}h^5Y^{(5)}(\xi_i)
\]

\[
|w_i| \geq \left( \frac{3}{3 + 5hL} \right)|w_{i+1}| + \left( \frac{3 - 3hL}{3 + 5hL} - 1 \right)|w_{i+2}| + \left( \frac{3 + hL}{3 + 5hL} - 1 \right)|w_{i-1}|
\]

\[
+ \left( \frac{3 + 4hL}{3 + 5hL} - 1 \right)|w_{i-2} - \frac{29}{30(3 + 5hL)}h^5Y^{(5)}(\xi_i)
\]

\[
|d_i| \geq \left( \frac{45}{45 + 24h^2L} \right)|d_{i+1}| - \left( \frac{45 - 38h^2L}{45 + 24h^2L} - 1 \right)|d_{i+2}|
\]

\[
+ \left( \frac{45 + 18h^2L}{45 + 24h^2L} - 1 \right)|d_{i-1} + \left( \frac{45 + 22h^2L}{45 + 24h^2L} - 1 \right)|d_{i-2} - \frac{5}{2(45 + 24h^2L)}h^5Y^{(5)}(\xi_i)
\]
Thus, as $h$ tends to zero, hence $\lim_{h \to 0} y'(x_{i+1}) = Y'(x_{i+1})$, $\lim_{h \to 0} y(x_{i+1}) = Y(x_{i+1})$, $\lim_{h \to 0} y'(x_{i+2}) = Y'(x_{i+2})$, and $\lim_{h \to 0} y(x_{i+2}) = Y(x_{i+2})$ satisfied Definition 1 thus the convergence of the 2PBEM4 is proved.

5. Numerical results

There are three tested numerical problems presented in this study. The previous studies by Wazwaz et al. [8] only produce analytical solution thus we compare our method, 2PBEM4 with Runge Kutta method order 4. All three problems are taken from [8].

Problem 1:

$$y''(x) = \frac{1}{2}y(x) + e^{-\frac{x}{2}} - 2e^{-x}, \ x \in [0, 1]$$

Boundary conditions: $y(0) = 0, y(1) = e^{-1}$

Exact solution: $y(x) = xe^{-x}$

Problem 2:

$$y''(x) = 1 + 2(1 + \frac{1}{8}x^2)\cos(\frac{1}{2}x) - 2\cos(\frac{1}{2}x)y(\frac{1}{4}x), \ x \in [0, \frac{\pi}{4}]$$

Boundary conditions: $y(0) = 1, y(\frac{\pi}{4}) = 1 + \frac{\sqrt{2}}{2} + \frac{\pi^2}{32}$

Exact solution: $y(x) = 1 + \frac{1}{2}x^2 + \sin(x)$

Problem 3:

$$y''(x) = (y^3(x) + y^2(x))y(\frac{1}{2}x), \ x \in [0, 1]$$

Boundary conditions: $y(0) = 1, y(1) = \frac{1}{2}$

Exact solution: $y(x) = \frac{1}{x+1}$

The following notations are used for Table 1-3:

- $h$ : Step size.
- Errors : Absolute errors at the specific points.
- MAXE : Maximum absolute errors.
- AVE : Average errors.
- ITN : Total iteration of guess.
- FCN : Total function call.
- TS : Total step.
- 2PBEM4 : Two Point Block Explicit Method order 4
- RK4 : Runge Kutta Method order 4

6. Discussions

Table 1-3 shows that our method, 2PBEM4 give more accurate results compare to RK4 because both maximum errors (MAXE) and average errors (AVE) of 2PBEM4 are smaller than RK4 as the step size, $h$ decreases. The total function calls (FCN) for 2PBEM4 are less than RK4 because RK4 is a one step method hence it needs four new functions for every iteration while 2PBEM4 is a multistep method hence it only needs one new function for every iteration. The total iteration steps (TS) for 2PBEM4 are also less compare to RK4 because 2PBEM4 is computed in block method where two solutions are calculated simultaneously while RK4 only calculate one solution for each iteration hence, RK4 needs more iteration steps to compute an approximate solution.
Table 1. The absolute errors of 2PBEM4 and RK4 for Problem 1

| x     | 2PBEM4  | RK4    | 2PBEM4  | RK4    | 2PBEM4  | RK4    | 2PBEM4  | RK4    |
|-------|---------|--------|---------|--------|---------|--------|---------|--------|
| 0.0   | 0.00E+00| 0.00E+00| 0.00E+00| 0.00E+00| 0.00E+00| 0.00E+00|        |        |
| 0.1   | 2.19E-04| 3.09E-04| 1.55E-07| 2.77E-05| 1.44E-10| 2.73E-06|        |        |
| 0.2   | 1.97E-04| 4.36E-04| 1.32E-07| 3.82E-05| 1.22E-10| 3.75E-06|        |        |
| 0.3   | 1.74E-04| 4.49E-04| 1.10E-07| 3.77E-05| 1.03E-10| 3.68E-06|        |        |
| 0.4   | 1.42E-04| 3.87E-04| 9.10E-08| 3.08E-05| 8.49E-11| 2.99E-06|        |        |
| 0.5   | 1.12E-04| 2.89E-04| 7.31E-08| 2.11E-05| 6.83E-11| 2.03E-06|        |        |
| 0.6   | 8.53E-05| 1.82E-04| 5.65E-08| 1.13E-05| 5.29E-11| 1.06E-06|        |        |
| 0.7   | 6.19E-05| 8.80E-05| 4.11E-08| 3.21E-06| 3.85E-11| 2.62E-07|        |        |
| 0.8   | 3.94E-05| 2.02E-05| 2.67E-08| 1.91E-06| 2.50E-11| 2.32E-07|        |        |
| 0.9   | 1.94E-05| 1.12E-05| 1.30E-08| 3.16E-06| 1.22E-11| 3.37E-07|        |        |
| 1.0   | 2.22E-16| 5.55E-17| 1.67E-16| 7.21E-16| 6.11E-16| 1.72E-15|        |        |

MAXE | 1.97E-04| 4.49E-04| 1.74E-07| 3.91E-05| 1.67E-10| 3.83E-06|        |        |
AVE  | 1.02E-04| 2.17E-04| 7.79E-08| 1.79E-05| 7.34E-11| 1.74E-06|        |        |
ITN  | 2        | 2       | 2        | 2       | 2        | 2       |        |        |
FCN  | 15       | 81      | 105      | 801     | 1005     | 8001    |        |        |
TS   | 7        | 10      | 52       | 100     | 502      | 1000    |        |        |

Table 2. The absolute errors of 2PBEM4 and RK4 for Problem 2

| x     | 2PBEM4  | RK4    | 2PBEM4  | RK4    | 2PBEM4  | RK4    | 2PBEM4  | RK4    |
|-------|---------|--------|---------|--------|---------|--------|---------|--------|
| 0.000 | 0.00E+00| 0.00E+00| 0.00E+00| 0.00E+00| 0.00E+00| 0.00E+00| 0.00E+00| 0.00E+00|
| 0.079 | 1.52E-04| 1.54E-03| 1.53E-07| 1.56E-04| 1.53E-10| 1.56E-05|        |        |
| 0.157 | 1.41E-04| 2.80E-03| 1.43E-07| 2.85E-04| 1.43E-10| 2.85E-05|        |        |
| 0.236 | 1.30E-04| 3.76E-03| 1.31E-07| 3.82E-04| 1.31E-10| 3.82E-05|        |        |
| 0.314 | 1.16E-04| 4.37E-03| 1.17E-07| 4.45E-04| 1.17E-10| 4.46E-05|        |        |
| 0.393 | 1.00E-04| 4.63E-03| 1.01E-07| 4.71E-04| 1.01E-10| 4.72E-05|        |        |
| 0.471 | 8.33E-05| 4.51E-03| 8.39E-08| 4.59E-04| 8.40E-11| 4.60E-05|        |        |
| 0.550 | 6.46E-05| 3.99E-03| 6.50E-08| 4.07E-04| 6.51E-11| 4.08E-05|        |        |
| 0.628 | 4.44E-05| 3.08E-03| 4.47E-08| 3.14E-04| 4.47E-11| 3.15E-05|        |        |
| 0.707 | 2.28E-05| 1.74E-03| 2.30E-08| 1.78E-04| 2.30E-11| 1.79E-05|        |        |
| 0.785 | 8.88E-16| 0.00E+00| 8.88E-16| 4.44E-16| 8.88E-16| 2.66E-15|        |        |

MAXE | 1.52E-04| 4.63E-03| 1.61E-07| 4.72E-04| 1.61E-10| 4.73E-05|        |        |
AVE  | 8.54E-05| 3.04E-03| 9.35E-08| 3.13E-04| 9.43E-11| 3.13E-05|        |        |
ITN  | 2        | 2       | 2        | 2       | 2        | 2       |        |        |
FCN  | 15       | 81      | 105      | 801     | 1005     | 8001    |        |        |
TS   | 7        | 10      | 52       | 100     | 502      | 1000    |        |        |

for each point. Due to these advantages in terms of FCN and TS, these shall give a lesser computation time for 2PBEM4 compare to RK4 and will reduce the time cost.

Other than that, Problem 1 and Problem 2 only need two iterations (ITN) for the Newton’s
Table 3. The absolute errors of 2PBEM4 and RK4 for Problem 3

| x   | 2PBEM4 | RK4  | 2PBEM4 | RK4  | 2PBEM4 | RK4  |
|-----|--------|------|--------|------|--------|------|
| 0.0 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 | 0.00E+00 |
| 0.1 | 3.85E-04 | 5.17E-04 | 6.23E-08 | 4.94E-05 | 2.54E-07 | 5.13E-06 |
| 0.2 | 3.82E-04 | 7.42E-04 | 2.71E-07 | 7.08E-05 | 5.20E-07 | 7.49E-06 |
| 0.3 | 2.99E-04 | 8.09E-04 | 6.09E-07 | 7.66E-05 | 8.07E-07 | 8.32E-06 |
| 0.4 | 2.16E-04 | 7.81E-04 | 9.64E-07 | 7.38E-05 | 1.12E-06 | 8.33E-06 |
| 0.5 | 1.04E-04 | 6.97E-04 | 1.34E-06 | 6.61E-05 | 1.47E-06 | 7.87E-06 |
| 0.6 | 5.99E-05 | 5.82E-04 | 1.75E-06 | 5.57E-05 | 1.85E-06 | 7.18E-06 |
| 0.7 | 4.14E-05 | 4.50E-04 | 2.20E-06 | 4.36E-05 | 2.27E-06 | 6.37E-06 |
| 0.8 | 1.73E-05 | 3.06E-04 | 2.68E-06 | 3.07E-05 | 3.24E-06 | 5.51E-06 |
| 0.9 | 8.11E-06 | 1.56E-04 | 3.12E-06 | 8.01E-05 | 3.80E-06 | 6.34E-06 |
| 1.0 | 2.85E-06 | 1.79E-06 | 3.79E-06 | 3.55E-05 | 3.78E-06 | 8.40E-06 |

MAXE 3.85E-04 8.09E-04 3.79E-06 7.67E-05 3.80E-06 8.40E-06
AVE 1.51E-04 5.04E-04 1.54E-06 4.93E-05 1.62E-06 6.34E-06
ITN 7 7 7 7 7 7
FCN 15 81 105 801 1005 8001
TS 7 10 52 100 502 1000

method to guess the initial value while Problem 3 needs more than two iterations because Problem 1 and Problem 2 are linear problems while Problem 3 is a nonlinear problem.

7. Conclusion

In this paper, it can be shown that the proposed method which is two point block explicit method order four (2PBEM4) is reliable to solve directly of second order DDEs with boundary conditions of pantograph delay due to all of the advantages of 2PBEM4 with comparison to the existing method, Runge Kutta method order 4 that have been discussed above.

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References

[1] Aziz N H B A 2015 *Block Multistep Methods For Solving First Order Retarded And Neutral Delay Differential Equations* (Universiti Putra Malaysia)
[2] Nevers K D and Schmitt K 1971 *Journal of Mathematical Analysis and Applications* 36 pp 588–597
[3] Bellen A and Zennaro M 1984 *Computing* 32 pp 307–318
[4] Agarwal R P and Chow Y M 1986 *Computational & Mathematics with Applications* 12A pp 1143–1153
[5] Bakke V L and Jackiewicz Z 1989 *Aplikace matematiky* 34 pp 1–17
[6] Qu R and Agarwal R P 1998 *Computational & Mathematics with Applications* 35 pp 121–135
[7] Bica A M 2011 *Computers & Mathematics with Applications* 62 pp 3829–3843 ISSN 0898-1221 URL http://www.sciencedirect.com/science/article/pii/S0898122111008005
[8] Wazwaz A M, Raja M A Z and Syam M I 2017 *Rom. Rep. Phys* 69 p 102
[9] Ola Fatunla S 1991 *International journal of computer mathematics* 41 pp 55–63
[10] Lambert J D 1973 *Computational methods in ordinary differential equations* (Wiley)