$B_c$ meson spectrum and hyperfine splittings in the shifted large-N-expansion technique

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Abstract

In the framework of potential models for heavy quarkonium, we compute the mass spectrum of the bottom-charmed $B_c$ meson system and spin-dependent splittings from the Schrödinger equation using the shifted-large-N expansion technique. The masses of the lightest vector $B_c^+$ and pseudoscalar $B_c$ states as well as the higher states below the threshold are estimated. Our predicted result for the ground state energy is $6253^{+15}_{-6}$ MeV and are generally in exact agreement with earlier calculations. The parameters of each potential are adjusted to obtain best agreement with the experimental spin-averaged data (SAD). Our results are compared with the observed data and with the numerical results obtained by other numerical methods.

I. INTRODUCTION

Recently, theoretical interest has risen in the study of the $B_c$ meson, the heavy $c\bar{b}$ quarkonium system with open charm and bottom quarks composed of two nonrelativistic heavy quarks. The spectrum and properties of the $c\bar{b}$ systems have been calculated various times in the past in the framework of heavy quarkonium theory [1]. Moreover, the recent discovery of the lightest vector $B_c^+$ meson [2] has inspired new theoretical interest in the subject [3-6]. For the mass of the lightest vector meson, the CDF Collaboration quotes $M_{B_c} = 6.40 \pm 0.39 \pm 0.13$ GeV. This state should be one of a number of states lying below the threshold for emission of $B$ and $D$ mesons. Such states are very stable in comparison with their counterparts in $c\bar{c}$ and $b\bar{b}$ systems. A particularly interesting quantity should be the hyperfine splitting that as for $c\bar{c}$ case seems to be sensitive to relativistic and subleading corrections in $\alpha_s$. For the above reasons it seems worthwhile to give a detailed account of the Schrödinger energies for $c\bar{c}$, $b\bar{b}$ and $c\bar{b}$ meson systems below the continuum threshold. Because of the success of the nonrelativistic potential model and the flavour independence of the $q_1\bar{q}_2$ potential, we choose a set of phenomenological and a QCD-motivated potentials. We insist upon
strict flavor-independence of its parameters. We also use a potential model that includes running coupling constant effects in both the spherically symmetric potential and the spin-dependent potentials to give a simultaneous account of the properties of the $c\bar{c}$, $b\bar{b}$ and $c\bar{t}$ systems. Since one would expect the average values of the momentum transfer in the various quark-antiquark states to be different, some variation in the values of the strong coupling constant and the normalization scale in the spin-dependent potential should be expected.

We have made this study mostly a fully treatment for the potentials used in the literature. In order to minimize the role of flavor-dependence, we use the same values for the coupling constant and the renormalization scale for each of the levels in a given system and require these values be consistent with a universal QCD scale.

In 1991 Kwong and Rosner [7] predicted the masses of the lowest vector (triplet) and pseudoscalar (singlet) states of the $B_c$ systems using an empirical mass formula and a logarithmic potential. Eichten and Quigg [1] gave a more comprehensive account of the energies and decays of the $B_c$ system that was based on the QCD-motivated potential of Buchmüller and Tye [8]. Gershtein et al. [9] also published a detailed account of the energies and decays of the $B_c$ system and used a QCD sum-rule calculations. Baldicchi and Prosperi [6] have computed the $c\bar{b}$ spectrum based on an effective mass operator with full relativistic kinematics. They have also fitted the entire light-heavy quarkonium spectrum. Fulcher [4] extended the treatment of the spin-dependent potentials to the full radiative one-loop level and thus included effects of the running coupling constant in these potentials. He also used the renormalization scheme developed by Gupta and Radford [10].

One of the important goals of the present work is to extend the shifted large-N expansion technique (SLNET) developed for the Schrödinger equation [11, 12] to reproduce the $c\bar{b}$ spectroscopy using a class of three static together with Martin and Logarithmic potentials which was already utilized for producing the spin-averaged data (SAD) self-conjugate ($q\bar{q}$) mesons and the (SAD) non-self conjugate ($q_1\bar{q}_2$) mesons [13]. We also extend our work using an improved QCD-motivated potential proposed by Buchmüller and Tye [8].

The contents of this article is as follows: in section II, we present the solution of the
Schrödinger equation using the SLNET for the non-self conjugate $q_1q_2$ mass spectrum. In section III we present all the potentials used in this work. A brief discussion and conclusion appear in section IV.

II. WAVE EQUATION

In this section we shall consider the N-dimensional space Schrödinger equation for any spherically symmetric potential $V(r)$. If $\psi(r)$ denotes the Schrödinger’s wave function, a separation of variables $\psi(r) = Y_{\ell,m}(\theta, \phi)u(r)/r^{(N-1)/2}$ gives the following radial equation (in units $\hbar=1$)[11,12]

$$\left\{-\frac{1}{2\mu} \frac{d^2}{dr^2} + \frac{[\kappa - (1-a)][\kappa - (3-a)]}{8\mu r^2} + V(r)\right\}u(r) = E_{n,\ell}u(r),$$

(1)

where $\mu = (m_{q_1}m_{q_2})/(m_{q_1} + m_{q_2})$ is the reduced mass for the two interacting particles. Here $E_{n,\ell}$ denotes the Schrödinger binding energy, and $\kappa = N + 2\ell - a$, with $a$ representing a proper shift to be calculated later on and $\ell$ is the angular quantum number. We follow the shifted $1/N$ or $1/\kappa$ expansion method [12] by defining

$$V(r) = \frac{\kappa^2}{Q} \left[ V(r_0) + \frac{V'(r_0)}{k^{1/2}} r_0 x + \frac{V''(r_0)}{2k} \frac{r_0^2 x^2}{k} + \cdots \right],$$

(2)

and also the energy eigenvalue expansion [12,13]

$$E_{n,\ell} = E_0 + E_1/\kappa + E_2/\kappa^2 + E_3/\kappa^3 + O\left(1/\kappa^4\right),$$

(3)

where $x = \kappa^{1/2}(r/r_0 - 1)$ with $r_0$ is an arbitrary point where the taylor expansions is being performed about and $Q$ is a scale to be set equal to $\kappa^2$ at the end of calculations. Inserting equations (2) and (3) into equation (1) yields

$$\left\{-\frac{1}{2\mu} \frac{d^2}{dx^2} + \frac{1}{2\mu} \left(\frac{\kappa}{4} - \frac{(2-a)}{2} + \frac{(1-a)(3-a)}{4 \kappa}\right)\right\} \times \left(1 - \frac{2x}{k^{1/2}} + \frac{3x^2}{k} - \cdots\right)$$

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\[ + \frac{r_0^2}{Q} \left( V(r_0) + \frac{V'(r_0) r_0 x}{k^{1/2}} + \frac{V''(r_0) r_0^2 x^2}{2k^2} + \cdots \right) \phi_{n_r}(x) = \xi_{n_r} \phi_{n_r}(x), \]  

where the final analytic expression for the \(1/k\) expansion of the energy eigenvalues appropriate to the Schrödinger particle is [11]

\[ \xi_{n_r} = \frac{r_0^2}{Q} \left[ E_0 + \frac{E_1}{k} + \frac{E_2}{k^2} + \frac{E_3}{k^3} + O \left( \frac{1}{k^4} \right) \right], \]

where \(n_r\) is the radial quantum number. Here, we formulate the SLNET (expansion as \(1/k\)) for the nonrelativistic motion of spinless particle bound in spherically symmetric potential \(V(r)\). The resulting eigenvalue of the \(N\)-dimensional Schrödinger equation is written as [12]

\[ \xi_{n_r} = \frac{k}{k} \left[ \frac{1}{8\mu} + \frac{r_0^2 V(r_0)}{Q} \right] + \left[ (n_r + \frac{1}{2}) \omega - \frac{(2 - a)}{4\mu} \right]
+ \frac{1}{k} \left[ \frac{(1 - a)(3 - a)}{8\mu} + \alpha^{(1)} \right] + \alpha^{(2)}, \]

where \(\alpha^{(1)}\) and \(\alpha^{(2)}\) are the expressions given by Imbo et al [11]. Comparing equation (5) with (6) yields

\[ E_0 = V(r_0) + \frac{Q}{8\mu r_0^2}; \]

\[ E_1 = \frac{Q}{r_0^2} \left[ \left( n_r + \frac{1}{2} \right) \omega - \frac{(2 - a)}{4\mu} \right], \]

\[ E_2 = \frac{Q}{r_0^2} \left[ \frac{(1 - a)(3 - a)}{8\mu} + \alpha^{(1)} \right], \]

\[ E_3 = \frac{Q}{r_0^2} \alpha^{(2)}. \]

Here the quantity \(r_0\) is chosen so as to minimize the leading term, \(E_0\) [12,13]. That is,

\[ \frac{dE_0}{dr_0} = 0 \quad \text{and} \quad \frac{d^2E_0}{dr_0^2} > 0. \]

Therefore, \(r_0\) satisfies the relation

\[ Q = 4\mu r_0^3 V'(r_0), \]
and to solve for the shifting parameter $a$, the next contribution to the energy eigenvalue $E_1$ is chosen to vanish [11] so that the second- and third-order corrections are very small compared with the leading term contribution. The energy states are calculated by considering the leading term $E_0$ and the second-order and third-order corrections, it implies that

$$a = 2 - 2(2n_r + 1)\mu\omega,$$

with

$$\omega = \frac{1}{4\mu} \left[ 3 + \frac{r_0V'(r_0)}{V''(r_0)} \right]^{1/2}.$$

Once $r_0$ is being determined, with the choice $k = \sqrt{Q}$ which rescales the potential, we derive an analytic expression that satisfies $r_0$ in equations (12)-(14). We have defined the analytic expression for the $1/k$ expansion of the energy eigenvalues (3) and determined the shifting parameter requiring $E_1 = 0$ [12, 13]. For the Coulomb potential, considered as a testing case, the results are found to be strongly convergent and highly accurate. The calculations of the energy eigenvalues were carried out up to the second order correction. Therefore, the Schrödinger binding energy to the third order is

$$E_{n,\ell} = E_0 + \frac{1}{r_0^2} \left[ \frac{(1-a)(3-a)}{8\mu} + \alpha^{(1)} + \frac{\alpha^{(2)}}{k} + O \left( \frac{1}{k^2} \right) \right].$$

Once the problem is collapsed to its actual dimension $N = 3$, it simply rests the task of relating the coefficients of our equation to the one-dimensional anharmonic oscillator in order to read the energy spectrum. For the $N = 3$ physical space, the equation (1) restores its three-dimensional form, one obtains

$$1 + 2\ell + (2n_r + 1) \left[ 3 + \frac{r_0V'(r_0)}{V''(r_0)} \right]^{1/2} = [8\mu r_0^3 V'(r_0)]^{1/2}.$$

Once $r_0$ is being determined through equation (16), the Schrödinger binding energy of the $q_1q_2$ system in equation (15) becomes relatively simple and straightforward. We finally obtain the total Schrödinger mass binding energy for spinless particles as

$$M(q_1q_2) = m_{q_1} + m_{q_2} + 2E_{n,\ell}.$$
where \( n = n_r + 1 \) is the principal quantum number. As stated before in [11-13], for a fixed \( n \) the computed energies become more accurate as \( \ell \) increases. This is expected since the expansion parameter \( 1/N \) or \( 1/\bar{k} \) becomes smaller as \( \ell \) becomes larger since the parameter \( \bar{k} \) is proportional to \( n \) and appears in the denominator in higher-order correction.

### III. SOME POTENTIAL MODELS

#### A. Static potentials

The \( c\bar{b} \) system that we investigate in the present work is often considered as nonrelativistic system, and consequently our treatment is based upon Schrödinger equation with a Hamiltonian

\[
H_o = -\frac{\nabla^2}{2\mu} + V(r) + V_{SD},
\]

(18)

where \( V_{SD} \) is the spin-dependent term having the simple form

\[
V_{SD} \rightarrow V_{SS} = \frac{32\pi \alpha_s}{9m_q m_{q2}} \delta^3(r) \vec{s}_1 \cdot \vec{s}_2.
\]

(19)

The spin dependent potential is simply a spin-spin part and this would enable us to make some preliminary calculations of the energies of the lowest two S-states of the \( c\bar{b} \) system. The potential parameters in this section are all strictly flavor-independent. The potential parameters are fitted to the low-lying energy levels of \( c\bar{c} \) and \( b\bar{b} \) systems. The strong coupling constant \( \alpha_s \) is fitted to the observed charmonium hyperfine splitting of 117 MeV. The numerical value of \( \alpha_s \) is dependent on the potential form and found to be compatible to the other measurements of reference [1,4,6-7,9]. The hyperfine observed splitting in charmonium fixes \( \alpha_s \) for each potential. The perturbative part of such a quantity was evaluated at the lowest order in \( \alpha_s \). Baldicchi and Prosperi [6] used the standard running QCD coupling expression.
\[ \alpha_s(Q) = \frac{4\pi}{(11 - \frac{2}{3}n_f) \ln \left( \frac{Q^2}{\Lambda^2} \right)}. \] (20)

with \( n_f = 4 \) and \( \Lambda = 0.2 \, \text{GeV} \) cut at a maximum value \( \alpha_s(0) = 0.35 \), to give the right \( J/\psi - \eta_c \) splitting and to treat properly the infrared region. Details on their numerical work are given in Ref. [6]. Whereas Brambilla and Vairo [3] took in their perturbative analysis \( 0.26 \leq \alpha_s(\mu = 2 \text{GeV}) \leq 0.30 \).

The central potentials in (18) include a class of a static potentials of the general form

\[ V(r) = -ar^{-\beta} + br^\beta + c, \quad 0 < \beta \leq 1, \] (21)

previously proposed by Lichtenberg [14] where the parameters \( a \) and \( b \) are positive while \( c \) may be of either sign. These static quarkonium potentials are monotone nondecreasing, and concave functions which satisfy the condition

\[ V''(r) > 0 \quad \text{and} \quad V''(r) \leq 0. \] (22)

This comprise the following potentials:

1. **Cornell potential**

The QCD-motivated Coulomb-plus-linear potential (Cornell potential) [15]

\[ V_C(r) = -\frac{a}{r} + br + c, \] (23)

with the adjustable parameters as

\[ [a, b, c] = [0.52, 0.1756 \, \text{GeV}^2, -0.8578 \, \text{GeV}]. \] (24)

Such a potential has the Coulomb-like behavior at low distance is supposed to represent the short range gluon exchange and the term confining the quarks rises linearly at large distances is supposed to represent the confinement string tension. The main drawback of this potential is that the \( c\bar{c} \) and \( b\bar{b} \) states lie in an intermediate region of quark separation where neither limiting forms of (22) should be valid.
2. **Song-Lin potential**

This phenomenological potential was proposed by Song and Lin [16] having the form

\[
V_{SL}(r) = -ar^{-1/2} + br^{1/2} + c,
\]

with the adjustable parameters

[\[a, b, c\] = \[0.923 \text{ GeV}^{1/2}, 0.511 \text{ GeV}^{3/2}, c = -0.798 \text{ GeV}\].

The characteristic feature of this potential may be traced in Ref. [16].

3. **Turin potential**

Lichtenberg and his co-workers [17] suggested such a potential which is an intermediate between the Cornell and Song-Lin potentials. This potential [17] has the form

\[
V_T(r) = -ar^{-3/4} + br^{3/4} + c,
\]

with its adjustable parameters

[\[a, b, c\] = \[0.620 \text{ GeV}^{1/4}, 0.304 \text{ GeV}^{7/4}, -0.823 \text{ GeV}\].

4. **Martin potential**

The phenomenological power-law potential of the form

\[
V_M(r) = b(r \times 1 \text{GeV})^{0.1} + c,
\]

is labeled as Martin’s potential [18] with the parameters values

[\[b, c\] = \[6.898 \text{ GeV}^{1.1}, -8.093 \text{ GeV}\].

(potential units are also in GeV).
5. Logarithmic potential

A Martin’s power-law potential reduces [19] into a form

\[ V_L(r) = b \ln(r \times 1 GeV) + c, \]

with

\[ [b, c] = [0.733 \text{ GeV}, -0.6631 \text{ GeV}] . \]

The potential forms in (29), and (31) were used by Eichten and Quigg and all of these potential forms were also used for \( \psi \) and \( \Upsilon \) data probing \( 0.1 \text{ fm} < r < 1 \text{ fm} \) region.

B. QCD-motivated potentials

1. Igi-Ono potential

The Coulomb-plus-linear potential has been successful in describing the spectrum of existing quarkonium systems. Moreover, the same is true for logarithmic potential as well as for power law potentials [8]. These potentials all describe well the shape of the intermediate region. Neither the long-distance nor the short-distance nature of the interquark potential is probed by upsilon and charmonium systems. Other systems depend on the short- and long-distance parts of the potential as in toponium system. The potentials for some systems should account properly for the running \( \alpha_s \) at the different distances, the \( \alpha_s \) and the string constant that describe the long-distance behavior can vary independently. Furthermore, the \( \Lambda_{QCD} \) used to evaluate \( \alpha_s \) is related to a specific renormalization scheme so that comparison with other calculations is possible. This leads to expanding \( \alpha_s \) to at least 2-loop order and use of nontrivial methods in the interpolation between long-and short-distance behaviors. The interquark potential at short distances has been calculated to 2-loop calculations in the
modified minimal-subtraction (\(\overline{MS}\)) scheme [20]. Together with the 2-loop expression for \(\alpha_s\), one has [8,21]

\[
V(r) = -\frac{4\alpha_s(\mu)}{3r} + \left[ 1 + \frac{\alpha_s(\mu)}{2\pi}(b_0 \ln \mu r + A) \right],
\]

and

\[
\alpha_s(\mu) = \frac{4\pi}{b_0 \ln(\mu^2/\Lambda_{\overline{MS}}^2)} \left[ 1 - \frac{b_1}{b_0} \ln(\mu^2/\Lambda_{\overline{MS}}^2) \right],
\]

with

\[
[b_0, b_1, A] = \left[ 11 - \frac{2}{3} n_f, 102 - \frac{38}{3} n_f, b_0 \gamma_E + \frac{31}{6} - \frac{5}{9} n_f \right].
\]

Here \(n_f\) is the number of flavors with mass below \(\mu\) and \(\gamma_E = 0.5772\) is the Euler’s number. The renormalization scale \(\mu\) is usually chosen to be \(1/r\) to obtain a simple form for \(V(r)\). The singularity in \(\alpha_s(1/r)\) at \(r=1/\Lambda_{\overline{MS}} = 5\ GeV^{-1} \approx 1\ fm\) for \(\Lambda_{\overline{MS}} = 200\ MeV\). This singularity can be removed by the substitution

\[
\ln \frac{1}{r^2 \Lambda_{\overline{MS}}^2} \rightarrow f(r) = \ln \left[ \frac{1}{r^2 \Lambda_{\overline{MS}}^2} + b \right].
\]

The constant \(b\) is an adjustable parameter of the potential and will not affect the perturbative part of the potential. Hence, setting \(n_f = 4\) in equation (35), the one-gloun exchange part of the interquark potential simply takes the form

\[
V_{OGE}^{(n_f=4)}(r) = -\frac{16\pi}{25} \frac{1}{r f(r)} \left[ 1 - \frac{462 \ln f(r)}{625} + \frac{2\gamma_E + \frac{53}{76}}{f(r)} \right],
\]

where the function \(f(r)\) can be read off from (36). Furthermore, the long distance interquark potential grows linearly leading to confinement as

\[
V_L(r) = ar.
\]

Igi and Ono [21] proposed a potential whose general form

\[
V^{(n_f=4)}(r) = V_{OGE}^{(n_f=4)} + ar + dre^{-gr},
\]
so as to interpolate smoothly between the two parts. They added phenomenologically a term $d r e^{-9r}$ to the potential so that to adjust the intermediate range behavior by which the range of $\Lambda_{\overline{\text{MS}}}$ is extended keeping linearly rising confining potential. Hence, the $\Lambda_{\overline{\text{MS}}}$ runs from 100 to 500 $MeV$ keeping a good fit to the $c\bar{c}$ and $b\bar{b}$ spectra. Thereby, the potential with $b = 20$ is labeled as type I, the one with $b = 5$ is labeled as type II, and that one with $d = 0$ and $b = 19$ is labeled as type III.

2. Improved Chen-Kuang potential

Chen and Kuang [22] proposed two improved potential models so that the parameters therein all vary explicitly with $\Lambda_{\overline{\text{MS}}}$ so that these parameters can only be given numerically for several values of $\Lambda_{\overline{\text{MS}}}$. Such potentials have the natural QCD interpretation and explicit $\Lambda_{\overline{\text{MS}}}$ dependence both for giving clear link between QCD and experiment and for convenience in practical calculation for a given value of $\Lambda_{\overline{\text{MS}}}$. It has the general form

$$V^{(n_f=4)}(r) = kr - \frac{16\pi}{25} \frac{1}{rf(r)} \left[ 1 - \frac{462}{625} \ln f(r) + \frac{2\gamma_E + \frac{53}{75}}{f(r)} \right],$$

(40)

where the string tension is related to Regge slope by $k = \frac{1}{2\pi\alpha'}$. The function $f(r)$ in (40) can be read off from

$$f(r) = \ln \left[ \frac{1}{\Lambda_{\overline{\text{MS}}}^r} + 4.62 - A(r) \right]^2,$$

(41)

and

$$A(r) = \left[ 1 - \frac{\Lambda_{\overline{\text{MS}}}^r}{4 \Lambda_{\overline{\text{MS}}}^r} \right]^{\frac{1}{4}} \exp \left\{ - \left[ 15 \left( \frac{\Lambda_{\overline{\text{MS}}}^r}{\Lambda_{\overline{\text{MS}}}^r} - 1 \right) \Lambda_{\overline{\text{MS}}}^r \right]^2 \right\},$$

(42)

with parameters values

$$[k, \alpha', \Lambda_{\overline{\text{MS}}}^r] = [0.1491 \text{ GeV}^2, 1.067 \text{ GeV}^{-2}, 180 \text{ MeV}].$$

(43)

The details of this potential can be traced in Ref. [22].
IV. DISCUSSION AND CONCLUSION

We solve the Schrödinger equation for various potentials to determine the position of the $1S$ spin-averaged data (SAD) for $c\bar{c}$, $b\bar{b}$, and then for $c\bar{b}$ systems. The hyperfine splitting of the ground state is given by (19). The hyperfine splitting observed in the charmonium family [23]

$$M(J/\psi) - M(\eta_c) = 117MeV,$$

(44)

fixes the strong coupling constant $\alpha_s$ for each potential. For simplicity and for the sake of comparison we neglect the variation of $\alpha_s$ with momentum in (20) to have a common spectra for all states and scale the splitting of $c\bar{b}$ and $b\bar{b}$ from the charmonium value in (44). The effective constant value, fixed by the described way, has been applied to the description of not only the $c\bar{c}$ system, but also the $b\bar{b}$, and possibly $c\bar{b}$ quarkonia. The consideration of the variation of the effective Coulomb interaction constant becomes especially essential for the $\Upsilon$ particle, for which $\alpha_s(\Upsilon) \neq \alpha_s(\psi)$. Thus, in calculating the splittings of the $c\bar{b}$ spectra, we have to take into account the $\alpha_s(\mu)$ dependence on the reduced mass of the heavy quarkonium instead of $\alpha_s(Q)$ for the reasons stated in Ref. [3]. That is, the QCD coupling constant $\alpha_s$ in (19)-(20) is defined in the Gupta-Radford (GR) renormalization scheme [10]

$$\alpha_s = \frac{6\pi}{(33 - 2n_f) \ln \left( \frac{\mu}{\Lambda_{GR}} \right)},$$

(45)

in which $\Lambda_{GR}$ is related to $\Lambda_{\overline{MS}}$ by

$$\Lambda_{GR} = \Lambda_{\overline{MS}} \exp \left[ \frac{49 - 10n_f/3}{2(33 - 2n_f)} \right].$$

(46)

The treatment of our model with momentum dependence form (20) would increase the accuracy, it probably give very close results and might reproduce the experimental values equally well within the errors.

Table I reports our prediction for the Schrödinger mass spectrum of the four lowest $c\bar{b}$ $S$–states together with the first three $P$– and $D$–states for various potentials. If we use
the $\alpha_s$ determined for the $c\bar{c}$ and $b\bar{b}$ systems by other authors [1,4,6-7,9], we predict their energy masses to within a few $MeV$ to the calculated SAD [13,17,24]. Since the model is spin independent and as the energies of the singlet states of quarkonium families have not been measured, a theoretical estimates of these unknown levels introduces uncertainty into the calculated SAD. Its worthwhile to note that SAD is defined as the average mass of the $(s = 1, \ell = 1)$ states in the form 

$$SAD(nP_i) = \frac{1}{9} [5M(n^3P_2) + 3M(n^3P_1) + M(n^3P_0)]$$

and for $(s = 1, \ell = 0)$ states by

$$SAD(nS_j) = \frac{1}{4} [3M(n^3S_1) + M(n^1S_0)]$$

in which the SAD $S$ level gives the weight of only $1/4$ to the unknown singlet level and $3/4$ to the known triplet level.

Instead of showing the calculated masses in $GeV$ as in Ref. [6], it is useful to report the spectrum for heavy meson masses in $MeV$ obtained by our formalism in numerical form to four significant figures. Obviously, this trend provides a measure of the accuracy in reproducing the experimental and the calculated SAD as one can expect in this match to a nearest few $MeV$. It has been shown the possibility of producing $c\bar{b}$ mesons in $e^+e^-$ and hadron-hadron colliders [25,26], so that the study of the $c\bar{b}$ mesons is not merely academic. It is clear that the differences between the spectra predicted by different potentials are not large.

Moreover, the calculated fine and hyperfine splitting values of the vector and pseudoscalar masses of the $c\bar{b}$ system for the two-lowest $S-$ states are also presented in Table I. Predictions for the $c\bar{b}$ ground-state masses depend little on the potential. The $B_c$ and $B_c^*$ masses and splittings lie within the ranges quoted by Kwong and Rosner [7] in their survey of techniques for estimating the masses of the $c\bar{b}$ ground state. Thus our results in Table I are very similar to those presented in Table I of Eichten and Quigg [1] and also to that presented by Fulcher [4]. The results obtained with the Song-Lin and the Martin potential in all cases fall between the extremes defined by the other potentials. Here, we report the range of the coupling constant we take in our analysis $0.220 \leq \alpha_s \leq 0.313$ for all potentials and $0.264 \leq \alpha_s \leq 0.313$ for the class of static potentials in our study as shown in Table I. Overmore, our predictions to the $c\bar{b}$ masses of the lowest S-wave (singlet and triplet) and also their splittings appear
together with those estimated using various methods by many authors are also shown in Table II. Larger discrepancies among the various methods occur for the the ground and excited states [6]. As noted in Table II, Bambilla and Vairo [3] calculated the maximum final result of \((M_{B^*})_{\text{pert}} = 6326^{+29}_{-9}\) MeV, the upper limit corresponds to the choice of parameters \(\Lambda_{n_f=3}^{\overline{MS}} = 350\) MeV and \(\mu = 1.2\) GeV, while the lower limit to \(\Lambda_{n_f=3}^{\overline{MS}} = 250\) MeV and \(\mu = 2.0\) GeV as the best approximation to their perturbative calculation. It’s worthwile to note from Table II, that the SLNET estimations fall in the range demonstrated by other authors.

The fitted parameters of the potential of type I and II which were proposed by Igi and Ono are listed in Table III. Our predictions for the \(c\bar{b}\) mass spectra are reported in Table IV for the type I (with \(b = 20\)) together with the type II (with \(b = 5\)). Moreover, the singlet and triplet masses together with the hyperfine splittings predicted for the two types of this potential are also reported in Table V. We, hereby, tested acceptable parameters for \(\Lambda_{\overline{MS}}\) from 100 to 500 MeV for the type I and type II Igi-Ono potentials to produce the \(c\bar{b}\) masses and their splittings. Small discrepancies between our prediction and SAD experiment [24] can be seen for higher states and such discrepancies are probably seen for any potential model and it might be related to the threshold effects or quark-gluon mixings. The fitted set of parameters for the Igi-Ono potential (type III ) presented by Ref. [21] are also tested in our method with \(b = 19\) and \((\Lambda_{\overline{MS}} = 300\) MeV and also 390 MeV) and then \(b = 16.3\) and \(\Lambda_{\overline{MS}} = 300\) MeV which seems to be more convenient than \(\Lambda_{\overline{MS}} = 500\) MeV used by other authors [8]. Results of this study are also presented in Table VI. It is clear that the overall study seems likely to be good and the reproduced masses of states are also reasonable. Once the experimental masses of the \(c\bar{b}\) spectra becomes available one may be able to sharpen analysis by adjusting potential parameters and then choosing the mostly convenient value of \(\Lambda_{\overline{MS}}\). We see that the quark masses \(m_c\) and \(m_b\) are sensitive to the variation of \(\Lambda_{\overline{MS}}\), their values increase and explained in the following way. The absolute value of the short-range potential decreases with increasing \(\Lambda_{\overline{MS}}\). In order to reproduce experimental masses we need larger quark masses for larger values of \(\Lambda_{\overline{MS}}\). This situation becomes completely different if
we are allowed to add an additional constant to (39) and reduce the effect of the exponential term therein for the higher states. However, we have not also attempted such possibility in this paper for the sake of comparison with the other authors.

We have found the $q_1q_2$ potentials which reproduce the experimental masses of the $c\bar{b}$ states for various values of $\Lambda_{\overline{MS}}$. Using this model, we see that the experimental $c\bar{b}$ splittings can be reproduced for $\Lambda_{\overline{MS}} \sim 200 - 400$ MeV and $b = 20$ while $\Lambda_{\overline{MS}} = 100$ and $500$ MeV are clearly ruled out. Taking the preferred value of $\Lambda_{\overline{MS}} = 400$ MeV, we can predict the splittings in exact agreement to several MeV with the other formalisms (c.f., Table I of Ref.[6]). We have selected some preferred values for $\Lambda_{\overline{MS}}$ and $b$ as they provide the best close fine and hyperfine splittings to the other works as shown in Table VI.

The predicted $c\bar{b}$ spectra obtained from the Chen-Kuang potential are also listed in Table VI. We see that for states below the threshold, the deviations of the predicted spectra from the experimental SAD are within several MeV. We find that $m_c$ and $m_b$ are insensitive to the variation of $\Lambda_{\overline{MS}}$ for this potential. This is consistent with the conventional idea that, for heavy quarks, the constituent quark mass is close to the current quark mass which is $\Lambda_{\overline{MS}}$ independent. The $\Lambda_{\overline{MS}}$ dependence of the Chen-Kuang potential is given in (41)-(42) and the predicted spectra is obtained for various values of $\Lambda_{\overline{MS}} = 100, 180$ and $375$ MeV. Numerical calculations show that this potential is insensitive to $\Lambda_{\overline{MS}}$ in the range from 100 to 375 MeV, and as $\Lambda_{\overline{MS}}$ runs from 375 to 500 MeV, the potential becomes more sensitive. The obtained $n^1S_0$ and $n^3S_1$ splittings for the $c\bar{b}$ system in the Chen-Kuang potential are also listed in Table VI. For a certain range of $\Lambda_{\overline{MS}}$ the agreement is good. Moreover, the fine and hyperfine splittings of the S-states in the $c\bar{b}$ system predicted by Igi-Ono and Chen-kuang potentials are listed in Table VII for some proper parameters. They are considerably smaller than the corresponding values $\Delta_{1S}(c\bar{b}) = 76$ MeV, and $\Delta_{2S}(c\bar{b}) = 42$ MeV predicted by the quadratic formalism. Moreover, Chen-Kuang [22] predicted $\Delta_{1S}(c\bar{b}) = 49.9$ MeV, and $\Delta_{2S}(c\bar{b}) = 29.4$ MeV for their potential with $\Lambda_{\overline{MS}} = 200$ MeV in which the last splitting is almost constant as $\Lambda_{\overline{MS}}$ increases. Our predictions for $\Delta_{1S}(c\bar{b}) = 68$ MeV, and $\Delta_{2S}(c\bar{b}) = 35$ MeV for the Chen-Kuang potential with $\Lambda_{\overline{MS}}$ runs from 100 into 375 MeV. We also find
$\Delta_{1S}(cb) = 67\, MeV$, and $\Delta_{2S}(cb) = 33\, MeV$ for the Igi-Ono potential with $\Lambda_{\overline{MS}} = 300\, MeV$ and $b = 16.3$. The present model has the following features: (1) The present potential predicts smaller $\Delta_{1S}$ and $\Delta_{2S}$ than the other potentials do for $cb$ system and the present $\Delta_{1S}$ and $\Delta_{2S}$ do not depend on $\Lambda_{\overline{MS}}$ more sensitively (2) The experimental $cb$ splitting can be reproduced for the preferred range of $\Lambda_{\overline{MS}}$ runs from 100 into 375 $MeV$. Table VII reports our results using SLNET compared to other formalisms.

In this paper, we have developed the SLNET in the treatment of the $cb$ system using group of static and QCD-motivated potentials. For such potentials the method looks quite attractive as it yields highly accurate results. The convergence of this method seems to be very fast as the higher corrections to energy have lower contribution. It is interesting to note that the scope of this method can be extended to more realistic potentials.

In this respect, in reproducing the SAD, we used the same fitted parameters of the other authors for the sake of comparison and also for the sake of testing the accuracy of our approach. Here, we would expect much better agreement to experimental data in case of fitting our own parameters properly. Finally, we comment that the fitted parameters of any potential are model-dependent in any study.

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REFERENCES

[1] S. Godfrey and N. Isgur, Phys. Rev. D 32, 189 (1985); E.J. Eichten and C. Quigg, ibid. 49, 5845 (1994); E. Bagan, H.G. Dosch, P. Gosdzinsky, S. Narison, and J.M. Richard, Z. Phys C 64, 57 (1994); R. Roncaglia, A. Dzierba, D.B. Lichtenberg, and E. Predazzi, Phys. Rev. D 51, 1248 (1995).

[2] CDF Collaboration, F. Abe et al., Phys. Rev. Lett. 81, 2432 (1998); CDF Collaboration, F. Abe et al., Phys. Rev. D 58, 112004 (1998); OPAL Collaboration, K. Ackerstaff et al., Phys. Lett. B 420, 157 (1998); ALEPH Collaboration, R. Barate et al., ibid. 402, 231 (1997); DELPHI Collaboration, P. Abreu et al., ibid. 398, 207 (1997).

[3] N. Brambilla and A. Vairo, Phys. Rev. D 62, 094019 (2000); P. Colangelo and F. De Fazio, ibid. 61, 034012 (2000); P. Colangelo, G. Nardalli, and N. Paver, Z. Phys. C 57, 43 (1993).

[4] L.P. Fulcher, Phys. Rev. D 60, 074006 (1999); ibid. 60, 074006 (1999); ibid. 44, 2079 (1991).

[5] C.T.H. Davis, K. Hornbostel, G.P. Lepage, A.J. Lidsey, J. Shigemitsu, and J. Sloan, Phys. Lett. B 382, 131 (1996).

[6] M. Baldicci and G.M. Prosperi, Phys. Lett. B 436, 145 (1998); ibid. Phys. Rev. D 62, 114024 (2000); ibid. Fiz. B 8, 251 (1999).

[7] W. Kwong and J. Rosner, Phys. Rev. D 44, 212 (1991).

[8] W. Buchmüller and S. Tye, Phys. Rev. D 24, 132 (1981).

[9] S. Gershtein et al., Phys. Rev. D 51, 3613 (1995).

[10] S. Gupta and S. Radford, Phys. Rev. D 24, 2309 (1981); ibid. 25, 2690 (1982); S. Gupta, S. Radford, and W. Repko, Phys. Rev. D 26, 3305 (1982); ibid. 34, 201 (1986).

[11] T. Imbo, A. Pagnamenta and U. Sukhatme, Phys. Rev. D 29, 1669 (1984); H. Chris-
tiansen, L.N. Epele, H. Fanchiotti, and C.A. Garcia Canal, Phys. Rev. A 40, 1760 (1989).

[12] S. Ikhdair, O. Mustafa and R. Sever, Tr. J. Phys. 16, 510 (1992); ibid. 17, 474 (1993).

[13] S.M. Ikhdair and R. Sever, Z. Phys. C 56, 155 (1992); ibid. 58, 153 (1993).

[14] D. B. Lichtenberg et al., Phys. Lett. 193, 95 (1987).

[15] E. Eichten et al., Phys. Rev. Lett. 34, 369 (1975); ibid. 36, 500 (1976); ibid. Phys. Rev. D 17, 3090 (1979); ibid. 21, 203 (1980).

[16] S. Xiaotong and L. Hefen, Z. Phys. C 34, 223 (1987).

[17] D.B. Lichtenberg, E. Predazzi, R. Roncaglia, M. Rosso and J.G. Will, Z. Phys. C 41, 615 (1989); ibid. 46, 75 (1990).

[18] M. Martin, Phys. Lett. 93 B, 338 (1980); ibid. 100 B, 511 (1980).

[19] C. Quigg and J.L. Rosner, Phys. Lett. 71 B, 153 (1977); ibid. Phys. Rep. 56, 167 (1979).

[20] W.A. Bardeen et al., Phys. Rev. D 18, 3998 (1978); W.J. Marciano, ibid. 29, 580 (1984). A. Billoire, Phys. Lett. 92 B, 343 (1980).

[21] K. Igi and S. Ono, Phys. Rev. D 33, 3349 (1986).

[22] Y.-Q. Chen and Y.-P. Kuang, Phys. Rev. D 46, 1165 (1992).

[23] Particle Data Group, J.J. Hernández et al., Phys. Lett. B 239, 1 (1990); K. Hikasa et al., Phys. Rev. D 45, S1 (1992).

[24] D.E. Groom et al., The Eur. Phys. J. C 15, 1 (2000).

[25] C.-H. Cheng and Y.-Q. Chen, High Energy Phys. Nucl. Phys. (to be published); Institute of Theoretical Physics, Academia Sinica, Reports Nos. AS-ITP-91-19 and CCAST-91-21 (unpublished).
[26] S.S. Gershtein, A.K. Likhoded, and S.R. Slabospitsky, Int. J. Mod. Phys. A 6, 2309 (1991); V.V. Kiselev et al., Yad. Fiz. 49, 1100 (1989) [Sov. J. Nucl. Phys. 49, 682 (1989)]

[27] M. Baker, J.S. Ball, and F. Zachariasen, Phys. Rev. D 45, 910 (1992); R. Roncaglia et al., Indiana University Report No. IUHET 270, January 1994 (unpublished), *ibid.* D 32, 189 (1985); E. Bagan et al., CERN Report No. TH. 7141/94 (unpublished),

[28] UKQCD Collaboration, H.P. Shanahan et al., Phys. Lett. B 453, 289 (1999).
TABLE I. The $c\bar{b}$ masses and hyperfine splittings ($\Delta_{nS}^*$

| States | [6,24] Cornell | Song-Lin | Turin | Martin | Logarithmic |
|--------|----------------|----------|-------|--------|-------------|
| $\alpha_s =$ | 0.313 | 0.264 | 0.286 | 0.251 | 0.220 |
| $m_c (GeV) =$ | 1.840 | 1.820 | 1.790 | 1.800 | 1.500 |
| $m_b (GeV) =$ | 5.232 | 5.199 | 5.171 | 5.174 | 4.905 |
| $M(c\bar{b})$ | 6315 | 6315 | 6306 | 6307 | 6301 | 6317 |
| $1S$ | 6334 | 6335 | 6325 | 6326 | 6319 | 6334 |
| $1^3S_1$ | 6258 | 6253 | 6248 | 6249 | 6247 | 6266 |
| $1^1S_0$ | 6873 | 6888 | 6875 | 6880 | 6892 | 6903 |
| $2^1S_0$ | 6883 | 6897 | 6884 | 6889 | 6902 | 6911 |
| $\Delta_{1S}^*$ | 42 | 37 | 34 | 36 | 37 | 31 |
| $2S$ | 7246 | 7271 | 7209 | 7246 | 7236 | 7225 |
| $2^3S_1$ | 7587 | 7455 | 7535 | 7483 | 7448 |
| $2^1S_0$ | 6772 | 6743 | 6733 | 6731 | 6730 | 6754 |
| $3S$ | 7154 | 7138 | 7104 | 7123 | 7125 | 7127 |
| $4S$ | 7464 | 7371 | 7428 | 7398 | 7375 |
| $1P$ | 7043 | 7003 | 6998 | 6998 | 7011 | 7027 |
| $2P$ | 7367 | 7340 | 7284 | 7320 | 7311 | 7301 |
| $3P$ | 7636 | 7510 | 7588 | 7536 | 7502 |

$^*\Delta_{nS} = M(n^3S_1) - M(n^1S_0)$. 

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TABLE II. The predicted $\bar{c}b$ masses of the lowest S-wave and its splitting compared with the other authors (all in $MeV$).

| Work                  | $M_{B_c}(1^1S_0)^*$ | $M_{B_c}(1^3S_1)$ | $\Delta_{1S}$ |
|-----------------------|---------------------|-------------------|----------------|
| Eichten et al. [1]    | 6258 ± 20           |                   |                |
| Colangelo and Fazio [3]| 6280                | 6350              |                |
| Baker et al. [27]     | 6287                | 6372              |                |
| Roncaglia et al. [27] |                     | 6320 ± 10         |                |
| Godfrey et al. [1]    | 6270                | 6340              |                |
| Bagan et al. [1,27]   | 6255±20             | 6330 ± 20         |                |
| Bambilla et al. [3]   |                     | 6326±20           |                |
| Baldicchi et al. [6]  | 6194 ~ 6292         | 6284 ~ 6357       | 65 ≤ $\Delta_{1S}$ ≤ 90 |
| SLNET†                | 6253±13             | 6328±7            | 69 ≤ $\Delta_{1S}$ ≤ 80 |
| SLNET‡                | 6258±8              | 6333±2            |                |
| SLNET                 | 6310†               |                   |                |

* The experimental mass of this singlet state is presented in [2].
† Averaging over the five values in Table I.
‡ We treat Eichten and Quigg’s results [1] in the same manner.
†† Our best estimation for the center-of-gravity triplet state.
| $\Lambda_{\overline{MS}}$ (GeV) | $a$ (GeV$^2$) | $g$ (GeV) | $d$ (GeV$^2$) | $m_c$ (GeV) | $m_b$ (GeV) |
|-------------------------------|--------------|-----------|--------------|-------------|-------------|
| 0.1†                          | 0.1733       | 0.3076    | 0.4344       | 1.134       | 4.563       |
| 0.2                           | 0.1587       | 0.3436    | 0.2550       | 1.322       | 4.731       |
| 0.3                           | 0.1443       | 0.3280    | 0.0495       | 1.471       | 4.868       |
| 0.4                           | 0.1387       | 2.903     | 0.582        | 1.515       | 4.910       |
| 0.5                           | 0.1391       | 2.955     | 1.476        | 1.514       | 4.911       |
| 0.1‡                          | 0.1762       | 0.2753    | 0.4720       | 1.120       | 4.551       |
| 0.2                           | 0.1734       | 0.3479    | 0.5362       | 1.267       | 4.684       |
| 0.3                           | 0.1615       | 0.4482    | 0.6020       | 1.416       | 4.815       |
| 0.4                           | 0.1389       | 0.6219    | 0.5632       | 1.604       | 4.986       |
| 0.5                           | 0.1137       | 1.0029    | 0.7368       | 1.748       | 4.118       |

† Type I potential.

‡ Type II potential.
TABLE IV. The $c$-$b$ mass spectra predicted for various $\Lambda_{\overline{MS}}$ using Igi-Ono (type I and II) potential (all in $MeV$).

| States | Ref.[6,24] $\Lambda_{\overline{MS}}$ | $100$ | $200$ | $300$ | $400$ | $500$ |
|--------|-----------------------------------|-----|-----|-----|-----|-----|
| $b = 20$ |                                   |     |     |     |     |     |
| $1S$   | 6327                              | 6355| 6342| 6336| 6347| 6349 |
| $2S$   | 6906                              | 6941| 6928| 6907| 6911| 6923 |
| $3S$   | 7246                              | 7290| 7266| 7270| 7272| 7274 |
| $1P$   | 6754                              | 6781| 6768| 6759| 6763| 6751 |
| $2P$   | 7154                              | 7170| 7155| 7151| 7153| 7143 |
| $1D$   | 7028                              | 7055| 7041| 7030| 7031| 7019 |
| $2D$   | 7367                              | 7360| 7351| 7353| 7354| 7342 |

$\Lambda_{\overline{MS}}$ = 5

| States | Ref.[6,24] $\Lambda_{\overline{MS}}$ | $100$ | $200$ | $300$ | $400$ | $500$ |
|--------|-----------------------------------|-----|-----|-----|-----|-----|
| $1S$   | 6327                              | 6357| 6347| 6342| 6326| 6316 |
| $2S$   | 6906                              | 6940| 6921| 6936⁺| 6937| 6930 |
| $3S$   | 7246                              | 7284| 7300| 7262| 7272⁺| 7238 |
| $1P$   | 6754                              | 6782| 6766| 6763| 6749| 6746 |
| $2P$   | 7154                              | 7168| 7161| 7160| 7139| 7136 |
| $1D$   | 7028                              | 7055| 7038| 7038⁺| 7026| 7022 |
| $2D$   | 7367                              | 7361| 7346| 7340| 7335| 7347⁺ |

⁺ Carried out to the first order.

⁺⁺ Carried out up to the second order correction.
TABLE V. The $\bar{c}b$ mass spectra and $\Delta_nS$ for various $\Lambda_{MS}$ calculated by using Igi-Ono potential with $\alpha_s = 0.250$ (all in MeV).

| States $\Lambda_{MS}$ | 100  | 200  | 300  | 400  | 500  | 100  | 200  | 300  | 400  | 500 |
|----------------------|------|------|------|------|------|------|------|------|------|-----|
| $1S$                 | 6327 | 6329 | 6316 | 6310 | 6321 | 6323 | 6331 | 6321 | 6316 | 6300 |
| $1^1S_0$             | 6276 | 6263 | 6256 | 6268 | 6273 | 6277 | 6271 | 6265 | 6250 | 6239 |
| $1^3S_1$             | 6347 | 6334 | 6328 | 6339 | 6339 | 6349 | 6338 | 6332 | 6317 | 6307 |
| $\Delta_{1S}$        | 71   | 71   | 72   | 72   | 67   | 72   | 67   | 67   | 67   | 68  |
| $2S$                 | 6906 | 6915 | 6902 | 6881 | 6885 | 6897 | 6914 | 6895 | 6910 | 6911 |
| $2^1S_0$             | 6888 | 6876 | 6856 | 6861 | 6873 | 6888 | 6869 | 6883 | 6883 | 6878 |
| $2^3S_1$             | 6924 | 6911 | 6889 | 6893 | 6905 | 6923 | 6904 | 6920 | 6920 | 6912 |
| $\Delta_{2S}$        | 36   | 35   | 33   | 32   | 32   | 35   | 35   | 37   | 37   | 34  |

*Here we cite Ref. [6].
TABLE VI. The $c\bar{b}$ mass spectra and $\Delta_nS$ using Igi-Ono (type III) and Chen-Kuang potentials.

| $\Lambda_{MS}$ | 300 | 300 | 390 | 100 | 180 | 375 |
|----------------|-----|-----|-----|-----|-----|-----|
| $b = 16.3$    | 19  | 19  |     |     |     |     |
| $m_c = 1.506^*$ | 1.478$^\dagger$ |     |     |     |     |     |
| $m_b = 4.897^*$ | 4.876$^\dagger$ |     |     |     |     |     |
| $M(c\bar{b})$ |     |     |     |     |     |     |
| $1S$           | 6309$^\dagger$ | 6337 | 6298 | 6323 | 6323 | 6323 |
| $1^3S_1$       | 6326 | 6354 | 6318 | 6340 | 6340 | 6340 |
| $1^1S_0$       | 6258 | 6287 | 6238 | 6272 | 6272 | 6272 |
| $\Delta_1S$   | 67  | 67  | 81  | 68  | 68  | 68  |
| $2S$           | 6880 | 6898 | 6878 | 6879 | 6879 | 6879 |
| $2^3S_1$       | 6889 | 6906 | 6886 | 6888 | 6888 | 6888 |
| $2^1S_0$       | 6855 | 6874 | 6853 | 6853 | 6853 | 6853 |
| $\Delta_2S$   | 33  | 33  | 33  | 35  | 35  | 35  |
| $3S$           | 7247 | 7262 | 7255 | 7257 | 7257 | 7257 |
| $4S$           | 7553 | 7569 | 7564 | 7570 | 7570 | 7570 |
| $1P$           | 6725 | 6749 | 6738 | 6723 | 6723 | 6723 |
| $2P$           | 7124 | 7142 | 7136 | 7126 | 7126 | 7126 |
| $1D$           | 6997 | 7018 | 7014 | 6992 | 6992 | 6992 |
| $2D$           | 7328 | 7345 | 7342 | 7332 | 7332 | 7332 |

$^*$. Mass (in GeV) fitted for Igi-Ono (type III) potential.

$^\dagger$. Mass (in GeV) fitted for Chen-Kuang potential.

$^\ddagger$. Here we cite Ref. [6].
TABLE VII. The fine and hyperfine splittings in our work compared with that in other works.

| States | Quadratic* | Linear* | Fulcher* | Lattice* | C-Q† | I-O† |
|--------|------------|---------|----------|----------|------|------|
| \(b =\) |            |         |          |          |      | 16.3 |
| \(\Lambda_{\overline{MS}} =\) |            |         |          |          | 100  | 375  | 300 |
| Fine splittings |            |         |          |          |      |      |
| \(M(2S) - M(1S)\) | 558† | 533  | 579  | 672 ± 120 | 556  | 571  |
| \(M(3S) - M(1S)\) | 931 | 899  | 934  | 938  |
| \(M(2P) - M(1P)\) | 382 | 376  | 403  | 399  |
| \(M(2D) - M(1D)\) | 324 | 321  | 340  | 331  |
| \(b =\) | 20† | 5† | 19† | 19† | 16.3† |
| \(\Lambda_{\overline{MS}} =\) | 400 | 100  | 390  | 300  | 300  |
| \(M(2S) - M(1S)\) | 558 | 564  | 583  | 580  | 561  | 571  |
| \(M(3S) - M(1S)\) | 931 | 925  | 927  | 957  | 927  | 938  |
| Hyperfine splittings |            |         |          |          |      |      |
| \(\Delta_{1S}\) | 76 | 62  | 55  | 41 ± 20 | 68  | 67  |
| \(\Delta_{2S}\) | 42 | 33  | 32  | 30 ± 8 | 35  | 33  |

* Here we cite Ref.[6].
† We used SLNET.
‡ Splitting masses (in MeV).