BRANCHED HOLOMORPHIC CARTAN GEOMETRIES AND
CALABI–YAU MANIFOLDS

INDRANIL BISWAS AND SORIN DUMITRESCU

Abstract. We introduce the concept of a branched holomorphic Cartan geometry. It
generalizes to higher dimension the definition of branched (flat) complex projective structure
on a Riemann surface introduced by Mandelbaum [Ma1]. This new framework is much more
flexible than that of the usual holomorphic Cartan geometries. We show that all compact
complex projective manifolds admit a branched flat holomorphic projective structure. We
also give an example of a non-flat branched holomorphic normal projective structure on
a compact complex surface. It is known that no compact complex surface admits such
a structure with empty branching locus. We prove that non-projective compact simply
connected Kähler Calabi–Yau manifolds do not admit any branched holomorphic projective
structures. The key ingredient of its proof is the following result of independent interest:
If \( E \) is a holomorphic vector bundle over a compact simply connected Kähler Calabi–Yau
manifold, and \( E \) admits a holomorphic connection, then \( E \) is a trivial holomorphic vector
bundle, and any holomorphic connection on \( E \) is trivial.

Contents

1. Introduction
2. Holomorphic Cartan geometry and branched holomorphic Cartan geometry
   2.1. Holomorphic Cartan geometry
   2.2. Developing curves
   2.3. Branched holomorphic Cartan geometry
   2.4. The developing map
3. Examples of branched holomorphic Cartan geometries
   3.1. The standard model
   3.2. Flat Cartan geometries
   3.3. Construction of branched holomorphic Cartan geometries
   3.4. Branched flat affine and projective structures
   3.5. Branched normal holomorphic projective structure on complex surfaces
4. A criterion
5. Holomorphic projective structure on parallelizable manifolds

2010 Mathematics Subject Classification. 53B21, 53C56, 53A55.
Key words and phrases. Complex projective structure, branched Cartan geometry, Calabi–Yau manifold,
complex surface.
1. Introduction

The uniformization theorem for Riemann surfaces asserts that any Riemann surface is isomorphic either to the projective line $\mathbb{C}P^1$, or to a quotient of $\mathbb{C}$, or of the unit disk in $\mathbb{C}$, by a discrete group of projective transformations (lying in the Möbius group $\text{PGL}(2, \mathbb{C})$).

In particular, any Riemann surface $X$ admits a holomorphic atlas with coordinates in $\mathbb{C}P^1$ and transition maps in $\text{PGL}(2, \mathbb{C})$. This defines a \textit{(flat) complex projective structure} on $X$.

Complex projective structure on Riemann surfaces were introduced in connection with the study of the second order ordinary differential equations on complex domains and they had a very major role to play in understanding the framework of uniformization theorem [Gu, StG].

The complex projective line acted on by the Möbius group is a geometry in the sense of Klein’s Erlangen program in which he proposed to study Euclidean, affine and projective geometries in the unifying frame of the homogeneous model spaces $G/H$, where $G$ is a finite dimensional Lie group and $H$ a closed subgroup in $G$.

Following Ehresmann [Eh], a manifold $X$ is locally modelled on a homogeneous space $G/H$, if $X$ admits an atlas with charts in $G/H$ and transition maps given by elements in $G$ using its left-translation action on $G/H$. Any $G$-invariant geometric feature of $G/H$ will have an intrinsic meaning on $X$.

Elie Cartan generalized Klein’s homogeneous model spaces to \textit{Cartan geometries} (or \textit{Cartan connections}) (see definition in Section 2.1). We recall that these are geometrical structures infinitesimally modelled on the homogeneous spaces $G/H$. A Cartan geometry associated to the affine (respectively, projective) geometry is classically called an affine (respectively, projective) connection. A Cartan geometry on a manifold $X$ is equipped with a curvature tensor (see definition in Section 2.1) which vanishes exactly when $X$ is locally modelled on $G/H$ in the sense of Ehresmann [Eh]. In such a situation the Cartan geometry is called \textit{flat}.

In this article we study holomorphic Cartan geometries on compact complex manifolds of complex dimension at least two. Contrary to the situation of Riemann surfaces, holomorphic Cartan geometries in higher dimension are not always flat. Moreover, for a compact complex manifold, to admit a holomorphic Cartan geometry is a very stringent condition: indeed most of the compact complex manifolds do not admit any holomorphic Cartan geometry.

In [KO], Kobayashi and Ochiai proved that compact complex surfaces admitting a holomorphic projective connection are biholomorphic either to the complex projective plane $\mathbb{C}P^2$, or to a quotient of an open set in $\mathbb{C}P^2$ by a discrete group of projective transformations acting properly and discontinuously on it. In particular, such a compact complex surface also admits a flat complex projective structure (modelled on $\mathbb{C}P^2$). In this list of compact complex surfaces admitting (flat) complex projective structures, the only \textit{projective} ones are $\mathbb{C}P^2$, ...
abelian varieties (and their unramified finite quotients) and quotients of the ball (complex hyperbolic plane).

Another source of inspiration for this paper is the work of Mandelbaum [Ma1, Ma2] who introduced and studied branched affine and projective structures on Riemann surfaces. According to his definition, branched projective structures on Riemann surfaces are given by some holomorphic atlas where local charts are finite branched coverings on open sets in \( \mathbb{C}P^1 \) and transition maps lie in PGL(2, \( \mathbb{C} \)). Such structures arise naturally in the study of conical hyperbolic structures, and also when one consider ramified coverings.

Here we define a more general notion of branched holomorphic Cartan geometry on a complex manifold \( X \) (see Definition 2.1), which is valid also in higher dimension and encompass non-flat geometries. We show that the notion of curvature continues to hold, and in fact the curvature vanishes exactly when there is a holomorphic atlas where local charts are branched holomorphic maps to the model \( G/H \). Two local charts agree up to the action on \( G/H \) of an element in \( G \). The geometric description of the flat case follows the description in the usual case: there exists a branched holomorphic developing map from the universal cover of \( X \) to the model \( G/H \) which is a local biholomorphism away from a divisor. This developing map is equivariant with respect to the monodromy homomorphism (which is a group homomorphism from the fundamental group of \( X \) into \( G \), unique up to post-composition by inner automorphisms of \( G \)).

This new notion of branched Cartan geometry is much more flexible than the usual one. For example, all compact complex projective manifolds admit a branched flat holomorphic projective structure (see Proposition 3.1).

We also prove that there exist branched normal holomorphic projective connections (see definition in Section 2.3) on compact surfaces which are not flat (see Proposition 3.4). This is not the case for holomorphic projective connections with empty branching set, meaning any normal projective structure on a compact complex surface is known to be automatically flat [Du3].

The following is proved in Theorem 6.2.

If \( E \) is a holomorphic vector bundle over a compact simply connected Kähler Calabi–Yau manifold, and \( E \) admits a holomorphic connection, then \( E \) is a trivial holomorphic vector bundle equipped with the trivial connection.

This result, which is of independent interest, is related to the classification of branched holomorphic Cartan geometries on Calabi–Yau manifolds. It yields Corollary 6.3 that asserts the following:

1. Any branched holomorphic Cartan geometry of type \( G/H \), with \( G \) complex affine Lie group, on a compact simply connected (Kähler) Calabi–Yau manifold is flat. Consequently, the model \( G/H \) of the Cartan geometry must be compact.

2. Non-projective compact simply connected Kähler Calabi–Yau manifolds do not admit any branched holomorphic projective structure.

The structure of this paper is as follows. Section 2 introduces the main notation and definitions. Section 3 gives interesting examples of branched holomorphic Cartan geometries
and contains the proofs of Proposition 3.1 and Proposition 3.4. In Section 4 we give a criterion (Theorem 4.1) for the existence of a branched holomorphic Cartan geometry. In Section 5 we study holomorphic projective structures on compact parallelizable manifolds. Section 6 deals with branched holomorphic Cartan geometries on Calabi–Yau manifolds, and it contains the proofs of Theorem 6.2 and Corollary 6.3.

2. Holomorphic Cartan geometry and branched holomorphic Cartan geometry

2.1. Holomorphic Cartan geometry. We first recall the definition of a holomorphic Cartan geometry.

Let $G$ be a connected complex Lie group and $H \subset G$ a connected complex Lie subgroup. The Lie algebras of $H$ and $G$ will be denoted by $\mathfrak{h}$ and $\mathfrak{g}$ respectively.

Let $X$ be a connected complex manifold and $f : E_H \to X$ (2.1) a holomorphic principal $H$–bundle on $X$. Let

$$E_G := E_H \times^H G \xrightarrow{f_G} X$$

(2.2)

be the holomorphic principal $G$–bundle on $X$ obtained by extending the structure group of $E_H$ using the inclusion of $H$ in $G$. So, $E_G$ is the quotient of $E_H \times G$ where two points $(c_1, g_1), (c_2, g_2) \in E_H \times G$ are identified if there is an element $h \in H$ such that $c_2 = c_1h$ and $g_2 = h^{-1}g_1$. The projection $f_G$ in (2.2) is induced by the map $E_H \times G \to X$, $(c, g) \mapsto f(c)$, where $f$ is the projection in (2.1). The action of $G$ on $E_G$ is induced by the action of $G$ on $E_H \times G$ given by the right–translation action of $G$ on itself. Let $\text{ad}(E_H) = E_H \times^H \mathfrak{h}$ and $\text{ad}(E_G) = E_G \times^G \mathfrak{g}$ be the adjoint vector bundles for $E_H$ and $E_G$ respectively. We recall that $\text{ad}(E_H)$ (respectively, $\text{ad}(E_G)$) is the quotient of $E_H \times \mathfrak{h}$ (respectively, $E_G \times \mathfrak{g}$) where two points $(z_1, v_1)$ and $(z_2, v_2)$ are identified if there is an element $g \in H$ (respectively, $g \in G$) such that $z_2 = z_1g$ and $v_1$ is taken to $v_2$ by the automorphism of the Lie algebra $\mathfrak{h}$ (respectively, $\mathfrak{g}$) given by automorphism of the Lie group $H$ (respectively, $G$) defined by $y \mapsto g^{-1}yg$. We have a short exact sequence of holomorphic vector bundles on $X$

$$0 \to \text{ad}(E_H) \xrightarrow{i_1} \text{ad}(E_G) \to \text{ad}(E_G)/\text{ad}(E_H) \to 0.$$  (2.3)

The holomorphic tangent bundle of a complex manifold $Y$ will be denoted by $TY$. Let

$$\text{At}(E_H) = (TE_H)/H \to X$$

and

$$\text{At}(E_G) = (TE_G)/G \to X$$

be the Atiyah bundles for $E_H$ and $E_G$ respectively; see [At]. Let

$$0 \to \text{ad}(E_H) \xrightarrow{i_2} \text{At}(E_H) \xrightarrow{q_H} TX \to 0$$

(2.4)

and

$$0 \to \text{ad}(E_G) \xrightarrow{i_0} \text{At}(E_G) \xrightarrow{q_G} TX \to 0$$

(2.5)

be the Atiyah exact sequences for $E_H$ and $E_G$ respectively; see [At]. The projection $q_H$ (respectively, $q_G$) is induced by the differential of the map $f$ (respectively, $f_G$) in (2.1) (respectively, (2.2)). A holomorphic connection on a holomorphic principal bundle is defined
to be a holomorphic splitting of the Atiyah exact sequence associated to the principal bundle $At$. Therefore, a holomorphic connection on $E_G$ is a holomorphic homomorphism

$$\psi : At(E_G) \longrightarrow \text{ad}(E_G)$$

such that $\psi \circ \iota_0 = \text{Id}_{\text{ad}(E_G)}$.

A holomorphic Cartan geometry on $X$ of type $G/H$ is a pair $(E_H, \theta)$, where $E_H$ is a holomorphic principal $H$–bundle on $X$ and

$$\theta : At(E_H) \longrightarrow \text{ad}(E_G)$$

is a holomorphic isomorphism of vector bundles such that $\theta \circ \iota_2 = \iota_1$ (see (2.4) and (2.3) for $\iota_2$ and $\iota_1$ respectively). Therefore, we have the following commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{ad}(E_H) & \stackrel{\iota_2}{\longrightarrow} & At(E_H) & \stackrel{q_H}{\longrightarrow} & TX & \longrightarrow & 0 \\
\| & & \downarrow{\theta} & & \downarrow{\phi} & & \\
0 & \longrightarrow & \text{ad}(E_H) & \stackrel{\iota_1}{\longrightarrow} & \text{ad}(E_G) & \longrightarrow & \text{ad}(E_G)/\text{ad}(E_H) & \longrightarrow & 0
\end{array}
$$

[Sh, Ch. 5]; the above homomorphism $\phi$ induced by $\theta$ is evidently an isomorphism.

We can embed $\text{ad}(E_H)$ in $At(E_H) \oplus \text{ad}(E_G)$ by sending any $v$ to $(\iota_2(v), -\iota_1(v))$ (see (2.4), (2.3)). The Atiyah bundle $At(E_G)$ is the quotient $(At(E_H) \oplus \text{ad}(E_G))/\text{ad}(E_H)$ for this embedding. The inclusion of $\text{ad}(E_G)$ in $At(E_G)$ in (2.3) is given by the inclusion $\iota_1$ or $\iota_2$ of $\text{ad}(E_G)$ in $At(E_H) \oplus \text{ad}(E_G)$ (note that they give the same homomorphism to the quotient bundle $(At(E_H) \oplus \text{ad}(E_G))/\text{ad}(E_H)$).

Given a holomorphic Cartan geometry $(E_H, \theta)$ of type $G/H$ on $X$, the homomorphism

$$At(E_H) \oplus \text{ad}(E_G) \longrightarrow \text{ad}(E_G), \quad (v, w) \mapsto \theta(v) + w$$

produces a homomorphism

$$\theta' : At(E_G) = (At(E_H) \oplus \text{ad}(E_G))/\text{ad}(E_H) \longrightarrow \text{ad}(E_G)$$

which satisfies the condition that $\theta' \circ \iota_0 = \text{Id}_{\text{ad}(E_G)}$, meaning $\theta'$ is a holomorphic splitting of (2.5). Therefore, $\theta'$ is a holomorphic connection on the principal $G$–bundle $E_G$.

The curvature $\text{Curv}(\theta')$ of the connection $\theta'$ is a holomorphic section

$$\text{Curv}(\theta') \in H^0(X, \text{ad}(E_G) \otimes \Omega^2_X),$$

where $\Omega^i_X := \bigwedge^i(TX)^*$.

The Cartan geometry $(E_H, \theta)$ is called normal if

$$\text{Curv}(\theta') \in H^0(X, \text{ad}(E_H) \otimes \Omega^2_X)$$

[Sh, Ch. 8, § 2, p. 338].

The Cartan geometry $(E_H, \theta)$ is called flat if

$$\text{Curv}(\theta') = 0$$

[Sh, Ch. 5, § 1, p. 177]. Consequently, flat Cartan geometries are normal.

If $(E_H, \theta)$ is a holomorphic Cartan geometry, then the isomorphism $\theta$ can be interpreted as a $g$–valued holomorphic 1–form $\beta$ on $E_H$ satisfying the following three conditions:
(1) the homomorphism $\beta : TE_H \rightarrow E_H \times g$ is an isomorphism,
(2) $\beta$ is $H$–equivariant with $H$ acting on $g$ via conjugation, and
(3) the restriction of $\beta$ to each fiber of $f$ coincides with the Maurer–Cartan form associated to the action of $H$ on $E_H$.

(See [Sh].)

2.2. Developing curves. This subsection is based on [Br]. Consider $f_G = E_G \rightarrow X$ and a second projection map $h_G : E_G \rightarrow G/H$ which associates to each class $(c, g) \in E_G$ the element $gH \in G/H$.

The differentials of the projections $f_G$ and $h_G$ map the horizontal space of the connection $\theta'$ at $(c, g) \in E_G$ isomorphically onto $Tf_G(c)_X$ and onto $Tg_H(G/H)$ respectively. This defines an isomorphism between $T_{f_G(c)}X$ and $T_{g_H}(G/H)$. Hence a Cartan geometry provides a family of 1-jets identifications of the manifold $X$ with the model space $G/H$.

Also, the connection $\theta'$ defines a way to lift differentiable curves in $X$ to $\theta'$-horizontal curves in $E_G$. Moreover this lift is unique once one specifies the starting point of the lifted curve.

2.3. Branched holomorphic Cartan geometry.

Definition 2.1. A branched holomorphic Cartan geometry on $X$ of type $G/H$ is a pair $(E_H, \theta)$, where $E_H$ is a holomorphic principal $H$–bundle on $X$ and
$$\theta : At(E_H) \rightarrow ad(E_G)$$
is a holomorphic homomorphism of vector bundles, such that following three conditions hold:

1. $\theta$ is an isomorphism over a nonempty open subset of $X$, and
2. $\theta \circ \iota_2 = \iota_1$ (see (2.1) and (2.3)).

In other words, we have a commutative diagram
$$0 \rightarrow \text{ad}(E_H) \overset{\iota_2}{\rightarrow} At(E_H) \overset{\theta}{\rightarrow} TX \rightarrow 0$$

and
$$0 \rightarrow \text{ad}(E_H) \overset{\iota_1}{\rightarrow} \text{ad}(E_G) \rightarrow \text{ad}(E_G)/\text{ad}(E_H) \rightarrow 0$$
of holomorphic vector bundles on $X$, such that $\theta$ is an isomorphism over a nonempty open subset of $X$; the homomorphism $\phi$ in (2.6) is induced by $\theta$.

Let $U \subset X$ be the nonempty open subset over which $\theta$ is an isomorphism. From the commutativity of (2.6) it follows that $\phi$ is an isomorphism exactly over $U$.

Lemma 2.2. The complement $X \setminus U$ is a divisor.

Proof. Let $d$ be the complex dimension of $X$. The homomorphism $\phi$ in (2.6) produces a homomorphism
$$\bigwedge^d \phi : \bigwedge^d TX \rightarrow \bigwedge^d (\text{ad}(E_G)/\text{ad}(E_H)),$$
so $\bigwedge^d \phi$ is a holomorphic section of the line bundle $(\bigwedge^d (\text{ad}(E_G)/\text{ad}(E_H))) \otimes \Omega^d_X$. The homomorphism $\phi$ is an isomorphism exactly on the complement of the divisor associated to this
section $\bigwedge^d \phi$ of $(\bigwedge^d (\text{ad}(E_G) / \text{ad}(E_H))) \otimes \Omega^d_X$. Since $\phi$ is an isomorphism exactly over $U$, the complement $X \setminus U$ coincides with the support of the divisor associated to the above section $\bigwedge^d \phi$.

**Definition 2.3.** The divisor associated to the above section $\bigwedge^d \phi$ of $(\bigwedge^d (\text{ad}(E_G) / \text{ad}(E_H))) \otimes \Omega^d_X$ will be called the *branching divisor* for the branched holomorphic Cartan geometry $(E_H, \theta)$ on $X$.

As in the case of usual Cartan geometries, consider the homomorphism

$$\text{At}(E_H) \oplus \text{ad}(E_G) \longrightarrow \text{ad}(E_G), \quad (v, w) \mapsto \theta(v) + w.$$ 

Since $\text{At}(E_G) = (\text{At}(E_H) \oplus \text{ad}(E_G)) / \text{ad}(E_H)$, the above homomorphism produces a holomorphic connection

$$\theta' : \text{At}(E_G) \longrightarrow \text{ad}(E_G) \quad (2.8)$$

on $E_G$.

We will call a branched holomorphic Cartan geometry $(E_H, \theta)$ to be *normal* if

$$\text{Curv}(\theta') \in H^0(X, \text{ad}(E_H) \otimes \Omega^2_X).$$

We will call a branched holomorphic Cartan geometry $(E_H, \theta)$ to be *flat* if

$$\text{Curv}(\theta') = 0.$$

If $(E_H, \theta)$ is a branched holomorphic Cartan geometry, then the homomorphism $\theta$ can be interpreted as a $\mathfrak{g}$–valued holomorphic 1–form $\beta$ on $E_H$ satisfying the following three conditions:

1. the homomorphism $\beta : TE_H \longrightarrow E_H \times \mathfrak{g}$ is an isomorphism over a nonempty open subset of $E_H$,
2. $\beta$ is $H$–equivariant with $H$ acting on $\mathfrak{g}$ via conjugation, and
3. the restriction of $\beta$ to each fiber of $f$ coincides with the Maurer–Cartan form associated to the action of $H$ on $E_H$.

2.4. **The developing map.** Let $X$ be a simply connected complex manifold (it need not be compact), and let $E_H$ be a holomorphic principal $H$–bundle on $X$. Consider the holomorphic principal $G$–bundle $E_G := E_H \times^H G \longrightarrow X$. Let

$$\theta : \text{At}(E_H) \longrightarrow \text{ad}(E_G)$$

be a branched holomorphic Cartan geometry on $X$ of type $G/H$ such that the associated connection $\theta'$ on $E_G$ (constructed in (2.8)) is flat.

Since $E_G$ is a flat principal $G$–bundle over a simply connected manifold, we conclude that $E_G$ is trivializable using the flat connection. Note that the trivialization of $E_G$ is not quite unique. Once we fix a point $z_0$ of $E_G$, there is a unique isomorphism $\xi_{z_0}$ of $E_G$ with the trivial principal $G$–bundle $X \times G \longrightarrow X$ satisfying the following two conditions:

1. $\xi_{z_0}$ takes the connection $\theta'$ on $E_G$ to the trivial connection on the trivial principal $G$–bundle $X \times G$, and
(2) $\xi_{z_0}(z_0) = (f_G(z_0), e)$, where $f_G$ is the projection (as in (2.2)) of $E_G$ to $X$, and $e \in G$ is the identity element.

If we replace $z_0$ by another point $z'_0$ of $E_G$, then there is a unique element $g$ of $G$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
E_G & \xrightarrow{\xi_{z_0}} & E_G \\
\downarrow & & \downarrow \\
X \times G & \xrightarrow{L_g} & X \times G
\end{array}
\]

where $L_g$ is the diffeomorphism $(x, h) \mapsto (x, gh)$ of $X \times G$.

Fix an element $z_0 \in E_G$. Identify $E_G$ with the trivial principal $G$–bundle $X \times G \rightarrow X$ using $z_0$. Using this identification, the reduction $E_H$ of $E_G$ becomes a holomorphic reduction of structure group of $X \times G \rightarrow X$ to the subgroup $H \subset G$. Now we observe that any holomorphic reduction of $X \times G \rightarrow X$ to $H$ is given by a holomorphic map

$$\varpi : X \rightarrow G/H.$$ 

To see this, let $p_H : G \rightarrow G/H$ be the quotient map. Then

$$(\text{Id}_X \times p_H)^{-1}((\text{graph}(\varpi))) \subset X \times G$$

is a reduction of structure group to $H$, where $\text{graph}(\varpi) \subset X \times (G/H)$ is the graph of $\varpi$ consisting of all points of the form $\{(x, \varpi(x))\}_{x \in X}$. Conversely, any holomorphic reduction of structure group of $X \times G \rightarrow X$ to $H$ gives a holomorphic map from $X$ to $G/H$.

Let

$$\varpi : X \rightarrow G/H$$

be the holomorphic map for the reduction $E_H$ of $E_G = X \times G$ to $H$. This map $\varpi$ is a developing map for the branched holomorphic Cartan geometry $\theta$. If we replace $z_0$ by another point of $E_G$, then the developing map gets composed with a left–translation of $G/H$ by an element of $G$.

From the definition of a branched holomorphic Cartan geometry of type $G/H$ it follows that $\varpi$ is a local biholomorphism over the complement of the branching divisor.

3. Examples of branched holomorphic Cartan geometries

3.1. The standard model. We recall the standard (flat) Cartan geometry of type $G/H$.

Set $X = G/H$. Let $F_H$ be the holomorphic principal $H$–bundle on $X$ defined by the quotient map $G \rightarrow G/H$; we use the notation $F_H$ instead of $E_H$ because it is a special case which will play a role later. Identify the Lie algebra $\mathfrak{g}$ with the right–invariant vector fields on $G$. This produces an isomorphism of $\text{At}(F_H)$ with $\text{ad}(F_G)$ and hence a Cartan geometry of type $G/H$ on $X$ (we use the notation $F_G$ instead of $E_G$ for the same reason as above). Equivalently, the tautological holomorphic $\mathfrak{g}$–valued 1–form on $G = F_H$ satisfies all the three conditions needed to define a Cartan geometry of type $G/H$ (see the last paragraph of Section 2.1).

The above holomorphic $\mathfrak{g}$–valued 1–form on $G = F_H$ will be denoted by $\theta_{G,H}$. 
3.2. **Flat Cartan geometries.** A $(E_H, \theta)$ holomorphic Cartan geometry of type $G/H$ is flat if and only if it is locally isomorphic to $(F_H, \theta_{G,H})$ \[Sh\] Ch. 5, § 5, Theorem 5.1].

If a complex manifold $X$ admits a flat holomorphic Cartan geometry of type $G/H$, then $X$ admits a covering by open subsets $U_i$ and biholomorphisms onto open subsets of $G/H$

$$\phi_i : U_i \rightarrow G/H$$

such that each transition map

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)$$

is, on each connected component, the restriction of an automorphism of $G/H$ given by the left-translation action of an element $g_{ij} \in G$ \[Sh\] Ch. 5, § 5, Theorem 5.2].

Following Ehresmann, \[Eh\], one classically defines then a *monodromy homomorphism* $\rho$ from the fundamental group $\pi_1(X)$ of $X$ into $G$ and a developing map $\delta : \tilde{X} \rightarrow G/H$ which is a $\pi_1(X)$-equivariant local biholomorphism from the universal cover $\tilde{X}$ into the model $G/H$.

The same strategy was used in Section 2.4 to adapt the proof to the branched case.

3.3. **Construction of branched holomorphic Cartan geometries.** Let $X$ be a connected complex manifold and

$$\gamma : X \rightarrow G/H$$

a holomorphic map such that the differential

$$d\gamma : TX \rightarrow T(G/H)$$

is an isomorphism over a nonempty subset of $X$.

The above condition on $d\gamma$ is equivalent to the condition that $\dim X = \dim(G/H)$ with $\gamma(X)$ containing a nonempty open subset of $G/H$. Note that in general, the homomorphism $d\gamma$ is always an isomorphism over an open subset of $X$, which may be empty.

Set $E_H$ to be the pullback $\gamma^*F_H$. Note that we have a holomorphic map $\eta : E_H \rightarrow F_H$ which is $H$–equivariant and fits in the commutative diagram

$$\begin{array}{ccc}
E_H & \xrightarrow{\eta} & F_H \\
\downarrow & & \downarrow \\
X & \xrightarrow{\gamma} & G/H
\end{array}$$

Then $(E_H, \eta^*\theta_{G,H})$ defines a branched Cartan geometry of type $G/H$ on $X$.

To describe the above branched Cartan geometry in terms of the Atiyah bundle, first note that $\text{At}(\gamma^*F_H)$ coincides with the subbundle of the vector bundle $\gamma^*\text{At}(F_H) \oplus TX$ given by the kernel of the homomorphism

$$\gamma^*\text{At}(F_H) \oplus TX \rightarrow \gamma^*T(G/H), \quad (v, w) \mapsto \gamma^* q_{G,H}(v) - d\gamma(w),$$

where $q_{G,H} : \text{At}(F_H) \rightarrow T(G/H)$ is the natural projection (see (2.3)), and

$$d\gamma : TX \rightarrow \gamma^*T(G/H)$$
is the differential of $\gamma$. Consider the standard Cartan geometry $\theta_{G,H} : \text{At}(F_H) \to \text{ad}(F_G)$ of type $G/H$ on the quotient $G/H$. The restriction of the homomorphism

$$\gamma^* \text{At}(F_H) \oplus TX \to \gamma^* \text{ad}(F_G), \quad (a, b) \mapsto \gamma^* \theta_{G,H}(a)$$

to $\text{At}(\gamma^* F_H) \subset \gamma^* \text{At}(F_H) \oplus TX$ is a homomorphism

$$\text{At}(\gamma^* F_H) \to \text{ad}(\gamma^* F_G) = \gamma^* \text{ad}(F_G) = \text{ad}(E_G),$$

which defines a branched holomorphic Cartan geometry of type $G/H$ on $X$.

The divisor of $X$ over which the above branched Cartan geometry of type $G/H$ on $X$ fails to be a Cartan geometry evidently coincides with the divisor over which the differential $d\gamma$ fails to be an isomorphism.

The curvature of the holomorphic connection on $E_G$ associated to the above branched Cartan geometry of type $G/H$ on $X$ vanishes identically. Indeed, this follows immediately from the fact that the standard Cartan geometry $\theta_{G,H}$ is flat. In particular, this branched Cartan geometry on $X$ is normal.

The developing map for this flat branched Cartan geometry on $X$ is the map $\gamma$ itself.

Conversely, let $X$ be a complex manifold endowed with a branched flat holomorphic Cartan geometry with branching divisor $D$. Then Section 2.4 (which is an adaptation in the branched case of the proof of Theorem 5.2 in [Sh, Ch. 5, §5]) shows that $X$ admits a covering by open (simply connected) subsets $U_i$ such that for each $i$ there exists a holomorphic map $\phi_i : U_i \to G/H$ which is a local biholomorphism on $U_i \setminus (U_i \cap D)$; moreover, for every pair $i, j$, each connected component of the overlap $U_i \cap U_j$, there exists a $g_{ij} \in G$ such that $g_{ij} \circ f_j = f_i$ on the entire connected component.

Then the Ehresmann method [EH] (based by analytic continuation of charts along paths) defines a monodromy morphism $\rho : \pi_1(X) \to G$ and a developing map $\delta : \tilde{X} \to G/H$ which is a local biholomorphism away from the pull-back of $D$ to the universal cover.

### 3.4. Branched flat affine and projective structures.

Let us recall the standard model $G/H$ of the affine geometry.

Consider the semi-direct product $\mathbb{C}^d \rtimes \text{GL}(d, \mathbb{C})$ for the standard action of $\text{GL}(d, \mathbb{C})$ on $\mathbb{C}^d$. This group $\mathbb{C}^d \rtimes \text{GL}(d, \mathbb{C})$ is identified with the group of all affine transformations of $\mathbb{C}^d$. Set $H = \text{GL}(d, \mathbb{C})$ and $G = \mathbb{C}^d \rtimes \text{GL}(d, \mathbb{C})$.

A holomorphic affine structure (or equivalently holomorphic affine connection) on a complex manifold $X$ of dimension $d$ is a holomorphic Cartan geometry of type $G/H$. This terminology comes from the fact that the bundle $F_H$ will be automatically isomorphic to the holomorphic frame bundle of $X$ and the form $\theta_{G,H}$ defines a holomorphic connection in the holomorphic tangent bundle of $X$. Conversely, any holomorphic connection in the holomorphic tangent bundle of $X$ uniquely defines a holomorphic Cartan geometry of type $G/H$ (where $G$ and $H$ are as above). This connection is torsion-free exactly when the Cartan geometry is normal [MM]. For more details on the equivalence between the several definitions of a holomorphic affine connection (especially with the one seeing the connection as
an operator $\nabla$ acting on local holomorphic vector fields and satisfying the Leibniz rule), the reader is referred to [MM, Sh].

A branched holomorphic Cartan geometry of type $\mathbb{C}^d \times \mathrm{GL}(d, \mathbb{C})/\mathrm{GL}(d, \mathbb{C})$ will be called a branched holomorphic affine structure or a branched holomorphic affine connection.

We also recall that a holomorphic projective structure (or a holomorphic projective connection) on a complex manifold $X$ of dimension $d$ is a holomorphic Cartan geometry of type $\mathrm{PGL}(d+1, \mathbb{C})/Q$, where $Q \subset \mathrm{PGL}(d+1, \mathbb{C})$ is the maximal parabolic subgroup that fixes a given point for the standard action of $\mathrm{PGL}(d+1, \mathbb{C})$ on $\mathbb{C}P^d$ (the space of lines in $\mathbb{C}^{d+1}$). In particular, there is a standard holomorphic projective structure on $\mathrm{PGL}(d+1, \mathbb{C})/Q = \mathbb{C}P^d$. Locally a holomorphic projective connection is an equivalence class of holomorphic affine connections, where two affine connections are considered to be equivalent if they admit the same unparametrized geodesics. The projective connection is normal exactly when it admits a local representative which is a torsionfree affine connection [MM, OT].

We will call a branched holomorphic Cartan geometry of type $\mathrm{PGL}(d+1, \mathbb{C})/Q$ a branched holomorphic projective structure or a branched holomorphic projective connection.

**Proposition 3.1.** Every compact complex projective manifold admits a branched flat holomorphic projective structure.

**Proof.** Let $X$ be a compact complex projective manifold of complex dimension $d$. Then there exists a finite surjective algebraic, hence holomorphic, morphism 

$$\gamma : X \longrightarrow \mathbb{C}P^d.$$

Indeed, one proves that the smallest integer $N$ for which there exists a finite morphism $f$ from $X$ to $\mathbb{C}P^N$ is $d$. If $N > d$, then there exists $P \in \mathbb{C}P^N \setminus f(X)$; now consider the projection $\pi : \mathbb{C}P^N \setminus \{P\} \longrightarrow \mathbb{C}P^{N-1}$. The fibers of $\pi \circ f$ must be finite (otherwise $f(X)$ would contain a line through $P$, hence $P$). Since $\pi \circ f$ is a proper morphism with finite fibers, it must be finite.

Now we can pull back the standard holomorphic projective structure on $\mathbb{C}P^d$ using the above map $\gamma$ to get a branched holomorphic projective structure on $X$. \qed

**Proposition 3.2.**

(i) Simply connected compact complex manifolds do not admit any branched flat holomorphic affine structure.

(ii) Simply connected compact complex manifolds admitting a branched flat holomorphic projective structure are Moishezon.

**Proof.** (i) If, by contradiction, a simply connected compact complex manifold $X$ admits a branched flat holomorphic affine structure, then the developing map $\delta : X \longrightarrow \mathbb{C}^d$ is holomorphic and nonconstant, which is a contradiction.

(ii) If $X$ is a simply connected manifold of complex dimension $d$ admitting a branched flat holomorphic projective structure, then its developing map is a holomorphic map $\delta : X \longrightarrow \mathbb{C}P^d$ which is a local biholomorphism away from a divisor $D$ in $X$. Thus, the algebraic dimension of $X$ must be $d$; consequently, $X$ is Moishezon. \qed
Since any given compact Kähler manifold is Moishezon if and only if it is projective, one gets the following:

**Corollary 3.3.** Non-projective simply connected Kähler manifolds do not admit any branched flat holomorphic projective structure.

In particular, non-projective $K3$ surfaces do not admit any branched flat holomorphic projective structure.

3.5. Branched normal holomorphic projective structure on complex surfaces. In [KO], Kobayashi and Ochiai classified all compact complex surfaces admitting a holomorphic projective structure (connection). All of them happen to be isomorphic to quotient of open subsets of $\mathbb{C}P^2$ by discrete subgroups of $\text{PGL}(3, \mathbb{C})$ acting properly and discontinuously. Consequently, all of them also admit a flat holomorphic projective structure. Among those surfaces, the only projective ones are the following: $\mathbb{C}P^2$, surfaces covered by the ball and the abelian varieties (and their finite unramified quotients).

Moreover, it is known that every normal projective structure (connection) on a compact complex surface is automatically flat [Du3].

**Proposition 3.4.** There exists branched holomorphic projective structures on compact complex surfaces which are normal, but not flat.

**Proof.** Let $Y$ be a compact connected Riemann surface of genus at least two. Fix two holomorphic 1–forms $\alpha_1, \alpha_2 \in H^0(X, \Omega^1_X)$ that are linearly independent. Set $X = Y \times Y$. Let $Q \subset \text{PGL}(3, \mathbb{C})$ be the maximal parabolic subgroup that fixes the point $(1, 0, 0) \in \mathbb{C}P^2$ for the standard action of $\text{PGL}(3, \mathbb{C})$ on $\mathbb{C}P^2$. Set $H = Q$ and $G = \text{PGL}(3, \mathbb{C})$.

Let $E_H = X \times H \xrightarrow{f} X$ be the trivial holomorphic principal $H$–bundle on $X$. So, the corresponding holomorphic principal $G$–bundle $E_G$ is the trivial holomorphic principal $G$–bundle $X \times G \rightarrow X$. The adjoint vector bundles $\text{ad}(E_H)$ and $\text{ad}(E_G)$ are the trivial vector bundles $X \times \mathfrak{h} \rightarrow X$ and $X \times \mathfrak{g} \rightarrow X$ respectively. The trivialization of $E_H$ produces a trivial holomorphic connection on $E_H$. This connection defines a holomorphic splitting of the Atiyah exact sequence in (2.4). Hence we have

$$\text{At}(E_H) = \text{ad}(E_G) \oplus TX = (X \times \mathfrak{h}) \oplus TX.$$ 

Now let

$$\theta : \text{At}(E_H) \rightarrow \text{ad}(E_G) = X \times \mathfrak{g}$$

be the holomorphic homomorphism which over any point $(y_1, y_2) \in Y \times Y = X$ is defined by

$$(w, (v_1, v_2)) \mapsto w + \begin{pmatrix} 0 & 0 & \alpha_1(y_1)(v_1) \\ \alpha_1(y_1)(v_1) & 0 & 0 \\ \alpha_2(y_2)(v_2) & 0 & 0 \end{pmatrix}, \quad w \in \mathfrak{h}, \ v_i \in T_{y_i}Y.$$ 

Note that the Lie algebra $\mathfrak{g}$ is the space of $3 \times 3$ complex matrices of trace zero, while $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ consisting of matrices $(a_{i,j})^3_{i,j=1}$ with complex entries such that
Let $\theta'$ be the holomorphic connection on $E_G = X \times G \rightarrow X$ associated to $\theta$ (see (2.8)). To describe $\theta'$, let $D_0$ denote the trivial holomorphic connection on $E_G = X \times G$ given by its trivialization. Let 

$$p_i : X = Y \times Y \rightarrow Y, \quad i = 1, 2$$

be the projection to the $i$–th factor. Then we have 

$$\theta' = D_0 + \begin{pmatrix} 0 & 0 & p_i^* \alpha_1 \\ p_i^* \alpha_1 & 0 & 0 \\ p_2^* \alpha_2 & 0 & 0 \end{pmatrix};$$

note that $\text{ad}(E_G) = X \times g$, and 

$$\begin{pmatrix} 0 & 0 & p_i^* \alpha_1 \\ p_i^* \alpha_1 & 0 & 0 \\ p_2^* \alpha_2 & 0 & 0 \end{pmatrix} \in H^0(X, \text{ad}(E_G) \otimes \Omega^1_X)$$

because the diagonal entries are zero. Therefore, the curvature $\text{Curv}(\theta')$ of the connection $\theta'$ has the following expression:

$$\text{Curv}(\theta') = \begin{pmatrix} 0 & 0 & p_i^* \alpha_1 \\ p_i^* \alpha_1 & 0 & 0 \\ p_2^* \alpha_2 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & p_i^* \alpha_1 \\ p_i^* \alpha_1 & 0 & 0 \\ p_2^* \alpha_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} (p_i^* \alpha_1) \wedge (p_2^* \alpha_2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (p_2^* \alpha_2) \wedge (p_i^* \alpha_1) \end{pmatrix}$$

Hence we have 

$$\text{Curv}(\theta') \in H^0(X, \text{ad}(E_H) \otimes \Omega^2_X).$$

So the branched projective structure $(E_H, \theta)$ constructed above in normal. But we have $\text{Curv}(\theta') \neq 0$. \hfill \[ \square \]

We don’t know whether (non-projective) compact complex surfaces admitting branched holomorphic projective structures are exactly those admitting a branched flat holomorphic projective structure.

4. A CRITERION

Let $X$ be a compact connected Kähler manifold of complex dimension $d$ equipped with a Kähler form $\omega$. Chern classes will always mean ones with real coefficients. For a torsionfree coherent analytic sheaf $V$ on $X$, define 

$$\text{degree}(V) := (c_1(V) \cup \omega^{d-1}) \cap [X] \in \mathbb{R}.$$  \hspace{1cm} (4.1)

The degree of a divisor $D$ on $X$ is defined to be $\text{degree}(\mathcal{O}_X(D))$. The degree of a general coherent analytic sheaf on $X$ is the degree of its torsionfree quotient.

Fix an effective divisor $D$ on $X$. Fix a holomorphic principal $H$–bundle $E_H$ on $X$.

**Proposition 4.1.** If $\text{degree}(\Omega_X^1) - \text{degree}(D) \neq \text{degree}(\text{ad}(E_H))$, then there is no branched holomorphic Cartan geometry of type $G/H$ on $X$ with branching divisor $D$ (see Definition 2.3). In particular, if $D \neq 0$ and $\text{degree}(\Omega_X^1) \leq \text{degree}(\text{ad}(E_H))$, then there is no branched holomorphic Cartan geometry of type $G/H$ on $X$ with branching divisor $D$. 

Proof. Let \((E_H, \theta)\) be a branched holomorphic Cartan geometry of type \(G/H\) on \(X\) with branching divisor \(D\). Consider the homomorphism \(\bigwedge^d \phi\) in (2.7). Since \(D\) is the divisor for the corresponding holomorphic section of the line bundle \(\bigwedge^d (\text{ad}(E_G)/\text{ad}(E_H))) \otimes \Omega^1_X\), we have

\[
\text{degree}(D) = \text{degree}(\left(\bigwedge^d (\text{ad}(E_G)/\text{ad}(E_H))) \otimes \Omega^1_X\right))
= \text{degree}(\text{ad}(E_G)) - \text{degree}(\text{ad}(E_H)) + \text{degree}(\Omega^1_X).
\]  

(4.2)

Recall that \(E_G\) has a holomorphic connection \(\theta'\) corresponding to \(\theta\). It induces a holomorphic connection on \(\text{ad}(E_G)\). Hence we have \(c_1(\text{ad}(E_G)) = 0\) [At, Theorem 4], which implies that \(\text{degree}(\text{ad}(E_G)) = 0\). Therefore, from (4.2) it follows that

\[
\text{degree}(\Omega^1_X) - \text{degree}(D) = \text{degree}(\text{ad}(E_H)).
\]  

(4.3)

Hence there is no branched holomorphic Cartan geometry of type \(G/H\) on \(X\) with branching divisor \(D\) if we have \(\text{degree}(\Omega^1_X) - \text{degree}(D) \neq \text{degree}(\text{ad}(E_H))\).

If \(D \neq 0\), then \(\text{degree}(D) > 0\). Hence in that case (4.3) fails if we have \(\text{degree}(\Omega^1_X) \leq \text{degree}(\text{ad}(E_H))\). \(\square\)

Corollary 4.2.

(i) If \(\text{degree}(\Omega^1_X) < 0\), then there is no branched holomorphic affine structure on \(X\).

(ii) If \(\text{degree}(\Omega^1_X) = 0\), then all branched holomorphic affine structures on \(X\) are actually holomorphic affine structures.

Proof. Set \(H = \text{GL}(d, \mathbb{C})\) and \(G = \mathbb{C}^d \rtimes \text{GL}(d, \mathbb{C})\). Recall that a branched holomorphic affine structure on \(X\) is a branched holomorphic Cartan geometry on \(X\) of type \(G/H\), where \(H\) and \(G\) are as above. Let \((E_H, \theta)\) be a branched holomorphic affine structure on the compact Kähler manifold \((X, \omega)\) of dimension \(d\). The homomorphism

\[
\text{M}(d, \mathbb{C}) \otimes \text{M}(d, \mathbb{C}) \longrightarrow \mathbb{C}, \quad A \otimes B \longmapsto \text{Trace}(AB)
\]

is nondegenerate and \(\text{GL}(d, \mathbb{C})\)-invariant. In other words, the Lie algebra \(\mathfrak{h}\) of \(H = \text{GL}(d, \mathbb{C})\) is self-dual as an \(H\)-module. Hence we have \(\text{ad}(E_H) = \text{ad}(E_H)^*\), in particular, the equality

\[
\text{degree}(\text{ad}(E_H)) = 0
\]

holds.

As noted before, for a nonzero effective divisor \(D\) we have \(\text{degree}(D) > 0\). Therefore, the corollary follows from Proposition 4.1. \(\square\)

Remark 4.3. Let \(X\) be a rationally connected compact complex manifold. The proof of Theorem 4.1 in [BM] extends to branched Cartan geometries on \(X\). In other words, any branched Cartan geometry of type \(G/H\) on \(X\) is flat and it is given by a holomorphic map \(X \longrightarrow G/H\) (see Section 3.3). This implies that \(G/H\) is compact.
5. Holomorphic projective structure on parallelizable manifolds

A complex manifold is called \textit{parallelizable} if its holomorphic tangent bundle is holomorphically trivial. We recall that, by a theorem of Wang \cite{Wa}, compact complex parallelizable manifolds are isomorphic to quotients $G/\Gamma$ of complex Lie groups $G$ by a \textit{cocompact} lattice $\Gamma \subset G$ (recall that cocompact (or normal) lattices are those for which the quotient is compact). Such a quotient is known to be Kähler if and only if $G$ is abelian.

All the compact complex parallelizable manifolds admit a holomorphic affine structure (connection) given by the trivialization of the holomorphic tangent bundle (by right-invariant vector fields). As soon as $G$ is non-abelian the holomorphic affine connection for which right-invariant vector fields are parallel have non-vanishing torsion and, consequently, it is not flat.

We will prove the following:

\textbf{Proposition 5.1.} Let $G$ be a complex semi-simple Lie group and $\Gamma$ a cocompact lattice in $G$. Then the quotient $G/\Gamma$ does not admit any branched flat affine structure.

The following lemma will be needed in the proof of Proposition 5.1

\textbf{Lemma 5.2.} Let $G$ be a complex semi-simple Lie group and $\Gamma$ a cocompact lattice in $G$. Then any branched holomorphic Cartan geometry on $X = G/\Gamma$ has an empty branching set.

\textbf{Proof.} Assume, by contradiction, that the branching set is not empty. Then, by Lemma 2.2 the branching set must be a divisor in $X$. On the other hand, it is known, \cite{HM}, that $G/\Gamma$ contains no divisor, which is a contradiction. \hfill \square

Now we go back to the proof of Proposition 5.1

\textbf{Proof of Proposition 5.1} Assume, by contradiction, that $X = G/\Gamma$ admits a branched flat affine structure. Using Lemma 5.2 the branching set must be empty. Consider then the holomorphic affine connection $\nabla$ in the holomorphic tangent bundle $TX$ associated the holomorphic flat affine structure. If $d$ is the complex dimension of $X$, denote by $(V_1, V_2, \ldots, V_d)$ a family of globally defined holomorphic vector fields on $X$ trivializing $TX$ (the $V_i$’s descend from right-invariant vector fields on $G$). For any $i, j$, the holomorphic vector field $\nabla_{V_i} V_j$ is also globally defined on $X$ and must be a linear combination of $V_i$’s with constant coefficients. It now follows that the pull-back of $\nabla_{V_i} V_j$ to $G$ is a right-invariant vector field. This implies that the pull-back to $G$ of $\nabla$ is right-invariant. But it is known, \cite{Du5}, that a semi-simple complex Lie group does not admit translation invariant holomorphic flat affine structures, which is a contradiction. \hfill \square

The simplest example is that of compact quotients of $\text{SL}(2, \mathbb{C})$ by lattices $\Gamma$: they do not admit any branched flat holomorphic affine structure. However, as we will see, they admit flat holomorphic projective structures.

Indeed, the Killing quadratic form on the Lie algebra of $\text{SL}(2, \mathbb{C})$ is nondegenerate. It endows the complex manifold $\text{SL}(2, \mathbb{C})$ with a right-invariant \textit{holomorphic Riemannian metric} in the sense of the following definition.
**Definition 5.3.** A holomorphic Riemannian metric on $X$ is a holomorphic section

$$g \in H^0(X, S^2((TX)^*))$$

such that for every point $x \in X$ the quadratic form $g(x)$ on the fiber $T_xX$ is nondegenerate.

A holomorphic Riemannian metric on a complex manifold of dimension $d$ is a holomorphic Cartan geometry of the type $G/H$, where $H$ is the complex orthogonal group $O(d, \mathbb{C})$ and $G$ is the semi-direct product $\mathbb{C}^d \rtimes O(d, \mathbb{C})$ for the standard action of $O(d, \mathbb{C})$ on $\mathbb{C}^d$ [Sh, Ch. 6].

As in the Riemannian and pseudo-Riemannian setting, one associates to a holomorphic Riemannian metric $g$ a unique holomorphic affine connection $\nabla$. This connection $\nabla$, called the Levi–Civita connection for $g$, is uniquely determined by the following two properties:

- $\nabla$ is torsionfree, and
- the holomorphic tensor $g$ is parallel with respect to $\nabla$.

The curvature of this Levi–Civita connection $\nabla$ vanishes identically if and only if $g$ is locally isomorphic to the standard flat model $dz_1^2 + \ldots + dz_n^2$, seen as a homogeneous space for the group $G = \mathbb{C}^d \rtimes O(d, \mathbb{C})$.

The holomorphic Riemannian metric on $SL(2, \mathbb{C})$ coming from the Killing quadratic form is bi-invariant (since the Killing quadratic form is invariant under the adjoint action of $SL(2, \mathbb{C})$). It has nonzero constant sectional curvature [Gh]. Since the Levi–Civita connection of a metric of constant sectional curvature is known to be projectively flat, this endows $SL(2, \mathbb{C})$ with a bi-invariant flat holomorphic projective structure. For more details about the geometry of holomorphic Riemannian metrics one can see [Gh, Du1, DZ].

Interesting exotic deformations of parallelizable manifolds $SL(2, \mathbb{C})/\Gamma$ were constructed by Ghys in [Gh].

The above mentioned deformations in [Gh] are constructed by choosing a group homomorphism

$$u : \Gamma \longrightarrow SL(2, \mathbb{C})$$

and considering the embedding $\gamma \longmapsto (u(\gamma), \gamma)$ of $\Gamma$ into $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ (acting on $SL(2, \mathbb{C})$ by left and right translations). Algebraically, the action is given by:

$$(\gamma, x) \in \Gamma \times SL(2, \mathbb{C}) \longrightarrow u(\gamma^{-1})x\gamma \in SL(2, \mathbb{C}).$$

It is proved in [Gh] that, for $u$ close enough to the trivial morphism, $\Gamma$ acts properly and freely on $SL(2, \mathbb{C})$ and the quotient $M(u, \Gamma)$ is a compact complex manifold (covered by $SL(2, \mathbb{C})$). In general, these examples do not admit parallelizable manifolds as finite covers. Moreover, for generic $u$, the space of all holomorphic global vector fields on them is trivial. All manifolds $M(u, \Gamma)$ inherit a flat holomorphic projective structure (coming from the bi-invariant projective structure constructed above). Moreover, any small deformation of the manifold $SL(2, \mathbb{C})$ is isomorphic to $M(u, \Gamma)$ for some $u$ [Gh].

Therefore, we get the following:

**Theorem 5.4 (Ghys).** Complex compact parallelizable manifolds $SL(2, \mathbb{C})/\Gamma$ and their small deformations admit a flat holomorphic projective structure.
It is not known whether for generic homomorphisms $u$, complex manifolds $M(u, \Gamma)$ admit any other flat holomorphic projective structure apart from the standard one (that descends from the bi-invariant flat holomorphic projective structure on $\text{SL}(2, \mathbb{C})$ constructed above).

For some non-generic homomorphisms $u$, complex manifolds $M(u, \Gamma)$ also admit holomorphic Riemannian metrics with nonconstant sectional curvature $[\text{Gh}]$. The associated holomorphic projective structures on those manifolds are not flat.

Recall here the main result in [DZ]:

**Theorem 5.5 (DZ).** Let $M$ be a compact complex threefold endowed with a holomorphic Riemannian metric. Then $M$ admits a finite unramified covering bearing a holomorphic Riemannian metric of constant sectional curvature (and hence has the associated flat holomorphic projective structure).

Now we will describe the global geometry of holomorphic projective structures on complex parallelizable manifolds. Let us first prove the following.

**Lemma 5.6.** Consider a holomorphic projective connection on a compact complex manifold $X$ with trivial canonical bundle. Then $X$ admits a holomorphic affine connection $\nabla$ which is projectively isomorphic to the given holomorphic projective connection.

**Proof.** Let $X = \bigcup U_i$ be an open covering of $X$ such that on each $U_i$ there exists a holomorphic affine connection $\nabla_i$ projectively equivalent to the given projective connection. Let $\omega$ be a global nontrivial holomorphic section of the canonical bundle (it is trivial by assumption). On each $U_i$, there exists a unique holomorphic affine connection $\tilde{\nabla}_i$ projectively equivalent to $\nabla_i$ satisfying the condition that $\omega$ is parallel with respect to $\tilde{\nabla}_i$ [OT, Appendix A.3]. By uniqueness, these $\tilde{\nabla}_i$’s agree on the overlaps of the $U_i$’s and define a global holomorphic affine connection on $X$ projectively equivalent to the original holomorphic projective connection (for a different proof one can also combine two results in [Gu, p. 96] and [KO, p. 78–79]). □

The following proposition is proved using Lemma 5.6.

**Proposition 5.7.** Let $G$ be a complex Lie group of dimension $d$ and $\Gamma$ a lattice in $G$. Then $X = G/\Gamma$ admits a flat holomorphic projective structure if and only if there exists a Lie group homomorphism $\tilde{i} : \tilde{\text{G}} \rightarrow \text{PGL}(d+1, \mathbb{C})$, where $\tilde{\text{G}}$ is the universal cover of $G$, such that $\tilde{i}(\tilde{\text{G}})$ acts with an open orbit on the standard model $\mathbb{C}P^d$.

Note that the condition in the statement of Proposition 5.7 is equivalent to the existence of a Lie algebra homomorphism $\tilde{i}$ from the Lie algebra of $G$ into the Lie algebra of $\text{PGL}(d+1, \mathbb{C})$, such that the image of $\tilde{i}$ intersects trivially the Lie subalgebra of the stabilizer $Q$ of a point in $\mathbb{C}P^d$. A classification of those complex Lie algebras admitting such a homomorphism is done in [Ka] (see also [Ag] for the real case).

**Proof of Proposition 5.7.** First assume that there exists a group homomorphism $i : \tilde{\text{G}} \rightarrow \text{PGL}(d+1, \mathbb{C})$ such that $i(\tilde{\text{G}})$ acts on $\mathbb{C}P^d$ with an open orbit $O \subset X$. Fix a point $o \in O$,
and consider the map 
\[ \pi : \tilde{G} \rightarrow O \]
defined by \( \pi(g) = i(g) \cdot o \) for all \( g \in \tilde{G} \). This map \( \pi \) is a covering and the pull-back of the flat holomorphic projective structure on \( O \) through \( \pi \) is a right-invariant flat holomorphic projective structure on \( \tilde{G} \). This flat holomorphic projective structure on \( \tilde{G} \) descends to the quotient \( X = \tilde{G}/\tilde{\Gamma} \), where \( \tilde{\Gamma} \) is the inverse image of \( \Gamma \) in the universal covering \( \tilde{G} \) of \( G \).

To prove the converse, assume that \( G/\Gamma \) is equipped with a flat holomorphic projective structure. By Lemma 5.6, there exists a holomorphic affine connection \( \nabla \) on \( G/\Gamma \) which is projectively equivalent to the given flat holomorphic projective structure. The proof of Proposition 5.1 shows that the pull-back of \( \nabla \) to \( G \) is a right-invariant holomorphic affine connection. In particular, the pull-back of the initial flat holomorphic projective structure to \( G \) is right-invariant. It follows that the Lie algebra of \( G \) acts locally projectively on the standard projective model \( \mathbb{C}P^d \). Since the model is simply connected, this local action extends to a projective locally free global action of \( \tilde{G} \) on \( \text{PGL}(d+1, \mathbb{C}) \). This gives the required Lie group homomorphism \( i \).

It may be remarked that the Lie group homomorphism \( i \) in the statement of Proposition 5.7 extends the monodromy homomorphism \( \rho : \tilde{\Gamma} \rightarrow \text{PGL}(d+1, \mathbb{C}) \) to a Lie group homomorphism \( i \). The projective structures with this property are called homogeneous.

In order to see that \( \text{SL}(2, \mathbb{C}) \) admits actions as in the statement of Proposition 5.7, consider the irreducible linear action of \( \text{SL}(2, \mathbb{C}) \) on the vector space of homogeneous polynomials of degree 3 in two variables (by linearly changing the variables). The projectivization of this linear action gives a projective action of \( \text{SL}(2, \mathbb{C}) \) on \( \mathbb{C}P^3 \) with an open orbit, namely the \( \text{SL}(2, \mathbb{C}) \)-orbit of those polynomials that are product of three distinct linear forms (recall that the projective action of \( \text{SL}(2, \mathbb{C}) \) on the projective line \( \mathbb{C}P^1 \) is transitive on the set of triples of distinct points).

6. Calabi–Yau manifolds and branched Cartan geometries

In this section we are interested in understanding branched holomorphic Cartan geometries on Calabi–Yau manifolds.

Recall that Kähler Calabi–Yau manifolds are compact complex Kähler manifolds with the property that the first Chern class (with real coefficients) of the holomorphic tangent bundle vanishes. By Yau’s theorem proving Calabi’s conjecture, those manifolds admit Kähler metrics with vanishing Ricci curvature [Ya]. Compact Kähler manifolds admitting a holomorphic affine connection have vanishing real Chern classes [At]; it was proved in [IKO] using Yau’s result that they must admit finite unramified coverings which are complex tori. It was proved in [BM] (see also [Du2, Du4]) that Calabi–Yau manifolds bearing a holomorphic Cartan geometry admit finite unramified covers by complex tori. We extend here this result to branched holomorphic Cartan geometries.

**Theorem 6.1.** A compact (Kähler) Calabi–Yau manifold \( X \) bearing a branched holomorphic affine structure admits a finite unramified covering by a complex torus.
Proof. Since $c_1(TX) = 0$, part (ii) in Corollary 4.2 implies that the branched holomorphic affine structure on $X$ is actually a holomorphic affine structure (connection). Hence $X$ admits a finite unramified covering by a complex torus [IKO].□

**Theorem 6.2.** Let $X$ be a compact simply connected Kähler manifold such that $c_1(TX) = 0$. Let $E$ be a holomorphic vector bundle on $X$ equipped with a holomorphic connection. Then $E$ is a trivial holomorphic vector bundle and $D$ is the trivial connection on it.

Proof. The theorem of Yau says that $X$ admits a Ricci–flat Kähler metric [Ya]. Fix a Ricci–flat Kähler form $\omega$ on $X$. The degree of a torsionfree coherent analytic sheaf on $X$ will be defined using $\omega$ (as in (4.1)). From the given condition that $\omega$ is Ricci–flat it follows that the tangent bundle $TX$ is polystable. Since $TX$ is polystable with $c_1(TX) = 0$, and $E$ admits a holomorphic connection, it follows that $E$ is semistable [Bi, p. 2830].

Note that $c_i(E) = 0$, $i \geq 1$, because $E$ admits a holomorphic connection [At, p. 192–193, Theorem 4]. In particular, we have degree($E$) = 0.

We will now recall a theorem of Simpson in [Si2]. Let $(Y, \omega_Y)$ be a compact Kähler manifold of dimension $m$. The works of Corlette and Simpson, [Co], [Si1], give a natural bijective correspondence between the complex vector bundles on $Y$ with irreducible flat connection and stable Higgs bundles $(V, \varphi)$ on $Y$ with degree($V$) = 0 = $ch_2(V) \wedge \omega_Y ^{m-2}$ (see [Si2, p. 20, Corollary 1.3]). It should be mentioned that if $(U, D)$ is a complex vector bundle on $Y$ with an irreducible flat connection, and $(V, \varphi)$ is the polystable Higgs bundles on $Y$ corresponding to it, then the holomorphic vector bundles on $Y$ underlying $U$ and $V$ need not coincide in general. They do coincide when \( \varphi = 0 \). It should be mentioned that \( \varphi = 0 \) if and only if the corresponding flat connection $D$ is unitary. In [Si2], Simpson extended this correspondence to connections which are not necessarily irreducible and Higgs bundles not necessarily polystable. He proved an equivalence of categories between the following two:

1. The category of complex vector bundles $U$ on $Y$ with a flat connection $D$.
2. The category of semistable Higgs bundles $(V, \varphi)$ on $Y$ with degree($V$) = 0 = $ch_2(V) \wedge \omega_Y ^{m-2}$ and satisfying the condition that $V$ admits a filtration of holomorphic subbundles such that each subbundle in the filtration is preserved by $\varphi$, and each successive quotient for this filtration equipped with the Higgs field induced by $\varphi$ is polystable with degree zero.

(See [Si2, p. 36, Lemma 3.5].) When $Y$ is a complex projective manifold, and the cohomology class of $\omega_Y$ is rational, Simpson improved the above equivalence. For a complex projective polarized manifold $Y$ there is an equivalence of categories between the following two:

1. The category of complex vector bundles $U$ on $Y$ with a flat connection $D$.
2. The category of semistable Higgs bundles $(V, \varphi)$ on $Y$ with degree($V$) = 0 = $ch_2(V) \wedge \omega_Y ^{m-2}$.

(See [Si2, p. 40, Corollary 3.10].) In both these equivalences of categories the holomorphic vector bundles underlying $U$ and $V$ do not coincide in general. But they indeed coincide if $\varphi = 0$.

Therefore, setting $\varphi = 0$ in the first equivalence of categories we get the following:
A holomorphic vector bundle $U$ on a compact Kähler manifold $(Y, \omega_Y)$ admits a flat holomorphic connection if $U$ admits a filtration of holomorphic subbundles such that each successive quotient $Q$ for the filtration is polystable with degree$(Q) = 0 = ch_2(Q) \wedge \omega_Y^{m-2}$.

Similarly, setting $\varphi = 0$ in the second equivalence of categories we get the following:

A holomorphic vector bundle $U$ on a complex polarized projective manifold admits a flat holomorphic connection if $U$ is semistable with degree$(U) = 0 = ch_2(U) \wedge \omega_Y^{m-2}$.

Consequently, if the Calabi–Yau manifolds $X$ in the theorem is projective, and the cohomology class of $\omega$ is rational, then that $E$ admits a flat holomorphic connection, because $E$ is semistable with vanishing Chern classes. Since $X$ is simply connected all flat bundles on $X$ are trivial. Therefore, $E$ is the trivial vector bundle. Since $H^0(X, \Omega^1_X) = 0$, the trivial holomorphic vector bundle has exactly one holomorphic connection, namely the trivial connection. Hence $D$ is the trivial connection on $E$.

We will now address the general Kähler case.

Let $V \subset E$ be a polystable subsheaf such that

- degree$(V) = 0$, and
- the quotient $E/V$ is torsionfree.

The second condition implies that $V$ is reflexive. Since $E$ is semistable, and $V$ is polystable with degree$(V) = 0 = \deg(E)$, it follows that $E/V$ is semistable with degree$(E/V) = 0$.

Let $d$ be the complex dimension of $X$. Let the ranks of $V$ and $E/V$ be $r$ and $s$ respectively. Since $V$ and $E/V$ are semistable, we have the Bogomolov inequality

\[
((2r \cdot c_2(V) - (r - 1)c_1(V)^2) \cup \omega^{d-2}) \cap [X] \geq 0,
\]

\[
((2s \cdot c_2(E/V) - (s - 1)c_1(E/V)^2) \cup \omega^{d-2}) \cap [X] \geq 0
\]

[BM, Lemma 2.1].

We will show that the inequalities in (6.1) and (6.2) are equalities. Denote the sheaf $E/V$ by $W$. We have

\[
2(r + s)c_2(V \oplus W) - (r + s - 1)c_1(V \oplus W)^2
\]

\[
= 2(r + s)(c_2(V) + c_2(W) + c_1(V)c_1(W)) - (r + s - 1)(c_1(V)^2 + c_1(W)^2 + 2c_1(V)c_1(W))
\]

\[
= \frac{r + s}{r}(2rc_2(V) - (r - 1)c_1(V)^2) + \frac{r + s}{s}(2rc_2(W) - (s - 1)c_1(W)^2)
\]

\[
- \frac{s}{r}c_1(V)^2 - \frac{r}{s}c_1(W)^2 + 2c_1(V)c_1(W)
\]

\[
= \frac{r + s}{r}(2rc_2(V) - (r - 1)c_1(V)^2) + \frac{r + s}{s}(2rc_2(W) - (s - 1)c_1(W)^2) - \frac{1}{sr}(s \cdot c_1(V) - r \cdot c_1(W))^2.
\]

On the other hand, $c_1(V \oplus W) = c_1(E) = 0$, so

\[
\frac{r + s}{r}((2r \cdot c_2(V) - (r - 1)c_1(V)^2) \cup \omega^{d-2}) \cap [X] + \frac{r + s}{s}((2r \cdot c_2(W) - (s - 1)c_1(W)^2) \cup \omega^{d-2}) \cap [X]
\]

\[
- \frac{1}{sr}((s \cdot c_1(V) - r \cdot c_1(W))^2 \cup \omega^{d-2}) \cap [X] = 0.
\]
From Hodge index theorem (see [Vo, Section 6.3]) it follows that
\[-\frac{1}{sr}((s \cdot c_1(V) - r \cdot c_1(W))^2 \cup \omega^{d-2}) \cap [X] \geq 0.\]
Therefore, from (6.3) we conclude that the inequalities in (6.1) and (6.2) are equalities.

Since \((2r \cdot c_2(V) - (r - 1)c_1(V)^2) \cup \omega^{d-2}) \cap [X] = 0\), from [BS, p. 40, Corollary 3] we conclude that \(V\) is a polystable vector bundle admitting a projectively flat unitary connection. Therefore, projective bundle \(P(V)\) for \(V\) is given by a representation of \(\pi_1(X)\) in \(PU(r)\). As \(X\) is simply connected, we conclude that the projective bundle \(P(V)\) is trivial. Hence
\[V = L^{\oplus r},\] (6.4)
where \(L\) is a holomorphic line bundle on \(X\). We have
\[\text{degree}(L) = 0,\]
because \(\text{degree}(V) = 0\).

Now assume that \(V\) is preserved by the connection \(D\) on \(E\). Then \(V\) is a subbundle of \(E\), and the quotient \(E/V\) has a holomorphic connection \(D_1\) induced by \(D\). Consequently, we may repeat the above arguments for \((E/V, D_1)\) and get a subsheaf \(V_1 \subset E/V\) which is a direct sum of line bundles of degree zero (as in (6.4)). Again assume that \(V_1\) is preserved by \(D_1\) and repeat the argument. In this way we get a filtration of \(E\) by subbundles such that each successive quotient is a polystable vector bundle of degree zero. As explained before, a theorem of Simpson implies that \(E\) admits a flat holomorphic connection. Since \(X\) is simply connected, this implies that \(E\) is holomorphically trivial. A trivial holomorphic vector bundle on \(X\) has exactly one holomorphic connection because \(H^0(X, \Omega^1_X) = 0\) (recall that \(X\) is simply connected). Therefore, a trivial holomorphic vector bundle on \(X\) has only the trivial connection.

Now assume the opposite, namely that \(V\) is not preserved by the connection \(D\) on \(E\). Consider the holomorphic section of \(\text{Hom}(V, E/V) \otimes \Omega^1_X\) given by \(D\); it is nonzero because \(V\) is not preserved by \(D\). Let
\[\delta : TX \longrightarrow \text{Hom}(V, E/V)\] (6.5)
be the homomorphism given by this section.

The rank of \(\text{Hom}(V, E/V)\) is \(rs\). We have
\[\text{degree}(\text{Hom}(V, E/V)) = r \cdot \text{degree}(E/V) - s \cdot \text{degree}(V) = 0.\] (6.6)
We have \(V^*\) to be semistable because \(V\) is semistable. Now, since both \(V^*\) and \(E/V\) are semistable, it follows that \(\text{Hom}(V, E/V) = (E/V) \otimes V^*\) is semistable [AB, Lemma 2.7]; recall that \(V\) is locally free, so \((E/V) \otimes V^*\) is torsionfree. Since \(\text{Hom}(V, E/V)\) is semistable of degree zero (shown in (6.6)), and \(TX\) is a polystable vector bundle of degree zero, we conclude that the image \(\delta(TX)\) in (6.5) is also a polystable vector bundle of degree zero; here we are using the fact that the image of a polystable sheaf in a semistable sheaf of same slope (= degree/rank) is also polystable of the common slope.

Let \(t\) be the rank of \(U := \delta(TX)\). We have
\[(2rs \cdot c_2(\text{Hom}(V, E/V)) - (rs - 1)c_1(\text{Hom}(V, E/V))^2) \cup \omega^{d-2}) \cap [X]\]
\[= ((2r \cdot c_2(V) - (r - 1)c_1(V)^2) \cup \omega^{d-2}) \cap [X] + ((2s \cdot c_2(E/V) - (s - 1)c_1(E/V)^2) \cup \omega^{d-2}) \cap [X] = 0.\]

This implies that
\[((2t \cdot c_2(U) - (t - 1)c_1(U)^2) \cup \omega^{d-2}) \cap [X] = 0,
\]
because the Bogomolov inequality holds for both \(U\) and \(\text{Hom}(V, E/V)/U\). Indeed, the Bogomolov inequality holds for all three terms in the short exact sequence
\[0 \rightarrow U \rightarrow \text{Hom}(V, E/V) \rightarrow \text{Hom}(V, E/V)/U \rightarrow 0\]
and furthermore it is an equality for \(\text{Hom}(V, E/V);\) hence the Bogomolov inequality is an equality for both \(U\) and \(\text{Hom}(V, E/V)/U\).

Again from [BS, p. 40, Corollary 3] we conclude that \(P(U)\) admits a flat connection. Hence \(U\) is of the form
\[U = N^{\oplus t},\]
where \(N\) is a holomorphic line bundle on \(X\) of degree zero.

Since \(TX\) is polystable, the quotient bundle \(U\) is a direct summand of \(TX\). This implies that \(U\) is a subbundle of \(TX\). Hence we have a holomorphic decomposition
\[TX = N \oplus N',\]  
where \(N\) is a holomorphic line bundle on \(X\) of degree zero, and the rank of \(N'\) is \(d - 1\).

A result of Beauville [Be, Theorem A] associates to any holomorphic splitting
\[TX = U_1 \oplus U_2 \oplus \ldots \oplus U_j\]
a corresponding decomposition \(X = X_1 \times X_2 \times \ldots \times X_j\), with \(X_i\) simply connected Calabi–Yau manifolds, such that \(U_i = \pi_i^*(TX_i)\), where \(p_i : X \rightarrow X_i, 1 \leq i \leq j,\) are the canonical projections. Now from (6.7) we conclude that \(X\) is a product of Calabi–Yau manifolds with one factor of dimension one. But there is no simply connected compact Calabi–Yau manifold of complex dimension one. Therefore, we get a contradiction. This completes the proof. \(\square\)

**Corollary 6.3.**

(i) Any branched holomorphic Cartan geometry of type \(G/H\), with \(G\) complex affine Lie group, on a compact simply connected (Kähler) Calabi–Yau manifold is flat. Consequently, the model \(G/H\) of the Cartan geometry must be compact.

(ii) Non-projective compact simply connected (Kähler) Calabi–Yau manifolds do not admit branched holomorphic projective structures.

Recall that \(G\) is a complex affine Lie group if it admits a linear representation \(\rho : G \rightarrow \text{GL}(N, \mathbb{C})\), for some \(N\), with discrete kernel. Complex semi-simple Lie groups are complex affine (see Theorem 3.2, chapter XVII, in [Ho]).

**Proof.** Let \(X\) be a simply connected Calabi–Yau manifold endowed with a branched holomorphic Cartan geometry of type \(G/H\), with \(G\) complex affine Lie group.
Let $\rho : G \to \text{GL}(N, \mathbb{C})$ be a linear representation of $G$ with discrete kernel. The corresponding Lie algebra representation $\rho' : \text{Lie}(G) \to \text{M}(N, \mathbb{C})$ is an injective.

Consider the principal bundle $E_G$ and the associated holomorphic vector bundle $E_\rho$ of rank $N$ on $X$ for the above homomorphism $\rho$. Then $E_\rho$ inherits a holomorphic connection $\theta_\rho'$ and, by Theorem 6.2, this connection $\theta_\rho'$ must be flat. Since the curvature of $\theta_\rho'$ is the image of the curvature of the connection $\theta'$ of $E_G$ through $\rho'$, and $\rho'$ is injective, it follows that $\theta'$ is also flat.

Proof of (i): As shown above, Theorem 6.2 implies that the associated holomorphic connection $\theta'$ of $E_G$ must be flat. Consequently, the Cartan geometry of type $G/H$ is flat. The developing map $\delta : X \to G/H$ is a branched holomorphic map. This implies that $\delta(X) = G/H$ is compact.

Proof of (ii): This follows from part (i) and Corollary 3.3.

ACKNOWLEDGEMENTS

We thank the referee for pointing out [Br]. This work was carried out while the first author was visiting the Université Côte d’Azur. He thanks the Université Côte d’Azur and Dieudonné Department of Mathematics for their hospitality.

REFERENCES

[Ag] Y. Agaoka, Invariant flat projective structures on homogeneous spaces, *Hokkaido Math. Jour.* **11** (1982), 125–172.
[AB] B. Anchouche and I. Biswas, Einstein-Hermitian connections on polystable principal bundles over a compact Kähler manifold, *Amer. Jour. Math.* **123** (2001), 207–228.
[At] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.
[BS] S. Bando and Y.-T. Siu, Stable sheaves and Einstein-Hermitian metrics, in: *Geometry and analysis on complex manifolds*, pp. 39–50, World Sci. Publishing, River Edge, NJ, 1994.
[Be] A. Beauville, Complex manifolds with split tangent bundle, *Complex Analysis and Algebraic Geometry*, A volume in memory of Michael Schneider. ed.: Thomas Peternell and Frank-Olaf Schreyer, 61–70, de Gruyter, 2000.
[Bi] I. Biswas, Vector bundles with holomorphic connection over a projective manifold with tangent bundle of nonnegative degree, *Proc. Amer. Math. Soc.* **126** (1998), 2827–2834.
[BM] I. Biswas and B. McKay, Holomorphic Cartan geometries, Calabi–Yau manifolds and rational curves, *Diff. Geom. Appl.* **28** (2010), 102–106.
[Br] R. Bryant, https://mathoverflow.net/questions/229569/intuition-for-the-cartan-connection-and-rolling-without-slipping-in-cartan-geo.
[Co] K. Corlette, Flat G-bundles with canonical metrics, *Jour. Diff. Geom.* **28** (1988), 361–382.
[Du1] S. Dumitrescu, Homogénéité locale pour les métriques riemanniennes holomorphes en dimension 3, *Ann. Institut. Fourier* **57** (2007), 739–773.
[Du2] S. Dumitrescu, Structures géométriques holomorphes sur les variétés compactes, *Ann. Sci. Éc. Norm. Sup.* **34** (2001), 557–571.
[Du3] S. Dumitrescu, Connexions affines et projectives sur les surfaces complexes compactes, *Math. Zeit.* **264** (2010), 301–316.
[Du4] S. Dumitrescu, Killing fields of holomorphic Cartan geometries, *Monatsh. Math.* **161** (2010), 145–154.
[Du5] S. Dumitrescu, Une caractérisation des variétés parallélisables compactes admettant des structures affines, *Com. Ren. Acad. Sci. Paris* **347** (2009), 1183–1187.
S. Dumitrescu and A. Zeghib, Global rigidity of holomorphic Riemannian metrics on compact complex 3-manifolds, *Math. Ann.* **345** (2009), 53–81.

C. Ehresmann, Sur les espaces localement homogènes, *L’Enseign. Math.* **35** (1936), 317–333.

E. Ghys, Déformations des structures complexes sur les espaces homogènes de *SL(2, C)*, *Jour. Reine Angew. Math.* **468** (1995), 113–138.

R. C. Gunning, *On uniformization of complex manifolds: the role of connections*, Princeton Univ. Press, 1978.

G. P. Hochschild, *Basic theory of algebraic groups and Lie algebras*, Graduate Texts in Mathematics, 75. Springer-Verlag, New York-Berlin, 1981.

A. T. Huckleberry and G. A. Margulis, Invariant analytic hypersurfaces, *Invent. Math.* **71** (1983), 235–240.

M. Inoue, S. Kobayashi and T. Ochiai, Holomorphic affine connections on compact complex surfaces, *Jour. Fac. Sci. Univ. Tokyo*, Sect. IA Math. **27** (1980), 247–264.

H. Kato, Left invariant flat projective structures on Lie groups and prehomogeneous vector spaces, *Hokkaido Math. Jour.* **42** (2012), 1–35.

S. Kobayashi and T. Ochiai, Holomorphic projective structures on compact complex surfaces, *Math. Ann.* **249** (1980), 75–94.

R. Mandelbaum, Branched structures on Riemann surfaces, *Trans. Amer. Math. Soc.* **163** (1972), 261–275.

R. Mandelbaum, Branched structures and affine and projective bundles on Riemann surfaces, *Trans. Amer. Math. Soc.* **183** (1973), 37–58.

R. Molzon and K. P. Mortensen, The Schwarzian derivative for maps between manifolds with complex projective connections, *Trans. Amer. Math. Soc.* **348** (1996), 3015–3036.

V. Ovsienko and S. Tabachnikov, *Projective Differential Geometry Old and New*, 165, Cambridge Univ. Press, 2005.

R. W. Sharpe, *Differential Geometry: Cartan’s Generalization of Klein’s Erlangen Program*, Graduate Text Math., 166, Springer-Verlag, New York, Berlin, Heidelberg, 1997.

C. T. Simpson, Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization, *Jour. Amer. Math. Soc.* **1** (1988), 867–918.

C. T. Simpson, Higgs bundles and local systems, *Inst. Hautes Études Sci. Publ. Math.* **75** (1992), 5–95.

H. P. de Saint Gervais, *Uniformization of Riemann Surfaces. Revisiting a hundred year old theorem*, E.M.S., 2016.

C. Voisin, *Théorie de Hodge en géométrie algébrique complexe*, Cours Spécialisés, Collection SMF, 2002.

H.-C. Wang, Complex parallelisable manifolds, *Proc. Amer. Math. Soc.* **5** (1954), 771–776.

S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I, *Comm. Pure Appl. Math.* **31** (1978), 339–411.

**School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India**

E-mail address: indranil@math.tifr.res.in

**Université Côte d’Azur, CNRS, LJAD, France**

E-mail address: dumitres@unice.fr