Exact analytical solution of Quantum Discord for Generalized Werner-Like Non-X states

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The Generalized Werner-Like states (GWLs) are a class of non-X states in which the exchange operator is replaced for a generic one-rank projector in the Werner states. We obtained an exact analytical expression of Quantum Discord for these states. The optimization problem involved is solved by giving an analytical expression, in exact form for the conditional entropy. We compared the Quantum Discord (QD) with the Entanglement of Formation (EoF) for the same states. The pure states of GWLs with equal concurrence have the same QD and EoF.

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I. INTRODUCTION

Quantum correlations lie in the foundation of quantum mechanics and are the heart of quantum information science. They are important to study the differences between the classical and quantum worlds because, in general, the quantum systems can be correlated in ways inaccessible to classical objects. The research on quantum correlation measures were initially developed on the entanglement-separability paradigm [1] (and the references therein). However, it is well known that entanglement does not account for all quantum correlations and that even correlations of separable states are not completely classical. Entanglement is an inevitable feature of not only quantum theory but also any non-classical theory [2], and this is necessary for emergent classicality in all physical theories. The study of quantum correlation quantifiers other than entanglement, such as the Quantum Discord (QD), has a crucial importance for the full development of new quantum technologies because it is more robust than entanglement against the effects of decoherence [3–6], and can be among others a resource in quantum computation [7–9], quantum non-locality [10], quantum key distribution [11], remote state preparation [12], quantum cryptography [13] and quantum coherence [14].

Experimentally it is difficult to prepare pure states, the researcher must do a thorough examination of the system to know the possible pure states to which the system can access. In general, the states are mixed since they characterize the interaction of the system with its surrounding environment. The study of the quantum information properties of mixed states is more complicated and less understood that the pure states. The set of Werner states [15] (Ws) is an important type of mixed states, derived in 1989, which plays a fundamental role in the foundations of quantum mechanics and quantum information theory. Since these states admit a hidden variable model without violating Bell’s inequalities, then the correlation measured that are generated with these states can also be described by a local model, despite of being entangled. Moreover, these states are used as quantum channels with noise that do not maintain the additivity, they are also employed in the study of deterministic purifications [16]. On the other hand, the Werner-Popescu states [17] (WPs), the Quasi-Werner states [18] (QWs) and the Bell Werner-Like state [19] (BWLs), also called noise singlets, for bipartite system of qubits, are mixing states maximally entangled and have been studied widely as a fundamental resource for the quantum information processing, and also in the study of non-local properties in quantum mechanics. These mixing states are the most natural generalization of the GWLs. The GWLs (for detail see section II) are a family of mixed states, obtained by the convex sum between a maximally mixed state and by Henderson and Vedral [22], is a more general concept to measure quantum correlations than quantum entanglement, since separable mixed states can have nonzero QD. This measures the fraction of the pairwise mutual information that is locally inaccessible in a multipartite system (for detail see section IV). The QD is also called the \textit{locally inaccessible information} (LII) [23], since the QD measured on one partition is the information of the system that is inaccessible to an observer in other partition. In this context, quantum measurements only provide information on the partition measured, however, simultaneously they introduce disturbance and destroy the coherence in the system. One of the problems QD has is the minimization process involved for the calculation of the conditional entropy. Until now, the QD only has been obtained for

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a few special classes of two-qubit X–states [24, 27], and generally this is determined numerically [28]. Yao and collaborators [29] has evaluated numerically the QD for a special class of non–X states when the Bloch vectors are orthogonal vectors. This class of states cannot be written as a GWLs, since in the representation of Fano-Bloch both states do not match. Recently, Huang [30] obtained a precise mathematical characterization of the computational difficulty of EoF and QD. In particular, he proved that computing a large class of entanglement measures (including, but not limited to, EoF) and computing QD are NP-complete and NP-hard in some particular cases. The QD is not always larger than the entanglement, and there is not clear evidence of the relationship between entanglement and quantum discord [31, 52], in general, since they seem to capture different properties of the states. The principal aim of this paper is to derive analytical solutions of QD for the GWLs built with GBps, and compare the QD with a measure of entanglement, specifically the EoF.

This paper is organized as follows. A detailed review of the GWLs is given in Sec. II. In Sec. IV we present an analytical approach to obtain the exact solutions of the QD for the GWLs, while in Sec. III we determine the EoF for GWLs. In Sec. V we evaluate the QD and EoF for several states GBps, using four discrete state with different concurrences, here show the behavior of the EoF and QD with mixing parameter. In this section, also we prove that the QD is a monotonous function of the concurrence of the GBps. Finally, in the Secs. VI and VII we present the analysis drawn from our results and conclusions of work. Additionally, three appendices are included which contain the calculation of the projective measure on pure state, the calculation of the conditional entropy for the GWLs and the calculation of the critical point for mixing parameter where Bell’s inequality is violated.

II. GENERALIZED WERNER-LIKE NON–X STATES

Let \(|ij\rangle\) be the computational bases in space \(\mathcal{H}_2 \otimes \mathcal{H}_2\), where \(\{ij\} = \{00, 01, 10, 11\}\) and \(\mathcal{H}_2\) is the Hilbert space of dimension two. The GBps \(|\psi\rangle\) is given by,

\[
|\psi\rangle = z_1 |00\rangle + z_2 |01\rangle + z_3 |10\rangle + z_4 |11\rangle .
\] (1)

The complex numbers \(z_i\) (with \(i = 1, 2, 3, 4\)) are those that verify the normalization condition \(\sum_i |z_i|^2 = 1\). The GBps can be represented by a \(2 \times 2\) matrix whose elements are obtained with the components of the pure state [31], in accordance with

\[
\hat{\mathcal{W}}_\psi = [(ij|\psi\rangle)_{2 \times 2} = \begin{bmatrix} 00|\psi\rangle \langle 01|\psi\rangle & 01|\psi\rangle \langle 11|\psi\rangle \end{bmatrix} = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}
\] (2)

The normalization condition of GBps \(|\psi\rangle\) in term of matrix \(\hat{\mathcal{W}}_\psi\), it is written as \(\text{tr} \left[ \hat{\mathcal{W}}_\psi \hat{\mathcal{W}}_\psi^\dagger \right] = 1\) (see appendix A for details). The one-rank projector built with the GBps \(|\psi\rangle\) or the density matrix of the GBps \(|\psi\rangle\) is denoted as

\[
\hat{P}_\psi = \langle \psi | \psi \rangle = \begin{bmatrix} z_1^2 & z_1 z_2 & z_1 z_3 & z_1 z_4 \\ z_1 z_2 & z_2^2 & z_2 z_3 & z_2 z_4 \\ z_1 z_3 & z_2 z_3 & z_3^2 & z_3 z_4 \\ z_2 z_3 & z_2 z_4 & z_3 z_4 & z_4^2 \end{bmatrix} .
\] (3)

The \(\bar{z}_i\) shown in the expression [33] are the conjugate complex of \(z_i\), with \(i = 1, 2, 3, 4\).

On the other hand, the bipartite Ws of qubits are self-adjoint operators, bounded and of class trace that act onto the composite space \(\mathcal{H}_2 \otimes \mathcal{H}_2\), formed by

\[
\rho_W(p) = \frac{1-p}{4} \hat{1}_4 + \frac{p}{4} \hat{P}_4 = \frac{1-p}{4} \hat{1}_4 - \hat{P}_{\Phi-}
\]

\[
= \frac{1}{4} \begin{bmatrix} 1 + p & 0 & 0 & 0 \\ 0 & 1 - p & 2p & 0 \\ 0 & 2p & 1 - p & 0 \\ 0 & 0 & 0 & 1 + p \end{bmatrix} ,
\] (4)

being \(|\Phi-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle\) a Bell state, and \(p\) is the mixing parameter with \(p \in [-1, \frac{1}{3}]\). Its range of values guarantees the positivity of the Ws. Furthermore, \(\hat{P}_4\) is the exchange operator defined by

\[
\hat{P}_4 = \sum_{i,j=0}^3 |ij\rangle \langle ji| = \hat{1}_4 - 2\hat{P}_{\Phi-} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ,
\] (5)

The Ws, given in the expression [14], are states–X invariant under any local unitary operator of the form \(U \otimes \bar{U}\), they admit a model of hidden variables [13] if \(p \in [-1, \frac{1}{2}]\), being still entangled.

The GWLs are one-parametric family of mixed states, obtained by the convex sum between a maximally mixed state (also called unpolarized state) and a one-rank projector built with the GBps, given by expression [14]. In other words, the GWLs is a generalization of the Ws when exchange the operator \(\frac{1}{2} \hat{P}_4\) by the projector \(\hat{P}_\psi\), and furthermore exchange the mixing parameter \(p\) by \(-p\). Then, the density matrix of fourth order for the GWLs has the form:

\[
\rho_{GWL}(\psi, p) = \frac{1-p}{4} \hat{1}_4 + p \hat{P}_\psi = \begin{bmatrix} \frac{1-p}{4} + |p| z_1 |^2 & p z_1 z_2 \langle z_1 z_2 & p z_1 z_3 \langle z_1 z_3 & p z_1 z_4 \langle z_1 z_4 \\ p z_2 z_1 \langle z_2 z_1 & \frac{1-p}{4} + |p| z_2 |^2 & p z_2 z_3 \langle z_2 z_3 & p z_2 z_4 \langle z_2 z_4 \\ p z_3 z_1 \langle z_3 z_1 & p z_3 z_2 \langle z_3 z_2 & \frac{1-p}{4} + |p| z_3 |^2 & p z_3 z_4 \langle z_3 z_4 \\ p z_4 z_1 \langle z_4 z_1 & p z_4 z_2 \langle z_4 z_2 & p z_4 z_3 \langle z_4 z_3 & \frac{1-p}{4} + |p| z_4 |^2 \end{bmatrix} .
\] (6)
The range of variation of the mixing parameter $p$ is now $\frac{1}{2} \leq p \leq 1$, which guarantees the positivity of the GWLs. The parameter $p$, considered in the expression \[ 4 \], is understood as a probability when the range of variation is $0 \leq p \leq 1$. In this case the GWLs represents a convex sum of the density matrix of the GWPs $|\psi\rangle$ and non-coherent density matrix of an unpolarized state (white noise), with probabilities $p$ and $1-p$, respectively. The WPs, QWs and BWLs are obtained where the Bell states $|\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\Phi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ are used to built the projector $\hat{P}_\psi$ in the expression \[ 6 \]. One difference between the states \[ 1 \] and \[ 6 \] is that the Ws are X states while that the GWLs are not–X states in general, unless $z_1 = z_4 = 0$ or $z_2 = z_3 = 0$; like the WPs, QWs and BWLs. Others fundamental difference is that $\hat{F}$ is an involutive operator ($\hat{F}^2 = \mathbb{I}$) while $\hat{P}_\psi$ is an idempotent operator ($\hat{P}_\psi^2 = \hat{P}_\psi$), so that the GWLs generate in principle different correlations that the Ws, since the replacing of $\frac{1}{\sqrt{2}}\hat{P}_\psi$ by $\hat{P}_\psi$ in the expression \[ 4 \] makes that $\rho_W(p)$ do not be unitarily equivalent to $\rho_{GWL}(\psi,p)$, for this reason the GWLs looses the invariance under local unitary transformations. Nevertheless, the GWLs and Ws are connected by the transformation \[ \rho_{GWL}(\Phi_{-},-p) = \frac{1+\rho}{4} \mathbb{I}_4 - p \hat{P}_{\Phi_{-}} = \rho_W(p). \] (7)

This equality is exact only in four dimensions. In other dimensions it is impossible to obtain the equality \[ 7 \]. But any unitary transformation applied on GWLs leaves them invariant in shape, without changing the mixing parameter, this is

\[ \rho_{GWL}(\psi,p) \xrightarrow{\mathcal{U}} \hat{U} \rho_{GWL}(\psi,p) \hat{U}^\dagger = \frac{1-p}{4} \mathbb{I}_4 + p \hat{P}_{\psi}, \] (8)

where $|\psi\rangle = \hat{U} |\psi\rangle$. The Ws changed by no local unitary transformations are called Werner Derivative states (WDs), and these states lead to a type of GWLs. The study of local and nonlocal properties is done in reference \[ 33 \], but this study is incomplete since it only considers a particular class of unitary transformations. Therefore, all the correlations contained in the WDs are present in the GWLs.

### III. ENTANGLEMENT OF FORMATION FOR THE GWLS

A good measure to quantify the entanglement of a pure state $|\psi\rangle$ is the von-Neumann entropy \[ 33 \], since a pure state can be constructed from a set of maximally entangled singlet states and the number of these states is proportional to the entropy of the reduced states of any partition \[ 33 \]. However, the von-Neumann entropy is not a good measure of the degree of entanglement for mixed states because there are product states whose partitions may have entropies different from zero, for example, $p = \rho_1 \otimes \rho_2$ with $S[\rho_1] \neq 0$. In order to quantify the degree of entanglement in arbitrary bipartite state

Wootters \[ 32, 36 \] proposed the EoF, given by

\[ \text{EoF}[\rho] = H_2 \left( \frac{1+\Delta_p}{2} \right), \quad \text{with} \quad \Delta_p \overset{\text{def}}{=} \sqrt{1-C^2[\rho]}. \] (9)

The function $H_2(z) = -z \log_2(z) - (1-z) \log_2(1-z)$ shown in the equation \[ 9 \] is the Shannon binary entropy function, and $C[\rho]$ is the concurrence function of the state $\rho$, defined as $C[\rho] \overset{\text{def}}{=} \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4})$. The $\lambda_i$'s are the eigenvalues of the positive operator $\rho \rho^\dagger$, arranged in decreasing order. The operator $\rho \rho^\dagger$ is the spin-flip operation on the conjugate of the state $\rho$, i.e. $\rho^\dagger = (\sigma_y \otimes \sigma_y) \rho (\sigma_y \otimes \sigma_y)$, being $\mathbb{I}$ the conjugate complex of $\rho$. In the case of a pure state $|\psi\rangle$, the spin-flip operation onto the conjugate complex of the state is given by $\rho^\dagger = (\sigma_y \otimes \sigma_y) |\psi\rangle \langle \psi| (\sigma_y \otimes \sigma_y) = |\psi\rangle \langle \psi|$, so that $\rho \rho^\dagger = |\psi\rangle \langle \psi| |\psi\rangle \langle \psi|$, and the characteristic equation $\rho \rho^\dagger |\lambda\rangle = \lambda |\lambda\rangle$ leads to $\lambda = |\langle \psi| \psi\rangle|^2$, after projecting this equation on $|\psi\rangle$. Also, the determinant of $|\psi\rangle \langle \psi|$ is zero and therefore $\rho \rho^\dagger$ has a null eigenvalue with multiplicity three which corresponds to the ortogonal projection to the state $|\psi\rangle$. In this sense, $\sqrt{\lambda_1} = |\langle \psi| \psi\rangle|$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$, being the concurrence for a pure state $|\psi\rangle$

\[ C[|\psi\rangle] = |\langle \psi| \psi\rangle| = |\langle \psi| \sigma_y \otimes \sigma_y \psi\rangle|, \]

\[ = 2 |z_1 z_4 - z_2 z_3| = 2 |\text{det} \hat{W}_\psi|. \]

With the aim of obtaining the EoF for the GWLs, and compare it with the QD of same state, we calculate the Wootters concurrence given in the equation \[ 9 \]. For the case of GWLs, the spin-flip operation applied on the conjugate complex of the states defined by equation \[ 6 \] is

\[ \tilde{\rho}_{GWL}(\psi,p) = \frac{1-p}{4} \mathbb{I}_4 + p |\tilde{\psi}\rangle \langle \tilde{\psi}| \equiv \rho_{GWL}(\tilde{\psi},p), \] (11)

while

\[ \rho_{GWL}(\psi,p) \tilde{\rho}_{GWL}(\psi,p) = \left( \frac{1+\rho}{4} \right)^2 \mathbb{I}_4 + \hat{A}, \]

\[ \hat{A} = p^2 C[|\psi\rangle|\psi\rangle \langle \psi| + \frac{1-p}{4} \left( |\tilde{\psi}\rangle \langle \tilde{\psi}| + |\tilde{\psi}\rangle \langle \tilde{\psi}| \right). \] (13)

Here we have replaced $|\psi\rangle \langle \psi|$ by $C[|\psi\rangle|\psi\rangle \langle \psi|$, where $\phi$ is the argument of $|\psi\rangle \langle \psi|$. On the other hand, the eigenvectors of the matrix $\hat{A}$ are equal to the eigenvectors of $\rho_{GWL}(\psi,p) \tilde{\rho}_{GWL}(\psi,p)$, so we will focus on finding the eigenvalues of this matrix. It is clear from equation \[ 13 \] that the domain of $\hat{A}$ can be expanded as linear combinations of the pure states $|\psi\rangle$ and $|\tilde{\psi}\rangle$, which means that the eigenvectors of $\hat{A}$ can be written as $|\lambda\rangle = \Lambda_1 |\psi\rangle + \Lambda_2 |\tilde{\psi}\rangle$. Projecting the equation of eigenvectors $\hat{A} |\lambda\rangle = \lambda |\lambda\rangle$ on the states $|\psi\rangle$ and $|\tilde{\psi}\rangle$ we obtain an equation system for $|\langle \psi| \lambda\rangle|$ and $|\langle \tilde{\psi}| \lambda\rangle|$, from which a straightforward calculation yields

\[ \left[ \begin{array}{c} \frac{p(1-p)}{4} - \lambda \\ \frac{p(1+3p)}{4} - \lambda \end{array} \right] = 0, \]

\[ \left[ \begin{array}{c} C[|\psi\rangle]|\psi\rangle \langle \psi| - i \phi \\ C[|\psi\rangle]|\psi\rangle \langle \psi| + i \phi \end{array} \right] \]

\[ = \lambda. \] (14)
To determine a solution other than the trivial one, we impose that the determinant of the equation system is zero and obtain the following eigenvalue equation
\[
\lambda^2 - \frac{p^2(1 - 2\Delta^2_{\psi}) + \rho}{2} + \frac{p^2(1 - p)^2}{16} \Delta^2_{\psi} = 0, \quad (15)
\]
From this equation, two eigenvalues are determined. The other two eigenvalues of the operator \(\hat{A}\) that correspond to the eigenvectors expanded into \(\{\psi, |\psi\rangle\}\) are zero because \(\det(\hat{A}) = 0\). Finally, the eigenvalues of equation (15) are in decreasing order
\[
\lambda_1 = (\frac{1-p}{4})^2 + p(\frac{(1-p)+2pC^2(\psi)}{4}) \sqrt{\frac{(1+p)^2-4p^2\Delta^2_{\psi}}{(1+p)^2-4p^2\Delta^2_{\psi}}}, \quad (16a)
\]
\[
\lambda_2 = (\frac{1-p}{4})^2 + p(\frac{(1-p)+2pC^2(\psi)}{4}) \sqrt{\frac{(1+p)^2-4p^2\Delta^2_{\psi}}{(1+p)^2-4p^2\Delta^2_{\psi}}}, \quad (16b)
\]
and
\[
\lambda_3 = \lambda_4 = (\frac{1-p}{4})^2, \quad (16c)
\]
so that the concurrence for GWLs is given by
\[
C[\rho_{\text{GWL}}] = \max \left\{ 0, \sqrt{\lambda_1} - \sqrt{\lambda_2 - \frac{1-p}{4}} \right\}. \quad (17)
\]
Matching \(\sqrt{\lambda_1}\) and \(\sqrt{\lambda_2 + \frac{1-p}{4}}\) we get the value of the largest mixing parameter \(p_c\), from which the EoF is zero, thereby the critical mixing parameter is given by
\[
p_c = \frac{1}{1 + 2C[|\psi\rangle]} \quad (18)
\]
This quantity is a critical value that limits the border between entanglement and separability of GWLs. In other words, the GWLs are separable when \(-\frac{1}{4} \leq p \leq p_c\) and entangled when \(p_c < p \leq 1\). So that, the critical value \(p_c\) decreases monotonously with the increase of the concurrence of the GBPs \(|\psi\rangle\). In particular, for BWLS or WPs we have the usual result [13], namely, they are entangled if \(1/3 < p \leq 1\) and classically correlated if \(-1/3 \leq p < 1/3\), since \(|\psi\rangle\) is maximally entangled, i.e. \(C[|\psi\rangle] = 1\). When the pure state \(|\psi\rangle\) is a product state \(C[|\psi\rangle] = 0\) then all the GWLs are a convex sum of product states. In effect, taking \(|\psi\rangle = |A\rangle \otimes |B\rangle\) result that
\[
\rho_{\text{GWLS}}(A \otimes B, p) = \sum_{i,j=0}^1 \frac{1-p^2}{4} \hat{P}_i \otimes \hat{P}_j + p^2 \hat{P}_A \otimes \hat{P}_B \quad (19)
\]
where \(\hat{P}_i \equiv |i\rangle \langle i|\) (with \(i = 0, 1, A, B\)) for value of mixing parameters where the GWLs are separable. However, not all the projectors in the convex sum are orthogonal, so there would be correlations not necessarily classic.

IV. QUANTUM DISCORD OF GWLS

The fundamental amount for the study of quantum information, in terms of its uncertainty, is the von-Neumann entropy $\mathcal{S}$. Namely, when uncertainty grows the state contains less information. According to Wilde [37] this quantity measures the expected value of quantum information content. This quantity is defined in bits as $\mathcal{S}[\rho] \overset{\text{def}}{=} -\text{tr}[\rho \log_2 \rho] = -\sum_i \lambda_i \log_2 \lambda_i$, where the $\lambda_i$’s are the eigenvalues of the density operator $\rho$. For pure states ($\rho = \hat{\psi}$) the von-Neumann entropy is zero, because the density operator is a one-rank projector and its eigenvalues are $\lambda_1 = 1$ and the rest are zeros; thus, the information contained in a pure state is maximal. The maximal uncertainty, in dimension four, is represented by a maximally mixed states ($\frac{1}{4}\hat{1}_4$), with a value for the von-Neumann entropy of 2, in bits; because it is eigenvalues are all $\lambda_1 = \frac{1}{4}$ and the rest are zeros. Then, in bipartites systems of qubits one has $0 \leq \mathcal{S}[\rho] \leq 2$. Thus, the entropy for the GWLs and Ws given in (9) and (11) will be bounded between these two values, being zero when $p = 1$ and $p = -1$ and maximum when $p = 0$, respectively. The GWLs given in the equation (11), has a simple eigenvalue given by $\frac{1+p}{4}$, and three degenerates eigenvalues with value $\frac{1-p}{4}$, this allows to obtain the von-Neumann entropy, given by
\[
\mathcal{S}_{AB}[\psi, p] = \mathcal{S}[\rho_{\text{GWLS}}] = -\text{tr}[\rho_{\text{GWLS}} \log_2 \rho_{\text{GWLS}}]
\]
\[
\mathcal{S}_{AB}[\psi, p] = 2 - \frac{3(1+p)}{4} \log_2(1-p) - \frac{1+p^2}{4} \log_2(1+3p)
\]
\[
\mathcal{S}_{AB}[\psi, p] = 2 - \frac{1}{4} \log_2 \left( \frac{(1+3p)^{p+1}}{(1-p)^{p-1}} \right). \quad (20)
\]
This expression is independent of the values $z_i$ of GBPs $|\psi\rangle$, in addition to being a monotonic function of the mixing parameter $p$. Is clear from (20) that the information provided by the GWLs is minimal (maximum entropy) when $p = 0$, while that the information is maximal (minimum entropy) when $p = 1$.

To determine the quantum information of each partition of the system $\mathcal{H}_2 \otimes \mathcal{H}_2$ contained in GWLs, given the equation (11), is sufficient to take their partial traces, so that we have to
\[
\rho_A^{\text{GWLS}} = \text{tr}_B[\rho_{\text{GWLS}}] = \frac{1-p}{2} \hat{1}_2 + p \hat{\Psi}_\psi \hat{\Psi}_\psi^\dagger. \quad (21a)
\]
\[
\rho_B^{\text{GWLS}} = \text{tr}_A[\rho_{\text{GWLS}}] = \frac{1-p}{2} \hat{1}_2 + p (\hat{\Psi}_\psi^\dagger) (\hat{\Psi}_\psi^\dagger)^\dagger. \quad (21b)
\]
Where $\hat{\Psi}_\psi^\dagger$ is the transposed matrix of $\hat{\Psi}_\psi$. In this context the transpose operation connects the density operator of both partitions and this operation does not modify the eigenvalues of the reduced states. For this reason, the expressions (21a) and (21b) show that the entropies of the reduced states are equal, so that
\[
-\text{tr}[\rho_A^{\text{GWLS}} \log_2 \rho_A^{\text{GWLS}}] = -\text{tr}[\rho_B^{\text{GWLS}} \log_2 \rho_B^{\text{GWLS}}],
\]
\[
\mathcal{S}_A(\psi, p) = \mathcal{S}_B(\psi, p) = H_2 \left( \frac{1+p\Delta_{\psi}}{2} \right). \quad (22)
\]
Since $\hat{\Psi}_\psi \hat{\Psi}_\psi^\dagger$ has two eigenvalues give by $\frac{1}{2} (1 \pm \Delta_{\psi})$. When $p = 1$ in the equation (22) one has the EoF of pure state $|\psi\rangle$, given in the equation (9). On the other
hand, when \( p = 0 \) the entropy of the reduced state is maximal, take the value of one bit, which corresponds to a maximally mixed state in \( \mathcal{H}_2 \).

In order to quantify the conditional entropy, a projective measurement is required. We performed this measurement on the partition \( A \) of the bipartite system, in accordance with

\[
\hat{\Pi}_m^A = \hat{\Pi}_m \otimes \hat{1}_2 = \frac{1}{2} \left[ \hat{1}_2 + (-1)^m \hat{n} \cdot \hat{\sigma} \right] \otimes \hat{1}_2,
\]

and projective measurement made on the partition \( B \) is given by

\[
\hat{\Pi}_m^B = \hat{1}_2 \otimes \hat{\Pi}_m = \hat{1}_2 \otimes \frac{1}{2} \left[ \hat{1}_2 + (-1)^m \hat{n} \cdot \hat{\sigma} \right],
\]

with \( m = 0, 1 \). Here \( \hat{n} = \sin(2\theta) \cos(\phi) \hat{i} + \sin(2\theta) \sin(\phi) \hat{j} + \cos(2\theta) \hat{k} \) is a unitary vector on the Bloch sphere, and \( \hat{\sigma} = \sigma_x \hat{i} + \sigma_y \hat{j} + \sigma_z \hat{k} \) is the Pauli vector; while the set \( \{ \hat{i}, \hat{j}, \hat{k} \} \) is canonical basis of the Euclidean space \( \mathbb{R}^3 \). After this local measurement on the density matrix \( \rho_{GWL} \), given in the equation \( 36 \), the state of the system becomes a hybrid quasi-classical state \( \| \), where both partitions have same functional expression to a GWLs, in a two dimensional space. This, using the Lüder rule \( [38] \) for the partition \( X \) we have that

\[
\rho_{GWL} \xrightarrow{\hat{n}_X} \rho_{GWL|\Pi_X^A} = \frac{(\hat{\Pi}_m^X)^\dagger \rho_{GWL}(\hat{\Pi}_m^X)}{p_m^X},
\]

with \( X = A, B \) and \( p_m^X \) correspond to the probability of reaching that post-measurement states in the partition \( X \). This probability can be evaluated (see appendix [A]) as

\[
p_m^X = \langle \hat{\Pi}_m^X \rangle_{\rho_{GWL}} = tr \left[ \hat{\Pi}_m^X \rho_{GWL} \right] = \frac{1-p}{2} + p \langle \hat{\Pi}_m^X \rangle.
\]

Where \( \langle \hat{\Pi}_m^X \rangle \) is the transition probability of the GBPs to the state \( \hat{\Pi}_m^X \), which are evaluated for both partitions as

\[
\langle \hat{\Pi}_m^A \rangle = tr \left[ \hat{\omega}_A^\dagger \hat{\Pi}_m^A \hat{\omega}_A \right],
\]

\[
\langle \hat{\Pi}_m^B \rangle = tr \left[ \hat{\omega}_B^\dagger \hat{\Pi}_m^B \hat{\omega}_B \right].
\]

Then, the mixing states obtained with the rule Lüders \( [24] \) for both partitions (see appendix [A]) are give by

\[
\rho_{GWL|\Pi_m^A} = \hat{\Pi}_m \otimes \left\{ \frac{1-x_m(p)}{2} \hat{1}_2 + x_m(p) \hat{\bar{P}}_B \right\},
\]

\[
\rho_{GWL|\Pi_m^B} = \left\{ \frac{1-y_m(p)}{2} \hat{1}_2 + y_m(p) \hat{\bar{P}}_A \right\} \otimes \hat{\Pi}_m,
\]

where \( \hat{\bar{P}}_A \equiv \{ \hat{\psi}_A \} \left( \hat{\psi}_A \right) \) and \( \hat{\bar{P}}_B \equiv \{ \hat{\psi}_B \} \left( \hat{\psi}_B \right) \) are one-rank projectors, defined by

\[
\hat{P}_A = \sum_{i,j} \frac{ \langle i | (\hat{\omega}_A^\dagger) (\hat{1}_2 + (-1)^m \hat{n} \cdot \hat{\sigma}) (\hat{\omega}_A^\dagger) | j \rangle \langle j | \rangle i \rangle}{tr[\hat{\omega}_A^\dagger (\hat{1}_2 + (-1)^m \hat{n} \cdot \hat{\sigma}) \hat{\omega}_A] - (i | i \rangle)^2},
\]

\[
\hat{P}_B = \sum_{i,j} \frac{ \langle i | (\hat{\omega}_B^\dagger) (\hat{1}_2 + (-1)^m \hat{n} \cdot \hat{\sigma}) (\hat{\omega}_B^\dagger) | j \rangle \langle j | \rangle i \rangle}{tr[\hat{\omega}_B^\dagger (\hat{1}_2 + (-1)^m \hat{n} \cdot \hat{\sigma}) \hat{\omega}_B] - (i | i \rangle)^2}.
\]

The quantities \( x_m(p) \) and \( y_m(p) \) that appear in the equations \( 27 \) are equivalents to new mixing parameters of the GWLs in the partition \( B \) and \( A \), respectively. These are given by (see appendix [A])

\[
x_m(p) = \frac{p \langle \hat{P}_m^A \rangle_{\psi}}{\frac{1-p}{2} + p \langle \hat{P}_m^A \rangle_{\psi}},
\]

\[
y_m(p) = \frac{p \langle \hat{P}_m^B \rangle_{\psi}}{\frac{1-p}{2} + p \langle \hat{P}_m^B \rangle_{\psi}},
\]

Noteworthy, that \( x_m(p) \) and \( y_m(p) \) are an injective functions of the mixing parameter \( p \), so \( x_m(p) \) and \( y_m(p) \) present the same variation range of \( p \). It is important to see as well that the two \( x_m(p) \) or \( y_m(p) \) are not independent, since the sum over all probabilities \( \sum_m \langle \hat{P}_m^A \rangle_{\psi} = 1 \) with \( X = A, B \) impose a restriction on the mixing parameters of the reduced states. This restrictions is given by

\[
\sum_m \frac{x_m(p)}{1-x_m(p)} = \sum_m \frac{y_m(p)}{1-y_m(p)} = \frac{2p}{1-p}.
\]

Of the expressions indicated in \( 21, 24, 27 \) and \( 28 \) it is clear that the results of the measurement process in the partition \( A \) and \( B \) are built with the matrix \( \hat{W} \) and \( \hat{W}^T \), respectively. In this context, the transpose operation gathered with exchange operator connects to the post-measurement mixing states of both partitions.

Until now, a projective measurement on the partitions \( A \) or \( B \) projects the system into the statistical ensembles \( \{ p_m^A, \rho_{AB|\Pi_m^A} \} \) or \( \{ p_m^B, \rho_{AB|\Pi_m^B} \} \), respectively, quantifies the information in the unmeasured partition by means of the quantum conditional entropy, given respectively by

\[
S_A|\Pi_m^A(\rho_{AB}) = \min_{\Pi_m^A} \langle \Pi_m^A \rangle_{\rho} S_A|\Pi_m^A(\rho_{AB}),
\]

\[
S_B|\Pi_m^B(\rho_{AB}) = \min_{\Pi_m^B} \langle \Pi_m^B \rangle_{\rho} S_B|\Pi_m^B(\rho_{AB}),
\]

where \( S_A|\Pi_m^A(\rho_{AB}) \) and \( S_B|\Pi_m^B(\rho_{AB}) \) are the von-Neumann entropy of the partition \( A \) and \( B \) of \( \rho_{AB} \) obtained after the projective measurements \( \Pi_m^B \) or \( \Pi_m^A \), respectively. The entropy might give different results depending on the basis choice, a minimization is taken over all possible one-rank measurements so that minimization chooses the measurement of a partition that extracts as much information as possible of the other partition. The entropy after of the measurement in the partition \( A \) is it given by

\[
S_B|\Pi_m^A(\psi, p) = \min_{\Pi_m^A} \sum_m p_m^A S_B|\Pi_m^A(\psi, p), = \frac{1}{2} \min_{\Pi_m^A} \sum_m \frac{1-p}{1-x_m(p)} H_2 \left( \frac{1+x_m(p)}{2} \right).
\]

Here the probability \( p_m^A \) is replaced by the expression \( 27 \), while the probability \( \langle \hat{\Pi}_m^A \rangle_{\psi} \) is written in terms
we showed that the process of minimizing the conditional entropy is usually the optimization of the conditional entropy $S_{B|\Pi_m^A}$, over all projective measurements. However, in the Appendix [3] we showed that the process of minimizing for conditional entropy consists in finding the values of $x_m(p)$ that minimize the probability $\langle \Pi_m^A \rangle$. Such that the conditional entropy of the partition B have the form

$$S_{B|\Pi_m^A}(\psi, p) = F_p(x_0) + F_p(x_1), \quad (33)$$

where

$$F_p(x) = \frac{1-p}{2(1-x^2)} H_2 \left( \frac{1+x}{2} \right), \quad (34)$$

while the values of $x_0$ and $x_1$ minimize and maximize the probability $\langle \Pi_m^A \rangle$, in the equation [29a], respectively (see appendix [3]). Namely, $x_0$ is obtained when the probability $\langle \Pi_m^A \rangle$ is minimized,

$$x_0 = \frac{p\langle \Pi_m^A \rangle_{\min} - \frac{1-p}{2} \langle \Pi_0^A \rangle_{\min}}{1 \ - \ \frac{1-p}{2} \langle \Pi_0^A \rangle_{\min}}, \quad (35a)$$

but $x_1$ is obtained from to relation [39], finding that

$$x_1 = \frac{2p - (1+p)x_0}{1 + p - 2x_0} = \frac{p(1 - \langle \Pi_m^A \rangle_{\min})}{1 - p + p(1 - \langle \Pi_0^A \rangle_{\min})} = \frac{p\langle \Pi_m^A \rangle_{\max}}{1 - \frac{1-p}{2} \langle \Pi_0^A \rangle_{\max}}. \quad (35b)$$

The probability $\langle \Pi_m^A \rangle$ presents oscillations around 1/2 with amplitude $A_{\psi}$ (see appendix [3]), thereby the value of $\langle \Pi_0^A \rangle_{\min} = \frac{1}{2} - A_{\psi}$, and $x_0$ and $x_1$ can be written as

$$x_0 = \frac{p(1 - 2A_{\psi})}{1 - 2pA_{\psi}}, \quad x_1 = \frac{p(1 + 2A_{\psi})}{1 + 2pA_{\psi}}. \quad (36)$$

A straightforward calculation show that

$$A_{\psi} = \frac{1}{2} \sum_{i=1}^{n} \left( \text{tr} \left[ \hat{\mathbb{W}}_i^T \hat{\mathbb{M}}_i \hat{\mathbb{W}}_i \right] \right) = \frac{1}{2} \Delta_{\psi}. \quad (37)$$

The equation [39] is an exact analytical expression for the conditional entropy after a measurement in partition A. The aforementioned procedure can be applied to obtain the conditional entropy $S_{A|\Pi_n^B \psi}(\psi, p)$, after a measurement in partition B. The same result is obtained, except that instead of the matrix $\hat{\mathbb{W}} \psi$, its transpose is used. In addition, the mixing parameter $x_m(p)$ must be replaced by $y_m(p)$, namely,

$$S_{A|\Pi_n^B \psi}(\psi, p) = \min_{\Pi_n^B \psi} \sum_m p_m S_{A|\Pi_n^B \psi}(\psi, p), \quad (38)$$

giving

$$S_{A|\Pi_n^B \psi}(\psi, p) = \frac{1}{2} \min_{\Pi_n^B \psi} \sum_m \frac{1-p}{1-y_m(p)} H_2 \left( \frac{1+y_m(p)}{2} \right). \quad (39)$$

This implies that the minimized condition entropy take the form (see appendix [13])

$$S_{A|\Pi_n^B \psi}(\psi, p) = F_p(y_0) + F_p(y_1), \quad (40)$$

furthermore

$$B_{\psi} = \frac{1}{2} \sum_{i=1}^{3} \left( \text{tr} \left[ \hat{\mathbb{W}}_i^T \hat{\mathbb{W}}_i \right] \right)^2 = \frac{1}{2} \Delta_{\psi}. \quad (41)$$

It is important to indicate that the value of $B_{\psi}$ is coincident with the value of $A_{\psi}$, and both quantity are monotonous functions of the concurrence of the GBps $|\psi\rangle$. For this reason the conditional entropy of both partitions are the same, and as well this amounts are monotonous functions of the concurrence of the GBps.

Finally, the QD or LII is defined as the difference between the total correlation (or mutual information) and classical correlations (or conditional mutual information) coded in the same state. The quantum mutual informations or total correlation is a measure of how much information grows in a bipartite system when partitions are observed together. This quantity is defined as

$$I_{AB} = S[\rho_A] + S[\rho_B] - S[\rho_{AB}].$$

The classical correlations or conditional mutual information measured in the partition A and B are written as $J_{AB} = S[\rho_B] - S[\rho_{AB}]$ and $J_{AB} = S[\rho_B] - S[\rho_{AB}]$. These quantities measure the gain of information in the partition when the other is measured. Then the QD or LII of any state $\rho_{AB}$, when performing measured on the partition A, can be written as

$$\delta_{AB}(\rho_{AB}) = \frac{I_{AB}(\rho_{AB}) - J_{AB}(\rho_{AB})}{S[\rho_A]} = S[\rho_A] - S[\rho_{AB}] + S[\rho_{AB}], \quad (41a)$$

and when performing measured on the partition B it is given by

$$\delta_{AB}(\rho_{AB}) = \frac{I_{AB}(\rho_{AB}) - J_{AB}(\rho_{AB})}{S[\rho_B]} = S[\rho_B] - S[\rho_{AB}] + S[\rho_{AB}], \quad (41b)$$

Generally the QD of mixing states is asymmetric, i.e. $\delta_{AB} \neq \delta_{BA}$. This allow to study the average of LII, defined by $\delta_{A|B} = (\delta_{AB} + \delta_{BA})/2$, and the balance of LII, defined as $\delta_{A|B}^{-} = (\delta_{AB} - \delta_{BA})/2$ (see reference [39]). Nevertheless, a straightforward calculation showed that [39] and [40] are coincident (see appendix [13], being
equals the QD of the GWLs in both partitions, therefore the balance of LII is zero and the average of LII is same that the QD for GWLs. If we take the explicit forms of the entropies given in the equation (41) we can obtain the exact analytical expressions for the GWLs, being

\[
\delta_{AB}(\rho, p) = \delta_{AB}(\rho, p) = \delta_{AB}(\rho, p) = -2 + \frac{1}{2} \log_2 \left[ \frac{(1+2p\Delta_{\omega})}{(1-p\omega)} \right] + H_2 \left( \frac{1+p\Delta_{\omega}}{2} \right) + \frac{1-p\Delta_{\omega}}{2} H_2 \left( \frac{1+p(1-2\Delta_{\omega})}{2(1-p\Delta_{\omega})} \right) + \frac{1+p\Delta_{\omega}}{2} H_2 \left( \frac{1+p(1+2\Delta_{\omega})}{2(1+p\Delta_{\omega})} \right)
\]

The QD is zero when \( p = 0 \) in the expression (42), and the QD is coincident with the EoF of GWPs when \( p = 1 \). In these cases it is take into account that \( H_2(\frac{1}{2}) = 1 \) and \( H_2(1) = 0 \). Then, the QD is symmetrical and also is a monotonous function of the concurrence \( C(|\psi\rangle) \) of the GBPs. So, all the GBPs with the same concurrence have equal QD, forming equivalence classes.

V. EXAMPLES

In order to illustrate the behavior of the QD of the GWLs, and its dependence with the mixing parameter \( p \) and the concurrence of GBPs, we consider the four pure states shown below

\[
\rho_{GWL}(\psi_1, p) = \frac{1}{4} \begin{bmatrix}
16 - 9p & 3\sqrt{35}p & \sqrt{35}p & 7p \\
3\sqrt{35}p & 16 + 29p & 15p & 3\sqrt{35}p \\
\sqrt{35}p & 3\sqrt{35}p & 15p - 11p & \sqrt{35}p \\
7p & 3\sqrt{35}p & \sqrt{35}p & 16 - 9p
\end{bmatrix}
\]

\[
\rho_{GWL}(\psi_2, p) = \frac{1}{4} \begin{bmatrix}
10 - p & 3\sqrt{6}p & 6\sqrt{6}p & -3p \\
3\sqrt{6}p & 10 - 4p & 12p & -\sqrt{6}p \\
6\sqrt{6}p & 3\sqrt{6}p & 2(5 + 7p) - 2\sqrt{6}p & -3p \\
-3p & -\sqrt{6}p & -2\sqrt{6}p & 10 - 9p
\end{bmatrix}
\]

\[
\rho_{GWL}(\psi_3, p) = \frac{1}{4} \begin{bmatrix}
9 & 9\sqrt{2}p & -6\sqrt{2}p & -3p \\
9\sqrt{2}p & 9(1 + p) & -12p & -3\sqrt{2}p \\
-6\sqrt{2}p & -12p & 9 - p & 2\sqrt{2}p \\
-3p & -3\sqrt{2}p & 2\sqrt{2}p & 9 - 8p
\end{bmatrix}
\]

For that a GWLs to present the form of a X–state the matrix \( \mathcal{W}_\psi \) of the pure state \( |\psi\rangle \) should have the form of one diagonal or antidiagonal matrix; any other way, the GWLs are non–X states. However, the EoF and QD of GWLs only depend of the concurrence of the GBPs and it does not depend on the topology that the mixing states possesses. For example, the GWLs built with the pure state

\[
|\psi_2\rangle = \frac{\sqrt{2 + \sqrt{3}}}{2} |01\rangle + \frac{\sqrt{2 - \sqrt{3}}}{2} |10\rangle
\]

have a density matrix in form of non–X state but their QD, as well as the EoF, are the same obtained with the pure state \( |\psi_2\rangle \) given in (43b).

The QD for the states indicated in the equations (43a) are sketched in the Fig.11 together with the EoF calculated by the equation (9). Finally, let us study the effect of incorporating a local phase into the pure state with which the GWLs are built, for this we consider the following state

\[
|\psi_0\rangle = -\frac{\sqrt{2}}{6} e^{i\phi_1} |00\rangle + \frac{\sqrt{2}}{3} e^{i\phi_2} |01\rangle + \frac{\sqrt{2}}{3} e^{i\phi_3} |10\rangle + \frac{\sqrt{2}}{6} e^{i\phi_4} |11\rangle,
\]

its concurrence can be written as

\[
C(|\psi_0\rangle) = 2 \left| 3 e^{i(\phi_2 + \phi_3)} + e^{i(\phi_1 + \phi_4)} \right|
\]
it can be observed that the QD is a monotonous function that grows with the increase of the concurrence of GBps, but the QD of these states is not a monotonous function of its own EoF for all values of the mixing parameter. The EoF and QD are coincident only in three values of the mixing parameter. The EoF and QD for GWLs with a discrete pure state $|\Psi_+\rangle$, $|\psi_1\rangle$ (line blue dashed), $|\psi_2\rangle$ (magenta dotted line) and $|\psi_3\rangle$ (dotted line) and $|\psi_4\rangle$ (dot-dashed red line) those concurrences are equal to $C_{\text{max}} = 1$, $C_3 = \frac{3}{4}$, $C_2 = \frac{1}{3}$ and $C_1 = \frac{1}{2}$, respectively.

$$C[|\psi_6\rangle] = \frac{3}{2} \left| 3 + e^{i\phi} \right| = \frac{3}{2} \left| 1 + 3e^{-i\phi} \right|$$

where $\phi \equiv \phi_1 + \phi_2 - (\phi_2 + \phi_3)$. The value of the concurrence given in the equation (46) does not change if $\phi$ is changed by $-\phi$, so that the QD and EoF are same if the phase take any position in the component of pure state $|\psi_5\rangle$. Now the QD is function of mixing parameter and angle of phase $\phi$, whose graph is shown in the Fig. 2.

VI. RESULTS

In Fig. 1 it can be observed that the QD is a monotonous function that grows with the increase of the concurrence of GBps, but the QD of these states is not a monotonous function of its own EoF for all values of the mixing parameter. The EoF and QD are coincident only in three values of the mixing parameter $p = 0$, $p = p_i$ and $p = 1$, where $p_i$ is the value of the mixing parameter for which the EoF intercepts with the QD, it is found numerically (since the equation that determines the value of $p_i$ is transcendental) and are reported in Table 1. When $p > p_i$, we have that $EoF(p, \phi) > \delta_{AB}(\psi, p)$ while that $EoF(p, \phi) < \delta_{AB}(\psi, p)$ when $p < p_i$. We note that in the interval $p > p_i$ the QD and EoF are very close to each other, but they present more discrepancies in the interval $p_i < p < p_c$ beging $p_c$ the critical value of the mixing parameter that limits the border between entanglement and separability of GWLs, which is obtained in the equation (18). In Fig. 1 we observed that the GWLs is entanglement when $p > p_c$. The values of $p_c$ for the GWLs, built with the GBsp given in (33), are showed in Table 1. In the interval $-\frac{1}{3} \leq p \leq p_i$ the GWLs contain mixing states that maintain a correlation between the partitions of system, which is not associated with entanglement; in this sense it is said that QD presents quantum correlations that go beyond entanglement.

VII. CONCLUSION

An exact analytical solution of QD for Generalized Werner-Like non-X states have bee found. The optimization process involved in minimizing the conditional entropy is solved in an analytical form. The QD obtained is symmetric and increases with the concurrence of the GBps with which is built the GWLs. The maximum value obtained for QD is only for WPs, QWs and BWLs. The WS and GWLs present diferent entanglement and QD, and only are coincident when the transformation $\{\}$ applies.

TABLE I: Critical values $p_c$, the intersection point $p_i$ and the mixing parameter $p_b$ where Bell inequality is violated

| System              | $|\psi_1\rangle$ | $|\psi_2\rangle$ | $|\psi_3\rangle$ | $|\Psi_+\rangle$ |
|---------------------|------------------|------------------|------------------|-----------------|
| Value $p_c$         | 2/3              | 1/2              | 2/5              | 1/3             |
| Value $p_i$         | 0.919            | 0.888            | 0.878            | 0.879           |

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Appendix A: Projective measurement onto pure state and GWLs

Let $U = [U_{ij}]$ be unitary transformation, the bases $\{|\pi_m\rangle\}$ is unitarily equivalent to the computational bases $\{|i\rangle\}$ if $\{|\pi_m\rangle\} = \sum U_{mi} |i\rangle$. The projector, associated to these measurement are

$$\hat{\Pi}_m = |\pi_m\rangle \langle \pi_m| = \sum_{ij} U_{im}\overline{U}_{jm} |i\rangle \langle j|,$$  \hspace{1cm} (A1)

where $\overline{U}_{jm}$ is the complex conjugate of $U_{jm}$. The projectors associated to local projective measurement in the partition $A$ of a bipartite system are

$$\hat{\Pi}_m^A = \hat{\Pi}_m \otimes \hat{1} = \sum_{ijk} U_{im}\overline{U}_{jm} |ik\rangle \langle jk|,$$  \hspace{1cm} (A2)

where the identity operator $\hat{1}$ has been replaced by the sum of projectors $\sum_k |k\rangle \langle k|$. On the other hand, any pure state $|\psi\rangle$ that belongs to $\mathcal{H} \otimes \mathcal{H}$ can be written in terms of computational basis as

$$|\psi\rangle = \sum_{ij} \psi_{ij} |ij\rangle \quad \text{with} \quad \sum_{ij} \psi_{ij} \overline{\psi}_{ij} = 1.$$  \hspace{1cm} (A3)

In order to simplify our results we define the matrix $\hat{\mathbb{W}}_\psi$, whose elements are $\psi_{ij}$, so the normalization condition can be written as

$$\sum_{ij} \psi_{ij} \overline{\psi}_{ij} = 1 \Rightarrow \sum_{ij} [\hat{\mathbb{W}}_\psi]_{ij} [\hat{\mathbb{W}}_\psi]^\dagger_{ij} = 1$$

$$\sum_{ij} [\hat{\mathbb{W}}_\psi]_{ij} [\hat{\mathbb{W}}_\psi]^\dagger_{ij} = 1 \Rightarrow \sum_{ij} [\hat{\mathbb{W}}_\psi]_{ij} [\hat{\mathbb{W}}_\psi]^\dagger_{ij} = 1$$

$$\therefore \text{tr} [\hat{\mathbb{W}}_\psi [\hat{\mathbb{W}}_\psi]^\dagger] = \text{tr} [\hat{\mathbb{W}}_\psi [\hat{\mathbb{W}}_\psi]^\dagger] = 1.$$  \hspace{1cm} (A4)

The representation of a pure state in terms of the density matrix is given by the following rank-one projector,

$$|\psi\rangle \langle \psi| = \sum_{ijkl} \psi_{ij} \overline{\psi}_{kl} |ij\rangle \langle kl|.$$  \hspace{1cm} (A5)

the reduced states are obtained by taking partial trace over both partitions, thus, for partition $A$ we have that

$$\rho_A(\psi) = \text{tr}_B [ |\psi\rangle \langle \psi|] = \sum_{ijkl} \psi_{ij} \overline{\psi}_{kl} \text{tr}_B [ |ij\rangle \langle kl|]$$

$$= \sum_{ijkl} \psi_{ij} \overline{\psi}_{kl} \delta_{jk} |i\rangle \langle k| = \sum_{ijk} \psi_{ij} \overline{\psi}_{kj} |i\rangle \langle k|.$$  \hspace{1cm} (A6)

and for partition $B$ we have,

$$\rho_B(\psi) = \text{tr}_A [ |\psi\rangle \langle \psi|] = \sum_{ijkl} \psi_{ij} \overline{\psi}_{kl} \text{tr}_A [ |ij\rangle \langle kl|]$$

$$= \sum_{ijkl} \psi_{ij} \overline{\psi}_{kl} \delta_{ik} |j\rangle \langle l| = \sum_{ij} \psi_{ij} \overline{\psi}_{ij} |j\rangle \langle l|.$$  \hspace{1cm} (A7)

This shows that partition $B$ can be accessed through the transpose operation. On the other hand, the probability of obtaining a result after the local projective measurement $\{A2\}$ when the system is initially in the pure state $|\psi\rangle$ is given by

$$\langle \hat{\Pi}_m^A \rangle_\psi = \langle \psi | \hat{\Pi}_m^A | \psi \rangle = \sum_{ijkl} \overline{\psi}_{ik} \psi_{jk} U_{im} \overline{U}_{jm},$$  \hspace{1cm} (A8)

in which equation $\{A2\}$ has been used. The last expression can be written in matrix form as,

$$\langle \hat{\Pi}_m^A \rangle_\psi = \sum_{ijkl} \overline{\psi}_{ik} \psi_{jk} U_{im} \overline{U}_{jm}$$

$$\sum_{ijkl} [\hat{\mathbb{W}}^T_\psi]_{kij} [\hat{\mathbb{W}}_\psi]_{xij} = \text{tr} [\hat{\mathbb{W}}^T_\psi [\hat{\mathbb{W}}_\psi]^\dagger]$$

$$\langle \hat{\Pi}_m^A \rangle_\psi = \text{tr} [\hat{\mathbb{W}}^T_\psi [\hat{\mathbb{W}}_\psi]^\dagger] \hat{\Pi}_m = \langle \hat{\Pi}_m \rangle_{\rho(\psi)},$$  \hspace{1cm} (A9)

where we have used the expressions $\{A1\}$ and $\{A6\}$. If the measurement is performed on partition $B$ then

$$\langle \hat{\Pi}_m^B \rangle_\psi = \langle \psi | \hat{\Pi}_m^B | \psi \rangle = \text{tr} [\hat{\mathbb{W}}^T_\psi [\hat{\mathbb{W}}_\psi]^\dagger] \hat{\Pi}_m.$$  \hspace{1cm} (A10)

Now, we perform a projective measurement on partition $A$ of the pure state $|\psi\rangle$. In order to do this, we apply Lüders rule $\{38\}$ to pure state $|\psi\rangle$, obtaining

$$|\psi\rangle \langle \psi| = \hat{\Pi}_m^A \langle \hat{\Pi}_m^A \rangle^\dagger_\psi \langle \hat{\Pi}_m \rangle_{\rho(\psi)},$$  \hspace{1cm} (A11)

$$= \frac{1}{\langle \hat{\Pi}_m^A \rangle_\psi} \sum_{ijkl} \overline{\psi}_{jk} \psi_{ij} U_{im} \overline{U}_{jm} \overline{U}_{jk} |ij\rangle \langle kl|.$$  \hspace{1cm} (A12)
Where we have defined
\[ \rho_B|\Pi_A^a = \sum_{ij} \frac{\langle i| \hat{W}_\psi^j|\hat{W}_\psi|j \rangle}{\text{tr} \left[ \hat{W}_\psi^j \Pi_A^a \hat{W}_\psi^j \right]} |i \rangle \langle i| = |\hat{\psi}_B \rangle \langle \hat{\psi}_B| . \] (A13)

In the equation (A13) it has been replaced in the equation (A10). We can show that (A13) is a pure state, since it is a one-rank projector operator. A straightforward calculation leads to \( \text{tr} [\rho_B|\Pi_A^a] = 1 \) and \( \rho_B|\Pi_A^a = \rho_B|\Pi_A^{a*} \).

In the case of projective measurement in the partition \( B \) the results are similar, except for the transpose operation in the matrix \( \hat{W}_\psi \).

Finally, we perform a local projective measurement on the GWLs (6) in the partition \( A \). According to the equation (29a) we have
\[ \rho_{\text{GWL}|\Pi_A^m} = \frac{(\hat{\Pi}_m^A) \rho_{\text{GWL}} (\hat{\Pi}_m^A)^\dagger}{p_m^A} \]
\[ \rho_{\text{GWL}|\Pi_A^{a*}} = \frac{1}{p_m^A} \left[ \frac{1}{p_m^A} \hat{\Pi}_m^A + p |\psi\rangle \langle \psi| (\hat{\Pi}_m^A)^\dagger \right] \]
\[ \rho_{\text{GWL}|\Pi_A^{a}} = \frac{1}{p_m^A} \left[ \frac{1}{p_m^A} \hat{\Pi}_m^A + p \langle \hat{\Pi}_m^A |\psi\rangle \langle \psi| \right] \]
\[ \rho_{\text{GWL}|\Pi_A^{a*}} = \hat{\Pi}_m \otimes \frac{1 - p}{4p_m^A} \hat{\Pi}_m^A + p \langle \hat{\Pi}_m^A |\psi\rangle \langle \psi| \rangle \rangle \] (A14)

Where we have used Eqs. (A11), (A12) and (A13). Defining the mixing parameter in the partition \( B \) as
\[ x_m(p) = \frac{p}{p_m^A} = \frac{\langle \hat{\Pi}_m^A |\psi\rangle}{\langle \hat{\Pi}_m^A |\psi\rangle}, \] (A15)

While the term that accompanies the identity matrix in (29a) can be written as
\[ 1 - p = \frac{1}{4p_m^A} \frac{1 - p}{\langle \hat{\Pi}_m^A |\psi\rangle} \] (A16)

As shown in equation (27a). This result is very important since the projective measurement does not alter the structure of the GWLs, but modifies the mixing parameter \( p \) by \( x_m(p) \).

Appendix B: Calculation of conditional entropy for Werner-like states

For the optimization process, it is convenient to define the equation (31), which is a positive and monotonically increasing function of the mixing parameter \( x_m(p) \) of partition \( B \). So that the conditional entropy (22) is given by
\[ S_B|\Pi_A^m)(\psi,p) = \min_{\{\Pi_A^m\}} \sum_m F (x_m(p)). \] (B1)

The minimum is obtained when there is a set of value for the mixing parameter \( x_m(p) \) such that the function \( F \) is minimal, subject to restriction (30). For the case \( n = 2 \), it is sufficient to find the value \( x_m \) for which \( F \) is minimal, while \( x_m \) is obtained from (30). Deriving \( F(z_m) \) with respect to \( z_m \) and after a simple calculation, we can obtain
\[ dF(x_m) = - \frac{(1 - p) \log_2 \left( \frac{1}{2} + \frac{1}{2} x_m \right)}{2(1 - x_m)^2} \] (B2)

Using the values of \( x_m \) given in (29a), it is easy to show that
\[ dF(x_m) = \log_2 \left( \frac{1}{2} + \frac{1}{2} x_m \right) \] (B3)

It is clear from (B3) that the process of minimizing the conditional entropy is relegated to finding the values of \( x_m \) that minimize the probability \( (\hat{\Pi}_m^A |\psi\rangle \) which in turn minimize the function \( F(x_m) \). This probability presents oscillations around the uniform distribution, which allows us to evaluate its minimum quickly. Considering the local projective measurement (29a) and after straightforward calculation, we obtain the simplified result
\[ \langle \hat{\Pi}_m^A |\psi\rangle = \frac{1}{2} \left[ 1 + (\sigma_2)_{\rho_A}(\psi) \cos(\theta + m\pi) \right. \]
\[ \left. + (e^{i \phi} \sigma_x)_{\rho_A}(\psi) \sin(\theta + m\pi) \right] \] (B4)

where \( \rho_A(\psi) \) is given by (A10) and the explicit expressions for the coefficients of the trigonometric functions are
\[ (\sigma_z)_{\rho_A}(\psi) = \sqrt{|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2}, \] (B5a)
\[ (e^{i \phi} \sigma_x)_{\rho_A}(\psi) = 2 \text{Re} \left[ (z_1 \bar{z}_3 - z_2 \bar{z}_4)e^{-i \phi} \right]. \] (B5b)

Taking into account that \( \text{Re} \left[ z e^{i \phi} \right] \leq |z| \) and
\[ 2|z_1 \bar{z}_3 - z_2 \bar{z}_4| = \sqrt{(\sigma_x)_{\rho_A}(\psi) + (\sigma_y)_{\rho_A}(\psi)}, \] we have that the amplitude of the oscillations presented in (B4) is given by
\[ A_\psi = \frac{1}{2} \sqrt{\langle \sigma_z \rangle_{\rho_A}^2(\psi) + \langle \sigma_x \rangle_{\rho_A}^2(\psi) + \langle \sigma_y \rangle_{\rho_A}^2(\psi)}, \]
\[ \text{with} \sum_{i=1}^3 |\sigma_i|_{\rho_A}^2(\psi) = \frac{1}{2} \sum_{i=1}^3 \text{tr} |\sigma_i \rho_A(\psi)|^2, \]
\[ = \frac{1}{2} \sum_{i=1}^3 \text{tr} \left[ \sigma_i \bar{W}_\psi^i \bar{W}_\psi \right]^2 = \frac{1}{2} \sum_{i=1}^3 \text{tr} \left[ \bar{W}_\psi^i \sigma_i \bar{W}_\psi \right]^2, \]
\[ = \frac{1}{2} \sqrt{|| \psi ||^4 - C || \psi ||^2} = \frac{1}{2} \Delta_\psi. \] (B6)
Where it has been used using the equation \(\text{(A6)}\). This result coincides with \(\text{(37)}\). So the minimum probability value is

\[
\langle \tilde{P}_{m}^{A} \rangle_{\psi}^{\text{min}} = \frac{1}{2} - A_{\psi} = \frac{1}{2} \left(1 - \Delta_{\psi}\right).
\]

(S7)

Sintetizing, the value that minimize the function \(F(x_{m}(p))\), and therefore minimize the conditional entropy \(\text{(B1)}\), is given by \(\text{(35a)}\).