Can the Decoherent Histories Description of Reality be Considered Satisfactory?

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Abstract
We discuss some features of the Decoherent Histories approach. We consider four assumptions, the first three being in our opinion necessary for a sound interpretation of the theory, while the fourth one is accepted by the supporters of the DH approach, and we prove that they lead to a logical contradiction. We discuss the consequences of relaxing any one of them.

1 Introduction.
The Decoherent Histories (DH) approach of Griffiths [1, 2], Omnès [3, 4] and Gell–Mann and Hartle [3, 5, 6] has attracted in recent years a lot of attention since it seemed to yield a solution to the conceptual and interpretative problems of standard quantum mechanics (SQM) without requiring relevant changes of the formalism. This feature is not shared by other attempts to work out a quantum theory without observers like hidden variable theories [10, 11] which need additional parameters besides (or in place of) the wave function to characterize the state of an individual physical system, or by the dynamical reduction models [12, 13, 14, 15] which accept that Schrödinger’s equation must be modified.

The general structure of the theory can be summarized as follows: let \( S \) be a physical system which at the initial time \( t_0 \) is associated to the statistical operator \( W(t_0) = W \), and let \( U(t, t') \) be the unitary operator describing its evolution. One then chooses \( n \) arbitrary times \( t_1 < t_2 < \ldots < t_n \), and for each of them (let us say \( t_m \)) one considers an exhaustive set \( \{P_{\alpha m}\} \) of mutually exclusive projection operators:

\[
\sum_{\alpha m} P_{\alpha m} = 1, \quad P_{\alpha m} P_{\beta m} = \delta_{\alpha m, \beta m} P_{\alpha m}.
\]
One also considers the following projection operators:

\[ Q^m = \sum_\alpha \pi_\alpha^m P^\alpha_m, \]  

(2)

where \( \pi_\alpha^m \) take the values 0 or 1.

One history is then defined by the sequence of times \( t_1, t_2, \ldots, t_n \) and a corresponding sequence of projection operators, each of them taken from the family \( \{ Q^m \} \) \( (m = 1, 2, \ldots, n) \), defined as in (2):

\[ \text{His}^{(\alpha)} = \{(Q_{1}^{\alpha_1}, t_1), (Q_{2}^{\alpha_2}, t_2), \ldots, (Q_{n}^{\alpha_n}, t_n)\}. \]  

(3)

When the operators appearing in eq. (3) belong to the exhaustive set \( \{ P^\alpha_m \} \), \( \text{His}^{(\alpha)} \) is said to be maximally fine-grained. Consideration of the operators \( \{ Q^m \} \) corresponds to taking into account coarse-grained histories. A family of histories is a set whose elements are all histories having the form (3), i.e. all maximally fine-grained histories and their coarse-grainings. For a given family one then considers what is usually denoted as the decoherence functional:

\[ D(\alpha, \beta) = \text{Tr}[P_1^{\alpha_1}U(t_n, t_{n-1})P_{n-1}^{\alpha_{n-1}}U(t_{n-1}, t_{n-2})\ldots U(t_1, t_0)W \nonumber \]

\[ U(t_1, t_0)\ldots U(t_{n-1}, t_{n-2})P_{n-1}^{\beta_{n-1}}U(t_n, t_{n-1})P_n^{\beta_n}], \]  

(4)

in which the projection operators \( P_1^{\alpha_1}, \ldots, P_n^{\alpha_n} \) characterize the history \( \text{His}^{(\alpha)} \) and the projection operators \( P_1^{\beta_1}, \ldots, P_n^{\beta_n} \) another history \( \text{His}^{(\beta)} \). A family of histories is said to be decoherent if and only if:

\[ D(\alpha, \beta) = \delta_{\alpha, \beta}D(\alpha, \alpha), \]  

(5)

i.e. iff the decoherence functional vanishes when the two maximally fine-grained histories \( \text{His}^{(\alpha)} \) e \( \text{His}^{(\beta)} \) do not coincide. When they coincide, the expressions \( D(\alpha, \alpha) \) are assumed to define a probability distribution over the maximally fine-grained histories of the decoherent family. In this case one can also consider the expressions corresponding to \( D(\alpha, \alpha) \) in which the coarse grained operators replace the fine grained ones and assume that they give the probability of the coarse-grained histories. We will say that a history is decoherent if it belongs to at least one decoherent family.

As it should be clear from this presentation, the theory, at its fundamental level, does not attach any particular role either to measurement processes (even though it is perfectly legitimate to build up histories describing the unfolding with time of such processes and the occurrence of their outcomes), or to wave packet reduction, and represents an attempt to get rid of all those features which make fundamentally unsatisfactory the Copenhagen interpretation of SQM.

In this letter, we will discuss some interpretational issues of the DH approach, with special regard to “scientific realism”. In the literature analyses of a similar kind have been presented \[16, 17, 18, 19\] by some authors and the supporters of the DH approach have answered \[2, 20, 21\] to the remarks of the above quoted papers. We will add new arguments identifying precise problems that such an approach has to face: in particular, in

\[1\]Note that here we consider only maximally fine-grained histories.
the next section we will put forward four assumptions, the first three being, in our opinion, necessary for a realistic interpretation of the theory, while the fourth one is generally accepted by the supporters of the DH approach. We will then prove, in section 3, that such assumptions lead to a logical contradiction and thus they cannot hold simultaneously. In the final sections, we will analyze the consequences of relaxing any one of them. Though we are aware that the supporters of the DH approach will not subscribe all our assumptions, our argument helps to understand what can be done and, more important, what cannot be done within the DH approach.

2 Four assumptions.

Let us list explicitly our four assumptions, and discuss their conceptual status. For more details we refer the reader to [23].

a) Decoherent Families and Boolean Algebras. Among the proponents and the supporters of the Decoherent Histories, Omnès [3], and subsequently Griffiths [1], have suggested to equip any family of decoherent histories with an algebraic Boolean structure. For simplicity (and also since in what follows we will always make reference to families of this type) let us consider a family of histories characterized by only one time $t$, and, accordingly, by a unique exhaustive and exclusive set of projection operators $\{P_\alpha\}$ and their grainings $\{Q_\alpha^m\}$:

$$\text{His}^{(\alpha)} = \{(Q_\alpha, t)\}.$$  

In such a case, the logical connectives, the conjunction and the disjunction of two histories and the negation of one history, are defined in the following way:

$$\text{His}^{(\alpha)} \land \text{His}^{(\beta)} = \{(Q_\alpha \land Q_\beta, t)\} \quad Q_\alpha \land Q_\beta = Q_\alpha Q_\beta,$$

$$\text{His}^{(\alpha)} \lor \text{His}^{(\beta)} = \{(Q_\alpha \lor Q_\beta, t)\} \quad Q_\alpha \lor Q_\beta = Q_\alpha + Q_\beta - Q_\alpha Q_\beta,$$

$$\neg \text{His}^{(\alpha)} = \{(Q_\alpha^\perp, t)\} \quad Q_\alpha^\perp = 1 - Q_\alpha.$$  

We stress that the fact that any family can be equipped with a Boolean structure plays an essential role within the theory since it guarantees that one can use the rules of classical logic to deal with the histories belonging to a single decoherent family. This in turn implies that the same rules can be used to argue about the physical properties described by the histories, avoiding in this way all difficulties characterizing quantum logics. Since there is a general agreement about it, we will not discuss this feature any further.

b) Decoherent Histories and truth values. Let us restrict ourself to a specific family of histories. As already stated, it is one of the basic assumptions of the theory that, if the family satisfies the decoherence conditions, the diagonal elements $D(\alpha, \alpha)$ of the decoherence functional acquire the status of a probability distribution over the histories of the decoherent family. In connection with such an assumption one is naturally led to raise the question: probability of what? Of course, not probability of finding the system having the properties described by the history $\text{His}^{(\alpha)}$ if a measurement is performed, otherwise the theory would not represent an improvement of the Copenhagen interpretation of
Quantum Mechanics. The only possible answer, in order to have a sound theory, is that the probabilities refer to objective properties of the physical system, like classical probabilities. Only in this way one can hope to construct a realistic interpretation of Quantum Mechanics, as often advocated by the supporters of the DH approach.

To clarify this point, we can consider the probabilities of Classical Statistical Mechanics. Such a theory yields, in general, only a probability distribution over the sets of subsets of the phase space. Nevertheless, the theory allows to consider the physical system one is interested in as uniquely associated to a precise point in phase space at any given chosen time\(^2\). This association renders automatically true or false any statement concerning the properties of the system\(^3\); actually it is precisely this feature of Classical Statistical Mechanics that makes the theory compatible with a realistic attitude towards physical reality. For example, for a gas we usually know only its macroscopic thermodynamic properties like pressure, temperature, ..., or the average value of the microscopic ones; however, any statement concerning such properties has a precise truth value (in general unknown to us) which is uniquely determined by the point representing the actual state of the system.

On the other hand, in Standard Quantum Mechanics with the completeness assumption, this is no more possible: when a system is in a superposition of two states, one may not even think that it possesses one of the properties described by those two states. That is why the Copenhagen interpretation is not a realistic one; and that is why one needs the projection postulate in order to actualize the quantum potentialities through measurement processes.

Thus, in order to avoid giving up the request of realism, the probabilities of the DH approach must be the analogous of classical probabilities: this means that to every decoherent history it must be possible to associate a precise truth value 1 or 0, even though in general we do not know which one is the right one, like in classical statistical mechanics.

Obviously, this assignment of truth values to histories can be expressed formally by an appropriate homomorphism \(h\) from the histories of any single decoherent family onto the set \(\{0, 1\}\) which must satisfy the conditions making legitimate to resort to classical reasoning when dealing with such histories:

\[
\begin{align*}
    h[\text{His}^{(\alpha)} \land \text{His}^{(\beta)}] &= h[\text{His}^{(\alpha)}] \land h[\text{His}^{(\beta)}],
    \\
    h[\text{His}^{(\alpha)} \lor \text{His}^{(\beta)}] &= h[\text{His}^{(\alpha)}] \lor h[\text{His}^{(\beta)}],
    \\
    h[\neg \text{His}^{(\alpha)}] &= h[\text{His}^{(\alpha)}] \perp = 1 - h[\text{His}^{(\alpha)}].
\end{align*}
\]

In simpler terms, the homomorphism must preserve the logical operations of conjunction, disjunction and negation. For instance if a history is true the fact that the correspondence \(h\) be an homomorphism satisfying the above relations implies that its negation is false; if one history is true and a second history is false, then their conjunction is false, while

\(^2\)Actually, the theory specifies that the most complete characterization of the state of a physical system is just given by the assignment of such a point in phase space. This is nothing but the completeness requirement of the theory.

\(^3\)Note that the mentioned properties might also refer to a certain graining, such as “the energy of a molecule lies within such and such an interval...”
their disjunction is true. As already remarked, the homomorphic nature of \( h \) guarantees that classical logic can be used within a single family of decoherent histories and that the truth values associated to the elements (i.e., the histories) of the boolean algebras (i.e., the decoherent families) obey classical rules.

c) **Histories belonging to different families.** According to the proponents of the DH approach, it is one of the firm points of the theory that one cannot compare different histories belonging to decoherent families which are incompatible among themselves, i.e., such that there does not exist a decoherent family which can accommodate all of them. Thus any conclusion one derives from such histories is neither true nor false; it is simply devoid of any meaning. One cannot however avoid raising the following question: when the same history belongs to different decoherent families (which are generally incompatible), should one require that its truth value be the same or should one allow it to change when he changes the family? Or is this question meaningless, within the theory? We remember that we previously said that any decoherent history should have a truth value and, as such, should be related to some “element of reality” when it is true. Then, if one accepts that the truth value of the same history depends on the decoherent family to which it is considered to belong, one has to face an extremely embarrassing situation: if he looks at the history from the perspective of a given decoherent family, then it may turn out (e.g.) to be true, i.e. to represent properties objectively possessed by the physical system at the times characterizing the history. But, alternatively, if he considers the same history as belonging to a second decoherent family (different from the previous one) then it may turn out to be false, i.e., it identifies physical properties which are not possessed by the physical system at the considered times. Nor can one say that the question we have posed is not legitimate (unless he denies from the beginning the very existence of truth values), since when we are talking about decoherent histories, we are talking about physical properties that systems possess or fail to possess, and it is important to know whether these properties are objective or depend in some way upon our choice of the family. We believe that it is unavoidable to assume that the truth value of a single history cannot depend from the decoherent family one is considering. It seems to us that this assumption is fundamental in order to have a realistic interpretation of physical processes. Anyway, we will further comment on this crucial point in the final Sections.

d) **How many families of decoherent histories can be considered?** One of the main difficulties that the theory had to face since its appearance is the following: are all families of decoherent histories equally legitimate to describe objective properties of physical systems or should one introduce some criterion limiting the number of acceptable families to few, or even to only one of them? The fundamental reasons for which this problem has to be faced are the following. First, the very existence of incompatible decoherent families gives rise to various difficulties of interpretation; as already remarked different histories belonging to incompatible families, when considered separately can be assumed to describe correctly the properties of a physical system, while it is forbidden to consider them together. This feature of the theory seems absolutely natural to the supporters of the Decoherent Histories, but it is a source of worries for the rest of the
scientific community. Secondly, there are many families (actually the majority of them) which, in spite of the fact that they satisfy the decoherence condition, cannot be endowed by any direct physical meaning: how can then one consider them as representing objective properties of physical systems? In spite of these difficulties, some supporters [2] insist in claiming that there are no privileged families. Accordingly, we take (for the moment) the same point of view and we assume that any decoherent family has to be taken into account. Actually, in the proof of our theorem, we will limit our considerations to very few and quite reasonable families, and we will by no means need to resort to the consideration of exotic histories to derive our conclusions.

In the next section we prove that the Decoherent Histories approach of quantum mechanics, to avoid logical inconsistencies, requires to give up at least one of the previous four assumptions. We will not exhibit the general derivation of such a conclusion (which has been given in [23]), but we will prove our theorem with reference to a quite simple example which is sufficient to make clear the crucial lines of our reasoning.

3 An explicit example proving that the Decoherent Histories approach is incompatible with the four previous assumptions.

Let us focus our attention on a quite simple physical system, i.e. two spin $1/2$ particles. We take into account only the spin degrees of freedom and we suppose that the Hamiltonian does not involve the spin variables (so that one can consider it as identically equal to zero — the quantum state of the system does not change with time). Let us consider the spin operators $\sigma_1^x, \sigma_1^y, \sigma_1^z$ (in units of $\hbar/2$) for particle 1, and $\sigma_2^x, \sigma_2^y, \sigma_2^z$ for particle 2.

We take now into account the following table of nine spin operators for the composite system:

$$
\begin{array}{ccc}
\sigma_1^1 & \sigma_2^2 & \sigma_1^1\sigma_2^2 \\
\sigma_2^1 & \sigma_1^2 & \sigma_1^1\sigma_2^1 \\
\sigma_2^2 & \sigma_1^1 & \sigma_1^1\sigma_2^2 \\
\end{array}
$$

This set of operators has been first considered by Peres [24] and Mermin [25], to investigate the unavoidable contextuality of any deterministic hidden variable theory. Their argument is quite straightforward: if one assumes that the specification of the hidden variables determine per se which one of the two possible values (+1 and -1) these operators “possess”, one gets a contradiction. In fact, since the product of the three commuting (and thus compatible) operators of each line and of the first two columns is
the identity operator (which must obviously assume the value 1 for any choice of the hidden variables) while the product of the three commuting operators of the last column equals minus the identity operator, no acceptable assignment of values (+1 and -1) to the nine operators can be made. The way out from this difficulty is also well known: one has to accept the contextual nature of possessed properties, meaning that the truth value of (e.g.) the statement “this observable has the value +1” is not uniquely determined by the complete specification of the system under consideration but it depends on the overall context. In the case under consideration this means that the truth value of the considered statement might (and actually for at least one of them must) depend on the fact that the considered observable is measured together with the others compatible observables appearing in the same line, or together with the others compatible observables of the column to which it belongs. This fact is considered as puzzling by some people and absolutely natural by others [26]. In any case, the way out does not lead to inconsistencies since some of the operators appearing in the considered line and column do not commute among themselves. It is therefore impossible to perform simultaneously the two sets of experiments. We would like to stress the crucial fact that the ambiguity about the truth values is here directly associated to actual physically different situations. In the words of the authors of [26] this fact reflects little more than the rather obvious observation that the result of an experiment should depend upon how it is performed!

We now work out an argument related to the one just mentioned, which however has a completely different conceptual status since it deals with a theoretical scheme in which there are no hidden variables; besides, we will always deal with projection operators of a set of observables such that the resulting families are decoherent — this makes the proof more lengthy than in the previous case.

We consider six families of decoherent histories all of them being one–time histories referring to the same time instant \( t > t_0 \) (\( t_0 \) being the initial time) and to the same initial state described by a given statistical operator (which we do not need to specify). Being one–time histories the corresponding families are characterized by one exhaustive set \( \{ P_{m=1}\} \) of mutually exclusive projection operators plus their coarse-grainings and they turn out to be automatically decoherent. Let us characterize them in a precise way:

- **Family A.** The histories of this family make reference to the properties of the observables \( \sigma_1^x, \sigma_2^x \) and \( \sigma_1^x \sigma_2^x \). Since such operators commute with each other one can characterize the maximally fine–grained histories of the family as those associated to the projection operators on their common eigenmanifolds. Let us list the common eigenstates, the corresponding eigenvalues and the associated projection operators and histories:

  a) The first eigenstate is:

  \[ |1x^+\rangle \otimes |2x^+\rangle \quad \implies \quad \begin{cases} +1 & \sigma_1^x \\ +1 & \sigma_2^x \\ +1 & \sigma_1^x \sigma_2^x \end{cases} \]

  the associated projection operator is \( P_{1x^+2x^+} \) and the history corresponding to it will be denoted as \( \text{His}[1x^+2x^+] \).

  b) The second one is:
\[ |1x^+\rangle \otimes |2x^-\rangle \implies \begin{cases} +1 & \sigma_x^1 \\ -1 & \sigma_x^2 \\ -1 & \sigma_x^1 \sigma_x^2 \end{cases} \]

whose associated projection operator is \( P_{1x^+2x^-} \) and the corresponding history \( \text{His}[1x^+2x^-] \).

c) The third eigenstate is:

\[ |1x^-\rangle \otimes |2x^+\rangle \implies \begin{cases} -1 & \sigma_x^1 \\ +1 & \sigma_x^2 \\ -1 & \sigma_x^1 \sigma_x^2 \end{cases} \]

whose associated projection operator is \( P_{1x^-2x^+} \) and the corresponding history \( \text{His}[1x^-2x^+] \).

d) Finally, the fourth common eigenstate is:

\[ |1x^-\rangle \otimes |2x^-\rangle \implies \begin{cases} -1 & \sigma_x^1 \\ -1 & \sigma_x^2 \\ +1 & \sigma_x^1 \sigma_x^2 \end{cases} \]

whose associated projection operator is \( P_{1x^-2x^-} \) and the corresponding history \( \text{His}[1x^-2x^-] \).

Besides the four histories we have just listed it is useful, for our future purposes, to take into account the two following coarse–grained histories:

\[
\begin{align*}
\text{His}[(xx)^+] &= \text{His}[1x^+2x^+] \lor \text{His}[1x^-2x^-] \quad (6) \\
\text{His}[(xx)^-] &= \text{His}[1x^+2x^-] \lor \text{His}[1x^-2x^+] \quad (7)
\end{align*}
\]

Obviously, the first of these histories is associated to the projection operator \( P_{1x^+2x^+} + P_{1x^-2x^-} \). Note that if this history is true, then the property possessed by the system referring to the operator \( \sigma_x^1 \sigma_x^2 \) is the one corresponding to the eigenvalue \(+1\), while, if it is false, it is the one corresponding to the eigenvalue \(-1\). The second coarse–grained history is associated to the projection operator \( P_{1x^+2x^-} + P_{1x^-2x^+} \), and it corresponds to the negation of the history \( \text{His}[(xx)^+] \).

• Family B. It deals with properties related to the operators \( \sigma_y^1, \sigma_y^2 \) \( \in \sigma_y^1 \sigma_y^2 \). The game is strictly analogous to the previous one: the basic histories being \( \text{His}[1y^+2y^+] \) associated to the projection operator \( P_{1y^+2y^+} \); \( \text{His}[1y^+2y^-] \) associated to the projection operator \( P_{1y^+2y^-} \); \( \text{His}[1y^-2y^+] \) associated to the projection operator \( P_{1y^-2y^+} \), and, finally, \( \text{His}[1y^-2y^-] \) associated to the projection operator \( P_{1y^-2y^-} \). We will also deal with the two coarse–grained histories:

\[
\begin{align*}
\text{His}[(yy)^+] &= \text{His}[1y^+2y^+] \lor \text{His}[1y^-2y^-] \quad (8) \\
\text{His}[(yy)^-] &= \text{His}[1y^+2y^-] \lor \text{His}[1y^-2y^+] \quad (9)
\end{align*}
\]

which are associated to the projection operator \( P_{1y^+2y^+} + P_{1y^-2y^-} \) and to the eigenvalue \(+1\) of the operator \( \sigma_y^1 \sigma_y^2 \); and to the projection operator \( P_{1y^+2y^-} + P_{1y^-2y^+} \), corresponding
to the negation of the previous history, respectively.

- **Family C.** The relevant commuting operators are $\sigma_x^1\sigma_y^2$, $\sigma_y^1\sigma_x^2$ and $\sigma_z^1\sigma_z^2$, their common eigenstates and the corresponding eigenvalues are:
  a) The first one is:

  \[
  \frac{1}{\sqrt{2}}[|1z^+\rangle \otimes |2z^+\rangle + i|1z^-\rangle \otimes |2z^-\rangle] \implies \begin{cases} +1 & \sigma_x^1\sigma_y^2 \\ +1 & \sigma_y^1\sigma_x^2 \\ +1 & \sigma_z^1\sigma_z^2 \end{cases}
  \]

  The associated projection operator is $P_{(xy^+)(yx^+)(zz)^+}$ and the corresponding history $\text{His}[(xy^+)(yx^+)(zz)^+]$.
  b) The second one is:

  \[
  \frac{1}{\sqrt{2}}[|1z^+\rangle \otimes |2z^+\rangle - i|1z^-\rangle \otimes |2z^-\rangle] \implies \begin{cases} -1 & \sigma_x^1\sigma_y^2 \\ -1 & \sigma_y^1\sigma_x^2 \\ +1 & \sigma_z^1\sigma_z^2 \end{cases}
  \]

  The associated projection operator is $P_{(xy^-)(yx^-)(zz)^+}$ and the corresponding history $\text{His}[(xy^-)(yx^-)(zz)^+]$.
  c) The third is:

  \[
  \frac{1}{\sqrt{2}}[|1z^+\rangle \otimes |2z^-\rangle + i|1z^-\rangle \otimes |2z^+\rangle] \implies \begin{cases} -1 & \sigma_x^1\sigma_y^2 \\ +1 & \sigma_y^1\sigma_x^2 \\ -1 & \sigma_z^1\sigma_z^2 \end{cases}
  \]

  The associated projection operator is $P_{(xy^-)(yx^+)(zz)^-}$ and the corresponding history $\text{His}[(xy^-)(yx^+)(zz)^-]$.
  d) Finally, the fourth one is:

  \[
  \frac{1}{\sqrt{2}}[|1z^+\rangle \otimes |2z^-\rangle - i|1z^-\rangle \otimes |2z^+\rangle] \implies \begin{cases} +1 & \sigma_x^1\sigma_y^2 \\ -1 & \sigma_y^1\sigma_x^2 \\ -1 & \sigma_z^1\sigma_z^2 \end{cases}
  \]

  The associated projection operator is $P_{(xy^+)(yx^-)(zz)^-}$ and the corresponding history $\text{His}[(xy^+)(yx^-)(zz)^-]$.

  We will also consider the following six coarse-grained histories:

  \[
  \begin{align*}
  \text{His}[(xy)^+] & = \text{His}[(xy)^+(yx)^+(zz)^+] \lor \text{His}[(xy)^+(yx)^-(zz)^-], \\
  \text{His}[(xy)^-] & = \text{His}[(xy)^-(yx)^-(zz)^+] \lor \text{His}[(xy)^-(yx)^+(zz)^-], \\
  \text{His}[(yx)^+] & = \text{His}[(xy)^+(yx)^+(zz)^+] \lor \text{His}[(xy)^-(yx)^+(zz)^-], \\
  \text{His}[(yx)^-] & = \text{His}[(xy)^-(yx)^-(zz)^+] \lor \text{His}[(xy)^+(yx)^-(zz)^-], \\
  \text{His}[(zx)^+] & = \text{His}[(xy)^+(yx)^+(zz)^+] \lor \text{His}[(xy)^-(yx)^+(zz)^-], \\
  \text{His}[(zx)^-] & = \text{His}[(xy)^-(yx)^-(zz)^+] \lor \text{His}[(xy)^+(yx)^-(zz)^-].
  \end{align*}
  \]
According to the above definition we have:

\[
\text{His}[(zz)^+] = \{ \text{His}[(xy)^+] \land \text{His}[(yx)^+] \} \lor \{ \text{His}[(xy)^-] \land \text{His}[(yx)^+] \}, \quad (16)
\]

\[
\text{His}[(zz)^-] = \{ \text{His}[(xy)^-] \land \text{His}[(yx)^+] \} \lor \{ \text{His}[(xy)^+] \land \text{His}[(yx)^-] \}, \quad (17)
\]

and, obviously, the corresponding relations hold for their images under the homomorphisms.

**Family D.** It accommodates the operators \( \sigma_x^2, \sigma_y^2 \) and \( \sigma_x^1 \sigma_y^2 \). The four maximally fine-grained histories are: \( \text{His}[1x^+2y^+] \) whose associated projection operator is \( P_{1x^+2y^+} \); \( \text{His}[1x^+2y^-] \) whose associated projection operator is \( P_{1x^+2y^-} \); \( \text{His}[1x^-2y^-] \) whose associated projection operator is \( P_{1x^-2y^+} \); and finally history \( \text{His}[1x^-2y^-] \) whose associated projection operator is \( P_{1x^-2y^-} \). We will also deal with the two following coarse–grained histories:

\[
\text{His}[(xy)^+] = \text{His}[1x^+2y^+] \lor \text{His}[1x^-2y^-], \quad (18)
\]

\[
\text{His}[(xy)^-] = \text{His}[1x^+2y^+] \lor \text{His}[1x^-2y^-]. \quad (19)
\]

As it is evident these histories are the same as those ((10) and (11)) appearing in Family C. In fact they are associated to the projection operators on the eigenmanifolds of the operator \( \sigma_x^1 \sigma_y^2 \) corresponding to the eigenvalues \( +1 \) and \( -1 \), respectively. According to assumption c), since these are the same histories, also their truth values will be the same.

**Family E.** It deals with the operators \( \sigma_y^1, \sigma_x^2 \) and \( \sigma_y^1 \sigma_x^2 \). The four maximally fine-grained histories are: \( \text{His}[1y^+2x^+] \), whose associated projection operator is \( P_{1y^+2x^+} \); \( \text{His}[1y^+2x^-] \) whose associated projection operator is \( P_{1y^+2x^-} \); \( \text{His}[1y^-2x^+] \) whose associated projection operator is \( P_{1y^-2x^+} \); and finally the history \( \text{His}[1y^-2x^-] \) whose associated projection operator is \( P_{1y^-2x^-} \). As usual we will also consider two coarse–grained histories:

\[
\text{His}[(yx)^+] = \text{His}[1y^+2x^+] \lor \text{His}[1y^-2x^-], \quad (20)
\]

\[
\text{His}[(yx)^-] = \text{His}[1y^+2x^-] \lor \text{His}[1y^-2x^+]. \quad (21)
\]

In this case these two histories coincide with the two coarse–grained histories ((12) and (13)) belonging to Family C, since they are identified by the same projection operators. Accordingly the corresponding truth values must be the same.

**Family F.** This is the last family we will take into account and it is associated to the operators \( \sigma_x^1 \sigma_x^2, \sigma_y^1 \sigma_y^2 \) and \( \sigma_x^1 \sigma_x^2 \). Once more the common eigenstates are:

\[
\frac{1}{\sqrt{2}}[|1z^+\rangle \otimes |2z^+\rangle + |1z^-\rangle \otimes |2z^-\rangle] \quad \Longrightarrow \quad \begin{cases} +1 & \sigma_x^1 \sigma_x^2 \\ -1 & \sigma_y^1 \sigma_y^2 \\ +1 & \sigma_x^1 \sigma_x^2 \end{cases}
\]

whose associated projection operator is \( P_{(xx)^+(yy)^-(zz)^+} \) and the corresponding history \( \text{His}[(xx)^+(yy)^-(zz)^+] \),

\[
\frac{1}{\sqrt{2}}[|1z^+\rangle \otimes |2z^+\rangle - |1z^-\rangle \otimes |2z^-\rangle] \quad \Longrightarrow \quad \begin{cases} -1 & \sigma_x^1 \sigma_x^2 \\ +1 & \sigma_y^1 \sigma_y^2 \\ +1 & \sigma_x^1 \sigma_x^2 \end{cases}
\]
whose associated projection operator is \( P_{(xx)^-(yy)^+(zz)^+} \) and the corresponding history \( \text{His}[(xx)^-(yy)^+(zz)^+] \),

\[
\frac{1}{\sqrt{2}} [\ket{1z^+} \otimes \ket{2z^-} + \ket{1z^-} \otimes \ket{2z^+}] \implies \begin{cases} 
+1 \sigma_x^1 \sigma_y^2 \\
+1 \sigma_y^2 \sigma_z^1 \\
-1 \sigma_z^1 \sigma_x^2
\end{cases}
\]

whose associated projection operator is \( P_{(xx)^+(yy)^+(zz)^-} \) and the corresponding history \( \text{His}[(xx)^+(yy)^+(zz)^-] \),

\[
\frac{1}{\sqrt{2}} [\ket{1z^+} \otimes \ket{2z^-} - \ket{1z^-} \otimes \ket{2z^+}] \implies \begin{cases} 
-1 \sigma_x^1 \sigma_y^2 \\
-1 \sigma_y^2 \sigma_z^1 \\
-1 \sigma_z^1 \sigma_x^2
\end{cases}
\]

whose associated projection operator is \( P_{(xx)^-(yy)^-(zz)^-} \) and the corresponding history \( \text{His}[(xx)^-(yy)^-(zz)^-] \). We will also take into account the six following coarse–grained histories:

\[
\text{His}[(xx)^+] = \text{His}[(xx)^+(yy)^-(zz)^+] \lor \text{His}[(xx)^+(yy)^+(zz)^-], \quad (22)
\]

\[
\text{His}[(xx)^-] = \text{His}[(xx)^-(yy)^+(zz)^+] \lor \text{His}[(xx)^-(yy)^-(zz)^-], \quad (23)
\]

coinciding with those appearing in Family A,

\[
\text{His}[(yy)^+] = \text{His}[(xx)^-(yy)^-(zz)^+] \lor \text{His}[(xx)^-(yy)^+(zz)^-], \quad (24)
\]

\[
\text{His}[(yy)^-] = \text{His}[(xx)^+(yy)^-(zz)^+] \lor \text{His}[(xx)^-(yy)^-(zz)^-], \quad (25)
\]

coinciding with those appearing in Family B,

\[
\text{His}[(zz)^+] = \text{His}[(xx)^+(yy)^-(zz)^+] \lor \text{His}[(xx)^-(yy)^+(zz)^-], \quad (26)
\]

\[
\text{His}[(zz)^-] = \text{His}[(xx)^+(yy)^+(zz)^+] \lor \text{His}[(xx)^-(yy)^-(zz)^-], \quad (27)
\]

which coincide with those appearing in family C. Note that the above relations imply:

\[
\text{His}[(zz)^+] = \{\text{His}[(xx)^+] \land \text{His}[(yy)^-]\} \lor \{\text{His}[(xx)^-] \land \text{His}[(yy)^+]\}, \quad (28)
\]

\[
\text{His}[(zz)^-] = \{\text{His}[(xx)^+] \land \text{His}[(yy)^+]\} \lor \{\text{His}[(xx)^-] \land \text{His}[(yy)^-]\}, \quad (29)
\]

and that, obviously, the corresponding relations hold between their images under the homomorphism.

Given these premises we can prove our theorem. Let us consider the history \( \text{His}[1x^+2x^+] \) belonging to family A, and let us assume that the spin component of particle 1 along the \( x \) axis possesses the value +1 and that the same hold for the spin of particle 2. This means that the history \( \text{His}[1x^+2x^+] \) is true: \( h\{\text{His}[1x^+2x^+]\} = 1 \), and that the three histories \( \text{His}[1x^-2x^+] \), \( \text{His}[1x^+2x^-] \) and \( \text{His}[1x^-2x^-] \) are false: \( h\{\text{His}[1x^-2x^+]\} = 0 \), \( h\{\text{His}[1x^+2x^-]\} = 0 \) and \( h\{\text{His}[1x^-2x^-]\} = 0 \). The truth values of the histories

\[4\text{Of course, any other choice of the eigenvalues of the two spin operators will lead to the same contradiction as the one we will derive.} \]
His[(xx)^+] e His[(xx)^-] are then uniquely determined by the properties of the homomorphism \( h \):

\[
\begin{align*}
h\{\text{His}[(xx)^+]\} & = h\{\text{His}[1x^+2x^+] \lor \text{His}[1x^-2x^-]\} = \\
& = h\{\text{His}[1x^+2x^+]\} \lor h\{\text{His}[1x^-2x^-]\} = \\
& = 1 \lor 0 = 1, \\
h\{\text{His}[(xx)^-]\} & = h\{\text{His}[1x^+2x^-] \lor \text{His}[1x^-2x^+]\} = \\
& = h\{\text{His}[1x^+2x^-]\} \lor h\{\text{His}[1x^-2x^+]\} = \\
& = 0 \lor 0 = 0.
\end{align*}
\]

The conclusion of our analysis can be summarized in the following table:

| His[1x^+2x^+] | His[1x^+2x^-] | His[1x^-2x^+] | His[1x^-2x^-] | His[(xx)^+] | His[(xx)^-] |
|---------------|---------------|---------------|---------------|-------------|-------------|
| 1             | 0             | 0             | 0             | 1           | 0           |

Now we take into account Family B and, without paying any attention to the conclusions we have reached arguing within the previous family, we suppose that particle 1 has its spin pointing along the positive direction of the axis \( y \), while particle 2 has its spin pointing in the negative direction of the same axis. We get then another table:

| His[1y^+2y^+] | His[1y^+2y^-] | His[1y^-2y^+] | His[1y^-2y^-] | His[(yy)^+] | His[(yy)^-] |
|---------------|---------------|---------------|---------------|-------------|-------------|
| 0             | 1             | 0             | 0             | 0           | 1           |

Analogous procedures can be applied to Family D:

| His[1x^+2y^+] | His[1x^+2y^-] | His[1x^-2y^+] | His[1x^-2y^-] | His[(xy)^+] | His[(xy)^-] |
|---------------|---------------|---------------|---------------|-------------|-------------|
| 0             | 1             | 0             | 0             | 0           | 1           |

and to Family E:

| His[1y^+2x^+] | His[1y^+2x^-] | His[1y^-2x^+] | His[1y^-2x^-] | His[(yx)^+] | His[(yx)^-] |
|---------------|---------------|---------------|---------------|-------------|-------------|
| 1             | 0             | 0             | 0             | 1           | 0           |

We come now to discuss Family C. As already remarked it contains two histories His[(xy)^+] and His[(xy)^-] which coincide with two histories belonging to Family D; according to assumption c) of Section 2 they must have the same truth values. An analogous argument holds for the histories His[(yx)^+] e His[(yx)^-]. By considering the truth values
of all these histories and taking into account relations (16) and (17), we can deduce the truth values of the histories His[(xz)+] and His[(xz)−]:

\[
\begin{align*}
\text{His}[(xz)^+] &= h\{\text{His}[(xy)^+] \land \text{His}[(yx)^+]] \lor \text{His}[(xy)^-] \land \text{His}[(yx)^-]]\rangle \\
&= [h\{\text{His}[(xy)^+] \land \text{His}[(yx)^+]] \lor \\
&\lor h\{\text{His}[(xy)^-] \land h\{\text{His}[(yx)^-]]\}]
\end{align*}
\]

(30)

\[
\begin{align*}
\text{His}[(xz)^-] &= h\{\text{His}[(xy)^-] \land \text{His}[(yx)^+]] \lor \text{His}[(xy)^+] \land \text{His}[(yx)^-]]\rangle \\
&= [h\{\text{His}[(xy)^-] \land \text{His}[(yx)^+]] \lor \\
&\lor h\{\text{His}[(xy)^+] \land h\{\text{His}[(yx)^-]]\}]
\end{align*}
\]

(31)

As one should have expected the two truth values are opposite, since the two considered histories are mutually exclusive and exhaustive. In this way we have identified the truth table for the histories of Family C:

| His[(xy)+] | His[(xy)-] | His[(yx)+] | His[(yx)-] | His[(xz)+] | His[(xz)-] |
|------------|------------|------------|------------|------------|------------|
| 0          | 1          | 1          | 0          | 0          | 1          |

The last step consists in performing a similar analysis for Family F. As already remarked its two histories His[(xx)+] e His[(xx)−] coincide with histories belonging to Family A and, according to assumption c), must have the same truth values, 1 and 0, respectively. The same holds for the histories His[(yy)+] and His[(yy)−], which coincide with two histories belonging to Family B. Just as in the previous case, taking into account the relations (28) and (29), we can then evaluate the truth values of the two histories His[(xz)+] and His[(xz)−]:

\[
\begin{align*}
\text{His}[(xz)^+] &= h\{\text{His}[(xx)^+] \land \text{His}[(yy)\rangle] \lor [\text{His}[(xx)^-] \land \text{His}[(yy)^+]]\rangle \\
&= [h\{\text{His}[(xx)^+] \land \text{His}[(yy)^-]] \lor \\
&\lor h\{\text{His}[(xx)^-] \land h\{\text{His}[(yy)^+]]\}]
\end{align*}
\]

(32)

\[
\begin{align*}
\text{His}[(xz)^-] &= h\{\text{His}[(xx)^+] \land \text{His}[(yy)^+]] \lor [\text{His}[(xx)^-] \land \text{His}[(yy)^-]]\rangle \\
&= [h\{\text{His}[(xx)^+] \land \text{His}[(yy)^+]] \lor \\
&\lor h\{\text{His}[(xx)^-] \land h\{\text{His}[(yy)^-]]\}]
\end{align*}
\]

(33)

We can then exhibit the truth table for the histories of Family F:

| His[(xx)+] | His[(xx)-] | His[(yy)+] | His[(yy)-] | His[(xz)+] | His[(xz)-] |
|------------|------------|------------|------------|------------|------------|
| 1          | 0          | 0          | 1          | 1          | 0          |
Comparing the two last truth tables one sees that the Families C and F attribute opposite truth values to the two histories His[(zz)+] and His[(zz)−]: if one limits his considerations to Family C, then we can claim with certainty that both particles have their spin antiparallel along the z axis, on the contrary, if we take into consideration Family F, then we must conclude that the two particles have their spin parallel with respect to the same axis.

To avoid being misunderstood we stress once more that we have used only the four assumptions listed in the previous section to derive the above contradiction; in particular we have never made statements involving different histories belonging to incompatible decoherent families.

4 An answer to Griffiths’ objections.

R. Griffiths\textsuperscript{5} has repeatedly expressed (in private correspondence) his disappointment with this paper, claiming that in developing our argument we violate one of the fundamental rules of the DH approach, the one we have already mentioned and which we shall refer to as the single family rule in what follows. According to such a rule any reasoning must employ a single family of decoherent histories. Since in the crucial example of Section 3 we resort to the consideration of six incompatible decoherent families and we combine, in a way or another, the conclusions drawn within the six families to derive a contradiction, we violate the single family rule; consequently, in Griffiths’ opinion, our line of thought is wrong.

Griffiths has also repeatedly called our attention on an example which has been exhaustively discussed in the literature. In his opinion such an argument is strictly similar to ours (\cite{21} and references therein) but it is much simpler and it should allow to better understand the essential role of the single family rule as well as its implications\textsuperscript{6}. The rule and its implications are, in Griffiths’ opinion, accepted both by the supporters as well as by the opponents of the DH approach.

In brief the example of ref. \cite{21} which we present here in a slightly modified version used by Griffiths in our correspondence goes as follows: one considers three orthogonal states $|A\rangle$, $|B\rangle$ and $|C\rangle$, and the two states:

$$|\varphi\rangle = \frac{1}{\sqrt{3}} [ |A\rangle + |B\rangle + |C\rangle ],$$

$$|\psi\rangle = \frac{1}{\sqrt{3}} [ |A\rangle + |B\rangle - |C\rangle ].$$

Let $t_0 < t_1 < t_2$ be three time instants and suppose that the dynamics is trivial, i.e. $H = 0$. One then considers the following three histories:

$$\text{His}^{(1)} = \{ (|\varphi\rangle\langle\varphi|, t_0), (1, t_1), (|\psi\rangle\langle\psi|, t_2) \}, \quad \text{(36)}$$

$$\text{His}^{(2)} = \{ (|\varphi\rangle\langle\varphi|, t_0), (|A\rangle\langle A|, t_1), (1, t_2) \}, \quad \text{(37)}$$

$$\text{His}^{(3)} = \{ (|\varphi\rangle\langle\varphi|, t_0), (|B\rangle\langle B|, t_1), (1, t_2) \}. \quad \text{(38)}$$

\textsuperscript{5}Private communication.

\textsuperscript{6}Goldstein \textsuperscript{9} has considered a similar but slightly more elaborated example. The following remarks hold both for Griffiths’ as well as for Goldstein’s examples.
It is then possible to prove that there is a decoherent family $A$ such that the two histories $\text{His}^{(1)}$ and $\text{His}^{(2)}$ belong to it and, moreover, within it:

$$\frac{p(\text{His}^{(1)} \land \text{His}^{(2)})}{p(\text{His}^{(1)})} = 1 \quad \text{so that} \quad \text{His}^{(1)} \Rightarrow \text{His}^{(2)},$$

$p()$ being the probability distribution characterizing the histories of the family $A$.

There is also a second family of decoherent histories $B$ such that both $\text{His}^{(1)}$ and $\text{His}^{(3)}$ belong to it and, within it, one has:

$$\frac{p(\text{His}^{(1)} \land \text{His}^{(3)})}{p(\text{His}^{(1)})} = 1 \quad \text{so that} \quad \text{His}^{(1)} \Rightarrow \text{His}^{(3)}.$$ 

(40)

Obviously, in the above equation $p()$ is the probability distribution characterizing the histories of $B$.

Finally, there is a third decoherent family $C$ which accomodates $\text{His}^{(2)}$ and $\text{His}^{(3)}$. Obviously, such histories are mutually exclusive since the states $|A\rangle$ and $|B\rangle$ are orthogonal: this implies that they cannot correspond simultaneously to physical properties of the system under consideration.

Let us now suppose that the history $\text{His}^{(1)}$ is true. According to (39) we can conclude that also $\text{His}^{(2)}$ is true, and, according to (40) that $\text{His}^{(3)}$ must also be true. This, however, cannot happen since the two considered histories are mutually exclusive.

Why is this argument not correct? As remarked by the DH supporters and in particular by Griffiths, the argument violates the single family rule since the conclusion requires the consideration of three different decoherent families $A$, $B$ and $C$, which are incompatible with each other. The line of reasoning leading to the contradiction is forbidden by the rules of the DH approach.

Even though we believe, with d’Espagnat [16, 17], that the single family rule cannot be considered so natural and free from puzzling aspects as the DH supporters seem to believe, we are perfectly aware that, at the purely formal level, Griffiths’ criticisms concerning the just discussed example are legitimate. Does this imply that the same conclusion holds also for our example of Section 3? We believe that this is not the case:

- Since Griffiths has repeatedly stated (in his papers and in his correspondence with us) that decoherent histories refer to objective properties of physical systems, that they are the beables of the DH approach, then he must accept that they have precise truth values. The very existence of a truth–functional for the histories of a decoherent family amounts simply to the formal expression that such histories speak of objective properties of physical systems:

| Decoherent histories refer to **objective** properties of physical systems. |
|------------------------------------------------------------------------------------------------|
| They are given a precise truth value: 0 or 1. |

15
Moreover if we want that within a decoherent family the usual classical rules can be used (once more Griffiths himself, as well as Omnès, have repeatedly stressed the necessity of this feature), then we must accept that such a truth–functional be a homomorphism:

Inside a single decoherent family classical logics holds.

\[ \Downarrow \]

The truth–valuation is a homomorphism.

All this has been described and discussed in details in Section 2. Thus, it seem useless to us that Griffiths insists, in his correspondence, that within his theory decoherent histories speak of objective properties of physical systems, but that no homomorphism of the kind we have just envisaged exists. Such an attitude is contradictory: either decoherent histories make reference to objective properties, but then one must unavoidably accept the existence of a truth valuation (which must be an homomorphism if classical rules must hold) for them, or one denies the very possibility of considering a truth valuation, but then the histories lose any objective physical meaning. He can make the choice he prefers.

- We stress that the existence of a truth value for the histories of a decoherent family (if one considers them as referring to objectively possessed properties) preceeds logically (actually ontologically) the assignement of a probability distribution to them. Actually, the probability distribution makes reference to our knowledge about the physical system which has a fundamentally contingent character, in the precise sense that it depends on the information we have at a considered time instant. This cannot change the fact that the physical system has objective properties (accordingly, the histories are true if they account for such properties and are false if they do not) which are completely independent from the probabilistic (epistemic) informations we have about the system. Since the properties — and not the probability — correspond (in a realistic position as the one advocated by Griffiths) to physical reality, they must play a primary role within the theory and should be the objects of interest for the scientist.

- Consideration of the Kochen and Specker theorem raises the problem of whether the mathematical formalism of quantum mechanics allows a consistent assignement of truth values to the projection operators of the Hilbert space, i.e. whether they can be considered as representing objective properties of physical systems, indipendentely of the probabilities one attaches to such projection operators. Accordingly, the theorem deals with the algebraic properties of projection operators (with specific reference to their non commutative nature) but it never takes into consideration the probabilities of the formalism. According to our previous analysis it should be clear why Kochen and Specker have chosen this line of approach: if a theory pretends to speak of properties objectively possessed by physical systems, such properties, just
because they are *objective* cannot depend on our probabilistic knowledge which, in general, has a *subjective* character.

Our example of Section 3 can be considered as a simple transcription of the Kochen and Specker theorem (in the version of Peres and Mermin) in the language of decoherent histories. Our aim is to prove that also within such a theoretical scheme one cannot attribute too many *objective* properties to physical systems, just as in standard quantum mechanics. From this point of view, it has to be stressed that in our reasoning we *never* make reference to the probability distributions which can be attached to the histories of the six considered families. Vice versa, in Griffiths’ example *the whole argument is based in a fundamental way on the consideration of the probability distribution*, as it is evident from equations (39) and (40). This is an important difference between the two arguments which, in our opinion, Griffiths has not been able to grasp, in spite of the fact that it is evident that one cannot derive a Kochen and Specker–like contradiction by resorting to the example he takes into account. This is not a purely formal difference; it has precise implications, as we are going to discuss.

- The “single family rule”, as we have already stated, is the “*fundamental principle of quantum reasoning*” [2]. It states that:

  Any reasoning must employ a single family of decoherent histories.

Griffiths himself has made very clear what he means by the expression “*quantum reasoning*” [2]:

> The type of quantum reasoning we shall focus ... is that in which one starts with some information about a system, known or assumed to be true, and from these *initial data* tries to reach valid *conclusions* which will be true if the initial data are correct.

As one can easily grasp from his last papers [4], when he uses the expression “*quantum reasoning*” he has actually in mind a reasoning of exclusively probabilistic nature (due to the fact that we can have only a probabilistic knowledge about physical systems), which allows to manipulate probabilities to derive new informations. From this point of view the single family rule might appear as a reasonable request: actually, since the “quantum reasoning” has a fundamentally probabilistic nature, and since the probabilities depend from the considered family (just as in classical mechanics they depend on the graining one choses, and different grainings correspond to different probability assignments) it seems natural that “quantum reasoning” (in the above sense) depends from the decoherent family one considers.

However, as repeatedly stressed, if one of the basic assumptions of the theory is that histories refer to objective properties of physical systems, then beside a fundamentally probabilistic “quantum reasoning” based on our knowledge about the system,

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7Obviously the independence of objective properties from probability distributions cannot be complete, since, e.g., in the case in which the theory attaches probability 1 to a property, then it must be possessed.
there must be a second type of reasoning based on the properties objectively possessed by physical systems, independently from our precise knowledge of them. This fact, we stress it once more, is imposed by the very statement that the decoherent histories speak of objective properties of physical systems. This is precisely the reasoning at the basis of the Kochen and Specker theorem (even though their interest was directed to hidden variable theories). And this is precisely the reasoning of our Section 3. The two ways of reasoning are fundamentally different and there is no reason for which the single family rule, which might hold for the first line of thought, must obviously hold also for the second. On the contrary, it cannot hold for the second one since assuming such a rule within this perspective amounts simply to assume that the physical properties are not objective, being related to the family which one chooses.

The argument of Section 3 is different, not only for its formal aspects, but for its very essence from those which have been considered in the literature, and, in particular, from the one which is repeatedly mentioned by Griffiths. For these reasons, our argument cannot simply be dismissed by invoking acritically the single family rule, as it has happened up to now.

5 Conclusions.

The conclusion of our investigation should be obvious: if one wants to entertain the Decoherent Histories point of view, he must give up at least one of the previous assumptions. Let us discuss a little bit more what happens if we relax one of them.

• If one gives up the request that any decoherent family be endowed with a boolean structure, then he is giving up the possibility of using classical reasoning within such a family, loosing in this way the nicest feature of the theory and the very reason to consider it. Since, as stated before, nobody seems to contemplate this possibility, we do not discuss it any further.

• One could give up the second assumption, stating that not every decoherent history has a truth value. This is, in our opinion, a very dangerous move: in fact, giving a truth value to a decoherent history is not simply a formal act, but it means that we are establishing a precise correspondence between such a history and some objective physical properties. If we deny any truth value to the history, then we deny such correspondence, and the history becomes just an empty statement devoid of any physical meaning. In Classical Statistical Mechanics, all events in phase space are given a truth value, because they all correspond to particular physical properties, even if one in general knows only their probability distributions. In Standard Quantum Mechanics, on the other hand, no truth–value assignment exists in general, and in fact the quantum projection operators do not correspond in general to any physical property possessed by systems for the simple reason that quantum systems do not have actual but only potential properties before a measurement process is performed. So if we assume that some histories have no truth
value, then we must accept that they are meaningless from the physical point of view. Of course, this is not a problem, but then the theory has to tell us which histories have a truth value, and which do not, i.e. which correspond to physical properties (and then have a precise ontological status) and which do not (and, as such, are only empty statements devoid of any ontological meaning): without any such prescription, the theory would be incomplete.

Omnès [4, 5], for example, has tried to give a precise answer to the previous question: specifically, he has proposed a criterion for truth which is independent from the families, and which also eliminates the problem of the existence of families describing senseless properties for classical macroscopic objects. Unfortunately, Dowker and Kent [18] have shown that his proposal is not tenable.

• Assumption c) seems to us impossible to give up. In fact, let us recall the argument concerning the impossibility of considering, within hidden variable theories, the values of the observables of the table at the beginning of Section 3 as uniquely determined by the hidden variables (or equivalently, as objectively possessed). There, we have mentioned that the only consistent way out from this embarrassing situation derives from accepting that the truth values of statements concerning the predictions of the theory about the outcomes of measurements depend from the whole context. In particular, different truth values are necessarily associated to different and incompatible measurement procedures, i.e., to different physical situations. In the case of the Decoherent Histories the situation is radically different. In fact, they do not speak of measurement outcomes but of properties possessed independently of any procedure to test them. Therefore, within such a conceptual framework to make the truth value of a precise history dependent from the family to which it is considered to belong seems to us logically unacceptable: it would be better to keep the Copenhagen interpretation.

• If we decide to give up assumption d), then we recognize that the theory as it stands is not complete, because the decoherence condition by itself does not select the proper families to be used for describing physical systems, and we have to find new criteria in order to complete the theory. This fact does not mark by itself the definitive failure of the program: it simply points out that the theory needs to be enriched by new assumptions apt to identify the family, or the families, which are physically significant. This, however, is not an easy task, and our example throws a precise and disquieting light on the difficulties one will meet in trying to consistently implement such ideas. In fact we can raise the question: which one (or ones) of the six families summarized in the table at the beginning of Section 3 should be discarded? Which criterion could one use making some of these families acceptable and forbidding the consideration of the remaining ones, given the fact that they have a quite similar conceptual status and they speak of analogous properties of our system?

To conclude, our analysis shows that the DH approach, as it stands (whatever inter-

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8Of course, if one accepts that some histories have no truth value (i.e. he gives up assumption b)), then assumption c) becomes meaningless for those histories.
interpretation one decides to subscribe), is either incomplete or does not meet the requirements for a “realistic” description of the physical world, the very reason for which it has been proposed.

Acknowledgments

We acknowledge useful discussions with A. Kent and R. Griffiths.

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