Continuous Flow Model of a Historical Battle: A Fresh Look at Pickett’s Charge

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Abstract
A continuous flow model of infantry behavior, based on conservation of individuals and tracking of subunit identity, has been developed in sufficient detail that it can now be applied to a realistic simulation of a historical battle. Pickett’s charge during the 1863 Battle of Gettysburg, Pennsylvania in the U.S. Civil War was chosen as an initial application of the model. This scenario is a good test of the current mathematical model because many modern military tactics were employed, in a context where the action took place on foot or horseback, and the historical map and troop numbers are available. Compared to a discrete agent model, the flow model was found to better capture the interaction of the forces with the terrain and each other. A brigade-level simulation, faithful to the details of the historical events, was performed. The main source of asymmetry in the numbers of casualties was found to be the inability of the Confederate forces to use effective ranged fire while they were moving. Comparison of simulations with and without terrain effects showed that they slow the pace of battle and favor the defenders, exposing the attackers to heavy ranged fire for an extended period. A statistical analysis of possible outcomes for an ensemble of 1000 randomized perturbations of the baseline brigade-level scenario was carried out. Consistent with historical events, it was found that only 6\% of the scenarios resulted in an outcome that could be considered a Confederate victory.

1 Introduction

Conventional military simulations (wargames) and related game-like simulations typically use discrete agents, essentially virtual game pieces, to model groups of individuals and machinery. This aggregation of discrete entities into groups can be very arbitrary, and it may discard much complex and important individual behavior.

Continuous agents (probability densities of individuals) could potentially overcome these difficulties. A continuous flow model could more realistically portray group mo-
tion, provide decision makers with a more intuitive understanding of the momentum of a conflict, and become a useful tool for training and decision making.

These ideas have been explored to a limited extent in previous literature. An early effort to model discrete entities as a flow was made by Lighthill and Whitham [1], who developed a fluid dynamic model of traffic flow. Protopopescu et al. [2] and Fields [3] extended these ideas to combat modeling. Hughes [4] adapted the Lighthill and Whitham [1] model to human crowds, and later applied the approach to modeling the 1415 Battle of Agincourt [5]. Spradlin and Spradlin [6] examined different kinds of Lanchester [7] fire models in the context of continuous units. Adapting swarming and schooling models from mathematical biology, Keane [8] implemented non-local effects through convolution integrals in order to enforce unit formations. González and Villena [9] created an advanced version of a reaction-diffusion model [3], in which motion responsive to enemy force concentration was implemented.

In the context of infantry combat, the models proposed in these previous studies over-emphasize the fluid aspects of unit motion, and they lack an efficient means to represent tight infantry formations. In Poggie et al. [10] we introduced a new continuous flow model of infantry behavior in a military conflict. The model incorporated conservation of individuals, effects of crowd density and surface inclination on walking speed, and Lanchester [7] combat models for both close-in and ranged fire. A new approach to tracking subunit identity was introduced to implement unit movement and orientation to better represent infantry formations.

In the present work, we have brought in refinements to the model, including discrete artillery units, an overlay layer to represent different kinds of terrain on the map (vegetation, fences, buildings, etc.), improvements to movement and ranged fire, a list of orders for each unit, unit morale, and checks for retreat and pressing an attack.

With this new approach, a battle flow model can be used for efficient and realistic simulations. We illustrate this point by applying the method to a historical battle.

1.1 Application to Pickett’s Charge

As an initial historical application of the model, we chose the Battle of Gettysburg, Pennsylvania in the U.S. Civil War, focusing on the famous Pickett’s charge of July 3, 1863. This application is a good test of our current mathematical model because many modern military tactics were employed in a context where the action took place on foot or horseback.

Furthermore, the details of the historical battle are relatively well known. Maps and numbers for soldiers and equipment are available from historical records. Maps in Gottfried [11] and Laino [12] provide a reconstruction of the timeline and unit motion through the battle. Busey and Martin [13] have published a comprehensive tabulation of the forces present.

The American Civil War (1861–1865) pitted the agricultural South (Confederacy, Red) against the industrial North (Union, Blue). In a dispute that arose over whether newly acquired western territories would allow slavery, Southern states seceded from the United States starting in December 1860 [14]. Fighting began at Fort Sumter, South Carolina in April 1861. Union successes in 1864 eventually led to Confederate surrender in 1865, with a cost of at least million combined casualties (about 3% of the
population). Nonetheless, at the time of the Battle of Gettysburg (July 1–3, 1863), the final outcome of the war was very much in doubt.

In spring 1863, the Confederate forces initiated an invasion of southern Pennsylvania. The Union army moved to position itself between the invaders and the threatened Washington, DC. The two sides met at the crossroads of Gettysburg, PA.

The Battle of Gettysburg was the largest battle ever fought in North America. On the first two days of fighting (July 1–2), each army suffered tremendous losses with inconclusive results. On the third day (July 3), the armies were positioned on opposite ridgelines known as Seminary Ridge and Cemetery Ridge. The ridges ran approximately north-south, and were separated by a flat valley about 1 km wide. The valley was occupied by several farms, with fenced fields, drainage ditches, and a road running north-south.

At 13:00 on July 3, the Confederate artillery opened fire, and the Union guns replied. The Union artillery commander soon silenced his guns to conserve ammunition. The Confederate artillery continued to fire, using up much of its ammunition.

Sometime after 14:00, Confederate infantry left the shelter of Seminary Ridge and began a frontal assault. (A simplified map of the situation is given in Figure 1.) Union guns opened fire across the valley. After marching a kilometer under large caliber shelling, the advancing Confederate infantry reached the fence-lined Emmitsburg Road. They had to stop to tear down the fences, under bombardment, and when they got moving again, they did so in a staggered fashion without their earlier cohesion. (We replicate this behavior in the model, as described later.) Additional Union guns to the north and south turned on the attacking mass, enfilading it.

With the enemy approximately 300 m away, and now properly charging (captured in the model as faster forward motion), the sheltered Union infantry opened fire. The Union command continued to send reinforcements, and even had units to the left and right advance slightly, adding more enfilading fire. These effects combined to funnel the Confederate attack into a narrower, denser mass. Since infantry of that era could not simultaneously fire and move, Confederate troops did not fire until within about 100 m of the defenders.

Small groups of Southern soldiers reached Union lines, notably near the now-famous copse near the center of the Union lines. Just to the north, at The Angle, a few broke through the Northern line. Each time, Union troops rallied and threw the attackers back. By 15:30, the assault petered out, and disorganized attackers withdrew.

1.2 Previous Simulations

A few previous wargame-type studies have addressed the Battle of Gettysburg. Tillman and Engle examined Ewell’s decision not to attack Culp’s Hill on July 1, 1863, using the Janus combat simulation software. That software provides stochastic simulation of combat in three dimensions and time. It is a discrete agent model, and each entity in this specific simulation represented either 80 infantrymen or 2 artillery pieces. Comparing four alternative scenarios, they concluded that Ewell’s decision was correct: he did not have the forces to take the hill, and even with plausible additional forces he could not have held the hill for long.
Armstrong and Sodergren [18] modeled Pickett’s Charge using the Lanchester aimed fire equations (square law), testing sensitivity of the outcome to changes in troop strength within the historical uncertainty and changes in Confederate tactics. There is no map or terrain in their zero dimensional model, only a phased interaction (artillery barrage, Confederate advance, skirmish) of aggregated units. They concluded that the Confederates could plausibly have taken the Union position under a different tactical approach, but that the Confederates lacked sufficient forces to exploit such a victory. They write: “Thus, it seems that while Lee could have captured The Angle via a better barrage and a larger charge, adding more men to the initial assault would have added significant risk to his ability to exploit a breakthrough and achieve the larger victory he was seeking.”

With the use of a model based on continuous densities of individuals, our approach differs from these previous discrete agent simulations. Our results agree in some ways with Armstrong and Sodergren [18], but in including the effects of terrain and unit flow, and particularly the interaction with fences, we capture new behavior in the model and come to a slightly different conclusion.

2 Mathematical Model

The mathematical model used in the present work is an extension of the continuous flow model introduced in Poggie et al. [10]. The basis of the model lies in conservation of individuals and tracking of subunit identity. The pattern of flow of a unit across the map is a result of the interaction of impediments of the environment with the unit’s efforts to maintain formation in following its ordered path.

2.1 Unit Motion

A fundamental assumption of the model is that individuals can be tracked; they do not spontaneously appear or disappear. Individuals are aggregated into units (here brigades or armies), which will be enumerated with the subscript \( i = 1, 2, \ldots, N \). The local density of members of each unit is governed by a conservation equation [4] of the form:

\[
\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i u_i) = -\omega_i
\]  

(1)

where \( t \) is time, \( \nabla \) is the spatial (\( \mathbf{x} \)) gradient operator, the density of individuals per area for Unit-\( i \) is \( \rho_i \), the walking velocity is \( u_i \), and the rate of casualties per area is \( \omega_i \).

The velocity consists of a combination of directed motion and diffusion:

\[
u_i = V_i - \frac{D_i}{\rho_i} \nabla \rho_i
\]  

(2)

For the present work, diffusion is retained as an option in the computer code, but the diffusion coefficient has been set to \( D_i = 0 \) because it did not seem to contribute to the realism of the simulation. (The military tactics of the Civil War era involved tight formations; diffusion tends to make a formation gradually spread out.) The directed
motion is affected by local crowd density, ground slope, proximity to goal location, terrain type, and unit orders, summarized as:

$$V_i = V_m \cdot f(\rho) \cdot g(s) \cdot h(r) \cdot T(x) \cdot d_i$$  \hspace{1cm} (3)

Here $V_m$ is the maximum walking speed, $\rho = \sum_i \rho_i$ is the local total density, $d_i$ is the walking direction, $s = d_i \cdot \nabla H$ is the ground slope in the direction of walking (directional derivative), and $r$ is the distance from the destination. The variable $H(x)$ is the local elevation, obtained from a topographical map of the area of interest. The variable $T(x)$ represents the effect of local terrain type.

Speed is assumed to slow with increasing crowd density according to the formula:

$$f(\rho) = -6 \left( \frac{\rho}{\rho_m} \right)^5 + 15 \left( \frac{\rho}{\rho_m} \right)^4 - 10 \left( \frac{\rho}{\rho_m} \right)^3 + 1$$  \hspace{1cm} (4)

The values for the parameters in Eqs. (3)–(4), $V_m = 1.4$ m/s and $\rho_m = 5.6$ m$^{-2}$, are based on data on pedestrian walking speed under crowded conditions [19]. Equation (4) is chosen to obtain a smooth transition from walking at full speed, $f(\rho) = 1$ and $V = V_m$, under very low density of individuals, to no motion, $f(\rho) = 0$ and $V = 0$, at a maximal crowd density of $\rho_m$. A discussion of some of the possible alternative options for this function is given in Poggie et al. [10].

Given the relatively gentle slope of the Gettysburg terrain, walking speed is assumed to be affected by terrain according to the following simple formula:

$$g(s) = \begin{cases} 
\frac{1}{1 + 3.5s} & \text{if } s > 0 \text{ (uphill)}, \\
1 & \text{if } s \leq 0 \text{ (downhill)}. 
\end{cases}$$  \hspace{1cm} (5)

This model assumes that walking downhill does not significantly affect speed. Additional discussion of this function is given in Poggie et al. [10].

Fences, roads, buildings, and local vegetation can affect the speed of infantry motion. This effect is accounted for by a map overlay layer, denoted by the function $T(x)$. The overlay function is intended not only to represent physical impediments to motion, but also to account for psychological effects of an obstructed field of view and small sub-grid scale motions that are not resolved in the simulation. The elevation $H(x)$ and terrain $T(x)$ fields for the Gettysburg battlefield will be discussed later, in the context of Figure [1].

Unit formations are implemented through a subunit identity function [10]. The identity of a subunit is tracked using its position in the initial formation. The following equation tracks the origin of a subunit $\xi_i$:

$$\frac{D\xi_i}{Dt} = \frac{\partial \xi_i}{\partial t} + (u_i \cdot \nabla) \xi_i = 0$$  \hspace{1cm} (6)

Making an analogy to fluid mechanics, this formulation is equivalent to setting the material derivative to zero; identity does not change following the flow.

Self-aware subunits can have individual goals and position relative to the unit’s centroid. To define the goals, we use a goal potential related to the current distance from a goal position:

$$\phi_i(x, \xi_i) = |x - z_i(\xi_i)|^2$$  \hspace{1cm} (7)
Here $x$ is the current position and $z_i$ is the desired position for a particular subunit identified by $\xi_i$. The subunit walks in the direction that minimizes the goal potential:

$$d_i = -\frac{\nabla \phi_i}{|\nabla \phi_i|}$$  (8)

If a subunit is near the target location (about 3 grid cells), the speed is reduced as an error function of distance. This allows each part of the unit to eventually come to rest at a stable fixed position. If a subunit is very far from the goal (about 40 cells), the units march double time to catch up. This behavior is captured by the function:

$$h(r) = \text{erf} \left( \frac{r}{1.5 \Delta s} \right) + \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{r - 40 \Delta s}{10 \Delta s} \right) \right]$$  (9)

where $r = |x - z_i|$ and $\Delta s$ is the grid (map) resolution.

We set the goal through translation and rotation from the initial position:

$$z_i = A (\xi_i - y_{1,i}) + y_{2,i}$$  (10)

$$A = \begin{bmatrix} a(t) \cos \theta(t) & -b(t) \sin \theta(t) \\ a(t) \sin \theta(t) & b(t) \cos \theta(t) \end{bmatrix}$$  (11)

Here $y_{1,i}$ represents the initial center for Unit-$i$ and $y_{2,i}$ is the current target location of the center. By varying $y_{2,i}$ with time, the center of the unit can be made to follow an arbitrary, curved path. The variable $\theta$ establishes rotation; it too can vary with time. The parameters $a$ and $b$ control stretching along the coordinate axes; these allow the shape of the formation to change. Although the transformation specifies rigid translation and rotation, the resulting motion of the unit is fluid, not rigid. Each subunit will do its best to reach the goal position, encountering different conditions along the way.

## 2.2 Combat Model

Combat is divided into two categories: close-in combat and ranged combat. Close-in combat is intended to represent hand-to-hand combat and fire at a sufficiently close range that the attacker can pick an individual target. Ranged fire represents fire at a sufficient range that the target is the general enemy unit and not an individual. The notation $\omega_i = \omega_i' + \omega_i''$ will be used to designate these two parts of the losses.

Close-in combat is modeled according to the Lanchester area fire model [7, 20]. The rate of fire is proportional to the local concentration of attackers $\rho_j$. The number of available targets is proportional to the local density of defenders $\rho_i$. Summing the result over all enemy units, the loss rate for Unit-$i$ due to close-in combat is:

$$\omega_i' = \rho_i \sum_j k_{ij} \rho_j$$  (12)

where $k_{ij}$ is a proportionality coefficient for each pair of combatants. The summation could potentially also include friendly fire ($k_{ij} \neq 0$ for $i = j$). In general, the proportionality coefficient is not necessarily symmetric ($k_{ij} \neq k_{ji}$); the effectiveness of the
fire of one unit is not necessarily the same as another. Nonetheless, in the absence of detailed information on the subject, we assume a baseline value of $k_{ij} = 5.0 \times 10^{-2} \text{ m}^2/\text{s}$ for all units, without friendly fire ($k_{ij} = 0$ for $i = j$). This figure is on the same order as those employed in previous studies with Lanchester models [21], and it has given plausible results in our tests. An additional tapering of close-in combat for very small densities, $\rho_i < 10^{-4}\rho_m$ was used to eliminate unrealistic combat in regions where there were essentially no forces.

Several options could be considered for ranged fire, including a highly detailed model in which each attacking subunit had a different target subunit. Such a model is awkward and inefficient to implement in a computer program employing domain decomposition to achieve parallel speedup, primarily because the attacker and target may lie in different domains, with their information held on separate computer processes. The data would need to be exchanged via message passing. For the present work, therefore, we chose to implement ranged fire by treating all units as discrete, aggregated entities. Under this model, only the data summarizing the status of the unit as a whole need to be exchanged. An aggregated attack is applied to the defender, and losses are spread across the defender’s constituents.

For ranged fire under this approach, each attacking unit must first choose a target unit. Here we choose the nearest enemy centroid within an angular window of $\pm 45^\circ$ from the current bearing of the attacking unit.

The loss rate for Unit-$i$ due to ranged fire is given by:

$$\omega''_i = \rho_i \sum_j k'_{ij} f_1 f_2 f_3 f_4 f_5 R_j$$

where $R_j = \iint \rho_j \, dA$ is the total number of elements or individuals in each attacking force. For the brigade-level simulations (Section 3.2), the ranged fire efficiency for infantry was taken to be $k'_{ij} = 8.0 \text{ s}^{-1}$ and that of artillery was $k'_{ij} = 16.0 \text{ s}^{-1}$.

Several factors are considered that affect the efficiency of ranged fire: target direction, target range, attacker and defender bearing, elevation difference, and attacker motion. The range function is:

$$f_1(r_{ij}) = \begin{cases} 1 & \text{if } r_{ij} < R_0, \\ \left(\frac{R_0}{r_{ij}}\right)^2 & \text{if } r_{ij} \geq R_0. \end{cases}$$

where $r_{ij}$ is the distance between the attacker and the defender. The parameter $R_o$ is a characteristic range, taken to be 100 m for infantry rifle fire and 1200 m for artillery fire. For infantry units, firing to the front should be more effective than firing to the side. The historical infantry units were trained and organized to primarily fire to the front; getting a clear shot to the side might be difficult. Considering the angle $\alpha$ between the target direction and the attacker bearing (both angles as seen from the attacker’s centroid), this effect is implemented with:

$$f_2(\alpha) = \exp\left(-2\alpha^2/\alpha_c^2\right)$$

With this function, shooting straight ahead ($\alpha = 0^\circ$) is effective, whereas shooting at $\alpha = \alpha_c$ is highly ineffective. Here the characteristic angle parameter is taken to be $\alpha_c = 90^\circ$. 


Similarly, relative orientation should have an effect. Flanking fire should be more effective than head-on fire. If $\beta$ is the angular difference between defender bearing and attacker bearing (again, as seen from the unit centroid), the effect of relative orientation is taken as:

$$f_3(\beta) = (3 + \cos \beta)/2$$

(16)

Specifically, opposing units facing toward each other ($\beta = 180^\circ$) gives the baseline of $f_3 = 1.0$, flanking fire ($\beta = 90^\circ$) gives an improved effectiveness of $f_3 = 1.5$, and attacking an enemy’s rear ($\beta = 0^\circ$) doubles aimed fire effectiveness, $f_3 = 2.0$.

Shooting up or down should also diminish accuracy. Taking $\theta$ to be the angle of fire relative to horizontal, we implement the effect of elevation difference as:

$$f_4(\theta) = \exp\left(-2{\theta^2}/\theta_r^2\right)$$

(17)

For the present work, the elevation reference angle is taken to be $\theta_r = 30^\circ$. In future implementations of the model, we hope to improve the realism of this effect by including a line-of-sight check. That is, an improved model would check for terrain in the way of a rifle shot, but perhaps not for an artillery shot.

A moving attacker should be less effective in ranged fire. The effect of attacker speed $V$ is:

$$f_5(V) = \exp\left(-50V/V_r\right)$$

(18)

where $V_r = V_m$ is a reference speed. This model provides a strong penalty for shooting while moving. Stationary defenders have a strong advantage. This choice seems reasonable for Civil War era infantry combat.

### 2.3 Artillery

Since the artillery units considered for the present model of Pickett’s Charge consisted of at most 15 guns, the continuous flow model did not seem appropriate. Thus, a mixed model was employed, with special discrete units selected to represent artillery. These are traditional wargame units.

Although the framework exists in our computer code for moving discrete units, the artillery units were specified to be immobile on the time scale of the simulation. Further, we did not allow losses for artillery units, nor did they participate in close-in combat. Again, these simplifications could be relaxed in future versions of the model.

For the purposes of computing infantry motion around the artillery units, the latter were assigned an effective density distribution. The peak value was taken to be $0.1\rho_m$ with Gaussian decay away from the center with a characteristic scale of 20 m.

### 2.4 Unit Orders and Breakpoints

The computer code includes a facility to provide each unit with a list of orders. For the Confederate units, there were three actions: rotate to face the direction of intended motion, translate to bring the unit centroid to a specified location, and rotate to face the nearest enemy. For the Union units, the list of orders was to wait until a Confederate unit was within 500 m, rotate, translate, and face the enemy. For two Union units (combined 8 Ohio and 126 New York regiments, and Stannard’s brigade), the orders were
modified to flank the charging Confederates with a combined translation and rotation motion. (See Figure 1 and Table 1 for the units involved in the simulations.)

The algorithm attempted to have the units carry out their orders in a smooth manner. In other words, \( y_{2,i} \) in Eq. (10) was set to attempt a slow march of 0.6 m/s and \( \theta(t) \) was set to attempt to complete a rotation of the formation at a comparable rate. A tapering function slowed down the ordered translation and rotation as the desired position and formation were approached. Characteristic values of 10 m and 10° defined the onset of the tapering.

These variables represent goals for the unit as a whole; the subunits are not necessarily able to keep up, given the constraints of terrain, crowding, and so on. As shown in the examples, the result is an irregularity in distribution of forces. In this way, the continuous flow model adds realism over models based on discrete unit motion.

A provision to modify the orders to either retreat or press an attack is also included. At the beginning of each time step, the current fractional losses \( F_i \) of each unit were calculated. A morale variable was then computed as follows:

\[
M_i = M_i^0 - \frac{F_i}{F_{i,m}}
\]

where the initial morale is \( M_i^0 \) and the reference fractional losses were set to \( F_{i,m} = 0.35 \). Then, for each unit, a series of checks was performed. First, the nearest enemy for the given unit was found in a sector of \( \pm 45° \) from the unit’s bearing. If that closest visible enemy was in retreat, the selected unit was given a morale increment of 0.5. If a visible Union unit was retreating, Confederate units with positive morale were ordered to press the attack, pursuing the enemy beyond their original orders. For negative morale, units of both sides withdrew to their initial positions.

### 2.5 Numerical Implementation

Simple boundary conditions, consistent with the mathematics of the model, were used at the edge of the map. For flow in, the density \( \rho_i \) was set to a very small value and the identity \( \xi_i \) was set to the local coordinates. For flow out, the appropriate boundary conditions would be extrapolation of the values near the boundary. Nonetheless, to avoid having units, or portions of units, leave the map, the directed velocity component normal to the boundary \( V \cdot n \) was tapered to zero around the border of the map. The width of this boundary region was 2% of the map dimension.

The main computer program was implemented in modern, object-oriented Fortran. Spatial derivatives of the convective terms in Eq. (1) and Eq. (6) were computed using simple, second-order upwind differencing based on the directed velocity for the density equation and the total velocity for the subunit identity equation. Employing a harmonic limiter [22] worked well to avoid numerical oscillations. Other spatial derivative terms were calculated with second-order central differences. Time integration was carried out using second-order Runge-Kutta time stepping.

Higher-order and implicit time stepping schemes were tested, as well as central spatial differences with explicit numerical dissipation, but these approaches did not appear to offer any significant advantages over the methods selected here.
Table 1: Brigades / regiments participating in Pickett’s Charge, based on the data of Busey and Martin [13].

| Ref. No. | Side | Brigade Leader | Init. Morale | Est. Strength |
|----------|------|----------------|--------------|--------------|
| 1        | Union| OH 120        | 0.7          | 141          |
| 2        | Union| Willard       | 0.7          | 1030         |
| 3        | Union| Smyth         | 0.7          | 828          |
| 4        | Union| Webb          | 0.7          | 693          |
| 5        | Union| Hell          | 0.7          | 669          |
| 6        | Union| Stone         | 0.8          | 745          |
| 7        | Union| Harrow        | 0.7          | 831          |
| 8        | Union| Stannard      | 0.7          | 1715         |
| 9        | Union| Cross         | 0.7          | 632          |
| 10       | Union| Kelly         | 0.7          | 399          |
|          |      | Total         |              | 8036         |
| 11       | Conf.| Brockenbrough | 0.8          | 829          |
| 12       | Conf.| Lane          | 0.7          | 1203         |
| 13       | Conf.| Davis         | 0.8          | 1404         |
| 14       | Conf.| Lawrence (Scales) | 0.7 | 879         |
| 15       | Conf.| Marshall (Pettigrew) | 0.8 | 1495         |
| 16       | Conf.| Fry (Archer)  | 0.8          | 739          |
| 17       | Conf.| Armistead     | 1.0          | 1223         |
| 18       | Conf.| Garnett       | 1.0          | 824          |
| 19       | Conf.| Kemper        | 1.0          | 1163         |
| 20       | Conf.| Lang (Perry)  | 0.7          | 437          |
| 21       | Conf.| Wilcox        | 0.7          | 1205         |
|          |      | Total         |              | 11481        |

The computational cost of the model is relatively high compared to discrete agent approaches with an equal number of units; the model is highly efficient, however, compared to a discrete model with enough agents to capture the subunit motion. To achieve a reasonable run time (in terms of wall clock), multi-level parallel calculation methods were employed. At the large scale, domain decomposition was employed. The map was split into overlapping tiles, with each tile assigned to a computational process. Communication of data in the overlapping boundary regions (fringe cells or ghost cells) was implemented using the MPI library. Each tile corresponded to an MPI rank, and messages were exchanged using persistent sends and receives.

At a finer level of parallelism, outer loops were multi-threaded using the OpenMP library. Inner loops were automatically vectorized by the compiler (tested with Intel ifort and GNU gfortran).

3 Pickett’s Charge Scenario

The present calculations are intended to represent the events of the afternoon of July 3, 1863. The situation is illustrated by the historical Gettysburg maps of Figure 1 and the units involved are summarized in Tables 1, 2.

The initial configuration of units is presented with labels in Figure 1. The color contours show the local elevation and the dark contour lines represent the overlay layer. Confederate forces are shown in red, Union in blue. The oval contours are brigades or regiments; the red dots are artillery units.

On the Confederate lines, the Heth / Pettigrew division (Brockenbrough, Davis, Marshall, and Fry brigades) is initially stationed approximately along the crest of Seminary Ridge. Pender’s division (Lane and Lowrance) lies behind them. Pickett’s division (Armistead, Garnett, and Kemper) stands to the south, and Anderson’s division (Lang and Wilcox) stands close by. In front of the infantry, several artillery positions form two lines.
Figure 1: Battlefield details, based on the maps of Gottfried [11] and Laino [12]. (a) Initial configuration. Oval contours represent densities of infantry, dots artillery; Confederate forces in red, Union in blue. (b) Overlay layer labeled with various features.
Table 2: Artillery units participating in Pickett’s Charge, based on the data of Busey and Martin [15].

| Ref. No. | Side | Battery Leader | Guns |
|----------|------|----------------|------|
| 22       | Union | Bancroft / Wilkeson | 6    |
| 23       | Union | Mason / Eakin | 6    |
| 24       | Union | Edgell | 4    |
| 25       | Union | Hill | 4    |
| 26       | Union | Norton | 6    |
| 27       | Union | McCartney | 6    |
| 28       | Union | Woodruff | 6    |
| 29       | Union | Milton / Bigelow | 6    |
| 30       | Union | Arnold | 6    |
| 31       | Union | Cushing | 6    |
| 32       | Union | Brown / Prentice | 6    |
| 33       | Union | Rotty | 4    |
| 34       | Union | Daniels | 6    |
| 35       | Union | Thomas | 6    |
| 36       | Union | Hart | 4    |
| 37       | Union | Phillips | 6    |
| 38       | Union | Thompson | 5    |
| 39       | Union | Rask | 2    |
|          |       | Total | 95   |

37 Conf. | Subscriber | 4    |
38 Conf. | McGowan | 4    |
39 Conf. | Bronson / Zimmerman | 4    |
40 Conf. | Johnson | 4    |
41 Conf. | Marye | 4    |
42 Conf. | Rose | 6    |
43 Conf. | Wingfield | 5    |
44 Conf. | Graham | 4    |
45 Conf. | Wyant | 4    |
46 Conf. | Brooke | 4    |
47 Conf. | Wadell | 4    |
48 Conf. | Woolfolk | 4    |
49 Conf. | Blount | 4    |
50 Conf. | Buckeye | 4    |
51 Conf. | Macon | 4    |
52 Conf. | Stirling | 6    |
53 Conf. | Robertson | 3    |
54 Conf. | Nesbitt | 3    |
55 Conf. | Miller | 3    |
56 Conf. | Taylor | 4    |
57 Conf. | Gilbert | 4    |

| Feature               | Minimum 7 (m) | Characteristic Scale (cm) |
|-----------------------|---------------|---------------------------|
| Woods                 | 0.35          | —                         |
| Orchard               | 0.55          | —                         |
| Corn                  | 0.60          | —                         |
| Grain                 | 0.65          | —                         |
| Building              | 0.05          | 20.0                      |
| Post and rail fence   | 0.05          | 10.0                      |
| Worm fence            | 0.10          | 10.0                      |
| Stone wall            | 0.30          | 10.0                      |
| Road                  | 1.00          | 10.0                      |
The Union forces occupy a defensive position along a north-south line along Cemetery Ridge. The combined 8 Ohio / 126 New York unit lies to the west of Emmitsburg Road. Hays’ division (Willard and Smyth) hold the northern end of the line. In the middle is Gibbon’s division (Webb, Hall, and Harrow) and the Doubleday / Rowley division (Stone and Stannard). To the south lies Caldwell’s division (Cross and Kelly).

Figure 1b shows the elevation and the terrain overlay layer used to represent the effects of fences, roads, buildings, and local vegetation. Conditions for the map overlay layer were based on the maps provided in Gottfried [11] and Laino [12]. These historical conditions are relatively well-known because of the availability of the Warren [23] Map, created through a survey conducted a few years after the end of the US Civil War. In processing the map data, regions of vegetation (corn, grain, orchard, woods) were defined by polygons, and the overlay function value was taken to be constant inside each polygon.

Buildings were characterized by a point; the overlay function was defined by a central peak with a decaying distribution (error function) away from the center with a characteristic scale. These appear as blue dots in Figure 1b. An example is the dot located approximately at the coordinates (1.5 km, 1.6 km) on the map; this represents the house and barn of the Bliss Farm.

Similarly, roads and various kinds of fences (post and rail, stone, worm) were defined by line segments with decay in \( T(x) \) with distance from the line. The values selected for the overlay function for various terrain features are listed in Table 3. The strong fences on either side of Emmitsburg Road were significant obstacles during the historical battle.

The overlay layer was smoothed to obtain numerical stability; abrupt jumps in \( T(x) \) led to numerical oscillations. For a 384 \( \times \) 309 numerical grid (8 m resolution), three passes of a nine-point Laplacian averaging stencil provided sufficient smoothness without excessive distortion of the map.

A fundamental input to the model is the number of soldiers and artillery pieces participating in the fighting related to Pickett’s Charge. Busey and Martin [13] have published a comprehensive tabulation of the forces present at the Battle of Gettysburg. Very accurate numbers are available for June 30, when both sides carried out a routine roll call. Good figures of the forces engaged and lost (killed, wounded, captured, or missing) have been assembled for the overall three-day battle from accounts made in the aftermath. Nonetheless, the forces available on the afternoon of July 3, after two days of fighting, are unknown, and the numbers must be estimated.

The numbers quoted in the literature for the forces participating in the fighting related to Pickett’s Charge vary. In a review of the literature, Armstrong and Sodergren [18] found estimates (in round numbers) of 5800–8000 Union and 10500–13000 Confederate infantry engaged, with, respectively, casualties of 1300–2300 and 5900–6500 for each side.

For the present calculations, we used figures from Busey and Martin [13], with additional input from Gottfried [11] 24, Hessler et al. [25], and Laino [12] to judge which units participated. A force estimate was made for each unit at the start of the Pickett’s Charge event, based on known forces and losses for the overall battle, and history of combat on the previous two days of the battle.

Table 1 provides details of the brigades and certain regiments participating in Pick-
ett’s Charge. The first column indicates the reference number used in our modeling, roughly corresponding to position north to south on the map in Figure 1. The next column provides the brigade leader, to identify the unit. An exception is the first unit, which represents a combination of parts of the 8 Ohio and 126 New York regiments that were positioned to the west of Emmitsburg Road, apart from the main Union lines. The table also provides an associated initial morale score $M^0_i$, based on historical information for the date of last combat for that unit. We assume that a unit that had engaged in recent combat would have lower morale.

Again, Busey and Martin [13] provide accurate numbers for forces present at a roll call on June 30, 1863, and good approximations of engaged forces on the first day of the Gettysburg battle and of losses sustained in the whole conflict. The forces that took part in the Pickett’s Charge scenario must be estimated. The final column in Table 1 is our estimate of those forces.

The corresponding data for artillery batteries are presented in Table 2. The artillery units are identified by battery leader. Some of the Union batteries (Units 34 and 35) have been aggregated for simplicity. Again, the numbers were taken from Busey and Martin [13]. The selection of batteries that participated in the Pickett’s Charge scenario is based on Hessler et al. [25] and on our own judgement.

The present approach does not require exact figures for the forces engaged in the scenario. It is sufficient to have a good approximation. In our methodology, that approximation provides the baseline for a study of the sensitivity of the outcome on the parameters used as input in the model. (See the discussion associated with Figure 5.)

According to the maps presented in Gottfried [24] and Laino [12] all units were in formation by 13:00 on July 3. By 14:30, the brigades commanded by Brockenbrough, Davis, Marshall, and Fry were marching toward the northern end of the Union lines, and the unit composed of the 8 Ohio and 126 New York regiments was swinging out to flank the Confederates. Meanwhile, Kemper and Garnett were taking their brigades toward the southern part of the Union lines. By 14:50, Brockenbrough was withdrawing, but Lane, Lowrance, and Armistead had begun to advance. Kemper and Garnett were shifting northward. On the Union side, Stannard was beginning a maneuver to flank the Confederate attack. At 15:15, most of the Confederate units had crossed Emmitsburg Road; only Lane and Lowrance remained to the west. Later, at 15:30, the Confederate charge was losing its momentum, and many units were retreating. By 15:50 Lang and Wilcox had made a late crossing of Emmitsburg Road for an attack; the rest of the Confederates were in full retreat.

Some of the motion of the Confederate attackers may be hard to reconstruct. Reardon [26] notes (p. 21) historical accounts of “the unusual indirect approach march that the Virginians [Pickett’s division] employed,” and quotes a survivor as stating that they “moved alternately by the front & by the left flank under a most deadly fire of infantry and artillery.” The present model does not attempt to match this apparent staircase path; diagonal straight lines with appropriate waypoints are employed.

### 3.1 Army Level of Aggregation

As an initial demonstration of the mathematical model, we consider the Pickett’s Charge scenario at an army level of aggregation. There are four entities in this version of the
Figure 2: Results at the army level of aggregation. Oval contours represent densities of infantry, dots aggregated artillery; Confederate forces in red, Union in blue. (a) Initial state, 0.0 min. (b) Red crossing Emmitsburg Road, 20.0 min. (c) Initial contact, 30.0 min. (d) Red retreats on suffering heavy casualties, 40.0 min. Video available at https://engineering.purdue.edu/~jpoggie/battle_flow_model/armies.mp4
Figure 3: Effect of terrain at the army level of aggregation. Oval contours represent densities of infantry, dots aggregated artillery; Confederate forces in red, Union in blue. (a) Elevation and terrain effects included. (b) Perfectly flat landscape. Video available at [https://engineering.purdue.edu/~jpoggie/battle_flow_model/armies_noTerrain.mp4](https://engineering.purdue.edu/~jpoggie/battle_flow_model/armies_noTerrain.mp4)
model: Union infantry, Union artillery, Confederate infantry, and Confederate artillery.

The initial force strength is based on aggregation of the brigade-level data (Tables 1–2). At the start of the battle simulation, the Union army had 8036 infantrymen and 95 artillery pieces, whereas the Confederate army had 11481 infantrymen and 86 artillery pieces. The computational grid consisted of $384 \times 309$ points, or 8 m map resolution. The time step was 1.0 s, with a total run of 3600 steps, or 1.0 h of simulated time.

The model parameters used for this case were slightly different than those employed for the brigade-level simulation described in the next section. The initial morale was taken as $M_0^i = 1.0$ for all units. The ranged fire efficiency for infantry and artillery were, respectively, reduced to $k_{ij}^e = 2.0 \text{ s}^{-1}$ and $k_{ij}^s = 4.0 \text{ s}^{-1}$.

Figure 2 shows results at selected times in the simulation. Again, the color contours are the local elevation, the dark lines are contours of the overlay function, red contours give the density of Confederate forces, and blue contours the density of Union forces. The two solid dots, initially at the army centers, represent the aggregated artillery of each side.

Figure 2(a) shows the initial conditions (0.0 min); the armies are seen to be spread across the two ridges. With the start of the simulation, the Confederate army begins marching toward the Union position. The Union forces hold their positions until the Confederates are close, at which time they move forward to a defensive position at the fences and stone walls that mark the edge of the field. (This motion is used as an indicator in the model; it is not intended to be historically accurate.) Initially exchanging artillery and ranged musket fire, the armies come into close contact at 30.0 min, Figure 2(c). At this stage, the close-in combat model dominates the casualty rate. After taking heavy casualties, the Confederates are in full retreat by 40.0 min, Figure 2(d).

The model predicts 1736 Union and 6115 Confederate casualties by the end of the battle. These figures are well within the range of the historical numbers of 1300–2300 Union and 5900–6500 Confederate casualties quoted by Armstrong and Sodergren [18].

The case was re-run with the elevation and overlay layer effects turned off, to test the effects of these features on the outcome of the scenario. Figure 3 compares results for the two cases near the time of army contact in the simulation. As might be expected, the terrain effects are seen to produce the wrinkling of the unit density contours that is not present for the perfectly flat terrain case. This prediction is an important feature of the continuous flow model; to capture the same effect in a discrete model would require a very large number of agents.

A more striking result was a change in the total number of casualties in the absence of terrain, a total of about 1945 Union and 5819 Confederate. The Union forces take about 200 more casualties, and the Confederates 300 fewer, under the conditions of perfectly flat terrain. The terrain effects are seen to slow the pace of battle, and they favor the defenders. With terrain effects included, the slow progress of the Confederate troops toward the Union lines exposes the attacking Confederates to heavy ranged fire for an extended period.
3.2 Brigade Level of Aggregation

The scenario for the main set of calculations was carried out at the brigade level of aggregation. The computational grid, time step, and simulation duration were the same as those used for the army-level simulation. Selected results are shown in Figure 4. The initial conditions are shown in the top left subfigure at time 0.0 min. The Confederates begin their march eastward at the start of the simulation. At about the 18 min mark, the 8 Ohio / 126 New York unit and Stannard’s brigade swing around to flank the charging Confederates. At 25.0 min (top right in Figure 4), a group of Confederates has developed a force concentration on the western side of Emmitsburg Road. Pickett’s division, having been delayed crossing the road, heads northeast toward The Angle. At 33.0 min (bottom left in Figure 4), some of the Confederates press their attack at The Angle, but other units have taken such casualties that they begin to withdraw. By 43.0 min, many of the Confederate brigades are in full retreat (bottom right in Figure 4).

For this baseline scenario, the model predicts 2534 Union and 3620 Confederate casualties by the end of the battle, slightly out of the range of the historical figures collected by Armstrong and Sodergren [18]. The model parameters could be adjusted to make the predictions fall in the middle of the estimates of the historical figures, but our approach is to instead address variation of the model parameters in a systematic manner.

To assess the sensitivity of the model to changes in the initial conditions and model parameters, a statistical analysis of the outcome for randomized inputs was carried out. As with any historical event, there is a level of uncertainty in the historical details of the Pickett’s Charge scenario. Furthermore, as noted above, even eyewitness accounts of the battle have been colored by the personal and political interests of the survivors [26]. All of the inputs to a model of the battle must thus be considered to lie within an uncertainty band that may be larger than one would hope. Both the value and validity of the results are measured against the uncertainties in the literature.

To generate variations on the baseline course of events (Case 0), a program was written in the Python language to generate randomized inputs. The algorithm employed the PCG-64 pseudo-random number generator. For repeatability, a set of 1000 cases was generated using a seed based on the case number (Cases 1–1000). All significant initial conditions and model parameters were randomized, as follows.

Initial positions and goal positions were randomized using a normal distribution with a standard deviation of 100 m. (Thus, about 95% of the randomly generated values lie within ±200 m of the baseline positions.) Similarly, the initial and final bearings were randomized with a standard deviation of 10°. The parameters characterizing the initial and final shapes of the formation were randomized using a uniform distribution ranging between 80% and 120% of the baseline figure. A similar approach was applied to the morale parameters, initial forces, and ordered marching speed.

To test the effect of strong variations in the effect of the parameters characterizing combat (close-in combat factor, ranged-fire combat factor, and weapon range), these parameters were perturbed by factors of two, that is, with a uniform distribution between 50% and 200% of the baseline value. The changes in combat effectiveness were applied evenly across all units; under this perturbation, no unit gained an advantage or disadvantage relative to the others.
Figure 4: Results at the brigade level of aggregation (Case 0). Oval contours represent densities of infantry, dots artillery; Confederate forces in red, Union in blue. (a) Initial state, 0.0 min. (b) Initial contact, 25.0 min. (c) Heavy combat near The Angle, 33.0 min. (d) Red retreats after suffering heavy casualties, 43.0 min. Video available at https://engineering.purdue.edu/~jpoggie/battle_flow_model/brigades_case0.mp4
Figure 5: Statistical analysis of 1000 cases with randomized initial conditions and model parameters. (a) Time histories of total forces for each case. (b) Initial forces for each case. (c) Final forces for each case.
The results are shown in Figure 5. Time histories of the total Union (blue lines) and Confederate (red lines) forces (an ensemble forecast) are shown in Figure 5(a) for Cases 0–1000. The results reflect slow losses from artillery fire during the initial charge, followed by more rapid losses beginning at about the 25 min mark when the units close to rifle range.

Figure 5(b) presents a plot of initial Union versus Confederate forces; this is a kind of slice at time zero of the time-history plot. Each dot represents the initial condition of a particular case. The perturbation of \( \pm 20\% \) in the initial forces is apparent here.

The key results are presented in Figure 5(c), which shows Union versus Confederate forces at the end of the 60 min of simulated time. Again, this is a kind of slice of the time-history plot at the end of the battle, and each dot represents the end state of one of the cases. The baseline Case 0 results in about 5600 Union and 7900 Confederate survivors (about 2500 and 3600 casualties, respectively). With the variation in the results under the strong perturbation of the model inputs, the number of survivors lies in a range of about \( \pm 2500 \) for each side, the scatter in Figure 5(c). Given these high levels of casualties, these results are hard to interpret by themselves.

To clarify the outcome and facilitate interpretation, a set of victory criteria was established based on the number of Confederate soldiers in retreat at the end of the 60 min of simulated time. If 50% or more of the surviving Confederate forces were in retreat, the result was considered a conclusive Union victory. If 10% or less of the total Confederate forces were in retreat, the result was considered a Confederate victory. Results in between these thresholds were considered a thin Union victory.

In Figure 5(c), the outcomes are color coded by these victory criteria. The lack of possible outcomes in favor of the Confederates is immediately apparent. Under these criteria, about half (49.8%) of the scenarios resulted in a conclusive Union victory. Another, almost equal, percentage (44.6%) of outcomes resulted in a thin Union victory. Only 5.7% of the cases could be considered Confederate victories. In the context of our model, then, the Confederate forces had little chance of victory in the Pickett’s Charge battle.

4 Discussion

A continuous flow model of infantry behavior [10] has been developed in sufficient detail that it can now be applied to a realistic simulation of a historical battle. The model is based on conservation of individuals and tracking of subunit identity. Extensions to the model include artillery units, an overlay layer to represent different kinds of terrain on the map, improvements to movement and ranged fire, a list of orders for each unit, unit morale, and checks for retreat and pressing an attack.

Pickett’s charge during the 1863 Battle of Gettysburg, Pennsylvania in the U.S. Civil War was chosen as an initial application of the model. This scenario is a good test of the current mathematical model because many modern military tactics were employed, in a context where the action took place on foot or horseback. Furthermore, detailed maps and figures for troop numbers are available for the historical battle.

An initial simulation at the army level of aggregation was carried out with and without terrain effects. Elevation, buildings, vegetation, and particularly fences slowed the
advance of the Confederate attack when terrain was included. The resulting differences in troop density and position had consequences for both close-in and ranged combat. With terrain effects included, the slow progress of the Confederate troops toward the Union lines exposed the attacking Confederates to heavy ranged fire for an extended period, and resulted in higher Confederate casualties.

The results illustrate the strength of the continuous flow model in representing the effect of terrain on infantry movement. To capture the same effect in a discrete model would require an unworkably large number of agents. Compared to a discrete agent model with the same number of units, we judge that the continuous flow model better captures the interaction of the forces with the terrain and each other.

The main case study was a brigade-level simulation of the conflict. This was intended to be as faithful as possible to the historical events, and that aim appears to be achieved. The main source of asymmetry in the numbers of casualties was the inability of the Confederate forces to use effective ranged fire while they were moving. Again, the terrain slowed the pace of battle and favored the defenders, exposing the attackers to heavy ranged fire for an extended period.

A statistical analysis of possible outcomes for randomized perturbations of the baseline brigade-level scenario was carried out. These perturbations encompassed all the initial conditions and all the significant model parameters characterizing combat. Using the number of Confederate units in retreat at the end of the 60 min simulation as a threshold, it was found that only 6% of the scenarios resulted in an outcome that could be considered a Confederate victory.

Our results differ somewhat from those Armstrong and Sodergren [18] obtained using a Lanchester model. Those authors concluded that the Confederates could plausibly have taken the Union position under a different tactical approach.

In the context of the present mathematical model, which better represents the spatial evolution of the conflict, no reasonable concentration of Confederate forces had sufficient numerical advantage to break through and hold a section of the Union lines in a militarily significant manner. Given the limited advantage in numbers held by the Confederates, over-concentration of their forces (say on The Angle) would leave their flanks open to Union counterattack. They must assign at least some forces to each part of the Union lines. Given that situation, it is almost impossible for them to achieve the 3:1 local numerical advantage that is held to be required for victory [27] [28].

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