Shape theory via polar decomposition

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Abstract
This work proposes a new model in the context of statistical theory of shape, based on the polar decomposition. The non isotropic noncentral elliptical shape distributions via polar decomposition is derived in the context of zonal polynomials, avoiding the invariant polynomials and the open problems for their computation. The new polar shape distributions are easily computable and then the inference procedure can be studied under exact densities. As an example of the technique, a classical application in Biology is studied under three models, the usual Gaussian and two non normal Kotz models; the best model is selected by a modified BIC criterion, then a test for equality in polar shapes is performed.

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1 Introduction

Matrix variate statistical shape analysis has been extensively studied in the last two decades by a number of approaches: via QR decomposition (Goodall and Mardia (1993)); via SVD decompositions (Goodall (1991), Le and Kendall (1993), Díaz-García et al. (1997), Díaz-García et al. (2003)); via affine transformations (Goodall and Mardia (1993), Díaz-García et al. (2003), Caro-Lopera et al. (2009)); among many others methods (Dryden and Mardia (1998) and the references therein). However, the polar decomposition has not been included yet in the context of shape theory.

According to the transformation, we say that the shape of an object is all geometrical information which remains after filtering out translation, scale, rotation, reflection, uniform share, etc, from an original figure comprised in $N$ landmarks in $K$ dimension. Statistical shape theory study the mean shape of populations in presence of randomness.

Some of the classical works (Goodall and Mardia (1993)) assume an isotropic Gaussian model for the landmark matrix in order to obtain shape densities expanded in known polynomials, such as zonal polynomials (James (1964), Muirhead (1982)); then, generalisations for matrix variate shape theory under elliptical models appeared via the SVD method (Díaz-García et al. (2003)) and via the affine technique (Caro-Lopera et al. (2009)). However, in order to obtain zonal polynomials a partial non isotropy was assumed, otherwise, considering a full non isotropy, the densities are expanded in terms of invariant polynomials (Davis (1980)), which are non available for large degrees.

Now, the isotropic assumption, say $\Theta = \mathbf{I}_K$ for an elliptical shape model of the form

$$\mathbf{X} \sim \mathcal{E}_{N \times K}(\mu_X, \Sigma_X, \Theta, h),$$

restricts substantially the correlations of the landmarks in the figure and it is non appropriate for applications. So, we expect the non isotropic model, with any positive definite matrix $\Theta$, as the best model for considering all the possible correlations among the anatomical (geometrical or mathematical) points.

This work solves that problem and sets the non isotropic noncentral elliptical shape distributions via polar decomposition in the context of zonal polynomials, avoiding the invariant polynomials and the open problems for their computation. The new shape distributions are easily computable and then the inference procedure can be studied under exact densities.

In section 2 the so termed polar shape coordinates are introduced and the main mathematical tools are studied in order to obtained the polar size and shape density. Then the polar shape density is derived in section 3. Section 4 studies the central case and the cor-
responding invariance under the family of elliptical distributions. Finally, section 5 gives explicit densities and performs inference with three models, the Gaussian and two non-normal Kotz models.

2 Polar size-and-shape distribution

Consider a full non isotropy (non singular) elliptical model

\[ X \sim E_{N \times K}(\mu_X, \Sigma_X, \Theta, h), \]

with generator function \( h(\cdot) \).

In order to avoid the referred problem of invariant polynomials consider the following procedure: Let

\[ X \sim E_{N \times K}(\mu_X, \Sigma_X, \Theta, h), \]

if \( \Theta^{1/2} \) is the positive definite square root of the matrix \( \Theta \), i.e. \( \Theta = (\Theta^{1/2})^2 \), with \( \Theta^{1/2} : K \times K \), see Gupta and Varga (1993, p. 11), and noting that

\[ X\Theta^{-1/2}X' = X(\Theta^{-1/2}\Theta^{-1/2})^{-1}X' = X\Theta^{-1/2}(X\Theta^{-1/2})' = ZZ', \]

where

\[ Z = X\Theta^{-1/2}, \]

then

\[ Z \sim E_{N \times K}(\mu_Z, \Sigma_X, I_K, h) \]

with \( \mu_Z = \mu_X \Theta^{-1/2} \), see Gupta and Varga (1993, p. 20).

And we arrive at the classical starting point in shape theory where the original landmark matrix is replaced by \( Z = X\Theta^{-1/2} \) (see Goodall and Mardia, 1993, for example). Then we can proceed as usual, removing from \( Z \), translation, scale, rotation in order to obtain the shape of \( Z \) (or \( X \)) via QR, SVD, or polar decompositions, for example.

In this paper we consider a new system of shape coordinates, the polar shape coordinates \( u \) of \( X \) which are constructed as follows:

\[ LX = Y = RH = rWH = rW(u)H. \]

The matrix \( L \) is as usual an \((N - 1) \times N\) Helmert submatrix and we assume that

\[ Y \sim E_{N-1 \times K}(\mu, \Sigma \otimes I_K, h), \quad \mu = L\mu_X, \quad \Sigma = L\Sigma_XL'. \]
Under this approach, \( Y = RH \) is the polar decomposition, where \( R : N - 1 \times N - 1 \) is a positive definite matrix and \( H \in V_{N - 1, K} \).

It is important to note that only under \( n = N - 1 \) the polar approach is valid.

Recall that for the singular value decomposition \( Y = PLQ' \), the polar decomposition of \( Y \) is given by \( Y = RH \), where \( R = PLP' \) and \( H = PQ' \).

So, we start with a known result, see Cadet (1996):

**Lemma 2.1.** Let \( Y : N - 1 \times K \), then there exist \( R : N - 1 \times N - 1 \) a positive definite matrix and \( H \in V_{N - 1, K} \) such that \( Y = RH \) and

\[
(dY) = \prod_{i<j}^{N-1} (L_i + L_j)(dR)(HdH'),
\]

with \( L = \text{diag}(L_1, \ldots, L_{N-1}) \) and \( R = PLP' \) i.e. \( L_i = \lambda_i(R) \).

So the main result of the section follows:

**Theorem 2.1.** The polar size-and-shape density is

\[
f_R(R) = \frac{\pi^{(N-1)K}}{2^{-N+1}N\Gamma_{N-1} [\frac{4}{\pi}] |\Sigma|^\frac{K}{2}} \sum_{t=0}^{\infty} \sum_{\kappa} h^{(2t)} \left[ \frac{\text{tr} (\Sigma^{-1}R^2 + \Omega)}{t!} \right] C_{\kappa} \left( \Omega \Sigma^{-1} R^2 \right)^{\frac{K}{2}} h^{(t)} \left( -2\mu' \Sigma^{-1} R H \right)^t (HdH').
\] (1)

**Proof.** The density of \( Y \) with \( \Omega = \Sigma^{-1} \mu \Theta^{-1} \mu' \) is

\[
f_Y(Y) = \frac{1}{|\Sigma|^\frac{K}{2}} h \left[ \text{tr} (\Sigma^{-1}YY' + \Omega) - 2 \text{tr} \Sigma^{-1}Y \mu' \right].
\]

If the decomposition \( Y = RH \) is performed and the Lemma 2.1 is applied, then the joint density of \( R \) and \( H \) remains

\[
f_{R,H}(R, H) = \frac{\prod_{i<j}^{N-1} (L_i + L_j)}{\Sigma^{[K/2]}} h \left[ \text{tr} (\Sigma^{-1}R^2 + \Omega) - 2 \mu' \Sigma^{-1} R H \right].
\]

Assuming that \( h(\cdot) \) can be expanded in a convergent power series, see Fang and Zhang (1990), i.e.

\[
h(a + v) = \sum_{t=0}^{\infty} \frac{h^{(t)}(a)}{t!} v^t,
\]

hence

\[
f_{R,H}(R, H) = \frac{\prod_{i<j}^{N-1} (L_i + L_j)}{\Sigma^{[K/2]}} \sum_{t=0}^{\infty} \frac{h^{(t)} \left( \text{tr} (\Sigma^{-1}R + \Omega) \right)}{t!} \left[ \text{tr} (-2\mu' \Sigma^{-1} R H) \right]^t (HdH').
\]
So, the marginal of \( R \) is

\[
f_R(R) = \prod_{i < j} \frac{(L_i + L_j)}{|\Sigma|^{K/2}} \sum_{t=0}^{\infty} h^{(t)} \left[ \text{tr} \left( \Sigma^{-1} R + \Omega \right) \right] \int_{\mathbb{V}_{N-1,K}} \left[ \text{tr} \left( -2\mu' \Sigma^{-1} RH \right) \right]^t (HdH').
\]

The integral equals zero when \( t \) is odd, then by \( \text{James} \) (1964, eq. (22))

\[
\int_{\mathbb{V}_{N-1,K}} \left[ \text{tr} \left( -2\mu' \Sigma^{-1} RH \right) \right]^{2t} (HdH') = \frac{2^{N-1} \pi^{(N-1)K}}{\Gamma_{N-1} \left[ \frac{K}{2} \right]} \sum_{\kappa} \left( \frac{\kappa}{2} \right)_t 4^t \left( \frac{\kappa}{2} \right)_t C_\kappa \left( \Omega \Sigma^{-1} R^2 \right).
\]

Noting that \( \left( \frac{1}{2} \right)_t = \frac{1}{t!} \), so

\[
f_R(R) = \frac{\pi^{(N-1)K}}{2^{N+1} \Gamma_{N-1} \left[ \frac{K}{2} \right]} \sum_{\kappa} \sum_{t=0}^{\infty} h^{(2t)} \left[ \text{tr} \left( \Sigma^{-1} R^2 + \Omega \right) \right] C_\kappa \left( \Omega \Sigma^{-1} R^2 \right) \frac{t! \left( \frac{1}{2} \right)_t}{\left( \frac{1}{2} \right)_t}.
\]

3 Polar shape density

Now, observe that \( R : N - 1 \times N - 1, R > 0 \), contains \( (N - 1)N/2 \) different coordinates \( (r_{ij} = r_{ji}) \). Let \( v(R) \) the vector consisting of the different elements \( r_{ij} \), taken column by column. Then the polar shape matrix \( W \), can be written as:

\[
v(W) = \frac{1}{r} v(R), \quad r = \| R \| = \sqrt{\text{tr} R^2} = \| Y \|.
\]

Then by \( \text{Muirhead} \) (1982), Theorem 2.1.3, p. 55:

\[
(dv(W)) = r^m \prod_{i=1}^{m} \sin^{m-i} \Theta_i d\Theta_i \wedge dr, \quad m = \frac{N(N - 1)}{2} - 1,
\]

which will denoted as

\[
(dW) = r^m J(u) \prod_{i=1}^{m} d\Theta_i \wedge dr.
\]

Thus:

**Theorem 3.1.** The polar shape density is

\[
f_W(W) = \frac{2^{N-1} \pi^{(N-1)K}}{\Gamma_{N-1} \left[ \frac{K}{2} \right]} \sum_{\kappa} \sum_{t=0}^{\infty} \frac{C_\kappa \left( \Omega \Sigma^{-1} W^2 \right) t! \left( \frac{1}{2} \right)_t}{\left( \frac{1}{2} \right)_t} \int_0^{\infty} r(N-1)^2 + 2t - 1 h^{(2t)} \left[ r^2 \text{tr} \Sigma^{-1} W^2 + \text{tr} \Omega \right] \left( dr \right).
\]
Proof. The density of $\mathbf{R}$ is

$$f_{\mathbf{R}}(\mathbf{R}) = \frac{\pi^{(N-1)K} \prod_{i<j} (L_i + L_j)}{2^{-N+1} \Gamma_{N-1} \left( \frac{K}{2} \right) |\Sigma|^{\frac{K}{2}}} \sum_{t=0}^{\infty} \sum_{\kappa} h^{(2t)} \left[ \text{tr} (\Sigma^{-1} \mathbf{R}^2) \right] C_\kappa \left( \Omega \Sigma^{-1} \mathbf{R}^2 \right) \frac{1}{t!} \left( \frac{1}{4} K \right)_\kappa.$$

Let be $\mathbf{W}(\mathbf{u}) = \mathbf{R}/r$, then the joint density function of $\mathbf{W}(\mathbf{u})$ and $r$ is given by

$$f_{r, \mathbf{W}(\mathbf{u})}(r, \mathbf{W}(\mathbf{u})) = \frac{2^{N-1} \pi^{(N-1)K}}{\Gamma_{N-1} \left( \frac{K}{2} \right) |\Sigma|^{\frac{K}{2}}} \prod_{i<j} r (\lambda_i + \lambda_j) \times \sum_{t=0}^{\infty} \sum_{\kappa} h^{(2t)} \left[ \text{tr} (r^2 \Sigma^{-1} \mathbf{W}^2 + \Omega) \right] C_\kappa \left( r^2 \Omega \Sigma^{-1} \mathbf{W}^2 \right) \frac{1}{t!} \left( \frac{1}{4} K \right)_\kappa,$$

with $m = N(N - 1)/2 - 1$. Let $\lambda_i = \lambda_i(\mathbf{W})$ of $\mathbf{W}$, so if $L_i = \lambda_i(\mathbf{R})$, thus $L_i = r \lambda_i$. Also note that:

1. $C_\kappa \left( r^2 \Omega \Sigma^{-1} \mathbf{W}^2 \right) = r^{2 \kappa} C_\kappa \left( \Omega \Sigma^{-1} \mathbf{W}^2 \right)$,
2. $\prod_{i<j} r (\lambda_i + \lambda_j) = r^{(N-1)(N-2)/2} \prod_{i<j} (\lambda_i + \lambda_j)$,
3. $h^{(2t)} \left[ \text{tr} (r^2 \Sigma^{-1} \mathbf{W}^2 + \Omega) \right] = h^{(2t)} \left[ r^{2t} \text{tr} \Sigma^{-1} \mathbf{W}^2 + \text{tr} \Omega \right]$.

Collecting powers of $r$ as $r^{m+2t+(N-1)(N-2)/2 - r^{(N-1)^2+2t-1}}$, the marginal of $\mathbf{W}$ is

$$f_{\mathbf{W}}(\mathbf{W}) = \frac{2^{N-1} \pi^{(N-1)K}}{\Gamma_{N-1} \left( \frac{K}{2} \right) |\Sigma|^{\frac{K}{2}}} \prod_{i<j} (\lambda_i + \lambda_j) J(\mathbf{u}) \times \sum_{t=0}^{\infty} \sum_{\kappa} C_\kappa \left( \Omega \Sigma^{-1} \mathbf{W}^2 \right) \frac{1}{t!} \left( \frac{1}{4} K \right)_\kappa \times \int_0^{\infty} r^{(N-1)^2+2t-1} h^{(2t)} \left[ r^{2t} \text{tr} \Sigma^{-1} \mathbf{W}^2 + \text{tr} \Omega \right] (dr).$$

Remark 3.1. Given that $\mathbf{H} \in V_{N-1,K}$, we cannot classify the polar shape densities by including or excluding reflections as in the QR shape distribution cases.

4 Central case

The central case of the elliptical polar shape densities follows easily:

Corollary 4.1. The central polar size-and-shape density is given by

$$f_{\mathbf{R}}(\mathbf{R}) = \frac{2^{N-1} \pi^{(N-1)K}}{\Gamma_n \left( \frac{K}{2} \right) |\Sigma|^{\frac{K}{2}}} \prod_{i<j} (L_i - L_j) \times h \left[ \text{tr} \Sigma^{-1} \mathbf{R}^2 \right]$$
Proof. Just take $\mu = 0$ in Theorem 2.1 and use $h^{(0)}(\cdot) = h(\cdot)$.

And finally, we have that

Corollary 4.2. The central polar shape density is invariant under the elliptical family and it is given by

$$f_W(W) = \frac{2^{N-2}\pi^{((N-1)K-N)/2}}{N-1 \left[ \frac{K}{2} \right]} \prod_{i,j}^{N-1} (\lambda_i + \lambda_j) J(u) \left( \mbox{tr} \Sigma^{-1} W^2 \right)^{-\frac{(N-1)^2}{2}}$$

Proof. It is straightforward from Theorem 3.1. Take $\mu = 0$, and use $h^{(0)}(\cdot) = h(\cdot)$, then

$$f_W(W) = \frac{2^{N-1}\pi^{(N-1)K}}{\Gamma_{N-1} \left[ \frac{K}{2} \right]} \prod_{i,j}^{N-1} (\lambda_i + \lambda_j) \int_0^\infty r^{(N-1)^2-1} h \left[ r^2 \mbox{tr} \Sigma^{-1} W^2 \right] (dr)$$

Let be $s = \left( \mbox{tr} \Sigma^{-1} W^2 \right)^{\frac{1}{2}} r$, so $ds = \left( \mbox{tr} \Sigma^{-1} W^2 \right)^{\frac{1}{2}} (dr)$, and

$$\int_0^\infty \left( \frac{s}{\left( \mbox{tr} \Sigma^{-1} W^2 \right)^{\frac{1}{2}}} \right)^{(N-1)^2-1} h \left( s^2 \right) \frac{ds}{\left( \mbox{tr} \Sigma^{-1} W^2 \right)^{\frac{1}{2}}} = \left( \mbox{tr} \Sigma^{-1} W^2 \right)^{-\frac{(N-1)^2}{2}} \int_0^\infty s^{N(N-1)-1} h \left( s^2 \right) (ds) = \frac{\Gamma \left[ (N-1)^2 \right]}{2\pi^{\frac{(N-1)^2}{2}} \left( \mbox{tr} \Sigma^{-1} W^2 \right)^{\frac{(N-1)^2}{2}}}.$$

Then

$$f_W(W) = \frac{\pi^{(N-1)(K-N)/2} \Gamma \left[ \frac{(N-1)^2}{2} \right]}{2^{-N+2} \Gamma_{N-1} \left[ \frac{K}{2} \right]} \prod_{i,j}^{N-1} (\lambda_i + \lambda_j) J(u) \left( \mbox{tr} \Sigma^{-1} W^2 \right)^{-\frac{(N-1)^2}{2}}.$$

5 Some particular models

Finally, we give explicit shapes densities for some elliptical models.

The Kotz type I model is given by

$$h(y) = \frac{R^{T-1} + \frac{K(N-1)}{2} \Gamma \left( \frac{K(N-1)}{2} \right)}{\pi^{K(N-1)/2} \Gamma \left( T - 1 + \frac{K(N-1)}{2} \right)} y^{T-1} \exp \{-Ry\}.$$

So, the corresponding $k$-th derivative follows from

$$\frac{d^k}{dy^k} y^{T-1} \exp \{-Ry\} = (-R)^k y^{T-1} \exp \{-Ry\} \left\{ 1 + \sum_{m=1}^{k} \binom{k}{m} \prod_{i=0}^{m-1} (T - 1 - i) (-Ry)^{-m} \right\},$$

see Caro-Lopera et al. (2009).
It is of interest the normal case, i.e. when $T = 1$ and $R = \frac{1}{2}$, here the derivation is straightforward from the general density.

The required derivative follows easily, it is,

$$h^{(k)}(y) = \frac{R^{K(N-1)/2}}{\pi^{(N-1)/2}} (-R)^k \exp(-Ry)$$

and replacing

$$\int_0^\infty r^{(N-1)^2+2t-1} h^{(2t)} \left[ r^2 \text{tr} \Sigma^{-1} \mathbf{W}^2 + \text{tr} \Omega \right] dr$$

$$= \frac{R^{K(N-1)/2} \text{etr}\{-R\Omega\}}{2\pi^{(N-1)/2} (\text{tr} R \Sigma^{-1} \mathbf{W}^2)^{\frac{(N-1)^2}{2}+t}} \Gamma \left[ (N-1)^2 + t \right],$$

in

$$f_{\mathbf{W}}(\mathbf{W}) = \frac{2^{N-1} \pi^{(N-1)/2} \prod_{i<j} (\lambda_i + \lambda_j) J(u)}{\Gamma_{N-1} \left[ \frac{K}{2} \right] \Phi} \sum_{i=0}^{\infty} \sum_{\kappa} C_\kappa \left( R^2 \Omega \Sigma^{-1} \mathbf{W}^2 \right)$$

$$= \sum_{i=0}^{\infty} \frac{1}{t! (\text{tr} R \Sigma^{-1} \mathbf{W}^2)^t} \sum_{\kappa} C_\kappa \left( R^2 \Omega \Sigma^{-1} \mathbf{W}^2 \right) \left( \frac{1}{2} \right)_\kappa .$$

we have proved that

**Corollary 5.1.** The Gaussian polar shape density is

$$f_{\mathbf{W}}(\mathbf{W}) = \frac{2^{N-2} J(u) \prod_{i<j} (\lambda_i + \lambda_j)}{R^{-K(N-1)/2} \Gamma_{N-1} \left[ \frac{K}{2} \right] \Phi} \text{etr}\{-R\Omega\}$$

$$\times \sum_{i=0}^{\infty} \frac{1}{t! (\text{tr} R \Sigma^{-1} \mathbf{W}^2)^t} \sum_{\kappa} C_\kappa \left( R^2 \Omega \Sigma^{-1} \mathbf{W}^2 \right) \left( \frac{1}{2} \right)_\kappa .$$

Finally, we propose the result for the Kotz type I model

$$h(y) = \frac{R^{T-1+K(N-1)/2} \Gamma \left( \frac{K(N-1)}{2} \right) y^{T-1} \exp(-Ry)}{\pi^{K(N-1)/2} \Gamma \left( T - 1 + \frac{K(N-1)}{2} \right) y^{T-1}}.$$
Corollary 5.2. The Kotz type I polar shape density is

\[
f_W(W) = \frac{2^{N-1} \prod_{i<j} (\lambda_i + \lambda_j) J(u)}{\Gamma_{N-1} \left[ \frac{K}{2} \right] \left| \Sigma \right|^\frac{K}{2} \text{etr} \{ R \Omega \}} \sum_{t=0}^{\infty} \sum_{\kappa} \frac{C_\kappa (\Omega \Sigma^{-1} W^2)}{t! \left( \frac{1}{2} K \right)_\kappa} \\
\times \frac{R^{T-1} + \frac{K(N-1)}{2} + 2t \Gamma \left( \frac{K(N-1)}{2} \right)}{\Gamma \left( T - 1 + \frac{K(N-1)}{2} \right)} \\
\times \left\{ \sum_{i=0}^{\infty} \frac{1}{i!} \prod_{u=0}^{i-1} (T - 1 - u) \times \frac{(\text{tr} \Omega)^{T-1-i \Gamma} \left[ \frac{(N-1)^2}{2} + i + t \right]}{2R^{\frac{(N-1)^2}{2}+i+t}(\text{tr} \Sigma^{-1} W^2)^{\frac{(N-1)^2}{2}+t}} \right\}.
\]

Proof. The corresponding \( k \)-th derivative follows from

\[
\frac{d^k}{dy^k} y^{T-1} \exp \{ -Ry \} = (-R)^k y^{T-1} \exp \{ -Ry \} \left\{ 1 + \sum_{m=1}^{k} \binom{k}{m} \left[ \prod_{i=0}^{m-1} (T - 1 - i) \right] (-R)^{-m} \right\},
\]

(see Caro-Lopera et al. (2009)), and the corresponding polar shape density is obtained after
some simplification like

\[
 f_{W}(W) = \frac{2^{N-1} \pi^{\frac{(N-1)K}{2}} \prod_{i<j} (\lambda_i + \lambda_j) J(u)}{\Gamma N-1 \left( \frac{K}{2} \right) \left| \Sigma \right|^{\frac{K}{2}}} \sum_{t=0}^{\infty} \sum_{\kappa} C_\kappa \left( \Omega \Sigma^{-1} W^2 \right) \\
 \times \int_0^{\infty} r^{(N-1)2 + 2t - 1} J(u) \left[ r^2 \Sigma^{-1} W^2 + tr \Omega \right] dr \\
 \times \frac{2^{N-1} \prod_{i<j} (\lambda_i + \lambda_j) J(u)}{\Gamma N-1 \left( \frac{K}{2} \right) \left| \Sigma \right|^{\frac{K}{2}} \text{etr} \left( R \Omega \right)} \sum_{t=0}^{\infty} \sum_{\kappa} C_\kappa \left( \Omega \Sigma^{-1} W^2 \right) \\
 \times \frac{R^{T-1 + K(N-1)2 + 2t N} \Gamma \left( K(N-1)2 \right)}{\Gamma \left( T - 1 + K(N-1)2 \right)} \right) \\
 \times \left\{ \frac{1}{t!} \prod_{u=0}^{i-1} \left( T - 1 - u \right) \frac{\left( tr \Omega \right)^{T-1-i} \Gamma \left( \frac{(N-1)2}{2} + i + t \right)}{2R^{(N-1)2 + 2t + tr \Sigma^{-1} W^2}^{\frac{(N-1)2}{2} + t}} \\
 + \frac{2t}{m} \left[ \prod_{i=0}^{m-1} \left( T - 1 - i \right) \right] (-R)^{-m} \\
 \times \frac{1}{t!} \prod_{u=0}^{i-1} \left( T - 1 - m - u \right) \frac{\left( tr \Omega \right)^{T-1-m-i} \Gamma \left( \frac{(N-1)2}{2} + t + i \right)}{2R^{(N-1)2 + 2t + tr \Sigma^{-1} W^2}^{\frac{(N-1)2}{2} + t}} \right\} \square
\]

5.1 Example: Mouse Vertebra

This experiment has been studied in the Gaussian case by Dryden and Mardia (1998) and some references there in. Here we study this data under three different models, the usual normal and two non normal Kotz models, the best distribution will determined by applying a modified BIC criterion.

We start with the isotropic version of the Gaussian polar density given in corollary 5.1

\[
 f_{W}(W) = \frac{J(u)}{2^{K(N-1)2 - (N-1)2 + N+2 \Gamma N-1 \left( \frac{K}{2} \right) \Sigma K(N-1)-2(N-1)2} \left( \frac{\mu' \mu}{2\Sigma^2} \right)} \\
 \times \Gamma \left( \frac{(N-1)2}{2} + t \right) \sum_{t=0}^{\infty} \sum_{\kappa} C_\kappa \left( \frac{1}{2\Sigma^2} \mu' \Sigma^{-1} W^2 \mu \right) .
\]

Consider now two Kotz models from corollary 5.2 for \( T = 2 \) and \( R = \frac{1}{2} \), we obtain after some simplification that
\[ f_W(W) = \frac{2^{N^2-K(N-1)+1}}{2^{-(N-1)^2+K(N-1)}} J(u) \prod_{i<j} (\lambda_i + \lambda_j) \]

\[ \times \left\{ \frac{K(N-1)(K(N-1)+2)\Gamma_{N-1} \left[ \frac{N}{2} \right]}{K(N-1)\Gamma_{N-1} \left[ \frac{N}{2} \right]} \right\} \exp \left( -\frac{\mu^T \Sigma^{-1} \mu}{2} \right) \]

\[ \times \sum_{t=0}^{\infty} \left\{ \left( -2t + 4t^2 - 4t \text{tr} \left( \frac{\mu^T \mu}{2\Sigma^2} \right) + 2t \text{tr}^2 \left( \frac{\mu^T \mu}{2\Sigma^2} \right) \right) / t! \right\} \]

\[ \times \sum_{\kappa} C_\kappa \left( \frac{N}{2} \frac{\mu^T W \mu}{\Sigma^2} \right) \left( \frac{1}{2} K \right)^\kappa. \]

and, for \( T = 3 \) and \( R = \frac{1}{2} \), the corresponding non Gaussian isotropic model is given by

\[ f_W(W) = \frac{2^{N^2-K(N-1)-1}}{2^{-(N-1)^2+K(N-1)}} J(u) \prod_{i<j} (\lambda_i + \lambda_j) \]

\[ \times \left\{ \frac{K(N-1)\Gamma_{N-1} \left[ \frac{N}{2} \right]}{K(N-1)\Gamma_{N-1} \left[ \frac{N}{2} \right]} \right\} \exp \left( -\frac{\mu^T \Sigma^{-1} \mu}{2} \right) \]

\[ \times \sum_{t=0}^{\infty} \left\{ \left( -2t + 4t^2 - 4t \text{tr} \left( \frac{\mu^T \mu}{2\Sigma^2} \right) + 2t \text{tr}^2 \left( \frac{\mu^T \mu}{2\Sigma^2} \right) \right) / t! \right\} \]

\[ \times \sum_{\kappa} C_\kappa \left( \frac{N}{2} \frac{\mu^T W \mu}{\Sigma^2} \right) \left( \frac{1}{2} K \right)^\kappa. \]

We contrast these three models via the modified BIC criterion, they will be applied to the data of two groups (small and large) of mouse vertebra, an experiment very detailed in Dryden and Mardia (1998).

The likelihood based on the exact densities require the computation of the above series, a carefully comparison with the known hypergeometric of one matrix argument indicates that these distributions can be obtained by a suitable modification of the algorithms of Koev and Edelman (2006).

In order to decide which the elliptical model is the best one, different criteria have been employed for the model selection. We shall consider a modification of the BIC statistic as discussed in Yang and Yang (2007), and which was first achieved by Rissanen (1978) in a coding theory framework. The modified BIC is given by:

\[ BIC^* = -2\mathcal{L}(\bar{\mu}, \bar{\Sigma}^2, h) + n_p (\log(n+2) - \log 24), \]

where \( \mathcal{L}(\bar{\mu}, \bar{\Sigma}^2, h) \) is the maximum of the log-likelihood function, \( n \) is the sample size and \( n_p \) is the number of parameters to be estimated for each particular shape density.
Table 1: Grades of evidence corresponding to values of the \( BIC^* \) difference.

| \( BIC^* \) difference | Evidence |
|-------------------------|----------|
| 0–2                    | Weak     |
| 2–6                    | Positive |
| 6–10                   | Strong   |
| > 10                   | Very strong |

As proposed by Kass and Raftery (1995) and Raftery (1995), the following selection criteria have been employed for the model selection.

In order to apply the above densities we need to restrict the number of landmarks in such way that \( \min(K, N - 1) = N - 1 \), so in the mouse vertebra data we must select 3 landmarks of the original 6 points. In the following example we consider the landmarks 1, 2 and 6 which corresponds to the widest part of the vertebra.

The maximum likelihood estimators for location and scale parameters associated with the small and large groups are summarised in the following table:

Table 2: Maximum likelihood estimators.

| Group | \( BIC^* \) | \( \bar{\mu}_{11} \) | \( \bar{\mu}_{12} \) | \( \bar{\mu}_{21} \) | \( \bar{\mu}_{22} \) | \( \sigma^2 \) |
|-------|-------------|-----------------|-----------------|-----------------|-----------------|-------------|
| Small | \( K: T = 2 \) | \( -129.5719 \) | \( -4.5992 \) | \( -92.1556 \) | \( -18.2941 \) | \( 3.0906 \) | \( 28.1767 \) |
|       | \( K: \bar{T} = 3 \) | \( -139.2333 \) | \( -5.6873 \) | \( -50.9245 \) | \( -20.4493 \) | \( 4.8584 \) | \( 27.6280 \) |
| Large | \( K: T = 2 \) | \( -204.2375 \) | \( -28.0776 \) | \( -73.1440 \) | \( -32.0698 \) | \( 13.1131 \) | \( 75.7733 \) |
|       | \( K: \bar{T} = 2 \) | \( -167.2111 \) | \( -26.0985 \) | \( -82.8228 \) | \( -36.3675 \) | \( 13.3209 \) | \( 75.9727 \) |

According to the modified BIC criterion, the Kotz model with parameters \( T = 2, R = \frac{1}{2} \) and \( s = 1 \) is the most appropriate for the small group, instead the Kotz distribution with parameters \( T = 3, R = \frac{1}{2} \) and \( s = 1 \) models the large group. There is a very strong difference between these models and the classical normal in this experiment.

Let \( \mu_1 \) and \( \mu_2 \) be the mean polar shape of the small and large groups, respectively. We test equal mean shape under the best models, and the likelihood ratio (based on \( -2 \log \Lambda \approx \chi^2 \)) for the test \( H_0 : \mu_1 = \mu_2 \) vs \( H_a : \mu_1 \neq \mu_2 \), provides the p-value 0.99, which means that
there extremely evidence that the mean shapes of the two groups are equal.

For any elliptical model we can obtain the polar shape density, however a nontrivial problem appears, the 2t-th derivative of the generator model, which can be seen as a partition theory problem. For the general case of a Kotz model \( s \neq 1 \), and another models like Pearson II and VII, Bessel, Jensen-logistic, we can use formulae for these derivatives given by Caro-Lopera et al. (2009). The resulting densities have again a form of a generalised series of zonal polynomials which can be computed efficiently after some modification of existing works for hypergeometric series (see Koev and Edelman (2006)), thus the inference over an exact density can be performed, avoiding the use of any asymptotic distribution, and the initial transformation avoids the invariant polynomials of Davis (1980), which at present seems can not be computable for large degrees.

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