Hopf Algebroids, Bimodule Connections and Noncommutative Geometry

Aryan Ghobadi
Queen Mary University of London
School of Mathematics, Mile End Road
London E1 4NS, UK
Email: a.ghobadi@qmul.ac.uk

Abstract

We construct new examples of left bialgebroids and Hopf algebroids, arising from noncommutative geometry. Given a first order differential calculus $\Omega^1$ on an algebra $A$, with the space of left vector fields $\mathfrak{X}^1$, we construct a left $A$-bialgeroid $B\mathfrak{X}^1$, whose category of left modules is isomorphic to the category of left bimodule connections over the calculus. When $\Omega^1$ is a pivotal bimodule, we construct a Hopf algebroid $H\mathfrak{X}^1$ over $A$, by restricting to a subcategory of bimodule connections which intertwine with both $\Omega^1$ and $\mathfrak{X}^1$ in a compatible manner. Assuming the space of 2-forms $\Omega^2$ is pivotal as well, we construct the corresponding Hopf algebroid $D\mathfrak{X}$, for flat bimodule connections, and recover Lie-Rinehart Hopf algebroids as a quotient of our construction in the commutative case. We use these constructions to provide explicit examples of Hopf algebroids over noncommutative bases.

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1 Introduction

The relationship between Hopf algebroids and Hopf algebras is analogous to that of groupoids and groups and since the discovery of significant Hopf algebras or quantum groups in the 1980s, there were several attempts to define an analogous notion of Hopf algebroids or quantum groupoids [24, 39, 40]. Today, the different formulations of these structures are well understood [8], and there exists an extensive literature [7,9,16,19,34,35], generalising various properties of Hopf algebras to the setting of Hopf algebroids. Despite this, there continues to be a shortage of examples of Hopf algebroids with noncommutative base algebras. Since classically Lie algebroids and groupoids arise naturally in differential geometry [26], we choose to tackle this problem in the setting of noncommutative differential geometry [5]. In the same spirit, Lie-Rinehart algebras [32] are regarded as algebraic generalisations of Lie algebroids and their universal enveloping algebras provide a family of example of Hopf algebroids [21,22,31], with applications to differential geometry [17, 18] and differential equations [19]. However, the Lie-Rinehart construction is limited to the commutative setting, whereas we obtain a Hopf algebroid associated to any unital algebra $A$ equipped
with a pivotal first order differential calculus of 1-forms. From an algebraic perspective, Hopf algebroids and bialgebroids \([34, 36]\) lift the closed monoidal structure of the category of \(A\)-bimodules, over a possibly noncommutative algebra \(A\). Meanwhile, bimodule connections \([10]\) were introduced to provide a subcategory of connections over a noncommutative space, with a monoidal structure, which lifts that of \(A\)-bimodules. We construct the left bialgebroid representing this category in Theorem 3.5. Assuming the space of 1-forms is a pivotal bimodule, as defined in Section 4.2, we construct the Hopf algebroid representing a subcategory of bimodule connections which lifts the closed monoidal structure of the category of \(A\)-bimodules. When provided with a space of 2-forms which has a pivotal structure compatible with that of the space of 1-forms, we extend this construction to a Hopf algebroid representing flat bimodule connection. In the commutative setting, a Lie-Rinehart algebra with a finitely generated and projective space of vector fields contains all the aforementioned data and its corresponding Hopf algebroid can be recovered as a quotient of our construction in the flat case. In Sections 4.4 and 5.2, we utilise our constructions to provide several explicit examples of Hopf algebroids over noncommutative algebras.

While a single definition for bialgebroids has now been accepted, several definitions of Hopf algebroids have been explored. A bialgebra is called a Hopf algebra if it admits a linear endomorphism called the \textit{antipode}, which lifts the inner homs of \(\mathcal{VEC}\) to its module category. The corresponding generalisation for Hopf algebroids is that of Schauenburg \([34]\). However, a Schauenburg Hopf algebroid does not need to admit an antipode and an example of such a Hopf algebroid was presented in \([23]\). The alternative versions, which involve an antipode are that of Lu \([24]\) and Bőhm, and Szlachányi \([9]\), the latter of which are examples of Schauenburg Hopf algebroids. The Hopf algebroids constructed here satisfy Schauenburg’s axioms, however in Theorems 4.13 and 5.5, we present a criterion for each of our examples to admit invertible antipodes, in the sense of Bőhm, and Szlachányi. In Section 2.1, we review the relevant definitions of closed monoidal categories and the theory of Hopf algebroids.

The flavour of noncommutative geometry we employ here is that of noncommutative Riemannian geometry, as presented in \([5]\), which is somewhat different from, but not incompatible with, Connes’ more well known approach \([12]\) coming out of spectral triples and cyclic homology. The algebra of continuous functions on a manifold is replaced by an arbitrary algebra \(A\) and the additional data of 1-forms on the manifold is replaced by an \(A\)-bimodule \(\Omega^1\) and a linear map \(d : A \rightarrow \Omega^1\) satisfying the Leibnitz rule, as in Definition 2.4. To capture a more complete picture of geometry, we would require the additional data of higher differential forms. However, our constructions up to Section 5 only require a first order differential calculus. We review the relevant definitions and provide several examples of such structures in Section 2.2.

An important tool in geometry is to understand vector bundles over a manifold. The Serre-Swan theorem tells us that this is the same as looking at finitely generated projective modules over the algebra of smooth functions on the manifold. In differential geometry, one would like to understand differentiation on smooth bundles, which translates to viewing covariant derivatives on these modules. The algebra of smooth functions on a manifold is commutative, and any left module over this algebra can be viewed as a bimodule, with the same left action acting on the right. In particular, one can tensor connections over the algebra. Over a noncommutative algebra however, one must distinguish between left and right connections and there is no natural monoidal structure on either category. To overcome this issue, one must look at left (or right) bimodule connection which consist of a bimodule \(M\), instead of a left module, with a left connection \(\nabla : M \rightarrow \Omega^1 \otimes M\) and a bimodule map \(\sigma : M \otimes \Omega^1 \rightarrow \Omega^1 \otimes M\) called
an $\Omega^1$-intertwining, satisfying compatibility conditions which are presented in Definition 2.13. As demonstrated in [10], the category of left bimodule connections has a monoidal structure, with the tensor of two bimodules with such data having a natural left connection and a compatible $\Omega^1$-intertwining. Bimodule connections originally arose in [3, 14] and have continued to be of interest in noncommutative geometry [4, 3, 5, 15, 29, 30].

Classically, vector fields over the manifold are dual to the space of 1-forms. However, in the noncommutative case, the bimodule $\Omega^1$ can have a left dual bimodule $\chi^1$ or a right dual bimodule $\bar{\Omega}^1$. In [3], given compatible bimodule connections on $\Omega^1$ and $\chi^1$, the algebra $TX^1_A$ is defined by an associative product on $T_A\chi^1$, such that the action of elements in $X^\otimes n$, captures local geometry and the action of vector fields. In Proposition 6.15 of [5], it is demonstrated that the category of left $TX^1_A$-modules is isomorphic to the category of left connections over the calculus. Hence, as an algebra $TX^1_A$ is independent of the choice of bimodule connection on $\Omega^1$, up to isomorphism. We review this construction and the relevant definitions in Section 2.3.

In Section 3.2 we construct a left $A$-bialgebroid $BX^1_A$ whose category of left modules is isomorphic to the category of left bimodule connections, $A\mathcal{M}_A$. We first construct a smaller bialgebroid in Section 3.1 whose category of modules is isomorphic to the category of $A$-bimodules with $\Omega^1$-intertwinings, $A\mathcal{M}_A^{\Omega^1}$. We denote this algebra by $B(\Omega^1)$ and construct $BX^1_A$ as a quotient of the free product of $B(\Omega^1)$ and $TX^1_A$ by the relevant relations. In Section 3.4 we describe $BX^1_A$ by generators and relations for several differential calculi.

The authors of [3] conclude by stating that a bialgebroid or Hopf algebroid structure on $TX^1_A$ would be desirable, while a coproduct does not seem to be available. It is well-known that a Hopf algebra $H$, comes equipped with the structure of a commutative algebra in the center of the category of left $H$-modules. A similar phenomenon was conjectured in [3], since $TX^1_A$ was found to have a commutative algebra structure in the center of the monoidal category $A\mathcal{E}_A$. While $TX^1_A$ does not admit a bialgebroid structure, it is a subalgebra of the bialgebroid $BX^1_A$ whose representations form $A\mathcal{E}_A$. In Section 3.3 we recover the lax braiding making $TX^1_A$ an object in the monoidal center in [3], by restricting the coproduct of $BX^1_A$ to $TX^1_A$.

Although the category of left bimodule connections is monoidal, it does not lift the closed structure of $A\mathcal{M}_A$. In Section 4.1 we consider bimodule connections with invertible $\Omega^1$-intertwinings. In this case, left and right bimodule connections correspond (Remark 4.4) and it is the first step towards obtaining a closed monoidal category of connections. Consequently, we construct the bialgebroids $IB(\Omega^1)$ and $IB\chi^1$, which represent the category of bimodules with invertible $\Omega^1$-intertwinings and that of invertible bimodule connections, respectively. To obtain a closed monoidal category, we require $\Omega^1$ to be pivotal. We say a bimodule is pivotal if its left and right dual bimodules are isomorphic. In other words, the space of left vectorfields, $\chi^1$, and that of right vectorfields, $\bar{\Omega}^1$, are isomorphic. For a commutative algebra, any left module is a pivotal bimodule when considered as a bimodule. In Section 4.2 we show that several examples of differential calculi which are of interest, such as quiver calculi, bivariant calculi on Hopf algebras and calculi admitting a quantum Riemannian metric, all have a pivotal structure.

In Section 4.3 we construct a quotient of $IB(\Omega^1)$, $H(\Omega^1)$ so that it admits a bijective antipode. Any bimodule with an invertible $\Omega^1$-intertwining map has two induced intertwinings with bimodules $\chi^1$ and $\bar{\Omega}^1$, (41) and (40). Since $\Omega^1$ is pivotal, the additional relations present in $H(\Omega^1)$ make the induced $\chi^1$-intertwinings on $H(\Omega^1)$-
modules, inverses. We construct $H\mathcal{X}^1$ as the quotient of $IB\mathcal{X}^1$ by the same relations and observe that it admits a Hopf algebroid structure, Theorem 4.12. In Theorem 4.13 we demonstrate that $H\mathcal{X}^1$ admitting an antipode is equivalent to the existence of a suitable linear map $\gamma : \mathcal{X}^1 \to A$, which satisfies condition 46.

When provided with the space of 2-forms, $\Omega^2$, one can define the notion of curvature for connections and what it means for a connection to have zero curvature or to be flat. In Chapter 6 of [5], a quotient of $T\mathcal{X}^1_X$ called $\mathcal{D}_A$ is constructed to represent the category of flat connections. However, to obtain a monoidal category of flat bimodule connections one needs to assume that the $\Omega^1$-intertwinings of the connections extend to $\Omega^2$-intertwinings. After briefly reviewing this theory in Section 5 we construct the corresponding quotient of $H\mathcal{X}^1$ for flat bimodule connections and denote it by $\mathcal{D}X$. Theorem 5.4. In Theorem 5.5 we provide a criterion for when $\mathcal{D}X$ admits an invertible antipode in the sense of Böhm, and Szlachányi.

In Section 5.3 we review our construction in the commutative setting. A Lie-Rinehart algebra consists of a commutative algebra $A$ and a Lie algebra $(\mathcal{X}^1, [,])$, such that $\mathcal{X}^1$ is an $A$-module and $A$ is a $\mathcal{X}^1$-module satisfying additional compatibility conditions. When $\mathcal{X}^1$ is finitely generated and projective, with $\Omega^1$ as its dual module, the data of a Lie-Rinehart algebra translates exactly to $\Omega^1$ being a first order calculus over $A$ and the calculus extending to $\Omega^2 = \bigwedge^2(\Omega^1)$, where $\bigwedge^2(\Omega^1)$ is the exterior power of $\Omega^1$ as an $A$-module. We review this correspondence and show that the universal enveloping algebra of $(A, \mathcal{X}^1)$ is isomorphic to $\mathcal{D}_A$. More generally, if $A$ is commutative and $\Omega^1$ is a symmetric bimodule, $T\mathcal{X}^1_X$ has a natural Hopf algebroid structure. We remark that both Hopf algebroid structures of $T\mathcal{X}^1_X$ and $\mathcal{D}_A$, can be recovered as quotients of $H\mathcal{X}^1$ and $\mathcal{D}X$, in the commutative and Lie-Rinehart settings, respectively.

In Sections 3.4 and 4.4 we provide several examples of left bialgebroids and Hopf algebroids, respectively, in terms of generators and relations. For any finite quiver $\Gamma = (V, E)$, we construct a Hopf algebroid over the algebra $\mathcal{K}(V)$, which contains the quiver path algebra as a subalgebra. We describe the structure of $H\mathcal{X}^1$ over a base Hopf algebra, for an arbitrary bicovariant calculus and calculate an explicit example for the group algebra of the Dihedral group of order 6, $\mathcal{C}D_6$. Other examples include derivation calculi on any algebra and a specific inner calculus over the algebra of complex 2-by-2 matrices $M_2(\mathbb{C})$. In Section 5.2 we construct $\mathcal{D}X$ explicitly in the cases of finite quivers with no loops and $\mathcal{C}D_6$.

2 Preliminaries

Notation. Throughout this work, $\mathcal{K}$ will denote a field and $A$ an algebra over this field. When necessary we denote the multiplication of $A$ by $\cdot : A \otimes A \to A$ and otherwise we denote $a \cdot b$ by $ab$ for brevity, where $a, b \in A$. We use the notation $[a, b] = ab - ba$ for the commutator of two elements $a, b$. For a vectorspace $V$, $TV$ will denote the free associative algebra $\mathcal{K} \oplus V \oplus V \otimes_\mathcal{K} V \oplus \ldots$ over the vectorspace $V$. If $R$ and $S$ are two algebras, $R \ast S$ will denote the free product of associative algebras $R$ and $S$. We will denote actions of an algebra $A$ on its (left) module $M$, by $\cdot aM$, where $a \in A$ and $m \in M$, unless otherwise noted. We denote the category of $A$-bimodules by $A\mathcal{M}_A$ and the category of vectorspaces by $\mathcal{V}ECT$. For any algebra $R$ and an $R$-bimodule $M$, $TRM$ will denote the free monoid generated by $M$ in $A\mathcal{M}_A$, which is defined on the vectorspace

$$TRM = R \oplus M \oplus (M \otimes_R M) \oplus (M \otimes_R M \otimes_R M) \oplus \ldots$$
For a natural number $n$, we denote $M^\otimes_R M^\otimes_R \cdots \otimes_R M$ for $n$ copies of $M$, by $M^\otimes_R^n$. Throughout this work $\otimes$ will denote the tensor product over the algebra $A$ and $\otimes_K$ the tensor product over $K$. We use Sweedler’s notation for coproducts of coalgebras and $R|R$-corings $(C, \Delta, \epsilon)$: for an element $c$, $\Delta(c) = c_{(1)} \otimes c_{(2)}$, where the right hand side is a sum of elements of the form $c_{(1)} \otimes c_{(2)}$ in $C \otimes C$. We denote the 2-by-2 matrix with 1 in the $(i,j)$-th position and zeros elsewhere by $E_{ij}$. All sums $\sum_i$ will be taken over a free index $i$ with values in a finite set. We have omitted $\sum_i$ when the sum is taking place over dual bases arising from coevaluation maps and whenever such terms appear with free indeces, summation is implicit.

### 2.1 Bialgebroids and Hopf Algebroids

We briefly recall the theory of monoidal categories and refer the reader to [25] for additional details. We call $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ a monoidal category where $\mathcal{C}$ is a category, $1$ an object of $\mathcal{C}$, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ a bifunctor and $\alpha : (id_C \otimes id_C) \otimes id_C \to id_C \otimes (id_C \otimes id_C)$, $\lambda : 1 \otimes id_C \to id_C$ and $\rho : id_C \otimes 1 \to id_C$ natural isomorphisms satisfying coherence axioms as presented in Chapter VII of [25]. In what follows $\alpha, \lambda, \rho$ will all be trivial isomorphisms, hence we will avoid discussing them. The main examples of monoidal categories which we consider here, are the category of vectorspaces over a field and the category of bimodules over an algebra.

A functor $F : \mathcal{C} \to \mathcal{D}$ between monoidal categories is said to be (strong) monoidal if the exists a natural (isomorphism) transformation $F_2(-, -) : F(-) \otimes F(-) \to F(- \otimes -)$ and a (isomorphism) morphism $F_0 : 1 \otimes \to F(1)$ satisfying

\[
F_2(X \otimes Y, Z)(F_2(X, Y) \otimes id_{F(Z)}) = F_2(X, Y \otimes Z)(id_{F(X)} \otimes F_2(Y, Z))\alpha_{F(X), F(Y), F(Z)}
\]

\[
F(r)F_2(X, 1)id_{F(X)} = id_{F(X)} = F(l)F_2(1, X)id_{F(X)}
\]

where we have omitted the subscripts denoting the ambient categories, since they are clear from context. If $F$ has a left adjoint, it is said to be part of a comonoidal adjunction, and the resulting monad on $\mathcal{D}$ is called a bimonad. Although, we do not use bimonads directly, we are viewing bialgebroids as an example of bimonads and refer the reader to [8,11].

An algebra or monoid in a monoidal category $\mathcal{C}$ consists of a triple $(M, \mu, \eta)$, where $M$ is an object of $\mathcal{C}$ and $\mu : M \otimes M \to M$ and $\eta : 1 \otimes \to M$ are morphisms in $\mathcal{C}$ satisfying $\mu(id_M \otimes \eta) = id_M = \mu(\eta \otimes id_M)$ and $\mu(id_M \otimes \mu) = \mu(\mu \otimes id_M)\alpha_{M, M, M}$. A coalgebra or comonoid in $\mathcal{C}$ can be defined by simply reversing the arrows in the definition of a monoid.

For an object $X$ in a monoidal category $\mathcal{C}$, we say an object $^\vee X$ is a left dual of $X$, if there exist morphisms $ev_X : ^\vee X \otimes X \to 1$ and $coev_X : 1 \to X \otimes ^\vee X$ such that

\[
(ev_X \otimes id_{^\vee X})(id_{^\vee X} \otimes coev_X) = id_{^\vee X}, \quad (id_X \otimes ev_X)(coev_X \otimes id_X) = id_X
\]

In such a case, we call $X$ a right dual for $^\vee X$. Furthermore, a right dual of an object $X$ is denoted by $X^\vee$, with evaluation and coevaluation maps denoted by $ev_X : X \otimes X^\vee \to 1$ and $coev_X : 1 \to X^\vee \otimes X$, respectively. The category $\mathcal{C}$ is said to be left (right) rigid or autonomous if all objects have left (right) duals. If a category is both left and right rigid, we simply call it rigid. We call a category $\mathcal{C}$ left (right) closed if for any object $X$ there exists an endofunctor $[X, -]^l$ (resp. $[X, -]^r$) on $\mathcal{C}$ which is right adjoint to $- \otimes X$ (resp. $X \otimes -$). By definition $[X, -]^l, [X, -]^r : C^{op} \times C \to \mathcal{C}$ are
bifunctors. If a category is left and right closed, we call it closed. Observe that if $X$ has a left (right) dual $X^\vee$ (resp. $X^\wedge$), the functor $- \otimes X^\vee$ (resp. $X^\wedge \otimes -$) is left adjoint to $- \otimes X$ (resp. $X \otimes -$) and $X^\vee$ (resp. $X^\wedge$) is unique up to isomorphism. Furthermore, if $X$ has a left (right) dual, $X^\vee \cong [X, 1_\mathbb{C}]^l$ (resp. $X^\wedge \cong [X, 1_\mathbb{C}]^r$). We have adopted the notation of \cite{1} here, and what we refer to as a left closed structure is referred to as a right closed structure in various other sources \cite{19, 34}.

It is well known that strong monoidal functors preserve dual objects i.e. $F(X^\vee) \cong F(X)$ with $F_0 F(ev) F_2 (X^\vee, X)$ and $F_2^{-1} (X, X^\vee) F(coev) F_0^{-1}$ acting as the evaluation and coevaluation morphisms for $F(X^\vee)$. For left (right) closed monoidal categories $\mathcal{C}$ and $\mathcal{D}$, we say a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is left (right) closed if the canonical morphism $F(X, Y)_\mathcal{D} \to [F(X), F(Y)]_{\mathcal{D}}$ is an isomorphism for any pair of objects $X, Y$ in $\mathcal{C}$.

Before introducing bialgebroids, we briefly recall the theory of Hopf algebras. An algebra $A$ is said to have a bialgebra structure if $(A, \delta, \nu)$ is a coalgebra in the category of vectorspaces satisfying $\delta(a_1) a_2 = (aa')_1 \otimes (aa')_2$ for any $a, a' \in A$, where $\delta(a) = a_1 \otimes a_2$ by Sweedler’s notation. There are three additional axioms involving 1 and $\nu$, which can be found in Chapter 4 \cite{8}. The coproduct $\delta$ of a Hopf algebra is usually denoted by $\Delta$, but we choose to reserve $\Delta$ for the coproduct of bialgebroids. The category of left $A$-modules for a bialgebra $A$ has a natural monoidal structure which makes the forgetful functor $A \mathcal{M} \to \mathcal{VEC}$ a strong monoidal functor. A bialgebra is called a Hopf algebra, if there exists an anti-multiplicative linear map $S : A \to A$ satisfying $S(a_1) a_2 = a_1 S(a_2) = \nu(a) 1_A$ for any $a \in A$. The map $S$ is called the antipode and exists if and only if the forgetful functor $A \mathcal{M} \to \mathcal{VEC}$ is left closed. Moreover, $S$ is bijective if and only if the forgetful functor is closed.

For an algebra $A$, the opposite algebra $A^{op}$ is the algebra structure defined on $A$ by $(\pi(b)a) = b\pi(a)$, where we denote elements of the opposite ring with a line above i.e $a, b \in A$ and $\pi, \bar{\pi} \in A^{op}$. It is a well-known fact that $A$-bimodules correspond to left $A \otimes_K A^{op}$-modules, where $A^{op} = A^{op} \otimes_K A^{op}$ is called the enveloping algebra of $A$. More concretely, there exists an equivalence of categories, between the category of $A$-bimodules $A \mathcal{M}$ and that of left $A^{op}$-modules $A^{op} \mathcal{M}$. Hence, we use $A^{op} \mathcal{M}$ and $A \mathcal{M}$ interchangeably. We will denote elements of $A^{op} = A \otimes_K A^{op}$ by $\overline{ab}$ where $a \in A$ and $\overline{b} \in A^{op}$.

The category of $A$-bimodules has a natural monoidal structure by tensoring bimodules over the algebra $A$, denoted by $\otimes$, and the algebra $A$ regarded as an $A$-bimodule acting as the unit object. It is well known that a bimodule has a left (right) dual in the monoidal category $A \mathcal{M}$ if and only if it is finitely generated and projective, fgp for short, as a right (left) $A$-module. A straightforward proof is presented in Proposition 3.8 of \cite{5}. In particular, $A \mathcal{M}$ is closed with

$$
[M, N]^l := \text{Hom}_A(M, N), \quad [\overline{a\overline{b}}](m) = a f(bm), \quad f \in \text{Hom}_A(M, N)
$$

$$
[M, N]^r := A \text{Hom}(M, N), \quad [\overline{a\overline{g}}](m) = g(ma)b, \quad g \in A \text{Hom}(M, N)
$$

where $\overline{ab} \in A^{op}$ and $A \text{Hom}(M, N)$ and $A \text{Hom}(M, N)$ denote the vectorspaces of right and left $A$-module morphisms from $M$ to $N$, respectively. Explicitly, the units and counits of the adjunctions for the left and right closed structures, are given by

$$
\varphi^M_N : N \longrightarrow A \text{Hom}(M, N \otimes M), \quad \varphi^M_N : \text{Hom}_A(M, N) \otimes M \longrightarrow N
$$

$$
[f] : \text{Hom}_A(M, N \otimes M) \longrightarrow N
$$

$$
\varepsilon^M_N : \text{Hom}_A(M, N \otimes M) \longrightarrow N
$$

$$
\varepsilon^M_N : f \otimes m \longmapsto f(m)
$$
any additive left adjoint functor
the reference, but here we refer to [8]. The Eilenberg-Watts theorem [37] tells us that $M$ for any pair of $A$-bimodules $M$ and $N$. Consequently, for a right or left fgp bimodule $M$, we identify $\gamma M$ by $\text{Hom}_A(M, A)$ and $M^\vee$ by $\text{Hom}(M, A)$.

The notation for Hopf algebroids and bialgebroids varies quite a bit depending on the reference, but here we refer to [8]. The Eilenberg-Watts theorem [37] tells us that any additive left adjoint functor $F : A^e M \to A M$ is isomorphic to a functor $A \otimes A^e -$, where $B$ is an $A^e$-bimodule. For an $A^e$-bimodule $B$ we denote the functor $A \otimes A^e -$ by $B \boxtimes - : A M A \to A M A$. This functor absorbs the bimodule structure via its right $A^e$-action and produces new bimodule actions via its left $A^e$-action. Explicitly, for an $A$-bimodule $M$:

$$B \boxtimes M = B \otimes_K M / \{(br) \otimes_K m - b \otimes_K (rms) \mid m \in M, r, s \in A, b \in B\}$$

$$r(b \boxtimes m)s = (r \boxtimes b) \boxtimes m \quad \forall m \in M, \forall r, s \in A, \forall b \in B$$

An $A^e$-bimodule $B$, can be considered as an $A$-bimodule either by its right or left $A^e$-action, and we denote the latter $A$-bimodule by $|B|$. We continue to adapt the notation of [8] and recall the following definitions from Chapter 5.

**Definition 2.1.** Let $A$ be an algebra and $B$ an $A^e$-bimodule.

(I) An $A^e$-ring structure on $B$ consists of a $K$-algebra structure $(\mu, 1_B)$ on $B$ with an algebra homomorphism $\eta : A^e \to B$, such that the $A^e$-bimodule structure on $B$ is induced by the algebra homomorphism $\mu(\eta \otimes_K \text{id}_B)$ coincides with the left action of $A^e$ and $\mu(\text{id}_B \otimes_K \eta)$ with the right action of $A^e$. Equivalently, an $A^e$-ring structure on $B$ consists of $A^e$-bimodule maps $\mu_{AA} : B \otimes_A B \to B$ and $\eta_{A^e} : A^e \to B$, which provide $B$ with the structure of a monoid in the category of $A^e$-bimodules.

(II) An $A|A|$-coring structure on $B$ consists of bimodule maps $\Delta : |B| \to |B| \otimes |B|$ and $\epsilon : |B| \to A$ satisfying

$$b_{(1)} \otimes (b_{(2)})_{(1)} \otimes (b_{(2)})_{(2)} = (b_{(1)})_{(1)} \otimes (b_{(1)})_{(2)} \otimes b_{(2)} \quad (2)$$

$$\epsilon(b_{(1)})b_{(2)} = b = \epsilon(b_{(2)})b_{(1)} \quad (3)$$

$$\Delta(br) = (b_{(1)}r) \boxtimes b_{(2)\bar{r}} \quad (4)$$

$$\epsilon(br) = \epsilon(b) \quad (5)$$

for any $b \in B$ and $r, s \in A$, where $\Delta(b) = b_{(1)} \otimes b_{(2)}$ is denoted by Sweedler's notation. Conditions (2), (3) are equivalent to $(|B|, \Delta, \epsilon)$ being a comonoid in the category of $A$-bimodules.

(III) A left $A$-bialgebroid structure on $B$ consists of an $A^e$-ring structure $(\mu, \eta)$ and an $A|A|$-coring structure $(\Delta, \epsilon)$ on $B$ satisfying

$$(bb')_{(1)} \otimes (bb')_{(2)} = b_{(1)}b'_{(1)} \otimes b_{(2)}b'_{(2)}, \quad (6)$$

$$\Delta(1_B) = 1_B \otimes 1_B \quad (7)$$

$$\epsilon(1_B) = 1_A \quad (8)$$

$$\epsilon(bb') = \epsilon(b \epsilon(b')) = \epsilon(\delta(b')) \quad (9)$$
for any \( b, b' \in B \), where \( 1_B = \eta(1_A) \).

From the above axioms for an \( A \rightrightarrows A \)-coring \( B \), one can deduce that the image of \( \Delta \) lands in

\[
B \times_A B := \left\{ \sum_i b_i \otimes b'_i \in |B \otimes |B| \middle| \sum_i b_i \pi \otimes b'_i = \sum_i b_i \otimes b'_i a, \ a \in A \right\}
\]

Bialgebroids are often defined with reference to \( B \times_A B \), the Takeuchi \( \times \)-product \([35]\), and often called \( \times \)-bialgebras. The equivalence of the above definition and the more popular variation is present in both \([8, 9]\).

Any \( A' \)-ring \( B \) comes equipped with an algebra map \( \eta : A' \rightarrow B \), therefore by restriction of scalars, any \( B \)-module is equipped with an \( A \)-bimodule structure and there exists a forgetful functor \( U : B \rightrightarrows A \rightarrow A \rightrightarrows A \). In fact, \( B \rightrightarrows B \rightrightarrows A \rightarrow A \rightrightarrows A \) form a free/forgetful adjunction and a left action of \( B \) on a bimodule \( M, B \rightrightarrows M \rightarrow M \) factors through an \( A \)-bimodule map \( B \rightrightarrows M \rightarrow M \). In this setting, \( B \) has the additional structure of an \( A \)-bialgebroid, if and only if \( U \) is strong monoidal. In particular, the map

\[
\Delta_{M,N} : B \rightrightarrows (M \otimes N) \rightarrow (B \rightrightarrows M) \otimes (B \rightrightarrows N)
\]

is well-defined and a bimodule map, for any pair of bimodules \( M, N \). Hence, if \((M, \rightrightarrows_M)\) and \((N, \rightrightarrows_N)\) are \( B \)-modules, the \( B \)-action on \( M \otimes N \) is defined by the composition \( \rightrightarrows_M \otimes \rightrightarrows_N \Delta_{M,N} \). Moreover, the counit \( \epsilon \) provides the monoidal unit \( A \), with a \( B \)-action \( \epsilon_0 : B \rightrightarrows A \rightarrow A \) defined by \( b \rightrightarrows a \mapsto \epsilon(ba) \).

We must point out that the theory described above is not symmetric. A right \( A \)-bialgebroid structure on \( B \) arises when we ask the category of right \( B \)-modules to be monoidal so that the forgetful functor \( M_B \rightarrow A \rightrightarrows A \) is strong monoidal.

There have been several variations of the Hopf condition for bialgebroids to mimic the Hopf condition for bialgebras. The choice which interests us, is to say a bialgebroid \( B \) is Hopf when the forgetful functor \( B \rightrightarrows A \rightrightarrows A \) is closed. This would be the case for Schauenburg Hopf algebroids as introduced in \([34]\). A class of such Hopf algebroids are those introduced by Böhm-Szlachányi \([9]\), which admit an antipode-like map.

**Definition 2.2.**

(I) A Schauenburg Hopf algebroid or \( \times \)-Hopf algebra structure on \( B \) consists of an \( A \)-bialgebroid structure as above, such that the maps

\[
\beta : B \rightrightarrows_A \rightarrow B \rightrightarrows B \quad \quad \quad \beta : B \rightrightarrows B \rightarrow B \rightrightarrows B
\]

\[
b \rightrightarrows_B b' \mapsto b_{(1)} \otimes b_{(2)} b' \quad \quad \quad b \rightrightarrows b' \mapsto b_{(1)} b' \rightrightarrows b_{(2)}
\]

where we define

\[
B \rightrightarrows_B = B \rightrightarrows_K B / \{ b \rightrightarrows_K b' - b \otimes_K s b' | b, b' \in B, s \in A^{op} \}
\]

\[
B \rightrightarrows_B = B \rightrightarrows_K B / \{ b \rightrightarrows_K b' - b \otimes_K r b | b, b' \in B, r \in A \}
\]

\[
B \rightrightarrows_B = B \rightrightarrows_K B / \{ b \rightrightarrows_K b' - b \otimes_K s b | b, b' \in B, s \in A \}
\]

are invertible.

(II) A Böhm-Szlachányi Hopf algebroid structure on \( B \) consists of an \( A \)-bialgebroid structure as above and an anti-algebra automorphism \( S : B \rightarrow B \) satisfying

\[
S(\eta(\pi)) = \eta(a)
\]
generalized calculus is often called a require the surjectivity condition. If \( \ker(A) \) for any \( a, b \) for all \( b \in B \) and \( \eta \in A^op \).

**Note.** In what follows, we will simply write \( a \) and \( \eta \) to refer to \( \eta(a) \) or \( \eta(\eta) \) as the images of elements \( a \in A \) and \( \eta \in A^op \) in an \( A^e \)-ring \( B \). This is not an abuse of notation, since the multiplication and the module structure of an \( A^e \)-ring \( B \) coincide.

If \( B \) is a Schauenburg Hopf algebroid and \( \beta, \vartheta \) are invertible, we denote \( \beta^{-1}(b \circ 1) = b_{[+]} \otimes_{A^{op}} b_{[-]} \) and \( \vartheta^{-1}(1 \circ b) = b_{[+]} \circ b_{[-]} \). In this case, the closed structure of \( A \mathcal{M}_A \) is lifted to \( B \mathcal{M} \) via the following \( B \)-actions:

\[
B \boxtimes \text{Hom}_A(M, N) \to \text{Hom}_A(M, N) \quad B \boxtimes \text{Hom}_A(M, N) \to A \text{Hom}(M, N)
\]

\[
b \boxtimes f \mapsto (m \mapsto b_{(+)}f(b_{(-)}m)) \\
b \boxtimes g \mapsto (m \mapsto b_{(+)}g(b_{(-)}m))
\]

for any pair of \( A \)-bimodules \( M, N \). Equivalently, \( \vartheta^{-1} \) and \( \beta^{-1} \) can be recovered, if one has a well-defined actions of \( B \) on the inner homs, such that the units and counits presented in (1) are \( B \)-module morphisms. For a left bialgebroid \( B \), this is precisely what it means for the forgetful functor \( B \mathcal{M} \to A \mathcal{M}_A \) to be closed.

If \( B \) is a Böhm-Szlachányi Hopf algebroid with an invertible antipode \( S : B \to B \) then the inverses of \( \beta, \vartheta \) are given by

\[
\beta^{-1}(b \circ b') = S^{-1}(S(b)(b')) \otimes_{A^{op}} S(b) b' \tag{13}
\]

\[
\vartheta^{-1}(b \circ b') = S^{-1}(b')(b) \otimes S^{-1}(b')(b) \tag{14}
\]

Finally, we refer the reader to Chapter 5 of [5] and [9] for further details on these elementary facts. We conclude by presenting the following Theorem which motivates our work when looking at the category of bimodule connections:

**Theorem 2.3.** [35] For an algebra \( A \) and an abelian monoidal category \( C \), if \( F : C \to A \mathcal{M}_A \) is an additive functor with a left adjoint \( G \), such that \( FG : A \mathcal{M}_A \to A \mathcal{M}_A \) has a right adjoint, then \( F \) is (closed) strong monoidal if and only if \( C \) is equivalent to \( B \mathcal{M} \) for a left (Hopf) bialgebroid \( B \).

From this point onwards, we only consider left bialgebroids and left Hopf algebroids, when referring to bialgebroids or Hopf algebroids.

### 2.2 Noncommutative Geometry Framework and Examples

Here we provide a brief introduction to noncommutative Riemannian geometry as presented in [5]. In particular, all details and proofs relating to the examples presented here can be found in Chapter 1 of [5].

**Definition 2.4.** By a (first order) differential calculus over an algebra \( A \), we refer to an \( A \)-bimodule \( \Omega^1 \) along with a linear map \( d : A \to \Omega^1 \) satisfying \( d(ab) = (da)b + a(db) \), for any \( a, b \in A \).

In [5] and most of the literature, the additional condition \( \Omega^1 = \text{Span}_K \{ adb \mid a, b \in A \} \) (the surjectivity condition) is also required. If this property does not hold, \( \Omega^1, d \) is often called a generalized calculus [34]. However, in what will follow, we do not require the surjectivity condition. If \( \kappa(d) = K \cdot 1 \), where \( 1 \) is the unit of algebra
A, we say the calculus is connected. Every algebra has a natural largest connected differential calculus, namely the universal calculus \( \Omega^1_{uni} = \ker(\cdot) \subset A \otimes_k A \), with differential \( da = 1 \otimes a - a \otimes 1 \). Any first order differential calculus satisfying the surjectivity condition arises as a quotient of the universal calculus.

**Example 2.5.** [Classical Example] Let \( M \) be a smooth manifold, \( A = C^\infty(M) \) the algebra of smooth functions on \( M \), \( \Omega^1 \) the space of 1-forms and \( d : A \to \Omega^1 \) the usual differential on smooth functions. In this case, \( A \) is commutative and \( \Omega^1 \) has a bimodule structure where the left and right module structure agree.

We say a differential calculus is called inner if there exists an element \( \theta \in \Omega^1 \) such that \( da = \theta(a) \). Notice that even over a commutative algebra \( A \), inner calculi are only possible because we are not requiring \( \Omega^1 \) to have the same left and right module structure.

**Example 2.6.** [Finite Quivers] Let \( V \) be a finite set, and \( A = K(V) = \{ f : V \to K \} \) be the algebra of functions on \( V \). There exists a natural basis for \( A \), namely \( \{ f_p \mid p \in V \} \), where \( f_p(q) = \delta_{p,q} \) for any \( p, q \in V \). In fact, \( A \) is the finite dimensional algebra with a complete set of idempotents \( T := \{ f_p \mid p \in V \} \) as its basis and is thereby semisimple. Any \( A \)-bimodule \( M \) decomposes as \( M = \bigoplus_{p \in V} V_p \) such that \( f_p m f_q = \delta_{p,q} \delta_{q,m} m \), for \( m \in V_p \). Hence, a bimodule over \( A \) corresponds to the choice of a directed graph or quiver, on the set of points \( V \): for a set of edges \( E \subset V \times V \), and an edge \( e \in E \), we denote its corresponding basis element in \( \Omega^1 \) by \( \epsilon_e \), so that

\[
\rho \Omega^1 q = \text{Span}_K \{ \epsilon_e \mid s(e) = p, t(e) = q \}
\]

where \( s, t : E \to V \) are the usual source and target maps. The differential structure is defined by

\[
d f = \sum_{e \in E} [f(t(e)) - f(s(e))] \epsilon_e
\]

The calculus is inner with \( \theta = \sum_{e \in E} \epsilon_e \). The surjectivity condition holds if and only if no edge has the same source and target and two points have at most one edge between them.

If \( \Omega^1 \) is a left (right) free module over \( A \) with a basis of cardinality \( n \), we say \( \Omega^1 \) is left (right) parallelised with left (right) cotangent dimension \( n \). If \( \Omega^1 \) is both left and right parallelised, we call it simply parallelised. Although our work does not require \( \Omega^1 \) to be parallelised, such bimodules facilitate our calculations when producing examples.

**Example 2.7.** [Derivation Calculus] First order differential calculi on \( \Omega^1 = A \), regarded as an \( A \)-bimodule, are just derivations \( d : A \to A \) i.e. endomorphisms \( d \) satisfying the Leibnitz rule as presented in Definition 2.4.

**Example 2.8.** [\( M_2(\mathbb{C}) \)] The complete moduli of surjective first order calculi for the algebra of 2-by-2 matrices \( A = M_2(\mathbb{C}) \) has been described in Example 1.8 of [38]. An example of such calculi is \( \Omega^1 = M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \) as a free bimodule, equipped with an inner calculus by \( \theta = E_{12} \oplus E_{21} \).

It is well known that bicovariant calculi [38] or Hopf bimodules over Hopf algebras are parallelised. In particular, a Hopf module \( \Omega^1 \) over a Hopf algebra \( (A, \delta, \nu, s) \) is free as a right \( A \)-module and decomposes as \( \Omega^1 \cong A \otimes_k A \), for a particular subspace \( A \subset \Omega^1 \). Under this decomposition, the right \( A \)-action arises from \( A \), solely. The left
A-action on $\Omega^1$ arises by considering $\Lambda \otimes K A$ as the tensor of two left $A$-modules, where $\Lambda$ has an induced left $A$-action $\triangleright$ defined by $a \triangleright \lambda = a_{(1)} \lambda s(a_{(2)})$, for any $a \in A$ and $\lambda \in \Lambda$. Consequently, the left action of $\Omega^1$ translates to

$$b(\lambda \otimes K a) = b_{(1)} \triangleright \lambda \otimes K b_{(2)} a = b_{(1)} \lambda s(b_{(2)}) \otimes K b_{(3)} a$$

As we will see in Section 4.2, $\Omega^1$ is free as a left $A$-module, in a symmetric manner. A Hopf bimodule has compatible $A$-bimodule and $A$-bicomodule structures, which give rise to the above structure. In particular, $\Lambda = \{\omega \in \Omega^1 \mid \delta_R(\omega) = \omega \otimes K 1\}$, where $\delta_R$ denotes the right $A$-coaction. The left coaction of $\Omega^1$ restricts to $\Lambda$ and along with the left action $\triangleright$, make $\Lambda$ a left Yetter-Drinfeld module. A first order differential calculus $\Omega^1$ over a Hopf algebra is called a bicovariant differential calculus if $\Omega^1$ is a Hopf bimodule. Bicovariant differential calculi which satisfy the surjectivity condition are in bijection with $Ad$-stable left ideals of $A^+ = \ker(\nu)$. For further detail on bicovariant calculi, we refer the reader to Section 2.3 of [5] and conclude with a particular example of bicovariant calculi over a Hopf algebra.

**Example 2.9.** [Group Algebra [28]] Given a group $G$, a left module $G$-module $(\Lambda, \triangleright)$ and a 1-cocycle $\zeta \in Z^1(G, \Lambda)$ i.e. a map $\zeta : \mathbb{K} G \rightarrow \Lambda$ such that $\zeta(gh) = g \triangleright \zeta(h) + \zeta(g)$, there is a corresponding differential calculus $\Omega^1 = \Lambda \otimes K G$ over the group algebra $\mathbb{K} G$ with the differential defined by

$$d(g) = \zeta(g) \otimes_K g$$

The calculus is inner if and only if $\zeta$ is exact i.e. there exists an element $\theta \in \Lambda$ such that $\zeta(g) = g \triangleright \theta - \theta$. When $G$ is finite and $|G|$ is invertible in $K$, then the calculus is always inner with $\theta = \frac{1}{|G|} \sum_{g \in G} \zeta(g)$.

When looking at the classical case, first order differential calculus only contains the data for 1-forms. To capture a true generalisation of classical geometry one must consider the space of all differential forms.

**Definition 2.10.** A differential graded algebra or DGA on an algebra $A$ is a graded algebra $(\Omega^\bullet = \oplus_{n \geq 0} \Omega^n, \wedge)$ with $\Omega^0_0 = A$ and a differential $d : \Omega^n \rightarrow \Omega^{n+1}$ such that $d^2 = 0$ and $d(\omega \wedge \rho) = (d\omega) \wedge \rho + (-1)^n \omega \wedge (d\rho)$, where $\rho \in \Omega^\bullet$ and $\omega \in \Omega^n$, hold for all $n \geq 0$.

If a DGA is generated by $A$ and $dA$, we refer to it as an exterior algebra on $A$. Observe that given a DGA $(\Omega^\bullet, \wedge)$ on $A$, $(\Omega^1, d)$ form the data for a first order differential calculus. Conversely, every first order calculus $(\Omega^1, d)$ can be extended to an exterior algebra on $A$ called its maximal prolongation, such that any exterior algebra on $A$, which agrees with $(\Omega^1, d)$ on its first grading and differential, is a quotient of the maximal prolongation by a differential ideal. Further details can be found in Section 1.5 of [5].

### 2.3 Connections

**Definition 2.11.** If $(\Omega^1, d)$ is a differential calculus on the algebra $A$, by a left connection or left covariant derivative, we mean a left $A$-module $M$ and a linear map $\nabla : M \rightarrow \Omega^1 \otimes M$ satisfying

$$\nabla(am) = a\nabla(m) + da \otimes m$$

for all $a \in A$ and $m \in M$.
A right connection can be described similarly, as a right $A$-module $M$ with a linear map $\nabla : M \to M \otimes \Omega^1$ satisfying $\nabla(ma) = \nabla(m)a + m \otimes da$. The category of left (right) connections on a differential calculus which has left (right) connections $(M, \nabla_M)$ as objects and left (right) module maps $f : M \to N$ satisfying $(\text{id}_M \otimes f)\nabla_M = \nabla_N f$ (resp. $(f \otimes \text{id}_M)\nabla_M = \nabla_N f$), as morphisms between $f : (M, \nabla_M) \to (N, \nabla_N)$, is denoted by $\mathcal{A}\mathcal{E}$ (resp. $\mathcal{E}_A$).

A natural question which arises is when can one describe $\mathcal{A}\mathcal{E}$ as modules over an algebra. This question was answered in Chapter 6 of [3]. When $\Omega^1$ is right fgp, we denote $X^1 := \Omega^1$ with $ev : X^1 \otimes \Omega^1 \to A$ and $\text{coev} : A \to \Omega^1 \otimes X^1$ as the respective evaluation and coevaluation maps for dual bimodules, as described in Section 2.1. The bimodule $X^1$ can be thought of as the space of vector fields on the noncommutative space, since it is dual to the space of 1-forms. In this setting, $\mathcal{A}\mathcal{E} \equiv T_{X^1^*} M$, where $T_{X^1^*}$ is the associative algebra defined as

$$T_{X^1^*} = A \ast T^1_X / \langle ax - xa, x\bullet a - xa - ev(x, da) \mid a \in A, x \in X^1 \rangle$$

where $\bullet$ denotes the associative product in $A \ast T^1_X$ and a left $T_{X^1^*}$-module $M$ has a left $A$-module structure by restriction of scalars. Hence, the action of $T_{X^1^*}$ on $M$ restricts to a map $\triangleright : T_{X^1^*} \otimes M \to M$ and the corresponding left connection $\nabla : M \to \Omega^1 \otimes M$ is defined by

$$\nabla = (\text{id}_M \otimes \triangleright)(\text{coev} \otimes \text{id}_M)$$

Conversely, any left connection $(M, \nabla)$ induces an action of $T_{X^1^*}$ on $M$, with the action of $A$ agreeing with the left $A$-module structure on $M$ and the action of elements of $X^1$ being defined by $(ev \otimes \text{id}_M)(\text{id}_{X^1} \otimes_X \nabla)$.

**Remark 2.12.** The ideal quotiented out from $A \ast T^1_X$ demonstrates that we can describe $T_{X^1^*}$ via an associative product on $T_{X^1}$. We have an isomorphism of vector spaces

$$X^1 \otimes_X X^1 \cong (X^1 \otimes X^1) \oplus \text{Span}\{xa \otimes y - x \otimes_K ay \mid x, y \in X^1, a \in A\} \quad (18)$$

If $x \otimes_K y = \sum_i x_i \otimes_K y_i \oplus \sum_j (w_j a_j \otimes_K z_j - w_j \otimes_K a_j z_j)$ by the above decomposition, then $x \bullet y = \sum_i x_i y_i + \sum_j ev(w_j, da_j)z_j$ in $T_{X^1^*}$. Extending this idea to iterated products of elements of $X^1$, we can organise $T_{X^1^*}$ as an associative product on $T_{X^1}$. In [3], $T_{X^1}$ is presented as associative product on the vector space $T_{X^1}$ to begin with. However, the multiplication of elements of $X^1 \otimes m$ and $X^1 \otimes n$ are defined iteratively, by requiring $\Omega^1$ and $X^1$ to have compatible bimodule connections. This description of $T_{X^1^*}$ is meant to encode the classical action of vector fields. Since we are only interested in $T_{X^1^*}$ as an algebra and $T_{X^1^*}$ is independent of the choice of bimodule connection on $X^1$, up to isomorphism, the above definition is satisfactory. But we must emphasise that arranging $T_{X^1^*}$ as a product on $T_{X^1}$, as above, will not produce the same product as the method of [3] via bimodule connection, but an isomorphic one.

**Definition 2.13.** If $(\Omega^1, d)$ is a differential calculus on the algebra $A$, by a left bimodule connection, we mean an $A$-bimodule $M$ and a linear map $\nabla : M \to \Omega^1 \otimes M$ such that there exists a bimodule map $\sigma : M \otimes \Omega^1 \to \Omega^1 \otimes M$ satisfying

$$\nabla(am) = a \nabla(m) + da \otimes m, \quad \nabla(ma) = \nabla(m)a + \sigma(m \otimes da)$$

for all $a \in A$ and $m \in M$. 
A right bimodule connection is defined symmetrically as a bimodule $M$ with a right connection $\nabla$ and a left $\Omega^1$-intertwining $\sigma : \Omega^1 \otimes M \to M \otimes \Omega^1$ satisfying $\nabla(a m) = a \nabla(m) + \sigma(da \otimes m)$ for all $m \in M$ and $a \in A$. Observe that a left bimodule connection structure on a bimodule does not imply the existence of a right bimodule connection structure. The category of left bimodule connections on a differential calculus, which has left bimodule connections $(M, \nabla_M, \sigma)$ as objects and bimodule maps $f : M \to N$ satisfying $(\text{id}_N \otimes f)\nabla_M = \nabla_N f$ and $\sigma_N(f \otimes \text{id}_N) = (\text{id}_M \otimes f)\sigma_M$ as morphisms $f : \alpha \to \beta$, is denoted by $\underline{A}\mathcal{E}_A$. The category of right bimodule connections is defined symmetrically and denoted by $\underline{A}\mathcal{E}_A$.

For a surjective calculus, a triple $(M, \nabla, \sigma)$ being a left bimodule connection is a property for a given bimodule $M$ with a left connection $\nabla$ and $\sigma$ is not additional data. Although we do not focus on surjective calculi, we comment on the features of our construction in the surjective setting in Remark 3.6. In the classical setting, where $A$ is a commutative algebra and we regard any left module as a bimodule with the right action coinciding with the left action, every left connection is a left bimodule connection.

The benefit of working with bimodule connections is that the category of bimodule connections comes equipped with a forgetful functor $U : \underline{A}\mathcal{E}_A \to \underline{A}M_A$ which sends a triple $(M, \nabla_M, \sigma_M)$ to its underlying bimodule $M$. Furthermore, the described monoidal structure on $\underline{A}\mathcal{E}_A$ applies the usual bimodule tensor product on the underlying bimodules of the bimodule connections. In other words, $U$ is strong monoidal. By Theorem 2.3, $\underline{A}\mathcal{E}_A$ can be written as the category of modules over a bialgebroid if and only if it is abelian and $U$ is co-continuous and has a left adjoint. This is the case when $\Omega^1$ is right fgp.

## 3 Bialgebroids Representing Bimodule Connections

Before we construct the bialgebroid representing $\underline{A}\mathcal{E}_A$, we must look at the category of bimodules which intertwine with $\Omega^1$ and construct the bialgebroid representing this category.

### 3.1 Category of Intertwining Modules

Let $\Omega^1$ be a right fgp $A$ bimodule and $\mathcal{X}^1$ be its left dual with $\text{coev} : A \to \Omega^1 \otimes \mathcal{X}^1$ and $\text{ev} : \mathcal{X}^1 \otimes \Omega^1 \to A$ as described in Section 2.1. Denote $\text{coev}(1) = \sum_i \omega_i \otimes x_i$ so that $\sum_i \omega_i \otimes x_i = \sum_i \omega_i \otimes x_i a = \text{coev}(a)$ holds for any $a \in A$. 

Proposition 3.2. The algebra $\sigma$ map

Proof.

Definition 3.1. For an $A$ bimodule $\Omega^1$, we define the category of $\Omega^1$-intertwined bimodules to have pairs $(M, \sigma_M)$, where $M$ is an $A$-bimodule along with a bimodule map $\sigma_M : M \otimes \Omega^1 \to \Omega^1 \otimes M$, as objects and $f : M \to N$ bimodule maps satisfying $\sigma_N(f \otimes \text{id}_{\Omega^1}) = (\text{id}_{\Omega^1} \otimes f)\sigma_N$ as morphisms. We denote this category by $A\mathcal{M}_A^{\Omega^1}$.

Let $\mathfrak{M} = \mathfrak{X}^1 \otimes \mathfrak{K} \Omega^1$, then $\mathfrak{M}$ has a $A^e$-bimodule structure:

$$\overline{ab}(x, \omega)b\overline{a} = (AXB, B\omega A')$$

for any $a, a', b, b' \in A$, where we denote arbitrary elements of $\mathfrak{M}$ by $(x, \omega)$. Hence, define $B(\Omega^1) := T_A(\mathfrak{X}^1 \otimes \mathfrak{K} \Omega^1)$ as an algebra and denote its multiplication by $\bullet$ so that

$$(a \bullet (x, \omega)) = (ax, \omega), \quad (x, \omega) \bullet a = (xa, \omega) \quad (19)$$

$$\overline{a}(x, \omega) = (x, aw), \quad \overline{a}(x, \omega) = (x, aw) \quad (20)$$

hold for $(x, \omega) \in \mathfrak{M}$ and $a \in A$. Equivalently, $B(\Omega^1)$ is isomorphic to the quotient of the algebra $T(\mathfrak{M} \oplus A)$ by the ideal generated by the set of relations (19) and (20), for all $(x, \omega) \in \mathfrak{M}$ and $a \in A$.

To obtain a bialgebroid structure on $B(\Omega^1)$, we define the coproduct and counit for elements of $A^e$ and $\mathfrak{M}$, and extend them multiplicatively to $B(\Omega^1)$ by $\Delta(m \bullet n) = m(1) \bullet m(1) \otimes m(2) \bullet n(2)$ and $\epsilon(m \bullet n) = \epsilon(m) \bullet \epsilon(n)$.

$$\Delta(a \overline{b}) = a \overline{b} \otimes \overline{b} \quad (21)$$

$$\Delta((x, \omega)) = (x, \omega) \otimes (x, \omega) \quad (22)$$

$$\epsilon(a \overline{b}) = ba \quad \epsilon((x, \omega)) = ev(x, \omega) \quad (23)$$

for $a \overline{b} \in A^e$ and $(x, \omega) \in \mathfrak{M}$.

Proposition 3.2. The algebra $B(\Omega^1)$ along with $\Delta, \epsilon$ has a left $A^e$-bialgebroid structure.

Proof. It is easy to see that $\Delta$ and $\epsilon$ are well defined with respect to relations (19) and (20). We must also check that $\Delta$ and $\epsilon$ are bimodule maps:

$$\Delta(a \bullet (x, \omega)) = \Delta((ax, \omega)) = a \bullet (x, \omega) \otimes (x, \omega) = a \Delta((x, \omega))$$

$$\epsilon(a \bullet (x, \omega)) = ev(ax, \omega) = ae((x, \omega))$$

$$\epsilon((x, \omega)) = ev(x, \omega) = ev(x, \omega)$$

where $(x, \omega) \in \mathfrak{M}$ and $a \in A$. Now check that $(B(\Omega^1), \Delta, \epsilon)$ is an $A|A$-coring. Coassociativity (22) and the counit condition (23) follow easily by the definition of $\Delta, \epsilon$ on the generators and are left to the reader. We briefly check (21) and (24) for $a \overline{b} \in A^e$ and $(x, \omega) \in \mathfrak{M}$.

$$\Delta((x, \omega) \bullet a \overline{b}) = \Delta((xa, bw)) = (x, \omega) \bullet (x, \omega) \bullet (x, \omega)$$

$$\epsilon((x, \omega) \bullet a) = ev(xa \otimes \omega) = ev(xa \otimes \omega)$$

Since $\Delta$ and $\epsilon$ are well-defined on the generators and (24) holds, they can be extended multiplicatively to an $A|A$-coring structure on $B(\Omega^1)$. By defining the comultiplication and counit multiplicatively, $B(\Omega^1)$ automatically satisfies the bialgebroid axioms. □
Notice that for \((x_i, \omega_i) \in \mathfrak{M}\), where \(1 \leq i \leq n\),
\[
\epsilon\left((x_1, \omega_1)(x_2, \omega_2)\cdots(x_n, \omega_n)\right) = \text{ev}^{(n)}(x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes \omega_n \otimes \cdots \otimes \omega_1)
\]
where \(\text{ev}^{(n)}\) is defined iteratively by \(\text{ev}^{(n+1)} = \text{ev}(\text{id}_{\mathfrak{X}^1} \otimes \text{ev}^{(n)} \otimes \text{id}_{\Omega}^1)\) and \(\text{ev}^{(1)} = \text{ev}\).

**Theorem 3.3.** There exists an isomorphism of categories \(B(\Omega^1)\mathcal{M} \cong \mathcal{A}_A^\mathfrak{M}^1\).

**Proof.** Any \(B(\Omega^1)\)-module \(M\) has an induced \(A\)-bimodule structure, by restriction of scalars to \(A^e\). Moreover, \(M\) has an induced \(\Omega^1\)-intertwining \(\sigma\) defined by
\[
\sigma(m \otimes \omega) = \omega_i \otimes (x_i, \omega)m
\]
for \(m \in M\) and \(\omega \in \Omega^1\). The left column of relations in (19) and (20) imply that the map \(\sigma\) is well defined, while the right column of relations make \(\sigma\) a bimodule map. Conversely, an \(A\)-bimodule \(M\) with an \(\Omega^1\)-intertwining map, \(\sigma\), has an induced action of \(B(\Omega^1)\) defined on the generators of the algebra by
\[
(x, \omega)m = (\text{ev} \otimes \text{id}_M)(x \otimes \sigma(m \otimes \omega)), \quad (ab)m = amb
\]
where \((x, \omega) \in \mathfrak{M}, ab \in A^e\) and \(m \in M\). The two correspondences described are each others inverses and their functoriality follows easily. \(\square\)

### 3.2 Mutation of \(T\mathfrak{X}_1\) for Bimodule Connections

In this section we construct the bialgerboid whose category of left modules recovers the category of left bimodule connections, \(\mathcal{E}_A\). Any left bimodule connection in \(\mathcal{E}_A\) is a bimodule with an \(\Omega^1\)-intertwining and a left connection. Hence, every left bimodule connection has an induced \(B(\Omega^1)\)-action and a \(T\mathfrak{X}_1\)-action arising from its \(\Omega^1\)-intertwining and left connection, respectively. The only additional data defining a left bimodule connection, is how its left connection and right \(A\)-action interact. We define \(B\mathfrak{X}_1\) to be the quotient of algebra \(T(\mathfrak{M} \oplus \mathfrak{X}_1 \oplus A^e)\) by the ideal generated by the set of relations (19), (20) and

\[
\begin{align*}
\text{a} \cdot x &= ax \\
x \cdot \text{a} &= xa + \text{ev}(x, da) \\
\text{x} \cdot \omega &= \overline{\partial}x + (x, da)
\end{align*}
\]

for all \(x \in \mathfrak{X}_1, \omega \in \Omega^1, a \in A\). Equivalently, \(B\mathfrak{X}_1\) is the quotient of the free product of algebras \(B(\Omega^1) \ast T\mathfrak{X}_1\) by the ideal which the set of relations (24), (25) and (26) generate.

We extend the coproduct and counit of \(B(\Omega^1)\) to \(B\mathfrak{X}_1\) by defining it on elements of \(\mathfrak{X}_1\) and extending them multiplicatively to \(B\mathfrak{X}_1\), by \(\Delta(m \* n) = m_{(1)} \* n_{(1)} \otimes m_{(2)} \* n_{(2)}\) and \(\epsilon(m \* n) = \epsilon(m \epsilon(n))\):
\[
\Delta(x) = x \otimes 1 + (x, \omega) \otimes x_i \quad \epsilon(x) = 0
\]
for \(x \in \mathfrak{X}_1\).

**Lemma 3.4.** The coproduct \(\Delta\) and counit \(\epsilon\) are well-defined maps on \(B\mathfrak{X}_1\) and provide \(B\mathfrak{X}_1\) with a left \(A\)-bialgebroid structure.
Proof. Since we have defined $\Delta$ and $\epsilon$ on the generators of the algebra and extended them multiplicatively to the rest of the algebra, we must first check if they are well-defined:

\[
\begin{align*}
\Delta(a \ast x) &= \Delta(ax) = a \ast x_1 \otimes x_2 = a \ast x \otimes 1 + a \ast (x, \omega_i) \otimes x_i = \Delta(ax) \\
\Delta(x \ast a) &= x_1 \ast a \otimes x_2 = x \ast a \otimes 1 + (x, \omega_i) \ast a \otimes x_i \\
&= \Delta(xa) + \text{ev}(x \otimes da) \otimes 1 = \Delta(xa + \text{ev}(x \otimes da)) \\
\Delta(\overline{m} \ast x) &= x_1 \otimes x_2 \ast \overline{m} = x \ast \overline{m} + (x, \omega_i) \otimes x_i \ast \overline{m} \\
&= \Delta(x \ast \overline{m} - (x, da))
\end{align*}
\]

The map $\Delta$ being a bimodule morphism follows from the above calculations. Now, we check that $\epsilon$ is well-defined:

\[
\begin{align*}
\epsilon(a \ast x) &= \epsilon(a \ast (x)) = 0 = \epsilon(ax) \\
\epsilon(x \ast a) &= \epsilon(x \ast (a)) = \epsilon(xa + \text{ev}(x \otimes da)) = \epsilon(xa) + \epsilon(\text{ev}(x \otimes da)) \\
\epsilon(x \ast \overline{m}) &= \epsilon(x \ast (\overline{m})) = \epsilon(x \ast (a)) = \epsilon(x, da) = \epsilon((x, da)) \\
&= \epsilon(x \ast \overline{m}) + \epsilon((x, da)) = \epsilon(x, da)
\end{align*}
\]

It also follows that $\epsilon$ is a bimodule map. Now we demonstrate coassociativity \(^3\) and the counit condition \(^3\)

\[
(\Delta \otimes \text{id}_{B^X}) \Delta(x) = x \otimes 1 \otimes 1 + (x, \omega_i) \otimes x_i \otimes 1 \\
+ (x, \omega_j) \otimes (x_j, \omega_i) \otimes x_i = (\text{id}_{B^X} \otimes \Delta) \Delta(x) \\
\epsilon(x_1 \ast x_2) = \epsilon(x_1) + \epsilon((x, \omega_i))x_i = x \\
\epsilon(x_2 \ast x_1) = \epsilon((1)x + \epsilon(x_i))(x, \omega_i) = x
\]

The other coring axioms are easy to check and are left to the reader. The bialgebroid axioms hold since we defined the coproduct and counit multiplicatively. \(\qed\)

**Theorem 3.5.** There exists an isomorphism of categories \(_B^X \cdot M \cong \mathcal{E}_A\).

**Proof.** By restriction of scalars to $B(\Omega^1)$ and Theorem \(^3\) a $B^X$-module $M$ has an $A$-bimodule structure and an induced $\Omega^1$-intertwining defined by $\sigma(m \otimes \omega) = \omega_i \otimes (x_i, \omega)m$. As described in Section \(^3\) by restriction to $T^X_\ast$, the module $M$ has a left connection defined by $\nabla(m) = \sum \omega_i \otimes x_i m$. The induced connection $\nabla$ is a left bimodule connection

\[
\nabla(ma) = \omega_i \otimes x_i (ma) = \omega_i \otimes (x_i \ast \overline{m}) m
\]

\[
= \omega_i \otimes (\overline{m} \ast x_i) m + \omega_i \otimes (x_i, da) m
\]

\[
= \omega_i \otimes (x_i, m) a + \sigma(m \otimes da) = \nabla(m)a + \sigma(m \otimes da)
\]

for all $a \in A$ and $m \in M$. Functoriality follows easily and the functor in the opposite direction is formed by realising that the induced $T^X_\ast$ and $B(\Omega^1)$ actions for a bimodule connection satisfy relation \(^3\) and induce an action of $B^X$. \(\qed\)

**Remark 3.6.** When the calculus is surjective, a triple $(M, \nabla, \sigma)$ being a left bimodule connection is a property for a given bimodule $M$ with a left connection $\nabla$ i.e. the $\Omega^1$-intertwining $\sigma$ is not additional data and either exists or not. We observe that in this case the generators of the form $X \otimes \Omega^1$ are made redundant in the definition of
$B\mathfrak{X}^1$, because $\Omega^1$ is spanned by elements of the form $adb$, where $a, b \in A$, and for any $(x, adb) \in \mathfrak{X}^1 \otimes_\mathbb{K} \Omega^1$ we have

$$(x, adb) = (x \otimes \overline{b} \otimes x) \pi = [x, \overline{b}] \pi$$

Thereby, $B\mathfrak{X}^1$ reduces to a quotient of $A^e \ltimes T\mathfrak{X}^1$ with relations of $T\mathfrak{X}^1_*$, (22), (23) and relations arising from $[x, \xi] \pi$ being regarded as elements of $\mathfrak{X}^1 \otimes_\mathbb{K} \Omega^1$. We do this reduction for Example 3.17.

### 3.3 $T\mathfrak{X}^1_*$ as a Central Commutative Algebra in $\mathcal{E}_A$

In this section we consider the $A$-bimodule structure on $T\mathfrak{X}^1_*$ which arises from $A$ being a subalgebra of $T\mathfrak{X}^1_*$. In [27], $T\mathfrak{X}^1_*$ is presented with the additional structure of a commutative algebra in the lax center of $\mathcal{E}_A$. We briefly recall the definition of the center of a monoidal category from [27].

If $(C \otimes 1_\otimes, \alpha, \lambda, r)$ is a monoidal category as described in Section 2.1, then the (lax) center of $C$ has pairs $(X, \tau)$ as objects, where $X$ is an object in $C$ and $\tau : X \otimes \_ \rightarrow \_ \otimes X$ is a natural (transformation) isomorphism satisfying

$$\tau_{1_\otimes} = l^{-1}_X r_X, \quad (id_M \otimes \tau_N)(\tau_{X,M,N} = \alpha_{M,N,X} \tau_{M \otimes N}$$

and morphisms $f : X \rightarrow Y$ of $C$ satisfying $(id_C \otimes f) \tau = \nu(f \otimes id_C)$, as morphism $f : (X, \tau) \rightarrow (Y, \nu)$. We denote the lax center and center by $Z^{lax}(C)$ and $Z(C)$, respectively. This construction is often referred to as the Drinfeld-Majid center. The lax center is also referred to as the prebraided or weak center. The (lax) center has a monoidal structure via

$$(X, \tau) \otimes (Y, \nu) := (X \otimes Y, (\tau \otimes id_Y)(id_X \otimes \nu))$$

and $(1_\otimes, l^{-1}_X r_X)$ acting as the monoidal unit, so that the forgetful functor to $C$ is strong monoidal.

First we observe that if we restrict the coproduct $\Delta$ to $T\mathfrak{X}^1_*$, we obtain a map

$$\overline{\Delta} : T\mathfrak{X}^1_* \rightarrow (\mathfrak{M} \otimes T\mathfrak{X}^1_*) \oplus T\mathfrak{X}^1_* \otimes 1$$

where $\mathfrak{M}$ is the ideal generated by elements of $\mathfrak{M}$ in $B\mathfrak{X}^1$. Notice that we are abusing notation here and should be writing $T\mathfrak{X}^1_*$ instead of 1. However, we do this to emphasise that the image of the map is 1 in $T\mathfrak{X}^1_*$. For any bimodule $M$, we can restrict $\Delta_{M,A}$, as described in [10], to $T\mathfrak{X}^1_*$:

$$\overline{\Delta}_M : T\mathfrak{X}^1_* \otimes M \rightarrow (\mathfrak{M} \otimes M) \otimes (T\mathfrak{X}^1_* \otimes M) \otimes 1$$

Observe that $\overline{\Delta}_M$ is in fact an $A$-bimodule morphism. This is because $(B\mathfrak{X}^1 \otimes M) \otimes (B\mathfrak{X}^1 \otimes A)$ is the image of $\Delta_{M,A}$ and $B\mathfrak{X}^1 \otimes A = B\mathfrak{X}^1/\{b \otimes a \cdot b = b \cdot a \otimes 1 \}$ for any $b \in B\mathfrak{X}^1$ and $m \in M$.

$$\Delta_{M,A}(b \otimes ma) = \Delta_{M,A}(b \otimes m) = (b_{(1)} \otimes m) \otimes b_{(2)} \cdot a$$

holds and $\overline{\Delta}_M$ is an $A$-bimodule morphism.

Consequently, for any $B\mathfrak{X}^1$-module $(M, \triangleright : B\mathfrak{X}^1 \otimes M \rightarrow M)$, the composition

$$\lambda_M : T\mathfrak{X}^1_* \otimes M \xrightarrow{\overline{\Delta}_M} ((\mathfrak{M} \otimes M) \otimes (T\mathfrak{X}^1_* \otimes M) \otimes 1 \xrightarrow{\triangleright \otimes id_{\mathfrak{X}^1_*}} M \otimes T\mathfrak{X}^1_*$$
is an $A$-bimodule map. Recall that the algebra $TX^1$ has a natural $A$-bimodule structure due to $A$ being its subalgebra, which makes $TX^1$ a left $A^\ell$-module. We can extend this left $A^\ell$-action on $TX^1$ to a left $B^X^1$-module structure, where the elements of $TX^1$ act by the multiplication of the algebra, and the action of the ideal $(\mathfrak{m})$ is zero. Equivalently, as a left bimodule connection we obtain the triple $(TX^1, \sum_i \omega_i \otimes x_i \cdot - , 0)$. Consequently, $\lambda_M$ becomes a morphisms of bimodule connections i.e. $\lambda_M$ respects the $B^X^1$-action since the coproduct respects multiplication by $[\mathfrak{m}]$. Furthermore, for any morphism of left bimodule connections $f : M \to N$, the right square below commutes

$$
\begin{array}{ccc}
TX^1 \otimes M & \xrightarrow{\Delta_M} & (\mathfrak{m} \otimes M) \otimes TX^1 \oplus (TX^1 \otimes M) \otimes 1 \\
\downarrow{id_{TX^1} \otimes f} & & \downarrow{(id_{TX^1} \otimes f) \otimes id_{TX^1}} \\
TX^1 \otimes N & \xrightarrow{\Delta_M} & (\mathfrak{m} \otimes N) \otimes TX^1 \oplus (TX^1 \otimes N) \otimes 1 \\
\end{array}
$$

and thereby $\lambda_N(id_{TX^1} \otimes f) = (f \otimes id_{TX^1}) \lambda_M$. This implies that

$$
\lambda : TX^1 \otimes id_{\mathcal{E}_A} \to id_{\mathcal{E}_A} \otimes TX^1
$$

is a natural transformation. It follows directly from the definition of $\Delta_M$, the coassociativity of $\Delta$, $[\mathfrak{m}]$, and the counit condition, $[\mathfrak{m}]$, that $\lambda$ satisfies the braiding conditions $[28]$.

**Theorem 3.7.** [Theorem 8.2 $[3]$] The triple $(TX^1, \sum_i \omega_i \otimes x_i \cdot - , 0)$ along with braiding $\lambda$ becomes an object in the lax center $Z^{lax}(\mathcal{E}_A)$.

The braiding presented for the left bimodule connection $(TX^1, \sum_i \omega_i \otimes x_i \cdot - , 0)$ in $[3]$, coincides with our definition of $\lambda$ on the elements of $X^1$ and $A$, and is extended iteratively for their basis of $TX^1$ and ultimately gives the same braiding. Additionally, in $[3]$, $TX^1$ forms a commutative algebra with the braiding $\lambda$ i.e. $\cdot (\lambda_{TX^1}) = \cdot$. This follows from the image of $\Delta_M$ on the right component being the identity i.e. the diagram

$$
\begin{array}{ccc}
TX^1 \otimes M & \xrightarrow{id_{TX^1} \otimes 1} & TX^1 \otimes M \otimes 1 \\
\downarrow{\Delta_M} & & \downarrow{0 \otimes id_{TX^1} \otimes M \otimes 1} \\
(\mathfrak{m} \otimes M) \otimes TX^1 \oplus (TX^1 \otimes M) \otimes 1 & & TX^1 \otimes M \otimes 1
\end{array}
$$

commutes. When $M = TX^1$, the action of $\mathfrak{m}$ on $TX^1$ is zero and

$$
\begin{array}{ccc}
TX^1 \otimes TX^1 & \xrightarrow{\cdot} & TX^1 \otimes TX^1 \\
\downarrow{\Delta_{TX^1}} & & \downarrow{0 \otimes id_{TX^1} \otimes \cdot \otimes id_{TX^1}} \\
(\mathfrak{m} \otimes TX^1) \otimes TX^1 \oplus (TX^1 \otimes TX^1) \otimes 1 & & TX^1 \otimes TX^1 \otimes 1
\end{array}
$$

commutes.

The author would like to point out that although the above description answers why $TX^1$ appears as a commutative algebra in the lax center of $\mathcal{E}_A$ and provides a
framework for the work presented in [3], it does not seem to relate to previous work on bialgebroids. As demonstrated in [11], central commutative algebras should be viewed equivalent to Hopf comonads. However, the resulting comonad is not a part of the picture below.

\[ B(\Omega^1)\mathcal{M} \cong A\mathcal{M}_A^{O^1} \xrightarrow{\Delta} A\mathcal{M}_A \]

The forgetful functor \( \mathcal{M}_A^1 \rightarrow A\mathcal{M}_A^{O^1} \) does not appear to have a left adjoint. In other words, \( B\mathcal{X}^1 \) does not arise as the composition of two bimonads as defined in [11]. It is also not an extension by a central commutative algebra, as described in Section 3.4.7 of [7], since \( T\mathcal{X}_\bullet^1 \) is not a commutative algebra in the center of \( A\mathcal{M}_A^O^1 \).

### 3.4 Examples of Bialgebroids

Now we present several examples of left bialgebroids by generators and relations, arising from the differential calculi presented in Section 2.2. In the examples below we will not repeat how the coproduct and counit are defined on elements of \( A\mathcal{L} \) in \( B\mathcal{X}^1 \), since they follow from the bialgebroid axioms.

**Example 3.8.** [Derivation Calculus] Recall that for any derivation \( d \) on an algebra \( A \), we regard \( \Omega^1 = A \) as a bimodule, so that \( \mathcal{X}^1 = A \), where the evaluation morphism is given by multiplication and the coevaluation morphism is given by \( \text{coev}(1) = 1 \otimes 1 \).

It is easy to see that \( T\mathcal{X}_\bullet^1 \) is isomorphic to

\[ T\mathcal{X}_\bullet^1 = A \ast \mathbb{K}[D]/(D \ast a = a \ast D + da \mid a \in A) \]

where \( D = 1 \in \mathcal{X}^1 \). In this case we say the algebra factorizes as \( A\mathcal{L}^{\mathbb{K}[D]} \) under the commuting relations \( D \ast a = a \ast D + da \), for \( a \in A \). The bialgebroid \( B\mathcal{X}^1 \) has the additional generator \( F = 1 \otimes_{\mathbb{K}} 1 \in \mathcal{X}^1 \otimes_{\mathbb{K}} \Omega^1 \) and factorizes as \( A\mathcal{L}^\mathcal{L}^1 \) with the commutation relations

\[ [D, a] = da, \quad [D, \overline{a}] = \overline{da}, \quad [F, a] = [F, \overline{a}] = 0 \]

where \( a \in A \). The coproduct and counit are defined on the generators by \( \Delta(D) = D \otimes 1 + 1 \otimes D \), \( \Delta(F) = F \otimes F \) with \( \epsilon(D) = 0 \) and \( \epsilon(F) = 1 \).

**Example 3.9.** [Finite Quivers] Example 2.2 provided a setting for differential geometry on a finite quiver \( \Gamma = (V, E) \), with \( A = \mathbb{K}(V) \) and \( \Omega^1 = \bigoplus_{e \in E} \mathbb{K}\overline{e} \). Consequently, \( \mathcal{X}^1 = \bigoplus_{e \in E} \mathbb{K}\overline{e} \) where \( f^e \overline{e} = \delta_{p,t(e)} \delta_{q,s(e)} \overline{e} \). In this case \( T\mathcal{X}_\bullet^1 = \mathbb{K}(f^e, \overline{e} \mid p \in \Gamma, e \in E) / \mathcal{U} \) where \( \mathcal{U} \) is the ideal generated by relations

\[ f_p \overline{q} = \delta_{p,q} f_q, \quad f_p \overline{e} = \delta_{p,t(e)} \overline{e} \]

\[ \overline{e} f_p = \delta_{p,q} \overline{q} - f_t(e) \]

for all \( e \in E \) and \( p, q \in V \). In Lemma 4.1 of [20], it was pointed out that a left connection over this calculus corresponds to a quiver representation in the classical sense [2]. We can explain this by observing that the quiver path algebra \( \mathbb{K}\Gamma \), whose module category recovers the category of quiver representations, is isomorphic to \( T\mathcal{X}_\bullet^1 \).

The quiver algebra \( \mathbb{K}\Gamma \) has the same generators, however it has \( \overline{e} \cdot f_p = \delta_{p,q(e)} \overline{e} \) as
a relation instead of (20). There exists an isomorphism of algebras \( K\Gamma \to TX^1 \) define by
\[
f_p \mapsto f_p, \quad \epsilon \mapsto \epsilon - f_t(e)
\]
Hence, the bialgebroid \( BX^1 \) is the quotient of \( K\Gamma \langle \bar{f}_p, (\bar{e}_1, \bar{e}_2) \mid p \in S, \ e_1, e_2 \in E \rangle \) by the additional relations
\[
\bar{f}_p \bar{f}_q = \delta_{p,q} \bar{f}_q, \quad f_p \bar{f}_q = \bar{f}_q f_p
\]
\[
(\bar{e}_1, \bar{e}_2) f_p \bar{f}_q = (\bar{e}_1, \bar{e}_2) \delta_{p,s(e_1)} \delta_{q,s(e_2)}
\]
\[
f_p \bar{f}_q (\bar{e}_1, \bar{e}_2) = (\bar{e}_1, \bar{e}_2) \delta_{p,t(e_1)} \delta_{q,t(e_2)}
\]
\[
\bar{e}_1 \bar{e}_2 = \bar{f}_q \bar{e}_2 + \sum_{e \in E, t(e) = q} (\bar{e}_1, \bar{e}) - \sum_{e \in E, s(e) = q} (\bar{e}_1, \bar{e}_2)
\]
and the coproduct and counit are defined by
\[
\Delta((\bar{e}_1, \bar{e}_2)) = \sum_{e \in E} (\bar{e}_1, \bar{e}) \otimes (\bar{e}, \bar{e}_2), \quad \epsilon((\bar{e}_1, \bar{e}_2)) = \delta_{e_1, e_2} f_t(e_1)
\]
\[
\Delta(\bar{e}_1) = \bar{e}_1 \otimes 1 + \sum_{e \in E} (\bar{e}_1, \bar{e}) \otimes (\bar{e} + f_t(e)), \quad \epsilon(\bar{e}_1) = -f_t(e_1)
\]
for all \( e_1, e_2 \in E \) and \( p, q \in V \).

**Example 3.10.** [\( M_2(\mathbb{C}) \)] For the calculus of Example 2.8, we denote elements \( 1 \oplus 0 \) and \( 0 \oplus 1 \) in \( \Omega^1 \) by \( s \) and \( t \), respectively. Hence, \( X^1 \) is a free bimodule with \( f_s \) and \( f_t \) as the dual basis to \( s \) and \( t \). The algebra \( TX^1 \) was described in Chapter 6 of [23], and factorises as \( A.\mathcal{C}(f_s, f_t) \) with commutation relations
\[
f_s a = a f_s + [E_{12}, a], \quad f_t a = a f_t + [E_{21}, a]
\]
The bialgebroid \( BX^1 \) factorises as \( A.\mathcal{C}(f_i, \gamma_j \mid i, j \in \{s, t\}) \) with additional relations
\[
f_i [\bar{s}] = a f_i + [E_{12}, a] \gamma_s + [E_{21}, a] \gamma_t, \quad [\gamma_j, a^i] = 0
\]
for \( i, j \in \{s, t\} \). The coproduct and counit are defined by
\[
\Delta(f_i) = f_i \otimes 1 + i \gamma_s \otimes f_s + i \gamma_t \otimes f_t \quad \epsilon(f_i) = 0
\]
\[
\Delta(\gamma_j) = i \gamma_s \otimes i \gamma_j + i \gamma_t \otimes i \gamma_j \quad \epsilon(\gamma_j) = \delta_{i,j}
\]
for \( i, j \in \{s, t\} \). The calculus in this case is surjective with \( s = (dE_{12})E \) and \( t = (dE_{12})E \), where \( E = E_{11} - E_{22} \). By Remark 3.6, generators of the form \( i \gamma_j \) become redundant:
\[
i \gamma_s = E [f_i, E_{21}], \quad i \gamma_t = -E [f_i, E_{12}]
\]
where \( i \in \{s, t\} \). Thereby, \( BX^1 \) factorises as \( A.\mathcal{C}(f_i, f_t) \) with the \( TX^1 \) relations as above and the additional relations
\[
[f_i, [\bar{s}]] = E [E_{12}, a] [f_i, E_{21}] + E [E_{21}, a] [f_i, E_{12}]
\]
for all \( i \in \{s, t\} \) and \( a \in A \).
Example 3.11. [Hopf Bimodules] If $(A, \delta, \nu, s)$ is a Hopf algebra and $\Omega^1 = \Lambda \otimes_{K} A$ a Hopf bimodule, then $\Omega^1$ being right fgp is equivalent to $\Lambda$ being a finite dimensional vectorspace, with basis $\lambda_i^{(n)}$. Hence, $X^1 \cong A \otimes_{K} \Lambda^*$ is free as a left module, where $\Lambda^*$ is the dual vectorspace to $\Lambda$ with dual basis $\{f_i^{(n)}\}_{i=1}^n$. Here, $\Lambda^*$ has an induced right $A$-action corresponding to the left $A$-action of $\Lambda$ defined by $f \circ a = f(\lambda \cdot a)$ for all $f \in \Lambda^*$ and $a \in A$. In this case, $TX^1_\bullet$ was described in Chapter 6 of [53] and factorises as $A \cdot T \Lambda^*$ with commutation relation
\[
f_i \cdot a = a^{(2)}_i \cdot f_i \circ a^{(1)} + \partial^i(a)
\]
where $\partial^i(a) = ev(1 \otimes f_i \otimes da)$. The $A^e$-bimodule $X^1 \otimes_{K} \Omega^1$ is free as a left $A^e$-module and isomorphic to $A^e \otimes_{K} (\Lambda^* \otimes_{K} \Lambda)$. We denote the basis of $\Lambda^* \otimes_{K} \Lambda$ by $(f_i, \lambda_j)$. Hence, the bialgebroid $BX^1$ factorises as $A . T \Sigma$ where $\Sigma = \Lambda^* \oplus (\Lambda^* \otimes_{K} \Lambda)$, with additional commutation relations
\[
(f_i, \lambda_j) \cdot a b = a^{(2)}(b^{(2)} \cdot (f_i \circ a^{(1)}, b^{(1)} \cdot \lambda_j))
\]
and
\[
[f_i, f_j] = \sum_{j=1}^n \partial(a_i \cdot (f_i, \lambda_j))
\]
for all $1 \leq i, j \leq n$. The coproduct and counit are given by
\[
\Delta(f_i) = f_i \otimes 1 + \sum_{j=1}^n (f_i, \lambda_j) \otimes f_j, \quad \epsilon(f_i) = 0
\]
\[
\Delta((f_i, \lambda_j)) = \sum_{k=1}^n (f_i, \lambda_k) \otimes (f_k, \lambda_j), \quad \epsilon((f_i, \lambda_j)) = \delta_{i,j}
\]
for all $1 \leq i, j \leq n$.

Example 3.12. [C$D_6$] Let $D_6$ denote the Dihedral group with 6 elements with presentation $(a, b \mid a^3 = b^2 = 1, a^2 b = ba)$ and $\Lambda$ its 2-dimensional irreducible complex representation with basis $\xi, \tau$ defined by
\[
a \triangleright \xi = \frac{1}{2}(\xi + \sqrt{3} \tau), \quad b \triangleright \xi = \xi, \quad a \triangleright \tau = \frac{1}{2}(\sqrt{3} \xi + \tau), \quad b \triangleright \tau = -\tau
\]
Recall from Example 2.29 that we obtain an inner calculus on $A = \mathbb{C}D_6$, by taking $\theta = \xi + \tau$, so that $d : \mathbb{C}D_6 \to \Omega^1$ satisfies
\[
d(a) = \frac{1}{2}[-(1 + \sqrt{3})\xi + (\sqrt{3} - 1)\tau] \otimes \mathbb{C} a, \quad d(b) = -2\tau \otimes \mathbb{C} b
\]
Consequently, $\Lambda^*$ has a dual basis to $\Lambda$, denoted by $f_\xi, f_\tau$ and $TX^1_\bullet$ factorises as $A . \mathbb{C}(f_\xi, f_\tau)$ with commutation relations
\[
f_\xi \cdot a = \frac{1}{2} a \cdot (f_\xi - \sqrt{3} f_\tau) - \frac{1}{2}(1 + \sqrt{3})a, \quad f_\xi \cdot b = b \cdot f_\xi
\]
\[
f_\tau \cdot a = \frac{1}{2} a \cdot (\sqrt{3} f_\xi + f_\tau) + \frac{1}{2}(\sqrt{3} - 1)a, \quad f_\tau \cdot b = -b \cdot f_\tau - 2b
\]
The resulting bialgebroid $BX^1$ factorises as $A^e . \mathbb{C}(f_\xi, f_\tau, \xi \cdot \gamma_\xi, \tau \cdot \gamma_\tau, \gamma_\xi \cdot \gamma_\tau)$ with additional relations
\[
[f_\xi, f_\tau] = -\frac{1}{2}(1 + \sqrt{3}) a \circ (\xi \cdot \gamma_\xi + \frac{1}{2}(\sqrt{3} - 1)a \circ \gamma_\tau, \quad [f_\xi, f_\tau] = -2b \circ \gamma_\tau
\]
Proof. First, observe that since the forgetful functor from $\mathcal{A}\mathcal{M}_A$ to $\mathcal{M}_A$ is strong monoidal, if $(N, \tau)$ is a left dual of $(M, \sigma)$, then $N \cong ^\vee M$ and $M$ is right fgp. Furthermore, the evaluation and coevaluation morphisms $ev$ and $coev$ must commute with the intertwining maps i.e.

$$
eval{i} \circ \id_{\Omega^i} = (\id_{\Omega^i} \circ \eval{)}(\tau \otimes \id_M)(\id_N \otimes \sigma)$$

for $i \in \{\xi, \tau\}$. The coproduct and counit take the form of

$$
\Delta(f_i) = f_i \otimes 1 + i \gamma_{\xi} \otimes f_{\xi} + i \gamma_{\tau} \otimes f_{\tau} \quad \epsilon(f_i) = 0
$$

$$
\Delta(i_{\gamma_{ij}}) = i_{\gamma_{\xi}} \otimes \xi_{\gamma_{ij}} + i_{\gamma_{\tau}} \otimes \tau_{\gamma_{ij}} \quad \epsilon(i_{\gamma_{ij}}) = \delta_{i,j}
$$

for $i, j \in \{\xi, \tau\}$.

4 Hopf Algebroids for Pivotal Calculi

We would like the monoidal category of connections which we consider to lift the closed monoidal structure of $\mathcal{A}\mathcal{M}_A$. In this a situation, if a bimodule with such a connection is right (left) fgp, its dual bimodule $\,^\vee M$ (resp. $M^\vee$) will have an induced connection making it left (right) dual to the original connection in this monoidal category of connections. In Section 3.4.2 of [5], several statements are presented, demonstrating that if $M$ is a right (left) fgp bimodule with a left (right) bimodule connection $(M, \nabla, \sigma)$ such that $\sigma$ is invertible, then $\,^\vee M$ (resp. $M^\vee$) has a compatible right (left) bimodule connection structure. The subcategory of invertible bimodule connections, with left bimodule connections with invertible $\Omega^1$-intertwinings as objects, is hence considered as a nicer category to work with. In particular, left and right bimodule connections with invertible intertwining morphisms $\Omega^1$-intertwinings as objects, is hence considered as a nicer category to work with. In particular, left and right bimodule connections with invertible intertwining morphisms coincide. However, the category of invertible bimodule connections is not closed: given a right fgp bimodule $M$ with an invertible left bimodule connection $(M, \nabla, \sigma)$, its left dual bimodule $\,^\vee M$ will have a right bimodule connection structure denoted by $(\,^\vee M, \,^\vee \nabla, \,^\vee \sigma)$, but the $\Omega^1$-intertwining $\sigma^\sharp$ is not necessarily invertible. In fact, there is a natural way of defining connections on inner homs of invertible bimodule connections, but to obtain the correct closed monoidal category lifting the structure of $\mathcal{A}\mathcal{M}_A$, we must find a subcategory of $\mathcal{A}\mathcal{M}_A^{\Omega^1}$ which lifts the closed structure of $\mathcal{A}\mathcal{M}_A$.

4.1 Invertible Bimodule Connections

To agree with [5], we denote the category of invertible bimodule connections i.e. the subcategory of $\mathcal{J}\mathcal{E}_A$, where objects $(M, \nabla, \sigma)$ have invertible $\Omega^1$-intertwinings $\sigma$, by $\mathcal{J}\mathcal{E}_A^{\Omega^1}$. Furthermore, we denote the subcategory of $\mathcal{A}\mathcal{M}_A^{\Omega^1}$ of bimodules with invertible $\Omega^1$-intertwinings by $\mathcal{A}\mathcal{I}\mathcal{M}_A^{\Omega^1}$. It should be clear that $\mathcal{A}\mathcal{I}\mathcal{M}_A^{\Omega^1}$ is a monoidal subcategory of $\mathcal{A}\mathcal{M}_A^{\Omega^1}$.

Lemma 4.1. An object of $\mathcal{A}\mathcal{M}_A^{\Omega^1}$, $(M, \sigma)$ has a (right) left dual, if and only if $M$ is (left) right fgp and $\sigma$ is invertible.

Proof. First, observe that since the forgetful functor from $\mathcal{A}\mathcal{M}_A^{\Omega^1}$ to $\mathcal{M}_A$ is strong monoidal, if $(N, \tau)$ is a left dual of $(M, \sigma)$, then $N \cong \,^\vee M$ and $M$ is right fgp. Furthermore, the evaluation and coevaluation morphisms $ev$ and $coev$ must commute with the intertwining maps i.e.

$$
eval{i} \circ \id_{\Omega^i} = (\id_{\Omega^i} \circ \eval{)}(\tau \otimes \id_M)(\id_N \otimes \sigma)$$

\[\]
\[ \id_{\Omega^1} \otimes \coev = (\tau \otimes \id_M)(\id_N \otimes \sigma)(\coev \otimes \id_{\Omega^1}) \]

From the above equations, it is easy to check that the morphism \((\id_M \otimes \Omega^1) \otimes \text{ev}) (\id_M \otimes \tau \otimes \id_M) (\coev \otimes \id_{\Omega^1} \otimes \id_M)\) becomes the inverse of \(\sigma\). Conversely, if \(M\) is right fgp with left dual \(\gamma M\) and \(\sigma\) is invertible, we define \((\gamma M, \sigma^2)\) by

\[ \sigma^2 = (\text{ev} \otimes \id_{\Omega^1} \otimes \gamma M)(\id_{\gamma M} \otimes \sigma^{-1} \otimes \id_{\gamma M})(\id_{\gamma M} \otimes \Omega^1 \otimes \coev) \quad (31) \]

so that \((\gamma M, \sigma^2)\) is left dual to \((M, \sigma)\) in \(\mathcal{A}\mathcal{M}_{\mathcal{A}}^{\Omega^1}\), via \(\text{coev} \) and \(\text{ev}\).

For \(\mathcal{A}\mathcal{M}_{\mathcal{A}}^{\Omega^1}\) to be representable, we need the additional requirement for \(\Omega^1\) to be left fgp as well as right fgp, with its right dual bimodule denoted by \(\Omega^1\). Let \(\text{coev} : A \rightarrow \Omega^1 \otimes \Omega^1\) and \(\text{ev} : \Omega^1 \otimes \Omega^1 \rightarrow A\) denote the respective coevaluation and evaluation maps and denote \(\text{coev}(1) = \sum_j y_j \otimes \rho_j\). Parallel to Section 3.1, we consider \(\Omega^1 \otimes \Omega^1\) as an \(A^e\)-bimodule via \((32)\), so that \(\mathcal{T}_A(\Omega^1 \otimes \Omega^1)\)-modules have the structure of \(A\)-bimodules \(M\) with a bimodule map \(\Omega^1 \otimes M \rightarrow M \otimes \Omega^1\). We denote this category by \(\Omega^1\mathcal{M}_{\mathcal{A}}^A\) and observe that \(\mathcal{T}_A(\Omega^1 \otimes \Omega^1)\mathcal{M} \approx \Omega^1\mathcal{M}_{\mathcal{A}}^A\). This can be proved in a completely symmetric manner to the arguments in Section 3.1. Consequently, the bialgebroid whose module category is isomorphic to \(\mathcal{A}\mathcal{M}_{\mathcal{A}}^{\Omega^1}\) is a quotient of the free product of algebras \(B(\Omega^1)\) and \(\mathcal{T}_A(\Omega^1 \otimes \Omega^1)\) by an ideal which imposes the induced intertwinings with \(\Omega^1\) to be inverses.

For a bimodule \(M\), when necessary we distinguish bimodule morphisms \(\Omega^1 \otimes M \rightarrow M \otimes \Omega^1\) and \(M \otimes \Omega^1 \rightarrow \Omega^1 \otimes M\) by referring to them by \(\text{left}\) and \(\text{right}\) \(\Omega^1\)-intertwinings. Otherwise, we refer to both morphisms as \(\Omega^1\)-intertwinings and the domain and codomain of morphisms will be clear from context.

Let \(\mathfrak{F} := (X^1 \otimes \Omega^1) \oplus (\Omega^1 \otimes \Omega^1)\) as a vectorspace and \(R := \mathcal{T}_A(\mathfrak{F})\) as an algebra, where the \(A^e\) bimodule structure of \(\mathfrak{F}\) is defined as follows

\[ a^{(x, \omega)} b = (a^{(x, \omega)} b) = (a_{x b}, b') \quad (32) \]
\[ a^{(p, y)} b = (a_{p y}, b) \quad (33) \]

where \(a, a', b, b' \in A\), \((x, \omega) \in X^1 \otimes \Omega^1\) and \((p, y) \in \Omega^1 \otimes \Omega^1\). It is easy to check that the bialgebroid structures of \(\mathcal{T}_A(X^1 \otimes \Omega^1)\) and its symmetric counterpart \(\mathcal{T}_A(\Omega^1 \otimes \Omega^1)\), lift to \(R\) multiplicatively. Alternatively, we can view \(R\) as the free product of \(A^e\)-algebras \(\mathcal{T}_A(X^1 \otimes \Omega^1)\) and \(\mathcal{T}_A(\Omega^1 \otimes \Omega^1)\). From this point of view, it is easy to see that the free product of two \(A^e\)-algebras with \(A\)-bialgebroid structures will have a natural \(A\)-bialgebroid structure: modules over the free product algebra are simply \(A\)-bimodules with actions from both algebras and the tensor of two such bimodules over \(A\) will have an induced action from both bialgebroids, which induces an action of the free product algebra. Ultimately, the coproduct and counit induced on the free product algebra, from the categorical point of view, extend the coproduct and counit of each bialgebroid to the free product algebra, multiplicatively.

We define \(IB(\Omega^1)\) as the quotient of algebra \(R\) by the set of relations

\[ (\omega, y)(x, \omega) = \text{ev}(\omega \otimes y) \quad (34) \]
\[ (x, \rho)(\omega, y) = \text{ev}(x \otimes \omega) \quad (35) \]

for any \(x \in X, \omega \in \Omega^1, y \in \Omega^1\).

\textbf{Lemma 4.2.} The bialgebroid structure of \(R\) descends to a well defined bialgebroid structure on \(IB(\Omega^1)\).
Proof. Since the bialgebroid structure on \( R \) is defined by multiplicatively, we only need to check that the comultiplication and counit are well defined on its quotient \( IB(\Omega^1) \).

To do this we look at the relations generating the ideal quotiented from \( R \). For relation (34) we demonstrate this by the following calculations

\[
\Delta((\omega, y)\bullet(x, \omega)) = (\omega, y_j)\bullet(x, \omega_k) \otimes (\rho_j, y)\bullet(x_k, \omega) \\
= \text{ev}(\omega_k \otimes y_j) \otimes (\rho_j, y)\bullet(x_k, \omega) = 1 \otimes \text{ev}(\omega_k \otimes y_j) \otimes (\rho_j, y)\bullet(x_k, \omega) \\
= 1 \otimes (\omega_k, y)\bullet(x_k, \omega) = 1 \otimes \text{ev}(\omega \otimes y) = \Delta \text{ev}(\omega \otimes y)
\]

and

\[
\epsilon((\omega, y)\bullet(x, \omega)) = \text{ev}(\omega, \text{ev}(x, \omega) \otimes y) = \text{ev}(\omega \otimes y) = \epsilon(\text{ev}(\omega \otimes y))
\]

where \( y \in \mathfrak{Y}^1 \) and \( \omega \in \Omega \). The morphisms \( \Delta, \epsilon \) being well defined for relation (35), follows similarly and is left to the reader.

\[\square\]

Theorem 4.3. There is an isomorphism of categories \( IB(\Omega^1), M \cong _A \mathcal{M}^{\Omega_1}_A \).

Proof. For a \( IB(\Omega^1) \)-module \( M \), we can obtain left and right \( \Omega^1 \)-intertwinings \( \sigma, \tau \) on \( M \) by restriction of scalars to subalgebras \( T_A(\mathcal{X}^1 \otimes \mathcal{X} \Omega^1) \) and \( T_A(\mathcal{Y}^1 \otimes \mathcal{Y} \Omega^1) \):

\[
\sigma(m \otimes \omega) = \omega_i \otimes (x_i, \omega)m, \quad \tau(\omega \otimes m) = (\omega, y_j)m \otimes \rho_j
\]

For any \( m \otimes \omega \in M \otimes \Omega^1 \),

\[
\tau \sigma(m \otimes \omega) = \tau(\omega_i \otimes (x_i, \omega)m) = (\omega, y_j)\bullet(x_i, \omega)m \otimes \rho_j
\]

\[
= \text{ev}(\omega, y_j)m \otimes \rho_j = \text{ev}(\omega, y_j) \otimes \rho_j = m \otimes \omega
\]

holds by relation (34). Similarly, \( \sigma \tau = \text{id}_{M \otimes \Omega^1} \) follows from relation (35). The converse statement follows by looking at the induced actions of \( T_A(\mathcal{X}^1 \otimes \mathcal{X} \Omega^1) \) and \( T_A(\mathcal{Y}^1 \otimes \mathcal{Y} \Omega^1) \) on the underlying \( A \)-bimodule of any object \((M, \sigma)\) in \(_A \mathcal{M}^{\Omega_1}_A \). This gives rise to an action of \( R \) on \( M \) and by the calculation above relations (34) and (35) annihilate \( M \), making \( M \) an \( IB(\Omega^1) \)-module.

\[\square\]

We can obtain the left bialgebroid \( IB\mathcal{X}^1 \) whose module category recovers left bimodule connections with invertible \( \Omega^1 \)-intertwinings, as the quotient of the free product of \( T \mathcal{X}^1 \star IB(\Omega^1) \), by the relations (24), (34), (35).

Remark 4.4. By symmetry, we can describe the category of right connections, \( \mathcal{E}_A \), as left modules over the algebra \( T \mathcal{Y}^1 \), which is defined as the quotient of the algebra \( A^{op} \star T \mathcal{Y}^1 \) by relations

\[
\begin{align*}
\pi y &= ya, & y \pi &= ay + \text{ev}(da \otimes y)
\end{align*}
\]

for \( y \in \mathcal{Y}^1 \) and \( \pi \in A^{op} \). In Lemma 3.70 of [3], it is noted that a left bimodule connection \((M, \nabla, \sigma)\) with invertible \( \sigma \), has an induced right bimodule connection structure with \((M, \sigma^{-1} \nabla, \sigma^{-1})\). We can view this as \( T \mathcal{Y}^1 \), being isomorphic to the subalgebra of \( IB\mathcal{X}^1 \) generated by

\[
y \mapsto (\omega_i, y) \bullet x_i, \quad \pi \mapsto \pi
\]

for \( y \in \mathcal{Y}^1 \) and \( \pi \in A^{op} \).
As explained in Theorem 4.3, the relations (34) and (35) imply that the intertwining map on a $IB(\Omega^1)$-module $M$ defined via $\sigma(m,\omega) = \sum \omega_i \otimes (x_i, \omega)m_i$, is invertible. To do this we had to add a number of generators to the algebra $(\Omega^1 \otimes_k (\Omega^1)^\vee)$ and impose some minor relations, (34) and (35), on their interaction with the previous generators. However, as mentioned before $\sigma$ being invertible for a right fgp bimodule, does not make $\sigma^\vee$ and $AIBM_{\Omega^1}^\vee$ does not lift the closed structure of $AIBM_A$. We need a suitable subcategory where $\sigma^\vee : M \otimes (\Omega^1)^\vee \rightarrow (\Omega^1)^\vee \otimes M$ is invertible as well. Hence, we need to translate this condition to $(\sigma^\vee)^\vee : M \otimes (\Omega^1)^{\vee \vee} \rightarrow (\Omega^1)^{\vee \vee} \otimes M$ being invertible. We can impose this condition on the bialgebroid $IB(\Omega^1)$, by adding generators of the form $(\Omega^1)^{\vee \vee} \otimes_k (\Omega^1)^{\vee \vee}$ and similar relations to (34) and (35). On the other hand, $(\sigma^\vee)^\vee$ will not necessarily be invertible, and we will have to repeat the process infinitely. Instead, in the next section we focus on the case where $\Omega^1 \cong (\Omega^1)^{\vee \vee}$ so that all the generators required already exist in $R$ and by imposing the correct relations the arguments mentioned become cyclic.

4.2 Pivotal Modules

Definition 4.5. We say a bimodule $M$ is a pivotal bimodule if there exists a bimodule isomorphism $\gamma M \cong M^{\vee}$, or equivalently $M \cong M^{\vee \vee}$.

Many familiar examples of differential calculi are pivotal bimodules. In the classical case, if $A$ is commutative and $\Omega^1$ has the same left and right $A$-actions, then $\gamma \Omega^1 \cong A Hom(\Omega^1, A)$ and $(\Omega^1)^{\vee} \cong Hom_A(\Omega^1, A)$ are naturally isomorphic.

Example 4.6. [Quantum Riemannian Metric] We say a differential calculus $\Omega^1$ on algebra $A$ has a quantum metric if $\Omega^1$ is self-dual i.e. $\gamma \Omega^1 \cong (\Omega^1)^{\vee}$ as an $A$-bimodule with evaluation and coevaluation maps $ev, coev$ satisfying

$$(ev \otimes id_{\Omega^1})(id_{\Omega^1} \otimes coev) = id_{\Omega^1} = (id_{\Omega^1} \otimes ev)(coev \otimes id_{\Omega^1})$$

In this case, $g = coev(1)$ is called a quantum metric for the calculus.

Of course any free bimodule such as the calculus over $M_2(\mathbb{C})$, presented in Example 2.3 is also pivotal and self dual.

Example 4.7. [Finite Quivers] Any quiver calculus as described in Example 2.6 is pivotal. Recall that $X^1 = \text{span}\{\overline{e} | e \in E\}$, where $f \overline{e} g = f(t(e))\overline{e} g(s(e))$ for any pair $f, g \in \mathbb{K}(V)$. The evaluation and coevaluation maps are given by

$$coev(1) = \sum_{e \in E} \overline{e} \otimes \overline{e}, \quad ev(\overline{e_1} \otimes \overline{e_2}) = \delta_{e_1,e_2}f_{t(e_1)}$$

$$coev(1) = \sum_{e \in E} \overline{e} \otimes \overline{e}, \quad ev(\overline{e_1} \otimes \overline{e_2}) = \delta_{e_1,e_2}f_{s(e_1)}$$

for any $e_1, e_2 \in E$, so that $X^1$ is both left dual and right dual to $\Omega^1$.

Not every parallelised calculus is pivotal. However, the class of bicovariant calculi over Hopf algebras have this additional property:

Example 4.8. [Hopf Bimodules] Recall that a Hopf bimodule $\Omega^1$ for a Hopf algebra $A$, decomposes as a free right module $\Lambda \otimes_k A$. When the antipode of $A$, $s$, is invertible,
we utilise the following isomorphism to move between free right \( A \)-modules and free left \( A \)-modules:

\[
\Phi : A \otimes_\mathbb{K} A \rightarrow A \otimes_\mathbb{K} A \\
\Phi^{-1} : A \otimes_\mathbb{K} A \rightarrow A \otimes_\mathbb{K} A \\
\lambda \otimes_\mathbb{K} a \mapsto a(2) \otimes_\mathbb{K} s^{-1}(a(1)) \triangleright \lambda \\
a \otimes_\mathbb{K} \lambda \mapsto a(1) \triangleright \lambda \otimes_\mathbb{K} a(2)
\]

Observe that the left \( A \)-action translates to \( \Phi(b \triangleright (\lambda \otimes_\mathbb{K} a)) = ba \otimes_\mathbb{K} \lambda \), making \( \Omega^1 \) free as a left \( A \)-module as well with \( \Omega^1 \equiv A \otimes_\mathbb{K} \Lambda \) and \((\Omega^1)^\vee \equiv \Lambda^* \otimes_\mathbb{K} A \). We denote elements of \( A \otimes_\mathbb{K} \Lambda \) and \( \Lambda^* \otimes_\mathbb{K} A \) by \( a \otimes_\mathbb{K} \lambda \) and \( f \otimes_\mathbb{K} a \), respectively. Observe that as bimodules:

\[
b(a \otimes_\mathbb{K} \lambda) = ba \otimes_\mathbb{K} \lambda \quad (a \otimes_\mathbb{K} \lambda)b = (ab(2) \otimes_\mathbb{K} s^{-1}(b(1)) \triangleright \lambda)
\]

\[
(f \otimes_\mathbb{K} a)b = f \otimes_\mathbb{K} ab \\
b(f \otimes_\mathbb{K} a) = (f \triangleleft s^{-1}(b(1)) \otimes_\mathbb{K} b(2)a)
\]

for \( b \in A \). Hence, the evaluation and coevaluation morphisms for \( \Omega^1 \) are calculated as follows

\[
\text{coev}(1) = \sum_{i=1}^n (\lambda_i \otimes_\mathbb{K} 1) \otimes (1 \otimes_\mathbb{K} f_i), \quad \text{ev}((a \otimes_\mathbb{K} f) \otimes (\lambda \otimes_\mathbb{K} b)) = abf(\lambda)
\]

\[
\text{coev}(1) = \sum_{i=1}^n (f_i \otimes_\mathbb{K} 1) \otimes (1 \otimes_\mathbb{K} \lambda_i), \quad \text{ev}((a \otimes_\mathbb{K} \lambda) \otimes (f \otimes_\mathbb{K} b)) = abf(\lambda)
\]

where \( \lambda \in \Lambda, f \in \Lambda^* \) and \( a, b \in A \). Furthermore, \( \Omega^1 \) is pivotal and the isomorphism between \( \vee(\Omega^1) \) and \( (\Omega^1)^\vee \) is provided by

\[
\begin{align*}
\vee(\Omega^1) = A \otimes_\mathbb{K} \Lambda^* & \longleftrightarrow \Lambda^* \otimes_\mathbb{K} A = (\Omega^1)^\vee \\
(a \otimes_\mathbb{K} f) & \mapsto \sum_{i=1}^n (fi \otimes_\mathbb{K} a(2))f(s((\lambda_i)_{(1)}a(3)) \triangleright (\lambda_i)_{(0)}) \\
\sum_{i=1}^n (a(2) \otimes_\mathbb{K} f_i)f(s^{-2}(a(1)(\lambda_i)_{(1)} \triangleright (\lambda_i)_{(0)})) & \longleftrightarrow (f \otimes_\mathbb{K} a)
\end{align*}
\]

where \( \lambda_{(1)} \otimes_\mathbb{K} \Lambda_{(0)} = \delta_L(\lambda) \) denotes the left coaction of \( \Lambda \) as a Yetter-Drinfeld module. The category of Hopf modules over a Hopf algebra \( A \), has a natural monoidal structure lifting that of \( A \)-bimodules. In particular, when the antipode \( s \) of \( A \) is invertible, the category of Hopf bimodules has a braided monoidal structure and is monoidal equivalent to the category of left Yetter-Drinfeld modules \([6]\). Using this equivalence and the fact that in a braided monoidal category, left and right duals of an object are isomorphic, we obtain the above isomorphism.

### 4.3 Resulting Hopf Algebroid Structure

From this point onwards we assume that \( \Omega^1 \) is a pivotal bimodule and modify our notation from previous sections. We denote evaluation and coevaluation maps as before, but with applying the isomorphism \( \mathcal{X}^1 \cong \Omega^1 \) so that

\[
\text{coev} : A \rightarrow \Omega^1 \otimes \mathcal{X}^1, \quad \text{coev}(1) = \sum_i \omega_i \otimes x_i, \quad \text{ev} : \mathcal{X}^1 \otimes \Omega^1 \rightarrow A \\
\text{coev} : A \rightarrow \mathcal{X}^1 \otimes \Omega^1, \quad \text{coev}(1) = \sum_j y_j \otimes \rho_j, \quad \text{ev} : \Omega^1 \otimes \mathcal{X}^1 \rightarrow A
\]
With this notation we define $H(\Omega^1)$ to be the quotient of $IB(\Omega^1)$ by the additional relations
\begin{align}
(y_j, \omega) \bullet (\rho_j, x) &= \text{ev}(x, \omega) \\
(\omega, x_i) \bullet (x, \omega_i) &= \text{ev}(\omega, x)
\end{align}
(38) (39)
for any $x \in \mathfrak{X}^1$ and $\omega \in \Omega^1$.

**Lemma 4.9.** The comultiplication and counit of $IB(\Omega^1)$, are well-defined on the quotient algebra, $H(\Omega^1)$, and give rise to an $A$-bialgebroid structure on $H(\Omega^1)$.

*Proof.* The proof is completely symmetric to that of Lemma 4.2 and is left to the reader. □

By Theorem 4.3 an $IB(\Omega^1)$-module can be viewed as an $A$-bimodule with an invertible $\Omega^1$-intertwining $\sigma : M \otimes \Omega^1 \rightarrow \Omega^1 \otimes M$. Hence, we can translate the additional relations in $B\mathfrak{X}^1$, to the maps
\begin{align}
(\text{ev} \otimes \text{id}_{M \otimes \mathfrak{X}^1})(\text{id}_{\mathfrak{X}^1} \otimes \sigma \otimes \text{id}_{\mathfrak{X}^1})(\text{id}_{\mathfrak{X}^1 \otimes M} \otimes \text{coev}) : \mathfrak{X}^1 \otimes M \rightarrow M \otimes \mathfrak{X}^1 & \quad (40) \\
(\text{id}_{\mathfrak{X}^1 \otimes M} \otimes \text{id}_{\mathfrak{X}^1} \otimes \sigma^{-1})(\text{id}_{\mathfrak{X}^1} \otimes \sigma)(\text{id}_{M \otimes \mathfrak{X}^1}) : M \otimes \mathfrak{X}^1 \rightarrow \mathfrak{X}^1 \otimes M & \quad (41)
\end{align}
being each others inverses. Notice that when $M$ is right fgp, the second map being invertible is equivalent to $\sigma^\ast$ being invertible, which is what we desire in a closed subcategory of $\mathcal{A}\mathcal{M}\mathcal{A}^{\Omega^1}$. If $\Omega^1$ were not pivotal, we would have to write $\Omega^1$ instead of $\mathfrak{X}^1$ in the second morphism, and the two morphisms could not be inverses.

**Theorem 4.10.** The category of $H(\Omega^1)$-modules is isomorphic to the category of $A$-bimodules with invertible $\Omega^1$-intertwining maps $\sigma$, such that bimodule maps (40), (41) are inverses. We denote this category by $\mathcal{A}\mathcal{M}\mathcal{A}^{\Omega^1}$.

*Proof.* Under the correspondence described in Theorem 4.3, an $H(\Omega^1)$-module $M$ has an induced invertible $\Omega^1$-intertwining $\sigma$. By recalling the definition of $\sigma$, the morphisms (40) and (41) translate to
\begin{align}
(\text{ev} \otimes \text{id}_{M \otimes \mathfrak{X}^1})(x \otimes \sigma(m \otimes \omega_i) \otimes x_i) &= (x, \omega_i)m \otimes x_i \\
(\text{id}_{\mathfrak{X}^1 \otimes M} \otimes \text{coev})(y_j \otimes \sigma^{-1}(\rho_j \otimes m) \otimes x) &= y_j \otimes (\rho_j, x)m
\end{align}
respectively, for any $x \in \mathfrak{X}^1$ and $m \in M$. In this form, the morphisms being inverses follows directly from (38) and (39). The converse direction also follow trivially. □

In the above paragraph, we already hinted at the fact that the left (right) duals, of right (left) fgp bimodules with $\Omega^1$-intertwinings in $\mathcal{A}\mathcal{I}\mathcal{M}\mathcal{A}^{\Omega^1}$, will have invertible $\Omega^1$-intertwinings. We now show that in fact $\mathcal{A}\mathcal{I}\mathcal{M}\mathcal{A}^{\Omega^1}$ is closed and $H(\Omega^1)$ is a Schauenburg Hopf algebroid. In fact, $H(\Omega^1)$ admits an invertible antipode and has the form of a Böhm-Szlachányi Hopf algebroid.

**Theorem 4.11.** The map $S : H(\Omega^1) \rightarrow H(\Omega^1)$ is defined by
\begin{align}
S(a) &= \overline{a}, \quad S((x, \omega)) = (\omega, x) \\
S(\overline{a}) &= a, \quad S((\omega, x)) = (x, \omega)
\end{align}
for $a \in A$, $\omega \in \Omega^1$ and $x \in \mathfrak{X}^1$ and extended anti-multiplicatively to $H(\Omega^1)$. The map $S$ is a well-defined anti-algebra automorphism of algebra $H(\Omega^1)$, with $S^{-1} = S$ and satisfies the conditions in Definition 2.2 (II).
Proof. We have defined $S$ on the generators of the algebra, and must verify that $S$ is well-defined by looking at the relations. Notice that relations (32) and (33) are symmetric under $S$ and is well-defined on relation (34) due to relation (39).

$$S((\omega_i, x)(x_i, \omega)) = S((x_i, \omega)) \star S((\omega_i, x))$$

$$= (\omega, x_i) \star (x, \omega_i) = ev(\omega, x) = S(\overline{ev}(\omega, x))$$

where $x \in \mathcal{X}^1$ and $\omega \in \Omega^1$. Similar arguments apply for the other relations and one can conclude that $S$ is well defined and by definition $S = S^{-1}$. Since the image of the coproduct falls in the Takeuchi $\times$-product, we only need to check the antipode conditions (13) and (14) on the generators of the bialgebroid. For generators $(x, \omega) \in \mathcal{X}^1 \otimes \Omega^1$,

$$S((x, \omega)(1)) \star (x, \omega)(2) \circ S((x, \omega)(1)) = (\omega, x)(1) \star (x, \omega)(2)$$

$$= (\omega, y_j) \star (x, \omega) \circ (\rho_j, x) = \overline{ev}(\omega, y_j) \circ (\rho_j, x)$$

$$= 1 \circ \overline{ev}(\omega, y_j) \star (\rho_j, x) = 1 \circ (\omega, x) = 1 \circ S(x, \omega)$$

and

$$S^{-1}(x, \omega)(1) \circ S^{-1}(x, \omega)(2)(1) \circ (x, \omega)(1) = (\omega, x)(1) \circ (\omega, x)(1) \star (x, \omega)$$

$$= (\omega, y_j) \circ (\rho_j, x) \star (x, \omega_i) = (\omega, y_j) \otimes \overline{ev}(\rho_j, x)$$

$$= \overline{ev}(\rho_j, x) \star (\omega, y_j) \circ 1 = (\omega, x) \circ 1 = S^{-1}(x, \omega) \circ 1$$

hold. A symmetric argument applies for generators of the form $(\omega, x) \in \Omega^1 \otimes \mathcal{X}^1$. □

Using the antipode we can describe the closed structure of $\mathcal{X}^1 \otimes \mathcal{M}^\otimes A$, which lifts that of $A \mathcal{M}^\otimes_A$. For a pair of $H(\Omega^1)$-modules $M$ and $N$, we recover the action of $H(\Omega^1)$ by (16):

$$[(x, \omega)f](m) = (x, \rho_j)f((\omega, y_j)m), \quad [(\rho, y)f](m) = (\rho, x)\overline{f}((y, \omega_i)m)$$

$$[(x, \omega)g](m) = (y_j, \omega)g((\rho_j, x)m), \quad [(\rho, y)g](m) = (\omega_i, y)g((x, \rho)m)$$

for any $m \in M, (x, \omega) \in \mathcal{X}^1 \otimes \Omega^1, (\rho, y) \in \Omega^1 \otimes \mathcal{X}^1$, $f \in \text{Hom}_A(M, N)$ and $g \in A \text{Hom}(M, N)$.

Notation. We have used the notation $[hf](m) = h_{(+)f}(h_{(-)}m)$ to distinguish between $[hf](m)$, where $[hf]$ is the morphism obtained by $h \in H(\Omega^1)$ acting on the morphism $f \in \text{Hom}_A(M, N)$ and $hf(m)$, where $h$ acts on $f(m)$ as an element of $N$. In what follows, we will continue to adapt this notation.

Now we look at bimodule connections whose underlying intertwinings belong to $\mathcal{X}^1 \otimes \mathcal{M}^\otimes A$. At this point it should be clear that to do this we need to take the quotient of $IBX^1$ by the ideal generated by the set of relations (45) and (49). We denote this algebra by $HX^1$. From the arguments in Lemmas 4.2 and 4.9, it follows that the resulting algebra carries down the left $A$-bialgebroid structure of $BX^1$. Moreover, $HX^1$ is a Schauenburg Hopf algebroid. Observe that in order to demonstrate this, we only need to prove that the category of $HX^1$-modules lifts the closed structure of $A \mathcal{M}^\otimes_A$. Since we have already described the action of $H(\Omega^1)$ on $\text{Hom}_A(M, N)$ and $A \text{Hom}(M, N)$, we only need to present a well-defined action of elements of $\mathcal{X}^1$ in $HX^1$, or in particular a connection on $\text{Hom}_A(M, N)$ and $A \text{Hom}(M, N)$.
Theorem 4.12. For $H \mathcal{X}^1$-modules $M, N$, we can extend the actions of $H(\Omega^1)$ on the inner homs, to actions of $H \mathcal{X}^1$ by defining the action of elements $x \in \mathcal{X}^1$ by

\begin{align*}
[xf](m) &= x(f(m)) - (x, \rho_j) f((\omega_i, y_j) \bullet x, m) \\
[xg](m) &= y_j g((\rho_j, x)m) - g((\omega_i, y_j) \bullet x, m)
\end{align*}

(44)

where $m \in M, f \in \text{Hom}_A(M, N)$ and $g \in \text{AHom}(M, N)$, so that the closed monoidal structure of $A^1 \mathcal{M}$, we can extend the actions of $\mathcal{A}^1 \mathcal{M}$ lifts to the category of $H \mathcal{X}^1$-modules.

Proof. We must first check that the $H \mathcal{X}^1$-actions defined above are well defined. We then proceed to showing that the units and counits of the adjunctions providing the closed structure of $A^1 \mathcal{M}$, are $H \mathcal{X}^1$-module morphisms. Since the actions of elements in $\mathcal{X}^1 \otimes \mathcal{X}^1$ and $\mathcal{X}^1 \otimes \Omega^1$ are lifted from $H(\Omega^1)$, we only need to check these facts on the generators of the form $x \in \mathcal{X}^1$. In particular, we only need to look at relations (24), (25) and (26) for the $H \mathcal{X}^1$-action to be well-defined:

\begin{align*}
[(ax)f](m) &= a(x(f(m))) - (x, \rho_j) f((\omega_i, y_j) \bullet x, m) \\
[(xa)f](m) &= x(a(f(m))) - (x, \rho_j) a(f((\omega_i, y_j) \bullet x, m))
\end{align*}

(45)

\begin{align*}
[(x\bullet \tau)f](m) &= x(f(am)) - (x, \rho_j) f((\omega_i, y_j) \bullet x, am) \\
[(x\bullet \tau)f](m) &= x(f(am)) - (x, \rho_j) f((\omega_i, y_j) \bullet x, am)
\end{align*}

(46)

where $a \in A, x \in \mathcal{X}^1$ and $f \in \text{Hom}_A(M, N)$. Similarly for $g \in \text{AHom}(M, N)$, we note that the right $A$-action on $M$ arises from the action of $A^{\text{op}} \subset H \mathcal{X}^1$.

\begin{align*}
[(ax)g](m) &= y_j g((\rho_j, x) \bullet \tau m) - g((\omega_i, y_j) \bullet x, m) \\
[(xa)g](m) &= x_j g((\rho_j, x) \bullet \tau m) - g((\omega_i, y_j) \bullet x, m)
\end{align*}

(47)

\begin{align*}
[(x\bullet \tau)g](m) &= y_j \bullet \tau g((\rho_j, x)m) - \tau g((\omega_i, y_j) \bullet x, m) \\
[(x\bullet \tau)g](m) &= y_j \bullet \tau g((\rho_j, x)m) - \tau g((\omega_i, y_j) \bullet x, m)
\end{align*}

(48)

Hence, the actions of $H(\Omega^1)$ on the inner homs extend to well-defined actions of $H \mathcal{X}^1$. We now show that the the unit and counit, $\varepsilon^M$ and $\varepsilon^N$, of the adjunction $- \otimes M \dashv \text{Hom}_A(M, -)$, respect the $H \mathcal{X}^1$-actions. Let $x \in \mathcal{X}^1, f \in \text{Hom}_A(M, N), m \in M$ and $n \in N$.

\begin{align*}
[x^M](m) &= x_n(m) - (x, \rho_j) n((\omega_i, y_j) \bullet x, m) \\
=[x^M](m) &= x_n(m) - (x, \rho_j) n((\omega_i, y_j) \bullet x, m)
\end{align*}

(49)

\begin{align*}
\varepsilon^M(x(f \otimes m)) &= \varepsilon^M((x, \omega_i) f \otimes x, m) = [x](m) \\
&+ [x, \omega_i](x, m) = x(f(m)) - (x, \rho_j) f((\omega_i, y_j) \bullet x, m)
\end{align*}

(50)
Similarly, we look at the unit and counit, $\Theta^M$ and $\Pi^M$, of the adjunction $M \otimes - \dashv \text{Hom}(M, -)$ respecting the $H \mathcal{X}^1$-actions. Let $x \in \mathcal{X}^1, g \in \text{AHom}(M, N), m \in M$ and $n \in N$.

\[
[x\Theta^M_N(n)](m) = y_jg_n((\rho_j, x)m) - g_n(y_j\bullet(\rho_j, x)m) \\
= y_j((\rho_j, x)m \otimes n) - y_j\bullet(\rho_j, x)m \otimes n \\
= y_j(\rho_j, x)m \otimes n + (y_j, \omega_i)(\rho_j, x)m \otimes x_n - y_j\bullet(\rho_j, x)m \otimes n \\
= \text{ev}(x, \omega_i)m \otimes x_n = m \otimes xn = [\Theta^M_N(xn)](m)
\]

\[
\Pi^M_N(x(m \otimes g)) = \Pi^M_N(xm \otimes (x, \omega_i)m \otimes x, g) \\
= g(xm) + [x, g]((x, \omega_i)m) = g(xm) + y_jg((\rho_j, x_i)\bullet(\omega_i, x_i)m) \\
- g(y_j\bullet(\rho_j, x_i)\bullet(\omega_i, x_i)m) = g(xm) + y_jg(\text{ev}(\rho_j \otimes x)m) \\
- g(y_j\text{ev}(\rho_j \otimes x)m) = g(xm) + xg(m) - g(xm) \\
+ \text{ev}(y_j \otimes \text{ev}(\rho_j \otimes x))(m) - g(\text{ev}(y_j \otimes \text{ev}(\rho_j \otimes x)))m \\
= g(xm) = x\Pi^M_N(m \otimes g)
\]

Proof: 

The Hopf algebroid $H \mathcal{X}^1$ is not expected to admit an antipode in general. For the existence of an antipode, we require a linear map $\Upsilon : \mathcal{X}^1 \to A$ satisfying

\[
\Upsilon(xa) = \Upsilon(x)a + \text{ev}(x \otimes da), \quad \Upsilon(ax) = a\Upsilon(x) + \text{ev}(da \otimes x) \quad (46)
\]

for any $x \in \mathcal{X}^1$ and $a \in A$. In fact, the existence of such a map is equivalent to $H \mathcal{X}^1$ admitting an antipode.

**Theorem 4.13.** The Hopf algebroid $H \mathcal{X}^1$ admits an invertible antipode if and only if there exists a linear map $\Upsilon : \mathcal{X}^1 \to A$ satisfying (46). In particular, if such $\Upsilon$ exists, the maps $S$ and $S^{-1}$ defined by

\[
S(x) = - (\omega_i, x) \bullet x_i - \Upsilon(x) \quad (47)
\]

\[
S^{-1}(x) = - (y_j + \Upsilon(y_j)) \bullet (\rho_j, x) \quad (48)
\]

for $x \in \mathcal{X}^1$, extend $S$ and $S^{-1}$ from Theorem 4.7 to well-defined anti-algebra morphisms on $H \mathcal{X}^1$ and are inverses. Furthermore, they satisfy the conditions in Definition 2.2(II).

**Proof.** ($\Rightarrow$) First we recall the following elementary fact stated in [23]: if a Hopf algebroid admits an antipode $S : H \mathcal{X}^1 \to H \mathcal{X}^1$ as defined in Definition 2.2(II), then

\[
a \triangleleft h = e(S(h) \bullet a), \quad a \in A, \ h \in H \mathcal{X}^1
\]

defines a right action of the algebra $H \mathcal{X}^1$ on $A$, such that the action $A^{op} \subset H \mathcal{X}^1$ coincides with left multiplication i.e. $a_1 \triangleleft a_2 = a_2a_1$ for $a_1, a_2 \in A$. Hence, we define the map $\Upsilon : \mathcal{X}^1 \to A$ by $\Upsilon(x) := - e(S(x)) = - 1 \triangleleft x$. It is then straightforward to check that (46) holds:

\[
\Upsilon(ax) = - e(S(ax)) = - e(S(x) \bullet \omega) = - e(S(x) \bullet a) = - e(S(\omega \bullet x))
\]
Hence, we only have to check relations (24), (25) and (26). First we demonstrate this for $S$:

$$S(a \star x) = S(x) \star a = -(\omega_i, x) \star x_i \star a - \Upsilon(x) \star a$$

$$= -((\omega_i, ax) \star x_i - (\omega_i, x) \star (x_i, da) - a \Upsilon(x))$$

$$S(x \star a) = \overline{a} \star S(x) = -((\omega_i, xa) \star x_i - \Upsilon(x)a)$$

$$S(x \star \overline{a}) = a \star S(x) = -(\omega_i, x) \star x_i - a \Upsilon(x) = -((\omega_i, x) \star x_i - \Upsilon(x)a)$$

and for $S^{-1}$:

$$S^{-1}(a \star x) = S^{-1}(x) \star a = -(y_j + \Upsilon(y_j)) \star (p_j, x) \star a = S^{-1}(ax)$$

$$S^{-1}(x \star a) = \overline{a} \star S^{-1}(x) = -((y_j + \Upsilon(y_j)) \star (p_j, x))$$

$$S^{-1}(x \star \overline{a}) = a \star S^{-1}(x) = -a \star (y_j + \Upsilon(y_j)) \star (p_j, x)$$

where $a \in A$ and $x \in \mathcal{X}^1$. We must also check that $S$ and $S^{-1}$ are inverse. Let $x \in \mathcal{X}^1$.

$$S^{-1}(x) = -S^{-1}(x) \star (x, \omega_i) - \Upsilon(x)$$

$$= (y_j + \Upsilon(y_j)) \star (p_j, x) \star (x, \omega_i) - \Upsilon(x)$$

$$SS^{-1}(x) = -(x, p_j) \star \left(S(y_j) + \Upsilon(y_j)\right)$$

$$= (x, p_j) \star \left((\omega_i, y_j) \star x_i + \Upsilon(y_j)\right) - (x, p_j) \Upsilon(y_j)$$

$$= (x, p_j) \star (\omega_i, y_j) \star x_i = \Upsilon(x \otimes \omega_i)x_i = x$$

Since the coproduct falls in the Takeuchi product, we only need to verify axioms (13) and (14) on the generators of the bialgebroid. Let $x \in \mathcal{X}^1$.

$$S(x_1) \star x_2 \circ S(x_1) = (\omega_i, x) \star x_1 \circ (\omega_i, x) (\omega_i, x) \star x_1 = S(x_1) \circ S(x_2)$$

$$= (\omega_i, y_j) \star x_i \circ (p_j, x) - (\omega_i, x) \star x_i \circ S(\omega_i, x) \star x_i$$

$$- 1 \circ \Upsilon(x) = -(\omega_i, y_j) \star (x_i, x_k) \circ (p_j, x) \star x_k - 1 \circ \Upsilon(x)$$
bimodule and the Hopf algebroid

\( H \)

with the relations of a collection of elements

d \[\text{Derivation Calculus} \]

Recall the bialgebroid constructed in Example 4.15.

\[\{ \text{with basis} \ \Omega \ \text{admit antipodes. In particular, if the calculus} \]

As a corollary of Theorem 4.13, several of the Hopf algebroids constructed here will

\( \text{product, counit and antipode are extended as follows} \)

\( \text{In the algebraic manipulations above, both properties of} \) 46 \( \text{have been used but the}

additional terms have been omitted. \)

For any pair of \( HX^1 \)-modules \( M \) and \( N \), one can easily check that the induced connections on the inner homs \( \text{Hom}_A(M, N) \) and \( \text{AHom}(M, N) \) calculated via the antipode, \( (15) \), agrees with those presented in Theorem 4.12. In particular, the terms including \( \Upsilon \) cancel out in the calculation of \( (15) \).

\textbf{Remark 4.14. In the classical theory of Hopf algebras, if a bialgebra admits an antipode, the antipode is unique. However, as demonstrated by the above theorem, this is not true for Hopf algebroids. In fact, one can add any bimodule morphism} \( \phi : X^1 \to A \)

to \( \Upsilon \) and \( \Upsilon + \phi \) will again satisfy 46.

\textbf{4.4 Examples of Hopf Algebroids}

As a corollary of Theorem 4.13 several of the Hopf algebroids constructed here will admit antipodes. In particular, if the calculus \( \Omega^1 \) is a finitely generated free \( A \)-bimodule with basis \( \{ f_i \}_{i=1}^n \) for \( X^1 \), then \( \Upsilon(\sum_i a_i f_i) = \sum_i ev(da_i \otimes f_i) \) satisfies (46), for any collection of elements \( a_i \in A \).

\textbf{Example 4.15. [Derivation Calculus]} \text{Recall the bialgebroid constructed in Example 3.8 for a derivation} \( d : A \to A \). \text{To obtain} \( HX^1 \), \text{a new generator} \( E = (1, 1) \in \Omega^1 \otimes_X X^1 \) \text{is added and the new relations are equivalent to} \( F \cdot E = 1 = E \cdot F \). \text{Hence,} \( HX^1 = A^e \cdot X(D, F, F^{-1}) \) \text{with the commutation relations in Example 3.8. The} \text{coproduct, counit and antipode are extended as follows} \)

\( \Delta(F^{-1}) = F^{-1} \otimes F^{-1}, \quad \epsilon(F^{-1}) = 1 \)

\( S(D) = -F^{-1}D \quad S(F) = F^{-1} \quad S(F^{-1}) = F \)

\textbf{Example 4.16. [M}_2(\mathbb{C})] \text{For the differential calculus of Example 2.8} \), \( \Omega^1 \text{is a free bimodule and the Hopf algebroid} \) \( HX^1 \text{factorises as} \ A^e \cdot \mathbb{C}

\( \{ f_{i, j}, \kappa_j, \gamma_j \mid i, j \in \{ s, t \} \} \) \text{with the relations of} \ BX^1 \text{presented in Example 3.10 and additional relations} \)

\( [\kappa_j, a \kappa_j] = 0 \)

\( z \gamma_j \cdot \kappa_j + \zeta \gamma_j \cdot \kappa_j = \delta_{i, j} = z \gamma_s \cdot \kappa_s + \zeta \gamma_t \cdot \kappa_t \)

\( z \kappa_j \cdot z \gamma_j + \zeta \kappa_j \cdot z \gamma_j = \delta_{i, j} = z \kappa_s \cdot \gamma_s + \zeta \kappa_t \cdot \gamma_t \)
for all $i, j \in \{s, t\}$ and $a^e \in A^e$. The coproduct, counit and antipode extend similarly by

$$
\Delta(k_j) = \iota_{k_s} \otimes \iota_{k_j} + \iota_{k_t} \otimes \iota_{k_j}, \quad \epsilon(k_j) = \delta_{i,j}
$$

$$
S(f_i) = -\iota_{k_s} \cdot f_s - \iota_{k_t} \cdot f_t, \quad S(\gamma_j) = \iota_{k_s}, \quad S(k_j) = \iota_{k_i}
$$

for all $i, j \in \{s, t\}$.

**Example 4.17.** [Finite Quiver] For a finite quiver $\Gamma = (V, E)$, we described $B\mathcal{X}^1$ as an extension of the quiver path algebra $k\Gamma$ in Example 3.9. The resulting Hopf algebroid on $k(V)$ can also be described with relation to the quiver path algebra and additional generators as

$$
(\vec{e}_1, \vec{e}_2) \mapsto (\vec{e}_1, \vec{e}_2) \mid p \in S, \ e_1, e_2 \in E
$$

with the relations presented in Example 3.9 and additional relations

$$
\sum_{e \in E} (\vec{e}_1, \vec{e}) \cdot (\vec{e}_2, \vec{e}) = f_{e_1} \cdot \delta_{e_1, e_2}, \quad \sum_{e \in E} (\vec{e}, \vec{e}_1) \cdot (\vec{e}_2, \vec{e}) = f_{e_2} \cdot \delta_{e_2, e_2}
$$

for all $e_1, e_2 \in E$ and $p, q \in V$. The coproduct and counit of the new generators are given by

$$
\Delta((\vec{e}_1, \vec{e}_2)) = \sum_{e \in E} (\vec{e}_1, \vec{e}_1) \otimes (\vec{e}, \vec{e}_2), \quad \epsilon((\vec{e}_1, \vec{e}_2)) = \delta_{e_1, e_2} f_{e_1}
$$

for any $e_1, e_2 \in E$. In fact $H \mathcal{X}^1$ admits an antipode since the map $\Upsilon : \mathcal{X}^1 \to A$ defined by $\Upsilon((\vec{e})) = f_{e_1} - f_{e_2}$, for $e \in E$, satisfies (40). Translating this data in terms of $k\Gamma$, the antipode takes the form

$$
S(\vec{e}_1) = -\sum_{e \in E} (\vec{e}, \vec{e}_1) \cdot e - \sum_{e \in E} (\vec{e}, \vec{e}_1) - f_{e_1}
$$

$$
S((\vec{e}_1, \vec{e}_2)) = (\vec{e}_2, \vec{e}_1), \quad S((\vec{e}_1, \vec{e}_2)) = (\vec{e}_2, \vec{e}_1)
$$

for any $e_1, e_2 \in E$.

**Example 4.18.** [Bicovariant Calculi] If $A$ is a Hopf algebra and $\Omega^1$ a bicovariant calculus over $A$, then as demonstrated in Example 3.7, $\Omega^1$ is free as a left $A$-module so that $\Omega^1 \otimes_k (\Omega^1)^* \cong A^e \otimes_k (\Lambda \otimes_k \Lambda^*)$ as a left $A^e$-module. Hence the Hopf algebroid $H \mathcal{X}^1$ factorises as $A^e T\mathcal{M}$ where $\mathcal{M} = \Lambda^e \otimes (\Lambda^e \otimes_k \Lambda) \otimes (\Lambda \otimes_k \Lambda^*)$, with the relations present in Example 4.17 and additional commutation relations

$$
(\lambda_1, f_k) \ast \mathcal{B} = a_{(2)}(a_{(1)}) \ast (s^{-1}(a_{(1)}) \triangleright \lambda_j, f_j \triangleleft s^{-1}(b_{(1)}))
$$

$$
\sum_{i=1}^{n} (\lambda_1, f_k) \ast (f_i, \lambda_j) = \delta_{j,k} = \sum_{i=1}^{n} (f_j, \lambda_i) \ast (\lambda_k, f_i)
$$

$$
\sum_{i=1}^{n} (f_i, \lambda_j) \ast (s^{-2}(\lambda_i)(-1) \triangleright (\lambda_i)(0), f_k) = f_k(s^{-2}(\lambda_j)(-1) \triangleright (\lambda_j)(0))
$$
for all $1 \leq j, k \leq n$ and $a, b \in A$. The coproduct and counit of $B\mathcal{X}^1$ extend to $H\mathcal{X}^1$ by

$$
\Delta((\lambda_i, f_j)) = \sum_{k=1}^{n} (\lambda_i, f_k) \otimes (\lambda_k, f_j) \quad \epsilon((\lambda_i, f_j)) = \delta_{i,j}
$$

for all $1 \leq i, j \leq n$.

**Example 4.19.** $[\mathbb{C}D_0]$ For the differential calculus of Example 3.12, the left connection on $\Lambda$ is trivial. Hence, the Hopf algebroid $H\mathcal{X}^1$ over the group algebra $\mathbb{C}D_0$, factorises as $A^e:\mathbb{C}\langle f_i, i, j \rangle$ with the relations of $B\mathcal{X}^1$ as presented in Example 3.12 and additional relations

$$
\begin{align*}
\xi i \bullet a &= \frac{1}{2}(\xi i + \sqrt{3} \gamma i), \quad \tau i \bullet a = \frac{1}{2}(\sqrt{3} \xi i + \gamma i) \\
\iota \xi j \bullet \iota \tau \xi j &= \frac{1}{2}(\xi \xi - \sqrt{3} \gamma \xi i), \quad \iota \tau \xi j \bullet \iota \gamma \xi j = \frac{1}{2}(\sqrt{3} \xi \xi + \gamma \xi j)
\end{align*}
$$

for $i, j \in \{\xi, \tau\}$. The coproduct and counit extend as

$$
\Delta(\iota i \xi ) = \iota \xi \otimes \xi i + \tau i \otimes \gamma i, \quad \epsilon(\iota i \xi ) = \delta_{i,j}
$$

for $i, j \in \{\xi, \tau\}$.

## 5 Flat Bimodule Connections

Classically, the curvature on connections is defined using the Lie bracket on vector fields or alternatively, the exterior derivative from the space of 1-forms to the space of 2-forms. In this section, we assume $d: A \to \Omega^1$ is part of a dga $\Omega^*$. However, we only require the bimodule $\Omega^2$ and linear maps $d: \Omega^1 \to \Omega^2$ and $\wedge: \Omega^1 \otimes \Omega^1 \to \Omega^2$, satisfying the relevant properties, as additional data. We briefly recall the definitions of curvature, flat connections and the sheaf of differential operators from [5].

If $(M, \nabla)$ is a left connection, then the curvature of $\nabla$ is a map $R_M: M \to \Omega^2 \otimes M$ defined by

$$
R_M = (d \otimes \text{id}_M - \text{id}_M \otimes \nabla) \nabla
$$

We say $(M, \nabla)$ is a flat left connection, if $R_M = 0$ and denote the subcategory of flat left connections in $\mathcal{E}$ by $\mathcal{F}$.

In Chapter 6 of [5], the category $\mathcal{F}$ is shown to be isomorphic to the category of modules over an algebra, $D_A$, when $\Omega^2$ is right fgp. We denote the left dual bimodule of $\Omega^2$ by $\mathcal{X}^2$ and denote the respective coevaluation and evaluation maps by $\text{coev}$ and $\text{ev}$ and denote $\text{coev}(1) = \sum x_i^2 \otimes \omega_i^2$. In this case, $\mathcal{F} \cong D_A M$, where $D_A$ is the algebra obtained as the quotient of $TX^*_A$ by the ideal generated by relations

$$
\text{ev}(x^2 \otimes d \omega_i) x_i = \text{ev}(x^2 \otimes \omega_i \wedge \omega_k) x_k x_j = 0
$$
for all $x^2 \in \mathfrak{X}^2$. It is easy to verify that the mentioned ideal annihilating a $T\mathfrak{X}^1$-module is equivalent to the induced connection on the module being flat [Corollary 6.24 [5]].

Although one could quotient out the algebra $B\mathfrak{X}^1$ by the same relations, (49), and discuss bimodule connection which have a flat left connection, the tensor product of two such connections will not have zero curvature. To discuss a monoidal category of flat bimodule connections, we must assume the bimodule connections are extendable. We say an $\Omega^1$-intertwining map $\sigma : M \otimes \Omega^1 \to \Omega^1 \otimes M$ is extendable if there exists an $\Omega^2$-intertwining map $\sigma_2 : M \otimes \Omega^2 \to \Omega^2 \otimes M$ such that the equation

$$ (\wedge \otimes \text{id}_M)(\text{id}_{\Omega^1} \otimes \sigma)(\sigma \otimes \text{id}_{\Omega^1}) = \sigma_2(\text{id}_M \otimes \wedge) \quad (50) $$

holds as an equality of bimodule morphisms with domain $M \otimes \Omega^1 \otimes \Omega^1$ and codomain $\Omega^2 \otimes M$. An additional condition is required when the calculus is not surjective. The equation

$$ (\wedge \otimes M)(\text{id}_{\Omega^1} \otimes \sigma)(\nabla \otimes \text{id}_{\Omega^1}) + (\text{id}_{\Omega^1} \otimes \nabla)\sigma = (d \otimes \text{id}_M)\sigma - \sigma_2(\text{id}_M \otimes d) \quad (51) $$

must hold for linear maps with domain $M \otimes \Omega^1$ and codomain $\Omega^2 \otimes M$. This condition appears implicitly in Lemma 4.12 of [5] and is said to be equivalent to the curvature being a right module morphism. However, if the calculus is not surjective, this is an additional condition. The subcategory of left bimodule connections which are flat, extendable and satisfy condition (51), is a monoidal subcategory of $\mathcal{A}^{\mathfrak{A}}$ and is denoted by $\mathcal{A}^{\mathfrak{A}}$. This is discussed in Section 4.5.1 of [5]. To obtain the bialgebroid whose category of modules is isomorphic to $\mathcal{A}^{\mathfrak{A}}$, we must adjoin additional generators of the form $\mathfrak{X}^2 \otimes \mathfrak{X}^2$ to $B\mathfrak{X}^1$, to induce $\Omega^2$-intertwinings and quotient out the corresponding relations for flatness (49), extendability (53) and the additional condition (52). However, the category $\mathcal{A}^{\mathfrak{A}}$ will again not lift the closed structure of $\mathcal{A}^{\mathfrak{A}}$. Instead, we will look at the relevant closed monoidal subcategory of flat bimodule connections in $\mathcal{A}^{\mathfrak{A}}$, and the construction of the relevant bialgebroid for $\mathcal{A}^{\mathfrak{A}}$ will also be implicitly present in our work.

### 5.1 Hopf Algebroid $\mathcal{D}\mathfrak{X}$ in Flat Case

The closed subcategory of flat bimodule connections with extendable $\Omega^1$-intertwining which we would like to consider, should lift the closed structure of $\mathcal{A}^{\mathfrak{A}}$. Since the extendability condition adds an underlying $\Omega^2$-intertwining to our connection, the underlying $\Omega^2$-intertwining of such bimodules must belong to the appropriate closed subcategory of $\Omega^2$-intertwinings. Hence, as for $\Omega^1$-intertwinings in Section 4.3, we assume $\Omega^2$ is left and right fgp and pivotal as an $A$-bimodule. We denote the relevant coevaluation and evaluation maps between $\Omega^2$ and its left and right dual $\mathfrak{X}^2$, by $\text{coev}$, $\text{cev}$, $\text{ev}$ and $\text{pev}$. We utilise the following notation $\text{coev}(1) = \sum_i x_i \otimes \omega_i \in \mathfrak{X}^2 \otimes \Omega^2$ and $\text{cev}(x^2) = \sum_i r_i x_i \otimes y_i \in \Omega^2 \otimes \mathfrak{X}^2$.

Additionally, we require $\wedge$ to be a pivotal bimodule morphism i.e. for any $x^2 \in \mathfrak{X}^2$, the equation

$$ \text{ev}(x^2 \otimes \omega_i \wedge \omega_j)x_j \otimes x_i = y_i \otimes y_j \text{cev}(\rho_i \wedge \rho_j \otimes x^2) \quad (52) $$

holds for elements of $\mathfrak{X}^1 \otimes \mathfrak{X}^1$. Since both $\Omega^1$ and $\Omega^2$ are both pivotal, $\wedge$ provides two bimodule morphisms from $\mathfrak{X}^2$ to $\mathfrak{X}^1 \otimes \mathfrak{X}^1$, presented on either side of the equation above, and condition (52) requires these two bimodule morphisms to be equal.

We note that the free product of two Hopf algebras over an algebra $A$, as $A^\times$-algebras will again be a Hopf algebroid over $A$. Since modules over the free product
are just $A$-bimodules with additional actions of each algebra, the action of both Hopf algebroids on tensor products and inner homs simply lift to the category of modules over the free product. Hence, We obtain a new Hopf algebroid by considering the free product of $A \otimes_{X} A^{op}$-algebroids $H \otimes_{X}^{1}$ and $H(\Omega^{2})$ and denote it by $F$. We define $D X$ as the quotient of $F$ by the ideal generated by relations $49$ and

$$\text{ev}(x^{2} \otimes \omega_{i} \wedge \omega_{j})[x_{j} \bullet (x_{i}, \omega)] + (x_{j}, \omega) \bullet x_{i} = \text{ev}(x^{2} \otimes d\omega_{i})(x_{i}, \omega) - (x^{2}, d\omega)$$

(53)

and

$$\text{ev}(x^{2} \otimes \omega_{i} \wedge \omega_{j})(x_{j}, \rho) \bullet (x_{i}, \omega) = (x^{2}, \omega \wedge \rho)$$

(54)

$$\text{ev}(\rho_{i} \wedge \rho_{j} \otimes x^{2})(\omega, y_{j}) \bullet (\rho, y_{j}) = (\omega \wedge \rho, x^{2})$$

(55)

for all $x^{2} \in X^{2}$ and $\omega, \rho \in \Omega_{1}^{1}$.

Firstly, note that an $F$-module $M$ is an $A$-bimodules with a left bimodule connection $(M, \nabla, \sigma)$, so that $(M, \sigma)$ lies in $X_{A}^{1} IM_{A}^{1}$ and an invertible $\Omega^{2}$-intertwining $\sigma_{2}$ such that $(M, \sigma_{2})$ lies in $X_{A}^{1} IM_{A}^{2}$. By constructing $\sigma$ and $\sigma_{2}$ for an $F$-module, as described in Theorem 3.3, we can deduce that the annihilation of the module by relations $53$ and $55$ is equivalent to $\sigma$ and $\sigma^{-1}$ extending to $\sigma_{2}$ and $\sigma_{2}^{-1}$, respectively. Since $\wedge$ is a pivotal morphism and $54$ holds, relations $53$ and $55$ are equivalent to relations

$$(y_{i}, \rho) \wedge (y_{j}, \omega) \text{ev}(\rho_{j} \wedge \rho_{i} \otimes x^{2}) = (x^{2}, \omega \wedge \rho)$$

(56)

$$(\omega, x_{i}) \bullet (\rho, x_{j}) \text{ev}(x^{2} \otimes \omega_{i} \wedge \omega_{j}) = (\omega \wedge \rho, x^{2})$$

(57)

respectively. Recall that any $H(\Omega^{1})$-module has a pair of induced $X^{1}$-intertwining, $41$ and $40$, which are inverses. Relations $56$ and $57$ annihilating an $F$-module, are equivalent to the induced $X^{1}$-intertwinnings on the module, extending to the corre- sponding $X^{2}$-intertwinnings.

Relation $55$ annihilating an $F$-module, is equivalent to the induced bimodule connection and intertwinedings of the $F$-module satisfying the additional condition $51$. We previously noted that, when the calculus in question is surjective i.e. $\Omega_{1}$ is generated by elements of the form $bda$ where $a, b \in A$, then relation $55$ follows from $49$ and $54$.

$$0 = 0_{bda} = (\text{ev}(x^{2} \otimes d\omega_{i}) \bullet x_{i} - \text{ev}(x^{2} \otimes \omega_{i} \wedge \omega_{k}) \bullet x_{k} \bullet x_{j}) \bullet bda$$

$$= \text{ev}(x^{2} \otimes d\omega_{i}) \bullet x_{i} + \text{ev}(x^{2} \otimes d\omega_{i})(x_{i}, da) - \text{ev}(x^{2} \otimes \omega_{i} \wedge \omega_{k}) \bullet x_{k} \bullet x_{j}$$

$$- \text{ev}(x^{2} \otimes \omega_{j} \wedge \omega_{k}) \bullet [x_{k} \bullet (x_{j}, da) + (x_{k}, da) \bullet x_{j}] \bullet bda$$

$$= \text{ev}(x^{2} \otimes d\omega_{i})(x_{i}, da) - \text{ev}(x^{2} \otimes \omega_{i} \wedge \omega_{k})(x_{k} \bullet (x_{j}, da) + (x_{k}, da) \bullet x_{j})$$

$$= \text{ev}(x^{2} \otimes \omega_{j} \wedge \omega_{k})(x_{k}, bda) - \text{ev}(x^{2} \otimes \omega_{j} \wedge \omega_{k})(x_{k} \bullet (x_{j}, bda) + (x_{k}, bda) \bullet x_{j})$$

$$= \text{ev}(x^{2} \otimes \omega_{j} \wedge \omega_{k})(x_{k}, bda) - (x^{2}, db \wedge da)$$

for any $x^{2} \in X^{2}$ and $a, b \in A$.

Remark 5.1. If $\wedge : \Omega^{1} \otimes \Omega^{1} \rightarrow \Omega^{2}$ splits as a bimodule map, we do not need to add- tional generators to $H \otimes_{X}^{1}$ to capture the intertwining map extending. In other words, when $\wedge$ is surjective, the relations imposed in $D X$, describe the additional generators of $H(\Omega^{2})$ in terms of elements of $H \otimes_{X}^{1}$. Additionally when $\wedge$ splits, the extendabilty conditions would simply be equivalent to relations

$$\text{ev}(x^{2} \otimes \omega_{i} \wedge \omega_{j}) \bullet (x_{j}, \rho) \bullet (x_{i}, \omega) = 0 = \text{ev}(\rho_{i} \wedge \rho_{j} \otimes x^{2})(\omega, y_{j}) \bullet (\rho, y_{j})$$

}
5 FLAT BIMODULE CONNECTIONS

on $H^X$, for all $\omega \wedge \rho \in \ker(\wedge)$.

**Notation.** From this point onwards, whenever the action of elements of $A$ and $A^{op}$ agrees with a module action on elements in the algebras constructed, we avoid writing $\ast$ for brevity. For example, for elements $a \in A$, $x \in X^1$ and $\omega \in \Omega^1$, we simply write $ax$ and $a(x, \omega)$ instead of $a \ast x$ and $a(x, \omega)$, respectively.

**Theorem 5.2.** The algebra $DX$ inherits the bialgebroid structure of $F$.

**Proof.** We only need to check that the comultiplication and counit of $F$ are well defined on its quotient $DX$. We first look at the comultiplication and the extendibility relations. Let $x^2 \in X^2$ and $\omega, \rho \in \Omega^1$ and consider relation (53):

$$\Delta(\mathbf{ev}(x^2 \otimes \omega_i \wedge \omega_k) \ast x_h \ast x_j) = \mathbf{ev}(x^2 \otimes \omega_j \wedge \omega_k)[x_h \ast x_j \otimes 1 +$$

$$+ x_h \ast (x_j, \omega_i) \otimes x_i + (x_h, \omega_i) \ast x_j \otimes x_i + (x_h, \omega_i) \ast (x_j, \omega_k) \otimes x_i \ast x_m]$$

$$= \mathbf{ev}(x^2 \otimes d\omega_i) \ast x_i \otimes 1 + (x^2, \omega_m \wedge \omega_i) \otimes x_i \ast x_m$$

$$+ [\mathbf{ev}(x^2 \otimes d\omega_i)(x_i, \omega_i) - (x^2, d\omega_i)] \otimes x_i$$

$$= \Delta(\mathbf{ev}(x^2 \otimes d\omega_i)x_i) + (x^2, \omega_i^2) \otimes [\mathbf{ev}(x^2 \otimes \omega_m \wedge \omega_i)x_i \ast x_m$$

$$- \mathbf{ev}(x^2 \otimes d\omega_i)x_i] = \Delta(\mathbf{ev}(x^2 \otimes d\omega_i)x_i)$$

where $x^2 \in X^2$. To check relation (59) itself, let $x^2 \in X^2$ and $\omega \in \Omega^1$:

$$\Delta(\mathbf{ev}(x^2 \otimes \omega_i \wedge \omega_j)[x_i \ast (x_i, \omega) + (x_j, \omega) \ast x_i]) = \mathbf{ev}(x^2 \otimes d\omega_i \ast x_i) + (x^2, \omega_i) \ast x_i$$

$$+ (x^2, \omega_i) \ast x_i \otimes (x_i, \omega) + (x^2, \omega_i) \ast (x_i, \omega) \ast x_i \ast x_i$$

$$= \Delta(\mathbf{ev}(x^2 \otimes d\omega_i)x_i) - (x^2, \omega_i) \otimes [x_i \ast (x_i, \omega) - (x^2, d\omega_i)]$$

$$+ [\mathbf{ev}(x^2 \otimes d\omega_i)(x_i, \omega_i) - (x^2, d\omega_i)] \otimes (x_i, \omega)$$

$$= \Delta(\mathbf{ev}(x^2 \otimes d\omega_i)(x_i, \omega_i) - (x^2, d\omega_i))$$

For the counit to be well-defined, all computations follow in a straightforward manner.

Let $x^2 \in X^2$ and $\omega, \rho \in \Omega^1$:

$$\epsilon(\mathbf{ev}(x^2 \otimes d\omega_i) \ast x_i) = 0 = \epsilon(\mathbf{ev}(x^2 \otimes \omega_i \wedge \omega_k) \ast x_h \ast x_j)$$

$$\epsilon(\mathbf{ev}(x^2 \otimes \omega_i \wedge \omega_j)[x_j \ast (x_i, \omega) + (x_j, \omega) \ast x_i])$$

$$\mathbf{ev}(x^2 \otimes \omega_i \wedge \omega_j) \ast (x_j, \omega) \ast x_i] = 0$$

$$= \mathbf{ev}(x^2 \otimes \omega_i \wedge d\omega_i \ast x_i \otimes (x_i, \omega) + \mathbf{ev}(x^2 \otimes d\omega_i) \ast (x_i, \omega) + \mathbf{ev}(x^2 \otimes d\omega_i) \ast (x_i, \omega))$$

$$= -\mathbf{ev}(x^2 \otimes (d\omega_i) \ast (x_i, \omega)) + \mathbf{ev}(x^2 \otimes d\omega_i) \ast (x_i, \omega)$$
\[ e \left( \text{ev}(x^2 \otimes d\omega)(x_i, \omega) - (x^2, d\omega) \right) \]

\[ e(\text{ev}(\rho_1 \wedge \rho_2 \otimes x^2) \ast (\omega, y_i) \ast (\rho, y_j)) = e(\text{ev}(\rho \otimes y_i) \otimes y_i) \text{ev}(\rho_1 \wedge \rho_2 \otimes x^2) \]

\[ = e(\text{ev}(x^2 \otimes \omega)) = \epsilon((\omega \wedge \rho, x^2)) \]

\[ \epsilon(\text{ev}(x^2 \otimes \omega_i \wedge \omega_j) \ast (x_i, \rho) \ast (x_j, \omega)) = \text{ev}(x^2 \otimes \omega_i \wedge \omega_j) \text{ev}(x_i \otimes \text{ev}(x_i \otimes \omega)) \rho) = e(x^2, \omega \wedge \rho) = \epsilon((x^2, \omega \wedge \rho)) \]

To prove that \( D\mathcal{X} \) has a Hopf algebroid structure we need to describe some additional nontrivial relations which hold in \( D\mathcal{X} \).

**Lemma 5.3.** The following additional relations hold in \( D\mathcal{X} \):

\[ \text{ev}(d\rho_1 \otimes x^2)(\omega_i, y_j) \ast x_i + \text{ev}(\rho_1 \wedge \rho_2 \otimes x^2)(\omega_k, y_m) \ast x_k \ast (\omega_l, y_n) \ast x_l = 0 \]  

(58)

\[ \text{ev}((\rho_1 \wedge \rho_2 \otimes x^2)(\omega_i, y_j) \ast x_i + (\omega, y_j) \ast (\omega_i, y_n) \ast x_l) \]

\[ = (d\omega, x^2) - \text{ev}(d\rho_1 \otimes x^2)(\omega, y_i) \]

(59)

\[ (\omega_i, y_j) \ast (\rho_1, x_j) \text{ev}(x^2 \otimes \omega_i \wedge \omega_j) + (\omega_i, x_i) \text{ev}(x^2 \otimes d\omega_i) \]

\[ = \text{ev}(\omega \otimes \text{ev}(y_i^2 \otimes d\omega_i)x_i)(\rho_2^2, x^2) - \text{ev}(\omega \otimes \text{ev}(y_i^2 \otimes \omega_j \wedge \omega_k) x_k x_j \ast (\rho_2^2, x^2) + \text{ev}(\omega \otimes \text{ev}(y_i^2 \otimes x_j \wedge \omega_k) x_k x_j \ast (\rho_2^2, x^2)) \]

(60)

for all \( x^2 \in \mathfrak{X}^2 \) and \( \omega \in \Omega^1 \).

**Proof.** Let \( x^2 \in \mathfrak{X}^1 \). We prove identity (58) holds, using relations (34) in \( H(\Omega^2) \), (54), (34) in \( H(\Omega^1) \), (53) and (59), respectively:

\[ \text{ev}(d\rho_1 \otimes x^2)(\omega_i, y_j) \ast x_i + \text{ev}(\rho_1 \wedge \rho_2 \otimes x^2)(\omega_k, y_m) \ast x_k \ast (\omega_l, y_n) \ast x_l \]

\[ = (\omega_i^2, x^2) \ast \text{ev}(x_i^2 \otimes \omega_i \wedge \omega_j)(\omega_k, y_m) \ast x_k \ast (\omega_l, y_n) \ast x_l \]

\[ + (\omega_i^2, d\rho_1) \ast (\omega_l, y_n) \ast x_l \]

\[ = (\omega_i^2, x^2) \ast \text{ev}(x_i^2 \otimes \omega_i \wedge \omega_j)(\omega_l, y_n) \ast x_l - \text{ev}(x_i^2 \otimes \omega_i \wedge \omega_j) x_j \ast (\rho_1, x_j) \ast (\omega_l, y_n) \ast x_l \]

\[ + (\omega_i^2, x^2) \ast \text{ev}(x_i^2 \otimes d\omega_i) x_i = 0 \]

Let \( \omega \in \Omega^1 \). Identity (59) follows from relations (34) in \( H(\Omega^2) \) and (54), (34) in \( H(\Omega^1) \) and (53).

\[ \text{ev}(\rho_1 \wedge \rho_2 \otimes x^2)(\omega_i, y_m) \ast x_i \ast (\omega, y_n) \ast (\omega_i, y_n) \ast x_l \]

\[ = (\omega_i^2, x^2) \ast \text{ev}(x_i^2 \otimes \omega_i \wedge \omega_j)(\omega_k, y_m) \ast x_k \ast (\omega_l, y_n) \ast x_l \]

\[ + (\omega_i, x_i) \ast (\omega_l, y_n) \ast x_l \]

\[ = (\omega_i^2, x^2) \ast \text{ev}(x_i^2 \otimes \omega_i \wedge \omega_j)(\omega_l, y_n) \ast x_l + x_l \text{ev}(x_i^2 \otimes \omega_j) \]

\[ = (\omega_i^2, x^2) \ast \text{ev}(x_i^2 \otimes \omega_j \wedge \omega_k)(\omega_l, y_n) \ast x_l + x_l \text{ev}(x_i^2 \otimes \omega_j) \]

\[ - \text{ev}(x_i \otimes d\omega_i)(x_i \otimes \omega_j) \]

\[ = (\omega_i^2, x^2) \ast \text{ev}(x_i^2 \otimes \omega_j \wedge \omega_k)(\omega_l, y_n) \ast x_l + x_l \text{ev}(x_i \otimes \omega_j) \]

\[ = (\omega_i^2, x^2) \ast \text{ev}(x_i^2 \otimes \omega_j \wedge \omega_k)(\omega_l, y_n) \ast x_l + x_l \text{ev}(x_i \otimes \omega_j) \]

\[ = (\omega_i^2, x^2) \ast \text{ev}(x_i^2 \otimes \omega_j \wedge \omega_k)(\omega_l, y_n) \ast x_l + x_l \text{ev}(x_i \otimes \omega_j) \]

\[ = (\omega_i^2, x^2) \ast \text{ev}(x_i^2 \otimes \omega_j \wedge \omega_k)(\omega_l, y_n) \ast x_l + x_l \text{ev}(x_i \otimes \omega_j) \]
\[
\begin{align*}
&= (\omega^2, x^2) \cdot \text{ev}(x^2 \otimes \omega_j \wedge \omega_k) \cdot \text{ev}(x_j \otimes \omega) \\
&= (\omega^2, x^2) \left[ \text{ev}(x^2 \otimes \omega_j) \cdot (x_j, \rho_n) - (x^2, d\rho_n) \right] \cdot (\omega, y_n) \\
&= - (\omega^2, x^2) \cdot \text{ev}(x^2 \otimes \omega_j \wedge \text{dev}(x_j \otimes \omega)) \\
&\quad + (\omega^2, x^2) \cdot \text{ev}(x^2 \otimes (d\omega) \cdot \text{ev}(x_1, \omega)) - \text{ev}(x^2 \otimes \omega_1 \wedge \text{dev}(x_1 \otimes \omega)) \\
&= - \text{ev}(x^2 \otimes d\rho_n)(\omega, y_n) + (d\omega, x^2)
\end{align*}
\]

We prove identity (60) by a similar manipulation, using relations (38) in \(H(\Omega^2)\), (54), (39) in \(H(\Omega^1)\) and (53):

\[
\begin{align*}
&(\omega, x_i) \cdot \text{ev}(x^2 \otimes \omega_i \wedge \omega_j) + (\omega, x_i) \cdot \text{ev}(x^2 \otimes d\omega_i) \\
&= (\omega, x_i) \cdot \text{ev}(x^2 \otimes (\omega^2, \omega) \wedge \omega_j) \cdot (\rho^2, x^2) + (\omega, x_i) \cdot (\omega^2, \omega) \cdot (\rho^2, x^2) \\
&= (\omega, x_i) \cdot \text{ev}(x^2 \otimes \omega_i \wedge \omega_j) \cdot (\rho^2, x^2) \\
&\quad + (\omega, x_i) \cdot (\omega^2, \omega) \cdot (\rho^2, x^2) \\
&= (\omega, x_i) \cdot \text{ev}(x^2 \otimes (d\omega) \cdot \text{ev}(x_1, \omega)) - \text{ev}(x^2 \otimes \omega_1 \wedge \text{dev}(x_1 \otimes \omega)) \\
&\quad + \text{ev}(x^2 \otimes d\rho_n)(\omega, y_n) + (d\omega, x^2)
\end{align*}
\]

\[\square\]

**Theorem 5.4.** The bialgebroid \(DX\) has a Hopf algebroid structure.

**Proof.** We have an induced action of \(H X^1\) and \(H(\Omega^2)\) on the inner homs by Theorems 4.11 and 4.12. Hence, we only need to check whether the relations imposed on \(F\) fall in the annihilator of the induced actions on the inner homs of \(DX\)-modules. If the relations for \(DX\) annihilate the inner homs, then the unit and counits for the adjunctions are \(F\)-module morphisms and automatically become \(DX\)-module morphisms, thereby making \(DX\) a Hopf algebroid.

We check the above for relation (54) and leave the similar calculation for (55) to the reader. Let \(M\) and \(N\) be \(DX\)-modules, \(f \in \text{Hom}_A(M, N)\), \(x^2 \in X^2\) and \(\omega, \rho \in \Omega^1\). We show that relation (54) is annihilated for the induced action on \(\text{Hom}_A(M, N)\), by using (54) for \(N\) and relation (55) for \(M\):

\[
\begin{align*}
\text{ev}(x^2 \otimes \omega \wedge \omega_j)(x_j, \rho) \cdot (x_1, \omega) f (m) \\
&= \text{ev}(x^2 \otimes \omega_j \wedge \omega_k) \cdot (x_j, \rho_n) f (x_i, \rho_n) \cdot (\omega, y_n) m \\
&= (x^2, \rho_n \wedge \rho_n) f ((\omega, y_n) \cdot (\rho, y_n) m) \\
&= (x^2, \rho^2) \text{ev}(\rho_n \wedge \rho_n \otimes y^2_n) f ((\omega, y_n) \cdot (\rho, y_n) m) \\
&= (x^2, \rho^2) f ((\omega \wedge \rho, y^2_n) m) = (x^2, \omega \wedge \rho) f (m)
\end{align*}
\]

What remains to be checked is that for \(DX\)-modules \(M\) and \(N\), the induced connections on \(\text{Hom}_A(M, N)\) and \(_A\text{Hom}(M, N)\) are flat and satisfy the additional condition (51). To show that (49) annihilates \(\text{Hom}_A(M, N)\), we use the identities (49) and (53).
for \(N\) and (58) for \(M\). Let \(f \in \text{Hom}_A(M, N)\), \(m \in M\) and \(x^2 \in \mathcal{X}^2\):
\[
\begin{align*}
&\left[ ev(x^2 \otimes d\omega) x_i, f - ev(x^2 \otimes \omega_i \wedge \omega_j) x_j \bullet x_i f \right](m) \\
= &\left[ ev(x^2 \otimes d\omega) x_i f(m) - ev(x^2 \otimes d\omega)(x_i, p_i) f(\omega_i, y_i) \bullet x_i m \right] \\
&- \left[ ev(x^2 \otimes \omega_i \wedge \omega_j) x_j \bullet x_i f(m) \\
&+ \left[ ev(x^2 \otimes \omega_i \wedge \omega_j)(x_j, p_j)(x_i, p_j) f(\omega_i, y_j) \bullet x_i m \right] \\
&- \left[ ev(x^2 \otimes \omega_i \wedge \omega_j)(x_j, p_j)(x_i, p_j) f(\omega_i, y_i) \bullet x_i m \right] \\
= &\left[ x^2, d\rho f(\omega_i, y_i) \bullet x_i m \right] - \left[ x^2, \rho f(\omega_i, y_i) \bullet x_i m \right] \\
&+ \left[ x^2, \rho f(\rho_m \wedge \rho_i \otimes \omega_i m)(\omega_i, y_i) \bullet x_i m \right] = 0
\end{align*}
\]

To show that (53) annihilates \(\text{Hom}_A(M, N)\), we use the identities (53) for \(N\) and (59) for \(M\). Let \(f \in \text{Hom}_A(M, N)\), \(m \in M\), \(x^2 \in \mathcal{X}^2\) and \(\omega \in \Omega^1\):
\[
\begin{align*}
&\left[ ev(x^2 \otimes \omega \wedge \omega_j)(x_j \bullet x_i, \omega) + (x_j, \omega \bullet x_i) f \right](m) \\
= &\left[ ev(x^2 \otimes \omega \wedge \omega_j)(x_j \bullet x_i, \omega) + (x_j, \omega \bullet x_i) f \right](\omega_i, y_i) m \\
&- \left[ ev(x^2 \otimes \omega \wedge \omega_j)(x_j \bullet x_i, \omega) + (x_j, \omega \bullet x_i) f \right](\omega_i, y_i) m \\
&- \left[ ev(x^2 \otimes \omega \wedge \omega_j)(x_j \bullet x_i, \omega) + (x_j, \omega \bullet x_i) f \right](\omega_i, y_i) m \\
= &\left[ x^2, d\rho f(\omega_i, y_i) \bullet x_i m \right] - \left[ x^2, d\rho f(\omega_i, y_i) \bullet x_i m \right] \\
&- \left[ x^2, \rho f(\rho_m \wedge \rho_i \otimes \omega_i m)(\omega_i, y_i) \bullet x_i m \right] = 0
\end{align*}
\]

Now we demonstrate that (49) annihilates \(\text{AHom}(M, N)\). Let \(g \in \text{AHom}(M, N)\), \(m \in M\) and \(x^2 \in \mathcal{X}^2\):
\[
\begin{align*}
&\left[ ev(x^2 \otimes d\omega) x_i g - ev(x^2 \otimes \omega_i \wedge \omega_j) x_j \bullet x_i g \right](m) \\
= &\left[ yg(\rho_i, x_i) ev(x^2 \otimes d\omega) m - yg(\rho_i, x_i) ev(x^2 \otimes d\omega) m \right] \\
&- \left[ yg(\rho_i, x_i) ev(x^2 \otimes d\omega) m - yg(\rho_i, x_i) ev(x^2 \otimes d\omega) m \right] \\
&+ \left[ yg(\rho_i, x_i) ev(x^2 \otimes d\omega) m - yg(\rho_i, x_i) ev(x^2 \otimes d\omega) m \right] \\
&= \left[ x^2 \bullet \omega, d\rho \bullet y(\rho_m \wedge \rho_i \otimes \omega_i m) \right] = 0
\end{align*}
\]

Since \(g\) is a left \(A\)-module morphism, then for any \(x \in \mathcal{X}^1\) and \(a \in A\),
\[
x g(am) - g((xa)m) = (xa)g(m) - g((xa)m)
\]
holds, where the terms \(ev(x \otimes da)g(m)\) cancel each other. Going back to our calculation, we utilise identity (60) and relation (49) for \(M\):
\[
\begin{align*}
&\left[ ev(x^2 \otimes d\omega) x_i g - ev(x^2 \otimes \omega_i \wedge \omega_j) x_j \bullet x_i g \right](m)
\end{align*}
\]
Since $\wedge$ is a pivotal module morphism, (52), then
\[
y_i \cdot y_m \cdot ev(p_m \land p_l \otimes x^2) = y_i \cdot ev(y_m \otimes d \cdot ev(p_m \land p_l \otimes x^2)) + ev(x^2 \otimes \omega_i \land x_j \cdot x_i) + ev(y_m \otimes d \cdot ev(p_m \otimes ev(x^2 \otimes \omega_i \land x_j \cdot x_i)) x_i
\]
holds for any $x^2 \in X^2$. Using this fact and relation (49) for $N$, we see that all terms in our calculation cancel out:
\[
\begin{align*}
&= ev(g_i \otimes \omega_i) x_i g((\rho_i^2, x^2)m) - ev(y_i^2 \otimes \omega_i \land \omega_k) x_k g(x_j \cdot (\rho_i^2, x^2)m) + ev(y_i \otimes d \cdot ev(p_m \otimes y_i^2))(\rho_i^2, x^2)m \\
&\quad - (y_i \cdot ev(y_m \otimes d \cdot ev(p_m \land p_l \otimes y_i^2)) + ev(y_i^2 \otimes \omega_i \land x_j \cdot x_i) g((\rho_i^2, x^2)m) \\
&\quad - ev(y_m \otimes d \cdot ev(p_m \otimes y_i^2))(\rho_i^2, x^2)m + y_i \cdot ev(y_m \otimes d \cdot ev(p_m \otimes ev(y_i^2 \otimes \omega_i \land x_k))) x_j g((\rho_i^2, x^2)m) \\
&\quad + ev(y_i \otimes d \cdot ev(p_m \otimes y_i^2))(\rho_i^2, x^2)m + ev(y_i \otimes d \cdot ev(p_m \otimes ev(y_i^2 \otimes \omega_i \land x_j \cdot x_i))) x_j \cdot (\rho_i^2, x^2)m \\
&\quad - (y_i \cdot ev(y_m \otimes d \cdot ev(p_m \land p_l \otimes y_i^2)) + ev(y_i^2 \otimes \omega_i \land x_j \cdot x_i) g((\rho_i^2, x^2)m) \\
&\quad + ev(y_i \otimes d \cdot ev(p_m \otimes y_i^2))(\rho_i^2, x^2)m + ev(y_i \otimes d \cdot ev(p_m \otimes ev(y_i^2 \otimes \omega_i \land x_j \cdot x_i))) x_j \cdot (\rho_i^2, x^2)m \\
&\quad = ev(y_i \otimes d \cdot ev(p_m \land p_l \otimes x^2)) - ev(y_i^2 \otimes \omega_i \land x_j \cdot x_i) g((\rho_i^2, x^2)m) = 0
\end{align*}
\]

It remains to show that relation (53) annihilates $\mathcal{A}\text{Hom}(M, N)$. For this computation we use the facts mentioned above about left $A$-module morphisms and $\wedge$ being pivotal, in addition to identity (60) annihilating $M$ and relation (53) annihilating $N$. Let $g \in \mathcal{A}\text{Hom}(M, N)$ and $m \in M$:
\[
\begin{align*}
&= ev(g_i \otimes \omega_i) x_i g((\rho_i^2, x^2 \otimes \omega_i \land x_j \cdot x_i \cdot x_j \cdot x_i, \omega^2)m) - ev(x^2 \otimes d \cdot ev(p_m \land \omega_j) x_j \cdot x_i \cdot x_i, \omega^2)m \\
&\quad - (y_m, \omega) g(y_m \otimes (\rho_m, x_i) \otimes (\rho_m, x_j) \otimes ev(x^2 \otimes \omega_i \land x_j \cdot x_i \cdot x_i, \omega^2)m) \\
&\quad + y_m \cdot (y_m, \omega) g((\rho_m, x_i) \otimes (\rho_m, x_j) \otimes ev(x^2 \otimes \omega_i \land x_j \cdot x_i \cdot x_i, \omega^2)m) \\
&\quad - (y_m, \omega) g((\rho_m, x_i) \otimes y_m \cdot \rho_m \otimes ev(x^2 \otimes \omega_i \land x_j \cdot x_i \cdot x_i, \omega^2)m) \\
&\quad - (y_m, \omega) g(y_m \cdot ev(p_m \land \rho_m \otimes y_i^2))(\rho_i^2, x^2)m \\
&\quad - (y_m, \omega) g(y_m \cdot ev(p_m \land \rho_m \otimes y_i^2))(\rho_i^2, x^2)m
\end{align*}
\]
Relation (63) arises from relation (53) where \( x \) holds for any proof.

The Hopf algebroid

Theorem 5.5. We now extend this result to the sense of Böhm and Szlachányi. We now extend this result to \( \mathcal{D} \).

In Theorem 4.13, we provided a criterion for when \( H \) admits an antipode in the sense of Böhm and Szlachányi. We now extend this result to \( \mathcal{D} \).

Theorem 5.5. The Hopf algebroid \( \mathcal{D} \) is a Böhm-Szlachányi Hopf algebroid, if and only if there exists a linear map \( \Upsilon : \mathcal{X} \rightarrow A \) satisfying (76) and additional relations

\[
\Upsilon(\omega(x^2 \otimes \omega, x)) = 0 \quad (62)
\]

\[
\omega(x^2 \otimes \omega, x) = \omega(x^2 \otimes \omega, x) - \omega(x^2 \otimes \omega, x) = \omega(x^2 \otimes \omega, x) \quad (63)
\]

hold for any \( x^2 \in \mathcal{X}^2 \) and \( \omega \in \Omega^1 \).

Proof. (\( \Rightarrow \)) The argument is similar to that of Theorem 4.13. If \( \mathcal{D} \) were to admit an antipode, \( S \), we can use it to recover \( \Upsilon \) by \( \Upsilon(x) = -S(x) \) for \( x \in \mathcal{X}^1 \). Hence, \( \Upsilon \) would satisfy relations (49) and additional relations arising from the flat relation (49) and the additional condition (53). Let \( x^2 \in \mathcal{X}^2 \), then relation (62) arises directly from relation (49):

\[
0 = -\epsilon(S(\omega(x^2 \otimes \omega, x)) - \omega(x^2 \otimes \omega, x)) = \Upsilon(\omega(x^2 \otimes \omega, x)) - \epsilon(S(x) \cdot S(\omega(x^2 \otimes \omega, x)))
\]

Relation (63) arises from relation (53) where \( \omega \in \Omega^1 \):

\[
\omega(x^2 \otimes \omega, x) = \omega(x^2 \otimes \omega, x) - \omega(x^2 \otimes \omega, x) = \omega(x^2 \otimes \omega, x)
\]
5 FLAT BIMODULE CONNECTIONS

\[ + \text{ev}(\text{dev}(\omega \otimes \text{ev}(x^2 \otimes \omega_j \wedge \omega_k) x_k) \otimes x_j) \]

\( (\Leftarrow) \) We assume that such a linear map \( \Upsilon \) satisfying (46), (62) and (63) exists. A consequence of (63) which we use during the proof is that for any \( x^2 \in \mathcal{X} \):

\[ y \text{ev}(dp_l \otimes x^2) - \text{ev}(x^2 \otimes d\omega_l) x_l - y \text{ev}(\text{dev}(p_l \otimes \text{ev}(x^2 \otimes \omega_j \wedge \omega_k) x_k) \otimes x_j) \]

\[ = \Upsilon(\text{ev}(x^2 \otimes \omega_j \wedge \omega_k) x_k) x_j + \text{ev}(x^2 \otimes \omega_j \wedge \omega_k) x_k \Upsilon(x_j) \]

holds. We extend the antipode \( S \) of \( H \mathcal{X}^1 \) and \( H(\Omega^2) \) as defined in Theorems 4.12 and 4.11 to \( \mathcal{D} \mathcal{X} \) and need to show that the antipode \( S \) well-defined on \( \mathcal{D} \mathcal{X} \). In particular, we need to check relations (49) and (53). Let \( x \in \mathcal{X}^2 \). We prove that \( S \) is well-defined for (49) by first applying the properties of \( \Upsilon \), then applying identity (58) and using the fact that \( \wedge \) satisfies (52), as we did in the proof of Theorem 5.4

\[ -S(\text{ev}(x^2 \otimes d\omega_l) x_l - \text{ev}(x^2 \otimes \omega_j \wedge \omega_k) x_k x_j) \]

\[ = (\omega, \text{ev}(x^2 \otimes d\omega_l) x_l) x_l + \Upsilon(\text{ev}(x^2 \otimes d\omega_l) x_l) + \Upsilon(x_j) + \Upsilon(\text{ev}(x^2 \otimes \omega_j \wedge \omega_k) x_k) x_l + \Upsilon(x_j) \]

\[ + (\omega, x_j) x_j \Upsilon(\text{ev}(x^2 \otimes \omega_j \wedge \omega_k) x_k) x_l + \Upsilon(x_j) \]

\[ + (\omega, x_j) x_j \Upsilon(\text{ev}(x^2 \otimes \omega_j \wedge \omega_k) x_k) x_l + \Upsilon(x_j) \]

Let \( \omega \in \Omega^1 \). We prove \( S \) is well-defined for relation (53) by using the properties of \( \Upsilon \), then applying identity (59) and using the fact that \( \wedge \) satisfies (52)

\[ -S(\text{ev}(x^2 \otimes \omega_j \wedge \omega_k) [x_j \Upsilon(x_j, \omega) + (x_j, \omega) x_l] - \text{ev}(x^2 \otimes d\omega_l) x_l, \omega) \]

\[ = (\omega, x_l) \Upsilon(x_l, \omega) \text{ev}(x^2 \otimes \omega_j \wedge \omega_k) x_k x_j \]

\[ + (\omega, x_j) \Upsilon(\text{ev}(x^2 \otimes \omega_j \wedge \omega_k) x_k x_j \]

\[ = (\omega, y \text{ev}(d\rho_l \otimes x^2) x_l, \omega) \]

\[ = (\omega, y \text{ev}(d\rho_l \otimes x^2) x_l, \omega) \]

\[ = (\omega, y \text{ev}(d\rho_l \otimes x^2) x_l, \omega) \]

We also need to check relations (49) and (53) for the inverse of the antipode \( S^{-1} \). We prove that \( S^{-1} \) is well-defined for (49) by using identity (60) and relations (57), (49) and \( \wedge \) satisfies (52), respectively:

\[ -S^{-1}(\text{ev}(x^2 \otimes d\omega_l) x_l - \text{ev}(x^2 \otimes \omega_j \wedge \omega_k) x_k x_j) \]
Finally, we prove that the manipulation used previously for the FLAT BIMODULE CONNECTIONS = \( \Upsilon \) yields:

\[
\begin{align*}
&(y_i + \Upsilon(y_i)) \bullet ((p_i, \text{ev}(x^2 \otimes \omega_i) x_i) + (p_i, x_j) \bullet y_i \bullet (p_i, \text{ev}(x^2 \otimes \omega_j \wedge \omega_k) x_k)) \\
&\quad + (y_i + \Upsilon(y_i)) \bullet (p_i, x_j) \bullet \Upsilon(y_i) \bullet (p_i, \text{ev}(x^2 \otimes \omega_j \wedge \omega_k) x_k) \\
&= \text{ev}(y_i^2 \otimes \omega_i) x_i \bullet (p_i^2, x^2) - \text{ev}(y_i^2 \otimes \omega_j \wedge \omega_k) x_k \bullet x_j \bullet (p_i^2, x^2) \\
&\quad + \text{ev}(y_i \otimes \text{ev}(p_i \otimes \text{ev}(y_i^2 \otimes \omega_j \wedge \omega_k) x_k) x_j) (p_i^2, x^2) \\
&\quad + \Upsilon(\text{ev}(y_i^2 \otimes \omega_i) x_i)(p_i^2, x^2) - \Upsilon(\text{ev}(y_i^2 \otimes \omega_j \wedge \omega_k) x_k) x_j \bullet (p_i^2, x^2) \\
&\quad + \Upsilon(\text{ev}(y_i^2 \otimes \omega_j \wedge \omega_k) x_k) x_j (p_i^2, x^2) \\
&\quad + (y_i + \Upsilon(y_i)) \bullet \text{ev}(p_i \otimes \Upsilon(y_i) \bullet \text{ev}(y_i^2 \otimes \omega_j \wedge \omega_k) x_k) x_j (p_i^2, x^2) \\
&= (y_i \otimes \text{ev}(p_i \otimes \text{ev}(y_i^2 \otimes \omega_j \wedge \omega_k) x_k) x_j) (p_i^2, x^2) + \Upsilon(\text{ev}(y_i^2 \otimes \omega_j \wedge \omega_k) x_k) x_j (p_i^2, x^2) \\
&\quad - \Upsilon(\text{ev}(y_i^2 \otimes \omega_j \wedge \omega_k) x_k) x_j \bullet (p_i^2, x^2) + \Upsilon(\text{ev}(y_i^2 \otimes \omega_j \wedge \omega_k) x_k) x_j \bullet (p_i^2, x^2) = 0
\end{align*}
\]

Finally, we prove that \( S^{-1} \) is well-defined for relation (53) by using identity (60), the manipulation used previously for \( \wedge, (52) \), and the properties of \( \Upsilon, (46) \) and (65).
Remark 5.6. In the proof of Theorem 5.5, it is implicit that Υ arises from a right action of DX on A, where the action of elements of H(Ω^1) and H(Ω^2) agree with the counit. As mentioned previously if the calculus is surjective, then relation (50) follows from the flatness relation (49). Hence, to obtain a right action of DX on A, one would only need to check the flatness condition, which translates to (62) for Υ and condition (63) would follow.

5.2 Examples

In this section we calculate DX explicitly in the cases of finite quivers and the bicovariant calculus of Example 3.12 on C Da. Observe that for any first order calculus there can be several choices for Ω^2, but for our construction we require Ω^2 to be a pivotal bimodule and ∧ a pivotal bimodule morphism, satisfying (53).

Example 5.7. [Finite Quivers] For simplicity we assume the finite quiver Γ = (V, E), does not have any loops i.e. there no edge e ∈ E has the same source and target. There are several choices of Ω^2 for the calculus on finite quivers Γ = (V, E) [Proposition 1.40 [3]]. Here we take Ω^2 to be the quotient of Ω^1 ⊗ Ω^1 by the sub-bimodule spanned by sums \( \sum s(e_1) = p, t(e_2) = q \) \( \overrightarrow{c_1} \otimes \overrightarrow{c_2} \) corresponding to each pair of vertices p, q ∈ V. The bimodule morphism ∧ is the natural projection \( \Omega^1 \otimes \Omega^1 \rightarrow \Omega^2 \) and the differential \( d: \Omega^1 \rightarrow \Omega^2 \) is defined by

\[
d\overrightarrow{c_1} = \sum_{e \in E} \overrightarrow{c} \wedge \overrightarrow{c_1} - \sum_{e \in E} \overrightarrow{c_1} \wedge \overrightarrow{c}
\]

for any \( \overrightarrow{c_1} \in \Omega^1 \). The left and right dual of \( \Omega^2 \) is the quotient of \( \Omega^1 \otimes \Omega^1 \), by the same relations \( \sum s(e_1) = p, t(e_2) = q \) \( \overrightarrow{c_1} \otimes \overrightarrow{c_2} \) corresponding to each pair of vertices p, q ∈ V. To define a pair of evaluation and coevaluation maps, we nominate a 2-step \((a_{p,q}, b_{p,q}) \in E \times E\) for each pair of vertices p, q ∈ V, such that \( s(a_{p,q}) = p, t(b_{p,q}) = q \) and \( s(b_{p,q}) = t(b_{p,q}) \). We denote the set of nominated 2-steps by \( N = \{(a_{p,q}, b_{p,q}) \in E \times E \mid p, q \in V\} \). Notice that for any pair of vertices p, q ∈ V,

\[
(\overrightarrow{c_1} \wedge \overrightarrow{c_2})_{a_{p,q}}^{b_{p,q}} = \sum_{(e_1, e_2) \in E_2} -\overrightarrow{c_1} \wedge \overrightarrow{c_2}
\]

Thereby \( \Omega^2 \) is spanned by elements \( \overrightarrow{c_1} \wedge \overrightarrow{c_2} \) whose underlying 2-steps are not nominated and lie in \( E_2 = \{(e_1, e_2) \in E \times E \mid t(e_1) = s(e_2)\} \setminus N \). With this basis, we can describe the coevaluation and evaluation maps by

\[
\text{coef}(1) = \sum_{(e_1, e_2) \in E_2} \overrightarrow{c_1} \wedge \overrightarrow{c_2} \otimes \overrightarrow{c_2} \wedge \overrightarrow{c_1}, \quad \text{ev}(\overrightarrow{c_2} \wedge \overrightarrow{c_1} \otimes \overrightarrow{c_3} \wedge \overrightarrow{c_4}) = \delta_{e_1, e_3} \delta_{e_2, e_4} f(t(e_2))
\]

for any \( \overrightarrow{c_1} \wedge \overrightarrow{c_2} \in \Omega^2 \) and \( \overrightarrow{c_2} \wedge \overrightarrow{c_1} \in \Omega^2 \). It is trivial to check that ∧ is a pivotal bimodule morphism. By Remark 5.1, since ∧ is surjective and splits, we do not need to add any additional generators to \( H \Omega^1 \) which was constructed in Example 4.17. We only need to quotient out the additional relations for extendability. First notice that by (54) the elements defining the action of \( \Omega^2 \otimes \Omega^2 \) will be given by

\[
(\overrightarrow{c_2} \wedge \overrightarrow{c_1} \wedge \overrightarrow{c_3} \wedge \overrightarrow{c_4}) := (\overrightarrow{c_2} \wedge \overrightarrow{c_1}) \bullet (\overrightarrow{c_3} \wedge \overrightarrow{c_4}) - (\overrightarrow{c_{p,q}} \wedge \overrightarrow{c_3}) \bullet (\overrightarrow{c_{p,q}} \wedge \overrightarrow{c_3})
\]
where \( p = s(e_1) \) and \( q = t(e_2) \). By a similar deduction via (53), the extendability relations which we quotient out \( H \mathcal{X}^1 \) by are
\[
\sum_{s(e_3) = v, t(e_4) = w} (\bar{e}_2, \bar{e}_4) \cdot (e_1, e_3) - (\bar{b}_{p,q}, e_3) \cdot (\bar{b}_{p,q}, e_4) = 0
\]
for all pair of vertices \( v, w \in V \) and any \( \bar{e}_2, \bar{e}_4 \in \mathcal{X}^2 \), where \( p = s(e_1) \) and \( q = t(e_2) \).

The flatness relation (49) reduces to relations for all \( \mathcal{X}^1 \) and \( \mathcal{X}^2 \), where \( p = s(e_1) \) and \( q = t(e_2) \). However, this is in terms of elements of \( T \mathcal{X}_e^1 \) and we have described \( H \mathcal{X}^1 \) in terms of \( \mathcal{K} \Gamma \). In terms of the quiver path algebra, the relations translate to
\[
\bar{b}_{p,q} \cdot \bar{a}_{p,q} = \bar{e}_2 \cdot \bar{e}_1
\]
holding for all \( \bar{e}_2, \bar{e}_1 \in \mathcal{X}^2 \), where \( p = s(e_1) \) and \( q = t(e_2) \). Similarly, The additional condition (53) reduces to
\[
\bar{e}_2 \cdot (e_1, e_3) + (\bar{e}_2, e_3) \cdot e_1 - \bar{b}_{p,q} \cdot (\bar{a}_{p,q}, e_3) - (\bar{b}_{p,q}, e_3) \cdot \bar{b}_{p,q} - (\bar{b}_{p,q}, e_3) = 0
\]
for all \( e_2, e_3 \in \mathcal{X}^2 \) and \( e_1 \in \Omega^1 \), where \( p = s(e_1) \) and \( q = t(e_2) \). Not only is \( DX \) a Hopf algebroid, but it also admits an antipode. One can show that \( \Upsilon \) as defined for \( H \mathcal{X}^1 \) in Example 2.17 satisfies the conditions presented in Theorem 5.5. Recall \( \Upsilon(e) = f_s(e) - f_t(e) \) for any \( e \in E \). For a 2-step \( (e_1, e_2) \in E \) with \( p = s(e_1) \) and \( q = t(e_2) \), condition (62) translates to
\[
\mathcal{U}(\bar{e}_2) - \bar{b}_{p,q} + \mathcal{U}(\bar{b}_{p,q}) \cdot \bar{e}_1 = f_s(e_2) - f_s(b_{p,q}) - f_t(e_1) + f_t(b_{p,q})
\]
which is trivially equal to zero. Condition (63) also follows from a straightforward calculation for the four nontrivial cases where \( \omega \) is one of \( e_1, e_2, a_{p,q} \) or \( b_{p,q} \).

Example 5.8. [Finite Groups] The calculus presented for a finite group algebra \( \mathbb{K}G \) in Example 2.9 can be extended by setting \( \Omega^2 = \Lambda^2 \otimes \mathbb{K}G \), where \( \Lambda^2 \) is the exterior power of the vector space \( \Lambda \). The differential \( d : \Omega^1 \rightarrow \Omega^2 \) is defined as \( d(\lambda \otimes \gamma g) = \lambda \odot \gamma (g) \otimes \mathbb{K}g \). The left action of \( \mathbb{K}G \) on \( \Omega^2 \) is induced the action on the tensor product of left Yetter-Drinfeld modules and is described by
\[
g \triangleright (\lambda_1 \otimes \lambda_2 \otimes \mathbb{K}g) = (g \triangleright \lambda_1) \otimes (g \triangleright \lambda_2) \otimes \mathbb{K}gh
\]
where \( g \in G \) and \( \lambda_1 \otimes \lambda_2 \otimes \mathbb{K}h \in \Lambda^2 \otimes \mathbb{K}G \). Since \( \Lambda \) is finite dimensional, \( \Omega^2 \) is a finitely generated free right module. Additionally, by construction \( \Omega^2 \) is a Hopf bimodule and by Example 2.8 it is a pivotal bimodule. It is a straightforward calculation to check that \( \Lambda \) is a pivotal bimodule morphism. Hence, we can construct \( DX \) the calculus on the Dihedral group \( D_e \) described in Example 5.12. Since \( \Lambda \) is surjective, the generators of the from \( \Omega^2 \otimes \mathcal{X}^2 \) are redundant: for \( i, j, k, l \in \{ \xi, \tau \} \)
\[
(f_j \otimes f_i, k \otimes l) = i^j \otimes j^k \cdot i^j \otimes j^k
\]
Hence $D\mathcal{X}$ reduces to imposing the relevant extendability relations

$$\gamma_i \gamma_j - j \gamma_i \gamma_k = \gamma_{ijk} - j \gamma_i \gamma_k + \gamma_{jkl}$$

$$j \gamma_i \gamma_j - j \gamma_i \gamma_k = \gamma_{ijk} - j \gamma_i \gamma_k + j \gamma_{jkl}$$

for all $i, j, k, l \in \{\xi, \tau\}$, on $H\mathcal{X}^1$ constructed in Example 4.19. The flat condition then translates to

$$f_\xi \cdot f_\tau = f_\tau \cdot f_\xi$$

Since the calculus is surjective, the additional condition (53) follows directly.

### 5.3 Commutative Case and Lie-Rinehart Algebras

In this section, we assume that the algebra $A$ is commutative and recover several known Hopf algebroid structures in the commutative setting, as quotients of $H\mathcal{X}^1$ and $D\mathcal{X}$.

When $A$ is a commutative algebra, the ordinary category of connections $\mathcal{C}$ is well known to have a monoidal closed structure. Since $A^{op} \cong A$, every left $A$-module has a natural $A$-module structure with the right and left actions agreeing. We call such a natural symmetric bimodule (referred to as the classical case in [1]), every symmetric bimodule with symmetric product of connections and inner homs, for any pair of left connections $A$ and $(39)$ all become trivial. We recover the induced action of $T\mathcal{X}$, the Hopf algebroid structure of $T\mathcal{X}_{\text{det}}$ can be recovered by viewing it as the quotient of $H\mathcal{X}^1$ by relations

$$a = a, \quad (x, \omega) = ev(x \otimes \omega), \quad (\omega, x) = ev(\omega \otimes x)$$

where $a \in A$, $x \in \mathcal{X}^1$, and $\omega \in \Omega^1$. First, observe that when the calculus is surjective, second pair of relations follow from $a = a$ holding for all $a \in A$:

$$a = a \Rightarrow (x, da) = [x, a] = [x, a] = ev(x \otimes da)$$

$$\Rightarrow ev(\omega, y) = ev(\omega, y) = (\omega, y)(x, \omega) = (\omega, y)ev(x \otimes \omega) = (\omega, y)$$

Secondly, notice that under these relations, the Hopf relations on $H\mathcal{X}^1$. (54), (55), (58) and (59) all become trivial. We recover the induced action of $T\mathcal{X}$ on the usual tensor product of connections and inner homs, for any pair of left connections $M$ and $N$:

$$x(m \otimes n) = xm \otimes n + m \otimes xn, \quad [xf](m) = xf(m) - f(xm)$$

where $f \in \text{Hom}_A(M, N), m \in M$ and $m \otimes n \in M \otimes N$. Notice that left inner homs and right inner homs agree for symmetric bimodules.

In Section 2.4 of [1] the semi-classical case is considered, where $A$ is a commutative algebra and $\Omega^1$ is a surjective calculus, not necessarily assumed to be symmetric, while the connections are still regarded as symmetric bimodules with invertible bimodule connections. The author then recovers the induced connections on inner homs of this category of connections, by noting that this is possible when the $\Omega^1$-intertwinings of the bimodule connections in consideration are invertible, Theorem 2.4.2.2 [1]. One can
deduce from the calculations above that for a surjective calculus, the \( \Omega^1 \)-intertwining of invertible bimodule connection on symmetric bimodule will be forced to be the flip map. Additionally, the Hopf algebroid representing the category in consideration would be the quotient of \( H \mathcal{X}^1 \) by relations \( a = \pi \) for all \( a \in A \). The author of \cite{11} only needed to discuss invertible bimodule connections on symmetric bimodules since the additional Hopf conditions (33) and (39), hold immediately when quotienting \( IB \mathcal{X}^1 \) by the relation \( a = \pi \). In other words, the quotients of \( IB \mathcal{X}^1 \) and \( H \mathcal{X}^1 \) by the relation \( a = \pi \) are isomorphic, and produce the Hopf algebroid in question. Observe that when \( \Omega^1 \) is not symmetric, the quotient will not necessarily be isomorphic to \( T \mathcal{X}^1 \), but the additional relations \( \text{ev}(xa \otimes \omega) = \text{ev}(x \otimes \omega)a \) and \( a\text{ev}((\omega \otimes x) = \text{ev}(\omega a \otimes x) \) arise from the relations of \( IB(\Omega^1) \). These additional relations can be seen to arise directly when we require the flip map and its inverse, between \( \Omega^1 \) and a symmetric bimodule, to be bimodule maps.

To understand \( \mathcal{D} \mathcal{X} \) in the commutative setting, we first recall the definition of Lie-Rinehart algebras and their associated family of Hopf algebroids from \cite{21, 22}. A pair \((A, \mathcal{X}^1)\) is called a Lie-Rinehart algebra if \( A \) is a commutative algebra, \( \mathcal{X}^1 \) an \( A \)-module with a linear maps \([,] : \mathcal{X}^1 \otimes \mathcal{X}^1 \to \mathcal{X}^1 \) and \( \tau : \mathcal{X}^1 \to \text{Der}(A) \) such that

(I) \([,] \) is antisymmetric

(II) \([,] \) satisfies the Jacobi identity

(III) \([x, y](a) = x(y(a)) - y(x(a)) \) for all \( x, y \in \mathcal{X}^1 \) and \( a \in A \)

(IV) \((ax)(b) = a(x(b)) \) for all \( x \in \mathcal{X}^1 \) and \( a, b \in A \)

(V) \([x, ay] = x(a)y + a[x, y] \) for all \( x, y \in \mathcal{X}^1 \) and \( a \in A \)

where for any \( x \in \mathcal{X}^1 \) and \( a \in A \), we abuse notation and denote \( \tau(x)(a) \) by \( x(a) \).

Observe that axioms (I) and (II) make \((\mathcal{X}^1, [ , ])\) a Lie algebra and (III) simply states that \( \tau : \mathcal{X}^1 \to \text{Der}(A) \) is a Lie algebra morphism, where the Lie bracket on \( \text{Der}(A) \) is defined by \([\phi, \psi] = \phi \psi - \psi \phi \) for \( \phi, \psi \in \text{Der}(A) \).

The universal enveloping algebra of a Lie-Rinehart algebra \((A, \mathcal{X}^1)\), denoted by \( V(A, \mathcal{X}^1) \), was described by Rinehart in \cite{32}. Originally, this algebra was defined as the universal enveloping algebra of a Lie structure on \( A \oplus \mathcal{X}^1 \). Alternatively, one can formulate \( V(A, \mathcal{X}^1) \) as the quotient of the free algebra \( A \ast T \mathcal{X}^1 \) by relations

\[
am = ax, \quad ax = x\bullet a + x(a), \quad x\bullet y = y\bullet x + [x, y]
\]

for all \( x, y \in \mathcal{X}^1 \) and \( a \in A \). It is now common knowledge that \( V(A, \mathcal{X}^1) \) admits a Hopf algebroid structure \cite{22, 34}, which induces the same actions described in \cite{65}.

The principle geometric example this construction is generalising is the algebra of differential operators on a smooth manifold: if \( A \) is the algebra of smooth functions on a smooth finite dimensional manifold and \( \mathcal{X}^1 \) the Lie algebra of smooth vector fields on the manifold, then \( V(A, \mathcal{X}^1) \) is isomorphic to the algebra of differential operators on the manifold and \((A, \mathcal{X}^1)\)-modules or equivalently \( V(A, \mathcal{X}^1)\)-modules are known to be equivalent to the usual notion of flat connections \cite{17}.

Let \( \mathcal{X}^1 \) be a fgp \( A \)-module with dual \( \Omega^1 \). Since \( \mathcal{X}^1 \) is a symmetric bimodule, it does not matter, whether we ask \( \mathcal{X}^1 \) to be left or right fgp and \( \Omega^1 \) will also be symmetric as a bimodule. We have a bijection between linear maps \( \tau : \mathcal{X}^1 \to \text{Der}(A) \) satisfying (IV) and first order calculi on \( \Omega^1 \) i.e. linear maps \( d : A \to \Omega^1 \) satisfying the Leibnitz rule :

\[
x(a) = \text{ev}(x \otimes da) \iff d(a) = x_i(a)\omega_i
\]
where we denote the evaluation and coevaluation maps as in previous sections. In this setting the linear map \([\cdot] : \mathcal{X}^1 \otimes \mathcal{X}^1 \rightarrow \mathcal{X}^1\) allows one to extend the calculus to \(\Omega^2 = \Lambda^2(\Omega^1)\), where \(\Lambda^2(\Omega^1)\) is the exterior power of \(\Omega^1\) as an \(A\)-module. If \([\cdot]\) is antisymmetric and \((V)\) holds, then define \(d : \Omega^1 \rightarrow \Omega^2\) by

\[
d\omega = \sum_{i<j} [x_i, (ev(x_j \otimes \omega)) - x_j, (ev(x_i \otimes \omega)) - ev([x_i, x_j] \otimes \omega)] \omega_i \wedge \omega_j
\]

for \(\omega \in \Omega^1\). In particular, condition \((V)\) makes \(d\) into a well-defined map with \(\Lambda^2(\Omega^1)\) as its codomain. By definition \(d\) satisfies the Leibnitz rule, \(d(a \omega) = da \wedge \omega + ad \omega\), but \(d\) extending the differential of the calculus i.e. \(d^2 = 0\), is equivalent to \((III)\) holding. Since \(\Omega^1\) is a fgp module, then \(\Lambda^\bullet(\Omega^1)\) is also fgp with \(\mathcal{X}^2 = \Lambda^2(\mathcal{X}^1)\) as its dual, and the coevaluation and evaluation maps defined by \(\text{corev}(1) = \omega_1 \wedge \omega_2 \otimes x_i \wedge x_{i+2}\) and

\[
\text{ev}(x \wedge y \otimes \rho) = \text{ev}(x \otimes \omega) \text{ev}(y \otimes \rho) - \text{ev}(y \otimes \omega) \text{ev}(x \otimes \rho)
\]

respectively, where we use the notation \(i_1\) and \(i_2\) to denote the sum over indices such that \(i_1 < i_2\). In this setting, observe that the sheaf of differential operators \(D_A\), as defined in [4], which is the quotient of \(TX^1_A\) by the flat relation (49), is exactly isomorphic to \(V(A, \mathcal{X}^1)\): for any \(x \wedge y \in \mathcal{X}^2\), we can use identities \([x, y]a = x(a)y - [x, ay]\) and \([x, y]a = y(a)x + [xa, y]\) to expand the relation (49).

\[
0 = \text{ev}(x \wedge y \otimes d\omega) \bullet x_i = \text{ev}(x \wedge y \otimes \omega_j \wedge \omega_k) \bullet x_i \bullet x_j
\]

As mentioned previously, the Hopf algebroid structure of \(V(A, \mathcal{X}^1)\) induces the same actions described in (65). These actions can be recovered from the actions of \(DX\) after quotienting out the relations (64). Note that under these relations, the additional relation in \(DX\), (51), holds trivially and by the above calculation, this quotient is exactly isomorphic to \(D_A\) and \(V(A, \mathcal{X}^1)\).

Observe the statement of Theorem 5.5 mirrors that of Proposition 3.11 in [20]. In [20], it is proven that \(V(A, \mathcal{X}^1)\), as a left Hopf algebroid, admits an antipode in the sense of Böhm and Szlachányi, if and only if there exists a right action of \(V(A, \mathcal{X}^1)\) on \(A\). In Theorems 4.13 and 5.5 we have demonstrated that \(DX\) admits an antipode if and only if \(A\) has a right action of \(DX\), and thereby recovered the map \(\Upsilon : \mathcal{X}^1 \rightarrow A\). In particular, our construction of the antipode presented in Theorem 4.13 agrees with the antipode presented in [20], after applying relations (64). It is a consequence of the work in [20] and Theorem 3 of [18] that Lie-Rinehart algebras admit an antipode when \(\mathcal{X}^1\) is finitely generated and projective. It is not clear to the author, whether \(DX\) also admits an antipode for any Lie-Rinehart algebra \((A, \mathcal{X}^1)\), although \(H\mathcal{X}^1\) admits an antipode with \(\Upsilon(x) = x_i (ev(x \otimes \omega_i))\). We must also note that all examples of Hopf algebroids which do not admit antipodes, presented in [23, 33], do
REFERENCES

not cross over to our work, since $\mathcal{X}^1$ is not finitely generated and projective in these examples.

Note that while $V(A, \mathcal{X}^1)$ is often constructed as the universal enveloping algebra of a certain Lie algebra, the Jacobi identity (II) holding is not required to define $V(A, \mathcal{X}^1)$. From the point of view of differential forms, flat connections only need $\Omega^1$ and $\Omega^2$ in the dga, to be defined. The Jacobi identity holding is actually equivalent to the calculus extending to $\Omega^3 = \bigwedge^3(\Omega^1)$. This is additional datum which we do not require for our construction. Furthermore, while in the commutative case the existence of $d : \Omega^1 \to \Omega^2$ is equivalent to the existence a Lie-like bracket on $\mathcal{X}^1$, in the noncommutative case, this does not necessarily provide an asymmetric map on $\mathcal{X}^1 \otimes \mathcal{X}^1$. We refer the reader to Section 6.1 of [5] for a brief discussion on this.

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