Some relations following from the decomposition formula for one multidimensional Lauricella hypergeometric function

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Fundamental solutions for a class of multidimensional elliptic equations with several singular coefficients were constructed recently. These fundamental solutions are directly connected with multiple Lauricella hypergeometric function and the decomposition formula is required for their investigation which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables. In this paper, some relations following from the decomposition formula for one multidimensional Lauricella hypergeometric function are determined.

**Key words:** multidimensional elliptic equation with several singular coefficients; fundamental solutions; multiple Lauricella hypergeometric functions; decomposition formula; summation formula.

We consider the equation

$$F^{(m,n)}_{\alpha}(u) := \sum_{i=1}^{m} u_i x_i + \sum_{j=1}^{n} \frac{2\alpha_j}{x_j} u_{x_j} = 0$$

in the region $R^m_n = \{ x : x_1 > 0, x_2 > 0, ..., x_n > 0 \}$, where $x = (x_1, ..., x_m)$, $m \geq 2$, $0 \leq n \leq m$; $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_j$ are real numbers with $0 < 2\alpha_j < 1$, $j = 1, n$.

Fundamental solutions of the equation were constructed recently. In fact, the fundamental solutions of the equation (1) can be expressed in terms of Lauricella’s hypergeometric function in $n$ variables, that is, the Lauricella multivariable hypergeometric function

$$F_A^{(n)}(a, b_1, ..., b_n; A_1, ..., A_n; z_1, ..., z_n)$$

defined by

$$F_A^{(n)}(a, b_1, ..., b_n; c_1, ..., c_n; z_1, ..., z_n) = F_A^{(n)}\left[ \begin{array}{c} a, b_1, ..., b_n \\ c_1, ..., c_n \end{array} \right] z_1, ..., z_n$$

$$= \sum_{p_1, ..., p_n=0}^{\infty} \frac{(a)_{p_1+...+p_n}}{(c_1)_{p_1} ... (c_n)_{p_n}} \frac{(b_1)_{p_1} ... (b_n)_{p_n}}{p_1! ... p_n!} z_1^{p_1} ... z_n^{p_n}, \quad \sum_{i=1}^{n} |z_i| < 1,$$ (2)

where $c_i \neq 0, -1, -2, ..., i = 1, n$ and $(\kappa)_\nu$ denotes the general Pochhammer symbol or the shifted factorial, since

$$(1)_m = m! (m \in N_0 := N \cup \{0\}; N = \{1, 2, 3, ...\}),$$

which is defined in terms of the familiar Gamma function, by

$$(\kappa)_0 = 1, (\kappa)_\nu = \frac{\Gamma (\kappa + \nu)}{\Gamma (\kappa)}, \nu \in N.$$

We thus obtain the following fundamental solutions:

$$q_k(x, \xi) = \gamma_k \prod_{i=1}^{k} (x_i \xi_i)^{1-2\alpha_i} e^{-2\alpha_k} F_A^{(n)}\left[ \begin{array}{c} \tilde{\alpha}_k, 1 - \alpha_1, ..., 1 - \alpha_k, \alpha_{k+1}, ..., \alpha_n \\ 2 - 2\alpha_1, ..., 2 - 2\alpha_k, 2\alpha_{k+1}, ..., 2\alpha_n; \sigma_1, ..., \sigma_n \end{array} \right],$$ (3)
where

$$\alpha_k = \frac{m}{2} + k - 1 - \sum_{i=1}^{k} \alpha_i + \sum_{i=k+1}^{n} \alpha_i,$$

$$\gamma_k = 2^{2\alpha_k - m} \frac{\Gamma(\alpha_k)}{\pi^{m/2}} \prod_{i=k+1}^{n} \frac{\Gamma(\alpha_i)}{\Gamma(2\alpha_i)} \prod_{j=1}^{k} \frac{\Gamma(1 - \alpha_j)}{\Gamma(2 - 2\alpha_j)}, \quad k = 0, n;$$

$$\sigma_k = 1 - \frac{r_k^2}{R^2}, \quad r^2 = \sum_{i=1}^{m} (x_i - \xi_i)^2, \quad r_k^2 = (x_k + \xi_k)^2 + \sum_{i=1, i \neq k}^{m} (x_i - \xi_i)^2, k = 1, n.$$

Here \( \sum_{i=1}^{l} \) is to be interpreted as zero, when \( l = 0 \) or \( l = n \), and \( \sum_{i=l+1}^{l} \) is to be interpreted as one, when \( l = 0 \) or \( l = n \).

For a given multiple hypergeometric function, it is useful to find a decomposition formula which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables. Burchnall and Chaundy [1, 2] systematically presented a number of expansion and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. For example, the Appell function

$$F_2(a, b_1, b_2; c_1, c_2; x, y) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b_1)_i(b_2)_j}{(c_1)_i(c_2)_j} \frac{x^i y^j}{i! j!}$$

has the expansion [1]

$$F_2(a, b_1, b_2; c_1, c_2; x, y) = \sum_{i=0}^{\infty} \frac{(a)_i (b_1)_i (b_2)_i}{i! (c_1)_i (c_2)_i} x^i y^i F(a+i, b_1+i; c_1+i; x) F(a+i, b_2+i; c_2+i; y),$$

where

$$F(a, b; c; z) = F\left[\frac{a}{c}; \frac{b}{c}; z\right] = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i i!} z^i$$

is Gaussian hypergeometric function [3].

The Burchnall-Chaundy method, which is limited to functions of two variables, is based on the following mutually inverse symbolic operators [1]

$$\nabla(h) = \frac{\Gamma(h) \Gamma(\delta_1 + \delta_2 + h)}{\Gamma(\delta_1 + h) \Gamma(\delta_2 + h)}, \quad \Delta(h) = \frac{\Gamma(\delta_1 + h) \Gamma(\delta_2 + h)}{\Gamma(h) \Gamma(\delta_1 + \delta_2 + h)}, \quad (4)$$

where \( \delta_1 = x \frac{\partial}{\partial x} \) and \( \delta_2 = y \frac{\partial}{\partial y} \).

In order to generalize the operators \( \nabla(h) \) and \( \Delta(h) \), defined in (4), A.Hasanov and H.M.Srivastava [5, 6] introduced the operators

$$\nabla_{z_1, z_2, \ldots, z_n}(h) = \frac{\Gamma(h) \Gamma(\delta_1 + \ldots + \delta_n + h)}{\Gamma(\delta_1 + h) \Gamma(\delta_2 + \ldots + \delta_n + h)} \Gamma(\delta_1 + \delta_2 + \ldots + \delta_n + h), \quad (5)$$

$$\Delta_{z_1, z_2, \ldots, z_n}(h) = \frac{\Gamma(\delta_1 + h) \Gamma(\delta_2 + \ldots + \delta_n + h)}{\Gamma(h) \Gamma(\delta_1 + \delta_2 + \ldots + \delta_n + h)} \Gamma(\delta_1 + \ldots + \delta_n + h), \quad (6)$$
where \( \delta_k = z_k \frac{\partial}{\partial z_k} \), with the help of which they managed to find decomposition formulas for a whole class of hypergeometric functions in several variables. For example, the hypergeometric Lauricella function \( F_A^{(n)} \), defined by formula \([2]\) has the decomposition formula \([5]\)

\[
F_A^{(n)}(a, b_1, \ldots, b_n; c_1, \ldots, c_n; z_1, \ldots, z_n) = \sum_{m_2, \ldots, m_n=0}^{\infty} \frac{(a)_{m_2 + \ldots + m_n} (b_1)_{m_2 + \ldots + m_n} (b_2)_{m_2 + \ldots + m_n} \cdots (b_n)_{m_2 + \ldots + m_n} z_1^{m_1} \cdots z_n^{m_n}}{m_2! \cdots m_n! (c_1)_{m_2 + \ldots + m_n} (c_2)_{m_2 + \ldots + m_n} \cdots (c_n)_{m_2 + \ldots + m_n}} \cdot F(a + m_2 + \ldots + m_n, b_1 + m_2 + \ldots + m_n; c_1 + m_2 + \ldots + m_n; z_1) \cdot F(a + (n-1)(a + m_2 + \ldots + m_n, b_2 + m_2, \ldots, b_n + m_n; c_2 + m_2, \ldots, c_n + m_n; z_2, \ldots, z_n), n \in \mathbb{N}\setminus\{1\}. (7)
\]

However, due to the recurrence of formula (7), additional difficulties may arise in the applications of this expansion. Further study of the properties of operators \((5)\) and \((6)\) showed that formula (7) can be reduced to a more convenient form.

**Lemma 1** \([3]\). The following decomposition formula holds true at \( n \in \mathbb{N}\setminus\{1\} \)

\[
F_A^{(n)}(a, b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_n; z_1, \ldots, z_n) = \sum_{m_{i,j}=0}^{\infty} \frac{(a)_{A(n,n)} (b_{k,n})_{B(k,n)} z_k^{B(k,n)}}{m_{i,j}! \prod_{k=1}^{n} (c_k)_{B(k,n)}} F(a + A(k,n), b_1 + B(k,n); c_2 + B(k,n); z_k), (8)
\]

where

\[
A(k,n) = \sum_{i=2}^{k+1} m_{i,j}, B(k,n) = \sum_{i=2}^{k} m_{i,k} + \sum_{i=k+1}^{n} m_{k+1,i}. (9)
\]

The formula (8) is proved by the method mathematical induction \([3]\).

It should be noted here that the sum \( \sum_{k=1}^{n} B(k,n) \) has the parity property, which plays an important role in the calculation of the some values of hypergeometric functions. In fact, by virtue of equality

\[
\sum_{k=2}^{n} \sum_{i=2}^{k} m_{i,k} = \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} m_{k+1,i}
\]

we obtain

\[
\sum_{k=1}^{n} B(k,n) = 2 \sum_{k=2}^{n} m_{i,k} = 2 \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} m_{k+1,i}. (10)
\]

We present some simple properties of the functions \( A(k,n) \) and \( B(k,n) \), defined by the formula (9):

\[
A(n+1, n+1) - B(n+1, n+1) = A(n,n), \quad (11)
\]

\[
A(k+1, k+1) - B(k+1, k+1) = A(k,n) - B(k,n) + m_{2,n+1} + \ldots + m_{k,n+1}. \quad (12)
\]

Those properties are easily proved if we proceed from the definitions of functions \( A(k,n) \) and \( B(k,n) \).
Lemma 2. Let $a, b_1, \ldots, b_n$ are real numbers with $a = 0, -1, -2, \ldots$ and $a > b_1 + \ldots + b_n$. Then the following summation formula holds true at $n \in \mathbb{N}\backslash\{1\}$

$$
\sum_{m_{i,j}=0}^{\infty} \frac{(a)_{A(n,n)}}{m_{ij}!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n)} (a - b_k)_{A(k,n) - B(k,n)}}{(a)_{A(k,n)}} = \Gamma \left( a - \sum_{k=1}^{n} b_k \right) \frac{\Gamma^{n-1}(a)}{\prod_{k=1}^{n} \Gamma(a - b_k)}.
$$

(13)

Note that if we put $n = 2$ in the formula (13), then

$$
F(b_1, b_2; a; 1) = \sum_{m_{22}=0}^{\infty} \frac{(b_1)_{m_{22}} (b_1)_{m_{22}}}{(e)_{m_{22}} m_{22}!} = \frac{\Gamma(a - b_1 - b_2) \Gamma(a)}{\Gamma(a - b_1) \Gamma(a - b_2)},
$$

(14)

that is, the formula (13) is a natural generalization of the well-known summation formula for the Gauss hypergeometric function.

The proof of Lemma 2 is carried out by the method of mathematical induction.

From equality (14) it follows that the formula (13) is valid for $n = 2$.

Now we denote the left side of the formula (13) by

$$
T_n(a, b_1, \ldots, b_n) := \sum_{m_{i,j}=0}^{\infty} \frac{(a)_{A(n,n)}}{m_{ij}!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n)} (a - b_k)_{A(k,n) - B(k,n)}}{(a)_{A(k,n)}}
$$

and considering fair equality

$$
T_n(a, b_1, \ldots, b_n) = \Gamma \left( a - \sum_{k=1}^{n} b_k \right) \frac{\Gamma^{n-1}(a)}{\prod_{k=1}^{n} \Gamma(a - b_k)}.
$$

we will prove that

$$
T_{n+1}(a, b_1, \ldots, b_{n+1}) = \Gamma \left( a - \sum_{k=1}^{n+1} b_k \right) \frac{\Gamma^{n}(a)}{\prod_{k=1}^{n+1} \Gamma(a - b_k)}.
$$

(15)

For this aim we will put

$$
T_{n+1}(a, b_1, \ldots, b_{n+1}) = \sum_{m_{i,j}=0}^{\infty} \frac{(a)_{A(n+1,n+1)}}{m_{ij}!} \prod_{k=1}^{n+1} \frac{(b_k)_{B(k,n+1)} (a - b_k)_{A(k,n+1) - B(k,n+1)}}{(a)_{A(k,n+1)}}
$$

and show the validity of the recurrence relation

$$
T_{n+1}(a, b_1, \ldots, b_{n+1}) = \prod_{k=1}^{n+1} \left[ \frac{\Gamma(a) \Gamma(a - b_k - b_{n+1})}{\Gamma(a - b_{n+1}) \Gamma(a - b_k)} \right] T_n(a - b_{n+1}, b_1, \ldots, b_n).
$$

(16)

This process consists of $n$ steps. A detailed look at the first step.

By virtue of the equalities

$$
\sum_{m_{i,j}=0}^{\infty} = \sum_{m_{i,j}=0}^{\infty} \sum_{m_{1,n+1}=0}^{\infty} = \sum_{m_{i,j}=0}^{\infty} \sum_{m_{1,n+1}=0}^{\infty} \sum_{m_{i,j}=0}^{\infty} \sum_{m_{1,n+1}=0}^{\infty} \sum_{m_{1,n+1}=0}^{\infty}\sum_{(2 \leq i \leq \leq n+1)}^{\infty}.
$$

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and the properties of functions $A(k, n)$ and $B(k, n)$ (see formulas (11) and (12)), the right side of equality

\[
T_{n+1}(a, b_1, ..., b_{n+1}) = \sum_{m_{i,j}=0}^{\infty} \frac{(a)_{A(n+1,n+1)}}{m_{i,j}!} \prod_{k=1}^{n+1} \frac{(b_k)_{B(k,n+1) - A(k,n+1) - B(k,n+1)}}{(a)_{A(k,n+1)}}
\]

it is easy to convert to the form

\[
T_{n+1}(a, b_1, ..., b_{n+1}) = \sum_{m_{i,j}=0}^{\infty} \frac{(a-b_{n+1})_{A(n,n)}}{m_{i,j}!} \prod_{k=1}^{n+1} \frac{(b_k)_{B(k,n+1) - A(k,n) - B(k,n) + m_{2,n+1} + ... + m_{n,n+1}}}{(a)_{A(k,n) + m_{2,n+1} + ... + m_{n,n+1}}}
\]

where

\[
S(k, n) = \sum_{m_{n+1,n+1}=0}^{\infty} \frac{(b_n + B(n,n))_{m_{n+1,n+1}+1}(b_{n+1} + m_{2,n+1} + ... + m_{n,n+1})_{m_{n+1,n+1}+1}}{m_{n+1,n+1}+1!(a+A(n,n)+m_{2,n+1}+...+m_{n,n+1})^m_{n+1,n+1}+1}.
\]

It is easy to notice that

\[
S(k, n) = F(b_n + B(n,n), b_{n+1} + m_{2,n+1} + ... + m_{n,n+1}; a + A(n,n) + m_{2,n+1} + ... + m_{n,n+1}; 1).
\]

Applying now the summation formula (13) to the last equality after elementary transformations we get

\[
T_{n+1}(a, b_1, ..., b_{n+1}) = \sum_{m_{i,j}=0}^{\infty} \frac{(b_n)_{B(n,n)}}{m_{i,j}!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n) + m_{2,k+1,n+1} + m_{2,n+1} + ... + m_{k,n+1} - A(k,n) - B(k,n) - m_{2,n+1} + ... + m_{k,n+1})}{(a)_{A(k,n) + m_{2,n+1} + ... + m_{k+1,n+1}}}
\]

For definiteness, we denote the result of the first step of the process under consideration by $T_{n+1}(a, b_1, ..., b_{n+1})$. We continue the process of proving the recurrence relation (16). In each next step, having consistently repeated the reasoning carried out in the first step, we get

\[
T_{n+1}(a, b_1, ..., b_{n+1}) = \frac{\Gamma^s(a)}{\Gamma^s(a-b_{n+1})} \prod_{k=n-s+1}^{n} \frac{\Gamma(a-b_k - b_{n+1})}{\Gamma(a-b_k)}
\]
\[
\sum_{m_{ij}=0}^{\infty} \frac{1}{m_{ij}!} \prod_{k=n-s+1}^{n} \frac{(b_k)_{B(k,n)} (a - b_k - b_{n+1})_{A(k,n)} - B(k,n)}{(a - b_{n+1})_{A(k,n)}} \]

\[
\sum_{m_{ij}=0}^{\infty} \frac{(a - b_{n+1})_{A(n,n)} (b_{n+1})_{A(n,n)} - B(k,n)}{m_{ij}!} \]

and in the last step

\[
T_{n+1}^{(n)} (a, b_1, \ldots, b_{n+1}) = \frac{\Gamma^n (a)}{\Gamma^n (a - b_{n+1})} \prod_{k=1}^{n} \frac{\Gamma (a - b_{n+1} - b_k)}{\Gamma (a - b_k)} \]

that is

\[
T_{n+1}^{(n)} (a, b_1, \ldots, b_{n+1}) = \frac{\Gamma^n (a)}{\Gamma^n (a - b_{n+1})} \prod_{k=1}^{n} \frac{\Gamma (a - b_{n+1} - b_k)}{\Gamma (a - b_k)} T_n (a - b_{n+1}, b_1, \ldots, b_n).
\]

Thus, the validity of the ratio (16) is established. By the induction hypothesis, from the (16) follows the equality

\[
T_n (a - b_{n+1}, b_1, \ldots, b_n) = \Gamma \left( a - b_{n+1} - \sum_{k=1}^{n} b_k \right) \frac{\Gamma^{n-1} (a - b_{n+1})}{\prod_{k=1}^{n} \Gamma (a - b_{n+1} - b_k)}.
\]

Substituting the last expression in (16) we get equality (13). Q.E.D.

**Lemma 3.** The following equality

\[
\lim_{j_1, \ldots, j_n \to 0} \sum_{k_1, \ldots, k_n} \frac{1}{z_1^{b_1} \cdots z_n^{b_n}} F_A^{(n)} \left( a, b_1, \ldots, b_n; c_1, \ldots, c_n; 1 - \frac{1}{z_1}, \ldots, 1 - \frac{1}{z_n} \right)
\]

\[
= \frac{1}{\Gamma(a)} \Gamma \left( a - \sum_{k=1}^{n} b_k \right) \prod_{k=1}^{n} \frac{\Gamma (c_k)}{\Gamma (c_k - b_k)}
\]

is valid.

**Proof.** By virtue of the decomposition formula (8) we obtain

\[
F_A^{(n)} \left( a, b_1, \ldots, b_n; c_1, \ldots, c_n; 1 - \frac{1}{z_1}, \ldots, 1 - \frac{1}{z_n} \right) = \sum_{m_{ij}=0}^{\infty} \frac{\Gamma^{n-1} (a, b_1, \ldots, b_n)}{m_{ij}!}
\]

\[
\cdot \prod_{k=1}^{n} \frac{\Gamma \left( a - b_{n+1} - b_k \right)}{\Gamma (c_k - b_k)} \left( 1 - \frac{1}{z_k} \right) \frac{B(k,n)}{n} \cdot \frac{\Gamma (c_k)}{\Gamma (c_k - b_k)} F \left( a + A(k,n), b_k + B(k,n); c_k + B(k,n); 1 - \frac{1}{z_k} \right).
\]
Applying now the familiar autotransformation formula
\[ F(a, b; c; x) = (1 - x)^{-b} F\left(c - a, b; \frac{x}{x - 1}\right) \]
to each hypergeometric function included in the sum (18), we get
\[ F_A^{(n)}\left(a, b_1, \ldots, b_n; c_1, \ldots, c_n; 1 - \frac{1}{z_1}, \ldots, 1 - \frac{1}{z_n}\right) = z_1^{b_1} \cdots z_n^{b_n} \sum_{m, j = 0}^{\infty} \frac{(a)_{A(n, n)} m_{ij}!}{(z_1 \cdots z_n)^n} \]
\[ \cdot \prod_{k=1}^{n} \left( \frac{(b_k)_{B(k, n)}}{(c_k)_{B(k, n)}} (z_k - 1)^{B(k, n)} F\left(c_k - a + B(k, n) - A(k, n), b_k + B(k, n); 1 - z_k\right) \right). \]

Next, we calculate the limit
\[ \lim_{z_k \to 0, k = 1, \ldots, n} z_1^{-b_1} \cdots z_n^{-b_n} F_A^{(n)}\left(a, b_1, \ldots, b_n; c_1, \ldots, c_n; 1 - \frac{1}{z_1}, \ldots, 1 - \frac{1}{z_n}\right) \]
and the resulting expression we apply the summation formula (14), with the result that we obtain the equality (17). Q.E.D.

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