Drinfeld twist is applied to the Lie algebra $gl(2)$ so that a two-parametric deformation of it is obtained, which is identical to the Jordanian deformation of the $gl(2)$ obtained by Aneva et al. The same twist element is applied to deform the Lie superalgebra $sl(1/2)$, since the $gl(2)$ is embedded into the $sl(1/2)$. By making use of the FRT-formalism, we construct a deformation of the Lie supergroup $SL(1/2)$. 

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†Presented at the 7th International Colloquium "Quantum Groups and Integrable Systems", Prague, 18–20 June 1998.
1 Introduction

It is well known that deformation of Lie groups or Lie algebras is not unique. As for \( gl(2) \), its multiparametric quantum deformation is classified recently [1]. In this note, a two-parametric deformation of \( gl(2) \) by the Drinfeld twist [2] is considered and it is shown that the deformed \( gl(2) \) is identical to the one obtained before [4]. The twist for \( gl(2) \) used here is also applicable to deform the Lie superalgebra \( sl(1/2) \), since \( gl(2) \subset sl(1/2) \). The universal R-matrix for the deformed \( sl(1/2) \) is constructed according to the method of twisting. This enables us, using the FRT-formalism, to construct a deformation of the Lie supergroup \( SL(1/2) \). The same scenario has been carried out for a one-parametric deformation of \( sl(2) \) and \( osp(1/2) \) (Note that \( sl(2) \subset osp(1/2) \)) [3]. The present work follows the line adapted in [3].

The deformation by the Drinfeld twist has been developed in recent years. For example, multiparametric twists for the Drinfeld-Jimbo deformation of simple Lie algebras [8], one-parametric twist for \( sl(2) \) [9], and twist for Poincaré algebra [10], Heisenberg algebra [11], esoteric quantum groups [12], \( sl(N) \) [13], and \( osp(1/2) \) [3,14] have been considered.

2 Drinfeld twist

This section is devoted to a brief review of the Drinfeld twist. Drinfeld develops it in his study of quasi Hopf algebras, we however restrict ourselves to ordinary Hopf algebras throughout this note.

Let \( A \) be a Hopf algebra with coproduct \( \Delta_0 \), counit \( \epsilon_0 \) and antipode \( S_0 \). Let \( F \) be an invertible element in \( A \otimes A \) satisfying the conditions

\[
\begin{align*}
(\epsilon_0 \otimes id)(F) &= (id \otimes \epsilon_0)(F) = 1, \\
F_{12}(\Delta_0 \otimes id)(F) &= F_{23}(id \otimes \Delta_0)(F),
\end{align*}
\]

then we obtain a new Hopf algebra \( H \) with the same algebraic relations as \( A \). However, Hopf algebraic mappings of \( H \) are different, they are twisted by the twist element \( F \). The coproduct, the counit and the antipode for \( H \) are given by

\[
\begin{align*}
\Delta &= F\Delta_0 F^{-1}, & \epsilon &= \epsilon_0, & S &= uS_0u^{-1},
\end{align*}
\]

where \( u = m(id \otimes S_0)(F) \).

When the algebra \( A \) has a universal R-matrix \( R_0 \), the universal R-matrix \( R \) for \( H \) is given by

\[
R = F_{21}R_0F^{-1}.
\]

This shows that if \( A \) is cocommutative, then \( H \) is a triangular Hopf algebra. This is the case that we would like to consider, since we shall start with a Lie algebra.
3 Twist for $gl(2)$

The Lie algebra $gl(2)$ has elements $Z, H,$ and $X_{\pm}$ and defined by the relations

$$[H, X_{\pm}] = \pm 2X_{\pm}, \quad [X_+, X_-] = H, \quad [Z, \bullet] = 0. \quad (3.1)$$

A twist element for $gl(2)$ is given by

$$F = \exp \left( \frac{g}{2h} \sigma \otimes Z \right) \exp \left( -\frac{1}{2} H \otimes \sigma \right), \quad (3.2)$$

with

$$\sigma \equiv -\ln(1 - 2hX_+),$$

where $h$ and $g$ are deformation parameters.

According to (2.2), the coproduct and the antipode are deformed and they read

$$\Delta(Z) = Z \otimes 1 + 1 \otimes Z,$$

$$\Delta(H) = H \otimes e^\sigma + 1 \otimes H + \frac{g}{h}(1 - e^\sigma) \otimes Ze^\sigma,$$

$$\Delta(X_+) = X_+ \otimes 1 + e^{-\sigma} \otimes X_+,$$

$$\Delta(X_-) = X_- \otimes e^\sigma + 1 \otimes X_- - hH \otimes e^\sigma H - \frac{h}{2} H(H + 2) \otimes e^\sigma(e^\sigma - 1)$$

$$+ g(e^\sigma - 1) \otimes Ze^\sigma H + g(H - e^\sigma + 1) \otimes Ze^\sigma$$

$$+ g(e^\sigma - 1)(H + e^\sigma + 1) \otimes Z e^{2\sigma}$$

$$- \frac{g^2}{2h}(e^\sigma - 1) \otimes Z^2 e^\sigma - \frac{g^2}{2h}(e^\sigma - 1)^2 \otimes Z^2 e^{2\sigma},$$

$$S(Z) = -Z,$$

$$S(H) = -He^{-\sigma} + \frac{h}{2} Z(e^{-\sigma} - 1),$$

$$S(X_+) = -Xe^\sigma,$$

$$S(X_-) = - \left\{ X_- + \frac{h}{2} H^2(e^{-\sigma} + 1) - hH(e^{-\sigma} - 1) - gHZe^{-\sigma}$$

$$+ gZ(e^{-\sigma} - 1) + \frac{g^2}{2h}(e^{-\sigma} - 1)Z^2 \right\} e^{-\sigma}. \quad (3.3)$$

The universal R-matrix for the non-cocommutative coproduct given above reads

$$R = \exp \left( \frac{g}{2h} Z \otimes \sigma \right) \exp \left( -\frac{1}{2} \sigma \otimes H \right) \exp \left( \frac{1}{2} H \otimes \sigma \right) \exp \left( -\frac{g}{2h} \sigma \otimes Z \right). \quad (3.4)$$

A counit is necessary for a Hopf algebra, it is undeformed, that is, $\epsilon(Z) = \epsilon(H) = \epsilon(X_{\pm}) = 0.$ Thus we arrive to the definition

**Definition 1.** The triangular Hopf algebra generated by $\{Z, H, X_{\pm}\}$ satisfying the relations (3.1) and (3.3) is said to be the two-parametric deformation of $U(gl(2))$ by twisting or $U_{h,g}(gl(2))$. 

3
Let us take the particular non-linear combinations of generators,

\[
\begin{align*}
A &= Z, \\
H' &= e^{-\sigma/2}H, \\
X &= \frac{1}{2\hbar}\sigma, \\
Y &= e^{-\sigma/2}\left(X_+ + \frac{\hbar}{2}H^2\right) - \frac{\hbar}{8}e^{\sigma/2}(e^{-\sigma} - 1).
\end{align*}
\]  

(3.5)

Then \(A, H', X\) and \(Y\) satisfy the commutation relations,

\[
\begin{align*}
[X, Y] &= H', \\
[H', X] &= 2\sinh \frac{\hbar X}{h}, \\
[H', Y] &= -Y(\cosh \hbar X) - (\cosh \hbar X)Y, \\
[A, \bullet] &= 0.
\end{align*}
\]  

(3.6)

The Hopf algebra mappings for these generators are given by

\[
\begin{align*}
\Delta(A) &= A \otimes 1 + 1 \otimes A, \\
\Delta(H') &= H' \otimes e^{\hbar X} + e^{-\hbar X} \otimes H' - \frac{2g}{\hbar} \sinh \hbar X \otimes A e^{\hbar X}, \\
\Delta(X) &= X \otimes 1 + 1 \otimes X, \\
\Delta(Y) &= Y \otimes e^{\hbar X} + e^{-\hbar X} \otimes Y - \frac{g^2}{\hbar} \sinh \hbar X \otimes A^2 e^{\hbar X} + gH' \otimes A e^{\hbar X}, \\
\epsilon(\bullet) &= 0, \\
S(A) &= -A, \\
S(X) &= -X, \\
S(H') &= -e^{\hbar X} H' e^{-\hbar X} - \frac{2g}{\hbar} (\sinh \hbar X) A, \\
S(Y) &= -e^{\hbar X} Y e^{-\hbar X} + \frac{g^2}{\hbar} (\sinh \hbar X) A^2 + g e^{\hbar X} H' A e^{-\hbar X}.
\end{align*}
\]

Therefore the algebra generated by \(\{A, H', X, Y\}\) is nothing but the one introduced in \([4]\). The authors of \([4]\) define the algebra so as to be dual to the Jordanian matrix quantum group \(GL_{h,g}(2)\) \([5, 6]\).

4 Twist for \(sl(1/2)\)

The Lie superalgebra \(sl(1/2)\) has four even and four odd elements denoted by \(Z, H, X_\pm, v_\pm, \bar{v}_\pm\), respectively. They satisfy the relations

\[
\begin{align*}
[H, X_\pm] &= \pm 2X_\pm, \\
[X_+, X_-] &= H, \\
[Z, \bullet] &= 0,
\end{align*}
\]
It is easily seen from the above relations that the even elements \{Z, H, X_\pm\} form a \(gl(2)\) subalgebra and the universal enveloping algebra \(\mathcal{U}(sl(1/2))\) is generated by \{Z, H, X_\pm, v_\pm, \bar{v}_\pm\}. The observation of \(gl(2) \subset sl(1/2)\) implies that the twist element for \(gl(2)\) can be used to twist \(sl(1/2)\). Using (3.3), the twisted coproduct for the odd elements are given by

\[
\Delta(\bar{v}_+) = \bar{v}_+ \otimes e^{-\sigma/2} + \exp\left(-\frac{\sigma}{2h}\right) \otimes \bar{v}_+,
\]
\[
\Delta(v_-) = v_- \otimes e^{\sigma/2} + \exp\left(\frac{\sigma}{2h}\right) \otimes v_- + hH \exp\left(\frac{\sigma}{2h}\right) \otimes v_+ e^\sigma - gv_+ e^\sigma \otimes Z e^{\sigma/2} - g(e^\sigma - 1) \exp\left(\frac{\sigma}{2h}\right) \otimes Z v_+ e^\sigma,
\]
and the antipode is

\[
S(\bar{v}_+) = -\bar{v}_+ \exp\left(\frac{1}{2h}(h + g)\sigma\right)
\]
\[
S(v_-) = -(v_- - hH v_+ + g v_+(1 + Z)) \exp\left(-\frac{1}{2h}(h + g)\sigma\right).
\]

The counit is undeformed and given by \(\epsilon(\bar{v}_+) = \epsilon(v_-) = 0\). The Hopf algebra mappings for even elements have already been given in (3.3).

**Definition 2.** The triangular Hopf algebra generated by \{Z, H, X_\pm, \bar{v}_+, v_-\} satisfying the relations (4.1), (4.2) and (4.3) is said to be the two-parametric deformation of \(\mathcal{U}(sl(1/2))\) by twisting or \(\mathcal{U}_{h,g}(sl(1/2))\).

The universal R-matrix for \(\mathcal{U}_{h,g}(sl(1/2))\) is same as the one for \(\mathcal{U}_{h,g}(gl(2))\). Noting that the fundamental representation of \(\mathcal{U}_{h,g}(sl(1/2))\) is same as \(sl(1/2)\), we obtain the R-matrix in the fundamental representation of \(\mathcal{U}_{h,g}(sl(1/2))\), and see that it is a direct sum of four matrices

\[
R = (1) \oplus \tilde{R} \oplus \tilde{R}^{-1} \oplus \bar{R},
\]

where

\[
\tilde{R} = \begin{pmatrix}
1 & 2g \\
0 & 1
\end{pmatrix}, \quad \bar{R} = \begin{pmatrix}
1 & h + g & -h - g & h^2 - g^2 \\
0 & 1 & 0 & h - g \\
0 & 0 & 1 & -h + g \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

The matrix \(\bar{R}\) is the R-matrix (3.4) in the fundamental representation of \(gl(2)\).
5 Two-parametric deformation of $SL(1/2)$

Using the R-matrix \([4]\) and \(Z_2\) graded version of FRT-formalism \([8]\), a matrix quantum supergroup dual to \(U_{h,g}(sl(1/2))\) can be constructed.

Introducing a $3 \times 3$ supermatrix

\[
M = \begin{pmatrix} e & \Psi \\ \Theta & T \end{pmatrix},
\]

where

\[
\Psi = (\xi, \eta), \quad \Theta = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}, \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

and $e, T$ are even elements and $\Psi, \Theta$ are odd elements, the FRT-formalism guarantees that the commutation relations for the entries of $M$ are given by RMM-relation and their Hopf algebra mappings are given by

\[
\Delta(M) = M \otimes M, \quad \epsilon(M) = I_3, \quad S(M) = M^{-1}. \quad (5.1)
\]

The commutation relations for the entries of $M$ read

\[
e\Psi = \Psi e\tilde{R}, \quad \tilde{R}e\Theta = \Theta e, \quad \tilde{R}eT = Te\tilde{R},
\]

\[
(\xi\Psi \eta\Psi) = -(\Psi\xi \Psi\eta)\tilde{R}, \quad \tilde{R} \begin{pmatrix} \gamma \Theta \\ \delta \Theta \end{pmatrix} = - \begin{pmatrix} \Theta \gamma \\ \Theta \delta \end{pmatrix}, \quad (5.2)
\]

The last relation shows that the submatrix $T$ satisfy the same algebraic relations as the $GL_{h,g}(2)$ \([6]\). As in \([3]\), a determinant for $T$ is defined by $detT = ad - bc - (h+g)ac$, then it is not difficult to see that $detT$ is not a center of deformed $SL(1/2)$ and the noncommutativity is independent of $h$

\[
[\Omega, \ detT] = 0, \quad \Omega = e, \xi, \delta, c \]

\[
[\eta, \ detT] = -2g \xi detT, \quad [\gamma, \ detT] = 2g \delta detT,
\]

\[
[a, \ detT] = [detT, d] = 2g c detT, \quad [b, \ detT] = 2g \{(detT)d - a(detT)\}.
\]

Assuming that the $detT$ is invertible, the explicit form of the inverse matrix of $T$ can be obtain (see \([3]\) for the formula).

We define a superdeterminant for the quantum supermatrix $M$ by

\[
\text{sdet}M = (detT)^{-1}(e - \Psi T^{-1}\Theta). \quad (5.3)
\]
This has the same form as the undeformed case (it is also called Berezinian in undeformed case). Direct computation shows that the \( sdetM \) commute with all elements of the deformed \( SL(1/2) \) so that we can safely set \( sdetM = 1 \). A coproduct and a counit for \( M \) are obvious from (5.1), however, an antipode is not. It is necessary to assume that the combination \( e - \Psi T^{-1} \Theta \) has a inverse. Then the inverse matrix of \( M \) is given by

\[
M^{-1} = \begin{pmatrix} 1 & 0 \\ -T^{-1} \Theta & I_2 \end{pmatrix} \begin{pmatrix} \left( e - \Psi T^{-1} \Theta \right)^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\Psi T^{-1} \\ 0 & I_2 \end{pmatrix},
\]

(5.4)

where \( I_2 \) is the \( 2 \times 2 \) unit matrix.

**Definition 3.** An algebra generated by the entries of \( M \) satisfying (5.2), (5.1), (5.4) and \( sdetM = 1 \) is said to be the two-parametric deformation of \( Fun(SL(1/2)) \) or \( SL_{h,g}(1/2) \).

We finally give some remarks. The twist element \( \mathcal{F} \) (3.2) is also applicable to deform \( osp(1/2) \oplus u(1) \), since \( gl(2) \subset osp(1/2) \oplus u(1) \). The universal R-matrix for the obtained algebra is used to deform the supergroup \( OSp(1/2) \otimes U(1) \). The inclusion of a odd element of \( sl(1/2) \) into a twist element may be possible. The inclusion of an odd elements is found for \( osp(1/2) \) recently [14].

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