Knotted Portals in Virtual Reality

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Abstract

This article describes a software for visualization of branched covers of knots based on an idea by Bill Thurston [Thu12] to view a knot as a “portal” to other universes. Our implementation allows users to explore these knotted portals either on a screen or in virtual reality using a head-mounted device with room-tracking.

This allows users not only to see these glued different worlds, but experience them by being able to walk through the portals.

This software can be used to enable students to learn about knots, gluing, (branched) covers, or just to have a fun look at portals.

1 Introduction

In a video titled “Knots to Narnia” [Thu12], Bill Thurston presents an approach to “visualize” the cyclic branched cover of a knot by interpreting the knot as a portal to other universes. He demonstrates this using a wire to create different life-sized knotted portals. The wire is “magical” and, when its ends are joined, creates a “rip in the fabric of the universe”, creating a portal from our world to a parallel world called “Narnia” in reminiscence of the novels by C.S. Lewis. He then proceeds to explain the phenomena arising in the context of such portals by walking through this wire portal (see Fig. 1).

The object being studied is a cyclic branched cover of order 2 of a knot. This means that a knot defines a gluing of several sheets of \( \mathbb{R}^3 \) by viewing it as a branching curve, analogous to branching points in the construction of, for example, the complex logarithm (see Fig. 4).

![Figure 1: Thurston stepping through a portal generated by the unknot from Earth to Narnia.](image)

While this representation of branched covers of knots is fascinating, it still requires quite a lot of imagination to be able to picture this portal even for simple cases. This gave the motivation to implement this vision as a

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1 The video was recorded by Tony Phillips as he asked topologists to do “demos” with knots. To his knowledge, Thurston was the first to illustrate this phenomenon of a branched world in this way.
virtual reality software, giving users the possibility to not only see these portals but actually be able to walk through them as Thurston did.

In this paper, we describe the implementation of this software and a description of the mathematics involved in the construction of the portals as well as the group structures given by them.

2 Project history

Previous attempts include the software “Polycut” by Ken Brakke [Bra]. This software was designed “for visualizing multiple universes connected by a certain kind of wormhole”, with the purpose of illustrating “the author’s contention that soap films are best viewed as minimal cuts in covering spaces”. In the software, the user can view different knots and links and some of their branched covers as differently colored regions, as well as soap films, which are the minimal surfaces separating the sheets.

We wanted to achieve something different, as our goal was to give a real “world” instead of just colors, as well as achieve a virtual reality experience.

There was an attempt to achieve this by porting Ken Brakke’s code to CAVE virtual reality technology by George Francis, Alison Ortony, Elizabeth Denne, Stuart Levy and John Sullivan during the illiMath2001 research program, but remained unfruitful, to quote: “Though a complete solution to this visualization problem still eludes us, extensive geometrical documentation and evaluation of extant software was undertaken this summer and presented as a PME talk at MathFest, Madison, WI.”.

Figure 2: The author stepping through a portal given by the unknot.

In this project, we achieved our goal through a new software called KnotPortal by using a combination of a game engine and a head-mounted virtual reality device capable of room-scale tracking (see Fig. 2). In our software, the user can move around in a fully immersive experience featuring different real worlds. It is adaptable as new knots can easily be added, and a non-VR version can be used if a VR-headset is not available.

3 Motivation

Besides the relevance given through the fact that the study of branched coverings was the first approach to knot theory, there are many other issues motivating this project.

Despite many different attempts to visualize knot theory, the focus most of the time on the approach of investigating the knot itself. This is in contrast to the approach often taken in mathematics, where the object studied is the complement of the knot. This is, however, not so easily visualized.

Also, this exploration of branched coverings via the movement of one’s body is a perfect example of a meaningful application of embodied mathematics, as already displayed by Thurston himself.
4 Software

The software was created with Unity3D \cite{Unity17}, the virtual reality gear is HP Mixed Reality\footnote{This is not to be confused with augmented reality; Mixed Reality is just the brand name Microsoft has given its virtual reality technology.}. Scripts are in C\# or in DirectX 9-style HLSL. The groups determining the connection of the worlds were computed with the help of GAP \cite{GAP19}.

4.1 Input

The software is given a knot through some parametrization, as well as a group multiplication table, for example generated with GAP. Examples for knot parametrizations together with group multiplication tables are given in Sec. 6. The software further needs a map on which “cone segment” (see below) gets assigned which group element, the generator-to-cone map.

4.2 The setting up of the cut surface

At the start of the program, the following steps are carried out.

1. Build all needed worlds
2. Set up a camera in each world, moving and rotating as the player camera moves and rotates.
3. Let each camera render to a full-screen sized texture, and assign the textures to the post-processing shader.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{heegaard_cone.png}
\caption{The Heegaard cone construction for the trefoil knot. The cone has three self-intersecting lines, resulting in three cone segments (as depicted in \cite{Sti80}).}
\end{figure}

Then, in the first world, we apply the cone construction from \cite{Hee98}, see Fig. 3. The goal is to provide a cut surface for the gluing of the worlds. This is analogous to the cut line given in the construction of the domain of the complex logarithm in Fig. 4. In our case, we cut from the branch curve to “infinity” (in the implementation a point sufficiently far away), and glue together the different worlds along the cutting surface.
1. The knot is placed in the world as a tubular mesh around a Catmull-Rom non self intersecting closed spline, given the control points from the discretized parametrization.

2. A point \( p \) is chosen, from which a normal knot projection is obtained.

3. A cone is built from this point by building a mesh formed by the triangles obtained through filling all line segments from \( p \) to every start and end of the line segments of the knot. This results in a sort of cone, possibly self-intersecting.

4. The cone is cut along the intersections, leading to a number of mesh pieces. These are duplicated and the duplicated has its normals flipped to give a backside.

5. Each “cone segment” is assigned a generator of the group according to the provided generator-to-cone map. Its backside gets assigned the inverse of the generator.

Now, in each frame, if the knot is visible, perform the following steps on the CPU:

1. Transform the knot’s anchor points from world space into screen space.

2. Using the line segments, divide the screen space into polygonal regions by an algorithm of \([\text{dBCvKO08}]\).

3. Find a central point in each region using a C# port of the “polylabel” algorithm from \(\text{https://github.com/mapbox/polylabel}\) to find the pole of inaccessibility of the region.

4. Raycast each point from the camera, multiplying the current world generator with every generator from a cone segment encountered along the way. In this way, build a map assigning a generator to each polygonal screen region.

Then run the following steps in the post-processing shader:

1. For each pixel, perform an optimized \(3\) point-in-polygon test.

2. Assign the pixel the pixel from the camera texture of the world corresponding to the polygon’s generator.

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\(^4\)Optimized by first checking if the pixel lies in a bounding box around the polygon, or in a circle of small enough radius around the pole of inaccessibility of the region.
4.3 Player teleportation
In each frame, raycast from the players old position to his new one. Multiply the current world generator with every cone segment’s generator encountered by the raycast, giving the new world. Teleport the player to the point in the same place, but the new world.

This implies that in contrast to expectation, teleportation occurs much later (or earlier, depending on the direction of approach to the knot) as one might think. It does not happen as one “passes through the portal”, but as one passes through the cut surfaces, i.e. the cone segments, which are the “real” portal.

5 Mathematics background
5.1 History of branched coverings
Knots are ubiquitous objects in our world, and applications of knot theory range from understanding why headphones get tangled spontaneously [RS07] to phenomena in quantum physics. Although knots are found throughout human history such as the famous Gordian Knot, their modern mathematical study began in the 18th century by Vandermonde [Van71] and rised together with topology [Prz07]. The first applications of known mathematical methods to knots came with Poincaré’s Analysis Situs [Poi95]. Heegaard used topological methods to compute the 2-fold branch cover of the trefoil knot [Hee98], but did not use the result to discriminate the trefoil from the unknot, as this now central problem of knot theory was not of interest to him and was only proved by Tietze in 1908 using the fundamental group [Sti80, p. 226]. He used the cover to construct “Riemann spaces”, analog to the construction of Riemann surfaces in one dimension higher [Sti12].

Alexander then proved in 1920 proved in [Ale20] that “Every closed orientable triangulable n-manifold M is a branched covering of the n-dimensional sphere”. The theory was even further developed when [HLM83] provided a universal knot, a knot such that every 3-manifold is a branched cover of it.\footnote{For a more complete history, consult [ACI17].}

6 Example cases
These cases all describe branched covers of order 2, i.e. the knot as the branching curve has order 2. So a path going around a knot segment twice is back in the same world (sheet) it started in.

6.1 Unknot
For the unknot $K$, the knot group is $\pi_1(\mathbb{S}^3 \setminus K)$ which is $\pi_1(\mathbb{S}^1 \times D^2) \cong \mathbb{Z}$ with presentation $\langle a \rangle$. Adding the relation $a^2$ to include the fact that we are looking at covers of order two, the presentation is $\langle a | a^2 \rangle$. This is thus a two-fold covering with deck transformation group $\mathbb{Z}_2$, or equivalently the (Coxeter) group $A_1$.

The unknot is represented in the software through the parametric equations
\[
\begin{pmatrix}
0.8 \sin t \\
1.5 \cos t \\
0
\end{pmatrix}
\]
, generates $|A_1| = 2$ worlds, and has 1 portal. The group multiplication matrix of $A_1$ is $\begin{pmatrix}
e & a \\
a & e
\end{pmatrix}$. As the cone associated to this knot has no self-intersections, the generator-to-cone map is trivial, assigning every cone segment the group element $a$.

6.2 Twisted Unknot
This case is of course the same as the unknot from a knot theoretical standpoint.

As for the implementation, the knot is given by
\[
\begin{pmatrix}
2 \sin(t+1) \\
3 \sin(t+1) \cos(t+1) \\
\sin t
\end{pmatrix}
\]
, but as there are two portals leading to the same world, the generator-to-cone map assigns $a$ to both cone segments.
6.3 Trefoil knot

For the trefoil knot $K$, the knot group is $\langle a, b \mid a^3 = b^2 \rangle$ as the trefoil knot is the $(2, 3)$ torus knot \[ Sti80 \]. Alternatively, it can be given by $\langle x, y \mid xy = yx \rangle$ \[ Rol03, p. 61 \]. By using $xy = yx \cong xyxyxy = xyxyxy \cong yxyxyx = (xy)^3 \cong yxyxx = (xy)^3 \cong (yxx)^2 = (xy)^3$ we can see the isomorphism between the two presentations. Adding the relations $x^2$ and $y^2$, we obtain the presentation $\langle a, b \mid (xy)^3, x^2, y^2 \rangle$. This is the dihedral group of the triangle, and a Coxeter group with Coxeter matrix $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$. The group order 6 implies the construction of 6 worlds from this knot. In general, the $r$-fold branched covering of the torus knots of type $(p, q)$ is a Brieskorn manifold $M(p, q, r)$, the intersection of the 5-sphere $S^5$ in $\mathbb{C}^3$ with the equation given through $z_1^p + z_2^q + z_3^r = 1$. \[ PAAI18 \].

In KnotPortal, the trefoil knot is represented through the parametric equations

$$\begin{pmatrix} \sin t + 2 \sin 2t \\ \cos t - 2 \cos 2t \\ - \sin 3t \end{pmatrix}$$

The group multiplication matrix of $S_3$ is

$$\begin{pmatrix} e & a & b & c & d & f \\ a & e & d & c & f & b \\ c & d & b & f & e & a \\ d & c & a & b & f & e \\ f & b & c & e & a & d \end{pmatrix}$$

The generator-to-cone map assigns the elements $a$, $b$, and $c$ to the three cone segments, respectively.

6.4 Figure eight knot

The presentation of the figure eight knot is $\langle x, y \mid x^{-1} y x y^{-1} = y x^{-1} y x \rangle$ \[ Rol03, p. 58 \]. Again adding the relations $x^2$ and $y^2$, one obtains $\langle x, y \mid (xy)^5, x^2, y^2 \rangle$, which is again a Coxeter group, namely $H_2$, which is of order 10. This knot thus generates 10 worlds.

In the software, it is represented through

$$\begin{pmatrix} (2 + \cos 2t) \cos 3t \\ (2 + \cos 2t) \sin 3t \\ \sin 4t \end{pmatrix}$$

The group multiplication table is

$$\begin{pmatrix} a & b & c & d & e & f & g & h & i & j \\ b & a & d & c & f & e & h & g & j & i \\ c & j & e & b & g & d & i & f & a & h \\ d & i & f & a & h & c & j & e & b & g \\ e & h & g & j & i & b & a & d & c & f \\ f & g & h & i & j & a & b & c & d & e \\ g & f & i & h & a & j & c & b & e & d \\ h & e & j & g & b & i & d & a & f & c \\ i & d & a & f & c & h & e & j & g & b \\ j & c & b & e & d & g & f & i & h & a \end{pmatrix}$$

6.5 Solomon’s Seal knot

The presentation of its group is $\langle x, y \mid x y x y x y^{-1} x^{-1} y^{-1} x^{-1} y^{-1} \rangle$ \[ Liv93 \]. After adding the relations for the generators, the order two covering group of this knot is thus the same as for the Figure eight knot.

6.6 Square knot/Granny knot

The presentation of the covering group is $\langle x, y, z \mid x y x = y x y, x z x = z x z \rangle$ \[ Rol03, p. 62 \]. Together with the results
6.7 Hopf Link

Each of the branching curves gives a generator, and the two commute, so the deck transformation group is \( \langle a, b | a^2, b^2, (ab)^2 \rangle \). This group is a Coxeter group with matrix \( \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \), which is \( \mathbb{Z}_2^2 \), or equivalently, \( A_1^2 \). This results in 4 worlds and 3 portals.

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