KMS STATES ON PIMSNER ALGEBRAS ASSOCIATED WITH C*-DYNAMICAL SYSTEMS

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ABSTRACT. We examine the theory of the KMS states on Pimsner algebras arising from multivariable unital C*-dynamical systems. As an application we show that Pimsner algebras of piecewise conjugate classical systems attain the same KMS states, even though it is still an open problem whether or not they are ∗-isomorphic.

1. INTRODUCTION

In the most part of this paper we investigate the structure of the KMS states on Pimsner algebras associated with C*-dynamical systems. The motivation for this work is two-fold. First of all we are inspired by the growing interest on the structure of the KMS states that involves: the Cuntz algebra [24], Cuntz-Krieger algebras [8], Hecke algebras [1], C*-algebras associated with subshifts [22], Pimsner algebras [17], the Toeplitz algebra of $\mathbb{N} \times \mathbb{N}^\times$ [19, 18], C*-algebras of dilation matrices [20], and C*-algebras of self-similar actions [21]. Secondly we are interested in analyzing further equivalence relations of classical systems [7, 15]. As an application of our analysis we show that Pimsner algebras of classical piecewise conjugate systems admit the same KMS states in a rather strong sense. This is achieved without showing that the Pimsner algebras are ∗-isomorphic, which remains an open problem [7, 15].

Given $d$ ∗-endomorphisms $\alpha_i$ of a C*-algebra $A$ there is a Toeplitz-Pimsner algebra $\mathcal{T}(A, \alpha)$ and a Cuntz-Pimsner algebra $\mathcal{O}(A, \alpha)$ that one can form. Both of them admit an action $\sigma$ of $\mathbb{R}$ induced by the gauge action. We show that there is a phase transition at the inverse temperature $\beta = \log d$ in the following sense: (a) for $\beta \in (-\infty, \log d)$ there are no KMS states (Proposition 3.1); (b) for $\beta \in (\log d, +\infty)$ there is an affine weak*-homeomorphism from the simplex of the tracial states on $A$ onto the $(\sigma, \beta)$-KMS states on $\mathcal{T}(A, \alpha)$ (Theorem 3.3). Moreover there is an affine weak*-homomorphism from the simplex of the tracial states (resp. states) on $A$ onto the $(\sigma, \beta)$-KMS states on $\mathcal{T}(A, \alpha)$ (Propositions 3.4 and 3.5). By using that $\mathcal{O}(A, \alpha)$ is a quotient of $\mathcal{T}(A, \alpha)$ we show that the same results pass over for the states on $A$ that vanish on the ideal $(\cap_{i=1}^d \ker \alpha_i)^\perp$ (Theorem 3.6 and Corollary 3.8). However when $(A, \alpha)$ is injective then...
there are no \((\sigma, \beta)\)-KMS states on \(\mathcal{O}(A, \alpha)\) for all \(\beta > \log d\) (Proposition 3.7).

Different phenomena appear for the critical temperature \(\beta = \log d\), making a unification hard to achieve. Our results in this case are of existential nature and do not yield a parametrization. For example if \(A = C(X)\) and \(d = 1\) then \(\mathcal{O}(A, \text{id}) = \mathcal{O}(X \times \mathbb{T})\) admits plenty of \((\sigma, 0)\)-KMS states, i.e. tracial states. For \(A = C(X)\) and \(d > 1\) we have that any state on \(A\) induces a \((\sigma, \log d)\)-KMS state on \(\mathcal{O}(A, \text{id})\); in particular if \(A = C\) and \(d > 1\) then \(\mathcal{O}(C, \text{id}) = \mathcal{O}_d\) has a unique \((\sigma, \log d)\)-KMS state \([24, 9, 2]\). We show that there are (finite dimensional) cases where the \((\sigma, \log d)\)-KMS state is unique and other (finite dimensional) cases where there are more than one (Remark 3.14). In addition we give a sufficient and necessary condition for \(\mathcal{O}(A, \alpha)\) to have KMS states at \(\log d\) when \(d > 1\) (Proposition 3.12). In the case where \(A = C(X)\) and \(d > 1\) we derive that \(\mathcal{O}(A, \alpha)\) attains \((\sigma, \log d)\)-KMS states (Corollary 3.13). This is achieved by using the operator that averages over the actions \(\alpha_i\), and it is inspired by the Perron-Frobenius type theory of Matsumoto, Watatani and Yoshida \([22]\). The case where \(d = 1\) is reduced to finding tracial states on a usual C*-crossed product by using the tail adding techniques of \([12, 14]\).

The analysis of Laca and Neshveyev \([17]\) on Pimsner algebras makes also use of a Perron-Frobenius type operator. Moreover it concerns actions of \(\mathbb{R}\) beyond the ones inherited from the gauge action. Their approach is rather illuminating but it seems hard to be adopted in our specific example to provide a parametrization of the KMS states. It is the recent works of Laca and Raeburn \([19]\), and Laca, Raeburn, Ramagge and Whittaker \([21]\) that supply us with the efficient tools and algorithms for our purposes.

The tensor algebra of a C*-dynamical system in the sense of Muhly and Solel \([23]\) consists of the lower triangular non-involutive part of \(T(A, \alpha)\). The author and Katsoulis \([15]\) have shown that isometric isomorphisms of the tensor algebras is a complete invariant for unitary equivalent systems when \(A = C(X)\). On the other hand Davidson and Katsoulis \([7]\) show that piecewise conjugacy is implied by algebraic isomorphism of the tensor algebras. The converse is known to hold in specific cases (e.g. \(n = 2, 3\)) and gives stronger isomorphisms by isometric maps \([7]\). The combination of \([7]\) and \([15]\) suggests that the converse in full generality could be achieved by proving that unitary equivalence and piecewise conjugacy are equivalent. We note here that unitary equivalent systems are piecewise conjugate and it is the converse of this fact that concerns us.

It is immediate that unitary equivalent systems admit *-isomorphic Pimsner algebras. Therefore an examination of the aforementioned problem can be conducted within the class of C*-algebras; in particular by analyzing the invariants of C*-algebras. The form of the KMS states that we obtain is

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\(^*\) We inform the reader that \([17]\) follows the definition of Pimsner \([25]\), and the results require a small reformulation to agree with the modern language of C*-correspondences that followed the work of Katsura \([16]\).
accommodating for this task. As an application we show that Pimsner algebras of piecewise conjugate systems admit the same theory of KMS states (Corollary 3.19). This gives evidence that unitary equivalence and piecewise conjugacy may be equivalent. However the same conclusion is derived for any equivalence relation that respects the orbit of a point in the sense of Lemma 3.18. Even more this conclusion can be reformulated to cover non-classical systems as well: if \((A, \alpha)\) and \((C, \gamma)\) are dynamical systems of the same multiplicity and there is a \(*\)-isomorphism \(\hat{\phi}: A \to C\) such that \(\{\tau a \omega(a) \mid w \in F_+^d, |w| = m\} = \{\tau \hat{\phi}^{-1} \gamma_w \hat{\phi}(a) \mid w \in F_+^d, |w| = m\}\) for all \(a \in A\) and \(m \in \mathbb{Z}_+\) then \(\mathcal{T}(A, \alpha)\) and \(\mathcal{T}(C, \gamma)\) (resp. \(\mathcal{O}(A, \alpha)\) and \(\mathcal{O}(C, \gamma)\)) admits the same KMS states in the strong sense of Corollary 3.13.

2. Preliminaries

2.1. Kubo-Martin-Schwinger states. Let \(\sigma: \mathbb{R} \to \text{Aut}(A)\) be an action on a C*-algebra \(A\). Then there exists a norm-dense \(\sigma\)-invariant \(*\)-subalgebra \(A_{an}\) of \(A\) with the following property: for every \(a \in A_{an}\) the function \(\mathbb{R} \ni t \mapsto \sigma_t(a) \in A\) is analytically continued to an entire function \(\mathbb{C} \ni z \mapsto \sigma_z(a) \in A\) (see [3, Proposition 2.5.22]). A state \(\psi\) of \(A\) is called a \((\sigma, \beta)\)-KMS state if it satisfies

\[
\psi(ab) = \psi(b \sigma_i(b)),
\]

for all \(a, b\) in a norm-dense \(\sigma\)-invariant \(*\)-subalgebra of \(A_{an}\). If \(\beta = 0\) or if the action is trivial then a KMS state is a tracial state on \(A\).

The KMS condition follows as an equivalent for the existence of particular continuous functions. More precisely, for \(\beta > 0\) let \(D = \{z \in \mathbb{C} \mid 0 < \text{Im}(z) < \beta\}\); then a state \(\psi\) is a \((\sigma, \beta)\)-KMS state if and only if for any pair \(a, b \in A\) there exists a complex function \(F_{a,b}\) that is analytic on \(D\) and continuous (hence bounded) on \(\overline{D}\) such that

\[
F_{a,b}(t) = \psi(a \sigma_i(b)) \quad \text{and} \quad F_{a,b}(t + i\beta) = \psi(\sigma_i(b)a),
\]

for all \(t \in \mathbb{R}\) (see [4, Proposition 5.3.7]).

A state \(\psi\) of a C*-algebra \(A\) is called a ground state if the function \(z \mapsto \psi(a \sigma_z(b))\) is bounded on \(\{z \in \mathbb{C} \mid \text{Im}z > 0\}\) for all \(a, b\) inside a dense analytic subset \(A_{an}\) of \(A\). A state \(\psi\) of \(\mathcal{N}\mathcal{T}(A, \alpha)\) is called a KMS\(_\infty\) state if it is the \(w^*\)-limit of \((\sigma, \beta)\)-KMS states as \(\beta \to \infty\). The reader should be aware of the fact that this distinction is not apparent in [3, 4] and is coined in [19, 21].

2.2. C*-dynamical systems. A C*-dynamical system \((A, \alpha)\) of multiplicity \(d\) consists of \(d\) \(*\)-endomorphisms \(\alpha_i\) of a C*-algebra \(A\). In particular when \(A = C_0(X)\) for a locally compact Hausdorff space \(X\) then each \(\alpha_i\) is identified by a proper continuous map \(\sigma_i: X \to X\). In this case we will call \((A, \alpha) \equiv (X, \sigma)\) a multivariable classical system. We will say that \((A, \alpha)\) is unital if \(A\) has a unit and every \(\alpha_i\) is unital for \(i = 1, \ldots, d\). We will say that \((A, \alpha)\) is non-degenerate if all the \(\alpha_i\) are non-degenerate for \(i = 1, \ldots, d\). We will say that \((A, \alpha)\) is injective if \(\bigcap_{i=1}^d \ker \alpha_i = (0)\).
There is a C*-correspondence construction associated with \((A, \alpha)\) that gives rise to several elements we will use. It is not necessary for our purposes to review the whole theory of C*-correspondences. Instead we will give the appropriate definitions in terms of \((A, \alpha)\).

The required ideal in \(A\) for the covariant representations coined by Katsura [16] is defined by

\[
J_{(A, \alpha)} = (\cap_{i=1}^{d} \ker \alpha_i)^\perp.
\]

Therefore \((A, \alpha)\) is injective if and only if \(J_{(A, \alpha)} = A\). We denote by \(T(A, \alpha)\) the Toeplitz-Pimsner algebra related to \((A, \alpha)\), i.e. the universal C*-algebra generated by the formal monomials \(v_w a\) with \(a \in A\) and \(w \in \mathbb{F}_+^d\), such that \([v_1, \ldots, v_d]\) is a row isometry and \(a v_i = v_i \alpha_i(a)\) for all \(a \in A\) and \(i = 1, \ldots, d\). There is a well known construction that gives a faithful representation of \(T(A, \alpha)\). Let \(\pi: A \to \mathcal{B}(H)\) be a faithful representation of \(A\). On the Hilbert space \(K = H \otimes \ell^2(\mathbb{F}_+^d)\) let the orbit representation

\[
\tilde{\pi}(a) = \text{diag}\{\pi \alpha_i(a) | w \in \mathbb{F}_+^d\}, \text{ for all } a \in A,
\]

where \(\overline{w} = i_m \ldots i_1 = i_1 \ldots i_m\) is the reversed word of \(w = i_m \ldots i_1\). By defining \(V_i \xi \otimes e_w = \xi \otimes e_{iw}\) we form a row isometry \([V_1, \ldots, V_d]\) and the representation

\[
a \mapsto \tilde{\pi}(a) \quad \text{and} \quad v_i \mapsto V_i
\]

lifts to a faithful representation of \(T(A, \alpha)\) onto the C*-algebra \(\mathcal{C}(\tilde{\pi}, V)\) generated by \(V_w \tilde{\pi}(a)\) for all \(a \in A\) and \(w \in \mathbb{F}_+^d\). From now on let us fix such a family \(\tilde{\pi}, \{V_i\}_{i=1}^{d}\). Since \(A\) embeds isometrically in \(T(A, \alpha)\) we will drop the use of \(\tilde{\pi}\).

We denote by \(O(A, \alpha)\) the Cuntz-Pimsner algebra related to \((A, \alpha)\), i.e. the quotient of \(T(A, \alpha)\) by the ideal generated by the differences

\[
a - \sum_{i=1}^{d} V_i \alpha_i(a) V_i^* = a(I - \sum_{i=1}^{d} V_i V_i^*), \text{ for all } a \in J_{(A, \alpha)}.
\]

We denote by \(q: T(A, \alpha) \to O(A, \alpha)\) the quotient map. Since \(A\) embeds isometrically inside \(O(A, \alpha)\) we will make the identification \(a \equiv q(a)\) for all \(a \in A\). Moreover we denote by \(S_\mu = q(V_\mu)\) for all \(\mu \in \mathbb{F}_+^d\). Therefore \(O(A, \alpha)\) is the universal C*-algebra generated by the formal monomials \(s_w a\) with \(a \in A\) and \(w \in \mathbb{F}_+^d\), such that \([s_1, \ldots, s_d]\) is a row isometry, \(a s_i = s_i \alpha_i(a)\) for all \(a \in A\) and \(i = 1, \ldots, d\), and

\[
a(I - \sum_{i=1}^{d} s_i s_i^*) = 0, \text{ for all } a \in J_{(A, \alpha)}.
\]

In particular, when \((A, \alpha)\) is injective then \(O(A, \alpha)\) is the universal C*-algebra generated by the \(s_i a\) such that \([s_1, \ldots, s_d]\) is a row unitary and \(a s_i = s_i \alpha_i(a)\) for all \(a \in A\) and \(i = 1, \ldots, d\).

When \((A, \alpha)\) is unital then \(1 \in A\) is also the unit for \(T(A, \alpha)\) and \(O(A, \alpha)\). If \((A, \alpha)\) is not unital then we form the unitization \((A^{(1)}, \alpha^{(1)})\) so that \(\alpha^{(1)}_i(a + \)
\( \lambda = \alpha_i(a) + \lambda \). We denote by \( A^{(1)} = A + C \) the unitization of \( A \), even when \( A \) has a unit. We make the convention that \( A^{(1)} = A \) only when \( (A, \alpha) \) is unital. Since the unitization is an extension of the original system we get that \( T(A, \alpha) \) can be realized as a C*-subalgebra of \( T(A^{(1)}, \alpha^{(1)}) \). Indeed consider a Fock representation \( (\pi, \{V_i\}_{i=1}^d) \) of \( (A^{(1)}, \alpha^{(1)}) \), take \( (\pi |_{A}, \{V_i\}_{i=1}^d) \) and use the gauge invariant uniqueness theorem for the Toeplitz-Pimsner algebras. On the other hand, when \( A \) is not unital then \( A \) is an essential ideal of \( A^{(1)} \), and when \( A \) is unital then \( A^{(1)} = A \oplus C \). Therefore we obtain that \( J(A, \alpha) = J(A^{(1)}, \alpha^{(1)}) \). By the gauge invariant uniqueness theorems we have that \( O(A, \alpha) \) is a C*-subalgebra of \( O(A^{(1)}, \alpha^{(1)}) \) respectively. We remark that we make use of the gauge invariant uniqueness theorems in their general form as proven by Katsura [16]. The reader is referred also to [13] for an alternative proof and an overview on the subject.

The author with Katsoulis [14] developed a method of dilating a non-injective \( (A, \alpha) \) to an injective \( (C, \gamma) \) such that \( O(A, \alpha) \) is a full corner of \( O(C, \gamma) \). Let us review this construction. Let the direct sum C*-algebra \( C = C_0 \oplus (\oplus_{w \in F^+} C_w) \)

where \( C_0 = A \) and \( C_w = A/J(A, \alpha) \) for all \( w \in F^d \). Note that \( C_0 \) is not to be mistaken with \( C_∅ \). We then define the *-endomorphisms \( \gamma_i : C \to C \) such that

\[
(\gamma_1(a, (c_w)))_{\mu} = \begin{cases} 
\alpha_1(a) & \text{if } \mu = 0, \\
q_J(a) & \text{if } \mu = ∅, \\
c_1^n & \text{if } \mu = 1^{n+1}, n \geq 0, \\
c_w & \text{if } \mu = 1w, \mu \neq 1^n, n \geq 1, \\
0 & \text{otherwise},
\end{cases}
\]

where \( q_J : A \to A/J(A, \alpha) \), and

\[
(\gamma_i(a, (c_w)))_{\mu} = \begin{cases} 
\alpha_i(a) & \text{if } \mu = 0, \\
c_w & \text{if } \mu = iw, \\
0 & \text{otherwise},
\end{cases}
\]

for \( i = 2, \ldots, d \). For \( d = 2 \) this is depicted in the following diagram
where the solid arrows represent $\gamma_1$ and the broken arrows $\gamma_2$, with the understanding that we don’t write the cases where the elements are sent to zero. When $(A, \alpha)$ is non-degenerate then $(C, \gamma)$ is non-degenerate as well; however when $(A, \alpha)$ is unital then $(C, \gamma)$ is no longer unital. We call $(C, \gamma)$ the injective dilation of $(A, \alpha)$. (Here the $*$-endomorphisms $\{\gamma_n\}_{n=1}^d$ should not be confused with the gauge action $O$ on the Pimsner algebras.)

In particular when $d = 1$ we write $\alpha_1 = \alpha$; then $C = A \oplus c_0(A/\ker \alpha \perp)$ and

$$\gamma(a, (c_n)) = (\alpha(a), a + \ker \alpha \perp, (c_n)).$$

In turn we may extend the injective system $(C, \gamma)$ of multiplicity 1 to the direct limit automorphic system $(\tilde{C}, \tilde{\gamma})$ given by the diagram

$$\begin{array}{cccccccc}
C & \gamma & \rightarrow & C & \gamma & \rightarrow & C & \gamma & \rightarrow & \cdots & \rightarrow & \tilde{C} \\
\downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \tilde{\gamma} \\
C & \gamma & \rightarrow & C & \gamma & \rightarrow & C & \gamma & \rightarrow & \cdots & \rightarrow & \tilde{C}
\end{array}$$

Then we have that $O(C, \gamma) = O(\tilde{C}, \tilde{\gamma}) = \tilde{C} \times_{\mathbb{Z}} \mathbb{Z}$. The system $(\tilde{C}, \tilde{\gamma})$ is called the automorphic dilation of $(A, \alpha)$. The interested reader is addressed to [12] for the pertinent details.

Finally we mention that we will write $|\mu| = n$ for the length of a word $\mu = \mu_n \ldots \mu_1 \in \mathbb{F}_d^\perp$.

### 3. KMS states on Pimsner algebras

#### 3.1. The Toeplitz-Pimsner algebra

Given the gauge action $\{\gamma_n\}_{n=1}^d$ of $\mathcal{T}(A, \alpha)$ we let $\sigma : \mathbb{R} \rightarrow \text{Aut}(\mathcal{T}(A, \alpha))$ be the action with $\sigma_t = \gamma_{e^{it}}$. The linear span of the monomials $V_\mu a V^*_\nu$, with $\mu, \nu \in \mathbb{F}_d^\perp$ and $a \in A$, is dense in $\mathcal{T}(A, \alpha)$, and

$$\sigma_t(V_\mu a V^*_\nu) = e^{i(|\mu| - |\nu|)t} V_\mu a V^*_\nu.$$ 

Since the function $t \mapsto \sigma_t(V_\mu a V^*_\nu)$ extends to the entire function $z \mapsto e^{z(|\mu| - |\nu|)} V_\mu a V^*_\nu$, then the $(\sigma, \beta)$-KMS condition for a state $\psi$ is translated into

$$\psi(V_\mu a V^*_\nu \cdot V_\kappa b V^*_\lambda) = \psi(V_\kappa b V^*_\lambda \cdot \sigma_t V_\mu a V^*_\nu)$$

$$= e^{-(|\mu| - |\nu|)\beta} \psi(V_\kappa b V^*_\lambda \cdot V_\mu a V^*_\nu)$$

for all $a, b \in A$ and $\mu, \nu, \kappa, \lambda \in \mathbb{F}_d^\perp$. Let us begin with the following conditions on the form of KMS states.

**Proposition 3.1.** Let $(A, \alpha)$ be a unital $C^*$-dynamical system with multiplicity $d$.

(i) If $\beta < \log d$ then there are no $(\sigma, \beta)$-KMS states.

(ii) If $\beta \geq \log d$ and $\beta > 0$, then $\psi$ is $(\sigma, \beta)$-KMS state if and only if $\psi|_A$ is tracial and $\psi(V_\mu a V^*_\nu) = \delta_{\mu, \nu} e^{-|\mu| \beta} \psi(a)$, for all $a \in A$ and $\mu, \nu \in \mathbb{F}_d^\perp$. 

Proof. Let the unit $1 \in A$, which is also a unit for $T(A, \alpha)$. Then we obtain

$$1 = \psi(1) \geq \sum_{k=1}^{d} \psi(V_i V_i^*) = \sum_{k=1}^{d} \psi(V_i^* \sigma_i \beta(V_i)) = e^{-\beta} \sum_{i=1}^{d} \psi(1) = e^{-\beta} d,$$

where we used that $1 - \sum_{i=1}^{d} V_i V_i^* \geq 0$; thus $\beta \geq \log d$.

For item (ii) fix $\beta \geq \log d$ such that $\beta > 0$ and suppose first that $\psi$ is a $(\sigma, \beta)$-KMS state. Since $\sigma_{\ell}|_{A} = \text{id}_A$ we obtain that $\psi|_{A}$ is tracial. Furthermore we have that

$$\psi(V_m a V_n^*) = \psi(a V_n^* \sigma_{i \beta}(V_m)) = e^{-|\mu| \beta} \psi(a V_n^* V_m).$$

If we show that $\psi(a V_n^*) = \psi(a V_m^*) = 0$ for all $a \in A$ and $\mu \neq \emptyset$ then the above computation yields $\psi(V_m a V_n^*) = \delta_{\mu, \nu} e^{-|\mu| \beta} \psi(a)$. To this end suppose that $\mu = \mu \ldots \mu$; then

$$\psi(V_m a) = e^{-\beta} \psi(V_{\mu} V_{\mu} \ldots V_{\mu} a V_{\mu}) = e^{-\beta} \psi(V_{\mu} V_{\mu} \ldots V_{\mu} V_{\mu} \alpha_{\mu_n}(a)).$$

Induction implies that

$$\psi(V_m a) = e^{-|\mu| \beta} \psi(V_{\mu} \ldots V_{\mu} \alpha_{\mu_n}(a)) = e^{-|\mu| \beta} \psi(V_{\mu} \alpha_{\mu_n}(a)),$$

and consequently for $k = l \cdot n$ we obtain

$$|\psi(V_m a)| = (e^{-|\mu| \beta})^k |\psi(V_{\mu} \alpha_{\mu}(a))| \leq (e^{-|\mu| \beta})^k \|a\|.$$

Since $\beta > 0$, taking the limit over $k$ yields $\psi(V_m a) = 0$. By taking adjoints we also obtain that $\psi(a V_n^*) = 0$. Moreover we have

$$\psi(a V_m^*) = \psi(V_m a_{\sigma_{i \beta}}(a)) = \psi(V_m a) = 0,$$

which shows that a $(\sigma, \beta)$-KMS state must have the form of the statement.

Conversely suppose that $\psi$ is as in the statement for some $\beta \geq \log d$. We will verify the KMS condition. To this end let $f = V_m a V_n^*$ and $g = V_b b V_{\lambda}^*$. A direct computation shows that

$$fg = \begin{cases} V_m a V_n^* b V_{\lambda}^* & \text{when } \nu = \kappa \kappa', \\ V_m a V_{\kappa'} b V_{\lambda}^* & \text{when } \kappa = \nu \lambda', \\ 0 & \text{otherwise,} \end{cases}$$

$$= \begin{cases} V_m a \alpha_{\nu \lambda'}(b) V_{\lambda'}^* & \text{when } \nu = \kappa \kappa', \\ V_{\mu} \alpha_{\kappa \lambda'}(a) b V_{\lambda}^* & \text{when } \kappa = \nu \kappa', \\ 0 & \text{otherwise.} \end{cases}$$

In a similar way we have that

$$gf = \begin{cases} V_b b \alpha_{\lambda \nu'}(a) V_{\nu \lambda'}^* & \text{when } \lambda = \mu \lambda', \\ V_{\mu} \alpha_{\mu \nu'}(b) a V_{\nu}^* & \text{when } \mu = \lambda \mu', \\ 0 & \text{otherwise.} \end{cases}$$

We aim to show that $\psi(f g) = e^{-|\mu| - |\nu|} \beta \psi(g f)$. 

First we deal with the case where $\nu = \kappa \nu'$ and $\lambda = \mu \lambda'$. By assumption we have that

$$\psi(fg) = \delta_{\mu,\lambda
u'} e^{-|\nu|\beta} \psi(a\alpha_{\nu'}(b)),$$

and

$$\psi(gf) = \delta_{\kappa,\mu \lambda'} e^{-|\kappa|\beta} \psi(b\alpha_{\lambda'}(a)) = \delta_{\kappa,\mu \lambda'} e^{-|\kappa|\beta} \psi(\alpha_{\lambda'}(a)b),$$

since $\psi|_A$ is tracial.

**Claim.** Under the assumption that $\nu = \kappa \nu'$ and $\lambda = \mu \lambda'$, we have that $\mu = \lambda \nu'$ if and only if $\kappa = \nu \lambda'$. Each of them implies that $\nu' = \lambda' = 0, \mu = \lambda$ and $\kappa = \nu$.

**Proof of the Claim.** If $\mu = \lambda \nu'$ then we obtain that $\lambda = \mu \lambda' = \lambda \nu' \lambda'$. Therefore $\nu' = \lambda' = 0$ and as a consequence we obtain that $\nu = \kappa \nu' = \kappa$, thus $\kappa = \nu \lambda'$. The converse follows by symmetry which completes the proof of the claim.

Therefore if $\mu = \lambda \nu'$ or $\kappa = \nu \lambda'$ then we have that $\psi(fg) = e^{-|\nu|\beta} \psi(ab)$ and

$$e^{-(|\mu| - |\nu|)\beta} \psi(gf) = e^{-|\mu|\beta} e^{-|\kappa|\beta} \psi(ab) = e^{-|\nu|\beta} \psi(ab),$$

since $|\mu| - |\nu| + |\kappa| = |\mu|$. Otherwise we get that $\psi(fg) = 0 = \psi(gf)$ which satisfies the KMS condition trivially.

Secondly we deal with the case where $\nu = \kappa \nu'$ and $\mu = \lambda \mu'$. By assumption we have that

$$\psi(fg) = \delta_{\mu,\lambda \nu'} e^{-|\nu|\beta} \psi(a\alpha_{\nu'}(b)),$$

and

$$\psi(gf) = \delta_{\nu,\kappa \mu'} e^{-|\kappa|\beta} \psi(\alpha_{\mu'}(b)a) = \delta_{\nu,\kappa \mu'} e^{-|\nu|\beta} \psi(\alpha_{\mu'}(b)),$$

since $\psi|_A$ is tracial.

**Claim.** Under the assumption that $\nu = \kappa \nu'$ and $\mu = \lambda \mu'$, we have that $\mu = \lambda \nu'$ if and only if $\nu = \kappa \mu'$. Each of them implies that $\nu' = \mu'$.

**Proof of the Claim.** If $\mu = \lambda \nu'$ then $\lambda \mu' = \lambda \nu'$, which implies that $\mu' = \nu'$; thus $\nu = \kappa \nu' = \kappa \mu'$. The converse follows by symmetry and the proof of the claim is complete. Therefore if $\mu = \lambda \nu'$ or $\nu = \kappa \mu'$ we obtain that $\psi(fg) = e^{-|\nu|\beta} \psi(a\alpha_{\nu'}(b))$ and

$$e^{-(|\mu| - |\nu|)\beta} \psi(gf) = e^{-|\mu|\beta} e^{-|\kappa|\beta} \psi(a\alpha_{\mu'}(b)) = e^{-|\nu|\beta} \psi(a\alpha_{\mu'}(b)).$$

Otherwise we have that $\psi(fg) = 0 = \psi(gf)$ and the KMS condition is satisfied trivially.

The case where $\kappa = \nu \kappa'$ reduces to the previous computation by substituting the roles of $f$ and $g$ by $g^*$ and $f^*$ respectively, for which we have just showed that the KMS equation holds. Finally in the above computations we also established that if $\nu \neq \kappa \nu'$ or $\kappa \neq \nu \kappa'$ then we cannot have that
$\lambda = \mu \lambda'$ and $\kappa = \nu \lambda'$, or that $\mu = \lambda \mu'$ and $\kappa \mu' = \nu$. In these cases the KMS condition is satisfied trivially.

We continue with our examination when $\beta > \log d$. We leave the case $\beta = \log d$ for later. Nevertheless we remark that the condition $\beta \geq \log d > 0$ is satisfied when $d > 1$, hence Proposition 3.2 holds in this case. On the other hand if $d = 1$ and $\beta = \log d = 0$ then by definition a $(\sigma, 0)$-KMS state is a tracial state.

**Proposition 3.2.** Let $(A, \alpha)$ be a unital $C^*$-dynamical system of multiplicity $d$ and let $\beta > \log d$. Then for every tracial state $\tau$ of $A$ there exists a $(\sigma, \beta)$-KMS state $\psi_\tau$ of $\mathcal{T}(A, \alpha)$ such that

$$
\psi_\tau(V_\mu a V_\nu^*) = \delta_{\mu, \nu} \cdot (1 - e^{-\beta d}) \cdot \sum_{m=0}^{\infty} e^{-(m+|\mu|)\beta} \sum_{|w|=m} \tau \alpha_{\pi_\tau}(a),
$$

for all $a \in A$ and $\mu, \nu \in \mathbb{F}_d^+$. 

**Proof.** Fix a tracial state $\tau$ of $A$ and let $(H_\tau, \pi_\tau, \xi_\tau)$ be the GNS construction associated with $\tau$. Then the Fock representation $(\tilde{\pi}_\tau, V_\tau)$ on $H_\tau \otimes \ell^2(\mathbb{F}_d^+)$ defines a representation of $\mathcal{T}(A, \alpha)$. For every word $\kappa \in \mathbb{F}_d^+$ let the vector states

$$
\psi_\kappa(f) = \langle (V_\tau \times \tilde{\pi}_\tau)(f) \xi_\tau \otimes e_\kappa, \xi_\tau \otimes e_\kappa \rangle, \text{ for all } f \in \mathcal{T}(A, \alpha).
$$

We define the functional $\psi_\tau: \mathcal{T}(A, \alpha) \to \mathbb{C}$ by

$$
\psi_\tau(f) := (1 - e^{-\beta d}) \sum_{m=0}^{\infty} e^{-m \beta} \sum_{|\kappa|=m} \psi_\kappa(f).
$$

Then $\psi_\tau$ is well defined as a norm limit of vector states. Moreover $\psi_\tau$ is positive since every finite sum is so as a sum positive vector states. If $f = 1_A$ then we get $\psi_\kappa(1_A) = 1$ hence

$$
\psi_\tau(1_A) = (1 - e^{-\beta d}) \sum_{m=0}^{\infty} e^{-m \beta} \sum_{|\kappa|=m} \psi_\kappa(1_A)
$$

$$
= (1 - e^{-\beta d}) \sum_{m=0}^{\infty} e^{-m \beta} d^m
$$

$$
= (1 - e^{-\beta d}) \sum_{m=0}^{\infty} (e^{-\beta d})^m = 1,
$$

since by assumption $e^{-\beta d} < 1$. Thus $\psi_\tau$ is a state on $\mathcal{T}(A, \alpha)$. 

Next we show that \( \psi \) is as in the statement. For \( f = V_\mu a V^{*}_\nu \) with \( \mu \neq \nu \) we directly verify that \( \psi_\kappa(f) = 0 \). For \( f = V_\mu a V^{*}_\mu \) we have that
\[
\psi_\kappa(f) = \begin{cases} 
\sum |\kappa| = m \, e^{-m \beta} \sum \psi_\kappa(f) & \text{when } \kappa = \mu \kappa', \\
0 & \text{otherwise,}
\end{cases}
\]
Therefore we obtain that
\[
\psi_\tau(f) = (1 - e^{-\beta d}) \sum_{m=0}^\infty e^{-m \beta} \sum_{|\kappa| = m} \psi_\kappa(f)
\]
Finally we show that \( \psi_\tau \) is a \((\sigma, \beta)\)-KMS state. A direct computation shows that
\[
\psi_\tau(a) = e^{-|\mu| \beta} \sum_{|w| = m} \tau a_w(a)
\]
which equals \( \psi_\tau(V_\mu a V^{*}_\mu) \), and the proof is then completed by Proposition 3.1.

In the following theorem we show that the \((\sigma, \beta)\)-KMS states are exactly of this form when \( \beta > \log d \).

**Theorem 3.3.** Let \( (A, \alpha) \) be a unital \( C^* \)-dynamical system of multiplicity \( d \) and \( \beta > \log d \). Then there is an affine weak*-homeomorphism \( \tau \mapsto \psi_\tau \) from the simplex of the tracial states on \( A \) onto the simplex of the \((\sigma, \beta)\)-KMS states on \( T(A, \alpha) \) with
\[
\psi_\tau(V_\mu a V^{*}_\mu) = \delta_{\mu, \nu} \cdot (1 - e^{-\beta d}) \sum_{m=0}^\infty e^{-(m+|\mu|) \beta} \sum_{|w| = m} \tau a_w(a)
\]
for all \( a \in A \) and \( \mu, \nu \in F^d_+ \).

**Proof.** The fact that \( \tau \mapsto \psi_\tau \) is an affine weak*-continuous mapping follows by the standard arguments of [21, Proof of Theorem 6.1].

First we show that the mapping is onto. To this end, given a \((\sigma, \beta)\)-KMS state \( \varphi \) of \( T(A, \alpha) \) we will construct a tracial state \( \tau \) of \( A \) such that \( \varphi = \psi_\tau \). By Proposition 3.1 it suffices to show that \( \varphi(a) = \psi_\tau(a) \) for all \( a \in A \). In
what follows the key is to isolate a projection in $T(A, \alpha)$ that commutes with $A$. To this end we will use the projection

$$P = I - \sum_{i=1}^{d} V_i V_i^* \in T(A, \alpha).$$

Indeed recall that $aV_i V_i^* = V_i \alpha_t(a)V_i^* = V_i V_i^* a$, hence $Pa = aP$ for all $a \in A$. Moreover

$$\varphi(P) = 1 - \sum_{i=1}^{d} \varphi(V_i V_i^*) = 1 - \sum_{i=1}^{d} e^{-\beta} \varphi(V_i^* V_i) = 1 - e^{-\beta} d.$$

We aim to show that the function $\varphi_P : A \to \mathbb{C} : a \mapsto \varphi(P a P \varphi)^{\alpha \beta}$ gives the appropriate tracial state on $A$. Since $\sigma_{\beta} (P) = P$ and $\sigma_{\beta}(a) = a$ we have that $\varphi(PabP) = \varphi(bPPa) = \varphi(PbaP)$. Thus $\varphi_P$ is indeed a tracial state on $A$. Let the elements

$$p_m = \sum_{k=0}^{m} \sum_{|w|=k} V_w P V_w^* \in T(A, \alpha).$$

By definition $(1 - \sum_{i=1}^{k} V_i V_i^*) V_w = 0$, hence $PV_u V_w P = 0$ when $w \neq u$. Therefore each $p_m$ is a projection. Furthermore we obtain

$$\varphi(p_m) = \sum_{k=0}^{m} \sum_{|w|=k} \varphi(V_w P V_w^*) = \sum_{k=0}^{m} \sum_{|w|=k} e^{-|w|\beta} \varphi(P)$$

$$= (1 - e^{-\beta} d) \sum_{k=0}^{m} \sum_{|w|=k} e^{-|w|\beta} = (1 - e^{-\beta} d) \sum_{k=0}^{m} (e^{-\beta} d)^k$$

$$= 1 - (e^{-\beta} d)^m.$$

Thus $\lim_{m} \varphi(p_m) = 1$ and by [20, Lemma 7.3] we get that $\lim_{m} \varphi(p_m \alpha p_m) = \varphi(f)$ for all $f \in T(A, \alpha)$. In particular for $a \in A$ we get that

$$\varphi(a) = \lim_{m} \varphi(p_m \alpha p_m)$$

$$= \lim_{m} \sum_{k=0}^{m} \sum_{l=0}^{m} \sum_{|w|=k} \sum_{|u|=k} \varphi(V_w P V_w^* a V_u P V_u^*)$$

$$= \lim_{m} \sum_{k=0}^{m} \sum_{l=0}^{m} \sum_{|w|=k} \sum_{|u|=k} e^{-|w|\beta} \varphi(P V_w^* a V_u P V_u^* V_u P)$$

$$= \lim_{m} \sum_{k=0}^{m} \sum_{|w|=k} e^{-|w|\beta} \varphi(P V_w^* a V_w P)$$

$$= \lim_{m} \sum_{k=0}^{m} \sum_{|w|=k} e^{-|w|\beta} \varphi(P \alpha \pi(a) P) = \sum_{k=0}^{\infty} \sum_{|w|=k} e^{-|w|\beta} \varphi_P \alpha \pi(a).$$
which shows that $\varphi = \psi_\tau$ for $\tau = \varphi_\mu$.

Finally we show that the mapping is one-to-one. To this end it suffices to show that if $\psi_\tau$ is the state associated with a tracial state $\tau$ of $A$ as in Proposition 3.1 then $(\psi_\tau)_p(a) = \tau(a)$. Indeed if this is true then $\psi_\tau = \psi_{\tau'}$ will imply that $\tau = \tau'$. A direct computation shows that

$$\psi_\tau(PaP) = \psi_\tau(a) - \sum_{i=1}^d \left( \psi_\tau(V_i^*VaV_i^*) - \psi_\tau(aV_iV_i^*) \right) + \sum_{i,j=1}^d \psi_\tau(V_i^*aV_j^*V_j^*)$$

$$= \psi_\tau(a) - \sum_{i=1}^d \left( e^{-\beta} \psi_\tau(V_i^*aV_i) - e^{-\beta} \psi_\tau(V_i^*aV_i) + e^{-\beta} \psi_\tau(V_i^*aV_i) \right)$$

$$= \psi_\tau(a) - \sum_{i=1}^d e^{-\beta} \psi_\tau a_i(a),$$

where we have used that $\psi_\tau(V_i^*aV_i) = e^{-\beta} \psi_\tau(V_i^*aV_i)$, $\psi_\tau(aV_iV_i^*) = e^{-\beta} \psi_\tau(V_i^*aV_i)$, and that $\psi_\tau(V_i^*aV_j^*V_j^*) = \delta_{i,j} e^{-\beta} \psi_\tau(V_i^*aV_i)$. By the definition of $\psi_\tau$ we then obtain

$$\psi_\tau(PaP) = (1 - e^{-\beta}d) \sum_{m=0}^\infty \left( \sum_{|w|=m} \tau a_{\overline{m}}(a) - \sum_{i=1}^d \sum_{|w|=m} \tau a_{\overline{i}w}(a) \right)$$

$$= (1 - e^{-\beta}d) \sum_{m=0}^\infty \left( \sum_{|w|=m} \tau a_{\overline{w}}(a) - \sum_{i=1}^d \sum_{|w|=m+1} \tau a_{\overline{i}w}(a) \right) = \tau(a),$$

which ends the proof.

We continue with the examination of the ground states.

**Proposition 3.4.** Let $(A, \alpha)$ be a unital $C^*$-dynamical system. Then the mapping $\tau \mapsto \psi_\tau$ with

$$\psi_\tau(V_\mu aV_\nu^*) = \begin{cases} \psi(a) & \text{for } \mu = \emptyset = \nu, \\ 0 & \text{otherwise,} \end{cases}$$

is an affine weak*-homomorphism from the state space $S(A)$ onto the space of the ground states on $T(A, \alpha)$.

**Proof.** First we show that if $\psi$ is a ground state then it is of the aforementioned form. Suppose that $\mu \neq \emptyset$; by assumption the map

$$r + it \mapsto \psi(V_\mu \sigma_{r+it}(aV_\nu^*)) = e^{-i|\nu|r} e^{i|\nu|t} \psi(V_\mu aV_\nu^*),$$

must be bounded when $t > 0$, for all $a \in A$. Therefore $\psi(V_\mu aV_\nu^*) = 0$ in this case. When $\nu = \emptyset$ but $\mu \neq \emptyset$, then the function

$$r + it \mapsto \psi(a^* \sigma_{r+it}(V_\mu^*)) = e^{-i|\mu|r} e^{i|\mu|t} \psi(a^* V_\mu^*)$$

must be bounded when $t > 0$, for all $a^* \in A$. Therefore $\psi(a^* V_\mu^*) = 0$ in this case. When $\mu = \emptyset$ and $\nu = \emptyset$, then the function

$$r + it \mapsto \psi(a^* \sigma_{r+it}(aV_\nu^*)) = e^{-i|\nu|r} e^{i|\nu|t} \psi(a^* V_\nu^*)$$

must be bounded when $t > 0$, for all $a^* \in A$. Therefore $\psi(a^* V_\nu^*) = 0$ in this case.

We then conclude that $\psi_\tau = \psi_\mu$. Therefore

$$\psi_\tau(V_\mu aV_\nu^*) = \psi_\mu(aV_\nu^*) = \psi(a)$$

for $\mu = \emptyset = \nu$, and

$$\psi_\tau(V_\mu aV_\nu^*) = 0$$

for $\mu \neq \emptyset$. Therefore

$$\psi_\tau(V_\mu aV_\nu^*) = \begin{cases} \psi(a) & \text{for } \mu = \emptyset = \nu, \\ 0 & \text{otherwise,} \end{cases}$$

is an affine weak*-homomorphism from the state space $S(A)$ onto the space of the ground states on $T(A, \alpha)$. 

The proposition follows.

We now show that if $\psi$ is a ground state then it is of the aforementioned form. Suppose that $\mu \neq \emptyset$; by assumption the map

$$r + it \mapsto \psi(V_\mu \sigma_{r+it}(aV_\nu^*)) = e^{-i|\nu|r} e^{i|\nu|t} \psi(V_\mu aV_\nu^*),$$

must be bounded when $t > 0$, for all $a \in A$. Therefore $\psi(V_\mu aV_\nu^*) = 0$ in this case. When $\nu = \emptyset$ but $\mu \neq \emptyset$, then the function

$$r + it \mapsto \psi(a^* \sigma_{r+it}(V_\mu^*)) = e^{-i|\mu|r} e^{i|\mu|t} \psi(a^* V_\mu^*)$$

must be bounded when $t > 0$, for all $a^* \in A$. Therefore $\psi(a^* V_\mu^*) = 0$ in this case. When $\mu = \emptyset$ and $\nu = \emptyset$, then the function

$$r + it \mapsto \psi(a^* \sigma_{r+it}(aV_\nu^*)) = e^{-i|\nu|r} e^{i|\nu|t} \psi(a^* V_\nu^*)$$

must be bounded when $t > 0$, for all $a^* \in A$. Therefore $\psi(a^* V_\nu^*) = 0$ in this case.

We then conclude that $\psi_\tau = \psi_\mu$. Therefore

$$\psi_\tau(V_\mu aV_\nu^*) = \psi_\mu(aV_\nu^*) = \psi(a)$$

for $\mu = \emptyset = \nu$, and

$$\psi_\tau(V_\mu aV_\nu^*) = 0$$

for $\mu \neq \emptyset$. Therefore

$$\psi_\tau(V_\mu aV_\nu^*) = \begin{cases} \psi(a) & \text{for } \mu = \emptyset = \nu, \\ 0 & \text{otherwise,} \end{cases}$$

is an affine weak*-homomorphism from the state space $S(A)$ onto the space of the ground states on $T(A, \alpha)$. 

The proposition follows.
must be bounded for \( t > 0 \). This implies that \( \psi(a^*V_\sigma) = 0 \). Taking adjoints yields \( \psi(V_\mu a) = 0 \).

Conversely let \( \psi \) be a state on \( T(A, \alpha) \) that satisfies the condition of the statement. Then for \( f = V_\mu aV_\sigma^* \) and \( g = V_\nu bV_\lambda^* \) we compute

\[
|\psi(f\sigma_{r+it}(g))|^2 = |e^{it(\kappa-\lambda)(r+it)}\psi(fg)|^2 \\
= e^{-|\kappa-\lambda|t}\psi(fg)|^2 \\
\leq e^{-|\kappa-\lambda|t}\psi(f^*f)\psi(g^*g) \\
\leq e^{-|\kappa-\lambda|t}\psi(V_\mu a^*aV_\sigma^*)\psi(V_\lambda b^*bV_\lambda^*).
\]

When \( \nu \neq \emptyset \) or \( \kappa \neq \emptyset \) then the above expression is 0. When \( \nu = \lambda = \emptyset \) then

\[
|\psi(f\sigma_{r+it}(g))|^2 = e^{-|\kappa|t}\psi(V_\mu aV_\nu b)|^2 = e^{-|\kappa|t}\psi(V_\mu a\alpha_{\nu}/\emptyset(a)b)|^2,
\]

which is zero when \( \mu \neq \emptyset \) or \( \lambda \neq \emptyset \). Finally when \( \mu = \nu = \kappa = \lambda = \emptyset \) then \( \psi(f\sigma_{r+it}(g)) = \psi(ab) \), which is bounded for all \( a, b \in A \).

To end the proof it suffices to show that every state on \( A \) gives rise to a ground state on \( T(A, \alpha) \). Fix \( \tau \in S(A) \) and let \( (H_\tau, \pi_\tau, \xi_\tau) \) be the associated GNS representation. Then for the Fock representation \( (\pi_\tau, V_\tau) \) we define the state

\[
\psi(f) = \langle f\xi_\tau \otimes e_{\emptyset}, \xi_\tau \otimes e_{\emptyset} \rangle,
\]

for all \( f \in T(A, \alpha) \).

It is readily verified that \( \psi(a) = \tau(a) \) for all \( a \in A \). For \( f = V_\mu aV_\sigma^* \) we compute

\[
\psi(V_\mu aV_\sigma^*) = \langle V_{\tau,\mu}\pi_\tau(a)V_{\tau,\sigma}\xi_\tau \otimes e_{\emptyset}, \xi_\tau \otimes e_{\emptyset} \rangle \\
= \langle \pi_\tau(a)V_{\tau,\nu}\xi_\tau \otimes e_{\emptyset}, V_{\tau,\mu}\xi_\tau \otimes e_{\emptyset} \rangle \\
= \delta_{\mu,\emptyset}\delta_{\nu,\emptyset}\langle \pi_\tau(a)\xi_\tau, \xi_\tau \rangle \\
= \begin{cases} 
\tau(a) & \text{when } \mu = \emptyset = \nu, \\
0 & \text{otherwise,}
\end{cases} \\
= \begin{cases} 
\psi(a) & \text{when } \mu = \emptyset = \nu, \\
0 & \text{otherwise,}
\end{cases}
\]

and the proof is complete. \( \blacksquare \)

We continue with the analysis of the KMS\(_\infty\) states on \( T(A, \alpha) \).

**Proposition 3.5.** Let \( (A, \alpha) \) be a unital \( C^* \)-dynamical system. Then the mapping \( \tau \mapsto \psi_\tau \) with

\[
\psi_\tau(V_\mu aV_\sigma^*) = \begin{cases} 
\tau(a) & \text{for } \mu = \emptyset = \nu, \\
0 & \text{otherwise,}
\end{cases}
\]

is an affine weak*-homomorphism from the tracial state space \( T(A) \) onto the space of the KMS\(_\infty\) states on \( T(A, \alpha) \).
Proof. Let $\psi_\beta$ be $(\sigma, \beta)$-KMS states on $T(A, \alpha)$ for $\beta > \log d$ that converge in the w*-topology to $\psi$. By Proposition 3.1 we obtain that $\psi|_A$ is a tracial state on $A$ and that

$$\psi_\beta(V_\mu a V_\nu^*) = \delta_{\mu, \nu} e^{-|\mu| \beta} \psi(a)$$

which tends to zero when $\beta \to \infty$.

Conversely, let $\psi_\tau$ be as in the statement with respect to a tracial state $\tau$ of $A$. Let $\psi_{\tau, \beta}$ be as defined in Proposition 3.2, i.e.

$$\psi_{\tau, \beta}(V_\mu a V_\nu^*) = \delta_{\mu, \nu} \cdot (1 - e^{-\beta d}) \cdot \sum_{m=0}^\infty e^{-m|\mu| \beta} \sum_{|w|=m} \tau_{\pi}(a).$$

By the w*-compactness we may choose a sequence of such states that converges to a state, say $\psi$. By definition $\psi$ is then a KMS$_\infty$ state, and we aim to show that $\psi_\tau = \psi$. When $\mu \neq \emptyset$ or $\nu \neq \emptyset$ then we get that

$$\psi(V_\mu a V_\nu^*) = \lim_{\beta \to \infty} \psi_{\tau, \beta}(V_\mu a V_\nu^*) = 0 = \psi_\tau(V_\mu a V_\nu^*),$$

as in the preceding paragraph. When $\mu = \nu = \emptyset$ we obtain

$$\psi_{\tau, \beta}(V_\mu a V_\nu^*) = (1 - e^{-\beta d}) \cdot \sum_{m=0}^\infty e^{-m|\mu| \beta} \sum_{|w|=m} \tau_{\pi}(a)$$

$$\quad = (1 - e^{-\beta d}) \cdot \left( \tau(a) + \sum_{m=1}^\infty e^{-m|\mu| \beta} \sum_{|w|=m} \tau_{\pi}(a) \right).$$

However we have that

$$|\sum_{m=1}^\infty e^{-m|\mu| \beta} \sum_{|w|=m} \tau_{\pi}(a)| \leq \|a\| \cdot \sum_{m=1}^\infty e^{-m|\mu| \beta} d^m$$

$$\quad = \|a\| (-1 + (1 - e^{-\beta d})^{-1}).$$

Taking $\beta \to \infty$ yields that the quantity $\sum_{m=1}^\infty e^{-m|\mu| \beta} \sum_{|w|=m} \tau_{\pi}(a)$ tends to zero. Trivially $\lim_{\beta \to \infty} (1 - e^{-\beta d}) = 1$ and we obtain $\lim_{\beta \to \infty} \psi_{\tau, \beta}(a) = \tau(a)$. Therefore

$$\psi(V_\mu a V_\nu^*) = \psi(a) = \lim_{\beta \to \infty} \psi_{\tau, \beta}(a) = \tau(a) = \psi_\tau(a) = \psi_\tau(V_\mu a V_\nu^*)$$

which completes that $\psi = \psi_\tau$ on $T(A, \alpha)$. The last part follows in the same way as in the proof of Proposition 3.4.

3.2. The Cuntz-Pimsner algebra. The gauge action on $O(A, \alpha)$ induces an action of $R$ on $O(A, \alpha)$ as in the case of $T(A, \alpha)$. We will denote it by the same symbol $\sigma$. (The reason being that) the gauge actions on $T(A, \alpha)$ and $O(A, \alpha)$ intertwine the quotient map $q: T(A, \alpha) \to O(A, \alpha)$ on the ideal generated by

$$a \cdot (I - \sum_{i=1}^d V_i V_i^*),$$

for all $a \in J_{A, \alpha}$. 


Therefore the \((\sigma, \beta)\)-KMS states on \(\mathcal{O}(A, \alpha)\) define \((\sigma, \beta)\)-KMS states on \(\mathcal{T}(A, \alpha)\). We aim to show that there is actually a bijection. As we observed in the introduction we cannot apply [17] directly in order to obtain the following theorem.

**Theorem 3.6.** Let \((A, \alpha)\) be a unital \(C^*\)-dynamical system of multiplicity \(d\) and \(\beta > \log d\). Then there is an affine weak*-homeomorphism \(\tau \mapsto \varphi_\tau\) from the simplex of the tracial states on \(A\) that vanish on \(J_{(A, \alpha)}\) onto the simplex of the \((\sigma, \beta)\)-KMS states on \(\mathcal{O}(A, \alpha)\) such that

\[
\varphi_\tau(S_\mu a S_\nu^*) = \delta_{\mu, \nu} \cdot (1 - e^{-\beta}d) \cdot \sum_{m=0}^{\infty} e^{-(m+|\mu|)\beta} \sum_{|w|=m}^{\infty} \tau_{\alpha_{\overline{w}}}(a),
\]

for all \(a \in A\) and \(\mu, \nu \in \mathbb{F}_d^d\).

**Proof.** Let \(\tau\) be a tracial state on \(A\) and let \(\psi_\tau\) be as in Proposition 3.2. For convenience let us write \(M = (1 - e^{-\beta}d)\) and compute

\[
\psi_\tau\left(a \sum_{i=1}^{d} (I - V_i V_i^*)\right) = \psi_\tau(a) - \sum_{i=1}^{d} \psi_\tau(a V_i V_i^*) = \psi_\tau(a) - \sum_{i=1}^{d} \psi_\tau(V_i \alpha_i(a) V_i^*) = M \cdot \sum_{m=0}^{\infty} e^{-(m+|\mu|)\beta} \sum_{|w|=m}^{\infty} \tau_{\alpha_{\overline{w}}}(a) -
\]

\[
- \sum_{i=1}^{d} M \cdot \sum_{m=0}^{\infty} e^{-(m+1)\beta} \sum_{|w|=m}^{\infty} \tau_{\alpha_{\overline{w}}}(\alpha_i(a)).
\]

In particular we have that

\[
\sum_{m=0}^{\infty} e^{-(m+1)\beta} \sum_{|w|=m}^{\infty} \tau_{\alpha_{\overline{w}}}(\alpha_i(a)) = \sum_{m=0}^{\infty} e^{-(m+1)\beta} \sum_{|w|=m}^{\infty} \tau_{\alpha_{\overline{w}}}(a) = \sum_{m=1}^{\infty} e^{-m\beta} \sum_{|w|=m, w \notin i \mathbb{F}_d^d} \tau_{\alpha_{\overline{w}}}(a).
\]

Therefore we get that

\[
\sum_{m=0}^{\infty} e^{-m\beta} \sum_{|w|=m}^{\infty} \tau_{\alpha_{\overline{w}}}(a) - \sum_{m=0}^{\infty} e^{-(m+1)\beta} \sum_{|w|=m}^{\infty} \tau_{\alpha_{\overline{w}}}(\alpha_i(a)) =
\]

\[
= \tau(a) + \sum_{m=1}^{\infty} e^{-m\beta} \sum_{|w|=m, w \notin i \mathbb{F}_d^d} \tau_{\alpha_{\overline{w}}}(a).
\]

Inductively we obtain that

\[
\psi_\tau\left(a \sum_{i=1}^{d} (I - V_i V_i^*)\right) = \tau(a), \text{ for all } a \in A.
\]
As a consequence, if \( \tau \) vanishes on \( J_{(A, \alpha)} \) then \( \psi_\tau \) vanishes on \( \ker q \) and a \( (\sigma, \beta) \)-KMS state on \( \mathcal{O}(A, \alpha) \) is defined by \( \varphi_{\tau}q = \psi_\tau \). Conversely if \( \varphi_{\tau} \) is as in the statement then let \( \psi_\tau = \varphi_{\tau}q \) and the above equation shows that \( \tau \) vanishes on \( J_{(A, \alpha)} \).

**Proposition 3.7.** Let \( (A, \alpha) \) be an injective unital C*-dynamical system of multiplicity \( d \). Then \( \mathcal{O}(A, \alpha) \) does not attain \( (\sigma, \beta) \)-KMS states for \( \beta \neq \log d \).

**Proof.** If \( (A, \alpha) \) is injective then \( J_{(A, \alpha)} = A \), hence \( \sum_{i=1}^{d} S_i S_i^* = 1 \) in \( \mathcal{O}(A, \alpha) \). Proceed as in the proof of Proposition 3.1 to obtain an estimation for \( \beta \) by using the \( S_i \) in the place of \( V_i \). However now we obtain equality which shows that \( \beta = \log d \). 

We conclude with the analogues for the ground states and the KMS\(_\infty\) states on \( \mathcal{O}(A, \alpha) \). The proofs are left to the reader. We remark that in the case of the KMS\(_\infty\) states one has to make use of Theorem 3.6.

**Corollary 3.8.** Let \( (A, \alpha) \) be a unital C*-dynamical system. Then the mapping \( \tau \mapsto \varphi_{\tau} \) with

\[
\varphi_{\tau}(S_\mu a S_\nu^*) = \begin{cases} 
\tau(a) & \text{for } \mu = \emptyset = \nu, \\
0 & \text{otherwise},
\end{cases}
\]

is an affine weak*-homomorphism from the simplex of the states \( S(A) \) (resp. the tracial states \( T(A) \)) that vanish on \( J_{(A, \alpha)} \) onto the simplex of the ground states (resp. KMS\(_\infty\) states) on \( \mathcal{O}(A, \alpha) \).

### 3.3. KMS states at \( \beta = \log d \)

We continue with the examination of the \( (\sigma, \log d) \)-KMS states. First we note that it suffices to examine such states just for \( \mathcal{O}(A, \alpha) \).

**Proposition 3.9.** Let \( (A, \alpha) \) be a unital C*-dynamical system. Then \( \psi \) is a \( (\sigma, \log d) \)-KMS state on \( T(A, \alpha) \) if and only if it factors through a \( (\sigma, \log d) \)-KMS state on \( \mathcal{O}(A, \alpha) \).

**Proof.** If \( \psi \) is a \( (\sigma, \log d) \)-KMS state on \( T(A, \alpha) \) then

\[
1 = \psi(1) = \psi(V_i^* V_i) = e^{-\log d} \psi(V_i V_i^*) = \frac{1}{d} \psi(V_i V_i^*),
\]

for all \( i = 1, \ldots, d \). By the Cauchy-Schwartz inequality we then obtain

\[
|\psi(a(1 - \sum_{i=1}^{d} V_i V_i^*))|^2 \leq \psi(a^* a) \cdot \psi((1 - \sum_{i=1}^{d} V_i V_i^*)^*(1 - \sum_{i=1}^{d} V_i V_i^*))
= \psi(a^* a) \cdot \psi(1 - \sum_{i=1}^{d} V_i V_i^*)
\]

for all \( a \in A \), since \( 1 - \sum_{i=1}^{d} V_i V_i^* \) is a projection in \( T(A, \alpha) \). However

\[
\psi(1 - \sum_{i=1}^{d} V_i V_i^*) = 1 - \sum_{i=1}^{d} \psi(V_i V_i^*) = 0
\]
which shows that $\psi(a(1 - \sum_{i=1}^d V_i V_i^*)) = 0$ for all $a \in A$ and in particular for $a \in J_{(A,\alpha)}$. Hence $\psi$ vanishes on $\ker q$ and thus defines a state $\varphi$ on $\mathcal{O}(A,\alpha)$. The converse follows by the fact that the actions of $\mathbb{R}$ on the Pimsner algebras intertwine $q$, and the proof is complete.

As for providing existence it suffices to restrict to the injective case. We isolate this fact in the following remark.

**Remark 3.10.** Suppose that $(A,\alpha)$ is a unital system that is not injective. Then we can use the tail adding technique as in the work of the author with Katsoulis [14] in order to obtain the injective dilation $(C,\gamma)$ of the same multiplicity. In particular we have that $A \subseteq C$ and that $p\gamma_i(a)p = \alpha_i(a)$ for all $a \in A$ and $p = 1 \in A$. In addition $\mathcal{O}(A,\alpha)$ is proven to be a full corner of $\mathcal{O}(C,\gamma)$ by the projection $p$. By construction the system $(C,\gamma)$ is not unital ($C$ does not have a unit). Nevertheless if $(C^{(1)},\gamma^{(1)})$ is the unitization then we have that $\mathcal{O}(C,\gamma)$ (and consequently $\mathcal{O}(A,\alpha)$) is a $C^*$-subalgebra of $\mathcal{O}(C^{(1)},\gamma^{(1)})$. All these inclusions are canonical in the sense that if $(\pi_i,\{S_i\}_{i=1}^d)$ defines a faithful representation of $\mathcal{O}(C^{(1)},\gamma^{(1)})$, then $(\pi|_C,\{S_i\}_{i=1}^d)$ defines a faithful representation of $\mathcal{O}(C,\gamma)$ and $(\pi|_A,\{S_i p\}_{i=1}^d)$ defines a faithful representation of $\mathcal{O}(A,\alpha)$. As a consequence the gauge action is compatible with the inclusions.

**Proposition 3.11.** Let $(A,\alpha)$ be a unital $C^*$-dynamical system and let $(C,\gamma)$ be its injective dilation as in [14]. Then a $(\sigma,\log d)$-KMS state on $\mathcal{O}(C,\gamma)$ or $\mathcal{O}(C^{(1)},\gamma^{(1)})$ defines a $(\sigma,\log d)$-KMS state on $\mathcal{O}(A,\alpha)$ by restriction.

**Proof.** Recall that $\mathcal{O}(A,\alpha)$ is a (full) corner of $\mathcal{O}(C,\gamma)$ and the inclusion respects the gauge action, hence the action $\sigma$.

Therefore without loss of generality we assume henceforth that $(A,\alpha)$ is an injective and unital $C^*$-dynamical system. We examine separately the cases $d = 1$ and $d > 1$. Motivated by [22] we obtain the following result for $d > 1$.

**Proposition 3.12.** Let $(A,\alpha)$ be an injective unital $C^*$-dynamical system of multiplicity $d > 1$. We write $\tau \in AVT(A,\alpha)$ for the $\tau \in T(A)$ such that

$$\tau(a) = \frac{1}{d} \sum_{i=1}^d \tau \alpha_i(a), \quad \text{for all } a \in A,$$

and we write $\varphi \in AVT(\mathcal{O}(A,\alpha)^\gamma)$ for the $\varphi \in T(\mathcal{O}(A,\alpha)^\gamma)$ such that

$$\varphi(f) = \frac{1}{d} \sum_{i=1}^d \varphi(S_i^* f S_i), \quad \text{for all } f \in \mathcal{O}(A,\alpha)^\gamma.$$

Then there is an affine weak*-homeomorphism between the three simplices of $AVT(A,\alpha)$, of $AVT(\mathcal{O}(A,\alpha)^\gamma)$, and that of the $(\sigma,\log d)$-KMS states on $\mathcal{O}(A,\alpha)$. 
**Proof.** Recall that the fixed point algebra is $O(A, \alpha)^\gamma = \overline{\text{span}}_{\mu, \nu} A_n$ where $A_n = \text{span}\{S_\mu aS_\nu^* \mid |\mu|, |\nu| \leq n\}$. On the other hand let the C*-subalgebras

$$B_n = \text{span}\{S_\mu aS_\nu^* \mid |\mu| = |\nu| = n\}$$

of $O(A, \alpha)^\gamma$ so that $A_n = \cup_{m=0}^n B_m$. Since the system is injective and unital we may use that $1 = \sum_{i=1}^d S_i S_i^*$ to obtain

$$B_{m-1} \ni S_\mu aS_\nu^* = S_\mu a \cdot 1 \cdot S_\nu^* = \sum_{i=1}^d S_\mu \alpha_i(a) S_{\nu i}^* \in B_m.$$ Consequently $A_n = B_n$ for all $n \in \mathbb{Z}_+$ and $O(A, \alpha)^\gamma$ is the inductive limit of the $B_n$ for $n \in \mathbb{Z}_+$ via the connecting *-homomorphisms

$$B_{n-1} \ni S_\mu aS_\nu^* \mapsto \sum_{i=1}^d S_\mu \alpha_i(a) S_{\nu i}^* \in B_n.$$ In what follows we will use this form of $O(A, \alpha)^\gamma$.

For $\tau \in \text{AVT}(A, \alpha)$ we define the extension $\varphi_n$ on $B_n$ by

$$\varphi_n(S_\mu aS_\nu^*) := \frac{1}{d^n} \sum_{|w|=|\mu|} \tau(S_w^* S_\mu aS_\nu^* S_w) = \frac{1}{d^n} \delta_{\mu, \nu} \tau(a).$$

Every $\varphi_n$ is positive being the average on the diagonal of $B_n$. The family $\{\varphi_n\}_n$ is compatible with the directed sequence on the $B_n$, because

$$\varphi_{n+1}\left(\sum_{i=1}^d S_\mu \alpha_i(a) S_{\nu i}^*\right) = \sum_{i=1}^d \frac{1}{d^{n+1}} \delta_{\mu_i, \nu i} \tau(\alpha_i(a))$$

$$= \delta_{\mu, \nu} \frac{1}{d^n} \sum_{i=1}^d \frac{1}{d} \tau(\alpha_i(a)) = \varphi_n(S_\mu aS_\nu^*).$$

Therefore $\{\varphi_n\}_n$ defines a positive functional $\varphi_\tau : O(A, \alpha)^\gamma \to \mathbb{C}$. The unit $1 \in A$ is mapped to $f_n := \sum_{|\nu|=n} S_\nu S_\nu^*$ via the inclusion $A \hookrightarrow B_n$. The computation

$$\varphi_n(f_n) = \frac{1}{d^n} \sum_{|w|=|\mu|} \sum_{|v|=|\nu|} \tau(S_w^* S_v S_\nu^* S_w) = \frac{1}{d^n} \sum_{|w|=|\mu|, |v|=n} \delta_{w, v} \tau(1) = 1$$

shows that each $\varphi_n$ is a state and consequently $\varphi$ is a state. In addition, for the elements $S_\mu aS_\nu^*, S_\nu bS_\lambda^* \in B_n$ we have

$$\varphi_n(S_\mu aS_\nu^* \cdot S_\nu bS_\lambda^*) = \delta_{\nu, \kappa} \varphi_n(S_\mu aS_\lambda^*) = \delta_{\nu, \kappa} \delta_{\mu, \lambda} \frac{1}{d^n} \tau(ab)$$

which is symmetrical (since $\tau$ is tracial) and implies that every $\varphi_n$ is tracial. Thus $\varphi_\tau$ is a tracial state on $O(A, \alpha)^\gamma$. Now let $f = S_\mu aS_\nu^* \in O(A, \alpha)^\gamma$ and
compute
\[
\frac{1}{d} \sum_{i=1}^{d} \varphi_\tau(S_i^* f S_i) = \frac{1}{d} \sum_{i=1}^{d} \varphi_\tau(S_i^* S_i \alpha_i(a S_i^*))
\]
\[
= \frac{1}{d} \sum_{i=1}^{d} \delta_{\mu, \mu'} \delta_{\tau, \tau'} \varphi_\tau(S_{\mu'}^* \alpha_{\mu'}(a S_{\mu'}^*))
\]
\[
= \frac{1}{d} \sum_{i=1}^{d} \delta_{\mu, \mu'} \delta_{\tau, \tau'} \delta_{\mu, \mu'} \frac{1}{d |\mu|-1} \tau(a)
\]
\[
= \frac{1}{d |\mu|} \tau(a) = \varphi_\tau(f)
\]
which shows that \( \varphi_\tau \in AVT(O(A, \alpha)\gamma) \). Due to the form of \( \varphi_\tau \) we obtain that the mapping \( \tau \mapsto \varphi_\tau \) is one-to-one. In order to show injectivity fix \( \varphi \in AVT(O(A, \alpha)\gamma) \) and compute
\[
\varphi(a) = \frac{1}{d} \sum_{i=1}^{d} \varphi(S_i^* a S_i) = \frac{1}{d} \sum_{i=1}^{d} \varphi(S_i^* S_i \alpha_i(a)) = \frac{1}{d} \sum_{i=1}^{d} \varphi \alpha_i(a),
\]
hence \( \tau = \varphi|_A \in AVT(A, \alpha) \).

For the second part fix \( \varphi_\tau \in AVT(O(A, \alpha)\gamma) \) for some \( \tau \in AVT(A, \alpha) \) and define the state \( \psi_\tau = \varphi_\tau E: O(A, \alpha) \to \mathbb{C} \) where \( E \) is the conditional expectation onto the fixed point algebra. Then \( \psi|_A = \tau \) is tracial and
\[
\psi_\tau(S_{\mu} a S_{\mu'}^*) = \delta_{|\mu|, |\mu'|} \varphi_\tau(S_{\mu} a S_{\mu'}^*) = \delta_{|\mu|, |\mu'|} \delta_{\mu, \mu'} \frac{1}{d |\mu|-1} \tau(a) = \delta_{\mu, \mu'} \frac{1}{d |\mu|} \psi_\tau(a).
\]
By Proposition 3.1 we obtain that \( \psi_\tau \) is a \( (\sigma, \log d) \)-KMS state. Conversely if \( \psi \) is a \( (\sigma, \log d) \)-KMS state let \( \tau = \psi|_A \). Then \( \tau \) is tracial and a similar computation as before yields \( \tau \in AVT(A, \alpha) \). By the form of \( \psi \) we obtain that the mapping \( \tau \mapsto \psi_\tau \) is an affine isomorphism.

Furthermore due to the formulas of \( \varphi_\tau \) and \( \psi_\tau \) we obtain that the affine isomorphisms are weak*-homeomorphisms.

**Corollary 3.13.** Let \( (X, \sigma) \) be a classical system of multiplicity \( d > 1 \) such that \( X \) is a compact Hausdorff space and \( \bigcup_{i=1}^{d} \sigma_i(X) = X \). Then \( O(X, \sigma) \) attains \( (\sigma, \log d) \)-KMS states.

**Proof.** The only reason we assume that \( A = C(X) \) is to ensure that the states are automatically tracial. Then the result follows by Proposition 3.12 and the ideas of [22, Section 4]. In short, let the positive contractive map \( \lambda: A \to A \) such that \( \lambda(a) = \frac{1}{d} \sum_{i=1}^{d} \alpha_i(a) \). The spectral radius of \( \lambda \) equals \( 1 \in \mathbb{R} \) because \( (\lambda - \text{id}_A)(1_A) = 0 \) so that \( 1 \in \mathbb{C} \) is in the spectrum of \( \lambda \). Let us denote by \( \lambda^\#: A^\# \to A^\# \) the induced operator on the continuous linear functionals of \( A \). Since \( \text{sp}(\lambda) = \text{sp}(\lambda^\#) \) we have that the spectral radius of \( \lambda^\# \) is \( 1 \in \mathbb{R} \). Therefore for the resolvent \( R^\#(t) = (t - \lambda^\#)^{-1} \) of \( \lambda^\# \) there
is a $\varphi_0 \in A^#$ such that $\|R^t(t)\varphi_0\|$ is unbounded as $t \downarrow 1$. By the Jordan decomposition we may assume that $\varphi_0$ is a state and let the states

$$\varphi_n = \frac{R^t(t)\varphi_0}{\|R^t(t)\varphi_0\|} \quad \text{for } n \geq 1.$$ 

By weak*-compactness let $\varphi$ be an accumulation point of $(\varphi_n)$ for which we obtain that $\varphi = \lambda^\# \varphi = \varphi \lambda$. Therefore $\varphi = \frac{1}{d} \sum_{i=1}^d \varphi \alpha_i$ and by assumption $\varphi$ is a tracial state; consequently $\varphi \in AVT(A,\alpha)$.

**Remark 3.14.** There is a case where the $(\sigma, \log d)$-KMS state on Corollary 3.13 is unique. Suppose that $A$ is commutative and finite dimensional, and let $\lambda: A \to A$ defined by $\lambda(a) = \frac{1}{d} \sum_{i=1}^d \alpha_i(a)$. We say that $\lambda$ is irreducible if $\lambda(J) \subseteq J$ holds only for the trivial ideals $J$ of $A$. In this case the system $(A, \alpha)$ is injective and $\lambda$ corresponds to a non-negative irreducible matrix. By the Perron-Frobenius Theorem then the eigenvalue $1 \in \mathbb{C}$ of Corollary 3.13 is simple and thus $\varphi$ is unique.

Finally we prove the existence of $(\sigma, \log d)$-KMS states when $d = 1$. In this case the KMS states coincide with the tracial states by definition.

**Proposition 3.15.** Let $\alpha: \mathbb{Z}_+ \to \text{End}(A)$ be a unital $C^*$-dynamical system and let $\hat{\gamma}: \mathbb{Z}_+ \to \text{Aut}(\hat{C})$ be its automorphic dilation (see Subsection 2.2). For any tracial state $\tau$ of $\hat{C}$ there exists a tracial state $\psi$ of $O(A, \alpha)$ such that

$$\psi(S_n a S_m^*) = \delta_{n,m} \tau \hat{\gamma}^{-n}(a) \quad \text{for all } a \in A, \, n, m \in \mathbb{Z}_+.$$ 

**Proof.** Recall that $O(A, \alpha)$ is a full corner of $\hat{C} \rtimes \hat{\gamma} \mathbb{Z}$ [12, 14]. Hence a tracial state on the crossed product defines by restriction a tracial state on $O(A, \alpha)$. If $\tau$ is a tracial state on $\hat{C}$ then let $\psi := \tau E: \hat{C} \rtimes \hat{\gamma} \mathbb{Z} \to \mathbb{C}$ where $E$ is the conditional expectation of the crossed product. It is then readily verified that $\psi|_{O(A, \alpha)}$ satisfies the condition of the statement.

**Remark 3.16.** The difference between Proposition 3.15 and Corollary 3.13 boils down to having an explicit formula that defines the states. A review of the proof of Proposition 3.1 highlights this phenomenon. Even though the tracial states of Proposition 3.15 satisfy $\tau = \tau \gamma$ (where $(C, \gamma)$ is the injective dilation of $(A, \alpha)$) these are far from being the only choices. For example if $A = \mathbb{C}$ then $O(A, \alpha) = \mathbb{C}(T)$ and obviously there are plenty of other tracial states than the trivial one.

### 3.4. Applications

Let $(A, \alpha) \equiv (X, \sigma)$ and $(C, \gamma) \equiv (Y, \rho)$ be two classical systems on the compact Hausdorff spaces $X$ and $Y$. There are several notions of equivalences for $(X, \sigma)$ and $(Y, \rho)$.

We say that $(X, \sigma)$ and $(Y, \rho)$ are **unitarily equivalent** if there is a homeomorphism $\phi: X \to Y$ and a unitary $U \in M_{d_\sigma, d_\rho}(C(Y))$ that intertwines the diagonal actions

$$\text{diag}_{\phi \sigma}(f) := \text{diag}(f \phi \sigma_i | i = 1, \ldots, d_\sigma)$$
and
\[ \text{diag}_{\rho_0}(f) := \text{diag}(f \rho_i \phi \mid i = 1, \ldots, d). \]
As a consequence the multiplicities \( d_\sigma \) and \( d_\rho \) must coincide. Unitary equivalence of \((X, \sigma)\) and \((Y, \rho)\) is in fact unitary equivalence of the associated C*-correspondences. Therefore unitarily equivalent systems have isomorphic tensor algebras and Pimsner algebras (and so the Pimsner algebras attain the same KMS-theory). Recall that the tensor algebra of \((A, \alpha)\) in the sense of Muhly and Solel [23] is the closed subalgebra of \( T(A, \alpha) \) generated by \( v_w a \) with \( a \in A \) and \( w \in \mathbb{F}_+^d \). The author and Katsoulis [15] showed a converse: unitary equivalence is a complete invariant for isometric isomorphic tensor algebras. Such a result does not hold for the Cuntz-Pimsner algebras. For \( d = 1 \) unitary equivalence is just conjugacy of the systems and Hørmander and Parry [11] have given a counterexample of a homeomorphism that is not conjugate to its inverse (however the crossed products are *-isomorphic).

On the other hand the systems are called piecewise conjugate if they have the same multiplicities and there is a homeomorphism \( \phi: X \to Y \) such that for every \( x \in X \) there is a neighborhood \( U \) of \( x \) and a permutation \( \pi \) on \( d \) symbols so that
\[ \rho_i(\phi|_U) = \phi \sigma_{\pi(i)}|_U, \text{ for all } i = 1, \ldots, d. \]
Piecewise conjugacy is weaker than unitary equivalence and let us include a short proof for completeness of the discussion.

**Proposition 3.17.** If \((A, \alpha) \equiv (X, \sigma)\) and \((C, \gamma) \equiv (Y, \rho)\) are unitarily equivalent then they are piecewise conjugate.

**Proof.** Suppose that there is a homeomorphism \( \phi: X \to Y \) and a unitary \( U = [u_{ij}] \in M_d(C(Y)) \) such that
\[ \text{diag}_{\phi \sigma}(f) U = U \text{diag}_{\rho_0}(f), \text{ for all } f \in C(Y). \]
By substituting \( \rho_i \) with \( \phi^{-1} \rho_i \phi \) we may assume that \( X = Y \) and \( \phi = \text{id} \). Fix a point \( x \in X \) and set \( U^{(1)} = U \). Since \( U(x) = [u_{ij}(x)] \) is a unitary in \( M_d(C) \) there is at least one \( u_{ij} \) such that \( u_{ij}(x) \neq 0 \). Choose a neighborhood \( U_1 \) of \( x \) such that \( u_{ij}|_{U_1} \neq 0 \). By using this element perform the first step of the Gaussian elimination as in [15, Lemma 3.3] to obtain a second invertible matrix \( U^{(2)} \). Again by invertibility there is a neighborhood \( U_2 \) of \( x \) and an element in the second row of \( U^{(2)} \) that is nowhere zero on \( U_2 \). Use this element to perform Gaussian elimination on \( U_1 \cap U_2 \). Inductively and by using the last step of [15, Lemma 3.3] we finally obtain a permutation \( \pi \) on \( d \) symbols and \( d \) elements, say \( v_{ii} \), such that
\[ \text{diag}_{\sigma_{\pi(i)}|_{U}}(f) \cdot \text{diag}_{v_{ii}|_{U}}(d)_{i=1}^{d} = \text{diag}(v_{ii}|_{U})_{i=1}^{d} \cdot \text{diag}_{\rho_0}(f), \]
where \( U = U_1 \cap U_2 \cap \cdots \cap U_d \). Furthermore \( v_{ii}(x) \neq 0 \) for all \( x \in U \). Therefore the equation \( f \sigma_{\pi(i)}|_{U} \cdot v_{ii}|_{U} = v_{ii}|_{U} \cdot f \rho_i \) implies that \( \sigma_{\pi(i)}|_{U} = \rho_i|_{U} \) for all \( i = 1, \ldots, d. \)
Davidson and Katsoulis [7] introduced the notion of piecewise conjugacy and showed that it is an invariant for algebraic isomorphism of tensor algebras. The converse also was shown in several cases, including the cases of \( n = 2, 3 \), where algebraic isomorphism is replaced by the stronger isometric isomorphism. In fact this is proven by showing that piecewise conjugacy implies unitary equivalence in these specific cases. It remains an open problem whether the converse of Proposition 3.17 is true (or not) in full generality.

In particular it is not known whether the Pimsner algebras of piecewise systems are \(*\)-isomorphic for any multiplicity. Nevertheless, we aim to show that they have the same KMS-theory. We begin with the following lemma.

**Lemma 3.18.** Suppose that \((A, \alpha) \equiv (X, \sigma)\) and \((C, \gamma) \equiv (Y, \rho)\) are piecewise conjugate by the homeomorphism \( \phi: X \to Y \). If \( \hat{\phi}: C \to A \) is the implemented \(*\)-isomorphism then

\[
\sum_{|w|=m} \tau_{\alpha \pi \tau} = \sum_{|w|=m} \tau_{\hat{\phi}^{-1} \gamma \pi \phi}
\]

for all \( \tau \in S(A) \) and \( m \in \mathbb{Z}_+ \).

**Proof.** By substituting \( \rho_i \) with \( \phi^{-1} \rho_i \phi \) we may assume that \( X = Y \) so that for every \( x \in X \) there is a neighborhood \( \mathcal{U} \) of \( x \) and a permutation \( \pi \) on \( d \) symbols such that \( \rho_i|_{\mathcal{U}} = \sigma_{\pi(i)}|_{\mathcal{U}} \) for all \( i = 1, \ldots, d \). In particular for any \( x \in X \) we obtain

\[
\{ \sigma_i(x) \mid i = 1, \ldots, d \} = \{ \rho_i(x) \mid i = 1, \ldots, d \}.
\]

For the inductive step suppose that for any \( x \in X \) we have that

\[
\{ \sigma_w(x) \mid w \in F_+^d, |w| = n \} = \{ \rho_w(x) \mid w \in F_+^d, |w| = n \}
\]

holds for all \( n \leq m \). Let a word \( \mu \in F_+^d \) of length \( m + 1 \) such that \( \mu = i_0w \) with \( |w| = m \). For the point \( y = \sigma_w(x) \) we have that

\[
\sigma_\mu(x) = \sigma_{i_0}(y) \in \{ \rho_i(y) \mid i = 1, \ldots, d \}.
\]

On the other hand by the inductive hypothesis we have that

\[
y = \sigma_w(x) \in \{ \rho_{w'}(x) \mid w' \in F_+^d, |w'| = m \},
\]

therefore

\[
\rho_{i_0}(y) = \rho_{i_0}\sigma_w(x) = \rho_{i_0}\rho_{w'}(x) \in \{ \rho_\nu(x) \mid \nu \in F_+^d, |\nu| = m + 1 \}.
\]

Thus we obtain that

\[
\sigma_\mu(x) \in \{ \rho_\nu(x) \mid \nu \in F_+^d, |\nu| = m + 1 \}.
\]

Since \( \mu \) was arbitrary and by symmetry we have that

\[
\{ \sigma_w(x) \mid w \in F_+^d, |w| = m + 1 \} = \{ \rho_w(x) \mid w \in F_+^d, |w| = m + 1 \},
\]

for any \( x \in X \). Consequently we obtain that

\[
\{ \sigma_w(x) \mid w \in F_+^d, |w| = m \} = \{ \rho_w(x) \mid w \in F_+^d, |w| = m \},
\]

for all \( m \in \mathbb{Z}_+ \).
Let $\tau$ be a state on $A$ that is a finite convex combination of pure states, i.e. $\tau = \sum_{k=1}^{N} \lambda_k \text{ev}_{x_k}$ with $\sum_{i=1}^{N} \lambda_k = 1$. Then we compute
\[
\sum_{|w|=m} \tau_{\alpha} w = \sum_{k=1}^{N} \lambda_k \sum_{|w|=m} \text{ev}_{x_k} w = \sum_{k=1}^{N} \lambda_k \sum_{|w|=m} \text{ev}_{\sigma_w(x_k)} = \sum_{|w|=m} \tau_{\gamma} w
\]
for all $m \in \mathbb{Z}_+$. The proof is completed by taking limits of such states $\tau$.

**Corollary 3.19.** Suppose that $(A, \alpha) \equiv (X, \sigma)$ and $(C, \gamma) \equiv (Y, \rho)$ are piecewise conjugate by the homeomorphism $\phi: X \to Y$ and let $\hat{\phi}: C \to A$ be the induced $\ast$-isomorphism. Then the affine weak*-homeomorphism $S(A) \to S(C): \tau \mapsto \tau_{\hat{\phi}}$ lifts to an affine weak*-homeomorphism between the KMS states on $T(A, \alpha)$ and $T(C, \gamma)$. In particular if $\beta > \log d$ and $\tau \mapsto \psi_\tau$ is the parametrization obtained by Theorem 3.3 then the diagram

\[
\begin{array}{c}
\tau \\
\downarrow \psi_\tau \\
\tau_{\hat{\phi}} \downarrow \psi_{\tau_{\hat{\phi}}}
\end{array}
\]

is commutative in the sense that
\[
\psi_{\tau_{\hat{\phi}}}(V_{\mu}cV_{\nu}^*) = \psi_\tau(V_{\mu}\hat{\phi}(c)V_{\nu}^*), \text{ for all } c \in C \text{ and } \nu, \mu \in \mathbb{F}_d^d.
\]
The same holds for the parametrization of Proposition 3.12 for $\beta = \log d > 0$, and for the parametrization of Theorem 3.6 on the KMS states on $O(A, \alpha)$ and $O(C, \gamma)$.

**Proof.** Without loss of generality we may assume that $A = C$ and $\hat{\phi} = \text{id}$ by substituting every $\rho_i$ by $\phi^{-1} \rho_i \phi$. Note that this yields a unitary equivalence between $(Y, \rho)$ and $(X, \phi^{-1} \rho \phi)$ which implies that the Pimsner algebras are $\ast$-isomorphic. This $\ast$-isomorphism is given by
\[
V_{\mu}cV_{\nu}^* \mapsto V_{\mu}\hat{\phi}(c)V_{\nu}^*, \text{ for all } c \in C \text{ and } \mu, \nu \in \mathbb{F}_d^d,
\]
and thus respects the statement. Furthermore by piecewise conjugacy we have that
\[
\sigma_i(x) \in \bigcup_{i=1}^{d} \rho_i(x)
\]
for all $x \in X$ and $i = 1, \ldots, d$, therefore
\[
\bigcup_{i=1}^{d} \sigma_i(X) = \bigcup_{i=1}^{d} \rho_i(X).
\]
As a consequence we have that $J_{(A, \alpha)} = J_{(A, \gamma)}$. Hence it suffices to prove the claim for the Toeplitz-Pimsner algebras.

For $d = 1$ piecewise conjugacy coincides with unitary equivalence and so the Pimsner algebras coincide. When $d > 1$ then the analysis on the ground
states and the KMS∞ states implies the required. In particular notice that
the ground states and the KMS∞ states coincide.

Let \( \beta \geq \log d > 0 \) and let \( \tau \in S(A) \). Then by Lemma 3.18 we have that
\[
\sum_{|w| = m} \tau \alpha_w(a) = \sum_{|w| = m} \tau \gamma_w(a), \quad \text{for all } a \in A.
\]
Note that \( \tau \) defines both a \((\sigma, \beta)\)-KMS state on \( \mathcal{T}(A, \alpha) \) and a \((\sigma, \beta)\)-KMS state on \( \mathcal{T}(A, \gamma) \). Let \( \psi_{\tau, \alpha} \) be as in Proposition 3.2 for \( \mathcal{T}(A, \alpha) \) and let \( \psi_{\tau, \gamma} \) be as in Proposition 3.2 for \( \mathcal{T}(A, \gamma) \). We then obtain
\[
\psi_{\tau, \alpha}(V_{\mu} a V_{\nu}^*) = \delta_{\mu, \nu} \cdot (1 - e^{-\beta d}) \cdot \sum_{m=0}^{\infty} e^{-(m+|\mu|)\beta} \sum_{|w| = m} \tau \alpha_w(a)
= \delta_{\mu, \nu} \cdot (1 - e^{-\beta d}) \cdot \sum_{m=0}^{\infty} e^{-(m+|\mu|)\beta} \sum_{|w| = m} \tau \gamma_w(a)
= \psi_{\tau, \gamma}(V_{\mu} a V_{\nu}^*),
\]
for all \( V_{\mu} a V_{\nu}^* \in \mathcal{T}(A, \alpha) \). Similarly, if \( \tau \in AVT(A, \alpha) \) then
\[
\tau = \frac{1}{d} \sum_{i=1}^{d} \tau \alpha_i = \frac{1}{d} \sum_{i=1}^{d} \tau \gamma_i,
\]
hence \( \tau \in AVT(A, \gamma) \) which completes the proof.

**Remark 3.20.** It is evident that any equivalence relation between dynamical systems (even non-classical) that implies an equation as that of Lemma 3.18 automatically produces a result similar to Corollary 3.19. In particular one just needs to check the equation of Lemma 3.18 just for pure states. The proof follows in the same way as above and it is left to the interested reader.

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