Soliton Equations with Self-Consistent Sources

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Abstract

We consider some soliton equations with self-consistent sources. A brief review of main SESCS is presented. In particular we construct the Heisenberg ferromagnetic equation with self-consistent sources (HFESCS) which is integrable. The corresponding Lax representation is presented. Some properties of HFESCS are analyzed. The relation between soliton equations with self-consistent potentials and soliton equations with self-consistent sources is studied.

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1 Introduction

Soliton equations with self-consistent sources (SESCS) were proposed in [1]. They describe nonlinear waves and have important applications in plasma physics, hydrodynamics, solid state physics etc. As example, we can mention that one of important SESCS - the nonlinear Schrödinger equation with self-consistent sources (NLSESCS), describes the nonlinear interaction of an ion acoustic wave in a two component homogeneous plasma with the electrostatic highfrequency wave [2]. Other important SESCS, the KP equation with self-consistent sources (KPESCS) represents the interaction of a short wave packet propagating on the x-y plane with the long wave at some angle to each other (see [3] and the references therein). As the third example, we can mention that the interaction of long and short capillary-gravity waves describes by the the KdV equation with self-consistent sources (KdVESCS) [4]. In such models, the sources may change some properties of the physical system, for example, the velocities of the solitons. In literature, some explicit solutions of SESCS like: solitons, positons and negatons were obtained. We note that for the given a soliton equation, its version with self-consistent sources is not unique. Some SESCS and their properties were studied in the literature (see e.g. refs. [5]-[7] and references therein).

In this paper, we present the main SESCS with their Lax representations (LR) in continuous, discrete and dispersionless cases. Then we give the Heisenberg ferromagnetic equation with self-consistent sources (HFESCS). This HFESCSC is integrable.

The structure of the paper is as follows. In Sec.2, we present a brief review of the main SESCS for three cases - continuous, discrete and dispersionless. Then in Sec.3, we construct the integrable HFESCSC and its some properties are analysesed. In Sec. 4, we present some integrable generalizations of HFESCS. In Sec. 5, we study the relation between soliton equations with self-consistent potentials and soliton equations with self-consistent sources. We summarize the main results of this paper in Sec. 5.

2 Brief review of soliton equations with self-consistent sources

To make the paper self-contained, we first briefly recall the main soliton equations with self-consistent sources. Then we will give the HFESCS, its Lax reresentation (LR), gauge equivalent counterpart and some generalizations.

2.1 Continuous (1+1)-dimensional equations

2.1.1 KdV equation with self-consistent sources

The KdVESCS looks like (see e.g. [8])

\[ u_t + 6uu_x + u_{xxx} + 4 \sum_{j=1}^{n} \varphi_j \varphi_{j,x} = 0, \tag{2.1a} \]

\[ \varphi_{j,xx} + (\lambda_j + u) \varphi_j = 0, \quad j = 1, \ldots, n, \tag{2.1b} \]
where $\lambda_j$ = real constants. The corresponding LR is given by

$$
\phi_{xx} + (\lambda + u)\phi = 0,
$$

(2.2a)

$$
\phi_t = A_n(\lambda, u, \Phi_n)\phi,
$$

(2.2b)

where

$$
A_n(\lambda, u, \Phi_n)\phi = u_x\phi + (4\lambda - 2u)\phi_x + \sum_{j=1}^{n} \frac{\phi_j}{A_j - \lambda} W(\phi_j, \phi).
$$

Here $W(\phi_j, \phi) \equiv \phi_j\phi_x - \phi_j,x\phi$ is the usual Wronskian determinant and $\Phi_n = (\phi_1, \ldots, \phi_n)$.

### 2.1.2 mKdV equation with self-consistent sources

The mKdV equation with self-consistent sources (mKdVSCS) reads as [9]

$$
u_t = \frac{u_{xxx}}{4} - \frac{3}{2} u^2 u_x + \sum_{j=1}^{N} (\phi_{1j}\phi_{2j})_x,
$$

(2.3a)

$$
\phi_{1j,x} = \lambda_j\phi_{2j} - u\phi_{1j}, \quad \phi_{2j,x} = \lambda_j\phi_{1j} + u\phi_{2j}, \quad j = 1, \ldots, N.
$$

(2.3b)

### 2.1.3 NLS equation with self-consistent sources

The nonlinear Schrodinger equation with self-consistent sources (NLSESCS) is given by [10]

$$
q_t = -i(q_{xx} - 2q^2r) + \sum_{j=1}^{m} \left[ (\phi^{(1)}_j)^2 + (\phi^{(1)}_j,x)^2 \right] + \sum_{j=1}^{n} (\phi^{(1)}_j)^2,
$$

(2.4c)

$$
r_t = i(r_{xx} - 2qr^2) + \sum_{j=1}^{m} \left[ (\phi^{(2)}_j)^2 + (\phi^{(2)}_j,x)^2 \right] + \sum_{j=1}^{n} (\phi^{(2)}_j)^2.
$$

(2.4d)

The corresponding LR reads as

$$
\psi_x = U(\lambda, q, r)\psi, \quad \psi_t = V(\lambda, q, r)\psi + \sum_{j=1}^{m} \left[ \frac{H(\phi_j)}{\lambda - \lambda_j} + \frac{H(\phi^{(1)}_j)}{\lambda - \lambda^{(1)}_j} \right] \psi + \sum_{j=1}^{n} \frac{H(\phi_j)}{\lambda - \zeta_j} \psi.
$$

(2.5)

Let us also now we present two reductions of the NLSESCS.

1) Let

$$
r = q^*, \quad \lambda_j = -\lambda_j^* \quad \phi_j^* = \pm S_+ \phi_j, \quad j = 1, \ldots, m,
$$

(2.6a)

$$
\text{Re} \zeta_j = 0, \quad \phi_j^{(2)*} = \phi_j^{(1)} \equiv w_j, \quad j = 1, \ldots, n.
$$

(2.6b)

Then we obtain the following equations

$$
\varphi_{j,x} = U(\lambda_j, q, q^*)\varphi_j, \quad j = 1, \ldots, m,
$$

(2.7a)

$$
w_{j,x} = \zeta_j w_j + qw_j^*, \quad (\text{Re} \zeta_j = 0), \quad j = 1, \ldots, n,
$$

(2.7b)
\[ q_t = i(2|q|^2q - q_{xx}) + \sum_{j=1}^{m} \left[ (\varphi_j^{(1)})^2 + (\varphi_j^{(2)*})^2 \right] + \sum_{j=1}^{n} w_k^2 \quad (2.7c) \]

with LR

\[ \psi_x = U(\lambda, q, q^*)\psi, \quad \psi_t = V(\lambda, q, q^*)\psi + \sum_{j=1}^{m} \left[ \frac{H(\varphi_j)}{\lambda - \lambda_j} + \frac{H(S_+ \varphi_j)}{\lambda + \lambda_j^*} \right] \psi + \sum_{j=1}^{n} \frac{H((w_j, w_j^*)^T)}{\lambda - \zeta_j} \psi. \quad (2.8) \]

ii) Now we take the reduction

\[ r = -q^*, \quad \lambda'_j = -\lambda_j^* \quad \varphi'_j = \pm iS^+ \varphi_j, \quad j = 1, \ldots, m. \quad (2.9) \]

Then we come to the system

\[ \varphi_{j,x} = U(\lambda_j, q, -q^*)\varphi_j, \quad j = 1, \ldots, m, \quad (2.10a) \]

\[ q_t = i(-2|q|^2q - q_{xx}) + \sum_{j=1}^{m} \left[ (\varphi_j^{(1)})^2 - (\varphi_j^{(2)*})^2 \right] \quad (2.10b) \]

with the LR of the form

\[ \psi_x = U(\lambda, q, -q^*)\psi, \quad \psi_t = V(\lambda, q, -q^*)\psi + \sum_{j=1}^{m} \left[ \frac{H(\varphi_j)}{\lambda - \lambda_j} - \frac{H(S_- \varphi_j)}{\lambda + \lambda_j^*} \right] \psi. \quad (2.11) \]

2.1.4 Camassa-Holm equation with self-consistent sources

The Camassa-Holm equation with self-consistent sources (CHESCS) is defined as follows

\[ q_t = -J \left( \frac{\delta H_0}{\delta q} - 2 \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta q} \right) \]

\[ = -(q \partial_x + \partial_q)(u + 2 \sum_{j=1}^{N} \lambda_j \varphi_j^2) \]

\[ = -2qu_x - uq_x + \sum_{j=1}^{N} (-8\lambda_j q \varphi_j \varphi_{j,x} - 2\lambda_j q_x \varphi_j^2), \quad (2.12a) \]

\[ \varphi_{j,xx} = (\lambda_j q + \frac{1}{4})\varphi_j, \quad j = 1, \ldots, N. \quad (2.12b) \]

Equivalently this system can be written as

\[ q_t = -2qu_x - uq_x + \sum_{j=1}^{N} [\varphi_j^2]_x - (\varphi_j^2)_{xxx}, \quad (2.13a) \]

\[ \varphi_{j,xx} = (\lambda_j q + \frac{1}{4})\varphi_j, \quad j = 1, \ldots, N. \quad (2.13b) \]

The LR of the CHESCS has the form

\[ \varphi_{xx} = (\lambda q + \frac{1}{4})\varphi, \quad (2.14a) \]

\[ \varphi_t = -\frac{1}{2}B_x \varphi + B \varphi_x, \quad (2.14b) \]
where $f(\varphi_j)$ is some undetermined function of $\varphi_j$. From these equations we get

$$\lambda q_t = LB + \lambda (2B_x q + B q_x),$$

(2.15)

where $L = -\frac{1}{2} \partial^3 + \frac{1}{2} \partial$. So finally we obtain

$$\lambda q_t = \frac{1}{2} \sum_{j=1}^{N} \frac{\alpha_j}{\lambda - \lambda_j} [f''' \varphi_{jxxx} + 3(f'' \varphi_j - f') \varphi_{jxx} + \lambda_j q_j (f' \varphi_j - 2f)]$$

+ $[-2qu_x - uq_x + \sum_{j=1}^{N} \beta_j (2q \varphi_j f' + q_x f)] \lambda - \frac{1}{2} \sum_{j=1}^{N} \beta_j [f'' \varphi_{jxx} + (3f'' \varphi_j + f')]$

$$\times (\lambda_j q + \frac{1}{4} \varphi_{jx} + \lambda_j f' q_x \varphi_j - f' \varphi_j) + \sum_{j=1}^{N} \alpha_j (q_x f + 2q f' \varphi_{jxx}).$$

(2.16)

### 2.1.5 Degasperis-Procesi equation with self-consistent sources

The Degasperis-Procesi equation with self-consistent sources (DPESCS) has the form [12]

$$m_t = -um_x - 3u_x m - \frac{1}{6} \sum_{j=1}^{n} \partial(1 - \partial^2)(4 - \partial^2)(\lambda_j q_j r_j),$$

(2.17a)

$$q_{j,xxx} = q_{j,x} - m \lambda_j q_j,$$

(2.17b)

$$r_{j,xxx} = r_{j,x} + m \lambda_j r_j, \quad j = 1, \ldots, n.$$  

(2.17c)

The LR of the DPESCS reads as

$$\psi_{xxx} = \psi_x - m \lambda \psi,$$

(2.18a)

$$\psi_t = -\frac{1}{\lambda} \psi_{xx} - u \psi_x + (u_x + \frac{2}{3\lambda}) \psi$$

$$+ \left( \sum_{j=1}^{n} \frac{\lambda \lambda_j^2}{6 \lambda_j^2 - \lambda^2} (3 \lambda (q_j r_{j,xx} - q_{j,xxx} r_j) - 4 \lambda_j q_j r_j - 2 \lambda_j (q_j r_j)_{xx}) \right) \psi$$

$$+ \left( \sum_{j=1}^{n} - \frac{1}{2} \frac{\lambda \lambda_j^2}{\lambda_j^2 - \lambda^2} (\lambda (q_j r_{j,x} - q_{j,x} r_j) + \lambda_j (q_j r_j)_x) \right) \psi$$

$$+ \left( \sum_{j=1}^{n} \frac{\lambda \lambda_j^3}{\lambda_j^3 - \lambda^3} q_j r_{j} \right) \psi_{xx}.$$  

(2.18b)

### 2.1.6 AKNS equation with self-consistent sources

The AKNS equation with self-consistent sources (AKNSESCS) is given by [10]

$$q_t = -i(q_{xx} - 2q^2 r) + \sum_{j=1}^{n} (\varphi_j^{(1)})^2,$$

(2.19)

$$r_t = i(r_{xx} - 2q r^2) + \sum_{j=1}^{n} (\varphi_j^{(2)})^2,$$

(2.20)
\[
\varphi_{j,x} = \begin{pmatrix} -\lambda_j & q \\ r & \lambda_j \end{pmatrix} \varphi_j, \quad j = 1, \ldots, n,
\]
(2.21)

where \(\varphi_j = (\varphi_j^{(1)}, \varphi_j^{(2)})^T\). The LR for the AKNSSCS looks like
\[
\psi_x = U \psi,
\]
(2.22)
\[
\psi_t = R^{(n)} \psi,
\]
(2.23)
where
\[
R^{(n)} = V + \sum_{j=1}^{n} \frac{H(\varphi_j)}{\lambda - \lambda_j}
\]
(2.24)
and
\[
U = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}, \quad V = i \begin{pmatrix} -2\lambda^2 + qr & 2\lambda q - qx \\ 2\lambda r + rx & 2\lambda^2 - qr \end{pmatrix}, \quad H(\varphi_j) = \frac{1}{2} \begin{pmatrix} -\varphi_j^{(1)} \varphi_j^{(2)} & (\varphi_j^{(1)})^2 \\ -(\varphi_j^{(2)})^2 & \varphi_j^{(1)} \varphi_j^{(2)} \end{pmatrix}.
\]
(2.25)

### 2.2 Continuous (2+1)-dimensional equations

#### 2.2.1 KP equation with self-consistent sources

The KP equation with self-consistent sources (KPESCS) reads as [13]

\[
[u_{1,t} - 3u_1 u_{1,x} - \frac{1}{4} u_{1,xxx} + \sum_{i=1}^{N} (q_i r_i)_x]_x - \frac{3}{4} u_{1,yy} = 0,
\]
(2.26a)
\[
q_{i,y} = q_{i,xx} + 2u_1 q_i,
\]
(2.26b)
\[
r_{i,y} = -r_{i,xx} - 2u_1 r_i, \quad i = 1, \ldots, N.
\]
(2.26c)

Its LR is given by
\[
\psi_y = \psi_{xx} + 2u_1 \psi,
\]
(2.27a)
\[
\psi_t = \psi_{xxx} + 3u_1 \psi_x + \frac{3}{2} (u_{1,x} + (\partial^{-1} u_{1,y})) \psi + \sum_{i=1}^{N} q_i \partial^{-1} (r_i \psi).
\]
(2.27b)

#### 2.2.2 mKP equation with self-consistent sources

The mKP equation with self-consistent sources (mKPESCS) reads as [14]

\[
u_t + u_{xxx} + 3\alpha^2 D^{-1}(u_{yy}) - 6\alpha D^{-1}(u_y) u_x - 6u^2 u_x + 4 \sum_{i=1}^{N} (\Psi_i \Phi_i)_x = 0,
\]
(2.28a)
\[
\alpha \Psi_{i,y} = \Psi_{i,xx} - 2u \Psi_{i,x},
\]
(2.28b)
\[
\alpha \Phi_{i,y} = -\Phi_{i,xx} - 2u \Phi_{i,x}.
\]
(2.28c)

Its LR looks like
\[
\alpha \psi_{1,y} = \psi_{1,xx} - 2u \psi_{1,x},
\]
(2.29a)
\[ \psi_{1,t} = (A_1(u)\psi_1) + T^1_N(\Psi, \Phi)\psi_1, \quad T^1_N(\Psi, \Phi)\psi_1 = -4 \sum_{i=1}^{N} \Psi_i \int \Phi_i\psi_{1,x} dx \] 

(2.29b)

and

\[ \alpha \psi_{2,y} = -\psi_{2,xx} - 2u\psi_{2,x}, \]

(2.30a)

\[ \psi_{2,t} = (A_2(u)\psi_2) + T^2_N(\Psi, \Phi)\psi_2, \quad T^2_N(\Psi, \Phi)\psi_2 = 4 \sum_{i=1}^{N} \Phi_i \int \Psi_i\psi_{2,x} dx. \]

(2.30b)

### 2.2.3 Davey-Stewartson equation with self-consistent sources

The Davey-Stewartson equation with self-consistent sources (DSESCS) reads as [15]

\[ iu_t + u_{xx} + u_{yy} + u(v_{xx} + v_{yy}) = \delta[i(\Psi_1\Psi_2 + \Psi_1\Psi_1) - 0.25(\Phi_1\Phi_2 + \Phi_1\Phi_1)], \]

(2.31)

\[ v_{xy} - 2|u|^2 = 0, \]

(2.32)

\[ \Phi_{jxy}\Phi_j - \Phi_{jjx}\Phi_{jj} + 0.5|u|^2\Phi_j^2 = 0, \]

(2.33)

\[ \Phi_{jxy}\Psi_j - \Psi_{jx}\Psi_{jj} + |u|^2\Psi_j^2 = 0, \]

(2.34)

where \( j = 1, 2, 3, 4. \)

### 2.2.4 Ishimori equation with self-consistent sources

The Ishimori equation with self-consistent sources was constructed in [16].

### 2.3 Discrete equations

#### 2.3.1 Toda Lattice Equation with Self-Consistent Sources

The Toda lattice equation with \( N \) self-consistent sources (TLSCS) is given by [17]

\[ v_t = v(p^{(-1)} + \sum_{j=1}^{N} \phi_{j-}^{(-1)} \phi_{j+}^{(-1)}) - v(p + \sum_{j=1}^{N} \phi_{j-} \phi_{j+}), \]

(2.35)

\[ p_t = v(1 + \sum_{j=1}^{N} \phi_{j-}^{(-1)} \phi_{j+}) - v^{(1)}(1 + \sum_{j=1}^{N} \phi_{j-} \phi_{j+}^{(1)}), \]

(2.36)

\[ L\phi_{j+} = \lambda_j \phi_{j+}, \]

(2.37)

\[ L^*\phi_{j-} = \lambda_j \phi_{j-}, \quad j = 1, \ldots, N. \]

(2.38)

The corresponding LR reads as

\[ L\psi = v^{(1)}\psi^{(1)} + p\psi + \psi^{(-1)} = \lambda\psi \]

(2.39)

\[ -\psi_t = v^{(1)}\psi^{(1)} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} v^{(1)}\phi_{j-}^{(1)} \left( \phi_{j+}^{(1)} - \phi_{j+}\psi^{(1)} \right) \]

(2.40)
\[ L\phi_j^+ = \lambda_j \phi_j^+, \quad (2.41) \]

\[ L^* \phi_j^- = \lambda_j \phi_j^-, \quad j = 1, \ldots, N. \quad (2.42) \]

After the transformation

\[ v := \exp(x^{(-1)} - x), \quad p := x_t - \sum_{j=1}^N \phi_j^+ \phi_j^- , \quad (2.43) \]

we have

\[ x_{tt} = \exp \left( x^{(-1)} - x \right) \left( 1 + \sum_{j=1}^N \phi_j^+ \phi_j^- (1) \right) - \exp \left( x - x^{(1)} \right) \left( 1 + \sum_{j=1}^N \phi_j^{(1)} \phi_j^- \right) + \sum_{j=1}^N (\phi_j^+ \phi_j^-)_t. \quad (2.44) \]

### 2.3.2 Discrete KP equation with self-consistent sources

The discrete KP equation with self-consistent sources were studied in [18].

### 2.4 Dispersionless equations

#### 2.4.1 Dispersionless KP equation with self-consistent sources

The dispersionless KP equation with self-consistent sources (dKPSCS) has the form [13]

\[ (u_t - 3uu_x + \sum_{i=1}^N v_i)_x = \frac{3}{4} u_{yy}, \quad (2.45a) \]

\[ p_{iy} = (p_i^2 + 2u)_x, \quad (2.45b) \]

\[ v_{iy} = 2(v_i p_i)_x, \quad i = 1, \ldots, N. \quad (2.45c) \]

It is the compatibility of the following equations

\[ p_y = (p^2 + 2u)_x = 2pp_x + 2u_x, \quad (2.46a) \]

\[ p_t = (p^3 + 3up + 3w + \sum_{i=1}^N \frac{v_i}{p - p_i})_x = 3p^2 p_x + 3(Up)_x + 3w_x + \sum_{i=1}^N \frac{v_i}{p - p_i} - \sum_{i=1}^N \frac{v_i(p_x - p_{ix})}{(p - p_i)^2}, \quad (2.46b) \]

where \( w_x = \frac{1}{2} u_y. \)

#### 2.4.2 Dispersionless mKP equation with self-consistent sources

The dispersionless mKP equation with self-consistent sources (dmKPESCS) can be written as [19]

\[ 2v_t - \frac{3}{2} D_x^{-1}(v_{yy}) - 3v_x D_x^{-1}(v_y) + 3v^2 v_x - 2 \sum_{i=1}^N \frac{a_i}{p_i} v_x = 0, \quad (2.47a) \]

\[ a_{iy} = 2[a_i(p_i + v)]_x, \quad (2.47b) \]

\[ p_{iy} = (p_i^2 + 2vp_i)_x, \quad i = 1, \ldots, N. \quad (2.47c) \]
3 Integrable Heisenberg ferromagnetic equation with self-consistent sources

One of the important soliton equations is the Heisenberg ferromagnet equation (HFE) which can be written as [21, 22]

\[ iS_t + 0.5[S, S_{xx}] = 0, \]

where

\[ S = \left( \begin{array}{cc} S_3 & S^- \\ S^+ & -S_3 \end{array} \right), \quad S^+ S^- + S_3^2 = 1. \]

(3.49)

It is integrable. In literature, some integrable and not integrable generalizations of the HFE (3.48) were studied (see [23-46] and references therein). Below we present the one of the integrable generalizations of the HFE (3.43) namely the integrable Heisenberg ferromagnetic equation with self-consistent sources (HFESCS).

3.1 The equation

Let us consider the Myrzakulov-XCIX (M-XCIX) equation [35-37]. It can be written in the different forms. One of its form reads as

\[ iS_t^+ + S^+ S_{3xx} - S_{xx}^+ S_3 + \frac{2}{a} (S^+ \psi_1 \psi_2 - S_3 \psi_2^2) = 0, \]

(3.50)

\[ iS_t^- - (S^- S_{3xx} - S_{xx}^- S_3) - \frac{2}{a} (S^- \psi_1 \psi_2 + S_3 \psi_1^2) = 0, \]

(3.51)

\[ iS_{3t} + 0.5(S^- S_{xx}^+ - S_{xx}^- S^+) + \frac{1}{a} (S^- \psi_2^2 + S_3 \psi_1^2) = 0, \]

(3.52)

\[ \psi_{1x} - ia(S_3 \psi_1 + S^- \psi_2) = 0, \]

(3.53)

\[ \psi_{2x} - ia(S^+ \psi_1 - S_3 \psi_2) = 0, \]

(3.54)

where \( a \) is a complex constant, \( \psi = (\psi_1, \psi_2)^T \) and \( S^+ S^- + S_3^2 = 1 \). Note that this system we can rewrite equivalently as

\[ iS_t^+ + (S^+ S_{3x} - S_{xx}^+ S_3 - ia^{-2} \psi_2^2)_x = 0, \]

(3.55)

\[ iS_t^- - (S^- S_{3x} - S_{xx}^- S_3 + ia^{-2} \psi_1^2)_x = 0, \]

(3.56)

\[ iS_{3t} + 0.5 (S^- S_{xx}^+ - S_{xx}^- S^+) - 2ia^{-2} \psi_1 \psi_2)_x = 0, \]

(3.57)

\[ \psi_{1x} - ia(S_3 \psi_1 + S^- \psi_2) = 0, \]

(3.58)

\[ \psi_{2x} - ia(S^+ \psi_1 - S_3 \psi_2) = 0. \]

(3.59)

The above system we can interpretate as the HFESCS.
3.2 The Lax representation

The LR of the M-XCIX equation looks like

$$\Phi_x = U \Phi,$$
(3.60)

$$\Phi_t = V \Phi,$$
(3.61)

where

$$U = -i \lambda S, \quad S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \quad V = -2i \lambda^2 S + \lambda SS_x + \left( \frac{i}{\lambda + a} - \frac{i}{a} \right) \begin{pmatrix} \psi_1 \psi_2 & -\psi_1^2 \\ \psi_2^2 & -\psi_1 \psi_2 \end{pmatrix}. \quad (3.62)$$

3.3 Gauge equivalent counterpart

The gauge equivalent counterpart of the M-XCIX equation is given by

$$iq_t + q_{xx} + 2q^2 r + 2i\varphi_1^2 = 0,$$
(3.63)

$$ir_t - r_{xx} - 2qr^2 + 2i\varphi_2^2 = 0,$$
(3.64)

$$\varphi_1 x - ia\varphi_1 - q\varphi_2 = 0,$$
(3.65)

$$\varphi_2 x + r\varphi_1 + ia\varphi_2 = 0,$$
(3.66)

where $\varphi = (\varphi_1, \varphi_2)^T$. It is the AKNSSCS (2.19)-(2.21) for the one source case [10]. The LR of the AKNSES reads as [10]

$$\psi_x = U \psi,$$
(3.67)

$$\psi_t = V \psi,$$
(3.68)

where

$$U = \begin{pmatrix} -\lambda & q \\ -r & \lambda \end{pmatrix}, \quad V = i \begin{pmatrix} -2\lambda^2 + qr & 2\lambda q + q_x \\ -2\lambda r + r_x & 2\lambda^2 - qr \end{pmatrix} + \frac{i}{\lambda + a} \begin{pmatrix} -\varphi_1 \varphi_2 & \varphi_1^2 \\ -\varphi_2^2 & \varphi_1 \varphi_2 \end{pmatrix}. \quad (3.69)$$

4 Some generalizations of the HFESCS

If we have $N$ sources, then the HFESCS (3.44)-(3.48) takes the form

$$iS^+_t + S^+ S_{3xx} - S^+_{xx} S_3 + 2 \sum_{j=1}^N a_j^{-1} (S^+ \psi_{1j} \psi_{2j} - S_3 \psi_{2j}^2) = 0,$$
(4.70)

$$iS^-_t - (S^- S_{3xx} - S^-_{xx} S_3) - 2 \sum_{j=1}^N a_j^{-1} (S^- \psi_{1j} \psi_{2j} + S_3 \psi_{1j}^2) = 0,$$
(4.71)

$$iS^+_t + 0.5(S^- S^+_{xx} - S^-_{xx} S^+) + \sum_{j=1}^N a_j^{-1} (S^- \psi_{2j}^2 + S^+ \psi_{1j}^2) = 0,$$
(4.72)

$$\psi_{1xx} - ia_j (S_3 \psi_{1j} + S^- \psi_{2j}) = 0,$$
(4.73)
\[
\psi_{2jx} - ia_j(S^+\psi_{1j} - S_3\psi_{2j}) = 0.
\] (4.74)

We can rewrite this equation also as

\[
iS_t^+ + \left(S^+ S_{3x} - S_x^+ S_3 - i \sum_{j=1}^N a_j^{-2} \psi_{2j}^2 \right)_x = 0,
\] (4.75)

\[
iS_t^- - \left(S^- S_{3x} - S_x^- S_3 + i \sum_{j=1}^N a_j^{-2} \psi_{1j}^2 \right)_x = 0,
\] (4.76)

\[
iS_{3t} + 0.5 \left(S^- S_{3x}^+ - S_x^- S^+ - 2i \sum_{j=1}^N a_j^{-2} \psi_{1j} \psi_{2j} \right)_x = 0,
\] (4.77)

\[
\psi_{1jx} - ia_j(S_3\psi_{1j} + S^-\psi_{2j}) = 0,
\] (4.78)

\[
\psi_{2jx} - ia_j(S^+\psi_{1j} - S_3\psi_{2j}) = 0.
\] (4.79)

The LR of this generalized HFESCS has the form

\[
\Phi_x = -U\Phi,
\] (4.80)

\[
\Phi_t = V\Phi,
\] (4.81)

where

\[
U = -i\lambda S, \quad S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \quad V = -2i\lambda^2 S + \lambda SS_x + \sum_{j=1}^N \left(\frac{i}{\lambda + a_j} - \frac{i}{a_j}\right) \begin{pmatrix} \psi_{1j} \psi_{2j} & -\psi_{1j}^2 \\ \psi_{2j}^2 & -\psi_{1j} \psi_{2j} \end{pmatrix}.
\] (4.82)

5 Relation between soliton equations with self-consistent potentials and soliton equations with self-consistent sources

As well-known, there are exist several soliton equations with self-consistent potentials. It is interesting to note that these equations are related with SESCS. Here we demonstrate these relations in two examples.

5.1 The M-XCIX equation

The standard form of the M-XCIX equation reads as [35]-[37]

\[
iS_t + 0.5[S, S_{xx}] + a^{-1}[S, W] = 0,
\] (5.83)

\[
iW_x + a[S, W] = 0.
\] (5.84)

Let us solve the equation (5.78). Its solution (or one of the solutions) we can write as

\[
W = \begin{pmatrix} \psi_1 \psi_2 & -\psi_1^2 \\ \psi_2^2 & -\psi_1 \psi_2 \end{pmatrix},
\] (5.85)

where \(\psi_j\) are the solutions of the system (3.47)-(3.48). In terms of \(S^+, S^-, S_3\) and \(\psi_j\), the M-XCIX equation (5.77)-(5.78) takes the form (3.44)-(3.48).
5.2 The NLS-Maxwell-Bloch equation

Now we return to the AKNSSCS (3.57)-(3.60). Let us introduce 3 new functions as

\[ p = -\varphi_1^2, \quad k = \varphi_2^2, \quad \eta = -\varphi_1\varphi_2. \]  

(5.86)

Then in terms of these new functions the AKNSSCS (3.57)-(3.60) takes the form

\[ iq_t + q_{xx} + 2q^2r - 2ip = 0, \]  

(5.87)

\[ ir_t - r_{xx} - 2qr^2 - 2ik = 0, \]  

(5.88)

\[ p_x - 2iap - 2q\eta = 0, \]  

(5.89)

\[ k_x + 2iak - 2r\eta = 0, \]  

(5.90)

\[ \eta_x + rp + qk = 0, \]  

(5.91)

It is nothing but the nonlinear Schrödinger-Maxwell-Bloch equation (see e.g. [35]-[37] and references therein). Now let us consider the reductions \( k = \delta\bar{p}, \quad r = \delta\bar{q}, \quad \delta = \pm 1. \) Then the nonlinear Schrödinger-Maxwell-Bloch equation takes the form

\[ iq_t + q_{xx} + 2\delta|q|^2q - 2ip = 0, \]  

(5.92)

\[ p_x - 2iap - 2q\eta = 0, \]  

(5.93)

\[ \eta_x + \delta(\bar{q}p + q\bar{p}) = 0. \]  

(5.94)

6 Conclusion

In this paper, we have briefly review of soliton equations with self-consistent sources. We have also presented their Lax representations where we can. Then the integrable extension of the HFE namely the HFE with self-consistent sources are presented. Its Lax representation and gauge equivalent counterpart are also presented. Finally the integrable HFESCS with \( N \) - sources with its LR are given. Finally we consider the relation between soliton equations with self-consistent potentials and soliton equations with self-consistent sources.

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