Elements of Finite Order in Lie Groups and Discrete Gauge Symmetries

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ABSTRACT

We apply Kac’s theory of elements of finite order (EFO) in Lie groups to the description of discrete gauge symmetries in various supersymmetric grand unified models. Taking into account the discrete anomaly cancellation conditions, we identify the EFO which generate certain matter parities in the context of the supersymmetric $SO(10)$ and $E_6$ models.

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1. Introduction.

The study of discrete symmetries has found many applications in particle physics. Among their prominent phenomenological issues, let us mention that they arise in solar neutrino models, and provide a constrained structure to the quark and lepton mass matrices, suppression of FCNC, and elimination of terms in the superpotential which could lead to pathological phenomena such as a too rapid proton decay. However, global discrete symmetries experience drawbacks at the Planck scale, because they do not survive some gravitational (e.g. wormhole) effects. (For a review, see Ref. [1]). This, in addition to the theorists’ prejudice in favor of local symmetries, provides a natural reason to promote them to “discrete gauge symmetries” (DGS). According to the Krauss-Wilczek scenario, DGS appear as the remnants of a given continuous gauge group [2]. (It has been discussed recently that CP could be interpreted as a DGS [3].)

The simplest example consists in the spontaneous breakdown of a $U(1)$ gauge symmetry by a Higgs field of charge $Nq \neq 0$. If the $U(1)$ charge of every particle of the model is an integer multiple of $q$ (not necessarily the charge of one of the fields) then a $Z_N$ discrete symmetry survives. Similarly, here we consider only discrete abelian groups, although the elements of finite order of a Lie group generate nonabelian discrete groups as well. It is not always possible to gauge any discrete symmetry because, for instance, of the requirement of anomaly cancellation [4, 5]. Constraints have been discussed and classified e.g. in Refs. [6-8].

When the discrete group is a cyclic group $Z_N$, its action on a multiplet is made manifest through the assignment of complex roots of unity to the fields of the multiplet. These complex numbers can be seen as the eigenvalues of an automorphism of finite order of the gauge group acting on the fields representation. Given a continuous gauge group $G$ (e.g. a grand unified group) which contains the remnant DGS, the action of the discrete group on a multiplet of $G$ is the same as the action of an automorphism of an element of finite order $X \in G$ (i.e. $X^N = Id_G$, for some $N$). Obviously, for any automorphism $\phi$ of $X$ which acts on a multiplet of $G$, we have

$$\phi^N(X) = X,$$

so that $\phi$ provides eigenvalues to the fields, and thus defines a $Z_N$ symmetry. Here, whether the DGS are contained or not in a $U(1)$ is immaterial. As for the usual case where the $U(1)$ subgroup of $G$ must be not embedded into the standard
model gauge group, here the set of elements of finite order must not be contained in the standard model.

Here we restrict \( \phi \) to be an inner automorphism of \( G \) (i.e. \( \phi \) has the form \( \phi_Y(X) = YXY^{-1} \), for some \( Y \in G \)), and use Kac’s formalism of elements of finite order (EFO), which has known important developments and applications [9, 11] since its original formulation in Ref. [10]. The theory of EFO has provided valuable tools in calculating with representations of Lie groups. For example, their “characters” provide an efficient and practical way for determining the irreducible components of the tensor product of irreducible representations. Also, they can be used to generate signatures of representations of noncompact real forms. (For other applications, see Refs. [9-11]).

For our purpose, the main interest in EFO theory is that they provide a systematic way to assign eigenvalues of finite order (i.e. of type \( \exp\left(\frac{2\pi i}{N}k\right) \), where \( k \) is an arbitrary integer, and \( N \) is a positive integer) to the weight vectors of a representation or, equivalently, to the various superfields in a multiplet. In the mathematics literature, the EFO is said to provide a “\( Z_N \) grading” of the representation, that is, a decomposition of the representation space,

\[
V = V_0 + V_1 + \cdots,
\]

where the subspace \( V_l \) corresponds to the eigenvalue \( \exp\left(\frac{2\pi i}{N}l\right) \), along with a decomposition of the Lie algebra,

\[
g = g_0 + g_1 + \cdots,
\]

with \( g_k \) associated to \( \exp\left(\frac{2\pi i}{N}k\right) \), such that

\[
g_k \cdot V_l = V_{k+l},
\]

where \( g_k \) and \( V_l \) represent any of their respective elements. Therefore we would like to underline the fact that, in the context of a grand unified model, a discrete symmetry provides a grading of the unification Lie algebra as well as the representation. This is analogous to the fact that the time reversal operation leaves unchanged the angular momenta \( J \) and space translation operators \( P \), but multiply by \(-1\) the time translation \( H \) and Lorentz boosts generators \( K \), thus providing the Poincaré algebra with a \( Z_2 \) grading:

\[
[g_\mu, g_\nu] = g_{\mu+\nu \mod 2}; \; g_0 = \{J, P\}; \; g_1 = \{H, K\}.
\]
Another example consists of the gradings provided by the action of “generalized charge conjugation operators”, introduced in Ref. [12]. Apart from being naturally connected with the existence of finite groups in Lie groups, EFO allows one to determine the “congruence class” of an irreducible representation, which has been considered in Ref. [13] as a criterion for discrete gauge $R$-parity to survive at low energy. (We shall consider this point later in the paper). Thus we might expect the theory of EFO to be a useful tool for the investigation of discrete symmetries in various contexts of particle physics.

The purpose of this paper is to bring attention to Kac’s theory of EFO for the study of DGS in grand unified theories. As an example, we identify some generalized matter parities with EFO. This method could certainly find applications in other areas of particle physics, given its natural relationship with discrete groups. In Section 2, we provide an elementary introduction to the theory of EFO, with emphasis on the points that we need. The method is restricted to the diagonal representative of a conjugacy class of EFO. As an illustration of the method, in Section 3 we identify the EFO associated to anomaly free DGS which ensure proton stability in supersymmetric models. We make some concluding remarks in Section 4. As we will mention in this paper, the method in its present form does not allow one to identify all the discrete subgroups of a Lie group, although an elegant extension going into that direction seems possible. In fact, all the discrete subgroups of a Lie group are generated by EFO, in general non-diagonal. Also, since we are considering subgroups of a Lie group, the present method cannot describe discrete symmetries which are not in some (grand unification) Lie group. Therefore, the discrete groups that we describe must be the remnants of some grand unified groups.

2. Overview of EFO theory.

In this section, we recall some basic facts about the theory of EFO. (Details can be found in Refs. [9-11]). An EFO is an element of a Lie group $G$ (which we assume to be simply connected), and acts on its Lie algebra via the adjoint representation. An element $X \in G$ is of order $N$ if it is the smallest positive integer such that

$$X^N = Id_G.$$  \hspace{1cm} (1)

This implies, in particular, that the inner automorphisms (like any automorphism) of the Lie algebra $g$ of $G$, $\phi(x) = X x X^{-1}, x \in g$ are of finite order (i.e.
\( \phi^M(x) = x \), for \( M \) some divisor of \( N \). Hence, \( \phi \) splits \( g \) as \( g = \bigoplus_{k=0}^{N-1} g_k \), according to
\[
\phi(x_k) = \exp \left( \frac{2\pi i}{N} k \right) x_k, \tag{2}
\]
where \( x_k \) lies in the vector subspace \( g_k \) of \( g \). This decomposition is a \( \mathbb{Z}_N \) grading of \( g \) \([9, 10]\). The cyclic group \( \mathbb{Z}_N \) is generated by \( X \).

For example, the element of \( SU(3) \),
\[
X = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{3}
\]
is of order 2 (i.e. \( X^2 = Id_{SU(3)} \)), and acts on the Lie algebra \( su(3) \) as
\[
X \begin{pmatrix} h_1 \\ e_{-\alpha_1} \\ e_{-\alpha_2} \end{pmatrix} = \begin{pmatrix} h_1 \\ e_{-\alpha_1} \\ e_{-\alpha_2} \end{pmatrix}, \quad X^{-1} = \begin{pmatrix} h_1 & -e_{\alpha_1} & e_{\alpha_1+\alpha_2} \\ -e_{\alpha_1} & h_2 & -e_{\alpha_2} \\ e_{-\alpha_1+\alpha_2} & -e_{\alpha_2} & -(h_1 + h_2) \end{pmatrix}, \tag{4}
\]
so that \( su(3) \) is partitioned into two components,
\[
su(3)_0 = \begin{pmatrix} h_1 \\ 0 \\ h_2 \end{pmatrix}, \quad su(3)_1 = \begin{pmatrix} 0 \\ e_{-\alpha_1} \\ e_{\alpha_2} \end{pmatrix}. \tag{5}
\]
The elements of component \( su(3)_0 \) belong to the eigenvalue \((-1)^0\) in (4), and those of \( su(3)_1 \), to eigenvalue \((-1)^1\).

We have described the decomposition (or grading) of a Lie algebra. We need also to consider the action of EFO on representations. For a representation \( (\rho, V) \) of \( G \), such that \( v' = \rho(X) \cdot v, \quad X \in G \) being an EFO, and \( v, v' \in V \), we have the analog of (1),
\[
\rho(X)^{N(\rho)} = Id_{End(V)}, \tag{6}
\]
where \( N(\rho) \) is some divisor of \( N \). (From now on, we consider representations of Lie algebras, rather than Lie groups). As for the Lie algebras, eigenvalues are thus
assigned to representation vectors $v$ belonging to the representation space $V$ of $g$ as follows,

$$\rho(X) \cdot v_k = \exp \left( \frac{2\pi i}{N} k \right) v_k,$$

where $v_k$ is an element of the representation subspace $V_k$. In that case, the representation space is graded compatibly with the grading of $g$ obtained previously. This means that if $x$ is in the subspace $g_k$ of $g$, and $v$ is in the subspace $V_l$ of $V$, then $v' = \rho(x) \cdot v$ is either zero, or in the representation subspace $V_{k+l} \mod N$.

For example, the element (3) of $SU(3)$ partitions the three-dimensional irreducible representation of $su(3)$ as

$$X \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a \\ b \\ -c \end{pmatrix} \rightarrow V_0 = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \quad V_1 = \begin{pmatrix} a \\ 0 \\ c \end{pmatrix}. \quad (8)$$

This $Z_2$ grading is compatible with that of $su(3)$ found in (5), that is,

$$su(3)_0 \cdot V_0 = V_0, \quad su(3)_0 \cdot V_1 = V_1, \quad su(3)_1 \cdot V_0 = V_1, \quad su(3)_1 \cdot V_1 = V_0.$$

In practice, we are much more interested in the decomposition (8) of the representation than in the Lie algebra itself (5). It is such a grading which achieves the phenomenological virtues mentioned in the introduction.

We now turn to the description of Kac’s algorithm and notation [9-11]. Different elements of $G$ which belong to the same conjugacy class have the same order. Hereafter the objects of interest are the conjugacy classes of EFO, and we shall give a way to obtain their unique diagonal representative. For every element $X$ of the Lie group $G$ there is a unique element $x \in F$ (where $F$ denotes the fundamental region of the affine Weyl group acting on the Cartan subalgebra of the Lie algebra $g$) such that $X$ is conjugate to $\exp(2\pi ix)$. If $X$ is an element of finite order, we denote by $s$ the point in $F$ which represents the conjugacy class of $X$. Hence, every conjugacy class of EFO meets $F$ in exactly one point. According to Kac’s notation, the point $s$ is labelled by a set of weighted barycentric coordinates

$$s = [s_0, \ldots, s_r],$$

where the greatest common divisor of the $s_i$ is 1, and $r$ is the rank of $G$. The sets $s$ are in one-to-one correspondence with the conjugacy classes of EFO in $G$, and
the numbers $s_0, \ldots, s_r$ are associated with the extended Coxeter-Dynkin diagram. From now on, the term EFO will actually designate a conjugacy class of EFO, unless stated otherwise.

The useful fact is that every EFO $s$ has within its conjugacy class a unique diagonal representative which acts on the representation subspace $V^\lambda$ which belongs to the weight $\lambda = \sum_{i=1}^r c_i \alpha_i$ (the $c_i$ being rational numbers, and $\alpha_i$, simple roots of $g$) as

$$v_\lambda \rightarrow v'_\lambda = \exp \left( \frac{2\pi i}{M} < \lambda, s > \right) v_\lambda,$$

with $< \lambda, s > = \sum_{i=1}^r c_i s_i$. The exact order of $X$ is given by $N = MC$, where $M$ and $C$ depend on $g$ and are given in Table 6 of Ref. [9]. To summarize, a conjugacy class of EFO (denoted by an array $s$) provides each weight vector (i.e. particle) in some irreducible representation of $G$ with an eigenvalue given by the exponential in (9). In other words, equation (9) provides the diagonal representative of the class $s$.

For example, consider once again $SU(3)$. An EFO of $SU(3)$ is specified by $s = [s_0, s_1, s_2]$, and its order is $N = MC$, where

$$M = s_0 + s_1 + s_2, \quad C = \frac{3}{\gcd(3; s_1 + 2s_2)}.$$

The only (conjugacy class of) element of order two is given by $s = [0, 1, 1]$, for which $M = 2$ and $C = 1$. The action of this EFO on the three-dimensional fundamental representation of highest weight $(1, 0)$, with the three weight vectors

$$\lambda_1 = (2\alpha_1 + \alpha_2)/3, \; \lambda_2 = (-\alpha_1 + \alpha_2)/3, \; \lambda_3 = (-\alpha_1 - 2\alpha_2)/3,$$

is given, using (9), by

$$v_{\lambda_1} \rightarrow -v_{\lambda_1}, \; v_{\lambda_2} \rightarrow v_{\lambda_2}, \; v_{\lambda_3} \rightarrow -v_{\lambda_3}.$$

Written as a matrix, this is nothing but (3). In the same way, the adjoint representation (i.e. $su(3)$ itself) has its basis elements graded as

$$su(3)_0 = \{h_1, h_2, e_{\pm(\alpha_1 + \alpha_2)}\}, \; su(3)_1 = \{e_{\pm\alpha_1}, e_{\pm\alpha_2}\},$$

in accordance to (5).
Below we shall find all EFO of order 2 and 3, in supersymmetric grand unification models $SO(10)$ and $E_6$. We denote each EFO by $s_{(i)}$ (not to be confused with its components $[s_0, \ldots, s_r]$). Suppose that there are $m$ EFO $s_{(1)}, \ldots, s_{(m)}$ of order $N$. They lead $a$ $priori$ to $N^m$ different discrete actions on multiplets. An EFO $s_{(j)}$ acts on the vector belonging to the weight $\lambda$ as (see Refs. [9-11]):

$$v_\lambda \rightarrow v'_\lambda = \exp \left( \frac{2\pi i}{M} < \lambda, s_{(j)} > \right) v_\lambda.$$ (11)

Given $m$ EFO: $s_{(1)}, \ldots, s_{(m)}$, we shall denote the product of their eigenvalues by $s_{(1)}^{n_1} \cdots s_{(m)}^{n_m}$ (where the $n_i$'s are nonnegative integers), and its action is given by

$$v_\lambda \rightarrow v'_\lambda = \left( e^{\frac{2\pi i}{M} \lambda, s_{(1)} >} \right)^{n_1} \cdots \left( e^{\frac{2\pi i}{M} \lambda, s_{(m)} >} \right)^{n_m} v_\lambda,\nonumber$$

$$= \exp \left( \frac{2\pi i}{M} (n_1 < \lambda, s_{(1)} > + \cdots + n_m < \lambda, s_{(m)} >) \right) v_\lambda, \quad (12)$$

for all the weights $\lambda$ of the representation.

The product $s_{(1)}^{n_1} \cdots s_{(m)}^{n_m}$ allows one to identify a discrete symmetry acting on a representation space. Let us mention that the symmetries obtained here are just those obtained from inner automorphisms of a Lie algebra, from elements of finite order in the corresponding Lie group. Hence we do not find all the discrete components admitted by a Lagrangian invariant under the Lie group. In order to get the complete set of discrete symmetries, one should also consider the action of outer automorphisms of the Lie algebra. Such a study would require the use of the present algorithm by taking an EFO in a group larger than $G$ (i.e. in which $G$ is contained. In practice, we could take $G \times G$, for example). As mentioned by Moody-Patera-Sharp [11], any discrete group of $G$ consists of elements of finite order, so that any element of that finite group can be identified with elements of finite order in $G$. However the prescription described previously only allows to describe the diagonal representatives of conjugacy classes of EFO. The problem of identifying systematically all the discrete groups in some Lie group by means of EFO has not been considered in the literature yet. Here we shall not pursue this problem further and hence, our list of DGS is not complete.
Next we will take into account various constraints in order to obtain pertinent discrete symmetries. For instance, if the symmetry is color-blind then the three weight vectors corresponding to each quark flavor must have the same eigenvalue. Also, one must take into account anomaly cancellation constraints [6, 7]. This is discussed in the next section.

3. Application to matter parities.

In this section, we interpret the grading described in the previous section as being the result of a discrete symmetry. Specifically, we consider EFO contained in some unification Lie group (in a supersymmetric model) which acts on some multiplets. We must impose some restrictions upon the possible gradings. For instance, if a discrete symmetry is color-blind, that is, if it acts in the same way upon the various colors of a fixed quark flavor, then we must consider EFO (or products of EFO) which provide the same eigenvalue to the different colors of a given quark. Also, since a DGS is a relic of a spontaneously broken continuous gauge symmetry, then it should satisfy certain anomaly cancellation conditions. That imposes further constraints on the possible EFO. We consider all of these constraints in the present section.

A $Z_N$ symmetry acts on the fields (or superfields) as

$$\Psi_k \rightarrow \Psi'_k = \omega_N^{(k)} \Psi_k,$$

$$= \exp \left( \frac{2\pi i}{N} \alpha_k \right) \Psi_k,$$

where the $\alpha_k$ are some charges, which can be identified with the product of EFO eigenvalues (12) as

$$\alpha_k = N \sum_{l=1}^{m} n_l < \lambda, s_{(l)},$$

where the field $\Psi_k$ corresponds to the weight vector $\lambda$. Like the authors of Ref. [14], we consider the three generators $\omega = R, A$ and $L$, where

$$R_N = \exp \left( \frac{2\pi i}{N} I_3^R \right), \quad A_N = \exp \left( \frac{2\pi i}{N} Y_A \right), \quad L_N = \exp \left( \frac{2\pi i}{N} L \right).$$

In other terms, the charges $\alpha$ in (13) are $I_3^R, Y_A$ and $L$. The charge $I_3^R$ corresponds to the third component of a right-handed weak-isospin, $Y_A$ couples the quarks and
leptons like the $E_6$ generator corresponding to the (Cartan) diagonal generator of the $SU(2)$ subgroup appearing in the decomposition $E_6 \supset SU(6) \times SU(2)$, and $L$ is the standard lepton number. (Their charges for the standard superfields are listed in Table I.) A general element $g_N$ of a $Z_N$ discrete symmetry can be written:

$$g_N = R_N^m \times A_N^n \times L_N^p, \quad m, n, p = 0, \ldots, N - 1,$$

so that $R, A, L$ form a “basis” of DGS.

As mentioned in [14], these symmetries are further constrained by the imposition of discrete anomaly cancellation conditions [6]. Necessary (but not sufficient) conditions are given in Ref. [6]: (1) cubic $Z_3^N$; (2) mixed $Z_N$-graviton-graviton [15]; and (3) mixed $Z_N - SU(M) - SU(M)$ anomalies. We consider discrete symmetries which are remnants of some grand unification model (a good reference about grand unified theories is Ref. [16]). For example, the “matter parity” is added in the minimal supersymmetric extension of the standard model (MSSM) in order to eliminate both the baryon- and the lepton-number violating terms and lead to an acceptable rate of proton decay [17]. The concept of matter parity has been extended to that of “generalized parity” in Refs. [18] (see also Ref. [14]). The scheme of Ref. [2] suggests that these DGS are subgroups of a $U(1)$. In turn, the $U(1)$ is a subgroup of a grand unified Lie group, but not contained in the standard model gauge group. Here, we do not need the intermediate $U(1)$. The existence of DGS in various grand unified models has been investigated in Ref. [19].

In their study of dimension four operators in the superpotential, Ibáñez and Ross [14] have found various types of symmetries, out of which only two generalized parities are discrete anomaly free (with the minimal content of the supersymmetric standard model). One is (1) the standard $R$-parity (which is a $Z_2$ symmetry given by $R_2$ in (15)), and the other is (2) the $Z_3$ symmetry generated by $B_3 = R_3 L_3$ [14]. (There are other anomaly free $Z_3$ symmetries ($R_3, L_3, R_3 L_3^2$) but they require the existence of additional fermions with fractional $Z_3$ charge, so that the actual symmetry is rather –at least– $Z_9$ [8].)

We now study the presence of the discrete symmetries (15) generated by EFO in the supersymmetric grand unified models $SO(10)$ and $E_6$. We identify the conjugacy classes of EFO which provide us with some generators of the Ibáñez-Ross [14] classification of discrete symmetries. The flavor content of the weight systems of interest for $SO(10)$ and $E_6$ is in the Tables I, II and V of Ref. [19].
order $N$ of an EFO is given by $N = MC$. For $SO(10)$, an EFO $\mathbf{s} = [s_0, s_1, \ldots, s_5]$ has
\[
M = s_0 + s_1 + 2s_2 + 2s_3 + s_4 + s_5,
\]
and
\[
C = \frac{4}{\gcd(4; 2s_1 + 2s_3 + 3s_4 + 5s_5)}.
\]
For $E_6$, an EFO is specified by $\mathbf{s} = [s_0, \ldots, s_6]$ and has
\[
M = s_0 + s_1 + 2s_2 + 3s_3 + 2s_4 + s_5 + 2s_6,
\]
and
\[
C = \frac{3}{\gcd(3; s_1 - s_2 + s_4 - s_5)}.
\]
$M$ is the order of the action on the adjoint representation, i.e. $N(\rho)$ in (6) for $\rho = \text{adjoint representation}$. Values of $M$ and $C$ for other simple Lie groups are given in Table 6 of Ref. [10].

The first point we must investigate is to find all the EFO of a given order $N$. Let us first consider the $Z_2$ discrete symmetries. For $SO(10)$ there are three EFO of order two, namely
\[
\mathbf{s}_{(1)} = [0, 0, 0, 1, 1, 1], \quad \mathbf{s}_{(2)} = [0, 0, 1, 0, 0, 0], \quad \mathbf{s}_{(3)} = [0, 1, 0, 0, 0, 0],
\]
as can be verified by using (17) and (18). In order to find the products (12) that lead to the generators $R, A, L$ of (15), one could, in principle, solve (14) for $n_l$, but notice that this is not a simple algebraic system, because both sides of the equation actually read modulo $N$. The product $\mathbf{s}_{(1)}\mathbf{s}_{(2)}\mathbf{s}_{(3)}$ provides the discrete symmetry $R_2$. In other words, each $\mathbf{s}_{(i)}$ corresponds to an element of order two of $SO(10)$ (with its action on the weight vectors given by (9)), and the product is the actual product of the three group elements, represented by diagonal matrices.

For $E_6$, the situation is less interesting. There are two EFO of order two:
\[
\mathbf{s}_{(1)} = [0, 1, 0, 0, 1, 0], \quad \mathbf{s}_{(2)} = [0, 0, 0, 0, 0, 1].
\]
Unlike the case of $SO(10)$, they do not reproduce any of the generator in (14). At best, their product provides the discrete symmetry $R_2$, but after multiplying the Higgs fields $h, h'$ by $-1$. 

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We now turn to the $Z_3$ discrete symmetries. For $SO(10)$, there are six EFO of order three:

\begin{align}
&\mathbf{s}(1) = [1, 2, 0, 0, 0, 0], \quad \mathbf{s}(2) = [1, 0, 1, 0, 0, 0], \\
&\mathbf{s}(3) = [0, 1, 0, 0, 0, 0], \quad \mathbf{s}(4) = [0, 1, 0, 2, 0], \\
&\mathbf{s}(5) = [0, 1, 0, 0, 1, 0], \quad \mathbf{s}(6) = [1, 0, 0, 1, 1].
\end{align}

(23)

The discrete symmetry $\alpha_A$ is given by the products:

\[ s_2 s_3^2 s_4^2 s_5 s_6, \quad s_1 s_2 s_3^2 s_4 s_5, \quad s_1 s_2^2 s_3 s_4 s_5. \]

The symmetry $\alpha_R$ corresponds to any of the two possibilities:

\[ s_1 s_2 s_3 s_6, \quad s_1 s_2^2 s_3^2 s_4 s_5. \]

For $E_6$, there are five EFO of order three:

\begin{align}
&\mathbf{s}(1) = [0, 0, 0, 1, 0, 0], \quad \mathbf{s}(2) = [1, 0, 0, 0, 0, 1], \\
&\mathbf{s}(3) = [1, 1, 0, 0, 1, 0], \quad \mathbf{s}(4) = [0, 1, 1, 0, 0, 0], \\
&\mathbf{s}(5) = [0, 0, 0, 1, 1, 0].
\end{align}

(24)

We have found that $s_1 s_2 s_3^2 s_4 s_5$ provides exactly the discrete symmetry $\alpha_R$. The discrete symmetry $\alpha_A$ is given by the product $s_1 s_2^2 s_3 s_4^2 s_5$. (Obviously, we can form combinations like $\alpha_A \alpha_R = s_1 s_5$, using the property $s_3^3 = Id$.)

In all cases, the generator $L$ does not correspond to any EFO. As explained before, Kac’s notation provide the diagonal representative of a conjugacy class of EFO, so that the missing DGS could be generated by non-diagonal elements of finite order. This is not further studied here. This restriction of the method prevents us from finding the symmetries which forbid FCNC, since the most general form of such symmetries is

\[ S_N = R_N^m \times \left( L_N^j \right)^{p_j} \times A_N, \]

(25)

where $L^j$ corresponds to the three standard lepton numbers. In principle, one could need just to consider three copies of the fermions representations, with the same element of finite order generating $L_N^j$. (Generation dependent DGS are discussed in Ref. [20].) Hence, of the two anomaly free generalized parities $R_2$
and $R_3L_3$ [14], only $R_2$ can be found by Kac’s method in its current form, in the $SO(10)$ model.

To close this section, let us recall from Ref. [10] the relation between congruence classes of representations and the theory of EFO. Having in mind the criterion stated in Ref. [13], based on the congruence classes of representations, let us say that in the present framework, they can be obtained from the EFO associated to a central element of the Lie group $G$. In turn, a central element is given by an EFO whose entry is at a “tip” of the extended Coxeter-Dynkin diagram, that is, a node which has numerical mark equal to 1. For $SO(10)$, the tips are at $s_0$, $s_1$, $s_4$, and $s_5$. For $E_6$ they are $s_0$, $s_1$ and $s_5$. Indeed, the center of $SO(10)$ contains four elements and there are four congruence classes. The center of $E_6$ consists of three elements, and they describe the three congruence classes. Therefore, we expect Martin’s “surprisingly mild conditions” [13] to appear as natural consequences when DGS are considered in the formalism of EFO.

4. Concluding remarks.

Our purpose in doing this work is to bring attention to Kac’s theory of EFO as a mathematical concept to understand formally and generate elegantly discrete symmetries in a Lie group. We have applied the concept to the DGS in some grand unified models, although the procedure could be applied to a much larger class of problems in particle physics, where discrete groups are often encountered.

We have found that one cannot generate all the anomaly free discrete symmetries, such as they appear in Ref. [14]. This is so because the method describes only the action of the diagonal representatives of conjugacy classes of EFO rather than individual elements of finite order, which could lead to all (abelian and nonabelian) discrete subgroups of a Lie group. Therefore, a possible avenue of research is to extend Kac’s EFO theory so that one could describe as well non-diagonal elements of a conjugacy class. Another possible extension would be to treat systematically the action of EFO through outer automorphisms in addition to the inner automorphisms. Thus, although the present method encompasses many DGS which are not anomaly free, it does not allow to find the complete set of anomaly free DGS. If such extensions could be achieved, then the present concepts might display some elegant properties when applied to particle physics. It might be possible that we get such nice properties also by changing the assignment of particles to vectors in the representation (so that, for instance, one could
predict the EFO which give the various parities discussed in Section 3.

An interesting question, related to the generation of discrete groups by EFO, is if the EFO theory provides an elegant algorithm to tell whether or not a given EFO belongs to some subgroup. For instance, in the context of some grand unified theory, it is relevant to know whether the discrete group generated by an EFO belongs to the standard model subgroup. Put the inverse way, given a subgroup (e.g. the standard model) it would be interesting to describe, through EFO, the complete set of discrete symmetries which are complementary to this subgroup in a larger Lie group (e.g. grand unified gauge group). There is an indication that the presence of zeros in the array $s$ can be identified with the subgroup associated to the corresponding nodes in the Coxeter-Dynkin diagram. This question has not been answered in general [J. Patera, private communication].

Another limitation of the present procedure is that it concerns only the discrete symmetries which are contained in some Lie group. In other words, one cannot get the discrete symmetries of some Lagrangian which are not in the continuous group of symmetry of the Lagrangian. In principle, the method can be applied to other grand unified models, like $SU(5), SU(4) \times SU(2)^2, SU(3)^3$, etc. or as large as $E_8$ or $SU(16)$.

Other applications in high energy physics have not been explored yet.

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### Table I. $Z_N$ charges of generators $R, A, L$.

|      | $q$ | $u^c$ | $d^c$ | $l$ | $e^c$ | $h$ | $h'$ |
|------|-----|-------|-------|-----|-------|-----|------|
| $\alpha_R$ | $0$ | $-1$  | $1$   | $0$ | $1$   | $1$ | $-1$ |
| $\alpha_A$ | $0$ | $0$   | $-1$  | $-1$| $0$   | $0$ | $1$  |
| $\alpha_L$ | $0$ | $0$   | $0$   | $-1$| $1$   | $0$ | $0$  |