GLOBAL EXISTENCE OF GEOMETRIC ROUGH FLOWS

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Abstract. In this paper we consider rough differential equations on a smooth manifold \((M)\). The main result of this paper gives sufficient conditions on the driving vector-fields so that the rough ODE’s have global (in time) solutions. The sufficient conditions involve the existence of a complete Riemannian metric \((g)\) on \(M\) such that the covariant derivatives of the driving fields and their commutators to a certain order (depending on the roughness of the driving path) are bounded. Many of the results of this paper are generalizations to manifolds of the fundamental results in [5].

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1. Introduction

1.1. Overview. If \([0, T] \times \mathbb{R}^N \ni (t, y) \rightarrow Y_t(t) \in \mathbb{R}^N\) is a smooth time dependent vector field on \(\mathbb{R}^N\), it is natural to consider solving for \(y : [0, T] \rightarrow \mathbb{R}^N\) and \((t_0, y_0) \in [0, T] \times \mathbb{R}^N\) the ordinary differential equation,

\[
\dot{y}(t) = \dot{Y}_t(y(t)) \quad \text{with} \quad y(t_0) = y_0,
\]

where the dot indicates derivatives in \(t\). The following theorem is then well known and easy to prove.

**Theorem 1.1.** If there exists \(a, b \in (0, \infty)\) such that

\[
\left| \dot{Y}_t(y) \right| \leq a + c |y| \quad \forall \quad (t, y) \in [0, T] \times \mathbb{R}^N,
\]

then Eq. (1.1) has a unique solution.

In the setting of stochastic differential equations one wishes to solve (1.1) in the case where \(Y_t\) is a random vector field which typically is rough in \(t\) and in particular no longer differentiable in \(t\). Such equations have been under active investigation ever since Itô’s pioneering work \([28, 29]\) and use the random structure of \(Y\) in order to give meaning to Eq. (1.1). More recently in his pioneering work on rough paths, Terry Lyons \([33, 35, 36]\) was able to show that one may use the statistical properties of \(Y\) in order to “enhance” \(Y\) with added information (see Example 1.22 below) where the degree of enhancement needed increases with the lack of differentiability of \(t \rightarrow Y_t\). Once the enhanced \(Y\) is found, Lyons was able to dispense with the randomness and again make deterministic sense of Eq. (1.1). In particular he was able to prove the analogue of Theorem 1.1 when \(c = 0\). The main goal of this paper is to give a version of Lyons’ rough differential equation (RDE for short) existence theorem when \(\mathbb{R}^N\) is replaced by a smooth manifold \(M\) and the size of \(\dot{Y}_t\) is now measured by a Riemannian metric \((g)\) on \(M\). Along the way we will also see that boundedness condition on \(Y\) (i.e. \(c = 0\)) in Lyons existence theorem may be considerably weakened. For some history and other results on relaxing the \(c = 0\) condition, see \([32]\).

This paper was inspired and highly influenced by Bailleul’s paper \([5]\). In fact, this paper grew out of an unsuccessful attempt to understand the unbounded vector-field existence result stated in \([5\textnormal{, Theorem 4.1.}]\). After finishing the first draft of this paper the author discovered \([2]\) which contains similar results to those in this paper when \(M\) is a Euclidean or more generally a Banach space. More recently I have been alerted to the work of Brault and Lejay \([8, 9]\) and Martin Weidner \([42]\). In principle Brault and Lejay develop a more general version of the results in Section 2 of this paper. However, there seems to be a mistaken assertion in Eq. (21) on page 11 of \([8]\) which will likely require a rewriting of their results. [I do suspect most of the results in \([8]\) are morally true.] The existence results in \([42]\) are quite analogous to those given in this paper including the idea of using a conformal change of the metric to extend the theory away from bounded vector fields,
compare [42, Proposition 5.2] with Proposition 6.10. However, some of the
details of the proofs are different. In particular, Weidner makes use of a
number of localization arguments in order exploit known RDE results on \(\mathbb{R}^N\)
while in this paper all of the proofs are intrinsic. Moreover, the proofs given
here follow Bailleul’s method in [3] which does not require knowing previous
RDE results on \(\mathbb{R}^N\). Other than using some ordinary differential equation
estimates on manifolds developed in [18] (some of these same estimates are
also in [42]), this paper is essentially self-contained.

Before describing the results in this paper precisely, let us first rephrase
(see Theorem 1.6) what it means to solve Eq. (1.1) in such a way that easily
generalizes to the rough path setting. We begin by fixing some notation.

**Notation 1.2.** Throughout this paper, we let \(M\) be a smooth finite dimen-
sional manifold and \(\Gamma (TM)\) be the linear space of smooth vector fields on
\(M\). Typically \(g\) will denote a Riemannian metric on \(M\), \(\nabla = \nabla^g\) be the
associated Levi-Civita covariant derivative, \(R = R^g\) be the curvature tensor of
\(\nabla\), and \(d (\cdot, \cdot) = d_g (\cdot, \cdot)\) be the length metric associated to
\(g\).

We will also abuse notation and use \(d\) to denote a fixed integer in \(\mathbb{N}\).
This should not cause confusion as distance function will always come with
arguments.

**Definition 1.3.** A \(d\)-dimensional dynamical system on \(M\) is a linear
map, \(\mathbb{R}^d \ni w \rightarrow V_w \in \Gamma (TM)\).

A \(d\)-dimensional dynamical system on \(M\) is completely determined by
knowing \(\{ V_{e_j} \}_{j=1}^d \subset \Gamma (TM)\) where \(\{ e_j \}_{j=1}^d\) is the standard basis for \(\mathbb{R}^d\).

The object of this paper is to describe necessary conditions for a “rough
ordinary differential equations” associated to \(V\) (RODE for short) to have
global (in time) solutions. These conditions will be in the form of the exis-
tence of a Riemannian metric, \(g\), on \(M\) which is complete and “controls” the
size of \(V\) “appropriately.” The following two theorems serves as warm-up to
the general results to be stated in subsection 1.3 below.

**Notation 1.4.** Given a dynamical system, \(V\), and Riemannian metric, \(g\),
on \(M\), let

\[
|V|_M := \sup_{w \in \mathbb{R}^d : |w| = 1} \sup_{m \in M} |V_w (m)|_g
\]

and

\[
|\nabla V|_M := \sup_{w \in \mathbb{R}^d : |w| = 1} \sup_{v \in TM : |v| = 1} |\nabla_v V_w|_g.
\]

**Theorem 1.5.** Suppose that \(V\) is a dynamical system and \(g\) is a com-
plete Riemannian metric on \(M\) such that \(|\nabla V|_M < \infty\), then for every
\(x \in C^1 ([0, T], \mathbb{R}^d)\), \(s \in [0, T]\), and \(m \in M\), there exists \(\sigma \in C^1 ([0, T], M)\)
such that

\[
\dot{\sigma} (t) = V_{\dot{x}(t)} (\sigma (t)) \text{ with } \sigma (s) = m.
\]
Proof. For a proof of this classical theorem the reader may refer to [18 Corollary 2.12] with $Y_t := V_{\dot{z}(t)}$. A more general form of this theorem may also be found in Lemma 6.3 of the Appendix 6.

If $V$ is a dynamical system satisfying the conclusions of Theorem 1.5, let $[0,T]^2 \times M \ni (t,s,m) \mapsto \varphi_{t,s}^x (m) \in M$ be the flow map defined by requiring,

$$\frac{d}{dt} \varphi_{t,s}^x (m) = V_{\dot{z}(t)} (\varphi_{t,s}^x (m)) \text{ with } \varphi_{s,s}^x (m) = m. \quad (1.4)$$

We will usually abbreviate the previous equation by writing

$$\frac{d}{dt} \varphi_{t,s}^x = V_{\dot{z}(t)} \circ \varphi_{t,s}^x \text{ with } \varphi_{s,s}^x = Id_M. \quad (1.5)$$

In the next theorem, we will give an alternate characterization of the flow $\varphi_{t,s}^x$ which is suitable for defining $\varphi^x$ where $x \in C^1 ([0,T], \mathbb{R}^d)$ is replaced by much rougher paths. In the hypothesis of Theorem 1.6 we will not only require that Eq. (1.3) holds but that also if $|V|_M < \infty$. Later in the introduction we will (based on the results of Appendix 6) see that by replacing $g$ by an appropriate conformally equivalent metric we may remove this added restriction on $V$.

**Theorem 1.6.** Suppose that $V$ is a dynamical system and $g$ is a complete Riemannian metric on $M$ such that both Eqs. (1.2) and (1.3) are satisfied. Then for every $x \in C^1 ([0,T], \mathbb{R}^d)$, $\varphi_{t,s}^x \in \text{Diff } (M)$, $\varphi^x$ is multiplicative (i.e. $\varphi_{t,s}^x \circ \varphi_{u,u}^x = \varphi_{t,u}^x$ for all $s,t,u \in [0,T]$), and there exists $K < \infty$ such that

$$d \left( \varphi_{t,s}^x (m), e^{V_{x,s,t}} (m) \right) \leq K |t - s|^2 \forall \ s,t \in [0,T] \text{ and } m \in M, \quad (1.6)$$

where

$$x_{s,t} := x(t) - x(s) \forall \ s,t \in [0,T].$$

Conversely, if $\{ \varphi_{t,s} \}_{t,s \in [0,T]} \subset \text{Diff } (M)$ is a multiplicative and there exists $K < \infty$ such that Eq. (1.6) holds, then $\varphi_{t,s}^x = \varphi_{t,s}^x$.

Proof. Suppose that $\varphi_{t,s}^x$ is defined as in Eq. (1.5). The estimate in Eq. (1.6) is now a direct consequence of [18 Theorem 4.11] applied with $\kappa = 1$. The fact that $\varphi_{t,s}^x \in \text{Diff } (M)$ and $\varphi^x$ is multiplicative is a standard property of smooth flows, see for example, [18 Theorem 2.14] for a more detailed summary of such flows.

Conversely, if $\varphi_{t,s} \in \text{Diff } (M)$ is multiplicative and satisfies the estimate in Eq. (1.6) and $f \in C^\infty (M)$, then

$$|f (\varphi_{t,s} (m)) - f (m) + V_{x_{s,t}} f (m)| \leq C |t - s|^2.$$

Dividing this estimate by $|t - s|$ implies,

$$\left| \frac{f (\varphi_{t,s} (m)) - f (m)}{t - s} - V_{x_{s,t}} f (m) \right| \leq C |t - s|$$
and upon letting $t \to s$, gives

$$\frac{d}{dt} |_{t=s} f (\varphi_{t,s}(m)) = \left( V_{\dot{x}(s)} f \right)(m).$$

Since the previous equality is valid for all $f \in C^\infty (M)$, we conclude that $\frac{d}{dt} \varphi_{t,s}(m)$ exists and

$$\frac{d}{dt} \varphi_{t,s}(m) = V_{\dot{x}(s)}(m).$$

Then using the multiplicative property of $\varphi_{t,s}$ it follows that

$$\frac{d}{dt} \varphi_{t,s}(m) = \frac{d}{dt} \left( \varphi_{t+\varepsilon,t} \circ \varphi_{t,s}(m) = V_{\dot{x}(t)}(\varphi_{t,s}(m)) \right),$$

i.e. $\varphi_{t,s}$ satisfies flow ODE in Eq. (1.4). $\square$

A basic idea of rough paths and Ballieu [5] is to generalize Eq. (1.6) so as to allow for much rougher paths, $x$, see Theorem 1.27. The rest of this introduction is devoted to stating the main results of this paper. As usual in rough path theory, the rougher $x$ becomes the more extra information one must enhance $x$ with in order to give meaning to Eq. (1.4). The next subsection introduces the basic rough path language we will need in this paper. The summary of the main theorems of the paper will follow in subsection 1.3.

1.2. Basic Notations. Although our presentation here will be self-contained, the reader’s wishing for more background on rough paths may consult the monographs [22, 23, 34, 37] and the survey article of [31]. The reader may also wish to consult [11–13,19] for the beginnings of general theory of rough paths on manifolds. For a more detailed description of the algebra presented here, see [18] Subsection 1.2.

**Definition 1.7 (Tensor Algebras).** Let $T(\mathbb{R}^d) := \bigoplus_{k=0}^\infty [\mathbb{R}^d] \otimes^k$ be the tensor algebra over $\mathbb{R}^d$ so the general element of $\omega \in T(\mathbb{R}^d)$ is of the form

$$\omega = \sum_{k=0}^\infty \omega_k \text{ with } \omega_k \in (\mathbb{R}^d) \otimes^k \text{ for } k \in \mathbb{N}_0$$

where we assume $\omega_k = 0$ for all but finitely many $k$. Multiplication is the tensor product and associated to this multiplication is the Lie bracket,

$$(1.7) \quad [A, B]_{\otimes} := A \otimes B - B \otimes A \text{ for all } A, B \in T(\mathbb{R}^d).$$

A good reference for nilpotent Lie algebras and related material is [39] although everything we need will be described here and explained in more detail when needed in Section 3 below.

**Definition 1.8 (Free Lie Algebra).** The free Lie algebra over $\mathbb{R}^d$ will be taken to be the Lie-subalgebra, $F(\mathbb{R}^d)$, of $(T(\mathbb{R}^d), [\cdot, \cdot]_{\otimes})$ generated by $\mathbb{R}^d$. 
Remark 1.9. If \((g, [\cdot, \cdot])\) is a Lie algebra and \(V \subset g\) is a subspace, then using Jacobi’s identity one easily shows that Lie sub-algebra \((\text{Lie}(V))\) of \(g\) generated by \(V\) may be described as;

\[
\text{Lie}(V) = \text{span} \cup_{k=1}^{\infty} \{ \text{ad}_{v_1} \ldots \text{ad}_{v_{k-1}} v_k : v_1, \ldots, v_k \in V \},
\]

where \(\text{ad}_A := [A, B]\) for all \(A, B \in g\). As a consequence of this remark it follows that \(F(R^d)\) is a \(\mathbb{N}_0\)-graded Lie algebra with

\[
F(R^d) = \bigoplus_{k=0}^{\infty} F_k(R^d) = F(R^d) \cap [R^d]^\otimes_k \subset F(R^d).
\]

According to this grading, if \(A \in F(R^d)\) we let \(A_k \in F_k(R^d)\) denote the projection of \(A\) into \(F_k(R^d)\).

The spaces \(T(R^d)\) and \(F(R^d)\) are infinite dimensional. We are going to be most interested in the finite dimensional truncated versions of these algebras.

Definition 1.10 (Truncated Tensor Algebras). Given \(\kappa \in \mathbb{N}\), let

\[
T^{(\kappa)}(R^d) := \bigoplus_{k=0}^{\kappa} [R^d]^\otimes_k \subset T(R^d)
\]

which is algebra under the multiplication rule,

\[
AB = \sum_{k=0}^{\kappa} (AB)_k = \sum_{k=0}^{\kappa} \sum_{j=0}^{k} A_j \otimes B_{k-j} \quad \forall A, B \in T^{(\kappa)}(R^d)
\]

and a Lie algebra under the bracket operation, \([A, B] := AB - BA\) for all \(A, B \in T^{(\kappa)}(R^d)\).

Notation 1.11. Let \(\pi_{\leq \kappa} : T(R^d) \to T^{(\kappa)}(R^d)\) and \(\pi_{> \kappa} := I_{T(R^d)} - \pi_{\leq \kappa} : T(R^d) \to \bigoplus_{k=\kappa+1}^{\infty} [R^d]^\otimes_k\) be the projections associated to the direct sum decomposition,

\[
T(R^d) = T^{(\kappa)}(R^d) \oplus \bigoplus_{k=\kappa+1}^{\infty} [R^d]^\otimes_k.
\]

Further let

\[
\mathfrak{g}^{(\kappa)} = \bigoplus_{k=1}^{\kappa} [R^d]^\otimes_k
\]

which is a two sided ideal as well as a Lie sub-algebra of \(T^{(\kappa)}(R^d)\).

With this notation the multiplication and Lie bracket on \(T^{(\kappa)}(R^d)\) may be described as,

\[
AB = \pi_{\leq \kappa} (A \otimes B) \quad \text{and} \quad [A, B] = \pi_{\leq \kappa} [A, B]_{\otimes}.
\]

Notation 1.12 (Induced Inner product). The usual dot product on \(R^d\) induces an inner product, \(\langle \cdot, \cdot \rangle\) on \(T^{(\kappa)}(R^d)\) uniquely determined by requiring
For any \( v, w \in \mathbb{R}^d \) and \( 1 \leq k \leq \kappa \). We let \( |A| := \sqrt{\langle A, A \rangle} \) denote the associated Hilbertian norm of \( A \in T^{(\kappa)}(\mathbb{R}^d) \).

It turns out to often be more convenient to measure the size of \( A \in g^{(\kappa)} \) using the following "homogeneous norms.”

**Definition 1.13 (Homogeneous norms).** For \( A \in g^{(\kappa)} \subset T^{(\kappa)}(\mathbb{R}^d) \), let

\[
N(A) := \max_{1 \leq k \leq \kappa} |A_k|^{1/k}.
\]

**Definition 1.14 (Free Nilpotent Lie algebra).** The **step \( \kappa \)** **free nilpotent Lie algebra** on \( \mathbb{R}^d \) may then be realized as the Lie sub-algebra, \( F^{(\kappa)}(\mathbb{R}^d) \), of \( (T^{(\kappa)}(\mathbb{R}^d), [\cdot, \cdot]) \) generated by \( \mathbb{R}^d \subset T^{(\kappa)}(\mathbb{R}^d) \).

Again, a simple consequence of Remark 1.9 is that, as vector spaces, \( F^{(\kappa)}(\mathbb{R}^d) = \pi_{\leq \kappa}(F(\mathbb{R}^d)) \) and \( F^{(\kappa)}(\mathbb{R}^d) \) is graded as

\[
F^{(\kappa)}(\mathbb{R}^d) = \oplus_{k=0}^{\kappa} F^{(\kappa)}_k(\mathbb{R}^d)
\]

where

\[
F^{(\kappa)}_k(\mathbb{R}^d) := F^{(\kappa)}(\mathbb{R}^d) \cap \left[ \mathbb{R}^{d} \right]^{\otimes k} \subset F^{(\kappa)}(\mathbb{R}^d) \text{ for } 1 \leq k \leq \kappa.
\]

It is not difficult to show (see see [18, Section 1.2]) using the universal properties of the tensor algebra that the following notation is well defined.

**Notation 1.15.** Given a dynamical system, \( V : \mathbb{R}^d \to \Gamma(TM) \), let \( V^{(\kappa)} : F^{(\kappa)}(\mathbb{R}^d) \to \Gamma(TM) \) be the unique dynamical system such that

\[
V_A^{(\kappa)} := LV_{w_1} \cdots LV_{w_{j-1}} V_{w_j}
\]

whenever \( A = \text{ad}_{w_1} \cdots \text{ad}_{w_{j-1}} w_j \) with \( 1 \leq j \leq \kappa \) and \( w_i \in \mathbb{R}^d \). [For \( A \in F^{(\kappa)}(\mathbb{R}^d) \), we will usually simply write \( V_A \) for \( V_A^{(\kappa)} \).]

**Remark 1.16.** It is **not** in general true that \( V^{(\kappa)} := V|_{F^{(\kappa)}(\mathbb{R}^d)} : F^{(\kappa)}(\mathbb{R}^d) \to \Gamma(TM) \) is a Lie algebra homomorphism. In order for this to be true we must require that \( LV_{v_n} \cdots LV_{v_1} V_0 = 0 \) for all \( \{ a_j \}_{j=0}^{\kappa} \subset \mathbb{R}^d \), i.e. \( \{ V_a : a \in \mathbb{R}^d \} \) should generate a step-\( \kappa \) nilpotent Lie sub-algebra of \( \Gamma(TM) \).

We now need to introduce a number of semi-norms on vector fields and dynamical systems on \( M \).
Notation 1.17 (Tensor Norms). If $X \in \Gamma(TM)$ and $m \in M$, let

$$|X|_m := |X(m)|_g$$

$$|\nabla X|_m := \sup_{|v_m| = 1} |\nabla_{v_m} X|_g,$$

$$|\nabla^2 X|_m = \sup_{|v_m| = 1 = |w_m|} |\nabla^2_{v_m \otimes w_m} X|_g,$$

$$|R(X, \cdot)|_m := \sup_{|v_m| = 1 = |w_m|} |R(X(m), v_m) w_m|_g,$$ and

$$H_m(X) := |\nabla^2 X|_m + |R(X, \cdot)|_m.$$ 

We further let $|X|_M = \sup_{m \in M} |X|_m$, \ldots, $H_M(X) := \sup_{m \in M} H_m(X)$.

Definition 1.18 (Dynamical system semi-norms). When $V : \mathbb{R}^d \to \Gamma(TM)$ is a dynamical system of $\kappa \in \mathbb{N}$, let

$$|V^{(\kappa)}|_M := \left\{ |V_A|_M : A \in F^{(\kappa)}(\mathbb{R}^d) \text{ with } |A| = 1 \right\},$$

$$|\nabla V^{(\kappa)}|_M := \left\{ |\nabla V_A|_M : A \in F^{(\kappa)}(\mathbb{R}^d) \text{ with } |A| = 1 \right\}, \text{ and}$$

$$H_M(V^{(\kappa)}) := \left\{ H_M(V_A) : A \in F^{(\kappa)}(\mathbb{R}^d) \text{ with } |A| = 1 \right\}.$$ 

Any of these expressions are allowed to take plus infinity as a value.

In order to introduce rough paths we need to go from Lie algebras to Lie groups. Let $G^{(\kappa)}(\mathbb{R}^d) := 1 + \mathfrak{g}^{(\kappa)} \subset T^{(\kappa)}(\mathbb{R}^d)$ which forms a group under the multiplication rule of $T^{(\kappa)}(\mathbb{R}^d)$. In fact, $G^{(\kappa)}$ is a Lie group with Lie algebra, $\text{Lie}(G^{(\kappa)}) = \mathfrak{g}^{(\kappa)}$ and the exponential map,

$$\text{Lie}(G^{(\kappa)}) \ni \xi \to e^\xi = \sum_{k=0}^{\kappa} \frac{\xi^k}{k!} \in G^{(\kappa)}(\mathbb{R}^d)$$

is a diffeomorphism where $\xi^k := \pi_{\leq \kappa} (\xi^{\otimes k})$ inside of $T^{(\kappa)}(\mathbb{R}^d)$. We will mostly only use the following subgroup of $G^{(\kappa)}(\mathbb{R}^d)$.

Definition 1.19 (Free Nilpotent Lie Groups). For $\kappa \in \mathbb{N}$, let $G^{(\kappa)}_{\text{geo}}(\mathbb{R}^d) \subset G^{(\kappa)}$ be the simply connected Lie subgroup of $G^{(\kappa)} = 1 \oplus_{k=1}^{\kappa} [\mathbb{R}^d]^{\otimes k}$ whose Lie algebra is $F^{(\kappa)}(\mathbb{R}^d)$. This subgroup is a step-$\kappa$ (free) nilpotent Lie group which we refer to as the \textbf{geometric sub-group} of $G^{(\kappa)}$.

It is well known as a consequence of the Baker-Campel-Dynken-Hausdorff formula (see for example [18 Proposition 3.12]) that the exponential map,

$$F^{(\kappa)}(\mathbb{R}^d) \ni \xi \to e^\xi = \sum_{k=0}^{\kappa} \frac{\xi^k}{k!} \in G^{(\kappa)}_{\text{geo}}(\mathbb{R}^d),$$
is a diffeomorphism. Furthermore that inverse, log, of this diffeomorphism may be computed using

$$\log (1 + \xi) = \sum_{k=1}^{\kappa} \frac{(-1)^{k+1}}{k} \xi^k.$$  

**Definition 1.20** (Hölder geometric rough paths). For any $T \in (0, \infty)$, let

$$\Delta_T := \left\{ (s, t) \in [0, T]^2 : 0 \leq s \leq t \leq T \right\}.$$  

Given $\alpha \in \left( \frac{1}{\kappa + 1}, \frac{1}{\kappa} \right]$, $X \in C \left( \Delta_T, G^{(\kappa)}_{geo}(\mathbb{R}^d) \right)$ is an $\alpha$-Hölder geometric rough path if;

1. $X_{s,s} = 1$ for all $s \in [0, T]$,
2. $X_{st}X_{tu} = X_{su}$ for all $0 \leq s \leq t \leq u \leq T$, and
3. there is a constant $C < \infty$ such that

$$N \left( X_{s,t} \right) \leq C |t - s|^\alpha \text{ for all } (s,t) \in \Delta_T.$$  

**Example 1.21** (Smooth Rough paths). If $x \in C^1([0, T], \mathbb{R}^d)$, $X^0_{st} := 1$, and for $1 \leq k \leq \kappa$,

$$X^k_{st} := \int_{s \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \leq t} dx(\sigma_1) \otimes dx(\sigma_2) \otimes \cdots \otimes dx(\sigma_k)$$

for all $(s, t) \in \Delta_T$, then $X := \sum_{k=0}^\kappa X^k_{st} \in G^{(\kappa)}_{geo}$ is an 1-Hölder geometric rough path. In fact if $C = \max_{s \in [0,T]} |\dot{x}(t)|$, then

$$|X^k_{st}| \leq \int_{s \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \leq t} |dx(\sigma_1)| \cdot |dx(\sigma_2)| \cdots |dx(\sigma_k)|$$

$$= \frac{1}{k!} \left( \int_s^t |dx(\sigma)| \right)^k \leq \frac{C^k}{k!} |t - s|^k.$$  

It is known that every $\alpha$-Hölder rough path is certain limit of a sequence of such smooth rough paths, see [23, Section 8.6].

**Example 1.22** (Enhanced Brownian Motion). If $\{B_t\}_{t \geq 0}$ is an $\mathbb{R}^d$-valued Brownian motion, let $B_{s,t} := B_t - B_s$, and

$$B^2_{s,t} := \int_s^t (B_\sigma - B_s) \otimes \delta B_\sigma$$

where $\delta B$ denote the Stratonovich differential of $B$. Then

$$B_{s,t} := 1 + B_{s,t} + B^2_{s,t} \in G^{(2)}_{geo}$$

is almost surely a $\alpha$-Hölder geometric rough path for any $0 < \alpha < \frac{1}{2}$. For this example and other Gaussian process examples, see [23, Section 13.2 and Chapter 15] and the references therein.
Notation 1.23. Given a geometric rough path as in Definition 1.20, we extend \( X_{s,t} \) to all \((s,t) \in [0,T]^2\) by
\[
X_{s,t} := X_{t,s}^{-1} \quad \text{when} \quad 0 \leq t \leq s \leq T.
\]

Remark 1.24. From items 1. and 2. of Definition 1.20, if we let \( g_t := X_{0,t} \), then
\[
g_0 = 1 \in G^{(\kappa)}(\mathbb{R}^d) \quad \text{and for} \quad 0 \leq s \leq t \leq T,
\]
\[
g_t = X_{0,t} = X_{0,s} X_{st} = g_s X_{st} \implies X_{st} = g_s^{-1} g_t.
\]
Thus we see it is natural to extend the definition of \( X \) using Eq. (1.12)
\[
X_{s,t} := g_s^{-1} g_t \quad \text{for all} \quad (s,t) \in [0,T]^2.
\]
This extension satisfies, \( X_{s,t} = X_{t,s}^{-1} \) for all \( s,t \in [0,T] \) and hence is consistent with Notation 1.23. From Eq. (1.12), it is simple to verify \( X_{s,t} X_{t,u} = X_{su} \) for all \( s,t,u \in [0,T] \). Moreover, as a consequence of Corollary 7.4, we also have
\[
|X_{s,t}^k| = |X_{t,s}^k| \leq C |t-s|^\alpha \quad \text{for} \quad 1 \leq k \leq \kappa \quad \text{and} \quad (s,t) \in [0,T]^2.
\]

Assumption 1. Throughout this paper we assume that \( 0 < \alpha \leq 1 \) and \( \kappa \in \mathbb{N} \) always satisfy
\[
\theta := \alpha (\kappa + 1) > 1.
\]

1.3. Statement of the Main Results. For the rest of this paper let \( X \) be a Hölder rough path as in Definition 1.20 and \( V \) be a \( d \)-dimensional dynamical system on \( M \) as in Definition 1.3. When \( d(\cdot,\cdot) \) the Riemannian distance function associated to a Riemannian metric, \( g \), on \( M \) and \( f,g : M \to M \) are two maps and \( U \) is a subset of \( M \), we let
\[
d_U(f,g) := \sup_{m \in U} d(f(m),g(m)).
\]
We will mostly use this definition with \( U = M \).

Assumption 2. Throughout this paper we assume that \( V_A \in \Gamma(TM) \) is complete for all \( A \in F^{(\kappa)}(\mathbb{R}^d) \).

A standard condition that guarantees a vector field, \( Y \in \Gamma(TM) \), is complete is to assume there is a complete metric \( g \) on \( M \) such that
\[
\sup_{m \in M} \frac{|Y(m)|_g}{1 + d(o,m)} < \infty
\]
where \( o \) is a fixed point in \( M \) and \( d \) is the length metric associated to \( g \). See Lemma 6.3 and Examples 6.4 and 6.5 of Appendix 1 for a review of this fact along with some extensions to this type of result. It should also be noted that if \( |\nabla Y|_M < \infty \), then Eq. (1.15) holds, see for example [18, Lemma 2.10].
Define 1.25 (Approximate flows). If $X$ is an $\alpha$-Hölder rough path and $V : \mathbb{R}^d \to \Gamma (TM)$ is a dynamical system satisfying Assumption 2, let

$$\mu_{t,s} = \mu^X_{t,s} := e^{V^{(\kappa)}_{log(X_{s,t})}} \in \text{Diff} (M) \forall (s,t) \in [0,T]^2.$$ \hfill (1.16)

The next simple proposition indicates the importance of $\mu_{t,s}$.

**Proposition 1.26 (Nilpotent Flows).** If $\{ V_a : a \in \mathbb{R}^d \} \subset \Gamma (TM)$ generates a step-$\kappa$ Nilpotent Lie sub-algebra then $V^{(\kappa)} : \mathbb{F}^{(\kappa)} (\mathbb{R}^d) \to \Gamma (TM)$ is a Lie-algebra homomorphism. Moreover, $\mu_{t,s}$ is multiplicative (has the flow property),

$$\mu_{t,s} \circ \mu_{s,r} = \mu_{t,r} \forall (s,t) \in [0,T]^2.$$ \hfill (1.17)

**Proof.** This follows from basic Lie theoretic considerations. Here is a proof based on [18, Corollary 4.7] which under the given assumptions states that

$$e^{V^{(\kappa)}_B} \circ e^{V^{(\kappa)}_A} = e^{V^{(\kappa)}_{\log(e^A e^B)}} \text{ for all } A, B \in \mathbb{F}^{(\kappa)} (\mathbb{R}^d).$$

Taking $A = \log (X_{r,s})$ and $B = \log (X_{s,t})$ in this identity while using

$$\log(e^A e^B) = \log (X_{r,s} X_{s,t}) = \log (X_{r,t})$$
gives the flow identity in Eq. \hfill (1.17).

When $\{ V_a : a \in \mathbb{R}^d \}$ does not generate a step-$\kappa$ Nilpotent Lie sub-algebra, the flow property in Eq. \hfill (1.17) will no longer hold. Nevertheless, the following main theorem of this paper gives necessary conditions on $V$ so that there is a unique "flow" on $M$ close to $\mu_{t,s}$.

**Theorem 1.27 (Global Existence).** Suppose that $V : \mathbb{R}^d \to \Gamma (TM)$ is a dynamical system and $g$ is a complete metric on $M$ such that $\| V^{(\kappa)} \|_M + \| \nabla V^{(\kappa)} \|_M < \infty.$ Then Assumption 2 holds (so $\mu_{t,s}$ in Definition 1.25 is well defined) and there exists a unique function $\varphi \in C \left( [0,T]^2 \times M,M \right)$ such that:

(1) $\varphi_{t,t} = \text{Id}$ for all $t \in [0,T],$

(2) $\varphi_{t,s} \circ \varphi_{s,r} = \varphi_{t,r} \forall (s,t) \in [0,T]^2,$ and

(3) there exists a constant $C < \infty$ such that

$$d_M (\varphi_{t,s}, \mu_{t,s}) \leq C \| t - s \|_\theta \forall (s,t) \in [0,T]^2,$$

where $\theta = (\kappa + 1) \alpha > 1$ as in Assumption 7. Moreover, $C (K) := \sup_{(s,t) \in [0,T]^2} \text{Lip}_K (\varphi_{t,s}) < \infty$ for all compact subsets, $K$, of $M$.

Stochastic variants (of one kind or another) of the estimate in Eq. \hfill (1.18) occur frequently, for example see \hfill [7,14,15,26,27,40,41] for a few examples. Theorem 1.27 gives a notion of solution to rough differential equations which is championed by Bailleul, see for example \hfill [2,5]. However, it should be

1 Items 1, and 2, imply that $\varphi_{t,s}$ is invertible with inverse given by $\varphi_{s,t}$ for all $(s,t) \in [0,T]^2$. Thus each $\varphi_{t,s} : M \to M$ is a locally Lipschitz homeomorphism for all $s,t \in [0,T].$
noted that a precursor to this approximate flow notion of solution already appeared in Davie in [17] which gives a similar definition at the level of paths. The next proposition shows that the notion of solution to rough differential equations used in Theorem 1.27 gives a solution in the sense of Davie. For further information in this direction the reader is referred to [10] and [2].

**Proposition 1.28 (Path characterization of solutions).** If we fix \( m_0 \in M \) and let \( y_t := \varphi_{t,0}(m_0) \), then \( y \) satisfies

\[
|f(y_t) - (V_{X_{t},t} f)(y_s)| \leq C(f) |t - s|^\theta
\]

where \( f \) is a smooth function on \( M \), \( \theta = (\kappa + 1) \alpha > 1 \) as in Assumption 1, and \( C(f) \) depends on the bounds on \( f \) and its derivatives to order \( 2\kappa \overline{\kappa} \) over a compact neighborhood of \( y[0,T] \).

**Proof.** Let us show \( y \) satisfies the above estimate. To see this we observe that

\[
f(y_t) = f \circ \varphi_{t,s}(y_s) = f \circ \mu_{t,s}(y_s) + O\left(|t - s|^\theta\right)
\]

and

\[
f \circ e^{V_{\log(X_{st})}(y_s)} = \sum_{k=0}^{\kappa} \frac{1}{k!} \left( V_{\log(X_{st})}^{(\kappa)k} f \right)(y_s) + O\left(|t - s|^\theta\right)
\]

Finally, \( \sum_{k=0}^{\kappa} \frac{1}{k!} V_{\log(X_{st})}^{k} = V_B \) where

\[
B = \sum_{k=0}^{\kappa} \frac{1}{k!} |\log(X_{st})|^{\otimes k} = \pi_{\leq \kappa} \sum_{k=0}^{\kappa} \frac{1}{k!} |\log(X_{st})|^{\otimes k} + O\left(|t - s|^\theta\right)
\]

\[
e^{\log(X_{st})} + O\left(|t - s|^\theta\right) = X_{st} + O\left(|t - s|^\theta\right)
\]

where \( O\left(|t - s|^\theta\right) \in \oplus_{k=\kappa+1}^{2\kappa} [\mathbb{R}^d]^{\otimes k} \). Combining these estimates shows

\[
f(y_t) = (V_{X_{t},t} f)(y_s) + O\left(|t - s|^\theta\right)
\]

\( \square \)

Our next goal is to remove the hypothesis in Theorem 1.27 that \( |V^{(\kappa)}|_M < \infty \). We begin with the following simple lemma.

**Lemma 1.29.** Suppose \((M, g, o)\) is a pointed Riemannian manifold, \( \kappa \in \mathbb{N} \), and \( V : \mathbb{R}^d \to \Gamma(TM) \) is a dynamical system on \( M \). If \( |\nabla V^{(\kappa)}|_M < \infty \) and
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$$C_\kappa (V, o) := \max \left( |\nabla V^{(\kappa)}|_M, |V^{(\kappa)}(o)| \right) < \infty,$$

then

$$|V_A(m)| \leq \left( |V^{(\kappa)}(o)| + d(o,m) |\nabla V^{(\kappa)}|_M \right) |A|$$

$$\leq C_\kappa (V, o) |A| (1 + d(o,m))$$

for all $$A \in F^{(\kappa)}(\mathbb{R}^d)$$ and $$m \in M.$$  

Proof. From [18, Lemma 2.10] with $$X(m) := V_A(m)$$ and $$p = o$$ states that

$$||V_A(m)|| − ||V_A(o)|| \leq |\nabla V_A|_{d(o,m)}.$$ 

The estimates in Eq. (1.19) are now an elementary consequence of the previous inequality. □

**Notation 1.30.** Let $$(M,o)$$ be a pointed manifold. To each Riemannian metric, $$g,$$ on $$M,$$ let $$\bar{g}$$ be the continuous Riemannian metric defined by

$$\bar{g}(v,w) = \left( \frac{1}{1 + d(o,m)} \right)^2 g(v,w) \ \forall \ v,w \in T_m M \text{ and } m \in M$$

and let $$\bar{d}$$ be the length metric associated to the continuous metric.

**Corollary 1.31.** Suppose that $$V : \mathbb{R}^d \to \Gamma(TM)$$ is a dynamical system satisfying Assumption[3] and there exists a complete metric, $$g,$$ on $$M$$ such that

$$|\nabla V^{(\kappa)}|_M < \infty.$$ 

Then there exists a unique function $$\varphi \in C([0,T]^2 \times M,M)$$ such that;

1. $$\varphi_{t,t} = Id$$ for all $$t \in [0,T],$$
2. $$\varphi_{t,s} \circ \varphi_{s,r} = \varphi_{t,r} \ \forall \ 0 \leq r \leq s \leq t \leq T,$$ and
3. there exists a constant $$C < \infty$$ such that

$$\bar{d}_M(\varphi_{t,s},\mu_{t,s}) = \sup_{m \in M} \bar{d}(\varphi_{t,s}(m),\mu_{t,s}(m))$$

$$\leq C |t−s|^\theta \ \forall \ (s,t) \in [0,T]^2,$$

where $$\bar{d}$$ is as in Notation 1.30.

Proof. This is an immediate consequence of Theorem[1.27] in conjunction with Corollary 6.9 and Proposition 6.10 of Appendix 6 below. □

1.4. Examples.

**Example 1.32.** Let $$M = G$$ be a Lie group with $$\mathfrak{g} := T_e G$$ being the Lie algebra and define $$V_\xi = \tilde{\xi} \in \Gamma(TG)$$ for all $$\xi \in \mathfrak{g}.$$ Then $$V$$ satisfies the hypothesis of Theorem[1.27] relative to any left invariant Riemannian metric on $$G.$$

\[\text{From Lemma 1.29, the assumption that } |\nabla V^{(\kappa)}|_M < \infty \text{ implies } V_A(m) \text{ has at most linear growth in } m \text{ for all } A \in F^{(\kappa)}(\mathbb{R}^d).\]
Example 1.33. Let $M = \mathbb{R}^D$ with the standard Euclidean Riemannian metric. If $V_i \in \Gamma(T\mathbb{R}^D)$ are bounded vector fields on $\mathbb{R}^D$ with bounded derivative to sufficiently high order, then $V$ satisfies the hypothesis of Theorem 1.27.

The proofs of the next proposition and corollary may be found in Section 5 at the end of the paper.

Proposition 1.34. Suppose that $(M^d, g)$ is a complete Riemannian manifold, $T\mathbb{R}^d$ is parallelizable, and $V : \mathbb{R}^d \to \Gamma(TM)$ is a dynamical system such that

\begin{equation}
(1.20) \quad g(V_a(m), V_b(m)) = a \cdot b \text{ for all } a, b \in \mathbb{R}^d \text{ and } m \in M.
\end{equation}

Further let $Q(a, b) \in C^\infty(M, \mathbb{R}^d)$ is determined by

\begin{equation}
(1.21) \quad [V_a, V_b] = V_{Q(a, b)} \text{ for all } a, b \in \mathbb{R}^d.
\end{equation}

[Notice that $Q : \mathbb{R}^d \times \mathbb{R}^d \to C^\infty(M, \mathbb{R}^d)$ is a skew-symmetric bilinear map.]

If $V_{a_1} \ldots V_{a_k} Q(a, b) \in C^\infty(M, \mathbb{R}^d)$ is bounded for all $1 \leq k \leq \kappa - 1$, $a, b \in \mathbb{R}^d$, and $a_j \in \mathbb{R}^d$, then $V$ satisfies the hypothesis of Theorem 1.27.

We end the introduction with some necessary conditions that the rough version of Cartan’s rolling map has global in time solutions. Let $(M^d, g)$ be a Riemannian manifold of dimension $d$, $O(d)$ be the Lie group of orthogonal $d \times d$ matrices, and $so(d)$ be the Lie algebra of $O(d)$ consisting of recall skew symmetric $d \times d$ - matrices. In order to describe Cartan’s rolling map we need to recall the following orthogonal frame bundle notations.

Notation 1.35 (Orthogonal frame bundle). Let $\pi : O(M) \to M$ be the principal bundle of orthogonal frames, i.e.

\[ O(M) = \cup_{m \in M} O_m(M) = \cup_{m \in M} [\pi^{-1}(\{m\})] \]

where for each $m \in M$, $O_m(M)$ is the set of isometries, $u : \mathbb{R}^d \to T_m M$. To each $A \in so(d)$, let $A^* \in \Gamma(TO(M))$ be the vertical vector field defined by

\[ A^*(u) := \frac{d}{dt} |_0 u e^{tA} \text{ for all } u \in O(M) \]

and to each $a \in \mathbb{R}^d$ let $B_a$ be the horizontal vector field defined by

\[ B_a(u) := \frac{d}{dt} |_0 \frac{1}{t} (\sigma) u \in T_u O(M) \]

where $\frac{1}{t} (\sigma) u$ denote parallel translation along any smooth path, $\sigma(t) \in M$, such that $\sigma(0) = ua \in T_m M$.

Given a smooth path $x(t) \in \mathbb{R}^d$ and $u_o \in O_{m_o}(M)$, Cartan’s rolling of $x$ onto $M$ is the path $\sigma(t) = \pi(u(t)) \in M$ where $u(t) \in O(M)$ satisfies the ODE

\begin{equation}
(1.22) \quad \dot{u}(t) = B_{\dot{x}(t)}(u(t)) \text{ with } u(0) = u_o \in O(M).
\end{equation}
Thus we want to consider the existence of solutions to the rough version of Eq. (1.22). We will in fact consider the more general ODE for $u(t) \in O(M)$;

$$\dot{u}(t) = B_{\dot{x}(t)}(u(t)) + A^*(u(t))$$

with $u(0) = u_0 \in O(M)$.

where now $(x(t), A(t)) \in \mathbb{R}^d \times so(d)$ is a given path.

**Corollary 1.36.** Suppose that $(M, g)$ is a complete Riemannian manifold with bounded geometry to order $\kappa - 1$, i.e. $\nabla^k R$ is bounded for $0 \leq k \leq \kappa - 1$, then for any $\alpha$- Hölder rough path, $X_{s,t} \in G_{geo}^{(\kappa)}(\mathbb{R}^d \times so(d))$ with $\theta = \alpha(\kappa + 1) > 1$, the rough version of Eq. (1.23) has a unique solution defined for all $t \in [0, T]$.

The proof of this corollary will amount to showing that the dynamical system, $V : \mathbb{R}^d \times so(d) \to \Gamma(\text{T}O(M))$, defined by

$$V_{(a,A)}(u) := B_{\dot{a}}(u) + A^*(u)$$

satisfies the hypothesis of Proposition 1.34 under the given bounded geometry assumptions.

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2. **A metric space almost multiplicative function theorem**

This section is devoted to a version of the Lyons’ almost multiplicative function theorem in the context of Bi-Lipschitz maps on an abstract complete metric space which generalizes those in [5, Theorem 2.1], [21], [20], [25], and [33] in reverse chronological order. As mentioned in the introduction this topic is also taken up in [8,9] and [42]. Ideas of this form for smooth function of time are already prevalent in numerical and functional analysis literature, see for example the review article [16].

2.1. **Function Space Metrics.**

**Remark 2.1.** Suppose that $(X, \tau)$ is a topological space, $(M, d)$ is a complete metric space, and $\{f_n\}_{n=1}^\infty \subset C(X, M)$ satisfy

$$\lim_{n,m \to \infty} \sup_{x \in X} d(f_n(x), f_m(x)) = 0.$$ 

Then, as is well known, there exists $f \in C(X, M)$ such that

$$\lim_{n \to \infty} \sup_{x \in X} d(f(x), f_n(x)) = 0.$$ 

**Notation 2.2.** When $(M, d)$ is a metric space let $\text{Homeo}(M)$ denote those $f \in C(M, M)$ which are homeomorphisms. Also for $f, g \in C(M, M)$, let $d_M(f, g) := \sup_{m \in M} d(f(m), g(m))$ as in Eq. (1.14).
Definition 2.3. We say \( f \in C(M, M) \) is **Lipschitz** if there exists \( K = K(f) < \infty \) such that
\[
d(f(m), f(m')) \leq K d(m, m') \quad \forall m, m' \in M.
\]
We denote the best such constant by \( \text{Lip}(f) \), i.e.
\[
\text{Lip}(f) := \sup_{m \neq m'} \frac{d(f(m), f(m'))}{d(m, m')}
\]
We will write \( \text{Lip}(f) = \infty \) if \( f \) is not Lipschitz.

Remark 2.4. If \( f, g, g_1, g_2 \in C(M, M) \), then
\[
d_M(g_1 \circ f, g_2 \circ f) \leq d_M(g_1, g_2)
\]
while for \( m \in M \),
\[
d(f \circ g_1(m), f \circ g_2(m)) \leq \text{Lip}(f) d(g_1(m), g_2(m)) \leq \text{Lip}(f) d_M(g_1, g_2)
\]
and hence
\[
d_M(f \circ g_1, f \circ g_2) \leq \text{Lip}(f) \cdot d_M(g_1, g_2).
\]
Also for \( m, m' \in M \),
\[
d(f \circ g(m), f \circ g(m')) \leq \text{Lip}(f) d(g(m), g(m')) \leq \text{Lip}(f) \text{Lip}(g) d(m, m')
\]
from which it follows that
\[
\text{Lip}(f \circ g) \leq \text{Lip}(f) \text{Lip}(g).
\]

More generally we have the following extensions of these results.

Proposition 2.5. Suppose that \( f_j, g_j \in C(M, M) \) for \( 1 \leq j \leq n \), then
\[
d_M(f_n \circ \cdots \circ f_1, g_n \circ \cdots \circ g_1)
\leq \sum_{k=1}^{n} [\text{Lip}(f_n \circ \cdots \circ f_{k+1}) \wedge \text{Lip}(g_n \circ \cdots \circ g_{k+1})] d_M(f_k, g_k)
\]
and in particular,
\[
d_M(f_1 \circ \cdots \circ f_n, g_1 \circ \cdots \circ g_n) \leq \sum_{k=1}^{n} [\text{Lip}(f_n \circ \cdots \circ f_{k+1})] d_M(f_k, g_k).
\]
The convention used above is that \( \text{Lip}(f_n \circ \cdots \circ f_{k+1}) = 1 \) when \( k = n \).

Proof. The proof is by induction on \( n \) where in the case \( n = 1 \), there is nothing to prove. The induction step is as follows;
\[
d_M(f_n \circ \cdots \circ f_1, g_n \circ \cdots \circ g_1)
\leq d_M(f_n \circ \cdots \circ f_2 \circ f_1, f_n \circ \cdots \circ f_2 \circ g_1)
\quad + d_M(f_n \circ \cdots \circ f_2 \circ g_1, g_n \circ \cdots \circ g_2 \circ g_1)
\leq \text{Lip}(f_n \circ \cdots \circ f_2) d_M(f_1, g_1) + d_M(f_n \circ \cdots \circ f_2, g_n \circ \cdots \circ g_2).
\]
By interchanging the roles of $f$ and $g$ we further can show, 

$$d_M (f_n \circ \cdots \circ f_1, g_n \circ \cdots \circ g_1) \leq \text{Lip} (g_n \circ \cdots \circ g_2) d_M (f_1, g_1) + d_M (f_n \circ \cdots \circ f_2, g_n \circ \cdots \circ g_2)$$

which combined with the previous inequality allows us to conclude that

$$d_M (f_n \circ \cdots \circ f_1, g_n \circ \cdots \circ g_1) \leq \text{Lip} (f_n \circ \cdots \circ f_2) \wedge \text{Lip} (g_n \circ \cdots \circ g_2) d_M (f_1, g_1) + d_M (f_n \circ \cdots \circ f_2, g_n \circ \cdots \circ g_2)$$

and the induction step is complete. \hfill \Box

2.2. **Oriented Partitions.** For $s, t \in [0, T]$, let

$$J(s, t) := [\min(s, t), \max(s, t)]$$

be the interval between $s$ and $t$.

**Definition 2.6** (Oriented Partitions). If $(s, t) \in [0, T]^2$ with $s \neq t$, let $P(s, t)$ denote the **oriented partitions**, $\pi$, of $J(s, t)$ where $\pi \in P(s, t)$ iff $\pi = (s_k)_{k=0}^n = (s_0, \ldots, s_n)$ is an ordered subset of $J(s, t)$ such that

$$s = s_0 < s_1 < \cdots < s_n = t \text{ when } t > s$$

and

$$t = s_n < s_{n-1} < \cdots < s_0 = s \text{ when } s > t.$$ 

We further let $\{\pi\} := \{s_k : 0 \leq k \leq n\}$ be the unordered points in $\pi$ and $\#(\pi) = n$ which is the same as the number of connected components of $J(s, t) \setminus \{\pi\}$. [We will often abuse notation and simply refer to an element, $\pi \in P(s, t)$, as an oriented partition of $J(s, t)$ or more simply as a partition of $J(s, t)$.]

**Notation 2.7.** For $\varepsilon \in (0, 1/2]$ and $s, t \in [0, T]$, let $J_\varepsilon(s, t)$ be the **middle** $1-2\varepsilon$ **subinterval** of $J(s, t)$ defined by

$$J_\varepsilon(s, t) := J(s + \varepsilon(t-s), t - \varepsilon(t-s)).$$

[If we let $J_\varepsilon = J_\varepsilon(0, 1) = [\varepsilon, 1 - \varepsilon]$, then we may express $J_\varepsilon(s, t)$ as $J_\varepsilon(s, t) := s + (t-s)J_\varepsilon$.]

![Figure 1. The middle (1-2\varepsilon) fraction subinterval when s < t. The figure for s > t is similar.](image)

**Definition 2.8** ($\varepsilon$-special partitions). Let $s, t \in [0, T]$ and $\varepsilon \in (0, 1/2]$, $\pi \in P(s, t)$ be a partition of $J(s, t)$. We say a point $u \in \pi$ is **$\varepsilon$-special** if $u \in J_\varepsilon(s, t) \cap \pi$ and then we define the notion of $\pi$ being an **$\varepsilon$-special partition** inductively on $n = \#(\pi) \geq 1$ as follows.
(1) If \( n = 1 \) then \( \pi \) is \( \varepsilon \) special for any choice of \( 0 < \varepsilon \leq 1/2 \).

(2) Assuming that \( \varepsilon \)-special has been defined for partitions with \( \#(\pi) = n \) for some \( n \geq 2 \) and suppose \( \pi = (s_0, s_1, \ldots, s_{n+1}) \in \mathcal{P}(s, t) \) with \( \#(\pi) = n + 1 \), then \( \pi \) is \( \varepsilon \)-special if there exists a \( 1 \leq p \leq n \) such that: 1) \( u := s_p \in \pi \cap J_{\varepsilon}(s, t) \), 2) \( \pi_{\leq u} := (s_0, s_1, \ldots, s_p) \) is an \( \varepsilon \)-special partition of \( J(s,u) \), and 3) \( \pi_{\geq u} := (s_p, s_{p+1}, \ldots, s_{n+1}) \) is an \( \varepsilon \)-special partitions of \( J(u,t) \).

**Notation 2.9** (Uniform Partitions). For \((s,t) \in [0,T]^2\) and \( n \in \mathbb{N} \), let \( \pi^n(s,t) = (s_0, \ldots, s_n) \in \mathcal{P}(s,t) \) where

\[
(2.2) \quad s_i := s + \frac{i}{n}(t-s) \quad \text{for} \quad 0 \leq i \leq n
\]

be the uniform partition of \( J(s,t) \) with \( n \) equal subdivisions. We also let \( \pi^{(n)}(s,t) := \pi^{2^n}(s,t) = (s_0, \ldots, s_{2^n}) \in \mathcal{P}(s,t) \) where

\[
(2.3) \quad s_i := s + i2^{-n}(t-s) \quad \text{for} \quad 0 \leq i \leq 2^n
\]

so that \( \pi^{(n)}(s,t) \) is the uniform partition of \( J(s,t) \) with \( 2^n \)-subdivisions.

**Example 2.10.** For \((s,t) \in [0,T]^2\) and \( n \in \mathbb{N} \) the partition \( \pi^{(n)}(s,t) \) in Eq. \((2.3)\) \( \varepsilon = 1/2 \)-special. There are also many sub-partitions of \( \pi^{(n)}(s,t) \) which are still \( 1/2 \)-special. For example if \( n \geq 2 \), then

\[
\pi = \{s\} \cup \left\{s + (t-s)k2^{-n} : 2^{(n-1)} \leq k \leq 2^n\right\}
\]

is still \( 1/2 \) uniform.

**Lemma 2.11.** For \((s,t) \in [0,T]^2\) and \( n \in \mathbb{N} \), the uniform partition, \( \pi^n = \pi^n(s,t) \), of \( J(s,t) \) is \( \varepsilon = 1/3 \)-special for any \( n \in \mathbb{N} \).

*Proof.* The proof is by induction on \( n = \#(\pi) \). For \( n = 1 \), \( \pi^1 = \{s,t\} \) and there is nothing to prove. For \( n = 2 \) the interior point of \( \pi^2 \) is \( u = s + (t-s)/2 \) is the mid-point of \( \pi^2 \) and hence \( \pi^2 \) is \( 1/2 \)-special. For \( n = 3 \), we let \( u = s + (t-s)/3 \in J_{1/3}(s,t) \) so that \( u \) is \( 1/3 \)-special and so by \( n = 1 \) and \( n = 2 \) cases already discussed, \( \pi^3 \) is \( 1/3 \)-special. For the induction step, suppose we have shown, for some \( n \in \mathbb{N} \), that every uniform partition, \( \pi \) of any compact interval \( \#(\pi) \leq n \) is \( 1/3 \)-special. Then for \( \pi^n = \pi^{n+1}(s,t) \in \mathcal{P}(s,t) \), we let

\[
u = \left\{ \begin{array}{ll}
s + (t-s)\frac{n}{2} & \text{if} \quad n \text{ is odd} \\
s + (t-s)\frac{n/2}{n+1} & \text{if} \quad n \text{ is even}\end{array} \right. \in \pi^{n+1}.
\]

So when \( n \) is odd \( u \) is \( 1/2 \)-special and when \( n \) is even, \( u \in \varepsilon_n := \frac{1}{2 \pi n+1} \)-special. As \( \frac{d}{dn} \frac{n}{n+1} = \frac{1}{(n+1)^2} > 0 \) it follows that \( \varepsilon_n \geq \varepsilon_2 = 1/3 \) so that in all cases \( u \) is at least \( 1/3 \)-special. The result now follows by applying the induction hypothesis applied to remaining uniform partitions of \( J(s,u) \) and \( J(u,t) \) respectively. \qed
Definition 2.12. For \( \theta > 1 \) and \( \varepsilon \in (0, 1/2] \), let

\[
\gamma(\varepsilon, \theta) := \max_{x \in J_\varepsilon(0,1)} \left[ x^\theta + (1 - x)^\theta \right] = \varepsilon^\theta + (1 - \varepsilon)^\theta < 1.
\]

The point of \( \varepsilon \)-special partitions is that in the arguments below we will often arrive at an estimate for a quantity, \( Q \), of the form

\[
Q \leq k \left[ |t - u|^\theta + |u - s|\theta \right] = k \left[ \beta^\theta + (1 - \beta)^\theta \right] |t - s|\theta
\]

where \( u \in \pi \setminus \{s, t\} \) and \( \beta := |t - u| / |t - s| \). If we now further assume that \( u \in \pi \cap J_\varepsilon(s, t) \) then we know \( \beta \in J_\varepsilon(0, 1) \) and we will have

\[
Q \leq k \max_{\beta \in J_\varepsilon(0,1)} \left[ \beta^\theta + (1 - \beta)^\theta \right] |t - s|^\theta = k \gamma(\varepsilon, \theta) |t - s|^\theta.
\]

The pre-factor, \( \gamma(\varepsilon, \theta) \), being less than one will play a crucial role in the arguments to follow.

2.3. Approximate Flows.

Definition 2.13 (Pre-flows). A pre-flow is a function, \( (s, t, m) \rightarrow \mu_{t,s}(m) \) in \( C\left( [0, T]^2 \times M, M \right) \), such that \( \mu_{t,s} \in \text{Homeo}(M) \), \( \mu_{s,t} = \mu_{t,s}^{-1} \), and \( \mu_{t,t} = \text{Id}_M \) for all \( s, t \in [0, T] \).

Notation 2.14. Suppose that \( \mu \in C\left( [0, T]^2 \times M, M \right) \), \( s, t \in [0, T] \), and \( \pi = (s_0, \ldots, s_n) \in \mathcal{P}(s, t) \) is an (oriented) partition \( J(s, t) \). For any \( 0 \leq l < k \leq n \), let

\[
\mu^\pi_{s_k, s_l} := \mu_{s_k, s_{k-1}} \circ \cdots \circ \mu_{s_{l+2}, s_{l+1}} \circ \mu_{s_{l+1}, s_l}.
\]

and in particular,

\[
\mu^\pi_{t,s} = \mu_{s_n, s_{n-1}} \circ \cdots \circ \mu_{s_2, s_1} \circ \mu_{s_1, s_0}.
\]

Notation 2.15. For \( n \in \mathbb{N} \) and \( (s, t) \in [0, T]^2 \) let

\[
\mu^{\pi_n}_{t,s} := \mu_{t,s}^{\pi_n(s, t)} \text{ and } \mu^{(n)}_{t,s} = \mu^{\pi_n(s, t)}
\]

where \( \pi_n(s, t) \) and \( \pi^{(n)}(s, t) = \pi^{2n}(s, t) \) are the uniform partition of \( J(s, t) \) as in Notation 2.3.

The next lemma is a fairly direct extension of Lemma 2.4 in [5].

Lemma 2.16 (Local Trotter bounds). Let \( \mu \in C\left( [0, T]^2 \times M, M \right) \) be a pre-flow as in Definition 2.13. Assume there exists \( \theta > 1 \), \( c < \infty \), a continuous increasing function, \( k : [-T, T] \rightarrow [0, \infty) \) such that \( k(0) := \lim_{t \to 0} k(t) = 0 \) and for all \( (s, t) \in [0, T]^2 \),

\[
\text{Lip}(\mu_{t,s}) \leq (1 + k(t - s)) \text{ and } \sup_{u \in J(s,t)} d_M(\mu_{tu} \circ \mu_{us}, \mu_{ts}) \leq c |t - s|^\theta.
\]
Let \( \varepsilon \in (0, 1/2] \) be given and for \( \delta \in (0, T] \), let \( k^* (\delta) := \max (k (\delta), k (-\delta)) \).

If \( \delta > 0 \) is chosen so that

\[
p (\delta) := \gamma (\theta, \varepsilon) + k^* (\delta) < 1
\]

and \( L \geq \frac{c}{1-p(\delta)} \), then for all \( s, t \in [0, T] \) with \( |t - s| \leq \delta \) and any \( \varepsilon \)-special partition \( (\pi) \) of \( J (s, t) \),

\[
d_M (\mu_{t,s}^\pi, \mu_{t,s}) \leq L |t - s|^\theta.
\]

Proof. The proof is by induction on \( r (\pi) = \# (\pi) - 1 \). When \( r = 0 \), \( \mu_{t,s}^\pi = \mu_{t,s} \) and there is nothing to prove. If \( r = 1 \), then \( \pi = (s, u, t) \) if \( s < t \) or \( \pi = (t, u, s) \) if \( t < s \) for some \( u \in J_\varepsilon (s, t) \). It then follows by assumption that

\[
d_M (\mu_{t,s}^\pi, \mu_{t,s}) = d_M (\mu_{t,u} \circ \mu_{u,s}, \mu_{t,s}) \leq c |t - s|^\theta
\]

from which it follows that we are going to need to choose \( L \geq c \).

Suppose that the estimate in Eq. \((2.8)\) is known to hold for all \( s, t \in [0, T] \) and all \( \varepsilon \)-special partitions \( \pi \in \mathcal{P} (s, t) \) with \( r (\pi) \leq r_0 \) for some \( r_0 \in \mathbb{N} \) and suppose that \( \pi \) is an \( \varepsilon \)-special partition of \( J (s, t) \) with \( \# (\pi) = r_0 + 1 \). By definition of \( \pi \) being \( \varepsilon \)-special, there exists \( u \in \pi \cap J_\varepsilon (s, t) \) such that \( \pi \cap J (s, u) \) and \( \pi \cap J (u, t) \) are \( \varepsilon \)-special partitions of \( J (s, u) \) and \( J (u, t) \) respectively. Using \( \mu_{t,s}^\pi = \mu_{t,u}^\pi \circ \mu_{u,s}^\pi \), the triangle inequality, the assumptions, and the induction hypothesis it follows that

\[
d_M (\mu_{t,s}^\pi, \mu_{t,s}) = d_M (\mu_{t,u} \circ \mu_{u,s}, \mu_{t,u} \circ \mu_{u,s})
\]

\[
\leq d_M (\mu_{t,u} \circ \mu_{u,s}, \mu_{t,u} \circ \mu_{u,s}) + d_M (\mu_{t,u} \circ \mu_{u,s}, \mu_{t,u} \circ \mu_{u,s})
\]

\[
\leq d_M (\mu_{t,u} \circ \mu_{u,s}, \mu_{t,u} \circ \mu_{u,s}) + \text{Lip} (\mu_{t,u}) d_M (\mu_{u,s}^\pi, \mu_{u,s})
\]

\[
\leq L |t - u|^\theta + (1 + k (t - u)) L |u - s|^\theta + c |t - s|^\theta
\]

\[
= \left[ |t - u|^\theta + |u - s|^\theta \right] L + k (t - u) |u - s|^\theta L + c |t - s|^\theta
\]

\[
\leq [\gamma (\theta, \varepsilon) + k (t - s)] L |t - s|^\theta + c |t - s|^\theta.
\]

We now choose \( \delta > 0 \) so that \( p (\delta) := \gamma (\theta, \varepsilon) + k^* (\delta) < 1 \) and then choosing \( L \) sufficiently large (i.e. \( L \geq \frac{c}{1-p(\delta)} \)), such that \( p (\delta) L + c \leq L \), it will follow for \( (s, t) \in [0, T]^2 \) with \( |t - s| < \delta \), that

\[
d_M (\mu_{t,s}^\pi, \mu_{t,s}) \leq (p (\delta) L + c) |t - s|^\theta \leq L |t - s|^\theta
\]

which completes the inductive step.

The next theorem extends the previous lemma by removing the restriction on \( s \) and \( t \) being close to one another.
Theorem 2.17 (Global product bounds). Let \( \mu \in C\left([0, T]^2 \times M, M\right) \) be as in Lemma 2.16, i.e. \( \mu \) satisfies the estimates in Eq. (2.6). Then for all \( \varepsilon \in (0, 1/2] \) there exists \( L \varepsilon < \infty \) such that
\[
(2.10) \quad d_M(\mu^\pi_{t,s}, \mu_{t,s}) \leq L \varepsilon |t-s|^{\theta} \quad \text{for all } s, t \in [0, T].
\]

Proof. Let \( K := \max_{|t| \leq T} k(t) \), \( \delta = \delta(\varepsilon) \) be as in Lemma 2.16, \( s, t \in [0, T] \) with \( \alpha := t-s \neq 0 \), and suppose that \( \pi \) is an \( \varepsilon \)-special partition of \( J(s, t) \).

Since \( \pi \) is \( \varepsilon \)-special, there exists \( u \in J_\varepsilon(s, t) \cap \pi \), i.e. \( u \in \pi \) and \( u \) lies between \( s + \varepsilon \alpha \) and \( t - \varepsilon \alpha \) and therefore,
\[
|t - u| \leq |t - (s + \varepsilon \alpha)| = |\alpha| (1 - \varepsilon) \quad \text{and}
|s - u| \leq |s - (t - \varepsilon \alpha)| = |\alpha| (1 - \varepsilon).
\]

Let us now assume that \( |t - s| (1 - \varepsilon) \leq \delta \), i.e.
\[
|\alpha| = |t - s| \leq \delta_1 := (1 - \varepsilon)^{-1} \delta.
\]

Then for \( u \in J_\varepsilon(s, t) \cap \pi \) as above we will have \( |t - u| \leq \delta \) and \( |s - u| \leq \delta \) and hence by the triangle inequality and Lemma 2.16,
\[
d(\mu^\pi_{t,s}, \mu_{t,s}) = d(\mu^\pi_{t,u} \circ \mu^\pi_{u,s}, \mu_{t,u})
\leq d(\mu^\pi_{t,u} \circ \mu^\pi_{u,s}, \mu_{t,u} \circ \mu_{u,s}) + d(\mu_{t,u} \circ \mu_{u,s}, \mu_{t,s})
\leq \text{Lip}(\mu_{t,u}) d(\mu^\pi_{u,s}, \mu_{u,s}) + d(\mu^\pi_{t,u}, \mu_{t,u}) + d(\mu_{t,u} \circ \mu_{u,s}, \mu_{t,s})
\leq (1 + K) L |u - s|\theta + L |t - u|\theta + c |t - s|\theta \leq L_1 |t - s|\theta
\]
where \( L_1 = (2 + K) L + c \).

We may now repeat this same argument with \( \delta \) replaced by \( \delta_1 \) to find, for \( s, t \in [0, T] \) with
\[
|t - s| \leq \delta_2 := (1 - \varepsilon)^{-1} \delta_2 = (1 - \varepsilon)^{-2} \delta,
\]
that
\[
d(\mu^\pi_{t,s}, \mu_{t,s}) \leq \text{Lip}(\mu_{t,u}) d(\mu^\pi_{u,s}, \mu_{u,s}) + d(\mu^\pi_{t,u}, \mu_{t,u}) + d(\mu_{t,u} \circ \mu_{u,s}, \mu_{t,s})
\leq (1 + K) L_1 |u - s|\theta + L_1 |t - u|\theta + c |t - s|\theta \leq L_2 |t - s|\theta
\]
where \( L_2 := (2 + K) L_1 + c \). Hence, by induction, if \( s, t \in [0, T] \) with \( |t - s| \leq \delta_n = (1 - \varepsilon)^{-n} \delta \), then
\[
d(\mu^\pi_{t,s}, \mu_{t,s}) \leq L_n |t - s|\theta
\]
where \( L_n \) is defined inductively by \( L_n = (2 + K) L_{n-1} + c \) with \( L_0 = L \). To complete the proof we need only use this estimate with \( n \) sufficiently large so that \( \delta_n = (1 - \varepsilon)^{-n} \delta \geq T \). \( \Box \)

Lemma 2.18. Suppose \( \mu \in C\left([0, T]^2 \times M, M\right) \) is a pre-semi group and for some \( \varepsilon \in (0, 1/2] \), there exists a \( \delta > 0 \) and \( C < \infty \) such that \( \text{Lip}(\mu^\pi_{t,s}) \leq C \) for all \( (s, t) \in [0, T]^2 \) with \( |t - s| \leq \delta \) and \( \varepsilon \)-special partitions, \( \pi \), of \( J(s, t) \). Then there exists a constant \( C' < \infty \) such that \( \text{Lip}(\mu^\pi_{t,s}) \leq C' \) for
all \((s, t) \in [0, T]^2\) and all \(\varepsilon\)-special partitions, \(\pi\), of \(J(s, t)\), i.e. we may drop the restriction that \(|t - s| \leq \delta\).

**Proof.** The proof is very similar to the proof of Theorem 2.17. Let \(\delta_1 = (1 - \varepsilon)^{-1} \delta\) and \((s, t) \in [0, T]^2\) with \(|t - s| \leq \delta_1\) and \(\pi\) be an \(\varepsilon\)-special partition of \(J(s, t)\). As in the proof of Theorem 2.17, for \(u \in J(s, t) \cap \pi\) (which exists as \(\pi\) is \(\varepsilon\)-special) we have both \(|u - s| \leq \delta\) and \(|t - u| \leq \delta\) and hence

\[
\operatorname{Lip}(\mu^\pi_{t,u}) = \operatorname{Lip}(\mu^\pi_{t,u} \circ \mu^\pi_{u,s}) \leq \operatorname{Lip}(\mu^\pi_{t,u}) \cdot \operatorname{Lip}(\mu^\pi_{u,s}) \leq C^2.
\]

We may now repeat this procedure with \(\delta_2 = (1 - \varepsilon)^{-1} \delta_1 = (1 - \varepsilon)^{-2} \delta\) and \((s, t) \in [0, T]^2\) with \(|t - s| \leq \delta_2\) and \(\pi\) is an \(\varepsilon\)-special partition of \(J(s, t)\) in order to find \(\operatorname{Lip}(\mu^\pi_{t,u}) \leq C^4\). Continuing in this way inductively, if \(|t - s| \leq \delta_n = (1 - \varepsilon)^{-n} \delta\) and \(\pi\) is an \(\varepsilon\)-special partition of \(J(s, t)\), then

\[
\operatorname{Lip}(\mu^\pi_{t,u}) \leq C^{2^n} < \infty.
\]

It then follows that \(\operatorname{Lip}(\mu^\pi_{t,u}) \leq C^n\) for all \((s, t) \in [0, T]^2\) where \(C^n := C^{2^n}\) provided we choose \(n \in \mathbb{N}\) so that \(\delta_n = (1 - \varepsilon)^{-n} \delta \geq T\). \(\square\)

The next two corollaries are easy consequences of Proposition 2.5.

**Corollary 2.19.** Suppose that \(\mu_{t,s}\) and \(\nu_{t,s}\) are two pre-semigroups on \(M\), see Definition 2.13. If \(\pi = \{s_k\}_{k=0}^n\) is a partition of \(J(s, t)\), then

\[
d(\mu^\pi_{t,s}, \nu^\pi_{t,s}) \leq \sum_{k=1}^{n} \left( \operatorname{Lip}(\mu^\pi_{t,s_k}) \wedge \operatorname{Lip}(\nu^\pi_{t,s_k}) \right) d(\mu_{s_k, s_{k-1}}, \nu_{s_k, s_{k-1}}).
\]

**Corollary 2.20.** Let \(\pi = (s_0, \ldots, s_n) \in \mathcal{P}(s, t)\) be a partition of \(J(s, t)\) and to each \(1 \leq k \leq n\) let \(\Lambda_k \in \mathcal{P}(s_{k-1}, s_k)\) be a partition of \(J(s_{k-1}, s_k)\) and let \(\pi^\ast \in \mathcal{P}(s, t)\) be the unique oriented partition of \(J(s, t)\) such that \(\{\pi^\ast\} = \cup_{k=1}^{n} \{\Lambda_k\}\). Then

\[
d(\mu^\pi_{t,s}, \mu^\pi_{t,s}) \leq \sum_{k=1}^{n} \operatorname{Lip}(\mu^\pi_{t,s_k}) d(\mu_{s_k, s_{k-1}}, \mu_{s_k, s_{k-1}}).
\]

**Proof.** Again Eq. (2.11) follows from Proposition 2.5 along with the following two identities,

\[
\mu^\pi_{t,s} = \mu_{s_0, s_{n-1}} \circ \cdots \circ \mu_{s_2, s_1} \circ \mu_{s_1, s_0} \quad \text{and} \quad \mu^\pi_{t,s} = \mu^{\Lambda_n}_{s_n, s_{n-1}} \circ \cdots \circ \mu^{\Lambda_2}_{s_2, s_1} \circ \mu^{\Lambda_1}_{s_1, s_0}.
\]

\(\square\)

**Corollary 2.21.** Let \(\pi = (s_0, \ldots, s_n) \in \mathcal{P}(s, t)\) be an oriented partition of \(J(s, t)\) and

\[
\pi^\ast = (s_0, s_1^*, s_1, s_2^*, s_2, \ldots, s_{n-1}^*, s_{n-1}, s_n)
\]

where \(s_k^*\) is chosen arbitrarily to lie between \(s_{k-1}\) and \(s_k\) for \(1 \leq k \leq n\). Then

\[
d(\mu^\pi_{t,s}, \mu^\pi_{t,s}) \leq \sum_{k=1}^{n} \operatorname{Lip}(\mu^\pi_{t,s_k}) d(\mu_{s_k, s_{k-1}}, \mu_{s_k, s_{k-1}}) \circ \mu_{s_k, s_{k-1}}.
\]
Proof. This is a very special case of Corollary 2.20 where $\Lambda_k := (s_{k-1}, s_k)$ for $1 \leq k \leq n$. □

Corollary 2.22. Suppose $\mu \in C \left( [0,T]^2 \times M, M \right)$ is a pre-flow and there exists $C < \infty$ such that $\text{Lip} \left( \mu_{t,s}^{(n)} \right) \leq C$ and and $L < \infty$ such that Eq. (2.10) holds for all $(s,t) \in [0,T]^2$ and all $\frac{1}{2}$-special partitions of $J(s,t)$. Let $n \in \mathbb{N}$ and $\pi_n := \pi^n(s,t)$ be an oriented uniform partition of $J(s,t)$ and $\pi_n^* := \bigcup_{k=1}^{n} \Lambda_k$ where for each $1 \leq k \leq n$, $\Lambda_k = \pi^n_{k-1} (s_{k-1}, s_k) \in \mathcal{P}(s_{k-1}, s_k)$ is a uniform partition of $J(s_{k-1}, s_k)$ with any number of subdivisions, $n_k$, which can depend on $k$. Then

$$d \left( \mu_{t,s}^{(n)}, \mu_{t,s}^{(n)} \right) = d \left( \mu_{t,s}^{(n)}, \mu_{t,s}^{(n)} \right) \leq LC \left| t - s \right| \left( \frac{1}{n} \right)^{\theta - 1}$$

Proof. By Corollary 2.20 along with Eq. (2.10) and the assumption that $\sup_{s,t \in [0,1]} \text{Lip} \left( \mu_{t,s}^{(n)} \right) \leq C$, we find

$$d \left( \mu_{t,s}^{(n)}, \mu_{t,s}^{(n)} \right) \leq \sum_{k=1}^{n} \text{Lip} \left( \mu_{t,s}^{(n)} \right) d \left( \mu_{s_{k-1}, s_k}, \mu_{s_{k-1}, s_k} \right) \leq \sum_{k=1}^{n} \text{Lip} \left( \mu_{t,s}^{(n)} \right) L \left| s_k - s_{k-1} \right|^{\theta} \leq LC \sum_{k=1}^{n} \left| \frac{t - s}{n} \right|^{\theta} = LC \left| t - s \right| \left( \frac{1}{n} \right)^{\theta - 1}.$$ 

□

Corollary 2.23. If $\mu \in C \left( [0,T]^2 \times M, M \right)$ is a pre-flow satisfying the assumptions in Corollary 2.22 and (as usual) $\mu_{t,s}^{(n)} := \mu_{t,s}^{(n)}$, then

$$\lim_{n,k \rightarrow \infty} \sup_{s,t \in [0,T]} d \left( \mu_{t,s}^{(n)}, \mu_{t,s}^{(k)} \right) = 0,$$

i.e. $\left\{ \mu_{t,s}^{(n)} (m) \right\}_{n=1}^{\infty}$ is Cauchy uniformly in $(t,s,m) \in [0,T]^2 \times M$.

Proof. By applying Eq. (2.12) with $n$ replaced by $2^n$ and $\pi_{2^n} := \pi_{2^n}^{n+1} (s,t) \in \mathcal{P}(s,t)$ allows us to conclude,

$$\sup_{s,t \in [0,T]} d \left( \mu_{t,s}^{(n)}, \mu_{t,s}^{(n+1)} \right) \leq LCT \left( \frac{1}{2^n} \right)^{\theta - 1}.$$ 

As $\theta > 1$, the right member of this equation is summable over $n \in \mathbb{N}$ which is sufficient to prove Eq. (2.13). □

We now come to the main result of this section which can be viewed as a generalization of [3] Theorem 2.1 to complete metric spaces. The history of such results in the context of rough paths starts with Lyon’s “almost multiplicative function” theorem (see [33] and [34] Theorem 3.2.1, p. 41).
Successive generalizations / alternative formulations of Lyon’s result may be found in Gubinelli [25], Feyel and de la Pradelle [20], Feyel, de la Pradelle and Mokobodski [21], and likely many other references.

**Theorem 2.24 (AMF theorem – Existence of \( \varphi \)).** Suppose, as in Corollary 2.22, that \( \mu \in C \left([0,T]^2 \times M, M\right) \) is a pre-flow and there exists \( C < \infty \) such that \( \text{Lip}(\mu^t_s) \leq C \) and \( L < \infty \) such that Eq. (2.10) holds for all \((s,t) \in [0,T]^2\) and all \( \frac{1}{3}\)-special partitions of \( J(s,t) \). If we further assume that \((M,d)\) is a complete metric space, then there exists a unique \( \varphi \in C \left([0,T]^2 \times M, M\right) \) such that:

1. \( \varphi_{t,t} = \text{Id}_M \) for all \( t \in [0,T] \),
2. \( \varphi_{t,s} \circ \varphi_{s,r} = \varphi_{t,r} \) for all \( s,t,r \in [0,T] \), and
3. \( \varphi \) is close to \( \mu \), i.e.

\[
(2.14) \quad d(\varphi_{t,s}, \mu_{t,s}) \leq L |t - s|^\theta \quad \text{for all } s,t \in [0,T].
\]

Note that items 1. and 2. above implies that \( \varphi_{t,s} \in \text{Homeo}(M) \) for all \( s,t \in [0,T] \). It is also true that \( \varphi_{t,s} : M \to M \) is Lipschitz with

\[
(2.15) \quad \text{Lip}(\varphi_{t,s}) \leq C \quad \text{for all } s,t \in [0,T].
\]

**Proof.** By Corollary 2.23 and the completeness of \( M \), there exists \( \varphi \in C \left([0,T]^2 \times M, M\right) \) such that

\[
\lim_{n \to \infty} \sup_{s,t \in [0,T]} d(\mu^{(n)}_{t,s}, \varphi_{t,s}) = 0.
\]

From Eq. (2.10) we have

\[
d_M(\mu^{(n)}_{t,s}, \mu_{t,s}) \leq L |t - s|^\theta.
\]

Passing to the limit as \( n \to \infty \) in this inequality then gives Eq. (2.14). Similarly, if \( x,y \in M \), then

\[
d(\varphi_{t,s}(y), \varphi_{t,s}(x)) = \lim_{n \to \infty} d(\mu^{(n)}_{t,s}(y), \mu^{(n)}_{t,s}(x)) \leq \liminf_{n \to \infty} \text{Lip}(\mu^{(n)}_{t,s}) \leq C
\]

which gives Eq. (2.15). To finish the proof we must show \( \varphi \) is multiplicative and \( \varphi \) is unique which we do in four parts.

1. \( \varphi_{t,s} = \varphi_{t,u} \circ \varphi_{u,s} \) when \( u \) is between \( s \) and \( t \). Let \( s,t \in [0,T] \) and suppose that \( u = u_s := s + (t - s) a 2^{-p} \) for some \( p \in \mathbb{N} \) and \( 0 < a < 2^p \). Let \( b = 2^p - a \), \( n \in \mathbb{N} \) and consider the \( 2^{-n} \) uniform partitions

\[
\pi_\leq = \pi^{2^n}(s,u) = (s_\leq(k))_{k=0}^n \in \mathcal{P}(s,u) \quad \text{and}
\pi_\geq = \pi^{2^n}(u,t) = (s_\geq(k))_{k=0}^n \in \mathcal{P}(u,t)
\]
Upon noting that 
\[ \pi_0 = \pi_{a^2} = \pi_{2^n(2^n + 1)} \]
and 
\[ \lim_{n \to \infty} \pi_\infty = \pi_{2^n} \]
we further divide each subinterval of \( \pi_0 \) into \( a \) (b) equal pieces, that is for each \( 1 \leq k \leq 2^n \), let
\[
\Lambda_k := \left( s + (t - s) \frac{a (k - 1) + j}{2^{p+n}} \right)_{j=0}^{a} \in \mathcal{P} (s < (k - 1), s < (k)) \quad \text{and} \quad \Lambda'_k := \left( u + (t - s) \frac{b (k - 1) + j}{2^{p+n}} \right)_{j=0}^{b} \in \mathcal{P} (s > (k - 1), s > (k)).
\]

If \( \pi_x^* \) and \( \pi_x^- \) are constructed as in Corollary 2.22:\footnote{Another way to view this construction is to start with the uniform partition, \( \pi := \pi_{2^n} (s, t) \), which is the “join” of the uniform partitions \( \pi_x = \pi_x^a (s, u) \in \mathcal{P} (s, u) \) and \( \pi_x = \pi_x^b (u, t) \in \mathcal{P} (u, t) \). We further subdivide each partition interval of \( \pi_x \) and \( \pi_x^* \) into \( 2^n \) pieces in order to construct the uniform partitions, \( \pi_x^* = \pi_{2^n(2^n + 1)} (s, u) \) and \( \pi_x^- = \pi_{2^n(2^n + 1)} (u, t) \). The join of these two partition is then \( \pi^* = \pi_{2^n(2^n + 1)} (s, t) = \pi_{2^n(2^n + 1)} (s, u) \) oriented uniform partitions. Note that size of each of these subdivisions are \( (t - s) a^{2^{-p}} / a = (t - s) 2^{-(p+n)} \) and \( (t - s) b^{2^{-p}} / b = (t - s) 2^{-(p+n)} \) respectively.}

\[
d \left( \mu_{t,u}^{(n)} \bigotimes \mu_{t,u}^{(n)} \right) \leq C L \left( \frac{1}{2^n} \right)^{\theta - 1} |t - u|^\theta
\]
and
\[
d \left( \mu_{u,s}^{(n)} \bigotimes \mu_{u,s}^{(n)} \right) \leq C L \left( \frac{1}{2^n} \right)^{\theta - 1} |u - s|^\theta.
\]

Upon noting that \( \mu_{t,u}^{(n+p)} \bigotimes \mu_{u,s}^{(n+p)} = \mu_{t,u}^{(n)} \bigotimes \mu_{t,u}^{(n)} \), \( \mu_{t,u}^{(n+p)} \bigotimes \mu_{t,u}^{(n)} \), and \( \mu_{u,s}^{(n)} \bigotimes \mu_{u,s}^{(n)} \) we find,
\[
d \left( \mu_{t,u}^{(n+p)} \bigotimes \mu_{t,u}^{(n)} \bigotimes \mu_{u,s}^{(n)} \right) = d \left( \mu_{t,u}^{(n+p)} \bigotimes \mu_{u,s}^{(n)}, \mu_{t,u}^{(n)} \bigotimes \mu_{u,s}^{(n)} \right) \\
\leq \text{Lip} \left( \mu_{t,u}^{(n)} \right) d \left( \mu_{t,u}^{(n)}, \mu_{u,s}^{(n)} \right) + d \left( \mu_{t,u}^{(n)}, \mu_{u,s}^{(n)} \right) \\
\leq C^2 L \left( \frac{1}{2^n} \right)^{\theta - 1} |u - s|^\theta + C L \left( \frac{1}{2^n} \right)^{\theta - 1} |t - u|^\theta \\
\rightarrow 0 \text{ as } n \to \infty.
\]

Since also, \( \lim_{n \to \infty} d \left( \mu_{t,u}^{(n+p)} \bigotimes \varphi_{t,s}^{(n)} \right) = 0 \), and
\[
d \left( \mu_{t,u}^{(n)} \bigotimes \varphi_{t,u}^{(n)} \bigotimes \varphi_{u,s}^{(n)} \bigotimes \varphi_{u,s}^{(n)} \right) \leq \text{Lip} \left( \mu_{t,u}^{(n)} \right) d \left( \mu_{u,s}^{(n)}, \varphi_{u,s}^{(n)} \right) + d \left( \mu_{t,u}^{(n)}, \varphi_{t,u}^{(n)} \right) \\
\leq C d \left( \mu_{u,s}^{(n)}, \varphi_{u,s}^{(n)} \right) + d \left( \mu_{t,u}^{(n)}, \varphi_{t,u}^{(n)} \right) \rightarrow 0 \text{ as } n \to \infty.
\]
we conclude that
\[
d(\varphi_{t,s}, \varphi_{t,u} \circ \varphi_{u,s}) \leq d\left(\mu_{t,s}^{(n+p)}, \varphi_{t,s}\right) + d\left(\mu_{t,u}^{(n+p)} \circ \mu_{u,s}^{(n+p)}, \varphi_{t,u} \circ \varphi_{u,s}\right)
\]
\[
+ d\left(\mu_{t,u}^{(n)} \circ \mu_{u,s}^{(n)}, \varphi_{t,u} \circ \varphi_{u,s}\right) \rightarrow 0 \text{ as } n \rightarrow \infty,
\]
i.e. \(\varphi_{t,s} = \varphi_{t,u} \circ \varphi_{u,s}\). As this identity holds for a dense set of \(u\) between \(s\) and \(t\) and since \(\varphi\) is continuous in all of its variables, we conclude that \(\varphi_{t,s} = \varphi_{t,u} \circ \varphi_{u,s}\) for all \(u\) between \(s\) and \(t\).

(2) \(\varphi_{t,s} \in \text{Homeo}(M)\) and \(\varphi^{-1}_{t,s} = \varphi_{s,t}\) for all \(s, t \in [0, T]\). For each \(n \in \mathbb{N}\) it is easy to verify that
\[
\mu_{t,s}^{2n} \circ \mu_{s,t}^{2n} = I_{DM} = \mu_{s,t}^{2n} \circ \mu_{t,s}^{2n}.
\]
Passing to the limit as \(n \rightarrow \infty\) in this identity then shows
\[
\varphi_{t,s} \circ \varphi_{s,t} = I_{DM} = \varphi_{s,t} \circ \varphi_{t,s}
\]
which completes the proof of this step.

(3) \(\varphi\) is multiplicative. Let \(\alpha_t := \varphi_{0,t} \in \text{Homeo}(M)\) for all \(t \in [0, T]\). Making use of items 1. and 2., if \(0 \leq s \leq t \leq T\), then \(\alpha_t = \varphi_{t,s} \circ \alpha_s\) and hence \(\varphi_{t,s} = \alpha_t \circ \alpha_s^{-1}\). By interchanging the roles of \(s\) and \(t\), if \(0 \leq t \leq s \leq T\), then \(\varphi_{s,t} = \alpha_s \circ \alpha_t^{-1}\). Taking the inverse of this equation shows (again using item 2.) that \(\varphi_{t,s} = \varphi_{s,t}^{-1} = \alpha_t \circ \alpha_s^{-1}\). Thus for all \(s, t \in [0, T]\) we have \(\varphi_{t,s} = \alpha_t \circ \alpha_s^{-1}\) and from this identity, if \(r, s, t\) are arbitrary points in \([0, T]\), then
\[
\varphi_{t,s} \circ \varphi_{s,r} = \alpha_t \circ \alpha_s^{-1} \circ \alpha_s \circ \alpha_r^{-1} = \alpha_t \circ \alpha_r^{-1} = \varphi_{t,r}.
\]

(4) Uniqueness of \(\varphi\). Suppose that \(\psi \in C\left([0, T]^2 \times M, M\right)\) also satisfies items 1.–3. in the statement of the theorem. Then for \(s, t \in [0, T]\) and \(u\) between \(s, t\) we have
\[
d(\psi_{t,s}, \varphi_{t,s}) = d(\psi_{t,u} \circ \psi_{u,s}, \varphi_{t,u} \circ \varphi_{u,s})
\]
\[
\leq d(\psi_{t,u} \circ \psi_{u,s}, \varphi_{t,u} \circ \psi_{u,s}) + d(\varphi_{t,u} \circ \psi_{u,s}, \varphi_{t,u} \circ \varphi_{u,s})
\]
\[
\leq d(\psi_{t,u}, \varphi_{t,u}) + \text{Lip}(\varphi_{t,u}) d(\psi_{u,s}, \varphi_{u,s})
\]
\[
(2.16) \leq d(\psi_{t,u}, \varphi_{t,u}) + 2LC |s - u|^{\theta}.
\]
We now let \(\pi_n = \pi^n(s, t) = (s_k)_{k=0}^n \in \mathcal{P}(s, t)\) be the oriented uniform partition of \(J(s, t)\). Taking \(u = s_1\) in Eq. (2.16) shows,
\[
d(\psi_{t,s}, \varphi_{t,s}) \leq d(\psi_{t,s_1}, \varphi_{t,s_1}) + 2LC \left|\frac{t - s}{n}\right|^{\theta}.
\]
Iterating this procedure then shows,
\[ d(\psi_{t,s}, \varphi_{t,s}) \leq d(\psi_{t,s_2}, \varphi_{t,s_2}) + 2LC \left| \frac{t-s}{n} \right|^\theta + 2LC \left| \frac{t-s}{n} \right|^\theta \]
\[ \vdots \]
\[ \leq n \cdot 2LC \left| \frac{t-s}{n} \right|^\theta = 2LC |t-s|^\theta \cdot \frac{1}{n^{\theta-1}} \rightarrow 0 \text{ as } n \rightarrow \infty \]
which shows \( \psi_{t,s} = \varphi_{t,s} \).

We may improve on the uniqueness proof of \( \varphi \) in order to give a continuity statement of \( \varphi \) relative to \( \mu \).

**Theorem 2.25** (Continuity theorem). Suppose that \( \mu_{t,s}, \nu_{t,s} \in \text{Homeo}(M) \) are two approximate flows satisfying the assumptions of Theorem 2.24 and \( \varphi_{t,s}, \psi_{t,s} \in \text{Homeo}(M) \) are the unique multiplicative functions such that
\[ d(\varphi_{t,s}, \mu_{t,s}) \leq L |t-s|^\theta \text{ and } d(\psi_{t,s}, \nu_{t,s}) \leq L |t-s|^\theta \text{ for all } s, t \in [0, T]. \]
If there exists \( \varepsilon > 0 \) and \( 0 < \alpha < 1 \) such that
\[ d(\mu_{t,s}, \nu_{t,s}) \leq \varepsilon |t-s|^\alpha \text{ for all } s, t \in [0, T], \]
then
\[ d(\varphi_{t,s}, \psi_{t,s}) \leq \begin{cases} 
3 \varepsilon \cdot |t-s|^\alpha & \text{if } |t-s| \leq \left( \frac{\varepsilon}{L} \right)^{1/ \beta} \\
6CL^{1-\alpha} \varepsilon^{\frac{\theta-1}{\theta-\alpha}} |t-s| & \text{if } |t-s| > \left( \frac{\varepsilon}{L} \right)^{1/ \beta}.
\end{cases} \]

If we further assume that \( 0 \leq \varepsilon < 1 \), then
\[ d(\varphi_{t,s}, \psi_{t,s}) \leq \max \left( 6CL^{1-\alpha} T^{1-\alpha}, 3 \right) \varepsilon^{\frac{\theta-1}{\theta-\alpha}} |t-s|^\alpha \forall s, t \in [0, T] \]
and in the special case where \( \beta = 1 \) (i.e. \( \theta = 1 + \alpha \)), this inequality becomes,
\[ d(\varphi_{t,s}, \psi_{t,s}) \leq \max \left( 6C (LT)^{1-\alpha}, 3 \right) \varepsilon^\alpha |t-s|^\alpha \forall s, t \in [0, T]. \]

**Proof.** Let \( \beta := \theta - \alpha > 0 \). From the triangle inequality and the given estimates it follows that
\[ d(\varphi_{t,s}, \psi_{t,s}) \leq d(\varphi_{t,s}, \mu_{t,s}) + d(\mu_{t,s}, \nu_{t,s}) + d(\nu_{t,s}, \psi_{t,s}) \]
\[ \leq 2L |t-s|^\theta + \varepsilon |t-s| = \left[ 2L |t-s|^\beta + \varepsilon \right] |t-s|^\alpha. \]
We now define \( \delta > 0 \) so that \( L\delta^\beta = \varepsilon \), i.e. \( \delta = (\varepsilon/L)^{1/\beta} \). Then from the above inequality,
\[ d(\varphi_{t,s}, \psi_{t,s}) \leq 3 \varepsilon \cdot |t-s|^\alpha \text{ when } |t-s| \leq \delta = \left( \frac{\varepsilon}{L} \right)^{1/ \beta}. \]
If \( 0 \leq s < t \leq T \) with \( |t-s| > \delta \), choose \( n \in \mathbb{N} \) and \( 0 \leq r < \delta \) so that \( |t-s| = n\delta + r \) and then let \( s_k = s + k\delta \) for \( 0 \leq k \leq n \) and \( s_{n+1} = t = s_n + r \).
By the multiplicative property of \( \phi \) and \( \psi \) and Corollary 2.19, it follows that
\[
\begin{align*}
d(\varphi_{t,s}, \psi_{t,s}) &= d(\varphi_{s_{n+1}, s_n} \circ \ldots \varphi_{s_1, s_0}, \psi_{s_{n+1}, s_n} \circ \ldots \psi_{s_1, s_0}) \\
&\leq C \sum_{k=1}^{n+1} \text{Lip} (\varphi_{s_k, s_{k-1}}) d(\varphi_{s_k, s_{k-1}}, \psi_{s_k, s_{k-1}}) \\
&\leq 3C (n+1) \varepsilon \delta^\alpha.
\end{align*}
\]

Since
\[
1 \leq n = \frac{|t-s| - r}{\delta} \leq \frac{|t-s|}{\delta}
\]
we find,
\[
\begin{align*}
d(\varphi_{t,s}, \psi_{t,s}) &\leq 3C \left( \frac{|t-s|}{\delta} + 1 \right) \varepsilon \delta^\alpha = 3C (|t-s| + \delta) \varepsilon \delta^{\alpha-1} \\
&\leq 6C L^{1-\alpha} |t-s| \varepsilon^{(\theta-1)/\beta} = 6C L^{1-\alpha} |t-s| \varepsilon^{(\theta-1)/\beta}
\end{align*}
\]
which along with Eq. (2.20) completes the proof of Eq. (2.17). Using
\[
|t-s| = |t-s|^{1-\alpha} |t-s|^{\alpha} \leq T^{1-\alpha} |t-s|^{\alpha},
\]
in Eq. (2.21) implies,
\[
(2.22) \quad d(\varphi_{t,s}, \psi_{t,s}) \leq 6C L^{1-\alpha} T^{1-\alpha} \varepsilon^{(\theta-1)/\beta} |t-s|^{\alpha} \quad \text{when} \quad |t-s| > \delta.
\]

Since \( (\theta - 1)/ (\theta - \alpha) < 1 \), if \( 0 \leq \varepsilon < 1 \), then \( \varepsilon \leq \varepsilon^{(\theta-1)/(\theta-\alpha)} \) and so the estimates in Eq. (2.18) follows directly from Eqs. (2.20) and (2.22). \( \square \)

**Notation 2.26** (Mesh size). If \( s, t \in [0, T] \) and \( \pi = (s_0, \ldots, s_n) \in \mathcal{P} (s, t) \), let \( |\pi| \) be the mesh size of \( \pi \) defined by
\[
(2.23) \quad |\pi| := \max_{1 \leq k \leq n} |s_k - s_{k-1}|.
\]

**Proposition 2.27.** Continuing the assumptions and notation in Theorem 2.24, then for \( s, t \in [0, T] \)
\[
\varphi_{t,s} = \lim_{\pi \in \mathcal{P}(s,t) : |\pi| \to 0} \mu_{t,s}^\pi \quad \text{uniformly in} \quad (s, t) \in [0, T]^2.
\]

More precisely, if \( s, t \in [0, T] \) and \( \pi \in \mathcal{P} (s, t) \), then
\[
(2.24) \quad d(\varphi_{t,s}, \mu_{t,s}^\pi) \leq CL |\pi|^{\theta-1} |t-s|.
\]

**Proof.** Let \( \pi = (s_0, \ldots, s_n) \in \mathcal{P} (s, t) \) be an oriented partition of \( J (s, t) \). By Corollary 2.19 the estimates in Eqs. (2.14) and (2.15), and the fact that
Given two functions, \( f(x) \) and \( g(x) \), depending on some parameters indicated by \( x \), we write \( f(x) \lesssim g(x) \) if there exists a constant, \( C(\kappa) \), only possibly depending on \( \kappa \) so that \( f(x) \leq C(\kappa) g(x) \) for the allowed values of \( x \). Similarly we write \( f(x) \asymp g(x) \) if both \( f(x) \lesssim g(x) \) and \( g(x) \lesssim f(x) \) hold.

**Notation 3.2.** For \( \lambda \geq 0 \) and \( m, n \in \mathbb{N} \) with \( m < n \), let

\[
Q_{[m,n]}(\lambda) := \max \left\{ \lambda^k : k \in \mathbb{N} \cap [m, n] \right\} = \max \left\{ \lambda^m, \lambda^n \right\} \quad \text{and} \quad Q_{(m,n)}(\lambda) := \max \left\{ \lambda^k : k \in \mathbb{N} \cap (m, n) \right\} = \max \left\{ \lambda^{m+1}, \lambda^n \right\}.
\]

For \( m, n \in \mathbb{N} \) with \( m < n \), it is not difficult to verify

\[
Q_{(m,n)}(\lambda + \mu) \asymp Q_{[m,n]}(\lambda) + Q_{[m,n]}(\mu) \quad \text{for} \quad \mu, \lambda \geq 0.
\]

**Theorem 3.3** (Corollary 4.15]). If \( A, B \in F(\kappa) (\mathbb{R}^d) \), then (with \( N(\cdot) \) as in Definition 1.13)

\[
d_M\left( e^{V_B} \circ e^{V_A}, e^{V_{\log(cA\cdot B)}} \right) \leq K_0 \cdot N(A) N(B) Q_{[\kappa-1,2(\kappa-1)]} (N(A) + N(B))
\]

and

\[
d_M\left( e^{V_B} \circ e^{V_A}, e^{V_{\log(cA\cdot B)}} \right) \leq K_0 \cdot Q_{(\kappa,2\kappa]} (N(A) + N(B)).
\]
where for a suitable, \( k < \infty \),
\[
\mathcal{K}_0 = ke^{k|\nabla (\kappa)|_{Q_{1,\kappa}}(N(A) \vee N(B))}. \left[ \left| V(\kappa) \right|_{M} \left| \nabla V(\kappa) \right|_{M} \right]
\]

We further will make use the following Riemannian metric on \( TM \).

**Definition 3.4 (Riemannian metric on \( TM \)).** Given a smooth curve, \( v(t) \in TM \) with \( v(0) \in T_{\sigma(t)}M \) where \( \sigma = \pi \circ v \) is a smooth curve on \( M \), let
\[
g^{TM} (\dot{v}(0), \dot{v}(0)) := |\dot{v}(0)|^2_g + |\nabla_t v(t) |_{t=0}^2_g
\]
where \( \nabla_t v(t) \) is the covariant derivative of \( v \) along \( \sigma \). We further let \( d^{TM} : TM \times TM \to [0, \infty) \) be the associated length metric associate to \( g^{TM} \). [See [18, Section 5] for a more detailed discussion of the definitions and properties of \( g^{TM} \) and \( d^{TM} \).]

**Notation 3.5.** For \( f, g \in C^\infty (M, M) \) and a subset \( U \subset M \), let
\[
d^U_{TM} (f_*, g_*) = \sup_m \sup \{ d^TM (f_* v_m, g_* v_m) : v_m \in T_m M \text{ with } |v_m| = 1 \},
\]
where \( f_* : TM \to TM \) denotes the differential of \( f \), i.e.
\[
f_* \dot{\sigma} (0) := \frac{d}{dt} |0 f (\sigma(t)) \in T_{\sigma(0)} M.
\]

The relationship between \( \text{Lip}(f) \) and \( f_* \) is (see [18 Lemma 2.9])
\[
(3.1) \quad \text{Lip}(f) = |f_*|_M := \sup \{|f_* v| : v \in TM \text{ with } |v| = 1\}.
\]

**Theorem 3.6 ([18 Corollary 8.5]).** If \( A, B \in F^{(\kappa)} (\mathbb{R}^d) \), then
\[
d^M_{TM}(e_*^V_B, 1d^{TM}) \leq \left[ \left| V(\kappa) \right|_{M} + \left| \nabla V(\kappa) \right|_{M} e^{\left| \nabla (\kappa) \right|_{M} |B|} \right] |B|,
\]
and there exists \( \mathcal{K}_1 \) such that
\[
d^M_{TM} \left( [e^{V_B \circ e^{V_A}}]_* e_*^{V_{\log (e^{\lambda A} B)}} \right) \leq \mathcal{K}_1 \cdot N (A) N (B) Q_{(\kappa - 1, 2(\kappa - 1))} (N (A) + N (B)).
\]

where
\[
\mathcal{K}_1 = \mathcal{K}_1 \left( \left| V(\kappa) \right|_{M}, \left| \nabla V(\kappa) \right|_{M}, H_M \left( V(\kappa) \right), N (A) \vee N (B) \right).
\]

In the proof of the next Corollary 4.5 we will use the following two properties of \( d^{TM} \):
\[
(3.2) \quad ||v| - |w|| \leq d^{TM} (v, w) \quad \forall \ v, w \in TM, \quad \text{and}
\]
\[
(3.3) \quad d^{TM} (\lambda v, \lambda w) \leq (\lambda \vee 1) d^{TM} (v, w) \quad \forall \ \lambda \geq 0 \text{ and } v, w \in TM,
\]
see [18 Proposition 5.9] and [18 Theorem 5.11] respectively. We also need the following basic estimates for complete vector field, \( X \in \Gamma (TM) \).

**Proposition 3.7 ([18 Corollary 2.27]).** If \( X \in \Gamma (TM) \) is complete, then
\[
(3.4) \quad \text{Lip} (e^X) = \left| e^X_* \right|_M \leq e^{\left| \nabla X \right|_M}.
\]
satisfying Assumption 2 and such that

\[ H_M (X) \text{ is as in Notation 1.17.} \]

4. Proof of Theorem 1.27

In this section we fix \( \alpha \in \left( \frac{1}{\kappa+1}, \frac{1}{\kappa} \right) \), \( 0 < T < \infty \), an \( \alpha \)-Hölder geometric rough path \( X_{s,t} \in C^{(\kappa)}_{\text{geo}} (\mathbb{R}^d) \), and \( \mu_{t,s} := e^{V_{\log(X_{st})}} \in \text{Diff} (M) \). Note that by definition of a \( \alpha \)-Hölder rough path, there exists \( c < \infty \) such that

\[ N(X_{s,t}) \leq c \vert t - s \vert^\alpha \text{ for all } (s,t) \in [0,T]^2. \]

We also let \( \theta := \alpha (\kappa + 1) > 1 \) as in Eq. (1.13).

**Theorem 4.1.** Suppose that \( V : \mathbb{R}^d \to \Gamma (TM) \) is a dynamical system satisfying Assumption 3 and \( g \) is a (not necessarily complete) metric on \( M \) such that \( \left\vert V^{(\kappa)} \right\vert_M + \left\vert \nabla V^{(\kappa)} \right\vert_M < \infty \) (see Definition 1.18). Then

\[ d_M (\mu_{t,s}, Id_M) \leq \left\vert V^{(\kappa)} \right\vert_M \left\vert \log(X_{s,t}) \right\vert \leq c \left\vert V^{(\kappa)} \right\vert_M Q_{1,\kappa} (c \vert t - s \vert^\alpha) \]

and there exists a constant, \( k = k (\kappa) < \infty \) such that

\[ d_M (\mu_{t,s} \circ \mu_{s,r}, \mu_{t,r}) \leq k c^k \left\vert \nabla V^{(\kappa)} \right\vert_M Q_{1,\kappa} (2k c_\kappa \vert t - r \vert^\alpha) \left\vert V^{(\kappa)} \right\vert_M \left\vert \nabla V^{(\kappa)} \right\vert_M Q_{(\kappa,2\kappa)} (2k c_\kappa \vert t - r \vert^\alpha). \]

**Note well:** curvature does not enter these bounds!

**Proof.** Let \( A = \log (X_{r,s}) \) and \( B = \log (X_{s,t}) \) in which case

\[ \Gamma (A, B) = \log (e^{A} e^{B}) = \log (X_{r,s} X_{s,t}) = \log (X_{r,t}), \]

\[ N (A) = N (\log (X_{r,s})) \leq c_\kappa \cdot N (X_{r,s}) \leq c_\kappa \cdot k \vert s - r \vert^\alpha, \]

and

\[ N (A) \vee N (B) \leq k c_\kappa \vert t - r \vert^\alpha. \]

Thus as an application of Theorem 3.3

\[ d_M (\mu_{t,s}, Id_M) \leq \left\vert V^{(\kappa)} \right\vert_M \left\vert \log (X_{s,t}) \right\vert \leq \left\vert V^{(\kappa)} \right\vert_M Q_{1,\kappa} (N (\log (X_{s,t}))) \]

and

\[ d_M (\mu_{t,s} \circ \mu_{s,r}, \mu_{t,r}) = d_M (e^{VB} \circ e^{VA}, e^{V_{\Gamma (A, B)}}) \]

\[ \leq k c^k \left\vert \nabla V^{(\kappa)} \right\vert_M Q_{1,\kappa} (N (A) \vee N (B)) \left\vert V^{(\kappa)} \right\vert_M \left\vert \nabla V^{(\kappa)} \right\vert_M Q_{(\kappa,2\kappa)} (N (A) \vee N (B)) \]

\[ \leq k c^k \left\vert \nabla V^{(\kappa)} \right\vert_M Q_{1,\kappa} (2k c_\kappa \vert t - r \vert^\alpha) \left\vert V^{(\kappa)} \right\vert_M \left\vert \nabla V^{(\kappa)} \right\vert_M Q_{(\kappa,2\kappa)} (2k c_\kappa \vert t - r \vert^\alpha). \]

\[ \square \]

**Notation 4.2.** Let \( \Gamma_c (TM) \) denote the smooth compactly supported vector fields on \( M \).
Theorem 4.3 (Localizing Approximates). Let us continue the notation and assumptions in Theorem 4.1. Assume that $g$ is a complete Riemannian metric on $M$ (i.e. $(M, d)$ is complete) and $K$ is a compact subset of $M$, i.e. $K$ is closed and bounded. Then there exists $\tilde{V} : \mathbb{R}^d \to \Gamma_c(TM)$ such that $\tilde{\mu}_{t,s}^n = \mu_{t,s}^n$ on $K$ for all $n \in \mathbb{N}$ and $(s, t) \in [0, T]^2$.

Proof. Let $\theta := (\kappa + 1) \alpha > 1$ as in Assumption [1]. Let $K_1$ be the compact subset of $M$ containing $K$ defined by

$$K_1 := \left\{ \mu_{t,s}(m) : (s, t) \in [0, T]^2 \text{ and } m \in K \right\}.$$ 

By Theorem 4.1 and Theorem 2.17 there exists $C < \infty$ such that

$$d_M(\mu_{t,s}^n, \mu_{t,s}) \leq C |t - s|^\theta \text{ for all } (s, t) \in [0, T]^2.$$ 

Now let

$$K_2 := \left\{ m \in M : d(m, K_1) \leq C \left[T^\theta \land 1\right] \right\}$$

which is again closed and bounded and hence compact. Moreover we have $\mu_{t,s}^n(m) \in K_2$ for all $m \in K$, $(s, t) \in [0, T]^2$, and $n \in \mathbb{N}$. Lastly let

$$R := \max_{(s, t) \in [0, T]^2} |\tilde{V}_{log(x_{s,t})}|_M \text{ and } K_3 := \left\{ m \in M : d(m, K_2) \leq R \right\}.$$ 

We then have $\mu_{t,s}(K_2) \subset K_3$ for all $(s, t) \in [0, T]^2$. Let $\varphi \in C_c^\infty(M, [0, 1])$ such that $\varphi = 1$ on a neighborhood of $K_3$ and define $\tilde{V} := \varphi \tilde{V}$. We will finish the proof by showing $\tilde{\mu}_{t,s}^n = \mu_{t,s}^n$ on $K$ for all $n \in \mathbb{N}$ and $(s, t) \in [0, T]^2$.

If $m \in K_2$ and $0 \leq \tau \leq 1$, then

$$d(e^{\tilde{V}_{log(x_{s,t})}}(m), m) \leq \tau |\tilde{V}_{log(x_{s,t})}|_M \leq R$$

and so $e^{\tilde{V}_{log(x_{s,t})}}(m) \in K_3$ for all $0 \leq \tau \leq 1$. Therefore,

$$\frac{d}{d\tau} e^{\tilde{V}_{log(x_{s,t})}}(m) = \tilde{V}_{log(x_{s,t})} \left( e^{\tilde{V}_{log(x_{s,t})}}(m) \right) = \frac{\partial}{\partial \tau} \tilde{V}_{log(x_{s,t})} \left( e^{\tilde{V}_{log(x_{s,t})}}(m) \right) \text{ for } 0 \leq \tau \leq 1.$$ 

From this we conclude that

$$e^{\tilde{V}_{log(x_{s,t})}}(m) = e^{\tilde{V}_{log(x_{s,t})}}(m) \forall m \in K_2 \text{ and } 0 \leq \tau \leq 1$$

and in particular this implies that $\tilde{\mu}_{t,s} = \mu_{t,s}$ on $K_2$ for all $(s, t) \in [0, T]^2$.

This shows $\tilde{\mu}_{t,s}^n = \mu_{t,s}^n$ on $K$ for all $(s, t) \in [0, T]^2$ when $n = 1$. We now finish proof by induction on $n$. If $m \in K$, $n \in \mathbb{N}$, and

$$u := t - \frac{1}{n + 1}(t - s),$$

then $\mu_{u,s}^n(m) \in K_2$ and so using the induction hypothesis and the fact that $\tilde{\mu}_{t,s} = \mu_{t,s}$ on $K_2$ we find,

$$\tilde{\mu}_{t,s}^{n+1}(m) = \tilde{\mu}_{t,u} \circ \mu_{u,s}^n(m) = \tilde{\mu}_{t,u} \circ \mu_{u,s}^n(m)$$

$$= \mu_{t,u} \circ \mu_{u,s}^n(m) = \mu_{t,s}^{n+1}(m).$$

$\square$
Theorem 4.4. Let $V : \mathbb{R}^d \to \Gamma (TM)$ be a dynamical system such that
\[
\left| V^{(\kappa)} \right|_M + \left| \nabla V^{(\kappa)} \right|_M + H_M \left( V^{(\kappa)} \right) < \infty,
\]
let $1/(\kappa + 1) < \alpha \leq 1/\kappa$, $X_{s,t} \in G^{(\kappa)}_{\text{geo}}(\mathbb{R}^d)$ be an $\alpha$-Hölder geometric rough path in $\mathbb{R}^d$, and for $(s, t) \in [0, T]^2$, let
\[
\mu_{t,s} := e^{V \log (X_{s,t})} \in \text{Diff} (M)
\]
as in Definition 1.25. Then $\mu_{t,s}$ is a good approximate flow, i.e.
\[
d_M (\mu_{t,s}, Id_{TM}) \leq C \left( \left| V^{(\kappa)} \right|_M, \left| \nabla V^{(\kappa)} \right|_M \right) |t - s|^\alpha \quad \text{and}
\]
\[
d_M^{TM} (\mu_{t,s} \mu_{s,t}, \mu_{t,s}) \leq K \left( \left| V^{(\kappa)} \right|_M, \left| \nabla V^{(\kappa)} \right|_M, H_M \left( V^{(\kappa)} \right) \right) |t - r|^\theta,
\]
where $\theta = (\kappa + 1) \alpha$ as Assumption 1. The constant $C$ and $K$ also depend on $\sup_{(s,t) \in [0,T]^2} |\log (X_{s,t})|$. We suppress this dependence as we view the geometric rough path, $X$, as being fixed.

Proof. As in the proof of Theorem 4.1, let $A = \log (X_{r,s})$ and $B = \log (X_{s,t})$, so that $\Gamma (A, B) = \log (X_{r,t})$,
\[
\mu_{t,s} = e_s^{V_B}, \quad \mu_{s,t} = e_s^{V_A}, \quad \text{and} \quad \mu_{t,s} = e_s^{V (A, B)}.
\]
Thus by Theorem 3.6
\[
d_M^{TM} (\mu_{t,s} \mu_{s,t}, Id_{TM}) \leq C \left( \left| V^{(\kappa)} \right|_M, \left| \nabla V^{(\kappa)} \right|_M, e^{\left| V^{(\kappa)} \right|_M |\log (X_{s,t})|} \right) \cdot |\log (X_{s,t})|
\]
\[
\leq C \left( \left| V^{(\kappa)} \right|_M, \left| \nabla V^{(\kappa)} \right|_M \right) |t - s|^\alpha
\]
and
\[
d_M^{TM} (\mu_{t,s} \mu_{s,t}, \mu_{t,s}) \leq K \cdot (N (A) \vee N (B))^{\kappa + 1} \leq K |t - r|^\theta.
\]

\[\Box\]

Corollary 4.5. Under the same assumptions of Theorem 4.4, if $\varepsilon \in (0, 1/2)$ is given, there exists a $\delta > 0$ and $L < \infty$ such that for any $(s, t) \in [0, T]^2$
\[
|t - s| \leq \delta \quad \text{and any} \ \varepsilon\text{-special partitions, } \pi, \ \text{of } J (s, t),
\]
\[
d_M^{TM} (\mu_{t,s} \mu_{s,t}) \leq L |t - s|^\theta.
\]

where $\theta = (\kappa + 1) \alpha > 1$ as in Eq. (1.13).
Proof. When $\pi = \{s < u < t\}$, Eq. (4.1) holds with $L = C$ as in Theorem 4.4. Thus we will need to take $L \geq c$. The proof will now be completed by induction on $n = \#(\pi)$. Let $\pi$ be an $\varepsilon$-special partition of $J(s,t)$ and $u \in J_\varepsilon(s,t) \cap \pi$ which exists as $\pi$ is $\varepsilon$-special. Then for a unit vector, $v_m \in T_{m,M}$,

$$d^TM(\mu_{t,ss}^\pi v_m, \mu_{t,ss} v_m) = d^TM(\mu_{t,us}^\pi \mu_{u,ss}^\pi v_m, \mu_{t,ss} v_m) \leq d^TM(\mu_{t,us}^\pi \mu_{u,ss}^\pi v_m, \mu_{t,us} \mu_{u,ss}^\pi v_m)$$

(4.2)

By Theorem 4.4,

$$d^TM(\mu_{t,us} \mu_{u,ss}^\pi v_m, \mu_{t,ss} v_m) \leq C |t - s|^\theta.$$

By Eq. (3.3) and the induction hypothesis,

$$d^M(\mu_{t,us}^\pi, \mu_{t,us} \mu_{u,ss}^\pi v_m) \leq (1 + |\mu_{u,ss}^\pi v_m|) d^M(\mu_{t,us}^\pi, \mu_{t,us}) \leq (1 + |\mu_{u,ss}^\pi v_m|) L |t - u|^\theta.$$

Now by Eq. (3.2), Theorem 4.4 and the induction hypothesis,

$$|\mu_{u,ss}^\pi v_m| \leq |v_m| + d^M(\mu_{u,ss}^\pi v_m, v_m) \leq 1 + d^M(\mu_{u,ss}^\pi v_m, \mu_{u,ss} v_m) + d^M(\mu_{u,ss} v_m, v_m) \leq 1 + L |u - s|^\theta + \hat{C} |u - s|^\alpha$$

(4.3)

which combined with the previous estimate shows

$$d^M(\mu_{t,us}^\pi \mu_{u,ss}^\pi v_m, \mu_{t,us} \mu_{u,ss}^\pi v_m) \leq \left(1 + \hat{C} |u - s|^\alpha + L |u - s|^\theta \right) L |t - u|^\theta.$$

Thus we have shown so far that

$$d^M(\mu_{t,ss}^\pi v_m, \mu_{t,ss} v_m) \leq \left(1 + \hat{C} |u - s|^\alpha + L |u - s|^\theta \right) L |t - u|^\theta + C |t - s|^\theta$$

(4.4)

By Proposition 3.8 with $X = V_{\log(X,u,t)}$, $v_m \rightarrow \mu_{u,ss}^\pi v_m$ and $w_p \rightarrow \mu_{u,ss} v_m$ we have

$$d^M(\mu_{t,us}^\pi \mu_{u,ss}^\pi v_m, \mu_{t,us} \mu_{u,ss}^\pi v_m) \leq e^{\|q\nabla V_{\log(X,u,t)}\|_M} [1 + H_M(V_{\log(X,u,t)}) |\mu_{u,ss} v_m|] d^M(\mu_{u,ss}^\pi v_m, \mu_{u,ss} v_m)$$

$$\leq e^{c|t - u|^\alpha} [1 + c |u - t|^\alpha |\mu_{u,ss} v_m|] d^M(\mu_{u,ss}^\pi v_m, \mu_{u,ss} v_m) \leq e^{c|t - u|^\alpha} [1 + c |u - t|^\alpha |\mu_{u,ss} v_m|] L |u - s|^\theta$$

$$\leq e^{c|t - u|^\alpha} \left[1 + c |u - t|^\alpha \left(1 + \hat{C} |u - s|^\alpha \right) \right] L |u - s|^\theta \leq (1 + \hat{C} |t - s|^\alpha) L |u - s|^\theta$$
wherein we have used Eq. (3.2) along with Theorem 4.4 to conclude
$$|\mu_{u,s}v_m| \leq |v_m| + d(\mu_{u,s}v_m, v_m) = 1 + C|u-s|^\alpha.$$ Combining this result with Eq. (4.4) shows
$$d^{TM}(\mu_{\pi t,s}^*v_m, \mu_{t,s}^*v_m) \leq \gamma(\varepsilon, \theta)|t-s|^\theta + \left(\tilde{C}|u-s|^\alpha + L|u-s|^\theta\right)L|t-u|^\theta
+ C|t-s|^\theta + \tilde{c}|t-s|^\alpha L|u-s|^\theta
\leq \left(\gamma(\varepsilon, \theta) + \tilde{C} + \tilde{c}\right)\delta^\alpha + \delta^\theta L + C|t-s|^\theta.$$

We now choose $\delta > 0$ so small that $\gamma(\varepsilon, \theta) + C\delta^\alpha < 1$, for example, choose $\delta$ so that $\gamma(\varepsilon, \theta) + C\delta^\alpha \leq 1 + \gamma(\varepsilon, \theta)$.

Then we want to choose $L$ so that
$$\alpha(\varepsilon, \theta)L + 2\delta^\theta L^2 + C \leq L \iff L \geq \frac{1}{1 - \alpha(\varepsilon, \theta)}\left[C + 2\delta^\theta L^2\right].$$

We do this by requiring
$$L = \frac{2}{1 - \alpha(\varepsilon, \theta)}C$$
and then shrink $\delta$ so that $2\delta^\theta L^2 \leq C$. For these choices we have
$$\alpha(\varepsilon, \theta)L_1 + 2\delta^\theta L^2 + C \leq L$$
and the induction step is complete.

□

Corollary 4.6 (Lip-bounds). Under the same assumptions of Theorem 4.4, to each $\varepsilon \in (0, 1/2]$, there exists $C = C(\varepsilon) < \infty$ such that $\text{Lip}(\mu_{t,s}^\pi) \leq C < \infty$ for all $(s, t) \in [0, T]^2$ and all $\varepsilon$-special partitions, $\pi$, of $J(s, t)$.

Proof. By Eq. (3.2) and Corollary 4.5 if we assume that $(t-s) \leq \delta$, then (using Eq. (3.1))
$$\text{Lip}(\mu_{t,s}^\pi) = |\mu_{t,s}^\pi|_M \leq |\mu_{t,s}|_M + d_M^{TM}(\mu_{t,s}^\pi, \mu_{t,s})
\leq |\mu_{t,s}|_M + L|t-s|^\theta.$$ Moreover by Proposition 3.7
$$|\mu_{t,s}|_M \leq e^{\left|\nabla V_{\log(x,s,t)}\right|_M} \leq e^{C|t-s|^\alpha}.$$ Thus it follows that
$$\text{Lip}(\mu_{t,s}^\pi) \leq e^{C\delta^\alpha} + L\delta^\theta < \infty.$$
We may now remove the restriction that \((t - s) \leq \delta\) by an application of Lemma 2.18.

We are now ready to prove Theorem 1.27 which we split into two parts; 1) existence is subsection 4.1 and 2) uniqueness in subsection 4.2.

**Theorem 4.7** (Global Existence). Suppose that \(V : \mathbb{R}^d \to \Gamma(TM)\) is a dynamical system satisfying Assumption 2 and there exists a complete metric, \(g\), on \(M\) such that \(\left| V^{(\alpha)} \right|_M + \left| \nabla V^{(\alpha)} \right|_M < \infty\). Then there exists a unique function \(\varphi \in C \left( [0, T] \times M, M \right)\) such that:

1. \(\varphi_{t,s} = Id\) for all \(t \in [0, T]\),
2. \(\varphi_{t,s} \circ \varphi_{s,r} = \varphi_{t,r} \quad \forall \ 0 \leq r \leq s \leq t \leq T\), and
3. there exists a constant \(C < \infty\) such that

\[
d_{M} (\varphi_{t,s}, \mu_{t,s}) \leq C |t - s|^\alpha \quad \forall \ (s, t) \in [0, T]^2.
\]

Moreover, \(C (K) := \sup_{(s, t) \in [0, T]^2} \mathrm{Lip}_K (\varphi_{t,s}) < \infty\) for all compact subsets, \(K\), of \(M\).

### 4.1. Existence proof for Theorem 1.27

Suppose that \(V : \mathbb{R}^d \to \Gamma(TM)\) is dynamical system of complete vector fields and \(g\) is a Riemannian metric on \(M\) so that \(\left| V^{(\alpha)} \right|_M + \left| \nabla V^{(\alpha)} \right|_M < \infty\). As usual let

\[
\mu_{t,s} := e^{V_{\log}(X_{s,t})}.
\]

By Proposition 3.7 we know that

\[
\mathrm{Lip} (\mu_{t,s}) = \left| e_{s} V_{\log}(X_{s,t}) \right|_M \leq e \left| \nabla V_{\log}(X_{s,t}) \right|_M \leq e \left| \nabla V^{(\alpha)} \right|_M |\log(X_{s,t})|
\]

\[
\leq e^{|\nabla V^{(\alpha)}|_M |t-s|^{\alpha}} = 1 + O \left(|t - s|^{\alpha}\right).
\]

From Theorem 4.1 we know

\[
d_{M} (\mu_{t,s} \circ \mu_{s,r}, \mu_{t,r}) \leq K |t - r|^{\alpha} \quad \text{for all} \ 0 \leq r \leq s \leq t \leq T.
\]

where \(K = K \left( \left| V^{(\alpha)} \right|_M , \left| \nabla V^{(\alpha)} \right|_M \right)\). We can now apply the Trotter approximation bounds in Lemma 2.16 along with the large-time extension Theorem 2.17 to conclude, for ever \(\varepsilon \in (0, \frac{1}{2}]\), there exists and \(L_{\varepsilon} < \infty\) such that

\[
d_{M} (\mu_{t,s}^{\varepsilon}, \mu_{t,s}) \leq L_{\varepsilon} |t - s|^{\theta} \quad \text{for all} \ (s, t) \in [0, T]^2.
\]

Let us now further assume that \(g\) is a complete metric and choose compact subsets, \(\{K_N\}^\infty_{N=1}\), of \(M\) so that \(K_N \subset K_{N+1}\) for all \(N \in \mathbb{N}\) and \(K_N \uparrow M\) as \(N \uparrow \infty\). By Theorem 4.3 there exists \(V^N \in \Gamma_{\varepsilon} (TM)\) so that if \(\mu_{N,t,s} := e^{V_{\log}^N(X_{s,t})}\), then

\[
\mu_{N,t,s} = \mu_{t,s}^N \quad \text{on} \ K_N \quad \text{for all} \ n \in \mathbb{N}.
\]

We further have by Corollary 4.6 that to each \(\varepsilon \in (0, 1/2]\), there exists \(C = C_N (\varepsilon) < \infty\) such that \(\mathrm{Lip} (\mu_{N,t,s}^N) \leq C < \infty\) for all \((s, t) \in [0, T]^2\) and all \(\varepsilon\)-special partitions, \(\pi\), of \(J (s, t)\). These assertions verify the assumptions
of Theorem 2.24 from which it follows that \( \varphi_{N,t,s} = \lim_{n \to \infty} \mu_n^{2^n} \) exists uniformly on \( M \) and moreover, \( \varphi_{N,t,s} \in C(M,M) \) is a semi-group such that

\[
d(\varphi_{N,t,s}, \mu_{N,t,s}) \leq L_{1/3}(N) |t - s|^\theta.
\]

If \( \hat{N} > N \), then we will have

\[
\varphi_{\hat{N},t,s} = \lim_{n \to \infty} \mu_{\hat{N},t,s}^{2^n} = \lim_{n \to \infty} \mu_{t,s}^{2^n} = \lim_{n \to \infty} \mu_{N,t,s}^{2^n} = \varphi_{N,t,s} \text{ on } K_N.
\]

This shows that it is well defined to put \( \varphi_{t,s} := \varphi_{N,t,s} \) on \( K_N \) and moreover we have in fact shown that \( \varphi_{t,s} = \lim_{n \to \infty} \mu_n^{2^n} \) uniformly on compact subsets of \( M \). Passing to the limit in Eq. (4.5) shows that

\[
d(\varphi_{t,s}, \mu_{t,s}) \leq L_{1/3} |t - s|^\theta \text{ for all } (s,t) \in [0,T]^2.
\]

Choose \( \hat{N} \) sufficiently large such that \( \cup_{0 \leq s,t \leq T} \varphi_{N,t,s} \) is a compact subset of \( K_{\hat{N}} \) so that if \( m \in K_N \), then

\[
\varphi_{t,s} \circ \varphi_{s,r}(m) = \varphi_{t,s} \circ \varphi_{N,s,r}(m) = \varphi_{\hat{N},t,s} \circ \varphi_{N,s,r}(m)
\]

\[
= \varphi_{\hat{N},t,s} \circ \varphi_{\hat{N},s,r}(m) = \varphi_{\hat{N},t,r}(m) = \varphi_{t,r}(m).
\]

As \( N \in \mathbb{N} \) was arbitrary and \( K_N \uparrow M \) as \( N \uparrow \infty \) we conclude that \( \varphi_{t,s} \circ \varphi_{s,r} = \varphi_{t,r} \). Thus we have proved the existence assertion in Theorem 1.27.

4.2. Uniqueness proof for Theorem 1.27. One way to prove uniqueness is to show, with \( \psi_{t,s} \) satisfies items 1.-3. of Theorem 1.27 for \( m \in M \) that \( x_t := \psi_{t,0}(m) \) (or more generally \( x_t := \psi_{t,s}(m) \) for \( t \geq s \)) satisfies the RDE as described in Proposition 1.28 and that solutions in this sense are unique. This would then show that \( \psi_{t,s}(m) = \varphi_{t,s}(m) \) where \( \psi \) is any other solution as described in Theorem 1.27. We wish to avoid developing the path-wise notion of solutions in this paper and so we will try to use a variant of proof given in Theorem 2.24 instead.

Let \( (s,t,m) \to \psi_{t,s}(m) \) be a continuous function which satisfies 1.-3. of Theorem 1.27 and \( \varphi_{t,s} \) be the solution we have already constructed. Then by the triangle inequality we know that

\[
d_M(\varphi_{t,s}, \psi_{t,s}) \leq 2C |t - s|^\theta \forall (s,t) \in [0,T]^2.
\]

Given a compact subset, \( K \subset M \), let

\[
K' = \left[ \bigcup_{(s,t) \in [0,T]^2} \psi_{t,s}(K) \right] \cup \left[ \bigcup_{(s,t) \in [0,T]^2} \varphi_{t,s}(K) \right]
\]

which is still compact since it is the union of two compact sets. Let \( c := c(K') := \sup_{(s,t) \in [0,T]^2} \text{Lip}_K'(\varphi_{s,t}) < \infty \). Next suppose \( (s,t) \in [0,T]^2, n \in \mathbb{N}, \) and \( s_k := s + \frac{k(t-s)}{n} \) for \( 0 \leq k \leq n \). We then find, using the notation in
Eq. (1.14) with \( U = K \) and \( U = \psi_{s_1,s_0}(K) \), that
\[
d_K(\psi_{t,s_1}, \varphi_{t,s_1}) = d_K(\psi_{t,s_1} \circ \psi_{s_1,s_0}, \varphi_{t,s_1} \circ \varphi_{s_1,s_0}) \\
\leq d_K(\psi_{t,s_1} \circ \psi_{s_1,s_0}, \varphi_{t,s_1} \circ \varphi_{s_1,s_0}) + d_K(\varphi_{t,s_1} \circ \psi_{s_1,s_0}, \varphi_{t,s_1} \circ \varphi_{s_1,s_0}) \\
\leq d_{\psi_{s_1,s_0}}(K)(\psi_{t,s_1}, \varphi_{t,s_1}) + c d_K(\psi_{s_1,s_0}, \varphi_{s_1,s_0}) \\
\leq d_{\psi_{s_1,s_0}}(K)(\psi_{t,s_1}, \varphi_{t,s_1}) + c 2C |s_1 - s_0|^{\theta}.
\]

Similarly,
\[
d_{\psi_{s_1,s_0}}(K)(\psi_{t,s_1}, \varphi_{t,s_1}) \\
= d_{\psi_{s_1,s_0}}(K)(\psi_{t,s_2} \circ \psi_{s_2,s_1}, \varphi_{t,s_2} \circ \varphi_{s_2,s_1}) \\
\leq d_{\psi_{s_1,s_0}}(K)(\psi_{t,s_2} \circ \psi_{s_2,s_1}, \varphi_{t,s_2} \circ \varphi_{s_2,s_1}) + d_{\psi_{s_1,s_0}}(K)(\varphi_{t,s_2} \circ \psi_{s_2,s_1}, \varphi_{t,s_2} \circ \varphi_{s_2,s_1}) \\
\leq d_{\psi_{s_2,s_1}}(s_1)(K)(\psi_{t,s_2}, \varphi_{t,s_2}) + c d_K(\psi_{s_2,s_1}, \varphi_{s_2,s_1}) \\
\leq d_{\psi_{s_2,s_1}}(K)(\psi_{t,s_2}, \varphi_{t,s_2}) + c 2C |s_2 - s_1|^{\theta}
\]
and continuing in this way, we may show by induction that
\[
d_K(\psi_{t,s}, \varphi_{t,s}) \leq d_{\psi_{s_m,s_0}}(K)(\psi_{t,s_m}, \varphi_{t,s_m}) + c 2C \sum_{k=1}^{m} |s_k - s_{k-1}|^{\theta}.
\]

Taking \( m = n \) and then letting \( n \to \infty \) in the previous equation shows,
\[
d_K(\psi_{t,s}, \varphi_{t,s}) \leq c 2C \sum_{k=1}^{n} |s_k - s_{k-1}|^{\theta} \\
= c(K) 2Cn \left| \frac{t-s}{n} \right|^{\theta} \to 0 \text{ as } n \to \infty.
\]
This shows \( \psi_{t,s} = \varphi_{t,s} \) on \( K \) and as \( K \) was arbitrary, the uniqueness proof is complete.

5. Proofs of Proposition 1.34 and Corollary 1.36

Recall that if \( (M,g) \) is a Riemannian manifold, then the Koszul formula for the Levi-Civita Covariant derivative is;
\[
2g(\nabla_X Y, Z) = X[g(Y,Z)] + Y[g(X,Z)] - Z[g(X,Y)] \\
+ g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X).
\]

(5.1)

**Corollary 5.1.** Suppose that \( (M^d,g) \) is a Riemannian manifold such that \( TM \) is parallelizable and \( V: \mathbb{R}^d \to \Gamma(TM) \) is a dynamical system satisfying Eq. (1.20) as in Proposition 1.34 and \( Q(a,b) \in C^\infty(M,\mathbb{R}^d) \) is determined by \( [V_a, V_b] = V_{Q(a,b)} \) for all \( a,b \in \mathbb{R}^d \), see Eq. (1.21). Then the Levi-Civita covariant derivative, \( \nabla \), on \( TM \) satisfies
\[
\nabla_{V_a} V_b = \frac{1}{2} V_{Q(a,b)-Q^a_- b - Q^b_- a},
\]
where \( Q_a \in C^\infty(M,\text{End}(\mathbb{R}^d)) \) is defined by \( Q_a b = Q(a,b) \) and \( Q^a_- \) denotes the matrix transpose of \( Q_a \).
Proof. Since $g (V_a, V_b) = a \cdot b$ is constant on $M$ for each $a, b \in \mathbb{R}^d$, the Koszul formula in Eq. (5.1) shows for $a, b, c \in \mathbb{R}^d$ that,

\[
2g (\nabla_a V_b, V_c) = g ([V_a, V_b], V_c) - g ([V_a, V_c], V_b) - g ([V_b, V_c], V_a)
\]

\[
= g (V_{Q(a,b)}, V_c) - g (V_{Q(a,c)}, V_b) - g (V_{Q(b,c)}, V_a)
\]

\[
= Q (a, b) \cdot c - Q (a, c) \cdot b - Q (b, c) \cdot a
\]

\[
= Q (a, b) \cdot c - Q^t a \cdot Q^t b - c \cdot Q^t a
\]

\[
= g \left( V_{Q(a,b)} - Q^t a \cdot V_{Q(b,c)} - Q^t b \cdot V_c \right)
\]

from which Eq. (5.2) follows. \qed

Proof of Proposition 1.34. If $f \in C^\infty (M, \mathbb{R}^d)$ is a bounded function such that $V_{a_1} \ldots V_{a_k} f$ is bounded for all $a_j \in \mathbb{R}^d$ and $k \in \mathbb{N}_0$, then for $a \in \mathbb{R}^d$ we have

\[
[V_a, V_j] = V_{Q(a,j)} + V_{V_{a,j}} = V_{a,j} + Q (a, j).
\]

The assumptions given in Proposition 1.34 imply that $g := V_a f + Q (a, f) \in C^\infty (M, \mathbb{R}^d)$ is a bounded function such that $V_{a_1} \ldots V_{a_k} g$ is bounded for all $a_j \in \mathbb{R}^d$ and $1 \leq k \leq \kappa - 1$. We also have from Corollary 5.1 that

\[
\nabla_a V_j f = V_{a,j} f + \frac{1}{2} V_{Q(a,j)} - Q^t a \cdot V_{a,j} = V_g
\]

where

\[
g := V_{a,j} f + \frac{1}{2} (Q (a, f) - Q^t a \cdot Q^t a)
\]

is again a bounded function such that $V_{a_1} \ldots V_{a_k} g$ is bounded for all $a_j \in \mathbb{R}^d$ and $1 \leq k \leq \kappa - 1$. Thus it follows by induction that $L_{V_{a_1}} L_{V_{a_2}} \ldots L_{V_{a_{k-1}}} V_{a_k}$ is a bounded vector field such that $\nabla_{V_{a_k}} \left( L_{V_{a_1}} L_{V_{a_2}} \ldots L_{V_{a_{k-1}}} V_{a_k} \right)$ is bounded for all $1 \leq k \leq \kappa$ and $a, a_j \in \mathbb{R}^d$. \qed

Proof of Corollary 1.36. We begin by recalling some frame bundle basics which may be found in [30 Chapter III]. Associate to a tensor field, $G \in \Gamma \left( T^* (M)^\otimes k \otimes \text{End} (TM) \right)$, the function

\[
\tilde{G} : O (M) \rightarrow Hom \left( \left[ \mathbb{R}^d \right]^\otimes k, \text{End} \left( \mathbb{R}^d \right) \right)
\]

defined by

\[
\tilde{G} (u) [a_1 \otimes \cdots \otimes a_k] = u^{-1} G (ua_1 \otimes \cdots \otimes ua_k) u.
\]

If $A \in so (d)$, then a chain rule computation shows,

\[
(5.3) \quad \left( A^* \tilde{G} \right) (u) = - \text{ad}_A \tilde{G} (u) + \tilde{G} (u) D_A
\]

where

\[
(5.4) \quad D_A [a_1 \otimes \cdots \otimes a_k] := \sum_{j=1}^k a_1 \otimes \cdots \otimes a_{j-1} \otimes Aa_j \otimes a_{j+1} \cdots \otimes a_k.
\]
Similarly using the definition of $\nabla G$ one also shows $B(\cdot) \tilde{G} = \tilde{\nabla} G$. The dynamical system, $V : \mathbb{R}^d \times so(d) \to \Gamma (TO (M))$ as in Eq. (1.24), trivializes $TO (M)$ and we let $g^{O(M)}$ to be the unique Riemannian metric on $O(M)$ such that $V$ is isometric when $so(d)$ is equipped with the Hilbert-Schmidt inner product defined by

$$ A \cdot C := \text{tr} (A^* C) = - \text{tr} (AC). $$

Again referring to [30, Chapter III], if $A, C \in so(d)$ and $a, c \in \mathbb{R}^d$, then the following commutator formulas hold;

$$ [A^*, C^*] = [A, C]^*, \quad [A^*, B_a] = B_{Aa}, \quad \text{and} \quad [B_a, B_c] = - \tilde{R}_{a,c}, $$

where as above, $\tilde{R}_{a,c} (u) := \tilde{R} (u) a \otimes c = u^{-1} R (ua, uc) u$. Using these commutator formulas it follows that $[V_{(a,A)}, V_{(c,C)}] = Q_{((a,A),(c,C))}$ where

$$ Q_{((a,A),(c,C))} := (Ac - Ca, [A, C]) - \left( 0, \tilde{R}_{a,c} \right). $$

Moreover, from the derivative formulas in Eqs. (5.3) and (5.4), it follows that

$$ V_{(a_1, A_1)} \cdots V_{(a_k, A_k)} Q_{((a,A),(c,C))} $$

depends linearly on $(\tilde{R}, \nabla \tilde{R}, \ldots, \tilde{\nabla}^k \tilde{R})$. Hence the expressions in Eq. (5.5) are bounded functions on $O(M)$ for $0 \leq k \leq \kappa - 1$ provided $|\nabla^j R|_M < \infty$ for $0 \leq j \leq \kappa - 1$. The proof is then complete by an application of Proposition 1.34 and Theorem 1.27.

**6. Appendix: Variational Estimates**

**Lemma 6.1.** If $\sigma \in AC ([0, T], M)$ and $o \in M$ is fixed, then $[0, T] \ni t \to d (o, \sigma (t))$ is absolutely continuous and moreover,

$$ \left| \frac{d}{dt} d (o, \sigma (t)) \right| \leq |\dot{\sigma} (t)| \text{ for a.e. } t. $$

**Proof.** By the (reverse) triangle equality for metrics and the definition of $d$ in terms of minimization over absolutely continuous paths we find,

$$ |d (o, \sigma (t)) - d (o, \sigma (s))| \leq d (\sigma (s), \sigma (t)) \leq \ell_g (|J_{(s,t)}|) \leq \int_s^t |\dot{\sigma} (\tau)| d\tau. $$

This suffices to prove the claimed results in the lemma.

**Corollary 6.2.** Suppose that $\rho : [0, \infty) \to (0, \infty)$ is a continuous function,

$$ G (s) := \int_0^s \frac{1}{\rho (\sigma)} d\sigma, $$

$o \in M$ is fixed, and $\sigma \in AC ([0, T], M)$ satisfies

$$ |\dot{\sigma} (t)| \leq \rho (d (o, \sigma (t))) \text{ for a.e. } t \in [0, T], $$

for $0 \leq k \leq \kappa - 1$. The proof is then complete by an application of Proposition 1.34 and Theorem 1.27.
then
\[
G(d(o, \sigma(t))) \leq G(d(o, \sigma(0))) + t.
\]

Proof. By Lemma 6.1 and the assumption in Eq. (6.1) we have,
\[
\frac{d}{d\tau} d(o, \sigma(\tau)) \leq \left| \frac{d}{d\tau} d(o, \sigma(\tau)) \right| \leq |\dot{\sigma}(\tau)| \leq \rho(d(o, \sigma(\tau))) \quad \text{for a.e. } \tau.
\]

Since \(G\) is locally Lipschitz, \(\tau \rightarrow G(d(o, \sigma(\tau)))\) is still absolutely continuous and
\[
\frac{d}{d\tau} G(d(o, \sigma(\tau))) = \frac{1}{\rho(d(o, \sigma(\tau)))} \frac{d}{d\tau} d(o, \sigma(\tau)) \quad \text{for a.e. } \tau.
\]
Thus dividing Eq. (6.3) by \(\rho(d(o, \sigma(\tau)))\) and integrating the result shows
\[
G(d(o, \sigma(t))) - G(d(o, \sigma(0))) = \int_0^t \frac{1}{\rho(d(o, \sigma(\tau)))} \frac{d}{d\tau} d(o, \sigma(\tau)) d\tau \leq \int_0^t 1 d\tau = t
\]
from which the result follows. \(\square\)

Lemma 6.3. If \((M, g)\) is a complete Riemannian manifold and \(\rho\) and \(G\) are as in Corollary 6.2 such that
\[
\lim_{s \to \infty} G(s) = \int_0^\infty \frac{1}{\rho(\sigma)} d\sigma = \infty
\]
which guarantees that \(G : [0, \infty) \to [0, \infty)\) is bijective. If \(Y \in \Gamma(TM)\) satisfies the bound \(|Y(m)|_g \leq \rho(d(o,m))\) for all \(m \in M\), then \(Y\) is complete and we have the estimate,
\[
d(o, e^{tY}(m)) \leq G^{-1}(G(d(o,m)) + t) \quad \forall \ t \geq 0.
\]
Proof. Let \(\sigma : [0, T) \to M\) denote a solution to \(\dot{\sigma}(t) = Y(\sigma(t))\). Then by Corollary 6.2
\[
G(d(o, \sigma(t))) \leq G(d(o, \sigma(0))) + t
\]
which is equivalent to Eq. (6.4) since \(G\) is bijective and increasing and bijective. From this expression we see that \(\sigma(t)\) can not explode and this shows \(Y\) is complete. The completeness of \((M, g)\) is used here to imply closed bounded sets are compact. \(\square\)

There are many other possible examples for \(\rho\). We will give two here.

Example 6.4 (Gronwall Estimates). The standard example for \(\rho\) is \(\rho(\sigma) = C(1 + \sigma)\) for some \(C < \infty\) in which case
\[
G(s) = C^{-1} \ln(1 + s),
\]
\[
G^{-1}(u) = e^{Cu} - 1,
\]
\[ G^{-1}(G(s) + t) = G^{-1}\left(C^{-1}\ln(1 + s) + t\right) = e^{C(C^{-1}\ln(1 + s) + t)} - 1 = (1 + s)e^{Ct} - 1. \]

Thus the estimate in Eq. (6.4) becomes the usual Bellman-Gronwall type estimate,
\[ d(o, e^{tY}(m)) \leq (1 + d(o, m))e^{Ct} - 1 \quad \forall \; t \geq 0. \]

**Example 6.5** (Double Exponential Growth Estimates). In this example let us suppose that
\[ \rho(\sigma) = C(1 + \sigma)(1 + \ln(1 + \sigma)). \]
After making the change of variables, \( u = \ln(1 + \sigma) \), one shows
\[ G(s) = C^{-1}\ln(1 + \ln(1 + s)), \quad \text{and} \]
\[ G^{-1}(u) = \exp(\exp(\exp(Cu) - 1)) - 1. \]

Using these expressions the estimate in Eq. (6.4) becomes,
\[ d(o, e^{tY}(m)) \leq e^{[e^{Ct} - 1]}(1 + d(o, m))e^{Ct} - 1. \]

Our next goal is to understand the previous estimates in terms of conformal changes of the Riemannian metric \( g \). We begin with the following basic lemma.

**Lemma 6.6.** Suppose \((M, g, o)\) is a pointed Riemannian manifold, \( \rho \in C^1([0, \infty), (0, \infty)) \), and \( \tilde{g} \) is the continuous metric on \( M \) defined by
\[ \tilde{g}(v, w) = \rho^2(d(o, m))g(v, w) \]
for all \( v, w \in T_mM \) and \( m \in M \).

If \( p \in M \) and there exists a \( g \)-length minimizing geodesic, \( \sigma : [0, 1] \to M \), joining \( o \) to \( p \), then
\[ d_{\tilde{g}}(o, p) = \int_0^{d(o, p)} \rho(u) \, du \]
and moreover,
\[ d_{\tilde{g}}(o, p) = \ell_{\tilde{g}}(\sigma), \quad \text{where for any } \gamma \in AC([0, 1], M), \]
\[ \ell_{\tilde{g}}(\gamma) := \int_0^1 \sqrt{\tilde{g}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt = \int_0^1 \rho(d(o, \gamma(t))) \, |\dot{\gamma}(t)|_g \, dt. \]

**Proof.** Let \( AC_{o,p}([0, 1], M) \) denote those paths, \( \gamma \in AC([0, 1], M) \), such that \( \gamma(0) = o \) and \( \gamma(1) = p \). For any \( \gamma \in AC_{o,p}([0, 1], M) \), we have
\[ \ell_{\tilde{g}}(\gamma) \geq \int_0^1 \rho(d(o, \gamma(t))) \, |\dot{\gamma}(t)|_g \, dt \]
\[ \geq \int_0^1 \rho(d(o, \gamma(t))) \frac{d}{dt} d(o, \gamma(t)) \, dt = \int_0^{d(o, p)} \rho(u) \, du, \]
wherein the inequality is a result of Lemma 6.1. Thus we conclude that
\[ \hat{d}(o, p) = \inf \{ \ell_{\tilde{g}}(\gamma) : \gamma \in AC_{o,p}([0, 1], M) \} \geq \int_0^{d(o, p)} \rho(u) \, du. \]
If we now further assume that there exists a $g$-length minimizing geodesic, \( \sigma (t) \), joining \( o \) to \( p \), then \( |\dot{\sigma} (t)|_g = d (o, p) \) and \( d (o, \sigma (t)) = td (o, p) \) for \( 0 \leq t \leq 1 \) and so for this \( \sigma \),

\[
\ell_{\tilde{g}} (\sigma) = \int_0^1 \rho (d (o, \sigma (t))) |\dot{\sigma} (t)|_g \, dt
\]

\[
= \int_0^1 \rho (td (o, p)) d (o, p) \, dt = \int_0^{d (o, p)} \rho (u) \, du
\]

which completes the proof.

Our next goal is to smooth out the conformal factor, \( \rho^2 (d (o, m)) \), used Lemma 6.6 by replacing \( d (o, \cdot) \) by an appropriate smooth approximation. The next lemma explains how one can do this in Euclidean space.

**Lemma 6.7.** For every \( \varepsilon > 0 \), let \( |x|_\varepsilon := \sqrt{|x|^2 + \varepsilon^2} \) for \( x \in \mathbb{R}^d \). Then \( |\cdot|_\varepsilon \in C^\infty (\mathbb{R}^d \to [0, \infty)) \) satisfies, for all \( x, y \in \mathbb{R}^d \), the estimates:

\[
|\cdot|_\varepsilon \in C^\infty (\mathbb{R}^d \to [0, \infty))
\]

(6.5) \( |x| \leq |x|_\varepsilon \leq |x| + \varepsilon \), and

(6.6) \( |x|_\varepsilon - |y|_\varepsilon \leq |x - y| \).

**Proof.** Equation (6.5) easily follows from the fact that \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \) for \( a, b \geq 0 \). If we let \( g_\varepsilon (t) := \sqrt{t^2 + \varepsilon^2} \) for \( t \in \mathbb{R} \), then

\[
|g_\varepsilon (t) - g_\varepsilon (s)| \leq \int_s^t |\dot{g}_\varepsilon (\tau)| \, d\tau = \int_s^t \left| \frac{\tau}{\sqrt{\tau^2 + \varepsilon^2}} \right| \, d\tau \leq |t - s|
\]

for all \( s, t \in [0, \infty) \). Taking \( t = |x| \) and \( s = |y| \) in this inequality shows

\[
|x|_\varepsilon - |y|_\varepsilon \leq ||x| - |y|| \leq |x - y|
\]

which verifies Eq. (6.6).

According to Greene and Wu [24, Section 2], [38], and [1, Theorem 1] the analogue of Lemma 6.7 is valid on any Riemannian manifold \((M, g)\). What is shown in these references is that for any \( r > 0 \) and \( \delta \in C (M, (0, \infty)) \), there exists \( f \in C^\infty (M, \mathbb{R}) \) such that \( \text{Lip} (f) \leq 1 + r \) and

\[
|d (o, p) - f (p)| \leq \delta (p)
\]

for all \( p \in M \).

If we now fix \( \varepsilon > 0 \) and take \( \delta (p) = \frac{1}{2} \varepsilon \) and \( h (p) := f (p) + \frac{1}{2} \varepsilon \), then \( \text{Lip} (h) = \text{Lip} (f) \leq 1 + r \) and the above displayed inequality is equivalent to

\[
d (o, p) \leq h (p) \leq d (o, p) + \varepsilon \text{ for all } p \in M.
\]

We now fix \( \varepsilon, \delta > 0 \) and \( h \in C^\infty (M, \mathbb{R}) \) so that

(6.7) \( d (o, p) \leq h (p) \leq d (o, p) + \varepsilon \) and \( \text{Lip} (h) \leq 1 + r \).

**Proposition 6.8** (Conformal change of metrics). If \((M, g)\) is a Riemannian manifold with Levi-Civita derivative, \( \nabla \), and \( \tilde{g} = e^{2\varepsilon} g \), then the associated Levi-Civita covariant derivative, \( \nabla \), is given by

\[
\tilde{\nabla} X Y = \nabla_X Y + X \varphi \cdot Y + Y \varphi \cdot X - g (X, Y) \nabla \varphi
\]
where \( \nabla \varphi \) is being used to denote the gradient of \( \varphi \) relative to the metric \( g \).

**Proof.** Using

\[
X [\bar{g} (Y, Z)] = X [e^{2\varphi} g (X, Y)] = 2 [X \varphi] \bar{g} (Y, Z) + e^{2\varphi} [X g (Y, Z)]
\]

and similar expressions for \( Z \bar{g} (X, Y) \) and \( Y \bar{g} (X, Z) \), the Koszul formula \( \bar{\nabla} X Y \) and \( \bar{\nabla} X Y \) may be used to show,

\[
2 \bar{g} (\bar{\nabla} X Y, Z) = e^{2\varphi} 2g (\nabla X Y, Z) + 2 [X \varphi] \bar{g} (Y, Z) + 2 [Y \varphi] \bar{g} (X, Z) - [Z \varphi] \bar{g} (X, Y)
\]

\[
= 2 \bar{g} (\nabla X Y, Z) + 2 [X \varphi] \bar{g} (Y, Z) + 2 [Y \varphi] \bar{g} (X, Z) - g (Z, \nabla \varphi) e^{2\varphi} g (X, Y)
\]

which easily implies Eq. (6.8). \( \square \)

**Corollary 6.9.** Let \( \varepsilon, r > 0 \) be given and choose \( h \in C^\infty (M, [0, \infty)) \) such that Eq. (6.7) holds and define \( \bar{g} \) to be the metric on \( M \) given by

\[
\bar{g} (v, w) = \left[ \frac{1}{1 + h (m)} \right]^2 g (v, w) \text{ for } v, w \in T_m M \text{ and } m \in M.
\]

If \( (M, g) \) is a complete Riemannian manifold, then;

(1) for all \( p \in M, \)

\[
\frac{1}{1 + \varepsilon} \ln (1 + d (o, p)) \leq \bar{d} (o, p) \leq \ln (1 + d (o, p)),
\]

(2) and \( (M, \bar{g}) \) is still a complete Riemannian manifold.

**Proof.** Let

\[
\bar{g} (v, w) = \left[ \frac{1}{1 + d (o, p)} \right]^2 g (v, w) \text{ for } v, w \in T_p M \text{ and } p \in M.
\]

From Eq. (6.7) it follows that

\[
\frac{1}{1 + d (o, p)} \leq \frac{1}{1 + d (o, p) + \varepsilon} \leq \frac{1}{1 + h (p)} \leq \frac{1}{1 + d (o, p)} \forall \ p \in M
\]

and therefore

\[
\frac{1}{1 + \varepsilon} \sqrt{\bar{g}} \leq \sqrt{\bar{g}} \leq \sqrt{\bar{g}}
\]

and hence

\[
\frac{1}{\sqrt{1 + \varepsilon}} \bar{d} \leq \bar{d} \leq \bar{d}
\]

where \( \bar{d} (o, p) = \ln (1 + d (o, p)) \) in this case. This proves item (1).

For the proof of item (2), we observe that Eq. (6.9) shows that \( d \) and \( \bar{d} \) have the same notions of bounded sets. As \( d \) and \( \bar{d} \) generate the same topology, \( d \) and \( \bar{d} \) also have the same notions of closed sets. Thus a subset, \( A \subset M \) is \( d \)-closed and bounded iff \( A \) is \( d \)-closed and bounded iff \( A \) is compact since \( (M, g) \) is complete. Combining all of these facts shows \( (M, \bar{g}) \) is still complete. \( \square \)
Proposition 6.10. Let us continue the notation used in Corollary 6.9 and let $Y \in \Gamma (TM)$. Then $Y$ is $\bar{g}$ bounded iff there exists $\bar{C} < \infty$ such that

$$\tag{6.10} |Y (m)|_g \leq \bar{C} \left(1 + d(o, m)\right) \forall m \in M.$$ 

Let us now suppose that $Y$ satisfies the estimate in Eq. (6.10). Under this assumption, we have

$$\tag{6.11} \|\bar{\nabla} Y\|_{\bar{g}, M} := \sup_{|v|_g = 1} |\nabla v Y|_g < \infty$$

ciaff

$$\tag{6.12} |\nabla Y|_{\bar{g}, M} := \sup_{|u|_g = 1} |\nabla u Y|_g < \infty.$$ 

Moreover, if $|\nabla Y|_{\bar{g}, M} < \infty$, then both estimates in Eq. (6.10) and (6.11) hold.

Proof. Suppose that $Y \in \Gamma (TM)$, then $Y$ is $\bar{g}$ bounded iff

$$\tag{6.13} \infty > C := \sup_{m \in M} |Y (m)|_{\bar{g}} = \sup_{m \in M} \left(\frac{1}{1 + h(m)} |Y (m)|_g\right),$$

i.e. if there exists $C < \infty$ such that

$$\tag{6.14} |Y (m)|_{\bar{g}} \leq C \left(1 + h(m)\right) \forall m \in M$$

and it is easily seen that this is equivalent to the existence of $\bar{C} < \infty$ such that Eq. (6.10) holds.

Now let $Y \in \Gamma (TM)$ satisfies Eq. (6.10) or equivalently Eq. (6.14) for some $C < \infty$. Let us note for $v_m \in T_m M$ that

$$1 = |v|_g = \frac{1}{1 + h(m)} |v_m|_g \iff |v_m|_g = 1 + h(m).$$

Thus if we let

$$u_m := \frac{1}{1 + h(m)} v_m,$$

then $|u_m|_g = 1$ and

$$|\nabla v_m Y|_g = \frac{1}{1 + h(m)} |\nabla v_m Y|_g = |\nabla u_m Y|_g$$

from which it follows that

$$|\nabla Y|_{\bar{g}, M} = \sup_{|v|_g = 1} |\nabla v_m Y|_g.$$ 

Next we use Proposition 6.8 with

$$\varphi = \ln \left(\frac{1}{1 + h}\right) = -\ln (1 + h)$$

and

$$w_m \varphi = -\frac{1}{1 + h(m)} w_m h = -\frac{1}{1 + h(m)} g(h(m), w_m)$$
in order to see that
\[
\overline{\nabla}_{u_m} Y = \nabla_{u_m} Y + u_m \varphi \cdot Y (m) + (Y \varphi) (m) \cdot u_m - g (u_m, Y (m)) \nabla \varphi (m)
\]
\[
= \nabla_{u_m} Y - K (u_m, Y (m))
\]
where
\[
K (u_m, Y (m)) = \frac{1}{1 + \bar{h} (m)} \left[ (u_m h) Y (m) + (Y h) (m) \cdot u_m - g (u_m, Y (m)) \nabla h (m) \right]
\]
which satisfies,
\[
|K (u_m, Y (m))| \leq (1 + r) 3C,
\]
with C as in Eq. (6.13). Thus we see, under the condition that \( C < \infty \), that
\[
\infty > |\overline{\nabla} Y|_{g, M} = \sup_{|u|_g = 1} |\nabla_{u_m} Y|_g \iff \sup_{|u|_g = 1} |\nabla_{u_m} Y| < \infty.
\]
Conversely if \( |\nabla Y|_{g, M} := \sup_{|u|_g = 1} |\nabla_{u_m} Y| < \infty \), then for \( m \in M \) and \( \sigma (t) \) an absolutely continuous curve joining \( o \) to \( m \) we have
\[
|Y (m)|_g = \left| \left( / (\sigma)^{-1} \right) Y (0) \right|_g = \left| \left( / (\sigma)^{-1} \right) Y (\sigma (1)) \right|_g
\]
\[
= |Y (o) + \int_0^1 / (\sigma)^{-1} \nabla \sigma (t) Y dt|_g \leq |Y (o)|_g + \int_0^1 |\nabla \sigma (t) Y |_g \ dt
\]
\[
\leq |Y (o)|_g + |\nabla Y|_{g, M} \int_0^1 |\sigma (t)| \ dt = |Y (o)|_g + |\nabla Y|_{g, M} \ell_g (\sigma).
\]
Taking the infimum over all such paths shows that
\[
|Y (m)|_g \leq |Y (o)|_g + |\nabla Y|_{g, M} d_g (o, m).
\]
and we have shown that in fact \(|\nabla Y|_{g, M} < \infty\) implies both Eq. (6.10) and (6.11) hold.

\[\square\]

7. Appendix: Rough Path Basics

Let \( 0 < T < \infty \), \( x (\cdot) \in C^1 ([0, T], \mathbb{R}^d) \), and for \( s, t \in [0, T] \) let \( X_{s,t} \in G_{\beta \gamma \alpha} (\mathbb{R}^d) \) denote the solution to the ODE,

\[
\frac{d}{dt} X_{s,t} = X_{s,t} \dot{x} (t) \quad \text{with} \quad X_{s,s} = 1.
\]

If we let \( g (t) := X_{0,t} \), then
\[
\dot{g} (t) = g (t) \dot{x} (t) \quad \text{with} \quad g (0) = 1
\]
and we easily see that \( X_{s,t} = g (s)^{-1} g (t) \) and in particular, if \( s, t, u \in [0, T] \), then
\[
X_{s,t} X_{t,u} = g (s)^{-1} g (t) g (t)^{-1} g (u) = g (s)^{-1} g (u) = X_{s,u}.
\]
Proposition 7.1. If \( X_{s,t}^{(\kappa)} \in G_{\text{ge}}^{(\kappa)}(\mathbb{R}^d) \) is as in Eq. (7.1) and for all \( k \in \mathbb{N} \) we let
\[
(7.2) \quad X_{s,t}^k := \int_s^t dt_k \int_s^{t_{k-1}} dt_{k-1} \cdots \int_s^{t_2} dt_1 \dot{x}(t_1) \otimes \cdots \otimes \dot{x}(t_k),
\]
then
\[
(7.3) \quad X_{s,t}^{(\kappa)} = 1 + \sum_{k=1}^{\kappa} X_{s,t}^k.
\]
We may also write \( X_{s,t}^k \) as
\[
(7.4) \quad X_{s,t}^k = \int_{s \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq t} dx(t_1) dx(t_2) \cdots dx(t_k) \text{ if } t \geq s
\]
and
\[
(7.5) \quad X_{s,t}^k = (-1)^k \int_{t \leq t_1 \leq t_2 \leq \cdots \leq t_{k-1} \leq t_k \leq s} dx(t_k) dx(t_{k-1}) \cdots dx(t_1) \text{ if } t \leq s.
\]

Proof. We will show, for all \( 0 \leq m \leq \kappa \), that
\[
(7.6) \quad X_{s,t} = 1 + \sum_{k=1}^{m} \int_s^t dt_1 \int_{s \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq t} dt_{k-1} \cdots dt_1 \dot{x}(t_k) \cdots \dot{x}(t_1) + R_m(s,t)
\]
where
\[
(7.7) \quad R_m(s,t) = \int_s^t dt_1 \int_{s \leq t_1 \leq t_2 \leq \cdots \leq t_{m-1} \leq t} dt_{m-1} \cdots \int_{s \leq t_1 \leq t_2 \leq \cdots \leq t_{m-1} \leq t} dt_1 X_{s,t_{m+1}} \dot{x}(t_{m+1}) \dot{x}(t_m) \cdots \dot{x}(t_1)
\]
and by convention, \( \sum_{k=1}^{m} (\ldots) \equiv 0 \). For \( m = 0 \), Eq. (7.6) reads,
\[
X_{s,t} = 1 + R_0(s,t) = 1 + \int_s^t dt_1 X_{s,t_1} \dot{x}(t_1)
\]
which holds true because the fundamental theorem of calculus combined with the ODE in Eq. (7.1). To complete the inductive proof we use the identity,
\[
X_{s,t_{m+1}} = 1 + \int_s^{t_{m+1}} dt_{m+2} X_{s,t_{m+2}} \dot{x}(t_{m+2})
\]
in Eq. (7.7) to find,
\[
R_m(s,t) = \int_s^t dt_1 \int_{s \leq t_1 \leq t_2 \leq \cdots \leq t_{m-1} \leq t} dt_{m-1} \cdots \int_{s \leq t_1 \leq t_2 \leq \cdots \leq t_{m-1} \leq t} dt_1 X_{s,t_{m+1}} \dot{x}(t_{m+1}) \dot{x}(t_m) \cdots \dot{x}(t_1)
\]
\[
+ R_{m+1}(s,t).
\]
Taking \( m = \kappa \) in Eq. (7.6) while using \( R_\kappa(s,t) \equiv 0 \) in \( T^{(\kappa)}(\mathbb{R}^d) \), gives,
\[
X_{s,t} = 1 + \sum_{k=1}^{\kappa} \int_s^t dt_1 \int_{s \leq t_1 \leq t_2 \leq \cdots \leq t_{k-1} \leq t} dt_k \dot{x}(t_k) \cdots \dot{x}(t_1)
\]
and then relabeling the \((t_1, \ldots, t_k)\) to \((t_k, \ldots, t_1)\) in each term gives Eq. (7.3). The identities in Eqs. (7.4) and (7.5) are fairly simple rewrites of Eq. (7.2). For example if \(t \leq s\), the limits in each of the iterated integrals go from larger times to smaller time and so switching each of these limits gives rise to the factor \((-1)^k\). The relationship between all of times \((t_1, \ldots, t_k)\) when \(t \leq s\) are \(t \leq t_k \leq t_{k-1} \leq \cdots \leq t_1 \leq s\) and so

\[
X_{s,t}^k = (-1)^k \int_{t \leq t_k \leq t_{k-1} \leq \cdots \leq t_1 \leq s} dx (t_1) dx (t_2) \cdots dx (t_k).
\]

Lastly relabeling the \((t_1, \ldots, t_k)\) to \((t_k, \ldots, t_1)\) in this identity gives the second identity in Eq. (7.4).

**Notation 7.2.** For \(1 \leq k < \infty\), let \(\sigma_k : [\mathbb{R}^d]^{\otimes k} \to [\mathbb{R}^d]^{\otimes k}\) be the isometric isomorphism uniquely determined by

\[
\sigma_k [v_1 \otimes v_2 \otimes \cdots \otimes v_k] = v_k \otimes \cdots \otimes v_2 \otimes v_1 \forall v_1, v_2, \ldots, v_k \in \mathbb{R}^d.
\]

**Corollary 7.3.** If \(0 \leq s \leq t \leq T\) and \(1 \leq k \leq \kappa\), then \(X_{t,s}^k = (-1)^k \sigma_k X_{s,t}^k\) or equivalently stated,

\[
X_{t,s} = 1 + \sum_{k=1}^\kappa (-1)^k \sigma_k X_{s,t}^k.
\]

**Proof.** From Eqs. (7.4) and (7.5),

\[
X_{s,t} = 1 + \sum_{k=1}^\kappa \int_{s \leq t_k \leq t_{k-1} \leq \cdots \leq t_1 \leq t} dx (t_k) \cdots dx (t_2) dx (t_1)
\]

and

\[
X_{t,s} = 1 + \sum_{k=1}^\kappa (-1)^k \int_{s \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq t} dx (t_k) \cdots dx (t_2) dx (t_1)
\]

\[
= 1 + \sum_{k=1}^\kappa (-1)^k \int_{s \leq t_k \leq t_{k-1} \leq \cdots \leq t_1 \leq t} dx (t_1) dx (t_2) \cdots dx (t_k)
\]

\[
= 1 + \sum_{k=1}^\kappa (-1)^k \sigma_k X_{s,t}^{(k)}.
\]

**Corollary 7.4.** If \(g = 1 + \sum_{k=1}^\kappa g_k \in G^{(\kappa)}_{geo}(\mathbb{R}^d)\), then

\[
g^{-1} = 1 + \sum_{k=1}^\kappa (-1)^k \sigma_k g_k.
\]

**Proof.** By Chow’s theorem, to each \(g \in G^{(\kappa)}_{geo}(\mathbb{R}^d)\), there exists a path \(\cdot \in C^1([0,1], \mathbb{R}^d)\) such that \(g = X_{0,1}\). Therefore,

\[
g^{-1} = X_{1,0} = 1 + \sum_{k=1}^\kappa (-1)^k \sigma_k X_{0,1}^{(k)} = 1 + \sum_{k=1}^\kappa (-1)^k \sigma_k g_k.
\]
Corollary 7.5. If $g \in G_{\text{geo}}^{(\kappa)}(\mathbb{R}^d)$, then $|g_k| = |[g^{-1}]_k|$ for $1 \leq k \leq \kappa$ and in particular $N(g) = N(g^{-1})$.

The next result provides motivation for the constructions used in this paper. Let $V : \mathbb{R}^d \to \Gamma(TM)$ be a dynamical system, $x(\cdot) \in C^1([0,T], \mathbb{R}^d)$, and $\varphi_{t,s} \in \text{Diff}(M)$ denote the solution to

$$\dot{\varphi}_{t,s} = V \circ \varphi_{t,s} \text{ with } \varphi_{s,s} = \text{Id}_M.$$  

In the proof below we will make use of the simple observation that

$$\frac{d}{d\sigma} X_{\kappa,s,t} = -\dot{x}(\sigma) X_{\kappa,s,t}^{-1} \text{ for any } \kappa \in \mathbb{N}.$$  

Theorem 7.6. If $f \in C^{\infty}(M)$, then for any $s, t \in [0,T]$ and $\kappa \in \mathbb{N}_0$,

$$f \circ \varphi_{t,s} = V_{X_{\kappa,s,t}} f - \int_s^t \left( \frac{d}{d\sigma} X_{\kappa+1,\sigma,t} \right) \circ \varphi_{\sigma,s} d\sigma$$

$$= V_{X_{\kappa,s,t}} f + \int_s^t \left( \frac{d}{d\sigma} X_{\kappa+1,\sigma,t} f \right) \circ \varphi_{\sigma,s} d\sigma.$$  

Proof. When $\kappa = 0$, Eq. (7.12) states

$$f \circ \varphi_{t,s} = f + \int_s^t (V_{\dot{x}(\sigma)} f) \circ \varphi_{\sigma,s} d\sigma$$

which holds by the fundamental theorem of calculus applied to the differential identity,

$$\frac{d}{dt} f \circ \varphi_{t,s} = (V_{\dot{x}(t)} f) \circ \varphi_{t,s} \text{ with } f \circ \varphi_{s,s} = f.$$  

For the inductive step we use the fundamental theorem of calculus along with the chain rule to conclude,

$$-V_{X_{\kappa+1,s,t}} f = \left( V_{X_{\kappa+1,s,t}} f \right) \circ \varphi_{\sigma,s}|_{\sigma=s} = \int_s^t \left( V_{X_{\kappa+1,s,t}} f \right) \circ \varphi_{\sigma,s} d\sigma$$

$$= \int_s^t \left( \frac{d}{d\sigma} X_{\kappa+1,\sigma,t} f \right) \circ \varphi_{\sigma,s} d\sigma + \int_s^t \left( \frac{d}{d\sigma} X_{\kappa,s,t} f \right) \circ \varphi_{\sigma,s} d\sigma$$

$$= \int_s^t \left( \frac{d}{d\sigma} X_{\kappa+1,\sigma,t} f \right) \circ \varphi_{\sigma,s} d\sigma - \int_s^t \left( \frac{d}{d\sigma} X_{\kappa+2,\sigma,t} f \right) \circ \varphi_{\sigma,s} d\sigma.$$  

Putting this identity into Eq. (7.11) gives,

$$f \circ \varphi_{t,s} = V_{X_{\kappa,s,t}} f + V_{X_{\kappa+1,s,t}} f - \int_s^t \left( \frac{d}{d\sigma} X_{\kappa+1,\sigma,t} f \right) \circ \varphi_{\sigma,s} d\sigma$$

$$= V_{X_{\kappa,s,t}} f - \int_s^t \left( \frac{d}{d\sigma} X_{\kappa+2,\sigma,t} f \right) \circ \varphi_{\sigma,s} d\sigma$$

which completes the inductive proof.
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