Deformations of Differential Calculi

J. Madore
Laboratoire de Physique Théorique et Hautes Energies*
Université de Paris-Sud, Bât. 211, F-91405 Orsay

J. Mourad
GPS, Université de Cergy Pontoise
Site de St. Martin, F-95302 Cergy Pontoise

A. Sitarz
Johannes Gutenberg Universität
Institut für Physik, D-55099 Mainz

Abstract

It has been suggested that quantum fluctuations of the gravitational field could give rise in the lowest approximation to an effective noncommutative version of Kaluza-Klein theory which has as extra hidden structure a noncommutative geometry. It would seem however from the Standard Model, at least as far as the weak interactions are concerned, that a double-sheeted structure is the phenomenologically appropriate one at present accelerator energies. We examine here to what extent this latter structure can be considered as a singular limit of the former.

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*Laboratoire associé au CNRS, URA D0063
1 Motivation

It has been suggested (Madore & Mourad 1995) that quantum fluctuations of the gravitational field (Deser 1957, Isham et al. 1971) could give rise in the lowest approximation to an effective noncommutative version (Madore, 1990) of Kaluza-Klein theory which has as extra hidden structure a noncommutative geometry described by, for example an algebra $M_n$ of $n \times n$ complex matrices (Dubois-Violette et al. 1989, 1991). From the Standard Model it would seem however, at least as far as the weak interactions are concerned, that the phenomenologically most appropriate extra structure at accelerator energies would be one based on the algebra $\mathbb{H} \times \mathbb{C}$ (Connes & Lott 1992, Iochum & Schücker 1994). It would be possible to reconcile the implications of the quantum fluctuations with the experimental facts if one could show that in a consistent natural way one of the former structures could be deformed into the latter. We shall study here a simpler problem; we shall consider the algebra $\mathbb{C} \times \mathbb{C}$ as a singular contraction of the algebra $M_2$.

The algebra $\mathbb{C} \times \mathbb{C}$ is the algebra of complex-valued functions on the space of two points. The algebra $M_2$ can be also considered as describing the two-point structure, but in a symmetric way. The adjoint action of the group $SO_3$ on $M_2$ is the analogue of rotational symmetry in ordinary geometry. The contraction does not respect this symmetry; it is obviously broken by a specific choice of two points. One can imagine the algebra $M_2$ as describing a very fuzzy sphere with only two distinct regions. One can consider the contraction as a singular deformation of the sphere; its effect is to squeeze the sphere to a line and the two regions onto the end points. The algebra $\mathbb{C} \times \mathbb{C}$ can be also considered as describing a classical spin system which can take only 2 values. The corresponding quantum spin system is described by the algebra $M_2$ (Bratteli & Robinson 1989). So $\mathbb{C} \times \mathbb{C}$ can be considered as a limit of $M_2$ when $\hbar \to 0$.

The 2-point model and the $SO_3$-symmetric model depend each on a differential calculus over the corresponding algebra. It is our purpose here to show that the contraction can be extended to a map between the corresponding differential calculi. In the next two sections we recall briefly the two models and in Section 4 we define the contraction. In Section 5 we present our conclusions.

2 The noncommutative algebra

Let $\lambda_a$, for $1 \leq a \leq 3$, be an anti-hermitian basis of the Lie algebra of the special linear group $SL_2$. The product $\lambda_a \lambda_b$ can be written in the form

$$\lambda_a \lambda_b = \frac{1}{2} C^c_{ab} \lambda_c - \frac{1}{2} g_{ab}. \quad (2.1)$$

The structure constants $C^c_{ab}$ can be chosen real and the $g_{ab}$ can be expressed in terms of them by the equation

$$g_{ab} = -\frac{1}{4} C^c_{ad} C^d_{bc}. \quad (2.2)$$

We shall lower and raise indices with $g_{ab}$ and its inverse $g^{ab}$. The tensor $C_{abc}$ is completely antisymmetric. The $\lambda_a$ can be chosen such that the $g_{ab}$ are the components of the euclidean metric in 3 dimensions. For example they can be written in terms of the Pauli matrices $\sigma_a$ as $\lambda_a = -(i/\sqrt{2}) \sigma_a$. The structure constants are given then by $C_{123} = \sqrt{2}$. The matrix algebra $M_2$ is generated by the $\lambda_a$ as an algebra.
The derivations
\[ e_a = m \text{ad} \lambda_a \] (2.2)
form a basis over the complex numbers for the set \( \text{Der}(M_2) \) of all derivations of \( M_2 \). The mass parameter \( m \) has been included so that they have the correct physical dimensions. We recall that the adjoint action is defined on the element \( f \in M_2 \) by \( e_a f = m \text{ad} \lambda_a(f) = m [\lambda_a, f] \). Any element \( X \) of \( \text{Der}(M_2) \) can be written as a linear combination \( X = X^a e_a \) of the \( e_a \) where the \( X^a \) are complex numbers. The 3-dimensional vector space \( \text{Der}(M_2) \) is a Lie-algebra, the Lie algebra of the group \( SU_2 \); it is the analogue of the Lie algebra of global vector fields on a smooth manifold. In particular the derivations \( e_a \) satisfy the commutation relations \( [e_a, e_b] = m C_{abc} e_c \).

We define the 1-forms \( \Omega^1(M_2) \) over \( M_2 \) just as one does in the commutative case (Dubois-Violette 1988). We define \( df \) for \( f \in M_2 \) by the formula
\[ df(e_a) = e_a f. \] (2.3)
This means in particular that
\[ d\lambda^a(e_b) = m [\lambda_b, \lambda^a] = -m C_{abc} \lambda^c. \]
The set of \( d\lambda^a \) constitutes a system of generators of \( \Omega^1(M_2) \) as a left or right module but it is not a convenient one. For example \( \lambda^a d\lambda^b \neq d\lambda^b \lambda^a \). There is a better system of generators \( \theta^a \) completely characterized by the equations
\[ \theta^a(e_b) = \delta^a_b. \] (2.4)
The \( \theta^a \) are related to the \( d\lambda^a \) by the equations
\[ d\lambda^a = m C_{abc} \lambda^b \theta^c, \quad \theta^a = m^{-1} \lambda_b \lambda^a d\lambda^b. \] (2.5)
Because of the relation (2.4) we have
\[ \theta^a \theta^b = -\theta^b \theta^a, \quad \lambda^a \theta^b = \theta^b \lambda^a. \]
The products here are defined by
\[ (\theta^a \theta^b)(e_c, e_d) = \frac{1}{2} \delta^{ab}_{cd}, \quad (\lambda^a \theta^b)(e_c) = \lambda^a \theta^b(e_c), \quad (\theta^b \lambda^a)(e_c) = \theta^b(e_c) \lambda^a. \] (2.6)
The \( \theta^a \) generate an exterior algebra \( \wedge^* \) of dimension \( 2^3 \) and they satisfy the same structure equations as the components of the Maurer-Cartan form on the special linear group \( SL_2 \):
\[ d\theta^a = -\frac{1}{2} m C_{abc} \theta^b \theta^c. \] (2.7)
Using these relations it is easy to see that the algebra \( \Omega^*(M_2) \) is equal to the tensor product of \( M_2 \) and \( \wedge^* \):
\[ \Omega^*(M_2) = M_2 \otimes \wedge^*. \] (2.8)
It is therefore of dimension \( 4 \times 2^3 \).

The homology of the complex \( (\Omega^*(M_2), d) \) satisfies the isomorphisms
\[ H^2(M_2) \simeq H^1(M_2) = 0, \quad H^3(M_2) \simeq H^0(M_2) = \mathbb{Z}, \quad H^p(M_2) = 0, \quad p \geq 4. \] (2.9)
In the absence of a possible definition of homology groups this can be considered as a form of Poincaré duality. An arbitrary topological space $V$ has homology $H_*(V)$ as well as cohomology $H^*(V)$ which, if $V$ is a smooth manifold of dimension $n$, are isomorphic: $H_*(V) \cong H^{n-*}(V)$. Using the (co)homology Chern characters this isomorphism can be expressed in terms of the algebra $\mathcal{C}(V)$ of smooth functions on $V$ as an isomorphism of $K^*(\mathcal{C}(V))$ onto $K_*(\mathcal{C}(V))$ and as such generalized to a property of arbitrary algebras. Connes (1995) has stressed the importance of the role of this version of Poincaré duality as a necessary condition in distinguishing noncommutative geometries which can be considered as ‘smooth’ from those which are only ‘topological’. By ‘smooth’ we mean here something stronger, an algebra whose differential calculus is based on derivations.

From the generators $\theta^a$ we can construct the 1-form

$$\theta = -m\lambda_a \theta^a = -\frac{1}{2} \lambda_a d\lambda^a = \frac{1}{2} d\lambda_a \lambda^a. \quad (2.10)$$

It follows directly from the definitions that the exterior derivative $df$ of an element $f \in M_2$ can be written in terms of a commutator with $\theta$:

$$df = -[\theta, f]. \quad (2.11)$$

This is not true however for an arbitrary element of $\Omega^*(M_2)$. From (2.5) and (2.7) it follows that

$$d\theta + \theta^2 = 0. \quad (2.12)$$

As a left or right $M_2$-module $\Omega^1(M_2)$ is free with three generators but from (2.11) one sees that as a $M_2$-bimodule $\Omega^1(M_2)$ is generated by $\theta$ alone.

It is interesting to note that the differential algebra $\Omega^*(M_2)$ can be imbedded in a larger algebra in which there is an element $\theta$ such that (2.11) is satisfied for all elements of $\Omega^*(M_2)$. For the details we refer to Madore (1995). In general any differential calculus $\Omega^*(\mathcal{A})$ over an arbitrary associative algebra $\mathcal{A}$ can be enlarged by addition of an element $\theta$ such that (2.11) is satisfied for all elements of $\Omega^*(\mathcal{A})$. In the case of the de Rham calculus over a smooth manifold the extra element can be chosen to be the phase of the Dirac operator (Connes 1994).

One defines a Yang-Mills potential to be an anti-hermitian element $\omega \in \Omega^1(M_2)$. We can write it using $\theta$ as

$$\omega = \theta + \phi. \quad (2.13)$$

The unitary elements $U_2$ of the algebra $M_2$ can be considered as the group of gauge transformations. For $g \in U_2$ we have $\omega \mapsto g^{-1} \omega g + g^{-1} dg$. It is clear that in particular $\theta \mapsto \theta$ and therefore that $\phi \mapsto g^{-1} \phi g$. Expand $\phi$ in terms of the basis $\theta^a$: $\phi = \phi_a \theta^a$. It follows from (2.12) that the field strength $F$ of the potential $\omega$ is given by

$$F = d\omega + \omega^2 = \frac{1}{2} F_{ab} \theta^a \theta^b, \quad F_{ab} = [\phi_a, \phi_b] - m C^{c}_{ab} \phi_c. \quad (2.14)$$

Let $\mathcal{C}(\mathbb{R}^4)$ be the algebra of smooth functions on space-time. Using $\Omega^*(M_2)$ one can construct a differential calculus over the algebra $\mathcal{C}(\mathbb{R}^4) \otimes M_2$ which describes a fuzzy version of space-time. The electromagnetic lagrangian in the extended space can be written

$$\mathcal{L} = \frac{1}{4} \text{Tr}(F_{\alpha \beta} F^{\alpha \beta}) + \frac{1}{2} \text{Tr}(D_\alpha \phi_a D^\alpha \phi^a) - V(\phi), \quad (2.15)$$
where the Higgs potential $V(\phi)$ is given by

$$V(\phi) = -\frac{1}{4} \text{Tr}(F_{ab}F^{ab}).$$

(2.16)

The characteristic mass scale is the scale $m$ introduced in (2.2). Since the group is $U_2$ there are four gauge bosons $A^0, A^a$. In the 'broken phase' the Higgs kinematical term yields a mass term

$$\mathcal{L}_m = \frac{1}{2} m^2 A^αa A^βb \text{Tr}([\lambda_α, \lambda_β][\lambda_a, \lambda_b])g^{cd}.$$  

(2.17)

From (2.1) we see that the gauge bosons acquire masses given by

$$m_0 = 0, \quad m_a = 2m.$$  

(2.18)

For more details we refer to Dubois-Violette et al. (1989).

To construct the lagrangian (2.15) we used in an essential way a metric on the extra algebraic structure we added to space-time; it appears as the last factor in (2.17). We chose this metric to be the Killing metric defined in (2.1). This is the only metric with respect to which the derivations (2.2) are Killing derivations. We could have chosen however another one, for example that given by

$$\tilde{g}_{ab} = \text{diag}(1, 1, \epsilon^2).$$

(2.19)

We would find then the mass spectrum

$$\tilde{m}_0 = 0, \quad \tilde{m}_1 = \tilde{m}_2 = \sqrt{2}\epsilon^{-1}m, \quad \tilde{m}_3 = 2m.$$  

(2.20)

The two modes $A^1$ and $A^2$ decouple in the limit $\epsilon \to 0$.

From the second term in the lagrangian (2.15) we see that if we use the metric (2.19) then we must renormalize the amplitudes of the scalar fields:

$$\tilde{\phi}_1 = \phi_1, \quad \tilde{\phi}_2 = \phi_2, \quad \tilde{\phi}_3 = \epsilon^{-1}\phi_3.$$  

(2.21)

The non-vanishing components of $F_{ab}$ are given by

$$\tilde{F}_{23} = \epsilon[\tilde{\phi}_2, \tilde{\phi}_3] - m C^{123}\tilde{\phi}_1,$$
$$\tilde{F}_{31} = \epsilon[\tilde{\phi}_3, \tilde{\phi}_1] - m C^{231}\tilde{\phi}_2,$$
$$\tilde{F}_{12} = [\tilde{\phi}_1, \tilde{\phi}_2] - \epsilon m C^{312}\tilde{\phi}_3,$$

(2.22)

and the Higgs potential is given by

$$V(\phi) = -\frac{1}{2} \text{Tr}(\epsilon^{-2}\tilde{F}_{23}^2 + \epsilon^{-2}\tilde{F}_{31}^2 + \tilde{F}_{12}^2).$$

(2.23)

In the 'broken phase' the Higgs field $\phi$ acquires the non-vanishing vacuum value $\phi = -\theta$ and therefore from (2.13) the modes are described by the coefficients in the expansion of the gauge potential:

$$\omega = (\omega_0^a \lambda_a + \frac{i}{\sqrt{2}} \omega_0^b)\theta^b.$$  

(2.24)
If we renormalize as above and retain only quadratic terms we find

\[
V(\phi) \simeq - \frac{1}{2} m^2 \text{Tr} (\epsilon^{-2} c_1^2 + \epsilon^{-2} c_2^2 + c_3^2)
\]

(2.25)

with

\[
c_1 = (\epsilon C^a_{2b} \tilde{\omega}^b_3 - C^a_{3b} \tilde{\omega}^b_2 + C^1_{23} \tilde{\omega}^0_1) \lambda_a + \frac{i}{\sqrt{2}} C^1_{23} \tilde{\omega}^0_1,
\]

\[
c_2 = (C^a_{3b} \tilde{\omega}^b_1 - \epsilon C^a_{1b} \tilde{\omega}^b_3 + C^2_{31} \tilde{\omega}^0_2) \lambda_a + \frac{i}{\sqrt{2}} C^2_{31} \tilde{\omega}^0_2,
\]

\[
c_3 = (C^a_{1b} \tilde{\omega}^b_2 - C^a_{2b} \tilde{\omega}^b_1 + \epsilon C^3_{12} \tilde{\omega}^0_3) \lambda_a + \frac{i}{\sqrt{2}} C^3_{12} \tilde{\omega}^0_3.
\]

(2.26)

That is, we have to leading order in \(\epsilon\)

\[
\epsilon^2 m^{-2} V(\phi) \simeq 2(\tilde{\omega}^1_1 + \tilde{\omega}^2_1)^2 + 2(\tilde{\omega}^0_1 - \tilde{\omega}^2_1)^2 + (\tilde{\omega}^3_2)^2 + (\tilde{\omega}^0_2)^2 + (\tilde{\omega}^0_3)^2.
\]

(2.27)

We find then that

\[
\tilde{\omega}^1_2 + \tilde{\omega}^2_2 \rightarrow 0, \quad \tilde{\omega}^3_2 \rightarrow 0, \quad \tilde{\omega}^0_1 \rightarrow 0,
\]

\[
\tilde{\omega}^1_1 - \tilde{\omega}^2_2 \rightarrow 0, \quad \tilde{\omega}^3_2 \rightarrow 0, \quad \tilde{\omega}^0_2 \rightarrow 0.
\]

(2.28)

in the limit \(\epsilon \rightarrow 0\). But to within a gauge transformation we have

\[
\tilde{\omega}^3_2 = \tilde{\omega}^3_2, \quad \tilde{\omega}^3_1 = \tilde{\omega}^1_1, \quad \tilde{\omega}^1_2 = \tilde{\omega}^2_2.
\]

(2.29)

Therefore in the limit we have

\[
\omega \rightarrow \tilde{\omega}^1_1 (\lambda_1 \theta^1 + \lambda_2 \theta^2) + \chi_3 \theta^3
\]

(2.30)

where we have set

\[
\chi_3 = \frac{i}{\sqrt{2}} \tilde{\omega}^0_3 + \tilde{\omega}^3_1 \lambda_3.
\]

(2.31)

There remain 3 modes, a real scalar field \(\omega^1\) and a real scalar doublet \(\chi_3\). If we impose a reality condition and reduce the algebra \(M_2\) to the algebra \(H\) of quaternions the field \(\tilde{\omega}^0_3\) will not be present and \(\chi_3\) will be a singlet.

### 3 The commutative algebra

The algebra \(M_2\) has a natural \(\mathbb{Z}_2\) grading \(M_2 = M_2^+ \oplus M_2^-\) with the unit matrix and \(\lambda_3\) even and \(\lambda_1\) and \(\lambda_2\) odd. Let \(\eta\) be an antihermitean odd matrix with

\[
\eta^2 = -1
\]

(3.1)

and define the differential \(d\) of an arbitrary element \(\alpha \in M_2\) by

\[
d\alpha = -[\eta, \alpha]
\]

(3.2)

with a graded commutator. From (3.1) it follows that

\[
d\eta + \eta^2 = 0.
\]

(3.3)
The unit on the right is the unit in $M_2$ considered as a 2-form. Equation (3.3) is to be compared with (2.12). For all $p \geq 0$ we set

$$
\Omega^{2p}(\mathbb{C} \times \mathbb{C}) = M_2^+, \quad \Omega^{2p+1}(\mathbb{C} \times \mathbb{C}) = M_2^-.
$$

(3.4)

Then $\Omega^*(\mathbb{C} \times \mathbb{C})$ is a differential calculus over the algebra $\mathbb{C} \times \mathbb{C} = M_2^+$. One defines a Yang-Mills potential to be an anti-hermitian element $\omega \in \Omega^1(\mathbb{C} \times \mathbb{C}) = M_2^-$. We can write it using $\eta$ as

$$
\omega = \eta + \phi.
$$

(3.5)

The unitary elements $U_1 \times U_1$ of the algebra $M_2^+$ can be considered as the group of gauge transformations. For $g \in U_1 \times U_1$ we have $\omega \mapsto g^{-1} \omega g + g^{-1} dg$. It is clear that in particular $\eta \mapsto \eta$ and therefore that $\phi \mapsto g^{-1} \phi g$. It follows from (3.3) that the field strength $F$ of the potential $\omega$ is given by

$$
F = d\omega + \omega^2 = \phi^2 + 1.
$$

(3.6)

Using $\Omega^*(\mathbb{C} \times \mathbb{C})$ one can construct a differential calculus over $\mathcal{C}(\mathbb{R}^4) \otimes (\mathbb{C} \times \mathbb{C})$, the algebra of functions on a double-sheeted space-time. The electromagnetic lagrangian in the extended space can be written

$$
\mathcal{L} = \frac{1}{4} \text{Tr}(F_{\alpha\beta} F^{\alpha\beta}) + \frac{1}{2} \mu^2 \text{Tr}(D_\alpha \phi D^\alpha \phi) + \frac{1}{4} \mu^4 \text{Tr}((\phi^2 + 1)^2).
$$

(3.7)

The parameter $\mu^{-1}$ has the dimensions of length and is a measure of the distance between the two sheets of space-time. Since the group is $U_1 \times U_1$ there are two gauge bosons $A^0, A^3$. In the ‘broken phase’ they have masses given by

$$
m_0 = 0, \quad m_3 = \mu.
$$

(3.8)

For more details we refer to Connes & Lott (1990), or to Coquereaux et al. (1991, 1993). See also Dubois-Violette et al. (1991).

Contrary to (2.15) the lagrangian (3.7) does not involve a metric on the extra algebraic structure, which does not in fact possess one. Comparing (3.7) with the lagrangian (2.15) equipped with the metric defined by (2.19) we find that they have the same gauge-boson spectrum in the limit $\epsilon \to 0$ provided we set

$$
\mu = 2m.
$$

(3.9)

## 4 The contraction

Let $\mathcal{A}$ be an associative algebra and $\mathcal{A}_\epsilon$ a 1-parameter family of such algebras with $\mathcal{A}_0 = \mathcal{A}$. Then $\mathcal{A}_\epsilon$ is a deformation of $\mathcal{A}$. Deformations of associative algebras have been studied in general by Gerstenhaber (1964); all regular deformations of a simple algebra are trivial. We are interested in the algebra $\mathbb{C} \times \mathbb{C}$ as a singular contraction of $M_2$ and in the extension of the contraction to a map from the differential calculus of Section 2 into that of Section 3. In defining the contraction it is convenient to use the universal calculus.

Over any arbitrary associative algebra $\mathcal{A}$ one can construct (Karoubi 1983) the universal calculus $\Omega_u^*(\mathcal{A})$. One defines $\Omega_u^1(\mathcal{A})$ to be the kernel of the multiplication map which takes $\mathcal{A} \otimes \mathcal{A}$ into $\mathcal{A}$ and for each $p \geq 2$ one sets

$$
\Omega_u^p(\mathcal{A}) = \Omega_u^1(\mathcal{A}) \otimes \cdots \otimes \mathcal{A} \Omega_u^1(\mathcal{A})
$$

(4.1)
where the tensor product on the right contains \( p \) factors. The differential \( d_u \) which takes \( \mathcal{A} \) into \( \Omega_u^1(\mathcal{A}) \) is given, for arbitrary \( a \in \mathcal{A} \), by
\[
d_a a = 1 \otimes a - a \otimes 1. \tag{4.2}
\]
It can be extended to a map of \( \Omega_u^p(\mathcal{A}) \) into \( \Omega_u^{p+1}(\mathcal{A}) \) by Leibniz’s rule. There is a projection \( \phi \) of \( \Omega_u^*(\mathcal{A}) \) onto every other differential calculus over \( \mathcal{A} \) given by
\[
\phi(d_u a) = da. \tag{4.3}
\]
Let \( T^* \) be the tensor calculus over the vector space spanned by the \( \theta^a \). Then the universal differential calculus over \( M_2 \) is given by \( \Omega^*(M_2) = M_2 \otimes_C T^* \). Comparing this with (2.8) we see that there is a canonical imbedding,
\[
\Omega^*(M_2) \hookrightarrow \Omega_u^*(M_2), \tag{4.4}
\]
and that for \( p = 1 \) this imbedding is an isomorphism:
\[
\Omega^1(M_2) \simeq \Omega_u^1(M_2). \tag{4.5}
\]
If we introduce the element
\[
\theta_u = -\frac{1}{2} \lambda_a d_u \lambda^a \tag{4.6}
\]
then the isomorphism can be written as
\[
\phi(\theta_u) = \theta. \tag{4.7}
\]
The calculus (3.4) we constructed in Section 3 can in fact be identified with the universal calculus over \( \mathbb{C} \times \mathbb{C} \):
\[
\Omega^*(\mathbb{C} \times \mathbb{C}) \simeq \Omega_u^*(\mathbb{C} \times \mathbb{C}). \tag{4.8}
\]
If we introduce the element
\[
\eta_u = -\lambda_3 d_u \lambda_3 \tag{4.9}
\]
then the isomorphism can be written as
\[
\phi(\eta_u) = \eta. \tag{4.10}
\]
If we extend \( \phi \) to \( \Omega_u^2(\mathbb{C} \times \mathbb{C}) \) we find from (3.1) that
\[
\phi((d_u \lambda_3)^2) = -2, \tag{4.11}
\]
where the right-hand side is considered as an element of \( \Omega^2(\mathbb{C} \times \mathbb{C}) = M_2^+ \).

From general arguments it is known that any deformation of the algebraic structure of \( M_2 \) can be expressed as a deformation of the generators. For each \( \epsilon \geq 0 \) consider the change of basis
\[
\lambda'_a = (\epsilon \lambda_1, \epsilon \lambda_2, \lambda_3). \tag{4.12}
\]
Equation (2.1) can be rewritten in terms of the new basis with
\[
C'^{12}_{23} = C^{12}_{23}, \quad C'^{23}_{12} = C^{23}_{12}, \quad C'^{13}_{12} = \epsilon^2 C^{13}_{12}
\]
and
\[ g'_{ab} = \text{diag}(\epsilon^2, \epsilon^2, 1). \] (4.13)

The quantity \( C'_{abc} = g'_{ad}C'_{dcb} \) remains completely antisymmetric as it must by general arguments. Let \( \mathcal{A}_\epsilon \) be the algebra generated by the \( \lambda'_a \). Then \( \mathcal{A}_\epsilon = M_2 \) for all \( \epsilon > 0 \) but the sequence has the algebra \( \mathbb{C} \times \mathbb{C} \) as a singular limit. There are two steps involved here. The first is the singular contraction \( \epsilon \to 0 \) which leaves two nilpotent elements \( (\lambda'_1, \lambda'_2) \). To obtain \( \mathbb{C} \times \mathbb{C} \) one must quotient with respect to the ideal (the radical) generated by these elements.

Under the deformation the metric \( \tilde{g}_{ab} \) defined in (2.19) becomes
\[ \tilde{g}'_{ab} = \epsilon^2 \text{diag}(1, 1, 1). \]

So the deformation of the algebra (4.12) coupled with the deformation (2.19) yields again a diagonal metric. This establishes the relation between the two deformations (2.19) and (4.12).

We must show that this contraction can be lifted to a contraction of \( \Omega^*(M_2) \) into \( \Omega^*(\mathbb{C} \times \mathbb{C}) \) which respects the action of the differentials:
\[
\begin{array}{ccc}
\Omega^*(M_2) & \xleftarrow{\psi} & \Omega^*(\mathbb{C} \times \mathbb{C}) \\
\downarrow d & & \downarrow d \\
\Omega^*(M_2) & \xrightarrow{\epsilon \to 0} & \Omega^*(\mathbb{C} \times \mathbb{C})
\end{array}
\] (4.14)

It is obvious that the contraction of the algebra induces a contraction of the corresponding universal calculus,
\[ \Omega_u^*(M_2) \xleftarrow{\psi} \Omega_u^*(\mathbb{C} \times \mathbb{C}), \] (4.15)

which respects the action of the differentials \( d_u \). We noticed also in (4.4) that the forms over \( M_2 \) can be considered as elements of the universal algebra. The extension (4.14) is therefore uniquely determined by the original contraction of the algebra as a composition of (4.4), (4.15) and (4.8).

The inverse of (4.15) is an imbedding of \( \Omega_u^*(\mathbb{C} \times \mathbb{C}) \) into \( \Omega_u^*(M_2) \) which coupled with the projection of \( \Omega_u^*(M_2) \) onto \( \Omega^*(M_2) \) yields a homomorphism of differential algebras
\[ \Omega^*(\mathbb{C} \times \mathbb{C}) \xrightarrow{\psi} \Omega^*(M_2), \] (4.16)

under which
\[ \psi(\eta) = -m(\lambda_1 \theta^1 + \lambda_2 \theta^2) \] (4.17)

and therefore
\[ \psi(\eta^2) = \epsilon^2 m^2 C_{12}^3 \lambda_3 \theta^1 \theta^2 = -m^2 \lambda_3 d \theta^3 = \frac{1}{2} (d \lambda_3)^2. \] (4.18)

Using (4.11) we see that this is compatible with (3.1). We can conclude from (4.17) that \( \psi \) restricted to \( \Omega^1(\mathbb{C} \times \mathbb{C}) \) and to \( \Omega^2(\mathbb{C} \times \mathbb{C}) \) is a monomorphism and that, since the \( \theta^a \) anticommute,
\[ \psi(\Omega^p(\mathbb{C} \times \mathbb{C})) = 0, \quad p \geq 3. \]

If we compare (2.10) with (4.17) we find that
\[ \theta = \psi(\eta) - m \lambda_3 \theta^3. \] (4.19)
The second term commutes with the elements of $M_2^+$ and so (4.19) is compatible with (2.11) and (3.2). One sees also that because of (4.18), (2.12) is compatible with (3.3).

One can use the homomorphism $\psi$ to construct a new differential algebra $\Omega^*(\mathbb{C} \times \mathbb{C})$ over $\mathbb{C} \times \mathbb{C}$ given by

$$
\Omega^p(\mathbb{C} \times \mathbb{C}) = \Omega^p(\mathbb{C} \times \mathbb{C}), \quad p \leq 2;
$$

$$
\Omega^p(\mathbb{C} \times \mathbb{C}) = 0 \quad p \geq 3,
$$

which one can then extend to an algebra $\Omega^{*''}(\mathbb{C} \times \mathbb{C})$ by adding a 1-form $\eta^3$ with the relations

$$
\lambda_3 \eta^3 = \eta^3 \lambda_3, \quad \eta^3 \eta = -\eta \eta^3, \quad (\eta^3)^2 = 0, \quad d\eta^3 = -2\lambda_3.
$$

Notice that if $\lambda_3$ is considered as a 2-form then $d\lambda_3 = 0$ in $\Omega^*(\mathbb{C} \times \mathbb{C})$. The differential algebra $\Omega^{*''}(\mathbb{C} \times \mathbb{C})$ is more similar to that introduced by Connes & Lott over the algebra $\mathbb{H} \times \mathbb{C}$ than to the original $\Omega^*(\mathbb{C} \times \mathbb{C})$. The homomorphism $\psi$ can be extended to $\Omega^{*''}(\mathbb{C} \times \mathbb{C})$ by setting

$$
\psi(\eta^3) = m\theta^3
$$

and (4.19) can be rewritten as

$$
\theta = \psi(\eta - \lambda_3 \eta^3).
$$

The contraction of the potential $\omega$ and the associated field strength $F$ of Section 2 onto those of Section 3 is in principle uniquely and well defined. However under the deformation each individual mode has to be renormalized so that the coefficient in the kinetic term remains constant. In other words the coefficient of the modes has to be written using the normalized basis $\theta^a$ of the 1-forms and these do not vanish under the contraction. In fact $\theta^1$ and $\theta^2$ are singular and $\theta^3$ has a nonvanishing limit. From (4.17) we see that (2.30) can be written as

$$
m\omega \rightarrow m\chi_3 \theta^3 - \tilde{\omega}_1 \psi(\eta).
$$

Although $\chi_3$ is a function with values in $\mathbb{C} \times \mathbb{C}$ the second term of the right-hand side of (4.20) is not the image of an element of $\Omega^1(\mathbb{C} \times \mathbb{C})$ under $\psi$. The image of the 2-point model of Connes & Lott and Coquereaux under the homomorphism $\psi$ is found to be equal to the singular deformation (2.19) of the model of Dubois-Violette et al. with the addition of a real scalar gauge-invariant doublet $\chi_3$. If one uses the differential calculus $\Omega^{*''}(\mathbb{C} \times \mathbb{C})$ one can rewrite (4.22) as

$$
m\omega \rightarrow \psi(\chi_3 \eta^3 - \tilde{\omega}_1 \eta).
$$

The 2-point model with the new differential calculus is identical to the singular deformation (2.19) of the model of Dubois-Violette et al.

5 Conclusions

We have shown that the 1-forms of the universal differential calculus over $\mathbb{C} \times \mathbb{C}$ can be considered as a singular limit of a sequence of 1-forms of a differential calculus which is ‘smooth’ in the sense that it is based on the derivations of an algebra. Equivalently we have shown that to within a real scalar doublet the 2-point model of Connes & Lott and Coquereaux is a singular contraction of a model defined on a ‘smooth’ geometry. An obvious extension would be to investigate to what extent the extended model of Connes & Lott, using the algebra $\mathbb{H} \times \mathbb{C}$, can be obtained as a singular limit of geometries which are ‘smooth’.
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