Dipaths of length at least double the minimum outdegree

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Abstract

A special case of a conjecture by Thomassé is that any oriented graph with minimum outdegree $k$ contains a dipath of length $2k$. For the sake of proving whether or not a counterexample exists, we present reductions and establish bounds on both connectivity and the number of vertices in a counterexample.

We further conjecture that in any oriented graph with minimum outdegree $k$, every vertex belongs to a dipath of length $2k$.

1. Introduction

Bang-Jensen and Gutin [2] and Sullivan [5] note that Stéphan Thomassé has conjectured that any digraph with minimum outdegree $k$ and girth $g$ contains a dipath of length $k(g - 1)$, which, according to Sullivan, implies the Caccetta-Häggkvist conjecture. Bang-Jensen and Gutin and Sullivan further note that this conjecture remains open for $g = 3$. Bondy and Murty [3] present the following special case of this conjecture:

**Conjecture 1.** Any oriented graph with minimum outdegree $k$ contains a dipath of length $2k$.

**Conventions 1.** We use the term oriented graph to refer to an orientation of a finite simple graph, i.e. one containing no loops or multiple edges.

We use the term counterexample to refer to a counterexample to conjecture 1, i.e. an oriented graph with minimum outdegree $k$, that contains no dipath of length greater than $2k - 1$.

We use the term cycle to refer exclusively to directed cycles.

We use $k$ to refer to minimum outdegree, unless stated otherwise.

An oriented graph with minimum outdegree $k$ must contain at least $2k + 1$ vertices, and the oriented graph with the required outdegree on $2k + 1$ vertices must be an orientation of $K_{2k+1}$. Thus, any oriented graph containing a hamiltonian dipath necessarily contains a dipath of length at least $2k$.

For results relating to outdegree, indegree, and number of vertices (i.e. order), see Bang-Jensen and Gutin [2] and Jackson [5].
In this note we describe various reductions to simplify the oriented graphs under consideration. Furthermore, we establish bounds relating to connectivity and order. We also present a stronger version of conjecture 1.

**Conjecture 2.** All vertices in an oriented graph with minimum outdegree $k$ belong to a dipath of length $2k$. If a vertex is not a cut vertex, then it is also the initial vertex of a dipath of length $2k$.

2. Reductions

**Theorem 1.** If an oriented graph with minimum outdegree $k$ necessarily contains a directed path of length $2k$, then the following reductions will not affect the existence of such a path.

**Step 1.** If any vertices have outdegree greater than $k$, we delete arcs until all vertices have outdegree of exactly $k$. The resulting digraph may not necessarily be connected.

**Step 2.** A strongly connected digraph is one such that any vertex in such a component is reachable by any other vertex, and a strong component of a digraph is a maximal subdigraph containing vertices that can all reach one another [3].

We identify minimal subsets of vertices such that no vertex in the subset dominates any vertex outside of the subset. Each of subset $X$ must be a strong component, i.e. the outdegree of any vertex subset $Y$ of $X$ must be non-zero, because otherwise $X$ would not be minimal. By theorem 3.6 in Bondy and Murty [3], each vertex of $X$ must be reachable by any other vertex of $X$.

We delete any vertex not belonging to a minimal strong component. All of the components remaining after step 2 will also have minimum outdegree $k$, so each, if the conjecture holds, each of them will contain a dipath of length $2k$. Therefore, we need consider only one component and can treat it as an oriented graph.

One obvious consequence is that the oriented graph obtained through these steps contains no sources.

**Proof.** The deletion of vertices and/or arcs cannot increase the length of any dipath. Therefore, as long as the minimum outdegree is preserved, the digraph obtained through these deletions should also contain a dipath of length $2k$ if conjecture 1 holds.

**Conventions 2.** Unless otherwise stated, for the remainder of this note we will assume that any oriented graph mentioned has had the above reduction performed.

For an oriented graph with minimum outdegree of at least 1 obtained through these reductions, we can delete exactly one outgoing arc from each vertex and obtain a digraph with minimum outdegree $k - 1$. This digraph is not necessarily connected or strongly connected, but we can repeat step 2 to obtain an oriented graph.

It is always possible to select at least one vertex $v$ of an oriented graph $D$ with outdegree $k$ and then and create a subdigraph with minimum outdegree $k - 1$ that does not contain $v$. Delete each arc dominating $v$, which decreases by one the outdegree of each vertex dominating $v$. $v$ is now a source and can be deleted.
3. Specific cases

Theorem 2. Any vertex in an oriented graph $D$ with minimum outdegree 1 belongs to a dipath of length at least 2.

Proof. Let $x$ be a vertex in $D$. $x$ must dominate some vertex $y$, and $y$ must dominate another vertex $z$. The $xz$-dipath has length of 2. □

Claim 1: For any dipath $p$ of maximum length for a given oriented graph $D$, each vertex that is part of $p$ can be the initial vertex of a dipath of equal or greater length if and only if that dipath contains $v$.

Proof. Every vertex dominating an initial vertex of a dipath of maximum length in $D$ must be part of all such dipaths originating from that vertex, for otherwise, that directed path could be extended by making one of the vertices dominating the initial vertex into the new initial vertex of a path. Similarly, the terminal vertex of such a dipath must dominate only other vertices belonging to that path.

Theorem 3. Any vertex in an oriented graph $D$ with outdegree 2 belongs to a dipath of length at least 4.

Proof. Let $D$ be an oriented graph with outdegree 2. Select a vertex $v$, which must dominate some two vertices $x$ and $y$.

Because $D$ contains no sources and is strongly connected, there exists a vertex $z$ such that $z$ dominates $v$ and there exists a $vz$-dipath.

Let $p$ be a $vz$-dipath of minimum length. Without loss of generality, let it be $(v, x, ..., z)$. This dipath contains at least two arcs.

$p$ has minimum length, so it does not contain $y$. If $y$ dominates a vertex $r$ that does not belong to $p$, then the dipath $(x, ..., z, v, y, r)$ has length of at least 4.

It is possible that $y$ dominates only vertices that belong to $p$. However, because $p$ has minimum length, $y$ can only dominate vertices belonging to $p$ if they are $x$ and the third vertex of $p$, $w$, which is the one dominated by $x$. $w$ and $z$ may be the same vertex.

In that case, there is also a minimum $vz$-dipath $(v, y, ..., z)$. Furthermore, the other vertex $q$ dominated by $x$ cannot be part of that dipath, for otherwise it would not be minimal. Therefore, the dipath $(y, ..., z, v, x, q)$ has length of at least 4. □

Lemma 1. Any oriented graph $D$ with minimum outdegree $k$ contains a cycle of length at least $k + 2$. Jackson [5] has proven essentially the same claim for minimum indegree $k$. Using similar reasoning, we provide the following short proof.

Proof. Let $v$ be a vertex in an oriented graph with minimum degree $k$. Proceed to construct a dipath. At each step, select one vertex, dominated by the current terminal vertex, that is not already part of the path. This dipath will continue to grow until one reaches a vertex $x$ that dominates $k$ vertices on the path. Because $x$ does not dominate the vertex dominating it, it is part of a cycle of length at least $k + 2$ that contains vertices from the dipath.
Corollary 1. In a counterexample, any two cycles of length at least $k + 2$ must share at least two vertices.

Proof. If they share exactly one vertex, it is possible to construct a dipath containing all of their vertices, which has length of at least $2k + 2$.

If they share no vertices, then it is possible to construct a dipath containing all of their vertices and any vertices on the shortest dipath connecting one cycle to another, which has length of at least $2k + 3$.

Theorem 4. For $k = 3$, there exists a dipath of length 6.

Proof. There exists a cycle of length at least 5. If the longest cycle has length $\geq 7$, then it contains a dipath of length 6.

Assume the longest cycle has length 6. At least one vertex on the cycle must dominate a vertex not on the cycle, and the dipath consisting of all the cycle vertices and the other dominated vertex has length 6.

Next, assume the longest cycle has length 5, and label the vertices $a, b, c, d, e$, which each dominating the next, and $e$ dominating $a$. At least one vertex $v$ on that cycle must dominate a vertex not part of that cycle, so let $a$ dominate $v$. Furthermore, $v$ cannot be dominated by any other vertex not on the cycle, because otherwise that would create a path of length 6. $v$ cannot dominate $b$, for otherwise there would exist a longer cycle. Therefore, $v$ must dominate $c, d, e$.

Furthermore, the vertices on the cycle must dominate at least $15 - \binom{5}{2} = 5$ vertices not on the cycle, so at least two other vertices on the cycle must dominate different vertices not on the cycle, which means that every vertex on the cycle must be dominated by at least one vertex not on the cycle. Thus, at least two consecutive vertices on the cycle must dominate different vertices. W.l.o.g., let $b$ dominate $y$. $vcdesab$ is a dipath of length 6.

4. Bounds on order and connectivity

Theorem 5. An oriented graph $D$ with outdegree $k$ greater than or equal to 2 that contains a cut vertex also contains a dipath of length at least $2k + 2$.

Theorem 6. Any counterexample must have minimum outdegree of at least 4 and must be at least 3-vertex-connected.

Proof of theorem 5. Let $v$ be the cut vertex, and let $X$ and $Y$ be disjoint nonempty vertex subsets of $D - v$, with $X \cup Y = D - v$, and with the additional requirement that vertices in $X$ only dominate other vertices in $X$ and vertices in $Y$ only dominate other vertices in $Y$.

Because $D$ is strongly connected, $v$ dominates at least one vertex in both $X$ and $Y$ and is dominated by at least one other vertex in both $X$ and $Y$.

As described in the previous section, it is possible to form a subdigraph $D'$ of $D$ with minimum outdegree $k - 1$ that does not contain $v$. $X'$ and $Y'$ are the resulting subdigraphs on vertices of $X$ and $Y$ respectively. They are not connected to one another, and they themselves are not necessarily connected. However, each is a digraph with minimum outdegree $k - 1$, and, as such, each contains a cycle of length at least $k + 1$, cycle $C_x$ in $X'$ and $C_y$ in $Y'$. These cycles are
also present in \( D \). Furthermore, each cycle either contains a vertex dominating \( v \), or there exists a dipath containing only one vertex from the cycle that reaches a vertex dominating \( v \). Similarly, each cycle contains a vertex dominated by \( v \), or there exists a dipath containing only one vertex from the cycle that begins from a vertex dominated by \( v \) and terminating in a vertex of that cycle.

Therefore, it is possible to form a directed path consisting of at least the vertices of \( C_x, v \), and the vertices \( C_y \). Such a dipath has length of at least \( 2k + 2 \). \( \blacksquare \)

**Proof of theorem 6.** Any counterexample to conjecture 1 must be at least 3-vertex-connected. If there is a cut vertex, the longest dipath will be of length at least \( 2k + 2 \). If the counterexample is 2-vertex-connected, we can delete one vertex in a minimal cut, and the resulting subdigraph will have minimum degree \( k - 1 \). This digraph also contains a cut vertex, so it has a dipath of length at least \( 2k \). \( \blacksquare \)

**Corollary 2.** A counterexample cannot be path-mergeable.

**Proof.** Bang-Jensen [2] has shown that a path-mergeable digraph contains a hamiltonian cycle if and only if it is strong and the underlying graph does not have a cut vertex.

As for order, we establish that any possible counterexample for a given value of \( k \) must have an order that falls within a finite range of possible values. If such counterexamples exist, there must exist one having the smallest order, so the deletion and any vertex and the replacement of the arcs dominating that vertex by one arc from each of its in-neighbors dominating any distinct vertex that it does not already dominate must result in a dipath of length at least \( 2k \). We call this process “rewiring.”

As stated in the introduction, the oriented graph with minimum outdegree \( k \) that has the smallest order possible is an orientation of \( K_{2n+1} \). By the elementary fact that all tournaments contain a hamiltonian path, such an oriented graph must contain a path of length \( 2k \).

**Theorem 7.** An oriented graph on at least \( (2k - 1)^k + 2 \) vertices contains a dipath of length at least \( 2k \).

**Proof.** This oriented graph is strongly connected, so any vertex must be reachable from another vertex. However, because of the outdegree is \( k \), a graph can reach at most \( x^k \) other vertices at a distance equal to or less than \( x \). A vertex can thus reach at most \( (2k - 1)^k \) other vertices at a distance less than \( 2k \). \( \blacksquare \)

**Lemma 2.** It is always possible to add one vertex so that only maximal dipaths passing through that vertex are increased by one.

**Proof.** Add a vertex \( z \). Select any other vertex \( x \) and one of its outneighbors \( y \). Delete \( xy \) and add \( xz \) and \( yz \). Then, add one arc from \( z \) to the \( k - 1 \) other vertices dominated by \( z \). The addition of \( z \) in this manner makes it possible to extend every dipath including \( x \) by exactly one arc originating from \( z \). The length of dipaths not including \( x \) cannot be similarly extended.

**Theorem 8.** If there is a counterexample, there must exist one such that all vertices belong to a maximal dipath of length \( 2k - 1 \).
**Proof.** If a counterexample exists, its number of vertices must exist within the bounds established. Therefore, there must be some counterexample on a maximum number of vertices, which means that the addition of one more vertex must create a path of length 2k.

By the lemma above, if there were some vertex not part of maximal dipath of length 2k − 1, we could add an additional vertex in a way such that it does not lead to the formation of any dipath of length 2k, which contradicts the assumption that D is the counterexample on a maximum number of vertices. ■

5. **Concluding remarks on a complete proof**

Proof of either of the following two conjectures would immediately prove conjecture 1.

**Conjecture 3.** It is always possible to remove a vertex and rewire the arcs that dominate it without increasing the length of the dipath of maximum length.

This conjecture implies that there exists no counterexample of smallest order.

**Conjecture 4.** There exists a way to remove one arc from every vertex and decrease the max path length by at least 2.

This conjecture implies that, for any counterexample of minimum outdegree k, there exists a counterexample of minimum outdegree k − 1, which contradicts theorems 2 and 3.

One consideration that may be productive to explore further is the restriction that certain vertices can only dominate or be dominated by certain others. Initial vertices of a dipath of length 2k − 1 can only be dominated by vertices on that dipath, and terminal vertices of such a dipath can only dominate vertices on that dipath.

Jackson [5] proves that in any oriented graph with minimum outdegree and indegree equal to k, there exists a dipath of length 2k. We conjecture that may be some contradiction arising from the requirement for a counterexample that there always be one vertex with outdegree less than k and one with outdegree greater than k.

Another possible route could involve an investigation of “growing” the oriented graph in a manner similar to the method of proof described in theorems 2 and 3. Such a proof might proceed by induction. For a probabilistic investigation, we recommend the construction of a graph through the use of an algorithm that assigns a certain probability to whether a given vertex dominates a vertex that is already part of the oriented graph or one that is not. The increase in dipath length as a function of time may yield useful information.

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