Invariants Associated to Orthogonal $\epsilon$-constants

Darren Glass

Abstract

In this paper we use the theory of $\epsilon$-constants associated to tame finite group actions on arithmetic surfaces to define a Brauer group invariant $\mu(\mathcal{X}, G, V)$ associated to certain symplectic motives of weight one. We then discuss the relationship between this invariant and $w_2(\pi)$, the Galois theoretic invariant associated to tame covers of surfaces.

1 Introduction and Background

Brauer group invariants associated to motives have been long studied by mathematicians. In the case of orthogonal motives of even weight, these invariants were first examined by Frohlich in [F]. Deligne used these Brauer group invariants and their relationship with certain $\epsilon$-constants in order to give a proof of the Frohlich-Queyrut Theorem in [D2] by showing that certain global orthogonal root numbers were one by interpreting the associated local orthogonal root numbers as Stiefel Whitney classes and then using the local root numbers to define an element of order two in the Brauer group of $\mathbb{Q}$. Similar results and methods have been done by Saito (in [S2], for example) and others. In this paper, we will define a Brauer group invariant associated to certain motives which are symplectic and have weight one.

In order to construct the relevant motives we first define $\mathcal{X}$ to be an arithmetic surface of dimension 2 which is flat, regular, and projective over $\mathbb{Z}$. Throughout this paper we will assume that $f : \mathcal{X} \rightarrow Spec(\mathbb{Z})$ is the structure morphism. Define $G$ to be a finite group which acts tamely on $\mathcal{X}$. In other words, for each closed point $x \in \mathcal{X}$, the order of the inertia group of $x$ is relatively prime to the residue characteristic of $x$. Let $\mathcal{Y}$ be
the quotient scheme $\mathcal{X}/G$, which we will assume is regular, and that for all finite places $v$ the fiber $\mathcal{Y}_v = (\mathcal{X}_v)/G = Y \otimes_{\mathbb{Z}} (\mathbb{Z}/p(v))$ has normal crossings and smooth irreducible components with multiplicities relatively prime to the residue characteristic of $v$. Finally, let $V$ be a representation of $G$ over $\mathbb{Q}$.

We wish to define a class $\mu(\mathcal{X}, G, V)$ in the global Brauer group $H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$ which will depend only on $\mathcal{X}, G,$ and $V$. In particular, $\mu(\mathcal{X}, G, V)$ will be the element whose local invariant at the place $v$ is given by the sign of $\epsilon_v(D', V)$, where $D'$ is an appropriately chosen horizontal canonical divisor on $\mathcal{Y}$. In other words, $O_Y(D' + Y_T)$ is isomorphic to $\omega_{Y/Z}(Y^\text{red}_S)$ where $Y_T$ and $Y_S$ are unions of vertical fibers of $\mathcal{Y}$.

In addition to being a canonical divisor, we must impose an additional condition on our choice of $D'$ in order for $\mu(\mathcal{X}, G, V)$ to be well-defined. To do this we recall that Proposition 3.10 of [G] showed that if $F'$ and $G'$ are components of $\mathcal{Y}^\text{red}_v$ and $D'$ is a canonical divisor in the above sense, then there is a canonical isomorphism between $O_{F'}(D' \cap F')$ and $\omega_{F'}(F' \cap G')$. This isomorphism maps the global section $1 \in \Gamma(O_{F'}(D' \cap F'))$ to an element $\gamma \in \Gamma(\omega_{F'}(F' \cap G'))$ such that $\gamma$ has a simple pole at all $x \in F' \cap G'$. Define $a_x$ to be the residue of $\gamma$ at these points $x$. We wish to choose $D'$ so that all of the residue terms $a_x$ are equal to 1 and we will call such a choice of $D'$ a nice divisor. Nice divisors exist due to a moving lemma which we will make explicit in Section 2. We can now state the main theorem.

**Theorem 1.1.** In the case where $V$ is an orthogonal representation of dimension equal to zero and trivial determinant and where $D'$ is a nice divisor, the local constant $\epsilon_v(D', V)$ is independent of the choice of $D'$. In particular, the element $\mu(\mathcal{X}, G, V)$ in the global Brauer group $H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$ whose local invariant at the place $v$ is given by the sign of $\epsilon_v(D', V)$ is well-defined.

Section two of this paper will prove theorem [1]. Section three recalls the definition of the Galois theoretic invariant $w_2(\pi)$ defined by Cassou-Nogues, Erez and Taylor in [CNET] which comes from a situation similar to the one we are working in and proves a connection between the two invariants. Finally, in section four we prove the moving lemma which we used in section two in order to show that nice divisors always exist.

The author would like to thank Ted Chinburg for his insightful comments and suggestions, in particular on the proof in section four.
2 Definition of $\mu(\mathcal{X}, G, V)$

In [G], the author proves the following result about orthogonal $\epsilon$-constants associated to tame finite group actions on surfaces (We refer the reader to [D1], [CEPT1], and [G] for all relevant definitions.):

**Theorem 2.1.** If $V$ is an orthogonal virtual representation of degree zero and trivial determinant then for all finite places $v$ of $\mathbb{Q}$ we have the following formula:

$$
\frac{\epsilon(\mathcal{Y}_v, V)}{\epsilon(D'_v, V)} = \prod_i \det(V_i) \prod \epsilon_{0,z}(C_i, V_i) \epsilon_{0,z}(C_i, V_{i2}) \epsilon(z, V)
$$

where $Z$ is the set of crossing points of the components of the fiber $\mathcal{Y}_v$ and $D'$ is chosen to be a canonical horizontal divisor on $\mathcal{Y}$.

By canonical we mean that $D'$ is a horizontal divisor on $\mathcal{Y}$ such that $D' + \mathcal{Y}_T = K_\mathcal{Y} + \mathcal{Y}_S^{\text{red}}$, where $\mathcal{Y}_S^{\text{red}}$ is the sum of the reductions of the fibers of $\mathcal{Y}$ at the places in the set $S$ of bad primes, $\mathcal{Y}_T$ is the sum of the (necessarily reduced) fibers of $\mathcal{Y}$ over the places in $T$. We also wish to choose $D'$ so that it intersects the non-smooth fibers $\mathcal{Y}_v$ of $\mathcal{Y}$ transversally at smooth points on the reduction of $\mathcal{Y}_v$.

Given any such horizontal divisor $D'$ it is possible to find a nice divisor which is close to it due to the following moving lemma, which is proven in section 4. In particular, this shows that nice divisors $D'$ always exist and therefore that our class $\mu(\mathcal{X}, G, V)$ will be well-defined.

**Lemma 2.2.** There exists a meromorphic function $h$ on $\mathcal{Y}_v^{\text{red}}$ such that the divisor of $h$ intersects the special fibers $\mathcal{Y}_v^{\text{red}}$ transversally at smooth points away from $D'_v$ and such that $h$ takes on prescribed values at the singular points of $\mathcal{Y}_v^{\text{red}}$. In particular, given a horizontal divisor $D'$ as in the previous section, the divisor $D' + \text{div}(h)$ will have residue maps equal to one at the crossing points of components of $\mathcal{Y}_v^{\text{red}}$.

**Proof of theorem 2.1.** For any fixed place $v$ of $\mathbb{Q}$ and any component $C_i$ of $\mathcal{Y}_v^{\text{red}}$, it follows from the definition that $\delta_{v, C_i}$ is the class which corresponds to the element $\delta = (\oplus 0) \oplus (\oplus a_x) \in (\oplus_{x \in C_i \cap Z} \mathbb{Z}) \oplus (\oplus_{x \in C_i \cap Z} K^*/U^1_x)$. This term is independent of the choice of nice divisor and it follows that the right hand side of the equation in Theorem 2.1 is as well. Furthermore, it is clear that $\epsilon(\mathcal{Y}_v, V)$ is independent of our choice of $D'$. Thus, it follows from the
theorem that \( \epsilon_v(D', V) \) is independent of the choice of \( D' \). Next, we note that \( \epsilon_{v,0}(D', V) \) must also be independent of our choice of \( D' \) (by Lemma 3.3 of \([G]\), for example). Therefore it must be the case that \( \epsilon_v(D', V) = \epsilon_{v,0}(D', V)\epsilon(D'_v, V) \) is independent of the choice of \( D' \). The product of all of the \( \epsilon_v(D', V) \) is equal to \( \epsilon(D', V) \), which must be equal to one from the theorem of Fröhlich and Queyrut \([FQ]\). This tells us that we can define an element \( \mu(\mathcal{X}, G, V) \) in \( H_2(Q, \mathbb{Z}/2\mathbb{Z}) \) by setting the local component at the prime \( v \) to be equal to the sign of \( \epsilon_v(D', V) \). ■

3 The connection to \( w_2(\pi) \)

Let \( \pi : \mathcal{X} \to \mathcal{Y} \) be a tamely ramified cover of degree \( n \), where \( \mathcal{X} \) and \( \mathcal{Y} \) are regular schemes and \( \mathcal{Y} \) is connected. Furthermore, we must make the technical assumption that the ramification indices are all odd. Cassou-Nogues, Erez, and Taylor use Grothendeick’s equivariant cohomology theory to define an invariant \( w_2(\mathcal{X}/\mathcal{Y}) = w_2(\pi) \in H^2(\mathcal{Y}_{et}, \mathbb{Z}/2\mathbb{Z}) \) associated to this situation. Their definition generalizes to define classes \( w_i(\pi) \) which lie in \( H^i(\mathcal{Y}_{et}, \mathbb{Z}/2\mathbb{Z}) \) for all positive integers \( i \), but in this thesis we will only be interested in \( w_2 \). These terms are generalized Stiefel-Whitney classes, and are obtained by pulling back the universal Hasse-Witt classes defined by Jardine using classifying maps related to a quadratic form \( E \). The precise definition of \( E \) uses the existence of a locally free sheaf \( D_{\mathcal{X}/\mathcal{Y}}^{-1/2} \) whose square is the inverse different of the covering \( \mathcal{X}/\mathcal{Y} \). In this section, we will consider the relationship between the class \( \mu(\mathcal{X}, G, V) \) lying in \( H^2(Q, \mathbb{Z}/2\mathbb{Z}) \) which we defined in theorem 1.1 and \( w_2(\pi) \).

In \([CNET]\) the following equality in \( H^2(\mathcal{Y}_{et}, \mathbb{Z}/2\mathbb{Z}) \), which is an analog of a theorem of Serre, is proved:

\[
w_2(\pi_*(D_{\mathcal{X}/\mathcal{Y}}^{-1/2}, Tr_{\mathcal{X}/\mathcal{Y}})) = w_2(\pi) + (2) \cup (d_{\mathcal{X}/\mathcal{Y}}) + \rho(\mathcal{X}/\mathcal{Y})
\]

where \( \rho(\mathcal{X}/\mathcal{Y}) \) is defined by the ramification of \( \mathcal{X}/\mathcal{Y} \), \( d_{\mathcal{X}/\mathcal{Y}} \) is the function field discriminant, and the left hand side of the equation is the Hasse-Witt invariant associated to the square-root of the inverse different bundle. Note that if we look at the one-dimensional version of this formula the middle term on the right hand side becomes trivial. Therefore, in the case of étale covers of curves the formula reduces to

\[
w_2(\pi) = w_2(\pi_*(D_{\mathcal{X}/\mathcal{Y}}^{-1/2}, Tr_{\mathcal{X}/\mathcal{Y}})) = w_2(\pi_*(D_{\mathcal{X}/\mathcal{Y}}^{-1/2}, Tr_{\mathcal{X}/\mathcal{Y}})) = w_2(E)
\]
where \( w_2(E) \) is the second Hasse-Witt invariant associated to the square root of the inverse different, as described in detail in [CNET].

Let \( D' \) be a choice of a canonical divisor on \( Y \) in the sense of the previous sections, and let \( i : D' \hookrightarrow Y \) be the natural inclusion. An étale covering of \( Y \) naturally restricts to give an étale covering of \( D' \). We now have the following natural maps

\[
i^* : H^2(Y_{et}, \mathbb{Z}/2\mathbb{Z}) \to H^2(D'_{et}, \mathbb{Z}/2\mathbb{Z})
\]

\[
res : H^2(D'_{et}, \mathbb{Z}/2\mathbb{Z}) \to H^2_{et}(\mathbb{Q}(D'), \mathbb{Z}/2\mathbb{Z}) = H^2_{gal}(\overline{\mathbb{Q}}/\mathbb{Q}(D'), \mathbb{Z}/2\mathbb{Z})
\]

\[
cor : H^2_{gal}(\overline{\mathbb{Q}}/\mathbb{Q}(D'), \mathbb{Z}/2\mathbb{Z}) \to H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})
\]

where the latter two maps are restriction and corestriction in the sense of Serre (for details see Chapter VII of [Se]). Composing these maps gives a natural map

\[
H^2(Y_{et}, \mathbb{Z}/2\mathbb{Z}) \to H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})
\]

We denote the image of the class \( w_2(\pi) \in H^2(Y_{et}, \mathbb{Z}/2\mathbb{Z}) \) under this map by \( \tilde{w}_2(\pi) \in H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) \). At first glance it appears as though this element may depend on our choice of canonical divisor \( D' \). However, we want to show that it does not depend on this choice and furthermore that the element \( \tilde{w}_2(\pi) \) is connected in a natural way to the element \( \mu(\chi, G, V) \). Recall that \( \mu(\chi, G, V) \) is defined by letting the local invariant at the place \( v \) be given by the sign of \( \epsilon(D'_v, V) \). This also appears to depend on the choice of \( D' \) but turns out to be independent of choice.

The first natural observation is that the class \( \mu(\chi, G, V) \) depends on the choice of a representation \( V \) of \( G \) while \( \tilde{w}_2(\pi) \) does not. The natural representation to consider is the regular representation of the group \( G \), which we denote by \( R \). In particular, the nicest possible theorem comparing the invariants would say that \( \mu(\chi, G, R) = \tilde{w}_2(\pi) \). However, we have only shown that \( \mu(\chi, G, V) \) is a well-defined class in the case where \( V \) is of dimension zero and of trivial determinant, neither of which holds for \( R \). So instead of setting \( V = R \), we consider the representation \( V = R - \text{det}(R) - T^{n-1} \), where \( \text{det}(R) \), the determinant of the regular representation, is a character whose order is either one or two, \( T \) is the trivial representation and \( n \) is the degree

5
of the cover $X/Y$. This choice of $V$ is an orthogonal representation, and it has trivial determinant and dimension 0. We can now prove the following theorem:

**Theorem 3.1.** Assume that we are in the above situation, and in particular that $V = R - \det(R) - T^{n-1}$. Let $Y_1/Y$ be either the trivial cover or the subcover of $X/Y$ of degree 2, depending on whether $\det(R)$ is of order 1 or 2 respectively. Then as classes in $H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$, we have the equality

$$\mu(X, G, V) = \tilde{w}_2(X/Y) - \tilde{w}_2(Y_1/Y) - (n - 1)\tilde{w}_2(Y/Y)$$

The proof of this theorem relies on the interpretation of each side of the equation as a Stiefel-Whitney class. In particular, Cassou-Nogues, Erez, and Taylor show that the element $w_2(\pi)$ is the Hasse-Witt invariant associated to the full covering of surfaces. Thus, when we restrict the class to the one-dimensional divisor $D'$ we see that the element $i^*(w_2(\pi)) \in H^2(D'_et, \mathbb{Z}/2\mathbb{Z})$ is equal to the Stiefel-Whitney class associated to the form $E' = (D_{D'/D}^{-1/2}, Tr_{D/D'})$ on the canonical divisor $D'$ of $Y$. This follows as a generalization to étale cohomology of results of Fröhlich in \([F]\), which allow us to associate the class $i^*(w_2(\pi))$ to $G$-extensions of the ring of integers of the residue field of the generic point of $D'$.

Next we make use of the results of Deligne which allow us to interpret local Stiefel-Whitney classes in terms of local root numbers. In particular, the following lemma is shown in \([D2]\):

**Lemma 3.2.** Let $d = 1$, so that the fibers $X_v$ and $Y_v$ are all one dimensional schemes. Furthermore, let $V$ be an orthogonal virtual representation of dimension zero and trivial determinant. Under these hypotheses, the local root number $W(V_v) = \text{sign}(\epsilon_v(Y, V))$ is equal to $e^{2\pi i cl(sw_v)}$, where $sw_v$ is the local Stiefel-Whitney class, and $cl(sw_v) \in \{0, 1/2\} \subset \mathbb{Q}/\mathbb{Z}$.

In other words, in characteristic not equal to two, the sign of the $\epsilon$-constants $\epsilon_v(D', V)$ of the representation on the one dimensional horizontal divisor $D'$ are determined by whether or not the classes $w_2(\pi)$ are trivial in the Brauer group, and $\epsilon_v(D', V)$ is automatically positive when $v = 2$. It turns out that these are exactly the terms that are coming up in the computation of the class of Cassou-Nogues, Erez, and Taylor.

In particular, $\epsilon_v(D', V) = \epsilon_v(D', R)\epsilon_v(D', \det(R))$ is the same as the local Hasse-Witt invariants. However, we are working with étale covers of curves.
and so from the results of [CNET] discussed above, these Hasse-Witt invariants are simply the images of the appropriate classes $w_2(\pi)$. This proves Theorem 3.1. ■

4 Proof of Lemma 2.2

The proof of Lemma 2.2 involves a generalized version of Bertini’s Theorem. For now, let us assume that $X$ is a smooth curve defined over an infinite field $k$ and let us choose a finite set of points $p_1, \ldots, p_m$ on $X$. We define the divisor $p = \sum_i p_i$. Furthermore, let us choose constants $c_i$ which lie in the residue field $k(p_i)$ of the points $p_i$. Finally, let us choose $\Lambda$ to be an effective very ample divisor on $X$ of large degree which is supported off of $p$. We look at the group of global sections $H^0(X, \mathcal{O}_X(\Lambda))$ and let $f_0, \ldots, f_t$ be a basis of this group. This basis gives us a projective embedding from $X$ into $\mathbb{P}^t_k$ whose projective coordinates we will write as $x_0, \ldots, x_t$.

We wish to prove that there exist linear forms $l_0$ and $l_1$ in the variables $x_i$ such that the following properties hold:

1. If $H_i$ is the hyperplane defined by $l_i = 0$ in $\mathbb{P}^t_k$ then $H_i \cap X$ is a finite set of closed points which is regular and disjoint from $\{p_1, \ldots, p_m\}$. Furthermore, we wish to choose the $l_i$ so that $H_1 \cap H_2 \cap X$ is empty.

2. It follows from (1) that the function $l_1/l_0|_X$ is in $\mathcal{O}_{X,p_i}$ for each $i$. However, we further wish to specify that the image of $l_1/l_0$ in $k(p_i)$ is the given constant $c_i$.

The classical version of Bertini’s Theorem (Theorem II.8.18 of [H]) tells us that there exist linear forms $l_0$ so that $H_0$ satisfies condition (1). We now fix one choice of such an $l_0$, and we will attempt to construct an $l_1$ so that the pair satisfies properties (1) and (2). We begin by looking at the set $V$ consisting of all linear forms such that $\{l_0, l_1\}$ satisfy condition (2). In other words,

$$V = \{l = a_0x_0 + \ldots + a_ix_i \mid \forall j, \frac{l}{l_0}|_X(p_j) = c_j \in k(p_j)\}$$

This $V$ will be an affine space over $k$. Furthermore, because we chose the divisor $\Lambda$ to have high degree it follows from a Riemann-Roch argument that $V$ is of codimension $m$ inside of $H^0(X, \mathcal{O}_X(\Lambda))$. 7
For each point $x \in X$, we now define a set $V_x \subseteq V$ which consists of all linear forms $l \in V$ so that the hyperplane defined by $l = 0$ has contact order $> 1$ at $x$. In other words, $V_x$ will consist of those linear forms who do not intersect $X$ nicely at the point $x$. We can again use the Riemann-Roch theorem to show that for almost all choices of $x$, we get that the dimension of $V_x$ is equal to $\dim V - 2$.

Let $U = X - \{p_1, \ldots, p_m\}$ so that $U$ is an affine curve, and define $T \subseteq U \times V$ to be the set of all pairs $(x, l)$ such that $x \in U$ and $l \in V_x$. We have seen that the projection map $\pi : T \to U$ is surjective and for almost all $x \in U$ (in particular for those points such that $k(x) = k$), we see that the fiber $\pi^{-1}(x)$ is an affine space whose dimension is equal to $\dim V - 2$. In particular, this shows that $T$ is irreducible and that the dimension of $T$ is equal to $\dim V - 1$. But this shows us that the natural projection map $\gamma : T \to V$ must not be surjective.

In particular, we can choose some element $l_1 \in V$ which is not in the image of $\gamma$. In particular, the hyperplane $H$ defined by $\{l_1 = 0\}$ is such that $H \cap U$ is regular and, since $l_1 \in V$, we know that $l_1/l_0(p_i) = c_i \in k(p_i)$, and thus that $l_1$ and $l_0$ satisfy conditions (1) and (2) above.

So far, we have only made the argument for the case where $X$ is a smooth curve. However, as long as $X$ is a reduced curve with smooth irreducible components which have normal crossings, then the same argument will hold as long as we include these crossing points in the set of $\{p_i\}$. Instead of using the normal Riemann-Roch theorem we will now use the version for singular curves described on p.298 of [H].

In order to prove lemma 2.2 we apply this generalized version of Bertini’s theorem to $X = \mathcal{Y}_v^{\text{red}}$. Specifically, choose the set of points $\{p_1, \ldots, p_m\}$ to include the crossing points of components of $\mathcal{Y}_v^{\text{red}}$ as well as the points in $D' \cap \mathcal{Y}_v^{\text{red}}$. The above argument then allows us to find a meromorphic function $h$ where we can specify the values of the function $h = l_1/l_0$ at the crossing points of $\mathcal{Y}_v^{\text{red}}$ so that the residues that come up when we consider $D'' = D' + \text{div}(h)$ are all equal to one and $D''$ intersects $\mathcal{Y}_v^{\text{red}}$ in the way we want. ■

References

[CEPT1] Chinburg, T.; Erez, B.; Pappas, G.; Taylor, M. $\epsilon$-constants and the Galois structure of de Rham cohomology. Ann. of Math. (2) 146
(1997), no. 2, 411–473.

[CNET] Cassou-Nogues, P.; Erez, B.; Taylor, M. Invariants of a quadratic form attached to a tame covering of schemes. Colloque International de Thiorie des Nombres (Talence, 1999). J. Thior. Nombres Bordeaux 12 (2000), no. 2, 597–660.

[D1] Deligne, P. Les constantes des équations fonctionnelles des fonctions $L$. Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 501–597. Lecture Notes in Math., Vol. 349, Springer, Berlin, 1973

[D2] Deligne, P. Les constantes locales de l’équation fonctionnelle de la fonction $L$ d’Artin d’une représentation orthogonale. Invent. Math. 35 (1976), 299–316

[F] Fröhlich, A. Orthogonal representations of Galois groups, Stiefel-Whitney classes and Hasse-Witt invariants. J. Reine Angew. Math. 360 (1985), 84–123.

[FQ] Fröhlich, A.; Queyrut, J. On the functional equation of the Artin $L$-function for characters of real representations. Invent. Math. 20 (1973), 125–138.

[G] Glass, D. $\epsilon$-Constants and Orthogonal Representations, preprint

[H] Hartshorne, R. Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[sS] Saito, S. Functional equations of $L$-functions of varieties over finite fields. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 31 (1984), no. 2, 287–296.

[tS] Saito, T. $\epsilon$-factor of a tamely ramified sheaf on a variety. Invent. Math. 113 (1993), no. 2, 389–417.

[tS2] Saito, T. The sign of the functional equation of the $L$-function of an orthogonal motive. Invent. Math. 120 (1995), no. 1, 119–142.

[Se] Serre, J-P. Local fields. Translated from the French by Marvin Jay Greenberg. Graduate Texts in Mathematics, 67. Springer-Verlag, New York-Berlin, 1979. viii+241 pp