Evolution equations: Frobenius integrability, conservation laws and travelling waves

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Abstract
We give new results concerning the Frobenius integrability and solution of evolution equations admitting travelling wave solutions. In particular, we give a powerful result which explains the extraordinary integrability of some of these equations. We also discuss ‘local’ conservation laws for evolution equations in general and demonstrate all the results for the Korteweg–de Vries equation.

Keywords: geometric partial differential equations, Korteweg–de Vries, Frobenius integrability, Vessiot theory, solvable structures, solitons

1. Introduction
Travelling wave solutions of nonlinear partial differential equations (PDEs) have been studied extensively. Our aim is to present a geometric perspective on these solutions, viewed as the projection to the configuration space of the common level sets of functions on a jet bundle. In particular, these common level sets are the integral manifolds of a certain Frobenius integrable distribution, the Vessiot distribution.

Our recent work [10, 11], encodes a system of PDEs as an exterior differential system and we have provided a systematic approach to solving second order linear and nonlinear PDEs in the presence symmetries (known as solvable structures), a generalization of the usual reduction by symmetry process, and based on the Vessiot distribution. We present an application of this technique to find travelling wave solutions of evolution PDEs. In doing so we give a new result (theorem 3.1) identifying necessary criteria for the sort of integrability enjoyed by the Korteweg–de Vries soliton.

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The layout of this paper is as follows: in section 2 we review our approach to the geometric treatment of PDEs. We define solvable structures and discuss their utility for Frobenius integrable distributions. In this context we give a new result which will be important in discussing travelling wave solutions of evolution equations. In section 3 we discuss the travelling wave solutions of general nonlinear evolution PDEs and we present new results about their integrability. In section 4 we give a new presentation of the Korteweg–de Vries soliton as an example. In section 5 we describe the relationship between conserved densities, constants of the motion and first integrals of the Vessiot distribution.

2. Solvable structures and PDEs

Consider a system of PDEs \[ G^a(x^i, u^j, u_{i_1}^j, \ldots, u_{i_{k-1}}^j) = 0. \] The subscripts \( 1 \leq i_1 \leq \cdots \leq i_k \leq m \) are used to specify the partial derivatives of \( u^j \), where \( k \) is the maximum order of the system. We denote the total space by \( \mathbb{X} \times \mathbb{U} \) as the product of the spaces \( \mathbb{X} \) and \( \mathbb{U} \) of the independent and dependent variables respectively. Usually \( \mathbb{X} = \mathbb{R}^m \) and \( \mathbb{U} = \mathbb{R}^n \).

A system of PDEs can be regarded as a submanifold \( S \) of the jet bundle \( J^k(\mathbb{R}^m, \mathbb{R}^n) \), and so the associated contact Pfaffian system on \( J^k \) induces a Pfaffian system on \( S \) by restriction. In this way we can think of the PDE as a manifold with a Pfaffian system. The solutions of the PDE correspond to \( m \)-dimensional integral manifolds of the Pfaffian system on which \( \text{dx}^I \wedge \cdots \wedge \text{dx}^m \neq 0 \). This inequality is usually called an independence condition and simply requires that our integral submanifolds suitably project to the graph space \( \mathbb{X} \times \mathbb{U} \).

The contact Pfaffian system \( \Omega^k(X, U) \) consists of all one-forms whose pull-back by a prolonged section of \( \mathbb{X} \times \mathbb{U} \) vanishes. Locally it is spanned by the contact forms

\[ \theta_I := du_I - \sum_{i=1}^m u_I^j dx^i, \]

where \( I \) is a multi-index of order less than or equal to \( k - 1 \).

The dual distribution \( \Omega^k(X, U) \) consists of all vector fields annihilated by contact forms \( \Omega^k(X, U) \). A straightforward calculation [5] shows that it is generated by

\[ V^{(k)}_i := \partial_i + \sum_{j=1}^n \sum_{0 \leq |I| < k} u_I^j \partial_{u^j}, \quad 1 \leq i \leq m, \]

\[ V^{(k)}_j := \partial_{u^j}, \quad |I| = k, \quad 1 \leq j \leq n. \] (2)

The Vessiot distribution [4, 12] is essentially the restriction of this dual distribution to the submanifold defined by our differential equation condition \( G = 0 \). In general, the Vessiot distribution is not Frobenius integrable.

Vessiot [12] has given an algorithm for constructing all the Frobenius integrable sub-distributions of any given distribution. If the distribution is not Frobenius integrable, then it is interesting to ask for the sub-distributions which are Frobenius integrable. He looks for generic sub-distributions that satisfy the algebraic constraint called by him involutions of degree \( r \) and then shows that maximal such involutions can be deformed so as to be Frobenius integrable. He does this via the Cauchy–Kowalevski theorem. We have given an equivalent method [10, 11] to find the largest integrable Vessiot sub-distributions (reduced Vessiot distributions). We showed how to use a solvable structure (see below) to integrate a PDE in...
the original coordinates, but most importantly, we showed how to impose a solvable structure on a PDE so as to determine particular largest solvable sub-distributions of the Vessiot distribution. For more details see [10, 11].

So the first problem is to locate the largest integrable sub-distributions which satisfy the independence condition. Then we apply the integrating factor technique [8] to integrate these integrable Vessiot sub-distributions. The paper by Sherring and Prince [8] extends Lie’s approach to integrating a Frobenius integrable distribution by using solvable structures. First we need some basic definitions (see for example [2]).

Definition 2.1. A differential p-form Ω is *simple or decomposable* if it is the wedge product of p one-forms.

Definition 2.2. A *constraint* one-form θ for differential form Ω is any one-form satisfying θ ∧ Ω = 0, which implies Y ∧ θ = 0, ∀ Y ∈ ker Ω.

Definition 2.3. A *characterizing form* for an p-dimensional (vector) distribution D is a form on Mn of degree (n – p) which is the exterior product of (n – p) constraint forms.

Definition 2.4. Let Ω ∈ Ωp(Mn) for some p > 1 be decomposable, Ω is *Frobenius integrable* if dΩ = λ ∧ Ω. Equivalently, D = ker Ω is Frobenius integrable, that is, closed under the Lie bracket. Note that dim (ker Ω) = n – p, since Ω is simple.

A solvable structure is then defined as follows:

Definition 2.5. Let Ω be a characterizing p-form for an (n – p) dimensional distribution D on a manifold Mn. An ordered set of p linearly independent vector fields \{X_1, ..., X_p\} with dim(Sp \{X_1, ..., X_p\} ⊕ D) = n forms a solvable structure [8] for Ω (equivalently D) if the sequence of simple forms

\[ Ω, X_1ω, ..., X_{p−1}ω, ..., X_pω \]

satisfies

\[ \mathcal{L}_{X_1}Ω = ℓ_1Ω, \]
\[ \mathcal{L}_{X_2}(X_1ω) = ℓ_2(X_1ω), \]
\[ \vdots \]
\[ \mathcal{L}_{X_p}(X_{p−1}, ..., X_pω) = ℓ_p(X_{p−1}, ..., X_pω), \]

for some smooth functions ℓ_1, ..., ℓ_p on Mn.

The Lie symmetry determination software package dimsym [7] can be used to locate solvable structures for distributions.

Our main tool to integrate a Frobenius integrable distribution using solvable structures is the following theorem from [8]:

Theorem 2.6. Let Ω be a decomposable k-form on a manifold Mn, and let Sp(\{X_1, ..., X_k\}) be a k-dimensional distribution on an open U ⊆ Mn satisfying X_iω = 0 everywhere on U. Further suppose that Sp(\{X_{i+1}, ..., X_k\} ∪ ker Ω) is integrable for some j < k and that X_i is a symmetry of Sp(\{X_{i+1}, ..., X_k\} ∪ ker Ω) for i = 1, ..., j.
Put \( \sigma^i := X_1, \ldots, \hat{X}_i, \ldots, X_k, \Omega \), where \( \hat{X}_i \) indicates that this argument is missing and 
\( \omega^i := \frac{\omega^i}{\Omega} \) for \( i = 1, \ldots, k \) so that \( \{ \omega^1, \ldots, \omega^k \} \) is dual to \( \{ X_1, \ldots, X_k \} \). Then 
\( d \omega^1 = 0; \ d \omega^2 = 0 \mod \omega^1; \ d \omega^3 = 0 \mod \omega^1, \omega^2; \ldots; \ d \omega^k = 0 \mod \omega^1, \ldots, \omega^{k-1}, \) 
so that locally 
\[
\omega^1 = d \gamma^1,
\omega^2 = d \gamma^2 - X_1(\gamma^2)d \gamma^1,
\omega^3 = d \gamma^3 - X_2(\gamma^3)d \gamma^2 - \left( X_1(\gamma^3) - X_2(\gamma^3)X_1(\gamma^2) \right) d \gamma^1,
\vdots
\omega^i = d \gamma^i \mod d \gamma^1, \ldots, d \gamma^{i-1},
\]
for some \( \gamma^1, \ldots, \gamma^i \in \mathbb{F}(T^*U). \) The system \( \{ \omega^{i+1}, \ldots, \omega^k \} \) is integrable modulo \( d \gamma^1, \ldots, d \gamma^i \) 
and locally \( \Omega = \gamma^0 d \gamma^1 \land d \gamma^2 \land \ldots \land d \gamma^i \land \omega^{i+1} \land \ldots \land \omega^k \) for some \( \gamma^0 \in \mathbb{F}(T^*U). \) 
Each \( \gamma^i \) is uniquely defined up to the addition of arbitrary functions of \( \gamma^1, \ldots, \gamma^{i-1}. \)

We will now demonstrate the utility of a closed characterizing \( p \)-form, \( \Omega \), on \( M^n \). Firstly 
note that for \( p < n \) the closure of \( \Omega \) implies its Frobenius integrability. As shown in the 
proposition below, in the presence of a solvable structure this closure allows the immediate 
identification of closed constraint one-forms (factors of \( \Omega \)) for the distribution in question. (So 
far as we know this is a new result.)

**Proposition 2.7.** Let \( \Omega \) be a closed characterizing \( p \)-form for a distribution \( D \) on \( M^n \) with 
\( p < n \). Let \( \{ X_1, \ldots, X_p \} \) be a solvable structure for \( \Omega \) with the additional properties that 
\[
\mathcal{L}_{X_1} \Omega = 0,
\mathcal{L}_{X_p}(X_p, \Omega) = 0,
\vdots
\mathcal{L}_{X_i}(X_2, \ldots, X_{i-1}, X_p, \Omega) = 0.
\]
Then the forms in the following sequence are all simple and closed and the last is a closed 
one-form factor of \( \Omega \):
\[
\Omega, X_p, \ldots, X_2, \omega_1, \ldots, \omega_p, \Omega.
\]
Moreover, if \( \Omega(X_p, \ldots, X_1) \) is not constant (it is necessarily non-zero) then 
\[
d \left( \Omega(X_p, \ldots, X_1) \right) \wedge (X_p, \ldots, X_1, \Omega) = 0,
\]
effectively integrating the last closed form in the sequence.

**Proof.** Note that simple forms are exactly those with maximal (and constant) dimension 
kernels. If \( \omega \) is a simple form and \( Z \) is not in its kernel then \( Z \omega \) is simple because its kernel is 
dimension one greater than that of \( \omega \). So each form in the above sequence is simple. 
The closure of each form follows from the Lie derivative formula: 
\[
\mathcal{L}_{X} \alpha = X \cdot d \alpha + d(X \cdot \alpha)
\]
and the closure of \( \Omega \). For example, 
\[
d(X_p, \Omega) = \mathcal{L}_{X_p} \Omega - X_p \cdot d \Omega = 0.
\]

**Remarks.** In the best case, where every ordering of \( 1, \ldots, p \) produces a solvable structure 
from \( \{ X_1, \ldots, X_p \} \), this proposition produces all the closed one-form factors of \( \Omega \) and we 
explicitly find the integral manifolds of the corresponding distribution.
In the following section, we will consider an evolution PDE of one dependent variable and two independent variables and explain the application of our technique developed in [10, 11] to find the travelling wave solutions.

3. Evolution PDE and travelling wave solution

Suppose we have an evolution PDE of order \( k \) of one dependent variable \( u \) and two independent variables \( t, x \) given by

\[
u_t = F(t, x, u, u_x, u_{xx}, u_{xxx}, \ldots),
\]

for some smooth function \( F \). The embedded codimension one submanifold

\[
S := \{(t, x, u, u_t, u_x, u_{xt}, u_{xx}, \ldots) \in J^k(\mathbb{R}^2, \mathbb{R}) \mid u_t - F(t, x, u, u_x, u_{xx}, \ldots) = 0\},
\]

is a subset of \( J^k(\mathbb{R}^2, \mathbb{R}) \). The local solution of the PDE is described by the map

\[
i: (t, x, u, u_t, u_x, u_{xt}, \ldots) \rightarrow (t, x, u, F, u_x, u_{xt}, u_{xx}, \ldots).
\]

We are interested in travelling wave solutions [3], \( u(x, t) = f(x - ct) \), of the evolution equation (3). Equivalently,

\[
u_t + cu_x = 0.
\]

We add equation (4) and its differential consequences, via the RIF algorithm [1, 6], as extra conditions to (3). These differential consequences are

\[
u_{tx} = -cu_{xx}, \quad u_{tt} = c^2u_{xx}, \quad u_{ttt} = -c^3u_{xxx}, \quad u_{tttx} = c^2u_{xxx}, \quad u_{txx} = -cu_{xxx}, \ldots.
\]

The map \( i \) becomes

\[
i: (t, x, u, u_t, u_x, u_{xt}, u_{xx}, \ldots) \rightarrow (t, x, u, F, u_x, u_{xt}, -cu_{xx}, c^2u_{xxx} \ldots).
\]

The \( k \)th order pulled-back (by \( i^* \)) contact system \( D^+_N \) on \( J^k(\mathbb{R}^2, \mathbb{R}) \) is generated by

\[
\theta^1 = du + \frac{F}{c}(dx - cdt),
\]

\[
\theta^2 = dF + cu_x(dx - cdt),
\]

\[
\vdots
\]

\[
\theta^{[A]} := du_A - u_t(dx - cdt), \quad A = (x, \ldots, x), \quad 3 \leq |A| \leq k,
\]

with standard characterizing form \( \Omega_0 = \theta^1 \wedge \ldots \wedge \theta^k \). The next step is to determine the Frobenius integrability of this contact system, equivalently of \( \Omega_0 \), i.e.

\[
d\theta^a \wedge \Omega_0 = 0 \iff \theta^a \wedge d\Omega_0 = 0, \quad a = 1, \ldots, k.
\]

Once we have a Frobenius integrable Vessiot distribution its integration follows, for example, using solvable symmetry structures (see theorem 2.6). Projection of these integral submanifolds to \( \mathbb{R}^2 \times \mathbb{R} \) produces graphs of solutions of our PDE.

The next result is key to deciding whether an evolution equation has a Frobenius integrable Vessiot distribution without any reduction. It explains why, for example, the Korteweg–de Vries, is so amenable to solution as we will see.

**Theorem 3.1.** Let \( \Omega_0 \) be the standard characterizing \( k \)-form for an evolution PDE (3) on \( J^k(\mathbb{R}^2, \mathbb{R}) \), \( k \geq 3 \) with the travelling wave ansatz (4).
\( (1) \) \( \Omega_\theta \) is Frobenius integrable if and only if
\[
d F \wedge \phi \equiv 0 \mod \{ du, du_{xx}, ..., du_A \} \quad |A| = k \iff F_i + cF_x = 0
\]
and
\( (2) \) \( d\Omega_\theta = 0 \) if and only if \( \Omega_\theta \) is Frobenius integrable and \( F_{\phi_{k+1}} = 0 \), where \( \phi := dx - cdt \) and \( |A - 1| = k - 1 \).

**Proof.** First of all note that, with \( |A| = k \),
\[
d F \in \text{Sp} \{ dt, dx, du, du_{xx}, ..., du_A \}
\]
and that
\[
d \theta^1 = \frac{1}{c} d F \wedge \phi, \quad d \theta^2 = c du_{xx} \wedge \phi, \quad d \theta^{B_1} = -du_B \wedge \phi, \quad 2 < |B| \leq k.
\]
Then
\[
\begin{align*}
(1) \quad &d \theta^1 \wedge \Omega_\theta = 0, \quad d \theta^2 \wedge \Omega_\theta = 0, \quad d \theta^{B_1} \wedge \Omega_\theta = 0, \quad 2 < |B| < k \\
&\text{and} \\
(2) \quad &d \theta^1 \wedge \Omega_\theta = -du_A \wedge \phi \wedge du \wedge d F \wedge du_{xx} \wedge ... \wedge du_{A-1}.
\end{align*}
\]
Hence \( \Omega_\theta \) is Frobenius integrable if and only if
\[
d F \wedge \phi \equiv 0 \mod \{ du, du_{xx}, ..., du_A \} \iff F_i + cF_x = 0.
\]

(2) Now
\[
d \Omega_\theta = d \theta^1 \wedge \theta^2 \wedge ... \wedge \theta^k + ... + (-1)^{k-1} \theta^1 \wedge \theta^2 \wedge ... \wedge \theta^{k-1} \wedge d \theta^k,
\]
and every term except the last is trivially zero. The last term is
\[
\begin{align*}
&(-1)^k du \wedge d F \wedge du_{xx} \wedge ... \wedge du_{A-2} \wedge du_A \wedge \phi \\
&= (-1)^k du \wedge (F_i dx + F_i dt) \wedge du_{xx} \wedge ... \wedge du_{A-2} \wedge du_A \wedge \phi \\
&\quad + (-1)^k du \wedge F_{\phi_{k+1}} du_{A-1} \wedge du_{xx} \wedge ... \wedge du_{A-2} \wedge du_A \wedge \phi,
\end{align*}
\]
hence the closure result. \( \square \)

**Example 1.** Consider the evolution equation
\[
u_t = u_{xxx}
\]
along with the travelling wave ansatz (4). \( D_V^* \) is spanned by
\[
\begin{align*}
\theta^1 &:= du + \frac{1}{c} u_{xx} (dx - cdt), \quad \theta^2 := du_{xxx} + cu_{xx} (dx - cdt), \\
\theta^3 &:= du_{xx} - u_{xxx} (dx - cdt),
\end{align*}
\]
and \( D_V \) is spanned by
\[
\begin{align*}
V_1 &:= \frac{\partial}{\partial t} + u_{xxx} \frac{\partial}{\partial u} - cu_{xx} \frac{\partial}{\partial u_{xx}} + c^2 u_{xxx} \frac{\partial}{\partial u_{xxx}}, \\
V_2 &:= \frac{\partial}{\partial x} - \frac{1}{c} u_{xxx} \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_{xx}} - cu_{xx} \frac{\partial}{\partial u_{xxx}}.
\end{align*}
\]
\( D_V^* \) is Frobenius integrable with \( d\Omega_\theta = 0 \). A solvable structure for \( D_V \) (found with \texttt{dimsym} [7]) is \( \{ X_3, X_2, X_1 \} \) where
\[ X_1 := \frac{\partial}{\partial t}, \quad X_2 := \frac{\partial}{\partial x}, \quad X_3 := u \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_{xx}} + u_{xxx} \frac{\partial}{\partial u_{xxx}}. \]

It can be immediately seen that \( [X_u, V] = 0, [X_1, X_2] = 0 = [X_1, X_3] \) and \( [X_2, X_3] = X_2 \). A simple calculation shows that

\[ \mathcal{L}_{X_1} \Omega_\theta = 0, \quad \mathcal{L}_{X_2} \Omega_\theta = 0, \quad \mathcal{L}_{X_3} \Omega_\theta = 3 \Omega_\theta. \]

Proposition 2.7 indicates that \( X_1, X_2, \Omega_\theta \) is a closed one-form factor of \( \Omega_\theta \). This integrates to the following first integral of \( D_V \) (see the section 4 for a definition):

\[ f^4 := cu_{xx}^2 + u_{xxx}^2. \]

It is a simple matter to scale \( X_3 \) to \( \tilde{X}_3 := (f^4)^{-\frac{1}{2}}X_3 \) so that \( \mathcal{L}_{\tilde{X}_3} \Omega_\theta = 0 \) and \( [X_1, \tilde{X}_3] = 0, [X_2, \tilde{X}_3] = (f^4)^{-\frac{1}{2}}X_2 \), maintaining the solvable structure \( \{\tilde{X}_3, X_1, X_2\} \). From proposition 2.7 we immediately recover a further independent closed one-form, namely \( \tilde{X}_3, X_1, X_2 \) even though \( \{\tilde{X}_3, \tilde{X}_1, \tilde{X}_2\} \) is not itself a solvable structure. As a result we have another first integral of \( D_V \), namely

\[ f^2 := cu + u_{xx}. \]

Using theorem 2.6 we obtain a third integral

\[ f^3 := x - ct + c^{-\frac{1}{2}} \tan^{-1}\left( c^{-\frac{1}{2}} \frac{u_{xxx}}{u_{xx}} \right) \]

so that the integral manifolds of \( D_V \) are the common levels sets of \( f^1, f^2, f^3 \) and the solutions of (6) is the 3 parameter family obtained by eliminating all the derivatives from these codimension 3 level set equations and solving for \( u \).

### 4. The Korteweg–de Vries equation

The Korteweg–de Vries equation

\[ u_t = -uu_x + u_{xxx}, \tag{7} \]

produces the canonical example of a nonlinear solitary travelling wave. Applying the travelling wave ansatz described in the previous section we find that the third order pulled-back contact system \( D_V^* \) on \( J^1(\mathbb{R}^2, \mathbb{R}) \) is generated by

\[ \theta^1 := du + \frac{F}{c} (dx - cdt), \]
\[ \theta^2 := dF + cu_{xx} (dx - cdt), \]
\[ \theta^3 := du_{xx} - u_{xxx} (dx - cdt), \]

with \( F = \frac{c}{c - u} u_{xxx} \).

From theorem 3.1 \( D_V^* \) is clearly Frobenius integrable, that is

\[ d\theta^3 \wedge \Omega_\theta = 0. \]

Moreover, and again from theorem 3.1, \( d\Omega_\theta = 0 \) so that locally

\[ \Omega_\theta = df^1 \wedge df^2 \wedge df^3. \]
The corresponding Vessiot distribution $D_V$ is generated by

$$V_1 \equiv \frac{\partial}{\partial t} + F \frac{\partial}{\partial u} - cu_{xxx} \frac{\partial}{\partial u_{xx}} + \frac{c}{(c-u)^2} \left( (c-u)^2 u_{xx} - u_{xxx}^2 \right) \frac{\partial}{\partial u_{xxx}}.$$  

$$V_2 \equiv \frac{\partial}{\partial x} - F \frac{\partial}{\partial u} + u_{xxx} \frac{\partial}{\partial u_{xx}} - \frac{1}{(c-u)^2} \left( (c-u)^2 u_{xx} - u_{xxx}^2 \right) \frac{\partial}{\partial u_{xxx}}.$$  

The common level sets of $f^\alpha$ project to graphs of travelling wave solutions. There are various ways to find the $f^\alpha$, for example we could use a solvable structure and proposition 2.7 since $\Omega_2$ is closed. However, the KdV problem does not admit a projectable solvable structure. As it happens it is a simple matter to find two of the one-form factors of $\Omega_2$ directly and in a way that connects to known results. For the sake of convenience, we re-label the variables

$$y_1 \equiv x - ct, \quad y_2 \equiv u, \quad y_3 \equiv F, \quad y_4 \equiv u_{xx}.$$  

Now find the closed one-form factors, $dH$, of $\Omega_2$:

$$dH \wedge \Omega_2 = 0.$$  

Equation (8) implies

$$\frac{\partial H}{\partial y_1} - \frac{1}{c} \frac{\partial H}{\partial y_2} - cy_3 \frac{\partial H}{\partial y_3} - \frac{1}{c} (c-u) y_3 \frac{\partial H}{\partial y_4} = 0.$$  

So we need integral curves of

$$X \equiv \frac{\partial}{\partial y_1} - \frac{1}{c} \frac{\partial}{\partial y_2} - cy_3 \frac{\partial}{\partial y_3} - \frac{1}{c} (c-u) y_3 \frac{\partial}{\partial y_4}.$$  

This gives the well known ODE

$$y_2'' + (y_2 - c)y_2' = 0,$$  

where $' \equiv \frac{d}{dy_1}$, $y_1 \equiv x - ct$, $y_2 \equiv u$. That is

$$u'' + (u - c)u' = 0.$$  

This leads directly to closed forms for two of the first integrals:

$$f^1 = u'' + u \left( \frac{1}{2} u - c \right) = u_{xx} + u \left( \frac{1}{2} u - c \right),$$  

$$f^2 = u'^2 + \frac{1}{3} u^2 (u - c) - 2 f^1 u = u_{xx}^2 + \frac{1}{2} u^2 (3c - u) - 2 u u_{xx}.$$  

Restricting to an arbitrary common level set of $f^1$, $f^2$ gives

$$u'' + u \left( \frac{1}{2} u - c \right) = K, \quad u'^2 + \frac{1}{3} u^2 (u - c) - 2 f^1 u = L.$$  

Applying the usual boundary conditions at infinity forces $K = 0$, $L = 0$ and gives the soliton solution

$$u(x, t) = 3c \; \text{sech}^2 \left( \frac{1}{2} \sqrt{c} (x - ct) + M \right).$$  

In this expression $M$ is the value of the third integral, $f^3$, which is famously indeterminate in the absence of boundary conditions.
5. Conserved quantities and constants of the motion

In this section we will demonstrate the construction of constants of the motion for evolution equations. In general these constants of the motion will not apply to all solutions but for those which share one or more first integrals.

To make things concrete, consider a second order evolution PDE of one dependent variable \( u \) and two independent variables \( t, x \) of the form

\[
\frac{d^2u}{dt^2} = F(x, t, u, u_x, u_{xx}).
\]  

(10)

For the moment we do not assume the travelling wave ansatz (4).

We give the following definitions:

**Definition 5.1.** A first integral of an integrable distribution \( D := \text{Sp}\{X_1, \ldots, X_p\} \) on \( M^n \) is a smooth, non-constant (local) function on \( M^n \) with \( X_f \neq 0 \) for all \( a \).

**Definition 5.2.** A conservation law of (10),

\[
\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0,
\]

involves two suitably integrable functions of \( t, x, u, u_x, u_{xx} \), the density \( T \) and the flux \( X \). The resulting equation

\[
\frac{d}{dt} \int_b^a Tdx = 0
\]

identifies \( \int_b^a Tdx \) as a constant of the motion. The integrand \( T \) is called a conserved density.

Suppose that, as described in section 2, we have a particular and possibly reduced, integrable Vessiot distribution, \( D_{\text{Vess}} \), for our evolution equation, spanned by \( V_1, V_2 \) on \( J^2(\mathbb{R}^2, \mathbb{R}) \). Without loss, they look like,

\[
V_1 = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + \ldots, \quad V_2 = \frac{\partial}{\partial t} + F \frac{\partial}{\partial u} + \ldots.
\]

The integral manifolds \( S \) of \( D_{\text{Vess}} \) are the common level sets of three independent first integrals \( f^a \), so that \( V_a(f^a) = 0 \) and \( df^1 \wedge df^2 \wedge df^3 \neq 0 \) (at least locally on \( S \)). We emphasize that these integrals are specific to the particular solution of (10) determined by \( D_{\text{Vess}} \). By construction, \( V_1, V_2 \) project to the total derivatives \( \frac{d}{dt} \), \( \frac{d}{dx} \) along the projected solutions on the total (graph) space \( \mathbb{R}^2 \times \mathbb{R} \).

Suppose that

\[
\mathcal{N} := \{ p \in S : f^a(p) = c^a \}
\]

is a common level set of the \( f^a \). Then \( V_1, V_2 \) are tangent to \( \mathcal{N} \) and let \( \gamma_1, \gamma_2 \) be integral curves of \( V_1, V_2 \) through some point \( p \in \mathcal{N} \). Then \( \gamma_1, \gamma_2 \) project by \( \pi : J^2(\mathbb{R}^2, \mathbb{R}) \rightarrow \mathbb{R}^2 \times \mathbb{R} \) to curves on the solution surface \( N := \pi(\mathcal{N}), u = u(t, x; c^a) \), and

\[
\left( \frac{\partial}{\partial t} + u_x \frac{\partial}{\partial u} \right)_{\pi \circ \gamma_1}, \quad \left( \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} \right)_{\pi \circ \gamma_2}
\]

are tangent to these projected curves on \( N \). See figure 1.
Now consider the meaning of

\[ I := \int_a^b G \, dx \quad \text{with} \quad G = G(x, t, u, \ldots, u_{xx}, \ldots). \]

For the solution corresponding to \( D_{\psi_{1t}} \)

\[ \int_a^b G \, dx = \int_a^b G \circ \gamma_2 \, dx, \]

that is, the integral of \( G \) on \( S \) along an integral curve \( \gamma_2 : [a, b] \to N \) of \( V_2 \) from

\[ \gamma_2(a) = (t, a, u(t, a), \ldots, u_{xx}(t, a, u(t, a))) \]
to \( (t, b, u(t, b), \ldots, u_{xx}(t, b, u(t, b))) = \gamma_2(b). \)

Thus the functional \( I \) is an integral of the restriction of \( G \) to a solution surface and of course all the integral curves of \( V_2 \) project to curves on solution surfaces.

Moreover, the total time derivative of \( I \) has the meaning \( \left( \int_a^b G \circ \gamma_2 \, dx \right) \), because total time differentiation is just the projection of the directional derivative in the direction of \( V_1 \).
Now $[V_1, V_2] = 0$ by construction, so

$$V_1\left(\int_a^b G \circ \gamma_2 \, dx\right) = \int_a^b V_1(G) \circ \gamma_2 \, dx.$$  

Clearly, if $G$ is a composite of $f^\alpha$ then $V_1(G) = 0$. Further, $G$ then takes a single constant value, $l$ say, on the integral manifold of $D_{Vred}^a_{\gamma_2}$ on which $\gamma_2$ lies, so

$$\int_a^b G \circ \gamma_2 \, dx = l(b - a),$$  

and

$$V_1\left(\int_a^b G \circ \gamma_2 \, dx\right) = 0.$$  

Typical conservation laws look like

$$\frac{d}{dt}\int_a^b G(x, y, u, \ldots, u_{xxx}, \ldots) \, dx = 0,$$

and so any composite $G$ of the $f$’s produces a ‘constant of the motion’ for the solution determined by these $f$’s.

As an example consider all regular solutions of the Korteweg–de Vries equation satisfying the travelling wave ansatz (4). They all have first integral $f^1 := u_{xx} + u\left(\frac{1}{2}u - c\right)$ and so

$$\int_a^b u_{xx} + u\left(\frac{1}{2}u - c\right) \, dx$$

is a constant of the motion for some suitable interval $[a, b]$.

Returning to the general situation for $G : S \to \mathbb{R}$, where $G$ is not a composite of first integrals, but where we continue to consider the solution determined by our particular $D_{Vred}^a$,

$$\frac{d}{dt}\int_a^b G \, dx = V_1\left(\int_a^b G \circ \gamma_2 \, dx\right) = \int_a^b V_1(G) \circ \gamma_2 \, dx.$$

The integral curve $\gamma_2 : [a, b] \to \hat{N}$ represents an element of a family of curves obtained by varying the value of the $t$ co-ordinate of the point $\gamma_2(a) \in S$. If $g_t : S \to S$ is the one-parameter group generated by $V_1$ then these curves are the images $g_t \circ \gamma_2$ of $\gamma_2$. Because $[V_1, V_2] = 0$, this family of curves lies on an integral manifold of $D_{Vred}^a$ that is a lifted solution of the PDE and a common level set of the $f^\alpha$. See figure 2.

In the situation, where

$$\frac{d}{dt}\int_a^b G \, dx = 0,$$

we have

$$V_1\left(\int_a^b G \circ \gamma_2 \, dx\right) = 0 \equiv \int_a^b V_1(G) \circ \gamma_2 \, dx.$$  

The interpretation is that the integral is independent of the integral curve $g_t \circ \gamma_2$ along which the integral is performed. This does not imply that $V_1(G) = 0$. In other words, conserved densities are not necessarily first integrals but first integrals are always conserved densities.
A simple example of this is the well-known conserved density \( T_3 = u^3 + \frac{1}{2}u_x^2 \) of the Korteweg–de Vries equation. For all travelling wave solutions

\[
\frac{d}{dt} \int_{-a}^{a} T_3 \, dx = \int_{-a}^{a} V_1(T_3) \, dx = -c \int_{-a}^{a} \frac{du^3}{dx} \, dx = 0,
\]

since \( V_1(T_3) = V_1(u^3) = -cV_2(u^3) \), while \( T_3 \) is clearly not a first integral.

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