A new Menon-type identity derived from group actions

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Abstract

In this short note, we give a new Menon-type identity involving the sum of element orders and the sum of cyclic subgroup orders of a finite group. It is based on applying the weighted form of Burnside’s lemma to a natural group action.

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Key words: Menon’s identity, weighted Burnside’s lemma, group action, sum of element orders, sum of cyclic subgroup orders.

1 Introduction

One of the most interesting arithmetical identities is due to P.K. Menon [5].

Menon’s identity. For every positive integer \( n \) we have

\[
\sum_{a \in \mathbb{Z}_n^*} \gcd(a - 1, n) = \varphi(n) \tau(n),
\]

where \( \mathbb{Z}_n^* \) is the group of units of the ring \( \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}, \) \( \gcd(, \) \) represents the greatest common divisor, \( \varphi \) is the Euler’s totient function and \( \tau(n) \) is the number of divisors of \( n. \)

There are several approaches to Menon’s identity and many generalisations. One of the methods used to prove Menon-type identities is based on the Burnside’s Lemma concerning group actions (see e.g. [5][10]). In what follows, we will use a generalization of this result, called the Weighted Form of Burnside’s Lemma (see e.g. [2]).
**Weighted Form of Burnside’s Lemma.** Given a finite group $G$ acting on a finite set $X$, we denote

$$
\text{Fix}(g) = \{ x \in X : g \circ x = x \}, \forall g \in G.
$$

Let $R$ be a commutative ring containing the rationals and $w : X \rightarrow R$ be a weight function that is constant on the distinct orbits $O_{x_1}, \ldots, O_{x_k}$ of $X$. For every $i = 1, \ldots, k$, let $w(O_{x_i}) = w(x)$, where $x \in O_{x_i}$. Then

$$
\sum_{i=1}^{k} w(O_{x_i}) = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in \text{Fix}(g)} w(x).
$$

Note that the Burnside’s Lemma is obtained from (1) by taking the weight function $w(x) = 1, \forall x \in X$.

Next we will consider a finite group $G$ of order $n$ and the functions

$$
\psi(G) = \sum_{g \in G} o(g) \quad \text{and} \quad \sigma(G) = \sum_{H \in C(G)} |H|,
$$

where $o(g)$ is the order of $g \in G$ and $C(G)$ is the set of cyclic subgroups of $G$. Also, for every divisor $m$ of $n$, we will denote $G_m = \{ g \in G : g^m = 1 \}$.

Our main result is stated as follows.

**Theorem 1.** Under the above notations, we have

$$
\sum_{a \in \mathbb{Z}_n^*} \psi(G_{\gcd(a-1,n)}) = \varphi(n) \sigma(G).
$$

Clearly, (2) gives a new connection between the above functions $\psi(G)$ and $\sigma(G)$. We remark that an alternative way of writing (2) is

$$
\sum_{a \in \mathbb{Z}_n^*} \sum_{d \mid \gcd(a-1,n)} d \phi(d) n_d(G) = \varphi(n) \sigma(G),
$$

where $n_d(G)$ denotes the number of cyclic subgroups of order $d$ in $G$, for all $d$ dividing $n$.

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1 For more details concerning these functions, we refer the reader to [1, 3] and [4], respectively.
For $G = Z_n$, Theorem 1 leads to the following corollary.

**Corollary 2.** We have

$$\sum_{a \in Z_n^*} \psi(Z_{gcd(a-1,n)}) = \varphi(n) \sigma(n), \quad (4)$$

where $\sigma(n)$ is the sum of divisors of $n$.

Finally, since $\psi(Z_n) \geq \frac{q}{p+1} n^2$, where $q$ and $p$ are the smallest and the largest prime divisor of $n \geq 2$ (see the proof of Lemma 2.9(2) in [3]), from (4) we infer the following inequalities.

**Corollary 3.** We have

$$\frac{q}{p+1} \frac{1}{\varphi(n)} \sum_{a \in Z_n^*} \gcd(a-1,n)^2 \leq \sigma(n) \leq \frac{1}{\varphi(n)} \sum_{a \in Z_n^*} \gcd(a-1,n)^2. \quad (5)$$

## 2 Proof of Theorem 1

Let $Z_n^* = \{a \in \mathbb{N} : 1 \leq a \leq n, \gcd(a,n) = 1\}$ be the group of units (mod $n$). The natural action of $Z_n^*$ on $G$ is defined by

$$a \circ g = g^a, \forall (a,g) \in Z_n^* \times G.$$

Then two elements of $G$ belong to the same orbit if and only if they generate the same cyclic subgroup. This shows that the weight function $w : G \rightarrow \mathbb{R}$, $w(g) = o(g), \forall g \in G$, is constant on the distinct orbits $O_{g_1}, ..., O_{g_k}$ of $G$. Thus we can apply the Weighted Form of Burnside’s Lemma.

First of all, we observe that $w(O_{g_i}) = o(g_i) = |\langle g_i \rangle|, \forall i = 1, ..., k$, and therefore the left side of (1) is $\sigma(G)$.

Next we will prove that $\text{Fix}(a) = G_{gcd(a-1,n)}$, for any $a \in Z_n^*$. Indeed, if $g \in \text{Fix}(a)$ then $g^a = g$, that is $g^{a-1} = 1$. Since $|G| = n$, we also have $g^n = 1$. Consequently, $g^{gcd(a-1,n)} = 1$, i.e. $g \in G_{gcd(a-1,n)}$. The converse inclusion is obvious.

Now (1) becomes

$$\sigma(G) = \frac{1}{\varphi(n)} \sum_{a \in Z_n^*} \sum_{g \in G_{gcd(a-1,n)}} o(g) = \frac{1}{\varphi(n)} \sum_{a \in Z_n^*} \psi(G_{gcd(a-1,n)}),$$

as desired. \qed
References

[1] H. Amiri, S.M. Jafarian Amiri and I.M. Isaacs, *Sums of element orders in finite groups*, Comm. Algebra **37** (2009), 2978–2980.

[2] F. Harary and E.M. Palmer, *Graphical enumeration*, Academic Press, New York, 1973.

[3] M. Herzog, P. Longobardi and M. Maj, *An exact upper bound for sums of element orders in non-cyclic finite groups*, J. Pure Appl. Algebra **222** (2018), 1628–1642.

[4] T. De Medts and Tărnăuceanu, *Finite groups determined by an inequality of the orders of their subgroups*, Bull. Belg. Math. Soc. Simon Stevin **15** (2008), 699–704.

[5] P.K. Menon, *On the sum* \( \sum (a - 1, n)\) \((a, n) = 1\), J. Indian Math. Soc. **29** (1965) 155-163.

[6] I. M. Richards, *A remark on the number of cyclic subgroups of a finite group*, Amer. Math. Monthly **91** (1984), 571–572.

[7] B. Sury, *Some number-theoretic identities from group actions*, Rend. Circ. Mat. Palermo **58** (2009), 99-108.

[8] M. Tărnăuceanu, *A generalization of Menon’s identity*, J. Number Theory **132** (2012), 2568-2573.

[9] M. Tărnăuceanu, *The number of cyclic subgroups of finite abelian groups and Menon's identity*, Bull. Aust. Math. Soc. **101** (2020), 201-206.

[10] L. Tóth, *Menon’s identity and arithmetical sums representing functions of several variables*, Rend. Semin. Mat. Univ. Politec. Torino **69** (2011), 97-110.

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