Cutting out the cosmological middle man: 
General Relativity in the light-cone coordinates

Ermis Mitsou¹, Giuseppe Fanizza², Nastassia Grimm¹, Jaiyul Yoo¹,³

¹Center for Theoretical Astrophysics and Cosmology, University of Zurich, CH–8057, Zürich, Switzerland
²Instituto de Astrofísica e Ciências do Espaço, Faculdade de Ciências da Universidade de Lisboa, Edifício C8, Campo Grande, P-1740-016, Lisbon, Portugal
³Physics Institute, University of Zurich, Winterthurerstrasse 190, CH-8057, Zürich, Switzerland

E-mail: ermitsou@physik.uzh.ch, gfanizza@fc.ul.pt, ngrimm@physik.uzh.ch, jyoo@physik.uzh.ch

Abstract: Analytical computations in relativistic cosmology can be split into two sets: time evolution relating the initial conditions to the observer’s light-cone and light propagation to obtain observables. Cosmological perturbation theory in the FLRW coordinates constitutes an efficient tool for the former task, but the latter is dramatically simpler in light-cone-adapted coordinates that trivialize the light rays towards the observer world-line. Here we point out that time evolution and observable reconstruction can be combined into a single computation that relates directly initial conditions to observables. This is possible if one works uniquely in such light-cone coordinates, thus completely bypassing the FLRW “middle-man” coordinates. We first present in detail these light-cone coordinates, extending and generalizing the presently available material in the literature, and construct a particularly convenient subset for cosmological perturbation theory. We then express the Einstein and energy-momentum conservation equations in these coordinates at the fully non-linear level. This is achieved through a careful 2+1+1 decomposition which leads to relatively compact expressions and provides good control over the geometrical interpretation of the involved quantities. Finally, we consider cosmological perturbation theory to linear order, paying attention to the available gauge symmetries and gauge-invariant quantities.
Contents

1 Introduction & summary 2

2 The light-cone coordinates and the conformal Fermi subset 5
  2.1 Basics 5
  2.2 The interpretation of $N$ 6
  2.3 The temporal gauge and rigid conformal time parametrization 7
  2.4 Regularity conditions close to the observer 8
  2.5 The non-rotational observational gauge 9
  2.6 One final simplification 10
  2.7 Full gauge recap: the LCCF coordinates 10

3 Geometrical preliminaries 11
  3.1 $d + 1$ foliation 11
  3.2 Gauss-Codazzi-Mainardi equations 13

4 GR in LC 15
  4.1 $4 \rightarrow 3 + 1$ 15
  4.2 $3 + 1 \rightarrow 2 + 1 + 1$ 17
  4.3 The LC shift vector 18
  4.4 The GR equations 18

5 Conformal fields 20

6 Linear cosmological perturbation theory 22
  6.1 Background 22
  6.2 Perturbations 23
  6.3 Gauge-invariant variables 27
  6.4 Linearized equations of motion 29
  6.5 Road-map to analytical solutions 32

A Regularity conditions 33

B Perfect fluid matter $3 + 1$ decomposition 36

C Raw linearized equations of motion 38
1 Introduction & summary

The standard approach for analytical calculations in cosmology is perturbation theory around the homogeneous and isotropic solution. In the overwhelming majority of works in the literature, the space-time of that solution is described by the Friedmann-Lemaître-Robertson-Walker (FLRW) class of metrics

\[ ds^2 = -N^2(t) \, dt^2 + a^2(t) \, dl^2, \]  

(1.1)

where \( N \) is the lapse function, whose choice determines the time parametrization, \( a \) is the scale factor and \( dl^2 \) is the line-element of a 3-dimensional maximally symmetric Euclidean space. The form (1.1) has the convenient property of making explicit the isometries of that space-time. Further choosing “simple” spatial coordinates for \( dl^2 \), the fluctuations on top of this idealized solution can be decomposed in the spectral/harmonic fashion, thus revealing the independent degrees of freedom at the linear level. This approach is therefore optimal for solving perturbatively the equations of motion of a given theory. We will refer to this class of coordinate systems, i.e. those in which the metric is (1.1) plus small-amplitude fluctuations, as the “FLRW coordinates”.

In relativistic cosmology, solving equations of motion is not sufficient for comparing theory and observation – one must also solve further differential equations in order to obtain the observable quantities: e.g. redshift, luminosity distance, galaxy number density, etc. Unlike the equations of motion, which can be split into evolution equations and purely spatial (“constraint”) equations, the equations determining the cosmological observables are of light-like nature, as they control the propagation of photons along the past light-cone all the way up to the observer point. Therefore, in the FLRW coordinates the perturbative solutions for observables involve integrals along the past light-cone of the solutions of the equations of motion – the higher the order in perturbations, the larger the number of nested integrals. The corresponding literature is abundant, with well-established results at the linear level and also an important amount of work on non-linear effects in the past decade.¹

The analytical expressions for the solutions of cosmological observables are fairly complicated already at linear order, essentially due to the plethora of terms corresponding to distinct relativistic effects, but nevertheless tractable in practice. However, they often become discouragingly complicated at non-linear orders in perturbations, making their numerical computation tedious and prone to mistakes, thus motivating the consideration of alternative approaches. Interestingly, it is one of the conceptual foundations of General Relativity (GR) – the independence of physics on the choice of coordinate system – that provides the most dramatic simplification. To be specific, consider a set of coordinates denoted by \( \{ t, w, \theta^a \} \), with \( a \in \{ 1, 2 \} \), and the following line-element

\[ ds^2 = -2N \Upsilon dt \, dw + \Upsilon^2 dw^2 + \gamma_{ab} \left( d\theta^a - U^a dw \right) \left( d\theta^b - U^b dw \right). \]  

(1.2)

We further require that \( N, \Upsilon > 0 \), that \( \gamma_{ab} \) is a Euclidean metric and that the \( t, w \) = constant hypersurfaces have spherical topology, so that the \( \theta^a \) are angular coordinates. As we will show, any space-time can be brought to this form by an appropriate coordinate transformation, at least in some region of the full manifold. Equation (1.2) therefore corresponds to a coordinate choice, not a restriction to a particular class of space-times, and we will refer to it as the “light-cone” coordinates (LC). The particular case \( N = 1 \) is already known as the “geodesic light-cone” coordinates (GLC) [2].

¹See [1] for a large selection of references. This effort in pushing analytical calculations to higher order in perturbation theory is motivated by the upcoming order-of-magnitude increase in the quantity and quality of observational data. Indeed, in order to correctly interpret this data, one requires theoretical predictions that match the precision set by the observational uncertainties.
Clearly, no choice of functions \{N, \Upsilon, U^a, \gamma_{ab}\} allows one to obtain the FLRW form (1.1), because of the presence of the off-diagonal term \(\sim dt\, dw\) and the absence of the \(\sim dt^2\) term. Therefore, this choice of coordinates is radically different from the ones that are usually considered in cosmology, as it cannot be obtained by simply adding some small fluctuations to (1.1).

So how does (1.2) simplify the computation of observables? The answer is two-fold:

- The past light-cone of the observer, on which all of the observable information lies, is simply given by a hypersurface of the form \(w = \text{constant}\).

- The angular coordinates of the incoming photons on such light-cones are constant all along their trajectory, i.e., the light-rays reaching the observer are straight lines, just as in flat space-time.

The first point implies that the LC system is adapted to the “input-output” aspect of cosmology: the input data consists of initial conditions at some \(t = \text{constant}\) hypersurface in the early universe (e.g., after inflation or recombination), while the output data consists of the observable information, now also lying at a single coordinate value hypersurface \(w = \text{constant}\). The second point is the one which simplifies the actual computation of the observables: their expressions no longer contain integrals along the line of sight, since these integrals were precisely taking into account the bending and deformation of incoming light-beams due to curvature in the FLRW coordinates [3–6]. Thus, in LC coordinates the observables depend solely on fields evaluated at the source and the observer points, meaning that the computational cost of solving some differential equations (multiple nested integrals) is dramatically reduced down to some algebraic operations and possibly a few derivatives. Finally, from the second point we also infer that the LC coordinates can only cover a finite patch around the observer point, because they break down as soon as the first caustic of incoming light-rays occurs. However, this is not a problem for perturbative computations, since the fluctuation fields are defined on the background space-time which can be covered globally.

Until now the use of LC coordinates has been restricted to the following “hybrid” approach: work with FLRW coordinates to compute time-evolution from the early universe to the observer’s light-cone (e.g., FLRW-longitudinal gauge) and then switch to (G)LC through a coordinate transformation to compute the observables [6–16]. The rationale behind this approach is that each coordinate system is the most suited for the task at hand: FLRW coordinates simplify the description of time evolution, while LC coordinates simplify observable reconstruction. However, in proceeding so one is essentially trading the complication of solving the observable differential equations in the FLRW coordinates for the complication of performing the coordinate transformation between FLRW and LC. As a result, the computational complexity is not drastically reduced, but is merely displaced to a different stage of the overall calculation.

Our central observation here is that the FLRW coordinates appear as an unnecessary “middle man” in the approach described above. Indeed, the \(t\) coordinate of the LC system (1.2) parametrizes space-like Cauchy surfaces, so one can very well describe time evolution directly within that system. The aim of this paper is to express the minimal set of equations of motion that one requires in relativistic cosmology in the LC coordinates. Writing down equations in a given coordinate system is a long but straightforward work. However, if one proceeds in a brute-force manner, then one quickly realizes that the complexity of the resulting expressions renders them very hard to handle, especially if one takes into account the fact that we are interested in higher-order perturbation theory. Instead, here we will employ standard tools of differential geometry to obtain relatively compact expressions involving mathematical objects with a clear geometrical interpretation and readily solvable at the linearized level. The procedure can be summarized in the following three steps:
• We perform a standard 3 + 1 space/time decomposition of the manifold (also known as “ADM” decomposition \([17]\)), in which the \(N\) field of (1.2) appears as the ADM lapse function.

• We perform a further 2 + 1 angles/radius decomposition of the space-like hypersurfaces, thus leading to a 2 + 1 + 1 angles/radius/time decomposition of the full manifold.

• We identify the particular choice of ADM shift vector \(N^i\) which leads to the LC form (1.2). The norm of \(N^i\) is the limit case \(N_i N^i = N^2\) in which \(\partial_t\) becomes light-like, while the direction of \(N^i\) is set to the radial direction of the 2 + 1 decomposition, thus leading to light-cone hypersurfaces.

In particular, from the last point it is clear that (1.2) is a choice of coordinates, since the freedom of choosing the ADM shift is part of the freedom of choosing the coordinate system.

The equations that we will decompose as described above are the Einstein and energy-momentum conservation equations. We will proceed in great detail, so that the same procedure can be repeated for other equations of motion that one encounters in cosmology, such as the ones of other fields (e.g. scalar and vector).\(^2\) The end-product of the paper is a set of evolution equations, for gravity and matter, and purely spatial “constraint” equations. We will derive them at the fully non-linear level, but we will also provide their expression to linear order in perturbation theory around the homogeneous and isotropic solution, leaving higher order results for future work. With these equations, given some initial data at an early \(t = t_i\) hypersurface, one can evolve this information up to some future time after the observation event \(x_o\) and then simply “read-off” the observables by collecting field values on the corresponding past light-cone \(w = w_o\). One can understand these equations as the fusion of the differential equations describing time-evolution and observable reconstruction, so that solving them amounts to solving both problems simultaneously. Importantly, the linearized equations are not much harder to solve than their counterparts in the FLRW coordinates, as we show by explicitly solving part of them and providing a road-map for solving the rest. As a result, there is indeed a clear gain in efficiency. Finally, another original aspect of this paper will be to explore the consequences, and also highlight the practical advantages, of allowing \(N \neq 1\).

The paper is organized as follows. In section 2 we provide the necessary information in order to properly define and handle the LC coordinates, proving in particular the claims made in the introduction. The original part of this section is the discussion of the \(N \neq 1\) generalization of the already known GLC case, including its potential for more convenient gauge choices, and the generalization of the regularity conditions at the observer to arbitrary observer dynamics, i.e. including acceleration and rotation. In section 3 we present the geometrical tools that we will use to derive the desired equations, i.e. \(d + 1\) decomposition and the Gauss-Codazzi-Mainardi equations. The reader who is familiar with this machinery can skip that section, although we recommend a quick look to get acquainted with our conventions and definitions. Next, in section 4 we express the Einstein and energy-momentum conservation equations in LC form, that is, through a 2+1+1 angles/radius/time decomposition of the space-time manifold and a specific choice of the ADM shift vector. Finally, in section 5 we perform a field redefinition that simplifies the cosmological perturbation theory, which we then consider in section 6 to linear order. There we provide the linearized equations around the homogeneous and isotropic background, paying attention to the available gauge symmetries, and show how to solve these equations.

\(^2\)The case of Boltzmann distributions would require some extra structure, but would make use of the same geometrical construction.
2 The light-cone coordinates and the conformal Fermi subset

2.1 Basics

Let us start by writing explicitly the inverse metric components of (1.2)

\[ g^{tt} = -N^{-2}, \quad g^{tw} = -N^{-1} \Upsilon^{-1}, \quad g^{ta} = -N^{-1} \Upsilon^{-1} U^a, \quad g^{ww} = 0, \quad g^{wa} = 0, \quad g^{ab} = \gamma^{ab}, \]

where \( \gamma^{ab} \) is the inverse of \( \gamma_{ab} \). We first note that the \( t = \) const. hypersurfaces are space-like, because their normal vector \( \sim g^\mu_\nu \partial_\nu t \equiv g^\mu t \) has negative norm \( g^\mu_\nu g^\mu t g^\nu t \equiv g^{tt} < 0 \), so \( t \) is a time coordinate. On the other hand, the \( w = \) const. hypersurfaces are light-like, because their normal vector \( \sim g^\mu_\nu \partial_\nu w \equiv g^\mu w \) has vanishing norm \( g^\mu_\nu g^\mu w g^\nu w \equiv g^{ww} = 0 \). Taking into account the requirement that the \( t, w = \) const. hypersurfaces have spherical topology, we have that the \( w = \) const. hypersurfaces are light-cones and are therefore attached to some central world-line in space-time, henceforth simply referred to as the “observer”. One must choose between future and past light-cones, otherwise we over-parametrize space-time, and in this context the obvious choice is past. The observer world-line is described by a relation between the two coordinates \( \{ t, w \} \)

\[ w = w_o(t), \]

since the angular coordinates are not defined there, by definition. Also, the function \( w_o(t) \) must be monotonic for each past light-cone to be uniquely labeled. In general we will use a “\( o \)” subscript to denote evaluation at the observer world-line.

We next define the following future-oriented normal vector field to the \( w = \) const. hypersurfaces

\[ k^\mu := -g^\mu_\nu \partial_\nu w \equiv -g^{tw} = N^{-1} \Upsilon^{-1} \left( 1, 0, 0 \right)_\mu, \quad g\left( k, k \right) = 0, \]

and note that, for light-like vectors, being “normal” to a surface in the above sense actually means being tangent to it.\(^3\) Further observing that the \( k^\mu \) vector field is geodesic

\[ k^\nu \nabla_\nu k^\mu = 0, \]

we conclude that it can be interpreted as the 4-momentum vector of the incoming photons to the observer, up to a constant factor. One can then also verify that

\[ k^\mu \partial_\mu \theta^a = g^{wa} = 0, \]

meaning that the angular coordinates of these incoming photons are constant all along their trajectory. Therefore, the LC coordinates trivialize the light-propagation along the \( w = \) const. past light-cones.

Let us now observe that the line-element (1.2) does not correspond to a complete gauge fixing, i.e. one can still perform some coordinate transformations without altering its form. These are given by the full time-reparametrization freedom

\[ t \rightarrow t'(t, w, \theta), \]

\( ^3\)To understand this intuitively, consider for simplicity 2 + 1 Minkowski space-time with trivial coordinates \( g_{\mu\nu} = \text{diag}(-1,1,1) \) and pick the space-like surface \( t = \) const., so that its normal vector is the time-like \((1,0,0)\). Visually, performing a boost in some arbitrary direction leads to the surface and normal vector getting bent towards each other in that direction. In the limit of infinite boosting the two objects become tangent to each other, while maintaining their relative normality, which is possible because they are now a light-like hypersurface and a light-like vector.
and the light-cone and light-cone-dependent angle reparametrizations

\[ w \to w'(w), \quad \theta^a \to \theta'^a(w, \theta), \quad (2.7) \]

respectively. The former (2.6) is due to the fact that \( N \) is the lapse function of the ADM decomposition, which is therefore completely free to choose. In the GLC case, where \( N = 1 \), the freedom (2.6) is reduced down to \( t \to t + \text{constant} \). The other two transformations (2.7) do not involve a full space-time function and are therefore "residual" gauge transformations.4

Let us also mention another closely related class of coordinate systems which predates GLC, the so-called "observational coordinates" [18] (see also [1, 19]), where the time coordinate \( t \) is traded for a space-like coordinate. Thus, instead of slicing space-time into light-cones and spatial hypersurfaces, one slices it into light-cones and time-like cylinders. Here, however, we are interested in describing evolution in time of \( t = \text{constant} \) data, so the appropriate system is LC.

2.2 The interpretation of \( N \)

The freedom of choosing \( N \) is related to the different ways of slicing space-time into \( t = \text{constant} \) hypersurfaces. To see this, we define their future-oriented unit-normed normal vector

\[ n^\mu := \frac{-g^{\mu \nu} \partial_\nu t}{\sqrt{-g_{tt}}} = (N^{-1}, Y^{-1}, Y^{-1} U^a)\mu, \quad g(n, n) \equiv -1, \quad (2.8) \]

and note that its evolution in time is entirely controlled by \( N \)

\[ n^\nu \nabla_\nu n^\mu \equiv N^{-1} (g^{\mu \nu} + n^\mu n^\nu) \partial_\nu N. \quad (2.9) \]

Physically, the vector field \( n^\mu \) can be thought of as the 4-velocity field of the family of test masses. Consistently with the fact that motion at constant \( w \) means light-like motion, we see that the future time-like motion (2.8) requires a strictly positive \( w \) component \( n^w > 0 \). Moreover, that equation also allows us to interpret \( U^a \) as the angular velocity of said observer family. The vector field in (2.9) is then the family’s 4-acceleration, meaning that \( N \) gives us access to a broad set of possible dynamics. In particular, GLC corresponds to the case where the family is in free-fall \( N = \text{constant} \), hence the "geodesic" part of the name. The value \( N = 1 \) is then obtained through a constant rescaling of \( t \), thus making that variable the proper time of the family.

The member of that family that sits at the tip of the light-cones is nothing but the "observer" defined through (2.2), i.e. the actual observer involved in the cosmological observations. In the works employing the GLC system so far, one usually considers the rest of the members of that family to play the role of the sources involved in cosmological observations. As a result, in GLC both observer and sources are in free-fall, while in the more general LC case their dynamics is controlled by \( N \). Here we will choose not to perform this identification for the sources, so that \( N \) will be free to choose either for simplifying some particular computations or for simplifying some particular observable.5

This generalized description of sources does not spoil at all the advantages of the LC coordinates, i.e. the observables are still local functions of the fields. For instance, since the \( k^\mu \) field (2.3) denotes the 4-momentum of incoming photons on the light-cone, the redshift of a given source with 4-velocity

---

4See [15] for a detailed discussion of these freedoms in the GLC context.

5For instance, one can choose \( N \) such that the corresponding \( t \) variable coincides with a monotonic observable such as redshift or cosmological distances, or at least such that one of these quantities has zero fluctuations in cosmological perturbation theory.
$V_s^\mu(t, w, \theta)$ observed by an observer sitting at the central point with 4-velocity $n_o^\mu(w)$ is given by the integral-free expression (in LC coordinates)

$$1 + z(t, w, \theta) := \frac{(k_\mu V_s^\mu)(t, w, \theta)}{(k_\mu n_o^\mu)(w)} \equiv \Upsilon_o(w) V_s^w(t, w, \theta). \tag{2.10}$$

In the special case where the $V^\mu$ values are taken out of the $n^\mu$ field (2.8), one recovers the well-known result from the GLC literature

$$1 + z(t, w, \theta) \rightarrow \frac{\Upsilon_o(w)}{\Upsilon(t, w, \theta)}, \tag{2.11}$$

which therefore holds for generic $N$. The already known expressions for the GLC angular diameter and luminosity distances [3, 15], Jacobi map [5] and galaxy number density [6] will be generalized to arbitrary $V^\mu$ and $N$ in future work.

### 2.3 The temporal gauge and rigid conformal time parametrization

Let us next look at the light-cone reparametrization freedom in (2.7), which amounts to choosing the function introduced in (2.2) that describes the location of the observer world-line. We will consider for definiteness the simplest choice for cosmology that is the “temporal gauge” [15]

$$w_o(t) = t, \tag{2.12}$$

so that the observer world-line is located at $w = t$. Let us now see how this translates as a condition on the metric components. We first note that, under a light-cone reparametrization $w \rightarrow w'(w)$ we have in particular

$$\Upsilon^{-1} - \frac{dw'}{dw} \Upsilon^{-1}, \tag{2.13}$$

so that the gauge can be fixed through a condition on $\Upsilon$. More precisely, at the observer point the function $\Upsilon(t, w, \theta)$ has no angular dependence (see below) and reduces to a function of $t$ alone $\Upsilon_o(t) := \Upsilon(t, w_o(t))$. Alternatively, if we use the inverse function $w_o^{-1}$, then we can define a function of $w$ only $\Upsilon_o(w) := \Upsilon(w_o^{-1}(w), w)$, which is exactly the amount of information we can manipulate with $w \rightarrow w'(w)$ given (2.13). We are therefore looking for a condition on $\Upsilon_o$. Denoting by $x_o^\mu(\tau)$ the observer world-line, where $\tau$ is their proper time, the fact that $n_o^\mu$ is the observer 4-velocity

$$\frac{dx_o^\mu}{d\tau}(\tau) = n^\mu(x_o(\tau)), \tag{2.14}$$

along with the temporal gauge

$$x_o^t(\tau) = x_o^w(\tau), \tag{2.15}$$

implies that

$$\Upsilon_o(t) = N_o(t), \tag{2.16}$$

which is therefore the expression of the temporal gauge in terms of the metric components. In particular, note that the temporal gauge is not consistent with the GLC gauge $N = 1$ if $\Upsilon_o \neq 1$. This is actually the case for the global description of the cosmological homogeneous and isotropic solution where $\Upsilon$ is the scale factor $a(t)$. The temporal gauge then implies that we will be working with the conformal time parametrization in that context.

Finally, note that (2.16) is a condition on $\Upsilon_o$, but $N$ is still free to choose thanks to the time reparametrization symmetry (2.6). In particular, $N_o(t)$ is controlled by the transformations of the
form $t \rightarrow f(t)$, and one must simply be careful to combine these with $w \rightarrow f(w)$ in order to preserve the temporal gauge. This freedom can be fixed within cosmological perturbation theory by setting

$$N_0(t) = a(t),$$

(2.17)
to all orders in the perturbations, i.e. setting to zero the scale factor fluctuations at the observer, which is why we will name this the “rigid conformal time” parametrization. Thus, from now we will use the notation $a(t) := N_0(t) = \Upsilon_a(t)$, i.e. even when working at the fully non-linear level, with this quantity then reducing to the scale factor when doing cosmological perturbation theory. In practice, this simply means that when we will perturb $N = a + \delta N$ around the cosmological background, we will impose $\delta N_0 = 0$, and so on for $\Upsilon$.

### 2.4 Regularity conditions close to the observer

With the temporal gauge the quantity

$$r := w - t,$$

(2.18)
controls the spatial distance to the observer world-line if $t$ is kept constant, i.e. it is a “radius” coordinate. Assuming a regular space-time at the observer (e.g. no black hole), we can therefore express the metric components through a series of the form

$$g_{\mu\nu}(t, w, \theta) = \sum_{n=0}^{\infty} g_{(n)}^{(0)}(t, \theta) r^n.$$  

(2.19)
The reason for considering such an expansion is that the spherical nature of the LC coordinates constrains the first few coefficients $g_{(n)}^{(0)}(t, \theta)$. This is essentially due to the consistency requirement that no angular direction can be privileged at the origin $w = t$, assuming that space-time is regular there. A first regularity condition is that a tensor of the form $T_{a_1...a_n}d\theta^{a_1}...d\theta^{a_n}$ starts at order $r^n$, because otherwise the invariant quantity $T_{a_1...a_n}d\theta^{a_1}...d\theta^{a_n}$ would not be well-defined at the observer. Here this implies

$$U^{(0)}_a \equiv 0, \quad \gamma_{ab}^{(0)} \equiv \gamma_{ab}^{(1)} \equiv 0,$$

(2.20)
where the angular indices are displaced using $\gamma_{ab}$, so $U_a := \gamma_{ab} U^b$. But there are also further regularity conditions on the first few non-trivial terms of each series. Some of them are intuitive, e.g. for angular scalars the zeroth order term cannot depend on angles, since it survives in the limit of zero distance to the observer, so here

$$N^{(0)}(t, \theta) = \Upsilon^{(0)}(t, \theta) = a(t),$$

(2.21)
as already discussed in the previous subsection. However, most of the regularity conditions are not so transparent and must therefore be derived, as we do in detail in appendix A, using a generalization of the method employed in [20]. To express these conditions we need to define a few objects: a time-dependent parametrization of the unit-sphere in Euclidean space $\hat{X}(t, \theta) := \{\hat{X}^i(t, \theta)\}_{i=1,2,3}$, i.e.

$$\hat{X} \cdot \hat{X} \equiv 1,$$

(2.22)
and the corresponding induced metric and volume form on the unit-sphere

$$g_{ab} := \partial_a \hat{X} \cdot \partial_b \hat{X}, \quad g_{ab} := \sqrt{\hat{q}} \varepsilon_{ab},$$

(2.23)
respectively. The simplest case is

$$\hat{X} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \quad g_{ab} = \text{diag}(1, \sin^2 \theta), \quad g_{ab} = \sin \theta \varepsilon_{ab}.$$
but we will not commit to a particular $\hat{X}(t, \theta)$ map yet. In terms of these quantities, the remaining regularity conditions read:

$$U_a^{(1)} = 0, \quad \gamma_{ab}^{(2)} = a^2 q_{ab},$$

(2.25)

for the first non-trivial order and

$$N_a^{(1)} = a^2 \hat{\alpha} \cdot \hat{X},$$

(2.26)

$$\Upsilon_a^{(1)} = a^2 \left[2Z + \hat{\alpha} \cdot \hat{X}\right],$$

(2.27)

$$U_a^{(2)} = -a^3 \left[\partial_a Z + \hat{q}_{ab} \partial_b \left(W + \hat{\omega} \cdot \hat{X}\right)\right],$$

(2.28)

$$\gamma_{ab}^{(3)} = 2a^3 \left[ \left(Z + \hat{\alpha} \cdot \hat{X}\right) q_{ab} + \hat{q}_{(a} D_{b)} \partial_c W \right],$$

(2.29)

for the second one. Here $\hat{\alpha}(t)$ and $\hat{\omega}(t)$ are the acceleration and angular velocity vectors of the observer in their proper reference frame, respectively, $W(t, \theta) \sim \hat{q}^{ab} \partial_a \hat{X} \cdot \partial_b \hat{X}$ fully captures time-dependence of $\hat{X}(t, \theta)$, $Z(t, \theta)$ is also an undetermined function and $D_a$ is the covariant derivative associated to $q_{ab}$. Evaluating (2.9) at the observer position one obtains $\delta_{ab}^a \hat{\alpha} \cdot \hat{X}$, which is consistent with the interpretation of this vector as the 4-acceleration of the observer. Equations (2.21), (2.25) and (2.26) to (2.29) are the complete set of regularity conditions, i.e. there are no more constraints to higher orders.

### 2.5 The non-rotational observational gauge

We now consider the angular reparametrization freedom in (2.7) and note that it must be fixed on physical grounds. Out of all the possible parametrizations of the sky, only a small subset corresponds to the angles that the actual observer uses in practice, since these angles are defined only up to a global rotation of the sky, not the full 2d diffeomorphism group. In particular, correlation functions and spectral analysis on the sphere implicitly make use of these “observed” angular coordinates. Moreover, these coordinates are usually defined with respect to distant (“at infinity”) reference objects in the sky, thus rotating in time so as to compensate the spinning of the observer $\hat{\omega}(t)$. This extra requirement then determines the angular coordinates up to a time-independent rotation in the sky. The result is called the “non-rotating observational gauge” [15] and has been constructed explicitly in [20] for the $N = 1$ and $\hat{\omega} = 0$ case.

Once the regularity conditions (2.25) are given, it is easy to see how that gauge can be imposed. Following the same logic we use to fix $\Upsilon_a(t)$ through $w \rightarrow w'(w)$, here too the freedom $\theta^a \rightarrow \theta'^a(w, \theta)$ is equivalent to $\theta^a \rightarrow \theta'^a(t, \theta)$, since $w = t$ at the observer, meaning that we can choose the map $\hat{X}(t, \theta)$ freely. Since $\gamma_{ab}^{(2)} = a^2(t) q_{ab}(t, \theta)$ is the angular geometry infinitesimally close to the observer, it constitutes their “sky”, so we must choose a $\hat{X}(t, \theta)$ that reproduces the standard 2-metric

$$q_{ab} = \text{diag}(1, \sin^2 \theta)_{ab}.$$  

(3.30)

This fixes the function $\hat{X}(t, \theta)$ up to time-dependent rotations, which is exactly the freedom we need to compensate the observer’s rotation. Setting without loss of generality $\hat{\omega}(t) = \omega(t)(0, 0, 1)$, we can consider the rotating generalization of (2.24)

$$\hat{X}(t, \theta) = \left(\sin \theta \cos \left(\varphi - \int^t dt' a(t') \omega(t')\right), \sin \theta \sin \left(\varphi - \int^t dt' a(t') \omega(t')\right), \cos \theta\right),$$

(3.31)

\[\text{The condition (2.25) invalidates the argumentation and conclusion of [21], which is based on the erroneous assumption that } \gamma_{ab}^{(2)} \text{ can have a non-constant curvature, due to neglecting the regularity conditions at the observer.}\]
which leads to the same 2-metric \((2.30)\), but also (see appendix A)
\[
W = -\vec{\omega} \cdot \vec{X},
\]
which compensates indeed the effect of the observer’s rotation in the metric components \((2.28)\)
\[
U_a^{(2)} = -a^3 \partial_a Z,
\]
and also simplifies \((2.29)\) down to
\[
\gamma^{(3)}_{ab} = 2a^3 \left[ Z + \vec{\alpha} \cdot \vec{X} \right] q_{ab}.
\]
Finally, in the less usual case where one defines the angles with respect to earth-fixed references, the choice is \((2.24)\), so that \(W = 0\) and \(U_a^{(2)} = -a^3 \left[ \partial_a Z + \vec{\omega} \cdot \hat{q}_{ab} \partial_b \hat{X} \right] \).

### 2.6 One final simplification

The gauges imposed so far are already known in the GLC context. As we will now show, however, the consideration of \(N \neq 1\) leads to a very convenient and natural gauge choice, which is not reachable when \(N = 1\). To see this, let us go back to the coordinate transformation performed in appendix A, which relates the generalized Fermi normal coordinates \(\{T, \vec{X}\}\) to the LC ones \(\{t, w, \theta\}\). After imposing the regularity conditions, the temporal part of the transformation \((A.2)\) reduces to
\[
T(t, w, \theta) = \int t(a(t') \, dt' + \frac{1}{2} \dot{a}(t) - a^2(t) Z(t, \theta) \right) r^2 + \mathcal{O}(r^3),
\]
where we have used \((A.39)\). The fact that the zeroth and first orders are completely determined is due to the fact that we have fixed \(N^{(0)} = a\) and that \(N^{(1)}\) is determined by the observer’s acceleration \((2.26)\). We then see that the second-order coefficient is essentially \(Z\). Since we have only considered the first two non-trivial orders in the computation of appendix A, we do not know a priori whether \(Z\) is free to choose or determined. Going to the next order, one can verify that \(Z\) is indeed free to choose, because it must satisfy an equation which involves, not surprisingly, the free function \(N^{(2)}(t, \theta)\). Thus, \(Z\) parametrizes the freedom of performing time-reparametrizations at that order, while remaining within the LC class of coordinates. Put differently, \(Z\) can be brought to any desired function through a time-reparametrization of order \(r^2\), and doing so simply amounts to fixing \(N^{(2)}\). Given the regularity conditions \((2.26)\) to \((2.29)\), a natural choice is then
\[
Z = 0.
\]
It is now clear why this is not possible to enforce in the GLC case, where \(N^{(2)}\) is fixed to zero. Indeed, in [20] the authors showed that \(Z\), or more precisely their analogue of \(\Upsilon^{(1)}\), is fully determined by some components of the Riemann tensor at the observer position and in the Fermi coordinates. In contrast, here an analogous relation arises, but also involving the free function \(N^{(2)}\). Setting \((2.36)\) therefore determines \(N^{(2)}\) in terms of the Riemann tensor components in the Fermi coordinates, i.e. this is not a regularity condition, but just means that \(N^{(2)}\) is no longer free to choose.

### 2.7 Full gauge recap: the LCCF coordinates

Starting with the generic LC coordinates \((1.2)\), we have further specified these coordinates through successive conditions, either for computational convenience, or out of physical requirements. The first
kind of conditions are the temporal gauge (2.16), rigid conformal time parametrization (2.17) and also (2.36), while the second kind is the non-rotating observational gauge (2.30) and (2.31). All of these conditions can be collectively expressed through the form of the lowest-order coefficients of the metric

\[ N = a \left[ 1 + ar\vec{\alpha} \cdot \hat{X} + \mathcal{O}(r^2) \right], \]

\[ \Upsilon = a \left[ 1 + ar\vec{\alpha} \cdot \hat{X} + \mathcal{O}(r^2) \right], \]

\[ U_a = \mathcal{O}(r^3), \]

\[ \gamma_{ab} = a^2 r^2 \left[ 1 + 2ar\vec{\alpha} \cdot \hat{X} \right] \text{diag}(1, \sin^2 \theta)_{ab} + \mathcal{O}(r^4), \]

(2.37) \]

with the extra information that \( a(t) \) is the scale factor in cosmological perturbation theory and (2.31). In particular, note that all the non-trivial information enters exclusively through the \( \Upsilon \) combination

\[ N = \Upsilon + \mathcal{O}(r^2), \]

\[ \gamma_{ab} = \Upsilon^2 r^2 \text{diag}(1, \sin^2 \theta)_{ab} + \mathcal{O}(r^4), \]

(2.41)
and that the observer dynamics \( \{\vec{\alpha}, \vec{\omega}\} \) enter only through the combination \( \vec{\alpha} \cdot \hat{X}[\vec{\omega}] \). As for (2.39), it leads to the welcome feature of an observer 4-velocity (2.8) with trivial angular part

\[ n^a_\sigma = (a, a, 0, 0)\mu, \]

(2.42)
which is also not the case in GLC in general [20]. Observe next that, for a free-falling observer \( \vec{\alpha} = 0 \), the first two non-trivial orders of each metric component are the ones of the homogeneous and isotropic solution, which is clearly reminiscent of the Fermi normal coordinates, although in the present LC context. To make this relation precise, we trade the \( w \) coordinate for \( r := w - t \) and then the set \( \{r, \theta^a\} \) for the corresponding Cartesian coordinates \( \vec{x} \), to find that the line-element becomes

\[ ds^2 = a^2(t) \left[ -dt^2 + d\vec{x}^2 + \mathcal{O}(r^2) \right]. \]

(2.43)

These are the conformal Fermi normal coordinates [22], which differ from the usual Fermi normal coordinates by the fact that \( a(t) \) has been factored out. It therefore makes sense to refer to the LC coordinates supplemented by the extra conditions (2.37) to (2.40) as the (generalized) “light-cone conformal Fermi” coordinates, or “LCCF” for simplicity. The only leftover freedom is the choice of \( N^{(n=2)}(t, \theta) \) functions, which is still the bulk of the time-reparametrization freedom. In section 6.3 we will see that these functions can be chosen such that the redshift, or some cosmological distance, is given by their background value to all orders in cosmological perturbation theory, or such that the equations of motion take a particularly natural form, analogous to the FLRW-longitudinal gauge.

3 Geometrical preliminaries

3.1 \( d + 1 \) foliation

We consider a \( D \)-dimensional manifold \( \mathcal{M} \) with metric \( g \) of arbitrary signature and let \( d := D - 1 \). We then invoke a local coordinate system of the form \( \{y, x^i\} \), thus locally slicing (or “foliating”) \( \mathcal{M} \) in \( y = \text{const.} \) hypersurfaces which we denote by \( \Sigma_y \). They all share the same topology \( \Sigma_y \simeq \Sigma \), so we locally have \( \mathcal{M} \simeq \mathbb{R} \times \Sigma \). The line-element decomposes as follows

\[ ds^2 = sN^2dy^2 + h_{ij} \left( dx^i - sN^i dy \right) \left( dx^j - sN^j dy \right), \quad s = \pm 1, \]

(3.1)
where \( N, N^i \) and \( h_{ij} \) are respectively known as the “lapse function”, “shift vector” and “d-metric”, and \( s \) allows us to consider both the time-like and space-like cases simultaneously.\(^7\) The lapse and shift form the unit normal vector to \( \Sigma_y \)

\[
n(n, n) = s. \tag{3.2}
\]

A useful expression for computations is

\[
n_\mu := g_{\mu\nu} n^\nu = sN_\mu. \tag{3.3}
\]

To see that \( n \) is indeed the normal vector to \( \Sigma_y \), note that the tangent space of the latter is generated by the \( d \) vectors \( \partial_i \) and

\[
g(n, \partial_i) \sim \delta_i^y \equiv 0. \tag{3.4}
\]

We see that \( N \) and \( N^i \) capture the mismatch between the normal vector to \( \Sigma_y \) and the vector \( \partial_y \) which is parallel to motion along \( y \), i.e. when the \( x^i \) are held fixed (by definition of a partial derivative). Finally, to interpret \( h_{ij} \), we first define

\[
h_{\mu}^\nu := \delta_{\mu}^\nu - sn_\mu n^\nu, \tag{3.5}
\]

which is the projector onto the tangent space of \( \Sigma_y \)

\[
h_{\mu}^\nu n^\nu \equiv 0, \quad h_\mu^\nu h_\nu^\nu \equiv h_\mu^\nu, \quad h_\mu^\mu \equiv d. \tag{3.6}
\]

The spatial components of \( h_{\mu\nu} \) are the \( d \)-metric components \( h_{ij} \), so the latter is the induced metric on \( \Sigma_y \). Note also that the spatial components of \( h^{\mu\nu} \), i.e. \( h^{ij} \), coincide with the components of the inverse matrix of \( h_{ij} \), so the notation is consistent.

Now if we change the way we slice \( \mathcal{M} \), we change \( n \). However, in this construction the slicing information is contained in the coordinate choice and, in particular, in the way the \( y \) and \( x^i \) coordinates are split. As a consequence, \( n \) does not transform as a vector under all coordinate transformations. This is clear from the fact that its components are made of tensor components (3.2). We then see that it transforms as a vector

\[
n^{\mu}(x') = \frac{\partial x'^\mu}{\partial x^\nu} n^\nu(x), \tag{3.7}
\]

only under the following subgroup of coordinate transformations

\[
y \to y'(y), \quad x^i \to x'^i(y, x), \tag{3.8}
\]

in which case the metric components transform as follows

\[
N'(y', x') = \frac{dy}{dy'} N(y, x),
\]

\[
N'^i(y', x') = \frac{dy}{dy'} \left[ \frac{\partial x'^i}{\partial x^j} N^j(y, x) - s \frac{\partial x'^j}{\partial y} \right], \tag{3.9}
\]

\[
h'^{ij}(y', x') = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} h^{kl}(y, x).
\]

The second equation in (3.8) reparametrizes each slice \( \Sigma_y \) independently, i.e. we perform a different \( x^i \)-coordinate transformation for each \( y \) value. In contrast, the first transformation in (3.8) amounts

\(^7\)The choice of putting an \( s \) in front of \( N^i \) is only conventional, as it can always be reabsorbed in \( N^i \).
to a reparametrization of $y$ that is the same for all $x^i$, meaning that we are just reparametrizing the slices

$$\Sigma_y \to \Sigma_{y'(y)}.$$  

(3.10)

Thus, (3.8) is the largest subgroup preserving the slicing of $\mathcal{M}$, i.e. the $y = \text{const}$. hypersurfaces remain the same submanifolds of $\mathcal{M}$, as one could have expected from the fact that $n$ is invariant. For this reason, we will refer to (3.8) as the “slicing-preserving coordinate transformations” (SPCT). Consequently, any expression involving $n$ will only be invariant under this subgroup. Nevertheless, it is convenient to use $D$-dimensional notation anyway and keep referring to these objects as “tensors”.

One can next note that $N$ and $N^i$ can be chosen arbitrarily, i.e. they are pure gauge variables, since any values can be obtained by performing a generic coordinate transformation on the simplest choice

$$N_*=1, \quad N^i_* = 0,$$

(3.11)

which is known as the “synchronous gauge” in the time-like case. Indeed, denoting the corresponding coordinates by $(y_*, x^i_*)$ and performing an arbitrary coordinate transformation to some $(y, x^i)$, we get that $N$ and $N^i$ are precisely the information of the Jacobian matrix between the two systems

$$\partial_y y_* = N, \quad \partial_{y^i} x^i_* = s N^j \partial_{x^i} x^j_*,$$

(3.12)

and these can be chosen arbitrarily indeed. On the other hand, if we start with an arbitrary coordinate system $(y, x^i)$ with lapse $N$ and $N^i$, we can perform the following SPCT

$$y \to y, \quad x^i \to x^i_*(y, x),$$

(3.13)

so that, using (3.9), the resulting shift is

$$N^i \to \frac{\partial x^i}{\partial x^j} N^j - s \frac{\partial x^i}{\partial y} = 0.$$

(3.14)

In contrast, one cannot obtain $N \to 1$ by using a SPCT in general, because the multiplicative factor $dy/dy'$ in the transformation (3.9) does not depend on $x^i$. Thus, the information of the slicing lies exclusively in $N$, i.e. a choice of slicing amounts to a choice of $N$. On the other hand, $N^i$ controls how the points on a given slice $\Sigma_y$ are connected to the points of the next one $\Sigma_{y+d_y}$, i.e. it controls the way the slices are “glued” together.

### 3.2 Gauss-Codazzi-Mainardi equations

Having a $D$-dimensional tensor $h_{\mu\nu}$ representing the $d$-geometry of $\Sigma_y$ provides a straightforward way to express tensors on $\mathcal{M}$ in terms of tensors on $\Sigma_y$. One first defines the projection operation onto $\Sigma_y$

$$(T^\parallel)_{\rho_1...\rho_n} = h^\rho_{\rho_1}...h^\rho_{\rho_m}T^\rho_1...^\rho_m,$$

(3.15)

and refer to the tensors satisfying $T \equiv T^\parallel$, or equivalently $n \cdot T \equiv 0$, as “tangent” (to $\Sigma_y$) tensors. Working with a tangent vector as an example $X^\mu$, we stress that one must be careful with the position of the indices, because $n_\mu X^\mu \equiv 0$ implies

$$X^\mu \equiv 0, \quad \text{but} \quad X_y \equiv -s N^i X_i.$$

(3.16)

Nevertheless, the independent components lie in the spatial part and the two versions are consistently related by the $d$-metric

$$X_i \equiv g_{\mu i} X^\mu \equiv g_{iy} X^y + g_{ij} X^j \equiv h_{ij} X^j.$$

(3.17)
Thus, as long as we focus on the purely spatial components of tangent $D$-tensors, their position is irrelevant and they transform covariantly under SPCTs

$$T_{j_1 \ldots j_n}^{\ k_1 \ldots k_m} (y', x') = \frac{\partial x^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial x^{i_n}}{\partial x^{j_n}} \frac{\partial x'^{k_1}}{\partial x^{k_1}} \cdots \frac{\partial x'^{k_m}}{\partial x^{k_m}} T_{i_1 \ldots i_n}^{\ k_1 \ldots k_m} (y, x),$$

where $\frac{\partial x^j}{\partial x'^i}$ is the inverse matrix of $\frac{\partial x'^i}{\partial x^j}$. It then turns out that the space of such tensors is naturally endowed with a unique “tangent” covariant derivative $\nabla_T^\parallel$, defined by

$$\nabla_T^\parallel T := (\nabla T)^\parallel.$$

Indeed, in full analogy with the $D$-dimensional case, this is the only derivation that is compatible with the induced metric

$$\nabla_\mu h_{\mu\nu} \equiv 0,$$

and has zero torsion

$$\left[ \nabla_\mu^\parallel, \nabla_\nu^\parallel \right] \phi \equiv 0,$$

only now we also have the extra property

$$n^\mu \nabla_\mu^\parallel \equiv 0.$$

This allows one to implicitly define a Riemann tensor for $h_{\mu\nu}$ in the usual way

$$\left[ \nabla_\mu^\parallel, \nabla_\nu^\parallel \right] X^\rho =: (\mathcal{R}^\parallel)_\sigma^\rho_{\nu\mu} X^\sigma, \quad X \equiv X^\parallel.$$

where $\mathcal{R}^\parallel_{\mu\nu\rho\sigma}$ is explicitly tangent. One can then check that the spatial components of $\mathcal{R}^\parallel_{\mu\nu\rho\sigma}$ are equal to the Riemann tensor built out of $h_{ij}$

$$\mathcal{R}^\parallel_{ijkl} \equiv R_{ijkl}[h].$$

The $\mathcal{R}^\parallel_{\mu\nu\rho\sigma}$ tensor is therefore the “intrinsic” curvature of $\Sigma_y$, as it knows nothing about how these hypersurfaces are curved in the $n$ direction, i.e. in the ambient $\mathcal{M}$ space. This information is instead stored in the “extrinsic” curvature tensor

$$K_{\mu\nu} := h_\mu^\rho \nabla_\rho n_\nu \equiv \frac{1}{2} \mathcal{L}_n h_{\mu\nu},$$

where $\mathcal{L}_n$ is the Lie derivative in the $n$ direction

$$\mathcal{L}_n h_{\mu\nu} := n^\rho \partial_\rho h_{\mu\nu} + h_{\mu\nu} \partial_\rho n^\rho + h_{\mu\rho} \partial_\nu n^\rho,$$

and the form $\sim \mathcal{L}_n h_{\mu\nu}$ is obtained using the specific expression (3.2). This tensor is symmetric and tangent

$$K_{\mu\nu} \equiv K_{\nu\mu}, \quad n^\mu K_{\mu\nu} \equiv 0,$$

so all its independent information lies in $K_{ij}$. As in the case of the intrinsic curvature (3.24), here too the spatial components of (3.25) provide the direct relation to $h_{ij}$

$$K_{ij} \equiv \frac{1}{2N} (\partial_j + s \mathcal{L}_N) h_{ij},$$

---

8One also often finds the opposite sign convention for this definition.
where now $\mathcal{L}_N$ is the Lie derivative in the $N^i$ direction on $\Sigma_y$

$$\mathcal{L}_N h_{ij} := N^k \partial_ik h_{ij} + h_{kj} \partial_i N^k + h_{ik} \partial_j N^k \equiv D_i N_j + D_j N_i, \quad (3.29)$$

and $D_i$ is the covariant derivative made out of $h_{ij}$ on $\Sigma_y$. Note that the passage from (3.25) to (3.28) holds for any tangent $D$-tensor $T$

$$\mathcal{L}_N T_{i_1 \cdots i_m} = N^{-1} \left( \partial_{i_1} + s \mathcal{L}_N \right) T^{i_1 \cdots i_m}, \quad (3.30)$$

because the terms containing derivatives of the lapse are proportional to a contraction of $T$ and $n$. As a result, when acting on $d$-tensors, the operator $N^{-1} \left( \partial_y + s \mathcal{L}_N \right)$ is a covariant derivation under SPCTs (3.8). For instance, $K_{ij}$ transforms tensorially as

$$K_{ij}^\prime(y',x') = \partial_{x'i} \partial_{x'j} K_{kl}(y,x). \quad (3.31)$$

One last identity that is needed is the derivative of $n$ along itself

$$\nabla_n n_\mu = - s \nabla^\parallel_\mu \log N, \quad (3.32)$$

found using the specific expression (3.2). This is consistent with the fact that all the geometric information of the slicing lies in $N$ alone. In particular, $N = 1$ is known as “geodesic slicing”. With (3.32) and (3.25) we can express the derivative of $n_\mu$ as

$$\nabla_\mu n_\nu = (h_\mu^\rho + s n_\mu n^\nu) \nabla_\nu n_\nu \equiv K_{\mu\nu} - n_\mu \nabla^\parallel_\nu \log N. \quad (3.33)$$

We can now derive the Gauss-Codazzi-Mainardi equations. Expressing the right-hand side of (3.23) in terms of $\nabla_\mu$ and using $n_\nu \nabla_\mu X^\nu = - X^\nu \nabla_\mu n_\nu$, one obtains the Gauss-Codazzi equation

$$h_\alpha^\mu h_\beta^\nu h_\gamma^\rho h_\delta^\sigma R_{\alpha\beta\gamma\delta}[g] \equiv \nabla^\parallel_\mu K_{\nu\rho} - s [K_{\mu\rho} K_{\nu\sigma} - K_{\mu\sigma} K_{\nu\rho}], \quad (3.34)$$

which expresses the tangent part of the Riemann tensor of $\mathcal{M}$ in terms of the intrinsic and extrinsic curvatures of $\Sigma_y$. The full Riemann tensor information is then found by also considering parallel components, i.e. the Codazzi-Mainardi equations

$$h_\mu^\rho h_\nu^\sigma h_\alpha^\delta n^\gamma R_{\alpha\beta\gamma\delta}[g] \equiv \nabla^\parallel_\mu K_{\nu\rho} - \nabla^\parallel_\nu K_{\mu\rho}, \quad (3.35)$$

$$h_\mu^\rho n^\sigma n^\delta R_{\alpha\beta\gamma\delta}[g] \equiv - \mathcal{L}_n K_{\mu\nu} + K_{\mu\rho} K_\nu^\rho - s N^{-1} \nabla^\parallel_\mu \nabla^\parallel_\nu N, \quad (3.36)$$

which are found using

$$R_{\alpha\beta\gamma\delta} n^\delta \equiv R_{\gamma\delta\alpha\beta} n^\delta \equiv [\nabla_\alpha, \nabla_\beta] n_\gamma, \quad (3.37)$$

and expressing this in terms of $\nabla_n$ and $\nabla^\parallel_\mu$.

4 GR in LC

4.1 4 → 3 + 1

We now apply the foliation procedure described in the previous section to the case of interest, i.e. we pick $D = 4$ with a Lorentzian metric $g_{\mu\nu}$ and foliate the manifold with respect to the time-like direction $s = -1$. From now on we therefore denote the $y$ coordinate by “$t$”. The $n^\mu$ vector is thus
time-like and can therefore be interpreted as the 4-velocity of a family of observers, as discussed in section 2. The energy, momentum and stress tensor measured by that family is then

\[ E := n^\mu n^\nu T_{\mu\nu}, \quad P^\mu := -h^{\mu\sigma} n^\sigma T_{\rho\sigma}, \quad S^{\mu\nu} := h^{\mu\rho} h^{\nu\sigma} T_{\rho\sigma}, \]  

(4.1)

respectively, and the inverse decomposition can be compactly expressed as

\[ T_{\mu\nu}dx^\mu dx^\nu \equiv E (Nd^t)^2 - 2P^i (Nd^t) (dx^i + N^i dt) + S_{ij} (dx^i + N^i dt) (dx^j + N^j dt). \]  

(4.2)

The case where matter is a perfect fluid is treated in detail in the appendix B. Let us start by decomposing the Einstein equations. We first consider the non-trivial double trace of (3.34) to get the “time-time” component of the Einstein tensor

\[ n^{\mu} n^{\nu} G_{\mu\nu} \equiv \frac{1}{2} \left[ \mathcal{R}^\parallel + K^2 - K_{\mu\nu} K^{\mu\nu} \right], \]  

(4.3)

In terms of the fields on \( \Sigma_t \), the corresponding component of the Einstein equation thus reads

\[ E = \frac{1}{2} \left[ R[h] + K^2 - K_{ij} K^{ij} \right], \]  

(4.4)

where we use natural units \( c = 8 \pi G = 1 \). Next, we consider the non-trivial trace of (3.35) to get the “time-space” components

\[ h^{\mu\nu} n^{\rho} G_{\nu\rho} = \nabla^{\parallel}_\mu K^{\mu\nu} - (\nabla^{\parallel})^\mu K, \]  

(4.5)

so that the corresponding Einstein equation gives

\[ P_i = D_i K - D_j K^j. \]  

(4.6)

Finally, the “space-space” components of the Ricci tensor are found using (3.34) and (3.36)

\[ h^{\mu}_\mu h^{\sigma}_\sigma R_{\rho\sigma} \equiv g^{\alpha\beta} h^{\mu}_\mu h^{\sigma}_\sigma R_{\alpha\beta\sigma} \equiv h^{\alpha\beta} h^{\mu}_\mu h^{\sigma}_\sigma R_{\alpha\beta\sigma} - n^{\alpha} n^{\beta} h^{\mu}_\mu h^{\sigma}_\sigma R_{\alpha\beta\sigma} \]

\[ \equiv \mathcal{L}_n K_{\mu\nu} - 2K_{\mu\rho} K^{\rho\nu} + K_{\mu\nu} K + \mathcal{R}^\parallel_{\mu\nu} - N^{-1} \nabla^\parallel_\mu \nabla^\parallel_\nu N, \]  

(4.7)

so the corresponding components of the Einstein equation in the alternative form

\[ R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T, \]  

(4.8)

read

\[ (\partial_t - \mathcal{L}_N) K_{ij} = N \left[ 2K_{ik} K^{k}_{j} - K_{ij} K - R_{ij} [h] + S_{ij} - \frac{1}{2} h_{ij} (S - E) \right] + D_i D_j N. \]  

(4.9)

Along with (3.28)

\[ (\partial_t - \mathcal{L}_N) h_{ij} = 2N K_{ij}, \]  

(4.10)

these two equations form the dynamical part of the Einstein equations in first-order form if one considers \( K_{ij} \) as independent. Equations (4.4) and (4.6) are then constraint equations that are consistently preserved through evolution if they hold on the initial conditions. This set of equations could have been equivalently obtained by considering the ADM equations [17], trading the conjugate momenta \( \pi^{ij} \) for

\[ K_{ij} \equiv \frac{1}{\sqrt{h}} \left[ \pi_{ij} - \frac{1}{2} h_{ij} \pi \right], \]  

(4.11)
and using the Hamiltonian constraint (4.4) to modify the dynamical equation for \( K_{ij} \). The form considered here was originally obtained by York [23] and is usually referred to as the “standard ADM form” of the equations in the numerical relativity literature. As for the matter sector, we have the energy-momentum conservation equations \( \nabla_\mu T^{\mu\nu} = 0 \) that we can express in 3 + 1 form, i.e. as evolution equations for \( E \) and \( P_i \)

\[
\begin{align*}
(\partial_t - \mathcal{L}_N) E &= -D_j (N P^j) - P^i D_i N - N (KE + K_{ij} S^{ij}) , \\
(\partial_t - \mathcal{L}_N) P_i &= -D_j (N S^j_i) - ED_i N - NK P_i .
\end{align*}
\]

(4.12), (4.13)

Finally, note how the shift vector \( N^i \) enters only through the combination \( \partial_t - \mathcal{L}_N \) in all evolution equations (4.9), (4.10), (4.12) and (4.13). The Lie derivative is the generator of diffeomorphisms, so an evolution equation of the form \( \partial_t X = \mathcal{L}_X X + \ldots \) implies that, at every infinitesimal time step, one can perform an arbitrary infinitesimal 3-diffeomorphism \( x^i \rightarrow x^i - N^i(t, \vec{x}) \). Therefore, this is how the freedom of performing spatial reparametrizations \( x^i \rightarrow x^i(t, \vec{x}) \) manifests itself along time-evolution.

### 4.2 3 + 1 \( \rightarrow 2 + 1 + 1 \)

Now that our equations are expressed on the \( \Sigma_t \) manifolds, we further split \( x^i \rightarrow \{w, \theta^a\} \), where \( a \in \{1, 2\} \), thus foliating each \( \Sigma_t \) into \( w = \text{const.} \) surfaces \( S_{t,w} \). We will therefore now apply the Gauss-Codazzi-Mainardi formalism on \( \Sigma_t \) with \( s = 1 \), thus obtaining a “2 + 1 + 1” decomposition of our equations. Decomposing the line-element of \( \Sigma_t \) in the ADM fashion (3.1)

\[
d^2 := h_{ij} \, dx^i \, dx^j = Y^2 \, dw^2 + \gamma_{ab} (d\theta^a - U^a \, dw) \left( d\theta^b - U^b \, dw \right) ,
\]

(4.14)

the unit-normal vector to \( S_{t,w} \) is

\[
\nu := \frac{h^{wi}}{\sqrt{h^{ww}}} \, \partial_i \equiv Y^{-1} (\partial_w + U^a \partial_a) , \quad h(\nu, \nu) \equiv 1 ,
\]

(4.15)

and the projector onto \( S_{t,w} \) is

\[
\gamma_{ij} := h_{ij} - \nu_i \nu_j .
\]

(4.16)

Moreover, we choose the \( S_{t,w} \) surfaces to have spherical topology, so that \( w \) can be thought of as a radius on \( \Sigma_t \), while the \( \theta^a \) are angles. We can next define the extrinsic curvature of \( S_{t,w} \)

\[
C_{ij} := \gamma^k_{ij} D_k \nu_j \equiv \frac{1}{2} \mathcal{L}_\nu \gamma_{ij} ,
\]

(4.17)

so that, in full analogy with (3.25), its angular components read

\[
C_{ab} \equiv \frac{1}{2} T^{-1} (\partial_w + \mathcal{L}_U) \gamma_{ab} ,
\]

(4.18)

where here \( \mathcal{L}_U \) is the Lie derivative with respect to \( U^a \) on \( S_{t,w} \)

\[
\mathcal{L}_U \gamma_{ab} \equiv U^c \partial_c \gamma_{ab} + \gamma_{cb} \partial_a U^c + \gamma_{ac} \partial_b U^c \equiv \nabla_a U_b + \nabla_b U_a ,
\]

(4.19)

and \( \nabla_a \) is the covariant derivative made out of \( \gamma_{ab} \) on \( S_{t,w} \). The indices of tangent tensors to \( S_{t,w} \) are displaced using \( \gamma_{ab} \). It is also convenient to extract the components of \( K_{ij} \), \( P_i \) and \( S_{ij} \) under the \( S_{t,w} \) foliation

\[
\begin{align*}
\Theta := \nu^i \nu^j K_{ij} , \quad A_i := -2 \gamma^j_i \nu^k K_{jk} , \quad K_{ij} := \gamma^k_{ij} K_{kl} , \\
P := \nu^i P_i , \quad P_i := \gamma^j_i P_j ,
\end{align*}
\]

(4.20), (4.21)

and

\[
\begin{align*}
\Sigma := \nu^i \nu^j S_{ij} , \quad S_{ij} := \gamma^k_{ij} S_{kl} .
\end{align*}
\]

(4.22)
4.3 The LC shift vector

We are now in a position to describe the LC line-element (1.2) in this context. It simply amounts to
the following choice of shift vector
\[ N^i = -N \nu^i. \]  
(4.23)

With this, although \( n \) is time-like, and thus \( \Sigma_t \) is space-like, the vector \( \partial_t \) is light-like
\[ g(\partial_t, \partial_t) \equiv g_{tt} = 0, \]  
(4.24)

and radially oriented
\[ \partial_t \equiv N n + N^i \partial_i = N \left( n - \nu^i \partial_i \right). \]  
(4.25)

Thus, evolving along \( t \), while keeping \( w, \theta^a \) constant, means moving along a light-like direction. Since
the \( S_{t,w} \) submanifolds have spherical topology, the subset
\[ L_w := \bigcup_t S_{t,w}, \]  
(4.26)

therefore forms a light-cone. Note also that the shift condition (4.23) is preserved only if both sides transform as 3-vectors, meaning that it breaks the reparametrization freedom \( x_i \rightarrow \tilde{x}_i(t, \vec{x}) \) of the \( \Sigma_t \) slices down to the SPCTs of the \( S_{t,w} \) slices, which are nothing but the residual freedom (2.7).

Finally, with the shift now being fixed to (4.23), and equations (4.21) and (4.22), the energy-momentum tensor (4.2) reads
\[ T_{\mu\nu} dx^\mu dx^\nu \equiv E (N dt)^2 - 2 (N dt) [P (\Upsilon dw - N dt) + P_a (d\theta^a - U^a dw)] + \Sigma (\Upsilon dw - N dt)^2 + 2 S_a (\Upsilon dw - N dt) (d\theta^a - U^a dw) + S_{ab} (d\theta^a - U^a dw) (d\theta^b - U^b dw). \]  
(4.27)

4.4 The GR equations

We can now finally express the equations of GR in terms of the 2 + 1 + 1 fields. We start by projecting
(4.10) to get
\[ \dot{\Upsilon} = \Upsilon \left[ N \Theta - N' \right], \]  
(4.28)
\[ \dot{U}^a = N \left[ \Upsilon A^a - \nabla^a \Upsilon \right] + \Upsilon \nabla^a N, \]  
(4.29)
\[ \dot{\gamma}_{ab} = 2 N (K_{ab} - C_{ab}), \]  
(4.30)

where the dot denotes \( \partial_t \) and we have introduced the notation for tangent tensors to \( S_{t,w} \)
\[ X' := \mathcal{L}_\nu X \equiv \Upsilon^{-1} (\partial_w + \mathcal{L}_U) X, \]  
(4.31)

e.g.
\[ \mathcal{C}_{ab} \equiv \frac{1}{2} \gamma'_{ab}. \]  
(4.32)

From these equations we infer
\[ \dot{\nu}^i = - (N \Theta - N') \nu^i + N \left[ A^i - D^i{\parallel} \log \Upsilon \right] + D^i{\parallel} N, \quad \dot{\nu}_i = (N \Theta - N') \nu_i. \]  
(4.33)

We will also need the following relations
\[ \nu^i R_{ij} [h] \equiv - C^i - \mathcal{C}_{ab} C^{ab} - \Upsilon^{-1} \nabla^2 \Upsilon, \]  
(4.34)
These equations follow from the Gauss-Codazzi-Mainardi equations (3.34), (3.35) and (3.36) applied to the 2+1 split $h_{ij} \rightarrow \gamma_{ij} + \nu_i \nu_j$. Finally, we have the analogue of (3.32)

$$D_a \nu_t \equiv -D^b_\nu \log \gamma.$$  

(4.38)

With (4.9) and the above equations we can now compute the following time-derivatives

$$\dot{\Theta} = N \left[ -\Theta' + C' + \gamma^{-1} \left( \nabla^2 \gamma + A^a \nabla_a \gamma \right) - \Theta [\Theta + K] - \frac{1}{2} A_a A^a + C_{ab} C^{ab} + \frac{1}{2} (\Sigma - S + E) \right]$$

$$+ \left( \nabla_a \log \gamma - A_a \right) \nabla^a N + N^\nu,$$  

(4.39)

$$\dot{A}_a = N \left[ -A_a' + 2 \left( K_{ab}^b - \Theta \delta_{ab}^b \right) \nabla_b \log \gamma - K A_a + 2 \left( \nabla_b C_{ab} - \nabla_a C - S_a \right) \right]$$

$$+ 2 \left[ \left( C_{ab}^b - K_{ab}^b + \Theta \delta_{ab}^b \right) \nabla_b N - \nabla_a N' \right],$$  

(4.40)

$$\dot{K}_{ab} = N \left[ -K'_{ab} + C_{ab} - \Theta K_{ab} + 2 K_{ac} K_{ac}^c - K_{ab} K - 2 C_{ac} C_{ac}^c + C_{ab} C + \frac{1}{2} A_a A_b \right]$$

$$- A_a (\nabla_b \log \gamma + \gamma^{-1} \nabla_a \nabla_b \gamma + S_{ab} - \frac{1}{2} \gamma_{ab} (\dot{R} + \Sigma + S - E))$$

$$+ \nabla_a \nabla_b N + A_a (\nabla_b N) + C_{ab} N',$$  

(4.41)

As for the constraint equations (4.4) and (4.6)

$$E = \frac{1}{2} \left[ R - C^2 - C_{ab} C^{ab} + (K + 2 \Theta) K - K_{ab} K^{ab} - \frac{1}{2} A_a A^a \right] - \nabla^{-1} \nabla^2 \gamma - C',$$  

(4.42)

$$P = K_{ab} C_{ab} - \Theta C + \frac{1}{2} \nabla_a A^a + A^a \nabla_a \log \gamma + \kappa',$$  

(4.43)

$$P_a = \nabla_a (\Theta + K) - \nabla_b K_{ab} - (K_{ab}^b - \Theta \delta_{ab}^b) \nabla_b \log \gamma + \frac{1}{2} \left( C A_a + A_a' \right),$$  

(4.44)

and, finally, the energy-momentum conservation equations (4.12) and (4.13)

$$\dot{E} = -N \left[ E' + P' + C P + \nabla_a P^a + P^a \nabla_a \log \gamma \right]$$

$$+ (\Theta + K) E + \Theta \Sigma - A^a S_a + K_{ab} S^{ab} + 2 \left[ \Sigma P + P^a \nabla_a N \right],$$  

(4.45)

$$\dot{P} = -N \left[ (\Sigma' + C') + \Sigma + \nabla_a S^a + (P^a + 2 S^a) \nabla_a \log \gamma \right]$$

$$+ (2 \Theta + K) P - A^a P_a - C_{ab} S^{ab} - (E + \Sigma) N' + (P^a - S^a) \nabla_a N,$$  

(4.46)

$$\dot{P}_a = -N \left[ P'_a + C S_a + \nabla_a S'_a + \left[ S_a^b - (P + \Sigma) \delta_a^b \right] \nabla_b \log \gamma + (\Theta + K) P_a \right]$$  

(4.47)
\[-S_aN' - (E + P) \nabla_a N - S^b_b \nabla_b N.\]

Equations (4.28) to (4.30) and (4.39) to (4.44) form together the Einstein equations in LC form.

5 Conformal fields

In this section we introduce a new set of fields, the “conformal” ones, in terms of which the regularity conditions and the derivation of perturbative equations become both simpler and more transparent. They are given by are

\[\hat{N} := \Upsilon^{-1} N,\]
\[\hat{\Upsilon} := \log \Upsilon,\]
\[\hat{U}_a := \Upsilon^{-2} U_a,\]
\[\hat{\gamma}_{ab} := \Upsilon^{-2} \gamma_{ab},\]
\[\hat{\Theta} := \Upsilon \Theta,\]
\[\hat{A}_a := A_a,\]
\[\hat{K}_{ab} := \Upsilon^{-1} (K_{ab} - \Theta \gamma_{ab}),\]
\[\hat{E} := \Upsilon^2 E,\]
\[\hat{P}_a := \Upsilon P_a,\]
\[\hat{\Sigma} := \Upsilon^2 \Sigma,\]
\[\hat{S}_{ab} := S_{ab} - \Sigma \gamma_{ab},\]

and we use the convention of displacing the indices of hatted tensors with \(\hat{\gamma}_{ab}\). In terms of these fields the line-element (1.2) reads

\[ds^2 = e^{2\hat{\Upsilon}} \left[ -2\hat{N} dt dw + dw^2 + \hat{\gamma}_{ab} \left( d\theta^a - \hat{U}^a dw \right) \left( d\theta^b - \hat{U}^b dw \right) \right],\]

so now \(\hat{\Upsilon}\) enters as an overall conformal factor, while it simply disappears from the energy-momentum tensor (4.27)

\[T_{\mu\nu} dx^\mu dx^\nu \equiv \hat{E} \left( \hat{N} dt \right)^2 - 2 \left( \hat{N} dt \right) \left[ \hat{P} \left( dw - \hat{N} dt \right) + \hat{P}_a \left( d\theta^a - \hat{U}^a dw \right) \right] \]
\[+ \hat{\Sigma} \left[ \left( dw - \hat{N} dt \right)^2 + \hat{\gamma}_{ab} \left( d\theta^a - \hat{U}^a dw \right) \left( d\theta^b - \hat{U}^b dw \right) \right] + 2\hat{S}_a \left( dw - \hat{N} dt \right) \left( d\theta^a - \hat{U}^a dw \right) + \hat{S}_{ab} \left( d\theta^a - \hat{U}^a dw \right) \left( d\theta^b - \hat{U}^b dw \right).\]

Finally, because of the relations (2.41), the regularity conditions in the LCCF gauge (2.37) to (2.40) become extremely simple

\[\hat{N} = 1 + O(r^2),\]
\[\hat{\Upsilon} = \log a + a r \alpha \cdot \hat{X} + O(r^2),\]
\[\hat{U}_a = O(r^3),\]
\[ \dot{\gamma}_{ab} = r^2 \text{diag}(1, \sin^2 \theta)_{ab} + O(r^4). \] (5.19)

Let us now express the equations of the previous section in terms of the conformal fields. To that end, it is convenient to first write down some intermediary relations. We have

\[ C_{ab} \equiv e^{\tilde{\Psi}} \left[ \dot{\hat{C}}_{ab} + \dot{\gamma}_{ab} \dot{\hat{Y}} \right], \quad \mathcal{C} \equiv e^{-\tilde{\Psi}} \left[ \dot{\hat{C}} + 2 \dot{\hat{Y}} \right], \] (5.20)

where

\[ \dot{\hat{C}}_{ab} := \frac{1}{2} \dot{\gamma}_{ab}, \] (5.21)

and from now on the prime is defined by

\[ X' \rightarrow (\partial_w + \mathcal{L}_U) X, \] (5.22)

i.e. without the \( \Upsilon^{-1} \) factor. Note also that

\[ \dot{U}^a := \gamma^{ab} \dot{U}_b \equiv U^a, \] (5.23)

so that

\[ \mathcal{L}_U \equiv \mathcal{L}_\partial . \] (5.24)

The Christoffels of \( \gamma_{ab} \) and \( \dot{\gamma}_{ab} \) are related by

\[ \Gamma^c_{ab} \equiv \tilde{\Gamma}_{ab} + \delta_a^c \partial_b \hat{Y} + \delta_b^c \partial_a \hat{Y} - \dot{\gamma}_{ab} \delta^{cd} \partial_d \hat{Y}, \] (5.25)

so the quantities of interest for us are

\[ \mathcal{R} \equiv e^{-2\tilde{\Psi}} \left[ \hat{\mathcal{R}} - 2 \hat{\nabla}^2 \hat{Y} \right], \] (5.26)

\[ \nabla_b \hat{C}_{ab} - \nabla_a \hat{C} \equiv e^{-\tilde{\Psi}} \left[ \hat{\nabla}_b \hat{C} - \hat{\nabla}_a \hat{\hat{C}} + \hat{C}_a \hat{\nabla}_b \hat{Y} - \hat{\hat{C}}_a \hat{\nabla}_b \hat{Y} - \hat{\hat{Y}}' \hat{\nabla}_a \hat{Y} \right], \] (5.27)

\[ \nabla_a \nabla_b f \equiv \hat{\nabla}_a \hat{\nabla}_b f - 2 \hat{\nabla}_a f \hat{\nabla}_b \hat{Y} + \dot{\gamma}_{ab} \hat{\nabla}_c f \hat{\nabla}^c \hat{Y}, \] (5.28)

\[ \nabla^2 f \equiv e^{-2\tilde{\Psi}} \hat{\nabla}^2 f, \] (5.29)

where \( \hat{\mathcal{R}} \) and \( \hat{\nabla} \) are the Ricci tensor and covariant derivative of \( \dot{\gamma}_{ab} \) and \( f \) is a scalar on \( S_{t,w} \). The evolution equations become

\[ \dot{\dot{Y}} = \dot{N} \left[ \hat{\nabla} \left( \hat{\Theta} - \hat{Y}' \right) \right] - \dot{N}', \] (5.30)

\[ \dot{U}_a = \dot{N} \left[ \dot{\hat{A}}_a + 2 \left( \hat{\kappa}_a - \hat{C}_a \right) \dot{U}_b \right] + \nabla_a \dot{N} + 2 \dot{U}_a \dot{N}', \] (5.31)

\[ \dot{\gamma}_{ab} = 2 \dot{N} \left( \dot{\hat{C}}_{ab} - \dot{\hat{C}}_{ab} \right) + 2 \dot{\gamma}_{ab} \dot{\dot{N}}, \] (5.32)

\[ \dot{\hat{\Theta}} = \dot{N} \left[ -\dot{\Theta} + \dot{\hat{C}} + \dot{\hat{Y}}' \dot{\hat{C}} - 2 \dot{\Theta}^2 - \dot{\Theta} \dot{\hat{Y}} + \hat{\nabla}^2 \dot{\hat{Y}} \dot{\hat{Y}} + 3 \dot{\hat{\dot{Y}}}'' + \dot{\hat{C}}_{ab} \dot{\hat{C}}_{ab} - \frac{1}{2} \dot{\hat{A}}_a \dot{\hat{A}}^a \right. \\
\left. \quad + \frac{1}{2} \left( \dot{\hat{E}} - \dot{\hat{S}} - \dot{\hat{S}} \right) \right] + \left( \hat{\nabla}_a \dot{\hat{Y}} - \dot{\hat{A}}_a \right) \hat{\nabla}^a \dot{N} - \left( \dot{\hat{\Theta}} - \dot{\hat{Y}}' \right) \dot{\dot{N}}' + \dot{\dot{N}}'' \right], \] (5.33)

\[ \dot{\hat{A}}_a = \dot{N} \left[ -\dot{\hat{A}}_a - \left( \hat{\kappa} + 2 \hat{\Theta} \right) \dot{\hat{A}}_a + 2 \left( \hat{\nabla}_b \hat{C}^b_a - \hat{\nabla}^b_a \hat{C} + 2 \hat{\nabla}_a \hat{\hat{Y}} - 3 \hat{\dot{\hat{Y}}} \hat{\dot{\hat{Y}}} + 2 \dot{\hat{\dot{Y}}} \hat{\dot{\hat{Y}}} \dot{\hat{Y}} - \dot{\hat{\dot{Y}}} \hat{\dot{\hat{Y}}} \dot{\hat{Y}} - \dot{\hat{S}}_a \right) \right] \\
\left. \quad + 2 \left( \hat{\kappa}^b_a - \hat{\kappa}^b_a \right) \hat{\nabla}_b \dot{N} - 2 \hat{\nabla}_a \dot{N}', \right. \] (5.34)

\[ \dot{\hat{C}}_{ab} = \dot{N} \left[ -\dot{\hat{C}}_{ab} + \dot{\hat{\kappa}}_{ab} + 2 \dot{\hat{Y}} \hat{C}_{ab} + 2 \dot{\hat{\dot{\hat{Y}}}} \hat{C}_{ab} - \dot{\hat{\dot{\hat{Y}}}} \hat{C}_{ab} - 2 \dot{\hat{\dot{\hat{Y}}}} \hat{C}_{ab} + \hat{C}_{ab} + \frac{1}{2} \dot{\hat{A}}_a \dot{\hat{A}}_b \right. \]
\[ \begin{align*}
&+ 2 \hat{\nabla}_a \hat{\nabla}_b \hat{\Upsilon} - 2 \hat{\nabla}_a \hat{\Upsilon} \nabla_b \hat{\Upsilon} + \gamma_{ab} \left( - \hat{c}' - \hat{c}_{cd} \hat{\gamma}^{cd} + \frac{1}{2} \dot{A}_c \dot{A}_c - \frac{1}{2} \ddot{A} + 2 \dddot{\Upsilon}' - 2 \dddot{\Upsilon}'' \right) + \hat{S}_{ab} \right] \\
&+ \dot{N} \left[ \dot{K}_{ab} + \dot{\Theta}_{ab} \right] + \dot{A}(a \nabla_b \hat{N}) + \hat{\nabla}_a \hat{\nabla}_b \hat{N} - \gamma_{ab} \dot{\nabla}_b \hat{N},
\end{align*} \]

the constraint equations become

\[ \begin{align*}
\dot{E} &= \frac{1}{2} \left[ \dot{\mathcal{R}} - \dot{\mathcal{K}}_{ab} \dot{\mathcal{K}}^{ab} + \dot{\mathcal{K}}^2 - \dot{\mathcal{C}}_{ab} \dot{\mathcal{C}}^{ab} - \dot{\mathcal{C}}^2 \right] + 2 \dot{\mathcal{K}} \dot{\Theta} + 3 \dot{\Theta}^2 - \dot{\mathcal{C}}^2 - 2 \dot{\mathcal{C}} \dot{\Upsilon}', \\
- 2 \dot{\nabla}^2 \dot{\Upsilon} - \dot{\nabla}_a \dot{\nabla}_b \dot{\Upsilon} - 2 \dddot{\Upsilon}' - (\dot{\Upsilon})^2 - \frac{1}{4} \dot{A}_a \dot{\dot{A}}_a,
\end{align*} \]

and the energy-momentum conservation equations become

\[ \begin{align*}
\dot{\mathcal{P}} &= \dot{\mathcal{K}}_{ab} \dot{\mathcal{K}}^{ab} + \dot{\mathcal{K}}^2 + 2 \dot{\Theta} + \frac{1}{2} \dot{\nabla}_a \dot{\dot{A}}_a + A^a \dot{\nabla}_a \dot{\Upsilon} - 2 \dot{\Theta} \dot{\Upsilon}', \\
\dot{\mathcal{P}}_a &= - \nabla_b \dot{\mathcal{K}}_{ab} + \dot{\nabla}_a \left( \dot{\mathcal{K}} + 2 \dot{\Upsilon} \right) - 2 \left( \dot{\mathcal{K}}_{ab} + \delta_{ab} \dot{\Theta} \right) \nabla_b \dot{\Upsilon} + \frac{1}{2} \dot{A}_a + \frac{1}{2} \left( \dot{c} + 2 \dot{\Upsilon}' \right) \dot{A}_a,
\end{align*} \]

6 Linear cosmological perturbation theory

6.1 Background

In the LCCF coordinates the homogeneous and isotropic solution is given by

\[ \begin{align*}
N &= a(t), \quad \Upsilon = a(t), \quad U_a = 0, \quad \gamma_{ab} = a^2(t) r_k^2(r) q_{ab}(\theta),
\end{align*} \]

where

\[ \begin{align*}
r := w - t, \quad r_k(r) := \frac{\sin(\sqrt{k}r)}{\sqrt{k}}, \quad q_{ab} \, d\theta^a d\theta^b = d\theta^2 + \sin^2 \theta \, d\phi^2 =: d\Omega^2,
\end{align*} \]

with the latter indicating that we are also in the observational gauge (2.24). The line-element (1.2) thus reads

\[ \begin{align*}
ds^2 &= a^2 \left[ -2 dtdw + dw^2 + r_k^2(r) d\Omega^2 \right],
\end{align*} \]

so, in terms of \( r \) we recover the FLRW form

\[ \begin{align*}
ds^2 &= a^2 \left[ -dt^2 + dr^2 + r_k^2(r) d\Omega^2 \right],
\end{align*} \]

meaning that \( a \) is the scale factor, \( t \) is conformal time, \( r \) is the comoving distance to the observer and \( k \) is the spatial curvature. Using (4.28), (4.29), (4.30) and (4.18) we then find

\[ \begin{align*}
\Theta &= a^{-1} \mathcal{H}, \quad A_a = 0, \quad K_{ab} = a^{-1} \mathcal{H} \gamma_{ab}, \quad C_{ab} = a^{-1} \mathcal{K}_{ab}, \quad \mathcal{R} = \frac{2}{a^2 r_k^2},
\end{align*} \]
where we have defined the notation
\[ H := \frac{\dot{a}}{a}, \quad \mathcal{X} := \frac{r_k'}{r_k} \equiv \sqrt{k} \cot (\sqrt{k} r) \equiv \sqrt{\frac{1}{r_k^2} - k}, \] (6.6)
and the prime here reduces to differentiating with respect to \( w \) only since \( U_a \) vanishes. Note the useful identities
\[ \dot{r}_k \equiv -\mathcal{X}_r k, \quad r_k' \equiv \mathcal{X}_r k, \quad \dot{\mathcal{X}} \equiv \frac{1}{r_k^2}, \quad \mathcal{X}' \equiv -\frac{1}{r_k^2}. \] (6.7)

As for the matter sector
\[ E = \rho(t), \quad \Sigma = p(t), \quad S_a = 0, \quad S_{ab} = \gamma_{ab} p(t), \] (6.8)
where here \( \rho \) and \( p \) are the background total energy density and pressure. Let us now consider the conformal fields defined in the previous section. Their background values are
\[ \tilde{N} = 1, \quad \tilde{\Upsilon} = \log a, \quad \tilde{U}_a = 0, \quad \tilde{\gamma}_{ab} = r_k^2 q_{ab}, \] (6.9)
\[ \tilde{\Theta} = H, \quad \tilde{A}_a = 0, \quad \tilde{K}_{ab} = 0, \quad \tilde{C}_{ab} = \mathcal{X} \tilde{\gamma}_{ab}, \quad \tilde{R} = \frac{2}{r_k^2}, \] (6.10)
and
\[ \tilde{E} = a^2 \rho =: \tilde{\rho}, \quad \tilde{\Sigma} = a^2 p =: \tilde{p}, \quad \tilde{S}_a = 0, \quad \tilde{S}_{ab} = 0, \] (6.11)
so their first advantage is the fact that half of the tensors are now zero, meaning a simpler derivation of perturbative equations. The second advantage will be the absence of explicit \( a \) factors in the linearized equations, i.e. apart from the ones entering through \( H \). Finally, the non-trivial equations at the background level are (5.36), (5.33) and (5.39)
\[ 3 (H^2 + k) = \tilde{\rho}, \quad 2\dot{H} + H^2 + k = -\tilde{p}, \quad \dot{\tilde{\rho}} = -\tilde{H} (\tilde{\rho} + 3\tilde{p}), \] (6.12)
which are nothing but the Friedmann and energy conservation equations in terms of the conformal background variables, respectively.

### 6.2 Perturbations

Let us now consider fluctuations around the background solution. We will directly perform a Scalar-Vector-Tensor (SVT) decomposition with respect to the unit-sphere geometry \( q_{ab} \). In particular, we will use “\( D_a \)” to denote the corresponding covariant derivative and the indices of perturbative quantities will be displaced using \( q_{ab} \). Since this is a two dimensional geometry, the pure-vector components can be expressed in terms of pseudo-scalars, i.e.
\[ D_a h^a \equiv 0 \quad \Rightarrow \quad h_a \equiv \tilde{D}_a \tilde{h}, \] (6.13)
where
\[ \tilde{D}_a := \tilde{q}_a^b D_b, \] (6.14)
and
\[ \tilde{q}_{ab} := \sqrt{q} \varepsilon_{ab}, \] (6.15)
is the volume form on the unit-sphere. Therefore, a vector field decomposes as follows
\[ V_a = D_a \tilde{V} + \tilde{D}_a \tilde{V}, \] (6.16)
and we will generically use a bar and a tilde to distinguish between the pure-scalar and pure-vector parts. For tensors of rank higher than one, we first note that $q_{ab}$ allows one to eliminate all pure-trace components, while $\tilde{q}_{ab}$ allows one to eliminate any antisymmetric pair of indices through contraction, thus leaving only traceless and fully symmetric tensors to care about. One can then observe that pure-tensor components are zero because they obey as many constraints as their number of independent components. For instance, in the rank 2 tensor case

$$D_a h^{ab} = 0, \quad h_{a}^{a} = 0 \quad \Rightarrow \quad h^{ab} = 0. \quad (6.17)$$

Consequently, all non-trivial contributions come from the pure-scalar and pure-vector components. For instance, a traceless-symmetric rank 2 tensor field would first decompose as

$$T_{ab} = \bar{D}_{ab} \bar{T} + D_{(a} T_{b)}, \quad D_{a T}^{a} = 0, \quad (6.18)$$

where

$$\bar{D}_{ab} := D_{(a} D_{b)} - \frac{1}{2} q_{ab} D^{2}, \quad (6.19)$$

so that

$$T_{a} = \bar{D}_{a} \bar{T}, \quad (6.20)$$

and thus

$$T_{ab} = D_{ab} \bar{T} + \tilde{D}_{ab} \bar{T}, \quad (6.21)$$

where

$$\tilde{D}_{ab} := \tilde{D}_{(a} D_{b)} \equiv D_{(a} \tilde{D}_{b)}. \quad (6.22)$$

As a result, all perturbations are either scalars (barred) or pseudo-scalars (tilded) on the unit-sphere. Furthermore, if we decompose such a (pseudo-)scalar in the spherical harmonic basis on the sphere, then this amounts to decomposing the associated traceless-symmetric tensor of rank $s$ into spin-$s$ weighted spherical harmonics, since these are obtained by acting with $D_{a}$ and $\tilde{D}_{a}$ on $Y_{lm}$. The precise decomposition is

$$\phi(t, w, \theta) = (-1)^{s} \sum_{l=s}^{\infty} \frac{\sqrt{(l-s)!}}{(l+s)!} \sum_{m=-l}^{l} \phi_{lm}(t, w) Y_{lm}(\theta), \quad (6.23)$$

where the $s$-dependent factor is inherited by the unit-normalization of spin-$s$ spherical harmonics and thus leads to a simple relation between the scalar product of the tensors and the ones of the $\phi_{lm}$ components, e.g. in the vector case we have

$$\int d\Omega V_{a} V^{a} = \int d\Omega \left[D_{a} \bar{V} D^{a} \bar{V} + D_{a} \tilde{V} D^{a} \tilde{V}\right]$$

$$= - \sum_{l, l'=1}^{\infty} \frac{1}{\sqrt{l' (l+1)}} \sum_{m=-l}^{l} \sum_{m'=m-l'} \left[\bar{V}_{lm} \bar{V}_{l'm'} + \tilde{V}_{lm} \tilde{V}_{l'm'}\right] \int d\Omega Y_{lm} D^{2} Y_{l'm'}$$

$$= \sum_{l, l'=1}^{\infty} \frac{1}{\sqrt{l' (l+1)}} \sum_{m=-l}^{l} \sum_{m'=m-l'} \left[\bar{V}_{lm} \bar{V}_{l'm'} + \tilde{V}_{lm} \tilde{V}_{l'm'}\right] \int d\Omega Y_{lm} Y_{l'm'}$$

More precisely, the spin weighted spherical harmonics arise if one works with a dyad $e_{a}^{A}$ on the sphere, i.e. $q_{ab} e_{a}^{A} e_{b}^{B} = \delta_{AB}$, and the corresponding derivatives in these directions $D_{A} := e_{a}^{A} D_{a}$. For more details see section 5.11 of [1].
From this it is also clear why the \( l \) sum in (6.23) starts at \( l = s \), because for \( l < s \) the norm would be zero, meaning that \( l \) is the total angular momentum. Thus, \( \tilde{V}_{lm} \) corresponds to an “spin-1 \( E \)-mode”, \( \tilde{V}_{lm} \) is a “spin-2 \( B \)-mode”, while their tilded counterparts correspond to the respective “\( B \)-modes”. The \( E/\)\( B \)-mode distinction reflects the even/odd behavior under parity, which is a symmetry of the background solution, so the two sectors will be decoupled at the linear level. Finally, since all fluctuations are (pseudo-)scalars on the sphere, the angular derivatives can only enter the linear equations through the Laplacian combination \( D^2 \). Thus, decomposing the fields as in (6.23) amounts to replacing

\[
\phi(t, w, \theta) \rightarrow (-1)^s \sqrt{\frac{(l-s)!}{(l+s)!}} \phi_{lm}(t, w), \quad D^2 \rightarrow -l(l+1),
\]

in the equations. At the linear level all \( lm \) modes are therefore decoupled, leaving us with a set of linear partial differential equations in the two variables \( \{t, w\} \). Including non-linear orders will bring couplings between different \( lm \) values, but it will not change the fact that this is a system of 2-dimensional partial differential equations.

We are now ready to express the conformal fields in terms of fluctuations around the background solution

\[
\tilde{N} = 1 + r_k^2 \psi, \quad (6.26)
\]

\[
\tilde{T} = \log a + r_k^2 \phi, \quad (6.27)
\]

\[
\tilde{U}_a = r_k^2 \left( D_a \tilde{u} + \tilde{D}_a \tilde{u} \right), \quad (6.28)
\]

\[
\tilde{\gamma}_{ab} = r_k^2 \left[ q_{ab} (1 + 2 r_k^2 \chi) + 2 \left( D_a D_b \tilde{\chi} + \tilde{D}_{ab} \tilde{\chi} \right) \right], \quad (6.29)
\]

\[
\tilde{\Theta} = \tilde{\mathcal{H}} + r_k^2 \theta, \quad (6.30)
\]

\[
\tilde{A}_a = r_k^2 \left[ D_a \tilde{\alpha} + \tilde{D}_a \tilde{\alpha} \right], \quad (6.31)
\]

\[
\tilde{K}_{ab} = r_k^2 \left[ q_{ab} r_k^2 \kappa + D_a D_b \tilde{\kappa} + \tilde{D}_{ab} \tilde{\kappa} \right], \quad (6.32)
\]

\[
\tilde{E} = \tilde{\rho} (1 + r_k^2 \delta), \quad (6.33)
\]

\[
\tilde{\rho} = (\tilde{\rho} + \tilde{\rho} r_k^2 v), \quad (6.34)
\]

\[
\tilde{\rho}_a = (\tilde{\rho} + \tilde{\rho} r_k^2) r_k^2 \left[ D_a \tilde{v} + \tilde{D}_a \tilde{v} \right], \quad (6.35)
\]

\[
\tilde{\Sigma} = \tilde{\rho} + r_k^2 \sigma, \quad (6.36)
\]

\[
\tilde{S}_a = r_k^2 \left[ D_a \tilde{s} + \tilde{D}_a \tilde{s} \right], \quad (6.37)
\]

\[
\tilde{S}_{ab} = r_k^2 \left[ q_{ab} r_k^2 \pi + D_a D_b \tilde{\pi} + \tilde{D}_{ab} \tilde{\pi} \right]. \quad (6.38)
\]

Several remarks are in order here. First, note that we chose to introduce the tensor \( E \)-modes through the operator \( D_a D_b \), instead of \( \tilde{D}_{ab} \). The advantage is that the conformal 2-metric takes the form

\[
\tilde{\gamma}_{ab} = r_k^2 \left[ q_{ab} (1 + 2 r_k^2 \chi) + D_a X_b + D_b X_a \right], \quad (6.39)
\]

so that the spin-2 fields \( \tilde{\chi} \) and \( \tilde{\chi} \) enter as a \( (t, w) \)-dependent coordinate transformation, thus leaving \( \chi \) to control the curvature \( \mathcal{R} \) of the \( S_{t,w} \) surfaces. Second, note the non-trivial normalization of the
fluctuations, i.e. the presence of the $r_k^2$ factors, even in front of scalars. This choice is “natural” in the sense that the corresponding linearized equations will have no explicit $r_k$ factors, just as they will have no explicit $a$ factors thanks to our use of the conformal variables. Instead, $a$ and $r_k$ will only enter through the combinations $H$, $X$ and the background Laplacian operator on the $S_{t,w}$ surfaces

$$\Delta^{(2)} := \frac{1}{r_k^2} D^2.$$  \hspace{1cm} (6.40)

A priori, the only disadvantage is that the expansion in the distance to the observer starts at order $r^{-2}$, e.g.

$$\phi(t, w, \theta) = \infty \sum_{n=-2}^{\infty} \phi^{(n)}(t, \theta) r^n.$$ \hspace{1cm} (6.41)

Nevertheless, at least for the metric fluctuations, the first few orders are neutralized in a natural way by the regularity conditions in the LCCF gauge (5.16) to (5.19). In the free-falling observer case $\vec{\alpha} = 0$ we have simply

$$\psi^{(n<0)} \phi^{(n<0)} \chi^{(n<0)} = 0,$$ \hspace{1cm} (6.42)

for the spin-0 fields, and

$$\bar{u}^{(n<1)}, \tilde{u}^{(n<1)} = 0,$$ \hspace{1cm} (6.43)

for the spin-1 fields, and

$$\bar{\chi}^{(n<2)}, \tilde{\chi}^{(n<2)} = 0,$$ \hspace{1cm} (6.44)

for the spin-2 fields, i.e. the natural behavior that a spin-$s$ field starts at $O(r^s)$. The more general case $\vec{\alpha} \neq 0$ can then be obtained by simply shifting $\phi$

$$\phi \to \phi + \frac{\vec{\alpha}}{r} \cdot \vec{X}[\vec{\omega}].$$ \hspace{1cm} (6.45)

In fact, since $\vec{\alpha}$ and $\vec{\omega}$ are not constrained by the equation of motion, this substitution can be performed at the end, i.e. at the level of the cosmological observable expressions. As for the matter sector, there are no regularity constraints other than the obvious ones

$$\dot{\bar{E}}^{(0)} \equiv \dot{\bar{E}}^{(0)}(t), \hspace{0.5cm} \dot{\bar{p}}^{(0)} \equiv \dot{\bar{p}}^{(0)}(t), \hspace{0.5cm} \dot{\Sigma}^{(0)} \equiv \dot{\Sigma}^{(0)}(t), \hspace{0.5cm} \dot{\bar{P}}^{(0)} = 0, \hspace{0.5cm} \dot{\bar{S}}^{(0)} = 0, \hspace{0.5cm} \dot{\bar{S}}^{(n<1)} = 0,$$ \hspace{1cm} (6.46)

meaning that the matter fluctuations start at $O(r^{s-2})$. Moreover, note that the $\delta$ field in (6.33) is the relative fluctuation of the conformal energy density $\dot{\bar{E}}$, not of $E$ as is in the usual convention. On top of that, in the case where matter is a single perfect fluid (see appendix B) we have, at the linear level,\n
$$\pi, \bar{\pi}, \tilde{\pi}, \bar{s}, \tilde{s} = 0,$$ \hspace{1cm} (6.47)

so these five fields control the linear anisotropic stress tensor. The speed of sound $c_s$ in terms of the canonical variables is implicitly defined by $\delta \Sigma = c_s^2 \delta E + O(2)$, so in terms of the conformal fields $\Sigma$, $\dot{E}$ we obtain, to linear order,

$$\sigma = c_s^2 \dot{\rho}(\delta - 2\phi) + 2\dot{\rho}\phi,$$ \hspace{1cm} (6.48)

where here $c_s(t)$ is the background speed of sound. The perfect fluid is then entirely determined by the data $\{\delta, v, \bar{v}, \tilde{v}\}$ which can be evolved using the energy-momentum conservation equations.

Before we close this subsection, it is instructive to make the connection with the fluctuations in the FLRW coordinates. At the background level, the LC and FLRW metrics (6.3) and (6.4), respectively, are related by the reparametrization $w \to r + t$. Since the latter is a finite transformation, it can only
transforms like a tensor, as opposed to the gauge transformations of cosmological perturbation theory. Taking the linearized LC line-element (5.14)
\[
\frac{ds^2}{a^2} = -2 \left[ 1 + 2r_k^2(\psi + 2\phi) \right] dt dw + \left[ 1 + 2r_k^2\phi \right] dw^2 - 2r_k^2 \left[ D_a\hat{u} + \bar{D_a}\hat{u} \right] dr d\theta^a + r_k^2 \left[ q_{ab} \left( 1 + 2r_k^2(\chi + \phi) \right) + 2 \left( D_aD_b\bar{\chi} + \bar{D}_a\bar{\chi} \right) \right] d\theta^a d\theta^b + O(2),
\]
and replacing \( w \to r + t \), we find
\[
\frac{ds^2}{a^2} = - \left[ 1 + 2r_k^2(\psi + \phi) \right] dt^2 - 2r_k^2\psi dt dr - 2r_k^2 \left[ D_a\hat{u} + \bar{D_a}\hat{u} \right] dt d\theta^a + \left[ 1 + 2r_k^2\phi \right] dr^2 - 2r_k^2 \left[ D_a\hat{u} + \bar{D_a}\hat{u} \right] dr d\theta^a + r_k^2 \left[ q_{ab}(1 + 2r_k^2(\chi + \phi)) + 2 \left( D_aD_b\bar{\chi} + \bar{D}_a\bar{\chi} \right) \right] d\theta^a d\theta^b + O(2).
\]
The former is indeed a perturbation of the FLRW line-element in spherical coordinates (6.4) and in the conformal Fermi normal coordinates \([22]\) in particular. Its specificity is that the shift vector \( g_{0i} \) is determined by the rest of the fields. To get some intuition, let us consider flat space \( k = 0 \) in Cartesian coordinates, in which case
\[
\frac{g_{0i}}{a^2} = -\psi \bar{x}^i - (r^2\delta^{ij} - x^i x^j) \partial_j \bar{u} - \varepsilon^{ijk} x^j \partial_k \bar{u} + O(2), \quad r := |\bar{x}|.
\]
This expression is quite unusual, as it privileges the radial direction, and therefore does not correspond to any common gauge choice in the literature. In \([24]\) the authors showed how to obtain (6.51) by considering general perturbation theory and imposing the GLC gauge at the perturbative level.

### 6.3 Gauge-invariant variables

One clear lesson from cosmological perturbation theory in the FLRW coordinates is that the linearized equations become much simpler when expressed in terms of gauge-invariant field combinations. Moreover, this exercise provides a welcome consistency check of these equations by revealing their gauge invariance explicitly. Here the only freedom we have are the linearized time-reparametrizations that preserve the LCCF gauge
\[
t \to t + r_k^2T(t, w, \theta), \quad T^{(n<1)} = 0.
\]
As for the other two residual symmetries (2.7), the light-cone reparametrizations have been fixed to the temporal gauge, i.e. the observer sits at \( w = t \), while the \( w \)-dependent angular reparametrizations have been fixed to the non-rotating observational gauge (see section 2). In the case of the latter, however, this specification is not visible here, since we are using a covariant language to describe the \( S_{t, w} \) submanifolds, i.e. we did not need to specify the functions \( X(\theta), q_{ab}(\theta), \) or \( D_a \) etc. It will therefore also be convenient to build invariant quantities under that symmetry as well, which we can express as follows
\[
\theta^a \to \theta^a + q^{ab} \left[ D_b\hat{\Theta}(w, \theta) + \bar{D_b}\bar{\Theta}(w, \theta) \right]
\]
Implementing (6.52) and (6.53) as a gauge transformation of the metric fluctuations, i.e. as a coordinate transformation in (6.49) holding the background fixed, we obtain\(^\text{10}\)
\[
\psi \to \psi - \hat{T} - 2(T' + \bar{\chi}T),
\]
\(^\text{10}\)Note that for the quantities of spin greater than zero, which therefore enter the original fields through angular derivatives, the transformations are defined only up to an arbitrary function of time only.
\[
\phi \rightarrow \phi + T' + (2\mathcal{X} - \mathcal{H})T, \\
\bar{u} \rightarrow \bar{u} + \bar{\Theta}' - T, \\
\tilde{u} \rightarrow \tilde{u} + \tilde{\Theta}', \\
\chi \rightarrow \chi - T' - \mathcal{X}T, \\
\tilde{\chi} \rightarrow \tilde{\chi} - \tilde{\Theta}, \\
\bar{\chi} \rightarrow \bar{\chi} - \bar{\Theta}. 
\]

(6.55)

(6.56)

(6.57)

(6.58)

(6.59)

(6.60)

Note that, despite the fact that \( T \) starts at \( O(r) \) (see (6.52)), \( T^{(1)} \) affects the zeroth order of the spin-0 fields \( \{ \psi, \phi, \chi \} \) thanks to the derivatives and \( \mathcal{X} \) factors in (6.54), (6.55) and (6.58), meaning that this symmetry allow us to impose one full condition on these three fields. Going back to the discussion at the end of section 2.7, we can choose this condition to simplify a given observable. For instance, for a source with 4-velocity \( n^\mu \), so that the redshift is given by (2.11), setting \( \phi = 0 \) means that \( 1 + z = a \) to all orders in the perturbations. Another possibility is to trivialize the angular diameter distance, which is essentially controlled by the volume density \( \gamma^{1/4} \sim a \kappa \left[ 1 + r_k^2 (\phi + \chi) + \frac{1}{2} D^2 \tilde{X} \right] \) (see [15]).

As for the matter fluctuations, their transformation is obtained by proceeding similarly with the linearization of the energy-momentum tensor (5.15) and we obtain

\[
\delta \rightarrow \delta - \frac{i}{\hat{\rho}} T + 2 (T' + 2\lambda' T), \\
\sigma \rightarrow \sigma - \frac{i}{\hat{\rho}} T + 2\hat{\rho} (T' + 2\lambda' T), \\
v \rightarrow v + v' + 2\lambda' T, \\
\bar{v} \rightarrow \bar{v} + T, \\
\tilde{v} \rightarrow \tilde{v}, \\
\bar{\tilde{v}} \rightarrow \bar{\tilde{v}}, \\
\tilde{\tilde{v}} \rightarrow \tilde{\tilde{v}}, \\
\pi \rightarrow \pi, \\
\bar{\pi} \rightarrow \bar{\pi}, \\
\tilde{\pi} \rightarrow \tilde{\pi}.
\]

(6.61)

(6.62)

(6.63)

(6.64)

(6.65)

(6.66)

(6.67)

(6.68)

(6.69)

(6.70)

where \( \hat{\rho} \) and \( \hat{\rho} \) can be expressed in terms of \( \mathcal{H} \) and \( k \) using the background equations (6.12). We can therefore first form the invariant combinations under angular reparametrizations

\[
\bar{U} := \bar{u} + \bar{\chi}', \\
\tilde{U} := \tilde{u} + \tilde{\chi}', 
\]

(6.71)

and with them the fully invariant combinations

\[
\Psi := \psi - \hat{U} - 2 (\bar{U}' + \chi' \bar{U}), \\
\Phi := \phi + \bar{U}' + (2\mathcal{X} - \mathcal{H}) \bar{U}, \\
\Omega := \chi - \bar{U}' - \mathcal{X} \bar{U}, \\
\delta_U := \delta - \frac{i}{\hat{\rho}} \bar{U} + 2 (\bar{U}' + 2\lambda' \bar{U}), \\
\sigma_U := \sigma - \hat{\rho} \bar{U} + 2\hat{\rho} (\bar{U}' + 2\lambda' \bar{U}), \\
v_U := v + \bar{U}' + 2\lambda' \bar{U},
\]

(6.72)

(6.73)

(6.74)

(6.75)

(6.76)

(6.77)
\[ \bar{v} U := \bar{v} + \bar{U}. \]  

(6.78)

From this we see that working with these variables is tantamount to working in the gauge

\[ \bar{u} = -\bar{\chi}', \]  

(6.79)

since it means \( \bar{U} = 0 \), so that all of the above combinations reduce to the first term of each expression.

This gauge is therefore reminiscent of the longitudinal gauge in the FLRW coordinates, so we will refer to it as the “LCCF-longitudinal gauge”. As in the FLRW case, this gauge leads to the simplest form of the equations, i.e. the one with the minimal amount of time-derivatives and thus devoid of spurious degrees of freedom. Finally, the above transformations are consistent with the ones found in [24], where the authors work with general perturbations, if one further imposes the GLC conditions.

### 6.4 Linearized equations of motion

We can now linearize the equations of motion in conformal form, i.e. (5.30) to (5.41), around the cosmological background. The resulting expressions are given in appendix C. These are rather complicated, but become more transparent and amenable to resolution if we eliminate the “momenta” \( \{\theta, \bar{\alpha}, \bar{\kappa}, \bar{\kappa}, \bar{\bar{\kappa}}\} \) through equations (C.10) to (C.15) and express everything in terms of the gauge-invariant variables defined in the previous subsection. These are the equations we will display here.

However, before we do so, two remarks are in order about the equations of appendix C.

First, as mentioned previously, we observe that with the \( a \) factors involved in the conformal variable definition, along with the \( r_k \) factors in the normalization of the fluctuations, the linearized equations do not have any explicit dependence on \( a \) and \( r_k \). Rather, these quantities enter only through the combinations \( H \) and \( X \), respectively, and also the background angular Laplacian (6.40) for \( r_k \). Second, we see that the operator \( \partial_t \) often enters in the combination

\[ \dot{\phi} := \dot{\phi} + \phi', \]  

(6.80)

since this is the background form of the operator \( \partial_t - \mathcal{L}_N \) in the 3 + 1 equations. In our coordinates \( \{t, w, \theta^a\} \) the operator \( \partial_t \) denotes time derivation at constant light-cone \( w \), so (6.80) is the time derivative at constant spatial radius \( r \equiv w - t \). Indeed, trading the LC parametrization \( \{t, w\} \) for the FLRW one \( \{t, r\} \), the transformation of the partial derivatives

\[ dt \partial_t + dw \partial_w \rightarrow dt \partial_t + dr \partial_r, \]  

(6.81)

leads to

\[ \dot{\phi} \rightarrow \dot{\phi} - \partial_r \phi, \quad \partial_w \phi \rightarrow \partial_r \phi, \]  

(6.82)

so the prime operator is the same, i.e. it is simply re-interpreted as the derivative with respect to \( r \). Another convenient property of (6.80) is

\[ \dot{r}_k \equiv \bar{X} \equiv 0, \]  

(6.83)

since these are functions of \( r \) only. In what follows, we will use both \( \dot{\phi} \) and \( \dot{\phi} \), depending on what is the most convenient. We can now consider the linearized equations, starting with the gravitational evolution equations that are (C.16) to (C.19)

\[ \dot{\Psi}' + (2\bar{X} - \mathcal{H}) \left( \dot{\Psi} - \Psi' \right) - \left( 2\dot{\mathcal{H}} + \mathcal{H}^2 - 4\mathcal{H}\bar{X} + 2\bar{X}^2 - 2k \right) \Psi \]  

\[ + \bar{\Phi} + \dot{\mathcal{H}}\bar{\Phi} - 2\Phi'' - 8\bar{X}\Phi' - 4 (\bar{X}^2 - k) \Phi \]  

(6.84)
\[ -\Omega'' - 5\chi\Omega' - \left(3\chi^2 - 3k - \frac{1}{2}\Delta^{(2)}\right)\Omega = -\frac{1}{2}\left(\sigma_U + \Delta^{(2)}\bar{\pi}\right) - \pi , \]

\[ \Psi' + \chi\Psi - 2(\chi - \mathcal{H})\Psi' + 2(\mathcal{H}\chi + k)\Psi - 2\Phi'' - 6\chi\Phi' + 4k\Phi \]

\[ -\bar{\Omega} - 2\mathcal{H}\bar{\Omega} - \Omega'' - 4\chi\Omega' + \left(4k + \Delta^{(2)}\right)\Omega = -\pi , \]

\[ \bar{\Omega} + \bar{\Omega}' + 2\mathcal{H}\bar{\Omega} - \left(2\chi^2 + 2k + \Delta^{(2)}\right)\bar{\Omega} - \bar{Q}' = -2\bar{s} , \]

\[ \bar{Q} - \Psi - 2\Phi = \bar{\pi} , \]

\[ \bar{Q} + \bar{U} + 2(\mathcal{H} - \chi)\bar{U} = \bar{\pi} , \]

where \(\Delta^{(2)}\) is the background 2-dimensional Laplacian (6.40) and

\[ \bar{Q} := (\hat{\chi})^2 + 2\mathcal{H}\hat{\chi} , \quad \bar{Q} := (\hat{\chi})^2 + 2\mathcal{H}\hat{\chi} . \]

Next, we have the constraint equations (C.22) to (C.25)

\[ 3(\mathcal{H}^2 + k)\delta_U = 6\mathcal{H}\Phi - 2\left[\Phi'' + 6\chi\Phi' + \left(6\chi^2 - 2k + \Delta^{(2)}\right)\Phi\right] + 4\mathcal{H}\bar{\Omega} - 2\left[\Omega'' + 7\chi\Omega' + \left(9\chi^2 - k + \frac{1}{2}\Delta^{(2)}\right)\Omega\right] + 2\mathcal{H}\left[\Psi' + (4\chi - 3\mathcal{H})\Psi + \Delta^{(2)}\hat{\chi}\right] , \]

\[ 2\left(-\mathcal{H} + \mathcal{H}^2 + k\right)\nu_U = 2\left[\hat{\Phi}' + 2\chi\hat{\Phi} - \mathcal{H}(\Phi' + 2\chi\Phi)\right] + 2\left[\hat{\Omega}' + 3\chi\hat{\Omega}\right] - 2\mathcal{H}\Psi' - \left(4\mathcal{H}\chi + 2k + \frac{1}{2}\Delta^{(2)}\right)\Psi + \frac{1}{2}\Delta^{(2)}[\hat{\chi}' + 2\chi\hat{\chi}] , \]

\[ 2\left(-\mathcal{H} + \mathcal{H}^2 + k\right)\nu_U = 2\left[\hat{\Phi}' - \mathcal{H}\Phi\right] + \Omega + \frac{1}{2}\Psi' - (2\mathcal{H} - \chi)\Psi - \frac{1}{2}\hat{\chi}'' - 2\chi\hat{\chi}' - (\chi^2 + k)\hat{\chi} , \]

\[ 2\left(-\mathcal{H} + \mathcal{H}^2 + k\right)\nu = \frac{1}{2}\hat{\Omega}' + 2\chi\hat{\Omega} - \left(\chi^2 + k + \frac{1}{2}\Delta^{(2)}\right)\hat{\Omega} - \frac{1}{2}\hat{\chi}'' - 2\chi\hat{\chi}' - (\chi^2 + k + \frac{1}{2}\Delta^{(2)})\hat{\chi} . \]

and, finally, the energy-momentum conservation equations (C.26) to (C.29)

\[ 3(\mathcal{H}^2 + k)\left(\delta_U + \mathcal{H}\delta_U\right) = 2\left(\mathcal{H} - \mathcal{H}^2 - k\right)\left[\nu'U + \Psi' + 4\chi(\nu_U + \Psi) + \Delta^{(2)}(\nu_U + \hat{\chi}) + 2\Omega\right] - 6\mathcal{H}\left(5\mathcal{H}\delta_U - \Phi\right) - \mathcal{H}\left(3\sigma_U + 2\pi + \Delta^{(2)}\bar{\pi}\right) , \]

\[ -30- \]
\[ 2 \left( -\ddot{\mathcal{H}} + \dot{\mathcal{H}}^2 + k \right) (\dot{v}_U + 2\mathcal{H}v_U) = 2 \left( \ddot{\mathcal{H}} - 2\mathcal{H}' \right) \dot{v}_U - \sigma'_U - 2\mathcal{H}' \sigma_U - \Delta^{(2)} (\hat{s} - \mathcal{H}\hat{\pi}) + 2\lambda \pi \quad (6.96) \]

\[ -2 \left( \ddot{\mathcal{H}} + 2\mathcal{H}' + 2k \right) (\Phi' + 2\mathcal{H}\Phi) + 2 \left( \ddot{\mathcal{H}} - \mathcal{H}'^2 - k \right) (\Psi' + 2\lambda \Psi) , \]

\[ 2 \left( -\ddot{\mathcal{H}} + \dot{\mathcal{H}}^2 + k \right) (\dot{v}_U + 2\mathcal{H}v_U) = 2 \left( \ddot{\mathcal{H}} - 2\mathcal{H}' \right) \dot{v}_U - 4\mathcal{H}' \hat{s} - \left( \lambda^2 + k + \Delta^{(2)} \right) \hat{\pi} - \sigma_U - \pi \]

\[ -2 \left( \ddot{\mathcal{H}} + 2\mathcal{H}' + 2k \right) \Phi + 2 \left( \ddot{\mathcal{H}} - \mathcal{H}'^2 - k \right) \Psi , \quad (6.97) \]

\[ 2 \left( -\ddot{\mathcal{H}} + \dot{\mathcal{H}}^2 + k \right) (\dot{v} + 2\mathcal{H}v) = 2 \left( \ddot{\mathcal{H}} - 2\mathcal{H}' \right) \hat{v} - 4\mathcal{H}' \hat{s} - \left( \lambda^2 + k + \frac{1}{2} \Delta^{(2)} \right) \hat{\pi} . \quad (6.98) \]

Although simpler than the “raw” equations in appendix C, the equations of the previous subsection still need some massaging before we can solve them, or at least interpret them physically. So let us start with the gravitational evolution equations and let us use (6.88) to eliminate \( \dot{Q} \) from (6.86), obtaining

\[ \dot{\Psi} + 2 \left( \mathcal{H} - \lambda \right) \Psi - 2 \left( \Phi + \Omega \right)' - 4\mathcal{H}' \left( \Phi + \Omega \right) = 2\hat{s} - \hat{\pi}' . \quad (6.99) \]

Using this equation to eliminate \( \dot{\Psi} \) in (6.85) and then (6.89) to eliminate \( \dot{Q} \) in (6.87), we obtain the two completely decoupled wave equations

\[ \Box \Omega = -\pi - 2 \left( \hat{s}' + \lambda \hat{s} \right) + \hat{\pi}'' + \lambda \hat{\pi}' , \quad \Box \hat{U} = 2\hat{s} - \hat{\pi}' , \quad (6.100) \]

where

\[ \Box \phi := \dddot{\phi} - 2\mathcal{H}\dot{\phi} + \Delta^{(3)} \phi , \quad \Delta^{(3)} \phi := \phi'' + 2\lambda \phi' + \Delta^{(2)} \phi , \quad (6.101) \]

are the background scalar (conformal) d’Alembertian and Laplacian operators, respectively. Note that \( \Omega \) and \( \hat{U} \) are the only fields on which angular derivatives act in the evolution equations, once \( \dot{Q} \) and \( \dot{\Psi} \) have been eliminated, and that they are both sourced by anisotropic stress components exclusively. These fields therefore describe the \( E \) and \( B \)-mode gravitational wave excitations of GR, i.e. the degrees of freedom of the theory. Note that the first two non-trivial \( r \)-orders in each equation of (6.100) are constraints for \( \Omega^{(n<2)} \) and \( \hat{U}^{(n<3)} \)

\[ D^2 \Omega^{(0)} = -\pi^{(-2)} , \quad (2 + D^2) \Omega^{(1)} = -\pi^{(-1)} - 2\hat{s}^{(0)} , \quad (6.102) \]

and

\[ (2 + D^2) \hat{U}^{(1)} = 2\hat{s}^{(-1)} , \quad (6 + D^2) \hat{U}^{(2)} = 2\hat{s}^{(0)} - \hat{\pi}^{(1)} , \quad (6.103) \]

so the degrees of freedom are in \( \Omega^{(n<2)} \) and \( \hat{U}^{(n<3)} \). Assuming \( \Omega \) and \( \hat{U} \) solved, we are now left with the four fields \( \{ \Psi, \Phi, \chi, \hat{\chi} \} \), which do not carry any degrees of freedom. From the gravitational evolution equations we see that, a priori, we need to provide the initial conditions of seven fields \( \{ \Psi, \Phi, \chi, \hat{\chi}, \dot{\chi}, \dot{\hat{\chi}} \} \). The initial conditions of \( \{ \chi, \hat{\chi} \} \) are pure-gauge, since they can be chosen arbitrarily through the light-cone-dependent angular reparametrizations (6.59) and (6.60). Here this freedom is fixed to the non-rotating observational gauge, which implies a definite initial value for \( \{ \chi, \hat{\chi} \} \). However, this information is not required in order to evolve the equations, because these fields appear only through their time-derivatives \( \{ \dot{\chi}, \dot{\hat{\chi}} \} \), precisely because these are invariant under (6.59) and (6.60). Let us now see how the other five initial conditions \( \{ \Psi, \Phi, \chi, \hat{\chi}, \dot{\chi} \} \) are actually determined. We start with (6.94), which can be written as a Laplacian equation for \( r_k \hat{\chi} \)

\[ \frac{1}{r_k} \left( \Delta^{(3)} + 3k \right) \left[ r_k \left( \dot{\chi} + \hat{U} \right) \right] = \dot{U}' + 4\lambda \dot{U} - 4 \left( -\mathcal{H} + \mathcal{H}'^2 + k \right) \dot{v} , \quad (6.104) \]
thus determining that field in terms of the already solved $\tilde{U}$ and $\tilde{v}$. Next using (6.93) to eliminate $\Phi$ in (6.91), we obtain a Laplacian equation for $r^2_k \Phi$

$$\frac{1}{r^2_k} \left( \Delta^{(3)} - 3\mathcal{H}^2 \right) (r^2_k \Phi) = \frac{1}{4} \mathcal{H} (\Psi' + 10\mathcal{X} \Psi) + \mathcal{H} \left[ \frac{3}{4} \chi'' + 3\mathcal{X} \chi' + \left( \frac{3}{2} \chi^2 + \frac{3}{2} k + \Delta^{(2)} \right) \hat{\chi} \right]$$

$$+ \frac{1}{2} \mathcal{H} \hat{\Omega} - \Omega' - 7\mathcal{X} \Omega' - \left( 9\mathcal{X}^2 - \frac{1}{2} \Delta^{(2)} \right) \Omega$$

$$- \frac{3}{2} (\mathcal{H}^2 + k) \delta_U + 3\mathcal{H} \left( -\mathcal{H} + \mathcal{H}^2 + k \right) \bar{\nu}_U ,$$

which is the analogue of the usual Poisson equation in this context and therefore allows us to express $\Phi$ in terms of $\{\Psi, \hat{\chi}, \Omega\}$ and matter fields. Taking the ring derivative of this expression and using (6.88), (6.99), (6.100), (6.95) and (6.97) to eliminate $\{\hat{\chi}^o, \hat{\Psi}, \hat{\Omega}, \hat{\delta}_U, \hat{\nu}_U\}$, we obtain an equation that allows to similarly eliminate $\{\hat{\chi}, \hat{\Omega}, \bar{\nu}_U\}$, which are determined through (6.92) and (6.93). We have therefore used all of the constraint equations and there are no degrees of freedom left indeed.

### 6.5 Road-map to analytical solutions

The analytical solutions for specific types of matter will be given in future work, but let us conclude this section by providing some general remarks regarding that procedure. In the previous subsection we saw that we can rearrange the equations such that the operators acting on the fields are the familiar d’Alembertian $\square$ and Laplacian $\Delta^{(3)}$ in (6.101), respectively. The central ingredient for solving such equations are the eigenfunctions $f_{\omega,l}(r)$ of the Laplacian in spherical harmonic space

$$f''_{\omega,l}(r) + 2\chi(r) f'_{\omega,l}(r) - \frac{l(l+1)}{r^2(r)} f_{\omega,l}(r) = -\omega^2 f_{\omega,l}(r) .$$

(6.106)

Since this set of functions forms a basis, one can then use them to decompose the spatial dependence of every field

$$\phi(t, w, \theta) \sim \sum_{l=s}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta) \int \omega^2 d\omega \left[ \phi^+_{\omega,lm}(t) f^+_{\omega,l}(w-t) + \phi^-_{\omega,lm}(t) f^-_{\omega,l}(w-t) \right] ,$$

(6.107)

where we distinguish between the eigenfunctions that are regular (+) and diverging (-) at $r = 0$. The precise behavior is obtained by expanding around $r = 0$ and using of (6.106)

$$f^+_{\omega,l}(r) \sim r^l + \mathcal{O}(r^{l+1}) , \quad f^-_{\omega,l}(r) \sim r^{-l-1} + \mathcal{O}(r^{-l}) .$$

(6.108)

For the gravitational fields $\phi^-_{\omega,lm} = 0$, thanks to the regularity conditions in the LCCF gauge, but since the matter fluctuations start at $\mathcal{O}(r^{-2})$, we have to consider $f^-_{\omega,0}$ and $f^-_{\omega,1}$ in that sector. As an example, decomposing $\Omega$ as in (6.107) turns (6.100) into the ordinary differential equation

$$\tilde{\Omega}^{+}_{\omega,lm} + 2\mathcal{H} \tilde{\Omega}^{+}_{\omega,lm} + \omega^2 \Omega^{+}_{\omega,lm} = \text{matter} .$$

(6.109)

The domain of the eigenvalues $\omega$, i.e. the spectrum, and the eigenfunctions $f_{\omega,l}(r)$ depend on the value of the spatial curvature $k$. In the flat case we have $\omega \in \mathbb{R}^*$ and the spherical Bessel functions

$$f^+_{\omega,l}(r) = j_l(\omega r) , \quad f^-_{\omega,l}(r) = y_l(\omega r) ,$$

(6.110)
so that (6.107) becomes the spherical Fourier decomposition. In the open case \( k < 0 \) we also have \( \omega \in \mathbb{R}^+ \), but it is convenient to use the parametrization \( \omega^2 = -k \nu (2 - \nu) \), so that the IR region \( \omega^2 < -k \) is probed by \( \nu \in [0, 1] \), while the UV region \( \omega^2 > -k \) is probed by complex numbers of the form \( \nu = 1 + i \beta \), where \( \beta \in \mathbb{R}^+ \). The advantage is a simpler description of the eigenfunctions

\[
\begin{align*}
 f_{\nu, l}^+(r) &= \frac{2 F_1 \left[ \frac{\nu + 1}{2}, \frac{\nu + l + 1}{2}; \frac{1}{2} + l + 1; \tanh^2 \left( \sqrt{-k} r \right) \right]}{\cosh^\nu \left( \sqrt{-k} r \right)} \tanh^l \left( \sqrt{-k} r \right), \\
 f_{\nu, l}^-(r) &= \frac{2 F_1 \left[ \frac{\nu - 1}{2}, \frac{\nu - l}{2}; \frac{1}{2} - l; \tanh^2 \left( \sqrt{-k} r \right) \right]}{\cosh^\nu \left( \sqrt{-k} r \right)} \tanh^{-l-1} \left( \sqrt{-k} r \right),
\end{align*}
\]

where \( 2 F_1 \) is the hypergeometric function. Finally, in the closed case \( k > 0 \) we have a discrete spectrum \( \omega^2 = kn (2 + n) \), with \( n \in \mathbb{N} \), and

\[
\begin{align*}
 f_{n, l}^+(r) &= \frac{Q_{n+1/2}^{l+1/2} \left( \cos \left( \sqrt{k} r \right) \right)}{\sqrt{\sin \left( \sqrt{k} r \right)}}, \\
 f_{n, l}^-(r) &= \frac{P_{n+1/2}^{l+1/2} \left( \cos \left( \sqrt{k} r \right) \right)}{\sqrt{\sin \left( \sqrt{k} r \right)}},
\end{align*}
\]

where \( P \) and \( Q \) are the Legendre functions of the first and second kind, respectively. One must then also replace the \( \omega \) integral with a discrete sum over \( n \in [l, \infty] \) in (6.107), which is then the hyperspherical harmonics decomposition.

**Acknowledgments**

We are grateful to Giovanni Marozzi, Gabriele Veneziano and the participants of the second practitioner’s workshop on relativistic effects in cosmology (Zurich, 2019) for useful discussions, and especially to Fulvio Scaccabarozzi for collaboration at an early stage of this project. EM and JY are supported by a Consolidator Grant of the European Research Council (ERC-2015-CoG grant 680886), GF is supported by Fundação para a Ciência e a Tecnologia under the program Stimulus with the grant no. CEECIND/04399/2017/CP1387/CT0026 and NG and JY are supported by the Swiss National Science Foundation.

**A Regularity conditions**

Here we derive the non-trivial regularity conditions on the LC metric, i.e. equations (2.25) to (2.29). We employ the same procedure as in [20], but with an expansion at constant \( t \) instead of constant \( w \), and also generalize to arbitrary observer dynamics. We start by simply assuming that space-time is regular in the vicinity of the observer, so that there exists a coordinate system \((T, \vec{X})\) such that [25]

\[
d\delta^2 (T, \vec{X}) = - \left[ 1 + 2 \vec{A}(T) \cdot \vec{X} \right] dT^2 + 2 \left[ \vec{\Omega}(T) \times \vec{X} \right] \cdot dT \, d\vec{X} + d\vec{X}^2 + \mathcal{O}(X^2),
\]

so that \( T \) is the proper time of the observer, situated at \( \vec{X} = 0 \), \( \vec{A} \) is their acceleration 3-vector and \( \vec{\Omega} \) is their spin 3-vector. The higher-order terms \( \mathcal{O}(X^2) \) depend on the curvature tensor at the observer [26] and therefore on the specific space-time under consideration. Setting \( \vec{A} = \vec{\Omega} = 0 \), one recovers the Fermi normal coordinates [27].

We will now relate the coordinate system (A.1) to the LC system and, in doing so, obtain the regularity conditions on the LC metric components. We first expand the coordinate transformation functions around the observer as in (2.19)

\[
T(t, w, \theta) = T^{(0)}(t) + T^{(1)}(t, \theta) r + T^{(2)}(t, \theta) r^2 + \ldots,
\]

(A.2)
\[ \vec{X}(t, w, \theta) = \vec{X}^{(1)}(t, \theta) r + \vec{X}^{(2)}(t, \theta) r^2 + \ldots, \tag{A.3} \]

where we remind that \( r := w - t \). Inserting these two equations in (A.1), we obtain

\[
g_{tt} = -\left[ 1 + 2 \vec{A}(T) \cdot \vec{X} \right] (\partial_t T)^2 + 2 \left[ \vec{\Omega}(T) \times \vec{X} \right] \cdot \partial_t T \partial_t \vec{X} + (\partial_t \vec{X})^2 + \mathcal{O}[(w - t)^2],
\]

\[
g_{tw} = -\left[ 1 + 2 \vec{A}(T) \cdot \vec{X} \right] \partial_t T \partial_w T + \left[ \vec{\Omega}(T) \times \vec{X} \right] \cdot \left( \partial_t T \partial_w \vec{X} + \partial_w T \partial_t \vec{X} \right) + \partial_t \vec{X} \cdot \partial_w \vec{X} + \mathcal{O}[(w - t)^2],
\]

\[
g_{ta} = -\left[ 1 + 2 \vec{A}(T) \cdot \vec{X} \right] \partial_t T \partial_a T + \left[ \vec{\Omega}(T) \times \vec{X} \right] \cdot \left( \partial_t T \partial_a \vec{X} + \partial_a T \partial_t \vec{X} \right) + \partial_t \vec{X} \cdot \partial_a \vec{X} + \mathcal{O}[(w - t)^3],
\]

\[
g_{ww} = -\left[ 1 + 2 \vec{A}(T) \cdot \vec{X} \right] (\partial_w T)^2 + 2 \left[ \vec{\Omega}(T) \times \vec{X} \right] \cdot \partial_w T \partial_w \vec{X} + (\partial_w \vec{X})^2 + \mathcal{O}[(w - t)^2],
\]

\[
g_{wa} = -\left[ 1 + 2 \vec{A}(T) \cdot \vec{X} \right] \partial_w T \partial_a T + \left[ \vec{\Omega}(T) \times \vec{X} \right] \cdot \left( \partial_w T \partial_a \vec{X} + \partial_a T \partial_w \vec{X} \right) + \partial_w \vec{X} \cdot \partial_a \vec{X} + \mathcal{O}[(w - t)^3],
\]

\[
g_{ab} = -\left[ 1 + 2 \vec{A}(T) \cdot \vec{X} \right] \partial_a T \partial_b T + 2 \left[ \vec{\Omega}(T) \times \vec{X} \right] \cdot \partial_a T \partial_b \vec{X} + \partial_a \vec{X} \cdot \partial_b \vec{X} + \mathcal{O}[(w - t)^4].
\]

Next, we insert the expansions (A.2) and (A.3), using (2.19) and only the obvious conditions (2.20) and (2.21). We will consider only the first two non-trivial orders in each series, so that the resulting equations depend exclusively on the terms appearing in (A.1) and are therefore independent of the specific space-time under consideration. The leading order equations are

\[
0 = \left( \dot{T}^{(0)} - T^{(1)} \right)^2 - R^2, \tag{A.5}
\]

\[
0 = \left( \dot{T}^{(0)} - T^{(1)} \right) \partial_a T^{(1)} + R \partial_a R, \tag{A.6}
\]

\[
a^2 = \left( \dot{T}^{(0)} - T^{(1)} \right) T^{(1)} + R^2, \tag{A.7}
\]

\[
a^2 + \gamma_{ab}^{(2)} U_a^{(1)} U_b^{(1)} = -T_{(1)}^2 + R^2, \tag{A.8}
\]

\[
U_a^{(1)} = \frac{1}{2} \partial_a \left[ T_{(1)} - R^2 \right], \tag{A.9}
\]

\[
\gamma_{ab}^{(2)} = -\partial_a T^{(1)} \partial_b T^{(1)} + \partial_a \vec{X}^{(1)} \cdot \partial_b \vec{X}^{(1)}, \tag{A.10}
\]

where

\[
R^2 := \vec{X}^{(1)} \cdot \vec{X}^{(1)}. \tag{A.11}
\]

Combining (A.5) and (A.7) we find

\[
a^2 = \dot{T}^{(0)} \left( \dot{T}^{(0)} - T^{(1)} \right), \tag{A.12}
\]

which therefore implies that

\[
T^{(1)}(t, \theta) = T^{(1)}(t). \tag{A.13}
\]

Inserting this in (A.5), we then find

\[
R = \dot{T}^{(0)} - T^{(1)}, \quad \Rightarrow \quad R(t, \theta) = R(t), \tag{A.14}
\]

meaning that the vectors \( \vec{X}^{(1)}(t, \theta) \) generate a sphere with radius \( R(t) \). We can therefore decompose them as follows

\[
\vec{X}^{(1)}(t, \theta) \equiv R(t) \hat{X}(t, \theta), \quad \hat{X} \cdot \hat{X} = 1, \tag{A.15}
\]

where the \( \hat{X} \) allow us to construct the unit-sphere 2-metric \( g_{ab} \) and volume 2-form \( \bar{q}_{ab} \), defined in (2.23). With this (A.9) becomes \( U_a^{(1)} = 0 \), so that (A.5), (A.7) and (A.8) become

\[
T^{(1)} = 0, \quad a = R = \dot{T}^{(0)}, \tag{A.16}
\]
and, finally, (A.10) is $\gamma^{(2)}_{ab} = a^2q_{ab}$, i.e. we have obtained (2.25).

Now that all of the leading order equations are solved, we can consider the next-to-leading order. The corresponding expressions will contain $\dot{\hat{X}}$, so we must first express this quantity in a convenient way. We decompose $\dot{\hat{X}}$ in the basis \{\hat{X}, $\partial_a\hat{X}$\}, but since $\hat{X} \cdot \dot{\hat{X}} = 1$, we have $\hat{X} \cdot \dot{\hat{X}} = 0$ and thus

$$\dot{\hat{X}} \equiv aW^a\partial_a\hat{X},$$  \hspace{1cm} (A.17)

where $W^a(t, \theta)$ is an angular velocity vector field on the unit-sphere. It can be recovered through

$$W_a \equiv a^{-1}\dot{\hat{X}} \cdot \partial_a\hat{X},$$  \hspace{1cm} (A.18)

where we use $q_{ab}$ to displace angular indices. Defining the corresponding covariant derivative $D_a$, satisfying in particular

$$D_aD_b\hat{X} \equiv -q_{ab}\hat{X},$$  \hspace{1cm} (A.19)

and taking the divergence of (A.18), we find

$$D_aW^a \sim D_a\hat{X} \cdot D^a\hat{X} - 2\hat{X} \cdot \dot{\hat{X}} \equiv 0, \hspace{1cm} (A.20)$$

meaning that $W^a$ is a pure curl

$$W^a \equiv \tilde{q}^{ab}\partial_bW.$$  \hspace{1cm} (A.21)

We can now consider the next-to-leading order equations. By decomposing $\hat{X}^{(2)}$ in the basis \{\hat{X}, $\partial_a\hat{X}$\}

$$\hat{X}^{(2)} \equiv R^{(2)}\hat{X} + X^a_{(2)}\partial_a\hat{X},$$  \hspace{1cm} (A.22)

using the identities

$$\hat{X} \times \partial_a\hat{X} \equiv \tilde{q}^b_a\partial_b\hat{X}, \hspace{1cm} \partial_a\hat{X} \times \partial_b\hat{X} \equiv \tilde{q}_{ab}\hat{X},$$  \hspace{1cm} (A.23)

and taking into account the leading-order equations, we obtain

$$0 = \dot{\tilde{a}} + a^2\alpha - 2\left(R^{(2)} + T^{(2)}\right),$$  \hspace{1cm} (A.24)

$$0 = -a^2\tilde{q}^{ab}\partial_b(W + \omega) + \partial_a \left(R^{(2)} + T^{(2)}\right) + X^a_{(2)},$$  \hspace{1cm} (A.25)

$$\gamma^{(1)} = -\dot{\alpha} + 2T^{(2)} + 4R^{(2)},$$  \hspace{1cm} (A.26)

$$\gamma^{(1)} = 2R^{(2)},$$  \hspace{1cm} (A.27)

$$U^a_{(2)} = -a \left(\partial_aR^{(2)} + X^a_{(2)}\right),$$  \hspace{1cm} (A.28)

$$\tilde{\gamma}^{(3)}_{ab} = 2a \left[q_{ab}R^{(2)} + D_{(a}X^b_{(2)}\right],$$  \hspace{1cm} (A.29)

where

$$\alpha := \tilde{\alpha} \cdot \hat{X}, \hspace{1cm} \omega = \tilde{\omega} \cdot \hat{X},$$  \hspace{1cm} (A.30)

and

$$\tilde{\alpha}(t) := \tilde{A}(T^{(0)}(t)), \hspace{1cm} \tilde{\omega}(t) := \tilde{\Omega}(T^{(0)}(t)).$$  \hspace{1cm} (A.31)

We start by using (A.24) to eliminate $R^{(2)}$ and decompose $V_a$ and $X^a_{(2)}$ harmonically on the 2-sphere (see section 6.2 for details)

$$X^a_{(2)} \equiv \partial_aX^{(2)} + \tilde{q}^b_a\partial_b\hat{X}^{(2)}, \hspace{1cm} X^a_{(2)} \equiv \partial_aX^{(2)} + \tilde{q}^b_a\partial_b\hat{X}^{(2)},$$  \hspace{1cm} (A.32)
so that equations (A.25) to (A.29) lead to

\[ X^{(2)} = -\frac{1}{2} a^2 \alpha, \quad (A.33) \]
\[ \tilde{X}^{(2)} = a^2 (W + \omega), \quad (A.34) \]
\[ N^{(1)} = a^2 \alpha, \quad (A.35) \]
\[ \Upsilon^{(1)} = \dot{\alpha} + a^2 \alpha - 2 T^{(2)}, \quad (A.36) \]
\[ U_a^{(2)} = a \partial_a T^{(2)} - a^3 \tilde{q}_a^b \partial_b (W + \omega), \quad (A.37) \]
\[ \gamma_{ab}^{(3)} = a \left[ \dot{\alpha} + 2 a^2 \alpha - 2 T^{(2)} \right] q_{ab} + 2 a^3 \tilde{q}_c^a D_b \partial_c W. \quad (A.38) \]

This leads to equations (2.26) to (2.29) if we trade \( T^{(2)} \) for \( Z := a^{-2} \left[ \frac{1}{2} \dot{\alpha} - T^{(2)} \right] \).

To conclude this appendix, let us compare our derivation with the procedure employed in [20], where the authors focus on the GLC case \( N = 1 \) and also use the temporal gauge (2.16). There one considers an expansion of the form (2.19), but with coefficients that are functions of \((w, \theta)\) instead of \((t, \theta)\), i.e. it is an expansion in \( w - t \) at constant \( w \), instead of constant \( t \). This is fine, since \( w = t \) at the observer, but leads to a different expression of the regularity conditions than the ones obtained here. The reasons for choosing our approach instead are the following. First, being an expansion at constant space-like distance to the observer, instead of light-like as in [20], it is the same kind of expansion as in the generalized Fermi coordinates (A.1), thus leading to a more intuitive understanding of the mapping between the two systems. Second, and most importantly, our approach is more suited to cosmology, where \( \Upsilon \) is the scale factor \( a(t) \) up to fluctuations. Given the temporal gauge (2.16), or (2.21), in our approach we have the simple result \( \Upsilon_o(t) = N_o(t) \sim a(t) \), while in the approach of [20] the condition \( N = 1 \) implies \( \Upsilon_o(w) = 1 \), meaning that the \( a(t) \) information is disseminated in the expansion in \( t \).

B Perfect fluid matter 3 + 1 decomposition

Here we consider the case where the matter sector is made of a single perfect fluid. We can therefore express the energy-momentum tensor as

\[ T_{\mu \nu} = \rho V_\mu V_\nu + p (g_{\mu \nu} + V_\mu V_\nu), \quad V_\mu V^\mu = -1, \quad (B.1) \]

where \( V^\mu \) is the 4-velocity, while \( \rho \) and \( p \) are the rest-frame energy density and pressure, i.e. as measured by an observer family with 4-velocity \( V^\mu \), as opposed to \( E \) and \( S \) which are measured by the canonical observers with 4-velocity \( n^\mu \). In order to relate the rest-frame and canonical quantities, it is useful to define

\[ v^i := N^{-1} \left( \frac{V^i}{V^0} - \frac{n^i}{n^0} \right) \equiv N^{-1} \left( \frac{V^i}{V^0} + N^i \right), \quad W := -n_\mu V^\mu \equiv \left[ 1 - h_{ij} v^i v^j \right]^{-1/2}, \quad (B.2) \]

that are the 3-velocity of matter measured by the canonical observer \( n^\mu \) and the generalization of the Lorentz factor, respectively,

\[ h^{\mu \nu} V_\mu \equiv W v^i. \quad (B.3) \]
Plugging (B.1) inside (4.1) we then find

\[ E = W^2 (\rho + p) - p, \quad (B.4) \]
\[ P^i = W^2 (\rho + p) v^i, \quad (B.5) \]
\[ S^{ij} = W^2 (\rho + p) v^i v^j + ph^{ij}, \quad (B.6) \]

so that, in particular,

\[ S^{ij} = ph^{ij} + \frac{P^i P^j}{E + p}, \quad (B.7) \]

We can also invert the energy and velocity relations to get

\[ \rho = E - \frac{P^i P^i}{E + p}, \quad v^i = \frac{P^i}{E + p}, \quad (B.8) \]

Eqs. (4.12) and (4.13) are four equations for five independent variables \( E, P^i \) and \( p \), thanks to (B.7), so the equation of state that relates the rest-frame scalars \( p \equiv p(\rho(E, P)) \) closes the system.

As explained in detail in [28], in the presence of a single perfect fluid one does not actually need to solve the evolution equations (4.12) and (4.13), because \( E \) and \( P^i \) are fully determined by the gravitational fields through the constraint equations (4.4) and (4.6). With the latter and (B.7), we can then completely eliminate the perfect fluid variables in the evolution equations of gravity (4.9) and (4.10), which therefore become a closed system for \( h_{ij} \) and \( K_{ij} \) that can be evolved independently. The constraint equations then simply become a definition of the energy and momentum density that can be inferred from the gravitational field configurations, and these quantities automatically satisfy energy-momentum conservation (4.12) and (4.13). Note that this manipulation seems to be possible only in the context of perturbative cosmology where \( E + p \) has a non-zero “background” value, because it appears in the denominator in (B.7). However, as shown in [28], if the perfect fluid obeys the weak energy condition \( p \geq -\rho \), then

\[ S^{ij} = \begin{cases} ph^{ij} + \frac{P^i P^j}{E + p} & \text{if } E + p > 0 \\ 0 & \text{if } E + p = 0 \end{cases}, \quad (B.9) \]

so no problem arises in voids. In the presence of more than one fluid, or other types of matter content, this elimination still works, but only for one of the perfect fluids and the rest of matter must be given evolution equations to close the system.

Finally, note that one can also distinguish between the “mass” and “internal” (“temperature”) contributions to the energy density

\[ \rho = \rho_M + \rho_T, \quad (B.10) \]

which allows for a more general equation of state \( p \equiv p(\rho_M, \rho_T) \). This is especially interesting for the description of baryons, which have non-trivial thermodynamics \( p = p(\rho_T) \), as opposed to dark matter which can simply be treated with \( p = 0 \) at large enough scales. Having traded one field \( \rho \) for two new independent ones \( \rho_M \) and \( \rho_T \), we need an additional evolution equation to close the system and the natural choice is the conservation of mass, i.e.

\[ \nabla_\mu J^\mu = 0, \quad (B.11) \]

where

\[ J^\mu = \rho_M V^\mu, \quad (B.12) \]
is the mass 4-current. Just like $T^\mu\nu$, one can also decompose that current in the $n$-frame, to get the canonical mass density
\[ M := -n_\mu J^\mu \equiv W \rho_M , \] (B.13)
and 3-current
\[ h^i_\mu J^\mu = M v^i . \] (B.14)
The mass conservation equation then gives us the closing equation for the $E, P^i, M$ system
\[ (\partial_t - \mathcal{L}_\beta) M = -D_i \left( NM \frac{P^i}{E + p} \right) + NKM . \] (B.15)

Equations (4.12), (4.13) and (B.15), along with $p \equiv p(\rho_M, \rho_T)$, fully determine the perfect fluid dynamics. Having introduced one extra degree of freedom, however, these data can no longer be completely determined through the constraint equations (4.4) and (4.6) alone.

## C Raw linearized equations of motion

Here we will use the definition (6.71). A set of useful expressions to have at hand is
\[ \gamma^{ab} = r_k^{-2} \left[ q^{ab} \left( 1 - (2r_k^2 \chi + D^2 \hat{\chi}) \right) - 2 \left( \vec{D}_a \hat{\chi} + \vec{D}_b \hat{\chi} \right) \right] , \] (C.1)
\[ \bar{\Gamma}^{ab}_{\rho q} = \Gamma^{ab}_{\rho q} - 2 \bar{\Delta}^{ab}_{\rho q} \left( r_k^2 \hat{\chi} + \frac{1}{2} D^2 \hat{\chi} \right) \] (C.2)
\[ \bar{\mathcal{R}} = 2r_k^{-2} \left[ 1 - (2 + D^2) r_k^2 \hat{\chi} \right] , \] (C.3)
\[ \bar{C}_{ab} = r_k^2 \left[ q_{ab} \left( \chi' + \frac{1}{2} D^2 (\check{U} + 2 \chi \hat{\chi}) \right) + \bar{D}_a (\hat{U} + 2 \chi \hat{\chi}) \right] - \bar{D}_b (\check{U} + 2 \chi \hat{\chi}) \] (C.4)
\[ \bar{C}^{b}_{a} = \bar{\delta}^{b}_{a} \left[ \chi' + \frac{1}{2} D^2 \check{U} \right] + \bar{D}^{b}_{a} \hat{U} + \bar{D}^{a}_{b} \check{U} , \] (C.5)
\[ \check{\mathcal{K}} = 2r_k^2 \bar{\kappa} + D^2 \bar{\kappa} , \] (C.6)
\[ \check{S} = 2r_k^2 \pi + D^2 \bar{\pi} , \] (C.7)
and we have used the fact that, on a scalar $\phi$
\[ D^b D_{ab} \phi \equiv \frac{1}{2} D_a \left[ 2 + D^2 \right] \phi , \quad D^b D_{ab} \phi \equiv \frac{1}{2} D_a \left[ 2 + D^2 \right] \phi . \] (C.9)

With the above equations, the background equations (6.12) and the background angular Laplacian definition (6.40), equations (5.30) to (5.41) lead to the linear evolution equations
\[ \dot{\phi} + \phi' = \theta - \psi' - 2 \chi \psi + \mathcal{H} \psi , \] (C.10)
\[ \dot{\bar{u}} = \bar{\alpha} + \psi , \] (C.11)
\[ \dot{\check{u}} = \bar{\alpha} , \] (C.12)
\[ \dot{\chi} + \chi' = \kappa + \psi' + \mathcal{X} \psi , \] (C.13)
\[ \dot{\hat{\chi}} = \kappa - \check{U} , \] (C.14)
\[ \dot{\chi} = \kappa - \bar{U}, \quad (C.15) \]

\[
\dot{\theta} + \theta' + H\theta = \psi'' + (4\chi - H) \psi' + \left( H - 2H\chi + 2\lambda^2 - 2k \right) \psi + 2\phi'' + 8\lambda\phi' + 4(\lambda^2 - k) \phi
\]

\[
+ \chi'' + 5\lambda\chi' + \left( 3\lambda^2 - 3k - \frac{1}{2} \Delta^{(2)} \right) \chi + \frac{1}{2} \Delta^{(2)} (\bar{U}' + \chi \bar{U}) - \frac{1}{2} \left( \sigma + \Delta^{(2)} \bar{\pi} \right) - \pi, \quad (C.16) \]

\[
\dot{\alpha} + \alpha' + 2H\alpha = 2\left( \lambda^2 + k \right) \bar{U} - 2(\psi + 2\phi + \chi)' - 2\lambda(\psi + 2\phi + 2\chi) - 2\bar{s}, \quad (C.17) \]

\[
\dot{\alpha} + \alpha' + 2H\alpha = \left( 2\lambda^2 + 2k + \Delta^{(2)} \right) \bar{U} - 2\bar{s}, \quad (C.18) \]

\[
\dot{k} + \kappa' + 2H\kappa = -\psi'' - 3\chi \psi' + 2k\psi - 2\phi'' - 6\lambda\phi' + 4k\phi
\]

\[
- \chi'' - 4\lambda\chi' + \left( 4k + \Delta^{(2)} \right) \chi - \Delta^{(2)} (\bar{U}' + \chi \bar{U}) + \pi, \quad (C.19) \]

\[
\dot{\kappa} + \bar{k}' + 2H\bar{k} = \bar{U}' + 2\lambda \bar{U} + \psi + 2\phi + \bar{\pi}, \quad (C.20) \]

\[
\dot{\kappa} + \bar{k}' + 2H\bar{k} = \bar{U}' + 2\lambda \bar{U} + \bar{\pi}, \quad (C.21) \]

the linear constraint equations

\[
3(\lambda^2 + k) \delta = 6H\theta + 2H \left( 2\kappa + \Delta^{(2)} \bar{k} \right) - 2 \left[ \phi'' + 6\lambda\phi' + \left( 6\lambda^2 - 2k + \Delta^{(2)} \right) \phi \right]
\]

\[
- 2 \left[ \chi'' + 7\lambda\chi' + \left( 9\lambda^2 - k + \frac{1}{2} \Delta^{(2)} \right) \chi \right] - \Delta^{(2)} \left[ \bar{U}' + 3\lambda \bar{U} \right], \quad (C.22) \]

\[
2 \left( -H + \lambda^2 + k \right) \psi = 2(\theta' + 2H\theta) + 2(\kappa' + 3\lambda\kappa) + \Delta^{(2)}(\bar{k}' + \chi \bar{k}) - 2H(\phi' + 2\lambda\phi) + \frac{1}{2} \Delta^{(2)} \bar{\alpha}, \quad (C.23) \]

\[
2 \left( -H + \lambda^2 + k \right) \bar{\psi} = 2\theta + \kappa - 2H\phi + \frac{1}{2} (\bar{\alpha}' + 4\lambda \bar{\alpha}) - (\lambda^2 + k) \bar{\kappa}, \quad (C.24) \]

\[
2 \left( -H + \lambda^2 + k \right) \bar{\psi} = \frac{1}{2} (\bar{\alpha}' + 4\lambda \bar{\alpha}) - \left( \lambda^2 + k + \frac{1}{2} \Delta^{(2)} \right) \bar{\kappa}, \quad (C.25) \]

and the linear energy-momentum conservation equations

\[
3(\lambda^2 + k) \left( \dot{\delta} + \delta' + H\delta \right) = 2 \left( H - \lambda^2 - k \right) \left[ \psi' + 4\lambda\psi + 2\kappa + \Delta^{(2)} (\bar{\psi} + \bar{\kappa}) \right]
\]

\[
- 6H(\delta - \theta - H\psi) - H \left( 3\sigma + 2\pi + \Delta^{(2)} \bar{\pi} \right)
\]

\[
- 6 \left( \lambda^2 + k \right) \left( \psi' + 2\lambda\psi \right), \quad (C.26) \]

\[
2 \left( -H + \lambda^2 + k \right) (\dot{\psi} + \psi' + 2H\psi) = 2 \left( H - 2H\bar{\psi} \right) \psi - \sigma' - 2\lambda \sigma - \Delta^{(2)} \bar{s}
\]

\[
- 2 \left( H + 2H^2 + 2k \right) (\phi' + 2\lambda\phi)
\]

\[
+ 2 \left( H - \lambda^2 - k \right) (\psi' + 2\lambda\psi) + \chi \left( 2\pi + \Delta^{(2)} \bar{\pi} \right), \quad (C.27) \]

\[
2 \left( -H + \lambda^2 + k \right) (\dot{\psi} + \psi' + 2H\bar{\psi}) = 2 \left( H - 2H\bar{\psi} \right) \psi - \bar{s}' - 4\lambda \bar{s} - \left( \lambda^2 + k + \frac{1}{2} \Delta^{(2)} \right) \bar{\pi}_s - \pi - \sigma
\]

\[
+ 2 \left( H - \lambda^2 - k \right) \psi - 2 \left( H + 2H^2 + 2k \right) \phi, \quad (C.28) \]

\[
2 \left( -H + \lambda^2 + k \right) (\dot{\psi} + \psi' + 2H\bar{\psi}) = 2 \left( H - 2H\bar{\psi} \right) \psi - \bar{s}' - 4\lambda \bar{s} - \left( \lambda^2 + k + \frac{1}{2} \Delta^{(2)} \right) \bar{\pi}. \quad (C.29) \]

Note that for (C.16) we have used the Hamiltonian constraint (5.36) to eliminate \( E \) from (5.33).
References

[1] E. Mitsou and J. Yoo, *Tetrad formalism for exact cosmological observables*. 2019, 10.1007/978-3-030-50039-9, [1908.10757].

[2] M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, *Light-cone averaging in cosmology: Formalism and applications*, *JCAP* **07** (2011) 008 [1104.1167].

[3] I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, *Backreaction on the luminosity-redshift relation from gauge invariant light-cone averaging*, *JCAP* **04** (2012) 036 [1202.1247].

[4] I. Ben-Dayan, G. Marozzi, F. Nugier and G. Veneziano, *The second-order luminosity-redshift relation in a generic inhomogeneous cosmology*, *JCAP* **11** (2012) 045 [1209.4326].

[5] G. Fanizza, M. Gasperini, G. Marozzi and G. Veneziano, *An exact Jacobi map in the geodesic light-cone gauge*, *JCAP* **11** (2013) 019 [1308.4935].

[6] E. Di Dio, R. Durrer, G. Marozzi and F. Montanari, *Galaxy number counts to second order and their bispectrum*, *JCAP* **12** (2014) 017 [1407.0376].

[7] I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier and G. Veneziano, *Average and dispersion of the luminosity-redshift relation in the concordance model*, *JCAP* **06** (2013) 002 [1302.0740].

[8] G. Fanizza and F. Nugier, *Lensing in the geodesic light-cone coordinates and its (exact) illustration to an off-center observer in Lemaître-Tolman-Bondi models*, *JCAP* **02** (2015) 002 [1408.1604].

[9] G. Marozzi, *The luminosity distance-redshift relation up to second order in the Poisson gauge with anisotropic stress*, *Class. Quant. Grav.* **32** (2015) 045004 [1406.1135].

[10] E. Di Dio, R. Durrer, G. Marozzi and F. Montanari, *The bispectrum of relativistic galaxy number counts*, *JCAP* **01** (2016) 016 [1510.04202].

[11] G. Fanizza, M. Gasperini, G. Marozzi and G. Veneziano, *Time of flight of ultra-relativistic particles in a realistic Universe: a viable tool for fundamental physics?*, *Phys. Lett. B* **757** (2016) 505 [1512.08489].

[12] G. Fanizza, M. Gasperini, G. Marozzi and G. Veneziano, *A new approach to the propagation of light-like signals in perturbed cosmological backgrounds*, *JCAP* **08** (2015) 020 [1506.02003].

[13] G. Marozzi, G. Fanizza, E. Di Dio and R. Durrer, *CMB-lensing beyond the Born approximation*, *JCAP* **09** (2016) 028 [1605.08761].

[14] G. Marozzi, G. Fanizza, E. Di Dio and R. Durrer, *CMB-lensing beyond the leading order: temperature and polarization anisotropies*, *Phys. Rev. D* **98** (2018) 023535 [1612.07263].

[15] P. Fleury, F. Nugier and G. Fanizza, *Geodesic-light-cone coordinates and the Bianchi I spacetime*, *JCAP* **06** (2016) 008 [1602.04461].

[16] G. Fanizza, J. Yoo and S. G. Biern, *Non-linear general relativistic effects in the observed redshift*, *JCAP* **09** (2018) 037 [1805.05959].

[17] R. L. Arnowitt, S. Deser and C. W. Misner, *The Dynamics of general relativity*, *Gen. Rel. Grav.* **40** (2008) 1997 [gr-qc/0405109].

[18] G. F. R. Ellis, S. D. Nel, R. Maartens, W. R. Stoeger and A. P. Whitman, *Ideal observational cosmology*, *Phys. Rep.* **124** (1985) 315.

[19] F. Nugier, *Lightcone Averaging and Precision Cosmology*, Ph.D. thesis, UPMC, Paris (main), 2013. 1309.6542.

[20] G. Fanizza, M. Gasperini, G. Marozzi and G. Veneziano, *Observation angles, Fermi coordinates, and the Geodesic-Light-Cone gauge*, *JCAP* **01** (2019) 004 [1812.03671].
[21] E. Mitsou, F. Scaccabarozzi and G. Fanizza, *Observed Angles and Geodesic Light-Cone Coordinates*, *Class. Quant. Grav.* **35** (2018) 107002 [1712.05675].

[22] L. Dai, E. Pajer and F. Schmidt, *Conformal Fermi Coordinates*, *JCAP* **11** (2015) 043 [1502.02011].

[23] J. York, James W., *Kinematics and Dynamics of General Relativity*, in *Workshop on Sources of Gravitational Radiation*, pp. 83–126, 1978.

[24] G. Fanizza, G. Marozzi, M. Medeiros and G. Schiaffino, *The Cosmological Perturbation Theory on the Geodesic Light-Cone background*, 2009.14134.

[25] C. W. Misner, K. Thorne and J. Wheeler, *Gravitation*. W. H. Freeman, San Francisco, 1973.

[26] W.-T. Ni and M. Zimmermann, *Inertial and gravitational effects in the proper reference frame of an accelerated, rotating observer*, *Phys. Rev. D* **17** (1978) 1473.

[27] F. Manasse and C. Misner, *Fermi Normal Coordinates and Some Basic Concepts in Differential Geometry*, *J. Math. Phys.* **4** (1963) 735.

[28] D. Daverio, Y. Dirian and E. Mitsou, *A numerical relativity scheme for cosmological simulations*, *Class. Quant. Grav.* **34** (2017) 237001 [1611.03437].