UNITARY EXTENSION PRINCIPLE FOR NONUNIFORM WAVELET FRAMES IN $L^2(\mathbb{R})$

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Abstract. We study the construction of nonuniform tight wavelet frames for the Lebesgue space $L^2(\mathbb{R})$, where the related translation set is not necessary a group. The main purpose of this paper is to prove the unitary extension principle (UEP) and the oblique extension principle (OEP) for construction of multi-generated nonuniform tight wavelet frames for $L^2(\mathbb{R})$. Some examples are also given to illustrate the results.

1. Introduction

Wavelets have been extensively studied over the last few years and its role in both pure and applied mathematics is well known. It is not possible to give complete list of applications of wavelets, let us at least mention some [1, 2, 7, 8, 15, 16, 18, 23], also see many references therein. Wavelets in $L^2(\mathbb{R})$ are very efficient tools as it gives orthonormal basis for $L^2(\mathbb{R})$ in form of dilation and translation of finite numbers of function in $L^2(\mathbb{R})$ which is very simple and convenient form of basis for $L^2(\mathbb{R})$. Gabardo and Nashed [13] considered a generalization of Mallat’s classic multiresolution analysis (MRA), which is based on the theory of spectral pairs.

Definition 1.1. [13, Definition 3.1] Let $N \geq 1$ be a positive integer and $r$ be an odd integer relatively prime to $N$ such that $1 \leq r \leq 2N-1$, an associated nonuniform multiresolution analysis (abbreviated NUMRA) is a collection $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following properties:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,
(ii) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$,
(iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$,
(iv) $f(x) \in V_j$ if and only if $f(2Nx) \in V_{j+1}$,
(v) There exists a function $\phi \in V_0$, called the scaling function, such that the collection $\{\phi(x-\lambda)\}_{\lambda \in \Lambda}$, where $\Lambda = \{0, \frac{r}{N}\} + 2\mathbb{Z}$, is a complete orthonormal system for $V_0$.

Here, the translate set $\Lambda = \{0, \frac{r}{N}\} + 2\mathbb{Z}$ may not be a group. One may observe that the standard definition of a one-dimensional multiresolution analysis with dilation factor equal to 2 is a special case of NUMRA given in Definition 1.1. Related to the one-dimensional spectral pairs, Gabardo and Yu [14] considered sets of nonuniform wavelets in $L^2(\mathbb{R})$. For fundamental properties of nonuniform wavelets based on the spectral pair, we refer to [13, 14, 20].
Ron and Shen [17] introduced the unitary extension principle which gives the construction of a multi-generated tight wavelet frame for $L^2(\mathbb{R}^d)$, based on a given refinable function. Tight wavelet frames give a convenient way to represent a function in $L^2(\mathbb{R})$ in comparison of non-tight wavelet frames as in that case frame operator is constant multiple of identity operator in $L^2(\mathbb{R})$. Christensen and Goh in [6] generalized the unitary extension principle to locally compact abelian groups. They gave general constructions, based on B-splines on the group itself as well as on characteristic functions on the dual group. Motivated by the work of Gabardo and Nashed [13] for the construction of non-uniform wavelets, and application of frames in applied and pure mathematics, we study nonuniform wavelet frames for the Lebesgue space $L^2(\mathbb{R})$. Notable contribution in the paper is to introduce the unitary extension principle for the construction of multi-generated tight nonuniform wavelet frames of the form

$$\{\psi_{j,\lambda,\ell}\}_{j \in \mathbb{Z}, \lambda \in \Lambda} = \{(2N)^j \psi_n(2N)^j \gamma - \lambda\}_{j \in \mathbb{Z}, \lambda \in \Lambda}$$

in $L^2(\mathbb{R})$.

1.1. Overview and main results. The paper is organized as follows. In Section 2, we give basic notions, definitions and properties of operators related with nonuniform wavelet frames in $L^2(\mathbb{R})$. The general setup for nonuniform wavelet frame system in $L^2(\mathbb{R})$ is also given in Section 2. Section 4 gives some auxiliary results needed in the rest of the paper. The main results are given in Section 5. Theorem 5.1 gives the unitary extension principle (UEP) for the construction of multi-generated tight nonuniform wavelet frames for $L^2(\mathbb{R})$. The extended version of UEP (or oblique extension principle) for nonuniform wavelet frames for $L^2(\mathbb{R})$ can be found in Theorem 5.2. Some examples are given in Section 6 to illustrate our results.

1.2. Relation to existing work and motivation. Duffin and Schaeffer [12] introduced the concept of a frame for separable Hilbert spaces, while addressing some difficult problems from the theory of nonharmonic analysis. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. The norm induced by the inner product $\langle \cdot, \cdot \rangle$ is given by $\|f\| = \sqrt{\langle f, f \rangle}$, $f \in \mathcal{H}$. A family $\{f_k\}_{k=1}^\infty \subset \mathcal{H}$ is called a frame for $\mathcal{H}$, if there exist positive scalars $A_o \leq B_o < \infty$ such that for all $f \in \mathcal{H}$,

$$A_o \|f\|^2 \leq \sum_{k=1}^\infty |\langle f, f_k \rangle|^2 \leq B_o \|f\|^2. \quad (1.1)$$

The scalars $A_o$ and $B_o$ are called lower frame bound and upper frame bound, respectively. If it is possible to choose $A_o = B_o$, then we say that $\{f_k\}_{k=1}^\infty$ is a $A_o$-Parseval frame (or $A_o$-tight frame); and Parseval frame if $A_o = B_o = 1$. If only upper inequality in (1.1) holds, then we say that $\{f_k\}_{k=1}^\infty$ is a Bessel sequence sequence with Bessel bound $B_o$. If $\{f_k\}_{k=1}^\infty$ is a frame for $\mathcal{H}$, then $S : \mathcal{H} \to \mathcal{H}$ given by $Sf = \sum_{k=1}^\infty \langle f, f_k \rangle f_k$ is a bounded, linear and invertible on $\mathcal{H}$, and is called the frame operator. This gives the reconstruction formula of each vector $f \in \mathcal{H}$,

$$f = SS^{-1}f = \sum_{k=1}^\infty \langle S^{-1}f, f_k \rangle f_k.$$

Thus, each vector has an explicit series expansion which need not be unique. For application of frames in both pure and applied mathematics, we refer to book of Casazza and Kutyniok [3], Christensen [5] and Han [15]. Nowadays the theory of iterated function systems, quantum mechanics
and wavelets is emerging in important applications in frame theory, see [11, 21, 22] and many references therein. Very recent work on discrete frames of translates and discrete wavelet frames, and their duals in finite dimensional spaces can be found in [9, 10]. Wavelet frames in $L^2(\mathbb{R})$ are also very powerful tool for representing functions in $L^2(\mathbb{R})$ as sum of series of functions which are dilation and translation of finite number of functions in $L^2(\mathbb{R})$. It provides us convenient tool to expansion of functions in $L^2(\mathbb{R})$ of similar type as one that arise in orthonormal basis, however, wavelet frame conditions are weaker that makes wavelet frame more flexible. Nonuniform wavelet frames could be used in signal processing, sampling theory, speech recognition and various other areas, where instead of integer shifts nonuniform shifts are needed. In [19], Sharma and Manchanda gave necessary and sufficient conditions for nonuniform wavelet frames in $L^2(\mathbb{R})$.

Motivated by the work of Gabardo and Nashed [13] and Gabardo and Yu [14] in the study of nonuniform wavelets, we study frame properties of nonuniform wavelets in the Lebesgue space $L^2(\mathbb{R})$. We recall that the extension problems in frame theory has a long history. It is showed in [4] that the extension problem has a solution in the sense that “any Bessel sequence can be extended to a tight frame by adjoining a suitable family of vectors in the underlying space.” Ron and Shen introduced unitary extension principle for construction of tight wavelet frames in the Lebesgue space $L^2(\mathbb{R}^d)$. The unitary extension principle allows construction of tight wavelet frames with compact support, desired smoothness; and good approximation of functions. In real life application all signals are not obtained from uniform shifts; so there is a natural question regarding analysis and decompositions of this types of signals by a stable mathematical tool. Gabardo and Nashed [13] and Gabardo and Yu [14] filled this gap by the concept of nonuniform multiresolution analysis. In the direction of construction of Parseval frames from nonuniform multiwavelets systems, we develop a general setup and prove the unitary extension principle for construction of multi-generated nonuniform tight wavelet frames for $L^2(\mathbb{R})$. Ron and Shen [17] gave the unitary extension principle, where conditions for the construction of multi-generated tight wavelet frames for the Lebesgue space $L^2(\mathbb{R}^d)$ are based on a given refinable function.

2. Preliminaries

As is standard, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all integers, positive integers and real numbers, respectively. Throughout the paper, $N \in \mathbb{N}$, $r$ be an odd integer relative prime to $N$ such that $1 \leq r \leq 2N - 1$ and $\Lambda = \{0, \frac{r}{N}\} + 2\mathbb{Z}$. Notice that the discrete set $\Lambda$ is not always a group. The support of a function $\psi$ is denoted by $\text{Supp} \, \psi$, and defined as $\text{Supp} \, \psi = \text{clo} \left( \{x : \psi(x) \neq 0\} \right)$. Symbol $\overline{z}$ denote the complex conjugate of a complex number $z$. The conjugate transpose of a matrix $H$ is denoted by $H^\ast$. The characteristic function of a set $E$ is denoted by $\chi_E$. The spaces $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$ denote the equivalence classes of square-integrable functions and essentially bounded functions on $\mathbb{R}$, respectively. Next, we recall the Parseval identity. Let $\{e_k\}_{k \in \mathbb{Z}}$ be an orthonormal basis for a Hilbert space $H$. Then,

$$\sum_{k \in \mathbb{Z}} |\langle f, e_k \rangle|^2 = \|f\|^2, \quad f \in H \quad \text{(Parseval identity)}.$$

For $a, b \in \mathbb{R}$, we consider the following operators on $L^2(\mathbb{R})$.

$$T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad T_a f(\gamma) = f(\gamma - a) \quad \text{(Translation by $a$)},$$
Definition 2.1. Let \( \{\psi_1, \psi_2, \ldots, \psi_n\} \subset L^2(\mathbb{R}) \) be a finite set, where \( \psi_\ell \neq 0 \), \( 1 \leq \ell \leq n \). The family
\[
\{L^jT_\lambda \psi_\ell\}_{j, \lambda, \ell \in \Lambda} = \{(2N)^j \hat{\psi}_1(2N)^j \gamma - \lambda\}_{j, \lambda, \ell \in \Lambda}
\]
is called a nonuniform wavelet frame for \( L^2(\mathbb{R}) \), if there exist finite positive constants \( A \) and \( B \) such that
\[
A\|f\|^2 \leq \sum_{j, \lambda, \ell} \sum_{\ell=1}^n |(f, L^jT_\lambda \psi_\ell)|^2 \leq B\|f\|^2 \text{ for all } f \in L^2(\mathbb{R}).
\]

The Fourier transform of a function \( f \) is denoted by \( \mathcal{F}f \) or \( \hat{f} \), and defined as
\[
\mathcal{F}f = \hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \gamma} dx.
\]

For \( N \in \mathbb{N} \), \( j \in \mathbb{Z} \) and \( a \in \mathbb{R} \), by direct calculation, we have the following properties.
(i) \( L^j : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) is unitary map.
(ii) \( L^jT_a = T_{(2N)^{-j}a}L^j \).
(iii) \( \mathcal{F}L^j = L^{-j}\mathcal{F} \).
(iv) \( \mathcal{F}T_a = E_{-a}\mathcal{F} \).

3. The Nonuniform General Setup

In this section, we give a list of assumptions which will be used in the construction of Parseval nonuniform wavelet frames. To be precise, in formulation of the unitary extension principle there is long list of assumption, instead of writing each assumption again and again, we state all assumptions at once and call it nonuniform general setup: Let \( \psi_0 \in L^2(\mathbb{R}) \) be such that
(i) \( \hat{\psi}_0(2N\gamma) = H_0(\gamma)\hat{\psi}_0(\gamma), \ H_0(\gamma) \in L^\infty(\mathbb{R}); \)
(ii) \( \text{Supp} \hat{\psi}_0(\gamma) \subseteq [0, \frac{1}{4N}] \); and
(iii) \( \lim_{\gamma \rightarrow 0^+} \hat{\psi}_0(\gamma) = 1. \)

Further, let \( H_1, H_2, \ldots, H_n \in L^\infty(\mathbb{R}) \), and define \( \psi_1, \psi_2, \ldots, \psi_n \in L^2(\mathbb{R}) \) such that
\[
\hat{\psi}_\ell(2N\gamma) = H_\ell(\gamma)\hat{\psi}_0(\gamma), \ \ell = 1, 2, \ldots, n.
\]

Let \( H(\gamma) \) be a \((n+1) \times 1\) matrix given by
\[
H(\gamma) = \begin{bmatrix}
H_0(\gamma) \\
H_1(\gamma) \\
\vdots \\
H_n(\gamma)
\end{bmatrix} \in (n+1) \times 1
\]

Then, the collection \( \{\psi_\ell, H_\ell\}_{\ell=0}^n \) is called the nonuniform general setup.
4. Some Auxiliary Results

In this section, we give some auxiliary results that will be used in the sequel.

**Lemma 4.1.** For any \( f \in L^1(\mathbb{R}) \), the function \( Sf(\gamma) = \sum_{k \in \mathbb{Z}} f(\gamma + Nk) \) is well defined, \( N \)-periodic and belongs to \( L^1(0,N) \).

**Proof.** It is clear that \( Sf(\gamma) = \sum_{k \in \mathbb{Z}} f(\gamma + Nk) \) is \( N \)-periodic. For any \( f \in L^1(\mathbb{R}) \), we have

\[
\int_0^N \sum_{k \in \mathbb{Z}} |f(\gamma + Nk)| \, d\gamma = \int_\mathbb{R} |f(\gamma)| \, d\gamma < \infty.
\]

Thus, \( Sf(\gamma) \) is well defined a.e. on \( \mathbb{R} \), and also belongs to \( L^1(0,N) \). \( \square \)

**Lemma 4.2.** Assume that

(i) \( \psi_0 \in L^2(\mathbb{R}), \lim_{\gamma \to 0^+} \psi_0(\gamma) = 1 \) and \( \text{Supp} \ \hat{\psi}_0(\gamma) \subseteq [0, \frac{1}{N}] \);

(ii) \( f \in L^2(\mathbb{R}) \) such that \( \hat{f} \in C_c(\mathbb{R}) \).

Then, for any \( \epsilon > 0 \) there exist \( J \in \mathbb{Z} \) such that

\[
(1 - \epsilon)\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, L^jT_\lambda \psi_0 \rangle|^2 \leq (1 + \epsilon)\|f\|^2 \quad \text{for all } j \geq J.
\]

**Proof.** For any \( j \in \mathbb{Z} \), \((L^j \hat{f}) \hat{\psi}_0 \in L^1(\mathbb{R})\). Therefore, by Lemma 4.1, the function \( S(L^j \hat{f}) \hat{\psi}_0 \) is well defined. Further, for \( \gamma \in [0,N] \), we have

\[
S(L^j \hat{f}) \hat{\psi}_0 = \sum_{k \in \mathbb{Z}} \langle (L^j \hat{f}) \hat{\psi}_0, (\gamma - Nk) \rangle
= \sum_{k \in \mathbb{Z}} \langle (L^j \hat{f}), (\gamma - Nk) \hat{\psi}_0 \rangle (\gamma - Nk).
\]

Thus, \( S(L^j \hat{f}) \hat{\psi}_0 \) is bounded by finite linear combinations of translates of \( \hat{\psi}_0 \) and \( S(L^j \hat{f}) \hat{\psi}_0 \in L^2[0,N] \).

Note that

\[
\langle f, L^jT_\lambda \psi_0 \rangle = \langle \hat{f}, L^j \hat{T_\lambda \psi_0} \rangle = \langle \hat{f}, L^{-j}E_{-\lambda} \hat{\psi}_0 \rangle = \langle L^j \hat{f}, E_{-\lambda} \hat{\psi}_0 \rangle.
\]

Using \( \text{Supp} \ \hat{\psi}_0(\gamma) \subseteq [0, \frac{1}{N}] \) and \( \frac{1}{m} < \frac{1}{2} \), we compute

\[
\sum_{\lambda \in \Lambda} |\langle f, L^jT_\lambda \psi_0 \rangle|^2
= \sum_{\lambda \in \Lambda} |\langle L^j \hat{f}, E_{-\lambda} \hat{\psi}_0 \rangle|^2
= \sum_{\lambda \in \mathbb{Z}} |\langle L^j \hat{f}, E_{-\lambda} \hat{\psi}_0 \rangle|^2 + \sum_{\lambda \in (\mathbb{Z}/N) + 2\mathbb{Z}} |\langle L^j \hat{f}, E_{-\lambda} \hat{\psi}_0 \rangle|^2
= \sum_{m \in \mathbb{Z}} \left| \int_0^N S((L^j \hat{f}) \hat{\psi}_0)(\gamma) e^{2\pi i (2m)\gamma} \, d\gamma \right|^2 + \sum_{m \in \mathbb{Z}} \left| \int_0^N S((L^j \hat{f}) \hat{\psi}_0)(\gamma) e^{2\pi i (\frac{m}{N} + 2m)\gamma} \, d\gamma \right|^2
= \sum_{m \in \mathbb{Z}} \left| \int_0^\frac{1}{m} ((L^j \hat{f}) \hat{\psi}_0) e^{2\pi i (2m)\gamma} \, d\gamma \right|^2 + \sum_{m \in \mathbb{Z}} \left| \int_0^\frac{1}{m} ((L^j \hat{f}) \hat{\psi}_0) e^{2\pi i (\frac{m}{N} + 2m)\gamma} \, d\gamma \right|^2.
\]
Lemma 4.3. Suppose that

\[ \sum_{m \in \mathbb{Z}} \left| \int_0^{\frac{1}{2}} \left( (L^j f) \hat{\psi}_0 \right) e^{2\pi i (2m) \gamma} d\gamma \right|^2 + \sum_{m \in \mathbb{Z}} \left| \int_0^{\frac{1}{2}} \left( (L^j f) \hat{\psi}_0 \right) e^{2\pi i (\frac{\gamma}{2} + 2m) \gamma} d\gamma \right|^2 = \sum_{m \in \mathbb{Z}} \left| \int_0^{\frac{1}{2}} \left( (L^j f) \hat{\psi}_0 \right) e^{2\pi i (\frac{\gamma}{2} + 2m) \gamma} d\gamma \right|^2 \]  

(4.1)

Applying the Parseval identity on \( L^2(0, \frac{1}{2}) \) with respect to an orthonormal bases \( \{ \sqrt{2} e^{2\pi i (2m) \gamma} \} \) in (4.1), we obtain

\[ \sum_{\lambda \in \Lambda} |\langle f, L^j T_\lambda \psi_0 \rangle|^2 = \frac{1}{2} \int_0^{\frac{1}{2}} |(L^j \hat{f}) \hat{\psi}_0|^2 d\gamma + \frac{1}{2} \int_0^{\frac{1}{2}} |(L^j \hat{f}) \hat{\psi}_0|^2 d\gamma \]

\[ = \int_0^{\frac{1}{2}} |(L^j \hat{f}) \hat{\psi}_0|^2 d\gamma. \]  

(4.2)

Let \( \epsilon > 0 \) be given. Since \( \hat{\psi}_0(\gamma) \to 1 \) as \( \gamma \to 0^+ \), we can choose \( b \in ]0, \frac{1}{2}] \) so that

\[ (1 - \epsilon) \leq |\hat{\psi}_0(\gamma)|^2 \leq (1 + \epsilon), \quad 0 < \gamma < b. \]  

(4.3)

Choose \( J \in \mathbb{Z} \) large enough, so that \( \text{Supp} (L^j \hat{f}) \subseteq [-b, b] \) for all \( j \geq J \). Then, by (4.2), we have

\[ \sum_{\lambda \in \Lambda} |\langle f, L^j T_\lambda \psi_0 \rangle|^2 = \int_0^{b} |(L^j \hat{f}) \hat{\psi}_0|^2 d\gamma \text{ for all } j \geq J. \]  

(4.4)

By (4.3), (4.4) and the fact that \( L^j \) is unitary map, we have

\[ (1 - \epsilon)\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, L^j T_\lambda \psi_0 \rangle|^2 \leq (1 + \epsilon)\|f\|^2 \text{ for all } j \geq J. \]

Since the Fourier transform is unitary map, we get

\[ (1 - \epsilon)\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, L^j T_\lambda \psi_0 \rangle|^2 \leq (1 + \epsilon)\|f\|^2 \text{ for all } j \geq J. \]

This concludes the proof. \( \square \)

Lemma 4.3. Suppose that

(i) \( \psi_0 \in L^2(\mathbb{R}) \) satisfies \( \text{Supp} \hat{\psi}_0 \subseteq [0, \frac{1}{4}] \) and \( \hat{\psi}_0(2N \gamma) = H_0(\gamma) \hat{\psi}_0(\gamma) \), where \( H_0(\gamma) \in L^\infty(\mathbb{R}) \);

(ii) \( f \in L^2(\mathbb{R}) \) with \( \hat{f} \in C_c(\mathbb{R}) \), and \( H_1, H_2, \ldots, H_n \in L^\infty(\mathbb{R}) \) such that the \( (n + 1) \times 1 \) matrix

\[ H(\gamma) = \begin{bmatrix} H_0(\gamma) \\ H_1(\gamma) \\ \vdots \\ H_n(\gamma) \end{bmatrix}_{(n+1) \times 1} \]

satisfies \( H(\gamma)^* H(\gamma) = 1 \) a.e.;

(iii) \( \psi_1, \psi_2, \ldots, \psi_n \in L^2(\mathbb{R}) \) such that \( \hat{\psi}_\ell(2N \gamma) = H_\ell(\gamma) \hat{\psi}_0(\gamma), \ \ell = 1, 2, \ldots, n \).

Then

\[ \sum_{\ell=0}^{n} \sum_{\lambda \in \Lambda} |\langle f, L^j T_\lambda \psi_\ell \rangle|^2 = \sum_{\lambda \in \Lambda} |\langle f, L^j T_\lambda \psi_0 \rangle|^2. \]

Proof. For any \( j \in \mathbb{Z} \) and for any \( \ell = 0, 1, \ldots, n \), we have

\[ \langle f, L^j T_\lambda \psi_\ell \rangle = \langle L^{-j} f, L^{-1} T_\lambda \psi_\ell \rangle = \langle L^{-j} f, T_{(2N)\lambda} L^{-1} \psi_\ell \rangle \]
Using \( \text{Supp } \psi_0 \subseteq [0, \frac{1}{4m}] \), and Parseval identity on \( L^2(0, \frac{1}{4m}) \) with respect to orthonormal basis \( \{2\sqrt{N}e^{2\pi i(4Nm)\gamma}\}_{m \in \mathbb{Z}} \), we have

\[
\sum_{\lambda \in \Lambda} |(f, L^{-1}T_{\lambda}\psi_\ell)|^2 = \sum_{\lambda \in \mathbb{Z}} |(f, L^{-1}T_{\lambda}\psi_\ell)|^2 + \sum_{\lambda \in (\frac{1}{4m}+\mathbb{Z})} |(f, L^{-1}T_{\lambda}\psi_\ell)|^2
\]

\[
= \sum_{m \in \mathbb{Z}} \left| \sqrt{2N} \int_0^N S((L^j \hat{f})(\gamma)H_\ell(\gamma)\psi_0(\gamma))e^{2\pi i(2N\gamma)}e^{(2N\gamma)}d\gamma \right|^2
\]

\[
+ \left| \sqrt{2N} \int_0^N S((L^j \hat{f})(\gamma)H_\ell(\gamma)\psi_0(\gamma))e^{2\pi i(2N\gamma)}e^{(2N\gamma)}d\gamma \right|^2
\]  

(Using (4.5))

\[
= \frac{1}{2} \sum_{m \in \mathbb{Z}} \left| \int_0^{\frac{1}{4m}} (L^j \hat{f})(\gamma)H_\ell(\gamma)\psi_0(\gamma)e^{2\pi i(4Nm\gamma)}2\sqrt{N}d\gamma \right|^2
\]

\[
+ \frac{1}{2} \sum_{m \in \mathbb{Z}} \left| \int_0^{\frac{1}{4m}} (L^j \hat{f})(\gamma)H_\ell(\gamma)\psi_0(\gamma)e^{2\pi i(2m\gamma)}2\sqrt{N}d\gamma \right|^2
\]

\[
= \frac{1}{2} \int_0^{\frac{1}{4m}} |(L^j \hat{f})(\gamma)H_\ell(\gamma)\psi_0(\gamma)|^2 d\gamma + \frac{1}{2} \int_0^{\frac{1}{4m}} |(L^j \hat{f})(\gamma)H_\ell(\gamma)\psi_0(\gamma)|^2 d\gamma
\]

\[
\int_0^{\frac{1}{4m}} |(L^j \hat{f})(\gamma)H_\ell(\gamma)\psi_0(\gamma)|^2 d\gamma.
\]  

(4.6)

Since \( H(\gamma)^*H(\gamma) = 1 \) a.e., so \( H(\gamma) \) could be consider as an isometry from \( \mathbb{C}^1 \) into \( \mathbb{C}^{n+1} \). Using (4.6), we have

\[
\sum_{\ell=0}^n \sum_{\lambda \in \Lambda} |(f, L^{-1}T_{\lambda}\psi_\ell)|^2 = \sum_{\ell=0}^n \int_0^{\frac{1}{4m}} \left| (L^j \hat{f})(\gamma)H_\ell(\gamma)\psi_0(\gamma) \right|^2 d\gamma
\]

\[
\int_0^{\frac{1}{4m}} \left\| \begin{bmatrix} H_0(\gamma) \\ \vdots \\ H_n(\gamma) \end{bmatrix}_{(n+1)\times 1} \right\|_{\mathbb{C}^{n+1}}^2 d\gamma
\]

\[
= \int_0^{\frac{1}{4m}} \left\| \begin{bmatrix} (L^j \hat{f})(\gamma)\psi_0(\gamma) \end{bmatrix}_{1\times 1} \right\|_{\mathbb{C}^{n+1}}^2 d\gamma
\]

\[
= \int_0^{\frac{1}{4m}} \left\| (L^j \hat{f})(\gamma)\psi_0(\gamma) \right\|_{\mathbb{C}^{n+1}}^2 d\gamma.
\]  

(4.7)
Proof.\( \) Let \( \lambda \in \Lambda \), we can find an integer \( j > 0 \) such that \( \hat{f} \in C_c(\mathbb{R}) \), and let \( \epsilon > 0 \) be given. Then, by Lemma 4.2, we can find an integer \( j > 0 \) such that
\[
\sum_{\lambda \in \Lambda} |(f, L^j T_\lambda \psi_0)|^2 \leq (1 + \epsilon)\|f\|^2. \tag{4.10}
\]
Also, by Lemma 4.3, we have
\[
\sum_{\lambda \in \Lambda} |(f, L^{j-1} T_\lambda \psi_0)|^2 \leq \sum_{\lambda \in \Lambda} |(f, L^j T_\lambda \psi_0)|^2. \tag{4.11}
\]

Lemma 4.4. Let \( \{\psi_t, H_t\}_{t=0}^n \) be a nonuniform general setup, and let \( H(\gamma)^* H(\gamma) = 1 \). Then, the following holds.

(i) \( \{T_\lambda \psi_0\}_{\lambda \in \Lambda} \) is Bessel sequence with Bessel bound 1.

(ii) For any \( f \in L^2(\mathbb{R}) \),
\[
\lim_{j \to -\infty} \sum_{\lambda \in \Lambda} |(f, L^j T_\lambda \psi_0)|^2 = 0. \]

Proof. (i) : Let \( f \in L^2(\mathbb{R}) \) be such that \( \hat{f} \in C_c(\mathbb{R}) \), and let \( \epsilon > 0 \) be given. Then, by Lemma 4.2, we can find an integer \( j > 0 \) such that
\[
\sum_{\lambda \in \Lambda} |(f, L^j T_\lambda \psi_0)|^2 \leq (1 + \epsilon)\|f\|^2. \tag{4.10}
\]
Also, by Lemma 4.3, we have
\[
\sum_{\lambda \in \Lambda} |(f, L^{j-1} T_\lambda \psi_0)|^2 \leq \sum_{\lambda \in \Lambda} |(f, L^j T_\lambda \psi_0)|^2. \tag{4.11}
\]
Applying (4.11) $j$ times and using (4.10), we get
\[\sum_{\lambda \in \Lambda} |\langle f, T_{\lambda} \psi \rangle|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, L_{\lambda} T_{\lambda} \psi \rangle|^2 \leq (1 + \epsilon)\|f\|^2.\]

Since $\epsilon > 0$ was arbitrary, we have
\[\sum_{\lambda \in \Lambda} |\langle f, T_{\lambda} \psi \rangle|^2 \leq \|f\|^2.\]

Because this inequality holds on a dense subset of $L^2(\mathbb{R})$, it holds on $L^2(\mathbb{R})$. This proves (i).

(ii): Let $f \in L^2(\mathbb{R})$. Since $L_j$ is unitary map for all $j \in \mathbb{Z}$, by using (i), the family $\{L_j T_{\lambda} \psi_0\}_{\lambda \in \Lambda}$ is Bessel sequence with Bessel bound 1. For any $j \in \mathbb{Z}$ and for any bounded interval $I \subset \mathbb{R}$, we have
\[\sum_{\lambda \in \Lambda} |\langle f, L_{\lambda} T_{\lambda} \psi_0 \rangle|^2 \leq 2 \sum_{\lambda \in \Lambda} |\langle f \chi_I, L_{\lambda} T_{\lambda} \psi_0 \rangle|^2 + 2 \sum_{\lambda \in \Lambda} |\langle f (1 - \chi_I), L_{\lambda} T_{\lambda} \psi_0 \rangle|^2 \leq 2 \sum_{\lambda \in \Lambda} |\langle f \chi_I, L_{\lambda} T_{\lambda} \psi_0 \rangle|^2 + 2\|f (1 - \chi_I)\|^2.\]

Now, $\|f (1 - \chi_I)\|^2 \to 0$, if we choose $I$ to be sufficiently large. Therefore, we only need to show
\[\sum_{\lambda \in \Lambda} |\langle f \chi_I, L_{\lambda} T_{\lambda} \psi_0 \rangle|^2 \to 0 \text{ as } j \to -\infty.\]

Using the Cauchy-Schwarz’s inequality for integrals, we obtain
\[\sum_{\lambda \in \Lambda} |\langle f \chi_I, L_{\lambda} T_{\lambda} \psi_0 \rangle|^2 = (2N)^j \sum_{\lambda \in \Lambda} \left| \int_I f(\gamma) \overline{\psi_0((2N)^j \gamma - \lambda)} \, d\gamma \right|^2 \leq (2N)^j \|f\|^2 \sum_{\lambda \in \Lambda} \int_I \left| \psi_0((2N)^j \gamma - \lambda) \right|^2 \, d\gamma \]
\[= \|f\|^2 \sum_{\lambda \in \Lambda_{(2N)^j I - \lambda}} \int_I \left| \psi_0(\gamma) \right|^2 \, d\gamma. \tag{4.12}\]

Applying the Lebesgue dominated convergence theorem in (4.12), we have
\[\sum_{\lambda \in \Lambda} |\langle f \chi_I, L_{\lambda} T_{\lambda} \psi_0 \rangle|^2 \to 0 \text{ as } j \to -\infty.\]

Hence (ii) is proved. \qed

5. Unitary Extension Principle for Nonuniform Wavelet Frames

We begin this section with the UEP for nonuniform wavelet frames for $L^2(\mathbb{R})$.

**Theorem 5.1.** Let $\{\psi_\ell, H_\ell\}_{\ell=0}^n$ be a nonuniform general setup and $H(\gamma)^*H(\gamma) = 1$. Then, the nonuniform multiwavelets systems $\{L_{\ell} T_{\lambda} \psi_\ell\}_{\ell=1,2,...,n, \lambda \in \Lambda}$ constitutes a Parseval frame for $L^2(\mathbb{R})$.

**Proof.** Let $\epsilon > 0$ be given. Consider a function $f \in L^2(\mathbb{R})$ such that $\hat{f} \in C_c(\mathbb{R})$. By Lemma 4.2, we can choose $J > 0$ such that for all $j \geq J$,
\[(1 - \epsilon)\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, L_{\lambda} T_{\lambda} \psi_0 \rangle|^2 \leq (1 + \epsilon)\|f\|^2. \tag{5.1}\]

Using Lemma 4.3, we have
\[\sum_{\lambda \in \Lambda} |\langle f, L_{\lambda} T_{\lambda} \psi_0 \rangle|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, L_{\lambda} T_{\lambda} \psi_0 \rangle|^2 \leq (1 + \epsilon)\|f\|^2.\]
\[\begin{align*}
&= \sum_{j=0}^{n} \sum_{\lambda \in \Lambda} |\langle f, L^{j-1}T_{\lambda} \psi_{\ell} \rangle|^2 \\
&= \sum_{\lambda \in \Lambda} |\langle f, L^{j-1}T_{\lambda} \psi_{0} \rangle|^2 + \sum_{j=1}^{n} \sum_{\lambda \in \Lambda} |\langle f, L^{j-1}T_{\lambda} \psi_{\ell} \rangle|^2.
\end{align*}\] (5.2)

Applying Lemma 4.3 on \[\sum_{\lambda \in \Lambda} |\langle f, L^{j-1}T_{\lambda} \psi_{0} \rangle|^2,\] we get
\[\sum_{\lambda \in \Lambda} |\langle f, L^{j-1}T_{\lambda} \psi_{0} \rangle|^2 = \sum_{\lambda \in \Lambda} |\langle f, L^{j-2}T_{\lambda} \psi_{0} \rangle|^2 + \sum_{j=1}^{n} \sum_{\lambda \in \Lambda} |\langle f, L^{j-2}T_{\lambda} \psi_{\ell} \rangle|^2.\] (5.3)

By (5.2) and (5.3), we have
\[\sum_{\lambda \in \Lambda} |\langle f, L^{j}T_{\lambda} \psi_{0} \rangle|^2 = \sum_{\lambda \in \Lambda} |\langle f, L^{j-2}T_{\lambda} \psi_{0} \rangle|^2 + \sum_{j=1}^{n} \sum_{\lambda \in \Lambda} |\langle f, L^{j-2}T_{\lambda} \psi_{\ell} \rangle|^2.\]

Repeating the above arguments, for any \(m < j\), we have
\[\sum_{\lambda \in \Lambda} |\langle f, L^{j}T_{\lambda} \psi_{0} \rangle|^2 = \sum_{\lambda \in \Lambda} |\langle f, L^{m}T_{\lambda} \psi_{0} \rangle|^2 + \sum_{j=1}^{n} \sum_{\lambda \in \Lambda} |\langle f, L^{m}T_{\lambda} \psi_{\ell} \rangle|^2.\] (5.4)

It follows from (5.1) and (5.4) that for all \(j > J, m < j,\)
\[(1 - \epsilon)\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, L^{m}T_{\lambda} \psi_{0} \rangle|^2 + \sum_{j=1}^{n} \sum_{\lambda \in \Lambda} |\langle f, L^{m}T_{\lambda} \psi_{\ell} \rangle|^2 \leq (1 + \epsilon)\|f\|^2.\]

Letting \(m \to -\infty\) in above and using (ii) of Lemma 4.4, we have
\[(1 - \epsilon)\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, L^{p}T_{\lambda} \psi_{0} \rangle|^2 \leq (1 + \epsilon)\|f\|^2.\] (5.5)

Letting \(j \to \infty\) in (5.5), we have
\[(1 - \epsilon)\|f\|^2 \leq \sum_{\lambda \in \Lambda} \sum_{p=\infty}^{\infty} |\langle f, L^{p}T_{\lambda} \psi_{\ell} \rangle|^2 \leq (1 + \epsilon)\|f\|^2.\]

Since \(\epsilon > 0\) was arbitrary, we obtain
\[\sum_{\lambda \in \Lambda} \sum_{p=\infty}^{\infty} |\langle f, L^{p}T_{\lambda} \psi_{\ell} \rangle|^2 = \|f\|^2 \text{ for all } f \in L^2(\mathbb{R}),\]
as desired. \(\square\)

The next theorem gives the generalized (or oblique) extension principle for nonuniform wavelet frames in \(L^2(\mathbb{R})\). It gives the more flexible technique to construct nonuniform wavelet frames.

**Theorem 5.2.** Let \(\{\psi_{\ell}, H_{\ell}\}_{\ell=0}^{n}\) be a nonuniform general setup. Assume that there exist strictly positive function \(\theta \in L^\infty(\mathbb{R})\) for which
\[\lim_{\gamma \to 0^+} \theta(\gamma) = 1,\]
and
\[\theta(2N\gamma)|H_{\ell}(\gamma)|^2 + \sum_{\ell=1}^{n} |H_{\ell}(\gamma)|^2 = \theta(\gamma).\]

Then, \(\{L^{j}T_{\lambda} \psi_{\ell}\}_{\lambda \in \Lambda, \ell=1,\ldots,n}\) is a Parseval nonuniform wavelet frame for \(L^2(\mathbb{R})\).
Proof. Define \( \tilde{\psi}_0 \in L^2(\mathbb{R}) \) such that
\[
\tilde{\psi}_0(\gamma) = \sqrt{\theta(\gamma)} \hat{\psi}_0(\gamma).
\] (5.6)

Define functions \( \tilde{H}_0, \tilde{H}_1, \ldots, \tilde{H}_n \) as follows
\[
\tilde{H}_0(\gamma) = \sqrt{\frac{\theta(2N\gamma)}{\theta(\gamma)}} H_0(\gamma),
\]
\[
\tilde{H}_\ell(\gamma) = \sqrt{\frac{1}{\theta(\gamma)}} H_\ell(\gamma), \quad \ell = 1, 2, \ldots, n.
\]

Then, we have
\[
\hat{\tilde{\psi}}_0(2N\gamma) = \sqrt{\theta(2N\gamma)} \hat{\psi}_0(2N\gamma)
\]
\[
= \sqrt{\theta(2N\gamma)} H_0(\gamma) \hat{\tilde{\psi}}_0(\gamma)
\]
\[
= \sqrt{\theta(2N\gamma)} H_0(\gamma) \left( \frac{\psi_0(\gamma)}{\theta(\gamma)} \right)
\]
\[
= \sqrt{\frac{\theta(2N\gamma)}{\theta(\gamma)}} H_0(\gamma) \tilde{\psi}_0(\gamma)
\]
\[
= \tilde{H}_0(\gamma) \tilde{\psi}_0(\gamma),
\] (5.7)

and
\[
\lim_{\gamma \to 0^+} \hat{\tilde{\psi}}_0(\gamma) = \lim_{\gamma \to 0^+} \sqrt{\theta(\gamma)} \hat{\psi}_0(\gamma) = 1.
\] (5.8)

Since \( \{\psi_\ell, H_\ell\}_{\ell=0}^n \) is a nonuniform general setup, by (5.6), we have
\[
\text{Supp} \ \tilde{\psi}_0(\gamma) \subseteq \left[0, \frac{1}{4N}\right],
\] (5.9)

and
\[
\sum_{\ell=0}^n |\tilde{H}_\ell(\gamma)|^2 = |\tilde{H}_0(\gamma)|^2 + \sum_{\ell=1}^n |\tilde{H}_\ell(\gamma)|^2
\]
\[
= \frac{\theta(2N\gamma)}{\theta(\gamma)} |H_0(\gamma)|^2 + \sum_{\ell=1}^n \frac{|H_\ell(\gamma)|^2}{\theta(\gamma)}
\]
\[
= \frac{1}{\theta(\gamma)} \theta(\gamma)
\]
\[
= 1.
\] (5.10)

Thus
\[
\tilde{H}_\ell(\gamma) \in L^\infty(\mathbb{R}) \text{ for } \ell = 0, 1, \ldots, n.
\] (5.11)

Let \( \tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_n \in L^2(\mathbb{R}) \) be such that
\[
\tilde{\psi}_\ell(2N\gamma) = \tilde{H}_\ell(\gamma) \tilde{\psi}_0(\gamma), \quad \ell = 1, \ldots, n.
\] (5.12)

Define
\[ \tilde{H}(\gamma) = \begin{bmatrix} \tilde{H}_0(\gamma) \\ \tilde{H}_1(\gamma) \\ \vdots \\ \tilde{H}_n(\gamma) \end{bmatrix}_{(n+1) \times 1} \]

Then, by (5.7), (5.8), (5.9) and (5.11), the collection \( \{ \tilde{\psi}_\ell, \tilde{H}_\ell \}_{\ell=0}^n \) is a nonuniform general setup.

Using (5.10), we have

\[ \tilde{H}(\gamma)^* \tilde{H}(\gamma) = \sum_{\ell=0}^{n} |\tilde{H}(\gamma)|^2 = 1. \]

Hence, by Theorem 5.1, \( \{ L^j T\chi \tilde{\psi}_\ell \}_{j \in \mathbb{Z}, \lambda \in \Lambda} \) is a Parseval nonuniform wavelet frames for \( L^2(\mathbb{R}) \).

Next, we compute

\[ \hat{\psi}_\ell(2N\gamma) = H_\ell(\gamma) \hat{\psi}_0(\gamma) \]

\[ = \left( \tilde{H}_\ell(\gamma) \sqrt{\theta(\gamma)} \right) \left( \hat{\psi}_0(\gamma) \frac{1}{\sqrt{\theta(\gamma)}} \right) \]

\[ = \tilde{H}_\ell(\gamma) \hat{\psi}_0(\gamma) \]

\[ = \tilde{\psi}_\ell(2N\gamma). \]

This gives, \( \psi_\ell = \tilde{\psi}_\ell \). Hence, the system \( \{ L^j T\chi \tilde{\psi}_\ell \}_{j \in \mathbb{Z}, \lambda \in \Lambda} \) is a Parseval nonuniform wavelet frames for \( L^2(\mathbb{R}) \). \( \square \)

**Remark 5.3.** It is worth noticing that, when \( \theta = 1 \), Theorem 5.1 can be obtained from Theorem 5.2.

**Construction of nonuniform wavelet frame with two generators:** Computational effort reduces if we have less number of generator or window functions, so we wish to have as minimum numbers of generators as is it possible. In this direction, we have the following result as an application of Theorem 5.2.

**Corollary 5.4.** Let \( \psi_0 \in L^2(\mathbb{R}) \) such that

(i) \( \hat{\psi}_0(2N\gamma) = H_0(\gamma) \hat{\psi}_0(\gamma) \), where \( H_0(\gamma) \in L^\infty(\mathbb{R}) \);

(ii) \( \text{Supp} \hat{\psi}_0(\gamma) \subseteq [0, \frac{1}{2N}] \); and

(iii) \( \lim_{\gamma \to 0^+} \hat{\psi}_0(\gamma) = 1 \).

If we choose \( H_1(\gamma) = \sqrt{\theta(2N\gamma)} H_0(\gamma) i \), \( H_2(\gamma) = \sqrt{\theta(\gamma)} \), and \( \psi_1, \psi_2 \in L^2(\mathbb{R}) \) such that

\[ \hat{\psi}_\ell(2N\gamma) = H_\ell(\gamma) \hat{\psi}_0(\gamma), \ \ell = 1, 2. \]

Then

\[ \theta(2N\gamma) |H_0(\gamma)|^2 + |H_1(\gamma)|^2 + |H_2(\gamma)|^2 = \theta(\gamma). \]

Hence, by Theorem 5.2, \( \{ L^j T\chi \tilde{\psi}_\ell \}_{j \in \mathbb{Z}, \lambda \in \Lambda} \) form a Parseval nonuniform wavelet frame for \( L^2(\mathbb{R}) \).
6. Examples

This section gives some applicative examples of the UEP and its generalized version. The following example illustrates Theorem 5.1.

**Example 6.1.** Let $N = 2$, $r = 3$, and $\psi_0 \in L^2(\mathbb{R})$ be such that

$$\hat{\psi}_0(\gamma) = \frac{\sin(\gamma)}{\gamma} \chi_{[0, \frac{1}{8}]}(\gamma).$$

Then

(i) $\lim_{\gamma \to 0^+} \hat{\psi}_0(\gamma) = 1$;

(ii) $\text{Supp } \hat{\psi}_0 \subseteq [0, \frac{1}{8}]$; and

(iii) $\hat{\psi}_0(4\gamma) = \frac{\sin(4\gamma)}{4\gamma} \chi_{[0, \frac{1}{32}]}(\gamma) \chi_{[0, \frac{1}{8}]}(\gamma) = H_0(\gamma) \hat{\psi}_0(\gamma),$

where $H_0(\gamma) = \cos(\gamma) \cos(2\gamma) \chi_{[0, \frac{1}{32}]}(\gamma)$.

Let $H(\gamma) = \begin{bmatrix} H_0(\gamma) \\ H_1(\gamma) \\ H_2(\gamma) \\ H_3(\gamma) \end{bmatrix}.$

Then, $\{\psi_\ell, H_\ell\}_{\ell=0}^3$ is a nonuniform general setup such that

$H(\gamma)^* H(\gamma) = |H_0(\gamma)|^2 + |H_1(\gamma)|^2 + |H_2(\gamma)|^2 + |H_3(\gamma)|^2 = 1.$

Hence, by Theorem 5.1, $\{L^j T_\lambda \psi_\ell\}_{\lambda \in (0, \frac{1}{8j}), j \geq 2}$ is a nonuniform Parseval wavelet frame $L^2(\mathbb{R})$.

To conclude the paper, we illustrate Theorem 5.2 with the following example.

**Example 6.2.** Let $N = 2$, $r = 3$ and $\psi_0 \in L^2(\mathbb{R})$ be such that

$$\hat{\psi}_0(\gamma) = \chi_{[0, \frac{1}{8}]}(\gamma).$$

Then

(i) $\lim_{\gamma \to 0^+} \hat{\psi}_0(\gamma) = 1$;

(ii) $\text{Supp } \hat{\psi}_0(\gamma) \subseteq [0, \frac{1}{8}]$; and
(iii) $\psi_0(4\gamma) = \chi_{[0,\frac{1}{2}]}(4\gamma) = \chi_{[0,\frac{1}{2}]}(\gamma)\chi_{[0,\frac{1}{2}]}(\gamma) = H_0(\gamma)\psi_0(\gamma)$,

where $H_0(\gamma) = \chi_{[0,\frac{1}{2}]}(\gamma) \in L^\infty(\mathbb{R})$.

Let $\theta(\gamma) = 1$ and define $H_1(\gamma) = \chi_{\mathbb{R}\setminus[0,\frac{1}{2}]}$.

Then, the collection $\{\psi_t, H_\ell\}_{\ell=0}^1$ is a nonuniform general setup such that

$\theta(4\gamma)|H_0(\gamma)|^2 + |H_1(\gamma)|^2 = \theta(\gamma)$.

Hence, by Theorem 5.2, the nonuniform wavelet system $\{L^j\mathcal{T}_\lambda \psi_1\}_{j \in \mathbb{Z}, \lambda \in [0,\frac{1}{2}) + 2\mathbb{Z}}$ is a Parseval frame for $L^2(\mathbb{R})$.

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