The black di-ring: an inverse scattering construction

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Abstract
We use the inverse scattering method (ISM) to derive concentric non-supersymmetric black rings. The approach used here is fully five dimensional, and has the modest advantage that it generalizes readily to the construction of more general axi-symmetric solutions.
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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The black ring of Emparan and Reall [1] was the first concrete piece of evidence that in higher-dimensional gravity, the no-hair theorems of 3+1 dimensions need not apply. Their construction explicitly demonstrated that in an asymptotically flat spacetime with a given Arnowitt–Deser–Misner (ADM) mass and angular momentum, the geometry need not necessarily be that of the Myers–Perry black hole [2].

Emboldened by that discovery, a lot of recent work has been directed toward exploring black rings and related ideas [3]. One upshot of these investigations is that now we know that there is a continuous non-uniqueness for black hole solutions in higher dimensions. Concentric black rings (and the black Saturn [4, 6]) are an explicit way to realize this degeneracy, the idea being that you can distribute the angular momenta and the masses between the two black rings in a continuous way, while still keeping their total asymptotic values fixed.

Concentric supersymmetric black rings were first constructed in [7], and the restriction to supersymmetry was lifted in the work of [8]. The technique used in the latter relies on the clever observation that the problem can essentially be reduced to four dimensions, and then apply the formalism of [9]. A disadvantage of the lack of a genuinely five-dimensional derivation is that there is no immediate route that one can pursue in order to generalize this solution. For instance, to try to add more generic spins to the solution, or to generalize the construction to more generic Saturn-like solutions, we would have to tackle the genuinely five-dimensional problem. One purpose of this paper is to give a derivation of concentric rings...
based on the general formalism of the inverse scattering method (ISM), which does not rely on the reduction to four dimensions. The inverse scattering approach that we use here was first used in the context of higher-dimensional gravity in [10] and then further explored in various contexts in [11, 12].

The format of this paper is as follows. In the following section, we review the inverse scattering method and the use of Lax pairs for generating new solutions. Section 3 applies this formalism to the construction of multiple rings. Once the solution is at hand, we need to impose asymptotic flatness and the absence of certain singularities. These put some relations between the parameters in the solution. We conclude with some discussions and possible directions for future research.

2. The inverse scattering method: Lax pairs and solitons

In this section, we review the inverse scattering method as applied to the construction of axially symmetric vacuum solutions of Einstein’s equations. The formalism was developed in four dimensions by Belinski and Zakharov [13], a standard textbook is [14]. We will follow the presentation of the method, as given in [10], for five dimensions.

In 5D, axial symmetry implies the existence of three commuting Killing vector fields. The generic metric with these assumptions can be written as [15, 16]

\[ ds^2 = G_{ab}(\rho, z) \, dx^a \, dx^b + f(\rho, z)(d\rho^2 + dz^2) \quad (2.1) \]

where \( a, b = 1, 2, 3 \), and we are free to choose

\[ \det G = -\rho^2. \quad (2.2) \]

If we define two matrices,

\[ U \equiv \rho(\partial_\rho G)G^{-1}, \quad V \equiv \rho(\partial_z G)G^{-1}, \quad (2.3) \]

then Einstein’s equations take the form

\[ \partial_\rho U + \partial_z V = 0, \quad (2.4) \]
\[ \partial_\rho (\log f) = -\frac{1}{\rho} + \frac{1}{4\rho} \text{Tr}(U^2 - V^2), \quad (2.5) \]
\[ \partial_z (\log f) = \frac{1}{2\rho} \text{Tr}(UV). \quad (2.6) \]

The last two equations can be consistently integrated because the first equation is an integrability condition for them. So the problem is fully solved, once we fix \( G_{ab} \).

The inverse scattering method hinges on the fact that the equations that need to be solved, namely equations (2.2) and (2.4), can be thought of as the compatibility conditions for the following over-determined set of differential equations:

\[ D_\rho \Psi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2} \Psi, \quad D_z \Psi = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2} \Psi, \quad (2.7) \]

where

\[ D_\rho \equiv \partial_\rho + \frac{2\lambda \rho}{\lambda^2 + \rho^2} \partial_\rho, \quad D_z \equiv \partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2} \partial_z. \quad (2.8) \]

These equations comprise the Lax pair, \( \lambda \) is called the spectral parameter, and the generating matrix \( \Psi \) is such that \( \Psi(\lambda = 0, \rho, z) = G(\rho, z) \). The first step in the construction of new solutions is to start with a seed solution \( G_0 \), and then find the generating matrix \( \Psi_0 \) that
solves (2.7), with the appropriate $U_0$ and $V_0$. The generating matrix should satisfy the condition that $\Psi_0(\lambda = 0, \rho, z) = G_0(\rho, z)$. Now, we seek a new solution of the Lax pair in the form $\Psi = \chi \Psi_0$, where $\chi$ is called the dressing matrix. Once the dressing matrix is known, the new solution will be determined as $G(\rho, z) = \Psi(\lambda = 0, \rho, z)$.

We will be interested in finding dressing matrices that satisfy the ansatz

$$\chi = 1 + \sum_k \frac{R_k}{\lambda - \tilde{\mu}_k}, \quad (2.9)$$

where $k$ runs over $1, \ldots, n$: we say that we have an $n$-soliton dressing matrix. By imposing conditions on the analyticity structure of the poles in the $\lambda$-plane, it turns out that we can fix the $\tilde{\mu}$ to be

$$\tilde{\mu}_k = \pm \sqrt{\rho^2 + (z - a_k)^2 - (z - a_k)}, \quad (2.10)$$

where $a_k$ are real constants. We will refer to the positive sign pole as a soliton $\mu_k$, and the negative sign pole as an anti-soliton $\bar{\mu}_k$. In addition to $a_k$, we also need to specify $R_k$ (which are not constants) in order to fully specify the dressing matrix. It turns out, after some computation (we refer the interested reader to [14] for details), that this can be done by specifying $n$ constant vectors with components $m^{(k)}_a$. These are called the Belinski–Zakharov (BZ) vectors, and they have 3-components, as implied by the index $a$. Instead of writing down $R_k$ in terms of $m^{(k)}_a$, we will omit the intermediate steps and present the final solution (the metric $G$) after the $n$-soliton transformation. To do this, we first define new vectors $m^{(k)}_a$,

$$m^{(k)}_a = m^{(k)}_b \Psi^{-1}_0(\lambda = \tilde{\mu}_k, \rho, z)[_{ba}], \quad (2.11)$$

and the matrix $\Gamma$,

$$\Gamma_{kl} = \frac{m^{(k)}_a (G_0)_{ab} m^{(l)}_b}{\rho^2 + \tilde{\mu}_k \tilde{\mu}_l}. \quad (2.12)$$

In terms of these, the final metric will be written as

$$G_{ab} = (G_0)_{ab} - \sum_{kl} (G_0)_{ac} m^{(k)}_c (\Gamma^{-1})_{kl} m^{(l)}_d (G_0)_{db}. \quad (2.13)$$

Matrix multiplication along the $a, b, \ldots, \varepsilon$ indices is assumed.

The solution as written down in (2.13) does not always give rise to the appropriate normalization (2.2) for the final solution. Instead, for the above $n$-soliton transformation and choice of BZ vectors, one finds (see equation (8.27) in [14])

$$\det G = (-1)^n \rho^{2n} \left( \prod_{k=1}^n \tilde{\mu}_k^{-2} \right) \det G_0. \quad (2.14)$$

One way to overcome this difficulty is to only look at transformations which are of the following two-step form:

Step 1. Subtract solitons with the trivial BZ vectors. Trivial, in this context, means that the BZ vectors do not mix components of the diagonal seed metric that we start with.

Step 2. Add the same solitons back in the second step, but this time with the non-trivial BZ vectors.

The reason why this works is because the BZ vectors do not contribute to (2.14), only the solitons do. And the effect of the solitons in the first step is annulled by the second step, leaving us with $\det G = \det G_0 = -\rho^2$. The conformal factor $f$ associated with the final metric can be written as

$$f = f_0 \frac{\det(\Gamma_{kl})}{\det(\Gamma^{(00)})}. \quad (2.15)$$
where \( \Gamma_{kl}^{(0)} \) is obtained by ‘trivializing’ \( \Gamma_{kl} \), i.e., by setting the parameters that make the BZ vector non-trivial, to zero, in (2.12). The formalism presented here will become more transparent when we explicitly construct the solution in the following section.

3. The black di-ring

3.1. Seed solution and solitonic transformations

As described in the last section, the inverse scattering method uses certain multi-soliton transformations to generate new solutions of Einstein’s equations from old. So the trick is essentially to guess a seed solution, the BZ vectors, and the solitons: the formalism will then churn out the final solution.

A powerful way to handle stationary, axi-symmetric solutions was invented by Harmark [16] generalizing earlier work in four and higher [15] dimensions. The idea is that such a spacetime can be described by certain ‘rod configurations’. We can describe our solutions (both the seed and the final versions) using these rods. The seed solution for the black di-ring we take in the form given in figure 1. The construction can be extended straightforwardly\(^1\) to more rings by adding the same structure to the left.

Using the standard techniques of [16], we can read off the seed metric from the seed rod configuration:

\[
G_0 = \text{diag} \left\{ \frac{\mu_{1} \mu_{4}}{\mu_{2} \mu_{5} \mu_{7}}, \frac{\rho^2 \mu_{3} \mu_{6}}{\mu_{2} \mu_{5} \mu_{7}}, \frac{\mu_{2} \mu_{5} \mu_{7}}{\mu_{1} \mu_{4}} \right\}.
\]

(3.1)

The elements are the \( tt,\phi\phi,\psi\psi \) components, respectively. Clearly, this satisfies the normalization condition (2.2). To complete the description, we write down the conformal factor as well:

\[
f_0 = \frac{k^2 \mu_2 \mu_5 \mu_7 R_{12} R_{13} R_{14} R_{15} R_{17} R_{23} R_{24} R_{26} R_{31} R_{32} R_{35} R_{43} R_{45} R_{46} R_{47} R_{56} R_{57}}{\mu_1 \mu_4 R_{14}^2 R_{23} R_{27}^2 R_{36}^2 R_{57}^2 \prod_{i=1} R_{ii}}.
\]

(3.2)

Here \( k^2 \) is an integration constant and

\[
R_{ij} \equiv (\rho^2 + \mu_i \mu_j).
\]

(3.3)

We also define

\[
D_{ij} \equiv (\mu_i - \mu_j)
\]

(3.4)

for later convenience. Efficient computation of the conformal factor requires a formalism based on going to the complex plane, and is sketched in appendix E of [15].

\(^1\) In principle, the computational effort required to derive the final metric grows quickly as we increase the number of rings.
As described near the end of the previous section, we will subtract solitons and then add them back in, so that (2.2) is automatically respected. The intuition behind the choice of the seed and the solitons is based on the analysis of rod structures in the manner of Harmark (see [16]). In particular, the shapes and locations of the horizons can be determined from the rod structure, and that gives us a handle on the geometry without actually trying to analyze the forms of the metric functions.

The full solution generation process involves the following steps:

(1) Remove an anti-soliton at \(a_1\), with the trivial BZ vector \((1, 0, 0)\). This results effectively in multiplying \((G_0)_{tt}\) by \(-\frac{\rho^2}{\mu_1^2}\), upon direct application of (2.10)–(2.13).

(2) Remove another anti-soliton at \(a_4\), again with the trivial BZ vector \((1, 0, 0)\). This multiplies \((G_0)_{tt}\) by \(-\frac{\rho^2}{\mu_4^2}\).

(3) Pull out an overall factor of \(-\frac{\rho^4}{\mu_1 \mu_4}\) from the resulting metric. After we are done with the solitonic transformations, we will put this factor back in. This is a choice of convenience, and nothing prevents us from making that. The resulting metric after these three steps has the form

\[
\tilde{G}_0 = \text{diag}\left\{ \frac{1}{\mu_3 \mu_6}, -\frac{\mu_1 \mu_3 \mu_4 \mu_6}{\rho^2 \mu_2 \mu_5 \mu_7}, -\frac{\mu_2 \mu_5 \mu_7}{\rho^2}\right\}.
\]

This metric will be our seed for the next transformation, which involves two solitons. The generating matrix can be computed to be

\[
\Psi_0 = \text{diag}\left\{ \frac{1}{(\mu_3 - \lambda)(\mu_6 - \lambda)}, \frac{1}{(\mu_2 - \lambda)(\mu_5 - \lambda)(\mu_7 - \lambda)}, \frac{1}{-\lambda}\right\},
\]

where \(\bar{\mu}_i = -\frac{\rho^2}{\mu_i}\).

(4) Add two anti-solitons, one at \(a_1\) with the BZ vector \(m_0^{(1)} = (1, 0, c_1)\) and another at \(a_4\), with the BZ vector \(m_0^{(2)} = (1, 0, c_2)\), and perform a 2-soliton transformation to obtain \(\tilde{G}\).

(5) Absorb back the factor \(-\frac{\rho^4}{\mu_1 \mu_4}\) to obtain the final metric \(G\). The conformal factor \(f\) can be obtained from \(f_0\) using (2.15) with \(\Gamma_0 = \Gamma_{c_1 = c_2 = 0}\).

Once these transformations are done, we have the concentric ring solution, except that we still need to impose asymptotic flatness and the absence of certain singularities to make sure that the solution is regular and balanced. We will address this issue after writing down the explicit form of the metric.

### 3.2. The concentric ring solution

In this section, we write down the functions in the final metric for the concentric ring,

\[\mathrm{d}s^2 = G_{tt} \mathrm{d}t^2 + 2G_{t\psi} \mathrm{d}t \mathrm{d}\psi + G_{\psi\psi} \mathrm{d}\psi^2 + G_{\phi\phi} \mathrm{d}\phi^2 + f (\mathrm{d}\rho^2 + \mathrm{d}z^2),\]

before imposing regularity, etc. Here, the \(\phi\phi\)-component is the same as that of the seed metric:

\[G_{\phi\phi} = \frac{\mu_3 \mu_6 \rho^2}{\mu_2 \mu_5 \mu_7},\]

and the conformal factor,

\[f = \frac{A_1 + c_1^2 A_2 + 2c_1 c_2 A_3 + c_1^2 c_2^2 A_4 + c_2^2 A_5}{\mathcal{H}},\]
with
\[
A_1 = \mu_2^2 \mu_3^2 \mu_7 D_{14}^2 R_{12}^2 R_{15}^2 R_{16}^2 R_{17}^2 R_{23} R_{24} R_{26} R_{34} R_{35} R_{37} R_{45} R_{46} R_{47} R_{56} R_{67},
\]
\[
A_2 = \mu_2^2 \mu_3 \mu_4 \mu_6 D_{12}^2 D_{15}^2 R_{12} R_{14} R_{15} R_{23} R_{24} R_{26} R_{34} R_{35} R_{37} R_{45} R_{46} R_{47} R_{56} R_{67},
\]
\[
A_3 = \mu_1 \mu_2 \mu_3 \mu_4 \mu_6 \mu_7 \rho^2 D_{12} D_{15} D_{24} D_{45} \times R_{12} R_{14} R_{15} R_{23} R_{24} R_{26} R_{34} R_{35} R_{37} R_{45} R_{46} R_{47} R_{56} R_{67},
\]
\[
A_4 = \rho^4 \mu_2^2 \mu_3 \mu_4^2 \mu_5 \mu_7 D_{12}^2 D_{14}^2 D_{23} D_{24} D_{25} R_{12} R_{13} R_{24} R_{26} R_{35} R_{37} R_{45} R_{46} R_{67},
\]
\[
A_5 = \rho^2 \mu_1 \mu_2 \mu_4 \mu_6 \mu_7 \rho^2 D_{12}^2 D_{15}^2 D_{16}^2 D_{17}^2 D_{23} R_{12} R_{13}^2 R_{14} R_{15} R_{16}^2 R_{17} R_{23} R_{24} R_{26} R_{35} R_{37} R_{45} R_{46} R_{67},
\]
and
\[
\mathcal{H} = \mu_1 \mu_2 \mu_4 \mu_5 D_{14}^2 R_{13}^2 R_{14} R_{16} R_{17}^2 R_{23} R_{25}^2 R_{34} R_{36} R_{46} R_{47} R_{56}^2 \prod_{i=1}^{7} R_{i1},
\] (3.9)

The other components of the metric are
\[
G_{\mu \nu} = X_1 + c_1^2 X_2 + c_1^2 X_3 + 2 c_1 c_2 X_4 + c_1^2 c_2^2 X_5, \tag{3.10}
\]
\[
G_{\mu \psi} = -c_1 Y_1 - c_2 Y_2 + c_1^2 c_2 Y_3 + c_1 c_2^2 Y_4, \tag{3.11}
\]
\[
G_{\psi \psi} = Z_1 + c_1^2 Z_2 + c_1^2 Z_3 + 2 c_1 c_2 Z_4 + c_1^2 c_2^2 Z_5, \tag{3.12}
\]

with \( \Delta \equiv D_1 + c_1^2 D_2 + c_2^2 D_4 + 2 c_1 c_2 D_4 + c_1^2 c_2^2 D_5 \). The various functions are fully fixed by the following relations:
\[
X_1 = -\mu_1 \mu_4 D_1, \quad X_2 = \rho^2 \mu_1 \mu_4 D_2,
\]
\[
X_3 = \rho^2 \frac{\mu_1}{\mu_4} D_3, \quad X_4 = \rho^2 D_4, \quad X_5 = -\rho^4 \mu_1 \mu_4 D_5,
\]
\[
Z_1 = \mu_2 \mu_5 \mu_7 D_1, \quad Z_2 = -\frac{\mu_2^2 \mu_4 \mu_5 \mu_7}{\rho^2} D_2,
\]
\[
Z_3 = -\frac{\mu_2^2 \mu_4 \mu_5 \mu_7}{\rho^2} D_3, \quad Z_4 = -\frac{\mu_1 \mu_2 \mu_4 \mu_5 \mu_7}{\rho^2} D_4, \quad Z_5 = \frac{\mu_2^2 \mu_4 \mu_5 \mu_7}{\mu_5 \rho^2} D_5,
\] (3.13)

and the definitions
\[
D_1 = \mu_2^2 \mu_3^2 D_{14}^2 R_{12}^2 R_{13}^2 R_{15}^2 R_{16}^2 R_{17}^2 R_{23}^2 R_{26} R_{34}^2 R_{47},
\] (3.14)
\[
D_2 = \mu_1 \mu_2 \mu_3 \mu_6 \mu_7 \rho^2 D_{12}^2 D_{15}^2 R_{12}^2 R_{13}^2 R_{26} R_{34}^2 R_{45},
\] (3.15)
\[
D_3 = \mu_2 \mu_3 \mu_5 \mu_6 \mu_7 \rho^2 D_{12}^2 D_{15}^2 R_{13}^2 R_{14} R_{16}^2 R_{17},
\] (3.16)
\[
D_4 = \mu_1 \mu_2 \mu_3 \mu_4 \mu_6 \mu_7 \rho^2 D_{13} D_{14} D_{23} D_{45} R_{11} R_{12} R_{13} R_{14} R_{46} R_{47} R_{17},
\] (3.17)
\[
D_5 = \mu_1 \mu_2 \mu_3 \mu_6 \mu_7 \rho^2 D_{12}^2 D_{14}^2 D_{25}^2 D_{45},
\] (3.18)
\[
Y_1 = \mu_2 \mu_3 \mu_7 D_{12} D_{14} D_{15} R_{11} R_{13} R_{14} R_{36} R_{46} R_{17}^2,
\] (3.19)
\[
Y_2 = \mu_2 \mu_3 \mu_7 D_{12} D_{15} D_{45} R_{11} R_{13} R_{14} R_{46} R_{16}^2 R_{17}^2,
\] (3.20)
\[
Y_3 = \mu_2 \mu_3 \mu_7 D_{12} D_{14} D_{15} D_{26} R_{14} R_{13} R_{46} R_{47} R_{17},
\] (3.21)
\[
Y_4 = \mu_2 \mu_3 \mu_7 D_{12} D_{14} D_{15} D_{26} R_{11} R_{13} R_{14} R_{16} R_{17} R_{47},
\] (3.22)
To complete the solution we need to make sure that it is asymptotically flat and that there are no conical singularities. These conditions will generate various relations between the different parameters \((a_i, c_1, c_2, k)\) in the solution. This is what we turn to in the following sections.

3.3. Rod configuration and elimination of singularities

The rod structure [16] for the final solution is useful for understanding the horizons, and to see what conditions one has to impose on the parameters to make sure that there are no singularities. In this subsection, we discuss the spacelike rods because these are the ones that give rise to the conditions on the parameters. The rod structure is given in figure 2. The dots in the figure correspond to the locations of the singularities before we remove them. The final solution can be made completely regular and then the rod structure will not have the dots:

- The semi-infinite rod \((-\infty, a_2]\). The direction of the rod, which is defined as the eigendirection along which the final metric matrix \(G_{ab}(\rho = 0, z)\) has zero eigenvalues is \((0, 1, 0)\). In order to avoid a conical singularity at the location of the rod, the periodicity of the spacelike coordinate (here, that would be \(\phi\)) must be fixed according to the condition

\[
\Delta \phi = 2\pi \lim_{\rho \to 0} \sqrt{\frac{\rho^2 f}{G_{\phi \phi}}},
\]

We end up finding that when \(z < a_1\),

\[
\Delta \phi = 2\pi
\]

identically, and that when \(a_1 < z < a_2\),

\[
\Delta \phi = 2\pi \sqrt{\frac{c_1^2(a_2 - a_1)(a_5 - a_1)}{2(a_5 - a_1)(a_6 - a_1)(a_7 - a_1)}} = 2\pi.
\]

Note that the first equality in (2.25) is a direct result of imposing (2.23), while the second equality is a condition we are imposing on the parameters so that the period found in (2.24) matches that of (2.25). The period must be the same for all values of \(z\) for the metric to be continuous.

- The finite rod \([a_2, a_3]\) is timelike and corresponds to the outer black-ring horizon.

- The finite rod \([a_3, a_5]\). The direction of the rod is again \((0, 1, 0)\), and when \(a_3 < z < a_4\), to avoid conical singularities we need

\[
\Delta \phi = 2\pi \frac{|Y - Z c_1 c_2|}{\sqrt{X}} = 2\pi,
\]

\(\square\)
The way to see this is to note that there are no timelike rods adjacent to the semi-infinite one should be.

are dimensionful, but the conical deficit angles we have calculated are dimensionless as they are positive, but things go through essentially unchanged for other choices of sign, except for a minor caveat we will emphasize when we compute the ADM quantities. Note also that if one treats the tip of the $\psi$-rod at $a_7$ as the origin of the $z$-axis (which is allowed because of
translational invariance along $z$), then this means that the horizon starts away from the center of the geometry. Together with the fact that the system is axi-symmetric, similar arguments immediately lead us to the conclusion that there are two ring-shaped horizons, and that they are concentric. Our rod diagram can be compared to the rod diagrams for flat Minkowski space, Myers–Perry black hole, the black ring and the black Saturn, and they all fit together neatly.

3.4. Asymptotic flatness

It is possible to verify [16] that the asymptotic region is given by the conditions

$$\sqrt{\rho^2 + z^2} \to \infty,$$

with $z/\sqrt{\rho^2 + z^2}$ being finite. (3.32)

Introducing coordinates $r$ and $\theta$ according to [4],

$$\rho = \frac{1}{2} r^2 \sin 2 \theta, \quad z = \frac{1}{2} r^2 \cos 2 \theta,$$

(3.33)

the asymptotic limit is succinctly contained in $r \to \infty$. At infinity, we want the black di-ring metric to reduce to the form

$$d\mathbf{s}^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2 + r^2 \cos^2 \theta d\phi^2.$$  (3.34)

It is possible to check that $G_{tt}, G_{t\psi}, G_{\psi\psi}, G_{\phi\phi}$ go to the right limits as $r \to \infty$.

So far we have left the integration constant $k^2$ in the conformal factor $f$ to be arbitrary. In fact, the condition that

$$f(\rho^2 + z^2) \to dr^2 + r^2 d\theta^2$$

at infinity fixes $k^2 = 1$.

Once we impose all these conditions, we have a fully regular and balanced asymptotically flat di-ring. It is encouraging that the asymptotic flatness conditions do not result in too many further conditions on the parameters, for physical reasons. We elaborate on the counting of parameters in the following subsection.

3.5. Parameter counting

Let us count the number of parameters of the black di-ring. We have seven parameters $a_i$, two BZ parameters $c_i$ and $k^2$. So in total, we had ten parameters to begin with. Only the relative positions of $a_i$ matter because of translational invariance along $z$, so we can define [4]

$$L^2 = (a_7 - a_1)$$  (3.36)

as a convenient length scale. Together with $a_2, \ldots, a_6$, this leaves us with six variables. We saw above that $k^2$ is set to 1. $c_i$ are determined by $a_i$ according to (3.30) and (3.31), so we are still left with six independent parameters at this stage. The conical singularity constraints from the last section give rise to three more independent constraints, and so finally we end up with three independent parameters for the black di-ring.

Happily, this is what one would expect on general grounds. The di-ring should have two independent masses and two independent angular momenta, one each for each of the two rings. But one of these four can be scaled away because classical gravity is a conformal theory; see [5] for a nice discussion of this. So, indeed, we expect to have three independent parameters.

It should be noted that the three non-trivial constraints arising from the absence of conical singularities fix the parameters only implicitly. We have not been able to solve them analytically in a useful way. But the use of the metric, especially in the investigations of thermodynamical phases, etc, is bound to be numerical, so this is not a serious problem. In
particular, the fact that the constraints on the parameters is implicit should not be taken to mean that the constraints are inconsistent. The most direct way to demonstrate this is to find explicit values for $a_i$ which satisfy the constraints. To do this, first introduce the variables $z_i$ which are defined as

$$z_i = \frac{a_{i+1} - a_i}{L^2}.$$ (3.37)

Note that the sequence $0, z_1, z_2, z_3, z_4, z_5, 1$ is non-decreasing. We can rewrite the conical singularity elimination conditions of the last subsection in terms of these new variables. The advantage is that the overall scale $L$ drops off from all expressions, so we only have to deal with $z_i$. Now, we are left with three equations and five variables, and our aim is to show that there are no inconsistencies.

Generically, of course, such a system is well posed; our aim is to merely make sure that what we have is not some degenerate, inconsistent special case. This is easy to do numerically by starting with seeds for two of $z_i$ and solving for the remaining three using the constraint equations. The result is a consistent solution if and only if the resulting $z_i$ satisfy the non-decreasing property. When we do this, we find that there are indeed solutions. We present an example with the seed $z_1 = 0.3, z_2 = 0.4$, below:

$$\{z_1 = 0.3, z_2 = 0.4, z_3 = 0.678153, z_4 = 0.743009, z_5 = 0.832417\}.$$ (3.38)

It can be checked by direct substitution that these values solve the constraint equations (the scale $L$ does not affect this). More solutions can be found by a numerical scanning starting from this seed. A more exhaustive scanning strategy would be to systematically scan for $z_i$ between 0 and 1 using some appropriate bin size. Finding all interesting solutions is likely to require an adaptive bin-size scanning strategy, because we do not know the measure on the moduli space of $z_i$: in particular, it can have structures at various resolutions depending on where we are. A similar situation was encountered in [17] for the black Saturn as well. We strongly suspect that the space of solutions densely fills out at least part of the phase space considered in [5], but we leave the details for future work. It would also be interesting to see which of these phases go away, when we impose thermodynamic equilibrium between the two rings.

A more analytical, but less concrete, piece of evidence for existence of solutions is that there exist limits where we can reduce the solution to the single ring form. The fact that the well-known ring solution can be found in the boundary of the moduli space of our di-ring solutions is another indication that the moduli space is non-vacuous. Indeed, we can obtain the black ring of Emparan and Reall as a limit of our di-ring solution. A hint on how to do this can be found by comparing our final rod diagram with the black-ring rod diagram [16]: we set $a_2 = a_3$ and $a_1 = a_3$. After some massaging, the metric functions can be brought to the form of the black-ring metric as written in the coordinates presented in (A.7)–(A.10) in [4], if we do the following replacements: $c_1 \rightarrow c_2$ with the other subscripts renamed as $1 \rightarrow 4, 3 \rightarrow 7, 4 \rightarrow 6, 5 \rightarrow 5$. Here the left-hand sides correspond to the notations in [4] and the right-hand sides correspond to our notations. The singularity removal conditions also reduce to the corresponding conditions for the black ring.

We got the single ring in the above limit by (effectively) removing the outer black ring. An exactly analogous construction can be done by removing the inner black ring. We have checked that this also results in a single black-ring solution as expected.

\[2\] Note that we also have to use expressions (3.30) and (3.31) to solve for $c_i$. 

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3.6. ADM mass and angular momentum

The ADM mass and angular momentum of the solution can be computed using the metric functions, extending our results on asymptotic flatness. The basic idea is to expand the metric functions in the coordinates defined in section 3.3, and to identify the mass and angular momentum from the fall-offs; see section 4.3 of [16]. One way to simplify the computation is to go to infinity along the direction \( \theta = \frac{\pi}{4} \) so that we can set \( z = 0 \). Keeping track of the leading and sub-leading terms, once the dust settles we end up with

\[
G M_{\text{ADM}} = \frac{3\pi}{16(a_1 - a_4)^2} (4(a_1 - a_4)^2(a_6 - a_4 + a_5 - a_1) + (a_2 - a_1)(a_4 - a_2)(a_5 - a_1)(a_5 - a_4)c_1c_2),
\]

(3.39)

\[
G J_{\text{ADM}} = \pi \frac{(a_2 - a_1)(a_5 - a_1)c_1 + (a_4 - a_2)(a_5 - a_4)c_2}{2(a_4 - a_1)}.
\]

(3.40)

Considering the formidable form of the di-ring metric, one might get the impression that these expressions are rather simple. But one should remember that the conical deficit constraints and the relations relating \( c_i \) to \( a_i \) are yet to be applied to these relations, and this can only be done numerically. In this sense, the di-ring solution is more complicated than the Saturn solution.

It is intuitively clear from the expression for \( J \) that the choice of sign of \( c_i \) is directly related to the direction of rotation of each ring. In terms of the scale \( L \) that we introduced, \( GM \sim L^2 \), while \( GJ \sim L^3 \), which is expected both from general principles and also from the specific expressions obtained previously in the literature, e.g., for the case of the black Saturn.

It is also worth mentioning that the ADM mass presented above is manifestly positive (as it should be) as an automatic consequence of the ordering of the solitons, at least when \( c_i \) have identical signs. When \( c_i \) have opposite signs there is a tension between the two terms about the sign, and one needs to take care of the constraint equations as well, in determining the overall sign. Even when \( c_i \) have opposite signs, we have not found solutions with negative ADM mass. This might be an indication that there are not any such pathological solutions. But we have not searched systematically, so in the case such unacceptable solutions do exist, they will have to be ruled out by fiat\(^3\). It is plausible that mutual frame dragging between the rings puts constraints on allowed solutions when they are counter-spinning. A similar choice-of-sign restriction arose also in the case of the black Saturn.

4. Discussion

The purpose of this paper was to present a derivation of the black di-ring using the inverse scattering method. In this concluding section, we make some comments about our approach and about the di-ring solution.

The implementation of the inverse scattering method adopted here for the construction of the di-ring differs from the approach used in [11] for the construction of some other axially symmetric solutions. There, the condition on the determinant (2.2) was imposed by demanding that the solitonic transformations be limited to a \( 2 \times 2 \) block, and then renormalizing (2.14) appropriately. Instead, we keep the transformations general, following the idea presented in [10, 4]. The advantage of this approach is that it is sufficiently general to allow the possibility

\(^3\) Note that this extra restriction does not affect our previous general arguments on the existence of solutions, because this consideration applies only for a sub-sector of the solutions allowed by the constraints imposed by the absence of singularities.
of constructing more complicated axi-symmetric vacuum solutions: we hope to return to the construction of some of these generalizations in the future.

The black di-ring solution that we found is somewhat more complicated in its final form than the Saturn solution. This is expected, since the hole there is replaced here with another ring, and the latter is a more complicated object. Still, we found that the solution can be brought to a form that is numerically as tractable as the Saturn itself. This opens up the possibility of exploring questions regarding higher-dimensional black holes in the context of the black di-ring. One could also investigate the physics and thermodynamics of the di-ring solution. Similar analyses have been done for the black Saturn, where effects like frame dragging were explicitly checked. It would be interesting to see if there exists a parameter range where the two rings in our solution can be in thermodynamic equilibrium; see [5, 17]. Related questions are under investigation.

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