Invariance of interaction terms in new representation of self-dual electrodynamics

B.M. Zupnik

Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow Region, 141980, Russia; e-mail: zupnik@thsun1.jinr.ru

Abstract

A new representation of Lagrangians of 4D nonlinear electrodynamics is considered. In this new formulation, in parallel with the standard Maxwell field strength $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$, an auxiliary bispinor field $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$ is introduced. The gauge field strength appears only in bilinear terms of the full Lagrangian, while the interaction Lagrangian $E$ depends on the auxiliary fields, $E = E(V^2, \bar{V}^2)$. Two types of self-duality inherent in the nonlinear electrodynamics models admit a simple characterization in terms of the function $E$. The continuous $SO(2)$ duality symmetry between nonlinear equations of motion and Bianchi identities amounts to requiring $E$ to be a function of the $SO(2)$ invariant quartic combination $V^2 \bar{V}^2$. The discrete self-duality (or self-duality under Legendre transformation) amounts to a weaker condition $E(V^2, \bar{V}^2) = E(-V^2, -\bar{V}^2)$. This approach can be generalized to a system of $n$ Abelian gauge fields exhibiting $U(n)$ duality. The corresponding interaction Lagrangian should be $U(n)$ invariant function of $n$ bispinor auxiliary fields.

1 Introduction

This talk is based on our paper [1] (see, also Sec.3 of ref.[2]). It is well known that the on-shell $SO(2)$ ($U(1)$) duality invariance of Maxwell equations can be generalized to the whole class of the nonlinear electrodynamics models, including the famous Born-Infeld theory. The condition of $SO(2)$ duality can be formulated as a nonlinear differential equation for the Lagrangian of these theories [3, 4, 5]. This self-duality equation is equivalent to the known Courant-Gilbert equation, and its general solution can be reduced to an algebraic equation for the auxiliary real scalar variable and depends on an arbitrary real function of this variable [3, 6].

Note that only few examples of the exact solutions for self-dual Lagrangians $L(F_{mn})$ are known. The general self-dual solution can be analyzed in the representation with an additional auxiliary scalar field [6], however, the symmetry properties of this representation have not been studied.

We shall consider the new representation of self-dual electromagnetic Lagrangians using the auxiliary bispinor (tensor) fields which transform linearly with respect to the $U(1)$ duality group. In this representation, the self-duality condition is equivalent to the $U(1)$-invariance of the nonlinear self-interaction of auxiliary fields, while the bilinear free Lagrangian is not invariant. The Lagrangian involving only the Maxwell field strengths emerges as a result of eliminating the bispinor auxiliary fields by their algebraic equations.
of motion. More general nonlinear electrodynamics Lagrangians respecting the so-called discrete self-duality (or duality under Legendre transformation) also admit a simple characterization in terms of the self-interaction of the bispinor auxiliary fields. In this case it should be invariant with respect to a discrete self-duality transformation.

The linearly transforming bispinor auxiliary fields are very useful in the analysis of $U(n)$ duality-invariant systems of $n$ Abelian gauge fields. The corresponding Lagrangian is fully specified by the interaction term which is an $U(n)$ invariant function of $n$ auxiliary fields. The discrete self-dualities also amount to a simple restriction on the interaction function.

## 2 Self-dualities in nonlinear electrodynamics

Let us introduce the spinor notation for the Maxwell field strengths $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$ and the following the Lorentz-invariant complex variables:

$$\varphi = F^{\alpha\beta}F_{\alpha\beta}, \quad \bar{\varphi} = \bar{F}^{\dot{\alpha}\dot{\beta}}\bar{F}_{\dot{\alpha}\dot{\beta}}.$$  

In this representation, two independent invariants which one can construct out of the Maxwell field strength in the standard vector notation take the following form:

$$F^{mn}F_{mn} = 2(\varphi + \bar{\varphi}),$$

$$F_{mn}\bar{F}^{mn} = -2i(\varphi - \bar{\varphi}).$$

Here

$$F_{mn} = \partial_m A_n - \partial_n A_m, \quad \bar{F}^{mn} = \frac{1}{2}\varepsilon^{mnpq}F_{pq}.$$  

It will be convenient to deal with dimensionless $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$ and $\varphi, \bar{\varphi}$, introducing a coupling constant $f$, $[f] = 2$. Then the generic nonlinear Lagrangian is proportional to $f^{-2}$ and contains the dimensionless function

$$L(\varphi, \bar{\varphi}) = -\frac{1}{2}(\varphi + \bar{\varphi}) + L_{int}(\varphi, \bar{\varphi})$$

where $L_{int}(\varphi, \bar{\varphi})$ collects all possible self-interaction terms of higher-order in $\varphi, \bar{\varphi}$.

We shall use the following notation for the derivatives of the Lagrangian $L(\varphi, \bar{\varphi})$

$$P_{\alpha\beta}(F) \equiv i\partial L/\partial F^{\alpha\beta} = 2iF_{\alpha\beta}L_{\varphi},$$

$$L_{\varphi} = \partial L/\partial \varphi, \quad L_{\bar{\varphi}} = \partial L/\partial \bar{\varphi}.$$  

The nonlinear equations of motion have the following form in this representation:

$$\mathcal{E}_{\alpha\dot{\alpha}}(F) \equiv \partial_{\alpha}^{\dot{\alpha}}\bar{P}_{\dot{\alpha}\dot{\beta}}(F) - \partial_{\dot{\alpha}}^{\dot{\alpha}}P_{\alpha\beta}(F) = 0,$$

where $\partial_{\alpha\dot{\alpha}} = (\sigma^{\mu})_{\alpha\dot{\alpha}}\partial_{\mu}$. These equations, together with the Bianchi identities

$$\mathcal{B}_{\alpha\dot{\alpha}}(F) \equiv \partial_{\alpha}^{\dot{\alpha}}\bar{F}_{\dot{\alpha}\dot{\beta}} - \partial_{\dot{\alpha}}^{\dot{\alpha}}F_{\alpha\beta} = 0,$$

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constitute a set of first-order equations in which one can treat \( F_{\alpha\beta} \) and \( \bar{F}_{\dot{\alpha}\dot{\beta}} \) as unconstrained conjugated variables.

This set is said to be duality-invariant if the Lagrangian \( L(\varphi, \bar{\varphi}) \) satisfies certain nonlinear condition [3, 4, 5]. The precise form of this self-duality condition in the spinor notation is

\[
S[F, P(F)] \equiv F_{\alpha\beta}F_{\alpha\beta} + P_{\alpha\beta}P_{\alpha\beta} - \text{c.c.} = \varphi - \bar{\varphi} - 4[\varphi(L\varphi)^2 - \bar{\varphi}(L\bar{\varphi})^2] = 0. \tag{7}
\]

Using this condition, one can define the nonlinear transformations

\[
\delta\omega F_{\alpha\beta} = \omega P_{\alpha\beta}(F) \equiv 2i\omega F_{\alpha\beta}L\varphi, \quad \delta\omega P_{\alpha\beta}(F) = -\omega F_{\alpha\beta} \tag{8}
\]

where \( \omega \) is a real parameter. The set of equations (3), (6) is clearly invariant under these transformations.

Although the Lagrangian \( L(\varphi, \bar{\varphi}) \) satisfying (7) is not invariant with respect to transformation (8), one can still construct the \( SO(2) \) invariant function

\[
\delta\omega[L + \frac{i}{2}(P_{\alpha\beta}F^{\alpha\beta} - \bar{P}_{\dot{\alpha}\dot{\beta}}\bar{F}^{\dot{\alpha}\dot{\beta}})] = 0. \tag{9}
\]

The self-duality condition is equivalent to the following relations for the \( SO(2) \) variation of the Lagrangian:

\[
\delta\omega L \equiv \delta F_{\alpha\beta} \frac{\partial L}{\partial F_{\alpha\beta}} + \delta \bar{F}_{\dot{\alpha}\dot{\beta}} \frac{\partial L}{\partial \bar{F}_{\dot{\alpha}\dot{\beta}}} = -i\omega(P^2 - \bar{P}^2)
\]

\[
= \frac{i}{2}\delta\omega(P_{\alpha\beta}F^{\alpha\beta} - \bar{P}_{\dot{\alpha}\dot{\beta}}\bar{F}^{\dot{\alpha}\dot{\beta}}) = -i\omega(F^2 - \bar{F}^2). \tag{10}
\]

We shall consider the discrete self-duality condition as an invariance of the Lagrangian with respect to the Legendre transforms

\[
L(\varphi, \bar{\varphi}) \Rightarrow L(P^2, \bar{P}^2) = L(\varphi, \bar{\varphi}) + i(F^{\alpha\beta}P_{\alpha\beta} - \bar{F}^{\dot{\alpha}\dot{\beta}}\bar{P}_{\dot{\alpha}\dot{\beta}}) \tag{11}
\]

\[
F_{\alpha\beta} \Rightarrow P_{\alpha\beta} = i\partial L/\partial F^{\alpha\beta}, \quad \bar{F}_{\dot{\alpha}\dot{\beta}} \Rightarrow \bar{P}_{\dot{\alpha}\dot{\beta}}.
\]

It is easy to show that the \( SO(2) \) duality condition (7) guarantees the self-duality under Legendre transformation (see, e.g., [5]).

3 Linearly transforming auxiliary fields in a new representation of nonlinear electrodynamics

The recently constructed \( N = 3 \) supersymmetric extension of the Born-Infeld theory [2] suggests a new representation for the actions of nonlinear electrodynamics discussed in the previous Section.
Let us consider the auxiliary fields $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$ and the following generalization of the free Lagrangian:

$$\mathcal{L}_2(V, F) = \nu \bar{\nu} - 2 (V_{\alpha\beta} F_{\alpha\beta} + \bar{V}_{\dot{\alpha}\dot{\beta}} \bar{F}_{\dot{\alpha}\dot{\beta}}) + \frac{1}{2} (\varphi + \bar{\varphi}) ,$$

(12)

where $\nu \equiv V_{\alpha\beta} V^{\alpha\beta}$ and $\bar{\nu} \equiv \bar{V}_{\dot{\alpha}\dot{\beta}} \bar{V}^{\dot{\alpha}\dot{\beta}}$.

Eliminating $V_{\alpha\beta}$ by its algebraic equation of motion,

$$V_{\alpha\beta} = F_{\alpha\beta}, \quad \bar{V}_{\dot{\alpha}\dot{\beta}} = \bar{F}_{\dot{\alpha}\dot{\beta}},$$

(13)

we arrive at the free Maxwell Lagrangian.

Our aim will be to find a nonlinear extension of the free Maxwell Lagrangian using $L_2(V, F)$, such that this extension becomes the generic nonlinear Lagrangian $L^2(F^2, \bar{F}^2)$, eq. (3), after eliminating the auxiliary fields $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$ by their algebraic (nonlinear) equations of motion.

By Lorentz covariance, such a nonlinear Lagrangian has the following general form:

$$\mathcal{L}[V, F(A)] = \mathcal{L}_2[V, F(A)] + E(\nu, \bar{\nu}) ,$$

(14)

where $E$ is a real function encoding self-interaction. Varying the action with respect to $V_{\alpha\beta}$, we derive the algebraic relation between $V$ and $F(A)$ in this formalism

$$F_{\alpha\beta}(A) = V_{\alpha\beta}(1 + E_{\nu}) \quad \text{and c.c.} ,$$

(15)

where $E_{\nu} \equiv \partial E(\nu, \bar{\nu})/\partial \nu$. This relation is a generalization of the free equation (13) and it can be used to eliminate the auxiliary variable $V_{\alpha\beta}$ in terms of $F_{\alpha\beta}$ and $\bar{F}_{\dot{\alpha}\dot{\beta}}$. $V_{\alpha\beta} \Rightarrow V_{\alpha\beta}[F(A)]$ (see eq. (18) below). The second equation of motion in this representation, obtained by varying (14) with respect to $A_{\alpha\dot{\alpha}}$, has the form

$$\partial_{\dot{\alpha}}[F_{\alpha\beta}(A) - 2V_{\alpha\beta}] + \text{c.c.} = 0 .$$

(16)

After substituting $V_{\alpha\beta} = V_{\alpha\beta}[F(A)]$ from (15), eq. (16) becomes the dynamical equation for $F_{\alpha\beta}(A), \bar{F}_{\dot{\alpha}\dot{\beta}}(A)$ corresponding to the generic Lagrangian (3). Comparing (16) with (5) yields the relation

$$P_{\alpha\beta}(F) \equiv -2iF_{\alpha\beta}L_{\varphi} = i [F_{\alpha\beta} - 2V_{\alpha\beta}(F)] ,$$

(17)

$$V_{\alpha\beta}(F) = F_{\alpha\beta}G(\varphi, \bar{\varphi}) , \quad G = \frac{1}{2} - L_{\varphi} .$$

(18)

One should also add the relations:

$$G^{-1} = 1 + E_{\nu} , \quad \bar{G}^{-1} = 1 + E_{\bar{\varphi}} ,$$

(19)

which follow from (18).

A useful corollary of these formulas is the relation

$$\nu E_{\nu} = \frac{1}{4} \varphi (1 - 4L_{\varphi}^2) .$$

(20)
Until now we did not touch any issues related to self-dualities. A link with the consideration in the previous Section is established by Eq. (17) which relates the functions $P_{\alpha\beta}(F)$ and $V_{\alpha\beta}(F)$.

Substituting this into the $SO(2)$ duality condition (7) and making use of eq. (20) we find

\[
S[F, P(F)] = [F^{\alpha\beta} - V^{\alpha\beta}(F)]V_{\alpha\beta}(F) - c.c
= \nu E_\nu - \bar{\nu} E_{\bar{\nu}} = 0 .
\]  

(21)

The corresponding realization of the $SO(2)$ (or $U(1)$) transformations (8) in terms of $F_{\alpha\beta}$ and $V_{\alpha\beta}(F)$ is given by

\[
\delta_\omega V_{\alpha\beta} = -i\omega V_{\alpha\beta} ,
\]

(22)

\[
\delta_\omega F_{\alpha\beta} = i\omega [F_{\alpha\beta} - 2V_{\alpha\beta}] .
\]

(23)

It should be stressed that the duality transformation realizes linearly on the auxiliary fields of this representation.

It is important to emphasize that the new form (21) of the self-duality constraint (7) admits a transparent interpretation as the condition of invariance of $E(\nu, \bar{\nu})$ with respect to the $U(1)$ transformations (22)

\[
\delta_\omega E = 2i\omega(\bar{\nu}E_{\bar{\nu}} - \nu E_\nu) = 0 .
\]

(24)

The general solution of (21) is a function $\tilde{E}(a)$ which depends on the single real $U(1)$ invariant variable $a = \nu\bar{\nu}$ quartic in the auxiliary fields $V_{\alpha\beta}$ and $\bar{V}_{\dot{\alpha}\dot{\beta}}$

\[
E_{sd}(\nu, \bar{\nu}) = \tilde{E}(a) = \tilde{E}(\nu\bar{\nu}) .
\]

(25)

Thus we come to the notable result that the whole class of nonlinear extensions of the Maxwell action admitting the on-shell $SO(2)$ duality is parametrized by an arbitrary $SO(2)$ invariant real function of one argument $a = V^2\bar{V}^2$

\[
L_{sd}(F^2, \bar{F}^2) = \tilde{E}(a) + a(V^2 + \bar{V}^2) \left( \frac{d\tilde{E}}{da} \right)^2 .
\]

(26)

Finally, let us examine which restrictions on the interaction Lagrangian $E(\nu, \bar{\nu})$ are imposed by the requirement of the “discrete” self-duality with respect to the Legendre transform. For this we shall need a first-order representation of the Lagrangian (14)

\[
\hat{L}_2(V, P) \equiv L_2(V, F) + iF^{\alpha\beta}P_{\alpha\beta} - i\bar{F}^{\dot{\alpha}\dot{\beta}}\bar{P}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}(P^2 + \bar{P}^2) - V^2 - \bar{V}^2
- 2iP^\alpha V_{\alpha\beta} + 2i\bar{P}^{\dot{\alpha}\dot{\beta}}\bar{V}_{\dot{\alpha}\dot{\beta}} = L_2(iV, P) , \quad P_{\alpha\beta} = i(F_{\alpha\beta} - 2V_{\alpha\beta}) .
\]

(27)

Thus the Legendre transform of the quadratic Lagrangian $L_2(V, F)$ in the variables $F_{\alpha\beta}$ corresponds to the discrete $U(1)$ transformation $V_{\alpha\beta} \to i\bar{V}_{\dot{\alpha}\dot{\beta}} \equiv \bar{U}_{\dot{\alpha}\dot{\beta}}$. The Legendre transform of the whole Lagrangian (14) has the following form:

\[
\hat{L}(V, P) = L_2(U, P) + E(-U^2, -\bar{U}^2) .
\]

(28)
Comparing this dual Lagrangian with the original one (14), we observe that the necessary and sufficient condition of the discrete self-duality is the following simple restriction on the function $E$:

$$E(\nu, \bar{\nu}) = E(-\nu, -\bar{\nu}) .$$  (29)

### 4 $U(n)$ self-duality

Let us consider $n$ Abelian field-strengths

$$F^i_{\alpha\beta}, \quad \bar{F}^i_{\dot{\alpha}\dot{\beta}} ,$$  (30)

where $i = 1, 2 \ldots n$. As the first step, one can realize the group $SO(n)$ on these variables

$$\delta_\omega F^i_{\alpha\beta} = \xi^{ik} F^k_{\alpha\beta} , \quad \xi^{ki} = -\xi^{ik} .$$  (31)

This group is assumed to define an off-shell symmetry of the corresponding nonlinear Lagrangian $L(F^k, \bar{F}^k) = -\frac{1}{2}(F^i F^i) - \frac{1}{2}(\bar{F}^i \bar{F}^i) + L_{int}(F^k, \bar{F}^k)$.

The $U(n)$ self-duality conditions for the Lagrangian $L(F^k, \bar{F}^k)$ generalizing the $U(1)$ condition (1) have been analyzed in Refs.[7, 5]. In the spinor notation, these conditions read

$$A^{[kl]} = (F^k P^l) - (F^l P^k) - \text{c.c.} = 0 ,$$  (32)

$$S^{(kl)} = (F^k \bar{P}^l) + (P^k \bar{F}^l) - \text{c.c.} = 0 ,$$  (33)

where

$$P^k_{\alpha\beta}(F) \equiv i \frac{\partial L}{\partial F^k_{\alpha\beta}} , \quad \bar{P}^k_{\dot{\alpha}\dot{\beta}} \equiv -i \frac{\partial L}{\partial \bar{F}^k_{\dot{\alpha}\dot{\beta}}} ,$$  (34)

$$(F^k P^l) \equiv F^{k\alpha\beta} P^l_{\alpha\beta} , \quad (\bar{F}^k \bar{P}^l) \equiv \bar{F}^{k\dot{\alpha}\dot{\beta}} \bar{P}^l_{\dot{\alpha}\dot{\beta}} \text{ etc} .$$

The condition (32) amounts to the $SO(n)$ invariance of the Lagrangian and holds off shell. The second condition is the true analog of (4). It guarantees the covariance of the equations of motion for $F^k_{\alpha\beta}, \bar{F}^k_{\dot{\alpha}\dot{\beta}}$ together with Bianchi identities under the following nonlinear transformations:

$$\delta_\eta F^k_{\alpha\beta} = -\eta^{kl} P^l_{\alpha\beta}(F) ,$$  (35)

$$\delta_\eta \bar{F}^k_{\dot{\alpha}\dot{\beta}}(F) = \eta^{kl} \bar{F}^l_{\dot{\alpha}\dot{\beta}}$$

where $\eta^{kl} = \eta^{lk}$ are real parameters. On the surface of the condition (33) these transformations, together with (31), form the group $U(n)$. The particular solution of the $U(n)$ self-duality conditions (32),(33) constructed so far [7].

The $U(n)$ group structure becomes manifest after passing to new auxiliary variables:

$$V^k_{\alpha\beta}(F) \equiv \frac{1}{2} [F^k_{\alpha\beta} + i P^k_{\alpha\beta}(F)] , \quad \delta V^k_{\alpha\beta}(F) = \omega^{kl} V^l_{\alpha\beta}(F) ,$$  (36)

$$\omega^{kl} = \xi^{kl} + i \eta^{kl} , \quad \bar{\omega}^{kl} = -\omega^{kl} .$$

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Passing to an analog of the $V, F$ representation in the $U(n)$ case will allow us to find the general solution to (32), (33).

Let us define a new representation for the $SO(n)$ invariant nonlinear electrodynamics Lagrangians in terms of the Abelian gauge field strengths $F^k_{\alpha\beta}(A^k)$ and auxiliary fields $V^k_{\alpha\beta}$

$$L(V^k, F^k) = (V^k V^k) + (\bar{V}^k \bar{V}^k) - 2(V^k F^k) - 2(\bar{V}^k \bar{F}^k) + \frac{1}{2}(F^k F^k) + \frac{1}{2}(\bar{F}^k \bar{F}^k) + E(V^k, \bar{V}^k).$$  \hspace{1cm} (37)

The real Lagrangian of interaction $E(V, \bar{V})$ is $SO(n)$ invariant by definition.

In the general case of $E \neq 0$ the algebraic equation for $V_{\alpha\beta}$ is

$$F^k_{\alpha\beta} = V^k_{\alpha\beta} + \frac{1}{2} \frac{\partial E}{\partial V^k_{\alpha\beta}}.$$  \hspace{1cm} (38)

Using the relation

$$P^k_{\alpha\beta} = i[F^k_{\alpha\beta} - 2V^k_{\alpha\beta}],$$  \hspace{1cm} (39)

one can rewrite the $U(n)$ self-duality conditions (32) and (33) in this representation as follows

$$i(F^l V^k) - i(F^k V^l) - \text{c.c.} = 0,$$

$$F^l (V^k) + (F^k V^l) - 2(V^k V^l) - \text{c.c.} = 0.$$  \hspace{1cm} (40)

One can readily show that, after making use of the relation (38), these conditions can be brought into the form quite similar to the $U(1)$ self-duality condition (21)

$$V^k_{\alpha\beta} \frac{\partial E}{\partial V^l_{\alpha\beta}} - \bar{V}^k_{\alpha\beta} \frac{\partial E}{\partial \bar{V}^l_{\alpha\beta}} = 0.$$  \hspace{1cm} (41)

This constraint is none other than the condition of invariance of $E(V, \bar{V})$ with respect to the $U(n)$ transformations (36). The simplest example of the $U(n)$ self-dual action is defined by the quartic interaction of the auxiliary fields

$$E = (V^k V^l)(\bar{V}^k \bar{V}^l), \quad F^k_{\alpha\beta} = V^k_{\alpha\beta} + V^l_{\alpha\beta}(\bar{V}^k \bar{V}^l).$$  \hspace{1cm} (42)

The corresponding non-polynomial Lagrangian $L(F^k, \bar{F}^k)$ can be constructed from (37) using the solution $V^k(F, \bar{F})$ of the last algebraic equation.

Finally, let us notice that the condition of “discrete” self-duality in the general case is as follows

$$E(V^k_{\alpha\beta}, \bar{V}^k_{\alpha\beta}) = E(iV^k_{\alpha\beta}, -i\bar{V}^k_{\alpha\beta}).$$  \hspace{1cm} (43)

It is an obvious generalization of the $n = 1$ condition (28).

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References

[1] E.A. Ivanov and B.M. Zupnik, *New representation for Lagrangians of self-dual nonlinear electrodynamics*, Proceedings of XVI Max Born Symposium “Supersymmetries and quantum symmetries”, eds. E. Ivanov et al, p. 235, Dubna (2002); hep-th/0202203.

[2] E.A. Ivanov and B.M. Zupnik, *N=3 supersymmetric Born-Infeld theory*, Nucl. Phys. B 618 (2001) 3; hep-th/0110074.

[3] M.K. Gaillard and B. Zumino, *Nonlinear electromagnetic self-duality and Legendre transformation*, Duality and Supersymmetric Theories, eds. D.I. Olive et al, p. 33, Cambridge University Press, 1999; hep-th/9712103.

[4] G.W. Gibbons and D.A. Rasheed, *Electric-magnetic duality rotations in non-linear electrodynamics*, Nucl. Phys. B 454 (1995) 185; hep-th/9506033.

[5] S.M. Kuzenko and S. Theisen, *Nonlinear self-duality and supersymmetry*, Fortsch. Phys. 49 (2001) 273; hep-th/0007231.

[6] M. Hatsuda, K. Kamimura and S. Sekia, *Electric-magnetic duality invariant Lagrangians*, Nucl.Phys. B 561 (1999) 341 ; hep-th/9906103.

[7] P. Aschieri, D. Brace, B. Morariu and B. Zumino, *Nonlinear selfduality in even dimensions*, Nucl. Phys. B 574 (2000) 551; hep-th/9909021.