Quasinormal modes of hairy black holes caused by gravitational decoupling

Yi Yang, Dong Liu, Zheng-Wen Long,† Zhaoyi Xu
College of Physics, Guizhou University, Guiyang, 550025, China
(Dated: March 23, 2022)

Quasinormal modes (QNM) of hairy black holes caused by gravitational decoupling for the electromagnetic field and the scalar field perturbation are investigated. We have derived the equation of motion and effective potential in hairy black holes spacetime. The time-domain profiles of electromagnetic field and scalar field are obtained by the finite difference method which is improved to make it applicable to hair black hole, which indicates damping becomes slower as angular quantum number $l$ increases and the period of oscillation increases as angular quantum number $l$ increases.

The QNM frequency is calculated by the Prony method and the high-order WKB method, and the results show that the QNM frequency obtained by the Prony method is in good agreement with the result of the high-order WKB.

PACS: 04.70.Bw, 04.70.Dy, 98.80.Jk, 04.90.+e

I. INTRODUCTION

The general relativity shows that when a massive stellar collapses into a black hole, there are only three physical quantities: mass, angular momentum, and electric charge, which uniquely determine the properties of the black hole. All other information ("hair") disappeared [1, 2]. However, there may be some other physical quantities that describe black holes. For example, black holes may also have quantum hairs [3]. Many methods are used to evade the no-hair theorem [4–7]. Ovalle assumes that there are additional general sources described by the conserved energy-momentum tensor $\theta_{\mu\nu}$ [8–12]. This $\theta_{\mu\nu}$ can explain one or more fundamental fields, and its key property is that it is subject to gravity but does not directly interact with the matter of the black hole. Then, they obtained the hair black hole solution by gravity decoupling [13, 14]. In order to understand this kind of hair black hole more deeply, we will study its quasinormal modes (QNM).

Perturbing spacetime allows us to see the internal information of the black hole [15]. Vishveshwara [16] pointed out that when the background spacetime of a black hole is perturbed, its initial perturbation will gradually be dominated by a damped oscillation with a certain frequency. The frequency and damping characteristics of this oscillation are only related to the spacetime properties of the black hole, and are irrelevant to the initial oscillation. This special oscillation that occupies most of the perturbation evolution of the black hole is called the QNM. However, it should be noted that QNM is only a stage in the perturbation evolution of black holes. The complete black hole perturbation evolution process also includes the initial wave explosion phase and the tail phase. In the initial wave burst phase, the oscillation characteristics are determined by the perturbation source, while the tail phase is caused by the backscattering caused by the perturbation at asymptotically infinity. Physicists usually discover the physical characteristics of black holes by studying the quasinormal modes of the perturbation [17–22]. Through continuous development, people have proposed many methods to calculate QNM. The Pöschl-Teller potential approximation method [23, 24] is an earlier method for calculating the black hole’s QNM, but only when the effective potential of the black hole is similar to the Pöschl-Teller potential can a more accurate result be obtained, so its scope of application limited. The finite difference method (FDM) [25, 26] is a method that can be used to study QNM in an asymptotically flat spacetime background, but there may be calculation errors in the calculation, resulting in inaccurate results. The WKB method [27–31] assumes the spacetime perturbation potential of the black hole as a potential barrier, and uses the WKB approximation method in quantum mechanics to process the QNM of the black hole. Because of its very high accuracy, it is an important tool for studying low-frequency QNM. The paper will improve FDM and use high-order WKB to study the QNM of hairy black holes.
II. SCALAR FIELD AND ELECTROMAGNETIC FIELD PERTURBATION IN HAIR BLACK HOLES SPACETIME

Ovalle et al. used gravitational decoupling by minimal geometric deformation (MGD) \cite{32–35} to give a spherically symmetric black hole with scalar hairs. This method has two characteristics: extending simple solutions to more complex fields and decoupling some complex sources of gravity. At the same time, they assume that the system has a well-defined event horizon and the conserved energy-momentum tensor describing the additional source satisfies the strong energy condition or dominant energy condition outside the event horizon \cite{14}. The metric has the form

\[ ds^2 = e^\nu(r) dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]

where

\[ e^\nu = e^{-\lambda} = 1 - \frac{2M}{r} + \alpha e^{-r/(M-\alpha l_0/2)}, \]

with the $M$ is an asymptotic mass $M = M + \alpha l_0/2$. This solution is presented by using the extended geometric deformation and strong energy condition. Moreover, using the dominant energy condition, the “charged” hair black hole can be read as \cite{14}

\[ e^\nu = e^{-\lambda} = 1 - \frac{2M + \alpha l_0}{r} + \frac{Q^2}{r^2} - \frac{\alpha Me^{-r/M}}{r}. \]

We will study the QNM of this hair black hole. We only focus on the evolution of scalar fields and electromagnetic fields in this paper. Then we will derive the radial equation of motion and effective potential. The general covariant equation of scalar field can be expressed as

\[ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) = 0, \]

and the electromagnetic field can be read as

\[ \frac{1}{\sqrt{-g}} \partial_\mu (F_{\rho\sigma} g^{\mu\nu} g^{\sigma\rho} \sqrt{-g}) = 0, \]

where $F_{\rho\sigma} = \partial_\rho A^\sigma - \partial_\sigma A^\rho$ and $A_\mu$ is a vector potential.

Let us first consider the scalar field. According to the space-time line elements of the hair black hole, we expand the KG equation to get

\[ -\frac{\partial^2 \Psi}{e^\nu} + \frac{1}{r^2} \left( 2re^{\nu} \partial_\nu \Psi + r^2 e^{\nu} \partial_\nu r \Psi + r^2 e^\nu \partial_r^2 \Psi \right) + \frac{1}{r^2} \left( \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta \Psi + \frac{1}{\sin^2 \theta} \partial_\theta^2 \Psi \right) = 0, \]

where $e^\nu$ denote $\frac{d}{dr} e^{\nu(r)}$. Due to the symmetry of the spacetime background of the black hole, the scalar field can be assumed to be

\[ \Psi = \frac{\psi(r)}{r} Y_{lm}(\theta, \phi)e^{-i\omega t}. \]

Inserting it into Eq. (6), we can get the following equation

\[ \frac{d^2 \psi}{d\tau^2} + (\omega^2 - V(r)) \psi = 0, \]

where $\tau$ is the so-called tortoise coordinate, which is defined as $d\tau = 1/e^{\nu} dr$. The effective potential can be written as

\[ V(r) = \left( 1 - \frac{2M + \alpha l_0}{r} + \frac{Q^2}{r^2} - \frac{\alpha Me^{-r/M}}{r} \right) \left( l(l+1) + \frac{e^{\nu(r)}}{r^2} \right). \]

As everyone knows, electromagnetic field perturbation can be divided into odd perturbation and even perturbation. But we find that the effective potentials of odd perturbation and even perturbation are the same. For the perturbation of the electromagnetic field, its equation of motion can also be derived into Eq. (8), but its effective potential function in the hair black hole spacetime is

\[ V(r) = \left( 1 - \frac{2M + \alpha l_0}{r} + \frac{Q^2}{r^2} - \frac{\alpha Me^{-r/M}}{r} \right) \left( l(l+1) \right). \]
III. QNM OF HAIR BLACK HOLES

In order to facilitate the numerical calculation of QNM, the light cone coordinates are introduced

\[ u = t - \tau, \]
\[ v = t + \tau, \]  

(11)

where \( u \) and \( v \) are integral constants. When \( r > 2GM \), \( u \) describes the radial inward movement of light, and \( v \) describes the radial outward movement of light. Using this coordinates, the Eq. (8) can be expressed as

\[ \frac{\partial^2}{\partial u \partial v} \psi(u, v) = -\frac{1}{4} V(r) \psi(u, v). \]  

(12)

The time evolution operator can be expressed as

\[ \exp \left( \frac{\hbar}{\Delta^2} \right) = \exp \left( \frac{\hbar}{\Delta^2} + \frac{\hbar}{\Delta^2} \right) = \exp \left( \frac{\hbar}{\Delta^2} + \frac{\hbar}{\Delta^2} \right) - 1 + \frac{\hbar^2}{2} \left( \exp \left( \frac{\hbar}{\Delta^2} \right) + \exp \left( \frac{\hbar}{\Delta^2} \right) \right) \frac{\partial^2}{\partial u \partial v} + O \left( \hbar^4 \right). \]  

(13)

Using this operator and Taylor’s theorem, the Eq. (12) can be written discretely as [25, 26]

\[ \psi_N = \psi_E + \psi_W - \psi_S - \delta u \delta v V \left( \frac{\psi_E + \psi_E}{8} \right) + O \left( \Delta^4 \right). \]  

(14)

The points \( S, W, E, N \) are defined as follows: \( S = (u, v), W = (u + \Delta u, v), E = (u, v + \Delta v), N = (u + \Delta u, v + \Delta v) \). We discover the fact that the damping and oscillation process of scalar field and electromagnetic field perturbation are not sensitive to initial conditions. We use a Gaussian pulse with a width \( \sigma \) at the point \((v_0, u_0)\) as the initial pulse. The center of this Gaussian pulse is at \( v_c \), and the field is set to zero at \( u = u_0 \) and \( v = v_0 \), namely

\[ \psi(u = u_0, v) = \exp \left[ -\frac{(v - v_c)^2}{2\sigma^2} \right], \]
\[ \psi(u, v = v_0) = 0. \]  

(15)

After discretizing the equation and setting the initial conditions, the four-point difference method can be used for numerical calculation. On the \( u - v \) plane, the next point is calculated through the first three known points, so that the calculation can be continued until the value of all points on the \( u - v \) plane is calculated.

As long as our grid is large enough, we can get a good approximate solution of the wave equation at the horizon, where the QNM carries the information of the hair black hole.

In Fig. 1 and Fig. 2, we show the evolution of the electromagnetic field and the scalar field in the hair black hole, where the black line represents the QNM of Schwarzschild black hole \((M = 1, l = 1)\), the red line and the green line represent the QNM of the hair black hole for different angular quantum number. In these figures, we can clearly see the three stages. The first is the initial wave burst stage, which is mainly related to the initial perturbation source; the second stage is the QNM stage, which has nothing to do with the initial disturbance and mainly reflects the property of hair black hole spacetime; the third stage is the tailing stage. Moreover, the QNM of hair black hole has a slower damping rate than the Schwarzschild black hole, and the oscillation and damping of the wave function have a strong dependence on the angular quantum number \( l \) for hair black hole. It can be seen that the damping of the wave function becomes slower as \( l \) increases. The period of oscillation increases as \( l \) increases.

Apart from studying the time-domain profile, we also studied the QNM frequency. We use the Prony method to obtain the QNM frequency by the damped exponents [15, 36]

\[ \phi(t) \simeq \sum_{i=1}^{p} C_i e^{-i\omega_i t}. \]  

(16)

Meanwhile, we also used the 6th order WKB and 13th order WKB [37–39] to calculate the QNM frequency. We have listed the value of QNM frequency in Table I and Table II. The result of Table I is uncharged hair black hole for electromagnetic fields (EF) and scalar fields (SF), where \( \alpha = 0.4, M = 1, l_0 = 0.9 \). The result of Table II is charged hair black hole for electromagnetic fields (EF) and scalar fields (SF), where
FIG. 1: The QNM of the electromagnetic fields (left panel) and scalar fields (right panel) in the uncharged hairy black hole for different angular quantum number $l$ with $\alpha = 0.4, M = 1, l_0 = 0.9$.

FIG. 2: The QNM of the electromagnetic fields (left panel) and scalar fields (right panel) in the charged hairy black hole for different angular quantum number $l$ with $\alpha = 0.8, M = 1, l_0 = 1$.

| Fields | $l$ | Prony method | 6th order WKB | 13th order WKB |
|--------|----|--------------|---------------|---------------|
| EF     | 1  | 0.215866 − 0.077032i | 0.215711 − 0.0779736i | 0.215895 − 0.0777665i |
|        | 2  | 0.393312 − 0.080364i | 0.396080 − 0.0795818i | 0.396087 − 0.0795700i |
|        | 3  | 0.568753 − 0.082243i | 0.567928 − 0.0799969i | 0.567929 − 0.0799951i |
|        | 4  | 0.736743 − 0.082201i | 0.737201 − 0.0801633i | 0.737201 − 0.0801629i |
| SF     | 1  | 0.253824 − 0.087290i | 0.254604 − 0.0821300i | 0.254671 − 0.0820359i |
|        | 2  | 0.410575 − 0.083196i | 0.418750 − 0.0810530i | 0.418755 − 0.0810433i |
|        | 3  | 0.584903 − 0.082361i | 0.584015 − 0.0807452i | 0.584016 − 0.0807436i |
|        | 4  | 0.749507 − 0.082758i | 0.749681 − 0.0806157i | 0.749681 − 0.0806153i |

$\alpha = 0.8, M = 1, l_0 = 1$. The results show that the real and imaginary parts of QNM frequencies of hair black hole in the electromagnetic field increase with the increase of the angular quantum number. The real part of the scalar field increases with the increase of the angular quantum number, and the imaginary part decreases with the increase of the angular quantum number. This indicates that the scalar field damping more slowly than the electromagnetic field. It is interesting to note that the results of the Prony method are highly consistent with the results of higher-order WKB.
TABLE II: QNM frequencies of charged hairy black hole.

| Fields | l | Prony method | 6th order WKB | 13th order WKB |
|--------|---|--------------|---------------|---------------|
| EF     | 1 | 0.178213 − 0.0677685i | 0.178041 − 0.0674203i | 0.178179 − 0.0671907i |
|        | 2 | 0.329990 − 0.067544i | 0.328922 − 0.069094i | 0.328987 − 0.0690826i |
|        | 3 | 0.473809 − 0.067802i | 0.472461 − 0.0695438i | 0.472461 − 0.0695422i |
|        | 4 | 0.615197 − 0.067736i | 0.613667 − 0.0697259i | 0.613667 − 0.0697255i |
| SF     | 1 | 0.211926 − 0.0694625i | 0.210394 − 0.0712911i | 0.210526 − 0.0709812i |
|        | 2 | 0.348861 − 0.0687166i | 0.347823 − 0.0703753i | 0.34783 − 0.0703592i |
|        | 3 | 0.487107 − 0.0683673i | 0.485814 − 0.0701851i | 0.485815 − 0.0701832i |
|        | 4 | 0.625578 − 0.0680728i | 0.624020 − 0.0701113i | 0.624020 − 0.0701108i |

IV. SUMMARY

In summary, we study the QNM of hairy black holes caused by gravitational decoupling. By studying its perturbation in the case of electromagnetic field and scalar field perturbation, its time-domain profiles of QNM are calculated by our improved FDM, and the QNM frequency of this black hole is fitted according to the time-domain profiles, which are consistent with the results of the high-order WKB. The improved FDM is suitable for hair black holes. This technique may be applied to other black holes. For hairy black holes caused by gravitational decoupling, the evolution of QNM has similar behaviors in both the charged and uncharged hair black hole. QNM decreases with the increase of angular quantum number $l$. The QNM of a charged hair black hole has a lower frequency than the uncharged case. We find that the hair black hole has a slower damping rate compared to Schwarzschild black hole and the hair black hole can degenerate to a Schwarzschild black hole when $\alpha = 0$ and $Q = 0$. We expect our results to provide some direction for observing hairy black holes by gravitational decoupling in future experiments. As our work was nearing completion, we discovered an extremely valuable and similar work [40]. But we believe our work still has importance for gravitational decoupling theory.

Acknowledgments

This research was funded by the National Natural Science Foundation of China (Grant No.11465006 and 11565009) and the Natural Science Special Research Foundation of Guizhou University (Grant No.X2020068).

---

[1] R. Ruffini and J. A. Wheeler, Phys. Today 24, 30 (1971)
[2] S. Hawking, Commun. Math. Phys. 25, 152 (1972)
[3] S. W. Hawking, M. J. Perry, and A. Strominger, Phys. Rev. Lett. 116, 231301 (2016)
[4] K. G. Zloshchastiev, Phys. Rev. Lett. 94, 121101 (2005)
[5] T. P. Sotiriou and V. Faraoni, Phys. Rev. Lett. 108, 081103 (2012)
[6] A. Cisterna and C. Erces, Phys. Rev. D 89, 084038 (2014)
[7] G. Antoniou, A. Bakopoulos, and P. Kanti, Phys. Rev. Lett. 120, 131102 (2018)
[8] R. Casadio, J. Ovalle, Phys. Lett. B, 715, 251 (2012)
[9] J. Ovalle, F. Linares, Phys. Rev. D, 88, 104026 (2013)
[10] J. Ovalle, F. Linares, A. Pasqua, A. Sotomayor, Class. Quantum Grav., 30, 175019 (2013)
[11] J. Ovalle, L. A. Gergely, R. Casadio, Class. Quantum Grav., 32, 045015 (2015)
[12] J. Ovalle, Phys. Rev. D 95, 104019 (2017)
[13] J. Ovalle, R. Casadio, R. d. Rocha, A. Sotomayor and Z. Stuchlik, Eur. Phys. J. C, 78, 960 (2018)
[14] J. Ovalle, R. Casadio, E. Contreras and A. Sotomayor, Phys. Dark Univ. 31, 100744 (2021)
[15] R. A. Konoplya and A. Zhidenko, Rev. Mod. Phys. 83, 793 (2011)
[16] C. V. Vishveshwara, Nature 227, 936 (1970)
[17] J. Li, M. Hong and K. Lin, Phys. Rev. D 88, 064001 (2013)
[18] J. Li, K. Lin and N. Yang, Eur. Phys. J. C, 75, 131 (2015)
[19] H. Ma and J. Li, Chin. Phys. C 44, 095102 (2020)
[20] D. J. Liu, B. Yang, Y. J. Zhai and X. Z. Li, Class. Quant. Grav. 29, 145009 (2012)
[21] S. B. Chen and J. L. Jing, Class. Quant. Grav. 22, 533 (2005)
[22] S. B. Chen and J. L. Jing, Class. Quant. Grav. 22, 1129 (2005)
[23] H. J. Blome and B. Mashhoon, Phys. Lett. A 110, 231 (1984).
[24] G. Pöschl, E. Teller, Z. Physik 83, 143 (1933).
[25] C. Gundlach, R. H. Price, and J. Pullin, Phys. Rev. D 49, 883 (1994).
[26] C. Gundlach, R. H. Price, and J. Pullin, Phys Rev D 49, 890 (1993).
[27] S. Iyer and C. M. Will, Phys. Rev. D 35, 3621 (1987)
[28] B. F. Schutz and C. M. Will, Astrophys. J. Lett. 291, L33 (1985).
[29] A. Zhidenko, Class. Quant. Grav. 21, 273 (2004)
[30] K. D. Kokkotas and B. F. Schutz, Phys. Rev. D 37, 3378 (1988)
[31] R. A. Konoplya, Phys. Rev. D 68, 024018 (2003)
[32] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999)
[33] L. Randall and R. Sundrum, Phys. Rev. Lett 83, 4690 (1999)
[34] R. Casadio, J. Ovalle, R. da Rocha, Class. Quantum Grav. 32, 215020 (2015)
[35] J. Ovalle, Int. J. Mod. Phys. Conf. Ser. 41, 1660132 (2016)
[36] E. Berti, V. Cardoso, J. A. Gonzalez and U. Sperhake, Phys. Rev. D 75, 124017 (2007)
[37] R. A. Konoplya, A. Zhidenko and A. F. Zinhailo, Class. Quant. Grav. 36, 155002 (2019)
[38] R. A. Konoplya, A. Zhidenko,Phys. Rev. D 81, 124036 (2010)
[39] J. Matyjasek and M. Opala, Phys. Rev. D 96, 024011 (2017)
[40] R. T. Cavalcanti, R. C. de Paiva and R. da Rocha, [arXiv:2203.08740 [gr-qc]]