Research Article

Predicting Nonlinear Stiffness, Motion Range, and Load-Bearing Capability of Leaf-Type Isosceles-Trapezoidal Flexural Pivot Using Comprehensive Elliptic Integral Solution

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A leaf-type isosceles-trapezoidal flexural (LITF) pivot consists of two leaf springs located in the same plane [1]. The two leaf springs are arranged symmetrically and intersect at a virtual center of motion outside the pivot, as shown in Figure 1. The parallelogram flexure is a special type in which the two leaf springs intersect at infinity. LITF pivots have been utilized in many accurate mechanisms [2–4] due to their obvious advantages such as low cost, monolithic manufacturing, reduced weight, and smooth motion [5–7].

When delivering movement, the leaf springs of a LITF pivot undergo nonlinear large deflection that may carry one or two inflection points (where the resultant moment is equal to zero [8]), which complicates the accurate modeling of LITF pivots. The remote center location and stiffness of a LITF pivot are presented by the model of screw theory based on the small-deflection assumption [9], which limits the application range of the model. The two pseudorigid-body models with small-deflection assumption [10], i.e., a four-bar model and a pin-joint model, were proposed for the analysis of the moment-angle characteristics of LITF pivots subject to horizontal force and moment. However, the influence of vertical force to LITF pivots was neglected. The analytic models for stiffness and center shift were presented by using the beam constraint model (BCM) method [11], which can be used to solve the nonlinear characteristics of LITF pivots, i.e., the rotation angle is in the range of \(\pm 15^\circ\).

The efficiency of uniform-strength composite leaf springs under various loading conditions [12] was analyzed. Therefore, the accurate nonlinear analysis and load-bearing capability solution within the entire stress range are indispensable for the application of LITF pivots.

Because the leaf spring is so thin and flexible that the effects on axial elongation and shear are negligible, the elliptic integral solution is often considered to be the most accurate model for large deflection beams. Howell [13] presented the elliptic integral solutions for the large deflection beam with no inflection point. An elliptic integral solution for the beam with an inflection

1. Introduction

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Because the leaf spring is so thin and flexible that the effects on axial elongation and shear are negligible, the elliptic integral solution is often considered to be the most accurate model for large deflection beams. Howell [13] presented the elliptic integral solutions for the large deflection beam with no inflection point. An elliptic integral solution for the beam with an inflection
point was derived by Kimball and Tsai [14]. In our previous research [15], we developed the comprehensive elliptic integral solution to solve the large deflections of beams with multiple inflection points and subject to any kinds of load cases. Because each of the deflected leaf spring carry one or two inflection points, LITF pivots can be modeled by the comprehensive elliptic integral solution. The model can be used to solve the exact deflected shapes and nonlinear stiffness of LITF pivots subject to different loads. Through the stress analysis for deflected leaf springs, the maximum motion range and allowable loads of LITF pivots are solved.

The rest of this paper is organized as follows. In Section 2, the accurate kinetostatic model and stress check for LITF pivots are proposed. In Section 3, two examples are calculated to demonstrate the accuracy of the model for LITF pivots. The nonlinear stiffness and workspace evaluation of the two examples are then discussed. In Section 4, concluding remarks are presented.

2. Modeling

2.1. LITF Pivot. As shown in Figure 2, two springs (O1A and O2A, length L) of a LITF pivot intersect at point O and the angle between two leaf springs is 2β. The lengths of O1O2 and AB are w1 and w2, respectively. Letting N = w2/w1, when N ≠ 1, we then have

\[
\begin{align*}
\frac{w_1}{N-1} &= \frac{2L \sin \beta}{N-1}, \\
\frac{w_2}{N-1} &= \frac{2NL \sin \beta}{N-1}.
\end{align*}
\]

When N = 1, the LITF pivot becomes a parallelogram flexure, as shown in Figure 2(c), β = 0° and w1 = w2.

The global coordinate system OXY is established for the LITF pivot with the X axis oriented along O2O1 and the origin located at the midpoint of O2O1, as shown in Figure 2. The initial angle between leaf spring O1A and the X axis is θ1, and the angle between leaf spring O2B and the X axis is θ2. For N < 1, θ1 = 90° - β and θ2 = 90° + β. For N > 1, θ1 = 90° - β and θ2 = 90° + β.

The local coordinate systems O1X1Y1 and O2X2Y2 for leaf springs O1A and O2B are established with the origins placed at the fixed end and the X1 and X2 axes oriented along the leaf springs, respectively. The deflected end coordinates and the end angle of spring O1A with respect to the local coordinate O1X1Y1 are a1, b1, and θ1o, respectively. Similarly, the corresponding end coordinates and angle of O2B with respect to the local coordinate O2X2Y2 are denoted a2, b2, and θ2o, respectively. The horizontal displacement ΔX, vertical displacement ΔY, and rotation angle Δθ of the freedom for the LITF pivot in the global coordinate system are expressed as

\[
\begin{align*}
\Delta X &= \frac{1}{2}[(a_1 \cos \theta_1 + a_2 \cos \theta_2) - (b_1 \sin \theta_1 + b_2 \sin \theta_2)] \\
\Delta Y &= \frac{1}{2}[(a_1 \sin \theta_1 + a_2 \sin \theta_2) + (b_1 \cos \theta_1 + b_2 \cos \theta_2)] - L \cos \beta \\
\end{align*}
\]

The loop closure equations are given as

\[
\begin{align*}
\cos \theta_1 &- \sin \theta_1 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \cos \theta_2 &- \sin \theta_2 \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} - \begin{bmatrix} w_1 \\ 0 \end{bmatrix} \\
\sin \theta_1 &\cos \theta_1 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + w_2 \begin{bmatrix} \cos \Delta \theta \\ \sin \Delta \theta \end{bmatrix}.
\end{align*}
\]

Figure 3 shows the free-body diagrams for link AB and leaf springs O1A and O2B. When the pivot is subject to horizontal force Fx, vertical force Fy, and moment M at the midpoint C of link AB, the horizontal and vertical components of the end force and moment loaded at the O1A and O2B are F1x, F1y, M1o, and F2x, F2y, M2o, respectively. Applying the static equilibrium for link AB yields

\[
\begin{align*}
F_{1x} + F_{2x} + F_x &= 0 \\
-F_{1y} - F_{2y} + F_y &= 0 \\
-M_{1o} - M_{2o} + M - F_{1x}w_2 \sin \Delta \theta - F_{1y}w_2 \cos \Delta \theta &= 0
\end{align*}
\]

P1 and n1P1 are the components of F1x and F1y along O1Y1 and X1O1 and have

\[
\begin{bmatrix} F_{1x} \\ F_{1y} \end{bmatrix} = \begin{bmatrix} \cos \theta_1 & \sin \theta_1 \\ -\sin \theta_1 & \cos \theta_1 \end{bmatrix} \begin{bmatrix} n_1P_1 \\ P_1 \end{bmatrix}.
\]

Similarly, for O2B,

\[
\begin{align*}
\Delta X &= \frac{1}{2}[(a_1 \cos \theta_1 + a_2 \cos \theta_2) - (b_1 \sin \theta_1 + b_2 \sin \theta_2)] \\
\Delta Y &= \frac{1}{2}[(a_1 \sin \theta_1 + a_2 \sin \theta_2) + (b_1 \cos \theta_1 + b_2 \cos \theta_2)] - L \cos \beta \\
\Delta \theta &= \theta_{1o} - \theta_{2o}
\end{align*}
\]
2.2. **Comprehensive Elliptic Integral Solution.** Each of the two springs of a LITF pivot can be viewed as a cantilever beam subject to an end vertical force \( P \), an end horizontal force \( nP \), and an end moment \( M_2 \), as shown in Figure 4. The tip coordinates and tip angle of the deflected beam are denoted as \( a \), \( b \), and \( \theta_o \), respectively. Each deflected spring may carry \( m \) inflection points \( (m = 1 \text{ or } 2) \). The comprehensive solution [15] for the beam with inflection points, as summarized in the following, formulates the load parameters \((\kappa, n, \text{ and } P)\) and deflection parameters \((a, b, \text{ and } \theta_o)\) by introducing \( m \) as the shape parameter:

\[
\begin{align*}
\begin{cases}
F_{2x} = \cos \theta_2 \sin \theta_2 \\
F_{2y} = -\sin \theta_2 \cos \theta_2
\end{cases}
\begin{bmatrix}
m_2 P_2 \\
P_2
\end{bmatrix}.
\end{align*}
\]  

The deflection of each leaf spring can be modeled using the comprehensive elliptic integral solution summarized in the following section. The comprehensive elliptic integral solution for each leaf spring, together with the loop closure equation (3) and the static equilibrium equation (4), constitute the kinetostatic model for LITF pivots.

![Figure 2: Deformation diagram of LITF pivot. (a) \( N < 1 \). (b) \( N > 1 \). (c) \( N = 1 \).](image)

![Figure 3: Force analysis of the LITF pivot. (a) \( N \leq 1 \) and (b) \( N > 1 \).](image)

![Figure 4: Cantilever beam subject to combined force and moment loads.](image)

\[
\begin{align*}
\alpha &= \frac{S_a}{\sqrt{\eta}} f \\
\frac{a}{L} &= \frac{S_a}{a \eta^{3/2}} \left[- mf + 2me + \sqrt{2\eta} c \right] \\
\frac{b}{L} &= \frac{S_r}{a \eta^{3/2}} \left[ \eta f - 2mn + n \sqrt{2\eta} c \right]
\end{align*}
\]  


where $\alpha$ is defined as the force index (EI is the flexural rigidity of the beam),

$$\alpha = \sqrt{\frac{Pl^2}{EI}},$$

$k$ is the load ratio,

$$k = \frac{M^2}{2P EI},$$

and $S_r$ is the sign of the resulting moment at the fixed end of the beam,

$$S_r = \begin{cases} (−1)^m, & M_o ≥ 0, \\ (−1)^{m+1}, & M_o < 0. \end{cases}$$

Moreover,

$$\lambda = \sin \theta_o - n \cos \theta_o + k,$$  
$$\eta = \sqrt{1 + n^2},$$  
$$f = (−1)^mF(\gamma, t) - F(\gamma, t) + 2msF(t),$$  
$$e = (−1)^mE(\gamma, t) - E(\gamma, t) + 2msE(t),$$  
$$y_1 = \sin^{-1} \left[ \frac{\eta - \eta}{\sqrt{\lambda + \eta}} \right],$$  
$$y_2 = \begin{cases} \sin^{-1} \left[ \frac{\eta + \sin \theta_o - n \cos \theta_o}{\lambda + \eta} \right] - \pi + \phi < \theta_o \leq \phi, \\ -\sin^{-1} \left[ \frac{\eta + \sin \theta_o - n \cos \theta_o}{\lambda + \eta} \right] - 2\pi + \phi < \theta_o \leq -\pi + \phi, \end{cases}$$

$$t = \frac{\lambda + \eta}{2\eta},$$  
$$c = \sqrt{\lambda + n} - (−1)^m\sqrt{\lambda - \sin \theta_o + n \cos \theta_o}. $$

$$F(\gamma, t) = \int_0^\gamma \sqrt[3]{1 - \sin \frac{\delta}{t}} d\delta$$

The coordinates $(x, y)$ of an arbitrary point $A$ on the beam (shown in Figure 4) can be written as

$$x = \frac{S_r}{\alpha n^{1/2}} \left[ -m \eta f(\theta) + 2m \eta e(\theta) + \sqrt{2\eta} c(\theta) \right],$$  
$$y = \frac{S_r}{\alpha n^{1/2}} \left[ \eta f(\theta) - 2\eta e(\theta) + n \sqrt{2\eta} c(\theta) \right],$$

where $\theta$ is the deflected angle at point $A$ and

$$f(\theta) = (−1)^m(\theta) F(\gamma, t) - F(\gamma, t) + 2m(\theta)S_rF(t),$$  
$$e(\theta) = (−1)^m(\theta) E(\gamma, t) - E(\gamma, t) + 2m(\theta)S_rE(t),$$  
$$y = \begin{cases} \sin^{-1} \left[ \frac{\eta + \sin \theta - n \cos \theta}{\lambda + \eta} \right] - \pi + \phi < \theta \leq \phi, \\ -\sin^{-1} \left[ \frac{\eta + \sin \theta - n \cos \theta}{\lambda + \eta} \right] - 2\pi + \phi < \theta \leq -\pi + \phi, \end{cases}$$

$$c(\theta) = \sqrt{\lambda + n} - (−1)^m(\theta) \sqrt{\lambda - \sin \theta + n \cos \theta}.$$
3. Case Studies

In this section, a LITF pivot and a parallelogram flexure are employed as two cases to demonstrate the effectiveness of the comprehensive elliptic integral model. The parameters of the two pivots are given in Table 1, and the materials are polypropylene in which $E = 1.4 \text{ GPa}$ and $S_y = 34 \text{ MPa}$ [13].

3.1. Solution for LITF Pivot. The parameters of the LITF pivot are shown in Table 1. The lengths of link $O_1 O_2$ and $AB$ solved by equation (1) are $w_1 = 0.1167 \text{ m}$ and $w_2 = 0.0467 \text{ m}$, respectively. The deflected shapes of the pivot subject to different loads, the load-bearing capacity, and the corresponding motion range of the pivot will be discussed here.

3.1.1. Deflected Shapes under Different Loads. The deflected results of the pivot subject to different loads are obtained separately by using the comprehensive elliptic integral solution and a nonlinear finite element analysis (NFEA) model, as shown in Figures 6–10. For the NFEA model built with the ANSYS software, springs $O_1 A$ and $O_2 B$ are meshed into 100 elements with BEAM188, respectively, and the large displacement analysis option is turned on. BEAM188 is suitable for analyzing slender to moderately stubby beam structures. This element is based on the Timoshenko beam theory. Shear deformation is included. The results of the comprehensive elliptic integral solution agree well with NFEA.

For the LITF pivot subject to pure moment loaded at point C, the relationship between the rotation angle $\Delta \theta$ and the moment $M$ is shown in Figure 6. The LITF pivot reveals fine linearity for $\Delta \theta$ less than $10^\circ$ (the dashed line in Figure 6 expresses the linear approximation of the LITF pivot with small deformation). However, when $\Delta \theta$ is larger than $10^\circ$, the nonlinearity of the stiffness for this kind of pivot becomes remarkable. For $M = 0.49 \text{ N-m}$, the maximum curvature of the pivot occurring at point A is equal to $48.5519 \text{ m}^{-1}$, which is substituted into equation (16) to obtain the maximum stress, $\sigma_{\text{max}} = 33.986 \text{ MPa}$, close to the yield strength. Meanwhile, $\Delta \theta$ attains $27.04^\circ$, for which the deflected shape of the pivot is shown in Figure 7.

If $F_x = 5 \text{ N}$ and $M$ are loaded at point C simultaneously, the relationships of $\Delta X$, $\Delta Y$, and $\Delta \theta$ with $M$ before yield failure have slight nonlinearities, as shown in Figures 8–10. Otherwise, for $F_x = -5 \text{ N}$, $F_y = 0 \text{ N}$, and different $M$, the nonlinearities of the pivot become obvious. For $M$ from 0 to...
shown in Figures 8–10. The curve of $F_x = -5\, \text{N}$ and $F_y = 0\, \text{N}$ is intersected at $M = 0.2$ with that of $F_x = -5$ and $F_y = -5$, where the resultant moments for the rotation center $O$ are equal to zero, so that the pivot returns to the original position. The corresponding deflected shapes of the pivot for $F_x = -5\, \text{N}$, $F_y = 0\, \text{N}$, and $M = 0.5 \sim 0.5\, \text{N} \cdot \text{m}$, as shown in Figure 11, incline to the left and then to the right.

3.1.2. Workspace Evaluation. The stress of the deflected pivot solved by the kinetostatic model is checked by equation (16), and then the load-bearing capacity in different load cases and the motion range of the pivot are obtained.

(1) Horizontal Force and Moment. Figure 12 shows that the pivot subject to different $M$ and $F_x$ can bear a range of horizontal force. The arrows drawn in Figure 12 roughly mark the descending direction of the stress, and the covered area is the safe working region.

The pivot only subjected to horizontal force, i.e., $M = 0$, can bear the maximum horizontal force reaching $F_x = \pm 13.6\, \text{N}$, for which the corresponding rotation angles are $\Delta \theta = \pm 16.9^\circ$, as shown in Figure 13. With the incremental moment, the maximum positive horizontal force of the pivot gradually decreases and the anticlockwise rotation angle of link AB shows the increasing tendency.

For $M = 0.49\, \text{N} \cdot \text{m}$, the maximum stress of the pivot without horizontal force reaches the yield strength. When negative horizontal force and moment act on the pivot, the negative allowable horizontal force increases gradually and the corresponding angle decreases slightly with the increasing moment, as shown in Figures 12 and 13. For $M = 0.5\, \text{N} \cdot \text{m}$, the allowable negative force is $F_x = -24.12\, \text{N}$ and $\Delta \theta = -13.06^\circ$, as shown in Figure 14, and the maximum curvature also happens at point A.

The relative errors of the rotation angles between the comprehensive solution ($\Delta \theta_{\text{CS}}$) and the nonlinear finite element ($\Delta \theta_{\text{FEA}}$) results are expressed as

$$\text{Error} = \frac{\Delta \theta_{\text{CS}}}{\Delta \theta_{\text{FEA}}} - 1. \quad (17)$$

The errors of the positive rotation angles depicted in Figure 13 between the comprehensive solution and the nonlinear finite element results are less than 1.5%, which is shown in Figure 15.

(2) Vertical Force and Moment. For the pivot subject to $M$ and $F_y$, Figure 16 draws the maximum vertical force that the pivot subject to different moments can bear. Similarly, the declining direction of the stress is masked roughly by the arrows in Figure 16. Positive vertical force can counteract the rotation angle of the pivot caused by the moment. It should be noted that the tensile stress might lead to the failure of the pivot when positive vertical force reaches a certain value because the Bernoulli–Euler beam theory neglects the effect of axial elongation and the maximum positive vertical force cannot be predicted, the discussion of which is outside the scope of this paper.
The corresponding rotation angles subject to different moments and the maximum vertical forces depicted in Figure 16 are shown in Figure 17. The rotation angle slightly increases for $M < 0.2$ N·m and then decreases for $M > 0.2$ N·m. When $M = 0.2$ N·m, the allowable negative vertical force attains $F_y = -11.42$ N and the rotation angle is $\Delta \theta = 31.86^\circ$, for which the deflected shape agrees well with the result calculated by NFEA, as shown in Figure 18. The maximum stress of the pivot subject to $F_y = -11.42$ N and $M = 0.2$ N·m is $\sigma_{\text{max}} = 33.995$ MPa and occurs in the deflected spring $O_2 B$ shown by the diamond shape in Figure 18. When $M$ is greater than 0.49 N·m, the pivot subject to the negative force directly leads to the failure of the spring, so in this case, the pivot can only withstand the positive vertical force.

For $M = 0$ N·m, the pivot only subject to vertical force and the buckling of the spring may take place. For the buckled LITF pivot, the maximum bending stress $\sigma_{\text{max}}$ may be less than the yield strength of the material used, but the LITF pivot has been invalidated, so the maximum negative vertical force for $M = 0$ N·m is equal to the critical buckling force.

The buckled springs can be seemed as the fixed-guided beams with two inflection points ($m = 2$) that perhaps have two deformed shapes (I) and (II), as shown in Figure 19. The vertical displacement of the freedom is $\delta$, and the end slope of the buckled spring remains constant, i.e., $\theta_o = 0$. 
For the buckled springs, the coordinates of the free end are given as
\[
\begin{align*}
a &= L - \delta \cos \beta \\
b &= -\delta \sin \beta
\end{align*}
\] (18)

The vertical force \(F_y\) can be solved as
\[F_y = 2P(\sin \beta - n \cos \beta).\] (19)

Substituting \(m = 2, \theta_o \equiv 0,\) and equation (8) into equation (9) yields
\[
P = \frac{16EI}{L \sqrt{1 + n^2}} F^2(t),
\] (20)

\[
\begin{align*}
a &= \frac{n}{L} \left[ 1 - \frac{2E(t)}{F(t)} \right] \\
b &= \frac{1}{L} \left[ 1 - \frac{2E(t)}{F(t)} \right]
\end{align*}
\] (21, 22)

From equations (18), (21), and (22), \(n\) is
\[n = \frac{a}{b} = \frac{L - \delta \cos \beta}{\delta \sin \beta}.
\] (23)

Substituting equations (20) and (23) into equation (19) yields
\[
F_y = \frac{32EI}{L^2} \frac{\delta - L \cos \beta}{\sqrt{\delta^2 + L^2 - 2L\delta \cos \beta}} F^2(t).
\] (24)

When \(F_y\) reaches the critical buckling force, we have \(n \to \infty\) and \((a/L) \to 0,\) and then equation (21) reduces to \((F(t)/E(t)) \to 1\) and has
\[t = 0.\] (25)

We have \(F(t) = \pi/2;\) then, the critical buckling force \(F_{cr}\) from equation (24) is
\[
F_{cr} = -\frac{32EI \cos \beta}{L^2} F^2(0) = -\frac{8\pi^2EI \cos \beta}{L^2}.
\] (26)

Thus, for \(M = 0,\) the maximum negative vertical force of the pivot \(F_y\) is determined by the critical buckling force solved by equation (26) and equal to \(-41.0272\) N, as shown in Figure 16. When the pivot is loaded only by the vertical force, the leaf with two inflection points includes two deflection paths, which are shown in the left-hand leaf and right-hand leaf of Figure 19. The choice of the two solutions is decided by the processing factor of the leaf.

3.2. Parallelogram Flexure. A parallelogram flexure is a one-degree-of-freedom device that obtains accurate motion by the bending of the springs. Many authors have contributed to this problem; for example, Awtar et al. [18] proposed a beam constraint model and Dibiasio et al. [19] presented a pseudorigid-body model to simplify the derivation and
calculation. In the paper, the kinetostatic model is also suitable to analyze the parallelogram flexure.

For the parameters of the mechanism given in Table 1, the leaf springs $O_1A$ and $O_2B$ with one inflection point guide the motion of link $AB$ with minimal rotation. When $F_x$ is applied at point $C$, the horizontal displacement $\Delta X$ is obtained to arrive at a static equilibrium state, as shown in Figure 20. With increasing horizontal force, the nonlinear characteristics of the curve are gradually obvious and the rotation angle $\Delta \theta$ is slowly increasing, as shown in Figure 21. When $F_x = 8.5$ N is loaded at point $C$, $\Delta X = 0.032$ m and $\Delta \theta = -0.145^\circ$, for which the maximum curvature occurring at point $O_2$ is $K_{max} = 46.9396$ m$^{-1}$ and the maximum bending stress $\sigma_{max}$ solved by equation (16) is slightly less than $S_y$.

The rotation angle $\Delta \theta$ of link $AB$ is a parasitic error motion that is undesirable in response to the horizontal force $F_x$, which may be eliminated by an appropriate combination of moment $M$ or vertical force $F_y$ [18]. When $F_x$ and $M$ are loaded simultaneously at point $C$ to ensure $\Delta \theta = 0$, we have, from equations (4)–(6),

$$
\frac{M}{F_x} = \frac{M_{10}}{P_1}.
$$

(27)

For a parallelogram flexure because each deflected leaf spring carries one inflection point, where the resultant moment is equal to zero and the rotation angles at the fixed and free ends of each deflected leaf spring are both equal to zero, the inflection point occurs at the middle of the deflected spring, i.e., $x = a_1/2$. The moment at the inflection point is

$$
\frac{a_1}{2} P_1 + M_{10} = 0.
$$

(28)

Substituting equation (28) into equation (27) yields

$$
\frac{M}{F_x} = \frac{a_1}{2} = \frac{a_2}{2} = \frac{L + \Delta Y}{2}.
$$

(29)

For $F_y = 0$, the ratios in equation (29) during the intermediate stage are approximately constant, which agree well with the results of Ref. [18] equal to 0.5L, as shown in Figures 22 and 23. Then, the ratios between $M$ and $F_x$ are less than 0.5L with increasing horizontal force $F_x$ and the corresponding transverse stiffness gradually increases for protecting $\Delta \theta = 0$. As listed in Table 2, the corresponding moments and displacements for $F_x = 2 \sim 10$ N solved by the elliptic integral solution agrees well with the load-deflection relationship expressed in equation (29) and the deflected shapes of the pivot subject to $F_x$ and $M$ are shown in Figure 24.

When $F_x$ and $F_y$ are loaded at point $C$ simultaneously, $M$ is needed to ensure that $\Delta \theta = 0$. In this case, the inflection points also appear in the middle of the deflected leaf springs. However, if $F_y$ is a tensile force, the ratios between $M$ and $F_x$ are less than $a_1/2$ because of $n$ being less than zero. On the contrary, for $F_y$ as a pressure, the ratios between $M$ and $F_x$ are greater than $a_1/2$. The more obvious nonlinearity of the

**Figure 19:** Buckling deformation of the LITF pivot.

**Figure 20:** Plots of horizontal force $F_x$ vs horizontal displacement $\Delta X$.

**Figure 21:** Plots of rotation angle $\Delta \theta$ of parallelogram flexure subject to horizontal force $F_x$. 
pivot appears with increasing pressure $F_y$, as shown in Figure 23. Until the pressure reaches the critical buckling force calculated by equation (26) ($F_y = -47.3741\text{ N}$), the buckling of the pivot leads directly to failure.

4. Conclusions

The comprehensive elliptic integral solution was used for building the generalized model of LITF pivots and solving nonlinear deflection problems. For the LITF pivot, the accurate deflected shapes are described subject to different horizontal forces, vertical forces, and moments. Furthermore, based on the strength check and the analysis of the critical buckling force, motion range and load-bearing capability for the pivot are evaluated. For the parallelogram flexure, two cases for free rotation angle and constant rotation angle are discussed. The more accurate ratio between horizontal force and moment is proposed to ensure that the rotation angle remains constant. The analytical results for the maximum rotation angle of the LITF pivot subject to horizontal force and moment solved by the comprehensive elliptic integral solution are within 1.5 percent error compared to the finite element analysis results.

Data Availability

The calculation data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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