Pattern Formation by Robots with Inaccurate Movements

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Abstract

Arbitrary Pattern Formation is a fundamental problem in autonomous mobile robot systems. The problem asks to design a distributed algorithm that moves a team of autonomous, anonymous and identical mobile robots to form any arbitrary pattern $F$ given as input. In this paper, we study the problem for robots whose movements can be inaccurate. Our movement model assumes errors in both direction and extent of the intended movement. Forming the given pattern exactly is not possible in this setting. So we require that the robots must form a configuration which is close to the given pattern $F$. We call this the Approximate Arbitrary Pattern Formation problem. We show that with no agreement in coordinate system, the problem is unsolvable, even by fully synchronous robots, if the initial configuration 1) has rotational symmetry and there is no robot at the center of rotation or 2) has reflectional symmetry and there is no robot on the reflection axis. From all other initial configurations, the problem is solvable by 1) oblivious, silent and semi-synchronous robots and 2) oblivious, asynchronous robots that can communicate using externally visible lights.

1 Introduction

Distributed coordination of a team of autonomous mobile robots is a widely studied problem. Early investigations of these problems were experimental in nature with the main emphasis being on designing good heuristics. However, the last two decades have seen a flurry of theoretical studies on the computability and complexity issues related to distributed computing by such system of robots. These studies are aimed at providing provably correct algorithmic solutions to fundamental coordination problems and also to understand the minimal set of capabilities that the robots need to have in order to solve these problems. The traditional framework of theoretical studies considers a very weak model of autonomous robots: the robots are anonymous (they have no unique identifiers), homogeneous (they execute the same distributed algorithm), identical (they are indistinguishable by their appearance), disoriented (they have no common coordinate system), oblivious (they do not remember previous observations and calculations) and silent (they have no means of direct communication) computational entities that can freely move in the plane. However, the assumed mobility features of the robots in these studies are very strong. Two standard models regarding the movement of the robots are Rigid and Non-Rigid. In Rigid, if a robot $x$ wants to go to any point $y$, then it can move to exactly that point in one step. This means that the robots are assumed to be able to execute error-free movements...
in any direction and by any amount. Certain studies also permit the robots to move along curved trajectories. The algorithms in this model rely on the accurate execution of the movements and not robust to movement errors that real life robots are susceptible to. Furthermore, the error-free movements of the robot can theoretically give the robots remarkable computational powers as exhibited in [10]. A positional encoding technique was developed in [10] that allows a robot, that have very limited or no memory to store data, to implicitly store unbounded amount of information by encoding the data in the binary representation of its distance from another robot or some other object, e.g. the walls of the room inside which it is deployed. Exact movements allow the robots to preserve and update this data and solve complex problems which appear to be unsolvable by robots with limited or no memory, e.g. constructing a map of a complex art gallery by an oblivious robot. Obviously these techniques are impossible to implement in practice. On the other hand, for problems that we expect to be unsolvable by real life robots with certain restrictions in memory, communication etc., it may become difficult or impossible to theoretically establish a hardness or impossibility result due to the strong model. This creates a chasm between theory and practice. The Non-Rigid model assumes that a robot may stop before reaching its intended destination. However, there exists a constant $\delta > 0$ such that if the destination is at most $\delta$ apart, the robot will reach it; otherwise, it will move towards the destination by at least $\delta$. However, notice that 1) the movement is still error-free if the destination is close enough, i.e., within $\delta$, and 2) there is no error whatsoever in the direction of the movement even if the destination is far away. In [1], it was shown that these two properties allow robots to implement positional encoding even in the Non-Rigid model. This motivates us to consider a new movement model. Our movement model assumes errors in both direction and extent of the intended movement. Also, errors can occur no matter what the extent of the attempted movement is. The details of the model are presented in Section 2.

We study the Arbitrary Pattern Formation problem in this model. Arbitrary Pattern Formation is a fundamental robot coordination problem and has been extensively studied in the literature (See [2–9, 11–15] and references therein). The goal is to design a distributed algorithm that allows the robots to form any pattern $F$ given as input. This problem has been extensively studied in the literature in the Rigid and Non-Rigid model. However, the techniques used in these algorithms are not readily portable in our setting. For example, in most of these algorithms, the minimum enclosing circle of the configuration plays an important role. The center of the minimum enclosing circle is set as the origin of the coordinate system with respect to which the pattern is to be formed. So the minimum enclosing circle is kept invariant throughout the algorithm. The robots inside the minimum enclosing circle move to form the part of pattern inside the circle, without disturbing it. For the pattern points on the minimum enclosing circle, robots from the inside may have to move on to the circle. Also, the robots on the minimum enclosing circle, in order to reposition themselves in accordance with the pattern to be formed, will move along the circumference so that the minimum enclosing circle does not change. Notice that while moving along the circle, an error prone robot might skid off the circle. Also, when a robot from the inside attempts to move exactly on to the circle, it may move out of the circle due to the error in movement. In both cases, the minimum enclosing circle will change and the progress made by the algorithm will be lost. In fact, we face difficulty at a more fundamental level: exactly forming an arbitrary pattern is impossible by robots with inaccurate movement. Therefore, we consider a relaxed version of the problem called Approximate Arbitrary Pattern Formation where the robots are required to form an approximation of the input pattern $F$. We show that with no agreement in coordinate system, the problem is unsolvable, even by fully synchronous robots, if the initial configuration 1) has rotational symmetry and there is no robot at the center of rotation or 2) has reflectional symmetry and there is no robot on the reflection axis. From all other initial configurations, the problem is solvable by 1) oblivious, silent and semi-synchronous robots and 2) oblivious, asynchronous robots that can communicate using externally visible lights.
2 Model and Definition

2.1 Robot Model

The Robots A set of $n$ mobile computational entities, called robots, are initially positioned at distinct points in the plane. The robots are assumed to be anonymous (they have no unique identifiers that they can use in a computation), identical (they are indistinguishable by their physical appearance), autonomous (there is no centralized control) and homogeneous (they execute the same deterministic algorithm). The robots are modeled as points in the plane, i.e., they do not have any physical extent. The robots do not have access to any global coordinate system. Each robot has its own local coordinate system centered at its current position. There is no consistency among the local coordinate systems of the robots except for a common unit of distance. We call this the standard unit of distance.

Memory and Communication Based on the memory and communication capabilities, there are four models: OBLOT, LUMI, FSTA, and FCOM. In OBLOT, the robots are silent (they have no explicit means of communication) and oblivious (they have no memory of past observations and computations). In LUMI, the robots are equipped with lights which can assume a constant number of colors. The lights serve both as a weak explicit communication mechanism and a form of internal memory. In FSTA, a robot can only see the color of its own light, i.e., the light is internal and in FCOM, a robot can only see the light of other robots, i.e., the light is external. Therefore in FSTA, the robots are silent, but have finite memory, while in FCOM, the robots are oblivious, but have finite communication capability.

Look-Compute-Move Cycles The robots, when active, operate according to the so-called Look-Compute-Move cycles. In each cycle, a previously idle or inactive robot wakes up and executes the following steps. In the Look phase, the robot takes the snapshot of the positions (and their lights in case of LUMI and FCOM) of all the robots. Based on the perceived configuration (and its own light in case of LUMI and FSTA), the robot performs computations according to a deterministic algorithm to decide a destination point and (a color in case of the robots are silent, but have finite memory, while in FCOM, the robots are oblivious, but have finite communication capability.

Scheduler Based on the activation and timing of the robots, there are three types of schedulers considered in the literature. In FSYNC or fully synchronous, time can be logically divided into global rounds. In each round, all the robots are activated. They take the snapshots at the same time, and then perform their moves simultaneously. SSYNC or semi-synchronous coincides with FSYNC, with the only difference that not all robots are necessarily activated in each round. However, every robot is activated infinitely often. In ASYNC or asynchronous, there are no assumptions except that every robot is activated infinitely often. In particular, the robots are activated independently and each robot executes its cycles independently. The amount of time spent in Look, Compute, Move and inactive states is finite but unbounded, unpredictable and not same for different robots.

Movement of the Robots There are constants $0 \leq \lambda < 1$, $0 \leq \Delta < 1$, such that if a robot at $x$ attempts to move to $y$, then it will reach a point $z$ where $d(z, y) < \mu(x, y)d(x, y)$ where $\mu(x, y) = \min\left\{\Delta, \lambda d(x, y)\right\}$. Here $d(x, y)$ denotes (the numerical value of) the distance between the points $x$ and $y$ measured in the standard unit of distance. The movement of the robot will be along the straight line joining $x$ and $z$. Therefore, the distance error (i.e., the deviation from the intended distance to be travelled) of the robot is bounded by $error_d(d(x, y)) = \mu(x, y)d(x, y)$ and the angular error (i.e., the deviation from the intended trajectory) of the robot is bounded by $error_a(d(x, y)) = sin^{-1}\left(\frac{\mu(x, y)d(x, y)}{d(x, y)}\right) = sin^{-1}(\mu(x, y))$. Therefore, 1) $error_d(d(x, y))$ increases with $d(x, y)$, and 2) $error_a(d(x, y))$ increases with $d(x, y)$ up to a certain value, i.e., $sin^{-1}(\Delta)$ and then remains constant (See Fig. 1a.). So, $error_a(d(x, y)) \leq sin^{-1}(\Delta)$, for any $x, y$.
2.2 Definitions and Notations

We denote the configuration of robots by \( R = \{r_1, r_2, \ldots, r_n\} \) where each \( r_i \) denotes a robot as well as the point in the plane where it is situated. The input pattern given to the robots will be denoted by \( F = \{f_1, f_2, \ldots, f_n\} \) where each \( f_i \) denotes an element from \( \mathbb{R}^2 \).

Given two points \( x \) and \( y \) in the Euclidean plane, let \( d(x, y) \) denote the distance between the points \( x \) and \( y \) measured in the standard unit of distance. For three points \( x, y \) and \( c \), \( \angle_{x} c y \) (\( \angle_{y} x c \)) is the angle centered at \( c \) measured from \( x \) to \( y \) in the clockwise (resp. counterclockwise) direction. Also, \( \angle Dixy = \min(\angle_{x} c y, \angle_{y} x c) \). We denote by \( \text{line}(x, y) \) the straight line passing through \( x \) and \( y \).

By \( \text{seg}(x, y) \) (\( \text{encl}(x, y) \)) we denote the line segment joining \( x \) and \( y \) excluding (resp. including) the end points. If \( \ell_1 \) and \( \ell_2 \) be two parallel lines, then \( S(\ell_1, \ell_2) \) denotes the open region between these two lines. For any point \( c \) in the Euclidean plane and a length \( l \), \( C(c, l) = \{z \in \mathbb{R}^2 \mid d(c, z) = l\} \), \( B(c, l) = \{z \in \mathbb{R}^2 \mid d(c, z) < l\} \) and \( \overline{B}(c, l) = \{z \in \mathbb{R}^2 \mid d(c, z) \leq l\} = B(c, l) \cup C(c, l) \). If \( C \) is a circle then \( \text{encl}(C) \) and \( \text{encl}(C) \) respectively denote the open and closed region enclosed by \( C \). Also, \( \text{ext}(C) = \mathbb{R}^2 \setminus \text{encl}(C) \) and \( \text{ext}(C) = \mathbb{R}^2 \setminus \text{encl}(C) \). Hence, \( \text{encl}(C) = \text{encl}(C) \cup C \) and \( \text{ext}(C) = \text{ext}(C) \cup C \). Let \( x, y \) be two points in the plane and \( d(x, y) > l \). Suppose that the tangents from \( x \) to \( C(y, l) \) touches \( C(y, l) \) at \( a \) and \( b \). The \( \text{Cone}(x, B(y, l)) \) is the open region enclosed by \( B(y, l), \text{encl}(x, a) \) and \( \text{encl}(x, b) \), as shown in Fig. 1b. Also, \( \angle \text{Cone}(x, B(y, l)) = \angle axb \).

![Diagram](image)

**Property 1.** If \( x' \in \text{Cone}(x, B(y, l)) \setminus \overline{B}(y, l) \), then \( \text{Cone}(x', B(y, l)) \subseteq \text{Cone}(x, B(y, l)) \).

Given two points \( x \) and \( y \) in the Euclidean plane, we denote by \( F(x, y) \) the family of circles passing through \( x \) and \( y \). The center of all the circles lie on the perpendicular bisector of \( \text{encl}(x, y) \), say \( \ell \). Each point \( c \in \ell \) defines a unique circle of the family \( F(x, y) \), i.e., the circle from \( F(x, y) \) having center at \( c \). If \( C_1, C_2 \in F(x, y) \) and \( c_1, c_2 \) be their centers respectively, then \( (C_1, C_2)_{F(x, y)} \) and \( [C_1, C_2]_{F(x, y)} \) will denote respectively the family of circles \( \{C \in F(x, y) \mid c \in \text{seg}(c_1, c_2)\} \) where \( c \) is the center of \( C \) and \( \{C \in F(x, y) \mid c \in \text{encl}(c_1, c_2)\} \) where \( c \) is the center of \( C \).

**Property 2.** Let \( C_1, C_2 \in F(x, y) \), \( c_1, c_2 \) their centers respectively, and there is a closed half plane \( H \)
delimited by line$(x, y)$ such that $c_1, c_2 \in \mathcal{H}$. If $c$ is the mid-point of $\overline{encl}(x, y)$ and $d(c, c_1) \leq d(c, c_2)$, then $encl(C_1) \cap \mathcal{H} \subseteq encl(C_2) \cap \mathcal{H}$.

For a set $P$ of points in the plane, $C(P)$ and $c(P)$ will respectively denote the minimum enclosing circle of $P$ (i.e., the smallest circle $C$ such that $P \subseteq encl(C)$) and its center. The smallest enclosing circle $C(P)$ is unique and can be computed in linear time. For $P$, with $2 \leq |P| \leq 3$, $CC(P)$ is denotes the circumcircle of $P$ defined as the following. If $P = \{p_1, p_2\}$, $CC(P)$ is the circle having $\overline{encl}(p_1, p_2)$ as the diameter and if $P = \{p_1, p_2, p_3\}$, $CC(P)$ is the unique circle passing through $p_1, p_2$ and $p_3$.

**Property 3.** If $P' \subseteq P$ such that 1) $P'$ consists of two points or $P'$ consists of three points that form an acute angled triangle, and 2) $P \subseteq encl(CC(P'))$, then $CC(P') = C(P)$. Conversely, for any $P$, $\exists P' \subseteq P$ so that 1) $P'$ consists of two points or $P'$ consists of three points that form an acute angled triangle and 2) $CC(P') = C(P)$.

From Property 3 it follows that $C(P)$ passes either through two of the points of $P$ that are on the same diameter (antipodal points), or through at least three points so that some three of them form an acute or right angled triangle. Also, $C(P)$ does not change by adding points inside $encl(C(P))$ or deleting points in $encl(C(P))$. However, $C(P)$ may be changed by deleting points from $C(P)$. A point $p \in P$ is said to be critical if $C(P) \neq C(P \setminus \{p\})$. Obviously, $p \in P$ is critical only if $p \in C(P)$.

**Property 4.** If $|P \cap C(P)| \geq 4$ then there exists at least one point from $P \cap C(P)$ which is not critical.

**Property 5.** Let $p_1, p_2, p_3 \in P$ be three consecutive points in clockwise order on $C(P)$. If $\angle_{C} p_1 c(P) p_3 \leq \pi$, then $p_3$ is non-critical.

Consider all the concentric circles that are centered at $c(P)$ and have at least one point of $P$ on them. Let $C_i^j(P)$ ($C_i^j(P)$) denote the $i$th ($i \geq 1$) of these circles so that $C_i^j(P) \subset encl(C_i^j(P))$ ($C_i^j(P) \subset encl(C_{i+1}^j(P))$). We shall denote $c(P)$ by $C_0^0(P)$. So we have $C_1^1(P) = C(P)$ and if there is a point at $c(P)$, then $C_1^1(P) = c(P) = C_0^0(P)$.

We say that a configuration of robots $R$ is symmetry safe if one of the following three is true.

1. If there is some non-critical robot on $C(R)$, hence $|R \cap C(R)| \geq 3$, then
   - there is no robot at $c(R)$
   - $|R \cap C_1^1(R)| = 3$ and $|R \cap C_2^2(R)| = 1$
   - if $R \cap C_1^1(R) = \{r_1\}$ and $R \cap C_2^2(R) = \{r_2\}$, then $r_1, r_2, c(R)$ are not collinear.

2. If all robots on $C(R)$ are critical and $R \cap C(R) = \{r_1, r_2, r_3\}$, then $\Delta r_1 r_2 r_3$ is scalene, i.e., all three sides have different lengths.

3. If all robots on $C(R)$ are critical and $R \cap C(R) = \{r_1, r_2\}$, then
   - $|R \cap C_1^1(R)| = 1$
   - if $R \cap C_1^1(R) = \{r\}$, $r \notin line(r_1, r_2) \cup \ell$, where $\ell$ is the line passing through $c(R)$ and perpendicular to $line(r_1, r_2)$.

### 3 Basic Properties

#### 3.1 Approximate Arbitrary Pattern Formation

The **Arbitrary Pattern Formation** problem in its standard form is the following. Each robot of a team of $n$ robots is given a pattern $F$ as input. The input pattern $F$ is a list of $n$ distinct elements from $\mathbb{R}^2$. The problem asks for a distributed algorithm that guides the robots to a configuration that is similar to $F$ with respect to translation, reflection, rotation and uniform scaling. We refer to this version of the problem as the **Exact Arbitrary Pattern Formation** problem, highlighting the
fact that the configuration of the robots is required to be exactly similar to the input pattern. However, it is not difficult to see that Exact Arbitrary Pattern Formation is unsolvable in our model where the robot movements are inaccurate.

**Theorem 1.** Exact Arbitrary Pattern Formation is unsolvable by robots with inaccurate movements.

**Proof.** For simplicity, let \( n = 3 \). Suppose that the input pattern \( F \) is a triangle which is not similar to the initial configuration of the robots. Assume that the adversary activates exactly one robot at each round. Suppose that the pattern is formed at round \( i \). Let \( r_1 \) be the robot which is activated at this round, and \( r_2, r_3 \) are inactive. There are at most 4 points on the plane, say \( P_1, P_2, P_3, P_4 \), where \( r_3 \) needs to go to form the given pattern. But it is not possible to go to exactly at one of these points from any point \( P \notin \{P_1, P_2, P_3, P_4\} \) on the plane. So the pattern is not formed at round \( i \). The same arguments hold for any round that follows.  

Therefore, we introduce a relaxed version of the problem called the Approximate Arbitrary Pattern Formation. Intuitively, we want the robots to form a pattern that is close to the given pattern, but may not be exactly similar to it. Formally, the robots are given as input a pattern \( F \) and a number \( 0 < \epsilon < 1 \). The number \( \epsilon \) is small enough so that the distance between no two pattern points is less than \( \epsilon D \) where \( D \) is the diameter of \( C(F) \). Given the input \( (F, \epsilon) \), the problem requires the robots to form a configuration \( R = \{r_1, \ldots, r_n\} \) such that there exists an embedding (subject to translation, reflection, rotation and uniform scaling) of the pattern \( F \) on the plane, say \( P = \{p_1, \ldots, p_n\} \), such that \( d(p_i, r_i) \leq \epsilon D \) for all \( i = 1, \ldots, n \), where \( D \) is the diameter of \( C(P) \). In this case, we say that the configuration \( R \) is \( \epsilon \)-close to the pattern \( F \). Recall that the number \( \epsilon \) is such that the disks \( B(p_i, \epsilon D) \) are disjoint and the problem requires exactly one robot inside each disk. Furthermore, the movements should be collisionless.

### 3.2 Symmetries and Basic Impossibilities

We first present the concept of view (defined similarly as in [6]) of a point in a pattern or a robot in a configuration. The view if a point/robot can be used to determine whether the pattern/configuration is symmetric or asymmetric. Let \( \mathcal{R} = \{r_1, \ldots, r_n\} \) be a configuration of robots or a pattern of points. A map \( \varphi : \mathcal{R} \to \mathcal{R} \) is called an isometry or distance preserving if \( d(\varphi(r_i), \varphi(r_j)) = d(r_i, r_j) \) for any \( r_i, r_j \in \mathcal{R} \). \( \mathcal{R} \) is said to be asymmetric if \( \mathcal{R} \) admits only the identity isometry, and otherwise it is called symmetric. The possible symmetries that a symmetric pattern/configuration can admit are reflections and rotations.

For any \( r \in \mathcal{R} \), its clockwise view, denoted by \( V_\mathcal{C}(r) \), is a string of \( n + 1 \) elements from \( \mathbb{R}^2 \). Consider the polar coordinates of the points/robots in the coordinate system with origin at \( c(\mathcal{R}) \), \( c(\mathcal{R})r \) as the reference axis and the angles measured in clockwise direction. The first element of the string \( V_\mathcal{C}(r) \) is the coordinates of \( r \) and next \( n \) elements are the coordinates the \( n \) points/robots ordered lexicographically. The counterclockwise view \( V_\mathcal{C}(r) \) is defined analogously. Among \( V_\mathcal{C}(r) \) and \( V_\mathcal{C}(r) \), the one that is lexicographically smaller is called the view of \( r \) and is denoted as \( V(r) \).

**Property 6.**

1. \( \mathcal{R} \) admits a reflectional symmetry if and only if there exist two points \( r_i, r_j \in \mathcal{R} \), \( r_i, r_j \neq c(\mathcal{R}) \), not necessarily distinct, such that \( V_\mathcal{C}(r_i) = V_\mathcal{C}(r_j) \).

2. \( \mathcal{R} \) admits a rotational symmetry if and only if there exist two points \( r_i, r_j \in \mathcal{R} \), \( r_i \neq r_j \), \( r_i, r_j \neq c(\mathcal{R}) \), such that \( V_\mathcal{C}(r_i) = V_\mathcal{C}(r_j) \).

In a configuration, each robot can compute its view as well as the views of all other robots. Hence, these properties can be used by the robots to detect whether the configuration is symmetric or not.

A problem that is closely related is the Leader Election problem where a unique robot from the team is to be elected as the leader. The following theorem characterizes the configurations from where Leader Election can be deterministically solved.
Theorem 2. [6] Leader Election is deterministically solvable, if and only if the initial configuration $R$ does not have i) rotational symmetry with no robot at $c(R)$ or ii) reflectional symmetry with respect to a line $\ell$ with no robot on $\ell$.

We call the symmetries i) and ii) unbreakable symmetries. If a configuration does not have such symmetries, then the robots can use the views to elect a unique leader. It follows from Property 6 that if $R$ is asymmetric then the views of the robots are all different and hence the unique robot with lexicographically minimum view can be elected as the leader. If the configuration has rotational symmetry and there is a robot at $c(R)$, then that robot has the unique minimum view and can be elected as the leader. Now assume that the configuration is not asymmetric but does not have rotational symmetry. So it has reflectional symmetry with respect to a single line $\ell$. Notice that if a configuration has reflectional symmetry with respect to multiple lines then it has rotational symmetry. The robots on $\ell$ have different views as otherwise $R$ would admit another axis of symmetry. So the one among them that has minimum view can be elected as the leader.

It is known that Exact Arbitrary Pattern Formation is deterministically unsolvable by robots, even with Rigid movements, if the initial configuration has unbreakable symmetries. The same result holds for Approximate Arbitrary Pattern Formation.

Theorem 3. Approximate Arbitrary Pattern Formation is deterministically unsolvable, even in \texttt{LUMI + FSYNC} and with Rigid movements, if the initial configuration has unbreakable symmetries.

Proof. For any configuration of robots $R$, define $\gamma(r)$ for any $r \in R$ as $\gamma(r) = \Sigma_{r' \in R \setminus \{r\}} d(r, r')$. Let $R_0$ be an initial configuration of $n$ robots that has an unbreakable symmetry. For the sake of contradiction, assume that there is a distributed algorithm $\mathcal{A}$ that solves Approximate Arbitrary Pattern Formation for any input $(E, \epsilon)$ from this configuration, i.e., it forms a configuration that is $\epsilon$-close to $E$. Consider the following input pattern $F = \{f_1, f_2, \ldots, f_n\}$, where $f_1, f_2, f_3$ form an isosceles triangle with $d(f_1, f_2) = d(f_1, f_3) > d(f_2, f_3)$ and $f_4, \ldots, f_n$ are arranged on the smaller side of the triangle. If $d(f_1, f_2) = d(f_1, f_3)$ is sufficiently large compared to $d(f_2, f_3)$ and $\epsilon$ is sufficiently small, then for any configuration $R'$ of robots that is $\epsilon$-close to $F$, we have $\gamma(r_1) > \gamma(r)$ for all $r \in R' \setminus \{r_1\}$, where $r_1$ is the robot approximating $f_1$. This property can be used to elect $r_1$ as the leader. Hence, Approximate Arbitrary Pattern Formation can be used to solve Leader Election from the initial configuration $R_0$. This is a contradiction to Theorem 2.

So a necessary condition for solvability of Approximate Arbitrary Pattern Formation by robots with inaccurate movements is that the initial configuration must not have unbreakable symmetries. In Section 4, we show that this condition is also sufficient in \texttt{OBLOT + SSYNC} and \texttt{FCOM + ASYNC}.

3.3 Moving Through Safe Zone

In this section, we present some movement strategies that will be used several times in the main algorithm. Suppose that a robot needs to move to or close to some point in the plane. If the point is far away from the robot and it attempts to reach it in one step, the error would be very large and it will miss the target by a large distance. Furthermore, it may reach a point which makes the configuration to loose some desired property or due to the large deviation from the intended trajectory, it may collide with other robots. So the robot needs to move towards its target in multiple steps and move through a 'safe' region where it does not collide with any robot and the desired properties of the configuration are preserved.

We first discuss the following problem. Let $x_0$ and $y$ be two points in the plane so that $d(x_0, y) > l$. Suppose that a robot $r$ is initially at $x_0$ and the objective is that it has to move to a point inside $B(y, l)$ via a trajectory which should lie inside $\text{Cone}(x_0, B(y, l))$. A pseudocode description of the algorithm
that solves the problem is presented in Algorithm 1.

**Algorithm 1:**

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Input: A point \( y \) on the plane and a distance \( l \)
1: \( r \leftarrow \) myself
2: if \( d(r, y) > l \) then
3: \[ \frac{r}{d(r, y)} \geq \sin(\text{error}_a(d(r, y))) \]
4: Move to \( y \)
5: else \( p \leftarrow \) point on \( \text{seg}(r, y) \) so that \( \frac{r}{d(r, y)} = \sin(\text{error}_a(d(r, p))) \)
6: Move to \( p \)
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**Theorem 4.** Algorithm 1 is correct.

**Proof.** Initially we have \( r \) at point \( x_0 \) and \( d(x_0, y) > l \). First assume that \( \frac{r}{d(x_0, y)} \geq \sin(\text{error}_a(d(x_0, y))) \).

This implies that \( \angle(\text{Cone}(x_0, B(y, l))) = 2\sin^{-1}(\frac{r}{d(x_0, y)}) \geq 2\text{error}_a(d(x_0, y)) \), i.e., if \( r \) attempts to move to \( y \), its angular deviation will not take it outside \( \text{Cone}(x_0, B(y, l)) \). In this case, it will decide to move to \( y \). So it will move to a point \( z \) where \( d(z, y) < \mu(x_0, y)d(x_0, y) = \sin(\text{error}_a(x_0, y))d(x_0, y) \leq \frac{r}{d(x_0, y)}d(x_0, y) = l \). Hence, \( r \) gets inside \( B(y, l) \) in one step and clearly its trajectory is inside \( \text{Cone}(x_0, B(y, l)) \) as required. Now consider the case where \( \frac{r}{d(x_0, y)} < \sin(\text{error}_a(d(r, y))) \). Let \( \sin^{-1}(\frac{r}{d(x_0, y)}) = \theta_0 \). It will decide to move to a point \( p_1 \in \text{seg}(x_0, y) \) such that \( \text{error}_a(d(x_0, p_1)) = \theta_0 \).

So it will move to a point \( x_1 \in B(p_1, l_1) \) where \( l_1 = d(x_0, p_1)\sin(\theta_0) \). We have \( \text{Cone}(x_0, B(p_1, l_1)) \subseteq \text{Cone}(x_0, B(y, l)) \). Hence \( x_1 \in \text{Cone}(x_0, B(y, l)) \) and the trajectory is inside \( \text{Cone}(x_0, B(y, l)) \).

![Figure 2: An illustration of an execution of Algorithm 1](image)

If \( d(x_1, y) < l \), then we are done. If \( d(x_1, y) = l \), then \( r \) will decide to move to \( y \) and will get inside \( B(y, l) \) in one step. This is because it will reach a point \( z \) where \( d(z, y) < \mu(x_1, y)l \) \( \leq l \). So now assume that \( d(x_1, y) > l \). It will again check if \( \frac{r}{d(x_0, y)} \geq \sin(\text{error}_a(d(x_1, y))) \) or \( \frac{r}{d(x_1, y)} < \sin(\text{error}_a(d(x_1, y))) \). In the first case, it will get inside \( B(y, l) \) in one step and its complete trajectory is clearly inside \( \text{Cone}(x_0, B(y, l)) \). In the second case, it will decide to move to a point \( p_2 \in \text{seg}(x_1, y) \) such that \( \text{error}_a(d(x_1, p_2)) = \theta_1 \), where \( \theta_1 = \sin^{-1}(\frac{r}{d(x_1, y)}) \). It will reach a point \( x_2 \in B(p_2, l_2) \) where \( l_2 = d(x_1, p_2)\sin(\theta_1) \). We have \( \text{Cone}(x_1, B(p_2, l_2)) \subseteq \text{Cone}(x_1, B(y, l)) \). By
Property 1, we have $\text{Cone}(x_1, B(y, l)) \subset \text{Cone}(x_0, B(y, l))$. Hence $x_1 \in \text{Cone}(x_0, B(y, l))$ and the complete trajectory is inside $\text{Cone}(x_0, B(y, l))$.

Continuing like this, $r$ moves along a polygonal path $x_0 x_1 x_2 x_3 \ldots$ which lies inside $\text{Cone}(x_0, B(y, l))$. In each step, $r$ gets closer to $y$. If $\theta_i = \frac{1}{d(x_0, y)}$, we have $\theta_0 < \theta_1 < \theta_2 < \theta_3 < \ldots$. This implies that, if it cannot attempt to move to $y$ in the $i$th step ($i > 1$), the amount of distance it intends to move towards $y$ in that step is more than that in the $(i - 1)$th step. This implies that the minimum possible reduction in distance between $r$ and $y$ in the $i$th step is more than that in the $(i - 1)$th step. Therefore, after finitely many steps, it will get inside $B(y, l)$.

We now discuss some variants of the problem. They can be solved using the movement strategy of Algorithm 1 subject to some modifications.

1. Suppose that the robot $r$ is required to move inside, instead of a disk, some other region, say enclosed by some line segments and circular arcs. We can easily solve this problem using the same movement strategy, e.g., by fixing some disk $B(y, l)$ inside the region and following Algorithm 1. See Fig. 3a.

2. Now consider the situation where are the robot $r$, starting from $x_0$, have to get inside a disk $B(y, l)$, but there are some point obstacles that it needs to avoid. Let $\mathcal{O} \subset \mathbb{R}^2$ be the set of obstacles. However, there are no obstacles on $\text{seg}(x_0, y)$. Again a similar approach will work. Instead of $B(y, l)$, the robot $r$ just needs to consider $B(y, l')$ where $l' \in (0, l]$ is the largest possible length such that $\text{Cone}(r, B(y, l')) \cap \mathcal{O} = \emptyset$. See Fig. 3b.

3. Instead of point obstacles, now consider disk shaped obstacles. Assume that none of the obstacles intersect $\text{seg}(x_0, y)$. The same approach as in the previous problem would work here too. See Fig. 3c.

4. Now again consider point obstacles, but this time there might be some obstacles lying on $\text{seg}(x_0, y)$. Let $\mathcal{O}' = \mathcal{O} \cap \text{seg}(x_0, y)$. The robot will move to a point $x' \in \text{Cone}(r, B(y, l))$ so that there is no obstacle on $\text{seg}(x', y)$. For this, it will move so that it reaches a point in $B(y, l') \setminus \text{seg}(x_0, y)$ where $l' \in (0, l]$ is the largest possible length such that $\text{Cone}(r, B(y, l')) \cap (\mathcal{O} \setminus \mathcal{O}') = \emptyset$. See Fig. 3d.

## 4 The Main Algorithm

In this section, we present our main algorithm that solves the {approximate arbitrary pattern formation} problem from any initial configuration that does not have any unbreakable symmetries. Our algorithm works in $\text{OBLOT} + \text{SSYNC}$ and is sequential, which means that at any round at most one robot is instructed to move. We shall show in section 4.4 that this algorithm can be simulated in $\text{FCOM} + \text{ASYNC}$ using two colors. Therefore, we have the following result.

**Theorem 5.** {approximate arbitrary pattern formation} is solvable in $\text{OBLOT} + \text{SSYNC}$ and $\text{FCOM} + \text{ASYNC}$ from any initial configuration that does not have any unbreakable symmetries.

We now present our main algorithm for $\text{OBLOT} + \text{ASYNC}$. We assume that the initial configuration does not have any unbreakable symmetries. The algorithm works in three phases. In Phase 1, a configuration is produced which is asymmetric and all robots on its minimum enclosing circle are critical. Since the robots do not have access to any global coordinate system, there is no priori agreement regarding where and how the given pattern is to be formed on the plane. Therefore, to form the pattern, the robots first need to implicitly agree on some common coordinate system. The minimum enclosing circle of the configuration and the critical robots on it play a crucial role in establishing this agreement. So the minimum enclosing circle must remain invariant. Note that the minimum enclosing circle will remain invariant if the robots inside it move (while remaining inside) to form the pattern. However, the positions of the critical robots on the minimum enclosing circle may not be consistent.
(a) Starting from $x_0$, the robot has move inside the shaded region.

(b) Starting from $x_0$, the robot has move inside $B(y, l)$ avoiding point obstacles. There are no obstacles on the line segment joining $x_0$ and $y$.

(c) Starting from $x_0$, the robot has move inside $B(y, l)$ avoiding disk shaped obstacles. The line segment joining $x_0$ and $y$ does not intersect any obstacle.

(d) Starting from $x_0$, the robot has move inside $B(y, l)$ avoiding point obstacles. There are some obstacles on the line segment joining $x_0$ and $y$.

Figure 3: Some variants of Algorithm 1

with the pattern to be formed. So first the positions of the critical robots need to be adjusted to conform with the pattern to be formed. This is done in Phase 2. Then in Phase 3, the robots inside the minimum enclosing circle will reposition themselves to form the pattern.

The three phases are described in detail in the next three subsections. We introduce some Boolean predicates in Table 1 that will be used to describe the structure of the phases.
Table 1: Description of the variables used by the robots

| Variable | Description |
|----------|-------------|
| a        | \(R\) is an asymmetric configuration. |
| u        | \(R\) has an unbreakable symmetry. |
| c        | All robots on \(C(R)\) are critical. |
| s        | \(R\) is symmetry safe. |
| b        | The bounding structure is formed. |

4.1 Phase 1

The algorithm is in Phase 1 if \(\neg u \land (\neg a \lor \neg c)\) holds. The objective of this phase is to create a configuration in which \(a \land c\) holds, i.e., create an asymmetric configuration in which all robots on the minimum enclosing circle are critical. Phase 1 consists of three subphases: Subphase 1.1, Subphase 1.2 and Subphase 1.3. The algorithm is in Subphase 1.1 if \(\neg u \land \neg a\) holds, in Subphase 1.2 if \(a \land \neg s \land \neg c\) holds and in Subphase 1.3 if \(s \land \neg c\) holds. These subphases are described in detail in Sections 4.1.1, 4.1.2 and 4.1.3 respectively.

4.1.1 Subphase 1.1

The algorithm is in Subphase 1.1 if \(\neg u \land \neg a\) holds. Our objective is to create an asymmetric configuration, i.e., have \(a\). To describe the action of the robots in Subphase 1.1, we consider the following four cases.

Case 1 consists of the configurations in Subphase 1.1 where there is a robot at \(c(R)\). Now consider the cases where there is no robot at \(c(R)\). Notice that in this case, \(R\) cannot have a rotational symmetry because \(\neg u\) holds. So \(R\) has a reflectional symmetry with respect to a unique line \(\ell\). Since \(\neg u\) holds, there are robots on \(\ell\). If there is a non-critical robot on \(\ell\) then we call it Case 2. For the remaining cases where there is no non-critical robot on \(\ell\), we call it Case 3 if there are more than 2 robots on \(C(R)\) and Case 4 if there are exactly 2 robots on \(C(R)\).

Case 1. We have a robot \(r\) at \(c(R)\). In this case, \(r\) will move away from the center and all other robots will remain static. The destination \(y\) chosen by the robot \(r\) should satisfy the following the
conditions (See Fig. 4).

1. \(B(y, \mu(c(R), y)d(c(R), y)) \subset \text{encl}(C^2_{c}(R)) \setminus \{c(R)\}\)

2. \(B(y, \mu(c(R), y)d(c(R), y)) \cap \ell = \emptyset\) for any reflection axis \(\ell\) of \(R \setminus \{r\}\).

It is easy to see that such an \(y\) exists. Furthermore, \(r\) can easily compute such an \(y\).

**Lemma 1.** If the algorithm is in Subphase 1.1, Case 1 at some round, then after finitely many rounds we have a.

**Proof.** Suppose that \(r\) moves to a point \(z \in B(y, \mu(c(R), y)d(c(R), y))\). We show that the resulting configuration, say \(R'\), is asymmetric. Notice that condition 1 implies that \(r\) is not at \(c(R')\) and it is the unique robot closest to \(c(R')\). This implies the configuration has no rotational symmetry. For the sake of contradiction, assume that \(R'\) has reflectional symmetry with respect to a line \(\ell\). First assume that \(r\) is on \(\ell\). Then \(R' \setminus \{r\} = R \setminus \{r\}\) has reflectional symmetry with respect to \(\ell\). This cannot happen because of condition 2. So now assume that \(r\) is not on \(\ell\). This implies that there is another robot \(r'\) (its specular partner with respect to \(\ell\)) such that \(d(r, c(R')) = d(r', c(R'))\). This contradicts the fact that it is the unique closest robot to \(c(R')\). Hence \(R'\) does not have reflectional symmetry. Hence we conclude that \(R'\) is asymmetric.

**Case 2.** In this case, there is no robot at \(c(R)\), \(R\) has reflectional symmetry with respect to a unique line \(\ell\) and there is at least one non-critical robot on \(\ell\). Since \(\neg\)b holds, the views of the robots on \(\ell\) are all distinct. So let \(r\) be the non-critical robot on \(\ell\) with minimum view. Only \(r\) will move in this case. If \(r\) is at point \(x\), the destination \(y\) chosen by \(r\) should satisfy the following conditions (See Fig. 5).

1. \(x \in C^1_{c}(R)\), then \(\text{Cone}(x, B(y, \mu(x, y)d(x, y))) \subset \text{encl}(C^1_{c}(R)) \setminus \text{encl}(C^1_{c}^{-1}(R))\)

2. \(B(y, \mu(x, y)d(x, y)) \cap \ell = \emptyset\) for any reflection axis \(\ell\) of \(R \setminus \{r\}\).

Such points clearly exist and \(r\) can easily compute one.

![Figure 5: The movement in Subphase 1.1, Case 2.](image-url)
Proof. Suppose that \( r \) moves to a point \( z \in B(y, \mu(x, y)d(x, y)) \). Let \( R' \) be the new configuration. Since \( r \) was a non-critical robot \( C(R) = C(R') \). Notice that condition 1 implies that \( r \) is a unique robot at a distance \( d(z, c(R')) > 0 \) from \( c(R') \). This implies the configuration has no rotational symmetry. Similarly as in Lemma 1 we can show that \( R' \) has no reflectional symmetry. Hence we conclude that \( R' \) is asymmetric.

\[
\text{Case 3. We have no robot at } c(R), R \text{ has reflectional symmetry with respect to a unique line } \ell, \text{ there is no non-critical robot on } \ell \text{ and } C(R) \text{ has at least 3 robots on it. First we prove the following result.}
\]

\textbf{Lemma 3.} In Subphase 1.1, Case 3, the following are true.

1. There is no robot on \( \ell \cap \text{encl}(C(R)) \).
2. There is exactly one robot on \( \ell \).
3. If \( r \) is the unique robot on \( \ell \), then \( \frac{\pi}{2} < \max\{\angle rc(R)r'' \mid r'' \in R \cap C(R)\} < \pi \).

\textbf{Proof.} 1) This follows from the fact that any robot in \( \text{encl}(C(R)) \) is non-critical.

2) For the sake of contradiction assume that there are two robots on \( \ell \), i.e., two antipodal robots on \( \ell \cap C(R) \). Let these two robots be \( r \) and \( r' \). Let \( \ell' \) be the line perpendicular to \( \ell \) and passing through \( c(R) \). Let \( H \) and \( H' \) be the closed half-planes delimited by \( \ell' \) that contain \( r \) and \( r' \) respectively. By Property 5 and the fact that \( R \) has a reflectional symmetry with respect to \( \ell \), \( H \cap C(R) \) has no robot other than \( r \) because otherwise \( r \) is non-critical. Similarly, \( H' \cap C(R) \) has no robot other than \( r' \). This contradicts the fact that \( C(R) \) has at least 3 robots on \( C(R) \).

3) The robot \( r \) is on \( \ell \cap C(R) \). Let \( \max\{\angle rc(R)r'' \mid r'' \in R \cap C(R)\} = \theta \). We have \( \theta < \pi \) because of 2). We have \( \theta \neq \frac{\pi}{2} \) because then \( r \) is not critical. Finally, we cannot have \( \theta < \frac{\pi}{2} \) because of Property 3. \( \square \)

\textbf{Figure 6: The movement in Subphase 1.1, Case 3.}

Let \( r \) be the unique robot on \( \ell \). In fact it is on \( \ell \cap C(R) \). Let \( x \) denote its position. Let \( r_1, r_2 \) be the two robots (specular with respect to \( \ell \)) on \( C(R) \) such that the \( \angle rc(R)r_1 = \angle rc(R)r_2 = \max\{\angle rc(R)r'' \mid r'' \in R \cap C(R)\} \). Only \( r \) will move in this case and the chosen destination \( y \) should satisfy the following the conditions (See Fig. 6).
1. \( B(y, \mu(x, y)d(x, y)) \cap \ell = \emptyset \)

2. \( \text{Cone}(x, B(y, \mu(x, y)d(x, y))) \subset \text{ext}(C(R)) \cap \text{encl}(C') \) where \( C' \) is the largest circle from \( \{ C \in F(r_1, r_2) \mid R \subset \text{encl}(C) \} \)

3. \( B(y, \mu(x, y)d(x, y)) \cap C_i = \emptyset \), where \( C_i = C(r_i, d(r_1, r_2)), i = 1, 2 \).

4. \( B(y, \mu(x, y)d(x, y)) \subset \mathcal{S}(L_1, L_2) \)

Again, it is straightforward to see that such an \( y \) should exist and \( r \) can easily compute one.

**Lemma 4.** If the algorithm is in Subphase 1.1, Case 3 at some round, then after finitely many rounds we have \( a \).

**Proof.** Suppose \( r \) attempts to move to a point \( y \) as described and reaches a point \( P \). Let \( R' \) be the new configuration. Let \( C_1 = CC(\{ r_1, r_2, P \}) \). We shall prove that \( C_1 = C(R') \).

Let \( R'' = R \setminus \{ r, r_1, r_2 \} \). We observe that \( R'' \subset \text{encl}(C_0) \cap \text{encl}(C') \), where \( C_0 \) is the minimum enclosing circle of the initial configuration \( R \) and \( C' \) is as defined in the description. It is easy to see that for any \( C \in (C_0, C')_F(r_1, r_2) \), \( \text{encl}(C_0) \cap \text{encl}(C') \subset \text{encl}(C) \). Clearly \( C_1 \in (C_0, C')_F(r_1, r_2) \) and hence \( R'' \subset \text{encl}(C_1) \). Since \( P \in S(L_1, L_2) \), \( \Delta r_1 r_2 P \) is an acute angled triangle. Hence \( C_1 = C(R') \).

By condition 1 and 3, \( \Delta r_1 r_2 P \) is a scalene triangle. Hence \( R' \) is asymmetric. \( \square \)

**Case 4.** In this case, we have no robot at \( c(R) \). \( R \) has reflectional symmetry with respect to a unique line \( \ell \), there is no non-critical robot on \( \ell \) and \( C(R) \) has exactly 2 robots on \( C(R) \).

**Lemma 5.** In Subphase 1.1, Case 4, the following are true.

1. There is no robot on \( \ell \cap \text{encl}(C(R)) \).

2. There are two antipodal robots on \( \ell \).

**Proof.**

1) The same as in Lemma 3.

2) Since \( \neg b \) holds, \( \ell \) has at least one robot \( r \). By 1), it is on \( C(R) \). There is exactly one robot \( r' \) on \( C(R) \) other than \( r \) by our assumption. By Lemma 3, it must be diametrically opposite to \( r \), i.e., on \( \ell \). Hence \( r \) and \( r' \) are the antipodal robots on \( \ell \). \( \square \)

Let \( r \) and \( r' \) be the antipodal robots on \( \ell \). Since \( \neg b \) holds, the views of \( r \) and \( r' \) are different. So let \( r \) be the robot with minimum view. Only \( r \) will move in this case. Let \( \ell' \) be the line perpendicular to \( \ell \) and passing through \( r \). For each \( r'' \in R \setminus \{ r, r' \} \), consider the line passing through \( r'' \) and perpendicular to \( \text{seg}(r''r') \). Consider the points of intersection of these lines with \( \ell' \). Let \( P_1, P_2 \) (specular with respect to \( \ell \)) be the two of these points that are closest to \( \ell \). Let \( L_1, L_2 \) be the lines parallel to \( \ell \) and passing through \( P_1, P_2 \) respectively. Assuming that \( r \) is at point \( x \), the destination \( y \) chosen by \( r \) should satisfy the following conditions.

1. \( \text{Cone}(x, B(y, \mu d(x, y))) \subset \text{ext}(C(R)) \)

2. \( B(y, \mu(x, y)d(x, y)) \cap \ell = \emptyset \)

3. \( B(y, \mu(x, y)d(x, y)) \subset \mathcal{S}(L_1, L_2) \)

4. \( B(y, \mu(x, y)d(x, y)) \cap C(c, d(c, r')) = \emptyset \), where \( c = c(R \setminus \{ r, r' \}) \).

**Lemma 6.** If the algorithm is in Subphase 1.1, Case 4 at some round, then after finitely many rounds we have \( a \).
Proof. Let \( r, r' \) be the antipodal robots on \( \ell \) and \( r \) be the robot with minimum view among them. Let \( R' = R \setminus \{ r, r' \} \). Let \( A \) and \( B \) denote the positions of \( r \) and \( r' \) respectively. Let \( \ell' \) and \( \ell'' \) be the lines perpendicular to \( \ell \) and passing through \( A \) and \( B \) respectively. Let \( P_1, P_2 \) (specular with respect to \( \ell \) ) be the two points on \( \ell' \) as defined in the description of Case 4. Suppose that \( r \) decides to move to a point that satisfies 1-4 and reaches the point \( P \). Let \( H_1 \) and \( H_2 \) be the open half planes delimited by \( \ell \) that contain \( P_1 \) and \( P_2 \) respectively. Without loss of generality, assume that \( P \in H_1 \). The line parallel to \( \ell \) and passing through \( P \) intersects \( \ell' \) and \( \ell'' \) at \( C \) and \( D \) respectively. Let \( C_1 = CC([A, B]) \) (the red circle in Fig. 8), \( C_2 = CC([A, B, C]) \) (the green circle in Fig. 8), \( C_3 = CC([B, P]) \) (the orange circle in Fig. 8) and \( C_4 = CC([A, B, P_1]) \). We have \( C_1 = C(R) \) before the move by \( r \). We shall prove that after the move, we have \( C_3 = C(R) \). To show this we only need to prove that \( R' \subset encl(C_3) \).

Notice that \( C_1 \in F(A, B) \), \( C_2 \in F(A, B) \), \( C_4 \in F(A, B) \), \( C_2 \in F(B, D) \) and \( C_3 \in F(B, D) \). The definition of \( P_1 \) implies that \( R' \cap H_2 \subset encl(C) \) for any \( C \in (C_1, C_4)_{F(A, B)} \). Since \( C_2 \in (C_1, C_4)_{F(A, B)} \), we have \( R' \cap H_2 \subset encl(C_2) \). Also, \( R' \cap H_1 \subset encl(C_2) \) as \( encl(C_1) \cap H_1 \subset encl(C_2) \). So \( R' \subset encl(C_2) \). Now let \( H_3 \) be the open half plane delimited by \( \ell'' \) that contains \( R' \). So
\( R' \subset \text{encl}(C_2) \cap \mathcal{H}_3 \). But \( \text{encl}(C_2) \cap \mathcal{H}_3 \subset \text{encl}(C_3) \). Hence \( R' \subset \text{encl}(C_3) \), as required.

We have shown that \( C_3 \) is the minimum enclosing circle of the new configuration. Furthermore, \( C_3 \) has exactly two robots on it, i.e., \( r_1 \) and \( r_2 \). We have to show that the new configuration is asymmetric.

Let \( R'' \) denote the new configuration. Let \( L_1 = \text{line}(P,B) \) and \( L_2 \) the perpendicular bisector of \( \text{seg}(B,P) \). If \( R'' \) has rotational symmetry, then \( R' \) also has rotational symmetry and the center of their minimum enclosing circles must coincide. But \( c(R') \in \text{seg}(A,B) \) and \( c(R'') \in \text{seg}(P,B) \). This is a contradiction as \( \text{seg}(A,B) \cap \text{seg}(P,B) = \emptyset \). If the \( R'' \) has reflectional symmetry, then the axis of symmetry is either \( L_1 \) or \( L_2 \). If the \( R'' \) has reflectional symmetry with respect to \( L_1 \), then \( R' \) also has reflectional symmetry with respect to \( L_1 \). This implies that \( c(R') \in L_1 \), to be more precise, \( c(R') \in \text{seg}(P,B) \). This is a contradiction as \( c(R') \in \text{seg}(A,B) \) and \( \text{seg}(A,B) \cap \text{seg}(P,B) = \emptyset \).

So now assume that \( R'' \) has reflectional symmetry with respect to \( L_2 \). So \( R' \) also has reflectional symmetry with respect to \( L_2 \). Hence \( c(R') \in L_2 \). Since any point on \( L_2 \) is equidistant from \( B \) and \( P \), we have \( d(c(R'), B) = d(c(R'), P) \). But this contradicts condition 4. Therefore, we conclude that \( R'' \) is asymmetric.

\[
\begin{align*}
R' \subset \text{encl}(C_2) \cap \mathcal{H}_3. \text{ But } \text{encl}(C_2) \cap \mathcal{H}_3 \subset \text{encl}(C_3). \text{ Hence } R' \subset \text{encl}(C_3), \text{ as required.}
\end{align*}
\]

We have shown that \( C_3 \) is the minimum enclosing circle of the new configuration. Furthermore, \( C_3 \) has exactly two robots on it, i.e., \( r_1 \) and \( r_2 \). We have to show that the new configuration is asymmetric.

Let \( R'' \) denote the new configuration. Let \( L_1 = \text{line}(P,B) \) and \( L_2 \) the perpendicular bisector of \( \text{seg}(B,P) \). If \( R'' \) has rotational symmetry, then \( R' \) also has rotational symmetry and the center of their minimum enclosing circles must coincide. But \( c(R') \in \text{seg}(A,B) \) and \( c(R'') \in \text{seg}(P,B) \). This is a contradiction as \( \text{seg}(A,B) \cap \text{seg}(P,B) = \emptyset \). If the \( R'' \) has reflectional symmetry, then the axis of symmetry is either \( L_1 \) or \( L_2 \). If the \( R'' \) has reflectional symmetry with respect to \( L_1 \), then \( R' \) also has reflectional symmetry with respect to \( L_1 \). This implies that \( c(R') \in L_1 \), to be more precise, \( c(R') \in \text{seg}(P,B) \). This is a contradiction as \( c(R') \in \text{seg}(A,B) \) and \( \text{seg}(A,B) \cap \text{seg}(P,B) = \emptyset \).

So now assume that \( R'' \) has reflectional symmetry with respect to \( L_2 \). So \( R' \) also has reflectional symmetry with respect to \( L_2 \). Hence \( c(R') \in L_2 \). Since any point on \( L_2 \) is equidistant from \( B \) and \( P \), we have \( d(c(R'), B) = d(c(R'), P) \). But this contradicts condition 4. Therefore, we conclude that \( R'' \) is asymmetric.

\[
\begin{align*}
\text{Figure 8: Illustrations supporting the proof of Lemma 6}
\end{align*}
\]

\[4.1.2 \text{ Subphase 1.2}\]

Let us suppose that \( a \land \neg s \land \neg c \) holds. Our goal is to make the configuration symmetry safe. Since \( \neg c \) holds, the configuration will be symmetry safe if there is no robot at \( c(R) \), there is a unique closest and a unique second closest robot to \( c(R) \) and they are not collinear is with \( c(R) \). Now, if there is a robot at \( c(R) \) it will move in the same way as in Subphase 1.1, Case 1. Now consider the case, where there is no robot at \( c(R) \). If there is no unique robot closest to the center, then the non-critical robot on \( C_1^2(R) \) with the minimum view will decide to move to a point \( y \) such that \( y \) should satisfy the following conditions:

\[
\begin{align*}
1. \quad B(y, \mu d(c(R), y)) \subset \text{encl}(C_1^2(R)) \setminus \{c(R)\} \\
2. \quad B(y, \mu d(c(R), y)) \cap \ell = \emptyset \text{ for any reflection axis } \ell \text{ of } R \setminus \{r\}.
\end{align*}
\]

When there is a unique robot closest to the center, if there is only one robot on \( C_1^2(R) \), we are done. Otherwise, among all the non-critical robots lying on \( C_1^2(R) \), the one with the minimum view moves to a point \( y \) such that the following holds:
1. \( \text{Cone}(x, B(y, \mu d(c(R), y))) \subset \text{encl}(C_2(R)) \setminus \text{encl}(C_1(R)) \)

2. \( B(y, \mu d(c(R), y)) \cap \ell = \emptyset \) for any reflection axis \( \ell \) of \( R \setminus \{r\} \).

**Lemma 7.** If the algorithm is in Subphase 1.2, then after finitely many rounds we have \( s \).

**4.1.3 Subphase 1.3**

Let us suppose that, \( s \land \neg c \) holds. Our objective is to make the configuration \( a \land c \). As the configuration is asymmetric, there is a robot \( r_1 \) with minimum view among all the non-critical robots lying on \( C(R) \). Suppose that it is at \( x \). Then \( r_1 \) will decide move to a point \( y \) such that \( \text{Cone}(x, B(y, \mu d(x, y))) \cap R = \emptyset \) and \( \text{Cone}(x, B(y, \mu d(x, y))) \subset \text{ext}(C_2(R)) \setminus \text{encl}(C(R)) \). Clearly \( s \) holds in the resulting configuration. Clearly after finitely many rounds it is easy to see that \( s \land c \), and hence \( a \land c \) holds.

**Lemma 8.** If the algorithm is in Subphase 1.3, then after finitely many rounds we have \( a \land c \).

### 4.2 Phase 2

Let us first introduce some definitions. If Algorithm 2 is applied on the input pattern \( F \), then we obtain a set \( B_F \subseteq C(F) \cap F \) of pattern points such that if \( F' = (F \setminus \text{encl}(C(F))) \cup B_F \), then \( C(F') = C(F) \) and each point of \( B_F \) is a critical point of \( F' \). In other words, \( B_F \) is a minimal set of points of \( \subseteq C(F) \cap F \) such that \( C(C(B_F)) = C(F) \). By minimal set we mean that no proper subset of \( B_F \) has this property. Clearly by Property 3, \( B_F \) either consists of two antipodal points or three points that form an acute angled triangle. We call \( B_F \) the bounding structure of \( F \) (See Fig. 9a).

**Algorithm 2:**

```
Input : A pattern \( F = \{f_1, \ldots, f_n\} \)
1. Let \( C(F) \cap F = \{f_{j_1}, \ldots, f_{j_k}\} \), where \( j_1 < \ldots < j_k \)
2. \( B_F \leftarrow \{f_{j_1}, \ldots, f_{j_k}\} \)
3. for \( i = 1, \ldots, k \) do
4.   if \( f_{j_i} \) is non-critical in \( F \) then
5.     \( F \leftarrow F \setminus \{f_{j_i}\} \)
6.     \( B_F \leftarrow B_F \setminus \{f_{j_i}\} \)
7. Return \( B_F \)
```

We shall say that the bounding structure of \( F \) is formed by the robots if one of the following holds.

1. \( B_F \) has exactly two points, \( C(R) \) also has exactly two robots on it and \( R \) is symmetry safe.

2. \( B_F \) has exactly three points and \( C(R) \) also has exactly three robots on it. Let \( B_F = \{f_{i_1}, f_{i_2}, f_{i_3}\} \) and \( C(R) \cap R = \{r_1, r_2, r_3\} \). \( R \) is symmetry safe and furthermore, if \( \text{seg}(r_1, r_2) \) is the largest side of triangle formed by \( r_1, r_2, r_3 \) and \( \text{seg}(f_{i_1}, f_{i_2}) \) is a largest side of triangle formed by \( f_{i_1}, f_{i_2}, f_{i_3} \), there is an embedding \( f_i \mapsto P_i \) of \( F \) on the plane identifying \( \text{seg}(f_{i_1}, f_{i_2}) \) with \( \text{seg}(r_1, r_2) \) so that

   - \( r_3 \in B(P_{i_3}, \epsilon D) \), \( (D = \text{size of the embedded pattern } \{P_1, \ldots, P_n\}) \)
   - \( B(P_i, \epsilon D) \cap \text{encl}(C(r_1, r_2, r_3)) \neq \emptyset \) for all \( i \in \{1, \ldots, n\} \)

   The algorithm is in Phase 2 if \( a \land c \land \neg b \) holds. The objective is to have \( b \). We describe the algorithm for the following three cases.

#### 4.2.1 Case 1

Assume that \( C(R) \) has three robots and the bounding structure consists of three points. So the goal is to transform the triangle of the robots on \( C(R) \) so that the bounding structure of \( F \) is formed. Let \( C(R) \cap R = \{r_1, r_2, r_3\} \). Let \( T(r_1, r_2, r_3) \) denote the triangle formed by \( r_1, r_2, r_3 \). If \( T(r_1, r_2, r_3) \) is not scalene, then we shall make it using similar techniques from Subphase 1.1, Case 3. So now assume that \( T(r_1, r_2, r_3) \) is scalene. Let \( \text{seg}(r_1, r_2) \) be the largest side of \( T(r_1, r_2, r_3) \). In that case,
Figure 9: a) The input pattern $F$. The bounding structure $b_F$ consists of the blue pattern points. b)-c) The bounding structure is formed by the robots. d) To obtain a final configuration, each shaded region must have a robot inside it.

$r_3$ will be called the transformer robot. This robot will move to form the bounding structure of $F$. Let $L$ be the perpendicular bisector of $\overline{r_1r_2}$. Since no two sides of the triangle are of equal length, $r_3 \notin L$. Let $H$ be the open half-plane delimited by $L$ that contains $r_3$. Without loss of generality, assume that $r_1 \in H$. Let $L_1$ be the line parallel to $L$ and passing through $r_1$. Let $H'$ be the open half-plane delimited by $L'$ that contains $L$. Since $T(r_1, r_2, r_3)$ is acute angled, $r_3 \in H'$. Let $H''$ be the open half-plane delimited by $\text{line}(r_1, r_2)$ that contains $r_3$. Let $C_1 = C(r_1, \delta(r_1, r_2))$ and $C_2 = C(r_2, \delta(r_2, r_1))$. Since $\overline{r_1r_2}$ is the largest side of $T(r_1, r_2, r_3)$, $r_3 \in \text{encl}(C_1) \cap \text{encl}(C_2)$. If $C_3 = CC(r_1, r_2)$, then $r_3 \in \text{ext}(C_3)$ as $T(r_1, r_2, r_3)$ acute angled. Now take the largest side of the bounding structure $B_F$. In case of a tie, use the ordering of the points in the input $F$ to choose one of them. Embed the bounding structure $B_F$ on the plane identifying this side with $\overline{r_1r_2}$ so that the third point of the bounding structure is mapped to a point $P \in H \cap H''$. Since the bounding structure is acute angled, $P \in H'$. Also, $P \in \text{ext}(C_3)$ for the same reason. Furthermore,
since a largest side of the bounding structure is identified with $\overline{\text{seg}}(r_1, r_2)$, $P \in \text{encl}(C_1) \cap \text{encl}(C_2)$.

So we have $r_3 \in \mathcal{H} \cap \mathcal{H}' \cap \mathcal{H}'' \cap \text{encl}(C_1) \cap \text{encl}(C_2) \cap \text{ext}(C_3)$ (the blue region in Fig. 10a) and $P \in \overline{\mathcal{H}} \cap \mathcal{H}' \cap \mathcal{H}'' \cap \text{encl}(C_1) \cap \text{encl}(C_2) \cap \text{ext}(C_3)$.

We shall say that the transformer robot is **eligible to move** if $R \cap \text{encl}(C(R)) \subset \text{encl}(C_3) \cap \text{encl}(C_4)$, where $C_4$ is the circle passing through $r_1, r_2$ and point in $\mathcal{H}''$ where $C_1$ and $C_2$ intersect each other (the red region in Fig. 10b). The transformer robot will not move until it becomes eligible. So the robots in $\text{encl}(C(R))$ that are not in $\text{encl}(C_3) \cap \text{encl}(C_4)$ should move inside this region first. As discussed in Section 3.3, the robots will fix some specific disk $B(p, l) \subset \text{encl}(C_3) \cap \text{encl}(C_4)$ and move through the cone defined by it. The robots should move sequentially. So we ask the robot outside $\text{encl}(C_3) \cap \text{encl}(C_4)$ that is closest to $p$ to move first. If there are multiple such robots one with the minimum view is chosen. Recall that the configuration is asymmetric as $T(r_1, r_2, r_3)$ is scalene.

So now assume that $R \cap \text{encl}(C(R)) \subset \text{encl}(C_3) \cap \text{encl}(C_4)$, i.e., the transformer robot $r_3$ is eligible to move. Now $r_3$ will move in order to form the bounding structure. The movement of $r_3$ should be restricted inside the blue region $\mathcal{H} \cap \mathcal{H}' \cap \mathcal{H}'' \cap \text{encl}(C_1) \cap \text{encl}(C_2) \cap \text{ext}(C_3)$ as it moves inside $B(P, \epsilon D) \cap \mathcal{H} \cap \mathcal{H}' \cap \mathcal{H}'' \cap \text{encl}(C_1) \cap \text{encl}(C_2) \cap \text{ext}(C_3)$. As discussed in Section 3.3, $r_3$ will set a disk $B(P', l) \subset B(P, \epsilon D) \cap \mathcal{H} \cap \mathcal{H}' \cap \mathcal{H}'' \cap \text{encl}(C_1) \cap \text{encl}(C_2) \cap \text{ext}(C_3)$ (depending only on the shape of the region) and move through $\text{cone}(r_3, B(P', l))$ to get inside it. However, $\text{cone}(r_3, B(P', l))$
may not be completely contained inside the blue region. First $r_3$ may have to move so that $\text{seg}(r_3, P')$ is inside the blue region. Then $r_3$ can move accordingly as discussed in Section 3.3 by considering the region outside the blue region as obstacle.

![Figure 11: Illustration supporting the proof of Lemma 9.](image)

**Lemma 9.** If the algorithm is in Phase 2, Case-1 at some round, then after finitely many rounds we have $b$.

**Proof.** Clearly all the robots of $R \setminus \{r_1, r_2, r_3\}$ will move inside the red region (Shown in Fig.10b) in finitely many steps. Then $r_3$ becomes eligible to move. Recall that the movement of $r_3$ is restricted inside the blue region (Shown in Fig.10a). Notice that for any point $Q$ in the blue region, 1) $\Delta qr_1 r_2$ is acute angled, 2) $d(r_1, r_2)$ is the unique largest side of the triangle $\Delta qr_1 r_2$, 3) $R \setminus \{r_3\} \subseteq \text{encl} CC(Qr_1 r_2)$. This implies that $r_3$ remains the transformer robot during the movement. As discussed earlier $r_3$ will move inside $B(P, \epsilon D) \setminus H \setminus H' \setminus H'' \setminus \text{encl}(C_1) \setminus \text{encl}(C_2) \setminus \text{ext}(C_3)$ in finitely many steps, to form the bounding structure.

4.2.2 Case 2

In this case, $C(R)$ has three robots and the bounding structure consists of two points. Let $C(R) \cap R = \{r_1, r_2, r_3\}$ and $T(r_1, r_2, r_3)$ be the triangle formed by $r_1, r_2, r_3$. First the triangle $T(r_1, r_2, r_3)$ will be made scalene. Let $\text{seg}(r_1, r_2)$ be the largest side. Then $r_3$ will be the transformer robot. The plan is to move the transformer robot inward so that it is no longer on the minimum enclosing circle of the configuration. Let $C_1, C_2, C_3, H, H', H''$ denote the same as in Case 1. As before, we have $r_3 \in H \cap H' \cap H'' \cap \text{encl}(C_1) \cap \text{encl}(C_2) \cap \text{ext}(C_3)$ (the blue region in Fig. 12).

We shall say that the transformer robot is eligible to move if 1) $R \cap \text{encl}(C(R)) \subseteq \text{encl}(C_3)$ (the red region in Fig. 12) and 2) $R \setminus \{r_3\}$ is a symmetry safe configuration. The robots in $\text{encl}(C(R))$ will first move inside $\text{encl}(C_3)$. After that $C(R \setminus \{r_3\})$ will have only two robots, i.e., $r_1$ and $r_2$. So $R \setminus \{r_3\}$ will be symmetry safe if there is a unique robot closest to $O$, the midpoint of $r_1, r_2$, and it is not on $r_1, r_2$ or its perpendicular bisector. This can achieved easily. When $r_3$ becomes eligible to move, it will move inside the region $\text{encl}(C_3)$. We have to make sure that it does not collide with
other robots and its trajectory is inside the region $H \cap H' \cap H'' \cap \text{encl}(C_1) \cap \text{encl}(C_2) \cap \text{ext}(C)$ where $C = C_1^i(R \setminus \{r_3\})$. This can be done by the scheme described in 3.3.

![Figure 12: Illustrations for Phase 2, Case 2.](image)

**Lemma 10.** If the algorithm is in Phase 2, Case 2 at some round, then after finitely many rounds we have $b$.

**Proof.** After finitely many rounds $r_3$ will become eligible to move. Then, as in Case-1, $r_3$ will move through the blue region until it enters the red region, depicted in Fig. 12. While in the blue region $r_3$ will remain the transformer robot as in Case 1. After it enters the red region we have $CC(r_1, r_2) = C(R)$. Since $r_3 \in \text{ext}(C_1^i(R \setminus \{r_3\}))$, the configuration symmetry safe. Hence the bounding structure is formed. 

4.2.3 Case 3

In this case, $C(R)$ has two robots and the bounding structure consists of three points. Let $C(R) \cap R = \{r_1, r_2\}$. Here the strategy is to move outward one of the robots from $\text{encl}(C(R))$, say $r$, so that the minimum enclosing circle of the configuration is the circumcircle of $r, r_1$ and $r_2$. We shall call $r$ the transformer robot. The robot farthest from $c(R)$ will be chosen as the transformer robot. In case of a tie, it is broken using the asymmetry of the configuration.

Let $H$ be the open half plane delimited by $\text{line}(r_1, r_2)$ that contains $r$. Let $L_1$ and $L_2$ be the lines perpendicular to $\text{line}(r_1, r_2)$ and passing through respectively $r_1$ and $r_2$. Let $L$ be the perpendicular bisector of $\text{seg}(r_1, r_2)$. Without loss of generality, assume that $r \in S(L_1, L) \cup L$. Let $C_1 = C(r_1, d(r_1, r_2)), C_2 = C(r_2, d(r_2, r_1))$ and $C_3 = CC(r_1, r_2)$. Let $C_4$ be the largest circle from the family $\{C \in C(r_1, r_2) \mid \text{center of } C \text{ lies in } H \text{ and } R \subset \text{encl}(C)\}$. The algorithm asks $r$ to move into the region $\text{encl}(C_1) \cap \text{encl}(C_2) \cap \text{ext}(C_3) \cap \text{encl}(C_4) \cap H \cap S(L_1, L)$ (the blue region in Fig. 13). Again, this can be done by the scheme described in 3.3.

**Lemma 11.** If the algorithm is in Phase 2, Case 3 at some round, then after finitely many rounds we have $b$. 
Proof. Here the transformer robot $r$, which is a farthest robot from $c(R)$ has to get inside the blue region shown in Fig. 13. Until $r$ reaches there the minimum enclosing circle of the configuration remains unchanged and it remains the transformer robot since its movement is outward. When it moves to a point in the blue region we have $CC(r, r_1, r_2) = C(R)$. Hence the situation reduces to Case 1. So the bounding structure will be formed after finitely many rounds by Lemma 9.

4.2.4 Case 4

In this case, $C(R)$ has two exactly robots and the bounding structure also has exactly two points. So we have $\neg b$ because the configuration is not symmetry safe. The configuration can be easily made symmetry safe by previously discussed techniques.

Lemma 12. If the algorithm is in Phase 2, Case 4 at some round, then after finitely many rounds we have $b$.

4.3 Phase 3

The algorithm is in Phase 3 if $b$ holds. The objective of this phase is to form the pattern approximately. The termination condition of our algorithm is that $b$ holds (i.e., it is a Phase 3 configuration) and the configuration is $\epsilon$-close to $F$. Therefore, even if the initial configuration is $\epsilon$-close to $F$ (i.e., the pattern $F$ is already formed approximately), the algorithm will still go through the earlier phases to make $b$ hold and then approximately form the pattern while preserving $b$. The reason why we take this approach is because in general, even if the configuration is $\epsilon$-close to $F$, the robots may not be able to efficiently identify this. This is a basic difficulty of the problem. However, when $b$ holds there is a way fix a particular embedding of $F$ in the plane and then the only thing to check is whether there are robots close to each point of the embedding. For Phase 3, there are two cases to consider: $B_F$ has exactly two points (Case 1) and $B_F$ has exactly three points (Case 2).
4.3.1 Case 1

Let \( \{r_1, r_2\} = C(R) \cap R \). Let \( \ell = \text{line}(r_1, r_2) \) and \( \ell' \) be the line passing through \( c(R) \) and perpendicular to \( \ell \). Let \( r_1 \) be the unique robot that is closest to \( c(R) \) and in \( \text{encl}(C(R)) \setminus (\ell \cup \ell') \). We set a global coordinate system whose center is at \( c(R) \), \( X \) axis along \( \ell \), \( Y \) axis along \( \ell' \). The positive directions of \( X \) and \( Y \) axis are such that \( r_1 \) lies in the positive quadrant.

Perform a coordinate transformation (rotation) on the input pattern \( F \) so that the bounding structure is along the \( X \) axis. Let \( F' \) denote the input after this transformation. Consider the pattern points on \( C_1^F(F') \) except the points of the bounding structure. Reflect the pattern with respect to \( X \) axis or \( Y \) axis or both, if required, so that at least one of them is in the closed positive quadrant \( (X \geq 0, Y \geq 0) \). Let \( F'' \) denote the pattern thus obtained. Let \( \{f_0, f_1\} \) be the bounding structure of \( F'' \). So in \( F'' \), we have 1) \( f_0, f_1 \) on the \( X \) axis and 2) at least one point from \( C_1^F(F'') \cap (F \setminus \{f_j \}) \) in the closed positive quadrant. Each robot applies coordinate transformations on \( F \) and obtains the same pattern \( F'' \). Let \( f_i \) denote the first pattern point from \( C_1^F(F'') \cap (F \setminus \{f_j, f_k\}) \) that is in the closed positive quadrant.

The pattern \( F'' \) is mapped in the plane in the global coordinate system and scaled so that the bounding structure is mapped onto \( \text{seg}(r_1, r_2) \). These points are called the target points. The set of target points are denoted by \( T \). Let \( D \) be the diameter of \( C(T) \). Hence, \( D = d(r_1, r_2) \). Let \( t_i \) denote the target point corresponding to \( f_i \). Hence \( t_i \) is on \( C_1^F(T) \) and in the closed positive quadrant. Now we define the circle \( C_i \) as

- if \( t_i \in C_1^F(T) = c(T) \), then \( C_i = C(c(T), \epsilon D) \),
- if \( t_i \in C_1^F(T) = C(T) \), then \( C_i = C(c(T), (1 - \frac{\epsilon}{2})D) \),
- otherwise, \( C_i = C_1^F(T) \).

We shall say that a target point \( t \neq t_i \) is realized by a robot \( r \) if \( r \) is the unique closest robot to \( t \) and \( r \in B(t, \epsilon D) \cap \text{encl}(C_i) \). We shall say that \( t_i \) is realized by a robot \( r \) if all target points \( t \neq t_i \) are realized. \( r \) is the robot closest to \( t_i \) and \( r \in B(t, \epsilon D) \cap \text{encl}(C_i) \). Hence, if \( t_i \) is realized then it implies that all target points are realized, i.e., the given pattern is formed. We call this the final configuration.

The goal of this phase is to sequentially realize all the target points. For any non-final configuration in this phase, the target points can be partitioned as \( T = T_1 \cup T_2 \cup \{t_i\} \), where \( T_1 \) is the set of realized target points and \( T_2 \cup \{t_i\} \) are the unrealized target points.

This case works in the following way.

1. First the robot \( r_1 \) moves inside \( \text{encl}(C_i) \), if not already there. The movement should be such that \( s \) remains true.

2. When \( r_1 \) is inside \( \text{encl}(C_i) \), The robots from \( R \setminus \{r_1\} \) sequentially realize all the target points of \( T \setminus \{t_i\} \). During this, \( s \) should remain true and \( r_1 \) should remain as the unique robot closest to \( c(R) \).

3. When the target points of \( T \setminus \{t_i\} \) are realized, the robot \( r_1 \) will then realize \( t_i \). Again, \( s \) should remain true and \( r_2 \) should remain as the unique robot closest to \( c(R) \).

The movement strategy of 1 and 3 are straightforward. So we now discuss 2 in detail. The robots can be partitioned as \( R = R_1 \cup R_2 \cup \{r_1\} \), where \( R_1 \) is the set of robots realizing target points and \( R_2 \cup \{r_1\} \) are the rest. Notice that \( r_1 \) and \( r_2 \) are at the two target points corresponding to the bounding structure and hence \( r_1, r_2 \in R_1 \). So our goal here is to make \( T_2 = \emptyset, R_2 = \emptyset \). When \( T_2 \neq \emptyset, R_2 \neq \emptyset \), we choose the closest pair from the set \( R_2 \times T_2 \). In case of tie between say \( (r, t) \) and \( (r', t') \) with \( t \neq t' \), the target with lexicographically smaller coordinates (with respect to the global coordinate system) is chosen. In case of a tie between \( (r, t) \) and \( (r', t) \) the robot with lexicographically smaller coordinates is chosen. Let \( (r, t) \) be the chosen pair. We call the robot \( r \) the traveler and \( t \) its destination. The goal is for \( r \) to realize \( t \).
Figure 14: Illustrations for Phase 3, Case 1. In each row, the input pattern $F$ is shown on the left and a final configuration approximating $F$ is shown on the right. In each case, the bounding structure consists of the blue points and the green circle represents $C_1$. 
Let us denote the region \( B(t, \varepsilon D) \cap \overline{encl}(C(t)) \cap \overline{encl}(C(R)) \) as \( \mathcal{R}(t) \). Now \( r \) is either in \( \mathcal{R}(t) \) or not. In the first case, although \( r \in \mathcal{R}(t) \), there may be at least another robot \( r' \in \mathcal{R}(t) \) with \( d(r, t) = d(r', t) \). So it will move closer to \( t \) so that it realizes \( t \). So now we consider the later case where \( r \notin \mathcal{R}(t) \) and \( r \) has to get inside \( \mathcal{R}(t) \). \( \mathcal{R}(t) \) is either \( B(t, \varepsilon D) \) or some other region which can be of two types (a region bounded by two circular arcs or four circular arcs) as shown in Fig. 14f. As discussed in Section 3.3, in this case the robot will take some fixed disk \( B(i, l), i \in \mathcal{R}(t), l < \varepsilon D \) inside the region and try to move inside it. We set a fixed rule how the ball \( B(i, l) \) will be chosen and it depends only upon the shape of the region \( \mathcal{R}(t) \). We then call \( i \) the modified destination of \( r \). We use the terminology and the notation \( B(i, l) \) in general, i.e., even when \( \mathcal{R}(t) = (t, \varepsilon D) \), in which case \( B(i, l) = B(t, \varepsilon D) \). So the robot \( r \) has to get inside \( B(i, l) \). We now describe the algorithm.

Let \( S = \{ B(t', d(t', r')) \mid t' \in T_1 \text{ and } r' \text{ is the robot realizing } t' \cup \{ B(c(T), d(c(T), r_1)) \} \) be the set of disks around the points of \( T_1 \cup \{ c(T) \} \) with their radii being equal to their distances from their robots. The disks in \( S \) are obstacles that the robot needs to avoid while moving. If none of the disks from \( S \) intersect \( \text{seg}(r, t) \), then the robot can follow the movement strategy described in Section 3.3. Now assume that some disks from \( S \) are intersecting \( \text{seg}(r, t) \). If \( B(c, d(c, r')) \) is a disk such that \( c \) is either a target point from \( T_1 \) or \( c(T) \), then we say that \( r' \) is obstructing the traveler. First consider the case where all disks from \( S \) that intersect \( \text{Cone}(r, B(i, l)) \) have their centers not lying on \( \text{seg}(r, t) \). If \( B(c, d(c, r')) \) be such a disk, then \( r' \) will move closer to \( c \) so that the disk gets smaller and it does not intersect \( \text{seg}(r, t) \). If there are multiple such robots, then movements are sequentialized based on the view of their corresponding target points. Hence, after some rounds, there will not be any robot obstructing \( r \) and it can find a safe zone to move through. Now consider the case where the center of some disks from \( S \) is collinear with \( r \) and \( t \). In this case \( r \) will move to a point inside \( \text{Cone}(r, B(i, l)) \) so that there are no such collinearities as described in Section 3.3.

We have to show that \( r \) remains the traveler robot and \( t \) remains its target until it gets inside \( \mathcal{R}(t) \). Since \( r \) avoids the disks from \( S \), \( r \) does not become closest to any point from \( T_1 \cup \{ c(R) \} \). Therefore, all target points from \( T_1 \) remain realized by some other robot and \( r \) does not take the role of \( r_1 \). Recall that \( (r, t) \) was a closest pair from \( R_2 \times T_2 \) at first. After a move by \( r \), its distance from \( t \) reduces. So \( r \) remains the traveler. Furthermore, \( t \) remains its destination. To see this, assume for the sake of contradiction that \( r \) moves from \( x \) to \( y \) and we have \( d(y, t') < d(y, t) \) for some \( t' \in T_2 \). We have \( y \in \text{Cone}(x, B(t, l)) \subseteq \text{Cone}(x, B(t, \varepsilon D)) \). Let \( L \) be the perpendicular bisector of \( \text{seg}(t, t') \) and \( \mathcal{H} \) is the open half-plane delimited by \( L \) that contains \( t \). Since \( d(t, t') > 2 \varepsilon D \), \( B(t, \varepsilon D) \subset \mathcal{H} \) and hence \( \text{Cone}(x, B(t, \varepsilon D)) \subset \mathcal{H} \). This implies that \( y \in \mathcal{H} \) which contradicts our assumption that \( d(y, t') < d(y, t) \).

**Lemma 13.** If the algorithm is in Phase 3, Case 1 then we have a final configuration after finitely many rounds.

**4.3.2 Case 2**

In Case 1, the robot \( r_1 \) played crucial role in keeping \( b \) true throughout the execution and as a result having an agreement regarding the embedding of the pattern \( F \). In Case 2, the situation is simpler as the scalen triangle formed by the robots on \( C(R) \) determines a particular embedding of \( F \) as described in Section 4.2. So as long as the remaining robots stay inside \( \text{encl}(R) \), we have \( b \) and consequently an agreement regarding the embedding of \( F \). So these robots will sequentially move to get inside the regions \( B(t, \varepsilon D) \cap \text{encl}(C(R)) \), where \( t \) are the target point, while preserving \( b \). This will be achieved using the similar strategies as discussed in the previous case.

**Lemma 14.** If the algorithm is in Phase 3, Case 2 then we have a final configuration after finitely many rounds.

**4.4 Proof of Theorem 5**

Recall that a configuration with \( \neg u \land (\neg a \lor \neg c) \) is in Phase 1, a configuration with \( a \land c \land \neg b \) is in Phase 2, and a configuration with \( b \) is in Phase 3. It is easy to see that any configuration with \( \neg u \) belongs to one
of the three phases. Phase 1 terminates with a ∧ c which is either a Phase 2 or Phase 3 configuration. Phase 2 terminates with b which is a Phase 3 configuration. A final configuration is formed in Phase 3. Hence the algorithm solves the problem in \( \text{OBLOT} + \text{SSYNC} \) from any configuration which is \( \neg u \).

The main algorithm works in \( \text{OBLOT} + \text{SSYNC} \) and is sequential. This immediately gives an algorithm that works in \( \text{FCOM} + \text{ASYNC} \). This is because of the following result.

**Theorem 6.** Any sequential \( \text{OBLOT} + \text{SSYNC} \) algorithm can be simulated in \( \text{FCOM} + \text{ASYNC} \) using two colors.

To see this consider a sequential algorithm \( A \) that works in \( \text{OBLOT} + \text{SSYNC} \). \( A \) can be seen as a function that maps the snapshot taken by a robot to a movement instruction. We present an algorithm that simulates \( A \) in using two colors \{busy, idle\}. Initially the colors of all robots are set to idle. If any robot finds some robot with light set to busy, then it does nothing. Otherwise, it applies \( A \) on its snapshot (ignoring colors). If \( A \) returns a non-null move, it sets its light to busy and moves accordingly. If \( A \) returns a null move, it sets its light to idle (recall that it does not know what its present color is) and does not make any move.

### 5 Concluding Remarks

We have introduced a model for robots with inaccurate movements. We have presented algorithms for pattern formation in \( \text{OBLOT} + \text{SSYNC} \) and \( \text{FCOM} + \text{ASYNC} \). Devising an algorithm for \( \text{OBLOT} + \text{ASYNC} \) is an interesting direction for future research. Another direction would be to consider robots with physical extent.

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