DISCRETE SYMMETRY IN THE EPRL MODEL AND NEUTRINO PHYSICS

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ABSTRACT: In [1], we proposed a new interpretation of the EPRL quantization of the BC model for quantum general relativity using a monoidal functor we call the time functor.

In this preliminary draft we apply the theory of modules over monoidal functors [2] to the time functor, to propose an extension of the EPRL model which would include the standard model. This is motivated by recent advances in neutrino Physics.

Introduction. The algebraic structure of the EPRL model, and new approaches to constructing realistic matter fields.

This paper begins an attempt to construct a grand unified theory, using categorical quantum geometry. The idea is to extend the EPRL model for quantum general relativity using module categories and functors.

In [3] a new approach to the quantum theory of gravity was begun. The theory modelled spacetime on a simplicial complex, rather than a smooth manifold based on a point set continuum. Structural similarities between the algebra of tensor categories and the combinatorial topology of simplicial complexes were used to put a quantum geometry on the complex. This model had a number of successes, but did not have a good classical geometrical limit or Hilbert space on a space slice.

Aside from the well known renormalization problems of general relativity and the Planck length, this was motivated in part by the frustrating outcomes of attempts to get realistic matter fields from classical continuum geometries.

The question of extending the BC model to include realistic matter fields did not make much headway.

In [4], [5] [6] and [7], a new version of the BC model has been developed. The problems relating to the Hilbert space and the geometrical interpretation have been resolved.

In [1], the author described the mathematical form of the new model. The weak constraints were interpreted as given by a functor from Rep(SO(3)) to Rep(SO(4)) in the Euclidean signature, and to Rep(SO(3,1)) in the Lorentzian signature, which we called the time functor. The physical Hilbert space is just the preimage of the time functor.

An interpretation of the model was proposed in which the universe was in a topological state, which meant that the domain category of $F_t$ was also used to construct a three dimensional topological field theory. This suggests that the new version of quantum gravity is related to the categorical constructions of TQFTs in various dimensions [8], [22].

In this paper, we consider extensions of the EPRL model which seem to be promising ways to reproduce the standard model.
Our motivation for this is a suggestive coincidence between the theory of module functors over the time functor and recent developments in neutrino Physics. Recent attempts to model neutrino oscillations together with quark mixing matrices yield structures which also appear in the mathematical analysis of modules over $F_t$. In particular, in the work of Ma [9] the discrete group of symmetries of the tetrahedron and the exceptional Lie algebra $E_6$ [10] both appear in the mathematical foundation of a unified model which would explain the three generations with neutrino oscillations included. These two structures are linked in the correspondence between discrete subgroups of the rotation group and the simply laced Lie groups due to Mckay [12], which is closely related to the theory of module categories over the category of representations of $\text{Rep}(\text{SO}(3))$ and also of its quantum group deformation.

We can formulate this more suggestively by thinking of the data in a state sum as a substitution for the geometry and topology of the spacetime manifold. To include Yang-Mills fields and fermionic matter, we must find a categorical equivalent for bundles over the manifold. Bundles appear in more algebraic approaches to topology as modules over the base space; for example, the space of cross sections of a bundle forms a module over the function space of the manifold. Thus modules over the two categories plus functor out of which the EPRL model can be formed are a reasonable candidate for matter fields.

A module functor over the time functor would include module categories over its domain and range, and a functor analogous to the time functor between them. This is explained below. This would allow us to construct a state sum model analogous to the EPRL model from it. The image of the functors down from the module categories to the original pair would define a gravitational sector of the new model.

In the rest of this paper we give expositions of the mathematical structures we are considering, which we hope will be accessible to the quantum gravity community, and explore several possible ways to use them to produce a grand unified theory.

**The Time Functor**

The point of departure for the new model was the discovery of an elegant way to impose the simplicity constraints of the BC model weakly rather than strongly [4]. The EPR physical Hilbert space is composed of vectors in $\text{Rep}(\text{SO}(3))$ so the Hilbert space on a 3-manifold comes from the preimage of the relativistic representations labelling it under a functor.

The EPRL model has both a Euclidean and a Lorentzian signature version. Each can be described as a new type of functor, but they are quite different. In the Euclidean case, the functor

$$F_t : \text{RepSO}(3) \to \text{RepSO}(4)$$

can be defined by
\[ F_t(R_{2i}) = R_{i,i} \]

(where \( i \) is a half integer) and expanding linearly on direct sums. It follows from the elementary theory of spins that there exists a natural map

\[ M_{a,b} : F(a \otimes b) \rightarrow F(a) \otimes F(b). \]

\( M \) is an injection. Since we are in a category of Hilbert spaces, we can equally well define the projection \( M^* \) which goes backward. In the terminology of category theory, we can equally well say the functor is lax, using \( M^* \).

The functor \( F_t \) may be only weakly colax, however, by which we mean that the associators in \( \text{SO}(3) \) and \( \text{SO}(4) \) do not agree, so a correction factor must be included, connecting them in the coherence pentagon. We are still investigating if the correction can be made in the tensor operators of the category, so that the amended category is strictly colax.

In the Lorentzian signature case, the EPRL model proceeds by reducing the irreducible representations of the Lorentz group which are the building blocks of the theory into sums of representations of a suitable copy of \( \text{SU}(2) \) and selecting only the lowest spin representation which appears.

Since the Lorentz group is noncompact, the decompositions of tensor products are direct integrals. To make the following discussion mathematically rigorous, we will need a richer categorical description of \( \text{Rep}(\text{SO}(3,1)) \) in which objects are direct integrals of simple representations under general measures and morphisms include singular maps connecting \( L^2 \) and distributional objects like the theory of rigged Hilbert spaces, a further development of the theory of measured categories. We shall proceed somewhat formally in what follows.

Since an important aspect of the EPRL model is that it relates the Hamiltonian picture of loop quantum gravity, in which states are described by spin networks in space, with relativistic spin networks in spacetime; we want to think of it as a rule which assigns “spacetime” representations of \( \text{SL}(2, \mathbb{C}) \) to “space” representations of \( \text{SO}(3, \mathbb{R}) \). The proper mathematical expression of this is a functor

\[ F_\gamma : \text{Rep}(\text{SO}(3, R)) \rightarrow \text{Rep}(\text{SL}(2, C)); \]

Defined by:

\[ F_\gamma(R_k) = R(k, \gamma k). \]

which depends on the Immirzi parameter \( \gamma \). The case \( \gamma = 0 \) is a degenerate case in which much of what follows is incorrect.

This functor assigns to each irreducible representation \( R_k \) of \( \text{SO}(3, R) \) the irreducible \( R(k, \gamma k) \) of \( \text{SL}(2, C) \). Since the only morphisms between irreducibles in either category are multiples of the identity, the action of the functor on morphisms is immediate.
Considerably more subtle are the tensor and “renormalizability” properties of this functor, which are generalizations of the very special facts which make the Lorentzian EPRL model finite and physically interesting.

The action of the time functor on direct sums is straightforward. The behavior on tensor products is more subtle. The image under $F_\gamma$ of the tensor product of two objects injects into the tensor product of the two images.

\begin{equation}
F_\gamma(X \otimes Y) \rightarrow F_\gamma X \otimes F_\gamma Y.
\end{equation}

As we explained above, equation 1 is expressed by mathematicians by saying that the functor is (possibly weakly, as in the Euclidean case) colax.

However, the injection is improper, in the sense that a dirac delta function is an improper object of $L^2(R)$. This is because $\text{REP}(SL(2,C))$ contains representations labelled by a continuous parameter, and the tensor product of two representations is a direct integral in the sense of Mackey or Gelfand.

In order to make the construction of the time functor rigorous, we need a richer description of the category $\text{Rep}(SO(3,1))$ which would include distributional morphisms. This will require a further development of the theory of measured categories \[11\]. We therefore proceed somewhat formally in the following.

One might also like to take the formal projection dual to the injection in (1), but it is also too singular, and could be made rigorous only in the sense of a proper value. Further technical foundational work will be needed to formulate this.

$F_\gamma$ connects the discrete spectrum of areas in loop quantum gravity or 3d TQFT with the continuum of representations in spacetime models. The geometrical data on tetrahedra in the EPRL model correspond to intertwiners in $SO(3,\mathbb{R})$, lifted by the time functor, and not to general $SL(2,\mathbb{C})$ intertwiners.

Since the image category consists of infinite dimensional representations, there is no hope of the trace in the domain category going over via the functor to the range. Thus a naive evaluation of closed diagrams, such as the 15J symbols in the model is impossible. However, there is a renormalizable trace which is well defined on an ample set of diagrams in the domain category.

This renormalizable trace is just the multiple integral over $SL(2,\mathbb{C})$ \[3,4\], which played a crucial role in the finiteness of the BC model, and goes over to the new model. The renormalization just consists in dropping one of the integrations (it doesn’t matter which). The finiteness of the resulting integral expression is the key to the success of both models.

The ample set of diagrams on which the renormalized trace is finite includes the free graph on 4 or 5 vertices, which represent the tetrahedron and 4-simplex, and includes any diagram obtained by adding edges to any diagram already in the ample set \[13\].

We can provisionally define a claren functor (co-lax, amply renormalizable) between two tensor categories as one with an inclusion as in equation 1 for any pair of objects $X,Y$ in the first category, and a regularizable trace for the images
under the functor of an ample family of diagrams in the second category as defined above.

**Neutrino Oscillations and their implications for unified models**

Recent research has shown that the three different types of neutrinos are coupled. Surprisingly, their couplings involve large angles, unlike the mixing matrix for the quarks which involves smaller angles.

In attempting to explain this, it has been suggested [9] that the elementary particles, in addition to living in representations of Lie groups which explain their Yang-Mills charges, also live in representations of finite groups. In order to produce three weakly mixed pairs of quarks with widely differing masses and three strongly mixing neutrinos, the group needs to have three one dimensional representations $1, 1'$, and $1''$, and a three dimensional representation $3$. A search among finite groups yielded $A_4$, the symmetry group of the tetrahedron, as the natural candidate. Coupling matrices to Higgs sectors which are invariant under this group yield the tribimaximal mixing matrix, which has emerged from a long series of delicate experiments as a strong candidate for the neutrino sector. The discrete group symmetry is an advance in that the three generations of fermions are no longer just put in by hand.

Another line of research [10] has been in looking for grand unified groups for theories which include neutrino masses and neutrino mixing. It is difficult to get the quark and neutrino sectors to break in such different ways without two separate $U_1$’s which suggests a rank 6 group [10]. $E^6$ seems to be the most attractive candidate.

So putting this together, a combined symmetry under the discrete group $A_4$ and the Lie Group $E^6$ seems to be a likely setting for a grand unified theory which includes the latest neutrino Physics.

Can we find an explanation of the combination $A_4 \times E^6$ in the context of our proposal?

**Module categories and functors, and discrete subgroups**

In [22] it was observed that the direct sum and tensor product on $\text{Rep}(SO(3))$ satisfy the same axioms as sum and product in a ring. We phrased this by saying that it was a ring category. This was followed by the suggestion that module categories over it could be naturally be defined.

A **module category** structure on a category $M$ over a tensor category $A$ is given by a functor $\triangleright : A \times M \to M$ satisfying the usual consistency condition of a module.

In [2] this definition was augmented by a definition of a module over a tensor functor.

If $F : A \to B$ is a tensor functor, and $M$ and $N$ are module categories over $A$ and $B$ respectively, then a functor $\mu : M \to N$ is a module functor over $F$ if it intertwines the action functors on the two modules.
The definition in [2] also includes a coherence diagram (a pentagon); in the application to the time functor, this will need to be modified to include the correction factor in the associator, which we discussed above.

In [23] it was shown that the irreducible module categories over \( \text{Rep}(\text{SU}(2)) \) and its quantum deformation \( \text{Rep}(U_q \text{SU}(2)) \) are classified by simply laced Dynkin diagrams, i.e. correspond to the ADE series of simple Lie groups.

In the case of \( q \) a root of unity, the rank of the group must correspond to the level of the root.

This discovery was motivated by a series of discoveries in conformal field theory [24] which showed that conformal field theories built out of combinations of representations of the Kac-Moody algebra over \( \text{SU}(2) \) similarly corresponded to the ADE diagrams. The excitations of the extended CFT’s are particular combinations of the fundamental fields of the basic ones.

The classification of irreducible module categories over \( \text{Rep}(\text{SO}(3)) \) can be explained by elementary means. If we are given a discrete subgroup \( \Gamma \) of \( \text{SO}(3) \), then any representation of \( \text{SO}(3) \) can be considered as a representation of \( \Gamma \) by restriction. The tensor product in \( \text{Rep}(\Gamma) \) then gives us an action of \( \text{Rep}(\text{SO}(3)) \) on \( \text{Rep}(\Gamma) \). Furthermore, it is known that all module categories over \( \text{Rep}(\text{SO}(3)) \) are so constructed [23].

The operation of restriction of a representation to a subgroup has an adjoint called induction. The theory of induced representations allows us to regard objects in the model categories we are constructing as objects in the original representation categories of the Lie groups \( \text{SO}(3) \) etc (simple objects often get mapped to large direct sums). We want to use this to associate to excitations in the theory we build out of modules geometric variables in our “base” theory, i.e. to regard them as having gravitational fields.

The finite subgroups of \( \text{SO}(3) \) are well known. They consist of two infinite families, the cyclic and dihedral groups, and three special groups corresponding to the symmetries of the platonic solids (dual solids have the same symmetries. As abstract groups, these three are just \( A_4, S_4 \), and \( A_5 \). They are also referred to as T O and I (tetrahedron, octahedron, and icosahedron; whose symmetry groups they are) in the context of finite rotation groups.

(Now the classification of module categories over the representation category of the quantum group \( U_q(\text{SL}(2)) \), or equivalently of the corresponding Kac-Moody algebra requires more abstract arguments, but ends up with the same form, except for a restriction at \( q \) a root of unity [23]. Thus the extension of our program to \( q \)-deformed gravity [18] would be straightforward.)

So a module over the time functor would be given by a combination of discrete subgroups of \( \text{SO}(3) \) and \( \text{SO}(4) \) or \( \text{SO}(3,1) \) depending on the signature we consider, with a suitable functor connecting their representation categories. We discuss this below.

We can now see how the group \( A_4 \) appears naturally as a module category over \( \text{Rep}(\text{SO}(3)) \), the first step to connecting our module category program to particle Physics.

Now we would like to explain the ADE classification of the finite rotation groups, and in particular see if \( A_4 \) is somehow related to \( E_6 \) to unite the picture
which appears from neutrino Physics. This leads us to the Mckay correspondence in the next section.

**The classical and quantum Mckay correspondence. Applications to State Sum Models.**

In [12], Mckay discovered a mysterious relationship between the finite subgroups of SO(3), or more precisely, their double covers in SU(2), and the simply laced Lie groups, i.e., ADE in the well known classification. The representations of the finite groups correspond to the vertices of the Dynkin diagram of the corresponding Lie group, and the action of the two dimensional representation of SU(2) restricted to a representation of the finite group by the tensor product gives the edges of the diagram. To be more precise, they correspond to the extended diagram, which is the Dynkin diagram of the corresponding Kac-Moody algebra.

In this correspondence, the three non-planar finite rotation groups, T,O and I correspond to the exceptional algebras $E_6$, $E_7$, $E_8$. So we see the two pieces of symmetry which appear in the theory of elementary particles when we include neutrino oscillation Physics correspond to one another according to McKay!

The term “Mckay correspondence” has evolved in the literature to denote a rich variety of constructions which attempt to shed light on what at first seems to be an astonishing coincidence. The ADE classification of the representation categories of the quantum group $U_qSU(2)$ is sometimes called the “quantum Mckay correspondence.” Module categories can be thought of as a q-version of discrete subgroups.

A natural goal in this field of Mathematics has been to find a construction which would begin with the relevant finite group and directly construct the corresponding Kac-Moody algebra.

This has been accomplished in two ways. There is an algebraic construction due to I. Frenkel and his students [25] and a geometric one due to Nakajima [26].

Both of these constructions essentially involve construction a bosonic Fock space from the finite group. The Frenkel construction takes the sum of the category of representations of the wreath product of the finite group of all orders. The Nakajima construction begins with the quotient of a vector space by the action of the group, then considers symmetric products of all orders of the space with itself (configuration spaces). Both constructions give us the fundamental representation of the Kac-Moody algebra.

This suggests that in a state sum model based on a module category over Rep(So(3)) gauge field symmetries under $E^6$ could appear in the continuum limit as collective states on large faces composed of many elementary faces which had quantum states corresponding to the representations of the finite group T.

We do not yet understand the mathematical details of the construction of module functors well enough to say how the two kinds of representations would intertwine. There are still several possibilities for the geometry corresponding
to a module over the time functor. We outline the current state of our understanding below.

**Towards Picking a Theory**

A module over the time functor would consist of a discrete subgroup \(G\) of \(SO(3)\), another discrete subgroup \(H\) of \(SO(3,1)\) or \(SL(2,C)\), and a functor from \(\text{Rep } G\) to \(\text{Rep } H\) which intertwines the two actions and lifts the time functor. We have not yet completed the mathematical analysis of this condition, so this work is still partly programmatic. The details will be explored in a Mathematics paper.

The analogous problem in the Euclidean signature is easier to tackle. Since \(SO(4)\) splits into a left and right copy of \(SU(2)\) (factors of \(Z_2\) to one side), a finite subgroup \(H\) of \(SO(4)\) has two homomorphisms to finite subgroups of \(G_L, G_R\) of \(SO(3)\) [27]. In the simplest case, \(H\) is just the product of the two subgroups, and if they are isomorphic then sending a representation \(\gamma\) of \(G\) to the representation \(\gamma_L \otimes \gamma_R\) is clearly a module over \(F_t\).

The Lorentzian signature is more complex, and seems to offer more possibilities. A discrete subgroup of \(SL(2,C)\) is called a Kleinian group, and a rich literature on them exists.

So if \(G\) is the symmetry group of the tetrahedron in \(SO(3)\), and \(H\) is a Kleinian group, what condition would we expect to allow a module functor over \(F_t\) to connect them? The most obvious choice is for an injective homomorphism to exist \(G \rightarrow H\).

This turns out to be a mathematically interesting condition. It is equivalent to \(H\) having a “tetrahedral point” in hyperbolic space. In fact, the Kleinian group with the minimal hyperbolic covolume has such a point [28]. The implications of this choice for particle Physics still need to be explored.

Another possibility is just to let \(H= G= T\), i.e. to embed the subgroup \(T\) of \(SO(3)\) into \(SO(3,1)\) directly.

Now there are several ways we can imagine \(T \otimes E^6\) symmetry arising in these scenarios. One is to use just \(G= H= T\), and have the \(E^6\) symmetry arise in the continuum limit by the quantum McKay construction of Frenkel et. al. [25].

Another possibility is via a more complex Kleinian group. Kleinian groups of finite covolume correspond to Hyperbolic Kac-Moody algebras [29]. Some of these look like copies of \(E^6\) but with the copy of \(SU(2)\) at a node replaced by a nilpotent Lie algebra. Could this give rise to a symmetry breaking mechanism?

A more subtle possibility would be to use the q-deformed gravity proposal in [18]. As we explained, the module categories over quantum groups are very similar to the ones for the Lie groups, although they need more abstract treatment [29]. The suggestion in [1] to pass to a TQFT in order to include the cosmological constant as in [18] is therefore also possible, although the categorical quantum analog of Kleinian groups has not been studied yet.

So the suggestion we have made to construct new grand unified theories is as yet mostly a program, but a well defined one at least. We have at least commenced work on the necessary mathematical foundations for it.
Summary

The standard model of particle Physics resembles a jigsaw puzzle. The small number of symmetries and representations seem to fit together into a simple whole, but none of the attempts to find the whole have succeeded.

As usual when a puzzle is not solved, we begin to suspect that some pieces are missing. The intuition of the author is that discrete symmetry relating the maddening 3 generations may well be the piece.

If that is correct, tensor categories are a natural setting to unite Lie group and discrete group symmetries, and module categories are a nice bridge between them.

If one of the models we construct from Kleinian groups can explain why quarks and neutrinos come in different representations of SU(3) and T both, then the approach will become compelling.

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