Chern class and Riemann-Roch theorem for cohomology theory without homotopy invariance (preliminary version)

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Abstract

In this paper, we formulate axioms of certain graded cohomology theory and define higher Chern class maps following the method of Gillet [Gi1]. We will not include homotopy invariance nor purity in our axioms. It will turn out that the Riemann-Roch theorem without denominators holds for our higher Chern classes. We will also give an application to the computation of a higher regulator for Deligne-Beilinson cohomology.

1 Introduction

In his papers [G1] and [G2], Grothendieck defined Chern classes and characters

\[ c_i : K_0(X) \to \text{CH}^i(X), \quad \text{ch}_X : K_0(X) \to \text{CH}^*(X) \]

for a smooth variety \( X \) over a field \( k \), where \( K_0(X) \) (resp. \( \text{CH}^i(X) \)) denotes the Grothendieck group of vector bundles on \( X \) (resp. the Chow groups of algebraic cycles of codimension \( i \) on \( X \) modulo rational equivalence). Concerning the Chern character, he proposed the celebrated Grothendieck-Riemann-Roch theorem, which asserts that for a proper morphism \( f : Y \to X \) of smooth varieties over \( k \), the equality

\[ \text{ch}_X(f_*\alpha) \cdot \text{td}(T_X) = f_!(\text{ch}_Y(\alpha) \cdot \text{td}(T_Y)) \] (1.1.1)

holds in \( \text{CH}^*(X) \) for any \( \alpha \in K_0(X) \). Here \( \text{td}(T_X) \) denotes the Todd class of the tangent bundle \( T_X \) of \( X \), and \( f_* \) (resp. \( f! \)) denotes the push-forward of Grothendieck groups (resp. Chow rings). One immediately recovers the classical Riemann-Roch theorem for a smooth complete curve \( X \) of genus \( g \) with canonical divisor \( K \):

\[ \ell(D) - \ell(K - D) = \deg(D) - g + 1 \quad \text{for a divisor } D \text{ on } X, \]

by considering the case of the structure morphism \( X \to \text{Spec}(k) \).

In [Gi1], Gillet introduced certain axioms on graded cohomology theory \( \Gamma(\cdot) \) on a big Zariski site \( \mathcal{O}_{\text{Zar}} \) including homotopy invariance and purity. Concerning such cohomology
theory, he developed the general framework of universal Chern classes and characters, which endows with the Chern classes and characters for higher $K$-groups

$$C_{i,j} : K_j(X) \longrightarrow H^{2i-j}(X, \Gamma(i)), \quad \text{ch}_X : K_*(X) \longrightarrow \hat{H}^*(X, \Gamma(\bullet))_Q,$$

(1.1.2)

where $K_*(X)$ denotes the algebraic $K$-group $[Q]$ and $\hat{H}^*(X, \Gamma(\bullet))_Q$ denotes the direct product of the cohomology groups $H^j(X, \Gamma(i)) \otimes Q$ for all integers $i$ and $j$. He further extended the formula (1.1.1) to this last Chern character. The main aim of this paper is to extend Gillet’s results to graded cohomology theories which do not satisfy homotopy invariance or purity.

1.1 Setting and results

Let $\text{Sch}$ be the category consisting of schemes which are separated, noetherian, universally catenary and finite-dimensional, and morphisms of schemes. Let $\mathcal{C}$ be a subcategory of $\text{Sch}$ satisfying the following condition:

\begin{equation}
(*) \text{ If } f : Y \rightarrow X \text{ is smooth with } X \in \text{Ob}(\mathcal{C}), \text{ then } Y \in \text{Ob}(\mathcal{C}) \text{ and } f \in \text{Mor}(\mathcal{C}).
\end{equation}

We do not assume that $\mathcal{C}$ is closed under fiber products. Let $\Gamma(\bullet) = \{ \Gamma(n) \}_{n \in \mathbb{Z}}$ be a family of cochain complexes of abelian sheaves on the big Zariski site $\mathcal{C}_{\text{Zar}}$. Our axioms of admissible cohomology theory consist of the following four conditions (see Definition 2.5 below for details):

1. A morphism $\varrho : \mathbb{G}_m[-1] \rightarrow \Gamma(1)$ is given in $D(\mathcal{C}_{\text{Zar}})$. (From this $\varrho$, one obtains the first Chern class $c_1(L) \in H^2(X_{\text{Zar}}, \Gamma(1))$ of a line bundle $L$ on $X \in \text{Ob}(\mathcal{C})$.)

2. The Dold-Thom isomorphism, i.e., the projective bundle formula.

3. For a regular closed immersion $f : Y \hookrightarrow X$ of codimension $r$ in $\mathcal{C}$, push-forward morphisms

$$f_! : f_*\Gamma(i)_Y \longrightarrow \Gamma(i + r)_X[2r]$$

are given in $D(X_{\text{Zar}})$ and satisfy transitivity, projection formula and compatibility with the first Chern class. Here $D(X_{\text{Zar}})$ denotes the derived category associated with the additive category of unbounded complexes of abelian sheaves on $X_{\text{Zar}}$.

4. For a regular closed immersion $f : Y_* \hookrightarrow X_*$ of codimension 1 of simplicial objects in $\mathcal{C}_s$, push-forward morphisms

$$f_* : f_*\Gamma(i)_{Y_*} \longrightarrow \Gamma(i + 1)_{X_*}[2]$$

are given in $D((X_*)_{\text{Zar}})$ and satisfy the conditions analogous to (3).

The conditions (1)–(3) have been considered both by Gillet [Gi1] Definition 1.2 and Beilinson [Be] §2.3 (a)–(f). On the other hand, the last condition (4) have not considered in those literatures, which we will need to verify the Whitney sum formula for Chern classes of vector
bundles on simplicial schemes, cf. §4 below. See §3 for fundamental and important examples of admissible cohomology theories. We will define Chern class and character (1.1.2) for an admissible cohomology theory \( \Gamma(*) \), following the method of Gillet [Gi1].

As for the compatibility of the above axioms (1)–(4), the axiom (2) is compatible with (1) in the sense that the first Chern class of a hyperplane has been used in formulating (2). The push-forward morphisms in (3) and (4) are compatible with (1) by assumption. Moreover, we will prove the following compatibility assuming that \( \Gamma(*) \) is an admissible cohomology theory, cf. Corollary 7.6. Let \( E \) be a vector bundle of rank \( r \) on \( Y \in \text{Ob}(\mathscr{C}) \) and let \( X := \mathbb{P}(E \oplus 1) \) be the projective completion of \( E \), cf. (1.2.1). Then for the zero-section \( f : Y \to X \), we have

\[
  f_!(1) = c_r(Q)
\]

in \( H^{2r}(X_{\text{Zar}}, \Gamma(r)) \), where 1 denotes the unity of \( H^0(Y_{\text{Zar}}, \Gamma(0)) \) and \( Q \) denotes the universal quotient bundle on \( X \). This formula verifies the compatibility between the axioms (2) and (3), and plays an important role in our results on Riemann-Roch theorems:

**Theorem 1.1** (§9, §10) Let \( \Gamma(*) \) be an admissible cohomology theory on \( \mathscr{C} \), and let \( f : Y \to X \) be a projective morphism in \( \mathscr{C} \). Assume that \( X \) and \( Y \) are both regular.

(1) Assume that \( f \) satisfies the assumption (\( \#' \)) in Theorem [10.1] below. Then the formula (1.1.1) holds for \( \Gamma(*) \)-cohomology, i.e., the following diagram commutes:

\[
\begin{array}{ccc}
K_*(Y) & \xrightarrow{f_*} & K_*(X) \\
\text{ch}_Y(-) \cup \text{td}(T_f) & & \text{ch}_X \\
\hat{H}^*(Y_{\text{Zar}}, \Gamma(\bullet))_Q & \xrightarrow{f_*} & \hat{H}^*(X_{\text{Zar}}, \Gamma(\bullet))_Q.
\end{array}
\]

Here \( T_f \) denotes the virtual tangent bundle of \( f \), cf. §10 below, and \( f_! \) denotes the push-forward morphism that will be constructed in §7 below.

(2) Assume that \( f \) is a (regular) closed immersion of pure codimension \( r \geq 1 \) and satisfies the assumption (\( \# \)) in Theorem [2.7] below. Then the Riemann-Roch theorem without denominators holds for \( \Gamma(*) \)-cohomology, i.e., the following diagram commutes for any \( i, j \geq 0 \):

\[
\begin{array}{ccc}
K_j(Y) & \xrightarrow{f_*} & K_j(X) \\
\text{P}_{i-r,Y/X,j} & & \text{C}_{i,j,X} \\
H^{2(i-r)-j}(Y_{\text{Zar}}, \Gamma(i-r)) & \xrightarrow{f_*} & H^{2i-j}(X_{\text{Zar}}, \Gamma(i)),
\end{array}
\]

where \( \text{P}_{i-r,Y/X,j} \) denotes a mapping class defined by a universal polynomial \( P_{i-r,r} \) and universal Chern classes, cf. §9 below.

The Riemann-Roch theorem without denominators was first raised as a problem in [BGI] Exposé XVI §3, and proven by Jouanolou and Baum-Fulton-MacPherson for \( K_0 \) ([Jou] §1, [BFM] Chapter IV §5) and by Gillet for the Chern class maps (1.1.2) under the assumption
that $\Gamma(*)$ satisfies homotopy invariance and purity ([Gi1] Theorem 3.1). Theorem 1.1(2) removes the assumptions on homotopy invariance and purity.

This paper is organized as follows. In §2 we will formulate admissible cohomology theory, whose examples will be explained in §3. We will construct Chern classes of vector bundles, universal Chern class and character, higher Chern class and character, following the method of Gillet in §4–§6 below. The section 7 will be devoted to extending push-forward morphisms to projective morphisms in $\mathcal{C}$, which plays a key role in our proof of Riemann-Roch theorems. We will give an explicit construction of the Baum-Fulton-MacPherson polynomial in §8 for the convenience of the reader. After those preliminaries, we will prove Riemann-Roch theorems in §9–§10 and give an application in §11. In the final version we will further discuss a few more applications.

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1.2 Notation and conventions

In this paper, all schemes are assumed to be separated, noetherian of finite dimension and universally catenary. Unless indicated otherwise, all cohomology groups of schemes are taken over the Zariski topology.

For a scheme $X$, a closed subset $Z \subset X$ and a cochain complex $F^\bullet$ of abelian sheaves on $X$, we define the cohomology group $H^i_Z(X, F^\bullet)$ with support in $Z$ as the $i$-th cohomology group of the complex $\Gamma_Z(X, J^\bullet)$, where $J^\bullet$ denotes a $K$-injective resolution of $F^\bullet$, cf. [Sp], [Ho].

A projective morphism $f : Y \to X$ of schemes means a morphism which factors as follows for some integer $n \geq 0$:

$$Y \xleftarrow{i} \mathbb{P}^n \xrightarrow{p} X,$$

where $i$ is a closed immersion and $p$ is the natural projection, cf. [Ha2] p. 103. When $X$ is regular, a projective morphism $f : Y \to X$ in the sense of [GD1] 5.5.2 is projective in our sense by the existence of an ample family of line bundles on $X$, cf. [BGI] Exposé II Corollaire 2.2.7.1.

For a vector bundle $E$ over a scheme $X$, we define the projective bundle $\mathbb{P}(E)$ as

$$\mathbb{P}(E) := \text{Proj}(\text{Sym}^\bullet_{\mathcal{O}_X}(E^\vee)), \quad (1.2.1)$$

where $E$ denotes the locally free sheaf on $X$ represented by $E$ and $E^\vee$ means its dual sheaf over $\mathcal{O}_X$. We define the tautological line bundle $L^\text{aut}$ over $\mathbb{P}(E)$ as follows:

$$L^\text{aut} := \text{Spec}(\text{Sym}^\bullet_{\mathcal{O}_{\mathbb{P}(E)}}(\mathcal{O}(-1))), \quad (1.2.2)$$

where $\mathcal{O}(-1)$ denotes the $\mathcal{O}_{\mathbb{P}(E)}$-dual of the twisting sheaf $\mathcal{O}(1)$ of Serre. For a Cartier divisor $D$ on $X$, the line bundle over $X$ associated to $D$ means the line bundle

$$\text{Spec}(\text{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{O}_X(-D))), \quad (1.2.3)$$
which represents the invertible sheaf $\mathcal{O}_X(D)$ on $X$.

Let $\Delta$ be the simplex category, whose objects are ordered finite sets 

$$\{p\} := \{0, 1, 2, \ldots, p\} \quad (p \geq 0)$$

and whose morphisms are order-preserving maps. For integers $0 \leq i \leq p + 1$, let $d^i$ be the $i$-th coface map in $\Delta$:

$$d^i : \{p\} \longrightarrow \{p + 1\}, \quad j \mapsto \begin{cases} j & (0 \leq j < i) \\ j + 1 & (i \leq j \leq p). \end{cases}$$

For integers $0 \leq i \leq p$, let $s^i$ be the $i$-th codegeneracy map in $\Delta$:

$$s^i : \{p + 1\} \longrightarrow \{p\}, \quad j \mapsto \begin{cases} j & (0 \leq j \leq i) \\ j - 1 & (i < j \leq p + 1). \end{cases}$$

An arbitrary morphism in $\Delta$ is the composite of coface maps and codegeneracy maps.

**Definition 1.3** Let $\mathcal{B}$ be a category.

1. A **simplicial object** $X_\bullet$ in $\mathcal{B}$ is a functor $X_\bullet : \Delta^{op} \longrightarrow \mathcal{B}$.

   A morphism $\mathcal{B}$: $Y_\bullet \to X_\bullet$ of simplicial objects in $\mathcal{B}$ is a natural transform of such contravariant functors.

2. For a simplicial object $X_\bullet$ in $\mathcal{B}$ and a morphism $\alpha : \{p\} \to \{q\}$ in $\Delta$, we often write $\alpha^X : X_q \longrightarrow X_p \quad (X_p := X_\bullet(\{p\}))$

   for $X_\bullet(\alpha)$, which is a morphism in $\mathcal{B}$.

3. For a simplicial object $X_\bullet$ in $\mathcal{B}$, we often write $d^i$ (resp. $s^i$) for the $i$-th face morphism $(d^i)^X : X_{p+1} \to X_p$ (resp. the $i$-th degeneracy morphism $(s^i)^X : X_p \to X_{p+1}$).

**Definition 1.4** Let $X_\bullet$ be a simplicial scheme.

1. A **vector bundle** over $X_\bullet$ is a morphism $f : E_\bullet \to X_\bullet$ of simplicial schemes such that $f_p : E_p \to X_p$ is a vector bundle for any $p \geq 0$ and such that the commutative diagram

$$
\begin{array}{ccc}
E_q & \xrightarrow{f_q} & X_q \\
\alpha^E & \downarrow & \alpha^X \\
E_p & \xrightarrow{f_p} & X_p
\end{array}
$$

induces an isomorphism $E_q \cong \alpha^X E_p := E_p \times_{X_p} X_q$ of vector bundles over $X_q$ for any morphism $\alpha : \{p\} \to \{q\}$ in $\Delta$ (cf. [Gi2] Example 1.1).
(2) We say that a morphism \( f : Y_\star \to X_\star \) of simplicial schemes is a \textit{closed immersion} if \( f_p : Y_p \to X_p \) is a closed immersion for each \( p \geq 0 \). We say that a closed immersion \( f : Y_\star \to X_\star \) is \textit{exact}, if the diagram

\[
\begin{array}{ccc}
Y_q & \xrightarrow{f_q} & X_q \\
\downarrow{\alpha} & & \downarrow{\alpha X} \\
Y_p & \xrightarrow{f_p} & X_p
\end{array}
\]  

(1.4.1)

is cartesian for any morphism \( \alpha : [p] \to [q] \) in \( \Delta \). We say that a closed immersion \( f : Y_\star \to X_\star \) is \textit{regular} if it is exact and \( f_p : Y_p \to X_p \) is regular for each \( p \geq 0 \).

An effective Cartier divisor \( X_\star \) on \( Y_\star \) is a regular closed immersion \( X_\star \to Y_\star \) of pure codimension 1.

See Appendix \( \text{A} \) below for the definitions of Zariski site and cohomology of simplicial schemes.

## 2 Admissible cohomology theory

Let \( \mathcal{C} \) be as in \( \text{[1.1]} \). The aim of this section is to formulate the axioms of admissible cohomology theory in Definition \( \text{2.5} \) below.

**Definition 2.1 (Graded cohomology theory)** Let \( \Gamma(*) = \{ \Gamma(i) \}_{i \in \mathbb{Z}} \) be a family of complexes of abelian sheaves on \( \mathcal{C}_{\text{Zar}} \). We say that \( \Gamma(*) \) is a \textit{graded cohomology theory on} \( \mathcal{C} \), if it satisfies the following two conditions (cf. \( \text{[Gi1]} \) Definition 1.1):

(a) \( \Gamma(0) \) is concentrated in degrees \( \geq 0 \), and the 0-th cohomology sheaf \( \mathcal{H}^0(\Gamma(0)) \) is a sheaf of commutative rings with unity.

(b) \( \Gamma(*) \) is equipped with an associative and commutative product structure

\[
\Gamma(i) \otimes_{\mathcal{O}_{\mathcal{C}}} \Gamma(j) \longrightarrow \Gamma(i+j) \quad \text{in} \quad D(\mathcal{C}_{\text{Zar}})
\]

compatible with the product structure on \( \mathcal{H}^0(\Gamma(0)) \) stated in (a).

For a simplicial object \( X_\star \) in \( \mathcal{C} \), there is a natural restriction functor on the category of abelian sheaves

\[
\theta_{X_\star} : \text{Shv}_{\text{ab}}(\mathcal{C}_{\text{Zar}}) \longrightarrow \text{Shv}_{\text{ab}}((X_\star)_{\text{Zar}}),
\]

which sends a sheaf \( \mathcal{F} \) on \( \mathcal{C}_{\text{Zar}} \) to the sheaf \( (U \subset X_p) \mapsto \mathcal{F}(U) \) on \( (X_\star)_{\text{Zar}} \). See \( \text{[A]} \) 1 below for the definitions of the Zariski site and a Zariski sheaf on a simplicial scheme. This functor is exact and extends naturally to a triangulated functor on derived categories

\[
\theta_{X_\star} : D(\mathcal{C}_{\text{Zar}}) \longrightarrow D((X_\star)_{\text{Zar}}).
\]
Definition 2.2 Let $\Gamma(*)$ be a graded cohomology theory on $\mathcal{C}$ and let $X_\ast$ be a simplicial object in $\mathcal{C}$. For each $i \in \mathbb{Z}$, we define a complex $\Gamma(i)_{X_\ast}$ of abelian sheaves on $(X_\ast)_{\text{Zar}}$ by applying $\theta_{X_\ast}$ to the complex $\Gamma(i)$. We will often omit the indication of the functor $\theta_{X_\ast}$ in what follows.

Definition 2.3 (First Chern class) Let $\Gamma(*)$ be a graded cohomology theory on $\mathcal{C}$, and suppose that we are given a morphism $\varrho: O^\times[-1] \rightarrow \Gamma(1)$ in $D(\mathcal{C}_{\text{Zar}})$, where $O^\times$ means the abelian sheaf on $\mathcal{C}_{\text{Zar}}$ represented by the group scheme $\mathbb{G}_m$. Let $X_\ast$ be a simplicial object in $\mathcal{C}$, and let $L_\ast$ be a line bundle over $X_\ast$. There is a class $[L_\ast] \in H^1(X_\ast, O^\times)$ corresponding to $L_\ast$ (cf. [Gi2] Example 1.1). We define the first Chern class $c_1(L_\ast) \in H^2(X_\ast, \Gamma(1))$ as the value of $[L_\ast]$ under the map $H^1(X_\ast, O^\times) \rightarrow H^2(X_\ast, \Gamma(1))$.

The first Chern classes are functorial in the following sense:

Lemma 2.4 Let $\Gamma(*)$ be a graded cohomology theory on $\mathcal{C}$, and suppose that we are given a morphism $\varrho: O^\times[-1] \rightarrow \Gamma(1)$ in $D(\mathcal{C}_{\text{Zar}})$. Then for a morphism $f: Y_\ast \rightarrow X_\ast$ of simplicial objects in $\mathcal{C}$ and a line bundle $L_\ast$ over $X_\ast$, we have

$$c_1(f^*L_\ast) = f^*c_1(L_\ast) \quad \text{in} \quad H^2(Y_\ast, \Gamma(1)).$$

Here $f^*L_\ast$ denotes $L_\ast \times_X Y_\ast$, the inverse image of $E_\ast$ by $f$.

Proof. The assertion is obvious, because $\Gamma(1)$ is defined over the big site $\mathcal{C}_{\text{Zar}}$. □

Definition 2.5 (Admissible cohomology theory) We say that a graded cohomology theory $\Gamma(*)$ is an admissible cohomology theory on $\mathcal{C}$, if it satisfies the axioms (1) – (4) below. Compare with [Be] §2.3 (a)–(f), [Gi1] Definition 1.2.

1. (First Chern class) There exists a morphism $\varrho: O^\times[-1] \rightarrow \Gamma(1)$ in $D(\mathcal{C}_{\text{Zar}})$.

2. (Dold-Thom isomorphism) For a scheme $X \in \text{Ob}(\mathcal{C})$ and a vector bundle $E$ over $X$ of rank $r + 1$, the morphism

$$\gamma_E: \bigoplus_{j=0}^r \Gamma(i-j)X[-2j] \rightarrow Rp_*\Gamma(i)_P(E), \quad (x_j)_{j=0}^r \mapsto \sum_{j=0}^r \xi_j \cup p^*(x_j)$$

is an isomorphism in $D(X_{\text{Zar}})$. Here $p: \mathbb{P}(E) \rightarrow X$ denotes the projective bundle associated with $E$, cf. (1.2.1), and $\xi \in H^2(\mathbb{P}(E), \Gamma(1))$ denotes the first Chern class of the tautological line bundle, cf. Definition 2.3. See (1.2.2) for the definition of the tautological line bundle.
(3) (Push-forward for regular closed immersions) For a regular closed immersion $f : Y \hookrightarrow X$ of pure codimension $r$, there are push-forward morphisms
\[
f_! : f_* \Gamma(\mathcal{I})_Y \longrightarrow \Gamma(\mathcal{I} + r)_X[2r]
\]
in $D(X_{\text{Zar}})$, which satisfy the following four properties.

(3a) (Consistency with the first Chern class) When $r = 1$, the push-forward map
\[
f_! : H^0(Y, \mathcal{I}(0)) \longrightarrow H^2(X, \mathcal{I}(1))
\]
sends $1$ to the first Chern class of the line bundle over $X$ associated with $Y$, cf. Definition 2.3.

(3b) (Projection formula) The following diagram commutes in $D(X_{\text{Zar}})$:

\[
\begin{array}{ccc}
\Gamma(\mathcal{I})_X \otimes f_* \Gamma(\mathcal{I})_Y & \xrightarrow{id \otimes f_*} & \Gamma(\mathcal{I})_X \otimes \Gamma(\mathcal{I} + r)_X[2r] \\
 f^* \otimes id & \downarrow & \text{product} \\
 f_* \Gamma(\mathcal{I})_Y \otimes f_* \Gamma(\mathcal{I})_Y & \xrightarrow{\text{product}} & f_* \Gamma(\mathcal{I} + \mathcal{I})_Y \xrightarrow{f_*} \Gamma(\mathcal{I} + \mathcal{I} + r)_X[2r]. \\
\end{array}
\]

(3c) (Transitivity) For regular closed immersions $f : Y \hookrightarrow X$ and $g : Z \hookrightarrow Y$ in $\mathcal{C}$ of pure codimension $r$ and $r'$, respectively, the composite morphism
\[
(f \circ g)_* \Gamma(\mathcal{I})_Z \cong f_* g_* \Gamma(\mathcal{I})_Z \xrightarrow{g_*} f_* \Gamma(\mathcal{I} + r')_Y[2r'] \xrightarrow{f_*} \Gamma(\mathcal{I} + r + r')_X[2(r + r')]
\]
agrees with $(f \circ g)_*$.

(3d) (Base-change property) Let
\[
\begin{array}{ccc}
Y' & \xrightarrow{f'} & X' \\
\downarrow \oslash & \downarrow \circ & \downarrow \circ \\
Y & \xrightarrow{f} & X
\end{array}
\]
be a cartesian diagram in $\mathcal{C}$ such that $f$ and $f'$ are regular closed immersions of pure codimension $r$ and such that $g$ is a closed immersion or a smooth morphism. Then the following diagram commutes in $D(X_{\text{Zar}})$:

\[
\begin{array}{ccc}
R(g \circ f'), \Gamma(\mathcal{I})_{Y'} & \xrightarrow{f'_*} & Rg_* \Gamma(\mathcal{I} + r)_{X'}[2r] \\
Rf'_*(h^*) & \downarrow & \downarrow g^* \\
Rf_*(h^*) & \xrightarrow{f_*} & \Gamma(\mathcal{I} + r)_X[2r].
\end{array}
\]

(4) (Push-forward for simplicial schemes) For a simplicial object $X_\bullet$ in $\mathcal{C}$ and an effective Cartier divisor $f : Y_\bullet \hookrightarrow X_\bullet$ (see Definition 1.3(2)), there are push-forward morphisms
\[
f_! : f_* \Gamma(\mathcal{I})_Y \longrightarrow \Gamma(\mathcal{I} + 1)_{X_\bullet}[2] (i \geq 0)
\]
in $D((X_\bullet)_{\text{Zar}})$ which satisfy the properties analogous to (3a) and (3b).
Remark 2.6  (1) The axiom (4) is not considered in [Be] or [Gi1]. In fact, if \( \Gamma(*) \) satisfies homotopy invariance, then we need push-forward maps only for usual schemes to verify Theorem 4.2(3) below, cf. [Le] Part I, Chapter III §1.3.3.

(2) The axiom (4) implies the weak Gysin property in [Schn] p. 20 for simplicial schemes.

(3) We will need the properties (3c) and (3d) in the axiom (3) to verify Theorem 7.1. A key step is to extend push-forward morphisms to those for regular projective morphisms. See \([\mathcal Z]\\) below.

3 Examples of admissible cohomology theory

We give several fundamental and important examples of \( \mathcal C \) and \( \Gamma(*) \). The first four examples satisfy homotopy invariance, while the others do not.

3.1 Motivic complex

Let \( k \) be a field, and let \( \mathcal C \) be the full subcategory of \( \text{Sch} / k \) consisting of schemes which are smooth separated of finite type over \( k \). For \( i \in \mathbb{Z} \), we define \( \Gamma(i) \) on \( \mathcal C_{\text{Zar}} \) as follows:

\[
\Gamma(i) := \begin{cases} 
\mathbb{Z}(i) & (i \geq 0), \\
0 & (i < 0), 
\end{cases}
\]

where \( \mathbb{Z}(i) = C_*(\mathcal O_{\mathcal C}^{\text{et}})[i] \) denotes the motivic complex of Suslin-Voevodsky [SV] Definition 3.1, and \( C_\bullet \) denotes the singular complex construction due to Suslin. It is easy to see that \( \Gamma(*) = \{ \Gamma(i) \}_{i \in \mathbb{Z}} \) is a graded cohomology theory on \( \mathcal C \). We check that \( \Gamma(*) \) is an admissible cohomology theory. We define the morphism \( \varphi \) in the axiom 2.5(1) as the natural quasi-isomorphism \( \Gamma(1) \cong \mathcal O^\times[-1] \). By the comparison theorems in [Bl] Theorem 2.1, [FS] Proposition 12.1 and [V] Theorem 1, we have the following diagram of quasi-isomorphisms of complexes of abelian sheaves on \( X_{\text{Zar}} \) for each \( X \in \text{Ob}(\mathcal C) \):

\[
C_\bullet(\mathcal O_{\text{Zar}}(\mathbb{P}^i)/\mathcal O_{\text{Zar}}(\mathbb{P}^{i-1}))_X[-2i] \xrightarrow{\text{qis}} C_\bullet(\mathcal O_{\text{Zar}}(\mathbb{A}^i, 0))_X[-2i] \xrightarrow{\text{qis}} z^i(\mathbb{A}^i, \bullet)[-2i]
\]

Here \( z_{\text{equ}}(\mathbb{A}^i, 0) \) denotes the sheaf (on \( \mathcal C_{\text{Zar}} \)) of equi-dimensional cycles on \( \mathbb{A}^i \) of relative dimension 0, and \( z^i(\mathbb{X}, \bullet) \) (resp. \( z^i(\mathbb{X} \times \mathbb{A}^i, \bullet) \)) denotes the sheaf of complexes \( U \mapsto z^i(U, \bullet) \) (resp. \( U \mapsto z^i(\mathbb{A}^i_U, \bullet) \)) on \( X_{\text{Zar}} \), where \( z^i(U, \bullet) \) denotes Bloch’s cycle complex defining higher Chow groups of \( U \) [Bl]. The axiom 2.5(2) follows from the above diagram of quasi-isomorphisms and loc. cit. Theorem 7.1.

We check the axioms 2.5(3) and (4). For a regular closed immersion \( f : Y \hookrightarrow X \) of pure codimension \( r \) in \( \mathcal C \), we define the push-forward morphism \( f_! \) as the composite

\[
f_* \mathbb{Z}(i)_Y \cong f_* z^i(-Y, \bullet)[-2i] \xrightarrow{\text{loc}} z^{i+r}(-X, \bullet)[-2i] \cong \mathbb{Z}(i + r)_X[2r] \quad (3.1.1)
\]
in $D(X_{\text{Zar}})$, where the central arrow $f_{\text{cyc}}$ denotes the push-forward of cycles by $f$. The properties (3a)–(3d) of $f_{\text{cyc}}$ follows from the corresponding properties of $f_{\text{cyc}}$. For a simplicial object $X_*$ in $\mathcal{C}$, we define a complex $z^i(-X_*, \bullet)'$ of sheaves on $(X_*)_{\text{Zar}}$ as follows. For each $p \geq 0$, let $z^i(-X_p, \bullet)'$ be a subcomplex of $z^i(-X_p, \bullet)$ which is quasi-isomorphic to $z^i(-X_p, \bullet)$ and such that the pull-back maps

$$\beta^p : z^i(-X_p, \bullet)' \longrightarrow \alpha_* z^i(-X_{p+1}, \bullet)'$$

are defined for all faces and degeneracies $\beta : X_{p+1} \rightarrow X_p$. See [Lc] Chapter II Theorem 3.5.14 for the existence of such $z^i(-X_p, \bullet)'$. The datum $((z^i(-X_p, \bullet)'))_{p \geq 0}$ forms a complex $z^i(-X_*, \bullet)'$ of sheaves on $(X_*)_{\text{Zar}}$ (cf. Proposition [A,2(1)]), and the above diagram of quasi-isomorphisms yields an isomorphism

$$\mathbb{Z}(i)_{X_*} \cong z^i(-X_*, \bullet)' \text{ in } D((X_*)_{\text{Zar}}).$$

Finally for a regular closed immersion $f : Y_* \hookrightarrow X_*$ of simplicial objects in $\mathcal{C}$, one can define the desired push-forward morphism $f_!$ in a similar way as for (3.1.1).

### 3.2 Étale Tate twist

Let $n$ be a positive integer, and let $\mathcal{C}$ be the full subcategory of $\text{Sch}$ consisting of regular schemes over $\text{Spec}(\mathbb{Z}[n^{-1}])$. For $i \in \mathbb{Z}$, we define $\Gamma(i)$ on $\mathcal{C}_{\text{Zar}}$ as follows:

$$\Gamma(i) := \begin{cases} R\varepsilon_* \mu_n^{\otimes i} & (i \geq 0), \\ R\varepsilon_* (\mathcal{H}om(\mu_n^{\otimes -i}, \mathbb{Z}/n)) & (i < 0), \end{cases}$$

where $\mu_n$ denotes the étale sheaf of $n$-th roots of unity and $\varepsilon : \mathcal{C}_{\text{ét}} \rightarrow \mathcal{C}_{\text{Zar}}$ denotes the natural morphism of sites. Obviously $\Gamma(*) = \{\Gamma(i)\}_{i \in \mathbb{Z}}$ is a graded cohomology theory on $\mathcal{C}$. We define the morphism $\varrho$ in [2.5(1)] by the connecting morphism associated with the Kummer exact sequence on $\mathcal{C}_{\text{ét}}$

$$0 \longrightarrow \Gamma(1) \longrightarrow \mathcal{O}^\times \xrightarrow{n} \mathcal{O}^\times \longrightarrow 0.$$

The property [2.5(2)] follows from the homotopy invariance ([AGV] Exposé XV, Théorème 2.1) and the relative smooth purity.

We check that $\Gamma(*)$ satisfies the axioms [2.5(3) and (4), in what follows. Let $f : Y_* \hookrightarrow X_*$ be a regular closed immersion of pure codimension $r$ of simplicial schemes in $\mathcal{C}$. We use Gabber’s refined cycle class [FG] Definition 1.1.2

$$c_1_{X_0}(Y_0) \in H^2_{Y_0}(X_0, \Gamma(r)).$$

By the spectral sequence (cf. Proposition [A,3(3)] below)

$$E_1^{ab} = H^b_{Y_*}(X_*, \Gamma(r)) \Rightarrow H^{a+b}_{X_*}(\Gamma(r))$$

and the semi-purity in loc. cit. §8, we have

$$H^2_{Y_*}(X_*, \Gamma(r)) \cong \text{Ker}(d'_0 - d'_1 : H^1_{Y_0}(X_0, \Gamma(r)) \rightarrow H^0_{Y_1}(X_1, \Gamma(r))).$$
By the functoriality in loc. cit. Proposition 1.1.3 and the assumption that the square (1.4.1) is cartesian, we have

\[ d'_{0} \text{cl}_{X_{0}}(Y_{0}) = \text{cl}_{X_{1}}(Y_{1}) = d'_{1} \text{cl}_{X_{0}}(Y_{0}), \]

and consequently, \( \text{cl}_{X_{0}}(Y_{0}) \) belongs to \( \text{Ker}(d'_{0} - d'_{1}) \). We thus obtain a cycle class

\[ \text{cl}_{X_{0}}(Y_{0}) \in H^{2}_{q_{r}}(X_{*}, \Gamma(r)) \]

as the element corresponding to \( \text{cl}_{X_{0}}(Y_{0}) \). Since \( f'_{*} \Gamma(i)_{X_{*}} \simeq \Gamma(i)_{Y_{*}} \) on \( (Y_{*})_{\text{Zar}} \), the cup product with \( \text{cl}_{X_{0}}(Y_{0}) \) defines the desired push-forward morphism

\[ f'_{*} : f_{*} \Gamma(i)_{Y_{*}} \cong f'_{*} f^{*} \Gamma(i)_{X_{*}} \xrightarrow{\text{cl}_{X_{0}}(Y_{0}) \cup -} \Gamma(i + r)_{X_{*}}[2r] \quad \text{in} \quad D((X_{*})_{\text{Zar}}), \]

which satisfies the properties (3a), (3b). See loc. cit. Proposition 1.2.1 for (3c). The property (3d) follows from loc. cit. Proposition 1.1.4. Thus \( \Gamma(\ast) \) is an admissible cohomology theory on \( \mathcal{C} \).

### 3.3 Betti complex

Let \( \mathcal{C} \) be the full subcategory of \( \text{Sch}/\mathbb{C} \) consisting of schemes which are smooth separated of finite type over \( \mathbb{C} \). Let \( \mathcal{C}_{\text{an}} \) be the big analytic site associated with \( \mathcal{C} \). Let \( A \) be a subring of \( \mathbb{R} \) with unity. For \( i \in \mathbb{Z} \), we define \( \Gamma(i) \) on \( \mathcal{C}_{\text{Zar}} \) as follows:

\[ \Gamma(i) := R\varepsilon_{*}((2\pi \sqrt{-1})^{i}A), \]

where \( \varepsilon : \mathcal{C}_{\text{an}} \to \mathcal{C}_{\text{Zar}} \) denotes the natural morphism of sites. When \( A = \mathbb{Z} \), we define the morphism \( \varrho = \varrho_{\mathbb{Z}} \) in (2.5)(1) as the connecting morphism of the exponential exact sequence

\[ 0 \to 2\pi \sqrt{-1} \cdot \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^{\times} \to 0. \]

For a general \( A \), we define \( \varrho = \varrho_{A} \) as the composite morphism

\[ \varrho : \mathcal{O}^{\times}[\mathbb{Z}] \xrightarrow{\varrho_{\mathbb{Z}}} 2\pi \sqrt{-1} \cdot \mathbb{Z} \hookrightarrow 2\pi \sqrt{-1} \cdot A. \]

The axioms (2.5)(2)–(4) can be checked in a similar way as for (3.2).

### 3.4 Deligne-Beilinson complex

Let \( \mathcal{C} \) be as in (3.3). Let \( A \) be a subring of \( \mathbb{R} \) with unity. For \( i \in \mathbb{Z} \), we define \( \Gamma(i) \) on \( \mathcal{C}_{\text{Zar}} \) as follows:

\[ \Gamma(i) := \begin{cases} \Gamma_{\varrho}(i) & (i \geq 0), \\ R\varepsilon_{*}((2\pi \sqrt{-1})^{i}A) & (i < 0), \end{cases} \]

where \( \Gamma_{\varrho}(i) \) denotes the Deligne-Beilinson complex on \( \mathcal{C}_{\text{Zar}} \) in the sense of [EV] Theorem 5.5 and \( \varepsilon : \mathcal{C}_{\text{an}} \to \mathcal{C}_{\text{Zar}} \) denotes the natural morphism of sites, cf. (3.3). See loc. cit., Theorem 5.5 (b) and Proposition 8.5 (resp. [Ja1] §3.2) for the axiom (2.5)(1) and (2) (resp. (2.5)(3) and (4)).
3.5 Algebraic de Rham complex

Let \( k \) be a field, and let \( \mathcal{C} \) be the full subcategory of \( \text{Sch}/k \) consisting of schemes which are smooth separated of finite type over \( k \). For \( i \in \mathbb{Z} \), we define \( \Gamma(i) \) on \( \mathcal{C}_{\text{Zar}} \) as follows:

\[
\Gamma(i) := \begin{cases} 
\Omega^\bullet_{-/k} & (i \geq 0), \\
0 & (i < 0),
\end{cases}
\]

where \( \Omega^\bullet_{-/k} \) denotes the de Rham complex over \( k \) on \( \mathcal{C}_{\text{Zar}} \). We see that \( \Gamma(*) \) is an admissible cohomology theory on \( \mathcal{C} \), when we define \( \kappa \) in 2.5 (1) by logarithmic differentials. See [Ha1] Chapter II §2 for the axioms 2.5 (3) and (4).

3.6 Logarithmic Hodge-Witt sheaf

Let \( p \) be a prime number, and let \( \mathcal{C} \) be the full subcategory of \( \text{Sch} \) consisting of regular schemes over \( \text{Spec}(\mathbb{F}_p) \). Let \( n \) be a positive integer. For \( i \geq 0 \), we define \( \Gamma(i) \) on \( \mathcal{C}_{\text{Zar}} \) as follows:

\[
\Gamma(i) := \begin{cases} 
R^\varepsilon_* W_n^i \Omega^i_{\log}[-i] & (i \geq 0), \\
0 & (i < 0),
\end{cases}
\]

where \( W_n^i \Omega^i_{\log} \) denotes the étale subsheaf of the logarithmic part of the Hodge-Witt sheaf \( \Omega^i_{\log} \) on \( \mathcal{C}_{\text{ét}} \), cf. [Il], and \( \varepsilon : \mathcal{C}_{\text{ét}} \to \mathcal{C}_{\text{Zar}} \) denotes the natural morphism of sites. Then \( \Gamma(*) = \{\Gamma(i)\}_{i \in \mathbb{Z}} \) is an admissible cohomology theory on \( \mathcal{C} \), which we are going to check. We define the morphism \( \kappa \) in 2.5 (1) as the logarithmic differential map. See [Gr1] Chapter I Théorème 2.1.11 and [Sh] Theorems 2.1, 2.2 for the axiom (2). To verify the axioms (3) and (4), we construct the push-forward map for the regular closed immersion \( f : Y \hookrightarrow X \) of pure codimension \( r \) of simplicial schemes in \( \mathcal{C} \). For \( a \geq 0 \), let \( f_a : Y_a \hookrightarrow X_a \) be the \( a \)-th factor of \( f \). Note first that

\[
R^j f_a! \Gamma(i + r)_{X_a} \cong \begin{cases} 
0 & (j < i + 2r) \\
\mathcal{H}^j(\Gamma(i)_{Y_a}) & (j = i + 2r)
\end{cases}
\]

by loc. cit. Theorem 3.2 and Corollary 3.4, which immediately implies

\[
R^j f^! \Gamma(i + r)_{X_a} = 0 \quad \text{for} \quad j < i + 2r,
\]

cf. Propositions [A.2] (1), [A.3] (2). We denote the above isomorphism for \( j = i + 2r \) by \( (f_a)_! \). To show that the maps \( (f_a)_! \) for \( a \geq 0 \) give rise to an isomorphism

\[
\mathcal{H}^i(\Gamma(i)_{Y_a}) \cong R^{i+2r} f^! \Gamma(i + r)_{X_a},
\]

it is enough to check that the maps \( (f_a)_! \) are compatible with the simplicial structures of \( X_a \) and \( Y_a \), cf. Propositions [A.2] (1), [A.3] (2). One can easily check this by the local description of \( (f_a)_! \) in [Sh] p. 589 and the assumption that the square (1.4.1) is cartesian. Thus we obtain a morphism

\[
f_1 : f_* \Gamma(i)_{Y_a} \longrightarrow \Gamma(i + r)_{X_a}[2r] \quad \text{in} \quad D((X_a)_{\text{Zar}}).
\]

The properties (3a)–(3d) follows again from the local description in [Sh] p. 589.
3.7 \textit{p}-adic étale Tate twist

Let $B$ be a Dedekind ring of mixed characteristics, and put $S := \text{Spec}(B)$. Let $p$ be a prime number which is not invertible on $S$, and let $\mathscr{C}$ be the full subcategory of $\text{Sch}/S$ consisting of regular schemes $X$ which are flat of finite type over $S$ and satisfy the following condition:

- Let $B'$ be the integral closure of $B$ in $\Gamma(X, \mathcal{O}_X)$. Then for any closed point $x$ on $\text{Spec}(B')$ with $\text{ch}(x) = p$, the fiber $X \times_{\text{Spec}(B')} x$ is a reduced divisor with normal crossings on $X$.

Fix a positive integer $n$. For $i \in \mathbb{Z}$, we define

$$\Gamma(i) := R\varepsilon_\pi \mathcal{T}_n(i),$$

where $\varepsilon$ denotes the natural morphism of sites $\mathcal{C}_\text{ét} \to \mathcal{C}_\text{Zar}$, and $\mathcal{T}_n(i)$ denotes the $i$-th étale Tate twist with $\mathbb{Z}/p^n$-coefficients [Sat2] Definition 3.5, a bounded complex of sheaves on $\mathcal{C}_\text{ét}$. Then $\Gamma(*) = \{\Gamma(i)\}_{i \in \mathbb{Z}}$ is an admissible cohomology theory on $\mathcal{C}$. See [Sat2] Theorem 4.1 and Proposition 5.5 for the axioms [2.3] (2), (3a)--(3c) and (4). The property (3d) follows from the construction of the push-forward morphisms given there and the corresponding property in [3.2] above.

4 Chern class of vector bundles

In this section we define Chern classes of vector bundles over simplicial schemes following the method of Grothendieck and Gillet (cf. [Gi1] p. 144 Theorem 1, [Gi1] Definition 2.10), and prove Theorem 4.2 below. Let $\mathcal{C}$ be as in §1.1 and let $\Gamma(*)$ be an admissible cohomology theory on $\mathcal{C}$.

**Definition 4.1 (Chern class)** For a simplicial object $X_*$ in $\mathcal{C}$, we put

$$H^{2*}(X_*, \Gamma(*)) := \bigoplus_{i \geq 0} H^{2i}(X_*, \Gamma(i)),$$

which is a commutative ring with unity by the axioms [2.1](a), (b). For a vector bundle $E_*$ over $X_*$, we define the **Chern classes** of $E_*$

$$c(E_*) = (c_i(E_*))_{i \geq 0} \in H^{2*}(X_*, \Gamma(*))$$

as follows. Let $E_*$ be of rank $r$, and let $\pi$ be the natural projection $\mathbb{P}(E_*) \to X_*$. Let $L_\text{taut}^*$ be the tautological line bundle over $\mathbb{P}(E_*)$ and put $\xi := c_1(L_\text{taut}^*) \in H^2(\mathbb{P}(E_*), \Gamma(1))$. There is a Dold-Thom isomorphism

$$\bigoplus_{i=1}^r H^{2i}(X_*, \Gamma(i)) \cong H^{2r}(\mathbb{P}(E_*), \Gamma(r)), \quad (b_i)_{i=1}^r \mapsto \sum_{i=1}^r \xi^{r-i} \cup \pi^* (b_i) \quad (4.1.1)$$

by the axiom [2.3] (2) and Proposition [A.4] below. We define $c_0(E_*):=1 \in H^0(X_*, \Gamma(0))$ and define $(c_1(E_*), c_2(E_*), \ldots, c_r(E_*))$ as the unique solution $(c_1, c_2, \ldots, c_r)$ to the equation

$$\xi^r + \xi^{r-1} \cup \pi^*(c_1) + \cdots + \xi \cup \pi^*(c_{r-1}) + \pi^*(c_r) = 0$$

in $H^{2r}(\mathbb{P}(E_*), \Gamma(r))$. We define $c_i(E_*):=0$ for $i > r$. 

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Theorem 4.2 Let $X_\ast$ be a simplicial object in $\mathcal{C}$.

(1) (Normalization) We have $c_0(E_\ast) = 1$ and $c_i(E_\ast) = 0$ for $i > \text{rank}(E_\ast)$. If $E_\ast$ is a line bundle, then $c_1(E_\ast)$ defined here agrees with the first Chern class in Definition 2.2.2.

(2) (Functoriality) For a morphism $f : Y_\ast \to X_\ast$ of simplicial objects in $\mathcal{C}$ and a vector bundle $E_\ast$ over $X_\ast$, we have

$$c(f^*E_\ast) = f^*c(E_\ast),$$

where $f^*E_\ast$ denotes $E_\ast \times_X Y_\ast$, the inverse image of $E_\ast$ by $f$.

(3) (Whitney sum) For a short exact sequence $0 \to E'_\ast \to E_\ast \to E''_\ast \to 0$ of vector bundles over $X_\ast$, we have

$$c(E_\ast) = c(E'_\ast) \cup c(E''_\ast) \quad \text{in} \quad H^{2*}(X_\ast, \Gamma(\ast)).$$

(4) (Tensor product) For vector bundles $E_\ast$ and $E'_\ast$ over $X_\ast$, we have

$$\overline{c}(E_\ast \otimes E'_\ast) = \overline{c}(E_\ast) \star \overline{c}(E'_\ast)$$

$$\text{in} \quad \overline{H}^{2*}(X_\ast, \Gamma(\ast)) := \mathbb{Z} \times \{1\} \times \prod_{i \geq 1} H^{2i}(X_\ast, \Gamma(i)),$$

where $\overline{c}(E_\ast)$ denotes the augmented Chern class $(\text{rk}(E_\ast), c(E_\ast))$, and we endowed the set $\overline{H}^{2*}(X_\ast, \Gamma(\ast))$ with the $\lambda$-ring structure associated with the graded commutative ring $H^{2*}(X_\ast, \Gamma(\ast))$ (G2, Chapter I §3); $\star$ denotes the product of the $\lambda$-ring.

(5) The Chern classes $c(E_\ast)$ are characterized by the properties (1)–(3).

The properties (1) and (2) immediately follow from the definition of Chern classes and the functoriality in Remark 2.4. The property (4) follows from (3) and the splitting principle of vector bundles. The assertion (5) also follows from the splitting principle of vector bundles. The most important part of this theorem is the verification of the property (3), which we prove pursuing Grothendieck’s arguments in [G1] proof of Théorème 1, in what follows. We first prove the following lemma, where we cannot use the splitting bundle argument (cf. [Lc], Part I, Chapter III §1.3.3, [ILO], Exposé XVI Proposition 1.5) because we do not assume homotopy invariance:

Lemma 4.3 Let $E_\ast$ be a vector bundle of rank $r$ on $X_\ast$. Let $p : \mathbb{P}(E_\ast) \to X_\ast$ be the projective bundle associated with $E_\ast$, and let $L_\ast\text{taut}$ be the tautological line bundle over $\mathbb{P}(E_\ast)$. Suppose that we are given a filtration on $E_\ast$ by subbundles

$$E_\ast = E_\ast^0 \supset E_\ast^1 \supset \cdots \supset E_\ast^r = 0 \quad (r := \text{rank}(E_\ast))$$

such that the quotient $E_\ast^i / E_\ast^{i+1}$ is a line bundle for $0 \leq i \leq r - 1$. Then we have

$$\prod_{i=0}^{r-1} (c_i(L^\text{taut}) + p^*c_i(E_\ast^i / E_\ast^{i+1})) = 0 \quad \text{in} \quad H^{2r}(\mathbb{P}(E_\ast), \Gamma(r)).$$
Proof. Put $F_* := p^*E_* \otimes L_*^{\text{taut}}$, and let $s : \mathbb{P}(E_*) \to F_*$ be the composite morphism

$$ s : \mathbb{P}(E_*) \overset{1}{\longrightarrow} A^*_E(E_*) \longrightarrow p^*E_* \otimes L_*^{\text{taut}} = F_* , $$

where the central arrow is induced by the canonical inclusion $(L_*^{\text{taut}})^\vee \hookrightarrow p^*E_*$. Put

$$ F^i_* := p^*F^i_* \otimes L_*^{\text{taut}}, \quad G^i_* := F^i_*/F^{i+1}_* \quad \text{and} \quad V^i_* := s^{-1}(F^i_*) \quad \text{for} \quad 0 \leq i \leq r. $$

Note that $F^r_* = G^r_* = 0$ (as vector bundles on $\mathbb{P}(E_*)$) and that $F^{r-1}_* = G^{r-1}_*$ is a line bundle on $\mathbb{P}(E_*)$. When $X_0$ is a point (and $E_0 \cong A^r_0$), the degree 0 part of $s$ sends

$$ x = (b_1 : b_2 : \cdots : b_r) \longmapsto (b_1, b_2, \ldots, b_r) \otimes v $$

$$ \mathbb{P}(E_0) = \mathbb{P}(A^r_0) \longrightarrow F_0, $$

where $v$ denotes the dual vector of $(b_1, b_2, \ldots, b_r) \in (L_*^{\text{taut}})^\vee$. By this local description of $s$, we see the following (where $X_r$ is arbitrary):

- $V^i_r$ is smooth over $X_j$ for any $i \leq r - 1$ and $j \geq 0$, and $V^i_r$ is empty for any $j \geq 0$. In particular, $V^i_r$ is a simplicial object in $\mathcal{E}$ for each $i = 0, 1, \ldots, r$.

- For each $i = 0, 1, \ldots, r - 1$, $V^{i+1}_r$ is an effective Cartier divisor on $V^i_* (\text{cf. [GD2], Théorème 17.12.1})$, whose associated line bundle is isomorphic to $G^{i+1}_*|_{V^i_r}$.

Now let $f^i_* : V^i_* \to V^{i-1}_r$ (for $1 \leq i \leq r - 1$) be the natural closed immersion. Then we have

$$ \prod_{i=1}^r (c_1(L_*^{\text{taut}}) + p^*c_1(E^{i-1}_*/E^i_*)) = \prod_{i=1}^r c_1(G^{i-1}_*) $$

(Remark 2.4)

$$ = f^1_*(1 \cup f^{1*}(c_1(G^1_* \cup c_1(G^2_* \cup \cdots \cup c_1(G^{r-1}_*)))) $$

(the axioms 2.5(4))

$$ = f^1_*(1 \cup (c_1(G^1_*|_{V^1_*}) \cup c_1(G^2_*|_{V^1_*}) \cup \cdots \cup c_1(G^{r-1}_{*}|_{V^1_*}))) $$

(Remark 2.4)

$$ = f^1_*(1 \cup f^1_*(1 \cup \cdots \cup f^{r-1}_*(1 \cup c_1(G^{r-1}_{|_{V^{r-1}*}}))) \cdots ) $$

(by recurrence)

$$ = 0 $$

($G^{r-1}_{|_{V^{r-1}*}}$ is trivial)

as claimed. \qed

Proof of Theorem 4.2(3). Let $0 \to E'_* \to E_* \to E''_* \to 0$ be a short exact sequence of vector bundles over $X_*$. Let $\pi' : D'_* \to X_*$ and $\pi'' : D''_* \to X_*$ be the (simplicial) flag schemes of $E'_*$ and $E''_*$, respectively, and put

$$ D_* := D'_* \times_{X_*} D''_* , $$

which is identified with the flag scheme of $\pi''_*E'_*$ over $D''_*$. Let $\pi : D_* \to X_*$ be the natural projection. Since the pull-back map

$$ \pi^* : H^2(X_*, \Gamma(i)) \longrightarrow H^2(D_*, \Gamma(i)) $$

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is injective by (4.1.1), we may replace \((X_\ast, E_\ast, E'_\ast, E''_\ast)\) with \((D_\ast, \pi^*E_\ast, \pi^*E'_\ast, \pi^*E''_\ast)\) and assume that \(E_\ast\) has a filtration by subbundles

\[ E_\ast = E^0_\ast \supset E^1_\ast \supset \cdots \supset E^r_\ast = 0 \quad (r := \text{rank}(E_\ast)) \]

such that the quotient \(E^i_\ast/E^{i+1}_\ast\) is a line bundle for \(0 \leq i \leq r - 1\) and such that \(E^s_\ast = E'_\ast\) with \(s := \text{rank}(E''_\ast)\). In this case, \(c_i(E_\ast)\) agrees with the \(i\)-th elementary symmetric expression in

\[ \alpha_j := c_1(E_{j-1}^i/E^i_\ast) \quad \text{with} \quad 1 \leq j \leq r \]

by Lemma 4.3. Similarly, \(c_i(E''_\ast)\) (resp. \(c_i(E'_\ast)\)) is the \(i\)-th elementary symmetric expression in \(\alpha_1, \alpha_2, \ldots, \alpha_s\) (resp. \(\alpha_{s+1}, \alpha_{s+2}, \ldots, \alpha_r\)). The Whitney sum formula in question now follows from these facts. □

5 Universal Chern class and character

Let the notation be as in §4. In this section we review the construction of universal Chern classes due to Gillet [Gi1 §2, which will be given in (5.3.2) below.

Let \(X\) be a scheme which belongs to \(\text{Ob}(\mathcal{C})\). Let \(n\) be a non-negative integer, and let \(B_\ast \text{GL}_{n,X}\) be the classifying scheme of \(\text{GL}_{n,X}\), the general linear group scheme of degree \(n\) over \(X\). Applying the construction of Chern classes to the universal rank \(n\) bundle \(E_{\ast}^{univ}\) over \(B_\ast \text{GL}_{n,X}\), we obtain Chern classes

\[ c_i(E_{\ast}^{univ}) \in H^{2i}(B_\ast \text{GL}_{n,X}, \Gamma(i)) \quad (i \geq 0) \]

which are called the universal rank \(n\) Chern classes. We introduce the following notation:

**Definition 5.1** Let \(Y\) be a scheme, and let \(\text{Shv}(Y_{zar})\) (resp. \(\text{Shv}_*(Y_{zar})\)) be the category of sheaves of sets (resp. sheaves of pointed sets) on \(Y_{zar}\).

(1) We endow \(\Delta^{op}\text{Shv}_*(Y_{zar})\), the category of simplicial sheaves of pointed sets on \(Y_{zar}\), with the simplicial model structure [BG] Theorem 2, whose class of cofibrations (resp. fibrations, weak equivalences) are defined as that of monomorphisms (resp. global fibrations, morphisms which induce weak equivalences on stalks). We write \(\mathcal{H}o_*(Y)\) for its associated homotopy category.

(2) For a cochain complex \((\mathcal{F}^\bullet, d^\bullet)\) of abelian sheaves on \(Y_{zar}\) and an integer \(j \in \mathbb{Z}\), consider the following complex:

\[ \cdots \rightarrow \mathcal{F}^{j-2} \xrightarrow{d^{j-2}} \mathcal{F}^{j-1} \xrightarrow{d^{j-1}} \text{Ker}(d^j : \mathcal{F}^j \rightarrow \mathcal{F}^{j+1}), \]

which we regard as a chain complex with the most right term placed in degree 0. Taking the associated simplicial abelian sheaf to this complex (cf. [GJ] p. 162), we obtain a simplicial abelian sheaf on \(Y_{zar}\), which we denote by \(\mathcal{K}(\mathcal{F}^\bullet, j)\).
Let $B_*QPY$ be the nerve of the $Q$-category associated with the exact category $P_Y$ of locally free $\mathcal{O}_Y$-modules. Let $B_*\mathcal{Q}Y$ be the simplicial sheaf of pointed sets on $Y_{\text{Zar}}$ defined as

$$U \subset Y \text{ (open)} \mapsto B_*QPU,$$

which we call the $K$-theory space.

**Proposition 5.2 ([Gi1] p. 221)** For $X$ as before and $i \geq 0$, there is a canonical homomorphism

$$H^2(\mathcal{O}_X, \Gamma(i)) \longrightarrow \text{Mor}_{\mathcal{H}_{\text{univ}}(X)}(B_\ast GL_n(\mathcal{O}_X), \mathcal{H}(\Gamma(i)_X, 2i)).$$

Here $B_*GL_n(\mathcal{O}_X)$ denotes the simplicial sheaf of pointed sets on $X$ represented by $B_*GL_n,X$.

**Proof.** Let $\mathcal{I}^\bullet$ be a $K$-injective resolution of $\Gamma(i)$ on $\mathcal{O}_{\text{Zar}}$, cf. [Sp], [Ho]. Consider double complexes $C_\bullet, D_\bullet := \text{Mor}_{\text{Shv}(X_{\text{Zar}})}(B_\ast GL_n(\mathcal{O}_X), \mathcal{I}^\bullet,X)$. There is a functorial homomorphism $C^j,k \cong \text{Mor}_{\text{Shv}(\mathcal{I}/X_{\text{Zar}})}(B_\ast GL_n(\mathcal{O}), \mathcal{I}^k,J) \longrightarrow D^j,k$ for each $j \geq 0$ and $k \in \mathbb{Z}$, where we have used Yoneda’s lemma in the first isomorphism. So the proposition follows from the canonical isomorphisms for $\ell \in \mathbb{Z}$

$$H^\ell((C^\bullet)^\text{tot}) \cong H^\ell(B_*GL_{n,X}, \Gamma(i)),$n_{m,n} \colon GL_m(\mathcal{O}_X) \longrightarrow GL_n(\mathcal{O}_X), \quad A \mapsto \begin{pmatrix} A & O \\ O & I_{m-n} \end{pmatrix},$$

where $I_{m-n}$ denotes the identity matrix of degree $m - n$. This morphism induces a map

$$\kappa^\ast_{n,m} : \text{Mor}_{\mathcal{H}_{\text{univ}}(X)}(B_\ast GL_m(\mathcal{O}_X), \mathcal{K}) \longrightarrow \text{Mor}_{\mathcal{H}_{\text{univ}}(X)}(B_\ast GL_n(\mathcal{O}_X), \mathcal{K}),$$
which sends $c_{i,m}$ to $c_{i,n}$. Moreover, we have
\[
\text{Mor}_{\mathcal{H}_\bullet(X)}(B_*\mathcal{G}L(\mathcal{O}_X), \mathcal{H}) \cong \text{Mor}_{\mathcal{H}_\bullet(X)}(B_*\mathcal{G}L_n(\mathcal{O}_X), \mathcal{H}) \quad \text{for} \quad n \gg 0
\]
by stability (\cite{Gi1} Theorem 1.12, Proposition 1.17). Hence the class $c_{i,n}$ for a sufficiently large $n$ defines the desired mapping class $c_{i,\infty}$. Applying to $c_{i,\infty}$ the completion functor $\mathbb{Z}_\infty$ of Bousfield-Kan \cite{BK2}, we obtain a mapping class
\[
\mathbb{Z}_\infty(c_{i,\infty}) \in \text{Mor}_{\mathcal{H}_\bullet(X)}(\mathbb{Z}_\infty B_*\mathcal{G}L(\mathcal{O}_X), \mathbb{Z}_\infty \mathcal{H}(\Gamma(i)_X, 2i))
\]
(5.2.3)
We recall here the following fact (compare with \cite{Schl} Warning 2.2.9):

**Proposition 5.3** (\cite{Gi1} Proposition 2.15) There exists a canonical (functorial) morphism
\[
\gamma : \Omega B_*\mathcal{P}_X \longrightarrow \mathbb{Z}_\infty B_*\mathcal{G}L(\mathcal{O}_X) \quad \text{in} \quad \mathcal{H}_\bullet(X),
\]
where $\Omega B_*\mathcal{P}_X$ denotes the simplicial sheaf of pointed sets on $X_{Zar}$ defined as
\[
U \subset X \quad \text{(open)} \quad \mapsto \quad \Omega B_*\mathcal{P}_U,
\]
and $\Omega B_*\mathcal{P}_U$ denotes the loop space of $B_*\mathcal{P}_U$ (cf. \cite{Gi1} p. 33).

**Proof.** We are going to define $\gamma$ as the composite morphism in $\mathcal{H}_\bullet(X)$
\[
\Omega B_*\mathcal{P}_X \cong \mathbb{Z} \times \Omega_0 B_*\mathcal{P}_X \quad \overset{pr_2}{\longrightarrow} \quad \Omega_0 B_*\mathcal{P}_X \cong \mathbb{Z}_\infty B_*\mathcal{G}L(\mathcal{O}_X),
\]
where $\Omega_0 B_*\mathcal{P}_X$ denotes the connected component of $\Omega B_*\mathcal{P}_X$ containing the constant loop at 0. The last isomorphism is due to Quillen and Dror (cf. \cite{Schl} Theorem 2.2.9, \cite{Ge} Theorem 2.16), which is functorial in $X$.

We construct the most left isomorphism of (5.3.1) for the convenience of the reader, which is not described in \cite{Gi1} Proposition 2.15 explicitly. Fix an arbitrary open subset $U \subset X$. Let $\alpha_U : 0 \to \mathcal{O}_U$ (resp. $\beta_U : 0 \to \mathcal{O}_U$) be the morphism in $\mathcal{P}_U$ given by the following diagram in $P_U$:
\[
\begin{array}{ccc}
0 & \xrightarrow{id} & 0 \\
\gamma \downarrow & & \downarrow \gamma \\
\mathcal{O}_U & \to & \mathcal{O}_U
\end{array}
\]

and (resp. $0 \to \mathcal{O}_U \xrightarrow{id} \mathcal{O}_U$).

Let $\ell$ be the loop $(\beta_U)^{-1} * \alpha_U$ on $B_*\mathcal{P}_U$, leaving 0 for $\mathcal{O}_U$ along $\alpha_U$ and then coming back to 0 along $(\beta_U)^{-1}$. For $n \geq 0$ (resp. $n < 0$), we define a loop $\ell^n$ as the $n$-fold composition of $\ell$ (resp. the $(-n)$-fold composition of $\ell^{-1} = (\alpha_U)^{-1} * \beta_U$). By the composition of loops, we obtain a functorial morphism of (pointed) simplicial sets
\[
\mathbb{Z} \times \Omega_0 B_*\mathcal{P}_X \longrightarrow \Omega B_*\mathcal{P}_U, \quad (n, x) \mapsto \ell^n * x.
\]
Sheafifying this map, we obtain a morphism in $\Delta^{op}\text{Shv}_\bullet(X_{Zar})$
\[
\varphi : \mathbb{Z} \times \Omega_0 B_*\mathcal{P}_X \longrightarrow \Omega B_*\mathcal{P}_X,
\]
which is obviously functorial in $X$, and moreover, a weak equivalence in the sense of Definition 5.4(1). Indeed, $K_0(\mathcal{O}_{X,x})$ for $x \in X$ is generated by the above loop $\ell$ (i.e., $K_0(\mathcal{O}_{X,x}) \cong \mathbb{Z}$), and the stalk $\varphi_x$ is a weak equivalence by the theorem of Quillen mentioned before (cf. \cite{Gra} p. 228). Thus we obtain the desired isomorphism. \hfill \Box
Finally we define the universal Chern class as the composite morphism in $\mathcal{H}_\bullet(X)$

$$C_i : \Omega B_* \mathcal{D} \mathcal{P}_X \xrightarrow{\gamma} \mathbb{Z}_\infty B_* \mathcal{GL}(\mathcal{O}_X) \xrightarrow{\mathbb{Z}_\infty(c_i, \infty)} \mathbb{Z}_\infty \mathcal{H}(\Gamma(i)_X, 2i) \cong \mathcal{H}(\Gamma(i)_X, 2i),$$

(5.3.2)

where the last isomorphism is obtained from the fact that simplicial abelian groups are $\mathbb{Z}$-complete [BK1] 4.2. We next review the universal augmented total Chern class and the universal Chern character, which will be useful later.

**Definition 5.4** ([Gi1] Definition 2.27, 2.34) Let $X$ be a scheme which belongs to $\text{Ob}(\mathcal{C})$.

1. For $\mathcal{E}_* \in \text{Ob}(\Delta^{op}\text{Shv}_\bullet(X_{zar}))$, we define $H^{2i}(X, \mathcal{E}_*; \Gamma(i))$ (resp. $H^0(X, \mathcal{E}_*; \mathbb{Z})$) as the mapping class group $\text{Mor}_{\mathcal{H}_\bullet(X)}(\mathcal{E}_*, \mathcal{H}(\Gamma(i)_X, 2i))$ (resp. $\text{Mor}_{\mathcal{H}_\bullet(X)}(\mathcal{E}_*, \mathbb{Z})$).

2. We define the universal augmented total Chern class as

$$\tilde{C} := (\text{rk}, C_0, C_1, C_2, \ldots) \in H^0(X, \Omega B_* \mathcal{D} \mathcal{P}_X; \mathbb{Z}) \times \prod_{i \geq 0} H^{2i}(X, \Omega B_* \mathcal{D} \mathcal{P}_X; \Gamma(i)),$$

where $\text{rk}$ denotes the class of the rank function $\Omega B_* \mathcal{D} \mathcal{P}_X \to \mathbb{Z}$. Note that $C_0$ is the constant function with value 1.

3. (cf. [G2] Chapter I (1.29)) We define the universal Chern character as

$$\text{ch} := \text{rk} + \eta \left( \log \left( 1 + \sum_{i=1}^\infty C_i \right) \right) \in \prod_{i \geq 0} (H^{2i}(X, \Omega B_* \mathcal{D} \mathcal{P}_X; \Gamma(i)) \otimes \mathbb{Q}),$$

where $\eta = (\eta_i)_{i \geq 0}$ denotes the graded additive endomorphism

$$\eta_i(x_i) := \frac{(-1)^{i-1}}{(i-1)!} x_i \quad (x_i \in H^{2i}(X, \Omega B_* \mathcal{D} \mathcal{P}_X; \Gamma(i)) \otimes \mathbb{Q}).$$

For a morphism $f : Y \to X$ in $\mathcal{C}$, there is a commutative diagram in $\mathcal{H}_\bullet(Y)$

$$\begin{array}{ccc}
\Omega B_* \mathcal{D} \mathcal{P}_X & \xrightarrow{c_{i,Y}} & \mathcal{H}(\Gamma(i)_Y, 2i) \\
\downarrow f^* & & \downarrow f^* \\
\Omega B_* \mathcal{D} \mathcal{P}_Y & \xrightarrow{c_{i,Y}} & \mathcal{H}(\Gamma(i)_Y, 2i)
\end{array}$$

by Theorem 4.2(2) and the construction of $C_i$. One can also check a similar commutativity for the universal Chern character.

We end this section with the following definitions and facts, which will be useful later.

**Definition 5.5** Let $X$ be a scheme.
(1) We say that an object $J_\ast$ of $\Delta^{\text{op}}\text{Shv}_\ast(X_{\text{Zar}})$ is \textit{flasque}, if it is fibrant. We say that a morphism $\tau : \mathcal{E}_\ast \to J_\ast$ in $\Delta^{\text{op}}\text{Shv}_\ast(X_{\text{Zar}})$ is a \textit{flasque resolution} of $\mathcal{E}_\ast$, if $\tau$ is a trivial cofibration and $J_\ast$ is flasque.

(2) For a scheme $X$ and a closed subset $Z \subset X$, we define the functor
$$R \Gamma_Z(X, -) : \mathcal{H}_\ast(X) \longrightarrow \mathcal{H}_\ast$$

as $\mathcal{E}_\ast \mapsto \text{Fib}(\Gamma(X, J_\ast) \to \Gamma(X \setminus Z, J_\ast))$, where $\text{Fib}$ denotes the mapping fiber and $J_\ast$ denotes a flasque resolution of $\mathcal{E}_\ast$ in $\Delta^{\text{op}}\text{Shv}_\ast(X_{\text{Zar}})$.

(3) For a morphism $f : Y \to X$ of schemes, we define the functor
$$Rf_* : \mathcal{H}_\ast(Y) \longrightarrow \mathcal{H}_\ast(X)$$

as $\mathcal{E}_\ast \mapsto f_* J_\ast$, where $J_\ast$ denotes a flasque resolution of $\mathcal{E}_\ast$. When $f$ is a closed immersion, we define
$$Rf^! : \mathcal{H}_\ast(X) \longrightarrow \mathcal{H}_\ast(Y)$$

as $\mathcal{E}_\ast \mapsto f^* \text{Fib}(J_\ast \to j_* j^* J_\ast)$, where $j$ denotes the open immersion $X \setminus Y \hookrightarrow X$ and $\text{Fib}$ means the sheafified mapping fiber.

**Proposition 5.6** Let $f : Y \to X$ be a morphism of schemes, and let $j$ be an integer.

1. Let $\mathcal{F}^\bullet$ be a cochain complex of abelian sheaves on $Y_{\text{Zar}}$. Then we have
$$Rf_* \mathcal{K}(\mathcal{F}^\bullet, j) \cong \mathcal{K}(Rf_* \mathcal{F}^\bullet, j) \text{ in } \mathcal{H}_\ast(X).$$

2. Suppose that $f : Y \to X$ is a closed immersion, and let $\mathcal{G}^\bullet$ be a cochain complex of abelian sheaves on $X_{\text{Zar}}$. Then we have
$$Rf^! \mathcal{K}(\mathcal{G}^\bullet, j) \cong \mathcal{K}(Rf^! \mathcal{G}^\bullet, j) \text{ in } \mathcal{H}_\ast(Y).$$

**Proof.** The assertions follow from the following fact: For a chain complex of abelian Zariski sheaves on a scheme
$$J_\ast : \cdots \to J_r \to \cdots \to J_2 \to J_1 \to J_0$$

with each $J_i$ flasque (in the abelian sense) for $i > 0$, its associated simplicial abelian sheaf is flasque, cf. [GJ] Chapter III Lemma 2.9. □
6 Chern class and character for higher $K$-theory

Let the notation be as in §4. Let $X$ be a scheme which belongs to $\text{Ob}(\mathcal{C})$, and let $Z$ be a closed subset of $X$. For a simplicial sheaf of pointed sets $\mathcal{E}_* \in \text{Ob}(\Delta^{op}\text{Shv}_*(X_{\text{Zar}}))$ and a non-negative integer $j \geq 0$, we define

$$H^{-j}_Z(X, \mathcal{E}_*) := \pi_j(R\Gamma_Z(X, \mathcal{E}_*)) = \text{Mor}_{\mathcal{H}_{\text{est}}(X)}(\mathcal{S}_j^Z, \mathcal{E}_*),$$

where $\mathcal{S}_j^Z$ denotes the constant sheaf (of simplicial pointed sets) on $Z$ associated with the singular simplicial set of the $j$-sphere. See Definition 5.5 (2) for $R\Gamma_Z(X, -)$. We first note

Lemma 6.1 Let $\mathcal{F}^\bullet$ be a cochain complex of abelian sheaves on $X_{\text{Zar}}$. Then for $j \geq 0$ and $i \in \mathbb{Z}$, there is a canonical isomorphism

$$H^{-j}_Z(X, \mathcal{K}^i(\mathcal{F}^\bullet, i)) \cong H^{2i-j}_Z(X, \mathcal{F}^\bullet),$$

where the right hand side means the hypercohomology of $\mathcal{F}^\bullet$ with support in $Z$ in the sense of §7.2.

Proof. See [Gi1] Corollary 1.13. Note that the degree of the left hand side is not $j$ but $-j$, correctly. See also [BG] §2 Proposition 2. □

Let $K^j_Z(X) = K_j(X, X \smallsetminus Z)$ be the $j$-th algebraic $K$-group of $X$ with support in $Z$. We define the Chern class map

$$C^Z_{i,j,X} : K^j_Z(X) \to H^{2i-j}_Z(X, \Gamma(i))$$

(6.1.1) as the composite map

$$K^j_Z(X) = H^{-j}_Z(X, \Omega B_* \mathcal{D} \mathcal{P}_X) \xrightarrow{c_i} H^{-j}_Z(X, \mathcal{K}(\Gamma(i)_X, 2i)) \cong H^{2i-j}_Z(X, \Gamma(i)).$$

Here $c_i$ denotes the universal Chern class (5.3.2). We have used Lemma 6.1 in the last isomorphism. The map $C^X_{i,0,X}$, i.e., the case $j = 0$ and $Z = X$ agrees with $c_i$ for $X_\ast = X$ (constant simplicial scheme) defined in Definition 4.1.

Proposition 6.2 (1) $C^Z_{i,j}$ is contravariantly functorial in the pair $(X, Z)$, that is, for a morphism $f : X' \to X$ in $\mathcal{C}$ and a closed subset $Z' \subset X'$ with $f^{-1}(Z) \subset Z'$, the following diagram commutes:

$$\begin{array}{ccc}
K^j_Z(X) & \xrightarrow{f^*} & K^j_Z(X') \\
\downarrow C^Z_{i,j,X} & & \downarrow C^Z_{i,j,X'} \\
H^{2i-j}_Z(X, \Gamma(i)) & \xrightarrow{f^*} & H^{2i-j}_Z(X', \Gamma(i)).
\end{array}$$

(2) $C^Z_{i,j,X}$ is additive for $j > 0$.

Proof. (1) follows from the definition of $C^Z_{i,j,X}$ and the commutative diagram (5.4.1). The assertion (2) follows from Theorem 4.2 (3) and the arguments in [Gi1] Lemma 2.26. □
Remark 6.3 For the proof of Proposition 6.2, we need the framework of Chern classes of representations [Gi1] Definitions 2.1 and 2.10, which we omit in this paper because one can easily establish it under our setting by the same arguments as in loc. cit.

Definition 6.4  (1) We define $K^Z_*(X)$ as the direct sum of $K^Z_i(X)$ with $i \geq 0$, and define $\widehat{H}^*_Z(X, \Gamma(\ast))_Q$ as the direct product of $H^i_Z(X, \Gamma(n)) \otimes Q$ with $i, n \geq 0$.

(2) We define the Chern character $\text{ch}^Z_X: K^Z_*(X) \longrightarrow \widehat{H}^*_Z(X, \Gamma(\ast))_Q$ as the composite map

$$K^Z_j(X) \xrightarrow{\text{ch}} \prod_{i \geq 0} (H^i_Z(X, \mathcal{K}(\Gamma(i)X, 2i)) \otimes Q) \cong \prod_{i \geq 0} (H^{2i-j}_Z(X, \Gamma(i)) \otimes Q).$$

Here $\text{ch}$ denotes the universal Chern character defined in Definition 5.4 (3). We often write $\text{ch}_X$ for $\text{ch}^Z_X$.

Proposition 6.5 (1) $\text{ch}^Z_X$ is contravariantly functorial in the pair $(X, Z)$, that is, for a morphism $f : X' \rightarrow X$ in $\mathcal{C}$ and a closed subset $Z' \subset X'$ with $f^{-1}(Z) \subset Z'$, the following diagram commutes:

$$
\begin{array}{ccc}
K^Z_*(X) & \xrightarrow{f^*} & K^Z_*(X') \\
\downarrow{\text{ch}} & & \downarrow{\text{ch}_X'} \\
\widehat{H}^*_Z(X, \Gamma(\ast)) & \xrightarrow{f^*} & \widehat{H}^*_Z(X', \Gamma(\ast))
\end{array}
$$

(2) The Chern character $\text{ch}^Z_X$ is a ring homomorphism.

Proof. (1) follows from the definition of $\text{ch}^Z_X$ and the commutative diagram (5.4.1). The assertion (2) follows from Theorem 4.2 (4), Proposition 6.2 (2) and the arguments in [Gi1] Proposition 2.35.

7 Push-forward for projective morphisms

Let $\mathcal{C}$ and $\Gamma(\ast)$ be as in [4]. See [1, 2] for the definition of projective morphisms.

Definition 7.1 Let $f : Y \rightarrow X$ be a projective morphism in $\mathcal{C}$.

(1) By taking a factorization

$$f : Y \xleftarrow{g} \mathbb{P}^m_X \rightarrow X$$

with $g$ a closed immersion, we define the relative dimension of $f$ as the integer $m - \text{codim}(g)$. Because we deal with only universally catenary schemes, this number is independent of the factorization.
(2) We say that \( f \) is *regular* if \( f \) has a factorization as above for which \( g \) is a regular closed immersion. A regular projective morphism is a regular morphism in the sense of [FL] p. 86.

Now let \( \Gamma(*) \) be an admissible cohomology theory on \( \mathcal{C} \), and let \( f : Y \to X \) be a regular projective morphism in \( \mathcal{C} \). The main aim of this section is to construct push-forward morphisms in \( D(X_{zar}) \)

\[
f_t : Rf_*\Gamma(i + r)_{Y}[2r] \to \Gamma(i)_{X} \quad (i \in \mathbb{Z}) \tag{7.1.1}
\]

and prove Theorem 7.2 below, where \( r \) denotes the relative dimension of \( f \). The results in this section will play key roles in the following sections. Taking a factorization

\[
f : Y \xleftarrow{g} \mathbb{P}^n_X =: \mathbb{P}^n \to X
\]

of \( f \) such that \( g \) is a regular closed immersion, we are going to define the push-forward morphism \((7.1.1)\) as the composite

\[
Rf_*\Gamma(i + r)_{Y}[2r] \cong Rp_*Rg_*\Gamma(i + r)_{Y}[2r] \xrightarrow{Rp_*\psi} Rp_*\Gamma(i + m)_{P^n}[2m] \xrightarrow{p_*} \Gamma(i)_{X},
\]

where \( p_* \) denotes the composite of the Dold-Thom isomorphism and a projection

\[
p_* : Rp_*\Gamma(i + m)_{P^n}[2m] \cong \bigoplus_{j=0}^{m} \Gamma(i + m - j)_{X}[2(m - j)] \to \Gamma(i)_{X}.
\]

**Theorem 7.2** Let \( f : Y \to X \) be a regular projective morphism in \( \mathcal{C} \).

1. (Well-definedness) \( f_t \) does not depend on the choice of a factorization of \( f \). In particular, we have \( f_1 = f_2 \) when \( f \) is isomorphic to a natural projection \( \mathbb{P}^n_X \to X \).

2. (Projection formula) The following diagram commutes in \( D(X_{zar}) \):

\[
\begin{array}{ccc}
\Gamma(i)_X \otimes Rf_*\Gamma(j + r)_Y[2r] & \xrightarrow{\text{id} \otimes f_t} & \Gamma(i)_X \otimes \Gamma(j)_X \\
\uparrow f^* \otimes \text{id} & & \downarrow \text{product} \\
Rf_*\Gamma(i)_Y \otimes Rf_*\Gamma(j + r)_Y[2r] & \xrightarrow{\text{product}} & Rf_*\Gamma(i + j + r)_Y[2r] \xrightarrow{f} \Gamma(i + j)_X.
\end{array}
\]

3. (Transitivity) For another regular projective morphism \( f' : Z \to Y \) in \( \mathcal{C} \) of relative dimension \( r' \), the composite morphism

\[
R(f \circ f')_*\Gamma(i + r + r')_Z[2(r + r')] \cong Rf'_*Rf_*\Gamma(i + r + r')_Z[2(r + r')] \xrightarrow{f'_*f_t} Rf_*\Gamma(i + r)_Y[2r] \xrightarrow{f_1} \Gamma(i)_X
\]

agrees with \( (f \circ f')_1 \).
(4) **(Base-change property)** Let

\[
\begin{array}{c}
Y' \\ \downarrow \beta \\
Y \\ \downarrow f
\end{array} \quad \begin{array}{c}
X' \\ \downarrow \alpha
\end{array}
\]

be a commutative diagram in \( \mathcal{C} \) such that \( f \) and \( f' \) are regular projective morphisms of relative dimension \( r \) and such that \( \alpha \) and \( \beta \) are closed immersions. Let \( U \) be an open subset of \( Y \) for which the following square is cartesian:

\[
\begin{array}{c}
\beta^{-1}(U) \\ \downarrow f|_{\beta^{-1}(U)} \\
U \\ \downarrow f|_{U}
\end{array} \quad \begin{array}{c}
X' \\ \downarrow \alpha
\end{array}
\]

Then for a closed subset \( Z \subset Y \) contained in \( U \), the diagram

\[
\begin{array}{c}
R(\alpha \circ f')_* R\Gamma_{\beta^{-1}(Z)}(Y', \Gamma(i + r))[2r] \\ \downarrow \beta^* \\
Rf_* R\Gamma_Z(Y, \Gamma(i + r))[2r] \\ \downarrow f^*
\end{array} \quad \begin{array}{c}
\alpha_* R\Gamma_{\alpha^{-1}(f(Z))}(X', \Gamma(i)) \\
Rf_* R\Gamma_{\alpha^{-1}(f(Z))}(X, \Gamma(i))
\end{array}
\]

commutes in \( D(X_{\text{Zar}}) \).

**Remark 7.3**

(1) Applying Theorem 7.2 to the example in §3.6, we obtain push-forward morphisms of logarithmic Hodge-Witt sheaves for projective morphisms of regular schemes over \( \mathbb{F}_p \), which satisfy the properties listed above. This result answers the problem raised in [Sh] Remark 5.5 affirmatively, and we obtain the same compatibility as in loc. cit. Theorem 5.4 for \( W_n \Omega^q_{\log} \) with \( n \geq 2 \) as well.

(2) We do not need the axiom 2.5(4) to prove Theorem 7.2.

We first prepare the following lemma:

**Lemma 7.4** Let \( X \) be an object of \( \mathcal{C} \). Let \( m \) and \( n \) be non-negative integers, and put \( \mathbb{P}^m := \mathbb{P}^m_X, \mathbb{P}^n := \mathbb{P}^n_X \) and \( \mathbb{P}^m \times \mathbb{P}^n := \mathbb{P}^m \times_X \mathbb{P}^n \).

(1) Let \( p : \mathbb{P}^m \to X \) be the natural projection, and let \( s : X \to \mathbb{P}^m \) be a section of \( p \). Let \( \xi \in H^2(\mathbb{P}^m, \Gamma(1)) \) be the first Chern classes of the tautological line bundle on \( \mathbb{P}^m \). Then we have

\[
s_1(1) = \xi^m \quad \text{in} \quad H^{2m}(\mathbb{P}^m, \Gamma(m)),
\]

where \( s_1 \) denotes the push-forward map \( H^0(X, \Gamma(0)) \to H^{2m}(\mathbb{P}^m, \Gamma(m)) \).
(2) Put $N := mn + m + n$. Let $p' : \mathbb{P}^m \to X$ and $q : \mathbb{P}^N := \mathbb{P}^N_X \to X$ be the natural projections. Let $\psi : \mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ be the Segre embedding, and let $\pi : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m$ be the first projection. Then the diagram

$$
\begin{array}{c}
\frac{R(p \times p')_* \Gamma(i + m + n)_{\mathbb{P}^m \times \mathbb{P}^n} [2(m + n)]}{\pi_2}
\xrightarrow{\psi_*} Rq_* \Gamma(i + N)_{\mathbb{P}^N} [2N]
\end{array}
$$

is commutative in $D(X_{zar})$ for each $i \in \mathbb{Z}$.

(3) Let $g : Y \hookrightarrow X$ be a regular closed immersion of codimension $c$ in $\mathcal{C}$. Let $p_Y : \mathbb{P}^m_Y \to Y$ be the natural projection, and let $g' : \mathbb{P}^m_Y \hookrightarrow \mathbb{P}^m (= \mathbb{P}^m_X)$ be the regular closed immersion induced by $g$. Then the diagram

$$
\begin{array}{c}
g_* R_{p_Y} \Gamma(i + m)_{\mathbb{P}^m_Y} [2m]
\xrightarrow{R_{p_*} (g'_*)}
Rp_* \Gamma(i + m + c)_{\mathbb{P}^m} [2(m + c)]
\end{array}
$$

commutes in $D(X_{zar})$ for each $i \in \mathbb{Z}$.

(4) Let $Y \in \text{Ob}(\mathcal{C})$ be a scheme over $X$, and let $g : Y \hookrightarrow \mathbb{P}^m$ and $h : Y \hookrightarrow \mathbb{P}^n$ be regular closed immersions over $X$. Let $\varphi : Y \hookrightarrow \mathbb{P}^m \times \mathbb{P}^n$ be the closed immersion induced by $g$ and $h$. Then $\varphi$ is a regular closed immersion, and the diagram

$$
\begin{array}{c}
g_* \Gamma(i)_Y
\xrightarrow{R_{\pi_*} (\varphi)_*}
R\pi_* \Gamma(i + n + c)_{\mathbb{P}^m \times \mathbb{P}^n} [2(n + c)]
\end{array}
$$

commutes in $D((\mathbb{P}^m)_{zar})$ for each $i \in \mathbb{Z}$. Here $c$ denotes $\text{codim}(g)$, and $\pi : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m$ denotes the first projection.

**Proof of Lemma 7.4** (1) The assertion follows from the axioms in Definition 2.5(3), whose details are straight-forward and left to the reader.

(2) Let $s : X \to \mathbb{P}^m$ and $\xi \in H^2(\mathbb{P}^m, \Gamma(1))$ be as in (1). Let $\eta \in H^2(\mathbb{P}^n, \Gamma(1))$ (resp. $\zeta \in H^2(\mathbb{P}^N, \Gamma(1))$) be the first Chern classes of the tautological line bundle on $\mathbb{P}^m$ (resp. on $\mathbb{P}^N$). Fix a section $s' : X \to \mathbb{P}^n$ of $p'$, and let $\sigma : X \to \mathbb{P}^m \times \mathbb{P}^n$ be the morphism induced by $s$ and $s'$. By a similar argument as for (1), we see that

$$
\begin{align*}
\sigma(1) &= \pi_1^*(\xi^m) \cup \pi_2^*(\eta^n) \quad \text{in } H^{2(m+n)}(\mathbb{P}^m \times \mathbb{P}^n, \Gamma(m+n)), \\
(\psi \circ \sigma)_*(1) &= \zeta^N \quad \text{in } H^{2N}(\mathbb{P}^N, \Gamma(N)).
\end{align*}
$$

One can easily deduce the assertion from these facts.
(3) By the axioms in Definition 2.5 (3), the following diagram commutes in \( D(X_{zar}) \):

\[
g_\ast R_p \gamma (i + m)_{\mathbb{P}^m} [2m] \longrightarrow R_p \gamma (i + m + c)_{\mathbb{P}^m} [2(m + c)]
\]

\[
\bigoplus_{j=0}^m g_\ast (i + j)_{\mathbb{P}^{2j}} \longrightarrow \bigoplus_{j=0}^m \gamma (i + j + c)_{\mathbb{P}^{2(j + c)}},
\]

where the vertical isomorphisms are the Dold-Thom isomorphisms. The assertion follows from this fact.

(4) The first assertion follows from [BGI] Exposé VIII Corollaire 1.3. As for the second assertion, replacing \( P_m \) with \( X \), we may assume that \( m = 0 \) and that \( Y \) is a closed subscheme of \( X \) via \( g \). Then decomposing \( \varphi (= h) \) as \( Y \hookrightarrow \mathbb{P}^n \hookrightarrow \mathbb{P}^n_{\mathbb{X}} \), we see that the assertion is reduced to the results in (1) and (3), by the transitivity in Definition 2.5 (3c).

\[ \square \]

Proof of Theorem 7.2. We only give an outline of the proof here, whose details would be standard and left to the reader. We write \( P_m \) for the projective space \( P_m \) over \( X \) for simplicity.

(1) Suppose we are given two factorizations \( Y \overset{g}{\hookrightarrow} \mathbb{P}^m \overset{p}{\rightarrow} X \) and \( Y \overset{h}{\hookrightarrow} \mathbb{P}^n \overset{p'}{\rightarrow} X \) of \( f \).

There is a commutative diagram in \( \mathcal{C} \):

\[
\begin{array}{ccc}
Y & \overset{\varphi}{\leftarrow} & \mathbb{P}^m \\
\downarrow g & & \downarrow p \\
\mathbb{P}^n & \overset{\psi}{\leftarrow} & \mathbb{P}^{m+n} \\
\end{array}
\]

where \( \varphi \) denotes the closed immersion induced by \( g \) and \( h \), and \( \psi \) denotes the Segre embedding. The right vertical arrow \( q \) denotes the natural projection. Note that \( \varphi \) and \( \psi \) are both regular by [BGI] Exposé VIII Corollaire 1.3 and [GD2] Théorème 17.12.1, respectively. By Lemma 7.4 (2), (4) and the axioms in Definition 2.5 (3), we have

\[
p_\sharp \circ R_p (g h) = q_\sharp \circ R q (\psi \circ \varphi) : R f_\ast \gamma (i + r)_{\mathbb{P}^{2r}} \longrightarrow \gamma (i)_{\mathbb{X}},
\]

which implies the assertion.

(2) Fix a factorization \( Y \overset{g}{\hookrightarrow} \mathbb{P}^m \overset{p}{\rightarrow} X \) of \( f \). The projection formula holds for \( g \) by the axiom in Definition 2.5 (3b), and holds for \( p \) as well because the Dold-Thom isomorphism is compatible with the left multiplication by \( \Gamma (\ast)_{\mathbb{X}} \), cf. Definition 2.5 (2). The assertion follows from these facts.

(3) Taking factorizations \( f : Y \overset{g}{\hookrightarrow} \mathbb{P}^m \overset{p}{\rightarrow} X \) and \( f' : Z \overset{h}{\hookrightarrow} \mathbb{P}^n \overset{p'}{\rightarrow} Y \), one can easily deduce the assertion from Lemma 7.4 (3), (4). The details are left to the reader.

(4) Fix a factorization \( Y \overset{g}{\hookrightarrow} \mathbb{P}^m \overset{p}{\rightarrow} X \) of \( f \). There are cartesian squares

\[
\begin{array}{ccc}
\beta^{-1}(U) & \overset{\square}{\leftarrow} & \mathbb{P}^m_{\mathbb{X}} \\
\downarrow g \circ \iota & & \downarrow \iota \\
U & \overset{\square}{\rightarrow} & \mathbb{P}^m \\
\end{array}
\]

\[
\begin{array}{ccc}
& & X' \\
\downarrow & & \downarrow \iota \\
X & \overset{\square}{\rightarrow} & X
\end{array}
\]

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where the horizontal arrows of the left square are (locally closed) immersions. The assertion holds for the right square and a closed subset \( W \subset \mathbb{P}^m \) by the definition of push-forward morphisms. On the other hand, the assertion holds for the left square and a closed subset \( Z \subset Y \) contained in \( U \) by excision and the base-change property in Definition 2.5(3d). The assertion for \( f \) follows from these facts and the transitivity established in (3).

Corollary 7.5 Let \( Y \in \text{Ob}(\mathcal{E}) \) be a scheme which admits an ample family of invertible sheaves, and let \( \pi : E \to Y \) be a vector bundle of rank \( r + 1 \). Let \( p : \mathbb{P}(E) \to Y \) be the projective bundle associated with \( \Gamma \), cf. (1.2.1), which is projective in our sense by the assumption on \( Y \). Then for each \( i \in \mathbb{Z} \), the push-forward map \( p_! : R_p! \Gamma(i + r)_{\mathbb{P}(E)}[2r] \to \Gamma(i)_Y \) agrees with the composite of the Dold-Thom isomorphism and a projection

\[
p_! : R_p! \Gamma(i + r)_{\mathbb{P}(E)}[2r] \cong \bigoplus_{j=0}^r \Gamma(i + r - j)_Y [2(r - j)] \to \Gamma(i)_Y \quad \text{in } D(Y_{\text{Zar}}).
\]

Proof. By a standard hyper-covering argument, we may assume that \( E \cong \mathbb{A}^{r+1} \). Then the assertion follows from Theorem 7.2(1).

Corollary 7.6 Let \( Y \in \text{Ob}(\mathcal{E}) \) be a scheme which admits an ample family of invertible sheaves, and let \( \pi : E \to Y \) be a vector bundle of rank \( r \). Let \( p : X := \mathbb{P}(E \oplus 1) \to Y \) be the projective completion of \( E \), where 1 denotes the trivial line bundle over \( Y \). Let \( f : Y \hookrightarrow X \) be the zero section of \( p \), and let \( Q \) be the universal quotient bundle \( p^*(E \oplus 1)/(L^{\text{taut}})^Y \) on \( X \). Then the map \( f_* \) sends the unity \( 1 \in H^0(Y, \Gamma(0)) \) to \( c_r(Q) \in H^{2r}(X, \Gamma(r)) \).

Proof. Let \( \xi := c_1(L^{\text{taut}}) \in H^2(X, \Gamma(1)) \) be the first Chern class of the tautological line bundle over \( X \), and let \( i_\infty : \mathbb{P}(E) \hookrightarrow \mathbb{P}(E \oplus 1) = X \) be the infinite hyperplane. The Dold-Thom isomorphism in the axiom 2.5(2) (for both \( X \) and \( \mathbb{P}(E) \)) and the functoriality mentioned in Remark 2.4 imply that the kernel of the pull-back map

\[
i^*_\infty : H^{2r}(X, \Gamma(r)) \to H^{2r}(\mathbb{P}(E), \Gamma(r))
\]

is generated, over \( H^0(Y, \Gamma(0)) \), by the element

\[
\xi^r + \xi^{r-1} \cup p^*c_1(E) + \xi^{r-2} \cup p^*c_2(E) + \cdots + p^*c_r(E) = c_r(Q), \tag{7.6.1}
\]

where we have used Theorem 4.2(2), (3) for \( X_* = X \) to obtain the last equality. On the other hand, we have \( i^*_\infty \circ f_1 = 0 \), because \( f_1 \) factors through \( H^{2r}_Y(X, \Gamma(r)) \) and \( i^*_\infty \) factors through \( H^{2r}(X \setminus Y, \Gamma(r)) \). Therefore \( f_1(1) \) belongs to \( \text{Ker}(i^*_\infty) \) and we have

\[
f_1(1) = a \cdot c_r(Q) \quad \text{for some } a \in H^0(Y, \Gamma(0)).
\]

It remains to check \( a = 1 \). By the transitivity in Theorem 7.2(3), the claim is further reduced to the equality \( p_!(c_r(Q)) = 1 \), which follows from the equality (7.6.1), Theorem 7.2(2) and Corollary 7.5 for \( E \oplus 1 \) over \( Y \).
8 Construction of a universal polynomial

For an indeterminate $x$, we define

$$Z[x]^{\Diamond} := \{ f(x) \in \mathbb{Q}[x] \mid f(m) \in \mathbb{Z} \text{ for any } m \in \mathbb{Z} \}.$$  

Let $n$ and $r$ be integers with $n \geq 0$ and $r \geq 1$. In this section, we construct a universal polynomial

$$P_{n,r}(t_0, t_1, \ldots, t_n; u_1, u_2, \ldots, u_r) \in \mathbb{Z}[t_0]^{\Diamond}[t_1, \ldots, t_n; u_1, u_2, \ldots, u_r],$$

in an explicit way, modifying the construction of Fulton-Lang [FL] Chapter II §4. This polynomial agrees with the polynomial considered in [G2] Chapter I Proposition 1.5 up to signs of $u_j$’s with $j$ odd, and plays a central role in the Riemann-Roch theorem without denominators (cf. [Jou] §1). See also Proposition 8.4 and Theorem 9.1 below. We start with indeterminates $a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_r)$ and a power series

$$F_{n,r}(a, b) := \prod_{i=1}^{n} \prod_{j=0}^{r} \prod_{k_1<\cdots<k_j} (1 + a_i - b_{k_1} - b_{k_2} - \cdots - b_{k_j})^{(-1)^j},$$

which has constant term 1 and is symmetric in $a_1, a_2, \ldots, a_n$ and also in $b_1, b_2, \ldots, b_r$. It is well-known that $F_{n,r}(a, b) - 1$ is divisible by $b_1 b_2 \cdots b_r$ (cf. [FL] p. 44). We mention here a Chern-class-theoretic meaning of the power series $g_r(b) := F_{1,r}(0, b)$. Let $s_i$ be the $i$-th elementary symmetric expression in $b_1, b_2, \ldots, b_r$, and let $G_r(t_1, t_2, \ldots, t_r)$ be the power series satisfying

$$g_r(b) = 1 + s_r \cdot G_r(s_1, s_2, \ldots, s_r).$$

The following fact will be useful later in [9] below:

**Proposition 8.1** Let $\mathcal{C}$ and $\Gamma(*)$ be as in [2] and let $\pi : E \to X$ be a vector bundle of rank $r$ with $X \in \text{Ob}(\mathcal{C})$. Then we have

$$\sum_{i \geq 0} c_i(\lambda^{-1}(E^*)) = 1 + c_r(E) \cup G_r(c_1(E), c_2(E), \ldots, c_r(E)) \quad (8.1.1)$$

in the complete cohomology ring $\hat{H}^2(X, \Gamma(*)) := \prod_{i \geq 0} H^2(X, \Gamma(i))$, where we defined

$$\lambda^{-1}(E^*) := 1 - [E^*] + [\wedge^2 E^*] - \cdots + (-1)^r [\wedge^r E^*] \in K_0(X).$$

**Proof.** We may assume that $E$ is an successive extension of line bundles $L_1, L_2, \ldots, L_r$ by the splitting principle of vector bundles, cf. the axiom 2.3(2), Theorem 4.2(2). Put $\beta_j := c_1(L_j) \in H^2(X, \Gamma(1))$. Then the left hand side of (8.1.1) agrees with

$$\prod_{j=0}^{r} \left( \sum_{i \geq 0} c_i(\wedge^j E^*) \right)^{(-1)^j} = \prod_{j=0}^{r} \prod_{k_1<\cdots<k_j} (1 - \beta_{k_1} - \beta_{k_2} - \cdots - \beta_{k_j})^{(-1)^j} \quad (8.1.2)$$

by Theorem 4.2(3) for $X_* = X$. On the other hand, the right hand side of (8.1.1) agrees with that of (8.1.2) by the definition of $g_r(b)$. Thus we obtain the proposition. □
We next consider a power series
\[ J_{n,r}(a, b) := F_{n,r}(a, b) \cdot g_r(b)^{-n}, \]
which is symmetric in \( a_1, a_2, \ldots, a_n \) and in \( b_1, b_2, \ldots, b_r \) as well.

**Lemma 8.2**

1. \( J_{n,r}(a, b) - 1 \) is divisible by \( b_1 b_2 \cdots b_r \).

2. Let \( \sigma_j \) be the \( j \)-th elementary symmetric expression in \( a_1, a_2, \ldots, a_n \). Then we have
\[
(1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n) \cdot g_r(b) = J_{n,r}(a, b),
\]
where \( \cdot \) denotes the product of power series in the sense of \([G2]\) Chapter I §3 (1.16)–(1.17bis) defined by regarding \( a_j \) and \( b_j \) as of degree 1.

**Proof.** (1) follows from the fact that \( F_{n,r}(a, b) - 1 \) and \( g_r(b) - 1 \) are both divisible by \( b_1 b_2 \cdots b_r \). As for (2), we have
\[
(1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n) \cdot g_r(b) = \prod_{i=1}^{n} \prod_{j=0}^{r} \prod_{k_1 < \cdots < k_j} \left\{ \frac{1 + a_i - b_{k_1} - b_{k_2} - \cdots - b_{k_j}}{(1 + a_i)(1 - b_{k_1} - b_{k_2} - \cdots - b_{k_j})} \right\} = J_{n,r}(a, b)
\]
by the definition of the \( \cdot \)-product. \( \Box \)

**Definition 8.3** Let \( h_{n,r}(t_0, a, b) \) be the homogeneous component of degree \( n + r \), with respect to \( a \) and \( b \), of the power series
\[
J_{n,r}(a, b) \cdot g_r(b)^{t_0} = F_{n,r}(a, b) \cdot g_r(b)^{t_0-n} \in \mathbb{Z}[t_0] \hat{\otimes} \mathbb{Z}[a, b],
\]
where \( t_0 \) is of degree 0, and \( (1 + x)^y \) means the binary power series
\[
\sum_{i \geq 0} \binom{y}{i} x^i = 1 + y \cdot x + \frac{y(y-1)}{2} \cdot x^2 + \frac{y(y-1)(y-2)}{3!} \cdot x^3 + \cdots \in \mathbb{Z}[y] \hat{\otimes} [x].
\]
By Lemma 8.2(1), \( h_{n,r}(t_0, a, b) \) is divisible by \( b_1 b_2 \cdots b_r \). We finally define the desired polynomial \( P_{n,r} \) as that satisfying
\[
s_r \cdot P_{n,r}(t_0, \sigma_1, \sigma_2, \ldots, \sigma_n; s_1, s_2, \ldots, s_r) = h_{n,r}(t_0, a, b),
\]
where \( \sigma_j \) is as in Lemma 8.2(2), and \( s_j \) denotes the \( j \)-th elementary symmetric expression in \( b_1, b_2, \ldots, b_r \). Note that \( P_{n,r}(t_0, t_1, \ldots, t_{n}; u_1, u_2, \ldots, u_r) \) is weighted homogeneous of degree \( n \), provided that \( t_j \) and \( u_j \) are of degree \( j \). The first few examples are as follows:

- For \( t \geq 1 \),
\[
P_{0, r}(t_0; u_1, u_2, \ldots, u_r) = (-1)^{r-1}(r-1)! t_0, \quad (r \geq 1)
\]

- For \( r = 1 \),
\[
P_{1, 1}(t_0, t_1; u_1) = \frac{t_0(t_0+1)}{2} u_1 - t_1,
\]

- For \( r = 2 \),
\[
P_{1, 2}(t_0, t_1; u_1, u_2) = -t_0 u_1 + 2t_1,
\]

- For \( r = 3 \),
\[
P_{2, 1}(t_0, t_1, t_2; u_1) = \frac{t_0(t_0+1)(t_0+2)}{6} u_1^2 - (t_0+1)t_1 u_1 + t_1^2 - 2t_2.
\]
Proposition 8.4 Consider a series $1 + \tau_1 + \tau_2 + \cdots = 1 + \sum_{i \geq 1} \tau_i$, and suppose that $\tau_j$ and $u_j$ are of degree $j$. Then the weighted homogeneous component of degree $n$ of the series

$$\{ (1 + \tau_1 + \tau_2 + \cdots) \ast (1 + u_r \cdot G_r(u_1, u_2, \ldots, u_r)) \} \cdot (1 + u_r \cdot G_r(u_1, u_2, \ldots, u_r))^t_o$$

is $u_r \cdot P_{n-r,r}(t_0, \tau_1, \tau_2, \ldots, \tau_{n-r}; u_1, u_2, \ldots, u_r)$ (resp. zero) for $n \geq r$ (resp. for $1 \leq n < r$). Here $\ast$ denotes the product considered in Lemma 8.2. (2).

Proof. The assertion is a consequence of Lemma 8.2. The details are straight-forward and left to the reader. \qed

9 Riemann-Roch theorem without denominators

Let $\mathcal{E}$ and $\Gamma(\ast)$ be as in §4 and let $\mathcal{R}$ be the direct sum of $\mathcal{H}(\Gamma(i), 2i)$ with $i \geq 0$, which is a graded commutative ring object with unity in $\Delta^{op} Shv_\bullet(\mathcal{E}_{zar})$ by the existence of $K$-injective resolutions of complexes of abelian sheaves on $\mathcal{E}_{zar}$, cf. [Ho]. For $Z \in \text{Ob}(\mathcal{E})$, let $\mathcal{R}_Z$ be the restriction of $\mathcal{R}$ onto $Z_{zar}$, which is a graded commutative ring object with unity in $\Delta^{op} Shv_\bullet(Z_{zar})$.

Let $f : Y \hookrightarrow X$ be a regular closed immersion of codimension $r$ which belongs to $\mathcal{E}$. Using the universal polynomial $P_{n,r}$ constructed in [8] we define

$$P_{n,Y/X} := P_{n,r}(\text{rk}, C_1, C_2, \ldots, C_n; c_1(N_{Y/X}), c_2(N_{Y/X}), \ldots, c_r(N_{Y/X}))$$

in $\text{Mor}_{\mathcal{E}_{zar}}(\Omega B, \mathcal{P}_Y, \mathcal{H}(\Gamma(n)_Y, 2n))$ if $n \geq 0$. Here $c_i(N_{Y/X}) \in H^{2i}(Y, \Gamma^i)$ denotes the $i$-th Chern class of the normal bundle $N_{Y/X}$, and we have taken the polynomial of Chern classes with respect to the ring structure on $\mathcal{R}_Y$. See Definition 5.4 (2) and (5.3.2) for $\text{rk}$ and $C_i$, respectively. Note that $P_{n,Y/X}$ is well-defined, because the rank function has integral values. We define $P_{n,Y/X}$ as zero if $n < 0$. The main aim of this section is to prove a local version of Riemann-Roch theorem without denominators:

Theorem 9.1 Let $f : Y \hookrightarrow X$ be as above, and assume that $X$ and $Y$ are both regular and that $Y$ has pure codimension $r \geq 1$ on $X$. Assume further the following condition:

(\#) The blow-up of $X \times \mathbb{P}^1 := X \times_{\text{Spec}(\mathbb{Z})} \mathbb{P}_\mathbb{Z}^1$ along $Y \times \{\infty\}$ belongs to $\text{Mor}(\mathcal{E})$, and the zero section of the normal bundle $N_{Y/X}$ belongs to $\text{Mor}(\mathcal{E})$.

Then the following diagram commutes in $\mathcal{H}_\bullet(Y)$ for any $i \geq 0$:

$$\begin{array}{ccc}
\Omega B \cdot \mathcal{P}_Y & \xrightarrow{f_*} & Rf^! \Omega B \cdot \mathcal{P}_X \\
P_{n-r,Y/X} \downarrow & & \downarrow \text{c}^Y_{n-X} := Rf^!(C_i) \\
\mathcal{H}(\Gamma(i-r)_Y, 2(i-r)) & \xrightarrow{f_*} & Rf^! \mathcal{H}(\Gamma(i)_X, 2i),
\end{array}$$

where the upper horizontal arrow denotes the canonical isomorphism due to Quillen ([Q] §7). See Definition 5.5 (3) for $Rf^!$. \qed
Remark 9.2 Theorem 9.1 is a generalization of a theorem of Gillet [Gi1]. However, we have to note that his proof relies on an incorrect formula $C^Y(\mathcal{O}_Y) = j_!(\lambda^{-1}C(N))$ under the notation in loc. cit. Compare with (9.2.3) and (9.2.4) below.

Proof. We prove Theorem 9.1 in two steps by revising Gillet’s arguments in [Gi1] §3. We use the facts in Proposition 5.6 freely in what follows.

Step 1. Assume that $f$ is isomorphic to the zero section of the projective completion $\pi : \mathbb{P}(E \oplus 1) \to Y$ of a vector bundle $E \to Y$ of rank $r$, where $1$ denotes the trivial line bundle on $Y$. In this step we prove that the diagram (9.2.5) below is commutative, which is weaker than the assertion of the theorem and does not require the assumption ($\#)$. We first prove that the following diagram commutes in $\mathcal{H}_\bullet(Y)$:

$$
\begin{array}{ccc}
\Omega B_\bullet \mathcal{D} Y & \xrightarrow{\pi^*} & R \pi_* \Omega B_\bullet \mathcal{D} X \\
\downarrow f_* & & \downarrow \tilde{c}_X \\
n_{Y/X} & & n_{X/X}
\end{array}
$$

which is also weaker than the theorem. For $Z \in \text{Ob}(\mathcal{E})$, let $\mathcal{R}$ be as we defined in the beginning of this section. Let $\mathcal{R}^+_Z$ be the product of $\mathcal{R}(\Gamma(i)_Z, 2i)$ with $i \geq 1$, and put

$$A_Z := Z \times \{1\} \times \mathcal{R}^+_Z \in \text{Ob}(\Delta^{op}\text{Shv}(Z_{zar})),$$

which we endow with the ring structure associated with $\mathcal{R}^+_Z$ to obtain a ring object in $\mathcal{H}_\bullet(Z)$ ([G2] Chapter I §3). There is a commutative diagram in $\mathcal{H}_\bullet(Y)$:

$$
\begin{array}{ccc}
\Omega B_\bullet \mathcal{D} Y & \xrightarrow{\pi^*} & R \pi_* \Omega B_\bullet \mathcal{D} X \\
\downarrow f_* & & \downarrow \tilde{c}_X \\
n_{Y/X} & & n_{X/X}
\end{array}
$$

where $1_K$ denotes the unity of $K_0(Y)$, and $\ast$ (resp. $\bullet$) denotes the product structure on $L^\bullet$ (resp. $\Omega B_\bullet \mathcal{D} X$, cf. [Gi1] (2.31)). See Definition 5.4(2) for $\tilde{c}_X$. The square commutes by Theorem 4.2(4) and loc. cit. Lemma 2.32. The triangle commutes by the projection formula for $K$-theory, cf. [TT] Proposition 3.17. Let us remind here the following formulas:

$$
\begin{align*}
f_*(1_K) &= \lambda^{-1}(Q^\vee) \quad \text{in} \quad K_0(X), \quad \text{[FL] Chapter V Lemma 6.2} \quad \text{(9.2.3)} \\
f_!(1_\Gamma) &= c_r(Q) \quad \text{in} \quad H^{2r}(X, \Gamma(r)), \quad \text{(Corollary 7.6) (9.2.4)}
\end{align*}
$$

where $Q$ denotes the universal quotient bundle $\pi^*(E \oplus 1)/(L^{\text{aut}})^\vee$ on $X$ and $1_\Gamma$ denotes the unity of $H^0(Y, \Gamma(0))$. We have used the fact that a regular (noetherian and separated) scheme admits an ample family of invertible sheaves [BGG1] Exposé II Corollaire 2.2.7.1, in applying Corollary 7.6 By (9.2.2) and (9.2.3), $\tilde{c}_X \circ f_*$ agrees with the composite

$$
\begin{array}{ccc}
\Omega B_\bullet \mathcal{D} Y & \xrightarrow{\pi^*} & R \pi_* \Omega B_\bullet \mathcal{D} X \\
\downarrow \tilde{c}_X & & \downarrow \star \tilde{c}_X(\lambda^{-1}(Q^\vee)) \\
R \pi_* A_X & \xrightarrow{\star \tilde{c}_X(\lambda^{-1}(Q^\vee))} & R \pi_* A_X
\end{array}
$$

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The composite of \( f_s \) and \( C_{i,X} \) in (9.2.1) is this composite morphism followed by the projection to \( R\pi_*\mathcal{H}((\Gamma(i)_{X}, 2i)) \), which is zero if \( i < r \) by Propositions 8.1 and 8.4. As for the case \( i \geq r \), when we put

\[
P_{i-r, r}(\mathrm{rk}, C_X; \pi_i) := P_{i-r, r}(\mathrm{rk}, C_{i,X}, \ldots, C_{i-r,X}; \pi_{i-r}(Q), \ldots, \pi_r(Q))
\]

we have

\[
C_{i,X} \circ f_s = (c_r(Q) \cup P_{i-r, r}(\mathrm{rk}, C_X; \pi_i)) \circ \pi^* \quad (8.1, 8.4, \ \mathrm{rk}(\lambda_{-1}(Q^\vee)) = 0)
\]

where \( \pi^* = (f_1(1_r) \cup P_{i-r, r}(\mathrm{rk}, C_X; \pi_i)) \circ \pi^* \quad (\text{by (9.2.4)}) \)

\[
= f_1 \circ f^* P_{i-r, r}(\mathrm{rk}, C_X; Q) \circ f^* \circ \pi^* \quad (\text{the axiom 2.5(3b)})
\]

\[
= f_1 \circ P_{i-r,Y/X} \quad (5.4.1, f^* Q \cong N_{Y/X})
\]

Thus the diagram (9.2.1) is commutative. Finally we obtain a commutative diagram

\[
\begin{array}{ccc}
\Omega B_* \mathcal{D} \mathcal{P}_Y & \xrightarrow{f_*} & Rf_* \Omega B_* \mathcal{D} \mathcal{P}_X \\
\downarrow P_{i-r,Y/X} & & \downarrow \pi_{i,X} \\
\mathcal{H}(\Gamma(i)_{Y}, 2(i-r)) & \xrightarrow{\pi_i} & Rf_* \mathcal{H}(\Gamma(i)_{X}, 2i)
\end{array}
\]

from (9.2.1) and the transitivity in Theorem 7.2(3).

**Step 2.** We prove the theorem using the result of Step 1 and deformation to normal bundle. Let \( t : \text{Spec}(\mathbb{Z}) \to \mathbb{P}^1_{\mathbb{Z}} =: \mathbb{P}^1 \) be a morphism of schemes, and consider the following commutative diagram of schemes:

\[
\begin{array}{ccc}
Y & \xrightarrow{p} & Y \times \mathbb{P}^1 \\
\downarrow \xi_1 & & \downarrow \phi_1 \\
M_t & \xrightarrow{\phi_t} & X \\
\downarrow f & & \downarrow t \\
Y & \xrightarrow{\xi_t} & \text{Spec}(\mathbb{Z}).
\end{array}
\]

Here \( \phi \) denotes the blow-up of \( X \times \mathbb{P}^1 \) along \( Y \times \{\infty\} \), which is projective in our sense because \( X \times \mathbb{P}^1 \) is regular and admits an ample family of invertible sheaves. Note also that \( \phi \) is a morphism in \( \mathcal{C}_l \) by assumption. The arrow \( \xi \) is a closed immersion induced by \( h \), where we have used the fact that \( Y \times \{\infty\} \) is an effective Cartier divisor on \( Y \times \mathbb{P}^1 \). The vertical arrows are morphisms induced by \( t \). The arrow \( \xi_t \) (resp. \( \phi_t \)) is the pull-back of \( \xi \) (resp. \( \phi \)).

Note that we have

\[
M_t \cong \begin{cases} X & (t \neq \infty) \\
\mathbb{P}(N_{Y/X} \oplus 1) \cup \tilde{X} & (t = \infty),
\end{cases}
\]

where \( \tilde{X} \) denotes the blow-up of \( X \) along \( Y \). In particular, \( \xi_t \) (resp. \( \phi_t \)) is identical to \( f \) (resp. \( \text{id}_X \)) when \( t \neq \infty \). As for the case \( t = \infty \), it is well-known that \( \mathbb{P}(N_{Y/X} \oplus 1) \) meets \( \tilde{X} \) along the infinite hyperplane \( \mathbb{P}(N_{Y/X}) \) and that \( \xi_\infty \) factors through the zero section

\[
s : Y \to \mathbb{P}(N_{Y/X} \oplus 1) =: \mathbb{P}_\infty \quad (s(Y) \cap \mathbb{P}(N_{Y/X}) = \emptyset)
\]

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(see e.g. [FL] Chapter IV §5). Let \( g \) be the restriction of \( \varphi_\infty \) to \( \mathbb{P}_\infty \). We will prove the equality of morphisms

\[
C^Y_{i,X} \circ f_* = g_t \circ C^Y_{i,\mathbb{P}_\infty} \circ s_* : \Omega B_* \mathcal{D} \mathcal{P}_Y \to Rf^! \mathcal{K}(\Gamma(i)_X, 2i)
\]

(9.2.6) in \( \mathcal{M}_* \), where \( g \) denotes the restriction of \( \varphi_\infty \) to \( \mathbb{P}_\infty \) and the right hand side of this equality means the composite morphism

\[
\Omega B_* \mathcal{D} \mathcal{P}_Y \xrightarrow{s_*} R^1s_* \Omega B_* \mathcal{D} \mathcal{P}_{\mathbb{P}_\infty} \xrightarrow{c_n} R^1s_* \mathcal{K}(\Gamma(i)_{\mathbb{P}_\infty}, 2i) \xrightarrow{g} Rf^! \mathcal{K}(\Gamma(i)_X, 2i).
\]

We first check that (9.2.6) implies the theorem. Indeed, we have

\[
C^Y_{i,X} \circ f_* = g_t \circ C^Y_{i,\mathbb{P}_\infty} \circ s_* \quad \text{(by (9.2.6))}
\]

\[
= f_t \circ \pi_! \circ C^Y_{i,\mathbb{P}_\infty} \circ s_* \quad \text{(Theorem 7.2(3))}
\]

\[
= f_t \circ P_{1-r,Y/X} \quad \text{(by (9.2.5))}
\]

as claimed, where \( \pi \) denotes the projection \( \mathbb{P}_\infty \to Y \) and we have used the fact that \( N_{Y/X} \cong N_{Y/\mathbb{P}_\infty} \) in applying the result of Step 1.

We prove (9.2.6) in what follows. Noting that \( M \) belongs to \( \text{Ob}(\mathcal{C}) \) by (\#), consider composite morphisms

\[
\alpha : \Omega B_* \mathcal{D} \mathcal{P}_{Y \times \mathbb{P}_1} \xrightarrow{\xi_*} R^1\xi_* \Omega B_* \mathcal{D} \mathcal{P}_M \xrightarrow{C^M_{i,\mathbb{P}_1}} R^1\xi^! \mathcal{K}(\Gamma(i)_M, 2i) \xrightarrow{\phi_*} Rh^1 \mathcal{K}(\Gamma(i)_{Y \times \mathbb{P}_1}, 2i),
\]

\[
\beta_t : \Omega B_* \mathcal{D} \mathcal{P}_Y \xrightarrow{\nu_*} Rp_* \Omega B_* \mathcal{D} \mathcal{P}_{Y \times \mathbb{P}_1} \xrightarrow{\alpha} Rp_* Rh^1 \mathcal{K}(\Gamma(i)_{Y \times \mathbb{P}_1}, 2i) \xrightarrow{\phi_t^*} Rf^! \mathcal{K}(\Gamma(i)_X, 2i).
\]

One can easily check that

\[
\beta_0 = C^Y_{i,X} \circ f_* \quad \text{and} \quad \beta_\infty = g_t \circ C^Y_{i,\mathbb{P}_\infty} \circ s_*
\]

by the functoriality of Chern class maps (cf. Proposition 6.2(1)) and the base-change property in Theorem 7.2(4). Moreover, we have \( \phi_*^0 = \phi_*^\infty \), i.e., \( \beta_0 = \beta_\infty \). Indeed, we have

\[
Rp_* Rh^1 \mathcal{K}(\Gamma(i)_{Y \times \mathbb{P}_1}, 2i) \cong \bigoplus_{j=0,1} Rf^! \mathcal{K}(\Gamma(i - j)_X, 2(i - j))
\]

by the Dold-Thom isomorphism, cf. the axiom 2.5(2), and the pull-back of the tautological line bundle on \( X \times \mathbb{P}_1 \) onto \( X \times (\mathbb{P}_1 \setminus \{s\}) \) is trivial for any \( t \in \mathbb{P}_1(\mathbb{Z}) \), which imply that both \( \phi_*^0 \) and \( \phi_*^\infty \) agree with the natural projection. Thus we have

\[
C^Y_{i,X} \circ f_* = \beta_0 = \beta_\infty = g_t \circ C^Y_{i,\mathbb{P}_\infty} \circ s_*
\]

which completes the proof of the theorem. \( \square \)

As a direct consequence of Theorem 9.1 we obtain the following Riemann-Roch theorem without denominators:
Corollary 9.3 Under the setting of Theorem 9.1, the diagram

\[
\begin{array}{c}
K_j(Y) \xrightarrow{f_*} K_j^Y(X) \\
\downarrow \quad \downarrow \\
H^{2(i-r)-j}(Y, \Gamma(i-r)) \xrightarrow{f_*} H^{2i-2j}_Y(X, \Gamma(i))
\end{array}
\]

is commutative for any \(i, j \geq 0\), where \(P_{i-r,Y/X,j}\) denotes the composite map

\[
K_j(Y) \xrightarrow{P_{i-r,Y/X,j}} H^{2i-2j}_Y(X, \Gamma(i))
\]
and we have used Lemma 6.1 to obtain the last isomorphism.

10 Grothendieck-Riemann-Roch theorem

Let \(\mathcal{C}\) and \(\Gamma(*)\) be as in \[9\]. In this section, we prove Theorem 10.1 below. Let \(f : Y \to X\) be a projective morphism in \(\mathcal{C}\), and suppose that \(X\) and \(Y\) are regular. See Definition 6.4 for \(K_*(Y)\) and \(\hat{H}^*(X, \Gamma(*)) Q\). By taking a factorization

\[
f : Y \xrightarrow{g} \mathbb{P} := \mathbb{P}^m_X \to X
\]
with \(g\) a closed immersion, we define the virtual tangent bundle \(T_f\) of \(f\) as

\[
T_f := [g^*T_{\mathbb{P}/X}] - [N_{Y/\mathbb{P}}] \in K_0(Y),
\]
which is independent of the factorization of \(f\), cf. [FL] Chapter V Proposition 7.1. Note that the relative dimension defined in \[7\] is exactly the virtual rank of \(T_f\). We further define the Todd class \(\text{td}(T_f) \in \hat{H}^* (Y, \Gamma(*)) Q\) as

\[
\text{td}(T_f) := \text{td}(g^*T_{\mathbb{P}/X}) / \text{td}(N_{Y/\mathbb{P}})
\]

\[
= 1 + \frac{c_1(T_f)}{2} \quad + \frac{c_1(T_f)^2 + c_2(T_f)}{12} \quad + \frac{c_1(T_f)c_2(T_f)}{24}
\]

\[
+ \frac{-c_1(T_f)^4 + 4c_1(T_f)^2c_2(T_f) + c_1(T_f)c_3(T_f) - 3c_2(T_f)^2 - c_1(T_f)}{720} + \ldots,
\]

which is independent of the factorization of \(f\) as well.

Theorem 10.1 Let \(f : Y \to X\) be a projective morphism in \(\mathcal{C}\) with both \(X\) and \(Y\) regular. Assume the following condition:

\[\#\quad f : Y \to X\] is isomorphic to a projective space over \(X\), or there exists a decomposition \(Y \hookrightarrow \mathbb{P}^m_X \to X\) of \(f\) such that the blow-up of \(\mathbb{P}^m_X \times X\) along \(Y \times \{\infty\}\) belongs to \(\text{Mor}(\mathcal{C})\) and such that the zero section of the normal bundle \(N_{Y/\mathbb{P}^m_X}\) belongs to \(\text{Mor}(\mathcal{C})\).
Then the diagram

$$
\begin{array}{c}
K_*(Y) \xrightarrow{f_*} K^f_*(X) \\
\text{ch}_Y(-) \cup \text{td}(T_f) \downarrow \quad \downarrow \text{ch}^f_\ast
\end{array}
$$

is commutative, that is, for $\alpha \in K_*(Y)$ we have

$$
\text{ch}^f_\ast(f_*\alpha) = f_!(\text{ch}_Y(\alpha) \cup \text{td}(T_f)) \quad \text{in} \quad \hat{H}^\ast_\ast(Y, \Gamma(\bullet))_Q.
$$

Here $f_!$ denotes the push-forward morphism constructed in §7.

**Proof.** When $Y$ is a projective space over $X$, then the assertion follows from the Dold-Thom isomorphism (the axiom [2.5](2)) and the arguments in [FL] Chapter II Theorem 2.2 (see also loc. cit. Chapter V Theorem 7.3). Hence by loc. cit. Chapter II Theorem 1.1 and the same arguments as in Step 2 of the proof of Theorem 9.1 (we need the assumption ($\#'$) here), we have only to check the commutativity of the diagram

$$
\begin{array}{c}
K_*(Y) \xrightarrow{f_*} K_*(X) \\
\text{ch}_Y(-) \cup \text{td}(T_f) \downarrow \quad \downarrow \text{ch}_X
\end{array}
$$

assuming that $X \cong \mathbb{P}(E \oplus 1)$, the projective completion of a vector bundle $E \to Y$ of rank $r$ and that $f$ is isomorphic to the zero section of $\pi : \mathbb{P}(E \oplus 1) \to Y$. Let $Q$ be the universal quotient bundle on $X$. Then we have

$$
\text{ch}(f_*(1)) = \text{ch}(\lambda_{-1}(Q^-)) = c_*(Q) \cup \text{td}(Q)^{-1}
$$

(by [9.2.4])

$$
= f_!(1) \cup \text{td}(Q)^{-1} \quad (\text{[FL] Chapter I Proposition 5.3})
$$

$$
= f_!(1) \cup f_!\text{td}(Q)^{-1} \quad (\text{the axiom [2.5](3b)})
$$

$$
= f_!(\text{td}(T_f)) \quad (f_![Q] = [N_{Y/X}] = -T_f)
$$

in $\hat{H}^\ast_\ast(X, \Gamma(\bullet))_Q$. Therefore the diagram (10.1.1) commutes by the arguments in loc. cit. Chapter II Theorem 1.2 and the projection formula for $K$-theory and $\Gamma(\ast)$-cohomology. □

11 Computation via 1-extension

As an application of the Riemann-Roch theorem without denominators, we compute Chern classes using 1-extensions.

Let $\mathcal{C}$ be as in §3.3 Let $X \in \text{Ob}(\mathcal{C})$ be a proper smooth variety over $\mathbb{C}$ and let $Z \subset X$ be a reduced closed subscheme of pure codimension $r \geq 0$. Let $\Gamma_B(\ast)$ be the Betti complex...
with $A = \mathbb{Q}$ (cf. §3.3) and let $\Gamma_{\mathscr{O}}(*)$ be the Deligne-Beilinson complex with $A = \mathbb{Q}$ (cf. §3.4), which are both admissible cohomology theories on $\mathscr{O}$. We are concerned with the Chern class maps

\[ C^\text{sing,Y}_{i,j} : K^Y_{i,j}(X) \rightarrow H^{2i-j}_{Y}(X, \Gamma_B(i)), \]
\[ C^{\mathscr{O},Z}_{i,j} : K^Z_{i,j}(X) \rightarrow H^{2i-j}_{Z}(X, \Gamma_{\mathscr{O}}(i)), \]

where $Y$ is either $Z$ or $X$. We assume that $j \geq 1$ for simplicity (see Remark 11.3 below for the case $j = 0$), and put

\[ V := \text{Coker}(H^{2i-j-1}_{Z}(X, \Gamma_B(i)) \rightarrow H^{2i-j-1}_{Z}(X, \Gamma_B(i))). \]

We are going to compute the composite map

\[ C^{\mathscr{O},Z}_{i,j} : K^Z_{i,j}(X) \xrightarrow{\alpha_X} \text{Hom}_{D(MHS)}(\mathbb{Q}, R\Gamma_{X}(X, \Gamma_B(i))[j]) \xrightarrow{\tau} \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, H^{2i-j-1}(X, \Gamma_B(i))) \rightarrow \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, V) \]

(♠) in terms of the Chern class map of a regular dense open subset of $Z$, where MHS denotes the category of rational mixed Hodge structures and $\alpha_X$ denotes the canonical isomorphism given in the following lemma. The isomorphism $\tau$ is obtained from the fact that 2-extensions are trivial in MHS and the assumption that $j \geq 1$.

**Lemma 11.1** Let $X \rightarrow \text{Spec}(\mathbb{C})$ be as before, and let $T$ be a closed subscheme of $X$. Then for integers $i, j \geq 0$, there exists a canonical isomorphism

\[ \alpha_{X,T} : H^j_{T}(X, \Gamma_{\mathscr{O}}(i)) \xrightarrow{\sim} \text{Hom}_{D(MHS)}(\mathbb{Q}, R\Gamma_{T}(X, \Gamma_B(i))[j]) \]

fitting into a commutative diagram

\[
\begin{array}{ccc}
H^j_{T}(X, \Gamma_{\mathscr{O}}(i)) & \xrightarrow{\alpha_{X,T}} & H^j(X, \Gamma_{\mathscr{O}}(i)) \\
\downarrow & & \downarrow \alpha_{X} = \alpha_{X,X} \\
\text{Hom}_{D(MHS)}(\mathbb{Q}, R\Gamma_{T}(X, \Gamma_B(i))[j]) & \longrightarrow & \text{Hom}_{D(MHS)}(\mathbb{Q}, R\Gamma(X, \Gamma_B(i))[j]).
\end{array}
\]

**Proof.** For an open subset $U \subset X$, let $\mathbb{Q}(i)_{\mathscr{O},X,U}$ be the Deligne-Beilinson complex of $U$ on the analytic site $X_{an}$, cf. [EV] Definition 2.6. By the definition of $\Gamma_{\mathscr{O}}(i)$, there is a natural homomorphism of complexes

\[ \beta_{X,U} : R\Gamma(X_{an}, \mathbb{Q}(i)_{\mathscr{O},X,U}) \rightarrow R\Gamma(U_{zar}, \Gamma_{\mathscr{O}}(i)), \]

which is a quasi-isomorphism by loc. cit. Lemma 2.8. Let $\text{MHM}(X)$ be the category of mixed Hodge modules on $X$, cf. [Sa]. For $\mathcal{M} = (M, F^*, K_M) \in \text{Ob}(\text{MHM}(X))$, we define a complex $\mathcal{M}^!$ of abelian sheaves on $X_{an}$ as

\[ \mathcal{M}^! := \text{Cone}(K_M[-\dim X] \oplus F^0\text{DR}_X(M) \rightarrow \text{DR}_X(M))[-1], \]

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where $\text{DR}_X(M)$ and $F^0\text{DR}_X(M)$ denote the complexes (on $X_{an}$) with the most left term placed in degree 0

\[ \text{DR}_X(M) : M \rightarrow M \otimes \Omega^1_X \rightarrow \cdots \rightarrow M \otimes \Omega^n_X \rightarrow \cdots, \]

\[ F^0\text{DR}_X(M) : F^0M \rightarrow F^{-1}M \otimes \Omega^1_X \rightarrow \cdots \rightarrow F^{-q}M \otimes \Omega^n_X \rightarrow \cdots, \]

respectively. Because the assignment $H \mapsto H^\dagger$ is exact, this induces a functor

\[ (-)^\dagger : D^b(\text{MHM}(X)) \rightarrow D^b(X_{an}). \]

Note also that for an open immersion $\psi : U \hookrightarrow X$ there is a natural quasi-isomorphism of complexes (on $X_{an}$)

\[ \mathbb{Q}(i)_{\varphi, X, U} \xrightarrow{\text{qis}} (\mathbb{R}\psi_* \mathbb{Q}(i)_U)^\dagger, \]

where $\mathbb{Q}(i)_U$ denotes the Hodge module associated to the constant sheaf $(2\pi i)^i \mathbb{Q}$ on $U_{an}$. There is a diagram of quasi-isomorphisms of complexes

\[ R\Gamma(U_{Zar}, \Gamma_{\varphi}(i)) \xleftarrow{\text{qis}} R\Gamma(X_{an}, \mathbb{Q}(i)_{\varphi, X, U}) \xrightarrow{\text{qis}} R\Gamma(X_{an}, (\mathbb{R}\psi_* \mathbb{Q}(i)_U)^\dagger). \]

Considering this diagram for $U = X$ and $X \prec T$, we obtain an isomorphism

\[ \gamma_{X,T} : R\Gamma_T(X_{Zar}, \Gamma_{\varphi}(i)) \cong R\Gamma(X_{an}, (\mathbb{R}\phi^! \mathbb{Q}(i)_X)^\dagger) \quad \text{in} \quad D(\text{Ab}), \]

where $\phi$ denotes the closed immersion $T \hookrightarrow X$. Finally we define $\alpha_{X,T}$ as the composite

\[ H^j_T(X_{Zar}, \Gamma_{\varphi}(i)) \xrightarrow{\gamma_{X,T}} H^j(X_{an}, (\mathbb{R}\phi^! \mathbb{Q}(i)_X)^\dagger) \]

\[ \cong \text{Hom}_{D(\text{MHM}(X))}(\mathbb{Q}_X, \mathbb{R}\phi^! \mathbb{Q}(i)_X[q]) \]

\[ \cong \text{Hom}_{\text{MHS}}(\mathbb{Q}, R\Gamma_T(X_{an}, \mathbb{Q}(i)_X)[q]), \]

which obviously fits into the commutative diagram in the lemma.

We return to the setting of the beginning of this section. Let $\xi_Z$ be an element of $K'_j(Z)$, and put

\[ \chi_Z := C_{i,j}^{\text{sing}, Z}(f_*(\xi_Z)) \in H^{2i-j}_Z(X, \Gamma_B(i)), \]

where $f_*$ denotes the isomorphism $K'_j(Z) \cong K^j_{2}(X)$. There is a localization exact sequence of rational Betti cohomology

\[ 0 \rightarrow V \rightarrow H^{2i-j-1}(X \prec Z, \Gamma_B(i)) \xrightarrow{\delta} H^{2i-j}_Z(X, \Gamma_B(i)) \xrightarrow{\iota} H^{2i-j}(X, \Gamma_B(i)), \quad (11.1.1) \]

where $\delta$ (resp. $\iota$) denotes the connecting map (resp. the canonical map). By the assumption that $j \geq 1$, we have $\iota(\chi_Z) = 0$. Hence pulling-back this exact sequence by $\chi_Z$, that is, considering the fiber product in MHS

\[ E := \{(x, a) \in H^{2i-j-1}(X \prec Z, \Gamma_B(i)) \times \mathbb{Q} \mid \delta(x) = a \cdot \chi_Z \}, \]

we obtain a short exact sequence of rational mixed Hodge structures

\[ 0 \rightarrow V \rightarrow E \xrightarrow{\text{pr}_1} \mathbb{Q} \rightarrow 0, \]

which we denote by $\eta_Z$. The following results computing $C_{i,j}^{\varphi}(\xi_Z)$ have been used in a recent paper of the first author [A].

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\textbf{Theorem 11.2} Assume that $j \geq 1$ (see Remark 11.3 below for the case $j = 0$). Then

(1) The map $g_{i,j}^0$ in \textbullet sends $f_*(\xi_Z) \in K^0_j(X)$ to the class of $\eta_Z$, up to the sign $(-1)^j$.

(2) Let $Z_{\text{sing}}$ be the singular locus of $Z$, and put $Z^0 := Z \smallsetminus Z_{\text{sing}}$ and $X^0 := X \smallsetminus Z_{\text{sing}}$. Assume further that $2i - j < 2(r + 1)$. Then the sequence of Betti cohomology

\[ 0 \rightarrow V \rightarrow H^{2i-j-1}(X \smallsetminus Z, \Gamma_B(i)) \xrightarrow{\delta'} H^{2i-j}_{Z^0}(X^0, \Gamma_B(i)) \]  

is exact, where $\delta'$ is the composite of $\delta$ in (11.1.1) and a natural restriction map. Moreover, $\eta_Z$ is isomorphic to the pull-back of this exact sequence by the element

\[ g(P_{i-r, Z^0/X^0}(\xi_{Z^0})) \in H^{2i-j}_{Z^0}(X^0, \Gamma_B(i)), \]

where $g$ denotes the regular closed immersion $Z^0 \hookrightarrow X^0$ and $\xi_{Z^0}$ denotes the restriction of $\xi_Z$ to $K^0_j(Z^0) = K_j(Z^0)$. See \textbullet\ for the map $P_{i-r, Z^0/X^0}$.

\textbf{Proof.} (1) Put $m := 2i - j$ for simplicity, and consider the following big diagram:

\[ \begin{array}{ccc}
K^0_j(X) & \xrightarrow{a_1} & K_j(X) \\
C^0_{i,j} & \xrightarrow{(i)} & C^0_j \\
H^m_2(X, \Gamma_Z(i)) & \xrightarrow{a_2} & H^m(X, \Gamma_Z(i)) \\
\alpha_{X,Z} & \xrightarrow{(ii)} & \alpha_X \\
\Hom_{D(MHS)}(Q, R\Gamma_Z(X, \Gamma_B(i))[m]) & \xrightarrow{a_3} & \Hom_{D(MHS)}(Q, R\Gamma(X, \Gamma_B(i))[m]) \\
\Hom_{D(MHS)}(Q, (\tau_{\leq m} R\Gamma_Z(X, \Gamma_B(i))')[m]) & \xrightarrow{a_4} & \Hom_{D(MHS)}(Q, (\tau_{\leq m-1} R\Gamma(X, \Gamma_B(i)))[m]) \\
\Hom_{MHS}(Q, \text{Im}(\delta)) & \xrightarrow{a_5} & \text{Ext}^1_{MHS}(Q, V).
\end{array} \]

Here $\tau_{\leq m} R\Gamma_Z(X, \Gamma_B(i))'$ denotes the complex

\[ \text{Cone}(\tau_{\leq m} R\Gamma_Z(X, \Gamma_B(i)) \rightarrow H^m_2(X, \Gamma_B(i))/\text{Im}(\delta)[-m])[-1], \]

The arrows $a_1$, $a_2$ are canonical maps, and $a_3$ is induced by the canonical morphism

\[ \tilde{\tau} : R\Gamma_Z(X, \Gamma_B(i)) \rightarrow R\Gamma(X, \Gamma_B(i)). \]

The arrow $a_4$ denotes the morphism induced by $\tilde{\tau}$, and $a_5$ denotes the connecting map associated with the short exact sequence in MHS

\[ 0 \rightarrow V \rightarrow H^{m-1}(X \smallsetminus Z, \Gamma_B(i)) \xrightarrow{\delta} \text{Im}(\delta) \rightarrow 0. \]

The squares (i) and (iii) commute obviously, and the square (ii) commutes by the construction of $\alpha_{X,Z}$ and $\alpha_X = \alpha_{X,X}$ in Lemma 11.1. The square (iv) commutes up to the sign $(-1)^m$, cf. 38.
Lemma 9.5. The isomorphism $(\ast)$ (resp. $(\ast')$) in the left (resp. right) column follows from the fact that the Hodge $(0, 0)$-part of $H^n_Z(X, \Gamma_B(i))$ lies in $\text{Im}(\delta)$ (resp. the Hodge $(0, 0)$-part of $H^m(X, \Gamma_B(i))$ is zero). Finally, the composite of the left vertical columns sends $f_*(\xi_Z)$ to $\chi_Z$, and the composite of the right vertical columns agrees with the map $\varphi_d'$ in question. The assertion follows from these facts.

(2) Since we have $H^{2i-j}_Z(X, \Gamma_B(i)) = 0$ by the assumption on $i$ and $j$, the restriction map

$$H^{2i-j}_Z(X, \Gamma_B(i)) \longrightarrow H^{2i-j}_Z(X^\circ, \Gamma_B(i))$$

is injective, which implies the first assertion. The second assertion follows from Corollary 9.3 for Betti cohomology.

Remark 11.3 As for the case $j = 0$, one can easily modify the above arguments to obtain similar results for $\xi_Z \in K'_0(Z)$ satisfying $C_{i,0}^\text{sing}(f_*(\xi_Z)) = 0$ in $H^{2i}(X, \Gamma_B(i))$.

A Cohomology of simplicial schemes

In this appendix, we include the definitions of Zariski sheaves and cohomology on a simplicial scheme, for the convenience of the reader. Propositions A.2-A.4 stated below have been used in the main body of the paper.

A.1 Sheaves on simplicial schemes

We review the definition of Zariski site over a simplicial scheme (cf. [Fr] Definition 1.4).

Definition A.1 Let $X_\ast$ be a simplicial scheme.

(1) Let $\text{Ouv}/X_\ast$ be the category whose object is an open subset $U \subset X_p$ for some $p \geq 0$, and whose morphism is a commutative square

$$\begin{array}{ccc}
V & \longrightarrow & U \\
\downarrow & & \downarrow \\
X_q & \underset{\alpha_X}{\longrightarrow} & X_p
\end{array}$$

for some morphism $\alpha : [p] \rightarrow [q]$ in $\Delta$.

(2) We endow the category $\text{Ouv}/X_\ast$ with Zariski topology in such a way that a covering of a given object $U \rightarrow X$ is an open covering $\{U_i\}_{i \in I}$ of $U$. The resulting site is called the Zariski site over $X_\ast$ and denoted by $(X_\ast)_{\text{Zar}}$.

(3) An abelian presheaf on $(X_\ast)_{\text{Zar}}$ is a functor

$$\mathcal{F} : (\text{Ouv}/X_\ast)^{\text{op}} \rightarrow \text{Ab}.$$
(4) An abelian sheaf on \((X_*)_{\text{Zar}}\) is a presheaf \(\mathcal{F}\) which satisfies the gluing condition: for any object \(U \to X_p\) of \(\text{Ouv}/X_*\) and any open covering \(\{U_i\}_{i \in I}\) of \(U\), we have

\[
\mathcal{F}(U) \cong \ker \left( \mathcal{F}(pr_1) - \mathcal{F}(pr_2) : \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{(i,j) \in I \times I} \mathcal{F}(U_i \cap U_j) \right).
\]

We denote the category of abelian sheaves on \((X_*)_{\text{Zar}}\) by \(\text{Shv}^{\text{ab}}((X_*)_{\text{Zar}})\).

**Proposition A.2** Let \(X_*\) be a simplicial scheme.

1. To give a sheaf \(\mathcal{F}\) on \((X_*)_{\text{Zar}}\) is equivalent to giving a datum \(((\mathcal{F}_p)_{p \geq 0}, (\alpha^\sharp)_{\alpha \in \text{Mor}(\Delta)})\) consisting of sheaves \(\mathcal{F}_p\) on \((X_p)_{\text{Zar}}\) for \(p \geq 0\) and morphisms of sheaves on \((X_q)_{\text{Zar}}\)

\[
\alpha^\sharp : (\alpha^X)^* \mathcal{F}_p \to \mathcal{F}_q \quad \text{for} \quad \alpha \in \text{Mor}(\Delta([p], [q]))
\]

which satisfy \(\beta^\sharp \circ (\beta^X)^*(\alpha^\sharp) = (\beta \circ \alpha)^\sharp\) for \(\alpha : [p] \to [q]\) and \(\beta : [q] \to [r]\).

2. The category \(\text{Shv}^{\text{ab}}((X_*)_{\text{Zar}})\) is abelian with enough injective objects.

3. For each \(p \geq 0\), the functor

\[
(\cdot)_p : \text{Shv}^{\text{ab}}((X_*)_{\text{Zar}}) \to \text{Shv}^{\text{ab}}((X_p)_{\text{Zar}}), \quad \mathcal{F} \mapsto \mathcal{F}_p
\]

is exact and preserves injective objects, where \(\mathcal{F}_p\) denotes the restriction of \(\mathcal{F}\) to \((X_p)_{\text{Zar}}\).

**Proof.** See the remark after [Fr] Definition 2.1 for (1). See loc. cit. Proposition 2.2 for (2). See loc. cit. proof of Proposition 2.4 for (3). Although those arguments are written for the étale topology, the same arguments work for the Zariski topology as well. □

### A.2 Cohomological functors on simplicial schemes

Let \(X_*\) be a simplicial scheme and let \(Z_* \subset X_*\) be a simplicial closed subset. For an abelian sheaf \(\mathcal{F}\) on \((X_*)_{\text{Zar}}\), we define \(\Gamma_{Z_*}(X_*, \mathcal{F})\), the **global section of \(\mathcal{F}\) over \(X_*\) with support in \(Z_*\)**, as

\[
\Gamma_{Z_*}(X_*, \mathcal{F}) := \ker(d_0^\sharp - d_1^\sharp : \Gamma_{Z_0}(X_0, \mathcal{F}) \to \Gamma_{Z_1}(X_1, \mathcal{F}))
\]

We define the **\(i\)-th cohomology functor with support in \(Z_*\)**

\[
H^i_{Z_*}(X_*, -) : \text{Shv}^{\text{ab}}((X_*)_{\text{Zar}}) \to \text{Ab}
\]

as the \(i\)-th right derived functor of \(\Gamma_{Z_*}(X_*, -)\). Similarly for an object \((U \subset X_p)\) of \(\text{Ouv}/X_*\), we define

\[
H^i(U, -) : \text{Shv}^{\text{ab}}((X_*)_{\text{Zar}}) \to \text{Ab}
\]

as the \(i\)-th right derived functor of \(\Gamma(U, -)\).
For a morphism $f : X \rightarrow Y$ of simplicial schemes and an abelian sheaf $\mathcal{F}$ on $(X)_{\text{Zar}}$, we define the direct image $f_* \mathcal{F}$ on $(Y)_{\text{Zar}}$ by the assignment

$$f_* \mathcal{F} : (U \subset X_p) \mapsto \mathcal{F}(f_p^{-1}(U))$$

where $f_p : X_p \rightarrow Y_p$ denotes the $p$-th factor of $f$. We define the $i$-th higher direct image functor

$$R^i f_* : \text{Shv}_{ab}((X)_{\text{Zar}}) \rightarrow \text{Shv}_{ab}((Y)_{\text{Zar}})$$

as the $i$-th right derived functor of $f_*$. We define

$$g^! : \text{Shv}_{ab}((X)_{\text{Zar}}) \rightarrow \text{Shv}_{ab}((Z)_{\text{Zar}})$$

as the inverse image by $g$ of the sheaf $g_* \mathcal{F}$ on $(Z)_{\text{Zar}}$ as the inverse image by $g$ of the sheaf

$$\mathcal{F}_Z : (U \subset X_p) \mapsto \Gamma_{U \cap Z_p}(U, \mathcal{F})$$

on $(X)_{\text{Zar}}$. We define

$$R^i g^! : \text{Shv}_{ab}((X)_{\text{Zar}}) \rightarrow \text{Shv}_{ab}((Z)_{\text{Zar}})$$

as the $i$-th right derived functor of $g^!$. The following proposition shows us how to compute these derived functors.

**Proposition A.3** Let $f : X \rightarrow Y$ and $g : Z \hookrightarrow X$ be as before. Let $\mathcal{F}$ be an abelian sheaf on $(X)_{\text{Zar}}$.

1. Let $U \subset X_p$ be an object of $\text{Ouv}/X$. Then there is a natural isomorphism

$$H^i(U, \mathcal{F}) \cong H^i(U, \mathcal{F}_p)$$

which is functorial in $U$ and $\mathcal{F}$. Here $\mathcal{F}_p$ denotes $(\mathcal{F})_p$, cf. Proposition A.2(3), and the right hand side means the cohomology of $U$ in the usual sense.

2. Let $p \geq 0$ be a non-negative integer. Then both $R^i f_*$ and $R^i g^!$ commute with $(-)_p$, i.e., there are natural isomorphisms

$$(R^i f_* \mathcal{F})_p \cong R^i f_{p*}(\mathcal{F}_p) \quad \text{and} \quad (R^i g^! \mathcal{F})_p \cong R^i g_{p!}(\mathcal{F}_p),$$

which are functorial in $\mathcal{F}$. Here $f_p : X_p \rightarrow Y_p$ (resp. $g_p : Z_p \hookrightarrow X_p$) denotes the $p$-th factor of $f$ (resp. of $g$).

3. There is a functorial spectral sequence

$$E_1^{p,q} = H^q_{Z_p}(X_p, \mathcal{F}_p) \Rightarrow H^i_{Z}(X_*, \mathcal{F}).$$

**Proof.** The assertions (1) and (2) follow immediately from Proposition A.2(3) and the exactness of $(-)_p$, see [F] Proposition 2.4 for (3). □
By the existence of $K$-injective resolutions (cf. [Sp], [Ho]), the functors $H^i(X_*, -)$, $R^f_*$ and $R^g_!$ are extended naturally to the derived category of unbounded cochain complexes of abelian sheaves on $(X_*)_{Zar}$:

$$R\Gamma_{Zar}(X_*, -) : D((X_*)_{Zar}) \longrightarrow D(Ab),$$

$$Rf_* : D((X_*)_{Zar}) \longrightarrow D((Y_*)_{Zar}),$$

$$Rg_! : D((X_*)_{Zar}) \longrightarrow D((Z_*)_{Zar}).$$

**Proposition A.4 ([Gi1] Lemma 2.4)** Let $\mathscr{C}$ be as in [L] and let $\Gamma(*)$ be an admissible cohomology theory on $\mathscr{C}$. Let $E_\star \rightarrow X_\star$ be a vector bundle of rank $r + 1$, and let $p$ be the natural projection $\mathbb{P}(E_\star) \rightarrow X_\star$. Let $L^\text{aut}_\star$ be the tautological line bundle over $\mathbb{P}(E_\star)$ and put $\xi := c_1(L^\text{aut}_\star) \in H^2(\mathbb{P}(E_\star), \Gamma(1))$. Then there is a Dold-Thom isomorphism

$$\bigoplus_{j=0}^r H^{i-2j}(X_\star, \Gamma(n-j)) \cong H^i(\mathbb{P}(E_\star), \Gamma(n)), \quad (b_j)_{j=0}^r \mapsto \sum_{j=0}^r \xi^j \cup p^*(b_j).$$

Here the cup products are defined by the product structure on $\Gamma(*)$ and a $K$-injective resolution of $\Gamma(n)$.

**Proof of Proposition A.4** Since we have $H^i(\mathbb{P}(E_\star), \Gamma(n)) \cong H^i(X_\star, Rp_* \Gamma(n)_{\mathbb{P}(E_\star)})$, it is enough to check that the morphism

$$\bigoplus_{j=0}^r \Gamma(n-j)_{X_\star}[-2j] \longrightarrow Rp_* \Gamma(n)_{\mathbb{P}(E_\star)}, \quad (b_j)_{j=0}^r \mapsto \sum_{j=0}^r \xi^j \cup p^*(b_j)$$

is an isomorphism in $D((X_*)_{Zar})$, which follows from the axiom 2.5(2) and Proposition A.3(2). \qed

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