Demystifying the Holographic Mystique

D. V. Khveshchenko

Department of Physics and Astronomy, University of North Carolina, Chapel Hill, NC 27599

Thus far, in spite of many interesting developments, the overall progress towards a systematic study and classification of various ‘strange’ metallic states of matter has been rather limited. To that end, it was argued that a recent proliferation of the ideas of holographic correspondence originating from string theory might offer a possible way out of the stalemate. However, after almost a decade of intensive studies into the proposed extensions of the holographic conjecture to a variety of condensed matter problems, the validity of this intriguing approach remains largely unknown. This discussion aims at ascertaining its true status and elucidating the conditions under which some of its predictions may indeed be right (albeit, possibly, for a wrong reason).

Condensed Matter Holography: The Promise

Among the outstanding grand problems in condensed matter physics is that of a deeper understanding and classification of the so-called ‘strange metals’ or compressible non-Fermi liquid (NFL) states of the strongly interacting systems. However, despite all the effort and a plethora of the important and non-trivial results obtained with the use of the traditional techniques, this program still remains far from completion.

As an alternate approach, over the past decade there have been numerous attempts inspired by the hypothetical idea of holographic correspondence which originated from string/gravity/high energy theory (where it is known under the acronym AdS/CFT) to adapt its main concepts to various condensed matter (or, even more generally, quantum many-body) systems at finite densities and temperatures. Among the outstanding grand problems in condensed matter physics is that of a deeper understanding and classification of the so-called ‘strange metals’ or compressible non-Fermi liquid (NFL) states of the strongly interacting systems. However, despite all the effort and a plethora of the important and non-trivial results obtained with the use of the traditional techniques, this program still remains far from completion.

As an alternate approach, over the past decade there have been numerous attempts inspired by the hypothetical idea of holographic correspondence which originated from string/gravity/high energy theory (where it is known under the acronym AdS/CFT) to adapt its main concepts to various condensed matter (or, even more generally, quantum many-body) systems at finite densities and temperatures.1,2

In its original context, the ‘bona fide’ holographic principle postulates that certain $d + 1$-dimensional (‘boundary’) quantum field theories (e.g., the maximally supersymmetric $SU(N)$ gauge theory) may allow for a dual description in terms of a string theory which, upon a proper compactification, amounts to a certain $d + 2$-dimensional (‘bulk’) supergravity. Moreover, in the strong coupling limit (characterized in terms of the t’Hooft coupling constant $\lambda = g^2 N \gg 1$) and for a large rank $N \gg 1$ of the gauge symmetry group, the bulk description can be further reduced down to a weakly fluctuating gravity model which can even be treated semiclassically at the lowest (0th) order of the underlying $1/N$-expansion.

In the practical applications of the holographic conjecture, the partition function of a strongly interacting boundary theory with the Lagrangian $\mathcal{L}(\phi_a)$ would then be approached by a saddle-point (classical) value of the bulk action described by the Lagrangian $\mathcal{L}(g_{\mu\nu}, \ldots)$ which includes gravity and other fields dual to their boundary counterparts.1,2

$$Z[J] = \int \prod_{a=1}^{N} D\phi_a \exp(- \int dt d^d \bar{\phi} \mathcal{L}(\phi_a)) \approx \exp(- \int dr dt d^d \bar{\phi} \mathcal{L}(g_{\mu\nu}, \ldots))$$

(1)

evaluated with the use of a fixed background metric

$$ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + \sum_{ij} g_{ij} dx^i dx^j$$

(2)

while any quantum corrections would usually be neglected by invoking the small parameter $1/N$. Thus, considering that the task of solving a system of coupled Einstein-type differential equations can be fairly straightforward conceptually (albeit not necessarily technically), the holographic approach could indeed become a novel powerful tool for studying the strongly correlated systems and a viable alternative to the practically impossible problem of summing the entire perturbation series. Specifically, if proved valid, some of the broad ‘bottom-up’ generalizations of the original holographic conjecture known as ‘AdS/CMT’ (which, in many instances, should have been more appropriately called ‘non-AdS/non-CFT’) could indeed provide an advanced phenomenological framework for discovering new and classifying the already known types of the NFL behavior.

Thus far, however, a flurry of the traditionally detailed (hence, rarely concise) publications on the topic have generated not only a good deal of enthusiasm but some reservations as well. Indeed, the proposed ‘ad hoc’ generalizations of the original string-theoretical construction involve some of its most radical alterations, whereby most of its stringent constraints would have been abandoned in the hope of still capturing some key aspects of the underlying correspondence. This is because the target (condensed matter) systems generically tend to be neither conformally, nor Lorentz (or even translationally and/or rotationally) invariant and lack any supersymmetric (or even an ordinary) gauge symmetry with some (let alone, large) rank-$N$ non-abelian group.

Moreover, while sporting a truly impressive level of technical profress, the exploratory ‘bottom-up’ holographic studies have not yet helped to resolve such crucially important issues as:

- Are the conditions of a large $N$, (super)gauge symmetry, Lorentz/translation/rotational invariance of the boundary (quantum) theory indeed necessary for establishing a holographic correspondence with some weakly coupled (classical) gravity in the bulk?
- Are all the strongly correlated systems (or only a pre-
cious few) supposed to have gravity duals?
- What are the gravity duals of the already documented NFLs?
- Given all the differences between the typical condensed matter and string theory problems, what (other than the lack of a better alternative) justifies the adaptation ‘ad verbatim’ of the original (string-theoretical) holographic ‘dictionary’? and, most importantly:
- If the broadly defined holographic conjecture is indeed valid, then why is it so?

Considering that by now the field of CMT holography has grown almost a decade old, it would seem that answering such outstanding questions should have been considered more important than continuing to apply the formal holographic recipes to an ever increasing number of model geometries and then seeking some resemblance to the real world systems without a good understanding as to why it would have to be there in the first place. In contrast, the overly pragmatic ‘shut up and calculate’ approach prioritizes computational tractability over physical relevance, thus making it more about the method (which readily provides a plethora of answers but may struggle to specify the pertinent questions) itself, rather than the underlying physics.

On the other hand, there exist of course important ongoing efforts towards, both, constructing various ‘top-down’ holographic models as well as trying to derive holography from the already known concepts such as a renormalization procedure on the information-related tensor networks. However, the former approach formulated in terms of such objects as D-branes remains to be rather exotic and somewhat hard to connect to from the CMT perspective, whereas the latter one (which, in practice, amounts to a massive use of the Stratonovich and Trotter transformations combined with numerical solutions of the resulting flow equations of the functional RG-type) has yet to deliver a well-defined bulk geometry, other than the basic AdS with the dynamical $z = 1$ (or its Lifshitz modification with $z = 2$), that would be reminiscent of those metrics which are extensively utilized in the ‘bottom-up’ studies (see below).

Condensed Matter Holography: The Evidence

The circumstantial evidence that would be typically invoked in support of the general idea of holography includes such diverse topics as thermodynamics of black holes, hydrodynamics of quark-gluon plasma and unitary ultracold Fermi-gases, geometrization of quantum entanglement entropy, etc. However, while possibly attesting to the validity of some aspects of the holographic concept in general, those arguments may not be immediately pertinent to the specific condensed matter systems.

Therefore, for the broadly generalized holographic conjecture to prove relevant and gain a predictive power in the latter context its predictions would have to be systematically contrasted against experimental data as well as the results of other, more conventional, techniques providing a preliminary insight. Also, the holographic calculations would have to be carried out for (and allow for a cross-check between) a whole range of the thermodynamic and transport quantities, including specific heat, DOS, charge and spin susceptibilities, electrical, thermal, and spin conductivities, electron spectral function, etc.

By some (admittedly, risky) analogy with any evidence claimed to support, e.g., the ‘science’ of UFO and other paranormal phenomena - most (but not all) of which can be readily dismissed - in order to ascertain the true status of the CMT holography one would need to focus on (and identify the physical origin(s) of) those cases of factual agreement that can be deemed reliable and reproducible. It especially concerns those instances where the holographic predictions were reported to agree quantitatively with the results of some exact analytical or almost exact (e.g., Monte Carlo) calculations (see below).

At the ‘ad hoc’ level the holographic approach has already been opportunistically applied to a great variety of condensed matter systems which includes the ‘strange’ Fermi and Bose metals describing quantum-critical spin liquids, supersolids, quantum smectics and nematics, Mott insulators, (in)coherent conductors, itinerant (anti)ferromagnets, Quantum Hall effect, graphene and other Dirac/Weyl metals, multi-channel Kondo and other quantum impurity models, etc.

Routinely, the holographic calculations would be performed in the ‘mean-field’ approximation (i.e., at the 0th order of the would-be 1/N-expansion) and then compared to some selected sets of experimental data on the systems that tend to lack any supersymmetry, are characterized by the number of species $N \sim 1$ (such as spin, orbital, and/or valley components) and have only moderate (as opposed to very strong) interactions. The above caveats notwithstanding, however, such studies would often seek nothing short of (and occasionally claim to have found) a quantitative agreement with the data.

Also, many of the early CMT holographic calculations were carried out by ‘seeking where the lights are’ and utilizing just a handful of the historic black-hole solutions, the central among which is the Reissner-Nordstrom (RN) one that asymptotically approaches the $AdS_{d+2}$ (anti-de-Sitter) and $AdS_2 \times R^d$ geometries in the UV and IR limits, respectively, and is accompanied by the scalar potential $A_0 = \mu (1 - r/r_h)$. The emblackening factor
\begin{equation}
  f(r) = 1 - (1 + \mu^2)((r/r_h)^{d+1} + \mu^2(r/r_h)^{2d})
\end{equation}
incorporates the chemical potential $\mu$ and vanishes at the horizon of radius $1/r_h$ proportional to the Hawking temperature $T$ which is shared by the bulk and boundary degrees of freedom. Notably, this radius remains finite even when the temperature $T = (d + 1 - (d - 1)\mu^2)/4\pi r_h$ vanishes, thereby giving rise to the non-vanishing entropy $S (T \to 0) \neq 0$ and suggesting that the corresponding boundary theory could provide a description of some isolated ‘quantum impurity’, rather than a correlated many-body state with a
non-trivial spatial dispersion. Accordingly, the boundary fermion propagator \( G(\omega, k) \) demonstrates the behavior dubbed as 'semi-locally critical'\(^{12}\), which is characterized by a non-trivial frequency, yet a rather mundane momentum, dependence

\[
G(\omega, k) = 1/(a_k + b_k \omega^{2\eta_k})
\]

where \( a_k \), \( b_k \), and \( \eta_k \) are smooth functions of the momentum \( k \). In the space-time domain, the corresponding behavior

\[
G(\tau, x) \sim \exp(-S(\tau, x))
\]

is governed by the (semi)classical action

\[
S(\tau, x) = \sqrt{x^2 + (1 - 2\eta_k)^2 \ln^2 \tau}
\]

and is consistent with that of the spatially (almost) uncorrelated 'impurities', each of which exhibits a characteristic \( d = 0 \) quantum-critical scaling. Such NFL behavior indeed bears some superficial resemblance to that found in a certain class of the heavy-fermion compounds\(^8\) and DMFT calculations\(^9\). However, it is also plagued with such spurious features as potentially multiple Fermi surfaces, dispersionless peaks, and log-oscillating \( \omega \)-dependence\(^2\).

Given the multitude of the experimentally discovered NFLs, the 'locally-critical' scenario would seem to be much too limited to encompass more general types of the real-life NFL scenarios where, both, the \( x \)- and \( \tau \)- (or, correspondingly, \( \omega \)- and \( k \)-) dependencies of the propagator would be distinctly non-trivial. In light of that realization, the focus of the early holographic studies has gradually shifted towards a broader class of geometries, including such intrinsically non-Lorentz-invariant metrics as the Lifshitz, Shrodinger, helical Bianchi, etc. A particular attention has been paid to the static, diagonal, and isotropic 'hyperscaling-violating' (HV) metrics with the radial dependence of the form (up to a conformal equivalence),

\[
g_{tt} \sim -r^{2\theta/d-2}, \quad g_{rr} = g_{\theta\theta} \sim r^{2\theta/d-2}
\]

its finite-\( T \) version featuring the additional factor

\[
f = 1 - (r/r_h)^{d+z-\theta}.
\]

The dynamical exponent \( z \) controls the boundary excitation spectrum \( \omega \propto q^z \), while \( \theta \) quantifies a non-trivial scaling of the interval \( ds \rightarrow \lambda^{\theta/d} ds \), the scaling (albeit not Lorentz-) invariant case of the Lifshitz metric corresponding to \( \theta = 0 \). The physically sensible values of \( z \) and \( \theta \) must satisfy the 'null energy conditions'

\[
(d-\theta)(d(z-1)-\theta) \geq 0, \quad (z-1)(d+z-\theta) \geq 0
\]

signifying a thermodynamic stability of the corresponding geometry.

A proper choice of \( \theta \) determining the 'effective dimension' \( d_{\text{eff}} = d - \theta \) was discussed in the context of fermionic entanglement entropy which singles out the value \( \theta = d - 1 \), consistent with the notion of the Fermi surface as a \( d-1 \)-dimensional membrane in the reciprocal (momentum) space\(^{12}\).

The HV metrics arise among the solutions of various generalized gravity theories, including those with massive vector (Proca) fields, Horava gravity, as well as the Einstein-Maxwell-dilaton (EMD) theory which includes an additional neutral scalar ('dilaton') field\(^{11}\),

\[
L_{EMD} = \frac{1}{2} (R + \frac{d(d+1)}{L^2}) - \frac{1}{2} (\partial \phi)^2 - U(\phi) - \frac{V(\phi)}{2} F_{\mu\nu}^2
\]

where the dilaton potential \( U(\phi) \) and the effective gauge coupling \( V(\phi) \) are given by some (typically, exponential) functions of \( \phi \). This model is believed to be dual to a strongly interacting boundary theory at finite density and temperature which is deformed away from the hyperscaling limit by a relevant neutral scalar operator dual to the dilaton. The HV solutions of the coupled Einstein-Maxwell equations have also been obtained by taking into account a back-reaction of the fermionic matter fields on the background geometry\(^{12}\). It should be noted, though, that apart from a few exceptions\(^{15}\) such analyses were limited to the hydrodynamic (Thomas-Fermi) description of the fermions, while leaving out their more subtle (exchange and correlation) effects.

One purported success of the theory \( (8) \) (with \( U(\phi) \sim \phi^2 \) and \( V(\phi) = \text{const} \)) was a numerical fit to the experimentally observed power-law behavior of the mid-infrared optical conductivity\(^{14}\),

\[
\sigma(\omega) \sim \omega^{-2/3}
\]

in the normal state of the superconducting cuprates such as \( \text{BSCCO}^{16} \). Notably, though, such an agreement was found over less than half of a decade (\( 2 < \omega \tau < 8 \)) while the later studies did not confirm it\(^{17}\). Besides, it was also argued to be intermittent with the alternate 'universal', \( \sim \omega^{-1} \), behavior.

Another notable example is the holographic calculation\(^{17}\) of the numerical prefactor \( C \) in the empirical Homes relation between the superfluid density, critical temperature, and normal state conductivity\(^{14}\).

\[
\rho_s = CT_c \sigma(T_c^{0+})
\]

This relation is obeyed by a large variety of superconductors, including the weakly coupled (yet, sufficiently disordered) conventional ones where a justification for the use of holography would be rather hard to come by and where the Homes’ relation can be shown to hold within the scope of the traditional theory\(^{14}\). The holographic theory of Ref.\(^{17}\) claimed its success with finding the prefactor \( C \approx 6.2 \) to be close to the experimental values in the cuprates which happen to be \( C \approx 8.1 \) and 4.4 for the \( ab \)- and \( c \)-axes transport, respectively. Somewhat ironically, though, this would seem like a case where, in the absence of a compelling underlying reason, a perfect quantitative agreement could do more harm than good to the cause.

Arguably, a stronger case could have been made if the range of the possible values of \( C \) were unusually narrow, thus potentially hinting at some universality of the
result. This, however, does not seem to be the case. Moreover, the holographic calculation of Ref.\textsuperscript{27} does not readily reproduce any other empirical relations such as the Schneider's one\textsuperscript{20}

\[ \frac{T_c}{T_c} = 2 \sqrt{\frac{\rho_s}{\rho_s}} (1 - \frac{1}{2} \frac{\rho_s}{\rho_s}), \]  

(11)

thereby suggesting that the reported agreement with experiment may have been largely fortuitous.

Also, to further strengthen the case for holography, some works tend to use increasingly more and more complicated models with a larger number of fields while operating under the assumption that the IR physics should be fairly universal and insensitive to such details. The results, on the contrary, indicate that there is little universality, as by varying the contents of the bulk theory one can dramatically alter the boundary behavior - e.g., reproducing on demand the entire phase diagram of the cuprates with all the four main phases as well as the domains of their coexistence in the extended EM model with the additional vector and scalar fields which represent the competing orders\textsuperscript{21}

There has also been some effort on the experimental side, such as the report of measuring the shear viscosity-to-entropy density ratio in the cuprates\textsuperscript{22} which was found to be close to its celebrated holographic lower bound value\textsuperscript{23}

\[ \eta/S = 1/4\pi \]  

(12)

It should be noted, though, that instead of such genuine two-particle characteristic as the correlation function of the stress tensor’s component $T_{xy}$ in Ref.\textsuperscript{22} the viscosity $\eta$ was deduced from the single-particle electron spectral function. Furthermore, even the bound (12) itself is known not to be totally universal, its value becoming lower, e.g., in the absence of the spatial rotational invariance\textsuperscript{24}

\[ \eta_{zzzz}/S = (1/4\pi)(g_{zz}/g_{zz}) \]  

(13)

Yet another example of a successful application of the holographic approach was presented in Ref.\textsuperscript{25} where an impressive agreement was found between the holographically computed frequency-dependent (finite-temperature) conductivity $\sigma(\omega)$ and the Monte Carlo results for the Bose-Hubbard model whose critical behavior belongs to the universality class of the O(2)-symmetrical Wilson-Fisher critical point. Upon analytically continuing to the real frequencies this agreement gets progressively worse for $\omega < 2\pi T$, though\textsuperscript{25}. Most intriguingly, the original work of Ref.\textsuperscript{25} utilized the EM holographic model with a rather special higher-derivative term

\[ \Delta L = C_{abcd} F^{ab} F^{cd} \]  

(14)

where $C_{abcd}$ is the Weyl tensor, thus making one wonder as to the reasons behind a seemingly unique role of this model as the potential Bose-Hubbard’s dual. However, in the later Ref.\textsuperscript{26} nearly identical results were obtained with the use of a much less exotic EMD model with $U(\phi) = \phi^2$ and $V(\phi) = 1 + a\phi$. Thus, the previously reported agreement with the MC results (limited to $\omega > 2\pi T$) appears to be rather common, which observation takes away much of the intrigue surrounding the holographic model equipped with the term (14) and makes less pressing the need for understanding its otherwise inexplicably serendipitous success.

In light of the above, one would be led to the conclusion that at its current stage the CMT holography still lacks a true ‘smoking gun’ that could lend a firm support for this approach (albeit, possibly, in some reduced, rather than its most broad, overarching, form). Therefore, it is perhaps not too surprising that there is still no consensus, neither on the exact implications of the reports of some apparent agreement between the holographic predictions and the results of other approaches and/or experimental data, nor the general applicability and the principal limitations of the holographic approach itself.

**Emergent Geometry and ‘Holography Light’**

Regardless of those practices in the field of CMT holography that make it prone to criticism, its general idea is undeniably appealing. Indeed, it falls very much in line with some of the most profound paradigms in quantum many-body physics, including those of the bulk-edge correspondence and the notion of running couplings in the process of renormalization alongside the changing scale of energy/length/information. The former has recently received a lot of attention with the advent of topological insulators/superconductors and Dirac/Weyl (semi)metals, while the latter provides a standard framework for relating the bare (UV) to the effective (IR) physics.

Arguably, of all the holographic studies the most important is the quest into its possible underlying physical cause(s). This concerns, first and foremost, those situations where some precise - analytic or numerical - agreement has been found between the results of the holographic and some conventional approaches. One such recent example is provided by the random infinite-range fermion-hopping models that were studied in Ref.\textsuperscript{5}, drawing from the previous analyses of the random infinite-range spin-coupling systems\textsuperscript{26}.

These models also share a number of common features with the Kondo lattice\textsuperscript{27} and matrix\textsuperscript{28} ones, their unifying physical theme being that of a single ‘quantum impurity’ interacting self-consistently with a local bath. This behavior makes them potentially amenable to the description based on the holographic ‘semi-locally critical’ scenario of Refs.\textsuperscript{29}.

Indeed, in Refs.\textsuperscript{5} an emergent invariance under the group of reparametrizations was discovered and its relation to the ‘semi-local’ $AdS_3 \times R^d$ geometry was demonstrated by establishing an exact agreement between the
holographically computed single-particle propagators in this (non-fluctuating) geometry\textsuperscript{29} and the two- and four-point correlation functions of certain exactly solvable random fermion-hopping models.

Notably, such a perfect agreement was achieved without invoking a small $1/N$ parameter, thereby suggesting that the $1/N$-corrections (that would have been present in the original string-theoretical $AdS/CFT$ correspondence) may, in fact, be absent altogether. More precisely, although the large-$N$ limit indeed had to be taken in the random spin-coupling model of Ref.\textsuperscript{26} (after taking the limit of a large number $N$ of the lattice sites), the ‘semi-local’ behavior of its random fermion-hopping counterparts sets in already at $N = 1$ (but still for $N \gg \infty$).\textsuperscript{29}

Generalizing the results of Refs.\textsuperscript{2,2}, one can construct a whole family of ‘semi-local’ NFL regimes described by the propagator $G(\tau)$ which obeys the integral equations ($n = 2$ in Refs.\textsuperscript{2}

$$
\int G(\tau_1, \tau) \Sigma(\tau, \tau_2) d\tau = \delta(\tau_1 - \tau_2),
$$

$$
\Sigma(\tau_1, \tau_2) = \lambda G^n(\tau_1, \tau_2) G^{n-1}(\tau_2, \tau_1) \tag{15}
$$

that are manifestly invariant under an arbitrary change of variables $\tau = f(\sigma)$

$$
G(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/2n} G(\sigma_1, \sigma_2) \tag{16}
$$

To derive these equations from the (discretized) Hamiltonian of a random infinite-range fermion-hopping model

$$
H = \sum_{k=1}^{n} c_k \sum_{a_i, b_i} \psi^\dagger_{a_i} \cdots \psi^\dagger_{a_k} \psi_{b_1} \cdots \psi_{b_k} \tag{17}
$$

where the parameters $c_k$ need to be fine-tuned\textsuperscript{29}, while for the arbitrary values the system conforms to the generic behavior described by the $n = 2$ case. This is somewhat reminiscent of the situation with the exactly soluble $1d$ spin-$S$ chain model whose behavior is markedly non-generic for all $S > 1/2$.\textsuperscript{29}

Specifically, the solution of Eqs.(15) demonstrates an expressly NFL behavior

$$
G(\tau) \sim \tau^{-1/n} \tag{18}
$$

or, equivalently, $\Sigma(\omega) \sim \omega^{-1/1/n}$ which can also be reproduced holographically with the use of a massive bulk fermion subject to the $AdS_2 \times \mathbb{R}^d$ geometry and described by the action $L_f = i \psi \gamma^\mu \partial_\mu \psi - m \psi \bar{\psi}$ provided that the fermion mass $m$ is chosen as follows

$$
\Delta_\psi = \frac{1}{2} - \left( m^2 R^2 - q^2 E^2 \right)^{1/2} = 1/2 n \tag{19}
$$

where $R$ is the $AdS$ radius, $q$ is the fermion charge, and $E$ measures the particle-hole asymmetry\textsuperscript{5}. Thus, regardless of whether or not the ‘semi-locally critical’ scenario of Ref.\textsuperscript{2} is immediately applicable to any real-life materials, its predictions might still prove quite useful, as far as the properties of some properly crafted random spin-coupling and fermion-hopping models are concerned.

Notably, though, the dimension (19) arises in the so-called ‘unstable’ boundary CFT which is dual to $AdS_2\times\mathbb{R}^d$, while in the ‘stable’ one (to which the former is supposed to flow under a double-trace deformation) the operator dimension would be $\Delta_+ = \frac{1}{2} + (m^2 R^2 - q^2 E^2)^{1/2} > 1/2$, thus making the value (19) unattainable for any integer $n > 1$.

It is also worth noting that the behavior similar to Eq.(18) can be envisioned in the context of, e.g., sub-Ohmic spin-bath models which, incidentally, can be formulated in terms of the localized Majorana fermions. Among other things, it gives rise to the ubiquitous Lorentzian (‘Drude-like’) frequency dependencies of the various observables, their width being given by the inelastic phase relaxation rate.\textsuperscript{31} It might be interesting to explore such a possible connection further\textsuperscript{29}, as well as to reproduce Eq.(18) in the holographic models aspiring to describe the Kondo physics\textsuperscript{29}.

Also, while being characteristic of the purely classical (non-fluctuating) asymptotic near-horizon geometry\textsuperscript{27} the emergent reparametrization symmetry (15) does not rise to the same level as that in the elaborate constructs of the original (string-theoretical) holographic correspondence where the bulk supports a full-fledged quantum (super)gravity theory that only becomes classical in the large-$N$ limit.\textsuperscript{24} Nevertheless, it is quite remarkable that the quantum systems in question allow for some of their properties to be expressed in purely geometrical terms.

In that regard, emergent classical metrics and concomitant effective gravity-like descriptions are not that uncommon, the most remarkable example being provided by the intriguing relationship between quantum physics and classical geometry in the form of the famous extremal area law of entanglement entropy.\textsuperscript{34} To a somewhat lesser extent, same can be said about such connections found in thermodynamics of phase transitions (‘Fisher-Ruppeiner metric’)\textsuperscript{34}, quantum information and tensor network states\textsuperscript{35}, QHE hydrodynamics (‘Fubini-Study metric’)\textsuperscript{36}, Chern classes of the Bloch eigenstates of momentum\textsuperscript{37}, Berry phase of the adiabatic time evolution\textsuperscript{38}, etc.

Importantly, the emergent geometric structures such as effective metrics, curvatures, spin connections, etc. can indeed create the appearance of a limited form of some bulk-boundary correspondence (‘holography light’) that can be inadvertently mistaken for manifestations of the hypothetical full-fledged CFT holography.

Holographic Phenomenology: The Cuprates

Obtaining verifiable holographic predictions can be particularly instructive in those situations where experimental data exhibit robust scaling dependencies, thus hinting at a possible (near-)critical regime that might be amenable to a holographic scaling analysis.\textsuperscript{39} One such example is provided by the holography-inspired phe-
nomenologies of the cuprates which focused on the robust power-law behaviors of the longitudinal electrical conductivity, Hall angle, and magnetoresistivity that violates the conventional Kohler’s law\textsuperscript{40}
\begin{equation}
\sigma \sim T^{-1}, \quad \tan \theta_H \sim T^{-2}, \quad \Delta p/\rho \sim \rho^2 \tag{20}
\end{equation}
In the previously proposed scenario, Eqs.(20) were argued to indicate a possible existence of two distinctively different scattering times: \(\tau_t \sim 1/T\) and \(\tau_H \sim 1/T^2\) which characterize the longitudinal vs transverse\textsuperscript{41} or charge-symmetric vs anti-symmetric\textsuperscript{42} currents. Yet another insightful proposal was put forward in the framework of the marginal Fermi liquid phenomenology\textsuperscript{43}.

In contrast to the anomalous transport properties of the cuprates, their thermodynamic ones are more standard, including the Fermi-liquid-like specific heat \(C \sim T\) (except for a possible logarithmic enhancement\textsuperscript{44}). A recent attempt to rationalize such experimental findings was made in Refs.\textsuperscript{45,46}. Under the assumption of an underlying one-parameter scaling the dimensions of the observables were expressed in terms of the minimal set which includes the dynamical critical exponent \(z\) alongside the dimensions of the (mass) density \(\Delta_n\) and effective charge \(\Delta_e\), the two accounting for, roughly speaking, the wave function renormalization and vertex corrections, respectively.

The linear thermoelectric response is then described in terms of a trio of the fundamental kinetic coefficients
\begin{equation}
\mathbf{J} = \delta \mathbf{E} - \hat{\sigma} \nabla T, \quad \mathbf{Q} = T\hat{\kappa} \mathbf{E} - \hat{\kappa} \nabla T \tag{21}
\end{equation}
where the Onsager’s symmetry is taken into account and the off-diagonal entries in the \(2 \times 2\) matrices \(\sigma, \hat{\sigma}, \hat{\kappa}\) represent the Hall components of the corresponding conductivities.

To set up the scaling relations that reproduce Eqs.(20) and other experimentally observed algebraic dependencies one has to properly account for the time reversal and particle-hole symmetries. Also, one should be alerted to the fact that the exponents governing the temperature dependencies in the kinetic coefficients would be the same as those appearing in their frequency-dependent optical counterparts\textsuperscript{39}. However, the previous analyses based on the semiclassical kinetic equation show that such leading (minimal) powers of \(T\) may or may not actually survive, depending on whether or not the quasiparticle dispersion and Fermi surface topology conspire to yield comparable rates of the normal and umklapp inelastic scattering processes\textsuperscript{47}. It would appear, though, that in the cuprates, both, the multi-pocketed (in the under- and optimally-doped cases) as well as the extended concave (in the over-doped case) hole Fermi surfaces comply with such necessary conditions.

As an additional consistency check, the kinetic coefficients are expected to be consistent with the Fermi liquid relations \(S \sim (T/e\sigma)\delta \sigma/\delta \mu, \quad \nu_N \sim (T/eB)\delta \theta_H/\delta \mu\)
where the r.h.s. are proportional to the Fermi surface curvature but do not contain such single-particle characteristics as scattering time or effective mass and, therefore, might be applicable beyond the coherent quasiparticle regime.

It was found in\textsuperscript{48} that, somewhat surprisingly, most of the experimental data favor the rather mundane solution
\begin{equation}
z = 1, \quad \Delta_e = 0, \quad \Delta_n = 1 \tag{22}
\end{equation}
which imposes the following relation on the Seebeck (\(S\)), Hall Lorentz (\(L_H\)), and Nernst (\(\nu_N\)) coefficients (barring any Sondheimer-type cancellations)
\begin{equation} \[S\] + [\nu_N] - [L_H] = 0 \tag{23}\end{equation}
In the optimally doped cuprates, the experimentally measured thermopower, apart from a finite offset term, demonstrates a (negative) linear \(T\)-dependence\textsuperscript{49}. As regards the Hall Lorentz and Nernst coefficients, the data on the untwinned samples of the optimally doped \(YBa_2Cu_3O_7\) were fitted into a linear dependence, \(L_H \sim T\), whereas \(\nu_N\) was generally found to decrease with increasing \(T\), thus suggesting \([\nu_N] < 0\). Moreover, the Nernst signal increases dramatically with decreasing temperature, which effect has been attributed to the superconducting fluctuations and/or fluctuating vortex pairs whose (positive) contribution dominates over the quasi-particle one (the latter can be of either sign, depending on the dominant type of carriers) upon approaching \(T_c\). Besides, \(\nu_N\) turns out to be strongly affected by a proximity to the pseudogap regime and can even become anisotropic.

The ’Fermi-surfaced’ solution (22) should be compared with the significantly more exotic one obtained in Ref.\textsuperscript{45}
\begin{equation}
z = 4/3, \quad \Delta_e = -2/3, \quad \Delta_n = 2 \tag{24}
\end{equation}
where the thermoelectric coefficients \(S \sim T^{1/2}\), \(L_H \sim T\), \(\nu_N \sim T^{-3/2}\) do not obey Eq.(23). Also, in this case an agreement with the linear behavior of \(L_H\) reported in Ref.\textsuperscript{46} is guaranteed by imposing it as one of the constitutive relations. In spite of this predestined success, though, this scheme fails to reproduce the linear (up to a constant) thermopower, although it claimed the alternate \(S = a - bT^{1/2}\) dependence to provide an even better fit to the data\textsuperscript{48}.

Also, the solution (24) predicts \(C \sim T^{3/2}\) and, therefore, appears to be at obvious odds with the observed thermodynamic properties as well\textsuperscript{44}. Moreover, it predicts the linear in \(T\) (longitudinal) Lorentz ratio \(L = \kappa/T\sigma\) similar to its Hall counterpart, contrary to a constant \(L\), as suggested by the scenario (22).

Although an additional experimental effort is clearly called for in order to discriminate more definitively between the above predictions, it should be mentioned that more recently a slower-than-linear temperature behavior of the (electronic) Lorentz ratio has been reported\textsuperscript{50}.

Lastly, the solution (24) features \(\theta = 0\) and strong (infrared) charge renormalization (\(\Delta_e < 0\)). Since in the
presence of a well-defined \(d\)-dimensional Fermi surface
the ‘hyperscaling-violation’ parameter is expected to
coincide with its co-dimension, hence \(\theta = d - 1\),
the value \(\theta = 0\) hints at a point-like (‘Dirac’) Fermi surface (if
any). In contrast, Eq.(22) suggests a rather simple
physical picture where neither the Fermi liquid-like disper-
sion, nor the effective charge demonstrate any significant
renormalization. Thus, to construct a viable phenomeno-
logical description of the optimally doped cuprates one
might be able to do away without introducing the ad-
titional charge exponent \(\Delta_c\), contrary to the assertions
made in Refs.\textsuperscript{55,56}.

It might also be worth mentioning that another long-
time baffling, the underdoped, phase of the cuprates has
recently been downgraded to a potentially simpler orga-
nized state of matter with some of its properties being
Fermi-like\textsuperscript{52}.

Also, the recent attempt to match the exponents (24)
with the predictions of the semi-phenomenological ‘un-
particles’ model showed that it can only be achieved at
the expense of a further, multi-flavored, extension of that
scenario\textsuperscript{53}. However, the sheer complexity of the pro-
posed construction seems to indicate rather clearly that
the latter can hardly provide a natural (let alone, mini-
mal) theory of the anomalous transport in the cuprates.

Moreover, in the last of Refs.\textsuperscript{53} the method of fractional
derivatives was employed to generalize the Maxwell’s and
continuity equations,
\[
\partial_\mu F^{\mu\nu} = J^{\nu} \\
\partial_\mu J^{\mu} + \partial_0 \rho = 0
\] (25)

However, it can be readily seen that Eqs.(25) are only
consistent when the dimension of the vector potential
\([A_i] = [\partial_\mu] = \alpha\) equals \(z\) (which was chosen as \(z = 1\) in
Ref.\textsuperscript{22}), thus prohibiting it from taking any anomalous
value.

In fact, a gradual realization of how strained can be the
attempts to squeeze the holographic phenomenol-
y of the cuprates into the ‘Procrustes bed’ of the
two-parametric HV geometries may have already started
making its way into the holographic community\textsuperscript{54}.

Another viable candidate for applications of the
holographically-inspired scaling theory is provided by
the self-consistent strong-coupling solutions aimed at de-
scribing the properties of certain 2d and 3d antiferro-
magnetic metals. Analyzed under the assumption of a
momentum-independent NFL self-energy, \(\Sigma(\omega) \sim \omega^{1-\alpha}\),
it yields the critical exponents as sole functions of the
spatial dimension\textsuperscript{52}.
\[
\alpha = 1/2 - 1/z_b, \quad z_b = 4d/3, \quad z_f = 1/(1 - \alpha), \\
\nu = 1/(2 + z_b \alpha) (26)
\]

Notably, the hyperscaling relations are still obeyed, re-
sulting in the anomalous dependencies
\[
C \sim T^{1-\alpha}, \quad \sigma \sim \omega^{\alpha-1}, \quad \chi_s \sim T^{\alpha-1} \quad (27)
\]

In the presence of disorder these results are likely to be
modified, though\textsuperscript{56}.

Another recent work\textsuperscript{57} utilized the functional
RG technique, thus obtaining a different solution which
might be pertinent to the AFM metals: \(z_f = z_b = 3/2, \theta = 0, \nu = \gamma = 1\). Reproducing
these solutions and searching for the new ones\textsuperscript{29} poses
an interesting challenge and presents a important test
for the holographically inspired scaling theory of these
materials.

From AdS/CMT to Holographic Transport

A great many of the early works on CMT holography
utilized the standard holographic recipe for computing
electrical conductivity and other kinetic coefficients as
boundary limits of the ratios between the convergent (as-
associated with a response) and divergent (associated with a
source) terms in the solutions of the radial classical
equations\textsuperscript{5}. This prescription would result in, e.g., the
expression for the electrical conductivity
\[
\sigma(\omega) = \frac{1}{i\omega A} = \frac{1}{i\omega} r^{2-d} d\ln F_{ri}|_{r \to 0} (28)
\]
where the component of the field tensor \(F_{ri} = A +
Br^{d-1}/(d - 1)\) is a bulk dual of the boundary electric
current.

Moreover, in the absence of any dilaton accounting for the
scale dependent couplings and bringing about a depend-
ence on the radial holographic coordinate, the physical
observables become anomaly-free and scale-invariant,
hence unaffected by renormalization. In that case they
might be cast in a purely algebraic form in terms of the
horizon metric, as per the ‘membrane paradigm’\textsuperscript{5}. For
instance, the electrical conductivity and charge suscepti-
bility read
\[
\chi = \left( \int_0^{r_h} dr g_{rr} g_{tt}^{-1} \right)^{-1} \\
\sigma = \left( \frac{-g_{rr} g_{tt}}{g_{rr} g_{tt}} \right)^{1/2} g^{ii}|_{r_h}, (29)
\]
where \(g\) is the determinant of the metric. Per the Ein-
stein’s relation he two are related via the diffusion coef-
ficient, \(\sigma = \chi_s D\).

However, while it might be possible to predict such
general features as the exponents in the power-law fre-
quency dependences of the optical electrical and thermal
conductivities\textsuperscript{22} by using the formal prescriptions similar
to Eq.(28,29), the latter do not properly account for the
actual physical contents of the boundary theory and a
potentially intricate interplay between its different scat-
tering mechanisms.

A list of the proposed sources of current and/or mo-
mement relaxation (which is often confusingly referred
to as ‘dissipation’) employed in holography includes ran-
dom boundary and horizon potentials, helical (Bianchi
VI\textsubscript{0}-type) and spatially periodic (‘Q-lattice’) geo-
metries, massive gravity, axion fields, etc\textsuperscript{55}. 
Common to all the different approaches, though, is the unifying fact that the conductivity and other kinetic coefficients would be typically given by a sum of two terms

$$\sigma(\omega) = \sigma_0 + \frac{\epsilon^2 n^2}{\chi_{PP}(\gamma - i\omega)}$$  \hfill (30)$$

although in the 'top-down' constructions using the DBI action\textsuperscript{22} Eq.(30) would be replaced by $\sigma = \sqrt{\sigma_0^2 + \sigma_1^2}$.

In Eq.(30) the momentum susceptibility $\chi_{PP} = E + P$ is equal to the enthalpy density, and the first term corresponds to the (potentially, universal) diffusion-limited contribution to the conductivity that survives in the particle-hole symmetric limit of zero charge density ($n \to 0$). It is often associated with the processes of 'pair creation' by which a neutral system develops a non-vanishing (yet, finite) conductivity in the absence of momentum relaxation ($\gamma \to 0$), while the DC ($\omega \to 0$) conductivity of a finite density system is governed by the second term in (30), becoming infinite in the limit $\gamma \to 0$.

Moreover, in the case of Q-lattices and 1d periodic potentials (but not those that break translational invariance in all the spatial dimensions) the conductivity can still be expressed in terms of the horizon data, thus generalizing the case of zero charge density and no momentum relaxation. Interestingly, though, even for generic (‘non-Q’) lattices which break translational symmetry in all the spatial dimensions the conductivity can be obtained by solving some linearized, time-independent, and forced Navier-Stokes-type equations of an effective (charged and incompressible) fluid living on the horizon\textsuperscript{63}, thus extending the ‘membrane paradigm’ along the lines of the general concept of a fluid-gravity correspondence\textsuperscript{63}. The latter allows for a dual description of the bulk gravity theory in terms of the hydrodynamics of a certain boundary fluid whose stress-energy tensor acts as a source for the boundary metric. However, while unveiling yet another intrinsically geometric aspect of an underlying quantum dynamics, the fluid-gravity correspondence is clearly not identical to (albeit, possibly, far more general than) the original string-theoretical holography.

In certain axion models\textsuperscript{63}, the elastic rate can be readily computed although the resulting behavior $\gamma \sim \sqrt{m^2 + \Delta^2}/T$ can hardly remain physical in the $T \to 0$ limit. Remarkably, though, at $m = \sqrt{2R_h} = \sqrt{8\pi T}$ the linearized (classical gravitational) equations of motion appear to be exactly solvable, yielding the frequency- and momentum-dependent response functions $G(\omega, k)$ in a closed form and also signaling an emergent $SL(2, R) \times SL(2, R)$ symmetry\textsuperscript{64}.

In the more general situation, momentum conservation would be broken by some operator $O$ with the dimension $\Delta_O$ and the corresponding elastic scattering rate $\gamma$ can be expressed in terms of the spectral density $D(\omega, k) = Im < O(\omega, k)O(-\omega, -k)>$\textsuperscript{65}

$$\gamma = \frac{1}{d\chi_{PP}} \left| \frac{d}{d\omega} \frac{ImD(\omega, k)}{\omega} \right|_{\omega \to 0} \hfill (31)$$

In the EMD context, the following non-universal behavior was obtained by treating the pertinent 'random Ising magnetic field' disorder perturbatively\textsuperscript{66},

$$\sigma(T) \sim T^{2(z-1-\Delta_O)/z}$$  \hfill (32)$$

Also, the generalized Harris criterion for the relevance of disorder

$$\Delta < d + z - \theta/2 \quad \Delta < (d - \theta)/2 + z$$  \hfill (33)$$

gets saturated for $\Delta_O = (d - \theta)/2 + z$, resulting in the universal, yet $\theta$-dependent, behavior\textsuperscript{64}

$$\sigma \sim T^{(\theta - 2 - d)/z} \sim 1/(Sm^2)$$  \hfill (34)$$

where the concomitant scaling of the entropy density, $S \sim T^{(d - \theta)/z}$, has been taken into account. In Eq.(34) the second factor can be interpreted as the ‘thermal graviton mass’, $m \sim T^{1/z}$. Thus, it is only in the extreme ‘locally-critical’ limit $z \to \infty$ where the famous prediction $\sigma \sim 1/S$ of Ref.\textsuperscript{66} could be confirmed without any sleight of hands.

Moreover, should the relevant bosonic modes happen to be transverse and, therefore, protected from developing a mass on the grounds of unbroken gauge invariance (as it would be the case in any $SU(2)$- or $U(1)$-symmetric spin liquid state), the problem of computing the conductivity would require a full non-perturbative solution.

Notably, even in the HV model with $z = 3/2$ and $\theta = 1$ (which values satisfy the relation $z = 1 + \theta/d$ saturating the second of Eqs.(7)) proposed as a viable candidate for the bulk dual of the theory of 2d fermionic matter coupled to an overdamped bosonic (e.g., gauge) field\textsuperscript{20,66}, the conductivity exponent in Eq.(34) ($(\theta - 2 - d)/z = -2$) would be different from its target value ($-1$). This observation, too, might be pointing at some deeper problem with reproducing the linear normal state resistivity of the cuprates in the framework of the HV holographic models.

More recently, the main focus of the CMT holographic phenomenology began to shift towards developing a general 'holography-augmented' transport theory\textsuperscript{63,67,68}. This commendable effort strives to establish general relations between the transport coefficients and find their bounds (if any) that could remain valid regardless of the (in)applicability of the generalized holographic conjecture itself. Such relations are expected to hold in the hydrodynamic regime governed by strong inelastic interactions where the rates of momentum relaxation due to elastic disorder, lattice-assisted inelastic (Umklapp) electron-electron and electron-phonon (outside of the phonon drag regime) scattering are all much smaller than the universal inelastic rate $\Gamma \sim T$ of the normal collisions controlling the formation of a thermalized hydrodynamic state itself\textsuperscript{63,65}.

Notably, the onset of hydrodynamics is a distinct property of the strongly correlated systems, as it is unattainable in the standard FL regime where all the local (in the $k$-space) quasiparticle densities $n_k$ are nearly conserved.
The hydrodynamic regime does not set in for \( d = 1 \) either due to the peculiar \( 1d \) kinematics which facilitates the emergence of infinitely many (almost) conserved densities.

In the hydrodynamic regime, the holographic results have also been systematically compared to those of the hydrodynamics\(^{67}\) and memory matrix\(^{68}\) formalisms which do not rely on the existence of well-defined quasiparticles. The kinetic coefficients obtained by means of these alternate techniques appear to be similar to the holographic expressions (30), identifying the density \( n \) with the current-momentum susceptibility \( \chi_{JP} = \langle J \rangle P \gg n \) which quantifies the contribution to the current from ‘momentum drag’. Such formalism provides an elegant and physically transparent language for discussing, e.g., a decoupling of the electric current onto the coherent component parallel (in the functional sense) to the momentum \( P \) and the orthogonal, incoherent, one which has no overlap with \( P \) and is responsible for the universal conductivity of the neutral (particle-hole symmetric) system

\[
J = \frac{\chi_{JP}}{\chi_{PP}} P + J_{\text{incoh}} \tag{35}
\]

Although under a closer inspection the hydrodynamic results were found to be subtly different (beyond the leading order) from the earlier holographic ones, such differences have been reconciled by including the corrections to \( \sigma_{0} \) and the residue of the pole in Eq.(30) at \( \omega = -i \gamma \) (‘Drude weight’\(^{69,67}\)). This agreement between the holographic and hydrodynamic/memory matrix analyses was argued to provide an additional evidence supporting (at least, the transport-related aspects of) the former.

It should be noted, though, that the AC kinetic coefficients akin to Eq.(30) appear to have a strictly monotonic frequency dependence which is governed by the coherent zero-frequency Drude peak whose presence reflects the existence of a (nearly) conserved momentum which overlaps with the electrical current for \( n \neq 0 \).

On the other hand, a non-monotonic - or, else, strictly universal (frequency-independent) - behavior would be necessary in order for the conductivity to comply with the holographic sum rules proposed in Refs.\(^{69}\):

\[
\int_{0}^{\infty} (\sigma(\omega) - \sigma(\infty)) d\omega = 0
\]

\[
\int_{0}^{\infty} \frac{1}{\sigma(\omega)} - \frac{1}{\sigma(\infty)} d\omega = 0 \tag{36}
\]

These sum rules account for a transfer of the spectral weight from the coherent Drude peak to the incoherent high-frequency tail. Such a non-monotonic behavior develops at \( \omega \sim \Gamma \) and may or may not be detectable by the hydrodynamic/memory matrix analyses, though.

One of the most striking predictions of the memory matrix calculations is that of the reported absence of (many-body) localization and diffusion-dominated metallic transport in strongly interacting ‘strange metals’\(^{69}\). It would seem rather surprising, though, that while imposing robust conductivity bounds in \( d = 1 \) and 2 (where in the latter case the lower bound even seems comparable to the upper one), this approach fails to find such bounds in \( d > 2 \) where, according to the general wisdom, localization would be even less likely to set in.

At a deeper level, a fully quantum description of localization (or a lack thereof) still seems to remain out of reach in all the existing holographic treatments of disorder. In fact, any ‘mean-field’ approximation is likely to be in principle incapable of taking into account not only the localization-related phenomena (multiple acts of elastic scattering) but also the interference ones (multiple acts of consecutive elastic and inelastic scattering).

For one thing, the former type of the conductivity corrections would depend not only on the elastic scattering rate but also on some rather special inelastic ones (Cooperon/Diffusion phase-breaking), whereas the latter would typically acquire its temperature dependence via the fermion occupation factors. Neither source of the \( T \)-dependence can be readily envisioned in the variational approach of Ref.\(^{68}\), though. While suitable for analyzing classical percolation, this approach operates in terms of the Kirchoff’s law for a random network of classical resistors and employs the quadratic Thomson’s variational action akin to that for maximum entropy production. In light of such potential caveats it does not seem surprising that the results of Ref.\(^{68}\) suggest the diffusive metallic behavior even at weak repulsive interactions strengths in \( 2d \) - which would then be in conflict even with the perturbative Altshuler-Aronov theory.

Leaving out the remaining possibility of finding a better variational ansatz that could yield a lower conductivity than the simplest one of a constant current/density employed in Ref.\(^{68}\) and resulting in the above lower bound, it is worth noting that the scenario of Refs.\(^{68}\) might also be limited to the rather special class of models where no coupling between the translation symmetry-breaking (Stueckelberg) degrees of freedom and the Maxwell field is allowed.

To that end, a further investigation into the issue\(^{29}\) showed that, in contrast, no finite (let alone, universal) lower bound seems to exist in the more general models which include generic couplings between the axions with the expectation values \( \Phi_{i} \sim x_{\mu} \delta_{i}^{\mu} \) and vector and/or scalar fields

\[
\Delta L = \sum_{n=1}^{\infty} (g^{\mu\nu} \partial_{\mu} \Phi_{i} \partial_{\nu} \Phi_{i})^{n} [a_{n} F_{\mu\nu}^{2} + b_{n} \phi^{2}] \tag{37}
\]

Thus, conceivably, the scenario of ‘many-body delocalization’ proposed in Ref.\(^{68}\) is neither generic, nor generalizable beyond the scopes of the variational analysis. However, the results of Refs.\(^{29}\), too, should be taken cautiously, as, e.g., the second reference finds the conductivity to be potentially negative, depending on the strength of the couplings \( a_{n}, b_{n} \).

In that regard, although these and other predictions would certainly benefit from clear disclaimers about the...
limited range of the holographic parameters for which physically sensible predictions (e.g., positive conductivity) can be made, conspicuously, such limits do not readily follow from the phenomenological ‘bottom-up’ approach itself. In particular, it would be premature to apply any of the aforementioned holographic scenarios to the analysis of, e.g., the 2DEG systems demonstrating the apparent metal-insulator transition, complete with its peculiar scaling properties.  

**Holographic Transport: Dirac/Weyl materials**

An important example of the systems whose transport properties can be studied systematically and then compared to the holographic predictions is provided by the interacting Dirac/Weyl (semi)metals. In the past, the transport properties of (pseudo)relativistic systems have already been addressed in the context of, both, interacting 2d bosons and fermions. Such studies utilized the standard method of quantum kinetic equation which offers a viable insight that can be compared to the specific predictions of the general theory of holographic transport. The fermionic variant of this problem pertinent to the case of Coulomb-interacting graphene was studied in the so-called two-mode approximation where the emphasis was made on the conservation of charge and energy densities, as well as momentum density which is equivalent to the energy current up to the higher-order dissipative corrections (which feature is specific to the (pseudo)relativistic dispersion).

In the case of a weak Coulomb coupling $\alpha = e^2/hv << 1$ the equilibration rate of all other, non-conserved, modes would be of order $\Gamma_{\text{non-cons}} \sim \alpha^2 \ln \alpha / T$ which is higher than that of the conserved ones, $\Gamma_{\text{cons}} \sim \alpha^2 T$, thanks to the 2d kinematic (logarithmic) divergence of the Coulomb collision integral. Thus, the conserved modes can be singled out, while neglecting all those that undergo a faster relaxation.

It turns out, however, that the system possesses another (nearly)conserved density which is the ‘imbalance’ mode corresponding to the total number of electrons and holes (as opposed to their difference related to the charge density). Away from the neutral (particle-hole symmetric) regime its relaxation takes place through the Auger-type processes with the still slower rate of order $\Gamma_{\text{imb}} \sim T^4 / \mu^3$. Close to the neutrality point the results of the two- and three-mode analyses appear to differ somewhat, thus showing that the imbalance mode can indeed impact the low-energy hydrodynamic behavior.

It would, therefore, be desirable to carry out the three-mode calculations for the $SU(N)$-symmetric fermions with $N \gg 1$, focusing on the regime $1/N \ll \alpha^2 < 1$ where a comparison with the holographic predictions can be made.

Notably, the kinetic equation tends to yield the conductivity which develops the Drude peak followed by a dip at $\omega \sim \Gamma$, while no other peaks (nor zeros) emerge, in contrast to the holographic/hydrodynamic/memory matrix result. Besides, it remains to be seen as to how the explicitly computed corrections to the optical conductivity of graphene match with the general prediction:

$$\frac{\sigma(\omega)}{\sigma(\infty)} = 1 + \frac{C}{\ln \omega}$$

where the sum is taken over the operator expansion of the product of two current operators.

Another quantity of special interest would be the Lorentz ratio (ordinary, as well as the Hall one) as a function of frequency. In the DC limit and near the neutrality point this ratio is dramatically enhanced because the heat and charge currents are controlled by different mechanisms: momentum relaxation vs electron-hole pair creation. This prediction was found to be in a good agreement with the recent data on graphene showing a 20-fold enhancement (although the ratio $\eta/s \sim 10$ estimated in Ref. seems to indicate that even a free-standing graphene may not be that strongly coupled, after all).

It would also be interesting to generalize the analysis to $d = 2 + \epsilon$ dimensions where the effects of the Coulomb interactions are expected to be less dramatic and the separation between the slowly vs rapidly decaying modes becomes less pronounced.

Besides, one would need to introduce yet another nearly conserved density describing an imbalance between the numbers of quasiparticles in the vicinity of the two different Weyl points. This ‘chiral’ density incorporating (for $\epsilon = 1$) the effects of the 3d chiral anomaly gives rise to such concomitant transport phenomena as a negative magnetoresistance. Some of its underlying geometric aspects can be observed even at the semiclassical level. In terms of the Bloch curvature, $\Omega_k = \epsilon_{ij} < \partial \Psi / \partial k_i, \partial \Psi / \partial k_j >$ which affects the semiclassical equations of motion:

$$\frac{d\epsilon}{dt} = -\Omega_k \times \Omega_B, \quad \frac{dk}{dt} = -\partial \Psi / \partial k + \Omega_B \times d\epsilon$$

The current relaxation is then described by the equations:

$$\partial_\mu T^{\mu} = F^{\mu\nu} J_\nu - \gamma T^{\mu}, \quad \partial_\mu \gamma^{\mu} = \alpha \epsilon_{\mu\nu\lambda\rho} F^{\mu\nu} F^{\lambda\rho}$$

Reproducing these effects in their entirety would provide another check point for (and further advance) the holographic transport theory.

In that regard, one should also mention a series of works aimed at connecting holography to the more conventional field theoretical analysis. It seems, though, that the actual graphene and 3d Weyl/Dirac metals alike...
can hardly provide a viable playground for the CMT holography, as the effects of interactions in these materials are either mild or even outright weak. It would, therefore, be quite interesting to find some realization of their counterparts 'on steroids' in the condensed matter realm.

The extended multi-mode kinetic analysis of graphene may also allow for a deeper understanding of the role of viscosity in electron transport. The viscous terms become relevant for \( \eta > (enw)^2/\sigma \) where \( w \) is the width of the graphene sample and such analysis can even be performed in the framework of the Navier-Stokes equation, this time in the real space. Incidentally, this regime has recently been studied in the ultra-pure 2d metal PdCoO\textsubscript{281}, although the estimated ratio \( \eta/S \sim 10^6 \) can hardly allow one to view it as ideal fluid.

Also, the problem of a hot spot relaxation in graphene\textsuperscript{28} presents a specific example of the general 'quench'-type phenomena which, too, have been extensively addressed by the CMT holography\textsuperscript{28}. The 1d variant of this problem involves two reservoirs of different temperatures \( T_L \) and \( T_R \) which are brought into a thermal contact, its solution featuring a non-equilibrium stationary state characterized by a definite temperature \((T_LT_R)^{1/2}\) and constant energy flux (this solution does not seem to readily extend to the limit of either vanishing temperature, though). Moreover, this stationary state was shown to spread outward in the form of shock waves and was related to the Lorentz-boosted black-hole metric in the bulk, thus suggesting yet another explicit holographic connection.

To that end, it was conjectured in Refs.\textsuperscript{82} that a similar behavior would occur in higher dimensions as well. However, the recent analyses of the hot spot relaxation in 2d graphene\textsuperscript{74} suggest that a more involved scenario where, both, the resistive and viscous effects interfere with one another might be realized. Notably, the recent holographic work of Refs.\textsuperscript{83}, too, shows a more involved dynamics of the expanding stationary state.

**Analogue Holography Demonstrators**

The above examples of 'emergent geometry' prompt a systematic quest into the apparent holography-like relationships that would be governed by the already known, rather than some hypothetical, physics. To that end, the general holographic concept can benefit from the possibility of being simulated in various controlled 'analogue' environments.

One prospective design of a 'holography simulator' was proposed for implementation in flexible graphene and other semi-metallic monolayers (silicene, germanine, stanine, etc.)\textsuperscript{84}. Such stress-engineered desktop realizations of the system of 2d Dirac fermions in a curved geometry\textsuperscript{85} can also be grown on commensurate substrates (e.g., \( h-BN \)) in order to endow the bulk fermions with a finite mass via hybridization.

In Ref.\textsuperscript{84} a number of situations were discussed where a physical edge of the curved graphene flake which supports (almost) non-interacting massive Dirac fermions propagating in a curved 2d space exhibits a behavior that would be typically attributed to the effects of some phantom 1d interactions. In contrast to the aforementioned 'semi-local' scenario of Refs.\textsuperscript{82}, though, it is the momentum dependence of the 1d propagator \( G(\omega,k) \) that tends to become non-trivial.

Specifically, in graphene the artificial gauge and metric fields represent the elastic (phonon) degrees of freedom, their effective vector and scalar potentials

\[
A_0 \sim u_{xx} + u_{yy}, \quad A_x \sim u_{xx} - u_{yy}, \quad A_y \sim u_{xy} \tag{42}
\]

being composed of the components of the strain tensor\textsuperscript{86},

\[
u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i \partial_j h).
\]

Here \( u_i(x) \) and \( h(x) \) are the local in- and out-of-plane displacements of the monolayer, respectively, while the valley-specific fields \( A_\mu \) have opposite signs at the two different Dirac points. Computing the massive free Dirac fermion propagator in a curved space and taking its boundary limit one obtains the large-scale asymptotic behavior (4) governed by the geodesic action \( S_{hol}(\tau,x) \) for the background metric. For instance, in the case of a graphene flake shaped as a surface of rotation with the line element (here \( r \) and \( \phi \) are the normal polar coordinates)

\[
dl^2 = dr^2(1 + (\frac{\partial h(r)}{\partial r})^2) + r^2d\phi^2 \tag{43}
\]

with \( h(r) \sim (R/r)^\eta \) for \( r \geq a \) the 'warp factor' \( g(r) \sim 1/r^{(\eta+2)} \) diverges at small \( r \). One then obtains the geodesic action

\[
S(\tau,x) = m \sqrt{\tau^2 + (Rx)^{2/(\eta+1)}} \tag{44}
\]

which reveals an unconventional behavior of the boundary propagator as a function of the distance along the edge, thus suggesting the 'holographic' dynamical critical exponent \( z = \eta/(\eta + 1) \).

It is instructive to compare the asymptotic (4) with the action given by (44) with the propagator of 1d fermions interacting via a pairwise potential \( U(x) \sim 1/x^\zeta \) where \( \zeta < 1 \). Using the standard \( 1d \) bozimization technique and matching the large-\( x \) asymptotics, one finds that (44) can mimic the spatial decay of the 1d propagator in the presence of such interactions, provided that \( \eta = (1 - \zeta)/(1 + \zeta) \).

Another instructive example is provided by the line element

\[
dl^2_{\log} = dr^2 + R^2 \exp(-2(r/R)\lambda)d\phi^2 \tag{45}
\]

in which case the geodesic action reads

\[
S(\tau, x) = m \sqrt{\tau^2 + R^2(\ln x/a)^{2/\lambda}} \tag{46}
\]
For $\lambda = 1$ and at large $x$ the propagator decays algebraically, $G(0, x) \sim 1/x^{mR}$ which is reminiscent of the behavior found in the 1$d$ Luttinger liquids, while for $\lambda \neq 1$ Eq.(46) yields a variety of stretched/compressed exponential asymptotics which decay faster (for $\lambda < 1$) or slower (for $\lambda > 1$) than any power-law. For instance, by choosing $\lambda = 2/3$ one can simulate a faster-than-algebraic spatial decay, $G(0, x) \sim \exp(-const \ln^{3/2} x)$, in the 1$d$ Coulomb gas ($\sigma = 1$) which is indicative of the incipient 1$d$ charge density wave state.

For other values of $\lambda$ Eq.(46) reproduces the behavior in the boundary theory governed by the interaction $U(x) \sim (\ln x)^{(2/\lambda) - 3}/x$. Although the physical origin of such a bare potential would not be immediately clear, multiplicative logarithmic factors do routinely emerge in those effective 1$d$ interactions (‘double trace operators’) that are associated with various marginally (ir)relevant two-point operators. On the experimental side, the boundary correlations can be probed with such established techniques as time-of-flight, edge tunneling, and local capacitance measurements, thus potentially helping one to hone the proper analogue-holographic ‘dictionary’.

This holography-like correspondence once again suggests that some limited form of a bulk-boundary relationship might, in fact, be quite robust and hold regardless of whether or not the systems in question possess any particular symmetries, unlike in the original AdS/CFT construction. Naively, this form of correspondence can even be related to the Einstein’s equivalence principle (i.e., ‘curvature equals force’), according to which free motion in a curved space should be indistinguishable from the effect of a physical interaction (only, this time around, in the tangential direction).

As an alternate platform for doing analogue holography, optical metamaterials have long been considered as candidates for simulating such effects of general relativity as event horizons, redshift, black, white, and wormholes, inflation, Hawking radiation, dark energy, metric signatures transitions, ‘end-of-time’, and other cosmological scenarios.

However, the earlier proposals focused on the effective 3 + 1-dimensional metrics which requires some intricate engineering of the locked permittivity ($\epsilon$) and permeability ($\mu$) tensors.

$$\epsilon^{ij} = \mu^{ij} = g^{ij} \sqrt{-g}/|g_{00}| \quad (47)$$

More recently, it was proposed to use extraordinary (TM-polarized) monochromatic photons with the dispersion $\omega^2 = k_z^2/\varepsilon_{xy} + k_x^2/\varepsilon_{zz}$ for simulating the effective 2 + 1-dimensional metrics

$$g_{rr} = -\varepsilon_{xy}, \quad g_{tt} = g_{\phi\phi}/r^2 = -\varepsilon_{zz} \quad (48)$$

in the media with $\mu = 1$.

In the hyperbolic regime, $\epsilon_{xy} > 0$ and $\varepsilon_{zz} < 0$, the momentum component $k_z$ can then be thought of as an effective frequency, whereas $\omega$ plays the role of mass. In Ref.49 it was shown that the experimentally attainable two-component metamaterial configuration consisting of alternating metallic and insulating (flat or cylindrical) layers can simulate some of the HV geometries. The boundary propagator describing static spatial correlations of the optical field on the interface between the metamaterial and vacuum was found to behave as

$$< E_\omega(x)E_{-\omega}(0) > \sim \exp(-|\omega| x^{d/2}) \quad (49)$$

which should be contrasted to its counterpart in an isotropic medium with a (negative) dielectric constant, $< E_\omega(x)E_{-\omega}(0) > \sim \exp(-|\omega| x)$. Experimentally, such correlations can be studied by analyzing the statistics of a non-local optical field distribution with the use of holographic and speckle interferometry. Albeit still requiring a careful engineering of the dielectric medium, some hallmark features of the analogue holography could, in principle, be detected and then compared with, e.g., correlators of the vertex operators in the theory of a strongly self-interacting 2$d$ bosonic field, akin to that describing the thermodynamics of a fluctuating elastic membrane.

Lastly, one could further elaborate on yet another (historically, far more extensively studied) potential playground for analogue holography which is provided by the acoustic realizations of ‘emergent gravity’. By implementing such proposals one might also hope to establish a putative bulk-boundary correspondence (alongside its own ‘dictionary’) for a broader variety of the combined (pseudo)gravitational backgrounds, thereby gaining a better insight into the physical origin(s) of the apparent holography-like properties of various CMT and AMO systems.

Together with the systematic comparison between the predictions of the CMT holography and other, more traditional, approaches and/or experimental data it would be a necessary step towards vetting the intriguing, yet speculative, holographic ideas before the latter can be rightfully called a novel technique for tackling the ‘strange metals’ and other strongly correlated materials. In any event, though, it would seem rather unlikely that a hands-on expertise in string theory will become a mandatory prerequisite for understanding the cuprates.

The author gratefully acknowledges the support from and/or hospitality at the Aspen Center for Physics (funded by the NSF under Grant 1066293), the Galileo Galilei Institute (Florence), the Kavli Institute for the Physics and Mathematics of the Universe (Kashiwa), the Kavli Institute for Theoretical Physics (Beijing), the International Institute of Physics (Natal), and NORDITA (Stockholm) during the various stages of compiling these notes.
1. J. Polchinski, *Introduction to Gauge/Gravity Duality*, arXiv:1010.6134; J. McGreevy, *Holographic duality with a view toward many-body physics*, Adv. High Energy Phys. 2010, 723105 (2010); S. A. Hartnoll, *Lectures on holographic methods for condensed matter physics*, Class. Quant. Grav. 26, 224002 (2009); C. P. Herzog, J. Phys. A 42, 343001 (2009); S. Sachdev, *What can gauge-gravity duality teach us about condensed matter physics?*, Annual Review of Cond. Matt. Phys. 3, 9 (2012); Sebastian de Haro, Daniel R. Mayerson, Jeremy Butterfield, *Conceptual Aspects of Gauge/Gravity Duality*, arXiv:1509.09231.

2. Martin Ammon and Johanna Erdmenger, *Gauge/Gravity Duality: Foundations and Applications*, Cambridge University Press, 2015; Jan Zaanen, Yan Liu, Ya-Wen Sun, Koenraad Schalm, *Holographic Duality in Condensed Matter Physics*, Cambridge University Press, 2015.

3. S. Kobayashi, D. Mateos, S. Matsuura, R. C. Myers, and R. M. Thomson, *Holographic phase transitions at finite baryon density*, Journal of High Energy Physics 02 016 (2007), arXiv:hep-th/0611099; A. Karch and A. O'Bannon, *Metallic AdS/CFT*, Journal of High Energy Physics 09 024 (2007), arXiv:0705.3870; S. Kachru, A. Karch, and S. Yaida, *Holographic lattices, dimers and glasses*, Physical Review D 81 026007 (2010), arXiv:0909.2639; S. A. Hartnoll, J. Polchinski, E. Silverstein, and D. Tong, *Towards strange metallic holography*, Journal of High Energy Physics 04 120 (2010), arXiv:0912.1061; S. Ryu, T. Takayanagi, and T. Ugajin, *Holographic conductivity in disordered systems*, Journal of High Energy Physics 04 115 (2011), arXiv:1103.0668.

4. Sung-Sik Lee, *Quantum Renormalization Group and Holography*, JHEP 01 (2014) 076; Background independent holographic description: from matrix field theory to quantum gravity, JHEP10(2012)160; Holographic description of large N gauge theory, Nuclear Physics, B851, 143 (2011); ibid B 832, 567 (2010); Peter Lunts, Subhro Bhattacharjee, Jonah Miller, Erik Schnetter, Yong Baek Kim, Sung-Sik Lee, *Ab initio holography*, JHEP 1508:107,2015; Ching Hua Lee, Xiao-Liang Qi, *Exact holographic mapping in free fermion systems*, arXiv:1503.08592; Masanichi Miyaji, Tadashi Takayanagi, *Surface/State Correspondence as a Generalized Holography*, arXiv:1503.03542; Yu Nakayama, a-c test of holography vs quantum renormalization group, arXiv:1401.5257; Steven Jackson, Razieh Pourhasan, Hermann Verlinde, *Geometric RG Flow*, arXiv:1312.6914; Elias Kiritsis, Wenliang Li, Francesco Nitti, *Holographic RG flow and the Quantum Effective Action*, Fortsch. Phys. 62, No. 5-6, 389 (2014); Sung-Sik Lee, *Horizon as Critical Phenomenon*, arXiv:1603.08560.

5. S. Sachdev, *Bekenstein-Hawking Entropy and Strange Metals*, Phys. Rev. X 5, 041025 (2015), arXiv:1506.05111; Wenbo Fu, Subir Sachdev, *Numerical study of fermion and boson models with infinite-range random interactions*, arXiv:1603.05210; A. Kitaev, unpublished.

6. W. Witzczak-Krempa, E. Sorensen, S. Sachdev, —It the dynamics of quantum criticality via Quantum Monte Carlo and holography, Nature Physics 10, 361 (2014); W. Witzczak-Krempa, *Quantum critical charge response from higher derivatives: is more different?*, Phys.Rev. B89, 161114 (2014); Emanuel Katz, Subir Sachdev, Erik S. Sorensen, William Witzczak-Krempa, *Conformal field theories at non-zero temperature: operator product expansions, Monte Carlo, and holography*, arXiv:1409.3841; S. Bai, D.-W. Pang, *Holographic charge transport in 2+1 dimensions at finite N*, arXiv:1312.3351.

7. Kun Chen, Longxiang Liu, Youjin Deng, Lode Pollet, Nikolay Proko'ev, *Universal Conductivity in a Two-Dimensional Superfluid-to-Insulator Quantum Critical System*, Phys. Rev. Lett. 112, 030402 (2013).

8. S. A. Hartnoll, J. Polchinski, E. Sorensen, William Witczak-Krempa, *Entanglement Renormalization and Holography*, JHEP 01 (2012) 125; Brian Swingle, *Holographic Fermi Surfaces and Entanglement Entropy*, JHEP 01 (2012) 125; Emanuel Katz, Subir Sachdev, Erik S. Sorensen, William Witczak-Krempa, *Constructing holographic spacetimes using entanglement renormalization*, arXiv:1209.3304.

9. S. Sachdev, *Holographic metals and the fractionalized Fermi liquid*, Phys.Rev. Lett. 105, 151602 (2010); L. Huijse, S. Sachdev, *Fermi surfaces and gauge-gravity duality*, arXiv:1104.5022; S. Sachdev, *A model of a Fermi liquid using gauge-gravity duality*, Phys. Rev. D 84, 066009 (2011); L. Huijse, S. Sachdev, B. Swingle, *Hidden Fermi surfaces in compressible states of gauge-gravity duality*, arXiv:1112.0573; B. Swingle, L. Huijse, S. Sachdev, *Entanglement entropy of compressible holographic matter: loop corrections from bulk fermions*, arXiv:1305.3234; N. Ogawa, T. Takayanagi, T. Ugajin, *Holographic Fermi Surfaces and Entanglement Entropy*, JHEP 01 (2012) 125; Brian Swingle, *Entanglement Renormalization and Holography*, arXiv:0905.1317; Xi Dong, Sarah Har-
P. Kovtun, D.T. Son, A.O. Starinets, Viscosity in Strongly Interacting Quantum Field Theories from Black Hole Physics, Phys.Rev.Lett. 94 (2005) 111601

D.T. Son, A.O. Starinets, Viscosity in Strongly Interacting Quantum Field Theories from Black Hole Physics, Phys.Rev.Lett. 94 (2005) 111601

D.T. Son, A.O. Starinets, Minkowski-space correlators in AdS/CFT correspondence: recipe and applications, arXiv:hep-th/0205051

Anton Rebhan, Dominik Steineder, Violation of the Holographic Viscosity Bound in a Strongly Coupled Anisotropic Plasma, arXiv:1110.6825

Kiminad A. Mamo, Holographic RG flow of the shear viscosity to entropy density ratio in strongly coupled anisotropic plasma, JHEP 1210:070, 2012;

Robert C. Myers, Todd Sieren, William Witzczak-Krempa, A Holographic Model for Quantum Critical Responses, arXiv:1602.05599

Sachdev, Subir; Ye, Jinwu Gapless spin-fluid ground state in a random quantum Heisenberg magnet, Phys.Rev.Lett.70, 3339 (1993).

Georges, A.; Parcollet, O.; Sachdev, S. Quantum fluctuations of a nearly critical Heisenberg spin glass, 2001 Phys Rev B63, 4406; Georges, Antoine; Parcollet, Olivier; Sachdev, Subir, Mean Field Theory of a Quantum Heisenberg Spin Glass, 2000 Phys Rev Lett85, 840.

Dionysios Anninos, Sean A. Hartnoll, Liza Huijse, Victoria L. Martin, Large N matrices from a nonlocal spin system, arXiv:1412.1092 Cugliandolo, L. F.; Kurchan, J.; Parisi, G.; Ritort, F., Matrix Models as Solvable Glass Models 1995PhRvL..74.1012C; Masuku, Mthokozisi; Moulkwe, Mbavhalelo; Rodrigues, Joao P., Large N Matrix Hyperspheres and the Gauge-Gravity Correspondence, arXiv:1111.5785 Subir Sachdev, Strange metals and the AdS/CFT correspondence, J.Stat.Mech.1011:P11022; Javier M. Magan, Black holes as random particles: entanglement dynamics in infinite range and matrix models, arXiv:1601.04663.

C. Bonifield, T.Swartz, J.Schirmer, and D.V.Khveshchenko, in progress.

Takhtajan, L. A., The picture of low-lying excitations in the isotropic Heisenberg chain of arbitrary spins, 1982PhLA 87, 479; Babujian, H. M., Exact solution of the isotropic Heisenberg chain with arbitrary spins: Thermodynamics of the model, 1983NuPhB.215..317B; 1983 Exact solution of the one-dimensional isotropic Heisenberg chain with arbitrary spins S. 1982PhLA...90.479B.

Alexander Shirnirman, Yuriy Makhlkin, Spin-spin correlators in Majorana representation, Phys. Rev. Lett. 91, 207204 (2003); Robert S. Whitney, Yuriy Makhlkin, Alexander Shirnirman, Yuval Gefen, Geometric nature of the environment-induced Berry phase and geometric dephasing, Phys. Rev. Lett. 94, 070407 (2005); Pablo Schad, Alexander Shirnirman, Yuriy Makhlkin, Using Majorana spin-1/2 representation for the spin-boson model, arXiv:1504.05094; P. Schad, Yu, Makhlkin, B.N. Narozhny, G. Schn, A. Shirnirman, Majorana representation for dissipative spin systems, Annals of Physics 361, 401 (2015).

Paolo Benincasa, Alfonso V. Ramallo, Holographic Kondo Model in Various Dimensions, JHEP06(2012)133; Sarah Harrison, Shamit Kachru, Gonzalo Torroba, A maximally supersymmetric Kondo model, arXiv:1110.5325 Johanna Erdmenger, Carlos Hoyos, Andy O’Bannon, Jackson Wu, A Holographic Model of the Kondo Effect, JHEP12(2013)086, arXiv:1310.3271.

Johanna Erdmenger, Mario Flory, Max-Niklas Newrzella, Bending branes for DCFT in two dimensions , arXiv:1410.7811.

Johanna Erdmenger, Mario Flory, Carlos Hoyos, Max-Niklas Newrzella, Jackson M. S. Wu, Entanglement Entropy in a Holographic Kondo Model, arXiv:1511.03666

S. Ryu and T. Takayanagi, Phys.Rev.Lett. 96 (2006) 181602; hep-th/0603001 T. Takayanagi, Holographic dual of a boundary conformal field theory, Phys. Rev. Lett., vol. 107, p. 101602, Aug 2011.

Hiroaki Matsueda, Entanglement General Relativity from Fisher Information Metric, arXiv:1310.1831 Seyed Ali Hosseini Mansoori, Behrouz Mirza, Mohamadreza Fazel,
Javier Molina-Vilaplana, Information Geometry of Entanglement Renormalization for free Quantum Fields, JHEP09(2015)002; Yoichiro Hashizume, Hiroaki Matsueda, Information Geometry for Husimi-Temperley Model, arXiv:1407.2667; Xiao-Liang Qi, Exact holographic mapping and emergent space-time geometry, arXiv:1309.6282; Wei Li, Tadashi Takayanagi, Holography and Entanglement in Flat Spacetime, Phys.Rev.Lett.106:141301,2011; Xing Huang, Feng-Li Lin, Entanglement renormalization and integral geometry, arXiv:1507.04633; Masamichi Miyaji, Tokiro Numasawa, Noburo Shiba, Tadashi Takayanagi, Kento Watanabe, Gravity Dual of Quantum Information Metric, arXiv:1507.07555.

Bo Yang, Z. Papic, E. H. Rezayi, R. N. Bhatt, F. D. Drees Sels, Michiel Wouters, Quantum statistical gravity: time dilation due to local information in many-body quantum systems, arXiv:1602.05707.

Bo Yang, Z. Papic, E. H. Rezayi, R. N. Bhatt, F. D. M. Haldane; Band mass anisotropy and the intrinsic metric of fractional quantum Hall systems, Phys. Rev. B 85, 165318 (2012); Areg Ghazaryan, Tapash Chakraborty, Emergent Geometry Fluctuation in Quantum Confined Electron Systems, arXiv:1403.6485; Rudro R. Biswas, Semiclassical theory of viscosity in quantum Hall states, arXiv:1311.7149.

Titus Neupert, Claudio Chamon, Christopher Mudry, How to Measure the Quantum Geometry of Bloch Bands, Shunji Matsuura, Shinsei Ryu, Kento Watanabe, Li, and topological insulators, PhysRevB82, 245113; David Bauer, T. S. Jackson, Rahul Roy, Quantum geometry and stability of the fractional quantum Hall effect in the Hofstadter model, arXiv:1504.07185.

James M. Hickey, Sam Genway, Juan P. Garrahan, Dynamical phase transitions, time-integrated observables and geometry of states, Phys. Rev. B 89, 054301 (2014); Michael Kolodrubetz, Vladimir Gritsev, Anatiol Polkovnikov, Classifying and measuring the geometry of the quantum ground state manifold, Phys. Rev. B 88, 064304 (2013); Ali Mollabashi, Masahiro Nozaki, Shinsei Ryu, Tadashi Takayanagi, Holographic Geometry of cMERA for Quantum Quenches and Finite Temperature, JHEP03(2014)098.

D.V. Khveshchenko, Taking a critical look at holographic critical matter, arXiv:1404.7000, Lith. J. of Phys., 56, 208 (2015).

J.M. Harris, Y. F. Yan, P. Mati, N. P. Ong, P. W. Anderson, T. Kimura, and K. Kitazawa, Violation of Kohler’s Rule in the Normal-State Magnetoresistance of YBa2Cu3O7δ and La2Sr2-xCuO4 Phys. Rev. Lett. 75, 1391 (1995).

P.W. Anderson, all effect in the two-dimensional Luttinger liquid, Phys.Rev.Lett.67, 2092 (1991).

P.Coleman, A.J.Schofield, and A.M.Tsvelik, Phenomenological Transport Equation for the Cuprate Metals, Phys.Rev.Lett.76, 1324 (1996).

C.M.Varma, P. B. Littlewood, S. Schmitt-Rink, E. Abraham, and A. E. Ruichenstain, Phenomenology of the normal state of Cu-O high-temperature superconductors, Phys. Rev. Lett. 64, 497 (1990).

J.W.Loram, K.A. Mirza, J.M. Wade, J.R. Cooper, W.Y. Liang, The electronic specific heat of cuprate superconductors Physica C235-240, 134 (1994).

A.Karch, Conductivities for Hyperscaling Violating Geometries, JHEP06(2014)140; S.A.Hartnoll and A.Karch, Scaling theory of the cuprate strange metals, Phys. Rev. B 91, 155126 (2015).

D.V. Khveshchenko, Viable phenomenologies of the normal state of cuprates, Euro Phys. Lett. 111, 17005 (2015).

A. V. Chubukov, D. L. Maslov, V. I. Yudson, arXiv:1401.1461 Resistivity of a non-Galilean-invariant Fermi Liquid near Pomeranchuk Quantum Criticality, Phys. Rev. Lett. v. 106, 106403 (2011); H. K. Pal, V. I. Yudson, D. L. Maslov, Resistivity of non-Galilean-invariant Fermi- and non-Fermi liquids, Lith. J. Phys., 52, 142 (2012).

Mu-Yong Choi, J. S. Kim, Thermopower of high-Tc cuprates, Phys. Rev. B 59, 000192 (1999).

Y. Zhang, N. P. Ong, P. W. Anderson, D. A. Bonn, R. Liang, W. N. Hardy, Determining the Wiedemann-Franz ratio from the thermal Hall conductivity: Application to Cu and YBa2Cu3O6.95, Phys.Rev.Lett.84, 2219 (2000); Y.Wang, L.Li, and N.P.Ong, The Nextn effect in high-Tc superconductors, Phys. Rev. B73, 024510 (2006).

M. Matusiak and T. Wolf, Phys. Rev. B72, 054508(R) (2005); M.Matusiak, J.Hori, and T.Suzuki, Solid State Comm. 139, 376 (2006); M.Matusiak, K.Rogacki, and B.W.Veal, Euro Phys. Lett. 88, 47005 (2009).

B. Guertaurx, Universal scaling properties of extremal co-hesive holographic phases, JHEP 1401, 080 (2014); Charge transport in holography with momentum dissipation, ibid 1404, 181 (2014); B. Guertaurx, E. Kiritsis, Generalized Holographic Quantum Criticality at Finite Density, JHEP 1112, 036 (2011).

N. Barisic, M. K. Chan, M. J. Veit, C. J. Dowor, Y. Ge, Y. Tang, W. Tabis, G. Yu, X. Zhao, M. Greven, Hidden Fermi-liquid behavior throughout the phase diagram of the cuprates, arXiv:1507.07885; G. Grissonnanche, F. Laliberte, S. Dufour-Beausejour, M. Matusiak, S. Badoux, F. X. Tafti, B. Michon, A. Riopel, C. Yrch-Choiinere, J. C. Baglo, B. J. Ramshaw, R. Liang, D. A. Bonn, W. N. Hardy, S. Kramer, D. LeBoeuf, D. Graf, N. Doiron-Leyraud, L. Taillefer, Wiedemann-Franz law in the underdoped cuprate superconductor YBa2Cu3Oy, arXiv:1503.07572 [pdf]; G. Grissonnanche, F. Laliberte, S. Dufour-Beausejour, A. Riopel, S. Badoux, M. Caouette-Mansour, M. Matusiak, A. Junen-Fecteur, P. Bourgeois-Hope, O. Cyr-Choiinere, J. C. Baglo, B. J. Ramshaw, R. Liang, D. A. Bonn, W. N. Hardy, S. Kramer, D. LeBoeuf, D. Graf, N. Doiron-Leyraud, Louis Taillefer, Onset field for Fermi-surface reconstruction in the cuprate superconductor YBCO, arXiv:1508.05486.

Andreas Karch, Multiband models for field theories with anomalous current dimension, arXiv:1504.02478; Andreas Karch, Kridsanaphong Limtrakool, Philip W. Phillips, Unparticles and Anomalous Dimensions in the Cuprates, arXiv:1511.02868; Kridsanaphong Limtrakool, Philip W. Phillips, Anomalous Dimension of the Electrical Current in the Normal State of the Cuprates from the Fractional Aharonov-Bohm Effect, arXiv:1601.02340.

Andrea Amoretti, Matteo Baggioni, Nicodemo Magnoli, Daniele Musso, Chasing the cuprates with dilatonic dyons, arxiv: 1603.03029.

Elithu Abrahams, Joerg Schmalian, Peter Woelfle, Strong
coupling theory of heavy fermion criticality, Phys. Rev. B 90, 045105 (2014); Matthias Vojta, Ralf Bulla, Peter Wilke, Critical quasiparticles in single-impurity and Kondo lattice models, Eur. Phys. J. Special Topics 224, 1127 (2015); Peter Woelfle, Elihu Abrahams, Spin-flip scattering of critical quasiparticles and the phase diagram of YbB2Si2, arXiv:1506.08476.

T.R. Kirkpatrick, D. Belitz, Pre-asymptotic critical behavior and effective exponents in disordered metallic quantum ferromagnets, Phys. Rev. Lett. 113, 127203 (2014); Exponent relations at quantum phase transitions, with applications to metallic quantum ferromagnets, Phys. Rev. B 91, 214407 (2015); Quantum Correlations in Metals that Grow in Time and Space, arXiv:1506.01830.

Stefan A. Maier, Philipp Strack, Universality in antiferromagnetic strange metals, arXiv:1510.01351.

P. Kovtun, A. Ritz, Universal conductivity and central charges, Phys.Rev.D78, 066009 (2008); P. M. Hogan, A. G. Green, Universal non-linear conductivity near to an itinerant-electron ferromagnetic quantum critical point, Phys. Rev. B 78, 195104 (2008); N. Iqbal and H. Liu, Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm, Physical Review D79 025023 (2009), arXiv:0809.3808; S. Jain, Universal thermal and electrical conductivity from holography, JHEP 1011:092 (2010); Universal properties of thermal and electrical conductivity of gauge theory plasmas from holography, JHEP06(2010)023; Holographic electrical and thermal conductivity in strongly coupled gauge theory with multiple chemical potentials, JHEP 1003, 101 (2010).

S. Ryu, T. Takayanagi, and T. Ugajin, Holographic conductivity in disordered systems, Journal of High Energy Physics 04 115 (2011), arXiv:1103.6068.

Thomas Faulkner, Nabil Iqbal, Hong Liu, John McGreevy, David Vegh, Charge transport by holographic Fermi surfaces, Phys.Rev. D88, 045016 (2013); S. A. Hartnoll, D. M. Hofman, Locally critical umklapp scattering and holography, Phys.Rev.Lett.108, 241601 (2012); A. Donos, S. A. Hartnoll, Metal-insulator transition in holography, Nature Phys.9, 649 (2013); Phys.Rev.D86,124046 (2012); K. Balasubramanian, C.P. Herzog, Losing Forward Momentum Holographically, arXiv:1312.4953.

S. A. Hartnoll, J. E. Santos, Disordered horizons: Holography of randomly disordered fixed points, arXiv:1402.0872; J.-R.Sun, S.-Y. Wu, H.-Q. Zhang, Mimic the optical conductivity in disordered solids via gauge/gravity duality, Physics Lett. B 729 (2014) 177; D. Vegh. Holography without translational symmetry, arXiv:1301.0537; A. Donos and J. P. Gauntlett, Novel metals and insulators from holography, Journal of High Energy Physics 06 007 (2014), arXiv:1401.5077; M. Blake, D.Tong, Universal Resistivity from Holographic Massive Gravity, Phys. Rev. D 88, 106004 (2013); T. Andrade, B.Withers, A simple holographic model of momentum relaxation, arXiv:1311.3157; M. Blake, D. Tong, D. Vegh, Holographic Lattices Give the Graviton a Mass, Phys. Rev. Lett. 112, 071602 (2014); M.Blake and A.Donos, Quantum Critical Transport and the Hall Angle, arXiv:1406.1659; Sean A. Hartnoll, Theory of universal incoherent metallic transport, arXiv:1405.3651; Andrea Amoretti, Alessandro Braggio, Nicodemo Magnoli, Daniele Musso, Bounds on charge and heat diffusivities in momentum dissipating holography, JHEP 1507 (2015) 102; Matteo Baggioli, Oriol Pujolas, Holographic Polaron, the Metal-Insulator Transition and Massive Gravity, Phys. Rev. Lett. 114, 251602 (2015).

E. Banks, A. Donos, and J. P. Gauntlett, Thermoelectric DC conductivities and Stokes flows on black hole horizons, Journal of High Energy Physics 10 103 (2015), arXiv:1507.00234.

A. Donos and J. P. Gauntlett. The thermoelectric properties of inhomogeneous holographic lattices, Journal of High Energy Physics 01 035 (2015), arXiv:1409.6875.

A. Donos and J. P. Gauntlett. Navier-Stokes on black hole horizons and DC thermoelectric conductivity, Physical Review D92 121901 (2015), arXiv:1506.01360.

A. Donos, J. P. Gauntlett, T. Griffin, and L. Melgar. DC conductivity of magnetised holographic matter, Journal of High Energy Physics 01 113 (2016), arXiv:1511.00713.

S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, Nonlinear fluid dynamics from gravity, Journal of High Energy Physics 02 045 (2008), arXiv:0712.2456; M. Rangamani, Gravity and Hydrodynamics: Lectures on the fluid-gravity correspondence, Class.Quant.Grav., vol. 26, p. 224003, 2009; V. E. Hubeny, The Fluid/Gravity Correspondence: a new perspective on the Membrane Paradigm, Class.Quant.Grav., vol. 28, p. 114007, 2011; V. E. Hubeny, S. Minwalla, and M. Rangamani, The fluid/gravity correspondence in Black Holes in Higher Dimensions (G. Horowitz, ed.), Cambridge: Cambridge University Press, 2012.

R.A. Davison, Momentum relaxation in holographic massive gravity, Phys. Rev. D 88, 086003 (2013); Richard A. Davison, Blaise Goutéraux, Sean A. Hartnoll, Incoherent transport in clean quantum critical metals, arXiv:1507.07137.

Mike Blake, Momentum relaxation from the fluid/gravity correspondence, JHEP09(2015)010; Universal Charge Diffusion and the Butterfly Effect, arXiv:1603.08510; Richard A. Davison, Blaise Goutéraux, Dissecting holographic conductivities, JHEP 1509 (2015) 090; Momentum dissipation and effective theories of coherent and incoherent transport, JHEP 1501 (2015) 039.

A. Lucas, S. Sachdev, K. Schalm,Scale-invariant hyperscaling-violating holographic theories and the resistivity of strange metals with random-field disorder, arXiv:1401.7993; S. A. Hartnoll, Raghu Mahajan, Matthias Punk, Subir Sachdev, Transport near the Ising-nematic quantum critical point of metals in two dimensions, arXiv:1401.7012.

Zaanen, Jan (2004), Superconductivity: Why the temperature is high, Nature (London) 430 (6999): 512; Richard A. Davison, Koenraad Schalm, Jan Zaanen,Holographic duality and the resistivity of strange metals, Phys. Rev. B 89, 245116 (2014).

D.V. Khveshchenko, Searching for non-Fermi liquids under holographic light, Phys. Rev. B86, 115115 (2012).

Sean A. Hartnoll, Pavel K. Kovtun, Markus Mueller, Subir Sachdev, Theory of the Nernst effect near quantum phase transitions in condensed matter, and in dyonic black holes, Phys.Rev.B76:144502,2007; Keun-Young Kim, Kyung Ki Kim, Yunseok Seo, Sang-Jin Sin, Thermoelectric Conductivities at Finite Magnetic Field and the Nernst Effect, arXiv:1502.05380; Karl Landsteiner, Yan Liu, Yas-Wen Sun, Negative magnetoresistivity in chiral fluids and
holography, [arXiv:1410.6399] Mike Blake, Aristomenis Donos, Naladur Kohls, Magnetic response from holography, JHEP08(2015)124; Andrea Moretti, Alessandro Braggio, Nicola Maggiore, Nicola Domenico Magnoli, Daniele Musso, Analytic DC thermo-electric conductivities in holography with massive gravitons, Phys. Rev. D 91 (2015), 025002; Andrea Moretti, Daniele Musso, Magneto-transport from momentum dissipating holography, [arXiv:1502.02631]

Peter Jung, Achim Rosch, Lower bounds for the conductivities of correlated quantum systems, Phys. Rev. B 75, 245104 (2007); Andrew Lucas, Subir Sachdev, Conductivity of weakly disordered strange metals: from conformal to hyperscaling-violating regimes, Nuclear Physics B892, 239 (2015); Andrew Lucas, Conductivity of a strange metal: from holography to memory functions, JHEP 03, 071 (2015); Andrew Lucas, Subir Sachdev, Memory matrix theory of magnetotransport in strange metals, Phys. Rev. B 91, 195122 (2015); Andrew Lucas, Hydrodynamic transport in strongly coupled disordered quantum field theories, [arXiv:1506.02062]

S. Grozdanov, A. Lucas, S. Sachdev, and K. Schalm. —it Absence of disorder-driven metal-insulator transitions in simple holographic models, Physical Review Letters 115, 221601 (2015); [arXiv:1507.00003] S. Grozdanov, A. Lucas, and K. Schalm. Incoherent thermal transport from dirty black holes, [arXiv:1511.05970] Tatsuhiko N. Ikeda, Andrew Lucas, Yuichiro Nakai. Conductivity bounds in probe brane models, [arXiv:1601.07882]

X-H. Ge, S-J. Sin, and S-F. Wu. Lower bound of electrical conductivity from holography, [arXiv:1512.01917]

Matteo Baggioli, Oriol Pujolas, On holographic disorder-driven metal-insulator transitions, [arXiv:1601.07897] Blaise Goutéraux, Elias Kiritis, Wei-Jia Li. Effective holographic theories of momentum relaxation and violation of conductivity bound, [arXiv:1602.01067]

E. Abrahams, S. V. Kravchenko, M. P. Sarachik, Metallic behavior and related phenomena in two dimensions, Rev. Mod. Phys. 73, 251 (2001); S. V. Kravchenko, M. P. Sarachik, Metal-insulator transition in two-dimensional electron systems Rep. Prog. Phys. 67, 1 (2004); A metal-insulator transition in 2D: Established facts and open questions, [arXiv:0005.2968]

Spivak, B.; Kravchenko, S. V.; Kivelson, S. A.; Gao, X. A. P. Colloquium: Transport in strongly correlated two-dimensional electron fluids, 2010RvNP...82.1743S.

Kedar Damle, Subir Sachdev, Non-zero temperature transport near quantum critical points, Physical Review B 56, 8714 (1997); Subir Sachdev, Nonzero temperature transport near fractional quantum Hall critical points, Physical Review B 57, 7157 (1998); Kedar Damle, Subir Sachdev, Spin dynamics and transport in gapped one-dimensional Heisenberg antiferromagnets at nonzero temperatures, Physical Review B 57, 8307 (1998).

Markus Mueller, Lars Fritz, Subir Sachdev, Quantum-critical relativistic magnetotransport in graphene, Phys. Rev. B 78, 115406 (2008); Lars Fritz, Joerg Schmalian, Markus Mueller, Subir Sachdev, Quantum critical transport in clean graphene, Physical Review B 78, 085416 (2008); Lars Fritz, Quantum-Critical transport at a semimetal-to-insulator transition on the honeycomb lattice, Phys. Rev. B 83, 035125 (2011).

B.N. Narozhny, I.V. Gornyi, M. Titov, M. Schtt, A.D. Mirlin, Hydrodynamics in graphene: Linear-response trans-
Graphene, \[ \text{arXiv:1508.00836} \]

Galil, W. J. Moll, Pallavi Kushwaha, Nabhanila Nandi, Burkhard Schmidt, Andrew P. Mackenzie, Evidence for hydrodynamic electron flow in PdCoO, \[ \text{arXiv:1509.05691} \]

Julian Sonner, Andrew G. Green, Hawking Radiation and Non-equilibrium Quantum Critical Current Noise, PhysRevLett.109.091601; Denis Bernard, Benjamin Doyon, Non-equilibrium steady-states in conformal field theory, Ann. Henri Poincaré 16 (2015) 113; A hydrodynamic approach to non-equilibrium conformal field theories, \[ \text{arXiv:1507.07474} \]

Energy flow in non-equilibrium conformal field theory, J. Phys. A: Math. Theor. 45, 362001 (2012); M.J. Bhaseen, Benjamin Doyon, Andrew Lucas, Koenraad Schalm, Far from equilibrium flow in quantum critical systems, Nature Physics 11, 509 (2015); A. Kundu and S. Kundu, Steady-state Physics, Effective Temperature Dynamics in Holography, \[ \text{arXiv:1307.6607} \]

Karrasch, Ian and Moore, Non-equilibrium thermal transport and its relation to linear response, \[ \text{arXiv:1211.2236} \]

Ioannis Bakas, Kostas Skenderis, Benjamin Withers, Self-similarity of equilibrium strongly interacting systems from holography, \[ \text{arXiv:1512.09151} \]

Andrew Lucas, Koenraad Schalm, Benjamin Doyon, M. J. Bhaseen, Shock waves, rarefaction waves and non-equilibrium steady states in quantum critical systems, \[ \text{arXiv:1512.09037} \]

D.V.Khveshchenko, Simulating holographic correspondence in flexible graphene, Euro Phys. Lett., 104, 47002 (2013).

A.Iorio, Using Weyl symmetry to make Graphene a real lab for fundamental physics, \[ \text{arXiv:1207.6920} \]

Graphene: QFT in curved spacetimes close to experiments ; A.Iorio and G.Lambiase, The Hawking-Unruh phenomenon on graphene, \[ \text{arXiv:1108.2340} \]

M.Cvetic and G.W.Gibbons, Graphene and the Zermelo Optical Metric of the BTZ Black Hole, \[ \text{arXiv:1202.2938} \]

Pisin Chen, H. C. Rosu, Note on Hawking-Unruh effects in graphene, \[ \text{arXiv:1205.4039} \]

Antonio Capolupo, Giuseppe Vitiello, Probing Hawking and Unruh effects and quantum field theory in curved space by geometric invariants, \[ \text{arXiv:1311.2892} \]

M.A.Zubkov, G.E.Volovik, Emergent gravity in graphene, \[ \text{arXiv:1308.2249} \]

M. A. H. Vozmediano, M. I. Katsnelson, F. Guinea, Gauge fields in graphene, Physics Reports 496, 109 (2010); Fernando de Juan, Alberto Cortijo, Maria A. H. Vozmediano Dislocations and torsion in graphene and related systems, Nucl. Phys. B 828, 625 (2010); Fernando de Juan, Alberto Cortijo, Maria A. H. Vozmediano, Andres Cano, Aharonov-Bohm interferences from local deformations in graphene, \[ \text{arXiv:1105.0599} \]

M. I. Katsnelson, F. Guinea, M. A. H. Vozmediano, Gauge fields at the surface of topological insulators, \[ \text{arXiv:1105.6132} \]

A.L.Kitt, Vitor M. Pereira, Anna K. Swan, Bennett B. Goldberg, \nu, Phys. Rev. B 85, 115432 (2012); F. de Juan, M.Sturla, and M.A.H.Vozmediano, Phys.Rev.Lett.108, 227205 (2012); F. de Juan, J.L.Manes, and M.A.H.Vozmediano, Phys.Rev.B 87, 165131 (2013); M.Masir, D.Moldovan, and F.M.Peeters, \[ \text{arXiv:1304.0629} \]

M. Oliva-Leyva and G. A. Naumis, \[ \text{arXiv:1304.6682} \]

B. Reznik, Origin of the Thermal Radiation in a Solid-State Analog of a Black-Hole, \[ \text{arXiv:gr-qc/9703076} \]

Wanli Lu, JunFeng Jin, Huanyang Chen, Zhifang Lin, A simple design of an artificial electromagnetic black hole, 1003.5727; Miao Li, Rong-Xin Miao, Yi Pang , Casimir Energy, Holographic Dark Energy and Electromagnetic Metamaterial Mimicking de Sitter, 0910.3375; Miao Li, Yi Pang, Holographic de Sitter Universe, 1105.003; Rong-Xin Miao, Rui Zheng, Miao Li, Metamaterials Mimicking Dynamic Spacetime, D-brane and Noncommutativity in String Theory, 1005.5585; Tom G. Mackay, Akhlesh Lakhtakia , Towards a realization of Schwarzschild-(anti-)de Sitter spacetime as a particular metamaterial, 1102.1708; Towards a metamaterial simulation of a spinning cosmic string, 0911.4163; Tian-Ming Zhao, Rong-Xin Miao, Huge Casimir effect at finite temperature in electromagnetic Rindler space, 1110.1919; Dieter Brill, Black Holes and Wormholes in 2+1 Dimensions, \[ \text{arXiv:gr-qc/9904083} \]

Igor I. Smolyaninov, Yu-Ju Hung, Ehren Hvang, Experimental Modeling of Cosmological Inflation with Metamaterials, \[ \text{arXiv:1111.3300} \]

Igor I. Smolyaninov, Ehren Hvang, Evgenii Narimanov, Hyperbolic metamaterial interfaces: Hawking radiation from Rindler horizons and the "end of time", 1107.4053; Igor I. Smolyaninov, Evgenii E. Narimanov, Metric Signature Transitions in Optical Metamaterials, Phys.Rev.Lett.105:067402,2010; Igor I. Smolyaninov , Metamaterial "Multiverse", J.Optics 13:024004,2011; Metamaterial model of tachyonic dark energy, \[ \text{arXiv:1310.8155} \]

Analogue gravity in hyperbolic metamaterials, Phys. Rev. A 88, 033843 (2013); Extra-dimensional metamaterials: simple models of inflation and metric signature transitions, \[ \text{arXiv:1301.6060} \]

Modeling of causality with metamaterials, Journal of Optics 15, 025101 (2013); Extra-dimensional metamaterials: simple models of inflation and metric signature transitions, \[ \text{arXiv:1301.6060} \]

Analogue gravity in hyperbolic metamaterials, \[ \text{arXiv:1307.8431} \]

Igor I. Smolyaninov, Holographic duality in nonlinear hyperbolic metamaterials, 1401.3242.

D. V. Khveshchenko, Contrasting string holography to its optical namesake, Euro. Phys. Lett. 109, 61001 (2015).

M. Novello, M. Visser, and G. E. Volovik (eds.), Artificial black holes, World Scientific, Singapore (2002); R. Schutzhold and W.G. Unruh, (eds.), Quantum Analogues: From Phase Transitions to Black Holes and Cosmology, Springer Lecture Notes in Physics 718 (2007); C. Barcelo, S. Liberati and M. Visser, Analogue gravity, Living Rev. Rel. 8 (2005) 12, [Updated as Living Rev. Rel. 14 (2011) 3]; Survey of analogue spacetimes, Matt Visser, Lecture Notes in Physics Volume 870 (2013) 31-50.