Multiple Killing horizons and near horizon geometries

Marc Mars\(^1\), Tim-Torben Paetz\(^2\)\(\ast\) and José M M Senovilla\(^3\)

\(^1\) Instituto de Física Fundamental y Matemáticas, Universidad de Salamanca, Plaza de la Merced s/n, 37008 Salamanca, Spain
\(^2\) Gravitational Physics, University of Vienna, Boltzmanngasse 5, 1090 Vienna, Austria
\(^3\) Departamento de Física Teórica e Historia de la Ciencia, Universidad del País Vasco UPV/EHU, Apartado 644, 48080 Bilbao, Spain

E-mail: marc@usal.es (Marc Mars), tim-torben.paetz@univie.ac.at (Tim-Torben Paetz) and josemm.senovilla@ehu.es (José M M Senovilla)

Received 19 July 2018, revised 13 October 2018
Accepted for publication 24 October 2018
Published 19 November 2018

Abstract
Near horizon geometries with multiply degenerate Killing horizons \(\mathcal{H}\) are considered, and their degenerate Killing vector fields identified. We prove that they all arise from hypersurface-orthogonal Killing vectors of any cut of \(\mathcal{H}\) with the inherited metric—cuts are spacelike co-dimension two submanifolds contained in \(\mathcal{H}\). For each of these Killing vectors on a given cut, there are three different possibilities for the near horizon metric which are presented explicitly. The structure of the metric for near horizon geometries with multiple Killing horizons of order \(m \geq 3\) is thereby completely determined, and in particular we prove that the cuts on \(\mathcal{H}\) must be warped products with maximally symmetric fibers (ergo of constant curvature). The question whether multiple degenerate Killing horizons may lead to inequivalent near horizon geometries by using different degenerate Killings is addressed, and answered on the negative: all near horizon geometries built from a given multiple degenerate Killing horizon (using different degenerate Killings) are locally isometric.

Keywords: near horizon geometry, multiple Killing horizon, Killing vector
1. Introduction

In a recent paper [9] we have introduced the notion of a multiple Killing horizon (MKH) and have initiated a systematic study of its properties. In essence, multiple Killing horizons are null hypersurfaces which are simultaneously Killing horizons of two or more Killing vectors. The precise definition, recalled in the next section, is slightly more involved as one needs to take care of the fact that different generators can have different fixed points. Several general properties of multiple Killing horizons were obtained in [9]. In particular, one can attach a natural number $m \geq 2$ to each MKH, called order, which counts the number of its linearly independent Killing generators. The order of any MKH cannot be larger than the dimension of the spacetime where it lies, and examples exist for MKHs of any allowed order. Another important property of MKHs is that the surface gravity of each generator is constant, and that at most one (in an appropriate sense) can be different from zero. MKHs are called fully degenerate or non-fully degenerate depending on whether all its surface gravities vanish or not. The order of a fully degenerate MKH can be any number strictly smaller than the spacetime dimension, and again examples of any order exist [9].

From a physical point of view the existence of MKH might be relevant in view of Hawking radiation: the surface gravity associated to the horizon of a black hole can be interpreted as its temperature. In the case of a non-fully degenerate MKH one would be able to ascribe two temperatures to the black hole, whose meaning needs to be understood.

An example of paramount importance of MKH is the Killing horizon of a near horizon geometry (NHG) spacetime. These spacetimes are obtained by infinite zoom of the geometry around any Killing horizon with vanishing surface gravity. Thus, NHG describe the ‘focused’ geometry of degenerate Killing horizons. They turn out to be very interesting objects both from a geometric and from a physical perspective and have been extensively studied (see [6] and references therein). One of its general properties is the so-called ‘enhancement of symmetry’: in addition to the Killing vector associated to the original degenerate Killing horizon (which is preserved by the ‘zoom’ limit), any NHG always admits a second Killing vector [6, 8, 11] with respect to which the horizon is non-degenerate. This immediately turns all horizons associated to NHG spacetimes into multiple Killing horizons.

Now, what happens if the original horizon (before taking the near horizon limit) is a multiple Killing horizon itself? We have proved in [9] that any of its degenerate Killing generators survives to the limit. Hence, if the original multiple Killing horizon was fully-degenerate of order $m$, it follows that the near horizon limit has a non-fully degenerate Killing horizon of order $m + 1$. Correspondingly, if the original horizon was non-fully degenerate of order at least three, then the near horizon limit has at least the same order.

This fact raises the following natural question, posed in [9]. Since the original Killing horizon is degenerate with respect to more than one linearly independent Killing vector and any one of them can be used to perform the near horizon limit, are all the NHG one obtains by this process (locally) isometric to each other or not? If the answer were no, i.e. if the near horizon geometry depended on the choice of degenerate Killing generator, one could iterate the process and generate a potentially very large class of near horizon geometries starting from a single MKH. If, on the other hand, the answer is yes (the limit is independent of the choice of generator) one concludes that any degenerate Killing horizon (multiple or not) has a well-defined and unique near horizon geometry attached to it. One of the main objectives of this paper is to answer this question and prove that any degenerate Killing horizon (multiple or not) defines a unique near horizon geometry.

We emphasize that the uniqueness statement we prove here is completely unrelated to previous results on near horizon geometries, in particular to all uniqueness theorems of NHG
(or even of extremal isolated horizons) obtained under completely different hypotheses, which include field equations and/or extra symmetries, see e.g. [4, 5, 7]. Here we consider a priori different limiting procedures giving rise to a priori different NHG spacetimes and show that, a posteriori, all of them are locally isometric to each other.

The strategy we follow is to find the explicit coordinate change (i.e. local isometry) that transforms one near horizon geometry limit into another. Despite its apparent simplicity, the problem turns out to be substantially more involved than one could have expected. The key to success is the ability to obtain very explicit and fully general information on near horizon geometries admitting multiple Killing horizons of order at least three. Finding these results is our second main achievement and opens up the possibility of, eventually, finding a complete classification of all near horizon geometries admitting multiple Killing horizons with at least two degenerate generators. To be more specific, recall that near horizon geometries of dimension $n + 1$ are determined by a Riemannian manifold $(S, \gamma)$ of dimension $n - 1$ endowed with a one-form $s$ and a scalar $h$. We show that whenever the horizon of the near horizon geometry is multiple of order $m$ at least three, the geometry of $S$ is (locally) a warped product with fibers of dimension $m - 2$ of constant sectional curvature. In addition, there are $m - 2$ linearly independent hypersurface-orthogonal Killing vectors of $(S, \gamma)$ tangent to the fibers which, together with the trivial zero vector field, are in one-to-one correspondence with the degenerate Killing generators of the MKH. The one-form $s$ and scalar $h$ are completely and explicitly determined in terms of the geometric properties of any one of the non-trivial hypersurface-orthogonal Killing vectors of $(S, \gamma)$.

The plan of the paper is as follows. In section 2 we summarize the main results of [9] that are needed in this work. In section 3 we recall the limit process that leads to the near horizon geometry and discuss in which sense it is determined by the data $(S, \gamma, s, h)$. We also recall the key result in [9] showing that the near horizon limit does not reduce the order of multiple Killing horizons. We then find the equations that need to be satisfied by any degenerate Killing generator of the horizon. They involve the near horizon data $(S, \gamma, s, h)$ and the proportionality function $f$ on $S$ between generators. The full set of equations include the so-called master equation that must necessarily be satisfied by all MKH [9]. In the near horizon geometry case this is supplemented by two more equations involving $h$. We find the general solution of the full system and show that the solutions are related to hypersurface orthogonal Killing vectors in $(S, \gamma)$ (theorem 3 and lemma 1). In section 4 we exploit these results to prove that $(S, \gamma)$ is a warped product with fibers of constant curvature (theorem 5) which in particular means that if the MKH is of order $m \geq 3$ then $(S, \gamma)$ admits at least $(m - 1)(m - 2)/2$ linearly independent Killing vectors. The value of the constant curvature of those fibers is explicitly determined (theorem 6). Finally, we devote section 5 to proving that given any near horizon geometry spacetime with a multiple Killing horizon of order at least three, any choice of degenerate generator leads, via the standard near horizon limit, to a spacetime which is locally isometric to the original one (theorem 7). As already mentioned, we show this by finding explicitly the coordinate change that transforms one metric into another. We first find necessary geometric conditions that must be satisfied by the coordinate change, and which ultimately determines it in an essentially unique way. We then prove that this coordinate change indeed defines a (local) isometry between the two spacetimes. This step requires exploiting the explicit information about near horizon geometries with MKH obtained in previous sections. We finish the paper by stating and proving our main theorem, namely that to any multiple Killing horizon one can attach a unique near horizon geometry (theorem 8).
1.1. Notation

\((M, g)\) is a spacetime, that is, a connected, oriented and time-oriented \((n + 1)\)-dimensional Lorentzian manifold with metric \(g\) of signature \((-++, +, \ldots)\). All submanifolds will be without boundary and the topological closure of a set \(A\) is denoted by \(\overline{A}\). Given a vector (field) \(v\) in \(TM\), \(\nu\) denotes the metrically related one-form, and vice versa. Moreover, \(\mathcal{X}(N)\) denotes the set of smooth vector fields on a differentiable manifold \(N\). We use index-free and index notation. Lowercase Greek letters \(\alpha, \beta, \ldots\) are spacetime indices and run from 0 to \(n\). Capital Latin indices \(A, B, \ldots\) are co-dimension-2 submanifold indices running from 2 to \(n\). Small Latin indices \(i, j, \ldots\) will enumerate either (1) the different Killing vectors of multiple Killing horizons then taking values in \(\{1, \ldots, m\}\), where \(m \leq n + 1\), or (2) the hypersurface-orthogonal Killing vectors in \((S, \gamma)\) generating the maximally symmetric fibers, in which case they take values in \(\{1, \ldots, p\}\) with \(p \leq n - 1\).

2. Basics on multiple Killing horizons

We start by recalling the notions of Killing horizon and bifurcate Killing horizon of a spacetime \((M, g)\) of dimension at least two.

**Definition 1 (Killing horizon of a Killing \(\xi\)).** A Killing horizon of a Killing vector \(\xi\) of \((M, g)\) is a smooth embedded null hypersurface \(H_\xi\) such that \(\xi|_{H_\xi}\) is null, nowhere zero and tangent to \(H_\xi\). In case \(H_\xi\) has more than one connected component we also demand that the interior of its closure is a smooth connected hypersurface. The reason to allow for multiple connected components was discussed in [9]. Any Killing horizon \(H_\xi\) possesses the notion of surface gravity \(\kappa_\xi\) defined by

\[
\nabla_\xi \xi = \kappa_\xi \xi \quad \text{or equivalently} \quad \text{grad}(g(\xi, \xi)) = -2\kappa_\xi \xi. \quad (1)
\]

If \(\kappa_\xi\) vanishes, then \(H_\xi\) is called degenerate.

**Definition 2 (Bifurcate Killing horizon).** Let \(S\) be a connected co-dimension two spacelike submanifold of fixed points of a Killing vector \(\xi\). The set of points along all null geodesics orthogonal to \(S\) comprises a bifurcate Killing horizon [1, 3, 13] with respect to \(\xi\).

A bifurcate Killing horizon is composed of five pieces: two connected Killing horizons \(H_1^+\) and \(H_2^+\) to the future of \(S\)—not including \(S\)—, two more connected Killing horizons \(H_1^-\) and \(H_2^-\) to the past of \(S\), and \(S\) itself. Notice that \(H_1^+ \cup H_1^-\) is a Killing horizon according to the previous definition, and its closure (adding \(S\)) is a connected null hypersurface; and similarly for \(H_2^+ \cup H_2^-\).

In [9] we introduced the following class of Killing horizons.

**Definition 3 (Multiple Killing horizon (MKH)).** A multiple Killing horizon in \((M, g)\) is an embedded null hypersurface \(H\) such that there exist Killing horizons \(H_\xi\) associated to linearly independent Killing vectors \(\xi_i, i \in \{1, \ldots, m\}\) with \(m \geq 2\), with the property

\[
\overline{H} = \overline{H_{\xi_1}} = \cdots = \overline{H_{\xi_m}}.
\]

As proven in [9], the set of all Killing vectors in \((M, g)\) with a common multiple Killing horizon constitute a Lie sub-algebra, denoted by \(\mathcal{A}_{H}\), of the Killing Lie algebra. Its dimension
$m := \dim A_H \geq 2$ is called the order of the MKH. Sometimes we write ‘double, triple,’ etc, MKHs for $m = 2, 3$ etc.

The following fundamental theorem was proven in [9].

**Theorem 1.** $A_H$ contains an Abelian sub-algebra $A_{\text{deg}}^H$ of dimension at least $m - 1$ whose elements all have vanishing surface gravities. If $A_{\text{deg}}^H \neq A_H$, the remaining independent Killing vector (say $\xi$) in $A_H \setminus A_{\text{deg}}^H$ has a constant surface gravity $\kappa_\xi \neq 0$ and satisfies

$$[\xi, \eta] = -\kappa_\xi \eta, \forall \eta \in A_{\text{deg}}^H.$$  

(2)

Thus, there are two essentially inequivalent possibilities: fully degenerate MKH if $A_H = A_{\text{deg}}^H$, in which case its Lie algebra is Abelian and all surface gravities vanish; and non-fully degenerate MKHs, with an essentially unique non-zero surface gravity.

In general, the maximum possible dimension of $A_{\text{deg}}^H$ is $n = \dim(M) - 1$. Therefore, the maximum possible order of a MKH $H$ is $m = n$ for fully degenerate $H$, and $m = n + 1$ for non-fully degenerate $H$.

**3. Degenerate Killing vectors on near horizon geometries**

The near horizon geometry of a degenerate Killing horizon $H_\eta$ is usually defined as follows [6]: nearby $H_\eta$, with degenerate Killing vector $\eta$, local Gaussian null coordinates $(v, u, x^A)$ can be chosen such that the metric reads

$$g = 2dv \left( du + 2u \hat{s}_A dx^A + \frac{1}{2} u^2 \hat{h} dv \right) + \hat{\gamma}_{AB} dx^A dx^B$$

where $\hat{h}, \hat{s}_A$, and $\hat{\gamma}_{AB}$ are independent of $v$, the degenerate Killing reads $\eta = \partial_u$ and the degenerate Killing horizon has been placed at $H_\eta = \{ u = 0 \}$. Replacing $v \rightarrow v/\lambda$ and $u \rightarrow u\lambda$ here and taking the limit $\lambda \rightarrow 0$ one is led to the metric of its ‘near-horizon’ geometry

$$g_{\text{NHG}} = 2dv \left( du + 2u s_A dx^A + \frac{1}{2} u^2 h dv \right) + \gamma_{AB} dx^A dx^B$$

(3)

where $h = \hat{h}|_{u=0}$, $s_A = \hat{s}_A|_{u=0}$ and $\gamma_{AB} = \hat{\gamma}_{AB}|_{u=0}$. This is the ‘focused’ local geometry near $H_\eta$.

Note that a line element of the form (3) belongs to the Kundt class. As such this family of metrics has been studied and classified in the literature, see [2, 12] and the references given there.

**Remark 1.** The NHG of a degenerate Killing horizon $H_\eta$ can be intrinsically and geometrically defined as follows: pick up any co-dimension two submanifold $S \subset H_\eta$ (we call these cuts). Then

- $\gamma$ is the first fundamental form on $S$
- $s$ is the torsion one-form on $S$, defined by $s(V) := \ell(\nabla V \eta)$ for any $V \in \mathfrak{X}(S)$, where $\ell$ is uniquely determined by the conditions $g(\ell, V) = 0 \forall V \in \mathfrak{X}(S)$, $g(\ell, \ell) = 0$ and $g(\ell, \eta) = -1$.
- $h = 2\gamma^2(s, s) - \text{div} s + \frac{1}{2} R|_S - \frac{1}{2} \text{tr}_s \text{Ric}|_S$, \hfill
where $\gamma^\sharp$ is the contravariant metric associated to $\gamma$, div is the divergence on $(S, \gamma)$, $R$ is the scalar curvature and Ric the Ricci tensor of $(M, g)$, both pull-backed to $S$.

One can check that the scalar curvature $R$ of the metric $g$ coincides with the scalar curvature of the metric $g_{NHG}$ at every cut $S$, and similarly for the term $\text{tr}_\gamma \text{Ric} |_S$.

The construction of a near horizon geometry relies on Gaussian null coordinates associated to the degenerate Killing vector $\eta$. These coordinates cannot cover domains where $\eta$ has fixed points. Since we assume all spacetimes (in particular the NHG spacetime) to be connected, the Gaussian null coordinates leading to the NHG can cover at most one connected component of $\mathcal{H}_\eta$ and this is a proper subset of $\overline{\mathcal{H}}_\eta$ whenever the latter has fixed points of $\eta$. When the limit is performed to compute the NHG only this portion of $\mathcal{H}_\eta$ is considered whence the degenerate Killing vector $\eta = \partial_v$ of the NHG never has fixed points. The NHG does not see the rest of the original Killing horizon. The intrinsic definition in remark 1 has exactly the same limitation because the normalization condition $g(\ell, \eta) = -1$ cannot be fulfilled in domains where $\eta$ has zeros. In this paper we want to understand the NHG limit of multiple Killing horizons, so we must restrict to domains of the horizon which are connected and contain no fixed points of any of the degenerate Killing generators under consideration. Our results are valid only on those domains. To understand the global picture one would need to devise a way of defining NHG that allows for fixed points of the degenerate Killing. This is an interesting problem that deserves consideration but it is beyond the scope of this paper.

Any near-horizon geometry in the above sense possesses a non-fully degenerate MKH $\mathcal{H}_{NHG}$ because

1. The original degenerate Killing $\eta$ leads, after the limit, to a Killing vector which is also degenerate, and, by definition of the NHG, with $\mathcal{H}_{NHG} = \mathcal{H}_\eta$. This follows easily from the explicit local expression of the metric (3) as the Killing vector $\eta = \partial_v$ satisfies $g_{NHG}(\eta, \eta) = h^2$ so that $\eta$ is null on $\{u = 0\}$ and $\text{grad}(g(\eta, \eta)) = 0$, whence the surface gravity $\kappa_\eta$ vanishes, see (1).

2. The metric (3) always has another Killing vector given by [6, 8, 11]

$$\xi = v\partial_v - u\partial_u$$

which is null on, and tangent to, $\mathcal{H}_\eta = \{u = 0\}$ except at its set of fixed points $S_\xi \supset \{u = v = 0\}$. Thus, $\mathcal{H}_\xi = \mathcal{H}_\eta \setminus S_\xi$ is a Killing horizon for $\xi$ with several connected components but such that $\overline{\mathcal{H}}_\xi = \mathcal{H}_\eta = \mathcal{H}_{NHG}$, and therefore $\mathcal{H}_{NHG}$ is a MKH of order $m \geq 2$.

The commutator is

$$[\xi, \eta] = -\eta$$

hence theorem 1 implies that $\mathcal{H}_{NHG}$ is non-fully degenerate with $\kappa_\xi = 1$. Actually, any cut $S_{v_0} := \{u = 0, v = v_0\}$ of $\mathcal{H}_\eta$ is the bifurcation surface of a bifurcate Killing horizon with bifurcation Killing vector $\xi = v_0\eta$.

In [9] we established the following theorem.

**Theorem 2.** Let $\mathcal{H}$ be a multiple Killing horizon of order $m$ and $(M_{NHG}, g_{NHG})$ be the near-horizon geometry of a degenerate Killing vector $\eta \in \mathcal{A}_{\text{deg}}^\text{NHG}$. Then

(i) If $\mathcal{H}$ is fully degenerate, $(M_{NHG}, g_{NHG})$ admits a multiple Killing horizon $\mathcal{H}_{NHG}$ of order at least $m + 1$. 


(ii) If \( \mathcal{H} \) is non-fully degenerate and \( m \geq 3 \), then \( (\mathcal{M}_{\text{NHG}}, g_{\text{NHG}}) \) has a multiple Killing horizon \( \mathcal{H}_{\text{NHG}} \) of order at least \( m \).

Of course, the theorem also holds if \( \mathcal{H} \) is non-fully degenerate and of order \( m = 2 \), but then the result is trivial, as the MKH \( \mathcal{H}_{\text{NHG}} \) of all NHGs have \( m = 2 \) at least.

As briefly discussed in [9], this theorem raises the natural question of whether or not the NHG spacetime \( (\mathcal{M}_{\text{NHG}}, g_{\text{NHG}}) \) arising from a multiple Killing horizon \( \mathcal{H} \) is independent of the choice of \( \eta \in \mathcal{A}_{\mathcal{H}}^{\text{deg}} \). To address this problem, in the next subsection we identify the NHGs that possess a MKH of order \( m \geq 3 \) as well as their corresponding degenerate Killing vectors.

We have already mentioned above that the NHG taken w.r.t. a degenerate Killing vector \( \eta^{(1)} \) only takes (in general) a proper subset of the MKH \( \mathcal{H} \) into account. A second degenerate Killing vector \( \eta^{(2)} \) may have a different fixed point set on \( \mathcal{H} \) so that the NHGs computed from \( \eta^{(1)} \) and \( \eta^{(2)} \) work in general in different subsets of the MKH of the original spacetime. For this reason when analyzing (local) isometry of the NHGs we will consider connected portions of the MKH where both degenerate Killings have no fixed points.

### 3.1. Degenerate Killing vectors of MKH in NHGs

We start with a metric of type (3), which holds around a connected component of \( \mathcal{H}_{\eta} \), and derive the equations for the existence of degenerate Killing vector fields other than \( \eta = \partial_{\nu} \) there.

**Proposition 1.** Any Killing vector \( \zeta \) of the metric (3) which has (the appropriate dense subset of) \( \mathcal{H}_{\eta} = \{ u = 0 \} \) as degenerate Killing horizon must take the form

\[
\zeta = f \partial_{\nu} + \frac{u^2}{2} \Delta f \partial_{u} - u \text{grad} f
\]

where \( \Delta \) and \text{grad} are the Laplacian and gradient on any cut \( S_0 \subset \mathcal{H}_{\eta} \), and the function \( f \) satisfies the following relations:

\[
D_A D_B f = s_A D_B f + s_B D_A f,
\]

\[
D_A h D^A f = 2 hs^A D_A f,
\]

\[
h D_A f = 2 D^B f (D_B s_A - D_A s_B) + D_A (s^B D_B f) - 2 s_A s^B D_B f,
\]

where \( D_A \) is the covariant derivative on \( (S_0, \gamma) \).

Conversely, for any function \( f \) which solves (5)–(7) the vector field (4) belongs to \( \mathcal{A}_{\mathcal{H}_{\eta}}^{\text{deg}} \).

**Remark 2.** Equation (5) was found in full generality (for arbitrary MKHs) in [9] and called the master equation.

**Remark 3.** The degenerate Killing vector \( \zeta \) given in (4) has fixed points on \( \mathcal{H}_{\eta} \) if and only if the function \( f \) has zeros. As described above the NHG computed from \( \zeta \) is only defined where \( \zeta \) has no fixed points, whence we will be mainly interested in the subset \( \mathcal{H}_{\eta, \zeta} := \{ p \in \mathcal{H}_{\eta} : f(p) \neq 0 \} \). This will be relevant in section 5.
**Proof.** Let $\zeta \in A^\text{det}_{H^\eta}$, i.e. any degenerate Killing generator of $H^\eta$ in the metric (3), and set
\[
\zeta = f \partial_v + q \partial_u + \zeta^B \partial_B
\]
there. Then we know that
\[
q|_u = \zeta^B|_u = 0. \quad (8)
\]
It follows from theorem 1 that $[\eta, \zeta] = 0$ so that
\[
\partial_v f = \partial_v q = \partial_u \zeta^B = 0.
\]
Consider the Killing equations
\[
(\xi g)_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu \xi^\rho + g_{\rho\nu} \partial_\mu \xi^\rho = 0.
\]
Letting $\mu = u$ (with some abuse of notation), these relations for $\nu = u, v, A$ become respectively,
\[
\begin{align*}
\partial_v f &= 0 \quad \implies \quad f = f(x^A), \quad (9) \\
\partial_v q + 2u s_A \partial_u \zeta^A &= 0, \\
\partial_v f + \gamma_{AB} \partial_u \zeta^B &= 0. \\
\end{align*}
\]
Similarly, letting $\mu = v$, the equations for $\nu = v, A$ become respectively
\[
2u q h + u^2 \zeta^B \partial_u h = 0, \\
2q s_A + 2u \xi^B \partial_u s_A + \partial_v q + u^2 h \partial_A f + 2u s_B \partial_A \zeta^B = 0.
\]
Finally, for $\mu = A$ and $\nu = B$ we get
\[
\zeta^C \partial_C \gamma_{AB} + \gamma_{AC} \partial_B \zeta^C + \gamma_{CB} \partial_A \zeta^C + 2u s_A \partial_B f + 2u s_B \partial_A f = 0. \quad (14)
\]
Given (9) the function $f$ can be seen as a function on the cut $S_0$, and then $D_A f = \partial_v f$. Taking this into account together with (8), the solution of (11) reads
\[
\zeta^A = -u \gamma^{AB} D_B f = -u (\text{grad} f)^A
\]
and then (10) with (8) provides
\[
q = u^{-2} \gamma^A D_A f 
\]
while (12) becomes (6) and (13) becomes (7). The remaining equation (14) can now be written as
\[
-(L_{\text{grad}} f)_{AB} + 2s_A D_B f + 2s_B D_A f = 0
\]
which leads directly to (5). To finish the proof it is enough to note that the trace of (5) gives
\[
2s_B D_B f = \Delta f. \quad (16)
\]
For the converse one simply checks that all components (9)–(14) of the Killing equation are satisfied assuming that (5)–(7) hold.

Observe that the solution \( f = \text{const.} \) provides the original Killing \( \eta = \partial_v \). We can now derive the main result in this subsection.

**Theorem 3.** Let \( (M_{\text{NHG}}, g_{\text{NHG}}) \) be a near horizon geometry with metric (3). Then, the vector field (4), where \( f \) is a smooth non-constant scalar in \( S_0 \), is a degenerate Killing generator of the chosen connected component \( \mathcal{H}_\eta \) if and only if the following two conditions hold:

(i) The differential of \( f \) vanishes nowhere and the metric \( \gamma \) on a cut \( S_0 \subset \{ u = 0 \} = \mathcal{H}_\eta \) (and therefore on any such cut) admits a hypersurface-orthogonal Killing vector field

\[
\varsigma = \frac{Q(f)}{N} df, 
\]

where \( N := \gamma^2(df, df) \) is the square norm of \( df \) and \( Q(f) \) is a not identically zero solution of the system of ODEs

\[
\frac{dQ}{df} = Q(f)P(f), \quad \frac{d^2}{df^2} \left( \frac{dP}{df} + P^2 \right) + P \left( \frac{dP}{df} + P^2 \right) = 0.
\]

(ii) The torsion one-form \( s \) and metric coefficient \( h \) take the form

\[
s = \frac{1}{2} \left( \frac{dN}{N} - P df \right), \quad h = \frac{1}{2} N \left( \frac{dP}{df} + P^2 \right).
\]

In particular, \( s \) is closed.

**Proof.** Assume that the vector field (4) is a degenerate Killing vector of \( \{ u = 0 \} \) in \( (M_{\text{NHG}}, g_{\text{NHG}}) \). From proposition 1, equations (5)–(7) are satisfied. We first observe that (5) implies that if \( D_A f \) vanishes at a point then it vanishes everywhere. Since by assumption \( f \) is non-constant we conclude that \( df \neq 0 \) everywhere on \( S_0 \). Contracting (5) with \( D_A f \) one then derives

\[
s_A = \frac{1}{2} D_A (\ln N) - \frac{1}{2} \frac{D_A f}{N} \Delta f
\]

where we have used (16), and the definition of \( N \). Contracting here with \( D^i f \) gives

\[
D^i f D_A N = 2N \Delta f.
\]

A similar contraction of (7) using (16) provides

\[
h = \frac{1}{2} \frac{D^i f D_A (\Delta f) - (\Delta f)^2}{N}.
\]

Equation (20) can be rewritten as

\[
s = \frac{1}{2} d \ln N - \frac{1}{2} \frac{\Delta f}{N} df
\]

from where we derive
\[ ds = -\frac{1}{2} d \left( \frac{\Delta f}{N} \right) \wedge df. \]  

(24)

Contraction with \( \text{grad} f \) here leads to

\[ 2D^b f (D_b s_A - D_A s_b) = -D^b f D_b \left( \frac{\Delta f}{N} \right) D_A f + N D_A \left( \frac{\Delta f}{N} \right) \]

and introducing this into equation (7), making use of (22), (20) and (21), and after a little calculation, we get

\[ \frac{1}{N^2} \left[ D^b f D_b (\Delta f) - 2(\Delta f)^2 \right] D_A f = D_A \left( \frac{\Delta f}{N} \right). \]  

(25)

This equation has a right-hand side which is the exact differential of \( \Delta f / N \), while the left-hand side is proportional to \( df \), so

\[ \Delta f / N := P(f) \]

must be a function of \( f \). Thus (23) takes the form given in (19), and it follows from (24) that \( s \) is closed. Combining (22) with (25) and this notation we arrive at

\[ h = \frac{N}{2} \left( \frac{dP}{df} + P^2 \right) \]  

(26)

which is the second in (19). The remaining equation is (6), which on using (26) and (21) leads, after another computation, to the second in (18)

\[ \frac{d}{df} \left( \frac{dP}{df} + P^2 \right) + P \left( \frac{dP}{df} + P^2 \right) = 0. \]

Note that, introducing the function \( Q(f) \) as defined in the first of (18),

\[ s = \frac{1}{2} \frac{Q}{N} d \left( \frac{N}{Q} \right) \]  

(27)

so that the master equation becomes, after rearranging and dividing by \( N/Q \)

\[ D_A \left( \frac{Q}{N} D_A f \right) + D_B \left( \frac{Q}{N} D_B f \right) = 0. \]  

(28)

This states that (17) defines a hypersurface orthogonal Killing vector of \( (S_0, \gamma) \) and the only if part of the proof is completed.

For the converse, we assume that (i) and (ii) hold. The master equation (5) is satisfied due to (28). Concerning (6) and (7), we first observe that

\[ \varsigma(f) = \frac{Q}{N} \gamma^z (df, df) = Q \]

hence

\[ \varsigma(N) = 2 \varsigma^B (D_A D_B f) D^A f = 2D^A f D_A \varsigma(B D_B f) = 2D^A f D_A Q(f) = 2QPN \]  

(29)
where in the second equality we used that $\varsigma$ is a Killing vector. An equivalent way to state (29) is
\[ D^4 f D_A N = 2 P N^2 . \] (30)
Similarly, one has
\[ \varsigma^4 D_A f = \frac{1}{2} \left( \frac{D_A N}{N} - P(f) D_A f \right) D^4 f = \frac{1}{2} P N \] (31)
so that, using that $\varsigma$ is closed, the right-hand side of (7) becomes
\[ D_A (s^B D_B f) - 2 s_A s^B D_B f = D_A \left( \frac{1}{2} P N \right) - \left( \frac{D_A N}{N} - P D_A f \right) \frac{1}{2} P N \]
and (7) holds because $h$ is given by (19). Finally, we check (6):
\[ D^4 f D_A h - 2 h \varsigma^4 D_A f = \frac{1}{2} D^4 f D_A N \left( \frac{dP}{df} + P^2 \right) + \frac{1}{2} N D^4 f \frac{dP}{df} \left( \frac{dP}{df} + P^2 \right) D_A f \]
\[ - N \left( \frac{dP}{df} + P^2 \right) \frac{1}{2} N P = \frac{1}{2} N^2 \left[ \frac{d}{df} \left( \frac{dP}{df} + P^2 \right) + P \left( \frac{dP}{df} + P^2 \right) \right] = 0 \]
where in the second equality we used (30) and in the last one (18). In summary, equations (5)–(7) hold and, by proposition 1, the vector field (4) is a degenerate Killing generator of $\{ u = 0 \} \subset H_{\eta}$. This proves the converse, because the function $f$ is by assumption non-constant (in fact with nowhere zero gradient).

\[ \square \]

**Remark 4.** By theorem 3 $\varsigma$ is closed. Note, though, that under the conditions of this theorem, equations (27)–(31) and
\[ \Delta f = 2 \varsigma^4 D_A f = P N \]
hold true. Given that $N$ vanishes nowhere, this equation implies in particular that $P(f)$ is smooth everywhere on $S_0$. Therefore, the first equation in (19) combined with the first in (18) states that $\varsigma$ is exact on $S_0$.

It seems important to emphasize that to establish exactness of $\varsigma$ it is crucial that $\eta$ does, by definition of the NHG, not have fixed points on $H_{\text{NHG}}$. In [10] we will consider a similar setting, where, though, fixed points are possible, and in that case one can only deduce that $\varsigma$ is exact on cuts of $H_{\eta}$ as proper subsets of cuts of $H_{\text{NHG}}$. The reason for that is that the gauge (Gaussian null coordinates) becomes singular at the fixed points of $\eta$. This will be analyzed in more detail in [10].

In the next lemma, we find the most general solution of the ODE system (18).

**Lemma 1.** $Q(f)$ and $P(f)$ solve the system (18) if and only if, with $Q_0 \in \mathbb{R} \setminus \{0\}$, they belong to one of the following three exclusive cases:

(a) $P = 0$, $Q = Q_0$, and then $\varsigma = dN/(2N)$, $h = 0$.
(b) $P = 1/(f + c)$, $Q = Q_0(f + c)$ with $c \in \mathbb{R}$, and then

Class. Quantum Grav. 35 (2018) 245007
\[ s = \frac{1}{2} \left( \frac{dN}{N} - \frac{df}{f + c} \right), \quad h = 0. \]

(c) \( P = \frac{2(f + c)}{b + (f + c)^2} \) and \( Q = Q_0 [b + (f + c)^2] \) with \( b, c \in \mathbb{R} \). Then
\[ s = \frac{1}{2} \left( \frac{dN}{N} - \frac{2(f + c)}{b + (f + c)^2} df \right), \quad h = \frac{N}{b + (f + c)^2}. \tag{32} \]

**Proof.** The expressions for \( s \) and \( h \) in each case follow directly from (19) by simple substitution. First of all, \( P = 0 \) is clearly a solution of (18). In such case \( Q \) is a non-zero constant and we fall into case (a). Assume then that \( P \) is not identically zero and define \( W(f) \) by
\[ \frac{dP}{df} + P^2 := WP. \]

The second equation in (18) is then
\[ \frac{d}{df} (WP) + P^2 W = P \frac{dW}{df} + \frac{dP}{df} W + P^2 W = P \left( \frac{dW}{df} + W^2 \right) = 0. \]

It is immediate that the general solution of this ODE is either \( W = 0 \) or \( W = 1/(f + c) \) where \( c \) is a real constant. The first case corresponds to (b) because
\[ \frac{dP}{df} + P^2 = 0 \quad (\text{with} \quad P \neq 0) \quad \iff \quad P = \frac{1}{f + c} \]

and the integration of the first in (18) gives \( Q = Q_0 (f + c) \).

It remains the case \( W = (f + c)^{-1} \), which will be (c). We need to solve
\[ \frac{dP}{df} + P^2 = WP = \frac{P}{f + c}. \]

This is a Riccati equation and its solution is easily found by introducing here the first in (18) which yields
\[ \frac{d^2 Q}{df^2} = \frac{1}{(f + c) df} \quad \iff \quad \frac{dQ}{df} = 2Q_0 (f + c) \iff Q = Q_0 (b + (f + c)^2), \quad b \in \mathbb{R}. \tag{33} \]

The condition \( Q_0 \neq 0 \) is required because otherwise \( P = Q^{-1} \frac{dQ}{df} \) would vanish identically and we would fall into a previous case. From the expression (33) of \( Q \) one immediately finds \( P = 2(f + c)/(b + (f + c)^2) \) and that (32) holds. \( \square \)

**Remark 5.** As shown in remark 4, \( P(f) \) is smooth on \( S_0 \). This implies that \( f + c \) has a definite sign on \( S_0 \) in case (b) and that \( b + (f + c)^2 \) also has a definite sign in case (c). Consequently \( Q(f) \) vanishes nowhere and \( \varsigma \) has no fixed points. We can define a smooth positive function \( M \) on \( S_0 \) by
\[ N := Q^2 M. \]
This function is invariant under the flow of $\zeta$, that is, $\zeta(M) = 0$—equivalently it satisfies $D^{i}D_{i}M = 0$—as follows from

$$D^{i}D_{i}M = D^{i}D_{i}\left(\frac{N}{Q^{2}}\right) = \frac{2PN^{2}}{Q^{2}} - \frac{2N}{Q} \frac{dQ}{df} D_{i}M D^{i}f = 0.$$ 

This informs us that either $M$ is a constant or functionally independent of $f$, and thus either $dM = 0$ or $dM \wedge df \neq 0$.

**Definition 4.** Let $S_0$ be a cut of a connected component \{u = 0\} = $\mathcal{H}_f$ of the MKH of a NHG with local metric (3). By $\mathcal{A}_{S_0} \subset \mathcal{X}(S_0)$ we denote a collection of vector fields $\zeta \in \mathcal{X}(S_0)$ which are Killing vectors of $(S_0, \gamma)$ and take the form (17) with either $df = 0$ (yielding the zero vector field, which we call trivial) or with $df$ nowhere zero, $N = \gamma^{2}(df, df)$ and $(Q(f), P(f))$ solving (18) (ergo given by the explicit forms of lemma 1), such that (19) holds with fixed $s$ and $h$.

**Remark 6.** Note that $\mathcal{A}_{S_0}$ depends via (19) on $s$ and $h$. Different NHGs thus may select different Killing vectors $\zeta$, which may even produce $\mathcal{A}_{S_0}$s of different dimension (anticipating that $\mathcal{A}_{S_0}$ is a vector space, see proposition 2 below).

The following lemma shows that given any non-trivial $\zeta \in \mathcal{A}_{S_0}$ the functions $f$ and $Q(f)$ are defined uniquely up to a constant rescaling.

**Lemma 2.** Let $\zeta \in \mathcal{A}_{S_0}$ be non-trivial and let $f_{a} : S_0 \rightarrow \mathbb{R}$ and $Q_{a}(f_{a})$, $a = 1, 2$ be such that (17) and (18) hold. Then there exist constants $\alpha, \beta$ with $\alpha \neq 0$ such that $f_1 = \alpha f_2 + \beta$ and $Q_1(f_1) = \alpha Q_2(f_2)$. Furthermore, with obvious notations, $P_2(f_2) = \alpha P_1(f_1)$.

**Proof.** Let $Z_{a} := N_{a}/Q_{a}$ for $a \in \{1, 2\}$. Each $Z_{a}$ is nowhere zero and defined everywhere on $S_0$. From (27) it follows

$$s = \frac{1}{2Z_{1}}dZ_{1} = \frac{1}{2Z_{2}}dZ_{2} \implies Z_{1} = \alpha Z_{2}$$

where $\alpha$ is a non-zero constant. Since $\zeta = \frac{df_{1}}{Z_{1}} = \frac{df_{2}}{Z_{2}}$ it must be $df_{1} = \alpha df_{2}$ and hence $f_{1} = \alpha f_{2} + \beta$. Expression $\alpha Q_{2}(f_{2}) = Q_{1}(f_{1})$ is now immediate because $N_{1} = \alpha^{2}N_{2}$. The first in (18) for each $Q_{a}$ then gives $Q_{2}P_{2} = Q_{1}P_{1}$ which provides $P_{2}(f_{2}) = \alpha P_{1}(f_{1})$.

Observe that the invariance of $h$ follows easily from the second in (19).

Combining this with lemma 1 the following corollary follows by a simple computation.

**Corollary 1.** Letting $Q_{0}^{(a)}$, $c_{a}$ and $b_{a}$ be the constants defined by lemma 1 from the explicit form of $Q_{a}(f_{a})$ in each of the cases (note that the scaling transformation defined by $\alpha, \beta$ cannot change the case), the following scaling law is obtained

- Case (a): $Q_{0}^{(2)} = \frac{1}{\alpha} Q_{0}^{(1)}$,
- Case (b): $Q_{0}^{(2)} = Q_{0}^{(1)}$, $c_{2} = \frac{c_{1} + \beta}{\alpha}$,
- Case (c): $Q_{0}^{(2)} = \alpha Q_{0}^{(1)}$, $c_{2} = \frac{c_{1} + \beta}{\alpha}$, $b_{2} = \frac{b_{1}}{\alpha^{2}}$. 

13
As a consequence of these results, there exists a smooth non-zero function $Z$ such that the exact torsion one-form reads

$$ s = \frac{1}{Z} dZ $$  \hspace{1cm} (34)$$

and all non-trivial elements $\varsigma \in A_{S_0}$ can be represented by functions $f$, $Q(f)$ satisfying

$$ \gamma^2 (df, df) \frac{Q(f)}{Q(f)} = Z. $$

The function $Z$ is, itself, defined up to a constant rescaling but the point is that once $Z$ is fixed once and for all, this choice is made independently of $f$. Therefore, such a prescription freezes the scaling freedom defined by $\alpha$ in lemma 2. We shall make this choice from now on.

**Proposition 2.** $A_{S_0}$ is vector space contained in the set of hypersurface-orthogonal Killing vectors of $(S_0, \gamma)$.

**Proof.** That all elements of $A_{S_0}$ are hypersurface orthogonal Killing vectors is obvious, so we need to prove that they form a vector space. Any two non-trivial elements $\varsigma_1, \varsigma_2 \in A_{S_0}$ can be written as $\varsigma_1 = Z^{-1} df_1$, $\varsigma_2 = Z^{-1} df_2$, so that, for any $a_1, a_2 \in \mathbb{R}$, $\varsigma' := a_1 \varsigma_1 + a_2 \varsigma_2 = Z^{-1} d(a_1 f_1 + a_2 f_2)$ is a hypersurface-orthogonal Killing vector field of $(S_0, \gamma)$. We can assume that $f' := a_1 f_1 + a_2 f_2$ is non-constant (otherwise $\varsigma'$ is the zero vector and belongs to $A_{S_0}$ trivially). To prove that $\varsigma'$ belongs to $A_{S_0}$ observe that from the function $f' = a_1 f_1 + a_2 f_2$ we construct, using the expression (4), the vector field

$$ \zeta' = f' \partial_v + \frac{u^2}{2} \Delta f' \partial_u - \text{grad} f' = a_1 \zeta_1 + a_2 \zeta_2 $$

where $\zeta_1$ are the degenerate Killing vectors of the NHG generated by $\varsigma_1 \in A_{S_0}$. Therefore, $\zeta'$ is itself a degenerate Killing vector field of the NHG. The only if part of theorem 3 implies then that $Z$ must satisfy

$$ \frac{1}{Z} = \frac{Q'(f')}{\gamma^2 (df', df')}, $$

which was the only remaining condition for $\varsigma' \in A_{S_0}$. \hfill $\Box$

**Corollary 2.** For any pair of linearly independent $\varsigma_1, \varsigma_2 \in A_{S_0}$, the corresponding functions $f_1$ and $f_2$ are functionally independent everywhere on $S_0$: $df_1 \wedge df_2 \neq 0$.

**Proof.** From the proposition we know that $f' = a_1 f_1 + a_2 f_2$ gives rise to a degenerate Killing vector of type (4) for the MKH $H_\eta$ of $(M_{NHG}, g_{NHG})$. Theorem 3 then tells us that $df'$ cannot vanish anywhere on $S_0$, and thus $a_1 df_1 + a_2 df_2 \neq 0$ everywhere on $S_0$. In particular, $df_1 \wedge df_2 \neq 0$ on the whole $S_0$. \hfill $\Box$

### 4. Structure of near-horizon geometries with MKHs of order $m \geq 3$

The following theorem relates the dimension of $A_{S_0}$ to the order of the MKH $H_\eta$.

**Theorem 4.** Let $(M_{NHG}, g_{NHG})$ be a near horizon geometry with metric (3). The (necessarily multiple) Killing horizon $H_\eta$ has order $m \geq 2$ if and only if $A_{S_0}$ has dimension $m - 2$. 

Proof. The proof is a simple combination of previous results. Let $\xi_i, i = 1, \ldots, p := \dim A_0$ be a basis of $A_0$, and write $\xi_i = Z^{-1}d_{f_i}$. From corollary 2 follows that the $p + 1$ functions \{1, $f_1, \ldots, f_p$\} are linearly independent. For each such function we construct a vector field in $(M_{\text{NHG}}, g_{\text{NHG}})$ according to (4). By proposition 1 and theorem 3 this yields a collection of $p + 1$ linearly independent degenerate Killing generators of $H_0$. This implies that $p \leq m - 2$ where $m$ is the order of this multiple Killing horizon. The reverse inequality is proved similarly starting with a set of $m - 1$ linearly independent degenerate Killing generators of $H_0$. \hfill $\square$

As the maximum order of a MKH is given by the spacetime dimension $n + 1$, we have the following

**Corollary 3.** \(\dim (A_0) \leq n - 1\).

In view of lemma 1, expressions (19) for $h$ and $s$ are fully explicit. This allows us to find all possible NHGs with MKHs of order $m \geq 3$ in an explicit way locally. We establish the following fundamental result.

**Theorem 5.** Let $(M_{\text{NHG}}, g_{\text{NHG}})$ be a near horizon geometry with metric (3), and assume that the Killing horizon $H_0$ is multiple with order $m := 2 + p \geq 3$. Let $\xi_i \in A_0$ ($i, j, k = 1, \ldots, p$) be a set of linearly independent Killing vectors on any cut $S_0 \subset H_0$ which give rise, together with $\eta = \partial_0$ and via theorem 3, to $A_{\text{NHG}}$. For each $i$ let $Z, f_i$ and $Q_i(f_i)$ be functions on $S_0$ such that, with $N_i := \gamma^2(d_{f_i}, d_{f_i})$, $N_i = Q_i Z$ and $\xi_i = Z^{-1}d_{f_i}$ hold. Then, $s = 1/(2Z)dZ$ and one of the two following mutually exclusive cases holds:

\begin{enumerate}[(i)]
  \item $h = 0$ $\iff$ $\forall i, Q_i(f_i) = Q_0$ or $Q_i \neq 0, c_i \in \mathbb{R}$
  \item $h = Z$ $\iff$ $\forall i, Q_i = Q_0 (b_i + (f_i + c_i)^2)$, $Q_0 \neq 0, c_i, b_i \in \mathbb{R}$.
\end{enumerate}

In either case $(S_0, \gamma)$ is (locally) a warped product $S_0 = V \times \Sigma$ with metric

$$\gamma = \bar{\gamma} + \Omega g_\Sigma, \Omega : V \to \mathbb{R}$$

such that $(\Sigma, g_\Sigma)$ is a $p$-dimensional maximally symmetric Riemannian manifold of constant curvature $\varepsilon$.

Proof. From lemma 1 we know that if $h = 0$, then all Killing vectors $\xi_i$ must belong to either case (a) or case (b). This yields the expression in item (i) of the theorem. Observe that in this item (i), if $Q_i \in A_0$, and $\xi_i \in A_0$ belong to cases (a) and (b), respectively, then

$$Z = \frac{N_k}{Q_0} = \frac{N_j}{Q_0(b_j + (f_j + c_j)^2)}$$

must hold.

If $h$ is not identically zero, then all $\xi_i$ must belong to case (c) in the same lemma. Thus $Q_i = Q_0(b_j + (f_j + c_j)^2)$ for different in principle $Q_0$. Moreover $N_i = QZ$, combined with (32) shows that $h = Z$ in this case.

Next, we want to prove that $\gamma$ is a warped product metric with fibers of constant curvature. We start by showing that $A_0$ is in involution, i.e. that the commutators $[\xi_i, \xi_j]$ are linear combinations (with functions) of $\{Q_i\}$. Note first that from (31)
\[ D^{ij} f_i D_A Z = 2Z (D^{ij} f_j) s_A = Z P_i N_i, \]  
\[(36)\]

(the Einstein summation convention is not used for repeated indices \(i, j, \ldots\)), and from the master equation (5)
\[ D^{ij} f_i D_A D_B f_j = \frac{1}{2Z} D^{ij} f_i (D_A Z D_B f_j + D_B Z D_A f_j) = \frac{1}{2} P_i N_i D_B f_j + \frac{D_B Z}{2Z} D^{ij} f_i D_A f_j. \]  
\[(37)\]

Given that \( \varsigma_i = Z^{-1} df_i \) we compute
\[
[s_i, s_j]^B = \frac{1}{Z} D^{ij} f_i D_A \left( \frac{1}{Z} D^B f_j \right) - (i \leftrightarrow j) \\
= \frac{1}{Z^2} D^{ij} f_i \left( - \frac{1}{Z} D_A Z D^B f_j + D_A D^B f_j \right) - (i \leftrightarrow j) \\
= \frac{1}{Z^2} \left( - \frac{1}{2} N_i P_i D^B f_j + \frac{1}{2} N_j P_j D^B f_i \right) \\
= \frac{1}{2} \left( Q_i P_i s^B_i - Q_i P_j s^B_j \right) \]  
\[(38)\]

where in the third equality we inserted (36) and (37).

Recall that, according to corollary 2, \( \{s_i\} \) are not only linearly independent as vector fields, but even more, linearly independent at every point \( q \in S_0 \), so that the vector space \( T_q := \text{span}\{s_i\}_{q} \) has dimension \( p \). Thus, the collection \( \{T_q\} \) defines a distribution of dimension \( p \) which is in involution. By the Fröbenius theorem, \( S_0 \) can be foliated by injectively immersed integrable manifolds of dimension \( p \), whose tangent space is \( T_q \). Since we work locally on \( S_0 \), we may assume that these integral manifolds are embedded. We want to show that the induced metric is of constant curvature. Equivalently, we must show that the integral manifolds are maximally symmetric. To that aim define
\[ \varsigma_{ij} := f_j s_i - f_i s_j, \]
which are clearly tangent to the integral manifolds. It turns out that \( \varsigma_{ij} \) are Killing vectors of \((S_0, \gamma)\), as follows from
\[ D_A (\varsigma_{ij})_B = D_A (\varsigma_i)_B + f_j D_A (\varsigma_i)_B - D_A (\varsigma_j)_B - f_i D_A (\varsigma_j)_B = \frac{1}{Z} \left( D_A f_i D_B f_j - D_A f_j D_B f_i \right) + f_j D_A (\varsigma_i)_B - f_i D_A (\varsigma_j)_B = D_A (\varsigma_{ij})_B. \]

Moreover the Killing vectors \( \{\varsigma_i, \varsigma_{ij} (i < j)\} \) are linearly independent: take constants \( \{\alpha^i, \beta^i = \beta[j] \} \) satisfying
\[ 0 = \alpha^i \varsigma_i + \beta^i \varsigma_{ij} = \frac{1}{Z} \left( \alpha^i + 2 \beta^i f_i \right) df_i. \]

Since \( df_i \) are linearly independent at every point it must be \( \alpha^i + 2 \beta^i f_i = 0 \). But remember that the set \( \{1, f_i\} \) is functionally independent from where we conclude that \( \alpha^i = \beta^i = 0 \) proving that \( \{\varsigma_i, \varsigma_{ij} (i < j)\} \) is a set of linearly independent Killing vector fields. Hence, we have
\[ p + p(p - 1) \frac{2}{2} = p(p + 1) \frac{2}{2}. \]
linearly independent Killing vectors \( \{ \varsigma_i, \varsigma_j \} \). Since they are all tangent everywhere to \( p \)-dimensional manifolds, these spaces are maximally symmetric (and therefore, of constant curvature). Moreover, it must be the case that \( \{ \varsigma_i, \varsigma_j \} \) generate a Lie subalgebra of the Killing Lie algebra of \((S_0, \gamma)\). By a theorem due to Schmidt [14], or by noticing that \( \{ \varsigma_j \}_{j=1, \ldots, p} \) generate orthogonal hypersurfaces, the orbits admit orthogonal submanifolds, and thus the metric decomposes as the warped product (35).

It remains to show that, in item (ii), all constants \( Q_{0i} \) are equal to each other. This follows from the commutator (38) because in case (c) of lemma 1 one has

\[
[\varsigma_i, \varsigma_j] = Q_0(y) (f_i + c_i) \varsigma_j - Q_0(y) (f_j + c_j) \varsigma_i
\]

which can be rewritten as

\[
[\varsigma_i, \varsigma_j] + Q_0(y) c_ic_j - Q_0(y) c_jc_i = (Q_0(y) - Q_0(y)) f_i \varsigma_j.
\]

The lefthand side here is a Killing vector field, and thus the righthand side must be too. But given that \( \varsigma_j \) is a Killing vector itself and that \( df_i \neq 0 \), this can only happen if \( Q_{0i} = Q_{0j} \): finishing the proof.

The constant curvature of \((\Sigma, g_\varepsilon)\) can be explicitly found.

**Theorem 6.** Under the same hypotheses of theorem 5, if \( p > 1 \) the constant sectional curvature \( \varepsilon \) of the fibers \((\Sigma, g_\varepsilon)\) is given by

1. \( \varepsilon = -G^{kl} q_k q_l \) in item (i) of theorem 5 where

\[
q_i := \begin{cases} 0 & \text{if } \varsigma_i \text{ belongs to case (a)} \\ \frac{1}{2} Q_{0i} & \text{if } \varsigma_i \text{ belongs to case (b)} \end{cases}
\]

(39)

2. \( \varepsilon = Q_0 \left( Z - Q_0 G^{kl} (f_k + c_k)(f_l + c_l) \right) \) in item (ii) of theorem 5

where \( G^{kl} \) are the components of \( g^\sharp_\varepsilon \) in the basis \( \{ \varsigma_i \} \).

**Proof.** The curvature tensor of the orbits of a group of motions can be computed in terms of its Lie algebra structure constants and the metric, as proven in [15], see also [16] section 8.6. We apply a slightly simpler modification of that calculation to our situation. First of all, note that the set of hypersurface-orthogonal Killing vectors \( \{ \varsigma_i \} \) defines a frame, that is, a basis of \( \mathfrak{X}(S_0) \). Then, we wish to compute the curvature in this basis. The components of the metric in this basis will be denoted by

\[
G_{ij} := g_\varepsilon (\varsigma_i, \varsigma_j), G^{kl} g_{ij} = \delta^k_j.
\]

We start by computing the Levi-Civita connection \( \widetilde{D} \) of \((\Sigma, g_\varepsilon)\) in this basis, that is, \( D_{\varsigma_i} \varsigma_j \). Its anti-symmetric part in \( ij \) is given by the commutator (38). For the symmetric part, we use a formula derived in [15, 16] for arbitrary Killing vectors

\[
g_\varepsilon (\varsigma_i D_\varsigma \varsigma_j + D_\varsigma \varsigma_i) = g_\varepsilon ([\varsigma_i, \varsigma_j], \varsigma_i) + g_\varepsilon ([\varsigma_j, \varsigma_i], \varsigma_i) = Q_0 P_0 \mathcal{G}_{ij} - \frac{1}{2} Q_0 P_0 \mathcal{G}_{ij} = 0.
\]

where in the last equality we have used (38). Adding the symmetric and antisymmetric parts we deduce (summation on \( l \) and \( k \) is understood)
To compute the second derivative $\dddot{D}_s D'_s \zeta$ using this formula one needs to know the derivatives of $G_{ij}$ and $G^{ij}$ along $\zeta$. But these are easily found on using again (38) as

$$\zeta(G_{ik}) = \zeta(g_z(s_i, s_k)) = g_z(\xi_{s_i} G_{ik} s_k) + g_z(s_i, \xi_{s_k}) = \frac{1}{2} Q_i P_k \xi_{s_k} + \frac{1}{2} Q_k P_i \xi_{s_k} - Q_i P_k \xi_{s_k},$$

$$\zeta(G^{ik}) = -G^{ik} g^{lm} \zeta(G_{lm}).$$

The only remaining derivatives to be known are those of $Q_j P_j$, but for these it is convenient to separate the cases and consider their explicit form.

- For item (i) in the theorem, cases (a) and (b), we have $Q_j P_j = 2q_j$ are constants according to definition (39), and thus $\zeta_i(Q_j P_j) = 2\zeta_q(q_i) = 0$. Putting everything together and after a little calculation using (40) one can then easily obtain

$$D_s D'_s G_{sk} = D_s D'_s G_{sk} - D_s D'_s G_{sk} - D_s \zeta_i G_{k},$$

which proves 1. (It is easily seen with the used formulas that $\zeta_i(G^{lm} q_m q_n) = 0$ so that $G^{lm} q_m q_n$ is actually constant, as it must).

- For item (ii) in the theorem, that is for case (c), we have $Q_j P_j = 2Q_0(f_i + c_i)$ so that we need $\zeta_i(f_i)$. This can be derived from the following calculation

\[
d(\zeta_i(f_i)) = \xi_\zeta(\xi_f) = \xi_\zeta(Z \xi_i) = \zeta(Z) \xi_i + Z \xi_i \xi_i = 2Q_0 \xi_i + Z Q_0 \{ (f_i + c_i) \xi_i - \xi_i \} = Q_0 \{ (f_i + c_i) \xi_i + (f_i + c_i) \xi_i \}
\]

where (38) and

\[
\zeta_i(Z) = 2Q_0 Z(f_i + c_i)
\]

which comes from (36) have been used. This gives immediately

$$\zeta_i(f_i) = Q_0 \{ (f_i + c_i)(f_i + c_i) + c_i \}, c_i \in \mathbb{R}$$

(42)

where the constants $c_i$ are such that $c_i = b_i$. Observe that

$$G_{ij} = g_z(s_i, s_k) = Z^{-1} \zeta(f_i) = Z^{-1} Q_0 \{ (f_i + c_i)(f_i + c_i) + c_i \},$$

which in particular implies the symmetry $c_i = c_j$. Using this and putting everything together, a little longer calculation leads easily to

$$D_s D'_s G_{sk} = D_s D'_s G_{sk} - D_s \zeta_i G_{k}.$$

It is a matter of checking that the quantity $Q_0 \{ Z - Q_0 G^{ij}(f_i + c_i)(f_i + c_i) \}$ has zero derivative along all $\zeta_i$ on using (41) and (42). This finishes the proof.

For completeness, we now check that all Killing vectors of the form (4) related to elements $\zeta \in A_{Sl}$ according to theorem 3 necessarily commute with each other and with $\partial_a$, in agreement with theorem 1. To that end, take any two such Killing vectors

$$\zeta_a = f_a \partial_a + \frac{u^2}{2} \Delta f_a \partial_a - u \text{ grad } f_a$$

with $a \in \{1, 2\}$. Using that $Z \xi_i = \text{ grad } f_a$ for $Z$ as defined above, so that $\zeta_1(f_2) = \zeta_2(f_1)$ and $\Delta f_a = \zeta_a(Z)$, a direct computation provides for their commutator

$$\zeta_a = f_a \partial_a + \frac{u^2}{2} \Delta f_a \partial_a - u \text{ grad } f_a$$

with $a \in \{1, 2\}$. Using that $Z \xi_i = \text{ grad } f_a$ for $Z$ as defined above, so that $\zeta_1(f_2) = \zeta_2(f_1)$ and $\Delta f_a = \zeta_a(Z)$, a direct computation provides for their commutator.
\[
[\zeta_1, \zeta_2] = \frac{u^3}{2} Z \left( \zeta_2(\Delta f) - \zeta_1(\Delta f) \right) \partial_u + u^2 Z \left\{ Z [\zeta_1, \zeta_2] + \frac{1}{2} \zeta_1(Z) \zeta_2 - \frac{1}{2} \zeta_2(Z) \zeta_1 \right\}.
\]

The term proportional to \(u^2\) vanishes due to (36) and (38). Concerning the first term, proportional to \(u^3\), we note that \(\zeta_i\) are Killing vectors of \((S_0, \gamma)\), and thus they commute with the Laplacian \(\Delta\), which easily leads to \([\zeta_1, \zeta_2] = 0\), as expected.

### 5. Uniqueness of NHG for MKHs of order \(m \geq 3\)

Once we know the precise explicit form of the possible NHGs with MKHs of order \(m \geq 3\) we are ready to address the problem whether or not there can be several distinct NHGs arising from a MKH \(\mathcal{H}_{\text{MKH}}\) which admits at least two independent degenerate Killing vectors. More precisely, let \(\eta, \zeta \in \mathcal{A}_{\text{deg}}^{\text{NHG}}\) denote degenerate Killing vectors. We want to analyze whether there is a (local) isometry between the NHGs of \(\eta\) and \(\zeta\) associated to (each connected component of) the horizon \(\mathcal{H}_{\eta, \zeta} := \mathcal{H}_\eta \cap \mathcal{H}_\zeta \subset \mathcal{H}_{\text{MKH}}\).

Let \((\mathcal{M}_{\text{NHG}}, g_{\text{NHG}})\) be a near horizon geometry with metric (3), and assume that the Killing horizon \(\mathcal{H}_\eta = \{u = 0\}\) is multiple with order \(m \geq 3\). Then its Lie algebra \(\mathcal{A}_{\mathcal{H}_\eta}\) includes at least the following Killing vector fields

\[
\xi = v \partial_v - u \partial_u, \quad \eta = \partial_v, \quad \zeta = f \partial_v + \frac{u^2}{2} \Delta f \partial_u - u \text{ grad } f
\]

where the last one satisfies the relations proven in theorem 3, so that in particular \(f\) is a solution of (5). The NHG of this \((\mathcal{M}_{\text{NHG}}, g_{\text{NHG}})\) with respect to \(\eta\) is obviously itself, as follows from the intrinsic characterizations of \(h, s\) and \(\gamma\) given in remark 1. However, one can also construct another NHG for \((\mathcal{M}_{\text{NHG}}, g_{\text{NHG}})\) by using as degenerate Killing \(\zeta\) and by restricting the horizon to \(\mathcal{H}_{\eta, \zeta}\). It is a matter of checking that, for this Killing \(\zeta\), the corresponding intrinsic elements of remark 1 characterizing its NHG (call them \(S, H\) and \(\tilde{\gamma}\)) are such that \(\tilde{\gamma}\) is isometric to \(\gamma\) (and thus we will remove the tilde when not using a coordinate system) and the others are given by

\[
S|_{S_0} = s - \frac{df}{f} \bigg|_{S_0}, \quad H|_{S_0} = h - \frac{\Delta f}{f^2} + \frac{N}{f^2} \bigg|_{S_0}.
\]

Note that \(f\) does not vanish on \(\mathcal{H}_{\eta, \zeta}\).

By the standard near-horizon construction, there exist coordinates \(\{U, V, \gamma^A\}\) such that

\[
g_{\text{NHG}}(\zeta) = 2dV (dU + 2US_A d\gamma^A + \frac{1}{2} U^2 H dV) + \tilde{\gamma}_{AB} d\gamma^A d\gamma^B
\]

and \(H, S\) and \(\tilde{\gamma}\) have been extended off \(S_0\) as functions independent of \(U\) and \(V\).

This information is enough to prove the preliminary result that \(g_{\text{NHG}}(\zeta)\) admits a degenerate Killing generator for which the corresponding near horizon geometry brings us back to the original metric.

**Proposition 3.** The MKH \(\mathcal{H} := \{U = 0\}\) of the metric \(g_{\text{NHG}}(\zeta)\) has at least three Killing vectors given by

\[
\xi = V \partial_V - U \partial_U, \quad \tilde{\eta} = \partial_V, \quad \tilde{\zeta} = \frac{1}{F} \partial_V + \frac{U^2}{2} \left( \frac{1}{F} \right) \partial_U - U \text{ grad } \frac{1}{F}.
\]
where \( \tilde{\eta} \) and \( \tilde{\zeta} \) are degenerate and the function \( F \) is independent of \( U \) and \( V \) and satisfies 
\( F|_{S_0} = f. \) Moreover, the NHG of \( \mathcal{H}_{\tilde{\eta}, \tilde{\zeta}} \subset \mathcal{H} \) with respect to \( \tilde{\zeta} \) is the original metric \( g_{\text{NHG}} \) in (3).

**Proof.** From theorem 3 we know that the Killing vectors of \( g_{\text{NHG}}(\zeta) \) which are degenerate at \( \mathcal{H} \) other than \( \partial_V \) must solve equations (5)–(7) where now \( h \) and \( s \) must be substituted by their corresponding capital-letter versions. But given that \( f \) solves these equations (5)–(7) in their original form, it is easy to see that \( 1/f \) solves the new equations. It suffices to work on \( S_0, \) where \( F = f \)

\[
D_A D_B (1/f) - S_A D_B (1/f) - S_B D_A (1/f) \\
= -\frac{1}{f^2} D_A D_B f + \frac{2}{f^2} D_A D_B f - \left( s_A - \frac{1}{f} D_A f \right) \left( -\frac{D_B f}{f^2} \right) - \left( s_B - \frac{1}{f} D_B f \right) \left( -\frac{D_A f}{f^2} \right) \\
= -\frac{1}{f^2} (D_A D_B f - s_A D_B f - s_B D_A f) = 0.
\]

Similar, but a little longer calculations, using (5)–(7) prove that

\[
D^A (1/f) (D_A H - 2S_A H) = 0, \\
HD_A (1/f) = 2D^B (1/f) (D_B S_A - D_A S_B) + D_A (S^B D_B (1/f)) - 2S_A S^B D_B (1/f).
\]

In simpler words, \( 1/f \) is a solution of the corresponding equations, leading to the last Killing vector in the list given in the proposition. But this easily implies that repeating the NHG process just leads to the original metric given in (3).

We want to analyze whether (3) and (44) are isometric to each other or not. Our strategy will be to assume that they are (locally) isometric and find a set of necessary conditions that need to be satisfied, in particular regarding the explicit form of the coordinates \( \{u, v, x^A\} \). We will then confirm that this coordinate change indeed transforms one metric into the other.

Assume thus that the two spacetimes are locally isometric and that the isometry takes the cross section \( \{u = v = 0\} \) into the cross section \( \{U = V = 0\} \). The only Killing vector of \( g_{\text{NHG}} \) which is (i) a Killing generator of \( \mathcal{H}_{\eta} \), (ii) vanishes on \( S_0 \) and (iii) has surface gravity \( \kappa = 1 \) is \( \zeta \). Similarly, the only Killing vector of \( g_{\text{NHG}}(\zeta) \) satisfying the corresponding properties is \( \tilde{\xi} \). Thus, we are forced to identify \( \xi \) with \( \tilde{\xi} \).

The vector field \( \partial_V \) in \( g_{\text{NHG}}(\zeta) \) is by construction the Killing vector with respect to which we have performed the near horizon limit. Since in the original coordinates this vector is \( \zeta \), we are led to identify \( \zeta \) with \( \tilde{\eta} \). Concerning \( \eta \), this vector is a degenerate Killing generator of \( \mathcal{H}_{\tilde{\eta}} \). By the assumed isometry, it must also be a degenerate Killing generator of \( \mathcal{H}_{\tilde{\eta}, \tilde{\zeta}} \subset \mathcal{H} = \{U = 0\} \). Let \( m \) be the order of this multiple Killing horizon (we know that \( m \geq 3 \) by proposition 3) and \( \{\tilde{\eta}, \tilde{\zeta}, \zeta_{a}\} (a = 2, \cdots, m - 1) \) be a basis of degenerate Killing generators of \( \mathcal{H}_{\tilde{\eta}, \tilde{\zeta}} \). The assumed isometry forces the existence of constants \( \alpha, \beta, \beta^a \) such that

\[
\eta = \alpha \tilde{\eta} + \beta \tilde{\zeta} + \beta^a \zeta_a.
\]

We now use the proportionality \( \zeta|_{S_0} = f \eta|_{S_0} \) and let \( \tilde{f}_a \) be the functions on \( S_0 \) defined by \( \tilde{\zeta}_a|_{S_0} = \tilde{f}_a \partial_V \). Thus,

\[
\tilde{\eta}|_{S_0} = \partial_V|_{S_0} = \zeta|_{S_0} = f \eta|_{S_0} = f \left( \alpha \tilde{\eta} + \beta \tilde{\zeta} + \beta^a \zeta_a \right)|_{S_0} = f \left( \alpha + \frac{\beta}{f} + \beta^a \tilde{f}_a \right) \partial_V|_{S_0}.
\]
Since \( \{1, 1/f, f_a\} \) are functionally independent solutions of the master equation (5) in the metric \( g_{\text{NHG}}(\zeta) \), we conclude that the only possibility is \( \alpha = 0, \beta = 1 \) and \( \beta^\prime = 0 \) and we are forced to identify \( \eta \) and \( \zeta \), and thus \( \mathcal{H}_{\eta, \zeta} \) and \( \mathcal{H}_{\tilde{\eta}, \tilde{\zeta}} \).

We will have to deal with functions that agree on \( S_0 \) but are extended in two different ways off \( S_0 \), namely as functions independent of \( \{U, V\} \) and as functions independent of \( \{u, v\} \). A more geometric way to state this is that the set of points at equal value of \( x^A \) are different from the set of points at equal value of \( y^A \), even if these coordinates agree on \( S_0 \). The corresponding functions obtained by the two extensions are different and hence must be given different names. An example is the pair of functions \( f \) and \( F \), which agree on \( S_0 \) but have been extended so that \( f \) is independent of the coordinates \( \{u, v\} \) while \( F \) is independent of the coordinates \( \{U, V\} \). Note that this has nothing to do with the fact that either \( F \) or \( f \) can still be expressed in any coordinate system one wishes. The function \( h \) is independent of \( \{u, v\} \). The function that agrees with \( h \) on \( S_0 \) but is extended as a function independent of \( \{U, V\} \) will be denoted by \( \hat{h} \).

Similarly, we define \( \hat{H} \) as the function that agrees with \( H \) on \( S_0 \) and is extended as independent of \( \{u, v\} \). From the second in (43) one has

\[
\hat{H} = h - \frac{\Delta f}{f} + \frac{N}{f^2}.
\]

Our first step in the process of determining the coordinate change is to impose that the scalar products of various Killing fields must agree when computed with respect to each metric. Specifically it must be that \( g_{\text{NHG}}(\zeta)(\tilde{\eta}, \tilde{\eta}) = g_{\text{NHG}}(\zeta, \zeta) \), which after a simple calculation that uses (16) provides

\[
U^2H = u^2f^2 \left( h - \frac{\Delta f}{f} + \frac{N}{f^2} \right) = u^2f^2 \hat{H}.
\]

Similarly, the equality \( g_{\text{NHG}}(\zeta)(\tilde{\xi}, \tilde{\xi}) = g_{\text{NHG}}(\eta, \eta) \) yields

\[
u^2h = \frac{U^2\hat{h}}{F^2}.
\]

The equalities \( g_{\text{NHG}}(\xi, \xi) = g_{\text{NHG}}(\tilde{\xi}, \tilde{\xi}) \) and \( g_{\text{NHG}}(\xi, \eta) = g_{\text{NHG}}(\zeta, \tilde{\zeta}) \) yield, after another simple computation

\[
u(\nu h - 2) = UV(HUV - 2),
\]

\[
u(\nu h - 1) = U^2V \left( \frac{H}{F} - \frac{1}{2} \frac{\Delta}{F} \right) - \frac{U}{F}.
\]

Let us next find equations that must be satisfied by the change of coordinates between \( \{u, v, x^A\} \) and \( \{U, V, y^A\} \). The identification of \( \tilde{\eta} \) and \( \zeta \) leads to

\[
\tilde{\eta}(v) = \frac{\partial v}{\partial V} = \zeta(v) = f,
\]

\[
\tilde{\eta}(u) = \frac{\partial u}{\partial V} = \zeta(u) = \frac{1}{2} u^2 \Delta f,
\]

\[
\tilde{\eta}(x^A) = \frac{\partial x^A}{\partial V} = \zeta(x^A) = -u \text{ grad } f.
\]

Similarly, the identification of \( \xi \) and \( \tilde{\xi} \) implies
\[ v = \xi(v) = \xi(v) = V \frac{\partial v}{\partial V} - U \frac{\partial v}{\partial U} \quad (53) \]

\[ u = -\xi(u) = \xi(v) = -V \frac{\partial u}{\partial V} + U \frac{\partial u}{\partial U} \quad (54) \]

\[ 0 = \xi(x^A) = V \frac{\partial x^A}{\partial V} - U \frac{\partial x^A}{\partial U} \]

The last equation gives
\[ x^A = X^A(UV, y). \quad (55) \]

Let us integrate the pair (50)–(53). Inserting the second into the first yields the equivalent system
\[ U \frac{\partial v}{\partial U} + v = Vf, \quad \frac{\partial v}{\partial V} = f. \]

Defining \( C(U, V, y) \) by \( v = \frac{C}{U} \) this system becomes
\[ \frac{\partial C}{\partial U} = Vf(X(UV, y)), \quad \frac{\partial C}{\partial V} = Uf(X(UV, y)). \]

As a consequence \( U \partial_C C - V \partial_V C = 0 \) and hence \( C(U, V, y) \). The general solution is therefore
\[ v = \frac{1}{U} C(UV, y), \quad \frac{\partial C}{\partial (UV)} = f(X(UV, y)). \quad (56) \]

We next solve the pair (51)–(54). In terms of the function \( G(U, V, y) \) defined by \( u = U/G \), the system becomes
\[ \frac{\partial G}{\partial V} = -\frac{1}{2} U \Delta f, \quad 0 = V \frac{\partial G}{\partial V} - U \frac{\partial G}{\partial U}, \]

so that its general solution is
\[ u = \frac{U}{G(UV, y)}, \quad \frac{\partial G}{\partial (UV)} = -\frac{1}{2} \Delta f(X(UV, y)). \quad (57) \]

Observe that
\[ uv = \frac{C}{G}(UV, y). \]

Equation (48) together with this provides an expression for \( h \)
\[ h = \frac{G}{C} \left( \frac{G}{C} UV(UV H - 2) + 2 \right) \quad (58) \]

and the combination of (47) with (57) another one
\[ h = G^2 \frac{\dot{h}}{F^2}. \quad (59) \]

The combination of (58) with (59) provides a relation involving only coordinates on one side
\[ GC \frac{\dot{h}}{F^2} = \frac{G}{C} UV(UV H - 2) + 2. \quad (60) \]
Combining (49) with (58) gives another such relation
\[
\frac{1}{G} = C \hat{h} \frac{H}{F^2} - UV \left( \frac{H}{F} - \frac{1}{2} \Delta \left( \frac{1}{F} \right) \right) + \frac{1}{F}. \tag{61}
\]
From (60) and (61) we can thus get the two functions \( C \) and \( G \), given by
\[
G = \frac{F}{\Xi}, \tag{62}
\]
\[
C = \frac{F}{\hat{h}} \left[ UV \left( \hat{h} - \frac{1}{2} \Delta \frac{F}{\hat{h}} \right) - 1 + \Xi \right], \tag{63}
\]
where we have defined the abbreviation
\[
\Xi := \sqrt{\left( 1 + UV \frac{\Delta F}{2F} \right)^2 - U^2 V^2 \frac{\hat{h}}{F^2} N_F}
\]
where \( N_F \) is the squared norm of \( dF \) in the metric \( \hat{\gamma} \) of (44).

**Remark 7.** Expression (63) seems to have a problem if \( \hat{h} = 0 \), i.e., when the starting metric \( g_{\text{NHG}} \) has \( h = 0 \). However, it is easy to check that (63) has a well defined limit when \( \hat{h} \to 0 \), given by
\[
\hat{h} = 0 \implies C = \frac{UV F}{1 + UV \frac{\Delta F}{2F}} \left[ 1 + UV \left( \frac{\Delta F}{2F} - \frac{N_F}{2F^2} \right) \right]. \tag{64}
\]
All the previous formulas are given for general functions \( h, H, f \) and for general metric \( \gamma \) and one-forms \( s \) and \( S \). However, we know from theorem 5 that if the MKH in the NHG has order \( m \geq 3 \), then these objects take a very particular, explicitly known, form. Then, we have to take this into account and incorporate these explicit forms into the previous relations in order to find the sought isometry (coordinate change) between (3) and (44).

First of all we write down in a more explicit form the two spacetime metrics. The function \( f \) has nowhere vanishing gradient on \( S_0 \) and, according to theorem 5 the metric \( \gamma \) on \( S_0 \) takes the form
\[
\gamma = \hat{\gamma} + \frac{1}{N} df \otimes df = \hat{\gamma} + \frac{1}{Q^2(f) M} df \otimes df
\]
where in the second equality we used the function \( M \) defined by \( N = Q^2 M \) which was introduced in remark 5. Let \( \{ x^1 \} \) \( \{ A', B' \} \in \{ 2, \cdots, n-1 \} \) be a local coordinate system in the base manifold \( V \) of the warped product \( S_0 = V \times \Sigma \). Then \( \{ x^1, f \} \) is a coordinate system of \( S_0 \). From remark 5, \( M \) depends only on \( x^1 \) and \( M^{-1} \) is thus the warping function. Using the expressions for \( s \) and \( h \) given in theorem 3, the metric \( g_{\text{NHG}} \) takes the form (\( \otimes_y \) denotes symmetrized tensor product)
\[
g_{\text{NHG}} = 2dudv + 2d \ln(|QM|) \otimes_y dv + \frac{1}{2} u^2 MQ^2 \left( \frac{dP}{df} + P^2 \right) dv^2 + \hat{\gamma} + \frac{1}{Q^2 M} df^2. \tag{65}
\]
Concerning the metric \( g_{\text{NHG}}(\zeta) \), we first note that the expression of \( S \) and \( H \) on \( S_0 \) are, according to (43) and recalling that \( \Delta f = P(f) N = PQ^2 M \),
\[ S|_{S_0} = \frac{1}{2} \left( Pd F + \frac{dM}{M} - 2 \frac{df}{f} \right), \]

\[ H|_{S_0} = Q^2 M \left( \frac{1}{2} \left( \frac{dP}{df} + \frac{P^2}{f} \right) - \frac{P}{f} + \frac{1}{f^2} \right) := MK(f), \]

where the last expression defines the function of one variable \( K(f) \). The gradient of \( f \) in \( S_0 \) is clearly \( \text{grad} \ f = Q^2 M \partial_f \) and equation (52) for \( A = A' \) together with (55) readily implies that \( x^i(y) \) (i.e. independent of \( U, V \)). Without loss of generality we can choose the coordinate system \{ \( y \) \} on \( S_0 \) to be the same as \{ \( x \) \}. Hence \( x^i = y^i \) everywhere. This means in particular that the function \( M \) does not change by the coordinate transformation and the following expression holds everywhere (because they hold on \( S_0 \) and both sides are functions independent of \{ \( U, V \) \})

\[ N_F = MQ^2(F), \quad S = \frac{1}{2} \left( P(F)dF + \frac{dM}{M} - 2 \frac{df}{f} \right), \quad H = MK(F) \]

and the metric \( g_{\text{NHG}}(\zeta) \) is

\[ g_{\text{NHG}}(\zeta) = 2dud\nu + 2ud\ln \left( \frac{|M(F)|}{F^2} \right) \otimes_s d\nu + U^2 MK(F)d\nu^2 + \hat{\gamma} + \frac{dF^2}{MQ^2(F)}. \]  

From this point on we need to distinguish between the three possible cases according to lemma 1, that is cases (a) or (b) which satisfy \( h = 0 \), or case (c), for which \( h \neq 0 \).

5.1. The case (a)

This case satisfies \( P(f) = 0 \) and \( Q(f) = Q_0 \). By a trivial rescaling of \( M \) we may set \( Q_0 = 1 \) without loss of generality. The function \( K(f) \) in (66) is \( K(f) = 1/2f^2 \) and the metrics (65) and (67) to be compared become

\[ g_{\text{NHG}} = 2dud\nu + 2u d\ln M \otimes_s d\nu + \hat{\gamma} + \frac{dF^2}{M}, \]  

\[ g_{\text{NHG}}(\zeta) = 2dud\nu + 2ud\ln \left( \frac{M}{F^2} \right) \otimes_s d\nu + \frac{U^2 M}{F^2} d\nu^2 + \hat{\gamma} + \frac{dF^2}{M}. \]

Since \( \Delta F = 0, h = 0 \) (and hence \( \hat{h} = 0 \)), the function \( \Xi \) simplifies to \( \Xi = 1 \) so that (62) and (64) yield

\[ G = F, \quad C = UVF \left( 1 - \frac{UVM}{2F^2} \right). \]

From (56) and (57) the coordinate change is

\[ u = \frac{U}{F}, \quad v = VF \left( 1 - \frac{UVM}{2F^2} \right), \quad f = \frac{\partial C}{\partial(UV)} = F - \frac{UVM}{F}, \quad x^i = y^i. \]  

A straightforward computation shows that this coordinate change indeed transforms (68) into (69).
5.2. The case (b)

In this case we have

\[ Q(f) = Q_0(f + c), \quad P(f) = \frac{1}{f + c}. \]

Again a trivial rescaling of \( M \) allows one to set \( Q_0 = 1 \). The function \( K(f) \) is, from (66),

\[ K(f) = \frac{c(f + c)}{f^2}, \]

so the two metrics to be compared are

\[
\begin{align*}
g_{\text{NHG}} &= 2dudv + 2ud\ln(M|f + c|) \otimes \gamma_{AB} d\alpha^A d\alpha^B + \frac{1}{M(f + c)^2}df^2, \\
g_{\text{NHG}}(\zeta) &= 2dUdV + 2Ud\ln \left( \frac{M(F + c)}{F^3} \right) \otimes s dV + cU^2 \frac{M(F + c)}{F^3} dV^2 \\
&\quad + \gamma_{AB} d\alpha^A d\alpha^B + \frac{df^2}{M(F + c)^2}.
\end{align*}
\]

The function \( \Xi \) is now (given that \( \hat{h} = 0 \) and \( \Delta F = Q^2(F)P(F)M = (F + c)M \))

\[ \Xi = 1 + UV \frac{(F + c)M}{2F}. \]

The functions \( C \) and \( G \) are, from (64) and (62),

\[
\begin{align*}
C &= \frac{UV}{1 + UV \frac{(F + c)M}{2F}} \left( 1 - UV \frac{c(F + c)M}{2F} \right), \\
G &= \frac{F}{1 + UV \frac{(F + c)M}{2F}} \left( 1 - UV \frac{c(F + c)M}{2F} \right).
\end{align*}
\]

and the explicit coordinate change is obtained from (56) and (57) to be

\[
\begin{align*}
u &= \frac{U}{F} \left( 1 + UV \frac{(F + c)M}{2F} \right) , \\
v &= \frac{VF}{1 + UV \frac{(F + c)M}{2F}} \left( 1 - UV \frac{c(F + c)M}{2F} \right) , \\
f &= \frac{1}{\left( 1 + UV \frac{(F + c)M}{2F} \right)^2} \left( F - UVM \frac{c(F + c)(4F + UVM(F + c))}{4F^2} \right) , \\
x'^A &= y^A. \quad (73)
\end{align*}
\]

As before an explicit calculation shows that this coordinate change transforms (71) into (72).

5.3. The case (c)

Now we have \( Q(f) = b + (f + c)^2 \) (as before the multiplicative non-zero constant \( Q_0 \) can be absorbed in \( M \)) and \( P(f) = 2(f + c)/Q \). The functions \( h \) and \( K \) are, from (32) with \( N = Q^2M \) and (66),

\[
\begin{align*}
h &= M(b + (f + c)^2), \\
K(f) &= \frac{(b + c^2)(b + (f + c)^2)}{f^2}.
\end{align*}
\]

and the two metrics are now

\[
\begin{align*}
g_{\text{NHG}} &= 2dudv + 2ud\ln(M|b + (f + c)^2|) \otimes \gamma_{AB} d\alpha^A d\alpha^B + \frac{1}{M(b + (f + c)^2)^2}df^2, \\
&\quad + \gamma_{AB} d\alpha^A d\alpha^B + \frac{df^2}{M(b + (f + c)^2)^2}.
\end{align*}
\]
\[ g_{\text{NHG}}(\zeta) = 2dUdV + 2Ud\ln \left( \frac{M|b + (F + c)^2|}{F^2} \right) \otimes dV + U^2 M \frac{b + c^2}{F^2} \left( b + (F + c)^2 \right) dV^2 + \hat{\gamma}_{\alpha'\beta'} d\theta'^\alpha d\theta'^\beta + \frac{dF^2}{M(b + (F + c)^2)^2}. \] (75)

The coordinate change is now fairly complicated. The function \( \Xi \) is, after a calculation that uses \( N_F = M(b + (F + c)^2)^2 \) and \( \Delta F = 2(F + c)(b + (F + c)^2)M \),

\[ \Xi = \sqrt{1 + 2UV \frac{M(F + c)}{F} (b + (F + c)^2) - b(UV)^2} \left( b + (F + c)^2 \right)^2. \]

The function \( G \) is simply \( \frac{F}{\Xi} \) while \( C \) in (63) can be rewritten after an algebraic manipulation as

\[ C = \frac{F}{M(b + (F + c)^2)} (\Xi - 1) - cUV. \]

The explicit form of the coordinate change (56)–(57) turns out to be

\[ u = U \frac{\Xi}{F}, \quad v = -cV + \frac{F(\Xi - 1)}{UM(b + (F + c)^2)}, \]
\[ f = -c + \frac{1}{\Xi} \left( F + c - bUV \frac{M}{F} \left( b + (F + c)^2 \right) \right), \quad x'^{\alpha'} = y^{\alpha'}. \] (76)

Applying this coordinate change to the metric (74) is now fairly involved but one checks that indeed yields the metric (75).

Summarizing, we have proved the following theorem

**Theorem 7.** Let \((M_{\text{NHG}}, g_{\text{NHG}})\) be a near horizon geometry with metric (3) and assume that the Killing horizon \( \mathcal{H}_\eta = \{ u = 0 \} \) is multiple with order \( m \geq 3 \). Let \( \zeta \in A_{\mathcal{H}_\eta}^{\text{deg}} \) be independent of \( \eta \). Then, the NHG of \((M_{\text{NHG}}, g_{\text{NHG}})\) with respect to \( \zeta \) is locally isometric to \((M_{\text{NHG}}, g_{\text{NHG}}(\zeta))\) away from all fixed points of \( \zeta \). Moreover,

(i) If \( h = 0 \) then, in suitable coordinates, the metric \( g_{\text{NHG}} \) is either (68) or (71) and the coordinate changes are, respectively, (70) and (73).

(ii) If \( h \neq 0 \), then \( g_{\text{NHG}} \) can be written as (74) and the isometry is given by (76).

### 5.4. The main theorem

**Theorem 8.** Let \((M, g)\) be a spacetime containing a MKH \( \mathcal{H} \) with \( \dim(A_{\mathcal{H}}^{\text{deg}}) \geq 2 \) and let \( \eta, \zeta \in A_{\mathcal{H}}^{\text{deg}} \). Then, the NHGs of each connected component of \( \mathcal{H}_{\eta, \zeta} := \mathcal{H}_\eta \cap \mathcal{H}_\zeta \) with respect to \( \eta \) and \( \zeta \) are locally isometric.

**Proof.** When computing the NHGs of \((M, g)\) associated to \( \eta \) and \( \zeta \), respectively, one starts from different Gaussian null coordinates. However, it follows from remark 1 that the NHG is determined by the induced metric \( \gamma \), the torsion one-form \( s \) and the function \( h \) on any cut \( S \) of the connected component of \( \mathcal{H}_{\eta, \zeta} \). For the two Killing vectors these objects are related as described in (43). It follows that the NHG w.r.t. \( \zeta \) of \((M, g)\) coincides with the NHG w.r.t. \( \zeta \) of the NHG w.r.t. \( \eta \) of \((M, g)\).
Thus, the NHGs of \((M, g)\) with respect to \(\eta\) and \(\zeta\) are given by \(g_{\text{NHG}}(\eta)\), say (3), and \(g_{\text{NHG}}(\zeta)\), say (44), and theorem 7 gives the result at once.

Acknowledgments

MM acknowledges financial support under projects FIS2015-65140-P (Spanish MINECO/FEDER) and SA083P17 (Junta de Castilla y León). TTP acknowledges financial support by the Austrian Science Fund (FWF) P 28495-N27. JMMS is supported under Grants No. FIS2017-85076-P (Spanish MINECO/AEI/FEDER, UE) and No. IT956-16 (Basque Government).

ORCID iDs

Marc Mars https://orcid.org/0000-0002-9857-543X
Tim-Torben Paetz https://orcid.org/0000-0002-6053-400X
José M M Senovilla https://orcid.org/0000-0002-2700-9844

References

[1] Boyer R H 1969 Geodesic Killing orbits and bifurcate Killing horizons Proc. Roy. Soc. Lond. Ser. A 311 245–52
[2] Coley A, Hervik S, Papadopoulos G and Pelavas N 2009 Kundt spacetimes Class. Quantum Grav. 26 105016
[3] Kay B S and Wald R M 1991 Theorem on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon Phys. Rep. 207 49–136
[4] Kunduri H K and Lucietti J 2009 A classification of near-horizon geometries of extremal vacuum black holes J. Math. Phys. 50 082502
[5] Kunduri H K and Lucietti J 2009 Uniqueness of near-horizon geometries of rotating extremal \(\text{AdS}_4\) black holes Class. Quantum Grav. 26 055019
[6] Kunduri H K and Lucietti J 2013 Classification of near-horizon geometries of extremal black holes Living Rev. Relativ. 16 8
[7] Lewandowski J and Pawłowski T 2003 Extremal isolated horizons: a local uniqueness theorem Class. Quantum Grav. 20 587–606
[8] Lewandowski J, Szereszewski A and Waluk P 2016 Spacetimes foliated by nonexpanding and Killing horizons: higher dimension Phys. Rev. D 94 064018
[9] Mars M, Paetz T-T and Senovilla J M M 2018 Multiple Killing horizons Class. Quantum Grav. 35 155015
[10] Mars M, Paetz T-T and Senovilla J M M Multiple Killing horizons: the initial value formulation for \(\Lambda\)-vacuum (in preparation)
[11] Pawłowski T, Lewandowski J and Jezierski J 2004 Spacetimes foliated by Killing horizons Class. Quantum Grav. 21 1237–51
[12] Podol’skij S and Svarc R 2013 Explicit algebraic classification of Kundt geometries in any dimension Class. Quantum Grav. 30 125007
[13] Rácz I and Wald R M 1992 Extensions of spacetimes with Killing horizons Class. Quantum Grav. 9 2643–56
[14] Schmidt B G 1967 Isometry groups with surface-orthogonal trajectories Z. Naturforsch. 22a 1351
[15] Schmidt B G 1971 Homogeneous Riemannian spaces and Lie algebras of Killing fields Gen. Relativ. Gravit. 2 105–20
[16] Stephani H, Kramer D, MacCallum M A H, Hoenselaers C and Herlt E 2003 Exact Solutions of Einstein’s Field Equations (Cambridge Monographs on Mathematical Physics) 2nd edn (Cambridge: Cambridge University Press)