COMPACT KÄHLER MANIFOLDS WITH QUASI-POSITIVE SECOND
CHERN-RICCI CURVATURE

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ABSTRACT. Let $X$ be a compact Kähler manifold. We prove that if $X$ admits a smooth
Hermitian metric $\omega$ with quasi-positive second Chern-Ricci curvature $\text{Ric}^{(2)}(\omega)$, then $X$ is
projective and rationally connected. In particular, $X$ is simply connected.

CONTENTS

1. Introduction 1
2. Singular Hermitian metrics on vector bundles 4
3. Vanishing theorems for singular Hermitian metrics 5
4. The proof of main theorems 7
Reference 8

1. INTRODUCTION

The geometry of complex manifolds are characterized by various positivity notions
in complex differential geometry and algebraic geometry. Since the seminal works of
Mori and Siu-Yau on the solutions to Hartshorne conjecture and Frankel conjecture ([Mor79], [SY80]) on the characterizations of projective spaces, many remarkable
generalizations have been established, for instances, Mok’s uniformization theorem
on compact Kähler manifold with non-negative holomorphic bisectional curvature ([Mok88]) and the works of Campana, Demailly, Peternell and Schneider ([CP91],
[DPS94]) on the structure of projective manifolds with nef tangent bundles. For more
generic characterizations, we refer to [Mok88, CP91, DPS94, DPS01, WZ02, CT12,
LWZ13, Liu14, FLW17, HLT18, NZ19, Liu19, LOY19, LOY20] and the references therein.

The holomorphic sectional curvature also carries much geometric information
of complex manifolds. Indeed, thanks to the breakthrough work [WY16a] of Wu-
Yau, it is well-known that a compact Kähler manifold $X$ with negative or quasi-
negative holomorphic sectional curvature is algebraic and has ample canonical bundle ([TY17, WY16b, DT19]), which settles down a long-standing conjecture of S.-T. Yau
affirmatively. For more recent works on non-positive holomorphic sectional curvature,

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we refer to [Won81, HLW10, HLW16, HLWZ18, Nom18, Gue18, YZ19, WY20] and the references therein. On the other hand, in his A¹JProblem section, S.-T. Yau proposed the well-known conjecture [Yau82, Problem 47] that compact Kähler manifolds with positive holomorphic sectional curvature must be projective and rationally connected. Recently, we solved this conjecture affirmatively in [Yan18c] by introducing the concept of RC-positivity for abstract vector bundles, and many properties of such bundles are developed in [Yan18c, Yan18b, Yan18a, Yan18d]. For instance, we proved in [Yan18a] that a compact Kähler manifold with uniformly RC-positive tangent bundle must be projective and rationally connected. By using the ideas of RC-positivity and some deep analytical techniques in algebraic geometry, Shin-ichi Matsumura established in [Mat18a, Mat18b, Mat18c] a structure theorem for projective manifolds with non-negative holomorphic sectional curvature, which is analogous to fundamental works in [Mok88, CDP15, CH17, CH19] for manifolds with various non-negative properties (see also an approach in [HW15]). In the same spirit, Lei Ni and Fangyang Zheng introduced in [NZ18a, NZ18b] various notions of Ricci curvature and scalar curvature to obtain rational connectedness of compact Kähler manifolds.

In this paper, we investigate the geometry characterized by the first and second Chern-Ricci curvatures on Hermitian manifolds. Recall that for a Hermitian metric \( \omega \), its Chern curvature tensor has components \( R_{ijkl} \). The first Chern-Ricci curvature is

\[
\operatorname{Ric}^{(1)}(\omega) = \sqrt{-1} \left( \partial \bar{\partial} \log \det(\omega_{ij}) \right) dz^i \wedge d\zbar^j = -\sqrt{-1} \partial \bar{\partial} \log \det(h_{ij})
\]

and the second Chern-Ricci curvature is

\[
\operatorname{Ric}^{(2)}(\omega) = \sqrt{-1} \left( \partial \bar{\partial} \partial \bar{\partial} \log \det(h_{ij}) \right) dz^i \wedge d\zbar^j.
\]

When the Hermitian metric \( \omega \) is not Kähler, \( \operatorname{Ric}^{(1)}(\omega) \) and \( \operatorname{Ric}^{(2)}(\omega) \) are not necessarily the same. It is well-known that \( \operatorname{Ric}^{(1)}(\omega) \) represents the first Chern class of the complex manifold. However, the geometry of \( \operatorname{Ric}^{(2)}(\omega) \) is still mysterious.

Thanks to the celebrated Calabi-Yau theorem ([Yau78]), we know that a compact Kähler manifold \( X \) has a Hermitian metric with positive first Chern-Ricci curvature \( \operatorname{Ric}^{(1)}(\omega) \) if and only if \( X \) is Fano. As an analog, we proved in [Yan18c] that if a compact Kähler manifold admits a smooth Hermitian metric with positive second Chern-Ricci curvature \( \operatorname{Ric}^{(2)}(\omega) \), then \( X \) is projective and rationally connected. This result is also a generalization of the classical result of Campana [Cam92] and Kollár-Miyaoka-Mori [KMM92] that Fano manifolds are rationally connected. The main result of this paper is the following theorem.

**Theorem 1.1.** Let \( X \) be a compact Kähler manifold. If there exist a smooth Hermitian metric \( \omega \) on \( X \) and a smooth Hermitian metric \( h \) on the holomorphic tangent bundle \( TX \) such that

\[
\operatorname{tr}_\omega R^{(TX,h)} \in \Gamma(X, \End(TX))
\]

is quasi-positive. Then \( X \) is projective and rationally connected. In particular, \( X \) is simply connected.
Compact Kähler manifolds with quasi-positive second Chern-Ricci curvature

Xiaokui Yang

Here quasi-positive means non-negative everywhere and strictly positive at some point. We follow the ideas in [CDP15] and [GHS03] in the proof of Theorem 1.1, and the key new ingredient is an integration argument for singular Hermitian metrics instead of the pointwise maximum principle for RC-positive vector bundles employed in [Yan18c], since the latter does not work for manifolds with quasi-positive curvature tensors. It is easy to see that many compact Kähler manifolds with stable tangent bundle and positive slope can support smooth Hermitian metrics with positive or quasi-positive second Chern-Ricci curvature as required in Theorem 1.1 (e.g. [UY86, Don87]). On the other hand, the Kähler condition in Theorem 1.1 is necessary ([LY17, Section 6]). As a special case of Theorem 1.1, one has

**Corollary 1.2.** Let $X$ be a compact Kähler manifold. If $X$ admits a smooth Hermitian metric $\omega$ with quasi-positive second Chern-Ricci curvature $\text{Ric}^{(2)}(\omega)$, then $X$ is projective and rationally connected. In particular, $X$ is simply connected.

By using Corollary 1.2 and Yau’s theorem [Yau78], one has the following generalization of the result of Campana [Cam92] and Kollár-Miyaoka-Mori [KMM92].

**Corollary 1.3.** Let $X$ be a compact Kähler manifold. If $X$ admits a smooth Hermitian metric $\omega$ with quasi-positive first Chern-Ricci curvature $\text{Ric}^{(1)}(\omega)$, then $X$ is projective and rationally connected. In particular, $X$ is simply connected.

Similarly, for quasi-negative first Chern-Ricci curvature $\text{Ric}^{(1)}(\omega)$, one has

**Theorem 1.4.** Let $(X, \omega)$ be a compact Hermitian manifold with quasi-negative first Chern-Ricci curvature $\text{Ric}^{(1)}(\omega)$. If $X$ contains no rational curve, then $X$ is projective and $K_X$ is ample.

This result is a straightforward consequence of deep results in complex analytical and algebraic geometry ([Mor82, Siu84, Dem85, Kaw85, BCHM10, Cas13]), which would be of independent interest from the viewpoint of complex differential geometry. It is known that compact complex manifolds with quasi-negative (or quasi-positive) first Chern-Ricci curvature $\text{Ric}^{(1)}(\omega)$ are Moishezon ([Siu84, Dem85]), which are not necessarily projective (e.g. [MM07]). There are also many compact complex manifolds containing no rational curves, for instances, hyperbolic manifolds and Hermitian manifolds with non-positive holomorphic sectional curvature. Theorem 1.4 also provides another proof of [Lee18, Corollary 1.1] that compact Hermitian manifolds with non-positive holomorphic bisectional curvature and quasi-negative first Chern-Ricci curvature are projective manifolds with ample canonical bundles, which was established by purely analytical method.

The following conjecture on quasi-positive holomorphic sectional curvature is well-known and still widely open (e.g. [Yan18a, Conjecture 1.9]) and in the special case when $X$ is projective it was confirmed affirmatively by Matsumura ([Mat18b]).

**Conjecture 1.5.** Let $(X, \omega)$ be a compact Kähler manifold. If it has quasi-positive holomorphic sectional curvature, then $X$ is projective and rationally connected.

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2. SINGULAR HERMITIAN METRICS ON VECTOR BUNDLES

Let \( X \) be a complex manifold and \( \omega \) be a smooth Hermitian metric on \( X \). Locally, we can write the curvature tensor of \( (T_X, \omega) \) as

\[
R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{k}}}{\partial z^j \partial \bar{z}^l} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}k}}{\partial z^j} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^l},
\]

where \( \omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j \). The first Chern-Ricci curvature \( \text{Ric}^{(1)}(\omega) = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j \) has components \( R_{i\bar{j}}^{(1)} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 \log \det(g_{i\bar{j}})}{\partial z^j \partial \bar{z}^l} \). The second Chern-Ricci curvature is \( \text{Ric}^{(2)}(\omega) = \sqrt{-1} R_{i\bar{j}}^{(2)} dz^i \wedge d\bar{z}^j \) where \( R_{i\bar{j}}^{(2)} = g^{k\bar{l}} R_{k\bar{l}i\bar{j}} \). The scalar curvature \( s \) of the Chern connection is \( \text{tr}_\omega \text{Ric}^{(1)}(\omega) \) which is also the same as \( \text{tr}_\omega \text{Ric}^{(2)}(\omega) \).

Let \( E \to X \) be a holomorphic vector bundle and \( h \) be a smooth Hermitian metric on \( E \). The curvature \( R_E \) of the Chern connection \( \nabla \) on \( (E,h) \) has a similar formula

\[
R_{i\bar{j}k\bar{l}}^E = -\frac{\partial^2 h_{i\bar{k}}}{\partial z^j \partial \bar{z}^l} + h^{-\frac{s}{2}} \frac{\partial h_{i\bar{k}}}{\partial z^j} \frac{\partial h_{\bar{l}j}}{\partial \bar{z}^l}.
\]

We define \( R_{i\bar{j}}^{(1)} = h^{k\bar{l}} R_{i\bar{k}j\bar{l}} \) and \( R_{i\bar{j}}^{(2)} = g^{k\bar{l}} R_{k\bar{l}i\bar{j}} \). We also call \( \text{tr}_h R_E = \sqrt{-1} R_{i\bar{j}}^{(1)} dz^i \wedge d\bar{z}^j \) and \( \text{tr}_\omega R_E = \sqrt{-1} R_{i\bar{j}}^{(2)} dz^i \wedge d\bar{z}^j \) the first and second Chern-Ricci curvature of \( (E,h) \) with respect to the Hermitian manifold \( (X,\omega) \) respectively. When \( (E,h) = (TX,\omega) \), they are exactly the same as those curvatures of \( (X,\omega) \). A smooth Hermitian \( (1,1) \)-form \( A = \sqrt{-1} A_{i\bar{j}} dz^i \wedge d\bar{z}^j \) on \( X \) is call quasi-positive, if \( A_{i\bar{j}} \) is non-negative everywhere and positive at some point of \( X \). Similarly, we can define it for a tensor \( A \in \Gamma(X,\End(E)) \).

Singular Hermitian metrics on line bundles are introduced in [Dem92] by Demailly. Let \( L \) be a holomorphic vector bundle. A singular metric \( h^L \) on \( L \) can be written locally as \( h^L = e^{-L} \) for some \( \varphi \in L^1_{\text{loc}}(X,\mathbb{R}) \), and the curvature \( R^L = -\sqrt{-1} \partial \bar{\partial} \log h^L \) is defined in the sense of distributions. For singular Hermitian metrics on vector bundles, the definition would be very subtle, and we refer to [PT18, Section 2] for a detailed discussion (see also [dC98, Rau15, DWZZ20]). Recall that, for a smooth Hermitian metric \( h \) on \( E \), its curvature also takes the local form \( R_E = \bar{\partial} (h^{-1} \partial h) \). For the specified purpose in this paper, we only consider singular metrics such that \( h^{-1} \partial h \) is locally integrable, and then we can use standard theory on distributions (e.g. [Dem12]) to define the notion of weak positivity.

**Definition 2.1.** Let \( X \) be a complex manifold. A vector bundle \( E \) is called to have positive second Chern-Ricci curvature in the sense of distributions, if there exist a smooth metric \( \omega \) on \( X \) and a singular Hermitian metric \( h^E \) on \( E \) such that

\[
\text{tr}_\omega R_E \in \Gamma(X,\End(E))
\]

is strictly positive in the sense of distributions. The definition for non-negativity, quasi-positivity and etc. can be defined similarly.
Remark 2.2. Of course, the notion of second Chern-Ricci curvature can be defined in a broader way by using similar constructions as in \cite{dC98}. So far, it is not clear to the author whether the notions of Griffiths positivity or Nakano positivity for singular metrics defined in \cite{PT18, Rau15, DWZZ20} can imply the positivity of the second Chern-Ricci curvature in suitable sense, though it is obvious for smooth Hermitian metrics.

3. VANISHING THEOREMS FOR SINGULAR HERMITIAN METRICS

The main result of this section is the following theorem.

Theorem 3.1. Let $E$ be a holomorphic vector bundle over a compact complex manifold $X$. Suppose there exist a smooth Hermitian metric $\omega$ on $X$ and a smooth Hermitian metric $h$ on $E$ such that

$$\text{tr}_\omega R^{(E,h)} \in \Gamma(X, \text{End}(TX))$$

is quasi-positive. We have the following assertions.

1. Any invertible subsheaf $L$ of $O(\otimes^k E^*)$ ($k \geq 1$) is not pseudo-effective.
2. $\det E^*$ is not pseudo-effective.

The key difficulty in the proof of Theorem 3.1 is to deal with a line bundle $L$ which is only a subsheaf of $O(\otimes^k E^*)$. If it is indeed a subbundle, then its follows from a simple observation (c.f. \cite{CDP15}).

Lemma 3.2. The second Chern-Ricci curvature is decreasing in subbundles and increasing in quotient bundles.

It is well-known that the first Chern-Ricci curvature is not necessarily monotone as described in Lemma 3.2. The proof follows from a standard computation and we include details here since we need it for the distribution case.

Proof. Let $(E, h)$ be a Hermitian vector bundle and $S$ be a holomorphic subbundle of $E$. Let $r$ be the rank of $E$ and $s$ the rank of $S$. Without loss of generality, we can assume, at a fixed point $p \in X$, there exists a local holomorphic frame $\{e_1, \cdots, e_r\}$ of $E$ centered at point $p$ such that $\{e_1, \cdots, e_s\}$ is a local holomorphic frame of $S$. Moreover, we can assume that $h(e_\alpha, e_\beta)(p) = \delta_{\alpha\beta}$ for $1 \leq \alpha, \beta \leq r$. Hence, the curvature tensor of $S$ at point $p$ is

$$R^S_{\alpha\beta\gamma\delta} = -\frac{\partial^2 h_{\alpha\beta}}{\partial z^i \partial \bar{z}^j} + \sum_{\gamma=1}^s \frac{\partial h_{\alpha\gamma}}{\partial z^i} \frac{\partial h_{\gamma\beta}}{\partial \bar{z}^j}$$

where $1 \leq \alpha, \beta \leq s$. For any Hermitian metric $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ on $X$, we have

\begin{equation}
(\text{tr}_\omega R^E)|_S - \text{tr}_\omega R^S = \text{tr}_\omega (R^E|_S) - \text{tr}_\omega R^S = \sqrt{-1} \sum_{\alpha,\beta=1}^s g^a \left( \sum_{\gamma=s+1}^r \frac{\partial h_{a\gamma}}{\partial z^i} \frac{\partial h_{\gamma\beta}}{\partial \bar{z}^j} \right) e^a \otimes \bar{e}^\beta.
\end{equation}

It is easy to see that the right hand side of (3.1) is non-negative. The proof for quotient bundles is similar. \qed
Corollary 3.3. Let $E$ be a holomorphic vector bundle over a complex manifold $X$.

1. If $E$ has positive (resp, non-negative) second Chern-Ricci curvature, then each quotient bundle has positive (resp, non-negative) second Chern-Ricci curvature.
2. If $E$ has negative (resp, non-positive) second Chern-Ricci curvature, then each subbundle has negative (resp, non-positive) second Chern-Ricci curvature.
3. If $E$ has positive (resp, non-negative, quasi-positive, negative, non-positive, quasi-negative) second Chern-Ricci curvature, then so is $\otimes^k E$ for each $k \geq 1$.
4. If $E$ has positive (resp, non-negative, quasi-positive, negative, non-positive, quasi-negative) second Chern-Ricci curvature, then so is $\text{Sym}^k E$ (1 ≤ $p \leq \text{rk}(E)$).

Proof. We only need to prove (3). From the expression of the induced curvature formula of $(\otimes^k E, \otimes^h)$, one has

$$R(\otimes^k E, \otimes^h) = \otimes^k R(E, h) \in \Gamma \left( X, \Lambda^{1,1} X \otimes \text{End}(\otimes^k E) \right),$$

and

$$\text{tr}_\omega R(\otimes^k E, \otimes^h) = \text{tr}_\omega \left( \otimes^k R(E, h) \right) = \otimes^k \left( \text{tr}_\omega R(E, h) \right) \in \Gamma \left( X, \text{End}(\otimes^k E) \right).$$

Hence, the result follows. □

Remark 3.4. By standard distribution theory, one has similar results as in Lemma 3.2 and Corollary 3.3 for singular Hermitian metrics in Definition 2.1.

Lemma 3.5. Suppose that $E$ has non-negative second Chern-Ricci curvature in the sense of distributions and $L$ is a pseudo-effective line bundle. If $\sigma \in H^0(X, E^* \otimes L^*)$ is a non-trivial holomorphic section, then $\sigma$ does not vanish everywhere.

Proof. Let $\omega$ be a smooth Hermitian metric on $X$ and $h^E$ be a singular metric such that $\text{tr}_\omega R^E$ is non-negative in the sense of distributions. Since

$$\text{tr}_{\omega'} R^E = e^{-f} \text{tr}_\omega R^E,$$

we can assume further that $\omega$ is a Gauduchon metric ([Gau77]), i.e. $\partial\bar{\partial} \omega^{n-1} = 0$ where $\dim X = n$. Let $h^E$ be a singular metric on $L$ such that its curvature $\theta = -\sqrt{-1} \partial \bar{\partial} \log h^E \geq 0$ in the sense of distributions. For any $\sigma \in H^0(X, E^* \otimes L^*)$, we have

$$\text{tr}_\omega \left( \sqrt{-1} \partial \bar{\partial} |\sigma|^2_{h^E \otimes h^{L*}} \right) = |\nabla |\sigma|^2_{h^E \otimes h^{L*}} + (\text{tr}_\omega R^E \cdot h^{L*} + \text{tr}_\omega \theta \cdot h^{L*})(\sigma, \sigma).$$

Note that we are dealing with positive currents, and the product makes sense. An integration by part argument with respect to the Gauduchon metric $\omega$ shows that $\nabla |\sigma|^2 = 0$ a.e., and

$$(\text{tr}_\omega \sqrt{-1} \partial \bar{\partial}) |\sigma|^2_{h^E \otimes h^{L*}} \geq 0$$

in the sense of distributions. We know $|\sigma|^2_{h^E \otimes h^{L*}}$ is a global constant on the compact base $X$. If $\sigma$ is non-trivial, this constant is nonzero, i.e. $\sigma$ does not vanish everywhere. □

Corollary 3.6. Suppose that $E$ has non-negative second Chern-Ricci curvature in the sense of distributions, and $L$ is an invertible subsheaf of $\mathcal{O}(E^*)$. If $L$ is pseudo-effective, then $L$ is a line subbundle of $E^*$. 

6
Proof. Note that the subsheaf morphism $f : L \to \mathcal{O}(E^*)$ induces a nonzero section $\sigma \in H^0(X, E^* \otimes L^*)$. By Lemma 3.5, $\sigma$ does not vanish everywhere. This gives a trivial line subbundle $\mathcal{C}$ of the vector bundle $E^* \otimes L^*$, and so $L$ is a line subbundle of $E^*$.

Theorem 3.7. Suppose that $E$ has quasi-positive second Chern-Ricci curvature in the sense of distributions, and $L$ is a pseudo-effective line bundle. Then

\begin{equation}
H^0(X, E^* \otimes L^*) = 0.
\end{equation}

Proof. By using a similar argument as in the proof of Lemma 3.5, for any holomorphic section $\sigma \in H^0(X, E^* \otimes L^*)$, we deduce from (3.2) that the holomorphic section $\sigma$ vanishes on an open subset of $X$. By Aronszajn’s principle ([Aro57]), $\sigma$ is a zero section.

Proof of Theorem 3.1. Let $L$ be an invertible subsheaf of $\mathcal{O}(\otimes^k E^*)$ for some $k \geq 1$. Suppose to the contrary—that $L$ is pseudo-effective. By Corollary 3.3, $\otimes^k E$ has quasi-positive second Chern-Ricci curvature. By Corollary 3.6, the pseudo-effective invertible subsheaf $L$ is actually a line subbundle of $\otimes^k E^*$. It is easy to see that if $\otimes^k E$ has quasi-positive second Chern-Ricci curvature, then its dual bundle $\otimes^k E^*$ has quasi-negative second Chern-Ricci curvature. By decreasing principle in Lemma 3.2, the second Chern-Ricci curvature of $L$ is quasi-negative. Since $L$ is a line bundle, that means, the induced metric $h^L$ has quasi-negative Chern scalar curvature $s = \text{tr}_\omega(-\sqrt{-1} \partial \bar{\partial} \log h^L)$. Without loss of generality, we can assume $\omega$ is Gauduchon. Therefore

\begin{equation}
\int_X s\omega^n = n \int_X \omega_{BC}^1(L) \wedge \omega^{n-1} < 0.
\end{equation}

Hence, $L$ can not be pseudo-effective ([Yan19, Proposition 3.1],). This is a contradiction.

On the other hand, since $\det E^*$ is a subbundle of $\otimes^k E^*$ for some large $k$, we deduce $\det E^*$ can not be pseudo-effective.

4. The proof of main theorems

In this section, we prove Theorem 1.1, Corollary 1.3 and Theorem 1.4.

Proof of Theorem 1.1. We set $(E, h) = (TX, h)$. By Corollary 3.3 and Theorem 3.7, we have

$$H^0(X, (T^* X)^{\otimes k}) = 0.$$ 

It is well-known that for any $p$ satisfying $1 \leq p \leq \dim X$, $\Lambda^p T^* X$ is a direct summand of $(T^* X)^{\otimes k}$ for some large $k$. Therefore, we have

$$H^0_{\sigma}(X, \mathcal{C}) \cong H^0(X, \Lambda^p T^* X) = 0.$$ 

In particular, $H^2_{\sigma}(X, \mathcal{C}) = 0$. Since $X$ is Kähler, we deduce that $X$ is projective ([Kod54]). Now we following the ideas in [GHS03, Pet06, CDP15] to get the rational connectedness. By using Theorem 3.1, we conclude that $K_X = \det T^* X$ is not pseudo-effective. Thanks to [BDPP13], $X$ is actually uniruled. Let $\pi : X \dasharrow Z$ be the associated MRC fibration of $X$. After possibly resolving the singularities of $\pi$ and $Z$, we may assume that $\pi$ is a proper morphism and $Z$ is smooth. By [GHS03, Corollary 1.4], it follows that the target
of the MRC fibration is either a point or a positive dimensional variety which is not uniruled. Suppose \( X \) is not rationally connected, then \( \dim Z \geq 1 \) and \( Z \) is not uniruled. By using [BDPP13] again, \( K_Z \) is indeed pseudo-effective. Since \( K_Z = \det(T^*Z) \) is a direct summand of the vector bundle \( (T^*Z)^\otimes k \) for some large \( k \) and \( \mathcal{O}((T^*Z)^\otimes k) \) is a subsheaf of \( \mathcal{O}(T^*(X)^\otimes k) \), we obtain a pseudo-effective invertible subsheaf \( K_Z \) of \( \mathcal{O}(T^*(X)^\otimes k) \), which contradicts to part (1) of Theorem 3.1.

Proof of Corollary 1.3. By the celebrated Calabi-Yau theorem, there exists a smooth Kähler metric \( \omega_0 \) on \( X \) such that

\[
\text{Ric}^{(1)}(\omega_0) = \text{Ric}^{(1)}(\omega).
\]

Since \( \omega_0 \) is Kähler, one can deduce

\[
\text{Ric}^{(2)}(\omega_0) = \text{Ric}^{(1)}(\omega_0).
\]

Therefore, Corollary 1.3 follows from Corollary 1.2.

Proof of Theorem 1.4. By using the criterion of Siu-Demailly ([Siu84, Dem85]), we know \( X \) is a Moishezon manifold and \( K_X \) is big. On the other hand, by the deep result [BCHM10, Corollary 1.4.6], it is proved in [Cas13, Theorem 3.1] that a Moishezon manifold without rational curve must be projective. Moreover, by the base-point-free theorem and the relative cone theorem [Mor82, Kaw85, KMM92], one can deduce that \( K_X \) is ample since \( X \) is a projective manifold of general type without rational curve.

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Compact Kähler manifolds with quasi-positive second Chern-Ricci curvature

Xiaokui Yang

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Xiaokui Yang

Compact Kähler manifolds with quasi-positive second Chern-Ricci curvature

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