The Elliptic Representation of the General Painlevé 6 Equation

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1 Introduction

In this paper we study the elliptic representation of the sixth Painlevé equation

\[
\frac{d^2y}{dx^2} = \frac{1}{2} \left[ \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right] \left( \frac{dy}{dx} \right)^2 - \left[ \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right] \frac{dy}{dx} + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[ \alpha + \beta \frac{x}{y^2} + \gamma \frac{x-1}{(y-1)^2} + \delta \frac{x(x-1)}{(y-x)^2} \right],
\]

(PVI).

Though the elliptic representation of PVI has been known since R.Fuchs [8], in the literature there is no general study of its analytic implications. To fill this gap, we study here the analytic properties of the solutions in elliptic representation for all values of \(\alpha, \beta, \gamma, \delta\) and we derive their critical behavior close to the singular points \(x = 0, 1, \infty\). Moreover, we solve the connection problem for generic values of \(\alpha, \beta, \gamma, \delta\) and for the special (non-generic) case \(\beta = \gamma = 1 - 2\delta = 0\), which is important in 2-D topological field theory.

The six classical Painlevé equations were discovered by Painlevé [29] and Gambier [9], who classified all the second order ordinary differential equations of the type

\[
\frac{d^2y}{dx^2} = R \left( x, y, \frac{dy}{dx} \right)
\]

where \(R\) is rational in \(\frac{dy}{dx}, x\) and \(y\). The Painlevé equations satisfy the Painlevé property of absence of movable branch points and essential singularities. These singularities will be called critical points; for PVI they are \(0, 1, \infty\). The behavior of a solution close to a critical point is called critical behavior. A solution of the sixth Painlevé equation can be analytically continued to a meromorphic function on the universal covering of \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\). For generic values of the integration constants and of the parameters in the equation, it can not be expressed via elementary or classical transcendental functions. For this reason, it is called a Painlevé transcendent.

The Painlevé equations turned out to be important in physical models, since the works [24] [25] [26]. The reader may find a review of the physical applications in [13]. More recently, the Painlevé 6 equation found applications in topological field theory and Frobenius manifolds [4]. [5]

The first analytical problem with Painlevé equations is to determine the critical behavior of the transcendent at the critical points. Such a behavior must depend on two parameters, which are integration
constants. The second problem, called connection problem, is to find the relation between the couples of parameters at different critical points. The method of isomonodromic deformations developed in \[16\] was applied to the Painlevé 6 equation in \[15\], to solve such problems for a class of solutions of PVI with generic values of the parameters. The non-generic case $\beta = \gamma = 1 - 2\delta = 0$ is studied in \[10\] for its applications to topological field theory. Studies on the critical behavior can be also found in \[35\].

In the present paper we show that the elliptic representation is a valuable tool to study the critical behavior of the Painlevé 6 transcendents. We obtain results which include the results of \[13\] and extend the class of solutions to which they apply. On the other hand, we needed to use the isomonodromic deformation theory to solve the connection problem, to be formulated below, for the elliptic representation.

The elliptic representation was introduced by R. Fuchs in \[8\]. Let

$$\mathcal{L} := x(1-x) \frac{d^2}{dx^2} + (1-2x) \frac{d}{dx} - \frac{1}{4}. $$

be a linear differential operator and let $\wp(z; \omega_1, \omega_2)$ be the Weierstrass elliptic function of the independent variable $z \in \mathbb{P}^1$, with half-periods $\omega_1, \omega_2$. Let us consider the following independent solutions of the hyper-geometric equation $L\omega = 0$:

$$\omega_1(x) := \frac{\pi}{2} F \left( \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; x \right), \quad \omega_2(x) := \frac{\pi}{2} F \left( \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; 1-x \right),$$

where $F \left( \frac{1}{2}, \frac{1}{2}, 1; x \right)$ is the standard notation for the hyper-geometric function. R. Fuchs proved that the Painlevé 6 equation is equivalent to the following differential equation for a new function $u(x)$:

$$L(u) = \frac{1}{2x(1-x)} \frac{\partial}{\partial u} \left\{ 2\alpha \left[ \wp \left( \frac{u}{2}; \omega_1, \omega_2 \right) + \frac{1+x}{3} \right] - 2\beta \frac{x}{\wp \left( \frac{u}{2}; \omega_1, \omega_2 \right) + \frac{1+x}{3}} + 2\gamma \frac{1-x}{\wp \left( \frac{u}{2}; \omega_1, \omega_2 \right) + \frac{1+x}{3}} + (1-2\delta) \frac{x(1-x)}{\wp \left( \frac{u}{2}; \omega_1, \omega_2 \right) + \frac{1+x}{3}} \right\}$$

The connection to Painlevé 6 is given by the following representation of the transcendents:

$$y(x) = \wp \left( \frac{u(x)}{2}; \omega_1(x), \omega_2(x) \right) + \frac{1+x}{3}.$$

We review these facts in section 2.

The algebraic-geometrical properties of the elliptic representations where studied in \[22\]. Nevertheless, the analytic properties of the function $u(x)$ have not been studied so far, except for some special cases. The most simple case is $\alpha = \beta = \gamma = 1 - 2\delta = 0$. The function $u(x)$ is a linear combination of $\omega_1$ and $\omega_2$. This case was well known to Picard \[8\], and the critical behavior was studied in \[15\]. A more general case was studied in \[11\], for $\beta = \gamma = 1 - 2\delta = 0$ and $\alpha$ any complex number. The motivation of \[11\] was that this case is equivalent to the WDVV equations of associativity in 2-D topological field theory introduced by Witten \[36\], Dijkgraaf, Verlinde E., Verelinde H. \[2\] and it has applications to Frobenius manifolds \[4\] and quantum cohomology \[21\].

In this paper, we study the analytic properties of $u(x)$ and we compute the critical behavior of the transcendents for any value of $\alpha, \beta, \gamma, \delta$; moreover, we solve the connection problem in elliptic representation for generic $\alpha, \beta, \gamma, \delta$. For $\beta = \gamma = 1 - 2\delta = 0$.

### 1.1 Our results

#### 1.1.1 Local Representation

The equation $L(u) = 0$ has a general solution $u_0 = 2\nu_1 \omega_1 + 2\nu_2 \omega_2$, $\nu_1, \nu_2 \in \mathbb{C}$. We look for a solution of $L(u) = 0$ of the form $u(x) = 2\nu_1 \omega_1 + 2\nu_2 \omega_2(x) + u(x)$, where $u(x)$ is a perturbation of $u_0$. Let $C_0 := C \setminus \{0\}$, $C_0$ the universal covering and let $0 < r < 1$. We define the domains

$$\mathcal{D}(r; \nu_1, \nu_2) := \left\{ x \in C_0 \text{ such that } |x| < r, \left| \frac{e^{i\pi \nu_1}}{16^{1-\nu_2}} x^{1-\nu_2} \right| < r, \left| \frac{e^{i\pi \nu_1}}{16^{\nu_2}} x^{\nu_2} \right| < r \right\}$$

(2)
\[ D_0(r) := \left\{ x \in \mathbb{C}_0 \mid \text{such that } |x| < r \right\} \] (3)

We observe that the translations \( \nu_i \mapsto \nu_i + 2N_i, \ i = 1, 2, \ N_i \in \mathbb{Z} \) do not change a transcendent in the elliptic representation

\[ y(x) = \wp(\nu_1 \omega_1(x) + \nu_2 \omega_2(x) + v(x; \omega_1(x), \omega_2(x)) + \frac{1+x}{3}. \]

This is a consequence of the periodicity of the \( \wp \)-function. Therefore, one can take \( 0 \leq \Re \nu_i < 2, \ i = 1, 2. \) Nevertheless, we don't need to suppose such a range explicitly. Only in the case \( \Im \nu_2 = 0 \) we need to suppose that \( 0 \leq \nu_2 < 2. \) Finally, let us introduce the following expansion:

\[ v(x; \nu_1, \nu_2) := \sum_{n \geq 1} a_n x^n + \sum_{n \geq 0, m \geq 1} b_{nm} x^n \left[ e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{1-\nu_2} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm} x^n \left[ e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2} \right]^m \] (4)

**Theorem 1:** Let \( \nu_1, \nu_2 \) be two complex numbers.

I) For any complex \( \nu_1, \nu_2 \) such that \( \Re \nu_2 \neq 0 \) there exist a positive number \( r < 1 \) and a transcendent

\[ y(x) = \wp(\nu_1 \omega_1(x) + \nu_2 \omega_2(x) + v(x; \nu_1, \nu_2); \omega_1(x), \omega_2(x)) + \frac{1+x}{3} \]

such that \( v(x; \nu_1, \nu_2) \) is holomorphic in the domain \( D(r; \nu_1, \nu_2) \) and it is given by the expansion (4) which is convergent in \( D(r; \nu_1, \nu_2). \) The coefficients \( a_n, b_{nm}, c_{nm}, i = 1, 2, \) are certain rational functions of \( \nu_2. \) Moreover, there exists a positive constant \( M(\nu_2) \) such that

\[ |v(x; \nu_1, \nu_2)| \leq M(\nu_2) \left( |x| + \left| e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{1-\nu_2} \right| + \left| e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2} \right| \right) \text{ in } D(r; \nu_1, \nu_2) \] (5)

II) For any complex \( \nu_1 \) and real \( \nu_2, \) with the constraint \( 0 < \nu_2 < 1 \) or \( 1 < \nu_2 < 2, \) there exists a positive \( r < 1 \) and a transcendent

\[ y(x) = \wp(\nu_1 \omega_1(x) + \nu_2 \omega_2(x) + v(x; \nu_1, \nu_2); \omega_1(x), \omega_2(x)) + \frac{1+x}{3}, \text{ if } 0 < \nu_2 < 1 \]

or

\[ y(x) = \wp(\nu_1 \omega_1(x) + \nu_2 \omega_2(x) + v(x; -\nu_1, 2 - \nu_2); \omega_1(x), \omega_2(x)) + \frac{1+x}{3}, \text{ if } 1 < \nu_2 < 2 \]

such that \( v(x; \nu_1, \nu_2) \) and \( v(x; -\nu_1, 2 - \nu_2) \) are holomorphic in \( D_0(r), \) with convergent expansion (4) and bound (5) (for \( 1 < \nu_2 < 2 \) substitute \( \nu_1 \mapsto -\nu_1, \nu_2 \mapsto 2 - \nu_2). \)

Note that in the theorem

\[ \nu_2 \neq 0, 1 \]

We stress that in case II), if \( \nu_2 \) is greater that 2 or less then 0, we can always make a translation \( \nu_2 \mapsto \nu_2 + 2N \) to obtain \( 0 < \nu_2 < 2 \) (on the other hand, if \( -2N < \nu_2 < 2 - 2N, \) the formulae of case II) hold with the substitution \( \nu_2 \mapsto \nu_2 + 2N). \)

**Observation 1:** As a consequence of the theorem, for any \( N \in \mathbb{Z} \) and for any complex \( \nu_1, \nu_2 \) such that \( \Re \nu_2 \neq 0 \), there exists \( r_N < 1 \) and a transcendent

\[ y(x) = \wp(\nu_1 \omega_1(x) + [\nu_2 + 2N] \omega_2(x) + v(x; \nu_1, \nu_2 + 2N); \omega_1(x), \omega_2(x)) + \frac{1+x}{3} \text{ in } D(r; \nu_1, \nu_2 + 2N). \]

By periodicity of the \( \wp \)-function we re-write the transcendent as follows:

\[ y(x) = \wp(\nu_1 \omega_1(x) + \nu_2 \omega_2(x) + v(x; \nu_1, \nu_2 + 2N); \omega_1(x), \omega_2(x)) + \frac{1+x}{3} \text{ in } D(r; \nu_1, \nu_2 + 2N). \]

Moreover, we will show in section 8 that if a transcendent has the elliptic representation

\[ y(x) = \wp(\nu_1 \omega_1(x) + \nu_2 \omega_2(x) + v(x; \nu_1, \nu_2); \omega_1(x), \omega_2(x)) + \frac{1+x}{3} \]
in $D(r, \nu_1, \nu_2)$ for some $\nu_1, \nu_2$, $\Im \nu_2 \neq 0$, then for any integer $N$ there exists $\nu'_1$ (depending on $\nu_1, \nu_2$ and $N$) such that the transcendent has also the representation

$$y(x) = \varphi \left( \nu'_1 \omega_1(x) + \nu_2 \omega_2(x) + v(x; \nu_1, \nu_2 + 2N); \omega_1(x), \omega_2(x) \right) + \frac{1 + x}{3}$$

in $D(r, \nu'_1, \nu_2 + 2N)$.

**Observation 2:** Another consequence of the theorem is that for any complex $\nu_1, \nu_2$ such that $\Im \nu_2 \neq 0$ there exists $y(x) = \varphi \left( -\nu_1 \omega_1(x) + \left[ 2 - \nu_2 \right] \omega_2(x) + v(x; -\nu_1, 2 - \nu_2); \omega_1(x), \omega_2(x) \right) + \frac{1 + x}{3}$. Again we use the fact that the $\varphi$-function is periodic w.r.t. $2\omega_2$ and it is an even function. Therefore the transcendent becomes

$$y(x) = \varphi \left( \nu_1 \omega_1(x) + \nu_2 \omega_2(x) - v(x; -\nu_1, 2 - \nu_2); \omega_1(x), \omega_2(x) \right) + \frac{1 + x}{3}, \quad \text{in } D(r; -\nu_1, 2 - \nu_2)$$

Note that the series $-v(x; -\nu_1, 2 - \nu_2)$ is of the form

$$\sum_{n \geq 1} a_n x^n + \sum_{n \geq 0, m \geq 1} b_{nm} x^n \left[ e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{2-v_2} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm} x^n \left[ e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2 - 1} \right]^m$$

where we have re-named the constants $a_n, b_{nm}, c_{nm}$.

The domain $D(r; \nu_1, \nu_2 + 2N)$ can be written as follows:

$$(\Re \nu_2 + 2N) \ln \left| \frac{x}{16} \right| - \pi \Im \nu_1 - \ln r_N < \Im \nu_2 \arg x <$$

$$< (\Re \nu_2 - 1 + 2N) \ln \left| \frac{x}{16} \right| - \pi \Im \nu_1 + \ln r_N, \quad |x| < r_N$$

Therefore the domain $D(r, -\nu_1, 2 - \nu_2 - 2N)$ is

$$(\Re \nu_2 - 1 + 2N) \ln \left| \frac{x}{16} \right| - \pi \Im \nu_1 - \ln r_N < \Im \nu_2 \arg x <$$

$$< (\Re \nu_2 - 2 + 2N) \ln \left| \frac{x}{16} \right| - \pi \Im \nu_1 + \ln r_N, \quad |x| < r_N$$

We can draw their picture in the $(\ln |x|, \Im \nu_2 \arg x)$-plane. See figure [1].

### 1.1.2 Critical Behavior

It is possible to compute the critical behavior for $x \to 0$ of a transcendent of Theorem 1. For simplicity, we consider $x \to 0$ along the paths defined below. Let $\Im \nu_2 \neq 0$ and $V \in \mathbb{C}$. We define the following family of paths joining a point $x_0 \in D(r; \nu_1, \nu_2)$ to $x = 0$

$$\arg x = \arg x_0 + \frac{\Re \nu_2 - V}{\Im \nu_2} \ln \left| \frac{x}{|x_0|} \right|, \quad 0 \leq V \leq 1 \quad (6)$$

The paths are contained in $D(r; \nu_1, \nu_2)$. If $\Im \nu_2 = 0$ any regular path contained in $D_0(r)$ can be considered.

**Theorem 2:** Let $\nu_1, \nu_2$ be given.

If $\Im \nu_2 \neq 0$, the critical behavior of the transcendent $y(x) = \varphi(\nu_1 \omega_1 + \nu_2 \omega_2 + v(x; \nu_1, \nu_2); \omega_1, \omega_2) + (1 + x)/3$ when $x \to 0$ along the path [1] is:

For $0 < V < 1$:

$$y(x) = -\frac{1}{4} \left[ e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2} \left( 1 + O(|x^{\nu_2}| + |x^{1-\nu_2}|) \right) \right] \quad (7)$$

For $V = 0$:

$$y(x) = \left[ \frac{x}{2} + \sin^{-2} \left( -i\frac{\nu_2}{2} \ln \frac{x}{16} + \frac{\pi \nu_1}{2} \right) + \sum_{m \geq 1} c_m \left[ e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2} \right]^m \right] (1 + O(x)) \quad (8)$$
Figure 1: The domains \( D_1(r; \nu_1, \nu_2+2N) \) := \( D(r; \nu_1, \nu_2+2N) \), \( D_2(r; \nu_1, \nu_2+2N) \) := \( D(r; -\nu_1, 2-\nu_2-2N) \) and \( D_1(r; \nu_1, \nu_2+2[N+1]) \), \( D_2(r; \nu_1, \nu_2+2[N+1]) \) for arbitrarily fixed values of \( \nu_1, \nu_2, N \). They are represented in the plane \((\ln |x|, \Im \nu_2 \arg x + [\pi \Im \nu_1 + (\Re \nu_2 + 2N) \ln 16])\).
For $V = 1$:

$$y(x) = x \sin^2 \left( \frac{1 - \nu_2}{2} \ln \frac{x}{16} + \frac{\pi \nu_1}{2} + \sum_{m = 1}^{\infty} b_m \left[ e^{-i \pi \nu_1} \left( \frac{x}{16} \right)^{1-\nu_2} \right]^m \right) (1 + O(x)).$$

(9)

For $\nu_2$ real we have two cases. For $0 < \nu_2 < 1$, the transcendent $y(x) = \psi(\nu_1 \omega_1 + \nu_2 \omega_2 + v(x; \nu_1, \nu_2); \omega_1, \omega_2) + (1 + x)/3$ defined in $D_0(r)$ has behavior

$$y(x) = -\frac{1}{4} \left[ \frac{e^{i \pi \nu_1}}{16^{\nu_2-1}} \right] x^{\nu_2} \left( 1 + O(|x^{\nu_2}| + |x^{1-\nu_2}|) \right), \quad 0 < \nu_2 < 1$$

(10)

For $1 < \nu_2 < 2$, the transcendent $y(x) = \psi(\nu_1 \omega_1 + \nu_2 \omega_2 + v(x; -\nu_1, 2 - \nu_2); \omega_1, \omega_2) + (1 + x)/3$ defined in $D_0(r)$ has behavior

$$y(x) = -\frac{1}{4} \left[ \frac{e^{i \pi \nu_1}}{16^{\nu_2-1}} \right]^{-1} x^{2-\nu_2} \left( 1 + O(|x^{2-\nu_2}| + |x^{\nu_2-1}|) \right), \quad 1 < \nu_2 < 2$$

(11)

1.1.3 The Critical Points $x = 1, \infty$

Theorem 1 and 2 deal with the point $x = 0$. We now turn to the other critical points. Let us define $\omega_1^{(0)} := \omega_1, \omega_2^{(0)} := \omega_2; \omega_1^{(1)} := \omega_2, \omega_2^{(1)} := \omega_1$ and $\omega_1^{(\infty)} := \omega_1 + \omega_2, \omega_2^{(\infty)} := \omega_2$. We construct solutions

$$u(x) = \nu_1^{(1)} \omega_1^{(1)} + \nu_2^{(1)} \omega_2^{(1)} + v^{(1)}(x)$$

in a neighborhood of $x = 1$, and solutions

$$u(x) = \nu_1^{(\infty)} \omega_1^{(\infty)} + \nu_2^{(\infty)} \omega_2^{(\infty)} + v^{(\infty)}(x)$$

in a neighborhood of $x = \infty$. Let $P^1_i := P^1 \setminus \{i\}, i = 1, \infty$, and let $\widehat{P^1_i}$ be the universal covering.

**Theorem 3:** (In a Neighborhood of $x = 1$). For any complex $\nu_1^{(1)}, \nu_2^{(1)}$ such that $\Im \nu_2^{(1)} \neq 0$ there exists a transcendent $y(x) = \phi(\nu_1^{(1)} \omega_1^{(1)} + \nu_2^{(1)} \omega_2^{(1)} + v^{(1)}(x; \nu_1^{(1)}, \nu_2^{(1)}); \omega_1^{(1)}, \omega_2^{(1)})$ such that $v^{(1)}(x)$ is holomorphic in the domain

$$D(r; \nu_1^{(1)}, \nu_2^{(1)}) := \left\{ x \in \widehat{P^1} \text{ such that } |1 - x| < r, \left| e^{i \pi \nu_1^{(1)}} \left( \frac{1 - x}{16} \right)^{1-\nu_2^{(1)}} \right| < r, \left| e^{-i \pi \nu_1^{(1)}} \left( \frac{1 - x}{16} \right)^{\nu_2^{(1)}} \right| < r \right\}$$

where it has the convergent expansion:

$$v^{(1)}(x; \nu_1^{(1)}, \nu_2^{(1)}) = \sum_{n \geq 1} a_n (1 - x)^n + \sum_{n \geq 0, m \geq 1} b_{nm} (1 - x)^n \left[ e^{i \pi \nu_1^{(1)}} \left( \frac{1 - x}{16} \right)^{1-\nu_2^{(1)}} \right]^m +$$

$$+ \sum_{n \geq 0, m \geq 1} c_{nm} (1 - x)^n \left[ e^{-i \pi \nu_1^{(1)}} \left( \frac{1 - x}{16} \right)^{\nu_2^{(1)}} \right]^m$$

(12)

For any complex $\nu_1^{(1)}$ and real $\nu_2^{(1)}$ with the constraint $0 < \nu_2^{(1)} < 1$, there exists a sufficiently small $r$ and a transcendent $y(x) = \phi(\nu_1^{(1)} \omega_1^{(1)} + \nu_2^{(1)} \omega_2^{(1)} + v^{(1)}(x; \nu_1^{(1)}, \nu_2^{(1)}))$ such that $v^{(1)}(x; \nu_1^{(1)}, \nu_2^{(1)})$ is holomorphic in

$$D_0(r) := \left\{ x \in \widehat{P^1} \text{ such that } |1 - x| < r \right\}$$

6
where it has convergent expansion (12).

For any complex \(\nu_1^{(1)}\) and real \(\nu_2^{(1)}\) with the constraint \(1 < \nu_2^{(1)} < 2\), there exists a sufficiently small \(r\) and a transcendent \(y(x) = \varphi(\nu_1^{(1)} \omega_1^{(1)} + \nu_2^{(1)} \omega_2^{(1)} + v^{(1)}(x; -\nu_1^{(1)}, 2 - \nu_1^{(1)}))\) is holomorphic in \(D_0(r)\), where it has convergent expansion (12) with the substitution \((\nu_1^{(1)}, \nu_2^{(1)}) \rightarrow (-\nu_1^{(1)}, 2 - \nu_1^{(1)})\).

(In a Neighborhood of \(x = \infty\)). For any complex \(\nu_1^{(\infty)}\) and real \(\nu_2^{(\infty)}\) such that \(\Re \nu_2^{(\infty)} \neq 0\) there exists a transcendent \(y(x) = \varphi(\nu_1^{(\infty)} \omega_1^{(\infty)} + \nu_2^{(\infty)} \omega_2^{(\infty)} + v^{(\infty)}(x; \nu_1^{(\infty)}, \nu_2^{(\infty)}))\) is holomorphic with the constraint \(\forall \nu_1^{(\infty)}\) and real \(\nu_2^{(\infty)}\) with the constraint \(0 < \nu_2^{(\infty)} < 1\), there exists a sufficiently small \(r\) and a transcendent \(y(x) = \varphi(\nu_1^{(\infty)} \omega_1^{(\infty)} + \nu_2^{(\infty)} \omega_2^{(\infty)} + v^{(\infty)}(x; \nu_1^{(\infty)}, \nu_2^{(\infty)}))\) such that \(v^{(\infty)}(x; \nu_1^{(\infty)}, \nu_2^{(\infty)})\) is holomorphic in

\[
D_0(r) := \left\{ x \in \mathbb{P}_\infty^- \text{ such that } |x^{-1}| < r \right\}
\]

where it has convergent expansion:

\[
x^{\pm \nu^{(\infty)}(x; \nu_1^{(\infty)}, \nu_2^{(\infty)})} = \sum_{n \geq 1} d_n \left( \frac{1}{x} \right)^n + \sum_{n \geq 0, m \geq 1} b_{nm} \left( \frac{1}{x} \right)^n \left[ e^{-i\pi \nu_1^{(\infty)}} \left( \frac{16}{x} \right)^{1-\nu_2^{(\infty)}} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm} \left( \frac{1}{x} \right)^n \left[ e^{-i\pi \nu_1^{(\infty)}} \left( \frac{16}{x} \right)^{1-\nu_2^{(\infty)}} \right]^m
\]

(13)

For any \(\nu_1^{(\infty)}\) and real \(\nu_2^{(\infty)}\) with the constraint \(1 < \nu_2^{(\infty)} < 2\) there exists a sufficiently small \(r\) and a transcendent \(y(x) = \varphi(\nu_1^{(\infty)} \omega_1^{(\infty)} + \nu_2^{(\infty)} \omega_2^{(\infty)} + v^{(\infty)}(x; -\nu_1^{(\infty)}, 2 - \nu_1^{(\infty)}))\) is holomorphic in \(D_0(r)\) where it has convergent expansion (13) with the substitution \((\nu_1^{(\infty)}, \nu_2^{(\infty)}) \rightarrow (-\nu_1^{(\infty)}, 2 - \nu_1^{(\infty)})\).

Note: We have used the notations \(D(r; \nu_1, \nu_2)\) for the domains both at \(x = 0, x = 1\) and \(x = \infty\). We believe this will not confuse the reader, because it is always clear which is the critical point we are considering. Note that \(\nu_1^{(1)}\) comes with sign changed w.r.t. \(\nu_1\) at \(x = 0\); this is due to the definition of \(\omega_1^{(1)}\).

In section \ref{section:connection-problem} we also compute the critical behaviors for \(x \rightarrow 1\) and \(x \rightarrow \infty\).

\begin{subsection}{Connection Problem}

The elliptic representation allows us to obtained detailed information about the critical behavior of the Painlevé transcendents. On the other hand, the local analysis does not solve the connection problem. This is the problem of determining the critical behavior of a given transcendent at both \(x = 0, x = 1\) and \(x = \infty\). In our framework, we ask if a transcendent may have, at the same time, three representations

\[
y(x) = \varphi(\nu_1^{(0)} \omega_1^{(0)} + \nu_2^{(0)} \omega_2^{(0)} + v^{(0)}) + \frac{1 + x}{3}
\]

\[
= \varphi(\nu_1^{(1)} \omega_1^{(1)} + \nu_2^{(1)} \omega_2^{(1)} + v^{(1)}) + \frac{1 + x}{3}
\]

\[
= \varphi(\nu_1^{(\infty)} \omega_1^{(\infty)} + \nu_2^{(\infty)} \omega_2^{(\infty)} + v^{(\infty)}) + \frac{1 + x}{3}
\]

\end{subsection}
in the domains of Theorems 1 and 3. Moreover, we look for formulae which connect the three couples of parameters \((\nu_1^{(0)}, \nu_2^{(0)}), (\nu_1^{(1)}, \nu_2^{(1)}), (\nu_1^{(\infty)}, \nu_2^{(\infty)})\).

The connection problem may be solved using the method of isomonodromic deformations. Works on this problem are [13], [8] and [14]. The PVI is the isomonodromy deformation equation of a Fuchsian system of differential equations

\[
\frac{dY}{dz} = \left[ \frac{A_0(x)}{z} + \frac{A_2(x)}{z-x} + \frac{A_1(x)}{z-1} \right] Y
\]

The \(2 \times 2\) matrices \(A_i(x)\) \((i = 0, x, 1\) are labels) depend on \(x\) in such a way that the monodromy of a fundamental solution \(Y(z, x)\) does not change for small deformations of \(x\). They depend on the parameters \(\alpha, \beta, \gamma, \delta\) of PVI as follows:

\[
A_0(x) + A_1(x) + A_2(x) = -\frac{1}{2} \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}, \quad \text{eigenvalues of } A_i(x) = \pm \frac{1}{2} \theta_i, \quad i = 0, 1, x
\]

\[
\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad \beta = \frac{1}{2} \theta_1^2, \quad \gamma = \frac{1}{2} \theta_2^2, \quad \delta = \frac{1}{2}(1 - \theta_x^2)
\]

In section 8 we solve the connection problem for the elliptic representation for generic values of \(\alpha, \beta, \gamma, \delta\). More precisely, by generic case we mean:

\[
\nu_2^{(i)}, \theta_0, \theta_x, \theta_1, \theta_\infty \not\in \mathbb{Z}; \quad \pm 1 \pm \nu_2^{(i)} \pm \theta_1 \mp \theta_\infty, \quad \pm 1 \pm \nu_2^{(i)} \pm \theta_0 \mp \theta_x \not\in \mathbb{Z} \quad (14)
\]

The signs \(\pm\) vary independently. This is a technical condition which can be abandoned (except for \(\nu_2^{(i)} \not\in \mathbb{Z}\)) at the price of making the computations more complicated. For example, the non-generic case \(\beta = \gamma = 1 - 2\delta = 0\) and \(\alpha\) any complex number was analyzed in [11] for its relevant applications to Frobenius manifolds and quantum cohomology. We will review it in the paper.

To summarize the results for the generic case, we first observe that the critical behaviors provided by the elliptic representations along regular paths (except special directions for \(\mathcal{V} = 0, 1, \) see Theorem 2 and section 5.1) at \(x = 0, x = 1\) and \(x = \infty\) respectively are

\[
y(x) = a^{(0)} x^{\nu_2^{(0)}} (1 + \text{higher orders in } x), \quad x \to 0 \quad (15)
\]

\[
y(x) = 1 - a^{(1)} (1 - x)^{\nu_2^{(1)}} (1 + \text{higher orders in } (1 - x)), \quad x \to 1 \quad (16)
\]

\[
y(x) = a^{(\infty)} x^{1-\nu_2^{(\infty)}} (1 + \text{higher orders in } x^{-1}), \quad x \to \infty \quad (17)
\]

and the parameters \(\nu_1^{(i)}\) are given by

\[
e^{i\pi \nu_1^{(0)}} = -4a^{(0)} 16^{\nu_2^{(0)}} - 1, \quad e^{-i\pi \nu_1^{(1)}} = -4a^{(1)} 16^{\nu_2^{(1)}} - 1, \quad e^{i\pi \nu_1^{(\infty)}} = -4a^{(\infty)} 16^{\nu_2^{(\infty)}} - 1
\]

If \(\nu_2^{(i)}\) is real, the behavior is as above when \(0 < \nu_2^{(i)} < 1\). Otherwise, when \(1 < \nu_2^{(i)} < 2\) it is:

\[
y(x) = a^{(0)} x^{2-\nu_2^{(0)}} (1 + \text{higher orders in } x), \quad x \to 0 \quad (18)
\]

\[
y(x) = 1 - a^{(1)} (1 - x)^{2-\nu_2^{(1)}} (1 + \text{higher orders in } (1 - x)), \quad x \to 1 \quad (19)
\]

\[
y(x) = a^{(\infty)} x^{\nu_2^{(\infty)}-1} (1 + \text{higher orders in } x^{-1}), \quad x \to \infty \quad (20)
\]

with

\[
e^{-i\pi \nu_1^{(0)}} = -4a^{(0)} 16^{1-\nu_2^{(0)}}, \quad e^{i\pi \nu_1^{(1)}} = -4a^{(1)} 16^{1-\nu_2^{(1)}}, \quad e^{-i\pi \nu_1^{(\infty)}} = -4a^{(\infty)} 16^{1-\nu_2^{(\infty)}}
\]

Note that the ambiguity \(\nu_1^{(i)} \mapsto \nu_1^{(i)} + 2k, k\) integer, is natural, because \(\psi^{(i)}(x)\) does not change and the \(\varphi\)-function is periodic.

Let \(M_0, M_1, M_2\) be the monodromy matrices at \(z = 0, 1, x\), for a given basis in the fundamental group of \(\mathbb{P}^1 \setminus \{0, 1, x, \infty\}\). Such basis is chosen as in figure 2.
If
\[ \theta_0, \theta_x, \theta_1, \theta_\infty \not\in \mathbb{Z} \]
there is a one to one correspondence between a given choice of monodromy data \( \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0M_x), \text{tr}(M_0M_1), \text{tr}(M_1M_x) \) and a transcendent \( y(x) \). Namely:
\[
y(x) = y(x; \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0M_x), \text{tr}(M_0M_1), \text{tr}(M_1M_x))
\] (21)

We prove that such a transcendent has elliptic representations at \( x = 0, 1, \infty \), provided that \( \theta_0, \theta_x, \theta_1, \theta_\infty \) are functions of the monodromy data \( \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0M_x), \text{tr}(M_0M_1), \text{tr}(M_1M_x) \). Namely, we show that
\[
2 \cos(\pi \nu_2^{(0)}) = -\text{tr}(M_0M_x), \quad 2 \cos(\pi \nu_2^{(1)}) = -\text{tr}(M_1M_x), \quad 2 \cos(\pi \nu_2^{(\infty)}) = -\text{tr}(M_0M_1)
\] (22)

\[ a^{(i)} = a^{(i)}(\nu_2^{(i)}; \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0M_x), \text{tr}(M_0M_1), \text{tr}(M_1M_x)), \quad i = 0, 1, \infty \]

The formulas of \( a^{(i)} \) are quite long, so we do not write them here. They depend on the monodromy data through rational, trigonometric and \( \Gamma \)-functions. In particular, \( \nu_2^{(i)} \) enters explicitly. The procedure for computing such formulae is given in the Appendix. We note that the condition \( \nu_2^{(i)} \not\in \mathbb{Z} \) is equivalent to \( \text{tr}(M_iM_j) \neq \pm 2 \).

Conversely, we prove that a transcendent \( y(x) \) given by its elliptic representation close to one critical point is a transcendent \( 24 \) for some monodromy data. This follows from the consideration that the couple \( (\nu_1^{(i)}, \nu_2^{(i)}) \) is given at the critical point \( x = i \), and \( \theta_0, \theta_x, \theta_1, \theta_\infty \) are fixed by the equation PVI we are considering. From these data we can compute \( \text{tr}(M_0M_x), \text{tr}(M_1M_x), \text{tr}(M_0M_1) \). One of the traces is \(-2 \cos(\pi \nu_2^{(i)})\), the others depend on \( \nu_1^{(i)}, \nu_2^{(i)}, \theta_0, \theta_x, \theta_1, \theta_\infty \) through rational, trigonometric and \( \Gamma \)-functions. The formulae are rather long, so we refer the reader to the Appendix. In this way the transcendent \( 24 \) is obtained. From the monodromy data we compute the couples \( (\nu_1^{(i)}, \nu_2^{(i)}) \) at the other two critical points and we get the the elliptic representation of the initial transcendent at the other critical points. Therefore, the connection problem is solved.

Note that if we start from the elliptic representation at one critical point, say for example \( x = 0 \), then \( \nu_1^{(0)}, \nu_2^{(0)} \) are given. As explained above, we can compute the monodromy data and from them we compute \( \nu_2^{(j)} \) and \( a^{(j)} \) (then \( \nu_1^{(j)} \)) at the other two critical points. As already observed, the ambiguity \( \nu_1^{(j)} \rightarrow \nu_2^{(j)} + 2k (k \text{ integer}) \) does not change the elliptic representation. On the other hand, the ambiguities \( \nu_2^{(j)} \rightarrow \nu_2^{(j)} + 2N (N \text{ integer}), \nu_2^{(j)} \rightarrow -\nu_2^{(j)} \) and the ambiguity in the choice \( 0 \leq \Re \nu_2^{(j)} \leq 1 \) or \( 1 \leq \Re \nu_2^{(j)} \leq 2 \), which results from the cosines in \( 24 \), is due to the fact that the same transcendent has different elliptic representations in different domains (the choice of \( \nu_2^{(j)} \) determines the representation and the domain!).

To conclude the discussion of the generic case, some comments about our extension of previous known results are in order. The critical behavior for a class of solutions to the Painlevé 6 equation was found by Jimbo in [15] for generic values of \( \alpha, \beta, \gamma, \delta \). A transcendent in this class has behavior:
\[
y(x) = a^{(0)} x^{1-\sigma^{(0)}} (1 + O(|x|^\delta)), \quad x \rightarrow 0,
\] (23)
\[
y(x) = 1 - a^{(1)} (1 - x)^{1-\sigma^{(1)}} (1 + O(|1 - x|^\delta)), \quad x \rightarrow 1,
\] (24)
\[
y(x) = a^{(\infty)} x^{-\sigma^{(\infty)}} (1 + O(|x|^{-\delta})), \quad x \rightarrow \infty,
\] (25)
where \( \delta \) is a small positive number, \( a^{(i)} \) and \( \sigma^{(i)} \) are complex numbers such that \( a^{(i)} \neq 0 \) and
\[
0 \leq \Re \sigma^{(i)} < 1.
\] (26)

We remark that \( x \) converges to the critical points inside a sector with vertex on the corresponding critical point. The connection problem, i.e. the problem of finding the relation among the three pairs \( (a^{(i)}, a^{(i)}) \), \( i = 0, 1, \infty \), was solved in [15] for the above class of transcendents using the isomonodromy deformations theory. Actually, a transcendent in the class above coincides with a transcendent \( 24 \). In particular
\[
2 \cos(\pi \sigma^{(0)}) = \text{tr}(M_0M_x), \quad 2 \cos(\pi \sigma^{(1)}) = \text{tr}(M_1M_x), \quad 2 \cos(\pi \sigma^{(\infty)}) = \text{tr}(M_0M_1)
\]
and
\[ a^{(i)} = a^{(i)}(\sigma^{(i)}; \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0 M_x), \text{tr}(M_0 M_1), \text{tr}(M_1 M_x)), \quad i = 0, 1, \infty \]

For the formulas of \( a^{(i)} \) we refer to \[ \text{[80]} \]. The monodromy data are restricted by the following condition, equivalent to \[ \text{(26)} \]:
\[ |\text{tr}(M_i M_j)| \leq 2, \quad \Re\{\text{tr}(M_i M_j)\} \neq -2 \] (27)

As explained above, we have shown that the transcendents \[ \text{[24]} \] have elliptic representation. Therefore, Jimbo’s transcendents are included in our class of transcendents obtained by the elliptic representation. Observe that the behaviors \[ \text{[24–25]} \] are included in the behaviors \[ \text{[13–17]} \] (and \[ \text{[18–20]} \]), which hold for any \( \nu_2^{(i)} \notin (-\infty, 0) \cup \{1\} \cup [2, +\infty) \). Therefore, the condition \[ \text{(26)} \] is extended to any \( \sigma^{(i)} \in \mathbb{C} \) such that \( \sigma^{(i)} \notin (-\infty, 0) \cup \{1, +\infty\} \). This corresponds to the fact that we have solved the connection problem for any complex value of \( \text{tr}(M_i M_j) \) with the only constraint \( \text{tr}(M_i M_j) \neq \pm 2 \). This condition extends \[ \text{(27)} \].

To be more precise, the condition \( \nu_2^{(i)} \neq 1 \) is equivalent to \( \text{tr}(M_0 M_x) \neq 2 \) at \( x = 0 \); to \( \text{tr}(M_0 M_1) \neq 2 \) at \( x = 1 \); to \( \text{tr}(M_0 M_1) \neq 2 \) at \( x = \infty \). Nevertheless, in the case \( \text{tr}(M_i M_j) = 2 \) the critical behavior and the solution of the connection problem were achieved by Jimbo. Unfortunately, the condition \( \nu_2^{(i)} \neq 1 \) which we had to impose to study the elliptic representation (except for non-generic cases like \( \beta = \gamma = 1 - 2\delta = 0 \)) does not allow us to know the analytic properties and the critical behavior of the elliptic representation in this case. We expect that the properties of \( u(x) \) are such to exactly produce the critical behavior found by Jimbo for \( \text{tr}(M_i M_j) = 2 \), but we still have to cover this case.

The condition \( \nu_2^{(i)} \neq 0 \) (and 2), implies that we can not give the critical behaviors (and the elliptic representation) of \[ \text{[24]} \] at \( x = 0 \) for \( \text{tr}(M_0 M_x) = -2 \); at \( x = 1 \) for \( \text{tr}(M_1 M_x) = -2 \); at \( x = \infty \) for \( \text{tr}(M_0 M_1) = -2 \). To our knowledge, these cases have not yet been studied in the literature.

To conclude, the results of \[ \text{[15]} \] together with our extension provide the critical behaviors and the solution of the connection problem for the transcendents \[ \text{[24]} \] in the generic case for

for any value of \( \text{tr}(M_i M_j) \neq -2 \)

which corresponds to exponents
\[ \sigma^{(i)} \in \mathbb{C} \] such that \( \sigma^{(i)} \notin (-\infty, 0) \cup \{1, +\infty\} \).

We turn now to the special case \( \beta = \gamma = 1 - 2\delta = 0 \), important for its applications to topological field theory, Frobenius manifolds \[ \text{[8]} \] and quantum cohomology \[ \text{[19]} \text{[12]} \]. In this case is fully studied in \[ \text{[1]} \]. We can give a representation of \( u(x) \) in a domain which is wider than the generic case. Namely, at \( x = 0 \), the domain is
\[ D(r; \nu_1, \nu_2) := \left\{ x \in \mathbb{C}_0 \mid |x| < r, \quad \left| e^{-i\pi\nu_1} \left( \frac{x}{16} \right)^{2-\nu_2} \right| < r, \quad \left| e^{i\pi\nu_1} \left( \frac{x}{16} \right)^{\nu_2} \right| < r \right\} \]

In this domain \( v(x) \) is holomorphic with convergent expansion
\[ v(x) = \sum_{n \geq 1} a_n x^n + \sum_{n \geq 0, m \geq 1} b_{nm} x^n \left[ e^{-i\pi\nu_1} \left( \frac{x}{16} \right)^{2-\nu_2} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm} x^n \left[ e^{i\pi\nu_1} \left( \frac{x}{16} \right)^{\nu_2} \right]^m \]

If \( \nu_2 \) is real, the value \( \nu_2 = 1 \) is now allowed, namely, the constraint is \( \nu_2 \notin (-\infty, 0) \cup \{2, +\infty\} \). Therefore, by periodicity of the \( \wp \)-function we can assume \( 0 \leq \Re\nu_2 < 2, \nu_2 \neq 0 \). A similar result holds at \( x = 1 \) and \( x = \infty \) (see section \[ \text{8.2} \]).

According to \[ \text{[8]} \], we define \( 2 - x_0^2 := \text{tr} M_0 M_x, 2 - x_1^2 := \text{tr} M_1 M_x, 2 - x_\infty^2 := \text{tr} M_0 M_1 \). There is a one to one correspondence between triples \((x_0, x_1, x_\infty)\) (defined up to the change of two signs) and Painlevé transcendents, provided that at most one \( x_i \) is zero and not all the \( x_i \) are \( \pm 2 \) at the same time. Therefore we write \( y(x) = y(x; x_0, x_1, x_\infty) \). We show that one such transcendent has elliptic representations (half-periods are understood)
\[ y(x; x_0, x_1, x_\infty) = \wp\left( \nu_1^{(0)} \omega_1^{(0)}(x) + \nu_2^{(0)} \omega_2^{(0)}(x) + v^{(0)}(x; \nu_1^{(0)}, \nu_2^{(0)}) \right) + \frac{1 + x}{3} \]
\[ = \wp\left( \nu_1^{(1)} \omega_1^{(1)}(x) + \nu_2^{(1)} \omega_2^{(1)}(x) + v^{(1)}(x; \nu_1^{(1)}, \nu_2^{(1)}) \right) + \frac{1 + x}{3} \]  (28)
The formulae above have limits for
\[ e^{i \pi \nu_2(i)} = \frac{i \Gamma^4 \left(1 - \frac{\nu_2(0)}{2}\right)}{2 \sin(\pi \nu_2(0)) \Gamma^2 \left(\frac{3}{2} - \mu - \frac{\nu_2(0)}{2}\right) \Gamma^2 \left(\frac{1}{2} + \mu - \frac{\nu_2(0)}{2}\right)} \left[2(1 - e^{i \pi \nu_2(0)}) - \right. \\
\left. - f(x_0, x_1, x_\infty)(x_\infty^2 - e^{i \pi \nu_2(0)} x_1^2) \right] f(x_0, x_1, x_\infty) \]
where
\[ f(x_0, x_1, x_\infty) := \frac{4 - x_0^2}{x_1^2 + x_\infty^2 - x_0 x_1 x_\infty}, \quad \alpha = \frac{(2 \mu - 1)^2}{2} \]

Moreover, \( \exp{-i \pi \nu_1^0} \), \( \exp{i \pi \nu_1^\infty} \) are given by an analogous formula with the substitutions \((x_0, x_1, x_\infty) \mapsto (x_1, x_0, x_0 x_1 - x_\infty), \nu_2^0 \mapsto \nu_1^1 \) and \((x_0, x_1, x_\infty) \mapsto (x_\infty, -x_1, x_0 - x_1 x_\infty), \nu_2^0 \mapsto \nu_1^\infty \) respectively.

The most general choice of \( \nu_2 \) is \( 0 \leq \Re \nu_2 < 2 \). This corresponds to the fact that the transcendental \( y(x; x_0, x_1, x_\infty) \) also has three representations
\[ y(x; x_0, x_1, x_\infty) = \varphi(\tilde{v}_1^0 \omega_1^0(x) + \tilde{v}_2^0 \omega_2^0(x) + v^0(x; \tilde{v}_1^0, \tilde{v}_2^0)) + \frac{1 + x}{3} \]

where
\[ \cos \pi \nu_2(i) = \frac{x_i^2}{2} - 1, \quad 0 \leq \Re \nu_2(i) < 2, \quad i = 0, 1, \infty \]

The parameter \( \tilde{v}_1^0 \) is obtained by the formula
\[ e^{-i \pi \nu_1^0} = \frac{i \Gamma^4 \left(\frac{\tilde{v}_1^0}{2}\right)}{2 \sin(\pi \nu_1^0) \Gamma^2 \left(\frac{1}{2} - \mu + \frac{\nu_1^0}{2}\right) \Gamma^2 \left(-\frac{1}{2} + \mu + \frac{\nu_1^0}{2}\right)} \left[2(1 - e^{-i \pi \nu_1^0}) - \right. \\
\left. - f(x_0, x_1, x_\infty)(x_\infty^2 - e^{-i \pi \nu_1^0} x_1^2) \right] f(x_0, x_1, x_\infty). \]
exp\{i \pi \nu_1^1\}, \exp\{-i \pi \nu_1^\infty\} are given by an analogous formula with the substitutions \((x_0, x_1, x_\infty) \mapsto (x_1, x_0, x_0 x_1 - x_\infty), \tilde{v}_2^0 \mapsto \tilde{v}_1^1 \) and \((x_0, x_1, x_\infty) \mapsto (x_\infty, -x_1, x_0 - x_1 x_\infty), \tilde{v}_2^0 \mapsto \tilde{v}_1^\infty \) respectively.

The formulae above have limits for \( \nu_2 = 1, 1 \pm 2 \mu + 2m, m \text{ integer} \). They are listed in subsection 8.2.

Conversely, a transcendental
\[ y(x) = \varphi(\nu_1 \omega_1^0(x) + \nu_2 \omega_2^0(x) + v^0(x; \nu_1, \nu_2)) + \frac{1 + x}{3}, \quad \text{at } x = 0 \]

coincides with \( y(x; x_0, x_1, x_\infty) \), with the following monodromy data.

If \( 0 \leq \Re \nu_2 \leq 1 \):
\[ x_0 = 2 \cos \left(\frac{\pi}{2} \nu_2\right) \]

\[ (0) \]
where
\[ f(\nu_2, \mu) = -\frac{2 \sin^2 \left( \frac{\pi \nu_2}{2} \right)}{\cos(\pi \nu_2) + \cos(2\pi \mu)} , \quad G(\nu_2, \mu) = 4^{-\nu_2} 2 \Gamma \left( \frac{1}{2} - \mu - \frac{\nu_2}{2} \right) \Gamma \left( \frac{1}{2} + \mu - \frac{\nu_2}{2} \right) \]

If \( 1 \leq \Re \nu_2 < 2 \):
\[
x_0 = 2 \cos \left( \frac{\pi}{2} \nu_2 \right)
\]
\[
x_1 = \left[ \frac{e^{-i\frac{\pi}{2} \nu_1}}{4^{1-\nu_2} 2 f(\nu_2, \mu) G(\nu_2, \mu)} + \frac{4^{1-\nu_2} 2 G_1(\nu_2, \mu)}{e^{-i\frac{\pi}{2} \nu_1}} \right]
\]
\[
x_\infty = \left[ \frac{e^{i\frac{\pi}{2} (\nu_2 - \nu_1)}}{4^{1-\nu_2} 2 f(\nu_2, \mu) G(\nu_2, \mu)} + \frac{4^{1-\nu_2} 2 G_1(\nu_2, \mu)}{e^{i\frac{\pi}{2} (\nu_2 - \nu_1)}} \right]
\]
where
\[ G_1(\nu_2, \mu) = \frac{1}{4^{1-\nu_2} 2} \left( \frac{\nu_2}{2} \right)^2 \frac{\Gamma \left( \frac{\nu_2}{2} \right) \Gamma \left( \frac{1}{2} - \mu + \frac{\nu_2}{2} \right)}{\Gamma \left( \frac{1}{2} - \mu + \frac{\nu_2}{2} \right) \Gamma \left( -\frac{1}{2} + \mu + \frac{\nu_2}{2} \right)} \]

After computing the monodromy data, we can write the elliptic representations of \( y(x; x_0, x_1, x_\infty) \) at \( x = 1 \) and \( x = \infty \), namely \([23], [29]\). Since they are the elliptic representations at \( x = 1, x = \infty \) of (30), we have solved the connection problem for (30).

We observed that there is a one to one correspondence between Painlevé transcendents and triples of monodromy data \((x_0, x_1, x_\infty)\), defined up to the change of two signs, satisfying \( x_i \neq \pm 2, i = 0, 1, \infty, \) i.e. \( \nu_2 \neq 0 \) (and 2), and at most one \( x_j = 0 \). The cases when these conditions are not satisfied are studied in [23]. However, if \( x_i = \pm 2 \) (namely the trace is \(-2\)) the problem of finding the critical behavior at the corresponding critical point \( x = i \) is still open (except when all the three \( x_i \) are \( \pm 2 \); in this case there is a one-parameter class of solutions called Chazy solutions in [23]). We conclude that the results of our paper (and [11]), plus the results of [23] cover all the possible transcendents, except the special case when one or two \( x_i \) are \( \pm 2 \). We plan to cover this last case soon.

Finally, we expect that in all non-generic cases we can solve the connection problem and express the parameters \( \nu_1, \nu_2 \) in terms of monodromy data. From the conceptual point of view nothing should change with respect to [13] [7] [11] and the present paper; but the technical details may require a long time for computations.

2 The Elliptic Representation

We derive the elliptic form for the general Painlevé 6 equation. We follow [8]. Let
\[
u = \int_{-\infty}^{y} \frac{d\lambda}{\sqrt{\lambda(\lambda - 1)(\lambda - x)}} \tag{31}
\]
We observe that
\[
\frac{du}{dx} = \frac{\partial u}{\partial y} \frac{dy}{dx} + \frac{\partial u}{\partial x} = \frac{1}{\sqrt{y(y - 1)(y - x)}} \frac{dy}{dx} + \frac{\partial u}{\partial x}
\]
from which we compute
\[
\frac{d^2 u}{dx^2} + \frac{2 x - 1}{x(x - 1)} \frac{du}{dx} + \frac{u}{4 x(x - 1)} =
\]
\[
= \frac{1}{\sqrt{y(y - 1)(y - x)}} \left[ \frac{d^2 y}{dx^2} + \left( \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{y - x} \right) \frac{dy}{dx} - \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - x} \right) \left( \frac{dy}{dx} \right)^2 \right]
\]
\[ \frac{\partial^2 u}{\partial x^2} + \frac{2x - 1}{x(x-1)} \frac{\partial u}{\partial x} + \frac{u}{4x(x-1)} \]

By direct calculation we have:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{2x - 1}{x(x-1)} \frac{\partial u}{\partial x} + \frac{u}{4x(x-1)} = \frac{1}{2} \frac{\sqrt{y(y-1)(y-x)}}{x(x-1)} \frac{1}{(y-x)^2} \]

Therefore, \( y(x) \) satisfies PVI if and only if

\[ \frac{d^2 u}{dx^2} + \frac{2x - 1}{x(x-1)} \frac{du}{dx} + \frac{u}{4x(x-1)} = \frac{\sqrt{y(y-1)(y-x)}}{2x^2(1-x)^2} \left[ 2\alpha + 2\beta \frac{x}{y^2} + \gamma \frac{x - 1}{(y-1)^2} + \left( \delta - \frac{1}{2} \right) \frac{x(x-1)}{(y-x)^2} \right] \]  \( (32) \)

We invert the function \( u = u(y) \) by observing that we are dealing with an elliptic integral. Therefore, we write

\[ y = f(u, x) \]

where \( f(u, x) \) is an elliptic function of \( u \). This implies that

\[ \frac{\partial y}{\partial u} = \frac{\sqrt{y(y-1)(y-x)}}{2x^2(1-x)^2} \]

The above equality allows us to rewrite \( (32) \) in the following way:

\[ x(1-x) \frac{d^2 u}{dx^2} + (1-2x) \frac{du}{dx} - \frac{1}{4} u = \frac{1}{2x(1-x)} \frac{\partial}{\partial u} \psi(u, x), \]  \( (33) \)

where

\[ \psi(u, x) := 2\alpha f(u, x) - 2\beta \frac{x}{f(u, x)} + 2\gamma \frac{1-x}{f(u, x) - 1} + (1 - 2\delta)x \frac{x(x-1)}{f(u, x) - x} \]

The last step concerns the form of \( f(u, x) \). We observe that \( 4\lambda(\lambda - 1)(\lambda - x) \) is not in Weierstrass canonical form. We change variable:

\[ \lambda = t + \frac{1 + x}{3}, \]

and we get the Weierstrass form:

\[ 4\lambda(\lambda - 1)(\lambda - x) = 4t^3 - g_2 t - g_3, \quad g_2 = \frac{4}{3}(1-x+x^2), \quad g_3 := \frac{4}{27}(x-2)(2x-1)(1+x) \]

Thus

\[ \frac{u}{2} = \int_{\infty}^{y} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} \]

which implies

\[ f(u, x) = \mathcal{P} \left( \frac{u}{2}, \omega_1, \omega_2 \right) + \frac{1+x}{3} \]

We still need to explain what are the half periods \( \omega_1, \omega_2 \). In order to do that, we first observe that the Weierstrass form is

\[ 4t^3 - g_2 t - g_3 = 4(t - e_1)(t - e_2)(t - e_3) \]

where

\[ e_1 = \frac{2-x}{3}, \quad e_2 = \frac{2x-1}{3}, \quad e_3 = -\frac{1+x}{3}. \]

Therefore

\[ g := \sqrt{e_1 - e_2} = 1, \quad \kappa^2 := \frac{e_2 - e_3}{e_1 - e_3} = x, \quad \kappa'^2 := 1 - \kappa^2 = 1 - x \]

We identify \( e_1 = \varphi(\omega_1), e_2 = \varphi(\omega_1 + \omega_2), e_3 = \varphi(\omega_2) \). Therefore, the half-periods are

\[ \omega_1 = \frac{1}{g} \int_{0}^{1} \frac{d\xi}{\sqrt{(1-x^2)(1-\kappa^2 \xi^2)}} = \int_{0}^{1} \frac{d\xi}{\sqrt{(1-x^2)(1-\kappa'^2 \xi^2)}} = K(x) \]
The elliptic integral $K\bigg| \frac{1}{2} \bigg|$ whose convergent series for $|x| < 1$ is:

$$F\left(\frac{1}{2}, \frac{1}{2}, 1; x\right) = \sum_{n=0}^{\infty} \frac{[(\frac{1}{2})_n]^2}{(n!)^2} x^n.$$  

Namely:

$$K(x) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; x\right)$$

Moreover, $K(x)$ and $K(1 - x)$ are two linearly independent solutions of the hyper-geometric equation

$$x(1 - x)\omega'' + (1 - 2x)\omega' - \frac{1}{4}\omega = 0.$$  

Observe that for $|\arg(x)| < \pi$:

$$-\pi F\left(\frac{1}{2}, \frac{1}{2}, 1; 1 - x\right) = F\left(\frac{1}{2}, \frac{1}{2}, 1; x\right) \ln(x) + F_1(x)$$

where

$$F_1(x) := \sum_{n=0}^{\infty} \frac{[(\frac{1}{2})_n]^2}{(n!)^2} \left[ \psi(n + \frac{1}{2}) - \psi(n + 1) \right] x^n,$$

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z), \quad \psi\left(\frac{1}{2}\right) = -\gamma - 2 \ln 2, \quad \psi(1) = -\gamma, \quad \psi(a + n) = \psi(a) + \sum_{l=0}^{n-1} \frac{1}{a + l}.$$  

Therefore

$$\omega_2(x) = -\frac{i}{2} \left[ F\left(\frac{1}{2}, \frac{1}{2}, 1; x\right) \ln(x) + F_1(x) \right]$$

In the following we use sometimes the abbreviation $F(x)$ for $F\left(\frac{1}{2}, \frac{1}{2}, 1; x\right)$. The series of $F(x)$ and $F_1(x)$ converge for $|x| < 1$.

Let

$$\mathcal{L}(u) := x(1 - x) \frac{d^2 u}{dx^2} + (1 - 2x) \frac{du}{dx} - \frac{1}{4} u$$

The PVI equation becomes

$$\mathcal{L}(u) = \frac{1}{2x(1 - x)} \frac{\partial}{\partial u} \left\{ 2\alpha \left[ \varphi\left(\frac{u}{2}; \omega_1, \omega_2\right) - e_3 \right] - 2\beta \frac{x}{\varphi\left(\frac{u}{2}; \omega_1, \omega_2\right) - e_3} + + 2\gamma \frac{1 - x}{\varphi\left(\frac{u}{2}; \omega_1, \omega_2\right) - e_1} + (1 - 2\delta) \frac{x(1 - x)}{\varphi\left(\frac{u}{2}; \omega_1, \omega_2\right) - e_2} \right\}$$  

(34)

We recall that

$$\frac{1}{\varphi\left(\frac{u}{2}; \omega_1, \omega_2\right) - e_3} = \frac{1}{(e_1 - e_3)(e_2 - e_3)} \left[ \varphi\left(\frac{u}{2} + \omega_2\right) - e_3 \right]$$

$$\frac{1}{\varphi\left(\frac{u}{2}; \omega_1, \omega_2\right) - e_1} = \frac{1}{(e_1 - e_2)(e_1 - e_3)} \left[ \varphi\left(\frac{u}{2} + \omega_1\right) - e_1 \right]$$

$$\frac{1}{\varphi\left(\frac{u}{2}; \omega_1, \omega_2\right) - e_2} = \frac{1}{(e_1 - e_2)(e_3 - e_2)} \left[ \varphi\left(\frac{u}{2} + \omega_1 + \omega_2\right) - e_2 \right].$$

We observe also that $e_1 - e_3 = 1$, $e_2 - e_3 = x$, $e_1 - e_2 = 1 - x$. Therefore, (34) becomes:

$$\mathcal{L}(u) = \frac{1}{2x(1 - x)} \left[ 2\alpha \frac{\partial}{\partial u} \varphi\left(\frac{u}{2}; \omega_1, \omega_2\right) - 2\beta \frac{x}{\varphi\left(\frac{u}{2}; \omega_1, \omega_2\right) - e_3} + + 2\gamma \frac{\partial}{\partial u} \varphi\left(\frac{u}{2}; \omega_1, \omega_2\right) + (1 - 2\delta) \frac{\partial}{\partial u} \varphi\left(\frac{u}{2}; \omega_1, \omega_2\right) \right]$$  

(35)
3 Proof of Theorem 1

For technical reason it is convenient to introduce the following definitions:

\[ D_1(r; \nu_1, \nu_2) := \left\{ x \in \mathbb{C}_0 \text{ such that } |x| < r, \left| \frac{e^{-i\pi \nu_1}}{16^{1-\nu_2}} x^{1-\nu_2} \right| < r, \left| \frac{e^{i\pi \nu_1}}{16^{\nu_2}} x^{\nu_2} \right| < r \right\} \] (36)

\[ D_2(r; \nu_1, \nu_2) := \left\{ x \in \mathbb{C}_0 \text{ such that } |x| < r, \left| \frac{e^{-i\pi \nu_1}}{16^{2-\nu_2}} x^{2-\nu_2} \right| < r, \left| \frac{e^{i\pi \nu_1}}{16^{\nu_2-1}} x^{\nu_2-1} \right| < r \right\} \] (37)

Note that \( D_1(r; \nu_1, \nu_2) \equiv D(r; \nu_1, 2 - \nu_2) \) in (36) and \( D_2(r; \nu_1, \nu_2) \equiv D(r; -\nu_1, 2 - \nu_2) \). Let \( \tau(x) := \frac{\omega_2(x)}{\omega_1(x)} = \frac{1}{\pi} (\text{arg} x - i \ln \frac{|x|}{16}) + O(x) \)

be the modular parameter of the elliptic function, obtained expanding \( \omega_1(x), \omega_2(x) \) at \( x = 0 \). We expand the r.h.s of (33) in Fourier series. We are dealing with Weierstrass \( \wp \)-functions of the form

\[ \wp \left( \frac{x}{2} + \varepsilon_1 \omega_1 + \varepsilon_2 \omega_2 \right), \quad \varepsilon_i \in \{0, 1\}, \quad i = 1, 2 \]

As it is well known \[33\], the Fourier expansion of the \( \wp \)-function is

\[ \wp(z; \omega_1, \omega_2) = \left( \frac{\pi}{2\omega_1} \right)^2 \left[ -\frac{1}{3} + \frac{1}{\sin^2 \left( \frac{z}{2\omega_1} \right)} \right] + 8 \sum_{n \geq 1} \left( \frac{n \pi}{\omega_1} \right)^2 \left[ 1 - \cos \left( \frac{n \pi}{\omega_1} \right) \right], \quad \text{for } \Im \tau > \Im \left( \frac{\tau}{2\omega_1} \right) \]

Therefore, the expansion is possible provided that

\[ \Im \left( \frac{u(x)}{4\omega_1(x) + \varepsilon_1 + 2N_1} + \frac{\varepsilon_2 + 2N_2}{2} \tau(x) \right) < \Im \tau(x) \] (38)

We observe that for \( x \to 0 \)

\[ \Im \tau(x) = -\frac{1}{\pi} \ln \frac{|x|}{16} + O(x) \to +\infty \]

The condition (38) becomes

\[ -\Im \tau < \frac{1}{2} \Im \nu_1 \left[ + \frac{1}{2} \Im (\varepsilon_2 + 2N_2 + \Re \nu_2) \right] \Im \tau + \frac{1}{2} \Im \nu_2 \Re \tau + \Im \left( \frac{u}{2\omega_1} \right) < \Im \tau \] (39)

If \( \nu_2 \) is real, we divide by \( \Im \tau \), we let \( x \to 0 \) and we obtain

\[ -2 - 2N_2 - \varepsilon_2 < \nu_2 < 2 - 2N_2 - \varepsilon_2 \]

provided that \( v(x) \) is bounded. In general, (38) becomes

\[ (\Re \nu_2 + 2 + \varepsilon_2 + 2N_2) \ln \frac{|x|}{16} - \pi \Im \nu_1 + \Im \left( \frac{u}{2\omega_1} \right) \right] + O(x) \]

\[ \leq \Im \nu_2 \arg x \leq \]

\[ (\Re \nu_2 - 2 + \varepsilon_2 + 2N_2) \ln \frac{|x|}{16} - \pi \Im \nu_1 + \Im \left( \frac{u}{2\omega_1} \right) \right] + O(x) \] (40)

In the domain defined by the above condition, and \( |x| < 1 \), we do the Fourier expansion as follows

\[ \frac{\partial}{\partial u} \wp \left( \frac{x}{2} + \varepsilon_1 + 2N_1 \omega_1 + \varepsilon_2 + 2N_2 \omega_2 \right) = \]

\[ = \frac{\pi^3}{8\omega_1^3} \left\{ 8 \sum_{n \geq 1} \frac{n^2 e^{2\pi i n \tau}}{1 - e^{2\pi i n \tau}} \sin \left( \pi n \left( \frac{u}{2\omega_1} + \varepsilon_1 + (\varepsilon_2 + 2N_2) \tau \right) \right) - \frac{\cos \left( \frac{\pi}{2} \left( \frac{u}{2\omega_1} + \varepsilon_1 + (\varepsilon_2 + 2N_2) \tau \right) \right)}{\sin^3 \left( \frac{\pi}{2} \left( \frac{u}{2\omega_1} + \varepsilon_1 + (\varepsilon_2 + 2N_2) \tau \right) \right)} \right\} \]
\[
= \frac{i\pi^3}{2\omega^3} \left\{ \frac{e^{i\frac{f(x)}{2}} + e^{-i\frac{f(x)}{2}}}{[e^{i\frac{f(x)}{2}} - e^{-i\frac{f(x)}{2}}]^3} - \sum_{n \geq 1} \frac{n^2e^{2i\pi n \tau}}{1 - e^{2i\pi n \tau}} \left[ e^{inf(x)} - e^{-inf(x)} \right] \right\} \tag{41}
\]

where

\[
e^{if(x)} = \exp \left\{ i\tau \left[ (\nu_1 + \varepsilon_1) + (\nu_2 + \varepsilon_2 + 2N_2\tau(x) + \frac{\nu(x)}{\omega_1(x)} \right] \right\}
\]

Note that \( N_1 \) does not appear in the expansion. So, we can take \( N_1 = 0 \) in the following.

Now we make the further assumption that

\[
|e^{if(x)}| < 1 \tag{42}
\]

and we rewrite (41) as follows:

\[
\frac{i\pi^3}{2\omega^3} \left\{ \frac{e^{2i f(x)} + e^{i f(x)}}{[e^{i f(x)} - 1]^3} - \sum_{n \geq 1} \frac{n^2e^{i\pi n}[-(\nu_1 + \varepsilon_1) + (2 - \nu_2 - \varepsilon_2 - 2N_2\tau(x) - \frac{\nu(x)}{\omega_1(x)}]}{1 - e^{2i\pi n \tau}} \left[ e^{2inf(x)} - 1 \right] \right\}
\]

Due to (42), the denominator in the first term does not vanish, so that the expansion has no poles. The condition (42) is more restrictive than (40) and yields:

\[
(\Re \nu_2 + \varepsilon_2 + 2N_2) \ln \frac{|x|}{16} - \left\{ \pi \Im \nu_1 + \Im \left( \frac{\nu}{\omega_1} \right) \right\} + O(x)
\]

\[
\leq \Im \nu_2 \Im x \leq (\Re \nu_2 - \varepsilon_2 + 2N_2) \ln \frac{|x|}{16} - \left\{ \pi \Im \nu_1 + \Im \left( \frac{\nu}{\omega_1} \right) \right\} + O(x) \tag{43}
\]

We observe that, for any complex number \( C \),

\[
e^{i\pi C \tau(x)} = h(x)^C \left[ \frac{x}{16} \right]^C,
\]

where

\[
h(x) := e^C \left( \frac{4\pi}{\omega_1} + 4\ln 2 \right) = 1 + O(x), \quad x \to 0
\]

Therefore,

\[
\frac{\partial}{\partial u} \varphi \left( \frac{u}{2} + (\varepsilon_1 + 2N_1)\omega_1 + (\varepsilon_2 + 2N_2)\omega_2 \right) = \frac{i\pi^3}{2\omega^3} \left\{ \frac{A^2 + A}{(A - 1)^3} - \sum_{n \geq 1} \frac{n^2B^n}{1 - h(x)^n \left( \frac{x}{16} \right)^2n} \left( A^{2n} - 1 \right) \right\}
\]

where

\[
A = A \left( x, e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2 + \varepsilon_2 + 2N_2} e^{i\frac{\pi}{\omega_1}} \right) := h(x)^{\nu_2 + \varepsilon_2 + 2N_2} e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2 + \varepsilon_2 + 2N_2} e^{i\frac{\pi}{\omega_1}} \tag{44}
\]

\[
B = B \left( x, e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2 - \varepsilon_2 - 2N_2} e^{-i\frac{\pi}{\omega_1}} \right) := h(x)^{2\nu_2 - \varepsilon_2 - 2N_2} e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{2\nu_2 - \varepsilon_2 - 2N_2} e^{-i\frac{\pi}{\omega_1}} \tag{45}
\]

In the r.h.s. of equation (45), we have four \( \varphi \)-functions with different arguments. We recall that we can take \( N_1 = 0 \). As for \( N_2 \), it can be different in each \( \varphi \)-function. There are only two possibilities in order for the four resulting domains (43) to intersect: the \( \varphi \)-functions must be

\[
\varphi \left( \frac{u}{2} + 2N_2\omega_2 \right), \quad \varphi \left( \frac{u}{2} + \omega_1 + 2N_2\omega_2 \right), \quad \varphi \left( \frac{u}{2} + \omega_2 + 2N_2\omega_2 \right), \quad \varphi \left( \frac{u}{2} + 2N_2\omega_2 \right),
\]

and

\[
(a) : \quad N_2' = N_2
\]
or
\[ (b) : \quad N'_2 = N_2 - 1 \]

Thus, the r.h.s. of (33) is expanded in Fourier series if \(|x| < 1\), in the following domains.

Case (a):
\[
(\Re \nu_2 + 2N) \ln \frac{|x|}{16} - \left[ \pi \Im \nu_1 + \Im \left( \frac{\pi v}{\omega_1} \right) \right] + O(x) < \Im \nu_2 \arg x < \\
< (\Re \nu_2 - 1 + 2N) \ln \frac{|x|}{16} - \left[ \pi \Im \nu_1 + \Im \left( \frac{\pi v}{\omega_1} \right) \right] + O(x), \quad \text{for } N'_2 = N_2 =: N
\]

(46)

Case (b):
\[
(\Re \nu_2 - 1 + 2N) \ln \frac{|x|}{16} - \left[ \pi \Im \nu_1 + \Im \left( \frac{\pi v}{\omega_1} \right) \right] + O(x) < \Im \nu_2 \arg x < \\
< (\Re \nu_2 - 2 + 2N) \ln \frac{|x|}{16} - \left[ \pi \Im \nu_1 + \Im \left( \frac{\pi v}{\omega_1} \right) \right] + O(x), \quad \text{for } N'_2 + 1 = N_2 =: N
\]

(47)

Note that if \( \Im \nu_2 = 0 \) any value of \( \arg(x) \) is allowed, but dividing the previous expressions by \( \ln |x| \) and letting \( x \to 0 \) we have
\[-2N < \nu_2 < 1 - 2N \text{ for (a), } \quad 1 - 2N < \nu_2 < 2 - 2N \text{ for (b)}\]

provided that \( v(x) \) is bounded for \( x \to 0 \). Taking \([44] [45]\) into account in the two cases, we conclude that:

In case (a), for \( N'_2 = N_2 = N \), the following powers appear in the r.h.s. of (33):
\[ x^{2-\nu_2-2N}, \quad x^{\nu_2+2N}, \quad x^{1-\nu_2-2N}, \quad x^{1+\nu_2+2N} \]

The dominant powers are then
\[ x^{\nu_2+2N}, \quad x^{1-\nu_2-2N} \]

and we can write (33) as;
\[
\mathcal{L}(u) = \frac{1}{2x(1-x)} \mathcal{F} \left( x, e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{2-\nu_2-2N} e^{-i\pi \frac{v}{\omega_1}}, e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2+2N} e^{i\pi \frac{v}{\omega_1}} \right)
\]
on the domain defined by \(|x| < 1\) and (46), assuming that \( v(x) \) is bounded.

In case (b), for \( N'_2 + 1 = N_2 = N \), the following powers appear in the r.h.s. of (33):
\[ x^{2-\nu_2-2N}, \quad x^{\nu_2+2N}, \quad x^{3-\nu_2-2N}, \quad x^{\nu_2-1+2N} \]

The dominant powers are then
\[ x^{2-\nu_2-2N}, \quad x^{\nu_2-1+2N} \]

and we can write (33) as;
\[
\mathcal{L}(u) = \frac{1}{2x(1-x)} \mathcal{F} \left( x, e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{2-\nu_2-2N} e^{-i\pi \frac{v}{\omega_1}}, e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2-1+2N} e^{i\pi \frac{v}{\omega_1}} \right)
\]
on the domain defined by \(|x| < 1\) and (47), assuming that \( v(x) \) is bounded.

Let \( \epsilon < 1 \) be sufficiently small: in both cases, \( \mathcal{F}(x,y,z) \) is holomorphic for \(|x|, |y|, |z| < \epsilon \) and \( \mathcal{F}(0,0,0) = 0 \). In (a) we decompose
\[
\mathcal{F} \left( x, e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{1-\nu_2-2N} e^{-i\pi \frac{v}{\omega_1}}, e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2+2N} e^{i\pi \frac{v}{\omega_1}} \right) = \\
+ \mathcal{F} \left( x, e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{1-\nu_2-2N}, e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2+2N} \right)
\]
where
\[
\mathcal{G}\left(x, e^{-i\pi \nu_1} \left(\frac{x}{16}\right)^{1-\nu_2-2N}, e^{i\pi \nu_1} \left(\frac{x}{16}\right)^{\nu_2+2N}, v\right) :=
\]
\[
\mathcal{F}\left(x, e^{-i\pi \nu_1} \left(\frac{x}{16}\right)^{1-\nu_2-2N} e^{-i\pi \nu_1}, e^{i\pi \nu_1} \left(\frac{x}{16}\right)^{\nu_2+2N} e^{-i\pi \nu_1}\right) - \]
\[
-\mathcal{F}\left(x, e^{-i\pi \nu_1} \left(\frac{x}{16}\right)^{1-\nu_2-2N} e^{i\pi \nu_1} \left(\frac{x}{16}\right)^{\nu_2+2N} e^{i\pi \nu_1}\right)
\]

We note that \(\mathcal{L}(u) = \mathcal{L}(u_0 + 2v) = \mathcal{L}(u_0) + 2\mathcal{L}(v)\). Let us put
\[
w(x) := x \frac{d}{dx} v(x)
\]
Equation (35) becomes the system
\[
\begin{cases}
    x \frac{d}{dx} u = w \\
    x \frac{d}{dx} v = \Phi + \Psi
\end{cases}
\]
where
\[
\Phi = \Phi\left(x, e^{-i\pi \nu_1} \left(\frac{x}{16}\right)^{1-\nu_2-2N}, e^{i\pi \nu_1} \left(\frac{x}{16}\right)^{\nu_2+2N}\right) :=
\]
\[
= \frac{1}{4(1-x)^2} \mathcal{F}\left(x, e^{-i\pi \nu_1} \left(\frac{x}{16}\right)^{1-\nu_2-2N}, e^{i\pi \nu_1} \left(\frac{x}{16}\right)^{\nu_2+2N}\right)
\]
and
\[
\Psi = \Psi\left(x, e^{-i\pi \nu_1} \left(\frac{x}{16}\right)^{1-\nu_2-2N}, e^{i\pi \nu_1} \left(\frac{x}{16}\right)^{\nu_2+2N}, v, w\right) :=
\]
\[
= x(w + v/4) \frac{1}{1-x} + \frac{1}{4(1-x)^2} \mathcal{G}\left(x, e^{-i\pi \nu_1} \left(\frac{x}{16}\right)^{1-\nu_2-2N}, e^{i\pi \nu_1} \left(\frac{x}{16}\right)^{\nu_2+2N}, v\right)
\]
In the same way, in case (b) we obtain a similar system, where
\[
\Phi = \Phi\left(x, e^{-i\pi \nu_1} \left(\frac{x}{16}\right)^{2-\nu_2-2N}, e^{i\pi \nu_1} \left(\frac{x}{16}\right)^{\nu_2-1+2N}\right)
\]
\[
\Psi = \Psi\left(x, e^{-i\pi \nu_1} \left(\frac{x}{16}\right)^{2-\nu_2-2N}, e^{i\pi \nu_1} \left(\frac{x}{16}\right)^{\nu_2-1+2N}, v, w\right)
\]
By construction, in both cases \(\Phi(x, y, z)\) and \(\Psi(x, y, z, v, w)\) are holomorphic of their arguments for \(|x|, |y|, |z|, |v|, |w| < \epsilon\) for sufficiently small \(\epsilon < 1\) (assuming \(v(x)\) bounded in a neighborhood of \(x = 0\)).
Moreover
\[
\Phi(0, 0, 0) = \Psi(0, 0, 0, v, w) = \Phi(x, y, z, 0, 0) = 0
\]
We complete the proof of the theorem in case (a). Case (b) is analogous. We reduce the system of differential equations to a system of integral equations
\[
w(x) = \int_{L(x)} \frac{1}{s} \left\{ \Phi(s, e^{-i\pi \nu_1} \left(\frac{s}{16}\right)^{1-\nu_2-2N}, e^{i\pi \nu_1} \left(\frac{s}{16}\right)^{\nu_2+2N} + \right.\]
\[
\left. + \Psi(s, e^{-i\pi \nu_1} \left(\frac{s}{16}\right)^{1-\nu_2-2N}, e^{i\pi \nu_1} \left(\frac{s}{16}\right)^{\nu_2+2N}, v(s), w(s) \right\} ds\]
\[
v(x) = \int_{L(x)} \frac{1}{s} w(s) ds
\]
The point \(x\) and the path of integration are chosen to belong to the domain where \(|x|, |e^{-i\pi \nu_1} x^{1-\nu_2-2N}|, |e^{i\pi \nu_1} x^{\nu_2+2N}|, |w(x)|, |v(x)|\) are less than \(\epsilon\), in such a way that \(\Phi\) and \(\Psi\) are holomorphic. That such a domain is not empty will be shown below. In particular, we’ll show that if we require that \(|x| < r\),
such that \( |e^{-i\pi\epsilon} x^{1-\nu}| < r, |e^{i\pi\epsilon} x^{\nu^*}| < r \), where \( r < \epsilon \) is small enough, also the solutions of the integral equations \(|v(x)|\) and \(|w(x)|\) are less than \( \epsilon \). Such a domain is contained in \((R)\).

We choose the path of integration \( L(x) \) connecting 0 to \( x \), defined by

\[
\arg(s) = \arg(x) + \frac{\Re \nu_2 + 2N - \nu^*}{3\nu_2} \log \frac{|s|}{|x|}, \quad 0 < |s| \leq |x|, \quad \nu^* \in \mathbb{C}
\]

If \( x \) belongs to the domain \((R)\) in case (a) and to \((L)\) in case (b), than the path does not leave the domain when \( s \to 0 \), provided that \( 0 \leq \nu^* \leq 1 \) in case (a), or \( 1 \leq \nu^* \leq 2 \) in case (b). If \( 3\nu_2 = 0 \) we take the radial path \( \arg(s) = \arg(x), 0 < |s| \leq |x| \), namely \( \nu^* = \nu_2 \). The parameterization of the path is

\[
s = \rho \ e^{\frac{\arg x}{3\nu_2} \log \frac{|s|}{|x|}}, \quad 0 < \rho \leq |x|
\]

therefore

\[
|ds| = P(\nu_2 + 2N, \nu^*) \ d\rho, \quad P(\nu_2 + 2N, \nu^*) := \sqrt{1 + \left( \frac{\Re \nu_2 + 2N - \nu^*}{3\nu_2} \right)^2}
\]

For any complex numbers \( A, B \) we have

\[
\int_{L(x)} \frac{1}{|s|} \left( |s| + |As^{1-\nu_2-2N}| + |Bs^{\nu_2+2N}| \right)^n |ds| \leq \frac{P(\nu_2 + 2N, \nu^*)}{n \min(\nu^*, 1 - \nu^*)} \left( |x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}| \right)^n
\]

The quantity \( \min(\nu^*, 1 - \nu^*) \) is substituted by \( \min(\nu^*, 1, 2 - \nu^*) \) in case (b). To prove \((48)\) we observe that on \( L(x) \) we have

\[
|s^{\nu_2+2N+\alpha}| = |x^{\nu_2+2N+\alpha}| \frac{|s|^\nu^{\alpha}}{|x|^\nu^{\alpha}}, \quad \forall \alpha \in \mathbb{C}
\]

Therefore

\[
\int_{L(x)} \frac{1}{|s|} |s|^\alpha \left| As^{1-\nu_2-2N} \right| \left| Bs^{\nu_2+2N} \right|^k |ds| = \frac{|Ax^{1-\nu_2-2N}|}{|x| (1 - \nu^*)^j \nu^k} \frac{|Bx^{\nu_2+2N}|}{|x|^\nu^k} P(\nu_2 + 2N, \nu^*) \int_0^{|x|} d\rho \rho^{i + j (1 - \nu^*) + \nu^* k} = \frac{P(\nu_2 + 2N, \nu^*)}{(i + j + k) \min(\nu^*, 1 - \nu^*)} |x|^\nu^{(1 - \nu^*) j} \left| Ax^{1-\nu_2-2N} \right| \left| Bx^{\nu_2+2N} \right|^k
\]

from which \((48)\) follows, provided that \( 0 < \nu^* < 1 \). For \( 3\nu_2 = 0 \), this yields again \( 0 < \nu_2 < 1 \).

We observe that a solution of the integral equations is also a solution of the differential equations, by virtue of the following lemma:

**Lemma 1:** Let \( f(x) \) be a holomorphic function in the domain \( |x| < \epsilon, |Ax^{1-\nu_2-2N}| < \epsilon, |Bx^{\nu_2+2N}| < \epsilon \), such that \( f(x) = O(|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|) \), \( A, B \in \mathbb{C} \). Let \( L(x) \) be the path of integration defined above for \( 0 < \nu^* < 1 \) and

\[
F(x) := \int_{L(x)} \frac{1}{s} f(s) \ ds
\]

Then, \( F(x) \) is holomorphic on the domain and \( \frac{dF(x)}{dx} = \frac{1}{x} f(x) \).

The same holds for a function \( f(x) = O(|x| + |Ax^{2-\nu_2-2N}| + |Bx^{\nu_2-1+2N}|) \) in the domain \( |x| < \epsilon, |Ax^{2-\nu_2-2N}| < \epsilon, |Bx^{\nu_2-1+2N}| < \epsilon \) (\( L(x) \) being defined for \( 1 < \nu^* < 2 \)).

**Proof:** We give the proof in case (a). We choose a point \( x + \Delta x \) close to \( x \) and we prove that

\[
\int_{L(x)} - \int_{L(x+\Delta x)} = \int_{x+\Delta x} x \neq \Delta x, \quad \text{where the last integral is on a segment from} \ x \ \text{to} \ x + \Delta x.
\]

We consider a small disk \( U_R \) centered at \( x = 0 \) of small radius \( R < |x| \) and the points \( x_R := L(x) \cap U_R, x'_R := L(x+\Delta x) \cap U_R \). Since the integral of \( f/s \) on a finite close curve (not containing 0) is zero we have:

\[
\left( \int_{L(x)} - \int_{L(x+\Delta x)} \right) \int_x^{x+\Delta x} ds \frac{f(s)}{s} = \left( \int_{L(x_R)} - \int_{L(x'_R)} \right) \int_{x_R}^{x'_R} ds \frac{f(s)}{s}
\]

(49)
The last integral is on the arc $\gamma (x R, x R')$ from $x R$ to $x R'$ on the circle $|s| = R$. We have also kept into account the obvious fact that $L(x R)$ is contained in $L(x)$ and $L(x R')$ is contained in $L(x + \Delta x)$. We then take $R \to 0$ and we prove that the r.h.s. of (49) vanishes.

Taking into account that $f(x) = O(|x| + |Ax^{1-\nu^2-2N}| + |Bx^{\nu_2+2N}|)$ and (48) we have

$$\left| \int_{L(x R)} \frac{1}{s} f(s) \, ds \right| \leq \int_{L(x R)} \frac{1}{|s|} O(|s| + |As^{1-\nu^2-2N}| + |Bs^{\nu_2+2N}|) \, ds$$

$$\leq \frac{P(\nu_2 + 2 N, \nu^*)}{\min(\nu^*, 1 - \nu^*)} O(|x R| + |Ax^{1-\nu^2-2N}| + |Bx^{\nu_2+2N}|) = \frac{P(\nu_2 + 2 N, \nu^*)}{\min(\nu^*, 2 - \nu^*)} O(R^{\min(\nu^*, 1 - \nu^*)})$$

The last step follows from $|x^{\nu_2+2N}| = \frac{|x^{\nu_2+2N}|}{|x^{\nu^*}|} R^{\nu^*}$. So the integral vanishes for $R \to 0$. The same is proved for $\int_{L(x + \Delta x)}$. As for the integral on the arc we have

$$|\arg x R - \arg x R'| = |\arg x - \arg (x + \Delta x) + \frac{\Re \nu_2 + 2 N - \nu^*}{\Im \nu_2} \log \left| 1 + \frac{\Delta x}{x} \right|$$

or $|\arg x R - \arg x R'| = |\arg x - \arg (x + \Delta x)|$ if $\Im \nu_2 = 0$. This is independent of $R$, therefore the length of the arc is $O(R)$ and

$$\left| \int_{\gamma(x R, x R')} \frac{1}{|s|} f(s) \, ds \right| = O(R^{\min(\nu^*, 1 - \nu^*)}) \to 0 \text{ for } R \to 0$$

Now we prove a fundamental lemma.

**Lemma 2:** For any $N \in \mathbb{Z}$ and for any complex $\nu_1, \nu_2$ such that

$$\nu_2 \notin (-\infty, -2N] \cup \{1 - 2N\} \cup [2 - 2N, +\infty)$$

there exists a sufficiently small $r_N < 1$ such that the system of integral equations has a solution $v_1(x; \nu_1, \nu_2 + 2N)$ holomorphic in the domain $D_1(r_N; \nu_1, \nu_2 + 2N)$ defined in (39). Moreover, there exists a positive constant $M_1(\nu_2 + 2 N)$ depending on $\nu_2 + 2 N$ such that

$$|v_1(x; \nu_1, \nu_2 + 2N)| \leq M_1(\nu_2 + 2 N) \left( |x| + \frac{e^{-i \pi \nu_1}}{16^{1-\nu^2-2N}} x^{1-\nu^2-2N} + \frac{e^{i \pi \nu_1}}{16^{\nu_2+2N}} x^{\nu_2+2N} \right)$$

in $D_1(r; \nu_1, \nu_2)$.

The system of integral equations has another solution $v_2(x; \nu_1, \nu_2 + 2N)$ holomorphic in $D_2(r; \nu_1, \nu_2)$ defined in (47). There exists a positive constant $M_2(\nu_2 + 2 N)$ such that

$$|v_2(x; \nu_1, \nu_2 + 2N)| \leq M_2(\nu_2 + 2 N) \left( |x| + \frac{e^{-i \pi \nu_1}}{16^{2-\nu^2-2N}} x^{2-\nu^2-2N} + \frac{e^{i \pi \nu_1}}{16^{\nu_2+2N}} x^{\nu_2+2N} \right)$$

in $D_2(r; \nu_1, \nu_2)$.

Note that $D_i(r_N; \nu_1, \nu_2 + 2N) = D_i(r_N; \nu_1 + 2N, \nu_2 + 2N)$, $i = 1, 2$, for any $N_1 \in \mathbb{Z}$. To prove Lemma 2 we need some sub-lemmas

**Sub-Lemma 1:** Let $\Phi(x, y, z)$ and $\Psi(x, y, z, v, w)$ be two holomorphic functions of their arguments for $|x|, |y|, |z|, |v|, |w| < \epsilon$, satisfying

$$\Phi(0, 0, 0) = 0, \quad \Psi(0, 0, 0, v, w) = \Psi(x, y, z, 0, 0) = 0$$

Then, there exists a constant $c > 0$ such that:

$$|\Phi(x, y, z)| \leq c \left( |x| + |y| + |z| \right) \quad (50)$$
\[ |\Psi(x, y, z, v, w)| \leq c \left( |x| + |y| + |z| \right) \]  
(51)
\[ |\Psi(x, y, z, v_2, w_2) - \Psi(x, y, z, v_1, w_1)| \leq c \left( |x| + |y| + |z| \right) \left( |v_2 - v_1| + |w_2 - w_1| \right) \]  
(52)
for \(|x|, |y|, |z|, |v|, |w| < \epsilon\).

**Proof:** Let’s prove (51).

\[
\Psi(x, y, z, v, w) = \int_0^1 \frac{d}{d\lambda} \left( \lambda x, \lambda y, \lambda z, v, w \right) d\lambda
\]
\[
x \int_0^1 \frac{\partial \Psi}{\partial x}(\lambda x, \lambda y, \lambda z, v, w) d\lambda + y \int_0^1 \frac{\partial \Psi}{\partial y}(\lambda x, \lambda y, \lambda z, v, w) d\lambda + z \int_0^1 \frac{\partial \Psi}{\partial z}(\lambda x, \lambda y, \lambda z, v, w) d\lambda
\]

Moreover, for \(\delta\) small:

\[
\frac{\partial \Psi}{\partial x}(\lambda x, \lambda y, \lambda z, v, w) = \int_{|\zeta - \lambda x| = \delta} \frac{\Psi(\zeta, \lambda y, \lambda z, v, w)}{2\pi i} d\zeta
\]

which implies that \(\frac{\partial \Psi}{\partial x}\) is holomorphic and bounded when its arguments are less than \(\epsilon\). The same holds true for \(\frac{\partial \Psi}{\partial y}\) and \(\frac{\partial \Psi}{\partial z}\). This proves (51), \(c\) being a constant which bounds \(\frac{\partial \Psi}{\partial x}, \frac{\partial \Psi}{\partial y}, \frac{\partial \Psi}{\partial z}\). The inequality (51) is proved in the same way. We turn to (52). First we prove that for \(|x|, |y|, |z|, |v_1|, |v_2|, |w_1|, |w_2| < \epsilon\) there exist two holomorphic and bounded functions \(\psi_1(x, y, z, v_1, w_1, v_2, w_2), \psi_2(x, y, z, v_1, w_1, v_2, w_2)\) such that

\[
\Psi(x, y, z, v_2, w_2) - \Psi(x, y, z, v_1, w_1) = (v_2 - v_1) \psi_1(x, y, z, v_1, w_1, v_2, w_2) + (w_2 - w_1) \psi_2(x, y, z, v_1, w_1, v_2, w_2)
\]
(53)

In order to prove this, we write

\[
\Psi(x, y, z, v_2, w_2) - \Psi(x, y, z, v_1, w_1) =
\]
\[
= \int_0^1 \frac{d}{d\lambda} \Psi(x, y, z, \lambda v_2 + (1 - \lambda)v_1, \lambda w_2 + (1 - \lambda)w_1) d\lambda
\]
\[
= (v_2 - v_1) \int_0^1 \frac{\partial \Psi}{\partial v}(x, y, z, \lambda v_2 + (1 - \lambda)v_1, \lambda w_2 + (1 - \lambda)w_1) d\lambda +
\]
\[
+ (w_2 - w_1) \int_0^1 \frac{\partial \Psi}{\partial w}(x, y, z, \lambda v_2 + (1 - \lambda)v_1, \lambda w_2 + (1 - \lambda)w_1) d\lambda
\]
\[
= (v_2 - v_1) \psi_1(x, y, z, v_1, w_1, v_2, w_2) + (w_2 - w_1) \psi_2(x, y, z, v_1, w_1, v_2, w_2)
\]

Moreover, for small \(\delta\),

\[
\frac{\partial \Psi}{\partial v}(x, y, z, v, w) = \int_{|\zeta - v| = \delta} \frac{\Psi(x, y, z, \zeta, w)}{(\zeta - v)^2} \frac{dz}{2\pi i}
\]

which implies that \(\psi_1\) is holomorphic and bounded for its arguments less than \(\epsilon\). We also obtain \(\frac{\partial \Psi}{\partial v}(0, 0, 0, v, w) = 0\), then \(\psi_1(0, 0, 0, v_1, w_1, v_2, w_2) = 0\). The proof for \(\psi_2\) is analogous. We use (53) to complete the proof of (52). Actually, we observe that

\[
\psi_1(x, y, z, v_1, w_1, v_2, w_2) = \int_0^1 \frac{d}{d\lambda} \psi_1(\lambda x, \lambda y, \lambda z, v_1, w_1, v_2, w_2) d\lambda
\]
\[
= x \int_0^1 \frac{\partial \psi_1}{\partial x} d\lambda + y \int_0^1 \frac{\partial \psi_1}{\partial y} d\lambda + z \int_0^1 \frac{\partial \psi_1}{\partial z} d\lambda
\]

and we conclude as in the proof of (51).

\[
\square
\]

We solve the system of integral equations by successive approximations. We can choose any path \(L(x)\) such that \(0 < \nu^+ < 1\) in case (a), or \(1 < \nu^+ < 2\) in case (b). Here we complete the proof only for the case (a). Therefore, the solution \(v(x)\) we are going to find is \(v_1(x; v_1, v_2 + 2N)\). Case (b) is analogous and we don’t need to repeat the proof for it. It yields the function \(v_2(x; v_1, v_2 + 2N)\).
We choose $\nu^* = \frac{1}{2}$, therefore $\min\{\nu^*, 1 - \nu^*\} = \frac{1}{2}$. For convenience, we put

$$A := \frac{e^{-i\pi\nu_1}}{16\nu_2^{2N}}, \quad B := \frac{e^{i\pi\nu_1}}{16\nu_2^{2N}}$$

Therefore, for any $n \geq 1$ the successive approximations are:

$$v_0 = w_0 = 0$$

$$w_n(x) = \int_{L(x)} \frac{1}{s} \left\{ \Phi(s, A s^{1-\nu_2-2N}, B s^{\nu_2+2N}) + \Psi(s, A s^{1-\nu_2-2N}, B s^{\nu_2+2N}, v_{n-1}(s), w_{n-1}(s)) \right\} ds \quad (54)$$

$$v_n(x) = \int_{L(x)} \frac{1}{s} \ w_n(s) \ ds \quad (55)$$

**Sub-Lemma 2:** There exists a sufficiently small $\epsilon' < \epsilon$ such that for any $n \geq 0$ the functions $v_n(x)$ and $w_n(x)$ are holomorphic in the domain

$$D_1(\epsilon'; \nu_1, \nu_2 + 2N) := \left\{ x \in \mathbb{C} \ such \ that \ |x| < \epsilon', \ |Ax^{1-\nu_2-2N}| < \epsilon', \ |Bx^{\nu_2+2N}| < \epsilon' \right\}$$

They are also correctly bounded, namely $|v_n(x)| < \epsilon$, $|w_n(x)| < \epsilon$ for any $n$. They satisfy

$$|v_n - v_{n-1}| \leq \frac{(2c)^n(2P(\nu_2 + 2N))^2}{n!} (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^n \quad (56)$$

$$|w_n - w_{n-1}| \leq \frac{(2c)^n(2P(\nu_2 + 2N))^2}{n!} (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^n \quad (57)$$

where $P(\nu_2 + 2N) := P(\nu_2 + 2N, \nu^* = 1/2)$ and $c$ is the constant appearing in Sub-Lemma 1. Moreover

$$\frac{dv_n}{dx} = w_n$$

**Proof:** We proceed by induction.

$$w_1 = \int_{L(x)} \frac{1}{s} \Phi(s, A s^{1-\nu_2-2N}, B s^{\nu_2+2N}) \ ds, \quad v_1 = \int_{L(x)} \frac{1}{s} \ w_1(s) \ ds$$

It follows from Lemma 1 and (54) that $w_1(x)$ is holomorphic for $|x|, |Ax^{1-\nu_2-2N}|, |Bx^{\nu_2+2N}| < \epsilon$. From (58) and (50) we have

$$|w_1(x)| \leq \int \left\{ \frac{1}{|s|} \Phi(s, A s^{1-\nu_2-2N}, B s^{\nu_2+2N}) \right\} |ds|

\leq 2cP(\nu_2 + 2N)(|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|) \leq 6cP(\nu_2 + 2N)\epsilon' < \epsilon$$

on $D_1(\epsilon'; \nu_1, \nu_2 + 2N)$, provided that $\epsilon'$ is small enough. By Lemma 1, also $v_1(x)$ is holomorphic for $|x|, |Ax^{1-\nu_2-2N}|, |Bx^{\nu_2+2N}| < \epsilon$ and

$$\frac{dv_1}{dx} = w_1$$

By (58) we also have

$$|v_1(x)| \leq c(2P(\nu_2 + 2N))^2(|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|) \leq 12cP(\nu_2)^2\epsilon' < \epsilon$$

on $D_1(\epsilon'; \nu_1, \nu_2 + 2N)$, if $\epsilon'$ is small enough. Note that $P(\nu_2 + 2N) \geq 1$, so (57), (58) are true for $n = 1$. Now we suppose that the statement of the sub-lemma is true for $n$ and we prove it for $n + 1$. Consider:

$$|w_{n+1}(x) - w_n(x)| = \int_{L(x)} \frac{1}{s} \left[ \Psi(s, A s^{1-\nu_2-2N}, B s^{\nu_2+2N}, v_n, w_n) - \Psi(s, A s^{1-\nu_2-2N}, B s^{\nu_2+2N}, v_n, w_n) \right] \ ds$$

22
We see that the series converges uniformly in $D$ if they exist. We can also rewrite
\[ -\Psi(s, A^{1-\nu_2-2N}, B s^{\nu_2+2N}, v_{n-1}, w_{n-1}) \] ds\]

By (52) the above is
\[ \leq c \int_{L(x)} \frac{1}{|s|} \left( |A^{1-\nu_2-2N}| + |B s^{\nu_2+2N}| \right) \left( |v_n - v_{n-1}| + |w_n - w_{n-1}| \right) |ds| \]

By induction this is
\[ \leq 2c \frac{(2c)^n(2P(\nu_2 + 2N))^{2n}}{n!} \int_{L(x)} \frac{1}{|s|} \left( |A^{1-\nu_2-2N}| + |B s^{\nu_2+2N}| \right)^{n+1} |ds| \]
\[ \leq 2c \frac{(2c)^n(2P(\nu_2 + 2N))^{2n}}{n!} \frac{2P(\nu_2)}{n+1} \left( |x| + |A x^{1-\nu_2-2N}| + |B x^{\nu_2+2N}| \right)^{n+1} \]
\[ \leq \frac{(2c)^{n+1}(2P(\nu_2 + 2N))^{2(n+1)}}{(n+1)!} \left( |x| + |A x^{1-\nu_2-2N}| + |B x^{\nu_2+2N}| \right)^{n+1} \]

This proves (57). Now we estimate
\[ |v_{n+1}(x) - v_n(x)| \leq \int_{L(x)} |w_{n+1}(s) - w_n(s)| |ds| \]
\[ \leq \frac{(2c)^{n+1}(2P(\nu_2 + 2N))^{2(n+1)}}{(n+1)!} \int_{L(x)} \frac{1}{|s|} \left( |A^{1-\nu_2-2N}| + |B s^{\nu_2+2N}| \right)^{n+1} |ds| \]
\[ \leq \frac{(2c)^{n+1}(2P(\nu_2 + 2N))^{2(n+1)}}{(n+1)!} \left( |x| + |A x^{1-\nu_2-2N}| + |B x^{\nu_2+2N}| \right)^{n+1} \]
\[ \leq \frac{(2c)^{n+1}(2P(\nu_2 + 2N))^{2(n+1)}}{(n+1)!} \left( |x| + |A x^{1-\nu_2-2N}| + |B x^{\nu_2+2N}| \right)^{n+1} \]

This proves (56). From Lemma 1 we also conclude that $w_n$ and $v_n$ are holomorphic in $D(\epsilon', \nu_1, \nu_2)$ and
\[ x \frac{d v_n}{dx} = w_n \]

Finally we see that
\[ |v_n(x)| \leq \sum_{k=1}^{n} |v_{n+k} - v_{n+k-1}| \leq \exp\{2c(2P(\nu_2 + 2N))^2(|x| + |A x^{1-\nu_2-2N}| + |B x^{\nu_2+2N}|)\} - 1 \leq \exp\{2cP^2(\nu_2 + 2N)\} - 1 \]

and the same for $|w_n(x)|$. Therefore, if $\epsilon'$ is small enough we have $|v_n(x)| < \epsilon$, $|w_n(x)| < \epsilon$ on $D_1(\epsilon', \nu_1, \nu_2)$.

Note that $P(\nu_2 + 2N)$ grows as $N^2$, so $\epsilon'$ decreases as $N^{-2}$.

Let’s define
\[ v(x) := \lim_{n \to \infty} v_n(x), \quad w(x) := \lim_{n \to \infty} w_n(x) \]
if they exist. We can also rewrite
\[ v(x) = \lim_{n \to \infty} v_n(x) = \sum_{n=1}^{\infty} (v_n(x) - v_{n-1}(x)). \]

We see that the series converges uniformly in $D(\epsilon', \nu_1, \nu_2 + 2N)$ because
\[ |\sum_{n=1}^{\infty} (v_n(x) - v_{n-1}(x))| \]
From Sub-Lemma 2 we also have

\[
\sum_{n=1}^{\infty} \frac{(2c)^n (2P(\nu_2 + 2N))^{2n}}{n!} (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^n
\]

\[
= \exp\{8cP^2(\nu_2)(|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)\} - 1
\]

The same holds for \( w_n(x) \). Therefore, \( v(x) \) and \( w(x) \) define holomorphic functions in \( D_1(\epsilon', \nu_1, \nu_2 + 2N) \). From Sub-Lemma 2 we also have

\[
\int \frac{dv(x)}{dx} = w(x)
\]

in \( D_1(\epsilon', \nu_1, \nu_2 + 2N) \).

We show that \( v(x), w(x) \) solve the initial integral equations. The l.h.s. of (54) converges to \( w(x) \) for \( n \to \infty \). Let’s prove that the r.h.s. also converges to

\[
\int_{L(x)} \frac{1}{s} \left\{ \Phi(s, A s^{1-\nu_2-2N}, B s^{\nu_2+2N}) + \Psi(s, A s^{1-\nu_2-2N}, B s^{\nu_2+2N}, v(s), w(s)) \right\} \, ds.
\]

We have to evaluate the following difference:

\[
\left| \int_{L(x)} \frac{1}{s} \Psi(s, A s^{2-\nu_2-2N}, B s^{\nu_2+2N}, v(s), w(s)) \, ds - \int_{L(x)} \frac{1}{s} \Psi(s, A s^{2-\nu_2-2N}, B s^{\nu_2+2N}, v_n(s), w_n(s)) \, ds \right|
\]

By (52) the above is

\[
\leq c \int_{L(x)} \frac{1}{s} \left( |s| + |A s^{1-\nu_2-2N}| + |B s^{\nu_2+2N}| \right) (|v - v_n| + |w - w_n|) \, ds \quad (58)
\]

Now we observe that

\[
|v(x) - v_n(x)| \leq \sum_{k=n+1}^{\infty} |v_k - v_{k-1}|
\]

\[
= \sum_{k=n+1}^{\infty} \frac{(2c)^k (2P(\nu_2 + 2N))^{2k}}{k!} (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^k
\]

\[
\leq (|x| + |Ax^{2-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^{n+1} \sum_{k=0}^{\infty} \frac{(2c)^{k+1} P(\nu_2 + 2N)^{2(k+1)}}{(k + n + 1)!} (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^k
\]

The series converges. Its sum is less than some constant \( S(\nu_2 + 2N) \) independent of \( n \). We obtain

\[
|v(x) - v_n(x)| \leq S(\nu_2 + 2N) (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^{n+1}.
\]

The same holds for \( |w - w_n| \). Thus, (58) is

\[
\leq 2c S(\nu_2 + 2N) \int_{L(x)} \frac{1}{s} \left( |s| + |A s^{1-\nu_2-2N}| + |B s^{\nu_2+2N}| \right)^{n+2} \, ds
\]

\[
\leq \frac{2c S(\nu_2 + 2N) 2P(\nu_2 + 2N)}{n + 2} (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^{n+2}
\]

Namely:

\[
\left| \int_{L(x)} \frac{1}{s} \Psi(s, A s^{1-\nu_2-2N}, B s^{\nu_2+2N}, v(s), w(s)) \, ds - \int_{L(x)} \frac{1}{s} \Psi(s, A s^{1-\nu_2-2N}, B s^{\nu_2+2N}, v_n(s), w_n(s)) \, ds \right|
\]

\[
\leq \frac{2c S(\nu_2) 2P(\nu_2 + 2N)}{n + 2} (3\epsilon')^{n+2}
\]

In a similar way, the r.h.s. of (55) is

\[
\left| \int \frac{1}{s} (w(s) - w_n(s)) \, ds \right| \leq \frac{S(\nu_2 + 2N) 2P(\nu_2)}{n + 1} (3\epsilon')^{n+1}
\]

24
Therefore, the r.h. sides of (54), (55) converge on the domain $D(r, \nu_1, \nu_2)$ for $r < \min\{\epsilon', 1/3\}$. We note that $r = r_N$, namely it depends on $N$ in the same way as $\epsilon'$ does. Thus, it decreases as $N^{-2}$ as $N$ increases.

We finally observe that $|v(x)|$ and $|w(x)|$ are bounded on $D(r)$. For example

$$|v(x)| \leq (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|) \sum_{k=0}^{\infty} \frac{(2\nu)^{k+1}(2P(\nu_2))^{2(k+1)}}{(k+1)!} (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^k$$

where the sum of the series is less than a constant $M(\nu_2 + 2N)$. We have proved Lemma 2 for the case (a). Case (b) is analogous.

\[ \square \]

**Remark:** The domain $D_1(r_N, \nu_1, \nu_2)$ is

$$\left(\Re \nu_2 + 2N\right) \ln \frac{|x|}{16} - \pi \Im \nu_1 - \ln r_N < \Im \nu_2 \arg x <$$

< \left(\Re \nu_2 - 1 + 2N\right) \ln \frac{|x|}{16} - \pi \Im \nu_1 + \ln r_N, \quad |x| < r_N$$

while the domain $D_2(r_N, \nu_1, \nu_2)$ is

$$\left(\Re \nu_2 - 1 + 2N\right) \ln \frac{|x|}{16} - \pi \Im \nu_1 - \ln r_N < \Im \nu_2 \arg x <$$

< \left(\Re \nu_2 - 2 + 2N\right) \ln \frac{|x|}{16} - \pi \Im \nu_1 + \ln r_N, \quad |x| < r_N$$

We invite the reader to observe the $\ln r_N$ terms.

Also, note that for $\nu_2 \in \mathbb{R}$ the domain is specified by $\max \left\{ \left| x \right|, \left| \frac{i}{16} \nu_2 x^{1-\nu_2} \right|, \left| \frac{i}{16} \nu_1 x^{\nu_2} \right| \right\} < r$. We can make the notation easier by simply re-writing $|x| < r$ for sufficiently small $r$. So, we obtain the domain $D_0(r)$.

We have proved that the structure of the integral equations implies that $v(x)$ is bounded (namely $|v(x)| = O(r_N)$). The proof of Lemma 2 only makes use of the properties of $\Phi$ and $\Psi$, regardless of how these functions have been constructed.

In our case, $\Phi$ and $\Psi$ have been constructed from the Fourier expansion of elliptic functions. We see that the domain (16) contains $D_1(r_N; \nu_1, \nu_2 + 2N)$ and (17) contains $D_2(r_N; \nu_1, \nu_2 + 2N)$ because the term $\Im \frac{\nu_2}{A}$ in (16) and (17) is $O(r_N)$, while in $D_i(r, \nu_1, \nu_2 + 2N)$, $i = 1, 2$, the term $\ln r_N$ appear (see the Remark).

To conclude the proof of Theorem 1, we have to work out the series of $v(x)$. Let’s do that in case (a), namely for $v_1(x; \nu_1, \nu_2 + 2N)$. Case (b) is analogous. We observe that $w_1$ and $v_1$ are series of the type

$$\sum_{p, q, r \geq 0} c_{pqr}(\nu_2, \nu_2 + 2N) x^p (A^{1-\nu_2-2N})^q (Bx^{\nu_2+2N})^r$$

where $c_{pqr}(\nu_2 + 2N)$ is rational in $\nu_2$. This follows from

$$w_1(x) = \int_{L(x)} \Phi(s, As^{1-\nu_2-2N}, Bs^{\nu_2+2N}) \, ds,$$

and from the fact that $\Phi(x, A^{1-\nu_2-2N}, Bx^{\nu_2+2N})$ itself is a series (60) by construction, with coefficients which are rational functions of $\nu_2 + 2N$. The same holds true for $\Psi$. We conclude that $w_n(x)$ and $v_n(x)$ have the form (34) for any $n$. This implies that the limit $v(x)$ is also a series of type (60). We can reorder such a series. Consider the term

$$c_{pqr}(\nu_2) x^p (A^{1-\nu_2-2N})^q (Bx^{\nu_2+2N})^r,$$

and recall that by definition $B = \frac{1}{16}.^{1/2}$. We absorb $16^{-2n}$ into $c_{pqr}(\nu_2)$ and we study the factor

$$A^q - x^{p+(1-\nu_2-2N)q+(\nu_2+2N)r} = A^q - x^{p+q+(r-q)(\nu_2+2N)}$$
We have three cases:

1) \( r = q \), then we have \( x^{p+q} = x^n, \ n = p + q \).

2) \( r > q \), then we have \( x^{p+q} \left[ \frac{1}{A} x^{p+q+2N} \right]^{r-q} = x^n \left[ \frac{1}{A} x^{p+q+2N} \right]^m, \ n = p + q, \ m = q - r \).

3) \( r < q \), then we have \( A^{q-r} x^{p+q} \left[ A x^{1-r-2N} \right]^{q-r} = x^n \left[ A x^{1-r-2N} \right]^m, \ n = p + r, \ m = q - r \).

Therefore, the series of the type (59) can be re-written in the form of \( v(x) \) of Theorem 1. The series of \( v(x) \) is uniquely determined by the Painlevé equation, because it is constructed from \( \Phi \) and \( \Psi \) by successive approximations of the solutions of the integral equations.

This concludes the proof. We only remark that if \( \nu_2 \) is real, we make a translation \( \nu_2 \mapsto \nu_2 + 2N \) which yields \( 0 < \nu_2 < 1 \) or \( 1 < \nu_2 < 2 \) and all the formulae can be read for \( N = 0 \).

The statement of Theorem 1 as it follows from our proof is

Let \( \nu_1, \nu_2 \) be two complex numbers.

**I)** For any \( N \in \mathbb{Z} \) and for any complex \( \nu_1, \nu_2 \) such that

\[ \exists \nu_2 \neq 0 \]

there exist a positive number \( r_N < 1 \) and a transcendent

\[ y(x) = \varphi \left( \nu_1 \omega_1(x) + \nu_2 \omega_2(x) + v_1(x; \nu_1, \nu_2 + 2N); \ \omega_1(x), \omega_2(x) \right) + \frac{1 + x}{3} \]

such that \( v_1(x; \nu_1, \nu_2 + 2N) \) is holomorphic in the domain \( D_1(r_N; \nu_1, \nu_2 + 2N) \), where it has convergent expansion

\[ v_1(x; \nu_1, \nu_2 + 2N) = \sum_{n \geq 1} a_n^{(1)} x^n + \sum_{n \geq 0, m \geq 1} b_{nm}^{(1)} x^n \left[ e^{i \pi \nu_1 \left( \frac{x}{16} \right)^{1-\nu_2-2N}} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm}^{(1)} x^n \left[ e^{i \pi \nu_1 \left( \frac{x}{16} \right)^{\nu_2+2N}} \right]^m \]

There also exists a transcendent

\[ y(x) = \varphi \left( \nu_1 \omega_1(x) + \nu_2 \omega_2(x) + v_2(x; \nu_1, \nu_2 + 2N); \ \omega_1(x), \omega_2(x) \right) + \frac{1 + x}{3} \]

such that \( v_2(x; \nu_1, \nu_2 + 2N) \) is holomorphic in the domain \( D_2(r_N; \nu_1, \nu_2 + 2N) \), where it has convergent expansion

\[ v_2(x; \nu_1, \nu_2 + 2N) = \sum_{n \geq 1} a_n^{(2)} x^n + \sum_{n \geq 0, m \geq 1} b_{nm}^{(2)} x^n \left[ e^{-i \pi \nu_1 \left( \frac{x}{16} \right)^{2-\nu_2-2N}} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm}^{(2)} x^n \left[ e^{i \pi \nu_1 \left( \frac{x}{16} \right)^{\nu_2-1+2N}} \right]^m \]

For both transcendents, the coefficients \( a_n^{(i)}, b_{nm}^{(i)}, c_{nm}^{(i)} \), \( i = 1, 2 \), are rational functions of \( \nu_2 + 2N \).

On \( D_i(r_N; \nu_1, \nu_2 + 2N) \) there exists a positive constant \( M_i(\nu_2 + 2N) \) such that

\[ |v_1(x; \nu_1, \nu_2 + 2N)| \leq M_1(\nu_2 + 2N) \left( |x| + \left| e^{-i \pi \nu_1 \left( \frac{x}{16} \right)^{1-\nu_2-2N}} \right| + \left| e^{i \pi \nu_1 \left( \frac{x}{16} \right)^{\nu_2+2N}} \right| \right) \] \hspace{1cm} (60)

\[ |v_2(x; \nu_1, \nu_2 + 2N)| \leq M_2(\nu_2 + 2N) \left( |x| + \left| e^{-i \pi \nu_1 \left( \frac{x}{16} \right)^{2-\nu_2-2N}} \right| + \left| e^{i \pi \nu_1 \left( \frac{x}{16} \right)^{\nu_2-1+2N}} \right| \right) \] \hspace{1cm} (61)

**II)** For any complex \( \nu_1 \) and for any real \( \nu_2 \) such that

\[ 0 < \nu_2 < 1 \quad \text{or} \quad 1 < \nu_2 < 2 \]
there exists a positive $r < 1$ and a transcendent

$$y(x) = \psi(v_1 \omega_1(x) + v_2 \omega_2(x) + v(x; \nu_1, \nu_2); \omega_1(x), \omega_2(x)) + \frac{1 + x}{3}$$

such that $v(x; \nu_1, \nu_2)$ is holomorphic in $D_0(r; \nu_1, \nu_2)$, where it has the convergent expansion

$$v(x; \nu_1, \nu_2) = \sum_{n \geq 1} a_n x^n + \sum_{n \geq 0, m \geq 1} b_{nm} x^n \left[ e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{1-\nu_2} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm} x^n \left[ e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2} \right]^m$$

if $0 < \nu_2 < 1$; and

$$v(x) = \sum_{n \geq 1} a_n x^n + \sum_{n \geq 0, m \geq 1} b_{nm} x^n \left[ e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{2-\nu_2} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm} x^n \left[ e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{2-\nu_2} \right]^m$$

if $1 < \nu_2 < 2$. The coefficients are rational functions of $\nu_2$ and $|v(x)|$ is bounded as in [41], with $N = 0$, for $0 < \nu_2 < 1$ and as in [44], with $N = 0$, for $1 < \nu_2 < 2$.

The above statement is completely equivalent to the statement of Theorem 1 in the Introduction, due to Observation 1 and 2 there. In particular, $v_1(x; \nu_1, \nu_2) = v(x; \nu_1, \nu_2)$ of Theorem 1 and $v_2(x; \nu_1, \nu_2) = -v(x; -\nu_1, 2 - \nu_2)$

A technique similar to that used in the proof of Theorem 1 was first introduced by S. Shimomura in [44] for a class of functional equations.

4 Proof of Theorem 2

Theorem 2 is stated in the Introduction. Here we give its proof. Let $x_0 \in D(r; \nu_1, \nu_2)$. We expand $y(x)$ in Fourier series. This is possible on $D(r; \nu_1, \nu_2)$ as it is clear from the proof of Theorem 1. Let $f := \nu_1 + \nu_2 \tau + \frac{x}{\omega_1}$,

$$y(x) = \psi(\nu_1 \omega_1 + \nu_2 \omega_2 + v; \omega_1, \omega_2) + \frac{1 + x}{3} =$$

$$\left(\frac{\pi}{2\omega_1}\right)^2 \left\{ -\frac{1}{3} + \sin^{-2} \left( \frac{\pi}{2} f \right) + 8 \sum_{n \geq 1} \frac{ne^{2i\pi \tau}}{1 - e^{2i\pi \tau}} \left[ 1 - \cos (\pi n f) \right] \right\} + \frac{1 + x}{3} =$$

$$= \left(\frac{\pi}{2\omega_1}\right)^2 \left\{ -\frac{1}{3} - 4 \left[ e^{i\pi f} - e^{-i\pi f} \right]^2 + 4 \sum_{n \geq 1} \frac{ne^{2i\pi \tau}}{1 - e^{2i\pi \tau}} \left[ 2 - e^{i\pi n f} - e^{-i\pi n f} \right] \right\} + \frac{1 + x}{3}$$

Observe that for $x \to 0$ $\frac{x^2}{2\omega_1^2} = \frac{1}{\tau(x)^2} = 1 - \frac{x}{2} + O(x^2)$. We recall that $e^{i\pi \nu_2(x)} = h(x)^C (x/16)^C$, where $h(x) = 1 + O(x)$. We define

$$\mathcal{X}(x) := h(x) e^{i\pi \nu_1(x)} \left( \frac{x}{16} \right)^{2-\nu_2} e^{-i\pi \frac{\nu_2}{\tau(x)}}$$

and

$$\mathcal{Y}(x) := h(x)^{\nu_2} e^{i\pi \nu_1(x)} \left( \frac{x}{16} \right)^{\nu_2} e^{i\pi \frac{\nu_2}{\tau(x)}}$$

Thus

$$y(x) = \frac{x}{2} + O(x^2) + 4 \left( 1 - \frac{x}{2} + O(x^2) \right) \left\{ \sum_{n \geq 1} \frac{n \mathcal{X}^n(x)}{1 - h(x) e^{i\pi \nu_1(x) \tau(x)}} \left[ 2 \mathcal{Y}^n(x) - \mathcal{Y}^{2n}(x) - 1 \right] - \frac{\mathcal{Y}(x)}{1 - \mathcal{Y}(x)} \right\}$$

Note that the denominator $[1 - \mathcal{Y}(x)]^2$ in the last term does not vanish on $D$. 

27
The above formula immediately yields (9) and (10), (11). It is enough to recall that in $\mathcal{D}(r; \nu_1, \nu_2)$ the dominant powers are $x^{\nu_2}$, $x^{1-\nu_2}$. We also need to observe that
\[ |x^{\nu_2+\alpha}| = |x|^\nu_2 \frac{|x_0^{\nu_2+\alpha}|}{|x_0|^\nu_2}, \quad \forall \alpha \in \mathbb{C}. \]
and that $v(x) \to 0$ for $x \to 0$, according to (9). Therefore, (9) and (11), (11) follow simply taking the leading terms in the expansion.

As for (8), we observe that if $\nu = 1$, $v(x)$ does not vanish, namely:
\[ \phi(x) := \sum_{m \geq 1} b_{0m} \left[ e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{1-\nu_2} \right]^m \neq 0 \text{ as } x \to 0 \]
and $e^{i\pi \frac{\nu_2}{2}} = e^{2i\phi_1 (1 + O(x))}$. This implies that the dominant terms in the Fourier expansion are:
\[ y(x) = \left[ x^2 - 4e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2} e^{2i\phi(x)} - 4e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{2-\nu_2} e^{-2i\phi(x)} \right] (1 + O(x)) \]
from which (9) follows. Note that $x, x^{\nu_2}$ and $x^{2-\nu_2}$ are of the same order.

As for (8), $v(x)$ does not vanish when $\nu = 0$, because $x^{\nu_2} \neq 0$. Namely
\[ \psi(x) := \sum_{m \geq 1} c_{0m} \left[ e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2} \right]^m \neq 0 \text{ as } x \to 0 \]
In this case, it is convenient to keep the term $\sin^{-2}(\pi f/2)$. It is
\[ \sin^{-2} \left[ -i \frac{\nu_2}{2} \ln \frac{x}{16} + \frac{\pi \nu_1}{2} + \psi(x) + O(x) \right] = \sin^{-2} \left[ -i \frac{\nu_2}{2} \ln \frac{x}{16} + \frac{\pi \nu_1}{2} + \psi(x) \right] (1 + O(x)) \]
The last step is possible because $-i \frac{\nu_2}{2} \ln \frac{x}{16} + \frac{\pi \nu_1}{2} + \psi(x) \neq 0$ on $\mathcal{D}(r; \nu_1, \nu_2)$. So, (8) follows. Note that $\sin \left[ -i \frac{\nu_2}{2} \ln \frac{x}{16} + \frac{\pi \nu_1}{2} + \psi(x) \right] \neq 0$ on the domain, namely $y(x)$ in (8) does not have poles in $\mathcal{D}(r; \nu_1, \nu_2)$.

Remark: In (8) the $\sin^{-2}(\ldots)$ is never infinite on the domain. Actually, it is clear from the proof of Theorem 1 that the domains $\mathcal{D}_i(r; \nu_1, \nu_2 + 2N)$, $i = 1, 2$, where exactly chosen in such a way that $\sin(\ldots) \neq 0$. Therefore, the movable poles of the transcendent must lie outside $\mathcal{D}(r; \nu_1, \nu_2)$.

5 Special Cases

There are some special cases when the domain $\mathcal{D}(r; \nu_1, \nu_2)$ can be enlarged.

**FIRST CASE:** $\beta = 1 - 2\delta = 0$.

Theorem 1 can be stated as follows:

**Theorem 1 – special case $\beta = 1 - 2\delta = 0$:** For any complex $\nu_1, \nu_2$ with the constraint
\[ 0 < \nu_2 < 2 \quad \text{if } \nu_2 \text{ is real} \]
there exist a positive number $r < 1$ and a transcendent
\[ y(x) = \psi(\nu_1 \omega_1(x) + \nu_2 \omega_2(x) + v(x; \nu_1, \nu_2); \ \omega_1(x), \omega_2(x)) + \frac{1 + x}{3} \]
such that $v(x; \nu_1, \nu_2)$ is holomorphic in the domain
\[ \mathcal{D}(r; \nu_1, \nu_2) = \left\{ x \in \mathbb{C}_0 \mid |x| < r, \left| e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{2-\nu_2} \right| < r, \left| e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2} \right| < r \right\} \text{ if } \Im \nu_2 \neq 0 \]
or in the domain $D_0(r)$ if $0 < \nu_2 < 2$. It has convergent expansion:

$$v(x; \nu_1, \nu_2) = \sum_{n \geq 1} a_n x^n + \sum_{n \geq 0, m \geq 1} b_{nm} x^n \left[ e^{-i\nu_1} \left( \frac{x}{16} \right)^{2-\nu_2} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm} x^n \left[ e^{i\nu_1} \left( \frac{x}{16} \right)^{\nu_2} \right]^m$$

Note that the domain is larger than the general case and that $\nu_2 = 1$ is now allowed.

\textbf{Proof:} If we go back to the proof of Theorem 1 we see that in the equation (43) we only have the functions $\varphi(u/2 + \varepsilon_1 \omega_1 + 2N\omega_2)$. This means that we do not have $N_2$. Let us put $N_2 = N$. Then, we do the proof of Theorem 1 in the domain (43) with $N_2 = N$ and $\varepsilon_2 = 0$. A remarkable fact happens. While in proof for the general case we have to distinguish (for given $N$, $\nu_1$, $\nu_2$) between $D_1(r_N; \nu_1, \nu_2 + 2N)$ and $D_2(r_N; \nu_1, \nu_2 + 2N)$, in the present special case we prove the existence of $v(x)$ on the larger domain

$$D(r_N; \nu_1, \nu_2 + 2N) := \left\{ x \in \mathbb{C}_0 \mid |x| < r_N, \left| e^{-i\nu_1} \left( \frac{x}{16} \right)^{2-\nu_2} \right| < r, \left| e^{i\nu_1} \left( \frac{x}{16} \right)^{\nu_2+2N} \right| < r \right\} \quad (62)$$

where $v(x; \nu_1, \nu_2 + 2N)$ is represented as:

$$\sum_{n \geq 1} a_n x^n + \sum_{n \geq 0, m \geq 1} b_{nm} x^n \left[ e^{-i\nu_1} \left( \frac{x}{16} \right)^{2-\nu_2} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm} x^n \left[ e^{i\nu_1} \left( \frac{x}{16} \right)^{\nu_2+2N} \right]^m$$

We note that, if $\Im \nu_2 = 0$, the constraint $0 < \nu_2 < 2$ holds. Namely, now $\nu_2 = 1$ is allowed. When $\nu_2$ is real, $v(x)$ is equal to (63) with $N = 0$ and the domain is simply given by the condition $\max \{|x|, |e^{-i\nu_1} \left( \frac{x}{16} \right)^{2-\nu_2}|, |e^{i\nu_1} \left( \frac{x}{16} \right)^{\nu_2}| \} < r$, namely $|x| < r$ for $r$ small enough. This is the domain $D_0(r)$.

The critical behavior for $x \to 0$ can be studied along

$$\text{arg} x = \text{arg} x_0 + \frac{\Re \nu_2 - \mathcal{V}}{\Im \nu_2} \ln \frac{|x|}{|x_0|}$$

contained in $D(r; \nu_1, \nu_2)$ for $x_0 \in D(r; \nu_1, \nu_2)$ and $0 \leq \mathcal{V} \leq 2$. If $\Im \nu_2 = 0$ any path going to $x = 0$ is contained in $D_0(r)$. The critical behavior is obtained through the Fourier expansion. We give the result.

If $\Im \nu_2 \neq 0$, the transcendental $y(x) = \varphi(\nu_1 \omega_1 x + \nu_2 \omega_2 x + v(x; \nu_1, \nu_2) + (1 + x)/3$ has the following behaviors on $D(r; \nu_1, \nu_2)$:

For $0 < \mathcal{V} < 1$:

$$y(x) = -\frac{1}{4} \left[ \frac{e^{i\nu_1}}{16^{\nu_2-1}} \right] x^{\nu_2} \left( 1 + O(|x^{\nu_2}|) \right). \quad (63)$$

For $1 < \mathcal{V} < 2$:

$$y(x) = -\frac{1}{4} \left[ \frac{e^{i\nu_1}}{16^{\nu_2-1}} \right]^{-1} x^{2-\nu_2} \left( 1 + O(|x^{2-\nu_2}|) \right). \quad (64)$$

For $\mathcal{V} = 1$:

$$y(x) = x \sin^2 \left( i \frac{1 - \nu_2}{2} \ln \frac{x}{16} + \frac{\pi \nu_1}{2} \right) \left( 1 + O(x) \right). \quad (65)$$

For $\mathcal{V} = 0$:

$$y(x) = \left[ \frac{x}{2} + \sin^{-2} \left( i \frac{1 - \nu_2}{2} \ln \frac{x}{16} + \frac{\pi \nu_1}{2} \right) \right] \left[ \frac{e^{i\nu_1 \nu_2}}{16^{\nu_2-1}} \right] \left( 1 + O(x) \right). \quad (66)$$

For $\mathcal{V} = 2$:

$$y(x) = \left[ \frac{x}{2} + \sin^{-2} \left( i \frac{2 - \nu_2}{2} \ln \frac{x}{16} + \frac{\pi \nu_1}{2} \right) \right] \left[ \frac{e^{i\nu_1 \nu_2}}{16^{\nu_2-1}} \right] \left( 1 + O(x) \right). \quad (67)$$
For $\nu_2$ real, satisfying the constraints $0 < \nu_2 < 2$, the transcendent $y(x) = \varphi(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x;\nu_1,\nu_2)) + (1 + x)/3$ has the following behaviors on $D_0(r)$:

\[
y(x) = -\frac{1}{4} \left[ \frac{e^{i\pi\nu_1}}{16^{\nu_2-1}} \right] x^{\nu_2} \left( 1 + O(|x^{\nu_2}|) \right), \quad 0 < \nu_2 < 1 \quad (68)
\]

\[
y(x) = -\frac{1}{4} \left[ \frac{e^{i\pi\nu_1}}{16^{\nu_2-1}} \right]^{-1} x^{2-\nu_2} \left( 1 + O(|x^{2-\nu_2}|) \right), \quad 1 < \nu_2 < 2 \quad (69)
\]

\[
y(x) = x \sin^2 \left( \frac{\pi\nu_1}{2} \right) \left( 1 + O(x) \right), \quad \nu_2 = 1, \quad (\nu_1 \neq 0) \quad (70)
\]

**SECOND CASE:** $\alpha = \gamma = 0$.

Theorem 1 is restated as follows:

**Theorem 1 – special case $\alpha = \gamma = 0$:** For any complex $\nu_1, \nu_2$ with the constraint

\[-1 < \nu_2 < 1 \quad \text{if} \quad \nu_2 \text{ is real}
\]

there exist a positive number $r < 1$ and a transcendent

\[y(x) = \varphi(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x;\nu_1,\nu_2); \omega_1(x), \omega_2(x)) + \frac{1 + x}{3}
\]

such that $v(x;\nu_1,\nu_2)$ is holomorphic in the domain

\[D(r;\nu_1,\nu_2) = \left\{ x \in \mathbb{C}_0 \mid |x| < r, \quad \left| e^{-i\pi\nu_1} \left( \frac{x}{16} \right)^{1-\nu_2} \right| < r, \quad \left| e^{i\pi\nu_1} \left( \frac{x}{16} \right)^{\nu_2+1} \right| < r \right\}
\]

or in the domain $D_0(r)$ if $-1 < \nu_2 < 1$, where it has convergent expansion:

\[v(x;\nu_1,\nu_2) = \sum_{n \geq 1} a_n x^n + \sum_{n \geq 0, m \geq 1} b_{nm} x^n \left[ e^{-i\pi\nu_1} \left( \frac{x}{16} \right)^{1-\nu_2} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm} x^n \left[ e^{i\pi\nu_1} \left( \frac{x}{16} \right)^{\nu_2+1} \right]^m
\]

Note that now $\nu_2 = 0$ is allowed.

**Proof:** If we go back to the proof of Theorem 1, we see that in the equation (33) we only have the functions $\varphi(u/2 + \varepsilon \omega_1 + (2N_2^2 + 1)\omega_2)$. This means that we do not have $N_2$. This time, let us put $N_2' = N$. Therefore we do the proof of Theorem 1 for the domain (33) with $N_2' = N$ and $\varepsilon_2 = 1$. Therefore, on the (bigger) domain

\[D(r_N;\nu_1,\nu_2 + 2N) = \left\{ x \in \mathbb{C}_0 \mid |x| < r_N, \quad \left| e^{-i\pi\nu_1} \left( \frac{x}{16} \right)^{1-\nu_2-2N} \right| < r, \quad \left| e^{i\pi\nu_1} \left( \frac{x}{16} \right)^{\nu_2+1+2N} \right| < r \right\}
\]

there exist one $v(x)$, holomorphic with convergent expansion:

\[v(x;\nu_1,\nu_2 + 2N) = \sum_{n \geq 1} a_n x^n + \sum_{n \geq 0, m \geq 1} b_{nm} x^n \left[ e^{-i\pi\nu_1} \left( \frac{x}{16} \right)^{1-\nu_2-2N} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm} x^n \left[ e^{i\pi\nu_1} \left( \frac{x}{16} \right)^{\nu_2+1+2N} \right]^m
\]

In a sense, in the case $\beta = 1 - 2\delta = 0$ the disjoint union $D_1(r_N;\nu_1,\nu_2 + 2N) \cup D_2(r_N;\nu_1,\nu_2 + 2N)$ was replaced by the bigger domain (33). In the present case the disjoint union $D_1(r_N;\nu_1,\nu_2 + 2N) \cup D_2(r_{N+1};\nu_1,\nu_2 + 2[N + 1])$ is replaced by the bigger domain (71).

If $\nu_2$ is real, this time $\nu_2 = 0$ is allowed, but not $\nu_2 = 1$. Then, it is convenient to choose the constraint

\[-1 < \nu_2 < 1
\]
by doing \( \nu_2 \mapsto \nu_2 - 2 \) if \( 1 < \nu_2 < 2 \). The domain is simply specified by \( \max \{ |x|, |e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{1-\nu_2}| \}, \left| e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2+1} \right| < r \), namely \( |x| < r \) for \( r \) small enough. This is \( \mathcal{D}_0(r) \).

We compute the critical behaviors for \( x \to 0 \) along

\[
\arg x = \arg x_0 + \frac{\Re \nu_2 - \nu}{3 \nu_2} \ln \left| \frac{x}{x_0} \right|,
\]

which is contained in \( \mathcal{D}(r; \nu_1, \nu_2) \) if and only if \( x_0 \in \mathcal{D}(r; \nu_1, \nu_2) \) and \( -1 \leq \nu \leq 1 \). If \( \Im \nu_2 \neq 0 \) any path converging to \( x = 0 \) is contained in \( \mathcal{D}_0(r) \). We use Fourier expansion; the result is the following.

For \( 0 < \nu < 1 \):

\[
y(x) = -\frac{1}{4} \left[ \frac{e^{i\pi \nu_1}}{16^\nu_2 - 1} \right] x^{\nu_2} \left( 1 + O(|x|^{1-\nu_2}) \right).
\]

(72)

For \( -1 < \nu < 0 \):

\[
y(x) = -\frac{1}{4} \left[ \frac{e^{i\pi \nu_1}}{16^{\nu_2+1}} \right]^{-1} x^{-\nu_2} \left( 1 + O(|x|^{\nu_2+1}) \right).
\]

(73)

For \( \nu = 1 \):

\[
y(x) = x \sin^2 \left( \frac{1 - \nu_2}{2} \ln \left| \frac{x}{16} \right| + \frac{\nu \nu_1}{2} + \sum_{m \geq 1} b_{0m} \left[ \frac{e^{-i\pi \nu_1}}{16^\nu_2} \right]^m \right) \left( 1 + O(x) \right).
\]

(74)

For \( \nu = -1 \):

\[
y(x) = x \sin^2 \left( -\frac{\nu_2 + 1}{2} \ln \left| \frac{x}{16} \right| + \frac{\nu \nu_1}{2} + \sum_{m \geq 1} c_{0m} \left[ \frac{e^{i\pi \nu_1}}{16^\nu_2} \right]^m \right) \left( 1 + O(x) \right).
\]

(75)

For \( \nu = 0 \):

\[
y(x) = \left[ \frac{x}{2} + \sin \left( -\frac{\nu_2}{2} \ln \left| \frac{x}{16} \right| + \frac{x \nu_1}{2} \right) \right] \left( 1 + O(x) \right).
\]

(76)

For \( \nu_2 \) real, the behaviors on \( \mathcal{D}_0(r) \) are:

\[
y(x) = -\frac{1}{4} \left[ \frac{e^{i\pi \nu_1}}{16^\nu_2 - 1} \right] x^{\nu_2} \left( 1 + O(|x|^{1-\nu_2}) \right), \quad \text{if } 0 < \nu_2 < 1.
\]

(77)

\[
y(x) = -\frac{1}{4} \left[ \frac{e^{i\pi \nu_1}}{16^{\nu_2+1}} \right]^{-1} x^{-\nu_2} \left( 1 + O(|x|^{\nu_2+1}) \right), \quad \text{if } -1 < \nu_2 < 0.
\]

(78)

\[
y(x) = \left[ \frac{1}{\sin^2 \left( \frac{\nu_1}{2} \right)} + \frac{x}{2} \right] \left( 1 + O(x) \right), \quad \text{if } \nu_2 = 0, \ (\nu_1 \neq 0).
\]

(79)

6 Points \( x = 1, \infty \) – Comments to Theorem 3 and Critical behavior

Two independent solutions of the hyper-geometric equation \( \mathcal{L}(u) = 0 \) are the hyper-geometric function \( F(x; 1/2, 1/2, 1) \) and

\[
g \left( \frac{1}{2}, \frac{1}{2}, 1; x \right) := F_1(x) + F \left( \frac{1}{2}, \frac{1}{2}, 1; x \right) \ln x
\]

The following connection formulæ hold \( [57] \):

i) Connection 0 – 1

\[
F \left( \frac{1}{2}, \frac{1}{2}, 1; x \right) = -\frac{1}{\pi} g \left( \frac{1}{2}, \frac{1}{2}, 1; 1 - x \right), \quad |\arg(1 - x)| < \pi
\]
\[ g \left( \frac{1}{2}, \frac{1}{2}; 1; x \right) = -\pi F \left( \frac{1}{2}, \frac{1}{2}; 1; 1 - x \right), \quad |\arg x| < \pi \]

ii) Connection \(0 - \infty\)

\[ F \left( \frac{1}{2}, \frac{1}{2}; 1; x \right) = \frac{x^{-\frac{1}{2}}}{\pi} \left[ ig \left( \frac{1}{2}, \frac{1}{2}; 1; 1 + i\pi - \frac{x}{2} \right) + \pi F \left( \frac{1}{2}, \frac{1}{2}; 1; 1-x \right) \right], \quad -2\pi < \arg x < 0 \]

\[ g \left( \frac{1}{2}, \frac{1}{2}; 1; x \right) = x^{-\frac{1}{2}} g \left( \frac{1}{2}, \frac{1}{2}; 1; \frac{1}{x} \right), \quad |\arg x| < \pi \]

We recall that \(\omega_1(x)\) and \(\omega_2(x)\) are \(\pi/2\, F(1/2, 1/2, 1; x)\) and \(-i/2 \, g(1/2, 1/2; 1; x)\) respectively. This representation is convenient in a neighborhood of \(x = 0\). In this section we will use the notation \(\omega_1^{(0)}\) and \(\omega_2^{(0)}\) instead of \(\omega_1\) and \(\omega_2\). Namely

\[ \omega_1^{(0)}(x) = \frac{\pi}{2} F \left( \frac{1}{2}, \frac{1}{2}; 1; x \right), \quad \omega_2^{(0)}(x) = -\frac{i}{2} g \left( \frac{1}{2}, \frac{1}{2}; 1; x \right) \]

Taking into account the above connection formulae, we define in a neighborhood of \(x = 1\):

\[ \omega_1^{(1)}(x) := \omega_2^{(0)}(x) = \frac{i\pi}{2} F \left( \frac{1}{2}, \frac{1}{2}; 1; 1 - x \right), \quad |\arg x| < \pi \]

\[ \omega_2^{(1)}(x) := \omega_1^{(0)}(x) = -\frac{1}{2} g \left( \frac{1}{2}, \frac{1}{2}; 1; 1 - x \right), \quad |\arg(1 - x)| < \pi \]

We also define, in a neighborhood of \(x = \infty\):

\[ \omega_1^{(\infty)}(x) := \omega_1^{(0)}(x) + \omega_2^{(0)}(x) = \frac{\pi}{2} x^{-\frac{1}{2}} F \left( \frac{1}{2}, \frac{1}{2}; 1; \frac{1}{x} \right), \quad -\pi < \arg x < 0 \]

\[ \omega_2^{(\infty)}(x) := \omega_2^{(0)}(x) = -\frac{i}{2} x^{-\frac{1}{2}} g \left( \frac{1}{2}, \frac{1}{2}; 1; \frac{1}{x} \right), \quad |\arg x| < \pi \]

The above functions have branch cuts specified by the constrains on \(\arg x\). Once they are so defined, they are continued on the universal covering of \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\).

In the following, we use superscripts 0, 1, \(\infty\) for solutions \(u(x)\) of (35) represented around \(x = 0, 1, \infty\) respectively. At \(x = 0\) we looked for solutions

\[ \frac{u^{(0)}}{2} = \nu_1^{(0)} \omega_1^{(0)} + \nu_2^{(0)} \omega_2^{(0)} + v^{(0)} \]

and we found the representations of \(v^{(0)}(x)\) in Theorem 1. Now we look for a solution in a neighborhood of \(x = 1\) of the type

\[ \frac{u^{(1)}}{2} = \nu_1^{(1)} \omega_1^{(1)} + \nu_2^{(1)} \omega_2^{(1)} + v^{(1)} \]

and for a solution in a neighborhood of \(x = \infty\) of the type

\[ \frac{u^{(\infty)}}{2} = \nu_1^{(\infty)} \omega_1^{(\infty)} + \nu_2^{(\infty)} \omega_2^{(\infty)} + v^{(\infty)} \]

They yield Painlevé transcendentals \(g(x) = \varphi(u^{(1)}/2; \omega_1^{(0)}, \omega_2^{(0)})+(1+x)/3 \equiv \varphi(u^{(1)}/2; \omega_1^{(1)}, \omega_2^{(1)})+(1+x)/3\) and \(g(x) = \varphi(u^{(\infty)}/2; \omega_1^{(0)}, \omega_2^{(0)})+(1+x)/3 \equiv \varphi(u^{(\infty)}/2; \omega_1^{(\infty)}, \omega_2^{(\infty)})+(1+x)/3\).

The definitions of \(\omega_j^{(i)}\) are convenient because the proof of Theorem 1 can be repeated with no changes in a neighborhood of \(x = 1\) and \(\infty\). This yields the statement of Theorem 3 in the Introduction.
6.1 Critical Behaviors at $x = 1, \infty$

From Theorem 3 and the Fourier expansion of the Weierstrass function we obtain the critical behaviors of the transcendents of Theorem 3.

In a neighborhood of $x = 1$:

If $\Im \nu_2 = 0$, we let $x \to 1$ along any regular path. Otherwise, we consider the paths

$$\arg(1 - x) = \arg(1 - x_0) + \frac{\Re \nu_2^{(1)}}{\Im \nu_2^{(1)}} \ln \frac{|1 - x|}{|1 - x_0|}, \quad 0 \leq \nu \leq 1$$

joining $x_0$ to $x = 1$. They lie in $D(r; \nu_1^{(1)}, \nu_2^{(1)})$ if $x_0 \in D(r; \nu_1^{(1)}, \nu_2^{(1)})$.

Let $\nu_1^{(1)}, \nu_2^{(1)}$ be given. If $\Im \nu_2^{(1)} \neq 0$, the transcendent defined in Theorem 3 in $D(r; \nu_1^{(1)}, \nu_2^{(1)})$ has the following behaviors for $x \to 1$:

For $0 < \nu < 1$:

$$y(x) = 1 + \frac{1}{4} \left[ \frac{e^{-i \pi \nu_1^{(1)}}}{16 \nu_2^{(1)} - 1} \right] (1 - x)^{\nu_2^{(1)}} \left( 1 + O(|1 - x|^{\nu_2^{(1)} - 1}) \right). \quad (80)$$

For $\nu = 0$:

$$y(x) = 1 - \left\{ \frac{1 - x}{2} + \sin^{-2} \left( i \frac{\nu_2^{(1)}}{2} \ln \frac{1 - x}{16} + \frac{\pi \nu_1^{(1)}}{2} - i \sum_{m \geq 1} b_m \left[ e^{-i \pi \nu_1^{(1)}} \left( \frac{1 - x}{16} \right)^{\nu_2^{(1)}} \right]^m \right) \right\} (1 + O(1 - x)).$$

For $\nu = 1$:

$$y(x) = 1 - (1 - x) \sin^2 \left( i \frac{\nu_2^{(1)} - 1}{2} \ln \frac{1 - x}{16} + \frac{\pi \nu_1^{(1)}}{2} - i \sum_{m \geq 1} b_m \left[ e^{i \pi \nu_1^{(1)}} \left( \frac{1 - x}{16} \right)^{1 - \nu_2^{(1)}} \right]^m \right) (1 + O(1 - x)).$$

If $\Im \nu_2 = 0$, the transcendent defined on $D_0(r)$ has behavior (80) when $0 < \nu_2^{(1)} < 1$, or behavior

$$y(x) = 1 + \frac{1}{4} \left[ \frac{e^{-i \pi \nu_1^{(1)}}}{16 \nu_2^{(1)} - 1} \right]^{-1} (1 - x)^{2 - \nu_2^{(1)}} \left( 1 + O(|1 - x|^{2 - \nu_2^{(1)} - 1}) \right) \quad (81)$$

when $1 < \nu_2^{(1)} < 2$.

In a neighborhood of $x = \infty$:

If $\Im \nu_2^{(\infty)} = 0$, we let $x \to \infty$ along any path. Otherwise, we consider the paths

$$\arg x = \arg x_0 + \frac{\Re \nu_2^{(\infty)} - \nu}{\Im \nu_2^{(\infty)}} \ln \frac{|x|}{|x_0|}, \quad 0 \leq \nu \leq 1$$

joining $x_0 \in D(r; \nu_1^{(\infty)}, \nu_2^{(\infty)})$ to infinity.

Let $\nu_1^{(\infty)}, \nu_2^{(\infty)}$ be given. If $\Im \nu_2^{(\infty)} \neq 0$ the transcendent of Theorem 3 defined in $D(r; \nu_1^{(\infty)}, \nu_2^{(\infty)})$ has the following behaviors for $x \to \infty$:

For $0 < \nu < 1$:

$$y(x) = -\frac{1}{4} \left[ \frac{e^{i \pi \nu_1^{(\infty)}}}{16 \nu_2^{(\infty)} - 1} \right] x^{1 - \nu_2^{(\infty)}} \left( 1 + O(|x|^{-\nu_2^{(\infty)} - 1}) \right). \quad (82)$$
For \( V = 0 \):
\[
y(x) = (1 + O(x^{-1})) \left\{ \frac{1}{2} + x \sin^{-2} \left( -\frac{\nu_2^{(\infty)}}{2} \ln \frac{16}{x} + \frac{\pi \nu_2^{(\infty)}}{2} + \sum_{m \geq 1} c_{0m} \left[ e^{i \pi \nu_2^{(\infty)} \left( \frac{16}{x} \right)} \right]^m \right\}.
\]

For \( V = 1 \):
\[
y(x) = (1 + O(x^{-1})) \sin^2 \left( i \frac{1 - \nu_2^{(\infty)}}{2} \ln \frac{16}{x} + \frac{\pi \nu_2^{(\infty)}}{2} + \sum_{m \geq 1} b_{0m} \left[ e^{-i \pi \nu_2^{(\infty)} \left( \frac{16}{x} \right)} \right]^m \right\}.
\]

If \( \Im \nu_2 = 0 \), the transcendent defined on \( D_0(r) \) has behavior (82) when \( 0 < \nu_2^{(\infty)} < 1 \), or it has behavior
\[
y(x) = -\frac{1}{4} \left[ e^{i \pi \nu_2^{(\infty)} \left[ \frac{16}{x} \right]} \right]^{-1} x^{\nu_2^{(\infty)} - 1} \left( 1 + O(|x|^{\nu_2^{(\infty)} - 1}) \right)
\]
when \( 1 < \nu_2^{(\infty)} < 2 \).

### 7 Some Considerations on Analytic Continuation

We can easily study the effect of a small loop around a critical point on the transcendent of Theorem 1 and 3.

Consider a transcendent of Theorem 1
\[
y(x) = \phi \left( \nu_1 \omega_1^{(0)} + \nu_2 \omega_2^{(0)} + v(x; \nu_1, \nu_2); \omega_1^{(0)}, \omega_2^{(0)} \right) + \frac{1 + x}{3}
\]
defined on \( D(r; \nu_1, \nu_2) \). If \( \nu_2 \) is real, it is defined on \( D_0(r) \).

We do the loop \( x \mapsto xe^{2\pi i} \), where \( x \in D(r; \nu_1, \nu_2) \) (or \( D_0(r) \)), therefore \( |x| < 1 \). From the monodromy properties of \( F(1/2, 1/2, 1; x) \) and \( g(1/2, 1/2, 1; x) \) we have
\[
\omega_1^{(0)}(x) \mapsto \omega_1^{(0)}(xe^{2\pi i}) = \omega_1^{(0)}(x),
\]
\[
\omega_2^{(0)}(x) \mapsto \omega_2^{(0)}(xe^{2\pi i}) = \omega_2^{(0)}(x) + 2 \omega_1^{(0)}(x)
\]
We also have
\[
v(x; \nu_1, \nu_2) \mapsto v(xe^{2\pi i}; \nu_1, \nu_2) \equiv v(x; \nu_1 + 2 \nu_2, \nu_2)
\]
The last step of the above equalities follows from the explicit expansion of \( v(xe^{2\pi i}; \nu_1, \nu_2) \) given in Theorem 1, in the hypothesis that also \( xe^{2\pi i} \in D(r; \nu_1, \nu_2) \) or, which is the same thing, \( x \in D(r; \nu_1 + 2 \nu_2, \nu_2) \) [this always happens if \( \nu_2 \) is real]. Therefore
\[
y(x) \mapsto y(xe^{2\pi i}) =: y'(x) = \phi \left( \nu_1 \omega_1^{(0)}(x) + \nu_2 \omega_2^{(0)}(x) + 2 \omega_1^{(0)}(x) \right) + \frac{1 + x}{3}
\]
\[
\equiv \phi \left( (\nu_1 + 2 \nu_2) \omega_1^{(0)}(x) + \nu_2 \omega_2^{(0)}(x) + v(x; \nu_1 + 2 \nu_2, \nu_2); \omega_1^{(0)}(x), \omega_2^{(0)}(x) \right) + \frac{1 + x}{3}
\]
So, the effect of the loop is simply the transformation
\[
(\nu_1, \nu_2) \mapsto (\nu_1 + 2 \nu_2, \nu_2)
\]
From the above, we also obtain the critical behavior according to Theorem 2.

We remark that the above considerations hold only if both \( x \) and \( xe^{2\pi i} \) belongs to \( D(\nu_1, \nu_2) \) – namely \( x \in D(\nu_1 + 2 \nu_2, \nu_2) \). Otherwise, we can not represent \( v(xe^{2\pi i}; \nu_1, \nu_2) \) as \( v(x; \nu_1 + 2 \nu_2, \nu_2) \). This last case may actually occur when \( \Im \nu_2 \neq 0 \). If the point \( xe^{2\pi i} \) lies outside the domain it may be a pole of \( y(x) \).
The same procedure is applied to
\[
y(x) = \varphi\left(\nu_1^{(1)} \omega_1^{(1)}(x) + \nu_2^{(1)} \omega_2^{(1)}(x) + v(x; \nu_1^{(1)}, \nu_2^{(1)}); \omega_1^{(1)}, \omega_2^{(1)}\right) + \frac{1 + x}{3}
\]

Now we denote by \(D(r; \nu_1^{(1)}, \nu_2^{(1)})\) the domain around \(x = 1\). The effect of the loop \((1 - x) \mapsto (1 - x)e^{2\pi i}\), \(x \in D(r; \nu_1^{(1)}, \nu_2^{(1)})\) is
\[
(\nu_1^{(1)}, \nu_2^{(1)}) \mapsto (\nu_1^{(1)} - 2\nu_2^{(1)}, \nu_2^{(1)})
\]
if also \(x \in D(r; \nu_1^{(1)} - 2\nu_2^{(1)}, \nu_2^{(1)})\).

The procedure is applied to
\[
y(x) = \varphi\left(\nu_1^{(\infty)} \omega_1^{(\infty)}(x) + \nu_2^{(\infty)} \omega_2^{(\infty)}(x) + v(x; \nu_1^{(\infty)}, \nu_2^{(\infty)}); \omega_1^{(\infty)}, \omega_2^{(\infty)}\right) + \frac{1 + x}{3}
\]

Now \(D(r, \nu_1^{(\infty)}, \nu_2^{(\infty)})\) is the domain around \(x = \infty\). The effect of the loop \(x \mapsto xe^{-2\pi i}, x \in D(r; \nu_1^{(\infty)}, \nu_2^{(\infty)})\) is
\[
(\nu_1^{(\infty)}, \nu_2^{(\infty)}) \mapsto (\nu_1^{(\infty)} + 2\nu_2^{(\infty)}, \nu_2^{(\infty)})
\]
provided that also \(x \in D(r; \nu_1^{(\infty)} + 2\nu_2^{(\infty)}, \nu_2^{(\infty)})\).

## 8 Connection Problem

The analysis so far developed is only local. Two natural questions arise.

**Question 1.** This question makes sense in \(\nu_2\) is not real. Suppose that \(y(x)\) has the representation of Theorem 1 close to \(x = 0\): \(y(x) = \varphi(\nu_1 \omega_1 + \nu_2 \omega_2 + v(x; \nu_1, \nu_2)) + (1 + x)/3\) in \(D(r; \nu_1, \nu_2)\). We ask if \(y(x)\) may have a representation \(y(x) = \varphi(\nu_1' \omega_1 + \nu_2' \omega_2 + v(x; \nu_1, \nu_2 + 2N)) + (1 + x)/3\) in \(D(r; \nu_1', \nu_2 + 2N)\) for some integer \(N\). If this happens, we should be able to express \(\nu_1'\) as a function of \(\nu_1, \nu_2, N\).

**Question 2.** [Connection Problem]. May a given transcendent have three representations of Theorem 1 and Theorem 3 at \(x = 0, 1, \infty\) at the same time? If this happens, which is the relation between the three sets \((\nu_1^{(0)}, \nu_2^{(0)}), (\nu_1^{(1)}, \nu_2^{(1)}), (\nu_1^{(\infty)}, \nu_2^{(\infty)})\)?

The answer to this question is positive, but it can not be achieved with the method so far used, which only allows to obtain local results. For example, suppose we want to solve question 2) for \(y(x)\) represented as in Theorem 1. We can write
\[
y(x) = \varphi(\nu_1^{(0)} \omega_1^{(0)}(x) + \nu_2^{(0)} \omega_2^{(0)}(x) + v^{(0)}(x)) + \frac{1 + x}{3}
\]

\[
\equiv \varphi(\nu_2^{(1)} \omega_1^{(1)}(x) + \nu_1^{(1)} \omega_2^{(1)}(x) + v^{(1)}(x)) + \frac{1 + x}{3}
\]

(84)

There also exists a transcendent
\[
y(x) = \varphi(\nu_1^{(1)} \omega_1^{(1)}(x) + \nu_2^{(1)} \omega_2^{(1)}(x) + v^{(1)}(x)) + \frac{1 + x}{3}
\]

(85)

where \(v^{(1)}(x)\) is represented by Theorem 3. In particular, \(v^{(1)}(x)\) is bounded as \(x \to 1\). On the other hand, the function \(v^{(0)}\) is only known for small \(x\), and it may diverge as \(x \to 1\). In the best hypothesis, we may suppose that after the rescaling
\[
v^{(0)}(x) \mapsto v^{(0)}(x) - \delta \nu_1 \omega_1^{(1)}(x) - \delta \nu_2 \omega_2^{(1)}(x), \quad \delta \nu_1, \delta \nu_2 \in \mathbb{C}
\]

the new function \(v^{(0)} - \delta \nu_1 \omega_1^{(1)} - \delta \nu_2 \omega_2^{(1)}\) vanishes as \(x \to 1\) (this may happen, since the divergence of \(\omega_2^{(1)}\), which is \(\ln(1 - x)\) as \(x \to 1\), may cancel the divergence of \(v^{(0)}(x)\) as \(x \to 1\)). Therefore, [34] is a transcendent ([34]) where
\[
\nu_1^{(1)} = \nu_2^{(0)} + \delta \nu_1, \quad \nu_2^{(1)} = \nu_1^{(0)} + \delta \nu_2
\]
and
\[ v^{(1)}(x) = v^{(0)} - \delta \nu_1 \omega_1^{(1)} - \delta \nu_2 \omega_2^{(1)} \]

Unfortunately, we are not able to say if the above rescaling is possible using the local analysis. And if it is, we do not know \( \delta \nu_1, \delta \nu_2 \).

The answer to questions 1) and 2) can be obtained by the method of isomonodromic deformations (which, on the other hand, has some limitations in providing the local behavior).

### 8.1 Picard Solutions

Before answering Questions 1 and 2 in general, we recall that there is one case when the local analysis in elliptic representation becomes global, namely when \( v(x) = 0 \). This happens when \( \alpha = \beta = \gamma = 1 - 2\delta = 0 \). This case was already well known to Picard [30] and it was studied in [23]. The equation (35) reduces to the hypergeometric equation \( L(u) = 0 \), and it has general solution \( u(x)/2 = \nu_1 \omega_1^{(0)}(x) + \nu_2 \omega_2^{(0)}(x) \).

Therefore,
\[ y(x) = \wp(\nu_1 \omega_1^{(0)}(x) + \nu_2 \omega_2^{(0)}(x); \omega_1^{(0)}, \omega_2^{(0)}) + \frac{1 + x}{3} \]

Question 1) can be answered immediately. There is no need to prove Theorem 1 because we have no \( v(x) \)!

The function \( u(x)/2 = \nu_1 \omega_1^{(0)}(x) + \nu_2 \omega_2^{(0)}(x) \) in the argument of \( \wp \) is defined for any \( x \neq 0, 1, \infty \).

We obtain the critical behavior by Fourier-expanding
\[ \wp(\nu_1 \omega_1^{(0)}(x) + \nu_2 \omega_2^{(0)}(x); \omega_1^{(0)}, \omega_2^{(0)}) = \wp(\nu_1 \omega_1^{(0)}(x) + [\nu_2 + 2N] \omega_2^{(0)}(x); \omega_1^{(0)}, \omega_2^{(0)}) \]

For any fixed \( N \), the expansion is performed if \( |x| < 1 \) and
\[ \Im \left( \frac{\nu_1}{2} + \left( \frac{\nu_2}{2} + N \right) \tau(x) \right) < \Im \tau(x) \]

Namely:
\[ (\Re \nu_2 + 2 + 2N) \ln \frac{|x|}{16} + O(x) < \Im \nu_2 \arg x + \pi \Im \nu_1 < (\Re \nu_2 - 2 + 2N) \ln \frac{|x|}{16} + O(x) \]

(86)

Note that this defines a domain \( \mathcal{D}(r; \nu_1, \nu_2 + 2N) \) which contains the union of \( D_1(r < 1; \nu_1, \nu_2 + 2N), D_2(r < 1; \nu_1, \nu_2 + 2N), D_1(r < 1; \nu_1, \nu_2 + 2[N + 1]), D_2(r < 1; \nu_1, \nu_2 + 2[N + 1]) \) introduced in the proof of Theorem 1.

The critical behavior along
\[ \arg x = \arg x_0 + \frac{3\Re \nu_2 + 2N - \mathcal{V}}{3\Im \nu_2} \ln |x|, \quad -2 \leq \mathcal{V} \leq 2, \quad \Im \nu_2 \neq 0 \]

is:

For \( 0 < \mathcal{V} < 1 \)
\[ y(x) = -\frac{1}{4} \left[ \frac{e^{i\pi \nu_1}}{16^{\nu_2+2N-1}} \right] x^{\nu_2+2N} \left( 1 + O(x^{\nu_2+2N}) \right) \]

For \( 1 < \mathcal{V} < 2 \)
\[ y(x) = -\frac{1}{4} \left[ \frac{e^{i\pi \nu_1}}{16^{\nu_2+2N-1}} \right]^{-1} x^{2-\nu_2-2N} \left( 1 + O(x^{2-\nu_2-2N}) \right) \]

For \( \mathcal{V} = 1 \)
\[ y(x) = x \sin^2 \left( i \frac{1 - \nu_2 - 2N}{2} \ln \frac{x}{16} + \frac{\pi \nu_1}{2} \right) (1 + O(x)) \]

For \( \mathcal{V} = 0 \)
\[ y(x) = \left[ \frac{x}{2} + \sin^{-2} \left( -i \frac{\nu_2 + 2N}{2} \ln \frac{x}{16} + \frac{\pi \nu_1}{2} - i \frac{\nu_2 + 2N}{2} F_1(x) \right) \right] (1 + O(x)) \]
For $\mathcal{V} = 2$

$$ y(x) = \left[ \frac{x}{2} + \sin^{-2} \left( \frac{2 - \nu_2 - 2N}{2} \ln \frac{x}{16} + \frac{\nu_1}{2} + i \frac{2 - \nu_2 - 2N F_1(x)}{F(x)} \right) \right] (1 + O(x)) $$

For $-1 < \mathcal{V} < 0$: it is like $1 < \mathcal{V} < 2$ with $N \mapsto N + 1$.

For $-2 < \mathcal{V} < -1$: it is like $0 < \mathcal{V} < 1$ with $N \mapsto N + 1$.

For $\mathcal{V} = -1$: it is like $\mathcal{V} = 1$ with $N \mapsto N + 1$.

For $\mathcal{V} = -2$: it is like $\mathcal{V} = 0$ with $N \mapsto N + 1$.

We observe that the choice of $N$ is arbitrary, therefore the same transcendent has different critical behaviors on different domains $\mathcal{R}_N$ specified by different values of $N$. This answers Question 1.

Remark: Note that in the cases $\mathcal{V} = -2, 0, 2$, the $\sin^2(...)$ may vanish, therefore there may be (movable) poles. Actually, this happens because the domain now is bigger than that of the generic case since we did not impose that $\sin^2(...) \neq 0$, as we did in the proof of Theorem 1. This is also the reason why we have to keep $F_1(x)$ in the argument of $\sin^2(...) \in$ the Fourier expansion.

If $\Im \nu_2 = 0$, we choose the convention $0 \leq \nu_1 < 2$. The critical behavior for $0 < \nu_2 < 1$ is the same of the case $\Im \nu_2 \neq 0$ with $N = 0$ and $0 < \mathcal{V} < 1$; for $1 < \nu_2 < 2$ it is the same of the case $\Im \nu_2 \neq 0$ with $N = 0$ and $1 < \mathcal{V} < 2$. Finally,

$$ y(x) = (1 + O(x)) \left[ \frac{x}{2} + \sin^{-2} \left( \frac{\pi \nu_1}{2} \right) \right], \quad \text{if } \nu_2 = 1 $$

$$ y(x) = \left[ \frac{x}{2} + \sin^{-2} \left( \frac{\pi \nu_1}{2} \right) \right] (1 + O(x)), \quad \text{if } \nu_2 = 0, \quad \nu_1 \neq 0 $$

Question 2) can be answered. Actually

$$ y(x) = \wp \left( \nu_1 \omega_1^{(0)}(x) + \nu_2 \omega_2^{(0)}(x); \omega_1^{(0)}, \omega_2^{(0)} \right) + \frac{1 + x}{3} $$

$$ = \wp \left( \nu_2 \omega_1^{(1)}(x) + \nu_1 \omega_2^{(1)}(x); \omega_1^{(0)}, \omega_2^{(0)} \right) + \frac{1 + x}{3} $$

$$ = \wp \left( \nu_1 \omega_1^{(\infty)}(x) + (\nu_2 - \nu_1) \omega_2^{(\infty)}(x); \omega_1^{(0)}, \omega_2^{(0)} \right) + \frac{1 + x}{3} $$

Namely, the couples $(\nu_1^{(i)}, \nu_2^{(i)})$, $i = 0, 1, \infty$, are:

$$(\nu_1, \nu_2), \quad (\nu_2, \nu_1), \quad (\nu_1, \nu_2 - \nu_1)$$

8.2 Solution of the Connection Problem in a Non-Generic Case

We first give the solutions of Questions 1) and 2) in the special case $\beta = \gamma = 1 - 2\delta = 0$ studied in [1] and in [11], because of its connection to 2-D topological field theory and to Frobenius Manifolds [11] and quantum cohomology [4] [5] [12]. This sub-section mainly reviews the results of [11] and connects them to the framework of the present paper. In the next sub-section we will consider the generic case.

The Painlevé VI equation is the isomonodromic deformation equation of the fuchsian system [17]

$$ \frac{dY}{dz} = \left( \begin{array}{cc} A_0(z) & A_1(z) \\ z & z+1 \end{array} \right) + \left( \begin{array}{cc} A_2(z) \\ z-x \end{array} \right) Y \equiv A(z, x) Y $$

(87)

where $A_i(x)$ ($i = 0, 1, x$) are $2 \times 2$ matrices. They depend on $x$ in such a way that the monodromy matrices at $z = 0, 1, x$ do not change for small deformations of $x$. Moreover,

$$ A_0(x) + A_1(x) + A_x(x) = -\frac{1}{2} \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}, \quad \text{eigenvalues of } A_i(x) = \pm \frac{1}{2} \theta_i, \quad i = 0, 1, x $$

(88)
The constants $\theta_\nu, \nu = 0, 1, x, \infty$ are linked to the parameters of PVI:

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad \beta = -\frac{1}{2}\theta_0^2, \quad \gamma = \frac{1}{2}\theta_1^2, \quad \delta = \frac{1}{2}(1 - \theta_x^2)$$

A Painlevé transcendent is the solution $z = z(x)$ of $A_{12}(z, x) = 0$: actually $A_{12}(z, x) = \kappa(x)(z - y(x))/z(z - 1)(z - x)$, where the function $\kappa(x)$ does not interests us now (see [17]). The monodromy matrices give a representation of the fundamental group of $P^1\{0, 1, x, \infty\}$. They are usually denoted by $M_{\nu}, \nu = 0, 1, x, \infty$ and they correspond to counterclockwise loops around $z = 0, 1, x, \infty$ respectively. We order the loops as in figure 2 (an arbitrary base point is chosen). The monodromy matrices are not independent, because

$$M_1 M_x M_0 = M_\infty$$

Moreover

$$M_\infty \text{ is similar to } e^{2\pi i \begin{pmatrix} -\frac{\theta_0}{2} & 0 \\ 0 & \frac{\theta_x}{2} \end{pmatrix}} e^{2\pi i R}, \quad R = \begin{cases} 0 & \text{if } \theta_\infty \not\in \mathbb{Z} \\ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} & \text{if } 0 < \theta_\infty \in \mathbb{Z} \\ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} & \text{if } 0 > \theta_\infty \in \mathbb{Z} \end{cases}$$

where $b$ is a complex number. Following [7], let $\alpha = \frac{(2\mu - 1)^2}{2}$ (namely $\theta_\infty = 2\mu, \theta_0 = \theta_x = \theta_1 = 0$) and:

$$2 - x_0^2 = \text{tr } M_0 M_x, \quad 2 - x_1^2 = \text{tr } M_1 M_x, \quad 2 - x_\infty^2 = \text{tr } M_0 M_1.$$ 

They satisfy

$$x_0^2 + x_1^2 + x_\infty^2 - x_0 x_1 x_\infty = 4 \sin^2(\pi \mu).$$

(89)

For any fixed $\mu$, it is proved in [4] that there is a one to one correspondence between Painlevé transcendent and triples of monodromy data $(x_0, x_1, x_\infty)$, defined up to the change of two signs, satisfying $x_i \neq \pm 2, i = 0, 1, \infty$ and at most one $x_i = 0$. A transcendent in one to one correspondence to $(x_0, x_1, x_\infty)$ will be denoted $y = y(x; x_0, x_1, x_\infty)$.

Let $\sigma$ be a complex number such that $2 \cos(\pi \sigma) = \text{tr}(M_0 M_x)$, namely:

$$\cos \pi \sigma = 1 - \frac{x_\infty^2}{2}, \quad 0 \leq \mathfrak{R} \sigma \leq 1, \quad \sigma \neq 1$$

We define a domain in the universal covering $\widetilde{C_0}$ of $C \setminus \{0\}$, for small $|x|$, as follows: let $\epsilon$ be a small positive number, $0 < \sigma < 1$ a real number arbitrarily close to 1 and $\theta_1, \theta_2$ two real parameters. We define

$$D(\epsilon; \sigma) := \{x \in \widetilde{C_0} \text{ s.t. } |x| < \epsilon, \quad e^{-\theta_1 \mathfrak{R} \sigma} |x|^\sigma \leq |x^\sigma| \leq e^{-\theta_2 \mathfrak{R} \sigma}, \quad 0 < \sigma < 1\}.$$ 

(90)
If $0 \leq \sigma < 1$ we simply let $D(\epsilon; \sigma) := \{x \in \mathbb{C}_0 \text{ s.t. } |x| < \epsilon\}$. The critical behavior of $y = y(x; x_0, x_1, x_\infty)$ was obtained in [1] in the domains $D(\epsilon_n; \pm \sigma + 2n)$ for any integer $n$. We note that such domains can be rewritten as

$$\Re(\pm \sigma + 2n) \ln |x| + \vartheta_2 \Im(\pm \sigma) \leq \Im(\pm \sigma) \arg x < [\Re(\pm \sigma + 2n) - \check{\sigma}] \ln |x| + \vartheta_1 \Im(\pm \sigma), \quad |x| < \epsilon_n$$

As it is proved in [11], $\epsilon_n < 1$ is small (it decreases as $n$ increases) and $\Im(\pm \sigma) \vartheta_1$ decreases as $-\ln |n|$. The critical behavior in the domains $D(\epsilon_n; \pm \sigma + 2n)$ is:

$$y(x; x_0, x_1, x_\infty) = a(\pm \sigma + 2n; x_0, x_1, x_\infty) x^{1 - \left[\pm \sigma + 2n\right]}(1 + O(x^{\pm \sigma + 2n} + x^{1 - \left[\pm \sigma + 2n\right]}))$$

(91)

where

$$a(\sigma; x_0, x_1, x_\infty) = \frac{i16^\sigma \Gamma\left(\frac{\sigma + 1}{2}\right)^4}{8 \sin(\pi \sigma) \Gamma\left(1 - \mu + \frac{\sigma}{2}\right) \Gamma\left(\mu + \frac{\sigma}{2}\right)} \left[2(1 + e^{-i\pi \sigma}) - f(x_0, x_1, x_\infty)(x_\infty^2 + e^{-i\pi \sigma} x_1^2)\right] f(x_0, x_1, x_\infty)$$

and

$$f(x_0, x_1, x_\infty) := \frac{4 - x_0^2}{x_1^2 + x_\infty^2 - x_0 x_1 x_\infty}$$

(92)

The above formula for $a$ has some limit cases when $\sigma \to 0$, $\pm 2\mu + 2m$, $m$ integer. Namely:

I) For $x_0 = 0$.

$$\sigma = 0, \quad a = \frac{x_\infty^2}{x_1^2 + x_\infty^2}$$

provided that $x_1 \neq 0$ and $x_\infty \neq 0$.

II) For $x_0^2 = 4 \sin^2(\pi \mu)$ (and then $x_\infty^2 = -x_1^2 \exp(\pm 2\pi i \mu)$ as it follows from [83]), there are four sub-cases:

II1) $x_\infty^2 = -x_1^2 e^{2\pi i \mu}$

$$\sigma = 2\mu + 2m, \quad a = -\frac{1}{4x_1^2} \frac{16^{\sigma + 2m} \Gamma(\mu + m + \frac{1}{2})^4}{\Gamma(m + 1)^2 \Gamma(2\mu + m)^2}, \quad m = 0, 1, 2, ...$$

II2) $x_\infty^2 = -x_1^2 e^{2\pi i \mu}$

$$\sigma = 2\mu + 2m, \quad m = -1, -2, -3, ...$$

$$a = -\frac{\cos^4(\pi \mu)}{4\pi^4} 16^{\sigma + 2m} \Gamma(\mu + m + \frac{1}{2})^4 \Gamma(-2\mu - m + 1)^2 \Gamma(-m)^2 x_1^2$$

II3) $x_\infty^2 = -x_1^2 e^{2\pi i \mu}$

$$\sigma = -2\mu + 2m, \quad a = -\frac{1}{4x_1^2} \frac{16^{-2\mu + 2m} \Gamma(-\mu + m + \frac{1}{2})^4}{\Gamma(-2\mu + m + 1)^2 \Gamma(2m)^2}, \quad m = 1, 2, 3, ...$$

II4) $x_\infty^2 = -x_1^2 e^{2\pi i \mu}$

$$\sigma = -2\mu + 2m, \quad m = 0, -1, -2, -3, ...$$

$$a = -\frac{\cos^4(\pi \mu)}{4\pi^4} 16^{-2\mu + 2m} \Gamma(-\mu + m + \frac{1}{2})^4 \Gamma(2\mu - m)^2 \Gamma(1 - m)^2 x_1^2$$

Conversely, let be given a transcendent with behavior $y(x) = ax^{1-\sigma}(1 + \text{higher orders})$ as $x \to 0$, for given $\sigma$ and $a$. We define a triple $(x_0, x_1, x_\infty)$ by the formulae below:

$$x_0 = 2 \sin(\frac{\pi}{2} \sigma)$$

$$x_1 = i \left(\frac{\sqrt{a}}{f(\sigma, \mu) G(\sigma, \mu)} - \frac{G(\sigma, \mu)}{\sqrt{a}}\right)$$

$$x_\infty = \frac{e^{i\frac{\pi}{2}} \sqrt{a}}{f(\sigma, \mu) G(\sigma, \mu)} + \frac{G(\sigma, \mu)}{e^{i\frac{\pi}{2}} \sqrt{a}}$$

39
where

\[ f(\sigma, \mu) := \frac{2 \cos^2 \left( \frac{\pi \sigma}{2} \right)}{\cos(\pi \sigma) - \cos(2\pi \mu)} \equiv f(x_0, x_1, x_\infty), \quad G(\sigma, \mu) := \frac{4\pi \Gamma \left( \frac{\sigma + 1}{2} \right)^2}{2\Gamma(1 - \mu + \frac{\sigma}{2}) \Gamma(\mu + \frac{\sigma}{2})} \]

Again, there are limiting cases:

i) \( \sigma = 0 \)

\[
\begin{align*}
x_0 &= 0 \\
x_1^2 &= 2 \sin(\pi \mu) \sqrt{1 - a} \\
x_\infty^2 &= 2 \sin(\pi \mu) \sqrt{a}
\end{align*}
\]

ii) \( \sigma = \pm 2\mu + 2m \).

ii1) \( \sigma = 2\mu + 2m, \ m = 0, 1, 2, \ldots \)

\[
\begin{align*}
x_0 &= 2 \sin(\pi \mu) \\
x_1 &= -\frac{i}{2} \frac{16^{\nu - \nu_1} \Gamma(\mu + m + \frac{1}{2})^2}{\Gamma(\nu_1 + m + 1) \Gamma(\nu_2 + m)} \frac{1}{\sqrt{a}} \\
x_\infty &= i x_1 e^{-i\pi \mu}
\end{align*}
\]

ii2) \( \sigma = 2\mu + 2m, \ m = -1, -2, -3, \ldots \)

\[
\begin{align*}
x_0 &= 2 \sin(\pi \mu) \\
x_1 &= 2i \frac{\pi^2}{\cos^2(\pi \mu)} \frac{1}{\Gamma(\nu_1 + m + \frac{1}{2}) \Gamma(\nu_2 + m + 1) \Gamma(-2m - 1)} \sqrt{a} \\
x_\infty &= -ix_1 e^{i\pi \mu}
\end{align*}
\]

ii3) \( \sigma = -2\mu + 2m, \ m = 1, 2, 3, \ldots \)

\[
\begin{align*}
x_0 &= -2 \sin(\pi \mu) \\
x_1 &= -i \frac{16^{-\nu - \nu_1} \Gamma(-\mu + m + \frac{1}{2})^2}{\Gamma(-\nu_1 + m + 1) \Gamma(m)} \frac{1}{\sqrt{a}} \\
x_\infty &= ix_1 e^{i\pi \mu}
\end{align*}
\]

ii4) \( \sigma = -2\mu + 2m, \ m = 0, -1, -2, -3, \ldots \)

\[
\begin{align*}
x_0 &= -2 \sin(\pi \mu) \\
x_1 &= 2i \frac{\pi^2}{\cos^2(\pi \mu)} \frac{1}{\Gamma(-\nu_1 + m + \frac{1}{2}) \Gamma(\nu_2 + m) \Gamma(-m)} \sqrt{a} \\
x_\infty &= -ix_1 e^{-i\pi \mu}
\end{align*}
\]

In all the above formulae the sign of the square roots is arbitrary (a triple is define up to the change of two signs). The relation \( x_0^2 + x_1^2 + x_\infty^2 - x_0 x_1 x_\infty = 4 \sin^2(\pi \mu) \) is automatically satisfied. Note that \( \sigma \neq 1 \) implies \( x_0 \neq \pm 2 \).

It is proved in \[ \[ \] \] that if the triple \( (x_0, x_1, x_\infty) \) defined above is such that \( x_i \neq \pm 2 \ (i = 0, 1, \infty) \) and at most one \( x_i = 0 \), then there exists only one transcendent with behavior \( y(x) = ax^{1-\sigma}(1 + \text{higher orders}) \) as \( x \to 0 \) specified by the given \( a \) and \( \sigma \). It coincides with \( y(x; x_0, x_1, x_\infty) \).

Let us return to the elliptic representation. We observe that in the special case \( \beta = \gamma = 1 - 2\delta = 0 \) there is only \( \frac{d}{du} \varphi(u/2) \) in the r.h.s. of \[ \[ \] \]. Therefore, \( \varepsilon_1 = \varepsilon_2 = 0 \) in the proof of Theorems 1 and 3. As a consequence, not only at \( x = 0 \), but also at \( x = 1, \infty \) the domain \( D \) is larger. More precisely, at \( x = 0 \) the result of section \[ \[ \] \] applies: For any \( N \in \mathbb{Z} \), and for any complex \( \nu_1^{(0)}, \nu_2^{(0)} \), such that \( 0 < \nu_2^{(0)} < 2 \) if \( \nu_2^{(0)} \) is real, there exists a transcendent \( y = \varphi(\nu_1^{(0)} \omega_1^{(0)} + \nu_2^{(0)} \omega_2^{(0)} + v^{(0)}) + 1/\pi \) such that \( v^{(0)}(x) \) is holomorphic in

\[
\mathcal{D}(r_N; \nu_1^{(0)}, \nu_2^{(0)} + 2N) := \left\{ x \in \mathbb{C}_0 \mid |x| < r_N, \varepsilon^{i\pi \nu_1^{(0)}} \left( \frac{x}{16} \right)^{2-\nu_2^{(0)} - 2N} < r, \varepsilon^{i\pi \nu_1^{(0)}} \left( \frac{x}{16} \right)^{\nu_2^{(0)} + N} < r \right\}
\]
where it has the convergent expansion
\[
v(x; \nu_1^{(0)}, \nu_2^{(0)} + 2N) = \sum_{n \geq 1} a_n x^n + \sum_{n \geq 0, m \geq 1} b_{nm} x^n \left[ e^{-i \nu_1^{(0)}} \left( \frac{x}{16} \right)^{2 - \nu_2^{(0)} - 2N} \right]^m + \\
+ \sum_{n \geq 0, m \geq 1} c_{nm} x^n \left[ e^{i \nu_1^{(0)}} \left( \frac{x}{16} \right)^{\nu_2^{(0)} + 2N} \right]^m
\]

At \( x = 1 \). For any \( N \in \mathbb{Z} \), and for any complex \( \nu_1^{(1)}, \nu_2^{(1)} \), such that \( 0 < \nu_2^{(1)} < 2 \) if \( \nu_2^{(1)} \) is real, there exists a transcendent \( y = \varphi(\nu_1^{(1)} \omega_1^{(1)} + \nu_2^{(1)} \omega_2^{(1)} + v^{(1)}) + \frac{1}{1x} \) such that \( v^{(1)}(x) \) is holomorphic in

\[
\mathcal{D}(r_N; \nu_1^{(1)}, \nu_2^{(1)} + 2N) := \left\{ x \in \mathbb{P}_1^1 \text{ such that } |1 - x| < r_N, \quad \left| e^{i \nu_1^{(1)}} \left( \frac{1 - x}{16} \right)^{2 - \nu_2^{(1)} - 2N} \right| < r_N \right\}
\]

where it has a convergent expansion
\[
v^{(1)}(x; \nu_1^{(1)}, \nu_2^{(1)} + 2N) = \sum_{n \geq 1} a_n (1 - x)^n + \sum_{n \geq 0, m \geq 1} b_{nm} (1 - x)^n \left[ e^{i \nu_1^{(1)}} \left( \frac{1 - x}{16} \right)^{2 - \nu_2^{(1)} - 2N} \right]^m + \\
+ \sum_{n \geq 0, m \geq 1} c_{nm} (1 - x)^n \left[ e^{-i \nu_1^{(1)}} \left( \frac{1 - x}{16} \right)^{\nu_2^{(1)} + 2N} \right]^m
\]

At \( x = \infty \). For any \( N \in \mathbb{Z} \), and for any complex \( \nu_1^{(\infty)}, \nu_2^{(\infty)} \), such that \( 0 < \nu_2^{(\infty)} < 2 \) if \( \nu_2^{(\infty)} \) is real, there exists a transcendent \( y = \varphi(\nu_1^{(\infty)} \omega_1^{(\infty)} + \nu_2^{(\infty)} \omega_2^{(\infty)} + v^{(\infty)}) + \frac{1}{1x} \) such that \( v^{(\infty)}(x) \) is holomorphic in

\[
\mathcal{D}(r_N; \nu_1^{(\infty)}, \nu_2^{(\infty)} + 2N) := \left\{ x \in \mathbb{P}_\infty \text{ such that } |x^{-1}| < r_N, \quad \left| e^{i \nu_1^{(\infty)}} \left( \frac{16}{x} \right)^{2 - \nu_2^{(\infty)} - 2N} \right| < r_N \right\}
\]

where it has a convergent expansion
\[
x \cdot v^{(\infty)}(x; \nu_1^{(\infty)}, \nu_2^{(\infty)} + 2N) = \sum_{n \geq 1} a^{(1)}_n \left( \frac{1}{x} \right)^n + \sum_{n \geq 0, m \geq 1} b^{(1)}_{nm} \left( \frac{1}{x} \right)^n \left[ e^{-i \nu_1^{(\infty)}} \left( \frac{16}{x} \right)^{2 - \nu_2^{(\infty)} - 2N} \right]^m + \\
+ \sum_{n \geq 0, m \geq 1} c^{(1)}_{nm} \left( \frac{1}{x} \right)^n \left[ e^{i \nu_1^{(\infty)}} \left( \frac{16}{x} \right)^{\nu_2^{(\infty)} + 2N} \right]^m
\]

Note that we have used the same notations for the coefficients \( a_n, b_{nm}, c_{nm} \) both at \( x = 0, 1 \) and \( x = \infty \). This is for simplicity of notations, but they are different!

We are ready to identify the transcendents \( y(x; x_0, x_1, x_\infty) \) with the transcendents in elliptic representation. We know that for \( x \to 0 \) the transcendent \( y(x) = \varphi(\nu_1 \omega_1^{(0)} + \nu_2 \omega_2^{(0)} + v^{(0)}(x; \nu_1, \nu_2 + 2N)) + \frac{1}{1x} \) has behavior

\[
y(x) = -\frac{1}{4} \left[ \frac{e^{i \nu_1}}{16 \nu_2 + 2N - 1} \right] \nu_2^{2N} (1 + \text{higher orders}), \quad 0 < \nu < 1
\]
\[-\frac{1}{4} \left[ \frac{e^{i\pi \nu_1}}{16^{\nu_2-1}} \right]^{-1} x^2 - \nu_2 - 2N (1 + \text{higher orders}), \quad 1 < \nu < 2\]

On the other hand, a transcendent \( y(x; x_0, x_1, x_\infty) \) has behavior (91):

\[
y(x; x_0, x_1, x_\infty) = ax^{1-\nu} (1 + \text{higher orders})
\]

As we mentioned above, the critical behavior uniquely determines the transcendent. Therefore, the transcendent in the elliptic representation, for given \( \nu_1, \nu_2, N \), must coincide with some \( y(x; x_0, x_1, x_\infty) \) having the same critical behavior. Let us choose the convention

\[
0 \leq \Re \sigma \leq 1, \quad 0 \leq \Re \nu_2 < 2
\]

We take \( N = n = 0 \). A necessary condition for the transcendent to coincide is that

\[
D(\epsilon_0; \sigma) \cap D(\sigma_0; \nu_1, \nu_2) \neq \emptyset
\]

They actually coincide when they have the same critical behavior, namely if:

1) \( \sigma = 1 - \nu_2, \quad a = -\frac{1}{4} \left[ \frac{e^{i\pi \nu_1}}{16^{\nu_2-1}} \right], \quad \text{if } 0 \leq \Re \nu_2 \leq 1 \)

2) \( \sigma = \nu_2 - 1, \quad a = -\frac{1}{4} \left[ \frac{e^{i\pi \nu_1}}{16^{\nu_2-1}} \right]^{-1}, \quad \text{if } 1 \leq \Re \nu_2 < 2 \)

(recall that \( \sigma \neq 1 \) and \( \nu_2 \neq 0, 2 \)). Equivalently:

1) \( \nu_2 = 1 - \sigma, \quad e^{i\pi \nu_1} = -4a(\sigma) 16^{-\sigma} \quad \text{if } 0 \leq \Re \nu_2 \leq 1 \)

2) \( \nu_2 = 1 + \sigma, \quad e^{-i\pi \nu_1} = -4a(\sigma) 16^{-\sigma} \quad \text{if } 1 \leq \Re \nu_2 < 2 \)

We are now ready to answer Question 1) and Question 2). Actually, this is possible because the identifications 1) and 2) above allow us to express the parameters \( \nu_1, \nu_2 \) in terms of the triple of monodromy data \( (x_0, x_1, x_\infty) \) and \( \mu \).

We observe that transformation \( \sigma \rightarrow -\sigma \) is \( \sigma = 1 - \nu_2 \rightarrow \nu_2 - 2 \) in case 1), and \( \sigma = \nu_2 - 1 \rightarrow 1 - \nu_2 \) in case 2). Namely, in case 1) the behavior of \( y(x; x_0, x_1, x_\infty) \) is

\[
y(x; x_0, x_1, x_\infty) = a(\sigma; x_0, x_1, x_\infty)x^{1-\sigma} (1 + \text{higher orders}), \quad \text{in } D(\epsilon_0; \sigma)
\]

and

\[
y(x; x_0, x_1, x_\infty) = a(-\sigma; x_0, x_1, x_\infty)x^{1-\sigma} (1 + \text{higher orders}), \quad \text{in } D(\epsilon_0; -\sigma)
\]

This corresponds to the behaviors obtained in elliptic representation:

\[
y(x) = -\frac{1}{4} \left[ \frac{e^{i\pi \nu_1}}{16^{\nu_2-1}} \right] x^{2\nu_2} (1 + \text{higher orders}), \quad 0 < \nu < 1
\]

and

\[
y(x) = -\frac{1}{4} \left[ \frac{e^{i\pi \nu_1}}{16^{\nu_2-1}} \right]^{-1} x^{2-\nu_2} (1 + \text{higher orders}), \quad 1 < \nu < 2
\]

respectively, with \( \cos \pi \nu_2 = \frac{\nu_2}{2} - 1 \) and

\[
e^{i\pi \nu_1} = -\frac{i \Gamma^4 \left( 1 - \frac{\nu_2}{2} \right)}{2 \sin(\pi \nu_2) \Gamma^2 \left( \frac{3}{2} - \mu - \frac{\nu_2}{2} \right) \Gamma^2 \left( \frac{1}{2} + \mu - \frac{\nu_2}{2} \right)} \left[ 2(1 - e^{i\pi \nu_2}) - f(x_0, x_1, x_\infty)(x_\infty^{2} - e^{i\pi \nu_2} x_1^{2}) \right] f(x_0, x_1, x_\infty)
\]

In case 2), the behavior of \( y(x; x_0, x_1, x_\infty) \) is still

\[
y(x; x_0, x_1, x_\infty) = a(\sigma; x_0, x_1, x_\infty)x^{1-\sigma} (1 + \text{higher orders}), \quad \text{in } D(\epsilon_0; \sigma)
\]
and
\[
y(x; x_0, x_1, x_\infty) = a(-\sigma; x_0, x_1, x_\infty)x^{1-|\sigma|}(1 + \text{higher orders}), \quad \text{in } D(\epsilon_0; -\sigma)
\]

This corresponds to
\[
y(x) = -\frac{1}{4} \left[ \frac{e^{i\pi \nu _1}}{16^\nu _2 - 4} \right]^{-1} x^{2-\nu _2}(1 + \text{higher orders}), \quad 1 < \nu < 2
\]

and
\[
y(x) = -\frac{1}{4} \left[ \frac{e^{i\pi \nu _1}}{16^\nu _2 - 4} \right] x^{\nu _2}(1 + \text{higher orders}), \quad 0 < \nu < 1
\]

respectively, with \( \cos \pi \nu _2 = x_0^2 - 1 \) and
\[
e^{-i\pi \nu _1} = \frac{i \Gamma^4 \left( \frac{\nu _2}{2} \right)}{2 \sin(\pi \nu _2) \Gamma^2 \left( \frac{1 - \nu _2}{2} \right) \Gamma^2 \left( \frac{1 + \nu _2}{2} \right)} [2(1 - e^{-i\pi \nu _2}) - f(x_0, x_1, x_\infty)(x_\infty^2 - e^{-i\pi \nu _2} x_1^2)] f(x_0, x_1, x_\infty)
\]

**Answer to Question 1.** Let us study the effect of the shift \( \sigma \mapsto \sigma + 2n \) on the elliptic representation of \( y(x; x_0, x_1, x_\infty) \). \( y(x; x_0, x_1, x_\infty) \) has critical behaviors on the domains \( D(\epsilon_n; \sigma + 2n) \), with exponents \( \sigma + 2n \) and coefficients \( a = a(\sigma + 2n; x_0, x_1, x_\infty) \). The shift corresponds to \( \nu _2 \mapsto \nu _2 - 2n \) in case 1), and \( \nu _2 \mapsto \nu _2 + 2n \) in case 2). Using the properties of the \( \Gamma \)-function and (93),(94) we obtain, both for case 1) and case 2):
\[
e^{i\pi \nu _1} \big|_{\nu _2 + 2n} = e^{i\pi \nu _1} \big|_{\nu _2} \quad K(\nu _2, n)
\]

where
\[
K(\nu _2, n) := \left\{
\begin{array}{ll}
1, & n = 0 \\
\Pi_{k=1}^{n}[\nu _2 - 1 + 2\mu + 2(k-1)] \cdot [\nu _2 - 1 - 2\mu + 2(k-1)], & n > 0 \\
\Pi_{k=1}^{\nu _2}[\nu _2 - 1 + 2\mu - 2k] \cdot [\nu _2 - 1 - 2\mu - 2(k-1)], & n < 0
\end{array}
\right.
\]

Thus
\[
\nu _1 \big|_{\nu _2 + 2n} = \nu _1 \big|_{\nu _2} - \frac{i}{\pi} \ln K(\nu _2, n)
\]

In this way we have answered to Question 1): the representations
\[
y(x) = \varphi \left( [\nu _1 - \frac{i}{\pi} \ln K(\nu _2, N)] \omega _1 + \nu _2 \omega _2 + v(x; \nu _1 - \frac{i}{\pi} \ln K(\nu _2, N), \nu _2 + 2N) \right) + \frac{1 + x}{3}
\]

define the same transcendent on different domains \( D(\epsilon_N; \nu _1 - \frac{i}{\pi} \ln K(\nu _2, N), \nu _2 + 2N) \).

**Remark:** We now know the elliptic representation and the critical behavior of the transcendent \( y(x; x_0, x_1, x_\infty) \) on the union of the domains \( D(\epsilon_N; \nu _1 - \frac{i}{\pi} \ln K(\nu _2, N), \nu _2 + 2N), n \in \mathbb{Z} \) (for \( \Im \nu _2 \neq 0 \)). However, some regions in the \( (\ln |x|, \Im \nu _2 \arg x) \)-plane, for \( x \) close to 0, are not included in the union. In these regions the transcendent may have movable poles.

**Answer to Question 2.** In [7],[11] the connection problem was solved by showing that a given transcendent \( y(x; x_0, x_1, x_\infty) \) has critical behaviors:
\[
y(x; x_0, x_1, x_\infty) = a^{(0)} x^{1-\sigma^{(0)}} (1 + \text{higher orders in } x), \quad x \to 0
\]
\[
y(x; x_0, x_1, x_\infty) = 1 - a^{(1)} (1 - x)^{1-\sigma^{(1)}} (1 + \text{higher orders in } (1 - x)), \quad x \to 1
\]
\[
y(x; x_0, x_1, x_\infty) = a^{(\infty)} x^{\sigma^{(\infty)}} (1 + \text{higher orders in } x^{-1}), \quad x \to \infty
\]

where
\[
\cos(\pi \sigma^{(i)}) = 1 - \frac{x^2}{2'}, \quad 0 \leq \Re \sigma^{(i)} \leq 1, \quad \sigma^{(i)} \neq 1, \quad i = 0, 1, \infty
\]
The parameters $a^{(0)}$, $\sigma^{(0)}$ stand for $a, \sigma$ used before. $a^{(1)}, a^{(\infty)}$ are obtained by the formula (92) (or limit cases) with the substitutions $(x_0, x_1, x_\infty) \mapsto (x_1, x_0, x_0 x_1 - x_\infty)$, $\sigma^{(0)} \mapsto \sigma^{(1)}$ and $(x_0, x_1, x_\infty) \mapsto (x_\infty, -x_1, x_0 - x_1 x_\infty)$, $\sigma^{(0)} \mapsto \sigma^{(\infty)}$ respectively. The behavior (97) holds in a domain

$$D(M; \sigma^{(\infty)}) := \{x \in \mathbb{R} \setminus \{\infty\} \text{ s.t. } |x| > M, \ e^{-\theta_\infty \sigma^{(\infty)}} |x|^{-\delta} \leq |x^{-\sigma^{(\infty)}}| \leq e^{-\theta_\infty \sigma^{(\infty)}} \}$$

where $M > 0$ is sufficiently big; the behavior (96) holds in

$$D(\epsilon; \sigma^{(1)}) := \{x \in \mathbb{C} \setminus \{1\} \text{ s.t. } |1 - x| < \epsilon, \ e^{-\theta_\infty \sigma^{(1)}} |1 - x|^\delta \leq |(1 - x)^{\sigma^{(1)}}| \leq e^{-\theta_\infty \sigma^{(1)}} \}$$

At $x = 1$, $x = \infty$ we repeat the analysis we did above for $x = 0$, identifying $y(x; x_0, x_1, x_\infty)$ with a transcendent in elliptic representation with the same critical behaviors (by uniqueness of the critical behavior). Thus, we conclude that the transcendent $y(x; x_0, x_1, x_\infty)$ has a representation

$$y(x) = \varphi(\nu_1^{(0)} \omega_1^{(0)} + \nu_2^{(0)} \omega_2^{(0)} + \nu^{(0)}) + \frac{1 + x}{3}$$

at $x = 0$; it has a representation

$$y(x) = \varphi(\nu_1^{(1)} \omega_1^{(1)} + \nu_2^{(1)} \omega_2^{(1)} + \nu^{(1)}) + \frac{1 + x}{3}$$

at $x = 1$ and it has a representation

$$y(x) = \varphi(\nu_1^{(\infty)} \omega_1^{(\infty)} + \nu_2^{(\infty)} \omega_2^{(\infty)} + \nu^{(\infty)}) + \frac{1 + x}{3},$$

where the parameters $\nu_2^{(i)}$ are obtained from

$$\cos \pi \nu_2^{(i)} = \frac{x_2^{(i)}}{2} - 1, \ 0 \leq \Re \nu_2^{(i)} \leq 1 \ (\nu_2^{(i)} \neq 0), \ \text{or} \ 1 \leq \Re \nu_2^{(i)} < 2 \ i = 0, 1, \infty$$

and the parameter $\nu_1^{(0)}$ is obtained from (93) for the choice $0 \leq \Re \nu_2^{(0)} \leq 1$ and from (94) for the choice $1 \leq \Re \nu_2^{(0)} < 2$. Moreover, $\exp\{\pm i \nu_1^{(1)}\}$, $\exp\{\pm i \nu_1^{(\infty)}\}$ are given by formulae analogous to (93), (94) with the substitutions $(x_0, x_1, x_\infty) \mapsto (x_1, x_0, x_0 x_1 - x_\infty)$, $\nu_2^{(0)} \mapsto \nu_2^{(1)}$ and $(x_0, x_1, x_\infty) \mapsto (x_\infty, -x_1, x_0 - x_1 x_\infty)$, $\nu_2^{(0)} \mapsto \nu_2^{(\infty)}$ respectively.

Conversely, given

$$y(x) = \varphi(\nu_1 \omega_1^{(0)}(x) + \nu_2 \omega_2^{(0)}(x) + \nu^{(0)}(x; \nu_1, \nu_2)) + \frac{1 + x}{3}, \ \text{at } x = 0$$

it coincides with $y(x; x_0, x_1, x_\infty)$, with the following monodromy data.

If $0 \leq \Re \nu_2 \leq 1$:

$$x_0 = 2 \cos\left(\frac{\pi}{2} \nu_2\right)$$

$$x_1 = \left[ e^{-\nu_2} \frac{2 e^{i \frac{\pi}{4} \nu_1}}{f(\nu_2, \mu) G(\nu_2, \mu)} + \frac{G(\nu_2, \mu)}{4^{-\nu_2} 2 e^{i \nu_1}} \right]$$

$$x_\infty = \left[ e^{-\nu_2} \frac{2 e^{i \frac{\pi}{4} (\nu_1 - \nu_2)}}{f(\nu_2, \mu) G(\nu_2, \mu)} + \frac{G(\nu_2, \mu)}{4^{-\nu_2} 2 e^{i \nu_1}} \right]$$

where

$$f(\nu_2, \mu) = -\frac{2 \sin^2\left(\frac{\pi}{4} \nu_2\right)}{\cos(\pi \nu_2) + \cos(2\pi \mu)} \quad \text{and} \quad G(\nu_2, \mu) = 4^{-\nu_2} 2 \frac{\Gamma(1 - \frac{\nu_2}{2})^2}{\Gamma\left(\frac{3}{2} - \mu - \frac{\nu_2}{2}\right) \Gamma\left(\frac{1}{2} + \mu - \frac{\nu_2}{2}\right)}$$

If $1 \leq \Re \nu_2 < 2$:

$$x_0 = 2 \cos\left(\frac{\pi}{2} \nu_2\right)$$

$$x_1 = \left[ e^{-\nu_2} \frac{2 e^{i \frac{\pi}{4} \nu_1}}{4^{-\nu_2} 2 f(\nu_2, \mu) G_1(\nu_2, \mu)} + \frac{4^{-\nu_2} 2 G_1(\nu_2, \mu)}{e^{-i \frac{\pi}{4} \nu_1}} \right]$$

$$\frac{e^{-\nu_2}}{4^{-\nu_2} 2 f(\nu_2, \mu) G_1(\nu_2, \mu)} + \frac{4^{-\nu_2} 2 G_1(\nu_2, \mu)}{e^{-i \frac{\pi}{4} \nu_1}}$$
\[ x_\infty = \left[ \frac{e^{i\frac{\pi}{2}(\nu_2 - \nu_1)}}{4^{1-\nu_2} 2 f(\nu_2, \mu) G_1(\nu_2, \mu)} + \frac{4^{1-\nu_2}}{e^{i\frac{\pi}{2}(\nu_2 - \nu_1)}} \right] \]

where

\[ G_1(\nu_2, \mu) = \frac{1}{4^{1-\nu_2} 2 \Gamma(\frac{1}{2} - \mu + \frac{\nu_2}{2}) \Gamma(-\frac{1}{2} + \mu + \frac{\nu_2}{2})} \]

(The limit cases are left as an exercise for the reader). After computing the monodromy data, we can write the elliptic representations of \( y(x; x_0, x_1, x_\infty) \) at \( x = 1 \) and \( x = \infty \). These are the elliptic representations at \( x = 1, \infty \) of (100). Thus, we have solved the connection problem for (100).

We observed that there is a one to one correspondence between Painlevé transcendents and triples of monodromy data \((x_0, x_1, x_\infty)\), defined up to the change of two signs, satisfying \( x_i \neq \pm 2, \ i = 0, 1, \infty \) (i.e. \( \sigma^{(i)} \neq 1 \)) and at most one \( x_i = 0 \). The cases when these conditions are not satisfied are studied in (23). However, if \( x_i = \pm 2 \) (namely the trace is \( \mp 2 \)) the problem of finding the critical behavior at the corresponding critical point \( x = i \) is still open (except when all the three \( x_i \) are \( \pm 2 \): in this case there is a one-parameter class of solutions called Chazy solutions in [23]). We conclude that the results of our paper (together with [11]), plus the results of [23] cover all the possible transcendents, except the special case when one or two \( x_i \) are \( \pm 2 \). We plan to cover this last case soon.

### 8.3 Solution of the Connection Problem in the Generic Case

We solve the connection problem for the elliptic representation in the generic case:

\[ \nu_2, \theta_0, \theta_x, \theta_1, \theta_\infty \not\in \mathbb{Z}; \frac{\pm 1 \pm \nu_2 \pm \theta_1 \pm \theta_\infty}{2}, \frac{\pm 1 \pm \nu_2 \pm \theta_0 \pm \theta_x}{2} \not\in \mathbb{Z} \]

We show in the Appendix, following [13], that if \( \theta_0, \theta_x, \theta_1, \theta_\infty \not\in \mathbb{Z} \) there is a one to one correspondence between transcendents and the monodromy data \( \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0 M_x), \text{tr}(M_0 M_1), \text{tr}(M_1 M_x) \) of the fuchsian system (87). For this reason we write

\[ y(x) = y(x; \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0 M_x), \text{tr}(M_0 M_1), \text{tr}(M_1 M_x) ) \quad (101) \]

In the Appendix it is shown that for any value of \( \alpha, \beta, \gamma, \delta \), for any \( a \neq 0 \), for any complex \( \sigma \not\in (-\infty, 0] \cup [1, +\infty) \) and for any additional real parameters \( 0 < \bar{\sigma} < 1, \theta_1, \theta_2 \) there exists a sufficiently small \( \epsilon \) and a transcendental \( y(x; \sigma, a) \) in the domain \( D(\epsilon; \sigma) \) defined by (100). It has critical behavior

\[ y(x; \sigma, a) = ax^{1-\sigma}(1 + \text{higher orders}) \]

for \( x \to 0 \) along a regular path in \( D(\epsilon; \sigma) \).

In the Appendix it is also proved that in the generic case a transcendental (101) associated to the monodromy data \( \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0 M_x), \text{tr}(M_0 M_1), \text{tr}(M_1 M_x) \), coincides with a transcendental \( y(x; \sigma, a) \) when \( x \) lies in the domain \( D(\epsilon; \sigma) \). \( \sigma \) and \( a \) are obtained from the monodromy data by explicit formulae

\[ 2 \cos(\pi \sigma) = \text{tr}(M_0 M_x) \]

\[ a = a(\sigma; \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0 M_x), \text{tr}(M_0 M_1), \text{tr}(M_1 M_x)) \]

The formula for \( a \) is rather long and complicated, and we do not need its explicit form here; therefore we refer to the Appendix for its computation. Note that \( \sigma \) is defined up to \( \sigma \mapsto \pm \sigma + 2n, n \) integer. We can fix \( 0 \leq \Re \sigma \leq 1 \); thus, a transcendental (101) associated to the monodromy data coincides with the transcendents

\[ y(x; \pm \sigma + 2n, a(\pm \sigma + 2n; \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0 M_x), \text{tr}(M_0 M_1), \text{tr}(M_1 M_x))) \]

in the domains \( D(\epsilon_n; \pm \sigma + 2n) \) defined for sufficiently small \( \epsilon_n \) (note that only \( 0 < \sigma < 1 \) is allowed if \( \sigma \) is real, and in this case no translation \( \pm \sigma + 2n \) is possible). We observe that \( a \) depends on the monodromy data, but also explicitly on \( \sigma \). Namely, it changes if we do \( \sigma \mapsto \pm \sigma + 2n \) (being the monodromy data fixed). In the following, for brevity we write \( a = a(\sigma) \), understanding the dependence on the monodromy data.

We finally remark that in the Appendix the formula \( a = a(\sigma; \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0 M_x), \text{tr}(M_0 M_1), \text{tr}(M_1 M_x)) \) is derived for any \( \sigma \) such that \( \sigma \not\in (-\infty, 0] \cup [1, +\infty) \), but in [15] also the case \( \sigma = 0 \) is calculated. On
the other hand the cases $\Re \sigma < 0$ and $\Re \sigma \geq 1$ are not considered in \cite{13}, so we have included them in the Appendix.

Conversely, we can start from $y(x; \sigma, a)$. Namely, for given $\theta_0, \theta_x, \theta_1, \theta_\infty$ and $\sigma, a$, we can compute $\text{tr}(M_0 M_x)$, $\text{tr}(M_1 M_x)$, $\text{tr}(M_D M_1)$ as functions of $\sigma, \theta_\nu$ (\(\nu = 0, x, 1, \infty\)) by the formulae of the Appendix (in particular $\text{tr}(M_0 M x) = 2 \cos(\pi \sigma)$). So the transcendent $y(x; \sigma, a)$ associated to $\sigma, a$ will coincide with the transcendent \cite{101} associated to the monodromy data.

We claim the following:

Let $\sigma \notin (-\infty, 0] \cup [1, +\infty)$ and $a \neq 0$. Let $y(x)$ be a solution of $\text{PVI}$ in the generic case such that $y(x) = a x^{1-\sigma}(1 + \text{higher order terms})$ as $x \to 0$ in an open domain contained in $D(\epsilon; \sigma)$. Then, $y(x)$ coincides with $y(x; \sigma, a)$.

The claim is proved in Proposition A1 in the Appendix. As a consequence, a transcendent $y(x; \sigma, a)$ coincides with a transcendent in elliptic representation with the same critical behavior.

We fix the range of $\nu_2$ and $\sigma$ by $0 \leq \Re \nu_2 < 2$, $0 \leq \Re \sigma \leq 1$. If $\Im \nu_2 \neq 0$, we consider a transcendent

$$y(x) = \nu \left( \nu_1 \omega_1(x) + \nu_2 \omega_2(x) + v(x; \nu_1, \nu_2); \, \omega_1(x), \omega_2(x) \right) + \frac{1 + x}{3}$$

on the domain $D(\epsilon; \nu_1, \nu_2)$. For $x \to 0$ it has behavior

$$y(x) = -\frac{1}{4} \left[ \frac{e^{i \pi \nu_1}}{16^{\nu_2 - 1}} \right] x^{\nu_2} \left( 1 + \text{higher orders} \right).$$

Thus it coincides with

$$y(x; \sigma, a(\sigma)) = a(\sigma) x^{1-\sigma} \left( 1 + \text{higher orders} \right) \quad \text{if} \quad 0 \leq \Re \nu_2 \leq 1$$

or

$$y(x; -\sigma, a(-\sigma)) = a(-\sigma) x^{1+\sigma} \left( 1 + \text{higher orders} \right) \quad \text{if} \quad 1 \leq \Re \nu_2 \leq 2$$

where

$$\nu_2 = 1 - \sigma, \quad e^{i \pi \nu_1} = -4 a(\sigma) 16^{-\sigma} = -4 a(1 - \nu_2) 16^{\nu_2 - 1} \quad \text{in case} \quad (102)$$

$$\nu_2 = 1 + \sigma, \quad e^{i \pi \nu_1} = -4 a(-\sigma) 16^{\sigma} = -4 a(1 - \nu_2) 16^{\nu_2 - 1} \quad \text{in case} \quad (103)$$

We note that the corresponding domains $D(\epsilon; \nu_1, \nu_2)$ and $D(\epsilon; \sigma)$ have non-empty intersection, therefore Proposition A1 – namely the above claim – holds. If $\nu_2$ is real, we consider a transcendent

$$y(x) = \nu \left( \nu_1 \omega_1(x) + \nu_2 \omega_2(x) + v(x; \nu_1, \nu_2); \, \omega_1(x), \omega_2(x) \right) + \frac{1 + x}{3}$$

or

$$y(x) = \nu \left( \nu_1 \omega_1(x) + \nu_2 \omega_2(x) + v(x; -\nu_1, 2 - \nu_2); \, \omega_1(x), \omega_2(x) \right) + \frac{1 + x}{3}$$

$$= \frac{1}{4} \left[ \frac{e^{i \pi \nu_1}}{16^{\nu_2 - 1}} \right]^{\nu_2} x^{2 - \nu_2} \left( 1 + \text{higher orders} \right), \quad \text{if} \quad 1 < \nu_2 < 2$$

In both cases, they coincide with

$$y(x; \sigma, a(\sigma)) = a(\sigma) x^{1-\sigma} \left( 1 + \text{higher orders} \right), \quad 0 < \sigma < 1$$

where

$$\nu_2 = 1 - \sigma, \quad e^{i \pi \nu_1} = -4 a(\sigma) 16^{-\sigma} = -4 a(1 - \nu_2) 16^{\nu_2 - 1} \quad \text{in case} \quad (104)$$

$$\nu_2 = 1 + \sigma, \quad e^{-i \pi \nu_1} = -4 a(\sigma) 16^{-\sigma} = -4 a(\nu_2 - 1) 16^{1-\nu_2} \quad \text{in case} \quad (105)$$

The above identifications give $\nu_1$ and $\nu_2$ in terms of the monodromy data associated to $a, \sigma$. 

46
Conversely, we can start from \( y(x; \sigma, a) \) \((0 \leq \Re \sigma \leq 1, \sigma \neq 0, 1)\) and find its elliptic representation. This will be \( y(x) = \wp(\nu_1 \omega_1 + \nu_2 \omega_2 + v(x; \nu_1, \nu_2)) + \frac{1 + x}{3}, \) \(0 \leq \Re \nu_2 \leq 1 \) \((\nu_2 \neq 0, 1)\), with parameters

\[
\nu_2 = 1 - \sigma, \quad e^{i \pi \nu_1} = -4a(\sigma) 16^{-\sigma}
\]

Again \((\epsilon; \sigma)\) and \( D(r; \nu_1, \nu_2) \) (or in \( D_0(r) \) when \(3 \sigma = 0\)) have non empty intersection. Equivalently, according to the Observation 2 in the Introduction, \( y(x; \sigma, a) \) coincides with \( y(x) = \wp(\nu_1 \omega_1 + \nu_2 \omega_2(x) + v(x; -\nu_1, 2 - \nu_2)) + \frac{1 + x}{3}, \) \(1 \leq \Re \nu_2 \leq 2 \) \((\nu_2 \neq 1, 2)\), with parameters

\[
\nu_2 = 1 + \sigma, \quad e^{-i \pi \nu_1} = -4a(\sigma) 16^{-\sigma}
\]

The domain is now \( D(r; -\nu_1, 2 - \nu_2) \) which has non-empty intersection with \( D(\epsilon; \sigma) \).

**Answer to Question 2**. We are ready to solve the connection problem. In the Appendix we show, following \([13]\), that

\[
y(x; \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0 M_x), \text{tr}(M_0 M_1), \text{tr}(M_1 M_x)) =
\]

\[
= a^{(0)} x^{1-\sigma^{(0)}} (1 + \text{higher orders}), \quad x \to 0
\]

\[
= 1 - a^{(1)}(1 - x)^{1-\sigma^{(1)}} (1 + \text{higher orders}), \quad x \to 1
\]

\[
= a^{(\infty)} x^{\sigma^{(\infty)}} (1 + \text{higher orders}), \quad x \to \infty
\]

\((106)\)

as \( x \) converges to the critical points along regular paths contained in domains \( D(\epsilon; \sigma^{(0)}) \) defined in \((06)\), \( D(\epsilon; \sigma^{(1)}) \) defined in \((09)\), \( D(M; \sigma^{(\infty)}) \) defined in \((08)\) respectively. The parameters are computed in terms of the monodromy data by

\[
2 \cos(\pi \sigma^{(0)}) = \text{tr} M_0 M_x, \quad 2 \cos(\pi \sigma^{(1)}) = \text{tr} M_1 M_x, \quad 2 \cos(\pi \sigma^{(\infty)}) = \text{tr} M_0 M_1
\]

\(0 \leq \Re \sigma^{(i)} \leq 1, \quad \sigma^{(i)} \neq 0, 1, (i = 0, 1, \infty)\)

The coefficients \(a^{(0)}, a^{(1)}, a^{(\infty)}\) are also given in terms of the monodromy data by explicit formulae which are rather long to write here. The procedure for their computation is explained in the Appendix. Note that here we denote \( \sigma, a \) introduced before with \( \sigma^{(0)}, a^{(0)} \). By uniqueness of critical behavior (claim above – or Proposition A1), the elliptic representation at \( x = 1 \) of \((106)\) is \( y(x) = \wp(\nu_1^{(1)} \omega_1^{(1)} + \nu_2^{(1)} \omega_2^{(1)} + v^{(1)}(x; \nu_1^{(1)}, \nu_2^{(1)})) + \frac{1 + x}{3}, \) with

\[
\nu_2^{(1)} = 1 - \sigma^{(1)}, \quad e^{-i \pi \nu_1^{(1)}} = -4a^{(1)}(1 - \nu_2^{(1)}) 16^{\nu_2^{(1)} - 1}
\]

\((108)\)

At \( x = \infty \) the representation of \((107)\) is \( y(x) = \wp(\nu_1^{(\infty)} \omega_1^{(\infty)} + \nu_2^{(\infty)} \omega_2^{(\infty)} + v^{(\infty)}(x; \nu_1^{(\infty)}, \nu_2^{(\infty)})) + \frac{1 + x}{3}, \) with

\[
\nu_2^{(\infty)} = 1 - \sigma^{(\infty)}, \quad e^{i \pi \nu_1^{(\infty)}} = -4a^{(\infty)}(1 - \nu_2^{(\infty)}) 16^{\nu_2^{(\infty)} - 1}
\]

\((109)\)

Therefore, the solution of the connection problem is as follows. In the case \( \Im \nu_1^{(0)} \neq 0 \), let us start from an elliptic representation in \( D(r; \nu_1^{(0)}, \nu_2^{(0)}) \) close to \( x = 0 \):

\[
y(x) = \wp(\nu_1^{(0)} \omega_1^{(0)} + \nu_2^{(0)} \omega_2^{(0)} + v^{(0)}(x; \nu_1^{(0)}, \nu_2^{(0)})) + \frac{1 + x}{3}. \quad (110)\)

It must coincide with \( y(x; \sigma^{(0)}, a^{(0)}(\sigma^{(0)})) \) or \( y(x; -\sigma^{(0)}, a^{(0)}(-\sigma^{(0)})) \), according to the value of \( \Re \nu_2^{(0)} \), with parameters identified by the formulae which follow \((102)\) and \((103)\). In the case \( \Im \nu_2^{(0)} = 0 \), let us start from

\[
y(x) = \wp(\nu_1^{(0)} \omega_1^{(0)} + \nu_2^{(0)} \omega_2^{(0)} + v^{(0)}(x; \nu_1^{(0)}, \nu_2^{(0)})) + \frac{1 + x}{3}, \quad 0 < \nu_2^{(0)} < 1 \quad (111)\)

or

\[
y(x) = \wp(\nu_1^{(0)} \omega_1^{(0)} + \nu_2^{(0)} \omega_2^{(0)} + v^{(0)}(x; -\nu_1^{(0)}, 2 - \nu_2^{(0)})) + \frac{1 + x}{3}, \quad 1 < \nu_2^{(0)} < 2 \quad (112)\)

They must coincide with \( y(x; \sigma, a) \) with parameters identified by the formulae which follow \((104)\) and \((105)\) respectively. After these identifications, we compute monodromy data corresponding to \( \sigma^{(0)}, a^{(0)}, \).
and therefore we compute $\sigma^{(1)}$, $a^{(1)}$ and $\sigma^{(\infty)}$, $a^{(\infty)}$ from the monodromy data. Finally, we write the elliptic representation of (106), (107) with parameters (108) and (109) respectively. These are the elliptic representations of (110) (or (111), (112)) at $x = 1$ and $x = \infty$. This answers Question 2.

**Answer to Question 1.** As for Question 1, we recall that $y(x; \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0M_x), \text{tr}(M_0M_1)\text{tr}(M_1M_x))$ has representations $y(x; \pm \sigma + 2n, a(\pm \sigma + 2n)), n \in \mathbb{Z}$, on different domains. Let us consider the family of transcendent in elliptic representation (at $x = 0$)

$$y(x) = \varphi(\nu_1(N)\omega_1 + [\nu_2 + 2N] \omega_2 + v(x; \nu_1(N), \nu_2 + 2N)) + \frac{1 + x}{3}$$

$$\equiv \varphi(\nu_1(N)\omega_1 + \nu_2\omega_2 + v(x; \nu_1(N), \nu_2 + 2N)) + \frac{1 + x}{3}$$

where $\nu_1 = \nu_1(N)$ means that $\nu_1$ changes with $N$. They are the elliptic representations of the same transcendent in different domains if and only if they coincide with

$$y(x; \sigma - 2N, a(\sigma - 2N)) \quad \text{if } 0 \leq \Re \nu_2 \leq 1,$$

or

$$y(x; -\sigma - 2N, a(-\sigma - 2N)) \quad \text{if } 1 \leq \Re \nu_2 \leq 2,$$

where $y(x; \sigma - 2N, a(\sigma - 2N)), y(x; -\sigma - 2N, a(-\sigma - 2N))$, correspond to the same monodromy data (i.e. they are representations of $y(x; \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0M_x), \text{tr}(M_0M_1)\text{tr}(M_1M_x))$). We can explicitly compute (through the formulae of the Appendix) $\nu_1(N) = \text{function}(N, \nu_2, \nu_1(N = 0))$ from

$$e^{i\pi \nu_1(N)} = -4a(1 - \nu_2 - 2N)16^{\nu_2 + 2N - 1}$$

as we did in (102) and (103).

**Remark:** We have found the elliptic representations and the critical behaviors of

$$y(x; \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0M_x), \text{tr}(M_0M_1)\text{tr}(M_1M_x))$$

in the union of the domains $D(r_N; \nu_1(N), \nu_2 + 2N) (3\nu_2 \neq 0)$. However, some regions in the $(\ln |x|)$, $3\nu_2 \arg x)$-plane, for $x$ close to 0, are not included in the union. In these regions there may be movable poles (see figure 3).

In the generic case, we prove in the Appendix the one-to-one correspondence between transcendent and monodromy data. However, the condition $\sigma^{(i)} \neq 1$, namely $\nu_2^{(i)} = 0$, 2, implies that we can not give the critical behaviors (and the elliptic representation) of

$$y(x; \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0M_x), \text{tr}(M_0M_1)\text{tr}(M_1M_x))$$

at $x = 0$ for $\text{tr}(M_0M_x) = -2$, at $x = 1$ for $\text{tr}(M_1M_x) = -2$, at $x = \infty$ for $\text{tr}(M_0M_1) = -2$. These cases have still to be studied.

In the generic case we have also assumed $\sigma \not\in \mathbb{Z}$; nevertheless, in the case $\sigma^{(i)} = 0$ (tr$(M_iM_j) = 2$) the critical behavior and the solution of the connection problem can be found in the paper [15] by Jimbo. We note that the corresponding $\nu_2^{(i)}$ should be equal to 1, but unfortunately the condition $\nu_2^{(i)} \neq 1$ which we had to impose to study the elliptic representation (except for special cases like $\beta = \gamma = 1 - 2\delta = 0$) did not allow us to know the analytic properties and the critical behavior of the elliptic representation in this case. We expect that the properties of $u(x)$ are such to exactly produce the critical behavior found by Jimbo for $\sigma^{(i)} = 0$, but we still have to cover this case.

## 9 Appendix

We give a brief account of the solution of the connection problem for the generic PVI, following [15], with the extension of the values of monodromy data for which the results apply. In [13] the case $|\text{tr}(M_iM_j)| > 2$ is not considered, so we have to do it now. This extension is necessary to identify the transcendent with the elliptic representation. Such a generalization is proved exactly as in [13], to which
we refer for a detailed analysis of the non-generic case \( \beta = \gamma = 1 - 2\delta = 0 \) (already reviewed in section 8.2).

The Painlevé VI equation is the isomonodromic deformation equation of the fuchsian system (87). The Fuchsian system is obtained from the following Riemann-Hilbert problem. We fix

\[
\theta_0, \theta_x, \theta_1, \theta_\infty \notin \mathbb{Z}
\]

and monodromy matrices \( M_0, M_x, M_1 \) corresponding to the loops in the basis of figure 2 with eigenvalues \( \exp\{\pm i\pi\theta\_i\}, i = 0, x, 1 \) respectively. At infinity, the monodromy is \( M_1M_xM_0 \), and we require that this has eigenvalues \( \exp\{\pm i\pi\theta_\infty\} \). Having assigned the monodromy matrices and their eigenvalues, there exist \( 2 \times 2 \) invertible matrices \( C_0, C_x, C_1, C_\infty \) such that

\[
M_i = C_i^{-1} \exp \left\{ 2\pi i \left( \begin{array}{cc} \frac{\theta_i}{2} & 0 \\ 0 & -\frac{\theta_i}{2} \end{array} \right) \right\} C_i, \quad i = 0, x, 1 \quad (113)
\]

\[
M_1M_xM_0 = C_\infty^{-1} \exp \left\{ 2\pi i \left( \begin{array}{cc} -\frac{\theta_\infty}{2} & 0 \\ 0 & \frac{\theta_\infty}{2} \end{array} \right) \right\} C_\infty
\]

In order to construct the fuchsian system, we have to find a \( 2 \times 2 \) invertible matrix \( Y(z; x) \) holomorphic in \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \) with the assigned monodromy at \( z = 0, 1, x, \infty \). We also fix the indices of the Riemann-Hilbert problem, namely we require that \( Y(z; x) \) be normalized as follows:

\[
Y(z; x) = \begin{cases} 
(I + O((\frac{1}{z})) \left( \begin{array}{cc} \frac{\theta_0}{2} & 0 \\ 0 & \frac{\theta_\infty}{2} \end{array} \right) C_\infty, \quad z \to \infty \\
G_i(I + O(z - i)) (z - i) \left( \begin{array}{cc} \frac{\theta_i}{2} & 0 \\ 0 & -\frac{\theta_i}{2} \end{array} \right) C_i, \quad z \to i = 0, x, 1 
\end{cases} \quad (114)
\]

Here \( G_i \) are invertible matrices. The fuchsian system is obtained from

\[
A(z; x) := \frac{dY(z; x)}{dz} Y(z; x)^{-1} \Rightarrow A_i = G_i \left( \begin{array}{cc} \frac{\theta_i}{2} & 0 \\ 0 & -\frac{\theta_i}{2} \end{array} \right) G_i^{-1} \quad (115)
\]

The fuchsian system satisfies (88). We remark that we have to assume \( \theta_0, \theta_x, \theta_1, \theta_\infty \notin \mathbb{Z} \), otherwise the general form of \( Y(z; x) \) would be different from (114).

A \( 2 \times 2 \) Riemann-Hilbert always has solution (1). In our case, the solution is unique, up to diagonal conjugation \( A_i \to DA_iD^{-1} \), where \( D \) is any diagonal matrix. To prove this fact, let \( C_\nu, \bar{C}_\nu, \nu = 0, x, 1, \infty \), be such that

\[
C_\nu^{-1} \exp \left\{ 2\pi i \left( \begin{array}{cc} \frac{\theta_\nu}{2} & 0 \\ 0 & -\frac{\theta_\nu}{2} \end{array} \right) \right\} C_\nu = \bar{C}_\nu^{-1} \exp \left\{ 2\pi i \left( \begin{array}{cc} \frac{\theta_\nu}{2} & 0 \\ 0 & -\frac{\theta_\nu}{2} \end{array} \right) \right\} \bar{C}_\nu
\]

If follows that \( C_\nu \bar{C}_\nu^{-1} \) is any diagonal matrix. We denote \( Y \) and \( \bar{Y} \) the solutions (114) with \( C_\nu \) and \( \bar{C}_\nu \) respectively. Since \( C_\nu \bar{C}_\nu^{-1} \) is diagonal we conclude that \( Y \bar{Y}^{-1} \) is holomorphic at \( z = 0, 1, x, \infty \). Therefore, being holomorphic on \( \mathbb{P}^1 \), it is a constant diagonal matrix \( D := C_\infty \bar{C}_\infty^{-1} \). This proves uniqueness up to diagonal conjugation. We remark that if some \( \theta_\nu \) \( (\nu = 0, x, 1, \infty) \) is integer, the uniqueness may fail. The cases when this happens if \( \beta = \gamma = 1 - 2\delta = 0 \) are discussed in (2).

Once the Riemann-Hilbert problem is solved, we compute \( y(x) \) from \( A(z; x)_{12} = 0 \). It is clear that \( y(x) \) depends on the monodromy data \( M_0, M_x, M_1, \theta_\nu \) \( (\nu = 0, 1, x, \infty) \). Note that for any invertible matrix \( C \) we obtain the fuchsian system (113) from any \( Y(z; w)C \). The solution \( Y(z; w)C \) corresponds to the monodromy matrices \( C^{-1}M_iC \). This implies that \( y(x) \) depends on the invariants of \( M_i \) \( (i = 0, x, 1) \) with respect to the conjugation \( M_i \to C^{-1}M_iC \). They are traces and determinants of the products on the \( M_i \)'s. The determinants are 1, the traces of \( M_i \) are specified by the eigenvalues and the traces of the products \( M_iM_{i_2}...M_{i_n} \) for \( n > 2 \) are functions of the traces of \( M_iM_{i_2} \). Hence:

\[
y(x) = y(x; \theta_0, \theta_x, \theta_1, \theta_\infty, \text{tr}(M_0M_x), \text{tr}(M_0M_1), \text{tr}(M_1M_x))
\]
PVI is equivalent to the Schlesinger equations which ensure the isomonodromicity of the fuchsian system (116):

\[
\frac{dA_0}{dx} = \frac{[A_1A_0]}{x} \\
\frac{dA_1}{dx} = \frac{[A_1A_0]}{1-x} \\
\frac{dA_2}{dx} = \frac{[A_1A_0]}{x} + \frac{[A_1A_0]}{1-x}
\]  

(116)

The system (116) is a particular case of

\[
\frac{dA_\nu}{dx} = \sum_{\nu=1}^{n_2} [A_\mu, B_\nu] f_{\mu\nu}(x) \\
\frac{dB_\nu}{dx} = -\frac{1}{x} \sum_{\nu'1}^{n_2} [B_\nu, B_{\nu'}] + \sum_{\mu=1}^{n_1} [B_\nu, A_\mu] g_{\mu\nu}(x) + \sum_{\nu'1}^{n_2} [B_{\nu'}, B_{\nu'}] h_{\mu\nu'}(x)
\]  

(117)

where the functions \( f_{\mu\nu}, g_{\mu\nu}, h_{\mu\nu} \) are meromorphic with poles at \( x = 1, \infty \) and

\[
\sum_{\nu} B_\nu + \sum_{\mu} A_\mu = -A_\infty, \quad A_\infty := \begin{pmatrix} \frac{\vartheta_2}{2} & 0 \\ 0 & -\frac{\vartheta_2}{2} \end{pmatrix}
\]

System (116) is obtained for \( f_{\mu\nu} = g_{\mu\nu} = b_\nu/(a_\mu - x b_\nu), h_{\mu\nu} = 0, n_1 = 1, n_2 = 2, a_1 = b_2 = 1, b_1 = 0 \) and \( B_1 = A_0, B_2 = A_x, A_1 = A_1 \).

Let \( \vartheta \) be a complex number such that

\[
\vartheta \notin (-\infty, 0) \cup [1, +\infty)
\]

and let us consider the domain (16) with additional parameters \( \vartheta_1, \vartheta_2 \in \mathbb{R}, 0 < \vartheta < 1 \). In (16) the reader can find the proof of the following lemma, which is a generalization in \( D(\epsilon, \vartheta) \) of (1), page 262

**Lemma A1:** Consider matrices \( B_\nu^0 \) (\( \nu = 1, \ldots, n_2 \)), \( A_\mu^0 \) (\( \mu = 1, \ldots, n_1 \)) and \( \Lambda \), independent of \( x \) and such that

\[
\sum_{\nu} B_\nu^0 + \sum_{\mu} A_\mu^0 = -A_\infty \\
\sum_{\nu} B_\nu^0 = \Lambda, \quad \text{eigenvalues}(\Lambda) = \frac{\sigma}{2}, -\frac{\sigma}{2}, \quad \sigma \notin (-\infty, 0) \cup [1, +\infty).
\]

Suppose that \( f_{\mu\nu}, g_{\mu\nu}, h_{\mu\nu} \) are holomorphic if \( |x| < \epsilon' \), for some small \( \epsilon' < 1 \).

For any \( 0 < \vartheta < 1 \) and \( \vartheta_1, \vartheta_2 \) real there exists a sufficiently small \( 0 < \epsilon < \epsilon' \) such that the system (116) has holomorphic solutions \( A_\mu(x), B_\nu(x) \) in \( D(\epsilon, \vartheta) \) satisfying:

\[
||A_\mu(x) - A_\mu^0|| \leq C |x|^{1-\sigma_1} \\
||x^{-\Lambda} B_\nu(x) x^\Lambda - B_\nu^0|| \leq C |x|^{1-\sigma_2}
\]

Here \( C \) is a positive constant and \( \vartheta < \sigma_1 < 1 \)

**Lemma A2:** Let \( \theta_\infty \neq 0 \). Let \( r, s \) be two complex numbers not equal to zero. Let \( \hat{G}_0 \) be such that

\[
\hat{G}_0^{-1} \Lambda \hat{G}_0 = \text{diag} \left( \begin{pmatrix} \frac{\sigma}{2}, -\frac{\sigma}{2} \end{pmatrix} \right), \quad \sigma \neq 0
\]

The general solution of

\[
A_0^0 + A_\nu^0 + A_\nu^0 = \text{diag} \left( -\frac{\theta_\infty}{2}, \frac{\theta_\infty}{2} \right), \quad A_0^0 + A_\nu^0 = \Lambda, \quad \text{eigenvalues } A_i = \pm \frac{\theta_i}{2} (i = 0, 1, x)
\]

is

\[
A_0^0 = \hat{G}_0 \left( \begin{array}{cc}
\frac{\alpha(\beta+1-\gamma)}{\alpha-\beta} & s \\
\frac{\alpha\beta(\beta+1-\gamma)(\gamma-1-\alpha)}{(\alpha-\beta)^2} & \frac{\beta(\gamma-1-\alpha)}{\alpha-\beta}
\end{array} \right) \hat{G}_0^{-1} + \frac{\theta_0}{2},
\]

\[
A_\nu^0 = \hat{G}_0 \left( \begin{array}{cc}
\frac{\alpha(\gamma-1-\alpha)}{\alpha-\beta} & -s \\
-(\alpha^0_\nu)_{21} & \frac{\beta(\beta+1-\gamma)}{\alpha-\beta}
\end{array} \right) \hat{G}_0^{-1} + \frac{\theta_0}{2}
\]
As a corollary of Lemma A1, the limits

and

Theorem A1: 

Setting \( A(z; x)_{12} = 0 \) we obtain from Lemma A1 and Lemma A2:

\[
y(x) = -x \left( A_0 \right)_{12} (1 + O(|x|^{1-\delta}))
\]

where \( \Delta \) is a small positive number and \( x \to 0 \) in \( D(\epsilon; \sigma) \). Namely, we have the following theorem (for a detailed proof see [11]):

**Theorem A1:** Let \( \theta_\infty \neq 0 \). For any complex \( \sigma \notin (-\infty, 0] \cup [1, +\infty) \), for any complex \( a \neq 0 \), for any \( \vartheta_1, \vartheta_2 \in \mathbb{R} \) and for any \( 0 < \bar{\sigma} < 1 \), there exists a sufficiently small positive \( \epsilon \) and a transcendent \( y(x; \sigma, a) \) with behavior

\[
y(x; \sigma, a) = ax^{1-\sigma} (1 + O(|x|^\Delta)), \quad 0 < \Delta < 1,
\]

as \( x \to 0 \) along a regular path in \( D(\epsilon; \sigma) \).

The above local behavior is valid along any regular path, with the exception, which occurs if \( \exists \sigma \neq 0, \text{of the paths } \exists \sigma \arg(x) = \Re \sigma \log|x| + b \), where \( b \) is a constant such that the path is contained in \( D(\epsilon; \sigma) \): in this case the behavior is

\[
y(x; \sigma, a) = \sin^2 \left( \frac{i\sigma}{2} \ln x - \frac{i}{2} \ln(4a) - \frac{\pi}{2} \right) x (1 + O(|x|^\Delta))
\]

Note that we have excluded \( \sigma = 0 \) from Lemma A2 and therefore from Theorem A1. Such a case is however computed in [13] and so Theorem A1 holds also for \( \sigma = 0 \). On the other hand, in [13] only \( 0 \leq \Re \sigma < 1 \) is considered, thus Theorem A1 is a generalization of the result of [13].

We assume in the following that \( \vartheta_\nu \notin \mathbb{Z}, \nu = 0, x, 1, \infty \). We choose a solution of the fuchsian system

\[
\frac{dY}{dz} = \left[ A_0(x) \frac{A_x(x)}{z} + A_x(x) \frac{A_1(x)}{z-1} \right] Y
\]

normalized as follows

\[
Y(z, x) = \left( I + O \left( \frac{1}{z} \right) \right) z^{-\Lambda_\infty}, \quad z \to \infty, \quad A_\infty = \text{diag} \left( \frac{\theta_\infty}{2}, -\frac{\theta_\infty}{2} \right).
\]

As a corollary of Lemma A1, the limits

\[
\hat{Y}(z) := \lim_{x \to 0} Y(z, x), \quad \hat{\hat{Y}}(z) := \lim_{x \to 0} x^{-\Lambda} Y(x, z)
\]

exist when \( x \to 0 \) in \( D(\epsilon; \sigma) \). They satisfy

\[
\frac{d\hat{Y}}{dz} = \left[ \frac{A_0}{z-1} + \frac{A}{z} \right] \hat{Y}
\]

(123)
\[
\frac{d\tilde{Y}}{dz} = \left[ \frac{A_0}{z} + \frac{A_0}{z-1} \right] \tilde{Y}
\]

(124)

For the system (123), we choose a fundamental matrix solution normalized as follows

\[
\tilde{Y}_N(z) = \left( I + O\left(\frac{1}{z}\right) \right) z^{-A_0}, \quad z \to \infty
\]

(125)

\[
= (I + O(z)) z^A \tilde{C}_0, \quad z \to 0
\]

\[
= \hat{G}_1(I + O(z - 1)) (z - 1)^{-\theta} \tilde{C}_1, \quad z \to 1
\]

Where \( \hat{G}_1^{-1} A_0^0 \hat{G}_1 = \text{diag} \left( \frac{\theta}{2},-\frac{\theta}{2} \right) \). \( \hat{C}_0, \hat{C}_1 \) are connection matrices. For (124) we choose a fundamental matrix solution normalized as follows

\[
\tilde{Y}_N(z) = \left( I + O\left(\frac{1}{z}\right) \right) z^A, \quad z \to \infty
\]

(126)

\[
= \hat{G}_0(I + O(z)) z^{\theta} \tilde{C}_0, \quad z \to 0
\]

\[
= \hat{G}_1(I + O(z - 1)) (z - 1)^{-\theta} \tilde{C}_1, \quad z \to 1.
\]

Here \( \hat{G}_i^{-1} A_0^0 \hat{G}_i = \text{diag} \left( \frac{\theta}{2},-\frac{\theta}{2} \right) \) and \( \hat{C}_0, \hat{C}_1 \) are connection matrices. As it is proved in [15] and in [11], we have:

\[
\hat{Y}(z) = \tilde{Y}_N(z)
\]

\[
\hat{Y}(z) = \tilde{Y}_N(z) \hat{C}_0
\]

(127)

and, as a consequence of isomonodromicity, we have

\[
M_1 = \hat{C}_1^{-1} e^{2\pi i} \text{diag} \left( \frac{\theta}{2},-\frac{\theta}{2} \right) \hat{C}_1
\]

(128)

\[
M_0 = \hat{C}_0^{-1} e^{2\pi i} \text{diag} \left( \frac{\theta}{2},-\frac{\theta}{2} \right) \hat{C}_0 \hat{C}_0
\]

(129)

\[
M_x = \hat{C}_0^{-1} \hat{C}_1^{-1} e^{2\pi i} \text{diag} \left( \frac{\theta}{2},-\frac{\theta}{2} \right) \hat{C}_1 \hat{C}_0
\]

(130)

The connection matrices \( \hat{C}_0, \hat{C}_0, \hat{C}_1 \) can be computed explicitly because the 2 x 2 fuchsian systems (123) (124) can be reduced to the hyper-geometric equation.

**Lemma A3:** The Gauss hyper-geometric equation

\[
z(1 - z) \frac{d^2 y}{dz^2} + \left[ \gamma_0 - z(\alpha_0 + \beta_0 + 1) \right] \frac{dy}{dz} - \alpha_0 \beta_0 y = 0
\]

(131)

is equivalent to the system

\[
\frac{d\Psi}{dz} = \left[ \frac{1}{z} \begin{pmatrix} 0 & 0 \\
-\alpha_0 \beta_0 & -\gamma_0 \end{pmatrix} + \frac{1}{z - 1} \begin{pmatrix} 0 & 1 \\
0 & \gamma_0 - \alpha_0 - \beta_0 \end{pmatrix} \right] \Psi
\]

(132)

where \( \Psi = \begin{pmatrix} y \\
(z - 1) \frac{dy}{dz} \end{pmatrix} \).

**Lemma A4:** Let \( B_0 \) and \( B_1 \) be matrices of eigenvalues 0, 1 - \( \gamma \), and 0, \( \gamma - \alpha - \beta - 1 \) respectively, such that

\[
B_0 + B_1 = \text{diag}(-\alpha, -\beta), \quad \alpha \neq \beta
\]

Then

\[
B_0 = \begin{pmatrix}
\frac{\alpha(\beta+1-\gamma)}{\alpha-\beta} & \frac{\alpha(\beta+1-\gamma)}{\alpha-\beta} \\
\frac{\alpha(\beta+1-\gamma)(\gamma-1-\alpha)}{\alpha-\beta} & -\alpha (1 - \gamma)
\end{pmatrix}
\]

\[
B_1 = \begin{pmatrix}
\frac{\alpha(\gamma-\alpha-1)}{\alpha-\beta} & -\frac{\alpha(\gamma-\alpha-1)}{\alpha-\beta} \\
-(B_0)_{12} & \frac{\beta(\beta+1-\gamma)}{\alpha-\beta}
\end{pmatrix}
\]

52
for any $r_1 \neq 0$.

We leave the proof as an exercise (note that $\alpha, \beta, \gamma$ are not the coefficients of PVI. We apologize for using the same symbols). The following lemma connects Lemmas A3 and A4:

**Lemma A5:** The system (132) with
\[ \alpha_0 = \alpha, \quad \beta_0 = \beta + 1, \quad \gamma_0 = \gamma, \quad \alpha \neq \beta \]
is gauge-equivalent to the system
\[
\frac{dX}{dz} = \left[ \frac{B_0}{z} + \frac{B_1}{z-1} \right] X \tag{133}
\]
where $B_0, B_1$ are given in Lemma A4. This means that there exists a matrix
\[
G(z) = \left( \begin{array}{cc} \frac{1}{\beta-\alpha} & 1 \vspace{1mm} \\
\frac{1}{\beta-\alpha} & -\frac{1}{r_1} \end{array} \right)
\]
such that $X(z) = G(z) \Psi(z)$. It follows that (133) and the corresponding hyper-geometric equation (131) have the same fuchsian singularities $0, 1, \infty$ and the same monodromy group.

**Proof:** By direct computation. □

Note that the form of $G(z)$ ensures that if $y_1, y_2$ are independent solutions of the hyper-geometric equation, then a fundamental matrix of (133) may be chosen to be $X(z) = \left( \begin{array}{cc} y_1(z) & y_2(z) \\ * & * \end{array} \right)$.

Now we compute the monodromy matrices for the systems (123), (124) by reduction to an hyper-geometric equation. We assume $\sigma \notin \mathbb{Z}$

Let us start with (123). With the gauge $Y(1)(z) := z^{-\frac{\sigma}{\tau}}(z - 1)^{-\frac{\theta_1}{\tau}} Y_N(z)$ we transform (123) into
\[
\frac{dY^{(1)}}{dz} = \left[ A_0^0 - \frac{\theta_1}{\tau} z - 1 + \Lambda - \frac{\sigma}{\tau} I \right] Y^{(1)} \tag{134}
\]
We identify the matrices $B_0, B_1$ of Lemma A4 with $\Lambda - \frac{\sigma}{\tau}$ and $A_0^0 - \frac{\theta_1}{\tau}$. Therefore
\[
\alpha = \frac{\theta_\infty + \theta_1 + \sigma}{2}, \quad \beta = \frac{-\theta_\infty + \theta_1 + \sigma}{2}, \quad \gamma = 1 + \sigma
\]
(note that $\alpha - \beta = \theta_\infty \neq 0$). Also note that Lemma A2 for $\Lambda$ and $A_0^0$ follows from Lemma A4 applied to (134). The hyper-geometric equation connected to the present system through Lemma A3 has coefficients
\[
\begin{cases} 
\alpha_0 = \frac{\theta_\infty + \theta_1 + \sigma}{2} \\
\beta_0 = 1 + \frac{-\theta_\infty + \theta_1 + \sigma}{2} \\
\gamma_0 = \sigma + 1
\end{cases}
\]
Therefore, according to the standard theory of hyper-geometric equations, the generic case occurs when $\sigma, \theta_1, \theta_\infty \notin \mathbb{Z}$

Let $F(a, b, c; z)$ be the hyper-geometric function. For $\sigma \notin \mathbb{Z}$ we have two independent solutions of the hyper-geometric equation
\[
y_1^{(0)}(x) = F(\alpha_0, \beta_0, \gamma_0; z) \\
y_2^{(0)}(x) = z^{1-\gamma_0} F(\alpha_0 - \gamma_0 + 1, \beta_0 - \gamma_0 + 1, 2 - \gamma_0; z)
\]
For $\theta_1 \notin \mathbb{Z}$ we have also solutions
\[
y_1^{(1)}(z) = F(\alpha_0, \beta_0, \alpha_0 + \beta_0 + 1 - \gamma_0; 1 - z) \\
y_2^{(1)}(z) = (1 - z)^{\gamma_0 - \alpha_0 - \beta_0} F(\gamma_0 - \beta_0, \gamma_0 - \alpha_0, 1 + \gamma_0 - \alpha_0 - \beta_0; 1 - z)
\]
They are connected by
\[
[y_1^{(0)}, y_2^{(0)}] = [y_1^{(1)}, y_2^{(1)}] C_{01}
\]
\[
C_{01} := \begin{pmatrix}
\frac{\Gamma(\gamma_0 - \alpha_0 - \beta_0) \Gamma(\gamma_0)}{\Gamma(\gamma_0 - \alpha_0) \Gamma(\gamma_0 - \beta_0)} & \frac{\Gamma(\gamma_0 - \alpha_0 - \beta_0) \Gamma(2 - \gamma_0)}{\Gamma(1 - \alpha_0) \Gamma(1 - \beta_0)} \\
\frac{\Gamma(\alpha_0 + \beta_0 - \gamma_0) \Gamma(\gamma_0)}{\Gamma(\alpha_0 + \beta_0 - \gamma_0) \Gamma(\gamma_0)} & \frac{\Gamma(\alpha_0 + \beta_0 - \gamma_0) \Gamma(2 - \gamma_0)}{\Gamma(\alpha_0 + 1 - \gamma_0) \Gamma(\beta_0 + 1 - \gamma_0)}
\end{pmatrix}
\]

Note that
\[
Y^{(1)}(z) = \hat{G}_0 [I + O(z)] z^{\gamma_0 - 1} \hat{C}_0, \quad z \to 0
\]
\[
= \hat{G}_1 [I + O(z - 1)] (z - 1)^{\gamma_0 - 1} \hat{C}_1, \quad z \to 1
\]

From the behaviors of the hyper-geometric functions $y_i^{(0)}$ for $z \to 0$ and $y_i^{(1)}$ for $z \to 1$ and for a suitable choice of $\hat{G}_1$ we obtain
\[
Y^{(1)} = \begin{pmatrix} y_1^{(0)} & y_2^{(0)} \\ 0 & 0 \end{pmatrix} \hat{C}_0^{-1} \hat{C}_0 = \begin{pmatrix} y_1^{(1)} & y_2^{(1)} \\ 0 & 0 \end{pmatrix} \hat{C}_1
\]

Namely
\[
\hat{C}_1 (\hat{C}_0^{-1} \hat{C}_0)^{-1} = C_{01}
\]
This will be enough for our purposes (note that we did not use the hypothesis $\theta_\infty \notin \mathbb{Z}$).

We turn to the system (124). With the gauge $Y^{(2)}(z) := z^{-\sigma} (z - 1)^{-\theta} \hat{C}_0^{-1} \hat{Y}_N(z) \hat{C}_0$ we have
\[
\frac{dY^{(2)}}{dz} = \begin{bmatrix} B_0 & B_1 \\ z & z - 1 \end{bmatrix} Y^{(2)}
\]

where
\[
B_0 = \hat{C}_0^{-1} A_0^0 \hat{C}_0 - \frac{\theta_0}{\sigma}, \quad B_1 = \hat{C}_0^{-1} A_0^0 \hat{C}_0 - \frac{\theta_x}{\sigma}
\]

$B_0$ and $B_1$ are as in Lemma A4, with
\[
\alpha = -\sigma + \theta_0 + \theta_x, \quad \beta = \frac{\sigma + \theta_0 + \theta_x}{2}, \quad \gamma = 1 + \theta_0
\]

(Lemma A2 for $A_0^0$ and $A_0^0$ follows from Lemma A4 applied to the present case). The corresponding hyper-geometric equation has coefficients
\[
\begin{cases}
\alpha_0 = \frac{\sigma + \theta_0 + \theta_x}{2} \\
\beta_0 = 1 + \frac{\sigma + \theta_0 + \theta_x}{2} \\
\gamma_0 = 1 + \theta_0
\end{cases}
\]

Therefore, according to the standard theory of hyper-geometric equations, the generic case occurs for
\[
\theta_0, \theta_x, \sigma \notin \mathbb{Z}
\]

and we can choose three couples of independent solutions
\[
\begin{cases}
y_1^{(0)} = F(\alpha_0, \beta_0, \gamma_0; z) \\
y_2^{(0)} = z^{1 - \gamma_0} F(1 + \alpha_0 - \gamma_0, 1 + \beta_0 - \gamma_0, 2 - \gamma_0; z)
\end{cases}
\]
\[
\begin{cases}
y_1^{(1)} = F(\alpha_0, \beta_0, \alpha_0 + \beta_0 + 1 - \gamma_0; 1 - z) \\
y_2^{(1)} = (1 - z)^{\gamma_0 - \alpha_0 - \beta_0} F(\gamma_0 - \alpha_0 - \gamma_0, \gamma_0 - \beta_0, 1 - \alpha_0 - \beta_0; 1 - z)
\end{cases}
\]
\[
\begin{cases}
y_1^{(\infty)} = z^{1 - \alpha_0} F(\alpha_0, \alpha_0 - \gamma_0 + 1, \alpha_0 + 1 - \beta_0; \frac{1}{z}) \\
y_2^{(\infty)} = z^{1 - \beta_0} F(\beta_0, \beta_0 + 1 - \gamma_0, \beta_0 + 1 - \alpha_0; \frac{1}{z})
\end{cases}
\]

They are connected by
\[
[y_1^{(\infty)}, y_2^{(\infty)}] = [y_1^{(0)}, y_2^{(0)}] C_{\infty 0}
\]
\[
C'_{\infty 0} = \begin{pmatrix}
\frac{e^{-\pi i\alpha_0} \Gamma(1+\alpha_0-\beta_0)\Gamma(1-\gamma_0)}{\Gamma(1-\alpha_0)\Gamma(1+\alpha_0-\gamma_0)} & \frac{e^{-\pi i\beta_0} \Gamma(1+\beta_0-\alpha_0)\Gamma(1-\gamma_0)}{\Gamma(1-\beta_0)\Gamma(1+\beta_0-\gamma_0)} \\
e^{\pi i(\gamma_0-1-\alpha_0)} \frac{\Gamma(1+\alpha_0-\beta_0)\Gamma(\gamma_0-1)}{\Gamma(\alpha_0)\Gamma(\gamma_0-\beta_0)} & e^{\pi i(\gamma_0-1-\beta_0)} \frac{\Gamma(1+\beta_0-\alpha_0)\Gamma(\gamma_0-1)}{\Gamma(\beta_0)\Gamma(\gamma_0-\alpha_0)}
\end{pmatrix}
\]

From the above results, from the formulae (128), (129), (130), and from the cyclic properties of the trace we obtain

\[
Y^{(2)} = \begin{pmatrix}
y^{(\infty)}_1 & y^{(\infty)}_2 \\
\ast & \ast
\end{pmatrix} = \begin{pmatrix}
y^{(0)}_1 & y^{(0)}_2 \\
\ast & \ast
\end{pmatrix} \tilde{C}_0 \tilde{G}_0 = \begin{pmatrix}
y^{(1)}_1 & y^{(1)}_2 \\
\ast & \ast
\end{pmatrix} \tilde{C}_1 \tilde{G}_0
\]

Therefore

\[
\tilde{C}_0 \tilde{G}_0 \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+\sigma}{s} \end{pmatrix} = C'_{\infty 0}, \quad \tilde{C}_1 \tilde{G}_0 \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+\sigma}{s} \end{pmatrix} = C'_{\infty 1}
\]

We introduce the following matrices:

\[
m_0 := \begin{pmatrix} C'_{\infty 0} \begin{pmatrix} 1 & 0 \\ 0 & \frac{s}{1+\sigma} \end{pmatrix} \end{pmatrix}^{-1} e^{2\pi i \text{ diag}(\frac{s \theta_0}{2}, -\frac{s \theta_0}{2})} \begin{pmatrix} C'_{\infty 0} \begin{pmatrix} 1 & 0 \\ 0 & \frac{s}{1+\sigma} \end{pmatrix} \end{pmatrix} (135)
\]

\[
m_x := \begin{pmatrix} C'_{\infty 1} \begin{pmatrix} 1 & 0 \\ 0 & \frac{s}{1+\sigma} \end{pmatrix} \end{pmatrix}^{-1} e^{2\pi i \text{ diag}(\frac{s \theta_x}{2}, -\frac{s \theta_x}{2})} \begin{pmatrix} C'_{\infty 1} \begin{pmatrix} 1 & 0 \\ 0 & \frac{s}{1+\sigma} \end{pmatrix} \end{pmatrix} (136)
\]

\[
m_1 := C'_{\infty 1}^{-1} e^{2\pi i \text{ diag}(\frac{s \theta_1}{2}, -\frac{s \theta_1}{2})} C_{01} (137)
\]

From the above results, from the formulae (128), (129), (130), and from the cyclic properties of the trace we obtain

\[
\text{tr}(M_0 M_x) \equiv \text{tr}(m_0 m_x), \quad \text{tr}(M_1 M_x) \equiv \text{tr}(m_1 m_x), \quad \text{tr}(M_0 M_1) \equiv \text{tr}(m_0 m_1)
\]

We note that in order for the \( \Gamma \)-functions to be non-singular we have to require \((\pm \sigma \pm \theta_0 \pm \theta_x) / 2 \not\in \mathbb{Z}\). Therefore, to summarize we define the generic case to be

\[
\sigma, \theta_0, \theta_x, \theta_1, \theta_{\infty} \not\in \mathbb{Z}; \quad \frac{\pm \sigma \pm \theta_0 \pm \theta_x}{2}, \quad \frac{\pm \sigma \pm \theta_0 \pm \theta_x}{2} \not\in \mathbb{Z} \quad (138)
\]

Computing the traces, we find

\[
\text{tr}(M_0 M_x) = 2 \cos(\pi \sigma) \quad (139)
\]

\[
\text{tr}(M_1 M_0) = F_1(\sigma, \theta_0, \theta_x, \theta_1, \theta_{\infty}) \frac{1}{s} + F_2(\sigma, \theta_0, \theta_x, \theta_1, \theta_{\infty}) + F_3(\sigma, \theta_0, \theta_x, \theta_1, \theta_{\infty}) s \quad (140)
\]

\[
\text{tr}(M_x M_1) = -e^{-i\pi \sigma} F_1(\sigma, \theta_0, \theta_x, \theta_1, \theta_{\infty}) \frac{1}{s} + F_4(\sigma, \theta_0, \theta_x, \theta_1, \theta_{\infty}) - e^{i\pi \sigma} F_3(\sigma, \theta_0, \theta_x, \theta_1, \theta_{\infty}) s \quad (141)
\]

where \( F_i(\sigma, \theta_0, \theta_x, \theta_1, \theta_{\infty}) \) are long expressions which can be explicitly computed from (135), (136), (137) in terms of \( \Gamma \) functions (and trigonometric functions). We omit the explicit formulae because they keep some space; in any case, they are easily computed with a computer from (135), (136), (137). The first equation determines \( \sigma \) up to \( \sigma \mapsto \pm \sigma + 2n, n \in \mathbb{Z} \). Once \( \sigma \) is chosen, the last two equations determine

\[
s = \frac{e^{i\pi \sigma} \text{tr}(M_1 M_0) + \text{tr}(M_x M_1) - F_4 - e^{-i\pi \sigma} F_2}{(e^{i\pi \sigma} - e^{-i\pi \sigma}) F_1} \quad (142)
\]
CONNECTION PROBLEM: We explain how to solve the connection problem for the transcendent $y$ have the same asymptotic behavior as

$$A$$

Proof: Observe that both $A$ to the construction above (reduction to the hyper-geometric equation) $D$ on different domains

Proposition A1: Let us consider the generic case (138) and let $A$ be a solution of $A(0), A_1(x), A_2(x)$ be the matrices constructed from $y(x; \sigma, a)$ by means of the formulae of the Appendix C of [17]. It follows that $A_i(x)$ and $A_i^*(x), i = 0, 1, x,$ have the same asymptotic behavior as $x \to 0$. This is the behavior of Lemma A1. Therefore, according to the construction above (reduction to the hyper-geometric equation) $A_0(x), A_1(x), A_x(x)$ and $A_0^*(x), A_1^*(x), A_x^*(x)$ give the same monodromy. The solution of the Riemann-Hilbert problem for such monodromy is unique, up to diagonal conjugation of the fuchsian systems. Therefore $A_i(x)$ and $A_i^*(x), i = 0, 1, x$ are at most diagonally conjugated and $|A(z; x)|_{12} = 0$ and $|A^*(z; x)|_{12} = 0$ give $y(x) \equiv y(x; \sigma, a)$.

Remark: Formula (139) determines $\sigma$ up to $\sigma \to \pm \sigma + 2n, n \in \mathbb{Z}$. We can choose $0 \leq \Re \sigma \leq 1$ and therefore all solutions of (139) are $\pm \sigma + 2n, n \in \mathbb{Z}$. For the given monodromy data, the transcendent (143) has different representations

$$y(x; \sigma, a) = y(x; \theta_0, \theta_x, \theta_1, \theta_{\infty}, \text{tr}(M_0M_x), \text{tr}(M_0M_1), \text{tr}(M_1M_x))$$

(143)

on different domains $D(\epsilon_n; \pm \sigma + 2n)$ of theorem A1.

**Proposition A1:** Let us consider the generic case (138) and let $\sigma \notin (-\infty, 0] \cup [1, \infty)$ and $a \neq 0$. Let $y(x)$ be a solution of PVI such that $y(x) = ax^{1-\sigma}(1 + \text{higher order terms})$ as $x \to 0$ in an open domain contained in $D(\epsilon_n; \sigma)$ of theorem A1. Then, $y(x)$ coincides with $y(x; \sigma, a)$ of Theorem A1.

**Proof:** Observe that both $y(x)$ and $y(x; \sigma, a)$ have the same asymptotic behavior for $x \to 0$ in $D(\sigma)$. Let $A_0(x), A_1(x), A_x(x)$ be the matrices constructed from $y(x)$ and $A_0^*(x), A_1^*(x), A_x^*(x)$ constructed from $y(x; \sigma, a)$ by means of the formulae of the Appendix C of [17]. It follows that $A_i(x)$ and $A_i^*(x), i = 0, 1, x,$ have the same asymptotic behavior as $x \to 0$. This is the behavior of Lemma A1. Therefore, according to the construction above (reduction to the hyper-geometric equation) $A_0(x), A_1(x), A_x(x)$ and $A_0^*(x), A_1^*(x), A_x^*(x)$ give the same monodromy. The solution of the Riemann-Hilbert problem for such monodromy is unique, up to diagonal conjugation of the fuchsian systems. Therefore $A_i(x)$ and $A_i^*(x), i = 0, 1, x$ are at most diagonally conjugated and $|A(z; x)|_{12} = 0$ and $|A^*(z; x)|_{12} = 0$ give $y(x) \equiv y(x; \sigma, a)$.

**Connection Problem:** We explain how to solve the connection problem for the transcendent $y(x; \sigma, a)$.

$x = \infty$: Let

$$t := \frac{1}{x}, \quad y(x) := \frac{1}{t} \tilde{y}(t)$$

Then $y(x)$ solves PVI with parameters $\theta_0, \theta_x, \theta_1, \theta_{\infty}$ if and only if $\tilde{y}(t)$ solves PVI (with independent variable $t$) with parameters

$$(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_{\infty}) := (\theta_0, \theta_1, \theta_{\infty})$$

(144)

Namely, $\theta_1$ and $\theta_x$ are exchanged. From Theorem A1 if follows that there exists a solution $\tilde{y}(t; \sigma^{(\infty)}, a^{(\infty)}) = a^{(\infty)}t^{1-\sigma^{(\infty)}}(1 + O(t^{\Delta}))$ for $t \to 0$. Therefore, PVI in variable $x$ has a solution

$$y(x) = a^{(\infty)}x^{\sigma^{(\infty)}}(1 + O(x^{-\Delta})), \quad x \to \infty$$

in

$$D(M; \sigma^{(\infty)}; \theta_1, \theta_2, \tilde{\sigma}) := \{x \in \mathbb{C} \setminus \infty \text{ s.t. } |x| > M, e^{-\theta_1 \sigma^{(\infty)}}|x|^{-\tilde{\sigma}} \leq |x^{-\sigma^{(\infty)}}| \leq e^{-\theta_2 \sigma^{(\infty)}}0 < \tilde{\sigma} < 1\}$$

where $M > 0$ is sufficiently big and $0 < \Delta < 1$ is small.

$x = 1$: Now let

$$t := 1 - x, \quad y(x) := 1 - \tilde{y}(t)$$

Then $y(x)$ satisfies PVI with parameters $\theta_0, \theta_x, \theta_1, \theta_{\infty}$ if and only if $\tilde{y}(t)$ solves PVI (with independent variable $t$) with parameters

$$(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_{\infty}) := (\theta_1, \theta_x, \theta_0, \theta_{\infty})$$

(145)
Figure 3: The basis of loops for the system in $z$ and for the system in $\tilde{z}$. In parenthesis are the poles of $z$, out of parenthesis the poles of $\tilde{z}$. The left figure corresponds to $t = 1/x$, $\tilde{z} = tz$. The right figure corresponds to $t = 1 - x$, $\tilde{z} = 1 - z$. The dotted loops are for the system in $z$, the others for the system in $\tilde{z}$.

Namely, $\theta_0$ and $\theta_1$ are exchanged. From Theorem A1 if follows that there exists a solution $\tilde{y}(t; \sigma^{(1)}, a^{(1)}) = a^{(1)}t^{1-\sigma^{(1)}}(1 + O(t^\Delta))$ for $t \to 0$. Therefore, PVI in variable $x$ has a solution

$$y(x) = 1 - a^{(1)}(1 - x)^{1-\sigma^{(1)}}(1 + O((1-x)^\Delta))$$

in

$$D(\epsilon; \sigma^{(1)}; \theta_1, \theta_2, \tilde{\sigma}) := \{ x \in \mathbb{C}\backslash \{1\} \text{ s.t. } |1 - x| < \epsilon, \ e^{-\theta_1 \Im \sigma} |1 - x|^\sigma \leq |(1-x)^{\sigma^{(1)}}| \leq e^{-\theta_2 \Im \sigma}, \ 0 < \tilde{\sigma} < 1 \}$$

Let us examine how the monodromy data change.

$[x = \infty]:$ In the fuchsian system (121) we put $x = \frac{1}{t}$ and $\tilde{z} := tz$. We get

$$\frac{dY}{d\tilde{z}} = \begin{bmatrix} A_0 & A_x \\ \tilde{z} & \tilde{z} - 1 \end{bmatrix} Y$$

Therefore $z = 1, x$ have been exchanged and correspond to $\tilde{z} = t, 1$. The monodromy matrices $\tilde{M}_0, \tilde{M}_t, \tilde{M}_1$ corresponding to the loops ordered as in figure 3 with $t$ instead of $x$ are related to $M_0, M_1, M_x$ of (121) in the basis of figure 2.

$$M_0 = \tilde{M}_0, \ M_x = \tilde{M}_t^{-1}\tilde{M}_1\tilde{M}_t, \ M_1 = \tilde{M}_t$$

If we introduce the convenient notation

$$T_0 := \text{tr}(M_0M_x), \ T_1 := \text{tr}(M_1M_x), \ T_\infty := \text{tr}(M_0M_1)$$

we have

$$\begin{cases} T_0 = \text{tr}(\tilde{M}_0\tilde{M}_t^{-1}\tilde{M}_1\tilde{M}_t) \\ T_1 = \tilde{T}_1 \\ T_\infty = \tilde{T}_0 \end{cases}$$

where $\tilde{T}_1 = \text{tr}(\tilde{M}_1\tilde{M}_t)$, etc. Now we observe that

$$\text{tr}(\tilde{M}_0\tilde{M}_t^{-1}\tilde{M}_1\tilde{M}_t) = 4 \left[ \cos(\pi\tilde{\theta}_\infty) \cos(\pi\tilde{\theta}_1) + \cos(\pi\tilde{\theta}_0) \cos(\pi\tilde{\theta}_1) \right] - \text{tr}(\tilde{M}_1\tilde{M}_0) - \text{tr}(\tilde{M}_1\tilde{M}_t)\text{tr}(\tilde{M}_0\tilde{M}_t)$$

This follows from the identity

$$\text{tr}(AB) = \text{tr}(A)\text{tr}(B) - \text{tr}(AB^{-1}), \ A, B \ 2 \times 2 \text{ matrices}, \ \text{det}(B) = 1$$

57
and from
\[
\text{tr}(\hat{M}_1 \hat{M}_\sigma \hat{M}_0) = e^{i\pi \hat{\theta}_x} + e^{-i\pi \hat{\theta}_x}, \quad \text{tr}(\hat{M}_i) = e^{i\pi \hat{\theta}_i} + e^{-i\pi \hat{\theta}_i}, \quad i = 0, x, 1
\]

Therefore, recalling (144), we conclude that
\[
\begin{cases}
\hat{T}_0 = T_\infty \\
\hat{T}_1 = T_1 \\
\hat{T}_\infty = 4 \left[ \cos(\pi \theta_\infty) \cos(\pi \theta_1) + \cos(\pi \theta_x) \cos(\pi \theta_0) \right] - (T_0 + T_1 T_\infty)
\end{cases}
\]

Therefore the parameterization of \(\sigma^{(\infty)}\), \(a^{(\infty)}\) in terms of the monodromy data \(\theta_\nu\) \((\nu = 0, x, 1, \infty)\), \(T_0\), \(T_1\), \(T_\infty\) is obtained as follows. As for \(\sigma^{(\infty)}\) we get if from
\[
2 \cos(\pi \sigma^{(\infty)}) = \hat{T}_\infty
\]

As for \(a^{(\infty)}\), we compute if from the formulae (142), (118) with the substitutions ³
\[
\sigma \mapsto \sigma^{(\infty)}
\]
\[
\theta_x \mapsto \theta_1, \quad \theta_1 \mapsto \theta_x
\]
\[
T_0 \mapsto T_\infty, \quad T_\infty \mapsto 4 \left[ \cos(\pi \theta_\infty) \cos(\pi \theta_1) + \cos(\pi \theta_x) \cos(\pi \theta_0) \right] - (T_0 + T_1 T_\infty)
\]

\([x = 1]\): If we put \(x = 1 - t, z = 1 - \hat{z}\), the system (121) becomes
\[
\frac{dY}{dz} = \left[ \frac{A_1}{\hat{z}} + \frac{A_0}{\hat{z} - 1} + \frac{A_x}{\hat{z} - t} \right] Y
\]

Therefore \(z = 0, 1\) have been exchanged. The monodromy matrices \(\hat{M}_0, \hat{M}_t, \hat{M}_1\) are
\[
M_0 = \hat{M}_1, \quad M_x = \hat{M}_1 \hat{M}_t \hat{M}_1^{-1}, \quad M_1 = \hat{M}_1 \hat{M}_t \hat{M}_0 \hat{M}_t^{-1} \hat{M}_1^{-1}
\]

As above, this implies that
\[
\begin{cases}
\hat{T}_0 = T_1 \\
\hat{T}_1 = T_0 \\
\hat{T}_\infty = 4 \left[ \cos(\pi \theta_\infty) \cos(\pi \theta_1) + \cos(\pi \theta_x) \cos(\pi \theta_0) \right] - (T_\infty + T_0 T_1)
\end{cases}
\]

Therefore, we obtain \(\sigma^{(1)}\) from
\[
2 \cos(\pi \sigma^{(1)}) = T_1
\]

and \(a^{(1)}\) from the formulae (142), (118) with the substitutions ⁴:
\[
\sigma \mapsto \sigma^{(1)}
\]
\[
\theta_x \mapsto \theta_1, \quad \theta_1 \mapsto \theta_x
\]
\[
T_0 \mapsto T_1, \quad T_1 \mapsto T_0, \quad T_\infty \mapsto 4 \left[ \cos(\pi \theta_\infty) \cos(\pi \theta_1) + \cos(\pi \theta_0) \cos(\pi \theta_1) \right] - (T_\infty + T_0 T_1)
\]

This solves the connection problem for \(y(x; \sigma, a)\), which is the representation close to \(x = 0\) in \(D(\epsilon; \sigma)\) of \(y(x; \theta_0, \theta_\infty; \theta_1, \theta_\infty; \theta_0, T_0, T_\infty)\) for monodromy data computed through (138), (140), (144). From the monodromy data we can compute \(a^{(1)}, \sigma^{(1)}, a^{(\infty)}, \sigma^{(\infty)}\) as we just explained above. Therefore \(y(x; \sigma, a)\) has representations in some \(D(\epsilon; \sigma^{(1)})\) and \(D(M; \sigma^{(\infty)})\) at \(x = 1\) and \(x = \infty\) respectively, with parameters \(a^{(1)}, \sigma^{(1)}, a^{(\infty)}, \sigma^{(\infty)}\).

**NOTES**

NOTE 1: Frobenius manifolds are the geometrical setting for the WDVV equations and were introduced by Dubrovin in [3]. They are an important object in many branches of mathematics like singularity theory and reflection groups [22, 24, 3, 4], algebraic and enumerative geometry [19, 22].

NOTE 2: The limit of \(a\), for \(\theta_0, \theta_1, \theta_x \to 0, \theta_\infty = 2\mu\), computed from the formula (142), exists and it coincides with (142)!
NOTE 3: The substitutions we have obtained now reduce to \((x_0, x_1, x_\infty) \mapsto (x_\infty, -x_1, x_0 - x_1 x_\infty)\) in the non-generic case \(\beta = \gamma = 1 - 2\delta = 0\).

NOTE 4: The substitutions we have obtained now reduce to \((x_0, x_1, x_\infty) \mapsto (x_1, x_0, x_0 x_1 - x_\infty)\) in the non-generic case \(\beta = \gamma = 1 - 2\delta = 0\).

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