Hypersurfaces with constant principal curvatures in $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$

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Abstract

In this paper, we classify the hypersurfaces in $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$, $n \neq 3$, with $g$ distinct constant principal curvatures, $g \in \{1, 2, 3\}$, where $S^n$ and $H^n$ denote the sphere and hyperbolic space of dimension $n$, respectively. We prove that such hypersurfaces are isoparametric in those spaces. Furthermore, we find a necessary and sufficient condition for an isoparametric hypersurface in $S^n \times \mathbb{R} \subset \mathbb{R}^{n+2}$ and $H^n \times \mathbb{R} \subset L^{n+2}$ with flat normal bundle, having constant principal curvatures.

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1 Introduction

The study of hypersurfaces in product spaces has attracted the attention of many geometers in recent years. First the surfaces with constant mean curvature and more particularly the minimal surfaces in product spaces were studied in works of H. Rosenberg, W. Meeks and U. Abresch, [1], [13] and [19]. They were also studied by I. Onnis and S. Montaldo in [14], [15], [18] and B. Nelli in [16], between many others.

In [2] and [3], J. Aledo, J. Espinar and J. Gálvez described the surfaces with constant Gaussian curvature in $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$. Moreover, J. Espinar, J. Gálvez and H. Rosenberg, showed in [10] that a complete surface with constant positive extrinsic curvature in $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$ is a rotational sphere.

In order to unify the notations we are going to denote by $Q^n_c$ the sphere $S^n$, if $c = 1$ and the hyperbolic space $H^n$, if $c = -1.$
The rotational hypersurfaces in $Q^n_c \times \mathbb{R}$ were parametrized by F. Dillen, J. Fastenakels and J. Van Der Veken in [8] where they extended the work of M.P. do Carmo and M. Dajczer [9] about rotational hypersurfaces in space forms.

The hypersurfaces in $Q^n_c \times \mathbb{R}$ having a special field $T$ as a principal direction were locally classified by R. Tojeiro in [22]. The differentiable field $T$ and the differentiable function $\nu$ are defined by the equation

$$\frac{\partial}{\partial t} = df(T) + \nu \eta,$$

where $f$ is an immersion of a Riemannian $n$-dimensional manifold $M^n$ in $Q^n_c \times \mathbb{R}$ with unit normal vector field $\eta$ and $\partial/\partial t$ is an unitary vector field tangent to $\mathbb{R}$. More particularly, the hypersurfaces with constant angle, i.e., the hypersurfaces with constant function $\nu$, were also classified in [22].

As we can observe in [22, Proposition 4], the hypersurfaces in $Q^n_c \times \mathbb{R}$ have flat normal bundle as an isometric immersion into $E^{n+2}$ if and only if $T$ is a principal direction of $f$.

In [12], R. Tojeiro and F. Manfio classified locally the hypersurfaces in $Q^n_c \times \mathbb{R}$, $n \geq 3$, with constant sectional curvature.

Motivated by these results, in this paper we investigate hypersurfaces $f : M^n \to Q^n_c \times \mathbb{R}$ with constant principal curvatures.

It is well known that a hypersurface in a space form is isoparametric if and only if its principal curvatures are constant but this does not happen in other ambients, in general. For instance in [24] one can find examples of isoparametric hypersurfaces in complex projective spaces that do not have constant principal curvatures. See also G. Thorbergsson [21].

In order to analyze if, for hypersurfaces in $Q^n_c \times \mathbb{R}$, is true the equivalence between to be isoparametric and to have constant principal curvatures, we obtain a necessary and sufficient condition presented in Theorem 5.1 for such equivalence to occur in hypersurfaces that have $T$ as a principal direction. For this, we prove in Theorem 3.1 the existence of a local frame of differentiable principal directions, a result that has been used previously in the literature. We consider a hypersurface that has the field $T$ as a principal direction, construct its family of parallel hypersurfaces and relate their respective principal curvatures.

The main purpose of this work is the classification given by Theorem 8.4 of hypersurfaces of $Q^n_c \times \mathbb{R}$, $n \neq 3$, with $g$ distinct constant principal curvatures, $g \in \{1, 2, 3\}$. Initially, we obtain in Theorem 6.2 the classification of hypersurfaces in $Q^n_c \times \mathbb{R}$ that have constant principal curvatures contained in the class having $T$ as a principal direction. We prove some results related to the multiplicities of such curvatures such as Theorem 7.1 and Proposition 7.4 and finally we obtain Theorem 8.4.

2 Preliminaries

Let $Q^n_c$ denotes either the sphere $S^n$ or hyperbolic space $H^n$, according as $c = 1$ or $c = -1$, respectively. We consider

$$Q^n_c = \{(x_1, \ldots, x_{n+1}) \in E^{n+1}/cx_1^2 + x_2^2 + \cdots + x_{n+1}^2 = c\},$$

with $x_1 > 0$ if $c = -1$ and

$$E^{n+1} = \{(x_1, \ldots, x_{n+2}) \in E^{n+2}/x_{n+2} = 0\},$$
where we denote by $\mathbb{E}^{n+2}$ either the Euclidean space $\mathbb{R}^{n+2}$ or the Lorentzian space $\mathbb{L}^{n+2}$ of dimension $(n+2)$, according as $c = 1$ or $c = -1$, respectively. Here $(x_1, \ldots, x_{n+2})$ are the standard coordinates on $\mathbb{E}^{n+2}$ and the flat metric $\langle , \rangle$ in those coordinates is written as $ds^2 = c dx_1^2 + \ldots + dx_{n+2}^2$.

Given a hypersurface $f : M^n \to Q^n_+ \times \mathbb{R}$, let $\eta$ denote a unit vector field normal to $f$ and let $\partial / \partial t$ denote a unit vector field tangent to the second factor $\mathbb{R}$. We define the differentiable vector field $T \in TM^n$ and a smooth function $\nu$ on $M^n$ by

$$\partial / \partial t = df(T) + \nu \eta.$$  

(1)

Since $\partial / \partial t$ is a unit vector field, we have

$$\nu^2 + \|T\|^2 = 1.$$  

(2)

Let $\nabla$ be the Levi-Civita connection, $R$ be the curvature tensor of $M^n$ and let $A$ be the shape operator of $f$ with respect to $\eta$. The fact that $\partial / \partial t$ is parallel in $Q^n_+ \times \mathbb{R}$ yields for all $X \in TM^n$ that

$$\nabla_X T = \nu A X \quad \text{and} \quad X(\nu) = -\langle AX, T \rangle.$$  

(3)

(4)

Moreover, the Gauss and Codazzi equations are

$$R(X, Y)Z = (AX \wedge AY)Z + c((X \wedge Y)Z - \langle Y, T \rangle(X \wedge T)Z + \langle X, T \rangle(Y \wedge T)Z),$$  

(5)

and

$$(\nabla_X A)Y - (\nabla_Y A)X = c\nu(X \wedge Y)T,$$  

(6)

respectively, where $(X \wedge Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y$ and $X, Y, Z \in TM^n$.

**Remark 2.1.** By (4), if $\nu$ is constant and $T \neq 0$, then $T$ is a principal direction and the principal curvature associated to it is equal to 0.

Consider $F : M^n \to \mathbb{E}^{n+2}$ given by $F := i \circ f$ where $i : Q^n_+ \times \mathbb{R} \to \mathbb{E}^{n+2}$ is the inclusion map whose unit normal field $\xi$ satisfies $\langle \xi, \xi \rangle = c$. If $A_\xi$ is the Weingarten operator of the immersion $F$ with respect to the normal direction $\xi$, we obtain $A_\xi(T) = -\nu^2 T$ and $A_\xi(X) = -X$, for all $X \in [T]^\perp$ where $[T]^\perp = \{X \in TM^n / \langle X, T \rangle = 0\}$. Let $\widehat{\nabla}$ denote the Riemannian connection of $\mathbb{E}^{n+2}$.

**Proposition 2.2.** The following equalities hold for all $X \in TM^n$,

$$\widehat{\nabla}_X \xi = df(X) - \langle X, T \rangle \partial / \partial t,$$  

(7)

$$\nabla^\perp_X \xi = -\nu\langle X, T \rangle \eta,$$  

(8)

$$\nabla^\perp_X \eta = c\nu\langle X, T \rangle \xi.$$  

(9)

Two trivial classes of hypersurfaces of $Q^n_+ \times \mathbb{R}$ arise if either $T$ or $\nu$ vanishes identically. Both classes will appear in our results.

**Proposition 2.3.** [12 Proposition 1] Let $f : M^n \to Q^n_+ \times \mathbb{R}$ be a hypersurface.

(i) If $T$ vanishes identically, then $f(M^n)$ is an open subset of a slice $Q^n_+ \times \{t\}$.

(ii) If $\nu$ vanishes identically, then $f(M^n)$ is an open subset of a Riemannian product $M^{n-1} \times \mathbb{R}$, where $M^{n-1}$ is a hypersurface of $Q^n_+$. 

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In [23] one can find a theorem that classifies the totally geodesic hypersurface of $S^n \times \mathbb{R}$. The same result also holds for $\mathbb{H}^n \times \mathbb{R}$.

**Theorem 2.4.** [23, Theorem 3] Let $M^n$ be a totally geodesic hypersurface of $S^n \times \mathbb{R}$. Then $M^n$ is an open part of a hypersurface $S^n \times \{t_0\}$ for $t_0 \in \mathbb{R}$, or of a hypersurface $S^{n-1} \times \mathbb{R}$.

### 3 Existence of a frame of principal directions

In order to prove the results of the next sections, we will need the following theorem based on results that can be found in [7, Theorem 2.6] and [17]. It is very important since it shows the existence of a local frame of differentiable principal directions.

**Theorem 3.1.** Let $A$ be a symmetric tensor of type $(1, 1)$ in an oriented Riemannian manifold $M^n$, $n \geq 2$, with $g$ distinct eigenvalues $\lambda_1, \ldots, \lambda_g$ having constant multiplicities $m_1, \ldots, m_g$, respectively. Then for each point $p \in M$ there exist an orthonormal frame of differentiable eigenvalues $\{X_1, \ldots, X_n\}$ defined in a neighborhood $U$ of $p$ in $M$.

**Proof.** Without loss of generality we may suppose $\lambda_1 > \lambda_2 > \ldots > \lambda_g$. Let us consider the $g$ orthogonal distributions $D_{\lambda_i}$ with $i \in \{1, \ldots, g\}$ defined by

$$D_{\lambda_i}(p) = \{Y_p \in T_p M; AY_p = \lambda_i Y_p\}.$$ 

Given $p \in M$ let $U'$ be a neighborhood of $p$ in $M$ where are defined the differentiable fields $Y_1, \ldots, Y_n$ such that

$$\{Y^1_1(p), \ldots, Y^{m_1}_1(p)\} \text{ span the distribution } D_{\lambda_1}(p),$$

$$\{Y^2_{m_1+1}(p), \ldots, Y^{m_1+m_2}_2(p)\} \text{ span the distribution } D_{\lambda_2}(p), \ldots,$$

$$\{Y^g_{m_1+\ldots+m_g-1+1}(p), \ldots, Y^n_g(p)\} \text{ span the distribution } D_{\lambda_g}(p).$$

Let us define $X_1, \ldots, X_n$ in $U'$ by

$$X^1_i(x) = (A(x) - \lambda_2 I) \ldots (A(x) - \lambda_g I) Y^1_i(x), \text{ for } i \in \{1, \ldots, m_1\},$$

$$X^2_i(x) = -(A(x) - \lambda_1 I)(A(x) - \lambda_3 I) \ldots (A(x) - \lambda_g I) Y^2_i(x), \text{ for } i \in \{m_1+1, \ldots, m_1+m_2\}$$

and

$$X^k_i(x) = (-1)^{k-1} \prod_{j \neq k} (A(x) - \lambda_j I) Y^k_i(x),$$

for each $k \in \{3, \ldots, g\}$ and $i \in \{m_1+\ldots+m_{k-1}+1, \ldots, m_1+\ldots+m_k\}$, with $j \in \{1, \ldots, g\}$, where $I$ is the identity matrix of order $n$ and $x \in U'$.

Observe that those fields depend on $x$ in a differentiable form. This happens because the eigenvalues have constant multiplicity and so they are differentiable [17]. Moreover, as $\{X_1(p), \ldots, X_n(p)\}$ are linearly independent then $\{X_1(x), \ldots, X_n(x)\}$ are linearly independent for all $x$ in a neighborhood $U \subset U'$ of $p$ in $M$. Observe also that for each $x \in U$ the basis $\{X_1(x), \ldots, X_n(x)\}$ is positive since it has the same orientation as the basis $\{Y_1(x), \ldots, Y_n(x)\}$.

The characteristic polinomial of the operator $A(x)$ is given by

$$p(t, x) = (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \ldots (t - \lambda_g)^{m_g}.$$
and by Cayley-Hamilton theorem it follows that
\[(A(x) - \lambda_1 I)(A(x) - \lambda_2 I) \ldots (A(x) - \lambda_g I) = 0,\]
for each \(x \in U\). Then \((A(x) - \lambda_1 I)(A(x) - \lambda_2 I) \ldots (A(x) - \lambda_g I)Y_i(x) = 0\), for all \(i \in \{1, \ldots, n\}\) and \(x \in U\). In this way for all \(x \in U\),
\[(A(x) - \lambda_1 I)X_i^1(x) = (A(x) - \lambda_1 I)(A(x) - \lambda_2 I) \ldots (A(x) - \lambda_g I)Y_i^1(x) = 0,\]
for all \(i \in \{1, \ldots, m_1\}\),
\[(A(x) - \lambda_2 I)X_i^2(x) = -(A(x) - \lambda_1 I)(A(x) - \lambda_2 I) \ldots (A(x) - \lambda_g I)Y_i^2(x) = 0,\]
for all \(i \in \{m_1 + 1, \ldots, m_1 + m_2\}\),
\[(A(x) - \lambda_k I)X_i^k(x) = (-1)^{k-1}(A(x) - \lambda_1 I)(A(x) - \lambda_2 I) \ldots (A(x) - \lambda_g I)Y_i^k(x) = 0,\]
for all \(i \in \{m_1 + \ldots + m_{k-1} + 1, \ldots, m_1 + \ldots + m_k\}\). Then, for each \(x \in U\), we have that \(A(x)X_i^k(x) = \lambda_k X_i^k(x)\), for all \(i \in \{1, \ldots, n\}\) and \(k \in \{1, \ldots, g\}\).

By the Gram-Schmidt orthogonalization process we obtain orthonormalized sets \(\{X_i^1(x), \ldots, X_{m_1}^1(x)\}\); \(\{X_i^2(x), \ldots, X_{m_1+m_2}^2(x)\}\) and \(\{X_i^k(x), \ldots, X_{m_1+\ldots+m_k}^k(x)\}\) for all \(x \in U\) and \(k \in \{3, \ldots, g\}\). Moreover, let \(X_i^\alpha\) and \(X_j^\beta\) for \(i, j \in \{1, \ldots, n\}\) and \(\alpha, \beta \in \{1, \ldots, g\}\) with \(\alpha \neq \beta\). Then,
\[
(\lambda_\alpha - \lambda_\beta)\langle X_i^\alpha, X_j^\beta \rangle = \langle \lambda_\alpha X_i^\alpha, X_j^\beta \rangle - \langle X_i^\alpha, \lambda_\beta X_j^\beta \rangle = \langle AX_i^\alpha, X_j^\beta \rangle - \langle X_i^\alpha, AX_j^\beta \rangle = \langle AX_i^\alpha - AX_j^\alpha, X_j^\beta \rangle = 0.
\]
Since \(\lambda_\alpha \neq \lambda_\beta\), for \(\alpha \neq \beta\) we get \(\langle X_i^\alpha, X_j^\beta \rangle = 0\), for \(i, j \in \{1, \ldots, n\}\) and \(\alpha, \beta \in \{1, \ldots, g\}\), \(\alpha \neq \beta\). So we obtain a local orthonormal frame of differentiable eigenvalues \(\{X_1, \ldots, X_n\}\).

This leads to the following result.

**Corollary 3.2.** Let \(f: M^n \to \mathbb{Q}_c^n \times \mathbb{R}, n \geq 2\), be a hypersurface having \(g \geq 2\) distinct principal curvatures \(\lambda_1, \ldots, \lambda_g\), with constant multiplicities \(m_1, \ldots, m_g\), respectively. Then for each \(p \in M\), there exist an orthonormal frame of principal directions \(\{X_1, \ldots, X_n\}\) in a neighborhood \(U\) of \(p\) in \(M\).

In the next proposition we obtain some equations that will be useful in this paper.

**Proposition 3.3.** Let \(f: M^n \to \mathbb{Q}_c^n \times \mathbb{R}\) be a hypersurface having principal curvatures with constant multiplicity. Let \(\{X_1, \ldots, X_n\}\) be a frame of principal orthonormal directions and let \(\lambda_i\) be the principal curvature associated to \(X_i\). If \(T\) is a principal direction, \(X_n = \|T\|^{-1}T\) and \(\eta_\kappa = \eta - \nu \partial / \partial t\) then
\[
\hat{\nabla}_{X_i} \eta_\kappa = -\lambda_i df(X_i) + c\nu (X_i, T) \xi - X_i(\nu) \partial / \partial t, \quad (10)
\]
\[
X_i(\|T\|) = 0, \quad \text{for all } i \neq n, \quad \text{and} \quad X_n(\|T\|) = \nu \lambda_n, \quad (11)
\]
\[
X_i(\nu) = 0, \quad \text{for all } i \neq n, \quad \text{and} \quad X_n(\nu) = -\lambda_n \|T\|, \quad (12)
\]
\[
X_i(\pi_2 \circ f) = 0, \quad \text{for all } i \neq n \quad \text{and} \quad X_n(\pi_2 \circ f) = \|T\|. \quad (13)
\]
Proof. Observe that
\[ \bar{\nabla}_X \eta_Q = \bar{\nabla}_X (\eta - \nu \partial / \partial t) = \bar{\nabla}_X \eta - X_i(\nu) \partial / \partial t = -df(A_\eta X_i) + \nabla^\perp_X \eta - X_i(\nu) \partial / \partial t. \]

By (9) we get \( \bar{\nabla}_X \eta_Q = -\lambda_i df(X_i) + c v(X_i, T) \xi - X_i(\nu) \partial / \partial t. \)

Using equations (3) and (4), we get for all \( i \in \{1, \ldots, n\}, \)
\[ 2\|T\|X_i(\|T\|) = X_i(\|T\|^2) = X_i(T, T) = 2\langle \nabla_X T, T \rangle = 2\nu \lambda_i \langle X_i, T \rangle \]
and \( X_i(\nu) = -\langle A_\eta X_i, T \rangle = -\langle X_i, A_\eta T \rangle = -\lambda_n \langle X_i, T \rangle. \)

Moreover,
\[ X_i(\pi_2 \circ f) = d\pi_2(df(X_i)) = \pi_2 df(X_i) = \langle df(X_i), \partial / \partial t \rangle = \langle X_i, T \rangle. \]

\[ \square \]

4 Family of parallel hypersurfaces in \( S^n \times \mathbb{R} \) and \( H^n \times \mathbb{R} \)

Consider the hypersurfaces \( f: M^n \to Q^n_c \times \mathbb{R} \) and \( i: Q^n_c \times \mathbb{R} \to \mathbb{E}^{n+2} \) with normal fields \( \eta \) and \( \xi \), respectively such that \( \eta \) is unitary and \( \langle \xi, \xi \rangle = c \). Let \( F := i \circ f \), \( \pi_1: Q^n_c \times \mathbb{R} \to Q^n_c \) and \( \pi_2: Q^n_c \times \mathbb{R} \to \mathbb{R} \) be the canonical projections. Given \( t \in \mathbb{R} \), \( p \in M^n \) and \( v \in T_{f(p)}(Q^n_c \times \mathbb{R}) \) such that \( d_{f(p)} \pi_1(v) = v_1 \) and \( d_{f(p)} \pi_2(v) = v_2 \), the exponential map in \( Q^n_c \times \mathbb{R} \) is defined by
\[ \exp_{f(p)}(tv) = \left( C_c(\|v_1\|t)\pi_1(f(p)) + S_c(\|v_1\|t)\frac{v_1}{\|v_1\|}, \pi_2(f(p)) + tv_2 \right), \quad \text{if } v_1 \neq 0 \]
\[ \exp_{f(p)}(tv) = (\pi_1(f(p)), \pi_2(f(p)) + tv_2), \quad \text{if } v_1 = 0, \]
where
\[ C_c(s) = \begin{cases} \cos(s), & c = 1 \\ \cosh(s), & c = -1 \end{cases}, \quad S_c(s) = \begin{cases} \sin(s), & c = 1 \\ \sinh(s), & c = -1. \end{cases} \quad (14) \]

Take \( p \in M^n \), \( v \in T_{f(p)}(Q^n_c \times \mathbb{R}) \) and the curve \( \alpha: I \subset \mathbb{R} \to Q^n_c \times \mathbb{R} \) given by \( \alpha(t) = \exp_{f(p)}(tv) \). Observe that \( \alpha \) is a geodesic in \( Q^n_c \times \mathbb{R} \) that passes through the point \( \alpha(0) = (\pi_1(f(p)), \pi_2(f(p))) = f(p) \) and \( \alpha'(0) = (v_1, v_2) = v \).

From now on we will study the families of hypersurfaces that are parallel to a hypersurface having \( T \) as a principal direction. For this, take \( f: M^n \to Q^n_c \times \mathbb{R} \) a hypersurface that has \( T \) as a principal direction and all the principal curvatures with constant multiplicity. Let \( \{X_1, \ldots, X_n\} \) be a frame of orthonormal principal directions with \( X_n = \|T\|^{-1}T \). Observe that \( \xi \circ f = (\pi_1 \circ f, 0) \) and \( \eta_Q = \eta - \nu \partial / \partial t \) which implies that \( \|\eta_Q\| = \|T\| \neq 0 \). Then the hypersurfaces parallel to \( f \) are given by
\[ f_t = C_c(\|T\|t) \xi \circ f + S_c(\|T\|t)\|T\|^{-1}\eta_Q + (\pi_2 \circ f + tv) \partial / \partial t. \quad (15) \]

For all \( i \in \{1, \ldots, n\} \) we have
\[ df_t(X_i) = -ctS_c(\|T\|t)X_i(\|T\|)\xi + C_c(\|T\|t)\bar{\nabla}_{X_i} \xi + tC_c(\|T\|t)X_i(\|T\|)\|T\|^{-1}\eta_Q + S_c(\|T\|t)\bar{\nabla}_{X_i} \|T\|^{-1}\eta_Q + X_i(\pi_2 \circ f + tv) \partial / \partial t. \]
From (7), (10), (11) and (13) we obtain
\[ df_i(X_i) = (C_c(||T||t) - \lambda_i ||T||^{-1}S_c(||T||t)) df(X_i), \text{ for } i \neq n \] and
\[ df_i(X_n) = cvS_c(||T||t)(1 - t\lambda_n)\xi + (1 - t\lambda_n)(\nu^2C_c(||T||t) + ||T||^2) df(X_n) + (1 - t\lambda_n)(1 - C_c(||T||t))\nu||T||\eta. \] (17)

Then \( f_t \) is an immersion if \( C_c(||T||t) - \lambda_i ||T||^{-1}S_c(||T||t) \neq 0, \) for all \( i \in \{1, \ldots, n - 1\} \) and \( 1 - t\lambda_n \neq 0. \)

Observe that \( \eta_t \) given by
\[ \eta_t = -c||T||S_c(||T||t)\xi \circ f + C_c(||T||t)\eta_Q + \nu\partial/\partial t \] (18)
is a unit vector field normal to \( f_t. \)

Next result gives the relation between the principal curvatures of a hypersurface in \( \mathbb{Q}_c^n \times \mathbb{R} \) having \( T \) as principal direction and the principal curvatures of its parallel hypersurfaces.

**Proposition 4.1.** Let \( f : M^n \to \mathbb{Q}_c^n \times \mathbb{R} \) be a hypersurface having \( T \) as a principal direction and \( \lambda_i, i \in \{1, \ldots, n\} \) its principal curvatures. If \( f_t \) is a family of hypersurfaces parallel to \( f \) with principal curvatures \( \lambda^t_i, i \in \{1, \ldots, n\} \) then
\[ \lambda^t_i = \frac{c||T||S_c(||T||t) + \lambda_i C_c(||T||t)}{C_c(||T||t) - \lambda_i ||T||^{-1}S_c(||T||t)}, \text{ for } i \neq n, \] (19)
and
\[ \lambda^t_1 = \frac{1}{1 - t\lambda_n}. \] (20)

**Proof.** Let \( \{X_1, \ldots, X_n\} \) be an orthogonal frame of principal directions of \( f. \)

From (18), (7), (10), (11) and (12), we conclude that
\[ \tilde{\nabla}_X \eta_t = - (c||T||S_c(||T||t) + \lambda_i C_c(||T||t)) df(X_i), \text{ for } i \neq n \] and
\[ \tilde{\nabla}_X \eta_t = \{(1 - t\lambda_n)||T||C_c(||T||t) - \lambda_n S_c(||T||t)\} cv\xi + \{-\lambda_c ||T||^2 + \nu^2C_c(||T||t)\} - cv^2||T||S_c(||T||t)(1 - t\lambda_n)\} df(X_n) + \{cv||T||^2S_c(||T||t)(1 - t\lambda_n) - \nu||T||\lambda_n(1 - C_c(||T||t))\} \eta. \] (22)

Observe that if \( \{X_1, \ldots, X_n\} \) is an orthogonal frame of principal directions of \( f \) then it is also an orthogonal frame of principal directions of \( f_t. \)

From (16) and (17) we get
\[ \langle df_i(X_i), df_i(X_i) \rangle = (C_c(||T||t) - \lambda_i ||T||^{-1}S_c(||T||t))^2, \text{ for } i \neq n \] and
\[ \langle df_i(X_n), df_i(X_n) \rangle = (1 - t\lambda_n)^2. \] (23)

Moreover by using (16) and (21) we get for \( i \neq n, \)
\[ -\langle \tilde{\nabla}_X \eta_t, df_i(X_i) \rangle = (c||T||S_c(||T||t) + \lambda_i C_c(||T||t)) \]
\[ (C_c(||T||t) - \lambda_i ||T||^{-1}S_c(||T||t)). \] (25)

From (17) and (22) we conclude also that
\[ \langle \tilde{\nabla}_X \eta_t, df_i(X_n) \rangle = -\lambda_n(1 - t\lambda_n). \] (26)

Finally using (23) and (25) we show (19) and from (24) and (26) we obtain (20). \( \square \)

**Remark 4.2.** Since we are supposing that the principal curvatures \( \lambda_i, i \in \{1, \ldots, n\}, \) have constant multiplicity we conclude that the curvatures \( \lambda^t_i, i \in \{1, \ldots, n\}, \) also have constant multiplicity and by (17) they are differentiable.
5 A necessary and sufficient condition for an isoparametric hypersurface of $\mathbb{Q}^n_c \times \mathbb{R}$ having constant principal curvatures

Cartan proved in [4] that a necessary and sufficient condition for a family of parallel hypersurfaces in a Riemannian manifold to be isoparametric is that all hypersurfaces must have constant mean curvature.

In the next result we obtain the necessary and sufficient condition that an isoparametric hypersurface in $\mathbb{Q}^n_c \times \mathbb{R}$ having $T$ as a principal direction must satisfy to have constant principal curvatures.

**Theorem 5.1.** Let $f: M^n \to \mathbb{Q}^n_c \times \mathbb{R}$ be an isoparametric hypersurface having $T$ as a principal direction. Then $f$ has constant principal curvatures if and only if $\|T\|$ is constant.

**Proof.** Define the real valued function

$$u(t) = \sum_{i=1}^{n} \lambda_i^t.$$ 

Since $T$ is a principal direction we may use expressions (19) and (20). Observe that

$$\frac{\partial \lambda_i^t}{\partial t} = c\|T\|^2 + (\lambda_i^t)^2, \quad \text{for} \quad i \in \{1, \ldots, n-1\}. \tag{27}$$

If $f$ has constant principal curvatures then $\sum_{i=1}^{n} \lambda_i^k$, $1 \leq k \leq n$ is also constant. We have that $u'(t) = \sum_{i=1}^{n-1} c\|T\|^2 + (\lambda_i^t)^2 + (\lambda_n^t)^2$. Then

$$u'(0) = (n-1)c\|T\|^2 + \sum_{i=1}^{n} \lambda_i^2$$

and $\|T\|$ is constant.

On the converse if $\|T\|$ is constant the function $\nu$ is constant and by (i), $\lambda_n = 0$. Hence from (20), it follows that $\lambda_n^t = 0$. So $u(t) = \sum_{i=1}^{n-1} \lambda_i^t$ and consequently its derivative of order $k$ is $u^k(t) = \sum_{i=1}^{n-1} \frac{\partial^k \lambda_i^t}{\partial t^k}$. Observe that

$$\frac{\partial^2 \lambda_i^t}{\partial t^2} = 2c\|T\|^2 \lambda_i^t + 2(\lambda_i^t)^3. \tag{28}$$

Let us prove using the induction process that for odd $k$, $2 < k < n$ we get,

$$\frac{\partial^k \lambda_i^t}{\partial t^k} = u_{k,0} c^{\frac{k-1}{2}} \|T\|^{k+1} + u_{k,2} c^{\frac{k-1}{2}} \|T\|^{k-1}(\lambda_i^t)^2 + u_{k,4} c^{\frac{k-3}{2}} \|T\|^{k-3}(\lambda_i^t)^4 + \ldots + u_{k,k+1}(\lambda_i^t)^{k+1}, \tag{29}$$
where we denote by $u_{k,j}$ the $j$-th coefficient of the derivative of order $k$ of $\lambda_i^t$.

If $k$ is even, $2 \leq k < n$ we obtain

$$
\frac{\partial^k \lambda_i^t}{\partial t^k} = u_{k,0}c^k T^k \lambda_i^t + u_{k,1}c^k \frac{c^k}{2} T^k \lambda_i^t + u_{k,2}c^k \left(\lambda_i^t\right)^3 + u_{k,3}c^k \frac{c^k}{2} T^k \lambda_i^t + \ldots + u_{k,k+1} \left(\lambda_i^t\right)^{k+1},
$$

(30)

where $u_{k,0} = u_{k-1,1}$, $u_{k,1} = 2u_{k-1,2}$, $u_{k,2} = 3u_{k-1,3} + u_{k-1,1}$, $\ldots$, $u_{k,j} = (j+1)u_{k-1,j+1} + (j-1)u_{k-1,j-1}$, $\ldots$, $u_{k,k+1} = ku_{k-1,k}$. We point out that if $k$ is odd the index $j$ of $u_{k,j}$ is an even number and when $k$ is even the index $j$ of $u_{k,j}$ is odd.

By (27), we get $u_{1,2} = 1$. For $k = 2$ and using equation (28), we obtain

$$
\frac{\partial^2 \lambda_i^t}{\partial t^2} = 2cT^2 \lambda_i^t + 2(\lambda_i^t)^3 = 2u_{1,2}cT^2 \lambda_i^t + 2u_{1,3}(\lambda_i^t)^3 = u_{2,1}cT^2 \lambda_i^t + u_{2,3}(\lambda_i^t)^3,
$$

that satisfies equation (30).

By the induction hypothesis let us suppose that equations (29) and (30) hold for the index $k - 1$. We will show that they hold also for the index $k$.

If $k$ is an even number then $k - 1$ is an odd number and equation (29) holds, that is,

$$
\frac{\partial^{k-1} \lambda_i^t}{\partial t^{k-1}} = u_{k-1,0}c^k T^k \lambda_i^t + u_{k-1,1}c^k \frac{c^k}{2} T^k \lambda_i^t + u_{k-1,2}c^k \left(\lambda_i^t\right)^2 + \ldots + u_{k-1,k} \left(\lambda_i^t\right)^k.
$$

(31)

By deriving equation (31), with respect to the variable $t$, using equation (27), we get

$$
\frac{\partial^k \lambda_i^t}{\partial t^k} = 2u_{k-1,2}c^k T^{k-2} \lambda_i^t \frac{\partial \lambda_i^t}{\partial t} + 4u_{k-1,4}c^k \frac{c^k}{2} T^{k-2} \lambda_i^t \frac{\partial \lambda_i^t}{\partial t} + \ldots + ku_{k-1,1} \left(\lambda_i^t\right)^{k-1} \frac{\partial \lambda_i^t}{\partial t} + u_{k-1,1}c^k T^k \lambda_i^t + (2u_{k-1,2} + 4u_{k-1,4})c^k \frac{c^k}{2} T^{k-2} \lambda_i^t + \ldots + ku_{k-1,1} \left(\lambda_i^t\right)^{k+1} + ku_{k-1,1} \left(\lambda_i^t\right)^{k+1} + ku_{k-1,1} \left(\lambda_i^t\right)^{k+1} = u_{k,1}c^k T^k \lambda_i^t + u_{k,3}c^k \frac{c^k}{2} T^{k-2} \lambda_i^t + \ldots + u_{k,k+1} \left(\lambda_i^t\right)^{k+1}.
$$

Then equation (30) holds. In the same way it can be shown that equation (29) also holds.

Since $\lambda_n = 0$, we have that $u(0) = \sum_{i=1}^{n-1} \lambda_i = C_1$, with $C_1$ constant and

$$
u'(0) = \sum_{i=1}^{n-1} cT^2 + \lambda_i^2,
$$

which implies that $\sum_{i=1}^{n} \lambda_i^2 = C_2$, with $C_2$ constant. Hence

$$
u''(0) = (n - 1)2cT^2 \sum_{i=1}^{n-1} \lambda_i + 2 \sum_{i=1}^{n-1} \lambda_i^3 = (n - 1)2cT^2 C_1 + 2 \sum_{i=1}^{n-1} \lambda_i^3,
$$

and $\sum_{i=1}^{n} \lambda_i^3 = C_3$, with $C_3$ constant.
If $k$ is an even number we get
\[
\begin{align*}
 u^k(0) &= u_{k,1}c^2 \|T\|^k \sum_{i=1}^{n-1} \lambda_i + u_{k,3}c \frac{k-2}{2} \|T\|^{k-2} \sum_{i=1}^{n-1} (\lambda_i)^3 + u_{k,5}c \frac{k-4}{2} \|T\|^{k-4} \sum_{i=1}^{n-1} (\lambda_i)^5 \\
 &\quad + \ldots + u_{k,k+1} \sum_{i=1}^{n-1} (\lambda_i)^{k+1} \\
 &= u_{k,1}c^2 \|T\|^k C_1 + u_{k,3}c \frac{k-2}{2} \|T\|^{k-2} C_3 + u_{k,5}c \frac{k-4}{2} \|T\|^{k-4} C_5 + \ldots \\
 &\quad + u_{k,k+1} \sum_{i=1}^{n-1} (\lambda_i)^{k+1}.
\end{align*}
\]

Then $\sum_{i=1}^{n} \lambda_i^{k+1}$ is constant. In a similar way we obtain the same if $k$ is an odd number.

Finally we conclude that $\sum_{i=1}^{n} \lambda_i^k$, $1 \leq k \leq n$, is constant. Based on the demonstration of [6, Theorem 5.8], by Newton identity the coefficients of the characteristic polynomial of the Weingarten operator $A$ are also polynomials $\sum_{i=1}^{n} \lambda_i^k$, $1 \leq k \leq n$. Then the principal curvatures are also constant because they are the roots of the characteristic polynomial.

Next result for a hypersurface with constant angle $\nu \neq 1$, given by [22, Corollary 2], is proved using Theorem 5.1

**Corollary 5.2.** Let $f : M^n \to \mathbb{Q}_c^n \times \mathbb{R}$ be a hypersurface with constant angle and $T \neq 0$. Then $f$ is isoparametric if and only if the principal curvatures are constant.

**Proof.** By Remark 2.1, $T$ is a principal direction. Since $f$ has $\nu$ constant from (2) $\|T\|$ is also constant. Moreover by (1), $\lambda_n = 0$ and so by (20), $\lambda_n^0 = 0$.

Suppose that the principal curvatures $\lambda_i$ of $f$ are constant. By (19) and (20) the principal curvatures of the family $f_t$ are also constant and $f$ is isoparametric.

On the converse if $f$ is isoparametric, since $\|T\|$ is constant, by Theorem 5.1 the principal curvatures of $f$ are constant. \qed

## 6 Hypersurfaces in $\mathbb{Q}_c^n \times \mathbb{R}$ having constant principal curvatures and $T$ as a principal direction

In this section we classify the hypersurfaces of $\mathbb{Q}_c^n \times \mathbb{R}$ with constant principal curvatures having field $T$ as a principal direction. First, we state the following technical lemma.

**Lemma 6.1.** Let $a : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $a'(s) > 0$ and $a''(s) \neq 0$, for all $s \in I$. The solutions of the differential equation $a''(1 + (a')^2) - 3(a'')^2a' = 0$ are given by
\[
a(s) = -\frac{\sqrt{1 - (c_1s + c_2)^2}}{c_1} + c_3,
\]
where $c_1, c_2$ and $c_3$ are real constant, $0 < c_1s + c_2 < 1$ and $c_1 \neq 0$.  

Theorem 6.2. Let \( f : M^n \rightarrow \mathbb{Q}^n_c \times \mathbb{R}, n \geq 2 \) be a hypersurface having constant principal curvatures and \( T \) as a principal direction such that \( \nu(p) \neq 0 \), for all \( p \in M \). Then \( c = -1 \) and \( f \) is given locally by \( f(x, s) = h_s(x) + Bs\partial/\partial t, \) for some \( B \in \mathbb{R}, B > 0, \) where \( h_s \) is a family of horospheres in \( \mathbb{H}^n \). Moreover the principal curvature associated to the field \( T \) is equal to \( \frac{B}{\sqrt{1+B^2}} \) or \( -\frac{B}{\sqrt{1+B^2}} \).

Proof. By \cite{22}, Theorem 1] if \( T \) is a principal direction of \( f \) and \( \nu(p) \neq 0 \), for all \( p \in M \) then \( f \) is locally given by \( f : M^n = M^{n-1} \times I \rightarrow \mathbb{Q}^n_c \times \mathbb{R} \) with \( f(x, s) = h_s(x) + a(s)\partial/\partial t, \) where \( a : I \rightarrow \mathbb{R} \) is a differentiable function such that \( a'(s) > 0, \) for all \( s \in I. \) Moreover \( A_\eta X = -\frac{a'(s)}{b(s)} A^s X, \) for all \( X \in TM^{n-1}, \) where \( A^s \) is the shape operator of \( h_s \). In particular, \( A_\eta X_i = -\frac{a'(s)}{b(s)} \lambda_i^s(x) X_i, \) for the principal directions \( X_i \in TM^{n-1} \) of \( h \) and \( A_\eta T = \frac{a''(s)}{b^3(s)} T, \) where \( b(s) = \sqrt{1+a'(s)^2}. \) Therefore

\[
A_\eta(X_i) = \mu_i(x, s) X_i, \quad \text{with} \quad \mu_i(x, s) = -\frac{a'(s)}{b(s)} \lambda_i^s(x),
\]

for \( i \in \{1, \ldots, n-1\} \) and \( A_\eta T = \mu_n(x, s) T, \) with \( \mu_n(x, s) = \frac{a''(s)}{b^3(s)}. \)

It is known that the relation between the principal curvatures of a hypersurface of \( \mathbb{Q}^n_c \) and the principal curvatures of its parallel hypersurfaces is given by

\[
\lambda_i^s(x) = \frac{cS_c(s) + C_c(s)\lambda_i(x)}{C_c(s) - S_c(s)\lambda_i(x)}
\]

and therefore

\[
\mu_i(x, s) = -\frac{a'(s)}{b(s)} \left( \frac{cS_c(s) + C_c(s)\lambda_i(x)}{C_c(s) - S_c(s)\lambda_i(x)} \right), \quad i \in \{1, \ldots, n-1\}.
\]

Let us analyze under which conditions the functions \( \mu_i \) are constant. Observe that

\[
\frac{\partial \lambda_i^s}{\partial x} = \frac{\lambda_i'(x)}{(C_c(s) - S_c(s)\lambda_i(x))^2}
\]

and

\[
\frac{\partial \lambda_i^s}{\partial s} = c + (\lambda_i^s(x))^2.
\]

For \( i \in \{1, \ldots, n-1\}, \) \( \frac{\partial \mu_i}{\partial x} = -\frac{a'(s)}{b(s)} \frac{\partial \lambda_i^s}{\partial x} \) and \( \frac{\partial \mu_i}{\partial x} = 0 \) if and only if \( \frac{\partial \lambda_i^s}{\partial x} = 0. \)

Therefore, by (32), \( \mu_i \) is constant with respect to \( x \) if \( h \) is isoparametric, that is, if the functions \( \lambda_i \) are constant for all \( i \in \{1, \ldots, n-1\}. \)

Moreover,

\[
\frac{\partial \mu_i}{\partial s} = \frac{-a''b + a'b'}{b^2} \lambda_i^s - \frac{a'}{b} \frac{\partial \lambda_i^s}{\partial s} = -\frac{a''}{b^3} \lambda_i^s - \frac{a'}{b} (c + (\lambda_i^s)^2) = -\frac{a'' \lambda_i^s - a'b^2(c + (\lambda_i^s)^2)}{b^3}.
\]

Then, for \( i \in \{1, \ldots, n-1\}, \) \( \frac{\partial \mu_i}{\partial s} = 0 \) if and only if \( a'' \lambda_i^s + a'(1 + a'^2)(c + (\lambda_i^s)^2) = 0. \)
Thus, all solutions to equation (35) are given by

$$a''(1 + a'^2)(c + (\lambda^s_i)^2) = 0$$

and

$$a''(1 + a'^2) - 3a''a' = 0.$$  \hspace{1cm} (35)

As the function $a$ is differentiable of class $C^\infty$ we may consider just two cases: $a''(s) = 0$, for all $s \in I$ or $a''(s) \neq 0$ for all $s \in I$, restricting the interval $I \in \mathbb{R}$, if necessary.

**Case 1:** Suppose that $a''(s) = 0$ for all $s \in I$. Then $a''(s) = 0$ and consequently the equation (35) holds. By equation (34) we get $a'(1 + a'^2)(c + (\lambda^s_i)^2) = 0$. Since $a'(s) > 0$ we conclude that $c + (\lambda^s_i)^2 = 0$.

- If $c = 1$ then $1 + (\lambda^s_i)^2 \neq 0$. So this case cannot occur for $c = 1$.
- If $c = -1$ then $(\lambda^s_i)^2 = 1$. So $\lambda^s_i = \pm 1$ which implies that $\lambda_i = \pm 1$. Moreover, $\mu_n = 0$ and $\mu_i = \pm \frac{a'}{b}$ for $i \in \{1, ..., n-1\}$.

**Case 2:** Suppose that $a''(s) \neq 0$ for all $s \in I$. From Lemma 6.1 the solutions of the equation (35) are given by $a(s) = -\sqrt{1 - (c_1s + c_2)^2} + c_3$, where $c_1$, $c_2$ and $c_3$ are real constant with $c_1 \neq 0$. Let us certify if those solutions satisfy the equation (34). Observe that

$$a'(s) = \frac{c_1s + c_2}{\sqrt{1 - (c_1s + c_2)^2}} \quad \text{and} \quad a''(s) = \frac{c_1}{(1 - (c_1s + c_2)^2)^{\frac{3}{2}}}.$$  \hspace{1cm} (35)

Then

$$a'(1 + a'^2) = \frac{c_1s + c_2}{(1 - (c_1s + c_2)^2)^{\frac{3}{2}}}.$$  \hspace{1cm} (35)

Thus,

$$a''(1 + a'^2)(c + (\lambda^s_i)^2) = \frac{c_1}{(1 - (c_1s + c_2)^2)^{\frac{3}{2}}} \lambda^s_i + \frac{c_1s + c_2}{(1 - (c_1s + c_2)^2)^{\frac{3}{2}}}(c + (\lambda^s_i)^2).$$  \hspace{1cm} (35)

The solutions $a(s)$ of the equation (35) satisfy (34) if and only if

$$c_1 \lambda^s_i + (c_1s + c_2)(c + (\lambda^s_i)^2) = 0,$$  \hspace{1cm} (36)

for all $s \in I$.

Suppose that (36) holds, for all $s \in I$. If $c + (\lambda^s_i)^2 = 0$ then $c_1 \lambda^s_i = 0$, i.e., $\lambda^s_i = 0$, for all $s \in I$, since $c_1 \neq 0$. But,

$$\lambda^s_i(x) = \frac{cS_c(s) + C_c(s)\lambda_i(x)}{C_c(s) - S_c(s)\lambda_i(x)} = 0 \quad \text{implies that} \quad cS_c(s) + C_c(s)\lambda_i(x) = 0,$$

for all $s \in I$, which cannot occur. Then, in this case, it is not possible to have $c + (\lambda^s_i)^2 = 0$.  \hspace{1cm} (36)
Moreover, by deriving equation (36) we obtain
\[
\frac{\partial}{\partial s}\left(c_1\lambda_i^s + (c_1 s + c_2)(c + (\lambda_i^s)^2)\right) = c_1 \frac{\partial \lambda_i^s}{\partial s} + c_1 (c + (\lambda_i^s)^2) + 2(c_1 s + c_2)\lambda_i^s \frac{\partial \lambda_i^s}{\partial s} = 0.
\]
By equation (33), it follows that
\[
2c_1(c + (\lambda_i^s)^2) + 2(c_1 s + c_2)\lambda_i^s(c + (\lambda_i^s)^2) = 2(c + (\lambda_i^s)^2)(c_1 + (c_1 s + c_2)\lambda_i^s) = 0.
\]
Consequently, \(c_1 + (c_1 s + c_2)\lambda_i^s = 0\). Since \(c_1 s + c_2 > 0\), for all \(s \in I\) we have \(\lambda_i^s = -\frac{c_1}{c_1 s + c_2}\). Thus, in one way,
\[
\frac{\partial \lambda_i^s}{\partial s} = \frac{(c_1)^2}{(c_1 s + c_2)^2} = (\lambda_i^s)^2.
\]
But, on the other way, from (33) we have that \(\frac{\partial \lambda_i^s}{\partial s} = c + (\lambda_i^s)^2\). Consequently \(c = 0\) which cannot occur since we are considering only \(c = 1\) or \(c = -1\). So equation (36) does not hold and the solutions of the equation (35) are not solutions of the equation (34), with the condition \(a'''(s) \neq 0\), for all \(s \in I\).

Thus we conclude that \(a'' = 0\) and \(a(s) = Bs\) with \(B \in \mathbb{R}, B > 0\), since \(a'(s) > 0\) and \(\nu\) is constant, from [22, Corollary 2]. Therefore \(\lambda = \mu_n = 0\) and \(\mu_i = \frac{B}{\sqrt{1 + B^2}}\) or \(\mu_i = -\frac{B}{\sqrt{1 + B^2}}\), for \(i \in \{1, \ldots, n-1\}\).

Remark 6.3. From [12, Proposition 20], for \(n \geq 3\), and [22, Remark 7 (i)], for \(n = 2\), the hypersurfaces given in Theorem 6.2, are rotational hypersurfaces in \(\mathbb{H}^n \times \mathbb{R}\) for which the orbits are horospheres.

7 Multiplicities of the principal curvatures

In this section we discuss some results about the multiplicities of the principal curvatures of hypersurfaces in \(Q^c_n \times \mathbb{R}\).

Theorem 7.1. Let \(f: M^n \rightarrow Q^c_n \times \mathbb{R}, n \geq 2\) be a non umbilical hypersurface having constant principal curvatures with constant multiplicities and suppose that its function \(\nu \neq 0\). Then it has at least one principal curvature of multiplicity one.

Proof. Let \(\{X_1, X_2, \ldots, X_n\}\) be a local orthonormal frame field of principal directions of \(f\).

It is possible to write \(T = \sum_{i=1}^{n} b_i X_i\). As \(g\) is the number of distinct principal curvatures and \(f\) is non umbilical then \(g \geq 2\).

If \(n = 2\) then there exist two distinct principal curvatures and each one has multiplicity equal to 1.

If \(n = 3\) then \(g = 2\) or \(g = 3\). If \(g = 2\) one of the curvatures has multiplicity equal to 2 and the other one has multiplicity equal to 1. If \(g = 3\) each curvature has multiplicity equal to 1.
If \( n \geq 4 \) suppose that all the principal curvatures have multiplicity greater than or equal to 2. In this case \( 2 \leq g \leq \frac{n}{2} \), if \( n \) is an even number and \( 2 \leq g \leq \frac{n-1}{2} \) if \( n \) is an odd number. For a given \( \rho \in \{1, \ldots, g\} \) consider \( B_\rho = \{i \in \{1, \ldots, n\} / AX_i = \lambda_\rho X_i \} \). Observe that \( B_\rho \) has at least two elements.

For a given \( \rho \), consider the Codazzi equation,

\[
\nabla_{X_i} AX_j - \nabla_{X_j} AX_i - A[X_i, X_j] = c\nu(b_j X_i - b_i X_j),
\]

for \( i, j \in B_\rho \).

We have \( \nabla_{X_i} AX_j - \nabla_{X_j} AX_i = \lambda_\rho [X_i, X_j] = \lambda_\rho \sum_{k=1}^{n} \langle [X_i, X_j], X_k \rangle X_k \) and \( A[X_i, X_j] = \sum_{k=1}^{n} \langle [X_i, X_j], X_k \rangle AX_k \). Thus,

\[
\nabla_{X_i} AX_j - \nabla_{X_j} AX_i - A[X_i, X_j] = \sum_{k \not\in B_\rho} \langle [X_i, X_j], X_k \rangle (\lambda_\rho X_k - AX_k).
\]

From equations (37) and (38), we get

\[
\sum_{k \not\in B_\rho} \langle [X_i, X_j], X_k \rangle (\lambda_\rho - \lambda_k) X_k = c\nu b_j X_i + c\nu b_i X_j = 0.
\]

Since \( i, j \in B_\rho \) with \( i \neq j \), \( k \not\in B_\rho \) and \( \nu \neq 0 \) we should have \( b_i = b_j = 0 \) for all \( i, j \in B_\rho \), that is, \( T \) does not have components in the directions corresponding to principal curvatures whose multiplicities are greater than or equal to 2. So, assuming that there does not exist principal curvatures whose multiplicities are one, we conclude that \( T = 0 \). Finally we conclude that \( f(M^n) \) is an open subset of a slice \( \mathbb{Q}_c^n \times \{t\} \) and thus is totally geodesic. But this is against the hypothesis \( g \geq 2 \). So \( f \) has at least one principal curvature with multiplicity one.

\[\square\]

**Remark 7.2.** From the proof of the previous theorem we infer that \( T \) has no components in the directions whose correspondent curvatures have multiplicities greater than or equal to 2.

**Remark 7.3.** Theorem 7.1 holds also for \( \nu \equiv 0 \) if \( c = -1 \) since the corresponding curvature of the factor \( \mathbb{R} \) is \( \lambda = 0 \) and the others curvatures are non zero by [17]. Theorem 5. It is true also for \( \nu \equiv 0 \) and \( c = 1 \) if \( g = 2 \) and \( g = 3 \), excluding the cases when \( f(M^n) \) is an open set of \( M^{n-1} \times \mathbb{R} \) where \( M^{n-1} \) is a Cartan’s hypersurface for \( n \in \{7, 13, 25\} \).

**Proposition 7.4.** Let \( f : M^n \to \mathbb{Q}_c^n \times \mathbb{R}, n \geq 3 \) be a hypersurface with constant principal curvatures and respective multiplicities also constant having function \( \nu \neq 0 \). If just one principal curvature has multiplicity equal to one then the vector field \( T \) is a principal direction corresponding to that curvature. Moreover, all the curvatures having multiplicity greater than one do not vanish.

*Proof.* Let \( \{X_1, X_2, \ldots, X_n\} \) be a local orthonormal frame of principal directions of \( f \). Suppose, without loss of generality, that \( X_n \) is associated to \( \lambda \), i.e., \( AX_n = \lambda X_n \). By Remark 7.2 if \( \lambda \) is the only curvature of multiplicity one then \( T = bX_n \) where \( b : U \subset M^n \to \mathbb{R} \) is a
differentiable function defined on an open subset $U \subset M^n$, where the fields $X_1, X_2, \ldots, X_n$ are defined.

From the hypothesis we know that $g \geq 2$. For a given $\rho \in \{1, \ldots, g - 1\}$ let $B_\rho = \{i \in \{1, \ldots, n\} / AX_i = \mu_\rho X_i\}$ with $\mu_\rho \neq \lambda = \mu_\rho$. Observe that $B_\rho$ has at least two elements. Consider the Codazzi equation

$$\nabla_{X_i} AX_i - \nabla_{X_i} AX_n - A[X_n, X_i] = cv(b_i X_n - b_n X_i),$$

for $i \in B_\rho$. We have

$$\nabla_{X_i} AX_i - \nabla_{X_i} AX_n = \mu_\rho \nabla_{X_i} X_i - \lambda \nabla_X X_n \quad \text{and}$$

$$A[X_n, X_i] = \sum_{k=1}^n \langle \nabla_{X_k} X_i, X_k \rangle AX_k.$$  

Therefore from equations (40) and (41) we get

$$\nabla_{X_i} AX_i - \nabla_{X_i} AX_n - A[X_n, X_i] = \sum_{k=1}^n \langle \nabla_{X_k} X_i, X_k \rangle (\mu_\rho X_k - AX_k)$$

$$- \sum_{k=1}^n \langle \nabla_{X_k} X_n, X_k \rangle (\lambda X_k - AX_k).$$

Now using (39) we get

$$\sum_{k \not\in B_\rho} \langle \nabla_{X_k} X_i, X_k \rangle (\mu_\rho - \mu_k) X_k - \sum_{k \neq n} \langle \nabla_{X_k} X_n, X_k \rangle (\lambda - \mu_k) X_k - cv b_i X_n + cv b_n X_i = 0.$$  

Then

$$cv b - \langle \nabla_{X_i} X_n, X_i \rangle (\lambda - \mu) = 0, \quad \forall i \in B_\rho,$$  

and $b = 0 \Leftrightarrow \langle \nabla_{X_i} X_n, X_i \rangle = 0$ for $i \in B_\rho$.

By (3), $\nu \mu_\rho X_i = \nabla_{X_i} T = \nabla_{X_i} b X_n = X_i(b) X_n + b \nabla_{X_i} X_n$, $i \in \beta_\rho$, which implies that $\nu \mu_\rho \langle X_i, X_i \rangle = X_i(b) \langle X_n, X_i \rangle + b \langle \nabla_{X_i} X_n, X_i \rangle$, i.e., $b \langle \nabla_{X_i} X_n, X_i \rangle = \nu \mu_\rho$. Then $cv b \langle \nabla_{X_i} X_n, X_i \rangle = cv^2 \mu_\rho$ and by equation (42),

$$\langle \nabla_{X_i} X_n, X_i \rangle^2 (\lambda - \mu) = cv^2 \mu_\rho.$$  

Consequently

$$b = 0 \Leftrightarrow \langle \nabla_{X_i} X_n, X_i \rangle = 0, \quad \forall i \in B_\rho \Leftrightarrow \mu_\rho = 0.$$  

If $\mu_\rho = 0$ then $b = 0$ and $T = 0$, that is, $f(M)$ is totally geodesic, which is impossible since $g \geq 2$. Then $T$ is a principal direction and all the curvatures of multiplicity greater than one do not vanish.

From Theorem 6.2 we obtain the converse of Proposition 7.4 and next result is also true.

**Proposition 7.5.** Let $f : M^n \to \mathbb{Q}_c^n \times \mathbb{R}$, $n \geq 3$ with function $\nu \neq 0$ be a hypersurface having constant principal curvatures with constant multiplicities. Then the vector field $T$ is a principal direction if and only if there exist only one principal curvature of multiplicity one. Moreover, all the curvatures of multiplicity greater than one do not vanish.
8 Hypersurfaces of $\mathbb{Q}^n_c \times \mathbb{R}$ with constant principal curvatures for $g \in \{1, 2, 3\}$

In this section we present a result that classifies hypersurfaces with constant principal curvatures. For this we need some propositions.

**Proposition 8.1.** Let $f : M^2 \to \mathbb{Q}^n_c \times \mathbb{R}$ be a surface with two distinct constant principal curvatures $\lambda_1$ and $\lambda_2$. Let $\{X_1, X_2\}$ be an orthonormal frame of principal directions corresponding to $\lambda_1$ and $\lambda_2$. Consider $T = b_1 X_1 + b_2 X_2$, where $b_1, b_2 : M^2 \to \mathbb{R}$ are differentiable functions. Then

$$\lambda_1 \lambda_2 + 2cv^2 + \frac{c(\lambda_1 b_1^2 - \lambda_2 b_2^2)}{\lambda_2 - \lambda_1} + \frac{2v^2(b_1^2 + b_2^2)}{(\lambda_2 - \lambda_1)^2} = 0. \quad (43)$$

**Proof.** From Codazzi equation we get

$$\nabla_{X_1} A X_2 - \nabla_{X_2} A X_1 - A[X_1, X_2] = cv(b_2 X_1 - b_1 X_2). \quad (44)$$

Observe that

$$\nabla_{X_1} A X_2 = \lambda_2 \nabla_{X_1} X_2 = \lambda_2 (\nabla_{X_1} X_2, X_1) X_1,$$

since $X_1 \langle X_2, X_2 \rangle = 0$. In a similar way we get

$$\nabla_{X_2} A X_1 = \lambda_1 (\nabla_{X_2} X_1, X_2) X_2.$$

Thus,

$$A[X_1, X_2] = A(\nabla_{X_1} X_2 - \nabla_{X_2} X_1) = \lambda_1 (\nabla_{X_1} X_2, X_1) X_1 - \lambda_2 (\nabla_{X_2} X_1, X_2) X_2.$$

From equation (44),

$$(\lambda_2 - \lambda_1) (\nabla_{X_1} X_2, X_1) X_1 + (\lambda_2 - \lambda_1) (\nabla_{X_2} X_1, X_2) X_2 = cv(b_2 X_1 - b_1 X_2),$$

which implies, as $X_1$ and $X_2$ are linearly independent fields, that

$$(\lambda_2 - \lambda_1) \langle \nabla_{X_1} X_2, X_1 \rangle = cvb_2 \quad \text{and} \quad (\lambda_2 - \lambda_1) \langle \nabla_{X_2} X_1, X_2 \rangle = -cvb_1,$$

that is,

$$\nabla_{X_1} X_2 = \frac{cvb_2}{(\lambda_2 - \lambda_1)} X_1,$$

$$\nabla_{X_2} X_1 = \frac{-cvb_1}{(\lambda_2 - \lambda_1)} X_2. \quad (45)$$

Thus

$$\nabla_{X_1} X_1 = \frac{-cvb_2}{(\lambda_2 - \lambda_1)} X_2,$$

$$\nabla_{X_2} X_2 = \frac{cvb_1}{(\lambda_2 - \lambda_1)} X_1. \quad (46)$$

By (43), we get

$$\nabla_{X_1} T = \nabla_{X_1} (b_1 X_1 + b_2 X_2) = X_1(b_1) X_1 + b_1 \nabla_{X_1} X_1 + X_1(b_2) X_2 + b_2 \nabla_{X_1} X_2 = \nu \lambda_1 X_1 \quad \text{and}$$
\( \nabla_{X_2} T = \nabla_{X_2} (b_1 X_1 + b_2 X_2) = X_2(b_1)X_1 + b_1 \nabla_{X_2} X_1 + X_2(b_2)X_2 + b_2 \nabla_{X_2} X_2 = \nu \lambda_2 X_2. \)

Making the inner product of both equalities above with \( X_1 \) and \( X_2 \), we conclude

\[
X_1(b_2) = b_1 \langle \nabla_{X_1} X_2, X_1 \rangle = \frac{c \nu b_1 b_2}{\lambda_2 - \lambda_1},
\]

\[
X_2(b_1) = b_2 \langle \nabla_{X_2} X_1, X_2 \rangle = \frac{-c \nu b_1 b_2}{\lambda_2 - \lambda_1}.
\]

Therefore \( X_1(b_2) = -X_2(b_1) \). Moreover,

\[
X_1(b_1) + b_2 \langle \nabla_{X_1} X_2, X_1 \rangle = \nu \lambda_1,
\]

\[
X_2(b_2) + b_1 \langle \nabla_{X_2} X_1, X_2 \rangle = \nu \lambda_2.
\]

From equations (47) and (48) we obtain

\[
X_1(b_1) = \nu \lambda_1 - \frac{c \nu (b_2)^2}{\lambda_2 - \lambda_1},
\]

\[
X_2(b_2) = \nu \lambda_2 + \frac{c \nu (b_1)^2}{\lambda_2 - \lambda_1}.
\]

So, by (49),

\[
X_1(\nu) = -\lambda_1 b_1,
\]

\[
X_2(\nu) = -\lambda_2 b_2.
\]

Using now Gauss equation we obtain

\[
\langle R(X_1, X_2) X_2, X_1 \rangle = \lambda_1 \lambda_2 + c \nu^2.
\]

Observe that

\[
\nabla_{[X_1, X_2]} X_2 = \nabla(\nabla_{X_1} X_2 - \nabla_{X_2} X_1) X_2
\]

\[
= \nabla(\langle \nabla_{X_1} X_2, X_1 \rangle - \langle \nabla_{X_2} X_1, X_2 \rangle) X_2
\]

\[
= \langle \nabla_{X_1} X_2, X_1 \rangle \nabla_{X_1} X_2 - \langle \nabla_{X_2} X_1, X_2 \rangle \nabla_{X_2} X_2.
\]

Then

\[
\langle \nabla_{[X_1, X_2]} X_2, X_1 \rangle = \langle \nabla_{X_1} X_2, X_1 \rangle^2 + \langle \nabla_{X_2} X_1, X_2 \rangle^2.
\]

From equations (51) and (45), we obtain

\[
\langle \nabla_{X_1} \nabla_{X_2} X_2 - \nabla_{X_2} \nabla_{X_1} X_2, X_1 \rangle = \lambda_1 \lambda_2 + c \nu^2 + \frac{\nu^2 (b_1^2 + b_2^2)}{(\lambda_2 - \lambda_1)^2}.
\]

Observe, by equations (45) that \( \langle \nabla_{X_1} X_2, \nabla_{X_1} X_2 \rangle = 0 \) and so \( X_2 \langle \nabla_{X_1} X_2, X_1 \rangle = \langle \nabla_{X_2} \nabla_{X_1} X_2, X_1 \rangle \). In order to compute \( \langle \nabla_{X_2} \nabla_{X_1} X_2, X_1 \rangle \) we derive the first equality of (45) with respect to \( X_2 \) and use also the second equalities of equations (49) and (50) getting

\[
- \langle \nabla_{X_2} \nabla_{X_1} X_2, X_1 \rangle = \frac{-c \lambda_2 (\nu^2 - b_2^2)}{\lambda_2 - \lambda_1} + \frac{-\nu^2 b_1^2}{(\lambda_2 - \lambda_1)^2}.
\]
From equations (47) it follows that \( \langle \nabla_{X_2}X_2, \nabla_{X_1}X_1 \rangle = 0 \) and so \( X_1\langle \nabla_{X_2}X_2, X_1 \rangle = \langle \nabla_{X_1}X_2, X_2, X_1 \rangle \). In order to compute \( \langle \nabla_{X_1}X_2, X_2, X_1 \rangle \) we derive the second equality of (47) with respect to \( X_1 \) and use the first equation of (49) and (50) obtaining
\[
\langle \nabla_{X_1}X_2, X_1 \rangle = \frac{c\lambda_1(\nu^2 - b_1^2)}{\lambda_2 - \lambda_1} - \frac{\nu^2 b_2^2}{(\lambda_2 - \lambda_1)^2}.
\]
(54)

By summing equations (54) and (53), we get
\[
\langle \nabla_{X_1}X_2 - X_1, \nabla_{X_2}X_2, X_1 \rangle = -c\nu^2 + \frac{c(\lambda_2b_2^2 - \lambda_1b_1^2)}{\lambda_2 - \lambda_1} - \frac{\nu^2(b_1^2 + b_2^2)}{(\lambda_2 - \lambda_1)^2}.
\]
(55)

Finally from equations (55) and (52) we conclude that
\[
\lambda_1\lambda_2 + 2c\nu^2 + \frac{c(\lambda_1b_1^2 - \lambda_2b_2^2)}{\lambda_2 - \lambda_1} + \frac{2\nu^2(b_1^2 + b_2^2)}{(\lambda_2 - \lambda_1)^2} = 0.
\]

Next result shows that a minimal surface of \( \mathbb{Q}_c^2 \times \mathbb{R} \) with principal constant curvatures is totally geodesic.

**Corollary 8.2.** The minimal surfaces of \( \mathbb{Q}_c^2 \times \mathbb{R} \) with principal constant curvatures are totally geodesic.

**Proof.** Suppose that there exist a minimal surface with two distinct constant principal curvatures \( \lambda_2 = -\lambda_1 \). From Proposition 8.1 we get
\[
-\lambda_1^2 + 2c\nu^2 - \frac{c\lambda_1(b_1^2 + b_2^2)}{2\lambda_1} + \frac{2\nu^2(b_1^2 + b_2^2)}{4\lambda_1^2} = 0.
\]

We already know that \( \nu^2 + b_1^2 + b_2^2 = 1 \) and thus
\[
-\lambda_1^2 + 2c\nu^2 + \frac{c(\nu^2 - 1)}{2} + \frac{2\nu^2(1 - \nu^2)}{4\lambda_1^2} = 0,
\]
that is,
\[
\nu^4 - \nu^2(1 + 5c\lambda_1^2) + \lambda_1^2(2\lambda_1^2 + c) = 0.
\]
(56)

So we obtain a biquadratic equation on the variable \( \nu \), with constant real coefficients. If equation (56) has a solution then the function \( \nu \) is constant and consequently \( 0 = X_i(\nu) = -\langle AX_i, T \rangle = -b_i\lambda_i \), with \( i \in \{1, 2\} \). If \( \lambda_1 = 0 \) then \( \lambda_2 = -\lambda_1 = 0 \), but this cannot occur since we are assuming \( \lambda_1 \neq \lambda_2 \). Then \( b_1 = b_2 = 0 \) and \( T = 0 \). Thus there does not exist a minimal surface in \( \mathbb{Q}_c^2 \times \mathbb{R} \) with two distinct constant principal curvatures.

Proposition below shows that a hypersurface in \( \mathbb{Q}_c^n \times \mathbb{R} \), \( n \geq 4 \), with \( \nu \neq 0 \), that has three constant principal curvatures of constant multiplicities may not have two principal curvatures of multiplicity one.

**Proposition 8.3.** Let \( f: M^n \to \mathbb{Q}_c^n \times \mathbb{R}, n \geq 4 \), be a hypersurface with three constant distinct principal curvatures \( \lambda, \mu \), and \( \gamma \) of constant multiplicities and suppose that \( \nu(p) \neq 0 \), for all \( p \in M^n \). Then there do not exist two principal curvatures of multiplicity one.
Proof. Suppose there exist two principal curvatures $\lambda$ and $\mu$ of multiplicity one. Let \(\{X_1, X_2, \ldots, X_n\}\) be a frame of principal orthonormal directions such that \(AX_1 = \lambda X_1\), \(AX_2 = \mu X_2\) and \(AX_j = \gamma X_j\), for \(j \geq 3\). From Remark 7.2 we obtain \(T = b_1 X_1 + b_2 X_2\), where \(b_1, b_2: U \to \mathbb{R}\), \(U \subset M^n\) are differentiable functions.

From Codazzi equations, given in (3) we get
\[
\nabla_{X_1} AX_2 - \nabla_{X_2} AX_1 - A[X_1, X_2] = cv(b_2 X_1 - b_1 X_2),
\]
(57)
\[
\nabla_{X_1} AX_j - \nabla_{X_j} AX_1 - A[X_1, X_j] = -cvb_1 X_j, \text{ for each } j \in \{3, \ldots, n\},
\]
(58)
\[
\nabla_{X_2} AX_j - \nabla_{X_j} AX_2 - A[X_2, X_j] = -cvb_2 X_j, \text{ for each } j \in \{3, \ldots, n\},
\]
(59)
\[
\nabla_{X_\beta}\! AX_j - \nabla_{X_j} AX_\beta - A[X_\beta, X_j] = 0, \text{ for } j \in \{3, \ldots, n\} \text{ and } \beta \neq 1, 2, j.
\]
(60)

From equation (57), we obtain
\[
\mu \nabla_{X_1} X_2 - \lambda \nabla_{X_2} X_1 - A(\nabla_{X_1} X_2 - \nabla_{X_2} X_1) = cv(b_2 X_1 - b_1 X_2),
\]
that is,
\[
\sum_{k=1}^{n} \langle \nabla_{X_1} X_2, X_k \rangle (\mu I - A) X_k + \sum_{l=1}^{n} \langle \nabla_{X_2} X_1, X_l \rangle (A - \lambda I) X_l - cvb_2 X_1 + cvb_1 X_2 = 0.
\]

Thus,
\[
\langle \nabla_{X_1} X_2, X_1 \rangle (\mu - \lambda) - cvb_2 = 0,
\]
(61)
\[
\langle \nabla_{X_2} X_1, X_2 \rangle (\mu - \lambda) + cvb_1 = 0
\]
and
\[
\langle \nabla_{X_1} X_2, X_j \rangle (\mu - \gamma) + \langle \nabla_{X_2} X_1, X_j \rangle (\gamma - \lambda) = 0, \text{ for each } j \in \{3, \ldots, n\}.
\]
(62)

Proceeding analogously with equations (58), (59) and (60) from equation (58) we obtain for each \(j \in \{3, \ldots, n\},\)
\[
\langle \nabla_{X_1} X_j, X_1 \rangle (\gamma - \lambda) = 0,
\]
(63)
\[
\langle \nabla_{X_j} X_1, X_j \rangle (\gamma - \lambda) + cvb_1 = 0,
\]
(64)
\[
\langle \nabla_{X_j} X_1, X_\beta \rangle (\gamma - \lambda) = 0, \beta \neq 1, 2, j, \text{ and}
\]
(65)
\[
\langle \nabla_{X_1} X_j, X_2 \rangle (\gamma - \mu) + \langle \nabla_{X_j} X_1, X_2 \rangle (\mu - \lambda) = 0.
\]
(66)

By equation (59) we conclude for each \(j \in \{3, \ldots, n\}\) that
\[
\langle \nabla_{X_1} X_2, X_j \rangle (\gamma - \mu) = 0,
\]
(67)
\[
\langle \nabla_{X_j} X_2, X_j \rangle (\gamma - \mu) + cvb_2 = 0,
\]
(68)
\[
\langle \nabla_{X_j} X_2, X_\beta \rangle (\gamma - \mu) = 0, \beta \neq 1, 2, j, \text{ and}
\]
(69)
\[
\langle \nabla_{X_2} X_j, X_1 \rangle (\gamma - \lambda) + \langle \nabla_{X_j} X_2, X_1 \rangle (\lambda - \mu) = 0.
\]
(70)

By using equation (60) we get for each \(j \in \{3, \ldots, n\}\) and \(\beta \neq 1, 2, j,\)
\[
\langle \nabla_{X_\beta} X_j, X_1 \rangle - \langle \nabla_{X_j} X_\beta, X_1 \rangle = 0,
\]
(71)
\[
\langle \nabla_{X_\beta} X_j, X_2 \rangle - \langle \nabla_{X_j} X_\beta, X_2 \rangle = 0.
\]

From equations (63), (69) and (71), it follows that
\[
\langle \nabla_{X_\beta} X_j, X_1 \rangle = 0 \text{ and } \langle \nabla_{X_\beta} X_j, X_2 \rangle = 0.
\]
(72)
from equations (61) to (70) and (72) we conclude, for each \( j \in \{3, \ldots, n \} \), that

\[
\nabla X_1 X_1 = -\frac{c v b_2}{\mu - \lambda} X_2,
\]

\[
\nabla X_1 X_2 = \frac{c v b_2}{\mu - \lambda} X_1 + \sum_{j=3}^{n} \langle \nabla X_2 X_1, X_j \rangle \frac{(\lambda - \gamma)}{\mu - \gamma} X_j,
\]

\[
\nabla X_1 X_j = \langle \nabla X_j X_1, X_2 \rangle \frac{(\lambda - \mu)}{\gamma - \mu} X_1 + \sum_{\beta \neq 1,2,j} \langle \nabla X_1 X_j, X_\beta \rangle X_\beta,
\]

\[
\nabla X_2 X_1 = -\frac{c v b_1}{\mu - \lambda} X_2 + \sum_{j=3}^{n} \langle \nabla X_2 X_1, X_j \rangle X_j,
\]

\[
\nabla X_2 X_2 = \frac{c v b_1}{\mu - \lambda} X_1,
\]

\[
\nabla X_2 X_j = \langle \nabla X_j X_2, X_1 \rangle \frac{(\mu - \gamma)}{\gamma - \lambda} X_1 + \sum_{\beta \neq 1,2,j} \langle \nabla X_2 X_j, X_\beta \rangle X_\beta,
\]

\[
\nabla X_j X_1 = \langle \nabla X_j X_1, X_2 \rangle X_2 - \frac{c v b_1}{\gamma - \lambda} X_j,
\]

\[
\nabla X_j X_2 = \langle \nabla X_j X_2, X_1 \rangle X_1 - \frac{c v b_2}{\gamma - \mu} X_j,
\]

\[
\nabla X_j X_j = \frac{c v b_1}{\gamma - \lambda} X_1 + \frac{c v b_2}{\gamma - \mu} X_2 + \sum_{\beta \neq 1,2,j} \langle \nabla X_j X_j, X_\beta \rangle X_\beta,
\]

\[
\nabla X_\beta X_j = \langle \nabla X_\beta X_j, X_\beta \rangle X_\beta + \sum_{l \neq 1,2,j,\beta} \langle \nabla X_\beta X_j, X_l \rangle X_l.
\]

From equalities (3) and (4), for all \( X \in TM^n \), we get

\[
X_1(\nu) = -\lambda b_1,
\]

\[
X_2(\nu) = -\mu b_2,
\]

\[
X_j(\nu) = 0, \quad j \in \{3, \ldots, n \}.
\]

Moreover from

\[
\nabla X_1 (b_1 X_1 + b_2 X_2) = X_1 (b_1) X_1 + b_1 \nabla X_1 X_1 + X_1 (b_2) X_2 + b_2 \nabla X_1 X_2 = \nu \lambda X_1,
\]

we obtain

\[
X_1 (b_1) = \nu \lambda - \frac{c v b_2}{\mu - \lambda},
\]

\[
X_1 (b_2) = \frac{c v b_1 b_2}{\mu - \lambda},
\]

\[
b_2 \langle \nabla X_1 X_2, X_j \rangle = 0, \quad \text{for each } j \in \{3, \ldots, n \}.
\]

Analogously, deriving \( T \) with respect to \( X_2 \), we obtain

\[
\nabla X_2 (b_1 X_1 + b_2 X_2) = X_2 (b_1) X_1 + b_1 \nabla X_2 X_1 + X_2 (b_2) X_2 + b_2 \nabla X_2 X_2 = \nu \mu X_2,
\]

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and therefore,

\[ X_2(b_1) = -\frac{cvb_1b_2}{\mu - \lambda}, \]  
\[ X_2(b_2) = \nu\mu + \frac{cvb_1^2}{\mu - \lambda}, \]  
\[ b_1\langle \nabla X_2 X_1, X_j \rangle = 0, \quad \text{for each } j \in \{3, \ldots, n\}. \]  

On one hand using Gauss equation (5), we obtain

\[ K(X_1, X_2) = \lambda\mu + cv^2, \]  
\[ K(X_1, X_j) = \lambda\gamma + c(1 - b_1^2), \quad j \in \{3, \ldots, n\}, \]  
\[ K(X_2, X_j) = \mu\gamma + c(1 - b_2^2), \quad j \in \{3, \ldots, n\}. \]

On the other hand, we know that

\[ K(X_1, X_2) = \langle R(X_1, X_2)X_2, X_1 \rangle = \langle \nabla X_1 \nabla X_2 X_2 - \nabla X_2 \nabla X_1 X_2 - \nabla_{[X_1, X_2]} X_2, X_1 \rangle. \]

From equations (73) and (77) we get

\[ \langle \nabla X_1 \nabla X_2 X_2, X_1 \rangle = X_1 \left( \frac{cvb_1}{\mu - \lambda} \right), \]

and from (83) and (84),

\[ \langle \nabla X_1 \nabla X_2 X_2, X_1 \rangle = c\lambda \left( \frac{\nu^2 - b_1^2}{\mu - \lambda} - \frac{\nu^2b_2^2}{(\mu - \lambda)^2} \right). \]

By using now equations (74), (76), (83) and (88), we arrive to

\[ -\langle \nabla X_2 \nabla X_1, X_2 \rangle = \sum_{j=3}^n \langle \nabla X_2 X_1, X_j \rangle \left( \frac{\lambda - \gamma}{\mu - \gamma} \right) - X_2 \left( \frac{cvb_2}{\mu - \lambda} \right) \]

and so

\[ -\langle \nabla X_2 \nabla X_1, X_2 \rangle = \sum_{j=3}^n \langle \nabla X_2 X_1, X_j \rangle \left( \frac{\lambda - \gamma}{\mu - \gamma} \right) - \frac{c\mu(\nu^2 - b_2^2)}{\mu - \lambda} - \frac{\nu^2b_2^2}{(\mu - \lambda)^2}. \]  

Observe that

\[ \nabla_{[X_1, X_2]} X_2 = \frac{cvb_2}{\mu - \lambda} \nabla X_1 X_2 + \sum_{j=3}^n \langle \nabla X_2 X_1, X_j \rangle \left( \frac{\lambda - \gamma}{\mu - \gamma} \right) \nabla X_j X_2 \]

\[ + \frac{cvb_1}{\mu - \lambda} \nabla X_2 X_2 - \sum_{j=3}^n \langle \nabla X_2 X_1, X_j \rangle \nabla X_j X_2. \]

Thus,

\[ -\langle \nabla_{[X_1, X_2]} X_2, X_1 \rangle = -\frac{\nu^2(b_1^2 + b_2^2)}{(\mu - \lambda)^2} + \sum_{j=3}^n \langle \nabla X_2 X_1, X_j \rangle \langle \nabla X_j X_2, X_1 \rangle \left( 1 - \frac{\lambda - \gamma}{\mu - \gamma} \right). \]
From (38) and (80), we get
\[ \langle \nabla X_j X_2, X_1 \rangle = \langle \nabla X_2 X_j, X_1 \rangle \frac{(\gamma - \lambda)}{\mu - \lambda} \]
and so
\[ - \langle [X_1, X_2] X_2, X_1 \rangle = - \frac{\nu^2 (b_1^2 + b_2^2)}{(\mu - \lambda)^2} - \sum_{j=3}^{n} \langle \nabla X_j X_2, X_1 \rangle^2 \frac{(\gamma - \lambda)}{\mu - \lambda} \left( 1 - \frac{(\lambda - \gamma)}{\mu - \gamma} \right) . \]  
(95)

By summing (93) with (94) and (95) and comparing with (90), we get
\[ \frac{2\nu^2(1 - \nu^2)}{(\mu - \lambda)^2} + 2c\nu^2 + \lambda \mu + \frac{c(b_1^2 \lambda - b_2^2 \mu)}{\mu - \lambda} - \frac{2(\lambda - \gamma)}{\mu - \gamma} \sum_{j=3}^{n} \langle \nabla X_2 X_1, X_j \rangle^2 = 0. \]  
(96)

Proceeding analogously for $K(X_1, X_j)$ and $K(X_2, X_j)$ we conclude for each $j \geq 3$, that
\[ \frac{c\lambda(\nu^2 - b_1^2)}{\gamma - \lambda} - \frac{\nu^2}{\gamma - \lambda} \left( \frac{b_1^2}{\mu - \lambda} + \frac{b_2^2}{\mu - \lambda} \right) + \frac{\nu^2 b_2^2}{(\mu - \lambda)(\gamma - \mu)} - \lambda \gamma - c(1 - b_1^2) \]
\[ - \frac{2(\mu - \lambda)}{\gamma - \mu} \langle \nabla X_j X_1, X_2 \rangle^2 = 0, \]  
(97)
\[ \frac{c\mu(\nu^2 - b_2^2)}{\gamma - \mu} + \frac{\nu^2}{\gamma - \mu} \left( \frac{b_1^2}{\mu - \lambda} - \frac{b_2^2}{\gamma - \mu} \right) - \frac{\nu^2 b_1^2}{(\gamma - \lambda)(\mu - \lambda)} - \mu \gamma - c(1 - b_2^2) \]
\[ + \frac{2(\mu - \lambda)}{\gamma - \lambda} \langle \nabla X_j X_1, X_2 \rangle^2 = 0. \]  
(98)

From (89) we get $b_1 \langle \nabla X_j X_1, X_j \rangle = 0$, for each $j \in \{3, \ldots, n\}$. Suppose that $b_1(p) = 0$, for all $p \in U \subset M^n$. Then $T$ is a principal direction. As we are supposing $\nu \neq 0$ from Theorem 6.2 we get $c = -1$ and $g = 2$, which is against the hypothesis $g = 3$. So it must exist a $p_0$ such that $b_1(p_0) \neq 0$. Since the function $b_1$ is continuous there exist a neighborhood $V \subset U \subset M^n$ of $p_0$ such that $b_1(p) \neq 0$ for all $p \in V$. Thus $\langle \nabla X_j X_1, X_j \rangle = 0$ in $V$, for each $j \in \{3, \ldots, n\}$.

From equations (62) and (66) we conclude that
\[ \langle \nabla X_2 X_1, X_j \rangle = 0 \iff \langle \nabla X_j X_2, X_2 \rangle = 0 \iff \langle \nabla X_2 X_1, X_2 \rangle = 0, \quad \text{for each } j \in \{3, \ldots, n\}. \]

Therefore, from equations (96), (97) and (98) we obtain in $V$, respectively,
\[ \frac{2\nu^2(1 - \nu^2)}{(\mu - \lambda)^2} + 2c\nu^2 + \lambda \mu + \frac{c(b_1^2 \lambda - b_2^2 \mu)}{\mu - \lambda} = 0, \]  
(99)
\[ \frac{c\lambda(\nu^2 - b_1^2)}{\gamma - \lambda} - \frac{\nu^2 b_1^2}{(\gamma - \lambda)^2} + \frac{\nu^2 b_2^2}{(\gamma - \mu)(\gamma - \lambda)} - \lambda \gamma - c(1 - b_1^2) = 0, \]  
(100)
\[ \frac{c\mu(\nu^2 - b_2^2)}{\gamma - \mu} - \frac{\nu^2 b_1^2}{(\gamma - \mu)^2} + \frac{\nu^2 b_2^2}{(\gamma - \mu)(\gamma - \lambda)} - \mu \gamma - c(1 - b_2^2) = 0. \]  
(101)
Replacing $b_2$ by $1 - \nu^2 - b_1^2$ in equation (99), we get

$$-cb_1^2\frac{\lambda + \mu}{\mu - \lambda} = \lambda\mu + 2c\nu^2 - \frac{c\mu(1 - \nu^2)}{\mu - \lambda} + \frac{2\nu^2(1 - \nu^2)}{(\mu - \lambda)^2}.$$ 

With analogous arguments used in Proposition 8.2 we conclude that $\lambda + \mu \neq 0$. Then

$$b_1^2 = \nu^4 \frac{2c}{(\mu + \lambda)(\mu - \lambda)} + \nu^2\frac{(\mu - \lambda)(-3\mu + 2\lambda) - 2c}{(\mu + \lambda)(\mu - \lambda)} + \frac{\mu}{\mu + \lambda}(1 - c\lambda(\mu - \lambda)).$$

By summing (100) with (101) we get

$$c\nu^2\left(\frac{\lambda}{\gamma - \lambda} + \frac{\mu}{\gamma - \mu}\right) - c\lambda b_1^2 - \frac{c\mu b_1^2}{\gamma - \mu} - \frac{\nu^2 b_1^2}{\gamma - \lambda} - \frac{\nu^2 b_1^2}{(\gamma - \mu)^2} + \frac{\nu^2(1 - \nu^2)}{(\gamma - \mu)(\gamma - \lambda)} - \gamma(\mu + \lambda) - c(1 + \nu^2) = 0. \tag{103}$$

Now replacing $b_2$ by $1 - \nu^2 - b_1^2$ in (103) we get

$$\nu^4 \frac{(\mu - \lambda)}{(\gamma - \mu)^2(\gamma - \lambda)} + \nu^2\left(\frac{-(\mu - \lambda) + c\lambda(\gamma - \mu) + 2c\mu(\gamma - \mu)(\gamma - \lambda)}{(\gamma - \mu)^2(\gamma - \lambda)} - c\right) - \frac{c\gamma}{\gamma - \mu} - \gamma(\mu + \lambda) + cb_1^2\left(\frac{\mu - \lambda}{\gamma - \mu} - \frac{\lambda}{\gamma - \lambda}\right) + \nu^2 b_1^2\left(\frac{1}{(\gamma - \mu)^2} - \frac{1}{(\gamma - \lambda)^2}\right) = 0. \tag{104}$$

After replacing (102) in (104) we observe that the greatest power of $\nu$ in the last equation is $\nu^6$ whose coefficient is $\frac{1}{(\gamma - \mu)^2} - \frac{1}{(\gamma - \lambda)^2}(\mu + \lambda)(\mu - \lambda)$, which is against the hypothesis $\lambda \neq 0$. If that term is different from zero the equation has grade six and has constant real coefficients in the variable $\nu$. If not, i.e. if $(\gamma - \lambda)^2 = (\gamma - \mu)^2$ we have $\gamma - \lambda = \gamma - \mu$ or $\gamma - \lambda = -(\gamma - \mu)$. If $\gamma - \lambda = \gamma - \mu$ then $\lambda = \mu$ which is against the hypothesis $\lambda \neq 0$. If $\gamma - \lambda = -(\gamma - \mu)$ then $\mu - \lambda = 2(\gamma - \mu)$. Finally, the greatest power of $\nu$ would be of fourth order and the coefficient of $\nu^4$ is $\frac{1}{(\gamma - \mu)^2}$ which does not vanish. In that case we would have a biquadratic equation with real constant coefficients on the variable $\nu$.

Therefore, assuming the existence of two constant principal curvatures with multiplicity one and $\nu(p) \neq 0$ for all $p \in M^n$ the function $\nu$ must satisfy equation (104). In that case the function $\nu$ would be constant and consequently $T$ would be a principal direction. Now using Theorem 6.2 we would obtain $c = -1$ and $g = 2$, against the hypothesis $g = 3$.

Finally the conclusion is that there does not exist two principal constant curvatures with multiplicity one if we suppose also $\nu(p) \neq 0$, for all $p \in M^n$.

Taking into account the previous results we are ready to prove the local classification of hypersurfaces in $\mathbb{Q}^n_c \times \mathbb{R}$, $n \neq 3$, having constant principal curvatures and $g \in \{1, 2, 3\}$.

**Theorem 8.4.** Let $f : M^n \to \mathbb{Q}^n_c \times \mathbb{R}$ be a hypersurface with constant principal curvatures.

(i) If $g = 1$ and $n \geq 2$ then $f(M^n)$ is an open subset of $\mathbb{Q}^n_c \times \{t_0\}$, for any $t_0 \in \mathbb{R}$ or an open subset of a Riemannian product $M^{n-1} \times \mathbb{R}$. In the last case if $c = 1$, $M^{n-1}$ is a totally geodesic sphere in $\mathbb{S}^n$ and if $c = -1$, $M^{n-1}$ is a totally geodesic hyperplane in $\mathbb{H}^n$. 

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(ii) If \( g = 2 \) and \( n \geq 2 \) then \( c = -1 \) and \( f \) is locally given by \( f(x, s) = g_s(x) + Bs\partial/\partial t \), for some \( B \in \mathbb{R}, B > 0 \), with \( M^n = M^{n-1} \times I \), where \( g_s \) is a family of horospheres in \( \mathbb{H}^n \), or \( f(M^n) \) is an open subset of a Riemannian product \( M^{n-1} \times \mathbb{R} \). In the last case, if \( c = 1 \) then \( M^{n-1} \) is a non totally geodesic sphere in \( S^n \) and if \( c = -1 \), \( M^{n-1} \) is an equidistant hypersurface, a horosphere or a hypersphere in \( \mathbb{H}^n \).

(iii) Suppose that \( g = 3, n \geq 4 \) and the multiplicities of the principal curvatures are constant. Then if \( c = 1 \), \( f(M^n) \) is an open subset of a Riemannian product \( S^p(r) \times S^q(s) \times \mathbb{R} \), with \( n = p + q + 1 \) and \( r^2 + s^2 = 1 \) or an open subset of the product \( M^{n-1} \times \mathbb{R}, \) where \( M^{n-1} \) is a Cartan’s hypersurface with \( n \in \{4, 7, 13, 25\} \). If \( c = -1 \), \( f(M^n) \) is an open subset of the Riemannian product \( S^k \times \mathbb{H}^{n-k-1} \times \mathbb{R} \).

**Proof.** (i) Suppose that \( g = 1 \). If \( \nu \equiv 0 \) from Remark 24 we infer that the field \( T \) is a principal direction with corresponding principal curvature \( \lambda = 0 \). Then \( f \) an umbilical immersion implies that all the principal curvatures are zero, i.e. \( f \) is totally geodesic. In this case, \( f \) is an open subset of a Riemannian product \( M^{n-1} \times \mathbb{R} \), where \( M^{n-1} \) is a totally geodesic hypersurface in \( Q^c_n \).

Let us now suppose that \( \nu(p) \neq 0 \), for all \( p \in M \). Take \( \{X_1, X_2, ..., X_n\} \) a local orthonormal frame field of the immersion \( f \). We can write \( T = \sum_{i=1}^{n} b_i X_i \). From hypothesis we have \( AX_i = \lambda X_i \) for all \( i \in \{1, ..., n\} \) where \( \lambda \) is constant in \( \mathbb{R} \).

From Codazzi’s equations we get,

\[
\nabla_{X_i} AX_j - \nabla_{X_j} AX_i - A[X_i, X_j] = c\nu (b_j X_i - b_i X_j),
\]

which implies \( 0 = \lambda [X_i, X_j] - \lambda [X_i, X_j] = c\nu (b_j X_i - b_i X_j) \). Since \( \nu \neq 0 \) and the fields \( X_i, X_j \) are linearly independent for \( i \neq j \) it follows that \( b_i = b_j = 0 \) for all \( i, j \in \{1, ..., n\} \). Then \( T = 0 \) and \( f(M^n) \) is locally an open subset of \( Q^c_n \times \{t\} \), which proves item (i).

(ii) Suppose now \( g = 2 \). By 20 Proposition 2.2 the multiplicities of the principal curvatures are constant. Let us consider two subcases \( n = 2 \) and \( n \geq 3 \).

First case: \( n = 2 \)
Replace \( b_2^2 = 1 - \nu^2 - b_1^2 \) in equation (143) to get,

\[
-cb_1 \left( \frac{\lambda_1 + \lambda_2}{\lambda_2 - \lambda_1} \right) = \lambda_1 \lambda_2 + 2c\nu^2 - \frac{c\lambda_2(1 - \nu^2)}{\lambda_2 - \lambda_1} + \frac{2\nu^2(1 - \nu^2)}{(\lambda_2 - \lambda_1)^2}.
\]

From Corollary 8.2 we obtain \( \lambda_1 + \lambda_2 \neq 0 \) and so,

\[
b_1^2 = -c \left( \frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2} \right) \left\{ \lambda_1 \lambda_2 + 2c\nu^2 - \frac{c\lambda_2(1 - \nu^2)}{\lambda_2 - \lambda_1} + \frac{2\nu^2(1 - \nu^2)}{(\lambda_2 - \lambda_1)^2} \right\}. \tag{105}
\]

Replace also \( b_1^2 = 1 - \nu^2 - b_2^2 \) on equation (143) to get,

\[
-2\nu^4 + \nu^2 D + b_2^2 E = F, \tag{106}
\]

where \( D = 2c(\lambda_2 - \lambda_1)^2 - c\lambda_1(\lambda_2 - \lambda_1) + 2, E = -c(\lambda_2^2 - \lambda_1^2) \) and \( F = -\lambda_1 \lambda_2 (\lambda_2 - \lambda_1)^2 - c\lambda_1 (\lambda_2 - \lambda_1) \). Deriving equation (105) with respect to \( X_2 \) we get

\[
-8\nu^3 X_2(\nu) + 2\nu X_2(\nu) D + 2b_2 X_2(b_2) E = 0. \tag{107}
\]
If $\lambda_2 = 0$ from equation (50), $X_2(\nu) = 0$ and from equation (107), $b_2 X_2(b_2) = 0$. Then $b_2 = 0$ or the expression for $X_2(b_2)$ given by (49), $\nu = 0$ or $b_1 = 0$. If $\nu \equiv 0$, the field $T$ is a principal direction. If $\nu \not= 0$ then $b_2 = 0$ or $b_1 = 0$. Suppose that there exist $x_0$ such that $b_1(x_0) \not= 0$. Then by continuity there exist a neighborhood $V$ of $x_0$ such that $b_1(x) \not= 0$ for all $x \in V$. Then $b_2(x) = 0$ for all $x \in V$ and $T$ is a principal direction. Finally if $\lambda_2 = 0$ it follows that $T$ is a principal direction.

Suppose that $\lambda_2 \not= 0$. From (50), $b_2 = -\frac{X_2(\nu)}{\lambda_2}$ and from equation (107) we get

$$X_2(\nu) \left\{-8\nu^3 + 2\nu D - \frac{2E}{\lambda_2} X_2(b_2) \right\} = 0.$$ 

Therefore $X_2(\nu) = 0$ or $-8\nu^3 + 2\nu D - \frac{2E}{\lambda_2} X_2(b_2) = 0$.

If $X_2(\nu) = 0$ we obtain $b_2 = 0$ and $T$ is a principal direction. If not,

$$-8\nu^3 + 2\nu D - \frac{2E}{\lambda_2} \left(\nu \lambda_2 + \frac{c \nu (b_1)^2}{\lambda_2 - \lambda_1}\right) = 0.$$ 

So,

$$-8\nu^3 + 4\nu + 4c \nu (\lambda_2 - \lambda_1)^2 - 2c \nu \lambda_1 (\lambda_2 - \lambda_1) + 2c \nu (\lambda_2^2 - \lambda_1^2) + \frac{2\nu (\lambda_2 + \lambda_1)}{\lambda_2} b_1^2 = 0. \quad (108)$$

From (105) we get

$$\frac{2\nu (\lambda_2 + \lambda_1)}{\lambda_2} b_1^2 = -2c \nu \lambda_1 (\lambda_2 - \lambda_1) + 2\nu (1 - \nu^2) - \frac{4\nu^3 (\lambda_2 - \lambda_1)}{\lambda_2} - \frac{4c \nu^3 (1 - \nu^2)}{\lambda_2 (\lambda_2 - \lambda_1)}. \quad (109)$$

By replacing (109) in (108) it follows that

$$\frac{4c}{\lambda_2 (\lambda_2 - \lambda_1)} \nu^5 - \nu^3 \left(10 + 4 \frac{(\lambda_2 - \lambda_1)^2 + c}{\lambda_2 (\lambda_2 - \lambda_1)}\right) + 6\nu \left(1 + c(\lambda_2 - \lambda_1)^2\right) = 0.$$ 

Observe that $\frac{4c}{\lambda_2 (\lambda_2 - \lambda_1)} \neq 0$ and we obtain a polynomial of fifth grade on the variable $\nu$.

Note that $\nu \equiv 0$ is a solution for that equation and $T$ is a principal direction. If there are other solutions for that equation, also in that case $\nu$ is constant and $T$ is a principal direction.

In this way we conclude that in case $g = 2$ and $n = 2$, the field $T$ is a principal direction.

Let us analyze now what occurs if $\nu \equiv 0$ and $\nu \not= 0$.

If $\nu \not= 0$ the surface is locally given by Theorem 6.2. If $\nu = 0$ the surface is a cylinder over a curve, $f(M^2) = \alpha \times \mathbb{R}$, where $\alpha$ is a circle non totally geodesic in $S^2$, if $c = 1$, or $\alpha$ is a equidistant curve, a horocycle or a hyperbolic circle in $\mathbb{H}^2$, if $c = -1$.

Second case: $n \geq 3$

If $\nu(p) \not= 0$, for all $p \in M$, from Theorem 7.1 the immersion $f$ has at least one principal curvature with multiplicity one. As $n \geq 3$ and $g = 2$ there exist only one principal curvature with multiplicity 1 and from Proposition 7.4 $T$ is a principal direction corresponding
to that curvature. Then $f$ is given by Theorem 6.2. From Remark 6.3 it is a rotational hypersurface.

If $\nu \equiv 0$ then $f(M^n)$ is an open subset of the Riemannian product $M^{n-1} \times \mathbb{R}$, where $M^{n-1}$ is a hypersurface of $\mathbb{Q}^n_c$. Since the principal curvature corresponding to the factor $\mathbb{R}$ is null, using the classification of the isoparametric hypersurfaces in $\mathbb{Q}^n_c$, see [11] Theorem 5] and [5] p.4.4 the other curvature may not be zero. So $M^{n-1}$ must be an isoparametric umbilical and non totally geodesic hypersurface in $\mathbb{Q}^n_c$. This proves item (i).

(iii) Suppose now $\nu = 3$. According to the considered dimension, there are three possibilities for the multiplicities of principal curvatures: two curvatures with multiplicity 1, just one curvature with multiplicity one and all the curvatures with multiplicities $\geq 2$.

Let us analyze each case.

First case: Suppose that two of the curvatures have multiplicity one.

From Proposition 8.3 we don't have $\nu(p) \neq 0$, for all $p \in M^n$. Therefore $\nu \equiv 0$ and consequently $f(M^n)$ is an open subset of the Riemannian product $M^{n-1} \times \mathbb{R}$. Then one of the curvatures must be zero.

If $c = -1$, $\lambda = 0$ and the principal curvatures of an isoparametric hypersurface with $g = 2$ in $\mathbb{H}^n$ satisfy $\mu \gamma = 1$ and $\lambda = 0$ has multiplicity one. Then $f(M^n)$ is locally the product $S^1 \times \mathbb{H}^{n-2} \times \mathbb{R}$ or $S^{n-2} \times S^1 \times \mathbb{R}$.

If $c = 1$ and $n \geq 4$, $f(M^n)$ is an open subset $S^1 \times S^{n-2} \times \mathbb{R}$. Moreover for $n = 4$ it may also occur that $f(M^n)$ is locally given by $M^3 \times \mathbb{R}$, where $M^3$ is a tube over a Veronese surface in $S^4$.

Second case: Suppose that just one curvature has multiplicity 1.

If $\nu(p) \neq 0$, for all $p \in M^n$, from Proposition 7.4 the field $T$ is a principal direction and from Theorem 6.2 we get $c = -1$ and $g = 2$, which may not occur. Then $\nu \equiv 0$ and $f(M^n)$ is an open subset of the Riemannian product $M^{n-1} \times \mathbb{R}$. In this case the curvature with multiplicity 1 is $\lambda = 0$.

Now let us explicit $M^{n-1}$ if $c = -1$ and $c = 1$.

If $c = -1$, $f(M^n)$ is an open subset of $S^k \times \mathbb{H}^{n-k-1} \times \mathbb{R}$, with $k \geq 2$ and $n \geq 5$. If $c = 1$, $f(M^n)$ is an open subset of $S^k \times S^{n-k-1} \times \mathbb{R}$, with $k \geq 2$ and $n \geq 5$.

Third case: Suppose that all the curvatures have multiplicity $\geq 2$.

From Remark 7.2 we get $\nu \equiv 0$ and one the principal curvatures is $\lambda = 0$. Moreover, $f(M^n)$ is an open subset of the Riemannian product $M^{n-1} \times \mathbb{R}$.

Then if $c = -1$, by [11] Theorem 5], there does not exist isoparametric hypersurfaces, non totally geodesic in $\mathbb{H}^n$ with a principal curvature equal to zero. Otherwise, $\lambda = 0$ would have multiplicity one, which may not occur.

If $c = 1$, the isoparametric hypersurfaces in $S^n$ with $g = 3$, are the Cartan’s hypersurfaces. They have one principal curvature equal to zero. Then that case occur for $n \in \{7, 13, 25\}$ and $f(M^n)$ is locally given by the product $M^{n-1} \times \mathbb{R}$, where $M^{n-1}$ is a Cartan’s hypersurface in $S^n$.

Remark 8.5. The hypersurfaces classified in Theorem 8.4 have function $\nu$ constant and from Corollary 5.2 they are isoparametric in $\mathbb{Q}^n_c \times \mathbb{R}$.

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