LÉVY PROCESSES, MARTINGALES AND UNIFORM INTEGRABILITY

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ABSTRACT. We give equivalent conditions for the existence of generalized moments of a Lévy process \((X_t)_{t \geq 0}\). We show, in particular, that the existence of a generalized g-moment is equivalent to uniform integrability of \((g(X_t))_{t \in [0,1]}\). As an application, it turns out that certain functions of a Lévy process which are integrable and local martingales are already true martingales.

A generalized moment of a stochastic process \((X_t)_{t \geq 0}\) is an expression of the form \(E[g(X_t)]\). Such moments arise naturally when studying Markov semigroups. For Lévy processes, a necessary and sufficient condition based on the jumps and the integrability properties of the jump measure of the process is known (see Section 1). We will give a new proof of this result and add a few useful further equivalent conditions; in particular, \(E[g(X_t)]\) exists if, and only if, \(g(X_t)\) is uniformly integrable for bounded \(t\)-sets. Our arguments are based on Gronwall’s lemma and this technique can also be used (see Section 2) to show that certain functions of a Lévy process \((f(X_t))_{t \geq 0}\) which are both a local martingale and integrable, i.e. \(E|f(X_t)| < \infty\), are already proper martingales. In the last section (Section 3) we apply our results to get a short proof of the characterization of infinitely divisible lattice distributions and a ‘martingale’ criterion for the transience of Lévy processes.

Let us recall a few key concepts and techniques which will be needed later on. Most of our notation is standard or self-explanatory; in addition, we use \(|x|^p := \sum_{k=1}^d |x_k|^p\) with the usual modification if \(p = \infty\). We write \(\|f\|_{L^1(\mathbb{R}^d,g)} := \int_{\mathbb{R}^d} |f(x)| g(x) \, dx\) for the weighted \(L^1\)-norm (with positive and measurable weight function \(g : \mathbb{R}^d \to [0,\infty)\)) and \(L^1(\mathbb{R}^d,g) := \{f : \mathbb{R}^d \to \mathbb{R} \mid f \text{ measurable and } \|f\|_{L^1(\mathbb{R}^d,g)} < \infty\}\).

Lévy processes. A Lévy process \(X = (X_t)_{t \geq 0}\) is a stochastic process with values in \(\mathbb{R}^d\), stationary and independent increments and right-continuous sample paths with finite left-hand limits (càdlàg). Our standard references for Lévy processes are Sato [9] (for probabilistic properties) and Jacob [4] and [5] (for analytic aspects). It is well-known that a stochastic process \(X\) is a Lévy process if it has càdlàg paths and if its conditional characteristic function is of the form

\[
E\left[e^{i\ell \cdot (X_t - X_s)} \mid \mathcal{F}_s\right] = e^{-(t-s)\psi(\ell)}, \quad 0 \leq s \leq t, \quad \xi \in \mathbb{R}^d,
\]

where \(\mathcal{F}_s = \sigma(X_r, r \leq s)\) is the natural filtration of X. The characteristic exponent \(\psi : \mathbb{R}^d \to \mathbb{C}\) is uniquely determined by the Lévy–Khintchine formula

\[
\psi(\xi) = -ib \cdot \xi + \frac{1}{2} Q\xi \cdot \xi + \int_{\mathbb{R}^d \setminus \{0\}} \left(1 - e^{i\xi \cdot x} + i\xi \cdot x\mathbb{1}_{(0,1)}(|x|)\right) \nu(dx);
\]

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the Lévy triplet \((b, Q, \nu)\) where \(b \in \mathbb{R}^d\), \(Q \in \mathbb{R}^{d \times d}\) (a positive semidefinite matrix) and \(\nu\) (a Radon measure on \(\mathbb{R}^d \setminus \{0\}\) such that \(\int_{\mathbb{R}^d \setminus \{0\}} \min\{|x|^2, 1\} \nu(dx) < \infty\)) uniquely describe \(\psi\).

Using the characteristic exponent we can determine the infinitesimal generator \(A\) of the process \(X\) either as pseudo-differential operator

\[
Au(x) = -\psi(D)u(x) = \mathcal{F}^{-1}[-\psi \mathcal{F}u](x), \quad u \in \mathcal{D}(\mathbb{R}^d),
\]

where \(\mathcal{F}u(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\xi \cdot x} u(x) \, dx\) is the Fourier transform and \(\mathcal{D}(\mathbb{R}^d)\) is the Schwartz space of rapidly decreasing smooth functions, or as integro-differential operator

\[
(2) \quad Au(x) = Lu(x) + Ju(x) + Ku(x)
\]

where \(L, J\) and \(K\) are again linear operators: \(L\) is the local part, \(J\) takes into account the small jumps and \(K\) the large jumps, i.e.

\[
Lu(x) = b \cdot \nabla u(x) + \frac{1}{2} \nabla \cdot (Q \nabla u(x)),
\]

\[
Ju(x) = \int_{0<|y|<1} (u(x+y) - u(x) - y \cdot \nabla u(x)) \, \nu(dy),
\]

\[
Ku(x) = \int_{|y| \geq 1} (u(x+y) - u(x)) \, \nu(dy).
\]

The precise form of the domain \(\mathcal{D}(A)\) of \(A\) (as closed operator on the Banach space of continuous functions vanishing at infinity \((C_\infty^\infty(\mathbb{R}^d), \|\cdot\|_\infty)\) is not known; but both the test functions \(C_c^\infty(\mathbb{R}^d)\) and the Schwartz spaces \(\mathcal{D}(\mathbb{R}^d)\) are operator cores. On the other hand, the expression \(\mathcal{D}(A)\) has a pointwise meaning for every \(g \in C^2(\mathbb{R}^d)\) and we will continue to use the notation \(Ag(x)\) despite the fact that \(C^2(\mathbb{R}^d) \not\subseteq \mathcal{D}(A)\).

The transition semigroup \(\{P_t\}_{t \geq 0}\) corresponding to the generator \(A\) or the process \(X\) is given by \(P_t u(x) = \mathbb{E} [u(x + X_t)]\). Its adjoint, \(P_t^* u(x) = \mathbb{E} [u(x - X_t)]\) is the transition semigroup of the Lévy process \(-X = (-X_t)_{t \geq 0}\).

**Dynkin’s formula.** Let \((X_t)_{t \geq 0}\) be a Lévy process and denote by \(\mathcal{F}_t = \sigma(X_s, s \leq t)\) its natural filtration and \((A, \mathcal{D}(A))\) the infinitesimal generator. Dynkin’s formula states that for every \(u \in \mathcal{D}(A)\) and every stopping time \(\sigma\) with \(\mathbb{E} [\sigma] < \infty\) we have

\[
(3) \quad \mathbb{E} [u(X_\sigma + x)] - u(x) = \mathbb{E} \left[ \int_{(0,\sigma]} Au(X_s) \, ds \right].
\]

There are several ways to prove this result, e.g. using arguments from potential theory (as in [12] Proposition 7.31), semigroup theory (as in [8] Proposition VII.1.6) or by Itô’s formula. At the heart of the argument is the fact that

\[
M_t^{[u]} := u(X_t + x) - u(x) - \int_{(0,t]} Au(X_s) \, ds, \quad u \in \mathcal{D}(A),
\]

is an \(\mathcal{F}_t\)-martingale combined with a stopping argument. There are various ways to extend the class of functions \(u\) for which we have some kind of Dynkin’s formula. It is clear that formula \((3)\) can be extended to those functions \(u\) such that \(u(X_\sigma) \in L^1(\mathbb{P})\) and \(Au(X_{s\wedge \sigma}) \in L^1(\mathbb{P})\). Such moment estimates will be given below.

Here we need a **Dynkin inequality** which we are going to prove for positive \(g \in C^2(\mathbb{R}^d)\).

**Lemma 1 (Dynkin’s inequality).** Let \((X_t)_{t \geq 0}\) be a Lévy process with generator \((A, \mathcal{D}(A))\) and extend \(A\) using \((2)\) to \(C^2(\mathbb{R}^d)\). For every \(g \in C^2(\mathbb{R}^d)\) satisfying \(g(x) \geq 0\) and every stopping
time \( \sigma \) the following inequality holds

\[
E g(X_{t \wedge \sigma}) \leq g(0) + E \left[ \int_{(0,t \wedge \sigma)} |A g(X_s)| \, ds \right].
\]

**Proof.** Pick a cut-off function \( \chi_R \in C^\infty(\mathbb{R}^d) \) such that \( 1_{\mathbb{R}^d \setminus (0)} \leq \chi_R \leq 1_{\mathbb{R}^d \setminus (0)} \) for some \( R > 0 \). Since \( g \chi_R \in C^2(\mathbb{R}^d) \) we know that \( g \chi_R \in D(A) \), and we see that for any stopping time \( \sigma \) the process \( (M_{t \wedge \sigma}^{\chi_R})_{t \geq 0} \) is a martingale, hence

\[
E [(g \chi_R)(X_{t \wedge \sigma})] - g(0) = E \left[ \int_{(0,t \wedge \sigma)} A(g \chi_R)(X_s) \, ds \right].
\]

If we replace \( \sigma \) by the stopping time \( \sigma \wedge \tau_R \) where \( \tau_R = \inf \{ s \geq 0 \mid |X_s| \geq R \} \), then we can use the fact that \( |X_s| \leq R \) if \( s \in (0,t \wedge \sigma \wedge \tau_R) \). This implies, in particular, that \( \delta^\sigma(g \chi_R)(X_s) = \delta^\sigma g(X_s) \), and we see from the integro-differential representation (2) of \( A \)

\[
A(g \chi_R)(X_s) = L(g \chi_R)(X_s) + J(g \chi_R)(X_s) + K(g \chi_R)(X_s)
\]

\[
= b \cdot \nabla g(X_s) + \frac{1}{2} \nabla \cdot Q \nabla g(X_s) + \int_{|y| < 1} (g(X_s + y) - g(X_s) - y \cdot \nabla g(X_s)) \nu(dy)
\]

\[
+ \int_{|y| \geq 1} (g(X_s + y) - g(X_s)) \nu(dy);
\]

for the second equality observe that \( |X_s| \leq R, |X_s + y| \leq R + 1 \) for \( |y| < 1 \) and that \( L \) is a local operator. Since \( g \) is positive, hence \( g \chi_R \leq g \), we conclude that \( A(g \chi_R)(X_s) \leq Ag(X_s) \).

Inserting this into (6) gives

\[
E [(g \chi_R)(X_{t \wedge \sigma \wedge \tau_R})] - g(0) \leq E \left[ \int_{(0,t \wedge \sigma \wedge \tau_R)} Ag(X_s) \, ds \right] \leq E \left[ \int_{(0,t \wedge \sigma \wedge \tau_R)} |Ag(X_s)| \, ds \right].
\]

Since \( g \geq 0 \), we can use Fatou’s lemma on the left-hand side and get (5). \( \square \)

**Friedrichs mollifiers.** Let \( j : \mathbb{R}^d \to [0, \infty) \) be a \( C^\infty \)-function with compact support \( \text{supp} \ j \subseteq B_1(0) \) such that \( j(x) \) is rotationally symmetric and \( \int j(x) \, dx = 1 \). For every \( \varepsilon > 0 \) we define \( j_\varepsilon(x) := \varepsilon^{-d} j(x/\varepsilon) \), i.e., \( j_\varepsilon \) is again smooth, rotationally symmetric and satisfies \( \text{supp} \ j_\varepsilon \subseteq B_\varepsilon(0) \) and \( \int j_\varepsilon(x) \, dx = 1 \). For any locally bounded function \( g : \mathbb{R}^d \to \mathbb{R} \) the convolution

\[
g^\varepsilon(x) := j_\varepsilon \ast g(x) := \int g(x - y) j_\varepsilon(y) \, dy, \quad x \in \mathbb{R}^d,
\]

exists and defines a \( C^\infty \)-function. Moreover, \( \text{supp} \ g^\varepsilon \subseteq \text{supp} \ g + \text{supp} \ j_\varepsilon \subseteq \text{supp} \ g + B_\varepsilon(0) \).

The function \( g^\varepsilon \) is called Friedrichs regularization of \( g \).

**Submultiplicativity functions.** A function \( g : \mathbb{R}^d \to [0, \infty) \) is said to be submultiplicative if there exists a constant \( c = c(g) \in [1, \infty) \) such that

\[
\forall x, y \in \mathbb{R}^d \ : \ g(x + y) \leq cg(x)g(y).
\]

In order to avoid pathologies, we consider only measurable submultiplicative functions (1)

Every locally bounded submultiplicative function grows at most exponentially, i.e. there are constants \( a, b \in (0, \infty) \) such that \( g(x) \leq ae^{bx} \). Since \( 1 + g \) inherits submultiplicativity from \( g \), we may assume that \( g \geq 1 \). The following lemma shows that we can even assume that a submultiplicative function is smooth.

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(1) An example of a non-measurable submultiplicative function is \( g(x) = e^{a|x|}, x \in \mathbb{R} \), where \( a \) is a non-measurable solution to the functional equation \( a(x + y) = a(x) + a(y) \).
Lemma 2. Let \( g \) be a locally bounded submultiplicative function and \( g^\varepsilon \) its Friedrichs regularization. Then \( g^\varepsilon \in C^\infty \) is submultiplicative and it satisfies
\[
\forall x \in \mathbb{R}^d : \quad c_\varepsilon^{-1} g(x) \leq g^\varepsilon(x) \leq c_\varepsilon g(x)
\]
for some constant \( c_\varepsilon = c_\varepsilon g \).

Proof. Submultiplicativity follows immediately from the two-sided estimate \((8)\):
\[
g^\varepsilon(x + y) \leq c_\varepsilon g(x + y) \leq c_\varepsilon c_\varepsilon g(x)g(y) \leq c_\varepsilon^2 c_\varepsilon^2 g^\varepsilon(y).
\]
In order to see \((8)\), we use the definition of \( g^\varepsilon \) and fact that \( g \) is submultiplicative,
\[
g^\varepsilon(x) = \int g(x - y)j_\varepsilon(y) \, dy \leq c g(x) \int g(-y)j_\varepsilon(y) \, dy \leq c \sup_{|y| \leq \varepsilon} g(y)g(x)
\]
and
\[
g(x) = \int g(x)j_\varepsilon(y) \, dy \leq c \int g(x - y)g(y)j_\varepsilon(y) \, dy \leq c \sup_{|y| \leq \varepsilon} g(y)g^\varepsilon(x). \quad \square
\]

1. Generalized moments and uniform integrability

Let \((X_t)_{t \geq 0}\) be a Lévy process with triplet \((b, Q, \nu)\). The following moment result for a locally bounded submultiplicative functions \( g \) is well-known, cf. Sato [9] Theorem 25.3, p. 159:
\[
E[g(X_t)] < \infty \text{ for some (hence, all) } t > 0 \iff \int_{|y| \geq 1} g(y) \nu(dy) < \infty.
\]

Our aim is to show that this is also equivalent to a certain uniform integrability condition. Although we cast the statement and proof for Lévy processes, an extension to certain Lévy-type processes is possible; see Remark 4 below. We denote by \( J \) be the family of stopping times for the process \( X \) equipped with its natural filtration.

Theorem 3. Let \((X_t)_{t \geq 0}\) be a Lévy process with generator \( A \), transition semigroup \((P_t)_{t \geq 0}\), and triplet \((b, Q, \nu)\), and let \( g \) be a locally bounded submultiplicative function. The following assertions are equivalent:

a) \( E[g(X_t)] \) exists and is finite for some (hence, all) \( t > 0 \);
b) \( E[\sup_{s \leq t} g(X_s)] \) exists and is finite for some (hence, all) \( t > 0 \);
c) \( \{g(X_t)\}_{t \in J, \, s \leq t} \) is uniformly integrable for every fixed \( t > 0 \);
d) \( \sup_{s \in J, \, s \leq t} E[g(X_s)] \) is finite for every fixed \( t > 0 \);
e) \( \int_{|y| \geq 1} g(y) \nu(dy) < \infty \);
f) The adjoint semigroup \( P^*_t f(x) := E[f(x - X_t)] \) is a strongly continuous operator semigroup on the weighted \( L^1 \)-space \( L^1(\mathbb{R}^d, g) \).
g) The adjoint generator \( (A^* \phi)(x) := (A \phi)(-x) \) satisfies \( A^* \phi \in L^1(\mathbb{R}^d, g) \) for all \( \phi \in C_0^\infty(\mathbb{R}^d) \).
h) There exists a non-negative function \( \phi \in C_0^\infty(\mathbb{R}^d), \phi \not\equiv 0 \), such that \( A^* \phi \in L^1(\mathbb{R}^d, g) \).

If one (hence, all) of the conditions is satisfied, then there are constants \( c_i > 0, i = 1, 2, 3 \), such that
\[
E[g(X_t)] \leq c_1 e^{c_2 t}, \quad t \geq 0,
\]
and
\[
\|A^* \phi\|_{L^1(\mathbb{R}^d, g)} \leq c_3 \left( |b|_{\varepsilon_1} + |Q|_{\varepsilon_1} + \int_{|y| < 1} (1 + |y|^2) \nu(dy) + \int_{|y| \geq 1} g(y) \nu(dy) \right) \|\phi\|_{C_0^\infty(\mathbb{R}^d)}.
\]
**Remark 4.** There is a further equivalent condition if $x \mapsto g(|x|)$ is locally bounded, submultiplicative and $g(r)$ is increasing:

i) $E \left[ g\left( \sup_{s \leq t} |X_s| \right) \right] < \infty$ for some (hence, all) $t > 0$.

The direction $[\text{i} \Rightarrow \text{b}]$ follows from the assumption that $g$ is increasing. The implication $[\text{b} \Rightarrow \text{i}]$ does not need monotonicity since we have

$$g\left( \sup_{s \leq t} |X_s| \right) \leq \sup_{s \leq t} g\left( |X_s| \right)$$

at least if $x \mapsto g(|x|)$ is continuous. This can always be achieved by a Friedrichs regularization.

Let us also point out that we may replace $A^*$ and $P_t^*$ by $A$ and $P_t$ if either (the law of) $X_t$ is symmetric or if $g$ is even, i.e. $g(x) = g(-x)$.

The conditions $\text{d)}$ (for deterministic stopping times), $\text{e)}$, $\text{f)}$ can be found in Sato [9, Chapter 25]; $\text{b)}$ is due to Siebert [14] and variants of $\text{e)}$, $\text{h)}$ appear first in Hulanicki [3]; their proofs are cast in the language of probability on (Lie) groups. The streamlined proofs given in this paper are new.

Some of our arguments carry over to Lévy-type processes whose generators have bounded coefficients (see [1, p. 55] for the notation); in particular $[\text{e} \Rightarrow \text{f}]$ (using the alternative proof below) $\Rightarrow [\text{b} \Rightarrow \text{c}]$ becomes

$$\sup_{x \in \mathbb{R}^d} \int_{|y| \geq 1} g(y) \nu(x, dy) < \infty \Rightarrow \sup_{x \in \mathbb{R}^d} E^x[g(X_t - x)] < \infty$$

$$\Rightarrow \sup_{x \in \mathbb{R}^d} E^x[\sup_{s \leq t} g(X_s - x)] < \infty$$

$$\Rightarrow \{g(X_s - x)\}_{s \in \mathcal{F}, s \leq t} \text{ is unif. integrable}$$

$$\Rightarrow \sup_{s \in \mathcal{F}, s \leq t} E^x[g(X_s - x)] < \infty,$$

while $[\text{d} \Rightarrow \text{e}]$ only yields $\inf_{x \in \mathbb{R}^d} \int g(y) \nu(x, dy) < \infty$, and an additional condition of the type $\sup_{x \in \mathbb{R}^d} \int_{|y| \geq 1} g(y) \nu(x, dy) \leq C \inf_{x \in \mathbb{R}^d} \int_{|y| \geq 1} g(y) \nu(x, dy)$ is needed to get equivalences; this is partly worked out in [6].

In order to prove this theorem, we need a few preparations.

**Lemma 5.** Let $g$ be a locally bounded submultiplicative function. If $(X_t)_{t \geq 0}$ is a Lévy process, then

$$E\left[ \sup_{s \leq T} g(X_s) \right] = \kappa_T < \infty \quad \text{implies} \quad E\left[ \sup_{s \leq 2T} g(X_s) \right] \leq \kappa_T(1 + c\kappa_T) < \infty$$

and

$$E[g(X_t)] < \infty \text{ for some } t > 0 \quad \text{implies} \quad E[g(X_t)] < \infty \text{ for all } t > 0.$$

**Proof.** $1^\circ$ We have

$$E\left[ \sup_{s \leq 2T} g(X_s) \right] \leq E\left[ \sup_{s \leq T} g(X_s) \right] + E\left[ \sup_{T \leq s \leq 2T} g(X_s) \right] = \kappa_T + E\left[ \sup_{s \leq T} g(X_{s+T}) \right].$$
Since $g$ is submultiplicative, we see $g(X_{s+T}) \leq cg(X_{s+T} - X_T)g(X_T)$; moreover, $X_T$ and $(X_{s+T} - X_T)_{s \geq 0} \sim (X_s)_{s \geq 0}$ are independent and so

$$
E \left[ \sup_{s \leq T} g(X_{s+T}) \right] \leq cE \left[ \sup_{s \leq T} g(X_{s+T} - X_T) \right] E[g(X_T)] \\
\leq cE \left[ \sup_{s \leq T} g(X_s) \right] E[g(X_T)] \leq c\kappa_T^2.
$$

2° Let $t_0 > 0$ such that $E[g(X_{t_0})] < \infty$. Using the Markov property we see that for any $s < t_0$

$$
E \left[ g(X_{t_0}) \right] = E \left[ g(X_0 - X_s + X_s) \right] = \int_{\mathbb{R}^d} E \left[ g(X_s + y) \right] P(X_{t_0-s} \in dy).
$$

Thus, there is some $y$ such that $E[g(X_s + y)] < \infty$, and we conclude from the submultiplicative property that $E[g(X_s)] \leq cg(-y)E[g(X_s + y)] < \infty$ for all $s \leq t_0$. As before, we can now show that $E[g(X_{2t_0})] < \infty$ and, by iteration, we see that $E[g(X_t)] < \infty$ for all $t > 0$.

**Lemma 6.** Let $g$ be a locally bounded submultiplicative function and denote by $g^\varepsilon$ its regularization with a Friedrichs mollifier. If $A$ is the generator of a Lévy process given by (2), then

$$
|Ag^\varepsilon(x)| \leq C_\varepsilon \left( |b|_{\ell^1} + |Q|_{\ell^1} + \int_{y \neq 0} (1 \wedge |y|^2) \nu(dy) + \sup_{|y| \leq 1} g(y) + \int_{|y| \geq 1} g(y) \nu(dy) \right) g(x).
$$

Note that the constant $C_\varepsilon$ appearing in Lemma 6 is, in general, unbounded as $\varepsilon \to 0$.

**Proof.** Without loss of generality we can assume that $g \geq 1$. Otherwise we use $g + 1$ instead of $g$ and observe that $A(g + 1)^\varepsilon = A(g^\varepsilon + 1) = Ag^\varepsilon$. As in (2) we write $A = L + J + K$.

Observe that $|\partial^\alpha g^\varepsilon(x)| \leq c_{\alpha,\varepsilon}g(x)$ holds for every multi-index $\alpha \in \mathbb{N}_0^d$. This follows from

$$
|\partial^\alpha g^\varepsilon(x)| = \left| (\partial^\alpha j_\varepsilon) * g(x) \right| \leq \int |\partial^\alpha j_\varepsilon(y)| g(x - y) \, dy \leq cg(x) \sup_{|y| \leq \varepsilon} g(y) \int |\partial^\alpha j_\varepsilon(y)| \, dy.
$$

We can now estimate the three parts of $A$ separately. For the local part we use the above estimate with $|\alpha| = 1$ and $|\alpha| = 2$:

$$
|Lg^\varepsilon(x)| \leq \left( \sum_{i=1}^d |b_i|c_{\varepsilon,i} + \frac{1}{2} \sum_{i,k=1}^d |q_{ik}|c_{\varepsilon,i,k} \right) \cdot g(x) \leq c_\varepsilon (|b|_{\ell^1} + |Q|_{\ell^1}) \cdot g(x).
$$

The large-jump part is estimated using $|\alpha| = 0$ and the submultiplicativity of $g$:

$$
|Kg^\varepsilon(x)| \leq c_\varepsilon \int_{|y| \geq 1} g(x)g(y) + g(x) \nu(dy) = c_\varepsilon \int_{|y| \geq 1} (g(y) + 1) \nu(dy) \cdot g(x).
$$

We may replace this by the (strong) Markov property if $\sup_{y \in \mathbb{R}^d} \sup_{s \leq T} E[y|g(X_{s} - y)| < \infty$. For Lévy processes the first supremum is always trivial.
Using Taylor's formula with integral remainder term we can rewrite the part containing the small jumps and we see that
\[
\left| Jg^c(x) \right| = \left| \sum_{i,k=1}^d \int_{0<|y|<1} \int_0^1 \partial_i \partial_k g^c(x + ty) y_i y_k (1 - t) \, dt \, \nu(dy) \right| \\
\leq c_\varepsilon \sum_{i,k=1}^d c_{i,j,k} \int_{0<|y|<1} \int_0^1 g(x + ty) |y_i y_k| (1 - t) \, dt \, \nu(dy) \\
\leq c_\varepsilon \sum_{i,k=1}^d c_{i,j,k} \int_{0<|y|<1} \int_0^1 g(ty) |y_i y_k| (1 - t) \, dt \, \nu(dy) \cdot g(x) \\
\leq c_\varepsilon \sup_{|y| \leq 1} \int_{0<|y|<1} |y|^2 \, \nu(dy) \cdot g(x).
\]

If we combine these three estimates, the claim follows. \( \square \)

**Proof of Theorem 3.** We show (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) \( \Rightarrow \) (d) \( \Rightarrow \) (e) \( \Rightarrow \) (a) and (c) \( \Rightarrow \) (d) \( \Rightarrow \) (e). Through-out the proof we will assume that \( g \geq 1 \) and \( g \in C^2(\mathbb{R}^d) \). Otherwise we could replace \( g \) by its Friedrichs regularization \( g^\varepsilon \), see Lemma [2] and \( g + 1 \), resp., \( g^\varepsilon + 1 \).

**1** (a) \( \Rightarrow \) (b) If \( E[g(X_t)] \) is finite for some \( t > 0 \), then Lemma [5] shows that \( E[g(X_t)] \) is finite for all \( t > 0 \). For \( a, b > 0 \) we define a stopping time
\[
\sigma := \sigma_{a,b} := \inf \{ s \mid g(X_s) > cg(0)e^{a+b} \}
\]
and observe that we can use the subadditivity of \( g \) to get
\[
P \left( g(X_t) > e^a \right) \geq P \left( g(X_t) > e^a, g(X_\sigma) \geq cg(0)e^{a+b}, \sigma \leq t \right) \\
\geq P \left( g(X_\sigma - X_t) < g(0)e^b, g(X_\sigma) \geq cg(0)e^{a+b}, \sigma \leq t \right).
\]

The strong Markov property yields (for all \( t \leq T, T \) will be determined in the following step)
\[
P \left( g(X_t) > e^a \right) \geq \int_{\sigma \leq t} P \left( g(-X_{t-\sigma}) < g(0)e^b \right) P(d\omega) \\
\geq \inf_{r \leq t} P \left( g(-X_r) < g(0)e^b \right) \cdot P(\sigma \leq t) \\
\geq \frac{1}{2} P \left( \sup_{s \leq t} g(X_s) > cg(0)e^{a+b} \right).
\]

In the last estimate we use that \( \{ \sup_{s \leq t} g(X_s) > cg(0)e^{a+b} \} \subseteq \{ \sigma \leq t \} \). The factor \( \frac{1}{2} \) comes from the fact that \( g \) is locally bounded and \( \lim_{t \to 0} P(|X_t| > \varepsilon) = 0 \) (continuity in probability), which shows that there is some \( 0 < T \leq t_0 \) such that
\[
P \left( g(-X_t) < g(0)e^b \right) \geq \frac{1}{2} \quad \text{for all } t \leq T.
\]

This proves that for \( e^\gamma := cg(0)e^b \)
\[
P \left( \sup_{s \leq T} g(X_s) > e^{a+\gamma} \right) \leq 2P(g(X_T) > e^a).
\]

\(^{(1)}\) A direct proof of (e) \( \Rightarrow \) (a) is given in the next section.
We can now use the layer-cake formula to see that
\[
E \left[ \sup_{s \leq t} g(X_s) \right] = 1 + \int_1^\infty P \left( \sup_{s \leq t} g(X_s) > y \right) dy \\
= 1 + e^\gamma \int_{-\infty}^\infty P \left( \sup_{s \leq t} g(X_s) > e^\gamma e^a \right) e^a da \\
\leq 1 + e^\gamma \int_{-\infty}^0 e^a da + 2e^\gamma \int_0^\infty P (g(X_T) > e^a) e^a da \\
\leq e^\gamma + 2e^\gamma E [g(X_T)].
\]
Using Lemma 5 we see that \(E [\sup_{s \leq t} g(X_s)] < \infty\) for all \(n \in \mathbb{N}\), i.e., \(b\) holds for all \(t > 0\).

2° \(b) \Rightarrow c)\) If \(\sup_{s \leq t} g(X_s)\) is integrable for some \(t > 0\), then it is integrable for all \(t > 0\), cf. Lemma 5. For fixed \(t > 0\), let \(\sigma \in \mathcal{F}\) be a stopping time with \(\sigma \leq t\), then
\[
g(X_\sigma) \leq \sup_{r \leq t} g(X_r) \in L^1(\mathbb{P}).
\]
Consequently, the family \(\{g(X_\sigma)\}_{\sigma \in \mathcal{F}, \sigma \leq t}\) is dominated by the integrable random variable \(\sup_{r \leq t} g(X_r)\); hence, it is uniformly integrable.

3° \(c) \Rightarrow d)\) This is immediate from the fact uniform integrability implies boundedness in \(L^1\).

4° \(d) \Rightarrow e)\) We rearrange (7) and insert it into (6) to get
\[
E \left[ (g\chi_k)(X_{t \wedge \tau_k}) \right] = E \left[ \int_{[0,t \wedge \tau_k]} (L + J)g(X_s) \, ds \right] \\
= g(0) + E \left[ \int_{[0,t \wedge \tau_k]} \int_{|y| \geq 1} (g\chi_k(X_s + y) - g(X_s)) \nu(dy) \, ds \right] \\
= g(0) + E \left[ \int_{[0,t \wedge \tau_k]} \int_{|y| \geq 1} g\chi_k(X_s + y) \nu(dy) \, ds \right] - \nu(|y| \geq 1) \cdot E \left[ \int_{[0,t \wedge \tau_k]} g(X_s) \, ds \right].
\]
Now we use Lemma 6 for the Lévy generator \(L + J\) and the estimates
\[
E \left[ \int_{[0,t \wedge \tau_k]} g(X_s) \, ds \right] \leq \int_{[0,1]} E [g(X_s)] \, ds \leq t \sup_{s \leq t} E [g(X_s)]
\]
and
\[
E \left[ (g\chi_k)(X_{t \wedge \tau_k}) \right] \leq E \left[ g(X_{t \wedge \tau_k}) \right] \leq \sup_{\sigma \in \mathcal{F}, \sigma \leq t} E [g(X_\sigma)].
\]
Because of our assumption \(d)\) there is a constant \(C_t\), not depending on \(R\), such that
\[
C_t \geq E \left[ \int_{[0,t \wedge \tau_k]} \int_{|y| \geq 1} g\chi_k(X_s + y) \nu(dy) \, ds \right].
\]
Letting $R \to \infty$, Fatou’s lemma and yet another application of submultiplicativity yield

$$C_i \geq \frac{1}{c} \mathbb{E} \left[ \int_{\{\tau_i \}} \int_{|y| \geq 1} g(X_s + y) \nu(dy) \, ds \right]$$

$$\geq \frac{1}{c} \mathbb{E} \left[ \int_{\{\tau_i \}} \int_{|y| \geq 1} \frac{g(y)}{g(-X_s)} \nu(dy) \, ds \right]$$

$$\geq \frac{1}{c} \int_{|X_i| \leq R} \frac{dP}{g(-X_s)} \int_{|y| \geq 1} g(y) \nu(dy).$$

Since $g$ is locally bounded and $s \to X_s$ is càdlàg, $\mathbb{E}$ follows.

5° $[\Xi] \Rightarrow [\Theta]$ Let $\phi \in C_c^\infty(\mathbb{R}^d)$. The reflection $\tilde{\phi}(x) := \phi(-x)$ is again in $C_c^\infty(\mathbb{R}^d)$ and thus $\|A^* \tilde{\phi}\|_\infty < \infty$ by the representation of $A^*$ as an integro-differential operator (2) and Taylor’s formula. Choose $R > 0$ such that $\text{supp } \phi$ is contained in the ball $B_R(0)$, then

$$(A^* \tilde{\phi})(-x) = \int_{y \neq 0} \phi(x - y) \nu(dy), \quad |x| \geq 2R.$$ 

Since $g$ is bounded on compact sets, it follows that

$$\int_{\mathbb{R}^d} g(x) |(A^* \tilde{\phi})(x)| \, dx = \left( \int_{|x| \leq 2R} + \int_{|x| > 2R} \right) g(x) |(A^* \tilde{\phi})(x)| \, dx$$

$$\leq |B_{2R}(0)| \sup_{|x| \leq 2R} g(x) \|A^* \tilde{\phi}\|_\infty + \int_{|x| > 2R} \int_{y \neq 0} g(x) |\phi(x - y)| \nu(dy) \, dx.$$ 

It remains to show that the integral expression on the right-hand side is finite. By Tonelli’s theorem and a change of variables ($z = x - y$),

$$I := \int_{|x| > 2R} \int_{y \neq 0} g(x) |\phi(x - y)| \nu(dy) \, dx = \int \int \mathbb{1}_{|z+y| > 2R} \mathbb{1}_{|z| \leq R} |\phi(z)| g(z + y) \, dz \, \nu(dy),$$

where we use that $\text{supp } \tilde{\phi} \subseteq B_R(0)$. The elementary estimate $\mathbb{1}_{|z+y| > 2R} \mathbb{1}_{|z| \leq R} \leq \mathbb{1}_{|y| \geq R}$ and the submultiplicativity of $g$ now yield

$$I \leq c \left( \int_{|z| \leq R} |\phi(z)| g(z) \, dz \right) \left( \int_{|y| \geq R} g(y) \nu(dy) \right) < \infty;$$

the integrals are finite because of $[\Theta]$ and the local boundedness of $g$, $\phi$.

6° $[\Theta] \Rightarrow [\Phi]$ Trivial.

7° $[\Phi] \Rightarrow [\Xi]$ This part of the proof draws from a work by Hulanicki [3]. Take $\phi \in C_c^\infty(\mathbb{R}^d)$ non-negative such that $\phi \neq 0$ and $A^* \phi \in L^1(\mathbb{R}^d, g)$. Let us first assume that $g$ is bounded. Set

$$h(t) := \int_{\mathbb{R}^d} (P_t \phi)(-x) g(x) \, dx, \quad t \geq 0,$$

where $(P_t u)(x) := \mathbb{E}[u(x + X_t)]$ is the semigroup. We want to show that $|h'(t)| \leq C h(t)$ for some constant $C > 0$. An application of Tonelli’s theorem and a change of variables yield

$$\int_{\mathbb{R}^d} |P_t A\phi(-x)| g(x) \, dx \leq \mathbb{E} \left[ \int_{\mathbb{R}^d} |(A\phi)(y)| g(y + X_t) \, dy \right]$$

$$\leq c \mathbb{E} \left[ g(X_t) \right] \int_{\mathbb{R}^d} |(A^* \phi)(y)| g(y) \, dy < \infty;$$
the latter integral is finite because \( A^* \phi \in L^1(\mathbb{R}^d, g) \) and \( g \) is bounded. Moreover, Dynkin’s formula (3) entails that \( \frac{d}{dt} P_t \phi = P_t A \phi \). Consequently, we can use the differentiation lemma for parameter-dependent integrals, cf. [7] Proposition A.1, to obtain

\[
h'(t) = \int_{\mathbb{R}^d} \frac{d}{dt}(P_t \phi)(-x)g(x) \, dx = \int_{\mathbb{R}^d} (P_t A \phi)(-x)g(x) \, dx
\]

and, by the above estimate,

\[
|h'(t)| \leq c E[g(X_t)] \int_{\mathbb{R}^d} (A^* \phi)(y)|g(y)\, dy.
\]

The submultiplicativity of \( g \) gives

\[
\left( \int_{\mathbb{R}^d} \frac{\phi(x)}{g(x)} \, dx \right) g(y) \leq c \int_{\mathbb{R}^d} \phi(x)g(y - x) \, dx = c(\phi * g)(y)
\]

for all \( y \in \mathbb{R}^d \), i.e.

\[
(10) \quad g(y) \leq \frac{c}{\| \phi \|_{L^1(\mathbb{R}^d, 1/g)}}(\phi * g)(y);
\]

note that

\[
\| \phi \|_{L^1(\mathbb{R}^d, 1/g)} = \int_{\mathbb{R}^d} \frac{\phi(x)}{g(x)} \, dx \in (0, \infty)
\]

because \( g \geq 1 \) is locally bounded and \( \phi > 0 \) on a set of positive Lebesgue measure. Using (10) for \( y = X_t \), we get

\[
|h'(t)| \leq CE[(\phi * g)(X_t)] = C \int_{\mathbb{R}^d} E[\phi(X_t - x)]g(x) \, dx = Ch(t).
\]

Hence, by Gronwall’s lemma,

\[
h(t) \leq h(0)e^{\alpha t}, \quad t \geq 0,
\]

for some constant \( \alpha \geq 0 \). Invoking once more (10), we conclude that

\[
E[g(X_t)] \leq c'E[(\phi * g)(X_t)] = c'h(t) \leq c'h(0)e^{\alpha t}.
\]

So far, we assumed that \( g \) is bounded. For unbounded \( g \), we replace \( g \) by \( \min\{g, n\} \) – which is again submultiplicative – in the above estimates and find that

\[
E[\min\{g(X_t), n\}] \leq c''e^{\alpha t}
\]

for some constant \( c'' > 0 \) not depending on \( n \in \mathbb{N} \). Thus, by Fatou’s lemma, \( E[g(X_t)] \leq c''e^{\alpha t} \) for all \( t \geq 0 \).

8° [c] \( \Rightarrow [f] \). Let \( f \in L^1(\mathbb{R}^d, g) \). We see that

\[
\int_{\mathbb{R}^d} |P_t^* f(x)| \, g(x) \, dx \leq E \left[ \int_{\mathbb{R}^d} |f(x - X_t)| \, g(x - X_t + X_t) \, dx \right]
\]

\[
\leq cE[g(X_t)] \| f \|_{L^1(\mathbb{R}^d, g)}
\]

which shows that \( P_t^* : L^1(\mathbb{R}^d, g) \to L^1(\mathbb{R}^d, g) \) is continuous. Let \( \phi \in C_c(\mathbb{R}^d) \) and assume that \( \text{supp } \phi \) is contained in some ball \( B_R(0) \) with radius \( R > 0 \). We show that \( P_t^* \phi \to \phi \) in
$L^1(\mathbb{R}^d, g)$ as $t \to 0$. Since $\phi$ is uniformly continuous, we can pick $\epsilon = \epsilon(\eta) > 0$ in such a way that $|\phi(x + y) - \phi(x)| \leq \eta$ for all $x$. Thus,

$$\int_{\mathbb{R}^d} |P_t^\epsilon \phi(x) - \phi(x)| g(x) dx \leq E \left[ \int_{\mathbb{R}^d} |\phi(x - X_t) - \phi(x)| g(x) dx \right]$$

$$= E \left[ \int_{\mathbb{R}^d} |\phi(x - X_t) - \phi(x)| 1_{|X_t| < \epsilon} g(x) dx \right] + \int_{\mathbb{R}^d} E \left[ |\phi(x - X_t) - \phi(x)| 1_{|X_t| \geq \epsilon} g(x) dx \right]$$

$$\leq |B_{R^+}(0)| \sup_{|x| < \epsilon} g(x) : \eta + P(|X_t| \geq \epsilon)\|\phi\|_{L^1(\mathbb{R}^d, g)} + c E \left[ g(X_t) 1_{|X_t| \geq \epsilon} \right] \|\phi\|_{L^1(\mathbb{R}^d, g)}.$$}

Since $(g(X_t))_{t \geq 0}$ is uniformly integrable, $X_t \to 0$ in probability, and $\eta > 0$ is arbitrary, we obtain that $\|P_t^\epsilon \phi - \phi\|_{L^1(\mathbb{R}^d, g)} \to 0$ as $t \to 0$. Using that $C_c(\mathbb{R}^d)$ is a dense subset of $L^1(\mathbb{R}^d, g)$, we conclude that $(P_t^\epsilon)_{t \geq 0}$ is a strongly continuous operator semigroup on $L^1(\mathbb{R}^d, g)$.

9° $\mathbf{[\mathcal{F}]} \Rightarrow \mathbf{[a]}$. Let $\phi \in C_c(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, g)$ such that $\phi \geq 0$ and $\phi = 1$ on $B_1(0)$. Using $g \geq 1$ and the submultiplicative property of $g$, we see that

$$\|P_t^\epsilon \phi\|_{L^1(\mathbb{R}^d, g)} = E \left[ \int_{\mathbb{R}^d} (g(x + X_t)) \phi(x) dx \right] \geq \frac{1}{e} E \left[ g(X_t) \right] \int_{\mathbb{R}^d} \frac{\phi(x) dx}{g(x)};$$

this implies that $E \left[ g(X_t) \right] < \infty$. 

The proof of Theorem 3 contains the following moment result for Lévy processes with bounded jumps. Alternate proofs can be found in Sato [9, Theorem 25.3, p. 159] or [5, Lemma 8.2]. If we use in Step 3° the submultiplicative function $g(x) := e^{\beta |x|}$, $\beta > 0$, $x \in \mathbb{R}^d$, we get the following corollary.

**Corollary 7.** Let $Y = (Y_t)_{t \geq 0}$ be a Lévy process whose jumps are uniformly bounded. Then $Y$ has exponential moments, i.e. $E \left[ e^{\beta |Y_t|} \right] < \infty$ for all $\beta > 0$ and $t \geq 0$.

If $Y$ has a non-degenerate jump part, then moments of the type $E \left[ e^{\beta |X_t|^{1+\epsilon}} \right]$ do not exist, cf. [2, Theorem 3.3(c)].

2. **DOOB’S CONDITION [DL] FOR LÉVY PROCESSES**

Let $Y$ be a stochastic process and $\mathcal{T}$ be the family of all stopping times with respect to the natural filtration of $Y$. Recall that $Y$ satisfies the condition [DL] if for each fixed $t > 0$ the family $(Y_{t \wedge \sigma})_{\sigma \in \mathcal{T}}$ is uniformly integrable, i.e. if

$$\forall t > 0 : \limsup_{R \to \infty} \int_{\mathcal{T}} |Y_{t \wedge \sigma} - Y_{t \wedge \sigma \wedge R}| dP = 0.$$}

It is well known, cf. [8, Proposition IV.1.7, p. 124], that a local martingale is a martingale if, and only if, it is of class DL.

As a direct consequence of Theorem 3 we get the following characterization of the condition [DL] for functions of a Lévy process.

**Theorem 8.** Let $X = (X_t)_{t \geq 0}$ be a Lévy process with triplet $(b, Q, \nu)$ and $g$ a locally bounded submultiplicative function. The following assertions are equivalent.

a) The process $(g(X_t))_{t \geq 0}$ satisfies the condition [DL];

b) $E[g(X_t)]$ is finite for some $t > 0$;

c) $\int_{|y| \geq 1} g(y) \nu(dy) < \infty$.

**Corollary 9.** Let $X$ be a Lévy process and $f : \mathbb{R}^d \to \mathbb{R}$ such that $|f(x)| \leq g(x)$ for some locally bounded submultiplicative function $g$. If $E[g(X_t)] < \infty$ and if $(f(X_t))_{t \geq 0}$ is a local martingale, then $(f(X_t))_{t \geq 0}$ is a martingale.
Proof. Theorem 3 shows that \( g(X_t) \) is infinitely divisible if, and only if, there is a Lévy process \( A \) such that \( Af = 0 \), then \( f(X_t) \) is, by Itô’s formula, a local martingale. Corollary 9 thus gives a condition when this local martingale is a true martingale.

A direct proof that Theorem 3(c) entails (d) Sometimes it is useful to have a direct proof that existence of the moments of the Lévy measure give generalized moments for the process. The approach below gives a method using standard ‘household’ techniques from any course on Markov processes, notably Dynkin’s formula and Gronwall’s inequality; therefore it applies to more general (strong) Markov processes.

Alternative proof for Theorem 3(c) \(\Rightarrow\) (d) resp. Theorem 3(e) \(\Rightarrow\) (a). In view of Lemma 2 we may replace \( g \) by its regularization \( g^\varepsilon \). Combining Dynkin’s inequality (Lemma 11) and the estimate from Lemma 6 shows

\[
E \left[ g^\varepsilon(X_{t,\sigma}) \right] \leq g^\varepsilon(0) + C_{b,Q}^\varepsilon \mathbb{E} \left[ \int_{\{0,t,\sigma\}} g(X_s) \, ds \right].
\]

The constant \( C \) depends on \( \varepsilon \), the triplet \((b, Q, \nu)\) and on \( \int_{|y|\geq1} g(y) \nu(dy) \), see Lemma 6. If we replace \( \sigma \) by \( \sigma \wedge \tau_R \) with \( \tau_R = \inf\{s \geq 0 \mid |X_s| \geq R\} \) and set \( \kappa_R := \sup_{|y|\leq R} g(y) \), then we get

\[
E \left[ g^\varepsilon(X_{t,\sigma\wedge\tau_R}) \wedge \kappa_R \right] \leq g^\varepsilon(0) + C_{b,Q}^\varepsilon \mathbb{E} \left[ \int_{\{0,t,\sigma\wedge\tau_R\}} g(X_s) \wedge \kappa_R \, ds \right]
\]

\[
\leq g^\varepsilon(0) + C_{b,Q}^\varepsilon \int_{(0,t]} \mathbb{E} \left[ g^\varepsilon(X_{s\wedge\tau_R}) \wedge \kappa_R \right] \, ds.
\]

We may now appeal to Gronwall’s lemma and find

\[
E \left[ g^\varepsilon(X_{t,\sigma\wedge\tau_R}) \wedge \kappa_R \right] \leq g^\varepsilon(0)e^{C_{b,Q}^\varepsilon}
\]

and an application of Fatou’s lemma on the left-hand side proves \( E \left[ g^\varepsilon(X_{t,\sigma}) \right] < \infty \), and (d) follows, if we take \( \sigma \equiv t \).  

3. Martingale methods for Lévy processes

Let us give a further application of Theorem 3 to the recurrence/transience behaviour of Lévy processes. As a warm-up and in order to illustrate the method, we begin with a short proof for the characterization of infinitely divisible random variables taking values in a lattice; the result as such is, of course, well-known (Sato [9] Section 24). Recall that a random variable \( Y \) is infinitely divisible if, and only if, there is a Lévy process \( (X_t)_{t \geq 0} \) such that \( Y \sim X_1 \).

Theorem 10. Let \((X_t)_{t \geq 0}\) be a one-dimensional Lévy process with characteristic exponent \( \psi \) and triplet \((b, Q, \nu)\), see (1). The following assertions are equivalent (for fixed \( \beta \neq 0 \))

a) \( \psi(\beta) = i\alpha \) for some \( \alpha \in \mathbb{R} \).

b) \( \mathbb{E} \left[ e^{i \beta X_t} \right] = 1 \) for some, hence for all, \( t > 0 \).

c) \( X_{t_0} + \gamma \) takes values in the lattice \( 2\pi \mathbb{Z} \) for some \( t_0 > 0 \).

d) \( X_t + \alpha\beta^{-1}t \) takes values in the lattice \( 2\pi \mathbb{Z} \) for some \( \alpha \in \mathbb{R} \) and all \( t > 0 \).

e) \( \text{supp} \nu \subset 2\pi \mathbb{Z} \) and \( Q = 0 \) and \( b = -\alpha\beta^{-1} + 2\pi \beta^{-1} \sum_{|k|<\beta(2\pi)^{-1}} k\nu(2\pi \mathbb{Z} \setminus \{|k|<\beta(2\pi)^{-1}\}) \).
Remark 11. Our proof shows that there is also a connection between \( \gamma \) and \( \alpha \) in Theorem 10(c) and 3. If we use \( t = t_0 \) in 10(d), it holds that \( \gamma \in \alpha \beta^{-1} t_0 + 2 \pi \beta^{-1} Z \).

**Proof.** (a)⇒(b) This follows from \( E \left[ e^{i \beta X_t} \right] = e^{-t \psi(\beta)} \) for any \( t > 0 \).

(b)⇒(c) We fix \( t_0 > 0 \) such that \( E \left[ e^{i \beta X_{t_0}} \right] = 1 \) and apply Lemma 12 to \( X = \beta X_{t_0} \). This shows that there exists some \( \gamma \in \mathbb{R} \) such that \( X \) is supported in \( 2 \pi \beta^{-1} Z - \gamma \).

(c)⇒(d) Let \( t_0 > 0 \) be as in (c) and \( t > 0 \). We see that \( E \left[ e^{i \beta X_t} \right] = (e^{-t \psi(\beta)})^{t/t_0} \). As \( |e^{-t \psi(\beta)}| = 1 \), we know that \( \psi(\beta) \in \mathbb{R} \). Lemma 12 shows that \( X_t \) is supported in \( 2 \pi \beta^{-1} Z + i \psi(\beta) \beta^{-1} t \) for every \( t > 0 \).

(d)⇒(e) Let \( t > 0 \). A direct calculation shows that
\[
E \left[ e^{i \beta X_t} \right] = \sum_{k \in \mathbb{Z}} e^{i 2 \pi \beta^{-1} k} \mathbb{P}(X_t = 2 \pi \beta^{-1} k - \alpha \beta^{-1} t) = e^{-i \alpha t}.
\]

On the other hand, infinite divisibility entails
\[
e^{-i \alpha t} = E \left[ e^{i \beta X_t} \right] = e^{-t \psi(\beta)} \text{ for all } t > 0.
\]

This is only possible if \( \psi(\beta) = i \alpha \). Comparing this with the Lévy–Khintchine formula 11, we infer that \( b = -\alpha \beta^{-1} + 2 \pi \beta^{-1} \sum_{|k| < \beta(2 \pi)^{-1}} \nu(\{2 \pi \beta^{-1} k\}), Q = 0 \) and supp \( \nu \subset 2 \pi \beta^{-1} Z \).

(e)⇒(a) This follows from the Lévy–Khintchine formula 11.

The key step in the proof of Theorem 10 is the following lemma which is of independent interest.

**Lemma 12.** Let \( X \) be a real random variable. If there exists \( \beta \in \mathbb{R} \setminus \{0\} \) such that \( E \left[ e^{i \beta X} \right] = e^{i \theta} \) for some \( \theta \in \mathbb{R} \), then the distribution of \( X \) is supported on \( 2 \pi \beta^{-1} Z + \beta^{-1} \theta \).

**Proof.** Without loss of generality we may assume that \( \beta = 1 \), i.e. \( E \left[ e^{i X} \right] = 1 \) or \( E \left[ e^{i (X-\theta)} \right] = 1 \) for some \( \theta \in [0, 2 \pi) \). Let \( (X_n)_{n \in \mathbb{N}} \) be iid copies of the random variable \( X \) and define for every \( n \in \mathbb{N} \)
\[
Y_n := \frac{1}{2^n} \prod_{k=1}^n (1 + \cos(X_k - \theta)) = \prod_{k=1}^n \frac{1 + \cos(X_k - \theta)}{2}.
\]

Set \( \mathcal{F}_n := \sigma(X_1, \ldots, X_n) \) and note that \( (Y_n)_{n \in \mathbb{N}} \) is adapted to this filtration. We see that
\[
E[Y_n | \mathcal{F}_{n-1}] = E \left[ \prod_{k=1}^n \frac{1 + \cos(X_k - \theta)}{2} | \mathcal{F}_{n-1} \right] = Y_{n-1} \frac{1 + E \left[ \cos(X_n - \theta) \right]}{2} = Y_{n-1},
\]
so \( (Y_n)_{n \in \mathbb{N}} \) is a discrete-time martingale. As \( 0 \leq Y_n \leq 1 \) and \( EY_n = 1 \), we conclude that \( Y_n \) converges to 1 in \( L^1 \) and a.s. as \( n \to \infty \). Thus, \( Y_1 = E[1|\mathcal{F}_1] = 1 \) a.s., which implies that \( X - \theta \) takes only values in the lattice \( 2 \pi Z \). □

Assume now that \( (X_t)_{t \geq 0} \) is a Lévy process with which admits an exponential moment of the form \( E \left[ e^{\beta X_t} \right] < \infty \) for some \( \beta \neq 0 \). By Theorem 3, \( \int_{|y| \leq 1} e^{\beta y} \nu(dy) < \infty \), and it is easy to see from the Lévy–Khintchine formula 11 that the exponent \( \psi \) has a continuous continuation to all complex numbers \( \xi + i \eta \in \mathbb{C} \) with \( \xi \in \mathbb{R} \) and \( \eta \) between 0 and \( \beta \). In particular,
\[
E \left[ e^{\beta X_t} \right] = e^{-t \psi(-i \beta)}, \quad t > 0,
\]
which shows that the sets \( \{ \beta \in \mathbb{R} | E \left[ e^{\beta X_t} \right] = 1 \}, \{ \beta \in \mathbb{R} | \forall t > 0 : E \left[ e^{\beta X_t} \right] = 1 \} \) and \( \{ \beta \in \mathbb{R} | \psi(-i \beta) = 0 \} \) coincide.

**Theorem 13.** Let \( (X_t)_{t \geq 0} \) be a one-dimensional Lévy process and \( A := \{ \xi \in \mathbb{R} | E \left[ e^{\xi X_t} \right] = 1 \} \). If \( \beta \in A \setminus \{0\} \), then \( \beta X_t \) is transient.
Proof. Define $Y_t = e^{\beta X_t}$, with $\beta \in A \setminus \{0\}$. With a similar argument as in the proof of Lemma 12, we see that $Y_t$ is a martingale. Since $t \mapsto e^{\beta Y_t}$ is positive and right-continuous, the martingale convergence theorem shows that $\lim_{t \to \infty} Y_t = Y_\infty$ a.s. for some a.s. finite random variable $Y_\infty$. As $\beta X_t$ is again a Lévy process, we see that $e^{\beta X_t}$ can only converge if $\beta X_t \to -\infty$ as $t \to \infty$; thus, $\beta X_t$ cannot be recurrent. □

It is clear that Theorem 13 still holds for a $d$-dimensional Lévy process if we interpret $\xi X_t$ and $\beta X_t$ as scalar products with $\xi, \beta \in R^d$. By Cauchy’s inequality, $|\beta \cdot X_t|/|\beta| \leq |X_t|$, and so $\lim_{t \to \infty} |X_t| = \infty$, i.e. $(X_t)_{t \geq 0}$ is transient if $\beta \cdot X_t$ is transient.

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