Non-Universal Effects in Semi-Inclusive $B$ Decays

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We show that most spectra in the semileptonic decay $B \to X_u + l + \nu_l$, such as for example the distribution in the light-cone momentum $p_+ \equiv E_X - |\vec{p}_X|$ recently considered, do not have the same long-distance structure of the photon spectrum in the radiative decay $B \to X_s + \gamma$. On the contrary, the semileptonic distribution in the final hadron energy $E_X$ is connected to the radiative spectrum via short-distance factors only. The $E_X$ distribution also has a specific infrared structure known as the “Sudakov shoulder”. We also discuss an explicit check of the resummation formula for the semileptonic decays, based on a recent second-order computation.

1. Introduction

The main message of this talk is that, unlike what usually stated, most spectra in charmless semileptonic $B$ decays

$$B \to X_u + l + \nu_l,$$

such as the distribution in the light-cone momentum $p_+ \equiv E_X - |\vec{p}_X|$ recently considered $^1$, do not have the same long-distance structure of the photon spectrum in the radiative decays

$$B \to X_s + \gamma.$$

We are therefore in disagreement with the conclusions in $^1$, in which such a short-distance relation is instead claimed. The reason of the non-universality is a kinematical one: while in radiative decays $^2$ the hard scale $Q$ of the hadronic sub-process is fixed to the beauty mass $m_b$, in the semileptonic case one integrates instead over $Q$ up to $m_b$. That modifies the structure of the infrared logarithms in the semileptonic spectra at the subleading level.

We also show that the distribution of the hadron energy $E_X$ in the semileptonic decay is connected to the rare-decay photon spectrum via short-distance factors only. This semileptonic distribution also has a specific infrared structure known as the “Sudakov shoulder” $^2$$^3$$^4$.

Finally, we discuss an explicit check of the resummation formula for the semileptonic decays, based on a recent second-order computation, which is also a check of the basic relation between the hard scale $Q$ and the total hadron energy $E_X$:

$$Q = 2E_X.$$

The first preliminary question is whether beauty decays can be described with perturbation theory or they do not. Technically, the perturbative expansion is controlled by the QCD coupling $\alpha(m_b) \approx 0.22 \ll 1$, and is therefore a legitimate one. The real problem is however: is there a quantity which shows a good agreement with its perturbative prediction? It is natural to consider first inclusive quantities, which constitute “stronger” predictions of perturbative QCD (pQCD), because they are less sensitive to the non-perturbative hadron structure. In other words, if perturbation theory does not work for inclusive quantities, there is little hope that it will work for the spectra, even after adding some non-perturbative component — the shape function $^5$. Inclusive widths cannot be predicted with good accuracy because they are proportional to the fifth power of the ill-defined beauty mass and to $|CKM|$ combinations in principle unknown: $\Gamma \propto m_b^5|CKM|^2$. It is therefore convenient to look at ratios of widths, such as the semileptonic...
branching fraction: \( B_{SL} \equiv \Gamma_{SL}/\Gamma_{TOT} \). A recent second-order computation of the hadronic width shows a good agreement with the experimental value \( B_{exp} \approx 11\% \).  

Being the answer to the first question positive, the next question concerns the expected size of non-perturbative effects in the spectra. On dimensional grounds, one expects power-corrections \( \propto \Lambda/m_b \approx O(10\%) \), where \( \Lambda \) is a typical hadronic scale, which can be identified with \( \Lambda_{MS} \approx 200 \text{ MeV} \), or with the mass of a constituent quark \( m \approx 350 \text{ MeV} \), or with the mass of the \( \rho \) meson \( m_{\rho} \approx 770 \text{ MeV} \). We believe that a scale \( \Lambda \approx 0.5 \text{ GeV} \) is a reasonable choice. Clear peaks are visible in the hadron mass distributions in the radiative decay and in the semileptonic one, associated to the \( K^* \) and \( \rho \) final states respectively. Since these peaks cannot intrinsically be described in pQCD, one can take, as an estimate, just consider the emission of a soft gluon by the strange quark in (2). The invariant mass \( m_X = (p_s + k)^2 \approx m_b k_+ \), where \( k_+ \equiv k_0 + k_3 \) and we have taken the strange quark flying in the minus direction, so that \( p_s \approx 1/2 m_b (1; 0; 0; -1) \). The above equation shows an amplification of the soft effects in the jet mass, because a soft momentum component is multiplied by the hard scale. A soft gluon with momentum of the order of the hadronic scale, \( k_+ \approx \Lambda \), is certainly controlled by non-perturbative effects such as the structure of the \( B \) meson: that implies the slice \( m_X \approx \sqrt{m_b \Lambda} \) is non-perturbative. This is a quite distinct effect with respect to final-state hadronization, which is expected to be substantial in the resonance region only. The above equation for the \( m_X \) slice shows for example that in the top decay \( t \to X_b + W \) a final jet with a mass of \( \approx 10 \text{ GeV} \) cannot be described with “pure” pQCD! In general, we may identify the following kinematical regions:

1. \( m_X \approx Q \), i.e. the mass of the jet is of the order of the hard scale. The rate can be computed in fixed-order perturbation theory because there are no large infrared logarithms;

2. \( \sqrt{Q \Lambda} \ll m_X \ll Q \). The process is still perturbative but there are large infrared logarithms of the form

\[
\alpha^n \log^k \frac{Q^2}{m_X} \quad (k = 1, 2, \ldots, 2n),
\]

which need to be resummed to all orders \( n \) in order to have a consistent theoretical prediction;

3. \( \Lambda \ll m_X \leq O(\sqrt{\Lambda Q}) \). As discussed above, Fermi-motion effects, related to soft interactions, are substantial and have to be taken into account by introducing a non-perturbative function, the shape function;

4. \( \Lambda \approx m_{\Lambda} \). This is the exclusive or resonance region, where the rate is dominated by few channels and it is completely non-perturbative; to compute, one has to use for example lattice QCD.

One may ask: in which sense the decay becomes progressively more perturbative in the limit \( m_b \to \infty \)? The main peaks of the jet mass distributions, for example, occur in region \( \Lambda \) and are therefore non-perturbative even in the infinite mass limit. The answer to this question is that region \( \Lambda \) becomes progressively larger in the infinite mass limit and better separated from the Fermi motion region \( \alpha \).

2. Non-universality

The radiative decays \( \rho \) have a simpler long-distance structure than the semileptonic decays for illustrative purposes, let us neglect the beauty mass and assume that the top width is smaller than the life-time of a hadronic resonance.
That is because, in the radiative case, the tree-level process is the 2-body decay

$$b \rightarrow s + \gamma$$

having the large final hadron energy $$2E_X \approx m_b$$, where in the last member we have neglected the small strange mass $$m_s \ll m_b$$. As anticipated, the total hadron energy $$E_X$$ fixes the hard scale $$Q$$ in the decay according to eq. (3), so that $$Q \approx m_b$$. We are interested in the threshold region

$$m_X \ll E_X \leq m_b,$$

which can be considered a kind of “perturbation” of the tree-level process due to soft-gluon effects. The final hadron mass $$m_X$$, which vanishes in lowest order, remains indeed small in higher orders because of the condition (6), while the large tree-level hadron energy is only mildly increased by soft emissions.

In semileptonic decays, the tree-level process is instead the 3-body decay

$$b \rightarrow u + l + \nu$$

and the hadron energy $$E_X$$ (i.e. the energy of the up quark) can become substantially smaller than half of the beauty mass $$m_b/2$$. In other words, kinematical configurations are possible with $$E_X \approx m_b/2$$ as well as with $$E_X \ll m_b$$. Consider for example the kinematical configuration, in the $$b$$ rest frame, with the electron and the neutrino parallel to each other, having a large hadron energy, or the configuration with the leptons back to back, each one with an energy $$\approx m_b/2$$, having instead a small hadron energy. This fact is the basic additional complication in going from the radiative decays to the semileptonic ones: the hard scale is no more fixed by the heavy flavor mass but it depends on the kinematics according to eq. (3).

Before going on, let us give the general definition of a short-distance quantity in pQCD as far as threshold effects are concerned. We consider a process characterized by a hard scale $$Q \gg \Lambda$$, and by the infrared scales $$m_X^4/Q^2$$, $$m_X^2 \ll Q^2$$

--- the soft scale and the collinear scale respectively. If infrared effects are absent in the quantity under consideration, logarithmic terms of the form (4) must not appear in its perturbative expansion. The presence of such terms would indeed signal significant contributions from small momentum scales. That is because these terms originate from the integration of the infrared-enhanced pieces of the QCD matrix elements from the hard scale down to one of the infrared scales:

$$\int_{m_X^2/Q^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \quad \int_{m_X^2/Q^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \Rightarrow \log \frac{Q^2}{m_X^2}. \quad (8)$$

If perturbation theory shows a significant contribution from small momentum scales to some cross section or decay width, we believe there is no reason to think that the same should not occur in a non-perturbative computation.

On the contrary, a quantity such as a spectrum or a ratio of spectra is long-distance dominated if it contains infrared logarithmic terms in the perturbative expansion of the form (4). These terms must be resummed to all orders of perturbation theory in the threshold region (6). Our criterion to establish whether a quantity is short-distance or it is not is rather “narrow”: we believe it is a fundamental one and we use it systematically, i.e. we derive all its consequences. The consequences, as we are going to show, are in some cases not trivial.

Factorization and resummation of threshold logarithms in the radiative decays (2) leads to an expression for the event fraction of the form:

$$\frac{1}{\Gamma_r} \int_0^{t_s} \frac{d\Gamma_r}{dt_s} \ dt_s = C_r[\alpha(m_b)] \sum \{t_s; \alpha(m_b)\}$$

$$+ D_r[t_s; \alpha(m_b)], \quad (9)$$

where $$\Gamma_r$$ is the inclusive radiative width and $$t_s \equiv m_X^2/m_b^2$$ ($$x_r \equiv 2E_r/m_b = 1 - t_s$$). We have defined the following quantities, all having an expansion in powers of $$\alpha$$:

- $$C_r(\alpha)$$, a short-distance, process-dependent coefficient function, independent on the hadron variable $$t_s$$. The explicit expression of the first-order correction $$C_r^{(1)}$$ has been given in (3).
• $\Sigma(u; \alpha)$, the universal QCD form factor for heavy flavor decays, having a double expansion of the form:

$$
\Sigma[u; \alpha] = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{2n} \Sigma_{nk} \alpha^n \log^{k} \frac{1}{u} \\
= 1 - \frac{1}{2} \alpha C_F \log^2 \frac{1}{u} + \frac{7}{4} \alpha C_F \log \frac{1}{u} + \\
+ \frac{1}{8} \left( \frac{\alpha C_F}{\pi} \right)^2 \log^4 \frac{1}{u} + \cdots,
$$

(10)

where $C_F = 4/3$. In higher orders, as it is well known, $\Sigma$ contains at most two logarithms for each power of $\alpha$, coming from the overlap of the soft region and the collinear one in each emission and it has an exponential structure; \footnote{This form factor can be expressed in terms of the shape function $f$ via a universal coefficient function $C$ (not to be confused with $C_t$): $\Sigma[u;\alpha(Q)] = C(u;Q,\mu_F) \ast f(u;\mu_F)$, where the star denotes a convolution and $\mu_F \leq Q$ is a factorization scale. The coefficient function $C$ described hard collinear emissions and soft emissions with energies between the factorization scale $\mu_F$ and the hard scale $Q$. If we take for example $\mu_F = Q$, the first integral and the second one on the l.h.s. of eq. \ref{eq:10} are related to the shape function and to its coefficient function respectively.}

• $D_r(t_s; \alpha)$, a short-distance, process-dependent remainder function, not containing infrared logarithms and vanishing for $t_s \to 0$ as well as for $\alpha \to 0$.

Factorization and resummation of threshold logarithms in the semileptonic case is conveniently made starting with distributions not integrated over $E_X$, i.e. not integrated over the hard scale $Q$. The most general distribution in process \ref{eq:10}, a triple distribution, has been originally resummed in \cite{U. Aglietti} (see also \ref{eq:11} and \ref{eq:12}):

$$
\frac{1}{\Gamma} \int_0^u \frac{d^3 \Gamma}{dx dw du'} = C \left[ x, w; \alpha(w m_b) \right] x \\
\times \Sigma[u;\alpha(w m_b)] + D \left[ x, u, w; \alpha(w m_b) \right],
$$

(11)

where we have defined the following kinematical variables:

$$
x = \frac{2E_l}{m_b}, \quad w = \frac{Q}{m_b} \quad (0 \leq w \leq 2)
$$

(12)

and

$$
u = \frac{1 - \sqrt{1 - (2m_X/Q)^2}}{1 + \sqrt{1 - (2m_X/Q)^2}} \simeq \left( \frac{m_X}{Q} \right)^2.
$$

(13)

In the last member we have kept the leading term in the threshold region $m_X \ll Q$ only and $\Gamma$ is the total semileptonic width. Eq. \ref{eq:11} is a “kinematical” generalization of eq. \ref{eq:6} together with the quantities involved:

• $C \left[ x, w; \alpha \right]$, a coefficient function, dependent on the hadron energy and lepton one but independent on the hadron variable $w$;

• $\Sigma[u;\alpha(w m_b)]$, the universal QCD form factor for heavy flavor decays, which now is evaluated for a coupling with a general argument $Q = w m_b \leq m_b$;

• $D \left[ x, u, w; \alpha \right]$, a remainder function, depending on all the kinematical variables, not containing infrared logarithms and vanishing for $\alpha \to 0$ and $u \to 0$.

Many semileptonic spectra, such as the $p_+$ spectrum, are obtained integrating the resummed triple differential distribution \ref{eq:11} over the hadron energy, i.e. over the hard scale of the hadronic subprocess. The infrared logarithms entering the distribution in $p_+ \equiv p_+/m_b$ can be resummed by means of an effective form factor of the form \ref{eq:10}

$$
\Sigma_P \left[ p_+; \alpha(m_b) \right] \propto \int_0^\infty dw C_H \left[ w; \alpha(w m_b) \right] x \\
\times \Sigma \left[ p_+/w; \alpha(w m_b) \right],
$$

(14)

where we have omitted a constant factor and $C_H(w;\alpha) = 2w^2(3 - 2w) + O(\alpha)$ is a coefficient function. There is an integration over $w \in [0,1]$, which enters the first argument of the universal form factor $\Sigma$ as well as the argument of the QCD coupling. Hadronic subprocesses with different hard scales $Q = w m_b$ therefore contribute. By measuring the photon spectrum in the radiative decay \ref{eq:2}, one can only extract $\Sigma[u;\alpha(m_b)]$, while for the $p_+$ spectrum it is necessary to know $\Sigma[u;\alpha(w m_b)]$ with $0 \leq w \leq 1$. We conclude
therefore that it is not possible to derive the long-distance structure of the $\hat{p}_+$ spectrum from the radiative decay. Furthermore, according to the criterion discussed before, the ratio between the $\hat{p}_+$ spectrum and the radiative mass distribution evaluated for $t_s = \hat{p}_+$ is not a short-distance quantity: infrared logarithms only cancel in leading logarithmic approximation. The relation between the $\hat{p}_+$ spectrum and the radiative one therefore is not a true short-distance relation. These non-universality effects are of higher order in the log counting but not in the twist. Our conclusions are therefore in disagreement with those ones derived in [1]. Similar considerations can be repeated for the charged lepton spectrum and for the semileptonic distributions in the hadronic mass normalized to the hadronic energy or to the beauty mass [9].

3. Hadron energy spectrum

The hadron energy spectrum contains in first order logarithmic terms for $w > 1$ of the form

$$\alpha \log^2(w - 1) \text{ and } \alpha \log(w - 1),$$  

(15)

which formally diverge for $w \to 1^+$, inside the physical domain. These terms originate from the fact that at tree level the hadron energy is restricted to half the beauty mass:

$$E_X^{(0)} = E_u \leq \frac{m_b}{2},$$  

(16)

while it that go up to the whole beauty mass in higher orders:

$$E_X \leq m_b.$$  

(17)

Slightly above $m_b/2$ there are soft-gluon effects related to real emissions which cannot be cancelled by the virtual corrections, subject to the limitation [10]: that is the physical origin of the large logarithms in [10]. For $w \leq 1$ there are no large logarithms and therefore the usual fixed-order expansion holds:

$$\frac{1}{\Gamma} \frac{d\Gamma}{dw} = L^{(0)}(w) + \alpha L^{(1)}(w) + \cdots$$  

(18)

For $w > 1$ the large logarithms can be resummed by means of an expression of the form [4]:

$$\frac{1}{\Gamma} \frac{d\Gamma}{dw} = C_{W1}(\alpha) \left(1 - C_{W2}(\alpha) \Sigma[w - 1; \alpha(m_b)] + D_W(w; \alpha)\right).$$  

(19)

The following quantities enter the resummed spectrum:

1. $C_{W1}(\alpha)$, a first coefficient function determined imposing the continuity of the spectra [18] and [14] for $w \to 1, \mp 1$ respectively;

2. $C_{W2}(\alpha)$, a second coefficient function, determined imposing the consistency of the resummed expression with the fixed-order computation;

3. $\Sigma[w - 1; \alpha(m_b)]$, the universal QCD form factor entering the hadron mass distribution in the radiative case [2], resumming all the large logarithmic corrections for $w \to 1^+$, such as those in [10];

4. $D_W(w; \alpha)$, a remainder function, starting to $O(\alpha)$, free from infrared logarithms and vanishing for $w \to 1^+$.

By measuring the hadron energy spectrum slightly above $m_b/2$ one can determine the universal QCD form factor $\Sigma[w - 1; \alpha(m_b)]$, which also enters the radiative spectrum. Unlike previous cases, the relation between the semileptonic hadron energy spectrum and the radiative spectrum is a true short-distance relation.

A form analogous to [19] is also found in the resummation of the $C$-parameter above the 3-jet threshold [14]. The resummation of the double distribution in the hadron energy and in the charged lepton one involves a generalization of [19] [4].

4. Check of resummation formula

The resummed formula on the r.h.s. of eq. (11) has been derived with general effective-theory arguments, holding to any order in $\alpha$ [3,4]; one

\begin{footnote}
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The explicit expressions for the coefficient functions and the remainder function can be found in [4].
\end{footnote}
basically considers the infinite-mass limit for the beauty quark, $m_b \to \infty$, while keeping the hadronic energy $E_X$ and the hadronic mass $m_X$ fixed. The main result is that the hard scale $Q$ enters:

1. the argument of the infrared logarithms factorized in the QCD form factor $\Sigma[u; \alpha]$, $\log 1/u \cong \log Q^2/m_X^2$;

2. the argument of the running coupling $\alpha = \alpha(Q)$, from which the form factor $\Sigma[u; \alpha(Q)]$ depends — as well as the coefficient function and the remainder function do.

An explicit check of property 1 has been obtained verifying the consistency between the resummed expression on the r.h.s. of eq. (11) expanded up to $O(\alpha)$ and the triple distribution computed to the same order in [13]. Since the dependence of the coupling on the scale is a second-order effect, point 2 cannot be verified with the above computation. The possibility for instance that the hard scale $Q = w m_b$ in (11) is fixed instead by the beauty mass $m_b$, i.e. by $Q = m_b$, cannot be explicitly ruled out by comparing with the $O(\alpha)$ triple distribution, because $\alpha(w m_b) = \alpha(m_b) + O(\alpha^2)$. As far as we known, the only available second-order computation is that of the $O(\alpha^2 n_f)$ corrections to the distribution in the light-cone momentum $\hat{p}_+$. Expanding our resummed expression for the $\hat{p}_+$ spectrum up to second order and comparing the $O(n_f)$ parts of the logarithmic coefficients, we have found complete agreement with the explicit computation. If the coupling is evaluated instead for example at the beauty mass, we obtain different values of the coefficients, in disagreement with the Feynman diagram evaluation 6.

5. Conclusions

We have shown that there is no universality of long-distance effects between the radiative decay photon spectrum and many semileptonic decay distributions, such as for example the distribution in the light-cone momentum $p_+$. An exception is represented by the semileptonic distribution in the hadron energy $E_X$, which has exactly the same infrared structure of the radiative spectrum. These conclusions are reached by means of an all-order resummation of the threshold logarithms.

REFERENCES

1. A. Hoang, Z. Ligeti and M. Luke, Phys. Rev. D 71, 093007 (2005) (hep-ph/0502134v1).
2. S. Catani and B. Webber, J. High Energy Phys. 10, 005 (1997) (hep-ph/9710333).
3. U. Aglietti, Nucl. Phys. B 610, 293 (2001), (hep-ph/0104020v3).
4. U. Aglietti, G. Ferrera and G. Ricciardi, hep-ph/0507285v3.
5. G. Altarelli, et al., Nucl. Phys. B 208, 365 (1982); I. Bigi et al., Phys. Rev. Lett. 71, 496 (1993), (hep-ph/9304225).
6. A. Czarnecki, M. Shusarczyk and F. Tkachov, hep-ph/0511004v1.
7. U. Aglietti, M. Ciuchini and P. Gambino, Nucl. Phys. B 637, 427 (2002), (hep-ph/0204140).
8. A. Grozin and G. Korchemsky, Phys. Rev. D 53, 1378 (1996), (hep-ph/9411323); U. Aglietti and G. Ricciardi, Nucl. Phys. B 587, 363 (2000), (hep-ph/0003146); U. Aglietti, Nucl. Phys. Proc. Suppl. 96, 453 (2001), also in Montpellier 2000 Quantum Chromodynamics, (hep-ph/0009214); hep-ph/0102138, Phys. Lett. B 515, 308 (2001), (hep-ph/0103002).
9. U. Aglietti, G. Ferrera and G. Ricciardi, hep-ph/0509095v1.
10. U. Aglietti, G. Ferrera and G. Ricciardi, hep-ph/0509271v1.
11. R. Andersen and E. Gardi, hep-ph/0509360.
12. A. Czarnecki, M. Jezabek and J. Kühn, Acta Phys. Polon. B 20, 961 (1989).
13. F. De Fazio and M. Neubert, J. High Energy Phys. 06, 017 (1999) (hep-ph/9905351).
14. S. Catani, private communication.

6 A few days after paper [10] was put on spires, [11] also appeared, whose results on the $\hat{p}_+$ logarithmic coefficients are in agreement with [10].