CUT SELECTION FOR BENDERS DECOMPOSITION

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Abstract. In this paper, we present a new perspective on cut generation in the context of Benders decomposition. The approach, which is based on the relation between the alternative polyhedron and the reverse polar set, helps us to improve established cut selection procedures for Benders cuts, like the one suggested by Fischetti, Salvagnin, and Zanette [FSZ10]. Our modified version of that criterion produces cuts which are always supporting and, unless in rare special cases, facet-defining.

Our method can be parametrized by the selection of an objective vector in primal space. This can be used to leverage prior knowledge about the problem, as well as solution information obtained e.g. by a heuristic algorithm or from a previous iteration of the Benders decomposition algorithm.

Finally, we discuss our approach in relation to the state of the art in cut generation for Benders decomposition. In particular, we refer to Pareto-optimality and facet-defining cuts and observe that each of these criteria can be matched to a particular subset of parameterizations for our cut generation framework. As a consequence, our framework includes the method to generate facet-defining cuts proposed by Conforti and Wolsey [CW18].

1. Introduction

Consider a generic optimization problem with two subsets of variables $x$ and $y$ where $x$ is restricted to lie in some set $S \subseteq \mathbb{R}^n$ and $x$ and $y$ are jointly constrained by a set of $m$ linear inequalities. Such a problem can be written in the following form:

$$\begin{align*}
\min & \quad c^\top x + d^\top y \\
\text{s.t.} & \quad H x + Ay \leq b \\
& \quad x \in S \subseteq \mathbb{R}^n \\
& \quad y \in \mathbb{R}^k
\end{align*}$$

(1)

The interaction matrix $H \in \mathbb{R}^{m \times n}$ captures the influence of the $x$-variables on the $y$-subproblem: For fixed $x^*$, (1) reduces to an ordinary linear program with constraints $Ay \leq b - H x^*$.

We are interested in cases where the size of the complete problem (1) leads to infeasibly high computation times (or memory demands), but both the problem over $S$ and the problem resulting from fixing $x$ can separately be solved much more efficiently due to their special structures.

To deal with such problems, Benders [Ben62] introduced a method that works by iterating between these two “easier” problems:

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For a problem of the form (1), let the function \( z : \mathbb{R}^n \to \mathbb{R} \cup \{ \pm \infty \} \) represent the value of the optimal \( y \)-part of the objective function for a fixed vector \( x \):

\[
z(x) := \min_{y \in \mathbb{R}^k} \{ d^T y \mid Ay \leq b - Hx \}
\]

The corresponding epigraph of \( z \) is

\[
\text{epi}(z) = \left\{ (x, \eta) \in \mathbb{R}^n \times \mathbb{R} \mid \exists y \in \mathbb{R}^k : Ay \leq b - Hx, d^T y \leq \eta \right\}.
\]

Writing \( \text{epi}_S(z) := \text{epi}(z) \cap (S \times \mathbb{R}) \), this provides us with an alternative representation of (1):

\[
\min \{ c^T \eta \mid (x, \eta) \in \text{epi}_S(z) \}.
\]

This suggests the following iterative algorithm to solve the optimization problem (1):

Start by finding a solution \((x, \eta) \in S \times \mathbb{R}\) that minimizes \( c^T x + \eta \) without any additional constraints (add a generous lower bound for \( \eta \) to make the problem bounded). If \((x, \eta) \in \text{epi}(z)\), then \((x, \eta) \in \text{epi}_S(z)\) (since \( x \in S \)) and the solution is optimal. Otherwise, we add constraints violated by \((x, \eta)\) but satisfied by all \((x', \eta') \in \text{epi}(z)\) and iterate. This is of course just an ordinary cutting plane algorithm and the crucial question is how to select a \textit{separating inequality} in each iteration.

The original Benders algorithm uses \textit{feasibility cuts} (cuts with coefficient 0 for the variable \( \eta \)) and \textit{optimality cuts} (cuts with non-zero coefficient for the variable \( \eta \)), depending on whether the subproblem that results from fixing the \( x \)-variables is feasible or not (see, e.g., [VW10]). Fischetti, Salvagnin, and Zanette [FSZ10], on the other hand, presents a unified perspective that covers both cases: They begin by observing that the subproblem can be seen as a pure feasibility problem, represented by the set

\[
\left\{ y \in \mathbb{R}^k \mid Ay \leq b - Hx^*, d^T y \leq \eta^* \right\}.
\]

This polyhedron will be empty if and only if \((x^*, \eta^*) \notin \text{epi}(z)\) (see (3)) and any Farkas certificate for emptiness of (4) can be used to derive an additional valid inequality. The set of such certificates (up to positive scaling)

\[
P(x^*, \eta^*) := \left\{ \gamma, \gamma_0 \geq 0 \mid \gamma^T A + \gamma_0 d^T = 0, \gamma^T (b - Hx^*) + \gamma_0 \eta^* = -1 \right\}
\]

is called \textit{alternative polyhedron}. Thus \( P(x^*, \eta^*) = \emptyset \) if and only if \((x^*, \eta^*) \in \text{epi}(z)\) and every point \((\gamma, \gamma_0) \in P(x^*, \eta^*)\) induces an inequality \( \gamma^T (b - Hx) + \gamma_0 \eta \geq 0 \) that is valid for \( \text{epi}(z) \) but violated by \((x^*, \eta^*)\).

This characterization is very useful and has been demonstrated empirically to work well in [FSZ10]. However, it exposes some fundamental issues, which are demonstrated by the following example.

\textbf{Example 1} Consider the following optimization problem:

\[
\min \quad x + y \\
2x + y \geq 5 \\
\frac{1}{2}x + y \geq 3 \\
4x + 4y \geq 14
\]
Note that the constraint $4x + 4y \geq 14$ is redundant and does not support the feasible region. Suppose that we want to decompose the problem into its $x$-part and its $y$-part. To obtain the alternative polyhedron for a tentative master solution $(x^*, \eta^*)$, we rewrite the subproblem in the way of (4) as

$$
\begin{align*}
\begin{cases}
y \in \mathbb{R} \\
-y &\leq -5 - (-2x^*) \\
-y &\leq -3 - (-1/2x^*) \\
-4y &\leq -14 - (-4x^*) \\
y &\leq \eta^*
\end{cases}
\end{align*}
$$

We can then visualize the two-dimensional alternative polyhedron via its projection into the $\gamma_1-\gamma_3$-plane:
Writing the components of \((\gamma, \gamma_0)\) in order \((\gamma_1, \gamma_2, \gamma_3, \gamma_0)\), the three extremal points of \(P(0,0)\) are

\[
P_1 = \left( \frac{1}{5}, 0, 0, \frac{1}{5} \right)
\]

\[
P_2 = \left( 0, \frac{1}{3}, 0, \frac{1}{3} \right)
\]

\[
P_3 = \left( 0, 0, \frac{1}{14}, \frac{2}{7} \right).
\]

We can see that for each of these points, as shown by Gleeson and Ryan \([GR90]\), the set of inequalities for which the corresponding dual variable is positive, represents a minimal infeasible subsystem of \((7)\). Consequently, each extremal point yields one of the original inequalities as a cut. This notably includes the redundant inequality \(4x + 4y \geq 14\), which does not support the feasible region but is derived from the extremal point \(P_3\) in the alternative polyhedron.

A cut generated from the alternative polyhedron may thus be very weak, not even supporting the set epi(\(z\)). This is true even if we use a vertex of the alternative polyhedron and even if that vertex minimizes a given linear objective such as the vector \(\mathbf{1}\) as suggested in \([FSZ10]\). Furthermore, as a set of dual vectors, the alternative polyhedron is difficult to interpret and to relate to properties of the problem which might be known a priori.

In the following we present an improved approach for cut generation in the context of Benders decomposition. The approach is based on the relation between the alternative polyhedron as introduced above, which is commonly used for Benders cut generation, and the reverse polar set, originally introduced by Balas and Ivanescu \([BI64]\) in the context of transportation problems.

While the close similarity of the two sets is well-known in principle, we show that the alternative polyhedron can be viewed as an extended formulation of the reverse polar set and discuss the implications of that relation. A description of the former is much more readily available in the context of Benders decomposition, which makes it more useful as a basis for a cut generation routine. However, while all vertices of the alternative polyhedron possess some useful properties (their support corresponds to minimal infeasible subsystems of the Benders subproblem \([FSZ10; GR90]\)), those that correspond to vertices of the reverse polar set have additional advantages: They generate facet cuts, which in particular are always supporting.

Based on this insight, we develop a modified version of the cut generation procedure by Fischetti, Salvagnin, and Zanette \([FSZ10]\) that produces facet cuts for all but a subdimensional set of subproblem objectives without any additional computational effort. In addition, the new criterion is more robust with respect to the formulation of the problem. In particular it always generates supporting Benders cuts, which is not true for the original procedure proposed in \([FSZ10]\).

Furthermore, our method can be parametrized by the selection of an objective vector in primal space. This can be used to leverage information about the problem and its solutions, which may be available \(a \text{ priori}\) or may alternatively be generated during the course of the Benders decomposition algorithm, e.g. by a heuristic approach. Put into context of other well-known selection criteria, most notably Pareto-optimal or facet-defining cuts, each of these criteria can be matched to a particular subset of objective functions used in the context of our cut generation framework.
One particular choice of a parametrization can be obtained from a relative interior point of the feasible region. In this case one can show that the resulting cut-generating procedure is equivalent to the method proposed by Conforti and Wolsey [CW18], if applied to Benders Decomposition.

Using a computationally efficient representation of the relevant geometric objects, we use the theoretical criterion of [CL06] to obtain a practically useful method to generate facet-defining cuts. We thus provide an arguably simpler proof of the main result from [CW18]. It avoids some of the original proof’s technicalities and connects the approach more directly to previous work on cut selection, both within Benders decomposition (e.g.[FSZ10]) and more generally for separation from convex sets (e.g.[CL06]).

Before we proceed by investigating different representations of the set of possible Benders cuts, it is useful to record a general characterization of the set of normal vectors for cuts separating a point from \( \text{epi}(z) \) as defined in (3). In the following, we say that a halfspace \( H^\leq_{(\pi,\alpha)} := \{ x \in \mathbb{R}^n \mid \pi^\top x \leq \alpha \} \) is \( x \)-separating for a convex set \( C \subset \mathbb{R}^n \) and a point \( x \in \mathbb{R}^n \setminus C \) if \( x \notin H^\leq_{(\pi,\alpha)} \supset C \).

**Theorem 2.** Let \( z \) be defined as in (2) such that \( \text{epi}(z) \neq \emptyset \) and let \( (x^*, \eta^*) \), \( (\pi, \pi_0) \) \( \in \mathbb{R}^n \times \mathbb{R} \). Then \( (\pi, \pi_0) \) is the normal vector of a \((x^*, \eta^*)\)-separating halfspace for \( \text{epi}(z) \) if and only if there exists a vector \( \gamma \in \mathbb{R}^n_\geq \) satisfying

\[
(\pi^\top, \pi_0) \begin{pmatrix} x^* \\ \eta^* \end{pmatrix} - \gamma^\top b > 0
\]

(8)

\[
\gamma^\top A - \pi_0 d^\top = 0
\]

(9)

\[
\gamma^\top H = \pi^\top
\]

(10)

\[
\pi_0 \leq 0.
\]

(11)

**Proof.** Let \( h_{\text{epi}(z)} := \sup \{ c^\top x \mid x \in \text{epi}(z) \} \) be the support function of \( \text{epi}(z) \). The vector \((\pi, \pi_0)\) is the normal vector of a \((x^*, \eta^*)\)-separating halfspace for \( \text{epi}(z) \) if and only if

\[
0 < (\pi^\top, \pi_0) \begin{pmatrix} x^* \\ \eta^* \end{pmatrix} - h_{\text{epi}(z)}(\pi, \pi_0).
\]

(12)

By the definition of \( \text{epi}(z) \) (which is closed and polyhedral) and then by strong LP duality, we obtain

\[
h_{\text{epi}(z)}(\pi, \pi_0) = \max_{x \in \mathbb{R}^n, y \in \mathbb{R}^k} \left\{ \begin{pmatrix} \pi^\top, \pi_0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\} \begin{pmatrix} Ay \leq b - Hx \\ d^\top y \leq \eta \end{pmatrix}
\]

(13)

\[
= \min_{\gamma_0 \in \mathbb{R}_{\geq 0}, \gamma \in \mathbb{R}^m_{\geq 0}} \left\{ \gamma^\top b \right\} \begin{pmatrix} \gamma^\top A + \gamma_0 d^\top = 0 \\ \gamma^\top H = \pi^\top \\ -\gamma_0 = \pi_0 \end{pmatrix}.
\]

(14)

Note that in order for the equality \(-\pi_0 = \pi_0\) to hold and (14) to be feasible (and hence (13) to be bounded), we need that \( \pi_0 \leq 0 \).

For any \( \gamma \geq 0 \) satisfying the conditions (9) to (11), we thus have \( \gamma^\top b \geq h_{\text{epi}(z)}(\pi, \pi_0) \). Inequality (8) then implies that (12) is satisfied, which proves the claim. \( \square \)

As argued in the proof, any \( \gamma \) satisfying (9) to (11) is an upper bound for the support function \( h_{\text{epi}(z)} \) of \( \text{epi}(z) \). This means that once we obtain a certificate \( \gamma \) to prove that a
vector \((\pi, \pi_0)\) belongs to an \((x^*, \eta^*)\)-separating halfspace \(H^{\leq}_{((\pi, \pi_0), \alpha)}\), we immediately obtain a corresponding right hand side \(\alpha := \gamma^T b\).

Furthermore, the definition of the support function \(h_{\text{epi}(z)}\) immediately tells us when this right-hand side is actually optimal and the resulting halfspace supports \(\text{epi}(z)\):

**Remark 3.** Let \((x^*, \eta^*) \in \mathbb{R}^n \times \mathbb{R}\) and let \((\pi, \pi_0)\) be the normal vector of an \((x^*, \eta^*)\)-separating halfspace for \(\text{epi}(z)\). If \(\gamma\) minimizes \(\gamma^T b\) among all possible certificates in Theorem 2, then the halfspace \(H^{\leq}_{((\pi, \pi_0), \gamma^T b)}\) supports the set \(\text{epi}(z)\).

2. Benders Cuts from the Reverse Polar

While it would be sufficient for the approach from [FSZ10] to obtain an arbitrary \((x^*, \eta^*)\)-separating halfspace whenever the set in (4) is empty, the alternative polyhedron \(P(x^*, \eta^*)\) actually completely characterizes the set of all possible normal vectors of such halfspaces:

**Corollary 4.** The alternative polyhedron (5) completely characterizes all normal vectors of \((x^*, \eta^*)\)-separating halfspaces for \(\text{epi}(z)\). In particular:

a) Let \((\gamma, \gamma_0) \in P(x^*, \eta^*)\). Then \(\gamma^T H x - \gamma_0 \eta \leq \gamma^T b\) is violated by \((x^*, \eta^*)\), but satisfied by all \((x, \eta) \in \text{epi}(z)\).

b) Let \((\pi, \pi_0)\) be the normal vector of a \((x^*, \eta^*)\)-separating halfspace for \(\text{epi}(z)\). Then there exist \((\gamma, \gamma_0) \in P(x^*, \eta^*)\) and \(\lambda \geq 0\) such that \((\gamma^T H, -\gamma_0) = \lambda \cdot (\pi, \pi_0)\).

Observe, however, that in contrast to Remark 3, Corollary 4 does not guarantee that the cut generated from a point in the alternative polyhedron is supporting: A given vector \((\gamma, \gamma_0) \in P(x^*, \eta^*)\) does not necessarily minimize \(\gamma^T b\) among all points in \(P(x^*, \eta^*)\) which lead to the same cut normal. Indeed, as Example 1 shows, there are cases where this does actually occur and where even a cut generated from a vertex of the alternative polyhedron may not be supporting.

Alternatively, as argued by Cornuéjols and Lemaréchal [CL06], we can characterize the set of normals of separating cuts by the reverse polar set of \(\text{epi}(z) - (x^*, \eta^*)\), which is introduced in Balas and Ivanescu [BI64] and defined as follows:

**Definition 5.** Let \(C \subseteq \mathbb{R}^n\) be a convex set. Then the reverse polar set \(C^-\) of \(C\) is defined as

\[
C^- := \{ c \in \mathbb{R}^n \mid c^T x \leq -1 \text{ for all } x \in C \}.
\]

It is thus a subset of the polar cone

\[
\text{pos}(C)^\circ := \{ c \in \mathbb{R}^n \mid c^T x \leq 0 \text{ for all } x \in C \}.
\]

Note that by this definition, the reverse polar set is given by a, possibly infinite, intersection of halfspaces (an \(\mathcal{H}\)-representation). If \(C\) is a polyhedron, it is actually sufficient to consider halfspaces corresponding to vertices of \(C\). Nonetheless, even for a polyhedron \(C\), efficiently computing an \(\mathcal{H}\)-representation of \(C^-\) may not be possible in general as computing all the vertices of \(C\), when given in \(\mathcal{H}\)-representation, is NP-hard [Kha+08]). Given an explicit \(\mathcal{H}\)-representation of \(C\), there does exist however an extended formulation for \(C^-\) based on the coefficients from a convex combination of the vertices of \(C^-\), which can be easily obtained.

Even without an explicit \(\mathcal{H}\)-representation of the set \(\text{epi}(z)\) (which is itself known to us only by its extended formulation (3)), we can use Theorem 2 and the fact that the set defined by
Figure 2. The reverse polar set \((\text{epi}(z) - (x^*, \eta^*))^-\) and the corresponding polar cone (drawn in a coordinate system with \((x^*, \eta^*)\) as the origin). It can be seen that \((\text{epi}(z) - (x^*, \eta^*))^-\) is contained in the polar cone \(\text{pos}(\text{epi}(z) - (x^*, \eta^*))^o\) (indicated by the black solid lines) but offers a “richer” boundary from which we can choose cut normals. Specifically, for each vertex \(v\) of \((\text{epi}(z) - (x^*, \eta^*))^-\) there exists a facet of \(\text{epi}(z)\) with normal vector \(v\) and vice versa (see Theorem 22).

the inequalities (8) to (11) is homogenous, to easily obtain the following extended formulation of the reverse polar set, which is, as we will see, closely related to the alternative polyhedron:

\[
(\text{epi}(z) - (x^*, \eta^*))^- =\begin{cases}
(\pi, \pi_0) \in \mathbb{R}^{n+1} & \exists \gamma \geq 0 : \\
\left((\pi^\top, \pi_0)\right)\left(\frac{x^*}{\eta^*}\right) - \gamma^\top b \geq 1 \\
A^\top \gamma - \pi_0 d = 0 \\
H^\top \gamma = \pi \\
\pi_0 \leq 0
\end{cases}
\]

Note furthermore that, as a consequence of Remark 3, we can compute for any given normal vector \((\pi, \pi_0)\) a supporting inequality (if one exits) by solving the problem (13) or (14).

We thus have at our disposal two alternative characterizations of the set of possible normal vectors of \((x^*, \eta^*)\)-separating halfspaces: The alternative polyhedron and the reverse polar set. Despite their similarity, subtle differences exist between both representations that affect their usefulness for the generation of Benders cuts.

It should be noted at this point that we are not the first ones to notice the similarity between the approaches of Cornuéjols and Lemaréchal [CL06] and Fischetti, Salvagnin, and Zanette [FSZ10]. Indeed, the work of Cornuéjols and Lemaréchal [CL06] is explicitly cited in [FSZ10], albeit only in a remark about the possibility to exchange normalization and objective function in optimization problems over the alternative polyhedron (see Corollary 14 below). Fischetti, Salvagnin, and Zanette [FSZ10] focus on properties of the alternative polyhedron, as well as practical considerations about the implementation of Benders decomposition. They
list a number of interesting conclusions from practical experience, which to our knowledge
have not been written down clearly anywhere else and make the paper a very interesting read.
This is even more true for the more extensive (unpublished) draft [FSZ09].

However, neither [FSZ09] nor [FSZ10] goes into much detail regarding the exact relation
between the alternative polyhedron and the reverse polar, which underlies the approach
of Cornuéjols and Lemaréchal [CL06]. This is a main motivation for our work in this paper:
We will formulate the relation very precisely which allows us to gain some new insights about
the selection of Benders cuts and thus helps us to improve the performance of practical
implementations of the algorithm.

Before we proceed, we introduce a variant of the alternative polyhedron, the relaxed alter-
native polyhedron, which also appears in [GR90]. We will see that it is equivalent to the
original alternative polyhedron for almost all purposes, but can more easily be connected to
the reverse polar set:

**Definition 6.** Let a problem of the form (1) and a point \((x^*, \eta^*) \in \mathbb{R}^n \times \mathbb{R}\) be given. The relaxed alternative polyhedron \(P \leq (x^*, \eta^*)\) is defined as

\[
P \leq (x^*, \eta^*) := \left\{ \gamma, \gamma_0 \geq 0 \quad \left| \begin{array}{l}
\gamma^\top A + \gamma_0 d^\top = 0 \\
\gamma^\top (b - Hx^*) + \gamma_0 \eta^* \leq -1
\end{array} \right. \right\}.
\]

The following key theorem now becomes a trivial observation. However, to our knowledge,
the relation between the alternative polyhedron and the reverse polar set has not been made
explicit in a similar fashion before.

**Theorem 7.** Let \(z\) be defined as in (2) and \((x^*, \eta^*) \in \mathbb{R}^n \times \mathbb{R}\). Then

\[
(epi(z) - (x^*, \eta^*))^- = \begin{pmatrix} H^\top & 0 \\ 0 & -1 \end{pmatrix} \cdot P \leq (x^*, \eta^*).
\]

We revisit Example 1 to illustrate the above theorem.

**Example 1** (continued) In the situation of the optimization problem (6), we can use Theo-
rem 7 to obtain the reverse polar set

\[
(epi(z) - (0, 0))^- = \begin{pmatrix} H^\top & 0 \\ 0 & -1 \end{pmatrix} P \leq (0, 0)
\]

which is visualized below:
We can see that the point $P_3$, which lead to the non-supporting cut above, is mapped to the interior of the reverse polar set and will hence not appear as an extremal solution.

The fact, that in our example every vertex of the reverse polar set generates a supporting cut, is no coincidence: Indeed, for every point in the reverse polar set that is visible from the origin, the right hand side obtained from a valid $\gamma$ in (16) always yields a supporting cut. For points that are not visible from the origin, this is not necessarily true for all valid vectors $\gamma$, but even there a vector $\gamma$ exists that yields a supporting cut.

The following theorem provides us with a sufficient criterion for the first case (where any valid $\gamma$ yields a supporting cut). It is particularly useful in the context of cut-generating linear programs, which we will consider in Section 2.1 below.

Theorem 8. Let $(\omega, \omega_0) \in \mathbb{R}^n \times \mathbb{R}$ and let $(\pi, \pi_0) \in (\operatorname{epi}(z) - (x^*, \eta^*))^-$ be maximal with respect to the objective $(\omega, \omega_0)$ with $\omega^T \pi + \omega_0 \pi_0 < 0$. Furthermore, let $\gamma \in \mathbb{R}_{\geq 0}^m$ be a valid certificate for $(\pi, \pi_0)$ in (16). Then the halfspace $H_{((\pi, \pi_0), \gamma^T b)} \leq (\pi, \pi_0)$ supports $\operatorname{epi}(z)$.

Proof. By Remark 3, the statement is true if $\gamma$ minimizes $\gamma^T b$ among all possible certificates for the vector $(\pi, \pi_0)$ in Theorem 2. It is easy to verify that $\gamma$ is indeed a valid certificate for $(\pi, \pi_0)$ in Theorem 2. For a contradiction, we hence assume that it does not minimize $\gamma^T b$. Thus, let $\gamma' \geq 0$ be an alternative certificate for $(\pi, \pi_0)$ with $\gamma'^T b < \gamma^T b$. Then from (8) to (11) we obtain that $\gamma'^T A - \pi_0 d^T = 0$ and $\gamma'^T H = \pi^T$.

Furthermore,

$$\pi^T x^* + \pi_0 \eta^* - \gamma'^T b > \pi^T x^* + \pi_0 \eta^* - \gamma^T b \geq 1.$$
We can thus scale both \((\pi, \pi_0)\) and \(\gamma'\) by an appropriate factor \(\lambda \in (0, 1)\) to obtain that 
\[
\lambda \cdot (\pi, \pi_0) \in (\text{epi}(\omega) - (x^*, \eta^*))^-(\text{as certified by the vector } \lambda \cdot \gamma' \text{ in } (16)) \text{ and }
\]
\[
(\omega^T, \omega_0)(\lambda \cdot (\pi, \pi_0)) = \lambda \cdot (\omega^T \pi + \omega_0 \pi_0) > \omega^T \pi + \omega_0 \pi_0,
\]
a contradiction to the optimality of \((\pi, \pi_0)\). \(\square\)

The next section provides some answers as to when the conditions of the above theorem can be assumed to hold. Furthermore, we derive a method of using the original alternative polyhedron while guaranteeing that every cut that is generated corresponds to an optimal point in the reverse polar set in the sense of Theorem 8. In particular, this means that every cut that is generated will support epi\((z)\).

### 2.1. Cut-Generating Linear Programs

One way to select a particular cut normal from the reverse polar set or the alternative polyhedron is by maximizing a linear objective function over these sets. As both the reverse polar set and the relaxed alternative polyhedron are unbounded, however, there are objective directions for which no (finite) optimal solution exists. Cornuéjols and Lemaréchal [CL06, Theorem 2.3] establish the criteria on the objective function for which optimization problems over the reverse polar set are bounded. We have simplified the notation and rephrased the relevant parts of the theorem according to our terminology.

**Theorem 9.** Let \((x^*, \eta^*) \notin \text{epi}(z), (\omega, \omega_0) \in \mathbb{R}^n \times \mathbb{R}\), and
\[
z^* := \max \left\{ \omega^T \pi + \omega_0 \pi_0 \mid (\pi, \pi_0) \in (\text{epi}(z) - (x^*, \eta^*))^- \right\}.
\]
Then
\[
z^* \begin{cases} 
\leq 0 & \text{if } (\omega, \omega_0) \in \text{cl}(\text{pos}(\text{epi}(z) - (x^*, \eta^*))) \\
= \infty & \text{otherwise}.
\end{cases}
\]
Furthermore, if \((\omega, \omega_0) \in (\text{epi}(z) - (x^*, \eta^*))\), then \(z^* \leq -1\).

Note in particular that the last part of the above statement implies \(z^* < 0\) whenever \((\omega, \omega_0) \in \text{pos}(\text{epi}(z) - (x^*, \eta^*)) \setminus \{0\}\), which provides us with a large variety of objective functions for which \(\omega^T \gamma + \omega_0 \gamma_0 < 0\) in the optimal solution. By Theorem 8, this means that the cut which results from maximizing these objectives over the reverse polar set is guaranteed to be supporting. While a similar statement holds with respect to the relaxed alternative polyhedron, we can in fact use Theorem 7 to derive a much more precise relation between optimization problems over both sets.

**Theorem 10.** Let \(z\) be defined as in (2), \((x^*, \eta^*), (\omega, \omega_0) \in \mathbb{R}^n \times \mathbb{R}\) and
\[
(17) \quad (\bar{\omega}, \bar{\omega}_0)^T := (H \omega, -\omega_0)^T.
\]
Then \((\pi, \pi_0)\) is an optimal solution to the problem
\[
(18) \quad \max \left\{ \omega^T \pi + \omega_0 \pi_0 \mid (\pi, \pi_0) \in (\text{epi}(z) - (x^*, \eta^*))^- \right\}
\]
if and only if there exists \(\gamma^*\) such that \(H^T \gamma^* = \pi\) and \((\gamma^*, -\pi_0)\) is an optimal solution to the problem
\[
(19) \quad \max \left\{ \bar{\omega}^T \gamma + \bar{\omega}_0 \gamma_0 \mid (\gamma, \gamma_0) \in \text{cl}(x^*, \eta^*) \right\}.
\]
Furthermore, the objective values of both optimization problems are identical.
Proof. Let \((\pi, \pi_0)\) be an optimal solution to (18). By Theorem 7, there exists a vector \(\gamma^*\) with \(H^\top \gamma = \pi\) such that \((\gamma^*, -\pi_0) \in P^\leq(x^*, \eta^*)\). Now, let \((\gamma', \gamma_0')\) be an arbitrary point in \(P^\leq(x^*, \eta^*)\). By Theorem 7, \((H^\top \gamma', -\gamma_0') \in (\text{epi}(z) - (x^*, \eta^*))^-\) and thus from the optimality of \((\pi, \pi_0)\) for (18) we obtain
\[
\tilde{\omega}^\top \gamma' + \tilde{\omega}_0 \gamma_0' = (H \omega)^\top \gamma' - \omega_0 \gamma_0' = \omega^\top (H^\top \gamma') + \omega_0 (-\gamma_0') \\
\leq \omega^\top \pi + \omega_0 \pi_0 = (H \omega)^\top \gamma^* - \omega_0 (-\pi_0) = \omega^\top \gamma^* + \tilde{\omega}_0 (-\pi_0),
\]
which proves the optimality of \((\gamma^*, -\pi_0)\) for (19).

Similarly, let \((\gamma, \gamma_0)\) be an optimal solution to (19), \(\pi := H^\top \gamma\), and \(\pi_0 := -\gamma_0\), which means by Theorem 7, \((\pi, \pi_0) \in (\text{epi}(z) - (x^*, \eta^*))^-\).

Now, let \((\pi', \pi_0')\) be an arbitrary point in \(\text{epi}(z) - (x^*, \eta^*)))^-\). By Theorem 7, there exists \(\gamma'\) with \(H^\top \gamma' = \pi'\) such that \((\gamma', -\pi_0') \in P^\leq(x^*, \eta^*)\). Using the optimality of \((\gamma, \gamma_0)\) for (19), we obtain
\[
\omega^\top \pi' + \omega_0 \pi_0' = (H \omega)^\top \gamma' - \omega_0 (-\pi_0') = \omega^\top \gamma' + \tilde{\omega}_0 (-\pi_0') \\
\leq \omega^\top \pi + \omega_0 \pi_0 = \omega^\top H^\top \gamma - \omega_0 \gamma_0 = \omega^\top \pi + \omega_0 \pi_0,
\]
which proves the optimality of \((\pi, \pi_0)\) for (18). \(\square\)

Depending on the particular application, the structure of the matrix \(H\) can vary in many ways, but in line with our assumption that the \textit{master} problem should be significantly smaller than the \textit{subproblem}, it is reasonable to assume that \(H\) has many more rows than columns.

In this sense, we will henceforth use the relaxed alternative polyhedron as an extended formulation for the reverse polar set, which in particular is always polynomial in size. This will allow us to generate Benders cuts from points in the reverse polar set while algorithmically relying on the relaxed alternative polyhedron, an explicit description of which is generally trivial to obtain.

Note that the optimization problem stated in (19) is technically more general, since there is no reason to limit ourselves to objective functions of the form (17) \textit{a priori}. If we choose a different objective function, we still obtain a valid cut. However, since there may be no objective function \((\omega, \omega_0)\) such that the resulting cut normal is optimal for (18), we lose some of the properties associated with optimal solutions from the reverse polar set.

Indeed, this is the approach that Fischetti, Salvagnin, and Zanette [FSZ10] take: They use the problem in (19) with \(\tilde{\omega}_m = 0\) for all \(m\) which correspond to rows of zeros in the interaction matrix \(H\) and \(\tilde{\omega}_m = 1\) for all other \(m\), as well as \(\tilde{\omega}_0 = 1\) (or some other manual scaling factor). In general, there exists no vector \((\omega, \omega_0)\) such that this choice can be obtained by (17).

Before we return to Example 1 to illuminate this issue further, we note that optimization problems over the original and the relaxed alternative polyhedron are equivalent, provided that the optimization problem over the relaxed alternative polyhedron has a finite non-zero optimum:

**Remark 11.** Let \(z\) be defined as in (2) and let \((x^*, \eta^*) \in \mathbb{R}^n \times \mathbb{R}\). Let \((\tilde{\omega}, \tilde{\omega}_0) \in \mathbb{R}^m \times \mathbb{R}\) be such that \(\max\{\tilde{\omega}^\top \gamma + \tilde{\omega}_0 \gamma_0 | \gamma, \gamma_0 \in P^\leq(x^*, \eta^*)\} < 0\). Then the sets of optimal solutions for \(\tilde{\omega}^\top \gamma + \tilde{\omega}_0 \gamma_0\) over \(P^\leq(x^*, \eta^*)\) and \(P(x^*, \eta^*)\) are identical. Furthermore, every vertex of \(P^\leq(x^*, \eta^*)\) is also a vertex of \(P(x^*, \eta^*)\).

We now take a closer look at the role of objective functions in the context of Example 1:
Example 1 (continued) In the situation of the optimization problem (6), observe that the point \( P_3 \) actually minimizes the 1-norm over \( P(0,0) \) and is hence the unique result of the (unscaled) selection procedure from [FSZ10]. On the other hand, remember that the point \( P_3 \), which lead to the non-supporting cut above, is mapped to the interior of the reverse polar set and will hence never appear as an extremal solution.

Instead of dealing directly with the reverse polar set, we can, as observed in Theorem 10, almost always achieve the same result by optimizing over the alternative polyhedron. We only have to make sure that the objective function that we use can be written in the form \(( H \omega, -\omega_0 )^\top \). In our example, if we choose the objective function over the alternative polyhedron from the set

\[
\left\{ \left( \begin{pmatrix} -2 \\ -1/2 \\ -4 \end{pmatrix} \cdot \omega, -\omega_0 \right) \middle| \omega, \omega_0 \in \mathbb{R} \right\},
\]

then the point \( P_3 \in P(0,0) \) is never optimal with respect to an objective of the form \(( H \omega, -\omega_0 )\), just as \( \tilde{P}_3 = T \cdot P_3 \), being an internal point of \( \text{epi}(z)^- \), is never optimal for any linear objective over the reverse polar set.

One interesting difference between the alternative polyhedron and the reverse polar set, which can be verified using the above example, is their different behavior with respect to algebraic operations on the set of inequalities: If, for instance, we scale one of the inequalities by a positive factor, the reverse polar set remains unchanged (just as the feasible region defined by the set of inequalities). The alternative polyhedron, on the other hand, is distorted in response to the scaling of the system of inequalities. If an objective function is used which does not take this scaling into account, such as the vector of zeros and ones that Fischetti, Salvagnin, and Zanette [FSZ10] use, then the selected cut might change depending on the scaling factor. Even selecting a suitable manual scaling factor \( \tilde{\omega}_0 \) as mentioned above cannot fix this, since it cannot scale individual constraints against each other.

Before we proceed, we would like to sum up the main results from this section: With an appropriate choice of the objective function, we can solve any cut-generating optimization problem over the reverse polar set by solving a corresponding problem over the relaxed alternative polyhedron. In particular this means that we never need to obtain an explicit representation of the reverse polar set, which might not be readily available. As a consequence, it is sufficient in this context to focus on the selection of cut normals, since we automatically obtain the corresponding optimal right-hand side at no additional computational cost. This summary is captured in the following theorem:

**Theorem 12.** Let \( z \) be defined as in (2) and \((x^*, \eta^*)\), \((\omega, \omega_0)\) \(\in \mathbb{R}^n \times \mathbb{R} \), \((\tilde{\omega}, \tilde{\omega}_0) := (H \omega, -\omega_0)^\top \), and \((\gamma, \gamma_0)\) \(\in P(x^*, \eta^*) \) be maximal with respect to the objective \((\tilde{\omega}, \tilde{\omega}_0) \) such that \(\tilde{\omega}^\top \gamma + \tilde{\omega}_0 \gamma_0 < 0 \). Then the inequality \( \gamma^\top H x - \gamma_0^\top \eta \leq \gamma^\top b \) supports \( \text{epi}(z) \).

**Proof.** By Remark 11, if \((\gamma, \gamma_0)\) maximizes the objective \((\tilde{\omega}, \tilde{\omega}_0)\) over \( P(x^*, \eta^*) \), then it is also maximal within \( P^S(x^*, \eta^*) \). Using Theorem 10, this implies that \((\pi, \pi_0) := (H^\top \gamma, -\gamma_0) \) is an optimal solution with respect to the objective \((\omega, \omega_0)\) over the set \((\text{epi}(z) - (x^*, \eta^*))^- \). Furthermore, since both problems have the same objective value, we have that \( \omega^\top \pi + \omega_0 \pi_0 < 0 \).

Since \((\gamma, \gamma_0)\) \(\in P(x^*, \eta^*) \), we have that \( \gamma \) is a valid certificate for the vector \((\pi, \pi_0)\) in (16). By Theorem 8, this implies that the inequality \( \gamma^\top H x - \gamma_0^\top \eta \leq \gamma^\top b \) does indeed support \( \text{epi}(z) \). \( \square \)
Finally, to conclude our dictionary of cut-generating optimization problems, we derive an alternative representation of the optimization problem in (19), which will turn out to be much more useful in practice. For instance, the structure of the resulting problem will be very similar to the original subproblem, which makes it easy to use existing solution algorithms for the subproblem in a cut-generating program.

Cornuéjols and Lemaréchal [CL06, Theorem 4.2] prove that linear optimization problems over the reverse polar set can be evaluated in terms of the support function of the original set (in our case \( \text{epi}(z) - (x^*, \eta^*) \)). This can also be applied to the alternative polyhedron, as mentioned (without proof) by Fischetti, Salvagnin, and Zanette [FSZ10]. The following theorem generalizes Cornuéjols and Lemaréchal [CL06, Theorem 4.2] and makes a similar statement, which is applicable to a wider range of settings.

**Theorem 13.** Let \( K \subseteq \mathbb{R}^n \) be a cone and \( c_1, c_2 \in \mathbb{R}^n \). Consider the optimization problems

\[
\text{(21)} \quad \max \left\{ c_1^\top x \mid x \in K, c_2^\top x = -1 \right\}
\]

and

\[
\text{(22)} \quad \max \left\{ c_2^\top x \mid x \in K, c_1^\top x \geq 1 \right\}.
\]

Then the following hold:

a) If \( x^* \) is an optimal solution for (21) with objective value \( \xi > 0 \), then \( \frac{1}{\xi} \cdot x^* \) is an optimal solution for (22) with objective value \( \frac{1}{\xi} \).

b) Conversely, if \( x^* \) is an optimal solution for (22) with objective value \( \xi < 0 \), then \( -\frac{1}{\xi} \cdot x^* \) is an optimal solution for (22) with objective value \( -\frac{1}{\xi} \).

**Proof.** First, let \( x^* \) be an optimal solution of (21) with objective value \( \xi > 0 \). Then \( \frac{1}{\xi} \cdot x^* \in K \) and \( c_1^\top \left( \frac{1}{\xi} \cdot x^* \right) = 1 \). Hence the point \( \frac{1}{\xi} \cdot x^* \) is feasible for (22). Furthermore, its objective value is

\[
\text{(23)} \quad c_2^\top \left( \frac{1}{\xi} \cdot x^* \right) = \frac{1}{\xi} \cdot c_2^\top x^* = -\frac{1}{\xi}.
\]

To see that \( \frac{1}{\xi} \cdot x^* \) is indeed optimal, let \( x' \) be feasible for (22). We first claim that \( c_2^\top x' < 0 \): Suppose for a contradiction that \( c_2^\top x' \geq 0 \). Choose \( \varepsilon > 0 \) such that \( \varepsilon \cdot c_2^\top x' < 1 \). Then, \( c_2^\top (x^* + \varepsilon x') = c_2^\top x^* + \varepsilon \cdot c_2^\top x' =: \lambda \in [-1, 0] \). But this means that \( c_2^\top \frac{1}{\lambda}(x^* + \varepsilon x') = -1 \) and since \( x' \in K \) we furthermore have that \( \frac{1}{\lambda}(x^* + \varepsilon x') \in K \). Together, this implies that \( \frac{1}{\lambda}(x^* + \varepsilon x') \) is feasible for (21). But

\[
\text{(24)} \quad c_1^\top \frac{-1}{\lambda}(x^* + \varepsilon x') \geq \frac{-1}{\lambda} (c_1^\top x^* + \varepsilon) \geq c_1^\top x^* + \varepsilon > c_1^\top x^*,
\]

which contradicts the optimality of \( x^* \). This proves that, indeed, \( c_2^\top x' < 0 \).

Now, suppose that \( \mu := c_2^\top x' > \frac{-1}{\xi} \). As we have seen above, \( \mu < 0 \) and hence \( -\frac{1}{\mu} > \xi > 0 \). Since \( -\frac{1}{\mu} \cdot x' \in K \) and \( c_2^\top \left( -\frac{1}{\mu} \cdot x' \right) = -\frac{1}{c_2^\top x'} \cdot c_2^\top x' = -1 \), the point \( -\frac{1}{\mu} \cdot x' \) is feasible for (21) with objective value

\[
\text{(25)} \quad c_1^\top \left( -\frac{1}{\mu} \cdot x' \right) = -\frac{1}{\mu} c_1^\top x' \geq -\frac{1}{\mu} > \xi = c_1^\top x^*,
\]

again contradicting the optimality of \( x^* \).
For the reverse implication, let \( x^* \) be an optimal solution for (22) with objective value \( \xi < 0 \). Then \( -\frac{1}{\xi} \cdot x^* \in K \) and \( c_2^T (-\frac{1}{\xi} \cdot x^*) = -\frac{1}{\xi} \cdot c_2^T x^* = -1 \). Hence, \(-\frac{1}{\xi} \cdot x^*\) is feasible for (21) and its objective value is

\[
(26) \quad c_1^T (-\frac{1}{\xi} \cdot x^*) = -\frac{1}{\xi} \cdot c_1^T x^* \geq -\frac{1}{\xi}.
\]

To see that \(-\frac{1}{\xi} \cdot x^*\) is indeed optimal, suppose that there exists \( x' \) feasible for (21) with \( \mu := c_1^T x' > c_1^T (-\frac{1}{\xi} \cdot x^*) \geq -\frac{1}{\xi} > 0 \). Since \( \frac{1}{\mu} \cdot x' \in K \) and \( c_1^T (\frac{1}{\mu} \cdot x') = \frac{1}{\mu} c_1^T x' = 1 \), the point \( \frac{1}{\mu} \cdot x' \) is feasible for (22) with objective value \( c_2^T (\frac{1}{\mu} \cdot x') = \frac{1}{\mu} c_2^T x' = -\frac{1}{\mu} > \xi \), contradicting the optimality of \( x^* \).

Choosing \( c_1 := (Hx^* - b, -\eta^*) \), \( c_2 := (\tilde{\omega}, \tilde{\omega}_0) \) and \( K := \{ (\gamma, \gamma_0) \geq 0 | \gamma^T A + \gamma_0 d^T = 0 \} \), the following corollary immediately follows from Part a) of the above theorem:

**Corollary 14.** Let \((\tilde{\omega}, \tilde{\omega}_0) \in \mathbb{R}^m \times \mathbb{R}\) and let \((\gamma^*, \gamma_0^*)\) denote an optimal solution with value \( \xi > 0 \) for the problem

\[
(27) \quad \max_{\gamma, \gamma_0 \geq 0} \gamma^T (Hx^* - b) - \gamma_0 \eta^*
\]

\[
(28) \quad \gamma^T A + \gamma_0 d^T = 0
\]

\[
(29) \quad \tilde{\omega}^T \gamma + \tilde{\omega}_0 \gamma_0 = -1.
\]

Then \( \frac{1}{\xi} (\gamma^*, \gamma_0^*) \) is an optimal solution with value \(-\frac{1}{\xi}\) for

\[
(30) \quad \max \{ \tilde{\omega}^T \gamma + \tilde{\omega}_0 \gamma_0 | (\gamma, \gamma_0) \in P^\leq(x^*, \eta^*) \}.
\]

The structural similarity of (27) to (29) to the original problem becomes more apparent when we consider the dual problem:

**Corollary 15.** Let \((\tilde{\omega}, \tilde{\omega}_0) \in \mathbb{R}^m \times \mathbb{R}\) and \((\lambda, x, y)\) be an optimal solution for the problem

\[
(31) \quad \min \lambda
\]

\[
(32) \quad Ay \leq b - Hx^* - \lambda \tilde{\omega}
\]

\[
(33) \quad d^T y \leq \eta^* - \lambda \tilde{\omega}_0
\]

with \( \lambda > 0 \). Denote the corresponding dual solution by \((\gamma, \gamma_0)\). Then \( \frac{1}{\lambda} (\gamma, \gamma_0) \) is an optimal solution for

\[
(34) \quad \max \{ \tilde{\omega}^T \gamma + \tilde{\omega}_0 \gamma_0 | (\gamma, \gamma_0) \in P^\leq(x^*, \eta^*) \}
\]

with objective value \(-\frac{1}{\lambda}\).

Note that, together with our observations in the context of the definition of the alternative polyhedron (5), this means in particular that

a) whenever (31) to (33) has objective value 0, then the alternative polyhedron is empty and \((x^*, \eta^*) \in \text{epi}(z)\), and

b) whenever (31) to (33) is feasible with (finite) objective value greater than 0, then (18) and (19) have objective values strictly less than 0, which means that the requirements for Theorem 8 or Remark 11 are satisfied.

Finally, Corollary 15 exposes another interesting perspective on the restriction \((\tilde{\omega}, \tilde{\omega}_0)^T := (H\omega, -\omega_0)^T\) on the objective function (and hence the relation between optimization problems over the alternative polyhedron and the reverse polar set):
Remark 16. If \((\tilde{\omega}, \tilde{\omega}_0)^T := (H \omega, -\omega_0)^T\), then the optimization problem (31) to (33) becomes

\[
\min \lambda
\]
\[
Ay \leq b - H(x^* + \lambda \cdot \omega)
\]
\[
d^T y \leq \eta^* + \omega_0 \lambda
\]

Comparing the two optimization problems, both can be seen as a relaxation of the feasibility version of the original subproblem (4): They allow a solution to violate certain constraints, possibly (depending on the signs of entries in \(H\) and \(\omega\)) at the cost of strengthening others. In any case, a feasible solution for (4) is feasible for both problems with objective value 0.

The difference between the two is how exactly this relaxation is handled: In (31) to (33), it works on the level of individual inequalities by relaxing their right-hand sides, whereas in (35) to (37) it works on the level of the master solution \((x^*, \eta^*)\), allowing us to choose a possibly more advantageous value for the variable \(x\) itself.

3. Cut Selection

As we have seen in the previous section, Benders’ decomposition can be viewed as an instance of a classical cutting plane algorithm (Theorem 2). The Benders subproblem takes the role of the separation problem and the alternative polyhedron that is commonly used to select a Benders cut is a higher-dimensional representation (an extended formulation) of the reverse polar set, which characterizes all possible cut normals (Theorem 7).

Finally, Corollary 15 and Remark 16 show that selecting a cut normal by a linear objective over the reverse polar set or the alternative polyhedron can be interpreted as two different relaxations (31) to (33) and (35) to (37) of the original Benders feasibility subproblem (4). The latter relaxation is more general and coincides with the former for a particular selection of the objective function.

Cut selection is one of four major areas of algorithmic improvements for Benders decomposition that recent work has focused on (see, e.g., the very recent and extensive literature review in [Rah+17]). A number of selection criteria for Benders cuts have previously been explicitly proposed in the literature. Many of them also arise naturally from our discussion and analysis of the Benders decomposition algorithm above. We will first present these criteria in the way they typically appear in the literature and then try to link them to the reverse polar set and/or the alternative polyhedron.

3.1. Minimal Infeasible Subsystems. The work of Fischetti, Salvagnin, and Zanette [FSZ10] is based on the premise that “one is interested in detecting a ‘minimal source of infeasibility’” whenever the feasibility subproblem (4) is empty. They hence suggest to generate Benders cuts based on Farkas certificates that correspond to minimal infeasible subsystems (MIS) of (4). We define this criterion as follows:

Definition 17. Let \(z\) be defined as in (2) and let \((\pi, \pi_0) \in \mathbb{R}^n \times \mathbb{R}\). We say that \((\pi, \pi_0)\) satisfies the MIS criterion if there exists \((\gamma, \gamma_0) \geq 0\) such that \(\pi = H^T \gamma, \pi_0 = -\gamma_0\) and the inequalities which correspond to the non-zero components of \((\gamma, \gamma_0)\) form a minimal infeasible subsystem of (4).

Note that we have defined the MIS criterion as a property of a normal vector, rather than a property of a cut. The reason for this is that the cut normal is the only relevant choice to make, given that an optimal right-hand side for each cut normal is provided by Theorem 12. Accordingly, we will call any cut with a normal vector that satisfies the MIS criterion a MIS-cut.
Gleeson and Ryan [GR90] show that the set of \((\gamma, \gamma_0)\) that appear in the above definition is exactly (up to homogeneity) the set of vertices of the alternative polyhedron:

**Theorem 18.** Let \((x^*, \eta^*) \in \mathbb{R}^n \times \mathbb{R}\). For each vertex \(v\) of the (relaxed) alternative polyhedron \((5)\), the set of constraints corresponding to the non-zero entries of \(v\) forms a minimal infeasible subsystems of \((4)\), and vice versa.

This immediately provides a characterization of cut normals which satisfy MIS in terms of the alternative polyhedron, which is also used in [FSZ10]:

**Corollary 19.** Let \(z\) be defined as in \((2)\) and \((x^*, \eta^*) \in \mathbb{R}^n \times \mathbb{R}\). The vector \((\pi, \pi_0)\) satisfies the MIS criterion if and only if there is an extremal point \((\gamma, \gamma_0)\) of \(P(x^*, \eta^*)\) such that \((\pi, \pi_0) = (H^T \gamma, -\gamma_0)\).

Theorem 7 allows us to transfer the if-part of this characterization to the reverse polar set. The only-if-part is generally not true for the reverse polar set, i.e. there might be minimal infeasible subsystems that do not correspond to vertices of the reverse polar set (see, e.g., Example 1).

**Corollary 20.** Let \(z\) be defined as in \((2)\) and \((x^*, \eta^*) \in \mathbb{R}^n \times \mathbb{R}\). If \((\pi, \pi_0)\) is a vertex of \((\text{epi}(z) - (x^*, \eta^*))^-\), then it satisfies the MIS criterion.

Fischetti, Salvagnin, and Zanette [FSZ10] empirically study the performance of MIS-cuts on a set of multi-commodity network design instances. Their results suggest that MIS-based cut selection outperforms the standard implementation of Benders decomposition by a factor of at least 2-3. Furthermore, this advantage increases substantially when focusing on harder instances (e.g. those which could not be solved by the standard implementation within 10 hours).

### 3.2. Facet-defining Cuts

In cutting plane algorithms for polyhedra, facet-defining cuts are commonly considered to be very useful. They form the smallest family of inequalities which completely describe the target polyhedron. A cutting-plane algorithm that can separate (distinct) facet inequalities in each iteration is not necessarily computationally efficient, but at least it is automatically guaranteed to terminate after a finite number of iterations. Also in practical applications, facet cuts have turned out to be extremely useful, e.g. in the context of branch-and-cut algorithms for integer programs such as the Traveling Salesman Problem. This is why the description of facet-defining inequalities has been a large and very active area of research for decades (see [Bal75; NW88; Coo+98; KV08] and, as mentioned before, [CW18]).

Remember that a halfspace \(H_{((\pi, \pi_0), \alpha)}^{\leq}\) is facet-defining for a set \(C\) if \(C \subseteq H_{((\pi, \pi_0), \alpha)}^{\leq}\) and \(H_{((\pi, \pi_0), \alpha)}^{\leq} \cap C\) contains \(\dim(C)\) many affinely independent points. Analogously to the MIS criterion above, we define the FACET criterion for a normal vector in the context of Benders decomposition as follows:

**Definition 21.** Let \(z\) be defined as in \((2)\) and \((\pi, \pi_0) \in \mathbb{R}^n \times \mathbb{R} \setminus \{0\}\). We say that \((\pi, \pi_0)\) satisfies the FACET criterion if there exists \(\alpha \in \mathbb{R}\) such that \(H_{((\pi, \pi_0), \alpha)}^{\leq}\) is either facet-defining for \(\text{epi}(z)\) or the corresponding hyperplane \(H_{((\pi, \pi_0), \alpha)}\) contains \(\text{epi}(z)\).

Note that, in deviation from the definition of a facet-defining cut, we require that the halfspace supports at least \(\dim(C)\) affinely independent points. In other words, in the case where \(\text{epi}(z)\) is not full-dimensional, we also allow that \(\text{epi}(z)\) is entirely contained in the hyperplane which represents the boundary of \(H_{((\pi, \pi_0), \alpha)}^{\leq}\). In this situation, the comparison
of different cut normals is inherently difficult: If two cuts both support all of epi(z), which one should be preferred? In this sense, the criterion FACET captures arguably the strongest statement about a cut in relation to epi(z) that we can make in a given situation: In no case would we want to select a cut that supports neither a facet nor the entire set epi(z).

The following result was originally obtained by Balas [Bal98, Theorem 4.5] in his analysis of disjunctive cuts. It reappears in Cornuéjols and Lemaréchal [CL06, Theorem 6.2], using more familiar notation, but the latter contains a minor error in the case where the set P is subdimensional. We therefore re-prove a corrected version of the important parts of [CL06, Theorem 6.2] below, along the lines of their original proof.

**Theorem 22.** Let $P \subseteq \mathbb{R}^n$ be a polyhedron, $x^* \notin P$ and

$$r := \begin{cases} \dim(P) - 1, & x^* \in \text{aff}(P) \\ \dim(P), & x^* \notin \text{aff}(P). \end{cases}$$

Then, there exists an $x^*$-separating halfspace with normal vector $\pi \neq 0$ supporting an $r$-dimensional face of $P$ if and only if there exists a vertex $\pi^*$ of $\text{lin}(P - x^*) \cap (P - x^*)^-$ and some $\lambda > 0$ such that $\lambda \pi \in \pi^* + \text{lin}(P - x^*)^\perp$.

**Proof.** We begin by observing that $r + 1 = \dim(\text{lin}(P - x^*))$ for all $x^* \in \mathbb{R}^n$.

Since the halfspace with normal vector $\pi$ is $x^*$-separating, the property that $\pi$ supports an $r$-dimensional face of $P$ means there exist $r + 1$ affinely independent points in $P$ with $\pi^\top x = h_P(\pi) < \pi^\top x^*$. Let us denote these points by $x^1, \ldots, x^{r+1}$ and let $\pi' := \pi x - h_P(\pi)$. Then, $\pi' \in (P - x^*)^-$ and the inequalities $\langle \pi' \rangle^\top (x^i - x^*) \leq -1$ constitute a system of $r + 1$ linearly independent inequalities valid for $(P - x^*)^-$, which are all satisfied with equality by $\pi'$.

Let $\pi^*$ denote the orthogonal projection of $\pi'$ onto $\text{lin}(P - x^*)$. Since $\langle \pi^* \rangle^\top x = \langle \pi' \rangle^\top x$ for all $x \in \text{lin}(P - x^*)$, it holds in particular that $\pi^* \in (P - x^*)^-$ and that $\pi^*$ satisfies the same set of linearly independent inequalities with equality as $\pi'$ above. Furthermore, since $\langle x^i - x^* \rangle \in \text{lin}(P - x^*)$ for all $i \in [r+1]$, the point $\pi^*$ is indeed a vertex of $(P - x^*)^- \cap \text{lin}(P - x^*)$. Defining $\lambda := \frac{1}{\pi^\top x - h_P(\pi)}$, completes the proof of the forward direction of the statement.

For the backward direction, let $\pi^*$ be a vertex of $\text{lin}(P - x^*) \cap (P - x^*)^-$ and $\lambda \pi \in \pi^* + \text{lin}(P - x^*)^\perp$. Then, by the definition of the reverse polar set there exist $r + 1$ linearly independent points in $(P - x^*)$ such that $\pi^*$ satisfies the corresponding inequalities with equality. Denote these points by $x^1 - x^*, x^2 - x^*, \ldots, x^{r+1} - x^*$. Then, $(\lambda \pi)^\top x = (\pi^*)^\top x$ for all $x \in \text{lin}(P - x^*)$ and therefore $\lambda \pi \in (P - x)^-$ and $\lambda \pi$ exposes a face of $P$ containing the affinely independent points $x^1, \ldots, x^{r+1}$. The face is thus $r$-dimensional and, since $\lambda > 0$, it is supported by an $x^*$-separating halfspace with normal vector $\pi$.

Most notably, for the case where $P$ is full-dimensional the above theorem implies the following:

**Corollary 23.** Let $P \subseteq \mathbb{R}^n$ be a polyhedron with $\dim(P) = n$ and $x^* \notin P$. Then there exists an $x^*$-separating halfspace with normal vector $\pi$ supporting a facet of $P$ if and only if there exists a vertex $\pi^*$ of $(P - x^*)^-$ and $\lambda \geq 0$ such that $\lambda \pi = \pi^*$.

In this case, every cut generated from a vertex of the reverse polar set defines a facet of $\text{epi}(z)$. If an explicit $\mathcal{H}$-representation of the reverse polar set is available, we can thus easily obtain a facet-defining cut, e.g. by linear programming.
Note that since $P^\leq(x^*, \eta^*)$ is line-free, Theorem 7 implies that for every vertex of the reverse polar set there exists a vertex of the relaxed alternative polyhedron (and hence of the original alternative polyhedron) that leads to the same cut normal. In other words, if the normal of an $x^*$-separating halfspace satisfies the\textsc{facet} criterion, then it also satisfies the MIS criterion.

On the other hand, Theorem 7 is not sufficient to guarantee that selecting a vertex of the alternative polyhedron yields a facet-defining cut: As Example 1 shows, a vertex of $P^\leq(x^*, \eta^*)$, is not necessarily mapped to a vertex of the reverse polar set under the transformation from Theorem 7. This exposes a useful hierarchy of subsets of the alternative polyhedron according to the properties of the cut normals which they yield: It is easy to select a vertex of the alternative polyhedron, which guarantees that the resulting cut normal satisfies the MIS criterion, while the points that lead to cut normals satisfying the \textsc{facet} criterion constitute a subset of these vertices. The approach of selecting MIS-cuts may thus be viewed as a heuristic method to find \textsc{facet}-cuts.

Although cuts satisfying the MIS criterion in general do not satisfy the \textsc{facet} criterion, we can obtain some information on when this is the case in the situation of Theorem 10, i.e. if the objective function $(\bar{\omega}, \bar{\omega}_0)$ used to select the cut via problem (19) satisfies $(\bar{\omega}, \bar{\omega}_0) = (H\omega, -\omega_0)$ for some valid objective $(\omega, \omega_0)$ for problem (18).

In this case it turns out that we actually obtain a \textsc{facet}-cut for all objectives $(\omega, \omega_0)$ except those from a lower-dimensional subspace. More precisely, we can prove the following characterization of the relationship between extremal points of the alternative polyhedron and cut normals satisfying the \textsc{facet} criterion. This characterization is similar to [CW18, Proposition 6]:

**Theorem 24.** Let $z$ be defined as in (2), $(x^*, \eta^*) \in \mathbb{R}^n \times \mathbb{R} \setminus \text{epi}(z)$, and $(\omega, \omega_0) \in \text{cl}(\text{pos}(\text{epi}(z) - (x^*, \eta^*))$. Then, there exists an optimal vertex $(\gamma^*, \gamma_0^*) \in P^\leq(x^*, \eta^*)$ with respect to the objective function $(H\omega, -\omega_0)$ such that the resulting cut normal $(H^\top \gamma^*, -\gamma_0^*)$ satisfies the \textsc{facet} criterion.

*Proof.* Let $L := \text{lin}(\text{epi}(z) - (x^*, \eta^*))$ and observe that $L$ is orthogonal to the lineality space of $(\text{epi}(z) - (x^*, \eta^*))^\top$. For $a$, observe that from Theorem 9, the reverse polar set $(\text{epi}(z) - (x^*, \eta^*))^\top$ is line-free and we can therefore choose $(\pi, \pi_0)$ to be extremal in $(\text{epi}(z) - (x^*, \eta^*))^\top \cap L$. By Theorem 10, there exists $\gamma'$ with $H^\top \gamma' = \pi$ such that $(\gamma', -\pi_0)$ is an optimal solution to the problem

$$
\max \{(H\omega)^\top \gamma - \omega_0 \gamma_0 \mid (\gamma, \gamma_0) \in P^\leq(x^*, \eta^*)\}.
$$

Denote by $P^*$ the face of optimal solutions of (39) and observe that

$$(\gamma', -\pi_0) \in P^* \cap \{(\gamma, \gamma_0) \mid (H^\top \gamma, -\gamma_0) - (\pi, \pi_0) = 0\}
\subseteq P^* \cap \{(\gamma, \gamma_0) \mid (H^\top \gamma, -\gamma_0) - (\pi, \pi_0) \in L^\perp\}.
$$

Let $(\gamma^*, \gamma_0^*)$ be an extremal point of $P^* \cap \{(\gamma, \gamma_0) \mid (H^\top \gamma, -\gamma_0) - (\pi, \pi_0) \in L^\perp\}$ (which exists, since $P^\leq(x^*, \eta^*)$ is line-free). Then $(\gamma^*, \gamma_0^*)$ is obviously optimal for (39) and furthermore $(H^\top \gamma^*, -\gamma_0^*) = (\pi, \pi_0) + v$ with $v \in L^\perp$, which means by Theorem 22 that it satisfies the \textsc{facet} criterion.

It remains to show that $(\gamma^*, \gamma_0^*)$ is a vertex of $P^*$, thus showing that it is also a vertex of $P^\leq(x^*, \eta^*)$. To see this, let $(\gamma^1, \gamma_0^1), (\gamma^2, \gamma_0^2) \in P^*$ such that $(\gamma^*, \gamma_0^*) \in \text{relint}((\gamma^1, \gamma_0^1), (\gamma^2, \gamma_0^2))$. 

However, it follows that 

$$(\pi, \pi_0) + v = (H^T \gamma^*, -\gamma^*_0) \in \text{relint}([(H^T \gamma^1, -\gamma^*_0), (H^T \gamma^2, -\gamma^*_0)])$$

and by Theorem 7, $$[(H^T \gamma^1, -\gamma^*_0), (H^T \gamma^2, -\gamma^*_0)] \subseteq (\text{epi}(z) - (x^*, \eta^*))^{-}$$. As $$(\pi, \pi_0)$$ is extremal in the set $$(\text{epi}(z) - (x^*, \eta^*))^{-} \cap L$$, this implies that both $$((H^T \gamma^1, -\gamma^*_0), (H^T \gamma^2, -\gamma^*_0)) \in (\pi, \pi_0)+L^1$$ which in turn implies that $$(\gamma^1, \gamma^*_0), (\gamma^2, \gamma^*_0) \in P^* \cap \{(\gamma, \gamma_0) | (H^T \gamma, -\gamma_0) - (\pi, \pi_0) \in L^1\}$$. As $$(\gamma^*, \gamma^*_0)$$ is extremal in $$P^* \cap \{(\gamma, \gamma_0) | (H^T \gamma, -\gamma_0) - (\pi, \pi_0) \in L^1\}$$, we obtain that $$(\gamma^1, \gamma^*_0) = (\gamma^2, \gamma^*_0) = (\gamma^*, \gamma^*_0)$$, which proves extremality of $$(\gamma^*, \gamma^*_0)$$ in $$P^*$$.

In particular, the above theorem implies the following: If $$(\gamma^1, \gamma^*_0) \in P^= (x^*, \eta^*)$$ is an optimal extremal point with respect to the objective function $$(H \omega, -\omega_0)$$ such that the resulting valid normal $$(H^T \gamma^1, -\gamma^*_0)$$ does not satisfy the FACET criterion, then the optimal solution for maximizing $$(H \omega, -\omega_0)$$ over $$P^= (x^*, \eta^*)$$ is not unique. Furthermore, by Theorem 10, this implies that the same is true for maximizing the objective $$(\omega, \omega_0)$$ over $$(\text{epi}(z) - (x^*, \eta^*))^{-}$$.

We can summarize our results as follows: While any FACET-cut is also an MIS-cut, the reverse is not always true. However, if we optimize the objective $$(H \omega, -\omega_0)$$ over the alternative polyhedron, then there exists only a subdimensional set of choices for the vector $$(\omega, \omega_0)$$ for which the resulting cut might not satisfy the FACET criterion (those, for which the optimum over the reverse polar set is non-unique).

This suggests that these cases should be “rare” in practice, especially if we choose (or perturb) $$(\omega, \omega_0)$$ randomly from some full-dimensional set. This argument, why a cut obtained for a generic vector $$(\omega, \omega_0)$$ can be expected to be facet-defining, is identical to the concept of “almost surely” finding facet-defining cuts proposed by Conforti and Wolsey [CW18].

Looking back at Remark 16, this similarity should not come as a surprise: With $$(\omega, \omega_0) = (\bar{x} - x^*, \bar{\eta} - \eta^*)$$ for a point $$(\bar{x}, \bar{\eta}) \in \text{relint}(\text{epi}(z))$$, the resulting cut-generating LP is almost identical. In fact, the point $$(\bar{x}, \bar{\eta})$$ in this case takes the role of the point that the origin is relocated into in the approach from [CW18]. Observe, however, that while Conforti and Wolsey [CW18] require that point to lie in the relative interior of $$(\text{epi}(z))$$, we can actually expect a cut satisfying the FACET criterion from any $$(\omega, \omega_0)$$ for which the optimal objective over the reverse polar is strictly negative. By Theorem 9, one sufficient (but not necessary) criterion for this is to choose $$(\omega, \omega_0)$$ as above, even for an arbitrary point $$(\bar{x}, \bar{\eta}) \in \text{epi}(z)$$.

Finally, if one wants to be sure that a cut computed from the alternative polyhedron satisfies the FACET criterion, then, again choosing the objective to be of the form $$(H \omega, -\omega_0)$$, we can iteratively restrict the optimal set to a singleton, which always yields a cut that satisfies FACET. In fact, even a single iteration of the according process might be useful, since it reduces the dimension of the linear subspace, in which the objective function has to lie to obtain a cut that does not satisfy the FACET criterion, by at least one.

### 3.3. Pareto-Optimality.

The first systematic work on the general selection of Benders cuts to our knowledge was undertaken by Magnanti and Wong [MW81]. The paper, which has proven very influential and still being referred to regularly, focuses on the property of Pareto-optimality. It can intuitively be described as follows: A cut is Pareto-optimal if there is no other cut valid for $$(\text{epi}(z))$$ which is clearly superior, which dominates the first cut.

In this setting, any cut that does not support $$(\text{epi}(z))$$ is obviously dominated. Between supporting cuts, there is no general criterion for domination. If the cut normal $$(\pi, \pi_0)$$ satisfies $$(\pi_0 \neq 0$$, however, things become somewhat easier (this is also the case covered by [MW81]):
Figure 3. The dotted cut supports a facet of epi(z) and it supports epi_S(z), but it is still not Pareto-optimal. The solid cut supports a facet of epi_S(z) and is hence Pareto-optimal. The dashed cut is Pareto-optimal even though it does not support a facet of epi(z) (or epi_S(z)).

Definition 25. For a problem of the form (1), we say that an inequality \((\pi^\top, \pi_0)(x, \eta)^\top \leq \alpha\) with \(\pi_0 < 0\) is dominated by another inequality \((\pi'^\top, \pi'_0)(x, \eta)^\top \leq \alpha'\) if \(\pi'_0 < 0\) and

\[
\frac{\pi'^\top x - \alpha'}{-\pi'_0} \geq \frac{\pi^\top x - \alpha}{-\pi_0}
\]

for all \(x \in S\), with strict inequality for at least one \(x \in S\).

If \(\pi_0 < 0\) and \((\pi^\top, \pi_0)(x, \eta)^\top \leq \alpha\) is not dominated by any valid inequality for epi(z), then we call it Pareto-optimal.

Remember that the set \(S\) contains all points \(x \in \mathbb{R}^n\) that are feasible for an optimization problem of the form (1) if we ignore the linear constraints \(Hx + Ay \leq b\). By the above definition, a cut dominates another cut if the minimum value of \(\eta\) that it enforces is at least as good for all \(x \in S\) and strictly better for at least one \(x \in S\) (see fig. 3).

Analogously to the previous criteria, we define the Pareto criterion for a cut normal:

Definition 26. For a problem of the form (1) with \(z\) as defined as in (2), let \((\pi, \pi_0) \in \mathbb{R}^n \times \mathbb{R}\). We say that \((\pi, \pi_0)\) satisfies the Pareto criterion if there exists a scalar \(\alpha \in \mathbb{R}\) such that the inequality \((\pi^\top, \pi_0)(x, \eta)^\top \leq \alpha\) is Pareto-optimal.

This criterion is very reasonable: If a cut is not Pareto-optimal, then it can be replaced by a different cut, which is also valid for epi(z), but leads to a strictly tighter approximation. We would hence prefer to generate a stronger, Pareto-optimal cut right away.

The following theorem provides us with a characterization of Pareto-optimal cuts. It is based on the idea of [MW81, Theorem 1], which is formulated under the assumption that the subproblem is always feasible (which implies that \(\pi_0 < 0\) for any cut normal \((\pi, \pi_0)\)). While the original theorem is only concerned with sufficiency, we extend the result in a natural way to obtain a criterion that gives a complete characterization of Pareto-optimal cuts. We use the following separation lemma:
Lemma 27 ([Roc70]). Let $C \subseteq \mathbb{R}^n$ be a non-empty convex set and $K \subseteq \mathbb{R}^n$ a non-empty polyhedron such that relint($C$) $\cap K = \emptyset$. Then, there exists a hyperplane separating $C$ and $K$ which does not contain $C$.

Using this lemma, we can prove the following theorem:

Theorem 28. For a problem of the form (1), let $(\pi, \pi_0) \in \mathbb{R}^n \times \mathbb{R}$ with $\pi_0 < 0$. The inequality $(\pi^T, \pi_0)(x, \eta)^T \leq \alpha$ is Pareto-optimal if and only if $H_{((\pi, \pi_0), \alpha)}^{\leq}$ is a halfspace supporting epi($z$) in a point $(x^*, \eta^*) \in$ epi($z$) $\cap$ relint(conv($S$)) $\times \mathbb{R}$.

Proof. For the if part, suppose for a contradiction that the inequality $(\pi^T, \pi_0)(x, \eta)^T \leq \alpha$ is not Pareto-optimal, i.e. there exists some $\pi', \pi_0', \alpha'$ such that the inequality $(\pi'^T, \pi_0')(x, \eta)^T \leq \alpha'$ dominates the former inequality. This means that for all $x \in S$ (and hence all $x \in$ conv($S$)), it holds that

\[
\frac{\pi^T x - \alpha'}{-\pi_0'} \geq \frac{\pi^T x - \alpha}{-\pi_0}
\]

and furthermore

\[
\frac{\pi'^T x - \alpha'}{-\pi_0'} > \frac{\pi^T x - \alpha}{-\pi_0} \quad \text{for some } x \in S.
\]

Finally, since $H_{((\pi, \pi_0), \alpha)}^{\leq}$ supports epi($z$) in $(x^*, \eta^*)$,

\[
\frac{\pi'^T x^* - \alpha'}{-\pi_0'} \leq \eta^* = \frac{\pi^T x - \alpha}{-\pi_0} \leq \frac{\pi'^T x^* - \alpha'}{-\pi_0'}
\]

and hence equality must hold everywhere in the above inequality chain. Now, as $x^* \in$ relint(conv($S$)), we can choose $\lambda > 1$ such that $\tilde{x} := x + \lambda(x^* - \tilde{x}) \in$ conv($S$). But then

\[
\frac{\pi^T \tilde{x} - \alpha}{-\pi_0'} = (1 - \lambda) \frac{\pi^T x - \alpha}{-\pi_0'} + \lambda \frac{\pi'^T x^* - \alpha'}{-\pi_0'} \geq \frac{\pi^T x - \alpha}{-\pi_0'} \leq \frac{\pi'^T x^* - \alpha'}{-\pi_0'}
\]

contradicting (41).

For the only-if part, we first note that if $H_{((\pi, \pi_0), \alpha)}^{\leq}$ does not support epi($z$), then it is obviously dominated by $H_{((\pi, \pi_0), \alpha')}$ with $\alpha' := \alpha' + \epsilon$ for some $\epsilon > 0$. Therefore, let $H_{((\pi, \pi_0), \alpha)}^{\leq}$ be such that it supports epi($z$), but not in points from the set epi($z$) $\cap$ relint(conv($S$)) $\times \mathbb{R}$. Denote by $S^* := \{x \in \mathbb{R}^n \mid \exists \eta : (x, \eta) \in$ epi($z$) $\cap H_{((\pi, \pi_0), \alpha)}^{\leq}\}$ the set of points where $H_{((\pi, \pi_0), \alpha)}^{\leq}$ supports epi($z$).

Since relint(conv($S$)) $\cap S^* = \emptyset$, we can use Lemma 27 to obtain a hyperplane separating conv($S$) and $S^*$ which does not contain $S$. Hence, there exist $\pi^*, \alpha^*$ such that $\pi^T x \geq \alpha^*$ for all $x \in S^*$ and $\pi^T x \leq \alpha^*$ for all $x \in$ conv($S$), where the second inequality is strict for some $x \in$ conv($S$) and thus also for some $x^* \in S$.

Let $\epsilon > 0$, $\pi' := \pi - \epsilon \pi^*$ and $\alpha' := \alpha - \epsilon \alpha^*$. If $\epsilon$ is sufficiently small, then the inequality $(\pi'^T, \pi_0)(x, \eta)^T \leq \alpha'$ is valid for epi($z$): All $(x, \eta) \in$ epi($z$) with $x \notin S^*$ satisfy the original
inequality strictly and for all \( x \in S^* \),
\[
(\pi^T, \pi_0)(x, \eta)^\top - \alpha' = (\pi^T, \pi_0)(x, \eta)^\top - \alpha - \varepsilon \left( \pi^T x - \alpha^* \right) \\
\leq (\pi^T, \pi_0)(x, \eta)^\top - \alpha \leq 0,
\]
since the original inequality is valid for \( \text{epi}(z) \).

Finally, we claim that the inequality \((\pi^T, \pi_0)(x, \eta)^\top \leq \alpha\) is dominated by the inequality \((\pi^T, \pi_0)(x, \eta)^\top \leq \alpha'\): For all \( x \in S \), it holds that
\[
\frac{\pi^T x - \alpha'}{-\pi_0} = \frac{\pi^T x - \alpha}{-\pi_0} + \varepsilon \cdot \frac{\pi^T x - \alpha^*}{-\pi_0} \geq \frac{\pi^T x - \alpha}{-\pi_0}.
\]
Since the last inequality is strict for \( x^* \in S \), this proves the statement. \( \square \)

For the case where \( S \) is convex, the previous theorem immediately implies the following statement:

**Corollary 29.** If \( S \) is convex and \( H_{(\pi, \pi_0, \alpha)}^{\leq} \) supports a face \( F \) of \( \text{epi}_S(z) \) such that \( F \not\subset \text{relbd}(S) \times \mathbb{R} \), then \((\pi^T, \pi_0)(x, \eta)^\top \leq \alpha\) is Pareto-optimal.

Magnanti and Wong [MW81] also provide an algorithm that computes a Pareto-optimal cut by solving the cut-generating problem twice. While their algorithm is defined for the original Benders optimality cuts, it can be adapted to work with other cut selection criteria, as well. Sherali and Lunday [SL13] present a method based on multiobjective optimization to obtain a cut that satisfies a weaker version of Pareto-optimality by solving only a single instance of the cut-generating LP. Papadakos [Pap08] notes that, given a point in the relative interior of \( \text{conv}(S) \), a Pareto-optimal cut can be generated using a single run of the cut-generating problem. Also, under certain conditions on the problem, other points not in the relative interior allow this, as well. However, the approach suggested by the authors adds Pareto-optimal cuts independently from master- or subproblem solutions, together with subproblem-generated cuts, which are generally not Pareto-optimal. This means that the Pareto-optimal cuts which are added may not even cut off the current tentative solution. The upcoming Theorem 31 will lead to an approach that reconciles both objectives, generating cuts that are both Pareto-optimal and cut off the current tentative solution.

We use a result by Cornuéjols and Lemaréchal [CL06] on the set of points exposed by a cut normal \((\pi, \pi_0)\) to derive a method that always obtains a Pareto-optimal cut. The following lemma has been slightly generalized and rewritten to match our setting and notation, but it follows the general idea of Cornuéjols and Lemaréchal [CL06, Theorem 3.4].

**Lemma 30.** Let \((x^*, \eta^*) \in \mathbb{R}^n \times \mathbb{R} \setminus \text{epi}(z)\) and \((\omega, \omega_0) \in \text{pos}(\text{epi}(z) - (x^*, \eta^*))\) and let \((\pi, \pi_0)\) be optimal in \(Q := \text{epi}(z) - (x^*, \eta^*)\) with respect to the objective \((\omega, \omega_0)\). Then there exists \(\alpha \in \mathbb{R}\) such that \(H_{(\pi, \pi_0, \alpha)}^{\leq}\) supports \(\text{epi}(z)\) in
\[
(\bar{x}, \bar{\eta}) := \frac{(\omega, \omega_0)}{-h_Q(\omega, \omega_0)} + (x^*, \eta^*).
\]

**Proof.** The case of \((\omega, \omega_0) \in (\text{epi}(z) - (x^*, \eta^*))\) was proven by Cornuéjols and Lemaréchal [CL06, Theorem 3.4]. If \((\omega, \omega_0) \in \text{pos}(\text{epi}(z) - (x^*, \eta^*))\), then there is \(\mu > 0\) such that \(\mu \cdot (\omega, \omega_0) \in (\text{epi}(z) - (x^*, \eta^*))\). Note that if \((\pi, \pi_0)\) is optimal with respect to \((\omega, \omega_0)\), then
also with respect to \( \mu \cdot (\omega, \omega_0) \). Thus it follows from Cornuéjols and Lemaître [CL06, Theorem 3.4] that there exists \( \alpha \in \mathbb{R} \) such that \( H_{(\pi, \pi_0), \alpha} \) supports \( \text{epi}_S(z) \) in

\[
(\bar{x}, \bar{\eta}) := \frac{\mu \cdot (\omega, \omega_0)}{-h_Q(\mu \omega, \mu \omega_0)} + (x^*, \eta^*) = \frac{(\omega, \omega_0)}{-h_Q(\omega, \omega_0)} + (x^*, \eta^*).
\]

\( \square \)

We can now prove the theorem already mentioned above.

**Theorem 31.** Let \((x^*, \eta^*) \in S \times \mathbb{R}, (\omega, \omega_0) \in \text{relint}(\text{conv}(\text{epi}_S(z) - (x^*, \eta^*)))\), \((\pi, \pi_0)\) be optimal in \((\text{epi}(z) - (x^*, \eta^*))^-\) with respect to the objective \((\omega, \omega_0)\), and \(\pi_0 < 0\). Then \((\pi, \pi_0)\) satisfies the **Pareto criterion**.

**Proof.** Let \(Q := (\text{epi}(z) - (x^*, \eta^*))^-\) again and \(\lambda := -(h_Q(\omega, \omega_0))^{-1}\). Since, in particular, \((\omega, \omega_0) \in \text{epi}_S(z) - (x^*, \eta^*)\) it follows from the definition of the reverse polar set that \(h_Q(\omega, \omega_0) \leq -1\) and thus \(\lambda \in [0, 1]\).

For \((\bar{x}, \bar{\eta})\) from Lemma 30, we thus obtain that \(\bar{x}, \bar{\eta}) = \lambda ((\omega, \omega_0) + (x^*, \eta^*)) + (1 - \lambda)(x^*, \eta^*)\) is a convex combination of a point \((\omega, \omega_0) + (x^*, \eta^*) \in \text{relint}(\text{conv}(\text{epi}_S(z))) \subseteq \text{relint}(\text{conv}(S)) \times \mathbb{R}\) and \((x^*, \eta^*) \in S \times \mathbb{R}\). Therefore, \(\bar{x} \in \text{relint}(\text{conv}(S))\) and thus by Theorem 28 the cut defined by \((\pi, \pi_0)\) is Pareto-optimal.

\( \square \)

Note that the above theorem holds for all optimal solutions (and not only for extremal optimal solutions). As we know from Theorem 10, optimality with respect to \((\omega, \omega_0)\) over the reverse polar set is equivalent to optimality with respect to \((H \omega, -\omega_0)\) over the alternative polyhedron and hence Theorem 31 can be used with the (relaxed) alternative polyhedron, as well.

To conclude our results on Pareto-optimality, we would like to point out two observations concerning our results in relation to [Pap08]:

Firstly, Theorem 31 requires, a relative interior point of \(\text{conv}(\text{epi}_S(z) - (x^*, \eta^*))\), just as the original approach by Magnanti and Wong [MW81] required a relative interior point of \(S\). Naturally, since \(\text{conv}(\text{epi}_S(z) - (x^*, \eta^*))\) is a convex set, it holds that any convex combination of a relative interior point and any other point contained in the set is again a relative interior point. This statement is analogous to [Pap08, Theorem 8] and allows us to use an iterative update procedure for the point \((\omega, \omega_0)\) similar to the procedure used in the computational experiments in [Pap08]. Furthermore, as we have mentioned above, any Pareto-optimal cut generated using Theorem 31 takes into account the current tentative solution \((x^*, \eta^*)\), which makes sure that this solution is indeed cut off (which is not guaranteed for arbitrary Pareto-optimal cuts) and also refines the approximation of \(\text{epi}(z)\) in the proximity of the (infeasible) solution that yields the current lower bound.

Secondly, as Papadakos [Pap08] observed, supporting \(\text{epi}(z)\) in a relative interior point of \(\text{conv}(S)\) is sufficient, but not necessary to obtain a Pareto-optimal cut. Papadakos calls a point \(x\) a **Magnanti-Wong point**, if every cut that supports \(\text{epi}(z)\) in \(x\) is Pareto-optimal and they provide some additional criteria for when a point is a Magnanti-Wong point. In the context of the above theorem, this observation means that we obtain a Pareto-optimal cut whenever the point \(\bar{x}\) in the proof of Theorem 31 is a Magnanti-Wong point. For instance, since the convex hull of a Magnanti-Wong point and the relative interior of \(\text{conv}(S)\) consists of Magnanti-Wong points itself ([Pap08, Theorem 8]), it is quite common that parts of the relative boundary of \(\text{conv}(S)\) consist of Magnanti-Wong points. If this is true for the entire relative boundary, then any \((\omega, \omega_0) \in \text{conv}(\text{epi}_S(z) - (x^*, \eta^*))\) yields a Pareto-optimal cut.

The results from this section are summarized in Table 1.
Table 1. Guaranteed properties of the cut resulting from an extremal point in the alternative polyhedron which maximizes \((\tilde{\omega}, \tilde{\omega}_0)\) (under the assumption that a finite optimum exists). The checkmark in parentheses \(\checkmark\) indicates that the property is satisfied for all \((\tilde{\omega}, \tilde{\omega}_0)\) in the specified set except those from a specific sub-dimensional subset.

| \((\tilde{\omega}, \tilde{\omega}_0)\) | MIS | FACET | PARETO |
|-----------------------------------|-----|-------|--------|
| \(\in \mathbb{R}^{m+1}\)         | ✓   | ✓     |        |
| \(\in (H, -1) \cdot \mathbb{R}^{n+1}\) | ✓ (✓) | ✓     |        |
| \(\in (H, -1) \cdot \text{relint}(\text{conv}(\text{epi}S(z) - (x^*, \eta^*)))\) | ✓ (✓) | ✓     |        |

3.4. Simplified Formulations. The fact that extremal points of the alternative polyhedron do not necessarily correspond to extremal points of the reverse polar set (as pointed out in Section 3.2) hinges on the fact that the linear transformation from Theorem 7, which links the two polyhedra, is generally not a full-rank transformation. While we have developed techniques that allow us to overcome these distinctions and in most cases generate facet-defining cuts directly from the alternative polyhedron, they require some mathematical understanding of master- and subproblem as well as their connection via the interaction matrix \(H\).

We would therefore like to point out a special situation, which is related to a modeling technique which is occasionally used in the context of Benders decomposition to simplify the notation and implementation of the subproblem, as well as the computations required for cut generation (e.g., [GLM99; AC00; Pap08]):

The Benders subproblem that results from fixing the \(x\)-variables is equivalent to the optimization problem

\[
\min_{x, y} \quad d^T y \\
\text{s.t.} \quad Ay + Hx \leq b \\
\quad \quad \quad \quad x - x^* = 0 \\
\quad \quad \quad \quad y \in \mathbb{R}^k, x \in \mathbb{R}^n,
\]

in which this fixing is taken care of by the extra constraint \(x - x^* = 0\). This formulation theoretically increases the problem size, however most LP solvers are easily capable of reverting this blow-up by substituting the values of \(x^*\) during pre-processing. On the other hand, it nicely separates the complexity of the subproblem from the computation of Benders cuts: In the cut definition, only the objective function value and the dual variables of the constraints \(x - x^* = 0\) are required, all other dual variables can be ignored. Similarly, it is easy, e.g., to add constraints to the subproblem (as long as they do not introduce dependencies of additional master variables) without making any changes to the cut generation procedure.

While this technique can in certain cases make the practical implementation easier, we are not aware of any literature that discusses the implications of this technique in the context of cut selection or evaluates the performance impact of the technique. In the context of the cut selection procedure proposed by Fischetti, Salvagnin, and Zanette [FSZ10], one might be tempted to think of it as a limitation, since we can no longer choose an individual component of \(\tilde{\omega}\) for each coupling constraint in the original problem. Instead, we must choose a value for each coupling variable, which can be translated into values for each constraint by a transformation which depends on the entries of the interaction matrix \(H\). In particular, depending on the entries of \(H\), it might no longer be possible to realize the 0-1-objective vector proposed in [FSZ10].
On closer inspection however, it turns out that a side effect of the above transformation is that the interaction matrix for the new subproblem takes the form

\[
\begin{pmatrix}
0 \\
-I_n
\end{pmatrix}
\]

where \(I_n\) denote the \(n\)-dimensional identity matrix. This means that condition (17) on the objective function over the alternative polyhedron is satisfied by all objectives that have zero entries corresponding to the null rows of the above interaction matrix. This means that the apparent restriction of the choice of objective vectors \((\tilde{\omega}, \tilde{\omega}_0)\) coincides exactly with the restriction that is required to obtain cuts satisfying the FACET criterion in most cases.

In terms of the problem formulation from Corollary 15, this becomes

\[
\begin{align*}
\min \lambda \\
Ay + Hx & \leq b \\
x & = x^* - \lambda \cdot \tilde{\omega} \\
d^T y & \leq \eta^* - \lambda \tilde{\omega}_0
\end{align*}
\]

where \(\tilde{\omega}\) can be chosen arbitrarily and always satisfies the condition (17).

Observe that the entire theory presented in this paper could also be derived by restricting the problem formulation to the (blown-up) form (44) instead of restricting the subproblem objective in the way formulated in (17). As we have seen, however, this blow-up is unnecessary and our approach provided us with a clearer understanding of the underlying structure. Results with respect to (44) may thus be obtained as a special case.

3.5. Selection of Subproblem Objective. In each iteration of the Benders algorithm, we may choose a new subproblem objective \((\tilde{\omega}, \tilde{\omega}_0)\). As noted by Fischetti, Salvagnin, and Zanette [FSZ10], the approach originally used by Benders [Ben62] to generate optimality cuts corresponds to choosing \(\tilde{\omega} = 0, \tilde{\omega}_0 = 1\) (the original selection criterion used for feasibility cuts is unspecified and depends on the implementation of the solution algorithm). Fischetti, Salvagnin, and Zanette [FSZ10] also suggest a better selection criterion for general applications of Benders decomposition, which in the case of Corollary 15 corresponds to setting \(\tilde{\omega} = 1, \tilde{\omega}_0 = 1\). The rationale given by the authors is that this objective function gives preference to solutions where only a small number of constraints are active. The authors also note that \(\tilde{\omega}_0\) can be used as a “scaling factor taking into account a wider range of variable \(\eta\)”, but they do not go into details as to how it would be determined (beyond manually adapting it, e.g., to differences in order of magnitude between objective function values and the values of decision variables, which are known in advance).

A first criterion for the selection of an objective function vector is boundedness of the resulting subproblems: We can only hope to obtain a facet-defining cut from a vertex of the reverse polar set and while we could enforce boundedness (e.g., by using a sufficiently large bounding box), the depth of the cut approaches 0 as the objective value goes to infinity, hence a cut from a solution on the boundary of such a box will be of very low-depth. Furthermore, it would not support a facet of the set \(\text{epi}(\tilde{z})\), since it does not correspond to a vertex of the reverse polar set.

Theorem 31 allows us to choose an arbitrary point in the relative interior of \(\text{epi}_S(z)\) to generate an objective function that not only guarantees boundedness, but that always produces a Pareto-optimal cut that, in addition, is very likely to define a facet of \(\text{epi}_S(z)\), as well.

Finally, we can use any other objective function vector according to our selection criteria from Section 3. In most practical cases, we can use prior information about the problem
(e.g., monotonicity of the function $z(x)$) to choose an objective vector for which the subproblem is bounded (see also Theorem 9). In the rare case where a subproblem is indeed unbounded for a particular objective, we can easily recover by perturbing (or in the worst case replacing) the objective vector using one for which the subproblem is known (or guaranteed) to be bounded. In this case, we obviously have to re-solve the subproblem, which requires additional computational effort and hence should be taken care of to not happen too often.

4. Outlook

We conclude this paper by an outlook on interesting research questions in the context of cut selection for Benders decomposition.

In the context of a deeper empirical evaluation, some further choices with respect to parametrization of the algorithm as well as the implementation of certain subroutines would be interesting to analyze more deeply: In a generic implementation of Benders decomposition, upper bounds are computed solely to decide when the algorithm has converged and, if required, provide a feasible solution. By Theorem 9, any feasible solution can be used to derive a subproblem objective which satisfies the prerequisites of both Theorem 12 and Theorem 24. They thus result in the generation of a cut which is always supporting and often even supports a facet. Since information from a feasible solution can thus be used within the cut generation, it makes sense to investigate more closely the possibilities how such a solution can be obtained. This is likely to be very problem-specific, but some general ideas could be:

- How is the information from feasible solutions computed in different iterations best aggregated? Does it make sense to use e.g., a stabilization approach or a convex combination with some other choices for $(\omega, \omega_0)$, e.g., from previous iterations? This corresponds to the method used by Papadakos [Pap08] in their empirical study.
- More broadly, what different methods can be used to generate upper bounds and what effect do different upper bounds have on cut generation and the computational performance of the algorithm?

If a feasible solution is not available to be used as the basis for a subproblem objective, the cut-generating problem might be unbounded/infeasible. On the other hand, the approach from [FSZ10] with $\tilde{\omega} = 1$ yields a cut-generating LP that is always feasible, but the resulting cut might be weaker. How can both approaches be combined in a best-possible way? For instance, is choosing $\tilde{\omega} = H\omega + \varepsilon \cdot 1$ as the relaxation term and letting $\varepsilon$ go to zero a good choice?

Finally, the selection of a particular cut from a set of cuts satisfying the same quality criteria (e.g., that are all facet-defining) is notoriously difficult. Our approach provides a better geometric interpretation of the interaction between parametrization of the cut-generating LP and the resulting cut normals. How can a-priori knowledge about the problem be exploited in the context of this interaction? To what extent can a-priori knowledge about the problem (or information obtained through a fast preprocessing algorithm) be leveraged to improve the selection of a subproblem objective $(\omega, \omega_0)$?

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