A New Representation for the Solutions of Fractional Differential Equations with Variable Coefficients

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Abstract. A recent development in differential equations with variable coefficients by means of fractional operators has been a method for obtaining an exact solution by infinite series involving nested fractional integral operators. This solution representation is constructive but difficult to calculate in practice. Here, we show a new representation of the solution function, as a convergent series of single fractional integrals, which is computationally simpler and which we believe will quickly prove its usefulness in future computational work for applications. In particular, for constant coefficients, the solution is given by the Mittag-Leffler function. We also show some applications in Cauchy problems involving both time-fractional and space-fractional operators and with time-dependent coefficients.

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1. Introduction

Fractional differential equations (FDEs), describing relations between a function and its various derivatives of non-integer orders, have become a thriving field of research in the last few decades [19,28]. Several analytical methods for solving differential equations (ordinary or partial) can be expanded to the setting of FDEs, such as Mikusiński’s operational calculus [15,22,23], transform methods for partial differential equations [2,9], weak solutions and spectral theory [6,7,18], and regularity estimates [4,8,26]. One of the main reasons for studying FDEs is their wide-ranging applications in economics, physics, engineering, biology, etc., [16,17,32].

Just as with classical differential equations, there are different types of FDEs with different levels of difficulty. Linear differential equations are almost always easier than nonlinear ones, and those with constant coefficients are
easier than those where the coefficients are permitted to depend on one or more of the independent variables. Recently, a rigorous analytical method was applied to solve FDEs with variable coefficients. It has been applied, firstly to equations with fractional derivatives of Riemann–Liouville (RL) [20] and Caputo [27] type, later to equations with other types of fractional operators [10,11,29], and also to equations with time and space-fractional operators [3,30].

In the aforementioned papers, fractional ordinary differential equations with continuous variable coefficients were solved by means of an explicit representation for the solution, containing composition of RL fractional integrals, one inside another with multiplier functions. The resulting formulae are mathematically elegant but would be very difficult to calculate, even numerically, for a given problem with specific functions. Thus, the results in the literature so far are largely a mathematical curiosity, hard to apply in practice, even though the problems being solved (FDEs with variable coefficients) do have real applications. In this paper, we construct a new formula for the same solution function, which will be better suited to numerical calculations and real-world applications.

Specifically, we treat the special case (of the general Caputo FDE analysed in [27,29]) of the following equation with one variable coefficient:

$$\frac{d^\beta}{dt^\beta} y(t) + a(t)y(t) = b(t), \quad t \in [0, T],$$

(1)

where $T > 0$, $a, b \in C[0, T]$, $\beta \in \mathbb{C}$ with $\text{Re}(\beta) > 0$, and $\frac{d^\beta}{dt^\beta}$ is the Caputo fractional derivative (defined in a form that allows it to be applied on a suitably large function space, namely the expression (4)). In the previous work of [27], the unique solution of (1) with appropriate initial conditions was discovered to be

$$y(t) = \sum_{k=0}^{\infty} (-1)^k I_{0+}^\beta \left( a(t) I_{0+}^\beta \right)^k b(t).$$

Here, we seek a new representation of this solution function which does not involve arbitrarily long compositions of RL fractional integrals. In fact, we will show an alternative formula for the solution which is given by an infinite series of single RL fractional integrals of the forcing term $b(t)$, with the coefficient of this fractional integral depending just on the ordinary derivatives of the time-dependent coefficient $a(t)$. We note, however, that this paper is purely analytical; we can see heuristically that our formula will be much more amenable to calculation, but we have left actual numerical computations for a separate project by numerical specialists in the future.

The structure of the paper is given as follows. In Sect. 2, we provide definitions and results on fractional calculus and FDEs. In Sect. 3, we give the main results of the paper on the new representation of the solution of the FDE (1). In the special case of having a constant coefficient in (1), we also show that the representation of the solution is given by the Mittag-Leffler function, which is consistent with already known results. In Sect. 4, using the results obtained in the previous section, we give an analytical solution of a Cauchy
problem with time-dependent coefficient. In the end, some conclusions are given in Sect. 5.

2. Preliminaries

Below we shall recall the definitions and basic properties of the RL fractional integro-differential operators—for more details about these, see e.g. [31, Chapter 2] and [19, Chapter 2]. We shall also introduce the fractional differential equation to be analysed in this paper along with some rigorous facts about it. Finally, we shall introduce notations and basic facts concerning gamma functions and binomial coefficients, which will be used later in the paper.

2.1. Riemann–Liouville Fractional Integro-Differential Operators

Definition 2.1. ([31], formula (2.17)) Let $\alpha \in \mathbb{C}$, $\text{Re} (\alpha) > 0$, and let $f \in L^1[a, b]$. The (left-sided) RL fractional integral of $f$ is given by:

$$I^\alpha_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-u)^{\alpha-1} f(u) \, du, \quad t \in (a, b). \quad (2)$$

For $\text{Re}(\alpha), \text{Re}(\beta) > 0$ and $f \in L^1[a, b]$, we have the following semigroup relation [19,31]:

$$I^\alpha_{a+} I^\beta_{a+} f(t) = I^{\alpha+\beta}_{a+} f(t).$$

Definition 2.2. ([31], formula (2.32)) Let $\alpha \in \mathbb{C}$ ($\text{Re} \alpha \geq 0$), let $n = \lfloor \text{Re} \alpha \rfloor + 1$ so that $n - 1 \leq \text{Re} \alpha < n$, and let $f \in AC^n[a, b]$ where $-\infty < a < b < \infty$. The (left-sided) RL fractional derivative of $f$ is given by:

$$D^\alpha_{a+} f(t) = \left( \frac{d}{dt} \right)^n (I^{n-\alpha}_{a+} f)(t), \quad t \in (a, b). \quad (3)$$

In this paper, we shall follow Diethelm [5] in defining the Caputo derivative as follows, for $\text{Re}(\alpha) \geq 0$ and $n = \lfloor \text{Re} \alpha \rfloor + 1$ and $f \in AC^n[a, b]$:

$$C D^\alpha_{a+} f(t) = D^\alpha_{0+} \left( f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} (t-a)^j \right), \quad t \in (a, b). \quad (4)$$

Notice that, if $f \in AC^n[a, b]$, then the expression $C D^\alpha_{a+} f(t)$ of (4) is equivalent to the following formula which is often used as a definition for the Caputo derivative:

$$C D^\alpha_{a+} f(t) = I^{n-\alpha}_{a+} \left( \frac{d}{dt} \right)^n f(t), \quad t \in (a, b). \quad (5)$$

The existence of (5) is guaranteed by $f^{(n)} \in L^1[a, b]$, and the stronger condition $f \in C^n[a, b]$ gives continuity to the considered derivative. Since the definitions (4) and (5) are identical for any function $f$ such that (5) is defined, it is reasonable to use (4) as the definition of the Caputo fractional derivative on the larger class of functions $f$ such that (4) is defined, an extension of meaning of the Caputo derivative beyond the space $AC^n[a, b]$. Therefore, in the remainder of this paper, we shall refer to the operator defined by (4) as the Caputo fractional derivative, without further elaboration.
Finally, we give a fractional version of the Leibniz rule, various versions of which have been discussed in the literature. Results of this type have already proven useful in evaluating expressions involving nested fractional integrals with multiplier functions in between, e.g. in the 1972 paper of Love [21], although his work only covered the case where the multiplier is a power function. The following Lemma will be vital for our work below.

Lemma 2.3. ([24,25,28]) If \( u \in C[a, b] \) and \( v \) is an analytic function, then the following fractional Leibniz rule holds for any fractional integral of \( u(t)v(t) \):

\[
I_{a+}^\alpha (u(t)v(t)) = \sum_{n=0}^{\infty} \left(-\frac{\alpha}{n}\right) I_{a+}^{\alpha+n} u(t) \cdot \left(\frac{d}{dt}\right)^n v(t), \quad \text{Re}(\alpha) > 0.
\]

2.2. FDE with Variable Coefficients

The following multi-term FDE with time-dependent continuous variable coefficients was investigated and solved in [27], with a more general version in [12]:

\[
C D_0^{\beta_0} y(t) + \sum_{i=1}^{m} a_i(t) C D_0^{\beta_i} y(t) = b(t), \quad t \in [0, T], \quad m \in \mathbb{N}, \tag{6}
\]

with

\[
\left(\frac{d}{dt}\right)^k y(t) \bigg|_{t=+0} = c_k \in \mathbb{R}, \tag{7}
\]

for any \( k = 0, 1, \ldots, n_0 - 1 \) where \( a_i, b \in C[0, T] \), \( \text{Re}(\beta_0) > \text{Re}(\beta_1) > \ldots > \text{Re}(\beta_m) \geq 0 \) for \( n_i = \lfloor \text{Re}(\beta_i) \rfloor + 1 \) and \( i = 0, 1, \ldots, m-1 \). If \( \text{Re}(\beta_m) = 0 \) then we set \( \text{Im}(\beta_m) = 0 \) as well. As a special case of the problem above, equation (6) was also studied under homogeneous conditions:

\[
\left(\frac{d}{dt}\right)^k y(t) \bigg|_{t=+0} = 0, \quad k = 0, 1, \ldots, n_0 - 1. \tag{8}
\]

The solution found in [29] was given in the following special function space:

\[
C^{n_0-1,\beta_0}[0, T] := \left\{ y \in C^{n_0-1}[0, T] : C D_0^{\beta_0} y \in C[0, T] \right\},
\]

whose norm is

\[
\|y\|_{C^{n_0-1,\beta_0}[0, T]} = \sum_{k=0}^{n_0-1} \left\| \left(\frac{d}{dt}\right)^k y \right\|_{C[0, T]} + \left\| C D_0^{\beta_0} y \right\|_{C[0, T]}.
\]

According to [29, Theorem 3.8] and [29, Remark 2.7], we have the following result.

Theorem 2.4. The initial value problem given by (6) and (8), with the Caputo derivative operator as defined in (4), has a solution \( y \in C^{n_0-1,\beta_0}[0, T] \), which is unique and given by:

\[
y(t) = \sum_{k=0}^{\infty} (-1)^k I_{0+}^{\beta_0} \left(\sum_{i=1}^{m} a_i(t) I_{0+}^{\beta_0-\beta_i}\right)^k b(t). \tag{9}
\]
Notice that the representation (9) involves the composition of RL fractional integrals with function multipliers. In the case of constant coefficients, due to linearity and composition properties of RL fractional integrals, we can get a good enough representation by the Mittag-Leffler function [29, Theorem 4.3]. For the general case of time-dependent continuous variable coefficients, it seems that representation (9) cannot be improved, since the variable coefficients appear inside the composition of fractional integrals.

Here, we shall deal with the case of one variable coefficient, namely the following equation with \( \text{Re}(\beta) > 0 \) and \( a, b \in C[0, T] \):

\[
\mathcal{C}D^{\beta}_{0+} y(t) + a(t)y(t) = b(t), \quad t \in [0, T],
\]
with

\[
\left( \frac{d}{dt} \right)^k y(t) \bigg|_{t=+0} = 0, \quad k = 0, 1, \ldots, n - 1, \quad n = \lfloor \text{Re}(\beta) \rfloor + 1. \tag{11}
\]

From Theorem 2.4 above, it follows that the fractional differential equation (10) and (11) has a solution \( y \in C^{n-1,\beta}[0, T] \), which is unique, where \( n - 1 \leq \text{Re}(\beta) < n \), and it is represented by:

\[
y(t) = \sum_{k=0}^{\infty} (-1)^k I_{0+}^{\beta \cdot k} (a(t) I_{0+}^{\beta \cdot k}) b(t) = I_{0+}^{\beta \cdot k} \left( \sum_{k=0}^{\infty} (-1)^k (a(t) I_{0+}^{\beta \cdot k})^k b(t) \right).
\]

Our task in this paper is to show that this representation of the solution can be improved and given in a more explicit form, avoiding the composition of RL fractional integrals of \( (a(t) I_{0+}^{\beta \cdot k})^k \), which will be more suitable for explicit calculation and approximation.

### 2.3. Gamma Functions and Related Topics

**Definition 2.5.** ([1,34]) The gamma function \( \Gamma(z) \) is represented by

\[
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad \text{Re}(z) > 0.
\]

Throughout this paper, we will use the notation \( \binom{\gamma}{\beta} \) and \( \alpha! \) to denote the following expressions:

\[
\binom{\gamma}{\beta} = \frac{\gamma!}{\gamma!(\gamma - \beta)!} = \frac{\Gamma(\gamma + 1)}{\Gamma(\beta + 1)\Gamma(\gamma - \beta + 1)} \quad \text{and} \quad \gamma! = \Gamma(\gamma + 1),
\]
even when the numbers \( \gamma \) and \( \beta \) are not necessarily natural numbers. We will also frequently make use of the fact that

\[
\sum_{n=0}^{\infty} \frac{(\cdots)}{(n-k)!(N-n)!\cdots} = \sum_{n=k}^{N} \frac{(\cdots)}{n!(N-n)!\cdots},
\]
where the \( \cdots \) represent any possible quantity (constant or variable, depending on anything we want, but the same on both sides of the equation). This identity, reducing an infinite series to a finite one, is valid since dividing by a factorial of a negative integer always gives zero, due to the zeros of \( \frac{1}{\Gamma(z)} \).

We shall also frequently use the next lemma on series.
Lemma 2.6. Let $a, b, c \in \mathbb{C}$ be constants. The series
\[
\sum_{n=0}^{\infty} \frac{1}{(a - n)!(b - n)!(c + n)!n!}
\]
converges to the value
\[
\frac{(a + b + c)!}{a!b!(a + c)!(b + c)!},
\]
provided that either $\text{Re}(a + b + c) > -1$ or one of $a, b$ is in $\mathbb{N}$.

Proof. When $\text{Re}(a+b+c) > -1$, this result is a rewriting of Dougall’s formula for bilateral hypergeometric series [1, Theorem 2.8.2], which states that:
\[
\sum_{n=-\infty}^{\infty} \frac{(1 + a + n)!(1 + b + n)!}{(c + n)!(d + n)!} = \frac{\pi^2}{\sin(\pi a) \sin(\pi b)} \cdot \frac{(c + d - a - b)!}{(c - a)!(d - a)!(c - b)!(d - b)!},
\]
when $1 + \text{Re}(a + b) < \text{Re}(c + d)$. This result can be rewritten to our required result by replacing $a, b$ with $-a, -b$ and putting $d = 0$ (thus turning the bilateral series into a unilateral series from 0 to $\infty$) and using the reflection formula for the gamma function together with the following result which is a consequence of it:
\[
\frac{\Gamma(x + n)}{\Gamma(x)} = (-1)^n \cdot \frac{\Gamma(1 - x)}{\Gamma(1 - x - n)}, \quad x \in \mathbb{C}, \ n \in \mathbb{Z}.
\]
(This identity is easy to prove from writing each quotient of gamma functions as a finite product of terms: the left-hand side is $x(x+1)(x+2)\ldots(x+n-1)$ and the right-hand side similarly. This also justifies the validity of the identity for all $x \in \mathbb{C}$, without needing to avoid singularities of the gamma function.)

So the infinite series is convergent and the result follows in the case that $\text{Re}(a + b + c) > -1$.

If we suppose that one of $a, b$ is in $\mathbb{N}$, then the series is actually finite, since all but finitely many of the terms are zero. Fixing $a = N \in \mathbb{N}$, the result becomes
\[
\sum_{n=0}^{N} \frac{1}{(N - n)!(b - n)!(c + n)!n!} = \frac{(N + b + c)!}{N!b!(N + c)!(b + c)!},
\]
which we now know is true under the assumption $\text{Re}(N + b + c) > -1$. The variables $b$ and $c$ are still free in $\mathbb{C}$ subject to this restriction. Both sides of the equation (12) are analytic in $b$ and $c$, since the series is finite and therefore convergent, so we can use analytic continuation to deduce that (12) is valid for all $b, c \in \mathbb{C}$.

\[\square\]

3. Main Results
The genesis is the representation:
\[
y(t) = \sum_{k=0}^{\infty} (-1)^k I_{0+}^\beta \left(a(t) I_{0+}^\beta\right)^k b(t),
\]
where \( a(t), b(t) \in C[0, T] \) and \( \text{Re} (\beta) > 0 \). Here, in order to be able to use the fractional Leibniz rule of Lemma 2.3, we must also assume that \( a(t) \) is an analytic function. (The textbook of Miller & Ross [24, p. 97] mentions unpublished work of E. R. Love in which he weakened the analyticity assumption for the fractional Leibniz rule, but we have been unable to find this work, if indeed it was ever published.)

Let us use the following notation for the \( k \)-summand:

\[
S_k(t) := I_{0+}^{\beta} \left( a(t)I_{0+}^{\beta} \right)^k b(t).
\]

The goal of this section is to find a formula for the summand \( S_k(t) \), and hence for the solution function \( y(t) \), as an infinite series without involving any composition of fractional integrals.

### 3.1. Early Cases

Let us start by considering the first few cases of \( k \), in order to spot a pattern and write down a formula for general \( k \) to be proved by induction on \( k \).

**The case \( k = 0 \).** Here, the result is trivial:

\[
S_0(t) = I_{0+}^{\beta} b(t).
\]

**The case \( k = 1 \).** Here, the result follows from a single application of the fractional Leibniz rule:

\[
S_1(t) = I_{0+}^{\beta} \left( a(t)I_{0+}^{\beta} \right) b(t) = \sum_{n=0}^{\infty} \binom{-\beta}{n} a^{(n)}(t) \int_{0+}^{\beta + n} b(t)
\]

**The case \( k = 2 \).** Here, we need to use the fractional Leibniz rule and then the classical Leibniz rule:

\[
S_2(t) = I_{0+}^{\beta} \left( a(t)S_1(t) \right) = I_{0+}^{\beta} \left[ \sum_{n=0}^{\infty} \binom{-\beta}{n} a(t)a^{(n)}(t) \int_{0+}^{\beta + n} b(t) \right]
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{-\beta}{n} \binom{-\beta}{m} D^n a(t) \cdot I_{0+}^{\beta + n} b(t)
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{i=0}^{m} \binom{-\beta}{n} \binom{-\beta}{m} \binom{m}{i} D^i a(t) \cdot I_{0+}^{\beta + n} b(t).
\]

Putting \( p = n + m \), and rearranging the sums so that the \( i \)-sum is outside the \( m \)-sum, we have:

\[
S_2(t) = \sum_{p=0}^{\infty} \sum_{i=0}^{p} \sum_{m=i}^{p} \binom{-\beta}{p-m} \binom{-\beta}{m} \binom{m}{i} D^i a(t) \cdot I_{0+}^{\beta + p} b(t).
\]

The innermost sum (over \( m \)) can be simplified using Lemma 2.6:

\[
\sum_{m=i}^{p} \binom{-\beta}{p-m} \binom{-\beta}{m} \binom{m}{i} = \sum_{m=i}^{p} \frac{(-\beta)! (-\beta)!}{(-\beta - p + m)! (p - m)! (-\beta - m)! (m - i)! i!} = \sum_{n=0}^{p-i} \frac{(-\beta)! (-\beta)!}{(-\beta - p + n + i)! (p - n - i)! (-\beta - n - i)! n! i!}
\]
\[
\frac{(-\beta)!(-\beta)!}{i!} \sum_{n=0}^{\infty} \frac{1}{(p-i-n)!(-\beta-i-n)!(-\beta-p+i+n)!n!} = \frac{(-\beta)!(-\beta)!}{i!} \frac{(-2\beta-i)!}{(p-i)!(-\beta-i)!(-2\beta-p)!}.
\]

Therefore, re-labelling \( p \) as \( n \), we have the final answer for \( k = 2 \):

\[
S_2(t) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(-2\beta-i)!(-\beta)!}{(-2\beta-n)!(-\beta-i)!}(p-i)!n! D^i a(t) \cdot D^{n-i} a(t) \cdot \beta^{3+n} b(t).
\]

The case \( k = 3 \). Here, the manipulation is even more complicated. We start by applying the fractional Leibniz rule and then the classical Leibniz rule for a product of three functions:

\[
S_3(t) = I_{0+}^\beta \left( a(t) S_2(t) \right)
= I_{0+}^\beta \left[ \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(-2\beta-i)!(-\beta)!}{(-2\beta-n)!(-\beta-i)!}(p-i)!n! D^i a(t) \cdot D^{n-i} a(t) \cdot \beta^{3+n} b(t) \right]
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-\beta)!}{m!} \frac{(-2\beta-i)!(-\beta)!}{(-2\beta-n)!(-\beta-i)!}(p-i)!n! m! D^m a(t) \cdot D^i a(t) \cdot D^{n-i} a(t) \cdot \beta^{3+n+m} b(t)
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \sum_{r_1, r_2} \frac{(-2\beta-i)!(-\beta)!}{(-2\beta-n)!(-\beta-m)!(-\beta-i)!}(p-i)!n! m! r_1! r_2! D^i a(t) \cdot D^{i+r_2} a(t) \cdot D^{n-i+m-r_1-r_2} a(t) \cdot \beta^{3+n+m} b(t).
\]

Putting \( p = m+n \), and rearranging sums so that the \( m \)-series is the innermost one, we get:

\[
S_3(t) = \sum_{p=0}^{\infty} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \sum_{i=0}^{\infty} \sum_{m=r_1+r_2} \frac{(-2\beta-i)!(-\beta)!}{(-2\beta-p-m)!(-\beta-m)!(-\beta-i)!}(p-m-i)!r_1!r_2!(m-r_1-r_2)! D^i a(t) \cdot D^{i+r_2} a(t) \cdot D^{p-i-r_1-r_2} a(t) \cdot \beta^{3+p+r} b(t)
= \sum_{p=0}^{\infty} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \sum_{i=0}^{\infty} \sum_{m=r_1+r_2} \frac{1}{(-2\beta-i)!(-\beta-i)!} \frac{(-2\beta-i)!(-\beta)!}{(-\beta-i)!r_1!r_2!} D^i a(t) \cdot D^{i+r_2} a(t) \cdot D^{p-i-r_1-r_2} a(t) \cdot \beta^{3+p+r} b(t).
\]

By Lemma 2.6, the inner sum over \( m' \) is

\[
\frac{(-3\beta-r_1-r_2)!}{(-\beta-r_1-r_2)!} \frac{(-3\beta-p)!}{(-2\beta-i)!}.
\]
so we have

\[
S_3(t) = \sum_{p=0}^{\infty} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-3\beta - r_1 - r_2 - i)!(-\beta)!(-\beta)!}{(\beta - r_1 - r_2)!(p - r_1 - r_2 - i)!(\beta - r_1)!r_1!r_2!} \\
x D^{r_1}a(t) \cdot D^{i+r_2}a(t) \cdot D^{p-i-r_1-r_2}a(t) \cdot I_{0+}^{4\beta+p}b(t).
\]

Putting \( i_1 = r_1 \) and \( i_2 = i + r_2 \), this becomes

\[
S_3(t) = \sum_{p=0}^{\infty} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-3\beta - i_1 - i_2)!(-\beta)!}{(\beta - i_1 - i_2 + i)!(p - i_1 - i_2)!(\beta - i_1)!i_1!(i_2 - i)!} \\
x D^{i_1}a(t) \cdot D^{i_2}a(t) \cdot D^{p-i_1-i_2}a(t) \cdot I_{0+}^{4\beta+p}b(t)
\]

By Lemma 2.6 again, the inner sum over \( i \) is

\[
(-3\beta - i_1)!
\]

Therefore, re-labelling \( p \) as \( n \), we have the final answer for \( k = 3 \):

\[
S_3(t) = \sum_{n=0}^{\infty} \sum_{i_1+2i_2 \leq n} \left[ \sum_{i=0}^{\infty} \frac{(-3\beta - i_1 - i_2)!(-2\beta - i_1)!(\beta)!}{(\beta - i_1 - i_2 - i)!i_1!i_2!} \right] \\
x D^{i_1}a(t) \cdot D^{i_2}a(t) \cdot D^{n-i_1-i_2}a(t) \cdot I_{0+}^{4\beta+n}b(t).
\]

### 3.2. The General Case

Having solved the problem for the first few values of \( k \), we are now in a position to extrapolate the above results and guess a formula for general \( k \), which we can then prove by induction.

**Theorem 3.1.** For all \( k \geq 1 \), with all notation defined as above, we have

\[
S_k(t) = \sum_{n=0}^{\infty} \left[ \sum_{i_1+i_2+\cdots+i_k=n} \frac{(-\beta - n)!(\beta)!}{(-2\beta - i_1)!(\beta - i_1)!i_1!i_2!\cdots i_k!} \right] \\
x \frac{n!}{i_1!\cdots i_k!} a^{(i_1)}(t) \cdot \cdots a^{(i_k)}(t) \left( \frac{-\beta}{n} \right) I_{0+}^{(k+1)\beta+n}b(t).
\]

**Proof.** Let us prove it by induction on \( k \). The \( k = 1 \) case of (14) is

\[
S_1(t) = \sum_{n=0}^{\infty} \sum_{i=n}^{\infty} \frac{(-\beta - n)!}{(-\beta)!} \frac{n!}{i!} a^{(i)}(t) \left( \frac{-\beta}{n} \right) I_{0+}^{2\beta+n}b(t) = \sum_{n=0}^{\infty} a^{(n)}(t) \left( \frac{-\beta}{n} \right) I_{0+}^{2\beta+n}b(t),
\]

where the summation is over multiple \( n \) and \( i \), including both \( i = n \).
and the $k = 2$ case of (14) is
\[
S_2(t) = \sum_{n=0}^{\infty} \left[ \sum_{i_1+i_2=n} \frac{(-\beta-n)!(-2\beta-i_1)!}{(-\beta-i_1)!(-2\beta-n)!} \cdot \frac{n!}{i_1!i_2!} a^{(i_1)}(t)a^{(i_2)}(t) \right] \left( -\frac{\beta}{n} \right) I_0^{\beta+n} b(t)
\]
\[
= \sum_{n=0}^{\infty} \left[ \sum_{i=0}^{n} \frac{(-\beta-n)!(-2\beta-i)!}{(-\beta-i)!(-2\beta-n)!} \cdot \frac{n!}{i!} a^{(i)}(t)a^{(n-i)}(t) \right] \left( -\frac{\beta}{n} \right) I_0^{\beta+n} b(t),
\]
both of which are correct according to the work done above.

Now suppose the result holds for $S_k(t)$, and let us prove it for $S_{k+1}(t)$. Firstly, using the fractional Leibniz rule and then the classical Leibniz rule:

\[
S_{k+1}(t) = I_0^\beta (a(t)S_k(t))
\]
\[
= I_0^\beta \left[ \sum_{n,m=0}^{\infty} \frac{(-\beta)^n}{n!} I_0^{(k+1)\beta+n} b(t) \sum_{i_1+i_2+\cdots+i_k=n} \frac{n!}{i_1!\cdots i_k!} a^{(i_1)}(t)a^{(i_2)}(t) \cdots a^{(i_k)}(t) \right] \times \frac{(-\beta-n)!(-2\beta-i_1)!(-3\beta-i_1-i_2)!\cdots(-k\beta-i_1-\cdots-i_{k-1})!}{(-\beta-i_1)!(-2\beta-i_1-i_2)!\cdots(-(k-1)\beta-i_1-\cdots-i_{k-1})!}\frac{1}{(-\beta-n)!(-2\beta-i_1)!(-3\beta-i_1-i_2)!\cdots(-k\beta-i_1-\cdots-i_{k-1})!}
\]
\[
= \sum_{n,m=0}^{\infty} \frac{(-\beta)^n}{n!} I_0^{(k+2)\beta+n+m} b(t) \sum_{i_1+i_2+\cdots+i_k=m} \frac{n!}{i_1!\cdots i_k!} D^m [a^{(i_1)}(t)a^{(i_2)}(t) \cdots a^{(i_k)}(t)] \times \frac{(-\beta-n)!(-2\beta-i_1)!(-3\beta-i_1-i_2)!\cdots(-k\beta-i_1-\cdots-i_{k-1})!}{(-\beta-i_1)!(-2\beta-i_1-i_2)!\cdots(-(k-1)\beta-i_1-\cdots-i_{k-1})!}\frac{1}{(-\beta-n)!(-2\beta-i_1)!(-3\beta-i_1-i_2)!\cdots(-k\beta-i_1-\cdots-i_{k-1})!}
\]
\[
= \sum_{n,m=0}^{\infty} \sum_{i_1+i_2+\cdots+i_k=m} I_0^{(k+2)\beta+n+m} b(t) \sum_{i_1+i_2+\cdots+i_k=m} \frac{n!}{i_1!\cdots i_k!} a^{(r_1)}(t)a^{(i_1+r_2)}(t) \cdots a^{(i_k+r_{k+1})}(t) \times (-\beta)\left( -\beta \right)!(-2\beta-i_1)!(-3\beta-i_1-i_2)!\cdots(-k\beta-i_1-\cdots-i_{k-1})!
\]
Now, putting $p = n + m$ and then separating out $i_k$ and $r_{k+1}$ among the others, we have:

\[
S_{k+1}(t) = \sum_{p=0}^{\infty} \sum_{n=0}^{p} \sum_{i_1+i_2+\cdots+i_k=n} \frac{1}{i_1!\cdots i_k!r_1!\cdots r_{k+1}!} a^{(r_1)}(t)a^{(i_1+r_2)}(t) \cdots a^{(i_k+r_{k+1})}(t)
\]
\[
= \sum_{p=0}^{\infty} \sum_{n=0}^{p} \sum_{i_1+i_2+\cdots+i_k=n} \frac{1}{i_1!\cdots i_k!r_1!\cdots r_{k+1}!} a^{(r_1)}(t)a^{(i_1+r_2)}(t) \cdots a^{(i_k+r_{k+1})}(t) a^{(p-l-R)}(t)
\]
\[
\times \frac{1}{i_1! \ldots i_{k-1}!(n-I)!r_1! \ldots r_k!(p-n-R)!}
\]
\[
\times \frac{(-\beta)!(-\beta)!(-\beta)!(-3\beta - i_1 - i_2)! \ldots (-k\beta - i_1 - \cdots - i_{k-1}!)}
{(-\beta - i_1)!(-\beta - i_1 - i_2)! \ldots (-k(1)\beta - i_1 - \cdots - i_{k-1}!)\left((-k-\beta - n)!(-k-\beta - p + n)\right)}
\]
where we have written \( I = i_1 + \cdots + i_{k-1} \) and \( R = r_1 + \cdots + r_k \) for simplicity.
Now we can make the \( n \)-summation the innermost one and use Lemma 2.6 to say that
\[
\sum_{n=0}^{\infty} \frac{1}{n!(p-n-R)!(-k\beta - n)!(-\beta - p + n)!}
\]
\[
= \sum_{n'=0}^{\infty} \frac{1}{n'(p-n' - I - R)!(-k\beta - n'-I)!(-\beta - p + n' + I)!}
\]
\[
= \frac{(-\beta - R)!(-k\beta - I)!(-\beta - R)!(-k(1)\beta - p)!}{(p-I - R)!(-k\beta - I)!(-\beta - R)!(-k(1)\beta - p)!}.
\]
Re-labelling \( p \) as \( n \), and comparing the expression we have reached so far with the desired final result, we then see that it remains to prove for all \( n \geq 0 \) that:
\[
\sum_{i_1, \ldots, i_{k-1} \text{, } r_1, \ldots, r_k} \frac{(-\beta)!(-\beta)!(-3\beta - i_1 - i_2)! \ldots (-k\beta - i_1 - \cdots - i_{k-1}!)}{(-\beta - i_1)!(-\beta - i_1 - i_2)! \ldots (-k(1)\beta - i_1 - \cdots - i_{k-1}!)}
\]
\[
\times \frac{1}{i_1! \ldots i_{k-1}!r_1! \ldots r_k!(n-I-R)!}
\]
\[
= \sum_{j_1 + j_2 + \cdots + j_{k+1} = n} \frac{(-\beta)!(-3\beta - j_1 - j_2)! \ldots (-k\beta - j_1 - \cdots - j_{k})!}{(-\beta - j_1)!(-2\beta - j_1 - j_2)! \ldots (-k\beta - j_1 - \cdots - j_{k})!\left((-k+1)\beta - n\right)}
\]
\[
\times \frac{1}{j_1! \ldots j_{k+1}!} a^{(i_1)}(t) \ldots a^{(i_{k-1}+r_k)}(t)A^{(n-I-R)}(t)
\]
Writing \( j_1 = r_1, j_2 = i_1 + r_2, \ldots, j_k = i_{k-1} + r_k \), this becomes
\[
\sum_{j_1, \ldots, j_k \text{, } i_1 \leq j_2 \ldots, i_{k-1} \leq j_k} \frac{(-\beta)!(-\beta)!(-3\beta - i_1 - i_2)! \ldots (-k\beta - i_1 - \cdots - i_{k-1}!)}{(-\beta - i_1)!(-\beta - i_1 - i_2)! \ldots (-k(1)\beta - i_1 - \cdots - i_{k-1}!)}
\]
\[
\times \frac{1}{i_1! \ldots i_{k-1}!j_1!j_2! \ldots j_{k}!\left(n-j_1 - \cdots - j_{k}\right)!}
\]
\[
= \sum_{j_1, \ldots, j_k} \frac{(-\beta)!(-2\beta - j_1)!(-3\beta - j_1 - j_2)! \ldots (-k\beta - j_1 - \cdots - j_{k})!}{(-\beta - j_1)!(-2\beta - j_1 - j_2)! \ldots (-k\beta - j_1 - \cdots - j_{k})!\left((-k+1)\beta - n\right)}
\]
\[
\times \frac{1}{j_1! \ldots j_{k}!(n-j_1 - \cdots - j_{k})!} a^{(i_1)}(t) \ldots a^{(i_{k-1}+r_k)}(t).
\]
Therefore, it remains to prove the following for all $j_1, \ldots, j_k \geq 0$:

\[
\sum_{i_1, \ldots, i_{k-1}} (-\beta)!(-2\beta - i_1)!(-3\beta - i_1 - i_2)! \ldots (-k\beta - i_1 - \cdots - i_{k-2})! (-\beta - i_1)!(-2\beta - i_1 - i_2)! \ldots (-k\beta - i_1 - \cdots - i_{k-1})! \\
\times \frac{1}{(-\beta - j_1 - \cdots - j_k + i_1 + \cdots + i_{k-1})!i_{k-1}!(j_2 - i_1)! \ldots (j_k - i_{k-1})!} \\
= \frac{1}{(-\beta - j_1)!(-2\beta - j_1 - j_2)! \ldots (-k\beta - j_1 - \cdots - j_k)!j_2! \ldots j_k!},
\]

which is an identity between finite series, elementary albeit complicated to prove.

In the series on the left-hand side of (15), there are four denominator terms and no numerator terms involving $i_{k-1}$. By Lemma 2.6, we have

\[
\sum_{i_{k-1} = 0}^{\infty} \frac{1}{(-\beta - j_1 - \cdots - j_k + i_1 + \cdots + i_{k-1})!i_{k-1}!(j_2 - i_1)! \ldots (j_k - i_{k-1})!} \\
= \frac{1}{(-\beta - j_1)!(-2\beta - j_1 - j_2)! \ldots (-k\beta - j_1 - \cdots - j_k)!j_2! \ldots j_k!},
\]

so (15) is equivalent to

\[
\sum_{i_1, \ldots, i_{k-1}} (-\beta)!(-2\beta - i_1)!(-3\beta - i_1 - i_2)! \ldots (-k\beta - i_1 - \cdots - i_{k-2})! (-\beta - i_1)!(-2\beta - i_1 - i_2)! \ldots (-k\beta - i_1 - \cdots - i_{k-2})! \\
\times \frac{1}{(-\beta - j_1 - \cdots - j_k - i_1 + \cdots + i_{k-2})!i_{k-2}!(j_2 - i_1)! \ldots (j_{k-1} - i_{k-2})!} \\
= \frac{1}{(-\beta - j_1)!(-2\beta - j_1 - j_2)! \ldots (-k\beta - j_1 - \cdots - j_{k-2})!j_2! \ldots j_{k-2}!}.
\]

which is itself identical to (15) with $k$ replaced by $k - 1$. Therefore, by finite descent, it will be sufficient to prove (15) in the basic case $k = 2$. In this case, the equation (15) is

\[
\sum_i (-\beta)!(-\beta - i)! \cdot \frac{1}{(-\beta - j_1 - j_2 + i)!i!(j_2 - i)!} = \frac{(-2\beta - j_1)!}{(-\beta - j_1)!(-2\beta - j_1 - j_2)!j_2!},
\]

which follows from Lemma 2.6. Now we have completed the induction process, and the proof as well.

We now establish our most important result of the paper, which comes from Theorem 3.1.

**Theorem 3.2.** Let $a(t)$ be analytic and $b(t)$ be continuous on $[0, T]$, and let $\beta \in \mathbb{C}$ with $\text{Re}(\beta) > 0$ and $m = \lfloor \text{Re}(\beta) \rfloor + 1$. Then, the FDE (10) under the conditions (11) has a unique solution $y \in \mathcal{C}^{m-1, \beta}[0, T]$ given by the following convergent infinite series:

\[
y(t) = \mathcal{I}_{0+}^{\beta} b(t) + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} (-1)^k \left(\frac{-\beta}{n}\right) t^{(k+1)\beta+n} b(t) \\
\times \sum_{i_1 + i_2 + \cdots + i_k = n}
\]
\left(\frac{-\beta-n}{-\beta-i_1}(\frac{-3\beta-i_1-i_2}{-2\beta-i_1-i_2})\ldots(\frac{-k\beta-i_1-\ldots-i_{k-1}}{-(k-1)\beta-i_1-\ldots-i_{k-1}})\right)! \\
\times \frac{n!}{i_1!\ldots i_k!}a^{(i_1)}(t)\ldots a^{(i_k)}(t) \right].

3.3. Examples
In order to verify the consistency of our new results with some published results, let us take the special case of (10) where the variable coefficient is a constant, i.e. $a(t) = \lambda \in \mathbb{R}$. Thus, we consider

$$
\begin{cases}
C D_{0+}^{\beta} y(t) + \lambda y(t) = b(t), & t \in [0,T], \\
\left( \frac{d}{dt} \right)^k y(t) \bigg|_{t=+0} = 0, & k = 0, 1, \ldots, \lfloor \text{Re} \beta \rfloor, 
\end{cases}
$$

(16)

where $\text{Re} \beta > 0$, $\lambda \in \mathbb{R}$, and $b \in C[0,T]$.

The consequence of Theorem 3.2 in this case is that the FDE (16) has a unique solution $y \in C^{[\beta]}[0,T]$ given by:

$$
y(t) = \int_{t_0}^{t} (t-s)^{\beta-1} E_{\beta, \beta}(-\lambda(t-s)^{\beta})b(s) \, ds,
$$

(17)

where $E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$ is the two-parameter Mittag-Leffler function [14] for any complex parameters $\alpha, \beta$ with $\text{Re} \beta > 0$.

To see why, notice that, setting $a(t) = \lambda$ in Theorem 3.2, we have directly

$$
y(t) = \sum_{k=0}^{\infty} (-\lambda)^k I_0^{(k+1)\beta} b(t),
$$

since the second series given in Theorem 3.2 becomes zero for $n \neq 0$, while for $n = 0$ we get just $\lambda^k$. Now the desired representation follows straightforwardly using the definition of the Mittag-Leffler function.

The expression (17) is consistent with the results of [5, Theorem 7.2 and Remark 7.1] for the solution of the same differential equation. Therefore, we have verified the consistency of our results with those already known in the literature.

The above example is the simple case of a differential equation with constant coefficients. In order to emphasise the broader applicability of our results, let us also consider an example with non-constant coefficients:

$$
\begin{cases}
C D_{0+}^{\beta} y(t) + t^\alpha y(t) = t^\gamma, & t \in [0,T], \\
y(t) \bigg|_{t=+0} = 0, 
\end{cases}
$$

(18)

where $0 < \beta < 1$ and $\alpha, \gamma > 0$. According to [12, Example 6.2], the unique solution in $C^{[\beta]}[0,T]$ can be expressed as follows, using the generalised Kilbas–Saigo type Mittag-Leffler function of [19, (1.9.19)–(1.9.20)]:

$$
y(t) = I_{0+}^{\beta} (t^\gamma E_{\beta,1+\alpha/\beta,\gamma/\beta}(-t^{\alpha+\beta})) = I_{0+}^{\beta} \left( \sum_{k=0}^{\infty} t^{k\alpha+k\beta+\gamma} (-1)^k \prod_{j=0}^{k-1} \frac{(j(\beta+\alpha)+\gamma)!}{(j(\beta+\alpha)+\gamma+\beta)!} \right)
$$
\[
= \sum_{k=0}^{\infty} \left( -1 \right)^k t^{k\alpha + k\beta + \beta + \gamma} \frac{(k\alpha + k\beta + \gamma)!}{(k\alpha + k\beta + \beta + \gamma)!} \times \frac{\gamma!(\beta + \alpha + \gamma)!(2\beta + 2\alpha + \gamma)! \cdots ((k-1)\beta + (k-1)\alpha + \gamma)!}{(\gamma + \beta)!(2\beta + \alpha + \gamma)!(3\beta + 2\alpha + \gamma)! \cdots (k\beta + (k-1)\alpha + \gamma)!}.
\]

From our result here (Theorem 3.2), we have the following expression:

\[
y(t) = I_{\alpha+1}^{\beta} + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} (-1)^k \left( \frac{-\beta}{n} \right) I_{\alpha+1}^{(k+1)\beta + n} t^\gamma
\]

\[
\times \left[ \sum_{i_1 + i_2 + \cdots + i_k = n} \frac{(-\beta - n)!(-2\beta - i_1)!(-3\beta - i_1 - i_2)! \cdots (-k\beta - i_1 - \cdots - i_{k-1})!}{(-\beta - i_1)!(-2\beta - i_1 - i_2)! \cdots ((k-1)\beta - i_1 - \cdots - i_{k-1})!(-k\beta - n)!} \times \frac{n!}{i_1! \cdots i_k!} \cdot \frac{((\alpha - i_1)! \cdots (\alpha - i_k)!}{((\alpha - i_1)! \cdots (\alpha - i_k)!} \cdot \frac{(\gamma + (k+1)\beta + n)!}{(\gamma + (k+1)\beta + n)!} \right]
\]

Comparing the above two results, it is clear that the powers of \( t \) are consistent and we have the following identity between the coefficients:

\[
\frac{\gamma!(\beta + \alpha + \gamma)!(2\beta + 2\alpha + \gamma)! \cdots ((k-1)\beta + (k-1)\alpha + \gamma)!}{(\gamma + \beta)!(2\beta + \alpha + \gamma)!(3\beta + 2\alpha + \gamma)! \cdots (k\beta + (k-1)\alpha + \gamma)!((k+1)\beta + k\alpha + \gamma)!}
\]

\[
= \sum_{n=0}^{\infty} \frac{(\alpha)k\gamma!}{(\gamma + (k+1)\beta + n)!} \times \sum_{i_1 + i_2 + \cdots + i_k = n} \frac{1}{i_1! \cdots i_k!(\alpha - i_1)! \cdots (\alpha - i_k)!} \times \frac{(-\beta)!(-2\beta - i_1)!(-3\beta - i_1 - i_2)! \cdots (-k\beta - i_1 - \cdots - i_{k-1})!}{(-\beta - i_1)!(-2\beta - i_1 - i_2)! \cdots ((k-1)\beta - i_1 - \cdots - i_{k-1})!(-k\beta - n)!},
\]

valid for every \( k \in \mathbb{N} \) and for \( \alpha, \beta, \gamma > 0 \) with \( \beta < 1 \). (In the case \( k = 1 \), this identity reduces to the result of Lemma 2.6.)
4. Some Applications to Partial Differential Equations

The results of the previous section concern ordinary differential equations of fractional type, but they can also be applied to solve certain types of partial differential equations using both time-fractional and space-fractional operators. In this section, we give an analytical solution of a Cauchy type problem for a fractional partial differential equation with time-dependent coefficient. Normally, the solutions of such problems are found by applying numerical tools or different methods to approximate the solution. For instance, the general problems treated in [4,18,33] and references therein can be compared with our Cauchy type problems treated below. Finding the close form of the solution of an equation with a time-dependent variable coefficient, as we shall do in this section, will be potentially advantageous in the understanding of these problems.

First, we recall the Fourier transform, its inverse, and the fractional Laplacian, which will be used below.

Definition 4.1. As usual, the Fourier transform is defined by
\[ g(w) = (\mathcal{F}\phi)(w) = \hat{\phi}(w) = \int_{\mathbb{R}^n} e^{iwx} \phi(x) \, dx, \]
for a function \( \phi : \mathbb{R}^n \rightarrow \mathbb{C} \). Conversely, its inverse is defined by
\[ \phi(w) = (\mathcal{F}^{-1}g)(w) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iwx} g(x) \, dx. \]

Definition 4.2. The fractional Laplacian \((-\Delta)^\gamma\) is defined by [31, Chapter 5]:
\[ (\mathcal{F}(-\Delta)^\gamma f)(y) = |y|^{2\gamma}(\mathcal{F}f)(y), \quad y \in \mathbb{R}^n, \quad 0 < \gamma \leq 1/2. \] (19)

There are several equivalent forms to define the above operator, but in our analysis, the most convenient is the above one.

Theorem 4.3. Let \( T > 0, \ 0 < \alpha \leq 1, \ \beta > 0, \ m - 1 \leq \beta < m, \ r(x, \cdot) \in C[0,T], \)
and let \( \Psi(t) \in C[0,T] \) be analytic. We deal with the following equation:
\[
\begin{cases}
C \partial_t^\beta h(x,t) + \Psi(t)(-\Delta)^\alpha h(x,t) &= r(x,t), \quad t \in (0,T], \ x \in \mathbb{R}^n, \\
h(x,0) &= 0, \\
\partial_t h(x,0) &= 0, \\
\vdots \\
\partial_t^m h(x,0) &= 0,
\end{cases}
\] (20)

where \((-\Delta)^\alpha\) is the fractional Laplacian with respect to \( x \in \mathbb{R}^n \) and \( C \partial_t^\beta \) is the Caputo fractional derivative (defined according to (4) with \( a = 0 \)) with respect to \( t \in (0,T) \).

The solution of equation (20) is given by
\[
h(x,t) = \mathcal{I}_t^\beta r(s,t) + \pi^{n/2} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \left( -\frac{\beta}{n} \right) (-2^\alpha)^k \frac{\Gamma(\frac{ak+n}{2})}{\Gamma(-\alpha k/2)} \mathcal{I}_t^{(k+1)\beta+n} (r(s,t) * |s|^{-\alpha k-n})(x,t)
\]
\[
\sum_{i_1 + i_2 + \cdots + i_k = n} (-\beta - n)!(-2\beta - i_1)!(3\beta - i_1 - i_2)\ldots(-k\beta - i_1 - \cdots - i_{k-1})!
\]

\[
\sum_{i_1 + i_2 + \cdots + i_k = n} (-\beta - i_1)!(-2\beta - i_1 - i_2)\ldots(-k\beta - i_1 - \cdots - i_{k-1})!(k\beta - n)!
\]

\[
\frac{n!}{i_1! \ldots i_k!} \Psi^{(i_1)}(t) \ldots \Psi^{(i_k)}(t)
\]

\[\times \left[ 2^{\lambda+n} \pi^{\frac{n}{2}} \Gamma \left( \frac{\lambda+n}{2} \right) \rho^{-\lambda-n}, \quad \rho = \sqrt{\sum_{i=1}^{n} x_i}, \right.\]

\[
\frac{\Gamma \left( -\frac{\lambda}{2} \right)}{\Gamma \left( -\frac{\lambda+n}{2} \right)}
\]

Proof. The application of the Fourier transform with respect to the space variable on the original equation (20) transforms it into the following Cauchy problem in the \((s, t)\) domain:

\[
\begin{cases}
C \partial_t^\beta \hat{h}(s, t) + |s|^{2\alpha} \Psi(t) \hat{h}(s, t) = \hat{r}(s, t), & t \in (0, T], \ s \in \mathbb{R}^n, \\
\hat{h}(s, 0) = 0, & t = 0, \\
\partial_t \hat{h}(s, 0) = 0, & t = 0, \\
\vdots & \\
\partial_t^m \hat{h}(s, 0) = 0.
\end{cases}
\]

(21)

Here in (21), we have essentially an equation with variable coefficient \(|s|^{2\alpha} \Psi(t)\).

Hence, by using Theorem 3.2 with \(a(t) = |s|^{2\alpha} \Psi(t)\), we have that the system (21) has a solution, which is unique, in the space \(C^{\beta,m-1}[0, T]\) represented by:

\[
\hat{h}(s, t) = I_0^\beta \hat{r}(s, t) + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} (-1)^k (|s|^{2\alpha} \frac{\alpha k}{n}) \beta I_0^{(k+1)\beta+n} \hat{r}(s, t)
\]

\[
\times \left[ \sum_{i_1 + i_2 + \cdots + i_k = n} (-\beta - n)!(-2\beta - i_1)!(3\beta - i_1 - i_2)\ldots(-k\beta - i_1 - \cdots - i_{k-1})!
\]

\[
\times \frac{n!}{i_1! \ldots i_k!} \Psi^{(i_1)}(t) \ldots \Psi^{(i_k)}(t) \right].
\]

We then obtain the desired solution by the application of the inverse Fourier transform to the above expression for \(\hat{h}(s, t)\) and the following formula [13, p. 194, formula (2)]:

\[
\mathcal{F}(|s|^{2\lambda})(x) = 2^{\lambda+n} \pi^{\frac{n}{2}} \Gamma \left( \frac{\lambda+n}{2} \right) \rho^{-\lambda-n}, \quad \rho = \sqrt{\sum_{i=1}^{n} x_i},
\]

for all \(\lambda \in \mathbb{C}\) except \(\lambda = -n, -n - 2, \ldots\), at which points it has poles. \(\square\)

As a corollary of Theorem 4.3, considering the case \(\Psi(t) = \lambda \in \mathbb{R}\) in the equation (20), we get the following fractional Cauchy problem with a constant coefficient:
\[
\begin{cases}
C \partial_t^\beta h(x, t) + \lambda (-\Delta)^\alpha h(x, t) = r(x, t), \quad t \in (0, T], \ x \in \mathbb{R}^n, \\
h(x, 0) = 0, \\
\partial_t h(x, 0) = 0, \\
\vdots \\
\partial_t^m h(x, 0) = 0,
\end{cases}
\]

where \(0 < \alpha \leq 1, \ \beta > 0, \ m - 1 < \beta \leq m, \ r(x, \cdot) \in C[0, T], \) and \(\Psi(t) \in C[0, T].\)

Therefore
\[
h(x, t) = \int_0^t (t - u)^{\beta - 1} \mathcal{F}_s^{-1} \left\{ E_{\beta, \beta}(-\lambda |s|^{2\alpha} (t - u)^\beta) \hat{r}(s, t) \right\}(x, t) \, du,
\]

where the Mittag-Leffler function emerges in the same way as in Section 3.3 above, and where it is assumed that \(r(\cdot, t) \in L^1(\mathbb{R}^n)\) and \(E_{\beta, \beta}(-\lambda \cdot |t - s|^{2\alpha} (t - u)^\beta) \hat{r}(\cdot, t) \in L^1(\mathbb{R}^n).\)

### 5. Conclusions and Future Work

We have established new representations (by some convergent series) for the solutions to FDEs with variable time-dependent coefficients. These exact solutions had already been constructed in the previous work of [27], and their uniqueness in appropriate function spaces had been verified in [29], but the solutions constructed in the literature so far have been of a form which is very difficult to compute in practice, involving arbitrarily many nested fractional integrals with function multipliers in between. Our work here showcases a new formula which involves a single fractional integral in each summand, with a coefficient which depends only on the classical derivatives of the variable coefficient function. This is expected to be very useful in numerical calculations of solution functions for such equations, and we anticipate some new numerical work soon to validate and verify the efficiency of the formula that we have proved here. Since the current paper is purely analytic, such numerical computation will count as future work on this project.

To illustrate the impact of our obtained results, we have applied them, not only for ordinary differential equations with Caputo fractional derivatives and variable coefficients, but also for partial differential equations with both time-fractional and space-fractional derivative operators and with time-dependent coefficients. We have also verified that, in the case of constant coefficients, the results obtained are consistent with those already known in the literature [5] for this simple case.

The work in this paper has been under the assumption of homogeneous initial conditions. If we want to consider initial conditions different from zero, we can easily extend the results here, in a similar way as it was done in [27,29], to obtain a new version of Theorem 3.2 giving the solution of the problem (10) under general initial conditions, as well as a new version of Theorem 4.3 giving the solution of the problem (20) under general initial conditions. We leave these trivial variations of our results to be proved by an interested reader.
Many other extensions of these results are also possible. The differential equations considered here have featured only one fractional time derivative and one time-dependent coefficient. A more general class of FDEs was solved in the original work of [27], and we hope in the future to find a new series representation for that more general solution too, analogous to our work here. Furthermore, some other recent papers [10,11,29] have focused on extensions of the work of [27] to differential equations with other types of fractional operators, and these too can now be studied using the methods we have displayed in this paper. The results presented here have opened up many avenues for future research.

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Data availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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