RATE OF CONVERGENCE TO EQUILIBRIUM OF
SYMMETRIC SIMPLE EXCLUSION PROCESSES

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Abstract. We give bounds on the rate of convergence to equilibrium of the symmetric
simple exclusion process in $\mathbb{Z}^d$. Our results include the existent results in the literature.
We get better bounds and larger class of initial states via a unified approach. The method
includes a comparison of the evolution of $n$ interacting particles with $n$ independent ones
along the whole time trajectory.

1. Introduction.

The rate of convergence to equilibrium is one of the main problems in the theory of
Markov processes. It has recently attracted the attention of many authors in the context
of symmetric conservative particle systems in finite and infinite volume. In finite volume
the techniques used to obtain the rate of convergence to equilibrium rely mostly on the
estimation of the spectral gap of the generator. In general, one shows that the generator
of the particle system restricted to a cube of length $N$ has a gap of order $N^{-2}$ in any
dimension. This estimate together with standard spectral arguments permits to prove
that the particle system restricted to a cube of size $N$ decays to equilibrium in $L^2$ at
the exponential rate $\exp\{-ct/N^2\}$. This approach has been successfully extended to
the infinite volume setting and permitted to prove $L^2$-polynomial decay to equilibrium.
The method, however, does not give any information on the rate at which the system
converges to equilibrium starting from an arbitrary configuration or from arbitrary initial
distributions. In this article we consider the symmetric simple exclusion process in which
some explicit computations can be performed.

The symmetric simple exclusion process was introduced by Spitzer (1970). Informally
one can describe the process following the so called stirring representation. Fix a sym-
metric transition probability $p$ on $\mathbb{Z}^d : p(x, y) = p(y, x)$, $p(x, y) \geq 0$, $\sum_y p(x, y) = 1$
for all $x$ in $\mathbb{Z}^d$. We assume that the transition probability is translation invariant, $p(x, y) = p(0, y - x)$ and that it is indecomposable in the sense that for each $z$ in $\mathbb{Z}^d$, there exists an integer $n$ and a sequence $0 = z_0, \ldots, z_n = z$ such that $p(z_i, z_{i+1}) > 0$ for all $i = 0, \ldots, n - 1$. For each pair of sites $x, y$ such that $p(x, y) > 0$, consider a Poisson point process, denoted by $N_{x,y}(t)$, with rate $p(x, y)$. Assume that these processes are all independent. As initial state, fix some configuration $\eta$ with at most one particle per site. Thus, the configuration $\eta$ is a collection $\eta = \{\eta(x), x \in \mathbb{Z}\}$, where $\eta(x) = 1$ indicates that the site $x$ is occupied for the configuration $\eta$ and $\eta(x) = 0$ means that the site is empty. To obtain the state of the process from the initial states and from the Poisson point processes, we proceed as follows. Each time the Poisson process $N_{x,y}$ increases by 1, we interchange the variables $\eta(x)$ and $\eta(y)$. Notice that if both site $x$ and $y$ are occupied or if both are vacant before the jump of $N_{x,y}$, the configuration remains unchanged after the jump. In the other two cases the modification can be interpreted as the jump of a particle from the occupied site to the unoccupied site. Notice also that at each later time each site is occupied by at most one particle and that the total number of particles is conserved by the dynamics. This explains why this stochastic dynamics is called the symmetric simple exclusion process and in which sense it is conservative.

For $\rho \in [0, 1]$, denote by $\nu_\rho$ the Bernoulli product measure with density $\rho$. This is the probability measure on the configuration space $\{0, 1\}^{\mathbb{Z}^d}$ obtained by putting a particle at each site with probability $\rho$ independently of the other sites. Liggett proved that all the invariant measures for the symmetric simple exclusion process are convex combinations of the Bernoulli measures $\nu_\rho$ (cf. Liggett (1985)).

When one tries to study convergence to equilibrium for arbitrary initial configurations, the first issue is the very question: “what to prove?” The first attempt is to try to prove the existence of a function $h(t)$ that decreases to 0 as $t \uparrow \infty$ and the existence of a norm $\|f\|$ such that for all initial configuration $\eta$ chosen according to the invariant measure $\nu_\rho$ and for all cylinder function $f$

$$|\delta_\eta S(t)f - \nu_\rho f| \leq \|f\| h(t).$$

Here $\{S(t), t \geq 0\}$ stands for the semigroup of the symmetric simple exclusion process and $\delta_\eta$ for the probability measure concentrated on $\eta$ so that $\delta_\eta S(t)$ is the distribution of the process at time $t$ starting from $\eta$. Also, for a bounded function $f$ and a probability measure $\mu$, $\mu f$ stands for the expectation of $f$ with respect to $\mu$.

This is clearly not true as for any fixed $\epsilon > 0$ and any fixed $t$, one can always choose a set of configurations with $\nu_\rho$ positive probability for which

$$|\delta_\eta S(t)[\eta(0)] - \rho| \geq \epsilon.$$

Indeed, it is enough to consider configurations $\eta$ whose sites in a cube of length $t$ around the origin are all occupied. In general there is no hope to have uniform almost sure convergence in conservative systems. In spin flip systems the equilibrium is attained locally in an independent way for distant regions. For this reason one can hope to get almost sure convergence to equilibrium in spin-flip systems.
The second attempt is to fix a configuration \( \eta \), to choose the density \( \rho \) depending on the initial configuration: 
\[
\rho_t^n = \delta_n S(t) [\eta(0)]
\]
and to compute
\[
|\delta_n S(t) f - \nu \rho_t^n f| .
\]  
(1.1)

This approach cannot give a bound better than \( c(f) t^{-1/2} \) in any dimension. This is not satisfactory because, in view of the decay to equilibrium in \( L^2 \), one expects to obtain estimates of order \( t^{-d/2} \) in dimension \( d \). To check that this formulation cannot give bounds better than \( t^{-1/2} \), consider the configuration \( \eta \) such that 
\[
\eta(x) = 1 \quad \text{if and only if} \quad x_1 \geq 1.
\]
Here \( x_1 \) stands for the first coordinate of \( x \). In this case, a standard duality argument (that will be explained in section 2) gives that 
\[
\rho_t^n = P[X_t^0 \geq 1],
\]
where \( X_0 \) is a one dimensional symmetric random walk that starts from the origin. Fix the cylinder function 
\[
f(\eta) = \eta(e_1), \quad 1 \leq i \leq d,
\]
is the canonical basis of \( \mathbb{R}^d \). By duality and translation invariance, 
\[
\delta_n S(t) f = P[X_t^0 \geq 0].
\]
In particular, for this cylinder function, (1.1) is equal to 
\[
P[X_t^0 = 0],
\]
which is of order \( t^{-1/2} \).

At this point we have two possibilities. We may of course impose some regularity conditions on the initial configuration \( \eta \) (assume for instance that it is periodic) or to average the difference \( |\delta_n S(t) f - \nu \rho_t^n f| \) (or a power of this difference) with respect to some measure \( \nu \) that has asymptotic density \( \rho \) and some nice correlation properties. The other possibility is to take advantage that we are in the context of exclusion process, where all cylinder functions can be written as linear combinations of functions of type 
\[
\prod_{x \in A} \eta(x)
\]
for some finite set \( A \). In this case, a natural quantity to investigate is
\[
\delta_n S(t) \prod_{x \in A} \{\eta(x) - \rho_t^n(x)\},
\]
where \( \rho_t^n(x) = \delta_n S(t) \eta(x) \). These are the so-called \( v \)-functions introduced by Ferrari, Presutti, Scacciatelli and Vares (1991). A related quantity, also very natural in the context of the symmetric simple exclusion process is the difference
\[
\delta_n S(t) \prod_{x \in A} \eta(x) - \prod_{x \in A} \delta_n S(t) \eta(x).
\]  
(1.2)

We shall follow the two directions just mentioned. We shall first prove that the difference (1.2) can be expressed in terms of quantities related to independent random walks with transition rate \( p(x, y) \). This is the content of Theorem 2.2. This fact together with some elementary bounds on the transition probability of symmetric random walks will permit to obtain sharp estimates on the integral of powers of (1.1) with respect to translation-invariant measures that have density \( \rho \) and polynomial decaying correlations. This is the content of Theorem 2.1.

The comparison between \( n \) random walks interacting by exclusion and \( n \) independent random walks was studied by Bertein and Galves (1977), De Masi, Ianiro and Presutti (1982), De Masi and Presutti (1983), De Masi, Ianiro, Pellegrinotti and Presutti (1984),
Ferrari and Goldstein (1988), Ferrari, Presutti, Scacciatelli and Vares (1991) and Andjel (1994). We give a short and unified presentation which includes most of the above results.

A simple exclusion process in a finite box can be understood as a random walk in a finite set. Convergence to equilibrium for finite-state Markov processes have received new attention lately. We quote for instance Aldous (1983), Diaconis (1988), Diaconis and Stroock (1991). For the symmetric simple exclusion in a finite box, Quastel (1992) computed the spectral gap of the generator, which gives the $L^2$ exponential rate of convergence to equilibrium in finite volume. Using this result, Bertini and Zegarlinski (1998) proved polynomial $L^2$-convergence to equilibrium (time-correlation decay for the system in equilibrium) in infinite volume. Janvresse, Landim, Quastel and Yau (1999) proved a analogous result for the symmetric zero range process.

Cancrini and Galves (1995) obtained an upper bound for the rate of convergence of symmetric simple exclusion processes starting either from a periodic configuration or from a stationary measure satisfying mixing conditions. This result was extended to the one-dimensional nearest-neighbor zero-range process with rate $g(k) = 1\{k \geq 1\}$ by Galves and Guiol (1997).

2. Notation and results.

Let $p(x, y)$ be a symmetric, translation-invariant, irreducible, non-negative real matrix on $\mathbb{Z}^d$ such that

$$\sum_y |y|^2 p(0, y) < \infty.$$ 

To each pair of sites $(x, y)$ of $\mathbb{Z}^d$ attach a Poisson process of rate $p(x, y)$ denoted by $N_{x,y}(t)$. We adopt the following convention: $dN_{x,y}(t) = 1$ when there is an event of the corresponding Poisson process at time $t$. As a function of these Poisson processes, for $s < t$, we define a family of bijections of $\mathbb{Z}^d$ in itself, $\xi^s_t: \mathbb{Z}^d \to \mathbb{Z}^d$, such that

$$\xi^s_t(x) = x \quad \text{and} \quad d\xi^s_t = \sum_{x,y} \{(\xi^s_t)^{x,y} - \xi^s_t\}dN_{x,y}(t),$$

where for any bijection $\xi: \mathbb{Z}^d \to \mathbb{Z}^d$,

$$\xi^{x,y}(z) = \begin{cases} 
\xi(z) & \text{if } z \neq x, y, \\
\xi(y) & \text{if } z = x, \\
\xi(x) & \text{if } z = y
\end{cases}$$

and $N^s_{x,y}(t) = N_{x,y}(s + t) - N_{x,y}(s)$. $\xi^s$ is thus the identity at time $s$ and if there is a mark of the Poisson process $N_{x,y}$ at time $s + t$, $\xi^s(x)$ and $\xi^s(y)$ change their value in the following way: $\xi^s_t(x) = \xi^s_{t-}(y)$, $\xi^s_t(y) = \xi^s_{t-}(x)$. In particular, for any fixed $x$, the process $\xi^s_t(x)$ is a stochastic process that may have very long jumps even if the rates $p(\cdot, \cdot)$ are nearest-neighbor.

The simple exclusion process with initial configuration $\eta$ is defined as

$$\eta^\eta_t(x) = \eta(\xi^0_t(x)).$$
for all $t \geq 0$. Denote $S(t)f(\eta) = \mathbb{E}f(\eta_t)$. Here and below $\mathbb{P}$ and $\mathbb{E}$ stand for the probability and the expectation with respect to the Poisson point processes. The set of extremal invariant measures for this process is the set of Bernoulli product measures $\nu_\rho$ indexed by the density $\rho \in [0, 1]$. The main results of this article concern the convergence to those measures from different initial conditions. They are stated in the next theorem.

Fix $\alpha > 0$. We say that a measure $\nu$ on $\{0, 1\}^{\mathbb{Z}^d}$ has $\alpha$-decaying correlations if there exist a finite constant $C$ such that

$$\left| E_\nu[fg] - E_\nu[f]E_\nu[g] \right| \leq C \|f\|_\infty \|g\|_\infty \frac{1}{d(\Lambda_f, \Lambda_g)\alpha}$$

for every cylinder functions $f, g$. In this formula, $\Lambda_f, \Lambda_g$ stands for the support of the cylinder functions $f, g$ and $d(A, B)$ for the distance between two subsets $A, B$ of $\mathbb{Z}^d$.

**Theorem 2.1.** Let $\nu$ be a translation-invariant probability measure in $\{0, 1\}^{\mathbb{Z}^d}$ with density $\rho$ (that is $E_\nu[\eta(x)] = \rho$) and with $\alpha$-decaying correlations for some $\alpha > d$. Then, for each cylinder function $f$ there exists a constant $c(f)$ such that for all $t > 0$:

(a) Weak convergence:

$$|\nu S(t)f - \nu_\rho f| \leq c(f) (1 + t)^{-d/2}$$

(b) $L_p$ convergence for $1 \leq p \leq 2$:

$$\int \nu(d\eta) |\delta_\eta S(t)f - \nu_\rho f|^p \leq c(f) (1 + t)^{-dp/4}.$$  

(c) $L_p$ convergence for $p \geq 2$:

$$\int \nu(d\eta) |\delta_\eta S(t)f - \nu_\rho f|^p \leq c(f) (1 + t)^{-d/2}.$$  

**Remarks.** Part (a) in the theorem above improves the bound obtained by Cancrini and Galves (1995) in two directions: on the one hand, we only ask for polynomial decay of correlations in the initial measure, while [CG] requires exponential decay. On the other hand, in [CG] the upper bound includes a (spurious) log $t$ in the numerator. Part (b) generalizes a result obtained by Bertini and Zegarlinski (1996) in two senses: [BZ] proved (b) for $p = 2$ with product initial measure, while part (b) requires only the initial measure to have polynomial decay of correlations, which is of course satisfied by $\nu_\rho$, and part (b) is proved for $1 \leq p \leq 2$. Finally, since $\nu_\rho$ is an equilibrium measure, part (b) can be read as the time-decay of correlations of the system in equilibrium.

The proof of Theorem 2.1 is based on the self-duality of the symmetric simple exclusion process that we now explain. For $0 \leq s \leq t$, let $X^{x,t}_s$ denote the position at time $t - s$ of the particle sitting at $x$ at time $t$: $X^{x,t}_s = \xi^{t-s}_s(x)$. In contrast with $\xi^t(x)$, $X^{x,t}$ is a
random walk with transition rate \( p(\cdot, \cdot) \). Moreover, for any subset \( A \) of \( \mathbb{Z}^d \), \( \{X^x_t, x \in A\} \) evolves as symmetric exclusion random walks and

\[
\mathbb{P}[\eta^\eta_t(x) = 1, x \in A] = \mathbb{P}[\eta(X^x_t) = 1, x \in A]
\]

for every finite subset \( A \). This is the so-called self-duality relation of the symmetric simple exclusion process.

In the following theorem we compare the evolution of \( n \) particles interacting by exclusion with \( n \) independent particles. The proof is inspired in a similar result by Ferrari and Goldstein (1988), where exclusion processes with birth and deaths were considered, and a result by Ferrari, Presutti, Scacciatelli and Vares (1991). This can be seen as a probabilistic version of the “integration by parts formula”. See Proposition 8.1.7 of Liggett (1985) and display (1) of Andjel (1994), for instance.

**Theorem 2.2.** For any vector \( \underline{x} = (x_1, \ldots, x_n) \),

\[
\mathbb{E} \prod_{i=1}^{n} \eta^\eta_t(x_i) - \prod_{i=1}^{n} \mathbb{E} \eta^\eta_t(x_i) = - \int_0^t \sum_y \mathbb{P}(X^{x,t} = y) \sum_{i<j} p(y_i, y_j) (\rho^\eta_{t-s}(y_i) - \rho^\eta_{t-s}(y_j))^2 \prod_{k \neq i,j} \rho^\eta_{t-s}(y_k)
\]

for all \( t \) and any configuration \( \eta \). In the above formula the summation is carried over all \( \underline{y} = (y_1, \ldots, y_n) \) such that \( y_i \neq y_j \) for \( i \neq j \) and

\[
\rho^\eta_t(y) = \mathbb{E} \eta^\eta_t(y).
\]

In the next theorem we apply the previous result.

**Theorem 2.3.** Assume \( p(\cdot, \cdot) \) has a finite second moment.

(a) There exists a finite constant \( C \) such that for all \( t \geq 0 \),

\[
\sup_{\eta} \left| \mathbb{E} \prod_{i=1}^{n} \eta^\eta_t(x_i) - \prod_{i=1}^{n} \mathbb{E} \eta^\eta_t(x_i) \right| \leq C R_d(t),
\]

where \( R_1(t) = \log(1 + t)/\sqrt{1 + t} \), \( R_2(t) = \log(1 + t)/(1 + t) \) and \( R_d(t) = (1 + t)^{-1} \) for \( d \geq 3 \).

(b) For each positive function \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) let

\[
X^\varphi := \left\{ \eta : \text{ for all } x, y \in \mathbb{Z}^d, |x - y|^{-2}(\rho^\eta_t(x) - \rho^\eta_t(y))^2 \leq \varphi(t) \right\}.
\]
Then, for each decreasing function \( \varphi \) such that \( \varphi(t) \leq C_0(1 + t)^{-\frac{d+2}{2}} \) for some finite constant \( C_0 \), there exists a finite constant \( C_1 \) such that for all \( t \geq 0 \)

\[
\sup_{\eta \in X_{\varphi}} \left| \mathbb{E} \prod_{i=1}^{n} \eta_i^n(x_i) - \mathbb{E} \prod_{i=1}^{n} \eta_i^n(x_i) \right| \leq C_1 \frac{1}{(1 + t)^{d/2}} ;
\]

(c) Let \( \nu \) satisfy the conditions of Theorem 2.1. Then there exists a finite constant \( C \) such that for all \( t \geq 0 \)

\[
\nu \left| \mathbb{E} \prod_{i=1}^{n} \eta_i^n(x_i) - \mathbb{E} \prod_{i=1}^{n} \eta_i^n(x_i) \right| \leq C \frac{1}{(1 + t)^{d/2}} .
\]

Remarks. Part (a) was obtained by Ferrari, Presutti, Scacciatelli and Vares (1988). Let \( \eta \) be a periodic configuration on \( \mathbb{Z}^d \). It is easy to show that \( \eta \) belongs to \( X_{\varphi} \) for some \( \varphi \) satisfying the assumptions of Theorem 2.3 (b). Part (b) includes a result in Landim (1999), where the case of finite initial \( \eta \) is considered.

3. Proofs.

Before proving the theorems, we state an estimate that will be needed several times in the sequel. It is based on the classical negative correlations property of the symmetric exclusion process.

**Lemma 3.1.** Let \( \Phi : \mathbb{Z}^d \setminus \{0\} \to \mathbb{R}^+ \) be a summable function. Then, there exists a constant \( C \) such that for all set of different sites \( \{x_1, \ldots, x_n\} \) and for all \( t > 0 \),

\[
\sum_{j \neq k} \sum_{y_1, \ldots, y_n} \mathbb{P}[\{X_{t}^{x_1,t}, \ldots, X_{t}^{x_n,t}\} = \{y_1, \ldots, y_n\}] \Phi(y_j - y_k) \leq C (1 + t)^{-d/2} , \quad (3.1)
\]

where the second sum is carried over the set \( \{y_1, \ldots, y_n : y_a \neq y_b, \text{ for } a \neq b\} \).

**Proof.** If we fix \( j \neq k \) and sum over \( y_i \) for \( i \neq j, k \), we obtain that the left hand side of (3.1) is equal to

\[
\sum_{j \neq k} \sum_{y_j \neq y_k} \mathbb{P}[\{X_{t}^{x_1,t}, \ldots, X_{t}^{x_n,t}\} \supset \{y_j, y_k\}] \Phi(y_j - y_k) . \quad (3.2)
\]

The previous probability can be decomposed as

\[
\sum_{a \neq b} \mathbb{P}[\{X_{t}^{x_a,t}, X_{t}^{x_b,t}\} = \{y_j, y_k\}] .
\]
By the correlation inequality between symmetric exclusion random walks and symmetric independent random walks (cf. [L], Theorem VIII.1.7), each of theses probabilities is bounded above by

$$
\mathbb{P}[X^x_{t\cdot} \in \{y_j, y_k\}] \mathbb{P}[X^y_{t\cdot} \in \{y_j, y_k\}] = \{p_t(y_j - x_a) + p_t(y_k - x_a)\} \{p_t(y_j - x_b) + p_t(y_k - x_b)\}
$$

provided $p_t(x, y)$ stands for the probability of a continuous-time random walk with transition probability $p(\cdot, \cdot)$ to be at $y$ at time $t$ if it starts from $x$ at time 0 and provided $p_t(x)$ stands for $p_t(0, x)$. The sum (3.2) is thus bounded above by

$$
\sum_{a \neq b} \sum_{j \neq k} \sum_{y_j \neq y_k} \Phi(y_j - y_k) \{p_t(y_j - x_a) + p_t(y_k - x_a)\} \{p_t(y_j - x_b) + p_t(y_k - x_b)\}.
$$

Since $\Phi$ is summable and since $p_t(x)$ is bounded above by $C/(1 + t)^{d/2}$ for some finite constant that depends only on $p(\cdot, \cdot)$, this expression is bounded above by $C(n, p)(1 + t)^{-d/2}$, which concludes the proof of the lemma. \(\square\)

**Proof of Theorem 2.1 (a)** Let $\nu$ be a translation-invariant probability measure that has $\alpha$-decaying correlations. By duality, for any $n \geq 1$ and any distinct sites $x_1, \ldots, x_n$,

$$
\left| \mathbb{P}_\nu[\eta_t(x_i) = 1, i = 1, \ldots, n] - \rho^n \right|
= \left| \sum_{y_1, \ldots, y_n \atop y_i \neq y_k} \mathbb{P}[\{X^x_{t\cdot} \in \{y_1, \ldots, y_n\}\}] \{\nu(\eta(y_i) = 1, i = 1, \ldots, n) - \rho^n\} \right|.
$$

Since $\nu$ is translation-invariant with mean $\rho$ and has $\alpha$-decaying correlations, there exists a finite constant $C$ such that

$$
\left| \nu(\eta(y_i) = 1, i = 1, \ldots, n) - \rho^n \right| \leq C \sum_{j \neq k} \frac{1}{|y_j - y_k|^\alpha}.
$$

The right hand side of the previous expression is thus bounded above by

$$
C \sum_{j \neq k} \sum_{y_1, \ldots, y_n \atop y_i \neq y_k} \mathbb{P}[\{X^x_{t\cdot} \in \{y_1, \ldots, y_n\}\}] \frac{1}{|y_j - y_k|^\alpha}.
$$

To conclude the proof it remains to apply Lemma 3.1 for the function $\Phi(y) = |y|^{-\alpha}$. We can do this because in the above expression $y_j \neq y_k$. \(\square\)

**Proof of Theorem 2.2.** Fix $n$ distinct sites $x_1, \ldots, x_n$ and define $\{Y^{x_i,s}_{i,t} : i = 1, \ldots, n\}$ as a family of independent random processes with the same marginal distribution as $X^{x_i,s}_{t}$ respectively.
We realize the motion of the $Y$ process by considering a family of Poisson marks as the one defined by $N_{x,y}(t)$, called $N^0_{x,y}(t)$, independent of the precedent one as follows. Processes $X$ uses only the $N$ marks. Process $Y$ uses the $N$ marks if only one $Y$ particle is concerned by the jump. This means that if there is a $Y$ particle at $x$ and no $Y$ particle at $y$ at time $t$, then the process uses the marks of $N_{x,y}(t)$ and ignores the marks of $N^0_{x,y}(t)$. If the jump concerns two $Y$ particles, say particles $i$ and $j$ with $i < j$, then the $Y_i$ particle uses the $N$ marks and the $Y_j$ particle uses the $N^0$ marks to jump over the position of the $Y_i$ particle and the $N$ marks to jump to any other position. Thus if particle $Y_i$ is at $x$ and particle $Y_j$ at time $t$ with $i < j$, particle $Y_i$ uses the marks of $N_{x,y}(t)$ while particle $Y_j$ uses the marks of $N^0_{x,y}(t)$.

We just gave a coupling between a system of $n$ exclusion and $n$ independent particles. This means that we realized the two processes in the same probability space (the one generated by the product of the Poisson processes $(N_{x,y}(t), N^0_{x,y}(t); x, y \in \mathbb{Z}^d, t \geq 0)$) in such a way that the marginal distributions are those desired for both processes. We continue using $\mathbb{P}$ and $\mathbb{E}$ for the probability and expectation with respect to the product of the Poisson processes.

By definition of the symmetric simple exclusion process,

$$\mathbb{E} \left[ \prod_{i=1}^{n} \eta^*_t(x_i) \right] - \prod_{i=1}^{n} \mathbb{E} \left[ \eta^*_t(x_i) \right] = \mathbb{E} \left[ \prod_{i=1}^{n} \eta(X^{x_i,t}_t) \right] - \prod_{i=1}^{n} \eta(Y^{x_i,t}_t) .$$

Let $T_1$ be the first instant that two $Y$ particles occupy the same site. We say that a collision occurred at that time. Before the collision each $X$ particle occupies the same place of the corresponding $Y$ particle. For $s < t$ let

$$I(x_1, \ldots, x_n, s, t) = \prod_{i=1}^{n} \eta(X^{x_i,s}_s) - \prod_{i=1}^{n} \eta(Y^{x_i,s}_s) .$$

We want to compute the expectation of $I(x_1, \ldots, x_n, t) = I(x_1, \ldots, x_n, t, t)$. If $t < T_1$, $I(x_1, \ldots, x_n, t)$ is zero because the trajectories of the $X$ and $Y$ process coincide. On the other hand, since for all $0 \leq s < t$, $Y^{x_i,t}_t = Y^{x_i,s}_s, X^{x_i,t}_t = X^{x_i,s}_s, t - s$, on the set \{ $T_1 \leq t$ \},

$$I(x_1, \ldots, x_n, t) = \prod_{i=1}^{n} \eta(X^{x_i,s}_t, t - T_1) - \prod_{i=1}^{n} \eta(Y^{x_i,s}_t, t - T_1) . \quad (3.3)$$

Let

$$E_1 = \begin{cases} 1 & \text{if the collision at $T_1$ occurs due to a $N$ mark} \\ 0 & \text{if the collision at $T_1$ occurs due to a $N^0$ mark} \end{cases}$$

and let $Z_1 \in \{(i,j), 1 \leq i < j \leq n\}$ stand for the labels of the particles involved in the collision at time $T_1$:

$$Z_1 = (i, j) \quad \text{if and only if} \quad Y^{x_i,t}_{i,T_1} = Y^{x_j,t}_{j,T_1} \quad \text{and} \quad i < j .$$
Assume that the first collision is due to a collision between particles $i$ and $j$ with $i < j$. In this case, at time $T_1$, the position of particles $Y_{x_{x_k,t}}$ and $X_{x_{x_k,t}}$ coincide for $k \neq i, j$. Moreover, if the collision occurred due to a $N$ mark, i.e., due to a jump of particle $Y_i$ over particle $Y_j$, $Y_{x_{x_j,t}} = X_{x_{x_j,t}} = Y_{x_{x_j,t}}$. In the case where the collision occurred due to a $N^0$ mark, i.e., due to a jump of the $Y_j$ particle over $Y_i$, a similar identity holds with the roles of $i$ and $j$ exchanged. In particular, on the set $\{T_1 \leq t\}$,

$$
\prod_{k=1}^n \eta(Y_{x_{x_{x_{x_{k}},t-T_1}}}) = \sum_{i<j} 1\{Z_1 = (i,j)\} \prod_{k \neq \{i,j\}} \eta(Y_{x_{x_{x_{x_{k}},t-T_1}}}) \times \\
\times \left\{1\{E_1 = 0\} \eta(Y_{x_{x_{x_{x_{j}},t-T_1}}}) \eta(Y_{x_{x_{x_{x_{j}},t-T_1}}}) + 1\{E_1 = 1\} \eta(Y_{x_{x_{x_{x_{j}},t-T_1}}}) \eta(Y_{x_{x_{x_{x_{j}},t-T_1}}})\right\}.
$$

We now add and subtract in the right hand side $\eta(Y_{x_{x_{x_{x_{j}},t-T_1}}}) \eta(Y_{x_{x_{x_{x_{j}},t-T_1}}})$ to recover $\prod_{1 \leq k \leq n} \eta(Y_{x_{x_{x_{x_{k}},t-T_1}}})$. After this step we obtain that on the set $\{T_1 \leq t\}$,

$$
\prod_{k=1}^n \eta(Y_{x_{x_{x_{x_{k}},t-T_1}}}) = \prod_{k=1}^n \eta(Y_{x_{x_{x_{x_{k}},t-T_1}}}) + \sum_{i<j} 1\{Z_1 = (i,j)\} \prod_{k \neq \{i,j\}} \eta(Y_{x_{x_{x_{x_{k}},t-T_1}}}) \times \\
\times \left\{1\{E_1 = 0\} \eta(Y_{x_{x_{x_{x_{j}},t-T_1}}}) \eta(Y_{x_{x_{x_{x_{j}},t-T_1}}}) - \eta(Y_{x_{x_{x_{x_{j}},t-T_1}}}) \eta(Y_{x_{x_{x_{x_{j}},t-T_1}}})\right\} + 1\{E_1 = 1\} \eta(Y_{x_{x_{x_{x_{j}},t-T_1}}}) \eta(Y_{x_{x_{x_{x_{j}},t-T_1}}})\right\}.
$$

Denote the right hand side of this expression by $g(E_1, T_1, Z_1, X_{x_{x_{x_{x_{1}},t}}} = Y_{x_{x_{x_{x_{1}},t}}})$. It follows from this identity and (3.3) that

$$
I(x, t) = 1\{T_1 \leq t\}I(X_{x_{x_{x_{x_{1}},t}}} = Y_{x_{x_{x_{x_{1}},t}}}) - 1\{T_1 \leq t\}g(E_1, T_1, Z_1, X_{x_{x_{x_{x_{1}}}}} = Y_{x_{x_{x_{x_{1}},t}}})
$$

For a positive integer $\ell$, define $G_\ell = \sigma\{T_m, Z_m, X_{x_{x_{x_{x_{m}},t}}} : m = 1, \ldots, \ell\}$. Since $E_1$ is a Bernoulli random variable with parameter $1/2$ independent of $G_1$ and since the $Y$ particles evolve independently, by the strong Markov property on the set $1\{T_1 \leq t\}$ we get that

$$
\mathbb{E}(g(E_1, T_1, Z_1, X_{x_{x_{x_{x_{1}}}}}, Y_{x_{x_{x_{x_{1}},t}}})|G_1) = g(T_1, Z_1, X_{x_{x_{x_{x_{1}}}}}, \ldots, X_{x_{x_{x_{x_{1}},t}}})
$$
where
\[ g(u, \{i, j\}, x_1, \ldots, x_n) = \frac{1}{2} \left[ \mathbb{E} (\eta(Y_{x_i, t}^t)) - \mathbb{E} (\eta(Y_{x_j, t}^t)) \right]^2 \prod_{k \neq i, j} \mathbb{E} (\eta(Y_{k, t-u}^t)) \].

From the two previous identities, we obtain that
\[
\mathbb{E} (I(x_1, \ldots, x_n, t)|G_1) = 1\{T_1 \leq t\} I(X_{T_1}^{x_1, t}, \ldots, X_{T_n}^{x_n, t}, t-T_1) - 1\{T_1 \leq t\} g(T_1, Z_1, X_{T_1}^{x_1, t}, \ldots, X_{T_1}^{x_n, t}) .
\]

Repeating the argument for \( \ell \geq 2 \), we get the following expression for the expectation of \( I \):
\[
\mathbb{E} (I(x_1, \ldots, x_n, t)) = -\mathbb{E} \sum_{\ell=1}^{M(t)} g(T_1, Z_1, X_{T_1}^{x_1, t}, \ldots, X_{T_1}^{x_n, t}) ,
\]
where \( T_1, E_1, Z_1 \) inductively and \( M(t) = \sum_{\ell \geq 1} 1\{T_\ell \leq t\} \) is the number of collisions occurred by time \( t \). Notice that since \( g \) is positive, the above expression implies immediately that the distribution of \( \eta_t \) is a measure with negative correlations for any initial \( \eta \).

For \( i < j \), denote by \( T_{\ell}^{i,j} \) the instant of the \( \ell \)-th collision of particles \( i, j \) and by \( M^{i,j}(t) \) the number of collisions up to time \( t \) of particles \( i, j \) so that \( M(t) = \sum_{i < j} M^{i,j}(t) \) and
\[
\mathbb{E} (I(x_1, \ldots, x_n, t)) = -\sum_{i < j} \mathbb{E} \sum_{\ell=1}^{M^{i,j}(t)} g^{i,j}(T_\ell^{i,j}, X_{T_\ell^{i,j}}^{x_1, t}, \ldots, X_{T_\ell^{i,j}}^{x_n, t}) , \tag{3.4}
\]
where
\[
g^{i,j}(u, y) = \frac{1}{2} \left[ \mathbb{E} (\eta(Y_{x_i, t}^t)) - \mathbb{E} (\eta(Y_{x_j, t}^t)) \right]^2 \prod_{k \neq i, j} \mathbb{E} (\eta(Y_{k, t-u}^t)) . \tag{3.5}
\]

For \( i < j \), \( M^{i,j}(t) \) is a Poisson process with rate \( 2 \sum_y 1\{X_t = y\} p(y_i, y_j) \). Moreover, for any function \( h(u, x) \),
\[
\sum_{\ell=1}^{M^{i,j}(t)} h(T_\ell^{i,j}, X_{T_\ell^{i,j}}^{x_1, t}) = \int_0^t h(s, X_s^{x_1, t}) dM^{i,j}(s) .
\]

In particular, taking expectations,
\[
\mathbb{E} \sum_{\ell=1}^{M^{i,j}(t)} g^{i,j}(T_\ell^{i,j}, X_{T_\ell^{i,j}}^{x_1, t}) = 2 \int_0^t \sum_y p(y_i, y_j) \mathbb{P}(X_s^x = y) g^{i,j}(s, y) ds .
\]

This together with (3.4), (3.5) concludes the proof of the theorem. \( \square \)
Proof of Theorem 2.3. To show item (a) we use Theorem 2.2. Since we are taking supremum over \( \eta \), we can cancel the product \( \prod_{k \neq i,j} \rho^n_{t-s}(y_k) \). We get the following upper bound

\[
\sum_{i<j} \int_0^t \sum_{y_1, \ldots, y_n} p(y_i, y_j) \mathbb{P}(\{X_s^{x_1,t}, \ldots, X_s^{x_n,t}\} = \{y_1, \ldots, y_n\}) \times \rho^n_{t-s}(y_i) - \rho^n_{t-s}(y_j) \, ds
\]

for all \( s \geq 0 \). By Lemma 3.1 with \( \Phi(\cdot) = p(0, y)|y|^2 \), which is summable by hypothesis, (3.6) is bounded above by

\[
C \int_0^t (1 + s)^{-d/2} \frac{1}{1 + t - s} \, ds \leq C R_d(t).
\]

Here and below \( C \) stands for a finite constant that may change from line to line. This shows item (a). To show (b), again we can cancel the product \( \prod_{k \neq i,j} \rho^n_{t-s}(y_k) \) and need to bound (3.6) for \( \eta \) in \( X_\varphi \). For those \( \eta \)'s we have

\[
(\rho^n_{t-s}(y_i) - \rho^n_{t-s}(y_j))^2 \leq \varphi(t-s)|y_i - y_j|^2.
\]

Applying Lemma 3.1 with \( \Phi(\cdot) = p(0, y)|y|^2 \), we bound (3.6) by

\[
C \int_0^t (1 + s)^{-d/2} \varphi(t-s) \, ds \leq C(1 + t)^{-d/2}
\]

because \( \varphi(t) \) was assumed to be bounded above by \( C(1 + t)^{-(d+2)/2} \). This proves (b).

To show (c), as before, we need to compute the expectation with respect to \( \nu \) of (3.6). Hence using Fubini we need to compute \( \nu(\rho^n_{t-s}(y_i) - \rho^n_{t-s}(y_j))^2 \). Since \( \rho^n_{t}(y) = \sum_{z} \eta(z)p_t(y, z) \), developing the square we get

\[
\nu(\rho^n_{t-s}(y_i) - \rho^n_{t-s}(y_j))^2
\]

\[
= \sum_{z_1, z_2} \nu(\eta(z_1)\eta(z_2)) (p_t(y_i, z_1) - p_t(y_j, z_1))(p_t(y_i, z_2) - p_t(y_j, z_2))
\]

\[
= \rho \sum_{z} (p_t(y_i, z) - p_t(y_j, z))^2
\]

\[
+ \sum_{z_1 \neq z_2} (\nu \eta(z_1)\eta(z_2) - \rho^2) (p_t(y_i, z_1) - p_t(y_j, z_1))(p_t(y_i, z_2) - p_t(y_j, z_2))
\]

\[
+ \sum_{z_1 \neq z_2} \rho^2 (p_t(y_i, z_1) - p_t(y_j, z_1))(p_t(y_i, z_2) - p_t(y_j, z_2)).
\]
It follows from Lemma 3.2 below that the sum of the first and third line of the previous expression is bounded above by $C \rho (1 - \rho) |y_i - y_j|^2 (1 + t)^-(d+1)/2$, while the second, by Schwarz inequality, is bounded above by
\[
\sum_{z_1 \neq z_2} |\nu(z_1)^{1/2} - \rho| \{p_t(y_i, z_1) - p_t(y_j, z_1)\}^2.
\]
Since $\nu$ has $\alpha$-decaying correlations, by Lemma 3.2, this expression is bounded above by $C |y_i - y_j|^2 (1 + t)^-(d+1)/2$. Therefore, (3.6) is less than or equal to
\[
C(\rho) \sum_{i < j} \int_0^t \sum_{y_1, \ldots, y_n} p(y_i, y_j) |y_i - y_j|^2 \mathbb{P}(X^{x_t} = y) \frac{1}{(1 + t - s)(d+1)/2} ds.
\]
It remains to apply Lemma 3.1 to the function $\Phi(y) = p(0, y) |y|^2$ to obtain that the previous expression is bounded above by
\[
C(\rho) \int_0^t \frac{1}{(1 + s)^{d/2}} \frac{1}{(1 + t - s)(d+1)/2} ds.
\]
This concludes the proof of the theorem in dimension $d \geq 2$. In dimension 1, we need (3.7) instead of Lemma 3.2 to prove the estimate. □

We turn now to Theorem 2.1.

**Proof of Theorem 2.1.** We already proved part (a). In order to prove (b), fix $1 \leq p \leq 2$. It is enough to consider cylinder functions of type $f_A = \prod_{x \in A} \eta(x)$ for a finite subset $A$ of $\mathbb{Z}^d$. For such a cylinder function, $\delta_\eta S(t)f_A = \mathbb{E} \prod_{x \in A} \eta_t^\rho(x)$ and $\nu|f_A| = \rho|A|$, provided $|A|$ stands for the number of sites of $A$. Since $0 \leq f_A \leq 1$ and since $a^p \leq a$ for $0 \leq a \leq 1$, by Theorem 2.3 (c),
\[
\int \nu(d\eta) \left| \delta_\eta S(t)f_A - \nu_f |A| \right|^p \leq 2^{p-1} \int \nu(d\eta) \left| \prod_{x \in A} \mathbb{E} \eta_t^\rho(x) - \rho|A| \right|^p + C(1 + t)^{-d/2}
\]
for some finite constant $C$. Introducing intermediary terms, the first term on the right hand side is bounded above by
\[
C(A) \sum_{x \in A} \int \nu(d\eta) \left| \mathbb{E} \eta_t^\rho(x) - \rho \right|^p.
\]
Since $p \leq 2$, By Hölder inequality, the previous expression is bounded above by
\[
C(A) \sum_{x \in A} \left\{ \int \nu(d\eta) \left( \mathbb{E} \eta_t^\rho(x) - \rho \right)^2 \right\}^{p/2}.
\]
To show that this expression is bounded above by $C(A)(1 + t)^{-dp/4}$ it is enough to expand the square and to recall that $\nu$ has $\alpha$-decaying correlations and that $p_t(x)$ is bounded above by $Ct^{-d/2}$ uniformly in $x$. The proof of (c) is exactly the same. This concludes the proof of the theorem. □

We conclude this section with an estimate on the transition probability of symmetric random walks.
Lemma 3.2. Let \( \{X_t, t \geq 0\} \) be a random walk on \( \mathbb{Z}^d \) with transition probability \( p(\cdot, \cdot) \) satisfying the assumptions stated in the beginning of the article. Then, there exists a finite constant \( C \) such that

\[
\sum_{x \in \mathbb{Z}^d} |p_t(0, x + e_i) - p_t(0, x)| \leq C (1 + t)^{-1/2} \quad \text{and} \\
\sum_{x \in \mathbb{Z}^d} |p_t(0, x + e_i) - p_t(0, x)|^2 \leq C (1 + t)^{-(1+d)/2}.
\]

Here \( \{e_i, 1 \leq i \leq d\} \) stands for the canonical basis of \( \mathbb{R}^d \).

Proof. This result can be proved by a coupling argument indicated to us by E. Andjel or through the local central limit theorem. With slightly stronger assumptions on the moments of \( p(\cdot) \), the local central limit theorem gives better estimates (of type \( t^{-(d+2)/2} \)) for the second term. In dimension 1, with the assumptions of indecomposability and finite second moments, we have

\[
\sum_{x \in \mathbb{Z}} |p_t(0, x + 1) - p_t(0, x)|^2 \leq C (1 + t)^{-(1+\delta)} \tag{3.7}
\]

for some \( \delta > 0 \).

The coupling argument is as follows. Consider two random walks \( X^1_t, X^2_t \) with transition probability \( p(\cdot) \) on \( \mathbb{Z}^d \) and with initial states 0 and \( e_1 \). Assume that \( p(e_1) > 0 \). We couple these two random walks in the following way. If the process \( (X^1_t, X^2_t) \) is at \( (x^1, x^2) \), for \( y \neq e_1 \) it jumps to \( (x^1 + y, x^2 + y) \) at rate \( p(y) \), it jumps to \( (x^1 + e_1, x^2) \) at rate \( p(e_1) \) and it jumps to \( (x^1, x^2 + e_1) \) at rate \( p(e_1) \). We proceed in this way until they meet. From this time on, they jump together. With this coupling the difference \( X^1_t - X^2_t \) is a nearest-neighbor, symmetric, one-dimensional random walk with absorption at the origin that starts from \(-1\). A well known bound gives that the probability for this one-dimensional random walk to have not reached the origin before time \( t \) decays as \( t^{-1/2} \). This estimate permits to prove the lemma in the case \( p(e_1) > 0 \). In the other case, since the transition probability is assumed to be indecomposable, there exists an integer \( n \) and a sequence \( 0 = x_0, x_1, \ldots, x_n = e_1 \) such that \( p(x_i, x_{i+1}) > 0 \) and we may proceed in a similar way. Notice that this proof does not require any assumption on the moments of \( p \). \( \Box \)

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