The Parameterized Complexity Binary CSP for Graphs with a Small Vertex Cover and Related Results

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Abstract

In this paper, we show that Binary CSP with the size of a vertex cover as parameter is complete for the class W[3]. We obtain a number of related results with variations of the proof techniques, that include: Binary CSP is complete for W[2d+1] with as parameter the size of a vertex modulator to graphs of treedepth c, or forests of depth d, for constant c ≥ 1, W[t]-hard for all t ∈ N with treewidth as parameter, and hard for W[SAT] with feedback vertex set as parameter. As corollaries, we give some hardness and membership problems for classes in the W-hierarchy for List Colouring under different parameterisations.

1 Introduction

In this paper, we consider Binary CSP: we have a number of discrete variables, and for some pairs of variables a constraint, that tells which pairs of values are allowed for this pair of variables. The problem can be interpreted as a generalised form of graph colouring: each vertex chooses a colour from a list, and for each edge, we have a list of (ordered) pairs of colours allowed for the endpoints of the edge. We assume here that each variable can take a large number of values.

We look at the problem in a parameterized setting, and obtain a number of complexity theoretic results, classifying the problems complexity with respect to different parameterisations. Our first result is an unusual, and at first hand perhaps surprising completeness result:

Theorem 1. Binary CSP with the size of a minimum vertex cover as parameter is complete for the class W[3].

The W-hierarchy was introduced approximately thirty years ago in the work by Downey and Fellows that founded the field of parameterized algorithms and complexity. In this hierarchy, we have a collection of classes, including W[1], W[2], . . . , W[SAT], W[P]. Many problems are known to be complete for W[1] or for W[2]. For W[t] with a fixed integer t ≥ 3, so far, there are only very few examples of natural problems known to be complete for this class [2, 4, 5, 11, 12, 29].

The proof technique to obtain Theorem 1 can be generalised, e.g., to the following theorem.

Theorem 2. For each integer d ≥ 1, Binary CSP is complete for W[2d+1] with as parameter the size of a vertex modulator to a graph of treedepth at most d, and with as parameter the to a forest of depth at most d.

We also give a number of partial results, where we do not have matching hardness and membership proofs for classes in the W-hierarchy. An overview is given below.

We also have a number of corollaries for related problems, which can be seen as special cases of Binary CSP: List Colouring and Precolouring Extension. We discuss these below.

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The parameter ecology and vertex cover size  In the field of parameterized algorithms and complexity, a decision problem can be viewed as a parameterized problem in several different ways. Often, the considered parameter is a target value (like the size of a clique in the Clique problem). In many other cases, the parameter is a structural value of the input, e.g., the treewidth of a graph given as instance of a graph problem. Different structural parameters can give different complexity, and a systematic study reveals a rich structure, see [19]. A well studied graph parameter is this setting is the size of a minimum vertex cover. Many problems appear to be fixed parameter tractable or even have a kernel of polynomial size, when the minimum vertex cover size is used as parameter (e.g., [20, 22, 23] and [24]); in some cases, problems stay hard (for classes in the W-hierarchy, or even NP-hard) with small vertex cover size. The Binary CSP problem is an example of a problem that is known to be W[1]-hard with vertex cover as parameter [32, 22]; as discussed above, in this paper, we establish the precise complexity to be W[3]-completeness.

Related work: Binary CSP, List Colouring and Precolouring Extension  The Binary Constraint Satisfaction Problem is well studied, with studies ranging from practical computational experiments to complexity theoretic investigations. Several studies look at the complexity of Binary CSP under different structural assumptions on the graph formed by the binary constraints (called the Gaifman, or primal graph). We review here some of these results. In 1990, Freuder [24] showed that Binary CSP can be solved in \(O(n f(k))\) time for graphs of treewidth \(k\). A seminal result by Marx [31] showed that we cannot expect to replace treewidth here by another graph parameter that does not imply bounded treewidth. (For the precise statement, see [31].) It was long known that Binary CSP with the treewidth of the primal graph as parameter is W[1]-hard (see [27]); with additional parameterisations, the problem becomes fixed parameter tractable [27, 33]. Papadimitriou and Yannakakis [32] showed that Binary CSP is W[1]-complete when the number of variables (vertices in the graph) is the parameter.

A special case of Binary CSP is the List Colouring problem. Here, each vertex has a list of colours, and we ask if we can choose for each vertex a colour from its list such that for each edge, the colours of the endpoints of the edge are different. Fellows et al. [17] showed that List Colouring is W[1]-hard for graphs of bounded treewidth. This was improved by Fiala et al. [22], who showed that List Colouring with the size of a vertex cover as parameter is W[1]-hard. Recently, it was shown that List Colouring with treewidth as parameter is complete for the class XALP [9]; the class XALP are the parameterized problems that can be solved in \(n f(k)\log n\) space on a non-deterministic Turing Machine with additional access to a stack. The class appears to capture the complexity of several tree-structured problems. In [10], the parameterisation of List Colouring by pathwidth was shown to be complete for the class XNLP. XNLP is defined similarly as XALP, but without the additional stack. This class appears to capture several problems with a linear structure [10, 8, 16].

The Precolouring Extension problem is a further special case. Here, we are given a graph \(G\), a set of colours \(C\), and a colour for a subset of the vertices, and are asked if we can assign the other vertices also a colour from \(C\), such that for each edge, the colours of the endpoints differ.

A vertex modulator to a graph class \(\mathcal{G}\) is a set of vertices, that when removed give a graph in \(\mathcal{G}\). Vertex covers are a special type of vertex modulators, namely to graphs without edges, or independent sets. A feedback vertex set is another type of vertex modulator, now to graphs without cycles, i.e., to forests.

The related problems of List Colouring and Precolouring Extension for graphs with the size of a vertex or edge modulator to a clique was resolved in [3, 26, 28]. (One can observe directly that Binary CSP is NP-complete for cliques.)

A well known W[1]-complete problem is Multicolour Clique, introduced in [18]; we are
given a graph \( G = (V, E) \), and a colouring of the vertices with \( k \) colours, and ask if we can choose one vertex from each colour such that the chosen vertices form a clique (of size \( k \)). \( k \) is the parameter. We can turn an instance of MULTICOLOUR CLIQUE to an instance of BINARY CSP with \( k \) vertices as follows: take for each colour in \( G \) a vertex in \( H \), for each vertex in \( G \) a colour that the corresponding vertex in \( H \) can choose, and for each edge in \( G \) a pair of forbidden vertices in \( H \). This simple transformation with its reverse, give also a proof that BINARY CSP is \( W[1]\)-complete with the number of vertices as parameter. Generalisations of MULTICOLOUR CLIQUE were given in \([10, 9]\). A problem called CHAINED MULTICOLOUR CLIQUE was shown to be \( XNLP\)-complete \([10]\), and a problem called TREE-CHAIN multicolour CLIQUE was shown to be \( XALP\)-complete \([9]\). Using the same transformation as above, we obtain instances of BINARY CSP with parameterisations by respectively pathwidth plus degree, or bandwidth (\( XNLP\)-complete), and by treewidth plus degree, or tree-partition width plus degree (\( XALP\)-complete).

Table 1 gives an (incomplete) summary of several of the known results. One easily observes that LIST COLOURING with the number of vertices \( n \) as parameter has a polynomial kernel (and thus is in FPT): just remove all vertices whose list has size at least \( n \) (as they can always be coloured). If treewidth, pathwidth or tree partition width and degree are bounded, then one can remove all vertices whose list size is larger than their degree, and solve the remaining instance in linear time with dynamic programming. Tree partition width is an old graph parameter; it was introduced under the name strong treewidth by Seese in 1985 \([34]\), and gives a useful tree structure for graph algorithmic studies (e.g., \([6, 35]\).)

| Parameter                     | Binary CSP          | List Colouring     |
|-------------------------------|---------------------|--------------------|
| number of vertices            | \( W[1]\)-complete \([18, 32]\) | poly-kernel        |
| vertex cover                  | \( W[3]\)-complete \* | \( W[1]\)-hard \([22]\), in \( W[3]\) \* |
| feedback vertex set           | \( W[\text{SAT}]\)-hard, in \( W[P]\) \* | \( W[3]\)-hard, in \( W[P]\) \* |
| modulator to treedepth-\( d \) | \( W[2d + 1]\)-complete \* | \( W[2d + 1]\)-hard, in \( W[2d + 3]\) \* |
| modulator to depth \( d \)-forest | \( W[2d + 1]\)-complete \* | \( W[2d + 1]\)-hard, in \( W[2d + 3]\) \* |
| modulator to clique           | para-NP-complete    | FPT, poly-kernel \([3, 28]\) |
| treedepth                     | \( W[t]\)-hard \( \forall t \) \* in XL | \( W[t]\)-hard \( \forall t \in \mathbb{N} \) \* in XL |
| tree partition width          | \( XALP\)-complete \([9]\) | \( W[1]\)-hard, in XL \([7]\) |
| tree partition width + degree  | \( XALP\)-complete \([9]\) | FPT                 |
| pathwidth                     | \( XNLP\)-C \([10]\) | \( XNLP\)-C \([10]\) |
| bandwidth                     | \( XNLP\)-C \([10]\) | FPT                 |
| treewidth                     | \( XALP\)-C \([9]\) | \( XALP\)-C \([9]\) |
| treewidth + degree            | \( XALP\)-C \([9]\) | FPT                 |

Table 1: Complexity of BINARY CSP and LIST COLOURING. Results marked with \( * \) are shown in this paper. Some results without a reference are easy to obtain.

## 2 Preliminaries

In this paper, we denote the depth of a rooted tree as the maximum number of vertices on a path from root to leaf. A rooted forest is a collection of rooted trees. The depth of a rooted forest is the maximum depth of the trees in the forest \([3]\).

A rooted forest embedding of a graph \( G = (V, E) \), is a rooted forest \( F = (V, E_F) \) with the same vertex set as \( G \), such that for each \( vw \in E_F \), \( v \) is an ancestor of \( w \) or \( w \) is an ancestor of \( v \) in \( F \). (Note that the forest can contain edges that are not in \( G \).) The treedepth of a graph \( G = (V, E) \) is the maximum depth of of rooted forest embedding of \( G \).

\(^{1}\)The definitions of depth of a tree used in the literature can differ by one. Here we count the number of vertices, e.g., a tree consisting of a single vertex has depth 1.
Let \( G \) be a class of graphs. A vertex moderator to \( G \) for a graph \( G = (V, E) \) is a set of vertices \( W \subseteq V \), such that the graph, obtained from \( G \) by removing \( W, G' \setminus W = (V \setminus W \mid \{xy \in E \mid x, y \in V \setminus W\}) \), belongs to the class \( G \).

A vertex cover of a graph \( G = (V, E) \) is a set of vertices \( W \subseteq V \), such that every edge in \( E \) has at least one endpoint in \( W \). Note that a set of vertices is a vertex cover, if and only if it is a vertex moderator to the class of graphs without edges, or, equivalently, to the class of graphs with treedepth 1.

A feedback vertex set of a graph \( G \) is a vertex modulator to a forest, or, equivalently, a set of vertices that intersects each cycle in \( G \).

In Binary CSP, we are given a graph \( G = (V, E) \), a finite set of colours \( C \), for each vertex \( v \in V \) a set of colours \( C(v) \subseteq C \), and for each ordered pair of vertices \( v, w \in V \) with \( vw \in E \) an edge, a set of pairs of colours \( C(v, w) \subseteq C(v) \times C(w) \). We assume that this set of colour constraints is symmetric, in the sense that for each pair of adjacent vertices \( v, w \), when \((c, c') \in C(v, w)\), then \((c', c) \in C(w, v)\). The problem is whether there is an colouring \( f : V \to C \), such that for each \( v \in V \), \( f(v) \in C(v) \), and for each edge \( vw \in E \), \((f(v), f(w)) \in C(v, w)\).

In the List Colouring problems, we are given a graph \( G = (V, E) \), a finite set of colours \( C \), for each vertex \( v \in V \) a set (list) of colours \( C(v) \subseteq C \), and ask whether there exists a mapping of vertices to colours \( f : V \to C \), such that each vertex receives a colour from its list \((\forall v \in V : f(v) \in C(v))\), and for each edge \( xy \in E \), \( f(x) \neq f(y)\).

A Boolean formula is said to be \( t \)-normalised when it is the conjunction of disjunctions of conjunctions of \ldots of literals, with \( t \) levels of conjunctions or disjunctions. We only consider the case where \( t \geq 2 \), and assume that we start by conjunctions. Note that 2-normalised Boolean formulas are in Conjunctive Normal Form. One can easily note that we can also allow conjunctions and disjunctions with a single term. For instance, this implies that in our proofs, when modelling a problem as an instance of Weighted 3-Normalised Satisfiability, we can add a clause.

In the Weighted \( t \)-Normalised Satisfiability problem, we are given a \( t \)-normalised Boolean formula \( F \) on \( n \) variables, and an integer \( k \), and ask if we can satisfy \( F \) by setting exactly \( k \) of the variables to true, and all other variables to false.

We assume the reader to be familiar with the W-hierarchy, and parameterised reductions.

For more background, see e.g., [13, 14, 15]. We use the following result as starting point for membership proofs.

**Theorem 3** (Downey and Fellows, see [14, 15]). Let \( t \geq 2 \). Weighted \( t \)-Normalised Satisfiability is complete for \( W[t] \).

A \( t \)-normalised expression is said to be anti-monotone, if each literal is the negation of a variable. In the hardness proofs, we use the following result; using anti-monotone formulas gives simpler proofs.

**Theorem 4** (Downey and Fellows, see [14, 15]). Let \( t \geq 3 \) be odd. Weighted Anti-Monotone \( t \)-Normalised Satisfiability is complete for \( W[t] \).

Also, note that when we have a conjunction of literals as subformula in an instance of Weighted \( t \)-Normalised Satisfiability, then we can assume that at most \( k \) distinct literals in this conjunction are true — if not, then no setting with \( k \) true variables can satisfy this conjunction, thus the formula cannot be satisfied.

### 3 Hardness results

In this section, we give the proofs of the hardness results: \( W[3] \)-hardness for Binary CSP with vertex cover as parameter (Section 3.1), and its generalisation \( W[2d + 1] \)-hardness for Binary CSP with vertex modulator size to treedepth-\( d \) as parameter (Section 3.2).
3.1 Binary CSP on graphs with a small vertex cover

We start by giving the hardness proof for the parameterisation by vertex cover.

**Lemma 5.** Binary CSP with vertex cover as parameter is W[3]-hard.

**Proof.** Take an instance of Weighted 3-Normalised Anti-Monotone Satisfiability, i.e., we have a Boolean Formula $F$ that is the conjunction of disjunctions of conjunctions of negative literals, and ask if we can satisfy it by setting exactly $k$ variables to true. Suppose $x_1, \ldots, x_n$ are the variables used by $F$. Suppose $F$ is the conjunction of $r$ disjunctions of conjunctions of negative literals.

We build a graph $G = (V, E)$ as follows. $V$ consists of a set $W = \{w_1, \ldots, w_k\}$ of size $k$, and a set $I = \{v_1, v_2, \ldots, v_r\}$ of size $r$. $W$ will be the vertex cover of $G$, and $I$ will form an independent set. We add edges from each vertex in $W$ to each other vertex in the graph.

The colour set of a vertex $w \in W$ is $C(w) = \{x_1, \ldots, x_n\}$. Now consider a vertex $v_i \in I$, $i \in [1, r]$. We say that $v_i$ represents the $i$th disjunction of conjunctions of literals in $F$, i.e., each of the disjunctions in the formula is represented by one vertex in the independent set. Suppose that this disjunction has $t_i$ terms (each term is a conjunction of negative literals). We set $C(v_i) = [1, t_i]$. I.e., each colour of $v_i$ is an integer in $[1, t_i]$.

The intuition is as follows. We set a variable $x_i$ to true, if and only if exactly one vertex in $W$ is coloured by $x_i$. As all vertices in $W$ will get a different colour, we set in this way exactly $k$ variables to true. $F$ is the conjunction of $r$ disjunctions; each of these disjunctions is represented by one of the vertices $v_i \in I$. For each $v_i$, the disjunction represented by $v_i$ must be satisfied, so one of its terms must be satisfied. The colour of $v_i$ tells a satisfied term, i.e., if the colour of $v_i$ is $\alpha$, then the $\alpha$th term is satisfied. This is checked by looking at the edges from $v_i$ to the vertices in $W$.

We now give the colour constraints. For $w, w' \in W$, $w' \neq w'$, we have $C(w, w') = \{(x_i, x_j) | i \neq j\}$. I.e., we enforce that all vertices in $W$ have a different colour — this corresponds to setting exactly $k$ variables to true.

Now, consider a vertex $v_i \in I$ and $w \in W$. Suppose the $i$th disjunction of $F$ has $t_i$ terms. Recall that each such term is a conjunction of a number of negative literals, and that the colour of $v_i$ is an integer in $[1, t_i]$. We have that, for $j \in [1, t_i], j' \in [1, n]$: $(x_j, x_{j'}) \in C(v_i, w)$, if and only if for each literal $\neg x_j''$ that appears in the $j$th term of the $i$th disjunction of $F$, $j'' \neq j'$.

We call the thus created graph $G$, with colour lists for vertices and edges $C$.

**Proposition 6.** $F$ can be satisfied by setting exactly $k$ variables to true, if an only if the CSP-instance $G$ with colour lists $C$ has a solution.

**Proof.** Suppose $G$ can be satisfied by making $x_{i_1}, \ldots, x_{i_k}$ true, and all other literals false. Then colour the vertices in $W$ successively $x_{i_1}, \ldots, x_{i_k}$. The colour conditions between vertices in $W$ are thus satisfied.

Now consider a vertex $v_i \in I$. Consider the $i$th term $F_i$ of the (upper level) conjunction of $F$. This term must be satisfied by the truth assignment. Suppose the term is $F_i = F_{i,1} \lor \cdots \lor F_{i,t_i}$. At least one of the $F_{i,j}$'s must be satisfied by the truth assignment, say $F_{i,j'}$. Then colour $v_i$ by $j'$.

We can verify that the colour constraints for edges between $v_i$ and each $w_j$ are fulfilled. By assumption, $F_{i,j'}$ holds. It thus cannot contain a negative literal $\neg x_\alpha$, where $x_\alpha$ is set to true. So $w_j$ cannot have colour $x_\alpha$ when $\neg x_\alpha$ is a literal in $F_{i,j'}$.

Now, suppose $G$ has a colouring that satisfies all constraints. From the colour constraints between vertices in $W$, we see that all vertices in $W$ have a different colour. Set a variable $x_i$ to true, if and only if a vertex in $W$ has colour $x_i$, and otherwise, set it to false. We have thus set exactly $k$ variables to true.
Consider the ith term of the upper level conjunction of $F$. Suppose this term is $F_{i,1} \lor F_{i,2}$. Suppose $v_i$ is coloured $j$. For each negative literal $\neg x_\alpha$ in the conjunction $F_{i,j}$, by the colour constraints, we cannot have a vertex in $W$ coloured with $x_\alpha$, and thus $x_\alpha$ is set to false. Thus, the term $F_{i,j}$ is satisfied by the truth assignment, and thus $F_i$ is satisfied. As this holds for all terms of $F$, $F$ is satisfied.

From Proposition 6, we see that we have a parameterized reduction from Weighted 3-Normalised Satisfiability to Binary CSP with vertex cover as parameter. The result now follows from the W[3]-hardness of Weighted Anti-Monotone 3-Normalised Satisfiability [14, 15].

### 3.2 Binary CSP on graphs with a modulator to trees and forests of bounded depth

We can generalise the proof of Lemma 5 to modulators to trees and forests of bounded depth — notice that a vertex cover is a modulator to a forest with each tree of depth 1.

Interestingly, each increase of the depth of the trees by one corresponds to an increase in the W-hierarchy by two levels: this is because one level of depth in the tree or forest corresponds to a conjunction (looking at all children of a node) with a disjunction (the choice of a colour).

**Lemma 7.** Let $d \geq 2$ be an integer. Binary CSP with the size of a vertex modulator to forests of depth $d$ is W[2d + 1]-hard.

**Proof.** Take an instance of Weighted 2d + 1-normalised Anti-Monotone Satisfiability, i.e., we have formula $F$ that is a conjunction of disjunctions of conjunctions of . . . of conjunctions of negative literals, with $d + 1$ levels of conjunction, and $d$ levels of disjunction. Suppose the variables used in $X$ are $\{x_1, \ldots, x_n\}$.

Represent $F$ as a rooted tree $T$, with the root representing $F$, and each non-leaf representing a disjunction or conjunction, with the terms as children, and the literals as leaves. The depth of this tree is $2d + 2$. We also assume that $T$ is ordered, i.e., for each vertex with children, we have an ordering on the children (which thus allows us to talk about the ith child of a vertex.) For each vertex $v \in V$, write $F_v$ as the subformula of $F$ that corresponds to $v$.

Number the levels of the tree, with the root level 1, the children of the root level 2, etc. Note that the nodes in $T$ on odd levels represent a conjunction, and the nodes in $T$ on even levels represent a disjunction, except for the lowest level (which is $2d$, thus even), where the nodes represent literals. Now, build a forest $T'$, obtained by removing the root of $T$, and contracting all other nodes in $T$ on odd levels to their parent. I.e., only the nodes on odd levels of $T$ remain, and each is a child of their grandparent in $T$. Finally, remove all leaves (all nodes representing a literal). Note that $T$ is a forest with each tree of depth $d$.

The graph $G$ is formed by taking forest $T'$, and adding a set $W$ with $k$ vertices, with an edge from $W$ to each leaf in $T'$. Write $W = \{w_1, \ldots, w_k\}$. Also, we turn $W$ into a clique.

The role and colours of $W$ are as in the proof of Lemma 5. For each $w \in W$, $C(w) = \{x_1, \ldots, x_n\}$, and for $w, w' \in W$, $w \neq w'$, set $C(w, w') = \{x_i, x_{i'} | i \neq i'\}$.

We distinguish three types of vertices in $T'$. $R$ is the set of vertices in $T'$ that are the root of a tree in the forest. $L$ is the set of vertices in $T'$ that are a leaf. The set of other vertices in $T'$ (i.e., those that are not a root or leaf) is called $X$. Note that $F = \bigwedge_{v \in R} F_v$.

Each vertex in $T'$ represents a disjunction. For a vertex $v$, let $t_v$ be the number of terms of the disjunction, represented by $v$. (This equals the number of children of $v$ in $T$, but in general, $v$ will have much more children in $T'$.) The colour sets of vertices in $T'$ are:

- If $v \in R$, then $C(v) = [1, t_v]$. I.e., each colour is an integer in $[1, t_v]$; this integer points to a term of $F_v$. $F_v$ is a disjunction, and the colour of $v$ tells a term of this disjunction that is satisfied.
If \( v \in X \cup L \), then \( C(v) = [1, t_v] \cup \{ \square \} \). The colour either again is an integer in \([1, t_v]\) that points to a satisfied term of the disjunction \( F_v \), or is the value \( \square \), which stands for inactive. The inactive colour means that the \( F_v \) is not necessarily satisfied (it can be satisfied or not satisfied.) A vertex with colour in \([1, t_v]\) is said to be active.

Roots of the subtrees cannot have the colour \( \square \), so are always active.

For edges between vertices in \( L \) and \( W \), the colour constraints are similar as in the proof of Lemma 4 except that we also allow all pairs where \( v \) is inactive. Let \( v \in L \) and \( w \in W \). Note that \( F_v \) is a disjunction of conjunctions of negative literals. Say \( F_v = \bigvee_{i \in [1, t_v]} F_{v,i} \), with \( F_{v,i} \) a conjunction of negative literals. For \( i \in [1, t_v] \) and \( j \in [1, n] \), we have \( (i, x_j) \in C(v, w) \), if and only if \( \neg x_j \) is not part of the conjunction \( F_{v,i} \). In addition, we have \( (\square, x_j) \in C(v, w) \) for all \( j \in [1, n] \).

Now, we consider an edge \( vv' \in E \), with \( v, v' \in R \cup X \cup L \). Suppose \( v \) is the parent of \( v' \). \( C(v, v') \) consists of all pairs fulfilling one of the following conditions.

- If \( v \not\in R \), then \((\square, \square)\) is in \( C(v, v') \).
- For \( i \in [1, t_v] \), \( v' \) not a child of the \( i \)-th child of \( v \) in \( T \), then \((i, \square)\) is in \( C(v, v') \).
- For \( i \in [1, t_v] \), \( i' \in [1, t_{v'}] \), \( v' \) a child of the \( i \)-th child of \( v \) in \( T \), then \((i, i')\) is in \( C(v, v') \).

Claim 8. \( G \) has a colouring fulfilling the constraints, if and only if \( F \) can be satisfied by setting exactly \( k \) variables to true.

Proof. Suppose \( F \) can be satisfied by setting exactly \( k \) variables to true. Now, for each true variable, colour one vertex in \( W \) with that variable.

We say that all root nodes of trees in \( T' \) are active. Each active vertex is a disjunction that is satisfied by the variable setting. Top-down, we assign colours, as follows. If a vertex \( v \) is active, then at least one of the terms of the disjunction \( F_v \) must be satisfied. Choose a \( s_v \in [1, t_v] \) such that the \((s_v)\)-th term of \( F_v \), with \( v \) active, is satisfied. Colour \( v \) by \( s_v \). All vertices \( v' \) that are in \( T \) and the \((s_v)\)-th child of \( v \) are said to be active. These vertices precisely are the disjunctions that is represented by the \((s_v)\)-th child of \( v \) — as we assume that that child represents a satisfied conjunction, all its terms are also satisfied. Thus, by induction (top-down in the tree), active vertices represent satisfied subformulas. An inactive vertex is coloured by \( \square \).

One easily checks that all edges between vertices in \( T' \) fulfil the colour constraints.

An edge between a vertex \( v \in L \) and a vertex \( w \in W \) is always properly coloured when \( v \) has colour \( \square \). If \( v \) has a colour \( s_v \in [1, t_v] \), then \( v \) is active, hence \( F_v \) satisfied, and the \((s_v)\)-th term of \( F_v \) is satisfied. Thus, all literals in that term (which is a conjunction) are satisfied, and so the term cannot contain a literal \( \neg x_j \) with \( x_j \) set to true. So, \( w \) cannot have the colour \( x_j \).

Now, suppose \( G \) has a colouring satisfying all constraints. The constraints on \( W \) give that each vertex in \( W \) has a different colour from \( \{x_1, \ldots, x_n\} \). Set \( x_i \) to true, if and only if there is a vertex in \( W \) with colour \( x_i \). So, we have set \( k \) variables to true.

We claim that for each active vertex \( v \) in \( T \) (i.e., a vertex with colour different from \( \square \)), \( F_v \) is satisfied by the defined truth assignment. We proof this by induction, bottom-up in the tree. If \( v \in L \) is a leaf with colour \( s_v \in [1, t_v] \), then the \((s_v)\)-th term of \( F_v \) is satisfied: this term is a conjunction of negative literals. Consider a literal \( \neg x_i \) that appears in the term. If \( x_i \) would be true, then there is a vertex \( w \in W \) with colour \( x_i \), but then the edge \((v, w)\) would not satisfy the colour constraints. So, the \((s_v)\)-th term of \( F_v \) is a conjunction of satisfied literals, and thus the disjunction \( F_v \) is satisfied. If \( v \in X \cup R \) with colour \( s_v \in [1, t_v] \), then the \((s_v)\)-th term of \( F_v \) is a conjunction of terms. The colour constraints enforce that for all these terms their corresponding vertex is active (they are children of the \((s_v)\)-th child of \( v \) in \( T \)), and thus, the \((s_v)\)-th term of \( F_v \) is satisfied, and thus \( F_v \) is satisfied.
As roots of trees in $T'$ are always active (cannot have colour □), for each root $v$ of a tree in $T'$, we have $F_v$ is satisfied, so $F = \bigwedge_{v \in R} F_v$ is satisfied.

As $G$ and its colouring can be constructed in polynomial time, and we keep the same parameter $k$, the result follows from the fact that WEIGHTED $t$-NORMALISED ANTI-MONOTONE SATISFIABILITY is complete for W$[t]$ [14, 15].

**Corollary 9.** Binary CSP with as parameter the size of a vertex modulator to graphs of treedepth $d$ is W$[2d + 1]$-hard.

**Proof.** A tree with depth $d$ has treedepth at most $d$.

**Lemma 10.** Binary CSP with as parameter the size of a minimum feedback vertex set is W[ SAT]-hard.

**Proof.** We can use the same proof as for Lemma 7, but instead use WEIGHTED ANTI-MONOTONE SATISFIABILITY as problem to reduce from, and have no depth bound on the resulting tree or forest. WEIGHTED ANTI-MONOTONE SATISFIABILITY is W[ SAT]-complete [1].

### 4 Membership

In this section, we show that Binary CSP with vertex cover as parameter is in W[3] and that Binary CSP with the size of a vertex modulator to treedepth-$d$ belongs to the class W[2$d + 1$].

**Lemma 11.** Binary CSP with vertex cover as parameter is in W[3].

**Proof.** Suppose we are given an instance of Binary CSP, with a graph $G = (V, E)$, colour lists $C(v) \subseteq C$ for all $v \in V$, and $C(v, w) \subseteq C \times C$ for all edges $vw \in E$. Suppose $W \subseteq V$ is a vertex cover of $G$ of size $k$. Write $W = \{w_1, w_2, \ldots, w_k\}$.

We build a formula $F$ as follows. For each vertex $w \in W$ and each colour $c \in C(w)$, we take a Boolean variable $x_{w,c}$. The intuition is that $x_{w,c}$ is true, iff we colour $w$ with $c$.

We first give a formula $F^1(X)$, that ensures that each vertex has at least one colour.

$$F^1(X) = \bigwedge_{w \in W} \bigvee_{c \in C(w)} x_{w,c}$$

Note that when we set exactly $k$ variables to true and $F^1(X)$ holds, then for each $w \in W$, there is exactly one $c \in C(w)$ with $x_{w,c}$ true. We call this colouring of the vertices in $W$ the colouring given by $X$.

In the second step, we verify that the colouring given by $X$ does not create a colour conflict between vertices in $W$.

$$F^2(X) = \bigwedge_{w_1, w_2 \in W, w_1w_2 \in E} \bigvee_{(c, c') \in C(w_1, w_2)} x_{w_1,c} \wedge x_{w_2,c}$$

Note that, assuming $F^1(X)$ also holds, that $F^2(X)$ holds, if and only if the colouring given by $X$ does not create a conflict between vertices in $W$.

Our third formula has as argument a vertex $v \in V \setminus W$, and a colour $c \in C(v)$, and checks if we can colour $v$ by $c$ without creating a conflict with the colouring given by $X$. To keep the formula a single conjunction, we check for each vertex in $w$ and each colour $c'$ for $w$ that would conflict with $c$ that $w$ does not have colour $c'$.

$$F^3(X, v, c) = \bigwedge_{w \in W : vw \in E} \bigwedge_{c \in C(v), c' \in C(w), (c, c') \notin C(v, w)} \neg x_{w,c'}$$
Formula $F^4$ checks that all vertices in $V \setminus W$ can be coloured without creating a conflict with the colouring given by $X$.

$$F^4(X) = \bigwedge_{v \in V \setminus W} \bigvee_{c \in C(v)} F^3(X, v, c)$$

Finally, we define $F$:

$$F^4(X) = F^1(X) \land F^2(X) \land F^4(X)$$

From the discussion above, we see that $F(X)$ holds, if and only if $X$ gives a colouring of the vertices in $W$ that does not create a conflict between vertices in $X$, and each vertex in $V \setminus X$ can choose a colour without creating conflicts between vertices in $W$ and vertices in $X$. As $W$ is a vertex cover, $F(X)$ holds, if and only if the colouring of $W$ given by $X$ can be extended to the entire graph, i.e., if and only if the Binary CSP instance has a solution.

As each term $F^3(X, v, c)$ is a conjunction of literals, $F^4$ and $F$ are $3$-normalised. Observe that they have polynomial size. We can now conclude the result.

We can generalise the proof above to vertex modulators to treedepth-$d$.

**Lemma 12.** **Binary CSP** with the size of a vertex modulator to treedepth-$d$ is in $W[2d+1]$.

**Proof.** Assume an instance of Binary CSP, with a modulator set of vertices $W$ of size $k$, and a rooted forest embedding $T = (V \setminus W, F)$ of depth $d$. Write $W = \{w_1, \ldots, w_k\}$. We can assume that all edges in $F$ are also edges in $E$, otherwise, add the edge to $E$; for such a new edge $yz$, set $C(y, z) = C(y) \times C(z)$. (By allowing all pairs on new edges, the collection of valid colourings does not change.)

As in the previous proof, we take a Boolean variable $x_{w,c}$ for each $w \in W$, $c \in C(w)$. The first two steps are identical to that proof.

$$F^1(X) = \bigwedge_{w \in W} \bigvee_{c \in C(w)} x_{w,c}$$

$$F^2(X) = \bigwedge_{w_1, w_2 \in W} \bigvee_{(c, c') \in C(w_1, w_2)} x_{w_1,c} \land x_{w_2,c}$$

If $F^1(X)$ holds, then again for each $w \in W$, there is exactly one colour $c \in C(w)$ with $c(w)$ true; the thus obtained colouring of $W$ is again called the colouring given by $X$, and $F^2(X)$ holds, if and only if this colouring does not give a conflict between vertices in $W$.

For each $v \in V \setminus W$, let $A(v)$ be the set of ancestors of $v$, including $v$ itself; and let $B(v)$ be the set of children of $v$. Note that for each $v \in V \setminus W$, $|A(v)| \leq d$, as $T$ is a forest of depth at most $d$.

Let $f : A(v) \to C$ be a function that assigns a colour to each vertex in $A(v)$. We say that $f$ is conflict-free, iff for all $x \in A(v)$, $f(x) \in C(x)$, and for all $x, y \in A(v)$ with $xy \in E$, $(f(x), f(y)) \in C(x, y)$, i.e., the colouring satisfies locally the given constraints of the CSP instance.

For a vertex $v \in V \setminus W$, we let $\mathcal{F}_v$ be the set of all conflict-free functions $f : A(v) \to C$. Note that these sets do not depend on $X$ and can be computed in polynomial time for each vertex.

The next term, $F^5(X, v, f)$ is defined for a $v \in V \setminus W$ and $f \in \mathcal{F}_v$. It holds, if and only if the colouring $f$ does not create a conflict with a vertex in $W$ with the colouring given by $X$:

$$F^5(X, v, f) = \bigwedge_{x \in A(v), w \in W} \bigwedge_{c \in C(w), (f(x), c) \notin C(x, w)} \neg x_{w,c}$$

Suppose $x$ is a child of $v$ in $T$, and $f \in \mathcal{F}_v$, $c \in C(x)$. $f + x \to c$ is the function, that extends the domain of $f$ with the element $x$ and maps $x$ to $c$. Consider the following, recursive definition. It is defined for a $v \in V \setminus W$ and $f \in \mathcal{F}_v$.
\[ F^6(X, v, f) = F^5(X, v, f) \land \bigwedge_{y \in B(v)} \bigvee_{c \in C(y): (f + x \rightarrow c) \in F_y} F^6(X, y, f + x \rightarrow c) \]

\( F^6(X, v, f) \) holds, if and only if there is a colouring without conflicts for \( v \), the ancestors of \( v \), the descendants of \( v \), and the vertices in \( W \) such that \( X \) gives the colours for the vertices in \( W \), and \( f \) gives the colours for \( v \) and its ancestors. Or, in other words, if we can extend the colouring defined by \( X \) and \( f \) to the descendants of \( v \). That this is indeed the property expressed by \( F^6 \) can be shown with induction to the maximum distance to a leaf of vertices. For leaves, \( F^6(X, v, f) = F^5(X, v, f) \) and the property holds as leaves have no descendants. For non-leaves, the property holds if we can colour each child in \( B(v) \) with the subtree below the child.

The next formula checks that we can colour all vertices in \( V \setminus W \), given the colouring of \( W \) given by \( X \). For this, we check that we can choose a colour for each root, such that this colour can be extended to a colouring of the subtree below the root. Let \( R \) be the set of roots of \( T \). Note that a set \( A(r) \) for \( r \in R \) has size 1.

\[ F^7(X) = \bigwedge_{r \in R} \bigvee_{f \in F_r} F^6(X, r, f) \]

Finally, we can define \( F \):

\[ F(X) = F^1(X) \land F^2(X) \land F^7(X) \]

\( F(X) \) holds, if and only if the colouring of \( W \) given by \( X \) can be extended to a colouring of the entire graph while satisfying all colour constraints. What remains to be shown is that \( F \) is \( 2d + 1 \)-normalised, and that it can be computed in polynomial time.

First, notice that \( F^1 \) and \( F^2 \) are 3-normalised (where we sometimes take a single term as a conjunction or disjunction of 1 term). If \( v \) is a leaf, then each term of the form \( F^5(X, v, f) = F^6(X, v, f) \) is a conjunction of literals, i.e., 1-normalised. With induction, we have that if the maximum distance of a vertex \( v \) to a leaf is \( d' \), then each term \( F^6(X, v, f) \) is \((2d + 1)\)-normalised. As for each root, the distance to a leaf is at most \( d - 1 \), each term \( F^6(X, v, f) \) is \((2d - 1)\)-normalised, and thus \( F^7 \) and \( F \) are \((2d + 1)\)-normalised.

Second, we note that we can compute \( F^1 \), \( F^2 \), and all terms of the form \( F^5 \) in polynomial time, and thus also all terms of the form \( F^6 \) for leaves of \( T \). Suppose we have \( n \) vertices, and \( N = |C| \). For each \( v \in V \setminus W \), \( F_v \) has at most \( N^d \) elements, and as \( d \) is constant here, we can build all sets \( F_v \) in polynomial time. In order to build one term \( F^6(X, v, f) \), we need to build less than \( n \cdot N \) terms of the form \( F^6(X, y, f') \) for children \( y \) of \( v \). For each of these, the maximum distance of \( y \) to a leaf is one smaller than the maximum distance of \( v \) to a leaf. Thus, if the maximum distance of \( v \) to a leaf is \( d' \), then the time to compute one term \( F^6(X, v, f) \) is \( n^{2d'+O(1)} \cdot N^{2d'+O(1)} \). The maximum distance of a root to a leaf is \( d - 1 \), and \( F^7 \) has to compute at most \(|R| \cdot N \) terms, so the total time to build \( F \) is bounded by \( O(n^{2d+O(1)} N^{2d+O(1)}) \).

Thus, we have a parameterised reduction from Binary CSP with modulator to treedepth-\( d \) to Weighted (\( 2d + 1 \))-Normalised Satisfiability. Membership of the latter in \( W[2d + 1] \) (cf. Theorem 3) gives the result.

As forests of depth \( d \) have treedepth \( d \), we also have the following result. (A direct proof of this fact can be simpler than the proof above, as we can avoid the use of sets \( A(v) \).)

**Corollary 13.** Binary CSP with the size of a vertex modulator to a forest of depth \( d \) as parameter is in \( W[2d + 1]/f \).

Lemma 3 with Lemma 12 for the case \( d = 1 \) together give Theorem 1. Theorem 2 follows from Lemma 7 and Lemma 12.
Lemma 14. Binary CSP with the size of a minimum feedback vertex set as parameter is in $W[1]$. 

Proof. We can show that a problem is in $W[1]$ by giving a circuit, that has as input $n$ Boolean variables, that has polynomial size, and has an accepting input with exactly $k$ variables set to true, if and only if the instance of Binary CSP has a solution.

Suppose we are given a graph $G = (V, E)$, with colour constraints $C$ on vertices and edges. Suppose $W$ is a feedback vertex of $G$. Write $G \setminus W$ as forest $T$, and choose in each tree in $T$ a root.

For each $w \in W$, and $c \in C(w)$, we have a variable $x_{w,c}$, that corresponds to colouring $w$ with $v$. The circuit consists of two parts. One part has an output gate that is true, iff each vertex in $w$ gets a unique colour; the other part has an output gate that is true, iff given the colouring of $W$, we can colour the vertices in $T$. A final and-gate has as input the outputs of these parts, and gives the output of the circuit.

The first part is straightforward. In the second part, for $v \in V \setminus W$ a vertex in $T$, and $c \in C(v)$, define $y_{v,c}$ to be true, if and only if there is a colouring of the subgraph, consisting of $v$, $W$, and the descendants of $v$ fulfilling all constraints, that colours $v$ with $c$, and the vertices the vertices in $W$ as dictated by the setting of the input variables. Let $B(v)$ be the children of $v$ in $T$. One easily checks that the following recursive definition of $y_{v,c}$ is correct, and acyclic:

$$y_{v,c} = \bigwedge_{w \in W} \bigvee_{(c',c) \in C(v,w)} x_{w,c'} \bigwedge_{v' \in B(v)} \bigvee_{(c',c') \in C(v,v')} y_{v',c'}$$

Thus, we can build a circuit that computes all values $y_{v,c}$. The second part then outputs, with $R$ the set of roots in $T$

$$\bigwedge_{v \in R} \bigvee_{c \in C(v)} y_{v,c}$$

□

5 Corollaries

In this section, we give some simple corollaries of the results given above.

5.1 List Colouring

Corollary 15. Let $d \geq 2$. List Colouring is $W[2d - 1]$-hard and in $W[2d + 1]$ with the size of a vertex modulator to treedepth-$d$ as parameter.

Proof. Membership in $W[2d + 1]$ follows directly from the membership in $W[2d + 1]$ of Binary CSP, as List Colouring is a special case of Binary CSP.

For the other direction, we can use a well known and easy reduction from Binary CSP to List Colouring. Take an instance of Binary CSP. For each edge $vw \in E$, remove this edge, and instead add for each ‘forbidden’ pair $(c, c') \in C(v) \times C(w) \setminus C(v,w)$ a new vertex with an edge to $v$ and an edge to $w$, and give this new vertex the colour set $\{c, c'\}$. One easily sees that this gives an instance of List Colouring that is equivalent.

The operation increases the treedepth of a graph by at most one: use the same tree for the original vertices. For a new vertex $x$ with neighbours $v$ and $w$, we have that $v$ is an ancestor of $w$ or vice versa. Let $x$ be a child of the lower of these two vertices.

Thus we have a parameterised reduction from List Colouring with parameter a vertex modulator to treedepth $d$ to Binary CSP with parameter a vertex modulator to treedepth $d + 1$, and the result follows with Lemma 7. □
The case \( d = 1 \) gives the following direct consequence. \( W[1] \)-hardness of the problem was shown by Fiala et al. [22].

**Corollary 16.** List Colouring is in \( W[3] \) with the size of vertex cover as parameter.

The case \( d = 2 \) also gives an interesting corollary, by noting that a graph with treedepth 2 is a forest consisting of isolated vertices and stars (graphs of the form \( K_{1,r} \)).

**Corollary 17.** List Colouring is \( W[3] \)-hard with as parameter the size of a vertex modulator to forest of depth two, and thus for parameterisation by feedback vertex set size.

The complexity of List Colouring with treedepth as parameter is an intriguing open problem. From the results of this paper, we obtain the following corollary, but one can expect that List Colouring with treedepth as parameter will be hard for classes higher in the complexity hierarchies.

**Corollary 18.** For all integers \( t \geq 1 \), List Colouring and Binary CSP with treedepth as parameter are hard for \( W[t] \).

**Proof.** Notice that a graph with a vertex modulator of size \( k \) to a graph of treedepth \( d \) has treedepth at most \( d + k \). (Suppose we have a modulator \( W \) and forest \( T = (V \setminus W, E') \). Build a tree by taking a path with the vertices in \( W \) parameter are hard for \( W[d] \).

So, for each \( d \), we can use the transformation from Weighted \((2d + 1)\)-Normalised Satisfiability to Binary CSP, and go from a problem where we have to set \( k \) variables to true to a problem on a graph of treedepth \( k + d \). Thus, for each \( d \), this gives a proof that Binary CSP with treedepth as parameter is hard for \( W[2d + 1] \). From this, the result for List Colouring follows by the same observations as in the proof of Corollary 15.

### 5.2 Precolouring Extension

We now briefly discuss the Precolouring Extension problem. We are given a graph \( G = (V, E) \), a set of colours \( \mathcal{C} \), a subset \( W \subseteq V \), a partial colouring \( f : W \rightarrow \mathcal{C} \), and ask if we can extend \( f \) to a proper colouring of \( G \), i.e., is there an \( f' : V \rightarrow \mathcal{C} \), such that for all \( w \in W \) \( f'(w) = f(w) \), and for all edges \( xy \in E \) \( f(x) \neq f(y) \).

With well known or easy to obtain insights, we can see that Precolouring Extension is fixed parameter tractable with vertex cover as parameter. Suppose we have an instance \( G = (V, E), \mathcal{C}, W \subseteq V, f : W \rightarrow \mathcal{C} \). Suppose \( S \subseteq V \) is a vertex cover of \( G \). Write \( I = V \setminus S, k = |S| \). If \( |\mathcal{C}| \leq k \), then dynamic programming can be used to solve the problem [30]. Otherwise, each vertex in \( I \setminus W \) can be removed as it always has a colour. Now, build the equivalent List Colouring instance, by giving each vertex in \( V \setminus S \) the colour list obtained by removing all colours of precoloured neighbours from \( \mathcal{C} \). We have an equivalent instance, now of List Colouring, with a graph with at most \( k \) vertices; by removing all vertices with a list at least \( k \) long, we have an instance that can be described with \( O(k^2 \log k) \) bits.

A similar observation can be used to transform Precolouring Extension on graphs with a modulator of size \( k \) to treedepth \( d \) to instances of List Colouring on graphs a modulator of size \( k \) to treedepth \( d - 1 \). Suppose we have an instance \( G = (V, E), \mathcal{C}, W \subseteq V, f : W \rightarrow \mathcal{C} \), with a vertex set \( S \) such that \( G \setminus S \) has treedepth \( d \). Let \( T \) be a rooted forest embedding of \( G \setminus S \) of depth \( d \).

We distinguish the cases that \( |\mathcal{C}| \leq d + k \), and \( |\mathcal{C}| > d + k \). The treewidth of a graph with a vertex modulator to treedepth \( d \) is at most \( d + k \), and thus, in the former case, the problem is FPT in \( d + k \) [30].
Now assume $|C| > d + k$. Consider a leaf $v$ in $T$. If $v$ is not precoloured, i.e., $v \notin W$, then $v$ can be removed as it has degree at most $d + k$ (its ancestors in $T$ and the vertices in $W$ can be neighbours). Consider the transformation from a PRECOLOURING EXTENSION instance to a LIST COLOURING instance that removes all precoloured vertices, and for each precoloured vertex, their colour from the colour lists of all its neighbours. This gives an equivalent instance; when all leaves are precoloured, then all leaves from $T$ are removed by this transformation, and thus the depth of $T$ decreases by at least one. We can conclude the following.

**Corollary 19.** Let $d \geq 2$. PRECOLOURING EXTENSION is FPT with vertex cover as parameter, and $W[2d-3]$-hard and in $W[2d-1]$ with the size of a vertex modulator to treedepth-$d$ as parameter.

6 Conclusions

In this paper, we showed that BINARY CSP with vertex cover as parameter is complete for W[3], and gave a number of related results. The results of this paper illustrate that the complexity structure of problems in XP is rich, and a better understanding of the complexities of various problems in XP can reveal interesting and important insights. We end the paper with a few open problems.

- What is the complexity of LIST COLOURING with vertex cover as parameter? Currently, we know it is W[1]-hard and in W[3]; presumably at least one of these two results can be sharpened.
- Similarly, what is the complexity of LIST COLOURING with vertex modulator to treedepth-$d$ as parameter?
- Can we establish for which classes BINARY CSP and LIST COLOURING with the minimum size of a feedback vertex set are complete?
- Can we establish for which class BINARY CSP and LIST COLOURING with treedepth as parameter are complete? Note that there are simple (parameterized) reductions between these problems, in both directions, so the problems will be complete for the same class. It is easy to observe that the problems are in XL (employing a simple branching algorithm, following the tree, and keeping the choices for ancestors of the currently considered vertex in memory). It is unlikely that these problems are in the W-hierarchy, as they display some type of compositionality, cf. the discussion in [21, Section 4].

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