COSMOLOGY: TECHNIQUES AND OBSERVATIONS

Emilio Elizalde†
Department of Mathematics, Massachusetts Institute of Technology
77 Massachusetts Ave, Cambridge, MA 02139-4307, USA.

These lectures were addressed to nonspecialists willing to learn some basic facts, approaches, tools and observational evidence which conform modern cosmology. The aim is also to try to complement the many excellent treatises that exists on the subject (an exhaustive treatment being in any case impossible for lack of time, in the lectures, and of space here), instead of trying to cover everything in a telegraphic way. We start by recalling in the introduction a couple of philosophical questions that have always upset inquiring minds. We then present some original mathematical approaches to investigate a number of basic questions, as the comparison of two point distributions (each point corresponding to a galaxy or galaxy cluster), the use of non-standard statistics in the analysis of possible non-Gaussianities, and the use of zeta regularization in the study of the contributions of vacuum energy effects at the cosmological scale. And we also summarize a number of important issues which are both undoubtedly beautiful (from the physical viewpoint) and useful in present-day observational cosmology. To finish, the reader should be warned that, for the reasons already given and lack of space, some fundamental issues, as inflation, quantum gravity and string theoretical fundamental approaches to cosmology will not be dealt with here. A minimal treatment of any of them would consume more pages than the ones at disposal and, again, a number of excellent treatments of these subjects are available.

Keywords: Cosmology; Vacuum Energy; Zeta Functions.

1. Introduction

Cosmology is the study of the world we live in. It would be difficult to find a more noble endeavour for a human being —let aside perhaps from medicine— given our proven capacity to ‘understand’ things and events around us. To start, some already classic books, but very useful to everyone to get a basic knowledge of the cosmos are these in Ref. Here, let us just recall the several century long discussions, first on the issue of our Earth being flat or round, then about that same Earth being or not the center of the universe, and later the formulation of the universal law of gravitation by Newton, who extended thereby the observation of the behavior of an apple falling from a tree on the earth surface to build a universal law for the whole

* Lectures given at the “II International Conference on Fundamental Interactions”, Pedra Azul, Brazil, June 6-12, 2004.
† On leave from ICE, Consejo Superior de Investigaciones Científicas, and IEEC, Edifici Nexus, Gran Capitá 2-4, 08034 Barcelona, Spain. E-mail: elizalde@math.mit.edu, elizalde@ieec.fcr.es.
of the cosmos. This immediately rises to the thinking mind (at least) two questions.

- Why is the universe so ‘understandable’ to the human mind?
- And how comes that mathematics are so useful in the formulation of the laws of nature? Just recall the extreme simplicity of Newton’s formula, \( F = GMm/r^2 \), and the usefulness of Leibniz’s calculus. Is this not very remarkable?

Indeed, the question phrased by Eugene Wigner as that of the unreasonable effectiveness of mathematics in the natural sciences \(^2\) is an old and intriguing one. It goes back to Pythagoras and his school, ca. 550 BC ("all things are numbers"), even probably to the sumerians, and maybe to more ancient cultures, which left no trace. Immanuel Kant said that "the eternal mystery of the world is its comprehensibility" \(^a\) and Albert Einstein contributed also to this idea in his 1936 essay "Physics and Reality" where he elaborated on Kant’s statement. Also mathematical simplicity, and beauty, have remained for many years crucial ingredients when having to choose among different plausible possibilities.

Those are for sure profound and far reaching ideas by some of the people who established landmarks in the long way towards our present understanding of the Cosmos. However, one should be more humble and I would never dare to use words like "comprehend" or ‘understand’, but would rather replace them by describe, or modelize, in modern terms. Indeed, the fact e.g. that Newton’s formula is so extremely simple and far reaching does not mean at all that we understand the attraction of two massive bodies any better. Why do two bodies attract, and do not repel, each other? The only conclusion to be drawn is that Newton put us in possession of a very simple, useful, accurate, and universal model for ordinary gravity. In trying to answer questions as the last one and others, such as the nature of the mass of a body, and in trying to formulate an ultimately universal law for the whole Universe a lot of effort has been invested in the last 30 to 40 years. The results obtained have been extraordinary from the mathematical viewpoint (the new formalism itself and its mathematical applications, e.g. in algebraic geometry and knot theory). From the physical side the advance has been indeed consistent, but much less spectacular, and a unique theory of everything (TOE) is not in sight yet.

In the last decade considerable effort has been put in observational cosmology with very rewarding results as, just to mention two of them, the construction of the first maps of the cosmos (first including thousands and now millions of galaxies), and the discovery that the expansion of the universe we live in is accelerating. All indicates that this effort is going to continue, opening new perspectives for the awaited matching of the proposed mathematical models with the physical, observational results, and also (not less important) for new jobs for cosmology students.

\(^{a}\)The reader should pause and take some minutes to meditate such a profound and mysterious statement. The author cannot refrain from confessing that he had arrived to the same conclusion by himself alone, before reading Kant. When he first discovered Kant’s sentence, he was most deeply shocked.
These three lectures can only cover a very small part of what is known, or at least should be learnt, about cosmology. A personal bias is unavoidable and, far from trying to disguise it, the author’s purpose is to complement the many excellent treatises that exist on the subject, by touching upon some matters of his competence that are not so often discussed, and trying also to bring together things that appear disconnected in the existing literature. The first lecture will be on Mathematics as a tool to study the Universe. Among the new mathematics required to study physical processes since the appearance of Quantum Field Theory (QFT), there is the regularization or ‘summation’ of infinite series. In the cosmological setting and in brane and string theory or Quantum Gravity, the most elegant method to do this has proven to be zeta function regularization, introduced by S. Hawking in 1975 with exactly this purpose. A review of this technique will be presented in the first lecture, while a description of its uses in the calculation of the contribution of vacuum energy effects at the cosmological scale (e.g. in trying to explain the observed acceleration of the universe) will appear in the third. Also in the first lecture, an introduction of the formalism ordinarily used in Cosmology, as derived from the Einstein equations of General Relativity will be presented in some detail.

The second lecture is devoted to some aspects of observational cosmology. We start by reviewing a number of important issues which are both extremely beautiful (from the physical viewpoint) and useful in present-day observational cosmology, such as the Hubble law, the Sunyaev-Zeldovich effect, the appearance of the Lyman-alpha forest structures, and the two variants of the Sachs-Wolfe effect. Then we present in some detail the mathematical techniques used for the purpose of the study of the large scale structure of our Universe. These include the analysis of point distributions (each point corresponding to a galaxy or galaxy cluster) and the use of non-standard statistics in the analysis of possible non-Gaussianities of the cosmic microwave background (CMB) temperature and matter fluctuations. Finally, as anticipated already, in the third lecture we will present some simple model in order to explain how quantum vacuum effects at cosmological scale can possibly contribute to the present value of the cosmological constant. For this purpose a further elaboration of the zeta regularization techniques, including some formulas originally derived by the author, will be necessary.

There is no main conclusion to the paper, aside from the sum of the partial ones already suggested in the different sections. It is by now quite clear that the study of the Cosmos during the next decades will be most rewarding, not only intellectually or spiritually, but also in many more materialistic ways, for the many international collaborations already started or projected will undoubtedly require a large number of dedicated cosmologists in different tasks. Among them, to analyze and understand the observational results in the framework of existing and new ambitious theories, of which, for sure, there’ll still be plenty in the coming years.
Mathematics as a tool to study the Universe

We now elaborate on the issues raised in the Introduction.

2.1. Some disgression on divergent series

The fact that the infinite series
\[ s = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \] (1)
has the value \( s = 1 \) is nowadays clear to any school child. It was not so, in fact, for many centuries, as we can recall from Zeno of Elea’s paradox (or Zeno’s paradox of the tortoise and Achilles), transmitted by Aristotle and based on the pretended impossibility to do an infinite number of summations (or recurrent ‘jumps’ or steps of any kind, in a finite amount of time). In fact there are still modern versions of the Zeno paradox (e.g. the quantum Zeno paradox) which pop up now and then. We shall not discuss those here, but rather assume that the following process is clear to the reader: by taking the first term, \( 1/2 \), to the left what remains on the r.h.s. is just one half of the original series (i.e., \( 1/2 \) is a common factor), so that
\[ s - \frac{1}{2} = \frac{s}{2} \implies s = 1. \] (2)

Now, what is the sum of the following series?
\[ s = 1 + 1 + 1 + 1 + \cdots \] (3)

Again, any school child would answer immediately that \( s = \infty \). In fact, whatever \( \infty \) may be, everybody recognizes in this last expression the definition itself of the infinity: the piling of one and the same object, once and again, without end. Of course, this is true, but it is useless to modern Physics, being precise, since the advent of quantum fields. Calculations there are plagued with divergent series, and it is of no use to say that this series is divergent, and that other one is also divergent, and the other there too, and so on. One gets non-false but useless information in this way, since we do not see these infinites in Nature.

For years, there was the suspicion that one could try to give sense to divergent series. This has proven (experimentally!) to be true in Physics, but it were the mathematicians —many years before— who first realized this possibility. In fact, Leonard Euler (1707-1783) was convinced that “To every series one could assign a number”\(^\text{13}\) (that is, in a reasonable, consistent, and useful way, of course). Euler was unable to prove this statement in full, but he devised a technique (Euler’s summation criterion) in order to ‘sum’ large families of divergent series. His statement was however controverted by some other great mathematicians, as Abel, who said that “The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever”\(^\text{14}\). There is a classical treatise due to G.H. Hardy and entitled simply Divergent series\(^\text{15}\) that I highly recommend to the reader.

As always with modern Mathematics, one starts the attack on divergent series by invoking a number of axioms, like...
(i) If \( a_0 + a_1 + a_2 + \cdots = s \), then \( ka_0 + ka_1 + ka_2 + \cdots = ks \).

(ii) If \( a_0 + a_1 + a_2 + \cdots = s \), and \( b_0 + b_1 + b_2 + \cdots = t \), then
\[
(a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + \cdots = s + t.
\]

(iii) If \( a_0 + a_1 + a_2 + \cdots = s \), then \( a_1 + a_2 + \cdots = s - a_0 \).

A couple of examples.

(i) Using the third axiom we obtain that for the series
\[
s = 1 - 1 + 1 - 1 + \cdots,
\]
we have \( s = 1 - s \), and therefore \( s = 1/2 \). This value is easy to justify, since the series is oscillating between 0 and 1, so that 1/2 is the more 'democratic' value for it.

(ii) Using now the second axiom, we obtain that for the series
\[
t = 1 - 2 + 3 - 4 + \cdots,
\]
it turns out by subtracting it term by term from the former one that \( s - t = t \), and therefore \( t = s/2 = 1/4 \). Such result is already quite difficult to swallow. This is in common with most of the finites values that are obtained for infinite, divergent series.

But, what to do about our initial series \( 1 + 1 + 1 + \cdots \)? This one is most difficult to tame, and the given axioms do not serve to this purpose. But there is more to the axioms, which are only intended as a humble starting point. By reading Hardy's book one learns about a number of different methods that have been proposed and is good to know. They are due to Abel, Euler, Cesàro, Bernoulli, Dirichlet, Borel and some other mathematicians.\(^{b}\) The most powerful of them involve analytical continuation in the complex plane, as is the case of the so called zeta regularization method.

Thus, for instance, a series
\[
a_0 + a_1 + a_2 + \cdots
\] will be said to be Cesàro summable, and its sum to be the number \( s \), if the limit of partial sum means exists and gives \( s \), namely
\[
\exists \lim_{n \to \infty} \sum_{n=1}^{\infty} \frac{A_n}{n} = s, \quad A_n \equiv \sum_{j=1}^{n} a_j.
\] (5)

This criterion can be extended and gives rise to a whole family of criteria for Cesàro summability. On the other hand, the series before will be said to be Abel summable, and its sum to be the number \( s \),
\[
\sum_{n=0}^{\infty} a_n = s,
\] (6)

if the following function constructed as a power series
\[
f(x) \equiv \sum_{n=0}^{\infty} a_n x^n
\] (7)

\(^{b}\)Padé approximants should in no way be forgotten in this discussion.16
is well defined for $0 < x < 1$ and the limit when $x$ goes to 1 from the left exists and gives $s$, namely,

$$\exists \lim_{x \to 1^-} f(x) = s. \quad (8)$$

And similarly for the rest of the criteria, which are not equivalent, as one can check.\textsuperscript{15}

2.2. Zeta regularization in a nutshell

As advanced already, the regularization and renormalization procedures are essential issues of contemporary physics —without which it would simply not exist, at least in the form we know it.\textsuperscript{14} Among the different methods, zeta function regularization—which is obtained by analytic continuation in the complex plane of the zeta function of the relevant physical operator in each case—is maybe the most beautiful of all. Use of this method yields, for instance, the vacuum energy corresponding to a quantum physical system (with constraints of any kind, in principle). Assume the corresponding Hamiltonian operator, $H$, has a spectral decomposition of the form (think, as simplest case, in a quantum harmonic oscillator):

$$\{\lambda_i, \varphi_i\}_{i \in I},$$

being $I$ some set of indices (which can be discrete, continuous, mixed, multiple, . . .). Then, the quantum vacuum energy is obtained as follows:\textsuperscript{9}

$$\sum_{i \in I} (\varphi_i, H \varphi_i) = \text{Tr}_\zeta H = \sum_{i \in I} \lambda_i = \sum_{i \in I} \lambda_i^{-s}_{s = -1} = \zeta_H(-1), \quad (9)$$

where $\zeta_H$ is the zeta function corresponding to the operator $H$, and the equalities are in the sense of analytic continuation (since, generically, the Hamiltonian operator will not be of the trace class).\textsuperscript{5} Note that the formal sum over the eigenvalues is usually ill defined, and that the last step involves analytic continuation, inherent with the definition of the zeta function itself.

The method evolved from the consideration of the Riemann zeta function. This was introduced by Euler, from considerations of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots, \quad (10)$$

which is logarithmically divergent, and of the fact that putting a real exponent $s$ over each term

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots, \quad (11)$$

\textsuperscript{5}The reader should be warned that this $\zeta-$trace is actually no trace in the usual sense. In particular, it is highly non-linear, as often explained by the author elsewhere. Some colleagues are unaware of this fact, which has lead to important mistakes and erroneous conclusions too often.
then for \( s > 1 \) the series is convergent, while for \( s \leq 1 \) it is divergent. Euler called this expression, as a function of \( s \), the \( \zeta \)–function, \( \zeta(s) \), and found the following important relation

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left( 1 - \frac{1}{p^s} \right)^{-1},
\]

which is crucial for the applications of this function in Number Theory. By allowing the variable \( s \) to be complex, Riemann saw the relevance of this function (that now bears his name) for the proof of the prime number theorem\(^4\), and formulated thereby the \textit{Riemann hypothesis}, which is one of the most important problems (if not the most) in the history of Mathematics. More to that in the excellent review by Gelbart and Miller.\(^{18}\)

For the Riemann \( \zeta(s) \), the corresponding (complex) series converges absolutely for the (open) half of the complex plane to the right of the abscissa of convergence \( \text{Re} \, s = 1 \), while it diverges on the other side, but it turns out that it can be analytically continued to that part of the plane, being then everywhere analytic (and finite) except for the only, simple pole at \( s = 1 \) (Fig. 1).\(^9\) In more general cases, namely corresponding to the Hamiltonians which are relevant in physical applications,\(^{10,11}\) the situation is in essence quite similar, albeit in practice it can be rather more involved. A mathematical theorem exists, which assures that under very general conditions the zeta function corresponding to a Hamiltonian operator will be also meromorphic, with just a discrete number of possible poles, which are simple and extend to the negative side of the real axis.\(^f\)

The above picture already hints towards the use of the zeta function as a summation method. Let us consider two examples.

\( \text{(i) We interpret our starting series} \)

\[
s_1 = 1 + 1 + 1 + 1 + \cdots
\]

\( \text{as a particular case of the Riemann zeta function, e.g. for the value } s = 0. \) This value is on the left hand side of the abscissa of convergence (Fig. 1), where the series as such diverges but where the analytic continuation of the zeta function provides a perfectly finite value:

\[
s_1 = \zeta(0) = -\frac{1}{2}.
\]

So this is the value to be attributed to the series \( 1 + 1 + 1 + 1 + \cdots \).

\( ^4\)Which states that the number \( \Pi(x) \) of primes which are less or equal to a given natural number \( x \) behaves as \( x/\log x \), as \( x \to \infty \). It was finally proven, using Riemann’s work, by Hadamard and de la Vallée Poussin.

\( ^5\)Where it yields the harmonic series: there is no way out for this one.

\( ^f\)Although there are some exceptions to this general behavior, they correspond to rather twisted situations, and are outside the scope of this brief presentation.
To start, on the open half of the complex plane which is on the r.h.s of the abscissa of convergence $\Re s = 1$, $\zeta$ is defined as the absolutely convergent series: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. In the rest of the $s$–complex plane, $\zeta(s)$ is defined as the (unique) analytic continuation of the preceding function, which turns out to be meromorphic. Specifically, it is analytic everywhere on the complex plane except for one simple pole with residue equal to 1, which is at the point $s = 1$ (notice that it corresponds to the logarithmically divergent harmonic series, as already discussed).

(ii) The series

$$s_2 = 1 + 2 + 3 + 4 + \cdots$$

(15)

corresponds to the exponent $s = -1$, so that

$$s_2 = \zeta(-1) = -\frac{1}{12}.$$  

(16)

A couple of comments are in order.

- In a short period of less than a year, two distinguished physicists, A. Slavnov and F. Yndurain, gave seminars in Barcelona, about different subjects. It was remarkable that, in both presentations, at some point the speaker addressed the audience with these words: “As everybody knows, $1 + 1 + 1 + \cdots = -1/2$.”

- That positive series, as the ones above, can yield a negative result may seem uttermost nonsensical. However, it turns out that the most precise experiments

---

8Implying maybe: If you do not know this it is no use to continue listening. Remember by the way the lemma of the Pythagorean school: Do not cross this door if you do not know Geometry.
ever carried out in Physics do confirm such results. More exactly: models of regularization in QED built upon these techniques lead to final numbers which are in agreement with the experimental values up to the 14th figure. In recent experimental proofs of the Casimir effect, the agreement is also quite remarkable (given the difficulties of the experimental setup).

- The method of zeta regularization is based on the analytic continuation of the zeta function in the complex plane. Now, how easy is it to perform that? Will we need to undertake a lengthy complex plane computation every time? It turns out that this is not so. The result is immediate, in principle, once you know the appropriate reflection formula (also called functional equation) that your zeta function obeys. In the case of the Riemann zeta:

\[
\xi(s) = \xi(1-s), \quad \xi(s) \equiv \pi^{-s/2}\Gamma(s/2)\zeta(s).
\]

(17)

In practice these formulas are however not optimal for actual calculations, since they are ordinarily given in terms of power series expansions (as the Riemann zeta itself!). Fortunately, sometimes there are more clever expressions, that can be found, which converge exponentially fast, as the Chowla-Selberg formula and some others. Those give real power to the method of zeta regularization. More about this point in lesson 3, where a number of such expressions will be explicitly used in the calculation of the contribution of the quantum vacuum fluctuations to the cosmological constant.

3. Observational Cosmology: large scale structure

3.1. A landmark in observational cosmology

Redshift surveys of galaxies, being three-dimensional, do not suffer from projection effects of two-dimensional maps on the sky surface and are much more appropriate to obtain the true large scale structure of our Universe. However, not all the contribution to the redshift comes from the cosmological expansion (which defines the third dimension, along the line of sight), since there are also additional contributions coming from the peculiar velocities of the galaxy in question (attraction of other galaxies in a cluster, displacement of the cluster itself, etc.).

\[
\text{cz}_{\text{observed}} = \text{cz}_{\text{cosmological}} + v_{\text{peculiar}}.
\]

(18)

This can originate artifacts such as the “finger of God” effect in which clusters of galaxies appear as long fingers pointing radially towards the observer. It is not easy to correct for these effects, so one must be careful when trying to make sense of such structures and of three-dimensional maps in general.

The CfA redshift survey of de Lapparent, Geller and Huchra (1986, 1988) was a landmark. It was just “A slice of the Universe”, but for the first time, in a

\[\text{b}^\text{This formula is just an approximation, since as space is curved, for non-near galaxies the distance/redshift relation is non-linear.}\]
map of black dots where each dot corresponded to a whole galaxy, the large scale structure of our Universe (a map of our world) appeared in front of our eyes, for the very first time ever (Fig. 2). That survey, and the corresponding Southern Sky

![First CfA Strip](image)

Redshift Survey (da Costa et al., 1988)\textsuperscript{25} showed the by now familiar filaments and walls surrounding voids\textsuperscript{27} the “bubble-like” textures of the galaxy distribution, on scales where the galaxy-galaxy correlation function is negligible. The impact was immediate, and influenced a large amount of physicists working in different subjects, who tried to explain, modelize and even reconstruct the point distribution of the slice, in terms of more or less fundamental theories.

A paper by J. Ostriker, C. Thomson, and E. Witten\textsuperscript{28} tried to explain the voids and other structures as a consequence of string theory. I tried to address the much more technical (but in my view not less important) issue of how to perform the comparison of two point distributions. They would be, in the case under study, the observed galaxy distribution for the slice geometry and any point distribution obtained, say, from a simulation of a theoretical model which would pretend to yield ‘the same’ or a good approximation to the observed point map. The simple (but very difficult) question to be answered is just: how close are the observed map

Fig. 2. The first CfA redshift survey caused an immediate impact on the scientific community. It clearly showed that the distribution of galaxies in space was anything but random, with galaxies actually appearing to be distributed on surfaces, almost bubble like, surrounding large empty regions, or "voids." V. de Lapparent, M. Geller and J.P. Huchra, Astrophys. J. Lett. 302, L1 (1986) Smithsonian Astrophysical Observatory.
and the one obtained from a model? Of course, a rigorous answer can be given, from Statistics, in terms of the 2-point, 3-point, ..., $n-$point correlation functions of both point distributions. But it turns out that in practice higher order correlation functions are very difficult to compute for a large sample of points, and one has to find alternative, much more direct and optimized routes: the highest possible unbiased information from the lowest number of moments of the distribution. One of methods we considered are the so-called counts in cells. I suggested this as the starting point for a PhD Thesis to Enrique Gaztañaga, who had come to me at the appropriate time in search of a subject to work on. We did quite nice work together on these matters, later extended by Pablo Fosalba and Jose Barriga with considerable success.

Two important redshift surveys have been based on the IRAS catalogue (the IRAS survey and QDOT). The original surveys have now been extended to other slices and the original few thousands of galaxies of CfA are now transformed into the several millions monitored by the 2 Degree Field survey, 2dF, the Sloan Digital Sky Survey, SDSS, etc. The most recent results of the 2dF survey are depicted in Fig. 3. Comparison with Fig. 2 of the first CfA redshift survey shows the enormous progress in observational cosmology in the last 15 years. The observations seem to conclude that the structures of the large scale galaxy point distribution are essentially sheet-like, and that the scale of the sheets is limited only by the scale of the survey. The most remarkable feature, the so-called “great wall” (Geller and Huchra, 1989)\(^\text{35}\) has been seen to be enhanced by the selection function for the sample, but it also appears in other deep wide angle surveys. In any case, there has been much discussion about this issue. Great walls do not bound great voids, but seem to surround collections of smaller voids that are themselves bounded by not-so-great walls. It could be that the great walls are picked out and correlated by our own eyes to build a larger structure.\(^\text{26}\) Looking at N-body models suggests this kind of effect because it is easy for the brain to identify coherent structures on scales where there is no physical mechanism for generating structure.\(^\text{27}\)

Another very important survey is the APM Galaxy Survey, which contains positions, magnitudes, sizes and shapes for about three million galaxies selected from 269 UKST survey plates which were scanned using the APM Facility. The galaxies have apparent magnitudes with $17 < b_j < 20.5$ and are spread through the largest volume of the universe that has been surveyed to date. The picture to be seen in the APM Galaxy Survey page (http://www.nottingham.ac.uk/~ppzsjm/apm/apm.html) shows the galaxy distribution as a density map in equal area projection on the sky, centered on the South Galactic pole. Each pixel covers a small patch of sky 0.1 degrees on a side, and is shaded according to the number of galaxies within the area: where there are more

\(^1\)This was the seed and the beginning of our cosmology group at the IEEC/CSIC Institute in Barcelona. Presently we are involved in PLANCK\(^\text{29}\) High and Low Frequency Instruments,\(^\text{30}\) the Sloan SDSS,\(^\text{31}\) the APM\(^\text{32}\) the 2dF\(^\text{32}\) WMAP,\(^\text{34}\) etc.
Fig. 3. Final data release, of 30 June 2003, of the 2dF Galaxy Redshift Survey (Matthew Colless, Steve Maddox, John Peacock, et al., Anglo-Australian Observatory, http://magnum.anu.edu.au/~TDFgg/). The figure shows the map of the galaxy distribution produced from the completed survey.

galaxies, the pixels are brighter. Galaxy clusters, containing hundreds of galaxies closely packed together, are seen as small bright patches. The larger elongated bright areas are superclusters and filaments. These surround darker voids where there are fewer galaxies. The colours are coded according to the apparent magnitude of the galaxies in each pixel: fainter galaxies are shown as red, intermediate are shown as green and bright are show as blue. The more distant galaxies tend to be fainter, and also show less clustering, and so the maps has a generally uniform reddish background. The more nearby galaxies tend to be bright, and are more clustered, so the more prominent clusters of galaxies in the map tend to show up as blue. The small empty patches in the map are regions that we have excluded around bright stars, nearby dwarf galaxies, globular clusters and step wedges. It is very adviceable for the reader to look at this picture in http://www.nottingham.ac.uk/~ppzsjm/apm/apm.html by Steve Maddox, Will Sutherland, George Efstathiou and Jon Loveday.

3.2. The two pillars of observational cosmology, and other tools

The two major pillars of observational cosmology are, undoubtedly:

• the CMB temperature fluctuations, and
the fluctuations of the matter distribution (of galaxies), or density fluctuations.

From a technical viewpoint, observation gained an enormous thrust with the advent of multi-fiber spectrographs, with extremely complex astronomical instrument (as that of the 2dF): from the 5 to 10 redshifts per night that were produced with the old methods not so long ago one has got to the 100 to 2000 that can be taken now under good observation conditions.

3.2.1. Density fluctuations

The density fluctuations, \( \delta \), (generated e.g. by inflation) in a homogeneous universe of mean density \( \bar{\rho} \), are such that:

\[
\rho(x) = \bar{\rho} [1 + \delta(x)], \quad \delta = \frac{\rho - \bar{\rho}}{\bar{\rho}},
\]

where \( \rho(x) \) is the density around a given point, \( x \), and \( \delta(x) \) the corresponding fluctuation around it. In the fluid limit (non-crossing orbits) the evolution of fluctuations is then governed by the basic laws:

- the Friedmann equation,
- the continuity equation, and
- the Poisson equation,

which in Fourier space and in terms of the corresponding Fourier decomposition of the density fluctuations, namely,

\[
\delta(x) = \sum_k \delta_k e^{-ikx},
\]

result in the following fundamental equation:

\[
\frac{d^2 \delta_k}{d\eta^2} + H(\eta) \frac{d\delta_k}{d\eta} - \left[ \frac{3}{2} H(\eta) \Omega_M(\eta) - k^2 v_s^2 \right] \delta_k = 0,
\]

where \( v_s \) is the sound velocity in the fluid \( (v_s^2 = \partial p/\partial \rho) \), \( \eta \) the conformal time \((d\eta = dt/a)\), and

\[
\mathcal{H} \equiv \frac{d \ln a}{d\eta} = aH, \quad \Omega_M(\eta) \equiv \frac{\rho_M(\eta)}{\rho_c} = \frac{\Omega_M a^{-3}}{H^2},
\]

with \( H \) the Hubble constant at present, and similarly \( \Omega_M \).

There are two different regimes encompassing the solutions of the above equation, namely:

- (i) growing fluctuations, when the term in brackets is positive, and
- (ii) amortiguation, when it is negative, separated by the
- (iii) Jeans scale \( \lambda_J \), obtained when the term in brackets vanishes, that is, for

\[
k_J = \frac{2\pi}{\lambda_J}, \quad k_J^2 = \frac{3 \mathcal{H}^2 \Omega_M}{v_s^2}.
\]
3.2.2. Power spectrum

The power spectrum is the mean quadratic value of the fluctuation amplitude $k$ mode, that is

$$P(k) = \langle \delta_k^2 \rangle.$$ (24)

Owing to the fact that quantum fluctuations in the fundamental state are scale invariant, it turns out that the primordial spectrum, which is believed to have been originated by quantum fluctuations, must be scale invariant, too. This translates into a density spectrum of the kind:

$$P_0(k) = \langle \delta_k(0)^2 \rangle = A k^n, \quad n_s \simeq 1,$$ (25)

which is the Harrison-Zeldovich spectrum.

The Jeans instability produces a definite break of this behavior, at about

$$k_{\text{break}} \simeq 0.05 h/\text{Mpc}, \quad \lambda_{\text{break}} \simeq 60 \text{ Mpc}/h,$$ (26)

which is when matter domination starts to take over. Because of the uncertainty about the exact value of the Hubble constant (that is something around 100 km/s Mpc, it is very common to write this uncertainty as a coefficient, $h$, so that the true value is $H = 100 h \text{ km/s Mpc}$, and $h$ is a number somewhere between 0.6 and 0.8. This $h$ appears in many distance estimates, as in the expressions above.

3.2.3. Temperature spectrum

A similar analysis can be repeated for the temperature fluctuations of the CMB. However, the CMB is seen by us as a two-dimensional projection of the surface of last scattering, that is the moment, when the universe was about 300,000 years old, in that it was cool enough in order to suddenly become transparent to radiation. For this reason, it turns out that the Fourier decompositions is here a spherical harmonics decomposition, so that:

$$\delta_T(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} a_{\ell m} Y_{\ell m}(\theta, \varphi), \quad \ell = \frac{\pi}{\theta},$$ (27)

and the temperature power spectrum is given in terms of the coefficients

$$c_\ell = \langle a_{\ell m}^2 \rangle = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{+\ell} |a_{\ell m}|^2,$$ (28)

where the fair-sample hypothesis is used (namely, the mean over realizations equals the spatial mean).

The temperature power spectrum is the most basic reference tool used nowadays for the comparison of different models, in particular, the adjustment of the different peaks of the spectrum (the first being the acoustic peak). In the two following figures we see, respectively, an improved measurement of the angular power spectrum of temperature anisotropy in the CMB obtained from two recent analyses of
Fig. 4. A comparison of two recent analyses of observations with BOOMERanG. The agreement is generally very good, with the greatest variations at high $\ell$ where noise, rather than cosmic variance, dominates the errors. J.E. Ruhl, et al., Astrophys. J. 599, 786 (2003), Fig. 7.

observations with BOOMERanG (Fig. 4), and the one measured by Archeops (Fig. 5), which is conceived as a precursor of the PLANCK HFI instrument, using the same optical design and technology.

3.2.4. The two-point correlation function

Another tool impossible to dismiss was the definition of the two-point correlation function, by Totsuji and Kihara in 1969 as the excess function, $\xi(r)$, appearing in the following expression for the number of galaxies to be found in a volume $dV$ around the position $r$:

$$dP = n [1 + \xi(r)] dV,$$

(29)

$n$ being the average galaxy number density. The following scaling law is basic: the number galaxies (points) to be found in a sphere of radius $a$ behaves as a certain
power of $a$, for $a$ large enough, namely

$$N_{c,a} = n \int_0^a 4\pi r^2 [1 + \xi(r)] \, dr \propto a^{d_2}.$$  

(30)

In sky surveys, the value of $d_2$ has been seen to evolve from $d_2 = 2$ to $d_2 = 3$, when one goes from intermediate to larger (that is, $a > 30h^{-1}$ Mpc) scales.

Now, recall the behavior of the energy density, for large distance $a$, as a function of $a$, for a matter dominated and a radiation dominated universe, namely

$$\rho(a) \sim \rho_0 a^{-s} \quad \left\{ \begin{array}{ll}
  s = 3, & \text{matter dominated}, \\
  s = 4, & \text{radiation dominated}.
\end{array} \right.$$  

(31)

And from Friedmann’s equation

$$\frac{\dot{a}}{a} = \frac{8}{3} \pi G \rho_0 a^{-s} - \frac{k}{a^2},$$  

(32)

by taking an additional derivative, we obtain

$$\frac{\ddot{a}}{a} = -(s - 2) \frac{4}{3} \pi G \rho(a),$$  

(33)
which (remarkably) it is independent of $k$, and tells us at once that the expansion of the universe is decelerated by gravity. Also, that since we do not know of any form of energy with a behavior corresponding to a power $s < 2$, let aside of course from the cosmological constant, which has $s = 0$:

$$\rho_\Lambda = \frac{\Lambda}{8\pi G},$$

it turns out that the only possible causes of accelerated expansion can be the said cosmological constant or either a (still unknown) form of exotic matter or energy. In the case of the cosmological constant, $\Lambda$, it could come from $T_{\mu\nu}$ as a sort of vacuum contribution, so that

$$\rho(a) = \rho_{M,\text{now}} a^{-3} + \rho_{R,\text{now}} a^{-4} + \rho_\Lambda,$$

where $\text{now}$ refers to $a = 1$. Two kinds of different solutions to this puzzle have been proposed:

(i) to modify the curvature or geometric part of Einstein’s equations with the addition of terms of the sort $1/R^2$, $\ln R$, or other convenient functions of the curvature, $f(R)$, which sometimes produced the so-called phantom matter (sometimes giving rise to a new, future singularity, the big RIP, see, e.g., Ref. [39] and the references therein) or, alternatively,

(ii) to modify the matter part of the equations with the addition of true exotic matter or energy (see, e.g., Ref. [40] and the references therein).

### 3.3. A summary of cosmological facts and important effects

#### 3.3.1. The Olbers’ paradox (1757-1840)

Should be known to everybody but, just in case ... By looking at the sky, Olbers came to the thought that if the Universe were infinitely old and infinite in extent and stars could shine forever, then every direction you looked into would eventually end on the surface of a star: the whole sky should be as bright as the surface of the Sun.

Absorption by interstellar dust does not circumvent this paradox, since dust re-radiates whatever radiation it absorbs within a few minutes.

**Solution.** The Universe is not infinitely old and its expansion reduces the accumulated energy radiated by distant stars. Either one of these effects acting alone already solves the paradox.

#### 3.3.2. The Sunyaev-Zeldovich effect

Hot gas in clusters of galaxies distorts the spectrum of the CMB radiation. Hot electrons there scatter a small fraction of the CMB photons and replace them with slightly higher energy photons. That is, high energy electrons in the clusters and the homogeneity and isotropy of the CMB results in the CMB photons gaining
energy by Compton scattering. The difference between the CMB seen through the cluster and the unmodified CMB can be measured.

The effect, first described by R. Sunyaev and Ya.B. Zeldovich, is observed as a deficit of about 0.05% of CMBR photons, as the “missing” photons have been shifted to higher energy, with an increase of about 2%. Thus, the CMB radiation denounces the presence of galaxy clusters found in its way towards us.

This effect verifies the cosmological origin of the CMB. Moreover, combining radio with X-ray observations of cluster allows to determine:

- the distance of the cluster;
- the value of the Hubble constant $H_0$; and
- for very distant clusters, the value of the deceleration parameter.

3.3.3. The Sachs-Wolfe effect

Photons are subject to the influence of gravity (a GR effect). When passing through a higher (resp. lower) concentration of matter, they undergo a redshift (resp. a blueshift). R.K. Sachs and A.M. Wolfe where the first to realize this sort of effect should take place, under the form of perturbations of a cosmological model and angular variations of the cosmic microwave background. They also described an integrated form of the effect.

*The integrated Sachs-Wolfe effect.* Consists on the gravitational redshift induced by photons falling into and climbing out of regions of space with different matter density (potential wells), between the Earth and the surface of last scattering. In contrast, the non-integrated Sachs-Wolfe effect is only at the surface of last scattering itself.

3.3.4. The Lyman Alpha forest

Some spectacular pictures of the universe show that we actually live in a forest, made of “trees” of hydrogen gas which absorb light from distant objects. It leaves numerous absorption lines in a distant quasar’s spectra; the Ly-$\alpha$ forest. Remember that Lyman-alpha is the spectral line at 1216 Å, in the far ultraviolet, that corresponds to the transition of an electron between the two lowest energy levels of a hydrogen atom (with the rest of the transitions giving rise to the whole Lyman series).

Distant quasars get absorbed by many more clouds than nearby quasars. Quasars also emit a strong Lyman-alpha emission line. But the absorbing clouds all have smaller redshifts than the quasar since they have smaller distances. As a result the absorption lines are all on the blue or shorter wavelength side of the quasar emission line.
3.3.5. *The Tully-Fisher relation*

This is a promising distance determination technique (widely used in the 70’s and 80’s). Once improved, it is now one of the more accurate secondary distance indicators for the Universe. It relies on the relationship between the rate at which a spiral galaxy spins and its intrinsic luminosity: the faster a galaxy spins, the more luminous it is. Motion will cause a narrow line, e.g., a line due to some element like hydrogen, to be smeared out and to appear broad to the external observer.

The broader the line, the faster the galaxy is spinning. Centrifugal force and gravitational force are in balance

\[ \frac{v^2}{R} - \frac{GM_{\text{galaxy}}}{R^2} = 0. \]  

(36)

And this leads to a spin rate of the type:

\[ M_{\text{galaxy}} = \frac{v^2R}{G}. \]  

(37)

3.3.6. *The Hertzsprung-Russell diagram*

The HR diagram is a plot of all known stars, where on the horizontal axis one puts the star surface temperature (decreasing to the right), and on the vertical one the star luminosity. In the early 1900’s, Ejnar Herstzprung and Henry Norris Russell independently made the discovery that the luminosity of a star is related to its surface temperature. When luminosity versus temperature plots are made, stars do not fall randomly on the graph; rather they are confined to specific regions.

This tells you that there is some physical relationship between the luminosity and temperature of a star. Herstzprung and Russell made groupings of stars in the diagrams, according to the names: Main Sequence, Giants, Super-Giants, and White Dwarfs (these groups are referred to as *luminosity classes*). Stars spend most of their lives as main sequence stars.

The HR Diagram may be partially understood in terms of the luminosity for an object emitting thermal radiation:

\[ L \sim R^2T^4. \]  

(38)

If all objects in the HR diagram were the same size then all objects would lie along a diagonal line of slope = 2 in this logarithmic plot. Schematically, stars fall into regions above and below with respect to this line, which is actually deformed into a curved one containing the main sequence.

3.3.7. *The Hubble law*

States that radial velocities of galaxies are proportional to their distance (a Doppler shift caused by the cosmic expansion, Fig. 6). Namely:

\[ v = \frac{dD}{dt} = HD, \quad v = cz + \cdots, \]  

(39)
Fig. 6. A modern determination of Hubble’s law, by a empirical method that uses multicolor light curve shapes (MLCS) to estimate the luminosity, distance, and total line-of-sight extinction of Type Ia supernovae (SN Ia). A. Riess, W. Press, and R. Kirshner, Astrophys. J. 473, 88 (1996).

being the redshift

$$1 + z = \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \sqrt{\frac{1 + v/c}{1 - v/c}}$$

(40)
The value for the Hubble constant at present, obtained from observations by WMAP is

$$H_0 = 71 \pm 3.5 \text{ km/sec/Mpc}. \quad (41)$$

One should note that the Hubble law stated as $v = H D$, that is, as true for all values of $D$, even very large ones ($v > c$), must be modified in a curved spacetime, by establishing a chain of patches in each of which it is indeed applicable. In the way:

$$D_{\text{now}} = D_{\text{us to Z}} = D_{\text{us to A}} + D_{\text{A to B}} + \cdots + D_{X \text{ to Y}} + D_{Y \text{ to Z}}. \quad (42)$$

The relation between the Hubble law distance $D_{\text{now}}$, the velocity $v$ and the redshift $z$ is:

$$v = H_0 D_{\text{now}}, \quad D_{\text{now}} = \frac{c}{H_0} \ln(1 + z), \quad 1 + z = \exp(v/c). \quad (43)$$

While the Hubble law distance is (in principle) measurable, the need for helpers all along the chain of galaxies out to a distant galaxy (as described above) makes its use impractical.

### 3.3.8. Measurements of the distance

Other distances are measured more easily. One of them is

- **The angular size distance:**
  
  $$\theta = \frac{\text{size}}{D_A}, \quad \text{that is} \quad D_A = \frac{\text{size}}{\theta}, \quad (44)$$

  being here the size equal to the transverse extent of the object, and $\theta$ the angle (in radians) that it subtends on the sky.

- **The luminosity distance**, $D_L$, is defined through:

  $$\text{Flux} = \frac{\text{luminosity}}{4\pi D_L^2}. \quad (45)$$

  And the *light travel time*, by:

  $$D_{\text{ltt}} = c(t_0 - t_{em}). \quad (46)$$

- **Use of the Tully-Fisher relation.** This relation has been discussed before. The rotational velocity of a spiral galaxy is an indicator of its luminosity:

  $$L = \text{Const} \cdot V_{\text{rot}}^4. \quad (47)$$

  The rotational velocity is measured using an optical spectrograph or radio telescopes and thus gives us the luminosity. Combined then with the measured flux, this luminosity gives the distance.

  Let us consider two galaxies: a giant spiral and a dwarf spiral. Say the small galaxy is closer to the Earth, so that both cover the same angle on the sky (i.e., both have the same apparent brightness). But we observe that the distant galaxy...
E. Elizalde has greater rotational velocity, since the difference between the redshifted and the blueshifted sides is larger. Using this information, the relative distance of the two galaxies can be determined.

- The Faber-Jackson relation. The stellar velocity dispersion, \( s(v) \), of the stars in an elliptical galaxy is an indicator of its luminosity. In fact:

\[
L = \text{Const} \ s(v)^4.
\]

The velocity dispersion is measured using an optical spectrograph, and this gives us the luminosity. Combined with the measured flux, this luminosity gives the distance.

- Gravitational lens time delay. This gives also a measure of the relative distance. A quasar viewed through a gravitational lens turns into multiple images. But these light paths from the quasar to us have different lengths, that differ by

\[
D[\cos(q_1) - \cos(q_2)],
\]

where \( q \) is the deflection angle, and \( D \) the distance to the quasar.

And, since quasars are time variable sources, we can measure the path length difference by looking for time-shifted correlated variability in multiple images.

3.3.9. Age of the Universe

- Age of the chemical elements. Rubidium and strontium are usually found in rocks. But \( \text{Rb}^{87} \) decays into \( \text{Sr}^{87} \) (radiogenic) with a half-life of 47 billion years. \( \text{Sr}^{86} \) is not produced by any rubidium decay (non-radiogenic), so that it can be used to determine what fraction of \( \text{Sr}^{87} \) was produced by radioactive decay, by plotting the \( \text{Sr}^{87}/\text{Sr}^{86} \) ratio versus the \( \text{Rb}^{87}/\text{Sr}^{86} \) ratio.

- Earth. The oldest rocks are about 3.8 billion years old. The oldest meteorites are dated to be 4.56 billion years old.

- Radioactive dating of old stars. From the Thorium abundance in an old halo star: the \( \text{Th}/\text{Eu} \) (Europium) ratio in such star is 0.219 compared to 0.369 in the Solar System now. \( \text{Th} \) decays with half-life of 14.05 Gyr. This gives 15.6 ± 4.6 Gyr for the age, based on two stars (CS 22892-052 and HD 115444).

- Age of the older star clusters. When stars are burning hydrogen to helium in their cores, they fall on a single curve in the luminosity-temperature plot (the HR diagram, already discussed). From analysis of this behavior an age for our Universe greater than 12.07 Gyr, with 95% confidence, has been found.

- Ages of white dwarfs. In the globular cluster M4, it has been found to be 12.7 ± 0.7 Gyr. In 2004 Hansen et al. updated their analysis to find an age for M4 of 12.1 ± 0.9 Gyr, which is very consistent with the age of globular clusters from the main sequence.

- Age of the Universe. The current best value is 13.3 ± 0.2 billion years, from WMAP, but has been steadily increasing lately. Note however that the comoving radius of the universe is about 40 billion light-years (about a factor of 3 bigger).
3.3.10. The Universe is homogeneous and isotropic

The reader should not confuse homogeneity and isotropy. The pattern of a red brick wall (like the beautiful ones in Boston’s Beacon Hill) is an homogeneous but not isotropic one. On the contrary, the pattern of light rays emitted in any direction by a shining light in darkness is isotropic but not homogeneous.

Direct evidence for statistical homogeneity in the distribution of matter at sufficiently large scales came from the first accurate measurement of the galaxy two-point correlation function. Totsuji and Kihara (1969) solved this long-standing problem. In their own words: “The correlation function for the spatial distribution of galaxies in the universe is determined to be \((r_0/r)^1.8\), \(r\) being the distance between galaxies. The characteristic length \(r_0\) is 4.7 Mpc. This determination is based on the distribution of galaxies brighter than the apparent magnitude 19 counted by Shane and Wirtanen (1967). The reason why the correlation function has the form of the inverse power of \(r\) is that the universe is in a state of ‘neutral’ stability.” Deep physical insight into the gravitational many-body problem —usually a good way to short-circuit complicated mathematical formalism— led Totsuji and Kihara to their conclusion. Previous guesses at an exponential or Gaussian form for the correlation had been intensively discussed. With the new results, galaxy clustering could be considered to be a phase transition from a Poisson distribution to a correlated distribution, slowly developing on larger and larger scales as the universe expands.

The isotropic and homogeneous Universe case became much stronger after Penzias and Wilson announced the discovery of the Cosmic Microwave Background in 1965. In fact, as we now know, the deviations from homogeneity in the CMB radiation are of about a part in \(10^5\), what makes it too homogeneous and creates a severe problem when one wants to find some inhomogeneities, to serve as seeds for star and galaxy formation. Beautiful maps of the whole universe showing the temperature fluctuations of the CMB have been produced by WMAP (Fig. 7).

It is not easy to draw these maps. A lot of effects must be accounted for and conveniently subtracted from the image: the dipole contribution, the galaxy plane contribution, observational biases, etc.). The figure shows three different stages of the production of a final map (the cleaned one, top of Fig. 7).

The following plot (Fig. 8) shows that our universe approaches homogeneity (as measured now from the matter distribution) as big enough regions of the same are considered, of about or larger than 100 Mpc.

3.4. Short summary of inflation

Basic to cosmological observations, as those that already lead to the Big Bang model, is the consideration of a scale factor, \(a(t)\), to be taken e.g. as the distance between any pair of comoving objects (e.g. two distant galaxies), or even the curvature of the universe itself, if it is non-vanishing. The scale factor grows by an amount \(1 + Hdt\) during a time interval \(dt\), that is:

\[
D_G(t) = a(t)D_G(t_0),
\]  
(50)
Fig. 7. The linearly cleaned WMAP map (top), a Wiener filtered map (middle) and the raw map (bottom). All maps are shown in Mollweide projection in galactic coordinates with the galactic center \((l, b) = (0, 0)\) in the middle and galactic longitude \(l\) increasing to the left. M. Tegmark, A. de Oliveira-Costa, and A. Hamilton, Phys. Rev. D68, 123523 (2003).
with $D_G(t_0)$ being the distance to the galaxy $G$ right now, and $a(t)$ a universal scale factor that applies to all comoving objects. This law had to be changed for the description of the very beginning of the cosmos, owing to the serious problems of the original Big Bang model.

A. Starobinsky and A. Guth offered a solution to the flatness-oldness problem and to the horizon (or causality) problem of the old Big Bang theory, that was absolutely unable to explain them (together with some other, as the present absence of magnetic monopoles). In 1980, Alan Guth proposed a modification to the Big Bang theory, by suggesting that in the first moments of its life our universe inflated as if it had been the soapy membrane of a small bubble, that become gigantic in a small fraction of a second. Inflation is in fact a modification of the conventional Big Bang theory, proposing that the expansion of the universe was propelled by a repulsive gravitational force generated by an exotic form of matter. Although Guth’s initial proposal was flawed, this was soon overcome by the invention of “new inflation,” by Andrei Linde and independently by Andreas Albrecht and Paul
E. Elizalde

Steinhardt. “The bang was there, but it was not big,” said at some occasion A. Linde.49

Nowadays no self-respecting theory of the Universe is complete without a reference to inflation. But there is in the meantime such a large variety of versions that it would be impossible here to provide a minimally consistent account even of the basic ideas to encompass all them, thus the reader is referred to the excellent bibliography by the creators of the theory.50

Inflationary theory does not replace Big Bang theory, but adds an extra stage: before the Big Bang, the universe went through a period of extremely rapid expansion, growing by 30 orders of magnitude in a fraction of a second. It is difficult to imagine something becoming this large this quickly. In the words of Guth: “To picture a pea expanding to the size of the Milky Way more quickly than the blink of an eye.” When he came up with the theory of cosmic inflation, Guth was a 34-year old physicist at the Stanford Linear Accelerator Center, in the ninth year of a seemingly interminable career as a postdoctoral fellow. He was working on the problem of magnetic monopoles: the Big Bang model predicts an abundance of magnetic monopoles but none have ever been found.

A few years earlier, Linde had suggested that, in its early stages, the universe had undergone a series of phase transitions, accompanied by supercooling. Supercooling is seen quite often in phase transitions from one form of matter to another, such as water cooling to ice. In supercooling, water will remain liquid as it cools below 0, but at the slightest disturbance it will immediately freeze. Guth and Tye were working on the problem of how supercooling in the early universe would affect the production of magnetic monopoles. “So I went home one night and did that calculation and discovered that it would have a dramatic effect on the evolution of the universe,” Guth said once.13 The supercooled matter would cause gravity to reverse direction, so that objects would repel each other, resulting in exponential inflation. This would also make magnetic monopoles exceedingly rare. The impact of the theory was immediate.

One major puzzle solved by inflation is the fact that it explains the extreme homogeneity and isotropy of the universe, as observed by COBE and now with much greater precision by WMAP and several balloons. This is a highly improbable state viewed from Big Bang theory. In the inflationary scenario, however, stretching out a tiny, uniform universe exponentially results in a similarly uniform larger universe. Inflation also explains why parallel lines don’t cross — something everyone learns in school as a basic principle of Euclidean geometry. But other types of geometry are possible. The density of the universe determines whether it is open or closed. Theoretical calculations show that a universe coming from the usual Big Bang should be very curved, whereas scientific observations show the universe as flat and Euclidean. This “flatness” problem is also solved by inflation. For some interesting reference books see e.g. Ref. 51

One of the intriguing consequences of inflation is that quantum fluctuations in the early universe can be stretched to astronomical proportions, providing the
seeds for the large scale structure of the universe. The predicted spectrum of these fluctuations was calculated in 1982. One thinks of vacuum as empty and massless (with a density \(< 10^{-30}\) g/cc). Now, as we know from Quantum Field Theory (QFT), the vacuum is not empty but filled with virtual particles. These quantum fluctuations, once enormously enlarged by inflation, can be seen today as ripples in the cosmic background radiation, but the amplitude of these faint ripples is only about one part in \(10^5\). Nonetheless, these ripples were detected by the COBE satellite in 1992, and they have now been measured to much higher precision by the WMAP satellite and several balloons (like MAXIMA, DASI and BOOMERanG).

The properties of the radiation are found to be in excellent agreement with the predictions of the simplest models of inflation. Also, according to Guth and Farhi, with quantum tunneling it might be theoretically possible to ignite inflation in a hypothetical laboratory, thereby creating a new universe. The new universe, if it can be created, would not endanger our own universe. Instead it would slip through a wormhole and rapidly disconnect completely. And yet another intriguing feature of inflation is that almost all versions of inflation are eternal: once inflation starts, it never stops completely. Inflation has ended in our part of the universe, but very far away one expects that inflation is continuing, and will continue forever. Is it possible, then, that inflation is also eternal into the past? Recently Guth, Vilenkin and Bore 53 have shown that the inflating region of spacetime must have a past boundary, and that some new physics, perhaps a quantum theory of creation, would be needed to understand it.

The increasing precision of cosmological data sets is opening up new opportunities to test predictions from cosmic inflation. The impact of high precision constraints on the primordial power spectrum is expected to be important and the new generation of observations could provide real tests of the slow-roll inflation paradigm, as well as produce significant discriminating power among different slow-roll models. In particular, proposed next-generation measurements of the CMB temperature anisotropies, and specially polarization, as well as new Lyman-\(\alpha\) measurements could become practical in the near future. Relationships between the slope of the power spectrum and its first derivative are nearly universal among existing slow-roll inflationary models, and therefore these relationships can be tested on several scales with new observations. Among other things, this provides additional motivation for the measure of CMB polarization, to be accomplished with the PLANCK mission, in which our group is participating.

3.5. On the topology and curvature of space

The Friedmann-Robertson-Walker (FRW) model, which can be derived as the only family of solutions to the Einstein’s equations compatible with the assumptions of homogeneity and isotropy of space, is the generally accepted model of the cosmos (more details later in these lectures). But, as we surely know, the FRW is a family with a free parameter, \(k\), the curvature, that can be either positive, negative or zero
(the flat or Euclidean case). This curvature, or equivalently the curvature radius, $R$, is not fixed by the theory and should be matched with cosmological observations. Moreover, the FRW model, and Einstein’s equations themselves, can only provide local properties, not global ones, so they cannot tell about the overall topology of our world: is it closed or open? finite or infinite? Even being quite clear that it is, in any case, extremely large —and possibly the human species will never reach more than an infinitesimally tiny part of it— the question is very appealing to any of us. Note that all this discussion concerns only three dimensional space curvature and topology, time will not be involved.

3.5.1. On the curvature

Serious attempts to measure the possible curvature of the space we live in go back to Gauss, who measured the sum of the three angles of a big triangle with vertices on the picks of three far away mountains (Brocken, Inselberg, and Hohenhagen). He was looking for evidence that the geometry of space is non-Euclidean. The idea was brilliant, but condemned to failure: one needs a much bigger triangle to try to find the possible non-zero curvature of space. Now cosmologist have recently measured the curvature radius $R$ by using the largest triangle available, namely one with us at one vertex and with the other two on the hot opaque surface of the ionized hydrogen that delimits our visible universe and emits the CMB radiation (some 3 to 4 $\times 10^5$ years after the Big Bang). The CMB maps exhibit hot and cold spots (see Fig. 7). It can be shown that the characteristic spot angular size corresponds to the first peak of the temperature power spectrum, which is reached for an angular size of $5^\circ$ (approximately the one subtended by the Moon) if space is flat. If it has a positive curvature, spots should be larger (with a corresponding displacement of the position of the peak), and correspondingly smaller for negative curvature.

The joint analysis of the considerable amount of data obtained during the last years by the balloon experiments (BOOMERanG, MAXIMA, DASI), combined also with galaxy clustering data, have produced a lower bound for $|R| > 20h^{-1}$Gpc, that is, twice as large as the radius of the observable universe, of about $R_U \simeq 9h^{-1}$Gpc.

3.5.2. On the topology

Let us repeat that GR does not prescribe the topology of the universe, or its being finite or not, and the universe could perfectly be flat and finite. The simplest non-trivial model from the theoretical viewpoint is the toroidal topology (that of a tyre or a donut, but in one dimension more). This topology has been studied in depth by the author (more about that will come in the last lecture). Traces for the toroidal topology and more elaborated ones, as negatively curved but compact spaces, have been profusely investigated, and some circles in the sky with near identical temperature patterns were identified. And yet more papers appear from time to time proposing a new topology. However, to summarize all these efforts
and the observational situation, and once the numerical data are interpreted without bias (what sometimes was not the case, and led to erroneous conclusions), it seems at present that available data point towards a very large (we may call it infinite) flat space.

3.6. Expansion around a given probability density function

This section will be rather more technical. It is the starting point of some original work that we produced aimed at the study of matter and temperature fluctuations.

For any two probability density functions (pdfs), $f^{(1)}(x)$ and $f^{(2)}(x)$, with cumulants $\kappa^{(1)}_J$ and $\kappa^{(2)}_J$, it turns out that we can always write one of the pdfs in terms of the other one. In fact, the explicit formula is:

$$f^{(1)}(x) = \exp \left[ \sum_{J=0}^{\infty} (-1)^J \frac{\kappa^{(1)}_J - \kappa^{(2)}_J}{J!} \frac{d^J}{dx^J} \right] f^{(2)}(x).$$ (51)

Remember the definition of the cumulants, from the generating function

$$\ln \Phi(t) = \sum_{J=0}^{\infty} \frac{\kappa_J}{J!} (it)^J,$$ (52)

it turns out that the cumulants (or connected moments) are immediately related with the usual central moments of the distribution:

$$\kappa_1 = m, \quad \kappa_2 = \sigma^2, \quad \kappa_3 = \mu_3, \quad \kappa_4 = \mu_4 - 3\mu_2^2, \ldots$$ (53)

The proof of Eq. (51) is simple and I’ll just give it as a chain of equalities that the reader may check with some care.

**Proof:**

$$\Phi^{(1)}(t) = \langle e^{itx} \rangle = \int_{-\infty}^{\infty} e^{itx} f^{(1)}(x) \, dx = \int_{-\infty}^{\infty} e^{itx} \exp[ \sum_{J=0}^{\infty} \kappa^{(2)}_J (it)^J ] f^{(2)}(x) \, dx$$

$$= \int_{-\infty}^{\infty} f^{(2)}(x) \exp \left[ \sum_{J=0}^{\infty} \frac{\kappa^{(1)}_J - \kappa^{(2)}_J}{J!} (it)^J \right] e^{itx} \, dx$$

$$= \exp \left[ \sum_{J=0}^{\infty} \frac{\kappa^{(1)}_J - \kappa^{(2)}_J}{J!} (it)^J \right] \int_{-\infty}^{\infty} f^{(2)}(x) \, e^{itx} \, dx$$

$$= \exp \left[ \sum_{J=0}^{\infty} \frac{\kappa^{(1)}_J - \kappa^{(2)}_J}{J!} (it)^J \right]$$

$$= \exp \left[ \sum_{J=0}^{\infty} \frac{\kappa^{(1)}_J}{J!} (it)^J \right].$$
Fluctuations of the density field

They have been described before:

\[ \delta = \frac{\rho - \bar{\rho}}{\bar{\rho}} = \frac{\rho}{\bar{\rho}} - 1, \]  

(54)

where the values are taken smoothed over some fixed scale, \( R \). Here \( \rho \) is the value of the density field, and we have that

\[ 0 \leq \rho < +\infty, \quad -1 \leq \delta < +\infty. \]  

(55)

The fluctuation \( \delta = \delta(\vec{r}) \) is a stochastic field (\( \vec{r} \) position) while \( p(\delta) \) is the one-dim probability density function of this random variable. What is very important to note here is the fact that the density field fluctuation, \( \delta \) is a random variable which is bounded below by the value \(-1\) (and thus highly asymmetric), together with the fact that, on the other hand, the Gaussian pdf corresponds to a stochastic variable which is unbounded and symmetric.

Now the main problem is here to:

- Recover the full shape of \( p(\delta) \) from a few first order moments

where we understand that it is the optimal solution to this question the one that is most interesting: using the minimal number of moments we want to obtain the best possible representation of the pdf.

Its solution is in fact not unique. Several approaches haven been historically considered:

- The **Zeldovich approximation**, which
  - derives \( p(\delta) \) from analytic approximations;
  - reproduces important aspects of the non-linear dynamics, but it yields a poor approximation for the pdf and the moments.

- The **improved Zeldovich approximation**, which
  - is an exact non-linear perturbation theory for the moments;
  - is used for deriving the pdf from the Edgeworth expansion; and
  - has an accuracy proportional to the order of the cumulants involved.

- The **Edgeworth expansion**, which is obtained in terms of Gaussians and has been used to deal with matter pdfs and also with CMB fluctuations. Other uses of the Edgeworth expansion can be found in biology, economy and finance, and mathematical statistics. But it has some shortcomings, in practice, as the appearance of negative probability values, in particular it assigns \( \neq 0 \) probability to \( \delta < -1 \), as a consequence of the fact that a Gaussian pdf only makes sense for \( \sigma < 1 \).

- Alternative expansions are:
  - The **Gamma (or Pearson type-III)** pdf.
  - Other distributions.
3.8. Saddle point approximation and the Edgeworth expansion

Gravitational clustering from Gaussian initial conditions predicts that

$$<\delta_<^J>_c < \delta^2>_c^{J-1},$$

(56)
on large-scales (where the subindex $c$ indicates ‘connected’ terms, which correspond to the cumulants), thus the following ratios are most interesting, and widely used in cosmological analysis:

$$S_J \equiv \frac{\kappa_J}{\kappa_2} = \frac{<\delta_<^J>_c}{<\delta^2>_c^{J-1}}.$$  (57)

They are called hierarchical coefficients, being $S_3$ the Skewness and $S_4$ the Kurtosis.

To obtain an asymptotic expansion of $p(\delta)$ for small $\delta$, the Legendre transform is most appropriate:

$$\bar{\delta} \equiv \frac{d \psi(t)}{dt}, \quad G(\bar{\delta}) = \bar{\delta} t - \psi(t),$$

(58)

$$p(\delta) = \int_{\bar{\delta}=-\infty}^{\bar{\delta}=+\infty} \frac{G''(\delta)}{2\pi} \exp[-\delta G'(\delta) + \bar{\delta} G(\delta)] d\bar{\delta}.$$  (59)

It is dominated by stationary points of the exponential at $\delta = \bar{\delta}$. The saddle point approximation yields:

$$p(\delta) \approx \left[\frac{G''(\delta)}{2\pi}\right]^{1/2} \exp[-G(\delta)] \int_{-\infty}^{+\infty} \left[\frac{G''(\delta)}{2\pi}\right]^{1/2} \exp[-G(\delta)] d\delta.$$  (60)

From the generating function, $\psi(t)$,

$$\psi(t) = \sum_{n=2}^{\infty} \frac{\mu_n}{n!} t^n = \frac{1}{2} \sigma^2 t^2 + \frac{1}{6} S_3 \sigma^4 t^3 + \frac{1}{24} S_4 \sigma^6 t^4 + \cdots,$$  (61)

one readily has

$$G(\delta) \approx \left[\frac{1}{2} \delta^2 - \frac{S_3}{6} \delta^3 + \frac{1}{8} \left(\frac{S_3^2}{3} - \frac{S_4}{24}\right) \delta^4 + \mathcal{O}(\delta^5)\right] \sigma^{-2},$$  (62)

and finally

$$p(\nu) \approx \left\{1 + \frac{S_3}{3!} H_3(\nu) \sigma + \left[\frac{1}{4!} S_4 H_4(\nu) + \frac{10}{6!} S_3^2 H_6(\nu) \sigma^2\right] p_G(\nu) + \mathcal{O}\left(\sigma^3\right)\right\},$$  (63)

the $H_n(\nu)$ being Hermite’s polynomials:

$$H_3(\nu) = \nu^3 - 3\nu,$$
$$H_4(\nu) = \nu^4 - 6\nu^2 + 3,$$
$$H_5(\nu) = \nu^5 - 10\nu^3 + 15\nu,$$
$$H_6(\nu) = \nu^6 - 15\nu^4 + 45\nu^2 - 15,$$
$$H_7(\nu) = \nu^7 - 21\nu^5 + 105\nu^3 - 105\nu.$$
$H_3(\nu) = \nu^9 - 36\nu^7 + 378\nu^5 - 1260\nu^3 + 945\nu,$

(64)

This is the well-known (perturbative) Edgeworth series of a pdf up to third order. Higher orders in the Edgeworth series can be obtained by keeping higher orders in the Taylor expansion.

3.9. The Gamma —negative binomial or Pearson Type III (PT3)— distribution

It arises from the chi-square distribution with $N$ degrees of freedom when $1/\sigma^2 = N/2$ is taken to be a continuous parameter

$$
\phi(\delta) = \frac{(1 + \delta)^{\sigma^{-2} - 1}}{\sigma^{2\sigma} \Gamma(\sigma^{-2})} \exp\left(-\frac{1 + \delta}{\sigma^2}\right).
$$

(65)

The hierarchical coefficients are constant: $S_J = (J - 1)!$ Note that the variable is in this case bounded from below, and that we can adjust the parameters in order that the bound be placed at $-1$. Using now the general result obtained before, we can easily get an expansion of any pdf in terms of the Gamma distribution, as follows

$$
p(\mu) \equiv \sum_{n=0}^{\infty} c_n L_n^{(p-1)}(\mu) \phi(\mu),
$$

(66)

with coefficients

$$
c_n = \frac{n! \Gamma(p)}{\Gamma(n + p)} \int_0^{\infty} p(\mu)L_n^{(p-1)}(\mu) d\mu,
$$

(67)

being

$$
\phi(\mu) d\mu = \frac{1}{\Gamma(p)} \mu^{p-1} e^{-\mu} d\mu, \quad \mu = \frac{x - \alpha}{\beta} \geq 0.
$$

(68)

This is a three-parameter ($p, \alpha$ and $\beta$) family of distributions out of which only one, $p$, is relevant for normalized variables (such as the density fluctuations, $\delta$). Now everything is written in terms of the generalized Laguerre polynomials

$$
L_n^{(p-1)}(\mu) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+p-1}{n-k} \mu^k,
$$

(69)

which are, in particular,

$$
\begin{align*}
L_1^{(p-1)}(\mu) & = p - \mu, \\
L_2^{(p-1)}(\mu) & = \frac{p(p + 1)}{2} - (p + 1)\mu + \frac{\mu^2}{2}, \\
L_3^{(p-1)}(\mu) & = \frac{p(p + 1)(p + 2)}{6} - \frac{(p + 1)(p + 2)}{2} \mu + \frac{p + 2}{2} \mu^2 - \frac{\mu^3}{6}.
\end{align*}
$$

(70)
The coefficients \( c_n \) are found to be

\[
\begin{align*}
    c_0 &= 1, \quad c_1 = c_2 = 0, \\
    c_3 &= -\frac{\Gamma(p + 1)}{(p + 3)}(S_3 - 2!), \\
    c_4 &= \frac{\Gamma(p + 1)}{(p + 4)}[(S_4 - 3!) - 12(S_3 - 2!)], \\
    c_5 &= \frac{\Gamma(p + 1)}{(p + 5)}[-(S_5 - 4!) + 20(S_4 - 3!) - 120(S_3 - 2!)], \\
    \vdots \\
    c_n &= \frac{\Gamma(p + 1)}{(p + n)} \sum_{k=3}^{n} \left\{ (-1)^k a_{n,k} [S_k - (k - 1)!] ight. \\
    & \quad \left. + b_{n,k} [S_k - (k - 1)!]^2 + \cdots \right\} 
\end{align*}
\]

(71)

For a consistency check, it is clear that for \( S_k = (k - 1)! \) we recover the Gamma

\[
pdf: \quad p(\mu) = \phi(\mu).
\]

Finally

\[
p(\nu) = \left\{ 1 + \sum_{n=3}^{\infty} a^{n-2} F_n \right\}
\]
Let us now compare the Gamma and the Edgeworth expansions:

- The Gamma expansion recovers all the terms that appear in the Edgeworth expansion plus some corrective terms.
- The Gamma expansion has exponential tails and a better general behaviour than the Edgeworth expansion, both on the positivity of $p(\nu)$ and the variate itself, $\rho$ (Fig. 9).
- The behavior at the pick is also improved, as a detailed analysis shows (Fig. 10).

Fig. 10. Comparison of gravitational simulations with the second order Edgeworth and Gamma PDF expansions as a function of $\nu \equiv \delta/\sigma$. We use as parameters the measured values of $\sigma$, $S_3$ and $S_4$ as labeled in the Figure. The dotted, dashed and continuous lines correspond to the Gaussian, Gamma and Edgeworth distribution expansions. The inset shows a detail around the peak in linear scale.
3.10. Non-Gaussian bounds from CMB observations

A basic ingredient in order to understand the formation of large scale structures in our Universe is the distribution of initial conditions: Have fluctuations been generated in the standard inflationary epoch? or Do they require topological defects or more exotic assumptions? Generically, the first possibility leads to a Gaussian distribution, while the second leads to non-Gaussianities.

This issue can be addressed through:

- Present day Universe fluctuations traced by the galaxy distribution.
- The presence of anisotropies in the cosmic microwave background (CMB).

An important contribution to the uncertainties comes from the sample variance (i.e., the finite size of the observational sample). In fact a non-Gaussian signal can produce different sampling errors.

Now the problem is to:

- Place bounds on the degree of non-Gaussianity.

And the strategy to solve it consists in finding as many independent results as possible in order to have a large sampling over the underlying distribution, that is specifically, to study the sample variance of CMB experiments over independent sky regions or subsamples, and also to perform a Chi-square analysis taking different number of points and criteria.

In our pioneering analysis, performed some time ago, we started with a very few tests, namely Saskatoon, MAX, Phyton, MSAM, and ARGO, while subsequent work by our group and others has used lots of new data. The data points with their errors (horizontal ones corresponding to the window width) are displayed in Fig. 11.

What we found in our first study (which attracted considerable attention and was discussed in an editorial section of Science) was that, with the data at hand then, the Gaussianity hypothesis could be rejected at \( \sim 80\% \) confidence level. We were careful, however, to point out at the fact that we had still very poor statistics and that maybe systematic errors in the experiments could had been underestimated, among other considerations. In any case, we noted that by doubling systematic errors we still obtained \( \sim 60\% \) confidence. Subsequent analysis have diminished this predictions, although the case for possible non-Gaussianities is far from being closed. Still more date will be needed to settle this issue. For interesting references on the subject of statistical analysis of the large scale galaxy distribution, together with a review of redshift galaxy surveys and the most recent determination of the cosmological parameters from these analysis, see e.g. 61.
Fig. 11. Band power estimates of the \(r\text{ms}\) temperature anisotropy \(\delta T_l\) for observations given in our analysis. The vertical error bars show the (symmetrized) total errors in \(\delta T_l\) while the horizontal ones stand for the width of the windows. The dashed line is the best fit slope to the data, \(\delta T_l = (11/50)l + 18\). Continuous lines show the standard CDM model for two normalizations: \(Q_{\text{rms}} = 20\,\mu\text{K}\) (top), and \(18\,\mu\text{K}\) (bottom).

4. Vacuum energy and the cosmological constant

4.1. The cosmological constant

Our universe seems to be spatially flat and to possess a non-vanishing cosmological constant. Thus, Einstein’s ‘great mistake’ may turn out ultimately to be a great discovery, a necessary ingredient in order to explain the acceleration of the universe. In any case, for elementary particle physicists it constitutes (in the words of J. Bjørken) a great embarrassment, calculations there being off (when compared with physical facts) by the famous 120 orders of magnitude.

First, physicists tried to find a way to get rid of it (Coleman, Weinberg, Polchinski, ...), in the hope that it could be proven to be zero, what was hard enough. But now it turns out that it is non-vanishing, albeit very small, indeed a very peculiar quantity.

The cosmological constant has to do with cosmology, of course (through Einstein’s equations and the FRW universe obtained from them), but it has to do also with the local structure of elementary particle physics as the stress-energy density \(\mu\) of the vacuum

\[
L_{cc} = \int d^4x \sqrt{-g} \mu^4 = \frac{1}{8\pi G} \int d^4x \sqrt{-g} \lambda. \tag{73}
\]
In other words: two contributions appear, on the same footing
\[ \frac{\Lambda c^2}{8\pi G} + \frac{1}{\text{Vol}} \frac{\hbar c}{2} \sum_i \omega_i. \]  

(74)

4.2. From General Relativity to Cosmology

4.2.1. On the meaning of Einstein’s equations

Recall Einstein’s equations (formulated in 1915-17), including a cosmological constant \( \Lambda \):
\[ G_{\mu\nu} - \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}. \]  

(75)

with
\[ G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad R = R^\mu_\mu. \]  

(76)

These equations can be obtained from a variational principle, starting from an effective Einstein-Hilbert action
\[ S = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} (R - 2\Lambda) + \int d^4 x \sqrt{-g} L_{\text{mat}}. \]  

(77)

They have a very profound meaning.

On April this year I had to explain some couple of aspects of Special and General Relativity to the mixed TV audience of a Thursday evening. I decided to start with the most famous equation \( E = mc^2 \). “Look”, I said to the imposing camera in front of my face, “with this equation Einstein’s genius put on the same footing matter and energy, which had been always thought to be non-matching quantities. All of us have been told at primary school that apples and oranges don’t match: no way to add 3 apples and 4 oranges. Since, what the result would be? Seven, ... but seven what? However, we go to the grocer’s or the supermarket every week and we buy not only fruit, but all sort of different things, and the owner or the cashier just puts all items inside a bag and then ... he does it! precisely what we were told it was impossible to do: adds for us everything together and says ‘this makes $14.76’. But this is exactly what Einstein did: to find a conversion factor for the different quantities, which in his case was the velocity of light and for the cashier it’s just the price per pound of every item. Easy, isn’t it? Einstein, as the grocer’s, was not constrained by what we learn at school. In Einstein’s case this opened the way to the possibility of converting matter into energy, and vice-versa, what was soon put to test.

And then I went on, “Now let’s turn to GR. This is actually rather more difficult to grasp and no shop owner or cashier on Earth would have guessed the answer this time. Look, in the language of Cosmology, Einstein’s equation reads: \( \Omega_{\text{matter}} + \Omega_{\text{radiation}} + \Omega_\Lambda + \Omega_{\text{curvature}} = 1 \). This equation is not so widely known as the previous one, but is in no way less important. At first sight, you would say — alas, that’s again the same as before, only more items appear say, a computer, a car, a cellular phone, a futon, a T-shirt, ..., of course one can buy them all together at
the mall, no problem— But this wouldn’t be the whole truth. An important issue is missing from that argument, namely the last term, $\Omega_{\text{curvature}}$, which refers to the mathematical curvature of space-time itself. That’s very different from the rest of terms, since it means that the reference system itself gravitates, that there is no ‘outside reference system’. In other words and following with the same example as before, what Einstein did here was to put the grocer himself inside the bag together with the rest of the things we bought! Who now will do the sum for us? Actually the first to guess our Universe could behave in this remarkable way was Ernst Mach (1838-1916). And Einstein found out the precise equations to try to confirm such extraordinary idea. This is what Gravity Probe B is going to confirm, with very good precision, so that there can be no doubt that our Universe does in fact behave this way.” Then I went on, to explain frame warping and frame dragging.

What Einstein did, specifically, when building his equations:

- Geometry (curvature), radiation energy, matter, the cc, all are on the same footing and can be equated together. This is the mathematical concretion of Mach’s principle.
- $G_{\mu\nu}$ is a linear combination of the metric $g_{\mu\nu}$ and of first and second derivatives of the same.
- $T_{\mu\nu}$ is the energy-momentum tensor, and $\Lambda$ a (possible) cosmological constant.

Actually Einstein didn’t quite succeed in pining down in his theory of GR the whole content of Mach’s principle (see 65); but there is no doubt that remarkable glimpses of it are to be found in Einstein’s equations, namely frame warping and frame dragging by distributions of matter and rotating massive shells, respectively.

4.2.2. Gravity Probe B

The mission Gravity Probe B was launched by NASA on April 20, 2004, with the idea to try to see these two effects in great precision.

(i) Frame warping was proposed by deSitter in 1916, as the geodetic force a gyroscope would suffer in the presence of the space-time curvature induced by the presence of a mass. In the case of Gravity Probe B, which describes a polar orbit of 640 Km radius —the gyroscopes’ axis having been oriented towards a convenient guide star (IM Pegasi)— the calculated effect will be a displacement of the gyroscopes’ orbit (it won’t be exactly circular around the Earth) of 6.6 arcsec/year, with an expected error of less than $10^{-4}$.

(ii) Frame dragging was discovered by Lense and Thirring in 1918 as a gravitomagnetic force. It will be produced, in the case of Gravity Probe B, by the Earth’s (a massive body) rotation on the reference system, defined in this situation by the gyroscopes themselves. The orbit has been specifically chosen

---

1There is a nice essay from Frank Wilczek on that issue.
so that both effects on the gyroscopes are perpendicular. Frame dragging will result in the rotation axis of the gyroscopes trying to approach the Earth’s rotation axis by an amount of 42 milliarcsec/year, with an estimated error of $10^{-2}$ (equivalent to the section of a human hair seen from 15 Km distance, while the effect amounts to seeing the same hair from 400 m. Reportedly, this precision is still unprecedented in experimental observations. More details about this important mission can be found in its web page, where one can learn a lot about the physical meaning of GR, and this in the best possible way, namely, in the framework of an actual experiment.

Although the idea is very simple, and the first plans to launch such a satellite were met over 40 years ago, the technical difficulties involved are extraordinary and have postponed its final launch till a few months ago.

If the results confirm what almost every physicist believes, i.e. the validity and accuracy of GR, then there will be no way out but to admit that the mere notion of the existence of an ideal reference frame in the cosmos is absolutely erroneous. Any mathematical reference will also ‘gravitate’, that is, it will be unavoidably subject to the influence of all the masses in our Universe, and their rotation. In plain words, ‘the grocer himself will have to be put into the bag, indeed, and nobody will be able to do the sum for us.’ This is what we learn from looking at our Universe. If confirmed, these results will completely demolish Isaac Newton’s original formulation of the concept of absolute space, that was more clear to him than the purest of waters (and also to more one reputed philosopher, as I. Kant).

4.2.3. Solutions to Einstein’s equations

The Schwarzschild solution (1916) of Einstein’s equations reads

$$ds^2 = \left(1 - \frac{2MG}{r}\right)dt^2 - \left(1 - \frac{2MG}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2 \sin^2 \theta d\varphi^2.$$  \hspace{1cm} (78)

It was soon realized that it could describe a black hole, but not a entire universe. However, the Friedmann-Lemaître-Robertson-Walker (1935-36) solution (FRW), first found by A. Friedmann in 1922,

$$ds^2 = dt^2 - R^2(t) \left(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2 \sin^2 \theta d\varphi^2\right),$$  \hspace{1cm} (79)

with $k = 0, +1, -1$, not only can do so, but it is the only solution of Einstein’s equations (up to the constant $k$, the curvature) that can satisfy the requirements of homogeneity and isotropy, that our universe is (observationally) known to possess, to a high degree of accuracy. Now, in Cosmology the Friedmann equation is always written under the equivalent form:

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left[\Omega_R \left(\frac{a_0}{a}\right)^4 + \Omega_{NR} \left(\frac{a_0}{a}\right)^3 + \Omega_V + (1 - \Omega) \left(\frac{a_0}{a}\right)^2\right],$$  \hspace{1cm} (80)
with $\Omega_R$ being the radiation (or relativistic matter) content of our Universe, which satisfies the equation of state pressure = density/3, that is $p_R = \frac{1}{3}\rho_R$, being $\rho_R \propto a^{-4}$, and $a$ a typical distance (say the mean intergalactic one), as the Universe expands; the ordinary (non-relativistic) matter is $\Omega_{NR}$, and satisfies $p_{NR} = 0$, being $\rho_{NR} \propto a^{-3}$; the vacuum energy density, $\Omega_V$ is undistinguishable from the cosmological constant, with an equation of state $p_V = -\rho_V$, being here $\rho_V = \text{const}$. Finally, the equation has been normalized to one, so that the sum of the different contributions equals this number; in other words, $1-\Omega$ in the equation before (where $\Omega = \Omega_R + \Omega_{NR} + \Omega_V$) is the contribution of the geometry (of the curvature of the Universe, while $\Omega$ is ‘physical’ contribution), $\Omega_K$, which behaves as $\rho_K \propto a^{-2}$, and

$$\Omega_R + \Omega_{NR} + \Omega_K + \Omega_V = 1. \quad (81)$$

Presently, the contribution of $\Omega_K$ can be neglected, since from observations we obtain that $\Omega_K \simeq 0.0 \pm 0.1$, so that, as we explained before, we seem to live in a flat universe. Thus, one gets the famous cosmological triangle, which is a simple and intuitive graphical representation where one can read the proportions of the different terms in any proposed model for the present universe. The most plausible ones at present (e.g. from WMAP), yield a mere 4% for the entire ordinary matter+energy content (i.e., barions+photons, with just some ~0.05% for radiation), some 25% comes from dark or invisible matter (trapped mainly in galaxy clusters, and a fraction in galaxy halos) and the biggest part, around 70%, is an absolutely unknown called dark energy (those values are for an $h \simeq 0.72$).

Concerning dark matter, Zwicky noticed in 1933 already, that the gravitational action of the luminous matter was not enough in order to hold galaxy clusters together (could explain kpc structures at most.) Different kind of cold matter particles, with magnitudes that can differ in almost 100 orders, have been invoked by existing models, from axions and neutrinos to large planets (or ‘Jupiters’), and also theories deviating from ordinary Newtonian physics. A portion of particle dark matter is sure to exist: the mass coming from neutrinos could be already as large as the mass in visible stars. The lack of Newtonian matter is seen to occur at an extent range of distances, from the less than 1 kpc corresponding to dwarf spirals to the more than 100 Mpc in large clusters of galaxies. Twenty years ago, Milgrom made the remarkable observation that the need for dark matter in galaxies only arises when the Newtonian acceleration is less than a value $a_0 \simeq 0.3cH_0$. This is called now Milgrom’s law, and has given rise to a theory, the Modified Newtonian Dynamics (MOND). As we have seen before in some detail, of the known quantities only the cosmological constant has an equation of state which could contribute to the dark energy term (or, similarly, the proposed models of dynamic cc’s, as quintessence).
4.2.4. *In terms of the redshifts*

An equivalent (and even more commonly used) expression is in terms of the redshift, $z$ (taken in practice as the inverse of the cosmological time)

$$d_H(z) = H_0^{-1} \left[ \Omega_R(1+z)^4 + \Omega_{NR}(1+z)^3 + (1 - \Omega)(1+z)^2 + \Omega_V \right]^{-1/2}. \quad (82)\]

Limiting forms:

$$d_H(z) \sim \begin{cases} 
H_0^{-1} \Omega_R^{-1/2} (1+z)^{-2} & (z \gg z_{\text{eq}}), \\
H_0^{-1} \Omega_{NR}^{-1/2} (1+z)^{-3/2} & (z_{\text{eq}} \gg z \gg z_{\text{curv}}; \Omega_V = 0),
\end{cases} \quad (83)$$

Any of these equations describes the whole (thermal) history of our Universe. In fact, the different contributions are just simple monomials in $(1+z)$, so that in the origin of time, that is for $z$ large enough, the radiation term always dominates. As time went on, $z$ went down and for a certain value of $z$ (obtained when the first and second terms become exactly equal), there is a transition from a radiation dominated to a matter dominated epoch, then to a curvature dominated one, and so on; finally, the cc term takes over. These transitions occur at the values $z_{\text{eq}}$ and $z_{\text{curv}}$, obtained by equating the corresponding two terms in each case:

$$1 + z_{\text{eq}} \equiv \frac{\Omega_{NR}}{\Omega_R}, \quad 1 + z_{\text{curv}} \equiv \frac{1}{\Omega_{NR}} - 1; \quad (84)$$

- radiation dom. $\rightarrow$ matter dom. $\rightarrow$ curvature dom. $\rightarrow$ cc:
  - this is, in a single line, the whole thermal history of the Universe;
- $\lambda(z) = \lambda_0(1+z)^{-1} \rightarrow \lambda(z) > d_H(z)$, for $z > z_{\text{enter}}$, which is when general relativistic effects become important;
- $\lambda(t) = \lambda_0[a(t)/a_0], \quad \rho(t) = \rho_0[a_0/a(t)]^3$, those are the behaviors of the wave length an the density, as time elapses; from these behaviors we see that the mass of non-relativistic matter, $M(\lambda_0)$, inside a sphere of $r = \lambda_0/2$ is given by

$$M = \frac{4\pi}{3} \rho(t) \left[ \frac{\lambda(t)}{2} \right]^3 = \frac{4\pi}{3} \rho_0 \left( \frac{\lambda_0}{2} \right)^3 \quad (85)$$

so that:
- the co-moving scale $\lambda_0 \approx 1 \text{ Mpc}$ contains a typical *galaxy* mass;
- and $\lambda_0 \approx 10 \text{ Mpc}$ contains a typical *cluster* mass.

Moreover,

- At $z \gg z_{\text{enter}}$, one needs to consider General Relativity in full but only linear perturbation theory (non-linear effects are very small).
- For $z \ll z_{\text{enter}}$ we are in the non-linear epoch but only Newtonian gravity has to be taken into account.
- The role of general relativity calculations is thus to evolve initial (linear) perturbations up to $z \approx z_{\text{enter}}$ where the non-linear regime takes over.
And the *techniques* commonly employed in these processes are:

- Linear growth in the general relativistic regime.
- Gravitational clustering in the Newtonian theory.
- Linear perturbations in the Newtonian limit.
- The Zeldovich approximation.
- The spherical approximation, leading to the improved spherical collapse model.
- The use of scaling laws. Etc.

For details see, e.g., the following very good Refs. 72, 73, 75.

4.3. *The method of zeta-function regularization*

Hawking introduced this method as a basic tool for the regularization of infinities in QFT in a curved spacetime. The idea is the following: One could try to tame Quantum Gravity using the canonical approach, by defining an arrow of time and working on the space-like hypersurfaces perpendicular to it, with equal time commutation relations. Reasons against this:

(i) there are many topologies of the space-time manifold that are not a product $\text{R} \times M_3$;
(ii) such non-product topologies are sometimes very interesting;
(iii) what does it mean ‘equal time’ in the presence of Heisenberg’s uncertainty principle?

One thus turns naturally towards the path-integral approach:

$$<g_2, \phi_2, S_2 | g_1, \phi_1, S_1> = \int D[g, \phi] e^{i S[g, \phi]}, \quad \text{(86)}$$

where $g_j$ denotes the spacetime metric, $\phi_j$ are matter fields, $S_j$ general spacetime surfaces ($S_j = M_j \cup \partial M_j$), $D$ a measure over all possible ‘paths’ leading from the $j = 1$ to the $j = 2$ values of the intervening magnitudes, and $S$ is the action:

$$S = \frac{1}{16\pi G} \int (R - 2\Lambda)\sqrt{-g} \, d^4x + \int L_m \sqrt{-g} \, d^4x, \quad \text{(87)}$$

$R$ being the curvature, $\Lambda$ the cosmological constant, $g$ the determinant of the metric, and $L_m$ the Lagrangian of the matter fields. Stationarity of $S$ under the boundary conditions

$$\delta g|_{\partial M} = 0, \quad \vec{n} \cdot \delta g|_{\partial M} = 0, \quad \text{(88)}$$

leads to Einstein’s equations:

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi G T_{ab}, \quad \text{(89)}$$

$T_{ab}$ being the energy-momentum tensor of the matter fields, namely,

$$T_{ab} = \frac{1}{2\sqrt{-g}} \delta L_m \, \delta g^{ab}. \quad \text{(90)}$$
The path-integral formalism provides a way to deal ‘perturbatively’ with QFT in curved spacetime backgrounds. First, through a rotation in the complex plane one defines an Euclidean action:

\[ iS \longrightarrow -\hat{S}. \]  

(91)

One can also easily introduce the finite temperature formalism by the substitution \( t_2 - t_1 = i\beta \), which yields the partition function

\[ Z = \sum_n e^{-\beta E_n}. \]  

(92)

If one now adheres to the principle that the Feynman propagator is obtained as the limit for \( \beta \to \infty \) of the thermal propagator, we have shown, some time ago, that the usual principal-part prescription in the zeta-function regularization method (to be described below) need not be imposed any more as an additional assumption, since it beautifully follows from, and thus can actually be replaced, by this more general (and natural) principle.

Next comes the stationary phase approach (also called one-loop, or WKB), for calculating the path integral, which consists in expanding around a fixed background:

\[ g = g_0 + \bar{g}, \quad \phi = \phi_0 + \bar{\phi}, \]  

(93)

what leads to the following expansion in the Euclidean metric:

\[ \hat{S}[g, \phi] = \hat{S}[g_0, \phi_0] + S_2[\bar{g}, \bar{\phi}] + \cdots \]  

(94)

This is most suitably expressed in terms of determinants (for bosonic, resp. fermionic fields) of the kind (here \( A, B \) are the relevant (pseudo-)differential operators in the corresponding Lagrangian):

\[ \Delta_\phi = \det \left( \frac{1}{2\pi\mu^2} A \right)^{-1}, \quad \Delta_\psi = \det \left( \frac{1}{2\mu^2} B \right). \]  

(95)

4.4. A word on determinants

Many fundamental calculations of QFT reduce, in essence, to the computation of the determinant of some operator. One could even venture to say that, at one-loop order, any such theory reduces to a theory of determinants. The operators involved are ‘differential’ ones, as the normal physicist would say. In fact, properly speaking, they are pseudodifferential operators (ΨDO), that is, in loose terms ‘some analytic functions of differential operators’ (such as \( \sqrt{1 + D} \) or \( \log(1 + D) \), but not! \( \log D \)). This is explained in detail in Refs.\[77,78,79\]

Important as the concept of determinant of a differential or ΨDO may be for theoretical physicists (in view of what has just been said), it is surprising that this seems not to be a subject of study among function analysts or mathematicians in general. This statement must be qualified: I am specifically referring to determinants that involve in its definition some kind of regularization, very much
related to operators that are not trace-class. This piece of calculus—always involving regularization—falls outside the scope of the standard disciplines and even many physically oriented mathematicians know little or nothing about it. In a sense, the subject has many things in common with that of *divergent series* but has not been so deeply investigated and lacks any reference comparable to the very beautiful book of Hardy already mentioned. Actually, from this general viewpoint, the question of regularizing infinite determinants was already addressed by Weierstrass in a way that, although it has been pursued by some theoretical physicists with success, is not without problems—as a general method—since it ordinarily leads to non-local contributions that cannot be given a physical meaning in QFT. We should mention, for completion, that there are, since long ago, well established theories of determinants for degenerate operators, for trace-class operators in the Hilbert space, Fredholm operators, etc., but, again, these definitions of determinant do not fulfill all the needs mentioned above which arise in QFT.

Any high school student knows what a determinant is, in simple words, or at least how to calculate the determinant of a $3 \times 3$ matrix (and some of them, even that of a $4 \times 4$ one). But many one prominent mathematician will answer the question: *What is your favorite definition of determinant of a differential operator?* with: *I don’t have any*, or: *These operators don’t have determinants!* An even more ‘simple’ question I dare to ask the reader (which she/he may choose to ask to some other colleague on its turn) is the following: *What is the value of the determinant of minus the identity operator in an infinite dimensional space?* Followed by: *And that of the determinant $\prod_{n \in \mathbb{N}} (-1)^n$? Is it actually equal to the product of the separate determinants of the plus 1s and of the minus 1s?*

In this short note I will point out to specific situations, some of them having become common lore already and other that have appeared recently in the literature, concerning the concept of determinant in QFT, and I will try to give ‘reasonable’ answers to questions such as the last ones. As already mentioned, the mathematical theory of divergent series has been very fruitful in taming the infinities that have appeared in QFT, from the very beginning of its conception. Its role is very essential, at least in the first stage of the regularization/renormalization procedure. Euler and Borel summation methods, and analytic continuation techniques are there commonly used. But some difficulties exist that are inherent to the theory of divergent series (see, for instance). One of them is the well known fact that, sometimes, by using different schemes, different results are obtained. In a well posed physical situation, the ‘right’ one can then only be chosen after experimental validation. Another problem is to understand, in physical terms, what you are doing, while performing say an analytic continuation from one region of the complex plane to another. This has prevented e.g. the zeta function regularization procedure from getting general acceptance among common physicists.

The situation concerning infinite determinants is even worse, in a sense. There is no book on the subject to be compared, for instance, with the above mentioned one by Hardy and we see every day that dubious manipulations are being performed at
the level of the eigenvalues, that are then translated to the determinant itself and elevated sometimes to the category of standard results — when not of lore theorems. The first problem is the definition of the determinant itself. Let me quote in this respect from a famous paper by E. Witten: 82

The determinant of the Dirac operator is defined roughly as

$$\det D = \prod_i \lambda_i,$$

where the infinite product is regularized with (for example) zeta function or Pauli-Villars regularization. The zeta function definition of the determinant

$$\det \zeta D = \exp \left[ -\zeta D'(0) \right],$$

is maybe the one that has more firm mathematical grounds 83 In spite of starting from the identity: \( \log \det = \text{tr} \log \), it is known to develop the so called multiplicative anomaly: the determinant of the product of two operators is not equal, in general, to the product of the determinants (even if the operators commute!). This happens already with very simple operators (as two one-dimensional harmonic oscillators only differing in a constant term, Laplacians plus different mass terms, etc.). It may look incredible, at first sight, from the \( \text{tr} \log \) property and the additivity of the trace, but we must just take into account that the zeta trace is no ordinary trace (for it involves regularization), namely:

$$\text{Tr} \zeta D = \zeta D(-1),$$

so that \( \text{Tr} \zeta (A+B) \neq \text{Tr} \zeta A + \text{Tr} \zeta B \), in general. Not to understand this has originated a considerable amount of errors in the specialized literature — falsely attributed to misfunctions of the rigorous and elegant zeta function method!

As an example, consider the following commuting linear operators in an infinite-dimensional space, given in diagonal form by:

$$O_1 = \text{diag} (1, 2, 3, 4, \ldots), \quad O_2 = \text{diag} (1, 1, 1, 1, \ldots) \equiv I,$$

and their sum

$$O_1 + O_2 = \text{diag} (2, 3, 4, 5, \ldots).$$

The corresponding \( \zeta \)-traces are easily obtained:

$$\text{Tr} \zeta O_1 = \zeta R(-1) = -\frac{1}{12}, \quad \text{Tr} \zeta O_2 = \zeta R(0) = -\frac{1}{2},$$

$$\text{Tr} \zeta (O_1 + O_2) = \zeta R(-1) - 1 = -\frac{13}{12},$$

the last trace having been calculated according to the rules of infinite series summation (see e.g., Hardy 80). We observe that

$$\text{Tr} \zeta (O_1 + O_2) - \text{Tr} \zeta O_1 - \text{Tr} \zeta O_2 = -\frac{1}{2} \neq 0.$$
If this happens in such simple situation, involving the identity operator, one can easily imagine that any precaution one can take in manipulating infinite sums might turn out to be insufficient. Moreover, since the multiplicative anomaly—as has been pointed out before—originates precisely in the failure of this addition property for the regularized trace, we can already guess that it also can show up in very simple situations, in fact. The appearance of the multiplicative anomaly prevents, in particular, naive manipulations with the eigenvalues in the determinant, as re-orderings and splittings, what a number of physicists seem not to be aware of. All the above considerations may sound rather trivial, but actually they are not, and should be carefully taken into account before proceeding with the sort of manipulations of the eigenvalues and splittings of determinants that pervade the specialized literature.

4.5. **The zero point energy**

If $H$ is now the Hamiltonian corresponding to a physical, quantum system, the zero point energy is given by

$$<0|H|0>, \quad (103)$$

where $|0>$ is the vacuum state. In general, after normal ordering we’ll have:

$$H = \left( n + \frac{1}{2} \right) \lambda_n a_n a_n^\dagger, \quad (104)$$

and this yields for the vacuum energy:

$$<0|H|0> = \frac{\hbar c}{2} \sum_n \lambda_n. \quad (105)$$

(I won’t normally keep track of the $\hbar$’s and $c$’s that will be set equal to 1.) The physical meaning of this energy was the object of a very long controversy, involving many first-rate physicists, until the late Heindrik Casimir gave the explanation (over fifty years ago) that is widely accepted nowadays, and that’s the reason why the zero-point energy is usually associated to his name\textsuperscript{20}

Only in special cases will this sum be convergent. Generically one gets a divergent series, to be regularized by different means. The zeta-function method\textsuperscript{84}—which is best suited for doing these calculations\textsuperscript{84}—will interpret it as the value of the zeta function of $H$: $\zeta_H(s) = \sum_n \lambda_n^{-s}$, at $s = -1$ (we set $\hbar = c = 1$). Generically $\zeta_H(s)$ is only defined as an absolutely convergent series for $\text{Re } s > a_0$ ($a_0$ an abscissa of convergence),\textsuperscript{k} but it can be continued to the whole complex plane, with the possible appearance of poles as only singularities. If $\zeta_H(s)$ has no pole at $s = -1$ then we are done; if it hits a pole, further elaboration is necessary. That the mathematical result one thus gets coincides with great precision with the experimental one, constitutes another clear example of unreasonable effectiveness of mathematics\textsuperscript{2}.

\textsuperscript{k}Which in general it won’t be at $s = 1$, as for the case of the Riemann zeta function.
In fact things do not turn out to be so simple. One cannot assign a meaning to the absolute value of the zero-point energy, and any physical effect is an energy difference between two situations, such as a quantum field in curved space as compared with the same field in flat space, or one satisfying BCs on some surface as compared with the same in its absence, etc. This difference is the Casimir energy:

$$E_C = E_0^{BC} - E_0 = \frac{1}{2} (\text{Tr} \, H^{BC} - \text{Tr} \, H).$$

(106)

And here a very important problem appears, which has been the object of intermittent but continued discussion for some time: imposing mathematical boundary conditions (BCs) on physical quantum fields turns out to be a highly non-trivial act. This was investigated in much detail already in a paper by Deutsch and Candelas a quarter of a century ago. These authors quantized electromagnetic and scalar fields in the region near an arbitrary smooth boundary, and calculated the renormalized vacuum expectation value of the stress-energy tensor, to find that the energy density diverges as the boundary is approached. Therefore, regularization and renormalization did not seem to cure the problem with infinities in this case and an infinite physical energy was obtained if the mathematical BCs were to be fulfilled. However, the authors argued that surfaces have non-zero depth, and its value could be taken as a handy (dimensional) cutoff in order to regularize the infinities. This approach will be recovered later in this paper. Just two years after Deutsch and Candelas’ work, Kurt Symanzik carried out a rigorous analysis of QFT in the presence of boundaries. Prescribing the value of the quantum field on a boundary means using the Schrödinger representation, and Symanzik was able to show rigorously that such representation exists to all orders in the perturbative expansion. He showed also that the field operator being diagonalized in a smooth hypersurface differs from the usual renormalized one by a factor that diverges logarithmically when the distance to the hypersurface goes to zero. This requires a precise limiting procedure and point splitting to be applied. In any case, the issue was proven to be perfectly meaningful within the domains of renormalized QFT. In this case the BCs and the hypersurfaces themselves were treated at a pure mathematical level (zero depth) by using delta functions.

Recently, a new approach to the problem has been postulated. BCs on a field, $\phi$, are enforced on a surface, $S$, by introducing a scalar potential, $\sigma$, of Gaussian shape living on and near the surface. When the Gaussian becomes a delta function, the BCs (Dirichlet here) are enforced: the delta-shaped potential kills all the modes of $\phi$ at the surface. For the rest, the quantum system undergoes a full-fledged QFT renormalization, as in the case of Symanzik’s approach. The results obtained confirm those of Ref. in the several models studied albeit they do not seem to agree with those of Ref. They are also in clear contradiction with the ones quoted in the usual textbooks and review articles dealing with the Casimir effect, where no infinite energy density when approaching the Casimir plates has been reported.

Too often has it been argued that sophisticated regularization methods, as the zeta-function procedure, get rid of infinities in an obscure way (e.g. through analytic
continuation), so that, contrary to what happens with cut-offs, one cannot keep
trace of the infinities, which are cleared up without control, leading sometimes to
erroneous results. One cannot refute a statement of this kind rigorously, but it
should be noted that more once (if not always) the discrepancies between the result
obtained by using the zeta procedure and other —say cut-off like— approaches have
been proven to emerge from a misuse of zeta regularization, and not to stem from
the method itself. When employed properly, the correct results have been recovered
(for a good number of examples, see Refs. 9, 84, 89, 90, 11).

The expression above acquires a very important meaning as soon as one com-
pares different settings, e.g., one where some sort of boundary conditions are im-
posed to the vacuum (e.g., a pair of parallel plates, infinitely conducting, in the
vacuum corresponding to the electromagnetic field) with another situa-
tion where
the boundary conditions (the plates) are absent (they have been sent to infinity).
The difference yields a physically observable energy.

In general the sums appearing here are all divergent. They give rise to the most
primitive, but physically meaningful, examples of zeta function regula-
rization one
can think of. In fact, according to the definitions above:

\[<0 | H | 0 \rangle = \frac{1}{2} \zeta(-1).\] (107)

It is important to notice that the zero-point energy is something one always has
to keep in mind when considering any sort of quantum effect. Its contribution can
be in some cases negligible, even by several orders of magnitude (as seems to be the
case with sonoluminiscence effects), but it can be of a few percent (as in some laser
cavity effects), or even of some \(10^{-30}\%\) as in the case of several wetting phenomena
of alkali surfaces by Helium. Not to speak of the specifically devised experiments,
where it may account for the full result.

In the case of the calculation of the value of the cosmological constant, it is
immediate to see from the expressions considered before that:

\[<0 | T_{\mu\nu} | 0 \rangle = \frac{\Lambda}{8\pi G} + \frac{1}{2V} \sum_n \lambda_n,\] (108)

where \(V\) is the volume of the space manifold and the second term as a whole is
the vacuum energy density corresponding to the quantum field (or fields) we are
considering. Unless the first term (the cosmological constant), the vacuum energy
density is not a constant (it goes as \(a^{-4}\), \(a\) being a typical cosmological length).
However, this does not prevent the mixing of the two contributions when one con-
siders, e.g., ‘the presently observed value of the cosmological constant’. What we
have calculated is the second contribution for a scalar field of very low mass.

4.6. Quantum fluctuations of the cosmological vacuum energy

The issue of the cosmological constant has got renewed thrust from the observational
evidence of an acceleration in the expansion of our Universe, initially reported by
two different groups. There was some controversy on the reliability of the results obtained from those observations and on its precise interpretation, by a number of different reasons. Anyway, after new data has been gathered, there is presently reasonable consensus among the community of cosmologists that there is, in fact, an acceleration, and that it has the order of magnitude obtained in the above mentioned observations. In support of this consensus, the recently issued analysis of the data taken by the BOOMERAnG and MAXIMA-1 balloons have been correspondingly crossed with those from the just mentioned observations, to conclude that the results of BOOMERAnG and MAXIMA-1 can perfectly account for an accelerating universe and that, taking together both kinds of observations, one infers that we most probably live in a flat universe. As a consequence, many theoreticians have urged to try to explain this fact, and also to try to reproduce the precise value of the cosmological constant coming from these observations, in the available models.

Now, as crudely stated by Weinberg in a review paper, it is even more difficult to explain why the cosmological constant is so small but non-zero, than to build theoretical models where it exactly vanishes. Rigorous calculations performed in quantum field theory on the vacuum energy density, $\rho_V$, corresponding to quantum fluctuations of the fields we observe in nature, lead to values that are over 120 orders of magnitude in excess of the values allowed by observations of the space-time around us.

Rather than trying to understand the fine-tuned cancellation of such enormous values at this local level (a very difficult question that we are going to leave unanswered, and even unattended, here), in this section we will elaborate on a quite simple and primitive idea (but, for the same reason, of far reaching, inescapable consequences), related with the global topology of the universe and in connection with the possibility that a very faint, massless scalar field pervading the universe could exist. Fields of this kind are ubiquitous in inflationary models, quintessence theories, and the like. In other words, we do not pretend to solve the old problem of the cosmological constant, not even to contribute significantly to its understanding, but just to present an extraordinarily simple model which shows that the right order of magnitude of (some contributions to) $\rho_V$, in the precise range deduced from the astrophysical observations, e.g. $\rho_V \sim 10^{-10}$ erg/cm$^3$, are not difficult to get. To say it in different words, we only address here what has been termed by Weinberg the new cosmological constant problem.

In short, we shall assume the existence of a scalar field background extending through the universe and shall calculate the contribution to the cosmological constant coming from the Casimir energy density corresponding to this field for some typical boundary conditions. The ultraviolet contributions will be safely set to zero by some mechanism of a fundamental theory. Another hypothesis will be the existence of both large and small dimensions (the total number of large spatial coordinates will be always three), some of which (from each class) may be com-
pactified, so that the global topology of the universe will play an important role, too. There is by now a quite extensive literature both in the subject of what is the global topology of spatial sections of the universe and also on the issue of the possible contribution of the Casimir effect as a source of some sort of cosmic energy, as in the case of the creation of a neutron star. There are arguments that favor different topologies, as a compact hyperbolic manifold for the spatial section, what would have clear observational consequences. Other interesting work along these lines was reported in Ref. and related ideas have been discussed very recently in Ref. However, we differ from all those in several respects. To begin, the emphasis is put now in obtaining the right order of magnitude for the effect, e.g., one that matches the recent observational results. At the present stage, in view of the observational precision, it has no sense to consider the whole amount of possibilities concerning the nature of the field, the different models for the topology of the universe, and the different boundary conditions possible, with its effect on the sign of the force.

At this level, from our previous experience in these calculations and from the many tables (see, e.g., Refs. where precise values of the Casimir effect corresponding to a number of different configurations have been reported), we realize that the range of orders of magnitude of the vacuum energy density for the most common possibilities is not so widespread, and may only differ by at most a couple of digits. This will allow us, both for the sake of simplicity and universality, to deal with a most simple situation, which is the one corresponding to a scalar field with periodic boundary conditions. Actually, as explained in Ref. in detail, all other cases for parallel plates, with any of the usual boundary conditions, can be reduced to this one, from a mathematical viewpoint.

4.7. Two basic space-time models

Let us thus consider a universe with a space-time of one of the following types: \( \mathbb{R}^{d+1} \times T^p \times T^q, \mathbb{R}^{d+1} \times T^p \times S^q, \ldots \), which are actually plausible models for the space-time topology. A (nowadays) free scalar field pervading the universe will satisfy

\[
(-\Box + M^2)\phi = 0,
\]

restricted by the appropriate boundary conditions (e.g., periodic, in the first case considered). Here, \( d \geq 0 \) stands for a possible number of non-compactified dimensions.

Recall now that the physical contribution to the vacuum or zero-point energy \( \langle 0 | H | 0 \rangle \) (where \( H \) is the Hamiltonian corresponding to our massive scalar field and \( | 0 \rangle \) the vacuum state) is obtained on subtracting to these expression —with the vacuum corresponding to our compactified spatial section with the assumed boundary conditions— the vacuum energy corresponding to the same situation with the only change that the compactification is absent (in practice this is done by
conveniently sending the compactification radii to infinity). As well known, both of these vacuum energies are in fact infinite, but it is its difference

\[ E_C = \langle 0 | H | 0 \rangle_{R} - \langle 0 | H | 0 \rangle_{R \to \infty} \]  

(110)

(where \( R \) stands here for a typical compactification length) that makes physical sense, giving rise to the finite value of the Casimir energy \( E_C \), which will depend on \( R \) (after a well defined regularization/renormalization procedure is carried out).

In fact we will discuss the Casimir (or vacuum) energy density, \( \rho_C = E_C / V \), which can account for either a finite or an infinite volume of the spatial section of the universe (from now on we shall assume that all diagonalizations already correspond to energy densities, and the volume factors will be replaced at the end). In terms of the spectrum \( \{ \lambda_n \} \) of \( H \):

\[ \langle 0 | H | 0 \rangle = \frac{1}{2} \sum_n \lambda_n, \]  

(111)

where the sum over \( n \) is a sum over the whole spectrum, which involves, in general, several continuum and several discrete indices. The last appear typically when compactifying the space coordinates (much in the same way as time compactification gives rise to finite-temperature field theory), as in the cases we are going to consider. Thus, the cases treated will involve integration over \( d \) continuous dimensions and multiple summations over \( p + q \) indices (for a pedagogical description of this procedure, see Ref. [106]).

To be precise, the physical vacuum energy density corresponding to our case, where the contribution of a scalar field, \( \phi \) in a (partly) compactified spatial section of the universe is considered, will be denoted by \( \rho_\phi \) (note that this is just the contribution to \( \rho_V \) coming from this field, there might be other, in general). It is given by

\[ \rho_\phi = \frac{1}{2} \sum_\mathbf{k} \frac{1}{\mu} (k^2 + M^2)^{1/2}, \]  

(112)

where the sum \( \sum_\mathbf{k} \) is a generalized one (as explained above) and \( \mu \) is the usual mass-dimensional parameter to render the eigenvalues adimensional (we take \( \hbar = c = 1 \) and shall insert the dimensionfull units only at the end of the calculation). The mass \( M \) of the field will be here considered to be arbitrarily small and will be kept different from zero, for the moment, for computational reasons—as well as for physical ones, since a very tiny mass for the field can never be excluded. Some comments about the choice of our model are in order. The first seems obvious: the coupling of the scalar field to gravity should be considered. This has been done in all detail in, e.g., Ref. [107] (see also the references therein). The conclusion is that taking it into account does not change the results to be obtained here. Of course, the renormalization of the model is rendered much more involved, and one must enter a discussion on the orders of magnitude of the different contributions, which yields, in the end, an ordinary perturbative expansion, the coupling constant being...
finally re-absorbed into the mass of the scalar field. In conclusion, we would not gain anything new by taking into account the coupling of the scalar field to gravity. Owing, essentially, to the smallness of the resulting mass for the scalar field, one can prove that, quantitatively, the difference in the final result is at most of a few percent.

Another important consideration is the fact that our model is stationary, while the universe is expanding. Again, careful calculations show that this effect can actually be dismissed at the level of our order of magnitude calculation, since its value cannot surpass the one that we will get (as is seen from the present value of the expansion rate $\Delta R/R \sim 10^{-10}$ per year or from direct consideration of the Hubble coefficient). As before, for the sake of simplicity, and in order to focus just on the essential issues of our argument, we will perform a (momentaneously) static calculation. As a consequence, the value of the Casimir energy density, and of the cosmological constant, to be obtained will correspond to the present epoch, and are bound to change with time.

The last comment at this point would be that (as shown by the many references mentioned above), the idea presented here is not entirely new. However, the simplicity and the generality of its implementation below are indeed new. The issue at work here is absolutely independent of any specific model, the only assumptions having been clearly specified before (e.g., existence of a very light scalar field and of some reasonably compactified scales, see later). Secondly, it will turn out, in the end, that the only ‘free parameter’ to play with (the number of compactified dimensions) will actually not be that ‘free’ but, on the contrary, very much constrained to have an admissible value. This will become clear after the calculations below. Thirdly, although the calculation may seem easy to do, in fact it is not so. Some reflection identities, due to the author, will allow to be performed analytically.

### 4.8. The vacuum energy density and its regularization

To exhibit explicitly a couple of the wide family of cases considered, let us write down in detail the formulas corresponding to the two first topologies, as described above. For a $(p, q)$-toroidal universe, with $p$ the number of ‘large’ and $q$ of ‘small’ dimensions:

$$
\rho_\phi = \frac{\pi^{-d/2}}{2^d \Gamma(d/2) \prod_{j=1}^{\ell} a_j \prod_{h=1}^{\ell} b_h} \int_0^\infty dk \, k^{d-1} \sum_{n_p = -\infty}^{\infty} \sum_{n_q = -\infty}^{\infty} \left[ \sum_{j=1}^{p} \left( \frac{2\pi n_j}{a_j} \right)^2 + \sum_{h=1}^{q} \left( \frac{2\pi m_h}{b_h} \right)^2 + M^2 \right]^{1/2} \approx \frac{1}{a^p b^q} \sum_{n_p, n_q = -\infty}^{\infty} \left( \frac{1}{a^2} \sum_{j=1}^{p} n_j^2 + \frac{1}{b^2} \sum_{h=1}^{q} m_h^2 + M^2 \right)^{(d+1)/2+1},
$$

(113)
where the last formula corresponds to the case when all large (resp. all small) compactification scales are the same. In this last expression the squared mass of the field should be divided by $4\pi^2\mu^2$, but we have renamed it again $M^2$ to simplify the ensuing formulas (as $M$ is going to be very small, we need not keep track of this change). We also will not take care for the moment of the mass-dim factor $\mu$ in other places — as is usually done — since formulas would get unnecessarily complicated and there is no problem in recovering it at the end of the calculation. For a $(p$-toroidal, $q$-spherical)-universe, the expression turns out to be

$$\rho_\phi = \frac{\pi^{-d/2}}{2^d \Gamma(d/2)} \prod_{j=1}^p \frac{1}{a_j b_j} \int_0^\infty dk k^{d-1} \sum_{n_p=-\infty}^{\infty} \sum_{l=1}^{\infty} P_{q-1}(l) \times \left[ \sum_{j=1}^p \left( \frac{2\pi n_j}{a_j} \right)^2 + \frac{Q_2(l)}{b_j^2} + M^2 \right]^{1/2} \sim \frac{1}{a^p b^q} \sum_{n_p=-\infty}^{\infty} \sum_{l=1}^{\infty} P_{q-1}(l) \left( \frac{4\pi^2}{a^2} \sum_{j=1}^p n_j^2 + \frac{l(l+q)}{b^2} + M^2 \right)^{(d+1)/2+1},$$

(114)

where $P_{q-1}(l)$ is a polynomial in $l$ of degree $q - 1$, and where the second formula corresponds to the similar situation as the second one before. On dealing with our observable universe, in all these expression we assume that $d = 3 - p$, the number of non-compactified, 'large' spatial dimensions (thus, no $d$ dependence will remain).

As is clear, all these expressions for $\rho_\phi$ need to be regularized. We will use zeta function regularization, taking advantage of the very powerful equalities that have been derived by the author, and which reduce the enormous burden of such computations to the easy application of some formulas. For the sake of completeness, let us very briefly summarize how this works. We deal here only with the case when the spectrum of the Hamiltonian operator is known explicitly. Going back to the most general expressions of the Casimir energy corresponding to this case, namely Eq. (112), we replace the exponents in them with a complex variable, $s$, thus obtaining the zeta function associated with the operator as:

$$\zeta(s) = \frac{1}{2} \sum_k \left( \frac{k^2 + M^2}{\mu^2} \right)^{-s/2}. \quad (115)$$

The next step is to perform the analytic continuation of the zeta function from a domain of the complex $s$-plane with $\text{Re } s$ big enough (where it is perfectly defined by this sum) to the point $s = -1$, to obtain:

$$\rho_\phi = \zeta(-1). \quad (116)$$

The effectiveness of this method has been sufficiently described before (see, e.g., [9][10]). As we know from precise Casimir calculations in those references, no further subtraction or renormalization is needed in the cases here considered, in order to
obtain the physical value for the vacuum energy density (there is actually a subtraction at infinity taken into account, as carefully described above, but it is of null value, and no renormalization, not even a finite one, very common to other frameworks, applies here).

Using the formulas that generalize the well-known Chowla-Selberg expression to the situations considered above, Eqs. (113) and (114)—namely, multidimensional, massive cases—we can provide arbitrarily accurate results for different values of the compactification radii. However, as argued above we only aim here at matching the order of magnitude of the Casimir value and, thus, we shall just deal with the most simple cases of Eqs. (113) or (114), which yield the same orders of magnitude as the rest of them). Also in accordance with this observation, we notice that among the models here considered and which lead to the values that will be obtained below, there are in particular the very important typical cases of isotropic universes with the spherical topology. As all our discussion here is in terms of orders of magnitude and not of precise values with small errors, all these cases are included on equal footing. But, on the other hand, it has no sense to present a lengthy calculation dealing in detail with all the possible spatial geometries. Anyhow, all these calculations can indeed be done, and are very similar to the one here, as has been described in detail elsewhere. 104

For the analytic continuation of the zeta function corresponding to (113), we obtain:

\[
\zeta(s) = \frac{2\pi^{s/2+1}}{a^{(s+1)/2}b^{-(s-1)/2}\Gamma(s/2)} \sum_{m_h=-\infty}^{\infty} \sum_{h=0}^{p} \left( \frac{p}{h} \right) 2^h \sum_{n_h=1}^{\infty} \left( \sum_{k=1}^{h} m_k^2 + M^2 \right)^{(s-1)/4} \\
\times K_{(s-1)/2} \left[ \frac{2\pi a}{b} \left( \sum_{j=1}^{n_j} \left( \sum_{k=1}^{m_j} m_k^2 + M^2 \right) \right) \right],
\]

(117)

where \( K_{\nu}(z) \) is the modified Bessel function of the second kind. Having performed already the analytic continuation, this expression is ready for the substitution \( s = -1 \), and yields

\[
\rho_\phi = -\frac{1}{a^{p+1}} \sum_{h=0}^{p} \left( \frac{p}{h} \right) 2^h \sum_{m_h=1}^{\infty} \sum_{n_h=-\infty}^{\infty} \sqrt{\sum_{k=1}^{h} m_k^2 + M^2} \\
\times K_1 \left[ \frac{2\pi a}{b} \left( \sum_{j=1}^{n_j} \left( \sum_{k=1}^{m_j} m_k^2 + M^2 \right) \right) \right].
\]

(118)

Now, from the behaviour of the function \( K_{\nu}(z) \) for small values of its argument,

\[
K_{\nu}(z) \sim \frac{1}{2} \Gamma(\nu)(z/2)^{-\nu}, \quad z \to 0,
\]

(119)
we obtain, in the case when $M$ is very small,

$$
\rho_\phi = -\frac{1}{a^{p+1}} \left\{ M K_1 \left( \frac{2\pi a}{b} M \right) + \sum_{h=0}^{p} \frac{b^h}{h!} \sum_{n=1}^{\infty} \frac{M}{\sqrt{\sum_{j=1}^{n^2}}} n_j \right\} \times K_1 \left( \frac{2\pi a}{b} M \right) + O \left[ q \sqrt{1 + M^2} K_1 \left( \frac{2\pi a}{b} \sqrt{1 + M^2} \right) \right].
$$

(120)

At this stage, the only presence of the mass-dim parameter $\mu$ is as $M/\mu$ everywhere. This does not conceptually affect the small-$M$ limit, $M/\mu \ll b/a$. Using (119) and inserting now in the expression the $\hbar$ and $c$ factors, we finally get

$$
\rho_\phi = -\frac{\hbar c}{2\pi a^{p+1}} \left[ 1 + \sum_{h=0}^{p} \left( \frac{b}{h!} \right) 2^h \alpha \right] + O \left[ q K_1 \left( \frac{2\pi a}{b} \right) \right],
$$

(121)

where $\alpha$ is some finite constant, computable and under control, which is obtained as an explicit geometrical sum in the limit $M \to 0$. It is remarkable that we here obtain such a well defined limit, independent of $M^2$, provided that $M^2$ is small enough. In other words, a physically very nice situation turns out to correspond, precisely, to the mathematically rigorous case. This is moreover, let me repeat, the kind of expression that one gets not just for the model considered, but for many other cases, corresponding to different fields, topologies, and boundary conditions—aside from the sign in front of the formula, that may change with the number of compactified dimensions and the nature of the boundary conditions (in particular, for Dirichlet boundary conditions one obtains a value in the same order of magnitude but of opposite sign).

4.9. Numerical results

For the most common variants, the constant $\alpha$ in (121) has been calculated to be of order $10^2$, and the whole factor, in brackets, of the first term in (121) has a value of order $10^7$. This shows the value of a precise calculation, as the one undertaken here, together with the fact that just a naive consideration of the dependencies of $\rho_\phi$ on the powers of the compactification radii, $a$ and $b$, is not enough in order to obtain the correct result. Notice, moreover, the non-trivial change in the power dependencies from going from Eq. (120) to Eq. (121).

For the compactification radii at small scales, $b$, we shall simply take the magnitude of the Planck length, $b \sim l_P(\text{Planck})$, while the typical value for the large scales, $a$, will be taken to be the present size of the observable universe, $a \sim R_U$. With this choice, the order of the quotient $a/b$ in the argument of $K_1$ is as big as: $a/b \sim 10^{60}$. Thus, we see immediately that, in fact, the final expression for the vacuum energy density is completely independent of the mass $M$ of the field, provided this is very small (eventually zero). In fact, since the last term in Eq. (121) is exponentially vanishing, for large arguments of the Bessel function $K_1$, this contribution is zero,
for all practical purposes, what is already a very nice result. Taken in ordinary units
(and after tracing back all the transformations suffered by the mass term $M$) the
actual bound on the mass of the scalar field is $M \leq 1.2 \times 10^{-32}$ eV, that is, physi-
cally zero, since it is lower by several orders of magnitude than any bound coming
from the more usual SUSY theories — where in fact scalar fields with low masses of
the order of that of the lightest neutrino do show up, which may have observable
implications.

Table 1. Orders of magnitude of the vacuum energy density con-
tribution, $\rho_\phi$, of a massless scalar field to the cosmological con-
stant, $\rho_V$, for $p$ large compactified dimensions and $q = p + 1$ small
compactified dimensions, $p = 0, \ldots, 3$, for different values of the
small compactification length, $b$, proportional to the Planck length.

| $p$ | $p = 0$ | $p = 1$ | $p = 2$ | $p = 3$ |
|-----|----------|----------|----------|----------|
| $b = l_P$ | $10^{-13}$ | $10^{-6}$ | $1$ | $10^9$ |
| $b = 10l_P$ | $10^{-14}$ | $(10^{-8})$ | $10^{-3}$ | $10$ |
| $b = 10^2l_P$ | $10^{-15}$ | $(10^{-10})$ | $10^{-6}$ | $10^{-3}$ |
| $b = 10^3l_P$ | $10^{-16}$ | $(10^{-12})$ | $10^{-9}$ | $10^{-7}$ |
| $b = 10^4l_P$ | $10^{-17}$ | $10^{-14}$ | $(10^{-12})$ | $10^{-11}$ |
| $b = 10^5l_P$ | $10^{-18}$ | $10^{-16}$ | $10^{-15}$ | $10^{-15}$ |

By replacing all these values in Eq. (121), we obtain the results listed in Table 1
for the orders of magnitude of the vacuum energy density corresponding to a sample
of different numbers of compactified (large and small) dimensions and for different
values of the small compactification length in terms of the Planck length. Notice
again that the total number of large space dimensions is three, as corresponds to our
observable universe. As we see from Table 1 good coincidence with the observational
value for the cosmological constant is obtained for the contribution of a massless
scalar field, $\rho_\phi$, for $p$ large compactified dimensions and $q = p + 1$ small compactified
dimensions, $p = 0, \ldots, 3$, and this for values of the small compactification length,
$b$, of the order of 100 to 1000 times the Planck length $l_P$ (what is actually a very
reasonable conclusion, according also to other approaches).

To be noticed is the fact that full agreement is obtained only for cases where
there is exactly one small compactified dimension in excess of the number of large
compactified dimensions. We must point out that the $p$ large and $q$ small dimensions
are not all that are supposed to exist (in that case $p$ should be at least, and at
most, 3 and the other cases would lack any physical meaning). In fact, as we have
pointed out before, $p$ and $q$ refer to the compactified dimensions only, but there
may be other, non-compactified dimensions (exactly $3 - p$ in the case of the ‘large’
ones), what translates into a slight modification of the formulas above, but does not change the order of magnitude of the final numbers obtained, assuming the most common boundary conditions for the non-compactified dimensions (see e.g. [10] for an explanation of this technical point). In particular, the cases of pure spherical compactification and of mixed toroidal (for small magnitudes) and spherical (for big ones) compactification can be treated in this way and yield results in the same order of magnitude range. Both these cases correspond to (observational) isotropic spatial geometries. Also to be remarked again is the non-triviality of these calculations, when carried out exactly, as done here, to the last expression, what is apparent from the use of the generalized Chowla-Selberg formula. Simple power counting is absolutely unable to provide the correct order of magnitude of the results.

Dimensionally speaking, within the global approach adopted in the present paper everything is dictated, in the end, by the two basic lengths in the problem, which are its Planck value and the radius of the observable Universe. Just by playing with these numbers in the context of this precise calculation of the Casimir effect, we have shown that the observed value of $\rho_V$ may be remarkably well fitted, under general hypothesis, for the most common models of the space-time topology. Notice also that the most precise fits with the observational value of the cosmological constant are obtained for $b$ between $b = 100 l_P$ and $b = 1000 l_P$, with (1,2) and (2,3) compactified dimensions, respectively. The fact that the value obtained for the cosmological constant is so sensitive to the input may be viewed as a drawback but also, on the contrary, as a very positive feature of our model. For one, the Table 1 has a sharp discriminating power. In other words, there is in fact no tuning of a ‘free parameter’ in our model and the number of large compactified dimensions could have been fixed beforehand, to respect what we know already of our observable universe.

Also, it proves that the observational value is not easy at all to obtain. Table 1 itself proves that there is only very little chance of getting the right figure (a truly narrow window, since very easily we are off by several orders of magnitude). In fact, if we trust this value with the statistics at hand, we can undoubtedly claim —through use of the model— that the ones so clearly picked up by Table 1 are the only two possible configurations of our observable universe (together with a couple more coming from corresponding spherical compactifications). And all them correspond to a marginally closed universe, in full agreement too with other completely independent analysis of the observational data [73, 91, 92].

Many questions may be posed to the simple models presented here, as concerning the dynamics of the scalar field, its couplings with gravity and other fields, a possible non-symmetrical behaviour with respect to the large and small dimensions, or the relevance of vacuum polarization (see Ref. [110] concerning this last point). Above we have already argued that they can be proven to have little influence on the final numerical result (cf., in particular, the mass obtained for the scalar field in Ref. [107], extremely close to our own result, and the corresponding discussion there). From
the very existence and specific properties of the cosmic microwave radiation (CMB) —which mimics somehow the situation described (the ‘mass’ corresponding to the CMB is also in the sub-lightest-neutrino range)— we are led to the conclusion that such a field could be actually present, unnoticed, in our observable universe. In fact, the existence of scalar fields of very low masses is also demanded by other frameworks, as SUSY models, where the scaling behaviour of the cosmological constant has been considered.

Let us finally recall again that the Casimir effect is an ubiquitous phenomena. Its contribution may be small (as it seems to be the case, yet controverted, to sonoluminiscence), of some 10-30% (that is, of the right order of magnitude, as in wetting phenomena involving He in condensed matter physics). Here we have seen that it provides a contribution of the right order of magnitude, corresponding to our present epoch in the evolution of the universe. The implication that this calculation bears for the early universe and inflation is not clear from the final result, since it should be adapted to the situation and boundary conditions corresponding to those primeval epochs, what cannot be done straightforwardly. Work along this line is in progress.

Acknowledgments

The author is indebted to their former students and collaborators on some of the subjects of these lectures: Enrique Gaztañaga, Pablo Fosalba, Sergi R. Hildebrandt and Jose Barriga, as well as to the members of the Mathematics Department, MIT, and specially to Dan Freedman, for the continued hospitality. This investigation has been supported by DGICYT (Spain), project BFM2003-00620 and by the APM Service, MEC (Spain), grant PR2004-0126.

References

1. S. Weinberg, *The First Three Minutes* (Basic Books, NY, 1977); L. Krauss, *The fifth essence: the search for dark matter in the universe* (Basic Books, NY, 1989); L. Lederman and D.N. Schramm, *From quarks to the cosmos*, Scientific American Library (Freeman, NY, 1989); H. Pagels, *Perfect Symmetry* (M. Joseph, London, 1985); J.D. Barrow and F.J. Tipler, *The Anthropic Cosmological Principle* (Oxford UP, Oxford, 1986).
2. E. Wigner, Commun. Pure Appl. Math., 13, 1 (1960).
3. P. Ramond, *Field Theory, a Modern Primer* (Benjamin, Reading, Mass., 1981).
4. N. Birrell and P.C.W. Davies, *Quantum Fields in Curved Spaces* (Cambridge University Press, Cambridge, 1982).
5. I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, *Effective Action in Quantum Gravity* (IOP Publishing, Bristol, 1992).
6. S.W. Hawking, Commun. Math. Phys. 55 133 (1977); S.W. Hawking and W. Israel, Eds., *General Relativity, an Einstein Centenary Survey* (Cambridge University Press, 1979).
7. M. Kontsevich and S. Vishik, *Functional Analysis on the Eve of the 21st Century*. Volume 1, 173 (1993).
8. E. Elizalde, J. Comput. Appl. Math. 118, 125 (2000); E. Elizalde, J. Phys. A34, 3025 (2001).
9. E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko, and S. Zerbini, Zeta Regularization Techniques with Applications (World Scientific, Singapore, 1994); A.A. Bytsenko, G. Cognola, L. Vanzo and S. Zerbini, Phys. Reports 266, 1 (1996); K. Kirsten, Spectral functions in mathematics and physics (Chapman & Hall, London, 2001).
10. E. Elizalde, Ten Physical Applications of Spectral Zeta Functions (Springer-Verlag, Berlin, 1995).
11. A.A. Bytsenko, G. Cognola, E. Elizalde, V. Moretti and S. Zerbini, Analytic Aspects of Quantum Fields (World Scientific, Singapore, 2004).
12. B. Misra and E.C.G. Sudarshan, J. Math. Phys. 18, 756 (1977); A.P. Balachandran and S.M. Roy, Phys. Rev. Lett. 84, 4019 (2000).
13. H.W. Turnbull, The Great Mathematicians (New York University Press, NY, 1961); E.J. Alton, Encyclopedia of World Biography. (McGraw-Hill, NY, 1973) vol. 4, pp. 30-31.
14. E. Maor, To Infinity and Beyond: a Cultural History of the Infinite (Princeton University Press, 1991).
15. G.H. Hardy, Divergent Series (Oxford University Press, 1949).
16. G.A. Baker, Jr., and P. Graves-Morris, Padé Approximants (Cambridge University Press, 1996); C.M. Bender and S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers (McGraw-Hill, New York, 1978).
17. J. Collins, Renormalization: An Introduction to Renormalization, the Renormalization Group and the Operator-Product Expansion, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1984); W.D. McComb, Renormalization Methods: A Guide for Beginners (Oxford University Press, 2004).
18. S.S. Gelbart and S.D. Miller, Riemann’s zeta function and beyond, Bull. Amer. Math. Soc. (NS) 41 (2004) 59.
19. R.S. Van Dyck, Jr. Anomalous Magnetic Moment of Single Electrons and Positrons: Experiment. In Quantum Electrodynamics (Ed. T. Kinoshita) (World Scientific, Singapore, 1990) pp. 322-388; T. Kinoshita, Rept. Prog. Phys. 59, 1459 (1996).
20. H.B.G. Casimir, Proc. K. Ned. Acad. Wet. 51, 635 (1948).
21. S. K. Lamoreaux, Phys. Rev. Letters, 78, 5 (1997); U. Mohideen and A. Roy, Phys. Rev. Lett. 81 (1998) 4549; B.W. Harris, F. Chen, and U. Mohideen, Phys. Rev. A 62, 052109 (2000).
22. S. Chowla and A. Selberg, Proc. Nat. Acad. Sci. (USA), 35, 317 (1949).
23. E. Elizalde, J. Phys. A27, 3775 (1994); E. Elizalde, Commun. Math. Phys. 198, 83 (1998).
24. V. de Lapparent, M.J. Geller, and J. Huchra, Astrophys. J. Lett. 302, L1 (1986); V. de Lapparent, M.J. Geller, and J. Huchra, Astrophys. J. 332, 44 (1988).
25. L.N. da Costa et al., Astrophys. J. 327, 544 (1988).
26. G.R. Blumenthal, S.M. Faber, J.R. Primack, and M.J. Rees, Nature, 311, 517 (1984).
27. B.J.T. Jones, The Large Scale Structure of the Universe, in Observational and Physical Cosmology, eds, F. Sáchez, M. Collados and R. Rebolo (Cambridge Univ. Press, Cambridge, 1992).
28. J.P. Ostriker, C. Thompson, and E. Witten, Phys. Lett. B180, 231 (1986).
29. http://www.rssd.esa.int/index.php?project=Planck
30. http://sci.esa.int/science-e/www/object/index.cfm?fobjectid=34730
31. http://www.sdlss.org.
32. http://www.aao.gov.au/2df/
33. http://www-astro.physics.ox.ac.uk/~wjs/apm_survey.html
34. http://map.gsfc.nasa.gov/
35. M.J. Geller, and J. Huchra, Science 246, 897 (1989).
36. E. Gaztañaga, Curso de Astronomia y Astrofisica, UAB (2004) http://www.inaoep.mx/~gazta/
37. H. Totsuji and T. Kihara, Publ. Astron. Soc. Japan 21, 221 (1969).
38. S.M. Carroll, V. Duvvuri, M. Trodden, and M.S. Turner, Phys. Rev. D 70, 043528 (2004); D. Munshi, C. Porciani, and Y. Wang, Mon. Not. Roy. Astron. Soc. 349, 281 (2004).
39. E. Elizalde, S. Nojiri, S.D. Odintsov, Late-time cosmology in (phantom) scalar-tensor theory: dark energy and the cosmic speed-up, hep-th/0405034, to appear in Phys. Rev. D; G.W. Gibbons, Phantom Matter and the Cosmological Constant, hep-th/0302199.
40. R. R. Caldwell, A Phantom Menace, Phys. Lett. B 545, 23 (2002).
41. R.K. Sachs and A.M. Wolfe, Perturbations of a cosmological model and angular variations of the cosmic microwave background, Astrophys. J. 147, 73 (1967).
42. E.L. Wright, Cosmology Tutorial, http://www.astro.ucla.edu/~wright/cosmolog.html.
43. P. Schneider, Gravitational lensing as a probe of structure, XIV Canary Islands Winter School of Astrophysics “Dark Matter and Dark Energy in the Universe,” Tenerife, Spain (2003).
44. J. Cowan et al., Astrophys. J. 521, 194 (1999).
45. B. Hansen, R.M. Rich, R. Ibata, B.K. Gibson, and M.M. Shara, Astronomical J. 127, 2771 (2004).
46. W.C. Saslaw, The distribution of the galaxies: Gravitational clustering in Cosmology (Cambridge University Press, Cambridge, 1999)
47. S. Chastee, University of California-Santa Cruz, Stanford Report, January 22, 2003.
48. A.H. Guth, Phys. Rev. B 23, 347 (1981); A. Albrecht and P. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982); A. D. Linde, Phys. Lett. B 108, 389 (1982); A. D. Linde, Phys. Lett. B 129, 177 (1983).
49. A.H. Guth, The Inflationary Universe: The Quest for a New Theory of Cosmic Origins (Perseus Publishing, 1998); A. Linde, Particle Physics and Inflationary Cosmology (Harwood, NY, 1990); A.H. Guth, Inflation and the New Era of High-Precision Cosmology, lecture (physics@mit, Fall 2002).
50. L. Knox, Future Probes of the Primordial Scalar and Tensor Perturbation Spectra: Prospects from the CMB, Cosmic Shear and High-Volume Redshift Surveys, Davis Meeting on Cosmic Inflation, 2003, astro-ph/0304370. A.R. Liddle and D.H. Lyth, Cosmological inflation and large-scale structure (Cambridge University Press, 2000); E.W. Kolb and M.S. Turner, The early universe (Addison-Wesley, NY, 1990); J.V. Narlikar and T. Padmanabhan, Gravity, gauge theories, and quantum cosmology (Reidel, 1986).
51. E. Farhi and A.H. Guth, An obstacle to creating a universe in the laboratory, Phys. Lett. B 183, 149 (1987).
52. A. Borde, A.H. Guth, and A. Vilenkin, Phys. Rev. Lett. 90, 151301 (2003).
53. M. Tegmark, Measuring spacetime: from Big Bang to Black Holes; Science, 296, 1427 (2002).
54. N.J. Cornish et al., Phys. Rev. Lett. 92, 201302 (2004).
55. J.-P. Luminet et al., Nature, 425, 593 (2003).
57. M.G. Kendall, A. Stuart, and J.K. Ord, *Kendall’s advanced theory of statistics* (Oxford University Press, NY, 1991).

58. E. Gaztañaga, P. Fosalba, and E. Elizalde, Astrophys. J. **539**, 522 (2000); A. Romeo, E. Gaztañaga, J. Barriga, E. Elizalde, Mon. Not. Roy. Astron. Soc. **320**, 12 (2001); A. Romeo, E. Gaztañaga, J. Barriga, E. Elizalde, Int. J. Mod. Phys. **C10**, 687 (1999).

59. E. Gaztañaga, P. Fosalba, and E. Elizalde, Astrophys. J. **539**, 522 (2000); A. Romeo, E. Gaztañaga, J. Barriga, E. Elizalde, Mon. Not. Roy. Astron. Soc. **295**, L35 (1998); P. Fosalba, E. Gaztañaga, and E. Elizalde, *Gravitational Evolution of the Large-Scale Density Distribution: The Edgeworth and Gamma Expansions*, in Proceedings of the IGRAP meeting “Clustering at high redshift” (Marseille, France, June 1999).

60. H.K. Eriksen, et al., *Testing for Non-Gaussianity in the Wilkinson Microwave Anisotropy Probe Data: Minkowski Functionals and the Length of the Skeleton*, Astrophys. J. **612**, 64 (2004); D.L. Larson and B.D. Wandelt, *The Hot and Cold Spots in the WMAP Data are Not Hot and Cold Enough*, astro-ph/0404037.

61. V.J. Martínez and E. Saar, *Statistics of the Galaxy Distribution* (Chapman and Hall/CRC Press, Boca Raton, 2002); J.A. Peacock, *Large-scale surveys and cosmic structure*, Lectures delivered at the 2002 Tenerife Winter School, “Dark matter and dark energy in the universe”, astro-ph/0309240; O. Lahav and Y. Suto, *Measuring our universe with redshift surveys*, Living Reviews in Relativity 7, 8 (2004); O. Lahav and A.R. Liddle, *The Cosmological Parameters*, in “The Review of Particle Physics”, S. Eidelman et al. (Particle Data Group), Phys. Lett. **B592**, 1 (2004).

62. J.D. Bjørken, Phys. Rev. **D67**, 043508 (2003).

63. S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989); W. Fischler, I. Klebanov, J. Polchinski, and L. Susskind, Nucl. Phys. **B237**, 157 (1989); S. Coleman, Nucl. Phys. **B310**, 643 (1988); S. Coleman, Nucl. Phys. **B307**, 867 (1988); S. Weinberg, Phys. Rev. Lett. **59**, 2607 (1987); E. Baum, Phys. Lett. **B133**, 185 (1984); S. W. Hawking, in *Shelter Island II - Proceedings of the 1983 Shelter Island Conference on Quantum Field Theory and the Fundamental Problems of Physics*, ed. R. Jackiw et al. (MIT Press, Cambridge, 1995); Phys. Lett. **B134**, 403 (1984).

64. S. Weinberg, *Gravitation and Cosmology* (John Wiley and Sons, NY, 1972).

65. F. Wilczek, *Total Relativity: Mach 2004*, Physics Today, April 2004, pp. 10-11.

66. [http://einstein.stanford.edu/](http://einstein.stanford.edu/) [http://www.gravityprobeb.com/]

67. J. Lense and H. Thirring, *Über den Einfluss der Eigenrotation der Zentralkörper auf die Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie*, Phys. Z. **19**, 156 (1918).

68. K. Schwarzschild, *On the Gravitational Field of a Mass Point According to Einstein’s Theory*, Sitzungsberichte der Kniglich Preussischen Akademie der Wissenschaften zu Berlin, Phys.-Math. Klasse, 189 (1916).

69. A. Friedmann, *On the curvature of Space*, Zeitschrift für Physik, **21**, 326 (1924); G. Lematre, Ann. Soc. Sci. Bruxelles **A47**, 49 (1927); G. Lematre, Mon. Not. Roy. Astron. Soc. **91**, 483 (1931); H.P. Robertson, *Relativistic cosmology*, Rev. Mod. Phys. **5**, 62 (1933); H.P. Robertson, *Kinematics and World Structure*, Astrophys. J. **82**, 248 (1935); A.G. Walker, *On Milne’s Theory of World-Structure*, Proc. London Math. Soc. **42**, 90 (1936).

70. M. Milgrom, Astrophys. J. **270**, 365, 371 (1983); M. Kaplinghat and M.S. Turner, Astrophys. J. **569**, L19 (2002); [http://www.astro.umd.edu/~ssm/mond/](http://www.astro.umd.edu/~ssm/mond/)

71. J.P. Ostriker and P.J. Steinhardt, Nature **377**, 600 (1995); R. Caldwell, R. Dave and P.J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998); I. Zlatev, L. Wang and P.J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999); C. Armendariz-Picon, V. Mukhanov, P.J. Steinhardt, Phys. Rev. Lett. **85**, 4438 (2000).
72. P.J. Peebles, *Principles of Physical Cosmology* (Princeton UP, Princeton NJ, 1993); P.J. Peebles, *Large-Scale Structure of the Universe* (Princeton UP, Princeton NJ, 1980).

73. S.M. Carroll, Living Rev. Rel. **4**, 1 (2001).

74. S.M. Carroll, *Why is the Universe Accelerating?*, Contribution to *Measuring and Modeling the Universe*, Carnegie Observatories Astrophysics Series Vol. 2, ed. W. L. Freedman, astro-ph/0310342.

75. T. Padmanabhan, *Structure Formation in the Universe* (Cambridge UP, Cambridge, 1993).

76. K. Kirsten and E. Elizalde, Phys. Lett. **B365**, 72 (1995).

77. E. Elizalde, Commun. Math. Phys. **198**, 83 (1998).

78. A.P. Calderón and A. Zygmund, Am. J. Math. **79**, 801 (1957); Studia Math. **20**, 171 (1961); A.P. Calderón and R. Vaillancourt, Proc. Nat. Acad. Sci. U.S.A. **69**, 1185 (1972).

79. L. Hörmander, *The analysis of partial differential operators*, Vols I-IV (Springer, Berlin, 1983-85); F. Treves, *Introduction to pseudodifferential and Fourier integral operators*, Vols. I and II (Plenum, New York, 1980); M.E. Taylor, *Pseudodifferential operators* (Princeton University Press, Princeton, 1981); H. Lawson and M.L. Michelsohn, *Spin geometry* (Princeton University Press, Princeton, 1989).

80. G.H. Hardy, *Divergent Series* (Oxford University Press, Oxford, 1949).

81. T. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin, 1980).

82. E. Witten, *Supersymmetric index of three-dimensional gauge theory*, hep-th/9903005; E. Witten, Commun. Math. Phys. **121**, 351 (1989).

83. D.B. Ray, Adv. in Math. **4**, 109 (1970); D.B. Ray and I.M. Singer, Adv. in Math. **7**, 145 (1971); Ann. Math. **98**, 154 (1973).

84. E. Elizalde, J. Phys. **A34**, 3025 (2001); E. Elizalde, J. Comput. Appl. Math. **118**, 125 (2000); E. Elizalde, Commun. Math. Phys. **198**, 83 (1998); E. Elizalde, J. Phys. **A30**, 2735 (1997); K. Kirsten and E. Elizalde, Phys. Lett. **B365**, 72 (1995); E. Elizalde, J. Phys. **A27**, 3775 (1994); E. Elizalde, J. Phys. **A27**, L299 (1994); E. Elizalde, J. Phys. **A22**, 931 (1989); E. Elizalde and A. Romeo, Phys. Rev. **D40**, 436 (1989).

85. D. Deutsch and P. Candelas, Phys. Rev. **D20**, 3063 (1979).

86. K. Symanzik, Nucl. Phys. **B190**, 1 (1981).

87. R. L. Jaffe, *Unnatural acts: unphysical consequences of imposing boundary conditions on quantum fields*, hep-th/0307014; N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra and H. Weigel, *Casimir energies in light of quantum field theory*, hep-th/0207205; N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra and H. Weigel, Nucl. Phys. **B645**, 49 (2002); N. Graham, R. L. Jaffe and H. Weigel, Int. J. Mod. Phys. **A17**, 846 (2002).

88. V.M. Mostepanenko and N.N. Trunov, *The Casimir effect and its application* (Clarendon Press, Oxford, 1997); K.A. Milton, *The Casimir Effect: Physical manifestations of zero-point energy* (World Scientific, Singapore, 2001); M. Bordag, U. Mohideen and V.M. Mostepanenko, Phys. Rept. **353**, 1 (2001).

89. E. Elizalde, M. Bordag and K. Kirsten, J. Phys. **A31**, 1743 (1998); E. Elizalde, L. Vanzo and S. Zerbini, Commun. Math. Phys. **194**, 613 (1998); M. Bordag, E. Elizalde, K. Kirsten and S. Leseduarte, Phys. Rev. **D56**, 4896 (1997); M. Bordag, E. Elizalde and K. Kirsten, J. Math. Phys. **37**, 895 (1996); M. Bordag, E. Elizalde, B.eyer and K. Kirsten, Commun. Math. Phys. **179**, 215 (1996).

90. E. Elizalde, S. Naftulin and S.D. Odintsov, Phys. Rev. **D49**, 2852 (1994); E. Elizalde, S. Leseduarte, S.D. Odintsov, Phys. Rev. **D49**, 5551 (1994); E. Elizalde and S.D. Odintsov, Phys. Lett. **B303**, 240 (1993); E. Elizalde, S. Leseduarte, S.D. Odintsov
and Yu.I. Shil’nov, Phys. Rev. D 53, 1917 (1996); E. Elizalde, S. Nojiri and S.D. Odintsov, Phys. Rev. (Rapid Communications) D 59, 61501 (1999); E. Elizalde, S. Nojiri, S.D. Odintsov, and S. Ogushi, Phys. Rev. D 67, 063515 (2003); E. Elizalde, J.E. Lidsey, S. Nojiri and S.D. Odintsov, Phys. Lett. B 574, 1 (2003); E. Elizalde and A.C. Tort, Mod. Phys. Lett. A 19, 111 (2004); E. Elizalde and A.C. Tort, Phys. Rev. D 66, 045033 (2002); E. Elizalde and J. Quiroga Hurtado, Mod. Phys. Lett. A 19, 29 (2004); G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov and S. Zerbini, Mod. Phys. Lett. A 19, 1435 (2004); E. Elizalde, S. Nojiri, S.D. Odintsov, Late-time cosmology in (phantom) scalar-tensor theory: dark energy and the cosmic speed-up, Phys. Rev. D (2004), to appear, hep-th/0405034.

91. S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999).
92. A.G. Riess et al. [Hi-Z Supernova Team Collaboration], Astron. Journ. 116, 1009 (1998).
93. J.L. Tonry, et al., Astrophys. J. 594, 1 (2003); R.A. Knop et al., et al., Astrophys. J. 598, 102 (2003); G. Garavini, et al., Astronomical J. 128, 387 (2004).
94. A.G. Riess, The case for an accelerating universe from supernovae, astro-ph/0005229.
95. P. de Bernardis et al., Nature 404, 955 (2000).
96. S. Hanany et al., MAXIMA-1: A measurement of the cosmic microwave background anisotropy on angular scales of 10' to 5', astro-ph/0005123; A. Balbi et al., Constraints on cosmological parameters from MAXIMA-1, astro-ph/0005124.
97. S. Weinberg, Phys. Rev. D 61, 103505 (2000).
98. V. Sahni and A. Starobinsky, The case for a positive cosmological $\Lambda$–term, astro-ph/9904398.
99. I.L. Shapiro and J. Solà, Phys. Lett. B 475, 236 (2000).
100. T.R. Mongan, Gen. Rel. Grav. 33, 1415 (2001).
101. V. Blanloeil and B.F. Roukema, Eds., Cosmological Topology in Paris 1998, astro-ph/0010170. See also the entire Vol. 15 of Classical and Quantum Gravity (1998).
102. I.Yu. Sokolov, JETP Lett. 57, 617 (1993).
103. N.J. Cornish, D. Spergel, and G. Starkman, Class. Quant. Grav. 15, 2657 (1998); ibid, Phys. Rev. Lett. 77, 215 (1996); D. Müller, H.V. Fagundes, and R. Opher, Phys. Rev. D 63, 123508 (2001).
104. E. Elizalde, J. Math. Phys. 35, 3308 (1994); E. Elizalde, J. Math. Phys. 35, 6100 (1994).
105. T. Banks, M. Dine and A.E. Nelson, JHEP 06, 014 (1999).
106. E. Elizalde, Nuovo Cim. 104B, 685 (1989).
107. L. Parker and A. Raval, Phys. Rev. D 62, 083503 (2000).
108. E. Elizalde, Commun. Math. Phys. 198, 83 (1998); E. Elizalde, J. Phys. A 30, 2735 (1997).
109. E. Elizalde, J. Phys. A 27, L299 (1994).
110. V. Sahni and S. Habib, Phys. Rev. Lett. 81, 1766 (1998); L. Parker and A. Raval, Phys. Rev. D 60, 063512 and 123502 (1999).