Hamiltonian Formalism for Bose Excitations in a Plasma with a Non-Abelian Interaction

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Abstract—We have developed the Hamiltonian theory for collective longitudinally polarized colorless excitations (plasmons) in a high-temperature gluon plasma using the general formalism for constructing the wave theory in nonlinear media with dispersion, which was developed by V.E. Zakharov. In this approach, we have explicitly obtained a special canonical transformation that makes it possible to simplify the Hamiltonian of interaction of soft gluon excitations and, hence, to derive a new effective Hamiltonian. The approach developed here is used for constructing a Boltzmann-type kinetic equation describing elastic scattering of collective longitudinally polarized excitations in a gluon plasma as well as the effect of the so-called nonlinear Landau damping. We have performed detailed comparison of the effective amplitude of the plasmon–plasmon interaction, which is determined using the classical Hamilton theory, with the corresponding matrix element calculated in the framework of high-temperature quantum chromodynamics; this has enabled us to determine applicability limits for the purely classical approach described in this study.

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1. INTRODUCTION

It is shown in the theory of usual electron–ion plasma that weak turbulence of the plasma can be of two types (see, for example, [1]). Weak turbulence of the first type is caused by scattering of waves by plasma particles. Weak turbulence of the second type is due to decay, fusion, and scattering of waves off one another, which occur without energy exchange between particles and waves. In a number of publications [2–7], the kinetic equations for the simplest collective excitations (Langmuir plasmons) of the electron–ion plasma, which describe elastic scattering of plasmons off one another, were constructed and analyzed in detail.

At present, a certain interest is shown in the construction of kinetic description of the new fundamental state of matter (quark–gluon plasma that consists of asymptotically free quarks, antiquarks, and gluons; see, for example, review [8]), which is probably formed during ultrarelativistic heavy ion collisions. It is shown that in the high-temperature limit, a quark–gluon plasma is successfully described by the effective perturbation theory [9] reformulated in the terms of kinetic equations [10]. A gluon plasma (here, we will disregard for simplicity the presence of quarks and antiquarks) can be represented as a combination of two subsystems, viz., the subsystem of hard thermal gluons and the subsystem of soft plasma excitations, which exchange energy with each other. In a high-temperature gluon plasma, as well as in a usual electron–ion plasma, two types of collective plasma excitations exist, viz., transverse-polarized and longitudinal-polarized excitations (plasmons). In the absence of external chromomagnetic and chromoelectric fields, the color matrix of the number density of collective gluon excitations is diagonal; therefore, these excitations should be treated as colorless.

In [11], a kinetic description of the nonlinear interaction of colorless and color plasmons in the hard thermal loop approximation [9, 10] was developed. This approach is based on calculation of some effective currents generating these processes. Using these currents, the matrix elements of nonlinear interaction of an arbitrary (even) number of colorless plasmons are determined. In this study, we propose an alterna-
tive method for kinetic description of the nonlinear plasmon dynamics, which is based on the classical Hamiltonian formalism for systems with distributed parameters and which has been systematically developed by Zakharov [5–7] and Gitman and Tyutin [12]. In our case, this approach is based on the fact that equations describing a collisionless high-temperature plasma in the hard thermal loop approximation have the Hamiltonian structure that has been determined in [13–15]. This enables us to develop (at least for weakly excited states; see Conclusions) an independent approach to the derivation of the kinetic equation for soft longitudinally polarized gluonic plasma excitations. In the Hamiltonian approach, the matrix elements of the plasmon–plasmon interaction are obtained using special canonical transformations simplifying the plasmon interaction Hamiltonian.

This article has the following structure. In Section 2, we derive the fourth-order effective Hamiltonian operator \( \hat{H}_4 \) describing elastic scattering of two colorless plasmons off each other. In Section 3, we introduce plasmon distribution function \( N^i_k \) and analyze the fourth- and sixth-order correlation functions in plasmon creation and annihilation operators \( \hat{c}^\dagger_k \) and \( \hat{c}^a_k \). Section 4 is devoted to the derivation of the kinetic equation for soft gluon Boltzmann-type excitations with allowance for the Landau nonlinear damping effect for plasmons. Sections 5 and 6 are connected with the determination of the explicit form of three- and four-plasmon vertex functions using the hard thermal loop approximation and the approximation of the effective gluon propagator at the plasmon pole. In concluding Section 7, we outline possible ways for generalization of the Hamiltonian description to the case of a strongly excited gluon plasma.

In Appendix, we give all basic expressions for the effective gluon vertex functions and gluon propagator in the high-temperature approximation of hard thermal loops.

2. COLORLESS PLASMON INTERACTION HAMILTONIAN

Let us consider the application of the general Zakharov theory to a specific system (high-temperature gluon plasma) in the semi-classical approximation. The gauge field potentials describing the gluon field in the system are \( N_c \times N_c \) matrices in the color space and are defined in terms of \( A_\mu(x) = A^a_\mu(x) t^a \) with \( N_c^2 - 1 \) Hermitian generators \( t^a \) of the color \( SU(N_c) \) group in the fundamental representation. The field strength tensor \( F_{\mu\nu}(x) = F_{\mu\nu}^a(x) t^a \), where

\[
F_{\mu\nu}^a(x) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu
\]

obeys the Yang–Mills equation in the \( A_0 \) gauge:

\[
\partial_\mu F^{\mu\nu}(x) - ig[ A_\mu(x), F^{\mu\nu}(x) ] - \xi_0^{-1} n^\nu \n^\mu A_\nu(x) = -j^\nu(x),
\]

where \( \xi_0 \) is the gauge parameter in the given gauge. We will henceforth identify the four-vector \( n_\mu \) with global four-velocity \( u_\mu \) of the plasma. Color current \( j^\nu \) is defined conventionally:

\[
j^\nu(x) = g r^a \int d^4x' \delta(x-x') \operatorname{Tr}( T^a f_g(x, p)).
\]

Here, \( x = (t, \vec{x}) \) is the space–time variable of the initial dynamical system and \( (T^a)^{bc} \equiv -i f^{abc} \) is the color matrix in the adjoint representation. Gluon distribution function \( f_g = f_g(x, p) \) is an \( (N_c^2 - 1) \times (N_c^2 - 1) \) Hermitian matrix in the color space.

It is known that there exist two types of physical boson soft (transverse- and longitudinal-polarized) fields in an equilibrium hot quark–gluon plasma [8]. For simplicity, we confine our analysis only to processes involving longitudinally polarized plasma excitations, which are known as plasmons. These excitations are a purely collective effect of the medium, which has no analogs in the conventional quantum field theory. Let us consider the longitudinal part of the gauge field potential in the form of expansion

\[
\hat{A}_\mu^\nu(x) = \int \frac{d\vec{k}}{(2\pi)^3} \left( \frac{Z_\mu^j(\vec{k})}{2\alpha_k} \right)^{1/2} \times \{ \epsilon^\mu_{aj} \hat{a}_j^a e^{-i k \cdot x} + \epsilon^\mu_{aj} \hat{a}_j^a e^{i k \cdot x} \}, \quad k_0 = \alpha_k^j,
\]

where \( \epsilon^\mu_{aj} = \epsilon^\mu_{aj}(\vec{k}) \) is the polarization vector of a longitudinal plasmon; its explicit form depends on the choice of the gauge (in particular, in the \( A_0 \) gauge, this vector is defined by expression (5.6)). Factor \( Z(\vec{k}) \) is the residue of the effective gluon propagator at the plasmon pole. Coefficients \( \hat{a}_j^a \) and \( \hat{a}_j^a \) will be treated as quasiparticle creation and annihilation operators for plasmons obeying the commutation relations for Bose operators:

\[
[\hat{a}_k^a, \hat{a}_k^b] = [\hat{a}_k^a, \hat{a}_k^b] = 0,
\]

\[
[\hat{a}_k^a, \hat{a}_k^b] = \delta^{ab}(2\pi)^3 \delta(\vec{k} - \vec{k}').
\]

1 Color index \( a \) runs through values \( 1, 2, ..., N_c^2 - 1 \), while vector index \( \mu \) runs through values \( 0, 1, 2, 3 \). Everywhere in this article, we imply summation over repeated indices and use the system of units with \( \hbar = c = 1 \).
Multiphonon states are obtained by the multiple action of operator $\hat{a}_{k}^{\dagger a}$ on vacuum state $|0\rangle$, which obeys the following condition:

$$\hat{a}_{k}^{\dagger a}|0\rangle = 0.$$ 

Therefore, we refer as vacuum to the ground unexcited state of the system (i.e., the state without elementary collective excitations). In operators $\hat{a}_{k}^{\dagger a}$ and $\hat{a}_{k}^{a}$, only matrix elements corresponding to a change in the number of plasmons by unity differ from zero.

Let us write the quantum-mechanical analog of the Hamilton equation, namely, the Heisenberg equation for operator $\hat{a}_{k}^{\dagger a}$:

$$\frac{\partial \hat{a}_{k}^{\dagger a}}{\partial t} = [\hat{H}, \hat{a}_{k}^{\dagger a}].$$ (2.3)

Here, $\hat{H}$ is the Hamiltonian of the plasmon system, which is sum $\hat{H} = \hat{H}_{0} + \hat{H}_{\text{int}}$, where

$$\hat{H}_{0} = \int \frac{dk}{(2\pi)^{3}} \omega_{k} \hat{a}_{k}^{\dagger a} \hat{a}_{k}$$ (2.4)

is the Hamiltonian of noninteracting plasmons and $\hat{H}_{\text{int}}$ is interaction Hamiltonian. The dispersion relation $\omega_{k}^{a}$ for plasmons satisfies the following dispersion equation [16]:

$$\Re e'(\omega, k) = 0,$$ (2.5)

where

$$e'(\omega, k) = 1 + \frac{3\omega_{k}^{a}}{k^{2}} \left[ 1 - F\left(\frac{\omega_{k}^{a}}{|k|^{2}}\right) \right]$$

and

$$F(\chi) = \frac{\chi}{2} \ln \left| \frac{1 + \chi}{1 - \chi} \right| - i\pi \theta(1 - |\chi|)$$

is the longitudinal permittivity, $\omega_{k}^{a} = g^{2}N_{T}T^{a}/\hbar$; $T$ is the temperature of the system, and $g$ is the strong interaction constant. In the small amplitude approximation, the interaction Hamiltonian can be written as a formal integral-power series in $\hat{a}_{k}^{\dagger a}$ and $\hat{a}_{k}^{a}$:

$$\hat{H}_{\text{int}} = \hat{H}_{3} + \hat{H}_{4} + \ldots,$$

where the third- and fourth-order interaction Hamiltonians have the following structure:

$$\hat{H}_{3} = \int \frac{dk_{1}dk_{2}dk_{3}}{(2\pi)^{3}} F_{k,k_{1},k_{2}}(\omega_{k}^{a}\hat{a}_{k}^{\dagger a}\hat{a}_{k_{1}}^{\dagger a}\hat{a}_{k_{2}}^{a})$$

$$+ F_{k,k_{1},k_{2}}(\omega_{k}^{a}\hat{a}_{k}^{\dagger a}\hat{a}_{k_{1}}^{\dagger a}\hat{a}_{k_{2}}^{a})(2\pi)^{3}\delta(k - k_{1} - k_{2})$$

$$+ \frac{1}{3} \int \frac{dk_{1}dk_{2}dk_{3}}{(2\pi)^{3}} G_{k,k_{1},k_{2}}(\omega_{k}^{a}\hat{a}_{k}^{\dagger a}\hat{a}_{k_{1}}^{\dagger a}\hat{a}_{k_{2}}^{a})$$

$$+ \frac{1}{3} \int \frac{dk_{1}dk_{2}dk_{3}}{(2\pi)^{3}} G_{k,k_{1},k_{2}}(\omega_{k}^{a}\hat{a}_{k}^{\dagger a}\hat{a}_{k_{1}}^{\dagger a}\hat{a}_{k_{2}}^{a})(2\pi)^{3}\delta(k + k_{1} + k_{2}),$$ (2.6)

$$\hat{H}_{4} = \frac{1}{2} \int \frac{dk_{1}dk_{2}dk_{3}dk_{4}}{(2\pi)^{4}} T^{a_{1}a_{2}a_{3}a_{4}}_{k,k_{1},k_{2},k_{3},k_{4}}(\omega_{k}^{a}\hat{a}_{k}^{\dagger a}\hat{a}_{k_{1}}^{\dagger a}\hat{a}_{k_{2}}^{a})$$

$$(2\pi)^{3}\delta(k - k_{1} - k_{2} - k_{3})$$ (2.7)

and so on. Symbol $"^{\dagger a}a_{m}"$ indicates complex conjugation. In expression (2.7), we retained only the “essential” contribution in Zakharov’s terminology because resonance conditions

$$k + k_{1} + k_{2} + k_{3} = 0,$$

$$\omega_{k}^{a} + \omega_{k_{1}}^{a} + \omega_{k_{2}}^{a} + \omega_{k_{3}}^{a} = 0,$$

$$k = k_{1} + k_{2} + k_{3},$$

$$\omega_{k}^{a} = \omega_{k_{1}}^{a} + \omega_{k_{2}}^{a} + \omega_{k_{3}}^{a}$$

have no solutions for the plasmon spectrum defined by dispersion equation (2.5).

It should be noted that such a representation of the interaction Hamiltonian in the form of formal infinite power series in the creation and annihilation operators was considered in the monograph by Schwarz [17] based on the quantum field theory for scalar fields.

Coefficients $V^{a_{1}a_{2}}_{k,k_{1},k_{2}}, U^{a_{1}a_{2}}_{k,k_{1},k_{2}}$, and $T^{a_{1}a_{2}a_{3}a_{4}}_{k,k_{1},k_{2},k_{3},k_{4}}$ exhibit certain symmetry:

$$V^{a_{1}a_{2}}_{k,k_{1},k_{2}} = V^{a_{2}a_{1}}_{k,k_{1},k_{2}},$$ (2.8)

$$U^{a_{1}a_{2}}_{k,k_{1},k_{2}} = U^{a_{2}a_{1}}_{k,k_{1},k_{2}} = U^{a_{1}a_{2}}_{k_{1},k_{1},k_{2}},$$

$$T^{a_{1}a_{2}a_{3}a_{4}}_{k,k_{1},k_{2},k_{3},k_{4}} = T^{a_{1}a_{2}a_{4}a_{3}}_{k,k_{1},k_{2},k_{3},k_{4}} = T^{a_{1}a_{3}a_{2}a_{4}}_{k,k_{1},k_{2},k_{3},k_{4}}.$$ (2.9)

These coefficient functions determine specific properties of the medium (high-temperature gluon plasma in our case).

Let us consider the transformation from operators $\hat{a}_{k}^{\dagger a}$ to new operators $\hat{c}_{k}^{\dagger a}$:

$$\hat{a}_{k}^{\dagger a} = \hat{c}_{k}^{\dagger a} + \int \frac{dk_{1}dk_{2}}{(2\pi)^{3}} \left[ V^{a_{1}a_{2}}_{k,k_{1},k_{2}} \hat{c}_{k_{2}}^{a_{1}} \hat{c}_{k_{1}}^{a_{2}} + V^{a_{1}a_{2}}_{k,k_{2},k_{1}} \hat{c}_{k_{1}}^{a_{1}} \hat{c}_{k_{2}}^{a_{2}} \right]$$

$$+ \int \frac{dk_{1}dk_{2}}{(2\pi)^{3}} \left[ U^{a_{1}a_{2}}_{k,k_{1},k_{2}} \hat{c}_{k_{2}}^{a_{1}} \hat{c}_{k_{1}}^{a_{2}} + U^{a_{1}a_{2}}_{k,k_{2},k_{1}} \hat{c}_{k_{1}}^{a_{1}} \hat{c}_{k_{2}}^{a_{2}} \right]$$

$$+ \cdots + W^{a_{1}a_{2}a_{3}a_{4}}_{k,k_{1},k_{2},k_{3},k_{4}} \hat{c}_{k_{4}}^{a_{1}} \hat{c}_{k_{3}}^{a_{2}} \hat{c}_{k_{2}}^{a_{3}} \hat{c}_{k_{1}}^{a_{4}} + \cdots.$$ (2.10)

The canonicity conditions for this transformation,²

$$\int \frac{dk}{(2\pi)^{3}} \left[ \hat{c}_{k}^{\dagger a} \delta \hat{c}_{k}^{\dagger a} - \hat{c}_{k}^{\dagger a} \delta \hat{c}_{k}^{\dagger a} \right] = 0.$$
\[ \int dk' \left[ \frac{\delta_{\tilde{\alpha}} a_k}{\delta c_{k'}} - \frac{\delta_{\tilde{\alpha}} a_{k'}}{\delta c_k} \right] = \delta^{ab}(k - k') \]

impose certain limitations on the coefficient functions of series (2.10). Functions \( V^{(2)}_{k,k,k_2} \), \( V^{(3)}_{k,k,k_2} \) must satisfy conditions

\[
\begin{align*}
V^{(2)}_{k,k,k_2} &= -2V^{*}_{(1)a_1 a_2} \quad , \\
V^{(3)}_{k,k,k_2} &= V^{(3)}_{a_1 a_2 a_3} = V^{(3)}_{a_1 a_2 a_3} \quad ,
\end{align*}
\]

and functions \( W^{(i)}_{k,k,k_2} \), \( i = 1, ..., 4 \), satisfy conditions

\[
\begin{align*}
3W^{(1)}_{k,k,k_2} &= 4 \int \left| V^{*}_{(1)a_1 a_2} V^{(3)}_{(3)a_1 a_2 a_3} \right| \, dk' = -W^{*}_{(3)a_1 a_2 a_3} , \\
W^{(2)}_{k,k,k_2} &= 2 \int \left| V^{*}_{(1)a_1 a_2} V^{(2)}_{(2)a_1 a_2} \right| \, dk' = -W^{*}_{(2)a_1 a_2} , \\
W^{(3)}_{k,k,k_2} &= 2 \int \left| V^{*}_{(1)a_1 a_2} V^{(3)}_{(3)a_1 a_2 a_3} \right| \, dk' = W^{*}_{(3)a_1 a_2 a_3} , \\
W^{(4)}_{k,k,k_2} &= 4 \int \left| V^{*}_{(1)a_1 a_2} V^{(4)}_{(4)a_1 a_2 a_3 a_4} \right| \, dk' = -W^{*}_{(4)a_1 a_2 a_3 a_4} .
\end{align*}
\]

Because of specific features of dispersion equation (2.5) in a hot gluon plasma, resonance conditions

\[
\begin{align*}
|k - k_1| + |k + k_1 + k_2| &= 0 , \\
\omega_k &= \omega_k + \omega_{k_1} + \omega_{k_2} = 0 ,
\end{align*}
\]

(2.11)

have no solutions (i.e., the longitudinal plasmon spectrum is nondecaying). In this case, canonical transformation (2.10) makes it possible to exclude “insignificant” Hamiltonian \( \tilde{H}_3 \) (2.6) by just setting

\[
\begin{align*}
V^{(1)}_{k,k,k_2} &= -\frac{1}{\omega_k - \omega_{k_1} - \omega_{k_2}} (2\pi)^3 \tilde{\delta}(k - k_1 - k_2) , \\
V^{(3)}_{k,k,k_2} &= -\frac{1}{\omega_{k_1} + \omega_{k_2} + \omega_k} (2\pi)^3 \tilde{\delta}(k + k_1 + k_2) .
\end{align*}
\]

This exclusion procedure leads us to following structure of fourth-order effective Hamiltonian \( \tilde{H}_4 \):

\[
\tilde{H}_4 = \frac{1}{2} \int \left[ \frac{dk}{(2\pi)^3} \int \frac{dk_1}{(2\pi)^3} \int \frac{dk_2}{(2\pi)^3} \right] T^{a_1 a_2 a_3 a_4}_{k,k,k_1,k_2} \\
\times \tilde{c}_{k_1}^a \tilde{c}_{k_2}^a \tilde{c}_{k_3}^{a_3} \tilde{c}_{k_4}^{a_4} (2\pi)^3 \tilde{\delta}(k + k_1 - k_2 - k_3) ,
\]

where

\[
\begin{align*}
T^{a_1 a_2 a_3 a_4}_{k,k,k_1,k_2} &= T^{a_1 a_2 a_3}_{a_1 k,k,k_1 k_2} - 2 U^{b_1 a_1}_{-(k+k_1),k,k} U^{b_2 a_3}_{-(k+k_1),k,k} \\
&- 2 \frac{V^{a_1 a_2}_{k,k,k_1}}{\omega_{k+k_1} - \omega_k - \omega_{k_1}} - 2 \frac{V^{a_2 a_3}_{k,k,k_1}}{\omega_{k+k_1} - \omega_k - \omega_{k_1}} \\
&- 2 \frac{V^{a_3 a_4}_{k,k,k_1}}{\omega_{k+k_1} - \omega_k - \omega_{k_1}} - 2 \frac{V^{a_4 a_1}_{k,k,k_1}}{\omega_{k+k_1} - \omega_k - \omega_{k_1}} .
\end{align*}
\]

The determined effective amplitude has a simple diagrammatic interpretation shown in Fig. 1. Black square indicates amplitude \( \tilde{T}^{a_1 a_2 a_3 a_4}_{k,k,k_1,k_2} \). The first term on the right-hand side of Fig. 1 defines the direct interaction of four plasmons, which is generated by usual four- plasmon amplitude \( \tilde{T}^{a_1 a_2 a_3 a_4}_{k,k,k_1,k_2} \). The remaining terms are connected with the interaction generated by amplitudes \( U^{\alpha_1}_{k,k,k_1,k} \) and \( V^{a_1 a_2}_{k,k,k_1,k} \), with intermediate “virtual” oscillations. In our case, the conditions of smallness of amplitudes imply that

\[
|\tilde{T}^{(4)}_{k,k,k_1,k_2}|^2 \ll k \frac{\partial \omega_{k}}{\partial k} .
\]

(2.14)

Therefore, there exist two equivalent descriptions of the Hamilton system of colorless plasmons for the same physical processes. In the first case, we can use the original Hamiltonian\n
\[
\hat{H} = \hat{H}_0 + \hat{H}_3 + \hat{H}_4 + ... ,
\]

(2.15)

where \( \hat{H}_0, \hat{H}_3, \) and \( \hat{H}_4 \) are defined by expressions (2.4), (2.6), and (2.7), respectively; in the second case, we use Hamiltonian \( \tilde{H} \) obtained as a result of the nonlinear transformation of creation and annihilation Bose operators \( \hat{a}_k^\dagger \) and \( \hat{a}_k \):

\[
\tilde{H} = \tilde{H}_0 + \tilde{H}_4 + ... ,
\]

(2.16)

where, in turn

\[
\tilde{H}_0 = \int \frac{dk}{(2\pi)^3} \omega_k \hat{c}_k \hat{c}_k^\dagger ,
\]

and operator \( \tilde{H}_4 \) is defined by expression (2.12). The Heisenberg equations for operators \( \hat{a}_k^\dagger \) and \( \hat{a}_k \) have completely the same form (2.3) with corresponding Hamiltonians (2.15) and (2.16).

In connection with this construction, it is appropriate to mention the publication [19] close to the subject matter of our study, in which a new important concept of nonlinear \( f \)-oscillators has been intro-
duced. In [19], the problem of quantization of a harmonic oscillator was considered, in which the boson creation and annihilation operators were transformed nonlinearly into new creation and annihilation operators determining quantum $f$-oscillators. In this way, a new Hamiltonian with a quite nontrivial structure was obtained; this operator describes the same dynamics as the initial Hamiltonian, as observed in our case.

However, despite the closeness of the approaches proposed in the present study and in [19], they differ basically. In the approach considered in this section, creation and annihilation operators $(\hat{a}, \hat{a}^\dagger)$ and corresponding Hamiltonians (2.15) and (2.16) are connected by the canonical transformation that preserves the standard form of commutation relations (2.2). In the approach described in [19], the nonlinear transformations are noncanonical and, hence, the authors modified appropriately commutation relations of type (2.2) for preserving identity of the described dynamics. For this reason, it is impossible in our case to interpret nonlinear oscillations associated with boson operators just as oscillations with a specific energy dependence of the oscillation frequency as in the case of nonlinear $f$-oscillators (however, this fact may sometimes take place).

3. FOURTH-ORDER CORRELATION FUNCTION

Hamiltonian (2.12) describes elastic scattering of color plasmons by one another (i.e., $2 \rightarrow 2$ process).

The equations of motion for $\hat{c}_k^{a\dagger}$ and $\hat{c}_k^b$ are determined in this case by the corresponding Heisenberg equations:

$$
\frac{\partial \hat{c}_k^{a\dagger}}{\partial t} = i[\tilde{H}_0 + \tilde{H}_4, \hat{c}_k^{a\dagger}] = i\omega_k \hat{c}_k^{a\dagger}$$

$$-i \int \frac{dk_1 dk_2 dk_3}{(2\pi)^3} T_{k,k_1,k_2,k_3}^{a,a,a,a} \hat{c}_k^{a\dagger} \hat{c}_k^{a\dagger} \hat{c}_k^{a\dagger} \hat{c}_k^{a\dagger} \times (2\pi)^3 \delta(k + k_1 - k_2 - k_3),$$

(3.1)

For the case of $\delta$-oscillators, these equations can be written in the following form:

$$\frac{\partial \hat{c}_k^{a\dagger}}{\partial t} = i(\tilde{H}_0 + \tilde{H}_4, \hat{c}_k^{a\dagger}) = i\omega_k \hat{c}_k^{a\dagger}$$

$$+ i \int \frac{dk_1 dk_2 dk_3}{(2\pi)^3} T_{k,k_1,k_2,k_3}^{a,a,a,a} \hat{c}_k^{a\dagger} \hat{c}_k^{a\dagger} \hat{c}_k^{a\dagger} \hat{c}_k^{a\dagger} \times (2\pi)^3 \delta(k + k_1 - k_2 - k_3).$$

(3.2)

These exact equations in the absence of an external color field in the system enable us to determine the kinetic equation for the number density $N_k^{abl} = \delta^{ab} N_k^l$ of colorless plasmons.

If the set of waves for a low level of nonlinearity of (2.14) has random phases, this set can be described statistically by introducing correlation function

$$\langle \hat{c}_k^{a\dagger} \hat{c}_k^b \rangle = \delta^{ab} (2\pi)^3 \delta(k - k) N_k^l.$$  

(3.3)

It should be emphasized that the introduction of distribution function $N_k^l \equiv N(k, x, t)$ of quasiparticles (plasmons), which depends on plasmon momentum $h\vec{k}$ as well as on coordinate $x$ and time $t$, has sense only in the case when the number of plasmons varies slowly in space and time. This means that the variation of the function over distances on the order of wavelength $\lambda = 2\pi/\vec{k}$ and in time intervals on the order of wave period $T = 2\pi/\omega_k$ must be much smaller than function $N_k^l$ itself.

Proceeding from Heisenberg equations (3.1) and (3.2), we can determine the kinetic equation for plasmon number density $N_k^l$. For this purpose, we multiply Eqs. (3.1) and (3.2) by $\hat{c}_k^{b\dagger}$ and $\hat{c}_k^{a\dagger}$, respectively:

$$\frac{\partial \hat{c}_k^a}{\partial t} = -i\omega_k \hat{c}_k^a - i \int \frac{dk_1 dk_2 dk_3}{(2\pi)^3} T_{k,k_1,k_2,k_3}^{a,a,a,a} \hat{c}_k^{a\dagger} \hat{c}_k^{a\dagger} \hat{c}_k^{a\dagger} \hat{c}_k^{a\dagger} \times (2\pi)^3 \delta(k + k_1 - k_2 - k_3),$$

(3.1)

$$\frac{\partial \hat{c}_k^{a\dagger}}{\partial t} = i(\tilde{H}_0 + \tilde{H}_4, \hat{c}_k^{a\dagger}) = i\omega_k \hat{c}_k^{a\dagger}$$

$$+ i \int \frac{dk_1 dk_2 dk_3}{(2\pi)^3} T_{k,k_1,k_2,k_3}^{a,a,a,a} \hat{c}_k^{a\dagger} \hat{c}_k^{a\dagger} \hat{c}_k^{a\dagger} \hat{c}_k^{a\dagger} \times (2\pi)^3 \delta(k + k_1 - k_2 - k_3).$$

(3.2)
where
\[
\langle \hat{c}_k \hat{c}^*_k \hat{c}_k \hat{c}^*_k \rangle = \langle \hat{c}^*_k \hat{c}_k \hat{c}^*_k \hat{c}_k \rangle = \langle \hat{c}^*_k \hat{c}_k \hat{c}_k \hat{c}^*_k \rangle = \langle \hat{c}_k \hat{c}^*_k \hat{c}_k \hat{c}^*_k \rangle = \langle \hat{c}_k \hat{c}^*_k \hat{c}^*_k \hat{c}_k \rangle = \langle \hat{c}^*_k \hat{c}_k \hat{c}_k \hat{c}^*_k \rangle = \langle \hat{c}^*_k \hat{c}_k \hat{c}^*_k \hat{c}_k \rangle = \langle \hat{c}_k \hat{c}^*_k \hat{c}_k \hat{c}^*_k \rangle = \langle \hat{c}_k \hat{c}^*_k \hat{c}^*_k \hat{c}_k \rangle = \langle \hat{c}_k \hat{c}^*_k \hat{c}_k \hat{c}^*_k \rangle = \langle \hat{c}^*_k \hat{c}_k \hat{c}^*_k \hat{c}_k \rangle = \langle \hat{c}_k \hat{c}^*_k \hat{c}_k \hat{c}^*_k \rangle = \langle \hat{c}^*_k \hat{c}_k \hat{c}_k \hat{c}^*_k \rangle = \langle \hat{c}^*_k \hat{c}_k \hat{c}^*_k \hat{c}_k \rangle = \langle \hat{c}_k \hat{c}^*_k \hat{c}_k \hat{c}^*_k \rangle = \langle \hat{c}_k \hat{c}^*_k \hat{c}^*_k \hat{c}_k \rangle = \langle \hat{c}_k \hat{c}^*_k \hat{c}_k \hat{c}^*_k \rangle = \langle \hat{c}^*_k \hat{c}_k \hat{c}_k \hat{c}^*_k \rangle = \langle \hat{c}^*_k \hat{c}_k \hat{c}^*_k \hat{c}_k \rangle = \langle \hat{c}_k \hat{c}^*_k \hat{c}_k \hat{c}^*_k \rangle = \langle \hat{c}^*_k \hat{c}_k \hat{c}_k \hat{c}^*_k \rangle = \langle \hat{c}_k \hat{c}^*_k \hat{c}_k \hat{c}^*_k \rangle.
\]

By way of example, let us consider the explicit form of the contribution that generates the second term in the decomposition of correlation function (3.6):

\[
N_k^f N_k^l \delta(k_2 - k_3) \delta(k_3 - k_1) \int \mathcal{T}^{a_h a_h a_h a_h a_h a_h}_{k_h k_h k_h} N_k^f \, dk'.
\]

Comparing the last two expressions, we see that they have quite different structures.

Let us now consider the second six-point correlation function in expression (3.5). In this correlator, we write in explicit form only “regular” terms:
Consider the symmetry relations for scattering amplitude

\[
\tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} = \tilde{T}_{-k_3,-k_1,-k_4,k_2}^{\alpha_4 \alpha_3 \alpha_2 \alpha_1},
\]

and

\[
\tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} = \tilde{T}_{-k_4,-k_1,-k_3,k_2}^{\alpha_4 \alpha_3 \alpha_2 \alpha_1},
\]

we obtain the following equation for the fourth-order correlation function instead of \((3.5)\):

\[
\frac{\partial I_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}}{\partial t} = [\alpha_{k_1}^{\alpha_1} - \alpha_{k_2}^{\alpha_2} - \alpha_{k_3}^{\alpha_3} + \alpha_{k_4}^{\alpha_4}] I_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} + 6i \tilde{\omega}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} (N_{k_1} N_{k_2} N_{k_3} N_{k_4} + N_{k_1} N_{k_3} N_{k_2} N_{k_4}) \delta(k + k_1 - k_2 - k_3) \tag{3.8}
\]

\[
- N_{k_1} N_{k_2} N_{k_3} N_{k_4} (2\pi)^3 \delta(k + k_1 - k_2 - k_3).
\]

4. KINETIC EQUATION FOR GLUON EXCITATIONS

Let us now pass to the direct derivation of the kinetic equation for plasmons. The self-consistent set of equations \((3.4)\) and \((3.8)\) determines, in principle, the evolution of plasmon number density \(N_{k_1}^{\prime}\). However, we introduce one more simplification: in Eq. \((3.8)\), we disregard the term with the time derivative as compared to the term containing the difference in the eigenfrequencies of wave packets. Instead of relation \((3.8)\), we have

\[
\tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} = N_{k_1}^{\prime} N_{k_2}^{\prime} (2\pi)^3 \delta(k - k_2)
\]

\[
\times \delta(k_1 - k_3) + \delta(k - k_3) \delta(k_1 - k_2) + \frac{6}{\Delta \omega + i0} \tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} (N_{k_1}^{\prime} N_{k_2}^{\prime} N_{k_3}^{\prime} N_{k_4}^{\prime} + N_{k_1}^{\prime} N_{k_3}^{\prime} N_{k_2}^{\prime} N_{k_4}^{\prime})
\]

\[
- N_{k_1}^{\prime} N_{k_2}^{\prime} N_{k_3}^{\prime} N_{k_4}^{\prime} (2\pi)^3 \delta(k + k_1 - k_2 - k_3),
\]

where

\[
\Delta \omega \equiv \omega_1^{\prime} - \omega_2^{\prime} - \omega_3^{\prime} - \omega_4^{\prime}.
\]

Here, the first term on the right-hand side, which corresponds to completely uncorrelated waves (purely Gaussian fluctuations) is the solution to the homogeneous equation for fourth-order correlation function \(I_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}\). The second term determines the deviation of the four-point correlator from the Gaussian approximation for a low nonlinearity level of interacting waves.

We substitute the first term into the right-hand side of Eq. \((3.4)\) for \(N_{k_1}^{\prime}\):

\[
-i(2\pi)^3 N_{k_1}^{\prime} \int \frac{dk_k}{(2\pi)^3} N_{k_2}^{\prime} \tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \delta(k - k')
\]

\[
+ \tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \delta(k' - k) \tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \delta(k - k') - \tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \delta(k' - k) \tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \delta(k - k')
\]

\[
= -i2(2\pi)^3 \delta(k - k') N_{k_1}^{\prime} \int \frac{dk_k}{(2\pi)^3} N_{k_2}^{\prime} \tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \delta(k + k_1 - k_2 - k_3).
\]

Further, we substitute the second term into the right-hand side of Eq. \((3.4)\):

\[
-6i \int \frac{dk_k dk_2 dk_3}{(2\pi)^3} \tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \left( \frac{1}{\Delta \omega + i0} \right)
\]

\[
\times \tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} (2\pi)^3 \delta(k' + k_1 - k_2 - k_3)
\]

\[
\times (2\pi)^3 \delta(k + k_1 - k_2 - k_3) [N_{k_1}^{\prime} N_{k_2}^{\prime} N_{k_3}^{\prime} N_{k_4}^{\prime} + ...]
\]

\[
- \tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \left( \frac{1}{\Delta \omega + i0} \right) \tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} (2\pi)^3 \delta(k + k_1 - k_2 - k_3)
\]

\[
\times (2\pi)^3 \delta(k' + k_1 - k_2 - k_3) [N_{k_1}^{\prime} N_{k_2}^{\prime} N_{k_3}^{\prime} N_{k_4}^{\prime} + ...].
\]

Taking into account that

\[
\delta(k' + k_1 - k_2 - k_3) \delta(k + k_1 - k_2 - k_3) = \delta(k' - k) \delta(k + k_1 - k_2 - k_3),
\]

we can write the last expression in a more compact form:

\[
-6i(2\pi)^3 \delta(k - k') \int \frac{dk_k dk_2 dk_3}{(2\pi)^3}
\]

\[
\times (2\pi)^3 \delta(k + k_1 - k_2 - k_3) [N_{k_1}^{\prime} N_{k_2}^{\prime} N_{k_3}^{\prime} N_{k_4}^{\prime} + ...]
\]

\[
\times \left( \frac{\tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}}{\Delta \omega + i0} - \frac{\tilde{T}_{k_1,k_2,k_3,k_4}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}}{\Delta \omega - i0} \right)
\]

\[
\Delta \omega - i0.
\]
Further, performing the convolution of obtained expressions (3.4), (4.1), and (4.2) with $\delta^{ab}$, considering that
\[
\frac{1}{\Delta \omega + i0} - \frac{1}{\Delta \omega - i0} = -2i\pi \delta(\Delta \omega),
\]
and cancelling out factor $(2\pi)^4 \delta(k - k')$, we obtain the desired kinetic equation for colorless longitudinal gluon excitations:
\[
\frac{dN_k'}{dt} = \frac{4}{d_A^2} \int d\omega \left[ \frac{d^4k}{(2\pi)^3} N_k' \Im \left\{ \tilde{T}^{a_2}_k \tilde{T}^{a_1}_k \right\} \right]
+ 6 \int d\omega \frac{d^4k}{(2\pi)^3} \delta (\omega_k' + \omega_k - \omega_k') \delta(k + k - k) T^{a_2}_k \tilde{T}^{a_1}_k \tilde{T}^{a_2}_k \tilde{T}^{a_1}_k (\text{4.3})
\times (N_k' N_k' + N_k' N_k' N_k' - N_k' N_k' N_k' - N_k' N_k' N_k').
\]

In the limit of large occupation numbers of plasmon states $(N_k' \gg 1)$, the right-hand side of Boltzmann equation (4.4) is transformed into (4.3).

5. EXPLICIT FORM OF FUNCTION $T^{a_1a_2}_{k,k,k,k}$

It remains for us to determine the explicit form of vertex functions $T^{a_1a_2}_{k,k,k,k}$, $U^{a_1a_2}_{k,k,k,k}$, and $V^{a_1a_2}_{k,k,k,k}$ that appear in effective amplitude (2.13). In this section, we determine the form of function $T^{a_1a_2}_{k,k,k,k}$ in the hard thermal loop (HTL) approximation [8]. In [11], the probability of elastic scattering of two plasmons was determined in the HTL approximation:
\[
w_{\Pi}(k, k'; k_2, k_3) = 3M^{a_1a_2a_3}(k, k_1, -k_2, -k_3) \times M^{a_1a_2a_3}(k, k_1, -k_2, -k_3).
\]

Here, the matrix element of the four-plasmon decay has the following structure:
\[
M^{a_1a_2a_3}(k, k_1, -k_2, -k_3) = g^2 \left\{ \frac{Z(k)}{20k} \right\}^{1/2} \times \left\{ \frac{\tilde{a}^0(k)}{\sqrt{\tilde{a}^2(k)}} \right\}^{3} \prod_{j=1}^{3} \left( \frac{Z(k_j)}{20k_j} \right)^{1/2} \left( \frac{\tilde{a}^0_k}{\sqrt{\tilde{a}^2_k}} \right) \left( \text{5.2} \right)
\]
in turn, effective amplitude $\hat{T}^{a_1a_2a_3}_{k,k,k,k}$ is defined as
\[
\hat{T}^{a_1a_2a_3}_{k,k,k,k}(k, k_1, -k_2, -k_3) = \hat{f}^{a_a}_b f^{b_b_a_a}(k, k_1, -k_2, -k_3) \times \hat{f}^{a_a}_b f^{b_b_a_a}(k, k_1, -k_2, -k_3) \left( \text{5.3} \right)
\]
where $f^{abc}$ are antisymmetric structural constants of the color Lie algebra $su(N_c)$. Color factors in this expression are multiplied by the purely kinetic coefficients, viz., effective subamplitudes defined as...
The form of vertex functions \( \Gamma_{\mu\nu\lambda\sigma}(k, k_1, k_2, k_3) \) and \( \Gamma_{\mu\nu\lambda\sigma}(k, k_2, k_3) \), as well as of gluon propagator \( \bar{\phi}_\mu^\nu(k) \), is given in Appendix in the HTL approximation, (A.8)–(A.10). Two four-vectors

\[
\vec{u}_\mu(k) = \frac{k^2}{(k \cdot u)} (k_\mu - u_\mu(k \cdot u)) \quad \text{and} \quad \vec{v}_\mu(k) = k^2 u_\mu - k_\mu(k \cdot u)
\]

are the projectors onto the longitudinal direction of wavevector \( k \), written in the Lorentz-invariant form in the Hamilton and Lorentz gauge, respectively. Here, \( u^\mu \) is the four-velocity of the medium, which is \( u^\mu = (1, 0, 0, 0) \) in the rest system. Finally, four-vectors of form

\[
\left( \frac{Z_\mu(k)}{2\alpha_k} \right)^{1/2} \vec{u}_\mu(k) \equiv \frac{1}{2\alpha_k} \epsilon^\mu_\mu(k)
\]

on the right-hand side of Eq. (5.2) are conventional wavefunctions of a longitudinal physical gluon in the \( A_0 \) gauge, where factor \( \sqrt{Z_\mu(k)} \) ensures renormalization of the gluon wavefunction due to thermal effects. Factor \( 3 \) on the right-hand side of expression (5.1) accounts for three possible four-plasmon decay channels, which change the plasmon number density:

\[
g^* + g^*_1 \Leftrightarrow g_2^* + g_3^*, \quad g^* + g_2^* \Leftrightarrow g_1^* + g_3^*,
\]

\[
g^* + g_3^* \Leftrightarrow g_1^* + g_2^*.
\]

Considering two expressions (4.5) and (5.1) for the plasmon–plasmon scattering probability, we see that effective amplitude \( T^{a_1a_2a_3}_{k,k,k_1,k_2} \), defined by expression (2.13) should be identified (to within a numerical factor) with matrix element \( M^{a_1a_2a_3}(k, k_1, -k_2, -k_3) \), calculated using the high-temperature quantum field theory; i.e.,

\[
T^{a_1a_2a_3}_{k,k,k_1,k_2} = \left( \frac{d_4}{2} \right)^{1/2} M^{a_1a_2a_3}(k, k_1, -k_2, -k_3)
\]

From expressions for effective amplitudes (2.13) and (5.2), (5.3), we can immediately obtain the explicit form of amplitude \( T^{a_1a_2a_3}_{k,k,k_1,k_2} \), which appears as the coefficient function in the definition of fourth-order Hamiltonian \( \hat{H}_4 \):
Such a representation is single-valued. In view of complete antisymmetry of structure constants \( f^{a,a,b} \) in permutation of color indices, properties (2.8) immediately imply
\[
V_{k,k,k} = -V_{k,k,k}, \quad U_{k,k,k} = -U_{k,k,k}.
\] (6.2)

Further, using identity
\[
f^{a,a,b} f^{b,a,a} = -f^{a,a,b} f^{b,a,a} + f^{a,a,b} f^{b,a,a},
\]
for antisymmetric structure constants, we can reduce the left-hand side of relation (6.1) to form
\[
f^{a,a,b} f^{b,a,a} \left[ U_{-(k_1+k_2),k_2,k_2} U_{-(k_1+k_2),k_2,k_2} \right]_{\omega_k + \omega_k - \omega_k} + V_{k_1,k_2,k_2, k_2,k_2} V_{k_1,k_2,k_2, k_2,k_2} \right]_{\omega_k - \omega_k + \omega_k - \omega_k} + V_{k_1,k_2,k_2, k_2,k_2} V_{k_1,k_2,k_2, k_2,k_2} \right]_{\omega_k - \omega_k + \omega_k - \omega_k}
\]

\[
\frac{1}{2} \left( \frac{d_A}{2} \right)^{1/2} g^2 \left( \frac{e_{\mu}^i(k)}{\sqrt{20\omega_k}} \right) \prod_{l=1}^{3} \left( \frac{e_{\mu}^l(k)}{\sqrt{20\omega_k}} \right) \times \left[ *J_{\mu}^{\nu}(k, -k_3, -k_3) \right]_{\text{on-shell}}
\]
(6.3)

and second,
\[
V_{k_1,k_2,k_2, k_2,k_2} V_{k_1,k_2,k_2, k_2,k_2} \right]_{\omega_k - \omega_k + \omega_k - \omega_k}
\]

\[
\frac{1}{2} \left( \frac{d_A}{2} \right)^{1/2} g^2 \left( \frac{e_{\mu}^i(k)}{\sqrt{20\omega_k}} \right) \prod_{l=1}^{3} \left( \frac{e_{\mu}^l(k)}{\sqrt{20\omega_k}} \right) \times \left[ *J_{\mu}^{\nu}(k, -k_3, -k_3) \right]_{\text{on-shell}}
\]
(6.4)

On the left-hand sides of expressions (6.3) and (6.4), we have taken into account the evenness of the dispersion relation (i.e., \( \omega_i = \omega_i \)).

Further, at the second step in effective gluon propagators \( \tilde{Q}_{\mu,\nu} \), on the right-hand sides of relations (6.3) and (6.4) we retain only the terms with longitudinal projector \( \tilde{Q}_{\nu,\nu} \). For example, for first propagator \( \tilde{Q}_{\nu,\nu}(k + k_1) \), we perform substitution
\[
\tilde{Q}_{\nu,\nu}(k + k_1) \Rightarrow -\tilde{Q}_{\nu,\nu}(k + k_1) \Delta'(k + k_1)
\]
where the right-hand side, on account of relations (A.10) and (A.9), is given by explicit expression
\[
\frac{1}{\tilde{u}(k + k_1) \tilde{u}(k + k_1)} \frac{1}{(k + k_1)^2 - \Gamma(k + k_1)},
\]
(6.5)

analogous operations are performed for second propagator \( \tilde{Q}_{\nu,\nu}(k_2 - k_1) \). Near pole \( \omega \sim \omega_k \), longitudinal scalar propagator \( \Delta(k) = \Delta(\omega, k) \) behaves as (see, for example, [21])
\[
*\Delta'(\omega, k) = \frac{1}{\omega^2 - \omega_0^2 - \Delta'(\omega, k)} = \frac{Z(k)}{\omega^2 - \omega_0^2 - \Delta'(\omega, k)}.
\]

Using this approximation, we obtain, in particular, the following expressions for the first propagator:
\[
*\Delta'(k + k_1) = \frac{Z(k)}{20\omega_{k+k_1}} \times \frac{1}{\omega_k - \omega_{k+k_1} + \omega_{k+k_1} + \omega_{k+k_1}}
\]
(6.6)

and for the second propagator.
**7. CONCLUSIONS**

In this study, we have taken the first step in constructing the classical Hamiltonian formalism for describing processes of nonlinear interaction of soft gluon excitations in the Yang–Mills high-temperature field theory. We have constructed canonical transformation (2.10) in explicit form, which makes it possible to exclude third-order interaction Hamiltonian \( \dot{H}_3 \) (2.6) and to define in this way new effective interaction Hamiltonian \( \tilde{H}_4 \) (2.12) with gauge-invariant scattering amplitude \( T^\alpha_{\beta\gamma \delta} \). This interaction Hamiltonian determines a specific physical process, viz., elastic scattering of two colorless plasmons off each other. This scattering process dominates when the gauge field amplitude has order [11]

\[
|A_\alpha(x)| \sim \sqrt{gT} \quad \text{and} \quad N_k^{\alpha} \sim \frac{1}{g}
\]

which in fact corresponds to the level of thermal fluctuations in a hot gluon plasma. For this value of the gauge field amplitude at \( g \ll 1 \), plasmon number density \( N_k^{\alpha} \) is high, and the application of the purely classical description is justified. Moreover, the use of the linearized Boltzmann equation instead of exact equation (4.4) is justified for colorless plasmons because the Planck distribution, relative to which the deviation \( \delta N_k^{\alpha} \) of the phonon number density is measured, is of order

\[
N_k^{eq} \sim \frac{T}{\omega_k} \sim \frac{1}{g}
\]

In this case, we can state that the theory of plasmon–plasmon interaction for small amplitudes of soft excitations is linear, and the nonlinear effects associated with nonequilibrium fluctuations \( \delta N_k^{\alpha} \) of the plasmon number density can be treated as a perturbations.

The situation changes qualitatively when the system is strongly excited, which can occur in collisions of ultrarelativistic heavy ions in experiments with the Large Hadron Collider. For high intensities of excitations in a gluon plasma, it is necessary to consider next terms in the expansion of \( \dot{H}_{int} \). Since nonlinear
excitation processes involving an odd number of plasmons are forbidden, we can in principle to get rid of all "odd" interaction Hamiltonians \( \hat{H}_{2n+1}, n = 1, 2, \ldots \), by designing appropriately canonical transformations. In the limiting case of strong excitations, when

\[ |A_n(x)| \sim T \text{ and, accordingly, } N'_k \sim \frac{1}{g^2}, \]

these canonical transformations contain an infinite number of terms of any order in creation operators \( \hat{c}_k^a \) and annihilation ones \( \hat{c}_k^a \). In turn, this necessitates the inclusion of all higher-order plasmon elastic scattering processes \((3 \rightarrow 3, 4 \rightarrow 4, \ldots)\) on the right-hand side of kinetic equation (4.4) since all these processes are of the same order in interaction constant \( g \). Clearly, the procedure of linearization of the kinetic equation for plasmon number density \( N'_k \) becomes inapplicable in the given case, and we arrive here at the truly nonlinear theory of interaction of soft gluon excitations in a plasma with a non-Abelian type of interaction.

Thus, a nontrivial problem of constructing the explicit form of canonical transformations arises. These nonlinear canonical transformations must convert the original interaction Hamiltonian to a new effective form:

\[ \hat{H}_{\text{int}} \rightarrow \tilde{\hat{H}}_{\text{int}} = \tilde{\hat{H}}_4 + \tilde{\hat{H}}_6 + \ldots + \tilde{\hat{H}}_{2n+2} + \ldots. \]

However, the direct approach to determining the explicit form of required canonical transformations, which was used in this study, becomes ineffective in the attempt at the exclusion of even the next odd Hamiltonian \( \hat{H}_5 \) because of extremely cumbersome calculations.

For strongly excited states, when we are dealing with an infinite number of terms, a more adequate qualitatively new apparatus is required in the given situation (e.g., the introduction of a set of nonlocal canonical variables, which depend on an additional 3D unit vector as proposed in [22]). Another approach involves the use of relation

\[ A^a_{\mu}(k) = \tilde{A}^a_{\mu}(k) + \sum_{\nu} \tilde{A}^a_{\mu\nu}(k) J^{(2)\nu\mu}(A^{(0)}, A^{(0)}) + J^{(3)\nu\mu}(A^{(0)}, A^{(0)}, A^{(0)}) + \ldots, \]

(7.1)

where \( A^a_{\mu}(k) \) and \( \tilde{A}^a_{\mu}(k) \) are the interacting and free gauge fields of the system, and \( J^{(n)\nu\mu}(A^{(0)}, A^{(0)}, \ldots) \) are certain effective currents that are nonlinear functionals of the free field and are defined recurrently in the hard thermal loop approximation [11]. The coefficient functions in \( J^{(n)\nu\mu}(A^{(0)}, A^{(0)}, \ldots) \) are effective amplitudes of type (5.3). As the interacting field, we must take expression (2.1), while as the free field, we must take the expression of form

\[ A^{(0)\nu\mu}(x) = \int \frac{dk}{(2\pi)^3} \left( \frac{Z'(k)}{2\omega_k} \right)^{1/2} \{ \epsilon^\nu\mu_{\epsilon}\epsilon e^{-i\epsilon k} + \epsilon^\nu_{\epsilon}\epsilon\epsilon e^{i\epsilon k} \} \]

with operators \( \hat{c}_k^a \) and \( \hat{c}_k^{+a} \) that appear on the right-hand side of canonical transformations (2.10). Relation (7.1) in fact contains the required canonical transformation with any degree of accuracy if we use the appropriate approximations for propagators of type (6.6), (6.7) and vertex functions (6.8), (6.9), (5.8), etc. Relation (7.1) allows us to give a completely new interpretation of canonical transformations: transformations (2.10) determine a transition from noninteracting field \( A^{(0)\nu\mu}(k) \) to interacting field \( A^a_{\mu}(k) \), which takes into account all interaction effects in the medium. Analysis of this relationship requires separate consideration.

**APPENDIX**

**EFFECTIVE VERTICES AND GLUON PROPAGATOR**

In this Appendix, we consider the explicit form of vertex functions and gluon propagator in the high-temperature hard thermal loop (HTL) approximation [8, 9].

Effective three-gluon vertex

\[ \Gamma^{\nu\mu\lambda}(k, k_1, k_2) \equiv \Gamma^{\nu\mu\lambda}(k, k_1, k_2) + \delta\Gamma^{\nu\mu\lambda}(k, k_1, k_2) \]

(A.1)

is the sum of bare three-gluon vertex

\[ \Gamma^{\nu\mu\lambda}(k, k_1, k_2) = g^{\nu\mu}(k - k_1)^\rho + g^{\nu\lambda}(k_1 - k_2)^\rho + g^{\mu\lambda}(k_2 - k)^\rho \]

(A.2)

and the corresponding HTL correction

\[ \delta\Gamma^{\nu\mu\lambda}(k, k_1, k_2) = \frac{30\alpha_s}{4\pi} \int d\Omega \frac{\delta\Gamma^{\nu\mu\lambda}(k, k_1, k_2)}{4\pi k \cdot k_1 + i\epsilon} \times \left( \frac{\omega_k}{v \cdot k_1 - i\epsilon} - \frac{\omega_{k_1}}{v \cdot k_1 - i\epsilon} \right) \]

(A.3)

where \( v^\mu = (1, v), k + k_1 + k_2 = 0, \) and \( d\Omega \) is a differential solid angle. We consider below useful properties of the three-gluon HTL resummed vertex function for complex conjugation and permutation of momenta:

\[ (\ast\Gamma^{\mu\nu\lambda}(k_1 - k_2, k_2, k_1))^* \]

\[ = -\ast\Gamma^{\mu\nu\lambda}(k_1 + k_2, k_1 - k_2) \]

\[ = \ast\Gamma^{\mu\nu\lambda}(k_1 + k_2, -k_2, -k_1) \]

Further, effective four-gluon vertex

\[ \ast\Gamma^{\mu\nu\lambda\sigma}(k, k_1, k_2, k_3) \]

\[ \equiv \Gamma^{\mu\nu\lambda\sigma}(k, k_1, k_2, k_3) + \delta\Gamma^{\mu\nu\lambda\sigma}(k, k_1, k_2, k_3) \]

(A.5)

is the sum of bare four-gluon vertex

\[ \Gamma^{\mu\nu\lambda\sigma}(k, k_1, k_2, k_3) = 2g^{\mu\nu}(g^{\lambda\sigma} - g^{\nu\sigma})g^{\lambda\nu}g^{\mu\sigma} \]

(A.6)
and the corresponding HTL correction

\[ \delta \Gamma_{\mu \nu \lambda \sigma}^{k_1 k_2 k_3} = 3 \alpha s_{\rho i}^2 \int \frac{d \Omega}{4 \pi} \frac{v^\mu v^\nu v^\lambda v^\sigma}{v^2 + i \varepsilon} \]

\[ \times \left\{ \frac{1}{v \cdot (k + k_1) + i \varepsilon} \left( \frac{\omega_1}{v \cdot k_2 - i \varepsilon} - \frac{\omega_2}{v \cdot k_3 - i \varepsilon} \right) - \frac{1}{v \cdot (k + k_3) + i \varepsilon} \left( \frac{\omega_1}{v \cdot k_1 - i \varepsilon} - \frac{\omega_2}{v \cdot k_2 - i \varepsilon} \right) \right\} \]  

(A.7)

Finally, expression

\[ * \tilde{G}_{\mu \nu}^{k}(k) = -P_{\mu \nu}^{k}(k) * \Delta'(k) - \tilde{Q}_{\mu \nu}^{k}(k) * \Delta'(k) \]

\[ - \frac{\varepsilon_0}{(k \cdot u)^2} D_{\mu \nu}(k) \]  

(A.8)

is a gluon (retarded) propagator in the A_0 gauge, which is modified by effects of the medium. Here, “scalar” transverse and longitudinal propagators have form

\[ * \Delta'(k) = \frac{1}{k^2 - \Pi'(k)} \]

\[ * \Delta'(k) = \frac{1}{k^2 - \Pi'(k)} \]  

(A.9)

where

\[ \Pi'(k) = \frac{1}{2} \Pi^{\mu \nu}(k) P_{\mu \nu}^{k}(k) \]

\[ \Pi'(k) = \Pi^{\mu \nu}(k) \tilde{Q}_{\mu \nu}^{k}(k) \]

Polarization tensor \( \Pi_{\mu \nu}^{k}(k) \) in the HTL approximation has form

\[ \Pi^{\mu \nu}(k) = 3 \alpha s_{\rho i}^2 \left( u^\mu u^\nu - \omega \int \frac{d \Omega}{4 \pi} \frac{v^\mu v^\nu}{v \cdot k + i \varepsilon} \right) \]

and the longitudinal and transverse projectors are defined by expressions

\[ \tilde{Q}_{\mu \nu}^{k}(k) = \tilde{u}_\mu(k) \tilde{u}_\nu(k) \]

\[ P_{\mu \nu}^{k}(k) = g_{\mu \nu} - u_\mu u_\nu - \tilde{Q}_{\mu \nu}^{k}(k) \frac{(k \cdot u)^2}{k^2} \]  

(A.10)

respectively, where Lorentz-covariant four-vector \( \tilde{u}_\mu(k) \) is defined by formula (5.5).

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