Topologically enhanced harmonic generation in a nonlinear transmission line metamaterial

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Nonlinear transmission lines (NLTls) are nonlinear electronic circuits used for parametric amplification and pulse generation, and it is known that left-handed NLTls support enhanced harmonic generation while suppressing shock wave formation. We show experimentally that in a left-handed NLTL analogue of the Su-Schrieffer-Heeger (SSH) lattice, harmonic generation is greatly increased by the presence of a topological edge state. Previous studies of nonlinear SSH circuits focused on solitonic behaviours at the fundamental harmonic. Here, we show that a topological edge mode at the first harmonic can produce strong propagating higher-harmonic signals, acting as a nonlocal cross-phase nonlinearity. We find maximum third-harmonic signal intensities five times that of a comparable conventional left-handed NLTL, and a 250-fold intensity contrast between topologically nontrivial and trivial configurations. This work advances the fundamental understanding of nonlinear topological states, and may have applications for compact electronic frequency generators.
opological edge states—robust bound states guaranteed to exist at the boundary between media with topologically incompatible band structures—were first discovered in condensed matter physics. Recently, electronic LC circuits have emerged as a highly promising method of realizing these remarkable phenomena. Compared to other classical platforms like photonics, acoustics, and mechanical lattices, which have also been used to realize topologically nontrivial band structures and topological edge states, electronic circuits have several compelling advantages: extreme ease of experimental analysis; the ability to fabricate complicated structures via printed circuit board (PCB) technology; and the intriguing prospect of introducing nonlinear and/or amplifying circuit elements into the system, which can give rise to new phenomena.

One of the most interesting questions raised by the emergence of topologically nontrivial classical lattices is how topological edge states interact with nonlinear media. Previous studies have focused on nonlinearity-induced local self-interactions in the fundamental harmonic, which can give rise to solitons with anomalous plateau-like decay profiles in nonlinear SSH chains, or chiral solitons in two-dimensional lattices. It has also been suggested that topological edge states in nonlinear lattices could be used for robust traveling-wave parametric amplification, optical isolation, and other applications.

In this paper, we report on the implementation of a nonlinear SSH chain based on a left-handed nonlinear transmission line (NLTL), in which the topological edge state induces highly efficient harmonic generation.

The case of $\alpha = 1$ corresponds to a standard (non-dimerized) left-handed transmission line. This type of transmission line is characterized by having sites separated by capacitors, and connected to ground by inductors, rather than vice versa. Left-handed NLTLs have been shown to be useful for parametric amplification and pulse generation.

**Results**

**Circuit design.** The transmission line circuit is shown schematically in Fig. 1a. It contains inductors of inductance $L$ and capacitors of alternating (dimerized) capacitances $C_a$ and $C_b$. We will shortly treat the case where the $C_a$ capacitors are nonlinear (the $L$ and $C_b$ elements are always linear). First, consider the linear limit where $C_a$ is constant. We define the characteristic angular frequency $\omega_0 = (LC_a)^{-1/2}$, and the capacitance ratio $\alpha = C_a/C_b$. The characteristic frequency is $f_0 = \omega_0/2\pi \approx 19$ MHz, and the lattice has 40 sites. Two cases are shown: d $C_b$-type capacitors at the edge, for which the $\alpha > 1$ gap is topologically nontrivial; e $C_a$-type capacitors at the edge, for which the $\alpha > 1$ gap is trivial. Red dotted curves indicate the band-edge frequencies $f_0/\sqrt{2}$ and $\sqrt{\alpha/2}f_0$. Blue dashes indicate the operating regime of the nonlinear circuit, with $\alpha \approx 1.3$ in the linear (low-voltage) limit and $\alpha$ effectively increasing with voltage amplitude.
Let us treat the points adjacent to the capacitors as lattice sites, indexed by an integer \( k \), and close the circuit by grounding the edges [the left edge is the site labeled A in Fig. 1a]. Using Kirchhoff’s laws, we can show that a mode with angular frequency \( \omega \) satisfies (see Supplementary Note 1):

\[
\left( \mathcal{H} - \frac{1}{\alpha} \right) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\alpha} \\ 0 \\ \frac{1}{\alpha} \\ 0 \\ \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ \vdots \\ v_N \end{pmatrix} = \left( 1 - \frac{\omega^2}{\omega_s^2} \right) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ \vdots \\ v_N \end{pmatrix},
\]

(2)

where \( v_k \) denotes the complex voltage on site \( k \). The matrix \( \mathcal{H} \) has the form of the SSH Hamiltonian:

\[
\mathcal{H} = \begin{pmatrix} 0 & \frac{1}{\alpha} & 0 & \frac{1}{\alpha} & \vdots & \frac{1}{\alpha} \\ \frac{1}{\alpha} & 0 & \frac{1}{\alpha} & 0 & \vdots & \frac{1}{\alpha} \\ 0 & \frac{1}{\alpha} & 0 & \frac{1}{\alpha} & \vdots & \frac{1}{\alpha} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\alpha} & 0 & \frac{1}{\alpha} & 0 & \vdots & 0 \\ \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} & \vdots & \frac{1}{\alpha} \end{pmatrix}.
\]

Thus, the eigenfrequency modes of the circuit have a one-to-one correspondence with the SSH eigenstates.

The band diagram for the linear closed circuit is shown in Fig. 1d. The lack of an upper cutoff frequency is a characteristic of left-handed transmission lines\(^{40}\). There is a bandgap in the range \( \omega_s/\sqrt{2} < \omega < \sqrt{1/2} \omega_s \). For \( \alpha > 1 \), the bandgap contains edge states, which are zero-eigenvalue eigenstates of \( \mathcal{H} \) that can be characterized via a topological invariant derived from the Zak phase\(^1\). The edge state’s angular frequency is

\[
\omega_\text{es} = \sqrt{\alpha/(1+\alpha)} \omega_s.
\]

(4)

Note that the edge state is not at zero frequency, nor do they lie at precisely the middle of the bandgap; this is due to the aforementioned mapping from the circuit equations to the SSH model—specifically, the fact that \( \omega \) is not the eigenvalue in Eq. (2).

For \( \alpha < 1 \), there is a finite bandgap below \( \omega_s/\sqrt{2} \), which is topologically trivial and contains no edge states. If we swap the two types of capacitors, so that the \( C_0 \)-type capacitors are the ones at the edge, then the \( \alpha > 1 \) bandgap is trivial and the \( \alpha < 1 \) bandgap nontrivial, as shown in Fig. 1e.

Next, consider a nonlinear circuit with each \( C_0 \) capacitor consisting of a pair of back-to-back varactors. The nonlinear capacitance \( C_0 \) decreases with the magnitude of the bias voltage (the voltage between the end-points of the capacitor), as shown in Fig. 1c. For theoretical analyses, it is convenient to model this nonlinearity by

\[
a_\text{nl}(t) \equiv A + B[\Delta V(t)]^2,
\]

(5)

where \( a_\text{nl}(t) \equiv C_0/C_0(t) \), and \( \Delta V(t) \) is the bias voltage. The key feature of the nonlinearity is that at higher voltages, the effective value of \( \alpha \) increases. Depending on the chosen boundary conditions, this drives the circuit deeper into the topologically trivial or nontrivial regime.

**Experimental results.** The implemented NLTL, shown Fig. 1b, contains a total of 40 sites, or 20 unit cells. The linear circuit elements have \( L = 1.5 \mu \text{H} \) and \( C_0 = 47 \text{pF} \), so that \( \omega_s/2\pi = 19 \text{MHz} \). By fitting Eq. (5) to manufacturer data for the varactors at low bias voltages (see Supplementary Note 3), we obtain \( A = 1.32 \) and \( B = 0.51 \text{V}^{-2} \) (thus, in the linear limit, \( \alpha \approx 1.3 > 1 \)).

The fitted capacitance–voltage relation is shown in Fig. 1c.

We supply a continuous-wave sinusoidal input voltage signal, with tunable frequency \( f_\text{in} \) and amplitude \( V_\text{in} \), to either of the points labeled A and B in Fig. 1a. This allows us to study the cases corresponding to Fig. 1d, e, which we refer to as the “nontrivial” and “trivial” lattices, respectively (see Methods). In both cases, the input site is denoted as \( k = 0 \).

A typical set of measurement results is shown in Fig. 2a–c, for \( f_\text{in} = 16 \text{MHz} \) and \( V_\text{in} = 2.5 \text{V} \). On each site \( k \), the spectrum of the voltage signal is shown in Fig. 2c, with prominent peaks at odd harmonics (\( f_\text{in}, 3f_\text{in}, 5f_\text{in}, \ldots \)); even harmonics are suppressed due to the symmetry of the capacitance–voltage relation\(^{39}\). Focusing on the first and third harmonics, we define the respective peak values as \( |v_k^{(1)}| \) and \( |v_k^{(3)}| \), and use these to plot Fig. 2a, b. We verified that these experimental data agree well with results from the SPICE circuit simulator (see Supplementary Note 5).

From Fig. 2a, b, we see that the nontrivial and trivial lattices exhibit very different behaviors for both the first- and third-harmonic signals. First, consider the first-harmonic signal. In both lattices, there is an exponential decay away from the edge, but the decay is sharper in the nontrivial lattice, which may be attributed to the enhanced intensity arising from the coupling of the input signal to the topological edge state. As a quantitative measure of the localization of the first-harmonic signal, Fig. 2d, e shows the inverse participation ratio (IPR)

\[
\sum_k |v_k^{(1)}|^4 \left( \sum_k |v_k^{(1)}|^2 \right)^{-2} ; \text{ a larger IPR corresponds to a more localized profile.}^{43}
\]

We see that the IPR is substantially larger in the nontrivial lattice than in the trivial lattice, over a broad range of \( f_\text{in} \) and \( V_\text{in} \). The strong difference in localization is a key signature of nonlinearity: in the linear regime, a driving voltage on the edges of the nontrivial and trivial lattices would produce different overall amplitudes, but the same exponential decay profile (see Supplementary Note 1). It is interesting to note that the region of enhanced IPR, shown in Fig. 2d, closely resembles the nontrivial bandgap in Fig. 1d.

We can also see from Fig. 2a, c that strong higher-harmonic signals are present in the nontrivial lattice. Moreover, Fig. 2a indicates that the third-harmonic signal is extended, not localized to the edge. To understand this in more detail, we define

\[
\chi = \left\langle \frac{|v_k^{(3)}|^2}{|V_\text{in}|^2} \right\rangle / V_\text{in}^2.
\]

(6)

which quantifies the intensity of the third-harmonic signal relative to the input intensity at the first harmonic. Here, \( \langle \cdots \rangle \) denotes an average over the first ten lattice sites. Figure 3a, b plots the variation of \( \chi \) with \( f_\text{in} \) and \( V_\text{in} \). In the nontrivial circuit, the maximum value of the normalized intensity is \( \chi = 2.5 \) for \( f_\text{in} \approx 16 \text{MHz} \) and \( 1 \text{V} \leq V_\text{in} \leq 4 \text{V} \). The fact that \( \chi \) peaks over a relatively narrow frequency range, as shown in Fig. 3a, may be a finite-size effect: the high-frequency modes of the lattice form discrete sub-bands due to the finite lattice size [see Fig. 1d, e]. In computer simulations, we obtained a similar maximum value of \( \chi \approx 2.4 \) for the nontrivial lattice, whereas a comparable left-handed NLTL of the usual design (containing only identical nonlinear capacitances) has maximum \( \chi = 0.47 \) (see Supplementary Note 7).

The trivial lattice exhibits a much weaker third-harmonic signal. As indicated in Fig. 3c, for certain choices of \( f_\text{in} \) and \( V_\text{in} \), the value of \( \chi \) in the nontrivial lattice is 200 times that in the trivial lattice. Figure 3d plots the normalized third-harmonic signal intensities versus the site index \( k \), showing that they do not decay exponentially away from the edge. In the nontrivial lattice,
the normalized third-harmonic signal increases with $V_{in}$ (i.e., stronger nonlinearity).

**Discussion**

Our results point to a complex interplay between the topological edge state and higher-harmonic modes in the SSH-like NLTL. When a topological edge state exists in the linear lattice, it can be excited by an input signal at frequencies matching the bandgap of the linear lattice. The importance of the edge state is evident from the comparisons between the topologically trivial and nontrivial lattices (Figs. 2, 3). Note also that when the excitation frequency lies outside the linear bandgap, the two lattices behave similarly and the harmonic generation is relatively weak.

In the topologically nontrivial lattice, the resonant excitation generates third- and higher-harmonic signals that penetrate deep into the lattice, unlike the first-harmonic mode which is localized to the edge. Away from the edge, the higher-harmonic signals become stronger than the first harmonic, and hence dominate the effective value of the nonlinear $\alpha$ parameter. In the linear lattice, $\alpha$ is the parameter that drives the topological transition, and increasing $\alpha$ leads to a larger bandgap and hence a more confined edge state. In the nonlinear regime, Fig. 2 shows an order-of-magnitude increase in the third-harmonic signal amplitude in the nontrivial lattice, relative to the trivial lattice; this implies an effective increase in $\alpha$, and indeed we see that the first-harmonic mode profile is more strongly localized. A more localized edge state, in turn, produces a stronger response to an input signal.

The above interpretation is supported by a more detailed analysis of the coupled equations governing the different circuit mode harmonics (see Supplementary Notes 2–4). These equations involve an effective $\alpha$ parameter whose approximate value, in the $n$-th unit cell, is $\langle \alpha_n \rangle \approx A + 2B \sum_{m} |W_n^m|^2$, where $|W_n^m|$ is the $m$-th harmonic of the bias voltage on the nonlinear capacitor in the $n$-th unit cell, and $m = 1, 3, 5, \ldots$. We are able to show that propagating waves can be self-consistently realized for higher...
(m ≥ 3) harmonics in the presence of nonlinearity, even if the fundamental (m = 1) mode only has decaying solutions. The first-harmonic mode is localized to the edge, with localization length decreasing with (αn) in a manner similar to the linear SSH-like lattice. The generation of the higher-harmonic signals occurs mainly near the edge of the lattice, where the first-harmonic mode is largest. The nonlinearity-induced harmonic generation is aided by the well-known fact that the SSH edge state changes sign in each unit cell, corresponding to the fact that the gap closing in the SSH model takes place at the corner of the Brillouin zone. This feature increases the bias voltages across the nonlinear capacitors, which can thus exceed the values of the voltages at individual sites.

The input signal can also be applied to the middle of the lattice. In this context, it is interesting to note that when we choose to excite a single site in the bulk of an SSH-like lattice, the sections to either side of the excitation have different topological phases: either trivial on the left and nontrivial on the right, or vice versa, depending on the two possible choices of excitation site. If the source impedance is sufficiently low, the effect is similar to exciting independent chains to the left and right; thus, the enhanced higher-harmonic signal is preferentially emitted toward the topologically nontrivial side (see Supplementary Note 6).

The presence of higher-harmonic signals distinguishes our system from previous studies of nonlinear topological edge states, which were based on nonlinear self-modulation at a single harmonic. For instance, in a nonlinear SSH lattice where the coupling depends on the local intensity of a single mode, soliton-like edge states with anomalous mode profiles were predicted, and subsequently verified using a NLTL-like circuit. That circuit, unlike ours, had narrow frequency bands and thus did not support propagating higher-harmonic modes. Topological solitons based on nonlinear self-modulation are also predicted to exist in higher-dimensional lattices. In our case, the effective value of α away from the edge is dominated by the higher-harmonic signals; from the point of view of the first-harmonic mode, these act as a nonlocal nonlinearity, driving the entire lattice deeper into the topologically nontrivial regime, not just the sites with large first-harmonic intensity.

Our work opens the door to the application of topological edge states for enhancing harmonic generation, not just in transmission line circuits, but also a variety of other interesting systems. These include two-dimensional electronic lattices, where topological edge states have already been observed in the linear regime, and the unidirectional nature of the edge states may be even more beneficial for frequency-mixing. Higher dimensional circuit lattices may possess different thresholds for bulk propagation in different directions, with an extreme generalization being that of a corner mode circuit constructed in ref. Electronic circuits incorporating amplifiers and resistances may also be able to

**Fig. 3** Measured third-harmonic signal intensities in the nonlinear circuit. a, b Normalized third-harmonic signal intensity χ versus input frequency f0, and input voltage V0, for the a nontrivial lattice and b trivial lattice. Here, χ is derived from experimental data using the definition (6). c Ratio of the trivial and nontrivial intensities, χnon trivial/χtrivial, within the region indicated by boxes in a and b. d Measured third-harmonic intensities (normalized to the input signal) at different sites, for the three sets of input parameters indicated by stars in c: V0 = 0.5 V, 1.5 V, and 2.5 V, with fixed f0 = 16 MHz.
explore behaviors analogous to topological lasers44–48, combining topological states with both nonlinearity and non-Hermiticity. Finally, circuits containing varactors that are explicitly time-modulated may be suitable for generating synthetic dimensions to realize topological features in higher dimensions49–54.

Methods

Sample fabrication and experimental procedure. The NULT was implemented on a PCB (Seeed Tech. Co.), with each nonlinear capacitor consisting of a pair of back-to-back varactors (Skyworks Solutions, SMV1253-004LF). The transmission line, as fabricated, is topologically nontrivial, as shown in Fig. 1a. To probe the trivial circuit, we use a switch to add one sublattice unit cell at the rightmost end of the transmission line, and disconnect the leftmost C\textsubscript{0} and L in Fig. 1a. This yields a nontrivial circuit of same length, with the C\textsubscript{0} and C\textsubscript{1} capacitors swapped. A function generator (Tektronix AFG3102C) supplies the continuous-wave sinusoidal input voltage, and the voltages on successive lattice sites, k ≥ 1, are measured by an oscilloscope (Rohde & Schwarz RTE1024) in high-impedance mode. Numerical results were obtained using the SPICE circuit simulator.

Data availability

Raw experimental data and Python code used to generate all plots can be found at https://doi.org/10.21979/N9/I74ZP1. All other data are available from the authors upon reasonable request.

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References

1. Bernevig, B. & Hughes, T. Topological Insulators and Topological Superconductors (Princeton University Press, Princeton, New Jersey, 2013).
2. Liu, N., Owens, C., Sommer, A., Schuster, D. & Simon, J. Time- and angle-resolved dynamics in a topological circuit. Phys. Rev. X 5, 021031 (2015).
3. Albert, V. V., Glazman, L. I. & Jiang, L. Topological properties of linear circuit lattices. Phys. Rev. Lett. 114, 173902 (2015).
4. Lee, C. H. et al. Topolectric circuits. Commun. Phys. 1, 39 (2018).
5. Imhof, S. et al. Topolectric circuit realization of topological corner modes. Nat. Phys. 14, 925–929 (2018).
6. Goren, T., Plekhanov, K., Appas, F. & Le Hur, K. Topological Zak phase in strongly coupled LC circuits. Phys. Rev. B 97, 041106(R) (2018).
7. Zhu, W., Hou, S., Long, Y., Chen, H. & Ren, J. Simulating quantum spin Hall effect in the topological Lieb lattice of a linear circuit network. Phys. Rev. B 97, 075310 (2018).
8. Hadad, Y., Soric, I. C., Khanikaev, A. B. & Alù, A. Self-induced topological protection in nonlinear circuit arrays. Nat. Electron. 1, 178 (2018).
9. Serra-Garcia, M., Stürsunkn, R. & Huber, S. D. Observation of quadrupole transitions and edge mode topology in an LC network. Phys. Rev. B 99, 020304(R) (2019).
10. Wang, Z., Chong, Y. D., Joannopoulos, J. D. & Soljačić, M. Observation of unidirectional backscattering-immune topological electromagnetic states. Nature 461, 772 (2009).
11. Lu, L., Joannopoulos, J. D. & Soljačić, M. Topological photonics. Nat. Photonics 8, 821 (2014).
12. Khanikaev, A. B. & Shvets, G. Two-dimensional topological photonics. Nat. Photonics 11, 763 (2017).
13. Ozawa, T. et al. Topological photonics. arXiv:1802.04173 (2018).
14. Yang, Z. et al. Topological acoustics. Phys. Rev. Lett. 114, 114301 (2015).
15. Fleury, R., Khanikaev, A. B. & Alù, A. Floquet topological insulators for sound. Nat. Commun. 7, 11744 (2016).
16. He, C. et al. Acoustic topological insulator and robust one-way sound transport. Nat. Phys. 12, 1124 (2016).
17. Kane, C. L. & Lubensky, T. C. Topological boundary modes in isostatic lattices. Nat. Phys. 10, 39–45 (2014).
18. Huber, S. D. Topological mechanics. Nat. Phys. 12, 621 (2016).
19. Lee, C. H., Li, C., Jin, G., Liu, Y. & Zhang, X. Topological dynamics of gyroscopic and Floquet lattices from Newton’s laws. Phys. Rev. B 97, 085110 (2018).
20. Su, W. P., Schrieffer, J. R. & Heeger, A. J. Solitons in polyacetylene. Phys. Rev. Lett. 42, 1698 (1979).
21. Hadad, Y., Khanikaev, A. B. & Alù, A. Self-induced topological transitions and edge states supported by nonlinear staggered potentials. Phys. Rev. B 93, 155112 (2016).
22. Lumer, Y., Plotnik, Y., Rechtsman, M. C. & Segev, M. Self-localized states in photonic topological insulators. Phys. Rev. Lett. 111, 243905 (2013).
Author contributions

Y.W. designed the printed circuit board and performed the measurements. Y.W. and L.-J.L. analyzed the data. B.Z. and Y.D.C. supervised the project. Y.W., L.-J.L., C.H.L., B.Z., and Y.D.C. participated in the design and interpretation of the experiment, and all contributed substantially to the work.

Additional information

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