PRÜFER–LIKE CONDITIONS ON AN AMALGAMATED ALGEBRA ALONG AN IDEAL

CARMELO ANTONIO FINOCCHIARO

Abstract. Let \( f : A \longrightarrow B \) be a ring homomorphism and let \( b \) be an ideal of \( B \). In this paper we study Prüfer–like conditions in the amalgamation of \( A \) with \( B \) along \( b \), with respect to \( f \), a ring construction introduced in 2009 by D’Anna, Finocchiaro and Fontana.

In memory of my father

1. Introduction

Prüfer domains, introduced by H. Prüfer in \[35\], form a very relevant class of commutative rings. Throughout the years, this class was deeply studied by several authors (for a systematic study see \[17\]), so that many equivalent definitions of a Prüfer domain were given. For example, the notion of Prüfer domain globalizes the notion of valuation domain in a non local context. Moreover, the class of Prüfer domains is the natural generalization of the class of Dedekind domains in the non–Noetherian setting. Among the many equivalent conditions that make an integral domain \( A \) a Prüfer domain, we recall the following:

1. Every finitely generated ideal of \( A \) is projective.
2. \( A_p \) is a valuation domain, for each prime (maximal) ideal \( p \) of \( A \).
3. Every finitely generated ideal of \( A \) is locally principal.
4. If \( T \) is an indeterminate over \( A \), every polynomial \( f \in A[T] \) is a Gauss polynomial over \( A \) (i.e., \( c(fg) = c(f)c(g) \), for each polynomial \( g \in A[T] \), where \( c(f) \) denotes the content of the polynomial \( f \)).
5. Every nonzero finitely generated ideal of \( A \) is invertible.

In \[26\], the notion of Prüfer domain was generalized to arbitrary (commutative) rings, possibly with zerodivisors. Other important contributions to the study of the previous conditions in rings with zerodivisors.

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were given in [3], [14], [18], [24], [25], [30], [31], [32], [36], etc. On the other hand, recently, in [2], Bazzoni and Glaz showed, giving appropriate counterexamples, that none of the previous conditions is equivalent to the other, when \( A \) is a ring with zerodivisors. The fact that, in general, the rings satisfying previous Prüfer–like conditions are distinct classes of rings leads us to recall the following definition.

**Definition 1.1.** Let \( A \) be a ring.

1. \((P_1)\) \( A \) is a semi-hereditary ring if every finitely generated ideal of \( A \) is projective.
2. \((P_2)\) \( A \) has weak global dimension at most 1 if \( A_p \) is a valuation domain, for each prime (maximal) ideal \( p \) of \( A \).
3. \((P_3)\) \( A \) is an arithmetical ring if every finitely generated ideal of \( A \) is locally principal.
4. \((P_4)\) \( A \) is a Gauss ring if every polynomial \( f \in A[T] \) is a Gauss polynomial over \( A \).
5. \((P_5)\) \( A \) is a Prüfer ring if every regular and finitely generated ideal of \( A \) is invertible.

In [2], it is shown that, for each \( n \in \{1, 2, 3, 4, 5\} \), condition \((P_n)\) implies condition \((P_{n+1})\). More precisely, Bazzoni and Glaz proved that a ring \( A \) satisfies condition \((P_n)\) if and only if \( A \) satisfies condition \((P_{n+1})\) and the total ring of fractions \( \text{Tot}(A) \) of \( A \) satisfies condition \((P_n)\). Moreover, it is proved that, if \( \text{Tot}(A) \) is an absolutely flat ring, then conditions \((P_n)\) \((n \in \{1, 2, 3, 4, 5\})\) are equivalent on \( A \).

Recently, J. Boynton in [6] studied Prüfer–like conditions in pullbacks. The use of pullbacks and fiber products of ring homomorphisms is a very powerful tool to produce interesting examples (see [16], [19], [20]). Of particular interest are the pullbacks of the following type: let \( A \subseteq B \) be a ring extension such that \( A \) and \( B \) have a nonzero common ideal. In this case, call the conductor of \( B \) into \( A \) the largest nonzero common ideal to \( A \) and \( B \). It is well-known that the conductor of such a ring extension \( A \subseteq B \) is

\[
\mathfrak{c} := (B : A) := \{ x \in A : xB \subseteq A \}
\]

Thus, if \( \pi : B \to B/\mathfrak{c} \) is the canonical projection, then \( A \) is clearly the inverse image \( \pi^{-1}(A/\mathfrak{c}) \) of the subring \( A/\mathfrak{c} \) of \( B/\mathfrak{c} \).

In his paper, Boynton describes the transfer of Prüfer–like conditions on this kind of pullbacks, under the assumption that the conductor of \( A \subseteq B \) is a regular ideal of \( B \).

The aim of the present paper is to study Prüfer–like conditions on amalgamated algebras along ideals. More precisely, in [9] and [10], the authors have introduced the following new ring construction. Given a
ring homomorphism \( f : A \rightarrow B \) and an ideal \( b \) of \( B \), consider the subring
\[
A \rtimes^f b := \{(a, f(a) + b) : a \in A, b \in b\}
\]
of \( A \times B \), called the amalgamation of \( A \) with \( B \) along \( b \) with respect to \( f \). This construction generalizes the amalgamated duplication of a ring along an ideal (introduced and studied in \([8], [12], [13] \) and in \([33]\)). Moreover, several classical constructions (such as the \( A + XB[X] \), the \( A + XB[X] \) and the \( D + M \) constructions) can be studied as particular cases of the amalgamation (see \([9, \text{Examples } 2.5 \text{ and } 2.6]\)) and other classical constructions, such as the Nagata’s idealization (cf. \([34\text{, page 2}], [29, \text{Chapter VI, Section 25}]\)), and the CPI extensions (in the sense of Boisen and Sheldon \([5]\)) are related to it (see \([9, \text{Example } 2.7]\)). The level of generality chosen to define the amalgamation is due to the fact that the ring \( A \rtimes^f b \) may be studied in the frame of fiber product constructions. This allows to describe easily many algebraic properties of \( A \rtimes^f b \), in relation with those of \( A, B, b \) and \( f \).

Moreover, the ring \( A \) is always embedded into the ring \( A \rtimes^f b \), and the natural image of the ring \( A \) into \( A \rtimes^f b \) is a retract of \( A \rtimes^f b \) (see \([9, \text{Remark } 4.6 \text{ or Proposition } 4.7]\]). This will help us to describe the transfer of Prüfer–like conditions in the amalgamations.

The content of this paper is organized as follows: at the beginning, we prove that, under the assumption that the conductor of the ring extension \( A \rtimes^f b \subseteq A \times B \) is regular, the ring \( A \rtimes^f b \) satisfies Prüfer–like conditions \( (P_n) \), for \( n \in \{1, 2, 3, 4, 5\} \), only in the trivial cases.

Later, we investigate the general case (in which the conductor is not necessarily regular) and we provide sufficient and necessary conditions for \( A \rtimes^f b \) to satisfy conditions \( (P_n) \), for \( n \in \{1, 2, 3, 4, 5\} \).

The results of this paper form a part of author’s thesis. The author is grateful to Marco Fontana, Stefania Gabelli and Sarah Glaz for their helpful comments and suggestions.

2. Preliminaries

We begin with some terminology and notation. In the following, with the term ring we will mean a commutative ring with multiplicative identity. We will call an element of a ring \( A \) a regular element if it is not a zerodivisor, and set
\[
\text{Reg}(A) := \{a \in A : a \text{ is a regular element of } A\}
\]
Moreover, we will say that an ideal of \( A \) is a regular ideal if it contains a regular element of \( A \). As usual, we will denote be \( \text{Spec}(A) \) the set of
all prime ideals of $A$ and sometimes, but not always, it will be endowed with the Zariski topology.

We collect in the following proposition several properties of the ring construction $A \rtimes f b$, that follow easily from the definitions.

**Proposition 2.1.** ([2, Proposition 5.1]) Let $f : A \longrightarrow B$ be a ring homomorphism, $b$ an ideal of $B$ and let

$$A \rtimes f b := \{(a, f(a) + b) : a \in A, \ b \in b\}$$

The following statements hold.

1. Let $\iota := \iota_{A, f, b} : A \longrightarrow A \rtimes f b$ be the natural ring homomorphism defined by $\iota(a) := (a, f(a))$, for all $a \in A$. Then, $\iota$ is ring embedding, making $A \rtimes f b$ a ring extension of $A$ (with $\iota(A) = \Gamma(f) := \{(a, f(a)) : a \in A\}$ subring of $A \rtimes f b$).

2. Let $a$ be an ideal of $A$ and set $$a \rtimes f b := \{(a, f(a) + b) : a \in a, \ b \in b\}.$$ Then $a \rtimes f b$ is an ideal of $A \rtimes f b$, the composition of canonical homomorphisms $A \longrightarrow A \rtimes f b \longrightarrow A \rtimes f b / a \rtimes f b$ is a surjective ring homomorphism and its kernel coincides with $a$. Hence, we have the following canonical isomorphism:

$$\frac{A \rtimes f b}{a \rtimes f b} \cong \frac{A}{a}.$$ 

3. Let $p_A : A \rtimes f b \longrightarrow A$ and $p_B : A \rtimes f b \longrightarrow B$ be the natural projections of $A \rtimes f b \subseteq A \times B$ into $A$ and $B$, respectively. Then, $p_A$ is surjective and $\text{Ker}(p_A) = \{0\} \times b$. Moreover, $p_B(A \rtimes f b) = f(A) + b$ and $\text{Ker}(p_B) = f^{-1}(b) \times \{0\}$. Hence, the following canonical isomorphisms hold:

$$\frac{A \rtimes f b}{\{0\} \times b} \cong A \quad \text{and} \quad \frac{A \rtimes f b}{f^{-1}(b) \times \{0\}} \cong f(A) + b.$$

4. Let $\gamma : A \rtimes f b \longrightarrow (f(A) + b)/b$ be the natural ring homomorphism, defined by $(a, f(a) + b) \mapsto f(a) + b$. Then $\gamma$ is surjective and $\text{Ker}(\gamma) = f^{-1}(b) \times b$. Thus, we have the following natural isomorphisms

$$\frac{A \rtimes f b}{f^{-1}(b) \times b} \cong \frac{f(A) + b}{b} \cong \frac{A}{f^{-1}(b)}.$$

In particular, when $f$ is surjective we have

$$\frac{A \rtimes f b}{f^{-1}(b) \times b} \cong \frac{B}{b}.$$
**Definition 2.2.** Let $\rho : A \to C, \sigma : B \to C$ be ring homomorphisms. We recall that the following subring

$$\rho \times_C \sigma := \{(a, b) \in A \times B : \rho(a) = \sigma(b)\}$$

of $A \times B$ is usually called the fiber product of $\rho$ and $\sigma$.

**Proposition 2.3** ([10, Proposition 4.2]). Let $f : A \to B$ be a ring homomorphism, $b$ be an ideal of $B$. If $\pi : B \to B/b$ is the canonical projection and $\tilde{f} := \pi \circ f$, then $A \times_f b = \tilde{f} \times_{B/b} \pi$.

**Remark 2.4.** Let $f : A \to B$ be a ring homomorphism, $S$ be a multiplicative subset of $A$ and $b$ be an ideal of $B$. Consider the multiplicative subset $T := f(S) + b$ of $B$ and let $f_S : A_S \to B_T$ be the ring homomorphism induced by $f$. By a straightforward verification it is shown that $f_S^{-1}(b)A_S$. Moreover, for each ideal $\mathfrak{d}$ of $B$, it is immediate that $\mathfrak{d}B_T = B_T$ if and only if $f^{-1}(\mathfrak{d} + b) \cap S \neq \emptyset$. Thus, $B_T = \{0\}$ if and only if $f^{-1}(b) \cap S \neq \emptyset$.

If $p$ is a prime ideal of $A$ and $S := A \setminus p, T := S_p := f(S) + b$, we shall denote $f_S$ simply by $f_p$ and $bB_T$ simply by $b_{S_p}$.

The following result describes completely the prime spectrum of $A \times_f b$.

**Proposition 2.5.** ([10] Proposition 2.6] and [11] Propositions 4.1 and 4.2) We preserve the notation of Proposition 2.7. Set $X := \text{Spec}(A), Y := \text{Spec}(B), W := \text{Spec}(A \times_f b)$. For each prime ideal $p$ of $A$ and each prime ideal $q$ of $B$ not containing $b$, set

$$p' := \{(p, f(p) + b) : p \in p, b \in b\}$$

$$q' := \{(a, f(a) + b) : a \in A, b \in b, f(a) + b \in q\}.$$

Then, the following statements hold.

1. The map $p \mapsto p'$ establishes a closed embedding of $X$ into $W$, so its image, which coincides with $V(b_0)$, is homeomorphic to $X$.
2. The map $q \mapsto q'$ is a homeomorphism of $Y \setminus V(b)$ onto $W \setminus V(b_0)$.
3. The prime ideals of $A \times_f b$ are of the type $p'$ or $q'$, for $p$ varying in $X$ and $q$ in $Y \setminus V(b)$.
4. Let $p \in \text{Spec}(A)$. Then, $p'$ is a maximal ideal of $A \times_f b$ if and only if $p$ is a maximal ideal of $A$.
5. Let $q$ be a prime ideal of $B$ not containing $b$. Then, $q'$ is a maximal ideal of $A \times_f b$ if and only if $q$ is a maximal ideal of $B$.

In particular:

$$\text{Max}(A \times_f b) = \{p' : p \in \text{Max}(A)\} \cup \{q' : q \in \text{Max}(B) \setminus V(b)\}.$$
The ring $A \rtimes f b$ is local if, and only if, $A$ is local and $b \subseteq \text{Jac}(B)$. In this case, if $m$ is the maximal ideal of $A$, the maximal ideal of $A \rtimes f b$ is $m'$. In particular, if $A$ and $B$ are local rings and $b$ is a proper ideal of $B$, then $A \rtimes f b$ is a local ring.

(7) (a) If $q$ is a prime ideal of $B$ not containing $b$, then $(A \rtimes f b)_q$ is isomorphic to $B_q$.

(b) If $p$ is a prime ideal of $A$, consider the multiplicative subset $S_p := f(A \setminus p) + b$, and let $f_p : A_p \rightarrow B_{S_p}$ be the ring homomorphism induced by $f$. Then $(A \rtimes f b)_p'$ is isomorphic to $A_p \rtimes f p b_{S_p}$. In particular, if $p \nsubseteq f^{-1}(b)$, we have $B_{S_p} = \{0\}$ and thus $(A \rtimes f b)_p'$ is isomorphic to $A_p$.

Let $A$ be a ring and $p$ be a prime ideal of $A$. Recall that $(A, p)$ has the regular total order property if, for each pair of ideals $a_1, a_2$ of $A$, one at least of which is regular, the ideals $a_1 A_p, a_2 A_p$ are comparable.

The following characterization of Prüfer rings will be useful. We recall it here for the reader convenience.

**Theorem 2.6.** ([26, Theorem 13]) Let $A$ be a ring. The following conditions are equivalent.

(i) $A$ is a Prüfer ring.

(ii) If $a, b, c$ are ideals of $A$ and $b$ or $c$ is regular, then

$$a(b \cap c) = ab \cap ac.$$  

(iii) For each maximal ideal $m$ of $A$, $(A, m)$ has the regular total order property.

**Definition 2.7.** We say that a ring $A$ is a locally Prüfer ring if $A_m$ is a Prüfer ring, for each $m \in \text{Max}(A)$.

**Remark 2.8.** Let $A$ be a ring.

(a) By [32, Proposition 2.10], if $A$ is a locally Prüfer ring, then $A$ is a Prüfer ring.

(b) If $A$ is Gauss ring, then so is $A_m$, for each maximal ideal $m$ of $A$ (each localization of a Gauss ring is still a Gauss ring). It follows that $A$ is a locally Prüfer ring.

(c) Note that an example of a Prüfer and non locally Prüfer ring is given in [32, Example 2.11]. Moreover, as observed in [2, Example 3.8], if $K$ is a field and $T_1, T_2$ are indeterminates over $K$, then $K[T_1, T_2]/(T_1, T_2)^3$ is a local total ring of fractions (and thus a locally Prüfer ring) that is not a Gauss ring. Thus we have the following proper inclusions of classes of rings

$$\{\text{Semihereditary rings}\} \subsetneq \{\text{w.gl.dim} \leq 1\} \subsetneq \{\text{Arithmetical rings}\} \subsetneq$$
Remark 2.9. Let \( \{A_1, \ldots, A_r\} \) be a nonempty and finite collection of rings and let \( A := \prod_{i=1}^{r} A_i \). As noted by Bakkari in a recent preprint, posted on arXiv, for each \( n \in \{1, 2, 3, 4, 5\} \), \( A \) satisfies Prüfer–like condition \((P_n)\) if and only if \( A_i \) satisfies the same Prüfer–like condition \((P_n)\), for each \( i \in \{1, \ldots, r\} \).

3. Results when the conductor of the ring extension \( A \bowtie^f b \subseteq A \times B \) is regular

As noted in \cite{18}, Lemma 1.50], the conductor of the ring extension \( A \bowtie^f b \subseteq A \times B \) is \( c := f^{-1}(b) \times b \). The following results show that when \( c \) is a regular ideal of \( A \times B \) (i.e., if \( f^{-1}(b) \), \( b \) are regular ideals of \( A, B \), respectively), then \( A \bowtie^f b \) satisfies Prüfer–like conditions \((P_n)\) \((n \in \{1, 2, 3, 4, 5\})\) only in the trivial case.

Theorem 3.1. Let \( f : A \rightarrow B \) be a ring homomorphism and let \( b \) be an ideal of \( B \). If \( f^{-1}(b) \) and \( b \) are regular ideals, then the following conditions are equivalent.

(i) \( A \bowtie^f b \) is a Prüfer ring.

(ii) \( A, B \) are Prüfer rings and \( b = B \).

Proof. (ii) \( \Rightarrow \) (i). By (ii), \( A \bowtie^f b = A \times B \). Then, it suffices to apply \cite{22} Proposition 3].

(i) \( \Rightarrow \) (ii). Assume, by contradiction, that \( b \) is a proper ideal of \( B \), and pick a maximal ideal \( m \) of \( A \) containing \( f^{-1}(b) \). Consider the multiplicative subset \( S_m := f(A \setminus m) + b \) of \( B \). By Proposition \cite{25} (7), the localization of \( A \bowtie^f b \) at the maximal ideal

\[
\mathfrak{m}^f := \{(m, f(m) + b) : m \in m, b \in b\}
\]

is isomorphic to \( C := A_m \bowtie^f m \bowtie^f b_m \) \((f_m : A_m \rightarrow B_{S_m} \) is the ring homomorphism induced by \( f \)). Now, pick regular elements \( a_0 \in f^{-1}(b), b_0 \in b \). Then, in particular, \( a_1 := (a_0, b_0)A \bowtie^f b \) is a regular ideal of \( A \bowtie^f b \). Set \( a^* := a_0/1 \in A_m, b^* := b_0/1 \in B_T \). Obviously, \( a^*, b^* \) are regular elements. Since \( A \bowtie^f b \) is a Prüfer ring, \((A \bowtie^f b, \mathfrak{m}^f)\) has the regular total order property, by Theorem \cite{26} Thus, if \( a_2 := (a_0, 0)A \bowtie^f b \), the ideals

\[
(a^*, b^*)C = a_1C, \quad (a^*, 0) = a_2C
\]

are comparable. Since, in particular, \( b^* \neq 0 \), we have \((a^*, b^*)C \subseteq (a^*, 0)C\). It follows that \((a^*, 0)C \subseteq (a^*, b^*)C\). Thus, there exist elements \( \alpha \in A_m, \beta \in bB_{S_m} \) such that

\[
(a^*, 0) = (\alpha, f_m(\alpha) + \beta)(a^*, b^*)
\]

\( \subseteq \{\text{Gauss rings}\} \subseteq \{\text{Locally Prüfer rings}\} \subseteq \{\text{Prüfer rings}\} \)
Keeping in mind that \( a^* \) is regular, it follows that \( \alpha = 1 \). Then, \( b^*(1 + \beta) = 0 \), and thus \( \beta = -1 \), since \( b^* \) is regular. This implies \( bB_{sm} = B_{sm} \), and, by Remark 2.3, \( f^{-1}(b) \not\subseteq m \), a contradiction. Thus \( b = B \) and, consequently, \( A \ltimes^f B = A \times B \). Then, the remaining part of statement (ii) follows by \cite{22} Proposition 3].

**Corollary 3.2.** We preserve the notation of Proposition 2.1 and let \( n \in \{1, 2, 3, 4, 5\} \). If \( f^{-1}(b) \) and \( b \) are regular ideals, then the following conditions are equivalent.

(i) \( A \ltimes^f b \) satisfies Prüfer–like condition \((P_n)\) (resp. \( A \ltimes^f b \) is locally Prüfer).

(ii) \( A, B \) satisfy Prüfer–like condition \((P_n)\) (resp. \( A, B \) are locally Prüfer rings) and \( b = B \).

**Proof.** (ii)\( \implies \) (i). By (ii), \( A \ltimes^f b = A \times B \). Then, condition (i) follows by using Remark 2.9 \cite{22} Proposition 3] and definitions.

(i)\( \implies \) (ii). If \( A \ltimes^f b \) is locally Prüfer, then it is a Prüfer ring, by Remark 2.8(a). Thus \( b = B \), by Theorem 3.1. Moreover, it is immediately seen that \( A, B \) are locally Prüfer rings. Now, let \( n \in \{1, 2, 3, 4, 5\} \). If \( A \ltimes^f b \) satisfies Prüfer–like condition \( P_n \), then \( A \ltimes^f b \) is a Prüfer ring. Thus the conclusion follows by Theorem 3.1 and Remark 2.9. \( \square \)

**Corollary 3.3.** Let \( A \) be a ring and \( a \) be a regular ideal of \( A \). Consider the amalgamated duplication of \( A \) along \( a \)

\[ A \ltimes a := \{(a, a + \alpha) : a \in A, \alpha \in a\} \]

(see \cite{8, 12, 13}) and let \( n \in \{1, 2, 3, 4, 5\} \). Then, \( A \ltimes a \) satisfies Prüfer–like condition \((P_n)\) (resp. \( A \ltimes a \) is a locally Prüfer ring) if and only if \( A \) satisfies Prüfer–like condition \((P_n)\) (resp. \( A \) is a locally Prüfer ring) and \( a = A \).

**Proof.** Apply Corollary 3.2 keeping in mind \cite{9} Example 2.4]. \( \square \)

4. Results in the general case

**Lemma 4.1.** Let \( r : B \rightarrow A \) be a ring retraction, and \( T \) be an indeterminate over \( B \). If \( \sum_{i=0}^n b_i T^i \) is a Gauss polynomial over \( B \), then \( \sum_{i=0}^n r(b_i)T^i \) is a Gauss polynomial over \( A \).

**Proof.** It follows by the proof of \cite{11} Theorem 2.1(1)]. \( \square \)

**Proposition 4.2.** We preserve the notation of Proposition 2.1. If \( A \ltimes^f b \) is a Prüfer ring and \( f(\text{Reg}(A)) \subseteq \text{Reg}(B) \), then \( A \) is a Prüfer ring.
Proof. Let $T$ be an indeterminate over $A$ and $\alpha := (a_0, \ldots, a_n)$ be a regular and finitely generated ideal of $A$. Consider the polynomial $p(T) := \sum_{i=0}^{n} a_i T^i \in A[T]$. Pick a regular element $a \in \alpha$. Then, keeping in mind that $f(\text{Reg}(A)) \subseteq \text{Reg}(B)$, it is easily checked that $(a, f(a))$ is a regular element of the finitely generated ideal $\alpha^w := ((a_0, f(a_0)), \ldots, (a_n, f(a_n)))$ of $A \bowtie b$. Since $A \bowtie b$ is a Prüfer ring, it follows that $\alpha^w$ is an invertible ideal of $A \bowtie b$, and thus the polynomial $p_{\bowtie}(T) := \sum_{i=0}^{n} (a_i, f(a_i))T^i \in A \bowtie b[T]$, whose content is clearly $\alpha^w$, is a Gauss polynomial over $A \bowtie b$, by [36]. Let $p_{A} : A \bowtie b \rightarrow A$ be the projection $((a, f(a) + b) \mapsto a)$. Then we have $p(T) = \sum_{i=0}^{n} p_{A}((a_i, f(a_i)))T^i$. Since $p_{A}$ is a ring retraction ([9, Remark 4.6]), it follows that $p(T)$ is a Gauss polynomial over $A$, by Lemma 4.1. Thus its content, that is exactly the regular ideal $\alpha$, is invertible, by [31] Theorem 6]. This completes the proof. □

Remark 4.3. We preserve notation of Proposition 2.1. The fact that $A \bowtie b$ is a Prüfer ring does not imply, in general, that $A$ is a Prüfer ring. For an example, see [1] Example 2.3], keeping in mind [9 Remark 2.8].

The following result is obtained by modifying the proof of [4, Theorem 1].

Proposition 4.4. Let $\phi : A \rightarrow B$ be a surjective ring homomorphism. If $A$ is a Prüfer ring and $\text{Ker}(\phi)$ is a regular ideal of $A$, then $\alpha(\beta \cap \gamma) = \alpha \beta \cap \alpha \gamma$, for all ideals $\alpha, \beta, \gamma$ of $B$. In particular, $B$ is a Prüfer ring.

Proof. Let $\mathfrak{d} := \text{Ker}(\phi)$ and let $\alpha, \beta, \gamma$ be ideals of $B$. To prove the equality $\alpha(\beta \cap \gamma) = \alpha \beta \cap \alpha \gamma$, it suffices to show that $\alpha \beta \cap \alpha \gamma \subseteq \alpha(\beta \cap \gamma)$. If $\mathfrak{p} \in \alpha \beta \cap \alpha \gamma$, then there are elements $\mathfrak{p}_i \in \alpha, \mathfrak{b}_i \in \beta, \mathfrak{a}_j \in \alpha, \mathfrak{c}_j \in \gamma$, with $i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$, such that $\mathfrak{p} = \sum_{i=1}^{n} \mathfrak{p}_i \mathfrak{b}_i = \sum_{j=1}^{m} \alpha_j \mathfrak{c}_j$. For each $i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$, choose elements $a_i \in \phi^{-1}(\mathfrak{p}_i), b_i \in \phi^{-1}(\mathfrak{b}_i), \alpha_j \in \phi^{-1}(\mathfrak{a}_j), c_j \in \phi^{-1}(\mathfrak{c}_j)$, and set $a' := \phi^{-1}(a), b' := \phi^{-1}(b), c' := \phi^{-1}(c)$. If $x := \sum_{i=1}^{n} a_i b_i$, it is immediate that $x - \sum_{j=1}^{m} \alpha_j c_j \in \mathfrak{d}$. Therefore $x \in (a'c' + \mathfrak{d}) \cap (a'b')$. Keeping in mind Theorem 2.6 and the fact that $\mathfrak{d}$ is a regular ideal of $A$, we have

\[
(a'c' + \mathfrak{d}) \cap (a'b') = (a'c' \cap a'b') + (\mathfrak{d} \cap a'b') \subseteq (a'c' \cap a'(b' + \mathfrak{d})) + \mathfrak{d} = a'(c' \cap (b' + \mathfrak{d})) + \mathfrak{d}
\]

Thus, there are elements $a'^h \in a', b'^h \in b', d_h \in \mathfrak{d}$, with $h \in \{1, \ldots, r\}$ such that $b'^h + d_h \in c'$, for each $h$, and $x = \sum_{h=1}^{r} a'^h (b'^h + d_h) + d$, for some $d \in \mathfrak{d}$. It follows immediately that $\mathfrak{p} = \sum_{h=1}^{r} f(a'^h) f(b'^h) \in \alpha(\beta \cap \gamma)$. Now the first statement is clear. The fact that $B$ is a Prüfer ring follows by the previous statement and Theorem 2.6. □
Corollary 4.5. Preserve the notation of Proposition 2.1 and assume that \(A \ltimes J \mathfrak{b}\) is a Pr"ufer ring. Then the following statements hold.

1. If \(\{0\} \times \mathfrak{b}\) is a regular ideal of \(A \ltimes J \mathfrak{b}\), then \(A\) is a Pr"ufer ring.
2. If \(f^{-1}(\mathfrak{b}) \times \{0\}\) is a regular ideal of \(A \ltimes J \mathfrak{b}\), then \(f(A) + \mathfrak{b}\) is a Pr"ufer ring.

Proof. It sufficies to apply Propositions 2.1(3) and 4.4.

Now, we will give sufficient conditions to make \(A \ltimes J \mathfrak{b}\) a total ring of fractions (and, in particular, a Pr"ufer ring).

Proposition 4.6. Let \(A\) be a total ring of fractions (i.e. \(A = \text{Tot}(A)\)), \(f : A \rightarrow B\) be a ring homomorphism and \(\mathfrak{b}\) be an ideal of \(B\) contained in the Jacobson radical \(\text{Jac}(B)\) of \(B\). Assume that at least one of the following conditions hold.

(a) \(\mathfrak{b}\) is contained in \(f(A)\).
(b) \(\mathfrak{b}\) is a torsion \(A\)-module (with the \(A\)-module structure inherited by \(f\)).

Then \(A \ltimes J \mathfrak{b}\) is a total ring of fractions (and it is, in particular, a Pr"ufer ring).

Proof. Let \((a, f(a) + \mathfrak{b})\) be a non invertible element of \(A \ltimes J \mathfrak{b}\). The goal is to show that \((a, f(a) + \mathfrak{b})\) is a zerodivisor of \(A \ltimes J \mathfrak{b}\). Since \(\mathfrak{b} \subseteq \text{Jac}(B)\), by Proposition 2.5 it follows that \(\text{Max}(A \ltimes J \mathfrak{b}) = \{m' : m \in \text{Max}(A)\}\).

Thus, there exists a maximal ideal \(m\) of \(A\) such that \((a, f(a) + \mathfrak{b}) \in m'\), that is \(a \in m\). Since \(A\) is a total ring of fractions, it follows that \(a\) is a zerodivisor of \(A\). Hence, we can pick a nonzero element \(\alpha \in A\) such that \(\alpha a = 0\). The following two cases may occur.

- **Condition (a) holds.** If \(\alpha \in \text{Ann}_A(\mathfrak{b})\), then it follows immediately that \((a, f(a) + \mathfrak{b})(\alpha, f(\alpha)) = (0, 0)\). Otherwise, let \(\beta \in \mathfrak{b}\) be an element such that \(f(\alpha) \beta \neq 0\). Since \(\mathfrak{b} \subseteq f(A)\), there is an element \(x \in f^{-1}(\mathfrak{b})\) such that \(f(x) = \beta\). Of course, \(\alpha x \neq 0\) and \((\alpha x, 0) \in A \ltimes J \mathfrak{b}\), since \(\alpha x \in f^{-1}(\mathfrak{b})\). It follows \((a, f(a) + \mathfrak{b})(\alpha x, 0) = (0, 0)\).

- **Condition (b) holds.** Since \(\mathfrak{b}\) is a torsion \(A\)-module, there exists a regular element \(x_0 \in A\) such that \(f(x_0)\mathfrak{b} = 0\). Of course, \(\alpha x_0 \neq 0\), since \(\alpha \neq 0\). Then \((a, f(a) + \mathfrak{b})(\alpha x_0, f(\alpha x_0)) = (0, 0)\).

The conclusion is now clear.

Proposition 4.7. We preserve the notation of Proposition 2.1. The following statements hold
(1) If \(A \ltimes f b\) is an arithmetical ring, then \(A\) is an arithmetical ring.
(2) If \(A \ltimes f b\) is a Gauss ring, then \(A\) is a Gauss ring.

Proof. By [9, Remark 4.6], \(A\) is a ring retract of \(A \ltimes f b\), via the projection \(p_A : A \ltimes f b \longrightarrow A, ((a, f(a) + b) \mapsto a)\). Then, the conclusion follows by applying [1, Theorem 2.1(1) and Theorem 2.5]. □

Proposition 4.8. We preserve the notation of Proposition 2.7 and Remark 2.7. Assume that \(b_{S_m} = \{0\}\), for each \(m \in \text{Max}(A) \cap V(f^{-1}(b))\). Then, the following statements hold.

(1) If \(A\) is a locally Prüfer ring and \(B_n\) is a Prüfer ring, for each \(n \in \text{Max}(B) \setminus V(b)\), then \(A \ltimes f b\) is a locally Prüfer ring.
(2) If \(A\) is a Gauss ring and \(B_n\) is a Gauss ring, for each \(n \in \text{Max}(B) \setminus V(b)\), then \(A \ltimes f b\) is a Gauss ring.

Proof. By Proposition 2.5, we have
\[
\text{Max}(A \ltimes f b) = \{m' : m \in \text{Max}(A)\} \cup \\{\overline{n} : n \in \text{Max}(B) \setminus V(b)\}.
\]
Keeping in mind Proposition 2.5(7) and that \(b_{S_m} = \{0\}\), for each \(m \in \text{Max}(A) \cap V(f^{-1}(b))\), we have that \((A \ltimes f b)_{m'} = B_n\), for each \(n \in \text{Max}(B) \setminus V(b)\), and \((A \ltimes f b)_{n'} = A_m\), for each \(m \in \text{Max}(A)\). Then, statement (1) follows by definition. Statement (2) follows by noting that the property of being Gauss, for a ring, is local. □
Proposition 4.10. We preserve the notation of Proposition 2.4 and Remark 2.4. Assume that for each \( m \in \text{Max}(A) \cap V(f^{-1}(b)) \), either the map \( f_m : A_m \rightarrow B_{S_m} \) is surjective or \( f^{-1}(b)A_m \not= \{0\} \). Then, the following conditions are equivalent.

(i) \( A \otimes B \) is an arithmetical ring.
(ii) \( A \) is an arithmetical ring, \( B_{S_m} = \{0\} \), for each \( m \in \text{Max}(A) \cap V(f^{-1}(b)) \), and, for any \( n \in \text{Max}(B) \setminus V(b) \), the set of all the ideals of \( B_n \) is totally ordered by inclusion.

Proof. (i) \( \implies \) (ii). By [30, Theorem 1], the set of all ideals of each localization of \( A \otimes B \) at its maximal ideals is totally ordered by inclusion. Thus, Proposition 2.5(7) implies that in each localization \( B_n \) (\( n \in \text{Max}(B) \setminus V(b) \)) the set of all ideals is totally ordered by inclusion. Now, let \( m \) be a maximal ideal of \( A \) containing \( f^{-1}(b) \). By Proposition 2.3(7), the localization \( (A \otimes B)_{m'} \) is isomorphic to \( A_m \otimes B_{S_m} \). If \( \pi_m : B_{S_m} \rightarrow B_{S_m}/b_{S_m} \) is the canonical projection and \( \tilde{f}_m := \pi_m \circ f_m \), by Proposition 2.3 the ring \( A_m \otimes B_{S_m} \) is the fiber product of the ring homomorphisms \( f_m \) and \( \pi_m \). Keeping in mind that \( f_m^{-1}(b_{S_m}) = f^{-1}(b)A_m \) (Remark 2.3) and applying Proposition 2.3 it follows that \( b_{S_m} = \{0\} \). Thus, by Proposition 2.5(7), \( A_m \) is isomorphic to \( (A \otimes B)_{m'} \), for each maximal ideal \( m \) of \( A \). This proves that \( A \) is an arithmetical ring.

(ii) \( \implies \) (i). Apply [30, Theorem 1], Proposition 2.1(3) and the local structure of \( A \otimes B \) (Proposition 2.5(7)).

Proposition 4.11. We preserve the notation of Proposition 2.7 and Remark 2.4. Assume that, for each maximal ideal \( m \) of \( A \) containing \( f^{-1}(b) \), either the map \( f_m : A_m \rightarrow B_{S_m} \) is surjective or \( f^{-1}(b)A_m \not= \{0\} \). Then, the following conditions are equivalent.

(i) \( A \otimes B \) has weak global dimension at most 1.
(ii) \( A \) has weak global dimension at most 1, \( B_n \) is a valuation domain, for each \( n \in \text{Max}(B) \setminus V(b) \) and \( b_{S_m} = \{0\} \), for each \( m \in \text{Max}(A) \cap V(f^{-1}(b)) \).

Proof. (i) \( \implies \) (ii). By Proposition 2.5(7), we have \( (A \otimes B)_{m'} \cong B_n \), for any maximal ideal \( n \) of \( B \) not containing \( b \). Then, it follows, by definition, that \( B_n \) is a valuation domain for each \( n \in \text{Max}(B) \setminus V(b) \). Now, let \( m \) be a maximal ideal of \( A \) containing \( f^{-1}(b) \). Since, in particular, \( A \otimes B \) is an arithmetical ring, it follows \( b_{S_m} = \{0\} \), by Proposition 4.10. Thus, by Proposition 2.1(3), the localization \( A_m \) is isomorphic to the the valuation domain \( (A \otimes B)_{m'} \), for any maximal ideal \( m \) of \( A \). This proves that \( A \) has weak global dimension \( \leq 1 \).

(ii) \( \implies \) (i). Apply the local structure of \( A \otimes B \) (Proposition 2.5(7)). Note that (ii) implies (i), without any extra assumption.
To give conditions to make $A \ltimes f b$ a semi–hereditary ring, we want to use the following characterization.

**Theorem 4.12.** ([23 Corollary 4.2.19]) Let $A$ be a ring. Then, $A$ is semi–hereditary if and only if $A$ is coherent and the weak global dimension of $A$ is at most 1.

Let $\phi : A \to B$ be a ring homomorphism and let $M$ be a $B$–module. We shall denote by $\cdot \phi$ the scalar multiplication, induced by $\phi$, making $M$ an $A$–module.

**Lemma 4.13.** We preserve the notation of Proposition 2.1. If $b$ is a finitely generated $A$–module (with the $A$–module structure induced by $f$), then the ring embedding $\iota : A \to A \ltimes f b$ is finite.

**Proof.** Let $\{b_1, \ldots, b_n\} \subseteq b$ be a finite set of generators of the $A$–module $b$, and fix an element $(a, f(a) + b) \in A \ltimes f b$. Then, there exist elements $a_1, \ldots, a_n \in A$ such that $b = \sum_{i=1}^n a_i \cdot f b_i = \sum_{i=1}^n f(a_i)b_i$. It follows immediately that

$$(a, f(a) + b) = a \cdot (1, 1) + \sum_{i=1}^n a_i \cdot (0, b_i).$$

This proves that $\{(1, 1), (0, b_1), \ldots, (0, b_n)\} \subseteq A \ltimes f b$ is a finite set of generators of $A \ltimes f b$ as an $A$–module (with the structure induced by $\iota$), i.e. $\iota$ is finite. \[\square\]

**Proposition 4.14.** We preserve the notation of Proposition 2.1. Then, the following statements hold.

1. If $A \ltimes f b$ is a coherent ring, then $A$ is coherent.
2. If $A$ is a coherent ring and $b$ is a coherent $A$–module (with the structure induced by $f$), then $A \ltimes f b$ is a coherent ring.

**Proof.** Statement (1) follows by [9, Remark 4.6] and [23 Theorem 4.1.5].

(2). We begin by noticed that, since $b$ is, in particular, a finitely generated $A$–module, the ring embedding $\iota$ is finite, by Lemma 4.13. Now, let $p_A : A \ltimes f b \to A$, $p_B : A \ltimes f b \to B$ be the projections. Then, $p_A$ (resp. $p_B$) induces on $A$ (resp. $b$) a structure of $A \ltimes f b$–module. With these structures, we have the following short exact sequence

$$0 \to b \to A \ltimes f b \xrightarrow{\iota} A \to 0,$$

of $A \ltimes f b$–modules, where $i : b \to A \ltimes f b$ is defined by $\beta \mapsto (0, \beta)$, for each $\beta \in b$. Let $\iota : A \hookrightarrow A \ltimes f b$ be the ring embedding such that
(a, f(a)) \mapsto (a, f(a)), for each a \in A. On the $A \ltimes f^\oplus b$–module $b$, the map $\iota$ induces the following scalar multiplication

$$a \cdot \iota \beta := (a, f(a)) \cdot p_B \beta = p_B ((a, f(a))) \beta = f(a) \beta \quad (a \in A, \beta \in b)$$

It follows that the structure of $A$–module given to $b$ by $\iota$ is the same structure induced on $A \ltimes f^\oplus b$ by $f$. Since $\iota$ is finite and $b$ is a coherent $A$–module, by [27, Corollary 1.1] it follows that $b$ is a coherent $A \ltimes f^\oplus b$–module. Moreover, $\iota$ induces to the $A \ltimes f^\oplus b$–module $A$ the following scalar multiplication

$$a \cdot \iota \alpha := (a, f(a)) \cdot p_A \alpha = p_A ((a, f(a))) \alpha = a \alpha \quad (a, \alpha \in A)$$

Thus $\iota$ induces on $A$ its natural structure of module over itself. Since $A$, by assumption, is a coherent ring, it follows that it is a coherent $A \ltimes f^\oplus b$–module, again by [27, Corollary 1.1]. Then $A \ltimes f^\oplus b$ is a coherent $A \ltimes f^\oplus b$–module, by [7, Pag. 43, Exercise 11(a)], that is, $A \ltimes f^\oplus b$ is a coherent ring. □

**Corollary 4.15.** We preserve the notation of Proposition 2.1 and Remark 2.4, and assume that $b_{S_m} = \{0\}$, for each maximal ideal $m$ of $A$ containing $f^{-1}(b)$. If $A$ is a semi–hereditary ring (resp. semi–hereditary and Noetherian ring), $B_n$ is a valuation domain, for each $n \in \text{Max}(B) \setminus V(b)$ and $b$ is a coherent $A$–module (resp. finitely generated $A$–module), with the structure induced by $f$, then $A \ltimes f^\oplus b$ is a semi–hereditary ring.

**Proof.** Apply Theorem 4.12, Proposition 4.11(ii)⇒(i)) and Proposition 4.14, keeping in mind that, if $A$ is a Noetherian ring, an $A$–module is coherent if and only if it is finitely generated. □

Recall that an integral domain $A$ is almost Dedekind if $A_m$ is a DVR for each maximal ideal $m$ of $A$. Thus, in particular, an almost Dedekind domain is a Prüfer domain.

**Example 4.16.** Let $A$ be a non-Noetherian almost Dedekind domain having at least two distinct principal maximal ideals $m := (m), n := (n)$ (such a domain exists, see [17]), set $B := A/(m \cap n)$, let $f : A \to B$ be the canonical projection and set $b := m/(m \cap n)$. Trivially, $f^{-1}(b) = m$ and, since $f(n) \in S_m$, it follows that $b_{S_m} = \{0\}$. Let $\bar{n} := n/(m \cap n)$ be the unique maximal ideal of $B$ not containing $b$. Obviously, the localization $B_{\bar{n}}$ is isomorphic to the field $A/\bar{n}$. Moreover, the natural map $p : A \to b, a \mapsto f(am)$, is clearly $A$–linear, surjective and $\text{Ker}(p) = n$. This shows that $b$ is finitely presented as an $A$–module. Then, keeping
in mind that $A$ is a coherent ring, being it a Prüfer domain, and applying [7 Exercise 12 (a)(β)], it follows that $b$ is a coherent $A$–module. Then $A \blacktriangleright b$ is a semi–hereditary ring, by Corollary 4.15.

**Example 4.17.** Preserve the notation of Proposition 2.1. The fact that $A \blacktriangleright b$ is semi–hereditary does not imply, in general, that $b$ is coherent as an $A$–module and $b_{S_m} = \{0\}$, for each $m \in \text{Max}(A) \cap V(f^{-1}(b))$. For example, let $T$ be an indeterminate over $Q$, and let $A := Z$, $B := Q[T]$, $b := TQ[T]$, $f : A \longrightarrow B$ be the inclusion. Then $A \blacktriangleright b$ is isomorphic to the ring $Z + TQ[T]$, by [9 Example 2.5]. Moreover, by [28 Theorem 1.3], it follows easily that $A \blacktriangleright b$ is a Prüfer domain (i.e. a semi–hereditary domain). But, clearly, $b$ is not finitely generated as an $A$–module and $b_{S_m} \neq \{0\}$, for each $m \in \text{Max}(A)$.

**Corollary 4.18.** We preserve the notation of Proposition 2.1 and Remark 2.4. Assume that $b$ is a coherent $A$–module and that, for each $m \in \text{Max}(A) \cap V(f^{-1}(b))$, either $f_m$ is a surjective ring homomorphism or $f^{-1}(b)A_m \neq \{0\}$. Then, the following conditions are equivalent.

(i) $A \blacktriangleright b$ is a semi–hereditary ring.

(ii) $A$ is a semi–hereditary ring, $B_n$ is a valuation domain, for each $n \in \text{Max}(B) \setminus V(b)$ and $b_{S_m} = \{0\}$, for each $m \in \text{Max}(A) \cap V(f^{-1}(b))$.

**Proof.** (ii)$\implies$(i). It is the statement of Corollary 4.15.

(i)$\implies$(ii). By Proposition 4.14(1), $A$ is a coherent ring. Then, it suffices to apply Theorem 4.12 and Proposition 4.11 to complete the proof. \qed

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Dipartimento di Matematica – Università degli Studi Roma Tre, Largo San Leonardo Murialdo 1, 00146 Roma