Franz-Keldysh effect in strong-field QED

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We studied a QED analog of the Franz-Keldysh effect, and interplay between the non-perturbative (the Schwinger mechanism) and the perturbative particle production mechanism from the vacuum in the presence of a strong slow field superimposed by a weak field. We found that the Franz-Keldysh effect significantly affects the particle production: (i) the perturbative particle production occurs even below the threshold energy; (ii) the perturbative production becomes the most efficient just above the threshold energy; and (iii) an oscillating behavior appears in the production number above the threshold energy. We also found that these non-trivial changes are suppressed only weakly by powers of the critical field strength of QED, and thus it might be relevant even within the current experimental technologies.

I. INTRODUCTION

It was Dirac who first discovered a relativistic wave equation for electron, which is known as the Dirac equation today [1]. The Dirac equation admits infinitely negative energy states. This looks problematic because any state may fall into lower and lower energy states by emitting photons and thus there seem no stable states. This problem was resolved by Dirac himself by re-interpreting that negative energy states are all occupied in our physical vacuum (Dirac sea picture) [2]. Dirac’s interpretation suggests that our vacuum is not vacant space, but can be regarded as something like a semi-conductor with gap energy characterized by the electron mass scale. This implies that our vacuum exhibits non-trivial responses when exposed to external fields whose characteristic physical scale is larger than the gap energy, as semi-conductors do.

One of the most interesting responses is particle production from the vacuum in the presence of external electric fields. Roughly, there are two kinds of production mechanism, whose interplay is controlled by strength and frequency of the external field (or the Keldysh parameter) [3–6].

Namely, the first mechanism is the perturbative production mechanism, which occurs when the external field is weak but has high-frequency (i.e., energetic). This is an analog of the photo-absorption effect in semi-conductors. In this mechanism, the external field perturbatively kicks an electron filling the Dirac sea, and supplies energy. If the supplied energy (i.e., the frequency of the external field) is larger than the gap energy, the electron is kicked out to the positive energy band leaving a hole in the original negative energy state. Thus, a pair of an electron and a positron is produced. This mechanism is suppressed only weakly by powers of the coupling constant \( e \). Thus, it is not difficult to study the perturbative production mechanism with actual experiments (e.g. SLAC E144 experiment [7]).

The other mechanism is the non-perturbative production mechanism, which is the so-called Schwinger mechanism [8–10]. This can be understood as an analog of the electrical breakdown of semi-conductors (or the Landau-Zener transition [11–13]). This mechanism occurs when the external field is strong but has low-frequency. In the presence of strong electric field, the energy bands are tilted non-perturbatively, and a level crossing occurs. An electron filling the Dirac sea is now able to tunnel into the positive energy band, which results in spontaneous pair production of an electron and a positron. If the external field is slow enough, one may approximate the external field as a constant electric field. For this case, one can analytically derive a formula for the number of produced electrons (the Schwinger formula [10]) as

\[
\eta_{p,s}^{(\text{Sch})} = \frac{V}{(2\pi)^3} \exp \left[ -\frac{m_e^2 + p^2}{eE} \right].
\]

where \( p_L \) is transverse momentum with respect to the direction of the electric field. As apparent from Eq. (1), the Schwinger mechanism depends on the coupling constant \( e \) inversely in the exponential. This clearly shows the non-perturbative nature of the Schwinger mechanism.

Experimental verification of the Schwinger mechanism is very important and interesting because it opens up a novel way to unveil non-perturbative aspects of quantum electrodynamics (QED), which are one of the most unexplored areas of modern physics. Nevertheless, this has not been done yet. This is because the Schwinger mechanism is strongly suppressed by the exponential factor in Eq. (1). Thus, it requires extremely strong electric field \( eE_{cr} \equiv m_e^2 \sim \sqrt{10^{25}} \text{ W/cm}^2 \) to be manifest. Unfortunately, such a strong electric field is not available within our current experimental technologies. Indeed, the strongest laser that the current human beings at hand is HERCULES laser, which is as strong as \( eE \sim \sqrt{10^{24}} \text{ W/cm}^2 \) [15].

Upcoming intense laser facilities such as ELI [16] and HiPER [17] would reach \( eE \sim \sqrt{10^{22}} \text{ W/cm}^2 \), which is still weaker than the critical field strength \( E_{cr} \) by several orders of magnitude.

In recent years, there has been an increasing interest in corporative particle production mechanism between

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the perturbative and the non-perturbative mechanism. In particular, the dynamically assisted Schwinger mechanism \cite{18,22} is attracting much attention. The dynamically assisted Schwinger mechanism claims that the non-perturbative particle production (the Schwinger mechanism) by a strong slow electric field should be dramatically enhanced if one superimposes a weak fast (i.e., perturbative) electric field at the same time onto the vacuum. An intuitive explanation of this mechanism is the following: Firstly, a perturbative interaction kicks up an electron in the Dirac sea into the gap. Then, the electron inside of the gap is able to tunnel into the positive energy band easier because the tunneling length is reduced compared to the original length from the negative energy band. One of the striking results of the dynamically assisted Schwinger mechanism is the critical field strength $E_{cr}$ is reduced by several orders of magnitude by the perturbative kick. It is, thus, expected that this mechanism might be detectable even within the current experimental technologies. Although experimental verification of the dynamically assisted Schwinger mechanism is not equivalent to direct verification of the original Schwinger mechanism, it is still very interesting and important because it clearly involves non-perturbative aspects of QED.

Is there any other corporative particle production mechanism? If there is, it should serve as another powerful tool to investigate non-perturbative aspects of QED just like the dynamically assisted Schwinger mechanism does. In the area of semi-conductor physics, there is. This is the Franz-Keldysh effect \cite{23,24}. The Franz-Keldysh effect states that optical properties of bulk semi-conductors are significantly modified in the presence of a strong slow electric field. Namely, photo-absorption rate (i.e., the perturbative particle production rate) under a strong slow electric field becomes finite even below the gap energy, and exhibits an oscillating behavior above the gap energy. These non-trivial changes are caused by non-perturbative interactions between valence-band electrons and the strong electric field (which will be explained later in more detail in the language of QED). One of the most important features of the Franz-Keldysh effect is that its suppression is not so strong although it involves non-perturbative physics, which is usually strongly suppressed and is hard to study with experiments. Thanks to this advantage, the Franz-Keldysh effect in semi-conductors has been tested extensively by numerous experiments since 1960’s \cite{27,31}, and has many applications ranging from physics to industry (e.g. electro-absorption modulator, photo-detector, optical switching, etc).

Since semi-conductors are quite analogous to QED according to Dirac’s interpretation, it may be natural to ask if there is an analog of the Franz-Keldysh effect in QED. To the best of our knowledge, there exists no clear answer to this question. The purpose of this paper is to answer this question. Namely, we consider a situation such that a weak field is applied onto a strong slow field, and discuss how the perturbative particle production by

the weak field is modified by the strong field. We shall show that a QED analog of the Franz-Keldysh effect actually takes places, and non-trivial changes in the perturbative production number appear such as excess below the gap energy, and an oscillating behavior above the gap energy. We shall also see that the changes are suppressed only weakly by powers of the critical field strength, so that it might be relevant even within the current experimental technologies. In addition, we study interplay between the perturbative and the non-perturbative particle production mechanism as well to clarify the corporative nature of the Franz-Keldysh effect. Note that the weak field is not necessarily on-shell here. Thus, the weak field alone is able to produce particles from the vacuum by the perturbative mechanism. This situation is in contrast to stimulated pair production processes by an on-shell photon in the presence of strong fields (e.g. non-linear Breit-Wheeler process \cite{32}). An on-shell photon alone is not able to produce particles from the vacuum because of the energy conservation. Hence, such stimulated pair production processes cannot be regarded as a corporative particle production mechanism between the perturbative and the non-perturbative production mechanism, in which we are interested.

This paper is organized as follows: In Sec. II we explain a theoretical framework of this work. To be concrete, we consider a general situation, where a weak field is superimposed onto a strong slow field. We derive a general expression for the number of produced particles from the vacuum under those fields by developing perturbation theory, in which interactions due to the weak field are treated perturbatively but those due to the strong field are treated non-perturbatively. In Sec. III we consider a specific field configuration to quantitatively discuss the particle production mechanism in the presence of both strong and weak fields. Namely, we consider a constant homogeneous strong electric field and a monochromatic weak electric field. Based on our perturbation theory, we derive an analytical formula (without any approximations such as WKB approximation) for the number of produced particles for this particular field configuration. With this formula, we explicitly demonstrate how a QED analog of the Franz-Keldysh effect and the interplay between the non-perturbative and the perturbative particle production occur. Section IV is devoted to summary and discussion.

II. FORMALISM

In this section, we shall derive a formula for the number of particles produced from the vacuum in the presence of a strong slow field and a small perturbation on top of it. We first use the retarded-Green function technique to solve the Dirac equation perturbatively with respect to the perturbation, while we treat interactions due to the strong field non-perturbatively (Sec. II A). Then, we canonically quantize the field operator (Sec. II B), and
compute the in-vacuum expectation value of the number operator \( \langle \hat{N} \rangle \).

Note that we use the mostly minus metric \( g^{\mu\nu} = \text{diag}(+1,-1,-1,-1) \). Also, we work in the Heisenberg picture throughout this paper.

## A. Perturbative solution of the Dirac equation

We consider a situation such that an external gauge field \( A_\mu \) can be separated into two parts, i.e., a strong and slow field \( \bar{A}_\mu \) and a weak field \( A_\mu \), which is applied as a perturbation on top of the strong field \( \bar{A}_\mu \), as

\[
A_\mu = \bar{A}_\mu + A_\mu.
\]

(2)

We assume that the weak field \( A_\mu \) vanishes at the infinite past and future (adiabatic hypothesis) as

\[
A_\mu \xrightarrow{x^0 \to \pm \infty} 0.
\]

(3)

For simplicity, we adopt the temporal gauge fixing condition, i.e.,

\[
\bar{A}_\mu = (0, -\bar{A}), \quad A_\mu = (0, -A),
\]

(4)

where we introduced the three-vector potential \( \bar{A}, A \) as the spatial component of the corresponding gauge field.

Under the external field, the Dirac equation for a fermion field operator \( \hat{\psi} \) reads

\[
0 = [i\hat{\theta} - e\bar{A} - m] \hat{\psi} = [i\hat{\theta} - e\bar{A} - m] \hat{\psi} - e\hat{A}\hat{\psi},
\]

(5)

where \( e > 0 \) is the coupling constant and \( m \) is mass. Now, we shall solve the Dirac equation (5) perturbatively with respect to \( A_\mu \), while interactions between \( \bar{A}_\mu \) and \( \hat{\psi} \) are treated non-perturbatively. To this end, we introduce a retarded Green function \( S_R \) such that

\[
[i\hat{\theta}(x) - e\bar{A}(x) - m] S_R(x,y) = \delta^4(x-y),
\]

\[
S_R(x,y) = 0 \quad \text{for} \quad x^0 - y^0 < 0.
\]

(6)

Notice that \( S_R \) is fully dressed by the strong field \( \bar{A}_\mu \) (Furry picture [33]). With the Green function \( S_R \), one can write down a formal solution of the Dirac equation (5) perturbatively with

\[
\hat{\psi}(x) = \sqrt{Z}\hat{\psi}^{\text{in}}(x) + e \int d^4y S_R(x,y)\bar{A}(y)\hat{\psi}(y)
\]

\[
= \sqrt{Z} \left[ \hat{\psi}^{\text{in}}(x) + e \int d^4y S_R(x,y)\bar{A}(y)\hat{\psi}^{\text{in}}(y) + O(e^2) \right],
\]

(7)

where we used Eq. (6) and imposed a boundary condition for the field operator \( \hat{\psi} \) (Lehmann-Symanzik-Zimmermann (LSZ) asymptotic condition [34]) as

\[
0 = \lim_{x^0 \to -\infty} \left[ \hat{\psi} - \sqrt{Z}\hat{\psi}^{\text{in}} \right].
\]

(8)

Here, \( Z = 1 + \mathcal{O}(e) \) is a field renormalization constant and \( \hat{\psi}^{\text{in}} \) is a solution of the Dirac equation without the weak field \( A_\mu \) such that

\[
0 = [i\hat{\theta} - e\bar{A} - m] \hat{\psi}^{\text{in}}.
\]

(9)

## B. Annihilation/creation operators

One can define an annihilation/creation operator at in-state \( x^0 \to -\infty \) by canonically quantizing the asymptotic field operator \( \hat{\psi}^{\text{in}} \). To be more concrete, we first expand the asymptotic field operator \( \hat{\psi}^{\text{in}} \) in terms of a mode function \( \psi^{\text{in}} \) as

\[
\hat{\psi}^{\text{in}}(x) = \sum_s \int d^3p \left( \psi^{\text{in}}_{p,s} \hat{b}_{p,s}^\dagger + \psi^{\text{in}}_{p,s}^\dagger \hat{a}_{p,s} \right),
\]

(10)

with \( p \) and \( s \) being a label of canonical momentum and spin, respectively. Here, we normalize the mode function \( \psi^{\text{in}} \) by

\[
\int d^3x^\perp \psi^{\text{in}}_{p,s}(x)\psi^{\text{in}}_{p',s'}(x) = \delta^3(p-p')\delta_{ss'}
\]

\[
\int d^3x^\perp \psi^{\text{in}}_{p,s}(x)\psi^{\text{in}}_{p',s'}(x) = 0,
\]

(11)

and identify the positive/negative frequency mode if it approaches a plane wave with positive/negative frequency at \( x^0 \to -\infty \) as

\[
\lim_{x^0 \to -\infty} \psi^{\text{in}}_{p,s} = e^{\pm i\omega_p x^0} e^{i\mathbf{p} \cdot \mathbf{x}},
\]

(12)

where

\[
\omega_p = \sqrt{m^2 + \mathbf{p}^2}
\]

(13)

is on-shell energy, and \( \mathbf{P}^{\text{in}} \equiv \mathbf{p} - e\bar{A}(x^0 = -\infty) \) is kinetic momentum at \( x^0 \to -\infty \). Nextly, we impose the canonical commutation relation onto \( \hat{\psi}^{\text{in}} \). This is equivalent to quantize \( \hat{a}_{p,s}, \hat{b}_{p,s}^\dagger \) as

\[
\delta^3(p-p')\delta_{ss'} = \{ \hat{a}_{p,s}, \hat{b}_{p',s'}^\dagger \} = \{ \hat{b}_{p,s}, \hat{b}_{p',s'}^\dagger \},
\]

(14)

\[
\text{and} \quad \{ \hat{a}_{p,s}, \hat{a}_{p',s'}^\dagger \} = \{ \hat{b}_{p,s}, \hat{b}_{p',s'}^\dagger \} = 0.
\]

(15)

\[1\] Strictly speaking, the equality in Eq. (6) should be interpreted in a weak sense, i.e., the equality holds only after (products of) operators are sandwiched by states. This difference is not important in the following discussion since we are basically interested in the expectation value of number operator.
With this commutation relation, as usual, one can interpret $\hat{a}_{\mu, p, s}^\dagger (\hat{b}^\dagger_{\mu, p, s})$ as an annihilation (creation) operator of one particle (anti-particle) at in-state with quantum number $p$ and $s$.

In a similar manner, one can define an annihilation/creation operator at out-state $\hat{\psi}^\dagger$. Similar to $\hat{\psi}^\dagger$, we define an asymptotic field operator at out-state $\hat{\psi}^\dagger$ as a solution of the Dirac equation without the weak field $A_\mu$, 

$$0 = [i\partial - e\bar{A} - m] \hat{\psi}^\dagger,$$  

(15) 

with a boundary condition at $x^0 \to +\infty$ given by 

$$0 = \lim_{x^0 \to +\infty} \left[ i\hat{\psi} - \sqrt{Z} \hat{\psi}^\dagger \right].$$  

(16) 

We, then, expand the operator in terms of a mode function $\pm \hat{\psi}^\dagger_{p, s}$ as 

$$\hat{\psi}^\dagger (x) = \int d^3p \left[ +\hat{\psi}^\dagger_{p, s}(x)\hat{\psi}^\dagger_{p, s} + -\hat{\psi}^\dagger_{p, s}(x)\hat{\psi}^\dagger_{p, s} \right],$$  

(17) 

where we normalize the mode function $\pm \hat{\psi}^\dagger_{p, s}$ in the same manner as $\pm \hat{\psi}^\dagger_{p, s}$ (see Eq. (11)) as 

$$\int d^3x \pm \hat{\psi}^\dagger_{p, s} \mp \hat{\psi}^\dagger_{p', s'} = \delta^3 (p - p') \delta_{ss'},$$ 

$$\int d^3x \pm \hat{\psi}^\dagger_{p, s} \mp \hat{\psi}^\dagger_{p', s'} = 0.$$  

(18) 

The identification of positive/negative frequency mode is essentially the same as $\pm \hat{\psi}^\dagger_{p, s}$ (see Eq. (12)), but is now identified at $x^0 \to +\infty$ as 

$$\lim_{x^0 \to +\infty} \pm \hat{\psi}^\dagger_{p, s} \propto e^{\mp i\omega_{p, s} x^0} e^{\pm ip x},$$  

(19) 

where $P^0 \equiv p - e\bar{A}(x^0 = +\infty)$ is kinetic momentum at $x^0 \to +\infty$. Note that $\pm \hat{\psi}^\dagger_{p, s}$ is not necessarily identical to $\pm \hat{\psi}^\dagger_{p, s}$ in the presence of the strong external field. Since $\pm \hat{\psi}^\dagger_{p, s}$ obeys the same Dirac equation as $\pm \hat{\psi}^\dagger_{p, s}$ (Eqs. (9) and (15)), $\pm \hat{\psi}^\dagger_{p, s}$ can be written as a superposition of $\pm \hat{\psi}^\dagger_{p, s}$ and $\pm \hat{\psi}^\dagger_{p, s}$. In other words, the positive and negative frequency modes (i.e., particle and anti-particle modes) are mixed up with each other during the time-evolution due to the non-perturbative interactions between $A_\mu$ and $\psi$. This is one of the important differences from the standard perturbation theory without $A_\mu$, in which $\pm \hat{\psi}^\dagger_{p, s}$ and $\pm \hat{\psi}^\dagger_{p, s}$ are always identical. Finally, we impose the canonical commutation relation onto $\hat{\psi}^\dagger$, which quantizes $\hat{\psi}^\dagger$, $\hat{\psi}^\dagger$ as 

$$\delta^3 (p - p') \delta_{ss'} = \{ \hat{a}^\dagger_{p, s}, \hat{a}_{p', s'} \} = \{ \hat{b}^\dagger_{p, s}, \hat{b}_{p', s'} \},$$ 

(20) 

Thereby, we define an annihilation/creation operator at out-state.

The annihilation/creation operators at the different asymptotic times, $\hat{a}^\dagger_{\mu}$ and $\hat{\psi}^\dagger$, are not independent with each other: If the external fields are merely pure gauge fields $A_\mu \to \text{const.}$ (i.e., no electromagnetic fields), they are identical. This is a trivial situation and it is apparent that no particles are produced for this case. In contrast, for non-vanishing electromagnetic fields $A_\mu \neq \text{const.}$, or $A_\mu \neq \text{const.}$, they are no more identical but related with each other by a unitary transformation. We shall see below that this mismatch between the in- and out-state annihilation/creation operators results in particle production. Note that not only the non-perturbative interactions due to $A_\mu$, which result in the mixing of the mode functions $\pm \hat{\psi}^\dagger_{p, s}$, but also the perturbative interactions due to $A_\mu$ contribute to this mismatch of the annihilation/creation operators.

C. Particle production

The momentum distribution of produced particles (anti-particles) $n_{p, s}$ ($\bar{n}_{p, s}$) can be computed as an in-vacuum expectation value of the number operator at out-state,

$$n_{p, s} \equiv \langle \text{vac}; \text{in} | \hat{\psi}^\dagger_{p, s} \hat{\psi}_{p, s} | \text{vac}; \text{in} \rangle,$$

$$\bar{n}_{p, s} \equiv \langle \text{vac}; \text{in} | \hat{\psi}^\dagger_{p, s} \hat{\psi}_{p, s} | \text{vac}; \text{in} \rangle,$$  

(21) 

where $| \text{vac}; \text{in} \rangle$ is the in-vacuum state, which is a state such that it is annihilated by the annihilation operators at in-state as

$$0 = \hat{a}^\dagger_{\mu} | \text{vac}; \text{in} \rangle = \hat{b}^\dagger_{\mu} | \text{vac}; \text{in} \rangle.$$  

(22) 

We evaluate Eq. (21) in the lowest non-trivial order of $A_\mu$. To do this, we first rewrite $\hat{a}^\dagger_{\mu}$, $\hat{b}^\dagger_{\mu}$ in terms of $\hat{a}^\dagger_{\mu}$, $\hat{b}^\dagger_{\mu}$. By using Eq. (18), one can re-express $\hat{a}^\dagger_{\mu}$, $\hat{b}^\dagger_{\mu}$ as

$$\langle \hat{a}^\dagger_{p, s}, \hat{b}^\dagger_{p', s'} \rangle = \int d^3x \left[ + \hat{\psi}^\dagger_{p, s} \hat{\psi}_{p, s} - \hat{\psi}^\dagger_{p', s'} \hat{\psi}_{p', s'} \right].$$  

(23) 

Then, we use the boundary condition (16) and Eq. (7) to find
\[
\left( \frac{\delta n_{\mu,s}}{\delta A_{\mu,s}} \right) = \lim_{x^0 \to +\infty} \frac{1}{\sqrt{Z}} \int d^3x \left( \frac{\psi_{\mu,s}^{\text{out}}(x)}{-\psi_{\mu,s}^{\text{out}}(x)} \right) \hat{\psi}(x) = \lim_{x^0 \to +\infty} \int d^3x \left( \frac{\psi_{\mu,s}^{\text{out}}(x)}{-\psi_{\mu,s}^{\text{out}}(x)} \right) \left[ \psi_{\mu,s}^{\text{in}}(x) + e \int d^4y S_R(x,y) \mathcal{A}(y) \psi_{\mu,s}^{\text{in}}(y) + O(e^2) \right].
\]

By noting that the Green function \( S_R \) can be expressed in terms of the mode function \( \pm \psi_{\mu,s}^{\text{out}} \) as
\[
S_R(x,y) = -i\theta(x^0 - y^0) \sum_s \int d^3p \left[ +\psi_{\mu,s}^{\text{out}}(x) + \phi_{\mu,s}^{\text{out}}(y) + -\psi_{\mu,s}^{\text{out}}(x) - \phi_{\mu,s}^{\text{out}}(y) \right],
\]
one can evaluate Eq. \((24)\) in the first order of \( \mathcal{A}_\mu \) as
\[
\delta^{\text{out}(0)} = \delta^{\text{out}(0)} + \tilde{\delta}^{\text{out}(1)}, \quad \delta^{\text{out}(1)} = \delta^{\text{out}(1)} + \tilde{\delta}^{\text{out}(1)} + \tilde{\delta}^{\text{out}(1)} + \tilde{\delta}^{\text{out}(1)} + \tilde{\delta}^{\text{out}(1)} + \tilde{\delta}^{\text{out}(1)} + \tilde{\delta}^{\text{out}(1)} + \tilde{\delta}^{\text{out}(1)},
\]
where
\[
\delta^{\text{out}(0)} = \sum_{s'} \int d^3p' \left[ \frac{\int d^4x \psi_{\mu,s}^{\text{out}+}(x) \psi_{\mu,s'}^{\text{in}+}(x)}{\int d^4x \psi_{\mu,s'}^{\text{out}+}(x) \psi_{\mu,s}^{\text{in}+}(x)} \right] \delta_{\mu,s'}^{\text{out}(0)}, \quad \delta^{\text{out}(1)} = \sum_{s'} \int d^3p' \left[ \frac{-ie \int d^4x \psi_{\mu,s}^{\text{out}+}(x) \mathcal{A}(x) \psi_{\mu,s'}^{\text{in}+}(x)}{\int d^4x \psi_{\mu,s'}^{\text{out}+}(x) \psi_{\mu,s}^{\text{in}+}(x)} \right] \delta_{\mu,s'}^{\text{out}(1)},
\]
and
\[
\delta^{\text{out}(0)} = \sum_{s'} \int d^3p' \left[ \frac{\int d^4x \psi_{\mu,s}^{\text{out}+}(x) \psi_{\mu,s'}^{\text{in}+}(x)}{\int d^4x \psi_{\mu,s'}^{\text{out}+}(x) \psi_{\mu,s}^{\text{in}+}(x)} \right] \delta_{\mu,s'}^{\text{out}(0)}, \quad \delta^{\text{out}(1)} = \sum_{s'} \int d^3p' \left[ \frac{-ie \int d^4x \psi_{\mu,s}^{\text{out}+}(x) \mathcal{A}(x) \psi_{\mu,s'}^{\text{in}+}(x)}{\int d^4x \psi_{\mu,s'}^{\text{out}+}(x) \psi_{\mu,s}^{\text{in}+}(x)} \right] \delta_{\mu,s'}^{\text{out}(1)}.\]

An important point here is that, once interactions are switched on, the annihilation operators at out-state \( \tilde{\phi}^{\text{out}}, \tilde{\phi}^{\text{out}} \) differ from those at in-state \( \tilde{\phi}^{\text{in}}, \tilde{\phi}^{\text{in}} \) and always contain creation operators at in-state \( \tilde{\phi}^{\text{in}}, \tilde{\phi}^{\text{in}} \). Hence, the in-vacuum state is no more annihilated by the annihilation operators at out-state \( x^0 \neq \tilde{\phi}^{\text{out}}, \tilde{\phi}^{\text{out}} | \text{vac} \rangle \). This implies that the particle number \( n \propto |\tilde{\phi}^{\text{out}} | \text{vac} \rangle |^2, \tilde{n} \propto |\tilde{\phi}^{\text{out}} | \text{vac} \rangle |^2 \) become non-vanishing, i.e., particles are produced from the vacuum.

Now, we can explicitly write down a formula for the particle number. By substituting Eq. \((26)\) into Eq. \((21)\), we obtain
\[
n_{\mu,s} = \sum_{s'} \int d^3p' \left| \int d^4x \psi_{\mu,s}^{\text{out}+}(x) \psi_{\mu,s'}^{\text{in}+}(x) - e \int d^4x \psi_{\mu,s}^{\text{out}+}(x) \mathcal{A}(x) \psi_{\mu,s'}^{\text{in}+}(x) \right|^2,
\]
\[
\tilde{n}_{\mu,s} = \sum_{s'} \int d^3p' \left| \int d^4x \psi_{\mu,s}^{\text{out}+}(x) \psi_{\mu,s'}^{\text{in}+}(x) - e \int d^4x \psi_{\mu,s}^{\text{out}+}(x) \mathcal{A}(x) \psi_{\mu,s'}^{\text{in}+}(x) \right|^2.
\]

The first term in the bracket does not contain the weak field \( \mathcal{A}_\mu \) and it is completely determined by the non-perturbative interactions due to the strong field \( \mathcal{A}_\mu \). Thus, the first term is important for the non-perturbative particle production by the strong field (the Schwinger mechanism). On the other hand, the second term is important for the perturbative particle production by the weak field. This is because, for vanishing \( \mathcal{A}_\mu \), our formalism reduces to the standard perturbation theory without \( \mathcal{A}_\mu \), in which only the second term survives. However, it should be emphasized that our perturbation theory differs from the standard one because our fermion mode function \( \pm \psi_{\mu,s}^{\text{out}} \) is fully dressed by the strong field \( \mathcal{A}_\mu \). Thus, the second term depends on \( e \) and \( \mathcal{A}_\mu \) non-linearly. Note that our perturbation theory is valid no matter how slow or fast the weak field \( \mathcal{A}_\mu \) is as long as it is sufficiently slow or fast the weak field.
weaker than the strong one \( A_\mu \ll \bar{A}_\mu \).

### III. CONSTANT HOMOGENEOUS ELECTRIC FIELD + PERTURBATION

In this section, we consider a specific field configuration and discuss details of the particle production based on the perturbation theory developed in Sec. II. In Sec. III A, we first consider a case, in which the external fields \( A_\mu, \bar{A}_\mu \) are homogeneous in space, and the strong field \( \bar{A}_\mu \) is sufficiently slow so that it is well approximated by a constant electric field. For this case, one can analytically perform the integrations in Eq. (29) without any approximations to obtain a closed expression for the particle number. This enables us to better understand qualitative aspects of the particle production. In Sec. III B, we furthermore assume that the weak field is given by a monochromatic field, respectively. Note that \( \pm \psi_{p,s} \) are two eigenvectors of \( \gamma^0 \gamma^3 \) with eigenvalue one such that

\[
\gamma^0 \gamma^3 \Gamma_s = \Gamma_s, \quad \Gamma_s \Gamma_s' = \delta_{ss'}, (33)
\]

and the scalar functions \( A^{as}_p, B^{as}_p \) are

\[
\begin{align*}
A^{in}_p & = e^{-i \frac{p_0}{E}} e^{\frac{\pi m_\perp^2}{2 c E}} \frac{m_\perp}{2 c E} \left( \frac{i}{E} \right) - \frac{m_\perp}{2 c E} \left( \frac{i}{E} \right) \\
B^{in}_p & = -e^{-i \frac{p_0}{E}} e^{\frac{\pi m_\perp^2}{2 c E}} \left( \frac{i}{E} \right) + \frac{m_\perp}{2 c E} \left( \frac{i}{E} \right) \\
A^{out}_p & = -e^{i \frac{p_0}{E}} e^{\frac{\pi m_\perp^2}{2 c E}} \left( \frac{i}{E} \right) + \frac{m_\perp}{2 c E} \left( \frac{i}{E} \right) \\
B^{out}_p & = e^{i \frac{p_0}{E}} e^{\frac{\pi m_\perp^2}{2 c E}} \left( \frac{i}{E} \right) - \frac{m_\perp}{2 c E} \left( \frac{i}{E} \right)
\end{align*}
\]

(34)

where \( D_\nu(z) \) is the parabolic cylinder function\(^2\) and

\[
m_\perp \equiv \sqrt{m^2 + p_\perp^2}
\]

(35)

is transverse mass. \( p_\|, p_\perp \) are longitudinal and transverse momentum with respect to the direction of the electric field, respectively. Note that \( \pm \psi^{in}_{p,s} \) and \( \pm \psi^{out}_{p,s} \) are not linearly independent with each other, but are related with each other by

\[
\begin{pmatrix}
\pm \psi^{in}_{p,s} \\
\pm \psi^{out}_{p,s}
\end{pmatrix} = \begin{pmatrix}
\alpha_p & -\beta_p \\
\beta_p & \alpha_p
\end{pmatrix} \begin{pmatrix}
\pm \psi^{in}_{p,s} \\
\pm \psi^{out}_{p,s}
\end{pmatrix},
\]

(36)

where

\[
\begin{align*}
\alpha_p & = m_\perp \sqrt{2 \pi} \exp \left[ -\frac{\pi m_\perp^2}{2 c E} \right] \\
\beta_p & = \sqrt{2 \pi} \left( 1 - \frac{m_\perp^2}{2 c E} \right)
\end{align*}
\]

(37)

It should be stressed that \( |\alpha_p| \neq 1 \) and \( |\beta_p| \neq 0 \) if \( \vec{E} \neq 0 \). \( \alpha_p, \beta_p \) can be understood as an analog of the reflectance and the transmission coefficient in a barrier scattering.

\(^2\) In non-relativistic systems, the mode function in the presence of a constant electric field is expressed by the Airy function, not by the parabolic cylinder function. This is a slight difference between the Franz-Keldysh effect in QED and in semi-conductors.
problem in quantum mechanics, respectively. Thus, intuitively speaking, \(|\alpha_p| \neq 1\) \((|\beta_p| \neq 0)\) implies that particles in the Dirac sea are reflected by (tunneled into) the tilted gap in the presence of the strong electric field. This point plays an important role in the appearance of the Franz-Keldysh effect as we shall explain later.

With the use of Eqs. 32 and 34, one can evaluate the integrals in Eq. (29) analytically as

\[
\int d^3x \chi_{\bar{\psi}^\text{out}}\bar{\chi}\psi^\text{in}
= - \left( \int d^3x \chi_{\bar{\psi}^\text{out}}\bar{\chi}\psi^\text{in} \right)^* \\
= \delta_{ss'}\delta^3(p - p') \times \exp \left[ - \frac{\pi m^2}{2 eE} \right],
\]

and

\[
\begin{align*}
\int d^4x & \psi_{p,s}^\text{out} \bar{\psi}^\text{in}_{p',s'} \\
&= - \left( \int d^4x \psi_{p,s}^\text{out} \bar{\psi}^\text{in}_{p',s'} \right)^* \\
&= \delta_{ss'}\delta^3(p - p') \times \exp \left[ - \frac{\pi m^2}{2 eE} \right].
\end{align*}
\]

where the use is made of \(\delta^3(p = 0) = V/(2\pi)^3\) with \(V\) being the whole spatial volume. Equation (41) does not depend on \(s\) because electric fields do not distinguish spins. \(n_p = \bar{n}_{-p}\) holds because a particle and an antiparticle are always produced together as a pair from the vacuum, whose momentum and charge are zero. Note that we did not use any approximations (such as WKB approximation) in deriving Eq. (41).

In general, if there exists a “genuine” strong electric field which cannot be eliminated by any Lorentz transformations, \(|\alpha_p| \neq 1\) and \(|\beta_p| \neq 0\) hold. This implies that the Franz-Keldysh effect is a genuinely electrical effect. For example, strong plane waves, strong crossed fields, or strong magnetic field alone always gives \(|\alpha_p| = 1\) and \(|\beta_p| = 0\) no matter how strong it is, and hence the Franz-Keldysh effect never occurs.

1. Non-perturbative limit

The particle production becomes non-perturbative (i.e., the Schwinger mechanism occurs) if the weak electric field \(\bar{E}\) is so slow that \(\bar{E}\) is dominated by low-frequency modes \(\omega/\sqrt{eE} \ll 1\).

Indeed, by taking a limit of \(\omega/\sqrt{eE} \to 0\) in the integral of Eq. (41), we obtain

\[
n_{p,s} = n_{-p,s} \\
\sim \frac{V}{(2\pi)^3} \exp \left[ - \frac{\pi m^2}{2 eE} \right] + \frac{1}{2 eE} \int_0^\infty d\omega \frac{\bar{E}(\omega)}{E} \\
\sim \frac{V}{(2\pi)^3} \exp \left[ - \frac{\pi m^2}{2 eE} \right] + \frac{\pi m_1^2 \bar{E}(p/eE)}{2 eE}.
\]

In the last line, we used a mathematical trick \(\int_0^\infty d\omega e^{-i\omega t} \sim \pi \delta(t)\). In Eq. (42), the coupling constant \(e\) appears inversely in the exponential. This fact ensures that the particle production is actually non-perturbative.
for slow $\mathcal{E}$. Note that the distribution depends on $p_\parallel$ if $\mathcal{E}$ depends on time. Intuitively, the particle production occurs most efficiently at the instant when the longitudinal kinetic momentum $p_\parallel = p_\parallel + e\vec{E}x^0$ becomes zero, at which the energy cost to produce a particle is the smallest. Thus, the value of the weak field at $x^0 = -p_\parallel/e\mathcal{E}$ becomes important.

Equation (42) is consistent with the Schwinger formula for the non-perturbative particle production from a constant electric field. In fact, the Schwinger formula reads

$$n_{p,s}^{(Sch)} = \bar{n}_{p,s}^{(Sch)} = \frac{V}{(2\pi)^3} \exp \left[ -\frac{m^2}{e\mathcal{E}} \right] + \frac{\pi m^2}{2e\mathcal{E}} + O\left( \left( \frac{\mathcal{E}}{E} \right)^2 \right),$$

(43)

where $E = \hat{E} + \mathcal{E}$ is the total electric field strength. Thus, our formula (42) reproduces the Schwinger formula (43) up to $O((\mathcal{E}/E)^2)$ if one regards $\mathcal{E}(-p_\parallel/g\mathcal{E})$ as a constant. To reproduce $O((\mathcal{E}/E)^n)$-corrections ($n \geq 2$) correctly within our perturbation theory, one has to expand the annihilation operators (24) up to $n$-th order in $A_\mu$.

2. Perturbative limit

The perturbative particle production takes place if the weak electric field $\mathcal{E}$ is dominated by high-frequency modes. Indeed, by taking $\omega/\sqrt{e\mathcal{E}} \rightarrow \infty$ limit of the integrand in Eq. (41), we obtain

$$n_{p,s} = \bar{n}_{p,s} \sim \frac{V}{(2\pi)^3} \exp \left[ -\frac{m^2}{e\mathcal{E}} \right] + \frac{\pi m^2}{2e\mathcal{E}} \frac{e\mathcal{E}(2\omega_p)}{\omega_p^2}.$$  

(44)

Equation (44) is a superposition of the non-perturbative and the perturbative particle production. In fact, the second term does not contain the exponential factor, but just depends on $e$ linearly. Hence, it gives the perturbative particle production. The perturbative particle production does not depend on $E$, but solely depends on $\mathcal{E}$. The strong field $\hat{E}$ separately contributes to the scattering amplitude, and gives rise to the non-perturbative particle production. This is the first term in Eq. (44), which is independent of $\mathcal{E}$.

If $\hat{E}$ is smaller than the critical field strength $e\hat{E} \lesssim m_\perp$, the first term in Eq. (44) may be neglected because it is exponentially suppressed. Thus, Eq. (44) becomes purely perturbative as

$$n_{p,s} = \bar{n}_{p,s} \sim \frac{V}{(2\pi)^3} \frac{m^2}{4} \frac{e\mathcal{E}(2\omega_p)}{\omega_p^2}.$$  

(45)

Note that Eq. (45) reproduces the textbook formula for the perturbative particle production from a classical electric field [3, 37].

On the other hand, if $\mathcal{E}$ is super-critical $e\mathcal{E} \gtrsim m^2$, the first term in Eq. (45) becomes $O(1)$, which is superior to the second term $O(e\mathcal{E}/\omega_p^2)$. Then, Eq. (45) gives

$$n_{p,s} = \bar{n}_{p,s} \sim \frac{V}{(2\pi)^3} \exp \left[ -\frac{\pi m^2}{e\mathcal{E}} \right].$$  

(46)

This implies that the perturbative particle production by $\mathcal{E}$ is buried in the non-perturbative one by $\hat{E}$, and the particle production always looks non-perturbative no matter how slow or fast the weak field is. In other words, the interplay between the perturbative and the non-perturbative particle production becomes less manifest if $\mathcal{E}$ is super-critical.

B. Monochromatic wave as a perturbation

In this section, we consider an explicit example, in which the weak field is given by a monochromatic wave

$$\mathcal{E}(x) = \mathcal{E}_0 \cos \Omega x^0.$$  

(47)

With this configuration, we compute the momentum distribution $n_{p,s}$ and the total particle number $N \equiv \sum_s \int d^4p n_{p,s}$ to explicitly demonstrate how the interplay between the non-perturbative and the perturbative particle production occurs with changing the frequency $\Omega$. We also demonstrate that a QED analog of the Franz-Keldysh effect occurs. The Franz-Keldysh effect significantly lowers the threshold frequency for the perturbative particle production, and results in a characteristic oscillating pattern in $\Omega$-dependence.

1. Momentum distribution

By noting that the Fourier component $\tilde{\mathcal{E}}$ is sharply peaked at $\omega = \pm |\Omega|$ as

$$\tilde{\mathcal{E}}(\omega) = \pi \mathcal{E}_0 \left[ \delta(\omega - |\Omega|) + \delta(\omega + |\Omega|) \right],$$  

(48)

the formula for the number distribution (41) can be simplified as

$$n_{p,s} = \bar{n}_{p,s} = \frac{V}{(2\pi)^3} \exp \left[ -\frac{\pi m^2}{e\mathcal{E}} \right] \left[ 1 + \frac{\pi m^2}{2e\mathcal{E}} \frac{\mathcal{E}_0}{\mathcal{E}} \right]$$

$$\times \exp \left[ -\frac{i}{4} \frac{\Omega^2 + 4|\Omega|^2}{e\mathcal{E}} p_\perp \right] \times \left[ 1 - \frac{i}{2} \frac{m^2}{e\mathcal{E}} \right]^2.$$  

(49)

The number distribution (49) is plotted in Figs. 1 and 2. We also compared Eq. (49) with various other evaluations, i.e., the Schwinger non-perturbative formula (13).
the perturbative formula (45); and an exact result which is obtained by numerically solving the original Dirac equation (6).

For sub-critical field strength $e\tilde{E} \lesssim m_\perp^2$, the interplay between the non-perturbative and the perturbative particle production takes place (see the top panel of Fig. 1 and Fig. 2) as we already discussed analytically in Sec. 3.1.1. In the high-frequency region $\Omega/\sqrt{e\tilde{E}} \gg 1$, the particle production becomes the most efficient at $\Omega \sim 2\omega_p$. This implies that the production is dominated by the perturbative process. Indeed, the perturbative formula (45) for the monochromatic wave (47) is sharply peaked

$$n_{p,s}^{(pert)} = \frac{VT}{(2\pi)^3} \frac{e\tilde{E}_0^2}{8\omega_p^2 \omega_{p}^2} \delta(|\Omega| - 2\omega_p),$$

(50)

where we used $\delta(\omega = 0) = T/2\pi$ with $T$ being the whole time interval. Physically speaking, the location of the peak $\Omega = 2\omega_p$ in Eq. (50) can be understood as the threshold energy for one photon to create a pair of particles from the vacuum. On the other hand, in the low-frequency region $\Omega/\sqrt{e\tilde{E}} \ll 1$, the particle production becomes non-perturbative, and is consistent with the Schwinger formula for the total electric field (43).

Notice that the non-perturbative particle production is strongly suppressed by an exponential of $|e\tilde{E}|^{-1}$, but the perturbative one is only suppressed by powers of $e\tilde{E}$. Thus, the perturbative particle production is more abundant than the non-perturbative one for sub-critical field strength $e\tilde{E} \lesssim m_\perp^2$.

The structure of the perturbative peak at $\Omega \sim 2\omega_p$ (see the top panel of Fig. 1 and Fig. 2) is strongly modified by the strong field $\tilde{E}$. This is nothing but the Franz-Keldysh effect in QED. Indeed, in contrast to the naive perturbative formula (50), the perturbative peak in the figures is not a simple delta function strictly localized at the threshold energy, but it has non-trivial structure: (i) there is a tail below the threshold $\Omega \lesssim 2\omega_p$; (ii) the largest peak is located slightly above the threshold $\Omega \gtrsim 2\omega_p$; and (iii) above the threshold $\Omega > 2\omega_p$, the peak does not decrease monotonically but oscillates.

Here is an intuitive explanation why the Franz-Keldysh effect occurs (see also Fig. 3). The perturbative particle production from the vacuum occurs when a particle which is filling one of the negative energy states (the Dirac sea) is excited into one of the positive en-

FIG. 2. (color online) The number distribution $n_{p,s}$ as a function of the frequency $\Omega$ and the transverse mass $m_\perp$. $p_\parallel$ and $E_0$ are fixed as $p_\parallel/\sqrt{e\tilde{E}} = 0$ and $E_0 = \tilde{E}/100$. The blue line at $\Omega/\sqrt{e\tilde{E}} = 20$ shows the Schwinger formula (see Eq. (43)) for the non-perturbative field alone $\exp[-\pi m_\perp^2/e\tilde{E}]$. The red line at the bottom shows $\Omega = 2\omega_p$, at which the perturbative particle production is peaked (see Eq. (50)).
energy states. As the energy bands are non-perturbatively tilted by the strong electric field \( \bar{E} \), the probability density of a particle in the Dirac sea can tunnel into the gap. Once the particle tunneled into the gap, the threshold energy to excite the particle into a positive state is reduced. Thus, the perturbative particle production can occur even below the na"ive threshold (i). However, this does not necessarily imply that the particle production occurs most efficiently below the threshold. On the contrary, it should be slightly above the threshold in the presence of \( \bar{E} \). This is because only a part of the probability density can tunnel into the gap but the major part of it is reflected by the gap. Because of this reflection, the probability density takes its maximum slightly away from the gap, at which more energy is needed to excite the particle. Hence, the perturbative particle production becomes the most efficient slightly above the na"ive threshold energy (ii). Another important consequence of the reflection is that it mixes up the in-coming and out-going wave. Therefore, the probability density outside of the gap is no more uniform but oscillates in space. This results in the oscillating pattern in the distribution (iii) because the excitation energy at each local maximum of the probability density is different and larger excitation energy is needed for deeper local maxima. Note that the physical origin of (i) is quite similar to the dynamically assisted Schwinger mechanism. The quantum tunneling is assisted by the perturbative excitation in the dynamically assisted Schwinger mechanism, while the perturbative excitation is assisted by the quantum tunneling in the Franz-Keldysh effect.

For super-critical field strength \( e\bar{E} \gtrsim m^2 \) (see the bottom panel of Fig. 1 and Fig. 2), the non-perturbative production becomes so abundant no matter how slow of fast the frequency \( \Omega \) is. Thus, the perturbative produc-

tion is always buried in the non-perturbative one, and the interplay or the Franz-Keldysh effect is not manifest at first sight. This, however, does not imply that there is no interplay nor the Franz-Keldysh effect. Indeed, the distribution shows an oscillating behavior for large \( \Omega \), which is a reminiscent of the Franz-Keldysh oscillation (iii). Also, the production becomes smaller for large \( \Omega \), which is because the interplay takes place. For large \( \Omega \), the weak field \( \mathcal{E} \) and the strong field \( \bar{E} \) separately contribute to the scattering amplitude of the production process. As the weak field \( \mathcal{E} \) with large \( \Omega \) only gives a perturbative contribution, which is negligible to the non-perturbative one from \( \bar{E} \), the distribution is described well by the Schwinger formula for the strong field \( \bar{E} \) alone. On the other hand, for small \( \Omega \), not only \( \bar{E} \) but also \( \mathcal{E} \) contributes to the production process in a non-perturbative manner. Thus, the distribution is described by the Schwinger formula for the total field \( E = \bar{E} + \mathcal{E} \), which gives larger (if \( \mathcal{E}_0 > 0 \)) production compared to that for \( \bar{E} \) alone.

2. total number

The total number of produced particles \( N \) can be computed by integrating the spin \( s \) and the momentum \( p \) of the distribution \( n_{p,s} \) (49) as

\[
N = \sum_s \int d^3p \ n_{p,s}
= (e\bar{E})^2VT \times \frac{1}{e\bar{E}} \frac{1}{2\pi^2} \int dm_\perp m_\perp \exp \left[ -\frac{m_\perp^2}{e\bar{E}} \right]
\times \left[ 1 + \frac{\pi m^2}{2 e\bar{E}} i \frac{1}{F_1} \left( 1 - \frac{i}{2} \frac{m^2}{e\bar{E}} \right) \left( \frac{\mathcal{E}_0}{E} \right)^2 \right].
\]

Here, we neglected a term \( \propto \delta(\Omega) \) by assuming \( \Omega \neq 0 \), and thus there is no linear term \( (\mathcal{E}_0/E)^1 \) in the square bracket. Also, we evaluated the \( p_\perp \)-integration as \( \int dp_\perp = e\bar{E}T \). This is because the momentum \( p_\parallel \) and the time \( x^0 \)-integration are related with each other in the presence of a constant electric field \( \mathcal{E}_0 \). In fact, as we explained below Eq. (42), the particle production usually occurs at \( x^0 = -p_\parallel/e\bar{E} \). Thus, \( e\bar{E}dx^0 = -dp_\parallel \) should hold, which yields \( \int dp_\parallel = e\bar{E} \int dx^0 = e\bar{E}T \).

The total number (51) is plotted in Figs. 3 and 4. For comparison, Schwinger’s non-perturbative formula for the strong field \( \bar{E} \) (the integration of Eq. (43))

\[
A^{(\text{Sch})} = \sum_s \int d^3p \ n_{p,s}^{(\text{Sch})}
= (e\bar{E})^2VT \times \frac{1}{4\pi^3} \exp \left[ -\frac{m^2}{e\bar{E}} \right] ;
\]

FIG. 3. (color online) A schematic picture of the band structure of QED in the presence of a strong constant electric field \( \bar{E} \). The blue curve represents the probability density \( \psi^\dagger \psi \) of a particle in the Dirac sea. The black dashed line shows the band gap energy \( 2\omega_p \) between the band edges. The red (green) dashed line shows the energy needed to excite a particle in the Dirac sea which is tunneled into (reflected by) the tilted gap into a positive energy state.

the perturbative formula (the integration of Eq. (45))

\[ N^{(\text{pert})} = \sum_s \int d^3 p \, n_{p,s}^{(\text{pert})} \]

\[ = (e\bar{E})^2 VT \times \frac{1}{48\pi} \sqrt{1 - \frac{4m^2}{\Omega^2}} \]

\[ \times \left( 2 + \frac{4m^2}{\Omega^2} \right) \frac{\mathcal{E}_0}{\bar{E}}^2 \theta(\Omega - 2m); \quad (53) \]

and a superposition of them (the integration of Eq. (44))

\[ N^{(\text{Sch}+\text{pert})} = N^{(\text{Sch})} + N^{(\text{pert})}, \quad (54) \]

are also plotted in the figures. In Eq. (54), we dropped a cross term between Schwinger’s non-perturbative production and the perturbative one because it is negligible in the limit of \( T \to \infty \). Notice that the perturbative formula (53) has a cutoff at \( \Omega = 2m \), which is the minimum energy to create a pair of particles from the vacuum \( 2\omega_{p=0} = 2m \).

One can roughly understand the parameter dependence of the total number \( N \) (see Figs. 4 and 5) in terms of the interplay between the non-perturbative and perturbative particle production: For large frequency \( \Omega / \sqrt{e\bar{E}} \gg 1 \), the perturbative particle production by the weak field \( \bar{E} \) occurs. As the non-perturbative production by the strong field \( e\bar{E} \) is negligible for sub-critical field strength \( e\bar{E} \ll m^2 \), the perturbative process dominates the particle production and the total number is basically in agreement with the perturbative formula (53). Note, however, that there certainly exist small disagreements, which are nothing but the Franz-Keldysh effect and are discussed later in detail. On the other hand, the non-perturbative production for super-critical field strength \( e\bar{E} \gtrsim m^2 \) becomes so abundant that the perturbative production just gives a small correction to the non-perturbative one. Because of this small correction, the total number slightly deviates from the Schwinger formula for the strong field alone (52) and it is consistent with the sum of the Schwinger formula and the perturbative formula (54). For small frequency \( \Omega / \sqrt{e\bar{E}} \ll 1 \), the perturbative particle production does not take place and the production becomes purely non-perturbative. The total number becomes slightly larger than the Schwinger formula for the strong field alone (52) because not only the strong field but also the weak field contribute to the non-perturbative particle production. Indeed, Eq. (51) gives

\[ N \overset{[\Omega] \to 0}{\to} (e\bar{E})^2 VT \times \frac{1}{4\pi^3} \exp \left[ -\frac{\pi m^2}{e\bar{E}} \right] \]

\[ \times \left\{ 1 + \frac{1}{2} \pi \frac{m^2}{e\bar{E}} + \frac{\pi^2}{4} \left( \frac{m^2}{e\bar{E}} \right)^2 \right\} \left( \frac{\mathcal{E}_0}{\bar{E}} \right)^2, \quad (55) \]

which is actually larger than the Schwinger formula (52) by the factor of \( \mathcal{O}(e\bar{E})^2 \). Note that \( \mathcal{O}(e\bar{E})^2 \)-correction is absent in Eq. (55), although corrections to
the momentum distribution $n_{p,x}$ start from $O((\mathcal{E}/\bar{E})^4)$ (see Eq. (12)). This is because, for our monochromatic wave configuration, $O((\mathcal{E}/\bar{E})^4)$-correction in the total number becomes proportional to $\delta(\mathcal{E})$ after $p_{i,x}$-integration, and hence can be discarded for $\Omega \neq 0$. For general field configurations, $O((\mathcal{E}/\bar{E})^4)$-correction in the total number cannot be a delta function and has a finite value even for $\Omega \neq 0$, so that the correction should start from $O((\mathcal{E}/\bar{E})^4)$.

As pointed out, although our result is basically in agreement with the perturbative formula (53) or the sum with the Schwinger formula (54) in the high frequency regime $\Omega/\sqrt{\mathcal{E}}E \geq 1$, there certainly exist small disagreements between them. The disagreements are more clearly illustrated in Fig. 6 in which difference between our result and the sum of the Schwinger and the perturbative formula (54), $N - (N^{(\text{Sch})} + N^{(\text{pert})})$, near the threshold $\Omega \sim 2m$. $\mathcal{E}_0$ is fixed $\mathcal{E}_0 = \bar{E}/100$. Different colors distinguish the strength of the strong field $\bar{E}$.

As pointed out, although our result is basically in agreement with the perturbative formula (53) or the sum with the Schwinger formula (54) in the high frequency regime $\Omega/\sqrt{\mathcal{E}}E \geq 1$, there certainly exist small disagreements between them. The disagreements are more clearly illustrated in Fig. 6 in which difference between our result and the sum of the Schwinger and the perturbative formula (54), $N - (N^{(\text{Sch})} + N^{(\text{pert})})$, near the threshold $\Omega \sim 2m$ is plotted. The disagreements are nothing but the Franz-Keldysh effect. Namely, (i) the perturbative particle production occurs even below the threshold $\Omega \leq 2m$; (ii) the production number is slightly suppressed just above the threshold $\Omega \gtrsim 2m$; and (iii) the production number oscillates around the naive perturbative formula (53) above the threshold $\Omega > 2m$. The physical origin of this effect is completely the same as what we explained in Sec. III B i.e., the change of wave function due to the quantum tunneling and/or reflection by the tilted gap in the presence of strong electric field.

An important point of the Franz-Keldysh effect is that it is suppressed only weakly by powers of $m/\sqrt{\mathcal{E}}$ as can be seen from the figures. This is a big advantage from the experimental point of view. Indeed, typical QED non-perturbative processes such as the Schwinger mechanism are strongly suppressed by an exponential of $(m/\sqrt{\mathcal{E}})^{-1}$. This exponential suppression is so strong that it may be impossible to detect it with current experimental technologies whose typical physical scales are $\sqrt{\mathcal{E}}E \lesssim \sqrt{10^{-2} \times E_{\text{cr}}} = 10^{-1} \times m_e = O(10 \text{ keV})$, $V = O((1 \mu \text{m})^3)$, and $T = O(1 \text{ fsec})$. For example, $N^{(\text{Sch})}$ can be estimated as $N^{(\text{Sch})} \sim (4 \times 10^{-10}) \times T[\text{sec}]$, which implies that we need to wait extraordinary long time $\sim (1 \times 10^{38}) \text{ yr}$ to just detect one particle created by the Schwinger mechanism. However, the situation is much better for the Franz-Keldysh effect because of the weak power suppression. For example, Fig. 6 tells us that, for $E = 10^{-2} \times E_{\text{cr}}$ and $\mathcal{E}_0 = 10^{-2} \times \bar{E} = 10^{-4} \times E_{\text{cr}}$, the difference is the order of $N - (N^{(\text{pert})} + N^{(\text{Sch})}) \sim (1 \times 10^{4}) \times T[\sec]$.

IV. SUMMARY AND DISCUSSION

We studied an analog of the Franz-Keldysh effect in QED, and interplay between the non-perturbative (the Schwinger mechanism) and the perturbative particle production in the presence of a strong slow field and a weak perturbation on top of it.

In Sec. [11] we derived a general formula for the produced number of particles. Firstly, we used the retarded-Green function technique to solve the Dirac equation perturbatively with respect to the weak perturbation, while the interactions due to the strong field are treated non-perturbatively. We, then, employed the canonical quantization procedure in the presence of the strong field, and directly computed the in-vacuum expectation value of the number operator. The obtained formula (29) is written in terms of bi-linears between positive/negative frequency mode functions at in- and out-states, which are fully dressed by the strong field. This dressing enables us to study the Franz-Keldysh effect, which is a corporative effect between non-perturbative particle production mechanism due to the strong field and perturbative one due to the weak perturbation. Also, the formula (29) is valid no matter how fast or slow the weak perturbation is as long as the perturbation is sufficiently weaker than the strong field. Thus, the formula (29) is able to describe interplay between the perturbative particle production and the non-perturbative one (the Schwinger mechanism) with changing characteristic time-scale of the perturbation.

In Sec. [11] we considered a specific field configuration to discuss features of the particle production in more detail. To be concrete, we assumed that the strong field and the weak perturbation are given by a constant homogeneous electric field and a monochromatic wave, respectively. In this configuration, we analytically evaluated Eq. (29) without any approximations, and explicitly demonstrated how the interplay and the Franz-Keldysh effect occur. In particular, we found that the Franz-
Keldysh effect significantly affects the perturbative particle production mechanism: (i) the perturbative particle production occurs even below the threshold energy; (ii) the perturbative production becomes the most efficient just above the threshold energy; and (iii) a characteristic oscillating pattern appears in the production number above the threshold energy.

One of the most important features of the Franz-Keldysh effect is that it might be relevant even in the current experimental technologies. This is because the Franz-Keldysh effect is suppressed only weakly by powers of the critical field strength of QED (unlike the Schwinger mechanism, which is strongly suppressed exponentially). One of the possible experiments might be to measure the difference $\Delta N$ between the number of produced particles from the vacuum by a weak monochromatic wave with and without a strong electric field as suggested in Fig. 5. This is an analog of “modulation spectroscopy,” which is actually used in the area of condensed matter physics to detect the Franz-Keldysh effect in semi-conducting materials [38]. We expect that $\Delta N$ significantly deviates from zero near the threshold and exhibits characteristic patterns in the frequency dependence, e.g., an exponential tail below the threshold; a very sharp peak at the threshold; and an oscillation above the threshold.

The Franz-Keldysh effect may serve as a powerful tool to study non-perturbative aspects of QED. In particular, it is very useful to investigate the vacuum structure of QED. This is because the Franz-Keldysh effect occurs due to the change of wave function of particles filling the Dirac sea as we explained in Sec. III and the change is directly related to actual observables, i.e., frequency-dependence of the produced particle number. In fact, in the area of condensed matter physics, the Franz-Keldysh effect is experimentally used to precisely determine band structure of semi-conducting materials, and is very successful [38].

There are several possible future directions for this work. One direction is to consider more realistic field configurations, e.g., spatially inhomogeneous fields and/or perturbations; polarized perturbations; and inclusion of strong magnetic fields. Although we concentrated on the simplest situation in Sec. III (i.e., a constant homogeneous strong electric field and a weak monochromatic wave) for simplicity, our general formalism developed in Sec. I can be directly applicable to these more general situations. This is not only important for actual experiments, but also interesting from a phenomenological point of view. For example, recently it is argued in the context of the dynamically assisted Schwinger mechanism that spatial inhomogeneous perturbations dramatically change the production number [39, 40]. As the dynamically assisted Schwinger mechanism is also a corporative effect between strong fields and weak perturbations as the Franz-Keldysh effect is, we expect that similar changes should appear in the Franz-Keldysh effect as well. Inclusion of strong magnetic fields should also have significant impacts on the Franz-Keldysh effect because the Landau quantization comes into play. In fact, non-trivial changes are discussed in the area of condensed matter physics as a result of interplay between strong electric and magnetic fields [41, 44].

Another direction is to apply our perturbative canonical formulation to the dynamically assisted Schwinger mechanism [18–22]. Our formulation, or the resulting formula (41), may give us new insights on the dynamically assisted Schwinger mechanism because most of the existing studies rely on path-integral formulation such as the world-line instanton method [45–49]. One of the advantages of our perturbative formulation is that the production number can be expressed explicitly in terms of spectrum of weak perturbations (see Eq. (41)). This may enable us to better understand how spacetime-substructure of weak perturbations affect the dynamically assisted Schwinger mechanism. Indeed, a similar perturbative analysis was done recently in [50], which for the first time clarified the importance of high frequency modes of weak perturbations.

The last direction we would like to mention is applications to phenomenology. In particular, an application to heavy ion physics may be interesting. Just after a collision of ultra-relativistic heavy ions at RHIC and/or LHC, there appears very strong chromo-electromagnetic field (sometimes called “glamsa”), whose typical strength is $\mathcal{O}(1 \text{ GeV})$ [51, 52]. In addition to the glasma, there also exist jets, which are made up of high-energetic partons originating from initial hard collisions. Although the typical energy scale of jets are $\mathcal{O}(100 \text{ GeV})$, there are thousands of low-energetic jets (~a few GeV), which are called mini-jets. Thus, one may regard the system just after a collision as a superposition of strong field (glamsa) and weak perturbations on top of it ((mini)jets). This is essentially the same situation that we discussed in this paper. Thus, the Franz-Keldysh effect may take place. The Franz-Keldysh effect may change parton splitting functions, which might soften jet spectrum and help mini-jets to thermalize.

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