Floating of Extended States and Localization Transition in a Weak Magnetic Field

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We report results of a numerical study of non-interacting electrons moving in a random potential in two dimensions in the presence of a weak perpendicular magnetic field. We study the topological properties of the electronic eigenstates within a tight binding model. We find that in the weak magnetic field or strong randomness limit, extended states float up in energy. Further, the localization length is found to diverge at the insulator phase boundary with the same exponent \( \nu \) as that of the isolated lowest Landau band (high magnetic field limit).

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Recently there has been considerable interest in the fate of delocalized electronic states in a weak magnetic field in two dimensions (2D) \([1,2]\). In the limit of strong magnetic field, or equivalently weak randomness, it is believed that there exists a single critical energy within each Landau band where the localization length of electronic states diverges \([1,2]\). In contrast, one electron localization theory \([3]\) predicts that in the absence of magnetic field all states are localized in 2D. Consequently, it was argued \([4,5]\) that in the limit of weak magnetic field or strong randomness, where Landau bands merge together, these extended states do not disappear discontinuously but “float up”, tending to infinite energy in the \( B \to 0 \) limit \([1]\). Thus, for a given electron density (and hence finite Fermi energy \( E_F \)), for sufficiently low \( B \) all extended states are above \( E_F \) and the system becomes insulating. This scenario is crucial to the global phase diagram for the quantum Hall effect proposed by Kivelson et al. \([6]\) and has received strong experimental support \([7,8]\). Recently, however, based on numerical calculations of localization length on a tight binding model (TBM), Liu et al. \([9]\) concluded that extended states do not float and simply become localized as randomness increases. This issue is more clearly posed, and its resolution well described, by studying certain topological properties of the electronic eigenstates, as we shall see below.

A second issue of interest is the divergence of the localization length when approaching the insulator-quantum Hall phase transition. A previous numerical study \([10]\) performed on a random site TBM with a magnetic field suggested that the localization length exponent \( \nu \approx 0.8 \) in 2D at the localization transition point. Besides the fact that this value is much smaller than that at the transition between quantum Hall phases in the strong magnetic field limit \( \nu_H \approx 2.4 \), it violates the inequality \( \nu \geq 2/d \) \([11]\) which is widely believed to be satisfied in known random systems \([12]\). To address both these issues, a more clear-cut numerical method appears warranted.

In the presence of a magnetic field, electronic states exhibit interesting topological properties \([13–19]\). In particular, each state can be labeled by an integer called the Chern number, which is its boundary condition averaged Hall conductance, in units of \( e^2/h \) \([17,18]\). A state with nonzero Chern number carries Hall current and is necessarily extended. Thus by calculating the Chern numbers one is able to identify extended states unambiguously on finite size systems. This approach has proved very successful in addressing the localization problem in the lowest Landau band \([20]\). In this paper, we apply this approach to the TBM studied by Liu et al. and also by Ando \([21]\). Our results clearly support the “floating up” picture and are consistent with Thouless number calculations by Ando \([21]\). In fact, results of Liu et al. \([9]\) are also consistent with ours, but our interpretation of their results is somewhat different, as we discuss later.

We have also studied the dependence of the number and energies of extended states on system size. We find just like the case of individual Landau bands, the localization length diverges only at individual energies. In the high field (weak randomness) limit, the localization exponent is found to be the same as that of an isolated lowest Landau band, \( \nu_H \approx 2.4 \) \([6]\). For strong enough randomness we find that the localization length remains finite throughout the band and the number of states with nonzero Chern number goes to zero as the system size goes to infinity \([22]\). Using finite size scaling, we find the largest localization length of the system diverges as the critical randomness is reached with an exponent \( \nu_1 \) which is the same as \( \nu_H \), contrary to previous suggestion \([13]\) that the strong randomness exponent may be different from that in the lowest Landau levels. Thus our data show that \( \nu \) is a universal exponent for all spin polarized integer quantum Hall transitions, including the ultimate one to the insulating state.

We study the TBM on a square lattice with nearest neighbor hopping, a uniform magnetic field and random potential, described by the Hamiltonian:

\[
H = \sum_{mn} \{ -t(c_{m+1,n}^\dagger c_{m,n} + c_{m,n+1}^\dagger e^{i2\pi m} c_{m,n} + h.c.) \\
+ \epsilon_{m,n} c_{m,n}^\dagger c_{m,n} \},
\]

(1)

where the integers \( m \) and \( n \) are the \( x \) and \( y \) coordinates of the lattice site in terms of lattice constant, \( c_{m,n} \) is the fermion operator on that site, \( t \) is the hopping matrix ele-
ment which we set as the unit of energy from now on, and $\epsilon$ is the random potential ranging uniformly from $-W$ to $W$ (as in the Anderson model [23]). $\alpha$ is the amount of magnetic flux per plaquette in units of the flux quantum $\hbar e/\epsilon$. The Landau gauge $A = (0, Bx, 0)$ is used in Eq. (4). Here we concentrate on the case $\alpha = 1/N_f$, where $N_f$ is an integer. In this case, we have $N_f$ Landau subbands in the absence of random potential, and the lowest energy subbands map onto the lowest Landau levels in the limit $N_f \to \infty$, which is the continuum limit.

The Hall conductance of an individual eigenstate $|m\rangle$ can be obtained easily using the Kubo formula: \[ \sigma_{xy}^m = \frac{i e^2}{A} \sum_{n \neq m} \frac{\langle m|v_y|n\rangle\langle n|v_x|m\rangle - \langle m|v_x|n\rangle\langle n|v_y|m\rangle}{(E_n - E_m)^2}, \]

where $A$ is the area of the system, $v_x$ and $v_y$ are the velocity operators in the $x$ and $y$ directions respectively. For a finite system with the geometry of a parallelogram with periodic boundary conditions (torus geometry), $\sigma_{xy}^m$ depends on the two boundary condition phases $\phi_1$ and $\phi_2$. As shown by Niu et al., the boundary condition averaged Hall conductance takes the form \[ \langle \sigma_{xy}^m \rangle = \frac{1}{4\pi} \int d\phi_1 d\phi_2 \sigma_{xy}^m(\phi_1, \phi_2) = C(m)e^2/h, \] (2)

where $C(m)$ is an integer called the Chern number of the state $|m\rangle$. States with nonzero Chern numbers carry Hall current and are necessarily extended states [15].

Thus by numerically diagonalizing the Hamiltonian on a grid of $\phi_1$ and $\phi_2$, and calculating the Chern numbers of states of finite size systems by converting the integral in (2) to a sum over grid points, we are able to identify extended states unambiguously.

We have studied systems of square geometry with various size (from $3 \times 3$ to $15 \times 15$), strength of randomness ($W$) and magnetic field (equivalently, $N_f$). The number of samples explored for a given $W$ range from 2000 to 30 depending on system size. Most of our data were taken for $N_f = 3$. We do not, however, see any qualitative difference in behavior of the extended states, for systems with $N_f$ as large as 13. Hence we believe our results with $N_f = 3$ are generic and apply to the continuum limit $N_f \to \infty$.

Fig. 1 shows the density of states $[\rho(E)]$ and density of extended states with nonzero Chern numbers $[\rho_c(E)]$, for two different strength of randomness for 1/3 flux quantum per plaquette ($N_f = 3$) on a square of lattice size $9 \times 9$. For weak enough randomness ($W = 1.0$), the three Landau subbands are broadened by randomness, but are still well separated. We see there are extended states in all subbands, with their densities peaked essentially at the center of each subband. This is consistent with the previous study on individual Landau bands [24]. As randomness increases, the subbands further broaden and start to merge, as is seen for $W = 2.5$. In this case there are still three prominent peaks in $\rho(E)$ (we call them $E_1$, $E_2$, and $E_3$ respectively), which are (loosely) identified as centers of Landau subbands. $\rho_c(E)$, however, now looks very different: most of the extended states are near the center of the entire band ($E_2$) and there is no peak in $\rho_c(E)$ at $E_1$ or $E_3$, which are the centers of Landau subbands. There are nontrivial features in $\rho_c(E)$ which we discuss below, but it is clear from Fig. 1 that as the three subbands start to merge, the extended states in the lower and upper subbands move away from the centers of the subbands ($E_1$ and $E_3$) toward center of the band ($E_2$). This behavior is also seen in systems of $N_f$ as large as 13. We hence believe in the limit $N_f \to \infty$ (which can be mapped onto the continuum model), the extended states in the lowest subbands (which becomes Landau levels) float up toward the center of the band (which is at infinitely high energy relative to them in the continuum model). This provides unambiguous support for the floating up picture predicted theoretically [1] and seen experimentally [6].

The fact that the extended states in the lower and upper subbands float toward the center of the band as randomness increases may be understood in the following manner. In finite size systems, the Chern number of a state can change only when it becomes degenerate with another state under certain boundary conditions. This can be shown to occur only by tuning three parameters, including the two boundary condition angles plus the parameter characterizing the random potential. If such a degeneracy were to occur, the Chern numbers of the two states involved may change but their sum is conserved. Randomness tends to localize all states and annihilate the nonzero Chern numbers carried by the extended states. Thus states with nonzero Chern numbers of opposite signs “attract” each other and tend to move close in energy as randomness increases. It is believed that in the thermodynamic limit the localization length diverges and true current carrying (extended) states exist only at individual critical energies. For exactly the same reason, critical energies with total Chern numbers of opposite sign also “attract” each other as randomness increases. In the case of $N_f = 3$ systems, the total Chern numbers for the three subbands are 1, -2 and 1 respectively. Due to the “attraction”, we expect that as randomness is turned on, the extended states in the central subband with total Chern number $-2$ splits into two critical energies with total Chern number $-1$ each (by symmetry) and move toward the two band edges as randomness is increased further. Concurrently, the two critical energies of the upper and lower subbands with total Chern number +1 move away from the center of the subbands toward the center of the band. This is precisely what is seen in the
$\rho_c(E)$ at $W = 2.5$: There is a small dip at the center of the band indicating the splitting of the central critical energy; further, there are two less pronounced peaks from the two edge subbands, whose positions have clearly moved away from the corresponding peaks of $\rho(E)$.

Fig. 2 depicts the number of states with nonzero Chern number $N_c \equiv \int_{-\infty}^{\infty} \rho_c(E) dE$ versus the system size $N_s$ (number of sites), for different values of disorder $W$, for $N_f = 3$, on a double logarithmic plot. We find the plot is essentially linear for small $W$ up to $W \approx 3.0$, with slope $y = 0.79 \pm 0.01$ which is relatively independent of $W$ [24], indicating that $N_c \sim (N_s)^{\nu}$ in this region. This power law behavior is exactly what is expected [9,20] where there are individual critical energies $E_c^i$ in the vicinity of which the localization length diverges with a power law of the form $\xi(E) \sim |E - E_c|^{-\nu}$. In a finite system with linear size $L_s = \sqrt{N_s}$, states with $\xi(E) > L_s$ look extended. The number of such states goes like $N_c \sim N_s \rho(E_c)L_s^{-1/\nu} \sim N_s^{-1/2\nu}$, thus $y = 1 - 1/2\nu$. This gives $\nu = 2.4 \pm 0.1$, in agreement with the $\nu_H$ for lowest Landau band [24]. This suggests that $\nu$ is a universal exponent in all spin-polarized integer quantum Hall transitions.

For larger $W$, the dependence of $N_c$ on $N_s$ deviates from a power law and bends down as $N_s$ increases. This indicates that in this regime the two critical energies with total Chern number -1 have merged with the other two with Chern number +1: all extended states have disappeared and $\xi$ is finite throughout the band. For strong enough randomness and large $N_s$, $N_c$ decreases as $N_s$ increases; thus in the localized regime the average number of extended states per sample goes to zero in the thermodynamic limit. From the shape of the density of extended states and scaling of data we determine the critical randomness to be $W_c \approx 2.9 \pm 0.1$. For $W$ greater than but close to $W_c$, and large sizes $N_s$, $N_c$ is expected to take the scaling form: $N_c \sim N_s^y F(L_s/\xi_m) \sim N_s^y \tilde{F}(N_s^{1/(2\nu)}(W - W_c))$, where $\xi_m$ is the largest localization length in the system that diverges as $W_c$ is approached with exponent $\nu_f$. The best scaling is achieved with $\nu_f \approx 2.3$, assuming $W_c = 2.9$, and Fig. 3 shows the scaling function $F$. Taking into uncertainty in $W_c$ we estimate $\nu_f \approx 2.3 \pm 0.3$. This suggests that the localization length exponents are the same in both the localized and extended regimes, in contrast to a previous suggestion that they may be different [13]. The increasing negative slope of the scaling curve suggests that $N_c$ goes to zero faster than any power law as $N_s$ increases at large $N_s$.

We emphasize that the existence of this localized regime in the TBM is due to the facts that there exist critical energies with negative Chern numbers and the total Chern number of the system is zero. In the continuum however, the total Chern number of the critical energy of each Landau band is one, and there is no critical energy with negative Chern number at finite energy. Hence the extended states at these critical energies cannot annihilate their Chern numbers and become all localized as the randomness increases. They only float up and “disappear” at infinite energy. This becomes clear as one views the continuum system as the $N_f \to \infty$ limit of the TBM.

In the TBM, the natural energy scale is hopping $t$ (set to be 1 previously), and the zero point of energy is the center of the band. In the continuum however, the energy scale is Landau level spacing $h\omega_c$, and the zero point of energy is determined by identifying the center of the lowest energy band with energy $h\omega_c/2$. In terms of TBM parameters we have $h\omega_c = 4t/N_f$. Based on our data up to $N_f = 13$ we conclude that the critical randomness is almost $N_f$ independent and is about $W_c \approx 3t$, in agreement with Ando [21]. The energy at which the final merging and disappearance of critical energies measured from the bottom of the band is found to be of order $O(W_c)$, which is the only energy scale of the TBM at criticality. Therefore the number of Landau subbands below the lowest critical energy before it finally disappears is of order $O(W_c/h\omega_c) \propto N_f$. We hence conclude in the continuum limit ($h\omega_c$ finite, $N_f \to \infty$) the critical randomness strength ($W_c \propto N_f h\omega_c$) is infinite, and extended states all float up to infinite energy in the strong randomness (or weak magnetic field) limit.

Liu et al. [25] interpret the localization transition in the TBM as indication of disappearance of extended states in the continuum. As discussed above, the continuum limit of the TBM is subtle. They also find the energies of the extended states do not shift much relative to the center of the band ($E = 0$) and interpret it as evidence against floating. Our results for $N_f = 3$ is consistent with very little shift of $E_1$ and $E_3$ relative to $E_2$. However, critical energies clearly float away from peaks of $\rho(E)$ (which are roughly at the centers of Landau subbands), and more so relative to the bottom of the band. This is because as randomness increases, both the bottom of the band and peaks in $\rho(E)$ move downward. We believe this relative movement is clear indication of floating of extended states which survives the continuum limit.

In summary, we have found unambiguous numerical evidence, using a tight binding model (TBM) and considering the passage to the continuum limit, which indicates that extended states float up in energy toward infinity in the weak magnetic field limit in 2D in the continuum. For the TBM, this can be heuristically understood in terms of an “attraction” between states with opposite Chern numbers. There is a critical randomness strength in the TBM at which all states become localized. The localization length diverges with the same exponent as that of the isolated lowest Landau band (high field limit) when approaching this critical point.

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FIG. 1. Ensemble averaged density of states $\rho(E)$ and density of extended states $\rho_c(E)$ for two values of randomness $W$, for systems of size $9 \times 9$.

FIG. 2. Number of extended states $N_c$ versus system size $N_s$ for various $W$ on a double logarithmic scale. The solid line with slope $y = 0.79$ is a linear fit to the data for $W = 3.0$. 

FIG. 3. The scaling function $F(N_s^{1/(2\nu_i)}(W - W_c))$. 

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