A NOTE ON THE SQUEEZING FUNCTION

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Abstract. The squeezing problem on \( \mathbb{C} \) can be stated as follows. Suppose that \( \Omega \) is a multiply connected domain in the unit disk \( \mathbb{D} \) containing the origin \( z = 0 \). How far can the boundary of \( \Omega \) be pushed from the origin by an injective holomorphic function \( f : \Omega \to \mathbb{D} \) keeping the origin fixed?

In this note, we discuss recent results on this problem obtained by Ng, Tang and Tsai (Math. Anal. 2020) and by Gumenyuk and Roth (arXiv:2011.13734, 2020) and also prove few new results using a method suggested in one of our previous papers (Zapiski Nauchn. Sem. POMI 1998).

The squeezing problem. The squeezing function \( S_\Omega(z) \) of a planar domain \( \Omega \) is defined as follows. Suppose that \( \Omega \subset \mathbb{C} \) is such that there is an injective holomorphic function \( f(z) \) from \( \Omega \) to the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \). Let \( \mathcal{U}(\Omega) \) be the class of all such functions. Then \( S_\Omega : \Omega \to \mathbb{R} \) is defined by

\[
S_\Omega(z) = \sup\{ \text{dist}(0, \partial f(\Omega)) : f \in \mathcal{U}(\Omega), f(z) = 0 \}.
\]

According to [5], the squeezing function was first introduced in 2012 by Dong, Guan and Zhang [8] but the concept itself goes back to the work of Liu, Sun and Yau [4]. These authors defined \( S_\Omega \) and used it in a more general setting, namely, for classes of injective holomorphic mappings defined on domains \( \Omega \) in \( \mathbb{C}^n, n \geq 1 \).

The squeezing problem, i.e., the problem to find or characterize \( S_\Omega(z) \), is a difficult task, even in complex dimension 1. Any function \( f \in \mathcal{U}(\Omega) \) such that \( f(z) = 0 \) and \( S_\Omega(z) = \text{dist}(0, \partial f(\Omega)) \) will be called extremal for the squeezing problem (for the point \( z \in \Omega \)). In this case, the image \( f(\Omega) \) will be called an extremal domain.

For doubly connected domains \( \Omega \subset \mathbb{C} \), the squeezing function was identified by Ng, Tang and Tsai in 2020 [8]. To state the main result of [8], we first introduce necessary notations. As well known, see [4, Theorem 5.4], every domain \( \Omega \subset \mathbb{C} \) with a finite number of boundary continua \( \gamma_1, \ldots, \gamma_n, n \geq 2 \), can be mapped by \( w = \varphi_{a,k}(z), a \in \Omega, 1 \leq k \leq n \), conformally on \( \mathbb{D} \) slit along arcs \( C_{j,k} = C_{j,k}(a), j \neq k \), on the circles of radii \( 0 < r_{j,k} = r_{j,k}(a) < 1 \) centered at 0 so that \( \varphi_{a,k}(a) = 0 \), a non-degenerate boundary continuum \( \gamma_k \) corresponds to the unit circle \( \mathbb{T} = \partial \mathbb{D} \) and, for \( j \neq k \), \( \gamma_j \) corresponds to \( C_{j,k} \). Under the additional normalization \( \varphi_{a,k}'(a) > 0 \), that we assume in what follows, the mapping function \( \varphi_{a,k} \) is uniquely determined. If \( \gamma_k \) is a singleton, we put \( r_{j,k}(a) = 0 \) for all \( a \in \Omega \) and all \( j \neq k \). Without the uniqueness statement, the mapping property, discussed above, remains true for domains of any connectivity.

Theorem 1 ([8], [3]). Let \( \Omega \subset \mathbb{C} \) be a doubly connected domain with boundary components \( \gamma_1 \) and \( \gamma_2 \), at least one of which is non-degenerate. Then

\[
S_\Omega(z) = \max\{ r_{1,2}(z), r_{2,1}(z) \}.
\]

If \( r_{1,2}(z) < r_{2,1}(z) \), then \( \varphi_{z,1} \) is the unique (up to rotation about the origin) extremal function for the squeezing problem; if \( r_{1,2}(z) > r_{2,1}(z) \), then \( \varphi_{z,2} \) is the unique extremal function; and if \( r_{1,2}(z) = r_{2,1}(z) \), then both \( \varphi_{z,1} \) and \( \varphi_{z,2} \) are extremal.

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This theorem, except for the uniqueness part, was proved in [8] using the Loewner differential equation and tricky calculations with special functions. A simpler proof based on the potential theory, which also includes the proof of the uniqueness statement, was presented in [3].

In this note, we first show that Theorem 1 is immediate from Theorem 1.2 in our 1993 paper [10]. Then we discuss how the method used in [10] can be applied to study the squeezing problem in a more general setting.

Let $0 < r_1 < r_2 < 1$ and let $E$ be a compact subset in the closure $\overline{A}(r_1, r_2)$ of the annulus $A(r_1, r_2) = \{z : r_1 < |z| < r_2\}$. Let $D(E) = \{(D_1, D_2)\}$ be the set of pairs $(D_1, D_2)$ of non-overlapping domains in $\mathbb{D} \setminus E$, where $D_1$ is a simply connected domain containing the origin and $D_2$ is a doubly connected domain separating $E$ from $T$. In what follows, $m(D, z_0)$ stands for the reduced module of a simply connected domain $D$ with respect to the point $z_0 \in D$ and $m(D)$ stands for the module of a doubly connected domain $D$. For the definitions and properties of these moduli, we refer to Jenkins’s monograph [4] as the primary source and also to [3, 2, and 11]. Figure 1 illustrates our notations introduced above.

Consider the following module problem.

**Problem M.** Given nonnegative numbers $\alpha_1$ and $\alpha_2$, at least one of which is positive, identify all pairs $(D_1^*, D_2^*) \in D(E)$, which maximize the weighted sum of moduli

$$\alpha_1^2 m(D_1, 0) + \alpha_2^2 m(D_2)$$

over the set $D(E)$.

**Theorem 2** ([10, Theorem 1.2]). (1) There is a unique pair $(D_1^*, D_2^*) \in D(E)$ maximizing the sum (3) over the class $D(E)$. The domains $D_1^*$ and $D_2^*$ are, respectively, a circle domain and a ring domain of a quadratic differential $Q(z)\,dz^2$ defined on $\mathbb{D} \setminus E$, which is positive on $T$ and has a second order pole with circular local structure of trajectories at $z = 0$.

(2) Let $L(E)$ denote the free boundary of the module problem; i.e. $L(E) = (\partial D_1^* \cup \partial D_2^*) \cap (\mathbb{D} \setminus E)$. Then $L(E)$ consists of arcs of critical trajectories of $Q(z)\,dz^2$ and their endpoints in $\mathbb{D} \setminus E$.

(3) The following holds: (a) if $\alpha_1 \leq \alpha_2$, then $L(E) \subset \overline{D}_{r_2}$, where $D_r = \{z : |z| < r\}$; (b) if $\alpha_1 \geq \alpha_2$, then $L(E) \subset \overline{A}(r_1, 1)$; (c) if $\alpha_1 = \alpha_2$, then $L(E) \subset \overline{A}(r_1, r_2)$.

We note that Problem M is a particular case of Jenkins’s problem on extremal partitioning of Riemann surfaces, see [3, 6, 11]. Therefore, parts (1) and (2) of this theorem follow from Jenkins’s theorem on extremal partitioning; see [5, Theorem 1]. Part (3), which is essential for this paper, was proved in [10]. For convenience of the readers and because this result seems useful in the study of the squeezing function [1], a version of this proof will be presented at the end of this note.

**Proof of Theorem 2**. Using an auxiliary conformal mapping, we may assume without loss of generality that $\Omega$ is the unit disk $\mathbb{D}$ slit along a proper arc $C_{1,2}$ of the circle $\{z : |z| = r_{1,2}\}$, that $z = 0$, and that $r_{1,2} \geq r_{2,1}$. Consider Problem M with $\alpha_1 = \alpha_2 = 1$ for the set $E = C_{1,2}$. We call it Problem $P_1$. It follows from Grötzh’s lemma [4, Theorem 2.6] (or from Theorem 2(3) above) that $(\overline{D}_{1,2}, A(r_{1,2}, 1))$ is the extremal pair of domains for Problem $P_1$ and the complementary arc $C_{1,2} = \{z : |z| = r_{1,2}\} \setminus C_{1,2}$ is the free boundary of Problem $P_1$.

Now, suppose, by contradiction, that $S_\Omega(0) > r_{1,2}$. The latter implies that there is a function $f \in U(\mathbb{D} \setminus C_{1,2})$ such that $f(0) = 0$ and $\text{dist}(0, f(C_{1,2})) > r_{1,2}$. Let $\rho_1 = \inf\{|z| : z \in f(C_{1,2})\}$, $\rho_2 = \sup\{|z| : z \in f(C_{1,2})\}$. Consider Problem M, again with $\alpha_1 = \alpha_2 = 1$, for the set $f(C_{1,2}) \subset \overline{A}(\rho_1, \rho_2)$. We call it Problem $P_2$. Since Jenkins’s problem on extremal partitioning is conformally invariant (in the
sense that conformal mappings preserve extremal configurations), it follows that the pair \((f(D_{r_1,2}), f(A(r_{1,2},1)))\) is extremal for Problem \(P_2\). Also, conformal mappings preserve the module of a doubly connected domain. Hence,

\[
m(f(A(r_{1,2},1))) = m(A(r_{1,2},1)) = -\frac{1}{2\pi} \log r_{1,2}.
\] (4)

Now, it follows from Theorem 2(3) that the free boundary of Problem \(P_2\) lies in the annulus \(\bar{A}(\rho_1, \rho_2)\). This implies that \(f(A(r_{1,2},1)) \subset A(\rho_1,1)\). Since the module of a doubly connected domain increases under expansion, it follows that \(m(f(A(r_{1,2},1))) \leq m(A(\rho_1,1)) = -\frac{1}{2\pi} \log \rho_1\). Since \(r_{1,2} < \rho_1\), the latter inequality contradicts (4). Thus, our assumption leads to a contradiction and therefore we must have \(S_{\Omega}(0) = r_{1,2}\).

More results and questions. Let \(\Omega\) be a domain (of any connectivity) and let \(\gamma\) be a nondegenerate boundary continuum of \(\Omega\) separated from the rest of \(\partial \Omega\), i.e. such that \(\gamma \cap (\partial \Omega \setminus \gamma) = \emptyset\). If \(\Omega\) is finitely connected, the latter separation property always holds. Then, for each \(z \in \Omega\), we can consider a non-empty class \(U_{\gamma}(\Omega)\) of functions \(f \in U(\Omega)\) such that \(f(\gamma) = T\), in the sense of boundary correspondence. Then the function \(S_{\Omega,\gamma} : \Omega \to \mathbb{R}\) defined by

\[
S_{\Omega,\gamma}(z) = \sup \{\text{dist}(0, \partial f(\Omega)) : f \in U_{\gamma}(\Omega), f(z) = 0]\)

(5)
can be thought as the squeezing function toward the boundary continuum \(\gamma\). For this function we have the following result.

Lemma 1. Let \(\Omega\) and \(\gamma\) be as above and let \(a \in \Omega\). Suppose that there is a function \(f_{a,\gamma} \in U_{\gamma}(\Omega)\) such that \(f_{a,\gamma}(a) = 0\) and \(D \setminus f_{a,\gamma}(\Omega) \subset \{z : |z| = r\}, 0 < r < 1\). Then \(S_{\Omega,\gamma}(a) = r\).

Furthermore, \(f_{a,\gamma}\) is a unique (up to rotation about 0) function in \(U_{\gamma}(\Omega)\) that is extremal for the problem on the squeezing toward \(\gamma\) for the point \(a\).
Proof. Since the squeezing problem is conformally invariant, we may assume that \( \Omega \) is the disk \( D \) slit along a proper compact subset of the circle \( T_r = \{ z : |z| = r \} \), \( 0 < r < 1 \). Then our argument used in the proof of Theorem 3 shows that if \( f \in \mathcal{U}_r(\Omega) \) such that \( f(0) = 0 \) is extremal for the squeezing toward \( T_r \) problem, then \( D \setminus f(\Omega) \subset T_r \). Furthermore, same argument shows that \( f(D_r) = D_r \). Since \( f(0) = 0 \), the latter implies that \( f \) is a rotation about the origin. \( \square \)

Below, we assume that \( \Omega \) is a finitely-connected domain with \( n \geq 3 \) nondegenerate boundary continua \( \gamma_1, \ldots, \gamma_n \). In this case, the following properties are either exist in the literature or easy to prove:

1. **Existence of extremal functions.** For each \( z \in \Omega \) and \( 1 \leq k \leq n \), there is an extremal function for the problem on squeezing toward \( \gamma_k \); i.e. a function \( f_{z,k} \in \mathcal{U}_n(\Omega) \) such that \( f_{z,k}(z) = 0 \) and \( S_{\Omega, \gamma_k}(z) = \text{dist}(0, \partial f_{z,k}(\Omega)) \). Therefore, for each \( z \in \Omega \), there is a function \( f_{z} \in \mathcal{U}(\Omega) \) extremal for the squeezing problem on \( \Omega \).

2. **Continuity.** The functions \( S_{\Omega}(z) \) and \( S_{\Omega, \gamma_k}(z), k = 1, \ldots, n \), are continuous on \( \Omega \).

3. **Monotonicity of \( S_{\Omega, \gamma_k}(z) \) with respect to \( \gamma_n \).** For a fixed \( z \in \Omega \), \( S_{\Omega, \gamma_k}(z) \) is monotone with respect to \( \gamma_n \) in the following sense. Let \( \Omega' \neq \Omega \) be a finitely connected domain with boundary continua \( \gamma'_1, \ldots, \gamma'_{n'} \), \( 2 \leq n' \leq n \), such that \( a \in \Omega' \subset \Omega \) and \( \gamma'_k = \gamma_k \) for \( k = 1, \ldots, n' - 1 \). Then

\[
S_{\Omega', \gamma'_{n'}}(a) > S_{\Omega, \gamma_n}(a).
\]

4. **Non-monotonicity of \( S_{\Omega}(z) \) with respect to \( \Omega \).** For a fixed \( z \in \Omega \), \( S_{\Omega}(z) \) is not monotone as a function of the domain.

5. **Boundary values of \( S_{\Omega}(z) \).** \( S_{\Omega, \gamma_k}(z) \to 1 \) as \( z \to \gamma_k \) and therefore \( S_{\Omega}(z) \to 1 \) as \( z \to \partial \Omega \).

6. **Boundary values of \( S_{\Omega, \gamma_k}(z) \) on \( \gamma_j \), \( j \neq k \).** For each \( k \) and \( j \neq k \), there is \( 0 < c_{j,k} < 1 \) such that

\[
\limsup_{z \to \gamma_j} S_{\Omega, \gamma_k}(z) \leq c_{j,k}.
\]

7. **Separation property 1.** If \( c, 0 < c < 1 \), is sufficiently close to 1, then the level set \( \{ z \in \Omega : S_{\Omega, \gamma_k}(z) = c \} \) separates \( \gamma_k \) from \( \partial \Omega \setminus \gamma_k \).

8. **Separation property 2.** Let \( j \neq k \). The set \( \{ z \in \Omega : S_{\Omega, \gamma_j}(z) = S_{\Omega, \gamma_k}(z) \} \) separates \( \gamma_j \) from \( \gamma_k \) inside \( \Omega \).

1. The existence of extremal functions and therefore existence of extremal domains as well was established in [1, Theorem 2.1].

2. To prove the continuity property, one can use extremal functions \( f_{z,\Omega}(z) \) composed with the Moebius automorphisms of the unit disk \( \mathbb{D} \). The details are left to the interested reader.

3. To prove the monotonicity property of the squeezing function \( S_{\Omega, \gamma_k}(z) \) defined by [5], we consider the composition \( \varphi \circ f_{\Omega, \gamma_k} \) of the extremal function \( f_{\Omega, \gamma_k} \) with the function \( \varphi \) that is the Riemann mapping function from a simply connected domain with the boundary continuum \( \gamma'_n \), such that \( \varphi(0) = 0 \). Then the desired result follows from the Schwarz’s lemma.

4. To show that the monotonicity with respect to \( \Omega \) is absent, consider two simple examples. Let \( \Omega \) be a circularly slit disk \( \mathbb{D} \setminus \{ re^{i\theta} : |\theta| \leq \alpha \} \) with \( 0 < \alpha < \pi \). For \( 0 < r_1 < r, r < r_2 < 1 \), let \( \Omega_1 = \Omega \setminus [r_1, r], \Omega_2 = \Omega \setminus [r, r_2] \). Then the argument involving Theorem 2, as it was used in the proof of Theorem 1, shows that \( S_{\Omega_1}(0) < S_{\Omega_1}(0) < S_{\Omega_2}(0) \). Therefore, the monotonicity property of \( S_{\Omega}(z) \) as a function of the domain does not hold, in general.
(5) To show that $S_{Ω,γ_k}(z) → 1$ as $z → γ_k$, we may assume that $Ω ⊂ D$ and $γ_k = T$. Then the M"obius mapping $ϕ(ζ) = (ζ - z)/(1 - zζ)$ is in $U_{ε_k}(Ω)$ and it is easy to see that that $\text{dist}(0, \partial ϕ(Ω)) → 1$ as $|z| → 1$.

(6) Let $l$ be a Jordan curve in $Ω$ separating $γ_j$ from $\partial Ω\setminus γ_j$ and let $Ω_l$ be a doubly connected domain with boundary components $γ_j$ and $l$. It follows from the property (3) that $S_{Ω_l}(z) < S_{Ω_l}(z)$ for all $z ∈ Ω_l$.

Fix $z_0 ∈ Ω_l$. Let $m = m(Ω_l)$ be the module of the doubly connected domain $Ω_l$. Then there is a function $ψ$ that maps $Ω_l$ conformally onto the annulus $A(s, 1)$ with $s = e^{-2πm}$. In addition, we may assume that $ψ(z_0) = a$, $s < a < 1$. Furthermore, there exists a function $ψ$, such that $ψ(a) = 0$, which maps $A(s, 1)$ conformally onto the disk $D$ slit along $C$ on a circle $T_0$, with $0 < ρ < 1$, where $ρ = ρ(a)$ depends on $a$. The mapping $ψ$ is one of the well-studied canonical mappings. A particular property of $ψ$, we need here, was proved by E. Reich and S. E. Warschawski in 1960 [9, Lemma 3]. These authors showed that $ρ(a) = a$. This result implies that $a → s$ as $z_0 → γ_j$. It follows from Theorem 1 that $S_{Ω_l}(z_0) = ρ(a) = a$. This, being combined with property (3), implies that

$$\lim_{z_0 \to γ_j} \sup_{a \to s} \leq \lim_{a \to s} a = s = e^{-2πm} < 1,$$

as required.

(7) Let $c_k = \max_{j \neq k} \{c_{j,k}\}$, where $c_{j,k}$ introduced in part (6) above. Take $c$, $c_k < c < 1$, and consider the level set $L(c) = \{z ∈ Ω : S_{Ω,γ_k}(z) = c\}$. Since $S_{Ω, γ_k}$ is continuous on $Ω$ and has boundary value 1 on $γ_k$ and boundary values less than $c$ on other boundary continua, it follows that $L(c)$ separates $γ_k$ from the rest of $∂Ω$.

(8) The same argument, which was used in part (7), can be used to prove the separation property in question as well.

Let $Ω$ be an $n$-connected domain as above. Then, for each $z ∈ Ω$, there are $n$ functions $φ_{z,k}$, $1 ≤ k ≤ n$, and for each $k$ there are $n - 1$ radii $r_{j,k} = r_{j,k}(z)$ of circular arcs $C_{j,k} = C_{j,k}(z)$, as described above. An example of the domain $Ω$ and its image under one of the mappings $φ_{z,k}$ are shown in Figure 2. It was conjectured in [8] that

$$S_{Ω}(z) = \max_k \min_{j \neq k} r_{j,k}(z).$$

This conjecture, though it sounds plausible, was quickly disproved by Gumenyuk and Roth [3], whose engineering construction shows that for every $n ≥ 3$ there is an $n$-connected domain $Ω$, that is not a circularly slit disk, and there is a point $z ∈ Ω$ such that $S_{Ω}(z)$ is strictly greater than the right-hand side of (6). This shows that, in general, the family of domains extremal for the squeezing problem is not limited to the set of circularly slit disks. In [8] and [3], the question whether or
not there exist circularly slit disks of connectivity $n \geq 3$ that are extremal for the squeezing problem was left open. Thus, we complement results in [8] and [3] with the following theorem.

**Theorem 3.** For each $n \geq 3$, there is a domain $\Omega$, that is a circularly slit disk of connectivity $n$, such that $\Omega$ is the extremal domain, unique up to rotation about the origin, for the squeezing problem for the point $z = 0$.

**Proof.** Let $n \geq 3$, $0 < r < 1$, and $0 < \alpha < \pi/(n - 1)$. Let $\gamma_k = \{z = re^{i\theta} : |\theta - 2\pi(k - 1)/(n - 1)| \leq \alpha\}$ for $1 \leq k \leq n - 1$ and let $\gamma_0 = \mathbb{T}$. Let $\Omega = \Omega(n, r, \alpha)$ denote the disk $\mathbb{D}$ slit along the arcs $\Gamma_k$, $k = 1, \ldots, n - 1$.

It follows from Lemma 1 that $S_{\Omega, \gamma_n}(0) = r$. Furthermore, since $\Omega$ possesses $(n - 1)$-fold rotational symmetry, it follows that there is $\rho$, $0 < \rho < 1$, such that

$$S_{\Omega, \gamma_k}(0) = \rho \quad \text{for } k = 1, \ldots, n - 1.$$

We claim that, for fixed $n$ and $r$, $\rho = \rho(\alpha) \to 0$ as $\alpha \to 0$. To emphasize dependence on $\alpha$, we use notations $\Omega = \Omega(\alpha)$, $\gamma_k = \gamma_k(\alpha)$, etc. To prove the claim, we consider a circle $\Gamma_\varepsilon = \{z : |z - r| = \varepsilon\}$ with $\varepsilon > 0$ sufficiently small. If $\alpha$ is small, then $\Gamma_\varepsilon$ separates $\gamma_1(\alpha)$ from all other boundary components of $\Omega(\alpha)$. Let $D_\varepsilon(\alpha)$ denote the doubly connected domain bounded by $\Gamma_\varepsilon$ and $\gamma_1(\alpha)$. It is clear that $m(D_\varepsilon(\alpha)) \to \infty$ as $\alpha \to 0$. Since the module of a doubly connected domain is conformally invariant, it follows that for every $\delta > 0$ there is $\alpha_0 > 0$ such that $f(\Gamma_\varepsilon) \subset D_\varepsilon$, whenever $0 < \alpha < \alpha_0$ and $f \in \mathcal{U}_{\gamma_1(\alpha)}(\Omega(\alpha))$ is such that $f(0) = 0$. Since the images $f(\mathbb{T})$ and $f(\gamma_1(\alpha))$, $k = 2, \ldots, n - 1$, lie in the bounded component of $\mathbb{C} \setminus f(\Gamma_\varepsilon)$, the claim follows. Therefore, $\rho(\alpha) < r$ if $\alpha$ is small enough. Hence $S_{\Omega(\alpha)}(0) = r$ for all such $\alpha$ and $\Omega(\alpha)$ is the only extremal domain up to rotation about the origin. The proof is complete. \hfill \Box

**Example.** Given $a > 1$, let $\gamma_k = \gamma_k(a) = \{z = te^{2\pi i(k - 1)/3} : 1 \leq t \leq a\}$, $k = 1, 2, 3$. Then $\Omega = \mathbb{C} \setminus \bigcup_{k=1}^3 \gamma_k$ is a domain on $\mathbb{C}$ of connectivity 3. Let $\varphi_{0,1}$ map $\Omega$ conformally onto a circularly slit disk $\mathbb{D}$ such that $\varphi_{0,1}(\gamma_1) = \mathbb{T}$, $\varphi_{0,1}(0) = 0$, and $\varphi_{0,1}'(0) > 0$. Since $\Omega$ is invariant under rotation by angle $2\pi/3$ about 0, the symmetry principle implies that $\varphi_{0,1}(\gamma_2)$ and $\varphi_{0,1}(\gamma_3)$ are circular arcs lying on the same circle symmetrically to each other with respect to the real axis. Therefore, it follows from Theorem 3 that $\varphi_{0,1}$ is extremal for the squeezing toward $\gamma_1$ problem for the point 0. Since $\Omega$ is invariant under rotations by the angle $2\pi/3$, it follows that each of the other two functions, $\varphi_{0,2}$ and $\varphi_{0,3}$, where $\varphi_{0,k}$ maps $\gamma_k$ to the unit circle $\mathbb{T}$ such that $\varphi_{0,k}(0) = 0$, $k = 2, 3$, is also extremal for the squeezing problem toward, respectively, $\gamma_2$ or $\gamma_3$, for the point 0. Moreover, $S_{\Omega, \gamma_1}(0) = S_{\Omega, \gamma_2}(0) = S_{\Omega, \gamma_3}(0)$.

Thus, this example shows that there are domains $\Omega$ of connectivity $n > 2$ and points $z_0 \in \Omega$, such that every function $f$ extremal for the squeezing problem for $\Omega$ and $z_0$ maps $\Omega$ onto a circularly slit disk.

**Questions.** We finish this part with six questions for future study.

1. As Theorem 1 shows, if $\Omega$ is doubly connected, then every function $f$ extremal for the squeezing problem for some point $z_0 \in \Omega$ maps $\Omega$ onto a circularly slit disk. We suspect that this property characterizes doubly connected domains. So, the question: Is it true that if there is an open subset $G$ of $\Omega$, such that any function $f$ extremal for the squeezing problem for some $z \in G$ maps $\Omega$ onto a circularly slit disk, then $\Omega$ is doubly connected?

2. A point $z_0$ in a domain $\Omega$ with boundary continua $\gamma_1, \ldots, \gamma_n$, $n \geq 2$, is an equilibrium point for the squeezing problem if $S_{\Omega, \gamma_1}(z_0) = S_{\Omega, \gamma_2}(z_0) = \ldots = S_{\Omega, \gamma_n}(z_0)$. An annulus $A(r, 1)$ has the circle $\{z : |z| = \sqrt{r}\}$ as its set of equilibrium points. A domain $D_n(a) = \mathbb{C} \setminus \bigcup_{k=1}^n \{te^{2\pi i(k - 1)/n} : 1 \leq t \leq a\}$
with $a > 1$ and $n \geq 3$ has two equilibrium points, $z_1 = 0$ and $z_2 = \infty$. Let $D_n(a, R) = D_n(a) \cap \mathbb{D}_R$, where $R > a$. There is a unique value $R_0 > a$ such that $D_n(a, R)$ has exactly one equilibrium point $z_0 = 0$, while, for $R \neq R_0$, $D_n(a, R)$ does not have equilibrium points.

Is it true that every domain $\Omega$ of connectivity $n \geq 3$ has at most two equilibrium points?

(3) Are there circularly slit disks with boundary continua $\gamma_n = T$ and $\gamma_k \subset \mathbb{D}$, $k = 1, \ldots, n$, $n \geq 3$, that are extremal for the squeezing problem for $z = 0$, such that $\gamma_j$ and $\gamma_k$ lie on different circles when $j \neq k$?

(4) Are there domains $\Omega \subset \mathbb{D}$ of connectivity $n + m + 1$ with boundary continua $\gamma_{n+m+1} = T$ and $\gamma_k \subset \mathbb{D}$, $k = 1, \ldots, n + m$, such that $\gamma_k$ is an arc of a circle centered at $0$ when $1 \leq k \leq n$, $\gamma_k$ is not a circular arc when $n + 1 \leq k \leq n + m$, that are extremal for the squeezing problem for $z = 0$?

(5) Is it possible to characterize domains $\Omega$ of connectivity $n \geq 3$, that are extremal for the squeezing problem for a point $z_0 \in \Omega$, in terms of quadratic differentials similar to characterization of configurations of domains extremal for Jenkins’s problem [5] on extremal partitioning?

(6) Our proof of Theorem 2 on the extremal partitioning of the disk $\mathbb{D}$ with a compact barrier $E \subset \mathbb{D}$ can be extended to higher dimensions for the unit ball $B \subset \mathbb{R}^n$, $n \geq 3$, with a compact set $E \subset B$.

It would be interesting to know whether such a generalization can be used to study the squeezing problem in $\mathbb{C}^n$.

**Proof of Theorem 2.** Problem M is a particular case of Jenkins’s problem on extremal partitioning [5]. Therefore, parts (1) and (2) of Theorem 2 follow from [5, Theorem 1]. Furthermore, same Jenkins’s theorem implies that the metric $\rho_1(z)|dz| = \alpha_1^{-1}|Q(z)|^{1/2}|dz|$ is extremal for the module problem for the family of closed curves $\gamma \subset D_1^*$ separating $z = 0$ from $\partial D_1^*$ and the metric $\rho_2(z)|dz| = \alpha_2^{-1}|Q(z)|^{1/2}|dz|$ is extremal for the module problem for the family of closed curves $\gamma \subset D_2^*$ separating $T$ from $\partial D_2^* \setminus T$. Moreover, it implies that

$$\int_\gamma |Q(z)|^{1/2}|dz| \geq \alpha_2$$

(7)

for every Jordan curve $\gamma \subset \mathbb{D} \setminus E$, which separates $T$ from $E$ and $0$ and also for $\gamma = T$.

Let us consider the case $\alpha_1 \leq \alpha_2$, assuming that $E$ consists of a finite number of connected components. In this case $D_2^* \neq \emptyset$. Otherwise, $T$ would be on the boundary of $D_1^*$, which implies that

$$\int_T |Q(z)|^{1/2}|dz| < \alpha_1 \leq \alpha_2,$$

contradicting equation (7). Furthermore, since $E$ consists of a finite number of components, it follows that the free boundary $L(E)$ consists of a finite number of critical trajectories of $Q(z)dz^2$ and their endpoints in $\mathbb{D} \setminus E$.

Suppose by contradiction that

$$r_2 < \rho = \max\{|z| : z \in L(E)\} < 1.$$  

(8)

Since $L(E)$ consists of a finite number of analytic arcs, it follows that the intersection $L(E) \cap \mathbb{P}_\rho$ consists of a finite number of points $z_k$, $k = 1, \ldots, n$. It follows from the local structure of trajectories of $Q(z)dz^2$ near critical points, that each of the points $z_k$ is regular. This implies that, for all sufficiently small $\varepsilon > 0$, the set $L(E) \cap \tilde{A}(\rho - \varepsilon, \rho)$ consists of $n$ disjoint analytic arcs $s_k$, $k = 1, \ldots, n$, such that $s_k$ has its endpoints on the circle $T_{\rho - \varepsilon}$ and the point $z_k$ is an interior point of $s_k$. 

Let \( \tilde{s}_k \) denote the arc symmetric to \( s_k \) with respect to the circle \( T_{p-\varepsilon} \). For every \( \varepsilon \) small enough and each \( k \), the intersection \( \tilde{s}_k \cap L(E) \) is empty. In this case, there are \( n \) simply connected domains \( \Delta_k \), symmetric with respect to the circle \( T_{p-\varepsilon} \), such that \( \partial \Delta_k = \tilde{s}_k \cup s_k \), \( k = 1, \ldots, n \).

We claim that there is no \( k \) such that \( \Delta_k \subset D_2^* \). To prove this claim, suppose that \( \Delta_{k_j} \subset D_2^* \) for \( j = 1, \ldots, n_1, \ n_1 \leq n \). Let \( \partial^i D_2^* = \mathbb{T} \) and \( \partial^i D_2^* \) denote the boundary components of \( D_2^* \). Let \( D_2^* \) denote polarization of the doubly connected domain \( D_2^* \) (considered as a condenser whose plates are connected components of \( \mathbb{C} \setminus D_2^* \)) with respect to the circle \( T_{p-\varepsilon} \). In the case under consideration, the polarized domain \( D_2^* \) is a doubly connected domain having \( \mathbb{T} \) as one of its boundary components while the other boundary component of \( D_2^* \) is obtained from \( \partial^2 D_2^* \) by replacing the arcs \( s_{k_j} \) with the arcs \( \tilde{s}_{k_j}, \ j = 1, \ldots, n_1 \). For the definition and properties of polarization, we refer to [2, Chapter 3]. As well known, polarization increases the module of a doubly connected domain. Thus, in our case, we have

\[
m(D_2^*) < m(D_2^*),
\]

with the sign of strict inequality because \( D_2^* \) does not coincide with \( D_2^* \) up to reflection with respect to \( T_{p-\varepsilon} \). Therefore,

\[
\alpha_1^2 m(D_1^*, 0) + \alpha_2^2 m(D_2^*) < \alpha_1^2 m(D_1^*, 0) + \alpha_2^2 m(D_2^*).
\]

Since the pair \( (D_1^*, D_2^*) \) is admissible for Problem M, i.e. \( (D_1^*, D_2^*) \in \mathcal{D}(E) \), inequality [8] contradicts our assumption that the pair of domains \( (D_1^*, D_2^*) \) is extremal for Problem M.

Now we consider the case when \( \Delta_k \subset D_1^* \) for all \( k = 1, \ldots, n \). In this case, we consider a pair \( (D_1, D_2) \), where \( D_1 = D_1^* \setminus A(\rho - \varepsilon, \rho) \) is a simply connected domain and \( D_2 = D_2^* \cup A(\rho - \varepsilon, \rho) \cup T_\rho \) is a doubly connected domain. One can easily see that \( (D_1, D_2) \in \mathcal{D}(E) \). Let \( \rho_k(z) |dz| \) denote the extremal metric for the corresponding module problem for the domain \( D_k, k = 1, 2 \); see [4, Chapter II].

Since \( D_2^* \subset D_2 \), it follows that \( \rho_2(z) |dz| \) is admissible for the module problem for \( D_2^* \) but it is not extremal for this problem. Therefore,

\[
m(D_2^*) < \int \int_{D_2^*} \rho_2^2(z) \, dA - \int \int_{\cup_{k=1}^{n} \Delta_k} \rho_2^2(z) \, dA
\]

\[
= m(D_2) - \int \int_{\cup_{k=1}^{n} \Delta_k} \rho_2^2(z) \, dA.
\]

Next, we consider a metric \( \rho(z) |dz| \) on the domain \( D_1^* \) defined by

\[
\rho(z) = \begin{cases} 
\rho_1(z) & \text{if } z \in D_1, \\
\rho_2(z) & \text{if } z \in D_1^* \setminus D_1.
\end{cases}
\]

We claim that \( \rho(z) |dz| \) is admissible for the problem on the reduced module for the domain \( D_1^* \). To prove this, we consider an analytic Jordan curve \( \gamma \subset D_1^* \) separating 0 from \( \partial D_1^* \). If \( \gamma \subset D_1 \), then

\[
\int_{\gamma} \rho(z) |dz| = \int_{\gamma} \rho_1(z) |dz| \geq 1
\]

since \( \rho_1(z) |dz| \) is admissible for the problem on the reduced module of \( D_1 \).

Suppose now that \( \gamma \notin D_1 \). Then the circle \( T_{p-\varepsilon} \) divides \( \gamma \) into a finite number of arcs. By \( \tau_k, k = 1, \ldots, m \), we denote those of them, which lie in \( D_2 \). Let \( \sigma_k \subset D_1^* \) denote the closed arc of \( T_{p-\varepsilon} \) joining the endpoints of \( \tau_k, k = 1, \ldots, m \). It follows from Lemma 2, presented below, that

\[
\int_{\tau_k} \rho(z) |dz| \geq \int_{\sigma_k} \rho(z) |dz| \geq \int_{\sigma_k} \rho_1(z) |dz|.
\]
Let \( \tilde{\gamma} \) denote the curve obtained from \( \gamma \) by replacing the arcs \( \tau_k \) with the arcs \( \sigma_k, k = 1, \ldots, m \). Then \( \tilde{\gamma} \) is the curve in \( D_1 \), that is not Jordan in general, that separates 0 from \( \partial D_1 \). Hence,

\[
\int_{\tilde{\gamma}} \rho_1(z) |dz| \geq 1.
\]  

(12)

Combining equations (11) and (12), we conclude that

\[
\int_{\gamma} \rho(z) |dz| \geq \int_{\tilde{\gamma}} \rho_1(z) |dz| \geq 1.
\]  

(13)

Equations (10) and (13) imply that \( \rho(z)|dz| \) is an admissible metric for the problem on the reduced module in \( D_1^* \). Hence,

\[
m(D_1^*; 0) \leq \lim_{\epsilon \to 0} \left\{ \int_{D_1 \setminus D_\epsilon} \rho^2(z) \, dA + \frac{1}{2\pi} \log \epsilon \right\}
\]  

(14)

\[
= \lim_{\epsilon \to 0} \left\{ \int_{D_1 \setminus D_\epsilon} \rho^2(z) \, dA + \int_{\bigcup_{k=1}^N \Delta_k} \rho_2^2(z) \, dA + \frac{1}{2\pi} \log \epsilon \right\}
\]  

\[
= m(D_1, 0) + \int_{\bigcup_{k=1}^N \Delta_k} \rho_2^2(z) \, dA.
\]

Combining (9) and (14), we obtain the following inequalities:

\[
\alpha_1^2 m(D_1^*, 0) + \alpha_2^2 \rho(D_2^*) < \alpha_1^2 m(D_1, 0) + \alpha_2^2 \rho(D_2)
\]

\[
+ (\alpha_2^2 - \alpha_1^2) \int_{\bigcup_{k=1}^N \Delta_k} \rho_2^2(z) \, dA \leq \alpha_1^2 m(D_1, 0) + \alpha_2^2 \rho(D_2).
\]

The latter inequalities contradict to the assumption that the pair \((D_1^*, D_2^*)\) is extremal for Problem M. This contradiction shows that our assumption \( R > r_2 \) is wrong and therefore, \( L(E) \subset \bar{D}_{r_2} \), as required.

The latter inclusion is proved in the case when \( E \) consists of a finite number of components. In the general case, we approximate \( E \) with a sequence of compact sets \( E^j, j = 1, 2, \ldots \), such that \( E^{j+1} \subset E^j \) and such that \( E^j \) is bounded by appropriate level curves of Green’s function \( g_{D_0, E}(z, 0) \) of the domain \( D \setminus E \) with pole at 0. Then the required inclusion \( L(E) \subset \bar{D}_{r_2} \) will follow from the result already proved for sets \( E \) with finite number of connected components and from Carathéodory’s convergence theorem for simply connected and doubly connected domains.

The proof presented above, can be easily modified to show that if \( \alpha_1 \geq \alpha_2 \), then \( L(E) \subset A(r_1, 1) \).

\[ \square \]

**Lemma 2.** Let \( \sigma \) be an open arc on \( T \). Let \( D_k, k = 1, 2 \), be a doubly connected domain having the circle \( T \), \( 0 < r < 1 \), as one of its boundary components. Suppose that the other boundary component of \( D_1 \), call it \( \gamma_1 \), is such that \( l \subset \gamma_1 \) and \( D_1 \subset A(r, 1) \). Suppose further that the other boundary component of \( D_2 \), call it \( \gamma_2 \), is such that \( l \subset \gamma_2 \) and \( A(r, 1) \subset D_2 \). Let \( \rho_k(z)|dz| \) denote the extremal metric of the module problem in \( D_k, k = 1, 2 \). Then

\[
\rho_1(e^{i\theta}) \leq 1/2\pi \leq \rho_2(e^{i\theta}) \quad \text{for all } e^{i\theta} \in \sigma.
\]  

(15)

**Proof.** Let us prove the first inequality. Let \( f_1 \) maps \( D_1 \) conformally onto the annulus \( A(r_1, 1) \) such that \( f_1(\gamma_1) = T \). Since the module of a doubly connected domain increases under expansion, the following holds: \( r < r_1 < 1 \). In terms of the mapping function, the extremal metric can be expressed as follows (see [3] Chapter II):

\[
\rho_1(z)|dz| = \frac{1}{2\pi} \left| \frac{|f_1'(z)|}{|f(z)|} \right| |dz|.
\]
This shows that the desired result will follow if we prove that
\[ \text{meas}(f_1(\sigma)) \leq \text{meas}(\sigma) \] (16)
for every \( \sigma \subset \mathbb{T} \), if the assumptions of the lemma are satisfied.

To prove (16), consider a family of curves \( \Gamma_1(\sigma) = \{\gamma\} \) consisting of all rectifiable arcs \( \gamma \) joining \( \mathbb{T}_r \) and \( \sigma \) inside the domain \( D_1 \). Let \( \Gamma(s, \sigma) \) denote a similar family of curves in the annulus \( A(s, 1) \), \( 0 < s < 1 \). Since the module increases under expansion of the family of curves, we have

\[ \text{mod}(\Gamma_1(\sigma)) < \text{mod}(\Gamma(s, \sigma)) \] if \( D_1 \neq A(r,1) \). (17)

Now, suppose by contradiction that \( \text{meas}(f_1(\sigma)) > \text{meas}(\sigma) \). Since the module of a family of curves is conformally invariant and since a family of shorter curves has bigger module than a corresponding family of longer curves, we have the following:

\[ \text{mod}(\Gamma_1(\sigma)) = \text{mod}(\Gamma(r, f_1(\sigma))) > \text{mod}(\Gamma(f_1(\sigma))) \geq \text{mod}(\Gamma(s, \sigma)) \] (18)
contradicting (17). Thus, the inequality (16) is proved and therefore the first inequality in (15) holds. The proof of the second inequality in (15) follows the same lines. \( \square \)

Notice that (15) holds for doubly connected domains \( D_k \) having the circle \( \mathbb{T}_r \) of arbitrary small radius \( 0 < r < 1 \) as one of its boundary components. Therefore, taking the limit as \( r \to 0 \), we conclude that (15) remains valid if \( D_1 \) and/or \( D_2 \) is a simply connected domain containing \( 0 \). In the latter case, \( \rho_k(z)|dz| \) will denote the extremal metric for the problem on the reduced module \( m(D_k, 0) \).

In the case of simply connected domains, (15) also follows from the Loewner’s lemma or from Carleman’s expansion principle for the harmonic measure.

Remarks. (1) As Gumenyuk and Roth showed in [3], Theorem 1 on the extremality of circularly slit disks cannot be extended to domains of connectivity greater than two. This result sounds similar to the observation made in [10] about a possibility to extend Theorem 2 stated above in this note to the case of several barriers \( E_k \) contained in non-overlapping annuli \( A(r_k^1, r_k^2), 0 < r_k^1 \leq r_k^2 \leq r_k^3 \leq \cdots < r_k^n \leq r_k^n < 1 \); i.e. such that \( E_k \subset A(r_k^1, r_k^2) \). Precisely, if \( (D_1^*, \ldots, D_n^*) \) is the configuration of non-overlapping domains maximizing the weighted sum of moduli

\[ m(D_1,0)+\sum_{k=2}^n m(D_k), \]
then the free boundary \( L(E_1, \ldots, E_n) = \bigcup_{k=1}^n \partial D_k^* \cap (\mathbb{D} \setminus \bigcup_{k=1}^n E_k) \) is not necessarily contained in the union \( \bigcup_{k=1}^n A(r_k^1, r_k^2) \) of these annuli. Here, \( D_1 \subset \mathbb{D} \setminus \bigcup_{k=1}^n E_k \) is a simply connected domain containing \( 0 \) and \( D_k, k = 2, \ldots, n \), is a doubly connected domain in \( \mathbb{D} \setminus \bigcup_{k=1}^n E_k \) separating the set \( \bigcup_{j=1}^{k-1} E_j \) from the unit circle \( \mathbb{T} \) and the set \( \bigcup_{j=k+1}^n E_j \).

(2) Our proof of Theorem 2 relies on the technique, which uses weighted sums of moduli of free families of curves developed by J. A. Jenkins and others. As well known, a module of a doubly connected domain \( D \) with complementary components \( E_0 \) and \( E_1 \) is the reciprocal of the capacity, when \( D \) is considered as the field of the condenser with plates \( E_0 \) and \( E_1 \). This observation suggests that an approach utilizing properties of capacities and potential functions of condensers also can be used to study the squeezing problem. An example, demonstrating how this approach works, was given in the second proof of Theorem 1.3 in [10]. The proof of Theorem 2 in [3] is a nice demonstration how this approach based on the potential theory can be used in the context of the squeezing problem.
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