THE COMPLETE SOLUTION
TO BASS GENERALIZED JACOBIAN CONJECTURE

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Introduction

The Classical Jacobian Conjecture claims that any unramified endomorphism of a complex affine space is an automorphism, which means in more ordinary terms that for any integer \( n > 0 \), any polynomial map from \( \mathbb{C}^n \) to itself with an invertible jacobian function is itself invertible and its inverse is again a polynomial map (see for instance [7] and [1] or [3] for the ‘right” version of this conjecture in any characteristic). From this last point of view, this conjecture may be considered as a global version for polynomial maps of the classical Local Inversion Theorem, which explains largely the fascination that this conjecture exerts on generations of searchers for more than half a century.

In order to embed this conjecture in a geometric environment, where one could enjoy the beauty and the richness of tools of algebraic geometry and algebraic D-modules, as his paper [6] proves it, Hyman Bass proposed 25 years ago in [6], page 80 the following statement as the

**Generalized Jacobian Conjecture:**
Any unramified morphism from a complex irreducible affine and unirational variety whose invertible regular functions are all constant to a complex affine space of the same dimension is an isomorphism.

On the other hand, without any explicit connection with Bass conjecture, Victor Kulikov published in 1993 (see [18]) a non trivial construction of a complex irreducible rational and simply connected surface and an unramified morphism of geometric degree 3 (and hence which is not an isomorphism) from this surface to the complex affine space, without specifying if this surface is affine or not, or if its invertible regular functions are all constant or not.

The main aim of this paper is to bring this precision and thanks to this to expose the complete solution to Bass Generalized Jacobian Conjecture which turned to be true only in dimension one (see Theorem 1 below).

In order to make this precision as clear as possible, we introduce a family of irreducible affine and rational surfaces \( S(C_1, C_2, P) \) over any algebraically closed field \( K \), where \( C_1 \) and \( C_2 \) are irreducible curves of the projective plane over \( K \) and \( P \) a point of one of them. We also give a necessary and sufficient condition of factoriality for each surface of this family (see theorem 2 below) which is defined in such a way that it contains Kulikov surface and Ramanujam one (see the next section).

In the same aim, we give the proof of a general fact known by some algebraic geometers like V. Srinivas who told me about it, but which seems to be written nowhere in its whole generality in the abundant literature about algebraic geometry. This fact is a sufficient condition on the ground field of an irreducible simply connected and normal algebraic variety or on its Picard group in order that all its invertible regular functions are constant (see theorem 3 below).

Let us notice that according to the statement of the main Theorem 1 below, Kulikov morphisms are not counter-examples to the classical Jacobian Conjecture which keeps jealously and fiercely its more than 70 years old mystery.

We also deduce from this theorem some corollaries which bring some rays of light through the cloud of unknowing which still surrounds the notion of unramified or étale variety morphism even for the best experts of the subject, as the challenge of this conjecture proves it clearly.

A first mystery of these morphisms partially cleared up by these rays of light is the following. It is not difficult to see that the restriction to any sub-variety \( Z \subset X \) of an unramified morphism \( F \) from a variety \( X \) to another one \( Y \) is again an unramified morphism from \( Z \) to the Zariski closure of \( F(Z) \) in \( Y \) (it follows for instance from [5], Chapter VI, Proposition 3.5). Is the similar transfer from a variety \( X \) to any sub-variety \( Z \subset X \) true for any étale (i.e. unramified and flat) morphism \( F \) from \( X \) to \( Y \), at least when the varieties \( X, Y, Z \) are irreducible and non singular and \( Z \) is closed in \( X \)? According to [4], Lemma 3.4 and [1], Theorem
3, this question in the special case where \( X = Z \) is the complex affine space of dimension \( n \) is equivalent to the Classical Jacobian Conjecture in dimension \( n \). Unfortunately, Corollary 1 below brings a negative answer to this question in general when \( X \) and \( Y \) are assumed only to be irreducible and non singular. But the question remains open in the mentioned special case!

A second mystery of unramified or étale morphisms of variety partially illuminated by Theorem 1 is the following: if \( A \subset B \) is an extension of affine domains over an algebraically closed field \( \mathbb{K} \) such that \( A \) and \( B \) have the same invertible elements, \( A \) is factorial, and the canonical map from \( \text{Spec} \ B \) to \( \text{Spec} \ A \) is unramified, or equivalently étale, then among the multitude of primitive elements \( p \in B \) of the field extension induced by the extension \( A \subset B \), can we find a “normal” one, i.e. one \( p \) such that the sub-algebra of \( B \) generated by \( p \) is a normal ring? According to [1], Theorem 3, this question in the special case where \( \mathbb{K} = \mathbb{C} \) and \( A = B \) is the \( \mathbb{C} \)-algebra of polynomials in \( n \) indeterminates is equivalent to the Classical Jacobian Conjecture in dimension \( n \). Unfortunately again, Corollary 2 below brings a negative answer to this question in its generality, leaving it open in the mentioned special case!

A last mystery of unramified or étale morphisms partially lightened by Theorem 1 is the following: if \( A \subset B \) is an extension of factorial affine domains over an algebraically closed field \( \mathbb{K} \) such that \( A \) and \( B \) have the same invertible elements and the canonical morphism from \( \text{Spec} \ B \) to \( \text{Spec} \ A \) is unramified, or equivalently étale, is \( A \) multiplicatively closed in \( B \), or equivalently is each irreducible element of \( A \) also irreducible in \( B \)? According to [3], Theorem 3.11, this question in the special case where \( \mathbb{K} = \mathbb{C} \) and \( A = B \) is the \( \mathbb{C} \)-algebra of polynomials in \( n \) indeterminates, is again equivalent to the Classical Jacobian Conjecture in dimension \( n \). Unfortunately again, Corollary 3 brings a negative answer to this last question in its generality, but the mystery remains thick in the interesting special case!

In addition to the rays of light that the main theorem of the present paper projects on the mysteries of unramified or étale morphisms of algebraic varieties, for all algebraic geometers who seriously want to continue and deepen the program of “local study of schemes and schemes morphisms”, which is the achievement of Algebraic Geometry according to the structure of the height volumes of the “Bible of Algebraic Geometry” represented by the Treatise “Eléments de Géométrie Algébrique” of A. Grothendieck and J. Dieudonné, this main theorem is very useful to irrevocably refute false published explicit or implicit proofs of the Jacobian Conjecture.

A first example of such refutation is the one of the surprising proposition 18.3.1 of A. Grothendieck and J. Dieudonné themselves in the last volume [10] of the “Bible of Algebraic Geometry”, where they claim in particular that any unramified morphism of affine varieties is finite. Since any such a morphism from an affine variety to another normal one of the same dimension is étale, hence a covering, so this claim is in fact an implicit proof of the Jacobian Conjecture, according to the simple connectedness of complex affine spaces. On the other hand, according to the same connectedness and to the unramifiedness of each Kulikov morphism, Theorem 1 below is an irrevocable refutation of this surprising claim, even for an unramified morphism from a complex affine variety to a complex affine space. Another refutation of this claim, even for an unramified morphism from a complex affine curve to the complex affine line, is the corollary 5.2 of the paper [25] of Frans Oort. A third refutation of this claim, now for an unramified endomorphism of the affine plane over an algebraically closed field of positive characteristic, is the counter-example of P. Nousiainen to a conjecture of S. Wang indicated in the remark following Theorem 2.2 of [7].

A second example of a refutation by Theorem 1 below of a false published proof of the Jacobian Conjecture is the case of Therem 4 of the paper [28] of Hamet Seydi, a former thesis student of A. Grothendieck and the only mathematician, excepted Jean Dieudonné in [8], to have published a paper, not a book, with A. Grothendieck in [15]. Indeed, this Theorem 4 claims that any étale and surjective morphism between simply connected complex algebraic varieties is an isomorphism. On the other hand, it is easy to see that the complement of the image of an open morphism from an affine algebraic variety with only constant invertible regular functions to a affine factorial algebraic variety has a codimension greater than 1. So, if in addition this second variety is complex, non singular and simply connected, then this image also is simply connected. So, it follows from this Theorem 4 that any unramified endomorphism of a complex affine space is an open embedding. Since all the irreducible components of the complement of a dense affine subset of any variety have a codimension equal to 1 (see for instance [10], cor. 21.12.11), it follows from this Theorem 4, thanks to the two previous remarks on the complements of open subsets that any unramified endomorphism of a complex affine space is an automorphism. On the other hand, it also follows from this Theorem 4 and Theorem 1 below, thanks to the same remarks, that any Kulikov morphism is an isomorphism. However, this Theorem 4 is not the only false
one of the paper [28]. Its Theorem 6 also is refuted by the counter-example of the normalization morphism of
cusp, while the proof of its Theorem 1, claiming the truth of the Jacobian Conjecture in any dimension, is
invalidated by an unproved claim on the finiteness of the considered endomorphism of a complex affine space,
based on an unproved and false theorem I.21, p. 68 of G. Fisher in [12], for which we have a counter-example.

The attempt of rectification of the proof of Theorem 1 of [28] by its author in the introduction of his
second paper [29] on the subject is so refutable as his first attempt, because of a false application of a theorem
10.4.11 of A. Grothendieck and J. Dieudonné in [9] claiming in particular that any injective endomorphism
of an algebraic variety over an algebraically closed field is surjective, and because of the application of a false “Fundamental Lemma” of [29] refuted by the cited counter-example of Nousiainen to Wang conjecture.
The generalization of this “Fundamental Lemma” by “Lemma I.1” of [29] also is refuted both by Theorem 1
below and the cited Oort surjective unramified morphisms from a complex algebraic curve with only constant
invertible regular functions to the complex line. These cited last counter-examples are sufficient to refute
almost all claims of the prolific second paper [29] of Seydi.

An N-th example of refutation by Theorem 1 below of false published or pre-published proofs of the
Jacobian Conjecture is the main theorem 2.1 of the paper [23] of Susumu Oda, first published on ArXiv in 2003
before reaching its 33-th revision version in 2007, claiming that any unramified morphism from an irreducible
variety over an algebraically closed field of characteristic zero with only constant invertible regular functions to
an affine space of the same dimension over this field is an isomorphism. So this claim is irrevocably refuted by
Theorem 1 below, as well as by Oort cited theorem. The main theorem 4.8 of a second paper [24] of Susumu
Oda on the subject, first published on arXiv in 2007 before reaching its 49-th version in 2011, only added the
assumption of factoriality to the first algebraic variety of the main theorem of [23]. So this second main of Oda
is so irrevocably refuted by Theorem 1 below as this first one, and not by Oort cited theorem.

Before entering the subject, I would like to express my deep gratitude to Professors Arno van den Essen,
Hyman Bass, Victor Kulikov, V. Srinivas, Hans-Peter Kraft, and Harm Derksen for the fruitful conversations
that I had with them about Bass Generalized Jacobian Conjecture and Kulikov’s construction. I would also
like to thank all the participants of the convivial “Rencontre parisiennete autour de la Conjecture Jacobienne”
held at the University of Paris 6 on February 3, 1996, and during which the present solution has been exposed.
So, in spite of the relative oldness of the unpublished results presented in this solution, they are more actual
and useful than ever as “massive destruction arms” again false claims concerning unramified morphisms of
algebraic varieties.

**Kulikov surfaces and morphisms**

Let \( P = (1 : 1 : 1) \in \mathbb{P}_2 = \mathbb{P}_2(\mathbb{C}), (X_1, X_2, X_3) \) a system of indeterminates over \( \mathbb{C} \), \( Q_i = 3X_i^2 - X_1X_2 - X_1X_3 - X_2X_3 \) for \( 1 \leq i \leq 3 \), three quadratic forms defining three conics passing through \( P \), \( \phi \) the morphism
from \( \mathbb{P}_2 - \{ P \} \) to \( \mathbb{P}_2 \) whose homogeneous components are defined by the three previous forms, \( R \) the Zariski
closure in \( \mathbb{P}_2 \) of the set of ramification points of \( \phi \), which is the cubic with a node at \( P \) defined by the form
\[ \sum_{i \neq j} X_i^2X_j - 6X_1X_2X_3, \]
\( Q \) the generic linear combination with complex coefficients of the three previous forms, \( C \) the conic of \( \mathbb{P}_2 \) defined by \( Q \), passing by \( P \) and meeting transversely the cubic \( R \) at each point of
their intersection, and such that the image by \( \phi \) of the complement of \( C \) in \( \mathbb{P}_2 \) is contained in the complement
in \( \mathbb{P}_2 \) of a line \( L \) of \( \mathbb{P}_2 \), \( \sigma : \tilde{\mathbb{P}}_2 \to \mathbb{P}_2 \) the blowing-up the point \( P \) of \( \mathbb{P}_2 \), \( E \) the exceptional curve of \( \tilde{\mathbb{P}}_2 \), i.e.
\( \sigma^{-1}(P) \), \( \tilde{R} \) the strict transform of \( R \) by \( \sigma \), i.e. the irreducible curve of \( \tilde{\mathbb{P}}_2 \) such that \( \sigma^{-1}(R) = E \cup \tilde{R}, \tilde{C} \) the strict
transform of \( C \) by \( \sigma \), i.e. the irreducible curve of \( \tilde{\mathbb{P}}_2 \) such that \( \sigma^{-1}(C) = E \cup \tilde{C} \), and \( S \) the complement
of \( \tilde{R} \cup \tilde{C} \) in \( \mathbb{P}_2 \).

V. Kulikov proved in [18] that \( S \) is a rational and simply connected complex surface and that \( \phi \circ \sigma \) induces
an unramified morphism \( F_S \) of geometric degree 3 from \( S \) to \( \mathbb{C}^2 \cong \mathbb{P}^2 - L \).

We call such a \( S \) a “Kulikov surface” and such a \( F_S \) “the Kulikov morphism on \( S \)”.

Let us recall that the surface constructed by C.P. Ramanujam in [26] and known now as “Ramanujam
surface” is the complement, in the inverse image of \( P \) of \( \mathbb{P}_2 \) by the blowing up of one of its points \( P \), of the strict
transforms by this blowing up of an irreducible cubic of \( \mathbb{P}_2 \) with a cusp distinct from \( P \) and of an irreducible conic of \( \mathbb{P}_2 \) cutting transversely the cubic at \( P \) and meeting again the cubic at an unique other point with the
multiplicity 5. Ramanujam proved in [26] that his obviously non singular and rational surface is contractible,
more affine with only constant invertible regular functions (thanks for instance to [F], Corollary 2.5), factorial
(thanks for instance to [13], Theorem 1) and simply connected, but not simply connected at infinity, hence not
isomorphic to \( \mathbb{C}^2 \) (see [30] for more about this kind of surface).
We shall see in the following Theorem 1 that Kulikov surfaces share all the mentioned properties of Ramanujam surfaces with the eventual exception of the contractibility, thanks to the following Theorems 2 and 3.

**The surfaces $S(C_1, C_2, P)$ and their determinants**

Let $\mathbb{K}$ be an algebraically closed field, $\mathbb{P}_2 = \mathbb{P}_2(\mathbb{K})$, $C_1$ and $C_2$ two irreducible $\mathbb{K}$-algebraic curves of respective degrees $d_1$ and $d_2$, $P$ a point of one of them, $m_1$ and $m_2$ the respective multivities of $C_1$ and $C_2$ at $P$, and $M$ the matrix with lines $(d_1, m_1)$ and $(d_2, m_2)$, $\sigma: \mathbb{P}_2 \to \mathbb{P}_2$ the blowing-up of the point $P$ of $\mathbb{P}_2$, $E$ the exceptional curve of $\mathbb{P}_2$, i.e. $\sigma^{-1}(P)$, $\tilde{C}_1$ the strict transform of $C_1$ by $\phi$, i.e. the irreducible curve of $\mathbb{P}_2$ such that $\sigma^{-1}(C_1) = E \cup \tilde{C}_1$, and $\tilde{C}_2$ the strict transform of $C_2$ by $\sigma$, i.e. the irreducible curve of $\mathbb{P}_2$ such that $\sigma^{-1}(C_2) = E \cup \tilde{C}_2$.

We denote by $S(C_1, C_2, P)$ the complement of $\tilde{C}_1 \cup \tilde{C}_2$ in $\mathbb{P}_2$ and we call it “the surface deduced from $C_1$ and $C_2$ by blowing up at $P$”.

We denote by $\det(C_1, C_2, P)$ the determinant of the matrix $M$ and we call it “the determinant of the surface $S(C_1, C_2, P)$”.

**Theorem 1**

Any Kulikov surface is affine, non singular, rational, factorial, simply connected, but its fundamental group at infinity is infinite, and all its invertible regular functions are constant.

Hence, any Kulikov morphism gives a counter-example to Bass Generalised Jacobian Conjecture in any dimension greater than one, whereas this conjecture is true in dimension one.

**Proof**

Let $S$ be a Kulikov surface and $F_S$ the Kulikov morphism on $S$. It follows from [18] and the following theorems that $S$ is affine, non singular, rational, factorial, simply connected and that all its invertible regular functions are constant.

Let us now assume that $S$ isomorphic $\mathbb{C}^2$, and let $G$ be an isomorphism from $\mathbb{C}^2$ to $S$. So, $F_S \circ G$ is an unramified endomorphism of geometric degree 3 of $\mathbb{C}^2$, contrary to [22], Theorem 1.1. We deduce that $S$ is not isomorphic to $\mathbb{C}^2$. According to previous remarks and Theorem 2 bellow, the fundamental group at infinity of $S$ is infinite.

So, it is clear that $F_S$ gives a counter-example to Bass Generalized Jacobian Conjecture for any dimension greater than one.

Finally, let us consider a morphism $F: C \to \mathbb{C}$ satisfying the assumptions of Bass Conjecture. According Nagata’s refined version of Lüroth Theorem (see for instance [21], Theorem 4.12.2, p. 137), $C$ is rational. On the other hand, according to the unramifiedness of $F$ and the non singularity of $C$, $C$ is non singular (see for instance [6], Proposition 1.2 or more generally [27], Exposé I, Corollaire 9.11). So, $C$ is isomorphic to an open sub-variety of $\mathbb{C}$ (see for instance [16], Chapter 1, Exercice 6.1, p. 46). All invertible regular functions on $C$ being constant, it follows that this sub-variety of $\mathbb{C}$ is $\mathbb{C}$ itself, and hence that $F$ is an isomorphism, Q.E.D.

**Theorem 2**

Let $C_1$, $C_2$ be irreducible algebraic curves of the projective plane over an algebraically closed field $\mathbb{K}$ and $P$ a point of $C_1 \cup C_2$.

(i) $S(C_1, C_2, P)$ is a rational and non singular algebraic surface.

(ii) $S(C_1, C_2, P)$ is not affine if and only if each of $C_1$ and $C_2$ is a line (i.e. a curve of degree one) passing through $P$.

(iii) $S(C_1, C_2, P)$ is factorial if and only it is affine and $|\det(C_1, C_2, P)| = 1$.

(iv) If $\mathbb{K} = \mathbb{C}$, then $S(C_1, C_2, P)$ is isomorphic to $\mathbb{C}^2$ if and only if it is factorial, simply connected, and its fundamental group at infinity is finite.

**Proof**

1) Let us keep the notations of the definition of $S(C_1, C_2, P)$.

2) The statement (i) follows from the construction of $S(C_1, C_2, P)$.

3) Let us first remark that $\mathbb{P}_2$ being a ruled surface with invariant $e = 1$ (see for instance [16], Chapter V, example 2.11.5), any one of its irreducible curves distinct from its exceptional curve $E$ has a non-negative self-intersection number (it follows for instance from [16], Chapter V, Propositions 2.20 and 2.21).
3) Now, let $L$ be a line in $\mathbb{P}_2$ not passing by $P$ and $\tilde{L}$ its strict transform, i.e. inverse image by $\sigma$. For any divisor $D$ of an irreducible normal algebraic variety $X$, we denote by $\langle D \rangle$ its canonical image in the Picard group $\text{Pic} X$, and by $\sigma^*$ the canonical map from $\text{Pic} \mathbb{P}_2$ to $\text{Pic} \tilde{\mathbb{P}}_2$ induced by $\sigma$.

4) $\langle \tilde{L} \rangle = \sigma^*(\langle L \rangle)$ being a generator the group $\sigma^*(\text{Pic} \mathbb{P}_2)$ and $\text{Pic} \tilde{\mathbb{P}}_2$ being the direct sum of $\sigma^*(\text{Pic} \mathbb{P}_2)$ and $Z < E \subset \text{Pic} \tilde{\mathbb{P}}_2$, thanks to the splitness of the exact canonical sequence

$$Z < E \rightarrow \text{Pic} \tilde{\mathbb{P}}_2 \rightarrow \text{Pic} \mathbb{P}_2 \rightarrow \{0\}$$

we have for any irreducible curve $\tilde{C}$ of $\tilde{\mathbb{P}}_2$:

$$\langle \tilde{C} \rangle = (\deg \tilde{C}) \langle \tilde{L} \rangle - (\tilde{C}.E) < E >$$

(this follows for instance from [16], Chapter V, Example 1.4.2 and Propositions 3.2 and 3.6).

5) In particular, for $1 \leq i \leq 2$, since $\tilde{C}_i.E = m_i$ (see for instance [16], Chapter V, Corollary 3.7), we have:

$$\langle \tilde{C}_i \rangle = d_i < \tilde{L} > - m_i < E >$$

6) On the other hand we have:

$$\tilde{L}^2 = L^2 = 1, \tilde{L}.E = 0, E^2 = -1$$

(see for instance [16], Chapter V, Example 1.4.2 and Proposition 3.2)

7) So according to 5) and 6), the self-intersection number of the divisor $C_1 + C_2$, equal to $(d_1 + d_2)^2 - (m_1 + m_2)^2$, is not positive if and only if each of $C_1$ and $C_2$ is a line passing through $P$.

8) Similarly, according to 3), 4) and 6), for any irreducible curve $\tilde{C}$ of $\tilde{\mathbb{P}}_2$ distinct from $E$, with degree $d$ and intersection number $m$ with $E$, since $\tilde{C}^2 = d^2 - m^2$ is positive, so is $\tilde{C}.(\tilde{C} + \tilde{C}_2) = d(d_1 + d_2) - m(m_1 + m_2)$.

9) So the statement (ii) follows from 7) and 8) thanks to Nakai-Moishezon criterion of ampleness and Goodman criterion of affineness (see for instance [16], Chapter V, Theorem 1.10 and [17], Chapter II, Theorem 4.2).

10) Since the Picard group of the complement of an hypersurface of an irreducible non-singular variety is isomorphic to the residue group of the sub-group of the Picard group of the variety generated by the irreducible components of the hypersurface (it follows for instance from [H], Chapter II, Proposition 6.5 and Corollary 6.16), the statement (iii) follows from (i), (ii), and 5), thanks to the classical characterisation of the factoriality in terms of divisor class group (see for instance [16], Chapter II, Proposition 6.2).

11) Finally, the statement (iv) follows from the characterisation of the affine space over any algebraically closed field of characteristic 0 by the logarithmic Kodaira dimension (see for instance [20], Chapter I, Section 4, Theorem), thanks to the following Theorem 3 and the remark following the Theorem 1 of [14].

**Theorem 3**

Any invertible regular function on an irreducible normal simply connected variety over an algebraically closed field is constant if the characteristic of this field is 0 or if its divisor class group is trivial.

**Proof**

1) Let $K$ be such a field, $V$ such a variety, $\mathcal{O}_V$ the sheaf of regular functions on $V$, $K[U]$ the ring of regular functions on the open set $U \subset V$, $K[V]^*$ the group of invertible regular functions on $V$, and $T$ an indeterminate over $K$.

2) Let us assume that $K[V]^* \neq K^*$ and that the characteristic of $K$ or the divisor class group of $V$ is 0.

3) Let us remark that for any commutative domain $A$ with fractions field $K$, any element $a$ of $K$, any integer $n > 0$, and any indeterminate $X$ over $A$, the $A$-module $A[X]/(X^n - a)$ is without torsion, as it follows from the euclidian division of elements of $A[X]$ by $X^n - a$. So, the ring $A[X]/(X^n - a)$ is integral if and only if $X^n - a$ is irreducible in $K[X]$.

4) Now, according to classical sufficient conditions for such irreducibility (see for instance [L], Chapter VIII, Theorem 16 and Corollary 1), the liberty of the group $K[V]^*/K^*$ (see for instance [2], Proposition) and the assumption 2), there exists $f \in K[V]^*$ and an integer not divisible by the characteristic of $K$ such that $f$ does not a $n$-root in $K[V]^*$ and that for any affine open set $U \subset V$, the ring $K[U] \otimes_K K[T]/(1 \otimes T^n - f|U \otimes 1)$ is integral.
Corollary 3

Proof

So according to the choice of \( n \) and \( f \), \( W \) is an irreducible variety with the same dimension as \( V \) and \( \phi \) is unramified, hence étale thanks to the normality of \( V \) (see for instance [27], Exposé I, Corollary 9.11). So thanks to the obvious finiteness of \( \phi \), this one is an étale covering from the irreducible variety \( W \) to the irreducible simply connected variety \( V \), which means that \( \phi \) is an isomorphism.

7) Nevertheless, according to the definition of \( W \), \( f \) has a \( n \)-root in \( \mathbb{K}[V]^* \), contrary to the choice of \( f \).

8) So according to the absurdity of 2), we have the desired conclusion, Q.E.D.

Corollary 1

If \( S \) is a Kulikov surface, \( p \) a regular function on \( S \) which is a primitive element of the field extension induced by Kulikov morphism \( F_S \) on \( S \), \( S \) the variety \( \mathbb{C} \times S \), \( F_S \) the morphism from \( S \) to \( \mathbb{C}^3 \) such that \( F_S(x_0, x) = (x_0 + p(x), F_S(x)) \) for any \((x_0, x) \in \mathbb{C} \times S\), then \( F_S \) is an étale morphism which induces a non étale morphism \( \overline{F_S} \) from \( \{0\} \times S \) to the Zariski closure \( Z \) in \( \mathbb{C}^3 \) of the image of \( \overline{F_S} \).

Proof

1) Let us denote by \( \mathbb{C}[V] \) the ring of regular functions on the complex variety \( V \), \( \phi_S \) (resp. \( \phi_S^* \), resp. \( \overline{\phi_S} \)) the ring morphism induced by \( F_S \) (resp. \( F_S^* \), resp. \( \overline{F_S} \)), \( A \) the image of \( \phi_S \), \( B \) the ring of regular functions on \( S \), and \( X_0 \) an indeterminate over \( \mathbb{C} \).

2) The canonical map from \( \mathbb{C}[X_0] \otimes \mathbb{C}[\mathbb{C}^2] \) to \( \mathbb{C}[X_0] \otimes B \) induced by \( \phi_S \) being étale, so is \( \phi_S^* \), hence \( F_S^* \).

3) \( \mathbb{C}[\{0\} \times S] \) being canonically isomorphic to \( B \) and the canonic image of \( \overline{\phi_S}(\mathbb{C}[Z]) \) in \( B \) being \( A[p] \), the \( A \)-sub-algebra of \( B \) generated by \( p \), \( \overline{\phi_S} \) is étale if and only if \( B \) is étale over \( A[p] \).

4) But, according to the unramification of \( \phi_S \) proved in 2), \( \overline{\phi_S} \) also is unramified (see for instance [5], Chapter VI, Proposition 3.5).

5) So \( \overline{\phi_S} \) is étale if and only \( B \) is flat over \( A[p] \).

6) The conclusion follows from Theorem 1 and [11], Theorem 3.2, Q.E.D.

Corollary 2

If \( S \) is a Kulikov surface, \( \phi_S \) the ring morphism induced by Kulikov morphism on \( S \), \( A \) the image of \( \phi_S \), \( B \) the ring of regular functions on \( S \), and \( p \) any element of \( B \) which is a primitive element of the field extension induced by \( \phi_S \), then \( A[p] \), the \( A \)-sub-algebra of \( B \) generated by \( p \), is not normal (i.e. integrally closed), hence not unramified over \( A \).

Proof

It follows from theorem 1 and [11], Theorem 3.2, Q.E.D.

Corollary 3

With the same notations as in Corollary 2, the factorial sub-ring \( A \) of the factorial \( \mathbb{C} \)-affine domain \( B \) is not multiplicatively closed in \( B \), or equivalently there exists a prime element of \( A \) which is not prime in \( B \).

Proof

It follows from Theorem 1 and [3], Theorem 3.11, Q.E.D.

References

[1] K. ADJAMAGBO, On separable algebras over a U.F.D. and the Jacobian Conjecture in any Characteristic, in Automorphisms of Affine Spaces, A. van den Essen (ed.), 89-103, 1995, Kluwer Academic Publishers, Netherlands.

[2] K. ADJAMAGBO, Sur les morphismes injectifs et les isomorphismes des variétés algébriques affines, Communications in Algebra, 24 (3), 1117-1123 (1996).
[3] K. ADJAMAGBO, On isomorphisms of factorial domains and the Jacobian Conjecture in any Characteristic, Prépublication 91, Octobre 1996, Institut de Mathématiques de Jussieu, Université Paris 6 et Université Paris 7.

[4] K. ADJAMAGBO, A. van den ESSEN, Eulerian systems of partial differential equations and the Jacobian Conjecture, Journal of Pure and Applied Algebra 74 (1991) 1-15.

[5] A. ALTMAN, S. KLEIMAN, Introduction to Grothendieck Duality Theory, Lecture Notes in Mathematics 146, Springer-Verlag, Berlin, 1970.

[6] H. BASS, Differential Structure of étale Extensions of Polynomial Algebras, in Commutative Algebra, Proceedings of a Microprogram Held, June 15-July 2, 1987, M. Hoster, C. Hunecke, J.D. Sally, ed., Springer-Verlag, New York, 1989.

[7] H. BASS, E. CONNELL, D. WRIGHT, The Jacobian Conjecture: Reduction of Degree and formal Expansion of the Inverse, Bull. AMS 7(1982), 287-330

[8] J. DIEUDONNE, A. GROTHENDIECK, Critères différentielles de régularité pour les localisés des algèbres analytiques, J. Algebra 5, 1967, 305-324.

[9] A. GROTHENDIECK, J. DIEUDONNE, Eléments de Géométrie Algébrique, IV, Troisième Partie, Publications Mathématiques N° 28, 1966, Institut des Hautes Etudes Scientifiques.

[10] A. GROTHENDIECK, J. DIEUDONNE, Eléments de Géométrie Algébrique, IV, Quatrième Partie, Publications Mathématiques N° 32, 1967, Institut des Hautes Etudes Scientifiques.

[11] T. FUJITA, On the topology of non-complete surfaces, J. Fac. Sci. Univ. Tokyo 29 (1982) 503-566.

[12] G. FISCHER, Complex Analytic Geometry, Lecture Notes in Mathematics, Vol. 538, Springer-verlag, Berlin-New York, 1976.

[13] R. V. GURJAR, Affine varieties dominated by $\mathbb{C}^2$, Comment. Math. Helvetici 55(1980) 378-389.

[14] R. V. GURJAR, M. MIYANISHI, Affine surface with $\overline{\mathbb{R}} \leq 1$, in Algebraic Geometry and Commutative Algebra in Honour of Misayoshi NAGATA, 1987, 99-124.

[15] Alexander GROTHENDIECK, Hamet SEYDI, Platitude d’une adhérence schématique et lemme de Hironaka généralisé, Manuscripta Math. 5 (1971), 323-339.

[16] R. HARTSHORNE, Algebraic Geometry, G.T.M. 52, Springer-Verlag, 1977.

[17] R. HARTSHORNE, Ample Subvarieties of Algebraic Varieties, Lecture Notes in Math. 156, Springer-verlag, Berlin, 1970.

[18] V. KULIKOV, Generalised and local jacobian problems, Russian Acad. Sci. Izv. Math. Vol. 41, 1993, No. 2, 351-365.

[19] S. LANG, Algebra, Addison-Wesley, 8th Printing, 1985.

[20] M. MIYANISHI, Non-complete Algebraic Surfaces, Lecture Notes in Math. 857, Springer-Verlag, Berlin, 1981.

[21] M. NAGATA, Field Theory, Marcel Dekker, Inc., New York, 1977.

[22] S. YU. OREVKOV, On three-sheeted polynomial mappings of $\mathbb{C}^2$, Math. URSS Izv., Vol. 29, 1897, No. 3, 587-596.

[23] S. ODA, The Last Aproach to the Settlement of the Jacobian Conjecture, arXiv:math/0307080, 2003-2007.

[24] S. ODA, A Valuation Theoretic Approach to the Jacobian Conjecture, arXiv:0706.1138, 2007-2011.
[25] F. OORT, Units in number fields and function fields, Exposition. Math. 17 (1999), no. 2, 97-115.

[26] C. P. RAMANUJAM, A topological characterisation of the affine plane as an algebraic variety, Ann. of Math. 94 (1971) 69-88.

[27] A. GROTHENDIECK, Séminaire de Géométrie Algébrique, 1960-1961, I.H.E.S., 1960, Fascicule 1.

[28] H. SEYDI, La Conjecture Jacobienne, Rend. Sem. Mat. Messina, dere II, Vol. VI (1999), 175-179.

[29] H. SEYDI, La Conjecture Jacobienne II, Afr. diaspora J. Math. (N.S.) 10 (2010), no. 1, 87-121.

[30] M. G. ZAIDENBERG, On Ramanujam surfaces, $\mathbb{C}^{**}$-families, and exotic algebraic structures on $\mathbb{C}^n$. Trans. Moscow Math. Soc., 1994.