Research Article

Dynamical Behavior of Stochastic Markov Switching Hepatitis B Epidemic Model with Saturated Incidence Rate

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Received 18 January 2021; Accepted 24 December 2021; Published 17 January 2022

1. Introduction

Hepatitis B virus is a severe infectious disease that has emerged as one of the greatest threats to human health in the 21st century. An estimated 350 million people worldwide have been infected with hepatitis B virus [1]. The mathematical model to describe hepatitis B virus transmission and its control [2–5]. Recently, Khan et al. [6] investigated a hepatitis B epidemic model with saturated incidence rate:}

\[ \begin{align*}
\frac{dS}{dt} &= \Lambda - \frac{\alpha SI}{1 + \gamma I} - (\mu_0 + \nu)S, \\
\frac{dI}{dt} &= \frac{\alpha SI}{1 + \gamma I} - (\mu_0 + \mu_1 + \beta)I, \\
\frac{dR}{dt} &= \beta I + \nu S - \mu_0 R,
\end{align*} \]

with \( S(0) > 0, I(0) > 0, \) and \( R(0) > 0. \) In model (1), the birth rate is denoted by \( \Lambda. \) The transmission rate of hepatitis B is given by \( \alpha, \) while \( \mu_0 \) and \( \mu_1, \) respectively, demonstrated the natural and disease-induced death rates. Recovery rate is denoted by \( \beta, \) while the vaccination and saturation rates are \( \nu \) and \( \gamma, \) respectively. According to the theory in [6], model (1) always has the disease-free equilibrium \( E^0 = (S^0, 0, R^0), \) where the components are defined as \( S^0 = \Lambda/(\mu_0 + \nu), \) and \( R^0 = \Lambda(\mu_0(\mu_0 + \nu)). \) If \( R_0 < 1, \) \( E^0 \) is globally asymptotically stable. If \( R_0 > 1, \) \( E^0 \) is unstable and there exists an endemic equilibrium \( E^* = (S^*, I^*, R^*) \) which is globally asymptotically stable, where \( R_0 = \alpha\Lambda/(\mu_0 + \nu)(\mu_0 + \mu_1 + \beta). \)

In fact, epidemic models are inherently subject to a continuous spectrum of disturbances [7–11]. Many authors demonstrated that the white noise and colored noise have a great destabilizing influence on the epidemic transmission. Moreover, considering the effect of environment noise on the epidemic model has become a popular trend in controlling the spread of disease [12–16]. In this respect, some researches on stochastic hepatitis B virus models have been reported [17–19]. Particularly, in the epidemic model, the disease transmission rate \( \alpha \) represents an extremely important coefficient [16, 20]. In this paper, by taking into account the effect of continuous-time Markov chain on the transmission rate \( \alpha, \) we consider a...
stochastic analogue of the deterministic model (1):

\[
\begin{align*}
\text{d}S &= \left( \Lambda - \frac{\alpha(x(t)) S I}{1 + y I} - (\mu_0 + \nu) S \right) dt + \sigma_1(x(t)) S dB_1(t), \\
\text{d}I &= \left( \alpha(x(t)) S I - (\mu_0 + \mu_1 + \beta) I \right) dt + \sigma_2(x(t)) I dB_2(t), \\
\text{d}R &= (\beta I + \nu S - \mu_0 R) dt + \sigma_3(x(t)) R dB_3(t),
\end{align*}
\]

(2)

where \(B_i(t)\) are independent standard Brownian motions and \(\sigma_i^2\) stand for the intensities of \(B_i(t), i = 1, 2, 3\). \(x(t), t \geq 0\), is a right-continuous Markov chain on the complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with values in a finite space \(\mathcal{M} = \{1, 2, \ldots, N\}\) (see [21, 22]).

It is widely known that the stability of biomathematical model has always been a hot issue in recent years [23–26]. Compared with their corresponding deterministic cases, lots of stochastic models have no traditional positive equilibrium state. Consequently, the research of ergodic stationary distribution of s stochastic biomathematical model has been a research highlight. In addition, model (2) incorporates white noise as well as colored noise possessing important practical significance [27]. The main aim of this article is to prove the existence of stationary distribution for model (2). Above all, to guarantee existence and uniqueness of globally positive solution for model (2), we establish the following conclusion. Since the proof is standard, we omit it here.

**Lemma 1.** For any initial value \((S(0), I(0), R(0), \xi(0)) \in \mathbb{R}^4_+ \times \mathcal{M}\), there exists a unique positive solution \((S(t), I(t), R(t), \xi(t)) \in \mathbb{R}^4_+ \times \mathcal{M}\) of model (2) on \(t \geq 0\) almost surely (a.s.).

### 2. Existence of a Unique and Ergodic Stationary Distribution

**Theorem 2.** If \(R_0^* > 1\), where

\[
R_0^* = \frac{\sum_{k \in \mathcal{M}} \alpha_k \Lambda}{(\mu_0 + \nu + \sum_{k \in \mathcal{M}} \pi_k \sigma_1^2(k)/2)(\mu_0 + \mu_1 + \beta + \sum_{k \in \mathcal{M}} \pi_k \sigma_2^2(k)/2)}.
\]

(3)

then for any initial value \((S(0), I(0), R(0), \xi(0)) \in \mathbb{R}^4_+ \times \mathcal{M}\), model (2) has a unique stationary distribution which is ergodic.

**Proof.** In order to prove Theorem 2, we need to validate that the feasibility of (A1), (A2), and (A3) in Lemma 7 in the appendix holds. We have assumed (A1) holds in Section 1. To verify (A3), we need to find a nonnegative \(C^2\)-function \(V(S, I, R, k)\) and a compact set \(D_k \subset \mathbb{R}^4_+\) such that \(LV \leq -1\) for all \((S, I, R, k) \in (\mathbb{R}^4_+ \setminus D_k) \times \mathcal{M}\). Construct a \(C^2\)-function

\[
V(S, I, R, k) = M(-c_1 \ln S - c_2 \ln I + \rho(k) + (S + I + R)^{\rho+1} - \ln S - \ln I - \ln R = MV_1 + V_2 + V_3 + V_4 + V_5,
\]

(4)

where \(V_1 = -c_1 \ln S - c_2 \ln I + \rho(k), V_2 = (S + I + R)^{\rho+1}, V_3 = -\ln S, V_4 = -\ln I, V_5 = -\ln R, 0 < \rho < 2\mu_0/\max\{\sigma_i\}^2\), where \(\sigma_i = \max_{k \in \mathcal{M}} \{\sigma_i(k)\}\), and constants \(M, c_1, c_2, \text{compact set } D_k\) and function \(\rho(k)\) will be determined later. Employing Itô’s formula [28–34], we can get

\[
\mathcal{L}V_1 = -\frac{c_1 A}{S} + \frac{c_1 \alpha(k) I}{1 + y I} + c_1 \left( \mu_0 + \nu + \frac{1}{2} \sigma_1^2(k) \right) - \frac{c_1 \alpha(k) S}{1 + y I} + c_2 \left( \mu_0 + \mu_1 + \beta + \frac{1}{2} \sigma_2^2(k) \right)
\]

(5)

Choose \(M_1(k) = -3 \sqrt{\frac{c_1 \alpha(k) \Lambda}{c_1 \alpha(k) \Lambda}} + 1 + c_1(\mu_0 + \nu + (1/2) \sigma_1^2(k)) + c_2(\mu_0 + \mu_1 + \beta + (1/2) \sigma_2^2(k))\); on the basis of the irreducibility of generator matrix \(\Gamma\), one can find that for \(\Theta = (\Theta(1), \Theta(2), \ldots, \Theta(N))\), there exists \(\rho = (\rho(1), \rho(2), \ldots, \rho(N))^T\) satisfying the following Poisson system \(\Gamma \rho = (\sum_{k \in \mathcal{M}} \pi_k \Theta(k)) \Gamma - \Theta\). Let \(c_1\) and \(c_2\) satisfy

\[
c_1 \left( \mu_0 + \nu + \sum_{k \in \mathcal{M}} \pi_k \sigma_1^2(k)/2 \right) = c_2 \left( \mu_0 + \mu_1 + \beta + \sum_{k \in \mathcal{M}} \pi_k \sigma_2^2(k)/2 \right)
\]

(6)

Then,

\[
\mathcal{L}V_1 \leq -\sum_{k \in \mathcal{M}} \alpha_k \Lambda \left( \mu_0 + \nu + \sum_{k \in \mathcal{M}} \pi_k \sigma_1^2(k)/2 \right) \left( \mu_0 + \mu_1 + \beta + \sum_{k \in \mathcal{M}} \pi_k \sigma_2^2(k)/2 \right)
\]

(7)
where
\[
\lambda = R_b^5 - 1, \\
\varphi(I) = yI + c_i \tilde{\alpha}I,
\]
and set \( \tilde{\alpha} = \max_{k \in \mathcal{D}} \{ \alpha(k) \} \). Applying Itô’s formula, one can obtain
\[
\mathcal{L} V_2 = (\rho + 1)(S + I + R)^\rho (\Lambda - \mu S - (\mu_0 + \mu_1 + \mu_2) I - \mu R)
\]
\[
+ \frac{1}{2} \rho \rho (\rho + 1)(S + I + R)^{-1} (\sigma_1^2(k) S + \sigma_2^2(k) I^2)
\]
\[
+ \sigma_3^2(k) R^2 \leq (\rho + 1)(S + I + R)^\rho (\Lambda - \mu_0(S + I + R))
\]
\[
+ \max_{i=1,2,3} \{ \sigma_i^2 \} \frac{\rho}{2} (\rho + 1)(S + I + R)^{\rho - 1/2} = \Lambda (\rho + 1)(S + I + R)^\rho
\]
\[
- (\rho + 1) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \sigma_i^2 \} \right) (S + I + R)^{\rho - 1/2}
\]
\[
\leq B - \frac{1}{2} (\rho + 1) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \sigma_i^2 \} \right) (S + I + R)^{\rho - 1/2}
\]
\[
\leq B - \frac{1}{2} (\rho + 1) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \sigma_i^2 \} \right) (S + I + R)^{\rho - 1/2}
\]
\[
\leq \sup_{(S,I,R) \in \mathbb{R}^3} \left\{ \theta - \frac{1}{2} (\rho + 1) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \sigma_i^2 \} \right) (S + I + R)^{\rho - 1/2} \right\} < \infty.
\]
Denote
\[
\theta = B + (\mu_0 + \nu + (1/2) \sigma_1^2) + (\mu_0 + \nu + \beta + (1/2) \sigma_2^2)
\]
\[
+ (\mu_0 + (1/2) \sigma_3^2).
\]
By using Itô’s formula, we also have
\[
\mathcal{L} V_3 = -\frac{\Lambda}{S} + \frac{\alpha(k) I}{1 + yI} + \mu_0 + \nu + \frac{1}{2} \sigma_1^2(k),
\]
\[
\mathcal{L} V_4 = -\frac{\alpha(k) S}{1 + yI} + \mu_0 + \mu_1 + \beta + \frac{1}{2} \sigma_2^2(k),
\]
\[
\mathcal{L} V_5 = -\frac{\beta I}{R} - \frac{\nu S}{R} + \mu_0 + \frac{1}{2} \sigma_3^2(k).
\]
Hence, by (7), (9), and (12), we get
\[
\mathcal{L} V \leq -M \lambda + M \varphi(I) + \tilde{\alpha} I - \frac{\Lambda}{S} - \frac{\alpha S}{1 + yI} - \frac{\beta I}{R} - \frac{\nu S}{R} + \theta
\]
\[
- \frac{1}{2} (\rho + 1) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \sigma_i^2 \} \right) (S + I + R)^{\rho - 1/2},
\]
where \( \tilde{\alpha} = \min_{k \in \mathcal{D}} \{ \alpha(k) \} \). Here, we choose that the positive constant \( M \) satisfies the following inequality:
\[
-M \lambda + C \leq -2.
\]
For arbitrary \( \varepsilon > 0 \), define the following bounded closed set:
\[
D_{\varepsilon} = \left\{ \varepsilon \leq S \leq \frac{1}{\varepsilon}, I \leq \frac{1}{\varepsilon}, \hat{\varepsilon} \leq I \leq \frac{1}{\varepsilon}, \hat{\varepsilon} I \leq \frac{1}{\varepsilon} \right\},
\]
where \( \varepsilon \) satisfies the following conditions:
\[
-\frac{A}{\varepsilon} + K \leq -1,
\]
\[
-M \lambda + M \varphi(\varepsilon) + \tilde{\alpha} \varepsilon + C \leq -1,
\]
\[
-\frac{\beta}{\varepsilon} + K \leq -1,
\]
\[
-\frac{1}{2} (\rho + 1) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \sigma_i^2 \} \right) \frac{1}{\varepsilon^{\rho - 1/2}} + D \leq -1,
\]
\[
-\frac{1}{2} (\rho + 1) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \sigma_i^2 \} \right) \frac{1}{\varepsilon^{\rho - 1/2}} + E \leq -1,
\]
\[
-\frac{1}{2} (\rho + 1) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \sigma_i^2 \} \right) \frac{1}{\varepsilon^{\rho - 1/2}} + F \leq -1,
\]
where
\[
K = \sup_{(S,I,R) \in \mathbb{R}^3} \{ M \varphi(I) + \tilde{\alpha} I + C \},
\]
\[
D = \sup_{(S,I,R) \in \mathbb{R}^3} \left\{ M \varphi(I) + \tilde{\alpha} I + \theta - \frac{1}{2} (\rho + 1) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \sigma_i^2 \} \right) (I + R)^{\rho - 1/2} \right\},
\]
\[
E = \sup_{(S,I,R) \in \mathbb{R}^3} \left\{ M \varphi(I) + \tilde{\alpha} I + \theta - \frac{1}{2} (\rho + 1) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \sigma_i^2 \} \right) (S + R)^{\rho - 1/2} \right\},
\]
\[
F = \sup_{(S,I,R) \in \mathbb{R}^3} \left\{ M \varphi(I) + \tilde{\alpha} I + \theta - \frac{1}{2} (\rho + 1) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \sigma_i^2 \} \right) \left( S^{\rho - 1/2} + R^{\rho - 1/2} \right) \right\}.
\]
Furthermore,
\[
\mathbb{R}_+^3 \setminus D_{\varepsilon} = D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6,
\]
where

\[
D_1 = \{(S, I, R) \in \mathbb{R}^3, 0 < S < \varepsilon\},
\]
\[
D_2 = \{(S, I, R) \in \mathbb{R}^3, 0 < I < \varepsilon\},
\]
\[
D_3 = \{(S, I, R) \in \mathbb{R}^3, 0 < R < \varepsilon, S > \varepsilon, I > \varepsilon\},
\]
\[
D_4 = \{(S, I, R) \in \mathbb{R}^3, S > \frac{1}{\varepsilon}\},
\]
\[
D_5 = \{(S, I, R) \in \mathbb{R}^3, I > \frac{1}{\varepsilon}\},
\]
\[
D_6 = \{(S, I, R) \in \mathbb{R}^3, R > \frac{1}{\varepsilon}\}.
\]

(19)

Case 1. If \((S, I, R) \in D_1\), we derive that

\[
\mathcal{L}V \leq -\lambda_1 + \lambda_1 - \int \frac{\alpha S}{1 + y} - \frac{\beta I}{R} - \frac{\alpha S}{1 + y} + \theta
\]

\[
- \frac{1}{2} \left( \rho + 1 \right) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \bar{\sigma}_i^2 \} \right) \left( S^{\rho+1} + I^{\rho+1} + R^{\rho+1} \right)
\]

\[
\leq -\frac{\lambda_1}{\varepsilon} + K \leq -1.
\]

(20)

Case 2. If \((S, I, R) \in D_2\), we have

\[
\mathcal{L}V \leq -\lambda_1 + \lambda_1 - \int \frac{\alpha S}{1 + y} - \frac{\beta I}{R} - \frac{\alpha S}{1 + y} + \theta
\]

\[
- \frac{1}{2} \left( \rho + 1 \right) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \bar{\sigma}_i^2 \} \right) \left( S^{\rho+1} + I^{\rho+1} + R^{\rho+1} \right)
\]

\[
\leq -\frac{\lambda_1}{\varepsilon} + \theta \leq -1.
\]

(21)

Case 3. If \((S, I, R) \in D_3\), we compute

\[
\mathcal{L}V \leq -\lambda_1 + \lambda_1 - \int \frac{\alpha S}{1 + y} - \frac{\beta I}{R} - \frac{\alpha S}{1 + y} + \theta
\]

\[
- \frac{1}{2} \left( \rho + 1 \right) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \bar{\sigma}_i^2 \} \right) \left( S^{\rho+1} + I^{\rho+1} + R^{\rho+1} \right)
\]

\[
\leq -\frac{\beta}{\varepsilon} + K \leq -1.
\]

(22)

Case 4. If \((S, I, R) \in D_4\), we derive

\[
\mathcal{L}V \leq -\lambda_1 + \lambda_1 - \int \frac{\alpha S}{1 + y} - \frac{\beta I}{R} - \frac{\alpha S}{1 + y} + \theta
\]

\[
- \frac{1}{2} \left( \rho + 1 \right) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \bar{\sigma}_i^2 \} \right) \left( S^{\rho+1} + I^{\rho+1} + R^{\rho+1} \right)
\]

\[
\leq -\frac{1}{2} \left( \rho + 1 \right) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \bar{\sigma}_i^2 \} \right) \frac{1}{\varepsilon^{\rho+1}} + D \leq -1.
\]

(23)

Case 5. If \((S, I, R) \in D_5\), we conclude

\[
\mathcal{L}V \leq -\lambda_1 + \lambda_1 - \int \frac{\alpha S}{1 + y} - \frac{\beta I}{R} - \frac{\alpha S}{1 + y} + \theta
\]

\[
- \frac{1}{2} \left( \rho + 1 \right) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \bar{\sigma}_i^2 \} \right) \left( S^{\rho+1} + I^{\rho+1} + R^{\rho+1} \right)
\]

\[
\leq -\frac{1}{2} \left( \rho + 1 \right) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \bar{\sigma}_i^2 \} \right) \frac{1}{\varepsilon^{\rho+1}} + E \leq -1.
\]

(24)

Case 6. If \((S, I, R) \in D_6\), we have

\[
\mathcal{L}V \leq -\lambda_1 + \lambda_1 - \int \frac{\alpha S}{1 + y} - \frac{\beta I}{R} - \frac{\alpha S}{1 + y} + \theta
\]

\[
- \frac{1}{2} \left( \rho + 1 \right) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \bar{\sigma}_i^2 \} \right) \left( S^{\rho+1} + I^{\rho+1} + R^{\rho+1} \right)
\]

\[
\leq -\frac{1}{2} \left( \rho + 1 \right) \left( \mu_0 - \frac{\rho}{2} \max_{i=1,2,3} \{ \bar{\sigma}_i^2 \} \right) \frac{1}{\varepsilon^{\rho+1}} + F \leq -1.
\]

(25)

Then, we can obtain that for a sufficiently small \(\varepsilon\), \(LV < -1\) for any \((S, I, R) \in \mathbb{R}^3 \setminus \{D_2\}\). Therefore, we can verify (A3) in Lemma 7 of the appendix. On the other hand, the diffusion matrix \(D(x, k) = \text{diag} \{ \sigma_1^2(k)S, \sigma_2^2(k)I, \sigma_3^2(k)R^2 \}\) of model (2) is positive definite, which implies that condition (A2) in Lemma 7 holds. This completes the proof.  

Now, consider the corresponding model (2) without Markov switching:

\[
\begin{align*}
\frac{dS}{dt} &= \left( \lambda - \frac{\alpha S}{1 + y} - (\mu_0 + \nu)S \right) dt + \sigma_1 S dB_1(t), \\
\frac{dI}{dt} &= \left( \frac{\alpha S}{1 + y} - (\mu_0 + \mu + \beta)I \right) dt + \sigma_2 I dB_2(t), \\
\frac{dR}{dt} &= (\beta I + \nu S - \mu_0 R) dt + \sigma_3 R dB_3(t).
\end{align*}
\]

Define a parameter

\[
\bar{R}_0 = \alpha \int_0^\infty x \pi(x) dx \\
\mu_0 + \mu_1 + \beta + (\sigma_3^2/2),
\]

(27)

where

\[
\pi(x) = Q x^{-2 \left( 2(\mu_0 + \nu) \right) \sigma_1^2 \left( 2(\mu_0 + \nu) \right) \sigma_2^2 \left( 2(\mu_0 + \nu) \right) \sigma_3^2 \left( 2(\mu_0 + \nu) \right)} \\
- (2\sigma_1^2) \left( \left( \lambda x \right) + \left( \mu_0 + \nu \right) \right), \quad x \in (0, \infty).
\]

Similar to Theorem 3.1 in [35], it is easy to obtain the following result.

**Theorem 3.** Let \((S(t), I(t), R(t))\) be the solution of model (26). If \(R_0 < 1\), for any initial value \((S(0), I(0), R(0)) \in \mathbb{R}^3\),
In Theorem 2, we derive Remark 4.

and the distribution of model (2) for $k \in \mathcal{M} = \{1, 2, 3\}$. The initial value $S(0) = 0.8$, $I(0) = 0.7$, and $R(0) = 1.1$. Step size $\Delta t = 0.001$.

then the solution $(S(t), I(t), R(t))$ of model (26) satisfies

$$\lim_{t \to +\infty} I(t) = 0 \text{ a.s.},$$

and the distribution of $S(t)$ converges weakly to the measure which has the density

$$\pi(x) = Qx^{-2-((2(\mu_0+\nu))/\sigma_i^-)\sigma_i^- - 2((2(\mu_0+\nu))/\sigma_i^-)}/\sigma_i^- (\Lambda/x + (\mu_0 + \nu)), \quad x \in (0, +\infty),$$

where $Q$ is a constant such that $\int_0^\infty \pi(x)dx = 1$.

Remark 4. In Theorem 2, we derive $R_0^k = R_0$ when $a(k) \equiv a$ and $\sigma_i(k) \equiv 0$. This conclusion accords with practice.

3. Numerical Examples

In this section, we will test our theory conclusion by Milstein’s higher order method in [36].

Example 1. Let the generator of the Markov chain $\zeta_{ij}$ be

$$\Gamma = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1/4 & 1/4 & -1/2 \end{pmatrix},$$

in which $\zeta_{ij}$ is a right-continuous Markov chain taking value in $\mathcal{M} = \{1, 2, 3\}$. By solving the linear equation $\pi \Gamma = 0$, we obtain the unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, \pi_3) = (2/7, 3/7, 2/7)$. Choose parameters $\Lambda = 0.232$, $\gamma = 0.9$, $\mu_0 = 0.000232$, $\nu = 0.02$, $\mu_1 = 0.0000547$, $\beta = 0.12$, $\alpha(1) = 0.0013$, $\alpha(2) = 0.00129$, $\alpha(3) = 0.00132$, $\sigma_1(1) = 0.01$, $\sigma_2(1) = 0.01$, $\sigma_3(1) = 0.01$, $\sigma_1(2) = 0.011$, $\sigma_2(2) = 0.022$, $\sigma_3(2) = 0.055$, $\sigma_1(3) = 0.009$, $\sigma_2(3) = 0.019$, and $\sigma_3(3) = 0.063$. Then, $R_0^k = 1.2226 > 1$. In view of Theorem 2, there is a stationary distribution of model (2), and it is ergodic. Phase portrait of $(S(t), I(t), R(t))$ and histograms of $(S(t), I(t), R(t))$ are plotted in Figure 1.

Example 2. Select parameters $\Lambda = 0.232$, $\gamma = 0.9$, $\mu_0 = 0.000232$, $\nu = 0.02$, $\mu_1 = 0.0000547$, $\beta = 0.12$, $\alpha = 0.04$, $\sigma_1 = 0.1$, $\sigma_2 = 0.08$, and $\sigma_3 = 0.05$. By calculation, $R_0 = \alpha \Lambda / (\nu \beta)$.

Figure 1: $S(t)$, $I(t)$, and $R(t)$ have ergodic property. The pictures in (a) are Markovian chain. The pictures in (c) are the density functions of model (2) for $k \in \mathcal{M} = \{1, 2, 3\}$. The initial value $S(0) = 0.8$, $I(0) = 0.7$, and $R(0) = 1.1$. Step size $\Delta t = 0.001$. 
functions of the classes $S$: $\int R(t) \, dx = 1.16$, and $\int R(t) \, dx = 0.377 < 1$. It means that there exists a unique endemic equilibrium of determined model (1), which is globally asymptotically stable. Instead, in view of Theorem 3, we have $\lim_{t \to \infty} I(t) = 0$ a.s. and the distribution of $S(t)$ in model (26) converges weakly to the measure $\pi(x)$ (see Figure 2).

4. Concluding Remarks

The paper successfully investigates extinction and stationary distribution of a stochastic Markov switching hepatitis B epidemic model with saturated incidence rate. Besides the effect of Markovian switching on the deterministic SIRS epidemic models [37–39], pulse vaccination strategy (PVS) has been adopted to control the outbreaks and fastly tackle the spread of disease by wide areas [40]. In order to help future research, we propose the following definition related to SIR model by taking into account Markovian switching, impulse, and infinite delay.

**Definition 5.** Considering the following impulsive stochastic functional differential equation with Markovian switching (ISFDM),

$$
\begin{align*}
\text{d}Y(t) &= F_1(t, \xi(t), Y(t), \int_{-\infty}^{0} Y(t + \theta) \mu_1(\theta) \text{d}\theta) \, \text{d}t + F_2(t, \xi(t), Y(t), \int_{-\infty}^{0} Y(t + \theta) \mu_2(\theta) \text{d}\theta) \, \text{d}B(t), \\
& \text{for } t \neq t_k, \quad k \in \mathbb{N}, \\
Y(t_k^+) - Y(t_k^-) &= H_k Y(t_k), \quad k \in \mathbb{N},
\end{align*}
$$

where $Y(t + \theta), -\infty < \theta \leq 0$, represents $C_0$-value stochastic process, $C_g = \{ \psi \in C((\infty, 0]; \mathbb{R}^d) : \| \psi \|_{C_g} = \sup_{-\infty < \theta < 0} e^{\theta t} \| \psi(s) \| < +\infty \}, \quad g(s) = e^{-\| \psi(s) \|} = \sqrt{\psi_1^2(s) + \psi_2^2(s) + \cdots + \psi_d^2(s)},$ and $(\psi_1(s), \psi_2(s), \cdots, \psi_d(s)) \in \mathbb{R}^d$. $H_k > -1$, $\xi(t)$ denotes the regime switching [41, 42]. For $i = 1, 2$, $\mu_i(\theta)$ is a measure on $(-\infty, 0]$, $0 < t_1 < t_2 < \cdots$, $\lim_{k \to \infty} t_k = +\infty$. The initial condition $Y_0 \in C_g$ and $\zeta(0) = 0$, where $Y_0 = \theta = \{ \theta(\theta) : -\infty < \theta \leq 0 \}$.
Lemma 7 ([22]). If the following conditions are satisfied:

(A1) $\theta_{ij} > 0$ for any $i \neq j$.

(A2) For each $k \in \mathcal{M}$, $D(x, k) = (d_{ij}(x, k))$ is symmetric and satisfies $\lambda |\omega|^2 \leq (D(x, k)\omega, \omega) \leq \lambda^{-1} |\omega|^2$ for all $\omega \in \mathbb{R}^n$, with some constant $\lambda \in (0, 1)$ for all $x \in \mathbb{R}^n$.

(A3) There exists a nonempty open set $\mathcal{D}$ with compact closure, satisfying that, for each $k \in \mathcal{M}$, there is a nonnegative function $V(\cdot, k): \mathcal{D} \longrightarrow \mathbb{R}$ such that $V(x, k)$ is twice continuously differentiable and that for some $\alpha > 0$, $\mathcal{L}^2 V(x, k) \leq -\alpha, \ (x, k) \in \mathcal{D} \times \mathcal{M}$, then $(x(t), \xi(t))$ of system (A1) is positive recurrent and ergodic. That is to say, there exists a unique stationary distribution.

Data Availability

No data were used in this study.

Conflicts of Interest

The author declares that there are no competing interests.

Acknowledgments

This work was supported by grants from the Natural Science Foundation of Shandong Province of China (No. ZR2018MA023) and a Project of Shandong Province Higher Educational Science and Technology Program of China (No. J16L109).

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