Exponential Euler and Backward Euler Methods for Nonlinear Heat Conduction Problems

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Abstract—In this paper a variant of nonlinear exponential Euler scheme is proposed for solving nonlinear heat conduction problems. The method is based on nonlinear iterations where at each iteration a linear initial-value problem has to be solved. We compare this method to the backward Euler method combined with nonlinear iterations. For both methods we show monotonicity and boundedness of the solutions and give sufficient conditions for convergence of the nonlinear iterations. Numerical tests are presented to examine performance of the two schemes. The presented exponential Euler scheme is implemented based on restarted Krylov subspace methods and, hence, is essentially explicit (involves only matrix-vector products).

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1. INTRODUCTION AND PROBLEM SETTING

We consider a class of nonlinear heat conduction problems

\[
\frac{\partial}{\partial t} u(x, t) = \nabla \cdot (k(u) \nabla u(x, t)) + g(x, t), \quad u(x, 0) = u^0(x), \quad u(x, t)|_{\partial \Omega} = b(x, t),
\]

where \(x \in \Omega \subset \mathbb{R}^d\), \(d \in \{1, 2, 3\}\), \(t \in [0, T]\), \(\nabla u\) denotes the gradient of \(u(x, t)\), \(\nabla \cdot\) is the divergence operator and functions \(u^0(x), b(x, t)\) and \(g(x, t)\) are given. Furthermore, we have

\[
k(u) = k_0 u^\sigma,
\]

with given constants \(k_0 > 0\) and \(\sigma > 0\), and it holds

\[
b(x, t) \geq 0, \quad g(x, t) \geq 0, \quad u^0(x) \geq 0, \quad x \in \Omega, \quad t \geq 0.
\]

We assume that a suitable spatial discretization of initial-boundary problem (IVP) in (1) leads to initial-value problem

\[
y'(t) = -A(y(t))y(t) + g(t), \quad y(0) = v, \quad \text{with} \quad v \geq 0,
\]

where the matrix \(A(y) \in \mathbb{R}^{N \times N}\) is symmetric positive semidefinite for any \(y \in \mathbb{R}^N\) and its off-diagonal entries are nonpositive, i.e.,

\[
a_{ij}(y) \leq 0, \quad i \neq j, \quad \forall y \in \mathbb{R}^N.
\]

In (4) and throughout the paper we understand vector inequalities elementwise, i.e., \(v \geq 0\) means that all the entries of the vector \(v \in \mathbb{R}^N\) are nonnegative. Note that the function \(g : \mathbb{R} \to \mathbb{R}^N\) in (4) contains...
not only the values of the source function \( g(x, t) \) on the grid but also the grid contributions from the boundary conditions. These contributions will typically be of the form \(-a_{ij}(y)b(x, t)\) for some \( i \neq j \) and \( x \in \partial\Omega \). In view of (3), (5), we may assume that the values of the function \( g(t) \) are nonnegative vectors in \( \mathbb{R}^N \), i.e.,

\[
g(t) \geq 0, \quad t \geq 0.
\]

Furthermore, throughout the paper \((x, y) = y^T x, x, y \in \mathbb{R}^N\), denotes the standard inner product and, unless reported otherwise, \( || \cdot || \) is the Euclidean vector or matrix norm. We assume that IVP (4) has a unique solution and there is a constant \( L > 0 \) such that

\[
|A(u) - A(v)| \leq L ||u - v||, \quad \forall u, v \in \mathbb{R}^N.
\]

**Corollary 1.** Solution \( y(t) \) of the semidiscrete IVP (4) is entrywise nonnegative, i.e., \( y(t) \geq 0 \), \( t \geq 0 \) provided that conditions (5) and (6) hold.

**Proof.** The statement of the corollary follows from considering the explicit Euler scheme applied to (4):

\[
y^{n+1} = y^n - \Delta tA(y^n)y^n + g^n,
\]

where the superscript \( .^n \) denotes the time step number and \( \Delta t > 0 \) is the time step size. The last relation reads elementwise, for each entry \( y^{n+1}_i, i = 1, \ldots, N \), of the vector \( y^{n+1} \) as

\[
y^{n+1}_i = (1 - \Delta t a_{ii}(y^n))y^n_i - \sum_{j \neq i} a_{ij}(y^n)y^n_j + g^n_i.
\]

Since \( 1 - \Delta t a_{ii}(y^n) \geq 0 \) for sufficiently small \( \Delta t > 0 \) and due to (5) and (6), we have \( y^{n+1}_i \geq 0 \), \( n = 1, 2, \ldots \). For \( \Delta t \to 0 \) this nonnegative numerical solution converges to the unique solution of (4).  

\[\Box\]

**2. SOLUTION METHODS**

**2.1. The Backward Euler Method**

The backward Euler method applied to (4) reads

\[
\frac{y^{n+1} - y^n}{\Delta t} = -A(y^{n+1})y^{n+1} + g^{n+1}
\]

\[\Leftrightarrow (I + \Delta tA(y^{n+1}))y^{n+1} = y^n + \Delta tg^{n+1}, \quad n = 0, 1, 2, \ldots \]

(8)

Since the matrix \( A(y) \) is symmetric positive semidefinite for any \( y \in \mathbb{R}^N \), we have

\[
|| (I + \Delta tA(y))^{-1} || = \frac{1}{1 + \Delta t \lambda_{\min}(y)} \leq 1, \quad \forall y \in \mathbb{R}^N,
\]

(9)

with \( \lambda_{\min}(y) \geq 0 \) being the smallest eigenvalue of \( A(y) \). Hence, by rewriting the backward Euler scheme as \( y^{n+1} = (I + \Delta tA(y^{n+1}))^{-1}(y^n + \Delta tg^{n+1}) \) and taking the norm in the last relation, we see that the scheme (8) yields for any \( \Delta t > 0 \) a bounded solution, i.e.,

\[
||y^{n+1}|| \leq ||y^n|| + \Delta t||g^{n+1}||, \quad n = 0, 1, 2, \ldots .
\]

(10)

**Remark 1.** The boundedness properties stated in the paper imply stability for linear problems. For discussion on stability for nonlinear problems see, e.g., ([1], Sections I.2.3, I.2.8) and references therein.

Relation (8) is a system of nonlinear equations in \( y^{n+1} \). To solve the system, in ([2], App. 1, Ch. 2.11) the following iterative scheme is proposed:

\[
(I + \Delta tA(y^{(m)}))y^{(m+1)} = y^n + \Delta tg^{n+1}, \quad m = 0, 1, 2, \ldots
\]

(11)

where the superscript \((m)\) denotes the iteration number and we usually take \( y^{(0)} := y^n \). As argued in ([2], Appendix 1), the scheme is monotone for a certain finite-difference approximation of one-dimensional heat equation (1). Below we prove the monotonicity and convergence of the scheme for general, not necessarily one-dimensional heat equation.

**Corollary 2.** Assume the backward Euler method (8) in combination with iterative scheme (11) is applied to solve the IVP (4) with initial guess \( y^{(0)} = y^n \) and relations (5), (6) hold. Then for all time steps \( n = 0, 1, \ldots \) iterative scheme (8), (11)
1. is monotone, i.e., for any time step size $\Delta t > 0$ and all iterations $m = 0, 1, \ldots$

$$y^{(m+1)} \geq 0,$$ \hfill (12)

2. produces a sequence $y^{(m)}$ converging to solution $y^{n+1}$ of (8), i.e., $\|y^{n+1} - y^{(m)}\| \to 0$ as $m \to \infty$, provided that the time step size satisfies

$$0 < \Delta t < \frac{1}{L(\|y^n\| + \Delta t\|y^{n+1}\|)},$$ \hfill (13)

3. yields a bounded solution, i.e., for any time step size $\Delta t > 0$ and all iterations $m = 0, 1, \ldots$

$$\|y^{(m+1)}\| \leq \|y^n\| + \Delta t\|y^{n+1}\|.$$ \hfill (14)

**Proof.** To prove monotonicity property (12), we note that a matrix with nonpositive off-diagonal entries is positive semidefinite if and only if it is a (possibly singular) $M$-matrix ([3], Ch. 2.5). Therefore, for any $\Delta t > 0$ and any $y \in \mathbb{R}^N$ the matrix $I + \Delta t A(y)$ is a (nonsingular) $M$-matrix and its inverse is elementwise nonnegative: $(I + \Delta t A(y))^{-1} \geq 0$.

To prove convergence of iterations (11), we subtract relation (11) from (8) and define the error vector $e^{(m)} \equiv y^{n+1} - y^{(m)}$. Then, we obtain

$$(I + \Delta t A(y^{n+1}))y^{n+1} - (I + \Delta t A(y^{(m)}))y^{(m+1)} = 0,$$

$$e^{(m+1)} - \Delta t A(y^{(m)})y^{(m+1)} = -\Delta t A(y^{n+1})y^{n+1},$$

and, adding $\Delta t A(y^{(m)})y^{n+1}$ to both sides of the last equation,

$$e^{(m+1)} + \Delta t A(y^{(m)})y^{n+1} - \Delta t A(y^{(m)})y^{(m+1)} = \Delta t A(y^{(m)})y^{n+1} - \Delta t A(y^{n+1})y^{n+1},$$

$$(I + \Delta t A(y^{(m)}))e^{(m+1)} = \Delta t(A(y^{(m)}) - A(y^{n+1}))y^{n+1}.$$

Using (9), (7), (10), we can bound

$$\|e^{(m+1)}\| \leq \Delta t\|A(y^{(m)}) - A(y^{n+1})\|y^{n+1}\|$$

$$\leq \Delta t\|A(y^{(m)}) - A(y^{n+1})\|\|y^{n+1}\| \leq \Delta tL\|e^{(m)}\|\|y^{n+1}\|$$

$$\leq \Delta tL\|e^{(m)}\|\|y^{n+1}\| \leq \Delta tL\|e^{(m)}\|\|y^n\| + \Delta t\|y^{n+1}\|,$$

from which convergence can easily be seen provided (13) holds.

Finally, the norm estimate (14) can be obtained in the same way as the estimate (10), by taking the norm in the relations (11), (9).

In practice, the iterations (11) are stopped as soon as $m > 0$ (at least one iteration is done) and the residual of $y^{(m)}$ with respect to the nonlinear backward Euler equation (8)

$$r^{(m)} = y^n + \Delta t g^{n+1} - (I + \Delta t A(y^{(m)}))y^{(m)}$$

satisfies

$$\|r^{(m)}\| \leq \text{tol} \|y^n + \Delta t g^{n+1}\|,$$ \hfill (15)

with tol > 0 being a given tolerance value. We emphasize that the matrix in the residual expression is evaluated at $y^{(m)}$. 

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2.2. Nonlinear Exponential Euler Scheme

In this method, a numerical solution $y^{n+1} \approx y(t_{n+1})$ approximating solution $y(t)$ of (4) at $t_{n+1} = (n + 1)\Delta t$, $n = 0, 1, 2, \ldots$, is computed as follows:

\[
\begin{align*}
\text{solv IVP} \quad & \begin{cases} 
\tilde{y}'(t) = -A(\tilde{y}(t))\tilde{y}(t) + g^{n+1}, \quad t \in [t_n, t_n + \Delta t], \\
\tilde{y}(t_n) = y^n,
\end{cases} \\
\text{set} \quad & y^{n+1} := \tilde{y}(t_n + \Delta t),
\end{align*}
\]

where $g^{n+1} = g(t_{n+1})$ and $y^0 = v$. We solve IVP in (16) iteratively, by setting $\tilde{y}^{(0)}(t) \equiv y^n, t \in [t_n, t_n + \Delta t]$, and computing $\tilde{y}^{(m)}(t) \to \tilde{y}(t), m = 0, 1, 2, \ldots, t \in [t_n, t_n + \Delta t]$. At each iteration $m = 0, 1, \ldots$ we solve an IVP

\[
(\tilde{y}^{(m+1)}(t))' = -A_m(\tilde{y}^{(m+1)}(t))\tilde{y}^{(m+1)}(t) + g^{n+1}, \quad t \in [t_n, t_n + \Delta t],
\]

\[
\tilde{y}^{(m+1)}(t_n) = y^n,
\]

where $A_m(t) = A(\tilde{y}^{(m)}(t))$. Note that at the first iteration $m = 0$ the matrix is constant since the initial guess function $\tilde{y}^{(0)}(t)$ does not depend on time ($\tilde{y}^{(0)}(t) \equiv y^n$). Method (16) can be seen as a nonlinear variant of the exponential Euler scheme, see ([4], relation (1.6)).

We control convergence of the iterations (17) by checking residual of $\tilde{y}^{(m+1)}(t)$ with respect to ODE (ordinary differential equation) system $\tilde{y}' = -A(\tilde{y}(t))\tilde{y}(t) + g^{n+1}$:

\[
\tilde{r}^{(m+1)}(t) = -A(\tilde{y}^{(m+1)}(t))\tilde{y}^{(m+1)}(t) + g^{n+1} - (\tilde{y}^{(m+1)}(t))' = -A_m(t)\tilde{y}^{(m+1)}(t) + g^{n+1} + A_m(t)\tilde{y}^{(m+1)}(t) - g^{n+1} = [A_m(t) - A_{m+1}(t)]\tilde{y}^{(m+1)}(t).
\]

The iterations (17) are stopped as soon as the residual is small in norm at the final time $t = t_{n+1}$:

\[
\|\tilde{r}^{(m+1)}(t_{n+1})\| \leq \text{tol}\|A_{m+1}(t_{n+1})\tilde{y}^{(m+1)}(t_{n+1})\|,
\]

with tol > 0 being a given tolerance value.

It turns out that in practice for typical time step sizes $\Delta t$, solution $\tilde{y}^{(m+1)}(t)$ of (17) can be very well approximated by solution $y^{(m+1)}(t)$ of a simpler IVP

\[
(y^{(m+1)}(t))' = -A_m y^{(m+1)}(t) + g^{n+1}, \quad t \in [t_n, t_n + \Delta t],
\]

\[
y^{(m+1)}(t_n) = y^n,
\]

(19)

where the matrix is evaluated at $t = t_{n+1}$ and is kept constant: $A_m = A(y^{(m)}(t_{n+1}))$. As the next corollary shows, the deviation $\|\tilde{y}^{(m+1)}(t) - y^{(m+1)}(t)\|$ between the solutions of (17) and (19) can be estimated and easily computed in practice. To prove this result we first define an entire function

\[
\varphi(z) \equiv \frac{e^z - 1}{z}, \quad z \in \mathbb{C},
\]

with $\varphi(0) = 1$, and formulate the following lemma.

**Lemma 1 ([1], Section I.2.3).** Consider linear ODE system with variable coefficients

\[
y'(t) = -A(t)y(t) + g(t)
\]

and assume that there is a constant $\omega \in \mathbb{R}$ such that for the matrix exponential $\exp(-sA)$ holds $\|\exp(-sA)\| \leq e^{-s\omega}$, $s \in [0, T]$. Then,

\[
\|y(t)\| \leq e^{-t\omega}\|y(0)\| + \int_0^t e^{-(t-s)\omega}\|g(s)\|\,ds
\]

\[
\leq e^{-t\omega}\|y(0)\| + t\varphi(-t\omega)\max_{s \in [0,t]}\|g(s)\|, \quad t \in [0, T].
\]

**Proof.** See the last relation in ([1], Section I.2.3). \hfill \Box
Note that, for $t \geq 0$,
\[

t \varphi(-t \omega) = \begin{cases} 
  t, & \omega = 0, \\
  1 - e^{-t \omega}, & \omega > 0.
\end{cases}
\]  

We now prove the result concerning solutions of (17) and (19).

**Corollary 3.** For all time steps $n = 0, 1, \ldots$ and all iteration numbers $m = 0, 1, \ldots$, solution $\tilde{y}^{(m+1)}(t)$ of (17) and solution $y^{(m+1)}(t)$ of (19) satisfy
\[
\|\tilde{y}^{(m+1)}(t_n + \tau) - y^{(m+1)}(t_n + \tau)\| \leq \tau \varphi(-\tau \omega) \max_{s \in [t_n, t_n + \tau]} \left\| \left[ A_m - \tilde{A}_m(s) \right] y^{(m+1)}(s) \right\|, \\
\tau \in [0, \Delta t],
\]
where $\omega$ is such that
\[
\min_{(x,x) = 1, x \in \mathbb{R}^N} (\tilde{A}_m(t) x, x) \geq \omega \geq 0, \quad \forall t \in [t_n, t_n + \Delta t].
\]  

Note that $\omega \geq 0$ due to the assumption that $A(y)$ is positive semidefinite for all $y \in \mathbb{R}^N$.

**Proof.** Note that for $m = 0$ the matrix $\tilde{A}_m(t)$ is constant and equals $A_m$, so that IVPs (17) and (19) coincide and the estimate (23) trivially holds. Consider the case $m > 0$. Without loss of generality we give a proof for the first time step $n = 0$, $t \in [0, \Delta t]$. Substituting $y^{(m+1)}(t)$ in the ODE system $(\tilde{y}^{(m+1)}(t))' = -\tilde{A}_m(t)\tilde{y}^{(m+1)}(t) + g^{n+1}$, we obtain a residual $\tilde{r}(t)$ of $y^{(m+1)}(t)$,
\[
\tilde{r}(t) = -\tilde{A}_m(t)\tilde{y}^{(m+1)}(t) + g^{n+1} - (y^{(m+1)}(t))' = \left[ A_m - \tilde{A}_m(t) \right] y^{(m+1)}(t),
\]
and see that $y^{(m+1)}(t)$ solves a perturbed ODE system
\[
(y^{(m+1)}(t))' = -A_m(t)y^{(m+1)}(t) + g^{n+1} - \tilde{r}(t), \quad t \in [0, \Delta t].
\]
Subtracting this equation from the ODE system in (17), we arrive at an IVP for error function $\tilde{e}(t) = \tilde{y}^{(m+1)}(t) - y^{(m+1)}(t)$,
\[
\tilde{e}'(t) = -\tilde{A}_m(t)\tilde{e}(t) + \tilde{r}(t), \quad \tilde{e}(t) = 0, \quad t \in [0, \Delta t].
\]  

Condition (24) is equivalent to ([5], Section 1.5; [1], Section I.2.3)
\[
\| \exp(-t\tilde{A}_m(t)) \| \leq e^{-t \omega}, \quad t \geq 0.
\]  

Hence, applying the estimate (21) to (25), we have
\[
\|\tilde{e}(\tau)\| \leq \tau \varphi(-\tau \omega) \max_{\in [0, \tau]} \left\| \left[ A_m - \tilde{A}_m(s) \right] y^{(m+1)}(s) \right\|, \quad \tau \in [0, \Delta t],
\]
which is the sought after estimate (23) for the first time step $[0, \Delta t]$. \hfill \Box

Note that the error norm in (23) is zero at $t = t_n$ and the residual $\tilde{r}(t)$ is zero at $t = t_{n+1}$. Therefore, to approximately evaluate the right hand side of the estimate (23) in practice, we can compute $(s - t_n)\|A_m - \tilde{A}_m(s)\| y^{(m+1)}(s)$ at some point $s$ between $t_n$ and $t_n + \Delta t$, say at $s = t_n + \Delta t/2$. For practical values $\Delta t > 0$ this term often turns out to be negligibly small. If this is not the case we can decrease the time step size $\Delta t$ or solve (19) successfully on smaller time subintervals covering $[t_n, t_n + \Delta t]$, so that $\tilde{A}_m(t)$ is kept constant on these smaller subintervals and the deviation $\|\tilde{y}^{(m+1)}(t) - y^{(m+1)}(t)\|$ becomes smaller. In our experience, neglecting the large values $(s - t_n)\|A_m - \tilde{A}_m(s)\| y^{(m+1)}(s)$ is not disastrous and leads to an increase in the number of nonlinear iterations $m$. In the following we assume that $\Delta t > 0$ is chosen such that $\|\tilde{y}^{(m+1)}(t) - y^{(m+1)}(t)\|$ is sufficiently small.

Since both the matrix $A_m$ and the source function $g^{n+1}$ are constant in (19), iterations (19) can be written in an equivalent form
\[
y^{(m+1)}(t_n + \tau) = y^n + \tau \varphi(-\tau A_m(g^{n+1} - A_my^n), \quad \tau \in [0, \Delta t].
\]  

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Here \( \varphi(-\tau A) \) is the matrix function defined by the \( \varphi \) function (20), see, e.g., [6, 7]. In practice, we solve the IVP in (19) by computing \( y^{(m+1)}(t_n + \Delta t) \) in (27) with the restarted Krylov subspace method presented in [8].

The following result establishes monotonicity and convergence of the iterative exponential Euler method (16), (17).

**Corollary 4.** Assume the exponential Euler method (16) in combination with iterative scheme (17) is applied to solve the IVP (4) and relations (5), (6), (24) hold. For all time steps \( n = 0, 1, \ldots \) the iterative scheme (16), (17)

1. is monotone, i.e., for any time step size \( \Delta t > 0 \) and all iterations \( m = 0, 1, \ldots \)

\[
\tilde{y}^{(m+1)}(t) \geq 0, \quad t \in [t_n, t_n + \Delta t],
\]

2. produces a sequence \( \tilde{y}^{(m)}(t) \) converging to solution \( \tilde{y}(t) \) of (16), i.e., \( \max_{s \in [t_n, t_n + \Delta t]} ||\tilde{y}(s) - y^{(m)}(s)|| \to 0 \) as \( m \to \infty \), provided that the time step size \( \Delta t \) satisfies

\[
0 < \Delta t \varphi(-\Delta t \omega)L \max_{s \in [t_n, t_n + \Delta t]} ||\tilde{y}(s)|| < 1,
\]

where the function \( \varphi \) is defined in (20), (22) and \( \omega \) in (24).

3. yields a bounded solution, i.e., for any time step size \( \Delta t > 0 \) and all iterations \( m = 0, 1, \ldots \)

\[
||y^{(m+1)}(t \in \tau)|| \leq e^{-\tau \omega} ||y^n|| + \tau \varphi(-\tau \omega)||y^{n+1}||, \quad \tau \in [0, \Delta t],
\]

where \( \omega \) is defined in (24).

**Proof.** Without loss of generality we give the proof for \( n = 0 \), i.e., we consider the first time interval \([0, \Delta t]\) \( (t_n = 0) \). However, to clearly see the connection between the derivations in the proof and the corollary, we keep on writing superindices containing \( n \).

The monotonicity of iterations (17) can be established by observing that the iterative approximations \( \tilde{y}^{(m)}(t) \) solve IVP (17). For this IVP we can show nonnegativity of the solution in the same way as is done in the proof of Corollary 1.

To prove convergence of the iterations, we subtract the ODE system (17) from the ODE system (16) and obtain, for error function \( e^{(m+1)}(t) \equiv \tilde{y}(t) - y^{(m+1)}(t) \),

\[
(e^{(m+1)}(t))' = -A(\tilde{y}(t))\tilde{y}(t) + \tilde{A}_m(t)y^{(m+1)}(t),
\]

\[
(e^{(m+1)}(t))' = -A(\tilde{y}(t))\tilde{y}(t) + \tilde{A}_m(t)y^{(m+1)}(t) - \tilde{A}_m(t)\tilde{y}(t) + \tilde{A}_m(t)\tilde{y}(t),
\]

Applying to this ODE system estimate (21) and taking into account initial condition \( e^{(m+1)}(0) = 0 \) and relations (24), (26), we get

\[
||e^{(m+1)}(t)|| \leq t\varphi(-\tau \omega) \max_{s \in [0, t]} \left[ ||\tilde{A}_m(s) - A(\tilde{y}(s))|| \tilde{y}(s) \right]
\]

\[
= t\varphi(-\tau \omega) \max_{s \in [0, t]} \left[ ||A(y^{(m)}(s)) - A(\tilde{y}(s))|| \tilde{y}(s) \right]
\]

\[
\leq t\varphi(-\tau \omega) L \max_{s \in [0, t]} ||y^{(m)}(s) - \tilde{y}(s)|| \max_{s \in [0, t]} ||\tilde{y}(s)||, \quad t \in [0, \Delta t],
\]

where we use definition of \( \tilde{A}_m(t) \) and property (7). Since \( t\varphi(-\tau \omega) \) is a monotonically increasing function, we have

\[
\max_{s \in [0, \Delta t]} ||e^{(m+1)}(s)|| \leq \Delta t\varphi(-\Delta t \omega)L \max_{s \in [0, \Delta t]} ||e^{(m)}(s)|| \max_{s \in [0, \Delta t]} ||\tilde{y}(s)||,
\]
which implies \( \max_{s \in [0, \Delta t]} \| \epsilon^{(m+1)}(s) \| < \max_{s \in [0, \Delta t]} \| \epsilon^{(m)}(s) \| \) provided (29) holds.

The norm bound (30) can be obtained by applying estimate (21) to IVP (17).

**Remark 2.** Taking into account (22), we see that the convergence conditions (13) and (29) for the backward and exponential Euler schemes are very similar. The convergence condition for the exponential Euler scheme is less restrictive if \( \omega > 0 \).

3. NUMERICAL EXPERIMENTS

3.1. 1D Heat Equation

This test is considered in ([2], App. 1, Ch. 2.11). It is a one-dimensional problem (1) in domain \( \Omega = [0, 1] \) with \( k_0 = 0.5, \sigma = 2 \) and exact solution

\[
 u_{\text{exact}}(x, t) = \begin{cases} 
 \frac{\sigma c}{k_0} (ct - x)^{1/\sigma}, & \text{for } x \leq ct, \\
 0, & \text{for } x > ct,
\end{cases}
\]

where \( c \) is the heat wave speed (set to \( c = 1 \) in this test). We take initial and boundary conditions in (1) consistent with the exact solution and solve the problem on the time interval \( 0 \leq t \leq T = 0.5 \). The error values reported below are computed as

\[
 \frac{\| y^n - y_{\text{exact}}(T) \|}{\| y_{\text{exact}}(T) \|}, \quad n \Delta t = T,
\]

where the vector \( y_{\text{exact}}(T) \) contains the grid values of the exact solution function at final time \( T \). Furthermore, we set the accuracy tolerance in (15), (18) to \( \text{tol} = 10^{-2} \) and the tolerance for computing actions of the \( \varphi \) matrix function to \( 10 \times \text{tol} \). Taking more stringent tolerance values do not give a higher accuracy, as the error is determined by spatial discretization. The restarted Krylov subspace solver for evaluating the \( \varphi \) matrix function is employed with the maximal Krylov subspace dimension 30 (in most runs no restarts are necessary).

We first test monotonicity of both schemes by computing numerical solution for homogeneous Dirichlet boundary conditions and initial vector \( v \) with a single entry set to 1, and all the other entries being zero (cf. (4)). The solution to this problem, which can be seen as a numerical equivalent of the Green function, is shown in Fig. 1. As we see, both backward Euler method and exponential Euler method manage to produce nonnegative solutions for this test.

We now consider the regular test runs, with the boundary and initial conditions determined by the exact solution (32). A numerical solution and the corresponding error for the exponential Euler method are shown in Fig. 2.

In Table 1 nonlinear iteration numbers are shown for both methods for various spatial grids and time step sizes. For each grid the value of \( \max_{t \in [0, T]} \| A(y(t)) \|_1 \) is also shown in the table. The matvec (matrix-vector product) numbers for the exponential Euler method are the total numbers of the Krylov subspace iterations required to evaluate the \( \varphi \) matrix functions. The matvec values reported in the table for the backward Euler method have the following meaning. Solution of the linear system (11) in the backward Euler method can be replaced by a certain number of Chebyshev iterations. The number of Chebyshev iterations can then be chosen such that the scheme is stable (for linear problems) eventhough the linear systems (11) are solved approximately. This leads to the so-called explicit local iteration (LI) schemes, which are known to work well for heat conduction problems [9]. In the table the bracketed matvec values for the backward Euler method are the numbers of Chebyshev iterations needed in the monotone LI-M scheme [9]. As Chebyshev iterations require a singe matvec per iteration, the shown Chebyshev iteration numbers equal the matvec numbers in the LI-M method. We see that the exponential Euler scheme is more efficient in terms of the matvecs than the LI-M method.
Fig. 1. Numerical Green functions of the backward Euler (left) and exponential Euler (right) methods at time $T = 0.1$ computed with time steps $\Delta t = T$ (top) and $\Delta t = T/1000$ (bottom) on a uniform grid with $N = 128$ nodes.

Fig. 2. Solution (left) and error (right) of the exponential Euler scheme on the grid $N = 128$ at $T = 0.5$, computed with the time step size $\Delta t = 10^{-3}$ ($\Delta t \max_{t \in [0,T]} ||A(y(t))||_1 \approx 65$).
Table 1. 1D heat equation test problem. Results for the backward Euler (BE) and exponential Euler (EE) methods. The smallest matvec number for both methods and on each grid is underlined.

| $\Delta t$ | BE error, number of iterations (number of matvecs) | EE error, number of iterations (number of matvecs) |
|------------|-------------------------------------------------|--------------------------------------------------|
|            | $N = 128, \max_{t\in[0,T]} \|A(y(t))\|_1 \approx 6.5 \times 10^4$ |                                                   |
| 5e-05      | 4.07e-03, 10000 (35646)                         | 5.26e-03, 10039 (10926)                          |
| 1e-04      | 4.49e-03, 5000 (20206)                          | 5.63e-03, 5072 (8053)                           |
| 5e-04      | 7.95e-03, 1008 (7098)                           | 9.08e-03, 1118 (4247)                           |
| 1e-03      | 1.18e-02, 587 (5102)                            | 1.11e-02, 642 (3473)                            |
|            | $N = 256, \max_{t\in[0,T]} \|A(y(t))\|_1 \approx 2.6 \times 10^5$ |                                                   |
| 5e-05      | 2.33e-03, 10000 (50130)                          | 3.61e-03, 10073 (20584)                         |
| 1e-04      | 2.84e-03, 5000 (32482)                          | 4.66e-03, 5102 (15241)                          |
| 5e-04      | 6.65e-03, 1086 (13276)                          | 7.57e-03, 1142 (9318)                           |
| 1e-03      | 1.12e-02, 668 (9980)                            | 1.08e-02, 768 (7526)                            |

Table 2. 2D heat equation test problem. Results for the backward Euler (BE) and exponential Euler (EE) methods. The smallest matvec number for both methods and on each grid is underlined.

| $\Delta t$ | BE error, number of iterations (number of matvecs) | EE error, number of iterations (number of matvecs) |
|------------|-------------------------------------------------|--------------------------------------------------|
|            | $64 \times 64, \max_{t\in[0,T]} \|A(y(t))\|_1 \approx 3.0 \times 10^6$ |                                                   |
| 1e-06      | 1.24e-02, 5000 (12306)                          | 1.20e-02, 5000 (5079)                           |
| 5e-06      | 1.18e-02, 1011 (4266)                           | 1.16e-02, 1038 (1601)                           |
| 1e-05      | 1.17e-02, 536 (2668)                            | 1.17e-02, 613 (1806)                            |
| 5e-05      | 1.84e-02, 238 (2884)                            | 1.75e-02, 479 (4164)                            |
|            | $128 \times 128, \max_{t\in[0,T]} \|A(y(t))\|_1 \approx 1.3 \times 10^7$ |                                                   |
| 1e-06      | 7.44e-03, 5000 (20556)                          | 7.20e-03, 5000 (6135)                           |
| 5e-06      | 7.24e-03, 1032 (6654)                           | 7.51e-03, 1103 (4129)                           |
| 1e-05      | 7.65e-03, 613 (5814)                            | 8.52e-03, 732 (5210)                            |
| 5e-05      | 1.95e-02, 398 (9688)                            | 2.51e-02, 1045 (18403)                          |
|            | $256 \times 256, \max_{t\in[0,T]} \|A(y(t))\|_1 \approx 4.9 \times 10^7$ |                                                   |
| 1e-06      | 3.13e-03, 5000 (27700)                          | 3.34e-03, 5000 (12746)                          |
| 5e-06      | 4.44e-03, 1082 (12860)                          | 5.88e-03, 1202 (10800)                          |
| 1e-05      | 6.75e-03, 699 (13728)                           | 9.51e-03, 1067 (17955)                          |
| 5e-05      | 2.22e-02, 710 (34812)                           | 3.76e-02, 2759 (106906)                         |
3.2. 2D Heat Equation

We solve problem (1) in domain $\Omega = [0, 1] \times [0, 1]$ and we take $k_0 = 1$ and $\sigma = 2$ in (2). The initial and boundary conditions in (1) are taken such that problem (1) has exact solution

$$u_{\text{exact}}(x, y, t) = \frac{1}{\sqrt{t}} \max \left\{ 0; \frac{1.3 - \sqrt{x^2 + y^2}}{6} \right\}. \quad (34)$$

The time interval is chosen to be $[t_0, T] = [0.0001, 0.0051]$. The error values reported below are computed according to (33). Just as in the previous test, in (15), (18) the nonlinear accuracy tolerance is set to $\text{tol} = 10^{-2}$ and the tolerance for computing actions of the $\varphi$ matrix function to $10\text{tol}$. The maximal Krylov subspace dimension for evaluating the $\varphi$ matrix function is again 30.

In Table 2 the results for this 2D test are presented in the same form as for the previous test in Table 1. The results show the similar trend: the exponential Euler method appears to be more efficient in terms of required matvec numbers than the LI-M method (i.e., the backward Euler scheme combined with Chebyshev iterations to solve (11)).

4. CONCLUSIONS AND AN OUTLOOK TO FURTHER RESEARCH

For nonlinear heat conduction problems we have compared, theoretically and numerically, nonlinear iterative methods based on the backward Euler and on the exponential Euler schemes. Both methods are shown to produce monotone and bounded solutions and the underlying nonlinear iterations appear to have similar convergence properties. In the experiments both methods exhibit a robust performance. The proposed nonlinear exponential Euler method is an explicit scheme. Earlier, explicit schemes based on Chebyshev iterations (the so-called local iteration schemes) have been shown to work successfully for nonlinear heat conduction problems [9]. Preliminary estimates of computational work presented here suggest that the nonlinear exponential Euler outperforms the LI-M (local iteration modified) scheme. However, actual comparisons of the proposed exponential Euler method and the local iterations methods have yet to be done.

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