HERMITIAN ONE-MATRIX MODEL AND KP HIERARCHY

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Abstract. The partition functions of Hermitian one-matrix models are known to be tau-functions of the KP hierarchy. In this paper we explicitly compute the elements in Sato grassmannian these tau-functions correspond to, and use them to compute the \( n \)-point functions of Hermitian one-matrix model.

1. Introduction

Matrix models have been extensively studied by physicists and mathematicians because of its connections to many different fields and the numerous methods used to study them. See e.g. [16] for a comprehensive introduction for their history and various developments. See [3] for introduction to various connections to other branches of mathematics and physics, including Riemann zeta function, 2D quantum gravity, topological strings, moduli spaces of Riemann surfaces, conformal field theory, etc. For more comprehensive surveys on some of these topics, see e.g. [4] and [14]. For applications to the moduli spaces of algebraic curves, see e.g. [11, 20, 24, 25, 13].

An important problem in the study of matrix models is to compute their \( n \)-point correlations functions. One approach is first to recursively solve the loop equations (see e.g. [1]). Another approach is to apply the connection to integrable hierarchies. Since we are interested in the finite \( N \) case, the following two results are most relevant. First of all, the partition functions \( Z_N \) of Hermitian one-matrix models for all \( N \geq 1 \) form a tau-function of the Toda lattice hierarchy [23]. See e.g. [8]. This fact is recently used by Dubrovin and Yang [6] to develop a formula for the \( n \)-point functions. Secondly, for each \( N \), \( Z_N \) is a tau-function of the KP hierarchy. See [21, 19]. This is the point of departure of this paper. Since we use a different integrable hierarchy from that used by Dubrovin-Yang [6], our formula is different from theirs.

In an early work [27], for any tau-function of the KP hierarchy given by a formal power series, we have proved a formula for its associated \( n \)-point functions. This is based on Kyoto school’s approach to KP hierarchy: Sato grassmannian and boson-fermion correspondence. This formula will be recalled in §3.3. To apply it, one needs to convert the partition function which is originally an expression involving Newton power functions into an expression involving Schur functions. In general this is not an easy task. Fortunately in this case the partition function can be related to the representation theory of symmetric groups [12] (cf. §2.3), and also to the representation theory of unitary groups [14] (cf. §3.4). As a result, we get explicit expressions for the element in the Sato grassmannian to apply our formula for \( n \)-point functions.
The partition functions $Z_N$ gives a family of tau-functions of the KP hierarchy, one for each positive integer $N$. This can be made into a continuous family $Z_t$ by introducing the 't Hooft coupling constant. Our formula for $N$-point functions also works for this family, by replacing $N$ by $t$. This will be explained in §3.7.

KP hierarchy and representation theory also appear in the study of some geometric objects such as Hurwitz numbers, Mariñé-Vafa formula and open string invariants of conifolds. See e.g. [26]. We will report on their corresponding elements in Sato grassmannian and formulas for their $n$-point functions in a separate paper.

We arrange the rest of the paper in the following way. In §2 we recall some preliminaries of Hermitian matrix model, including the computations of correlators by fat graphs and by representation theory of symmetric groups. In §3 we explicitly compute the elements in Sato grassmannian that correspond to $Z_N$, and use them to compute the $n$-point functions. Our main result is formulated in Theorem 3.1 and Theorem 3.2. In an Appendix we present some concrete computational results for the $n$-point functions.

2. Preliminaries of Hermitian One-Matrix Model

In this Section we recall the computations in Hermitian matrix model by fat graphs and by representation theory of symmetric groups.

2.1. Formal quantum field theory. Let us recall the notion of a formal quantum field theory introduced in [28]. It consists of observable algebra and correlation functions. By an observable algebra we mean a commutative algebra $O$ with identity 1, whose elements are referred to as the observables. In our examples, an observable algebra is often an algebra of operators acting on some space of functions. The correlation functions are a sequence of homogeneous polynomial functions on $O$ with values in a commutative algebra $R$. When $O$ is generated by $\{O_i\}_{i \geq 0}$, where $O_0 = 1$, and

$$O_m O_n = O_n O_m,$$

for all $m, n \geq 0$, then the correlation functions are determined by the correlators $\langle O_{m_1}, \ldots, O_{m_n} \rangle \in R$. The normalized correlators $\langle O_{m_1}, \ldots, O_{m_n} \rangle' := \frac{\langle O_{m_1}, \ldots, O_{m_n} \rangle}{\langle O_0 \rangle}$ are defined by

$$\langle O_{m_1}, \ldots, O_{m_n} \rangle' := \frac{\langle O_{m_1}, \ldots, O_{m_n} \rangle}{\langle O_0 \rangle}.$$

The partition function is defined by:

$$Z := 1 + \sum_{n \geq 1} \langle O_{m_1}, \ldots, O_{m_n} \rangle' \frac{t_{m_1} \cdots t_{m_n}}{n!},$$

where $\{t_0, t_1, \ldots, t_n, \ldots\}$ are formal variables. The free energy $F$ is defined by:

$$F := \log Z.$$

The connected correlators are defined by:

$$\langle O_{m_1}, \ldots, O_{m_n} \rangle'_c := \frac{\partial^n F}{\partial t_{m_1} \cdots \partial t_{m_n}} \bigg|_{t_i = 0, i \geq 0}.$$

The $n$-point part of the free energy is defined by:

$$F^{(n)} := \frac{1}{n!} \sum_{m_1, \ldots, m_n} \langle O_{m_1}, \ldots, O_{m_n} \rangle'_c t_{m_1} \cdots t_{m_n}.$$
2.2. **Hermitian one matrix model as a formal quantum field theory.** We now explain that the Hermitian one matrix model can be regarded as a formal quantum field theory. We take the observable algebra to be the algebra $\Lambda$ of symmetric functions. We follow the notations in [15]. Denote by $\mathbb{H}_N$ the space of Hermitian $N \times N$ matrices. The algebra $S_N$ of $U(N)$-invariant polynomial functions on the space $\mathbb{H}_N$ is generated by $\text{tr}(M^n)$. We let $\Lambda$ act on $S_N$ as follows: For $g \in S_N$, define

$$
(p_n \cdot g)(M) := g^{-1} \text{tr}(M^n) \cdot g(M).
$$

For correlator we use Gaussian integrals on $\mathbb{H}_N$. On this space consider the Euclidean measure:

$$
dM = 2^{N(N-1)/2} \prod_{i=1}^{N} dM_{ii} \prod_{1 \leq i < j \leq N} d\text{Re}(M_{ij}) d\text{Im}(M_{ij}).
$$

For a polynomial function $f(M)$ in $M_{ij}$'s, define

$$
\langle f(M) \rangle_N := \frac{\int_{\mathbb{H}_N} dM f(M) e^{-\frac{1}{2g_s} \text{tr}(M^2)}}{\int_{\mathbb{H}_N} dM e^{-\frac{1}{2g_s} \text{tr}(M^2)}}.
$$

For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l)$, define

$$
\langle p_\lambda \rangle_N := g^{-l} \langle \text{tr}(M^{\lambda_1}) \cdots \text{tr}(M^{\lambda_l}) \rangle_N.
$$

Introduce formal variables $g_1, g_2, \ldots$, and let $g_{\lambda} = g_{\lambda_1} \cdots g_{\lambda_l}$. As usual, when $\lambda$ is the empty partition $\emptyset$, $p_{\emptyset} = 1$ and $g_{\emptyset} = 1$. Define the partition function by

$$
Z_N := \sum_{\lambda \in \mathcal{P}} \frac{1}{z_{\lambda}} \langle p_\lambda \rangle_N g_\lambda,
$$

where the summation is taken over the set $\mathcal{P}$ of all partitions. It can be formally written as follows:

$$
Z_N = \frac{\int_{\mathbb{H}_N} dM \exp \left( \sum_{n=1}^{\infty} \frac{g_{\lambda_n} - \delta_{n,2}}{ng_s} M^n \right)}{\int_{\mathbb{H}_N} dM \exp \left( -\frac{1}{2g_s} \text{tr}(M^2) \right)}.
$$

2.3. **Feynman expansion for Hermitian one-matrix model.** The correlators $\frac{1}{z_{\lambda}} \langle p_\lambda \rangle_N$ can be evaluated by the fat graph introduced by t’Hooft [22] based on Wick’s formula for Gaussian type integrals. See [2] for an exposition. One can check that:

$$
\langle M_{ij} \rangle_N = 0, \quad \langle M_{ij} M_{kl} \rangle_N = g_s \delta_{il} \delta_{jk}.
$$

In general, Wick’s formula gives for correlators of odd degree monomials

$$
\langle M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_{2n-1} j_{2n-1}} \rangle_N = 0,
$$

and for correlators of even degree monomials:

$$
\langle M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_{2n} j_{2n}} \rangle_N = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \langle M_{\sigma(1) j_{\sigma(1)}} M_{\sigma(2) j_{\sigma(2)}} M_{\sigma(2n-1) j_{\sigma(2n-1)}} \cdots M_{\sigma(2n) j_{\sigma(2n)}} \rangle_N.
$$
where the summation on the right-hand can be taken over the set of all possible ways of dividing \{1, \ldots, 2n\} into \(n\) pairs. For example, when \(n = 1\),

\[
\langle \frac{1}{2} \text{tr}(M^2) \rangle_N = \frac{1}{2} \sum_{i,j=1}^{N} (M_{ij} M_{ji})_N = \frac{1}{2} \sum_{i,j=1}^{N} g_s \delta_{ij} \delta_{jj} = \frac{1}{2} N^2 g_s,
\]

and when \(n = 2\),

\[
\langle \frac{1}{4} \text{tr}(M^4) \rangle_N = \frac{1}{4} \sum_{i_1, \ldots, i_4=1}^{N} (M_{i_1 i_2} M_{i_2 i_3} M_{i_3 i_4} M_{i_4 i_1})_N
\]

\[
= \frac{1}{4} \sum_{i_1, \ldots, i_4=1}^{N} (M_{i_1 i_2} M_{i_2 i_3})_N (M_{i_3 i_4} M_{i_4 i_1})_N
\]

\[
+ \frac{1}{4} \sum_{i_1, \ldots, i_4=1}^{N} (M_{i_1 i_2} M_{i_3 i_4})_N (M_{i_2 i_3} M_{i_4 i_1})_N
\]

\[
+ \frac{1}{4} \sum_{i_1, \ldots, i_4=1}^{N} (M_{i_1 i_2} M_{i_4 i_1})_N (M_{i_2 i_3} M_{i_3 i_4})_N
\]

\[
= \frac{1}{4} \sum_{i_1, \ldots, i_4=1}^{N} g_s \delta_{i_1 i_2} \delta_{i_2 i_3} \cdot g_s \delta_{i_3 i_1} \delta_{i_1 i_4}
\]

\[
+ \frac{1}{4} \sum_{i_1, \ldots, i_4=1}^{N} g_s \delta_{i_1 i_4} \delta_{i_2 i_3} \cdot g_s \delta_{i_2 i_1} \delta_{i_3 i_4}
\]

\[
+ \frac{1}{4} \sum_{i_1, \ldots, i_4=1}^{N} g_s \delta_{i_1 i_1} \delta_{i_2 i_4} \cdot g_s \delta_{i_2 i_4} \delta_{i_3 i_3}
\]

\[
= \frac{1}{2} g^2 s N^3 + \frac{1}{4} g^2 s N.
\]

The contributions to the correlators can be represented by fat graphs. For example, \(\text{tr}(M^4)\) can be represented by

\[
\begin{array}{c}
node1 \quad \node2 \\
\node3 \quad \node4
\end{array}
\]

The two terms that contributes to \(\langle \frac{1}{4} \text{tr}(M^4) \rangle_N\) correspond to the following two fat graphs: The first is a fat graph with one 4-valent vertex, two edges, and three boundary components, its automorphism group has order 2, and so its contribution is \(\frac{1}{2} N^3 g^2 s\):
the second is a fat graph with one 4-valent vertex, two edges, and one boundary components, its automorphism group has order 4, and so its contribution is $\frac{1}{4}Ng_s^2$.

Now suppose that $p_\lambda = p_1^{m_1} \cdots p_k^{m_k}$. Then one has

$$l(\lambda) = \sum_{i=1}^k m_i, \quad |\lambda| = \sum_{i=1}^k im_i, \quad z_\lambda = \prod_{i=1}^m i^{m_i} m_i!.$$  

Each $p_i$ is represented by a vertex where $i$ oriented bands are glued together. The contributions to

$$\langle p_\lambda \rangle = g_s^{-l(\lambda)} (\text{tr}(M))^{m_1} \cdots (\text{tr}(M^k))^{m_k} \rangle_N$$

is given by the following Fynman rules:

$$\sum_{\Gamma \in \Gamma^\lambda} \frac{w_\Gamma}{|\text{Aut}(\Gamma)|},$$

where the summation is taken over the set $\Gamma^\lambda$ of all fat groups $\Gamma$ obtained by gluing the $l(\lambda)$ atoms specified by $\lambda$, and for a fat graph $\Gamma$, $w_\Gamma$ takes the following form:

$$w_\Gamma = \prod_{v \in V(\Gamma)} w_v \cdot \prod_{e \in E(\Gamma)} w_e \cdot \prod_{f \in F(\Gamma)} w_f.$$  

Here $V(\Gamma)$ denotes the set of vertices of $\Gamma$, and for $v \in V(\Gamma)$,

$$w_v := g_s^{-1},$$  

$E(\Gamma)$ denotes the set of edges of $\Gamma$, and for $e \in E(\Gamma)$,

$$w_e := g_s;$$  

finally, $F(\Gamma)$ denotes the set of boundary components of the surface underlying $\Gamma$, and for $f \in F(\Gamma)$,

$$w_f := N.$$  

Therefore, we actually have

$$w_\Gamma = g_s^{\frac{1}{2}(E(\Gamma) - |V(\Gamma)|)} N^{\frac{1}{2}|F(\Gamma)|}.$$  

For $\Gamma \in \Gamma^\lambda$, it is clear that

$$|V(\Gamma)| = \sum_{i=1}^k m_i = l(\lambda), \quad |E(\Gamma)| = \frac{1}{2} \sum_{i=1}^k im_i = \frac{1}{2} |\lambda|,$$

and so we have

$$\langle p_\lambda \rangle_N = \sum_{\Gamma \in \Gamma^\lambda} \frac{1}{|\text{Aut}(\Gamma)|} g_s^{\frac{1}{2}|\lambda|-l(\lambda)} N^{\frac{1}{2}|F(\Gamma)|}. $$
For example, in degree two we have
\[
\frac{1}{2} \langle p_2 \rangle_N = \frac{1}{2} N^2,
\]
\[
\frac{1}{2!} \langle p_2^2 \rangle_N = \frac{1}{2} N g_s^{-1},
\]
in degree four,
\[
\frac{1}{4} \langle p_4 \rangle_N = (\frac{1}{2} N^3 + \frac{1}{2} N) g_s,
\]
\[
\frac{1}{3} \langle p_3 p_1 \rangle_N = N^2,
\]
\[
\frac{1}{2!2!} \langle p_2^2 \rangle_N = \frac{1}{4} N^2 + \frac{1}{8} N^4,
\]
\[
\frac{1}{2!2!} \langle p_2 p_1^2 \rangle_N = (\frac{1}{2} N + \frac{1}{4} N^3) g_s^{-1},
\]
\[
\frac{1}{4} \langle p_4 \rangle_N = \frac{1}{8} N^2 g_s^{-2}.
\]

There is a similar formula for connected correlators \( \langle \frac{1}{z} p_\lambda \rangle_N^c \):
\[
\langle \frac{1}{z} p_\lambda \rangle_N^c = \sum_{\Gamma \in \Gamma^c_\lambda} \frac{1}{|\text{Aut}(\Gamma)|} \frac{z_{\lambda = 2n}^{\lambda}}{\prod_{e \in E(\Gamma)} \tau_e} g_s^{\lambda - l(\lambda)} N^{l(\Gamma)} N^{l(F(\Gamma))},
\]
where \( \Gamma^c_\lambda \) is the set of connected fat graphs of type \( \lambda \). From this it is easy to see that \( \langle p_\lambda \rangle_N^c \big|_{g_s=1} \) is a polynomial in \( N \) with nonnegative integer as coefficients, this is because for \( \Gamma \in \Gamma^c_\lambda \),
\[
\frac{z_{\lambda = 2n}^{\lambda}}{|\text{Aut}(\Gamma)|} \in \mathbb{Z}.
\]

2.4. Calculations of the correlators by representation theory of symmetric groups. For higher degrees, it becomes very complicated to write down all the possible fat graphs and compute the orders of their automorphism groups. So it is desirable to find other methods to evaluate \( \langle p_\lambda \rangle_N \). We now recall the calculations of \( \langle p_\lambda \rangle_N \) by representation theory of the symmetric group due to Itzykson and Zuber [12].

First of all, given a fat graph \( \Gamma \), one can consider the (thin) graph \( \tilde{\Gamma} \) obtained from it by retracting each band to a one-dimensional edge. To get a fat graph \( \Gamma \) from a thin graph \( \tilde{\Gamma} \), it suffices to fix a cyclic ordering of the edges incident at vertex. Therefore, given a fat graph \( \Gamma \in \Gamma^\lambda \), where \( |\lambda| = 2n \), cut each edge of \( \tilde{\Gamma} \) in half, label the \( 2n \) half edges by \( 1, \ldots, 2n \). Using the cyclic ordering at each vertex, each such ordering determines a permutation \( \sigma \) in \( S_{2n} \) in the conjugacy class \( C_{\lambda} \). Each edge determines a transposition \( \tau_e = (i_e, j_e) \), where \( i_e, j_e \) are the labellings of two half-edges obtained by cutting \( e \). The involutions \( \{\tau_e\} \in E(\Gamma) \) are disjoint, let \( \tau = \prod_{e \in E(\Gamma)} \tau_e \). It is of type \( C_{2n} \). Now consider the labelling of the fat graph \( \Gamma \) induced from that of \( \tilde{\Gamma} \) by assigning the same number on each side of the half-band. By following the arrows along the boundary components of \( \Gamma \), one gets a partition of \( 2n \) of length \( |F(\Gamma)| \). From such considerations, Itzykson and Zuber [12] (2.7) obtained:
\[
\langle p_\lambda \rangle_N = \frac{1}{|C_{\lambda}|} g^{|\lambda|/2 - l(\lambda)} \sum_{\mu \in P_{2n}} \sum_{\tau \in C_{2n}} N^{l(\mu)}.
\]
So one needs to count the number \( N_{C_\lambda, C_\mu, C_\tau} \) of solutions of the following equation:

\[ \sigma \tau \in C_\mu, \quad \sigma \in C_\lambda, \quad \tau \in C_\tau. \]

Now we recall how to solve such an equation using representation theory.

For a finite group \( G \), given three conjugacy classes \( C_1, C_2, C_3 \), consider the number \( N_{C_1, C_2, C_3} \) of solutions

\[ g_1 g_2 g_3 = e, \quad g_i \in C_i, \quad i = 1, 2, 3. \]

Write \( \sigma_i = \sum_{g \in C_i} g \). This is an element in the center of the group ring \( \mathbb{C}G \). Hence, by Schur’s lemma, it acts a constant on any irreducible representation \( V_\rho \) of \( G \), and since

\[ \operatorname{tr} \sigma_i |_{V_\rho} = |C_i| \cdot \chi_\rho |_{C_i}, \]

where \( \chi_\rho \) is the character of \( V_\rho \), therefore,

\[ \sigma_i |_{V_\rho} = \frac{|C_i|}{\dim V_\rho} \chi_\rho |_{C_i}, \]

and so

\[ \sigma_1 \sigma_2 \sigma_3 |_{V_\rho} = \prod_{i=1}^{3} \frac{|C_i|}{\dim V_\rho} \chi_\rho |_{C_i}. \]

Now consider the action of \( \sigma_1 \sigma_2 \sigma_3 \) on the regular representation \( \mathbb{C}G \) of \( G \). On the one-hand, because

\[ \operatorname{tr} g |_{\mathbb{C}G} = |G| \cdot \delta_{g,e}, \]

we have

\[ \operatorname{tr} \sigma_1 \sigma_2 \sigma_3 |_{\mathbb{C}G} = |G| \cdot N(C_1, C_2, C_3); \]

on the other hand, using the decomposition

\[ \mathbb{C}G = \bigoplus_{\rho \in G^\vee} V_\rho \otimes \mathbb{C}^{\dim V_\rho}, \]

where \( G^\vee \) is the set of equivalence classes of irreducible representations of \( G \), by (28) one gets:

\[ \operatorname{tr} \sigma_1 \sigma_2 \sigma_3 |_{\mathbb{C}G} = \sum_{\rho \in G^\vee} (\dim V_\rho) \cdot \operatorname{tr} \sigma_1 \sigma_2 \sigma_3 |_{V_\rho} \]

\[ = \sum_{\rho \in G^\vee} \frac{1}{\dim V_\rho} \prod_{i=1}^{3} |C_i| \cdot \chi_\rho |_{C_i}. \]

Therefore,

\[ N_{C_1, C_2, C_3} = \frac{1}{|G|} \sum_{\rho \in G^\vee} \frac{1}{\dim V_\rho} \prod_{i=1}^{3} |C_i| \cdot \chi_\rho |_{C_i}. \]

This is a special case of the Burnside formula.

Let us recall some well-known facts from the representation theory of the symmetric groups to fix the notations. The conjugacy classes of the symmetric group \( S_n \) are in one-to-one correspondence with partitions \( \lambda \in \mathcal{P}_n \) (the set of partitions of \( n \)). Denote by \( C_\lambda \) the number of elements in the class with cycle type \( \lambda \). Each
element in this class consists of \( l \) disjoint cycles, with lengths \( \lambda_1, \ldots, \lambda_l \) respectively. The number of elements in this class is

\[
\frac{n!}{\prod_{i=1}^{l} \lambda_i} = \frac{n!}{1^{m_1} m_1! \cdots n^{m_n} m_n!}.
\]

The irreducible characters of \( S_n \) are also indexed by partitions \( \lambda \in \mathcal{P}_n \). Denote them by \( \chi^\lambda \). The value of \( \chi^\lambda \) on the conjugacy class \( C_\mu \) is denoted by \( \chi^\lambda_\mu \). They are all real numbers and they satisfy the orthogonality relations:

\[
\sum_{\lambda \in \mathcal{P}_n} \chi^\lambda_\mu \chi^\mu_\nu = \frac{n!}{z_\lambda} \delta_{\mu,\nu},
\]

\[
\frac{1}{n!} \sum_{\tau \in S_n} \chi^\lambda(\tau) \chi^\mu(\sigma \tau) = \frac{\chi^\lambda(\sigma)}{\chi^\mu(1^{2n})} \delta_{\lambda,\mu}.
\]

It is easy to see that \( C_\mu^{-1} = C_\mu \). By Burnside formula, one then gets [12 (2.7)]:

\[
\frac{1}{z_\lambda} \langle p_\lambda \rangle_N = g_s^{\left|\lambda\right|/2 - l(\lambda)} \sum_{\mu \in \mathcal{P}_{2n}} N^{l(\mu)} \sum_{\nu \in \mathcal{P}_{2n}} \frac{(2n)!}{z_\lambda^{2(n)}} \frac{\chi^\nu_\mu \chi^\nu_\lambda}{\chi^\nu_{(1^{2n})}} \chi^\lambda_{(1^{2n})}.
\]

With this formula, one can compute more correlators. For example, in degree six:

\[
\langle p_6 \rangle_N = (10N^2 + 5N^4)g_s^2,
\]

\[
\langle p_5p_1 \rangle_N = (5N + 10N^3)g_s,
\]

\[
\langle p_4p_2 \rangle_N = (4N + 9N^3 + 2N^5)g_s,
\]

\[
\langle p_4p_1^2 \rangle_N = 13N^2 + 2N^4,
\]

\[
\langle p_3^2 \rangle_N = (3N + 12N^3)g_s,
\]

\[
\langle p_3p_2p_1 \rangle_N = 12N^2 + 3N^4,
\]

\[
\langle p_3p_1^3 \rangle_N = (6N + 9N^3)g_s^{-1},
\]

\[
\langle p_2^3 \rangle_N = 8N^2 + 6N^4 + N^6,
\]

\[
\langle p_2^2p_1^2 \rangle_N = (8N + 6N^3 + N^5)g_s^{-1},
\]

\[
\langle p_2p_1^4 \rangle_N = (12N^2 + 3N^4)g_s^{-2},
\]

\[
\langle p_1^6 \rangle_N = 15N^3 g_s^{-3}.
\]
Combining with the examples in degree two and degree four in [23], the first few terms of the partition function are:

\[
Z_N = 1 + N^2 \cdot \frac{g_2}{2} + Ng_s^{-1} \cdot \frac{g_2^2}{2} + (N + 2N^3)g_s \cdot \frac{g_4}{4} + 3N^2 \cdot \frac{g_4g_1}{3} \\
+ (2N^2 + N^4) \cdot \frac{g_2^2}{8} + (2N + N^3)g_s^{-1} \cdot \frac{g_2g_4^2}{4} + 3N^2g_s^{-2} \cdot \frac{g_4^2}{4!} \\
+ (10N^2 + 5N^4)g_s^2 \cdot \frac{g_6}{6} + (5N + 10N^3)g_s \cdot \frac{g_5g_1}{5} \\
+ (4N + 9N^3 + 2N^5)g_s \cdot \frac{g_4g_2^2}{8} + (13N^2 + 2N^4) \cdot \frac{g_4g_1^2}{8} \\
+ (3N + 12N^3)g_s \cdot \frac{g_2^3}{18} + (12N^2 + 3N^4) \cdot \frac{g_3g_2}{6} \\
+ (6N + 9N^3)g_s^{-1} \cdot \frac{g_3g_2^3}{18} + (8N^2 + 6N^4 + N^6) \cdot \frac{g_5^3}{48} \\
+ (8N + 6N^3 + N^5)g_s^{-1} \cdot \frac{g_3g_1^2}{16} + (12N^2 + 3N^4)g_s^{-2} \cdot \frac{g_2g_1^4}{48} \\
+ 15N^3g_s^{-3} \cdot \frac{g_6}{720} + \cdots .
\]

Note

\[
Z_N = \sum_{\lambda} \frac{1}{\rho_{\lambda}} (\rho_{\lambda})_N \prod_i g_{\lambda_i}
= \sum_{\lambda} g_s^{(|\lambda|/2 - \ell(\lambda))} \sum_{\mu \in P_{2n}} N^{\ell(\mu)} \sum_{\nu \in P_{2n}} \frac{(2n)!}{\chi_{2\nu}(2\nu)^{2\mu}} \prod_i \rho_{\lambda_i}
= \sum_{\lambda} \sum_{\mu \in P_{2n}} N^{\ell(\mu)} \sum_{\nu \in P_{2n}} \frac{(2n)!}{\chi_{2\nu}(2\nu)^{2\mu}} \prod_i (\lambda_i/2 - 1) g_{\lambda_i}.
\]

So no information will be lost if one takes \(g_s = 1\), because one can recover \(Z_N\) from \(Z_N|_{g_s=1}\) by simply changing \(p_n\) to \(\lambda^{n/2 - 1}p_n\).

3. Closed Formula for \(n\)-Point Functions of Hermitian Matrix Model

In this section we use the fact that \(Z_N\) is a \(\tau\)-function of the KP hierarchy and apply the formula in [27] for \(n\)-point functions associated with a \(\tau\)-function of the KP hierarchy.

3.1. The \(n\)-point functions of Hermitian matrix model. After taking the logarithm of \(Z_N\), one gets the first few terms of the free energy \(F_N\) as follows:

\[
F_N = \frac{1}{2}N^2g_2 + \frac{1}{2}Ng_s^{-1}g_2^2 + (\frac{1}{2}N^3 + \frac{1}{4}N)g_sg_4 + N^2g_4g_1 + \frac{N^2}{4}g_2^2 + \frac{N}{2}g_s^{-1}g_2g_1^2 \\
+ (\frac{5N^2}{3} + \frac{5N^4}{6})g_s^2g_6 + (N + 2N^3)g_sg_5g_1 + (N^2 + N^3)g_sg_4g_2 + \frac{3N^2}{2}g_4g_1^2 \\
+ (\frac{N}{6} + \frac{2N^3}{3})g_sg_3^2 + 2N^2g_3g_2g_1 + \frac{N}{3}g_s^{-1}g_3g_1^3 + \frac{N^2}{6}g_2^3 + \frac{N}{2}g_s^{-1}g_2g_1^3 + \cdots.
\]
From this one sees that the first few terms of the \( n \)-point part of the free energy for \( n = 1, 2, 3 \) are

\[
F_{N}^{(1)} = \frac{1}{2} N^2 g_2 + \left( \frac{1}{2} N^3 + \frac{1}{4} N \right) g_3 g_4 + \left( \frac{5 N^2}{3} + \frac{5 N^4}{6} \right) g_5 + \cdots,
\]

\[
F_{N}^{(2)} = \frac{1}{2} N g_{-1}^2 + N^2 g_3 g_4 + \frac{N^2}{4} g_5^2 + (N + 2 N^3) g_5 g_6 + \cdots,
\]

\[
F_{N}^{(3)} = \frac{N}{2} g_{-1}^2 g_5 + \frac{3 N^2}{2} g_4 g_5^2 + 2 N^2 g_3 g_7 + \frac{N^2}{6} g_3^2 + \cdots.
\]

Since we are only interested in the coefficients of \( F_{N}^{(n)} \), we encode them in the \( n \)-point function defined by:

\[
G_{N}^{(n)} (\xi_1, \ldots, \xi_n) := \sum_{j_1, \ldots, j_n \geq 1} \left. \frac{\partial^n F_{N}}{\partial T_{j_1} \cdots \partial T_{j_n}} \right|_{g_s = 1} \xi_1^{-j_1-1} \cdots \xi_n^{-j_n-1},
\]

where

\[
T_n = \frac{g_n}{n}.
\]

For example,

\[
G_{N}^{(1)} (\xi_1) = N^2 \xi_1^{-3} + (2 N^3 + N) \xi_1^{-5} + (10 N^2 + 5 N^4) \xi_1^{-7} + \cdots,
\]

\[
G_{N}^{(2)} (\xi_1, \xi_2) = N \xi_1^{-2} \xi_2^{-2} + 3 N^2 (\xi_1^{-2} \xi_2^{-4} + \xi_1^{-4} \xi_2^{-2}) + 2 N^2 \xi_1^{-3} \xi_2^{-3} + 5 (N + 2 N^3) (\xi_1^{-2} \xi_2^{-6} + \xi_1^{-6} \xi_2^{-2}) + (4 N + 8 N^3) (\xi_1^{-3} \xi_2^{-5} + \xi_1^{-5} \xi_2^{-3}) + \left( 3 N + 12 N^3 \right) \xi_1^{-4} \xi_2^{-4} + \cdots,
\]

\[
G_{N}^{(3)} (\xi_1, \xi_2, \xi_3) = 2 N (\xi_1^{-2} \xi_2^{-2} \xi_3^{-3} + \xi_1^{-2} \xi_2^{-3} \xi_3^{-2} + \xi_1^{-3} \xi_2^{-2} \xi_3^{-2}) + 12 N^2 (\xi_1^{-2} \xi_2^{-2} \xi_3^{-5} + \xi_1^{-2} \xi_2^{-5} \xi_3^{-2} + \xi_1^{-5} \xi_2^{-2} \xi_3^{-2}) + 12 N^2 (\xi_1^{-2} \xi_2^{-3} \xi_3^{-4} + \xi_1^{-2} \xi_2^{-4} \xi_3^{-3} + \xi_1^{-3} \xi_2^{-2} \xi_3^{-4} + \xi_1^{-4} \xi_2^{-3} \xi_3^{-2} + \xi_1^{-3} \xi_2^{-4} \xi_3^{-3} + \cdots.
\]

One can recover \( F_{N}^{(n)} \) from \( G_{N}^{(n)} \) as follows:

\[
F_{N}^{(n)} = \frac{1}{n!} G_{N}^{(n)} (\xi_1, \ldots, \xi_n) |_{\xi_i^{-j_i} \to g_s / j_i}.
\]

Therefore, giving a closed formula for \( G_{N}^{(n)} \) is equivalent to giving a closed formula for \( F_{N}^{(n)} \).

It is not practical to compute the \( n \)-point functions of the Hermitian one-matrix model by either counting the fat graphs or computing the characters of all symmetric groups. Other methods have been developed for this purpose. See e.g. \[11\] \[13\] \[16\]. More recently, Dubrovin and Yang \[5\] have proved a formula for \( n \)-point functions of Hermitian one-matrix models based on the connection with Toda lattice hierarchy. We will present below a different formula based on connection with the KP hierarchy.
3.2. Partition function of Hermitian matrix model as $\tau$-function of KP hierarchy. It is well-known that the partition functions $\{Z_N\}_{n\geq 1}$ give a tau-function of the Toda lattice hierarchy of Ueno and Takasaki [23]. See e.g. Gerasimov et al [8]. This is the starting point for the computations of Dubrovin-Yang [6]. By a result of Shaw-Tu-Yen [21], $Z_N$ is a tau-function of the KP hierarchy with respect to $T_1, T_2, \ldots$, where $T_n = g_n/n$. (See also Mulase [19].) This fact is the point of departure for our computation for $n$-point functions of Hermitian matrix model.

3.3. Formula for $n$-point function associated with $\tau$-function of KP hierarchy. In an early work [27], we have obtained a formula for computing $n$-point functions associated with any tau-function (in formal power series) of the KP hierarchy. Let us recall this formula. Suppose that the tau-function corresponds to an element in Sata Grassmannian $U \in Gr^{(0)}$, given by a normalized basis

$$f_n = z^n + \sum_{m \geq 0} a_{n,m} z^{-m-1}$$

then after the boson-fermion correspondence the tau-function corresponds to:

$$|U\rangle = e^A |0\rangle,$$

where $A$ is a linear operator on the fermionic Fock space:

$$A = \sum_{m,n \geq 0} a_{n,m} \psi_{-m-1/2} \psi^*_{-n-1/2}.$$ 

Furthermore, for $n \geq 1$, the bosonic $n$-function associated to tau-function of the KP hierarchy is:

$$\sum_{j_1,\ldots,j_n \geq 1} \left. \frac{\partial^n F_U}{\partial T_{j_1} \cdots \partial T_{j_n}} \right|_{T=0} \xi_1^{-j_1-1} \cdots \xi_n^{-j_n-1} + \frac{\delta_{n,2}}{(\xi_1 - \xi_2)^2}$$

$$= (-1)^{n-1} \sum_{n\text{-cycles} \ i=1} A(\xi_{\sigma(i)}, \xi_{\sigma(i)+1}),$$

where $A(\xi_i, \xi_j)$ are the propagators:

$$A(\xi_i, \xi_j) = \begin{cases} \frac{1}{\xi_i - \xi_j} + A(\xi_i, \xi_j), & i < j, \\ A(\xi_i, \xi_j), & i = j, \\ \frac{1}{\xi_i - \xi_j} + A(\xi_i, \xi_j), & i > j. \end{cases}$$

In the above we have used the following notations:

$$A(\xi, \eta) = \sum_{m,n \geq 0} a_{n,m} \xi^{-m-1} \eta^{-n-1},$$

$$i_x, y = \left. \frac{1}{(x-y)^n} \right|_{x-\eta = k} \left( -\frac{n}{k} \right)^m x^{-n-k} y^k.$$ 

3.4. Partition function $Z_N$ in representation basis. In order to apply the formula in last subsection, we need to rewrite $Z_N$ in terms of Schur functions. We will first consider $Z_N|_{g_s=1}$, and regard $g_n$ as the Newton power functions $p_n$. Recall
Schur functions \(\{s_\lambda\}\) and the Newton functions \(\{p_\mu\}\) are related to each other by the Frobenius formula:

\[
p_\mu = \sum_\lambda \chi^\lambda_\mu s_\lambda, \tag{40}
\]

\[
s_\lambda = \sum_\mu \frac{\chi^\lambda_\mu}{z_\mu} p_\mu. \tag{41}
\]

Then we have

\[
Z_N|_{g_s=1} = \sum_\lambda \langle s_\lambda \rangle_N \cdot s_\lambda. \tag{42}
\]

By (23), we can derive the following formula:

\[
\langle s_\lambda \rangle_N = (2n - 1)! \frac{\chi^\lambda_{(2^n)}}{\chi^\lambda_{(12^n)}} \cdot \sum_{\mu \in P_{2n}} \chi^\lambda_\mu \frac{N^{l(\mu)}}{z_\mu}. \tag{43}
\]

Indeed,

\[
\langle s_\lambda \rangle_N = \sum_{\eta \in P_{2n}} \frac{\chi^\lambda_\eta}{z_\eta} \langle p_\eta \rangle_N
\]

\[
= \sum_{\eta \in P_{2n}} \frac{\chi^\lambda_\eta}{z_\eta} \sum_{\mu \in P_{2n}} N^{l(\mu)} \sum_{\nu \in P_{2n}} \frac{(2n)!}{z_\nu z_\mu} \frac{\chi^\nu_{(2^n)}}{\chi^\nu_{(12^n)}} \delta_{\lambda,\nu}
\]

\[
= \sum_{\mu \in P_{2n}} N^{l(\mu)} \sum_{\nu \in P_{2n}} \frac{(2n)!}{2^{n!}z_\mu} \frac{\chi^\nu_{(2^n)} \chi^\nu_{(12^n)}}{\chi^\nu_{(12^n)}} \delta_{\lambda,\nu}
\]

\[
= (2n - 1)! \frac{\chi^\lambda_{(2^n)}}{\chi^\lambda_{(12^n)}} \cdot \sum_{\mu \in P_{2n}} \chi^\lambda_\mu \frac{N^{l(\mu)}}{z_\mu}.
\]

By [15, Example 4, p. 45],

\[
\sum_{\mu \in P_{2n}} \chi^\lambda_\mu \frac{N^{l(\mu)}}{z_\mu} = s_\lambda|_{p_n=N} = \prod_{x \in \lambda} \frac{N + c(x)}{h(x)},
\]

where \(c(x)\) and \(h(x)\) denotes the content and the hook length of \(x\) respectively. For \(N\) large enough, the right-hand side is the dimension of the irreducible representation of \(U(N)\) indexed by \(\lambda\) (cf. [14]). In fact, the dimension of a representation of \(U(N)\) whose Young tableaux has rows of length \((\lambda_1, \lambda_2, \ldots, \lambda_l)\) is given by Weyl’s formula,

\[
\dim R^{U(N)}_\lambda = \frac{\prod_{1 \leq i < j \leq N} (h_i - h_j)}{\prod_{1 \leq i < j \leq N} (j - i)}. \tag{44}
\]
where \( h_i = N + \lambda_i - i \). One can separate the product into three cases:

\[
\dim R_\lambda^{U(N)} = \prod_{1 \leq i < j \leq l} \frac{(\lambda_i - i) - (\lambda_j - j)}{j - i} \cdot \prod_{1 \leq i < j \leq N} \frac{(\lambda_i - i) + j}{j - i} = \prod_{1 \leq i < j \leq l} \frac{(\lambda_i - i) - (\lambda_j - j)}{j - i} \cdot \prod_{1 \leq i \leq l, j = l+1} \frac{(\lambda_i - i + N)! (l - i)!}{(\lambda_i - i + l)! (N - i)!} = \prod_{1 \leq i < j \leq l} \frac{(\lambda_i - i) - (\lambda_j - j)}{j - i} \cdot \prod_{1 \leq i \leq l} \frac{(\lambda_i - i + l)! (N + j - i)!}{(\lambda_i - i + l)! (N + j - i)} = \prod_{x \in \lambda} \left( \frac{(N + c(x))}{h(x)} \right).
\]

Here in the last equality we have used Macdonald [15, p. 11, (4)].

Let \( R_\lambda \) be the irreducible representation of the symmetric group of \(|\lambda| := \sum_{i=1}^l \lambda_i\) objects, corresponding to the partition \( \lambda \). Then one has

\[
d_\lambda := \dim R_\lambda = \chi_{\lambda(1|1)}^\lambda = \prod_{v \in \lambda} h(v) |\lambda|!.
\]

As observed by Gross [9, Appendix A1] (see also Gross-Taylor [10, Appendix A]),

\[
\dim R_\lambda^{U(N)} = \frac{d_\lambda N^{\lambda|\lambda|}}{|\lambda|!} \prod_{v \in \lambda} \left( 1 + \frac{c(v)}{N} \right),
\]

where \( c(v) \) is the content of the box \( v \) in the Young diagram \( \lambda \): If \( v \) is at the \( i \)-th row and the \( j \)-column, then \( c(v) = j - i \). Therefore,

\[
\langle s_\lambda \rangle_N = \frac{(2n - 1)!}{(2n)!} \lambda(2n)^\lambda \prod_{v \in \lambda} (N + c(v)).
\]

Alternatively,

\[
\langle s_\lambda \rangle_N = c(\lambda) \dim R_\lambda^{U(N)} = c(\lambda) \cdot \frac{d_\lambda N^{\lambda|\lambda|}}{|\lambda|!} \prod_{v \in \lambda} \left( 1 + \frac{c(v)}{N} \right),
\]

where \( c(\lambda) \) is defined by:

\[
c(\lambda) := \frac{(2n - 1)! \lambda(2n)^\lambda}{\lambda(2n - 1)^\lambda}.
\]

It is interesting to note that the dimension formula for \( \dim R_\lambda^{U(N)} \) and \( \dim R_\lambda \) appear in both matrix model theory and large \( N \) Yang-Mills theory.
The explicit formula for the numbers $c(\lambda)$ was due to Di Francesco and Itzykson [5]. Define the set of $2n$ integers $f_i$ as follows

$$f_i = \lambda_i + 2n - i, \ i = 1, \ldots, 2n. \tag{50}$$

Following these authors, we will say that $\lambda$ is even if the number of odd $f_i$'s is the same as the number of even $f_i$'s. Otherwise, we will say that it is odd. One has the following result [5]:

$$c(\lambda) = \begin{cases} (-1)^{(n-1)/2} \frac{\prod_{f \text{ odd}} f!! \prod_{f' \text{ even}} (f' - 1)!!} {\prod_{f \text{ odd}, f' \text{ even}} (f-f')}, & \text{if } \lambda \text{ is even,} \\ 0, & \text{otherwise.} \end{cases} \tag{51}$$

We conjecture that any partition of an even number is even in the sense of [5]. For example,

$$\langle s(2) \rangle_N = \frac{1}{2} N(N+1),$$

$$\langle s(1^2) \rangle_N = -\frac{1}{2} N(N-1),$$

$$\langle s(4) \rangle_N = \frac{1}{8} N(N+1)(N+2)(N+3),$$

$$\langle s(3,1) \rangle_N = -\frac{1}{8} N(N+1)(N+2)(N-1),$$

$$\langle s(2,2) \rangle_N = \frac{1}{4} N^2(N+1)(N-1),$$

$$\langle s(2,1^2) \rangle_N = -\frac{1}{8} N(N+1)(N-1)(N-2),$$

$$\langle s(1^4) \rangle_N = \frac{1}{8} N(N-1)(N-2)(N-3),$$

$$\langle s(6) \rangle_N = \frac{1}{48} N(N+1)(N+2)(N+3)(N+4)(N+5),$$

$$\langle s(5,1) \rangle_N = -\frac{1}{48} N(N+1)(N+2)(N+3)(N+4)(N-1),$$

$$\langle s(4,2) \rangle_N = \frac{1}{16} N(N+1)(N+2)(N+3)(N-1)N,$$

$$\langle s(4,1^2) \rangle_N = -\frac{1}{24} N(N+1)(N+2)(N+3)(N-1)(N-2),$$

$$\langle s(3,3) \rangle_N = -\frac{1}{16} N(N+1)(N+2)(N-1)N(N+1),$$

$$\langle s(3,2,1) \rangle_N = 0,$$

$$\langle s(3,1,1,1) \rangle_N = \frac{1}{24} N(N+1)(N+2)(N-1)(N-2)(N-3),$$

$$\langle s(2,2,2) \rangle_N = \frac{1}{16} N(N+1)(N-1)N(N-2)(N-1),$$

$$\langle s(2,1,1,1) \rangle_N = -\frac{1}{16} N(N+1)(N-1)N(N-2)(N-3),$$

$$\langle s(2,1,1,1,1) \rangle_N = \frac{1}{48} N(N+1)(N-1)(N-2)(N-3)(N-4),$$

$$\langle s(1,1,1,1,1) \rangle_N = -\frac{1}{48} N(N-1)(N-2)(N-3)(N-4)(N-5).$$
To summarize,
\begin{equation}
Z_N|_{g_s=1} = \sum_{\lambda} c(\lambda) \cdot \prod_{x \in \lambda} \frac{(N + c(x))}{h(x)} \cdot s_{\lambda}.
\end{equation}

3.5. The partition function \(Z_N\) as a Bogoliubov transform. By the general theory developed in [27], we have
\begin{equation}
Z_N|_{g_s=1} = e^A|0\rangle
\end{equation}
in fermionic picture, where
\begin{equation}
A = \sum_{m,n \geq 0} A_{m,n} \psi_{-m-1/2}^* \psi_{-n-1/2}.
\end{equation}

In particular, \(Z_N|_{g_s=1}\) contains terms of the form \(A_{m,n} \psi_{-m-1/2}^* \psi_{-n-1/2}|0\rangle\), which after boson-fermion correspondence corresponds to
\begin{equation}
(-1)^n A_{m,n} s_{(m|n)}.
\end{equation}

where \((m|n)\) is a hook diagram in Frobenius notation: it is the partition \((m+1,1^n)\).

So one has
\begin{equation}
(-1)^n A_{m,n} = \langle s_{(m+1,1^n)} \rangle_N.
\end{equation}

The right-hand has been computed by Itzykson and Zuber [12]. When \(\lambda\) is a hook diagram of type \((p,q)\), i.e., \(\lambda = (q+1,1^p)\), such that \(p + q + 1 + q = 2n\),
\begin{equation}
\chi_{(2^n)}^{(q+1,p)} = (-1)^{(p+1)/2} \left[ \left( \frac{n-1}{p/2} \right) \right],
\end{equation}
where \([x]\) denotes the integral part of \(x\), and one has
\begin{equation}
\langle s_{(q+1,p)} \rangle_N = (-1)^{(p+1)/2} \left( \frac{n-1}{p/2} \right) \frac{(2n-1)!!}{(2n)!} [N]^{-p}.
\end{equation}

Here, to simplify the notations, we define for \(k \leq l\),
\begin{equation}
[N]_k^l := \prod_{j=k}^{l} (N + j).
\end{equation}

The basic properties of \([N]_k^l\) are
\begin{align}
[N - 1]_k^l &= [N]_{k-1}^{l-1}, \\
[N]_k^l - [N]_{k-1}^{l-1} &= (l - k + 1) \cdot [N]_{k-1}^{l-1} = (l - k + 1) \cdot [N - 1]_{k+1}^l.
\end{align}

Therefore, we get the following result:

**Theorem 3.1.** The fermionic representation of \(Z_N\) is given explicitly as follows:
\begin{equation}
Z_N|_{g_s=1} = \exp \left( \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!} \sum_{p=0}^{2n-1} (-1)^p \cdot (-1)^{(p+1)/2} \left( \frac{n-1}{p/2} \right) \cdot [N]_{-p}^{2n-1-p} \cdot \psi_{-(2n-p)-1/2} \psi_{-p-1/2}^* \right)|0\rangle.
\end{equation}
3.6. The n-point function for Hermitian matrix model. In last subsection we have seen that
\[
Z_N = \exp \left( \sum_{p,q \geq 0} A_{p,q} \psi_{-q-1/2} \bar{\psi}_{-p-1/2} \right) |0\rangle,
\]
where the coefficients \( A_{p,q} \) are explicitly given by:
\[
A_{p,q} = \begin{cases} 
(-1)^{p+[(p+1)/2]} \frac{(2n-1)!}{(n/2)!} \cdot [N]^{2n-1-p} & , \ p + q = 2n - 1, \\
0, & \text{otherwise.}
\end{cases}
\]
Hence one can apply our formula for n-point functions associated with tau-functions of KP hierarchy.

**Theorem 3.2.** The n-point function associated with \( Z_N \) is given by the following formula:
\[
G^{(n)}_N (\xi_1, \ldots, \xi_n) = (-1)^{n-1} \sum_{n \text{-cycles} i=1}^{n} \prod_{i=1}^{n} A(\xi_{\sigma(i)}, \xi_{\sigma(i+1)}) - \frac{\delta_{n,2}}{(\xi_1 - \xi_2)^2},
\]
where
\[
A(\xi, \eta) = \sum_{n \geq 1} \frac{(2n-1)!!}{(2n)!} \sum_{p=0}^{2n-1} (-1)^{p+[(p+1)/2]} \left( \begin{array}{c} n-1 \\ p/2 \end{array} \right) \cdot [N]^{2n-1-p} \cdot \xi^{p-1} \eta^{n-1-p-1}.
\]
The following are the first few terms of \( A(\xi, \eta) \):
\[
\begin{align*}
A(\xi, \eta) &= \frac{1}{2} [N]_0^1 \cdot \xi^{-1} \eta^{-2} + \frac{1}{2} [N]_1^0 \cdot \xi^{-2} \eta^{-1} + \frac{1}{8} [N]_2^0 \cdot \xi^{-1} \eta^{-4} \\
&+ \frac{1}{8} [N]_3^2 \cdot \xi^{-2} \eta^{-3} - \frac{1}{8} [N]_2^{-2} \cdot \xi^{-3} \eta^{-2} - \frac{1}{8} [N]_3^{-1} \cdot \xi^{-4} \eta^{-1} \\
&+ \frac{1}{48} [N]_0^5 \cdot \xi^{-1} \eta^{-6} + \frac{1}{48} [N]_1^{-4} \cdot \xi^{-2} \eta^{-5} - \frac{1}{24} [N]_2^{-2} \cdot \xi^{-3} \eta^{-4} \\
&- \frac{1}{24} [N]_3^{-3} \xi^{-4} \eta^{-3} + \frac{1}{48} [N]_4^{-1} \cdot \xi^{-5} \eta^{-2} + \frac{1}{48} [N]_5^{-0} \cdot \xi^{-6} \eta^{-1} + \cdots.
\end{align*}
\]
The one-point function is then
\[
G^{(1)}(\xi) = \sum_{n \geq 1} \frac{(2n-1)!!}{(2n)!} \sum_{p=0}^{2n-1} (-1)^{p+[(p+1)/2]} \left( \begin{array}{c} n-1 \\ p/2 \end{array} \right) \cdot [N]^{2n-1-p} \cdot \xi^{-2n-1}.
\]
The following are the first few terms:
\[
\begin{align*}
G^{(1)}(\xi) &= 1 \cdot N^2 \xi^{-3} + (2N^3 + N) \xi^{-5} + (5N^4 + 10N^2) \xi^{-7} \\
&+ (14N^5 + 70N^3 + 21N) \xi^{-9} \\
&+ (42N^6 + 420N^4 + 483N^2) \xi^{-11} \\
&+ (132N^7 + 2310N^5 + 6468N^3 + 1485N) \xi^{-13} \\
&+ (429N^8 + 12012N^6 + 66066N^4 + 56628N^2) \xi^{-15} + \cdots.
\end{align*}
\]
The coefficients are the Harer-Zagier numbers \( \epsilon_g(n) \) = number of ways to glue a 2n-gon to get a Riemann surface of genus \( g \):
\[
\epsilon_g(n) = \frac{(2n)!}{(n+1)!(n-2g)!} \cdot \text{coefficient of } x^{2g} \text{ in } \left( \frac{x/2}{\tanh(x/2)} \right)^{n+1}.
\]
This has been proved by Harer-Zagier [11] and Itzykson and Zuber [12]. Here we present another proof. Write $G^{(1)}(\xi) = \sum_{n \geq 1}(2n-1)!! \cdot b(n,N)\xi^{-2n-1}$. We now show that

\begin{equation}
(67) \quad b(n,N) = b(n,N-1) + b(n-1,N) + b(n-1,N-1).
\end{equation}

First note:

\begin{align*}
b(n,N) - b(n,N-1) &= \frac{1}{(2n)!} \sum_{p=0}^{2n-1} (-1)^{p+[(p+1)/2]} \binom{n-1}{[p/2]} \cdot ([N]^{2n-1-p} - [N-1]^{2n-1-p}) \\
&= \frac{1}{(2n)!} \sum_{p=0}^{2n-1} (-1)^{p+[(p+1)/2]} \binom{n-1}{[p/2]} \cdot [N]^{2n-2-p} \cdot (N + 2n - 1 - p - (N - 1 - p)) \\
&= \frac{1}{(2n-1)!} \sum_{p=0}^{2n-1} (-1)^{p+[(p+1)/2]} \binom{n-1}{[p/2]} \cdot [N]^{2n-2-p}.
\end{align*}

Now we apply the identity

\begin{equation*}
[N]^{2n-2-p} = [N]^{2n-2-p} + (2n-1) \cdot [N]^{2n-3-p}
\end{equation*}

recursively to get

\begin{equation}
(68) \quad b(n,N) - b(n,N-1) = \frac{1}{(2n-2)!} \sum_{j=0}^{2n-2} \sum_{p=0}^{j} (-1)^{p+[(p+1)/2]} \binom{n-1}{[p/2]} \cdot [N]^{2n-3-j} \\
+ \frac{1}{(2n-1)!} \sum_{p=0}^{2n-1} (-1)^{p+[(p+1)/2]} \binom{n-1}{[p/2]} \cdot [N]^{3-(2n-1)}.
\end{equation}

We now prove the following identity:

\begin{equation}
\sum_{p=0}^{j} (-1)^{p} \cdot (-1)^{[(p+1)/2]} \binom{n-1}{[p/2]} =
\begin{cases}
1, & j = 0, \\
(-1)^{j+[(j+1)/2]} \binom{n-2}{[j/2]} + (-1)^{j+1+[(j+1)/2]} \binom{n-2}{(j-1)/2}, & 1 \leq j \leq 2n-2, \\
0, & j = 2n-1.
\end{cases}
\end{equation}

This can be proved as follows. Denote the left-hand side by $U_j$ and the right-hand side by $V_j$. It is easy to see that

\begin{equation*}
U_0 = V_0 = 1, \quad U_1 = V_1 = 2, \quad U_{2n-1} = V_{2n-1} = 0.
\end{equation*}
For $2 \leq j \leq 2n - 2$, we have

\begin{align*}
U_j - U_{j-1} &= (-1)^j \cdot (-1)^{[(j+1)/2]} \binom{n-1}{[j/2]}, \\
V_j - V_{j-1} &= \left( (-1)^{j-1} \cdot (-1)^{[(j+1)/2]} \binom{n-2}{[j/2]} + (-1)^{j-1} \cdot (-1)^{[(j-1)/2]} \binom{n-2}{([j-1]/2)} \right) \\
&\quad - \left[ (-1)^{j-1} \cdot (-1)^{[(j)/2]} \binom{n-2}{([j]/2)} + (-1)^{j-2} \cdot (-1)^{[(j-1)/2]} \binom{n-2}{([j-2]/2)} \right] \\
&\quad + (-1)^{j-2} \cdot (-1)^{[(j-1)/2]} \left( \binom{n-2}{[j/2]} + \binom{n-2}{([j-1]/2)} \right) \\
&\quad = (-1)^j \cdot (-1)^{[(j+1)/2]} \binom{n-1}{[j/2]} \\
&\quad = U_j - U_{j-1}.
\end{align*}

This completes the proof of \(69\).

On the other hand, we have

\[
b(n-1, N) + b(n-1, N-1)
= \frac{1}{(2n-2)!} \sum_{p=0}^{2n-3} (-1)^p \cdot (-1)^{[(p+1)/2]} \binom{n-2}{[p/2]} \cdot \left( [N]_p^{2n-3-p} + [N-1]_p^{2n-3-p} \right)
= \frac{1}{(2n-2)!} \sum_{p=0}^{2n-3} (-1)^p \cdot (-1)^{[(p+1)/2]} \binom{n-2}{[p/2]} \cdot \left( [N]_p^{2n-3-p} + [N]_p^{2n-4-p} \right)
= \frac{1}{(2n-2)!} \sum_{p=0}^{2n-3} (-1)^p \cdot (-1)^{[(p+1)/2]} \binom{n-2}{[p/2]} \cdot \left( [N]_p^{2n-3-p} \right)
+ \frac{1}{(2n-2)!} \sum_{p=1}^{2n-2} (-1)^{p-1} \cdot (-1)^{[p/2]} \binom{n-2}{([p-1]/2)} \cdot \left( [N]_p^{2n-3-p} \right)
= b(n, N) - b(n, N-1).
\]

The last equality is just the result of combining \(68\) with \(69\).

In Harer and Zagier \(11\), the following result on the generating series of $\epsilon_g(n)$ is proved. Let

\[
C(n, k) := \sum_{0 \leq g \leq n/2} \epsilon_g(n) k^{n+1-2g}.
\]

Then \(C(n, k) = (2n-1)!c(n, k)\), where \(c(n, k)\) is defined by the generating series:

\[
1 + 2 \sum_{n=0}^{\infty} c(n, k) x^{n+1} = \left( \frac{1 + x}{1 - x} \right)^k,
\]

or by the recursion relations:

\[
c(n, k) = c(n, k-1) + c(n-1, k) + c(n-1, k-1)
\]
with the boundary conditions $c(0, k) = k$, $c(n, 0) = 0$. Hence by (67), $b(n, k) = c(n, k)$. Therefore, we have proved the following formula for $c(n, N)$:

\[
(73) \quad c(n, N) = \frac{1}{(2n)!} \sum_{p=0}^{2n-1} (-1)^{p+[(p+1)/2]} \binom{n-1}{p/2} [N]^{-p}^{2n-1-p}.
\]

Write $p = 2l$ or $p = 2l + 1$,

\[
(74) \quad c(n, N) = \frac{1}{(2n)!} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \left( [N]^{-2l-1}_{2l} + [N]^{-2l-2}_{-(2l+1)} \right).
\]

By numerical computations we have found the following identity:

\[
(75) \quad c(n-1, N) = \frac{1}{(2n)!} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \left( [N]^{-2l-1}_{2l} - [N]^{-2l-2}_{-(2l+1)} \right).
\]

By (74) this is equivalent to

\[
(76) \quad c(n, N) - c(n-1, N) = \frac{2}{(2n)!} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} [N]^{-2l-2}_{-(2l+1)}.
\]

By (72),

\[
c(n, N)
\]

\[=
\frac{1}{(2n)!} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \left( (N + 2n - 2l - 1) \cdot [N]^{-2l-2}_{2l} + [N]^{-2l-2}_{-(2l+1)} \right)
\]

\[=
\frac{2}{(2n)!} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} [N]^{-2l-2}_{-(2l+1)}
\]

\[+
\frac{1}{(2n-1)!} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \cdot [N]^{-2l-2}_{2l}.
\]

So we will prove

\[
(77) \quad c(n-1, N) = \frac{1}{(2n-1)!} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \cdot [N]^{-2l-2}_{2l},
\]

or by changing $n - 1$ to $n$,

\[
(78) \quad c(n, N) = \frac{1}{(2n+1)!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \cdot [N]^{-2l-2}_{2l}.
\]
Write the right-hand as \(a(n, N)\). We have
\[
a(n, N) - a(n, N - 1) = \frac{1}{(2n + 1)!} \sum_{l=0}^{n} (-1)^{l} \binom{n}{l} \cdot ([N]^{2n-2l} - [N - 1]^{2n-2l})
\]
\[
= \frac{1}{(2n)!} \sum_{l=0}^{n} (-1)^{l} \binom{n}{l} \cdot [N]^{2n-2l-1}
\]
\[
= \frac{1}{(2n)!} ([N]^{2n-1} + \frac{1}{(2n)!} \sum_{l=1}^{n} (-1)^{l} \binom{n}{l} \cdot [N]^{2n-2l-1})
\]
\[
= \frac{1}{(2n)!} (2n \cdot [N]^{2n-2} + [N]^{2n-2}) + \frac{1}{(2n)!} \sum_{l=1}^{n} (-1)^{l} \binom{n}{l} \cdot [N]^{2n-2l-1}
\]
\[
+ \frac{1}{(2n)!} \sum_{l=0}^{n} (-1)^{l} \binom{n}{l} \cdot [N]^{2n-2l-1}
\]
\[
= \frac{1}{(2n-1)!} ([N]^{2n-2} + [N - 1]^{2n-2}) + \frac{1}{(2n)!} (1 + (-1)^{1} \binom{n}{1}) [N]^{2n-3}
\]
\[
+ \frac{1}{(2n)!} \sum_{l=2}^{n} (-1)^{l} \binom{n}{l} \cdot [N]^{2n-2l-1}.
\]

By repeating this procedure we get:

\[
a(n, N) - a(n, N - 1)
\]
\[
= \frac{1}{(2n - 1)!} \sum_{l=0}^{n} \sum_{j=0}^{l} (-1)^{j} \binom{n}{j} \cdot ([N]^{2n-2l-2} + [N - 1]^{2n-2l-2})
\]
\[
+ \sum_{j=0}^{n} (-1)^{j} \binom{n}{j} \cdot [N]^{-1-2n}.
\]
\[
= \frac{1}{(2n - 1)!} \sum_{l=0}^{n} \sum_{j=0}^{l} (-1)^{j} \binom{n}{j} \cdot ([N]^{2n-2l-2} + [N - 1]^{2n-2l-2})
\]
\[
= a(n - 1, N) + a(n - 1, N - 1).
\]

In the above we have used the following identity for \(l = 0, \ldots, n:\)

\[
\sum_{j=0}^{l} (-1)^{j} \binom{n}{j} = \binom{n - 1}{l}.
\]

By checking the boundary values one then proves (78) and hence also (75).

The 2-point function is given by

\[
G_{N}^{(2)}(\xi_1, \xi_2) = \frac{A(\xi_1, \xi_2) - A(\xi_2, \xi_1)}{\xi_1 - \xi_2} - A(\xi_1, \xi_2) \cdot A(\xi_2, \xi_1).
\]
We claim that it is equal to \[-\text{in other words, if we set}\]

\[
\sum_{n \geq 1} \frac{(2n - 1)!!}{(2n)!} \cdot c(n - 1, N) \xi_2^{-2n} = \sum_{n \geq 1} (2n - 1) \cdot C(n - 1, N) \xi_2^{-2n},
\]

in other words, if we set \((-1)!! = 1\) and \(c(-1, N) = 1\), then one has

\[
\sum_{n=0}^{\infty} \frac{(2n - 1)!!}{(2k)!} \cdot c(n - 1, N) \xi_2^{-2n}
\]

\[
= \sum_{k=0}^{\infty} \frac{(2k - 1)!!}{(2k)!} \cdot [N]^{2k-1}_0 \cdot \xi_2^{-2k} \cdot \sum_{l=0}^{\infty} (-1)^l \frac{(2l - 1)!!}{(2l)!} \cdot [N]^{(2l-1)}_0 \cdot \xi_2^{-2l}.
\]
We need to prove the following identity:

\[
c(n - 1, N) = \frac{1}{(2n)!} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} (Nj)^{2n-2l-1} - \frac{1}{(2n)!} \prod_{j=-2(2n-1)}^{0} (N + j) - \frac{1}{(2n-1)!!} \sum_{k+l=n, k,l \geq 1} \frac{(2k-1)!!}{(2k)!!} [N]^{2k-1} \cdot (-1)^l (2l-1)!! [N]^{2k-2l-1} \cdot (2l)! [N]^{2n-2l-1} - \frac{1}{(2n)!} \sum_{k+l=n} \left( -1 \right)^l \binom{n}{k} [N]^{2k-1} \cdot [N]^{2n-2l-1}.
\]

We use (75) to get:

\[
c(n - 1, N) = \frac{1}{(2n)!} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} \left( [N]^{2n-2l-1} - [N]^{2n-2l-2} \right)
\]

\[
= \frac{1}{(2n)!} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} [N]^{2n-2l-1}
\]

\[
- \frac{1}{(2n)!} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} [N]^{2n-2l-2}
\]

\[
+ \frac{1}{(2n)!} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} [N]^{2n-2l-1}
\]

\[
+ \frac{1}{(2n)!} \sum_{l=0}^{n} (-1)^l \binom{n-1}{l-1} (N - 2l) \cdot [N]^{2n-2l-1}
\]

\[
+ \frac{1}{(2n)!} \sum_{l=0}^{n} (-1)^l \binom{n-1}{l} (N + 2n - 2l) \cdot [N]^{2n-2l-1}
\]

\[
= \frac{N}{(2n)!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} [N]^{2n-2l-1}
\]

\[
+ \frac{1}{(2n)!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} [N]^{2n-2l-1} - \frac{1}{(2n)!} \prod_{j=-(2n-1)}^{0} (N + j)
\]

\[
= \frac{N}{(2n)!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} [N]^{2n-2l-1}.
\]

Hence we have proved that

\[
(83) \quad c(n - 1, N) = \frac{N}{(2n)!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} [N]^{2n-2l-1}.
\]
The coefficient of $\xi_1^{-3}$ in $G_N^{(2)}$ is

\[
\sum_{n \geq 2} \frac{(2n-1)!!}{(2n)!} \cdot [N]^0_{n-1} \xi_2^{-(2n-1)} + \sum_{n \geq 2} \frac{(2n-1)!!}{(2n)!} (-1)^n \cdot [N]^0_{-(n-1)} \xi_2^{-(2n-1)} \\
+ \sum_{n \geq 2} \frac{(2n-1)!!}{(2n)!} \cdot [N]^0_{2n-2} \xi_2^{-(2n-1)} + \sum_{n \geq 2} \frac{(2n-1)!!}{(2n)!} (-1)^n \cdot [N]^1_{-(2n-2)} \xi_2^{-(2n-1)} \\
- \sum_{n \geq 1} \frac{(2n-1)!!}{(2n)!} \cdot [N]^0_{2n-1} \xi_2^{-(2n-1)} \\
- \sum_{m \geq 1} \frac{(2m-1)!!}{(2m)!} (-1)^{m-1} \cdot [N]_0^{-(2m-1)} \cdot \xi_2^{-2m} \\
+ \sum_{m \geq 1} \frac{(2m-1)!!}{(2m)!} (-1)^{m-1} \cdot [N]_1^{-(2m-2)} \cdot \xi_2^{-(2m-1)}.
\]

It can be rewritten in the following form:

\[
\sum_{n \geq 2} \frac{1}{n!2^n} \cdot [N]^0_{2n-1} \xi_2^{-(2n-1)} + \sum_{n \geq 2} \frac{1}{n!2^n} (-1)^n \cdot [N]^0_{-(n-1)} \xi_2^{-(2n-1)} \\
+ \sum_{n \geq 2} \frac{1}{n!2^n} \cdot [N]^0_{2n-2} \xi_2^{-(2n-1)} + \sum_{n \geq 2} \frac{1}{n!2^n} (-1)^n \cdot [N]^1_{-(2n-2)} \xi_2^{-(2n-1)} \\
+ \sum_{k,l \geq 1} (-1)^l \frac{1}{k!l!2^{k+l}} N(N-1) \cdot [N]^{2k-2}_{-(2l-1)} \cdot \xi_2^{-(2k+l)-1} \\
+ \sum_{k,l \geq 1} (-1)^l \frac{1}{k!l!2^{k+l}} \cdot N(N+1) \cdot [N]^{2k-1}_{-(2l-2)} \cdot \xi_2^{-(2k+l)-1} \\
= \sum_{n \geq 2} \frac{(2n-1)!!}{(2n)!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} [N]^{2n-2l-2}_{-(2l-1)} \xi_2^{-(2n-1)} \\
+ \sum_{n \geq 2} \frac{(2n-1)!!}{(2n)!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} [N]^{2n-2l-1}_{-(2l-2)} \xi_2^{-(2n-1)}.
\]
By \textsuperscript{[83]} we need to show that
\[
N(N-1)(2n-1)!! \frac{(2n-1)!}{(2n)!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} |N|^{2n-2l-2} \]
\[+ \quad N(N+1)(2n-1)!! \frac{(2n-1)!}{(2n)!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} |N|^{2n-2l-1} \]
\[= \quad 2(n-1)(2n-3)!! \cdot c(n-1, N) \]
\[= \quad 2(n-1)(2n-3)!! \cdot \frac{N}{(2n)!} \sum_{k+l=n} (-1)^l \binom{n}{k} \cdot |N|^{2k-1} \]

or equivalently,
\[
(2n-1) \cdot (N-1) \sum_{l=0}^{n} (-1)^l \binom{n}{l} |N|^{2n-2l-2} \]
\[+ \quad (2n-1) \cdot (N+1) \sum_{l=0}^{n} (-1)^l \binom{n}{l} |N|^{2n-2l-1} \]
\[= \quad 2(n-1) \cdot \sum_{l=0}^{n} (-1)^l \binom{n}{l} |N|^{2n-2l-1} \cdot \]

Denote by \(LHS\) and \(RHS\) the left-hand side and the right-hand side respectively. Then we have
\[
LHS - RHS = \sum_{l=0}^{n} (-1)^l \binom{n}{l} (2nN^2 + (2n - 4l)N + 8(n-1)l(n-l)) |N|^{2n-2l-2} \cdot
\]

This is easily shown to vanish by checking that for \(0 \leq l \leq n,
\[
(-1)^l \binom{n}{l} (2nN^2 + (2n - 4l)N + 8(n-1)l(n-l)) |N|^{2n-2l-2} \]
\[= \quad 2n \cdot (-1)^l \binom{n-1}{l} |N|^{2n-2l-2} - 2n \cdot (-1)^{l-1} \binom{n-1}{l-1} |N|^{2n-2l-2}N \]

From this we also get:
\[
\sum_{l=0}^{m} (-1)^l \binom{n}{l} (2nN^2 + (2n - 4l)N + 8(n-1)l(n-l)) |N|^{2n-2l-2} \]
\[= \quad 2n \cdot (-1)^m \binom{n-1}{m} |N|^{2n-2m-2} \cdot \]

To summarize, we have shown a surprising connection between the one-point function and the two-point function of the Hermitian one-matrix model:
\[
G_N^{(2)}(\xi_1, \xi_2) = \xi_1^{-2} \sum_{n \geq 1} (2n-1) \cdot C(n-1, N) \xi_2^{-2n} \]
\[+ \xi_1^{-3} \sum_{n \geq 2} (2n-2) \cdot C(n-1, N) \xi_2^{-(2n-1)-1} \cdot \cdot \cdot \]

(85)
\[
\text{where } G_N^{(1)}(\xi) = \sum_{n=1}^{\infty} C(n, N) \xi^{-n-1} \text{ is the one-point function computed by Harer and Zagier \textsuperscript{[11]}. We expect that further investigations of more terms in } G_N^{(2)} \text{ and of } G_N^{(n)} \text{ for } n > 2 \text{ will reveal more clearly the relationship to } G_N^{(1)}.\]
Remark 3.1. There are similar but different formulas for $c(n, N)$ in the literature. For example, \[16\] (6.5.22) reads:

\[(86)\]

$$c(n, N) = \sum_{j=0}^{n} \binom{n}{j} \left( \begin{array}{c} N \\ j+1 \end{array} \right) 2^j = \sum_{j=0}^{n} \binom{n}{j} \frac{2^j}{(j+1)!} \cdot [N]_{-j}$$

and \[16\] (6.5.29) reads:

\[(87)\]

$$c(n, N) = \frac{1}{2} \sum_{j_1 + j_2 = n+1} \binom{N}{j_1} \binom{N + j_2 - 1}{j_2} = \frac{1}{2} N \sum_{j_1 + j_2 = n+1} \frac{1}{j_1! j_2!} [N]^{j_2-1}_{-j_1+1}.$$  

3.7. A family of $\tau$-functions of the KP hierarchy by Hermitian matrix model. In the above we have seen that Hermitian one-matrix model defined by formal Gaussian integrals on the space $N \times N$ Hermitian matrices defines a tau-function $Z_N$ of the KP hierarchy, and we have presented a formula that computes the $n$-point functions as a formal power series whose coefficients are polynomials in $N$ with nonnegative integers as coefficients. Even though we have $N$ as a fixed positive integer to start with, by now we can treat it as a parameter that can take any real value. Hence by changing $N$ to $t$, we get a family $\{Z_t\}_{t \in \mathbb{R}}$ of $\tau$-functions of the KP hierarchy, whose associated $n$-point functions can be obtained by replacing $N$ by $t$ in all our formulas. This can be achieved in the framework of Hermitian matrix model by introducing the ’t Hooft coupling constant

\[(88)\]

$$t = N g_s.$$ 

In other words, one can take the coupling constant $g_s$ to be

\[(89)\]

$$g_s = \frac{1}{N} t,$$

and the formal matrix integral is then changed to:

\[(90)\]

$$Z_N = \frac{\int_{\mathbb{H}_N} dM \exp \left( N \text{tr} \sum_{n=1}^{N} \frac{g_{-n} g_{n-2}}{n!} M^n \right) }{\int_{\mathbb{H}_N} dM \exp \left( -\frac{N}{27} \text{tr}(M^2) \right)}.$$ 

The correlators for this model can be computed using the following change of variable:

\[(91)\]

$$N = g_s^{-1} t.$$ 

For example, in degree two we have

$$\langle p_2 \rangle_N = N^2 = t^2 g_s^{-2},$$

$$\langle p_1^2 \rangle_N = N g_s^{-1} = t g_s^{-2},$$

in degree four,

$$\langle p_4 \rangle_N = (N + 2N^3) g_s = t + 2t^3 g_s^{-2},$$

$$\langle p_3 p_1 \rangle_N = 3N^2 = 3t^2 g_s^{-2},$$

$$\langle p_2^2 \rangle_N = 2N^2 + N^4 = 2t^2 g_s^{-2} + t^4 g_s^{-4},$$

$$\langle p_2 p_1^2 \rangle_N = (2N + N^3) g_s^{-1} = 2t g_s^{-2} + t^3 g_s^{-2},$$

$$\langle p_1^4 \rangle_N = 3N^2 g_s^{-2} = 3t^2 g_s^{-4}.$$
and in degree six:

\[ \langle p_6 \rangle_N = (10N^2 + 5N^4)g_s^2 = 10t^2 + 5t^4g_s^{-2}, \]
\[ \langle p_5p_1 \rangle_N = (5N + 10N^3)g_s = 5t + 10t^3g_s^{-2}, \]
\[ \langle p_4p_2 \rangle_N = (4N + 9N^3 + 2N^5)g_s = 4t + 9t^3g_s^{-2} + 2t^5g_s^{-4}, \]
\[ \langle p_4p_1^2 \rangle_N = 13N^2 + 2N^4 = 13t^2g_s^{-2} + 2t^4g_s^{-4}, \]
\[ \langle p_5^2 \rangle_N = (3N + 12N^3)g_s = 3t + 12t^3g_s^{-2}, \]
\[ \langle p_3p_2p_1 \rangle_N = 12N^2 + 3N^4 = 12t^2g_s^{-2} + 3t^4g_s^{-4}, \]
\[ \langle p_3p_1^3 \rangle_N = (6N + 9N^3)g_s^{-1} = 6tg_s^{-2} + 9t^3g_s^{-4}, \]
\[ \langle p_2^3 \rangle_N = 8N^2 + 6N^4 + N^6 = 8t^2g_s^{-2} + 6t^4g_s^{-4} + t^6g_s^{-6}, \]
\[ \langle p_2p_1^2 \rangle_N = (8N + 6N^3 + N^5)g_s^{-1} = 8tg_s^{-2} + 6t^3g_s^{-4} + t^5g_s^{-6}, \]
\[ \langle p_2p_1 \rangle_N = (12N^2 + 3N^4)g_s^{-2} = 12t^2g_s^{-4} + 3t^4g_s^{-6}, \]
\[ \langle p_1^6 \rangle_N = 15N^3g_s^{-3} = 15t^3g_s^{-6}. \]

By (24),

\[ \langle \frac{1}{z_\lambda} p_\lambda \rangle_N = \sum_{\Gamma \in \mathcal{F}_\lambda} \frac{1}{| \text{Aut}(\Gamma)|} g_s^{2|\lambda|-l(\lambda)-|F(\Gamma)|} |t^{F(\Gamma)}|, \]

Denote by \( \Sigma_\Gamma \) the closed surface obtained from the fat graph \( \Gamma \) by filling \( |F(\Gamma)| \) discs along the boundary components. Then one has

\[ \frac{1}{2} |\lambda| - l(\lambda) - |F(\lambda)| = -\chi(\Sigma_\Gamma) = 2g(\Sigma_\Gamma) - 2, \]

where \( g(\Sigma_\Gamma) \) is the genus of \( \Sigma_\Gamma \).

An amazing fact is that the right-hand side is now independent of \( N \), it counts graphs on closed surfaces. Write

\[ Z_t := \sum_\lambda \sum_{\Gamma \in \mathcal{F}_\lambda} \frac{1}{| \text{Aut}(\Gamma)|} g_s^{-\chi(\Sigma_\Gamma)} \cdot t^{F(\Gamma)} \cdot p_\lambda, \]

then we have

\[ Z_t|_{g_s=1} = Z_N|_{g_s=1,N=t}. \]

In particular, \( Z_t|_{g_s=1} \) gives us a family of \( \tau \)-functions of the KP hierarchy. After replacing \( N \) by \( t \), the results in Theorem 3.1 and Theorem 3.2 also hold for \( Z_t \).

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A. Examples of n-Point Functions of Hermitian One-Matrix Models

In this Appendix we present some concrete examples of n-points functions of Hermitian one-matrix models computed by the formula in Theorem 3.2.

A.1. One-point function. For $n = 1$,

\begin{equation}
G_N^{(1)}(x) = A(x, x).
\end{equation}

The following are the first few terms:

\[ G_N^{(1)}(x) = N^2x^{-3} + (2N^3 + N)x^{-5} + (5N^4 + 10N^2)x^{-7} + (14N^5 + 70N^3 + 21N)x^{-9} + (42N^6 + 420N^4 + 483N^2)x^{-11} + (122N^7 + 2310N^5 + 6468N^3 + 1485N)x^{-13} + (429N^8 + 12012N^6 + 66066N^4 + 56628N^2)x^{-15} + (1430N^9 + 60060N^7 + 570570N^5 + 1169740N^3 + 225225N)x^{-17} + \cdots. \]

See A035309 of The On-Line Encyclopedia of Integer Sequences for more terms.

A.2. Two-point function. For $n = 2$,

\begin{equation}
G_N^{(2)}(x, y) = \frac{A(x, y) - A(y, x)}{x - y} - A(x, y) \cdot A(y, x).
\end{equation}

the following are the next few terms:

\[ G_N^{(2)}(x, y) = Nx^{-2}y^{-2} + 3N^2(x^{-4}y^{-2} + y^{-2}x^{-4}) + 2N^2x^{-3}y^{-3} + (10N^3 + 5N)(x^{-6}y^{-2} + x^{-2}y^{-6}) + (8N^3 + 4N)(x^{-5}y^{-3} + x^{-3}y^{-5}) + (12N^3 + 3N)x^{-4}y^{-4} + (35N^4 + 70N^2)(x^{-8}y^{-2} + x^{-2}y^{-8}) + (30N^4 + 60N^2)(x^{-7}y^{-3} + x^{-3}y^{-7}) + (45N^4 + 60N^2)(x^{-6}y^{-4} + x^{-4}y^{-6}) + (36N^4 + 60N^2)x^{-5}y^{-5} + (126N^5 + 630N^3 + 189N)(x^{-10}y^{-2} + x^{-2}y^{-10}) + (112N^5 + 560N^3 + 168N)(x^{-9}y^{-3} + x^{-3}y^{-9}) + (168N^5 + 630N^3 + 147N)(x^{-8}y^{-4} + x^{-4}y^{-8}) + (144N^5 + 600N^3 + 156N)(x^{-7}y^{-5} + x^{-5}y^{-7}) + (180N^5 + 600N^3 + 165N)x^{-6}y^{-6} + \cdots. \]
the next few terms are

\[ + (462N^6 + 4620N^4 + 5313N^2)(x^{-12}y^{-2} + x^{-2}y^{-12}) \]
\[ + (420N^6 + 4200N^4 + 4830N^2)(x^{-11}y^{-3} + x^{-3}y^{-11}) \]
\[ + (630N^6 + 5040N^4 + 4725N^2)(x^{-10}y^{-4} + x^{-4}y^{-10}) \]
\[ + (560N^6 + 4760N^4 + 4760)(x^{-9}y^{-5} + x^{-5}y^{-9}) \]
\[ + (700N^6 + 4900N^4 + 4795N^2)(x^{-8}y^{-6} + x^{-6}y^{-8}) \]
\[ + (600N^6 + 4800N^4 + 4770N^2)x^{-7}y^{-7} \]

and the next few terms are

\[ + (1716N^7 + 30030N^5 + 84084N^3 + 19305N)(x^{-14}y^{-2} + x^{-2}y^{-14}) \]
\[ + (7761N^5 + 15847N^3 + 27720N^3 + 17820N)(x^{-13}y^{-3} + x^{-3}y^{-13}) \]
\[ + (2376N^7 + 34650N^5 + 81774N^3 + 16335N)(x^{-12}y^{-4} + x^{-4}y^{-12}) \]
\[ + (2160N^7 + 32760N^5 + 80640N^3 + 16740N)(x^{-11}y^{-5} + x^{-5}y^{-11}) \]
\[ + (2700N^7 + 34650N^5 + 80640N^3 + 17145N)(x^{-10}y^{-6} + x^{-6}y^{-10}) \]
\[ + (2400N^7 + 33600N^5 + 80640N^3 + 16920N)(x^{-9}y^{-7} + x^{-7}y^{-9}) \]
\[ + (2800N^7 + 34300N^5 + 81340N^3 + 16695N)(x^{-8}y^{-8}) \]

and the next few terms are

\[ + (6435N^8 + 180180N^6 + 990990N^4 + 849420N^2)(x^{-16}y^{-2} + x^{-2}y^{-16}) \]
\[ + (6006N^8 + 168168N^6 + 924924N^4 + 792792N^2)(x^{-15}y^{-3} + x^{-3}y^{-15}) \]
\[ + (9009N^8 + 216216N^6 + 1027326N^4 + 774774N^2)(x^{-14}y^{-4} + x^{-4}y^{-14}) \]
\[ + (8316N^8 + 205128N^6 + 1003464N^4 + 778932N^2)(x^{-13}y^{-5} + x^{-5}y^{-13}) \]
\[ + (10395N^8 + 221760N^6 + 1011780N^4 + 783090N^2)(x^{-12}y^{-6} + x^{-6}y^{-12}) \]
\[ + (9450N^8 + 214200N^6 + 1008000N^4 + 781200N^2)(x^{-11}y^{-7} + x^{-7}y^{-11}) \]
\[ + (11025N^8 + 220500N^6 + 1014300N^4 + 781200N^2)(x^{-10}y^{-8} + x^{-8}y^{-10}) \]
\[ + (9800N^8 + 215600N^6 + 1009400N^4 + 781200N^2)x^{-9}y^{-9} \]
and the next few terms are
\[
+ (24310N^9 + 1021200N^7 + 9699690N^5 + 19885580N^3 + 3828825N)
\times (x^{-18}y^{-2} + x^{-2}y^{-18})
+ (22880N^9 + 960960N^7 + 9129120N^5 + 18715840N^3 + 3603600N)
\times (x^{-17}y^{-3} + x^{-3}y^{-17})
+ (34320N^9 + 1261260N^7 + 10540530N^5 + 19244940N^3 + 3378375N)
\times (x^{-16}y^{-4} + x^{-4}y^{-16})
+ (32032N^9 + 1201200N^7 + 10258248N^5 + 19139120N^3 + 3423420N)
\times (x^{-15}y^{-5} + x^{-5}y^{-15})
+ (40040N^9 + 1321320N^7 + 10480470N^5 + 19149130N^3 + 3468665N)
\times (x^{-14}y^{-6} + x^{-6}y^{-14})
+ (36960N^9 + 1275120N^7 + 10395000N^5 + 19145280N^3 + 3451140N)
\times (x^{-13}y^{-7} + x^{-7}y^{-13})
+ (43120N^9 + 1325940N^7 + 10461990N^5 + 19194560N^3 + 3433815N)
\times (x^{-12}y^{-8} + x^{-8}y^{-12})
+ (39200N^9 + 1293600N^7 + 10419360N^5 + 19163200N^3 + 3444840N)
\times (x^{-11}y^{-9} + x^{-9}y^{-11})
+ (44100N^9 + 1323000N^7 + 10478160N^5 + 19158300N^3 + 3455865N)
\times (x^{-10}y^{-10} + \cdots).
\]

A.3. Two-point function. For \( n = 3 \),

\[
G_N^{(3)}(x, y, z) = \left( \frac{1}{x - y} + A(x, y) \right) \left( \frac{1}{y - z} + A(y, z) \right) \left( \frac{1}{z - x} + A(z, x) \right) \
+ \left( \frac{1}{y - x} + A(y, x) \right) \left( \frac{1}{z - y} + A(z, y) \right) \left( \frac{1}{x - z} + A(x, z) \right).
\]

The first few terms of \( G^{(3)} \) are

\[
G_N^{(3)}(x, y, z) = 2N \left( \frac{1}{x^3y^2z^2} + \frac{1}{x^2y^3z^2} + \frac{1}{x^2y^2z^3} \right)
+ 12N^2 \left( \frac{1}{x^3y^3z^2} + \frac{1}{x^2y^3z^3} \right)
+ 12N^2 \left( \frac{1}{x^4y^3z^2} + \frac{1}{x^4y^2z^3} + \frac{1}{x^3y^4z^2} + \frac{1}{x^3y^2z^4} + \frac{1}{x^2y^4z^3} \right)
+ 8N^3 \frac{1}{x^3y^3z^3} + \cdots.
\]

To simplify notations, we write it in the following form:

\[
G_N^{(3)} = 2N[3, 2, 2] + 12N^2[5, 2, 2] + 12N^2[4, 3, 2] + 8N^2[3, 3, 3] + \cdots.
\]

The next few terms are

\[
+ (60N^3 + 30N)[7, 2, 2] + (60N^3 + 30N)[6, 3, 2] + (72N^3 + 24N)[5, 4, 2]
+ (48N^3 + 24N)[5, 3, 3] + (72N^3 + 24N)[4, 4, 3].
\]
\begin{align*}
+ & (280N^4 + 560N^2)[9, 2, 2] + (280N^4 + 560N^2)[8, 3, 2] \\
+ & (360N^4 + 540N^2)[7, 4, 2] + (240N^4 + 480N^2)[7, 3, 3] \\
+ & (360N^4 + 540N^2)[6, 5, 2] + (360N^4 + 480N^2)[6, 4, 3] \\
+ & (288N^4 + 480N^2)[5, 5, 3] + (432N^4 + 468N^2)[5, 4, 4].
\end{align*}

+ & (1260N^5 + 6300N^3 + 1890N)[11, 2, 2] + (1260N^5 + 6300N^3 + 1890N)[10, 3, 2] \\
+ & (1680N^5 + 6720N^3 + 1680N)[9, 4, 2] + (1120N^5 + 5600N^3 + 1680N)[9, 3, 3] \\
+ & (1680N^5 + 6720N^3 + 1680N)[8, 5, 2] + (1680N^5 + 6300N^3 + 1470N)[8, 4, 3] \\
+ & (1800N^5 + 6600N^3 + 1770N)[7, 6, 2] + (1440N^5 + 6000N^3 + 1560N)[7, 5, 3] \\
+ & (2160N^5 + 6660N^3 + 1350N)[7, 4, 2] + (1800N^5 + 6000N^3 + 1650N)[6, 6, 3] \\
+ & (2160N^5 + 6480N^3 + 1440N)[6, 5, 4] + (1728N^5 + 6336N^3 + 1440N)[5, 5, 5]
\end{align*}

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