Quasiopen sets, bounded variation and lower semicontinuity in metric spaces

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March 16, 2017

Abstract

In the setting of a metric space that is equipped with a doubling measure and supports a Poincaré inequality, we show that the total variation of functions of bounded variation is lower semicontinuous with respect to $L^1$-convergence in every 1-quasiopen set. To achieve this, we first prove a new characterization of the total variation in 1-quasiopen sets. Then we utilize the lower semicontinuity to show that the variation measures of a sequence of functions of bounded variation converging in the strict sense are uniformly absolutely continuous with respect to the 1-capacity.

1 Introduction

Let $(X, d, \mu)$ be a complete metric space equipped with a Radon measure $\mu$. The total variation of a function of bounded variation (BV function) $u$ in an open set $\Omega \subset X$ is defined by means of approximation with locally Lipschitz

\footnotesize

\textbf{Acknowledgments.} The research was funded by a grant from the Finnish Cultural Foundation. The author wishes to thank Nageswari Shanmugalingam for giving helpful comments on the manuscript.

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\textbf{2010 Mathematics Subject Classification:} 30L99, 31E05, 26B30.

\textbf{Keywords:} metric measure space, function of bounded variation, total variation, quasiopen set, lower semicontinuity, uniform absolute continuity
functions, that is,
\[ \|Du\| (\Omega) := \inf \left\{ \liminf_{i \to \infty} \int_{\Omega} g_{u_i} \, d\mu : u_i \in \text{Lip}_{\text{loc}}(\Omega), u_i \to u \text{ in } L^1_{\text{loc}}(\Omega) \right\}, \]
(1.1)
where each \( g_{u_i} \) is a 1-weak upper gradient of \( u_i \) in \( \Omega \); see Section 2 for definitions. From this definition it easily follows that the total variation is lower semicontinuous with respect to \( L^1 \)-convergence in open sets, that is, if \( U \subset X \) is open and \( u_i \to u \) in \( L^1_{\text{loc}}(U) \), then
\[ \|Du\| (U) \leq \liminf_{i \to \infty} \|Du_i\| (U). \]
(1.2)
For arbitrary (measurable) sets \( U \subset X \) we cannot define \( \|Du\| (U) \) simply by replacing \( \Omega \) with \( U \) in the definition of the total variation, because then the total variation would not yield a Radon measure, see Example 4.2. Instead, \( \|Du\| (U) \) is defined by means of approximation with open sets containing \( U \), following [26].

On the other hand, a set \( U \subset X \) is said to be \( 1 \)-quasiopen if for every \( \varepsilon > 0 \) there exists an open set \( G \subset X \) such that \( \text{Cap}_1(G) < \varepsilon \) and \( U \cup G \) is open. Quasiopen sets and related concepts of fine potential theory have been recently studied in the metric setting in e.g. [3, 4, 5, 6] in the case \( p > 1 \). See also the monographs [25] and [14] for the Euclidean theory and its history in the unweighted and weighted settings, respectively. In the case \( p = 1 \), analogous concepts have been recently studied in [20, 21, 24].

In this paper, we assume that the measure \( \mu \) is doubling and that the space supports a \((1, 1)\)-Poincaré inequality, and then we show that if \( U \subset X \) is a \( 1 \)-quasiopen set and \( \|Du\| (U) < \infty \), then the total variation \( \|Du\| (U) \) can be equivalently defined by replacing \( \Omega \) with \( U \) in (1.1). This is Theorem 4.3.

Using this result, we can then show that the lower semicontinuity (1.2) holds true also for every \( 1 \)-quasiopen set \( U \), if \( \|Du\| (U) < \infty \) and \( u_i \to u \) in \( L^1_{\text{loc}}(U) \). This is Theorem 4.5. Such a lower semicontinuity result may be helpful in solving various minimization problems, for example in the upcoming work [23].

The notion of uniform integrability of a sequence of functions \((g_i) \subset L^1(X)\) is often useful in analysis. This involves uniform absolute continuity with respect to the ambient measure. That is, for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( A \subset X \) with \( \mu(A) < \delta \), then \( \int_A g_i \, d\mu < \varepsilon \) for every \( i \in \mathbb{N} \).

The variation measure \( \|Du\| \) of a BV function \( u \) is, of course, not always absolutely continuous with respect to \( \mu \). On the other hand, it is a well-
known fact in the Euclidean setting that \(|Du|\) is absolutely continuous with respect to the 1-capacity \(\text{Cap}_1\). The proof of this fact is essentially the same in the more general metric setting, see [22, Lemma 3.9].

A sequence of BV functions \(u_i\) is said to converge strictly to a BV function \(u\) if \(u_i \to u\) in \(L^1(X)\) and \(|Du_i|(X) \to |Du|(X)\). Given such a sequence, we show that for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(A \subset X\) with \(\text{Cap}_1(A) < \delta\), then \(|Du_i|(A) < \varepsilon\) for every \(i \in \mathbb{N}\). In other words, the variation measures \(|Du_i|\) are uniformly absolutely continuous with respect to the 1-capacity. This is Theorem 5.6. The proof combines the previously discussed lower semicontinuity result with Baire’s category theorem.

2 Notation and definitions

In this section we introduce the notation, definitions, and assumptions used in the paper.

Throughout this paper, \((X, d, \mu)\) is a complete metric space equipped with a metric \(d\) and a Borel regular outer measure \(\mu\) that satisfies a doubling property, that is, there is a constant \(C_d \geq 1\) such that

\[
0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty
\]

for every ball \(B = B(x, r)\) with center \(x \in X\) and radius \(r > 0\). When we want to specify that a constant \(C\) depends on the parameters \(a, b, \ldots\), we write \(C = C(a, b, \ldots)\).

A complete metric space equipped with a doubling measure is proper, that is, closed and bounded sets are compact, see e.g. [2, Proposition 3.1]. For a \(\mu\)-measurable set \(A \subset X\), we define \(L^1_{\text{loc}}(A)\) to consist of functions \(u\) on \(A\) such that for every \(x \in A\) there exists \(r > 0\) such that \(u \in L^1(A \cap B(x, r))\). Other local spaces of functions are defined similarly. For any open set \(\Omega \subset X\), every function in the class \(L^1_{\text{loc}}(\Omega)\) is in \(L^1(\Omega')\) for every open \(\Omega' \subset \Omega\). Here \(\Omega' \subset \Omega\) means that \(\overline{\Omega'}\) is a compact subset of \(\Omega\).

For any set \(A \subset X\) and \(0 < R < \infty\), the restricted spherical Hausdorff content of codimension one is defined to be

\[
\mathcal{H}_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i \leq R \right\}.
\]

The codimension one Hausdorff measure of \(A \subset X\) is then defined to be

\[
\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A).
\]
The measure theoretic boundary \( \partial^* E \) of a set \( E \subset X \) is the set of points \( x \in X \) at which both \( E \) and its complement have positive upper density, i.e.

\[
\limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.
\]

The measure theoretic interior and exterior of \( E \) are defined respectively by

\[
I_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} = 0 \right\}
\]

and

\[
O_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} = 0 \right\}.
\]

Note that we always have a partitioning of the space into the disjoint sets \( \partial^* E, I_E, \) and \( O_E \).

By a curve we mean a rectifiable continuous mapping from a compact interval of the real line into \( X \). The length of a curve \( \gamma \) is denoted by \( \ell_\gamma \). We will assume every curve to be parametrized by arc-length, which can always be done (see e.g. [11, Theorem 3.2]). A nonnegative Borel function \( g \) on \( X \) is an upper gradient of an extended real-valued function \( u \) on \( X \) if for all curves \( \gamma \), we have

\[
|u(x) - u(y)| \leq \int_\gamma g \, ds,
\]

where \( x \) and \( y \) are the end points of \( \gamma \). We interpret \( |u(x) - u(y)| = \infty \) whenever at least one of \( |u(x)|, |u(y)| \) is infinite. We define the local Lipschitz constant of a locally Lipschitz function \( u \in \text{Lip}_{\text{loc}}(X) \) by

\[
\text{Lip} u(x) := \limsup_{r \to 0} \sup_{y \in B(x, r) \setminus \{x\}} \frac{|u(y) - u(x)|}{d(y, x)}.
\]

Then \( \text{Lip} u \) is an upper gradient of \( u \), see e.g. [7, Proposition 1.11]. Upper gradients were originally introduced in [15].

If \( g \) is a nonnegative \( \mu \)-measurable function on \( X \) and (2.3) holds for 1-almost every curve, we say that \( g \) is a 1-weak upper gradient of \( u \). A property holds for 1-almost every curve if it fails only for a curve family with zero 1-modulus. A family \( \Gamma \) of curves is of zero 1-modulus if there is a nonnegative Borel function \( \rho \in L^1(X) \) such that for all curves \( \gamma \in \Gamma \), the curve integral \( \int_\gamma \rho \, ds \) is infinite. Of course, by replacing \( X \) with a set
A ⊂ X and considering curves γ in A, we can talk about a function g being a (1-weak) upper gradient of u in A. A 1-weak upper gradient can always be perturbed in a set of µ-measure zero, see [2, Lemma 1.43], and so we understand it to be defined only µ-almost everywhere.

Given a µ-measurable set U ⊂ X, we consider the following norm

\[ \|u\|_{N^{1,1}(U)} := \|u\|_{L^1(U)} + \inf \|g\|_{L^1(U)}, \]

where the infimum is taken over all 1-weak upper gradients g of u in U. The substitute for the Sobolev space \( W^{1,1} \) in the metric setting is the Newton-Sobolev space

\[ N^{1,1}(U) := \{ u : \|u\|_{N^{1,1}(U)} < \infty \}. \]

We understand every Newton-Sobolev function to be defined everywhere in U (even though \( \| \cdot \|_{N^{1,1}(U)} \) is, precisely speaking, then only a seminorm). The Newton-Sobolev space with zero boundary values is defined as

\[ N^{1,1}_0(U) := \{ u|_U : u \in N^{1,1}(X) \text{ and } u = 0 \text{ in } X \setminus U \}. \]

Thus \( N^{1,1}_0(U) \) is a subclass of \( N^{1,1}(U) \), and it can also be considered as a subclass of \( N^{1,1}(X) \), as we will do without further notice.

It is known that for any \( u \in N^{1,1}_{\text{loc}}(U) \), there exists a minimal 1-weak upper gradient of u in U, always denoted by \( g_u \), satisfying \( g_u \leq g \) µ-almost everywhere in U, for any 1-weak upper gradient \( g \in L^1_{\text{loc}}(U) \) of u in U [2, Theorem 2.25]. For more on Newton-Sobolev spaces, we refer to [28, 2, 16].

The 1-capacity of a set \( A \subset X \) is given by

\[ \text{Cap}_1(A) := \inf \|u\|_{N^{1,1}(X)}, \]

where the infimum is taken over all functions \( u \in N^{1,1}(X) \) such that \( u \geq 1 \) in A. We know that \( \text{Cap}_1 \) is an outer capacity, meaning that

\[ \text{Cap}_1(A) = \inf \{ \text{Cap}_1(\Omega) : \Omega \supset A \text{ is open} \} \]

for any \( A \subset X \), see e.g. [2, Theorem 5.31]. For basic properties satisfied by the 1-capacity, such as monotonicity and countable subadditivity, see e.g. [2].

We say that a set \( U \subset X \) is 1-quasiopen if for every \( \varepsilon > 0 \) there exists an open set \( G \subset X \) such that \( \text{Cap}_1(G) < \varepsilon \) and \( U \cup G \) is open.

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, following [26]. See also e.g. [11, 8, 9, 10, 29] for
the classical theory in the Euclidean setting. Given a function $u \in L^1_{\text{loc}}(X)$, we define the total variation of $u$ in $X$ by

$$
\|Du\|(X) := \inf \left\{ \liminf_{i \to \infty} \int_X g_{u_i} \, d\mu : u_i \in \text{Lip}_{\text{loc}}(X), u_i \to u \text{ in } L^1_{\text{loc}}(X) \right\},
$$

where each $g_{u_i}$ is the minimal 1-weak upper gradient of $u_i$. We say that a function $u \in L^1(X)$ is of bounded variation, and denote $u \in BV(X)$, if $\|Du\|(X) < \infty$. By replacing $X$ with an open set $\Omega \subset X$ in the definition of the total variation, we can define $\|Du\|(\Omega)$. For an arbitrary set $A \subset X$, we define

$$
\|Du\|(A) = \inf \{ \|Du\|(\Omega) : A \subset \Omega, \Omega \subset X \text{ is open} \}.
$$

In general, if $A \subset X$ is an arbitrary set, we understand the statement $\|Du\|(A) < \infty$ to mean that there exists some open set $\Omega \supset A$ such that $u \in L^1_{\text{loc}}(\Omega)$ and $\|Du\|(\Omega) < \infty$. If $\|Du\|(\Omega) < \infty$, $\|Du\|\cdot$ is a finite Radon measure on $\Omega$ by [26, Theorem 3.4]. A $\mu$-measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X) < \infty$, where $\chi_E$ is the characteristic function of $E$. The perimeter of $E$ in $\Omega$ is also denoted by

$$
P(E, \Omega) := \|D\chi_E\|(\Omega).
$$

We have the following coarea formula from [26, Proposition 4.2]: if $\Omega \subset X$ is an open set and $u \in L^1_{\text{loc}}(\Omega)$, then

$$
\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(\{u > t\}, \Omega) \, dt. \tag{2.4}
$$

We will assume throughout the paper that $X$ supports a $(1,1)$-Poincaré inequality, meaning that there exist constants $C_P \geq 1$ and $\lambda \geq 1$ such that for every ball $B(x, r)$, every $u \in L^1_{\text{loc}}(X)$, and every upper gradient $g$ of $u$, we have

$$
\int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq C_P r \int_{B(x,\lambda r)} g \, d\mu,
$$

where

$$
u_{B(x,r)} := \int_{B(x,r)} u \, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.
$$
3 Preliminary results

In this section we consider certain preliminary results that we will need in proving the main theorems. We start with the following simple result concerning Newton-Sobolev functions with zero boundary values.

Lemma 3.1. Let $\Omega \subset X$ be an open set, let $u \in N^{1,1}(\Omega)$ with $-1 \leq u \leq 1$, and let $\eta \in N^{1,1}_0(\Omega)$ with $0 \leq \eta \leq 1$. Then $\eta u \in N^{1,1}_0(\Omega)$ with a 1-weak upper gradient $\eta g_u + |u| g_\eta$ (in $X$).

Here $g_u$ and $g_\eta$ are the minimal 1-weak upper gradients of $u$ and $\eta$ (in $\Omega$ and $X$, respectively). By [2, Corollary 2.21] we know that if $v \in N^{1,1}(X)$, then

$$g_v = 0 \text{ in } \{v = 0\} \quad (3.2)$$

($\mu$-almost everywhere, to be precise). Thus $g_v = 0$ outside $\Omega$, and so the function $\eta g_u + |u| g_\eta$ can be interpreted to take the value zero outside $\Omega$.

Proof. By the Leibniz rule, see [2, Theorem 2.15], we know that $\eta u \in N^{1,1}(\Omega)$ with a 1-weak upper gradient $\eta g_u + |u| g_\eta$ in $\Omega$. Moreover, $-\eta \leq \eta u \leq \eta \in N^{1,1}_0(\Omega)$, and then by [2, Lemma 2.37] we conclude $\eta u \in N^{1,1}_0(\Omega)$. Finally, by (3.2) we know that $\eta g_u + |u| g_\eta$ is a 1-weak upper gradient of $u \eta$ in $X$. $\square$

The following two lemmas describe two ways of enlarging a set without increasing the 1-capacity significantly.

Lemma 3.3 ([21, Lemma 3.1]). For any $G \subset X$ and $\varepsilon > 0$ there exists an open set $V \supset G$ with $\text{Cap}_1(V) \leq C_1(\text{Cap}_1(G) + \varepsilon)$ and $P(V, X) \leq C_1(\text{Cap}_1(G) + \varepsilon)$, for a constant $C_1 = C_1(C_d, C_P, \lambda) \geq 1$.

Proof. See [21, Lemma 3.1]; note that there was a slight error in the formulation, as the possibility $\text{Cap}_1(G) = 0$ was not taken into account, but this is easily corrected by adding an $\varepsilon$-term in suitable places. $\square$

Lemma 3.4. Let $G \subset X$ and $\varepsilon > 0$. There exists an open set $V \supset G$ with $\text{Cap}_1(V) \leq C_2(\text{Cap}_1(G) + \varepsilon)$ and a function $\eta \in N^{1,1}_0(V)$ with $0 \leq \eta \leq 1$ on $X$, $\eta = 1$ on $G$, and $\|\eta\|_{N^{1,1}(X)} \leq C_2(\text{Cap}_1(G) + \varepsilon)$, for some constant $C_2 = C_2(C_d, C_P, \lambda) \geq 1$.

Proof. By Lemma 3.3 we find an open set $V_0 \supset G$ with

$$\text{Cap}_1(V_0) \leq C_1(\text{Cap}_1(G) + \varepsilon) \quad \text{and} \quad P(V_0, X) \leq C_1(\text{Cap}_1(G) + \varepsilon).$$
By a suitable boxing inequality, see [12, Lemma 4.2], we find \( \{B(x_i, r_i)\}_{i=1}^{\infty} \) with \( r_i \leq 1 \) covering \( V_0 \), and

\[
\sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} \leq C_B(\mu(V_0) + P(V_0, X))
\]

for some constant \( C_B = C_B(C_d, C_P, \lambda) > 0 \). For each \( i \in \mathbb{N} \), take a \( 1/r_i \)-Lipschitz function \( 0 \leq f_i \leq 1 \) with \( f_i = 1 \) on \( B(x_i, 2r_i) \) and \( f_i = 0 \) on \( X \setminus B(x_i, 4r_i) \). Let \( f := \sup_{i \in \mathbb{N}} f_i \). By (3.2) and the fact that the local Lipschitz constant is an upper gradient, \( \chi_{B(x_i, 4r_i)}/r_i \) is a 1-weak upper gradient of \( f_i \). Hence the minimal 1-weak upper gradient of \( f \) satisfies \( g_f \leq \sum_{i=1}^{\infty} \chi_{B(x_i, 4r_i)}/r_i \), see e.g. [2, Lemma 1.28]. Then

\[
\int_X g_f \, d\mu \leq \sum_{i=1}^{\infty} \frac{\mu(B(x_i, 4r_i))}{r_i} \leq C_d \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i}
\]

\[
\leq C_d^2 C_B(\mu(V_0) + P(V_0, X))
\]

\[
\leq C_d^2 C_B(\text{Cap}_1(V_0) + P(V_0, X))
\]

\[
\leq 2C_d^2 C_B \text{Cap}_1(\text{Cap}_1(G) + \varepsilon).
\]

Moreover, since \( r_i \leq 1 \) for each \( i \in \mathbb{N} \),

\[
\int_X f \, d\mu \leq \sum_{i=1}^{\infty} \int_X f_i \, d\mu \leq \sum_{i=1}^{\infty} \frac{\mu(B(x_i, 4r_i))}{r_i} \leq 2C_d^2 C_B \text{Cap}_1(\text{Cap}_1(G) + \varepsilon).
\]

Let \( V := \bigcup_{i=1}^{\infty} B(x_i, 2r_i) \). Since \( f \geq 1 \) on \( V \), we get the estimate

\[
\text{Cap}_1(V) \leq \|f\|_{N^{1,1}(X)} \leq 4C_d^2 C_B \text{Cap}_1(\text{Cap}_1(G) + \varepsilon).
\]

On the other hand, for each \( i \in \mathbb{N} \), we can also take a \( 1/r_i \)-Lipschitz function \( 0 \leq \eta_i \leq 1 \) with \( \eta_i = 1 \) on \( B(x_i, r_i) \) and \( \eta_i = 0 \) on \( X \setminus B(x_i, 2r_i) \). Let \( \eta := \sup_{i \in \mathbb{N}} \eta_i \). Then \( \eta = 1 \) on \( V_0 \supseteq G \) and \( \eta = 0 \) on \( X \setminus V \), and similarly as for the function \( f \), we can estimate \( \|\eta\|_{N^{1,1}(X)} \leq 4C_d C_B C_1 \text{Cap}_1(G) \). Thus we can choose \( C_2 = 4C_d^2 C_B C_1 \).

The next lemma states that in the definition of the total variation, we can consider convergence in \( L^1(\Omega) \) instead of convergence in \( L^1_{\text{loc}}(\Omega) \).

**Lemma 3.5** ([18, Lemma 5.5]). Let \( \Omega \subset X \) be an open set and let \( u \in L^1_{\text{loc}}(\Omega) \) with \( \|Du\|_{(\Omega)} < \infty \). Then there exists a sequence \( (w_i) \subset \text{Lip}_{\text{loc}}(\Omega) \) with \( w_i - u \rightarrow 0 \) in \( L^1(\Omega) \) and \( \int_{\Omega} g_{w_i} \, d\mu \rightarrow \|Du\|_{(\Omega)} \).
Recall that \( g_{w_i} \) denotes the minimal 1-weak upper gradient of \( w_i \) (in \( \Omega \)). Note that above, we cannot write \( w_i \to u \) in \( L^1(\Omega) \), since the functions \( w_i, u \) are not necessarily in the class \( L^1(\Omega) \).

**Lemma 3.6** ([3, Lemma 9.3]). Every 1-quasiopen set is \( \mu \)-measurable.

In fact, this is proved for all \( 1 \leq p < \infty \) in the above reference, but we only need the case \( p = 1 \).

The coarea formula (2.4) states that if \( \Omega \subset X \) is an open set and \( u \in L^1_{\text{loc}}(\Omega) \), then

\[
\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(\{ u > t \}, \Omega) \, dt.
\]

If \( \|Du\|(\Omega) < \infty \), the above is true with \( \Omega \) replaced by any Borel set \( A \subset \Omega \); this is also given in [26, Proposition 4.2]. However, one can construct simple examples of non-Borel 1-quasiopen sets, so we need to verify the coarea formula for such sets separately. In doing this, we use the following lemma, which states that the total variation of a BV function is absolutely continuous with respect to the 1-capacity.

**Lemma 3.7** ([22, Lemma 3.9]). Let \( \Omega \subset X \) be an open set, and let \( u \in L^1_{\text{loc}}(\Omega) \) with \( \|Du\|(\Omega) < \infty \). Then for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( A \subset \Omega \) with \( \text{Cap}_1(A) < \delta \), then \( \|Du\|(A) < \varepsilon \).

**Proposition 3.8.** Let \( U \subset X \) be a 1-quasiopen set and suppose that \( \|Du\|(U) < \infty \). Then

\[
\|Du\|(U) = \int_{-\infty}^{\infty} P(\{ u > t \}, U) \, dt.
\]

**Proof.** Recall that implicit in the condition \( \|Du\|(U) < \infty \) is the requirement that there exists an open set \( \Omega \supset U \) such that \( u \in L^1_{\text{loc}}(\Omega) \) and \( \|Du\|(\Omega) < \infty \). Since \( U \) is 1-quasiopen, we can pick open sets \( G_i \subset X \) such that \( \text{Cap}_1(G_i) \to 0 \) and each \( U \cup G_i \) is an open set, and we can also assume that \( G_i \subset \Omega \) and \( G_{i+1} \subset G_i \) for each \( i \in \mathbb{N} \). Then by the coarea formula (2.4),

\[
\|Du\|(U \cup G_i) = \int_{-\infty}^{\infty} P(\{ u > t \}, U \cup G_i) \, dt.
\]

By Lemma 3.7, \( \|Du\|(U \cup G_i) \to \|Du\|(U) \) as \( i \to \infty \). Similarly, \( P(\{ u > t \}, U \cup G_i) \to P(\{ u > t \}, U) \) for every \( t \in \mathbb{R} \) for which \( P(\{ u > t \}, U \cup G_1) < \).
\[ \|Du\|_\infty(\Omega) = \inf \left\{ \liminf_{i \to \infty} \int_\Omega g_{u_i} \, d\mu : u_i \in \text{Lip}_\text{loc}(\Omega), u_i \to u \text{ in } L^1_\text{loc}(\Omega) \right\}, \]

where each \( g_{u_i} \) is the minimal 1-weak upper gradient of \( u_i \) in \( \Omega \). Moreover, by [2, Theorem 5.47], for any \( v \in N^{1,1}_\text{loc}(\Omega) \) and \( \varepsilon > 0 \) we can find \( w \in \text{Lip}_\text{loc}(\Omega) \) with \( \|v - w\|_{N^{1,1}(\Omega)} < \varepsilon \). Thus in the above definition we can equivalently assume that \( u_i \in N^{1,1}_\text{loc}(\Omega) \).

**Example 4.2.** Let \( X = \mathbb{R} \) (unweighted), let \( A := [0, 1] \), and let \( u := \chi_A \). Then by definition

\[ \|Du\|_\infty(A) = \inf \{ \|Du\|_\infty(\Omega) : \Omega \supset A, \ \Omega \text{ open} \} = 2. \]

On the other hand, the constant sequence \( u_i := 1 \) in \( A, i \in \mathbb{N} \), converges to \( u \) in \( L^1(A) \) and has upper gradients \( g_{u_i} = 0 \) in \( A \). This demonstrates that we cannot obtain \( \|Du\|_\infty(A) \) simply by writing (4.1) with \( \Omega \) replaced by \( A \). If we did define \( \|Du\|_\infty(D) \) in this way for all (\( \mu \)-measurable) sets \( D \subset \mathbb{R} \), then we would obtain \( \|Du\|_\infty(\mathbb{R}) = 2, \|Du\|_\infty(A) = 0, \) and \( \|Du\|_\infty(\mathbb{R} \setminus A) = 0 \), so that \( \|Du\| \) would not be a measure.

However, for 1-quasioopen sets we have the following.

**Theorem 4.3.** Let \( U \subset X \) be a 1-quasioopen set. If \( \|Du\|_\infty(U) < \infty \), then

\[ \|Du\|_\infty(U) = \inf \left\{ \liminf_{i \to \infty} \int_U g_{u_i} \, d\mu, u_i \in N^{1,1}_\text{loc}(U), u_i \to u \text{ in } L^1_\text{loc}(U) \right\}, \]

where each \( g_{u_i} \) is the minimal 1-weak upper gradient of \( u_i \) in \( U \).
Note that the condition \( u_i \to u \) in \( L^1_{\text{loc}}(U) \) means, explicitly, that for every \( x \in U \) there exists \( r > 0 \) such that \( u_i \to u \) in \( L^1(B(x, r) \cap U) \). In order for the formulation of the theorem to make sense, we need \( U \) to be \( \mu \)-measurable, which is guaranteed by Lemma 3.6.

First we prove the following weaker version.

**Proposition 4.4.** Let \( U \subset X \) be a 1-quasiopen set. If \( \Omega \supset U \) is an open set and \(-1 \leq u \leq 1 \) is a \( \mu \)-measurable function on \( \Omega \) with \( \| Du \|_{\Omega} < \infty \), then

\[
\| Du \|_{U} = \inf \left\{ \liminf_{i \to \infty} \int_{U} g_{u_i} \, d\mu, \, u_i \in N_{\text{loc}}^{1,1}(U), \, u_i \to u \text{ in } L^1_{\text{loc}}(U) \right\},
\]

where each \( g_{u_i} \) is the minimal 1-weak upper gradient of \( u_i \) in \( U \).

**Proof.** Denote the infimum in the statement of the theorem by \( a(u, U) \). Clearly \( a(u, U) \leq \| Du \|_{U} \), so we only need to prove that \( \| Du \|_{U} \leq a(u, U) \). We can assume that \( a(u, U) < \infty \). First assume also that \( u \in BV(X) \) with \(-1 \leq u \leq 1 \). Fix \( \varepsilon > 0 \). By Lemma 3.7, there exists \( \delta \in (0, \varepsilon) \) such that if \( A \subset X \) with \( \text{Cap}_1(A) < \delta \), then \( \| Du \|_{A} < \varepsilon \). Take a sequence \( (u_i) \subset N_{\text{loc}}^{1,1}(U) \) with \( u_i \to u \) in \( L^1_{\text{loc}}(U) \) and

\[
\liminf_{i \to \infty} \int_{U} g_{u_i} \, d\mu \leq a(u, U) + \varepsilon.
\]

By truncating, we can also assume that \(-1 \leq u_i \leq 1 \). Then take an open set \( G \subset X \) such that \( \text{Cap}_1(G) < \delta/C_2 \) and \( U \cup G \) is open. By Lemma 3.4 we find an open set \( V \supset G \) with \( \text{Cap}_1(V) < \delta \) and a function \( \eta \in N_0^{1,1}(V) \) with \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) on \( G \), and \( \| \eta \|_{N^{1,1}(X)} < \delta \).

By the definition of the total variation, we find a sequence \( (v_i) \subset N_{\text{loc}}^{1,1}(V) \) with \( v_i \to u \) in \( L^1_{\text{loc}}(V) \) and

\[
\| Du \|_{V} = \lim_{i \to \infty} \int_{V} g_{v_i} \, d\mu.
\]

We can again assume that \(-1 \leq v_i \leq 1 \), and then in fact \( v_i \to u \) in \( L^1(V) \). Define

\[
w_i := (1 - \eta) u_i + \eta v_i, \quad i \in \mathbb{N}.
\]

By the Leibniz rule, see [2, Theorem 2.15], \( (1 - \eta) u_i \) has a 1-weak upper gradient

\[
(1 - \eta) g_{u_i} + |u_i| g_{\eta}
\]
in $U$. By Lemma 3.1,
\[ \eta v_i \in N_0^{1,1}(V) \subset N^{1,1}(X) \subset N^{1,1}(U) \]
with a 1-weak upper gradient $\eta g_v + |v_i|g_{\eta}$ (in $X$, and thus in $U$). In total, $w_i$ has a 1-weak upper gradient
\[ g_i := (1 - \eta) g_u + \eta g_v + 2 g_{\eta} \]
in $U$. Next we show that in fact, $g_i$ is a 1-weak upper gradient of $w_i$ in $U \cup G$; note that while $g_u$ is only defined on $U$, $(1 - \eta) g_u$ is defined in a natural way on $U \cup G$, and similarly for the term $\eta g_v$.

Since $U$ is a 1-quasiopen set, it is also 1-path open, meaning that for 1-a.e. curve $\gamma$, the set $\gamma^{-1}(U)$ is a relatively open subset of $[0, \ell_{\gamma}]$, see [27, Remark 3.5]. Fix such a curve $\gamma$ in $U \cup G$, and assume also that the upper gradient inequality holds for the pair $(w_i, g_i)$ on any subcurve of $\gamma$ in $U$, and for the pair $(v_i, g_v)$ on any subcurve of $\gamma$ in $G$; by [2, Lemma 1.34(c)] this is true for 1-a.e. curve.

Now $[0, \ell_{\gamma}]$ is a compact set that is covered by the two relatively open sets $\gamma^{-1}(U)$ and $\gamma^{-1}(G)$. By the Lebesgue number lemma, there exists a number $\beta > 0$ such that every subinterval of $[0, \ell_{\gamma}]$ with length at most $\beta$ is contained either in $\gamma^{-1}(U)$ or in $\gamma^{-1}(G)$. Choose $m \in \mathbb{N}$ such that $\ell_{\gamma}/m \leq \delta$ and consider the subintervals $I_j := [j\ell_{\gamma}/m, (j + 1)\ell_{\gamma}/m]$, $j = 0, \ldots, m - 1$. If $I_j \subset \gamma^{-1}(U)$, then by our assumptions on $\gamma$,
\[ |w_i(j\ell_{\gamma}/m) - w_i((j + 1)\ell_{\gamma}/m)| \leq \int_{j\ell_{\gamma}/m}^{(j+1)\ell_{\gamma}/m} g_i(\gamma(s)) \, ds. \]
Otherwise $I_j \subset \gamma^{-1}(G)$. Recall that $\eta = 1$ on $G$. Then by our assumptions on $\gamma$,
\[ |w_i(j\ell_{\gamma}/m) - w_i((j + 1)\ell_{\gamma}/m)| = |v_i(j\ell_{\gamma}/m) - v_i((j + 1)\ell_{\gamma}/m)| \]
\[ \leq \int_{j\ell_{\gamma}/m}^{(j+1)\ell_{\gamma}/m} g_v(\gamma(s)) \, ds \]
\[ = \int_{j\ell_{\gamma}/m}^{(j+1)\ell_{\gamma}/m} g_i(\gamma(s)) \, ds. \]
Adding up the inequalities for $j = 1, \ldots, m - 1$, we conclude that the upper gradient inequality holds for the pair $(w_i, g_i)$ on the curve $\gamma$, that is,
\[ |w_i(0) - w_i(\ell_{\gamma})| \leq \int_0^{\ell_{\gamma}} g_i(\gamma(s)) \, ds. \]
Thus $g_i$ is a $1$-weak upper gradient of $w_i$ in the open set $U \cup G$. Next we show that $w_i \to u$ in $L^1_{\text{loc}}(U \cup G)$. Let $x \in U$. Since $u_i \to u$ in $L^1_{\text{loc}}(U)$, there is some $r > 0$ such that $u_i \to u$ in $L^1(B(x, r) \cap U)$. Moreover, $v_i \to u$ in $L^1(V)$, and by making $r$ smaller, if necessary, $B(x, r) \subset U \cup G$. Then

$$
\int_{B(x,r)} |w_i - u|\,d\mu \leq \int_{B(x,r)} |(1 - \eta)(u_i - u)|\,d\mu + \int_{B(x,r)} |\eta(v_i - u)|\,d\mu \\
\leq \int_{B(x,r) \cap U \setminus G} |u_i - u|\,d\mu + \int_{B(x,r) \cap V} |v_i - u|\,d\mu \\
\to 0 \quad \text{as } i \to \infty.
$$

On the other hand, if $x \in G$, then for some $r > 0$, $B(x, r) \subset G$. Then

$$
\int_{B(x,r)} |w_i - u|\,d\mu = \int_{B(x,r)} |v_i - u|\,d\mu \to 0.
$$

We conclude that $w_i \to u$ in $L^1_{\text{loc}}(U \cup G)$. Now by the definition of the total variation (recall (4.1) and the discussion after it)

$$
\|Du\|(U \cup G) \leq \liminf_{i \to \infty} \int_{U \cup G} g_i\,d\mu \\
\leq \liminf_{i \to \infty} \int_{U} g_{u_i}\,d\mu + \limsup_{i \to \infty} \int_{V} g_{v_i}\,d\mu + 2 \limsup_{i \to \infty} \int_{X} g_{\eta}\,d\mu \\
\leq a(u, U) + \varepsilon + \|Du\|(V) + 2 \int_{X} g_{\eta}\,d\mu \\
< a(u, U) + 4\varepsilon;
$$

recall that $\|Du\|(V) < \varepsilon$ since $\text{Cap}_1(V) < \delta$, and that $\|\eta\|_{N^{1,1}(X)} < \delta < \varepsilon$. In conclusion,

$$
\|Du\|(U) \leq \|Du\|(U \cup G) \leq a(u, U) + 4\varepsilon.
$$

Letting $\varepsilon \to 0$, the proof is complete in the case $u \in \text{BV}(X)$, $-1 \leq u \leq 1$.

Now we drop the assumption $u \in \text{BV}(X)$. By assumption, we have $-1 \leq u \leq 1$ on the open set $\Omega \supset U$, with $\|Du\|(\Omega) < \infty$. Take open sets $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega$ with $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$, and cutoff functions $\eta_j \in \text{Lip}_{c}(X)$ with $0 \leq \eta_j \leq 1$, $\eta_j = 1$ on $\Omega_j$, and $\eta_j = 0$ on $X \setminus \Omega_{j+1}$. Fix $j \in \mathbb{N}$. It is easy to check that $u\eta_j \in \text{BV}(X)$ for each $j \in \mathbb{N}$. Since $\Omega_j \cap U$ is a $1$-quasiopen
Before proving Theorem 4.3, we prove our second main result, which states that the total variation of BV functions is lower semicontinuous with respect to $L^1$-convergence in 1-quasiopen sets. In fact, we will use this to prove Theorem 4.3.

Theorem 4.5. Let $U \subset X$ be a 1-quasiopen set. If $\|Du\|(U) < \infty$ and $u_i \to u$ in $L^1_{\text{loc}}(U)$, then
\[
\|Du\|(U) \leq \liminf_{i \to \infty} \|Du_i\|(U).
\]

Proof. First assume that $E, E_i \subset X$, $i \in \mathbb{N}$, are $\mu$-measurable sets with $P(E, U), P(E_i, U) < \infty$ and $\chi_{E_i} \to \chi_E$ in $L^1_{\text{loc}}(U)$. For each $i \in \mathbb{N}$, the condition $P(E_i, U) < \infty$ means that we find an open set $\Omega_i \supset U$ such that $P(E_i, \Omega_i) < P(E_i, U) + 1/i < \infty$. Then by Lemma 3.5 for each $i \in \mathbb{N}$ we find a function $v_i \in \text{Lip}_{\text{loc}}(\Omega_i) \subset N^{1,1}_{\text{loc}}(\Omega_i)$ such that
\[
\|v_i - \chi_{E_i}\|_{L^1(\Omega_i)} < 1/i \quad \text{and} \quad \int_{\Omega_i} g_{v_i} \, d\mu < P(E_i, \Omega_i) + 1/i,
\]
where $g_{v_i}$ is the minimal 1-weak upper gradient of $v_i$ in $\Omega_i$. In particular, we have $v_i \in N^{1,1}_{\text{loc}}(U)$ with
\[
\|v_i - \chi_{E_i}\|_{L^1(U)} < 1/i \quad \text{and} \quad \int_{U} g_{v_i} \, d\mu < P(E_i, U) + 2/i,
\]
where $g_{v_i}$ is now the minimal 1-weak upper gradient of $v_i$ in $U$, which is of course at most the minimal 1-weak upper gradient of $v_i$ in $\Omega_i$. Now we clearly have $v_i \to \chi_E$ in $L^1_{\text{loc}}(U)$. Moreover, the condition $P(E, U) < \infty$ means that there exists an open set $\Omega \supset U$ such that $P(E, \Omega) < \infty$. Thus by Proposition 4.4
\[
P(E, U) \leq \liminf_{i \to \infty} \int_{U} g_{v_i} \, d\mu \leq \liminf_{i \to \infty} (P(E_i, U) + 2/i) = \liminf_{i \to \infty} P(E_i, U).
\]

Quasiopen sets do not form a topology, see Remark 9.1, but it is easy to see that the intersection of a 1-quasiopen set and an open set is 1-quasiopen.
Thus we have proved lower semicontinuity in the case of sets of finite perimeter. Then consider the function $u$. Note that it is enough to prove the lower semicontinuity for a subsequence. We have $u_i \to u$ in $L^1_{\text{loc}}(U)$, which means that for every $x \in U$ there exists $r_x > 0$ such that $u_i \to u$ in $L^1(B(x, r_x) \cap U)$. Consider the cover $\{B(x, r_x)\}_{x \in U}$. We know that the space $X$ is separable, see e.g. [2, Proposition 1.6], and this property is inherited by subsets of $X$. Thus $U$ is separable, and so it is also Lindelöf, meaning that every open cover of $U$ has a countable subcover, see [19, pp. 176–177]. Thus there exists a countable subcover $\{B(x_j, r_j)\}_{j \in \mathbb{N}}$ of $U$.

Consider the ball $B(x_1, r_1)$. We have $u_i \to u$ in $L^1(B(x_1, r_1) \cap U)$, and so by passing to a subsequence (not relabeled), for a.e. $t \in \mathbb{R}$ we have $\chi_{\{u_i > t\}} \to \chi_{\{u > t\}}$ in $L^1(B(x_1, r_1) \cap U)$, see e.g. [3, p. 188]. By a diagonal argument, we find a subsequence (not relabeled) such that for each $j \in \mathbb{N}$ and a.e. $t \in \mathbb{R}$ we have $\chi_{\{u_i > t\}} \to \chi_{\{u > t\}}$ in $L^1(B(x_j, r_j) \cap U)$. Since the balls $B(x_j, r_j)$ cover $U$, we conclude that for a.e. $t \in \mathbb{R}$, $\chi_{\{u_i > t\}} \to \chi_{\{u > t\}}$ in $L^1_{\text{loc}}(U)$.

We can assume that $\|Du_i\|(U) < \infty$ for all $i \in \mathbb{N}$, and so by Proposition 3.8 $\int_{t}^{\infty} P(\{u_i > t\}, U) \, dt < \infty$ for all $i \in \mathbb{N}$, and in particular, for each $i \in \mathbb{N}$ the mapping $t \mapsto P(\{u_i > t\}, U)$ is measurable, enabling us to use Fatou’s lemma. Moreover, we are able to use the lower semicontinuity for sets of finite perimeter proved above, because for a.e. $t \in \mathbb{R}$, $P(\{u_i > t\}, U) < \infty$ and $P(\{u_i > t\}, U) < \infty$ for all $i \in \mathbb{N}$. Indeed, now we use Proposition 3.8 the lower semicontinuity for sets of finite perimeter proved above, and Fatou’s lemma to obtain

$$\|Du\|(U) = \int_{-\infty}^{\infty} P(\{u > t\}, U) \, dt \leq \int_{-\infty}^{\infty} \liminf_{i \to \infty} P(\{u_i > t\}, U) \, dt \leq \liminf_{i \to \infty} \int_{-\infty}^{\infty} P(\{u_i > t\}, U) \, dt = \liminf_{i \to \infty} \|Du_i\|(U).$$

Knowing that the total variation is lower semicontinuous in a wider class of sets than just the open sets should prove useful in dealing with various minimization problems. In the upcoming work [23] we need lower semicontinuity of the total variation in the super-level sets of a given Newton-Sobolev function $w \in N^{1,1}(X)$. Such sets are 1-quasiopen since functions in the class $N^{1,1}(X)$ are 1-quasiconvex; see [6] for more on these concepts.
Finally, we give the proof of Theorem 4.3.

Proof of Theorem 4.3. Suppose that $\|Du\|(U) < \infty$. First suppose also that there exists $M > 0$ and an open set $\Omega \supset U$ such that $-M \leq u \leq M$ on $\Omega$, and $\|Du\|(\Omega) < \infty$. Again, denote by $a(u, U)$ the infimum in the statement of the theorem. It is obvious that a function $g$ is a 1-weak upper gradient of a function $v$ if and only if $g/M$ is a 1-weak upper gradient of $v/M$. Using this fact and Proposition 4.4, we obtain

$$\|Du\|(U)/M = \|D(u/M)\|(U) = a(u/M, U) = a(u, U)/M,$$

so that

$$\|Du\|(U) = a(u, U).$$

Then suppose we only have $\|Du\|(U) < \infty$. This means that there exists an open set $\Omega \supset U$ such that $u \in L^1_{\text{loc}}(\Omega)$ and $\|Du\|(\Omega) < \infty$. Define the truncations

$$u_M := \min\{M, \max\{-M, u\}\}, \quad M > 0,$$

and apply Theorem 4.5 and the first part of the current proof to obtain

$$\|Du\|(U) \leq \liminf_{M \to \infty} \|Du_M\|(U) = \liminf_{M \to \infty} a(u_M, U) \leq \liminf_{M \to \infty} a(u, U) = a(u, U).$$

Contrary to the case of open sets, lower semicontinuity can actually be violated in 1-quasiopen sets if the limit function is not a BV function. Thus the requirement $\|Du\|(U) < \infty$ in Theorem 4.5 is essential.

Example 4.6. Let $X = \mathbb{R}^2$ (unweighted). Denote the origin by 0, and let $U := \{0\}$. The set $B(0, r)$ is open for all $r > 0$, and it is easy to check that $\text{Cap}_1(B(0, r)) \leq 3\pi r$ for $0 < r \leq 1$. Thus $U$ is a 1-quasiopen set. Let

$$E := \bigcup_{i=1}^{\infty} \{(x_1, x_2) \in \mathbb{R}^2 : (2i + 1)^{-1} < x_1 < (2i)^{-1}\}.$$

It is well known that $P(E, B(0, r)) = \mathcal{H}^1(\partial^* E \cap B(0, r))$, where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure, see e.g. [1, Theorem 3.59]. Thus clearly $P(E, B(0, r)) = \infty$ for all $r > 0$, and so $P(E, U) = \infty$. Next, let

$$E_k := \bigcup_{i=1}^{k} \{(x_1, x_2) \in \mathbb{R}^2 : (2i + 1)^{-1} < x_1 < (2i)^{-1}\}, \quad k \in \mathbb{N}.$$
Then $\chi_{E_k} \to \chi_E$ even in $L^1_{\text{loc}}(\mathbb{R}^2)$ (and obviously in $L^1_{\text{loc}}(U)$). On the other hand, for any $k \in \mathbb{N}$ and $r < (2k+1)^{-1}$,

$$P(E_k, U) \leq P(E_k, B(0, r)) = 0,$$

since $E_k$ does not intersect the open set $B(0, r)$. Thus

$$P(E, U) > \lim_{k \to \infty} P(E_k, U),$$

that is, lower semicontinuity is violated.

Similarly we see that without the assumption $\|Du\|(U) < \infty$, Theorem 4.3 fails with the choice $u = \chi_E$, as the left-hand side is $\infty$ but the right-hand side is zero.

It would be interesting to know if the conclusions of Theorem 4.3 and Theorem 4.5 actually characterize 1-quasiopen sets.

**Open Problem.** Let $U \subset X$ be a $\mu$-measurable set such that the conclusion of Theorem 4.3 or the conclusion of Theorem 4.5 holds. Is $U$ then a 1-quasiopen set?

To conclude this section, we apply Lemma 3.4 to prove a somewhat different but quite natural characterization of 1-quasiopen sets, given in Proposition 4.10 below. For other characterizations of quasiopen sets, see [6]. First we take note of the following facts. By [12, Theorem 4.3, Theorem 5.1] we know that for any $A \subset X$,

$$\text{Cap}_1(A) = 0 \text{ if and only if } \mathcal{H}(A) = 0. \quad (4.7)$$

The following proposition follows from [22, Corollary 4.2] (which is originally based on [24, Theorem 1.1]).

**Proposition 4.8.** Let $u \in BV_{\text{loc}}(X)$ and let $\varepsilon > 0$. Then there exists an open set $G \subset X$ with $\text{Cap}_1(G) < \varepsilon$ such that $u^\wedge|_{X\setminus G}$ is real-valued lower semicontinuous and $u^\vee|_{X\setminus G}$ is real-valued upper semicontinuous.

Moreover, 1-quasiopen sets can be perturbed in the following way.

**Lemma 4.9.** Let $U \subset X$ be a 1-quasiopen set and let $A \subset X$ be $\mathcal{H}$-negligible. Then $U \setminus A$ and $U \cup A$ are 1-quasiopen sets.
Proof. Let \( \varepsilon > 0 \). Take an open set \( G \subset X \) such that \( \text{Cap}_1(G) < \varepsilon \) and \( U \cup G \) is an open set. By (4.7) we know that \( \text{Cap}_1(A) = 0 \), and since \( \text{Cap}_1 \) is an outer capacity, we find an open set \( V \supset A \) such that \( \text{Cap}_1(G) + \text{Cap}_1(V) < \varepsilon \). Now \( (U \setminus A) \cup (G \cup V) = (U \cup G) \cup V \) is an open set with \( \text{Cap}_1(G \cup V) < \varepsilon \), so that \( U \setminus A \) is a 1-quasiopen set. Similarly, \( (U \cup A) \cup (G \cup V) = (U \cup G) \cup V \) is an open set, so that \( U \cup A \) is also a 1-quasiopen set. \( \square \)

In the following, \( \Delta \) denotes the symmetric difference.

Proposition 4.10. Let \( U \subset X \). The following are equivalent:

(1) \( U \) is 1-quasiopen.

(2) There exists \( u \in N^{1,1}_\text{loc}(X) \) with \( \mathcal{H}(\{u > 0\} \Delta U) = 0 \).

(3) There exists \( u \in BV_\text{loc}(X) \) with \( \mathcal{H}(\{u^\wedge > 0\} \Delta U) = 0 \).

Proof.

- \( (1) \implies (2) \): Take a sequence of open sets \( G_i \subset X \) such that \( \text{Cap}_1(G_i) < 2^{-i} \) and each \( U \cup G_i \) is an open set. Then define \( v_i(x) := 2^{-i} \min\{1, \text{dist}(x, X \setminus (U \cup G_i))\}, \quad x \in X, \quad i \in \mathbb{N} \).

  By Lemma 3.4 there exist sets \( V_i \supset G_i \) and functions \( \eta_i \in N^{1,1}_\text{loc}(V_i) \) such that \( 0 \leq \eta_i \leq 1, \eta_i = 1 \) on \( G_i \), and \( \|\eta_i\|_{N^{1,1}(X)} \leq 2^{-i}C_2 \). Let \( u_i := (v_i - \eta_i)_+ \). Then \( 0 \leq u_i \leq 1, u_i = v_i > 0 \) on \( U \setminus V_i \), and \( u_i = 0 \) on \( G_i \). Since also \( u_i \leq v_i = 0 \) on \( X \setminus (U \cup G_i) \), in total \( u_i = 0 \) on \( X \setminus U \).

  For any bounded open set \( \Omega \subset X \), we have

\[
\|u_i\|_{N^{1,1}(\Omega)} \leq \|v_i\|_{N^{1,1}(\Omega)} + \|\eta_i\|_{N^{1,1}(\Omega)} \\
\leq \|v_i\|_{L^1(\Omega)} + \int_{\Omega} g_{v_i} \, d\mu + \|\eta_i\|_{N^{1,1}(X)} \\
\leq 2^{-i} \mu(\Omega) + 2^{-i} \mu(\Omega) + 2^{-i}C_2.
\]

Let \( u := \sup_{i \in \mathbb{N}} u_i \). Then \( \sup_{i \in \mathbb{N}} g_{u_i} \) is a 1-weak upper gradient of \( u \), see [2] Lemma 1.52. Thus \( \|u\|_{N^{1,1}(\Omega)} < \infty \), and so \( u \in N^{1,1}_\text{loc}(X) \). Moreover, \( u > 0 \) on \( U \setminus \bigcap_{i=1}^\infty V_i \), where \( \text{Cap}_1(\bigcap_{i=1}^\infty V_i) = 0 \) and thus \( \mathcal{H}(\bigcap_{i=1}^\infty V_i) = 0 \) by (4.7). On the other hand, \( u = 0 \) on \( X \setminus U \). Thus \( \mathcal{H}(\{u > 0\} \Delta U) = 0 \).
• (2) \implies (3): Take \( u \in N^{1,1}_{\text{loc}}(X) \) with \( \mathcal{H}(\{u > 0\} \Delta U) = 0 \). We know that \( u \) has a Lebesgue point at \( \mathcal{H}\text{-a.e.} \ x \in X \), see [17, Theorem 4.1, Remark 4.2] and (4.7). Thus \( u(x) = u^\wedge(x) \) at \( \mathcal{H}\text{-a.e.} \ x \in X \), and so \( \mathcal{H}(\{u^\wedge > 0\} \Delta U) = 0 \). Furthermore, \( N^{1,1}_{\text{loc}}(X) \subset \text{BV}_{\text{loc}}(X) \) by the discussion after (4.1). Thus \( u \in \text{BV}_{\text{loc}}(X) \).

• (3) \implies (1): Take \( u \in \text{BV}_{\text{loc}}(X) \) with \( \mathcal{H}(\{u^\wedge > 0\} \Delta U) = 0 \). By Proposition 4.8, there exist open sets \( G_i \subset X \) such that \( \text{Cap}_1(G_i) \to 0 \) and for each \( i \in \mathbb{N} \), \( u^\wedge|_{X \setminus G_i} \) is a lower semicontinuous function. Hence the set \( \{u^\wedge > 0\} \) is open in the subspace topology of \( X \setminus G_i \), and so the sets \( \{u^\wedge > 0\} \cup G_i \) are open (in \( X \)). We conclude that \( \{u^\wedge > 0\} \) is a 1-quasiopen set, and then by Lemma 4.9, \( U \) is also 1-quasiopen.

\[ \square \]

5 Uniform absolute continuity

In this section we use the lower semicontinuity result proved in the previous section to show that the variation measures of a sequence of BV functions converging in the strict sense are uniformly absolutely continuous with respect to the 1-capacity \( \text{Cap}_1 \).

First recall the following definition. Given a \( \mu \)-measurable set \( H \subset X \), a sequence of functions \( (g_i) \subset L^1(H) \) is said to be uniformly integrable if the following two conditions are satisfied. First, for every \( \varepsilon > 0 \) there exists a \( \mu \)-measurable set \( D \subset H \) with \( \mu(D) < \infty \) such that

\[
\int_{H \setminus D} g_i \, d\mu < \varepsilon \quad \text{for all } i \in \mathbb{N}.
\]

Second, for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( A \subset H \) is a \( \mu \)-measurable set with \( \mu(A) < \delta \), then

\[
\int_A g_i \, d\mu < \varepsilon \quad \text{for all } i \in \mathbb{N}.
\]

The second condition can be called the uniform absolute continuity of the measures \( g_i \mu \) with respect to \( \mu \). The variation measure of a BV function is usually not absolutely continuous with respect to \( \mu \), but according to Lemma 3.7, it is absolutely continuous with respect to the 1-capacity. Thus we can
analogously talk about the variation measures of a sequence of BV functions being uniformly absolutely continuous with respect to the 1-capacity.

Before stating our main theorem, we gather a few preliminary results. For these, we will also need the concept of BV-capacity, which is defined for a set \( A \subset X \) by

\[
\text{Cap}_{\text{BV}}(A) := \inf \|u\|_{\text{BV}(X)},
\]

where the infimum is taken over all \( u \in \text{BV}(X) \) such that \( u \geq 1 \) in a neighborhood of \( A \). By [12, Theorem 3.4] we know that if \( A_1 \subset A_2 \subset \ldots \subset X \), then

\[
\text{Cap}_{\text{BV}}\left( \bigcup_{i=1}^{\infty} A_i \right) = \lim_{i \to \infty} \text{Cap}_{\text{BV}}(A_i). \tag{5.1}
\]

On the other hand, by [12, Theorem 4.3] there is a constant \( C_{\text{cap}}(C_d, C_P, \lambda) \geq 1 \) such that for any \( A \subset X \),

\[
\text{Cap}_{\text{BV}}(A) \leq \text{Cap}_1(A) \leq C_{\text{cap}} \text{Cap}_{\text{BV}}(A). \tag{5.2}
\]

Thus the 1-capacity and the BV-capacity can often be used interchangeably, but the BV-capacity has the advantage that it is continuous with respect to increasing sequences of sets.

**Lemma 5.3.** Let \( \Omega \subset X \) be an arbitrary set. The space of sets \( A \subset \Omega \) with \( \text{Cap}_1(A) < \infty \), equipped with the metric

\[
\text{Cap}_1(A_1 \Delta A_2), \quad A_1, A_2 \subset \Omega,
\]

is a complete metric space if we identify sets \( A_1, A_2 \subset \Omega \) with \( \text{Cap}_1(A_1 \Delta A_2) = 0 \).

**Proof.** We know that \( \text{Cap}_1 \) is an outer measure, see e.g. [2, Theorem 6.7], and thus it is straightforward to check that \( \text{Cap}_1(\cdot \Delta \cdot) \) is indeed a metric. In particular, note that if \( A_1, A_2 \subset \Omega \) with \( \text{Cap}_1(A_1), \text{Cap}_1(A_2) < \infty \), then \( \text{Cap}_1(A_1 \Delta A_2) \leq \text{Cap}_1(A_1 \cup A_2) < \infty \), so the distance is always finite. To verify completeness, let \( \{A_i\}_{i \in \mathbb{N}} \) be a Cauchy sequence. We can pick a subsequence \( \{A_{i_j}\}_{j \in \mathbb{N}} \) such that \( \text{Cap}_1(A_{i_j} \Delta A_{i_{j+1}}) < 2^{-j} \) for all \( j \in \mathbb{N} \). It follows that \( \text{Cap}_1(A_{i_j} \Delta A_l) < 2^{-j+1} \) for all \( l > j \). Let

\[
A := \bigcap_{k=1}^{\infty} \bigcup_{l=k}^{\infty} A_{i_l},
\]

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so that $A \subset \Omega$. For a fixed $j \in \mathbb{N}$, we now have

\[
A_{ij} \setminus \bigcap_{k=1}^{\infty} \bigcup_{l=k}^{\infty} A_{ij} = A_{ij} \cap \bigcup_{k=1}^{\infty} \left( X \setminus \bigcup_{l=k}^{\infty} A_{ij} \right) = \bigcup_{k=1}^{\infty} \left( A_{ij} \cap \left( X \setminus \bigcup_{l=k}^{\infty} A_{ij} \right) \right) = \bigcup_{k=1}^{\infty} \left( A_{ij} \setminus \bigcup_{l=k}^{\infty} A_{ij} \right).
\]

Thus by (5.1) and (5.2),

\[
\text{Cap}_1(A_{ij} \setminus A) = \text{Cap}_1 \left( \bigcup_{k=1}^{\infty} \left( A_{ij} \setminus \bigcup_{l=k}^{\infty} A_{ij} \right) \right) \leq C_{\text{cap}} \text{Cap}_{BV} \left( \bigcup_{k=1}^{\infty} \left( A_{ij} \setminus \bigcup_{l=k}^{\infty} A_{ij} \right) \right) = C_{\text{cap}} \lim_{k \to \infty} \text{Cap}_{BV} \left( A_{ij} \setminus \bigcup_{l=k}^{\infty} A_{ij} \right) \leq C_{\text{cap}} \lim_{k \to \infty} \text{Cap}_1 \left( A_{ij} \setminus \bigcup_{l=k}^{\infty} A_{ij} \right) \leq C_{\text{cap}} \lim_{k \to \infty} \text{Cap}_1 \left( A_{ij} \setminus A_{ik} \right) \leq C_{\text{cap}} \lim_{k \to \infty} 2^{-j+1} = 2^{-j+1} C_{\text{cap}} \to 0 \text{ as } j \to \infty.
\]

Conversely,

\[
\text{Cap}_1(A \setminus A_{ij}) \leq \text{Cap}_1 \left( \bigcup_{l=j}^{\infty} A_{ij} \setminus A_{ij} \right) = \text{Cap}_1 \left( \bigcup_{l=j}^{\infty} (A_{ij+1} \setminus A_{ij}) \right) \leq \sum_{l=j}^{\infty} \text{Cap}_1 (A_{ij+1} \setminus A_{ij}) \leq \sum_{l=j}^{\infty} 2^{-l} = 2^{-j+1} \to 0 \text{ as } j \to \infty.
\]

Thus $\text{Cap}_1(A_{ij} \Delta A) \to 0$ as $j \to \infty$, and since $\{A_i\}_{i \in \mathbb{N}}$ is a Cauchy sequence, we have $\text{Cap}_1(A_i \Delta A) \to 0$ as $i \to \infty$. It is also clear that $\text{Cap}_1(A) < \infty$. \(\square\)
The following proposition, which follows from Proposition 4.8, provides many 1-quasiopen sets in which the lower semicontinuity result of the previous section can be applied; recall the definitions of the measure theoretic interior $I_E$ and the measure theoretic exterior $O_E$ from (2.1) and (2.2).

**Proposition 5.4 ([20, Proposition 4.2]).** Let $\Omega \subset X$ be an open set and let $E \subset X$ be a $\mu$-measurable set with $P(E, \Omega) < \infty$. Then the sets $I_E \cap \Omega$ and $O_E \cap \Omega$ are 1-quasiopen.

The 1-capacity has the following useful rigidity property.

**Lemma 5.5.** For any $A \subset X$, we have $\text{Cap}_1(I_A \cup \partial^* A) \leq \text{Cap}_1(A)$.

**Proof.** This follows by combining [20, Lemma 3.1] and [20, Proposition 3.8].

Now we give the main result of this section. The proof is partially based on Baire’s category theorem, similarly to the proof of the Vitali-Hahn-Saks theorem concerning uniformly integrable sequences of functions, see e.g. [1, Theorem 1.30].

**Theorem 5.6.** Let $\Omega \subset X$ be an open set, and suppose that $u_i \to u$ in $L^1_{\text{loc}}(\Omega)$ and $\|Du_i\|(\Omega) \to \|Du\|(\Omega)$, with $\|Du\|(\Omega) < \infty$ and $\|Du_i\|(\Omega) < \infty$ for all $i \in \mathbb{N}$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \subset \Omega$ with $\text{Cap}_1(A) < \delta$, then $\|Du_i\|(A) < \varepsilon$ for all $i \in \mathbb{N}$.

**Proof.** Fix $\varepsilon > 0$. By Lemma 3.7 there exists $\alpha > 0$ such that if $D \subset \Omega$ with $\text{Cap}_1(D) < C_1\alpha$, then $\|Du\|(D) < \varepsilon/2$. Fix $A \subset \Omega$ with $\text{Cap}_1(A) < \alpha$. By Lemma 3.3 we find an open set $V \supset A$ with $\text{Cap}_1(V) < C_1\alpha$ and $P(V, X) < C_1\alpha$. By Lemma 5.5 also $\text{Cap}_1(I_V \cup \partial^* V) < C_1\alpha$. By Proposition 5.4 $\Omega \cap O_V$ is a 1-quasiopen set, and thus by the lower semicontinuity Theorem 4.5 we get

$$\|Du\|(\Omega \cap O_V) \leq \liminf_{i \to \infty} \|Du_i\|(\Omega \cap O_V).$$

Since also $\|Du_i\|(\Omega) \to \|Du\|(\Omega)$, we have

$$\|Du\|(\Omega \setminus O_V) \geq \limsup_{i \to \infty} \|Du_i\|(\Omega \setminus O_V),$$

that is,

$$\|Du\|(\Omega \cap (I_V \cup \partial^* V)) \geq \limsup_{i \to \infty} \|Du_i\|(\Omega \cap (I_V \cup \partial^* V)).$$
But since \( \text{Cap}_1(I_V \cup \partial^* V) < C_1 \alpha \), we get

\[
\limsup_{i \to \infty} \| D u_i \|(\Omega \cap (I_V \cup \partial^* V)) < \varepsilon/2.
\]

Moreover, \( A \subset \Omega \cap V \subset \Omega \cap (I_V \cup \partial^* V) \), since \( V \) is open. In conclusion,

\[
A \subset \Omega \text{ and } \text{Cap}_1(A) < \alpha \implies \limsup_{i \to \infty} \| D u_i \|(A) < \varepsilon/2. \tag{5.7}
\]

Consider the metric space defined in Lemma 5.3. Define the sets

\[
A_k := \{ D \subset \Omega : \text{Cap}_1(D) < \infty \text{ and } \sup_{i \geq k} \| D u_i \|(D) \leq \varepsilon/2 \}, \quad k \in \mathbb{N}.
\]

We show that these sets are closed. Fix \( k \in \mathbb{N} \) and then fix \( i \geq k \). Let \( D \subset X \) with \( \text{Cap}_1(D) < \infty \). If \( D_n \in A_k, \ n \in \mathbb{N}, \) is a sequence with \( \text{Cap}_1(D_n \Delta D) \to 0 \), then since \( \| D u_i \| \) is absolutely continuous with respect to \( \text{Cap}_1 \), we have

\[
\| D u_i \|(D) \leq \liminf_{n \to \infty} (\| D u_i \|(D \setminus D_n) + \| D u_i \|(D_n)) = 0 + \liminf_{n \to \infty} \varepsilon/2 = \varepsilon/2.
\]

Since \( i \geq k \) was arbitrary, we have \( D \in A_k \), so \( A_k \) is closed.

Let

\[
\mathcal{Y} := \{ D \subset \Omega : \text{Cap}_1(D) < \alpha \}.
\]

By (5.7), \( \mathcal{Y} = \bigcup_{k=1}^{\infty} (A_k \cap \mathcal{Y}) \). Since \( \mathcal{Y} \) is an open subset of a complete metric space, Baire’s category theorem applies. Thus at least one of the sets \( A_k \) has nonempty interior in \( \mathcal{Y} \). That is, there exists \( D \in \mathcal{Y} \) and \( \tilde{\delta} > 0 \) such that every \( H \subset \Omega \) with \( \text{Cap}_1(H \Delta D) < \tilde{\delta} \) belongs to \( A_k \). Take any \( A \subset \Omega \) with \( \text{Cap}_1(A) < \tilde{\delta} \). Then

\[
\text{Cap}_1((D \cup A) \Delta D) < \tilde{\delta}
\]

and so

\[
\sup_{i \geq k} \| D u_i \|(A) \leq \sup_{i \geq k} \| D u_i \|(D \cup A) \leq \varepsilon/2 < \varepsilon.
\]

By Lemma 3.7, we find \( \tilde{\delta} > 0 \) such that if \( A \subset \Omega \) with \( \text{Cap}_1(A) < \tilde{\delta} \), then \( \| D u_i \|(A) < \varepsilon \) for all \( i = 1, \ldots, k - 1 \). Finally, we let \( \delta := \min\{\tilde{\delta}, \delta\} \). \( \square \)
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