Quantum phase transition and spontaneous symmetry breaking in a nonlinear quantum Rabi model

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The experimental advance on light-matter interaction into strong couplings has invalidated Jaynes-Cummings model and brought quantum Rabi model (QRM) to more relevance. The QRM only involves linear coupling via a single-photon process (SPP), while nonlinear two-photon process (TPP) is weaker and conventionally neglected. However, we find a contrary trend that enhancing the linear coupling might not suppress more the nonlinear effect but backfire to trigger some collapse of linear characters. Indeed, in strong SPP couplings a tiny strength of TPP may dramatically change properties of the system, like a symmetry spontaneous breaking. By extracting the ground-state phase diagram including both SPP and TPP, we find TPP in low frequency limit induces a quantum phase transition with continuity-discontinuity double faces, which split into two distinct transitions at finite frequencies and yields a triple point. Our analysis unveils a subtle SPP-TPP entanglement.

PACS numbers:

Introduction.– Recently significant efforts in experiments have pushed the exploration of fundamental quantum physics in light-matter coupling systems toward the (ultra-)strong coupling regime \[^{[1]}^{10}\]. This experimental enhancement on coupling strength has rendered the Jaynes-Cummings model less valid\[^{[1]}^{2}^{,}[5]\] and brought the quantum Rabi model (QRM) \[^{[1]}^{11}\] more to the front of investigation for light-matter interaction. Also theoretically the remarkable finding of integrability of the QRM \[^{[1]}^{12}\] has added a great fuel to heat up the interest \[^{[1]}^{13}^{–}[21],[25],[26]\] in the QRM which is particular for few-body quantum phase transition (QPT) \[^{[1]}^{17}^{,}[21],[25],[26]\], while QPTs usually occur in the thermodynamical systems. Indeed, there exists an universality of critical behavior for arbitrary particles that bridges the QPTs of the few-body limit and thermodynamical limit \[^{[2]}^{20}\].

Conventionally the QRM is a linear model which involves the coupling of a two-level or spin-half system and a bosonic mode via a single-photon process (SPP) of absorption and emission, while the nonlinear two-photon process (TPP) \[^{[2]}^{20}\], which involves a simultaneous emission or absorption of two photons \[^{[2]}^{27}\], is usually much weaker and not taken into account. Nowadays, the nonlinear process has been realized in different systems, e.g. Rydberg atoms \[^{[2]}^{28}^{,}[29]\] and quantum dots \[^{[2]}^{30}^{,}[32]\] in microwave cavities, and enhanced in trapped ions \[^{[2]}^{33}^{,}[34]\] and superconducting circuits \[^{[2]}^{35}\]. In such a situation, the issue of TPP-SPP competition arises. The traditional way to see the TPP effect is to suppress the SPP \[^{[2]}^{35}^{–}[37]\], while knowledge is lacking in wondering reversely the effect to strengthen the SPP, especially when QPT is relevant. Now that the enhancement of the SPP coupling is approaching QPT regime \[^{[2]}^{11}^{,}[10]\], the TPP-SPP competition calls for a new light in the context of QPT.

The model.– We consider the QRM with both SPP and TPP, \[^{[2]}^{H} = \omega a^\dagger a + \frac{\Omega}{2} \sigma_z g_1 \sigma_x [a^\dagger + a] + g_2 \sigma_z (a^\dagger)^2 + a^2\] , where \[^{[2]}^{a^\dagger} \text{(a)}\] is bosonic creation (annihilation) operator with frequency \[^{[2]}^{\omega}\], while \[^{[2]}^{\sigma_{x,y,z}}\] is the Pauli matrix with level splitting \[^{[2]}^{\Omega}\] and \[^{[2]}^{g_1,g_2}\] are the coupling strengths in the SPP and TPP. It is convenient to rotate around \[^{[2]}^{\sigma_y}\] so that the Hamiltonian takes the form

\[^{[2]}^{H} = \omega a^\dagger a + \frac{\Omega}{2} \sigma_z g_1 \sigma_x [a^\dagger + a] + g_2 \sigma_z (a^\dagger)^2 + a^2\] . (1)

In this form the \[^{[2]}^{\Omega}\] term effectively plays a role of tunneling between the spin-up and spin-down states along \[^{[2]}^{z}\] direction \[^{[2]}^{[19],[38]}\]. We adopt the notation in \[^{[2]}^{[1]}\] for further discussions.

Spontaneous symmetry breaking.– Intuitively one would think that enhancement of the SPP should suppress the effect of TPP, however we find the reality is con-
under simultaneous sign exchanges of effective displace-
ments wave function $|\psi\rangle$. The dashed line represents our analytic $g_{1c}$ (b) 3-dimensional (3D) view for $\langle \sigma_z \rangle$ with $\langle \sigma_z \rangle = 0$ at $g_2 = 0$ while there is a jump around $g_2 = 0$ for $g_1 > g_c$. (c-e) Evolution of the spin-down function $\psi^-$ versus $g_1$ at $g_2/g_1 = 0$, 0.01, 0.3. (f) Profile of $\psi^+$ and $\psi^-$ at $g_2/g_1 = 0, \pm 10^{-13}$ and $g_1 = 1.5 g_c$.

Monitoring the evolution of the wave function will make us more recognize the essential change in the na-
ture of quantum state. Indeed, as shown in Fig.1(c), in the absence of TPP, below (above) $g_c$ the system is in a single-branch state (double-branch state), which has one single wave packet (two separated wave packets) in the wave function $|\psi\rangle = \psi^+ |\uparrow\rangle + \psi^- |\downarrow\rangle$ with spin-$z$ components $\psi^\pm$. At $g_2 = 0$ the profile of $\psi^\pm (x)$ is symmetric under simultaneous sign exchanges of effective displace-
ment $x = (a^\dagger + a)/\sqrt{2}$ and spin, as in Fig.1(f), leading to vanishing $\langle \sigma_z \rangle$. In the presence of TPP, the situation is similar before the transition, but the transition leads to a broken-branch state in which only one branch survives for both $\psi^\pm$, as illustrated in Fig.1(d-f). Note here that broken-branch state occurs even for a tiny TPP strength as $g_2 = 10^{-10} g_c$.

We extract the phase diagram of $\langle \sigma_z \rangle$ by exact diagonalization in Fig.1(a). We see that $\langle \sigma_z \rangle$ is vanishing until the border $g_{1c} = g_c \sqrt{1 - g_2^2 / g_1^2}$ (red line), which is reduced by the TPP. Beyond the border $\langle \sigma_z \rangle$ jumps to finite values, and there is a sign change across the linear QRM line at $g_2 = 0$, while along this line $\langle \sigma_z \rangle$ remains zero which is discontinuous from the regions beside as afore-discussed. Such spontaneous symmetry breaking behavior can also be seen in other physical quantities, e.g. the displacement expectation $|a^\dagger + a|$. *Continuity-discontinuity double faces in transition.* The spin expectation $\langle \sigma_z \rangle$ yields another scenario. Figures 2(a,b) illustrate the behavior of $\langle \sigma_z \rangle$ for $\omega/\Omega = 0.001$. At $g_2 = 0$, the weak coupling regime remains flat as $\langle \sigma_z \rangle \sim -1$, while $\langle \sigma_z \rangle$ starts to rise beyond $g_{1c}$, around which the continuous change of $\langle \sigma_z \rangle$ indicates a quantum phase transition of second order. At finite $g_2$, the transition becomes discontinuous (first order) with a jump of $\langle \sigma_z \rangle$ at $g_{1c}$. Let us denote this jump by $\Delta^c_z$, while the counterpart in $\langle \sigma_z \rangle$ by $\Delta^s_z$. Although phase diagram of $\langle \sigma_z \rangle$ seems to show a "same" transition boundary $g_{1c}$, as $\langle \sigma_z \rangle$, a detailed comparison reveals a character different from $\langle \sigma_z \rangle$. On the other hand, by switching on $g_2$, for $\langle \sigma_z \rangle$ the transition at $g_{1c}$ evolves continuously from second order to first order so that $\Delta^c_z$ increases smoothly ($\Delta^c_z = 2 g_2^2 / (g_2^2 + g_1^2)$) as in Fig.2(b)), in a contrast to $\Delta^s_z$ which follows $\Delta^s_z = -2 g_2 g_1 / (g_2^2 + g_1^2)$ at finite $g_2$ but shows a jump upon turning on $g_2$. On the other hand, beyond $g_{1c}$ the dependence of $\langle \sigma_z \rangle$ on $g_2$ is continuous, only discontinuous in the derivative with respect to $g_1$ due to the cusp around $g_2 = 0$ (see Fig.2(c)), unlike $\langle \sigma_z \rangle$ which itself is already discontinuous. We find different nature underlying behind the double faces of discontinuity and continuity in $\langle \sigma_z \rangle$ and $\langle \sigma_z \rangle$. Actually there are two transitions to which $\langle \sigma_z \rangle$ and $\langle \sigma_z \rangle$
is scaled by its maximum along $g$. In increasing $g$ sufficiently, \(\langle g \rangle\) is shown in Figures 3(a,b) showing the wave-function evolution $\omega/\Omega$ at two boundaries, respectively, of single-, double- and broken-branch states. We denote the three distinct regimes, respectively in \(\{\Omega = 0\}\), by the spin-component weight which is related to spin expectation via $\rho_\pm = (1 + (\sigma_z))/2$ and $x_{0,\pm} = g_1/(1 \pm g_2^0)$, with $g_1^0 = \sqrt{2g_1}/\omega$ and $g_2^0 = 2g_2/\omega$, is the potential displacement. The color contrast of the $\tilde{x}_+$ map in Fig.3(c) clearly shows three regions with the two boundaries below a triple point. Indeed, by increasing $g_1$ at a fixed $g_2$ the value of $\tilde{x}_+$ is small and varies little in the first region, but starts to increase fast after entering the second region, and transits to the third region with a sign reversal. $\tilde{x}_+$ does not change sign in this $g_1$-$g_2$ quadrant but has a clear change of gradient at both boundaries, as indicated by the local peaks of $d\tilde{x}_-/(dg_1)$ in Fig.3(f) where one sees more clearly the triple point around $g_2 \sim 10^{-2}g_1$.

It should be mentioned that, although the second boundary $g_1^0$ opens a window within which the vanishing-$\langle \sigma_z \rangle$ scenario of linear SPP feature does not collapse in the presence of a nonlinear TPP, the limit value of $g_2$ to prevent this collapse is still very tiny, as shown by the negatively large logarithmic value of $g_2$ to base 10. This means that a small TPP perturbation might still be able to bring about a dramatic change to all the quantities that are sensitive to the second transition. Furthermore, from the deceasing of $\log (g_2^0/g_1)$ we see in a quantitative way that once going beyond the first boundary $g_1^0$, a stronger SPP will counterproductively be even more vulnerable to the TPP disturbance.

Underlying mechanisms. It will facilitate the understanding if we rewrite Hamiltonian \(\hat{H}\) in terms of the quantum harmonic oscillator \(a^\dagger = (\hat{x} - i\hat{p})/\sqrt{2}\), \(a = (\hat{x} + i\hat{p})/\sqrt{2}\), where $\hat{x}$ and $\hat{p}$ are position and momentum, as $H = \sum_{\sigma_z=\pm} (h^2/2m)(\sigma_z)^2/2\langle\sigma_z\rangle$, where $\langle\sigma_z\rangle = -\sigma_z$ and $+(-)$ labels the up (down) spin,

$$h^\pm = \omega(p^2/2m^\pm + v^\pm + \varepsilon_0), \quad v^\pm = v^\text{bp}_\pm + b_\pm$$

where $m^\pm = (1 \mp g_2^0)^{-1}$ is effective mass and $\varepsilon_0 = -[g_1^0/(1 - g_2^0) + 1]/2$ is a constant. Interestingly, apart from the asymmetric harmonic potential $v^\text{bp}_\pm = m^\pm \omega^2[x \pm x_{0,\pm}]^2/2$, which however has a degenerate frequency $\omega = \sqrt{1 - g_2^0}$, the TPP also leads to an effective bias field $b_\pm = \pm \sigma_0 g_2^0 g_2^0/(2(1 - g_2^0))$ to raise the degeneracy without an external field. We see that, although induced by the TPP, the bias $b_\pm$ becomes larger if one increases the SPP strength $g_1$. This entangled bias leads to different transition scenarios above and below the triple point.

In reality, whether degeneracy raising comes into final effect depends on what quantum state the system is in.
As indicated by $x_{0.1}$ in $v^{hp}_x$, one effect of the linear interaction $g_1$ is to separate the potentials. However, the relative large tunneling strength $\Omega/\omega$ at low frequencies prevents the true separation of spin-up and spin-down wave packets $\psi^+$ and $\psi^-$, as depicted in Fig.4(a), since larger wave-packet overlap gains more negative tunneling energy. As a result, the two spin components will stay together at the origin $x=0$ to form a single-branch state before transition. It happens that the harmonic potential at the origin, $v^{hp}_x(0) = (1 \mp g_2^2)g_1^2/[2(1 - g_2^2)]$, cancels with the bias energy $b_x$, so that $\psi^+$ and $\psi^-$ have a same potential. Thus, even a semiclassical consideration with $\tilde{p}^2 \to 0$ or a quantum energy, $\varepsilon_x = \langle \phi_x^+ | h | \phi_x^+ \rangle = (\omega - 1)\omega/2$ by undisplaced harmonic oscillator ground state $\phi_0^+$, yields a degenerate energy. This degeneracy leads to the vanishing $\langle \sigma_z \rangle$ in single-branch state.

Further enhancement of the SPP with larger $g_1$ will also enlarge the bias gap, as depicted in Fig.4(b), due to the entanglement of $g_1$ and $g_2$. Once the tunneling at the origin cannot afford the high potential cost in more separated potential $v^{hp}_x$, a transition occurs from single-branch state to broken-branch state at $g_1c$. However, in the presence of a weak TPP below the triple point, the energy competitions are more separated. The bias opening is slowed down by small $g_2$ so that the potential cost resulting from $g_1$-driven horizontal separation of $v^{hp}_x$ dominates first in competition with the tunneling, which leads to transition to double-branch state around $g_1c$. However, the two-side distribution due to the small bias.

Increasing $g_1$ deeper eventually becomes detrimental to the balance: $g_1$ not only separates $v^{hp}_x$ more thus leading to a faster decay in left-right tunneling, but also is enlarging the bias. This triggers the second transition between double- and broken-branch states. Around this transition, the final state $|\Psi\rangle = |\psi_{RL}\rangle + \delta_2|\psi_L\rangle$ is a superposition of the right/left states $\psi_{RL}$, with energy $\varepsilon_{RL}$, in same-side tunneling $\Omega_{\alpha\beta}$ and $\Omega_{\beta\beta}$, by a perturbation $\delta_2 = (\Omega_{\alpha\alpha} + \Omega_{\beta\beta})/(\varepsilon_L - \varepsilon_R)$ from left-right tunneling, which leads us to the analytic second boundary $g_{2c}$ at small $g_2$ and finite frequencies, located by $\delta_c \approx e^{-1}$:

$$g_{2c}(g_1) \approx \exp[-c^2g_{2c}(2\omega)] / \delta_c \approx \frac{g_1}{g_2}, \quad \frac{g_2}{g_1} = g_1, \quad \frac{g_2}{g_1} = g_2, \quad (3)$$

where $c = \sqrt{1 - g_1^{1.4}}$. The accuracy of $g_{2c}(\epsilon)$ is visible in Fig.3(d). The triple point may be estimated by $g_{2c}(\epsilon_{RL}) [4]$. We see that $g_{2c}$ is fully a quantum-mechanics effect, the left-right wave-packet overlap cannot be captured by a semiclassical consideration with a mass point $[25]$. Note that in the broken-branch state $|\psi_L\rangle$, the spin-up component has less weight than spin-down due to potential imbalance, thus leading to finite $\langle \sigma_z \rangle$, while in the double-branch state this spin imbalance cancels between $|\psi_R\rangle$ and $|\psi_L\rangle$ thus remaining in a vanishing $\langle \sigma_z \rangle$ as in the single-branch state. This is the reason why $\langle \sigma_z \rangle$ is sensitive to the second transition but responseless to the first one. As for $\langle \sigma_x \rangle$, the potential imbalance is opened in the transition from single-branch state to the double-branch state, which leads to a quick reduction of $\psi^+ - \psi^-$ overlap, giving a fast decrease of the spin flipping i.e. the $\langle \sigma_z \rangle$ amplitude at the first transition. But in shifting from double-branch state to broken-branch state the sum of leading same-side overlaps $S_{\alpha\alpha}$ and $S_{\beta\beta}$ is not affected, thus no sign of the second transition is observed in $\langle \sigma_z \rangle$. Concerning the low frequency limit, the much narrower wave packets relative to the packet distance result in an immediate decay of left-right overlap in the $g_1$-driven separation, thus the region between the two transitions becomes more invisible.

Conclusions.–We have seen that the TPP-SPP interplay leads to an entangled effective bias, which tends to raise spin degeneracy without an external field [40]. Strengthening SPP does not dwarf but enhances the role of TPP. In a strong SPP even a tiny strength of TPP can be crucial and lead to a spontaneous symmetry breaking behavior. At finite frequencies we unveil two successive transitions hidden in the weak TPP regime, with the first-order transition in strong TPP splitting into second-order and first-order-like ones below a triple point. Different groups of physical quantities are distinguished to be sensitive, respectively or simultaneously, to these transitions, thus useful for detections. The clarified mechanism shows a delicate competition of the TPP, SPP and tunneling. Our results are relevant for enhanced SPP
couplings in rapid experimental progress \cite{1,10} with the increasing interest in the TPP \cite{25,57}. We expect these phenomena might also leave imprints in other analysis such as dynamics \cite{17,33,43}, which could be a future work.

Acknowledgements Z.-J.Y. acknowledges the financial support of the Future and Emerging Technologies (FET) program under FET-Open Grant No. 618083 (CNTQC). L.C. and X.-M.S. acknowledge National Science Foundation of China (Grants No. 11325417 and No. 11674139). We are thankful to Prof. Hong-Gang Luo for valuable discussions.

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The expressions of $g_t$, $\Delta_x$, $\Delta_z$ in the semiclassical limit and $g_{tc}^{(n)}$ at finite frequencies can be derived in terms of an analytic variational method in strong couplings \cite{10} as well as variational polaron picture \cite{19,45}. For full $g_{-t}^2$ quadrants, $\langle \sigma_x \rangle$, $\langle \sigma_x (a_1 + a) \rangle$ are even for inversion of $g_1$ or $g_2$; $\langle \sigma_z \rangle$ and $\langle \sigma_z (a_1^2 + a^2) \rangle$ are even for $g_1$ and odd for $g_2$; $\langle a_1^2 + a \rangle$ is odd for both $g_1$ and $g_2$; $\langle a_1^2 + a \rangle$ is odd for $g_1$ (in the same ( exchanged) spin). Besides the $g_2 = 0$ line there is another point of spontaneous symmetry breaking at $g_1, g_2 = 0, g_t$ for $\langle a_1^2 + a \rangle$ and $\langle a_1^2 + a \rangle$.

Z.-J. Ying (unpublished).

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To estimate analytic triple point at low frequencies it needs to take squeezing effect into account which yields $g_{tc}^{(3)} \approx \sqrt{g_t^2 + \frac{x_0^2}{12}} (f^4 + 1 + f^{-4} + 24 f^{-4} \tilde{g}_t^2)$ where $f = \sqrt{1 + 432g_t^2}$, $\tilde{g}_t = g_t/\omega_c$, $\omega_c = x_c / \omega$ and $x_c = 1.8$, see Z.-J. Ying H.-G. Luo, H.-Q. Lin, and J. Q. You, arxiv (to appear).

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