THE NEW WEIGHTED INVERSE RAYLEIGH DISTRIBUTION
AND ITS APPLICATION

Demet Aydin

Abstract. In this study, a new weighted version of the inverse Rayleigh distribution based on two different weight functions is introduced. Some statistical and reliability properties of the introduced distribution including the moments, moment generating function, entropy measures (i.e., Shannon and Rényi) and survival and hazard rate functions are derived. The maximum likelihood estimators of the unknown parameters cannot be obtained in explicit forms. So, a numerical method has been required to compute maximum likelihood estimates. Finally, the daily mean wind speed data set has been analysed to show the usability of the new weighted inverse Rayleigh distribution.

Keywords: New weighted inverse Rayleigh distribution; Shannon entropy; hazard rate function; Fisher information matrix; wind speed data.

1. Introduction

The accuracy of procedures in the statistical analysis depends on the suitability of a distribution used in modeling a data set. Therefore, many statistical distributions have been proposed in the literature because it is very important to determine the distribution which provides the best fit to a data set.

One of the widely-used statistical distributions in the context of reliability studies is the inverse Rayleigh (IR) distribution introduced by Trayer [24]. Sherina and Oluyede [25] stated that the distribution of lifetimes of several types of experimental units can be modeled by the IR distribution. Various extensions of this distribution have been proposed in the literature: transmuted IR distribution [1], modified IR distribution [10], kumaraswamy IR distribution [21] and beta IR distribution [12].

On the other hand, the theory of weighted distributions introduced by Rao [17] and Fisher [3] provides a unifying approach to deal with the problems of model
specification and data interpretation (see [9]). There are more studies on weighted distributions and their applications in various fields including ecology and reliability (see [6], [7], [16], [14], [15], [19], [13] and [4] among the others). Fatima and Ahmad [8] also introduced a weighted IR (WIR) distribution with a single weight function \( w(x) = x^k \) where \( k \geq 0 \), and they studied several of its properties.

The objective of the paper is to introduce a new weighted version of IR distribution obtained by using two different weight functions and to discuss its basic characteristics.

The rest of the paper is organized as follows. The new WIR (NWIR) distribution is introduced in Section 2. Some of its statistical and reliability properties are given in Section 3. Equations of maximum likelihood estimates of parameters and a Fisher information matrix are obtained in Section 4. In Section 5, an application of the distribution to real data is presented. Finally, the paper ends with a conclusion.

2. The New Weighted Inverse Rayleigh Distribution

Suppose that \( X \) is a non-negative random variable with its probability density function (pdf), and \( w(x) \) is weight function where \( E(w(x)) < \infty \). The pdf of weighted distribution of \( X \) can be defined as

\[
f_w(x) = \frac{w(x) f(x)}{E(w(x))}.
\]

(2.1)

It should be noted that a general class of weight functions \( w(x) \) can be defined by

\[
w(x) = x^i e^{jx} F^k (x) (1 - F(x))^l,
\]

see [23]. Weight functions can be determined for a different combination of \( i, j, k \) and \( l \) values. If we take \( w(x) = x^i \), then the obtained distribution is called size-biased distribution, and it is length-biased distribution for \( i = 1 \).

Let \( X \) be a random variable with the IR distribution having the scale parameter \( \lambda \). The pdf and cumulative density function (cdf) of the IR distribution are given by

\[
f(x) = 2\lambda x^{-3} e^{-\lambda x}, \quad x > 0, \lambda > 0,
\]

\[
F(x) = e^{-\lambda x}, \quad x > 0, \lambda > 0,
\]

respectively. Now, substituting the multiplication of weighted functions, \( w_1(x) = x^{-\alpha} \) and \( w_2(x) = e^{-\alpha x^{-2}} \), and pdf of IR distribution in (2.1), the pdf of the NWIR distribution is defined by

\[
f_w(x) = \frac{w_1(x) w_2(x) f(x)}{E(w_1(x) w_2(x))} = \frac{2 (\alpha + \lambda)^{\frac{\alpha}{2}} + 1}{\Gamma \left( \frac{\alpha}{2} + 1 \right)} x^{-(\alpha + 3)} e^{-(\alpha + \lambda) x^{-2}}, \quad x > 0, \lambda > 0, \alpha > 0,
\]

(2.2)
where

\[
E\left(w_1(x)w_2(x)\right) = \int_0^\infty 2\lambda x^{-(\alpha+3)}e^{-(\alpha+\lambda)x^{-2}} \, dx
\]

\[
= \frac{\lambda \Gamma\left(\frac{\alpha}{2} + 1\right)}{(\alpha + \lambda)^{\frac{\alpha}{2}+1}} < \infty.
\]

It should be noted that the following transformation is applied in order to calculate \(E\left(w_1(x)w_2(x)\right)\)

\[
(2.3) \quad u = (\alpha + \lambda)x^{-2} \implies x = \frac{\alpha + \lambda}{u} \implies du = -2(\alpha + \lambda)x^{-3} \, dx.
\]

The corresponding cdf of the NWIR distribution is

\[
(2.4) \quad F_w(x) = \frac{\Gamma\left(\frac{\alpha}{2} + 1, \frac{x}{\alpha + \lambda}\right)}{\Gamma\left(\frac{\alpha}{2} + 1\right)}
= 1 - \frac{\gamma\left(\frac{\alpha}{2} + 1, \frac{x}{\alpha + \lambda}\right)}{\Gamma\left(\frac{\alpha}{2} + 1\right)}.
\]

Here \(\Gamma\left(\frac{\alpha}{2} + 1, \frac{x}{\alpha + \lambda}\right)\) is an upper incomplete Gamma function defined by

\[
\Gamma(a, x) = \int_x^\infty t^{a-1}e^{-t} \, dt.
\]

\[
\Gamma(a, x) = \Gamma(a) - \gamma(a, x),
\]

where \(\gamma(a, x)\) is a lower incomplete Gamma function as

\[
\gamma(a, x) = \int_0^x t^{a-1}e^{-t} \, dt.
\]

In FIG. 2.1, different pdf and cdf plots of the NWIR distribution are presented for the selected values of parameters \(\alpha\) and \(\lambda\). Now, let \(Y = (\alpha + \lambda)X^{-2}\), where \(X\) has the NWIR distribution with parameters \(\alpha\) and \(\lambda\). The pdf of the random variable \(Y\) becomes

\[
f(y) = \frac{1}{\Gamma\left(\frac{\alpha}{2} + 1\right)}y^{\frac{\alpha}{2}}e^{-y}
\]

for \(y > 0\). Thus, the random variable \(Y\) has a Gamma distribution shown as \(Y \sim \text{Gamma}\left(\frac{\alpha}{2} + 1, 1\right)\).
3. Statistical and Reliability Properties

In this section we consider some statistical and reliability properties of the NWIR distribution.

3.1. $r^{th}$ moments

If a random variable $X$ has the NWIR distribution with a scale parameter $\lambda$ and shape parameter $\alpha$, then the $r^{th}$ moment of the NWIR distributed random variable $X$ is obtained as

$$E(X^r) = \int_0^\infty \frac{2(\alpha + \lambda)^{\frac{r}{2}+1}}{\Gamma(\frac{r}{2} + 1)} x^{r-\alpha-3} e^{-(\alpha + \lambda)x^{-2}} dx.$$ 

In order to calculate $E(X^r)$, using the transformation in (2.3), we obtain

$$E(X^r) = (\alpha + \lambda)^{\frac{r}{2}} \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r}{2} + 1)}.$$ 

Hence, from the $r^{th}$ moment of the NWIR distribution, the first four moments can be easily calculated to obtain the mean, variance, coefficient of skewness and the coefficient of kurtosis of the NWIR distribution as follows

$$E(X) = (\alpha + \lambda)^{\frac{1}{2}} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2} + 1)},$$

$$E(X^2) = \frac{2(\alpha + \lambda)}{\alpha},$$

$$E(X^3) = (\alpha + \lambda)^{\frac{3}{2}} \frac{\Gamma(\frac{5}{2} + 1)}{\Gamma(\frac{3}{2} + 1)},$$

$$E(X^4) = \frac{2(\alpha + \lambda)^{\frac{3}{2}}}{\alpha^{\frac{3}{2}} \Gamma(\frac{5}{2} + 1)}.$$
The New Weighted Inverse Rayleigh Distribution

and

\[ E(X^4) = (\alpha + \lambda)^2 \frac{\Gamma \left(\frac{\alpha + 1}{2} \right) + 1}{\Gamma \left(\frac{\alpha + 1}{2} + 1\right)}. \]

### 3.2. Moment generating function

The moment generating function of the NWIR distribution is given as follows.

\[
M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} 2 (\alpha + \lambda)^{\frac{t^2}{2} + 1} \frac{\Gamma \left(\frac{\alpha + 1}{2} + 1\right)}{\Gamma \left(\frac{\alpha + 1}{2} + 1\right)} x^{-(\alpha + 3)} e^{-(\alpha + \lambda)x^{-2}} dx.
\]

By applying the Maclaurin series \( e^{tx} = \sum_{n=0}^\infty \frac{(tx)^n}{n!} \) and setting the transformation in (2.3), we finally get

\[
M_X(t) = \frac{1}{\Gamma \left(\frac{\alpha + 1}{2} + 1\right)} \sum_{n=0}^\infty \frac{t^n}{n!} (\alpha + \lambda)^{\frac{t^2}{2}} \Gamma \left(\frac{\alpha + 1}{2} + 1\right).
\]

### 3.3. Quantile function

The quantile function of the NWIR distribution is obtained by

\[
(3.1) \quad x_q = F_w^{-1}(q), 0 < q < 1,
\]

where \( F_w^{-1}(q) \) is the inverse of cdf in (2.4). The median of the NWIR distributed random variable \( X \) can be found by putting \( q = 0.5 \) in (3.1). \( F_w^{-1}(q) \) can be computed numerically via some mathematical and statistical software packages since it does not have a closed-form expression. Moreover, the equation in (3.1) can be used in order to generate a random number from the proposed distribution.

### 3.4. Mode

Now, the natural logarithm of the \( f_w(x) \) in (2.2) is given by

\[
(3.2) \quad \ln f_w(x) \propto - (\alpha + 3) \ln x - (\alpha + \lambda) x^{-2}.
\]

Using the differentiating equation (3.2) with respect to \( x \), we obtain as

\[
(3.3) \quad \frac{d}{dx} \ln f_w(x) = - (\alpha + 3) x^{-1} + 2 (\alpha + \lambda) x^{-3}.
\]
If the equation (3.3) is equal to 0 and solve for \( x \), then the mode of the NWIR distribution has the following expression

\[
X_M = \sqrt{\frac{2(\alpha + \lambda)}{\alpha + 3}}
\]

for \( \alpha > 0 \) and \( \lambda > 0 \). Note that \( f_w(x) \) is increasing when \( x \in (0, X_M) \) and is decreasing when \( x \in (X_M, \infty) \).

### 3.5. Shannon entropy

The statistical entropy introduced by Shannon [22] is defined as a measure of the information content associated with the outcome of a random variable (see [2]). The Shannon entropy of the NWIR distribution is expressed by

\[
I_S(\alpha, \lambda) = -E(\ln f_w(x)) = \ln \left( \frac{\Gamma \left( \frac{\alpha}{2} + 1 \right)}{2(\alpha + \lambda)^{\frac{3}{2} + 1}} \right) + (\alpha + 3) E(\ln x) + (\alpha + \lambda) E(\frac{x^2}{2}).
\]

To calculate \( E(\ln x) \), if we use the transformation in (2.3), then we have

\[
E(\ln x) = \frac{1}{2\Gamma \left( \frac{\alpha}{2} + 1 \right)} \int_0^\infty u^{\frac{\alpha}{2} - 1} (\ln (\alpha + \lambda) - \ln u) e^{-u} du
\]

\[
= \frac{1}{2} \left( \ln (\alpha + \lambda) - \Psi \left( \frac{\alpha}{2} + 1 \right) \right),
\]

where \( \Psi \) is a digamma function with

\[
\Psi (r) = \frac{d}{dr} \ln \Gamma (r) = \frac{\Gamma' (r)}{\Gamma (r)}, r > 0
\]

defined as the logarithmic derivative of the Gamma function. It is also well known that the derivative of \( \Gamma (r) \) is

\[
\Gamma' (r) = \int_0^\infty t^{r-1} (\ln t) e^{-t} dt.
\]

Substituting \( E(\frac{x^2}{2}) = \frac{\alpha + 1}{\alpha + \lambda} \) and (3.5) into (3.4), Shannon entropy of the NWIR distribution \( I_S(\alpha, \lambda) \) becomes

\[
I_S(\alpha, \lambda) = \ln \left( \frac{\Gamma \left( \frac{\alpha}{2} + 1 \right)}{2(\alpha + \lambda)^{\frac{3}{2} + 1}} \right) + \left( \frac{\alpha}{2} + 1 \right) + \frac{\alpha - 3}{2} \left( \ln (\alpha + \lambda) - \Psi \left( \frac{\alpha}{2} + 1 \right) \right).
\]
### 3.6. Rényi entropy

Rényi entropy considered by Rényi [18] is a generalization of the Shannon entropy. The Rényi entropy of the NWIR distribution is expressed by

\[
I_R(\delta) = \frac{1}{1 - \delta} \ln \int_0^\infty f_w^\delta (x) dx
\]

\[
= \frac{1}{1 - \delta} \ln \int_0^\infty \left( \frac{2 (\alpha + \lambda)^{\frac{\delta+1}{2}}}{\Gamma \left( \frac{\alpha}{2} + 1 \right)} x^{-(\alpha+3)} e^{-(\alpha+\lambda)x^2} \right)^{\delta} dx
\]

\[
= \frac{1}{1 - \delta} \ln \int_0^\infty \frac{2 (\alpha + \lambda)^{\frac{\delta+1}{2}}}{\Gamma \left( \frac{\alpha}{2} + 1 \right)} x^{-(\alpha+3)} e^{-(\alpha+\lambda)x^2} dx
\]

where \( \delta \neq 1 \) and \( \delta > 0 \). By using the transformation in (2.3), we obtain that

\[
I_R(\delta) = \frac{1}{1 - \delta} \ln \frac{2 (\alpha + \lambda)^{\frac{\delta}{2}+1}}{\Gamma \left( \frac{\alpha}{2} + 1 \right)} + \ln \int_0^\infty x^{-\delta(\alpha+3)} e^{-\delta(\alpha+\lambda)x^2} dx
\]

\[
= \frac{1}{1 - \delta} \left( \ln 2^{\delta-1} + \frac{1 - \delta}{2} \ln (\alpha + \lambda) - \delta \ln \Gamma \left( \frac{\alpha}{2} + 1 \right) \right)
\]

\[
+ \frac{1}{1 - \delta} \left( \ln \Gamma \left( \frac{\delta (\alpha + 3) - 1}{2} \right) - \delta (\alpha + 3) - \frac{1}{2} \ln \delta \right).
\]

### 3.7. Survival and hazard rate functions

The survival and hazard rate functions of the NWIR distribution are defined by

\[
S(x) = 1 - F_w(x) = \gamma \left( \frac{\alpha}{2} + 1, \frac{\alpha + \lambda}{x^2} \right) / \Gamma \left( \frac{\alpha}{2} + 1 \right),
\]

and

\[
H(x) = \frac{f_w(x)}{S(x)} = \frac{2 (\alpha + \lambda)^{\frac{\delta+1}{2}}}{\gamma \left( \frac{\alpha}{2} + 1, \frac{\alpha + \lambda}{x^2} \right)} x^{-(\alpha+3)} e^{-(\alpha+\lambda)x^2}
\]

for \( x > 0 \), respectively. In FIG. 3.1, the graphs of the survival and hazard rate functions, which are plotted against different values of the parameters \( \alpha \) and \( \lambda \), are demonstrated.

Then, to determine the behavior of the hazard rate function of the NWIR distribution, the lemma established by Glaser [5] is used. Now, we define

\[
\eta(x) = -\frac{f_w(x)}{f_w(x)} = (\alpha + 3) x^{-1} - 2 (\alpha + \lambda) x^{-3},
\]
and

\[ \eta'(x) = - (\alpha + 3) x^{-2} + 6 (\alpha + \lambda) x^{-4}, \]

where \( f_w(x) \) is derivative of pdf of the NWIR distribution with respect to \( x \). Thus, \( \eta'(x) = 0 \) provides when \( x_0 = \sqrt{\frac{2(\alpha + \lambda)}{\alpha + 3}} \) for \( \lambda > 0, \alpha > 0 \). Note that, \( \eta'(x) > 0 \) and \( \eta'(x_0) = 0 \) when \( 0 < x < x_0 \) and \( \eta'(x) < 0 \) when \( x > x_0 \). Therefore, the hazard rate function of the NWIR distribution is an upside down bathtub shape (see [19] and [23]).

![Fig. 3.1: Plots of the survival and hazard rate functions of the NWIR distribution where \( \alpha = 2, \lambda = 1 \) (green line); \( \alpha = 2, \lambda = 2 \) (blue line); \( \alpha = 5, \lambda = 3 \) (red line)](image)

### 3.8. Order statistics

Let \( X_{(1)}, X_{(2)}, \ldots, X_{(n)} \) be order statistics of a random sample \( X_1, X_2, \ldots, X_n \) from the NWIR distribution. It is well known that the pdf of \( r^{th} \) order statistic \( X_{(r)} \) \( (r = 1, 2, \ldots, n) \) is given as:

\[ f_{r,n}(x; \alpha, \lambda) = r \binom{n}{r} f(x) (F(x))^{r-1} (1 - F(x))^{n-r}. \]  \hspace{1cm} (3.6)

Applying the binomial series expansion of \( (1 - F(x))^{n-r} \) in (3.6), we get

\[ f_{r,n}(x; \alpha, \lambda) = \sum_{k=0}^{n-r} r \binom{n}{r} \binom{n-r}{k} (-1)^k f(x) (F(x))^{r+k-1}. \]  \hspace{1cm} (3.7)

After substituting (2.2) and (2.4) into (3.7), if we put the binomial series expansion of \( (F(x))^{r+k-1} \) in (3.7), then we have
The New Weighted Inverse Rayleigh Distribution

519

(3.8) \( f_{r,n}(x; \alpha, \lambda) = \sum_{k=0}^{n-r-r+k-1} \sum_{t=0}^{r+k-1} 2 (-1)^{r+k-1} \sum_{k=0}^{r+k-1} \left[ \frac{n-r}{r} \binom{n-r}{k} \binom{r+k-1}{t} \right] \left[ (\alpha + \lambda)^{\frac{n+k-1}{2}} (\frac{\alpha}{2} + 1, \frac{\alpha+\lambda}{2}) \right] \Gamma^{r+k} \left( \frac{\alpha}{2} + 1 \right) \left[ x^{-(\alpha+3)} e^{-x-(\alpha+\lambda)x^{-2}} \right]. \)

Thus, the pdfs of the smallest order statistic \( X_{(1)} \) and largest order statistic \( X_{(n)} \) can be obtained by writing the \( r = 1 \) and \( r = n \) in (3.8), respectively.

4. Estimation

Let \( \{X_1, X_2, \ldots, X_n\} \) be a random sample from the NWIR distribution. The log-likelihood function of the sample is

\[
\ln L(\alpha, \lambda | x) = n \ln 2 + n \left( \frac{\alpha}{2} + 1 \right) \ln (\alpha + \lambda) - n \ln \Gamma \left( \frac{\alpha}{2} + 1 \right) - (\alpha + 3) \sum_{i=1}^{n} \ln x_i - (\alpha + \lambda) \sum_{i=1}^{n} x_i^{-2}.
\]

By differentiating (4.1) with respect to parameters \( \alpha \) and \( \lambda \), we have normal equations as

\[
\frac{\partial \ln L(\alpha, \lambda | x)}{\partial \alpha} = \frac{n}{2} \ln (\alpha + \lambda) + n \left( \frac{\alpha}{2} + 1 \right) \frac{\alpha}{\alpha + \lambda} - \frac{n}{2} \Psi \left( \frac{\alpha}{2} + 1 \right) - \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} x_i^{-2} = 0
\]

\[
\frac{\partial \ln L(\alpha, \lambda | x)}{\partial \lambda} = n \left( \frac{\alpha}{2} + 1 \right) \frac{\alpha}{\alpha + \lambda} - \sum_{i=1}^{n} x_i^{-2} = 0,
\]

where \( \Psi \left( \frac{\alpha}{2} + 1 \right) = \frac{d}{d\alpha} \ln \Gamma \left( \frac{\alpha}{2} + 1 \right) = \frac{\Gamma'(\frac{\alpha}{2} + 1)}{\Gamma(\frac{\alpha}{2} + 1)}. \) Note that the solution of the equations in (4.2)-(4.3) gives maximum likelihood estimators \( \hat{\alpha} \) and \( \hat{\lambda} \) of parameters \( \alpha \) and \( \lambda \). However, they do not have a closed form solution, and we must use numerical methods to solve them. Now, to give asymptotically a lower bound for the covariance matrix of \( \hat{\alpha} \) and \( \hat{\lambda} \), the Fisher information matrix is provided as a minus expected value of the second-order partial derivatives of the log-likelihood function.
under the regularity conditions, see [11]. It is defined by

\[
I_n (\alpha, \lambda) = \begin{bmatrix}
- E \left( \frac{\partial^2 \ln L(\alpha, \lambda | x)}{\partial \alpha^2} \right) & - E \left( \frac{\partial^2 \ln L(\alpha, \lambda | x)}{\partial \alpha \partial \lambda} \right) \\
- E \left( \frac{\partial^2 \ln L(\alpha, \lambda | x)}{\partial \lambda \partial \alpha} \right) & - E \left( \frac{\partial^2 \ln L(\alpha, \lambda | x)}{\partial \lambda^2} \right)
\end{bmatrix},
\]

and the elements of the matrix are obtained as follows

\[
E \left( \frac{\partial^2 \ln L(\alpha, \lambda | x)}{\partial \alpha^2} \right) = \frac{n}{(\alpha + \lambda)} - \frac{n (\frac{\alpha}{\lambda} + 1)}{(\alpha + \lambda)^2} - \frac{n}{4} \Psi' \left( \frac{\alpha}{2} + 1 \right)
\]

\[
E \left( \frac{\partial^2 \ln L(\alpha, \lambda | x)}{\partial \lambda^2} \right) = - \frac{n}{(\alpha + \lambda)^2} - \frac{n (\frac{\alpha}{\lambda} - 1)}{(\alpha + \lambda)^2},
\]

where \( \Psi' \left( \frac{\alpha}{2} + 1 \right) \) is first derivative of \( \Psi \left( \frac{\alpha}{2} + 1 \right) \) with respect to \( \alpha \). Therefore, maximum likelihood estimators of parameters \( \alpha \) and \( \lambda \) have asymptotically normal distribution with mean vector \( \hat{\theta} \) and the covariance matrix \( I_n^{-1} (\alpha, \lambda) \) as

\[
\sqrt{n} \left( \hat{\alpha} - \alpha, \hat{\lambda} - \lambda \right) \to N_2 \left( \theta, I_n^{-1} (\alpha, \lambda) \right),
\]

where \( I_n^{-1} (\alpha, \lambda) \) is inverse of \( I_n (\alpha, \lambda) \).

5. An Application

In this section, we consider a real data set, which is the daily mean wind speed data for March, taken in 2015 from the Turkish Meteorological Services for Sinop, Turkey, to demonstrate the practicability of the proposed distribution over the IR and WIR (proposed by Fatima and Ahmad [8]) distributions, see Table 5.1.

| Wind Speed | 2.8 | 1.8 | 3.2 | 5.0 | 2.4 | 4.8 | 2.9 |
|------------|-----|-----|-----|-----|-----|-----|-----|
|            | 2.3 | 3.2 | 2.3 | 2.0 | 1.9 | 3.3 | 4.4 |
|            | 4.3 | 1.9 | 2.2 | 3.3 | 2.1 | 4.0 | 2.0 |
|            | 3.8 | 3.1 | 3.2 | 3.4 | 2.8 | 2.1 | 3.1 |

The Kolmogorov-Smirnov (K-S) test, which is one of the widely used goodness of fit tests, has been applied to verify that distributions fit to the real data set. The results of the K-S test indicate that the NWIR, WIR and IR distributions are suitable for modeling the data set since the computed K-S test values are less than theoretical K-S test value (K-S_{0.05,31} = 0.24), see Table 5.2.
Then, we determined which distribution better fits the real data set using model evaluating tests, i.e., the root mean square error (RMSE), the coefficient of determination ($R^2$), ln-likelihood ($\ln L$) and the Akaike information criterion (AIC).

The tests results demonstrate that the NWIR distribution gives a better fit to the data set compared to the WIR and IR distributions because it has minimum RMSE and AIC and maximum $R^2$ and $\ln L$ values among the other distributions (see Table 5.2 and FIG. 5.1). Additionally, it was observed that there is no difference between the fitting performances of the WIR and IR distributions for the wind speed data (see FIG. 5.1).

Table 5.2: The ML estimates of parameters and results of the K-S test, RMSE, $R^2$, $\ln L$ and AIC for the wind speed data

| Distribution | $\hat{\alpha}$ | $\hat{\lambda}$ | K-S | RMSE | $R^2$ | $\ln L$ | AIC |
|--------------|----------------|-----------------|------|-------|-------|--------|-----|
| NWIR         | 3.7934         | 17.1586         | 0.0971 | 0.0532 | 0.9687 | -41.2814 | 86.5629 |
| WIR          | 0.0100         | 7.1969          | 0.2398 | 0.1162 | 0.6691 | -48.7263 | 101.4525 |
| IR           | -              | 7.2331          | 0.2393 | 0.1158 | 0.6729 | -48.6648 | 101.3290 |

FIG. 5.1: Fitted plots and histogram for the data

6. Conclusion

In this study, a new weighted IR distribution based on two different weight functions has been introduced. Moments, the moment generating function, survival and hazard rate functions, order statistics and entropy measures of the new distribution have been derived. The estimating equations have been provided in order to obtain ML estimates of the individual parameters, and the Fisher information matrix has been derived in order to obtain approximate confidence intervals of the parameters. The relationship between the NWIR distribution and the Gamma distribution has also been proved.
The applicability and superiority of the proposed distribution over the WIR and IR distributions have been illustrated with real data. Therefore, the NWIR distribution can be considered as an alternative model for the statistical data analysis in wind speed studies and other fields.

REFERENCES

1. A. Ahmad, S. P. Ahmad and A. Ahmed: Transmuted inverse Rayleigh distribution: a generalization of the inverse Rayleigh distribution. Mathematical Theory and Modeling 4(7) (2014), 90–98.
2. S. F. Bush: Nanoscale Communication Networks. Artech House, 2010.
3. R. A. Fisher: The effects of methods of ascertainment upon the estimation of frequencies. Annals of Eugenics 6 (1934), 13–25.
4. M. E. Ghitany, F. Alqallaf, D. K. Al-Mutairi and H. A. Husain: A two-parameter weighted Lindley distribution and its applications to survival data. Mathematics and computers in Simulation 81(6) (2011), 1190–1201.
5. R. E. Glaser: Bathtub and related failure rate characterizations. Journal of American Statistical Association 75 (1980), 667–672.
6. R. C. Gupta and J. P. Keating: Relations for reliability measures under length biased sampling. Scandinavian Journal of Statistics 13 (1986), 49–56.
7. R. C. Gupta and S. N. Kirmani: The role of weighted distributions in stochastic modeling. Communications in Statistics-Theory and methods 19(9) (1990), 3147–3162.
8. K. Fatima and S. P. Ahmad: Weighted inverse Rayleigh distribution. International Journal of Statistics and Systems 12(1) (2017), 119–137.
9. J. X. Kersey: Weighted inverse Weibull and beta-inverse Weibull distribution. Georgia Southern University, 2010.
10. M. S. Khan: Modified inverse Rayleigh distribution. International Journal of Computer Applications 87(13) (2014), 28–33.
11. A. Klein and G. Mélard: Computation of the Fisher information matrix for time series models. Journal of Computational and Applied Mathematics 64(1–2) (1995), 57–68.
12. J. Leao, H. Saulo, M. Bourguignon, R. Cintra, R. Gó and G. Cordeiro: On some properties of the beta inverse Rayleigh distribution. Chilean Journal of Statistics 4(2) (2013), 111–131.
13. V. Leiva, A. Sanhueza and J. M. Angulo: A length-biased version of the Birnbaum-Saunders distribution with application in water quality. Stochastic Environmental Research and Risk Assessment 23(3) (2009), 299–307.
14. B. O. Oluyede: On inequalities and selection of experiments for length biased distributions. Probability in the Engineering and Informational Sciences 13(2) (1999), 169–185.
15. A. K. Nanda and K. Jain: Some weighted distribution results on univariate and bivariate cases. Journal of Statistical Planning and Inference 77(2) (1999), 169–180.
16. G. P. Patil: Encountered data, statistical ecology, environmental statistics, and weighted distribution methods. Environmetrics 2(4) (1991), 377–423.
17. C. R. Rao: *On discrete distributions arising out of methods of ascertainment*. The Indian Journal of Statistics, Series A, 27 (1965), 311–324.

18. A. Rényi: *On measures of information and entropy*. Statistics and Probability 1 (1961), 547–561.

19. R. Roman: *Theoretical properties and estimation in weighted Weibull and related distributions*. M. S. Thesis, Georgia Southern University, Georgia, 2010.

20. A. Saghir, S. Tazeem and I. Ahmad: *The length-biased weighted exponentiated inverted Weibull distribution*. Cogent Mathematics 3(1) (2016), 1267299.

21. M. Q. Shahbaz, S. Shahbaz and N. S. Butt: *The Kumaraswamy inverse Weibull distribution*. Pakistan Journal of Statistics and Operation Research 8(3) (2012), 479–489.

22. E. Shannon: *A mathematical theory of communication*. The Bell System Technical Journal 27(10) (1948), 379–423.

23. V. Sherina and B. O. Oluyede: *Weighted inverse Weibull distribution: statistical properties and applications*. Theoretical Mathematics and Applications 4(2) (2014), 1–30.

24. V. N. Trayer: *Doklady Acad. Nauk*, Belorus, 1964.

25. V. G. Voda: *On the inverse Rayleigh random variable*. Reports of Statistical Application Research 19 (1972), 13–21.

Demet Aydin
Sinop University
Department of Statistics
57000 Sinop, Turkey
daydin@sinop.edu.tr