Enclosed area distribution in percolation

Robert Ziff, University of Michigan, Ann Arbor

October 28, 2018

(Talk presented in the StatPhys22 conference in Bangalore, India, July 5-9 2004.)

Abstract

The number of two-dimensional percolation clusters whose external hulls enclose an area greater than $A$, in a system of area $\Omega$, behaves at the critical point as $C\Omega/A$ for large $A$, where $C = 1/8\pi\sqrt{3}$. Here we show that away from the critical point this factor is multiplied by a scaling function that is asymptotically proportional to a simple exponential $\exp(-A/A^*)$ where $A^*$ scales as $|p-p_c|^{-2\nu}$. The fit is better than for Kunz and Souillard sub-critical scaling, which would predict the asymptotic behavior $\exp(-(A/A^*)^{2/D})$ where $D = 91/48$ is the fractal dimension.

1 Percolation

In the percolation model, one considers a disordered system, constructed (for example) by taking a regular lattice and diluting it by making only a fraction $p$ of the sites or bonds conducting (occupied). One is concerned with the appearance of long-range connectivity in the system – the percolation transition – and the behavior near the point of that transition. For definiteness, we consider mainly bond percolation on a square lattice, and define

\[ s = \text{the number of wetted sites of a cluster.} \quad (1) \]

Isolated sites with no bonds attached correspond to $s = 1$. 

Bond percolation on a square lattice, showing the appearance of a percolating path from the top of the system to the bottom.

Following are pictures of a single system where $p$ is slowly increased (bonds are added) and clusters are merged (as in the Newman Ziff algorithm).
2 Traditional percolation distribution

What is the size distribution of clusters in percolation? According to scaling theory (e.g., Stauffer & Aharony 1994), the number of clusters of size $s$ per site of the lattice behaves as

$$n_s(p) = c_1 s^{-\tau} f(c_2(p - p_c)s^\sigma)$$  \hspace{1cm} (2)

where $\tau$ and $\sigma$, are universal exponents, and $f(x)$ is a universal function. However $c_1$ and $c_2$ are non-universal, but vary with the lattice and percolation type being considered. Here $f(0) = 1$ and $f(x) \rightarrow 0$ as $x \rightarrow \pm \infty$. The
scaling relation can also be written as

\[ n_s(p) = c_1 s^{-\tau} F_{\pm}(s/s^*) \]  

(3)

where \( s^* = s_\xi \) is a typical size and scales as \(|p - p_c|^{-1/\sigma}\). If a ratio of moments such as \( M_3/M_2 \) is used to define \( s^* \), then \( s^* \) will have a coefficient that differs above and below \( p_c \), and that difference will have to be included in the definition of \( F_{\pm}(z) \) in terms of \( f(z) \). In any case, \( F_{\pm}(x) \) is universal, but the non-universal coefficient \( c_1 \) remains.

Indeed, \( n_s \) can never directly be put into a completely universal form because it concerns the size \( s \) which is a lattice-level quantity and by its very nature non-universal.

### 3 Universal form of the size distribution

We have shown (Ziff, Lorenz, & Kleban 1999) that, at the critical point \( p_c \), the distribution of cluster size can be written in a completely universal form:

\[ N(\text{clusters whose enclosed area } \geq A) \sim \frac{C\Omega}{A} \]  

(4)

in a system of total area \( \Omega \), where \( C \) is a universal constant which for 2d percolation is given by (Cardy Ziff, 2003)

\[ C = \frac{1}{8\pi\sqrt{3}} \approx 0.0229720373 \]  

(5)

Results have also been found for the more general Potts model.

This result can also be phrased:

- The number of clusters per unit area \( \mathcal{A} \) whose enclosed area is greater than \( \mathcal{A} \) equals a universal number \( C \).

### 4 Proof of (5)

From the definition of \( n_s \) we have

\[ N(\text{clusters whose size } \geq s) = \Omega \int_s^\infty n_s ds \sim \Omega s^{1-\tau} \]  

(6)

Each cluster has one external hull which encloses an area

\[ A \sim r^2 \sim s^{2/D} \]  

(7)
where \( D \) is the fractal dimension of the cluster and \( r \) is a length scale. Thus, \( s \sim A^{D/2} \) and

\[
N\text{(clusters whose enclosed area} \geq A) \sim \Omega A^{(D/2)(1-\tau)} = \Omega/A
\]

by virtue of the hyperscaling relation

\[
\frac{d}{D} = \tau - 1
\]

Furthermore, the constant \( C \) must be universal, since \( N\text{(area} \geq A) \) is only a property of the larger clusters, or, in other words, a property of the universal percolation fractal.

5 Derivation of the exact value of \( C \)

Eq. (4) implies that the average depth, defined as the number of outer hulls crossed in going from a point to the edge of the system, scales as

\[
C \log(\Omega/A_0)
\]

where \( A_0 \) is a lower cutoff of the area (\( \sim \) the lattice spacing). Transforming conformally from an annulus to a rectangle using \( z \rightarrow \log z \), we find that this corresponds to a rectangular system with \( 8\pi C \) clusters crossing per unit length. But Cardy has shown that that number is \( 1/\sqrt{3} \), which implies

\[
C = \frac{1}{8\pi \sqrt{3}}
\]

as given above.

6 Numerical demonstration

We generate “rooted” hulls by carrying out a hull generating walk on a lattice, and stopping when the cluster closes or when an upper cutoff is reached. Using a (virtual) lattice of 65536 \( \times \) 65536, and a cutoff of \( 2^{22} = 4194304 \) steps, we add a weight to each walk of a factor of \( 1/(\text{no. steps}) \), and thus find an unbiased estimate of \( N\text{(area} = A) \) as long as \( A < \text{cutoff} \). We bin logarithmically,

\[
N(A, 2A) = N\text{(hulls w/enclosed area} \in (A, 2A))/\Omega
\]
which should behave asymptotically for large $A$ as

$$N(A, 2A) = C \left( \frac{1}{A} - \frac{1}{2A} \right) = \frac{C}{2A}$$

(13)

or $2AN(A, 2A) = C$ (or $2C$ for both interior and exterior hulls):

7 Off-critical behavior

Away from $p_c$, we expect

$$N(\text{hulls whose enclosed area } \geq A) \sim \frac{C\Omega}{A} G(A/A^*)$$

(14)

where $A^*$ = typical hull area and $G(x)$ is a scaling function that should be universal. One expects

$$A^* \sim \xi^2 \sim \frac{1}{|p - p_c|^{2\nu}}$$

(15)

where $\xi$ is the correlation length and $\nu = 4/3$ in 2d, since the area of a percolating cluster is not a fractal (even though its boundary is a fractal).

Here we plot the same quantity as before but at $p = 0.495$. The quantity $2AN(A, 2A)$ for both internal and external hulls approaches the value of $2C$ for $\log A \approx 5$ but then deviates for larger $A$.  

12
8 Average Area

We can define an average area by averaging $A^2$ over all clusters (normalizing by $\Omega$):

$$\langle A \rangle = \frac{1}{\Omega} \sum_A A^2 N(\text{area} = A)$$

$$\approx \frac{1}{\Omega} \int_0^\infty A^2 N(\text{area} = A) \, dA$$

$$= \frac{1}{\Omega} \int_0^\infty 2AN(\text{area} \geq A) \, dA$$

$$= 2C \int_0^\infty G(A/A^*) \, dA$$

$$= 2CA^* \int_0^\infty G(x) \, dx \quad (16)$$

so that

$$\langle A \rangle \propto A^* \quad (17)$$
Note that we do not use $\sum A N(\text{area}= A)$ to define the average area, because that quantity scales as $\ln A^*/A_0$, where $A_0$ is a lower cutoff (area of basic cell of the lattice). In fact, that quantity gives the average depth of crossed loops.

9 Numerical measurement of average area

We have verified that $\langle A \rangle$ indeed scales as $|p - p_c|^{-2\nu}$.

| $p$   | $\langle A \rangle$ | logarithmic slope |
|-------|---------------------|-------------------|
| 0.4   | 39.786              |                   |
| 0.42  | 70.351              | −2.554            |
| 0.44  | 148.060             | −2.587            |
| 0.455 | 313.931             | −2.612            |
| 0.46  | 427.332             | −2.618            |
| 0.465 | 607.705             | −2.620            |
| 0.47  | 912.729             | −2.639            |
| 0.475 | 1478.133            | −2.628            |
| 0.48  | 2670.927            | −2.646            |
| 0.485 | 5730.272            | −2.652            |
| 0.49  | 16841.970           | −2.659            |

Table. Average area size, per lattice site (both internal and external hulls) as a function of $p$. The logarithmic slope $\Delta \log A/\Delta \log(p_c - p)$, clearly approaches a value consistent with $-8/3$.

Here it is plotted on a log-log scale:
10 Higher-order behavior of the mean area

Assuming an exponent of $8/3$, we can find the higher-order behavior by plotting $(p_c - p)^{8/3} \langle A \rangle$ vs. $(p_c - p)^x$, and varying $x$ until we find a straight line, which occurs for $x = 4/3$: 

![Graph of Mean area $<A>$ vs. $(pc - p)$ on a log-log scale]

$y = -2.62907x - 2.39361$

$R^2 = 0.99996$
which implies

\[ \langle A \rangle = a_0(p_c - p)^{-8/3} + a_1(p_c - p)^{-4/3} \ldots \] (18)

11 Asymptotic behavior of \( G(x) \)

For large \( x \), we find \( G(x) \) decays as \( e^{-cx} \):
Because the precise definition of $A^*$ is arbitrary up to a constant, we can fix that constant by requiring that $G(x)$ is a simple exponential for large $x$:

$$G(x) \sim e^{-x} \quad x \gg 1$$  \hspace{1cm} (19)

which implies that

$$A^* = \langle A \rangle / 0.130$$  \hspace{1cm} (20)

so that

$$2C \int_0^\infty G(x) \, dx = 0.130$$  \hspace{1cm} (21)

12 Comparison to sub-critical percolation mass

For $s \gg s^*$, the distribution of mass $s$ in subcritical percolation is believed to behave as (Kunz and Souillard 1978):

$$n_s(p) \sim \frac{1}{s} e^{-as/s^*}$$  \hspace{1cm} (22)

- This is evidently consistent with our result only if $s \sim A$ (instead of $s \sim A^{D/2} = A^{91/96}$) for subcritical clusters
• Note also that \( n_s \) apparently has a more complicated form than \( N \), since \( n_s \) changes its leading power term from \( s^{-\tau} \) for \( s \ll s^* \), to \( s^{-1} \) for \( s \gg s^* \), while \( N \) always has the same leading behavior of \( A^{-1} \) for all \( A \).

13 Amplitude ratio of average areas

According to the universal of \( G(x) \), the amplitude ratio of

\[
\frac{\langle A \rangle^{\text{[ext]}}}{\langle A \rangle^{\text{[int]}}} = \frac{\int_0^\infty G(x)^{\text{[ext]}} \, dx}{\int_0^\infty G(x)^{\text{[int]}} \, dx}
\]

is also universal. (Note – there is only one \( A^* \) in the system, used for scaling for both internal and external clusters). The measurements suggest a value of about 180 for this ratio (note higher statistical error as \( p \rightarrow p_c \)):

14 Alternate form of scaling

We can also write \( N \) as

\[
N(\text{area} > A) = \frac{C\Omega}{A} g(\pm(A/A^*)^{1/2\nu})
\]

(24)
where $1/2\nu = 3/8$ and $\pm$ refers to external (+) and internal hulls (−). We write the argument of $g$ in this way because then it is proportional to $(p - p_c)A^{1/2\nu}$ and therefore $g(z)$ should be a single analytic function of $z$ for both positive and negative $z$, as in (2) for the size distribution. Here are our results for $p = 0.495$ for the scaling function $g(x)$:

\[ g(x) \]

\includegraphics{g(x).pdf}

15 Conclusions

We have shown

\[ N(\text{hulls whose enclosed area} \geq A) \sim \frac{C\Omega}{A} G(A/A^*) \] (25)

with $G(x)$ a unique universal function satisfying

\[ G(x) = 1 \quad x = 0 \] (26)

\[ G(x) \sim ce^{-x} \quad x \gg 1 \] (27)

where $c \approx \exp(0.439) = 1.55$ and $A^* = \langle A \rangle/0.130$.

16 Appendix: Scaling for $N(\text{area} = A)$

For $N_A$ = number of hulls whose area equals $A$, we can write

\[ N_A = \frac{C}{A^2} \mathcal{F}_\pm(A/A^*) = \frac{C}{A^2} \mathcal{F}(c_3(p - p_c)A^{1/2\nu}) \] (28)
where $1/2\nu = 3/8$ and $\mathcal{F}(x) = \mathcal{F}(x^{2\nu})$. This is not a universal form of the
distribution both because it involves clusters of exactly area $A$, and also
because the non-universal constant $c_3$ shows up in the argument of $\mathcal{F}$ (in the
second form). It is interesting however as the counterpoint of the scaling
form (2) of the percolation mass that is also analytic at $p = 0$.

Here we keep internal and external hulls separate. By duality, we have
the symmetry,

$$\mathcal{F}^{[\text{ext}]}(x) = \mathcal{F}^{[\text{int}]}(-x)$$

which allows us to the entire curve of $\mathcal{F}(x)$ by making measurements at
$p < p_c$ only, and using the internal hulls for $-x$ and the external hulls for
$x$. Here are the results for $p = 0.44$, where we plot $A^2N_A$ vs. $(p - p_c)A^{3/8}$,
alogous to (14):

17 Note

An inconsistent factor of 4 was used in the definition of the area, and this
error may not have been consistently corrected in all of the plots.
18 References

1. D. Stauffer and A. Aharony, *Introduction to Percolation Theory, 2nd. edition* (Taylor & Francis, London, 1994).

2. M. E. J. Newman and R. M. Ziff, Efficient Monte Carlo algorithm and high-precision results for percolation, Phys. Rev. Lett. 85 4104-4107 (2000), and Fast Monte Carlo algorithm for site or bond percolation, Phys. Rev. E 64, 016706 (2001).

3. J. Cardy and R. M. Ziff, Exact results for the universal area distribution in percolation, Ising, and Potts models, J. Stat. Phys. 110, 1-33 (2003).

4. R. M. Ziff, C. D. Lorenz, and P. Kleban, Shape-dependent universality in percolation, Physica A 266, 17-26 (1999).

5. H. Kunz and B. Souillard, Essential singularity in the percolation model, Phys. Rev. Lett. 40, 133-135 (1978).