Abelianized Fundamental Group of the Affine Space over a Finite Field and Big Witt Vectors in Several Variables

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Abstract

Let $X$ be a normal proper variety over a perfect field $k$. We describe abelian coverings of $X$ in terms of the functor $\text{HDiv}_X$ of principal relative Cartier divisors on $X$. If $k$ is finite, we obtain for the maximal abelian extension of the function field of $X$ the relation $\text{Gal}(\overline{K}_X^{ab}|K_X) = \text{Hom}_{\text{Ab}/k}(\overline{\text{HDiv}}_X, \mathbb{G}_m)$. As another application, we present the geometric abelianized fundamental group of the affine space $\mathbb{A}^n$ over a finite field by the group of big Witt vectors in $n$ variables, a generalization of the (usual) big Witt vectors.

Contents

0 Introduction 2

1 Universal Torsors under Affine Groups 3
  1.1 Principal Relative Cartier Divisors . . . . . . . . 3
  1.2 Rational Maps to Torsors under Affine Groups . . . 4
  1.3 Universal Affine Torsor with Modulus . . . . . . . 5

2 Geometric Class Field Theory 6
  2.1 Geometric Galois Group of a Function Field . . . . 6
  2.2 Fundamental Group of the Affine Space . . . . . . 7

3 Big Witt Vectors in Several Variables 9
0 Introduction

Let $X$ be a normal proper variety over a perfect field $k$. In order to classify abelian coverings of $X$ (in the sense of rational maps), Serre observed (basing on Lang’s work in the 1950’s, see e.g. [La]) that every abelian covering (not arising from an extension of the base field) is the pull-back of an isogeny. This leads to an explicit description of the class field theory of varieties over finite fields, using generalized Albanese varieties (see [Ru2, Thm. 0.4], [Ru3, Thm. 3.7]), if those are defined (this is the case when $X$ is smooth).

However, Serre’s construction of those isogenies (see [Se3, VI, § 2, No. 8, Prop. 7]) involves only affine algebraic groups. As a consequence, in the class field theory of varieties over finite fields one can replace the generalized Albanese varieties by their affinizations. These affinizations are universal objects for rational maps from $X$ to commutative affine algebraic groups. The structure of these universal affine groups is much simpler than the one of the generalized Albanese varieties. This results in a simplified description of class field theory of function fields over finite fields (given in this paper), for which we can reduce the assumptions on $X$: in particular, smoothness of $X$ can be dropped. The price one has to pay for getting rid of abelian varieties is that we might possibly lose control over the conductor.

Now suppose the base field $k$ is finite, let $\bar{k}$ be an algebraic closure. One of the main results of this approach is a description of the geometric Galois group of the maximal abelian extension of the function field of $X$ in terms of the functor $HDiv_X$ of principal relative Cartier divisors on $X$:

$$\text{Gal} \left( \frac{K_{\bar{k}}}{K_X} \right) \cong \text{Hom}_{\text{Ab}/k} \left( \text{HDiv}_X, \mathbb{G}_m \right)$$

(Theorem 2.2). For any compactification of the affine space $\mathbb{A}^n$ the generalized Albanese varieties coincide with their affinizations. Therefore in this case we still keep control over the conductor when applying our method. Thus we can use it in order to calculate the geometric abelianized fundamental group of $\mathbb{A}^n$: it is given by the group of $k$-valued points of the big Witt vectors in $n$ variables $\Lambda^n$:

$$\pi_1^{\text{ab}}(\mathbb{A}^n)^0 \cong \Lambda^n(k)$$

(cf. Theorem 2.3). Here $\Lambda^n$ is a generalization of the (usual) big Witt vectors, an additive version of which appeared in the context of algebraic K-theory, see [AGHL]. In order to preserve the geometric intuition we introduce here an independent multiplicative version: the ring of big Witt vectors in $n$ variables $\Lambda^n$ is the ring-scheme given by the $\mathbb{Z}$-functor

$$R \mapsto 1 + (t_1, \ldots, t_n) R[[t_1, \ldots, t_n]]$$
The affine $k$-group $\Lambda^n$ is Cartier dual to its completion at the identity $\hat{\Lambda}^n$ (Proposition 3.6), which is canonically isomorphic to the functor

$$R \mapsto \frac{R[t_1, \ldots, t_n]^*}{R^*}$$

(Point 3.5).

1. Universal Torsors under Affine Groups

Let $X$ be a proper variety over a perfect field $k$, not necessarily smooth or irreducible. In [Ru2, Section 2] we considered categories of rational maps from $X$ to torsors under commutative algebraic groups. In this note we consider categories of rational maps from $X$ to torsors under commutative affine groups. The construction of universal objects for the latter categories is simpler than for non-affine groups: while in the non-affine case we used duality of 1-motives with unipotent part, in the case of affine groups it is sufficient to consider the Cartier duality between affine groups and formal groups. Moreover, in the non-affine case the (difficult) group functor $\text{Div}_X$ of relative Cartier divisors was involved, while in the case of affine groups we can restrict to the (easier) subfunctor $\text{HDiv}_X$ of principal relative Cartier divisors.

1.1 Principal Relative Cartier Divisors

The $k$-group functor of principal relative Cartier divisors

$$\text{HDiv}_X : \text{Alg}/k \rightarrow \text{Ab}$$

is the subfunctor of principal elements of the $k$-group functor of relative Cartier divisors $\text{Div}_X$ from [Ru1, No. 2.1]. For the purpose of this note it is enough to consider the completion $\text{HDiv}_X : \text{Art}/k \rightarrow \text{Ab}$ of $\text{HDiv}_X$, which assigns to a finite $k$-algebra $R$ the group

$$\text{HDiv}_X(R) = \frac{\Gamma(X \otimes R, (\mathcal{K}_X \otimes_k R)^*)}{\Gamma(X \otimes R, (\mathcal{O}_X \otimes_k R)^*)} = \frac{(\mathcal{K}_X \otimes_k R)^*}{R^*},$$

i.e.

$$\text{HDiv}_X = \mathbb{G}_m(\mathcal{K}_X \otimes_k ?)/\mathbb{G}_m.$$

As this is a quotient of left-exact functors on $\text{Art}/k (= \text{formal } k\text{-groups})$, and the category of formal $k$-groups is abelian, we obtain as in [Ru2, Prop. 2.1]

Proposition 1.1. $\text{HDiv}_X$ is a commutative formal $k$-group.
1.2 Rational Maps to Torsors under Affine Groups

**Point 1.2.** A rational map $\varphi : X \dashrightarrow L$ to a commutative affine group $L$ induces a natural transformation $\tau_\varphi : L^\vee \rightarrow \hat{\text{HDiv}}_X$: let $$\langle ?, ? \rangle : L^\vee \times L(K_X)/L(k) \rightarrow G_m(K_X \otimes \_)/G_m$$ be the pairing obtained from Cartier duality, then we set $$\tau_\varphi = \langle ?, [\varphi] \rangle$$ where $[\varphi]$ is the class of $\varphi \in L(K_X)$ in $L(K_X)/L(k)$.

**Definition 1.3.** If $H$ is a formal subgroup of $\hat{\text{HDiv}}_X$, denote by $\text{Mr}_{\text{aff}}(X, H)$ the category of those rational maps for which the image of this induced transformation from Point 1.2 lies in $H$. If $k$ is an arbitrary perfect base field, we define $\text{Mr}_{\text{aff}}(X, H)$ via base change to an algebraic closure $\overline{k}$, as in this case we can identify a torsor with the group acting on it.

A simplification of the proof of [Ru2, Thm. 0.1] shows

**Theorem 1.4.** Let $H$ be a formal (resp. dual-algebraic formal) $k$-subgroup of $\hat{\text{HDiv}}_X$. The category $\text{Mr}_{\text{aff}}(X, H)$ admits a universal object $$\text{Lu}_{X,H}^{(1)} : X \dashrightarrow \text{Lu}_{X,H}^{(1)}.$$ Here $\text{Lu}_{X,H}^{(1)}$ is a torsor under an affine (resp. algebraic affine) commutative $k$-group $\text{Lu}_{X,H}^{(0)}$, which is given by the Cartier dual of $H$.

**Notation 1.5.** In the case $H = \hat{\text{HDiv}}_X$ the category $\text{Mr}_{\text{aff}}(X, \hat{\text{HDiv}}_X)$ is the category of all rational maps from $X$ to torsors under affine commutative groups. The universal object of this category is denoted by $\text{Lu}_{X}^{(1)} : X \dashrightarrow \text{Lu}_{X}^{(1)}$ (without any specification of $H$).

**Remark 1.6.** $\text{Lu}_{X,H}^{(1)}$ is generated by $X$ and $\text{Lu}_{X,H}^{(0)}$ is smooth. The rational map $\left(\text{Lu}_{X,H}^{(1)} : X \dashrightarrow \text{Lu}_{X,H}^{(0)}\right) \in \text{Mr}_{\text{aff}}(X, H)$ is characterized by the fact that the transformation $\tau_{\text{Lu}_{X,H}^{(1)}} : \text{Lu}_{X,H}^{(1)} \rightarrow \hat{\text{HDiv}}_X$ is the identity $H \xrightarrow{id} H$ (cf. [Ru2, Rmk. 2.18]).

**Remark 1.7.** The universal object $\text{Lu}_{X,H}$ is the affinization (see [DG, III, § 3, No. 8]) of a generalized Albanese variety $\text{Alb}_{X,F}$ from [Ru2], if $X$ is smooth, $H$ is dual-algebraic and $F$ is a dual-algebraic formal subgroup of $\text{Div}_X$ such that $H = F \cap \hat{\text{HDiv}}_X$. (This follows directly from the universal property of the affinization.)
Via Cartier duality and Galois descent we obtain the functoriality of \( \text{Lu}^{(i)}_{X,H} \) (cf. [Ru2 2.3.3]):

**Proposition 1.8.** Let \( H \subset \text{HDiv}_X \) be a formal (resp. dual-algebraic formal) \( k \)-subgroup. Let \( \psi : Y \to X \) be a morphism of proper varieties, such that \( \psi(Y) \) meets \( \text{Supp}(H) \) properly. Then \( \psi \) induces for every formal (resp. dual-algebraic formal) \( k \)-subgroup \( G \subset \text{HDiv}_Y \) containing \( \psi^*H \) a homomorphism of \( k \)-torsors \( \text{Lu}^{(i)}_{\psi,G,H} \) and a homomorphism of affine (resp. affine algebraic) commutative \( k \)-groups \( \text{Lu}^{(0)}_{\psi,G,H} \):

\[
\text{Lu}^{(i)}_{\psi,G,H} : \text{Lu}^{(i)}_{Y,G,H} \to \text{Lu}^{(i)}_{X,H} \quad \text{for } i = 1, 0.
\]

### 1.3 Universal Affine Torsor with Modulus

Let \( X \) be a proper variety over a perfect field \( k \), regular in codimension 1. Let \( D \) be an effective Cartier divisor on \( X \) (possibly non-reduced).

**Point 1.9.** In [KR] we assigned to a rational map from \( X \) to a torsor \( \varphi : X \to P \) an effective divisor \( \text{mod}(\varphi) \) on \( X \), the modulus of \( \varphi \). We define a filtration

\[
\text{HDiv}_X = \lim_{\to} D \text{HDiv}_X
\]

(here \( D \) ranges over all effective Cartier divisors on \( X \)) by formal subgroups \( H_{X,D} := F_{X,D} \cap \text{HDiv}_X \) of \( \text{HDiv}_X \), where \( F_{X,D} \) are the formal subgroups of \( \text{Div}_X \) from [Ru2, Def. 3.14]. The formal groups \( H_{X,D} \) are dual-algebraic by [Ru2, Prop. 3.15 and Lem. 1.17]. Moreover they satisfy the following property: If \( \varphi \) maps to a torsor under an affine algebraic commutative group, it holds \( \text{mod}(\varphi) \leq D \iff \text{im}(\tau_\varphi) \subset H_{X,D} \), see [Ru2, Lem. 3.16].

Using Theorem 1.4 this yields

**Theorem 1.10.** Let \( \text{Mr}_{aff}(X,D) \) denote the category of those rational maps \( \varphi : X \to P \) to torsors under affine algebraic commutative groups s.t. \( \text{mod}(\varphi) \leq D \). This category admits a universal object

\[
\text{Lu}^{(1)}_{X,D} : X \to \text{Lu}^{(1)}_{X,D}
\]

called the universal affine torsor of \( X \) of modulus \( D \). The affine algebraic commutative group \( \text{Lu}^{(0)}_{X,D} \) acting on \( \text{Lu}^{(1)}_{X,D} \) is given by the Cartier dual of the formal group \( H_{X,D} \).

Functoriality follows now from Proposition 1.8, cf. [Ru2] Prop. 3.22 and Cor. 3.23.
Proposition 1.11. Let $\psi : Y \to X$ be a morphism of smooth proper varieties. Let $D$ be an effective divisor on $X$ intersecting $\psi(Y)$ properly. Then $\psi$ induces a homomorphism of torsors $\hat{\text{Lu}}_{\psi,E,D}^{(i)}$ and a homomorphism of affine algebraic commutative groups $\hat{\text{Lu}}_{\psi,E,D}^{(0)}$, 

$$\hat{\text{Lu}}_{\psi,E,D}^{(i)} : \hat{\text{Lu}}_{Y,E}^{(i)} \to \hat{\text{Lu}}_{X,D}^{(i)}$$

for each effective divisor $E$ on $Y$ satisfying $E \geq (D - D_{\text{red}}) \cdot Y + (D \cdot Y)_{\text{red}}$, where $B \cdot Y$ denotes the pull-back of a Cartier divisor $B$ on $X$ to $Y$.

By Cartier duality and Galois descent we have

Corollary 1.12. If $D$ and $E$ are effective divisors on $X$ with $E \geq D$, then there are canonical surjective homomorphisms $\hat{\text{Lu}}_{X,E}^{(i)} \to \hat{\text{Lu}}_{X,D}^{(i)}$ for $i = 1, 0$, given by $\hat{\text{Lu}}_{\text{id}X,E,D}^{(i)}$.

Proposition 1.13. The universal object $\hat{\text{Lu}}_{X}^{(1)}$ (for the category of all rational maps to torsors under affine commutative groups) is a torsor under a pro-algebraic group $\hat{\text{Lu}}_{X}^{(0)}$.

Proof. $\hat{\text{Lu}}_{X}^{(0)}$ is the Cartier dual of $\overline{\text{Div}}_X = \lim \overline{\text{H}}_{X,D}$, which is an inductive limit of dual-algebraic formal groups $\overline{\text{H}}_{X,D}$ (Point 1.9). Thus $\hat{\text{Lu}}_{X}^{(0)} = \lim \hat{\text{Lu}}_{X,D}^{(0)}$ is the projective limit of algebraic affine groups $\hat{\text{Lu}}_{X,D}^{(0)}$.

2 Geometric Class Field Theory

Let $k = \mathbb{F}_q$ be a finite field, $\overline{k}$ an algebraic closure and $X$ an irreducible proper variety over $k$. We propose to classify abelian coverings of $X$ (in the sense of rational maps) by principal divisors on $X$.

2.1 Geometric Galois Group of a Function Field

Serre gives in [Se3, VI, § 2, No. 8] an explicit construction for any abelian covering (arising from a “geometric situation”) as pull-back of an isogeny of affine groups:

Point 2.1. Let $N$ be a finite group and $\psi : Y \to X$ a Galois covering (a rational map) with Galios group $N$, defined and Galois over $k$. Let $A_N$ be the affine space over $k$ defined by the condition that $A_N(\overline{k}) = \overline{k}[N]$ is the group-algebra of the group $N$. Let $G_N$ be the open $k$-subvariety of $A_N$, such that $G_N(\overline{k})$ is the set of invertible elements of $A_N(\overline{k})$. 

6
Then there exists a rational map $\varphi : X \to G_N/N$ over $k$ such that $Y \to X$ is isomorphic over $k$ to the pull-back of $G_N \to G_N/N$ via $\varphi$. (See [Se3] VI, § 2, No. 8, Prop. 7 for a proof.)

**Theorem 2.2.** Let $K_X^{ab}$ be the maximal abelian extension of the function field $K_X$ of $X$. The geometric Galois group $\text{Gal} \left( K_X^{ab} \mid K_X \right)$ is isomorphic to the group of $k$-valued points of $\text{Lu}_X$, which is the Cartier dual of $\text{HDiv}_X$:

$$\text{Gal} \left( K_X^{ab} \mid K_X \right) \cong \text{Lu}_X(k) = \left( \text{HDiv}_X \right)^{\vee}(k).$$

**Proof.** Consider a Galois covering $Y \to X$ as in Point 2.1, but suppose that the Galois group $N$ is abelian. Then $G_N/N$ is a commutative affine algebraic group, hence the rational map $\varphi : X \to G_N/N$ factors through $\text{HDiv}_X(D)$, where $D = \text{mod}(\varphi)$. Thus $Y \to X$ is the pull-back of an isogeny onto $\text{Lu}_X(D)$. As this isogeny defines an abelian extension of function fields, it is a quotient of the “$q$-power Frobenius minus identity” $\varphi := F_q - \text{id}$ (see [Se3] VI, No. 6, Prop. 6]). If $X_D \to X$ denotes the pull-back of $\varphi : \text{Lu}_X(D) \to \text{Lu}_X$, then $\text{Gal}(Y|X)$ is a quotient of

$$\text{Gal}(X_D|X) = \ker(\varphi) = \text{Lu}_X(D)(k).$$

Thus the system $\{X_D \to X\}_D$, where $D$ ranges over all effective Cartier divisors on $X$, is cofinal in the system of geometric abelian Galois coverings $\{Y \to X\}_{Y \mid X \text{ ab. geo.}}$. As the Galois groups $\text{Gal}(Y|X)$ are finite, we have a canonical isomorphism of projective limits (see [RZ] Lem. 1.1.9])

$$\text{Gal} \left( K_X^{ab} \mid K_X \right) \cong \lim_{Y \mid X \text{ ab. geo.}} \text{Gal}(Y|X) \cong \lim_D \text{Gal}(X_D|X) = \lim_D \text{Lu}_X(D)(k) = \text{Lu}_X(k).$$

The equality $\text{Lu}_X = \left( \text{HDiv}_X \right)^{\vee}$ is due to the construction of $\text{Lu}_X$ (Thm. 1.3).

### 2.2 Fundamental Group of the Affine Space

Finally we want to apply the tools developed so far to the class field theory of some given concrete variety. The affine space – a priori the easiest example one can imagine – is a welcome example in order to test our theory.

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1. Here we identify the torsor $\text{Lu}_X(D) := \text{Lu}_X^{(1)}(D)$ with the group $\text{Lu}_X^{(0)}(D)$ acting on it, since $k$ is finite and hence every $k$-torsor admits a $k$-rational point.
Let \( k = \mathbb{F}_q \) be a finite field. The abelianized fundamental group of the affine line \( \mathbb{A}^1 \) over \( k \) is supposed to be well-known since the generalized Jacobian of Rosenlicht is established. (Strangely I did not find this example in any text-book.) In this subsection we want to show that the abelianized geometric fundamental group of the affine space \( \mathbb{A}^n \) over \( k \) is just as easy as the case \( n = 1 \).

**Theorem 2.3.** Let \( \mathbb{A}^n \) be the \( n \)-dimensional affine space over \( k = \mathbb{F}_q \). Let \( t_1, \ldots, t_n \) be affine coordinates of \( \mathbb{A}^n \). Then

\[
\pi_1^{\text{ab}}(\mathbb{A}^n)^0 = \left( 1 + (t_1^{-1}, \ldots, t_n^{-1}) k[[t_1^{-1}, \ldots, t_n^{-1}]] \right)^\times.
\]

In other words: the abelianized geometric fundamental group of \( \mathbb{A}^n \) over \( \mathbb{F}_q \) is isomorphic to the group of \( \mathbb{F}_q \)-valued points of the big Witt vectors in \( n \) variables (see Section 3).

**Proof.** Consider \( \mathbb{A}^n \) as an open subvariety of \( X := (\mathbb{P}^1)^n \), the product of \( n \) copies of the projective line \( \mathbb{P}^1 \). Let \( S \subset X \) be the divisor at infinity such that \( \mathbb{A}^n = X \setminus S \). Since the Néron-Severi group \( \text{NS}_{\mathbb{P}^1} \) of \( \mathbb{P}^1 \) is torsion-free, the same is true for \( \text{NS}_X \) and it holds

\[
\Sigma(X) := \text{Hom} \left( \lim_{\rightarrow} \text{Hom}_{\mathbb{A}^1/k}(\mu_n, \text{NS}_X), \mathbb{Q}/\mathbb{Z} \right) = 0,
\]

hence for every effective divisor \( D \) on \( X \) we have

\[
\pi_1^{\text{ab}}(X, D)^0 = \text{Alb}_{X,D}(k)
\]
by [Ru3, Cor. 3.14].\(^2\) Since \( \text{Alb}_X = (\text{Pic}_X^0, \text{red})^\vee = 0 \) we have for every \( D \)

\[
\text{Alb}_{X,D} = \text{Lu}_{X,D}.
\]

Then

\[
\pi_1^{\text{ab}}(X \setminus S)^0 = \lim_{\rightarrow D} \pi_1^{\text{ab}}(X, D)^0 = \lim_{\rightarrow D} \text{Alb}_{X,D}(k)
\]

\[
= \lim_{\rightarrow D} \text{Lu}_{X,D}(k) = \left( \lim_{\rightarrow D} \mathcal{H}_{X,D} \right)^\vee(k)
\]

\[
= \left( \widehat{\text{HDiv}}_X^S \right)^\vee(k)
\]

\(^2\) For technical reasons the statement is formulated there only for \( \dim X \leq 2 \) but holds for arbitrary dimension. In the case \( X = (\mathbb{P}^1)^n \) and \( |D| = \bigcup_{i=1}^{n} \mathbb{P}^1 \times \ldots \times \{\infty\} \times \ldots \times \mathbb{P}^1 \) the necessary induction argument is particularly easy.
where \( \widehat{\text{HDiv}}^S_X \) denotes the functor of principal relative Cartier divisors with support in \( S \). Then \( \widehat{\text{HDiv}}^S_X \) is the functor that assigns to a finite \( k \)-ring \( R \) the group

\[
\widehat{\text{HDiv}}^S_X(R) = \frac{\Gamma(\mathcal{O}_A^n \otimes_k R)^*}{R^*} = \frac{R[t_1, \ldots, t_n]^*}{R^*}.
\]

Thus, as we will see in Point 3.5,

\[
\widehat{\text{HDiv}}^S_X = \widehat{\Lambda}(t_1, \ldots, t_n)
\]

is the completion at 1 of the \( k \)-group \( \Lambda(t_1, \ldots, t_n) \) of big Witt vectors in \( n \) variables, and this completion is the Cartier dual of \( \Lambda(t_1^{-1}, \ldots, t_n^{-1}) \) by Prop. 3.8. Then

\[
\left( \widehat{\text{HDiv}}^S_X \right)^\vee(k) = \Lambda(t_1^{-1}, \ldots, t_n^{-1})(k) = \left( 1 + (t_1^{-1}, \ldots, t_n^{-1}) k[[t_1^{-1}, \ldots, t_n^{-1}]] \right)^\times.
\]

\[\square\]

3 Big Witt Vectors in Several Variables

This section is devoted to a generalization of the ring-scheme of big Witt vectors (in one variable). Big Witt vectors in several variables can be expressed as an infinite product of big Witt vectors in one variable. We compute the Cartier dual of this group-scheme via reduction to the one-variable case.

Let \( k \) be a ring.

**Definition 3.1.** The group of big Witt vectors in \( n \) variables is the \( k \)-group defined by

\[
\Lambda^n := \Lambda(t_1, \ldots, t_n) := \ker \left( \mathbb{G}_m \left( ? \otimes k[[t_1, \ldots, t_n]] \right) \longrightarrow \mathbb{G}_m \right)
\]

w.r.t. the augmentation map

\[
k[[t_1, \ldots, t_n]] \longrightarrow k, \quad t_i \mapsto 0,
\]

i.e. \( \Lambda^n \) is the functor that assigns to a \( k \)-algebra \( R \) the group

\[
\Lambda^n(R) = \left( 1 + (t_1, \ldots, t_n) R[[t_1, \ldots, t_n]] \right)^\times.
\]

Every element \( \lambda \in \Lambda^n(R) \) has a unique decomposition

\[
\lambda = \prod_{\nu_1, \ldots, \nu_n} \left( 1 - r_{\nu_1} \cdots r_{\nu_n} t_1^{\nu_1} \cdots t_n^{\nu_n} \right).
\]
Remark 3.2. The case $n = 1$ is the $k$-ring of (usual) big Witt vectors $\Lambda := \Lambda^1$. The ring structure is given as follows: addition on $\Lambda$ is the multiplication of formal power series, while multiplication on $\Lambda$ is given by

$$\Lambda \times \Lambda \xrightarrow{\ast} \Lambda$$

$$\prod_{i \geq 1} (1 - a_i t^i), \prod_{j \geq 1} (1 - b_j t^j) \mapsto \prod_{i,j \geq 1} \left(1 - a_i^{(i,j)} b_j^{(i,j)} t^{i+j}ight)^{(i,j)}$$

where $(i, j) := \gcd(i, j)$, cf. [Bl] I, § 1, Prop. (1.1)].

**Proposition 3.3.** The group of big Witt vectors $\Lambda^n$ is canonically isomorphic to an (infinite) product of copies of $\Lambda$:

$$\Lambda^n \sim \prod_{\nu_1, \ldots, \nu_n \gcd(\nu_1, \ldots, \nu_n) = 1} \Lambda$$

**Proof.** Let $\nu := (\nu_1, \ldots, \nu_n)$ be a multi-index, write $t^\nu := t_1^{\nu_1} \cdot \cdots \cdot t_n^{\nu_n}$. Then we obtain a canonical isomorphism

$$\Lambda(t_1, \ldots, t_n) \cong \prod_{\nu \gcd(\nu) = 1} \Lambda(t^\nu)$$

$$\prod_{\nu} (1 - r_\nu t^\nu) = \prod_{\nu \gcd(\nu) = 1} \prod_{i} (1 - r_i t^{i\nu})$$

**Proposition 3.4.** The functor $\Lambda^n$ is a smooth connected unipotent commutative $k$-group and a $k$-ring.

**Proof.** Due to the decomposition of $\Lambda^n$ from Prop. 3.3 this follows from the properties of the (usual) big Witt vectors $\Lambda$. ■

Now let $k$ be a field.

**Point 3.5.** The completion at 1 of the big Witt vectors in $n$ variables is the formal $k$-subgroup of $\Lambda^n$

$$\hat{\Lambda}^n := \hat{\Lambda}(t_1, \ldots, t_n) \subset \Lambda(t_1, \ldots, t_n)$$

that assigns to a $k$-algebra $R$ the group

$$\hat{\Lambda}^n(R) = \left\{ 1 + \sum_{\nu_1, \ldots, \nu_n} r_{\nu_1, \ldots, \nu_n} t_1^{\nu_1} \cdots t_n^{\nu_n} \left| \begin{array}{c} r_{\nu_1, \ldots, \nu_n} \in \text{Nil}(R) \text{ for all } \nu_1, \ldots, \nu_n \\ r_{\nu_1, \ldots, \nu_n} = 0 \text{ for almost all } \nu_1, \ldots, \nu_n \end{array} \right. \right\}.$$
Then \( \hat{\Lambda}^n \) is canonically isomorphic to the \( k \)-group functor

\[
\hat{\Lambda}(t_1, \ldots, t_n) \cong \frac{\mathbb{G}_m(? \otimes k[t_1, \ldots, t_n])}{\mathbb{G}_m}.
\]

**Proposition 3.6.** The big Witt vectors \( \Lambda^n \) are Cartier dual to \( \hat{\Lambda}^n \), its completion at 1:

\[
(\Lambda^n)\check{\vee} \cong \hat{\Lambda}^n.
\]

The pairing is given by the multiplication \( \ast \) on \( \Lambda^n \) composed with an evaluation at \( t = 1 \):

\[
\hat{\Lambda}^n \times \Lambda^n \rightarrow \hat{\Lambda}^n \overset{t=1}{\rightarrow} \mathbb{G}_m \quad f, g \mapsto f \ast g \mapsto (f \ast g)(1).
\]

**Proof.** According to the decomposition of \( \Lambda^n \) from Prop. 3.3, we can reduce to the case of (usual) big Witt vectors \( \Lambda \). We have an isomorphism

\[
\Pi \varepsilon : \prod_{j \geq 1} W \overset{\sim}{\rightarrow} \Lambda, \quad (v_j)_j \mapsto \prod_j \text{Exp } (v_j, t^j)
\]

where \( \text{Exp} \) denotes the Artin-Hasse exponential, from an infinite product of copies of the ring of (small) Witt vectors \( W \) to the ring of big Witt vectors \( \Lambda \) (see [Dm, III, No. 1, Prop. on p. 53]). The Cartier dual of \( W \) is given by \( \hat{W} \), its completion at 0, and the pairing is

\[
\hat{W} \times W \rightarrow \hat{W} \overset{\text{Exp}}{\rightarrow} \hat{\Lambda} \overset{t=1}{\rightarrow} \mathbb{G}_m \quad v, w \mapsto v \cdot w \mapsto \text{Exp}(v \cdot w, t) \mapsto \text{Exp}(v \cdot w, 1)
\]

(see [DG, V, § 4, Cor. 4.6]). As the map \( \Pi \varepsilon \) is a homomorphism of \( k \)-rings (see [DG, V, § 5, Thm. 5.5]), the result follows.

**Proposition 3.7** (geometric description of Cartier duality). Let \( k \) be a perfect field, \( \bar{K} \) an algebraic closure. The Cartier duality between \( \hat{\Lambda} \) and \( \Lambda \) is expressed by the pairing \( \hat{\Lambda}(t) \times \Lambda(t^{-1}) \rightarrow \mathbb{G}_m \) determined by

\[
\hat{\Lambda}(t)(R) \times \Lambda(t^{-1})(R) \rightarrow \mathbb{G}_m(R \otimes \bar{K}) \quad f, g \mapsto \prod_{p \in |\mathbb{Z}(g')|} (f(p))_{g'}^v(f)
\]

where \( g' \in 1 + t^{-1}K[t^{-1}] \) is a truncation of \( g \) mod \( t^{-m}K[t^{-1}] \) for \( m \) sufficiently large, and \( \mathbb{Z}(g') \) is the divisor of zeroes of \( g' \).
Proof. Consider the Jacobian of $C = \mathbb{P}^1$ over $k$ with modulus $D = n[\infty]$. Its affine part is

$$L_{\mathbb{P}^1,n[\infty]}(k) = \frac{\mathcal{O}_{\mathbb{P}^1,\infty}^*}{k^* \times \left(1 + m_{\mathbb{P}^1,\infty}^n\right)} = \frac{1 + \mathfrak{m}_{\mathbb{P}^1,\infty}}{1 + t^{-1} \overline{k}[t^{-1}]} = 1 + m_{\mathbb{P}^1,\infty}^n.$$ 

Moreover, as $\text{Pic}_{\mathbb{P}^1} = 0$, we have $\text{Div}_0^0 = \text{HDiv}_{\mathbb{P}^1}$ and $\mathcal{F}_{\mathbb{P}^1,n[\infty]}^0 = \mathcal{H}_{\mathbb{P}^1,n[\infty]}$. Then

$$\hat{\Lambda}(t)(R) \cong \frac{R[t]^*}{R^*} = \text{HDiv}_{\mathbb{P}^1}(R) = \lim_{n} \mathcal{F}_{\mathbb{P}^1,n[\infty]}^0(R),$$

$$\Lambda(t^{-1})(\overline{k}) = 1 + t^{-1} \overline{k}[t^{-1}] = \lim_{n} L_{\mathbb{P}^1,n[\infty]}(\overline{k}).$$

According to [Ru3, Prop. 2.5, Point 2.7 and Point 2.4] the pairing $\hat{\Lambda}(t) \times \Lambda(t^{-1}) \to \mathbb{G}_m$ is thus given by the local symbol at infinity

$$\langle f, g \rangle_{\mathbb{P}^1,\infty} = (f, g)_{\infty}^{-1} = f(\text{div}(g'))$$

Here we can truncate $g$ as stated, since $f$ annihilates $m$th higher units if $f \in \mathcal{F}_{\mathbb{P}^1,m[\infty]}$ for some $m \in \mathbb{N}$ (see [KR] § 6, Prop. 6.4 (3))). Now the poles of $g \in 1 + t^{-1} \overline{k}[t^{-1}]$ lie all at 0, and for $f \in 1 + t \overline{k}[t]$ it holds $f(0) = 1$, hence

$$f\left(\text{div}(g')\right) = f(Z(g')) = \prod_{p \in |Z(g')|} f(p)^{v_p(f)}.$$ 

Combining the geometric description from Prop. 3.7 and the decomposition from Prop. 3.3 we obtain

Proposition 3.8. Over a perfect field $k$ we have the following duality:

$$\Lambda(t_1^{-1}, \ldots, t_n^{-1})^\vee = \hat{\Lambda}(t_1, \ldots, t_n)$$

and the pairing between $\Lambda(t_1^{-1}, \ldots, t_n^{-1})$ and $\hat{\Lambda}(t_1, \ldots, t_n)$ is induced by the local symbol map.

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3 The pairing $\langle \cdot, \cdot \rangle_{\mathbb{P}^1,\infty}$ here differs from the pairing considered in [Ru3] by an inversion in $\mathbb{G}_m$. 

12
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