Nonlinearity-assisted quantum tunnelling in a matter-wave interferometer

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Abstract
We investigate the nonlinearity-assisted quantum tunnelling and formation of nonlinear collective excitations in a matter-wave interferometer, which is realized by the adiabatic transformation of a double-well potential into a single-well harmonic trap. In contrast to the linear quantum tunnelling induced by the crossing (or avoided crossing) of neighbouring energy levels, the quantum tunnelling between different nonlinear eigenstates is assisted by the nonlinear mean-field interaction. When the barrier between the wells decreases, the mean-field interaction aids quantum tunnelling between the ground and excited nonlinear eigenstates. The resulting non-adiabatic evolution depends on the input states. The tunnelling process leads to the generation of dark solitons, and the number of the generated dark solitons is highly sensitive to the matter-wave nonlinearity. The results of the numerical simulations of the matter-wave dynamics are successfully interpreted with a coupled-mode theory for multiple nonlinear eigenstates.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Matter-wave interferometry involves coherent manipulation of the external or internal degrees of freedom of massive particles [1, 2]. Utilizing the well-developed techniques of trapping and cooling, the matter-wave interferometers have been realized with atomic Bose–Einstein condensates (BECs) [3–6]. Almost all BEC interferometers based on spatial interference measure the phase coherence by merging two initially separated condensates [3–6]. To recombine two condensates confined in a double-well potential, one has to transform the double-well potential into a single-well potential by decreasing the barrier height.

Intrinsic interparticle interactions in atomic condensates have stimulated various studies of the nonlinear behaviour of condensed atoms [7]. A balance between matter-wave dispersion and nonlinear interaction supports a number of nontrivial collective excitations, including
bright solitons in condensates with attractive interactions [8, 9] and dark solitons in condensates with repulsive interactions [10, 11]. In a harmonically trapped condensate with repulsive interparticle interactions, the nodes of excited nonlinear eigenstates correspond to dark solitons [12], so that the formation of dark solitons can be associated with populating excited states [12, 13]. Several methods of condensate excitation have been suggested, the most experimentally appealing ones involving time-dependent modifications of trapping potentials [14]. The operation of BEC interferometers and splitters based on the spatiotemporal Y- and X-junctions [15, 17] is greatly affected by the possibility of nonlinear excitations. The nonlinear excitations in BEC interferometers with repulsive interparticle interactions lead to the generation of dark solitons [16, 17], and can be utilized to enhance the phase sensitivity of the devices [17, 18].

The extensively explored mechanisms for population transfer between different eigenstates of a trapped BEC include non-adiabatic processes [15], Josephson tunnelling [19, 20] and Landau–Zener tunnelling [21–23], which are also responsible for the population transfer in linear systems. However, in a sharp contrast to linear systems, the quantum tunnelling between different nonlinear eigenstates can be assisted by the nonlinear mean-field interaction even in the absence of crossing (and avoided crossing) of the energy levels. Up to now, this peculiar type of quantum tunnelling remains poorly explored.

In this paper, we explore the intrinsic mechanism for the quantum tunnelling assisted by repulsive nonlinear mean-field interactions in a matter-wave interferometer. We consider the dynamical recombination process of a BEC interferometer, in which an initially deep one-dimensional double-well potential is slowly transformed into a single-well harmonic trap. Our numerical simulations, employing a time-dependent 1D mean-field Gross–Pitaevskii (GP) equation, show that multiple moving dark solitons are generated as a result of the nonlinearity-assisted quantum tunnelling between the ground and excited nonlinear eigenstates of the system, and the qualitative mechanism is independent of the particular shape of the symmetric double-well potential. Furthermore, the number of the generated dark solitons is found to be highly sensitive to the strength of the effective nonlinearity that in turn depends on the total number of condensed atoms and the atom–atom s-wave scattering length. The population transfer between different nonlinear eigenstates caused by the nonlinearity-assisted quantum tunnelling can be quantified by a coupled-mode theory for multiple nonlinear eigenstates of the system.

2. Model and numerical results

We consider a condensate under strong transverse confinement, \(m\omega^2_{\rho}(y^2 + z^2)/2\), so that the 3D mean-field model can be reduced to the following 1D model [24]:

\[
\frac{i\hbar}{\partial t} \Psi(x, t) = H_0 \Psi(x, t) + \lambda |\Psi(x, t)|^2 \Psi(x, t),
\]

where \(H_0 = -(\hbar^2/2m)(\partial^2/\partial x^2) + V(x, t)\), \(m\) is the atomic mass, \(\lambda > 0\) characterizes the effective nonlinearity which we assume to be repulsive and \(V(x, t)\) is an external potential. If the condensate order parameter \(\Psi(x, t)\) is normalized to one, the effective nonlinearity \(\lambda = 2N\alpha_s\omega_{\rho}\hbar\) is determined by the total number of atoms \(N\), the s-wave scattering length \(\alpha_s\), and the transverse trapping frequency \(\omega_{\rho}\) [25]. In what follows, we use the dimensionless version of the model equation obtained by choosing the natural units of \(m = \hbar = 1\).
We assume the time-dependent potential \( V(x, t) \) as a spatiotemporal Y-shape potential generated by the superposition of a 1D time-independent harmonic potential and a time-dependent Gaussian barrier (see figure 1):

\[
V(x, t) = \frac{1}{2} \omega^2 x^2 + B(t) \exp\left(-\frac{x^2}{2d^2}\right),
\]

where \( \omega \) is the trapping frequency, \( d \) is the barrier width, and the barrier height depends on time as follows:

\[
B(t) = \begin{cases} 
B_0 - \alpha t, & \text{for } t < B_0/\alpha, \\
0, & \text{for } t \geq B_0/\alpha,
\end{cases}
\]

where \( \alpha \) is the rate at which the barrier between the wells is ramped down. When the barrier height \( B(t) > \omega^2 d^2 \), the time-dependent potential is a double-well potential with two minima at \( x = \pm d \sqrt{2 \ln \left[ \frac{B_0}{(d^2\omega^2)} \right]} \). Thus, the 1D description is valid for weak longitudinal confinements satisfying \( \omega \sqrt{d^2} = \omega \sqrt{2 \ln \left[ B_0/(d^2\omega^2) \right]} \ll \omega_{\rho} \) and \( \omega \ll \omega_{\rho} \). To ensure the adiabatic evolution of the symmetric and antisymmetric initial eigenstates, the rate \( \alpha \) must be sufficiently small. In figure 2, we show the evolutions of the ground and first excited eigenstates of the system of the effective nonlinearity \( \lambda = 20 \) and ramping rate \( \alpha = 1/4 \). For such a small value of \( \alpha \), both the ground and first excited eigenstates of the initial double-well potential adiabatically evolve into the corresponding ground and first excited eigenstates of the final single-well potential. This means that the non-adiabatic effects are negligible for such a slowly varying process.

The usual double-well BEC interferometers involve condensates trapped in a double-well potential before recombination. Below, we consider the case of the initial state of the BEC being fully localized in a single well of a symmetric double-well potential with a sufficiently high barrier, so that there is no significant overlap between the Wannier states of the two wells. The fully localized initial state can be viewed as the equal-probability superposition of the ground and first excited eigenstates, so that it can be used to observe the interference of these.
two eigenstates [26]. For BECs trapped in such a deep potential, the mean-field ground and first excited states are degenerate or quasi-degenerate. Even for low barriers, as long as the tight-binding condition is still satisfied, the two-mode approximation will give the picture of a classical Bose–Josephson junction. In the framework of second quantization, the system obeys a two-site Bose–Hubbard Hamiltonian. In this fully quantum picture, a completely localized initial state corresponds to the highest excited state for repulsive interactions. This state exhibits degeneracy with the sub-highest excited state [27], which corresponds to the bistability in a classical Bose–Josephson junction [28].

Since initially there is no overlap between the two Wannier states, the quantum tunnelling between those states is negligible. As the barrier height gradually decreases, the overlap between two Wannier states becomes more significant. Then, both the quasi-degeneracy between the ground and first excited states in the mean-field picture and the quasi-degeneracy between the highest excited and sub-highest excited states in the quantum picture break down. The quantum tunnelling of the fully localized state becomes more pronounced as the barrier height is decreasing. Due to the very slow reduction in the barrier height, the kinetic energy stays small during the whole process, and the quantum tunnelling dominates the dynamics. However, the over-barrier hopping could occur in a rapidly varying process, in which case the kinetic energy can exceed the potential barrier.

To explore the dynamic evolution, we numerically integrate the GP equation with the well-developed operator-splitting procedure and the absorbing boundary conditions. In figure 3, we show the time evolution of the condensate density and phase for the trapping frequency $\omega = 0.2\pi$, the initial barrier height $B_0 = 20.0$, the barrier width $d = \sqrt{2}/2$, the ramping rate $\alpha = 1/4$ and different values of the effective nonlinearity $\lambda$. For the chosen small ramping down rate, all symmetric (antisymmetric) states of the deep double-well potential will adiabatically evolve into the corresponding ground (or excited) states of the single-well potential.
Evolution of the fully localized initial state strongly depends on the values of the effective nonlinearity $\lambda$. In the linear case ($\lambda = 0$), the fully localized initial state can be viewed as an equal-probability superposition of the symmetric and antisymmetric states, so that, according to the adiabatic theorem, the evolving state is always the equal-probability superposition of the ground and first excited states of the system. Due to the nonlinear interactions, the superposition principle becomes invalid, and the resulting behaviour can be interpreted as the coupled dynamics of the ground and multiple excited states of the nonlinear system. Akin to the linear systems, the quantum tunnelling appears once the quasi-degeneracy between the ground and the first excited state is broken, and gradually becomes significant with decreasing barrier height. The time scale on which the quantum tunnelling appears in the nonlinear system shortens with the growth of nonlinear interaction strength $\lambda$. The excited nonlinear states of the BEC in a single-well potential can be thought as stationary configurations of single or multiple dark solitons [12]. As a result of the population transfer to such excited modes, the condensate develops multi-peak distribution with significant phase gradients across density notches between the neighbouring peaks. These notches are dark or gray solitons with well-defined phase gradients close to $\pi$ (see figure 4).

The number of dark solitons formed in this process varies with the effective nonlinearity $\lambda$ (see figure 5). This dependence exhibits multiple plateaux as the effective nonlinearity changes, as shown in figure 5. Given the relationship between the nonlinear interaction...
Figure 4. Formation of dark solitons in the system with the effective nonlinearity $\lambda = 15$. Left: density distributions $|\Psi(x,t)|^2$ for different times. Right: phase distributions $\phi(x,t)$ for the corresponding density distributions.

Figure 5. Number of dark solitons generated in the condensate at $t = 80$ versus the effective nonlinearity parameter $\lambda$.

strength and the key parameters of the system, $\lambda = 2N\bar{a}_s\omega_{\perp}\hbar$, one can control the number of generated solitons by adjusting the s-wave scattering length with the Feshbach resonance, the total number of atoms in the condensate with initial preparation, and/or the transverse trapping frequency by tuning the transverse trapping field strength. The number of solitons remains unchanged for a long period of time before multiple collisions between solitons take place. The inelastic collisions lead to the radiation of small-amplitude waves, and after a large number of collisions the number of solitons oscillating in the trap changes.
3. Modal decomposition

To obtain the quantitative picture of the population transfer, we decompose an arbitrary state of our time-dependent system as \[18, 29\]

\[\Psi(x, t) = \sum_{j}^{N} C_j(t)\phi_j(x, t), \quad (4)\]

where \(\phi_j(x, t)\) is the \(j\)th stationary state for the nonlinear system with the potential \(V(x, t)\), which obeys the (dimensionless) equation

\[\mu_j(t)\phi_j(x, t) = \left[ -\frac{1}{2} \frac{d^2}{dx^2} + V(x, t) \right] \phi_j(x, t) + \lambda \phi_j^3(x, t). \quad (5)\]

Here, \(\mu_j(t)\) is the chemical potential for the \(j\)th stationary state. For each instantaneous form of the time-dependent potential, the nonlinear eigenstates \(\{\phi_j(x, t)\}\) form an orthogonal set, similarly to their linear counterparts \[18, 12, 29\]. Due to the nonlinear interparticle interactions, there exist additional stationary states (e.g., self-trapped states) which have no linear counterparts \[29, 30\]. Nevertheless, every stationary state of the nonlinear system can be composed of the orthogonal basis \(\{\phi_j(x, t)\}\). The nonlinear eigenstates of time-independent potentials are also time-independent. However, due to the violation of the superposition principle, the population transfer between different nonlinear eigenstates also occurs in time-independent systems \[29\]. This type of population transfer, which originates from the exchange collisions between atoms in different eigenstates, will be considered in detail below.

The population dynamics for different nonlinear eigenstates can be described by the evolution of the complex coefficients \(C_j(t)\), which obey a series of coupled first-order differential equations,

\[\frac{i}{\hbar} \frac{dC_j(t)}{dt} = \sum_{j}^{N} \left[ E_0^{l,j} + \sum_{k,k'} Q_{k,k'}^{l,j}(t)C_k(t)C_{k'}(t) \right] C_j(t). \quad (6)\]

Due to the conservation of the total number of particles, \(C_j(t)\) satisfy the normalization condition \(\sum_j |C_j(t)|^2 = 1\). Here, the linear coupling parameters are

\[E_0^{l,j}(t) = \int \phi_l^*(x, t)H_0\phi_j(x, t) \, dx, \quad (7)\]

and the nonlinear coupling parameters are

\[Q_{k,k'}^{l,j}(t) = \lambda \int \phi_l^*(x, t)\phi_k^*(x, t)\phi_k(x, t)\phi_j(x, t) \, dx. \quad (8)\]

For a spatially symmetric potential \(V(x, t) = V(-x, t)\), we have \(Q_{k,k'}^{l,j}(t) = 0\), and then \((k + k' + l + j)\) are odd integer numbers.

In our numerical simulations, we generalize the direct relaxation method for linear quantum systems \[31\] to calculate the eigenstates and their eigenvalues (chemical potentials) for our nonlinear system with different effective nonlinearities at any moment of time. Projecting the condensate wavefunction \(\Psi(x, t)\) onto the nonlinear eigenstates \(\phi_j(x, t)\), we find the population probabilities \(P_j(t) = |C_j(t)|^2 = |\int \phi_j^*(x, t)\Psi(x, t) \, dx|^2\) which depend on the time and the effective nonlinearity \(\lambda\).

In figure 6(a), we show the time evolution of the population probabilities \(P_j(t)\) for the effective nonlinearity \(\lambda = 2.0\). Here we only consider the four lowest eigenstates (i.e., \(N = 4\)), so that \(P_0\) is the ground-state population probability, \(P_j (j = 1, 2, 3)\) are the population probabilities of the \(j\)th excited state, and \(P_{\text{tot}} = P_0 + P_1 + P_2 + P_3\) is the total
probability of the first four eigenstates. For $t < 50$, the population probabilities keep almost unchanged. In the region of $50 < t < 80$, we observe a fast population transfer from the ground state to the first excited state. After the recombination of the two wells, i.e., for $t > 80$, the populations in different nonlinear eigenstates oscillate with time, even though the nonlinear system has a time-independent potential, time-independent eigenstates, and no energy degeneracy between the neighbouring eigenstates. This behaviour differs drastically from the linear dynamics, where populations in different eigenstates always remain unchanged. We find that at this value of the effective nonlinearity the total population probability $P_{\text{tot}}(t)$ in the first four eigenstates is always close to 1. The low-frequency population oscillations are dominated by the linear coupling between different modes and the high-frequency ones are due to the nonlinear cross-coupling of the nonlinear modes which corresponds to the exchange collision of atoms in different eigenstates.

For small $\lambda$, the dynamics of $P_j$ after the merging of the two wells can be approximately captured by the projection of the BEC state at the moment of the merging (here $t = 80$) onto the set of $N$ stationary nonlinear states $\phi_j(x)$ of the single-well potential $V_0(x)$ of $B(t) = 0$. This is confirmed in figure 6(b), where we employ the coupled-mode theory (6) with $N = 4$ eigenstates of $V_0(x)$ (cf figure 6(a)). The number of eigenstates, $N$, that must be considered in the coupled-mode theory, increases with the effective nonlinearity. The highest-order mode ($j = N$) of the harmonic potential $V_0(x)$ with a significant (nonzero) excitation probability $P_N$ at the merging time will therefore determine the number $N$ of dark solitons that are likely
to be formed. Figure 6(c) shows the excitation probabilities at $t = 80$ for the first six lowest eigenstates of $V_0(x)$ and different $\lambda$. By comparing the number of significantly excited states for different values of $\lambda$, one can see that the number of dark solitons formed is indeed approximately determined by the highest-excited nonlinear mode of the harmonic trap that is still sufficiently populated. For instance, $N = 1$ solitons are expected to form for $\lambda = 2$, and $N = 2$ for $\lambda = 10$ (cf figure 5).

4. Conclusions

We have explored the nonlinearity-assisted quantum tunnelling and formation of nonlinear collective excitations in the matter-wave interferometer based on a time-dependent double-well potential, dynamically reconfigured to form a single-well harmonic trap. In contrast to the Josephson tunnelling and Landau–Zener tunnelling, the nonlinearity-assisted quantum tunnelling is brought about by the nonlinear inter-mode population exchange scattering. The excitations caused by this type of tunnelling lead to the dark soliton generation in the process that differs dramatically from the phase imprinting [10, 11] or condensates collisions [16]. The number of generated solitons can serve as a sensitive measure of the degree of the nonlinearity in the system. With the well-developed techniques for preparing and manipulating condensed atoms in double-well potentials [3–6], loading the condensed atoms in one well of a deep double-well potential and adjusting the barrier height, the experimental observation of this effect seems feasible.

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Note added. After this manuscript was prepared for submission, the group of Peter Engels from the Washington State University reported the experimental observation of matter-wave dark solitons due to quantum tunnelling [32], in the process of sweeping a high potential barrier from one edge of the trap to the other.

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