RENORMALIZATION AMBIGUITIES IN CASIMIR ENERGY

Luiz C. de Albuquerque*

Instituto de Física, Universidade de São Paulo
P.O. Box 66318, 05389-970 São Paulo, SP, Brazil

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Abstract

Some questions were recently raised about the equivalence of two methods commonly used to compute the Casimir energy: the mode summation approach and the one-loop effective potential. In this respect, we argue that the scale dependence induced by renormalization effects, displayed by the effective potential approach, also appears in the MS method.

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*e-mail:claudio@fma1.if.usp.br
Casimir [1] showed that two neutral perfectly conducting parallel plates placed in the vacuum attract each other, due to zero-point oscillations of the electromagnetic field (field strength fluctuations). His starting point resembles an old idea of Euler and Heisenberg [2]. They used the zero-point oscillations of the Dirac field (charge fluctuations) in an external field to define an effective action to the electromagnetic field. The common feature is the summation of the energies associated with the vacuum fluctuations of the constrained quantum field to define the vacuum energy and the effective action. The effective action was later reinterpreted in terms of the generating functional of 1PI Green’s functions. The effective action at one-loop order is formally equivalent to the older definition of Euler and Heisenberg [3], and the vacuum energy expression obtained from it is also formally identical to the mode summation method used by Casimir.

The formal expression of the vacuum energy is ultraviolet (UV) divergent. The existence of infinite contributions in renormalizable field theories can be handled by the regularization/renormalization process. This procedure introduces two possible sources of ambiguities. The separation in a finite part plus an infinite contribution is arbitrary since many different subtractions are possible. The regularization procedure also introduces a dimensionfull parameter, the renormalization scale. The renormalization condition fix the finite part of the counterterms, and it may or it may not discard the renormalization scale. We deserve the name Casimir energy for the regularized vacuum energy satisfying some definite renormalization condition. The requirement of a zero vacuum energy for flat unconstrained Minkowski space-time is compatible with normal ordering concepts [4]. If this condition succeed in discard the infinite contributions and the renormalization scale, the Casimir energy is uniquely defined. This happens in the electromagnetic Casimir effect.
Otherwise the Casimir energy is ambiguous and, since the pressure on the plates is the unique measurable quantity, the result is non-predictive [5]. This may occurs for instance as a consequence of a mass term [6], or deviations from the plane geometry [5, 7, 8]. If the interest is not on the computation of Casimir forces, but in the contribution of the Casimir effect for other process, like the bag pressure in MIT Bag model calculations [8], this ambiguity signals the existence of a new, phenomenological, free parameter [5]. This observation is of importance in studies of liquid Helium films [9].

It was recently argued that the effective potential method is more reliable than the mode summation (MS) approach in the computation of the Casimir energy [10]: in some cases the MS approach may neglect a renormalization scale dependent contribution. The purpose of this paper is to show that the same scale dependence is expect in both methods (one-loop effective potential and MS). This indeed was shown in [8] for a massless scalar field. Here, a global analysis [11] is made of the massive scalar case for flat parallel plates. As a by-product, we also establish the analytical equivalence between regularization prescriptions of the MS formula and regulated definitions of the effective potential. We do not claim that the Casimir energy is unambiguous defined in all situations: since subtractions are necessary, the Casimir energy will depend on the subtraction employed.

Let \( \phi(x) \) be a real massive scalar field, with a \( \lambda \phi^4/4! \) self-coupling. The imposition of Dirichlet boundary conditions (DBC) in \( x_1 \), i.e. \( \phi(\vec{x}, x_1 = 0) = 0 = \phi(\vec{x}, x_1 = L) \), leads to the discrete set \( k_1 = \frac{2\pi n}{L} (n \in \mathcal{N}) \), plus plane waves in the other \( d = D - 1 \) dimensions. The DBC simulate the perfect conducting plates in the Casimir apparatus.

\( ^1D \) dimensional Euclidean space with positive signature. \( x^\mu = (t, x_1, \vec{x}_\perp), \vec{x}^\mu = (t, \vec{x}_\perp), \vec{k}^\mu = (k_0, \vec{k}_\perp) \).
The one-loop effective potential is given by the sum of the 1PI one-loop diagrams with zero external momenta. After summation [12], we have in the unconstrained space

\[ V^{(1)}(\bar{\phi}) = \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \left[ \ln(k^2 + M^2(\bar{\phi})) - \ln(k^2 + m^2) \right] + \delta V^{(1)}(\bar{\phi}), \]  

where \( M^2 = m^2 + \frac{\lambda}{2} \bar{\phi}^2 \), \( \bar{\phi} \) is a constant background field, and \( \delta V^{(1)}(\bar{\phi}) \) is the counterterm of order \( \hbar \). The second term in the square brackets comes from the normalization of the generating functional. The minimum of the effective potential gives the vacuum energy density [12, 13]. We study the case without spontaneous symmetry breaking (SSB), i.e. \( m^2 > 0 \) (the case with SSB cannot be discussed in a global framework [14]). Thus (the normalization factor and the counterterm are not written)

\[ \mathcal{E}_0(L) = V^{(1)}(\bar{\phi} = 0) = \frac{1}{2} \int_{(D-1)}^{(L)} \ln\left(\bar{k}^2 + \left(\frac{\pi n}{L}\right)^2 + m^2\right), \]  

where the DBC was imposed, and our notation is

\[ \int_{(D-1)}^{(L)} \equiv \frac{1}{L} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{d^{D-1} \bar{k}}{(2\pi)^{D-1}} = \text{Tr}_{k}^{(D)} |_{\text{DBC}}. \]  

\( \text{Tr}_{k}^{(D)} \) stands for the trace in the \( D \) dimensional momentum (\( k \))-space. Integration over \( k_0 \) gives the MS expression

\[ \mathcal{E}_0(L) = \frac{1}{2} \int_{(D-2)}^{(L)} \left(\bar{k}_\perp^2 + \left(\frac{\pi n}{L}\right)^2 + m^2\right)^{\frac{1}{2}}. \]  

\( \mathcal{E}_0 \) is the sum of all possible zero-point oscillations centered about \( \bar{\phi} = 0 \).

The equivalence showed above involves manipulations with ill-defined quantities. It is therefore necessary to give a mathematical definition to
the right-hand side (RHS) of equations (1) and (4). To start, the one-loop effective potential is written as

\[ V^{(1)}(\bar{\phi} = 0) = \frac{1}{2} \ln \left| \det_{k} \left( \frac{O}{\mu^2} \right) \right|_{\text{DBC}}, \tag{5} \]

where \( O(\bar{\phi}) = k^2 + m^2 \), and \( \mu \) is a mass scale used to normalize the determinant. Now a regulated version of eq.(5) is supplied by the generalized \( \zeta \) function technique [15],

\[ \det_{k} \left( \frac{O}{\mu^2} \right)_{\text{DBC}} = \exp \left[ -\frac{\partial}{\partial s} \zeta_{O}(s) \mid_{s=0} \right] \tag{6} \]

\[ = \left( \mu^2 \right)^{-\zeta_{O}(0)} \exp \left[ -\frac{\partial}{\partial s} \zeta_{O}(s) \mid_{s=0} \right]. \tag{7} \]

where

\[ \zeta_{O}(s) = \text{Tr}_{k}^{(D)} \left( \frac{O}{\mu^2} \right)^{-s} \mid_{\text{DBC}} \equiv \sum_{n}^{(L)} \left( \Lambda_n \mu^{-2} \right)^{-s} \tag{8} \]

is the generalized \( \zeta \) function associated with the \( D \)-dimensional operator \( \frac{O}{\mu^2} \), subjected to the DBC, with eigenvalues given by \( \mu^{-2} \Lambda_n = k^2 + \left( \frac{\pi n}{L} \right)^2 + m^2 \). \( \zeta_{O}(s) \) is the \( \zeta \) function of the operator \( O \), with the scale \( \mu \) factored out. Using equations (2), (6), and (7), we obtain the two equivalent forms

\[ E_{0}(L) = -\frac{1}{2} \frac{\partial}{\partial s} \zeta_{D}(s; \mu) \mid_{s=0} = -\frac{1}{2} \left[ \zeta_{D}'(0) + \zeta_{D}(0) \ln \mu^2 \right]. \tag{9} \]

An analytical continuation to the whole complex \( s \)-plane must be done before we take the limit \( s \to 0 \). It can be shown that for an elliptic, positive

Note that \( \ln \det A_{x,y} = \text{Tr}_{k}^{(D)} \ln A_{x,y} = \Omega_{D} \text{Tr}_{k}^{(D)} \ln A_{k} = \Omega_{D} \ln \det_{k} A_{k} \), where \( \Omega_{D} \) is a \( D \)-dimensional normalization volume, and \( A_{x,y} = (-\Box + m^2)\delta^{D}(x-y) \).

The simplified notation is: \( \frac{\partial}{\partial s} \zeta_{O}(s) \mid_{s=0} = \zeta_{D}'(0; \mu) \).
second-order operator, $\zeta_D(s)$ is a meromorphic function with only simple poles, in particular analytic at $s = 0$ \[13\]. Hence, $\mathcal{E}_0(L)$ defined in (9) is a finite quantity.

The scale contribution to the Casimir energy comes from the $\zeta(0)$ term in eq. (9). It is the absence of an analogous term in the naive MS formula, eq.(4), that leads Myers \[10\] to introduce the notion of “zero-point anomaly”. We will show below that the same scale behavior appears also in the MS approach. The key point is: by virtue of the formal relation derived above, a proper definition (i.e. regularization) of the effective potential leads to a definition of the MS formula eq.(4). To proceed along this way, we integrate over $k_0$ in eq. (8). This gives

$$
\zeta_D(s; \mu) = -\mu f(s) \left[ \frac{\partial}{\partial s} \frac{1}{\Gamma(s)} \int_{(D-2)} f^{(L)} \left[ \kappa_\perp^2 + \left( \frac{n}{L} \right)^2 + m^2 \right]^{1/2-s} \right]_{s=0},
$$

(10)

$$
f(s) = -\Gamma\left(s - \frac{1}{2}\right)/(2\sqrt{\pi}) \text{ is an irrelevant factor (} f(0) = 1 \text{). Now, using (9) we obtain for } \mathcal{E}_0(L) \text{ (plus counterterms):}
$$

$$
\mathcal{E}_0(L) = \frac{1}{2} \mu \frac{\partial}{\partial s} \left\{ \frac{1}{\Gamma(s)} \int_{(D-2)} f^{(L)} \left( \lambda_n \mu^{-2}\right)^{1/2-s} \right\}_{s=0} = \frac{1}{2} \mu \frac{\partial}{\partial s} \left\{ \frac{1}{\Gamma(s)} \zeta_{D-1} \left(s - \frac{1}{2}, \mu\right) \right\}_{s=0},
$$

(11)

where $\lambda_n = \kappa_\perp^2 + \left( \frac{n}{L} \right)^2 + m^2 \equiv \omega_n^2$ are the eigenvalues of the $(D - 1)$-dimensional (reduced) operator $\tilde{O} = \kappa_\perp + m^2$. We call $\zeta_{D-1}(s)$ the “reduced” $\zeta$ function.

For a function $G(s)$ analytic at $s = 0$, we use the approximation $1/\Gamma(s) \approx s + \gamma s^2 + O(s^3)$ to deduce $\frac{\partial}{\partial s} \frac{G(s)}{\Gamma(s)}|_{s=0} = G(0)$. Suppose that $\zeta_{D-1}(s - \frac{1}{2}, \mu)$ is analytic at $s = 0$; Then, we will obtain
\[ E_0(L) = \frac{1}{2} \mu \zeta_{D-1}(\frac{-1}{2}; \mu) = \frac{1}{2} \int_{(D-2)}^{L} \omega_n. \] (12)

This is exactly the non-regulated MS formula for the vacuum energy, eq.(4), in the present notation. It is clear that eq.(11) is another way to write eq.(9). However, formula (11) by itself is also a regularized expression for the vacuum energy in a MS like form (perhaps a strange one!). The UV divergences, associated with the high-frequency modes in (12), appear as poles in \( \zeta_{D-1}(s - \frac{1}{2}; \mu) \) when we take \( s \to 0 \). In general \( \zeta_{D-1}(s - \frac{1}{2}; \mu) \) is non regular at \( s = 0 \). Thus, we can interpret eq.(11) as a MS regularized formula for the vacuum energy.

A Mellin transform can be made to relate the generalized \( \zeta \) function to the trace of the heat kernel, \( Y(t) \equiv \text{tr} e^{-t\tilde{Q}t^{-2}} \). As is known [17] the heat kernel possess an asymptotic expansion for small \( t \). Using this expansion, we get the pole structure of \( \zeta_{D-1}(s; \mu) \) [18]

\[ \zeta_{D-1}(s; \mu) = \frac{1}{(4\pi)^{\frac{D+1}{2}} \Gamma(s)} \left\{ \sum_{j=0}^{\infty} \frac{C_j}{s - (\frac{D-1}{2} - j)} + F(s) \right\}. \] (13)

\( F(s) \) is an entire analytic function of \( s \). We see that \( \zeta_{D-1}(s; \mu) \) is a meromorphic function of \( s \), with simple poles at \( s = \frac{1}{2}, 1, \frac{3}{2}, ... \frac{D}{2} \) with residua given by the coefficients \( C_j \).

Using the asymptotic expansion (13) in eq. (11), we obtain

\[ E_0(L) = \frac{\mu}{2(4\pi)^{\frac{D+1}{2}}} \frac{\partial}{\partial s} \Gamma(s) \left\{ \sum_{j \neq \frac{D}{2}}^{\infty} \frac{C_j}{s - (\frac{D}{2} - j)} + F(s - \frac{1}{2}) + \frac{C_{\frac{D}{2}}}{s} \right\} \]

\[ = \tilde{E}_0(L) - \frac{\psi(1)}{2(4\pi)^{\frac{D}{2}}} \mu C_{\frac{D}{2}}(\mu). \] (14)

\(^4\text{Note that the entire expression on the RHS of (11) is analytic at } s = 0, \text{ as well as } \zeta_{D-1}(s; \mu)\)
where $\mathcal{E}_0(L)$ comes from the regular part inside the curly brackets in the first line. The combination $\mu C_{D/2}(\mu)$ do not depend on $\mu$ (Appendix A relates the various $\zeta$ functions appearing in the text).

To see the effect of a change of scale, $\mu \to \mu'$, eq. (8) can be used to obtain

$$\zeta_D(s; \mu') = \left(\frac{\mu'}{\mu}\right)^{2s} \zeta_D(s; \mu).$$

Using the equations (11) and (15), we have

$$\mathcal{E}_0(\mu') = \mathcal{E}_0(\mu) - \mu \frac{C_{D/2}(\mu)}{(4\pi)^{D/2}} \ln\left(\frac{\mu'}{\mu}\right).$$

The scale dependence is logarithmic and proportional to the coefficient $C_{D/2}$. This is already clear from eq.(9) since $\zeta_D(0) \propto \mu C_{D/2}(\mu)$. It is obvious from our approach that the same scale behavior is found both in the effective potential method (with the $\zeta$ function regularization) and in the MS approach (with the regularization prescription (11)). From another point of view, the $\zeta$ function definition of the effective potential is equivalent to the analytical regularization (11) of the MS formula.

For conformally invariant theories in flat space-time and flat parallel plates, $C_{D/2} = 0$ and $\mu$ disappears identically [8]. In the $\zeta$ function approach, one can define the Casimir energy by demanding that the vacuum energy disappears in the limit $L \to \infty$, thus fixing the finite part of the constant counterterm in [11] [14]. Another conventional choice, related to the previous one, is to define the Casimir energy (density) as [16]

$$\mathcal{E}_c = \mathcal{E}_0(L) - \mathcal{E}_0(L \to \infty).$$

This prescription is automatically achieved if we use the normalization factor in (11), calculated at $L \to \infty$. The renormalization condition (17) is physically
reasonable since it leads to a zero vacuum energy for flat unconstrained space-time, consistent with the usual normal ordering prescription.

In the general case we expect a dependence on $\mu$ in the Casimir energy. In the simple model discussed here the prescription \((17)\) (or the equivalent one) can not fix the scale for odd dimensional space-times:

$$
\begin{align*}
\mu C_D^{(+)}(\mu) &= (4\pi)^{D/2} \zeta_D^{(+)}(0) = \frac{(-1)^{D/2}}{(D/2)!} m^D, \\
\mu C_D^{(-)}(\mu) &= (4\pi)^{D/2} \zeta_D^{(-)}(0) = -\frac{(-1)^{(D-1)/2}}{(D-1)/2)! \sqrt{\pi m^{D-1}}/L,
\end{align*}
$$

where $+$ is for even and $-$ is for odd $D$-dimensional space-time, see eq. (A.8). The renomalization condition \((17)\) fails to give an unambiguous result for odd $D$ since $C_D^{(-)} \to 0$ as $L \to \infty$. $\zeta(0)$ is closely related to the trace anomaly \([18]\). For unconstrained fields or periodic boundary conditions (PBC), $\zeta_D^{(-)}(0) = 0$, even for $m \neq 0$ (indeed, $\zeta_D^{(-)}(0) = 0$ for $L \to \infty$). Hence, result \((18)\) seems to indicate a breakdown of scale invariance for odd $D$ induced by the DBC. This point deserves a further study, which however goes beyond the present purpose. The value of $\zeta_D^{(+)}(0)$ is the same as in the unconstrained space case ($L \to \infty$). This is also true in the PBC case for any space-time dimension (including the particular cases of a compactified space dimension, or finite temperature field theory in the Euclidean time). This fact is generalized in the statement that PBC do not introduce new ultraviolet structures in field theory, besides the usual ones. As eq.(18) indicates this is no longer true in the DBC case \([6]\).

Thus, to define a finite Casimir energy we have to impose a renormalization condition (as in eq.(17)), besides a regularization method. Of course, different renormalization conditions may lead to different (finite) Casimir en-
ergies. However, the logarithmic dependence displayed in eq.(16) is a general feature, within the class of analytical regulators discussed here\textsuperscript{5}. Many regularization prescriptions for the MS formula (4) are possible, and indeed are fairly used \[16\]. For instance, starting from the unregulated MS formula (12), Blau et al. \[8\] used the following analytical regularization (to be compared with eq. (11))

\[
\mathcal{E}_0(s) = \frac{1}{2\mu} \sum_{\ell} \left( \lambda_n \mu^{-2} \right)^{s-\frac{1}{2}} = \frac{1}{2\mu} \zeta_d \left( s - \frac{1}{2}; \mu \right). \tag{19}
\]

In this case, the pole in $s = 0$ do not cancel against $\Gamma(s)$, and the coefficient $C_D^2$ turns into an obstacle to give a finite Casimir energy. The total energy is finite, because of its effects over the gravitational field. Hence, the bare action must contain a term proportional to $C_D^2$. In the minimal subtraction scheme proposed in \[8\] the pole is simply removed, and the Casimir energy is defined by (P=principal value)

\[
\mathcal{E}_c \equiv \lim_{s \to 0} \frac{1}{2} \left\{ \mathcal{E}_0(+s) + \mathcal{E}_0(-s) \right\} \equiv \frac{\mu}{2} P \zeta_d \left( s - \frac{1}{2}; \mu \right). \tag{20}
\]

Although they used a completely different regularized MS expression (compare eq.(19) with eq.(11)) and more important, another renormalization condition (eq.(20) instead of eq.(17)), Blau et al. \[8\] derived the formula (16) for the scale dependence of the Casimir energy in the massless case. Clearly, the Casimir energy computed according equations (19) and (20) is not necessarily identical to the one computed using equations (11) and (17). For instance, in the model studied here a direct application of equations (19), (20) and (A.7) leads to

\textsuperscript{5}The same is true in the dimensional regularization method.
$$E_c = -\frac{1}{(4\pi)^{D/2}} \left[ g_+^\mu + 2 \left( \frac{m}{L} \right)^D \sum_{n=1}^{\infty} n^{-\frac{D}{2}} K_{\frac{D}{2}} (2mL^n) \right],$$

where

$$g_+^\mu = -\frac{\sqrt{\pi} m^{D-1}}{2L} \Gamma \left( -\frac{D-1}{2} \right) + \frac{(-1)^{\frac{D-1}{2}}}{(2L)^{D-1}} m^D \ln \frac{\mu}{m},$$

$$g_-^\mu = \frac{m^D}{2} \Gamma \left( -\frac{D}{2} \right) + \frac{(-1)^{\frac{D-1}{2}} \sqrt{\pi} m^{D-1}}{2L} \ln \frac{\mu}{m}.$$  

This result is to be contrasted with that obtained by an application of equations (11) and (17) [6]

$$E_c = -\frac{1}{(4\pi)^{D/2}} \left[ \tilde{g}_-^\mu + 2 \left( \frac{m}{L} \right)^D \sum_{n=1}^{\infty} n^{-\frac{D}{2}} K_{\frac{D}{2}} (2mL^n) \right],$$

where

$$\tilde{g}_-^\mu = g_-^\mu - \frac{m^D}{2} \Gamma \left( -\frac{D}{2} \right).$$

Finally, we will show that the MS regularization proposed in [8] has an analytical counterpart in the effective potential method. Using the formula

$$\ln \left( \frac{\tilde{b}}{a} \right) = \int_0^\infty \frac{dx}{x} [e^{-ax} - e^{-bx}],$$

we may recast $$V^{(1)}(\tilde{\phi})$$ in eq. (1) as

$$V^{(1)}(\tilde{\phi}) = -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \text{Tr}_k \left[ e^{-\tau(k^2 + M^2)} - e^{-\tau(k^2 + m^2)} \right].$$

This is essentially the Schwinger formula (SF) applied to the Euclidean effective potential [13]. We proved in [19] the equivalence between the SF and the MS formula. We will repeat only a few steps of the argument of [19], but now introducing a mass scale $$\mu$$.

The regularized version of the SF that will be used is [20]
\[ V^{(1)}(\bar{\phi} = 0) |_{DBC} = -\frac{1}{2} \int_0^{\infty} d\tau \, \tau^{s-1} \sum_{(D-1)} f^{(L)}(D) \, e^{-\tau(\bar{k}^2 + \frac{n^2 \pi^2}{L^2} + m^2)\mu^{-2}}, \]  

(26)

where \( s \) is large enough to make the integral well defined, and \( \mu \) is a mass scale. In this approach, we first compute the integral, then make an analytical continuation to the whole complex \( s \)-plane, and finally the limit \( s \to 0 \) is carefully taken. From eq. (26) we obtain

\[ \frac{\partial V^{(1)}}{\partial m^2} = \frac{1}{2\mu^2} \int_{(D-1)} f^{(L)}(D) \int_0^{\infty} d\tau \tau^s e^{-\tau(\bar{k}^2 + \frac{n^2 \pi^2}{L^2} + m^2)\mu^{-2}}, \]  

(27)

Using the definition of the Euler Gamma function, the integration over \( \tau \) is readily done. Then, we integrate over \( k_0 \) to obtain

\[ \frac{\partial V^{(1)}}{\partial m^2} = \frac{\Gamma\left(\frac{1}{2}\right)}{4\pi \mu^2} \sum_{(D-2)} f^{(L)}(D) \left[ \frac{\bar{k}_\perp^2 + \left(\frac{n \pi}{L}\right)^2 + m^2}{\mu^2} \right]^{\frac{1}{2} - s}, \]  

(28)

Integrating on \( m^2 \) and identifying \( \mathcal{E}_0 = V^{(1)}(\bar{\phi} = 0) \), we finally obtain (apart from an irrelevant additive constant)

\[ \mathcal{E}_0(s) = \frac{1}{2} g(s) \mu^2 \sum_{(D-2)} f^{(L)}(D) \left[ \frac{\bar{k}_\perp^2 + \left(\frac{n \pi}{L}\right)^2 + m^2}{\mu^2} \right]^{\frac{1}{2} - s}, \]  

(29)

where \( g(s) = \frac{1}{1-2s} \Gamma(s+\frac{1}{2}) \sqrt{\pi}. \) Taking \( g(0) = 1 \), we have

\[ \mathcal{E}_0(s) = \frac{1}{2} \mu^2 \sum_{(D-2)} f^{(L)}(D) \left( \lambda_n \mu^{-2} \right)^{\frac{1}{2} - s} = \frac{1}{2} \mu \zeta_d \left( s - \frac{1}{2} \right), \]  

(30)

which is just eq. (19). Hence, the analytical regularization used in the modified SF, eq. (26), is the equivalent in the effective potential method of the regularized MS formula of Blau et al. [8]. We can summarize: To each definition of the effective potential corresponds a regularized definition of the MS formula.
Appendix A

The generalized ζ function $\zeta_{O^2}(s) = \zeta_D(s; \mu)$ associated with the $D$-dimensional operator $O$ is related to the “reduced” ζ function $\zeta_{D-1}(s; \mu)$ by

$$\zeta_D(s; \mu) = \mu f(s) \frac{1}{\Gamma(s)} \zeta_{D-1}(s - \frac{1}{2}; \mu), \quad (A.1)$$

where $f(s)$ is defined below eq. (10). The ζ function $\zeta_D(s; \mu)$ possess an expansion of the same sort of the “reduced” $\zeta_{D-1}(s; \mu)$, see eq. (13). Let $\tilde{C}_j(\mu)$ be the coefficients of the expansion of $\zeta_D(s; \mu)$; a straightforward calculation gives

$$\zeta_D(0; \mu) = \frac{1}{(4\pi)^{D/2}} \tilde{C}_{D/2}(\mu),$$
$$\tilde{C}_{D/2}(\mu) = \mu C_{D/2}(\mu). \quad (A.2)$$

$\zeta_D(s)$ is the ζ function of the operator $O$. It is possible to relate $\zeta_D(s)$ and $\zeta_D(s; \mu)$. If $\tilde{C}_j$ are the coefficients of the expansion of $\zeta_D(s)$, it can be show that $\tilde{C}_{D/2}(\mu) = \tilde{C}_{D/2}$. Thus, by eq. (A.2) the combination $\mu C_{D/2}(\mu)$ do not depend on $\mu$:

$$\mu C_{D/2}(\mu) = (4\pi)^{D/2} \zeta_D(0). \quad (A.3)$$

To compute $\zeta_D(0)$, we integrate eq. (8) (without the scale $\mu$) to obtain

$$\zeta_D(s) = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(s - \frac{D-1}{2})}{\Gamma(s)} \frac{1}{L} \frac{1}{E_1^m} (s - \frac{D-1}{2}; \frac{\pi^2}{L^2}), \quad (A.4)$$

where we introduced the modified inhomogeneous Epstein function
\[ E_1^2(s; a) = \sum_{n=1}^{\infty} \left( a n^2 + c^2 \right)^{-s}, \]  
(A.5)

with \( a, c^2 > 0 \). The summation above converges for \( \Re s > \frac{1}{2} \).

The analytical continuation to the whole complex \( s \)-plane of \( E_1^2(s; a) \) is given by \[ E_1^2(\nu, a^2) = \frac{1}{2} c^{-2\nu} + \frac{1}{2} \frac{\sqrt{\pi}}{a} \frac{\Gamma(\nu - \frac{1}{2})}{\Gamma(\nu)} c^{1-2\nu} + 2 \frac{\sqrt{\pi}}{a} \left( \frac{ca}{\pi} \right)^{\nu-\frac{1}{2}} \frac{1}{\Gamma(\nu)} \sum_{n=1}^{\infty} n^{\nu-\frac{1}{2}} K_{\frac{1}{2}-s} \left( \frac{2\pi c n}{a} \right). \]  
(A.6)

Using (A.6) in (A.4), we obtain
\[
\zeta(s) = \frac{1}{(4\pi)^{D/2} L \Gamma(s)} \left[ -\frac{m^{D-1-2s}}{2} \Gamma(s - D - \frac{1}{2}) + \frac{L m^{D-2s}}{2\sqrt{\pi}} \Gamma(s - D - \frac{1}{2}) \right. \\
+ \left. \frac{2L}{\sqrt{\pi}} \left( \frac{m}{L} \right)^{D-s} \sum_{n=1}^{\infty} n^{s-D} K_{\frac{D}{2}-s} (2mL) \right]. \]  
(A.7)

The pole structure of \( \zeta(s) \) is given by \( \Gamma(s - D - \frac{1}{2}) \) (simple pole for even \( D \) and \( s \to 0 \)), and \( \Gamma(s - D - \frac{1}{2}) \) (odd \( D \) and \( s \to 0 \)). Since \( \Gamma(s \to 0) \to \infty \), \( \zeta(0) \) comes from the residue of the poles. Using \( z \Gamma(z) = \Gamma(z+1) \), we obtain (\( + \) corresponds to \( D \) even, and \( - \) to \( D \) odd)
\[
\zeta_D^{(+)}(0) = \frac{(-1)^{\frac{D}{2}}}{(4\pi)^{\frac{D}{2}} \left( \frac{D}{2} \right)!} m^D, \quad \zeta_D^{(-)}(0) = -\frac{1}{2} \frac{(-1)^{\frac{D+1}{2}}}{(4\pi)^{\frac{D+1}{2}} \left( \frac{D-1}{2} \right)!} \frac{m^{D-1}}{L}. \]  
(A.8)

Apart from a minus sign and a factor of 2, \( \zeta_D^{(-)}(0) \) resembles \( \zeta_D^{(+)}(0) \) for \( D - 1 \) space; the extra factor of \( L \) in \( \zeta_D^{(-)}(0) \) is seem to be necessary for dimensional reasons. For more details as well as a computation of \( \zeta_D(0) \) and \( \zeta_D'(0) \) for DBC in a thermal bath, see [6].
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