Modified Gravitational Theory and the Gravity Probe-B Gyroscope Experiment

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Abstract

A possible deviation from the precession of the Gravity Probe-B gyroscope predicted by general relativity is obtained in the nonsymmetric gravity theory. The time delay of radio signals emitted by spacecraft at planetary distances from the Sun, in nonsymmetric gravity theory is the same as in general relativity. A correction to the precession of the gyroscope would provide an experimental signature for the Gravity Probe-B gyroscope experiment. The Lense-Thirring frame-dragging effect is predicted to be the same as in GR.

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1 Introduction

The Gravity Probe-B spacecraft was launched on April 19, 2004, at Vandenberg Air Force Base, California, and is expected to produce experimental results that test the prediction of the relativistic precession of the onboard gyroscope. The polar orbit will be at a height of 640 km passing over every pole at 48.75 min.

A gravitational theory explanation of the acceleration of the expansion of the universe [1, 2, 3] and the observed flat rotation curves of galaxies was proposed [4, 5], based on the nonsymmetric gravitational theory (NGT) [6, 7, 8]. The equations of motion of a test particle in a static, spherically symmetric gravitational field modifies Newton’s law of acceleration for weak fields. It was shown [10] that the modified acceleration can account for the anomalous acceleration observed for the Pioneer 10 and 11 spacecraft [11].

In the following, we shall calculate a possible deviation from the predicted precession of the gyroscope obtained from general relativity (GR) using the test particle equations of motion in NGT. A previous calculation of such a deviation was based on an earlier version of NGT [6, 12]. In 1995, a new version of NGT was published...
in which the weak field linear approximation of the field equations was shown to be stable and the Hamiltonian was bounded from below leading to a stable vacuum state \[7, 8\]. We shall calculate the gyroscope motion from the 1995 version of NGT, using the static, spherically symmetric solution of the vacuum field equations and the equations of motion of test particles \[5\].

The gravitational constant at infinity is defined to be

\[
G \equiv G_\infty = G_0 \left( 1 + \sqrt{\frac{M_0}{M}} \right),
\]

where \(G_0\) is Newton’s gravitational constant, \(M\) is the mass of a physical source and \(M_0\) is a positive parameter. In the NGT phenomenology, there is also a range parameter \(r_0 = 1/\mu\), which is distance dependent in that it increases as we go from small to large gravitationally bound systems.

One important result derived from NGT is that photons move along null geodesics that predict the same bending of light and time delay effects for the Sun as in GR. The time delay of radio signals passing the limb of the Sun is shown in NGT to be the same for weak gravitational fields as in GR. This means that spacecraft measurements of the time delay of radio signals that determine the post-Newtonian parameter \(\gamma\) to be equal to unity to a high precision, do not prove that it is a foregone conclusion that the Gravity Probe-B experiment for the measurement of the gyroscope precession should agree with GR. The correction to the GR predicted value of the gyroscope’s precession, obtained from the NGT equations of motion of a spinning particle, could be detected by the sensitive Gravity Probe-B experiment. The frame-dragging effect predicted by NGT is the same as in GR.

2 The Equations of Motion of a Spinning Particle

The equations of motion of a test particle can be obtained from the NGT conservation laws \[9, 5\]:

\[
\frac{1}{2} (g_{\mu\rho} T_{\mu\nu,\rho} + g_{\rho\mu} T_{\nu\mu,\rho}) + [\mu\nu, \rho] T_{\mu\nu} = 0,
\]

where

\[
[\mu\nu, \rho] = \frac{1}{2} (g_{\mu\rho,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho})
\]

and \(T_{\mu\nu} = \sqrt{-g} T^{\mu\nu}\). The test particle source \(T_{\mu\nu}\) vanishes outside a narrow tube in four-dimensional spacetime that surrounds the coordinate point \(X^\mu\) of the particle. The motion of a monopole particle is obtained assuming that the dipole and higher moments of \(T_{\mu\nu}\) vanish.

We shall consider the motion of the test particle coupled to a skew symmetric source. This yields the equations of motion \[5\]:

\[
\frac{dw^\mu}{ds} + \left\{ \frac{\mu}{\alpha\beta} \right\} u^\alpha u^\beta = s^{(\mu\sigma)} f_{[\sigma\nu]} u^\nu,
\]

where

\[
G \equiv G_\infty = G_0 \left( 1 + \sqrt{\frac{M_0}{M}} \right),
\]
where $u^\mu = dx^\mu / ds$ is the 4-velocity of a particle and $s$ is the proper time along the path of the particle. Moreover, $g^{(\mu \alpha)} g_{(\nu \alpha)} = \delta^\mu_\nu$ and
\[
\left\{ \begin{array}{c} \mu \\ \alpha \beta \end{array} \right\} = \frac{1}{2} s^{(\mu \nu)} (\partial_\beta g_{(\alpha \sigma)} + \partial_\alpha g_{(\beta \sigma)} - \partial_\sigma g_{(\alpha \beta)}).
\] (5)
The notation is the same as in refs. [4, 5, 7, 8].

The spin angular momentum is defined by
\[
S^{\mu \nu} = \int d^3 x [(x^\mu - X^\mu) T^{\nu 0} - (x^\nu - X^\nu) T^{\mu 0}],
\] (6)
where $S^{\mu \nu}$ has the transformation properties of a tensor. We assume that the test particle is a pole-dipole particle, whereby the pole and dipole contributions of the integrated test particle source $T^{\mu \nu}$ do not vanish. The equations of motion of the spin are found to be
\[
\frac{DS^{\mu \nu}}{Ds} + u^\mu u_\gamma \frac{DS^{\nu \gamma}}{Ds} - u^\nu u_\gamma \frac{DS^{\mu \gamma}}{Ds} = 0,
\] (7)
where
\[
\frac{DS^{\mu \nu}}{Ds} \equiv \frac{dS^{\mu \nu}}{ds} + \left\{ \begin{array}{c} \mu \\ \alpha \beta \end{array} \right\} S^{\alpha \nu} u^\beta + \left\{ \begin{array}{c} \nu \\ \alpha \beta \end{array} \right\} S^{\mu \alpha} u^\beta.
\] (8)
We shall use the condition
\[
S^{\mu \nu} u_\nu = 0,
\] (9)
which leads to $u^i = 0 (i = 1, 2, 3)$ in the rest frame of the test particle and the condition $S^{0 \alpha} = 0$. From (9), we get
\[
\frac{DS^{\mu \nu}}{Ds} u_\nu + S^{\mu \nu} D u_\nu = 0.
\] (10)
Substituting (10) into (7), we get the equations of motion for the spin
\[
\frac{DS^{\mu \nu}}{Ds} = (u^\mu S^{\alpha \nu} - u^\nu S^{\mu \alpha}) \frac{D u_\alpha}{Ds},
\] (11)
where we used $s^{(\mu \nu)} u_\mu u_\nu = 1$. We have from Eq.(4) that
\[
\frac{D u_\mu}{Ds} = f_{[\mu \nu]} u^\nu,
\] (12)
where
\[
\frac{D u_\mu}{Ds} = \frac{d u_\mu}{ds} - \left\{ \begin{array}{c} \rho \\ \mu \beta \end{array} \right\} u^\beta u_\rho.
\] (13)
Substituting (12) into (11), we get
\[
\frac{DS^{\mu \nu}}{Ds} = (u^\mu S^{\nu \alpha} - u^\nu S^{\mu \alpha}) f_\alpha,
\] (14)
where $f_\alpha = f_{[\alpha \sigma]} u^\sigma$ and
\[
u^\alpha f_\alpha = 0.
\] (15)
In NGT the $f_\alpha$ acts as an external force that makes $DS^{\mu \nu} /Ds$ non-vanishing and this will affect the motion of the gyroscope orbiting Earth.
3 Experimental Prediction for the Gyroscope

From (13) we obtain

$$\frac{DS^{ij}}{Ds} = (u^iS^{j\alpha} - u^jS^{i\alpha})f_\alpha.$$  
(16)

We define the spatial vector $S$ in rectangular coordinates

$$S = (S^{23}, S^{31}, S^{12}) = (S^1, S^2, S^3).$$  
(17)

Here, $S$ is the spin angular momentum vector with respect to the point $r = (X^1, X^2, X^3)$. Eq. (15) tells us that the terms in (16) that involve $f_i$ are order $v|S||f|$, while the terms involving $f_0$ are of order $v^2|S||f|$ and can be neglected. Moreover, we have that $f_i \sim -f^i$. For a slowly moving gyroscope, we find

$$\frac{DS}{Dt} = S(v \cdot f) - f(v \cdot S).$$  
(18)

For large values of $r$ the metric line element is given by

$$ds^2 \equiv g(\mu\nu)dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 + \frac{2M}{r}\right)(dx^2 + dy^2 + dz^2),$$  
(19)

where $r = (x^2 + y^2 + z^2)^{1/2}$. The non-vanishing Christoffel symbols are

$$\begin{align*}
\left\{ i \right\}_{00} &= -\frac{1}{2} s^{(ik)}g_{00,k}, & \left\{ i0 \right\} &= \frac{1}{2} s^{00}g_{00,i}, \\
\left\{ j \right\}_{ik} &= \frac{1}{2} s^{(jl)}(g_{(il),k} + g_{(ik),l} - g_{(lk),i}).
\end{align*}$$  
(20)

From (8) and the left-hand side of (16), we have

$$\frac{DS}{dt} = \frac{dS}{dt} - \frac{2M}{r^3}(r \cdot v)S - \frac{M}{r^3}(r \times v) \times S - \frac{M}{r^3}(r \times W),$$  
(21)

where we define

$$W = (S^{10}, S^{20}, S^{30}).$$  
(22)

We have

$$(r \times v) \times S = v(r \cdot S) - r(v \cdot S).$$  
(23)

We obtain from (9) in isotropic coordinates

$$S^{10} \sim \left(1 + \frac{4M}{r}\right)(v_yS_z - v_zS_y),$$  
(24)

which gives

$$W \sim \left(1 + \frac{4M}{r}\right)(v \times S) \sim v \times S.$$  
(25)
We now obtain
\[ \mathbf{r} \times \mathbf{W} \sim \mathbf{r} \times (\mathbf{v} \times \mathbf{S}) = \mathbf{v}(\mathbf{r} \cdot \mathbf{S}) - \mathbf{S}(\mathbf{r} \cdot \mathbf{v}). \tag{26} \]

To our order of approximation, we have
\[ \frac{DS}{dt} = \frac{dS}{dt} - \frac{M}{r^3}(\mathbf{r} \cdot \mathbf{v}) + 2\mathbf{v}(\mathbf{r} \cdot \mathbf{S}) - \mathbf{r}(\mathbf{v} \cdot \mathbf{S}). \tag{27} \]

From (27), we finally have
\[ \frac{dS}{dt} = \frac{M}{r^3}(\mathbf{r} \cdot \mathbf{v}) + 2\mathbf{v}(\mathbf{r} \cdot \mathbf{S}) - \mathbf{r}(\mathbf{v} \cdot \mathbf{S}) + \mathbf{v}(\mathbf{S} \cdot \mathbf{f}) - \mathbf{f}(\mathbf{v} \cdot \mathbf{S}). \tag{28} \]

For weak gravitational fields, we have the equation of motion [5]:
\[ \frac{dv}{dt} = -\frac{Mr}{r^3} - \sigma \frac{\exp(-r/r_0)}{r^3} \left(1 + \frac{r}{r_0} \right), \tag{29} \]
where
\[ \sigma = \frac{\lambda s M_2}{3r_0^2}. \tag{30} \]

Here \( r_0 = 1/\mu \) is the range parameter associated with the “mass” of the skew field \( g_{\mu\nu} \), and \( \lambda \) and \( s \) denote the coupling strength of a test particle to matter and the coupling strength of the skew field \( g_{\mu\nu} \), respectively.

Let us consider the expansion for \( r \ll r_0 \):
\[ \exp(-r/r_0) \left(1 + \frac{r}{r_0} \right) = 1 - \frac{1}{2} \left( \frac{r}{r_0} \right)^2 + \frac{1}{3} \left( \frac{r}{r_0} \right)^3 - ... \tag{31} \]

Then, (29) gives for \( r \ll r_0 \):
\[ \frac{dv}{dt} \sim -\left[ \frac{Mr}{r^3} - \frac{\sigma r}{r^3} \left(1 - \frac{1}{2} \left( \frac{r}{r_0} \right)^2 \right) \right] \sim -\left( \frac{M^*}{r^3} \right) r, \tag{32} \]
where
\[ M^* = M - \sigma. \tag{33} \]

We shall now transform to the rest frame of the gyroscope. This yields [13, 12]:
\[ \mathbf{S}_{\text{rest}} = \left(1 + 2\frac{M}{r} \right)\mathbf{S} - \frac{1}{2} v^2 \mathbf{S} + \frac{1}{2} \mathbf{v}(\mathbf{S} \cdot \mathbf{f}). \tag{34} \]

Differentiating this with respect to time gives the time dependence of the spin vector in the rest frame of the gyroscope to lowest order:
\[ \frac{dS_{\text{rest}}}{dt} = \left(\frac{3M}{2r^3} - \frac{\sigma}{r^3} \right)[\mathbf{v}(\mathbf{r} \cdot \mathbf{S}_{\text{rest}}) - \mathbf{r}(\mathbf{v} \cdot \mathbf{S}_{\text{rest}})]. \tag{35} \]

Eq. (35) can be written as
\[ \frac{dS_{\text{rest}}}{dt} = \left(\frac{3M}{2r^3} - \frac{\sigma}{r^3} \right)(\mathbf{r} \times \mathbf{v}) \times \mathbf{S}_{\text{rest}}, \tag{36} \]
or, in the form
\[
\frac{dS_{\text{rest}}}{dt} = \Omega \times S_{\text{rest}},
\]
where
\[
\Omega = \left(\frac{3M}{2r^3} - \frac{\sigma}{r^3}\right)(r \times v).
\]

Let us take the gyroscope’s orbit to be a circle of radius \(r\) with unit normal orbital angular momentum vector \(\hat{J}\). The gyroscope’s velocity is given by
\[
v = -\left(\frac{M^*}{r^3}\right)^{1/2} (r \times \hat{J}).
\]

By using the relations
\[
r \times (r \times \hat{J}) = (r \cdot \hat{J})r - (r \cdot r)\hat{J} = -(r \cdot r)\hat{J} = -r^2 \hat{J},
\]
we obtain
\[
\Omega = \left(\frac{3M}{2r^3} - \frac{\sigma}{r^3}\right)\left(\frac{M}{r^3} - \frac{\sigma}{r^3}\right)^{1/2} r^2 \hat{J} \sim \left(\frac{3M}{2} - \sigma\right) \frac{M^{1/2}}{r^{5/2}} \hat{J}.
\]

For Earth, we choose
\[
\sqrt{(M_0)_{\oplus}} \leq 1.62 \times 10^{-7} \sqrt{M_{\odot}}
\]
and from this value and (I) we see that \(G \equiv G_\infty \sim G_0\).

The precession rate in NGT to the lowest order of approximation, averaged over a revolution for the Earth is given by (we reinstate \(G\) and \(c\)):
\[
\langle|\Omega|\rangle = B_{\oplus} \left(\frac{G_0 M_{\oplus}}{c^2 r^{5/2}}\right)^{1/2} = B_{\oplus} \left(\frac{G_0 M_{\oplus}}{c^2 R_{\oplus}^{5/2}}\right)^{1/2} \left(\frac{R_{\oplus}^{5/2}}{r^{5/2}}\right),
\]
where
\[
B_{\oplus} = \frac{3}{2} G_0 M_{\oplus} - \sigma_{\oplus}.
\]

Moreover, \(M_{\oplus}\) and \(R_{\oplus}\) denote the mass and the equatorial radius of Earth, respectively, and we have
\[
\sigma_{\oplus} = \frac{\lambda_{\oplus} s G_0^2 M_{\oplus}^2}{3 c^2 (r_0)^2_{\oplus}}.
\]

Setting \(\sigma_{\oplus} = 0\), we get for the GR prediction
\[
\langle|\Omega|\rangle_{\text{GR}} = \frac{3(G_0 M_{\oplus})^{3/2}}{2 c^2 R_{\oplus}^{5/2}} \left(\frac{R_{\oplus}}{r}\right)^{5/2} \left(\frac{R_{\oplus}}{r}\right)^{5/2} \text{arcsec/yr}.
\]

For the Gravity Probe-B drag-free spacecraft at a height of \(h = 640\) km above Earth, we obtain for GR:
\[
\langle|\Omega|\rangle_{\text{GR}} = 6.61 \text{arcsec/yr}.
\]
We see from (43) and (44) that the correction to GR reduces the gyroscope’s precession by the amount of the value of $\sigma_{\oplus}$. We expect to satisfy the bound

$$\sigma_{\oplus} < \frac{3G_0M_{\oplus}}{2},$$

which yields from (45):

$$\lambda_{\oplus} < \frac{9c^2(r_0)^2}{28G_0M_{\oplus}}.$$  

(48)

(49)

4 Frame Dragging Effect and Time Delay Observations

In NGT, we can include the effect of the Earth’s rotation to first order in the metric in rectangular isotropic coordinates [14]:

$$g(\mu\nu) = \begin{pmatrix}
-\left(1 + 2\frac{M}{r}\right) & 0 & 0 & -\frac{2Jy}{r^3}\\
0 & -\left(1 + 2\frac{M}{r}\right) & 0 & \frac{2Jx}{r^3}\\n0 & 0 & -\left(1 + 2\frac{M}{r}\right) & 0\\n-\frac{2Jy}{r^3} & \frac{2Jx}{r^3} & 0 & 1 - 2\frac{M}{r}
\end{pmatrix}.$$  

(50)

The precession due to the Lense-Thirring effect, which arises from the off-diagonal elements of the metric $g(\mu\nu)$, is identical to GR. The inclusion of the Earth’s rotation has no effect to this order on the diagonal elements of the metric.

The motion of massless photons is determined by the geodesic equation [5]:

$$\frac{du^\mu}{ds} + \left\{ \frac{\mu}{\alpha\beta} \right\} u^\alpha u^\beta = 0.$$  

(51)

To the order of approximation that we are considering, the total light time of travel of a radio signal emitted from a spacecraft in the solar, barycentric system is given by [11]:

$$t_2 - t_1 = \frac{r_{12}}{c} + \frac{2G_0M_{\oplus}}{c^3} \ln \left[ \frac{r_{1\odot} + r_{2\odot} + r_{12\odot}}{r_{1\odot} + r_{2\odot} - r_{12\odot}} \right] + \sum_{i} \frac{2G_0M_i}{c^3} \ln \left[ \frac{r_{1i} + r_{2i} + r_{12i}}{r_{1i} + r_{2i} - r_{12i}} \right],$$

(52)

where $M_i$ is the mass of a planet, an outer planetary system, or the Moon. Moreover, $r_{1\odot}$, $r_{2\odot}$ and $r_{12\odot}$ are the heliocentric distances to the point of the radio signal emission on Earth, to the point of signal reflection at the spacecraft, and the relative distance between these two points, respectively. The $r_{1i}$, $r_{2i}$ and $r_{12i}$ are distances relative to an ith body in the solar system, and $t_1$ refers to the transmission time at a tracking station on Earth, and $t_2$ refers to the reflection time at the spacecraft. This is the same result as predicted by GR. We conclude that accurate determinations of this time delay effect that agree with GR, using solar spacecraft probes, do not rule out the possibility of detecting a deviation from GR in the geodetic spin precession due to an NGT correction obtained from Eq.(43).
5 Conclusions

We have derived the equation of motion for a spinning particle and the geodetic precession of a gyroscope in the version of NGT published in 1995 [7, 8]. With the significant sensitivity of the Gravity Probe-B experiment, it is hoped that the NGT correction to the predicted GR gyroscope precession can be detected.

The time delay of a radio signal obtained from NGT agrees with the prediction of GR. Therefore, an accurate determination of the time delay of a radio signal from a spacecraft as it passes by close to the Sun, which competes with the accuracy of the determination of the Gravity Probe-B gyroscope’s precession, does not rule out a detection of a correction to the precession obtained from NGT. The predicted Lense-Thirring frame-dragging effect in NGT is the same as in GR.

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