INVERSE N-BODY SCATTERING WITH THE TIME-DEPENDENT HARTREE-FOCK APPROXIMATION

Michiyuki Watanabe
Department of Sciences, Okayama University of Science
1-1 Ridaicho, Kita-ku,
Okayama-shi 700-0005, Japan

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Abstract. We consider an inverse N-body scattering problem of determining two potentials—an external potential acting on all particles and a pair interaction potential—from the scattering particles. This paper finds that the time-dependent Hartree-Fock approximation for a three-dimensional inverse N-body scattering in quantum mechanics enables us to recover the two potentials from the scattering states with high-velocity initial states. The main ingredient of mathematical analysis in this paper is based on the asymptotic analysis of the scattering operator defined in terms of a scattering solution to the Hartree-Fock equation at high energies. We show that the leading part of the asymptotic expansion of the scattering operator uniquely reconstructs the Fourier transform of the pair interaction, and the second term of the expansion uniquely reconstructs the X-ray transform of the external potential.

1. Introduction.

1.1. Problem and result. Consider the quantum N-body systems of identical particles interacting pairwise by the two-body potential under an external potential acting on all particles. A typical example is N electrons in an atom with proton number Z at the nucleus. In that case, the external potential is the nucleus-electron attraction, and the two-body potential is the electron to electron repulsion. Inverse N-body scattering problems ask to determine the interaction potential and the external potential from the scattering states of particles. Such inverse problems have been extensively studied for N-body Schrödinger equations with no external potentials (Enss and Weder [5]; Novikov [16]; Wang [26, 27]; Vasy [24]; Uhlmann and Vasy [20–22]). The inverse scattering for the N-body Schrödinger equation in an external constant electric field was investigated by Weder [33]; Valencia and Weder [23].

Differently, Lemm and Uhlig [12] have investigated an inverse N-body problem by using Bayesian approach with the Hartree-Fock approximation. They gave a computationally feasible method of reconstructing an interaction potential from data by solutions to a stationary Hartree-Fock equation. Their work indicates that the Hartree-Fock approximation is also extremely useful as a way to investigate the inverse N-body problems.
The above mentioned works have focused only on recovering interactions. Since N-body systems is generally described by a non-relativistic Hamiltonian consisting of a one-body term with the kinetic energy and an external potential, and a two-body interaction term, the inverse problems of determining both the interaction potential and the external potential should be also investigated. However, little has been reported on the determination both the interaction potential and the external potential in the quantum N-body systems.

In this paper, we find that the time-dependent Hartree-Fock approximation for the inverse N-body scattering in quantum mechanics enables us to recover two potentials—an external potential acting on all particles and a pair interaction potential—from the scattering states with high-velocity initial states. This paper also proposes a new reconstruction procedure of recovering the two potentials.

Let us formulate our inverse problem and state our main result. We first recall that the n-dimensional N-body Schrödinger equation has the form:

$$i\frac{\partial}{\partial t} \Psi(t) = \tilde{H}_N \Psi(t),$$

$$\tilde{H}_N = \sum_{j=1}^{N} \left\{ \frac{1}{2} ( -i \nabla_{x_j} )^2 + V_{ext}(x_j) \right\} + \sum_{j<k}^{N} V_{int}(x_j - x_k),$$

where $i = \sqrt{-1}$, $x_j \in \mathbb{R}^n$, $V_{ext}(x_j)$ is an external potential and $V_{int}(x_j)$ is an interaction potential with $V_{int}(x_j) = V_{int}(-x_j)$. The Hartree-Fock approximation is known as the simplest one-body approximation. Writing the N-body wave function $\Psi(t) = \Psi(t, x_1, \cdots, x_N)$ with the Slater determinant

$$\Psi(t, x_1, \cdots, x_N) = (N!)^{-1/2} \det \left( u_j(t, x_k) \right)_{1 \leq j, k \leq N},$$

yields the one-body Schrödinger equation (see, e.g., Lubich [13, II.3.2])

$$i\frac{\partial u_j}{\partial t} = H(u_k)u_j,$$

$$H(u_k)u_j = \{ H_0 + V_{ext} + Q_H(x, u_k) \}u_j + \int_{\mathbb{R}^n} Q_F(x, y, u_k)u_j(t, y) dy,$$

for $1 \leq j, k \leq N$ and $j \neq k$, where $H_0 = -\frac{1}{2} \Delta = -\frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ and orbitals $u_j(t, x)$ is unknown functions in $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, and

$$Q_H(x, u_k) = \int_{\mathbb{R}^n} V_{int}(x - y) \sum_{k=1}^{N} \left| u_k(t, y) \right|^2 dy,$$

$$= V_{int} \sum_{k=1}^{N} \left| u_k(t, \cdot) \right|^2,$$

$$Q_F(x, y, u_k) = -V_{int}(x - y) \sum_{k=1}^{N} \bar{u_k}(t, y)u_k(t, x).$$

The non-linear Schrödinger equation (1) we study in this paper is called the Hartree-Fock equation (HF equation). The terms

$$Q_H(x, u_k)u_j(t, x)$$

and

$$\int Q_F(x, y, u_k)u_j(t, y) dy$$

yield the non-linear Schrödinger equation (1) we study in this paper is called the Hartree-Fock equation (HF equation). The terms

$$Q_H(x, u_k)u_j(t, x)$$

and

$$\int Q_F(x, y, u_k)u_j(t, y) dy$$
are called the Hartree term and the Fock term, respectively.

Next, we introduce some notations and assumptions on the potentials. Let \( W^{k,p}(\mathbb{R}^n) \) be the usual Sobolev space in \( L^p(\mathbb{R}^n) \). We abbreviate \( W^{k,2}(\mathbb{R}^n) \) as \( H^k(\mathbb{R}^n) \). The weighted \( L^2 \)-space is denoted as

\[
L^{2,s}(\mathbb{R}^n) = \left\{ u(x) : (1 + |x|^2)^{s/2} u(x) \in L^2(\mathbb{R}^n), s \in \mathbb{R} \right\}.
\]

Let \( C_0^\infty(\mathbb{R}^n) \) be the set of compactly supported smooth functions and \( S(\mathbb{R}^n) \) be the set of rapidly decreasing functions on \( \mathbb{R}^n \). The Fourier transform is denoted as

\[
(F u)(\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) \, dx.
\]

We define a function space \( S_0(\mathbb{R}^n) \) as

\[
S_0(\mathbb{R}^n) = \left\{ f \in S(\mathbb{R}^n) ; \hat{f} \in C_0^\infty(\mathbb{R}^n) \right\}.
\]

The multiplication operator with a fixed function \( V(x) \) is denoted as \( V \). The unitary group of the self-adjoint operator \( H_0 \) with a domain \( H^2(\mathbb{R}^n) \) is denoted as \( U_0(t) \) or \( e^{-itH_0} \). Then, solutions of the free Schrödinger equation \( i\partial_t v = H_0 v \) with initial data \( v(0) = f \) is written as \( v(t) = U_0(t)f = e^{-itH_0}f \). Consider solutions to the equation (1) with \( u_j(t) \rightharpoonup U_0(t)f^\pm_j \) as \( t \to \pm\infty \) in some function space. We term the solutions scattering solution and \( f^\pm_j \) scattering states. The scattering operator \( S \) assigns the free state \( U_0(t)f^-_j \) at \( t = -\infty \) to the free state \( U_0(t)f^+_j \) at \( t = +\infty \), or equivalently \( S : f^-_j \to f^+_j \). Our goal is to recover the external potential \( V_{ext}(x) \) and the interaction potential \( V_{int}(x) \) from the scattering operator \( S \).

Although we consider the three-dimensional inverse problem, to make it easy to explain a proof of our theorem, we will denote the spatial dimension by \( n \) throughout this paper.

Let potentials satisfy the following conditions.

**Assumption 1.1.** Let \( n = 3 \). We assume that the real-valued function \( V_{int}(x) \) has the following conditions:

(i) \( V_{int}(x) \geq 0 \) and

\[
|V_{int}(x)| \leq C|x|^{-2}, \quad \text{or} \quad V_{int} \in L^{n/2}(\mathbb{R}^n).
\]

(ii) \( \nabla V_{int} \in L^{n/2}(\mathbb{R}^n) \).

(iii) \( x \cdot \nabla V_{int} \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n) \) with \( 1 \leq \delta \).

(iv) \( V_{int}(-x) = V_{int}(x) \).

(v) \( |x|^2 V_{int}(x) \) is a non-increasing function of \( |x| \).

(vi) \( \sup_{x \in \mathbb{R}^n} (1 + |x|)^{1+\delta} |V_{int}(x)| < \infty \) for \( s > n/2 \).

**Assumption 1.2.** Let \( n = 3 \). We assume that the real-valued function \( V_{ext}(x) \) has the following conditions:

(i) \( V_{ext}(x) \geq 0 \).

(ii) \( V_{ext}(x) \) is a homogeneous function of degree \( -\gamma \) for \( |x| \geq \kappa \):

\[
V_{ext}(ax) = a^{-\gamma} V_{ext}(x) \quad \text{for} \quad a \geq 1, \gamma \geq 1 \text{ and some small } \kappa > 0,
\]

and \( V_{ext}(x) \) is a constant function in a neighborhood of the origin: \( V_{ext}(x) = C \) for \( |x| \leq \kappa \).

(iii) \( |x|^2 V_{ext}(x) \) is a non-increasing function of \( |x| \) except for a neighborhood of the origin.

(iv) \( \nabla V_{ext} \in L^\infty(\mathbb{R}^n) \) and \( \Delta V_{ext} \in L^n(\mathbb{R}^n) \).
Remark 1. The proof of the unique existence theorem on the scattering solutions of Schrödinger equations: requires the function \( V \) for \( \gamma \) homogeneous of degree \( \alpha \) with \( |\alpha| \leq \ell \),

\[
\sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{\delta/2} \left( \int_{|x-y| \leq 1} |D^\alpha V_{ext}(y)|^{p_0} \, dy \right)^{1/p_0} < \infty.
\]

Here we have denoted the integer part of \( x \) by \([x]\) and \( D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \), \( D_j = -i\partial/\partial x_j \).

Remark 2. The Assumption 1.2 is rather complicated condition. We therefore give another conditions on \( V_{ext}(x) \) simpler than Assumption 1.2.

Assumption 1.3. Let \( n = 3 \). We assume that the real-valued function \( V_{ext}(x) \) has the following conditions:

(i) \( V_{ext}(x) \geq 0 \).

(ii) For \( |x| \geq \kappa \), \( V_{ext}(x) \) is a continuously differentiable function and a homogeneous function of degree \(-\gamma\) with \( \gamma \geq 2 \). For \( |x| \leq \kappa \), \( V_{ext}(x) \) is a constant function.

(iii) For \( |\alpha| \leq 2 \) and \( \kappa > (3n)/2 + 3 \),

\[
|D^\alpha V_{ext}(x)| \leq \frac{C}{(1 + |x|)^\kappa}.
\]

The Assumption 1.2 implies the Assumption 1.3. Indeed, assuming that \( V_{ext} \) satisfies Assumption 1.3, and letting \( r = |x| \), we have

\[
V + \frac{1}{2} x \cdot \nabla V_{ext} = V + \frac{1}{2} r \partial_r V_{ext} = \frac{1}{2r} \partial_r (r^2 V_{ext}).
\]

This identity means that the condition \( V + \frac{1}{2} x \cdot \nabla V_{ext} \leq 0 \) implies the condition (iii) in Assumption 1.2. Recall the Euler’s homogeneous function theorem: if the function \( V(x) \) on \( \mathbb{R}^n \) is continuously differentiable function, then \( V(x) \) is a positively homogeneous of degree \( \gamma \) if and only if \( V(x) \) satisfies \( x \cdot \nabla V(x) = \gamma V(x) \). Then, in view of the Euler’s homogeneous function theorem, the condition (i) and (ii) in Assumption 1.3 gives \( x \cdot \nabla V_{ext} = -\gamma V_{ext} \leq 0 \), which implies

\[
V_{ext} + \frac{1}{2} x \cdot \nabla V_{ext} = V_{ext} - \frac{\gamma}{2} V_{ext} \leq 0
\]

for \( \gamma \geq 2 \). Hence, the condition (iii) in Assumption 1.2 is satisfied. Direct computations show that if \( V_{ext} \) has the condition (iii) in Assumption 1.3, then the function \( V_{ext}(x) \) satisfies conditions from (iv) to (vii) in Assumption 1.2.

As it turns out in Section 2, under the Assumption 1.1 and the Assumption 1.2, there exists a unique scattering solution \( u_j(t,x) \) of (1) with a condition \( u_j(t,x) \rightarrow \)
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\(e^{-itH_0} \varphi_j\) as \(t \to -\infty\) in \(L^2(\mathbb{R}^n)\) for any \(\varphi_j \in \mathcal{S}_0\), \(j = 1, \cdots, N\) sufficiently close to zero function. Put

(2)

\[
\psi_j(x) = (S \varphi)_j(x) := \varphi_j(x) + \frac{1}{i} \int_{\mathbb{R}} e^{itH_0} P_j(x, u) \, dt, \quad j = 1, 2, \cdots, N
\]

\[
P_j(x, u) = (QH(x, u) + V_{ext}(x)) u_j(t, x) + \int_{\mathbb{R}^n} QF(x, y, u_k) u_j(t, y) \, dy,
\]

where \(\varphi = (\varphi_j)_{1 \leq j \leq N}\) and \(u(t, x) = (u_j(t, x))_{1 \leq j \leq N}\) is the scattering solution to (1). Then it will be shown that \(u_j(x, t) \to e^{-itH_0} \psi_j(x)\) as \(t \to +\infty\) in \(L^2(\mathbb{R}^n)\). Therefore, the operator \(S\) defined as (2) represents a scattering operator for the HF equation (1).

The inverse problem considered in this paper is to determine the interaction and the external potentials from the scattering operator defined in terms of the scattering solution to the Hartree-Fock equation (1). Our main result is

**Theorem 1.1.** Let \(n = 3\). Assume that \(V_{int}(x)\) and \(V_{ext}(x)\) satisfy Assumption 1.1 and Assumption 1.2, respectively. Then the potentials \(V_{int}, V_{ext}\) are uniquely determined by \(S\).

We remark that our proof gives an explicit way to reconstruct the interaction and the external potentials from the asymptotic behavior of the function \(< (S - I)\Psi_v, \Psi_v >_{L^2} \) at \(|v| \to \infty\), where \(\Psi_v(x) = e^{iv \cdot x} \varphi(x)\) and \(< , >_{L^2}\) is the inner product in \(L^2(\mathbb{R}^n)\).

1.2. **Methods.** Because the high velocity limit (HVL) of the scattering operator (Enss and Weder [5]) makes it possible to recover the Schrödinger operator with the potentials, it has become an important tool for studying the inverse scattering problems for time-dependent Schrödinger equations (Weder [34]; Valencia and Weder [23], and references therein; Adachi and Maehara [3]; Adachi, Fujiwara and Ishida [1]; Adachi et al. [2]; Ishida [9]). According to recent researches for inverse nonlinear scattering ([31] and [32]), the method of the HVL also make it possible to recover nonlinearities. On the other hand, the small amplitude limit (SAL) of the scattering operator (Weder [35–44]) makes it possible to recover both the potential and nonlinearities. This method of the SAL, however, fails to reconstruct the interaction potential in the HF equation (1). Details of the method of SAL are described briefly as follows.

Because the HF equation (1) is a non-linear equation, our inverse problem is a non-linear inverse problem of recovering the linear part—zero-order coefficient—and non-linear part. Such non-linear inverse scattering problem have been extensively studied by Weder. It was proved that the SAL of the scattering operator uniquely determines coefficients—the coefficient of the linear part and coefficients of the power type non-linearity. In other words, this method of the SAL is an asymptotic analysis of the scattering operator \(S(\varepsilon \varphi)\) as \(\varepsilon \to 0\). In particular, Weder [35] showed that the Fréchet derivative of the scattering operator \(S\) uniquely determines the linear scattering operator for the Schrödinger operator \(H = H_0 + V_{ext}\). Thus, the non-linear inverse scattering problem of recovering the potential \(V_{ext}\) is reduced to the problem of recovering the Schrödinger operator \(H\) from the linear scattering operator.

This method of Weder was applied to inverse problems for Hartree equations ([28, 30]). Here, we briefly review the method to recover the coefficient function.
\( V_{\text{ext}} \) of the linear term in the case of \( N = 2 \) and \( u_1 = u_2 \) to the equation (1). Following Weder [35], the scattering operator \( S \) is defined in terms of wave operators \( W_{\pm} = \lim_{t \to \pm \infty} e^{itH} e^{-itH_0} \) for the Schrödinger operator \( H = H_0 + V_{\text{ext}} \):

\[
S = W_+^* S N W_-,
\]

\[
(SN \varphi)_j(x) = \varphi_j(x) + \frac{1}{i} \int_{\mathbb{R}} e^{itH} N_j(x, u) \, dt,
\]

\[
N_j(x, u) = Q_H(x, u_k) u_j(t, x) + \int_{\mathbb{R}^n} Q_F(x, y, u_k) u_j(t, y) \, dy,
\]

and the scattering operator \( S \) has expansion

\[
S(\varepsilon \varphi) = \varepsilon S_{V_{\text{ext}}} \varphi + O(\varepsilon^3)
\]

as \( \varepsilon \to 0 \) in \( H^1(\mathbb{R}^n) \), where \( S_{V_{\text{ext}}} \) denotes the scattering operator for the Schrödinger operator \( H \). We shall term this expansion \( \varepsilon \)-expansion”. This \( \varepsilon \)-expansion indicates that the scattering operator \( S \) uniquely determines the scattering operator \( S_{V_{\text{ext}}} \). As is well-known (see, e.g., [5]), the operator \( S_{V_{\text{ext}}} \) uniquely determines the external potential \( V_{\text{ext}} \). Then we can construct wave operators \( W_+ = \lim_{t \to -\infty} e^{itH} e^{-itH_0} \) and \( W_-^* \). Defining \( S_F \) as \( S_F = W_+ SW_-^* \), the small amplitude limit of the function \( \frac{1}{\varepsilon^2} (S_F - I)(\varepsilon \varphi) \) uniquely determines the interaction potential of the form \( V_{\text{int}}(x) = \lambda |x|^{-\sigma} \) (see [29]).

Recently, the interaction potential is successfully reconstructed by means of the method of the HVL of the scattering operator in the case of the Hartree-Fock equation (1) with no external potential \( V_{\text{ext}} \) ([32]). Our representation (2) permits the high-velocity analysis of the scattering operator to reconstruct both the interaction potential and the external potential. More precisely, analyzing the asymptotic expansion of the function \( <(S - I)\Psi, \Psi>_{L^2} \) at \( |v| \to \infty \) (\( v \)-expansion) shows that the leading term of the expansion uniquely determines the Fourier transform of the interaction potential \( V_{\text{int}} \) and the second term of the expansion uniquely determines the X-ray transform of the external potential \( V_{\text{ext}} \). This method of the \( v \)-expansion does not require construction of wave operators \( W_{\pm} \) to reconstruct the potentials. Thus the method of the \( v \)-expansion is simpler and less complicated than the method of the \( \varepsilon \)-expansion.

In order to prove that the operator defined as (2) is a scattering operator to the equation (1), we need a time-decay \( L^\infty \)-estimate on the solution to the Cauchy problem of the equation (1). Such estimate on the solution to the Hartree type equation has been extensively investigated for the case where the potential has the form \( \lambda |x|^{-\gamma} \) for some constants \( \lambda \) and \( \gamma \) (see, e.g., Wada [25]; Hayashi and Naumkin [7]; Hayashi and Ozawa [8]). Few researchers have addressed the problem of the time-decay \( L^\infty \)-estimate on the solution to the Hartree-Fock equation with general interaction and external potentials. In this paper, making use of the method of the pseudo-conformal conservation law (PCC method) and the Gagliardo-Nirenberg inequality we give a \( L^\infty \)-estimate on the solution to the equation (1). The PCC method requires the positivity of potentials and the restriction on a class of external potentials.

This paper is organized as follows: Section 2 proves that the operator \( S \) defined as (2) is the scattering operator for the Hartree-Fock equation (1), after proving a time-decay \( L^\infty \)-estimate on the solution to the Cauchy problem for the equation (1). We give the asymptotic expansion of the scattering operator with the high-velocity
initial states in Section 3. Section 4 is devoted to reconstructions of the interaction potential and the external potential.

2. Representation of the scattering operator. This section shows that the scattering operator for the Hartree-Fock equation (1) has a representation (2). We first recall that the unique existence of the scattering solution of Hartree equations with the external potential are studied in [28]. This result can be easily applicable to the Hartree-Fock equation (1). Assume that potentials $V_{\text{ext}}$ and $V_{\text{int}}$ satisfy Assumption 1.1 and Assumption 1.2, respectively. Let $\Phi$ in $L^2(\mathbb{R}^n)$ for $\varphi_j \in L^2(\mathbb{R}^n)$ with $\|\varphi_j\|_{L^2} \leq \varepsilon_0$, $j = 1, 2, \cdots, N$, has a unique solution

$$u_j \in W = L^3(\mathbb{R} : L^9(\mathbb{R}^n)) \cap L^\infty(\mathbb{R} : L^2(\mathbb{R}^n)), \quad \frac{1}{q} = \frac{1}{2} - \frac{2}{3n}.$$  

Moreover, there exists a unique $\psi_j \in L^2(\mathbb{R}^n)$ such that $u_j(t) \rightarrow e^{-it\mathbf{H}_0}\varphi_j$ as $t \rightarrow -\infty$ in $L^2(\mathbb{R}^n)$.

In what follows, because we are interested in the inverse scattering problem, we consider the scattering for high-velocity initial states. Let $\varphi \in \mathcal{S}_0(\mathbb{R}^n)$. Then the function $\Phi_v(x) = e^{iv\cdot x}\varphi(x)$ has a compact velocity support in the momentum space around $v$, due to the fact that $\hat{\Phi}_v(\xi) = \hat{\varphi}(\xi - v)$.

The main result of this section is

**Theorem 2.1.** Let $n = 3$. Assume that potentials $V_{\text{int}}$ and $V_{\text{ext}}$ satisfy Assumption 1.1 and Assumption 1.2, respectively. Let $u_j$, $j = 1, 2, \cdots, N$, be the scattering solutions to (1) with initial scattering states $\varphi_j \in \mathcal{S}_0$. Put

$$\psi_j = (S\varphi)_j(x) = \varphi_j(x) + \frac{1}{i} \int_\mathbb{R} e^{it\mathbf{H}_0} P_j(x, u) dt,$$

$$P_j(x, u) = (Q_H(x, u_k) + V_{\text{ext}}(x)) u_j(t, x) + \int_{\mathbb{R}^n} Q_F(x, y, u_k) u_j(t, y) dy.$$

Then we have

$$\|u_j(t) - e^{-it\mathbf{H}_0}\psi_j\|_{L^2} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$  

In Subsection 2.1, we prepare some lemmas to prove Theorem 2.1. Subsection 2.2 is devoted to state a $L^\infty$ estimate on the solution to the equation (1) and its proof. Theorem 2.1 is proved in Subsection 2.3.

2.1. Preliminary lemmas.

**Lemma 2.2** (Gagliardo-Nirenberg inequality). Let $q, r \leq \infty$ and let $j, m$ be any integers satisfying $0 \leq j < m$. Then for any $u \in W^{m, r}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, we have

$$\sum_{|\alpha| = j} \|D^\alpha u\|_{L^p} \leq M \sum_{|\beta| = m} \|D^\beta u\|_{L^q}^{1-a},$$

where $1/p = j/m + a(1/r - m/n) + (1 - a)/q$ for all $a \in [j/m, 1]$ with the following exception: if $m - j - (n/r)$ is a non-negative integer, then (4) is asserted for $a = j/m$, and where $M$ is a positive constant depending only on $n, m, j, q, r, a$.

The proof of Lemma 2.2 will be found in Friedman [6].

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Lemma 2.3. Let $1/q = 1/2 - 2/(3n)$. Assume that the potential $V_{\text{int}}$ satisfies Assumption 1.1. Then for any $u_j \in L^2(\mathbb{R}^n)$, $j = 1, 2, \cdots, N$, we have
\[
\|Q_H(\cdot, u_k)u_j\|_{L^2} + \left\|\int_{\mathbb{R}^n} Q_F(\cdot, y, u_k)u_j(y, t)\,dy\right\|_{L^2} \leq C\|u\|_{L^q}^3,
\]
where $C$ is a positive constant.

Proof. Following Mochizuki [14, Lemma 4.6], we obtain
\[
\|Q_H(\cdot, u_k)u_j\|_{L^2} \leq C \sum_{k=1, k \neq j}^N \|u_k\|_{L^{2\alpha}} \|u_k\|_{L^{2b}} \|u_j\|_{L^{2h}},
\]
\[
\left\|\int_{\mathbb{R}^n} Q_F(\cdot, y, u_k)u_j(y, t)\,dy\right\|_{L^2} \leq C \sum_{k=1, k \neq j}^N \|V_{\text{int}} * u_j u_k)\|_{L^2}
\leq \sum_{k=1, k \neq j}^N \|u_j\|_{L^{2\alpha}} \|u_k\|_{L^{2h}},
\]
where the positive constants $a, b, h$ satisfy $2a = 2b = 2h = q$, due to the Hölder’s inequality, and the Hardy-Littlewood-Sobolev inequality in the case where $V_{\text{int}}$ satisfies $|V_{\text{int}}(x)| \leq C|x|^{-2}$, or the Young’s inequality in the case where $V_{\text{int}}$ satisfies $V_{\text{int}} \in L^{n/2}(\mathbb{R}^n)$. In view of the inequality $\alpha^2 \beta < 2/3(\alpha^3 + \beta^3)$, one has
\[
\|Q_H(\cdot, u_k)u_j\|_{L^2} + \left\|\int_{\mathbb{R}^n} Q_F(\cdot, y, u_k)u_j(y, t)\,dy\right\|_{L^2} \leq C \left( \sum_{k=1, k \neq j}^N \|u_k\|_{L^q}^3 + \|u_j\|_{L^q}^3 \right)
\leq C\|u\|_{L^q}^3.
\]
This completes the proof.

Lemma 2.4. Let $n \geq 2$ and $s > 1$ and put $\Phi_v(x) = e^{iv \cdot x} \varphi(x)$. Assume that $q$ is a compact operator from $L^2(\mathbb{R}^n)$ to $L^{2,s}(\mathbb{R}^n)$. Then for any $v \in S_0$, there exist a positive constant $C$ such that
\[
\int_{-\infty}^{\infty} \|qU_0(t)\Phi_v\|_{L^2} \,dt \leq \frac{C}{|v|}
\]
for $|v|$ large enough.

The proof of Lemma 2.4 will be found in [5, Lemma 2.2] and its proof.

Let $\mathbf{u}(t)$ be the scattering solution and let $\Omega_-$ be the wave operator which assigns the free state $U_0(t)\mathbf{f}^-$ to the interacting state $\mathbf{u}(t) = U(t)\psi$ (see, e.g., Strauss [18]). In particular, $\Omega_- : H^1 \ni \mathbf{f}^- \to \mathbf{u}(0) \in H^1$.

Lemma 2.5. Let $n = 3$ and $\Phi_v(x) = (e^{iv \cdot x} \varphi_j(x))_{1 \leq j \leq N}, j = 1, 2, \cdots, N$. Assume that potentials $V_{\text{int}}$ and $V_{\text{ext}}$ satisfy Assumption 1.1 and Assumption 1.2, respectively. Then for any $\varphi_j \in S_0$, we have
\[
\left\|((\Omega_- - I)U_0(t)\Phi_v)_j\right\|_{L^2} = O(|v|^{-1})
\]
as $|v| \to \infty$ uniformly in $t \in \mathbb{R}$.

The proof of Lemma 2.5 is quite the same as [32, Lemma 2.3].
2.2. Time-decay estimate of solutions.

**Proposition 1.** Let \( n = 3 \) and \( a = \left(\frac{n}{2} + 1\right)^{-1}\left(\frac{n}{2}\right) = 3/4 \). Assume that potentials \( V_{int} \) and \( V_{ext} \) satisfy Assumption 1.1 and Assumption 1.2, respectively. Then the solutions \( u_j(t), j = 1, 2, \cdots, N \) of (1) with initial states \( \varphi_j \in S_0 \) satisfies

\[
\|u_j(t)\|_{L^\infty} \leq Ct^{-n/2}(\log t)^a,
\]

for \( t \geq e \), where \( C > 0 \).

Let us prepare for proving the Proposition 1. We denote \( V(t x) \) by \( V_t(x) \). Putting \( v_j(t) = (i t)^{n/2}e^{-i t|x|^2/2}u_j(t, t x) \) gives a equation (see e.g., Wada [25]):

\[
\begin{align*}
&\frac{i}{t} \partial_t v_j = -\frac{1}{2t^2} \Delta v_j + f_j(t, v), \\
&f_j(t, v) = V_{ext}^t(x)v_j(t) + \sum_{k=1}^N \{(V_{int}^t * |v_k|^2)v_j - (V_{int}^t * v_j v_k)v_k\}.
\end{align*}
\]

We note that the \( L^2 \)-conservation law for \( v_j(t) \) holds (see Isozaki [10]):

\[
\|v_j(t)\|_{L^2} = \|u_j(t)\|_{L^2} = \|u_j(0)\|_{L^2}, \quad j = 1, 2, \cdots, N.
\]

Thanks to the Gagliardo-Nirenberg inequality (Lemma 2.2) and (7), we have

\[
\begin{align*}
\|u_j(t)\|_{L^\infty} &\leq t^{-n/2}\|v_j(t)\|_{L^\infty} \\
&\leq Ct^{-n/2}\|v_j(t)\|_{L^2}^{1-\eta}\|\Delta v_j(t)\|_{L^2}^\eta \\
&= Ct^{-n/2}\|\Delta v_j(t)\|_{L^2}^\eta
\end{align*}
\]

for some \( C > 0 \). Therefore, estimating \( \|\Delta v_j(t)\|_{L^2} \) gives the proof of Proposition 1. In order to estimate \( \|\Delta v_j(t)\|_{L^2} \), we need

**Lemma 2.6.** Let \( n = 3 \). Assume that potentials \( V_{int} \) and \( V_{ext} \) satisfy Assumption 1.1 and Assumption 1.2, respectively. Then \( v_j(t) \) satisfies

\[
\sum_{j=1}^N \|\nabla v_j(t)\|_{L^2}^2 \leq C
\]

for some \( C > 0 \) and for any \( t > 0 \).

**Proof.** We first note that the pseudo-conformal conservation laws for \( v_j(t) \) (see, e.g., Cazenave [4, Section 7.2]) holds:

\[
\begin{align*}
\sum_{j=1}^N \|\nabla v_j(t)\|_{L^2}^2 + t^2G(t, v) &= \sum_{j=1}^N \|xu_j(0)\|_{L^2}^2 + \int_0^t s\Theta(s, v) ds, \\
&= \sum_{j=1}^N \|\nabla v_j(t)\|_{L^2}^2 + t^2G(t, v)
\end{align*}
\]
where
\[
G(t, v) = \sum_{j=1}^N \int_{\mathbb{R}^n} \frac{1}{2} V^t_{ext}(x) |v_j(t)|^2 dx + \sum_{j,k=1}^N \frac{1}{4} \int_{\mathbb{R}^n} (V^t_{int} \ast |v_k(t)|^2) |v_j(t)|^2 dx
- \sum_{j,k=1}^N \frac{1}{4} \int_{\mathbb{R}^n} (V^t_{int} \ast v_j(t)\overline{v_k(t)}) v_k(t)\overline{v_j(t)} dx,
\]
\[
\Theta(t, v) = \sum_{j=1}^N \int_{\mathbb{R}^n} \left( V^t_{ext}(x) + \frac{1}{2} x \cdot (\nabla V^t_{ext})(x) \right) |v_j(t)|^2 dx
+ \sum_{j,k=1}^N \int_{\mathbb{R}^n} \left( (V^t_{int} + \frac{1}{2} x \cdot \nabla V^t_{int}) \ast |v_k(t)|^2 \right) |v_j(t)|^2 dx
- \sum_{j,k=1}^N \int_{\mathbb{R}^n} \left( (V^t_{int} + \frac{1}{2} x \cdot \nabla V^t_{int}) \ast v_j(t)\overline{v_k(t)} \right) v_k(t)\overline{v_j(t)} dx.
\]

In view of the Assumption 1.2, Assumption 1.1 and the Cauchy-Schwarz inequality, we find that \(G(t, v) \geq 0\) and \(\Theta(t, v) \leq 0\) for \(t \geq 0\). In fact, we see from (7) that there exists \(\varepsilon_1 > 0\) such that \(\|v_j(t)\|_{L^2} = \|u_j(0)\|_{L^2} < \varepsilon_1\). The condition (ii) in Assumption 1.2 shows that
\[
\int_{\mathbb{R}^n} \left( V^t_{ext}(x) + \frac{1}{2} x \cdot (\nabla V^t_{ext})(x) \right) |v_j(t)|^2 dx
= C \int_{|x| \leq \kappa} |v_j(t)|^2 dx + \int_{|x| > \kappa} \left( V^t_{ext}(x) + \frac{1}{2} x \cdot (\nabla V^t_{ext})(x) \right) |v_j(t)|^2 dx
\leq C \varepsilon_1 + \int_{|x| > \kappa} \left( V^t_{ext}(x) + \frac{1}{2} x \cdot (\nabla V^t_{ext})(x) \right) |v_j(t)|^2 dx.
\]

This implies that there exists \(\kappa > 0\) such that \(\Theta(t, v) \leq 0\) for \(t \geq 0\).

It therefore follows from (9) that
\[
\frac{d}{dt} \left( \sum_{j=1}^N \|\nabla v_j(t)\|_{L^2}^2 + t^2 G(t, v) \right) = t\Theta(t, v) \leq 0
\]
for \(t \geq 0\), which implies that \(\sum_{j=1}^N \|\nabla v_j(t)\|_{L^2}^2 \leq C\). \(\square\)

**Proof of Proposition 1.** We are now in a position to prove Proposition 1. In the proof, we abbreviate the \(L^p\)-norm of a function \(f\) as \(\|f\|_p\). Applying \(\Delta\) to the equation (5), one has
\[
i \frac{\partial}{\partial t} \Delta v_j(t) = -\frac{1}{2\mu^2} \Delta^2 v_j(t) + \Delta f_j(t, v).
\]

Multiplying (10) by \(\Delta v_j\) and integrating the imaginary part over \(\mathbb{R}^n\), we have
\[
\frac{1}{2} \frac{d}{dt} \|\Delta v_j(t)\|_{L^2}^2 = \text{Im} \int_{\mathbb{R}^n} \Delta f_j(t, v)\Delta v_j(t) dx.
\]
Due to the fact that integrals

\[
\text{Im} \int_{\mathbb{R}^n} V_{\text{ext}}^t(x)|\Delta v_j(t)|^2 \, dx, \quad \text{Im} \sum_{j,k=1}^N \int_{\mathbb{R}^n} (V_{\text{int}}^t * |v_k(t)|^2)|\Delta v_j(t)|^2 \, dx,
\]

\[
\text{Im} \sum_{j,k=1}^N \int_{\mathbb{R}^n} (V_{\text{int}}^t * v_j(t)v_k^*(t))\Delta v_k(t)\Delta v_j(t) \, dx
\]

vanish, one gets

\[
\frac{1}{2} \sum_{j=1}^N \frac{d}{dt} \|\Delta v_j(t)\|^2_{L^2} = \sum_{\ell=1}^6 I_\ell(t),
\]

where

\[
I_1(t) = \text{Im} \sum_{j=1}^N \int_{\mathbb{R}^n} (\Delta V_{\text{ext}}^t)(x)v_j(t)\Delta v_j^*(t) \, dx,
\]

\[
I_2(t) = 2\text{Im} \sum_{j=1}^N \int_{\mathbb{R}^n} (\nabla V_{\text{ext}}^t)(x) \cdot \nabla v_j(t)\Delta v_j^*(t) \, dx,
\]

\[
I_3(t) = \text{Im} \sum_{j,k=1}^N \int_{\mathbb{R}^n} v_j(t)\Delta v_j^*(t) (\Delta (V_{\text{int}}^t * |v_k(t)|^2)) (x) \, dx,
\]

\[
I_4(t) = 2\text{Im} \sum_{j,k=1}^N \int_{\mathbb{R}^n} \Delta v_j^*(t) (\nabla v_j(t) \cdot \nabla (V_{\text{int}}^t * |v_k(t)|^2)) (x) \, dx,
\]

\[
I_5(t) = -\text{Im} \sum_{j,k=1}^N \int_{\mathbb{R}^n} v_k(t)\Delta v_j(t) (\Delta (V_{\text{int}}^t * v_j(t)v_k^*(t))) (x) \, dx,
\]

\[
I_6(t) = -2\text{Im} \sum_{j,k=1}^N \int_{\mathbb{R}^n} \Delta v_j^*(t) (\nabla v_k(t) \cdot \nabla (V_{\text{int}}^t * v_j(t)v_k^*(t))) (x) \, dx.
\]

We shall show that \(|I_1|, |I_2| \leq Ct^{-\gamma} \sum_{j=1}^N \|\Delta v_j(t)\|_{L^2}^2 \). Due to the Assumption 1.2 that \(V_{\text{ext}}(x)\) is a homogeneous function of degree \(-\gamma\), one has

\[
(\nabla V_{\text{ext}}^t)(x) = t^{-\gamma}(\nabla V_{\text{ext}})(x), \quad (\Delta V_{\text{ext}}^t)(x) = t^{-\gamma}(\Delta V_{\text{ext}})(x),
\]

and \(\nabla V_{\text{ext}} = \Delta V_{\text{ext}} = 0\) in the neighborhood of \(x = 0\). By using the Schwartz inequality, Hölder’s inequality, Gagliardo-Nirenberg inequality (Lemma 2.2) and
Lemma 2.6, we obtain

\[ |I_1(t)| \leq t^{-\gamma} \sum_{j=1}^{N} \|(\Delta V_{\text{ext}})v_j(t)\|_2 \|\Delta v_j(t)\|_2 \]

\[ \leq t^{-\gamma} \|\Delta V_{\text{ext}}\| \sum_{j=1}^{N} \|v_j(t)\|_2 \|\Delta v_j(t)\|_2 \]

\[ \leq Ct^{-\gamma} \|\Delta V_{\text{ext}}\| \sum_{j=1}^{N} \|\nabla v_j(t)\|_2 \|\Delta v_j(t)\|_2 \]

\[ \leq Ct^{-\gamma} \sum_{j=1}^{N} \|\Delta v_j(t)\|_2 \]

and

\[ |I_2(t)| \leq t^{-\gamma} \|\nabla V_{\text{ext}}\|_{\infty} \sum_{j=1}^{N} \|\nabla v_j(t)\|_2 \|\Delta v_j(t)\|_2 \]

\[ \leq Ct^{-\gamma} \sum_{j=1}^{N} \|\Delta v_j(t)\|_2. \]

We next claim that \( |I_\ell(t)| \leq Ct^{-1} \sum_{j=1}^{N} \|\Delta v_j(t)\|_2 \), \( \ell = 3, 4, 5, 6 \). It is easy to check that

\[
\int_{\mathbb{R}^n} v_j(t) \Delta \overline{v_j}(t) \left( \Delta (V_{\text{int}}^t \ast |v_k(t)|^2) \right) dx
\]

\[
= \int_{\mathbb{R}^n} v_j(t) \Delta \overline{v_j}(t) \sum_{m=1}^{3} (\partial_{x_m} V_{\text{int}}^t \ast \overline{v_k}(t) \partial_{x_m} v_k(t))(x) \, dx
\]

\[
+ \int_{\mathbb{R}^n} v_j(t) \Delta \overline{v_j}(t) \sum_{m=1}^{3} (\partial_{x_m} V_{\text{int}}^t \ast v_k(t) \partial_{x_m} \overline{v_k}(t))(x) \, dx.
\]

By using the Schwartz inequality, Hölder’s inequality, Young inequality and Gagliardo-Nirenberg inequality (Lemma 2.2), we have

\[
\left| \int_{\mathbb{R}^n} v_j(t) \Delta \overline{v_j}(t) \sum_{m=1}^{3} (\partial_{x_m} V_{\text{int}}^t \ast \overline{v_k}(t) \partial_{x_m} v_k(t))(x) \, dx \right|
\]

\[
\leq \sum_{m=1}^{3} \| \{ \partial_{x_m} V_{\text{int}}^t \ast \overline{v_k}(t) \partial_{x_m} v_k(t) \} v_j(t) \|_2 \|\Delta \overline{v_j}(t)\|_2
\]

\[
\leq \|\Delta \overline{v_j}(t)\|_2 \|v_j(t)\|_2 \sum_{m=1}^{3} \| \partial_{x_m} V_{\text{int}}^t \ast \overline{v_k}(t) \partial_{x_m} v_k(t) \|_n
\]
\[ \leq C \| \Delta \psi_j(t) \|_2 \| v_j(t) \| \frac{2n}{\pi} \sum_{m=1}^{3} \| \partial_{x_m} V_{int}^t \|_\frac{1}{2} \| \nabla \psi_j(t) \partial_{x_m} v_k(t) \|_\frac{n}{n-1} \]

\[ \leq C \| \Delta \psi_j(t) \|_2 \| v_j(t) \| \frac{2n}{\pi} \sum_{m=1}^{3} \| \partial_{x_m} V_{int}^t \|_\frac{1}{2} \| \nabla \psi_j(t) \|_\frac{2n}{n-2} \| \partial_{x_m} v_k(t) \|_2 \]

\[ \leq Ct^{-1} \| \Delta \psi_j(t) \|_2 \| \nabla v_j(t) \|_2 \| \nabla \psi_j(t) \|_2 \sum_{m=1}^{3} \| \partial_{x_m} V_{int}^t \|_\frac{1}{2} \| \partial_{x_m} v_k(t) \|_2 \]

\[ \leq Ct^{-1} \| \Delta v_j(t) \|_2, \]

which implies that \( |I_3(t)|, |I_5(t)| \leq Ct^{-1} \sum_{j=1}^{N} \| \Delta v_j(t) \|_2 \). Here we have used the fact that \( \| V^t \|_p = t^{-n/p} \| V \|_p \).

Similarly, one has

\[ \left| \int_{\mathbb{R}^n} \Delta \psi_j(t) \nabla v_j(t) \cdot \nabla ( (V_{int}^t * |v_k(t)|^2) ) (x) \, dx \right| \]

\[ \leq \int_{\mathbb{R}^n} \| \Delta \psi_j(t) \|_2 \| \nabla v_j(t) \|_2 \| V_{int}^t * v_k(t) \|_\infty \]

\[ \leq C \| \Delta \psi_j(t) \|_2 \| \nabla v_j(t) \|_2 \| V_{int}^t \|_\frac{2n}{n-2} \| v_k(t) \|_2 \]

\[ \leq C \| \Delta \psi_j(t) \|_2 \| \nabla v_j(t) \|_2 \| V_{int}^t \|_\frac{2n}{n-2} \| v_k(t) \|_2 \]

which implies that \( |I_4(t)|, |I_6(t)| \leq Ct^{-1} \sum_{j=1}^{N} \| \Delta v_j(t) \|_2 \).

In view of the inequalities for \( |I_\ell(t)|, \ell = 1, 2, \cdots, 6 \), we obtain a differential inequality:

\[ \frac{1}{2} \sum_{j=1}^{N} \frac{d}{dt} \| \Delta v_j(t) \|_2^2 \leq C(t^{-\gamma} + t^{-1}) \left( \sum_{j=1}^{N} \| \Delta v_j(t) \|_2^2 \right)^{1/2} \]

for \( t > 0 \), where \( C > 0 \) and \( \gamma \geq 1 \). This differential inequality implies that

\[ \sum_{j=1}^{N} \| \Delta v_j(t) \|_2 \leq C(\log t + 1) \leq C \log t \]

for \( t \geq e \). Hence, from (8), we obtain the desired estimate. \( \square \)

2.3. Proof of Theorem 2.1. Let \( u_j \in \mathcal{W} \) be a scattering solution to (1) with initial states \( e^{-itH_0} \phi_j \) at \( t = -\infty \). Because the scattering solution satisfies the integral equation

\[ u_j(t) = e^{-itH_0} \phi_j + \frac{1}{i} \int_{-\infty}^{t} e^{i(t-\tau)H_0} P_j(x, u), d\tau, \]
we obtain
\[(e^{-itH_0}\psi_j)(x) = (e^{-itH_0}\phi_j)(x) + \frac{1}{t} \int_\mathbb{R} e^{-i(t-\tau)H_0} P_j(x, u) d\tau \]
\[= u_j(t, x) + \frac{1}{t} \int_t^\infty e^{-i(t-\tau)H_0} P_j(x, u) d\tau.\]

Thanks to Proposition 1 and Lemma 2.3, for \(t \geq e\), one has
\[
\|u_j(t) - e^{-itH_0}\psi_j\|_{L^2} \leq \int_t^\infty \|e^{-i(t-\tau)H_0} V_{ext} u_j(\tau)\|_{L^2} d\tau
\]
\[+ \int_t^\infty \left\| e^{-i(t-\tau)H_0} \left( Q_H(\cdot, u_k)u_j(\tau) + \int_\mathbb{R}^n Q_F(x, y, u_k)u_j(y, \tau) dy \right) \right\|_{L^2} d\tau
\]
\[\leq \int_t^\infty \|V_{ext} u_j(\tau)\|_{L^2} d\tau + C \int_t^\infty \|u_j(\tau)\|_{L^2}^3 d\tau
\]
\[\leq C_1 \|V_{ext}\|_{L^2} \int_t^\infty \tau^{-3/2} (\log \tau)^3/4 d\tau + C_2 \int_t^\infty \|u_j(\tau)\|_{L^2}^3 d\tau
\]
\[\rightarrow 0 \quad \text{as} \ t \to \infty\]
for some \(C_1, C_2 > 0\), due to the fact that \(u_j \in L^3(\mathbb{R}; L^q)\) and
\[
\int_t^\infty \tau^{-3/2} (\log \tau)^3/4 d\tau < 27/4 \int_0^\infty e^{-\mu \mu^{3/4}} d\mu = 27/4 \Gamma \left( \frac{7}{4} \right),
\]
where \(\Gamma(s)\) is the Gamma function. This completes the proof. \(\square\)

3. **Asymptotics of the scattering operator.** Let \(\Phi_v(x) = e^{iv \cdot x} \phi(x)\). The components of the vector \(\Phi_v\) is denoted as \((\Phi_v)_j (j = 1, 2, \cdots, N)\). Put \(I_j(v) = i < \hat{S}_j(\hat{\Phi}_v), \hat{\Phi}_v >_{L^2}\). We consider the asymptotic behavior of the function \(I_j(v)\) as \(|v| \to \infty\). The X-ray transform of a function \(f\) is defined to be
\[
(Xf)(x, \theta) = \hat{f}(x, \theta) = \int_{-\infty}^{\infty} f(x + \theta t) dt,
\]
where \(x \in \mathbb{R}^n\) and \(\theta \in S^{n-1}\).

**Theorem 3.1.** Let \(n = 3\). Assume that potentials \(V_{int} and V_{ext}\) satisfy Assumption 1.1 and Assumption 1.2, respectively. Then for \(|v|\) sufficiently large and for any \(\varphi_j \in S_0\) with \(\|\varphi_j\|_{L^2}\) sufficiently small, \(j = 1, 2, \cdots, N\), the function \(I_j(v)\) has the expansion
\[
I_j(v) = \int_{\mathbb{R}^n} \overline{V_{int}(\xi)} H_j(\xi) d\xi + \frac{1}{|v|} < \overline{V_{ext}(\cdot, \hat{v})} \varphi_j, \varphi_j >_{L^2} + O(|v|^{-2})
\]
as \(|v| \to \infty\), where \(\hat{v} = v/|v| \in S^{n-1}\) and
\[
H_j(\xi) = \sum_{k=1}^N \int_\mathbb{R} \mathcal{F} \left( \|U_0(t)\varphi_k\|^2(\xi) \right) \mathcal{F} \left( \|U_0(t)\varphi_j\|^2(\xi) \right) dt
\]
\[= \sum_{k=1}^N \int_\mathbb{R} \mathcal{F} \left( \langle U_0(t)\varphi_j, U_0(t)\varphi_k \rangle \right) \langle U_0(t)\varphi_j, U_0(t)\varphi_k \rangle dt.
\]

**Proof.** In view of the representation of the scattering operator (2), we break the function \(I_j(v)\) in two parts:
\[
I_j(v) = I_j^{(0)}(v) + I_j^{(1)}(v),
\]
\[
I_j^{(0)}(v) = \int_{\mathbb{R}^n} \overline{V_{int}(\xi)} H_j(\xi) d\xi + \frac{1}{|v|} < \overline{V_{ext}(\cdot, \hat{v})} \varphi_j, \varphi_j >_{L^2} + O(|v|^{-2})
\]
\[
I_j^{(1)}(v) = \sum_{k=1}^N \int_\mathbb{R} \mathcal{F} \left( \|U_0(t)\varphi_k\|^2(\xi) \right) \mathcal{F} \left( \|U_0(t)\varphi_j\|^2(\xi) \right) dt
\]
\[= \sum_{k=1}^N \int_\mathbb{R} \mathcal{F} \left( \langle U_0(t)\varphi_j, U_0(t)\varphi_k \rangle \right) \langle U_0(t)\varphi_j, U_0(t)\varphi_k \rangle dt.
\]
where
\[ I_j^{(0)}(v) = \int_{\mathbb{R}} \langle N_j(\cdot, u), U_0(s)(\Phi_v)_{j}\rangle_{L^2} ds, \]
\[ I_j^{(1)}(v) = \int_{\mathbb{R}} \langle V_{\text{ext}}u_j(s), U_0(s)(\Phi_v)_{j}\rangle_{L^2} ds. \]

We know (see [32, subsection 2.2]) that the function \( I_j^{(0)}(v) \) can be expanded as
\[ I_j^{(0)}(v) = \int_{\mathbb{R}^n} \overline{V_{\text{int}}}H_j(\xi) d\xi + R_1(v) \]
with the estimate \( |R_1(v)| \leq C|v|^{-2} \) for some \( C > 0 \) and for \( |v| \) sufficient large.

We will claim that
\[ I_j^{(1)}(v) = \frac{1}{|v|} \left( \overline{V_{\text{ext}}}(\cdot, \overline{v}) \varphi_j, \overline{\varphi}_j \right)_{L^2} + O(|v|^{-2}) \]
as \( |v| \to \infty \). Let \( \Omega_- \) be a wave operator. Then we have
\[
I_j^{(1)}(v) = \left\langle \int_{\mathbb{R}} U_0(-t)V_{\text{ext}} \{ u_j(t) - U_0(t)(\Phi_v)_{\cdot}\} dt, (\Phi_v)_{j}\right\rangle_{L^2} \\
+ \left\langle \int_{\mathbb{R}} U_0(-t)V_{\text{ext}} U_0(t)(\Phi_v)_{j} dt, (\Phi_v)_{j}\right\rangle_{L^2} \\
= \left\langle \int_{\mathbb{R}} [(\Omega_- - I)(U_0(t)\Phi_v)]_{j} dt, V_{\text{ext}} U_0(t)(\Phi_v)_{j}\right\rangle_{L^2} \\
+ \frac{1}{|v|} \left\langle \overline{V_{\text{ext}}}(\cdot, \overline{v}) U_0(\tau/v)\varphi_j, U_0(\tau/v)\varphi_j \right\rangle_{L^2} \\
\leq \left\| [(\Omega_- - I)U_0(t)\Phi_v] \right\|_{L^2} \left\| \int_{\mathbb{R}} V_{\text{ext}} U_0(t)(\Phi_v)_{j} dt \right\|_{L^2} \\
+ \frac{1}{|v|} \left\langle \overline{V_{\text{ext}}}(\cdot, \overline{v}) U_0(\tau/v)\varphi_j, U_0(\tau/v)\varphi_j \right\rangle_{L^2}.
\]

Thanks to Lemma 2.4 and Lemma 2.5, the first term in (12) is estimated as \( O(|v|^{-2}) \) for \( |v| \) sufficiently large. We know (see [5]) that the second term in (12) is equal to
\[ \frac{1}{|v|} \left\langle \overline{V_{\text{ext}}}(\cdot, \overline{v})\varphi_j, \varphi_j \right\rangle_{L^2} + O(|v|^{-2}) \]
for \( |v| \) sufficiently large. The proof is completed.

4. **Reconstructions.** We complete the proof of Theorem 1.1 and give reconstruction formulas.

4.1. **Reconstruction of the interaction potential.** Let \( \Gamma \subset \mathbb{R} \) be a compact set and \( \Phi_v(x, \lambda) = e^{i\omega x}\varphi((\lambda + 1)x), \lambda \in \Gamma \). Put
\[ S_j^{\text{lim}}(\lambda) = \lim_{|v| \to \infty} i \left( ((S - I)\Phi_v(\cdot, \lambda))_{j}, (\Phi_v)_{j}(\cdot, \lambda) \right)_{L^2}, \quad j = 1, 2, \ldots, N. \]

In view of Theorem 3.1, we have for any \( \varphi_j \in S_0 \)
\[ S_j^{\text{lim}}(\lambda) = \int_{\mathbb{R}^n} \overline{V_{\text{int}}}H_j(\xi, \lambda) d\xi, \]
where
\[
H_j(\xi, \lambda) = \sum_{k=1}^{N} \int_{\mathbb{R}} \mathcal{F} ([U_0(t)\varphi_{k,\lambda}]^2)(\xi) \mathcal{F} ([U_0(t)\varphi_{j,\lambda}]^2)(\xi) \, dt \\
- \sum_{k=1}^{N} \int_{\mathbb{R}} \left| \mathcal{F} \left( (U_0(t)\varphi_{j,\lambda}) (U_0(t)\varphi_{k,\lambda}) \right)(\xi) \right|^2 \, dt.
\]

Here we have used the notation \( \varphi_{k,\lambda} = \varphi_k((\lambda + 1)x) \). Due to the fact (see [32, Theorem 1.12]) that the equation (13) is an integral equation of the first kind with a compact operator from \( H^k(\mathbb{R}^n) \) to \( L^2(\Gamma) \) for \( k > n/2 \), we can reconstruct \( \widetilde{V}_{\text{int}} \) from the scattering operator by using the theory of integral equations (see e.g., [11, Section 15.4]) or approximate techniques (see e.g., [15, Section 8.3]). For example, the Picard’s theorem allows us to obtain a reconstruction formula of \( \widetilde{V}_{\text{int}} \).

**Definition 4.1.** Let \( X \) and \( Y \) be Hilbert space, \( A : X \to Y \) be a compact linear operator, and \( A^* : Y \to X \) be its adjoint. Singular values of \( A \) is the non-negative square roots of the eigenvalue of non-negative self-adjoint compact operator \( A^*A : X \to X \). The singular system of \( A \) is the system \( \{\mu_n, \varphi_n, g_n\}, \, n \in \mathbb{N} \), where \( \varphi_n \in X \) and \( g_n \in Y \) are orthonormal sequences such that \( A\varphi_n = \mu_n g_n \) and \( A^*g_n = \mu_n \varphi_n \) for all \( n \in \mathbb{N} \).

We denote the null-space of the operator \( T \) by \( \mathcal{N}(T) \).

**Theorem 4.2.** Let \( n = 3 \). Assume that potentials \( V_{\text{int}} \) and \( V_{\text{ext}} \) satisfy Assumption 1.1 and Assumption 1.2, respectively. Then for any \( \varphi_j \in \mathcal{S}_0, \, j = 1, 2, \cdots, N \) the function \( S_j^{\text{lim}}(\lambda) \) is the \( L^2 \)-function on a compact set \( \Gamma \subset \mathbb{R} \). Moreover, letting \( \{\mu_n, \varphi_n, g_n\}, \, n \in \mathbb{N} \) be a singular system of the integral operator \( T \):

\[
(Tf)(\lambda) := \int_{\mathbb{R}^n} f(\xi)H_j(\xi, \lambda) \, d\xi,
\]

the Fourier transform of the interaction potential is reconstructed by the formula:

\[
\widetilde{V}_{\text{int}}(\xi) = \sum_{n=1}^{\infty} \frac{1}{\mu_n} \langle S_j^{\text{lim}}, g_n \rangle_{L^2(\Gamma)} \varphi_n
\]

if and only if \( S_j^{\text{lim}} \in \mathcal{N}(T^*)^\perp \) and satisfies

\[
\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} \left| \langle S_j^{\text{lim}}, g_n \rangle_{L^2(\Gamma)} \right|^2 < \infty.
\]

**Remark 3.** The procedure of constructing the singular system is as follows: Due to the fact that the operator \( T^*T \) is a self-adjoint compact operator on \( H^k(\mathbb{R}^n) \) for \( k > n/2 \), the operator \( T^*T \) has at least one eigenvalues different from zero and at most a countable set of eigenvalues accumulating only at zero. Let \( \{\phi_n\} \) denote an orthonormal sequences such that \( T^*T\phi_n = \mu_n^2 \phi_n \). Then we can define \( g_n = \mu_n^{-1} T^*\phi_n \).

Uniqueness of identifying \( \widetilde{V}_{\text{int}} \) follows from [32, Theorem 1.14]. Thus we conclude that one can uniquely determine \( \widetilde{V}_{\text{int}} \) from \( S \).
4.2. Reconstruction of the external potential. Assume that potentials $V_{\text{int}}$ and $V_{\text{ext}}$ satisfy Assumption 1.1 and Assumption 1.2. In addition, suppose that

$$
\int_{K}^{\infty} (1 + R) \| V_{\text{ext}}(x) F(|x| \geq R) \|_{L^{\infty}(\mathbb{R}^{n})} dR < \infty, \quad K > 0,
$$

where $F(A)$ is the characteristic function of $A \subseteq \mathbb{R}^{n}$. We note that the Assumption 1.3 includes the condition (14). Hence, the following also holds under the Assumption 1.1 and Assumption 1.3. In view of Theorem 3.1 with the help of Takiguchi [19, Proposition 3.2.], it is easy to verify that

$$
\lim_{|v| \to \infty} |v| \left\langle i \left( e^{-iv \cdot x} (S - I) \Phi_{\nu} \right)_{j} - \int_{\mathbb{R}} U_{0}(-t) N_{j}(x, U_{0}(t) \varphi) dt, \psi \right\rangle_{L^{2}} = \left\langle \tilde{V}_{\text{ext}}(\cdot, \tilde{v}) \varphi_{j}, \psi \right\rangle_{L^{2}}
$$

for any $\varphi_{j} \in \mathcal{S}_{0}$ and for any $\psi \in \mathcal{S}$. From Theorem 4.3, $\tilde{V}_{\text{int}}(\xi)$ is the known function. Therefore, $N_{j}(x, U_{0}(t) \varphi)$ is known function. Hence, the above identity shows that one can determine the X-ray transform of $V_{\text{ext}}$ from $S$ in the sense of the tempered distribution $\mathcal{S}'$. By using the inversion formula for the X-ray transform, we obtain a reconstruction formula of $V_{\text{ext}}$. More precisely, define the operator $I^{a}$, which is called the Riesz potential, as

$$
I^{a} f := \mathcal{F}^{-1}(|\xi|^{-a} \tilde{f}(\xi)), \quad a < n.
$$

We denote a hyperplane passing through the origin and orthogonal to $a \in \mathbb{S}^{n-1}$ by $\alpha^{\perp}$. Let $I_{\alpha}^{a}$ be the $(n-1)$-dimensional Riesz potential acting on the hyperplane $\alpha^{\perp}$. The adjoint of the X-ray transform is denoted as $X^{*}$:

$$
(X^{*} g)(x) = \int_{\mathbb{S}^{n-1}} g(\theta, x - (\theta \cdot x) \theta) d\sigma,
$$

where $d\sigma$ is the Lebesgue measure on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^{n}$. We know (see, e.g., Ramm-Katsevich [17, Theorem 2.6.2.]) that letting $f \in \mathcal{S}(\mathbb{R}^{n})$ and $g = X f$, one has for any $|a| < n$

$$
f = \frac{1}{2\pi |\mathbb{S}^{n-2}|} I^{-a} X^{*} I_{\alpha^{\perp}}^{a-1} g,
$$

where $|\mathbb{S}^{n-2}|$ is the surface area of $\mathbb{S}^{n-2}$. This inversion formula also holds for the tempered distribution $f \in \mathcal{S}'(\mathbb{R}^{n})$ and $g = X f \in \mathcal{S}'(T)$, where $T = \alpha^{\perp} \times \mathbb{S}^{n-1}$ (see Takiguchi [19]). Thus, we obtain

**Theorem 4.3.** Let $n = 3$ and $|a| < n$. Assume that potentials $V_{\text{int}}$ and $V_{\text{ext}}$ satisfy Assumption 1.1 and Assumption 1.2 with (14), respectively. Then for any $\varphi_{j} \in \mathcal{S}_{0}$, we have

$$
V_{\text{ext}} = \frac{I^{-a} X^{*} I_{\alpha^{\perp}}^{a-1}}{2\pi |\mathbb{S}^{n-2}|} \left| \varphi_{j} \right| \lim_{|v| \to \infty} |v| \left\langle i \left( e^{-iv \cdot x} (S - I) \Phi_{\nu} \right)_{j} - \int_{\mathbb{R}} U_{0}(-t) N_{j}(x, U_{0}(t) \varphi) dt \right\rangle
$$

in $\mathcal{S}'$.

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E-mail address: watanabe@xmath.ous.ac.jp