Warp products and (2+1) dimensional spacetimes

Soon-Tae Hong
Department of Science Education and Research Institute for Basic Sciences,
Ewha Womans University, Seoul 120-750, Republic of Korea

Yeji Kim
Department of Physics, Ewha Womans University, Seoul 120-750, Republic of Korea
(Dated: August 11, 2014)

We investigate the geometrical aspects of the extended (2+1) dimensional Banados-Teitelboim-Zanelli spacetimes in the multiply warped product scheme. To do this, we analyze the interior physical properties by constructing the explicit warp functions in these regions.

PACS numbers: 04.70.Bw; 04.60.Kz; 04.20.-q; 04.20.Dw; 04.20.Gz; 04.20.Jb; 04.50.Kd
Keywords: warped products; BTZ scalar theories; interior Ricci curvatures; event horizons

I. INTRODUCTION

Since the (2+1) dimensional Banados-Teitelboim-Zanelli (BTZ) [1–3] black hole has been proposed, significant interests in this black hole have been aroused with the novel discovery that the thermodynamics of higher dimensional black holes can be interpreted in terms of the BTZ solutions [4, 5]. There have been also tremendous advances in lower dimensional black holes related with the string theory since an exact conformal field theory describing a black hole in the (1+1) dimensional spacetime was proposed [6]. The BTZ black hole has been also generalized to possess the scalar tensor (ST) theories [7–11]. The Hawking and Unruh effects of the (2+1) dimensional black holes have been analyzed in terms of the global embedding Minkowski space (GEMS) approach [12–14] to produce the novel global higher dimensional flat embeddings of the (2+1) dimensional static, rotating and charged BTZ black holes and de Sitter spacetimes [15]. The (2+1) dimensional ST theories have been later analyzed [11] in terms of the GEMS scheme.

Now, we consider the ST theories in terms of the Lagrangian associated with scalar fields. To do this, we consider an action for the ST theories [9–11]:

\[ S = \int d^3x \sqrt{-g} [c(\chi)R - \omega(\chi)(\nabla \chi)^2 + V(\chi)], \quad (1.1) \]

where \( R \) is a scalar curvature, \( c(\chi) \) and \( \omega(\chi) \) are coupling functions for the scalar field \( \chi \) and \( V(\chi) \) is a potential function, respectively. For instance, the choice of \( c(\chi) = 1, \omega(\chi) = 0 \) and \( V(\chi) = 2\Lambda \) corresponds to the static BTZ black hole solution. For the case of \( c(\chi) = \chi \), we have the ST theories [9–11]. In (2+1)-dimensions, we take an ansatz for three-metric of the ST theories:

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\theta^2. \quad (1.2) \]

Using the Einstein equation obtained from the action in (1.1) and the metric in (1.2), we obtain [10]

\[ f'' + \frac{f'}{r} = \frac{2V}{\chi} - \frac{1}{\chi} \left[ f'\chi' + \frac{2}{r}(fr\chi')' \right], \quad (1.3) \]

\[ \chi'' + \omega(\chi)^2 = 0, \quad (1.4) \]

\[ f' = \frac{r^2V}{\chi} - \frac{1}{\chi}(fr^2\chi'), \quad (1.5) \]

where the prime denotes ordinary derivative with respect to \( r \). We next assume special forms of \( V(\chi) \) and \( \chi \) as in (2.1) and (2.2), respectively. From (1.4), we fix \( \omega \), and from (1.3) and (1.5) we obtain \( f(r) \). Here one notes that in this approach one cannot specify uniquely \( V(\chi) \) or \( \chi \) [10].

In this paper, we will consider two different cases with \( c(\chi) = \chi \) where gravitational forces are given by a mixture of the metric and the scalar field. Here we note that the scalar field will be just an ordinary one, not a phantom field. Exploiting the Lagrangian for the metric and scalar field, we will investigate the Einstein equations and warp product aspects of the ST theories in detail below. To do this, we will exploit an algorithm for roots of cubic and quartic equations associated with the warp products.
On the other hand, the warped product spacetimes [16, 17] have been attractive since they include classical examples of spacetime such as the Friedmann-Robertson-Walker manifold and the intermediate zone of the Reissner-Nordström (RN) black hole [18–22]. The interior Schwarzschild spacetime has been represented as a multiply warped spacetime with warping functions [23] to produce the Ricci curvature in terms of $f_1$ and $f_2$ for the multiply warped products of the form $M = R \times f_1 \cdot R \times f_2 \cdot S^2$. The interior RN-AdS spacetime has been also investigated by using the multiply warped product scheme [25]. In the (2+1) dimensional spacetime, the multiply warped product has been studied to investigate the interior solutions of the (2+1) BTZ black holes and the exterior solutions of the (2+1) de Sitter black holes [26]. In order to study nonsmooth curvatures associated with multiple discontinuities involved in the evolution of the universe, the multiply warped product has been also applied to the Friedmann-Robertson-Walker model [27].

In this paper, in order to investigate the physical properties inside the outer event horizon, we will analyze the multiply warped product manifold associated with the ST theories of the BTZ black holes containing the terms proportional to $1/r$ and $1/r^2$ as in (2.3) and (3.3), respectively. From now on we name them type-I and type-II, respectively. Exploiting the multiply warped product scheme, we will investigate the interior solutions for multiply warped functions in the (2+1) extended BTZ black holes, to explicitly obtain the Ricci and Einstein curvatures inside the outer event horizons of these metrics.

II. TYPE-I EXTENDED BTZ SPACETIME

Now, we consider the Lagrangian associated with an asymptotically constant scalar field [10]

$$\mathcal{L} = \sqrt{-g} \left[ \chi R - \frac{2}{1-\chi} (\nabla \chi)^2 + 2(3-3\chi + \chi^2)\Lambda \chi + \frac{8M}{27B^2} (1-\chi)^3 \right],$$

where

$$\chi = \frac{r}{r - 3B/2},$$

whose solution is given by

$$ds^2 = -\tilde{N}^2 dt^2 + \tilde{N}^{-2} dr^2 + r^2 d\phi^2,$$

$$\tilde{N}^2 = -M + \frac{MB}{r} + \frac{B^2}{r^2},$$

with $l^2 = 1/\Lambda$. Here $M$ is the positive mass parameter, $B$ in (2.3) is introduced to couple the scalar field $\chi$ to the metric $g_{\mu\nu}$ gravity and in that sense $B$ is meaningful. On the other hand, investigating the lapse functions $\tilde{N}^2$ and $N^2$ (for warp product case), one readily observe that the only curvature singularity is located at $r = 0$ [10]. One notes that the metric (2.3) looks like the Schwarzschild-AdS metric. If $\Lambda$ vanishes, the metric becomes exactly the same form as the four-dimensional Schwarzschild case. Variating the Lagrangian in (2.1) with respect to the metric, we obtain Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu},$$

where the energy-stress tensor is given by

$$T_{\mu\nu} = \frac{2}{\chi(1-\chi)} \nabla_\mu \chi \nabla_\nu \chi + \frac{1}{2} g_{\mu\nu} \left[ -\frac{2}{\chi(1-\chi)} (\nabla \chi)^2 + 2(3-3\chi + \chi^2)\Lambda + \frac{8M}{27B^2} (1-\chi)^3 \right].$$

Here we reinstate that the scalar field is an ordinary one, not a phantom field. Moreover, one can readily check that, for the case of warp product with $N^2$ in (2.9), we have the same Einstein equation with the same energy-stress tensor (2.5). Next, variation of the Lagrangian in (2.1) with respect to the scalar field $\chi$ yields

$$R - \frac{2}{(1-\chi)^2} (\nabla \chi)^2 + \frac{4}{1-\chi} \nabla^2 \chi + 6(1-\chi)^2 - \frac{8M}{9B^2} (1-\chi)^2 = 0.$$

A. Static case

Now, in order to investigate physical properties inside the black hole event horizon, we consider a multiply warped product manifold whose metric is of the form [23–26]

$$g = -d\mu^2 + \sum_{i=1}^{n} f_i^2 g_i,$$
where \( f_i \) are positive warping functions and \( g_i \) are the corresponding metrics, to extend the warped product spaces to richer class of spaces involving multiply products. In order to construct a multiply warped product manifold for the static type-I extended BTZ interior solution, we start with the three-metric inside the event horizon

\[
\sum_{i=1,2} g_i = \frac{X^2}{N^2} dt^2 - \frac{1}{N^2} dr^2 + r^2 d\phi^2
\]

with the lapse function for the interior solution

\[
N^2 = M - \frac{MB}{r} - \frac{r^2}{l^2}.
\]  

Here the parameters \( M, B \) and \( l \) are the same as those in (2.3). However, these parameters in (2.9) are defined inside the event horizon of the extended BTZ black hole. The roots of \( N^2 = 0 \) for arbitrary real value of \( r \) are given by

\[
r = r_1, \quad r = r_2 = \frac{1}{2} \left[ -r_1 + \left( r_1^2 + \frac{4MI^2B}{r_1} \right)^{1/2} \right] = -r_4, \\
r = r_3 = \frac{1}{2} \left[ -r_1 - \left( r_1^2 + \frac{4MI^2B}{r_1} \right)^{1/2} \right] = -r_5,
\]  

which shows that we do not have three positive roots. Here one notes that, since \( r \) is a radial distance, we have a constraint about \( r \), namely \( r > 0 \), if we exclude the curvature singularity at \( r = 0 \) in the range of \( r \). We also observe that for instance \( r_3 \) is negative for \( r_1 > 0 \) and thus we will use the factor \((r + r_5)(r_5 > 0)\) instead of \((r - r_3)\) to emphasize that \( r_3 \) is not a positive real root. We obtain an identity

\[
r_1 + r_2 + r_3 = 0.
\]  

Moreover, for a given nonzero \( B \), the mass parameter \( M \) is given in terms of the three roots as follows

\[
M = -\frac{r_1r_2r_3}{l^2B}.
\]  

For a vanishing \( B \), we also obtain \( M \) in terms of \( r_1 \) in (2.28) below. The lapse function can be now categorized in terms of the value of \( B \) and \( r_1 \).

(i) For \( 0 < B < B_1 \) with

\[
B_1 = \frac{2r_1}{MI^2}.
\]

and \( r_1 > 0 \) we find

\[
r_1 < \left( r_1^2 + \frac{4MI^2B}{r_1} \right)^{1/2} < 3r_1,.
\]

and thus we obtain two positive real roots \( r_1 \) and \( r_2 \) with \( r_1 > r_2 \) to yield

\[
N^2 = \frac{(r_1 - r)(r - r_2)(r + r_5)}{l^2r}
\]

which is well-defined in the region \( r_2 < r < r_1 \). Now we define a new coordinate \( \mu \) as follows

\[
d\mu^2 = N^{-2} dr^2,
\]

which can be integrated to yield

\[
\mu = \int_{r_2}^{r} dx \left( \frac{l^2x}{(r_1 - x)(x - r_2)(x + r_5)} \right)^{1/2},
\]

whose analytic solution is of the form

\[
\mu = \frac{2r_2l}{[l(r_2 + r_5)r_1]^{1/2}} \Pi(c_{11}, c_{12}, c_{13}) = G(r).
\]
Here the arguments \((c_{11}, c_{12}, c_{13})\) are given by

\[
c_{11} = \sin^{-1}\left(\frac{(r - r_2) r_1}{(r_1 - r_2) r}\right)^{1/2},
\]
\[
c_{12} = \frac{r_1 - r_2}{r_1},
\]
\[
c_{13} = \left(\frac{(r_1 - r_2) r_5}{(r_5 + r_2) r_1}\right)^{1/2},
\]
and \(\Pi(a, b, c)\) is the elliptic integral of the third kind defined as

\[
\Pi(a, b, c) = \int_0^a \frac{dx}{(1 - b \sin^2 x)(1 - c^2 \sin^2 x)^{1/2}}.
\]

Moreover, \(G(r)\) in (2.18) satisfies the following boundary condition

\[
\lim_{r \to r_2} G(r) = 0.
\]

One notes that \(dr/d\mu > 0\) implies that \(G^{-1}\) is a well-defined function.

(ii) For \(B \geq B_1\) and \(r_1 > 0\) we find two positive real roots \(r_1\) and \(r_2\) with \(r_1 \leq r_2\) to yield

\[
N^2 = \frac{(r_2 - r)(r - r_1)(r + r_5)}{l^2 r},
\]
which is well-defined in the region \(r_1 < r < r_2\). The remnant arguments are the same as those of the case (i), if we exchange \(r_1\) for \(r_2\) as in (2.15) and (2.22). We thus arrive at

\[
\mu = \frac{2r_1 l}{(r_1 + r_5) r_2^{1/2}} \Pi(c_{21}, c_{22}, c_{23}) = G(r).
\]

where

\[
c_{21} = \sin^{-1}\left(\frac{(r - r_1) r_2}{(r_2 - r_1) r}\right)^{1/2},
\]
\[
c_{22} = \frac{r_2 - r_1}{r_2},
\]
\[
c_{23} = \left(\frac{(r_2 - r_1) r_5}{(r_5 + r_1) r_2}\right)^{1/2}.
\]

Now, \(G(r)\) in (2.18) fulfills the boundary condition

\[
\lim_{r \to r_1} G(r) = 0,
\]
and \(dr/d\mu > 0\) indicates that \(G^{-1}\) is a well-defined function.

(iii) For \(B = 0\) and \(r_1 > 0\), which corresponds to the BTZ black hole, we find a positive real root \(r_1\) to yield

\[
N^2 = \frac{(r_1 - r)(r + r_1)}{l^2}
\]
where

\[
r_1 = M^{1/2} l,
\]
which is well-defined in the region \(0 < r < r_1\). We also obtain \(M\) as a function of \(r_1\):

\[
M = \frac{r_1^2}{l^2}.
\]

Now we define the coordinate \(\mu\) in (2.16) which can be integrated to yield

\[
\mu = \int_0^r dx \left(\frac{l^2}{(r_1 - x)(x + r_1)}\right)^{1/2},
\]
whose analytic solution is of the form

$$\mu = l \sin^{-1} \frac{r}{r_1} = G(r). \quad (2.30)$$

This form is consistent with the previous work for the BTZ spacetime [26]. Here $G(r)$ in (2.30) satisfies the boundary condition

$$\lim_{r \to 0} G(r) = 0 \quad (2.31)$$

and $dr/d\mu > 0$ implies that $G^{-1}$ is a well-defined function.

(iv) For $-B_2 < B < 0$ with

$$B_2 = \frac{r_1^3}{4Ml^2} \quad (2.32)$$

and $r_1 > 0$ we find

$$0 < \left( \frac{16M^2B}{r_1} \right)^{1/2} < r_1, \quad (2.33)$$

and thus we obtain a positive real root $r_1$ and two negative real roots $r_2$ and $r_3$ to produce

$$N^2 = \frac{(r_1 - r)(r + r_4)(r + r_5)}{l^2r}, \quad (2.34)$$

which is well-defined in the region $0 < r < r_1$. Now we define the coordinate $\mu$ in (2.16) which can be integrated to yield

$$\mu = \int_0^r dx \left( \frac{l^2x}{(r_1 - x)(r_1 + r_4)(r_1 + r_5)} \right)^{1/2}, \quad (2.35)$$

whose analytic solution is given by

$$\mu = \frac{2r_4l}{\sqrt{(r_1 + r_4)r_3}} \left[ \Pi(c_{31}, c_{32}, c_{33}) - F(c_{31}, c_{33}) \right] = G(r). \quad (2.36)$$

Here the arguments $(c_{31}, c_{32}, c_{33})$ are found to be

$$c_{31} = \sin^{-1} \left( \frac{(r_1 + r_4)r}{r_1(r + r_4)} \right)^{1/2},$$

$$c_{32} = \frac{r_1}{r_1 + r_4},$$

$$c_{33} = \left( \frac{r_1(r_5 - r_4)}{(r_1 + r_4)r_5} \right)^{1/2}. \quad (2.37)$$

Moreover, $\Pi(a, b, c)$ is given by (2.20) and $F(a, b)$ is the elliptic integral of the first kind defined as [29]

$$F(a, b) = \int_0^{a} \frac{dx}{(1 - b^2 \sin^2 x)^{1/2}}. \quad (2.38)$$

Moreover, $G(r)$ in (2.35) satisfies the boundary condition (2.31) and $dr/d\mu > 0$ implies that $G^{-1}$ is a well-defined function.

(v) For $B < -B_2$ and $r_1 > 0$ we find

$$\left( \frac{16M^2B}{r_1} \right)^{1/2} < 0, \quad (2.39)$$

and thus we obtain a positive real root $r_1$ and two imaginary roots $r_2$ and $r_3$ to produce

$$N^2 = \frac{(r_1 - r)(r^2 + r_1r - \frac{M^2B}{r_1})}{l^2r}. \quad (2.40)$$
Now we define the coordinate \( \mu \) in (2.16) which can be integrated to yield
\[
\mu = \int_0^r \frac{l^2 x}{(r_1 - x)(x^2 + r_1 x - \frac{M^2 B}{r_1})} \, dx = G(r),
\]
(2.41)
which is well-defined in the region \( 0 < r < r_1 \). Here, \( G(r) \) in (2.41) fulfills the boundary condition (2.31) and \( dr/d\mu > 0 \) implies that \( G^{-1} \) is a well-defined function. If a root of \( N^2 = 0 \) does not belong to the above categories, we cannot construct the coordinate \( \mu \).

Exploiting the coordinate \( \mu \) in (2.18), (2.23), (2.30), (2.36) and (2.41), we rewrite the metric (2.8) as warped products
\[
ds^2 = -d\mu^2 + f_1(\mu)^2 dt^2 + f_2^2(\mu)d\phi^2
\]
(2.42)
where
\[
f_1(\mu) = \left(M - \frac{MB}{G^{-1}(\mu)} - \frac{(G^{-1}(\mu))^2}{l^2}\right)^{1/2},
\]
\[
f_2(\mu) = G^{-1}(\mu).
\]
(2.43)

After some algebra using (2.18) and the warp products in (2.42), we obtain the following nonvanishing Ricci curvature components
\[
R_{\mu\mu} = -\frac{f_1''}{f_1} - \frac{f_2''}{f_2},
R_{tt} = \frac{f_1 f_1' f_2''}{f_2} + f_1 f_2'',
R_{\phi\phi} = \frac{f_1 f_2 f_2'}{f_1} + f_2 f_2''.
\]
(2.44)

Using the explicit expressions for \( f_1 \) and \( f_2 \) in (2.43), one obtains the Ricci curvature components
\[
R_{\mu\mu} = -\frac{2f_1'}{f_2} + \frac{3MB}{2f_2},
R_{tt} = \frac{2f_2 f_1'}{f_2} - \frac{3MBf_1^2}{2f_2^2},
R_{\phi\phi} = 2f_2 f_1' ,
\]
(2.45)
and the Einstein scalar curvature
\[
R = -\frac{6}{l^2},
\]
(2.46)
in the interior of the outer event horizon of the static type-I extended BTZ black hole.

**B. Rotating case**

Now we consider a multiply warped product manifold associated with the rotating type-I extended BTZ black hole inside the event horizon whose three-metric is given by
\[
ds^2 = N^2 dt^2 - N^{-2} dr^2 + r^2(d\phi + N^\phi dt)^2
\]
(2.47)
where the lapse and shift functions are found to become
\[
N^2 = M - \frac{MB}{r} - \frac{r^2}{l^2} - \frac{J^2}{4r^2},
N^\phi = -\frac{J}{2r^2},
\]
(2.48)
with an angular momentum \( J \). Note that four roots of the equation \( N^2 = 0 \) yield the lapse function in terms of the event horizons as follows

\[
N^2 = \frac{(r_+ - r)(r - r_1)(r - r_2)}{l^2 r^2} \tag{2.49}
\]

which, for the interior solution, is well-defined in the region \( r_- < r < r_+ \). The four roots \((r_+, r_-, r_1, r_2)\) of \( N^2 = 0 \) together with \( r_2 < r_1 < r_- < r_+ \) are analyzed in Appendix A. Because classification of roots of a quartic equation is highly nontrivial, we impose a restriction that \( r_- > 0 \) to proceed to obtain the coordinate \( \mu \).

Defining the coordinate \( \mu \) as in (2.16), we obtain

\[
\mu = \int_{r_-}^{r} dx \left( \frac{x^2}{(r_+ - x)(x - r_)(x - r_1)(x - r_2)} \right)^{1/2}, \tag{2.50}
\]
to yield the analytic solution of \( \mu \):

\[
\mu = \frac{2}{[(r_+ - r_1)(r_- - r_2)]^{1/2}} [(r_- - r_1) \Pi (c_{41}, c_{42}, c_{43}) + r_1 F(c_{41}, c_{43})] = G(r). \tag{2.51}
\]

Here \((c_{41}, c_{42}, c_{43})\) are given by

\[
c_{41} = \sin^{-1} \left( \frac{(r - r_1)(r_+ - r)}{(r_1 - r)(r_+ - r_1)} \right)^{1/2},
\]

\[
c_{42} = \frac{r_+ - r_-}{r_+ - r_1},
\]

\[
c_{43} = \left( \frac{(r_+ - r_-)(r_1 - r_2)}{(r_+ - r_1)(r_- - r_2)} \right)^{1/2}, \tag{2.52}
\]

\(\Pi(a, b, c)\) is the elliptic integral of the third kind defined in (2.20) and \( F(a, b) \) is given by (2.38). One notes that \( G(r) \) satisfies the boundary condition

\[
\lim_{r \to r_-} G(r) = 0, \tag{2.53}
\]

and \( dr/d\mu > 0 \) implies that \( G^{-1} \) is a well-defined function. In the vanishing angular momentum limit \( J = 0 \), the above solution (2.51) reduces to the static type-I extended BTZ spacetime case (2.18).

Exploiting the above coordinate (2.51), we obtain

\[
ds^2 = -d\mu^2 + f_1(\mu)^2 dt^2 + f_2^2(\mu)(d\phi + N^\phi dt)^2 \tag{2.54}
\]
to yield the metric of the warped product form (2.42) in comoving coordinates where one can replace [14, 26] \( d\phi + N^\phi dt \to d\phi \) to obtain the modified warping functions \( f_1 \) and \( f_2 \) as below

\[
f_1(\mu) = \left( M - \frac{MB}{G^{-1}(\mu)} - \frac{(G^{-1}(\mu))^2}{l^2} - \frac{J^2}{4(G^{-1}(\mu))^2} \right)^{1/2},
\]

\[
f_2(\mu) = G^{-1}(\mu). \tag{2.55}
\]

Here one notes that the detector locates in the comoving coordinates with the angular velocity \( d\phi/dt = -g_{t\phi}/g_{\phi\phi} = -N^\phi \) [14, 26]. We then arrive at the Ricci curvature components

\[
R_{\mu\mu} = -\frac{2f_1'}{f_2} + \frac{3MB}{2f_2^2} + \frac{J^2}{f_2^2},
\]

\[
R_{tt} = \frac{2f_1'^2}{f_2} - \frac{3MBf_1'}{2f_2^3} - \frac{J^2f_1^2}{f_2^3},
\]

\[
R_{\phi\phi} = 2f_2f_1'. \tag{2.56}
\]

Here one notes that there does not exist an additional term associated with the angular momentum \( J \) in the \( R_{\phi\phi} \) component since we have used the comoving coordinates. The Einstein scalar curvature is then given by

\[
R = \frac{6}{l^2} - \frac{J^2}{2f_2^2}, \tag{2.57}
\]
in the interior of the outer event horizon of the rotating type-I extended BTZ black hole. Note that in the \( J = 0 \) limit, the above Ricci components (2.56) and Einstein scalar curvature (2.57) reduce to the corresponding ones in (2.45) and (2.46) in the static type-I extended BTZ black hole, as expected.
III. TYPE-II EXTENDED BTZ SPACETIME

A. Static case

Now, we consider an asymptotically constant scalar field case with the Lagrangian [10]

$$\mathcal{L} = \sqrt{-g} \left[ \chi R - \frac{4\chi - 1}{2\chi(1 - \chi)}(\nabla \chi)^2 + \frac{M}{2L} + 6 \left( 2\Lambda - \frac{M}{2L} \right) \chi + 18 \left( -\Lambda + \frac{M}{4L} \right) \chi^2 + 2 \left( 4\Lambda - \frac{M}{L} \right) \chi^3 \right],$$  

(3.1)

where

$$\chi = \frac{r^2}{r^2 - 2L},$$  

(3.2)

which is associated with a metric of the form

$$ds^2 = -\tilde{N}^2 dt^2 + \tilde{N}^{-2} dr^2 + r^2 d\phi^2,$$

$$\tilde{N}^2 = -M + \frac{ML}{r^2} + \frac{r^2}{l^2}. $$  

(3.3)

Here $L$ in (3.3) is again introduced to couple the scalar field $\chi$ to the metric $g_{\mu\nu}$ gravity. Moreover, $L$ in $\tilde{N}^2$ of (3.3) is trivially related to $L$ in the modified lapse function $N^2$ for the warped product scheme since $N^2 = -\tilde{N}^2$ in (3.6). On the other hand, investigating the lapse functions $\tilde{N}^2$ and $N^2$ (for warped product case), one readily observe that the only curvature singularity is located at $r = 0$ [10]. Moreover, in the limit $L = (J^2/4M)$, the metric seems to be related to a rotating BTZ black hole. However, because the three-metric (3.3) does not contain a shift function, the metric does not allow such a rotating BTZ black hole. The rotating type-II extended BTZ spacetime case will be investigated in Section III-B below. Variating the Lagrangian in (3.1) with respect to the metric, we obtain the Einstein equation in (2.4) with the energy-stress tensor given by

$$T_{\mu\nu} = \frac{4\chi - 1}{2\chi^2(1 - \chi)} \nabla_\mu \chi \nabla_\nu \chi + \frac{1}{2} g_{\mu\nu} \left[ -\frac{4\chi - 1}{2\chi^2(1 - \chi)}(\nabla \chi)^2 + \frac{M}{2L} \chi \\
+ 6 \left( 2\Lambda - \frac{M}{2L} \right) + 18 \left( -\Lambda + \frac{M}{4L} \right) \chi + 2 \left( 4\Lambda - \frac{M}{L} \right) \chi^2 \right].$$  

(3.4)

Next, variation of the Lagrangian in (3.1) with respect to the scalar field $\chi$ produces

$$R + \frac{1 - 2\chi + 4\chi^2}{2\chi^2(1 - \chi)^2}(\nabla \chi)^2 + \frac{4\chi - 1}{\chi(1 - \chi)} \nabla^2 \chi + 6 \left( 2\Lambda - \frac{M}{2L} \right) + 36 \left( -\Lambda + \frac{M}{4L} \right) \chi + 6 \left( 4\Lambda - \frac{M}{L} \right) \chi^2 = 0. $$  

(3.5)

Now we consider a multiply warped product manifold associated with the static type-II extended BTZ spacetime inside the event horizon where three-metric (2.8) is given in terms of the lapse function

$$N^2 = M - \frac{ML}{r^2} - \frac{r^2}{l^2}. $$  

(3.6)

The roots of $N^2 = 0$ for arbitrary real value of $r$ are given by

$$r_+ = \left( \frac{Ml^2 + (M^2 l^4 - 4MLl^2)^{1/2}}{2} \right)^{1/2},$$

$$r_- = \left( \frac{Ml^2 - (M^2 l^4 - 4MLl^2)^{1/2}}{2} \right)^{1/2} = ir_1. $$  

(3.7)

We find an identity

$$r_+^2 + r_-^2 = Ml^2. $$  

(3.8)

For a given nonzero $L$, the mass parameter $M$ is given by the roots $r_+$ and $r_-$

$$M = \frac{r_+^2 r_-^2}{l^2 L}. $$  

(3.9)
For a vanishing $L$, we also obtain $M$ in (2.28) for the BTZ case. The lapse function can be classified in terms of the value of $L$ and $r_{\pm}$.

(i) For $0 < L < L_1$ with
\[ L_1 = \frac{Ml^2}{4}, \]  
(3.10)
we find two positive real roots $r_+$ and $r_-$ with $r_+ > r_-$ to produce
\[ N^2 = \frac{(r_+ - r)(r - r_+)(r + r_+)(r_+ + r_-)}{l^2r^2}, \]  
(3.11)
which, for the interior solution, is well-defined in the region $r_- < r < r_+$. Introducing the coordinate $\mu$ as in (2.16) we obtain
\[ \mu = \int_{r_-}^{r} dx \left( \frac{l^2x^2}{(r_+ - x)(x - r_+)(x + r_+)(x + r_-)} \right)^{1/2}, \]  
(3.12)
whose analytic solution is of the form
\[ \mu = l \sin^{-1} \left( \frac{r^2 - r_+^2}{r_+ - r_-} \right)^{1/2} = G(r). \]  
(3.13)
Here we have used the $r_{\pm}$ in (3.7). We note that $G(r)$ satisfies the boundary condition (2.53) and $dr/d\mu > 0$ implies that $G^{-1}$ is a well-defined function.

(ii) For $L = 0$, we have the BTZ case again. The ensuing argument is the same as that of (iii) of the static type-I, to yield the analytic solution (2.30) for $\mu$.

(iii) For $L < 0$, $r_+$ and $r_-$ are a positive real root and an imaginary one, respectively. The lapse function is then given by
\[ N^2 = \frac{(r_+ - r)(r + r_+)(r^2 + r_+^2)}{l^2r^2}, \]  
(3.14)
which, for the interior solution, is well-defined in the region $0 < r < r_+$. We introduce the coordinate $\mu$ as in (2.16) to obtain
\[ \mu = \int_{0}^{r} dx \left( \frac{l^2x^2}{(r_+ - x)(x + r_+)(x^2 + r_+^2)} \right)^{1/2} = G(r). \]  
(3.15)
We note that $G(r)$ satisfies the boundary condition (2.31) and $dr/d\mu > 0$ implies that $G^{-1}$ is a well-defined function. If a root of $N^2 = 0$ does not belong to the above categories, we cannot obtain the coordinate $\mu$.

Exploiting the above coordinate (2.30), (3.13) and (3.15), we can obtain the warped products (2.42) and the corresponding modified warping functions $f_1$ and $f_2$ given as below
\[ f_1(\mu) = \left( M - \frac{ML}{G^{-1}(\mu)^2} - \frac{(G^{-1}(\mu))^2}{l^2} \right)^{1/2}, \]
\[ f_2(\mu) = G^{-1}(\mu), \]  
(3.16)
to arrive at, in the interior of the outer event horizon of the static type-II extended BTZ spacetime, the Ricci curvature components,
\[ R_{\mu\mu} = - \frac{2f'_1}{f_2} + \frac{4ML}{f_2^2}, \]
\[ R_{tt} = \frac{2f'_1 f'_2}{f_2} - \frac{4ML f_1^2}{f_2^4}, \]
\[ R_{\phi\phi} = 2f_2 f'_1 \]  
(3.17)
and the Einstein scalar curvature
\[ R = - \frac{6}{l^2} - \frac{2ML}{f_2^2}. \]  
(3.18)
B. Rotating case

Next we consider a multiply warped product manifold associated with the rotating type-II extended BTZ spacetime inside the event horizon where three-metric (2.47) is given by the lapse and shift functions:

\[
N^2 = M - \frac{ML}{r^2} - \frac{r^2}{l^2} - \frac{J^2}{4r^2},
\]

\[
N^\phi = -\frac{J}{2r^2}.
\]  

(3.19)

The roots of \(N^2 = 0\) for arbitrary real value of \(r\) are given by

\[
r_+ = \sqrt{\frac{M^2 + (M^2l^4 - 4MLl^2 - J^2l^2)^{1/2}}{2}},
\]

\[
r_- = \sqrt{\frac{M^2 - (M^2l^4 - 4MLl^2 - J^2l^2)^{1/2}}{2}} = ir_2.
\]

(3.20)

We also obtain the identity (3.8). For a given nonzero \(L\), the mass parameter \(M\) is given in terms of the roots and the angular momentum

\[
M = \frac{r_+^2r_2^2 - \frac{1}{3}J^2l^2}{l^2L}.
\]

(3.21)

We also obtain the identity (3.8). The lapse function can be classified in terms of the value of \(L + \frac{J^2}{4M}\) and \(r_\pm\).

(i) For \(0 < L + \frac{J^2}{4M} < L_1\) with \(L_1\) in (3.10), we find two positive real roots \(r_+\) and \(r_-\) with \(r_+ > r_-\) to yield the lapse function of the form (3.14) which, for the interior solution, is well-defined in the region \(r_- < r < r_+\). Introducing the coordinate \(\mu\) as in (2.16) we obtain the coordinate \(\mu\) in (3.12). Here we note that \(r_+\) and \(r_-\) are different from that of the static case since we have a rotation term in (3.20). The analytic solution of (3.12) is given by

\[
\mu = l\sin^{-1}\left(\frac{r_2 - r_2}{r_2 - r_2}\right) = G(r).
\]

(3.22)

Here we have used the \(r_\pm\) in (3.20). We note that \(G(r)\) satisfies the boundary condition (2.53) and \(d\mu/dr > 0\) implies that \(G^{-1}\) is a well-defined function.

(ii) For \(L + \frac{J^2}{4M} = 0\), we obtain the BTZ case. The ensuing argument is again the same as that of (iii) of the static type-I, to produce the analytic solution (2.30) for \(\mu\) in the comoving coordinates.

(iii) For \(L + \frac{J^2}{4M} < 0\), \(r_+\) and \(r_-\) are a positive real root and an imaginary one, respectively. The lapse function is now given by

\[
N^2 = \frac{(r_+ - r)(r_+ + r_+)(r^2 + r_2^2)}{l^2r^2},
\]

(3.23)

which, for the interior solution, is well-defined in the region \(0 < r < r_+\). We introduce the coordinate \(\mu\) as in (2.16) to obtain

\[
\mu = \int_0^r dx \left(\frac{l^2x^2}{(r_+ - x)(x + r_+)(x^2 + r_2^2)}\right)^{1/2} = G(r).
\]

(3.24)

We note that \(G(r)\) satisfies the boundary condition (2.31) and \(dr/d\mu > 0\) implies that \(G^{-1}\) is a well-defined function. If a root of \(N^2 = 0\) does not belong to the above categories, we cannot construct the coordinate \(\mu\). Exploting the above coordinate (2.30), (3.22), and (3.24), we obtain the metric (2.54) and then we find the warped products (2.42) in the comoving coordinates where one can replace \(d\phi + N^\phi dt \rightarrow d\phi\) and the modified warping functions \(f_1\) and \(f_2\) are given as below

\[
f_1(\mu) = \left(M - \frac{ML}{(G^{-1}(\mu))^2} - \frac{(G^{-1}(\mu))^2}{l^2} - \frac{J^2}{4(G^{-1}(\mu))^2}\right)^{1/2},
\]

\[
f_2(\mu) = G^{-1}(\mu).
\]

(3.25)
to yield, in the interior of the outer event horizon of the rotating type-II extended BTZ spacetime, the Ricci curvature components,

\[
R_{\mu\nu} = -\frac{2f'_1}{f_2} + \frac{4ML}{f_2^2} + \frac{J^2}{f_2^4},
\]

\[
R_{tt} = \frac{2f_1^2f'_1}{f_2} - \frac{4MLf_2^2}{f_2^4} - \frac{J^2f_1^2}{f_2^4},
\]

\[
R_{\phi\phi} = 2f_2f'_1
\]

and the Einstein scalar curvature

\[
R = -\frac{6}{l^2} - \frac{2ML}{f_2^2} - \frac{J^2}{2f_2^4}.
\]

(3.26)

(3.27)

Note that in the \( J = 0 \) limit, the above Einstein scalar curvature (3.27) reduces to the corresponding one (3.18) in the static type-II extended BTZ spacetime case.

**IV. CONCLUSIONS**

We have studied the multiply warped product manifold associated with the ST theories of the BTZ black holes to evaluate the Ricci curvature components inside the black hole event horizons. Exploiting these Ricci curvatures, we have shown that all the Einstein scalar curvatures are identical both in the exterior and interior of the outer event horizons without discontinuities for these extended BTZ spacetimes.

To investigate the ST theories, we have considered two cases with \( c(\chi) = \chi \) in (1.1) where gravitational forces are given by a mixture of the metric and the scalar field. Here the scalar field is an ordinary one, not a phantom field. Using the Lagrangian for the metric and scalar field, we have investigated the Einstein equations and warp product aspects of the ST theories. To do this we have exploited the algorithm for roots of cubic and quartic equations associated with the warp products.

**Acknowledgments**

The author would like to thank the anonymous referees for helpful comments.

**Appendix A: Mathematical aspects of a quartic equation**

Now, we consider four roots of the lapse function in (2.48) which produces a quartic equation of the form

\[
x^4 + a_1x^2 + a_2x + a_3 = 0,
\]

(A.1)

where

\[
a_1 = -ML^2, \quad a_2 = MBf^2, \quad a_3 = \frac{1}{4}J^2f^2.
\]

(A.2)

Following the algorithm for treating the quartic equation [28], we obtain the four roots \((x_1, x_2, x_3, x_4)\) of the equation (A.1) given by

\[
x_1 = \frac{1}{2}p_1 + \frac{1}{2}p_2,
\]

\[
x_2 = \frac{1}{2}p_1 - \frac{1}{2}p_2,
\]

\[
x_3 = -\frac{1}{2}p_1 + \frac{1}{2}p_3,
\]

\[
x_4 = -\frac{1}{2}p_1 - \frac{1}{2}p_3,
\]

(A.3)
where
\[ p_1 = (y_0 - a_1)^{1/2}. \] (A.4)

For \( p_1 \neq 0 \) we obtain \(^1\)
\[ p_2 = \left( -p_1^2 - 2a_1 - \frac{2a_2}{p_1} \right)^{1/2}, \]
\[ p_3 = \left( -p_1^2 - 2a_1 + \frac{2a_2}{p_1} \right)^{1/2}, \] (A.5)

and for \( p_1 = 0 \) we arrive at
\[ p_2 = \left( -2a_1 + 2 \left( y_0^2 - 4a_3 \right)^{1/2} \right)^{1/2}, \]
\[ p_3 = \left( -2a_1 - 2 \left( y_0^2 - 4a_3 \right)^{1/2} \right)^{1/2}. \] (A.6)

Here \( y_0 \) is defined as a real root of the following cubic equation
\[ y^3 + b_1 y^2 + b_2 y + b_3 = 0, \] (A.7)
where
\[ b_1 = -a_1, \quad b_2 = -4a_3, \quad b_3 = 4a_1a_3 - a_2^2. \] (A.8)

The three roots of the cubic equation (A.7) are then readily given by (see Appendix B for details.)
\[ y_1 = q_1 + q_2 - \frac{1}{3} b_1, \]
\[ y_2 = -\frac{1}{2} (q_1 + q_2) + \frac{i\sqrt{3}}{2} (q_1 - q_2) - \frac{1}{3} b_1, \]
\[ y_3 = -\frac{1}{2} (q_1 + q_2) - \frac{i\sqrt{3}}{2} (q_1 - q_2) - \frac{1}{3} b_1, \] (A.9)
where
\[ q_1 = \left( q_3 + (q_3^2 + q_4^3)^{1/2} \right)^{1/3}, \]
\[ q_2 = \left( q_3 - (q_3^2 + q_4^3)^{1/2} \right)^{1/3}, \]
\[ q_3 = \frac{9b_1 b_2 - 27b_3 - 2b_4^2}{54}, \]
\[ q_4 = \frac{3b_2 - b_4^2}{9}. \] (A.10)

These three roots in (A.9) can be also rewritten in terms of trigonometric functions as follows
\[ y_1 = 2(-q_4)^{1/2} \cos \left( \frac{\psi}{3} \right) - \frac{1}{3} b_1, \]
\[ y_2 = 2(-q_4)^{1/2} \cos \left( \frac{\psi}{3} + \frac{2\pi}{3} \right) - \frac{1}{3} b_1, \]
\[ y_3 = 2(-q_4)^{1/2} \cos \left( \frac{\psi}{3} + \frac{4\pi}{3} \right) - \frac{1}{3} b_1, \] (A.11)

\(^1\) In the literature [28], there exist typos in the expressions for \( p_2 \) in (A.5), and for \( q_1 \) and \( q_2 \) in (A.10).
where $\psi$ satisfies the identities

$$
\cos \psi = \frac{q_3}{(-q_4^4)^{1/2}}, \\
\sin \psi = \left(1 + \frac{q_3^2}{q_4^4}\right)^{1/2}.
$$

(A.12)

Now, we consider the specific case of the static BTZ limit where the coefficients of the quartic equation (A.1) are given as

$$
a_1 = -M^2, \quad a_2 = 0, \quad a_3 = 0.
$$

(A.13)

From (A.9), one readily obtain the three roots of the cubic equation (A.7)

$$
y_1 = a_1, \quad y_2 = 0, \quad y_3 = 0.
$$

(A.14)

If we choose $y_0$ as in $y_0 = y_1 = a_1 = -M^2$, exploiting (A.3) we obtain the four roots of the quartic equation (A.1) as follows

$$
x_1 = 0, \quad x_2 = 0, \quad x_3 = M^{1/2}, \quad x_4 = -M^{1/2},
$$

(A.15)

which imply that $r_+ = x_3, \quad r_- = x_4, \quad r_1 = x_1$ and $r_2 = x_2$. Similarly for the choice of $y_0 = y_2 = y_3 = 0$, we again obtain the four roots of the equation (A.1)

$$
x_1 = M^{1/2}, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = -M^{1/2},
$$

(A.16)

to conclude that $r_+ = x_1, \quad r_- = x_4, \quad r_1 = x_2$ and $r_2 = x_3$. The mapping of $(r_+, r_-, r_1, r_2)$ onto $(x_1, x_2, x_3, x_4)$ depends on the choice of $y_0$ in $(y_1, y_2, y_3)$.

**Appendix B: Algorithm for roots of a cubic equation**

Next, we recapitulate the algorithm for finding solutions of a cubic equation appeared in [28]. To do this, we start with a cubic equation of the form

$$
y^3 + b_1y^2 + b_2y + b_3 = 0.
$$

(B.1)

Introducing a new variable

$$
z = y + \frac{1}{3}b_1,
$$

(B.2)

we find the following form

$$
z^3 + 3q_4z - 2q_3 = 0,
$$

(B.3)

where $q_3$ and $q_4$ are given by (A.10). Next, we readily check that

$$
z_1 = q_1 + q_2,
$$

(B.4)

with $q_1$ and $q_2$ being defined as in (A.10), is a root of the cubic equation (B.3). Moreover, we find that (B.3) is factorized as

$$
(z - z_1)(z^2 + z_1z + 3q_4) = 0,
$$

(B.5)

from which we obtain the other two roots of the from

$$
z_2 = -\frac{1}{2}(q_1 + q_2) + \frac{i\sqrt{3}}{2}(q_1 - q_2),
$$

$$
z_3 = -\frac{1}{2}(q_1 + q_2) - \frac{i\sqrt{3}}{2}(q_1 - q_2).
$$

(B.6)
Even though $z_2$ and $z_3$ possess an $i$ as shown in (B.6), this does not indicate anything about the numbers of real and complex roots, since $q_1$ and $q_2$ are themselves complex in general. In order to determine which roots are real or complex, we need to introduce a criterion parameter, namely the discriminant $D$ defined as

$$D = q_2^3 + q_4^3.$$  \hspace{1cm} (B.7)

If $D > 0$, we have one real root and two complex conjugates; if $D = 0$, we have three real roots and at least two roots are equal; if $D < 0$, we have three real roots and all roots are different. For our case of interest associated with (2.49), we need to have three different roots in the cubic equation. From now on, we will thus consider $D < 0$ case only. Keeping these roots $(z_1, z_2, z_3)$ in mind and using the definition (B.2), we readily find the roots (A.9) for the cubic equation in (A.7) or in (B.1). Moreover, we obtain the identities:

$$
\begin{align*}
    z_1 + z_2 + z_3 &= 0, \\
    z_1 z_2 + z_1 z_3 + z_2 z_3 &= 3q_4, \\
    z_1 z_2 z_3 &= 2q_3, \\
    z_1^2 + z_2^2 + z_3^2 &= -6q_4, \\
    z_1^3 + z_2^3 + z_3^3 &= 6q_3, \\
    z_1^4 + z_2^4 + z_3^4 &= 18q_4^2, \\
    z_1^5 + z_2^5 + z_3^5 &= -30q_3q_4.
\end{align*}
$$  \hspace{1cm} (B.8)

Finally, we formulate the above roots in terms of trigonometric function. To do this, we exploit the angle $\psi$ defined in (A.12). Here we again consider the negative discriminant case: $D < 0$ which is relevant to our cubic equation at hand. Exploiting the identities

$$q_3 \pm i(-q_3^2 - q_4^3)^{1/2} = (-q_4^{3/2}) e^{\pm i\psi},$$  \hspace{1cm} (B.9)

we arrive at the desired forms in (A.11).

---

[1] Banados, M., Teitelboim, C., and Zanelli, J. (1992). *Phys. Rev. Let.*** 69 1849.
[2] Martinez, C., Teitelboim, C., and Zanelli, J. (2000). *J. Phys. D*** 61 104013.
[3] Banados, M., Henneaux, M., Teitelboim, C., and Zanelli, J. (1993). *Phys. Rev. D*** 48 1506.
[4] Maldacena, J., and Strominger, A. (1998). *JHEP*** 9812 005.
[5] Sfetsos, K., and Skenderis, K. (1998). *Nucl. Phys. B*** 517 179.
[6] Witten, E. (1991). *Phys. Rev. D*** 44 314.
[7] Lemos, J.P.S. (1995). *Phys. Lett. B*** 353 46.
[8] Sa, P.M., Kleber, A., and Lemos, J.P.S. (1996). *Class. Quant. Grav.*** 13 125.
[9] Pimentel, L.O. (1989). *Class. Quant. Grav.*** 6 263.
[10] Chan, K.K.K. (1997). *Phys. Rev. D*** 55 3564.
[11] Hong, S.T., Kim, Y.W., and Park, Y.J. (2000). *Phys. Rev. D*** 62 064021.
[12] Deser, S., and Levin, O. (1997). *Class. Quant. Grav.*** 14 L163.
[13] Deser, S., and Levin, O. (1998). *Class. Quant. Grav.*** 15 L85.
[14] Deser, S., and Levin, O. (1999). *Phys. Rev. D*** 59 064004.
[15] Hong, S.T., Kim, Y.W., and Park, Y.J. (2000). *Phys. Rev. D*** 62 024024.
[16] Bishop, R.L., and O’Neill, B. (1969). *Trans. Am. Math. Soc.*** 145 1.
[17] Beem, J.K., Ehrlich, P.E., and Easley, K. (1996). *Global Lorentzian Geometry*, Marcel Dekker Pure and Applied Mathematics, New York.
[18] Reissner, H. (1916). *Ann. Phys.*** 50 106.
[19] Nordström, G. (1918). *Proc. K. Ned. Akad. Wet.*** 20 1238.
[20] Demers, J., Lafrance, R., and Meyers, R.C. (1995). *Phys. Rev. D*** 52 2245.
[21] Ghosh, A., and Mitra, P. *Phys. Lett. B* (1995). 357 295.
[22] Cognola, G., and Lecca, P. (1998). *Phys. Rev. D*** 57 1108.
[23] Choi, J. (2000). *J. Math. Phys.* 41 8163.
[24] Unal, B. (2000). *J. Geom. Phys.*** 34 287.
[25] Hong, S.T., Choi, J., and Park, Y.J. (2005). *Nonlin. Anal.*** 63 493.
[26] Hong, S.T., Choi, J., and Park, Y.J. (2003). *Gen. Rel. Grav.*** 35 2105.
[27] Choi, J., and Hong, S.T. (2004). *J. Math. Phys.*** 45 642.
[28] Weisstein, E.W. (2003). *CRC Concise Encyclopedia of Mathematics*, Chapman & Hall/CRC, New York.
[29] Gradshteyn, I.S., and Ryzhik, I.M. (2000). *Table of Integrals, Series and Products*, Academic Press, San Diego.