Many-body effects on out-of-plane phonons in graphene

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Abstract. We study the properties of out-of-plane phonons in the framework of the many-body theory of graphene. We investigate, in particular, the way in which the coupling to electron–hole excitations renormalizes the dispersion of the acoustic branch of out-of-plane phonons. We show that the effect of the charge polarization cuts off the quadratic dispersion at low energies, implying the absence of long-wavelength flexural phonons. This result holds in the low-energy Dirac theory of graphene, and it is confirmed by an analysis of the corrections to the interaction vertex beyond the random phase approximation (RPA). Furthermore, we show that the acoustic branch of out-of-plane phonons presents near the $K$ point a strong Kohn anomaly, which is much more pronounced than in the case of the in-plane phonons. The origin of the strong softening of the dispersion lies in the singular behaviour of the intervalley polarization at the threshold of electron–hole formation. This leads to a new branch of hybrid modes below the electron–hole continuum, with the potential to induce significant effects in the transport properties of graphene in the low-temperature regime.

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1. Introduction

The feasibility of isolating single layers of graphite [1]–[3] has led to a great deal of activity, as the new two-dimensional (2D) material (so-called graphene) has shown a number of remarkable electronic properties and potential applications [4]. Some of these have to do with the fact that the propagation of electrons in graphene is largely insensitive to the disorder of the samples, especially when that comes from impurities or defects with sizes larger than the C–C distance [5]. It has also been shown that the electrons should not feel much disturbance in their transmission through potential barriers in the case of normal incidence [6], illustrating the Klein paradox originally proposed for relativistic electrons. These properties actually arise from the peculiar low-energy dispersion of the electron system, properly described by a couple of Dirac fermions that account for conical valence and conduction bands meeting at the so-called Dirac points in the corners of the Brillouin zone.

From the point of view of many-body theory, the graphene quasiparticles have also an unusual stability. In the undoped system, the particular kinematic constraints of the massless dispersion actually prevent their decay into electron–hole pairs [7]. The threshold for the appearance of interband excitations has a linear dependence on their momentum $q$, with a slope given by the Fermi velocity $v_F$. On the other hand, $v_F|q|$ is also the maximum energy that can be released in the scattering of a quasiparticle with momentum transfer $q$. This means that, only in cases where the Coulomb interaction remains singular in the limit $q \to 0$, the usual computation of the quasiparticle decay rate from the $e–e$ interaction may lead to a nonvanishing result [8].

The current belief is that, in present experimental samples of graphene on a substrate, the main source of quasiparticle scattering comes from impurities placed away from the 2D layer. Typical substrates have in particular charged scatterers, whose long-range potential has a relevant effect on the quasiparticle decay. It has been proposed that these charged impurities have to be responsible for the linear dependence of the conductivity on carrier concentration [9]–[11]. Alternatively, if one were able to get rid of this kind of extrinsic disorder, there is the hope that the relevant quasiparticle scattering could come only from low-energy phonons. It is known, for instance, that in-plane acoustic phonons lead in graphene to a quasiparticle decay rate proportional to the Fermi energy and the temperature $T$ that translates into a linear dependence on $T$ of the resistivity [12]. This behaviour seems to be consistent with the experimental measures carried out in graphene on SiO$_2$, after subtracting the contribution of
the extrinsic-charged impurities [13]. The same analysis has also shown that the resistivity enters a different regime above a certain crossover temperature, with a marked nonlinear dependence on $T$ near the charge neutrality point.

The genuine 2D character of graphene also opens the possibility of having out-of-plane vibrations of the carbon lattice. It has been shown that their coupling to the in-plane phonons is what makes possible the existence of finite fluctuations normal to the 2D plane [14], with the consequent stability of the carbon layer at low temperature. The acoustic branch of out-of-plane phonons starts with a quadratic dependence of their energy on momentum, with a typical energy range related to the scale of the top of the optical branch at $\sim 0.1$ eV. These low-energy phonons may have a significant effect on quasiparticle scattering in graphene [15]. Their contribution to the quasiparticle decay rate is however attenuated by the fact that, in an ideal isolated graphene layer, the electron–phonon coupling for the out-of-plane vibrations has to be invariant under the reflection with respect to the graphene plane, thus implying a quadratic dependence on the phonon field. Recently, it has also been shown that these out-of-plane modes may suffer significant damping due to the effect of disorder and Coulomb interactions [16].

In this paper, we study the properties of the out-of-plane phonons in the framework of the many-body theory of graphene. Our main motivation is to clarify to what extent the very low-energy phonon modes are dressed by their coupling to electronic excitations. We will start by introducing the electron–phonon couplings through the conventional concept of deformation potential [17]. Then, we will investigate the way in which the coupling to electron–hole excitations renormalizes the dispersion of the acoustic branch of out-of-plane phonons. We will find out that the softness of these modes makes them very prone to being destabilized by the coupling to electron–hole excitations. One of the main results we obtain is that the quadratic dispersion of the acoustic branch turns out to be cut off at small momenta, implying the absence of long-wavelength flexural phonons. This conclusion is validated in the low-energy Dirac theory of graphene beyond the random phase approximation (RPA), after summing the dominant vertex corrections to the electron–phonon coupling.

The other relevant result is that, for large momenta close to the $K$ point at the corner of the Brillouin zone, the dispersion of the acoustic branch is also strongly renormalized, being softened in a much more pronounced way than in typical Kohn anomalies of the in-plane phonons [18, 19]. The origin of this effect lies in the singular behaviour of the intervalley polarization in graphene at the threshold of electron–hole formation. We will see that, at momenta connecting the two Dirac valleys in the $\pi$ bands, the coupling of the out-of-plane phonons to electron–hole pairs actually leads to a new branch of hybrid modes below the electron–hole continuum, with the potential to induce significant effects in the transport properties of graphene at very low energies [20].

### 2. Coupling of out-of-plane phonons to electron excitations

We first study the coupling of out-of-plane phonons to electrons in graphene, adopting a tight-binding approach. We take as a starting point the representation of the Bloch wavefunction for momentum $\mathbf{k}$ as a superposition of atomic wavefunctions $\phi(r)$ for the graphene $\pi$ orbitals

$$
\Psi_k(r) = \frac{1}{\sqrt{N}} \sum_{\alpha} \psi_{\alpha}(k) \sum_{\pi} e^{i\mathbf{k} \cdot \mathbf{R}_{\alpha}} \phi(r - \mathbf{R}_{\alpha}),
$$

(1)
\[ V(\mathbf{r}) = \sum_{n\alpha} v(\mathbf{r} - \mathbf{R}_{n\alpha}). \]  

In the tight-binding approximation, its matrix elements between Bloch states then take the form

\[ \langle \Psi_k(\mathbf{r})|V(\mathbf{r})|\Psi_k(\mathbf{r})\rangle = \frac{1}{N} \sum_{\alpha,\alpha'} \psi_{\alpha'}^*(\mathbf{k}') \psi_{\alpha}(\mathbf{k}) \sum_{n,n'} e^{i(\mathbf{k} \cdot \mathbf{R}_{n\alpha'} + \mathbf{k} \cdot \mathbf{R}_{n\alpha})} v(n'\alpha', n\alpha), \]  

where the atoms labelled by \( n\alpha \) and \( n'\alpha' \) are at least nearest neighbours and the atomic matrix element

\[ v(n'\alpha', n\alpha) = \int d^2r \phi^*(\mathbf{r} - \mathbf{R}_{n'\alpha'}) \sum_{n''\alpha''} v(\mathbf{r} - \mathbf{R}_{n''\alpha''}) \phi(\mathbf{r} - \mathbf{R}_{n\alpha}) \]  

is such that \( \mathbf{R}_{n'\alpha''} = \mathbf{R}_{n\alpha} \) or \( \mathbf{R}_{n'\alpha''} = \mathbf{R}_{n'\alpha'} \) when \( \mathbf{R}_{n\alpha} \neq \mathbf{R}_{n'\alpha'} \), or otherwise the atom at \( \mathbf{R}_{n'\alpha''} \) must be a nearest neighbour of that at \( \mathbf{R}_{n\alpha} = \mathbf{R}_{n'\alpha'} \).

The electron–phonon couplings can be obtained in this approach from the variation of the potential energy caused by the vibration of the carbon lattice. This gives rise to a deviation \( \mathbf{S}(\mathbf{R}_{n\alpha}) \) of each carbon atom with respect to its equilibrium position at \( \mathbf{R}_{n\alpha} \), which modifies in turn the potential by

\[ \delta V(\mathbf{r}) = -\sum_{n\alpha} \nabla v(\mathbf{r} - \mathbf{R}_{n\alpha}) \cdot \mathbf{S}(\mathbf{R}_{n\alpha}) + \frac{1}{2} \sum_{n\alpha} \mathbf{S}(\mathbf{R}_{n\alpha}) \cdot \nabla \nabla v(\mathbf{r} - \mathbf{R}_{n\alpha}) \cdot \mathbf{S}(\mathbf{R}_{n\alpha}) + \cdots \]  

The lattice vibration creates then a so-called deformation potential, which receives two different contributions in the tight-binding approximation. The on-site deformation potential \( \delta v_{\text{on}} \) comes from taking the atomic matrix element (4) with the orbitals at the same carbon atom,

\[ \delta v_{\text{on}} = -\int d^2r \phi(\mathbf{r}) \nabla v(\mathbf{r} - \mathbf{R}_{n'\alpha'}) \cdot (\mathbf{S}(\mathbf{R}_{n'\alpha''} + \mathbf{R}_{n\alpha}) - \mathbf{S}(\mathbf{R}_{n\alpha})) \phi(\mathbf{r}) + \frac{1}{2} \int d^2r \phi(\mathbf{r}) (\mathbf{S}(\mathbf{R}_{n'\alpha''})) \cdot \nabla \nabla v(\mathbf{r} - \mathbf{R}_{n'\alpha''} + \mathbf{R}_{n\alpha}) \cdot (\mathbf{S}(\mathbf{R}_{n'\alpha''} + \mathbf{R}_{n\alpha}) - \mathbf{S}(\mathbf{R}_{n\alpha})) \phi(\mathbf{r}) + \cdots \]  

while the off-site deformation potential \( \delta v_{\text{off}} \) is given by the matrix element between nearest-neighbour carbon atoms \( \mathbf{R}_{n\alpha} \) and \( \mathbf{R}_{n'\alpha'} \)

\[ \delta v_{\text{off}} = -\int d^2r \phi(\mathbf{r} - \mathbf{R}_{n'\alpha'}) (\nabla v(\mathbf{r} - \mathbf{R}_{n'\alpha'}) \cdot \mathbf{S}(\mathbf{R}_{n'\alpha''}) + \nabla v(\mathbf{r} - \mathbf{R}_{n\alpha}) \cdot \mathbf{S}(\mathbf{R}_{n\alpha})) \phi(\mathbf{r} - \mathbf{R}_{n\alpha}) + \cdots \]  

In order to obtain the expression of the electron–phonon couplings, we may write the displacement \( \mathbf{S}(\mathbf{R}_{n\alpha}) \) in terms of the polarization vector \( \mathbf{e}_\alpha^{(v)}(\mathbf{q}) \) for a given phonon branch \( v \),

\[ \mathbf{S}(\mathbf{R}_{n\alpha}) = \frac{1}{\sqrt{\rho \omega_0^{(v)}(\mathbf{q})}} e^{i\mathbf{q} \cdot \mathbf{R}_{n\alpha}} \mathbf{e}_\alpha^{(v)}(\mathbf{q}), \]  

where \( \rho \) is the mass density and \( \omega_0^{(v)}(\mathbf{q}) \) is the frequency of the phonon. Computing now the matrix element \( \langle \Psi_k(\mathbf{r})|\delta V(\mathbf{r})|\Psi_k(\mathbf{r})\rangle \), we extract the diagonal electron–phonon couplings \( g_{\alpha,\alpha} \)
from the on-site deformation potential

\[ g_{\alpha,\alpha}(k + q, k) = \frac{1}{\sqrt{\rho \omega_0(q)}} \psi_{\alpha}^*(k + q) \psi_{\alpha}(k) \sum_{j=1}^{3} (e^{iq\cdot u_j} e_{\alpha'}(q) - e_{\alpha}(q)) \cdot d(u_j), \tag{9} \]

\[ g^{(2)}_{\alpha,\alpha}(k + q + q', k) = -\frac{1}{\sqrt{\rho \omega_0(q)}} \frac{1}{\sqrt{\rho \omega_0(q')}} \psi_{\alpha}^*(k + q + q') \psi_{\alpha}(k) \]

\[ \times \sum_{j=1}^{3} (e^{iq'\cdot u_j} e_{\alpha'}(q') - e_{\alpha}(q')) d^{(2)}(u_j) (e^{iq\cdot u_j} e_{\alpha'}(q) - e_{\alpha}(q)), \tag{10} \]

where \( u_j \) are the vectors connecting the atom at \( R_{na} \) with its nearest neighbours, and

\[ d_\alpha(u) = \int d^2r \phi(r) \nabla v(r - u) \phi(r), \]
\[ d^{(2)}_{\alpha\beta}(u) = \int d^2r \phi(r) \nabla_{\alpha} \nabla_{\beta} v(r - u) \phi(r). \tag{11} \]

We will concentrate henceforth on the effects deriving from the on-site deformation potential, which exerts in general a stronger influence than the off-site counterpart. Regarding, in particular, the out-of-plane phonons, the relative significance of the couplings \( g \) and \( g^{(2)} \) may be different depending on the experimental conditions. In the case of suspended graphene samples, the deformation potential \( d \), as computed from the \( \pi \) carbon orbitals, is parallel to the carbon layer. Then, only the term \( (10) \) is relevant to describe the coupling of electrons to out-of-plane phonons. The picture is different, however, when graphene is lying on a substrate, as this always spoils to some extent the symmetry of the orbitals under the reflection with respect to the layer. In these conditions, the deformation potential \( d \) will get some component in the normal direction. If the graphene layer is strongly pinned to the substrate, flexural phonons may only arise for suitably short wavelengths, and the corresponding modes will couple through the one-phonon term \( (9) \). In instances where graphene is more loosely bound to the substrate, the two-phonon contribution \( (10) \) may however prevail. In what follows, we will investigate these two possibilities, assuming that the dominant coupling may be either \( g \) or \( g^{(2)} \) depending on the sample conditions.

3. Flexural phonons

3.1. Renormalization of flexural phonons by electron excitations

The softness of the bare dispersion of out-of-plane acoustic phonons means that they can be easily perturbed by the coupling to electronic excitations. The bare propagator of the phonons is

\[ D^{(0)}(q, \omega_q) = \frac{2 \omega_0(q)}{\omega_q^2 - \omega_0^2(q) + i\epsilon}, \tag{12} \]

with \( \omega_0(q) \approx \alpha q^2 \) for flexural modes. The dressed propagator \( D(q, \omega_q) \) must satisfy, in general, the self-consistent equation

\[ D(q, \omega_q) = D^{(0)}(q, \omega_q) + D^{(0)}(q, \omega_q) \Pi(q, \omega_q) D(q, \omega_q), \tag{13} \]
Figure 1. Lowest-order diagrams giving the momentum dependence of the phonon self-energy in models where the electron–phonon couplings depend (a) linearly and (b) quadratically on the phonon modes. Full and dashed lines stand for the propagation of electron and phonon fields, respectively.

where $\Pi(q, \omega_q)$ is the phonon self-energy. We will start by discussing the renormalization of the phonon dispersion in the RPA framework. We will make a separate discussion depending on whether the electron–phonon interaction is dominated by the one-phonon or the two-phonon coupling listed in equations (9) and (10).

We face first the situation where graphene is attached to some substrate, assuming that the coupling $g$ may have greater strength than $g^{(2)}$. The dominant contribution to the phonon self-energy will be given then by the diagram shown in figure 1(a). This corresponds essentially to the electron–hole polarization $\chi(q, \omega_q)$. In the RPA, we compute it in terms of the Coulomb potential $V_C(q) = e^2/2\kappa q$ and the bare electron–hole polarization $\chi(0)(q, \omega_q)$ as

$$\chi(q, \omega_q) = \frac{\chi(0)(q, \omega_q)}{1 - V_C(q)\chi(0)(q, \omega_q)}.$$  \hfill (14)

Combining (13) and (14), we obtain the propagator of the out-of-plane phonons dressed by any number of electron–hole bubbles:

$$D_{\text{RPA}} = \frac{D^{(0)}}{1 - D^{(0)} g^2 \chi^{(0)}(q, \omega_q)}.$$  \hfill (15)

The electron quasiparticles give rise in undoped graphene to a bare electron–hole polarization around each Dirac point [21]

$$\chi^{(0)}(q, \omega_q) = -\frac{q^2}{4\sqrt{v_F^2 q^2 - \omega_q^2}}.$$  \hfill (16)

The energy of the out-of-plane acoustic phonons is much lower than that of the electronic excitations. Then, for the purpose of finding the corrections to the phonon dispersion, we can set $\omega_q = 0$ in the polarization (16). The dispersion of the dressed phonons is obtained from the poles of the propagator (15). Thus, we obtain the equation

$$\omega_q^2 - \omega_0^2(q) + \omega_0(q) \frac{g^2}{v_F} \frac{|q|}{2 \left(1 + \frac{e^2}{8\kappa v_F}\right)} = 0.$$  \hfill (17)
We see that the self-energy corrections may give rise to a significant change in the phonon spectrum, but this will depend on the particular conditions of the graphene samples. In an ideally flat graphene layer, the coupling \( g \) would have a quadratic dependence on \( q \) at low momentum, apart from its dependence on phonon frequency, as follows from (9). However, real samples have distortions in the carbon lattice (ripples), inducing slight variations in the distance between nearest-neighbour atoms. Thus, \( g \) will pick up in general contributions that are nonvanishing in the limit \( q \to 0 \), and proportional to the square of the ratio between the C–C distance \( a \) and the local radius of curvature. Though this ratio may be small, these contributions will appear as the most relevant in (17), as they introduce the lowest power of \( q \) in the renormalization of the phonon dispersion. Assuming a momentum-independent average of \( g \) in real graphene samples in the limit \( q \to 0 \),

\[
g \approx \bar{g} - \frac{a|q|}{\omega_0(q)},
\]

we get from (17) the condition

\[
\omega_q^2 - \alpha^2 q^4 + \frac{\bar{g}^2}{2} \frac{a^2 |q|^3}{v_F} \frac{1}{1 + \frac{e^2}{8 \kappa v_F}} = 0.
\]

We see that equation (19) does not have a solution at small \( q \). This is actually consistent with the idea that graphene layers strongly bound to the substrate cannot have long-wavelength out-of-plane vibrations. A pole in the phonon propagator is only found for momenta

\[
|q| > \bar{g}^2 \frac{a^2}{2 \beta v_F \omega_0^2},
\]

with \( \beta = 1 + e^2 / 8 \kappa v_F \). We see, however, that the condition (20) pushes in general the momentum into a regime where the Dirac theory for graphene quasiparticles is not valid anymore. We can estimate the range of momenta satisfying (20) by recalling that the parameter \( \alpha \) is given in a tight-binding model (see equation (39) below) by

\[
\alpha = \frac{3}{4} \omega_0 a^2,
\]

where the energy scale \( \tilde{\omega}_0 \) must be such that the top of the optical branch corresponds to \( 6 \tilde{\omega}_0 \). We find that the minimum momentum \( q_c \) needed to satisfy (20) is anyhow a significant fraction of \( 1/a \). A precise determination of such a critical momentum requires therefore the computation of the polarization over the whole Brillouin zone, as will be undertaken later on.

We turn next to the situation where the graphene layer is not pinned to the substrate, so that the relevant electron–phonon interaction is given by the two-phonon coupling. To lowest order in \( g^{(2)} \), the momentum dependence of the phonon self-energy arises from the diagram represented in figure 1(b). This is now the convolution of the bare phonon propagator with the electron–hole polarization. By defining the parameter \( \bar{g}_2 \) in the limit of small momenta \( q \) and \( k \) of the phonons by

\[
g^{(2)} \approx \bar{g}_2 \frac{a^2 q \cdot k}{\sqrt{\omega_0(q)} \sqrt{\omega_0(k)}},
\]

we can approximate

\[
\Pi(q, \omega_q) \approx i \bar{g}_2^2 \frac{2}{\omega_0(q)} \int \frac{d^2 k}{(2\pi)^2} \int \frac{d\omega_k}{2\pi} a^4 (q \cdot k)^2 \frac{2}{\omega_k^2 - \omega_0^2(k) + i\epsilon} \chi(q - k, \omega_q - \omega_k).
\]
Using again the RPA expression (14) for the electron–hole polarization, we obtain for the real part of the self-energy

\[ \text{Re } \Pi(q, \omega_q) \approx -\frac{1}{16\pi^2} \tilde{g}_2^2 \frac{a^4 q^2}{\beta v_F \omega_0(q)} \int \frac{d^2 k}{\omega_0(k)} \cos^2(\phi) |q-k|, \] (24)

\( \phi \) being the angle between \( k \) and \( q \).

The renormalized phonon dispersion is obtained again from the zeros in the denominator of the propagator, which are given by

\[ \omega_q^2 - \omega_0^2(q) - 2\omega_0(q) \Pi(q, \omega_q) = 0. \] (25)

From this equation, we get to lowest order in \( q \)

\[ \omega_q^2 \approx -\mu^2 q^2 + \alpha^2 q^4, \] (26)

which implies that the phonon spectrum must be cut off at some critical momentum \( q_c \approx \mu/\alpha \).

The value of \( \mu \) can be obtained self-consistently, by computing the phonon self-energy with renormalized phonon frequencies

\[ \omega(q) \approx \sqrt{-\mu^2 q^2 + \alpha^2 q^4}. \] (27)

The quadratic corrections to the dispersion lead then to the equation

\[ \mu^2 = \frac{1}{8\pi \beta v_F} \int_{\mu/\alpha}^{k_c} \frac{a^4 k^3}{\sqrt{a^2 k^2 - \mu^2}} \] (28)

We observe that the integral in (28) depends on the high-energy cutoff limiting the range of the Dirac theory for graphene quasiparticles. To estimate the order of magnitude of \( \mu \), we will simply assume that \( k_c \) is some fraction of the large momentum \( 1/a \). Thus, we obtain from (28)

\[ \left( \frac{\mu}{\alpha} \right)^2 = \frac{r^3}{24\pi \beta v_F} \tilde{g}_2^2 \left( \frac{\mu}{\alpha} \right)^2 \left( k_c^2 + 2 \left( \frac{\mu}{\alpha} \right)^2 \right), \] (29)

where we have introduced the dimensionless ratio \( r = a v_F/\alpha \). This is a large parameter, of the order of \( \sim 10^2 \). Then, for reasonable values of the dimensionless coupling \( \tilde{g}_2/v_F^2 \), we always find that \( \mu/\alpha \) is actually of the same order of the high-energy cutoff \( k_c \). In these conditions, we can assert that there are no out-of-plane acoustic phonon modes within the low-energy Dirac theory of graphene, when this is investigated with the RPA.

Two questions arise then from the above conclusion. First, one may ask to what extent the equations (19) and (28) are modified when corrections to the polarization are taken into account beyond the RPA. Moreover, if there is some minimum momentum \( q_c \) for the phonon modes that lies beyond the range of validity of the low-energy Dirac theory, there is the question of its determination taking into account the whole electron dispersion over the Brillouin zone. We will conveniently address these problems in the two following subsections.

### 3.2. Renormalization beyond the RPA

The idea is to replace \( \chi^{(0)}(q, \omega_q) \) in the above analysis by an expression accounting at least by the dominant corrections to the bare polarization arising from the Coulomb interaction. As long as the Coulomb potential remains singular (as in undoped graphene), it gives rise to a number of diagrams that are enhanced by logarithmic factors with respect to the rest of contributions at each perturbative level [22]. The dominant contributions are given by the iteration of the...
scattering between the electron and the hole in the bubble diagram of the bare polarization, which gives rise to the ladder series depicted in figure 2. Then, it makes sense to sum all these dominant contributions, to see the effect of this nonperturbative approach for the one-particle irreducible polarization.

The analysis can be undertaken more precisely beginning with the irreducible vertex represented in figure 3, which we will consider for equal frequency of the in and out electrons. We will focus on the real part of that object, $\Gamma(k; q)$, computed when the in states are for instance in the valence band (below the Dirac point) and the out states are in the conduction band (above the Dirac point). We will denote the states with momentum $k$ in the lower band by $|k, -\rangle$, and those in the upper band by $|k, +\rangle$. It can be checked, in particular, that $\langle k, - | k + q, + \rangle \sim O(|q|)$ in the limit of small $q$. Thus, the first correction to the vertex from the Coulomb interaction has the leading behaviour at low momentum-transfer $q$ [23]

$$\Gamma_1(k; q) \approx \frac{e^2}{\kappa v_F} \int \frac{d^2p}{(2\pi)^2} \left| \frac{\langle p, - | p + q, + \rangle}{|p| + |p + q|} \right| \frac{1}{2|p - k|} \approx \frac{e^2}{64 \kappa v_F} \frac{|q|}{|k|}. \quad (30)$$

The important point is that, if we extract the leading $|q|$ behaviour as a common factor, there are diagrams that can be singled out at each level of the perturbative expansion, as they are enhanced by factors that diverge logarithmically in the limit $q \to 0$. Looking at the corrections $\Gamma_2(k; q)$ of order $e^4$, it can be checked that the ladder diagram gives among all of them the
dominant contribution. Always computing the leading behaviour as $q \to 0$, we find

$$\Gamma_2(k; q) \approx \frac{e^2}{\kappa v_F} \int \frac{d^2 p}{(2\pi)^2} \Gamma_1(p; q) \frac{1}{|p| + |p + q|} \frac{1}{2|p - k|}$$

$$\approx \frac{1}{64 \kappa v_F} \frac{e^2}{8\pi \kappa v_F} |q| \log \left( \frac{8|k|}{|q|} \right).$$

(31)

We can use (31) to compute in turn the corrections at the next level in perturbation theory, $\Gamma_3(k; q)$. The dominant contribution is given now by the ladder diagram of order $e^6$, which is enhanced by a factor $\log^2(8|k|/|q|)$. We have actually

$$\Gamma_3(k; q) \approx \frac{e^2}{\kappa v_F} \int \frac{d^2 p}{(2\pi)^2} \Gamma_2(p; q) \frac{1}{|p| + |p + q|} \frac{1}{2|p - k|}$$

$$\approx \frac{1}{64 \kappa v_F} \left( \frac{e^2}{8\pi \kappa v_F} \right)^2 |q| \log^2 \left( \frac{8|k|}{|q|} \right).$$

(32)

It is clear that this procedure to build the leading contribution at each perturbative level can be followed recursively. Moreover, the structure of the increasing degree of logarithmic divergence of the ladder diagrams is such that the ladder series can be easily summed up. In this leading logarithm approach, we find an approximation $\Gamma_{ladder}(k; q)$ to the vertex that takes the form (for $|q| \ll |k|$)

$$\Gamma_{ladder}(k; q) \approx \langle k, -|k + q, + \rangle + \frac{1}{64 \kappa v_F} \frac{|q|}{|k|} \frac{1}{\log \left( \frac{8|k|}{|q|} \right)}.$$

(33)

We can adopt this approach to carry out a similar sum of the ladder diagrams for the charge polarization. This amounts to taking the expression (33) for one of the vertices in the calculation of $\chi^{(0)}(q, 0)$. We thus obtain an approximation $\chi_{ladder}(q, 0)$ for the polarization in the limit of small $q$

$$\chi_{ladder}(q, 0) \approx -\frac{1}{4v_F} |q| - \frac{1}{64v_F} |q| \frac{e^2}{\kappa v_F} \sum_{s=\pm} \int \frac{d^2 p}{(2\pi)^2} \frac{(p + q, s) |p, -s \rangle}{|p| + |p + q|} \frac{1}{|p|} \frac{1}{\log \left( \frac{8|p|}{|q|} \right)}.$$

(34)

While the leading logarithm approximation for the vertex (33) leads to a singularity at $(e^2/(8\pi \kappa v_F)) \log(8|k|/|q|) = 1$, this is no obstacle for the calculation of the polarization, since the integral on the right-hand side of equation (34) can be defined by its principal value. We observe then that the corrections to the bare polarization $-|q|/4v_F$ range from negative values for $e^2/(4\pi \kappa v_F) < 1.1$ to positive values for the maximum couplings that can be reached in graphite in the absence of dielectric screening, $e^2/(4\pi v_F) \approx 2.2$. A plot of the corrections to the bare polarization is shown in figure 4. It is remarkable that, close to the typical couplings $e^2/(4\pi \kappa v_F) \approx 1$ for experimental samples of graphite on a substrate, the corrections change from one regime to the other, actually vanishing at some particular coupling constant. This means that, in general, we cannot expect significant corrections to the bare polarization from correlation effects in graphite. It is only in the case of suspended graphite, where the screening effects are greatly minimized, that the above ladder corrections can lead to a reduction of about 30% of the bare polarization.
Figure 4. Plot of the ladder series approximation to the polarization relative to $\chi^{(0)}(q, 0)$ in the limit $q \to 0$, as a function of the effective strength $e^2/(4\pi \kappa v_F)$ of the Coulomb interaction.

It is clear then that, after summing the dominant corrections to the polarization $\chi^{(0)}(q, 0)$ from ladder diagrams, its momentum dependence still has a leading behaviour $\approx -c|q|/v_F$, with a coefficient $c$ slightly different from $\frac{1}{2}$. In these conditions, we find that the strong renormalization of the out-of-plane phonon dispersion also holds in this nonperturbative framework beyond the RPA. This reinforces the conclusion that the flexural phonon modes should be suppressed at long wavelengths, as the approach we have developed above is rigorously valid in the limit of small momentum-transfer $q$.

3.3. Renormalization beyond the Dirac-cone approximation

It would be interesting to have a measure of the effect of the electron–hole excitations on the phonon dispersion beyond the region of small momenta where the conical dispersion for electron quasiparticles applies in graphene. This is particularly important to assert the existence of a momentum $q_c$ above which the out-of-plane acoustic phonon branch opens up. Equation (19) simply tells us that this may happen at some momentum given by a fraction of the inverse of the C–C distance $a$. For reasonable values of the electron–phonon coupling $g$, this places the value of $q_c$ away from the regime where the low-energy description in terms of Dirac fermions is valid in graphene. A similar problem arises in the model with the two-phonon coupling $g^{(2)}$, as we have seen that the critical momentum obtained from (29) scales with the high-energy cutoff of the Dirac theory.

We turn then to the computation of the electron–hole polarization over the Brillouin zone, starting from the general expression for undoped graphene

$$\chi^{(0)}(q, 0) = \frac{d^2 p}{(2\pi)^2} \frac{|\langle p, -|p+q, +\rangle|^2}{E^-(p) - E^+(p+q)},$$

(35)

where $E^\pm(p)$ corresponds to the dispersion of the $\pi$ and $\pi^*$ bands of graphene

$$E^\pm(p) = \pm t \sqrt{1 + 4 \cos^2(\sqrt{3}ap_x/2) + 4 \cos(3ap_x/2) \cos(3ap_y/2)},$$

(36)

and $|p, \pm\rangle$ represent the respective eigenstates in the two bands. Our purpose is to use (35) to determine the renormalization of the phonon dispersion, following some preferred direction.
from the centre to the boundary of the Brillouin zone. The connection with the result for $\chi^{(0)}(q, 0)$ in the low-energy Dirac theory is straightforward, since the computation of (35) in the limit $q \to 0$ leads to an expression that is not sensitive to the high-energy details of the bands. Placing a cutoff $\Lambda$ in the integral over $p$ and approximating $E^\pm(p) \approx \pm v_F|p|$ at the Dirac points, we obtain

$$
\chi^{(0)}(q, 0) = -\frac{|q|}{\pi^2 v_F} \int_0^{\Lambda/v_F|q|} dx \int_0^{2\pi} d\phi x \left( 1 - \frac{x + \cos \phi}{\sqrt{1 + x^2 + 2x \cos \phi}} \right) \left( \frac{1}{x + \sqrt{1 + x^2 + 2x \cos \phi}} \right)
$$

$$
\approx -\frac{|q|}{4v_F}, \quad \mathrm{for} \quad v_F|q| \ll \Lambda.
$$

We will see in the next section that the polarization about the large momentum from $\Gamma$ to $K$ gets instead a contribution from the high-energy scale of the bands, though the frequency and momentum dependences can be still computed in the framework of the low-energy Dirac theory.

The result of computing the electron–hole polarization using equation (35) has been represented along the $\Gamma K$ line in figure 5. The polarization turns out to have a monotonous trend in that direction, adopting an approximate linear behaviour around the $\Gamma$ and the $K$ point. The knowledge of $\chi^{(0)}(q, 0)$ can be used, together with the form of the bare phonon dispersion $\omega_0(q)$ over the Brillouin zone, to estimate the value $q_c$ at which the branch of flexural phonons opens up.

We remark at this point that a sensible approximation to $\omega_0(q)$, reproducing the quadratic dependence on $q$ around the $\Gamma$ point, must rely on a suitable choice for the potential energy $V_{ph}$ of the lattice vibrations. A convenient form of the potential is

$$
V_{ph} = \frac{1}{2} m_C \bar{\omega}_0^2 \sum_{n\alpha} \left( \sum_{j=1}^3 S(R_{n\alpha} + u_j) - 3S(R_{n\alpha}) \right)^2,
$$

Figure 5. Plot of the polarization $\chi^{(0)}(q, 0)$ (full line) as a function of momentum, from $\Gamma$ to the corner $K$ of the Brillouin zone. The dashed line corresponds to the linear behaviour of the polarization $\chi^{(0)}(q, 0)$ in the low-energy Dirac theory.
where $S(R_{mn})$ are the normal displacements of the carbon atoms, in the notation of section 2. Such a potential leads to a dispersion for the out-of-plane acoustic branch given by

$$\omega_0(q) = \tilde{\omega}_0 \left( 3 - \sqrt{1 + 4 \cos^2(\sqrt{3}a_p x / 2) + 4 \cos(\sqrt{3}a_p y / 2) \cos(3a_p y / 2)} \right).$$ (39)

At the $\Gamma$ point we recover the behaviour $\omega_0(q) \approx (3/4) \tilde{\omega}_0 a^2 q^2$, while at the $K$ point we have $\omega_0(q) \approx 3\tilde{\omega}_0 (1 - a|q|/2)$.

We can now rely on the RPA expression (15) to obtain the dressed phonon propagator from the polarization (35) and the bare phonon dispersion (39). The renormalized phonon dispersion $\omega(q)$ is found by looking again for the poles of the phonon propagator in the $\omega$ variable. We get in this way the momentum dependence represented in figure 6 for the acoustic phonon branch. We observe that this opens up at a finite momentum $q_c$ which is, in general, of the order of $\sim 1/a$, as anticipated by the analysis within the low-energy Dirac theory. The results refer to the model with the one-phonon coupling $g$, which is appropriate for the description of graphene on a substrate. Such a coupling should give then the relevant electron–phonon interaction in the short-wavelength regime where $q_c$ is found. Otherwise, we have taken the coupling $g$ as conveniently small but constant, bearing in mind the assumption that the small distortions of the carbon lattice (ripples) prevent it from vanishing at small momentum. The results in figure 6 provide evidence that the strong renormalization of the acoustic branch of out-of-plane phonons is not an effect requiring a large electron–phonon coupling. This supports the conclusion that flexural phonons may exist in graphene, but are suppressed above a certain critical wavelength where the dispersion is cut off due to the coupling to electron excitations.

4. Out-of-plane phonons at the $K$ point of graphene

In the preceding section it was enough to consider the momentum dependence of the static charge polarization for the purpose of assessing the corrections to the phonon dispersion, given the large difference between the energy scale of the phonons and the threshold for electron–hole

Figure 6. Plot of the bare phonon dispersion $\omega_0(q)$ given by (39) (dashed line) and the renormalized dispersion obtained from the dressed propagator (15) for momenta from $\Gamma$ to the $K$ point. The different full curves correspond, from top to bottom, to values of the effective electron–phonon coupling $g^2/v_F^2$ equal to 0.01, 0.02 and 0.03, with a common effective strength of the Coulomb interaction $e^2/4\pi \kappa v_F = 1.0$. 
creation. As the momentum approaches the corner of the Brillouin zone, however, a new branch of electron–hole excitations arises, with energy vanishing at the \( K \) point. This gives rise to interesting physics, since there is now a range where the energy of the phonons and that of the electron–hole pairs become comparable, so that new states may appear from the hybridization of the two kinds of excitations. The relevant effect turns out to be again a strong renormalization of the phonon dispersion, which can be understood from the development of a branch of bound states of phonon and electron–hole pair near the \( K \) point of graphene.

In order to find the new set of electron–hole excitations, we need to know the frequency and momentum dependence of the charge polarization near the \( K \) point. This object can be computed going back again to the low-energy Dirac theory of graphene, since the momentum-transfer creating the electron and the hole actually connects two Dirac valleys in the graphene \( \pi \) bands. Anyhow, the form of the polarization near the \( K \) point cannot coincide with the dependence (16) obtained within a given Dirac valley, as the expressions of the spinors corresponding to different valleys are not the same. Thus, if the Dirac Hamiltonian for the states in a given Dirac cone is given by

\[
H_0 = v_F \int d^2k \Psi^{(1)}(k)^\dagger \gamma^{(1)} \cdot k \Psi^{(1)}(k),
\]

with \( \gamma^{(1)} \equiv (\sigma_x, \sigma_y) \), the Dirac Hamiltonian for the states in the opposite cone in the Brillouin zone must be

\[
H'_0 = v_F \int d^2k \Psi^{(2)}(k)^\dagger \gamma^{(2)} \cdot k \Psi^{(2)}(k),
\]

with \( \gamma^{(2)} \equiv (\sigma_x, \sigma_y) \) [24]. This leads to a relevant modification in the overlap of the electron and hole states with respect to the expression \((p, -|p+q, +)\) used before within a single Dirac cone.

The frequency and momentum dependence of the polarization can be most conveniently computed using the Dirac propagators derived from (40) and (41). We also have to take into account that the coupling of the Dirac fermions to the out-of-plane acoustic phonons is given by the identity matrix in the space of two-component spinors. About the large momentum \( Q \) connecting the \( \Gamma \) and \( K \) points, we then have

\[
\text{Re} \chi^{(0)}_K(Q+q, \omega_q) = 4\pi \text{Tr} \int \frac{d^2p}{(2\pi)^2} \frac{d\omega_p}{2\pi} G^{(1)}(q-p, \omega_q-\omega_p) G^{(2)}(p, \omega_p),
\]

where \( G^{(a)}(p, \omega_p) = 1/(\omega_p - v_F \gamma^{(a)} \cdot p + i\epsilon) \).

The real part of \( \chi^{(0)}_K(Q+q, \omega_q) \) becomes

\[
\text{Re} \chi^{(0)}_K(Q+q, \omega_q) = 2\pi \left[ \frac{d^2p}{(2\pi)^2} \frac{d\omega_p}{2\pi} \frac{\omega_q - \omega_p + v_F \sigma \cdot (q-p)}{(\omega_q - \omega_p)^2 - v_F^2(q-p)^2} \right] \delta(\omega_p^2 - v_F^2p^2)
\]

\[
= 2\pi \sum_{s=\pm} \int \frac{d^2p}{(2\pi)^2} \frac{1}{|p|^2} \left[ \frac{p^2 + (q_x - p_x) p_x - (q_y - p_y) p_y}{q^2 - \omega_q^2/v_F^2 - 2q \cdot p + s2\omega_q |p|/v_F} \right].
\]

From equation (43), it becomes evident that the dependence of \( \chi^{(0)}_K(Q+q, \omega_q) \) is not isotropic with respect to the small deviations \( q \) about the large momentum \( Q \). This is possible as the vector \( Q \) already sets a preferred direction in momentum space. The anisotropy of the polarization can be also checked from the numerical computation of \( \chi^{(0)}_K(q, 0) \) using equation (35), when \( q \) goes...
polarization must reach a nonvanishing value at the large momentum

(44) around the corner \( K \) of the Brillouin zone. The integral in (43) still can be done analytically, giving the result

\[
\text{Re} \chi_{\pi}^{(0)}(Q + q, \omega_q) = \frac{1}{2\pi^2 v_F} \sum_{s=\pm} \int_0^{\infty} dp \\
\times \int_0^{2\pi} d\varphi \frac{2p^2 \sin^2 \varphi + q_s |p| \cos \varphi - q_s^2 |p| \sin \varphi - s\omega_q |p|/v_F}{q^2 - \omega_q^2/v_F^2 - 2q_s |p| \cos \varphi - 2q_s |p| \sin \varphi + s\omega_q |p|/v_F} \\
= \frac{1}{2\pi} \frac{1}{q^2 - \omega_q^2/v_F^2} \sum_{s=\pm} \text{Re} \int_0^{\infty} dp \\
\times \frac{4(p^2 - s\omega_q |p|/v_F)(q_s^2(q^2 - \omega_q^2/v_F^2) + q_s^2\omega_q^2/v_F^2) + (\omega_q^2/v_F^2)(q^2 - \omega_q^2/v_F^2)}{\sqrt{q^2 - \omega_q^2/v_F^2 + 4s\omega_q |p|/v_F - 4p^2}} \\
= -\frac{q_s^2 - \omega_q^2/v_F^2}{8q^2 - \omega_q^2}.
\]

As already mentioned, the result (44) is anisotropic as a consequence of having chosen \( Q \) to connect the \( \Gamma \) point to a given corner \( K \) of the Brillouin zone. We note in this respect that the form of the \( \gamma \) matrices has to be conveniently modified when the Dirac point is rotated by an angle of \( \pm 2\pi/3 \), despite the fact that the original and the transformed point are equivalent from the point of view of the underlying lattice. It can be checked that the analogue of the polarization (44) computed about the large momenta of the rotated points corresponds to replacing \( q_s^2 \) in the numerator of that expression by \( (q_s/2 + \sqrt{3}q_s/2)^2 \) and \( (q_s/2 - \sqrt{3}q_s/2)^2 \), respectively. The result making sense physically must be given by the average of the polarization for the three equivalent momenta, which leads to the isotropic result in the low-energy Dirac theory

\[
\chi^{(0)}(Q + q, \omega_q) = \frac{q^2 - 2\omega_q^2/v_F^2}{16q^2 - \omega_q^2}.
\]

The expression (45) turns out to be independent of the high-energy cutoff limiting the low-energy regime assumed in the Dirac theory. This has to do with the form in which the real part of \( \chi_{\pi}^{(0)}(Q + q, \omega_q) \) has been computed, adopting a principal value prescription that gets rid of a possible frequency and momentum-independent contribution. We know however, from the numerical computation of \( \chi^{(0)}(q, 0) \) integrating over the whole Brillouin zone, that the polarization must reach a nonvanishing value at the large momentum \( Q \), given by \( \approx -c'/ta^2 \) with \( c' \approx 1.722 \). The polarization (45) just misses that constant contribution, as it can be checked that its frequency and momentum dependence is in perfect agreement with the behaviour obtained from the numerical computation of \( \chi_{\pi}^{(0)}(Q + q, \omega_q) \). In what follows, we work therefore with the complete polarization

\[
\chi_{\pi}^{(0)}(Q + q, \omega_q) \approx -\frac{c'}{ta^2} + \frac{q^2 - 2\omega_q^2/v_F^2}{16q^2 - \omega_q^2}.
\]

We can now investigate the renormalization of the phonon dispersion at the \( K \) point, starting again from equation (15) in the framework of the RPA. In the present case, we have to
Figure 7. Plot of the renormalized phonon dispersion obtained from (48) as a function of the momentum $q$ measured from the $K$ point, for an effective electron–phonon coupling $g^2/v_F^2 = 0.02$. The representation excludes any branch in the region of the electron–hole continuum $\omega \geq v_F|q|$, which is above and very close to the left branch in the figure.

bear in mind that the Coulomb potential is attenuated at the large momentum $Q$, which makes its effect on the phonon propagator almost irrelevant. Actually we can rewrite (15) in the form

$$D_{RPA}(Q + q, \omega) \approx \frac{2 \omega_0(Q + q)(1 - \frac{e^2}{2\kappa |Q|}\chi_\pi(Q + q, \omega))}{\omega^2 - \omega_0^2(Q + q) + i\epsilon - ((\omega^2 - \omega_0^2(Q + q))\frac{e^2}{2\kappa |Q|} + 2\omega_0(Q + q)g^2)\chi_\pi(Q + q, \omega)}.$$  

(47)

The strength of the phonon-mediated interaction is given by the dimensionless coupling $g^2/v_F^2$, which can be estimated to be at least of the order of $\sim 0.01$ at the $K$ point. This is smaller than the strength of the Coulomb interaction in typical graphene samples, where $e^2/\kappa v_F \sim 1$. However, the latter enters in the denominator of (47) with a relative weight given by $\omega_0(Q)/4v_F|Q|$, which is of the order of $\sim 0.001$. Therefore, we see that the Coulomb interaction does not play a relevant role in dressing the phonon propagator at the large momentum transfer $Q$.

The renormalized phonon dispersion is obtained from the position of the zeros of the denominator in (47):

$$\omega^2 \approx \omega_0^2(Q + q) + 2\omega_0(Q + q)g^2 \left(-\frac{c'}{t a^2} + \frac{q^2 - 2\omega^2/v_F^2}{16\sqrt{v_F^2 q^2 - \omega^2}}\right).$$  

(48)

At the $K$ point, we can approximate $\omega_0(Q + q) \approx 3\omega_0(1 - a|q|/2 + \cdots)$. We observe anyhow that the singularity in the denominator of the charge polarization gives rise to a strong modification of the bare phonon dispersion. The behaviour of the renormalized phonon frequencies $\omega(Q + q)$, as obtained from the resolution of (48), is represented in figure 7. There we can see that the renormalized dispersion becomes drastically softened, with a vanishing energy of the phonon branch at the $K$ point. This effect remains qualitatively the same, irrespective of the value of the effective coupling $g^2/v_F^2$, as it only relies on the singularity of the intervalley polarization at $\omega = v_F|q|$. 

New Journal of Physics 11 (2009) 095015 (http://www.njp.org/)
The renormalized acoustic branch actually has an approximate linear dispersion that lies below the continuum of electron–hole excitations near the \( K \) point. This is a natural consequence of the fact that the new phonon branch arises from the hybridization of the phonon modes with electron–hole pairs. In a sense, one can think of the dressed phonons as bound states of the two types of excitations, with an energy that is therefore found below the threshold of electron–hole creation at \( \omega = v_F|q| \).

There is then the possibility to observe well-defined phonon modes of very low energy at the \( K \) point of graphene. It has been shown elsewhere that the approximate linear branch of out-of-plane phonons at the \( K \) point has to lead to a cubic energy dependence of the quasiparticle decay rate, competing with the quadratic behaviour expected from in-plane acoustic phonons over a certain energy range [20]. The renormalized dispersion of the out-of-plane acoustic phonons may thus have a significant impact on the transport properties of graphene, being possibly responsible for the nonlinear features observed in the temperature dependence of the resistivity close to the charge neutrality point.

5. Conclusion

In this paper, we have investigated the way in which the properties of out-of-plane phonons are renormalized by their coupling to electron–hole excitations in graphene. We have focused, in particular, on the case of the out-of-plane acoustic phonons, which have a very soft bare dispersion, with a quadratic dependence of energy on momentum. The existence of phonon modes of very low energy could be potentially dangerous for the stability of electron quasiparticles in graphene. However, due to the same softness of the modes, their low-energy dispersion can also be easily perturbed by the electron excitations, making pertinent the in-depth analysis of the many-body effects on the out-of-plane phonons.

We have shown that the effect of the charge polarization actually cuts off the spectrum of the out-of-plane acoustic modes at low momenta. This result holds in the framework of the low-energy Dirac theory of graphene, and it is also confirmed by an analysis of the corrections beyond the RPA. Thus, we have carried out the sum of the dominant contributions from Coulomb scattering to the electron–phonon vertex, finding out that the resulting nonperturbative effects do not introduce qualitative changes in the renormalization of the flexural phonons. The analysis of the phonon dispersion beyond the regime of applicability of the Dirac theory also shows that there is some critical momentum at which the acoustic branch of out-of-plane phonons opens up, going up thereafter for increasing values of the momentum.

The opening of the branch of flexural phonons at some critical momentum may lead to signatures in some other physical properties of the system, like the specific heat or the compressibility. Furthermore, it is also tempting to relate this kind of instability to the development of ripples in graphene. A careful analysis of the effect of corrugation would require, however, the formulation of the problem in a curved geometry or, alternatively, the description of the modulation of the graphene lattice by fictitious gauge fields, which is beyond the scope of the present paper. It would be interesting though to investigate the interplay between the instability of long-wavelength flexural phonons and the appearance of ripples that are usually seen as a reflection that the system is on the verge of structural instability.

We have also seen that the acoustic branch of out-of-plane phonons presents near the \( K \) point a very strong Kohn anomaly, which is much more pronounced than the usual effect for in-plane phonons [18, 19]. The mechanism for the softening of the phonon dispersion lies in the
singular character of the intervalley polarization of graphene at the threshold of electron–hole formation. The strong coupling between the out-of-plane phonons and the electron–hole pairs leads to a new branch of hybrid states right below the boundary of the electron–hole continuum. In the case of undoped graphene, their dispersion goes to zero energy at the $K$ point, opening therefore a relevant channel for the decay of electron quasiparticles. It is likely that part of the observed deviation from the linear temperature dependence of the resistivity in graphene has to do with the onset of the soft hybrid modes, whose contribution to the quasiparticle scattering rate turns out to be larger than that of the in-plane acoustic phonons above some crossover temperature $[20]$.

One may finally ask whether a similar strong Kohn anomaly could also appear in the optical branch of out-of-plane phonons. In this case, however, the different polarization vector of the phonons introduces a significant modification in the electron–phonon coupling. This is then proportional to $\sigma_3$ instead of the unit matrix in the space of the two-component spinors, which consequently modifies the electron–hole susceptibility. It can be checked that this object has then a frequency and momentum-dependence just proportional to $\sqrt{v_F^2q^2 - \omega^2}$, missing the kind of singularity essential to the above discussion. Such a dependence will still give rise to a Kohn anomaly of the type already observed in the dispersion of in-plane optical phonons, while it becomes clear that the stronger renormalization discussed above will be confined to the acoustic branch of out-of-plane phonons, susceptible to the singular form of the intervalley polarization in graphene.

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