Geometric phases for nonlinear coherent and squeezed states

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Abstract

The geometric phases for standard coherent states which are widely used in quantum optics have attracted considerable attention. Nevertheless, few physicists consider the counterparts of nonlinear coherent states, which are useful in the description of the motion of a trapped ion. In this paper, the non-unitary and non-cyclic geometric phases for two nonlinear coherent and one squeezed states are formulated, respectively. Moreover, some of their common properties are discussed, such as gauge invariance, non-locality and nonlinear effects. The nonlinear functions have dramatic impacts on the evolution of the corresponding geometric phases. They speed the evolution up or down. So this property may have an application in controlling or measuring geometric phase. For the squeezed case, when the squeezed parameter $r \to \infty$, the limiting value of the geometric phase is also determined by a nonlinear function at a given time and angular velocity. In addition, the geometric phases for standard coherent and squeezed states are obtained under a particular condition. When the time evolution undergoes a period, their corresponding cyclic geometric phases are achieved as well. And the distinction between the geometric phases of the two coherent states may be regarded as a geometric criterion.

1. Introduction

Recently, nonlinear coherent and squeezed states have drawn considerable attention from physicists. Filho and Vogel [1] defined the nonlinear coherent states as the right eigenstates of a generalized annihilation, whose concrete definition will be given in the following paragraph. They possessed not only some of the typical features of the standard coherent states but also the strong non-classical properties. Roy and Roy [2] used a displaced-operator technique to discover a new nonlinear coherent state. Its non-classical properties, such as quadrature squeezing, amplitude-squared squeezing and sub-Poissonian behaviour, were studied intensively. Subsequently, Kwek and Kiang [3] researched the nonlinear extension of the single-mode squeezed states and showed its sub-Poissonian behaviour.

The above paragraph reviewed the developments of the nonlinear coherent and squeezed states; now we pay attention to the evolution of geometric phases. In retrospect, it was discovered by Berry [4] in the context of adiabatic, unitary, cyclic evolution of a time-dependent quantum system. Besides the significance of physics, it also has a geometrical interpretation which is given by Simon [5]. It can be regarded as the holonomy of a line bundle $L$ over the space of parameters $M$ of the system, if $L$ is endowed with a natural connection.

Berry’s result was generalized by Aharonov and Anandan [6] via discarding the assumption of adiabaticity. The Aharonov and Anandan phase (A–A phase) could be obtained by the difference between the total phase and the dynamical one as well as determined by the loop integral of a natural connection on a $U(1)$ principle fibre bundle over projective Hilbert space.

Afterwards, the A–A phase was also generalized by Samuel and Bhandari [7]. Specifically speaking, depending on Pancharatnam’s earlier work [8], they found a more general phase in the context of the non-cyclic and non-unitary quantal evolution. However, their definition is an indirect one, which depends on a geodesic closing the initial and final points of the open path. Moreover, it is not manifestly gauge invariant. Soon Pati [9, 10] gave another definition without the need of geodesic, where a canonical one-form was defined and its line
integral gave the geometric phase. It is manifestly invariant under the phase and gauge transformation.

However, when the initial and final states are orthogonal, the above non-cyclic geometric phases become undefined. Manini and Pistolesi [11] introduced Abelian off-diagonal geometric phases under the condition of the adiabatic evolution. Mukunda et al [12] generalized the concepts without the confinement of adiabaticity. Furthermore, Kult et al [13] extended the conceptions to the non-Abelian cases.

Another line of development of the geometric phase is the generalization to mixed states. Uhlmann [14] discussed this issue in the mathematical context of purification. Sj¨oqvist et al [15] addressed the non-degenerate geometric phase in the non-cyclic and unitary evolution under the background of quantum interference. Some other extensions [16, 17] were also made.

Including the applications of the geometric phases in many areas of physics [18, 19], new perspectives such as topological quantum interference. Some other extensions [16, 17] were also made.

Based on the formulation given by Pati [9, 10], this paper aims at giving non-cyclic and non-unitary geometric phases of nonlinear coherent and squeezed states. The paper is organized as follows. Section 2 reviews nonlinear coherent and squeezed states as well as the non-unitary and non-cyclic geometric phase. In section 3, the geometric phases of the two nonlinear coherent states are calculated. Moreover, we analyse the distinction between the two geometric phases and some of their properties. In section 4, the geometric phase of the squeezed state is given. In addition, some of its features are discussed. At the end of this paper, a conclusion is drawn.

2. Reviews of nonlinear coherent and squeezed states and geometric phases

For nonlinear coherent states, the generalized annihilation and creation operators are introduced by

\[ A = af(N) , \quad A^\dagger = f(N)a^\dagger , \]

where \( a \) (\( a^\dagger \)) is the harmonic oscillator annihilation (creation) operator, \( N = a^\dagger a \), and \( f(x) \) is a reasonably well-behaved real function. \( A \) and \( A^\dagger \) satisfy the following nonlinear algebra:

\[ [A, A^\dagger] = (N + 1)f^2(N + 1) - Nf^2(N). \tag{1} \]

So we can construct the coherent operator, i.e. \( \exp(\alpha A^\dagger - \alpha^* A) \). However, because of equation (1), we cannot apply the BCH disentangling theorem to obtain the coherent operator [2]. So we construct two other operators \( B \) and \( B^\dagger \) which have the following commutation relations:

\[ [A, B^\dagger] = 1 \]

and

\[ [B, A^\dagger] = 1 , \]

where \( B = a^\dagger f(N)^{-1} \) and \( B^\dagger = f(N)^{-1}a^\dagger \). As a result, the coherent state operators [2] can be constructed as

\[ D(\beta) = \exp(\beta B^\dagger - \beta^* A) \]

and

\[ D(\beta) = \exp(\beta A^\dagger - \beta^* B) . \]

It is noteworthy that the above two operators are not unitary. When both of them act on the vacuum state, we can obtain two coherent states which are

\[ |\beta_1\rangle = D(\beta)(0) = \exp(\beta B^\dagger - \beta^* A)(0) \]

\[ = \exp\left(-\frac{1}{2}\beta^2\right) \exp(\beta B^\dagger) \exp(-\beta^* A)(0) \]

and

\[ |\beta\rangle = D(\beta)(0) = \exp(\beta A^\dagger - \beta^* B)(0) \]

\[ = \exp\left(-\frac{1}{2}\beta^2\right) \exp(\beta A^\dagger) \exp(-\beta^* B)(0) \]

via the BCH theorem. Both of them are represented in the energy eigenstates, i.e.

\[ |\beta_1\rangle = e^{-\frac{f(0)}{2} N} \sum_{n=0}^{\infty} \frac{\beta^n}{f(n)\sqrt{n!}} |n\rangle \tag{2} \]

and

\[ |\beta\rangle = e^{-\frac{f(0)}{2} N} \sum_{n=0}^{\infty} \beta^n \frac{f(n)!}{\sqrt{n!}} |n\rangle , \tag{3} \]

where \( f(n)! = f(1) \cdots f(n) \). If \( f(n) \) takes a particular form, the first nonlinear coherent state \(|\beta_1\rangle\) is regarded as the stationary state of a centre of mass motion of a trapped and bi-chromatically laser-driven ion far from the so-called Lamb–Dicke regime. By comparison with the standard coherent states, the nonlinear counterparts exhibit non-classical properties such as strong amplitude squeezing and self-splitting into two or more sub-states, which finally causes quantum interferences [1]. With respect to the second nonlinear coherent state \(|\beta\rangle\), it still displays various non-classical properties such as quadrature squeezing, amplitude-squared squeezing and sub-Poissonian behaviour [2].

Furthermore, Kwek and Kiang [3] pointed out that the squeezed operators could be defined as

\[ S_1(\zeta) = \exp\left[\frac{1}{2}(\zeta B^2 - A^2)\right] \]

and

\[ S(\zeta) = \exp\left[\frac{1}{2}(\zeta A^2 - B^2)\right] . \]

By the same procedure, when the above two operators act on the vacuum state, one can obtain the so-called squeezed states, i.e.

\[ |\zeta_1\rangle = S_1(\zeta)|0\rangle = \exp\left[\frac{1}{2}(\zeta B^2 - A^2)\right]|0\rangle \]

and

\[ |\zeta\rangle = S(\zeta)|0\rangle = \exp\left[\frac{1}{2}(\zeta A^2 - B^2)\right]|0\rangle . \]

If the above two states are expanded in the energy eigenstates, we can find that they coincide with each other, that is to say,

\[ |\zeta\rangle = |\zeta_1\rangle = N \sum_{n=0}^{\infty} \left(\frac{\sinh(r)}{2}\right)^n \frac{f(2n)!\sqrt{(2n)!}}{n!} |2n\rangle , \]

where
where \( N \) is an indefinite coefficient and \( \zeta = r e^{i \theta} \). By the way, the following formulas may be useful in expanding the squeezed states:
\[
\exp \left[ \frac{1}{2} (\zeta B^2 - \zeta^* A^2) \right] A \exp \left[ - \frac{1}{2} (\zeta B^2 - \zeta^* A^2) \right] = \cosh(r) A - e^{i \theta} \sinh(r) B^\dagger
\]
and
\[
\exp \left[ \frac{1}{2} (\zeta A^2 - \zeta^* B^2) \right] B \exp \left[ - \frac{1}{2} (\zeta A^2 - \zeta^* B^2) \right] = \cosh(r) B - e^{i \theta} \sinh(r) A^\dagger.
\]
Moreover, it is pointed out that unlike the normal squeezed states, the Mandel Q number of the nonlinear case is negative and as a consequence it exhibits sub-Poisson statistics.

The above paragraphs are about nonlinear coherent and squeezed states. And it is well known that the operators are non-unitary. Hence, to calculate the geometric phases, we must know the methods of calculating the non-unitary case, which are reviewed in the following paragraphs. Moreover, it is also applied to a non-cyclic evolution.

For a non-cyclic evolution, a way must be found to compare two different quantal states. According to the Pancharatnam connection [8], the total phase [9] can be formulated as
\[
\chi = \arg \left( \frac{\psi(0)}{\| \psi(0) \|} , \frac{\psi(\tau)}{\| \psi(\tau) \|} \right),
\]
where \( \| \psi(t) \| \) denotes the norm of the state vector \( |\psi(t)\rangle \), and \( |\psi(t)\rangle / \| \psi(t) \| \) represents \( |\psi(t)\rangle / \| \psi(t) \| \). And the dynamical phase can be written as
\[
\delta = - i \int_{\tau_0}^{\tau} \frac{\psi(\tau)}{\| \psi(\tau) \|} \frac{d\tau}{\| \psi(\tau) \|}.
\]
Hence, the geometric phase [23] reads
\[
\gamma = \chi - \delta.
\]
In order to spell out the geometric aspects of the above equation, Pati [9, 10] introduces the reference section which is
\[
|\chi(0)\rangle = \frac{\langle \psi(0)|\psi(0)\rangle}{\| \psi(0) \|} |\psi(0)\rangle = \frac{|\psi(t)\rangle|\psi(t)\rangle}{\| \psi(t) \|} |\psi(t)\rangle.
\]
In order to understand the above definition, we give a brief outline of fibre bundles. A quantum state \( \rho \) and a relative phase are represented by a point of projective Hilbert space \( \mathcal{P} \) and a circle \( U(1) \), respectively. Take the Cartesian product of \( \mathcal{P} \) and \( U(1) \) as well as define a projection map \( \Pi : \mathcal{L} \to \mathcal{P} \), where \( \mathcal{L} = \mathcal{P} \times U(1) \). Hence, we can regard the triple \( \mathcal{L}(\mathcal{P}, U(1), \Pi) \) as a fibre bundle, in which \( \mathcal{P} \) and \( U(1) \) are termed the base space and the fibre, respectively. Furthermore, the evolution of the quantum state \( \rho(t) \) can generate a curve in \( \mathcal{P} \). Then, another map \( s : \mathcal{P} \to \mathcal{L} \exists \) such that the image \( \Gamma(t) \) of the curve \( \rho(t) \) falls in the fibre over \( \rho \), i.e. \( \Pi \circ s = I d \rho \). And the map is called a section. It is easy to know that \( |\chi(0)\rangle \) is a section of the fibre bundle \( \mathcal{L} \). If \( |\chi(0)\rangle \) is going to be defined as a reference section, it has to satisfy the following condition that \( |\chi(0)\rangle |\chi(0)\rangle \) is always real and positive during the evolution; in other words, the two state vectors are in phase, which is in accordance with the Pancharatnam condition. Moreover, by the use of equation (6), the non-cyclic and non-unitary geometric phase [10] could be expressed as
\[
\gamma = i \int_{\tau_0}^{\tau} \left( \frac{d}{d\tau} |\chi(\tau)\rangle \right) d\tau \mod 2\pi.
\]
It is not very complex to verify that the geometric phase is independent of the choice of the phase of the quantum system state \( |\psi(t)\rangle \), reparametrization invariant and manifestly gauge invariant. Furthermore, the geometric phase could reduce to the Berry phase and A–A phase under an appropriate limit.

Up to now, we have already reviewed both the nonlinear coherent and squeezed states, as well as the non-cyclic and non-unitary geometric phases. Hence, we can research further the geometric phases of nonlinear coherent and squeezed states.

3. Geometric phase for the nonlinear coherent state

At any time \( t > 0 \), the first coherent state vector is given by
\[
|\beta_1(t)\rangle = e^{-|\beta|^2} \sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{f(n)!} e^{-i E_n t} |n\rangle,
\]
where \( E_n = \omega (\frac{1}{2} + n) \) and we set \( h = 1 \) for simplicity. In order to calculate the corresponding geometric phase, we must know the reference section (6). At first, let us calculate the first fraction of equation (6). Considering this problem in a polar coordinate, the first fraction is expressed as
\[
\langle \beta_1(t)|\beta_1(0)\rangle = e^{-i \chi} \exp \left[ i \arctan \left( \frac{\text{Im}(\beta_1(t) \beta_1(0)))}{\text{Re}(\beta_1(t) \beta_1(0))) \right) \right],
\]
where \( \chi \) is the total phase in accordance with equation (4). So it is necessary to obtain the inner product
\[
\langle \beta_1(t)|\beta_1(0)\rangle = e^{-|\beta|^2} \sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{f(n)!} |\beta|^{2n} \sin(E_n t).
\]
Substituting equation (10) into equation (9), one can obtain
\[
-\chi = \arctan \frac{\sum_{n=0}^{\infty} |\beta|^{2n}}{\sum_{n=0}^{\infty} |\beta|^{2n}} \sin(E_n t).
\]
Second, to seek for \( \| \beta_1(t) \| \), we can acquire it in this way, i.e.
\[
N_1 = \| \beta_1(t) \| = \sqrt{\langle \beta_1(t)|\beta_1(t)\rangle} = e^{-|\beta|^2} \sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{f(n)!} \sqrt{n!} \sin(E_n t),
\]
which is independent of time. Third, we obtain the reference section, which is
\[
|\chi(0)\rangle = \frac{1}{N_1} e^{-i \gamma} e^{-|\beta|^2} \sum_{n=0}^{\infty} \frac{|\beta|^{2n}}{f(n)!} \sqrt{n!} e^{-i E_n t} |n\rangle,
\]
via substituting equations (8), (9) and (12) into equation (6). It is in phase with the initial reference section \( |\chi(0)\rangle \), which
is in accordance with the Pancharatnam condition \[8\]. Fourth, the connection is calculated by the use of equation (12), which takes the form

\[
i(\chi(t)) = \frac{d\chi(t)}{dt} = \frac{d\chi}{dt} + \omega \left\{ \frac{1}{2} + |\beta|^2 \sum_{n=0}^{\infty} \frac{[|\beta|^2 n!]}{[f(n)+1]^n} \right\}. \tag{13}
\]

Finally, the non-unitary and non-cyclic geometric phase is reached, i.e.

\[
\gamma(t) = -\arctan \frac{\sum_{n=0}^{\infty} \frac{|\beta|^2 n!}{[f(n)+1]^n} \sin(E_n t)}{\sum_{n=0}^{\infty} \frac{|\beta|^2 n!}{[f(n)+1]^n} \cos(E_n t)} + \omega t \left\{ \frac{1}{2} + |\beta|^2 \sum_{n=0}^{\infty} \frac{[|\beta|^2 n!]}{[f(n)+1]^n} \right\} \mod 2\pi, \tag{14}
\]

by substituting equations (11) and (13) into equation (7). It is regarded as the result of parallel transport with the Pancharatnam connection \[8\], which uncover the geometric meaning.

Furthermore, let us discuss the above geometric phase. Except for the general properties of the geometric phase, such as reparametrization invariance and manifest gauge invariance under the gauge transformation \[|\chi(t)\rangle = e^{i\sum_{n=0}^{\infty} \frac{|\beta|^2 n!}{[f(n)+1]^n} \sin(E_n t)} |\chi(t)\rangle \tag{10}\], where \(n\) is an integer, let us discuss the other ones. To begin with, when \(f(n) = 1\), the nonlinear coherent state is reduced to the normal coherent state, so is the corresponding geometric phase. It takes this form

\[
\gamma = -\arctan \frac{\sum_{n=0}^{\infty} \frac{|\beta|^2 n!}{[f(n)+1]^n} \sin(E_n t)}{\sum_{n=0}^{\infty} \frac{|\beta|^2 n!}{[f(n)+1]^n} \cos(E_n t)} + \omega t \left\{ \frac{1}{2} + |\beta|^2 \sum_{n=0}^{\infty} \frac{[|\beta|^2 n!]}{[f(n)+1]^n} \right\} \mod 2\pi, \tag{15}
\]

which coincides with the result \[10\]. Moreover, if the quantum state undergoes the cyclic evolution, the geometric phase (14) becomes

\[
\gamma \left( \frac{2\pi}{\omega} \right) = 2\pi |\beta|^2 \sum_{n=0}^{\infty} \frac{[|\beta|^2 n!]}{[f(n)+1]^n} \sin(E_n t) + \omega t \left\{ \frac{1}{2} + |\beta|^2 \sum_{n=0}^{\infty} \frac{[|\beta|^2 n!]}{[f(n)+1]^n} \right\} \mod 2\pi.
\]

In addition, let us focus on the non-locality of equation (14). Pati \[10\] has proved that the geometric phase along the geodesic is equal to zero. Hence, if we join the end points by a geodesic, the line integral (7) can be converted to a surface integral by Stokes’s theorem. So the non-locality for this specific problem is unveiled. Finally, if a different nonlinear function \(f(n)\) is chosen, we will obtain a different geometric phase (14). In this paper, we choose

\[
f(n) = L^2_{n}(\eta^2) \left[ (n+1) L^2_{n}(\eta^2) \right]^{-1}, \tag{16}
\]

which is important to describe the motion of an ion in a harmonic potential and the interaction with two laser fields \[1\]. \(L^2_{n}\) are generalized Laguerre polynomials which are characterized by the Lamb–Dicke parameter \(\eta\). To get some ideas about the evolution of the geometric phase with respect to time \(t\), we show it in figure 1, where we choose \(|\beta|^2 = 1\) and \(\omega = \pi/4\). All of these curves vary nonlinearly with time in contrast with dynamical phases, which vary linearly with time. This is the main difference between the geometric phases and the dynamical ones. When \(\eta = 0\), \(f(n) = 1\), so that the geometric phase of nonlinear coherent states reduces to the standard case (15), which is shown in curve (a) in figure 1. It is illustrated that when \(\eta\) increases, the corresponding graph varies acutely. So we can conclude that the nonlinear functions tagged by \(\eta\) in this case affect the geometric phase dramatically. The different functions \(f(n)\) in accord with different parameter \(\eta\) speed the evolution of the geometric phase up or down, which make the evolution completely different. So \(\eta\) can be regarded as a nonlinear parameter which modulate the evolution of the corresponding geometric phase. Hence, it probably has potential applications in controlling or measuring the geometric phase.

In the above paragraphs, we have discussed the above non-unitary and non-cyclic geometric phase of the first nonlinear coherent state. Now let us study the counterpart of another nonlinear coherent state. For \(t > 0\), the time evolution of the state vector reads

\[
|\beta(t)\rangle = e^{-i\frac{\beta^2}{2} \sum_{n=0}^{\infty} \frac{|\beta|^2 n! f(n)^2}{\sqrt{n!}} t} e^{-iE_n t} |\psi\rangle.
\]

By similar calculation, we can obtain the geometric phase, which is

\[
\gamma(t) = -\arctan \frac{\sum_{n=0}^{\infty} \frac{|\beta|^2 n! f(n^2)}{n!} \sin(E_n t)}{\sum_{n=0}^{\infty} \frac{|\beta|^2 n! f(n^2)}{n!} \cos(E_n t)} + \omega t \left\{ \frac{1}{2} + |\beta|^2 \sum_{n=0}^{\infty} \frac{[|\beta|^2 n!]}{[f(n)+1]^n} \right\} \mod 2\pi. \tag{17}
\]

In short, similar properties with the above case are ignored. Let us concentrate on the special ones. At first, let us consider the distinction between the geometric phase of the first nonlinear coherent state (14) and the second one (17). It is not very hard to observe that \(f(n)\), which is the mark of the nonlinearity, is in the denominator position in the first case, whereas it becomes in the numerator position in the second case. So the
the two nonlinear squeezed states coincide with each other. Moreover, if geometric phase plays the pole of a criterion, which tell the geometric difference of the two coherent states. Thus, if equation (6) is also reduced to the normal case (15). In addition, while \( |\beta(t)| \) go through the cyclic evolution, the geometric phase (17) is transformed to be

\[
y(\frac{2\pi}{\omega}) = 2\pi |\beta|^2 \sum_{n=0}^{\infty} \frac{|\beta|^2 |f(n)|^2}{n^2} \frac{|f(n+1)|^2}{n^2} \mod 2\pi.
\]

Last but not least, again in order to get the ideas of equation (17), we show it in figure 2. For the sake of comparison with figure 1, we set the parameters \( \omega = \pi/4 \) and \( |\beta|^2 \), the same values as the previous case. Again, we can see the impact of \( f(n) \), which is shown in curves (b), (c) and (d). Taking a glance at curves (c) and (d), we can know nonlinear functions that exists the evolution of the geometric phase extremely slow. So in contrast with the geometric phase of the normal coherent state (15), it may be more convenient to measure and control the geometric phase precisely in experiments.

4. Geometric phase for the nonlinear squeezed state

In the previous section, the geometric phases of the two nonlinear coherent states are obtained. Now, in this section, we will research the counterpart of squeezed states. However, the two nonlinear squeezed states coincide with each other. So we can concentrate on one case, whose state vector of time evolution has the form

\[
|\xi(t)\rangle = N \sum_{n=0}^{\infty} \left( \frac{e^{i\chi}}{2} \right)^n \sqrt{\frac{(2n)!}{n!}} e^{-iE_{2n}t} |2n\rangle,
\]

where \( E_{2n} = \omega \left( \frac{1}{2} + 2n \right) \). First, let us focus on the first fraction of the reference section (6), which is expressed as

\[
\frac{\langle \xi(t) | \xi(0) \rangle}{|\xi(t)\rangle |\xi(0)\rangle \rangle} = e^{-i\chi} = \exp \left[ i \arctan \left( \frac{\text{Im}(\langle \xi(t) | \xi(0) \rangle)}{\text{Re}(\langle \xi(t) | \xi(0) \rangle)} \right) \right].
\]

where \( \chi \) represents the total phase. Hence, we are ready to calculate the inner product, i.e.

\[
\langle \xi(t) | \xi(0) \rangle = |N|^2 \sum_{n=0}^{\infty} \left( \frac{\tan(h)}{2} \right)^n \frac{|f(2n)!|^2}{(n!)^2} \frac{|f(2n+1)!|^2}{(n!)^2} \times \cos(h). (20)
\]

By the use of equations (19) and (20), we can obtain

\[
\frac{\sum_{n=0}^{\infty} \left( \frac{\tan(h)}{2} \right)^n \frac{|f(2n)!|^2}{(n!)^2} \frac{|f(2n+1)!|^2}{(n!)^2}}{\sum_{n=0}^{\infty} \left( \frac{\tan(h)}{2} \right)^n \frac{|f(2n)!|^2}{(n!)^2} \frac{|f(2n+1)!|^2}{(n!)^2}} \times \cos(h) = \frac{1}{2}.
\]

Second, let us concentrate on the denominator of the second fraction of the reference section (6), i.e. the norm of \( |\xi(t)\rangle \), which is

\[
N \equiv \frac{\langle \xi(t) | \xi(t) \rangle}{\sum_{n=0}^{\infty} \left( \frac{\tan(h)}{2} \right)^n \frac{|f(2n)!|^2}{(n!)^2} \frac{|f(2n+1)!|^2}{(n!)^2} \times \cos(h) \rangle = \frac{1}{2}.
\]

Third, substituting equations (18), (19) and (22) into equation (6), one can obtain the reference section

\[
|\chi_0(t)\rangle = e^{-i\chi} N \sum_{n=0}^{\infty} \left( \frac{\tan(h)}{2} \right)^n \frac{|f(2n)!|^2}{(n!)^2} \frac{|f(2n+1)!|^2}{(n!)^2} \times \frac{f(2n)! \sqrt{\frac{(2n)!}{n!}}}{n!} e^{-iE_{2n}t} |2n\rangle,
\]

which is in phase with \( |\chi_0(t)\rangle \). Fourth, by the use of equation (23), we can acquire the connection

\[
\frac{d}{dt} |\chi_0(t)\rangle = \frac{d}{dt} \sqrt{\frac{1}{2} + \tan^2(h)} \times \sum_{n=0}^{\infty} \left( \frac{\tan(h)}{2} \right)^n \frac{|f(2n)!|^2}{(n!)^2} \frac{|f(2n+1)!|^2}{(n!)^2} \frac{f^2(2n + 1) f^2(2n + 2)(2n + 1)}{\sum_{n=0}^{\infty} \left( \frac{\tan(h)}{2} \right)^n \frac{|f(2n)!|^2}{(n!)^2} \frac{|f(2n+1)!|^2}{(n!)^2}}
\]

Finally, via the integration of the above connection according to equation (7), the non-unitary and non-cyclic geometric phase can be achieved, i.e.

\[
y(t) = \chi + o(t) \left[ \frac{1}{2} + \tan^2(h) \right] \times \sum_{n=0}^{\infty} \left( \frac{\tan(h)}{2} \right)^n \frac{|f(2n)!|^2}{(n!)^2} \frac{|f(2n+1)!|^2}{(n!)^2} \frac{f^2(2n + 1) f^2(2n + 2)}{\sum_{n=0}^{\infty} \left( \frac{\tan(h)}{2} \right)^n \frac{|f(2n)!|^2}{(n!)^2} \frac{|f(2n+1)!|^2}{(n!)^2}} \mod 2\pi.
\]

It is the outcome of parallel transport according to the Pancharatnam connection [8].

From the above calculations, we have obtained the geometric phase (25); now let us discuss its features. Above all, when \( f(n) = 1 \), the above result reduced to the counterpart of the normal squeezed state, which is

\[
y(t) = \chi + o(t) \left[ \frac{1}{2} + \sin^2(h) \right] \mod 2\pi.
\]

Moreover, when the quantal state undergoes the cyclic evolution, we can obtain the corresponding geometric phase,
which takes this form
\[
\frac{2\pi}{\omega} = 2\pi \tanh^2(r) \\
\sum_{n=0}^{\infty} \left[ \frac{\ln(\eta)}{2} \right] 2n (2n+1) \left( \tanh(r) \right)^2 (2n+2) \\
\sum_{n=0}^{\infty} \left[ \frac{\ln(\eta)}{2} \right] 2n (2n+1) \left( \tanh(r) \right)^2 (2n+2)
\] mod $2\pi$.

Again, if $f(n)=1$, the unitary and cyclic geometric phase is reached, i.e.
\[
\frac{2\pi}{\omega} = 2\pi \sinh^2(r) \text{ mod } 2\pi.
\]
This is the area in phase space enclosed by the phase space trajectory. In addition, if we join the endpoints of the open curve by geodesics, by use of Stock’s theorem, the line integral (25) is converted to be the surface integral, which unmasks the non-locality of the geometric phase of the state. At last, if the same nonlinear function (16) is chosen and the parameters are set as $r = 1$ and $\omega = \pi/4$, we draw a graph of the geometric phase, equation (25), which is shown in figure 3. From them, we can make a conclusion that nonlinear functions have a dramatic effect on the geometric phase. The different functions in accordance with different $\eta$ make the velocity of the evolution of the geometric phases vary rapidly. On the other hand, it is very interesting to focus on the graphs of $\gamma(r)$ at a given angular velocity $\omega = \pi/4$ and a definite time $t = 0.5$ s, where $r$ is the squeezed parameter of equation (25). The curves are shown in figure 4. It is vividly illustrated that when $r \to \infty$, in other words $r$ is sufficiently large, the corresponding geometric phase may have a plateau, which is probably a common feature of the geometric phase of the squeezed states. However, the nonlinear function tagged by the present parameter $\eta$ affects the limiting value of the geometric phase, which is demonstrated by the curves (b), (c) and (d) in figure 4.

5. Conclusion
In summary, the non-cyclic and non-unitary geometric phases for nonlinear coherent and squeezed states are formulated, respectively. Furthermore, some of their properties are discussed such as gauge invariance, non-locality and nonlinear effects. The nonlinear functions have a strong effect on the evolution of the geometric phase. They speed the evolution up or down. So this property may have an application in controlling or measuring the geometric phase. For the squeezed case, when the squeezed parameter $r \to \infty$, the limiting value of the geometric phase is also modulated by the nonlinear function at a given time and angular velocity. In addition, the geometric phases for standard coherent and squeezed states are recovered if the nonlinear function $f(n) = 1$. When the time evolution undergoes a period, their corresponding cyclic geometric phases are achieved too.

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References
[1] de Matos Filho R L and Vogel W 1996 Phys. Rev. A 54 4560
[2] Roy B and Roy P 2000 J. Opt. B: Quantum Semiclass. Opt. 2 65
[3] Kwek L C and Kiang D 2003 J. Opt. B: Quantum Semiclass. Opt. 5 383
[4] Berry M 1984 Proc. R. Soc. A 392 45
[5] Simon B 1983 Phys. Rev. Lett. 51 2167
[6] Aharonov Y and Anandan J 1987 Phys. Rev. Lett. 58 1593
[7] Samuel J and Bhandari R 1988 Phys. Rev. Lett. 60 2339
[8] Pancharatnam S 1956 Proc. Indian Acad. Sci. A 44 247
[9] Pati A 1995 J. Phys. A: Math. Gen. 28 2087
[10] Pati A 1995 Phys. Rev. A 52 2576
[11] Manini N and Pistolesi F 2000 Phys. Rev. Lett. 85 3067
[12] Mukunda N, Arvind Chaturvedi S and Simon R 2001 Phys. Rev. A 65 012102
[13] Kult D 2007 Europhys. Lett. 78 60004
[14] Uhlmann A 1986 Rep. Math. Phys. 24 229
[15] Sjöqvist E, Pati A, Ekert A, Anandan J, Ericsson M, Oi D and Vedral V 2000 Phys. Rev. Lett. 85 2845
[16] Singh K, Tong D, Basu K, Chen J and Du J 2003 Phys. Rev. A 67 032106
| Reference | Details |
|-----------|---------|
| [17] | Tong D, Sjoqvist E, Kwek L and Oh C 2004 *Phys. Rev. Lett.* **93** 080405 |
| [18] | Shapere A and Wilczek F 1989 *Geometric Phases in Physics* (Singapore: World Scientific) |
| [19] | Bohm A, Mostafazadeh A, Koizumi H, Niu Q and Zwanziger J 2003 *The Geometric Phase in Quantum Systems* (Berlin: Springer) |
| [20] | Nayak C, Simon S, Stern A, Freedman M and Sarma S 2008 *Rev. Mod. Phys.* **80** 1083 |
| [21] | Bouchene M A and Abdel-Aty M 2009 *Phys. Rev. A* **79** 55402 |
| [22] | Bouchene M A, Abdel-Aty M and Mandal S 2010 *Phys. Rev. A* **82** 23409 |
| [23] | Mukunda N and Simon R 1993 *Ann. Phys.* **228** 205 |