Volume entropy

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Abstract
Building on a technical result by Brunnemann and Rideout on the spectrum of the volume operator in loop quantum gravity, we show that the dimension of the space of the quadrivalent diffeomorphism invariant states with no zero-volume nodes describing a region with total volume smaller than $V$ has finite dimension, bounded by $V \log V$. This implies that a notion of ‘volume entropy’ may be introduced on this state space, interpreted as the von Neumann entropy associated to the measurement of volume. However, it also becomes apparent that including the states with vanishing volume eigenvalues this entropy becomes divergent. We briefly discuss possible implications of this conundrum and difficulties arising for extending this analysis to higher valent nodes.

Keywords: spin networks, loop quantum gravity, spacetime entropy, volume operator, spacetime thermodynamics

1. Introduction

Thermodynamical aspects of the dynamics of spacetime have first been pointed out by Bekenstein’s introduction of an entropy associated to the horizon of a black hole [1]. This led to the formulation of the ‘laws of black holes thermodynamics’ by Bardeen, Carter, and Hawking [2] and to Hawking’s discovery of black role radiance, which has reinforced the geometry/thermodynamics analogy [3]. The connection between area and entropy suggests that it may be useful to treat aspects of space-time statistically at scales large compared to the Planck length [4], whether or not we expect the relevant microscopic elementary degrees of freedom to be simply the quanta of the gravitational field [5], or else. Black hole entropy, in particular, can be interpreted as cross-horizon entanglement entropy (see [6, 7] for recent results reinforcing this interpretation, and references therein), or—most likely equivalently—as the von Neumann entropy of the statistical state representing a macrostate with given

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horizon area. In the context of loop quantum gravity (LQG), this was considered in [8] and later extensively analyzed; for a recent review see [9, 10], see also [11, 12].

All such developments are based on the assignment of thermodynamic properties to space-time \textit{surfaces}. This association has motivated the holographic hypothesis: the conjecture that the degrees of freedom of a region of space are somehow encoded in its boundary.

In this paper, instead, we study statistical properties associated to spatial \textit{regions}. To this aim, we prove a finiteness result on the number of quantum states of gravity describing a region of finite volume. More precisely, we work in the context of LQG as defined in [13, 14] and prove that when the state space is restricted to include only states with non-zero volume eigenvalues, the dimension of the subspace describing a region of total volume smaller than \(V\) is finite. We give explicitly the upper bound of the dimension as a function of \(V\).

Thus, it follows that it is possible to define a von Neumann entropy for the quantum gravitational field on this restricted state space, associated to the volume of a region, and this entropy is finite.

The inclusion of states with zero-eigenvalues would automatically give rise to divergences as will become apparent in our analysis. The loop quantum gravity state space does contain these states, as the zero is in the spectrum of the volume operator. Nevertheless, it is surprising that merely not including states with zero eigenvalues gives rise to a finite von Neumann entropy.

This observation relies on a piece of information regarding the possible spin states, given a volume: in the four valent case these are finite in number as shown by Brunemann and Rideout in [15, 16]. Since a similar result, or a result on the non existence of such a bound, for higher valent nodes is to our knowledge absent in the literature, we restrict our analysis to the four valent case.

We caution that the analysis presented here is not directly applicable to other formulations of loop quantum gravity [17, 18], which would appear to present further complications not addressed in this article. First, the state space need not be confined to only quadrivalent graphs. As explained above, investigating the higher valent cases would require knowledge of whether an analog of the bound discovered by Brunemann and Rideout for the quadrivalent case exists or not. Second, the state space in [17, 18] is larger, as a result of defining diff-invariant spin-networks states through explicit embeddings in hypersurfaces, which can give rise to extra information such as the number of knots\(^2\).

Note that the existence of an entropy associated to bulk degrees of freedom of a spin network was already considered in [19–21]. In [20] the authors derive the BGS entropy of a graph with a particle living on it. However, volume information, the main observable to which we associate an entropy here, was not considered. In addition the authors note that the source of their entropy is not a genuine density matrix associated to the gravitational field, but rather a description of the matter energy levels in a given geometry. In [19] the entropy associated to the bulk degrees of freedom is suggested in the last sentence as an open possibility. In [21], differently than in the present work, the authors consider a fixed graph. Here, we sum the contribution of all graphs giving rise to a fixed volume.

2. Counting spin networks

Consider the measurement of the volume of a 3d spacelike region \(\Sigma\). The physical system measured is the gravitational field. In the classical theory, this is given by the metric \(q\) on \(\Sigma\):

\(^2\) We thank an anonymous referee for his insightful comments on these issues.
the volume is $V = \int_\Sigma \sqrt{\det q} \, d^3x$. In the quantum context, using the LQG formalism, the geometry of $\Sigma$ is described by a diffeomorphism invariant state in the kinematical Hilbert space $H_{\text{diff}}$. The volume of $\Sigma$ is described by a volume operator $\hat{V}$ on this state space. We refer to [13, 14] for details on basic LQG results and notation.

We restrict $H_{\text{diff}}$ to four-valent graphs $\Gamma$ where the nodes $n$ have non-vanishing (unoriented) volume $v_n$. The spin network states $|\Gamma, j_l, v_n\rangle \in H_{\text{diff}}$, where $j_l$ is the link quantum number or spin, form a countable, orthonormal basis of $H_{\text{diff}}$. (We disregard here eventual additional quantum numbers such as the orientation, that have no bearing on our result.) The intertwiner basis at each node is chosen so that the local volume operator $\hat{V}_n$, acting on a single node, is diagonal and is labelled by the eigenvalues $v_n$, of the node volume operator $\hat{V}_n$ associated to the node $n$.

$$\hat{V}_n |\Gamma, j_l, v_n\rangle = v_n |\Gamma, j_l, v_n\rangle.$$ (1)

The states $|\Gamma, j_l, v_n\rangle$ are also eigenstates of the total volume operator $\hat{V} = \sum_{n=1}^{N} \hat{V}_n$, where $N$ is the number of nodes in $\Gamma$, with eigenvalue

$$v = \sum_{n=1}^{N} v_n,$$ (2)

the sum of the node volume eigenvalues $v_n$.

We seek a bound on the dimension of the subspace $H_V$ spanned by the states $|\Gamma, j_l, v_n\rangle$ such that $v \leq V$. That is, we want to count the spin-networks with volume less than $V$. We do this by bounding the number $N_\Gamma$ of four valent graphs in $H_V$, the number $N_{\{j_l\}}$ of possible spin assignments, and the number of the volume quantum numbers assignments $N_{\{v_n\}}$ on each such graph. Clearly

$$\dim H_V \leq N_\Gamma N_{\{j_l\}} N_{\{v_n\}}.$$ (3)

Crucial to this bound is the analytical result on the existence of a volume gap in four-valent spin networks found in [15, 16] (see also [23]). The result is the following. In a node bounded by four links with maximum spin $j_{\text{max}}$ all non-vanishing volume eigenvalues are larger than

$$v_{\text{gap}} \geq \frac{1}{4\sqrt{2}} \ell_P^3 \gamma^\frac{3}{2} \sqrt{j_{\text{max}}}.$$ (4)

Where $\ell_P$ is the Planck constant and $\gamma$ the Immirzi parameter. Numerical evidence for equation (4) was first given in [22] and a compatible result was estimated in [24]. Since the minimum non-vanishing spin is $j = \frac{1}{2}$, this implies that

$$v_{\text{gap}} \geq \frac{1}{8} \ell_P^3 \gamma^\frac{3}{2} \equiv v_o.$$ (5)

From the existence of the volume gap, it follows that there is a maximum value of $N_\Gamma$, because there is a maximum number of nodes for graphs in $H_V$, as every node carries a minimum volume $v_o$. Therefore a region of volume equal or smaller than $V$ contains at most

$$n_V = \frac{V}{v_o}$$ (6)

nodes. Equation (4) bounds also the number of allowed area quantum numbers, because too large a $j_{\text{max}}$ would force too large a node volume. Therefore $N_{\{j_l\}}$ is also finite. Finally, since the dimension of the space of the intertwiners at each node is finite and bounded by the value of spins, it follows that also the number $N_{\{v_n\}}$ of individual volume quantum numbers
is bounded. Then (3) shows immediately that the dimension of $\mathcal{H}_V$ is finite. Let us bound it explicitly.

We start by the number of graphs. The number of nodes must be smaller than $n_V$, given in (6). The number $N_V$ of four-valent graphs with $n$ nodes is bounded by

$$N_V \leq n^{4n}$$

because each node can be connected to other $n - 1$ four times.

Equation (4) bounds the spins. Since we must have $V \geq v_{\text{gap}}$, we must also have

$$j \leq j_{\text{max}} \leq 32 \frac{V^2}{P^2} = \frac{1}{2} n_V^2.$$  

In a graph with $n$ nodes there are at most $4n$ links (the worst case being all boundary links), and therefore there are at most $\left(2j_{\text{max}} + 1\right)^{4n}$ spin assignments, or, in the large $j$ limit, $\left(2j_{\text{max}}\right)^{4n}$.

That is

$$N_{\{j\}} \leq \left(2j_{\text{max}}\right)^{4n} \leq n^{8n}.$$  

Finally, the dimension of the intertwiner space at each node is bounded by the areas associated to that node:

$$\dim K_{j_1,j_2,j_3,j_4} = \dim \text{Inv}_{SU(2)} (\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4})$$

$$= \min ((j_1 + j_2, j_3 + j_4) - \max ((j_1 - j_2), (j_3 - j_4))) + 1$$

$$\leq 2 \max (j_{\in\mathbb{N}}) + 1 \leq 4 \max (j_{\in\mathbb{N}})$$

with the last step following from $\max (j_{\in\mathbb{N}}) \geq 1/2$. Thus on a graph with $n$ nodes, the maximum number of combinations of eigenvalues is limited by:

$$N_{\{v\}} \leq (4j_{\text{max}})^n = 2^n n^{2n}.$$  

Combining equations (3), (7), (9) and (10), we have an explicit bound on the dimension of the space of states with volume less than $V = nV_v$:

$$\dim \mathcal{H}_V \leq (cnV_v)^{14n}$$

where $c$ is a number. For large $nV_v$ we can write

$$S_V \equiv \log \dim \mathcal{H}_V \leq 14 nV_v \log nV_v$$

which is the entropy associated to this Hilbert space. Explicitly

$$S_V \leq 14 \frac{V}{v_0} \log \frac{V}{v_0} \sim V \log V.$$  

In the large volume limit, when the eigenvalues become increasingly dense, this corresponds to a density of states $\nu(V) \equiv d(\dim \mathcal{H}_V)/dV$ similarly bounded

$$\nu(V) \leq 14 \left[ \log \left( \frac{cV}{v_0} \right) + 1 \right] \left( \frac{cV}{v_0} \right)^{14n}.$$  

### 3. Von Neumann proper volume entropy

In the previous section, we have observed that the dimension of the space of quantum states (with four-valent, finite-volume nodes) with total volume less than $V$ is finite. This result
implies that there is a finite von Neumann volume entropy associated to statistical states describing volume measurements.

The simplest possibility is to consider the micro-canonical ensemble describing the volume measurement of a region of space. That is, we take Volume to be a macroscopic (or thermodynamic, ‘coarse grained’) variable, and we write the corresponding statistical microstate that maximizes entropy. If the measured volume is in the interval \( I_V = [V - \delta V, V] \), with small \( \delta V \), then the corresponding micro-canonical state is simply

\[
\rho = \frac{\mathcal{P}_{V,\delta V}}{\dim \mathcal{H}_{V,\delta V}}
\]

where \( \mathcal{P}_{V,\delta V} \) is the projector on \( \mathcal{H}_{V,\delta V} \), namely the span of the eigenspaces of eigenvalues of the volume that are in \( I_V \).

Explicitly, the projector can be written in the form

\[
\mathcal{P}_{V,\delta V} \equiv \sum_{v \in I_V} |\Gamma, j_l, v \rangle \langle \Gamma, j_l, v |
\]

The von Neumann entropy of (15) is

\[
S = -\text{Tr}[\rho \log \rho] = \log \dim \mathcal{H}_{V,\delta V} < SV \sim V \log V.
\]

It is interesting to consider also a more generic state where \( \rho \sim p(V) \), for an arbitrary distribution \( p(V) \) of probabilities of measuring a given volume eigenstate with volume \( V \). For this state, the probability distribution of finding the value \( V \) in a volume measurement is

\[
P(V) = \nu(V) p(V)
\]

and the entropy can be written as the sum of two terms

\[
S = \int dV \nu(V) p(V) \log(p(V)) = S_P + S_{\text{Volume}}
\]

where the first

\[
S_P = -\int dV P(V) \log(P(V))
\]

is just the entropy due to the spread in the outcomes of volume measurements, while the second

\[
S_{\text{Volume}} \equiv S - S_P = \int dV P(V) \log(\nu(V))
\]

can be seen as a proper volume entropy. The bound found in the previous section on \( \nu(V) \), which indicates that \( \log(\nu(V)) \) grows less that \( V^2 \), shows that this proper volume entropy is finite for any distribution \( P(V) \) whose variance is finite. \( S_{\text{Volume}} \) can be viewed as the irreducible entropy associated to any volume measurement.

4. Lower bound

Let us now bound the dimension of \( \mathcal{H}_V \) from below. The crucial step for this is to notice the existence of a maximum \( \delta V \) in the spacing between the eigenvalues of the operator \( \hat{V}_V \). For instance, if we take a node between two large spins \( j \) and two \( \frac{1}{2} \) spins, the volume eigenvalues
have decreasing spacing, with maximum spacing for the lowest eigenvalues, of the order $v_0$. Disregarding irrelevant small numerical factors, let us take $v_0$ as the maximal spacing.

Given a volume $V$, let, as before, $n_V = V/v_0$ and consider spin networks with total volume in the interval $I_\nu = [(n - 1)v_0, n v_0]$. Let $N_m$ be the number of spin networks with $m$ nodes that have the total volume $\nu$ in the interval $I_\nu$. For $m = 1$, there is at least one such spin network, because of the minimal spacing. For $m = 2$, the volume $\nu$ must be split between the two nodes: $\nu = \nu_1 + \nu_2$. This can be done in at least $n - 1$ ways, with $\nu_1 \in I_p$ and $\nu_2 \in I_{n-p}$ and $p$ running from 1 to $n - 1$. This possibility is guaranteed again by the existence of the maximal spacing. In general, for $m$ nodes, there are

$$N_{n,m} = \binom{n - 1}{m - 1}$$

(23)

different ways of splitting the total volume among nodes. This is the number of compositions of $n$ in $m$ subsets. Finally, the number $m$ of nodes can vary between 1 and the maximum $n$, giving a total number of possible states larger than

$$N_n = \sum_{m=1}^{n} N_{n,m} = \sum_{m=1}^{n} \binom{n - 1}{m - 1} = 2^{n-1}.$$  

(24)

From which it follows that

$$\dim \mathcal{H}_V \geq 2^{n-1}.$$  

(25)

Can all these states be realised by inequivalent spin networks, with suitable choices of the graph and the spins? To show that this is the case, it is sufficient to recognise that there exists at least one (however peculiar) example of spin network for each sequence of $v_n$: given an arbitrary sequence of $v_n$, we can always construct a graph formed by a single one dimensional chain of nodes, each (except the two ends) with two links connecting to the adjacent nodes in the chain and two links in the boundary. All these spin networks exist and are non-equivalent to one another.

To be clear, there will be of course many more graphs for any given volume configuration, but given that we are looking for a lower bound, it is sufficient to find one example for every sequence of $v_n$. A graph composed by a one dimensional chain of nodes, each (except the two ends) with two links connecting to the adjacent nodes in the chain and two links in the boundary, leaves complete freedom to the assignment of volume to every node. Thus, for the purposes of extracting a lower bound we must count at least one graph for each sequence $v_n$.

Therefore we have shown that there are at least $2^{n-1}$ states with volume between $V - v_0$ and $V$. In the large volume limit we can write

$$\dim \mathcal{H}_V \geq 2^{n} = 2^n \pi$$

(26)

so that the entropy satisfies

$$c V \leq S \leq c' V \log V$$

(27)

with $c$ and $c'$ constants.

5. Higher valent graphs and zero volume eigenvalues

In this section we briefly discuss the case of higher valent graphs and the exclusion of states with zero volume eigenvalues.
The reason for restricting the above analysis to four-valent graphs, as outlined in the introduction, is that having restricted the state space to not include zero eigenvalues the bound on the number of spin configurations arises by considering the minimal non zero eigenvalue $\nu_{\text{gap}}$ in conjunction with the bound (4), used later in (8). To our knowledge, it is not currently known if a bound such as (4) applies to spin network states defined on higher-valent graphs, which leaves us unable to investigate whether the corresponding volume entropy could be finite [25].

The volume spectrum of LQG includes the vanishing eigenvalue. The interest in studying the restriction of the quadrivalent state space to states with non vanishing volume eigenvalues is justified à posteriori, from the observation underlined in this article that a von Neumann entropy associated to the 3-volume turns out to be finite because of the bound (4) discovered by Brunnemann and Rideout.

The problem with including zero eigenvalues is apparent from the fact that the right hand side of equations (6), (13) and (14) become divergent. The possibility of regularizing these divergences has not been considered here and is not obvious to us how this could be achieved.

We have seen however that a good definition of the entropy associated to volume measurements exists in a state space which may be a viable choice for a theory of quantum gravity. In the quadrivalent case, the only obstacle arises from having zero volume eigenvalues in the spectrum, the physical relevance of which, if any, is not clear. In particular, since the semiclassical limit corresponds to a limit of large quantum numbers, a theory built on a restricted state space as above would be expected to yield the same semiclassical limit as a theory which includes states with vanishing volume eigenvalues. On the other hand, considering at the classical level spacelike regions with zero 3-volume does not appear to be of any physical relevance.

6. Discussion

The geometrical entropy associated to surfaces of given Area plays a significant role in the current discussions of the quantum nature of spacetime. Here, we have shown that under suitable conditions, it is also possible to define a well behaved a von Neumann entropy associated to measurements of the volume of a region of space. We pointed out that the main obstacle for doing so arises from divergences directly linked to having a vanishing volume eigenvalue in the spectrum of the volume operator. The state space arising from restricting to states with non vanishing volume eigenvalues is a bona fide Hilbert space, expected to reproduce the same semiclassical limit. We have left open the question whether one should consider building a theory on this state space definition, using the well possessedness of volume entropy as a guide to the choice of state space, or attempt to tame these divergences.

We close with some speculative remarks, discussing possible physical roles played by the volume entropy studied in this article:

(i) In the classical low energy limit, volume and area are related by $V \sim A^{\frac{3}{2}}$. The volume entropy we have considered above is bounded by $S_V \sim V \log V \sim A^{\frac{3}{2}} \log A$, and thus may exceed the Bekenstein bound $S < S_A \sim A$. Volume entropy requires accessing the bulk, and the result is not necessarily the same for an outside observer. Therefore, this does not appear to violate the versions of the Bekenstein bound that only refer to external observables.

(ii) It has been recently pointed out that the interior of an old black hole contains surfaces with large volume [26, 27] and that the large volume inside black holes can play an important role in the information paradox [10, 28]. The results presented here may serve to quantify the corresponding interior entropy.
(iii) An interesting possibility is that a notion of entropy associated to the volume of space might provide an alternative to Penrose’s Weyl curvature hypothesis [29]. For the second principle of thermodynamics to hold, the initial state of the universe must have had low entropy. Penrose proposes to address this low entropy by taking into consideration the entropy associated to gravitational degrees of freedom. His hypothesis is that the degrees of freedom which have been activated to bring the increase in entropy from the initial state are the ones associated to the Weyl curvature tensor, which in his hypothesis was null in the initial state of the universe. A definition of the bulk entropy of space, which, as would be expected, grows with the volume, could perhaps perform the same role as the Weyl curvature degrees of freedom do in Penrose’s hypothesis: the universe had a much smaller volume close to its initial state, so the total available entropy was low—regardless of the matter entropy content—and has increased since then, just because for a space of larger volume we have a greater number of states describing its geometry.

(iv) Finally, it is intriguing to ponder whether the existence of a volume entropy, which would be larger for larger volumes, implies the existence of an entropic force driving to larger volumes. That is, could there be a statistical bias for transitions to geometries of greater volume? Generically, the growth of the phase space volume is a driving force in the evolution of a system: in a transition process, we sum over our states, so more available states for a given outcome imply greater probability of that outcome. Any concrete conclusions regarding this point would of course require the dynamics of the theory to be explicitly taken into account.

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