Mixed models as an alternative to Farima.

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Abstract

We construct a new process using a fractional Brownian motion and a fractional Ornstein-Uhlenbeck process of the Second Kind as building blocks. We consider the increments of the new process in discrete time and, as a result, we obtain a more parsimonious process with similar autocovariance structure to that of a FARIMA. In practice, variance of the new increment process is a closed-form expression easier to compute than that of FARIMA.

1 Introduction

Models, such as FARIMA or fractional exponential process (FEXP) may be adequate for modeling long and short dependencies observed in financial data [19][10]. In our paper, we introduce another process in discrete time, the mixed fractional Gaussian noise (mfGn) with similar autocovariance structure to the previous ones, i.e, its autocovariance function captures long and short correlations.

There is two main reasons for doing this. The first reason is to reduce the model risk introduced by incorrect calibration, i.e., parameters might be estimated with error, they may not be kept up-to-date, and so forth. Both models capture the short and long range dependencies. However a mfGn is more parsimonious model than a FARIMA one since for the former, we have to estimate only three parameters, Hurst and gamma parameters and the variance, but for the latter we have the AR and MA lag polinomials, the fractional integrated parameter and the variance, with a total of p+q+2 parameter.

The second reason is that, even theoretically, the autocovariance of a FARIMA process is well-known; it seems extremely difficult to implement computationally [5]. On the other hand, we want to implement a new model with an easy close-form expression for the autocovariance function. Its discrete version is more parsimonious and easy to compute with the consequent reduction of numerical errors involving calculations. For example, in the calculation of the risk of a position or the pricing of a financial instrument.
Finally, a discrete model depends on the time aggregation or systematic sampling. For example, if we assume a FARIMA process, follows a model of the type

$$\Phi(L)y_t = \Theta(L)\varepsilon_n$$

where $t = 0,1,2,\ldots$, $\Phi(L)$ and $\Theta(L)$ are lag polynomials and $\varepsilon_n$ is an error term. Conversely, the temporally aggregated series, $Y_T$, follows the model

$$\beta(L)Y_T = \xi(L)\varepsilon_n$$

where $T = 0,k,2k,\ldots$, $\beta(L)$ and $\xi(L)$ are aggregate lag polynomials and the operator $L$ is in $T$ time units, running in $kt$ periods. The variable $\varepsilon$ is an error term. In the case of a mfGn, the continuous time model, is not affected by the sampling frequency.

2 Fractional Autoregressive Integrated Moving Average

A time series $\tilde{X}_n$ is said to be a FARIMA($p$, $d$, $q$) process if it follows the equation

$$\Phi_p(L)(1-L)^d(\tilde{X}_n - \mu) = \Theta_q(L)\varepsilon_n$$

where $\varepsilon_n$ is a sequence of i.i.d. gaussian random variables.

Let $\Delta$ be the difference operator defined as $\Delta X_n = X_n - X_{n-1}$. Then the difference series $\Delta^d(\tilde{X}_n - \mu) = (1-L)^d(\tilde{X}_n - \mu)$ follows a stationary and invertible ARMA($p,q$) model with $L$ the lag operator, $d \in (-\frac{1}{2}, \frac{1}{2})$ the fractional integration parameter, and the AR polynomial, and the MA polynomial respectively given by

$$\Phi_p(L) = 1 - \phi_1L - \phi_2L^2 - \cdots - \phi_pL^p$$

$$\Theta_q(L) = 1 - \theta_1L - \cdots - \theta_qL^q$$

the AR polynomial, and the MA polynomial respectively.

The model has strong memory because the $\theta_i$ coefficients in its MA representation do not decay over time to zero, implying that the past shock $\varepsilon_i$ of the model has a permanent effect on the series.

2.1 Evaluation of FARIMA autocovariance function

As mention already in the introduction, the autocovariance of a FARIMA process is extremely difficult to implement computationally. For example,
a very simple procedure is to compute the autocovariances from the MA representation.

\[ Z_n = \Phi_p(L)^{-1}(1 - L)^{-d}\Theta_q(L)\varepsilon_n = \sum_{n=0}^{\infty} \phi_n^z \varepsilon_n \]  

(2)

with \( \psi_0 = 1 \).

Then, the autocovariance of a FARIMA process is:

\[ \gamma_k = \sum_{j=0}^{\infty} \psi_j^z \psi_{j+|k|}^z \sigma_\varepsilon^2 \]  

(3)

The drawback is that, because \( \psi_j \) declines hyperbolically, many terms are needed for an accurate approximation.

A seemingly simple alternative is to numerically integrate over the spectrum:

\[ \gamma_k = \int_{-\pi}^{\pi} f_z(\omega)e^{ix\omega} d\omega \]  

(4)

where the spectrum of the FARIMA process, \( f_z(\omega) \), is easily computed. However, numerical integration for each \( k \) does rapidly get prohibitively slow.

A computationally optimal autocovariance function of a FARIMA process for implementation is:

\[ \gamma_i = \sigma_\varepsilon^2 \sum_{k=-q}^{q} \sum_{j=1}^{p} \psi_k \zeta_j C(d, p + k - i, \rho_j), \]  

(5)

where \( \rho_1, \ldots, \rho_p \) are the roots (possibly complex) of the AR polynomial, and

\[ \psi_k = \sum_{s=|k|}^{q} \theta_s \theta_{s-|k|}, \quad \zeta_j^{-1} = \rho_j \left[ \prod_{i=1}^{p} (1 - \rho_i \rho_j) \prod_{m=1, m \neq j}^{p} (\rho_j - \rho_m) \right] \]

where \( \theta_0 = 1 \). \( C \) is defined as

\[ C(d, h, \rho) = \frac{\Gamma(1 - 2d) (d)_h}{\Gamma(1 - d)(1 - d)_h} \times \left[ \rho^{2p} F(d + h, 1; 1 - d + h; \rho) + F(d - h, 1; 1 - d - h; \rho - 1) \right] \]

Here \( \Gamma \) is the gamma function, \( \rho_j \) are the roots of the AR polynomial, and \( F(a, 1; c, \rho) \) is the hypergeometric function. See more technical details in [14] and [5].
3 Mixed Fractional Gaussian process

3.1 Auxiliary Processes

In this section, we introduce the processes use below. We follow mainly [10], [12], and [9]. We consider, throughout, some underlying complete probability space \((\Omega, \mathcal{F}, P)\) and denote by \(\mathcal{F}_t\) the sigma field representing the publicly available information at time \(t\). Typically, \(\mathcal{F}_t = \sigma(X_s : s \leq t)\), the sigma field generated by past and present values of the process in question \(X\), often called the history, up to and including time \(t\).

3.1.1 Fractional Gaussian Noise

To capture the long range dependence in the data we use a fractional Gaussian noise (fGn). First, we define fractional Brownian motion: The fractional Brownian motion (fBm) with Hurst parameter \(H \in (0, 1)\) is a Gaussian process \(B^H = \{B^H_t, t \in \mathbb{R}\}\) having the properties

(i) \(B^H_0 = 0\),

(ii) \(\mathbb{E}B^H_t = 0, t \in \mathbb{R}\),

(iii) \(\mathbb{E}B^H_t B^H_s = \frac{1}{2}(|t|^{2H}+|s|^{2H} - |t-s|^{2H})\), \(s, t \in \mathbb{R}\).

(iv) In the special case of \(H = \frac{1}{2}\). \(W\) denotes a standard Brownian motion with independent increments.

If \(B^H\) is fBm, then the increment sequence \(Z^H_k = B^H_{k+1} - B^H_k\) for \(k \in \mathbb{Z}\) is called fractional Gaussian noise.

**Proposition 3.1.** The process \(Z^H\) has the following properties

1. \(Z^H\) is stationary,

2. \(\mathbb{E}Z^H_k = 0\),

3. \(\mathbb{E}(Z^H_k)^2 = \sigma^2 = \mathbb{E}(Z^H_1)^2\)

4. The autocovariance function of the process \(Z^H\) is given by

\[
\gamma_k = \frac{\sigma^2}{2}(|k + 1|^{2H} - 2|k|^{2H} + |k - 1|^{2H})
\]

5. If \(\frac{1}{2} < H < 1\) then \(Z^H\) has long range dependence and \(\gamma_k > 0\).
3.1.2 Fractional Ornstein-Uhlenbeck process of the Second Kind

To capture the short range dependence, we use a process proposed by Kaarakka and Salminen [11]. Let $B^H = \{B^H_t : t \geq 0\}$ be a fractional Brownian motion with self-similarity parameter $H \in (0, 1)$ with the properties above.

We derive a new Gaussian process by means of Doob’s transform of $B^H$:

$$X^{(D,\alpha)}_t := e^{-\alpha t} B^H_{a_t}, \quad t \in \mathbb{R}$$

where $\alpha > 0$ and $a_t := a(t, H) := H e^{\alpha t/H}/\alpha$. The covariance function of $X^{(D,\alpha)}_t$ can be computed from definition (3.1.1) point 4. For $t > s$ we obtain

$$\mathbb{E}(X^{(D,\gamma)}_t X^{(D,\gamma)}_s) = \frac{1}{2} \left( \frac{H}{\alpha} \right)^{2H} \left( e^{\alpha (t-s)} + e^{-\alpha (t-s)} - e^{\alpha (t-s)} \left( 1 - e^{\alpha (t-s)} \right)^{2H} \right)$$

$X^{(D,\alpha)}_t$ is a stationary process. In particular, using (7) and the self-similarity property of the fractional Brownian motion, it may be proven that $X^{(D,\alpha)}_t$ is normally distributed with mean zero and variance $(H/\alpha)^{2H}$, for all $t$.

Consider next the process $Y^{(\alpha)}_t$ defined via

$$Y^{(\alpha)}_t := \int_0^t e^{-\alpha s} d\tilde{Y}^{(1)}_s$$

The process $Y^{(\alpha)}_t$ has stationary increments. Using $Y^{(\alpha)}$ the process $X^{(D,\alpha)}_t$ may be viewed as the solution of the equation

$$dX^{(D,\alpha)}_t = -\alpha X^{(D,\alpha)}_t dt + dY^{(\alpha)}_t, \quad (9)$$

with random initial value $X^{(0,\alpha)}_0 = B^H_{a_0} \sim \mathcal{N}(0, (H/\alpha)^{2H})$.

We now consider the Langevin SDE with $Y^{(1)}$ as the driving process:

$$dU^{(D,\gamma)}_t = -\gamma U^{(D,\gamma)}_t dt + d\tilde{Y}^{(1)}_t, \quad \gamma > 0, \quad (10)$$

The solution can be expressed as

$$U^{(D,\gamma)}_t = e^{-\gamma t} \int_{-\infty}^{\infty} e^{\alpha s} d\tilde{Y}^{(1)}_s = e^{-\gamma t} \int_{-\infty}^{t} e^{(\gamma-1)s} d\tilde{B}^{H}_{a_s}, \quad \gamma > 0,$$

where $\tilde{Y}^{(1)}_s$ is the two sided $Y^{(1)}$ process and $\alpha = 1$ in $a_t$.

**Definition 3.1.** The process $U^{(D,\gamma)}_t$ defined in (11) or, equivalently, via the SDE (10) is called the fractional Ornstein-Uhlenbeck process of second kind ($fOU_2$) with initial value $B^H_{H/\alpha}$.
Remark 3.1. By Proposition 3.11 in [11], the covariance of the process $U^{(D, \gamma)}_t$ decays exponentially and has short range dependence.

Remark 3.2. The process $U^{(D, \gamma)}$ has quadratic variation zero.

Proof. By proposition 3.4, [11], the sample paths of $U^{(D, \gamma)}$ are Hölder of order $\beta$ for $\forall \beta < H$. For $\frac{1}{2} < \beta < H$,

$$\left( U^{(D, \gamma)}_t - U^{(D, \gamma)}_s \right)^2 \leq K_T(\omega) |t - s|^{2\beta}. $$

Therefore, for any sequence $\pi_n$ of partitions of the interval $[0, T]$ such that $|\pi_n| \to 0$.

$$[U^{(D, \gamma)}, U^{(D, \gamma)}] = \mathbb{P} - \lim_{|\pi_n| \to 0} \sum_{t_k \in \pi_n} \left( U^{(D, \gamma)}_{t_k} - U^{(D, \gamma)}_{t_{k-1}} \right)^2$$

$$
\leq K_T(\omega) \lim_{|\pi_n| \to 0} \sum_{t_k \in \pi_n} |t_k - t_{k-1}|^{2\beta}
\leq K_T(\omega) \lim_{|\pi_n| \to 0} |\pi_n|^{2\beta - 1} T
= 0$$

almost surely as $n$ tends to infinity. \hfill $\Box$

Proposition 3.2. [11] The autocovariance of $U^{(D, \gamma)}$ has the kernel representation

$$
\mathbb{E}(U^{(D, \gamma)}_t U^{(D, \gamma)}_s) = H(2H - 1)H^{2H - 2} e^{-\gamma(t+s)} \int_{-\infty}^{-\infty} \int_{-\infty}^{-\infty} e^{(\gamma - 1 + \frac{H}{H})(u+v)} |e^{u/H} - e^{v/H}|^{2(1-H)} dm dv
$$

We give another expression in discrete time for the autocovariance function which may be used for computational calculations. To simplify the notation, we write $Q^{n,T}(k-j)$ at lag $(k-j)$ instead of $Q^{n,T}(t_k - t_j)$.

Proposition 3.3. Consider the time interval $[0, T]$ and an equidistant partition $\Pi := \{ t_i = \frac{iT}{n}; 0 \leq i \leq n \}$. Let $t_j, t_k \in \Pi$ with $j \leq k$. The autocovariance of $U^{(D, \gamma)}$ at lag $(k-j)$ is

$$
Q^{n,T}(k-j) = \mathbb{E}(U^{(D, \gamma)}_{t_j} U^{(D, \gamma)}_{t_k}) = H^{-2(\gamma - 1)H} H (2H - 1) e^{-\gamma(t_j + (k-j) - t_j)} \times
\left( \frac{H}{\gamma H} B((\gamma - 1)H + 1, 2H - 1) + \int_{H}^{m_j + (k-j) - t_j} m^{\gamma - 1} B(H/m; (\gamma - 1)H + 1, 2H - 1) dm \right)
$$

with $B(\cdot, \cdot)$ for the beta function and $B(\cdot; \cdot, \cdot)$ for the incomplete beta function with $B(1; \cdot, \cdot) = B(\cdot, \cdot)$.

Proof. See Appendix A. \hfill $\Box$
Corollary 3.1. Let denote the autocovariance of the increment process of \(U^{(D,\gamma)}\) at lag \(m\) by \(C^{n,T}(m)\). Then its autocovariance function takes the form:

\[
C^{n,T}(m) = 2Q^{n,T}(m) - [Q^{n,T}(m - 1) + Q^{n,T}(m + 1)]
\]

with \(0 \leq m \leq n - 1\).

**Proof.**

\[
C^{n,T}(m) = \mathbb{E} \left( \Delta U_{t_{m+1}}^{(D,\gamma)} \Delta U_{t_1}^{(D,\gamma)} \right) \\
= \mathbb{E} \left[ (U_{t_{m+1}}^{(D,\gamma)} - U_{t_m}^{(D,\gamma)}) (U_{t_1}^{(D,\gamma)} - U_{t_0}^{(D,\gamma)}) \right] \\
= \mathbb{E} \left( U_{t_{m+1}}^{(D,\gamma)} U_{t_1}^{(D,\gamma)} \right) + \mathbb{E} \left( U_{t_m}^{(D,\gamma)} U_{t_0}^{(D,\gamma)} \right) - \mathbb{E} \left( U_{t_m}^{(D,\gamma)} U_{t_1}^{(D,\gamma)} \right) \\
- \mathbb{E} \left( U_{t_{m+1}}^{(D,\gamma)} U_{t_0}^{(D,\gamma)} \right) \\
= 2Q^{n,T}(m) - [Q^{n,T}(m - 1) + Q^{n,T}(m + 1)]
\]

Remark 3.3. We give two examples for proposition (3.3) and corollary (3.1) respectively. Autocovariance function for a process \(U^{(D,0.3)}\) with parameter \(H = 0.9\), cf. Figure 1, Appendix B and autocovariance function for an increment process \(U^{(D,0.3)}\) with parameter \(H = 0.9\), cf. Figure 2, Appendix B.

3.2 Mixed Fractional Gaussian process

Now we are ready to construct a family of continuous processes \(X\) which is Gaussian and it has the following properties

(i) Let \(A = \{t_0, \ldots, t_n\}\) with \(0 = t_0 < \ldots < t_n = t\), be a partition of \([0,t]\) and \(\|A\| = \max_{1 \leq k \leq n} |t_k - t_{k-1}|\). Then the quadratic variation process \((X)_t = \lim_{\|A\| \to 0} \sum_{k=0}^{n} (X_{t_{k+1}} - X_{t_k})^2 = t\).

(ii) The corresponding increment process \(\Xi_{t_k} = X_{t_k} - X_{t_{k-1}}\) has a similar autocovariance structure to that of a FARIMA, i.e, it captures the short and long range dependences.

We construct this process as \(X_t = \sigma W_t + B_t^H + U_t^{(D,\gamma)}\) with \(H \in (\frac{1}{2}, 1)\) and \(\gamma > 0\). We assume the three processes are mutually independent and the fractional Gaussian noise \(B_t^H\) and the fractional Ornstein-Uhlenbeck process of the second kind \(U_t^{(D,\gamma)}\) may have different Hurst parameter. Then, its increment process is defined as

\[
\Xi_{t_k} = \sigma \Delta W_{t_k} + Z_{t_k}^H + \Delta U_{t_k}^{(D,\gamma)}
\]
Notice, that at first, the fractional Gaussian noise $Z^H_t$ and the increment fractional Ornstein-Uhlenbeck process of the second kind $\Delta u^{(D,\gamma)}_t$ may have different Hurst parameter. However, from a statistical view, ....

The fractional Gaussian noise process $Z^H_t$ captures the long range dependencies and its autocovariance function behaves asymptotically as a FARIMA. However, if the data contains strong short correlations, it fails to capture them, cf. Fig. ??.

To model the short range correlations, we add the increment fractional Ornstein-Uhlenbeck process of the second kind. As a result, we obtain a process with similar autocovariance structure as FARIMA or fractional exponential process.

In many applications in continuous time, such as in a delta hedging problem, we need to use Itô’s formula. However, we need to justify its use since $fBm$ and $fOU_2$ have both quadratic variation zero. Therefore, an increment Brownian motion $B_t$ is added so that the process $X$ has a continuous quadratic variation as $\langle X \rangle_t = \sigma^2 t$. Moreover, the increments of the Brownian process are independent so the autocovariance function of $Z$ does not change. Then, by proposition 3.4, the structure of its increment process $Z$ is similar to that of FARIMA process, see Figs. ?? and 8. The new process $Z$ is more parsimonious than that of a FARIMA with a consequent reduction of errors in model estimation and forecasting.

**Proposition 3.4.** Define $\Xi$ as a mixed fractional Gaussian process, i.e.,

$$\Xi_{tk} = \sigma \Delta B_{tk} + Z^H_{tk} + \Delta u^{(D,\gamma)}_{tk}$$

with $H > \frac{1}{2}$ and $\gamma > 0$ in an interval $[0, T]$. Then the variance is computed as

$$\mathbb{E}(\Xi_{tk})^2 = \mathbb{E}(W_{tk+1} - W_{tk})^2 + \mathbb{E} \left( B^H_{tk+1} - B^H_{tk} \right)^2 + \mathbb{E} \left( u^{(D,\gamma)}_{tk+1} - u^{(D,\gamma)}_{tk} \right)^2$$

**Remark 3.4.** Because $\Xi$ is stationary its variance may be written in terms of the first increment as

$$\mathbb{E}(\Xi_{t0})^2 = \sigma(t_1 - t_0) + (t_1 - t_0)^{2H} + 2 \left( Q^{n,T}(0) - Q^{n,T}(1) \right)$$

$$= \sigma t_1 + t_1^{2H} + 2 \mathbb{E} \left( u^{(D,\gamma)}_{t0} \right)^2 - 2 \mathbb{E} \left( u^{(D,\gamma)}_{t1} u^{(D,\gamma)}_{t0} \right).$$

By construction, we can apply Itô’s formula to our new process $X$. We give a definition of forward integral due to 3.

**Definition 3.2.** Let $t \leq T$ and $X : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a continuous process. The forward integral of a process $Y$ with respect to $X$ along equidistant $\pi_n$ partition of the interval $[0, T]$ such that $|\pi_n| \rightarrow 0$ is

$$\int_0^t Y_s dX_s := \lim_{n \rightarrow \infty} \sum_{k=1}^n Y_{tk} (X_{tk+1} - X_{tk}) \quad \text{with} \quad t_k \in \pi_n \quad (14)$$

when the $\mathbb{P}$-a.s limit exists.
If $X$ is a continuous process with continuous quadratic variation $\langle X \rangle_t$ such that $\langle X \rangle_t = \lim_{\|\Lambda\| \to 0} \sum_{k=0}^{n}(X_{t_{k+1}} - X_{t_k})^2$ $P$-a.s then we have the following Itô’s formula according to [13].

**Theorem 3.5.** Let $X : [0, \infty) \longrightarrow \mathbb{R}^1$ be a continuous function with continuous quadratic variation $\langle X \rangle_t$, and $f \in C^{1,2}([0, t] \times \mathbb{R})$ a twice differentiable real function. Then

$$f(X_t, t) = f(X_0, 0) + \int_0^t f_s(X_s, s) \, ds + \int_0^t f_x(X_s, s) \, dX_s + \frac{1}{2} \int_0^t f_{xx}(X_s, s) \, d\langle X \rangle_s$$

for any $t \geq 0$.

**Remark 3.6.** Note that the integral $\int_0^t f_x(X_s, s) \, dX_s$ is understood as a forward integral along the partition $\pi_n$.

4 Temporal aggregation

We shortly present the impact of temporal aggregation on a FARIMA process.

**Definition 4.1.** Let $n = mT$ with $m \geq 2$ and $L$ the lag operator, then the series

$$Y_T = \left( \sum_{i=0}^{m-1} L^i \right) y_{mT}$$

represents the $m$-period nonoverlapping aggregates of $y_n$.

A FARIMA($p,d,q$) process follows the equation

$$\Phi_p(L)(1 - L)^d y_n = \Theta_q(L) \varepsilon_n$$

The original process and the aggregated one are linked via a polynomial. We multiply both sides of equation (16) by the polynomial

$$\prod_{j=1}^{p} \left[ \frac{(1 - \delta^m L^m)}{(1 - \delta L^m)} \right]^{d+1} \left( \frac{1 - B^m}{1 - B} \right)^{d+1}$$

As a result, the aggregate series $Y_T$ follows a FARIMA($p,d,N$) with

$$N = \left( p + d + 1 + \frac{q - p - d - 1}{m} \right)$$

and autocovariance function as

$$\gamma_Y(j) = \left( \sum_{i=0}^{m-1} L^i \right)^{2d+1} \gamma_y(mj + (d + 1)(m - 1))$$
Conversely, the mixed fractional Gaussian noise has variance and auto-
covariance depending on the length interval $T$ and the sampling frequency $n$ as

$$\mathbb{E}\left( \Xi_{kT}^2 \right) = \frac{T}{n} + \left( \frac{T}{n} \right)^{2H} + \text{const}$$

and

$$\mathbb{E}\left( \Xi_{kT} \Xi_{jT} \right)^2 = \frac{1}{2} \left( \frac{T}{n} \right)^{2H} \left( |(k - j)| + 1 \right)^{2H} - 2|k - j|^{2H} + |(k - j) - 1|^{2H} + C^n,T(k - j)$$

with $H > \frac{1}{2}$ respectively.

With finite length aggregation, the autocovariance structure of the aggregates would depend on the exact autocovariance structure of the mixed fractional Gaussian noise.

5 Hedging and Expected Shortfall for Options

The need to quantify risk arises in many different contexts and has been
strongly motivated by the fear of systemic risk, i.e. the danger that problems
in a single financial institution may spill over and, in extreme situations,
disrupt the normal functioning of the entire financial system.

Lessons learned from the global banking crisis are now spotlight in a
review of risk management at all levels within financial institutions and
regulatory authorities. Solvency II, Europe’s risk-based reform of insurance
regulation, and Basel III, a global regulatory framework for banks on capital
adequacy, leverage ratios and liquidity standards, will fundamentally shift
the focus of the financial industry for many years to come. A central issue is
the measurement of risk. Among the existing approaches, Basel III mentions
Value-at-Risk (VaR) for raising capital requirements for the trading book
and complex securitisation exposures, and Solvency II the related notion of
expected shortfall (ES) is used in the definition of target capital. ES captures
the skewed and heavy-tailed pay-off functions.

Calculation of VaR and ES essentially consist of determining the loss
distribution function $F_X(x) = P(X \leq x)$, or functionals describing this
distribution function such as its mean, and variance. In order to achieve this,
a proper calibrated model is needed which captures the main features of the
dynamics of the value of a financial portfolio.

The effects of driving stochastic processes mixed fGn and FARIMA are
compared on the forecast of risk measures (VaR, ES) of a financial position.
5.1 Characterization of risk measures

In this paper, we pay attention to those measures applied in the framework of Basel III and Solvency II.

**Definition 5.1.** For a financial position $X$ with distribution $\mathbb{P}$, we define its Value-at-Risk at level $\alpha$ ($\text{VaR}^\alpha$) as

\[
\text{VaR}^\alpha(X) := -q_X(\alpha) = \inf \{ m \mid \mathbb{P}(X \leq m) \geq \alpha \} \tag{19}
\]

where $q_X(\alpha)$ is the quantile function of $X$. From the point of view of a practitioner, $\text{VaR}$ is the maximum loss he may expect over a given holding or horizon period, with a certain level of confidence.

However, the subadditivity property fails to hold for $\text{VaR}$ in general, so $\text{VaR}$ is not a coherent risk measure. For subadditivity measure diversification always leads to risk reduction, while for measures which fail this condition, diversification may increase in their value; cf. [1].

One possibility of a coherent measure which is defined in terms of $\text{VaR}$ would be Conditional Value at Risk or Expected Shortfall.

**Definition 5.2.** Let $X$ be the financial position on a specified time horizon $T$ and some specified probability level $\alpha \in (0, 1)$. The Expected Shortfall is then defined as

\[
\text{ES}^\alpha = \frac{1}{\alpha} \int_0^\infty \text{VaR}^p(X) \, dp \tag{20}
\]

**Remark 5.1.** If the distribution function of $X$ is continuous then it can be shown that $\text{ES}^\alpha = \mathbb{E}(X \mid X \leq \text{VaR}^\alpha)$.

5.2 Hedging and elimination of randomness

The dynamics of the price of an underlying asset of a derivative product may be modelled according to a mixed fractional Gaussian process as

\[
dS_t = \mu S_t dt + \sigma S_t dX_t \tag{21}
\]

Therefore, the dynamics of the option price is a function $F(S_t, t) \in C^{1,2}([0, t] \times \mathbb{R})$ and according to Itô’s formula

\[
F(S_t, t) = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial S_t} (\mu S_t dt + \sigma S_t dX_t) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 F}{\partial S_t^2} dt \\
= \sigma S_t \frac{\partial F}{\partial S_t} dX_t + \left( \mu S_t \frac{\partial F}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 F}{\partial S_t^2} + \frac{\partial F}{\partial t} \right) dt \tag{22}
\]

Let $\xi_t$ represent the value of a portfolio of one option with value $C(S_t, t)$ and $-\eta_t$ underlying stocks with price $S_t$. The minus sign of $\eta_t$ means we
hold a short position in the underlying asset. Therefore, the value of the portfolio at time $t$ is $\xi_t = C_t - \eta_t S_t$. We can write

$$d\xi_t = dC_t - \eta_t dS_t$$

$$= \sigma S_t \frac{\partial C}{\partial S_t} dX_t + \left( \mu S_t \frac{\partial C}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} + \frac{\partial C}{\partial t} \right) dt - \eta_t \left( \mu S_t dt + \sigma S_t dX_t \right)$$

$$= \sigma S_t \left( \frac{\partial C}{\partial S_t} - \eta_t \right) dX_t + \left( \mu S_t \left( \frac{\partial C}{\partial S_t} - \eta_t \right) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} + \frac{\partial C}{\partial t} \right) dt$$

where we have substituted equations (22) and (21) into $dC(S_t, t)$ and $dS_t$ respectively. Now, if

$$\eta_t = \frac{\partial C(S_t, t)}{\partial S_t}$$

we eliminate the randomness of the portfolio and by fact that $C(S_t, t)$ and $S_t$ are correlated implies that the option price will change by

$$dC_t = \eta_t dS_t$$

(23)

Example 5.1. Suppose we have a portfolio with one option and one stock with value $\xi_t = C_t + S_t$. Using the delta approximation (23) its value is a linear function of $S_t$ alone as $\xi_t = (\eta_t + 1)S_t$ and any change is given by $d\xi_t = (\eta_t + 1)dS_t$ at any $t$. We assume the distribution of the returns for the stock to be normal,

$$r_{u,t} \sim N(\mu_{u,t}, \sigma_{u,t}^2)$$

where $\mu_{u,t}$ is the conditional mean calculated according to equation (26) and $\sigma_{u,t}^2$ is the conditional variance calculated by equation (27). Therefore, the distribution of the returns of the portfolio is also normally distributed as

$$r_{p,t} \sim (\eta_t + 1)N(\mu_{u,t}, \sigma_{u,t}^2)$$

Let us denote the value at risk on the underlying by $VaR_u^\alpha$, where $\alpha$ is the confidence level, then

$$VaR_u^\alpha = \mu_{u,t} S_{t-1} + \sigma_{u,t} \Phi^{-1}(\alpha) S_{t-1}$$

(24)

Let us denote the value at risk on the portfolio by $VaR_p^\alpha$, where $\alpha$ is probability and recall that $VaR$ is a quantile of the loss distribution of the portfolio.
then:

\[
\begin{align*}
\alpha &= \mathbb{P}(\xi_t - \xi_{t-1} \leq VaR^\alpha_p) \\
&= \mathbb{P}((\eta_t + 1)(S_t - S_{t-1}) \leq VaR^\alpha_p) \\
&= \mathbb{P}((\eta_t + 1)e^{\eta_{u,t}} - 1 \leq VaR^\alpha_p) \\
&= \mathbb{P}((\eta_t + 1)S_{t-1}(e^{\eta_{u,t}} - 1) \leq VaR^\alpha_p) \\
&= \mathbb{P}\left(r_{u,t} - \mu_{u,t} \leq \log\left(\frac{1}{\eta_t + 1} \frac{VaR^\alpha_p}{S_{t-1}} + 1 \right) - \mu_{u,t}\right) \sigma_{u,t}
\end{align*}
\]

Using the normality assumption of returns

\[
\Phi^{-1}(\alpha) = \left[\log\left(\frac{1}{\eta_t + 1} \frac{VaR^\alpha_p}{S_{t-1}} + 1 \right) - \mu_{u,t}\right] \frac{1}{\sigma_{u,t}}
\]

for small \(\frac{1}{\eta_t + 1} \frac{VaR^\alpha_p}{S_{t-1}}\) we can use the approximation \(\log(1 + x) \approx x\).

Hence, the value at risk for one unit of the portfolio at confidence level \(\alpha\) is:

\[
VaR^\alpha_p = (\eta_t + 1)\sigma_{u,t}\Phi^{-1}(p)S_{t-1} + \mu_{u,t}(\eta_t + 1)S_{t-1} = (\eta_t + 1)VaR^\alpha_u
\]

If we assume that the underlying stock log price is modelled by a mixed fGn process [21] then the expected shortfall for the portfolio at level \(\alpha\) is

\[
ES^\alpha_p = -\left(\mu_{u,t} + \frac{1}{\alpha}\sigma_{u,t}\phi((\eta_t + 1)VaR^\alpha_u)\right)\xi_{t-1}.
\]

Now, \(\sigma_{u,t}\), which is a function of Hurst and gamma parameter, can be evaluated via proposition [34]. Note that we used the result that mixed fGn has continuous quadratic variation equal to \(\langle B \rangle_t\) so we were able to use Ito's formula to justify the linear approximation of the increment value of the option.

6 Backtesting study

We check if mixed models are good alternative to Farima models to forecast risk of financial data which exhibits short-long range dependences.

6.1 Prediction of conditional mean and variance at time \(n + k\)

We assume our random variables are jointly gaussian. We denote the best linear predictor of \(X_{n+k}\) as \(\hat{X}_{n+k} = \sum_{i=1}^{n} a_{i,k} X_i\) and use the mean squared error (MSE) as our criterio, \(\|X_{n+k} - \hat{X}_{n+k}\|_2^2 = E((X_{n+k} - \hat{X}_{n+k})^2)\). Assuming that the process is weakly stationary, let \(\tilde{x}_{n+k}\) denote the minimum mean square error linear predictor of \(x_{n+k}\) given the data \(\tilde{X}' = (x_1, \ldots, x_n)\), the mean \(\mu\) and the autocovariances \(\gamma_l\), with \(l = 0, \ldots, n - 1\).
\[ \hat{x}_{n+k} = \mu + g'_k \Gamma^{-1}_n (\bar{x} - \mu) \]  

(26)

where \( g'_k = (\gamma_{n+k-1}, \ldots, \gamma_k) \) and, by the law of total variance, the conditional variance for the forecast is

\[ V_{n+k} = \gamma_0 - g'_k \Gamma^{-1}_n g_k \]  

(27)

We compute the predictors of mean and variance by means of Durbin-Levinson algorithm [7].

6.2 Comparison of risk model performance.

Assessment of the accuracy of the expected shortfall forecasts should ideally be done by monitoring the performance of the model in the future. However, it is expected that violations are only observed rarely and a long period of time would be required. Backtesting is a procedure used to compare risk model performances over a period in the past.

In our study, we are not concerned with the estimation of the parameters of the models but to compare their performances. Therefore, we simulate data from a FARIMA model. This is two fold, first it allows us to control the dependence of the data and second use the FARIMA forecast of risk as a benchmark to evaluate the performance of the that of the mixed model.

We assume the model parameters are fixed except the gamma parameter which is calibrated so the predicted conditional variance of the mixed model approximates that of the FARIMA. We assume that the data is independent. This is questionable assumption since we are concerned with the correlation in the data but we may get an inside of the validation of the modeling as a first approximation. Future research with more formal test of violation ratios would be need to obtain a better conclusion.

We processed with the calculation of the autocovariance functions according to equations (5), (12) and proposition (2.1-4). The Expected Shortfall and Value at risk is then evaluated using section (2.1-2) of the general theory.

We analyze the results by means of violation ratios and volatility. If the return on a particular day exceeds the forecast, then we count it as a violation. Let \( \varsigma \) be a Bernoulli random variable with probability the risk level of the ES, where \( \varsigma_1 \) is the number of violations and \( \varsigma_0 \) is the number of non violations then the violation ratio is:

\[ \Psi = \frac{\varsigma_1}{\mathbb{E}(\varsigma)} \]  

(28)

As a result, numerical results are presented in the next Table. None of the models perform well...
7 Appendix A - Calculation of $fOU_2$ autocovariance

We start the calculation of the $fOU_2$ kernel representation from proposition \[\text{[1]}\].

Recall that the process $U^{(D,\gamma)}$ was defined in \[\text{[1]}\] as

$$U^{(D,\gamma)}_t = e^{-\gamma t} \int_{-\infty}^{t} e^{(\gamma-1)s} \, dZ_s$$

$$= H^{-(\gamma-1)H} e^{-\gamma t} \int_{0}^{a_t} s^{(\gamma-1)H} \, dZ_s$$

where $a_t := a(t, H) := He^{t/H}$ and $\gamma > 0$. A change of variable was made as $s = He^{\gamma/H}$.

To calculate the integral, we start defining the constant $C_1 \equiv H(2H-1)$ then

$$C_1 e^{-\gamma(t+s)} \int_{-\infty}^{t} \int_{-\infty}^{s} H^{2(H-1)} \frac{e^{(\gamma-1+1/H)(u+v)}}{e^{u/H} - e^{v/H}} \, du \, dv$$

$$= C_1 e^{-\gamma(t+s)} \int_{-\infty}^{t} \int_{-\infty}^{s} e^{(\gamma-1)(u+v)} \frac{e^{\frac{1}{H}(u+v)}}{e^{u/H} - e^{v/H}} \left(H - e^{u/H} - e^{v/H}\right)^{2(H-1)} \, du \, dv$$

$$= H^{-2(\gamma-1)H} C_1 e^{-\gamma(t+s)} \int_{-\infty}^{t} \int_{-\infty}^{s} H^{2(\gamma-1)H} e^{(\gamma-1)\frac{(u+v)H}{H}} e^{\frac{u}{H}} e^{\frac{v}{H}} \times \left(H e^{u/H} - He^{v/H}\right)^{2(H-1)} \, du \, dv$$

$$= H^{-2(\gamma-1)H} C_1 e^{-\gamma(t+s)} \int_{-\infty}^{t} \int_{-\infty}^{s} \left(H e^{\frac{u}{H}} e^{\frac{v}{H}}\right)^{(\gamma-1)H} e^{\frac{u}{H}} e^{\frac{v}{H}} \times \left(H e^{u/H} - He^{v/H}\right)^{2(H-1)} \, du \, dv$$

Next, we make a change of variable $m = He^{u/H}$ and $n = He^{v/H}$. The constant term is now $C_2 = H^{-2(\gamma-1)H} C_1$. Hence,

$$= C_2 e^{-\gamma(t+s)} \int_{0}^{\infty} \int_{0}^{\infty} (mn)^{(\gamma-1)H} |m-n|^{2(H-1)} \, dm \, dn$$

$$= C_2 e^{-\gamma(t+s)} \left( \int_{0}^{\infty} \int_{0}^{\infty} (mn)^{(\gamma-1)H} |m-n|^{2(H-1)} \, dm \, dn + \int_{a_s}^{\infty} \int_{0}^{\infty} (mn)^{(\gamma-1)H} |m-n|^{2(H-1)} \, dm \, dn \right)$$

$$= C_2 e^{-\gamma(t+s)} \left( 2 \int_{0}^{\infty} \int_{0}^{m} (mn)^{(\gamma-1)H} |m-n|^{2(H-1)} \, dm \, dn + \int_{a_s}^{\infty} \int_{0}^{\infty} (mn)^{(\gamma-1)H} |m-n|^{2(H-1)} \, dm \, dn \right)$$
We continue by making a second change of variable, \( \theta = \frac{n}{m} \), with the result

\[
C_2 e^{-\gamma(t+s)} \left( 2 \int_0^{a_s} m^{2\gamma H - 1} \int_0^{1} \theta^{(\gamma - 1)H} |1 - \theta|^{2(H-1)} d\theta dm \\
- \theta^{2(H-1)} d\theta dm + \int_{a_s}^{a_t} m^{2\gamma H - 1} \int_0^{\theta/m^{(\gamma - 1)H} |1 - \theta|^{2(H-1)} d\theta dm \right)
\]

\[
= C_2 e^{-\gamma(t+s)} \left( 2 B((\gamma - 1)H + 1, 2H - 1) \int_0^{a_s} m^{2\gamma H - 1} dm + \int_{a_s}^{a_t} m^{2\gamma H - 1} B(a_s/m; (\gamma - 1)H + 1, 2H - 1) dm \right)
\]

Finally, we obtain the desired result as

\[
E(U_1(D, \gamma), U_2(D, \gamma)) = H^{-2(\gamma - 1)H} H(2H - 1) e^{-\gamma(t+s)} \left( \frac{a_s^{2\gamma H}}{\gamma H} B((\gamma - 1)H + 1, 2H - 1) \right.
\]

\[
\left. + \int_{a_s}^{a_t} m^{2\gamma H - 1} B(a_s/m; (\gamma - 1)H + 1, 2H - 1) dm \right)
\]
Figure 1: Autocovariance effect of a fractional Ornstein-Uhlenbeck process of second kind $U_t^{(D,\gamma)}$ with parameters $\gamma = 0.1$ and $H = 0.9$ in the autocovariance function of a fractional Gaussian noise with parameter $H = 0.9$. 
Figure 2: Autocovariance effect of a fractional Ornstein-Uhlenbeck process of second kind $U^{(D,\gamma)}_t$ with parameters $\gamma = 1.5$ and $H = 0.7$ in the autocovariance function of a fractional Gaussian noise with parameter $H = 0.7$.

Figure 3: In the left picture, autocovariance functions of an increment fractional Ornstein-Uhlenbeck process of second kind $U^{(D,\gamma)}_t$ at different $H$ and fixed $\gamma = 0.1$. The second picture shows the autocovariances of the same process with fixed Hurst parameter $H = 0.7$ at different gammas $G$.

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