Generation of concurrence between two qubits locally coupled to a one dimensional spin chain

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We consider a generalized central spin model, consisting of two central qubits and an environmental spin chain (with periodic boundary condition) to which these central qubits are locally and weakly connected either at the same site or at two different sites separated by a distance $d$. Our purpose is to study the subsequent temporal generation of entanglement, quantified by concurrence, when initially the qubits are in an unentangled state. In the equilibrium situation, we show that the concurrence survives for a larger value of $d$ when the environmental spin chain is critical. Importantly, a common feature observed both in the equilibrium and the non-equilibrium situations while the latter is created by a sudden but global change of the environmental transverse field, is that the two qubits become maximally entangled for the critical quenching. Following a non-equilibrium evolution of the spin chain, our study for $d \neq 0$, indicates that there exists a threshold time above which concurrence attains a finite value. Additionally, we show that the number of independent decohering channels (DCs) is determined by $d$ as well as the local difference of the transverse field of the two underlying Hamiltonians governing the time evolution. The qualitatively similar behavior displayed by the concurrence for critical and off-critical quenches, as reported here, is characterized by analyzing the non-equilibrium evolution of these channels. The concurrence is maximum when the decoherence factor or the echo associated with the most rapidly DC decays to zero; on the contrary, the condition when the concurrence vanishes is determined non-trivially by the associated decay of one of the intermediate DCs. Analyzing the reduced density of a single qubit, we also explain the observation that the dephasing rate is always slower than the unentanglement rate. We further show that the maximally and minimally decohering channels show singular behavior which can be explained by invoking upon a quasi-particle picture.

PACS numbers:

I. INTRODUCTION

The notion of entanglement, that emerged from the pioneering work of Einstein, Podolsky and Rosen [1], is a key concept of quantum computation and quantum information theories [2–5]. Given the recent interest in the studies of quantum correlations in the context of quantum critical systems [6–9], there have been numerous efforts directed to understanding the connection between quantum information and quantum phase transitions (QPTs) [10, 11]. Entanglement is usually quantified through two quantum information theoretic measures: (i) concurrence [17–19], a separability based approach to measure the quantum correlation and (ii) quantum discord [20–22], a measurement based approach for estimating the non-classical correlations present in a bipartite system. There have also been numerous studies on the entanglement entropy which is another important tool to probe the entanglement between two blocks of a composite system obtained by measuring the von Neumann entropy associated with the reduced density matrix of one of the blocks [23].

It is now established that the effect of quantum criticality gets imprinted in the behavior of the ground state correlation which becomes maximum at the quantum critical point (QCP); for example, the concurrence can detect as well as characterize a QPT [10, 11]. On the other hand, the entanglement, arising due to the interaction between the system and its environment, leads to decoherence [24]. There exists a plethora of the studies investigating the effects induced by the environment on the quantum information processing [25]; simultaneously, the dynamical control of the decoherence is also being investigated extensively [26, 27].

The central spin model (CSM), consisting of a single qubit (spin-1/2) globally coupled to an environmental spin chain, is an important prototypical model to study the Loschmidt echo (LE), also known as the decoherence factor (DF) characterising the decoherence of the qubit; this has been studied for both equilibrium [28, 30] and non-equilibrium [31, 39] time evolution. Moreover, the concurrence [40] and the quantum discord [41] have been shown to satisfy the universal scaling law as predicted by the Kibble–Zurek argument [42, 43] when a parameter of the environmental Hamiltonian is driven linearly across a QCP. Additionally, a generalized central spin model (GCSM) where two spins are globally coupled to an environmental spin chain, with a periodic boundary condition (PBC), is also studied for probing the concurrence and quantum discord generated between the qubits when the composite system evolves in time [44, 46]; the concurrence generation is found to be maximum for the critical spin chain [47, 35]. In connection to the experimental studies, a QPT has already been observed with ultracold atoms in an optical lattice [49]. A possible realization of a one dimensional XY chain has also been proposed [50]. Furthermore, using NMR quantum simulator, it has been experimentally confirmed that the LE shows a dip at the QCP of a finite antiferromagnetic Ising spin chain,
thereby establishing it an ideal detector of a QCP \cite{51}.

Recently, there have been investigations \cite{52,53} of the unentangled (i.e., the decoherence) between two distant qubits initially entangled and connected to two different sites of the spin chain that evolves in time. The state transfer quality between two external qubits of a spin chain has also been investigated by analyzing the entanglement between them \cite{54}. Given the previous studies, we address the reverse question. Is there a temporal generation of entanglement between a pair of qubits, initially prepared in an unentangled state, connected at the same site and also two different sites (separated by a distance \(d\)) of the environmental spin chain? To address this particular issue, we consider two situations: (i) when the spin chain (chosen to be a one-dimensional transverse Ising chain) evolves temporally in time following the local coupling of the qubits, referred to as the equilibrium situation. (ii) There is an additional sudden global quench of the transverse field of the environmental Hamiltonian in addition to the local coupling, referred to as the non-equilibrium situation. To the best of our knowledge, ours is the first work to investigate the generation of the concurrence in the non-equilibrium situation using a GCSM with a local coupling.

We briefly summarize our main results at the outset: firstly, we are working in the weak coupling limit considering a PBC for the environmental spin chain. Our observation for the equilibrium situation is that the entanglement generation is of very small magnitude although, in the vicinity of the QCP of the environment, the concurrence becomes maximum and remains finite even when the two qubits are separated by a large distance. In the non-equilibrium situation, our investigation suggests that the concurrence, of much higher magnitude than the equilibrium case, can be induced by the global and sudden quench of the transverse field; this concurrence eventually decays with time. We explain this generic behavior of the concurrence by analyzing the echoes associated with different decohering channels (DCs) with time; we observe that the number of independent channels is dictated by the separation \(d\) and local difference of the transverse field of the two underlying Hamiltonian governing the time evolution. The decay of most rapidly DC is responsible for maximum amount of entanglement while the decay of the concurrence is non-trivially related to the decay of one of the intermediate decaying channel. For a finite separation between the qubits, we establish that the concurrence attains a non-zero value after a threshold time; this is attributed to the behavior of different DC up to the threshold time. Additionally, we show that the respective dephasing rate associated with each qubit is always slower than the unentanglement rate between the qubits. Furthermore, we characterize the distance dependent behavior of different DCs following a critical quench by making resort to a quasi-particles picture.

The paper is organized in the following manner. In Sec\textsc{II} we introduce the CGCM consisting of two qubit locally connected to two sites of an Ising chain with a transverse field. In parallel, we define the concurrence derived from the \(4 \times 4\) reduced density matrix of the two qubits obtained by tracing out the environmental degrees of freedom; this density matrix contains different LEs corresponding to different DCs associated with the environmental evolution. In Sec\textsc{III} the results obtained for the equilibrium situation are presented while in Sec\textsc{IV} we discuss the non-equilibrium behavior of the concurrence. We analyze the behavior of concurrence observed in equilibrium as well as non-equilibrium cases investigating the temporal evolution of different decohering channels. Finally, we make concluding remarks in Sec\textsc{VI}.

\section{II. MODEL}

We consider a GCSM in which two non-interacting qubits are connected by a local interaction to an environmental spin chain, chosen to be a one-dimensional ferromagnetic transverse Ising spin model, in such a way that the local transverse field of the environmental spin chain gets modified. The composite system, thus, is a generalization of the central spin model \cite{29}, in which a single spin-1/2 particle (qubit) is globally connected to all the spins of the environmental spin chain with an interaction Hamiltonian; the schematic diagram of the GCSM is shown in Fig. (1). The combined Hamiltonian \(H_T\), comprising of an environmental transverse Ising Hamiltonian \(H_E\) with \(N\) number of spins and interaction Hamiltonian \(H_{SE}\) of two qubits, is given by

\[
H_T = H_{SE} + H_E. \quad (1)
\]

Here,\[H_E = -J \sum_n \sigma_n^x \sigma_{n+1}^x - \lambda \sum_n \sigma_n^z, \quad (2)\]

where \(\sigma^z\)’s are the usual Pauli matrices. \(\lambda\) and \(J\) (set equal to unity below) are the transverse magnetic field and ferromagnetic cooperative interactions, respectively. We consider a PBC \(\sigma^z_{N+1} = \sigma^z_1\). The interaction Hamiltonian for two qubits \(A\) and \(B\) connected at different sites of the environment is given by\[H_{SE} = -\delta(|\uparrow\rangle\langle\uparrow|_A \otimes \sigma^z_p + |\downarrow\rangle\langle\downarrow|_B \otimes \sigma^z_q); \quad (3)\]

here, \(|\uparrow\rangle_{A,B}\) is an eigenstate of \(\sigma^z_{A,B}\) satisfying \(\sigma^z_{A,B} |\uparrow\rangle_{A,B} = |\uparrow\rangle_{A,B}\) while \(\sigma^z_{p,q}\) denote the environmental spin at \(p\) and \(q\)-th site, respectively; these sites are separated by a distance \(d\). \(\delta\) is the coupling strength and we shall work in the limit \(\delta \rightarrow 0\). Clearly, the interaction Hamiltonian \(\text{(3)}\) suggests interaction with the qubits modifies the local transverse field of the environment.

In order to study the generation of concurrence between two qubits, we take a completely unentangled (direct product) initial state given by:\[|\phi\rangle_{AB} = \frac{1}{2}(|\uparrow\rangle_A + |\downarrow\rangle_A) \otimes (|\uparrow\rangle_B + |\downarrow\rangle_B). \quad (4)\]
The initial state for the composite system is then given by $|\psi(\lambda_i, t = 0)\rangle = |\phi\rangle_{AB} \otimes |\eta(\lambda_i, t = 0)\rangle$, where $|\eta(\lambda_i, t = 0)\rangle$ is the initial ground state of the environmental Hamiltonian $H_E$ given in Eq. (1).

Focusing on the non-equilibrium situation, we consider a sudden quenching of the transverse field which is instantaneously changed from an initial value $\lambda_i$ to a final value $\lambda_f$ and study the subsequent temporal evolution of the composite system. We note that the equilibrium situation corresponds to $\lambda_f = \lambda_i$. Depending upon the state of the qubits, the interaction Hamiltonian leads to four channels of evolution for the environment. The channel Hamiltonians $H_{\alpha\beta}$ with $\lambda_f$ governing the dynamics are given by

$$H_{14}(\lambda_f) = H_E(\lambda_f),$$
$$H_{\uparrow\uparrow}(\lambda_f) = H_E(\lambda_f) - \delta(\sigma^x_1 + \sigma^x_q),$$
$$H_{\downarrow\uparrow}(\lambda_f) = H_E(\lambda_f) - \delta\sigma^y_q,$$
$$H_{\uparrow\downarrow}(\lambda_f) = H_E(\lambda_f) - \delta\sigma^y_q.$$  

(5)

The time-evolved state of the composite system is given by

$$|\psi(t)\rangle = \frac{1}{2} \left( |\uparrow\uparrow\rangle \otimes |\eta_{\uparrow\uparrow}(t)\rangle + |\downarrow\downarrow\rangle \otimes |\eta_{\downarrow\downarrow}(t)\rangle ight.$$
$$+ |\uparrow\downarrow\rangle \otimes |\eta_{\uparrow\downarrow}(t)\rangle + |\downarrow\uparrow\rangle \otimes |\eta_{\downarrow\uparrow}(t)\rangle \right),$$  

(6)

where $|\alpha\beta\rangle$ represents the state for two qubits and environmental evolved state $|\eta_{\alpha\beta}(t)\rangle$ is given by

$$|\eta_{\alpha\beta}(t)\rangle = e^{-iH_{\alpha\beta}t}|\eta(\lambda_i, t = 0)\rangle,$$  

(7)

where $\lambda_i$ is the initial homogeneous transverse field same for all sites i.e., $\lambda_n = \lambda_i$.

One can construct the reduced density matrix of the qubits by tracing out the environmental degrees of freedom from the composite density matrix constructed from $|\psi(t)\rangle$. The reduced density matrix for the two qubits system in the basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ is given by

$$\rho_s(t) = \frac{1}{4} \begin{bmatrix} 1 & d_{\uparrow\uparrow,\downarrow\downarrow} & d_{\uparrow\downarrow,\downarrow\uparrow} & d_{\downarrow\uparrow,\downarrow\downarrow} \\ d_{\uparrow\uparrow,\downarrow\downarrow}^* & 1 & d_{\uparrow\downarrow,\downarrow\uparrow}^* & d_{\downarrow\uparrow,\downarrow\downarrow}^* \\ d_{\uparrow\downarrow,\downarrow\uparrow}^* & d_{\uparrow\downarrow,\downarrow\uparrow}^* & 1 & d_{\downarrow\uparrow,\downarrow\downarrow}^* \\ d_{\downarrow\uparrow,\downarrow\downarrow}^* & d_{\downarrow\uparrow,\downarrow\downarrow}^* & d_{\downarrow\uparrow,\downarrow\downarrow}^* & 1 \end{bmatrix},$$

(8)

where $d_{\alpha\beta,\gamma\lambda}(t) = (\eta_{\alpha\beta}(t)|\eta_{\gamma\lambda}(t)\rangle$). The DFs or the echoes corresponding to different channels are $D_{\alpha\beta,\gamma\lambda}(t) = |d_{\alpha\beta,\gamma\lambda}(t)|^2$ and its explicit form is the following

$$D_{\alpha\beta,\gamma\lambda}(t) = |(\eta_{\alpha\beta}(t)e^{iH_{\alpha\beta}(\lambda_f)\gamma}\eta_{\alpha\beta}^\dagger(\lambda_f)|^2.$$  

(9)

Now, using the density matrix $\rho_s(t)$ given in Eq. (8), one can compute the concurrence between the two qubits. We shall follow the Wooter’s definition of concurrence given by

$$C(\rho_s) = \max(0, \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4),$$

(10)

where $\epsilon_i$’s are the square root of the eigenvalues in a descending order of the non-Hermitian matrix $M = \rho_s\rho_s^\dagger$ with $\rho_s$ defined as

$$\rho_s = (\sigma^y \otimes \sigma^y)\rho_s^\dagger(\sigma^y \otimes \sigma^y).$$

(11)

Therefore, one readily concludes that the concurrence between the two qubits are determined by the DF associated with the four channels.

Let us consider a generic Hamiltonian of a one-dimensional Ising chain in a site dependent transverse field $\lambda_n$ given by

$$H = -\sum_n (\sigma^x_n\sigma^x_{n+1} + \lambda_n\sigma^z_n).$$

(12)

One can obtain the initial Hamiltonian from the above Hamiltonian [12] by setting $\lambda_n = \lambda$ while for the final Hamiltonian $\lambda_n$ becomes different from $\lambda$ at those sites where the qubits are coupled. For the initial homogeneous case ($\lambda_n = \lambda$), the model in Eq. (12) has a QCP at $J = \lambda$ separating ferromagnetic (FM) and quantum paramagnetic (PM) phases. Using Jordan-Wigner transformations followed by Fourier transformation for a homogeneous and periodic chain, the energy spectrum for the Hamiltonian in Eq. (12) is obtained as [55, 56]

$$\varepsilon_q = \pm 2\sqrt{(\lambda + \cos q)^2 + \sin^2 q},$$

(13)

where $q$ is the momentum which takes discrete values given by $q = 2\pi m/N$ with $m = 0 \cdots N - 1$ for a finite system of length $N$.

In order to express the DF given in Eq. (9) in the fermionic representation one has to cast the Hamiltonian in the above basis following Jordan-Wigner transformation. The Hamiltonian in Eq. (12) can be described by a quadratic form in terms of spinless fermions $c_i$ and $c_i^\dagger$

$$H = \sum_{i,j} [c_i^\dagger A_{ij} c_j + \frac{1}{2} (c_i^\dagger B_{ij} c_j^\dagger + \text{h.c.})].$$

(14)
Here, $A$ is a symmetric matrix as $H$ is Hermitian and $B$ is an antisymmetric matrix which follows from the anti-commutation rules of $c_i$‘s. The elements of these matrices thus obtained are:

$$
\begin{align*}
A_{i,j} &= -(J\delta_{i,i+1} + J\delta_{i,j+1}) - 2\lambda_j\delta_{i,j}, \\
B_{i,j} &= -(J\delta_{i,i+1} - J\delta_{i,j+1}),
\end{align*}
$$

(15)

where $\lambda_j$ is the site dependent transverse field.

The Hamiltonian \[14\] can be written in the following form also

$$
H = \frac{1}{2} \Psi^\dagger \mathcal{H} \Psi,
$$

(16)

with $\Psi = (c_i^\dagger, c) = (c_1^\dagger, \ldots, c_N^\dagger, c_1, \ldots, c_N)$ and $\mathcal{H}$ is given by

$$
\mathcal{H} = \begin{bmatrix}
-A & -B \\
B & A
\end{bmatrix}.
$$

(17)

The above Hamiltonian can be diagonalized in terms of the normal mode spinless Fermi operators $\eta_k$ given by the relation \[55\].

$$
\eta_k = \sum_i (g_k(i)c_i + h_k(i)c_i^\dagger),
$$

(18)

where $g_k(i)$ and $h_k(i)$ are real numbers; $g_k(i)$ and $h_k(i)$ are obtained from the real matrices $g$ and $h$, respectively.

The unitary matrix $U$ that diagonalizes the Hamiltonian given in Eq. \[17\] can be constructed from $g_k(i)$ and $h_k(i)$

$$
U = \begin{bmatrix}
g & h \\
h & g
\end{bmatrix}.
$$

(19)

Using this Unitary operator one can also write the fermionic operator in terms of normal modes

$$
c_i = \sum_k (g_k(i)\eta_k + h_k(i)\eta_k^\dagger),
$$

(20)

In terms of the new operators $\eta_k$, the Hamiltonian in Eq. \[17\] takes the diagonal form,

$$
H = \sum_k \Lambda_k \left( \eta_k^\dagger \eta_k - \frac{1}{2} \right),
$$

(21)

with $\Lambda_k$ being the energy of different fermionic modes with index $k$.

In order to study the time evolution of concurrence $C$, we have to first compute the time-dependent DF that constitutes the reduced density matrix of two spins given in Eq. \[8\]. One can use the covariance matrix formalism to determine time evolution of DF associated with the different DCs governed by two different Hamiltonians \[53\] \[57\]. These different Hamiltonians $H_{\alpha\beta}$ shown in Eq. \[8\] are having different set of local transverse fields.

This formalism allows us to write the DF in the following way

$$
D_{\alpha\beta,\gamma\lambda}(t) = |\det(I - R_{\alpha\beta}(t) - R_{\gamma\lambda}(t))|^{1/2}.
$$

(22)

Here, $I$ is an identity matrix and $R_{\alpha\beta}(t)$‘s are the time evolved covariant matrices given by

$$
R_{\alpha\beta}(t) = e^{-iH_{\alpha\beta}t} R(0) e^{iH_{\alpha\beta}t}
$$

(23)

with $R(0) = \langle \eta(\lambda, t = 0)|\Psi\psi^\dagger|\eta(\lambda, t = 0)\rangle$, and its matrix form is given by

$$
R(0) = \begin{bmatrix}
\langle c^\dagger c \rangle & \langle c^\dagger c^\dagger \rangle \\
\langle cc \rangle & \langle cc^\dagger \rangle
\end{bmatrix} = \begin{bmatrix}
h^T h & h^T g \\
g^T h & g^T g
\end{bmatrix},
$$

(24)

where $T$ denotes the transpose of a matrix. This $2N \times 2N$ initial covariant matrix is composed of four blocks and all of these blocks are having the same dimension of $N \times N$.

### III. Equilibrium Study

![FIG. 2: (Color online) Plot (a) shows the equilibrium behavior of concurrence $C$ as a function of time with both the qubits connected at the same site i.e., $d = 0$, for different phases (FM and PM phase) including the QCP of the environmental chain; Plot (b) shows the variation of $C$ as a function of time when $d = 1$. In both the cases, the generation of entanglement is of small magnitude. We consider $N = 100$ and $\delta = 0.1$.](image)

In this section, we shall illustrate the equilibrium behavior of concurrence given in Eq. \[10\] as a function of time when the environment evolves along different DCs originated from the coupling to the qubit. In this case $\lambda_i = \lambda_f = \lambda$, for all the sites except the sites where the qubits are locally connected. Figure (2a) depicts the behavior of concurrence for $d = 0$ while Fig. (2b) shows it for $d = 1$. When the parameter value is chosen to be close to the quantum critical value ($\lambda = 0.99$), the concurrence initially grows as a function of time showing a prominent dip at $t = N/v = N/2$; this is because of the constructive interference of quasi-particles generated due
to the local connection of the qubit to the spin chain having group velocity \( v = 2 \) at the QCP. Thus, the finite size effect is manifested in this dip of the LE which is prominent for \( \lambda = 1 \). In the PM phase also, concurrence shows rapid fluctuations of small amplitude at around \( t = N/2 \); otherwise, it shows a time independent behavior. What is noteworthy that even if the qubits are initially unentangled, there is a generation concurrence only due to the local coupling during the temporal evolution of the composite system.

Additionally, maximum concurrence is generated when \( \lambda \) is close to the critical value (when there is a diverging length scale) rather than in the PM and FM phase. That the value of the concurrence attains maximum for \( \lambda \approx 1 \) is independent of the distance \( d \) between the qubits and hence is a universal observation. We however note that the maximum of the magnitude of the concurrence thus generated is very small in the equilibrium case in contrast to the non-equilibrium case to be discussed in the next section.

Furthermore, a non-zero concurrence survives when \( d \) increases within the small \( \delta \) limit; on the other hand, in the large \( \delta \) limit, concurrence is smaller compared to the small \( \delta \) case (see Fig. (3)). A large value of \( \delta \) makes the value of the local transverse field off-critical so that the correlation become short ranged and consequently the entanglement between two distant qubits is vanishingly small while for small \( \delta \) the value of the local transverse field stays critical and hence a long-range correlation exists in the environment.

IV. NON-EQUILIBRIUM STUDY

We shall now extend the previous studies to the situation in which the environmental spin chain undergoes a global sudden quenching, i.e., the transverse field \( \lambda \) is suddenly changed from an initial value \( \lambda_i \) to a final value \( \lambda_f \). In this non-equilibrium situation, two external qubits become more strongly entangled as compared to the earlier equilibrium situation. Results presented in Fig. (4a) suggest: (i) the concurrence generation is maximum when the spin chain is quenched to the QCP starting from the FM phase. (ii) Quenching within the same phase yields entanglement of smaller magnitude between the qubits as compared to the quenching between two different phases. Furthermore, concurrence remains non-zero for longer time if the quenching is performed within the same phase. Finally, there is a prominent peak in \( C \) appearing at time \( t = t^* \) that becomes smaller for higher quench amplitude. Additionally, there exists a secondary peak at \( t = t_2 \) after the primary peak at \( t^* \). On the other hand, we show in Fig. (4b) that \( t^* \) decreases with \( \delta \), in fact, is inversely proportional to \( \delta \) as shown in the inset. We note that the above features of \( C \) is qualitatively identical for all types of quenching protocols.

Figure (5) shows that for the critical quenching starting from the FM phase, \( C \) becomes maximum for \( d = 0 \) while for other cases with \( d \neq 0 \), it attains a finite value only after a threshold time \( t_{PH} \); this threshold time increases with the increasing \( d \). One can note that \( t^* \) attains a higher value for \( d = 0 \) as compared to the case \( d \neq 0 \); in the latter case, \( t^* \) almost remains constant. Additionally, we observe that \( C \) stays at non-zero vale for longer time for \( d = 0 \) as compared to \( d \neq 0 \) case.
V. INTERPRETATION USING CHANNEL ANALYSIS

In this section, we shall analyze the results presented in previous sections using the DF (or the LEs) associated with the different DCs which in turn lead to the generation of entanglement between the qubits which are initially unentangled. For this purpose, let us fix our notation first: \(L(\alpha \beta, \gamma \lambda) = |\det(I-R_{\alpha \beta}-R_{\gamma \lambda})|\); \(L(\downarrow \downarrow, \uparrow \uparrow) = \mu_1\); periodic boundary condition ensures that \(L(\downarrow \downarrow, \uparrow \downarrow) = L(\downarrow \uparrow, \downarrow \uparrow) = \mu_2\); \(L(\uparrow \uparrow, \downarrow \uparrow) = L(\uparrow \downarrow, \downarrow \downarrow) = \mu_3\) (hence, this is valid for all \(d\) as well as for equilibrium and non-equilibrium cases) and \(L(\downarrow \uparrow, \downarrow \downarrow) = \mu_4\). We therefore have to deal with these four DCs to analyze the temporal behavior of \(C\). We shall refer \(\mu_1\), \(\mu_2\), \(\mu_3\) and \(\mu_4\) as DCs in the subsequent discussion and figures. The schematic diagram as shown in Fig. (6) depicts different Hamiltonians \(H_{\uparrow \uparrow}\), \(H_{\uparrow \downarrow}\), \(H_{\downarrow \uparrow}\) and \(H_{\downarrow \downarrow}\) that govern the time evolution of four DCs.

Let us first investigate the behavior of individual channel in equilibrium scenario. When two qubits are coupled at the same site of the environmental chain i.e., \(d = 0\), only three of the four channels mentioned above are independent with corresponding DFs \(\sqrt{\mu_1}\), \(\sqrt{\mu_2}\) and \(\sqrt{\mu_3}\). Figure (7a,b,c) represent the temporal behavior of different DCs for \(d = 0\) when the environment is in the FM phase, at the QCP and in the PM phase, respectively. The other channel \(\mu_4\) does not have a dynamics and hence remains trivially unity for all time due to the fact that the initial state evolves with two identical Hamiltonians both with an additional transverse field \(\delta\) at one site. Furthermore, in the weak coupling limit \(\delta \to 0\), \(\mu_2 \simeq \mu_3\); therefore, one can approximately work with two independent DCs, \(\mu_1\) and \(\mu_2\). These observations lead us to the conclusion that for \(d = 0\) and within the weak coupling limit, the number of independent channels effectively depends on the corresponding difference of the local fields of two Hamiltonians those dictate the time evolution of the initial state; the differences in this case are \(\delta\) for \(\mu_2\) and \(2\delta\) for \(\mu_1\). The other notable point in Fig. (7d,e,f) is \(\mu_2\), calculated in different phases and at QCP, always remains at a higher value than that of the \(\mu_1\). This can be attributed to the fact that \(\mu_1\) exhibits a sharper short time fall than that of \(\mu_2\). This is a manifestation of the difference in local transverse field; in \(\mu_1\) it is of the order \(2\delta\) while \(O(\delta)\) in \(\mu_2\).

Let us now proceed to the case for \(d \neq 0\) and \(\delta \to 0\), where indeed we have three independent channels. Figure (7h,e,f) represent the temporal behavior of different DCs for \(d = 1\) when the environment is in the FM phase, at the QCP and in the PM phase, respectively. Interestingly, \(\mu_4\) in this case is not trivially unity as in the previous case of \(d = 0\); additionally, \(\mu_2\) deviates from \(\mu_3\) as the coupling strength \(\delta\) increases resulting in four independent channels for higher values of \(\delta\). One can see that in the FM phase \(\mu_2\) is always higher than \(\mu_1\) and \(\mu_4\) as shown in Fig. (7h,e,f). On the other hand, in the PM phase or at the QCP \(\mu_4 > \mu_2 > \mu_1\). One can hence conclude that for \(d \neq 0\) the channel \(\mu_4\) is maximally affected in the FM phase as compared to the QCP and PM phase. In this FM phase, \(\mu_1\) and \(\mu_4\) almost overlap with each other; \(\mu_1\) in the above phase shows prominent oscillations as one increases \(d\). In all the cases discussed above, \(\mu_1\) is maximally deviated from unity during its temporal evolution.

One can see from all cases shown the Fig. (7) that the DC \(\mu_2\) almost coincides with \(\mu_3\). This can be attributed to the fact that both of them are governed by the Hamiltonians which are deviated from each other in an identical way i.e., the local transverse field of the two underlying

\[
\begin{align*}
\mu_1 & \equiv |\langle \eta | \eta_{\downarrow \downarrow} \rangle| |\eta_{\downarrow \uparrow} \rangle |\eta_{\uparrow \uparrow} \rangle |\eta_{\uparrow \downarrow} \rangle|^4

\mu_2 & \equiv |\langle \eta | \eta_{\downarrow \downarrow} \rangle| |\eta_{\downarrow \uparrow} \rangle |\eta_{\uparrow \uparrow} \rangle |\eta_{\uparrow \downarrow} \rangle|^4

\mu_3 & \equiv |\langle \eta | \eta_{\downarrow \downarrow} \rangle| |\eta_{\downarrow \uparrow} \rangle |\eta_{\uparrow \uparrow} \rangle |\eta_{\uparrow \downarrow} \rangle|^4

\mu_4 & \equiv |\langle \eta | \eta_{\downarrow \downarrow} \rangle| |\eta_{\downarrow \uparrow} \rangle |\eta_{\uparrow \uparrow} \rangle |\eta_{\uparrow \downarrow} \rangle|^4
\end{align*}
\]
Hamiltonians are deviated by δ from each other. This is valid irrespective of the distance between the two qubits d.

To compare the dephasing rate of a single qubit with the temporal decay of the concurrence which is generated following a non-equilibrium evolution, we invoke upon the reduced density matrix that is obtained by tracing over one of the qubits from the density matrix of two qubits (8): \[ \rho_A(t) = \rho_B(t) = \frac{1}{2} \left[ \frac{1}{\sqrt{\mu_1^2 + \mu_2^2}} \begin{pmatrix} \sqrt{\mu_1^2 + \mu_2^2} & 0 \\ 0 & 1 \end{pmatrix} \right]. \] (25)

Here, \( S_{A(B)} \) denotes the dephasing factor (DP), incorporated in the off-diagonal terms of \( \rho_{A(B)} \), of the qubit \( A(B) \). DP quantifies the loss of coherence of a single qubit which was initially prepared in a pure state with another qubit; the decay time is determined by the dephasing rate. Remarkably, \( S_A = S_B = (\sqrt{\mu_1^2 + \mu_2^2})/\mu_1 \) and is completely independent of \( \mu_1 \) and \( \mu_4 \); this leads to an interesting consequence as we shall elaborate below.

Now, we focus on the non-equilibrium evolution of the channel choosing \( d = 0 \), first. Here, one has three independent channels unless \( \delta \to 0 \) when \( \mu_2 \) coincides with \( \mu_3 ; \mu_4 \) becomes trivially identity as in the equilibrium case. Figure (5a,b,c) show non-equilibrium temporal evolution of \( C \), DC and DP with \( d = 0 \) for the FM, the critical and the PM quenching while the initial state of the environment is in the FM phase, respectively. One can see that the primary peak of the concurrence occurs when \( \mu_1 \to 0 \) and the secondary peak when \( \mu_2 \mu_4 \to 0 \). On the other hand, concurrence becomes vanishingly small when \( \mu_2^{1/2} \to 0 \). Additionally, one can see that in the early time region \( \mu_2 \) almost overlaps with \( \mu_3 \) but in the course of the evolution, these two channels starts to behave differently leading to a visible deviation from each other. Recalling \( S_{A(B)} \) we find that there exists a finite coherence even long after the qubits become unentangled from each other; this implies that the dephasing rate is slower than the rate in which the qubits lose the entanglement. This observation is qualitatively explained as follows: the DCs \( \mu_2 \) and \( \mu_3 \) appear in the DP \( S_A \) and \( S_B \) in an additive manner; on the other hand, concurrence depends on the \( \mu \)'s in a complicated way. We therefore observe a long dephasing time \( T_D \) (dictating the decay of \( S_A \) and \( S_B \)) as compared to unentanglement time \( T_{UE} \) above which the concurrence vanishes between the two qubits. Additionally, \( T_D \gg T_{UE} \) for FM quenching where as \( T_D \) is comparable (of the same order) to \( T_{UE} \) for the PM quenching case and the quantum critical case. All the above observations are independent of the quenching path i.e., sudden quenching within the same phase or to QCP or to a different phase.

In parallel, Fig. (6a,b,c) represent the time evolution of \( C \), DC and DP with \( d = 40 \) for the FM, critical and the PM quenching while the initial state in the FM phase, respectively. For this \( d \not= 0 \) case, the \( \mu_4 \) exhibits improper oscillations with time. \( \mu_4 \) remains the minimally affected channel in the non-equilibrium situations like the equilibrium situations as it always lies close to unity. But

FIG. 7: (Color online) Equilibrium echo for different DCs \( \mu_1, \mu_2 \) and \( \mu_3 \) as a function of time for \( d = 0 \) are plotted when the spin chain is in the FM Phase (a), at the QCP (b) in the PM phase (c). We observe that \( \mu_3 \) superimposes on \( \mu_2 \) leading to two non-trivial independent channels as explained in the text. The above DCs along with \( \mu_4 \) are studied for the situation \( d = 1 \) in the following plots. There is an additional independent channel \( \mu_4 \) which is no longer trivially unity like earlier \( d = 0 \) case. Plot (d) is for the FM phase; the plot (e) and the plot (f) represent the situations when spin chain is quantum critical and in the PM phase, respectively. For both the cases we consider, \( N = 100, \delta = 0.01 \).

FIG. 8: (Color online) Non-equilibrium echos for different DCs, DP and \( C \) are plotted as a function of time with three different situations, quenching to the FM phase with \( \lambda_f = 0.8 \) (a), quenching to the PM phase with \( \lambda_f = 0.99 \) (b) and quenching to the PM phase where \( \lambda_f = 1.8 \) (c). Here, \( N = 100, \delta = 0.1, d = 0 \) and \( \lambda_0 = 0.2 \). \( \mu_4 \) displays the sharpest decay to zero among all the four channels. \( \mu_4 \) exhibits time independent behavior and \( \mu_3 \) vanishes more rapidly than \( \mu_2 \).

FIG. 9: (Color online) Time evolution of concurrence \( C \), DC and DP with \( d = 40 \) for the FM, critical and the PM quenching while the initial state of the environment is in the FM phase, respectively. One can see that the primary peak of the concurrence occurs when \( \mu_1 \to 0 \) and the secondary peak when \( \mu_2 \mu_4 \to 0 \). On the other hand, concurrence becomes vanishingly small when \( \mu_2^{1/2} \to 0 \). Additionally, one can see that in the early time region \( \mu_2 \) almost overlaps with \( \mu_3 \) but in the course of the evolution, these two channels starts to behave differently leading to a visible deviation from each other. Recalling \( S_{A(B)} \) we find that there exists a finite coherence even long after the qubits become unentangled from each other; this implies that the dephasing rate is slower than the rate in which the qubits lose the entanglement. This observation is qualitatively explained as follows: the DCs \( \mu_2 \) and \( \mu_3 \) appear in the DP \( S_A \) and \( S_B \) in an additive manner; on the other hand, concurrence depends on the \( \mu \)'s in a complicated way. We therefore observe a long dephasing time \( T_D \) (dictating the decay of \( S_A \) and \( S_B \)) as compared to unentanglement time \( T_{UE} \) above which the concurrence vanishes between the two qubits. Additionally, \( T_D \gg T_{UE} \) for FM quenching where as \( T_D \) is comparable (of the same order) to \( T_{UE} \) for the PM quenching case and the quantum critical case. All the above observations are independent of the quenching path i.e., sudden quenching within the same phase or to QCP or to a different phase.
the main difference with \( d = 0 \) case is that \( \mu_3 \) and \( \mu_2 \) almost always coincide with each other even when \( \delta \) is not vanishingly small. Therefore, here we have three independent channels \( \mu_1, \mu_2 \) and \( \mu_3 \). It shows that the primary peak of \( C \) occurs when \( \mu_1 \rightarrow 0 \) and the secondary peak is obtained when \( \mu_2 \mu_3 \simeq \mu_2^2 \rightarrow 0 \). The concurrence becomes vanishingly small when \( \mu_3 \rightarrow 0 \). In all the phases \( \mu_2^{1/2} \) and \( \mu_3^{1/2} \) remain finite even after concurrence vanishes. The other notable feature of this finite \( d \) case is that the \( T_{UE} \simeq T_{D}/2 \) for critical and PM quenching. \( T_{D} \gg T_{UE} \) for FM quenching case which has also been observed for \( d = 0 \) situation. One remarkable observation for \( d \neq 0 \) is that up to a threshold time \( t_{TH} \), concurrence remains zero and different DCs overlap with each other; for \( t < t_{TH} \), we see that \( \mu_4 \simeq \mu_1 \) and \( \mu_3^2 \simeq \mu_1 \) and after this threshold time the different DCs move away from each other except for the channels \( \mu_2 \) and \( \mu_3 \). \( C \) can only take a positive value after \( t_{TH} \) as shown in Fig. 5.

Comparing the results presented in Figs. [3] and Figs. [0], we note that for \( d = 0 \), \( H_{\uparrow \uparrow} \) determining the evolution of \( \mu_2 \) has a local transverse field modified by only \( \delta \) with respect to the final unperturbed Hamiltonian \( H_{\uparrow \downarrow} \) in which there is effect of coupling; on the other hand, \( H_{\uparrow \uparrow} \) in \( \mu_3 \) has a local field modified by \( 2 \delta \). Hence, \( \mu_2 \) and \( \mu_3 \) behave differently with time. One can infer that the dynamical evolution of \( \mu_2 \) matches with that of \( \mu_3 \) only in the \( \delta \rightarrow 0 \) limit as we have already mentioned earlier. Now, for \( d \neq 0 \), the underlying Hamiltonians \( H_{\uparrow \uparrow}, H_{\uparrow \downarrow} \) governing the dynamics of \( \mu_2 \) and \( \mu_3 \) are similar in the sense that both of them are having the identically modified transverse fields by an amount \( \delta \) at two different sites over the Hamiltonian \( H_{\uparrow \downarrow} \). This explains the observation that the temporal evolution \( \mu_2 \) and \( \mu_3 \), are identical and they fall on top of each other with time.

One can write the composite density matrix of two qubits in a generic form, valid for equilibrium as well as non-equilibrium situations, is given by

\[
\rho_s(t) = \frac{1}{4} \begin{bmatrix}
1 & \sqrt{\mu_3} & \sqrt{\mu_3} & \sqrt{\mu_1} \\
\sqrt{\mu_3} & 1 & \sqrt{\mu_2} & \sqrt{\mu_2} \\
\sqrt{\mu_3} & \sqrt{\mu_2} & 1 & \sqrt{\mu_2} \\
\sqrt{\mu_1} & \sqrt{\mu_2} & \sqrt{\mu_2} & 1
\end{bmatrix}.
\] (26)

The above density matrix can be reduced to a simplified form when \( d \neq 0 \) with \( \mu_2 = \mu_3 \). On the other hand, when \( d = 0 \) and \( \delta \rightarrow 0 \) one has \( \mu_4 = 1 \) and \( \mu_2 \rightarrow \mu_3 \), respectively.

The four eigenvalues obtained from \( \rho_s \hat{\rho}_s \) with \( \mu_2 = \mu_3 \) are given by

\[
\epsilon_1 = (-1 + \sqrt{\mu_4})^2, \quad \epsilon_2 = (-1 + \sqrt{\mu_1})^2,
\]

\[
\epsilon_{3,4} = \frac{1}{2} \left( 2 - 8\mu_3 + 2\sqrt{\mu_4} + \mu_4 + 2\sqrt{\mu_1} + \mu_1 \mp \sqrt{(\mu_4 - \mu_1)^2(-16\mu_3 + (2 + \sqrt{\mu_4} + \sqrt{\mu_1})^2)} \right)
\]

![FIG. 9: (Color online) DC, DP and C for \( d = 40 \) are plotted as a function of time with three different situations, quenching to the FM phase with \( \lambda_f = 0.8 \) (a), quenching to QCP with \( \lambda_f = 0.99 \) (b), quenching to the PM phase where \( \lambda_f = 1.8 \) (c). Here, \( N = 100, \delta = 0.1, \) and \( \lambda_i = 0.2. \) There exists a threshold time \( t_{TH} \) above which \( C \) attains a non-zero value. The dynamical behavior of \( \mu_2 \) coincides with that of \( \mu_3 \) through out the temporal evolution.](image)

![FIG. 10: (Color online) We plot \( C \) obtained from the reduced two qubits density matrix with \( \mu_1 = 1 \) as a function of \( \mu_3 \) and \( \mu_2 \) (a) and as a function of \( \mu_2 \) and \( \mu_3 \) (b). plot (a) indicate that the concurrence becomes maximum for \( \mu_1 = 0 \). Plot (b) suggests that \( C \) becomes maximum when \( \mu_2 \rightarrow 0 \) with a finite \( \mu_3 \) and vice versa; \( C \) is maximum for \( \mu_3 \mu_2 \rightarrow 0 \) except the case for \( (\mu_2 \rightarrow 0, \mu_3) \) and \( (\mu_3 \rightarrow 0, \mu_2) \). The above features hold true for any other value of \( \mu_4 \neq 1 \).](image)
is common when $\mu_4 < 1$ (for the $d \neq 0$ case) and $\mu_2 \neq \mu_3$ (for $d = 0$ case).

Now we shall examine the 2nd peak observed in $C$ when $\mu_2 \mu_3 \rightarrow 0$ for any value of $d$. In order to investigate this phenomena, we have to set $\mu_1 = 0$ in the reduced density matrix presented in Eq. (26). An analytic closed form expression of eigenvalues in terms of $\mu_2$, $\mu_3$ and $\mu_4$ can not be obtained in this case even with simplified situation $\mu_4 = 1$. We present the concurrence, numerically obtained, in Fig. (10) showing that the concurrence is almost a monotonic function of $\mu_2$ (while $\mu_3 = 0$) and $\mu_3$ (while $\mu_2 = 0$) except near the $\mu_2 = 1, \mu_3 = 0$ and $\mu_3 = 1, \mu_2 = 0$. Therefore, it is now clear that $C$ has a secondary maximum when $\mu_2 \mu_3 \rightarrow 0$ except near the point $\mu_2 = 0$ and $\mu_3 = 0$. This characteristics of concurrence is also seen for the case when $\mu_4 \neq 1$ (for finite $d$).

Our observation suggests that $\mu_4$ always stays at unity for $d = 0$ case but concurrence vanishes after the entanglement time $t_{1BE}$. One can therefore infer that concurrence becomes independent of $\mu_4$ and vanishes subsequently when $\mu_1 = \mu_2 = 0$; this can be easily seen by calculating the concurrence from the density matrix (26). Now, for the case $d = 0$, concurrence vanishes immediately after $\sqrt{\mu_3} \rightarrow 0$ as shown in Fig. (6,b,c). Under the small $\delta$ approximation, one can therefore conclude that $\mu_2 \simeq \mu_3$ is the killing channel which can destroy the concurrence. On the other hand, for $d \neq 0$, it is shown that concurrence vanishes after $\mu_3 \rightarrow 0$ instead of $\mu_3^{1/2}$ for $d = 0$ case. We can say that in the $\delta \rightarrow 0$ limit $\mu_2$ or $\mu_3$ is the killing channel for destroying the concurrence. Once $\mu_1 \rightarrow 0$, $\mu_2 \rightarrow 0$ and $\mu_3 \rightarrow 0$ then concurrence becomes independent of the other channel $\mu_4$. This can be seen in the temporal behavior of $C$ for $d = 0$ and $d \neq 0$, where $C$ vanishes in presence of a finite $\mu_4$.

We shall now explain the existence of a threshold time $t_{TH}$ (see Fig. (9,b,c)) in the light of above channel analysis. The reduced density matrix of the two qubits up to the threshold time $t_{TH}$ is given by

$$
\rho_s(t) = \frac{1}{4} \begin{bmatrix}
1 & \frac{\sqrt{\mu_3}}{\sqrt{3}} & \frac{\mu_3}{\sqrt{3}} & \frac{\mu_3}{\sqrt{3}} \\
\frac{\sqrt{\mu_3}}{\sqrt{3}} & 1 & \frac{\mu_3}{\sqrt{3}} & \frac{\mu_3}{\sqrt{3}} \\
\frac{\mu_3}{\sqrt{3}} & \frac{\mu_3}{\sqrt{3}} & 1 & \frac{\sqrt{\mu_3}}{\sqrt{3}} \\
\frac{\mu_3}{\sqrt{3}} & \frac{\mu_3}{\sqrt{3}} & \frac{\mu_3}{\sqrt{3}} & 1
\end{bmatrix},
$$

(28)

this is obtained from (26) by considering in the numerically observed behavior of DCs for $d \neq 0$ case, $\mu_4 = \mu_1$ and $\mu_2 = \mu_3$ and $\mu_4 = \mu_1$. One can compute the concurrence using the above density matrix (28). All the eigenvalues of $\rho_s \tilde{\rho}_s$ are same i.e., $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = (-1 + \mu_3)^2$. This yields zero concurrence when $t < t_{TH}$ up to which the above density matrix (28) is valid. After the threshold time, $\mu_4 \neq \mu_1$, $\mu_3 \neq \mu_1$ and hence concurrence becomes non-zero even if $\mu_2 = \mu_3$. This threshold time increases with distance and becomes maximum when the two qubits separated from each other maximally i.e., at $d = 50$.

Furthermore, we explore the behavior of four DCs $\mu_1$, $\mu_2$, $\mu_3$ and $\mu_4$ as a function of time by varying the distance between the two qubits. Figure (11,b,d) show that the $\mu_1$ and $\mu_4$ channels are sensitive to $d$. When the two qubits are at symmetric position (i.e., $d = 50$ for $N = 100$ and PBC), both the channels exhibit a singular behavior at $t = t_S = N/(4v)$. This singular behavior at $t_S$ does not exist for non-symmetric situations. It is also to be noted that additionally there is a revival time occurring at $t = t_R = N/(2v)$. The other two channels $\mu_2$ and $\mu_3$ are absolutely insensitive to distance and as a result $t_S = N/(4v)$ is no longer a special time scale for this channels even when $d = 50$ (see Fig. (11,b,c)). For these above two channels echo exhibits the singular behavior at $t = t_S = N/(2v)$ which is twice of the earlier singular time scale for $\mu_1$ and $\mu_4$. These observation shall now be analyzed in the light of the quasi-particle picture.

When the environmental spin chain is suddenly (and globally) quenched from the FM phase to the QCP, each of the environmental sites emit a pair of quasi-particles moving with opposite momentum in opposite directions. Now, these two quasi-particles meet at $t = t_R = N/(2v)$ after traversing half of the environmental chain and there is a constructive interference causing a partial revival of the initial state (see Fig. (1)). The channels $\mu_1$ and $\mu_4$ both involve two distinct Hamiltonians ($H_{\uparrow\uparrow}$, $H_{\uparrow\downarrow}$) and ($H_{\downarrow\uparrow}$, $H_{\downarrow\downarrow}$) which are different from each other in terms of the local transverse fields modified through the coupling of the qubits to the two sites, one can think of two extra separate emitters, located at these two sites with distance $d$ away from each other. Now, in the symmetric position $d = N/2$, quasi-particles need to travel only $d/2$ distance for such a revival to happen. Therefore, $t_S = d/(2v) = N/(4v)$ and $t_R = 2t_S$. (Over the former time, two quasi-particles travel $N/4$ while in the latter they traverse a length of $N/2$.) In the case for $d \neq 50$, the singular time scale does not appear as no constructive interference of two oppositely moving quasi-particles is not possible.

What is remarkable is that $\mu_2$ and $\mu_3$ do not exhibit the singular behavior at time $t = t_S$; their dynamical evolution is only governed by the revival time scale $t_R$ (see Fig. (11,b,d)). This is due to fact that one of the underlying Hamiltonians generates an extra pair of quasi-particles than that of the other Hamiltonian involved in $\mu_2$ and $\mu_3$ (i.e., $H_{\uparrow\downarrow}$ is different from $H_{\downarrow\downarrow}$ by a locally modified transverse field at a single site and same for ($H_{\uparrow\uparrow}$, $H_{\downarrow\downarrow}$)). Therefore, the quasi-particle from this single emitter has to travel a distance $N/2$ to partially recover the initial configuration even if the qubits are separated by a distance $d = N/2$. Therefore, the number of independent emitters, originated from the structure of the two underlying Hamiltonian that governs the dynamics, dictates the time scales of revival and singular behavior. These features are observed in the different channels of echo following a sudden quench.
In this paper, we have studied a GCSM where two qubits are locally connected to the environmental transverse Ising spin chain in such a way that the local transverse field of the environment gets modified. Working in the weak coupling limit, we explore the generation of the entanglement between the above pair of qubits, which are initially completely unentangled, both in equilibrium as well as non-equilibrium situations. In the former situation, the concurrence between them is very small in comparison to the non-equilibrium situation. However, the role of quantum criticality manifest in the behavior of concurrence as it becomes maximum at the QCP and survives even when \( d \) is large; this behavior persists even in the non-equilibrium situation. Additionally, in the latter situation the concurrence remains non-zero for longer time if the quenching is within the same phase. Furthermore, the time at which concurrence exhibits a primary peak is inversely proportional to the coupling strength. Remarkably, we show that there exists a threshold time choosing different values of \( d \). The singular and the revival behavior are explained in the text.

VI. CONCLUSION

In this paper, we have studied a GCSM where two qubits are locally connected to the environmental transverse Ising spin chain in such a way that the local transverse field of the environment gets modified. Working in the weak coupling limit, we explore the generation of the entanglement between the above pair of qubits, which are initially completely unentangled, both in equilibrium as well as non-equilibrium situations. In the former situation, the concurrence between them is very small in comparison to the non-equilibrium situation. However, the role of quantum criticality manifest in the behavior of concurrence as it becomes maximum at the QCP and survives even when \( d \) is large; this behavior persists even in the non-equilibrium situation. Additionally, in the latter situation the concurrence remains non-zero for longer time if the quenching is within the same phase. Furthermore, the time at which concurrence exhibits a primary peak is inversely proportional to the coupling strength. Remarkably, we show that there exists a threshold time choosing different values of \( d \). The singular and the revival behavior are explained in the text.

FIG. 11: (Color online) The temporal behavior of four DCs \( \mu_1 \) (a), \( \mu_2 \) (b), \( \mu_3 \) (c) and \( \mu_4 \) (d) are plotted as a function of time choosing different values of \( d \). Here, \( N = 100, \delta = 0.1 \) and \( \lambda_i(t) = 0.2(0.99) \). The singular and the revival behavior are explained in the text.

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