Meta-Percolation and Metal-Insulator Transition in Two Dimensions

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According to the scaling theory of localization, all quantum electronic states are localized in two-dimensional (2D) systems. One consequence of the theory is that there is no quantum percolation transition in 2D. However, in a real system at a finite temperature, electron phase coherence length is finite and the system is between quantum and classical. We find, in such a 2D system, a metal-insulator transition (MIT) caused by a novel type of percolation, meta-percolation. The relevance to recently observed 2D MIT is also discussed.

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There have been long lasting interests to understand localization problem in two-dimensional (2D) electron systems. According to the scaling theory of localization, there is no metal-insulator transition in non-interacting two-dimensional (2D) electron systems. One consequence of the theory is that there is no 2D quantum percolation transition. On the other hand, it is well known that there exists 2D classical percolation transition. Thus, it is interesting to understand that in a realistic electron system, where electron phase coherence length is finite, whether the system is quantum like, or classical like, or even more, unlike either, belongs to a new class.

In this paper, we study a 2D quantum percolation model with finite phase coherence length. The dephasing mechanism is introduced by attaching current-conserving voltage leads to the system, a method widely used for one-dimensional models in mesoscopic community. The conductance in such a system is derived and is found to contain both quantum and classical contributions. The conductance is calculated. Through the finite size scaling analysis, we show clear evidence of metal-insulator transition (MIT). The MIT is driven by a novel type percolation transition which is semi-quantum and semi-classical in nature. We call it meta-percolation. Finally, we discuss the possible relevance to the newly observed 2D MIT at zero magnetic field.

\[ \mathcal{H} = \sum_i \epsilon_i c_i^+ c_i + \sum_{<i,j>} t_{ij}(c_i^+ c_j + c_j^+ c_i) \]  

where \( ij \) denotes a pair of nearest-neighbor sites and \( \epsilon_i \) is the on-site energy. We choose \( \epsilon_i \) to be random, ranging from \(-W/2\) to \(W/2\) to model an on-site disorder. The nearest-neighbor hopping matrix elements \( t_{ij} \) is a random variable which assumes the values 1 (connected bonds) or 0 (empty bonds) with respective probabilities \( P \) and \( 1-P \). In order to introduce phase-breaking mechanism, we attach current-conserving voltage leads at random lattice sites (sites in Fig.1). The probability where a lattice site is attached with a voltage lead is denoted \( P_v \) and the hopping element between a voltage lead and a lattice site is denoted \( t_v \). The sole role of the voltage leads is to randomize the phases of incoming and outgoing wave-functions at the lead sites while maintain current conservation. Using Keldysh Green’s function formalism, it is straightforward to derive the multi-lead current-voltage relation:

\[ I_i = \sum_{i=1}^{N} \sigma_{ij} V_j \quad i = L, R, 1, 2, \cdots, N, \]  

where

\[ \sigma_{ij} = \frac{8e^2}{h} \text{Tr} [(T_{ia} G^{(a)}_{aa} T_{ai}) \rho_i (T_{ia} G^{(a)}_{aa} T_{aj}) \rho_j] \quad i \neq j \]  

and

\[ \sigma_{ii} = -\sum_{j \neq i} \sigma_{ij} \]  

In the above equations, \( \alpha \) denotes the vector space of the lattice system, \( T_{ai} \) is the transmission matrix, \( G^{(a)}_{aa} \) is the retarded (advanced) Green’s function of the lattice system, and \( \rho_i = \text{Im} g^a_i \) are the density of states matrices for the voltage leads and measurement leads \( L \) and \( R \), with \( g^a_i \) being the advanced Green’s function at ith lead. In order to satisfy current conservation, we require that the total current through each voltage lead to be zero,
namely, $I_i = 0$ when $i \neq L, R$. This restriction fixes the values for $V_i$. Under these conditions, we obtain the conductance between $L$ and $R$

$$g = \sigma_{LR}^i + \sigma_{LR}^i.$$  

(5)

Here $\sigma_{LR}^i$ is the direct conductance which can be obtained from Eq.(3) with $i = L$ and $j = R$, and $\sigma_{LR}^i$ is the indirect conductance,

$$\sigma_{LR}^i = -\text{Tr} \left[ \begin{array}{cc}
\sigma_{L1} & \cdots \\
\sigma_{L1} & \cdots \\
\vdots & \vdots \\
\sigma_{LN} & \cdots
\end{array} \right]^{-1} \left[ \begin{array}{cc}
\sigma_{11} & \cdots \\
\vdots & \vdots \\
\sigma_{1N} & \cdots
\end{array} \right].$$

(6)

The $\sigma_{LR}^i$ is the conductance of the conductance network obtained from the quantum lattice model. We should mention that although the expression for the direct conductance $\sigma_{LR}^d$ is identical for the pure quantum case without phase-breaking voltage leads, its value is different with voltage leads attached since the Green’s functions in the formula are affected by the presence of voltage leads.

We have calculated the total conductance $g$ numerically. In the following we present the results and discuss their implications. There are several parameters in our model as mentioned before: $P$ (probability for $t_{ij} = 1$), $P_v$ (probability for a voltage lead), $t_v$ (hopping between a voltage lead and the lattice system), $W$ (on-site disorder), and $E$ (electron energy). The results presented below are for $t_v = 0.1$, $W = 2$ and $E = 0$. We have done systematical studies by varying these three parameters, including zero on-site disorder with $W = 0$, and we found that the qualitative results remained unchanged.

In Fig.2(a) we show $g$ as a function of $P$ for a pure quantum case with $P_v = 0$. The calculation is done on an $L \times L$ square lattice. The different curves are for different $L$’s with $L = 10$ (solid line), $L = 20$ (dashed line) and $L = 30$ (long-dashed line). The important message from this plot is that for a given $P$, $g$ decreases with increasing system size $L$, indicating an insulating behavior. This result is consistent with the conclusion of scaling theory of localization, namely, in a disordered quantum system, all states are localized.

In Fig.2(b) we plot the same curves as in Fig.2(a), except with $P_v = 0.2$, i.e., $20\%$ of lattice sites are attached with phase-breaking voltage leads. The striking difference between the two plots is that in Fig.2(b), all curves cross at a single point $P_c \approx 0.62$. For $P > P_c$, $g$ increases with $L$, while it decreases with $L$ for $P < P_c$. Thus, the system is metallic above $P_c$ and insulating below $P_c$ with a MIT at $P = P_c$. At $P_c$, $g$ is independent of $L$, signifying a divergent length $\xi$ there. Therefore, the MIT is a second-order phase transition.

![FIG. 2. (a) Conductance $g$ as a function of the probability $P$ for a pure quantum system with size $L = 10$, 20 and 30. (b) Same as (a), except with $P_v = 0.2$. Inset: Plot $g$ vs. $P$ at the vicinity of $P_c$ for $L = 10$, 15, 20, 25, 30, 35 and 40.](image-url)

![What are the nature of the transition and the physical meaning of $\xi$? The phase coherent length $L_\phi$ defines a cut-off length scale beyond which the classical laws for calculating a conductance are valid. One can divide a system into blocks with area $L_\phi \times L_\phi$. Within each block, the conductance has to be determined quantum mechanically, i.e., localization physics plays an important role. The total conductance of the system is obtained by applying the classical laws to the block conductances. This defines a random conductance network problem. The localization length $\lambda_{loc}$ of the 2D quantum system, which is always finite, determines the average conductance and conductance fluctuations of each block. For the quantum percolation model studied in this work, $\lambda_{loc}$ depends on the probability $P$ for a fixed on-site disorder $W$. At a small $P$ such that $\lambda_{loc} \ll L_\phi$, the block conductance $g_{block} \sim e^{-L_\phi/\lambda_{loc}}$ and the system is an insulator. With increasing $P$ such that $\lambda_{loc} \approx L_\phi$, the average block conductance $g_{block}$ is of the order $e^2/h$. The detail distribution of $g_{block}$ is determined by the corresponding quantum Hamiltonian. For the quantum percolation model, $g_{block} = 0$ if inside the block there is no at least one connected quantum path within which every nearest-neighbor hopping $t_{ij} = 1$. Hence, the distribution function of $g_{block}$ always has a finite $\delta$-function like peak at $g_{block} = 0$, while it is a continuous function for](image-url)
$g_{\text{block}} > 0$. This defines a classical percolation problem in which $g_{\text{block}}$ may be zero or falls into a continuous distribution function. In this new conductance network, there exists a characteristic length $\xi$, which is the average size of the connected cluster. $\xi$ diverges at $P = P_c$, responsible for the fix point in Fig.2(b).

In Fig.3 we show the scaling properties of our data. We use the standard finite-size scaling analysis [8]. The conductance curves for system sizes $L$ and different $P$’s can be scaled to the metallic and insulating branches. Those curves with $P > P_c$ ($P_c = 0.62$) on the metallic side are scaled along the $L$-axis so that they coincide with a particular curve chosen here for $P = 0.70$. $\xi(P)$ is determined from curve shifting. The similar procedure is also applied to insulating side, in which case we scaled the curves to coincide with the curve for $P = 0.54$. We find that all the conductance $g$ for different size $L$ (ranging from 10 to 45) can be collapsed in a two-branch scaling function, $g(L) = f(L/\xi(P))$, where $\xi(P)$ diverges at $P_c$.

Fig.3(a) shows the scaling function and $\xi(P)$ is shown in the inset. We find that $\xi = |P - P_c|^{-\nu}$ with $\nu = 1.4 \pm 0.2$. On the insulating side, to a very good approximation, $g \propto e^{-L/\xi}$.

Another important scaling property is the $\beta$ function, defined as

$$\beta(g) = \frac{\partial \log g}{\partial \log L}. \quad (7)$$

Fig.3(b) shows $\beta(g)$. $\beta$ is negative for small $g$ and approaches a positive value for large $g$. In the inset, $\beta$ is plotted as a function of $\log g$ and we find that $\beta \propto \log(g/g_c)$ around $\beta \approx 0$, in common with other kinds of MIT [2].

In our model, the phase coherence length $L_\phi$ is controlled by the probability of voltage leads $P_v$ and the hopping between the a voltage lead and lattice system $t_v$. Although it is difficult to determine the explicit relation between $L_\phi$ and $P_v$ and $t_v$, it is evident that for a fixed $t_v$, $L_\phi$ decreases with increasing $P_v$ and approaches infinity (pure quantum system) at $P_v = 0$. We have carried out calculations for several values of $P_v$ for a fixed $t_v = 0.1$ to determine the metal-insulator transition points. Fig.4 presents the resulting MIT points $P_c$ versus $P_v$.

In the following, we make some remarks concerning our results.

(i) We find that at the transition point $P_c$ the critical conductance $g_c$ is of the order $e^2/h$. In our model $g_c$ depends on the on-site disorder $W$. At $W = 0$, $g_c \approx 2e^2/h$, the conductance for one conducting channel through the system with two spin degeneracy [3]. Thus, we associate the name meta-percolation to the conduction process at the transition point.

(ii) The meta-percolation proposed here is different from the conventional classical percolation transition. Close to the classical percolation point, the finite-size conductance scales like $g(L) = L^{-x} f(L/\xi)$ with non-zero exponent $x$. 

FIG. 3. (a) Scaling plot of $g$ vs. $L/\xi$ for $L = 10$, 12, 15, 18, 20, 25, 28, 30, 32, 35, 38, 40 and 45. Inset: $\xi$ vs. the probability $P$. (b) Plot $\beta$ function vs. $g$. Inset: $\beta$ vs. $\log g$ near $\beta = 0$.

FIG. 4. $P_c$ versus $P_v$. 

In the following, we make some remarks concerning our results.
Thus, \( g(L) \) for different \( L \) will never cross at a single point as for the case of meta-percolation shown in Fig.2(b).

(iii) The meta-percolation transition proposed in this work happens at zero temperature as long as the phase coherence length \( L_\phi \) is finite. However, for a realistic electron system where dephasing is caused by electron-electron or electron-phonon interaction, \( L_\phi \sim T^{-p} \) which diverges at \( T = 0 \), and in that case, the meta-percolation MIT can only occur at finite temperatures.

Before ending, we comment on the possible connection between the meta-percolation and the newly discovered MIT in 2D electron systems \[11\] \[27\]. It is well known that the dielectric constant of a 2D electron system becomes negative at low density where electron-electron interaction is important. A consequence of the negative dielectric constant, we suggest, could be the formation of a droplet state. The droplet state is a two-phase coexistence region of high density electron liquid and low density electron “gas”. Here we call it “gas” purely for the reason that its density is low. In fact, in the presence of impurities, the “gas” phase is a disordered Wigner crystal or a Wigner glass. The liquid phase has a better local conductivity compared to the “gas” phase because of its higher density. Thus, in the lattice model we have studied, the liquid phase corresponds to the region with nearest neighbor hopping \( t_{ij} = 1 \) while the ”gas” phase corresponds to \( t_{ij} = 0 \). At finite temperatures, electron phase coherence length is finite, thus the system is semi-quantum. In such a system, as we have shown, the meta-percolation can induce MIT and recently discovered 2D MIT may belong to this class. Unlike the true quantum phase transition, this kind of MIT disappears at \( T = 0 \) since electron phase coherence length is infinity there. However, if dephasing mechanism can be maintained at \( T = 0 \), for instance, by adding magnetic impurities in the system, then the meta-percolation MIT can survive even at zero temperature.

In summary, we have demonstrated that there is a novel type percolation (meta-percolation) in two dimensional systems. The meta-percolation can induce a metal-insulator transition and the transition is semi-quantum in nature.

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