A MARCINKIEWICZ MAXIMAL-MULTIPLIER THEOREM

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Abstract. For $r < 2$, we prove the boundedness of a maximal operator formed by applying all multipliers $m$ with $\|m\|_{V^r} \leq 1$ to a given function.

1. Introduction

Given an exponent $r$ and a function $f$ defined on $\mathbb{R}$, consider the $r$-variation norm

$$\|f\|_{V^r} = \|f\|_{L^\infty} + \sup_{N, \xi_0 < \cdots < \xi_N} \left( \sum_{i=1}^{N} |f(\xi_i) - f(\xi_{i-1})|^r \right)^{1/r},$$

where the supremum is over all strictly increasing finite length sequences of real numbers.

The classical Marcinkiewicz multiplier theorem states that if $r = 1$ and a function $m$ is of bounded $r$-variation uniformly on dyadic shells, then $m$ is an $L^p$ multiplier for $1 < p < \infty$ and

$$\|(m\hat{f})^{-}\|_{L^p} \leq C_{p,r} \sup_{k \in \mathbb{Z}} \|1_{D_k} m\|_{V^r} \|f\|_{L^p}, \tag{1.1}$$

where $D_k = [-2^{k+1}, -2^k) \cup (2^k, 2^{k+1}]$ and $\cdot, \cdot^{-}$ denote the Fourier-transform and its inverse. Later, Coifman, Rubio de Francia, and Semmes [2] (see also [8]) showed that the requirement of bounded 1-variation can be relaxed to allow for functions of bounded 2-variation, and in fact (1.1) holds whenever $r \geq 2$ and $|\frac{1}{r} - \frac{1}{2}| < \frac{1}{r}$.

The estimate [2] does not discriminate between multipliers of bounded 2-variation and those of bounded $r$-variation where $r < 2$, and so one might ask whether there is anything to be gained by controlling the variation norm of multipliers in the latter range of exponents.

Defining the maximal-multiplier operator

$$\mathcal{M}_r[f](x) = \sup_{m: \|m\|_{V^r} \leq 1} |(m\hat{f})^{-}(x)|, \tag{1.2}$$

where the supremum is over all functions in the $V^r$ unit ball, we have

**Theorem 1.1.** Suppose $1 \leq r < 2$ and $r < p < \infty$. Then

$$\|\mathcal{M}_r[f]\|_{L^p} \leq C_{p,r} \|f\|_{L^p}. \tag{1.3}$$

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The case \( r = 1 \) was observed independently by Lacey \[4\].

Note that in the definition of \( M_r \), each \( m \) is required to have finite \( r \)-variation on all of \( \mathbb{R} \) rather than simply on each dyadic shell as in (1.1). This is necessary for boundedness, as can be seen from the counterexamples of Christ, Grafakos, Honzík and Seeger \[1\].

Although the maximal operator (1.2) would seem to be fairly strong, we do not yet know of an application for the bound above. We will, however, quickly illustrate a strategy for its use that falls an (important) \( \epsilon \) short of success. Let \( \Psi \) be (say) a Schwartz function, and for each \( \xi, x \in \mathbb{R} \) and \( k \in \mathbb{Z} \), consider the \( 2^k \)-truncated partial Fourier integral

\[
S_k[f](\xi, x) = \text{p.v.} \int f(x-t)e^{2\pi i \xi t}\Psi(2^{-k}t)\frac{1}{t} \, dt.
\]

It was proven by Demeter, Lacey, Tao, and Thiele \[3\] that for \( q = 2 \) and \( 1 < p < \infty \),

\[
\| \sup_{\|g\|_{L^q} = 1} \left( \text{sup} \right) (S_k[f](\cdot, x) \hat{g})^* \|_{L^p} \leq C_{p,q} \| f \|_{L^p}.
\]

If we had the bound

\[
\| S_k[f](\xi, x)\|_{L^p(2^{-k}V_r^\epsilon)} \leq C_{p,r} \| f \|_{L^p}
\]

for some \( r < 2 \), then an application of Theorem 1.1 would give (1.4) for \( q > r \) by rather different means than \[3\]. In fact, one can see by applying the method in Appendix D of \[6\] that (1.5) holds for \( r > 2 \) and \( p > p' \). Unfortunately, it does fail for \( r \leq 2 \).

2. Proof of Theorem 1.1

The following lemma was proven in \[2\]; see also \[5\].

**Lemma 2.1.** Let \( m \) be a compactly supported function on \( \mathbb{R} \) of bounded \( r \)-variation for some \( 1 \leq r < \infty \). Then for each integer \( j \geq 0 \), one can find a collection \( \Upsilon_j \) of pairwise disjoint subintervals of \( \mathbb{R} \) and coefficients \( \{b_v\}_{v \in \Upsilon_j} \subset \mathbb{R} \) so that \( |\Upsilon_j| \leq 2^j \), \( |b_v| \leq 2^{-j/r} \| m \|_{V_r} \), and

\[
m = \sum_{j \geq 0} \sum_{v \in \Upsilon_j} b_v 1_v,
\]

where the sum in \( j \) converges uniformly.

The lemma above was applied in concert with Rubio de Francia’s square function estimate \[7\] to obtain (1.1). Here, we will argue similarly, exploiting the analogy between the Rubio de Francia square function estimate and the variation-norm Carleson theorem.

It was proven in \[7\] that for \( p \geq 2 \),

\[
\sup_{I} \| (\sum_{I \in \mathcal{I}} |(1_I \hat{f})|^2)^{1/2} \|_{L^p} \leq C_p \| f \|_{L^p},
\]

where the supremum above is over all collections of pairwise disjoint subintervals of \( \mathbb{R} \). Consider the partial Fourier integral

\[
S[f](\xi, x) = (1_{(\infty, \xi]} \hat{f})^*(x).
\]
It was proven in \[6\] that for \(s > 2\) and \(p > s'\),
\[
\|S[f](\xi, x)\|_{L^p(V_s^x)} \leq C_{p, s}\|f\|_{L^p}
\]
or, equivalently,
\[
(2.2) \quad \|\sup_I \left( \sum_{I \in I} |(1_I \hat{f})^{-s}| \right)^{1/s} \|_{L^p} \leq C_{p, s}\|f\|_{L^p}.
\]

Note that by standardizing limiting arguments, taking the supremum in \(1.2\) to be over all compactly supported \(m\) such that \(\|m\|_{V^r} \leq 1\) does not change the definition of \(M_r\). Applying the decomposition \(2.1\), we see that for any compactly supported \(m\) with \(\|m\|_{V^r} \leq 1\) we have
\[
|\hat{m}(f)(x)| \leq \sum_{j \geq 0} \sum_{\nu \in \nu_j} |b_{\nu}(1_{\nu} \hat{f})(x)|
\leq \sum_{j \geq 0} \sup_{\nu \in \nu_j} |b_{\nu}| \|\nu_j\|^{\frac{1}{s}} \left( \sum_{\nu \in \nu_j} |(1_{\nu} \hat{f})(x)|^s \right)^{1/s}
\leq C_{r, s} \sup_I \left( \sum_{I \in I} |(1_I \hat{f})(x)|^s \right)^{1/s},
\]
where, for the last inequality, we require \(s < r'\).

Provided that \(r < 2\) and \(p > r\) we can choose an \(s < r'\) with \(s > 2\) and \(p > s'\), giving \(2.2\) and hence \(1.3\).

The argument of Lacey \[4\] for \(r = 1\) follows a similar pattern, except with Marcinkiewicz’s method in place of \[2\] and the standard Carleson-Hunt theorem in place of \[6\].

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