Gravitomagnetic relativistic effects on turbulence

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Abstract

The dynamics of fluid-matter under the influence of gravitomagnetic fields are formulated and solved for the case of fully developed turbulence. Gravitomagnetic effects reduce the vortical complexity and nonlinearity of turbulence, even leading to its extinction within large volumes, and generate departures from Kolmogorov turbulence scalings, that are explained via a combination of dimensional and exact analysis arguments.

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INTRODUCTION

Understanding strongly nonequilibrium dynamics and emergence in basic field theories is a main task of statistical and nonlinear physics. In electrodynamics, an efficient way to tackle far-out of equilibrium processes is to focus on the macroscopic scales, and, by taking the magnetic and small-velocity limits of the theory, to formulate the magnetohydrodynamic (MHD) equations of motion. Strongly nonequilibrium states in these equations, known as MHD turbulence, are well understood, with many, fully-resolved numerical calculations having become available [1, 2]. At small enough temperatures for the formation of quantum mechanical, atomic bound states [3–5], we obtain low-energy electrodynamics [6], and MHD reduces to standard hydrodynamics and turbulence [7]. There are similar approaches for Newtonian gravitodynamics [8], but the problem of relativistic gravitational turbulence is not equally well developed.

Indeed, on one hand, turbulence is strongly nonlinear, and evolves over a continuum range of scales, which need to be well resolved to correctly capture its statistical structure. A key requirement is the resolution of the, all important, dissipation processes, whose efficiency peaks at high wavenumbers of the energy spectrum, where also most of the strain resides. On the other hand, the fully relativistic problem is so computationally complex, that calculations satisfying these standards are very difficult to perform. This research aims to formulate a hydrodynamic model of self-gravitating matter, which although not fully relativistic, retains some (conceptually important for gravitational theory) relativistic effects, known as gravitomagnetism. In this way, model generality is traded for computational quality. Another aim is to perform actual calculations with the simplified model, indicate particular gravitational effects on turbulence structure, and explain the resulted statistical phenomenology via scaling arguments.

Gravitomagnetism is far less well studied than “gravitelectricity” (Newtonian gravity) [9]. Indeed, in Galilean invariant theory of gravity, a rotating, massive sphere produces identical gravitational fields with a stationary one. But in relativity, and in the weak field, slow motion limit, the rotation of a massive sphere adds to the standard Schwarzschild field of a stationary sphere a gravitomagnetic element. A chief motive for this enquiry, is
the fact that turbulence, when seen as a strongly out of equilibrium, nonlinear system, is dominated by linear vortical structures (quasi-defects) [7], hence, rotating matter is the very essence of its physics. In other words, turbulence is the arena of the most complicated gravitomagnetic phenomena, which are the focus of this study.

Apart from their implications for gravity and turbulence theories, the results could, possibly, also benefit astrophysical investigations. Indeed, in neutron stars, gravitomagnetic effects were shown to affect precession rates by about 10% [12], and there is need to improve the quality of employed phenomenological turbulence models in relativistic hydrodynamics investigations of their differential rotation and mergers [10] [11]. Moreover, some of the insight into the structure of gravitomagnetic turbulence provided here, could inform astrophysical investigations of interstellar and inter-galactic medium turbulence, where it is important to understand turbulence vorticity dynamics [13] [14], and of large scale flow in galactic supercluster assemblies, where there is need for inclusion of nonlinear, self-consistent gravitodynamic effects [15] [18].

SELF-GRAVITATING PARTICLE SYSTEMS AT LARGE-SCALE, WEAK-FIELD AND SLOW-MOTION LIMITS

We aim to understand the large-scale dynamics of a microscopic theory describing the self-consistent interactions between gravity and a discrete system (dust) of $N$ spinless point particles (which, depending on the application setting, could be interstellar or inter-galactic medium particles, stars or galaxies) [9] [19] [20]

\[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} T^{\mu\nu}, \]

\[ \frac{d u_\mu^p}{d\tau} + \Gamma^\mu_{\nu\lambda} u_\nu^p u_\lambda^p = 0, \]

\[ T^{\mu\nu}(x^\lambda) = \sum_{p=1}^{N} m c \int d\tau \frac{\delta^{(4)}[x^\lambda - r_\lambda^p(\tau)]}{\sqrt{-g}} u_\mu^p u_\nu^p, \]

where $\sum$ indicates summation over $p = 1, ..., N$ particles in the system (Einstein summation convention not valid for $p$), $\mu, \nu, \lambda = 1, 2, 3, 4$, $m$ is the particle mass, $u_\mu^p = dr_\mu^p/d\tau$ is the particle four-velocity, $d\tau = \sqrt{-ds^2/c}$ is the proper time along particle trajectories,
\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu \] is the differential spacetime interval, \( g_{\mu\nu} \) is the metric tensor, \( g \) is the determinant of \( g_{\mu\nu} \), \( R^{\mu\nu} \) is the Ricci curvature tensor, \( R \) is the scalar curvature, \( T^{\mu\nu} \) is the energy-momentum tensor, and \( \Gamma^\mu_{\nu\lambda} \) are the Christoffel symbols. \( G \) is the gravitational constant and \( c \) is the speed of light. The first equation is the Einstein field equation that results from gauging the symmetry with respect to displacement transformations, and, since the particles are spinless, is a complete description of gravitational interactions in the system \[ 21, 23 \]. The second equation indicates that gravitational effects are globally equivalent to inertia in curved spacetimes or, locally, to the effect of a non-inertial reference frame in flat, special-relativistic spacetime, a feature that is going to play an important role in the derivation of the hydrodynamic model.

For large \( N \) typical of applications, it is practically difficult to calculate the evolution of the microscopic system over spacetime scales large enough to capture the phenomenology of fully developed turbulence. Hence, we coarse-grain the dynamics to obtain first the Einstein-Boltzmann system \[ 24, 26 \], and, further on, the Einstein-Navier-Stokes System (ENSS) of relativistic compressible fluid dynamics \[ 27, 31 \], where the Einstein equation is now the ensemble average of the corresponding microscopic equation. Notably, since gravitational systems are long-range interacting systems, the Boltzmann equation is not meant here in the sense of its familiar version for dilute gases, but in the general statistical mechanical sense of a kinetic equation with a collisional term. Then, the hydrodynamic approximation is valid at large times (in units of a relevant relaxation timescale). The latter depends on the particular system and modeling purposes \[ 32 \]. It could be the fast timescale \( \tau_v \) of violent relaxation, which is independent of the number of particles \( N \), or the longer collisional relaxation timescale \( \tau_c = N^\delta \) (where \( \delta \) is a system-dependent exponent). A typical \( \tau_c \) example is the Chandrasekhar relaxation timescale in stellar systems. Local equilibria based on \( \tau_v \) are solutions of the Vlasov equation, and those based on \( \tau_c \) of the Boltzmann equation \[ 24, 32 \]. Navier-Stokes type of diffusion has been applied to cosmological \[ 33 \], and interstellar medium dynamics \[ 30 \]. Although the ENSS system incorporates the physics of fully developed turbulence, the computational complexity is so high, that fully resolved calculations are not presently available. So, in this work, we proceed with a theoretical formulation in between Newton and Einstein gravities which retains significant relativistic effects, whilst allowing routine, fully resolved turbulence.
calculations.

In this context, we take the *weak field, slow motion* limit of ENSS, which reduces the Einstein equation to a type of gravitational Maxwell theory \[39\], and the relativistic compressible Navier-Stokes equation to its Newtonian counterpart. The latter have already been employed in *conjunction* with gravitational Maxwell theory in the study of accretion disks around massive astronomical objects (\[9\], page 327). A thorough discussion of the Newtonian Navier-Stokes limit is available in \[31\] (page 294). Hence, a simplified model is

\[
\begin{align*}
\partial_t E_i^g &= -4\pi G \rho, \\
\epsilon_{ijk} \partial_j E_k^g &= -\partial_t B_i^g, \\
\partial_t B_i^g &= 0, \\
\epsilon_{ijk} \partial_j B_k^g &= -\frac{16\pi G}{c^2} \rho u_i + \frac{4}{c^2} \partial_t E_i^g, \\
\partial_t \rho + \partial_i (\rho u_i) &= 0, \\
\partial_i (\rho u_i) + \partial_j (\rho u_i u_j) &= \rho (E_i^g + \epsilon_{ijk} u_j B_k^g) - \partial_i p + \partial_j \sigma_{ij},
\end{align*}
\]

where the last two equations are the compressible Navier-Stokes (NS) equations, \(i, j, k = 1, 2, 3\), \(\epsilon_{ijk}\) is the Levi-Civita symbol, \(E^g\) is the gravitoelectric field (i.e., the Newtonian gravitational field), \(B^g\) is the gravitomagnetic field, \(u\) is the fluid velocity, \(\rho\) is the density of fluid mass-energy, \(p\) is the scalar pressure, and \(\sigma_{ij}\) is the viscous stress tensor.

The corresponding scalar and vector potentials for \(E^g\) and \(B^g\) are the \(g_{00}\) and \(g_{0i}\) \((i = 1, 2, 3)\) \(g_{\mu\nu}\) components. The remaining six components \(g_{ij}\) become irrelevant, since they generally describe the geometries of curved three dimensional spaces (slices of constant \(ct\)), which, for *weak fields* and *low velocities* of interest here, become flat \((R^{ij} \equiv 0)\). Hence, the tensor-field theory of gravity is reduced to a vector one, but the causal structure of Minkowski spacetime (hence, also relativistic effects) are preserved in the guise of gravitomagnetism.

**GRAVITOMAGNETIC FLUID DYNAMICS EQUATIONS AND THEIR SCALING**

In the context of our investigation of self-gravitating, *homogeneous, isotropic* turbulence, the above system will be further simplified. Indeed, it is known that compressibility effects
cause deviations from Kolmogorov scalings of incompressible turbulence when turbulent eddies are in transonic or supersonic motion relative to each other \cite{34}. The dimensionless number quantifying this notion is the Mach number of turbulence \( M_t = \frac{u'}{c_s} \), where \( u' \) is the intensity of turbulent velocity fluctuations, and \( c_s \) is the medium’s speed of sound \cite{34}. In other words, \( M_t \) measures the ratio between turbulent kinetic energy and thermal energy, and the latter ought to be a significant proportion of the former, for compressibility effects to become important. There is observational evidence that many astrophysical flows could be treated as incompressible. For example, analysis of turbulent gas pressure maps in the intra-cluster medium revealed that pressure fluctuations are consistent with incompressible, Kolmogorov turbulence \cite{13,14}. This follows directly from the Mach number criterion, since, for the Coma cluster \cite{13,14}, it is estimated that \( \epsilon_{turb} \geq 0.1 \epsilon_{th} \), where \( \epsilon_{turb} \) is the kinetic energy density of turbulence, and \( \epsilon_{th} \) is the fluid’s thermal energy density, hence, the turbulent fluctuations are subsonic. These suggest that it would be useful to take, in the full model, the incompressible limit of the Navier-Stokes equation. That this is a meaningful limit of a relativistic theory has been demonstrated in many different works \cite{35–38}. This step allows also an important simplification of the gravity part of the equations: in a homogeneous, constant mass-energy density matter system, the gravitoelectric field \( E^g \) becomes dynamically irrelevant, hence, the time derivative of \( B^g \) becomes zero, the “displacement current” \( \frac{4}{c^2} \partial_t E^g_i \) can be dropped from the equation for the \text{curl} of \( B^g \), and \( E^g \) can be dropped from the NS equation. Notably, gravitodynamics is more nonlinear than electrodynamics, since, in the latter, electric current \( J^e \) includes the electric charge density \( \rho^e \) which differs from the fluid density \( \rho \), whilst, in the former, gravitational current \( J^g \) is identical to the fluid momentum. This identification of gravitational charge with inertial mass, or, as indicated above, of gravitational effects with inertia, allows the two \( B^g \) equations to form, together with the incompressible NS, a closed system of differential equations that can be autonomously solved,

\[
\begin{align*}
\partial_i B^g_i &= 0, \\
\epsilon_{ijk} \partial_j B^g_k &= -\frac{16\pi G}{c^2} \rho u_i, \\
\partial_i u_i &= 0, \\
\partial_i u_i &= \epsilon_{ijk} u_j B^g_k - \partial_i \left( \frac{p}{\rho} + \frac{u_j u_j}{2} \right) + \epsilon_{ijk} u_j \omega_k + \nu \partial_j \partial_j u_i.
\end{align*}
\]
It is important to note, that since we aim to study homogeneous, isotropic turbulence here, the above system refers exclusively to fluctuating quantities. Although the equations look similar to analogous equations in the electrodynamic magnetic limit [40, 41], their physical motivation is very different. In electrodynamics, there are negative and positive charges, hence, it is possible to realize the magnetic limit, by having approximately zero charge densities, but significant currents, which are responsible for neutralizing the charges [42]. In gravity, on the other hand, there cannot be zero mass-energy densities, but in homogeneous incompressible media, and in the weak-field, slow-motion limit, gravitomagnetism is, as explained above, all there is, and only gravitational effects induced by the flow of matter are observable. Certainly, gravitoelectric effects are very important in inhomogeneous incompressible systems, as are, for example, variable density (stratified) media, or accretion disks around massive central objects [9] (page 327).

To obtain a scaled system of equations, we define the constant \( \beta \equiv 16\pi G/c^2 \), and use it to scale \( B^g \) as \( \tilde{B}^g \equiv B^g/\sqrt{\beta \rho} \), and define the scaled gravitational current \( \tilde{J}^g_i = \epsilon_{ijk} \partial_j \tilde{B}^g_k \). Notably, \( \sqrt{\beta \rho} \) has units of cm\(^{-1} \), \( \tilde{B}^g \) has units of velocity cm s\(^{-1} \), and \( \tilde{J}^g \) has the units s\(^{-1} \) of flow vorticity \( \omega_i \equiv \epsilon_{ijk} \partial_j u_k \). Finally, by taking the curl of \( \epsilon_{ijk} \partial_j B^g_k \), we arrive at the scaled gravito-magneto-hydrodynamic (GMHD) equations

\[
\partial_j \partial_j \tilde{B}^g_i = \sqrt{\beta \rho} \omega_i, \\
\partial_t u_i = 0,
\]

\[
\partial_t u_i = -\partial_i \left( \frac{p}{\rho} + \frac{u_j u_i}{2} \right) + \epsilon_{ijk} u_j \omega_k - \epsilon_{ijk} \tilde{B}^g_j \tilde{J}^g_k + \nu \partial_j \partial_j u_i,
\]

where it is instructive to compare the equation for \( \tilde{B}^g_i \) with the equation for the velocity vector potential \( \psi \), \( \partial_j \partial_j \psi_i = -\omega_i \), where the difference in signs is due to the negative sign in the left hand side of the equation for the curl of \( B^g \). Notably, Lamb force \( \epsilon_{ijk} u_j \omega_k \) is the vector product of velocity with its vorticity, hence, since the gravitational current is the momentum, the novelty of gravitational effects (in comparison with Lamb force effects) depends on the relative orientation of \( \omega \) and \( B^g \) vectors. In this form of the NS equation, \( \epsilon_{ijk} \tilde{B}^g_j \tilde{J}^g_k \) term encodes genuine relativistic gravitational effects on flow structures. Indeed, it is straightforward to demonstrate [43] (page 89), that by combining Newtonian gravity with Lorentz transformations, and demanding identical physical predictions for different inertial frames, one “discovers” gravitomagnetism. In other words, the latter is a direct consequence
of the relativity principle. It is helpful to write the vorticity dynamics equation

\[ \partial_t \omega_i + u_j \partial_j \omega_i - \omega_j \partial_j u_i - \tilde{B}_j^g \partial_j \tilde{J}^g_i + \tilde{J}^g_j \partial_j \tilde{B}^g_i - \nu \partial_j \partial_j \omega_i = 0, \]

where the sum of the four inner terms could be succinctly written in terms of Lie derivatives as \( \mathcal{L}_u \omega - \mathcal{L}_{\tilde{B}} \tilde{J}^g \). It is straightforward then to define a gravitational interaction parameter \( N^g \), that measures gravitomagnetic effects in units of fluid inertia, \( N^g = |\mathcal{L}_{\tilde{B}} \tilde{J}^g|/|\mathcal{L}_u \omega| \). By inserting typical values of the various quantities in this expression, we obtain \( N^g = \ell_g \ell_h \beta \rho \), where \( \ell_g \) and \( \ell_h \) are length scales typical of gravitational and hydrodynamic-variable gradients correspondingly. The other important parameter, that measures nonlinear, inertial, nonequilibrium processes in units of linear, viscous, nonequilibrium processes, is the Reynolds number, \( Re = u' l / \nu \), where \( u' = \sqrt{\langle u^2 \rangle} \) is the turbulent intensity of \( u \), and \( l \) is the integral length scale which measures the size of turbulence-energy containing eddies [7].

**NUMERICAL METHODS AND FINITE PRECISION ARITHMETIC**

The model is solved with a staggered grid, fractional step, projection, finite volume, numerical method [44 45]. Spatial partial derivatives are computed with second order accurate schemes. An implicit, second order accurate in time, Crank-Nicolson (CN) scheme is applied to the viscous/diffusion terms, whilst all other terms evolve via an explicit, third order accurate in time, low storage Runge-Kutta (RK) method. The CN scheme is incorporated into the RK steps and the method becomes a hybrid RK/CN scheme. Flow incompressibility is enforced by projecting the velocity onto the space of divergence-free vector fields (Hodge projection). The time-steps are adaptive, limited by the Courant-Friedrichs-Lewy condition, and resolve the viscous processes in the flow. The gravitomagnetic field is computed self-consistently from the corresponding Poisson equation with Fast Fourier Transform methods. The algorithmic approximation of this numerical analysis adds finite-precision arithmetic round-off error to analytic truncation error: within the employed floating point number set \( \mathbb{F} \), the distance between 1 and the next larger floating point number is \( \epsilon_m = 0.222 \times 10^{-15} \). The smallest and largest numbers that can be represented are \( 2.2 \times 10^{-308} \) and \( 1.8 \times 10^{308} \) correspondingly. The algorithm arithmetic employs the round to nearest even rounding mode [46 47]. The computational domain is
a cube discretized into $256^3$ grid cells. In all calculations, the dissipation scales are fully resolved.

**HOMOGENEOUS, ISOTROPIC, GRAVITOMAGNETIC TURBULENCE**

First, we set up a steady-state, homogeneous, isotropic, pure NS turbulence with Taylor Reynolds number $Re_\lambda = u'\lambda/\nu \approx 80$, and then we switch on gravity. Here, $\lambda$ is the Taylor microscale $\lambda^2 = 15\nu \langle u_i u_i \rangle / 3\epsilon$, where $\epsilon = 2\nu \langle S_{ij} S_{ij} \rangle$ is the rate of turbulence energy dissipation, and $S_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ is the strain rate tensor [7]. To achieve steady state, viscous dissipation action is compensated by Lundgren’s linear forcing [48, 49]. $\lambda$ is a good representation of the length scale where most of turbulent strain takes place, so it is a good candidate for $\ell_h$, since the latter has to characterize vortex stretching. On the other hand, the solution of the Poisson equation for $\tilde{B}^g(x) = -\frac{\sqrt{\beta \rho}}{4\pi} \int \frac{\omega(x') dx'}{|x' - x|}$ indicates that $\tilde{B}^g$ is formed by the weighted sum of neighbouring-vorticity contributions, so it could be expected that gravitomagnetic gradients would scale with the largest correlation length in the system, which in turbulence case is the integral length scale $l$, that measures the size of large eddies. Hence, $N^g = l\lambda \beta \rho$. This intuition is fully supported by the computational solutions. Some important time scales are the viscous time scale $\tau_d = (\Delta x)^2/(6\nu)$, where $\Delta x$ is the computational grid size, the gravitational time scale $\tau_g^{N^g} = l/(\tilde{B}^g)'$, where $(\tilde{B}^g)' = \sqrt{\langle (\tilde{B}^g)^2 \rangle}$ is the turbulent intensity of $\tilde{B}^g$, and the time scale of energy containing motions, $\tau_e^{N^g} = l/u'$. To achieve a statistical steady state of turbulent gravity-matter interactions, we continue compensating viscous dissipation after enabling gravity. Starting from $N^g = 0.01$ and increasing the strength of gravitational effects, we find important effects close to $N^g = 10$, so we performed three extended-time calculations for $N^g = [10, 20, 40]$. Notably, $N^g$ is indicative of the coupling-strength between matter and gravity, which is a dynamical quality not to be confused with the strength of the resulting gravitational fields. The relations between the various time scales are $\tau_{10}^g = 9.8 \tau_d$, $\tau_{10}^g = 0.41 \tau_{e10}^g$, $\tau_{20}^g = 1.52 \tau_{10}^g$, and $\tau_{40}^g = 2.4 \tau_{10}^g$, $\tau_{20}^g = 1.37 \tau_{10}^g$, $\tau_{40}^g = 1.81 \tau_{10}^g$. All shown results correspond to steady states which are established after a transient period whose duration is inversely proportional to the coupling strength. In full support of our scalings, the solutions inform that, as $N^g$ increases,
FIG. 1: Steady state turbulence averages for $N^g=[10,20,40]$. Left: Ratio of Lamb vector magnitude over gravitational effect magnitude. Right: Cosine of angle between $\omega$ and $B^g$. $N^g$ increases from top to bottom curves. The shown time period corresponds to several dozens of $\tau^e_{N^g}$.

FIG. 2: Steady state turbulence averages for $N^g=[10,20,40]$. From left to right: vorticity, velocity and gravitomagnetic field mean squares. $N^g$ increases from top to bottom curves. The shown time period corresponds to several dozens of $\tau^e_{N^g}$. The shown quantities are scaled with the corresponding pure turbulence values.

so does the strength of gravitational effects relative to Lamb-force effects (Fig.1 left). Moreover, $B^g$ and $\omega$ tend to be antiparallel, and the intensity of this effect is proportional to $N^g$ (Fig.1 right). This indicates that gravity tends to neutralise the Lamb force, hence, to reduce the vortical complexity of turbulence, and make it less nonlinear. Indeed, the highest entstrophy and turbulence intensity values are associated with the weakest coupling (Fig.2 left and centre). In addition, due to reduction of the gravitational source (vorticity) levels, strong coupling leads to smaller values for the mean square of $\tilde{B}^g$ (Fig.2 right).

Of key importance are the velocity $E^{u}_k$, vorticity $E^{\omega}_k$, and gravitomagnetic-field $E^{\tilde{B}g}_k$
FIG. 3: Steady state turbulence for $N^g = 20$. From left to right: velocity, vorticity and gravitomagnetic field compensated spectra, exemplifying the corresponding $k^{-3/2}$, $k^{1/2}$ and $k^{-3.5}$ scalings.

spectra. Before enabling gravity, we verified the Kolmogorov scalings, $E^u_k \sim k^{-5/3}$ and $E^\omega_k \sim k^{1/3}$ for our pure turbulence calculation. Gravity induces new, GMHD scalings: $E^u_k \sim k^{-3/2}$, $E^\omega_k \sim k^{1/2}$, and $E^g_k \sim k^{-3.5}$ (Fig. 3). The spectra shown are for $N^g = 20$, since $N^g = 10$ turbulence is not equally representative of strong coupling effects, and $N^g = 40$ turbulence is not equally well resolved (albeit still satisfactorily). The computed scalings can be understood via dimensional and exact analysis arguments. For $E^u_k$, we can use a dimensional analytic relation due to Kraichnan [1] for the rate of kinetic energy dissipation (equal to the energy-flux in wavenumber space) $\epsilon = \tau_k (E^u_k)^2 k^4$. Here, $\tau_k$ is a time scale, which is associated with wavenumber $k$, and is characteristic of energy transfer processes from $k$ to higher wavenumbers. In Kolmogorov turbulence, $\tau_k = (E^M_k k^3)^{-1/2}$, i.e., the time scale of local in $k$ space turbulence eddies, which is indicative of their lifetime or the time-interval over which fluid motions of length scale $k^{-1}$ remain correlated. However, for a self-gravitating fluid with high $N^g$, $\tau_k$ ought to scale with $\tilde{B}^g$, since the latter would determine the decorrelation time via the gravitational forcing term in the NS equation. Hence, $\tau_k = (\tilde{B}^g k)^{-1}$, and inserting this into the expression for $\epsilon$, we obtain $E^u_k = (\epsilon \tilde{B}^g)^{1/2} k^{-3/2}$, which agrees with the computational result. Noting that $E^u_k$ units are $[E^u_k] = L^3 T^{-2}$, and that $[E^\omega_k] = LT^{-2}$, we obtain $E^\omega_k \sim E^u_k k^2$, hence, $E^\omega_k \sim k^{1/2}$, which also agrees with
FIG. 4: Vorticity isosurfaces for pure turbulence (left) and gravitational turbulence for $N^g = 20$ (right).

FIG. 5: Gravitational field (yellow) and vorticity (turquoise) isosurfaces in gravitational turbulence for $N^g = 20$.

The results. Next, it is straightforward to predict $\tilde{B}^g$ scaling: take the Fourier transform of $\tilde{B}^g$ equation, and square to obtain: $k^4|\tilde{B}^g(k)|^2 = \beta \rho |\tilde{\omega}(k)|^2$. Employing the definition $\int_0^\infty E^g_k dk = \iint |\tilde{\omega}(k)|^2 d^3k$, we deduce $|\tilde{\omega}(k)|^2 \sim k^{-3/2}$. This gives $|\tilde{B}^g(k)|^2 \sim k^{-11/2}$, and via the definition $\int_0^\infty E^g_k dk = \iint |\tilde{B}^g(k)|^2 d^3k$, we obtain $E^g_k \sim k^{-3.5}$, which accurately matches the computed value.

The morphologies of vorticity and gravitomagnetic fields present both surprising and deducible features (Figs 4,5). Pure turbulence vorticity isosurfaces drawn at 15% of maximum value (Fig 4, left) indicate the standard, predominantly linear, NS structures, spreading homogeneously over the whole domain. However, in the gravitational case, the vortex size is smaller, and, even when isosurfaces spanning the whole range of vorticity
levels are shown simultaneously (Fig.4 right), large volumes devoid of any vorticity are observed. This is explained by the tendency of the gravitomagnetic field to neutralize the Lamb force, and suppress turbulence. On the other hand, due to the vorticity source in the $\tilde{B}^g$ equation, $\tilde{B}^g$ and $\omega$ coexist in space, and, since the former involves nonlocal space averages of the latter, its isosurfaces form extended structures spanning the system’s domain (Fig.5). Here, vorticity isosurfaces are drawn at 15% of the maximum value, and magnetic field isosurfaces at 55% of maximum value. The $\tilde{B}^g$ morphology remains the same from the smallest isolevels up to 70% of maximum value, and its structures become localized only when, at sufficiently high field values, the corresponding high-vorticity source has very small support.

CONCLUSION

At hydrodynamic scales, relativistic gravitodynamics is characterized by three nonlinearities: (1) the nonlinearity of the Einstein equation for the field, (2) the nonlinearity of the Navier-Stokes equation for matter, and (3) the standard nonlinearity of interacting field theory emanating from matter-field coupling. Because of these nonlinearities, well resolved computations of relativistic self-gravitating fluids are too complex to perform. Hence, we formulated a far simpler problem here, which, nevertheless, retains genuine relativistic effects in the guise of gravitomagnetism. Indeed, by taking the weak-field, slow-motion limit, we removed the first nonlinearity, but preserved the other two. Although this appears to be a limitation, we still have a very demanding mathematical problem in our hands, especially when the fluid is turbulent. This is because turbulence is a unique example of nonlinearity that can become arbitrarily strong without “breaking” the underlying system (a fluid can sustain extremely large Reynolds numbers). Moreover, turbulence physics are dominated by Biot-Savart interactions between vortical structures, i.e., by vorticity, which also is the source of the gravitomagnetic field. In other words, the sources of gravity are the very structures that dominate turbulence physics. It was shown that the third nonlinearity tames the second. Indeed, enstrophy is intensified by vortex stretching and peaks at high wavenumbers, as a result of Lamb-force driven turbulence kinetic energy cascade. But as the cascade intensifies small-scale enstrophy, the latter generates a gravitomagnetic
field whose action on the fluid counterbalances the Lamb force driving the cascade. In other words, gravitomagnetism tends to linearize and damp out turbulence. At statistical equilibrium, the field levels are consistent with enstrophy intensification in turbulence that is allowed by the degree of flow nonlinearity reduction due to these field levels. The latter are inversely proportional to the field-matter coupling.

Future elaboration of GMHD vortex dynamics and detailed probing of gravity mediated vortex interactions in turbulence could be informative. Gravitomagnetic effects would generate novel coherent-structure formation mechanisms, and alter strain-rate tensor related statistics. Finally, the employment of advanced geometrical and topological methods for the characterization of gravitational and vorticity field structures would help indicate in a quantitative (rather than visual) way the differences between pure and gravitomagnetic turbulence flow patterns.

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