MORSE THEORY WITHOUT NONDEGENERACY

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ABSTRACT. We describe an extension of Morse theory to smooth functions on compact Riemannian manifolds, without any nondegeneracy assumptions except that the critical locus must have only finitely many connected components.

Let $M$ be a compact manifold (without boundary). Classical Morse theory [55] studies the topology of $M$ using a smooth real-valued function $f : M \rightarrow \mathbb{R}$ on $M$ with nondegenerate (and therefore isolated) critical points; the Morse inequalities relate the Betti numbers of $M$ to the numbers of critical points of $f$ with given Morse index. It is well known that it is possible to relax this nondegeneracy assumption, for example to allow the connected components of the critical locus to be submanifolds with nondegeneracy in the normal directions, so that $f$ is a Morse–Bott function [12, 13]. Our aim here is to remove the hypothesis of nondegeneracy, except for the much weaker requirement that the critical locus of $f$ has only finitely many connected components.

Morse theory has been generalised in many different ways in the decades since its introduction (cf. for example [6, 31, 45, 58, 70, 73, 75]), and there are several different approaches used to obtain Morse inequalities for suitable smooth functions $f : M \rightarrow \mathbb{R}$. In the main the different approaches involve the choice of a Riemannian metric $g$ on $M$, and the associated gradient flow $\text{grad}(f)$ of $f$, or at least some sort of pseudo-gradient vector field (and from this viewpoint Conley’s index theory [26, 63] is more general still, studying smooth flows which are not necessarily gradient flows).

(i) Attaching handles. The original approach of Morse [55] for $f : M \rightarrow \mathbb{R}$ with nondegenerate critical points was to study the topology of $f^{-1}(-\infty, a]$ as $a$ varies in $\mathbb{R}$, and to show that this is unchanged as $a$ increases except when $a$ passes through a critical value, when a handle is attached for each critical point $p$ with $f(p) = a$.

(ii) Morse stratifications. An approach which has been used to extend Morse theory to suitable smooth functions with non-isolated critical points [13, 14, 48] is to stratify $M$ according to the limiting behaviour of the gradient flow of $f$. Thus we get $M = \bigsqcup_{0 \leq j \leq k} S_j$, where each $S_j$ is a locally closed submanifold of $M$ which retracts onto its intersection $C_j$ with the critical set $\text{Crit}(f)$ for $f$, and where $U_j = \bigsqcup_{0 \leq i \leq j} S_i$ is open in $M$. For any field $F$, and under suitable orientability assumptions (which can be ignored if we use $\mathbb{Z}/2\mathbb{Z}$ coefficients), the Thom–Gysin sequences

$$\cdots \rightarrow H_{i+1-\text{codim} S_j}(S_j; F) \rightarrow H_i(U_{j-1}; F) \rightarrow H_i(U_j; F) \rightarrow H_{i-\text{codim} S_j}(S_j; F) \rightarrow \cdots$$

associated to the inclusions of $U_{j-1} = U_j \setminus S_j$ into $U_j$ (for $1 \leq j \leq k$), combined with homeomorphisms from neighbourhoods of $S_j$ in $U_j$ to neighbourhoods of the zero section in the normal
bundles to the strata $S_j$, can be used to derive Morse inequalities of the form

$$P_t(M) = \sum_{j=0}^k t^{\dim S_j} P_t(C_j) - (1 + t)R(t)$$

where $P_t(M) = \sum_{i\geq 0} t^i \dim H_i(M; \mathbb{F})$ is the Poincaré polynomial of $M$ and $R(t)$ is a polynomial with nonnegative coefficients. Here $\dim S_j = \text{ind}_f(C_j)$ is the Morse index of $f$ along $C_j$.

(iii) **Morse homology.** An approach which goes back to Milnor, Thom and Smale and which was reinvigorated by Witten [8, 67, 68, 65, 75] is to use a Morse function $f : M \to \mathbb{R}$ and a suitable Riemannian metric $g$ on $M$ to derive the Morse inequalities by defining a complex

$$\cdots \to C_{i+1}^{\text{Morse}}(f, g) \xrightarrow{\partial} C_i^{\text{Morse}}(f, g) \xrightarrow{\partial} C_{i-1}^{\text{Morse}}(f, g) \xrightarrow{\partial} \cdots$$

in terms of the critical set $\text{Crit}(f)$ and showing that its homology is isomorphic to the homology of $M$. Here Hodge theory can be used to describe the (de Rham) cohomology $H^j(M; \mathbb{F}) = (H_j(M; \mathbb{F}))^\ast$ in terms of harmonic forms on $M$. Motivated by ideas from supersymmetry, Witten used the smooth function $f : M \to \mathbb{R}$ to defined a modification $\Delta_t$ (depending on $t \in \mathbb{R}$) of the Laplacian $\Delta$, with $\Delta_0 = \Delta$, and studied the spectrum of $\Delta_t$ as $t \to \infty$. Solutions to $\Delta_t = 0$ as $t \to \infty$ localise where $df$ is small, i.e. near $\text{Crit}(f)$, and tunnelling effects lead to the definition of the Morse–Witten complex.

These approaches to the Morse inequalities have been extended in different ways from classical Morse (and Morse–Smale) functions to Morse–Bott and minimally degenerate functions [5, 7, 13, 14, 36, 48, 79], to Novikov inequalities for closed 1-forms [15, 16, 17, 18, 59, 60, 61], to allow $M$ to have boundary [2, 11, 23, 50, 51] and in suitable circumstances to be non-compact or infinite-dimensional [1, 4, 15, 16, 17, 18, 20, 28, 30, 32, 33, 34, 63]; some approaches using stratifications do not require $M$ to be a manifold [38, 62, 74, 78]. For the main results in this article we will assume that $M$ is a (finite-dimensional) compact Riemannian manifold without boundary, though we will briefly consider other situations. However our only hypothesis on $f : M \to \mathbb{R}$ will be that it is smooth and that the critical locus $\text{Crit}(f)$ has only finitely many connected components; then its set of critical values $\text{Critval}(f) = f(\text{Crit}(f))$ is a finite union of compact connected subsets of $\mathbb{R}$ and by Sard’s theorem has measure zero, so is finite.

In order to obtain such a generalisation of the classical Morse inequalities, we need more ingredients than those in (0.1). We will use a ‘system of Morse neighbourhoods’ for $f$ in the sense described below (or slightly more generally as given in Definition 1.4). For any smooth $f : M \to \mathbb{R}$ it is possible to choose a system of Morse neighbourhoods (see Proposition 1.10 below), and the resulting inequalities will be independent of this choice.

So let $f : M \to \mathbb{R}$ be a smooth function whose critical locus $\text{Crit}(f)$ has finitely many connected components. For $c \in \text{Critval}(f)$ let $\text{Crit}_c(f) = f^{-1}(c) \cap \text{Crit}(f)$ and let $D_c$ be the set of its connected components. Then $D = \bigcup_{c \in \text{Critval}(f)} D_c$ is the set of connected components of the critical set $\text{Crit}(f)$. A system of (strict) Morse neighbourhoods for $f$ is given by

$$\{ N_{C,n} : C \in D, \ n \geq 0 \}$$

such that if $C \in D_c$ and $n \geq 0$ then

(a) $N_{C,n}$ is a neighbourhood of $C$ in $M$ containing no other critical points for $f$, with $N_{C,n+1} \subseteq (N_{C,n})^\circ$ (where $(N_{C,n})^\circ = N_{C,n} \setminus \partial N_{C,n}$ is the interior of $N_{C,n}$) and $\bigcap_{m\geq 0} N_{C,m} = C$;
(b) \(\mathcal{N}_{C,n}\) is a compact submanifold of \(M\) with corners (locally modelled on \([0, \infty)^2 \times \mathbb{R}^{\dim M - 2}\)) and has boundary
\[
\partial \mathcal{N}_{C,n} = \partial_+ \mathcal{N}_{C,n} \cup \partial_- \mathcal{N}_{C,n}
\]
where \(\partial_\pm \mathcal{N}_{C,n}\) is a compact submanifold of \(M\) with boundary and
\[
\partial (\partial_+ \mathcal{N}_{C,n}) = \partial (\partial_- \mathcal{N}_{C,n}) = (\partial_+ \mathcal{N}_{C,n}) \cap (\partial_- \mathcal{N}_{C,n}) \subseteq f^{-1}(c);
\]
(c) the gradient vector field \(\nabla f\) on \(M\) associated to its Riemannian metric \(g\) satisfies
(i) \((\partial_+ \mathcal{N}_{C,n})^o = \partial_+ \mathcal{N}_{C,n} \setminus \partial (\partial_+ \mathcal{N}_{C,n}) \subseteq f^{-1}(c, \infty)\) with the restriction of \(\nabla f\) to \((\partial_+ \mathcal{N}_{C,n})^o\) pointing inside \(\mathcal{N}_{C,n}\), and
(ii) \((\partial_- \mathcal{N}_{C,n})^o = \partial_- \mathcal{N}_{C,n} \setminus \partial (\partial_- \mathcal{N}_{C,n}) \subseteq f^{-1}(-\infty, c)\) with the restriction of \(\nabla f\) to \((\partial_- \mathcal{N}_{C,n})^o\) pointing outside \(\mathcal{N}_{C,n}\).

An example of such Morse neighbourhoods is shown in Figure 1 in §1. Let \(P_t(\mathcal{N}_{C,n}, \partial_\pm \mathcal{N}_{C,n})\) denote the relative Poincaré polynomial \(\sum_{i \geq 0} t^i \dim_{\mathbb{R}} H_i(\mathcal{N}_{C,n}, \partial_\pm \mathcal{N}_{C,n}; \mathbb{F})\). We will see that the gradient flow \(\nabla f\), combined with excision, induces isomorphisms of relative homology
\[
\phi_i^{m,n} : H_i(\mathcal{N}_{C,n}, \partial_\pm \mathcal{N}_{C,n}; \mathbb{F}) \rightarrow H_i(\mathcal{N}_{C,m}, \partial_\pm \mathcal{N}_{C,m}; \mathbb{F})
\]
for \(m > n\).

**Remark 0.1.** Each \(C \in D\) is an isolated invariant set in the sense of Conley [26] for the gradient flow. Each neighbourhood \(\mathcal{N}_{C,n}\) is an isolating neighbourhood of \(C\) in Conley’s sense and the pair \((\mathcal{N}_{C,n}, \partial_\pm \mathcal{N}_{C,n})\) is an index pair. The homotopy type of this pair represents the Conley index of \(C\). It is shown in [26] that this index, and hence its homology \(H_*(\mathcal{N}_{C,n}, \partial_- \mathcal{N}_{C,n}; \mathbb{F})\), is independent of the choice of isolating neighbourhood.

By taking the limit as \(m \to \infty\) we can define vector spaces
\[
(0.2) \quad H_i(\mathcal{N}_{C,\infty}, \partial_\pm \mathcal{N}_{C,\infty}; \mathbb{F})
\]
with isomorphisms \(\phi_i^{m,n} : H_i(\mathcal{N}_{C,n}, \partial_\pm \mathcal{N}_{C,n}; \mathbb{F}) \rightarrow H_i(\mathcal{N}_{C,\infty}, \partial_\pm \mathcal{N}_{C,\infty}; \mathbb{F})\) compatible with \(\phi_i^{m,n}\) for all \(m, n, i\). Let
\[
P_i(\mathcal{N}_{C,\infty}, \partial_\pm \mathcal{N}_{C,\infty}) = \sum_{i \geq 0} t^i \dim_{\mathbb{R}} H_i(\mathcal{N}_{C,\infty}, \partial_\pm \mathcal{N}_{C,\infty}; \mathbb{R}) = \sum_{i \geq 0} t^i \dim_{\mathbb{R}} H_i(\mathcal{N}_{C,n}, \partial_\pm \mathcal{N}_{C,n}; \mathbb{R})
\]
for any \(n \geq 0\). Our main theorem Theorem 1.11, combined with Proposition 1.10, tells us that for each \(C \in D\) the vector space \(H_i(\mathcal{N}_{C,\infty}, \partial_\pm \mathcal{N}_{C,\infty}; \mathbb{R})\), up to canonical isomorphism (and hence also the Poincaré polynomial \(P_i(\mathcal{N}_{C,\infty}, \partial_\pm \mathcal{N}_{C,\infty})\)), is independent of the choice of system of strict Morse neighbourhoods and of the Riemannian metric on \(M\), and that \(M\) satisfies the descending Morse inequalities
\[
P_i(M) = \sum_{C \in D} P_i(\mathcal{N}_{C,\infty}, \partial_- \mathcal{N}_{C,\infty}) - (1 + t) R_i(t) \quad \text{where } R_i(t) \geq 0
\]
and the ascending Morse inequalities
\[
P_i(M) = \sum_{C \in D} P_i(\mathcal{N}_{C,\infty}, \partial_+ \mathcal{N}_{C,\infty}) - (1 + t) R_i(t) \quad \text{where } R_i(t) \geq 0.
\]
When \(f\) is Morse–Bott then the normal bundle to \(C \in D\) decomposes as \(TM|_C^+ \oplus TM|_C^-\) where the Hessian of \(f\) is positive definite on \(TM|_C^+\) and negative definite on \(TM|_C^-\). We can take \(\mathcal{N}_{C,n}\) to be the image under the exponential map determined by the Riemannian metric of a product of disc bundles in \(TM|_C^+\) and \(TM|_C^-\), and then \(\partial_\pm \mathcal{N}_{C,n}\) is the product of one disc
bundle with the boundary of the other. If these bundles are orientable then \( H_i(\mathcal{N}_{n,C}, \partial_\pm \mathcal{N}_{n,C}; \mathbb{R}) \) is isomorphic to \( H_{i-\text{rank}(TM)}(C; \mathbb{R}) \) and we recover the Morse inequalities (0.1).

**Remark 0.2.** When \( M \) is oriented then by Poincaré duality \( P_i(M) = i^{\dim M} P_i(1)(M) \), and it follows from Alexander-Spanier duality (cf. [42] Theorem 3.43) that \( P_i(\mathcal{N}_{n,C}, \partial_\pm \mathcal{N}_{n,C}) = \) is equal to \( i^{\dim M} P_i(1)(\mathcal{N}_{n,C}, \partial_\pm \mathcal{N}_{n,C}) \) for any \( n \geq 0 \). Hence \( P_i(\mathcal{N}_{\infty,C}, \partial_\pm \mathcal{N}_{\infty,C}) = i^{\dim M} P_i(1)(\mathcal{N}_{\infty,C}, \partial_\pm \mathcal{N}_{\infty,C}) \), and so the descending and ascending Morse inequalities are equivalent.

**Remark 0.3.** To prove the Morse inequalities we can weaken the assumption that there are only finitely many connected components of the critical locus of \( f \). It is enough to know that its set of critical values \( \text{Critval}(f) = f(\text{Crit}(f)) \) is an isolated (or equivalently finite) subset of \( \mathbb{R} \). Then the Morse inequalities are still true when \( D \) is replaced with \( (\text{Crit}(f) \cap f^{-1}(c) : c \in \text{Critval}(f)) \).

We will see that these Morse inequalities can be proved in several different ways which generalise the different approaches to the classical Morse inequalities. In particular they follow from a spectral sequence of multicomplexes which refines the information given by the vector module with basis \( \text{Crit} \) from a critical point \( q \) to \( p \) of index \( i \) to a critical point \( q \) of index \( i-1 \) given by the (finitely many) gradient flow lines from \( p \) to \( q \) (cf. for example [25]). The Morse–Witten complex is the differential module supported on \( \text{Crit} \) with basis \( \text{Crit} \) and differential

\[
\partial p = \sum_{q \in \text{Crit}(f)} n(p,q)q
\]

where \( n(p,q) \) is the number of flow lines from \( p \) to \( q \), counted with signs determined (canonically) by suitable choices of orientations. It is usually regarded as a complex via the grading by index, but we can also regard it as a differential module supported on the quiver \( \Gamma \), where \( \Gamma \) is graded by the function \( f \). From the viewpoint of the representation theory of quivers the Morse–Witten complex is a representation of \( \Gamma \) over \( \mathbb{F} \) such that each vertex is represented by a copy of \( \mathbb{F}_e \) each arrow by the identity on \( \mathbb{F} \) (up to a suitable sign) and the sum of all the arrows is the differential \( \partial \). Equivalently we can replace multiple arrows from \( p \) to \( q \) with just one, represented by the number of flow lines from \( p \) to \( q \) counted with signs. When we grade this representation using the homological degree and the critical value instead of the index, then we have a multicomplex supported on \( \Gamma \).

In our more general situation we can define a quiver \( \Gamma \) with vertices \( \{ v_C : C \in D \} \) labelled by the connected components \( C \) of \( \text{Crit}(f) \), and an arrow from \( C_+ \) to \( C_- \) for each connected component of

\[
\{ x \in M \cap \text{Crit}(f) : t \leq 0 \}
\]

that is closed in \( f^{-1}(f(C_-), f(C_+)) \). Here \( \psi_t(x) \) describes the downwards gradient flow for \( f \) from \( x \) at time \( t \). The analogue of the Morse–Witten complex is not in general a differential module supported on \( \Gamma \) with homology \( H_*(M; \mathbb{F}) \), but instead a spectral sequence of multicomplexes supported on the quiver \( \Gamma \) which abuts to \( H_*(M; \mathbb{F}) \), such that the vertex \( v_C \) is represented in the \( E_1 \) page of the spectral sequence by \( H_*(\mathcal{N}_{\infty,C}, \partial_\pm \mathcal{N}_{\infty,C}; \mathbb{F}) \).

The layout of this paper is as follows. In §1 we study systems of Morse neighbourhoods and give a first proof of the main theorem, Theorem 1.11; this generalises the proof of the classical
Morse inequalities via attaching handles. In §2 we describe another approach using Witten’s
deformation technique which was the original motivation for this project. Morse stratifications
and Morse double complexes with associated spectral sequences are defined in §3 and §4, giving
two further proofs. In §5 we consider spectral sequences of multicomplexes supported on
acyclic quivers. In §6 we introduce the quiver \( \Gamma \) described above, and the spectral sequence of
multicomplexes which abuts to \( H_*(M; \mathbb{F}) \) and has \( E_1 \) page given by \( H_*(N_{C,\infty}; \partial_{\pm}N_{C,\infty}; \mathbb{F}) \) for
\( C \in D \). §7 discusses extensions of our results to smooth functions on manifolds with boundary
or corners, allowing singular or infinite-dimensional spaces, the use of equivariant homology
and generalisations of the Novikov inequalities for closed 1-forms on \( M \), and describes some
applications.

The authors would like to thank Vidit Nanda, Graeme Segal, Edward Witten and Jon Woolf
for valuable discussions, and to dedicate this paper to the memory of Michael Atiyah, without
whom this collaboration would not have been possible. This work was supported in part by
AFOSR award FA9550-16-1-0082 and DOE award DE-SC0019380.

1. Systems of Morse neighbourhoods

Let \( M \) be a compact Riemannian manifold without boundary, and let \( f : M \to \mathbb{R} \) be a smooth
function on \( M \) whose critical locus \( \text{Crit}(f) \) has finitely many connected components, with its
set of critical values \( \text{Crit}(f) = \{ c_1, \ldots, c_p \} \).

In order to state our main result, we first define the notion of a system of Morse neighbour-
hoods for \( f \). The definition (see Definition 1.4 below) is slightly more general than that given
in the introduction; we will use the terminology ‘a system of strict Morse neighbourhoods’ for
the latter.

Definition 1.1. Let \( f : M \to \mathbb{R} \) be a smooth function on the compact Riemannian manifold \( M \)
such that the set \( D \) of connected components of the critical locus \( \text{Crit}(f) \) is finite. A system of
strict Morse neighbourhoods for \( f \) is given by

\[
\{ N_{C,n} : C \in D, \ n \geq 0 \}
\]

such that if \( f \) takes value \( c \) on \( C \in D \), and \( n \geq 0 \), then
(a) \( N_{C,n} \) is a neighbourhood of \( C \) in \( M \) containing no other critical points for \( f \), with \( N_{C,n+1} \subseteq \)
\( (N_{C,n})^\circ \) and \( \bigcap_{m \geq 0} N_{C,m} = C \);
(b) \( N_{C,n} \) is a compact submanifold of \( M \) with corners (locally modelled on \( [0, \infty)^2 \times \mathbb{R}^{\dim M-2} \))
and has boundary

\[
\partial N_{C,n} = \partial_+ N_{C,n} \cup_\partial \partial_- N_{C,n}
\]

where \( \partial_\pm N_{C,n} \) is a compact submanifold of \( M \) with boundaries and the corners of \( N_{C,n} \) are
given by

\[
\partial(\partial_+ N_{C,n}) = \partial(\partial_- N_{C,n}) = (\partial_+ N_{C,n}) \cap (\partial_- N_{C,n}) = f^{-1}(c) \cap \partial N_{C,n};
\]
(c) the gradient vector field \( \text{grad}(f) \) on \( M \) associated to its Riemannian metric \( g \) satisfies
(i) \( \partial_+ N_{C,n}^\circ = \partial_+ N_{C,n} \setminus (\partial_+ N_{C,n}) \subseteq f^{-1}(c, \infty) \) with the restriction of \( \text{grad}(f) \) to \( \partial_+ N_{C,n}^\circ \)
pointing inside \( N_{C,n} \), and
(ii) \( \partial_- N_{C,n}^\circ = \partial_- N_{C,n} \setminus (\partial_- N_{C,n}) \subseteq f^{-1}(-\infty, c) \) with the restriction of \( \text{grad}(f) \) to \( \partial_- N_{C,n}^\circ \)
pointing outside \( N_{C,n} \).

Remark 1.2. Analysis on manifolds with corners is not straightforward (cf. \([39, 46, 47, 53, 54]\))
but we will not need the subtleties of this theory.
Definition 1.3. We refer to a system of strict Morse neighbourhoods such that, for all \((C, n)\), the boundary \(\partial N_{C,n} = \partial_+ N_{C,n} \cup \partial_- N_{C,n}\) is a submanifold of \(M\) (and hence \(N_{C,n}\) is a submanifold with boundary), as a system of smooth Morse neighbourhoods.

Suppose that we have a system of strict Morse neighbourhoods for \(f\). Using the gradient flow \(\text{grad}(f)\), downwards on \(f^{-1}(c, \infty)\) and upwards on \(f^{-1}(-\infty, c)\) until \(\partial N_{C,n+1} \cup (f^{-1}(c) \cap N_{C,n})\) is reached, we obtain a retraction from \(N_{C,n}\) to \(\partial_+ N_{C,n+1} \cup (f^{-1}(c) \cap (N_{C,n} \setminus N_{C,n+1}))\), and so induces an isomorphism of relative homology groups

\[ H_i(N_{C,n}, \partial_- N_{C,n}; \mathbb{F}) \to H_i(N_{C,n+1} \cup (f^{-1}(c) \cap N_{C,n}), \partial_- N_{C,n+1} \cup (f^{-1}(c) \cap (N_{C,n} \setminus N_{C,n+1})); \mathbb{F}) \]

Excision ([42] Thm 2.20), together with the fact that \(f^{-1}(c) \cap N_{C,n}\) is diffeomorphic near \(f^{-1}(c) \cap \partial N_{C,n}\) to the product of \(f^{-1}(c) \cap \partial N_{C,n}\) with an interval, then gives isomorphisms from

\[ H_i(N_{C,n+1} \cup (f^{-1}(c) \cap N_{C,n}), \partial_- N_{C,n+1} \cup (f^{-1}(c) \cap (N_{C,n} \setminus N_{C,n+1})); \mathbb{F}) \]

to \(H_i(N_{C,n+1}, \partial_- N_{C,n+1}; \mathbb{F})\), and by composition we obtain isomorphisms of relative homology

\[ \phi_{m,n}^i : H_i(N_{C,n}, \partial_- N_{C,n}; \mathbb{F}) \to H_i(N_{C,m}, \partial_- N_{C,m}; \mathbb{F}) \]

when \(m > n\).

Next we will define a system of Morse neighbourhoods without the strictness condition. This allows the boundary of a Morse neighbourhood to decompose into three submanifolds instead of just two; in addition to \(\partial_+ N_{C,n}\) and \(\partial_- N_{C,n}\) which are transverse to the gradient flow of \(f\), another submanifold \(\partial_\perp N_{C,n}\) is allowed which is invariant under the gradient flow.

Definition 1.4. A system of Morse neighbourhoods for \(f\) is given by \(\{ N_{C,n} : C \in D, \ n \geq 0\} \) such that if \(n \geq 0\) and \(f\) takes value \(c\) on \(C \in D\) then

(a) \(N_{C,n}\) is a neighbourhood of \(C\) in \(M\) containing no points of \(\text{Crit}(f) \setminus D\), with \(N_{C,n+1} \subseteq (N_{C,n})^\circ\) and \(\bigcap_{m \geq 0} N_{C,m} = C\).
(b) \( \mathcal{N}_{C,n} \) is a compact submanifold of \( M \) with corners (locally modelled on \([0, \infty)^2 \times \mathbb{R}^{\dim M - 2}\)) and has boundary

\[
\partial \mathcal{N}_{C,n} = \partial_+ \mathcal{N}_{C,n} \cup \partial_- \mathcal{N}_{C,n} \cup \partial_{\bot} \mathcal{N}_{C,n}, \text{ with } \partial_+ \mathcal{N}_{C,n} \cap \partial_- \mathcal{N}_{C,n} \cap \partial_{\bot} \mathcal{N}_{C,n} = \emptyset,
\]

where \( \partial_+ \mathcal{N}_{C,n} \), \( \partial_- \mathcal{N}_{C,n} \) and \( \partial_{\bot} \mathcal{N}_{C,n} \) are compact submanifolds of \( M \) with boundaries forming the corners of \( \mathcal{N}_{C,n} \) while \( f \) is constant on \( (\partial_- \mathcal{N}_{C,n}) \cap (\partial_+ \mathcal{N}_{C,n}) = \partial(\partial_+ \mathcal{N}_{C,n}) \cap \partial(\partial_- \mathcal{N}_{C,n}) \), with value \( c \), and on each of \( \partial(\partial_+ \mathcal{N}_{C,n}) \cap \partial(\partial_- \mathcal{N}_{C,n}) = (\partial_+ \mathcal{N}_{C,n}) \cap (\partial_- \mathcal{N}_{C,n}) \):

(c) the gradient vector field \( \text{grad} (f) \) on \( M \) associated to its Riemannian metric \( g \) satisfies

(i) \( (\partial_+ \mathcal{N}_{C,n})^o = \partial_+ \mathcal{N}_{C,n} \setminus \partial(\partial_+ \mathcal{N}_{C,n}) \subseteq f^{-1}(c, \infty) \) with the restriction of \( \text{grad} (f) \) to \( (\partial_+ \mathcal{N}_{C,n})^o \) pointing inside \( \mathcal{N}_{C,n} \);

(ii) \( (\partial_- \mathcal{N}_{C,n})^o = \partial_- \mathcal{N}_{C,n} \setminus \partial(\partial_- \mathcal{N}_{C,n}) \subseteq f^{-1}(-\infty, c) \) with the restriction of \( \text{grad} (f) \) to \( (\partial_- \mathcal{N}_{C,n})^o \) pointing outside \( \mathcal{N}_{C,n} \);

(iii) \( \partial_{\bot} \mathcal{N}_{C,n} \) is contained in the union of the trajectories under the gradient flow \( \text{grad} (f) \) of \( f^{-1}(c) \cap \partial \mathcal{N}_{C,n} \), which is a submanifold of codimension one in (the smooth part of) \( f^{-1}(c) \).

**Remark 1.5.** A system of Morse neighbourhoods \( \{ \mathcal{N}_{C,n} : C \in D, n \geq 0 \} \) is strict if and only if \( \partial_{\bot} \mathcal{N}_{C,n} \) is empty, for each \( C \) and \( n \).

Finally it is useful to define another special type of Morse neighbourhoods; these will be called *cylindrical* Morse neighbourhoods. We will see that systems of cylindrical Morse neighbourhoods always exist, and that they allow us to build systems of strict Morse neighbourhoods.

**Definition 1.6.** A system of cylindrical Morse neighbourhoods for \( f \) is given by

\[
\{ \mathcal{N}_{C,n} : C \in D, n \geq 0 \}
\]

such that if \( f \) takes the value \( c \) on \( C \in D \) and if \( n \geq 0 \) then

(a) \( \mathcal{N}_{C,n} \) is a neighbourhood of \( C \in M \) containing no other critical points for \( f \), with \( \mathcal{N}_{C,n+1} \subseteq \mathcal{N}_{C,n}^o \) and \( \bigcap_{m \geq 0} \mathcal{N}_{C,m} = C \);

(b) \( \mathcal{N}_{C,n} \) is a compact submanifold of \( M \) with corners (locally modelled on \([0, \infty)^2 \times \mathbb{R}^{\dim M - 2}\)) and has boundary

\[
\partial \mathcal{N}_{C,n} = \partial_+ \mathcal{N}_{C,n} \cup \partial_- \mathcal{N}_{C,n} \cup \partial_{\bot} \mathcal{N}_{C,n}
\]

where \( \partial_+ \mathcal{N}_{C,n} \), \( \partial_- \mathcal{N}_{C,n} \) and \( \partial_{\bot} \mathcal{N}_{C,n} \) are compact submanifolds of \( M \) with boundary, and

(c) for some \( \delta_+ = \delta_+(c, n) > 0 \) and \( \delta_- = \delta_-(c, n) > 0 \)

(i) \( \partial_+ \mathcal{N}_{C,n} \subseteq f^{-1}(c + \delta_+) \) and \( \partial_- \mathcal{N}_{C,n} \subseteq f^{-1}(c - \delta_-) \);

(ii) \( \partial_{\bot} \mathcal{N}_{C,n} \) is the intersection of \( f^{-1}[c - \delta_-, c + \delta_+] \) with the trajectories under the gradient flow \( \text{grad} (f) \) of \( f^{-1}(c) \cap \partial \mathcal{N}_{C,n} \), which is a submanifold of codimension one in (the smooth part of) \( f^{-1}(c) \).

Now suppose that we have a system of cylindrical Morse neighbourhoods for \( f \) as above. By combining the gradient flow \( \text{grad} (f) \) downwards on \( f^{-1}[c + \delta_+(c, n + 1), \infty) \) and upwards on \( f^{-1}(-\infty, c - \delta_-(c, n + 1)) \), we obtain a retraction from \( \mathcal{N}_{C,n} \) onto

\[
\mathcal{N}_{C,n} \cap f^{-1}([c - \delta_-(c, n + 1), c + \delta_+(c, n + 1)]
\]

which takes \( \partial_- \mathcal{N}_{C,n} \cup (\partial_+ \mathcal{N}_{C,n} \cap f^{-1}([c, \infty))) \) to its intersection with \( f^{-1}([c - \delta_-(c, n + 1), c + \delta_+(c, n + 1))] \). Similarly the closure of the complement of \( \mathcal{N}_{C,n+1} \) in \( \mathcal{N}_{C,n} \cap f^{-1}([c - \delta_-(c, n + 1), c + \delta_+(c, n + 1))] \) is diffeomorphic via the gradient flow to the product with the interval \([c - \delta_-(c, n + 1), c + \delta_+(c, n + 1)]\)
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Figure 2. Cylindrical Morse neighbourhoods for \( f \) given locally by \( f(x, y) = xy(x-y) \).

\[ \delta_+(c, n+1) \] of the complement of \( \mathcal{N}_{C,n+1} \cap \mathcal{N}_{C,n} \cap f^{-1}(c) \), which in turn is diffeomorphic near \( \partial \mathcal{N}_{C,n+1} \cap f^{-1}(c) \) to the product of \( \partial \mathcal{N}_{C,n+1} \cap f^{-1}(c) \) with an interval. The retraction induces an isomorphism of relative homology groups from \( H_i(\mathcal{N}_{C,n}, \partial \mathcal{N}_{C,n}; F) \), where

\[ \partial_{-1} \mathcal{N}_{C,n} = \partial_{-1} \mathcal{N}_{C,n} \cup (\partial_{-1} \mathcal{N}_{C,n} \cap f^{-1}(-\infty, c]), \]

\[ H_i(\mathcal{N}_{C,n} \cap f^{-1}(c-\delta_-(c, n+1)), \partial_{-1} \mathcal{N}_{C,n} \cup (\partial_{-1} \mathcal{N}_{C,n} \cap f^{-1}[c-\delta_+(c, n+1), c]); F) \]

which is isomorphic to \( H_i(\mathcal{N}_{C,n+1} \cup (\mathcal{N}_{C,n} \cap f^{-1}(c)), \partial_{-1} \mathcal{N}_{C,n+1} \cup (\mathcal{N}_{C,n} \cap f^{-1}(c) \setminus \mathcal{N}_{C,n+1}); F) \). Using excision ([42] Thm 2.20) this is isomorphic to \( H_i(\mathcal{N}_{C,n+1}, \partial_{-1} \mathcal{N}_{C,n+1}; F) \). Thus by composition there are induced isomorphisms

\[ \phi_{m,n}^i : H_i(\mathcal{N}_{C,n}, \partial_{-1} \mathcal{N}_{C,n}; F) \to H_i(\mathcal{N}_{C,m}, \partial_{-1} \mathcal{N}_{C,m}; F) \]

when \( m > n \). When \( \mathcal{N} = \{ \mathcal{N}_{C,n} : C \in D, n \geq 0 \} \) is any system of Morse neighbourhoods, then combining this construction with that of (1.1) gives us isomorphisms of relative homology

\[ \phi_{m,n}^i : H_i(\mathcal{N}_{C,n}, \partial_{-1} \mathcal{N}_{C,n}; F) \to H_i(\mathcal{N}_{C,m}, \partial_{-1} \mathcal{N}_{C,m}; F) \]

when \( m > n \).

Remark 1.7. Suppose that \( \{ \mathcal{N}_{C,n} : C \in D, n \geq 0 \} \) is a system of Morse neighbourhoods for \( f \) with respect to a Riemannian metric \( g \) on \( M \), and that \( \{ \mathcal{N}_{C,n} : C \in D, n \geq 0 \} \) is a system of Morse neighbourhoods for \( f \) with respect to a Riemannian metric \( g' \) on \( M \). We have \( \mathcal{N}_{C,n} = C = \bigcap_{n \geq 1} \mathcal{N}_{C,n} \) and so, by compactness, for each \( n \) and \( C \) there exists \( m(C,n) \) such that

\[ \mathcal{N}_{C,m(C,n)} \subseteq (\mathcal{N}_{C,n})^e. \]

If \( m \geq m(C,n) \), then by constructing a metric which coincides with \( g \) in a neighbourhood of \( \partial \mathcal{N}_{C,n} \) and with \( g' \) in a neighbourhood of \( \partial \mathcal{N}_{C,m} \) and using the construction of the isomorphisms
given at (1.3), we obtain isomorphisms of relative homology
\[ \psi_i^{m,n} : H_i(\mathcal{N}_{C,n}^\prime, \partial_\pm \mathcal{N}_{C,n}^\prime; \mathbb{F}) \rightarrow H_i(\mathcal{N}_{C,m}^\prime, \partial_\pm \mathcal{N}_{C,m}^\prime; \mathbb{F}) . \]
These induce isomorphisms
\[ H_i(\mathcal{N}_{C,\infty}^\prime, \partial_\pm \mathcal{N}_{C,\infty}^\prime; \mathbb{F}) \rightarrow H_i(\mathcal{N}_{C,\infty}^\prime, \partial_\pm \mathcal{N}_{C,\infty}^\prime; \mathbb{F}) \]
when, as at (0.2), we define \( H_i(\mathcal{N}_{C,\infty}^\prime, \partial_\pm \mathcal{N}_{C,\infty}^\prime; \mathbb{F}) \) with isomorphisms \( \phi_i^{\infty,n} : H_i(\mathcal{N}_{C,n}^\prime, \partial_\pm \mathcal{N}_{C,n}^\prime; \mathbb{F}) \rightarrow H_i(\mathcal{N}_{C,\infty}^\prime, \partial_\pm \mathcal{N}_{C,\infty}^\prime; \mathbb{F}) \) compatible with \( \phi_i^{m,n} \) for all \( m, n, i \) by taking the limit as \( m \rightarrow \infty \) in (1.3).

Remark 1.8. Given any system of cylindrical Morse neighbourhoods \( \{\mathcal{N}_{C,n}^\prime : C \in D, n \geq 0\} \), we can modify each \( \mathcal{N}_{C,n}^\prime \) near \( \partial_\pm \mathcal{N}_{C,n}^\prime \) to obtain a system of strict Morse neighbourhoods
\[ \{\mathcal{N}_{C,n}^\prime : C \in D, n \geq 0\} \]
such that \( \mathcal{N}_{C,n}^\prime \subseteq \mathcal{N}_{C,n}^\prime \) and \( \mathcal{N}_{C,n}^\prime \cap f^{-1}(c) = \mathcal{N}_{C,n}^\prime \cap f^{-1}(c) \), with a homotopy equivalence taking the pair \( (\mathcal{N}_{C,n}^\prime, \partial_\pm \mathcal{N}_{C,n}^\prime) \) to the pair \( (\mathcal{N}_{C,n}^\prime, \partial_\pm \mathcal{N}_{C,n}^\prime) \) for each \( c, n \) and fixing \( f^{-1}(c) \) pointwise.

These induce isomorphisms from \( H_i(\mathcal{N}_{C,n}^\prime, \partial_\pm \mathcal{N}_{C,n}^\prime; \mathbb{F}) \) to \( H_i(\mathcal{N}_{C,n}^\prime, \partial_\pm \mathcal{N}_{C,n}^\prime; \mathbb{F}) \). Combining these and their inverses with the isomorphisms \( \phi_i^{m,n} \) defined at (1.2) above, we obtain the maps (1.1) for \( \{\mathcal{N}_{C,n}^\prime : C \in C, n \geq 1\} \), which are therefore isomorphisms.

If we wish, we can construct \( \mathcal{N}_{C,n}^\prime \) in such a way that the angle between \( \partial_+ \mathcal{N}_{C,n}^\prime \) and \( \partial_- \mathcal{N}_{C,n}^\prime \) is \( \pi \); then \( \partial_+ \mathcal{N}_{C,n}^\prime = \partial_+ \mathcal{N}_{C,n}^\prime \cup \partial_- \mathcal{N}_{C,n}^\prime \) is smooth and \( \mathcal{N}_{C,n}^\prime \) can be regarded as a submanifold of \( M \) with boundary, rather than a submanifold with corners. We can thus obtain a system of smooth Morse neighbourhoods (Definition 1.3). In addition we can choose \( \{\mathcal{N}_{C,n}^\prime : C \in C, n \geq 0\} \) so that \( f|_{\partial_\pm \mathcal{N}_{C,n}^\prime} \) is a Morse function on the interior of \( \partial_\pm \mathcal{N}_{C,n}^\prime \) (with minimum/maximum on the boundary \( \partial_- \mathcal{N}_{C,n}^\prime \cap \partial_+ \mathcal{N}_{C,n}^\prime \)).

We can also find a homotopy equivalence from \( \mathcal{N}_{C,n}^\prime \) to itself taking \( \partial_- \mathcal{N}_{C,n}^\prime \) to \( \partial_- \mathcal{N}_{C,n}^\prime \) so that
\[ (1.4) \quad H_i(\mathcal{N}_{C,n}^\prime, \partial_- \mathcal{N}_{C,n}^\prime; \mathbb{F}) \cong H_i(\mathcal{N}_{C,n}^\prime, \partial_- \mathcal{N}_{C,n}^\prime; \mathbb{F}) . \]

Remark 1.9. By applying Sard’s theorem ([64] Thm II.3.1) to the smooth function \( \|\text{grad}(f)\|^2 \) on the submanifold \( f^{-1}(c) \setminus \text{Crit}(f) \) of \( M \) for each \( c \in \text{Critval}(f) \), we can find a sequence \( (\epsilon_n(c))_{n \geq 1} \) of strictly positive real numbers which are regular values of \( \|\text{grad}(f)\|^2 \) on this submanifold and which tend to 0 as \( n \rightarrow \infty \). We can also choose disjoint open neighbourhoods \( \{U_C : C \in C_c\} \) in \( f^{-1}(c) \) of the critical sets \( C \in C_c \) contained in \( f^{-1}(c) \). There is some \( n_0 \equiv n_0(\epsilon) > 0 \) such that for each \( c \in \text{Critval}(f) \)
\[ \{x \in f^{-1}(c) : \|\text{grad}(f)(x)\|^2 \leq \epsilon_n(c)\} \subseteq \bigcup_{C \in C_c} U_C . \]
We can then construct a system of cylindrical Morse neighbourhoods \( \{\mathcal{N}_{C,n}^\prime : C \in C, n \geq 0\} \) such that
\[ \mathcal{N}_{C,n}^\prime \cap f^{-1}(c) = \{x \in f^{-1}(c) : \|\text{grad}(f)(x)\|^2 \leq \epsilon_{n_0+c}(c)\} \cap U_C . \]

The following proposition follows from Remarks 1.7, 1.8 and 1.9.

Proposition 1.10. Any smooth function \( f : M \rightarrow \mathbb{R} \) on a Riemannian manifold \( M \) whose critical locus \( \text{Crit}(f) \) has finitely many connected components has a system of strict Morse neighbourhoods. Moreover, if \( \mathcal{N} = \{\mathcal{N}_{C,n}^\prime : C \in C, \ n \geq 0\} \) is any system of Morse neighbourhoods for \( f \),
then the vector spaces $H_i(\mathcal{N}_{C,\infty}, \partial_-, \mathcal{N}_{C,\infty}; \mathbb{R})$, up to canonical isomorphism, and
$$P_i(\mathcal{N}_{C,\infty}, \partial_- \mathcal{N}_{C,\infty}) = \sum_{i \geq 0} t \dim_{\mathbb{F}} H_i(\mathcal{N}_{C,n}, \partial_- \mathcal{N}_{C,n}; \mathbb{F})$$
are independent of the choice of $n$, and also the choice of system of Morse neighbourhoods, and of the Riemannian metric on $M$, up to canonical isomorphism.

We can now state our generalised version of the Morse inequalities.

**Theorem 1.11.** Let $M$ be a compact Riemannian manifold without boundary, and suppose that $f : M \to \mathbb{R}$ is a smooth function whose critical locus $\text{Crit}(f)$ has finitely many connected components. Suppose also that $\{ \mathcal{N}_{C,n} : C \in C, n \geq 0 \}$ is a system of Morse neighbourhoods for $f : M \to \mathbb{R}$. Then the Betti numbers of $M$ satisfy the descending Morse inequalities
$$P_i(M) = \sum_{C \in C} P_i(\mathcal{N}_{C,\infty}, \partial_- \mathcal{N}_{C,\infty}) - (1 + t)R_i(t) \quad \text{where } R_i(t) \geq 0$$
and the ascending Morse inequalities
$$P_i(M) = \sum_{C \in C} P_i(\mathcal{N}_{C,\infty}, \partial_+ \mathcal{N}_{C,\infty}) - (1 + t)R_i(t) \quad \text{where } R_i(t) \geq 0.$$

**Proof.** The first proof we will give of these Morse inequalities follows the approach in the classical case given by attaching handles. By using the gradient flow we see that the topology of $f^{-1}(\infty, a]$ is unchanged as $a$ increases, except when $a$ passes through a critical value $C \in D_c$, and then disjoint Morse neighbourhoods $\mathcal{N}_{C,n}$ for $C \in D_c$ are attached along $\partial_+ \mathcal{N}_{C,n}$. Thus if $c_- = c - \delta_-$ and $c_+ = c + \delta_+$ where $\delta_+ > 0$ are sufficiently small, there is an isomorphism
$$H_i(f^{-1}(\infty, c_+], f^{-1}(\infty, c_-]; \mathbb{F}) \cong \bigoplus_{C \in D_c} H_i(\mathcal{N}_{C,\infty}, \partial_- \mathcal{N}_{C,\infty}; \mathbb{F})$$
induced by the gradient flow and excision (cf. Remark 1.7), and therefore a long exact sequence
$$\cdots \to \bigoplus_{C \in D_c} H_{i+1}(\mathcal{N}_{C,\infty}, \partial_- \mathcal{N}_{C,\infty}; \mathbb{F}) \xrightarrow{\delta_{i,c}} H_i(f^{-1}(\infty, c_-]; \mathbb{F}) \to H_i(f^{-1}(\infty, c_+]; \mathbb{F}) \to H_i(f^{-1}(\infty, c_-]; \mathbb{F}) \to \cdots$$
which tells us that $\dim H_i(f^{-1}(\infty, c_+]; \mathbb{F})$ is equal to
$$\dim H_i(f^{-1}(\infty, c_-]; \mathbb{F}) - \dim \text{im} \delta_{i,c} - \dim \text{im} \delta_{i-1,c} + \sum_{C \in D_c} \dim H_i(\mathcal{N}_{C,\infty}, \partial_- \mathcal{N}_{C,\infty}; \mathbb{F})$$
for each $i$ and $C$. By combining these long exact sequences for $c \in \text{Critval}(f)$ we obtain the descending Morse inequalities
$$P_i(M) = \sum_{C \in D} P_i(\mathcal{N}_{C,\infty}, \partial_- \mathcal{N}_{C,\infty}) - (1 + t)R_i(t) \quad \text{where } R_i(t) \geq 0;$$
here $R_i(t)$ is given by $\sum_{i \geq 0} t \sum_{c \in \text{Critval}(f)} \dim \delta_{i,c}$. The proof of the ascending Morse inequalities is similar. \[\square\]
Remark 1.12. As was noted in the introduction, when $M$ is oriented the descending Morse inequalities are equivalent to the ascending Morse inequalities

$$P_t(M) = \sum_{C \in D} P_t(\mathcal{N}_{C^\infty}, \partial_+ \mathcal{N}_{C^\infty}) - (1 + t)R_t(t) \quad \text{where } R_t(t) \geq 0$$

since by Poincaré duality $P_t(M) = t^{\dim M} P_{(1/t)}(M)$, and by Alexander-Spanier duality (cf. [42] Theorem 3.43) we have $P_t(\mathcal{N}_{C^\infty}, \partial_- \mathcal{N}_{C^\infty}) = t^{\dim M} P_{(1/t)}(\mathcal{N}_{C^\infty}, \partial_+ \mathcal{N}_{C^\infty})$.

2. Morse inequalities via Witten’s deformation technique

An alternative approach to proving the classical Morse inequalities (with real coefficients) was pioneered by Witten [75]. He made use of supersymmetric quantum mechanics, specifically a supersymmetric non-linear sigma model with target space $M$, which as before we assume to be a compact Riemannian manifold (without boundary). The Hilbert space of this theory is canonically isomorphic to the space of differential forms $\Omega^*(M)$ with the supercharges corresponding to the exterior derivative $d$ and codifferential $d^*$. The Hamiltonian is therefore the Laplacian (or Laplace-Beltrami operator) $\Delta = dd^* + d^*d$ on the space of differential forms.

This is a positive, essentially self-adjoint operator on the $L^2$ closure of the space of differential forms on $M$. The zero energy states are harmonic forms, and so, by the usual arguments from Hodge theory, the zero energy subspace is canonically isomorphic to the de Rham cohomology of $M$.

By adding a superpotential $tf$ (for a Morse function $f$ and constant $t > 0$), Witten deformed the theory, while preserving the supersymmetry, with the supercharges now $d^W_t = e^{-tf}d e^{tf}$ and $d^*W_t = e^{tf} d^* e^{-tf}$ respectively. Since the map $\omega \mapsto e^{tf} \omega$ is invertible, the cohomology, and hence the number of zero energy states, is unchanged by this deformation. The new Hamiltonian is given by the deformed Laplacian $\Delta_t = d^W_t d^*_t + d^*_t d^W_t$ and contains a potential term $t^2 \|\text{grad}(f)\|^2$ which means that, for $t \gg 1$, low energy states must be localised near critical points of $f$. As a result, the Hamiltonian can be approximated for $t \gg 1$ by a direct sum over supersymmetric harmonic oscillators associated to each critical point. Within this approximation, there exists a single zero energy state for each oscillator, which will be a $p$-form if the Hessian of the associated critical point has $p$ negative eigenvalues. Since the exact zero energy states must form a subspace of these approximate zero energy states, the Morse inequalities follow.

In this section, we outline a similar approach to rederive Theorem 1.11. This mirrors the strategy used in [20] to prove Novikov inequalities in the presence of ‘minimal degeneracy’ (cf. [48]) using a deformed Laplacian. We construct extended Morse neighbourhoods $\tilde{\mathcal{N}}_C$ by attaching cylindrical ends (on which the function grows quadratically) to smooth Morse neighbourhoods $\mathcal{N}_{C,n}$. It is enough to consider a single smooth Morse neighbourhood $\mathcal{N}_{C,nC}$ with $n_C$ fixed for each connected component $C \subseteq \text{Crit}(f)$.

As we shall see, the deformed $L^2$ cohomology of the extended Morse neighbourhoods is isomorphic to $H^*(\mathcal{N}_{C,nC}, \partial_- \mathcal{N}_{C,nC}; \mathbb{R})$. At low energies and large $t$, a deformed Laplacian $\Delta_t(M)$ acting on the manifold $M$ can be modelled by a direct sum of the same deformed Laplacian $\bigoplus_{C \in C} \Delta_t(\tilde{\mathcal{N}}_{C,nC})$ on a set of extended Morse neighbourhoods. As in Witten’s original argument, the zero energy states of the model Hamiltonian give an upper bound on the number of zero energy states of the deformed Laplacian and hence we will obtain another proof of Theorem 1.11.
Remark 2.1. The gradient flow of $f$ combined with local analysis near the corners $f^{-1}(c) \cap \partial \mathcal{N}_{C,nC}$ of $\mathcal{N}_{C,nC}$ can be used to show that there exists a smooth embedding

$$\phi : \partial \mathcal{N}_{C,nC} \times (1 - \epsilon, 1) \to \mathcal{N}_{C,nC}$$

for sufficiently small $\epsilon > 0$, such that

(a) $\phi\left(\partial \mathcal{N}_{C,nC} \times \{1\}\right) = \partial \mathcal{N}_{C,nC}$ and
(b) for all $x \in \partial \mathcal{N}_{C,nC}$ and $s \in (1 - \epsilon, 1]$, then $f(\phi(x, s)) = c + s^2 f_0(x)$, where $c = f(C)$ and $c + f_0(x)$ is the restriction of $f$ to $\partial \mathcal{N}_{C,nC}$.

Definition 2.2. Let $\mathcal{N}_{C,nC}$ be a smooth Morse neighbourhood (Definition 1.3) of some connected component $C \in D$ of the critical set $\text{Crit}(f)$. Then a corresponding extended Morse neighbourhood is

$$\tilde{\mathcal{N}}_{C,nC} = \mathcal{N}_{C,nC} \cup (\partial \mathcal{N}_{C,nC} \times (1 - \epsilon, \infty))$$

where we identify $\partial \mathcal{N}_{C,nC} \times (1 - \epsilon, 1]$ with its image in $\mathcal{N}_{C,nC}$ under a smooth embedding $\phi$ as in Remark 2.1. We shall refer to $T_C = \phi(\partial \mathcal{N}_{C,nC} \times (1 - \epsilon, 1])$ as a boundary region of $\mathcal{N}_{C,nC}$ and to $\tilde{T}_C = (\partial \mathcal{N}_{C,nC} \times (1 - \epsilon, \infty))$ as the extended boundary region of $\tilde{\mathcal{N}}_{C,nC}$.

The smooth function $f$ extends to this extended Morse neighbourhood by defining

$$f(x, s) = c + s^2 f_0(x)$$

where $c = f(C)$ for all $x \in \partial \mathcal{N}_{C,nC}$ and $s \in (1 - \epsilon, \infty)$. As before $c + f_0(x)$ is the restriction of $f$ to $\partial \mathcal{N}_{C,nC}$.

We can now construct a Riemannian metric $\tilde{g}^C$ on $\tilde{\mathcal{N}}_{C,nC}$ such that

(i) $\tilde{g}^C$ agrees with the metric $g$ upon restriction to $\mathcal{N}_{C,nC} \setminus T_C$;
(ii) on $\partial \mathcal{N}_{C,nC} \times [1 - 3\epsilon/4, \infty)$, we have

$$\tilde{g}^C = s^2 g|_{\partial \mathcal{N}_{C,nC}} \oplus g_R,$$

where $s \in [1 - 3\epsilon/4, \infty)$ and $g_R = ds^2$ is the standard metric on $\mathbb{R}$.

We can also choose a smooth metric $\tilde{g}$ on $M$ which agrees with $\tilde{g}^C$ within each Morse neighbourhood $\mathcal{N}_{C,nC}$, by modifying $g$ close to the Morse neighbourhoods $\mathcal{N}_{C,nC}$.

Remark 2.3. The Morse neighbourhoods $\{\mathcal{N}_{C,nC} : C \in C\}$ continue to satisfy the properties of Morse neighbourhoods with respect to this new metric $\tilde{g}$.

Definition 2.4. Following [20] let

$$\tilde{\Omega}_f^*(\tilde{\mathcal{N}}_C) = \{ \xi \in L^2 \Omega^*(\tilde{\mathcal{N}}_C) : d_f \xi \in L^2 \Omega^*(\tilde{\mathcal{N}}_C) \},$$

where $d_f = e^{-f} d e^f$ and $L^2 \Omega^*(\tilde{\mathcal{N}}_C)$ is the space of square integrable differential forms on $\tilde{\mathcal{N}}_C$ with square-integrability defined using the standard inner product of differential forms induced by the metric $\tilde{g}_C$.

Then the deformed $L^2$ cohomology $H^*(\tilde{\Omega}_f^*(\tilde{\mathcal{N}}_{C,n}), d_f)$ is defined to be the cohomology of the complex

$$0 \to \tilde{\Omega}_f^0(\tilde{\mathcal{N}}_C) \xrightarrow{d_f} \tilde{\Omega}_f^1(\tilde{\mathcal{N}}_C) \xrightarrow{d_f} \ldots \xrightarrow{d_f} \tilde{\Omega}_f^n(\tilde{\mathcal{N}}_C) \to 0.$$
Lemma 2.5. The deformed $L^2$ cohomology $H^*(\tilde{\Omega}^*_f(\tilde{\mathcal{N}}_C), d_f)$ is isomorphic to the relative de Rham cohomology $H^i(\mathcal{N}_{C,nc}, \partial_-\mathcal{N}_{C,nc})$.

Proof. The extended Morse neighbourhood $\tilde{\mathcal{N}}_C$ is a manifold with cylindrical end such that the closed one-form $d_f$ and the metric $\tilde{g}^C$ are both homogeneous of degree 2 at infinity. Hence, by Proposition 5.3 of [20], the deformed $L^2$ cohomology

$$H^*(\tilde{\Omega}^*_f(\tilde{\mathcal{N}}_{C,n}), d_f) \cong H^* (\tilde{\mathcal{N}}_{C,n}, f^{-1}((-\infty, b)))$$

for any $b$ just less than $f(C)$. But by retraction under gradient flow and excision this is in turn isomorphic to $H^i(\mathcal{N}_{C,nc}, \partial_-\mathcal{N}_{C,nc})$. \hfill \Box

Our next step is to construct a one-parameter family of deformed Laplacians $\Delta_t(\tilde{\mathcal{N}}_C)$ on each of the extended Morse neighbourhoods $\tilde{\mathcal{N}}_C$, together with a similar one-parameter family of Laplacians $\Delta_t(M)$ on the entire manifold $M$ and show that eigenstates of $\Delta_t(M)$ whose energy vanish if $t \to \infty$ are in one-to-one correspondence with zero energy states of $\oplus_C \Delta_t(\tilde{\mathcal{N}}_C)$.

This will require us to prove that there do not exist any non-zero energy states of $\oplus_C \Delta_t(\tilde{\mathcal{N}}_C)$ whose energy vanishes in the $t \to \infty$ limit. We therefore define the deformed Laplacians based not only on a deformed exterior derivative $d_t$, but also on a $t$-dependent metric $\tilde{g}^C_t$. By doing so, we will be able to show that the spectrum of $\Delta_t(\tilde{\mathcal{N}}_C)$ is independent of $t$, and hence any eigenstate of $\Delta_t(\tilde{\mathcal{N}}_C)$ that has zero energy in the $t \to \infty$ limit will also have zero energy at any finite $t$. Using basic Hodge theory combined with Lemma 2.5, the space of these zero energy states will be isomorphic to $H^i(\mathcal{N}_{C,nc}, \partial_-\mathcal{N}_{C,nc})$.

Let $\tau_t : \tilde{\mathcal{N}}_{C,nc} \to \tilde{\mathcal{N}}_{C,nc}$ form a one parameter family of diffeomorphisms for $t > 0$ with $\tau_t(p) = p$ for all $p \in \tilde{\mathcal{N}}_{C,nc} \setminus \bar{T}_C$, while

$$\tau_t(x, s) = (x, \xi_t(s)), $$

for all $(x, s) \in \bar{T}_C$, where the smooth monotonically-increasing function $\xi_t : \mathbb{R} \to \mathbb{R}$ satisfies $\xi_t(s) = s$ for $s \leq 1 - \epsilon/2$ and $\xi_t(s) = \sqrt{ts}$ for $s \geq 1 - \epsilon/4$.

For $s > 1 - \epsilon/4$ we have

$$\frac{1}{t} \tau_t^* \tilde{g} = \tilde{g} \quad \text{and} \quad \tau_t^* f = t(f - c) + c$$

where $c = f(C)$.

Definition 2.6. Let $d_t = e^{-\tau_t^* f} d e^{\tau_t^* f}$. The $t$-dependent Riemannian metric $\tilde{g}^C_t = \tau_t^* \tilde{g}^C$ induces a Hodge star operator $\star_t$, and hence a co-differential $d_t^* = (-1)^k e^{\tau_t^* f} \star_t^{-1} d \star_t e^{-\tau_t^* f}$, which is the adjoint of $d_t$ with respect to the inner product on $\Omega^*(\tilde{\mathcal{N}}_C)$ induced by the metric $\tilde{g}^C_t$. We can then define the deformed Laplacian

$$\Delta_t(\tilde{\mathcal{N}}_{C,nc}) = d_t^* d_t + d_t d_t^*.$$

Let $\tilde{\Omega}^*_t = \{ \omega \in L^2(\tilde{\mathcal{N}}_C) : \Delta_t \omega \in L^2(\Omega(\tilde{\mathcal{N}}_C)) \}$. Let $\tilde{\Delta}_t(\tilde{\mathcal{N}}_C)$ be the closure of the restriction of $\Delta_t(\tilde{\mathcal{N}}_{C,nc})$ to $\tilde{\Omega}^*_t$ in the $L^2$ completion $L^2(\Omega^*(\tilde{\mathcal{N}}_C))$ of $L^2(\Omega^*(\tilde{\mathcal{N}}_C))$.
Remark 2.7. Here square integrability with respect to the metric $\hat{g}^C$ is equivalent to square integrability with respect to $\tilde{g}^C$, since these metrics are bounded by constant positive scalar multiples of each other.

Using the diffeomorphism invariance of the exterior derivative, we see that

$$\tau^*_t(\Delta_1(\tilde{\mathcal{N}}_{C,n_c})f) = \Delta_1(\tilde{\mathcal{N}}_{C,n_c})(\tau^*_t f)$$

and so the spectrum of $\Delta_1(\tilde{\mathcal{N}}_{C,n_c})$ is independent of $t$. Since $\Delta_1(\tilde{\mathcal{N}}_{C,n_c})$ is an elliptic operator with discrete spectrum (Proposition 4.5 of [20]), then by standard Hodge theory arguments (Proposition 5.2 of [20]) and Lemma 2.5 we have

$$\text{Ker}(\Delta_1(\tilde{\mathcal{N}}_{C,n_c})) \cap \Omega^p(\tilde{\mathcal{N}}_{C,n_c}) = \text{Ker}(\Delta_1(\tilde{\mathcal{N}}_{C,n_c})) \cap \Omega^p(\tilde{\mathcal{N}}_{C,n_c}) = H^p((\tilde{\mathcal{N}}_{C,n_c}), d_f) = H^p(\mathcal{N}_{C,n_c}, \partial \mathcal{N}_{C,n_c}).$$

We can then define a deformed Laplacian on the manifold $M$ as follows.

Definition 2.8. Let the smooth function $f_t : M \to \mathbb{R}$ satisfy $f_t(p) = tf(p)$ for $p \in M \setminus \bigcup C \mathcal{N}_{C,n_c}$, while $f_t(x) = (t-1)c + \tau_t^* f$ for $x \in \mathcal{N}_{C,n_c}$. Let the metric $\hat{g}_t$ satisfy

$$\hat{g}_t(p) = t\tilde{g}(p)$$

for $p \in M \setminus \bigcup C \mathcal{N}_{C,n_c}$, while $\hat{g}_t = \tau_t^* \hat{g}$ on $\mathcal{N}_{C,n}$. Smoothness at $\partial \mathcal{N}_{C,n_c}$ follows from (2.1). Define $d_i = e^{-f_i}d\epsilon_i$, with the codifferential $d^*_i$ defined, analogously to Definition 2.6, as the adjoint of $d_i$ with respect to the inner product on $\Omega^*(M)$ induced by $\hat{g}_t$. Explicitly

$$d^*_i = (-1)^k e^{f_i} \ast^{-1} d \ast e^{-f_i},$$

where $\ast$ is the Hodge star operator induced by the metric $\hat{g}_t$. We therefore define

$$\Delta_i(M) = d_i d^*_i + d^*_i d_i.$$

As in Definition 2.6, we can also define $\tilde{\Delta}_i(M)$ to be the closure of $\Delta_i(M)$ in the $L^2$ completion $\tilde{\Omega}^*(M)$ of $\Omega^*(M)$.

On both $M$ and $\tilde{\mathcal{N}}_{C,n_c}$ $(\phi, \Delta, \phi) = (d_i \phi, d_i \phi) + (d^*_i \phi, d^*_i \phi) \geq 0$, for all states $\phi$, so both $\Delta_i(M)$ and $\Delta_i(\tilde{\mathcal{N}}_{C,n_c})$ are positive, densely-defined symmetric operators. It is well-known that their closures $\tilde{\Delta}_i(M)$ and $\tilde{\Delta}_i(\tilde{\mathcal{N}}_{C,n_c})$ are self-adjoint [19].

Remark 2.9. By the elliptic regularity theorem, if $\tilde{\Delta}_i(M) \phi = \lambda \phi$ (respectively $\tilde{\Delta}_i(\tilde{\mathcal{N}}_{C,n_c}) \phi = \lambda \phi$) for $\phi \in \Omega^*(M)$ (respectively $\phi \in L^2 \Omega^*(\tilde{\mathcal{N}}_{C,n_c})$), then $\phi \in \Omega^*(M)$ (respectively $\phi \in L^2 \Omega^*(\tilde{\mathcal{N}}_{C,n_c})$).

Definition 2.10. For each $C \in D$, let $J_C : \mathcal{N}_{C,n_c} \to [0, 1]$ be a smooth function such that for all $x \in \mathcal{N}_{C,n_c} \setminus T_C$ and for all $x = \phi(y, s) \in T_C$ with $s \leq 1 - \epsilon/4$ we have $J_C(x) = 1$, but for all $x = \phi(y, s) \in T_C$ with $s \geq 1 - \epsilon/8$ we have $J_C(x) = 0$. We shall also use $J_C$ to denote the functions $J_C : \tilde{\mathcal{N}}_{C,n_c} \to [0, 1]$ and $J_C : M \to [0, 1]$ that agree with $J_C : \mathcal{N}_{C,n_c} \to [0, 1]$ on $\mathcal{N}_{C,n_c}$ and are zero elsewhere. Let $J_0 : M \to [0, 1]$ satisfy

$$J_0 = \sqrt{1 - \sum C J_C^2},$$

while for all $C \in D$ we define $J_C : \tilde{\mathcal{N}}_{C,n_c} \to [0, 1]$ by

$$J_C = \sqrt{1 - J_C^2}.$$
Then \( \{J_C^2, J_C^2\} \) and \( \{J_C^2 \cup J_C^2 : C \in D\} \) form partitions of unity for \( \tilde{N}_{C,n_c} \) and \( M \) respectively.

**Remark 2.11.** For all \( \psi, \phi \in \Omega^*(\tilde{N}_{C,n_c}) \) and all \( C \in D \)

\[
(J_C \psi, \Delta_t(M) J_C \psi) = (J_C \phi, \Delta_t(\tilde{N}_{C,n_c}) J_C \psi),
\]

where on the left (respectively right) hand side \( J_C \psi \) and \( J_C \phi \) are treated as differential forms on \( M \) (respectively \( \tilde{N}_{C,n_c} \)) with support only in \( \tilde{N}_{C,n_c} \).

To complete the proof of Theorem 1.11, we need to show that the operator \( \bigoplus C \Delta_t \tilde{N}_{C,n_c} \) approximates the operator \( \Delta_t(M) \) at large \( t \) in the following sense:

**Lemma 2.12.** Let \( \tilde{\Delta}_{t,p} (\tilde{N}_{C,n_c}) \) and \( \tilde{\Delta}_{t,p} (M) \) be the restriction of \( \Delta_t (\tilde{N}_{C,n_c}) \) and \( \Delta_t (M) \) to \( p \)-forms. Moreover, let

\[
\tilde{\Delta}_{t,p}^D = \bigoplus C \in D \tilde{\Delta}_{t,p} (\tilde{N}_{C,n_c}) .
\]

Then for sufficiently large \( t \)

\[
\dim(\ker(\tilde{\Delta}_{t,p}^D)) = n_p
\]

where \( n_p \) is the number of eigenvalues of \( \tilde{\Delta}_{t,p}^M \) (counting multiplicities) with eigenvalue less than \( 1/\sqrt{t} \).

**Proof.** Let \( \psi_C \in \ker(\tilde{\Delta}_{t,p} (\tilde{N}_{C,n_c})) \) be normalised such that the inner product \( \langle \psi_C, \psi_C \rangle \) is 1. Given a Hamiltonian \( H \) which is the sum of a Laplace-Beltrami operator plus any first order differential operator and a set of functions \( \{J_i\} \) such that \( \sum_i J_i^2 = 1 \), the INS localisation formula (cf. [20] Lemma 8.2, [27, 70] Lemma 3.1) states that

\[
H = \sum_i J_i H J_i + \frac{1}{2} \sum_i [J_i, [J_i, H]] = \sum_i J_i H J_i - \sum_i \|d J_i\|^2 .
\]

From this we see that

\[
(J_C \psi_C, \tilde{\Delta}_t (\tilde{N}_{C,n_c}) J_C \psi_C) + (J_C \psi_C, \tilde{\Delta}_t (\tilde{N}_{C,n_c}) J_C \psi_C) = 2 \langle \psi_C, \|d J_C\|^2 \psi_C \rangle = O(1/t) ,
\]

where we have used the subscript \( t \) to indicate that the norm \( \|d J_C\| \) is defined using the metric \( \tilde{g}_C^t \). The last estimate follows because \( d J_C \) is independent of \( t \) and bounded and \( \tilde{g}_t = t \tilde{g} \) everywhere on the support of \( J_C \). Hence

\[
(J_C \psi_C, \tilde{\Delta}_t (\tilde{N}_{C,n_c}) J_C \psi_C) = O(1/t) .
\]

Since \( df \neq 0 \) and \( \tilde{g}_t = t \tilde{g} \) everywhere in the support of \( \tilde{J}_C \), we have

\[
\tilde{J}_C \tilde{\Delta}_t (\tilde{N}_{C,n_c}) \tilde{J}_C = t \tilde{J}_C \tilde{g}_t^{-1}(df, df) \tilde{J}_C + O(1) \geq \varepsilon t \tilde{J}_C^2 ,
\]

where the last inequality is true for sufficiently large \( t \), given any fixed \( \varepsilon < \inf_{\text{supp}(\tilde{J}_C)} \tilde{g}_t^{-1}(df, df) \).

Hence

\[
\varepsilon t \langle \tilde{J}_C \psi_C, \tilde{J}_C \psi_C \rangle \leq \langle \tilde{J}_C \psi_C, \tilde{\Delta}_t (\tilde{N}_{C,n_c}) \tilde{J}_C \psi_C \rangle = O(1/t)
\]

and

\[
(J_C \psi_C, J_C \psi_C) = 1 - \langle \tilde{J}_C \psi_C, \tilde{J}_C \psi_C \rangle = 1 - O(1/t^2) .
\]
However the support of $J_C$ lies entirely within $\mathcal{N}_C$. Hence

\begin{equation}
(2.7) \quad \frac{(J_C\psi_C, \Delta^M J_C\psi_C)}{(J_C\psi_C, J_C\psi_C)} = \frac{(J_C\psi_C, \tilde{\Delta}_i(\mathcal{N}_{C,n_C}) J_C\psi_C)}{(J_C\psi, J_C\psi_C)} = O(1/t).
\end{equation}

Let $k_{t,p} = \dim(\ker(\tilde{\Delta}^D_{t,p})) = \sum_{C \in D} \dim(\ker(\Delta_{t,p}(\mathcal{N}_{C,n_C}))).$ We showed that $k_{t,p}$ was finite in (2).

Let $\mu_n(\tilde{\Delta}_{t,p}(M))$ for $n \in \mathbb{Z}_{\geq 0}$ be defined by the following minimax formula

$$
\mu_n(\tilde{\Delta}_{t,p}(M)) = \sup_{\xi_1, \xi_2, \ldots, \xi_{n-1}} Q(\xi_1, \xi_2, \ldots, \xi_{n-1}; \tilde{\Delta}_{t,p}(M)),
$$

where $Q(\xi_1, \xi_2, \ldots, \xi_{n-1}; \tilde{\Delta}_{t,p}(M)) = \inf_{\psi} \langle (\psi, \tilde{\Delta}_{t,p}(M)\psi) \mid \psi \in D(\tilde{\Delta}_{t,p}(M)), \|\psi\| = 1, \forall i (\psi, \xi_i) = 0 \rangle$. By the spectral theory of self-adjoint operators,

$$
\mu_n = \min(E_n, \inf \sigma_{ess}(\tilde{\Delta}_{t,p}(M)),
$$

where $E_n$ is the $n$th eigenvalue (counting multiplicities) of $\tilde{\Delta}_{t,p}(M)$ (or $E_n = \infty$ if there are fewer than $n$ eigenvalues) and $\sigma_{ess}(\tilde{\Delta}_{t,p}(M))$ is the essential spectrum of $\tilde{\Delta}_{t,p}(M)$. (In fact, since $M$ is compact, $\tilde{\Delta}_{t,p}(M)$ has discrete spectrum). Since the space

$$
V = \text{span}\{ J_C\psi_C : \psi_C \in \ker(\tilde{\Delta}_{t,p}(\mathcal{N}_{C,n_C})) \ C \in D \}
$$

is $k_{t,p}$-dimensional and satisfies

$$
\sup_{\psi \in V, \|\psi\|=1} (\psi, \tilde{\Delta}_{t,p}(M)\psi) = \max_{C \in D} \sup_{\psi \in \ker(\Delta_{t,p}(\mathcal{N}_{C,n_C}))} \frac{(J_C\psi, \tilde{\Delta}_i(M) J_C\psi)}{(J_C\psi, J_C\psi)} = O(1/t),
$$

it follows that $\mu_{k_{t,p}} = O(1/t)$.

We now show that $\mu_{k_{t,p} + 1} > 1/\sqrt{t}$ for sufficiently large $t$. Let $V_{[0,2/\sqrt{t}]} \subseteq \overline{\Omega^P(M)}$ be the spectral subspace of $[0,2/\sqrt{t}]$ for the self-adjoint positive operator $\Delta_{t,p}(M)$. We assume that $\dim(V_{[0,2/\sqrt{t}]}) \geq k + 1$ and derive a contradiction for sufficiently large $t$.

Let $\phi \in V_{[0,2/\sqrt{t}]}$ have $(\psi, \psi) = 1$. By almost identical arguments to the ones above

$$
\sum_C (J_C\phi, \tilde{\Delta}_i(M) J_C\phi) + (J_0\phi, \tilde{\Delta}_i(M) J_0\phi) - 2 \sum_C (\phi, \|d J_C\|^2) \leq \frac{2}{\sqrt{t}}.
$$

Because $J_0\Delta_i(M) J_0 \geq \varepsilon t$ for sufficiently large $t$ at fixed sufficiently small $\varepsilon$,

$$
\sum_C (J_C\phi, J_C\phi) = 1 - (J_0\phi, J_0\phi) = 1 - O(1/t^{3/2})
$$

and

$$
\frac{(\Theta_C J_C\phi, \tilde{\Delta}^D_{t,p} \Theta_C J_C\phi)}{(\Theta_C J_C\phi, \Theta_C J_C\phi)} = \frac{\sum_C (J_C\phi, \tilde{\Delta}_i(M) J_C\phi)}{\sum_C (J_C\phi, J_C\phi)} = O(1/t^{1/2}).
$$

However the spectrum of $\tilde{\Delta}^D_{t,p}$ is discrete and independent of $t$. Hence, if $t$ is sufficiently large, then $2\sqrt{t}$ will be less than the minimum non-zero eigenvalue of $\tilde{\Delta}^D_{t,p}$. The only possibility would be to have

$$
\dim(\ker(\tilde{\Delta}^D_{t,p})) \geq k_{t,p} + 1,
$$
giving our desired contradiction. It follows that \( \mu_{k_i+1} \geq 2\sqrt{t} \) for sufficiently large \( t \) and \( \bar{\Delta}_{t,p}^D \) must have a discrete spectrum with exactly \( k_{i,p} \) eigenvalues below \( 1/\sqrt{t} \), which completes the proof. \(\square\)

Proof of Theorem 1.11 With Lemma 2.12 in hand, a proof of Theorem 1.11 follows straightforwardly. By standard Hodge-theoretic arguments, there is one zero energy state of \( \bar{\Delta}_{t,p}^M \) for each cohomology class for the deformed exterior derivative \( d_t \). However multiplication by \( e^{-tf} \) gives an isomorphism between this deformed cohomology and the ordinary de Rham cohomology. From this, together with Lemmas 2.5 and 2.12, we immediately obtain weak Morse inequalities

\[
(2.8) \quad P_t(M) = \sum_{C \in D} P_t(\mathcal{N}_{C,\infty}, \partial_+, \mathcal{N}_{C,\infty}) - S(t),
\]

where the coefficients of \( P_t(M) \) count the zero energy states of \( \bar{\Delta}_{t,p}(M) \), while \( P_t(\mathcal{N}_{C,\infty}, \partial_-, \mathcal{N}_{C,\infty}) \) counts the low energy states of \( \bar{\Delta}_{t,p}(M) \), so \( S(t) \) has non-negative coefficients.

The strong Morse inequalities (Theorem 1.11) also follow by standard arguments, which we sketch here. From a physics perspective, the strong inequalities arise because non-zero energy states in a supersymmetric system always come in equal energy pairs, one bosonic and one fermionic \([76]\). More specifically, let \( \phi \in \Omega^*(M) \) satisfy \( \Delta_i(M)\phi = E\phi \) for some \( E > 0 \). We can always write

\[
\phi = \frac{1}{E}(d_i^*d_i\phi + d_i d_i^*\phi).
\]

Since \( [d_i, \Delta_i(M)] = [d_i^*, \Delta_i(M)] = 0 \), \( d_i d_i^*\phi \) and \( d_i^* d_i \phi \) are respectively exact and co-exact eigenstates with the same eigenvalue. Hence we can always choose an eigenbasis for the low energy states of \( \Delta_i(M) \) such that every non-zero energy eigenstate is either exact or co-exact. Given an \( d_i \)-exact \( p \)-form eigenstate \( d\psi \), the \( (p-1) \)-form \( d_i^* d_i\psi \) is a co-exact eigenstate with the same eigenvalue. Similarly given a co-exact \( p \)-form eigenstate \( d_i^*\chi \), the \( (p+1) \)-form \( d_i d_i^*\chi \) is an exact eigenstate with the same eigenvalue.

We therefore obtain isomorphisms between the co-exact non-zero energy eigenspaces of \( p \)-forms and the exact non-zero energy eigenspaces of \( (p+1) \)-forms. It follows that we can rewrite (2.8) with \( S(t) = (1 + t)R(t) \) where \( R(t) = r_p t^p \) has the non-negative coefficients

\[
r_p = \dim \left( \text{Span} \left\{ \phi : \Delta_i(M)\phi = E\phi, \ d_i^*\phi = 0, \ 0 < E \leq 1/\sqrt{t} \right\} \right).
\]

This gives us an alternative route to Theorem 1.11. \(\square\)

3. Morse stratifications and Morse covers

This section generalises the construction of Morse stratifications, and the resulting proof of the Morse inequalities, to the situation where \( f : M \to \mathbb{R} \) is any smooth real-valued function on a compact Riemannian manifold \( M \) whose critical locus \( \text{Crit}(f) \) has finitely many connected components. It also associates to suitable systems of Morse neighbourhoods open covers of \( M \) (see Definition 3.6) and decompositions of \( M \) into submanifolds with corners (see Remark 3.7).

As before let \( D = \bigsqcup_{c \in \text{Critval}(f)} D_c \) be the finite set of connected components of \( \text{Crit}(f) \), where

\[
D_c = \{ C \in D : C \subseteq f^{-1}(c) \}.
\]
Let $\mathcal{N} = \{\mathcal{N}_{C,n} : C \in D, n \geq 0\}$ be a system of strict Morse neighbourhoods for $f$ satisfying $\mathcal{N}_{C,n} \cap \mathcal{N}_{C',n'} = \emptyset$ unless $C = C'$.

**Definition 3.1.** If $C \in D$ and $j \geq 0$ we will say that the downwards gradient flow $\{\psi_t(x) : t \geq 0\}$ (where $\psi_0(x) = x$) for $f$ from $x$ meets $\mathcal{N}_{C,j}$ if there is some $t_0 \geq 0$ such that $\psi_{t_0}(x) \in \mathcal{N}_{C,j}$. Let

$$W^+\mathcal{N}_{C,j} = \{x \in M : \text{the downwards gradient flow for } f \text{ from } x \text{ meets } \mathcal{N}_{C,j}\}$$

and

$$W^-\mathcal{N}_{C,j} = \{x \in M : \text{the upwards gradient flow for } f \text{ from } x \text{ meets } \mathcal{N}_{C,j}\}.$$

Similarly let

$$W^+\mathcal{N}_{C,j}^o = \{x \in M : \text{the downwards gradient flow for } f \text{ from } x \text{ meets } \mathcal{N}_{C,j}^o\}$$

and

$$W^-\mathcal{N}_{C,j}^o = \{x \in M : \text{the upwards gradient flow for } f \text{ from } x \text{ meets } \mathcal{N}_{C,j}^o\}.$$

**Lemma 3.2.** If $C \in D$ and $j \geq 0$ then

(i) $W^+\mathcal{N}_{C,j}$ is a locally closed submanifold of $M$ with corners, having interior $W^+\mathcal{N}_{C,j}^o$ boundary

$$\partial_-\mathcal{N}_{C,j} \cup \partial_+ W^+\mathcal{N}_{C,j},$$

where $\partial_+ W^+\mathcal{N}_{C,j}$ is the union of the upwards trajectories under the gradient flow for $f$ of $\partial_-\mathcal{N}_{C,j} \cap \partial_+\mathcal{N}_{C,j'}$, and corners given, as for $\mathcal{N}_{C,n}^o$, itself, by (the connected components of) the intersection $\partial_-\mathcal{N}_{C,j} \cap \partial_+\mathcal{N}_{C,j} = \partial_-\mathcal{N}_{C,j} \cap \partial_+\mathcal{N}_{C,j}$.

(ii) The downwards Morse flow induces a retraction of $W^+\mathcal{N}_{C,j}$ onto $\mathcal{N}_{C,j}$.

**Remark 3.3.** To extend Definition 3.1 and Lemma 3.2 to systems of Morse neighbourhoods which are not necessarily strict, we need to take $\partial_+ W^+\mathcal{N}_{C,j}$ to be the union of the upwards trajectories under the gradient flow for $f$ of the corners

$$(\partial_-\mathcal{N}_{C,j} \cap \partial_+\mathcal{N}_{C,j}) \cup (\partial_-\mathcal{N}_{C,j} \cap \partial_+\mathcal{N}_{C,j}) \cup (\partial_-\mathcal{N}_{C,j} \cap \partial_+\mathcal{N}_{C,j})$$

of $\mathcal{N}_{C,n}$ (or equivalently the upwards trajectories under the gradient flow of $(\partial_-\mathcal{N}_{C,j} \cap \partial_+\mathcal{N}_{C,j}) \cup \partial_+\mathcal{N}_{C,j}$). Then the corners of $W^+\mathcal{N}_{C,j}$ are (the connected components of) $\partial_-\mathcal{N}_{C,j} \cap \partial_+ W^+\mathcal{N}_{C,j} = (\partial_-\mathcal{N}_{C,j} \cap \partial_+\mathcal{N}_{C,j}) \cup (\partial_-\mathcal{N}_{C,j} \cap \partial_+\mathcal{N}_{C,j})$; the lower corners for $\mathcal{N}_{C,n}$.

Now if $x \in M$ then since $M$ is compact the downwards gradient flow $\{\psi_t(x) : t \geq 0\}$ for $f$ from $x$ has a limit point in $\text{Crit}(f)$, and so there is some $C \in D$ such that $\{\psi_t(x) : t \geq 0\}$ meets every Morse neighbourhood $\mathcal{N}_{C,n}$ of $C$. It follows from the definition of a system of Morse neighbourhoods that if $x \in \mathcal{N}_{C,n}$ then $\{\psi_t(x) : t \geq 0\}$ leaves $\mathcal{N}_{C,n}$ if and only if it meets the open subset $f^{-1}(-\infty,c)$ of $M$, where $c = f(C)$, and this happens if and only if it has no limit point in $C$. Thus for each $x \in M$ there is a unique $C \in D$ such that for every Morse neighbourhood $\mathcal{N}_{C,n}$ of $C$ the downwards gradient flow for $f$ from $x$ enters and never leaves $\mathcal{N}_{C,n}$.

**Definition 3.4.** If $C \in D$ let

$$W^+_C = \{x \in M : \text{for every } n \geq 0 \text{ the downwards gradient flow for } f \text{ from } x \text{ enters and never leaves the Morse neighbourhood } \mathcal{N}_{C,n}\}.$$

Similarly let

$$W^-_C = \{x \in M : \text{for every } n \geq 0 \text{ the upwards gradient flow for } f \text{ from } x \text{ enters and never leaves the Morse neighbourhood } \mathcal{N}_{C,n}\}.$$
Lemma 3.5. (i) $W^+_C = \bigcap_{j \geq 0} W^+ N_{C,j}^0$ and for any $j : D \to \mathbb{N}$ we have

$$M = \bigcup_{C \in D} W^+ N_{C,j(C)}^0.$$  

(ii) $M$ can be expressed as disjoint unions

$$M = \bigcup_{C \in D} W^+_C = \bigcup_{C \in D} W^-_C = \bigcup_{C, C_1 \in D} W^+_C \cap W^-_{C_1}$$

where $W^+_C \cap W^-_{C_1} = C$ and $W^+_C \cap W^-_{C_1}$ is empty unless $C = C'$ or $f(C') > f(C)$, and $W^+_C$ and $W^-_C$ are independent of the choice of system of Morse neighbourhoods.

(iii) For each $C, C' \in D$ the subsets $W^+_C, W^-_C$ and $W^+_C \cap W^-_{C'}$ of $M$ are locally closed with

$$\overline{W^+_C} \subseteq W^+_C \cup \bigcup_{C' \in D, f(C') > f(C)} W^+_{C'} \quad \text{and} \quad \overline{W^-_C} \subseteq W^-_C \cup \bigcup_{C' \in D, f(C') < f(C)} W^-_{C'}. $$

Proof. (i) and (ii) follow directly from the argument above. (iii) also follows since if $\psi_{t_0}(x) \in N^0_{C,n+1} \subseteq N^0_{C,n}$ for some $t_0 \in \mathbb{R}$ then $\psi_{t_0}(y) \in N^0_{C,n} \subseteq N^0_{C,n}$ for all $y$ in a neighbourhood of $x$. □

Definition 3.6. We will call $\{W^+_C : C \in D\}$ and $\{W^-_C : C \in D\}$ Morse stratifications of $M$. We will also call $\{W^+ N^0_{C,j(C)} : C \in D\}$ and $\{W^- N^0_{C,j(C)} : C \in D\}$ Morse covers of $M$.

Now define a strict partial order on $D$ by $C > C'$ if $f(C) > f(C')$, and extend it to a total order $>_{D}$ on $D$. For each $C \in D$ the subset

$$U_C = M \setminus \bigcup_{C' > C} U_{C'}$$

is open in $M$ and contains $W^+_C$ as a closed subset. There is then a long exact sequence of homology

$$(3.1) \quad \cdots \to H_{n+1}(U_C, U_C \setminus W^+_C; \mathbb{F}) \to H_n(U_C \setminus W^+_C; \mathbb{F}) \to H_n(U_C, U_C \setminus W^+_C; \mathbb{F}) \to H_n(U_C, U_C \setminus W^+_C; \mathbb{F}) \to \cdots.$$  

Moreover if $n$ is any natural number then $W^+ N_{C,n}^0$ is a neighbourhood of $W^+_C$ in $U_C$ and is a closed submanifold of $U_C$ with corners. Thus by excision

$$H_s(U_C, U_C \setminus W^+_C; \mathbb{F}) \cong H_s(W^+ N_{C,n}^0, W^+ N_{C,n}^0 \setminus W^+_C; \mathbb{F}).$$

The downwards gradient flow for $f$ induces a retraction from $W^+ N_{C,n}^0$ to $N_{C,n}^0$ and also a retraction from $W^+ N_{C,n}^0 \setminus W^+_C$ to $\partial N_{C,n}^0$. These induce an isomorphism

$$H_s(W^+ N_{C,n}^0, W^+ N_{C,n}^0 \setminus W^+_C; \mathbb{F}) \cong H_s(N_{C,n}^0, \partial N_{C,n}^0; \mathbb{F}).$$

The Morse inequalities (Theorem 1.11) now follow from the long exact sequence (3.1), as in the first proof given in §1.

Remark 3.7. Recall from Remark 1.9 that we can choose a system of cylindrical Morse neighbourhoods $\{N^0_{C,n} : C \in D, n \geq 0\}$ of the critical sets $C \in D$ such that if $c = f(C)$ then

$$N^0_{C,n} \cap f^{-1}(c) = \{x \in f^{-1}(c) \cap U_C : \|\text{grad}(f)(x)\|^2 \leq \varepsilon(C)_n\}$$

where $U_C$ is a fixed neighbourhood of $C$ in $M$ and $\varepsilon$ is a function from $D$ to the set of strictly decreasing sequences $(\varepsilon_1, \varepsilon_2, \ldots)$ of strictly positive real numbers converging to 0. This is possible by Sard’s theorem ([64] Thm II.3.1) applied to $\|\text{grad}(f)\|^2$ on $f^{-1}(c) \setminus \text{Crit}(f)$, which allows us to choose such sequences $\varepsilon(C)$ consisting of regular values of the smooth function.
\[\|\text{grad}(f)\|^2|_{f^{-1}(c) \cap \text{Crit}(f)}.\]

We then define \(\mathcal{N}_{C,n}^c\) to consist of those \(x \in f^{-1}[c - \delta, c + \delta]\) (for \(c = f(C)\) and \(\delta > 0\) sufficiently small) such that the gradient flow for \(f\) from \(x\) meets \(\mathcal{N}_{C,n}^c \cap f^{-1}(c)\). As in Remark 1.8 we can further choose a system \(\{\mathcal{N}_{C,n}^c : C \in D, n \geq 0\}\) of strict Morse neighbourhoods such that
\[\mathcal{N}_{C,n}^c \subseteq \mathcal{N}_{C,n}^c\] and \(\mathcal{N}_{C,n}^c \cap f^{-1}(c) = \mathcal{N}_{C,n}^c \cap f^{-1}(c)\)
for all \(c \in \text{Crit}(f)\), all \(C \in D_c\) and all \(n \geq 0\).

The gradient flow defines a diffeomorphism \(g_{c,c'}\) from \(f^{-1}(c) \setminus \text{Crit}(f)\) to an open subset of \(f^{-1}(c')\) for any \(c'\) sufficiently close to \(c\); if \(c < c'\) (respectively \(c' > c\)) then this open subset is
\[f^{-1}(c') \setminus \bigcup_{C \in D_c} W_C^+\]
(respectively \(f^{-1}(c') \setminus \bigcup_{C \in D_c} W_C^-\)). Conjugating \(\|\text{grad}(f)\|^2\) on \(f^{-1}(c) \setminus \text{Crit}(f)\) with the diffeomorphism \(g_{c,c'}\) defines a smooth function \(\|\text{grad}(f)\|^2_{c,c'}\) on this open subset of \(f^{-1}(c')\) (which extends to a continuous function on \(f^{-1}(c')\) by assigning the value 0 on \(\bigcup_{C \in D_c} W_C^c\)). Then
\[\mathcal{N}_{C,n}^c = \{x \in f^{-1}[c - \delta, c + \delta] : \|\text{grad}(f)\|^2_{c,f(x)} \leq \epsilon(C,n)\}.\]

Let \(\text{Crit}(f) = \{c_1, \ldots, c_P\}\) where \(c_1 < c_2 < \cdots < c_P\), and for any subset \(I\) of \(\{1, \ldots, P\}\) let
\[U_{I,c'} = f^{-1}(c') \setminus \bigcup_{C \in D_c: c' \leq f(C) \leq \min\{c_i : i \in I\}} W_C^+ \cup \bigcup_{C \in D_c: \max\{c_i : i \in I\} \leq f(C) \leq c'} W_C^-\]
Then for any \(c' \in f(M)\) and \(k \in \{1, \ldots, P\}\) there is a smooth function
\[\|\text{grad}(f)\|^2_{c,k,c'} : U_{k,c'} \to \mathbb{R}\]
obtained by conjugating \(\|\text{grad}(f)\|^2|_{f^{-1}(c_k)}\) with the gradient flow from \(U_{k,c'}\) to \(f^{-1}(c_k)\), which is a diffeomorphism onto an open subset of \(f^{-1}(c_k)\).

Similarly for any subset \(I\) of \(\{1, \ldots, P\}\) such that the open subset \(U_{I,c'}\) of \(f^{-1}(c')\) is nonempty, there is a smooth function
\[F_{I,c'} = (\|\text{grad}(f)\|^2_{c,i,c'})_{i \in I} : U_{I,c'} \to \mathbb{R}^{|I|}\]
and by Sard’s theorem the image in \(\mathbb{R}^{|I|}\) under \(F_{I,c'}\) of its critical set \(\text{Crit}(F_{I,c'})\) has Lebesgue measure 0. When \(c'\) and \(c''\) lie in the same connected component of \(\mathbb{R} \setminus \text{Crit}(f)\) then \(F_{I,c'}\) and \(F_{I,c''}\) are conjugate by a diffeomorphism and so \(F_{I,c'}(\text{Crit}(F_{I,c''}))\) equals \(F_{I,c''}(\text{Crit}(F_{I,c'}))\).

Thus up to conjugacy by a diffeomorphism there are only finitely many such functions \(F_{I,c'}\) to consider, and the union of finitely many subsets of Lebesgue measure 0 has Lebesgue measure 0. So we can choose a system of cylindrical Morse neighbourhoods \(\{\mathcal{N}_{C,n}^c : C \in D, n \geq 0\}\) of the critical sets \(C \in D\), and then a system \(\{\mathcal{N}_{C,n}^c : C \in D, n \geq 0\}\) of strict Morse neighbourhoods, as above, such that whenever \(f(C_i) = c_i\) for \(i \in I\) then if \((c_i)_{i \in I}\) lies in the image of \(F_{I,c'}\) it is a regular value of \(F_{I,c'}\). Thus near any \(f^{-1}(c')\) with \(c' \notin \text{Crit}(f)\) the subsets \(W^+\mathcal{N}_{C,n_i}^c\) are submanifolds (of codimension 0) with boundary which are invariant under the gradient flow and whose boundaries intersect transversely. So given any \([D : N]_{ij}\) by Lemma 3.5 we get a decomposition of \(M\) into submanifolds \(\mathcal{W}^+\mathcal{N}_{C,j(C)}^c \setminus \bigcup_{C \in D, f(C) > f(C)} \mathcal{W}^+\mathcal{N}_{C,\tilde{j}(\tilde{C})}^c\) with corners, meeting along submanifolds of their common boundaries, which can be regarded as approximations to the Morse stratifications.
4. DOUBLE COMPLEXES AND SPECTRAL SEQUENCES

In this section, given a smooth function \( f : M \to \mathbb{R} \) on a compact Riemannian manifold \( M \) whose critical locus \( \text{Crit}(f) \) has finitely many connected components, we will define a double complex with a filtration and associated spectral sequence which leads to another proof of Theorem 1.11.

Recall that a (homological) spectral sequence \( E \) of bigraded vector spaces over \( \mathbb{F} \) starting at \( r_0 \in \mathbb{N} \) is given by three sequences:
(i) for all integers \( r \geq r_0 \), a bigraded vector space \( E_r = \bigoplus_{p,q \in \mathbb{Z}} E_{r,p,q} \) called the \( r \)th page of the spectral sequence,
(ii) linear maps \( d_r : E_r \to E_r \) of bidegree \((-r, r-1)\) satisfying \( d_r \circ d_r = 0 \), called boundary maps or differentials, and
(iii) identifications of \( E_{r+1} \) with the homology of \( E_r \) with respect to \( d_r \).

Recall also that a double complex (respectively a first quadrant double complex) \( N \) over \( \mathbb{F} \) is given by a bigraded vector space \( N = \bigoplus_{p,q \in \mathbb{Z}} N_{p,q} \) (respectively \( N = \bigoplus_{p,q \in \mathbb{N}} N_{p,q} \)) over \( \mathbb{F} \) with two differentials \( \partial' : N \to N \) of bidegree \((-1, 0)\) and \( \partial' : N \to N \) of bidegree \((0, -1)\) satisfying \((\partial')^2 = 0 = (\partial')^2\) and \( \partial' \partial + \partial \partial' = 0 \) (cf. [10, 43, 72]). Then \( \partial = \partial' + \partial' \) satisfies \( \partial^2 = 0 \), and the total complex \( \text{Tot}(N) \) of \( N \) is given by
\[
\text{Tot}(N)_k = \bigoplus_{p+q=k} N_{p,q}
\]
with differential \( \partial \). A first quadrant double complex is the special case when \( \partial_j = 0 \) for \( j \geq 2 \) of a first quadrant multicomplex \( N \) over \( \mathbb{F} \) which is given by a bigraded vector space \( N = \bigoplus_{p,q \in \mathbb{N}} N_{p,q} \) with linear maps \( \partial_i : N \to N \) of bidegree \((-i, i-1)\) for \( i \geq 0 \) satisfying
\[
\sum_{i+j=n} \partial_i \partial_j = 0.
\]

Then \( \partial = \sum_{j \geq 0} \partial_j \) satisfies \( \partial^2 = 0 \), and the total complex \( \text{Tot}(N) \) of \( N \) is given by
\[
\text{Tot}(N)_k = \bigoplus_{p+q=k} N_{p,q}
\]
with differential \( \partial \).

Let \( C = \bigoplus_{k \in \mathbb{N}} C_k \) be a complex with differential \( d : C \to C \) of degree \(-1\) and a filtration \( F \) of \( C \) by subcomplexes
\[
0 = F_0 C \subseteq F_1 C \subseteq \cdots \subseteq F_L C = C.
\]

Then each \( C_k \) has a filtration \( 0 = F_0 C_k \subseteq F_1 C_k \subseteq \cdots \subseteq F_L C_k = C_k \) where \( F_\ell C_k = F_\ell C \cap C_k \) for \( 0 \leq \ell \leq L \), and there is also an induced filtration on the homology \( H(C) = \bigoplus_{k \in \mathbb{N}} H_k(C) \) of \( C \), where \( F_\ell H_k(C) \) is the image of the \( k \)th homology of \( F_\ell C \) under the map induced by the inclusion of \( F_\ell C \) in \( C \). Recall (from for example [52] Ch 16, Thm 5.4) that \( F \) determines a spectral sequence \( E_r = \bigoplus_{p,q \in \mathbb{N}} E_{r,p,q} \) with natural isomorphisms
\[
E_{r+1}^{1,p,q} \cong H_{p+q}(F_p C / F_{p-1} C)
\]
which abuts to the homology of \( C \) in the sense that there is some \( r_\infty \geq r_0 \) such that the differentials \( d_r \) are all zero when \( r \geq r_\infty \), giving natural isomorphisms \( E_{r_\infty} \cong E_{r_\infty+1} \cong \cdots \cong E_\infty \).
with

\[ E_{p,q}^\infty \cong F_p(H_{p+q}C)/F_{p-1}(H_{p+q}C). \]

**Definition 4.1.** Let \( \text{Critval}(f) = \{c_1, \ldots, c_p\} \) where \( c_1 < c_2 < \cdots < c_p \). Let \( b_k = (c_k + c_{k+1})/2 \) and

\[ A_k = f^{-1}(-\infty, b_k] \]

for \( 1 \leq k < p \).

**Remark 4.2.** The first proof of the Morse inequalities for \( f \) (Theorem 1.11) used the existence of a canonical isomorphism

\[ H_i(A_k, A_{k-1}; \mathbb{F}) \cong \bigoplus_{C \in D_k} H_i(N_{C,\infty}, \partial_- N_{C,\infty}; \mathbb{F}) \]

induced by the gradient flow of \( f \) and excision. Using the filtration of the chain complex \( C_s(M) \) of singular simplices on \( M \) by the sub-complexes \( C_s(A_k) \), we see that there is a spectral sequence \( E_{p,q}^r \) abutting to \( H_*(M; \mathbb{F}) \) with \( E_{p,q}^1 = H_{p+q}(A_p, A_{p-1}; \mathbb{F}) \) and boundary map taking \( [\xi] \in H_{p+q}(A_p, A_{p-1}; \mathbb{F}) \) where \( \xi \) is a chain in \( A_p \) with boundary in \( A_{p-1} \), to \( [\partial(\xi)] \in H_{p+q-1}(A_{p-1}, A_{p-2}; \mathbb{F}) \). The existence of this spectral sequence implies the Morse inequalities. We will see that there is a different (though related) spectral sequence arising from a double complex which also abuts to \( H_*(M; \mathbb{F}) \) and is easier to describe in terms of the spaces \( H_i(N_{C,\infty}, \partial_- N_{C,\infty}; \mathbb{F}) \).

Given a first quadrant double complex \( N = \bigoplus_{p,q \in \mathbb{N}} N_{p,q} \), as above, its total complex \( \text{Tot}(N) \) has two filtrations \( F^{(1)}_r \) and \( F^{(2)}_r \) defined by

\[ F^{(1)}_r \text{Tot}(N)_k = \bigoplus_{h \leq p} N_{h,k-h} \quad \text{and} \quad F^{(2)}_r \text{Tot}(N)_k = \bigoplus_{h \leq p} N_{k-h,h}. \]

These given us the ‘first and second spectral sequences’ \( E_{p,q}^{(1)} = \bigoplus_{p,q \in \mathbb{N}} E^{(1)}_{p,q} \) and \( E_{p,q}^{(2)} = \bigoplus_{p,q \in \mathbb{N}} E^{(2)}_{p,q} \) associated to the double complex, which both abut to the total complex \( \text{Tot}(N) \). The 0th page of the first spectral sequence is given by

\[ E_{p,q}^{(1)} = \bigoplus_{h \leq p} N_{h,p+q-h} / \bigoplus_{h \leq p} N_{h,p+q-h} = N_{p,q} \]

where the differential is the map on the quotient induced by \( \partial = \partial' + \partial'' \) or equivalently by \( \partial'' \). Thus

\[ E_{p,q}^{(1)} = H^s_q(N_{p,s}) \]

where \( H^s \) denotes homology with respect to the differential \( \partial'' \). Then the map induced by \( \partial'' \) on \( E_{p,q}^{(1)} \) is zero, so the map induced by \( \partial = \partial' + \partial'' \) is the same as that induced by \( \partial' \), and the second page of the first spectral sequence is given by

\[ E_{p,q}^{(2)} = H'_p H^s_q(N) \]

where \( H^s(N) \) is the homology of \( N \) with respect to the differential \( \partial'' \) and \( H'_p H^s_q(N) \) is the homology of \( H^s_q(N) \) with respect to the differential induced by \( \partial' \). The first few pages of the second spectral sequence have a similar description.

We can also consider a situation when the first quadrant double complex \( N = \bigoplus_{p,q} N_{p,q} \) has a filtration \( F \) by double complexes

\[ 0 = F_0 N \subseteq F_1 N \subseteq \cdots \subseteq F_L N = N. \]
Then the total complex $\text{Tot}(N)$ has an induced filtration given on $\text{Tot}(N)_k = \bigoplus_{p+q=k} N_{p,q}$ by

$$F_{\ell} \text{Tot}(N)_k = \text{Tot}(F_{\ell}N)_k = \bigoplus_{p+q=k} F_{\ell}N_{p,q}.$$  

This filtration determines a spectral sequence $E_1 = \bigoplus_{\ell,k} E_{\ell,k}$ abutting to $\text{Tot}(N)$ and satisfying

$$E_{1,\ell} = H_{\ell+k}(\bigoplus_{p+q=k} F_{\ell}N_{p,q} / F_{\ell-1}N_{p,q})$$

where the homology is taken with respect to the differential induced by $\partial = \partial' + \partial^s$ on the quotient $F_{\ell} \text{Tot}(N)/F_{\ell-1} \text{Tot}(N)$.

We will apply this construction to the Mayer–Vietoris double complex associated to a Morse cover $U^* = \{U_C : C \in D \}$ of $M$ as defined at Definition 3.6, or a Morse decomposition as described at Remark 3.7. Let $C_{q}(M)$ be the chain complex of singular simplices on $M$. The Mayer–Vietoris double complex of a cover $U^* = \{U_C : C \in D \}$ of $M$ abuts to the homology of $M$ and is given by

$$N_{p,q} = \bigoplus_{\sigma \in N_{p,q}(U^*)} C_q(\bigcap_{C \in \sigma} U_C)$$

where $N_{p}(U^*) = \{ \sigma \subseteq D : |\sigma| = p+1 \text{ and } \bigcap_{C \in \sigma} U_C \neq \emptyset \}$ defines the nerve of the cover $U^*$ (see [21] VII §4, [71]). If $\sigma = \{C_0, \ldots, C_p\}$ where $C_0 < \cdots < C_p$ for some fixed total order on $D$, the inclusions of $\bigcap_{C \in \sigma} U_C$ in $\bigcap_{C \in \sigma \setminus \{C_i\}} U_C$ induce chain maps $\partial_i$ from $N_{p,q}$ to $N_{p-1,q}$ and we define the differential $\partial'$ on $N$ by $\sum_{i=0}^{p-1} (-1)^i \partial_i$. The differential $\partial^s$ is given by the usual boundary map $N_{p,q} \to N_{p,q-1}$.

Note that if we choose the Morse cover $U^* = \{U_C : C \in D \}$ as at Remark 3.7 then the nerve $N_{p}(U^*)$ is independent of this choice; it is a combinatorial invariant of the smooth function $f$ on the compact Riemannian manifold $M$. The spaces $C_q(\bigcap_{C \in \sigma} U_C)$ are dependent on the choice of Morse cover, but this is not true of the spectral sequence associated to the filtration of the double complex $N$ given by

$$F_{\ell}N_{p,q} = \bigoplus_{\sigma \in N_{p,q}(U^*)} C_q(f^{-1}(-\infty, b_\ell] \cap \bigcap_{C \in \sigma} U_C).$$

This independence of the Morse cover follows from

Lemma 4.3. The relative homology $H_{k}(f^{-1}(-\infty, b_\ell] \cap \bigcap_{C \in \sigma} U_C, f^{-1}(-\infty, b_{\ell-1}] \cap \bigcap_{C \in \sigma} U_C; \mathbb{F})$ is given by $H_{k}(N_{C_{\infty}, \partial_{-}N_{C_{\infty}}}; \mathbb{F})$ if $\sigma = \{C\}$ where $C \in D_{\ell'}$, and is 0 otherwise.

Corollary 4.4. The $E_1$ page of the spectral sequence associated to the filtration of the Mayer–Vietoris double complex $N$ for the Morse cover $U^* = \{U_C : C \in D \}$ given by $F_{\ell}N_{p,q} = \bigoplus_{\sigma \in N_{p,q}(U^*)} C_q(f^{-1}(-\infty, b_\ell] \cap \bigcap_{C \in \sigma} U_C)$ is given by

$$E_{1,k,\ell} = \bigoplus_{C \in D_{\ell'}} H_{k+\ell}(N_{C_{\infty}, \partial_{-}N_{C_{\infty}}}; \mathbb{F}).$$

As before, let $C_{q}(M)$ be the chain complex of singular simplices on $M$ with coefficients in $\mathbb{F}$, and let $U^* = \{U_C : C \in D \}$ be a Morse cover as in Definition 3.6, where $U_C = \mathcal{W}^+_C N_{C,[C]}$ if $C \in D$
for some \( j : D \to \mathbb{N} \) such that \( j(C) \gg j(\bar{C}) \) when \( f(\bar{C}) > f(C) \). Let
\[
C^U_s(M) = \left\{ \sum_i n_i \sigma_i \in C_s(M) : \sigma_i \text{ has image contained in some element of } U^* \right\}.
\]

By [42] Prop 2.21, the inclusion \( C^U_s(M) \to C_s(M) \) is a chain homotopy equivalence. Moreover if \( A \subseteq M \) then there is an induced chain homotopy equivalence
\[
\frac{C^U_s(M)}{C^U_s(A)} \xrightarrow{\sim} \frac{C_s(M)}{C_s(A)}
\]
giving an isomorphism on relative homology \( H^U_j(M, A; \mathbb{F}) \cong H_j(M, A; \mathbb{F}) \). Similarly we obtain isomorphisms on relative homology \( H^U_j(\mathcal{N}_{C,j}, \partial_{\mathcal{N}_{C,j}}; \mathbb{F}) \cong H_j(\mathcal{N}_{C,j}, \partial_{\mathcal{N}_{C,j}}; \mathbb{F}) \) for each \( C \in D \) and \( j \), giving
\[
H_j(\mathcal{N}_{C,j}, \partial_{\mathcal{N}_{C,j}}; \mathbb{F}) \cong \left\{ \xi \in C^U_j(\mathcal{N}_{C,j}) : \partial \xi \in C^U_{j-1}(\partial_{\mathcal{N}_{C,j}}) \right\} \cap \left\{ \alpha + \beta : \alpha \in C^U_{j+1}(\mathcal{N}_{C,j}), \beta \in C^U_j(\mathcal{N}_{C,j}) \right\}.
\]

We would like to define \( \partial_{i\ell}^k : \bigoplus_{C \in D_{i\ell}} H_j(\mathcal{N}_{C,j}, \partial_{\mathcal{N}_{C,j}}; \mathbb{F}) \to \bigoplus_{C \in D_{i\ell}} H_{j-1}(\mathcal{N}_{C,j}, \partial_{\mathcal{N}_{C,j}}; \mathbb{F}) \) as \( 0 \) if \( \ell \geq k \) and for \( \ell \leq k \) as follows. Suppose \( \xi \in C^U_j(\mathcal{N}_{C,j}) \) with \( \partial \xi \in C^U_{j-1}(\partial_{\mathcal{N}_{C,j}}) \). Then we can write
\[
\partial \xi = \sum_{\ell=1}^{k-1} \sum_{C \in D_{i\ell}} \sigma_C
\]
where the image of \( \sigma_C \) for \( C \in D_{i\ell} \) lies in the intersection of \( \partial_{\mathcal{N}_{C,j}} \) with \( U_C \). We would like to set
\[
\partial_{i\ell}^k(\xi) = \sum_{C \in D_{i\ell}} [\sigma_C],
\]
where \( [\sigma_C] \in H_{j-1}(\mathcal{N}_{C,j}, \partial_{\mathcal{N}_{C,j}}; \mathbb{F}) \) is represented by the element of \( H_{j-1}(\mathcal{N}_{C,j}, \partial_{\mathcal{N}_{C,j}}; \mathbb{F}) \) given by flowing \( \sigma_C \) from \( \partial_{\mathcal{N}_{C,j}} \cap U_C \) until it meets \( \mathcal{N}_{C,j} \). These maps \( \partial_{i\ell}^k \) are not necessarily well defined from \( \bigoplus_{C \in D_{i\ell}} H_j(\mathcal{N}_{C,j}, \partial_{\mathcal{N}_{C,j}}; \mathbb{F}) \) to \( \bigoplus_{C \in D_{i\ell}} H_{j-1}(\mathcal{N}_{C,j}, \partial_{\mathcal{N}_{C,j}}; \mathbb{F}) \) as they stand. However for \( k - \ell = r \) they do define the \( E_{r+q} \)-page of a spectral sequence with
\[
E_{p,q}^1 = \bigoplus_{C \in D_{i\ell}} H_{p+q}(\mathcal{N}_{C,j}, \partial_{\mathcal{N}_{C,j}}; \mathbb{F}).
\]

This is the spectral sequence associated to the Mayer–Vietoris double complex of the cover \( U^* = \{ U_C : C \in D \} \) filtered as above.

**Remark 4.5.** As before let \( \psi_t(x) \) describe the downwards gradient flow for \( f \) from \( x \) at time \( t \). If \( j(C) \gg j(\bar{C}) \) when \( f(\bar{C}) > f(C) \) then \( \partial_{\mathcal{N}_{C,j}} \) only meets \( U_C \) when there are sequences \( C_0 = C, C_1, \ldots, C_m = \bar{C} \) in \( D \) and \( \ell_0 = \ell < \ell_1 < \cdots < \ell_m = k \) with \( f(C_p) = \{ c_{\ell_p} \} \) for \( 0 \leq p \leq m \), such that if \( 0 < p \leq m \) there is a connected component of
\[
\{ x \in M : \psi_t(x) : t \leq 0 \} \text{ has a limit point in } C_p \text{ and } \{ \psi_t(x) : t \geq 0 \} \text{ has a limit point in } C_{p-1}\}
\]
which is closed in \( f^{-1}(f(C_{p-1}), f(C_p)) \) (cf. Remark 3.7); note that if \( \ell_{p-1} = \ell - 1 \) then this subset is closed in \( f^{-1}(f(C_{p-1}), f(C_p)) \). When this subset is closed in \( f^{-1}(f(C_{p-1}), f(C_p)) \) then the maps \( \partial_{i\ell}^{p-1} \) described above are well defined from \( H_j(\mathcal{N}_{C_{p-1},j}, \partial_{\mathcal{N}_{C_{p-1},j}}; \mathbb{F}) \) to \( H_{j-1}(\mathcal{N}_{C_{p-1},j}, \partial_{\mathcal{N}_{C_{p-1},j}}; \mathbb{F}) \).
Note also that such a connected component must be contained in a connected component of the Morse stratum $W^r_{C_p}$ for $C_p$ (as defined at Definition 3.4) and a connected component of the Morse stratum $W^r_{C_{p+1}}$ for $C_{p+1}$. Thus we can decompose the maps $\partial^k_{\ell}$ according to these connected components (cf. Definition 6.6 below).

Remark 4.6. When $M$ is oriented the components $H_i(N_{C,\infty}, \partial_- N_{C,\infty}; \mathbb{F}) \to H_{i-1}(N_{C,\infty}, \partial_- N_{C,\infty}; \mathbb{F})$ of the maps $\partial^k_{\ell}$: $\bigoplus_{\ell \in D_{\ell}^k} H_i(N_{C,\infty}, \partial_- N_{C,\infty}; \mathbb{F}) \to \bigoplus_{\ell \in D_{\ell}^k} H_{i-1}(N_{C,\infty}, \partial_- N_{C,\infty}; \mathbb{F})$ (when well defined) can be described using the isomorphism

$$(H_i(N_{C,\infty}, \partial_- N_{C,\infty}; \mathbb{F}))^* \cong H_{\dim M-i}(N_{C,\infty}, \partial_+ N_{C,\infty}; \mathbb{F})$$

given by the intersection pairing between chains with boundaries in $\partial_- N_{C,C}$ meeting transversally (and therefore not meeting on $\partial N_{C,C}$). From this viewpoint the component mapping $H_i(N_{C,\infty}, \partial_- N_{C,\infty}; \mathbb{F})$ to $H_{i-1}(N_{C,\infty}, \partial_- N_{C,\infty}; \mathbb{F})$ is given by the bilinear pairing

$$H_i(N_{C,\infty}, \partial_- N_{C,\infty}; \mathbb{F}) \otimes H_{\dim M-i+1}(N_{C,\infty}, \partial_+ N_{C,\infty}; \mathbb{F}) \to \mathbb{F}$$

which takes a pair $(\xi, \eta)$ such that $\xi \in C_i(N_{C,j}C)$ with $\partial \xi \in C_{i-1}(\partial_- N_{C,j}C)$ and $\eta \in C_{\dim M-i+1}(N_{C,j\ell})$ with $\partial \eta \in C_{\dim M-i+1}(N_{C,j\ell})$, transports the boundary $\partial \xi$ of $\xi$ under the gradient flow until it meets $N_{C,j\ell+1}$ and takes the intersection pairing of this with $\eta \in C_{\dim M-i+1}(N_{C,j\ell})$. Equivalently this is given by an intersection pairing in a submanifold with corners of a moduli space of unparametrised flows. When

$$\{x \in M : \{\psi_t(x) : t \leq 0\} \text{ has a limit point in } \tilde{C} \text{ and } \{\psi_t(x) : t \geq 0\} \text{ has a limit point in } C\}$$

is closed in $f^{-1}(f(C), f(\tilde{C}))$ this intersection pairing is well defined (see Remark 3.7), but in general it is only well defined on a suitable page of the spectral sequence. In the Morse–Smale situation, when the vector spaces concerned are nonzero then this subset is always closed in $f^{-1}(f(C), f(\tilde{C}))$, and the intersection pairing counts flow lines in the usual way between critical points whose indices differ by one.

5. MULTICOMPLEXES SUPPORTED ON ACYCLIC QUIVERS

Let $\Gamma = (Q_0, Q_1, h, t)$ be an acyclic quiver (or equivalently a directed graph without oriented cycles). Here $Q_0$ and $Q_1$ are finite sets (of vertices and arrows respectively) and $h : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ are maps (determining the head and tail of any arrow).

Definition 5.1. The vertex span $R = F^{Q_0}$ and arrow span $A = F^{Q_1}$ of $\Gamma$ are the vector spaces of $F$-valued functions on $Q_0$ and $Q_1$, with bases identified with $Q_0$ and $Q_1$ via Kronecker delta functions. $R$ is a commutative algebra over $F$ under pointwise multiplication of functions, with the basis elements in $Q_0$ as idempotents, while $A$ is an $R$-bimodule via

$$(e \cdot f)(a) = e(pha) \quad \text{and} \quad (f \cdot e)(a) = f(a)e(\epsilon a)$$

for all $e \in R$, $f \in A$ and $a \in Q_1$. Any $R$-bimodule $M$ can be decomposed into sub-$R$-bimodules

$$M = \bigoplus_{e, \epsilon \in Q_0} M_{e, \epsilon} \quad \text{where} \quad M_{e, \epsilon} = eM\epsilon.$$ 

The path algebra of $\Gamma$ is the graded algebra

$$A = \bigoplus_{d \geq 0} A^d$$
where \( A^d = A \otimes_R A \otimes_R \cdots \otimes_R A \) and \( A^0 = R \). Here \( A^d \) has a basis given by the paths

\[
\{ a_1 \ldots a_d \mid a_1, \ldots, a_d \in \mathcal{Q}_1 \text{ and } t(a_k) = h(a_{k+1}) \text{ for } 1 \leq k < d \}
\]
of length \( d \) in \( \Gamma \), with multiplication given by concatenation where defined and 0 otherwise. Our assumption that \( \Gamma \) is acyclic implies that \( A^d = 0 \) when \( d \) is large enough, and therefore \( \dim_F A < \infty \).

Recall (from for example [29]) that a representation \( \rho \) of \( \Gamma \) over \( F \) is given by vector spaces \( V_w \) for \( w \in \mathcal{Q}_0 \) and linear maps \( \rho(a) : V_{h(a)} \to V_{t(a)} \) for \( a \in \mathcal{Q}_1 \). Any such representation \( \rho \) induces a representation \( V = \bigoplus_{w \in \mathcal{Q}_0} V_w \) of the path algebra \( A \) with a path \( a_1, \ldots, a_d \in A^d \) acting as the composition of the linear maps \( \rho(a_1), \ldots, \rho(a_d) \). Conversely any representation of the algebra \( A \) comes from a representation of \( \Gamma \). Let

\[
e_{\Gamma} = \sum_{e \in \mathcal{Q}_0} e \in R
\]

so that \( e_{\Gamma} b = b = b e_{\Gamma} \) for all \( b \in A \). If \( V = \bigoplus_{e \in \mathcal{Q}_0} V_e \) is a representation of \( \Gamma \) then a linear subspace \( W \) of \( V \) is a subrepresentation of \( \Gamma \) if and only if \( e_{\Gamma} W = W \).

**Remark 5.2.** If we also assume that \( \Gamma \) has no multiple arrows between the same two vertices, then we can make \( A \) into a differential graded algebra where the differential \( \delta a \) of an arrow from \( v \) to \( w \) is the sum of all the paths of length 2 from \( v \) to \( w \), and \( \delta \) is extended to paths via the Leibniz rule (cf. [40]). Let

\[
D_{\Gamma} = \sum_{a \in \mathcal{Q}_1} a \in A.
\]

Then \( \delta D_{\Gamma} = (D_{\Gamma})^2 \) is given by the sum of all paths of length 2 in \( \Gamma \); indeed \( (D_{\Gamma})^r \) is given by the sum of all paths of length \( r \) in \( \Gamma \) for any \( r \geq 1 \) and \( \delta((D_{\Gamma})^r) = 0 \) if \( r \) is even and \( (D_{\Gamma})^{r+1} \) if \( r \) is odd.

If we let \( \Gamma^r \) be the quiver with set of vertices \( \mathcal{Q}_0 \) and arrows given by paths of length \( r \) in \( \Gamma \), then its path algebra is \( \bigoplus_{d \geq r} A^d \) and \( D_{\Gamma^r} = (D_{\Gamma})^r \).

**Definition 5.3.** We will say that a subset \( \mathcal{Q}'_0 \) of \( \mathcal{Q}_0 \) defines a final subquiver (respectively an initial subquiver)

\[
\Gamma' = (\mathcal{Q}'_0, \mathcal{Q}'_1, h|_{\mathcal{Q}'_1}, t|_{\mathcal{Q}'_1}) \text{ where } Q'_1 = h^{-1}(Q'_0) \cap t^{-1}(Q'_0)
\]
of \( \Gamma \) if it satisfies the condition that \( v \in \mathcal{Q}'_0 \) whenever there exists \( w \in \mathcal{Q}'_0 \) and \( a \in \mathcal{Q}_1 \) with \( t(a) = w \) and \( h(a) = v \) (respectively \( t(a) = v \) and \( h(a) = w \)).

Note that if \( \Gamma_1 \) and \( \Gamma_2 \) are final subquivers of \( \Gamma \) then so are \( \Gamma_1 \cap \Gamma_2 \) and \( \Gamma_1 \cup \Gamma_2 \); the same is true for initial subquivers.

**Example 5.4.** If \( j \geq 0 \) then the subset of \( \mathcal{Q}_0 \) consisting of those \( v \in \mathcal{Q}_0 \) such that every path from \( v \) (respectively to \( v \)) in \( \Gamma \) has length at most \( j \) defines a final subquiver (respectively an initial subquiver) of \( \Gamma \).

**Example 5.5.** If \( d \in \mathbb{R} \) and \( f \) is any real-valued function on \( \mathcal{Q}_0 \) such that \( f(v) > f(w) \) whenever there is an arrow in \( \Gamma \) from \( v \) to \( w \), then the subset of \( \mathcal{Q}_0 \) consisting of those \( v \in \mathcal{Q}_0 \) such that \( f(v) < d \) (respectively \( f(v) > d \)) defines a final subquiver (respectively an initial subquiver) of \( \Gamma \).
Example 5.6. If \( v_0 \in Q_0 \) then the subset \( Q_0^{[v_0]} \) of \( Q_0 \) consisting of those \( v \in Q_0 \) such that there is a path from \( v_0 \) to \( v \) in \( \Gamma \) defines a final subquiver \( \Gamma^{[v_0]} \) of \( \Gamma \), and \( Q_0^{[v_0]} \cup \{ v_0 \} \) defines a final subquiver \( \Gamma^{[v_0]}_0 \). More generally for any \( r \in \mathbb{N} \) the subset \( Q_0^{[v_0>(r)]} = Q_0^{[v_0>(r-1)]} \) of \( Q_0 \) consisting of those \( v \in Q_0 \) such that there is a path from \( v_0 \) to \( v \) of length at least \( r \) in \( \Gamma \) defines a final subquiver \( \Gamma^{[v_0>(r)]} \) of \( \Gamma \). Similarly the subset \( Q_0^{[v_0,\leq(r)]} \) of \( Q_0 \) consisting of those \( v \in Q_0 \) such that there is a path from \( v \) to \( v_0 \) of length at most \( r \) in \( \Gamma \) defines an initial subquiver \( \Gamma^{[\leq(r)]} \) of \( \Gamma \).

Definition 5.7. Let \( V \) be a vector space over \( \mathbb{F} \). A (final) filtration \( F \) of \( V \) over the quiver \( \Gamma \) is given by a subspace \( V_0^F \) of \( V \) for every final subquiver \( \Gamma' \) of \( \Gamma \) such that
\[
V_0^F = \{0\} \quad \text{and} \quad V_{\Gamma}^F = V.
\]
\[
V_{\Gamma_1 \cap \Gamma_2}^F = V_{\Gamma_1}^F \cap V_{\Gamma_2}^F \quad \text{and} \quad V_{\Gamma_1 \cup \Gamma_2}^F = V_{\Gamma_1}^F + V_{\Gamma_2}^F
\]
while
\[
\Gamma_1 \subseteq \Gamma_2 \quad \text{implies} \quad V_{\Gamma_1}^F \subseteq V_{\Gamma_2}^F
\]
for all final subquivers \( \Gamma_1 \) and \( \Gamma_2 \) of \( \Gamma \). The associated \( \Gamma \)-graded vector space defined by the filtration is
\[
\text{gr}_F(V) = \bigoplus_{v \in Q_0} V_{\Gamma[\leq]}^F / V_{\Gamma[>]}^F.
\]

Remark 5.8. Initial filtrations can also be defined but we will not use these, so ‘filtration’ will mean a final filtration.

Remark 5.9. A representation \( V = \bigoplus_{w \in Q_0} V_w \) of \( \Gamma \) (or equivalently of \( A \)) has a natural filtration over \( \Gamma \), and the associated \( \Gamma \)-graded vector space can be thought of as the representation of \( \Gamma \) given by \( V \) but with all arrows represented by zero maps.

Definition 5.10. A chain complex \( C = \bigoplus_{n \in \mathbb{Z}} C_n \) with differential \( d : C \to C \) with \( d(C_n) \subseteq C_{n-1} \) is filtered over \( \Gamma \) if it is equipped with a filtration \( F \) of \( C \) such that, for every final subquiver \( \Gamma' \) of \( \Gamma \), the differential \( d \) maps the subspace \( C_{\Gamma'}^n \) of \( C \) into itself.

Definition 5.11. A multicomplex \( N \) supported on the quiver \( \Gamma \) is given by a representation
\[
N = \bigoplus_{w \in Q_0} \bigoplus_{n \in \mathbb{N}} N_{w,n}
\]
of \( \Gamma \) graded by \( \mathbb{N} \) with a differential \( d = d^{[0]} + d^{[1]} + d^{[2]} + \cdots \) on \( N \) such that
(i) if \( x \in N_{w,n} \) and \( a_1 \ldots a_d \in A^d \) is a path in \( \Gamma \) of length \( d \) then \( a_1 \ldots a_d x \in N_{w(a_1),n-d} \);
(ii) \( d^{[i]} \) maps \( N_{w,n} \) to the sum of the subspaces \( N_{w',n-1} \) such that there is a path of length \( i \) in \( \Gamma \) from \( w \) to \( w' \), and
(iii) \( e_i d^{[i]} = d^{[i]} e_i \)
for each \( n, i \geq 0 \) and \( w \in Q_0 \) (where \( N_{w,n} = 0 \) if \( n < 0 \)).
Its total complex is the complex with underlying vector space \( N \) and differential \( d \). The multicomplex will be called homogeneous of degree \( k \) if \( d^{[i]} = 0 \) unless \( i = k \).

When \( N \) is a multicomplex then the differential \( d \) preserves the natural filtration over \( \Gamma \) (see Remark 5.9 above), and since \( \Gamma \) is acyclic the equation
\[
0 = d^2 = (d^{[0]})^2 + (d^{[0]}d^{[1]} + d^{[1]}d^{[0]}) + (d^{[0]}d^{[2]} + (d^{[1]}d^{[2]} + d^{[2]}d^{[0]}) \cdots
\]
combined with (ii) implies that \( d^{[0]} \) is a differential, and that \( d^{[1]} \) restricted to the kernel of \( d^{[0]} \) is a differential, and that \( d^{[2]} \) restricted to the intersection of the kernels of \( d^{[0]} \) and \( d^{[1]} \) is a differential, and so on. Furthermore the kernel and image of \( d \) are invariant under \( e_{r} \) and hence they and the homology of the total complex inherit the structure of a representation of \( \Gamma \).

**Remark 5.12.** A multicomplex supported on \( \Gamma \) which is homogeneous of degree 0 is just a representation of \( \Gamma \) by complexes. A multicomplex supported on the trivial quiver \( A_{1} \) with one vertex and no arrows is always homogeneous of degree 0 and is just a complex. A multicomplex supported on the quiver \( \Gamma = A_{n} \) (given by \( \cdot \to \cdot \to \cdots \to \cdot \) with \( n \) vertices and \( n - 1 \) arrows), for some \( n \in \mathbb{N} \), can be relabelled so that it becomes a multicomplex in the usual sense. When a smooth function \( f : M \to \mathbb{R} \) on a Riemannian manifold \( M \) is Morse–Smale then we will see that its Morse–Witten complex can be regarded as a multicomplex over the quiver \( \Gamma \) defined at Definition 6.1 below which is homogeneous of degree 1; its differential is given by the sum \( D_{\Gamma} \) of all the arrows in \( \Gamma \) (cf. Remark 5.2 above). We can regard this quiver as graded by the function \( f \), in the sense that \( f \) takes a strictly greater value at the source of an arrow in \( \Gamma \) than at its target.

**Definition 5.13.** A spectral sequence \( E \) of multicomplexes over \( \mathbb{F} \) supported on \( \Gamma \) starting at \( r_{0} \in \mathbb{N} \) is given by the following data:

(i) for all integers \( r \geq r_{0} \), a multicomplex \( E_{r} \) supported on \( \Gamma^{r} \) with differential \( d_{r} \) given by \( D_{\Gamma^{r}} = (D_{\Gamma})^{r} \), called the \( r \)th page of the spectral sequence, and

(ii) identifications of \( E_{r+1} \) with the homology of \( E_{r} \) with respect to \( d_{r} \).

Here \( \Gamma^{r} \) and \( D_{\Gamma^{r}} = (D_{\Gamma})^{r} \) are defined as in Remark 5.2.

Since \( d_{r} = 0 \) when \( r \) is greater than the length of any path in \( \Gamma \), the \( r \)th page of such a spectral sequence is independent of \( r \) when \( r \gg 1 \).

**Definition 5.14.** A spectral sequence of multicomplexes supported on \( \Gamma \) is said to abut to a vector space \( W \) with a filtration over \( \Gamma \) if the \( E_{r} \) page for \( r \gg 1 \) is isomorphic to \( W \) as a vector space with filtration over \( \Gamma \).

Suppose that a chain complex \( C = \bigoplus_{n \in \mathbb{Z}} C_{n} \) with differential \( d : C \to C \) with \( d(C_{n}) \subseteq C_{n-1} \) is filtered over \( \Gamma \) by a filtration \( F \) which is preserved by \( d \). Then there is an induced differential \( d_{0} \) which is homogeneous of degree 0 on the associated \( \Gamma \)-graded vector space

\[
\text{gr}_{\Gamma,F}(C) = \bigoplus_{v \in Q_{1}} C^{F}_{\Gamma \uparrow \Gamma} \big/ C^{F}_{\Gamma \downarrow \Gamma},
\]

Just as in the classical case, this gives us the \( E_{0} \) page of a spectral sequence of multicomplexes over \( \Gamma \) which abuts to the homology of \( C \) with its induced filtration over \( \Gamma \). The \( E_{1} \) page of this spectral sequence is given by the homology of \( d_{0} \) with its induced \( \Gamma \)-grading and differential \( d_{1} \) induced by \( d_{0} \), where the representation of the arrows in \( \Gamma \) is determined by the requirement that \( d_{1} = D_{\Gamma} \). The \( E_{r} \) page is given for vertices \( w \in Q_{0} \) and integers \( n \in \mathbb{Z} \) by

\[
E_{w,n}^{r} = Z_{w,n}^{r} / (B_{w,n}^{r} + Z_{w,n-1}^{r-1}),
\]

where \( Z_{w,n}^{r} \) is the kernel of the linear map

\[
C_{n} \cap C_{\Gamma \uparrow w}^{F} \to C_{n-1} \cap C_{\Gamma \downarrow w}^{F} / C_{n-1} \cap C_{\Gamma \uparrow w}^{F}.
\]
induced by $d$, and $B_{w,n}^r$ is the intersection of $C_n \cap C_{n+1}^r$ with the image of $d$ restricted to $C_{n+1} \cap C_{n+1}^r$. The differential $d_r$ is induced by $d$, and the representation of the arrows in $\Gamma^*$ (that is, of the paths of length $r$ in $\Gamma$) is determined by the requirement that $d_r$ is given by $D_{r^*} = (D_r)^*$.

Remark 5.15. Suppose that we have another acyclic quiver $\tilde{\Gamma} = (\tilde{Q}_0, \tilde{Q}_1, \tilde{h}, \tilde{t})$ (again with no multiple arrows) which is a perturbation $\eta : \Gamma \rightarrow \tilde{\Gamma}$ of the quiver $\Gamma$, in the sense that each vertex of $\Gamma$ has split into finitely many vertices of $\tilde{\Gamma}$, and each path in $\Gamma$ has split into finitely many paths in $\tilde{\Gamma}$ in a compatible way. More precisely we require firstly a surjection $\eta_0 : \tilde{Q}_0 \rightarrow Q_0$, which allows us to identify the vertex span $R$ for $\Gamma$ with the subalgebra of the vertex span $\tilde{R}$ for $\tilde{\Gamma}$ spanned by the idempotents

$$\sum_{w \in \eta_0^{-1}(v)} w$$

for $v \in Q_0$.

and then $e_{\Gamma} = e_{\tilde{\Gamma}}$ and the path algebra $\tilde{A}$ for $\tilde{\Gamma}$ becomes an $R$-bimodule. Secondly there should be a surjection

$$\eta_1 : \{\text{paths in } \tilde{\Gamma}\} \rightarrow \{\text{paths in } \Gamma\}$$

such that $\eta_0(\tilde{h}(a_j)) = h(\eta_1(a_1 \ldots a_j))$ and $\eta_0(\tilde{t}(a_j)) = t(\eta_1(a_1 \ldots a_j))$ for every path $a_1 \ldots a_j$ in $\tilde{\Gamma}$, inducing a surjection $\tilde{A} \rightarrow A$ which respects the multiplication and $R$-bimodule structure but not necessarily the grading given by path-length.

In this situation every final subquiver $\Gamma' \subseteq \Gamma$ determined by $Q'_0 \subseteq Q_0$ defines a final subquiver $\eta^* \Gamma' \subseteq \Gamma$ determined by $\eta_0^{-1}(Q'_0)$. Thus any vector space $V$ with a filtration

$$F = \{V^F_\gamma : \gamma \text{ a final subquiver of } \Gamma\}$$

over $\Gamma$ has an induced filtration

$$\eta_* F = \{V^F_{\eta^*(\gamma)} : \gamma \text{ a final subquiver of } \Gamma\}$$

over $\Gamma$, which is refined by the filtration $F$ over $\Gamma$.

Now suppose that $N = \bigoplus_{w \in Q_0} \bigoplus_{n \in \mathbb{N}} N_{w,n}$ is a multicomplex supported over $\Gamma$, and its differential $d$ is homogeneous of degree 1. Then $N$ can be regarded as a representation $\rho$ of $\Gamma$ by writing

$$N = \bigoplus_{v \in Q_0} \left( \bigoplus_{w \in \eta_0^{-1}(v)} \bigoplus_{n \in \mathbb{N}} N_{w,n} \right)$$

and for any $a \in Q_1$ letting $\rho(a)$ be the linear combination

$$\sum_{\tilde{a} \in \eta_1^{-1}(a) \cap \tilde{A}} \lambda_{\tilde{a}} \tilde{a}$$

of the linear maps representing the elements of $\eta_1^{-1}(a)$ which have length one. In this way $N$ becomes a multicomplex supported over $\Gamma$, where the total complex of $N$ is the same whether we regard it as a multicomplex over $\tilde{\Gamma}$ or $\Gamma$.

By Remark 5.9 $N$ has a filtration $F$ over $\tilde{\Gamma}$, which is preserved by the differential on $N$, and hence $N$ has an induced filtration $\eta_* F$ over $\Gamma$, also preserved by the differential on $N$. There is an induced differential on the associated graded vector space

$$\text{gr}_{\eta_* F}(V) = \bigoplus_{v \in Q_0} V^F_{\eta^*(\Gamma' \supseteq v)} / V^F_{\eta^*(\Gamma' \supseteq v)}$$.
and as above, this gives us the $E_0$ page of a spectral sequence of multicomplexes over $\Gamma$ which abuts to the homology of the total complex of the multicomplex $N$ with its induced filtration over $\Gamma$.

6. SPECTRAL SEQUENCES OF MULTICOMPLEXES IN MORSE THEORY

As before let $f : M \to \mathbb{R}$ be a smooth function on a compact Riemannian manifold whose critical locus $\text{Crit}(f)$ has finitely many connected components, and let $D$ be the set of connected components of $\text{Crit}(f)$. In this section we will describe a spectral sequence of multicomplexes supported on a quiver which refines the information given by the vector spaces $H_i(N_{C,\infty}, \partial_-, N_{C,\infty}; \mathbb{F})$ and in the Morse–Smale situation reduces to the Morse–Witten complex.

When $(f, g)$ is Morse–Smale then $\text{Crit}(f) = D$ and if $C \in D$ then $H_i(N_{C,\infty}, \partial_-, N_{C,\infty}; \mathbb{F})$ is one-dimensional if $i$ is the index of the critical point in $C$ and otherwise is 0. Recall (from for example [25]) that in this situation there is an associated graph $\Gamma$ (more precisely a multigraph or quiver) embedded in $M$, with vertices given by the critical points of $f$ and arrows joining a critical point $p$ of index $i$ to a critical point $q$ of index $i - 1$ given by the (finitely many) gradient flow lines from $p$ to $q$. The Morse–Witten complex is the differential module with basis $\text{Crit}(f)$ and differential

$$\partial p = \sum_{q \in \text{Crit}(f)} n(p, q) q$$

where $n(p, q)$ is the number of flow lines from $p$ to $q$, counted with signs determined (canonically) by suitable choices of orientations. It is usually regarded as a complex via the grading by index (which is essentially the homological degree), but when we grade it using the homological degree and the critical value, then we have a multicomplex $N = \bigoplus_{p,q \in \mathbb{N}} N_{p,q}$ with $N_{p,q} = \bigoplus_{C \in D_q} H_{p+q}(N_{C,\infty}, \partial_-, N_{C,\infty}; \mathbb{F})$ which is supported on $\Gamma$, in the sense that the total differential $\partial$ is the sum of linear maps from $H_{p+q}(N_{C,\infty}, \partial_-, N_{C,\infty}; \mathbb{F})$ to $H_{q+r}(N_{C',\infty}, \partial_-, N_{C',\infty}; \mathbb{F})$ for $C \in D_q$ and $C' \in D_r$ of bidegree $(-i, i - 1)$, one for each arrow in $\Gamma$ from $C$ to $C'$. The spectral sequence described at the end of §5 contains much of the information in this (multi)complex, but it is packaged more efficiently in the Morse–Witten complex.

In the Morse–Smale situation a flow line from a critical point $p$ of index $i$ to a critical point $q$ of index $i - 1$ is a connected component of

$$\{ x \in M : \{ \psi_i(x) : t \leq 0 \} \text{ has a limit point in } \{ p \} \}$$

and

$$\{ x \in M : \{ \psi_i(x) : t \geq 0 \} \text{ has a limit point in } \{ q \} \},$$

which is always closed in $f^{-1}(f(q), f(p))$ (where $\psi_i(x)$ describes the downwards gradient flow for $f$ from $x$ at time $i$). In our more general situation we can define a quiver $\Gamma$ as follows.

**Definition 6.1.** The quiver $\Gamma$ associated to the smooth function $f : M \to \mathbb{R}$ on the Riemannian manifold $M$ has vertices $\{ v_C : C \in D \}$ labelled by the connected components $C$ of $\text{Crit}(f)$, and an arrow from $v_{C_+}$ to $v_{C_-}$ for each connected component of

$$\{ x \in M : \{ \psi_i(x) : t \leq 0 \} \text{ has a limit point in } C_+ \} \text{ and } \{ \psi_i(x) : t \geq 0 \} \text{ has a limit point in } C_- \}$$

that is closed in $f^{-1}(f(C_-), f(C_+))$.

**Remark 6.2.** When $(f, g)$ is Morse–Smale then the Morse–Witten complex of $f$ can be regarded as a multicomplex supported on $\Gamma$ in the sense of Definition 5.11, which is homogeneous of degree 1. When $f : M \to \mathbb{R}$ is smooth and $\text{Crit}(f)$ has finitely many connected components then a similar argument shows that there is a spectral sequence of multicomplexes supported on $\Gamma$ in the sense of Definition 5.13, where $C \in D$ is represented in the $E_1$ page by $H_s(N_{C,\infty}, \partial_-, N_{C,\infty}; \mathbb{F})$, for
and arrows from $C_k$ to $C_r$ are represented by the maps $\delta_k^r$ as in Remark 4.5. These are well defined at the $E_1$ stage for paths of length 1, at the $E_2$ stage there are well defined induced maps corresponding to paths of length two in $\Gamma$ and so on.

The analogue of the Morse–Witten complex is not in general a differential module which is a multicomplex supported on $\Gamma$ with homology $H_*(M; F)$, but instead the spectral sequence of multicomplexes supported on the quiver $\Gamma$ as described in Remark 6.2. $H_*(M; F)$ has a natural filtration over $\Gamma$ such that this spectral sequence abuts to $H_*(M; F)$ in the sense of Definition 5.14. One way to construct this spectral sequence of multicomplexes is to use the multicomplex given by the Morse–Witten complex for a Morse–Smale perturbation of the smooth function $f : M \to \mathbb{R}$. In order for this to work we need to describe the relative homology $H_*(\partial C, \infty; F)$ in Morse–Witten terms.

There are many descriptions of the relative (co)homology of a manifold with boundary which are appropriate for Morse theory [2, 11, 22, 23, 53, 66, 69] and many of them can be adapted to apply to manifolds with corners, whose boundaries decompose as the Morse neighbourhoods do, to give a description of the homology of the manifold relative to part of the boundary. We will use the approach of Laudenbach [51] and will avoid having to deal with corners by choosing a system of smooth Morse neighbourhoods (Remark 1.8).

Laudenbach considers a compact manifold $M$ with non-empty boundary, and defines a smooth real-valued function $f$ on $M$ to be Morse when its critical points lie in the interior of $M$ and are nondegenerate, and its restriction to the boundary is a Morse function in the usual sense; being Morse in this sense is generic among smooth functions on $M$. Then there are two types of critical points of the restriction of $f$ to $\partial M$: type N (for Neumann) when the gradient flow is pointing into $M$ and type D (for Dirichlet) when the gradient flow is pointing out of $M$. The homotopy type (and homology) of $\{ x \in M : f(x) \leq a \}$ may change when $a$ crosses a critical value of $f$ on the interior of $M$ or the value of a critical point of type N of $f|_{\partial M}$ but not when $a$ crosses the value of a critical point of type D of $f|_{\partial M}$. Similarly the homotopy type of $M$ relative to its boundary (and the relative homology $H_*(M, \partial M; F)$) may change when $a$ crosses a critical value of $f$ on the interior of $M$ or the value of a critical point of type D of $f|_{\partial M}$ but not when $a$ crosses the value of a critical point of type N of $f|_{\partial M}$.

**Definition 6.3.** Let $f : M \to \mathbb{R}$ be a Morse function in Laudenbach’s sense on a compact manifold $M$ with boundary. Let

- $C_k$ be the set of critical points of $f$ on the interior of $M$ with index $k$;
- $N_k$ be the set of critical points of $f$ on $\partial M$ of type N and index $k$;
- $D_k$ be the set of critical points of $f$ on $\partial M$ of type D and index $k - 1$.

Laudenbach proves

**Theorem 6.4.** (i) There is a differential on the free graded $\mathbb{Z}$-module generated by $C_s \cup N_s$ such that it is a chain complex whose homology is isomorphic to the homology of $M$.

(ii) There is a co-differential on the free graded $\mathbb{Z}$-module generated by $C_s \cup D_s$ such that it is a cochain complex whose cohomology is isomorphic to the relative cohomology $H^*(M, \partial M; \mathbb{Z}^\text{or})$ with coefficients twisted by the local system of orientations on $M$.

His proof of (i) involves using a pseudo-gradient vector field $X$ for $f$ which is adapted to the boundary in the sense that

a) $X \cdot f < 0$ except at the critical points in the interior of $M$ and the critical points of type N on $\partial M$;
b) $X$ points inwards along the boundary except near the critical points of type $N$ on $\partial M$ where it is tangent to the boundary;

c) if $p$ is a critical point in the interior of $M$, then $X$ is hyperbolic at $p$ and the quadratic form $X_{\text{lin}}^2 \cdot d^2 p$ (where $X_{\text{lin}}$ is the linear part of $X$ at $p$) is negative definite;

d) near any critical point $p$ of type $N$ for $f|_{\partial M}$ there are coordinates $x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$ on $M$ such that $M$ is given locally by $z \geq 0$ and $f$ is given locally by $f(x) = f(p) + z + q(y)$ where $q$ is a nondegenerate quadratic form and $X$ is tangent to the boundary, vanishing and hyperbolic at $p$ and $X_{\text{lin}}^2(q(y) + z^2)$ is negative definite;

e) $X$ is Morse–Smale, meaning that the global unstable manifolds and local stable manifolds are mutually transverse.

It is shown in [51] that a pseudo-gradient $X$ for $f$ adapted to the boundary always exists, that there is a differential on the free graded $\mathbb{Z}$-module generated by $C_k \cup N_k$ for $p \in C_k \cup N_k$ by

$$\partial p = \sum_{q \in C_{k-1} \cup N_{k-1}} m_{pq} q$$

where $m_{pq}$ is the number of flow lines from $p$ to $q$, counted with appropriate signs, and that the homology of the resulting complex is isomorphic to $H_q(M; \mathbb{F})$. The last statement is proved by

(i) showing that if $p$ is a critical point on $\partial M$ of type $N$, then $f$ and $X$ can be modified in an arbitrarily small neighbourhood $U$ of $p$ to a Morse function $f_1$ with the same Morse complex but with no critical point in $U \cap \partial M$ of type $N$ and instead one critical point in $U \cap \partial M$ of type $D$ (with the same index as $p$ as a critical point on $\partial M$) together with one interior critical point in $U \cap \partial M$ (with the same index as a critical point on the interior of $M$);

(ii) considering the case when there are no critical points of type $N$. In this case the Laudenbach complex for the homology of $M$ does not ‘see’ the boundary at all, in the sense that every flow line appearing in the differential connects critical points in the interior. Then standard Morse theory arguments show that it can be assumed without loss of generality that $f$ is weakly self-indexing, in the sense that for critical points $p$ and $q$ we have $f(p) > f(q)$ if and only if the index of $p$ is strictly greater than the index of $q$; this follows as in [51] §2.4 from

Lemma 6.5. Let $(f, X)$ be a Morse function and an adapted pseudo-gradient. Let $p$ and $q$ be two critical points with $f(p) > f(q)$. Assume that the open interval $(f(q), f(p))$ contains no critical value and that there are no connecting orbits from $p$ to $q$. Then there exists a path of Morse functions $f_t$ for $t \in [0, 1]$, with $f_0 = f$ and $f_1(p) < f_1(q) = f(q)$, such that $X$ is a pseudo-gradient for every $f_t$.

It is observed in [51] §2.4 that, when $f$ is weakly self-indexing and there are no critical points of type $N$ on the boundary, $M$ has the homotopy type of a CW-complex with a $k$-cell for each critical point of index $k$, and Laudenbach’s complex (6.1) is the corresponding cellular chain complex, whose homology is $H_q(M; \mathbb{F})$ by the cellular homology theorem.

Now let us return to the situation where $f : M \to \mathbb{R}$ is any smooth function whose critical locus has finitely many components, on the compact Riemannian manifold $M$. We can apply Laudenbach’s results to a perturbation of $f$ (or rather $-f$) restricted to suitably chosen Morse neighbourhoods. As at Remark 1.8, we can choose a system of strict Morse neighbourhoods $\{N_{C,n} : C \in D, n \geq 0\}$ such that $\partial N_{C,n} = \partial_+ N_{C,n} \cup \partial_- N_{C,n}$ is smooth, $f|_{\partial_+ N_{C,n}}$ is a Morse function and has no critical points on $\partial_+ N_{C,n} \cap \partial_- N_{C,n}$ and $f|_{\partial_+ N_{C,n}}$ (respectively $f|_{\partial_- N_{C,n}}$) achieves its minimum value (respectively its maximum value) precisely on $\partial_+ N_{C,n} \cap \partial_- N_{C,n}$. Then we can
perturb \( f \) in the interior of \( \mathcal{N}_{C,n} \) so that it is a Morse function on \( \mathcal{N}_{C,n} \) in Laudenbach’s sense, and moreover it is weakly self-indexing with every critical value strictly greater than the value of \( f \) at every critical point of its restriction to \( \partial_- \mathcal{N}_{C,n} \) and strictly less than the value of \( f \) at every critical point of its restriction to \( \partial_+ \mathcal{N}_{C,n} \). The critical points of type N for \( f \) (respectively type D for \(-f\)) on \( \partial \mathcal{N}_{C,n} \) are precisely those in \( \partial_+ \mathcal{N}_{C,n} \) while the critical points of type D for \( f \) (respectively type N for \(-f\)) are those in \( \partial_- \mathcal{N}_{C,n} \).

We will say that a pseudo-gradient vector field \( X \) for \( f \) on \( \mathcal{N}_{C,n} \) is doubly adapted to the boundary if

\( X \cdot f < 0 \) except at the critical points in the interior of \( \mathcal{N}_{C,n} \) and the critical points for the restriction of \( f \) to \( \partial \mathcal{N}_{C,n} \);

\( X \) points inwards along \( \partial_- \mathcal{N}_{C,n} \) except near the critical points on \( \partial \mathcal{N}_{C,n} \) where it is tangent to the boundary, and \( X \) points outwards along \( \partial_+ \mathcal{N}_{C,n} \) except near the critical points on \( \partial_+ \mathcal{N}_{C,n} \) where it is tangent to the boundary;

\( X \) is Morse–Smale.

By Laudenbach’s argument, there is a doubly adapted pseudo-gradient on the Morse neighbourhood \( \mathcal{N}_{C,n} \) and a differential on the free graded \( \mathbb{Z} \)-module generated by \( C_\pm \cup N_\pm \cup D_\pm \), which is given for \( p \in C_\pm \cup N_\pm \cup D_\pm \) by

\[
\partial p = \sum_{q \in C_{\pm 1} \cup N_{\pm 1} \cup D_{\pm 1}} m_{pq} q
\]

where \( m_{pq} \) is the number of flow lines from \( p \) to \( q \), counted with appropriate signs, and that the homology of the resulting complex is isomorphic to \( H_* (\mathcal{N}_{\Gamma}; \mathbb{F}) \).

Now instead of restricting the smooth map \( f : M \to \mathbb{R} \) to the Morse neighbourhoods \( \{ \mathcal{N}_{C,j(C)} : C \in D \} \) for some \( j : D \to \mathbb{N} \), let us consider \( f \) on \( M \) itself, but perturb it as above in the interior of each \( \mathcal{N}_{C,j(C)} \) to become a Morse function. Using a pseudo-gradient doubly adapted to the boundaries of the Morse neighbourhoods and given by the gradient flow of the perturbed function elsewhere, we obtain a differential determined by flow lines on the graded vector space with basis given (in the obvious modification of the notation above) by

\[
\bigcup_{C \in D} C_*^{[C]} \cup N_*^{[C]} \cup D_*^{[C]},
\]

with homology \( H_* (M; \mathbb{F}) \) and the structure of a multicomplex supported on a quiver \( \tilde{\Gamma} \). Here the choice of pseudo-gradient allows (broken) flow lines to cross \( \partial \mathcal{N}_{C,j(C)} \) only at critical points for the restriction of \( f \) to \( \partial \mathcal{N}_{C,j(C)} \).

The quiver \( \tilde{\Gamma} \) associated to the perturbed function \( \tilde{f} : M \to \mathbb{R} \) is a perturbation \( \Gamma \to \Gamma \) of the quiver \( \Gamma \) associated as at Definition 6.1 to the original smooth function \( f : M \to \mathbb{R} \), in the sense that each vertex of \( \Gamma \) has split into finitely many vertices of \( \tilde{\Gamma} \), and each path in \( \Gamma \) has split into finitely many paths in \( \tilde{\Gamma} \) in a compatible way. Indeed we can factor \( \Gamma \to \Gamma \) as \( \Gamma \to \Gamma' \to \Gamma \) where the quiver \( \Gamma' \) is defined as follows (cf. [9]).
Definition 6.6. The refined quiver $\Gamma'$ associated to the smooth function $f : M \to \mathbb{R}$ on the Riemannian manifold $M$ has vertices of three types: a vertex $v_C$ for each connected component $C$ of the critical locus $\text{Crit}(f)$ of $f$, a vertex $u_{G,C}$ for every connected component $G$ of the Morse stratum $W_C^+$ for $C$ (as defined at Definition 3.4) and a vertex $w_{H,C}$ for every connected component $H$ of the Morse stratum $W_C^-$ for $C$. The arrows are also of three types:

(i) there is an arrow from $u_{G,C}$ to $v_C$ for every $C \in D$ and every connected component $G$ of the Morse stratum $W_C^+$ for $C$,

(ii) there is an arrow from $v_C$ to $w_{H,C}$ for every $C \in D$ and every connected component $H$ of the Morse stratum $W_C^-$ for $C$,

(iii) there is an arrow from $w_{H,C_*}$ to $u_{G,C_*}$ for every $C_+$ and $C_-$ in $D$ such that there is a connected component of

$$\{x \in M : \{\psi_t(x) : t \leq 0\} \text{ has a limit point in } C_+ \text{ and } \{\psi_t(x) : t \geq 0\} \text{ has a limit point in } C_-\}$$

that is closed in $f^{-1}(f(C_-), f(C_+))$ and is contained in $G \cap H$.

The quivers $\Gamma$ and $\Gamma'$ only depend on the smooth function $f : M \to \mathbb{R}$ on the Riemannian manifold $M$, but the quiver $\Gamma$ depends on the perturbation $\tilde{f} : M \to \mathbb{R}$. Similarly the multicomplex supported on $\Gamma$ with underlying vector space $\mathbb{F}(\bigcup_{C \in D} C_i^C \cup N^C \cup \tilde{D}^C)$ (bi-graded by critical value and index/homological degree) depends on the choice of perturbation $\tilde{f} : M \to \mathbb{F}$ of $f : M \to \mathbb{R}$. However if we filter this multicomplex over $\Gamma$ as in §5, then we obtain a spectral sequence of multicomplexes supported on $\Gamma$. This spectral sequence is independent of the choice of perturbation $\tilde{f} : M \to \mathbb{F}$ of $f : M \to \mathbb{R}$ up to quasi-isomorphisms respecting the $\Gamma$-structure, and abuts to $H_*(M; \mathbb{F})$ in the sense of Definition 5.14. Moreover its $E_1$ page is the representation of $\Gamma$ given by vector spaces $H_*(N^C; \mathbb{F})$ for $C \in D$, and the differentials are as described in Remarks 6.2 and 4.5, or equivalently as described in Remark 4.6 with appropriate modifications using the local system $\mathbb{F}^{\text{or}}$ of orientations of $M$ when $M$ is not oriented.

Remark 6.7. For any pair $(f, g)$ where $g$ is a Riemannian metric on $M$ and $f : M \to \mathbb{R}$ is smooth with finitely many critical components, we get an acyclic quiver $\Gamma$ (with an $\mathbb{R}$-grading in the sense of [41] §2.1) and a filtration of the homology of $M$ over $\Gamma$, with a spectral sequence over $\Gamma$ abutting to $H_*(M; \mathbb{R})$. By analogy with the case for Morse and Morse–Bott functions, we can call $(f, g)$ ‘0-perfect’ if the spectral sequence degenerates at the $E_0$ page, and ‘1-perfect’ if it degenerates at the $E_1$ page; then 0-perfection corresponds to perfection in the traditional sense, while any Morse function is 1-perfect with the $E_1$ page given by its Morse–Witten complex. If $(f, g)$ is Morse–Smale then $\Gamma$ and the filtration and spectral sequence (up to quasi-isomorphism respecting the $\Gamma$-structure) are unchanged when we perturb $f$ and $g$ slightly. The space of all pairs $(f, g)$, where $g$ is a Riemannian metric on $M$ and $f : M \to \mathbb{R}$ is smooth with finitely many critical components, decomposes into open chambers in which $\Gamma$, the $\Gamma$-filtration of $H_*(M; \mathbb{R})$ and the spectral sequence are constant, separated by walls where ‘codimension-one accidents’ occur (cf. [51] §2.3) and more complicated situations where walls meet. Morse–Bott functions occupy an intermediate position: typically they lie where walls meet but it is possible to analyse at least some of the local structure of the space nearby [6, 7, 36]. This picture can be generalised to minimally degenerate Morse functions (see Definition 7.4 below), such as normsquares of moment maps, with appropriate metrics.
7. FURTHER EXTENSIONS AND APPLICATIONS

We have associated to a compact Riemannian manifold $M$ with smooth $f : M \to \mathbb{R}$, whose critical locus $\text{Crit}(f)$ has finitely many connected components, a quiver $\Gamma$, a refined quiver $\Gamma'$ and a spectral sequence of multicomplexes supported on $\Gamma$ involving the relative homology $H_*(\mathcal{N}_{C,n}, \partial_\ast \mathcal{N}_{C,n}; \mathbb{F})$ of Morse neighbourhoods, which abuts to the homology $H_*(M; \mathbb{F})$ of $M$. Using this (and in other ways) we have obtained Morse inequalities relating the dimensions of these homology groups. More generally we can define a Morse cobordism $\mathcal{M}$ as follows (cf. Definition 1.4), and extend our arguments to obtain similar descriptions for the relative homology $H_*(\mathcal{M}, \partial_\ast \mathcal{M}; \mathbb{F})$.

**Definition 7.1.** A Morse cobordism in dimension $n$ is given by a compact manifold $\mathcal{M}$ with corners (locally modelled on $[0, \infty)^2 \times \mathbb{R}^{\dim M - 2}$) and a Riemannian metric $g$, such that

(a) $\mathcal{M}$ has boundary $\partial \mathcal{M} = \partial_+ \mathcal{M} \cup \partial_- \mathcal{M} \cup \partial_1 \mathcal{M}$ where $\partial_+ \mathcal{M}$, $\partial_- \mathcal{M}$ and $\partial_1 \mathcal{M}$ are compact submanifolds of $\mathcal{M}$ with boundaries whose connected components form the corners of $\mathcal{M}$, and $\partial_+ \mathcal{M} \cap \partial_- \mathcal{M} \cap \partial_1 \mathcal{M} = \emptyset$;

(b) $f$ is locally constant on $(\partial_+ \mathcal{M}) \cap (\partial_- \mathcal{M}) = \partial(\partial_+ \mathcal{M}) \cap \partial(\partial_- \mathcal{M})$, and on each of $\partial(\partial_\pm \mathcal{M}) \cap \partial(\partial_1 \mathcal{M})$;

(c) the critical locus $\text{Crit}(f)$ has finitely many connected components, and every critical point $p$ of $f$ on $\mathcal{M}$ lies in the interior of $\mathcal{M}$ and satisfies

$$f(x) \leq f(p) \leq f(y)$$

for all $x \in \partial_- \mathcal{M}$ and $y \in \partial_+ \mathcal{M}$, with strict inequalities when $x \in (\partial_- \mathcal{M})^\circ$ and $y \in (\partial_+ \mathcal{M})^\circ$;

(d) the gradient vector field $\text{grad}(f)$ on $\mathcal{M}$ associated to its Riemannian metric $g$ satisfies

(i) the restriction of $\text{grad}(f)$ to $(\partial_+ \mathcal{M})^\circ$ points inside $\mathcal{M}$;

(ii) the restriction of $\text{grad}(f)$ to $(\partial_- \mathcal{M})^\circ$ points outside $\mathcal{M}$;

(iii) $\partial_\pm \mathcal{M}$ is invariant under the gradient flow $\text{grad}(f)$ (where defined);

(e) each connected component $Q$ of $\partial_1 \mathcal{M}$ has a diffeomorphism

$$\Psi_Q : (Q \cap \partial_+ \mathcal{M}) \times [b_-^Q, b_+^Q] \cong Q$$

whose composition with $f$ is projection onto the interval $[b_-^Q, b_+^Q]$, such that $\Psi_Q(x, d)$ is given by the intersection of the gradient flow from $x$ with $f^{-1}(d)$, and in particular $Q \cap \partial_+ \mathcal{M}$ and $Q \cap \partial_- \mathcal{M} = \Psi_Q((Q \cap \partial_+ \mathcal{M}) \times \{b_+^Q\}) \cong Q \cap \partial_1 \mathcal{M}$ are corners of $\mathcal{M}$.

**Definition 7.2.** The quiver associated to a Morse cobordism $\mathcal{M}$ as above has vertices of three types: a vertex $v_C$ for each connected component $C$ of the critical locus $\text{Crit}(f)$ of $f$ (on the interior of $\mathcal{M}$), a vertex $u_G$ for every connected component $G$ of $\partial_+ \mathcal{M}$ and a vertex $w_H$ for every connected component $H$ of $\partial_- \mathcal{M}$. There is an arrow from one vertex (labelled by $K = C, G$ or $H$) to another (labelled by $K' = C', G'$ or $H'$) for each connected component of

$$\{x \in M : \text{the trajectory of } x \text{ under } \text{grad}(f) \text{ flows up to } K \text{ and down to } K'\}$$

that has closed intersection with $f^{-1}(\sup f(K'), \inf f(K))$.

A Morse neighbourhood $\mathcal{N}_{C,n}$, for a smooth function $f : M \to \mathbb{R}$ whose critical locus $\text{Crit}(f)$ has finitely many connected components on a compact Riemannian manifold $M$, equipped with the restriction of $f$ and of the Riemannian metric, is a Morse cobordism in the sense of Definition 7.1. Moreover any Morse cobordism $\mathcal{M}$ can be decomposed into a union of Morse neighbourhoods and Morse cobordisms which are trivial in the sense that there are no (interior)
critical points, meeting along parts of their boundaries (in a way reminiscent of TQFTs). This enables us to construct as in §6 a spectral sequence of multicomplexes supported on the quiver given as in Definition 7.2 which abuts to the relative homology $H_\ast(M, \partial_\ast M; \mathbb{R})$.

Remark 7.3. Our arguments also extend to work for equivariant homology, and to obtain generalisations of the Novikov inequalities for a closed 1-form on $M$ (cf. [15, 16, 17, 18, 20, 59, 60, 61]).

Recall that a smooth function $f : M \to \mathbb{R}$ is a Morse–Bott function if the connected components $C \in D$ of its critical set $\text{Crit}(f)$ are submanifolds of $M$ and the Hessian of $f$ at any point of $C$ is nondegenerate in directions normal to $C$. In this situation $M = \bigcup_{C \in D} S_C$ where the Morse strata $S_C$ are given by

(7.1) \[ S_C = \{ x \in M : \text{the downwards gradient flow from } x \text{ has a limit point in } C \} \]

and Morse neighbourhoods $\mathcal{N}_{C,n}$ can be chosen as disc sub-bundles in the normal bundles to the Morse strata $S_C$ over neighbourhoods of $C$ in $S_C$ (with $\partial_\ast \mathcal{N}_{C,n}$ corresponding to the boundaries of the discs). Thus

$$ H_i(\mathcal{N}_{C,n}, \partial_\ast \mathcal{N}_{C,n}; \mathbb{R}) \cong H_{i-\text{codim}_S(S_C)}(C; \mathbb{R}) $$

giving us the classical Morse inequalities (0.1) as an immediate consequence of Theorem 1.11.

Morse–Bott homology has also been studied [7, 36, 43, 44, 79] in ways which are related to but not the same as our viewpoint in §6.

An early example of Morse theory applied to a smooth function which is not Morse–Bott was the norm-square of a moment map $\mu : M \to \mathfrak{k}^\ast$ for a Hamiltonian action of a compact Lie group $K$ on a compact symplectic manifold $M$ [48]. For any choice of invariant inner product on the Lie algebra $\mathfrak{k}$ of $K$ the norm-square $f = ||\mu||^2$ of the moment map is a $K$-equivariantly perfect minimally degenerate Morse function. More precisely, the set of critical points for $f = ||\mu||^2$ is a finite disjoint union of closed subsets $C \in D$ along each of which $f$ is minimally degenerate in the following sense.

Definition 7.4. A locally closed submanifold $\Sigma_C$ containing $C$ with orientable normal bundle in $M$ is a minimising (respectively maximising) submanifold for $f$ along $C$ if

1. the restriction of $f$ to $\Sigma_C$ achieves its minimum (respectively maximum) value exactly on $C$, and
2. the tangent space to $\Sigma_C$ at any point $x \in C$ is maximal among subspaces of $T_xM$ on which the Hessian $H_x(f)$ is non-negative (respectively non-positive).

If a minimising (respectively maximising) submanifold $\Sigma_C$ exists, then $f$ is called minimally (respectively maximally) degenerate along $C$. The smooth function $f$ is minimally (respectively maximally) degenerate if it is minimally (respectively maximally) degenerate along every $C \in D$. It is extremally degenerate if for each $C \in D$ it is either minimally or maximally degenerate along $C$.

In [48] it was shown that if $f$ is minimally degenerate then it induces a smooth stratification $\{ S_C : C \in D \}$ of $M$ where $S_C$ is the Morse stratum defined as at (7.1) for a suitable choice of Riemannian metric, and that there are thus Morse inequalities as in the Morse–Bott case. The stratum $S_C$ then coincides with $\Sigma_C$ near $C$.

Remark 7.5. Indeed suppose that $f : M \to \mathbb{R}$ is any extremally degenerate smooth function on a compact manifold $M$. Then just as in the Morse–Bott case, Morse neighbourhoods $\mathcal{N}_{C,n}$ can be chosen to be disc bundles in the normal bundle to an arbitrarily small neighbourhood
of $C$ (which retracts onto $C$) in the minimising or maximising manifold for $f$ along $C$. Here $\partial_\pm N_{C,n}$ corresponds to the boundary of the disc or the boundary of the neighbourhood (depending on whether there is a minimising or maximising manifold for $f$ along $C$). Thus $P_t(N_{C,\infty}, \partial_\pm N_{C,\infty}) = t^{\lambda_f(C)} P_t(C)$ where $\lambda_f(C)$ is the codimension (respectively dimension) of the minimising (respectively maximising) manifold along $C$. From Theorem 1.11, if the normal bundles are orientable, we obtain Morse inequalities of the classical form (0.1).

When $f = \|\mu\|^2$ is the normsquare of a moment map for a Hamiltonian action of a compact Lie group $K$ on a compact symplectic manifold then $f$ is minimally degenerate and there are also $K$-equivariant Morse inequalities. It was shown in [48] that these are in fact equalities: the normsquare of the moment map is equivariantly perfect. This was then used to obtain inductive formulas for the Betti numbers of symplectic quotients, and of quotient varieties arising in algebraic geometry. In their fundamental paper [3] Atiyah and Bott considered an infinite-dimensional version of the normsquare of a moment map given by the Yang–Mills functional arising in gauge theory over a compact Riemann surface. They conjectured that the Yang–Mills functional should induce an equivariantly perfect Morse stratification on an infinite-dimensional space of connections, but avoided the analytical difficulties created by the degeneracy of the functional, combined with the infinite-dimensionality, by using an alternative construction of the stratification. Their conjecture was proved by Daskalopoulos in [28] and extended by Witten in [77] to obtain intersection pairings.

Other applications of Morse theory in infinite-dimensional situations go back to Palais, Smale, Milnor and others, and of course more recently many arise in Floer theory [33, 34] (see [1, 4, 37] and references therein). Morse theory on singular spaces (cf. for example [38] and more recently [74]) and discrete versions of Morse theory (cf. [35, 49, 56, 57]) have also been studied.

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