Numerical Analysis on Ergodic Limit of Approximations for Stochastic NLS Equation via Multi-symplectic Scheme

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Abstract

We consider a finite dimensional approximation of the stochastic nonlinear Schrödinger equation driven by multiplicative noise, which is derived by applying a symplectic method to the original equation in spatial direction. Both the unique ergodicity and the charge conservation law for this finite dimensional approximation are obtained on the unit sphere. To simulate the ergodic limit over long time for the finite dimensional approximation, we discretize it further in temporal direction to obtain a fully discrete scheme, which inherits not only the stochastic multi-symplecticity and charge conservation law of the original equation but also the unique ergodicity of the finite dimensional approximation. The temporal average of the fully discrete numerical solution is proved to converge to the ergodic limit with order one with respect to the time step for a fixed spatial step. Numerical experiments verify our theoretical results on charge conservation, ergodicity and weak convergence.

AMS subject classification: 37M25, 60H35, 65C30, 65P10.

Key Words: stochastic Schrödinger equation, multiplicative noise, unique ergodicity, multi-symplectic scheme, weak error
1 Introduction

For the stochastic nonlinear Schrödinger (NLS) equation with a multiplicative noise in Stratonovich sense

\[ \begin{aligned}
  du &= i(\nabla u + \lambda |u|^2 u)dt + iu \circ dW, \\
  u(t, 0) &= u(t, 1) = 0, \quad t \geq 0, \\
  u(0, x) &= u_0(x), \quad x \in [0, 1]
\end{aligned} \]  

with \( \lambda = \pm 1 \), we consider the case that \( W \) is a real valued \( Q \)-Wiener process on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) with paths in \( H_0^1 := H_0^1(0, 1) \) with Dirichlet boundary condition. The Karhunen–Loève expansion of \( W \) is as follows

\[ W(t, x, \omega) = \sum_{k=0}^{\infty} \beta_k(t, \omega) Q^{\frac{1}{2}} e_k(x), \quad t \geq 0, \quad x \in [0, 1], \quad \omega \in \Omega, \]

where \( (e_k = \sqrt{2} \sin(k\pi x))_{k \geq 1} \) is an eigenbasis of the Dirichlet Laplacian \( \Delta \) in \( L^2 := L^2(0, 1) \) and \( (\beta_k)_{k \geq 1} \) is a sequence of independent real valued Brownian motions associated to the filtration \((\mathcal{F}_t)_{t \geq 0}\). In addition, the covariance operator \( Q \) is assumed to commute with the Laplacian and satisfies

\[ Q e_k = \eta_k e_k, \quad \eta_k > 0, \quad \forall k \in \mathbb{N}, \quad \eta := \sum_{k=1}^{\infty} \eta_k < \infty. \]

We refer to [9] for additional assumptions on the well-posedness of (1.1). It is shown that (1.1) is a Hamiltonian system with stochastic multi-symplectic structure and charge conservation law (see [7, 9, 11] and references therein). Structure-preserving numerical schemes have remarkable superiority to conventional schemes on numerically solving Hamiltonian systems over long time. As another kind of longtime behaviors, the ergodicity for this kind of conservative systems is an important and difficult problem which is still open. Motivated by [10], we study the ergodicity for a finite dimensional approximation (FDA) of the original equation instead.

In this paper, we investigate the ergodicity for a symplectic FDA of (1.1) and approximate its ergodic limit via a multi-symplectic and ergodic scheme. As we show that the FDA is charge conserved, without loss of generality, we consider the ergodicity in the finite dimensional unit sphere \( S \). There have been some papers considering the additive noise case with dissipative assumptions, and also some papers requiring the uniformly elliptic assumption on the whole space to ensure the unique ergodicity (see e.g. [3, 12, 13, 15, 16]). For the conservative FDA with a linear multiplicative noise, it has an uncertain nondegeneracy, which relies heavily on the solution. To overcome this difficulty, we construct an invariant control set \( \mathcal{M}_0 \subset S \), in which the FDA is shown to be nondegenerate. Together with the Krylov–Bogoliubov theorem and the Hörmander condition, we prove that the solution \( U \) possesses a unique invariant measure \( \mu_h \) (i.e., \( U \) is uniquely ergodic) with

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E} f(U(t)) dt = \int_{\mathcal{M}_0} f d\mu_h = \int_S f d\mu_h. \]

For many physical applications, the approximation of the invariant measure is of fundamental importance, especially when the invariant measure is unknown (see e.g. [1, 3–6, 13–16]). Some papers construct numerical schemes which also possess unique invariant measures, and then show the approximate error between invariant measures. For example, [6, 15] work with dissipative
systems driven by additive noise, and [16] considers elliptic SDEs with bounded coefficients and dissipative type condition. There is also some work concentrating on the approximation of the invariant measure, i.e., the approximation of the ergodic limit $\int_{S} f d\mu_h$, in which case the numerical schemes may not be uniquely ergodic. For instance, [3] approximates the invariant measure of stochastic partial differential equations with an additive noise based on Kolmogorov equation. [13] gives error estimates for time-averaging estimators of numerical schemes based on the associated Poisson equation and the assumption of local weak convergence order. Authors in [14] calculate the ergodic limit for Langevin equations with dissipations via quasi-symplectic integrators. There has been few results on constructing conservative and uniquely ergodic schemes to calculate the ergodic limit for conservative systems to our knowledge. We focus on the approximation of the ergodic limit via a multi-symplectic scheme, which is also shown to be uniquely ergodic. For a fixed spacial dimension, the local weak error of this fully discrete scheme (FDS) in temporal direction is of order two, which yields order one for the approximate error of the ergodic limit based on the associated Poisson equation (see also [4, 13]) and a priori estimates of the numerical solutions. That is,

$$\left| E \left[ \frac{1}{N} \sum_{n=0}^{N-1} f(U^n) - \int_{S} f d\mu_h \right] \right| \leq C_h \left( \frac{1}{T} + \tau \right).$$

The paper is organized as follows. In Section 2, we apply a symplectic semi-discrete scheme to the original equation to get the FDA, and show the unique ergodicity as well as the charge conservation law for the FDA. In Section 3, we present a multi-symplectic and ergodic FDS to approximate the ergodic limit, and show the approximate error based on a priori estimates and local weak error. In Section 4, the discrete charge evolution compared with those of Euler–Maruyama scheme and implicit Euler scheme, ergodic limit and global weak convergence order are tested numerically. Section 5 is the appendix containing proofs of some a priori estimates.

## 2 Unique ergodicity

In this section, we first apply the central finite difference scheme to (1.1) in spatial direction to obtain a FDA, which is also a Hamiltonian system. To investigate the ergodicity of this conservative system, we then construct an invariant control set $M_0 \subset S$ with respect to a control function introduced in Section 2.2. The FDA is proved to be ergodic in $M_0$ based on the Krylov–Bogoliubov theorem and the Hörmander condition.

### 2.1 Finite dimensional approximation (FDA)

Based on the central finite difference scheme and the notation $u_j := u_j(t), j = 1, \cdots, M$, we consider the following spatial semi-discretization

$$du_j = i \left[ \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + \lambda |u_j|^2 u_j \right] dt + i u_j \sum_{k=1}^{K} \sqrt{\eta_k} e_k(x_j) \circ d\beta_k(t)$$

with a truncated noise $\sum_{k=1}^{K} \sqrt{\eta_k} e_k(x) \beta_k(t), K \in \mathbb{N}$, a given uniform step size $h = \frac{1}{M+1}$ for some $M \leq K$ and $x_j = jh, j = 1, \cdots, M$. The condition $M \leq K$ here ensures the existence of the solution for the control function. Denoting vectors $U := U(t) = (u_1, \cdots, u_M)^T \in \mathbb{C}^M$, $\beta(t) = \sum_{k=1}^{K} \sqrt{\eta_k} e_k(x) \beta_k(t)$, we have
\[ (\beta_1(t), \cdots, \beta_K(t))^T \in \mathbb{R}^K, \text{ and matrices } F(U) = \text{diag}\{|u_1|^2, \cdots, |u_M|^2\}, \ E_k = \text{diag}\{e_k(x_1), \cdots, e_k(x_M)\}, \ \Lambda = \text{diag}\{\sqrt{n_1}, \cdots, \sqrt{n_K}\}, \ Z(U) = \text{diag}\{u_1, \cdots, u_M\}E_{MK}\Lambda, \]

\[ A = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -2 & 1 \end{pmatrix} \in \mathbb{R}^{M \times M}, \quad E_{MK} = \begin{pmatrix} e_1(x_1) & \cdots & e_K(x_1) \\ \vdots & \ddots & \vdots \\ e_1(x_M) & \cdots & e_K(x_M) \end{pmatrix}_{M \times K}, \]

then the FDA is in the following form

\[
\begin{cases}
\quad dU = i \left[ \frac{1}{h^2} AU + \lambda F(U)U \right] dt + iZ(U) \circ d\beta(t), \\
\quad U(0) = c_\ast (u_0(x_1), \cdots, u_0(x_M))^T,
\end{cases}
\]

where \(c_\ast\) is a normalized constant. The noise term in (2.1) has an equivalent Itô form

\[
iZ(U) \circ d\beta(t) = i \sum_{k=1}^{K} \sqrt{\eta_k} E_k U \circ d\beta_k(t) = -\frac{1}{2} \sum_{k=1}^{K} \eta_k E_k^2 U dt + i \sum_{k=1}^{K} \sqrt{\eta_k} E_k Ud\beta_k(t)
\]

\[ =: -\hat{E} U dt + i \sum_{k=1}^{K} \sqrt{\eta_k} E_k Ud\beta_k(t) \quad (2.2)\]

with \(\hat{E} = \frac{1}{2} \sum_{k=1}^{K} \eta_k E_k^2\). In the sequel, \(\| \cdot \|\) denotes the 2-norm for both matrices and vectors, which satisfies \(\|BV\| \leq \|B\|\|V\|\) for any matrices \(B \in \mathbb{C}^{m \times n}\) and vectors \(V \in \mathbb{C}^n\), \(m, n \in \mathbb{N}\). It is then easy to show that \(\|A\| \leq 4\), which is independent of the dimension \(M\).

**Proposition 2.1.** The FDA (2.1) possesses the charge conservation law, i.e.,

\[ \|U(t)\|^2 = \|U(0)\|^2, \quad \forall \ t \geq 0, \quad \mathbb{P}\text{-a.s.} \]

where \(\|U(t)\| = (\|P(t)\|^2 + \|Q(t)\|^2)^{1/2} = (\sum_{m=1}^{M} |p_m(t)|^2 + |q_m(t)|^2)^{1/2}\), \(P(t) = (p_1(t), \cdots, p_M(t))^T\) and \(Q(t) = (q_1(t), \cdots, q_M(t))^T\) are the real and imaginary parts of \(U(t)\) respectively.

**Proof.** Noticing that matrices \(A\) and \(F(U)\) are symmetric and the linear function \(Z(U)\) satisfies

\[
\bar{U}^T Z(U) = (\bar{u}_1, \cdots, \bar{u}_M) \begin{pmatrix} u_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_M \end{pmatrix} \begin{pmatrix} \sqrt{\eta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\eta_K} \end{pmatrix} = (|u_1|^2, \cdots, |u_M|^2)E_{MK} \begin{pmatrix} \sqrt{\eta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\eta_K} \end{pmatrix} \in \mathbb{R}^K, \quad (2.3)
\]

where \(\bar{U}\) denotes the conjugate of \(U\), we multiply (2.1) by \(\bar{U}^T\), take the real part, and then get the charge conservation law for \(U\). \(\square\)

In the sequel, without pointing out, all equations hold in the sense \(\mathbb{P}\)-a.s.

**Remark 1.** Eq. (1.1) can be rewritten into an infinite dimensional Hamiltonian system (see [11]). It is easy to verify that the central finite difference scheme (2.1) applied to (1.1) is equivalent to the symplectic Euler scheme applied to the infinite dimensional Hamiltonian form of (1.1), which implies the symplecticity of (2.1).
2.2 Unique ergodicity

As the charge of (2.1) is conserved shown in Proposition 2.1, without loss of generality, we assume that $U(0) \in \mathcal{S}$ and investigate the unique ergodicity of (2.1) on $\mathcal{S}$. As the nondegeneracy for (2.1) relies on the solution $U$ as a result of the multiplicative noise, the standard procedure to show the irreducibility and strong Feller property on the whole $\mathcal{S}$ do not apply. So we need to construct an invariant control set.

**Definition 1.** (see e.g. [2]) A subset $\mathcal{M} \neq \emptyset$ of $\mathcal{S}$ is called an invariant control set for the control system

$$
  d\phi = i \left[ \frac{1}{\hbar^2} A\phi + \lambda F(\phi)\phi \right] dt + i Z(\phi) d\Psi(t)
$$

(2.4)

of (2.1) with a differentiable deterministic function $\Psi$, if $\overline{\mathcal{O}^+(x)} = \overline{\mathcal{M}}$, $\forall x \in \mathcal{M}$, and $\mathcal{M}$ is maximal with respect to inclusion, where $\mathcal{O}^+(x)$ denotes the set of points reachable from $x$ (i.e., connected with $x$) in any finite time and $\overline{\mathcal{M}}$ denotes the closure of $\mathcal{M}$.

We state one of our main results in the following theorem.

**Theorem 2.1.** The FDA (2.1) possesses a unique invariant probability measure $\mu_h$ on an invariant control set $\mathcal{M}_0$, which implies the unique ergodicity of (2.1). Moreover,

$$
  \text{supp}(\mu_h) = \mathcal{S} \text{ and } \mu_h(\mathcal{S}) = \mu_h(\mathcal{M}_0) = 1.
$$

**Proof.**

**Step 1. Existence of invariant measures.**

From Proposition 2.1, we find $\pi_t(U(0), \mathcal{S}) = 1, \forall t \geq 0$, where $\pi_t(U(0), \cdot)$ denotes the transition probability (probability kernel) of $U(t)$. As the finite dimensional unit sphere $\mathcal{S}$ is tight, the family of measures $\pi_t(U(0), \cdot)$ is tight, which implies the existence of invariant measures by the Krylov–Bogoliubov theorem [8].

**Step 2. Invariant control set.**

Denoting $U = P + iQ$ with $P$ and $Q$ being the real and imaginary parts of $U$ respectively, we first consider the following subset of $\mathcal{S}$

$$
  \mathcal{S}_1 = \{ U = P + iQ \in \mathcal{S} : P > 0 \}.
$$

For any $t > 0$, $y, z \in \mathcal{S}_1$, there exists a differentiable function $\phi$ satisfying $\phi(s) = (\phi_1(s), \cdots, \phi_M(s))^T \in \mathcal{S}_1$, $s \in [0, t]$, $\phi(0) = y$ and $\phi(t) = z$ by polynomial interpolation argument. As $\text{rank}(Z(\phi(s))) = M$ for $\phi(s) \in \mathcal{S}_1$ and $M \leq K$, the linear equations

$$
  Z(\phi(s))X = -i\phi'(s) - \left[ \frac{1}{\hbar^2} A\phi(s) + \lambda F(\phi(s))\phi(s) \right]
$$

possess a solution $X \in \mathbb{C}^M$. As in addition $Z(\phi(s)) = \text{diag}\{\phi_1(s), \cdots, \phi_M(s)\} E_{MK} \Lambda$, where $\text{diag}\{\phi_1(s), \cdots, \phi_M(s)\}$ is invertible for $\phi(s) \in \mathcal{S}_1$, the solution $X$ depends continuously on $s$ and is denoted by $X(s)$. Thus, there exists a differentiable function $\Psi(\cdot) := \int_0^s X(s) ds$ which, together with $\phi$ defined above, satisfies the control function (2.4) with initial data $\Psi(0) = 0$. That is, for any $y, z \in \mathcal{S}_1$, $y$ and $z$ are connected, denoted by $y \leftrightarrow z$. The above argument also holds for the following subsets

$$
  \mathcal{S}_2 = \{ U = P + iQ \in \mathcal{S} : P < 0 \};
$$
\[ S_3 = \{ U = P + iQ \in S : Q > 0 \}; \]
\[ S_4 = \{ U = P + iQ \in S : Q < 0 \}. \]

For any \( y \in S_i, z \in S_j \) with \( i \neq j \) and \( i, j \in \{1, 2, 3, 4\} \), there must exist \( S_i, r_i \) and \( r_j \), satisfying \( r_i \in S_i \cap S_i \neq \emptyset \) and \( r_j \in S_j \cap S_i \neq \emptyset \) for some \( l \in \{1, 2, 3, 4\} \), such that \( y \leftrightarrow r_i \leftrightarrow r_j \leftrightarrow z \). Thus,
\[ M_0 := S_1 \cup S_2 \cup S_3 \cup S_4 = \{ U = P + iQ \in S : P \neq 0 \text{ or } Q \neq 0 \}, \]

with \( \overline{M_0} = S \), is an invariant control set for (2.4).

**Step 3. Uniqueness of the invariant measure.**

We rewrite (2.1) with \( P \) and \( Q \) according to its equivalent form in Itô sense and obtain

\[
d\left(\begin{array}{c}
P(t) \\
Q(t)
\end{array}\right) = \left(\begin{array}{cc}
-\dot{E} & -\frac{1}{\sqrt{M}} A - \lambda F(P, Q) \\
\frac{1}{\sqrt{M}} A - \lambda F(P, Q) & -\dot{E}
\end{array}\right) \left(\begin{array}{c}
P(t) \\
Q(t)
\end{array}\right) dt \\
+ \sum_{k=1}^{K} \sqrt{\eta_k} \left(\begin{array}{cc}
0 & -E_k \\
E_k & 0
\end{array}\right) \left(\begin{array}{c}
P(t) \\
Q(t)
\end{array}\right) d\beta_k(t)
\]

\[ =: X_0(P, Q) dt + \sum_{k=1}^{K} X_k(P, Q) d\beta_k(t). \tag{2.5} \]

To derive the uniqueness of the invariant measure, we consider the Lie algebra generated by the diffusions of (2.5)

\[ L(X_0, X_1, \cdots , X_K) = \text{span} \left\{ X_l, [X_i, X_j], [X_i, [X_i, X_j]], \cdots , 0 \leq i, j \leq K \right\}. \]

Choosing \( p_* = 0 \) and \( q_* = \frac{1}{\sqrt{M}} (1, \cdots , 1)^T \) such that \( z_* := p_* + iq_* \in S_4 \subset M_0 \), we derive that the following vectors

\[
X_k(p_*, q_*) = \sqrt{\eta_k} M \left(\begin{array}{c}
e_k(x_1) \\
\vdots \\
e_k(x_M) \\
0
\end{array}\right), \quad [X_0, X_k](p_*, q_*) = \sqrt{\eta_k} M \left(\begin{array}{c}
\frac{\dot{E}}{\sqrt{M}} e_k(x_1) \\
\vdots \\
\frac{\dot{E}}{\sqrt{M}} e_k(x_M) \\
\left(\frac{1}{\sqrt{M}} A + \frac{1}{M} I \right) e_k(x_1) \\
\vdots \\
\frac{1}{\sqrt{M}} A + \frac{1}{M} I e_k(x_M)
\end{array}\right)
\]

are independent of each other for \( k = 1, \cdots , M \), which hence implies the following Hörmander condition

\[ \dim L(X_0, X_1, \cdots , X_K)(z_*) = 2M. \]

Then there is at most one invariant measure with \( \text{supp}(\mu_h) = S \) according to [2]. Actually, according to above procedure, we obtain that Hörmander condition holds uniformly for any \( z \in M_0 \).

Combining the three steps above, we conclude that there exists a unique invariant measure \( \mu_h \) on \( M_0 \) for the FDA, with \( \mu_h(S) = \mu_h(M_0) = 1 \).

From the theorem above, we can find out that for some other nonlinearities, e.g. \( iF(x, |u|)u \) with \( F \) being some potential function, such that the equation still possesses the charge conservation law, we can still get the ergodicity of the finite dimensional approximation of the original
equation through the procedure above. The procedure could also be applied to higher dimensional Schrödinger equations with proper well-posed assumptions, but it may be more technical to verify the Hörmander condition.

**Remark 2.** According to the ergodicity of (2.1), we have

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E} f(U(t)) dt = \int_S f d\mu_h, \quad \forall \ f \in B_b(S), \quad \text{in } L^2(S, \mu_h),
\]

where \( B_b(S) \) denotes the set of bounded and measurable functions and \( \int_S f d\mu_h \) is known as the ergodic limit with respect to the invariant measure \( \mu_h \).

For more details, we refer to [8] and references therein.

3 Approximation of ergodic limit

A fully discrete scheme (FDS) with the discrete multi-symplectic structure and the discrete charge conservation law is constructed in this section, which could also inherit the unique ergodicity of the FDA. In addition, we prove that the time average of the FDS can approximate the ergodic limit \( \int_S f d\mu_h \) with order one with respect to the time step.

3.1 Fully discrete scheme (FDS)

We apply the midpoint scheme to (2.1), and obtain the following FDS

\[
\begin{align*}
U^{n+1} - U^n &= i \frac{\tau}{\hbar^2} A U^{n+\frac{1}{2}} + i \lambda \tau F(U^{n+\frac{1}{2}}) U^{n+\frac{1}{2}} + i Z(U^{n+\frac{1}{2}}) \delta_{n+1} \beta, \\
U^0 &= U(0) \in S,
\end{align*}
\]

(3.1)

where \( \tau \) denotes the uniform time step, \( t_n = n \tau, \ U^n = (u_1^n, \ldots, u_M^n) \in \mathbb{C}^M, \ U^{n+\frac{1}{2}} = \frac{U^{n+1} + U^n}{2} \) and \( \delta_{n+1} \beta = \beta(t_{n+1}) - \beta(t_n) \). For the FDS (3.1), which is implicit in both deterministic and stochastic terms, its well-posedness is stated in the following proposition.

**Proposition 3.1.** For any initial value \( U^0 = U(0) \in S \), there exists a unique solution \((U^n)_{n \in \mathbb{N}}\) of (3.1), and it possesses the discrete charge conservation law, i.e.,

\[ \|U^{n+1}\|^2 = \|U^n\|^2 = 1, \quad \forall \ n \in \mathbb{N}. \]

**Proof.** We multiply both sides of (3.1) by \( \overline{U^{n+\frac{1}{2}}} \), take the real part, and obtain the existence of the numerical solution by the Brouwer fixed-point theorem as well as the discrete charge conservation law.

For the uniqueness, we assume that \( X = (X_1, \ldots, X_M)^T \) and \( Y = (Y_1, \ldots, Y_M)^T \) are two solutions of (3.1) with \( U^n = z = (z_1, \ldots, z_M)^T \in S \). It follows that \( X, Y \in S \) and

\[
X - Y = i \frac{\tau}{\hbar^2} A \frac{X - Y}{2} + \frac{i \lambda \tau}{8} H(X, Y, z) + i Z \frac{X - Y}{2} \delta_{n+1} \beta,
\]

(3.2)

where

\[
H(X, Y, z) = \begin{pmatrix}
|X_1 + z_1|^2 (X_1 + z_1) - |Y_1 + z_1|^2 (Y_1 + z_1) \\
\vdots \\
|X_M + z_M|^2 (X_M + z_M) - |Y_M + z_M|^2 (Y_M + z_M)
\end{pmatrix}.
\]
Based on the fact that $|a|^2a - |b|^2b = |a|^2(a - b) + |b|^2(a - b) + ab(\overline{a} - \overline{b})$ for any $a, b \in \mathbb{C}$, we have

$$\Im((\overline{X} - \overline{Y})^TH(X, Y, z)) = \Im \left[ \sum_{m=1}^{M} (X_m + z_m)(Y_m + z_m)(\overline{X}_m - \overline{Y}_m)^2 \right]$$

with $\Im[V]$ denoting the imaginary part of $V$. Multiplying (3.2) by $(\overline{X} - \overline{Y})^T$, taking the real part, and we get

$$\|X - Y\|^2 = \frac{\lambda \tau}{8} \Im((\overline{X} - \overline{Y})^TH(X, Y, z))$$

$$\leq \frac{\tau}{8} \left( \max_{1 \leq m \leq M} |X_m + z_m||Y_m + z_m| \right) \|X - Y\|^2 \leq \frac{\tau}{2} \|X - Y\|^2,$$

where we have used the fact $X, Y, z \in \mathcal{S}$ and (2.3). For $\tau < 1$, we get $X = Y$ and complete the proof.

The proposition above shows that (3.1) possesses the discrete charge conservation law. Furthermore, (3.1) also inherits the unique ergodicity of the FDA and the stochastic multi-symplecticity of the original equation, which are stated in the following two theorems.

**Theorem 3.1.** The FDS (3.1) is also ergodic with a unique invariant measure $\mu_h^\tau$ on the control set $\mathcal{M}_0$, such that $\mu_h^\tau(\mathcal{S}) = \mu_h^\tau(\mathcal{M}_0) = 1$. Also,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(U^n) = \int_{\mathcal{S}} f \, d\mu_h^\tau, \quad \forall f \in B_b(\mathcal{S}), \quad \text{in } L^2(\mathcal{S}, \mu_h^\tau).$$

*Proof.* Based on the charge conservation law for $\{U^n\}_{n \geq 1}$, we obtain the existence of the invariant measure similar to the proof of Theorem 2.1.

To obtain the uniqueness of the invariant measure, we show that the Markov chain $\{U^{3n}\}_{n \geq 1}$ satisfies the minorization condition (see e.g. [12]). Firstly, Proposition 3.1 implies that for a given $U^n \in \mathcal{S}$, solution $U^{n+1}$ can be defined through a continuous function $U^{n+1} = \kappa(U^n, \delta_{n+1} \beta)$. As $\delta_{n+1} \beta$ has a $C^\infty$ density, we get a jointly continuous density for $U^{n+1}$. Secondly, similar to Theorem 2.1, for any given $y, z \in \mathcal{M}_0$, there must exist $i, j, k \in \{1, 2, 3, 4\}$ and $r_i, r_j \in \mathcal{M}_0$, such that

$$y \in \mathcal{S}_i, z \in \mathcal{S}_j, r_i \in \mathcal{S}_i \cap \mathcal{S}_k \quad \text{and} \quad r_j \in \mathcal{S}_j \cap \mathcal{S}_k.$$ 

As $\frac{y + r_i + r_j}{2} \in \mathcal{S}_i$ and $Z(\frac{y + r_i + r_j}{2})$ is invertible, $\delta_{3n+1} \beta$ can be chosen to ensure that

$$r_i - y = \frac{\tau}{R^2} A \frac{y + r_i}{2} + \frac{\lambda \tau F}{2} \frac{y + r_i}{2} + \frac{i Z(\frac{y + r_i}{2})}{2} \delta_{3n+1} \beta$$

holds, i.e., $r_i = \kappa(y, \delta_{3n+1} \beta)$. Similarly, based on the fact $\frac{r_i + r_j}{2} \in \mathcal{S}_k$ and $\frac{r_i + r_j}{2} \in \mathcal{S}_j$, we have $r_j = \kappa(r_i, \delta_{3n+2} \beta)$ and $z = \kappa(r_j, \delta_{3n+3} \beta)$. That is, for any given $y, z \in \mathcal{M}_0$, $\delta_{3n+1} \beta, \delta_{3n+2} \beta, \delta_{3n+3} \beta$ can be chosen to ensure that $U^{3n} = y$ and $U^{3(n+1)} = z$. Finally we obtain that, for any $\delta > 0$,

$$\mathbb{P}_3(y, B(z, \delta)) := \mathbb{P}(U^3 \in B(z, \delta)|U^0 = y) > 0,$$

where $B(z, \delta)$ denotes the open ball of radius $\delta$ centered at $z$. \hfill \square
The infinite dimensional system (1.1) has been shown to preserve the stochastic multi-symplectic conservation law locally (see i.e. [11])

\[
d_t (dp \wedge dq) - \partial_x (dp \wedge dv + dq \wedge dw) dt = 0
\]

with \( p, q \) denoting the real and imaginary parts of solution \( u \) respectively and \( v = p_x, w = q_x \) being the derivatives of \( p \) and \( q \) with respect to variable \( x \). We now show that this ergodic FDS (3.1) not only possesses the discrete charge conservation law as shown in Proposition 3.1 but also preserves the discrete stochastic multi-symplectic structure.

**Theorem 3.2.** The implicit FDS (3.1) preserves the discrete multi-symplectic structure

\[
\begin{align*}
\frac{1}{\tau} (dp_j^{n+1} \wedge dq_j^{n+1} - dp_j^n \wedge dq_j^n) - \frac{1}{h} (dp_j^{n+\frac{1}{2}} \wedge dv_{j+1}^{n+\frac{1}{2}} - dp_{j-1}^{n+\frac{1}{2}} \wedge dv_j^{n+\frac{1}{2}}) \\
- \frac{1}{h} (dq_j^{n+\frac{1}{2}} \wedge dw_{j+1}^{n+\frac{1}{2}} - dq_{j-1}^{n+\frac{1}{2}} \wedge dw_j^{n+\frac{1}{2}}) = 0,
\end{align*}
\]

where \( p_j^n, q_j^n \) denote the real and imaginary parts of \( u_j^n, v_j = \frac{1}{\tau} (p_j^n - p_{j-1}^n) \) and \( w_j = \frac{1}{h} (q_j^n - q_{j-1}^n) \).

**Proof.** Rewriting (3.1) with the real and imaginary parts of the components \( u_j^n \) of \( U^n \), we get

\[
\begin{align*}
\frac{1}{\tau} (q_j^{n+1} - q_j^n) - \frac{1}{h} (v_j^{n+\frac{1}{2}} - v_j^{n+\frac{1}{2}}) &= \left( (p_j^{n+\frac{1}{2}})^2 + (q_j^{n+\frac{1}{2}})^2 \right) p_j^{n+\frac{1}{2}} + p_j^{n+\frac{1}{2}} \zeta_j^K, \\
\frac{1}{\tau} (p_j^{n+1} - p_j^n) - \frac{1}{h} (w_j^{n+\frac{1}{2}} - w_j^{n+\frac{1}{2}}) &= \left( (p_j^{n+\frac{1}{2}})^2 + (q_j^{n+\frac{1}{2}})^2 \right) q_j^{n+\frac{1}{2}} + q_j^{n+\frac{1}{2}} \zeta_j^K, \\
\frac{1}{h} (p_j^{n+\frac{1}{2}} - p_j^{n+\frac{1}{2}}) &= v_j^{n+\frac{1}{2}}, \\
\frac{1}{h} (q_j^{n+\frac{1}{2}} - q_j^{n+\frac{1}{2}}) &= w_j^{n+\frac{1}{2}},
\end{align*}
\]

where \( \zeta_j^K = \sum_{k=1}^K \sqrt{\eta_k e_k(x_j)} \circ d\beta_k(t) \). Denoting \( z_j^{n+\frac{1}{2}} = (p_j^{n+\frac{1}{2}}, q_j^{n+\frac{1}{2}}, v_j^{n+\frac{1}{2}}, w_j^{n+\frac{1}{2}})^T \) and taking differential in the phase space on both sides of (3.3), we obtain

\[
\frac{1}{\tau} d \begin{pmatrix}
q_j^{n+1} - q_j^n \\
n_{j+1}^{n+1} - p_j^n
\end{pmatrix} + \frac{1}{h} d \begin{pmatrix}
-v_j^{n+\frac{1}{2}} \\
-w_j^{n+\frac{1}{2}} \\
-p_j^{n+\frac{1}{2}} \\
-q_j^{n+\frac{1}{2}}
\end{pmatrix} = \nabla^2 S_1(z_j^{n+\frac{1}{2}}) dz_j^{n+\frac{1}{2}} + \nabla^2 S_2(z_j^{n+\frac{1}{2}}) dz_j^{n+\frac{1}{2}} \zeta_j^K,
\]

where

\[
S_1(z_j^{n+\frac{1}{2}}) = \frac{1}{4} \left( (p_j^{n+\frac{1}{2}})^2 + (q_j^{n+\frac{1}{2}})^2 \right)^2 + \frac{1}{2} \left( v_j^{n+\frac{1}{2}} \right)^2 + \frac{1}{2} \left( w_j^{n+\frac{1}{2}} \right)^2
\]

and

\[
S_2(z_j^{n+\frac{1}{2}}) = \frac{1}{2} \left( p_j^{n+\frac{1}{2}} \right)^2 + \frac{1}{2} \left( q_j^{n+\frac{1}{2}} \right)^2.
\]

Then the wedge product between \( dz_j^{n+\frac{1}{2}} \) and (3.4) concludes the proof based on the symmetry of \( \nabla^2 S_1 \) and \( \nabla^2 S_2 \).
Lemma 1. For any initial value \( U^0 \in \mathcal{S} \) and \( \gamma \geq 1 \), if \( Q \in \mathcal{HS}(L^2, H^{\frac{3}{2} - \frac{1}{\gamma}}) \), then there exists a constant \( C \) such that the solution \((U^n)_{n \in \mathbb{N}}\) of (3.1) satisfies
\[
\mathbb{E} \left\| U^{n+1} - U^n \right\|^{2\gamma} \leq C(\tau^{2\gamma} h^{-4\gamma} + \tau^\gamma), \quad \forall n \in \mathbb{N},
\]
where \( \mathcal{HS}(L^{\gamma_1}, H^{\gamma_2}) \) denotes the space of Hilbert–Schmidt operators from \( L^{\gamma_1} \) to \( H^{\gamma_2} \).

Lemma 2. For any initial value \( U(0) \in \mathcal{S} \) and \( \gamma \geq 1 \), there exists a constant \( C \) such that the solution \( U(t) \) of (2.1) satisfies
\[
\mathbb{E}\| U(t_{n+1}) - U(t_n) \|^{2\gamma} \leq C(\tau^{2\gamma} h^{-4\gamma} + \tau^\gamma), \quad \forall n \in \mathbb{N}.
\]

The proofs of Lemmas above are given in the appendix for readers’ convenience.

3.2 Approximation of ergodic limit

To approximate the ergodic limit of (2.1) and get the approximate error, we give an estimate of the local weak convergence between \( U(\tau) \) and \( U^1 \), and the Poisson equation associated to (2.1) are also used (see [13]). Recall that the SDE (2.1) in Stratonovich sense has an equivalent Itô form
\[
dU = \left[ \frac{1}{h^2} AU + i\lambda F(U)U - \bar{E}U \right] dt + iZ(U)d\beta(t)
\]
\[
=:b(U)dt + \sigma(U)d\beta(t) \tag{3.5}
\]
based on (2.2). For any fixed \( f \in W^{4,\infty}(\mathcal{S}) \), let \( \hat{f} := \int_{\mathcal{S}} f d\mu_b \) and \( \varphi \) be the unique solution of the Poisson equation \( \mathcal{L}\varphi = f - \hat{f} \), where
\[
\mathcal{L} := b \cdot \nabla + \frac{1}{2} \sigma \sigma^T : \nabla^2
\]
denotes the generator of (3.5). It’s easy to find out that (3.5) satisfies the hypoelliptic setting (see e.g. [13]) according to the Hörmander condition in Theorem 2.1. Thus, \( \varphi \in W^{4,\infty}(\mathcal{S}) \) according to Theorem 4.1 in [13]. Based on the well-posedness of the numerical solution \((U^n)_{n \in \mathbb{N}}\) and the implicit function theorem, (3.1) can be rewritten in the following form
\[
U^{n+1} = U^n + \tau\Phi(U^n, \tau, h, \delta_{n+1}, \beta)
\]
for some function \( \Phi \). Denoting by \( D\varphi(u)\Phi_1 \) and \( D^k\varphi(u)(\Phi_1, \cdots, \Phi_k) \) the first and \( k \)-th order weak derivatives evaluated in the directions \( \Phi_j, j = 1, \cdots, k \) with \( D^k\varphi(u)(\Phi)^k \) for short if all the directions are the same in the \( k \)-th derivatives, then we have
\[
\varphi(U^{n+1}) = \varphi(U^n) + \tau \left[ D\varphi(U^n)\Phi_n + \frac{1}{2} \tau D^2\varphi(U^n)(\Phi_n)^2 \right] + \frac{1}{6} D^3\varphi(U^n)(\tau\Phi_n)^3 + R_n^\Phi
\]
\[
=:\varphi(U^n) + \tau \mathcal{L}\Phi \varphi(U^n) + \frac{1}{6} D^3\varphi(U^n)(\tau\Phi_n)^3 + R_n^\Phi, \tag{3.7}
\]
where $\Phi^n := \Phi(U^n, \tau, h, \delta_{n+1} \beta)$,

$$\mathcal{L}^n \phi(U^n) = D\phi(U^n)\Phi^n + \frac{1}{2}\tau D^2\phi(U^n)(\Phi^n)^2$$

and

$$R^n = \frac{1}{4!} D^4\phi(\theta_n)(\tau \Phi^n)^4$$

for some $\theta_n \in [U^n, U^{n+1}] := [u^n_1, u^{n+1}_1] \times \cdots \times [u^n_M, u^{n+1}_M]$. Adding (3.7) together from $n = 0$ to $n = N - 1$ for some fixed $N \in \mathbb{N}$, then dividing the result by $T = N\tau$, and noticing that $\mathcal{L}\phi(U^n) = f(U^n) - f$, we obtain

$$\frac{\phi(U^N) - \phi(U^0)}{N\tau} = \frac{1}{N} \left( \sum_{n=0}^{N-1} \left[ \mathcal{L}^n \phi(U^n) - \mathcal{L}\phi(U^n) \right] + \sum_{n=0}^{N-1} \mathcal{L}\phi(U^n) \right)$$

$$+ \frac{1}{\tau} \sum_{n=0}^{N-1} \frac{1}{6} D^3\phi(U^n)(\tau \Phi^n)^3 + \frac{1}{\tau} \sum_{n=0}^{N-1} R^n$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left[ \mathcal{L}^n \phi(U^n) - \mathcal{L}\phi(U^n) \right] + \frac{1}{6\tau} \sum_{n=0}^{N-1} D^3\phi(U^n)(\tau \Phi^n)^3$$

$$+ \frac{1}{N} \sum_{n=0}^{N-1} f(U^n) - f$$

$$+ \frac{1}{N} \sum_{n=0}^{N-1} R^n,$$

which shows

$$\left| \mathbb{E} \left[ \frac{1}{N} \sum_{n=0}^{N-1} f(U^n) - f \right] \right| \leq \frac{1}{N\tau} \mathbb{E} \left[ \phi(U^N) - \phi(U^0) \right] + \frac{1}{N\tau} \sum_{n=0}^{N-1} \mathbb{E} R^n$$

$$+ \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E} \left[ \mathcal{L}^n \phi(U^n) - \mathcal{L}\phi(U^n) + \frac{1}{6\tau} D^3\phi(U^n)(\tau \Phi^n)^3 \right] =: I + II + III. \quad (3.8)$$

The average $\frac{1}{N} \sum_{n=0}^{N-1} f(U^n)$ is regarded as an approximation of $\hat{f}$. We next begin to investigate the approximate errors by estimating $I$, $II$ and $III$ respectively.

According to the fact that $\phi \in W^{4, \infty}(S)$ and Lemma 1, we have

$$I \leq \frac{2\|\phi\|_{0, \infty}}{N\tau} \leq \frac{C}{T} \quad (3.9)$$

and

$$II \leq \frac{1}{N\tau} \sum_{n=0}^{N-1} \mathbb{E} \left[ \|\tau \Phi^n\|^4 \|D^4\phi\|_{L^\infty} \right] \leq \frac{C}{N\tau} \sum_{n=0}^{N-1} \mathbb{E} \left[ \|U^{n+1} - U^n\|^4 \right]$$

$$\leq \frac{C}{N\tau} \sum_{n=0}^{N-1} \left( \tau^4 h^{-8} + \tau^2 \right) \leq C \left( \tau^3 h^{-8} + \tau \right), \quad (3.10)$$

where $\|\phi\|_{\gamma, \infty} := \sup_{|\alpha| \leq \gamma, u \in S} |D^\alpha \phi(u)|$, $\gamma \in \mathbb{N}$.

It then remains to estimate the terms $III$. To this end, we need the estimate about the local weak convergence, which is stated in the following theorem. The proof of the following theorem is also given in the appendix.
Theorem 3.3. For a fixed spatial approximation (2.1), and for any initial value \( U^0 \in S \) and \( \varphi \in W^{4,\infty}(S) \), it holds under the condition \( Q \in H S(L^2, H^\frac{1}{2}) \) and \( \tau = O(h^4) \) that

\[
|E[\varphi(U(\tau)) - \varphi(U^1)]| \leq C_h \tau^2
\]

for some constant \( C_h = C(\varphi, \eta, h) \).

Now we are in the position of showing the approximation error between the time average of FDS and the ergodic limit of FDA.

Theorem 3.4. Under the assumptions in Theorem 3.3 and for any \( f \in W^{4,\infty}(S) \), there exists a positive constant \( C_h = C(f, \eta, h) \) such that

\[
\left| E\left[ \frac{1}{N} \sum_{n=0}^{N-1} f(U^n) - \bar{f} \right] \right| \leq C_h (\frac{1}{T} + \tau).
\]

Proof. Based on (3.8)–(3.10), it suffices to estimate term III. For any \( f \in W^{4,\infty}(S) \), we know from the statement above that the solution to the Poisson equation \( \mathcal{L} \varphi = f - \bar{f} \) satisfies \( \varphi \in W^{4,\infty}(S) \). Based on (3.7), Lemma 1 and the condition \( \tau = O(h^4) \), we have

\[
\varphi(U^1) = \varphi(U^0) + \tau \mathcal{L} \varphi(U^0) + \frac{1}{6} D^3 \varphi(U^0)(U^1 - U^0)^3 + O(\tau^2)
\]

\[
= \varphi(U^0) + \tau \mathcal{L} \varphi(U^0) + O(\tau^2),
\]

(3.11)

where \( \mathcal{E} \) means that the equation holds in expectation sense, and in the last step we have used the fact that

\[
D^3 \varphi(U^0)(U^1 - U^0)^3 = D^3 \varphi(U^0) \left( \frac{\tau}{h^2} AU^\frac{1}{2} + i \lambda \tau F(U^\frac{1}{2}) U^\frac{1}{2} + i Z(U^\frac{1}{2}) \delta_1 \beta \right)^3
\]

\[
= D^3 \varphi(U^0) \left( i Z(U^\frac{1}{2}) \delta_1 \beta \right)^3 + O(\tau^2 h^{-2} + \tau^2)
\]

\[
= D^3 \varphi(U^0) \left( \frac{i}{2} Z(U^1 - U^0) \delta_1 \beta + i Z(U^0) \delta_1 \beta \right)^3 + O(\tau^2 h^{-2} + \tau^2)
\]

\[
= O(\tau^2 h^{-2} + \tau^2)
\]

(3.12)

based on the linearity of \( Z \), Lemma 1 and that \( \mathbb{E} (i Z(U^0) \delta_1 \beta)^3 = 0 \). We can also get the following expression similar to (3.11) based on Taylor expansion and Lemma 2

\[
\varphi(U(\tau)) = \varphi(U^0) + \int_0^\tau \left( D \varphi(U^0) b(U(t)) + \frac{1}{2} D^2 \varphi(U^0) (\sigma(U(t)))^2 \right) dt
\]

\[
+ \int_0^\tau D \varphi(U^0) \sigma(U(t)) d\beta(t) + \frac{1}{6} D^3 \varphi(U^0)(U(\tau) - U^0)^3 + O(\tau^2)
\]

\[
= \varphi(U^0) + \int_0^\tau \tilde{\mathcal{L}} t \varphi(U^0) dt + O(\tau^2),
\]

(3.13)

where

\[
\tilde{\mathcal{L}} t \varphi(U^0) := D \varphi(U^0) b(U(t)) + \frac{1}{2} D^2 \varphi(U^0) (\sigma(U(t)))^2
\]

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and $\mathbb{E} \left[ \int_0^T D\varphi(U^n)\sigma(U(t))d\beta(t) \right] = 0$. Thus, subtracting (3.11) with (3.13), we derive

$$\left| \mathbb{E} \left[ \tau L^\Phi \varphi(U^n) - \int_0^T \tilde{L}_t \varphi(U^n)dt \right] \right| \leq \left| \mathbb{E} [\varphi(U(\tau)) - \varphi(U^1)] \right| + C\tau^2.$$  \hfill (3.14)

Noticing that

$$\left| \int_0^T \mathbb{E} \left[ \tilde{L}_t \varphi(U^n) - L\varphi(U^n) \right] dt \right| \leq \left| \int_0^T \mathbb{E} \left[ D\varphi(U^n) (b(U(t)) - b(U^n)) \right] dt \right| + \frac{1}{2} \left| \int_0^T \mathbb{E} \left[ D^2\varphi(U^n) (\sigma(U(t)) - \sigma(U^n), \sigma(U(t)) + \sigma(U^n)) \right] dt \right|$$  \hfill (3.15)

in which we have

$$\left| \mathbb{E} \left[ D\varphi(U^n) (b(U(t)) - b(U^n)) \right] \right| = \left| \mathbb{E} \left[ D^2\varphi(U^n) \left( \frac{1}{h^2} A (U(t) - U^n) + i\lambda \left( F(U(t)) U(t) - F(U^n) U^n \right) \right) \right] \right| \leq C(t h^{-2} + t)$$

for the first term in (3.15). In the last step, we have used the fact that $g(V) := F(V)V, \forall V \in \mathcal{S}$ is a continuous differentiable function which satisfies $|D^k g(V)| \leq C$ for $|V| \leq 1$ and $k \in \mathbb{N}$, and then replace $U(t) - U^n$ by the integral form of (2.1) to get the result. The second term in (3.15) can be estimated in the same way. Thus, we have

$$\left| \int_0^T \mathbb{E} \left[ \tilde{L}_t \varphi(U^n) - L\varphi(U^n) \right] dt \right| \leq C(\tau^2 h^{-2} + \tau^2).$$  \hfill (3.16)

We hence conclude based on (3.12), (3.14), (3.16) and Theorem 3.3 that

$$III = \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E} \left[ L^\Phi \varphi(U^n) - L\varphi(U^n) + \frac{1}{6\tau} D^3\varphi(U^n)(U^{n+1} - U^n)^3 \right] \right| \leq \frac{1}{\tau} \sup_{U^n \in \mathcal{S}} \left\{ \left| \mathbb{E} \left[ \tau L^\Phi \varphi(U^n) - \int_0^T \tilde{L}_t \varphi(U^n)dt \right] \right| + \left| \int_0^T \mathbb{E} \left[ \tilde{L}_t \varphi(U^n) - L\varphi(U^n) \right] dt \right| \right\} + C(\tau h^{-2} + \tau) \leq C_h \tau. \hfill (3.17)$$

Noticing that $\tau^3 h^{-8} = O(\tau)$ under the condition $\tau = O(h^4)$, from (3.9), (3.10) and (3.17), we finally obtain

$$\left| \mathbb{E} \left[ \frac{1}{N} \sum_{n=0}^{N-1} f(U^n) - \hat{f} \right] \right| \leq C_h \left( \frac{1}{T} + \tau \right).$$

\[ \square \]

**Remark 3.** Based on the theorem above and the ergodicity of (2.1), for a fixed $h$, we obtain

$$\left| \mathbb{E} \left[ \frac{1}{N} \sum_{n=0}^{N-1} f(U^n) - \frac{1}{T} \int_0^T f(U(t))dt \right] \right| \leq C_h (B(T) + \tau),$$

which implies that the global weak error is of order one, i.e.,

$$\left| \mathbb{E} \left[ f(U^n) - f(U(t)) \right] \right| \leq C_h (\bar{B}(t) + \tau), \quad t \in [n\tau, (n+1)\tau],$$

where $B(T) \to 0$ and $\bar{B}(T) \to 0$ as $T \to \infty$. On the other hand, a time independent weak error in turn leads to the result stated in Theorem 3.4.
4 Numerical experiments

In this section, numerical experiments are given to test several properties of scheme (3.1) with \( \lambda = 1 \), i.e., the focusing case. In the following experiments, we simulate the noise \( \delta_{n+1} \beta \) by \( \sqrt{\pi} \xi_n \) with \( \xi_n \) being independent \( K \)-dimensional \( N(0,1) \)-random variables, and choose \( \eta_k = k^{-4}, k = 1, \cdots, K \). In addition, we approximate the expectation by taking averaged value over 500 paths, and the proposed scheme, which is implicit, is numerically solved utilizing the fixed point iteration. In the sequel, we will use the notation \( ||U||^2 := \sum_{m=1}^M (|p_m|^2 + |q_m|^2) \) for \( U \in \mathbb{C}^M \) and \( \gamma \in \mathbb{N} \) with \( P = (p_1, \cdots, p_M)^T, Q = (q_1, \cdots, p_M)^T \) being the real and imaginary parts of \( U \). Notice that \( \| \cdot \|_2 = \| \cdot \| \).

![Figure 1: Charge evolution](image)

As we omit the boundary nodes in the simulation, as a result, we may choose the normalized initial value \( U^0 = c_s(U^0(1), \cdots, U^0(M))^T \) based on function \( u_0(x) \) satisfying \( U^0(m) = u_0(mh), m = 1, \cdots, M \), in which \( u_0(x) \) need not to satisfy the boundary condition in (1.1). Let \( u_0(x) = 1 \), and we get the normalized initial value \( U^0 \) satisfying \( \|U^0\| = 1 \), which is used in Figures 1, 3 and 4. We first simulate the discrete charge for the proposed scheme compared with Euler–Maruyama (EM) scheme and implicit Euler (IE) scheme, respectively. Figure 1 shows that the proposed scheme possesses the discrete charge conservation law \( \mathbb{E}\|U^n\|^2 = 1 \), which coincides with Proposition 3.1, while both the EM scheme and the IE scheme do not. As the EM scheme does not stable, whose solution will blow up in a short time, we choose the time step \( \tau \) small enough for the EM scheme in the experiments.

![Figure 1: Charge evolution](image)

As the ergodic limit \( \int_S f d\mu_h \) is unknown, to verify the ergodicity of the numerical solution, we simulate the time averages \( \frac{1}{N} \sum_{n=1}^N \mathbb{E}[f(U^n)] \) for the proposed scheme with the bounded function \( f \in C_b(S) \) being (a) \( f(U) = \|U\|^2 \), (b) \( f(U) = \sin(||U||^2) \) and (c) \( f(U) = e^{-\|U\|^2} \) in Figure 2, started from five different initial values \( U_l^0, 1 \leq l \leq 5 \). It is known from Theorem 3.1 that for almost every initial values \( U^0 \in S \), the time averages will converge to the same value, i.e. the ergodic limit. Thus, we choose five initial values

\[
U_l^0 = c_s(U_l^0(1), \cdots, U_l^0(M))^T, \ l = 1, \cdots, 5
\]

based on the following five functions

\[
u_{0,1}(x) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, \quad u_{0,2}(x) = 1, \quad u_{0,3}(x) = 2x,
\]

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\[ u_{0,4}(x) = \left( 1 - \sqrt{\frac{\pi}{2}} \left(\exp \frac{1}{4} - 1\right) \right) (1 - \exp (x(1 - x))), \]

\[ u_{0,5}(x) = c_{s} \text{sech} \left( \frac{x}{\sqrt{2}} \right) \exp \left( i \frac{x}{2} \right) \]

with \( U_{l}^{0}(m) = u_{0,l}(hm), 1 \leq m \leq M \) and \( c_{s} \) being normalized constants. The charge of all the initial functions equal one, and \( u_{0,4}(x) \) even satisfies the boundary condition in (1.1). Figure 2 shows that the proposed scheme started from different initial values converges to the same value with error no more than \( O(\tau) \) with \( h = 0.05 \) and \( \tau = 2^{-6} \), which coincides with Theorem 3.4.

![Figure 2](image_url)

**Figure 2**: The time averages \( \frac{1}{N} \sum_{n=1}^{N} E[f(U^{n})] \) for the proposed scheme with (a) \( f(U) = \|U\|_{3}^{3} \), (b) \( f(U) = \sin(\|U\|_{4}^{4}) \) and (c) \( f(U) = e^{-\|U\|_{4}^{4}} \) (\( \tau = 2^{-6}, h = 0.05, K = 30 \)).

![Figure 3](image_url)

**Figure 3**: The weak convergence order of \( \|E[f(U^{n}) - f(U(T))]\| \) with (a) \( f(U) = \|U\|_{3}^{3} \), (b) \( f(U) = \sin(\|U\|_{4}^{4}) \) and (c) \( f(U) = e^{-\|U\|_{4}^{4}} \) (\( \tau = 2^{-i}, 10 \leq i \leq 13, h = 0.05, T = 2^{-1}, K = 30 \)).

For a fixed \( h \), Figure 3 and 4 show the weak convergence order in temporal direction and the weak error over long time, respectively. Figure 3 shows that the proposed scheme is of order one in the weak sense for (a) \( f(U) = \|U\|_{3}^{3} \), (b) \( f(U) = \sin(\|U\|_{4}^{4}) \) and (c) \( f(U) = e^{-\|U\|_{4}^{4}} \) which coincides with the statement in Remark 3. Furthermore, based on the ergodicity for both FDS and FDA, the weak error is supposed to be independent of time interval when time is large enough. To verify this property, we simulate the weak error over long time in Figure 4 for (a) \( f(U) = \|U\|_{3}^{3} \), (b) \( f(U) = \sin(\|U\|_{4}^{4}) \) and (c) \( f(U) = e^{-\|U\|_{4}^{4}} \), and it shows that the weak error for the proposed scheme would not increase before \( T = 1000 \) while the weak error for the EM scheme would increase with time.
where we have used the fact that by the convexity of $S$

$$\left| e^{-\|U\|^2_4} \right| \leq 2$$

and Hölder’s inequality. In the last step of (5.1), noticing that, as $Q \in \mathcal{H}(L^2, H^\frac{3}{2}-1)$, that is, $\sum_{k=1}^{\infty} k^{3-2} \eta_k < \infty$, so $\eta_k = O(k^{-\frac{4}{7}+c})$ for any $c > 0$. Thus,

$$\sum_{k=1}^{\infty} k^{\gamma} = C \int_{1}^{\infty} k^{-\frac{4}{7}+c} \frac{k^{\gamma}}{2(2\gamma-1)} = C \int_{1}^{\infty} k^{\gamma} < \infty.$$

In conclusion,

$$E \left\| U^{n+1} - U^n \right\|^{2\gamma} \leq C \left( E \left\| \int_0^T A U^{n+\frac{1}{2}} \right\|^{2\gamma} + E \left\| \lambda \tau F(U^{n+\frac{1}{2}}) U^{n+\frac{1}{2}} \right\|^{2\gamma} + E \left\| Z(U^{n+\frac{1}{2}}) \delta_{n+1} \right\|^{2\gamma} \right) \leq C \frac{\tau^{2\gamma}}{h^{4\gamma}} \left\| U^{n+\frac{1}{2}} \right\|^{2\gamma} + C \tau^{2\gamma} + C \tau \leq C \left( \tau^{2\gamma} h^{-4\gamma} + \tau \right),$$

where we have used the fact that $\|A\| \leq 4$. 

Figure 4: The weak error $\|E[f(U^n) - f(U(T))]\|$ for (a) $f(U) = \|U\|^2_3$, (b) $f(U) = \sin(\|U\|^4_4)$ and (c) $f(U) = e^{-\|U\|^2_4}$ ($\tau = 2^{-12}$, $h = 0.05$, $T = 10^3$, $K = 30$).

5 Appendix

5.1 Proof of Lemma 1

As proved in Proposition 3.1 that $\|U^n\| = 1$, $\forall n \in \mathbb{N}$, for the nonlinear term, we have

$$E \left\| F(U^{n+\frac{1}{2}}) U^{n+\frac{1}{2}} \right\|^{2\gamma} = E \left( \sum_{m=1}^{M} \left| u_m^{n+\frac{1}{2}} \right|^6 \right)^{\frac{\gamma}{2}} \leq E \left( \sum_{m=1}^{M} \left| u_m^{n+\frac{1}{2}} \right|^2 \right)^{\frac{3\gamma}{2}} \leq E \left\| U^{n+\frac{1}{2}} \right\|^{6\gamma} \leq 1$$

by the convexity of $S$, i.e., $\|U^{n+\frac{1}{2}}\| \leq 1$, a.s. The noise term can be estimated as

$$E \left\| Z(U^{n+\frac{1}{2}}) \delta_{n+1} \right\|^{2\gamma} = E \left( \sum_{m=1}^{M} \left| u_m^{n+\frac{1}{2}} \right| \sqrt{\eta_k} \delta_{n+1} \beta_k \right)^{2\gamma} \leq E \left( \sum_{m=1}^{M} \left| u_m^{n+\frac{1}{2}} \right| \right)^{2\gamma} = E \left( \sum_{k=1}^{K} \left| \delta_{n+1} \beta_k \right| \right)^{2\gamma} \leq C \tau^{2\gamma}$$

by $|e_k(x_m)| \leq \sqrt{2}$ and Hölder’s inequality. In the last step of (5.1), noticing that, as $Q \in \mathcal{H}(L^2, H^\frac{3}{2}-1)$, that is, $\sum_{k=1}^{\infty} k^{3-2} \eta_k < \infty$, so $\eta_k = O(k^{-\frac{4}{7}+c})$ for any $c > 0$. Thus,

$$\sum_{k=1}^{\infty} \eta_k^{\frac{\gamma}{2(2\gamma-1)}} \leq C \sum_{k=1}^{\infty} k^{-\frac{4}{7}+c} \frac{k^{\gamma}}{2(2\gamma-1)} = C \sum_{k=1}^{\infty} k^{-\frac{1}{2(2\gamma-1)}} < \infty.$$
5.2 Proof of Lemma 2

From (2.1) and (2.2), based on Hölder’s inequality, we obtain

\[ \mathbb{E}\|U(t_{n+1}) - U(t_n)\|^{2\gamma} \]

\[ = \mathbb{E}\left( \int_{t_n}^{t_{n+1}} \left[ \frac{1}{h^2} AU + i\lambda F(U)U - \hat{E}U \right] dt + \int_{t_n}^{t_{n+1}} iZ(U)d\beta(t) \right)^{2\gamma} \]

\[ \leq C \left( \int_{t_n}^{t_{n+1}} \mathbb{E}\left\| \frac{1}{h^2} AU + i\lambda F(U)U - \hat{E}U \right\|^{2\gamma} dt \left( \int_{t_n}^{t_{n+1}} 1 \gamma dt \right)^{2\gamma - 1} \right. \]

\[ + \left. \mathbb{E}\left\| \int_{t_n}^{t_{n+1}} iZ(U)d\beta(t) \right\|^{2\gamma} \right) \]

\[ \leq C\tau^{2\gamma - 1} \left( \frac{1}{h^2} \right)^{2\gamma} \int_{t_n}^{t_{n+1}} \mathbb{E}\|U\|^{2\gamma} dt + C\tau^{2\gamma} + C\tau^\gamma \]

\[ \leq C(\tau^{2\gamma} h^{-4\gamma} + \tau^\gamma), \]

where we have used the boundedness of \( F(U) \) in \( S \) similar to that in Lemma 1. In the third step of the equation above, we also used

\[ \mathbb{E}\|\hat{E}U\|^{2\gamma} \leq C\mathbb{E}\left( \sum_{m=1}^{M} \left| \sum_{k=1}^{K} \eta_k e_k^2(x_m)u_m \right|^{2\gamma} \right) \]

\[ \leq C\mathbb{E}\left( \sum_{m=1}^{M} |u_m|^2 \left( \sum_{k=1}^{K} \eta_k \right)^{2\gamma} \right) \leq C\eta^{2\gamma} \mathbb{E}\|U\|^{2\gamma} \leq C \]

and

\[ \mathbb{E}\left\| \int_{t_n}^{t_{n+1}} iZ(U)d\beta(t) \right\|^{2\gamma} \leq C \left( \int_{t_n}^{t_{n+1}} \left( \mathbb{E}\|Z(U)\|_{HS}^{2\gamma} \right)^{\frac{1}{\gamma}} dt \right)^{\gamma} \]

\[ \leq C \left( \int_{t_n}^{t_{n+1}} \left( \mathbb{E}\left( \sum_{m=1}^{M} \sum_{k=1}^{K} |u_m e_k(x_m)\sqrt{\eta_k}|^2 \right)^{\frac{1}{\gamma}} dt \right)^{\gamma} \right)^{\gamma} \]

\[ \leq C \left( \int_{t_n}^{t_{n+1}} \left( \mathbb{E}\left( 2\eta\|U\|^2 \right)^{\frac{1}{\gamma}} dt \right)^{\gamma} \right)^{\gamma} \leq C\tau^\gamma \]

according to the Burkholder–Davis–Gundy inequality and the fact that the Hilbert–Schmidt operator norm \( \|Z(U)\|_{HS} = \|Z(U)\|_F \) with \( \| \cdot \|_F \) denoting the Frobenius norm.

5.3 Proof of Theorem 3.3

Based on Taylor expansion, Lemma 1 and 2, we obtain

\[ \mathbb{E} [\varphi(U(\tau)) - \varphi(U^1)] = \mathbb{E} [D\varphi(U^1)(U(\tau) - U^1) + O(\|U(\tau) - U^1\|^2)] \]

\[ = \mathbb{E} [D\varphi(U^0)(U(\tau) - U^1)] + \mathbb{E} [D^2\varphi(U^0)(U^1 - U^0, U(\tau) - U^1)] \]

\[ \mathbb{E} [\varphi(U(\tau)) - \varphi(U^1)] \leq \mathbb{E} |\varphi(U(\tau)) - \varphi(U^1)| \leq C\gamma. \]
We give the mild solution and discrete mild solution of (2.1) and (3.1) respectively,

\[
U(\tau) = e^{\frac{1}{\tau^2}A^\tau}U^0 + \int_0^\tau e^{\frac{1}{\tau^2}A(\tau-s)} \left( i\lambda F(U(s))U(s) - \dot{E}U(s) \right) ds
+ \int_0^\tau e^{\frac{1}{\tau^2}A(\tau-s)}iZ(U(s))d\beta(s),
\]

\[
U^1 = (I - \frac{i\tau}{2h^2}A)^{-1}(I + \frac{i\tau}{2h^2}A)U^0 + (I - \frac{i\tau}{2h^2}A)^{-1}i\lambda \tau F \left( U^{\frac{1}{2}} \right) U^{\frac{1}{2}}
+ (I - \frac{i\tau}{2h^2}A)^{-1}iZ \left( U^{\frac{1}{2}} \right) \delta_1 \beta.
\]

**Estimation of A.** Considering the difference between above equations, we have

\[
U(\tau) - U^1 = \left( e^{\frac{1}{\tau^2}A^\tau} - (I - \frac{i\tau}{2h^2}A)^{-1}(I + \frac{i\tau}{2h^2}A) \right) U^0
+ i\int_0^\tau \left[ e^{\frac{1}{\tau^2}A(\tau-s)} - (I - \frac{i\tau}{2h^2}A)^{-1} \right] \lambda F(U(s))U(s) ds
+ i\int_0^\tau (I - \frac{i\tau}{2h^2}A)^{-1} \lambda \left[ F(U(s))U(s) - F \left( U^{\frac{1}{2}} \right) U^{\frac{1}{2}} \right] ds
+ i\int_0^\tau \left[ e^{\frac{1}{\tau^2}A(\tau-s)} - (I - \frac{i\tau}{2h^2}A)^{-1} \right] Z(U(s))d\beta(s)
+ i\int_0^\tau \left( I - \frac{i\tau}{2h^2}A \right)^{-1} Z(U(s) - U^0)d\beta(s)
- \left[ \frac{i}{2} \left( I - \frac{i\tau}{2h^2}A \right)^{-1} Z(U^1 - U^0) \delta_1 \beta + \int_0^\tau e^{\frac{1}{\tau^2}A(\tau-s)} \dot{E}U(s) ds \right],
\]

=:a + b + c + d + e + f,

which, together with the fact that \( \mathbb{E}[D\varphi(U^0)d] = \mathbb{E}[D\varphi(U^0)e] = 0 \), yields that

\[
A = \mathbb{E} \left[ D\varphi(U^0)a \right] + \mathbb{E} \left[ D\varphi(U^0)b \right] + \mathbb{E} \left[ D\varphi(U^0)c \right] + \mathbb{E} \left[ D\varphi(U^0)\right] f
=:A_1 + A_2 + A_3 + A_4.
\]

Based on the estimates \( e^x - (1 - \frac{x}{2})^{-1}(1 + \frac{x}{2}) = O(x^3) \) for \( \|x\| < 1 \), and

\[
\left\| e^{\frac{1}{\tau^2}A(\tau-s)} - (I - \frac{i\tau}{2h^2}A)^{-1} \right\| \leq C \left( \frac{\tau}{h^2} \|A\| \right) \leq C\tau h^{-2}, \quad \forall s \in [0, \tau],
\]

(5.2)

we have

\[
|A_1| \leq C\|\varphi\|_{1,\infty}\|\tau h^{-2} A\| \|E\| U^0 \| \leq C\tau^3 h^{-6} \leq C\tau^2 h^{-2}
\]

(5.3)
under the condition $\tau = O(h^4)$, and
\[ |A_2| \leq C\|\varphi\|_{1,\infty} \int_0^T \|\tau h^{-2} A\| \|F(U(s))U(s)\|ds \leq C\tau^2 h^{-2}. \quad (5.4) \]

Term $A_3$ can be estimated based on Lemma 1 and 2.
\[ |A_3| = \left| \mathbb{E}\left[ D\varphi(U^0) \int_0^T \left( I - \frac{i\tau}{2h^2}A \right)^{-1} \left( (F(U(s))U(s) - F(U^0)U^0) \right) \right] \right| \]

in which we have known from the proof of Theorem 3.4 that
\[ F(U(s))U(s) - F(U^0)U^0 = g(U(s)) - g(U^0) \]
\[ = Dg(U^0)(U(s) - U^0) + \frac{1}{2} D^2 g(\theta(s))(U(s) - U^0)^2 \]
\[ = Dg(U^0) \left( \int_0^s \frac{1}{h^2} AU(r) + i\lambda F(U(r))U(r) - \hat{E} U(r)dr + \int_0^s Z(U(r))d\beta(r) \right) \]
\[ + \frac{1}{2} D^2 g(\theta(s))(U(s) - U^0)^2 \]

for some $\theta(s) \in [U^0, U(s)]$ and $s \in [0, \tau]$, and the same for the term $F\left( U^{1/2} \right)U^{1/2} - F(U^0)U^0$. Based on the fact that $\mathbb{E} \left[ Dg(U^0) \int_0^s Z(U(r))d\beta(r) \right] = 0$, we hence get
\[ |A_3| \leq C(\tau^2 h^{-2} + \tau^2) \quad (5.5) \]

similar to the proof of Lemma 2. Rewrite
\[ Z(U^1 - U^0)\delta_1 \beta = \left( \begin{array}{c}
    u_1^1 - u_1^0 \\
    \vdots \\
    u_M^1 - u_M^0
  \end{array} \right) E_{MK}\Lambda \delta_1 \beta \\
= \left( \begin{array}{cccc}
    \sum_{k=1}^K e_k(x_1) \sqrt{\eta_k} \beta_k & \cdots & \sum_{k=1}^K e_k(x_M) \sqrt{\eta_k} \beta_k
  \end{array} \right) (U^1 - U^0) \]

where $G$ satisfies that $\mathbb{E}[GU^0] = 0$. Utilizing that $\mathbb{E}[G(U^0)U^0] = 0$, we can rewrite term $A_4$ as
\[ A_4 = -\mathbb{E}\left[ D\varphi(U^0) \left( \frac{1}{2} \left( I - \frac{i\tau}{2h^2}A \right)^{-1} G(U^1 - U^0) + \int_0^T e^{\frac{1}{h^2}A(\tau-s)}\hat{E} U(s)ds \right) \right] \]
\[ = -\frac{1}{2} \mathbb{E}\left[ D\varphi(U^0) \left( I - \frac{i\tau}{2h^2}A \right)^{-1} G\left( \frac{i\tau}{h^2}AU^{1/2} + i\lambda \tau F(U^{1/2})U^{1/2} + iGU^{1/2} \right) \right] \]
\[ -\mathbb{E}\left[ D\varphi(U^0) \int_0^T e^{\frac{1}{h^2}A(\tau-s)}\hat{E} U(s)ds \right] \]
\[
\begin{align*}
&= \frac{\tau}{4h^2} \mathbb{E} \left[ D\varphi(U^0) \left( I - \frac{i\tau}{2h^2} A \right)^{-1} GA(U^1 - U^0) \right] \\
&\quad + \frac{1}{2} \lambda \tau \mathbb{E} \left[ D\varphi(U^0) \left( I - \frac{i\tau}{2h^2} A \right)^{-1} G \left( F(U_{\frac{1}{2}}) U_{\frac{1}{2}} - F(U^0) U^0 \right) \right] \\
&\quad + \frac{1}{4} \mathbb{E} \left[ D\varphi(U^0) \left( I - \frac{i\tau}{2h^2} A \right)^{-1} G^2(U^1 - U^0) \right] \\
&\quad + \mathbb{E} \left[ D\varphi(U^0) \left( I - \frac{i\tau}{2h^2} A \right)^{-1} \frac{1}{2} G^2 U^0 - \int_0^\tau e^{\frac{1}{h^2} A(\tau-s)} \hat{E} U(s) ds \right] \\
&=: A_{4,1} + A_{4,2} + A_{4,3} + A_{4,4},
\end{align*}
\]

in which, based on \( \mathbb{E}[G^2 U^0] = 0 \), \( A_{4,3} \) can be expressed as
\[
\frac{1}{4} \mathbb{E} \left[ D\varphi(U^0) \left( I - \frac{i\tau}{2h^2} A \right)^{-1} G^2 \left( \frac{i\tau}{h^2} AU_{\frac{1}{2}} + i\tau \lambda F(U_{\frac{1}{2}}) U_{\frac{1}{2}} + \frac{1}{2} G(U^1 - U^0) \right) \right].
\]

For any \( U \in \mathbb{C}^M \), we have
\[
\mathbb{E}\|GU\| = \mathbb{E}\|Z(U)\delta_1\beta\| \leq C\mathbb{E}\left( \|U\|^2 \left( \sum_{k=1}^K \sqrt{\eta_k \delta_1 \beta_k} \right)^2 \right)^{\frac{1}{2}} \leq C \tau^{\frac{1}{2}} \left( \mathbb{E}\|U\|^2 \right)^{\frac{1}{2}}.
\]

Hence \( \mathbb{E}\|G^3(U^1 - U^0)\| \leq C \tau^{\frac{1}{2}} \left( \mathbb{E}\|G^2(U^1 - U^0)\|^2 \right)^{\frac{1}{2}} \) can be further estimated based on (5.1) with \( \gamma = 4 \) under the condition \( Q \in \mathcal{HS}(L^2, \mathbb{H}_{\frac{1}{2}}) \), which together with Lemma 1 and \( \|U_{\frac{1}{2}}\| \leq 1 \) yields
\[
|A_{4,1} + A_{4,2} + A_{4,3}| \leq C(\tau^\frac{5}{2} h^{-4} + \tau^2 h^{-2} + \tau^2) \leq C(\tau^2 h^{-2} + \tau^2).
\]

For the term \( A_{4,4} \), we have
\[
\frac{1}{2} G^2 U^0 \equiv \frac{1}{2} \left( \begin{array}{c}
\sum_{k=1}^K e_k^2(x_1) \eta_k (\delta_1 \beta_k)^2 u_1^0 \\
\vdots \\
\sum_{k=1}^K e_k^2(x_M) \eta_k (\delta_1 \beta_k)^2 u_M^0
\end{array} \right), \quad \hat{E} U(s) = \frac{1}{2} \left( \begin{array}{c}
\sum_{k=1}^K e_k^2(x_1) \eta_k u_1(s) \\
\vdots \\
\sum_{k=1}^K e_k^2(x_M) \eta_k u_M(s)
\end{array} \right).
\]

Thus, we obtain
\[
A_{4,4} = \frac{1}{2} \mathbb{E} \left[ D\varphi(U^0) \left( I - \frac{i\tau}{2h^2} A \right)^{-1} \left( \begin{array}{c}
\sum_{k=1}^K e_k^2(x_1) \eta_k (\delta_1 \beta_k)^2 u_1^0 \\
\vdots \\
\sum_{k=1}^K e_k^2(x_M) \eta_k (\delta_1 \beta_k)^2 u_M^0
\end{array} \right) \right] \\
- \frac{1}{2} \mathbb{E} \left[ D\varphi(U^0) \int_0^\tau e^{\frac{1}{h^2} A(\tau-s)} \left( \begin{array}{c}
\sum_{k=1}^K e_k^2(x_1) \eta_k u_1(s) \\
\vdots \\
\sum_{k=1}^K e_k^2(x_M) \eta_k u_M(s)
\end{array} \right) ds \right] \\
= \frac{1}{2} \mathbb{E} \left[ D\varphi(U^0) \left( I - \frac{i\tau}{2h^2} A \right)^{-1} \left( \begin{array}{c}
\sum_{k=1}^K e_k^2(x_1) \eta_k ((\delta_1 \beta_k)^2 - \tau) u_1^0 \\
\vdots \\
\sum_{k=1}^K e_k^2(x_M) \eta_k ((\delta_1 \beta_k)^2 - \tau) u_M^0
\end{array} \right) \right]
\]
\begin{align}
+ \frac{1}{2} \mathbb{E} \left[ D\varphi(U^0) \int_0^\tau \left( \left( I - \frac{i\tau}{2h^2} A \right)^{-1} - e^{\frac{i\tau}{h^2} A (\tau - s)} \right) \left( \sum_{k=1}^K e_k^2(x_1) \eta_k u_1^0 \right) ds \right] \\
- \frac{1}{2} \mathbb{E} \left[ D\varphi(U^0) \int_0^\tau e^{\frac{i\tau}{h^2} A (\tau - s)} \left( \sum_{k=1}^K e_k^2(x_1) \eta_k \left( u_1(s) - u_1^0 \right) \right) ds \right], \quad (5.7)
\end{align}

where in the last step we have used the fact

\[ \left( \sum_{k=1}^K e_k^2(x_1) \eta_k \tau u_1^0 \right) = \int_0^\tau \left( \sum_{k=1}^K e_k^2(x_1) \eta_k u_1^0 \right) ds. \]

Noticing that the first term in (5.7) vanishes as \( \mathbb{E}(\delta_1 \beta_k)^2 = \tau \) and replacing \( U(s) - U^0 \) by the integral type of (2.1), then further calculation shows that

\[ |A_{4,4}| \leq C(\tau^2 h^{-2} + \tau^2) \quad (5.8) \]

based on (5.2) and the technique used in (5.5). We then conclude from (5.3)–(5.8) that

\[ |A| \leq C(\tau^2 h^{-2} + \tau^2) \leq C h^2. \quad (5.9) \]

**Estimation of \( C \).** Estimations of \( A_1 \) and \( A_2 \) show that

\[ \mathbb{E}\|a + b\|^2 \leq C(\tau^6 h^{-12} + \tau^4 h^{-4}) \leq C \tau^3. \quad (5.10) \]

Based on Hölder’s inequality, Itô isometry, Lemma 1 and 2, we have

\[ \mathbb{E}\|c + d\|^2 \leq C \tau \int_0^\tau \mathbb{E}\|U(s) - U^0\|^2 ds + \int_0^\tau C \tau^2 h^{-4} ds \leq C(\tau^3 h^{-4} + \tau^3) \quad (5.11) \]

and

\[ \mathbb{E}\|e\|^2 \leq C \mathbb{E} \left[ \int_0^\tau \left( \left( I - \frac{i\tau}{2h^2} A \right)^{-1} Z (U(s) - U^0) \right)^2_H ds \right] \leq C \tau^2. \quad (5.12) \]

Rewriting \( Z(U^1 - U^0) \delta_1 \beta = G \left( \frac{i\tau}{2h^2} AU^{1/2} + i\lambda \tau F(U^{1/2}) U^{1/2} + iG U^{1/2} \right) \), which together with the Hölder’s inequality and (5.1) yields

\[ \mathbb{E}\|f\|^2 \leq C(\tau^3 h^{-4} + \tau^2). \quad (5.13) \]

We then conclude from (5.10)–(5.13) and the condition \( \tau = O(h^4) \) that

\[ \mathbb{E}\|U(\tau) - U^1\|^2 \leq C \tau^2; \quad (5.14) \]

which yields

\[ |C| = O \left( (\mathbb{E}\|U^1 - U^0\|^4)^{\frac{2}{7}} (\mathbb{E}\|U(\tau) - U^1\|^2)^{\frac{2}{7}} + \mathbb{E}\|U(\tau) - U^1\|^2 \right) \leq C \tau^2. \quad (5.15) \]
Estimation of $B$. As for $B = \mathbb{E} \left[ D^2 \varphi(U^0) \left( U^1 - U^0, a + b + c + d + e + f \right) \right]$, according to the Hölder’s inequality, (5.10) and (5.11), we have

$$|\mathbb{E} \left[ D^2 \varphi(U^0) \left( U^1 - U^0, a + b + c + d \right) \right]| \leq C \left( \mathbb{E} \| U^1 - U^0 \|^2 \right)^{\frac{1}{2}} \left( \mathbb{E} \| a + b + c + d \|^2 \right)^{\frac{1}{2}} \leq C(\tau^2 h^{-2} + \tau^2).$$

Noticing that

$$\mathbb{E} \left[ D^2 \varphi(U^0) \left( U^1 - U^0, e + f \right) \right] = \mathbb{E} \left[ D^2 \varphi(U^0) \left( U^1 - U^0, i \int_0^\tau \left( I - \frac{i\tau}{2h^2} A \right)^{-1} Z(U(s) - U^1) d\beta(s) \right) \right]$$

$$+ \frac{1}{2} \mathbb{E} \left[ D^2 \varphi(U^0) \left( U^1 - U^0, i \left( I - \frac{i\tau}{2h^2} A \right)^{-1} Z(U^1 - U^0) \delta_1 \right) \right]$$

$$- \mathbb{E} \left[ D^2 \varphi(U^0) \left( U^1 - U^0, \int_0^\tau e^{i\frac{1}{2} A(\tau - s)} \tilde{E} U(s) ds \right) \right]$$

$$= B_1 + B_2 + B_3,$$

where $|B_1| \leq C\tau^2$ according to (5.14) and Lemma 1. Furthermore,

$$B_2 = \frac{1}{2} \mathbb{E} \left[ D^2 \varphi(U^0) \left( i \frac{\tau}{h^2} AU^\frac{1}{2} + i\tau \lambda F(U^\frac{1}{2}) U^\frac{1}{2}, i \left( I - \frac{i\tau}{2h^2} A \right)^{-1} Z(U^1 - U^0) \delta_1 \right) \right]$$

$$+ \frac{1}{2} \mathbb{E} \left[ D^2 \varphi(U^0) \left( iZ \left( \frac{U^1 - U^0}{2} \right) \delta_1 \beta, i \left( I - \frac{i\tau}{2h^2} A \right)^{-1} Z(U^1 - U^0) \delta_1 \right) \right]$$

$$+ \frac{1}{2} \mathbb{E} \left[ D^2 \varphi(U^0) \left( iZ(U^0) \delta_1, i \left( I - \frac{i\tau}{2h^2} A \right)^{-1} Z(U^1 - U^0) \delta_1 \right) \right]$$

$$= B_{2,1} + B_{2,2} + B_{2,3},$$

with $|B_{2,1} + B_{2,2}| \leq C(\tau^2 h^{-2} + \tau^2)$. Replacing $U^1 - U^0$ again by (3.1), we obtain

$$|B_{2,3}| \leq \frac{1}{2} \mathbb{E} \left[ D^2 \varphi(U^0) \left( iZ(U^0) \delta_1 \beta, i \left( I - \frac{i\tau}{2h^2} A \right)^{-1} Z \left( iZ(U^\frac{1}{2}) \delta_1 \beta \right) \delta_1 \right) \right]$$

$$+ C(\tau^2 h^{-2} + \tau^2)$$

$$\leq \frac{1}{2} \mathbb{E} \left[ D^2 \varphi(U^0) \left( iZ(U^0) \delta_1 \beta, i \left( I - \frac{i\tau}{2h^2} A \right)^{-1} Z \left( iZ(U^0) \delta_1 \beta \right) \delta_1 \right) \right]$$

$$+ C(\tau^2 h^{-2} + \tau^2)$$

$$= C(\tau^2 h^{-2} + \tau^2),$$

where in the last step we used the fact $\mathbb{E}[\delta_1^3] = 0$ and $U^0$ is $\mathcal{F}_0$-adapted. Also,

$$|B_3| \leq \left| \mathbb{E} \left[ D^2 \varphi(U^0) \left( i \frac{\tau}{h^2} AU^\frac{1}{2} + i\tau \lambda F(U^\frac{1}{2}) U^\frac{1}{2}, \int_0^\tau e^{i\frac{1}{2} A(\tau - s)} \tilde{E} U(s) ds \right) \right] \right|$$

$$+ \left| \mathbb{E} \left[ D^2 \varphi(U^0) \left( iZ(U^\frac{1}{2}) \delta_1 \beta, \int_0^\tau e^{i\frac{1}{2} A(\tau - s)} \tilde{E} \left( U(s) - U^0 \right) ds \right) \right] \right|$$
\[ + \left| \mathbb{E} \left[ D^2 \varphi(U^0) \left( iZ(U^0) \delta_1 \beta, \int_0^\tau e^{i \frac{1}{h^2} A(\tau-s)} \hat{\mathcal{E}} U^0 ds \right) \right] \right| \]
\[ \leq C(\tau^2 h^{-2} + \tau^2) + \frac{1}{2} \left| \mathbb{E} \left[ D^2 \varphi(U^0) \left( iZ(U^1 - U^0) \delta_1 \beta, \int_0^\tau e^{i \frac{1}{h^2} A(\tau-s)} \hat{\mathcal{E}} U^0 ds \right) \right] \right| \]
\[ \leq C(\tau^2 h^{-2} + \tau^2), \]
so we finally obtain
\[ |\mathcal{B}| \leq C(\tau^2 h^{-2} + \tau^2) \leq C h \tau^2, \]
which, together with (5.9), (5.15), completes the proof.

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