Gradient estimates for parabolic problems with Orlicz growth and discontinuous coefficients

Jehan Oh1 | Jihoon Ok2

1Department of Mathematics, Kyungpook National University, Daegu, Republic of Korea
2Department of Mathematics, Sogang University, Seoul, Republic of Korea

Correspondence
Jihoon Ok, Department of Mathematics, Sogang University, 35 Baekbeom-ro, Mapo-gu, Seoul 04107, Republic of Korea.
Email: jihoonok@sogang.ac.kr

Communicated by: P. Colli
Funding information
National Research Foundation of Korea, Grant/Award Number: 2020R1A4A1018190; Sogang University, Grant/Award Number: 202010022.01

We obtain Calderón-Zygmund-type estimates for parabolic equations with Orlicz growth, where nonlinearities involved in the equations may be discontinuous for the space and time variables. In addition, we consider parabolic systems with the Uhlenbeck structure.

KEYWORDS
degenerate parabolic equations, general growth, discontinuous coefficients, Calderón-Zygmund estimates, Orlicz spaces

MSC CLASSIFICATION
35K55; 35B65; 46E30

1 | INTRODUCTION

We study the local regularity theory for weak solutions to the following parabolic equations with a general Orlicz growth condition:

$$\partial_t u - \text{div} A(z, Du) = -\text{div} \left( \frac{g(|F|)}{|F|} F \right) \quad \text{in} \quad \Omega_I := \Omega \times I,$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is an open set, $I \subset \mathbb{R}$ is an interval, $z = (x, t) \in \Omega \times I = \Omega_I$, $u$ is a real valued function, and $Du \in \mathbb{R}^n$ is the gradient of $u$ with respect to the space variable $x$ (i.e., $Du = D_x u$). Here, $g:[0, \infty) \rightarrow [0, \infty)$ where $g \in C([0, \infty)) \cap C^1((0, \infty))$ and $g(0) = 0$ satisfies that

$$p - 1 \leq \frac{sg'(s)}{g(s)} \leq q - 1, \quad \forall s > 0 \quad \text{for some} \quad \frac{2n}{n+2} < p \leq q,$$

and $A : \Omega_I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ where $A(z, \cdot) \in C^1(\mathbb{R}^n \setminus \{0\})$ for all $z \in \Omega_I$ satisfies that

$$|A(z, \xi)| + |\xi| |D_\xi A(z, \xi)| \leq Ag(|\xi|)$$

and

$$D_\xi A(z, \xi) \eta \cdot \eta \geq v g'(|\xi|)|\eta|^2$$
for all \( z \in \Omega_t, \xi, \eta \in \mathbb{R}^n \setminus \{0\} \) and for some \( 0 < \nu \leq \Lambda \) where \( D_t A \) is the gradient of \( A \) with respect to \( \xi \). We further define the following:

\[
\varphi(s) := \int_0^s g(\sigma) \, d\sigma, \quad s \geq 0.
\] (1.5)

The prototype of (1.1) is the following \( g \)-Laplace type equation with a coefficient:

\[
\partial_t u - \text{div} \left( a(z) \frac{g(|Du|)}{|Du|} Du \right) = -\text{div} \left( \frac{g(|F|)}{|F|} F \right) \quad \text{(i.e.,} \quad A(z, \xi) = a(z) \frac{g(|\xi|)}{|\xi|} \xi),
\] (1.6)

where \( \nu \leq a(\cdot) \leq \Lambda \). In particular, if we set \( g(s) = s^p - 1 \) and \( a(\cdot) \equiv 1 \), it becomes a parabolic \( p \)-Laplace equation. The lower bound \( \frac{2n}{n+2} \) of \( p \) in (1.2) is generally assumed even in the regularity theory for parabolic \( p \)-Laplace problems (see the literature\(^1\)). Moreover, if \( 1 < p \leq \frac{2n}{n+2} \), even the boundedness of a weak solution requires an additional integrability assumption (see DiBenedetto\(^1\)).

Under the above setting, we say function \( u \) is a local weak solution to (1.1) if \( u \in L^\infty_{\text{loc}}(I, L^2_{\text{loc}}(\Omega)) \cap L^1_{\text{loc}}(I, W^{1,1}_{\text{loc}}(\Omega)) \) with \( \varphi(|Du|) \in L^1_{\text{loc}}(I, L^1_{\text{loc}}(\Omega)) \), and it satisfies

\[
-\int_{\Omega_t} u \partial_t \zeta \, dz + \int_{\Omega_t} A(z, Du) \cdot D\zeta \, dz = \int_{\Omega_t} \frac{g(|F|)}{|F|} F \cdot D\zeta \, dz
\] (1.7)

for all \( \zeta \in C_0^\infty(\Omega_t) \). Then, the main result of this paper is to prove that the following implication holds for every weak \( \Phi \)-function \( \psi = \psi(z) \) satisfying (aInc)\(_p\) and (aDec)\(_q\), for some \( 1 < p_1 \leq q_1 \) (see Section 2.2 for the definitions of the weak \( \Phi \)-function, (aInc) and (aDec)):

\[
\varphi([F]) \in L^\psi_{\text{loc}}(\Omega_t) \quad \Rightarrow \quad \varphi([Du]) \in L^\psi_{\text{loc}}(\Omega_t).
\] (1.8)

The results is obtained by obtaining a parabolic Calderón-Zygmund-type estimate under a suitable discontinuous assumption of the nonlinearity \( A \) for the \( z \) variable.

In the classical case in which \( \varphi(s) = s^p \), \( g(s) = ps^{p-1} \), and \( \psi(s) = s^q \), where \( p > \frac{2n}{n+2} \) and \( q > 1 \), the implication of (1.8) was proved by Acerbi and Mingione.\(^2\) The proof took advantage of the higher integrability in Kinnunen and Lewis\(^4\) and the Lipshitz regularity of parabolic \( p \)-Laplace problems in DiBenedetto and Friedman,\(^1,3\) and the so-called large-\( M \)-inequality principle. We also refer to other works\(^5-7\) regarding the Hölder continuity and Harnack’s inequality, and previous studies\(^8-10\) for Calderón-Zygmund-type estimates.

After determining these results for the classical case, research has been and still is being conducted to obtain the regularity theory for parabolic problems with Orlicz growth. Lieberman first studied the systematic regularity theory for elliptic problems with Orlicz growth in Lieberman\(^11\) and later considered parabolic problems with Orlicz growth in Lieberman.\(^12\) For further regularity results for parabolic problems with Orlicz growth, we refer to the literature\(^13-19\) and related references. The implication of (1.8) was proved in Cho\(^14\) for (1.1) with the special case \( A(z, \xi) = \frac{g(|\xi|)}{|\xi|} \xi \) (i.e., (1.6) with \( a(\cdot) \equiv 1 \)), and \( p \) in (1.2) is greater than or equal to 2. Hence, in this paper, we consider general nonlinearities that depend on \( z \) and moreover may be discontinuous for \( z \). We also refer to previous works\(^20-22\) for Calderón-Zygmund-type estimates in the Orlicz setting for elliptic problems.

We introduce our main result. A parabolic cylinder \( Q_{r, \rho}(z_0) \) where \( z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R} \) is denoted as \( Q_{r, \rho}(z_0) := B_r(x_0) \times (t_0 - \rho^2, t_0 + \rho^2) \), where \( B_r(x_0) \) is the open ball in \( \mathbb{R}^n \) with the center \( x_0 \) and radius \( r > 0 \). For simplicity, we write \( Q_r(z_0) = Q_{r, \rho}(z_0) \) and \( Q_r = Q_r(z_0) \) if the center is obvious. The main regularity assumption on \( A \) is the following.

**Definition 1.9.** Let \( \delta, R > 0 \). A \( : \Omega_t \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is \((\delta, R)\)-vanishing if the following holds for all \( Q_{r, \rho} \subset \Omega_t \) with \( r, \rho \in (0, R] \):

\[
\int_{Q_{r, \rho}} \theta(A; Q_{r, \rho})(z) \, dz \leq \delta,
\]

where

\[
\theta(A; Q_{r, \rho})(z) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \left| \frac{A(z, \xi) - (A(\cdot, \xi))_{Q_{r, \rho}}}{g(|\xi|)} \right|, \quad \text{and} \quad (A(\cdot, \xi))_{Q_{r, \rho}} = \int_{Q_{r, \rho}} A(z, \xi) \, dz.
\]
We note that from (1.3), \( \theta \leq 2\Lambda \) hence, the \((\delta, R)\)-vanishing condition implies that for any \( \kappa > 1 \),
\[
\int_{Q_{r,\rho}} [\theta(A; Q_{r,\rho})(z)]^k \, dz \leq (2\Lambda)^{k-1} \delta \quad \text{for all} \quad Q_{r,\rho} \subset \Omega_t \quad \text{with} \quad r, \rho \in (0, R].
\] (1.10)

In the special case in which \( A(z, \xi) = a(x)b(\xi)A_0(\xi) \), where \( z = (x, t) \), the \((\delta, R)\)-vanishing condition can be implied by the local BMO (bounded mean oscillation) conditions of \( a(\cdot) \) and \( b(\cdot) \). Next, we state the main theorem in this paper.

**Theorem 1.11.** Let \( g : [0, \infty) \to [0, \infty) \) with \( g \in C([0, \infty)) \cap C^1((0, \infty)) \) and \( g(0) = 0 \) satisfy (1.2), and \( A : \Omega_t \times \mathbb{R}^n \to \mathbb{R}^n \) with \( A(z, \cdot) \in C^1(\mathbb{R}^n \setminus \{0\}) \) for all \( z \in \Omega_t \) do (1.3) and (1.4). In addition, let \( F \in L^\infty_{loc}(\Omega_t; \mathbb{R}^n) \) with \( \varphi \) defined in (1.5), and \( \psi : [0, \infty) \to [0, \infty) \) be a weak \( \Phi \)-function satisfying \((\text{Inc})_p\), and \((\text{Dec})_{q_i}\) for some \( 1 < p_1 \leq q_i \) with constant \( L \geq 1 \). There exists small \( \delta = \delta(n, \Lambda, p, q, p_1, q_1, L, g(1), \psi(1)) > 0 \) such that if \( A \) is \((\delta, R_0)\)-vanishing for some \( R_0 > 0 \) and \( u \) is a local weak solution to (1.1), then we have the following implication:
\[
\varphi(|F|) \in W^s_{loc}(\Omega_t) \Rightarrow \varphi(|Du|) \in W^s_{loc}(\Omega_t)
\]
with the following estimate: for any \( Q_{2R} = Q_{2R}(z_0) \subset \Omega_t \) with \( R \leq R_0 \),
\[
\int_{Q_{\rho}} \psi(|Du|) \, dz \leq c \left[ \Psi \left( \int_{Q_{\rho}} \varphi(|Du|) \, dz + \int_{Q_{\rho}} \varphi(|F|) \, dz \right) \right] \int_{Q_{2\rho}} \varphi(|Du|) \, dz + c \int_{Q_{2\rho}} \psi(|F|) \, dz
\]
for some \( c = c(n, \Lambda, p, q, p_1, q_1, L, g(1), \psi(1)) > 0 \), where \( Q_R = Q_{y}(z_0), \Psi(s) := (\psi_1 o \varphi D^{-1})(s), \varphi_1(s) := \frac{\psi(s)}{s}, \text{and} \ D^{-1} \text{is the inverse of} \)
\[
D(s) := \min\{s^2, \varphi_2(s) \frac{n+2}{n+2} s^n\} = \min\{1, \varphi_2(s) \frac{n+2}{n+2}\} s^2, \text{where} \ \varphi_2(s) := \frac{\varphi(s)}{s^2}.
\] (1.12)

**Remark 1.13.** Regarding the function \( D \), since \( \varphi \) satisfies \((\text{Inc})_p\) with \( p > \frac{2n}{n+2} \), is strictly increasing; hence, the inverse \( D^{-1} \) is well-defined. In the case of the power functions, \( g(s) = ps^{-1} \) (hence, \( \varphi(s) = s^p \)) and \( \psi(s) = s^{q_i} \), we have \( D(s) = \min\{s^2, s^{-\frac{n+2}{n+2}}\} \) and, thus,
\[
\Psi(s) = \left( \max\{s^\frac{p}{2}, s^{\frac{2p}{2n+2}}\} \right)^{q_i-1}.
\]

Therefore, our result exactly implies the known results for the \( p \)-growth case (see, e.g., Bögelein et al.\(^23\)).

**Remark 1.14.** We present examples of a function \( g \) (and, hence, \( \varphi \) defined in (1.5)) which are not just power functions. Simple examples are \( g(s) = s^{p-1} \log(1 + s^p) \) and \( g(s) = \int_0^s \min\{u^{p-2}, u^{q_i-2}\} \, du \). The following example is quite complicated. Let \( p = 2 - \frac{3}{2n} \) and \( q = 2 + \frac{1}{2n} \), and set \( \kappa = (q - p)/3 > 0 \). We define the sequence \( s_k = 2^{\frac{k}{3}} \) for \( k = 0, 1, 2, \ldots \), and the function
\[
g(s) = \begin{cases} s^{\frac{p-1+\kappa}{2}} & 0 < s < 1, \\ s^{\frac{p-1}{2}} & 1 \leq s < 2, \\ s^{\frac{2k+2}{p-1}} & 2^{k+1} s^{\frac{2}{p-1}} \leq s < 2^{k+2}, \\ s^{\frac{2k+1}{p-1}} & 2^{k+1} s^{\frac{2}{p-1}} \leq s < 2^{k+2}. \\ \end{cases}
\]

It oscillates between degenerate and singular behavior (see Baroni and Lindfors\(^13\) for more details).

Finally, we shall introduce techniques used in the proof of the theorem. Our approach is based on the so-called \textit{large-M-inequality principle} that was used in the case in which \( \varphi(s) = s^p \) and \( \psi(s) = s^{q_i} \) in Acerbi and Mingione\(^2\) (we also refer to Mingione\(^24\) for its origin). In addition, we modify this principle in the Orlicz setting. A comparison estimate is required between the main Equation (1.1) and a homogeneous equation with a nonlinearity being independent of \( \xi \). To our best knowledge, a higher integrability result is essentially needed priori to obtain such a comparison estimate for partial differential equations with BMO-type discontinuous coefficients. Recently, Hästö and the second author of this paper proved the higher integrability for parabolic problems with Orlicz growth, hence this leads us to prove the main theorem. Moreover, we also use the Lipschitz regularity of the homogenous problem that has been proved in other works.\(^13,15\)
The remaining paper is organized as follows. Section 2 introduces the notation, Orlicz functions with related inequalities and function spaces, and preliminary results. In Section 3, we derive the comparison estimates. In Section 4, we prove Theorem 1.11. Finally, we briefly discuss the implication for parabolic systems with Orlicz growth in the final section, Section 5.

2  |  PRELIMINARIES

2.1  |  Notation

Let \( w = (y, r) \in \mathbb{R}^n \times \mathbb{R} \) and \( r > 0 \). Then, \( Q_r(w) = B_r(y) \times (r - r^2, r + r^2) \) is a usual parabolic cylinder with base \( B_r(y) := \{ x \in \mathbb{R}^n : |x - y| < r \} \). We further define a so-called intrinsic parabolic cylinder with \( \lambda > 0 \) and function \( \varphi : (0, \infty) \to (0, \infty) \) by

\[
Q^\lambda_r(w) := B_r(y) \times \left( r - \frac{r^2}{\varphi_2(\lambda)}, r + \frac{r^2}{\varphi_2(\lambda)} \right), \quad \text{where} \quad \varphi_2(s) := \frac{\varphi(s)}{s^2}.
\]

A function \( f : I \to \mathbb{R} \) with \( I \subset \mathbb{R} \) is almost increasing if, for some \( L \geq 1 \), \( f(t) \leq Lf(s) \) for every \( t, s \in I \) with \( t < s \). If we set \( L = 1 \), then we say that \( f \) is increasing. Similarly, we define an almost decreasing or decreasing function.

For an integrable function \( f : U \to \mathbb{R}^m \), \( U \subset \mathbb{R}^N \), we define \( (f)_U := f_{U, f}dz := \frac{1}{|U|} \int_U f dz \), where \( |U| \) is the Lebesgue measure of \( U \) in \( \mathbb{R}^N \).

We write \( f \lesssim g \) if \( f \leq cg \) for some \( c > 0 \), and \( f \approx g \) if \( c^{-1}g \leq f \leq cg \) for some \( c \geq 1 \).

2.2  |  Orlicz functions

Let \( \varphi : (0, \infty) \to [0, \infty) \) and \( p, q > 0 \). We introduce the following conditions:

(aInc)\(_p\) The map \( (0, \infty) \ni s \mapsto \varphi(s)/s^p \) is almost increasing with constant \( L \geq 1 \).

(aDec)\(_q\) The map \( (0, \infty) \ni s \mapsto \varphi(s)/s^q \) is almost decreasing with constant \( L \geq 1 \).

In particular, when \( L = 1 \) we use (Inc) and (Dec), instead of (aInc) and (aDec), respectively.

From the above definition, we directly deduce that if \( \varphi \) satisfies (aInc)\(_p\) and (aDec)\(_q\) for some \( p \leq q \) then

\[
c^dL^{-1}\varphi(s) \leq c^dLq(s) \leq \varphi(Cs) \leq C^dLq(s)
\]

for every \( s \in (0, \infty) \) and every \( 0 < c < 1 < C \). Moreover, if \( p_1 < p_2 \), or \( q_1 < q_2 \), then (aInc)\(_{p_2}\) implies (aInc)\(_{p_1}\), or (aDec)\(_{q_1}\) does (aDec)\(_{q_2}\). We use these properties many times later without explicit mention.

Next, we introduce the definition of a weak \( \Phi \)-function referring to Harjulehto and Hästö, 25, Definition 2.1.3

**Definition 2.1.** The function \( \varphi : [0, \infty) \to [0, \infty) \) is a weak \( \Phi \)-function if it is increasing with \( \varphi(0) = 0 \), \( \lim_{s \to 0^+} \varphi(s) = \lim_{s \to \infty} \varphi(s) = \infty \) and it satisfies (aInc)\(_1\). Moreover, if a weak \( \Phi \)-function is left-continuous and convex, then, it is a convex \( \Phi \)-function.

If \( \varphi \) is a weak \( \Phi \)-function or a convex \( \Phi \)-function, we write \( \varphi \in \Phi_\infty \) or \( \varphi \in \Phi_\infty \), respectively. We notice that \( \sqrt{\varphi(s^2)} \) need not be convex even if \( \varphi \) is convex, but the (aInc)\(_1\) property is conserved. For a weak \( \Phi \)-function, we define its conjugate function as follows:

\[
\varphi^*(r) := \sup_{s \in [0, \infty)} (rs - \varphi(s)), \quad r \in [0, \infty).
\]

The definition directly implies Young’s inequality:

\[
rs \leq \varphi(s) + \varphi^*(r), \quad r, s \in [0, \infty).
\]

We next introduce the properties of regular \( \Phi \)-functions.

**Proposition 2.3.** Let \( \gamma > 0 \) and \( 0 < p \leq q \). Suppose that \( \varphi \in \Phi_\infty \cap C^\gamma([0, \infty)) \).

(1) If \( \varphi' \) satisfies (aInc)\(_\gamma\), (aDec)\(_\gamma\), (Inc)\(_\gamma\), or (Dec)\(_\gamma\), then \( \varphi \) satisfies (aInc)\(_{\gamma+1}\), (aDec)\(_{\gamma+1}\), (Inc)\(_{\gamma+1}\), or (Dec)\(_{\gamma+1}\), respectively, with the same constant \( L \geq 1 \).
(2) If \( \varphi' \) satisfies \((a\text{Dec})\), with constant \( L \geq 1 \), then
\[
\varphi(s) \leq s \varphi'(s) \leq 2^{r+1} L \varphi(s) \text{ for all } s \in [0, \infty).
\]

(3) \( \varphi(s) \) satisfies \((Inc)_p\) and \((Dec)_q\) if and only if
\[
p \varphi(s) \leq s \varphi'(s) \leq q \varphi(s) \text{ for all } s \in [0, \infty). \tag{2.4}
\]

(4) \( \varphi^*(\varphi'(s)) \leq s \varphi'(s) \).

**Proof.** We only prove the statement in (3). The rest of the properties can be found in Hästö.\(^{26}\) Proposition 3.6

We first suppose that \( \varphi \) satisfies \((Inc)_p\) and \((Dec)_q\). Then, for \( s, h > 0 \) we have the following:
\[
\frac{(h+s)^p - 1}{h} \varphi(s) \leq \frac{\varphi(s+h) - \varphi(s)}{h} \leq \frac{(h+s)^q - 1}{h} \varphi(s).
\]

Passing \( h \to 0 \), we have the inequalities in (2.4). For the converse, from (2.4), \( \frac{d}{ds} \left( \frac{\varphi(s)}{s^p} \right) > 0 \) and \( \frac{d}{ds} \left( \frac{\varphi(s)}{s^q} \right) < 0 \), which imply \((Inc)_p\) and \((Dec)_q\), respectively. \( \square \)

From the above proposition, if \( g \) satisfies (1.2), then \( \varphi \) defined in (1.5) satisfies \((Inc)_p\) and \((Dec)_q\). From (2.2) and Proposition 2.3, for any \( \epsilon \in (0, 1) \),
\[
rs \lesssim \begin{cases} 
\epsilon \varphi(s) + c(\epsilon) \varphi^*(r), & r, s \in [0, \infty), \\
\epsilon \varphi^*(s) + c(\epsilon) \varphi(r), & r, s \in [0, \infty), 
\end{cases}
\] \( \tag{2.5} \)

for some constant \( c(\epsilon) = c(p, q, \epsilon) > 1 \).

For \( \varphi \in \Phi_w \) satisfying \((a\text{Inc})_p\) and \((a\text{Dec})_q\) for some \( 1 < p \leq q \) and \( \Omega \subset \mathbb{R}^n \), we define the Orlicz space as follows:
\[
L^\varphi(\Omega) := \left\{ f \in L^0(\Omega) : \int_\Omega \varphi(|f(x)|) \, dx < \infty \right\},
\]

where \( L^0(\Omega) \) is the set of measurable functions in \( \Omega \). For more properties for \( \Phi \)-functions and Orlicz spaces, we refer to Harjulehto and Hästö.\(^{25}\)

Next, we introduce a vector field \( V : \mathbb{R}^n \to \mathbb{R}^n \), which is defined by
\[
V(\xi) := \sqrt{g(|\xi|)} \frac{\xi}{|\xi|},
\]

where \( g \in C([0, \infty)) \cap C^1((0, \infty)) \) satisfies (1.2). For simplicity we assume that \( g(1) = 1 \). Then, we obtain the following relation between the above vector field and the function \( \varphi \) defined by (1.5).

**Lemma 2.6.** In the above setting, we have\(^{27}\)
\[
g'(|\xi_1| + |\xi_2|)|\xi_1 - \xi_2|^2 \approx |V(\xi_1) - V(\xi_2)|^2 \tag{2.7}
\]

and
\[
\varphi(|\xi_1 - \xi_2|) \lesssim \kappa |V(\xi_1) - V(\xi_2)|^2 + c(\kappa) \varphi(|\xi_2|) \tag{2.8}
\]

for all \( \xi_1, \xi_2 \in \mathbb{R}^n \) and \( \kappa > 0 \), where the hidden constants depend only on \( p, q \) and \( n \). In addition,
\[
|V(\xi)|^2 \approx \varphi(|\xi|) \tag{2.9}
\]

holds for all \( \xi \in \mathbb{R}^n \).
Let the vector field $A(z, \xi)$ satisfy (1.3) and (1.4) with (1.2). Then, from (1.4), we have the following monotonicity property of the vector field $A(z, \xi)$ with respect to the gradient variable $\xi$:

$$\langle [V(\xi_1) - V(\xi_2)]^2, g(\xi_1) + |\xi_1| \xi_1 - \xi_2 \rangle \lesssim (A(z, \xi_1) - A(z, \xi_2)) \cdot (\xi_1 - \xi_2)$$

(2.10)

for all $z \in \Omega$, and $\xi_1, \xi_2 \in \mathbb{R}^n$, where the hidden constants depend on $p$, $q$, $n$, and $v$.

### 2.3 Preliminary regularity results

In this subsection we introduce regularity results for homogeneous parabolic problems with Orlicz growth. The first result is the higher integrability of a weak solution.

**Theorem 2.11** (Higher integrability,16, Theorem 1.5). Let $\varphi : [0, \infty) \to [0, \infty)$ be a weak $\Phi$-function satisfying $(aInc)_p$ and $(aDec)_q$ with constant $L \geq 1$ and let $w$ be a weak solution to

$$w_t - \text{div} A(z, Dw) = 0 \text{ in } Q_r,$$

where $A : Q_r \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies that

$$|A(z, \xi)| \leq \Lambda \varphi(|\xi|) \text{ and } A(z, \xi) \cdot \xi \geq v \varphi(|\xi|)$$

for every $z \in Q_r$, $\xi \in \mathbb{R}^n \setminus \{0\}$ and for some $0 < \nu \leq \Lambda$. There exists $\sigma = \sigma(n, \nu, \Lambda, p, q, L, \varphi(1)) > 0$ such that $\varphi(|Dw|) \in L^{1+\sigma}(Q_r)$ with the following estimate:

$$\int_{Q_r} \varphi(|Dw|)^{1+\sigma} \, dz \leq c \left[ \left( \int_{Q_r} \varphi(|Dw|) \, dz \right)^\sigma \right] \left( \int_{Q_r} \varphi(|Dw|) \, dz \right)$$

for some $c = c(n, \nu, \Lambda, p, q, L, \varphi(1)) > 0$, where $D$ is given in (1.12) and $D^{-1}$ is the inverse of $D$.

The next result is the Lipschitz regularity for homogeneous equations with nonlinearity independent of $z$. For Cauchy–Dirichlet problems, we refer to Baroni and Lindfors.13, Theorem 1.2. Using this result with mollification, we have the following interior Lipschitz regularity for general problems.

**Theorem 2.12** (Lipschitz regularity). Let $g : [0, \infty) \to [0, \infty)$ with $g \in C([0, \infty)) \cap C^1([0, \infty))$ and $g(0) = 0$ satisfy (1.2), and $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ with $A_0 \in C^1(\mathbb{R}^n \setminus \{0\})$ do (1.3) and (1.4), replacing $A(z, \xi)$ with $A_0(\xi)$. If $v$ is a weak solution to

$$v_t - \text{div} A_0(Dv) = 0 \text{ in } Q_r,$$

then $v$ belongs to $L^\infty(Q_r)$ with the following estimate:

$$\|Dv\|_{L^\infty(Q_r)} \leq c \left( \int_{Q_r} [\varphi(|Dv|) + 1] \, dz \right)^{\max\left\{ \frac{1}{2}, \frac{\sigma}{p+2(q-2)} \right\}}$$

for some $c = c(n, \nu, \Lambda, p, q, g(1)) > 0$.

**Proof.** Let $0 < \epsilon < \min\{r^2, r\}$, $\eta_\epsilon$ be a standard mollifier, and $v_\epsilon = v \ast \eta_\epsilon$. Then we know that, up to a subsequence,

$$\begin{cases}
Dv_\epsilon & \to Dv \text{ a.e. in } Q_r \\
\varphi(|Dv_\epsilon|) & \to \varphi(|Dv|) \text{ in } L^1(Q_r) \text{ as } \epsilon \to 0.
\end{cases}$$

Hence, we have the following:

$$\int_{Q_r} \varphi(|Dv_\epsilon|) \, dx \leq 2 \int_{Q_r} \varphi(|Dv|) \, dx \text{ for ant sufficiently small } \epsilon > 0.$$
We next consider the weak solution \( h_\varepsilon \) to

\[
\begin{cases}
(h_\varepsilon)_t - \text{div} A_0(Dh_\varepsilon) = 0 \text{ in } Q_{2r}, \\
h_\varepsilon = v_\varepsilon \text{ on } \partial_0 Q_{2r}.
\end{cases}
\]

Then, by Baroni and Lindfors,\(^{13}\) Theorem 1.2 we have the following:

\[
\|Dh_\varepsilon\|_{L^\infty(Q_{1})} \leq c \left( \int_{Q_{2r}} \varphi(|Dh_\varepsilon|) + 1 \right) \frac{\max\{\frac{\varepsilon}{r}, \frac{\varepsilon^2}{r^2} \}}{\varepsilon - \varepsilon_0},
\]

where the constant \( c > 0 \) depends only on \( n, \nu, \Lambda, p, q, g(1) \) and is independent of \( \varepsilon \) and the boundary datum \( v_\varepsilon \). Next, we prove that \( Dh_\varepsilon \to Dv \) in \( L^q(Q_{2r}, \mathbb{R}^n) \) as \( \varepsilon \to 0 \), up to a subsequence. Then the proof is completed.

First, \( v_\varepsilon \) becomes a weak solution to the following parabolic problem:

\[
(v_\varepsilon)_t - \text{div}([A_0(Dv)]_\varepsilon) = 0 \text{ in } Q_{2r},
\]

where \([A_0(Dv)]_\varepsilon = A_0(Dv) + \eta_\varepsilon\). Then, from the previous two equations,

\[
\int_{Q_{2r}} (v_\varepsilon - h_\varepsilon)(v_\varepsilon - h_\varepsilon) \, dz + \int_{Q_{2r}} ([A_0(Dv)]_\varepsilon - A_0(Dh_\varepsilon)) \cdot (Dv_\varepsilon - Dh_\varepsilon) \, dz = 0
\]

(cf. Remark 3.4). The first integral is

\[
\int_{Q_{2r}} (v_\varepsilon - h_\varepsilon)(v_\varepsilon - h_\varepsilon) \, dz = \int_{Q_{2r}} \frac{1}{2} \frac{d}{dt} (v_\varepsilon - h_\varepsilon)^2 \, dz = \frac{1}{2} \int_{B_{2r}} (v_\varepsilon - h_\varepsilon)^2 \mid t=\varepsilon + (2r)^2 \, dx \geq 0.
\]

Hence, from (2.10) and (2.5),

\[
\int_{Q_{2r}} |V(Dv_\varepsilon) - V(Dh_\varepsilon)|^2 \, dz \leq c \int_{Q_{2r}} (A_0(Dv_\varepsilon) - A_0(Dh_\varepsilon)) \cdot (Dv_\varepsilon - Dh_\varepsilon) \, dz
\]

\[
\leq c \int_{Q_{2r}} (A_0(Dv_\varepsilon) - [A_0(Dv)]_\varepsilon) \cdot (Dv_\varepsilon - Dh_\varepsilon) \, dz
\]

\[
\leq c(\kappa_1) \int_{Q_{2r}} \varphi^*(|A_0(Dv_\varepsilon) - [A_0(Dv)]_\varepsilon|) \, dz + \kappa_1 \int_{Q_{2r}} \varphi(|Dv_\varepsilon - Dh_\varepsilon|) \, dz
\]

\[
\leq c(\kappa_1) \int_{Q_{2r}} \varphi^*(|A_0(Dv_\varepsilon) - A_0(Dv)|) \, dz + c(\kappa_1) \int_{Q_{2r}} \varphi^*(|[A_0(Dv)]_\varepsilon - A_0(Dv)|) \, dz
\]

\[
+ \kappa_1 \int_{Q_{2r}} [\varphi(|Dv_\varepsilon|) + \varphi(|Dh_\varepsilon|)] \, dz,
\]

where \( \kappa_1 \in (0, 1) \). From the preceding estimate, using (2.9), (1.4) and Proposition 2.3 (4), we first observe that

\[
\int_{Q_{2r}} \varphi(|Dh_\varepsilon|) \, dz \leq c \int_{Q_{2r}} \varphi(|Dv_\varepsilon|) \, dz \leq c \int_{Q_{2r}} \varphi(|Dv|) \, dz
\]

for any sufficiently small \( \varepsilon > 0 \). We next prove that the first integral on the right hand side in (2.13) approaches 0 as \( \varepsilon \to 0 \), up to a subsequence. Because \( \varphi^*([A_0(Dv_\varepsilon) - A_0(Dv)]_\varepsilon) \leq c \varphi(|Dv_\varepsilon|) \) and \( \varphi(|Dv_\varepsilon|) \to \varphi(|Dv|) \) in \( L^1(Q_{2r}) \), by a modified Lebesgue dominated convergence theorem, \( \varphi^*([A_0(Dv_\varepsilon) - A_0(Dv)]_\varepsilon) \to \varphi^*([A_0(Dv)]_\varepsilon) \) in \( L^1(Q_{2r}) \). Moreover, \( A_0(Dv_\varepsilon) \to A_0(Dv) \) a.e. in \( Q_{2r} \); thus, \( A_0(Dv_\varepsilon) \to A_0(Dv) \) in \( L^q(Q_{2r}, \mathbb{R}^n) \), up to a subsequence. Therefore, because \([A_0(Dv)]_\varepsilon \to A_0(Dv) \) in \( L^q(Q_{2r}, \mathbb{R}^n) \) and \( \kappa_1 \in (0, 1) \) is arbitrary,

\[
\lim_{\varepsilon \to 0} \int_{Q_{2r}} |V(Dv_\varepsilon) - V(Dh_\varepsilon)|^2 \, dz = 0.
\]
Finally, applying (2.8), we prove the following:

\[
\lim_{\epsilon \to 0} \int_{Q_\epsilon} \varphi(|Dh_\epsilon - Dv|) \, dz \leq \lim_{\epsilon \to 0} \int_{Q_\epsilon} \varphi(|Dh_\epsilon - Dv|) \, dz + \lim_{\epsilon \to 0} \int_{Q_\epsilon} \varphi(|Dv - Dv|) \, dz = 0.
\]

\[\square\]

### 3 COMPARISON

In this section, we derive comparison estimates. Assuming \(Q_4 = Q_4(0) \in \Omega_t\), we consider the following homogeneous problem:

\[
\begin{aligned}
\partial_t w - \text{div}A(z, Dw) &= 0 \quad \text{in} \quad Q_4, \\
\partial_p w &= u \quad \text{on} \quad \partial_p Q_4,
\end{aligned}
\]

where \(\partial_p Q_4\) is the parabolic boundary of \(Q_4\) and \(u\) solves Equation (1.1). Then, we obtain the following comparison estimate.

**Lemma 3.2** (First comparison estimate). Let \(w\) be the weak solution to (3.1). Then for any \(\epsilon > 0\), there exists a small \(\delta = \delta(n, \nu, \Lambda, p, q, \epsilon) > 0\) such that if

\[
\int_{Q_4} \left[ \varphi(|Du|) + \frac{1}{\delta} \varphi(|F|) \right] \, dz \leq 1
\]

then

\[
\int_{Q_4} |V(Du) - V(Dw)|^2 \, dz \leq \epsilon.
\]

**Remark 3.4.** We will use test functions involving the weak solution \(u\) in the weak formulation (1.7). However, the weak solution \(u\) to the parabolic equation in (1.1) may not be differentiable in the time variable \(t\). It is standard to consider Steklov averages to overcome this issue (see DiBenedetto\(^1\) and Baroni and Lindfors\(^13\)). This argument is now well-known in the field of parabolic PDEs; therefore, we abuse the notation \(\partial_t u\) without further explanation.

**Proof of Lemma 3.2.** We take \(\zeta = u - w\) as a test function in (1.1) and (3.1) using Steklov averages to obtain

\[
\int_{Q_4} (A(z, Du) - A(z, Dw)) \cdot (Du - Dw) \, dz \leq \int_{Q_4} \frac{g(|F|)}{|F|} F \cdot (Du - Dw) \, dz.
\]

Using (2.10) and Young’s inequality (2.5) with Proposition 2.3(4), we obtain the following:

\[
\int_{Q_4} |V(Du) - V(Dw)|^2 \, dz \\
\leq \int_{Q_4} g(|F|)(|Du| + |Dw|) \, dz \\
\leq \tau \int_{Q_4} \varphi(|Du|) \, dz + \tau \int_{Q_4} \varphi(|Dw|) \, dz + c(\tau) \int_{Q_4} \varphi(|F|) \, dz
\]

for any \(\tau \in (0, 1)\).
From (2.9), (2.10), (1.3), (3.5), Lemma 2.6 and Young's inequalities (2.5) with Proposition 2.3 (4),

\[
\int_{Q_4} \phi(|Dw|) \, dz \leq c \int_{Q_4} |V(Dw)|^2 \, dz \\
\leq c \int_{Q_4} A(z, Dw) \cdot Dw \, dz \\
\leq c \int_{Q_4} g(|Dw|)|Du| \, dz + \int_{Q_4} g(|Du|)(|Du| + |Dw|) \, dz \\
+ \int_{Q_4} g(|F|)(|Du| + |Dw|) \, dz \\
\leq \frac{1}{2} \int_{Q_4} \phi(|Dw|) \, dz + c \int_{Q_4} \phi(|Du|) \, dz + c \int_{Q_4} \phi(|F|) \, dz.
\]

Hence,

\[
\int_{Q_4} \phi(|Dw|) \, dz \leq c \int_{Q_4} \phi(|Du|) \, dz + c \int_{Q_4} \phi(|F|) \, dz.
\] (3.7)

Combining (3.6) and (3.7) yields

\[
\int_{Q_4} |V(Du) - V(Dw)|^2 \, dz \leq (c + 1)\tau \int_{Q_4} \phi(|Du|) \, dz + c(\tau) \int_{Q_4} \phi(|F|) \, dz \\
\leq (c + 1)\tau + c(\tau)\delta,
\]

where we used the assumption in (3.3). By setting \( \tau \in (0, 1) \) and \( \delta \in (0, 1) \) properly, we conclude that

\[
\int_{Q_4} |V(Du) - V(Dw)|^2 \, dz \leq \epsilon.
\]

We next consider the vector field \( A_0 : \mathbb{R}^n \to \mathbb{R}^n \), which is defined by

\[
A_0(\xi) := \int_{Q_3} A(z, \xi) \, dz = \frac{1}{|Q_3|} \int_{Q_3} A(z, \xi) \, dz,
\]

and the following reference problem:

\[
\begin{cases}
\partial_t v - \text{div} A_0(Dv) = 0 & \text{in } Q_3, \\
v = w & \text{on } \partial_0 Q_3,
\end{cases}
\] (3.8)

where \( w \) solves the problem in (3.1) and \( Q_3 = Q_3(0) \). Additionally, \( A_0 \) satisfies (1.3) and (1.4), replacing \( A(z, \xi) \) with \( A_0(\xi) \).

**Lemma 3.9** (Second comparison estimate). Let \( v \) be the weak solution to (3.8). Then for any \( \epsilon > 0 \), there exists a small \( \delta = \delta(n, \nu, \Lambda, p, q, \epsilon) > 0 \) such that if (3.3) holds and \( A \) is \((\delta, 3)\)-vanishing, then

\[
\int_{Q_3} |V(Dw) - V(Dv)|^2 \, dz \leq \epsilon
\]

and

\[
||Dv||_{L^{\infty}(Q_1)} \leq N_0
\] (3.10)

for some \( N_0 = N_0(n, \nu, \Lambda, p, q) \geq 1 \), where \( Q_1 = Q_1(0) \).

**Proof.** We test (3.1) and (3.8) with \( \zeta = w - v \) using Steklov averages (cf. Remark 3.4) to observe that

\[
\int_{Q_3} (A(z, Dw) - A_0(Dv)) \cdot (Dw - Dv) \, dz \leq 0.
\]
It follows that
\[
\int_{Q_3} (A_0(Dw) - A_0(Dv)) \cdot (Dw - Dv) \, dz \leq - \int_{Q_3} (A(z, Dw) - A_0(Dw)) \cdot (Dw - Dv) \, dz.
\]

As \( A_0(\cdot) \) is the mean of \( A(z, \cdot) \) over \( Q_3 \) with respect to the \( z \) variable, (2.10) holds, where \( A(z, \cdot) \) is replaced with \( A_0(\cdot) \). Using this finding and Definition 1.9, we obtain the following:

\[
\begin{align*}
\int_{Q_3} |V(Dw) - V(Dv)|^2 \, dz &\leq \int_{Q_3} |A(z, Dw) - A_0(Dw)| |Dw - Dv| \, dz \\
&\leq \int_{Q_3} \theta(A; Q_3)(z) g(|Dw|) |Dw - Dv| \, dz \\
&\leq \int_{Q_3} \theta(A; Q_3)(z) g(|Dw|) |Dw| \, dz + \int_{Q_3} \theta(A; Q_3)(z) g(|Dw|) |Dv| \, dz \\
&= I + II.
\end{align*}
\]

To estimate \( I \), we use Proposition 2.3, Hölder’s inequality and the higher integrability (Theorem 2.11) for \( Dw \):

\[
I = \int_{Q_3} \theta(A; Q_3)(z) g(|Dw|) |Dw| \, dz \\
\leq c \int_{Q_3} \theta(A; Q_3)(z) \varphi(|Dw|) \, dz \\
\leq c \left( \int_{Q_3} \theta^{\frac{1}{1+\sigma}} \, dz \right)^{\frac{\sigma}{1+\sigma}} \left( \int_{Q_3} \varphi(|Dw|)^{1+\sigma} \, dz \right)^{\frac{1}{1+\sigma}} \\
\leq c \left( \int_{Q_3} \theta^{\frac{1}{1+\sigma}} \, dz \right)^{\frac{\sigma}{1+\sigma}} \left( \varphi(D^{-1})(\int_{Q_3} \varphi(|Dw|) \, dz) \right)^{\frac{1}{\sigma}} \left( \int_{Q_3} \varphi(|Dw|) \, dz \right)^{\frac{1}{1+\sigma}}.
\]

It follows from the \( (\delta, 2) \)-vanishing condition of \( A \) with (1.10), (3.7) and (3.3) that

\[
I \leq c \delta^{\frac{\sigma}{1+\theta}} \left( \varphi(D^{-1})(c) \right)^{\frac{\sigma}{1+\theta}} c \frac{1}{1+\theta} \leq c \delta^{\frac{\sigma}{1+\theta}}.
\]

We also use Young’s inequality, properties of \( \varphi \) and \( \varphi^* \), Hölder’s inequality and the higher integrability for \( Dw \) to estimate \( II \) as follows:

\[
II = \int_{Q_3} \theta(A; Q_3)(z) g(|Dw|) |Dv| \, dz \\
\leq c(\tau) \int_{Q_3} \varphi^* \left( \theta g(|Dw|) \right) \, dz + \tau \int_{Q_3} \varphi(|Dv|) \, dz \\
\leq c(\tau) \max \left\{ \int_{Q_3} \theta^{\frac{1}{1+\sigma}} \varphi^* \left( g(|Dw|) \right) \, dz + \tau \int_{Q_3} \varphi(|Dv|) \, dz \\
\leq c(\tau) \left( \int_{Q_3} \theta^{\frac{1}{1+\sigma}} \, dz \right)^{\frac{\sigma}{1+\sigma}} \left( \int_{Q_3} \varphi(|Dw|)^{1+\sigma} \, dz \right)^{\frac{1}{1+\sigma}} \\
+ c(\tau) \left( \int_{Q_3} \theta^{\frac{1}{1+\sigma}} \, dz \right)^{\frac{\sigma}{1+\sigma}} \left( \int_{Q_3} \varphi(|Dw|)^{1+\sigma} \, dz \right)^{\frac{1}{1+\sigma}} + \tau \int_{Q_3} \varphi(|Dv|) \, dz.
\]
for any $\tau \in (0, 1)$. By similar argument, we obtain

$$II \leq c(\tau) \delta^{\frac{\nu}{\nu+\sigma}} + \tau \int_{Q_1} \varphi(|Dv|) \, dz.$$  

As in the proof of Lemma 3.2, we deduce that

$$\int_{Q_1} \varphi(|Dv|) \, dz \leq c \int_{Q_1} \varphi(|Dw|) \, dz \leq c. \quad (3.13)$$

Therefore, we observe that

$$II \leq c(\tau) \delta^{\frac{\nu}{\nu+\sigma}} + \tau \quad (3.14)$$

for any $\tau \in (0, 1)$.

Combining (3.11) with (3.12) and (3.14) results in the following:

$$\int_{Q_1} |V(Dw) - V(Dv)|^2 \, dz \leq c \delta \varpi \delta^{\frac{\nu}{\nu+\sigma}} + \tau.$$

By setting $\tau \in (0, 1)$ and $\delta \in (0, 1)$ properly, we conclude that

$$\int_{Q_1} |V(Dw) - V(Dv)|^2 \, dz \leq c.$$

Furthermore, the estimate in (3.10) follows directly from Theorem 2.12 and (3.13).

4 | PROOF OF THEOREM 1.11

We prove the main theorem starting with the following technical lemma.28

**Lemma 4.1.** Let $Y : [R_1, R_2] \to [0, \infty)$ be a bounded function. Suppose that for any $s_1$ and $s_2$, where $0 < R_1 \leq s_1 < s_2 \leq R_2$,

$$Y(s_1) \leq \theta Y(s_2) + \frac{P}{(s_2 - s_1)^\kappa} + Q$$

with $P, Q \geq 0$, $\kappa > 0$ and $\theta \in [0, 1)$. Then there exists $c = (\theta, \kappa) > 0$ such that

$$Y(R_1) \leq c(\theta, \kappa) \left[ \frac{P}{(R_2 - R_1)^\kappa} + Q \right].$$

**Proof of Theorem 1.11.** First, there exists $\tilde{\psi} \in \Phi_c \cap C^1([0, \infty))$ satisfying (Inc)$_p$, and (aDec)$_q$, such that $\psi \approx \tilde{\psi}$ (see H"ast"o and Ok16, Remark 2.6). Therefore, we assume that $\psi \in \Phi_c \cap C^1([0, \infty))$ without loss of generality. Moreover, we also assume that $g(1) = \psi(1) = 1$.

We let $Q_{2R} = Q_{2R}(z_0) \in \Omega_I$ with $R \leq R_0$ be fixed, where $R_0 > 0$ derives from the $(\delta, R_0)$-vanishing condition of $A$. For simplicity, we write $Q_\rho = Q_\rho(z_0), \rho \in (0, 2R]$. In addition, for $\rho > 0$ and $\lambda > 0$, we define

$$E(\rho, \lambda) := \{ z \in Q_\rho : |Du(z)| > \lambda \}.$$

Also, we define the following:

$$\lambda_0 := D^{-1} \left( \int_{Q_\lambda} \left[ \varphi(|Du|) + \frac{1}{\delta} \varphi(|F|) \right] \, dz \right), \quad (4.2)$$

where $\delta \in (0, 1)$, depending only on $n$, $\nu$, $\Lambda_p$, $p$, $q_1$, $q_1$, and $L$, is determined later (see below from (4.12)), and $D$ is defined by (1.12). From (Inc)$_p$ of $\varphi$ with $p > \frac{2n}{n+2}$,

$$C^\min \left\{ \frac{1}{2} \left( \frac{p(n+2)-2m}{2} \right) \right\} = \min \left\{ C^2, \frac{C^{p(n+2)-2m}}{2} \right\} \, D(s) \leq D(Cs) \quad (4.3)$$
for all $s > 0$ and $C \geq 1$.

We let $R \leq r_1 < r_2 \leq 2R$ and consider any $\lambda$ satisfying the following:

$$\lambda \geq \lambda_1 := \left(\frac{64R}{r_2 - r_1}\right)^{(n+2)} \max\left\{\frac{1}{2}, \frac{2}{\rho_\lambda^{n+2}}\right\} \lambda_0. \quad (4.4)$$

With this $\lambda$, we also define

$$\rho_\lambda := \min\{1, \varphi_2(\lambda)^{1/2}\}(r_2 - r_1) \leq R. \quad (4.5)$$

We notice that $Q^1_\rho(z) \subset Q_{r_1} \subset Q_{2R}$ for any $z \in E(r_1, \lambda)$ and $\rho \leq \rho_\lambda$, where $Q^1_\rho(z)$ is an intrinsic parabolic cylinder defined in Section 2.1. Then, we prove a Vitali type covering of the super-level set $E(r_1, \lambda)$ satisfying a balancing condition on each set.

**Lemma 4.6.** For each $R \leq r_1 < r_2 \leq 2R$ and $\lambda \geq \lambda_1$, there exist $w_i \in E(r_1, \lambda)$ and $\rho_i \in \left(0, \frac{C}{32}\right)$, $i = 1, 2, 3, \ldots$, such that $Q^1_{\rho_i}(w_i)$ are mutually disjoint,

$$E(r_1, \lambda) \setminus N \subset \bigcup_{i=1}^{\infty} Q^1_{\rho_i}(w_i)$$

for some Lebesgue measure zero set $N$.

$$\int_{Q^1_{\rho_i}(w_i)} \varphi(|Du|) + \frac{1}{\delta} \varphi(|F|) \, dz = \varphi(\lambda),$$

and

$$\int_{Q^1_{\rho_i}(w_i)} \varphi(|Du|) + \frac{1}{\delta} \varphi(|F|) \, dz < \varphi(\lambda) \text{ for all } \rho \in (\rho_i, \rho_{i+1}).$$

**Proof.** The proof is exactly the same as that in Hästö and Ok. However, for reader’s convenience, we provide the proof. For $w \in E(r_1, \lambda)$ and $\rho \in \left[\frac{C}{32}, \rho_\lambda\right)$, using (4.2), we obtain

$$\int_{Q^1_\rho(w)} \left[\varphi(|Du|) + \frac{1}{\delta} \varphi(|F|)\right] \, dz \leq \frac{|Q_{2\rho}|}{|Q^1_\rho(w)|} \int_{Q_{2\rho}} \left[\varphi(|Du|) + \frac{1}{\delta} \varphi(|F|)\right] \, dz$$

$$\leq \frac{|Q_{2\rho}|}{|Q^1_{\rho/32}(w)|} \varphi_2(\lambda) D(\lambda_0).$$

It follows from (4.5), (4.3), (4.4), and (1.12) that

$$|Q_{2\rho}| \varphi_2(\lambda) D(\lambda_0) \leq \left(\frac{64R}{r_2 - r_1}\right)^{n+2} \max\left\{1, \varphi_2(\lambda)^{-\frac{n+2}{2}}\right\} \varphi_2(\lambda) D(\lambda_0)$$

$$\leq D(\lambda) \max\left\{1, \varphi_2(\lambda)^{-\frac{n+2}{2}}\right\} \varphi_2(\lambda)$$

$$= \min\left\{1, \varphi_2(\lambda)^{-\frac{n+2}{2}}\right\} \lambda^2 \max\left\{1, \varphi_2(\lambda)^{-\frac{n+2}{2}}\right\} \varphi_2(\lambda) = \varphi(\lambda).$$

Therefore, we obtain

$$\int_{Q^1_\rho(w)} \left[\varphi(|Du|) + \frac{1}{\delta} \varphi(|F|)\right] \, dz \leq \varphi(\lambda) \text{ for all } \rho \in \left[\rho_i, \rho_{i+1}\right).$$

Moreover, from the parabolic Lebesgue differentiation theorem, we deduce that, for almost every $w \in E(r_1, \lambda)$,

$$\lim_{\rho \to 0^+} \int_{Q^1_\rho(w)} \left[\varphi(|Du|) + \frac{1}{\delta} \varphi(|F|)\right] \, dz \geq \varphi(|Du(w)|) > \varphi(\lambda).$$
As the map \( \rho \mapsto \int_{Q_{\rho}^j(w)} \left[ \varphi(|Du|) + \frac{1}{\delta} \varphi(|F|) \right] \, dz \) is continuous, there exists \( \rho_w \in \left( 0, \frac{\rho_i}{32} \right) \) such that

\[
\int_{Q_{\rho_w}^j(w)} \left[ \varphi(|Du|) + \frac{1}{\delta} \varphi(|F|) \right] \, dz = \varphi(\lambda)
\]

and

\[
\int_{Q_{\rho_w}^j(w)} \left[ \varphi(|Du|) + \frac{1}{\delta} \varphi(|F|) \right] \, dz < \varphi(\lambda) \quad \text{for all } \rho \in (\rho_w, \rho_i].
\]

Applying Vitali’s covering lemma for \( \{Q_{\rho_w}^j(w)\} \), we obtain the desired conclusion.

Continuing to prove Theorem 1.11, we set \( Q_i^{(j)} := Q_{2^j \rho_i}(w_i), j = 1, 2, 3, 4, 5 \), we observe from the previous lemma that

\[
|Q_i^{(j)}| \leq \frac{2}{\varphi(\lambda)} \int_{Q_{2^j \rho_i} \cap \{ |Du| > \frac{1}{2} \}} \varphi(|Du|) \, dz + \frac{2}{\delta \varphi(\lambda)} \int_{Q_{2^j \rho_i} \cap \{ |F| > \frac{1}{2} \}} \varphi(|F|) \, dz \quad \text{(4.7)}
\]

and

\[
\int_{Q_i^{(j)}} \left[ \varphi(|Du|) + \frac{1}{\delta} \varphi(|F|) \right] \, dz < \varphi(\lambda). \quad \text{(4.8)}
\]

Hereafter, we set \( \lambda \geq \lambda_1 \) and then consider the following rescaled functions and vector fields:

\[
g_{\lambda}(s) := \frac{g(\lambda s)}{g(\lambda)} \quad \text{for } s \geq 0,
\]

\[
A_{\lambda,i}(z, \xi) := \frac{1}{g(\lambda)} A(W_i, \lambda \xi) \quad \text{for } z \in Q_{\lambda}(0) \text{ and } \xi \in \mathbb{R}^n,
\]

\[
u_{\lambda,i}(z) := \nu(W_i) \quad \text{and} \quad F_{\lambda,i}(z) := \frac{F(W_i)}{\lambda} \quad \text{for } z \in Q_{\lambda}(0),
\]

where

\[
W_i = w_i + \left( 8 \rho_i, \frac{\lambda}{g(\lambda)} (8 \rho_i)^2 t \right) \quad \text{for } z = (x, t).
\]

Then, \( g_{\lambda} \) satisfies the condition in (1.2) with the same constants \( p \) and \( q \), and \( A_{\lambda,i}(z, \xi) \) satisfies (1.3) and (1.4) with \( \Omega_i = Q_{\lambda}(0) \) and is \((\delta, 2)\)-vanishing where \( g \) is replaced with \( g_{\lambda} \). Moreover, we observe that \( u_{\lambda,i} \) is a weak solution to

\[
\partial_t u_{\lambda,i} - \text{div} A_{\lambda,i}(z, D u_{\lambda,i}) = -\text{div} \left( \frac{g_{\lambda}(|F_{\lambda,i}|)}{|F_{\lambda,i}|} F_{\lambda,i} \right) \quad \text{in } Q_{\lambda}(0).
\]

By defining

\[
\varphi_{\lambda}(s) := \int_{0}^{s} g_{\lambda}(\sigma) \, d\sigma = \frac{1}{\lambda g(\lambda)} \varphi(\lambda s), \quad s \geq 0,
\]

from (4.8) where \( \lambda g(\lambda) \geq \varphi(\lambda) \) (see Proposition 2.3), we obtain the following:

\[
\int_{Q_{\lambda}(0)} \left[ \varphi_{\lambda}(|Du_{\lambda,i}|) + \frac{1}{\delta} \varphi_{\lambda}(|F_{\lambda,i}|) \right] \, dz = \frac{1}{\lambda g(\lambda)} \int_{Q_{\lambda}(0)} \left[ \varphi(|Du|) + \frac{1}{\delta} \varphi(|F|) \right] \, dz < 1.
\]
Here, we let \( \epsilon > 0 \) be sufficiently small, which is determined in (4.12) below. Then applying Lemmas 3.2 and 3.9, we can assert that there exist \( \delta = \delta(\epsilon) > 0 \), \( N_0 \geq 1 \), and a function \( \bar{v}_{\lambda,i} \) such that

\[
\int_{Q_{i(0)}} \left| V_\lambda(Du_{\lambda,i}) - V_\lambda(D\bar{v}_{\lambda,i}) \right|^2 \, dz \leq \epsilon \quad \text{and} \quad \|D\bar{v}_{\lambda,i}\|_{L^\infty} = \|Q_{i(0)}\| \leq N_0.
\]

where \( V_\lambda(\xi) = \frac{1}{\sqrt{2\pi\xi^2}} V(\lambda \xi) \) for \( \xi \in \mathbb{R}^n \). Here we remark that both \( \delta \) and \( N_0 \) are independent of \( \lambda \) and \( i \). Finally we set \( v_{\lambda,i}(z) = v_{\lambda,i}(x, t) := 8\rho_i \lambda \bar{v}_{\lambda,i} \left( \frac{x - y_i}{\lambda \rho_i}, \frac{g^i(t, t)}{\lambda \rho_i} \right) \), where \( w_i = (y_i, t_i) \). Then, we obtain

\[
\int_{Q_{i(0)}} \left| V(Du) - V(Dv_{\lambda,i}) \right|^2 \, dz \leq \lambda g(\lambda) \epsilon, \quad \text{and} \quad \|Dv_{\lambda,i}\|_{L^\infty} \leq N_0 \lambda. \tag{4.9}
\]

Next, for any \( \lambda > \lambda_1 \), we consider the upper-level sets \( E(r_1, c_0 N_0 \lambda) \) with \( c_0 := \frac{1}{10} (c_0 + 1)^2 \geq 1 \), where \( c_0 > 0 \) is the hidden constant in (2.8). By Lemma 4.6, the collection \( \{ Q_i^{(3)} \} \) covers \( E(r_1, \lambda) \setminus N \supset E(r_1, c_0 N_0 \lambda) \setminus N \) with \( |N| = 0 \), which yields

\[
\int_{E(r_1, c_0 N_0 \lambda)} \phi(|Du|) \, dz \leq \sum_{i=1}^{\infty} \int_{Q_i^{(3)} \cap \{|Du| > c_0 N_0 \lambda\}} \phi(|Du|) \, dz.
\]

Using the convexity and (Dec)_\phi of \( \phi \), and (2.8), we obtain the following:

\[
\phi(|Du|) \leq \frac{1}{2} \phi(2|Du - Dv_{\lambda,i}|) + \frac{1}{2} \phi(2|Dv_{\lambda,i}|) \\
\leq 2^{q-1} \phi(|Du - Dv_{\lambda,i}|) + 2^{q-1} \phi(|Dv_{\lambda,i}|) \\
\leq 2^{q-1} c_s |V(Dv_{\lambda,i}) - V(Du)|^2 + 2^{q-1} (c_s + 1) \phi(|Dv_{\lambda,i}|).
\]

It follows from (4.9) that, on the set \( Q_i^{(3)} \setminus \{|Du| > c_0 N_0 \lambda\} \),

\[
\phi(|Du|) \leq 2^{q-1} c_s |V(Du) - V(Dv_{\lambda,i})|^2 + 2^{q-1} (c_s + 1) \phi(N_0 \lambda) \\
\leq 2^{q-1} c_s |V(Du) - V(Dv_{\lambda,i})|^2 + \frac{1}{2} \phi(c_0 N_0 \lambda) \\
\leq 2^{q-1} c_s |V(Du) - V(Dv_{\lambda,i})|^2 + \frac{1}{2} \phi(|Du|),
\]

and, hence,

\[
\phi(|Du|) \leq 2^q c_s |V(Du) - V(Dv_{\lambda,i})|^2.
\]

Combining this with (4.9) results in

\[
\int_{E(r_1, c_0 N_0 \lambda)} \phi(|Du|) \, dz \leq 2^q c_s \lambda g(\lambda) \epsilon \sum_{i=1}^{\infty} |Q_i^{(3)}| \tag{4.10}
\]

Note that \( |Q_i^{(3)}| = 2^{3(n+2)} |Q_i^{(0)}| \) and that \( \lambda g(\lambda) \leq \phi(\lambda) \) by Proposition 2.3. Therefore, the estimates (4.10) and (4.7) lead to the following:
where the hidden constant depends only on $n$, $\nu$, $\Lambda$, $p$, and $q$. As $Q_i^{(0)} \subset Q_i(z_0)$. $i = 1, 2, \ldots$, are mutually disjoint, we conclude that, for any $\lambda \geq \lambda_1$,

$$\int_{E(r, c_0 N_0, \lambda)} \varphi(| Du|) \, dz \leq \varepsilon \int_{Q_2 \cap \{ | Du | > \frac{\lambda}{2} \}} \varphi(| Du|) \, dz + \varepsilon \sum_{i=1}^{\infty} \int_{Q_i \cap \{ | F | > \frac{\lambda}{4} \}} \varphi(| F|) \, dz. \tag{4.11}$$

We set

$$|Du|_k := \min\{|Du|, k\} \quad \text{for} \quad k \geq 0.$$

For $k > \lambda$, the inequality $|Du|_k > \lambda$ holds if and only if the inequality $|Du| > \lambda$ holds. Therefore, for any $R \leq r_1 < r_2 < 2R$,

$$\int_{Q_1} \frac{\psi(\varphi(|Du|_k))}{\varphi(|Du|_k)} \varphi(|Du|) \, dz = \int_{Q_1} \int_0^{[Du]_k} \frac{d}{d \lambda} \left( \frac{\psi(\varphi(\lambda))}{\varphi(\lambda)} \right) \varphi(|Du|) \, dz \, d\lambda$$

$$= \int_0^{[Du]_k} \int_{Q_1 \cap \{ |Du| > \lambda \}} \varphi(|Du|) \, dz \, d\lambda \left( \frac{\psi(\varphi(\lambda))}{\varphi(\lambda)} \right)$$

$$= \int_0^{[Du]_k} \int_{E(r, c_0 N_0, \lambda)} \varphi(|Du|) \, dz \, d\lambda \left( \frac{\psi(\varphi(c_0 N_0, \lambda))}{\varphi(c_0 N_0, \lambda)} \right)$$

$$\leq \int_0^{[Du]_k} \int_{E(r, c_0 N_0, \lambda)} \varphi(|Du|) \, dz \, d\lambda \left( \frac{\psi(\varphi(c_0 N_0, \lambda))}{\varphi(c_0 N_0, \lambda)} \right)$$

$$+ \int_{[Du]_k}^{[Du]_k} \int_{E(r, c_0 N_0, \lambda)} \varphi(|Du|) \, dz \, d\lambda \left( \frac{\psi(\varphi(c_0 N_0, \lambda))}{\varphi(c_0 N_0, \lambda)} \right)$$

$$=: I + II.$$

To estimate $I$, we apply (aDec)$_q$, of $\psi$ and (Dec)$_q$ of $\varphi$:

$$I \leq \frac{\psi(\varphi(c_0 N_0, \lambda_1))}{\varphi(c_0 N_0, \lambda_1)} \int_{Q_2} \varphi(|Du|) \, dz$$

$$\leq \left( \frac{64R}{r_2 - r_1} \right)^{(q_1 - 1)q(n+2) \max\{ \frac{1}{2}, \frac{2}{p(n+2) - 2n} \}} \psi_1(\varphi(\lambda_0)) \int_{Q_2} \varphi(|Du|) \, dz$$

$$\leq \left( \frac{R}{r_2 - r_1} \right)^{a_0} \psi_1(\varphi(\lambda_0)) \int_{Q_2} \varphi(|Du|) \, dz,$$

where $a_0 := (q_1 - 1)q(n + 2) \max\{ \frac{1}{2}, \frac{2}{p(n+2) - 2n} \}$. 

$$\int_{E(r, c_0 N_0, \lambda)} \varphi(|Du|) \, dz \leq \varepsilon \sum_{i=1}^{\infty} \int_{Q_i \cap \{ | F | > \frac{\lambda}{4} \}} \varphi(| F|) \, dz \leq \varepsilon \sum_{i=1}^{\infty} \int_{Q_i \cap \{ | F | > \frac{\lambda}{4} \}} \varphi(| F|) \, dz.$$
To estimate $\Pi$, we employ (4.11) and Fubini’s theorem:

\[
\Pi \leq c \int_0^{4k} \int_{Q_{r_k}(|Du| > \frac{3}{4})} \varphi(|Du|) \, dz \, d\varphi(\psi(c_0N_0 \lambda)) \frac{\varphi(c_0N_0 \lambda)}{\varphi(c_0N_0 \lambda)} \\
+ \frac{1}{\delta} \int_0^{\infty} \int_{Q_{r_k}(|F| > \frac{3}{4})} \varphi(|F|) \, dz \, d\varphi(\psi(c_0N_0 \lambda)) \frac{\varphi(c_0N_0 \lambda)}{\varphi(c_0N_0 \lambda)} \\
= c \int_{Q_{r_k}} \int_0^{4k} \varphi(|Du|) \, dz \, d\varphi(\psi(c_0N_0 \lambda)) \frac{\varphi(4c_0N_0 \lambda)}{\varphi(4c_0N_0 \lambda)} \\
+ \frac{1}{\delta} \int_{Q_{r_k}} \int_0^{\infty} \varphi(|F|) \, dz \, d\varphi(\psi(c_0N_0 \lambda)) \frac{\varphi(\frac{4}{5}c_0N_0 \lambda)}{\varphi(\frac{4}{5}c_0N_0 \lambda)} \\
= c \int_{Q_{r_k}} \int_0^{4k} \frac{\varphi(|Du|)}{\varphi(|Du|)} \varphi(|Du|) \, dz \, d\varphi(\psi(c_0N_0 \lambda)) \frac{\varphi(4c_0N_0 |F|)}{\varphi(\frac{4}{5}c_0N_0 |F|)} \\
\leq ce \int_{Q_{r_k}} \psi(|Du|) \, dz + ce \int_{Q_{r_k}} \varphi(|Du|) \, dz + ce \int_{Q_{r_k}} \varphi(|F|) \, dz.
\]

At this stage, we set $\epsilon = \epsilon(n, v, \Lambda, p, q, p_1, q_1, L) > 0$ so small that

\[
cc \leq \frac{1}{2}.
\]

Thus, $\delta$ is chosen. Combining the above estimates, we observe that

\[
\int_{Q_{r_k}} \psi(\varphi(|Du|_k)) \varphi(|Du|) \, dz \leq c \int_{Q_{r_k}} \frac{\psi(|Du|_k) \varphi(|Du|)}{\varphi(|Du|_k)} \varphi(|Du|) \, dz + c \int_{Q_{r_k}} \psi(\phi(\lambda_0)) \varphi(|Du|) \, dz \\
+ c \int_{Q_{r_k}} \varphi(\phi(|F|)) \, dz.
\]

Applying Lemma 4.1 yields the following:

\[
\int_{Q_{r_k}} \frac{\psi(|Du|_k)}{\varphi(|Du|_k)} \varphi(|Du|) \, dz \leq c \psi_1(\varphi(\lambda_0)) \int_{Q_{r_k}} \varphi(|Du|) \, dz + \int_{Q_{r_k}} \psi(\varphi(|F|)) \, dz.
\]

Letting $k \to \infty$ and using Fatou’s lemma, we conclude the following:

\[
\int_{Q_k} \psi(\varphi(|Du|)) \, dz \leq (\psi_1 \circ \phi)(\lambda_0) \int_{Q_{2k}} \varphi(|Du|) \, dz + \int_{Q_{2k}} \psi(\varphi(|F|)) \, dz \\
\leq (\psi_1 \circ \phi \circ D^{-1}) \left( \int_{Q_{2k}} \varphi(|Du|) + \varphi(|F|) \right) \int_{Q_{2k}} \varphi(|Du|) \, dz + \int_{Q_{2k}} \psi(\varphi(|F|)) \, dz,
\]

which completes the proof.
5 | PARABOLIC SYSTEM

We briefly discuss parabolic systems with a Uhlenbeck structure with a discontinuous coefficient. We consider the following parabolic system

\[ (u_i)_t - \text{div} \left( a(z) \frac{g(|Du|)}{|Du|} Du_i \right) = 0 \text{ in } \Omega_t, \quad i = 1, 2, \ldots, N, \tag{5.1} \]

where \( u = (u_1, \ldots, u_N), \quad N \geq 1, \quad g : [0, \infty) \to [0, \infty) \) with \( g \in C([0, \infty)) \cap C^1((0, \infty)) \) and \( g(0) = 0 \) satisfies

\[ p - 1 \leq \frac{sg'(s)}{g(s)} \leq q - 1, \quad \forall s \geq 0 \text{ for some } 2 \leq p \leq q. \tag{5.2} \]

Further, \( a : \Omega_t \to \mathbb{R} \) satisfies \( \nu \leq a(\cdot) \leq \Lambda \) for some \( 0 < \nu \leq \Lambda \). For the systems we only consider the degenerate case (i.e., \( p \geq 2 \) in (5.2)). In this case, the \((\delta, R)\)-vanishing condition is replaced with the following BMO-type condition on \( a(\cdot) \):

**Definition 5.3.** Let \( \delta, R > 0, a : \Omega_t \to \mathbb{R} \) is said to be \((\delta, R)\)-vanishing if

\[ \int_{Q_{r, \rho}} |a(z) - (a)_{Q_{r, \rho}}| \, dz \lesssim \delta \]

for all \( Q_{r, \rho} \subset \Omega_t \) with \( r, \rho \in (0, R] \).

Then, we state a result that is the counterpart of Theorem 1.11 for the system in (5.1).

**Theorem 5.4.** Let \( g : [0, \infty) \to [0, \infty) \) with \( g \in C([0, \infty)) \cap C^1((0, \infty)) \) and \( g(0) = 0 \) satisfy (5.2), and \( a : \Omega_t \to \mathbb{R} \) satisfy \( \nu \leq a(\cdot) \leq \Lambda \) for some \( 0 < \nu \leq \Lambda \). In addition, let \( F \in L^p_{\text{loc}}(\Omega_t) \) with \( \varphi \) defined in (1.5), and \( \psi : [0, \infty) \to [0, \infty) \) be a weak \( \Phi \)-function satisfying \((\text{Dec})_{p_1}\) and \((\text{Dec})_{q_1}\) for some \( 1 < p_1 \leq q_1 \) with the constant \( L \geq 1 \). There exists a small \( \delta = \delta(n, \nu, \Lambda, p, q, p_1, q_1, L, g(1), \psi(1)) > 0 \) such that if \( a(\cdot) \) is \((\delta, R_0)\)-vanishing for some \( R_0 > 0 \) and \( u \) is a local weak solution to (5.1), then we have the following implication

\[ \varphi(|F|) \in L^p_{\text{loc}}(\Omega_t) \Rightarrow \varphi(|Du|) \in L^p_{\text{loc}}(\Omega_t) \]

with the following estimate: for any \( Q_{2R} = Q_{2R}(z_0) \subset \Omega_t \) with \( R \leq R_0 \),

\[ \int_{Q_R} \psi(\varphi(|Du|)) \, dz \lesssim c \left[ \Psi \left( \int_{Q_{2R}} \left[ \varphi(|Du|) + \varphi(|F|) \right] \, dz \right) \right] \int_{Q_{2R}} \varphi(|Du|) \, dz \]

\[ + c \int_{Q_{2R}} \psi(\varphi(|F|)) \, dz \]

for some \( c = c(n, \nu, \Lambda, p, q, p_1, q_1, L, g(1), \psi(1)) > 0 \), where \( Q_R = Q_R(z_0) \) and \( \Psi \) is given in Theorem 1.11.

**Sketch of the proof.** The proof is exactly the same as in Theorem 1.11 except for the essential modification concerned with changing from single equations to systems. More precisely, in the proof, we replace \( A(z, \xi) \) where \( \xi \in \mathbb{R}^n \) with \( a(z) |\xi|^p - 2 \xi \) where \( z \in \mathbb{R}^{nN} \). Instead of Theorem 2.12, we take advantage of the Lipschitz regularity result for homogeneous parabolic systems in Cho, 14, Lemma 3.2 which is a modification of Diening et al. 15, Theorem 2.2. The higher integrability result 16, Theorem 1.5 (Theorem 2.11 in this paper) is proved for general parabolic systems. All analyses in Sections 3 and 4 are the same in both the equation and system cases and both the degenerate \((p \geq 2)\) and singular \((p < 2)\) cases. Therefore, by repeating Sections 3 and 4 and considering the replacement mentioned above, we prove the theorem.
Remark 3.5. In the above theorem, we only consider the degenerate case (i.e., $p \geq 2$ in (5.2)). The main reason is the lack of Lipschitz regularity when $p < 2$. In Diening et al., Theorem 2.2 the Lipschitz regularity for the parabolic $\varphi$-Laplace system (5.1), where $F \equiv 0$, $a(\cdot) \equiv 1$ and $\varphi$ is defined in (1.5), is proved under the assumption that the gradient of a weak solution is locally in $L^2$. This additional assumption is unclear from the definition of a weak solution when $p < 2$. We might also prove the theorem by replacing (5.2) with (1.2) and assuming that $|Du| \in L^2_{\text{loc}}(\Omega_t)$ (see Diening et al., Remark 1.1).

ACKNOWLEDGEMENTS

We thank the referee for useful comments. J. Oh is supported by NRF-2020R1A4A1018190. J. Ok is supported by the Sogang University Research Grant of 202010022.01.

CONFLICTS OF INTEREST

There are no conflicts of interest to this work.

ORCID

Jihoon Ok https://orcid.org/0000-0001-7848-9123

REFERENCES

1. DiBenedetto E. Degenerate Parabolic Equations. New York: Springer-Verlag; 1993. Universitext.
2. Acerbi E, Mingione G. Gradient estimates for a class of parabolic systems. Duke Math J. 2007;136(2):285-320.
3. DiBenedetto E, Friedman A. Regularity of solutions of nonlinear degenerate parabolic systems. J Reine Angew Math. 1984;349:83-128.
4. Kinnunen J, Lewis J. Higher integrability for parabolic systems of $p$-Laplacian type. Duke Math J. 2000;102(2):253-271.
5. DiBenedetto E, Friedman A. Hölder estimates for nonlinear degenerate parabolic systems. J Reine Angew Math. 1985;357:1-22.
6. DiBenedetto E, Gianazza U, Vespri V. Harnack estimates for quasi-linear degenerate parabolic differential equations. Acta Math. 2008;200(2):181-209.
7. DiBenedetto E, Gianazza U, Vespri V. Harnack’s Inequality for Degenerate and Singular Parabolic Equations. New York: Springer Monographs in Mathematics Springer; 2012.
8. Baroni P. Lorentz estimates for degenerate and singular evolutionary systems. J Differential Equations. 2013;255(9):2927-2951.
9. Bögelein V. Global gradient bounds for the parabolic $p$-Laplacian system. Proc Lond Math Soc. 2015;111(3):633-680.
10. Byun S, Ok J, Ryu S. Global gradient estimates for general nonlinear parabolic equations in nonsmooth domains. J Differential Equations. 2013;254(11):4290-4326.
11. Lieberman GM. The natural generalization of the natural conditions of Ladyzhenskaya and Ural’tseva for elliptic equations. Comm Partial Differential Equations. 1991;16(2-3):311-361.
12. Lieberman GM. Hölder regularity for the gradients of solutions of degenerate parabolic systems. Ukr Math Bull. 2006;3:352-373.
13. Baroni P, Lindfors C. The Cauchy-Dirichlet problem for a general class of parabolic equations. Ann Inst H Poincaré Anal Non Linéaire. 2017;34(3):593-624.
14. Cho Y. Calderón–Zygmund theory for degenerate parabolic systems involving a generalized $p$-Laplacian type. J Evol Equ. 2018;18(3):1229-1243.
15. Diening L, Scharle T, Schwarzacher S. Regularity for parabolic systems of Uhlenbeck type with Orlicz growth. J Math Anal Appl. 2019;472(1):46-60.
16. Hästö P, Ok J. Higher integrability for parabolic systems with Orlicz growth. ArXiv:1905.05577.
17. Hwang S, Lieberman G. Hölder continuity of bounded weak solutions to generalized parabolic $p$-Laplacian equations I: degenerate case. Electron J Differential Equations. 2015;287:32.
18. Hwang S, Lieberman G. Hölder continuity of bounded weak solutions to generalized parabolic $p$-Laplacian equations II: singular case. Electron J Differential Equations. 2015;288:24.
19. Yao F. Lorentz estimates for a class of nonlinear parabolic systems. J Differential Equations. 2019;266(4):2078-2099.
20. Byun S, Cho Y. Nonlinear gradient estimates for generalized elliptic equations with nonstandard growth in nonsmooth domains. Nonlinear Anal. 2016;140:145-165.
21. Cho Y. Global gradient estimates for divergence-type elliptic problems involving general nonlinear operators. J Differential Equations. 2018;264(10):6152-6190.
22. Verde A. Calderón–Zygmund estimates for systems of $\varphi$-growth. J Convex Anal. 2011;18(1):67-84.
23. Bögelein V, Duzaar F, Mingione G. Degenerate problems with irregular obstacles. J Reine Angew Math. 2011;650:107-160.
24. Mingione G. Calderón–Zygmund theory for elliptic problems with measure data. Ann Sc Norm Super Pisa Cl Sci. 2007;6(2):195-261.
25. Harjulehto P, Hästö P. Orlicz spaces and generalized Orlicz spaces. Lecture Notes in Mathematics, Vol. 2236. Cham: Springer;2019.
26. Hästö P, Ok J. Maximal regularity for local minimizers of non-autonomous functionals. to appear. ArXiv:1902.00261.
27. Diening L, Ettwein F. Fractional estimates for non-differentiable elliptic systems with general growth. *Forum Math.* 2008;20(3):523-556.
28. Giusti E. *Direct Methods in the Calculus of Variations.* River Edge, NJ: World Scientific Publishing Co., Inc.; 2003.

**How to cite this article:** Oh J, Ok J. Gradient estimates for parabolic problems with Orlicz growth and discontinuous coefficients. *Math Meth Appl Sci.* 2022;45(14):8718-8736. [https://doi.org/10.1002/mma.7845](https://doi.org/10.1002/mma.7845)