RECONSTRUCTION ALGEBRAS OF TYPE $D$ (I)

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Abstract. This is the second in a series of papers which give an explicit description of the reconstruction algebra as a quiver with relations; these algebras arise naturally as geometric generalizations of preprojective algebras of extended Dynkin diagrams. This paper deals with dihedral groups $G = D_{n,q}$ for which all special CM modules have rank one, and we show that all but four of the relations on such a reconstruction algebra are given simply as the relations arising from a reconstruction algebra of type $A$. As a corollary, the reconstruction algebra reduces the problem of explicitly understanding the minimal resolution (=G-Hilb) to the same level of difficulty as the toric case.

1. Introduction

It was discovered in [Wem07] that noncommutative algebra holds the key to obtaining some form of a McKay Correspondence for finite subgroups of $GL(2, \mathbb{C})$. More precisely if $G$ is a finite small cyclic subgroup of $GL(2, \mathbb{C})$ then the quiver of the endomorphism ring of the special CM $\mathbb{C}[x,y]^G$ modules (a ring build downstairs on the singularity) determines and is determined by the dual graph of the minimal resolution $\widetilde{X}$ of the singularity $\mathbb{C}^2/G$, labelled with self-intersection numbers. Since the dual graph of such a group is always a Dynkin diagram of type $A$, we call the noncommutative ring in question the reconstruction algebra of type $A$. The above is a correspondence purely on the level of the underlying quivers; it was further discovered that if we add in the extra information of the relations then in fact one can recover the whole space $\widetilde{X}$ (not just the dual graph) as a certain GIT quotient, and also that the reconstruction algebra describes the derived category of $\widetilde{X}$.

Following the work of Bridgeland [Bri02] and Van den Bergh [VdB04], these ideas were pursued further in [Wem08] where the above statement on the level of quivers was proved for all complex rational surface singularities. Two proofs of this fact were given, one non-explicit proof for the general case and one explicit proof which only covers the quotient case. It is perhaps important to emphasize two points. Firstly, although it was shown that the number of relations for the quiver can also be obtained from the intersection theory, the relations themselves were not exhibited. Secondly, the non-explicit proof tells us nothing about the special CM modules (for example what they are) and very little about their structure.

The motivation behind the above work, and indeed this paper, is the following general principle:
Principle 1.1. For any finite subgroup of $GL(2, \mathbb{C})$, given the dual graph of the minimal resolution of $\mathbb{C}^2/G$ (labelled with self intersection numbers) one can build a noncommutative algebra, the so called reconstruction algebra, which coincides with the endomorphism ring of the special CM modules. Using this ring we may extract the commutative minimal resolution both non-explicitly and explicitly, and also describe its derived category.

Much of this has now been proved, but of course to build an algebra from the dual graph we need to produce both a quiver and relations; so far we only have the quiver. Thus the main purpose of this paper is to provide the relations in the case of certain dihedral groups $D_{n,q}$ inside $GL(2, \mathbb{C})$. The companion paper [Wem09] deals with the remaining dihedral cases. Although this may be technical the relations are important; we shall see that the geometry of the minimal resolution echoes the similarity in the relations between different reconstruction algebras.

The basic philosophy is that the larger the fundamental cycle $Z_f$ the more the geometry resembles the SL(2, C) case, since the reconstruction algebra quiver and relations look and behave more like a preprojective algebra (see [Wem09, Section 5] for more details on this case). As $Z_f$ gets smaller and thus closer to being reduced, the more toric the geometry becomes since the reconstruction algebra quiver and relations begin to look and behave more like a reconstruction algebra of type $A$. Via the work of Wunram it is $Z_f$ which dictates the rank of the special CM modules, and different ranks induce slightly different algebra structures because polynomials factor in different ways.

The first step in this paper towards the goal of obtaining the relations is to fill in the gap in the classification of the specials CM modules left in [IW08]. Although the techniques in loc. cit could plausibly be used for dihedral groups, the AR quiver splits into cases and also the combinatorics are difficult; both factors make it hard to write down a general proof. The method used here is still quite combinatorially complex but it is at least possible to write down the proof. It also has the added philosophical benefit of not requiring any knowledge of the McKay quiver as instead we wrap the equations of the singularity around a noncommutative ring (as a quiver) and argue by a simple diagram chase that the modules corresponding to the vertices must be minimally 2-generated. Furthermore we explicitly obtain their generators. By [Wun88] this condition is both sufficient and necessary for a rank one CM module to be special.

Having explicitly obtained the specials and their generators, we then use this information to label the arrows in the known quiver in terms of polynomials involving $x$'s and $y$'s; our relations are then simply that ‘$x$ and $y$ commute’. From this we write down some obvious relations, and an easy argument using the known number of relations tells us that these are them all. Since there are choices for the generators of the special CM modules, in fact we obtain two different (but isomorphic) presentations of the endomorphism ring, corresponding to two different choices of such generators.

Once we have the quiver with relations, we are able to describe the moduli spaces of representations using explicit techniques. This is now very easy: since here the fundamental cycle $Z_f$ is reduced the dimension vector we use consists entirely of 1’s. Now (almost all) the relations for the reconstruction algebras in this paper are just the relations from the reconstruction algebra of type $A$, so the explicit extraction of the geometry comes almost entirely for free from the toric case.

We now describe the contents and structure of this paper in more detail: in Section 2 we define the groups $D_{n,q}$ and recall some of the known properties of the singularities $\mathbb{C}^2/D_{n,q}$. Furthermore we introduce and prove some combinatorics crucial to later arguments. In Section 3 we exhibit the special rank-1 CM modules for every group $D_{n,q}$ regardless of $Z_f$; when $Z_f$ is reduced these are all the special CM modules. Section 4 restricts to the case when $Z_f$ is reduced and in such a case we define $D_{n,q}$, the associated reconstruction algebra of type $D$, and show that it is isomorphic to the endomorphism ring of the special CM modules. In Section 5 we use this noncommutative algebra to exhibit explicitly the minimal resolution (which is equal to $D_{n,q}$-Hilb via a result of Ishii [Ish02]) in co-ordinates; we see that every
open set is a smooth hypersurface in \( \mathbb{C}^3 \), with equations given in terms of easy combinatorics. We also translate the co-ordinates of the open cover into ratios of the polynomial invariants downstairs. Section 6 provides some examples to illustrate the theory.

Note that the discovery of the specials and a similar open cover for dihedral groups has been discovered independently by Nolla de Celis [NdC09a, NdC09b] by using the McKay quiver and combinatorics of \( G \)-Hilb. However the benefits of using the reconstruction algebra over the McKay quiver is that it reduces the problem to the same level of difficulty as the toric case; thus from the viewpoint of the reconstruction algebra the geometry in this paper is not toric, but it may as well be.

Throughout, when working with quivers we shall write \( ab \) to mean \( a \) followed by \( b \). We work over the ground field \( \mathbb{C} \) but any algebraically closed field of characteristic zero will suffice.

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2. Dihedral groups and combinatorics

In this paper we follow the notation of Riemenschneider [Rie77].

**Definition 2.1.** For \( 1 < q < n \) with \( (n, q) = 1 \) define the group \( \mathbb{D}_{n,q} \) to be

\[
\mathbb{D}_{n,q} = \begin{cases} \langle \psi_{2q}, \tau, \varphi_{2(n-q)} \rangle & \text{if } n-q \equiv 1 \mod 2 \\ \langle \psi_{2q}, \tau \varphi_{4(n-q)} \rangle & \text{if } n-q \equiv 0 \mod 2 \end{cases}
\]

with the matrices

\[
\psi_k = \begin{pmatrix} \varepsilon_k & 0 \\ 0 & \varepsilon_k^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & \varepsilon_4 \\ \varepsilon_4 & 0 \end{pmatrix}, \quad \varphi_k = \begin{pmatrix} \varepsilon_k & 0 \\ 0 & \varepsilon_k \end{pmatrix}
\]

where \( \varepsilon_k \) is a primitive \( k \)-th root of unity.

The order of the group \( \mathbb{D}_{n,q} \) is \( 4(n-q)q \). The procedure to obtain the invariants \( \mathbb{C}[x,y]^{\mathbb{D}_{n,q}} \) is also well-known: firstly develop

\[
\frac{n}{n-q} = [a_2, \ldots, a_{e-1}]
\]

as a Jung-Hirzebruch continued fraction expansion. We fix this notation throughout the paper, noting the strange numbering. Now define series \( c_j, d_j \) and \( t_j \) for \( 2 \leq j \leq e \) by

\[
c_2 = 1 \quad c_3 = 0 \quad c_4 = 1 \quad c_j = a_{j-1}c_{j-1} - c_{j-2} \text{ for } 5 \leq j \leq e \\
d_2 = 0 \quad d_3 = 1 \quad d_4 = a_3 - 1 \quad d_j = a_{j-1}d_{j-1} - d_{j-2} \text{ for } 5 \leq j \leq e \\
t_2 = a_2 \quad t_3 = a_2 - 1 \quad t_4 = a_3(a_2 - 1) - 1 \quad t_j = a_{j-1}t_{j-1} - t_{j-2} \text{ for } 5 \leq j \leq e
\]

where the values to the right of the line exist only when \( e > 3 \), i.e. when \( n > q + 1 \), i.e. when the group \( \mathbb{D}_{n,q} \) does not lie inside \( SL(2, \mathbb{C}) \). Also define the series \( r_j \) for \( 2 \leq j \leq e \) by

\[
r_j = (n-q)t_j - q(e_2 + d_j) \text{ for } 2 \leq j \leq e
\]

Note by definition that \( r_j = a_{j-1}r_{j-1} - r_{j-2} \) for all \( 5 \leq j \leq e \) and note also that \( r_{e} = 0 \) and \( r_{e-1} = 1 \). Now after setting

\[
w_1 = xy \quad v_2 = x^{2q} + (-1)^{a_2}y^{2q} \quad v_3 = x^{2q} + (-1)^{a_2}y^{2q}
\]

we have the following result:

**Theorem 2.2.** [Rie77 Satz 2] The polynomials \( w_1^{2(n-q)} \) and \( w_t^r v_2^s d_1^t \) for \( 2 \leq t \leq e \) generate the ring \( \mathbb{C}[x,y]^{\mathbb{D}_{n,q}} \).
The main ingredient in the proof is Noether’s bound on the degree of the generators in characteristic zero; once this is used the proof degenerates into combinatorics. In this paper we shall need the following easy variant of the above: define
\[ w_2 = (x^3 + y^3)(x^3 + (-1)^{a_2}y^3) \]
\[ w_3 = (x^3 - y^3)(x^3 + (-1)^{a_2}y^3) \]

**Lemma 2.3.** The polynomials \( w_1^{2(n-q)} \) and \( w_1^{e_i} w_2^{c_i} w_3^{d_i} \) for \( 2 \leq t \leq e \) generate the ring \( \mathbb{C}[x,y]^{D_{n,q}} \).

For \( D_{n,q} \) throughout this paper fix the notation
\[ n_q = [\alpha_1, \ldots, \alpha_N] \]
as the Hirzebruch-Jung continued fraction expansion of \( \frac{n}{q} \). By Riemenschneider duality (see e.g. [Kid01 1.2] or [Wem07 2.1]) it is true that
\[ e - 2 = 1 + \sum_{i=1}^{N} (\alpha_1 - 2). \]

**Definition 2.4.** Define \( \nu < N \) to be the largest integer such that \( \alpha_1 = \ldots = \alpha_{\nu} = 2 \), or 0 if no such integer exists.

Now by [Bri68 2.11] the dual graph of the minimal resolution of \( \mathbb{C}^2/D_{n,q} \) is

```
\begin{center}
\begin{tikzpicture}[scale=0.8]
  \node (1) at (0,0) {\( -2 \)};
  \node (2) at (0,-1) {\( -2 \)};
  \node (3) at (1,-1) {\( -\alpha_1 \)};
  \node (4) at (2,-1) {\( \cdots \)};
  \node (5) at (3,-1) {\( -\alpha_{N-1} \)};
  \node (6) at (4,-1) {\( -\alpha_N \)};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (4);
  \draw (4) -- (5);
  \draw (5) -- (6);
\end{tikzpicture}
\end{center}
```

where the \( \alpha \)'s come from the Jung-Hirzebruch continued fraction expansion of \( \frac{n}{q} \) above. Notice that the only possible fundamental cycles \( Z_f \) for dihedral groups \( D_{n,q} \) are

```
\begin{center}
\begin{array}{cccccccc}
  1 & 1 & 1 & \cdots & 1 & 1 \\
  1 & 2 & \cdots & 2 & 1 & \cdots & 1
\end{array}
\end{center}
```

where (since by definition \( \nu < N \)) the number of 2's in the right hand picture is precisely \( \nu \). Thus \( \nu \) records the number of 2's in \( Z_f \).

**Definition 2.5.** [WunSS 1.2] Denote \( R = \mathbb{C}[x,y]^{D_{n,q}} \). An CM \( R \)-module \( M \) is called special if \( (M \otimes \omega_R)/\text{tors} \) is CM, where \( \omega_R \) is the canonical module of \( R \).

Wunram proved [WunSS 1.2] that there is a 1-1 correspondence between the non-free indecomposable special CM modules and the exceptional curves in the minimal resolution, and further that the rank of the special corresponding to a curve \( E_i \) is equal to the co-efficient of \( E_i \) in the fundamental cycle \( Z_f \). By the above discussion, this implies that for dihedral groups \( D_{n,q} \) there are precisely \( (N+2-\nu) \) non-free rank 1 special CM modules and \( \nu \) rank 2 special indecomposable CM modules. The rank 2 specials are known from [IW08 6.2]; in fact the classification of the specials for all finite subgroups of \( GL(2, \mathbb{C}) \) is complete with the exception of these \( (N+2-\nu) \) non-free rank 1 special CM modules in the dihedral cases.

To be more precise Wunram defined the specials using CM modules on the ring \( \mathbb{C}[[x,y]]^G \) where such modules are of the form \( \rho \otimes \mathbb{C} \mathbb{C}[[x,y]]^G \). In this paper we shall mainly be working with the \( \mathbb{C}[[x,y]]^G \)-modules \( \rho \otimes \mathbb{C} \mathbb{C}[[x,y]]^G \), i.e. we work in the non-complete case. We are mainly interested in computing the endomorphism ring in the non-complete case, but later we shall reduce this problem to the complete case since the associated graded ring of \( \text{End}_{\mathbb{C}[[x,y]]^G}(\bigoplus \rho \text{ special} (\rho \otimes \mathbb{C} \mathbb{C}[[x,y]]^G)) \) is \( \text{End}_{\mathbb{C}[[x,y]]^G}(\bigoplus \rho \text{ special} (\rho \otimes \mathbb{C} \mathbb{C}[[x,y]]^G)). \)

To find these specials, and thus finish the classification, we need the combinatorial series (as in Type A) together with some other combinatorial series:
Definition 2.6. For any integers $1 \leq m_1 < m_2$ with $(m_1, m_2) = 1$ we can associate to the continued fraction expansion $\frac{m_2}{m_1} = [\beta_1, \ldots, \beta_X]$ combinatorial series defined as follows:

1. the $i$-series, defined as
   
   $i_0 = m_2 \quad i_1 = m_1 \quad i_t = \beta_{i_{t-1}}i_{t-1} - i_{t-2}$ for $2 \leq t \leq X + 1$

2. the $j$-series, defined as
   
   $j_0 = 0 \quad j_1 = 1 \quad j_t = \beta_{j_{t-1}}j_{t-1} - j_{t-2}$ for $2 \leq t \leq X + 1$

3. The $l$-series, defined as
   
   $l_j = 2 + \sum_{p=1}^{j} (\beta_p - 2)$ for $1 \leq j \leq X$

4. The $b$-series. Define $b_0 := 1$, $b_{l_X-1} := X$, and further for all $1 \leq t \leq l_X - 2$ (if such $t$ exists), define $b_t$ to be the smallest integer $1 \leq b_t \leq X$ such that
   
   $t \leq \sum_{p=1}^{b_t} (\beta_p - 2)$.

Definition 2.7. Given a continued fraction expansion $\frac{m_2}{m_1} = [\beta_1, \ldots, \beta_X]$ we call the continued fraction expansion of $\frac{m_2}{m_2 - m_1}$ the dual continued fraction, and denote it by $[\beta_1, \ldots, \beta_X]^\ast$.

In type $A$ for the group $\frac{1}{2}(1, q)$ it was the $i$-series associated to the continued fraction expansion of $\frac{n}{q}$ that characterized the special CM modules; for the group $\mathbb{D}_{n,q}$ we shall see in Section 3 that the same is true, but with an extra subtlety involving $\nu$.

Throughout this section we shall be using the above combinatorial series for many different continued fraction inputs, thus to avoid confusion we now fix some notation.

Notation 2.8. For $\mathbb{D}_{n,q}$, throughout this paper we shall denote the combinatorial data in Definition 2.6 associated to the continued fraction expansion of $\frac{n}{q}$ by using the fonts and letters $i, j, l, b$. For all other continued fraction inputs we shall denote the combinatorial data in Definition 2.6 by using different fonts and letters.

We record some easy combinatorics.

Lemma 2.9. For any $\mathbb{D}_{n,q}$ with any $\nu$,

(i) $a_2 = \nu + 2$

(ii) $q = i_{\nu+1} + \nu(n - q)$

(iii) $r_2 = 2(n - q) - i_{\nu+1}$

(iv) $r_3 = (n - q) - i_{\nu+1} = i_\nu - 2i_{\nu+1}$

(v) $r_2 = 2r_3 + i_{\nu+1}$

Proof. (i) This is immediate by Riemenschneider duality.

(ii) This is trivial if $\nu = 0$, so we can assume that $\nu > 0$. This being the case it is easy to see that

\[ i_t = tq - (t - 1)n \text{ for all } 1 \leq t \leq \nu + 1 \]

since $\alpha_1 = \ldots = \alpha_\nu = 2$. In particular $i_{\nu+1} = (\nu + 1)q - \nu n$ and so the result is trivial.

(iii) This follows from (i) and (ii) since

\[ r_2 = a_2(n - q) - q(c_2 + d_2) = (\nu + 2)(n - q) - (i_{\nu+1} + \nu(n - q))(0 + 1) = 2(n - q) - i_{\nu+1}. \]

(iv) Immediate from (iii) since

\[ r_3 = (a_2 - 1)(n - q) - q(c_3 + d_3) = (a_2 - 1)(n - q) - q = r_2 - (n - q). \]

(v) Immediate from (iii) and (iv) above. \qed
Lemma 2.10. Take some continued fraction expansion \( \frac{p}{q} = [\beta_1, \ldots, \beta_X] \) and denote the combinatorial data from Definition 2.6 by \( \{I, J, L, B\} \). To the dual continued fraction \( [\beta_1, \ldots, \beta_X]' := [\gamma_1, \ldots, \gamma_Y] \) denote the combinatorial data by \( \{I, J, L, B\} \). Then

(i) \( B_t = L_t - 1 \) for all \( 1 \leq t \leq L_X - 1 \).
(ii) \( I_t = 1 + B_t + 1 \) for all \( 0 \leq t \leq L_X - 1 \).
(iii) \( J_{t+1} - J_t = J_{B_t} \) for all \( 0 \leq t \leq L_Y - 1 \).
(iv) \( J_{t+1} - J_t = J_{L_t - 1} \) for all \( 1 \leq t \leq L_X - 1 \).
(v) \( J_{B_t} = 1 + \sum_{p=1}^{B_t-1} J_p - 1 \) for all \( 1 \leq t \leq L_Y - 1 \).

Proof. (i) is an immediate consequence of Riemenschneider duality.
(ii) and (iii) are just a slight rephrasing of a result of Kidoh [Kid01, 1.3].
(iv) By duality, swapping bold and non-bold in (iii) gives \( J_{t+1} - J_t = J_{B_t} \) for all \( 0 \leq t \leq L_X - 1 \). The result then follows by (i).
(v) Follows immediately from (iv) since

\[
J_{B_t} - J_1 = (J_{B_t} - J_{B_t - 1}) + \ldots + (J_2 - J_1) = \sum_{p=1}^{B_t-1} J_{t_p - 1}.
\]

\( \Box \)

The following is known, and can be found in [BR78, p214].

Lemma 2.11. For \( D_{n, q} \) as above with \( \frac{n}{n-q} = [a_2, a_3, \ldots, a_{e-1}] \), then the \( r \) series is simply the \( i \)-series for the data \( [a_3 + 1, a_4, \ldots, a_{e-1}] \). More precisely, denoting the \( i \)-series of \( [a_3 + 1, a_4, \ldots, a_{e-1}] \) by \( I_0, I_1, \ldots \), we have \( r_k = I_{k-2} \) for all \( 2 \leq k \leq e \).

Proof. By definition \( \frac{n}{n-q} = [a_2, a_3, \ldots, a_{e-1}] = a_2 \cdot \frac{1}{[a_3, \ldots, a_{e-1}]} \) and so \( \frac{n-q}{a_2(n-q)-n} = [a_3, \ldots, a_{e-1}] \).
But combining Lemma 2.10 (iii), (iv) we know that \( r_3 = r_2 - (n - q) \) and further since \( r_2 = a_2(n) - q \) (by definition) we have

\[
\frac{r_2}{r_3} = \frac{a_2(n) - q}{a_2(n) - q - (n - q)} = \frac{n - q + a_2(n) - n}{a_2(n) - n} = [a_3 + 1, a_4, \ldots, a_{e-1}] = \frac{n-q}{a_2(n-q)-n}.
\]

The result now follows since \( r_4 = (a_3 + 1)r_3 - r_2 \) and \( r_t = a_{t-1}r_{t-1} - r_{t-2} \) for all \( 5 \leq t \leq e \). \( \Box \)

Note in particular this means \( r_{e-1} = 1 \) and \( r_e = 0 \).

Lemma 2.12. Consider \( D_{n, q} \). Then for all \( 2 \leq t \leq e - 2 \),

\[
r_{t+1} = r_{t+2} + i_{t_in}
\]

Proof. Denote the \( i \)-series associated to the following data as follows:

\[
\frac{n}{n-q} = [a_2, a_3, \ldots, a_{e-1}] \quad \text{by} \quad I_0, I_1, \ldots
\]

\[
[a_3, a_4, \ldots, a_{e-1}] \quad \text{by} \quad I_0, I_1, \ldots
\]

\[
[a_3 + 1, a_4, \ldots, a_{e-1}] \quad \text{by} \quad I_0, I_1, \ldots \quad \text{(as in Lemma 2.11)}
\]

Since \( \frac{i_{t+1}}{n} = [a_3 + 1, a_4, \ldots, a_{e-1}] \) it is clear that for \( 0 \leq j \leq e \)

\[
I_j = \begin{cases} i_0 + i_1 & \text{if } j = 0 \\ i_j & \text{if } j \geq 1 \end{cases} = \begin{cases} i_1 + i_2 & \text{if } j = 0 \\ i_{j+1} & \text{if } j \geq 1 \end{cases}
\]

Now by Lemma 2.11 for all \( 2 \leq k \leq e \) we have \( r_k = I_{k-2} \). Thus for \( 2 \leq t \leq e - 2 \), by the above and Lemma 2.11 applied to \( [\beta_1, \ldots, \beta_X] = [\alpha_1, \ldots, \alpha_N] \) we have

\[
r_{t+1} = I_{t-1} = i_t = i_{t+1} + i_{t-1} = I_t + i_{t-1} = r_{t+2} + i_{t-1}.
\]

\( \Box \)

The next lemma is the first key combinatorial lemma needed for the computation of the rank 1 special CM modules; the other is Lemma 2.13.
Lemma 2.13. Consider $\mathbb{D}_{n,q}$, then for all $\nu + 1 \leq t \leq N$

$$r_t = i_t - i_{t+1}.$$ 

Proof. Proceed by induction. Consider first the base case $t = \nu + 1$. Notice it is always true (by definition) that $l_{\nu+1} = \alpha_{\nu+1}$. Now to prove the base case requires 2 subcases:

(i) If $\alpha_{\nu+1} = 3$ then by Lemma 2.12(iv)

$$r_{\nu+1} = r_{\alpha_{\nu+1}} = r_3 = i_{\nu} - 2i_{\nu+1} = i_{\nu+1} - i_{\nu+2}$$

(ii) If $\alpha_{\nu+1} > 3$ then Lemma 2.12 and inductive hypothesis we have

$$r_{\nu+1} = r_3 - (l_{\nu+1} - 3)i_{\nu+1} = (i_{\nu} - 2i_{\nu+1} - (\alpha_{\nu+1} - 3)i_{\nu+1} = i_{\nu+1} - i_{\nu+2}$$

and so we are done. This proves the base case $t = \nu + 1$. If $\nu = N - 1$ we are done hence suppose $\nu < N - 1$, let $t$ be such that $\nu + 1 < t \leq N$ and assume that the result is true for smaller $t$. To prove the induction step again requires 2 subcases

(i) If $\alpha_t = 2$, then $l_t = l_{t-1}$ and so by inductive hypothesis

$$r_{l_t} = r_{l_{t-1}} = l_{t-1} - l_t = i_t - i_{t-1}.$$ 

(ii) If $\alpha_t > 2$ then by Lemma 2.12 and inductive hypothesis

$$r_{l_t} = r_{l_{t-1}} - (l_t - l_{t-1})i_t = (i_{t-1} - i_t) - (\alpha_t - 2)i_t = i_t - i_{t-1}.$$ 

□

Definition 2.14. Consider $\mathbb{D}_{n,q}$. Define for $\nu + 1 \leq k \leq N + 1$

$$\Delta_k = 1 + \sum_{t=\nu+1}^{k-1} c_t \quad \text{and} \quad \Gamma_k = \sum_{t=\nu+1}^{k-1} d_t$$

where the convention is that for $k = \nu + 1$ the sum is empty and so equals zero.

The following is the second key combinatorial lemma needed to determine the specials.

Lemma 2.15. Consider $\mathbb{D}_{n,q}$ for any $\nu$. Then for all $2 \leq t \leq e - 2$

$$c_{t+2} = c_{t+1} + \Delta_{b_t} \quad \text{and} \quad d_{t+2} = d_{t+1} + \Gamma_{b_t}.$$ 

Proof. (i) The $c$ statement. The trick is to interpret the $c$’s as the $j$-series associated to some continued fraction, then use the results of Lemma 2.10.

We firstly prove that the lemma holds in the case $t = 2$. Notice that $\Delta_{b_2} = 1$ since either $b_2 = \nu + 1$ and so the sum is empty (and so by convention zero), or $b_2 > \nu + 1$ in which case $l_{\nu+1} = \ldots = l_{b_2-1} = 3$ and so the sum is $c_3 + \ldots + c_3 = 0$. Thus $c_4 = c_3 + \Delta_{b_3}$ follows since $c_4 := 1$ and $c_3 := 0$. Hence the result is true for $t = 2$ thus we may assume that $e > 4$ and restrict our attention to the interval $3 \leq t \leq e - 2$.

Denote the $j$-series of $[a_4, \ldots, a_{e-1}]$ by $j$. It is clear from the definition of $c$ that

$$c_t = j_{t-3} \quad \text{for all } 3 \leq t \leq e.$$ 

To $[a_4, \ldots, a_{e-1}]^v$ denote the $j$ series by $b$, the $b$-series by $\mathbb{b}$ and the $l$ series by $L$. By Lemma 2.10(iii) applied to the data $[\beta_1, \ldots, \beta_X] = [a_4, \ldots, a_{e-1}]$

$$c_{t+2} - c_{t+1} = j_{t-1} - j_{t-2} = \mathbb{b}_{t-2} \quad \text{for all } 2 \leq t \leq e - 2.$$ 

On the other hand

$$\Delta_{b_t} = 1 + \sum_{p=\nu+1}^{b_t-1} c_p = 1 + \sum_{p=\nu+1}^{b_t-1} j_{t-3},$$

thus to prove the lemma we just need to show that

$$\mathbb{b}_{t-2} = 1 + \sum_{p=\nu+1}^{b_t-1} j_{t-3} \quad \text{for all } 3 \leq t \leq e - 2.$$
Now by Riemenschneider duality

\[ [a_4, \ldots, a_{e-1}]^\vee = \begin{cases} [\alpha_{b_2}, \alpha_{a_1+b_2}, \ldots, \alpha_N] & \text{if } \alpha_{b+1} = 3 \\ [\alpha_{b_2} - 2, \alpha_{a_1+b_2}, \ldots, \alpha_N] & \text{if } \alpha_{b+1} > 3 \end{cases} \]

from which it is easy to see in either case that

\[ \mathbb{B}_0 = 1 \quad \text{wheras } \mathbb{B}_t = b_{t+2} - (b_2 - 1) \quad \text{for all } 1 \leq t \leq e - 4 \]

and \( L_s = l_{(b_2-1)+s} - 2 \) for all \( 1 \leq s \leq N - (b_2 - 1) \). Hence

\[
1 + \sum_{p=v+1}^{b_t-1} \mathbb{J}_{p-3} = 1 + \sum_{p=b_2}^{b_t-1} \mathbb{J}_{p-3} = 1 + \mathbb{J}_{L_1+1} + \ldots + \mathbb{J}_{L_{(b_2-1)+t}+1} = 1 + \sum_{p=1}^{b_t-2} \mathbb{J}_{L_p} = \mathbb{B}_{t+2}
\]

for all \( 3 \leq t \leq e - 2 \) where the last equality holds by Lemma 2(iii) applied to the data \([\beta_1, \ldots, \beta_X] = [a_4, \ldots, a_{e-1}]\).

(ii) The \( d \) statement. The trick is again to interpret the \( d \) series as the \( j \)-series of something, but here it is a little bit more subtle.

Due to lack of suitable alternatives we recycle notation from the proof of (i): now denote the \( j \)-series of \([a_{\alpha_{b_2} - 1}, a_{a_1+b_2}, \ldots, a_{e-1}]\) by \( j \), and further for the dual continued fraction \([a_{\alpha_{b_2} - 1}, a_{a_1+b_2}, \ldots, a_{e-1}]^\vee\) denote the \( j \), \( b \) and \( l \) series by \( \mathbb{J} \), \( \mathbb{B} \) and \( \mathbb{L} \), respectively.

Now it is easy to see that \( d_3 = \ldots = d_{a_{b_2+1}} = 1 \) and further \( d_t = j_{(t+1)-a_{b_2+1}} \) for all \( \alpha_{b_2+1} + 1 \leq t \leq e \). Hence the result is certainly true (by the convention of the empty sum being zero) for the interval \( 2 \leq t \leq \alpha_{b_2+1} - 1 \). Thus we are done in the case \( \nu = N - 1 \) and also in the case \( e = 4 \). Thus we may assume that \( \nu < N - 1 \) and that \( e > 4 \), and concentrate on the interval \( (\alpha_{b_2+1} - 2) + 1 \leq t \leq e - 2 = (\alpha_{b_2+1} - 2) + (1 + \sum_{p=\nu+2}(\alpha_p - 2)) \).

By Lemma 2(ii) applied to the data \([\beta_1, \ldots, \beta_X] = [a_{\alpha_{b_2} - 1}, a_{a_1+b_2}, \ldots, a_{e-1}]\]

\[ d_{t+2} - d_{t+1} = j_{(t+3)-a_{b_2+1}} - j_{(t+2)-a_{b_2+1}} = \mathbb{J}_{b_{(t+2)-a_{b_2+1}+1}} \]

for all \( (\alpha_{b_2+1} - 2) + 1 \leq t \leq (\alpha_{b_2+1} - 2) + (1 + \sum_{p=\nu+2}(\alpha_p - 2)) \).

On the other hand

\[ \Gamma_{b_t} = \sum_{p=\nu+1}^{b_t-1} d_{l_p} = \sum_{p=\nu+1}^{b_t-1} j_{(l_p-a_{b_2+1})+1} \]

and so the result follows if we can show that

\[ \mathbb{B}_{(t+2)-a_{b_2+1}+1} = \sum_{p=\nu+1}^{b_t-1} j_{(l_p-a_{b_2+1})+1} \]

for all \( (\alpha_{b_2+1} - 2) + 1 \leq t \leq (\alpha_{b_2+1} - 2) + (1 + \sum_{p=\nu+2}(\alpha_p - 2)) \), i.e.

\[
\sum_{p=\nu+1}^{b_{(\alpha_{b_2+1} - 2)+t-1}} \mathbb{J}_{(l_p-a_{b_2+1})+1}
\]

for all \( 1 \leq t \leq 1 + \sum_{p=\nu+2}(\alpha_p - 2) \). Now by Riemenschneider duality

\[ [a_{\alpha_{b_2} - 1}, a_{a_1+b_2}, \ldots, a_{e-1}]^\vee = [\alpha_{b_2}, \ldots, \alpha_N] \]

and so \( L_y - 1 = 1 + \sum_{p=\nu+2}(\alpha_p - 2) \). Further it is easy to see that \( \mathbb{B}_0 = 1 \), \( \mathbb{B}_t = b_{(\alpha_{b_2+1} - 2)+t} - (\nu + 1) \) for all \( 1 \leq t \leq 1 + \sum_{p=\nu+2}(\alpha_p - 2) \) and \( L_t = l_{(\nu+1)+t} - a_{b_2+1} + 2 \) for all \( 1 \leq t \leq N - \nu \). This implies that the sum in (II) is simply

\[ \sum_{p=\nu+1}^{b_{(\alpha_{b_2+1} - 2)+t-1}} \mathbb{J}_{(l_p-a_{b_2+1})+1} = \mathbb{J}_1 + \mathbb{J}_{(\nu+2-a_{b_2+1})+1} + \ldots + \mathbb{J}_{(b_{\nu+2}-a_{b_2+1})+1} = 1 + \mathbb{J}_{L_1 + \ldots + \mathbb{J}_{L_{b_2-1}-1}} \]

and so by Lemma 2.10(v) applied to the data \([\beta_1, \ldots, \beta_X] = [a_{\alpha_{\nu+1} - 1}, a_{\alpha_{\nu+1} + 1}, \ldots, a_{\nu - 1}]\)

\[
\sum_{p=\nu+1}^{b(n_{\nu+1} - 2) + \nu - 1} j_{(t_p - \alpha_{\nu+1}) + 1} = 1 + \sum_{p=1}^{b_1 - 1} j_{t_p - 1} = b_t
\]

for all \(1 \leq t \leq L_Y - 1 = 1 + \sum_{p=\nu+2}^{N}(\alpha_p - 2)\), as required. \(\Box\)

3. Computation of the rank 1 specials

In this section we determine the rank 1 specials for any \(D_{n,q}\) and obtain their generators. We begin with the following simple lemma.

**Lemma 3.1.** Let \(G\) be a small finite subgroup of \(GL(2, \mathbb{C})\). For one-dimensional representations \(\rho, \sigma\), denote the corresponding rank 1 CM modules by \(S_\rho\) and \(S_\sigma\). Then as \(\mathbb{C}[x,y]^G\)-modules

\[
\text{Hom}_{\mathbb{C}[x,y]^G}(S_\rho, S_\sigma) \cong S_\rho \otimes S_\sigma^*.
\]

**Proof.** Since the group \(G\) is small, it is well known that \(\mathbb{C}[x,y]^#G \cong \text{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x,y])\) and \(\text{add}\mathbb{C}[x,y] = \text{CMC}[x,y]^G\). Consequently

\[
\text{proj}\mathbb{C}[x,y]^#G \cong \text{CMC}[x,y]^G
\]

is an equivalence of categories (for details see for example [Yos90, 10.9]) and thus

\[
\text{Hom}_{\mathbb{C}[x,y]^G}(S_\rho, S_\sigma) \cong \text{Hom}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x,y] \otimes_\mathbb{C} \rho, \mathbb{C}[x,y] \otimes_\mathbb{C} \sigma)
\]

\[
\cong \text{Hom}_{\mathbb{C}[x,y]}(\mathbb{C}[x,y] \otimes_\mathbb{C} \rho, \mathbb{C}[x,y] \otimes_\mathbb{C} \sigma)^G
\]

\[
\cong (\mathbb{C}[x,y] \otimes_\mathbb{C} (\sigma \otimes \rho^*))^G = S_\rho \otimes S_\sigma^*.
\]

\(\Box\)

We shall see that there is a strong relationship between some of the specials and the \(i\)-series associated with the continued fraction expansion of \(\frac{a}{q}\).

**Definition 3.2.** For \(1 \leq t \leq n - q\) define \(W_t\) to be the CM module containing \((xy)^t\).

We should make two remarks. Firstly the \(W_t\) are well defined since \((xy)^t\) is a relative invariant for the one-dimensional representation

| \(n - q\) odd | \(n - q\) even |
|----------------|----------------|
| \(\psi_{2q}\)  | \(\psi_{2q}\)  |
| \(\tau\)        | \(1\)          |
| \(\varphi_{2(n-q)}\) | \((-1)^t\)    |
|                 | \((-1)^t\)    |
| \(\varphi_{2(n-q)}\) | \(\varepsilon_{n-q}^t\) |

Secondly, the assumption \(t < n - q\) ensures that the \(W_t\) are all distinct representations. Now for any \(D_{n,q}\) by Lemma 2.10(iv) it is true that \(n_{\nu+1} < n - q\) and so we aim to prove that \(W_{i_{\nu+1}}, W_{i_{\nu+2}}, \ldots, W_{i_N}\) are special CM modules. By the previous discussion in Section 2 we just need to show that they are two-generated.

Denote \(R = \mathbb{C}[x,y]^G\). The next lemma is trivial but is used extensively.

**Lemma 3.3.** Consider a rank-1 CM \(R\)-module \(T\). Let \(f_1, f_2 \in T\) be such that every element of \(T\) may be written as \(Af_1 + Bf_2\) for some polynomials \(A\) and \(B\). Then \(T\) is generated as an \(R\)-module by \(f_1\) and \(f_2\).

**Proof.** Let \(a \in T\) be written as \(a = Af_1 + Bf_2\). We just need to show that we can replace \(A\) and \(B\) with polynomials inside \(R\), then the result obviously follows. Taking any element \(g\) of the group \(G\), since \(f_1, f_2 \in T\) we may act on \(a\) and then cancel the relative invariant scalars, leaving

\[
a = (g \cdot A)f_1 + (g \cdot B)f_2
\]
Thus summing over all group elements

\[ a = \frac{1}{|G|} (\sum_{g \in G} g \cdot A) f_1 + \frac{1}{|G|} (\sum_{g \in G} g \cdot B) f_2. \]

For technical reasons we split the proof that \( W_{i_0}, \ldots, W_{i_N} \) are special into two cases:

**Case 1:** \( 0 \leq \nu < N - 1 \), so \( \alpha_{\nu+1} \geq 3 \). By definition and Lemma 3.11 certainly we have the following maps between them

\[ W_{i_{\nu+1}} \xrightarrow{(xy)^{\nu+1-i_{\nu+2}}} W_{i_{\nu+2}} \quad \ldots \quad W_{i_{N-1}} \xrightarrow{(xy)^{N-1-i_N}} W_{i_N} \]

Now by Lemma 2.9 (iii) \( 2(n-q) = r_2 + i_{\nu+1} \) so since \( (xy)^{2(n-q)} \) is an invariant we also have a map \( (xy)^{r_2} : W_{i_{\nu+1}} \to R \). Since \( (xy)^{r_2} w_2 \) is an invariant this in turn means

(i) there is a map \( w_2 : R \to W_{i_{\nu+1}} \).

(ii) (since \( 2r_2 + i_{\nu+1} = r_2 \)) there is a map \( g := (xy)^{2r_2} w_2 : W_{i_{\nu+1}} \to R \).

But also \( (xy)^{r_1} w_2^{c_{i_1}} w_3^{d_{i_1}} \) is invariant, so by Lemma 2.13 we have, for each \( \nu + 1 \leq t \leq N - 1 \), a map

\[ w_2^{c_{i_1}} w_3^{d_{t-1}} : W_{i_t} \to W_{i_{t+1}} \]

and also a map \( w_2^{c_{i_N}} w_3^{d_{N-1}} : W_{i_{N-1}} \to R \). Thus we have justified all the maps in the following picture:

In general there will be more maps. If \( \alpha_{\nu+1} > 3 \) then for every \( 2 \leq t \leq \alpha_{\nu+1} - 2 \) add an extra map \( W_{i_{\nu+1}} \to R \) labelled \( (xy)^{r_2} w_2^{c_{i_1}} w_3^{d_{t-1}} \). For all such \( t \) it is true that \( b_t = \nu + 1 \) and so these maps go where they should since \( (xy)^{r_1 + 1} w_2^{c_{i_1}} w_3^{d_{t-1}} \) is an invariant and \( r_{t+1} = r_{t+2} + i_{b_t} \) by Lemma 2.12.

Now if \( s \) is such that \( \nu + 1 < s \leq N \) with \( \alpha_s > 2 \), then for every \( 1 \leq t \leq \alpha_s - 2 \) add an extra map \( W_{i_t} \to R \) labelled \( (xy)^{r_1 + 1} w_2^{c_{i_1} + 1} w_3^{d_{t-1} + 1} \). For all such \( t \) it is true that \( b_{t-2} + i_{s-1} = s \) and so by Lemma 2.12 \( r_{t-1} + i_{s-1} = r_{t-1} + i_{s-1} + i_{b_{t-2}} = r_{t+2} + i_{s-1} \). Thus these maps also go where they should since \( (xy)^{r_{t-1} + i_{s-1}} w_2^{c_{i_1} + 1} w_3^{d_{t-1} + 1} \) is an invariant.

**Notation 3.4.** We denote by \( D_1 \) the above quiver with all the extra arrows (if these extra arrows exist). We denote by \( D_2 \) the quiver obtained from \( D_1 \) by making the substitution \( w_2 \mapsto v_2 \) and \( w_3 \mapsto v_3 \) wherever \( w_2 \) and \( w_3 \) appear in the labels of the arrows in \( D_1 \).

In the next proposition the cycle pattern is the same as in type \( A \), but to write down the proof here is a little awkward since we need to rely heavily on the combinatorics from Section 2.

**Proposition 3.5.** At every vertex in the quiver \( D_1 \) all invariant polynomials exist as a sum of compositions of cycles at that vertex. Furthermore if we remove any one arrow this is no longer true. Both statements also hold for \( D_2 \).

**Proof.** We first prove the statements for \( D_1 \). It is clear by following the outside anticlockwise arrows that at every vertex there is a cycle labelled \( (xy)^{2(n-q)} \). By Lemma 2.3 we must justify why at each vertex there is also the cycle \( (xy)^{r_2} w_2^{c_{i_1}} w_3^{d_{t-1}} \) for all \( 2 \leq t \leq e \).

Firstly consider \( t = 2 \), i.e. \( (xy)^{r_2} w_2 \). At the vertex \( R \) we can see this by following the \( w_2 \) arrow pointing up on the left hand side then the \( (xy)^{r_2} \) pointing down; for the vertex
$W_{i_{v+1}}$ we go down via $(xy)^{r_2}$ first then up via $w_2$. For all other vertices follow all the anticlockwise arrows on the outside, except on the section between $W_{i_{v+1}}$ and $R$ where the $g = (xy)^{2r_3}w_2$ arrow should be followed; this composition gives $(xy)^{r_2}w_2$ by Lemma 2.9(v) since $r_2 = 2r_3 + i_{v+1}$.

Now consider $t = 3$, i.e. $(xy)^{r_3}w_3$. To view this we need two cases.

Case 1: $\alpha_{v+1} > 3$. In this case we need to justify why we can view $(xy)^{r_3}w_3$ at all vertices by following the path:

$$W_{i_{v+1}} \leftarrow W_{i_{v+2}} \leftarrow W_{i_{v+3}} \leftarrow \cdots W_{i_{N-1}} \leftarrow W_{i_N}$$

where in the section between $W_{i_{v+1}}$ and $R$ we follow the extra arrow out of $W_{i_{v+1}}$ labelled by $(xy)^{r_3}w_2^3w_3^{d_3}$. But the overall composition equals $(xy)^{r_3}w_2^3w_3^{d_3} = (xy)^{r_3}w_3$ since $\alpha_{v+1} > 3$ implies $b_2 = 2 + 1$ and so $r_3 = r_4 + i_{v+1}$ by Lemma 2.12.

Case 2: $\alpha_{v+1} = 3$. In this case $b_2$ is equal to the next $t$ such that $\alpha_t > 2$; in order to draw a picture assume that this is $\alpha_{v+3}$ then we need to justify why we can view $(xy)^{r_3}w_3$ as:

$$W_{i_{v+1}} \leftarrow W_{i_{v+2}} \leftarrow W_{i_{v+3}} \leftarrow \cdots W_{i_{N-1}} \leftarrow W_{i_N}$$

where in the above picture we have followed the extra arrow out of $W_{i_{v+3}}$ labelled by $(xy)^{r_3}w_2^3w_3^{d_3}$. Now $l_{v+2} = l_{v+1}$ since $\alpha_{v+2} = 2$, which equals $\alpha_{v+1} = 3$. Thus the extra arrow is really labelled $(xy)^{r_3}w_2^3w_3^{d_3} = (xy)^{r_3}w_3$ and so the composition through the extra arrow is simply $(xy)^{r_3}w_3$. But now $r_3 = r_4 + i_{v+3}$ by Lemma 2.12. This justifies $(xy)^{r_3}w_3$ at vertices $W_{i_{v+3}}, \ldots, W_{i_N}, R$ so we just need to justify the cycles at vertices $W_{i_{v+1}}$ and $W_{i_{v+2}}$ drawn in the above picture. But since $\alpha_{v+2} = 2$ this is trivial.

Thus in both cases, at all vertices we can see the invariant $(xy)^{r_3}w_3$. Now the proof continues in exactly the same way as the pattern in type A: for the convivence of the reader we do one more step, namely $(xy)^{r_3}w_2^3w_3^{d_4}$. Again we must split into cases:

Case 1: $\alpha_{v+1} > 4$. In this case we must justify why we can view $(xy)^{r_4}w_2^3w_3^{d_4}$ at all vertices by following the path:

$$W_{i_{v+1}} \leftarrow W_{i_{v+2}} \leftarrow W_{i_{v+3}} \leftarrow \cdots W_{i_{N-1}} \leftarrow W_{i_N}$$

where in the section between $W_{i_{v+1}}$ and $R$ we follow the extra arrow out of $W_{i_{v+1}}$ labelled by $(xy)^{r_3}w_2^3w_3^{d_4}$. But the overall composition equals $(xy)^{r_4}w_2^3w_3^{d_4}$ since $\alpha_{v+1} > 4$ implies $b_3 = \nu + 1$ and so $r_4 = r_5 + i_{v+1}$ by Lemma 2.12.
Case 2: $\alpha_{\nu+1} = 4$. How to see $(xy)^{r_\nu} w_2^c w_3^d$ at all vertices is similar to Case 2 above:

$$\begin{align*}
W_{i_{\nu+1}} & \overset{R}{\longrightarrow} W_{i_{\nu+2}} \overset{R}{\longrightarrow} W_{i_{\nu+3}} \\
W_{i_{\nu+1}} & \overset{R}{\longrightarrow} W_{i_{\nu+3}} \leftarrow W_{i_{\nu+2}} \leftarrow W_{i_{\nu+4}} \leftarrow \cdots \leftarrow W_{i_{N-1}} \leftarrow W_{i_N}
\end{align*}$$

where in the above picture we have followed the extra arrow out of $W_{i_{\nu+3}}$ labelled by $(xy)^{r_{1+i_{\nu+2}} w_2^c w_3^d}$. The justification is the same as above, except now $l_{\nu+2} = l_{\nu+1} = \alpha_{\nu+1} = 4$ so the extra arrow is really labelled $(xy)^{r_{1+i_{\nu+2}} w_2^c w_3^d}$. Consequently the composition through the extra arrow is simply $(xy)^{r_{1+i_{\nu+2}} w_2^c w_3^d}$, but $r_1 = r_5 + i_{\nu+3}$ in this case by Lemma 2.12. The justification for the cycles at $W_{i_{\nu+1}}$ and $W_{i_{\nu+2}}$ is identical to Case 2 above.

Case 3: $\alpha_{\nu+1} = 3$. This case now depends on whether $\alpha_{\nu+3} > 3$ or not.

Subcase 1: $\alpha_{\nu+3} > 3$. We must justify why $(xy)^{r_\nu} w_2^c w_3^d$ can be viewed as the composition of paths

$$\begin{align*}
W_{i_{\nu+1}} & \overset{R}{\longrightarrow} W_{i_{\nu+2}} \overset{R}{\longrightarrow} W_{i_{\nu+3}} \\
W_{i_{\nu+1}} & \overset{R}{\longrightarrow} W_{i_{\nu+3}} \leftarrow W_{i_{\nu+2}} \leftarrow W_{i_{\nu+4}} \leftarrow \cdots \leftarrow W_{i_{N-1}} \leftarrow W_{i_N}
\end{align*}$$

where the left extra arrow out of $W_{i_{\nu+3}}$ is labelled $(xy)^{r_{1+i_{\nu+2}} w_2^c w_3^d}$ whereas the right extra arrow out of $W_{i_{\nu+3}}$ is labelled $(xy)^{r_{2+i_{\nu+2}} w_2^c w_3^d}$. Now $l_{\nu+2} = l_{\nu+1} = \alpha_{\nu+1} = 3$ in this case, so the left hand composition equals $(xy)^{r_{2+i_{\nu+2}} w_2^c w_3^d}$ whereas the right hand composition equals $(xy)^{r_{3+i_{\nu+3}} w_2^c w_3^d}$. Now $\alpha_{\nu+1} = 3$ and $\alpha_{\nu+3} > 2$ implies that $b_2 = \nu + 3$ and so by Lemma 2.15 we see that $c_4 = c_3 + \Delta_{\nu+3}$ and $d_4 = d_3 + \Gamma_{\nu+3}$. Also $\alpha_{\nu+3} > 3$ implies $b_3 = \nu + 3$ and so $r_3 = r_5 + i_{\nu+3}$ by Lemma 2.12. Hence both compositions represent $(xy)^{r_\nu} w_2^c w_3^d$.

Subcase 2: $\alpha_{\nu+3} = 3$. We must now travel to the next $\alpha > 2$; to draw a picture suppose $\alpha_{\nu+4} > 2$ then we must justify why the invariant can be viewed as

$$\begin{align*}
W_{i_{\nu+1}} & \overset{R}{\longrightarrow} W_{i_{\nu+2}} \overset{R}{\longrightarrow} W_{i_{\nu+3}} \overset{R}{\longrightarrow} W_{i_{\nu+4}} \\
W_{i_{\nu+1}} & \overset{R}{\longrightarrow} W_{i_{\nu+3}} \leftarrow W_{i_{\nu+2}} \leftarrow W_{i_{\nu+4}} \leftarrow \cdots \leftarrow W_{i_{N-1}} \leftarrow W_{i_N}
\end{align*}$$

But this is true again by Lemmas 2.12 and 2.15.

This finishes the justification for the invariant $(xy)^{r_\nu} w_2^c w_3^d$. The proof now continues in this fashion. It is worth stressing that the cycle pattern is exactly the same as for the reconstruction algebra of type $A$, where Lemmas 2.9, 2.12 and 2.15 are used to make sure that the combinatorics match. Since by Lemma 2.3 the collection $(xy)^{r_\nu} w_2^c w_3^d$ for all $2 \leq t \leq e$ generate $\mathbb{C}[x, y]^{D_{\nu+3}}$ it follows that we can see, at each vertex, all invariants. If we remove any one arrow from $D_1$ then by inspection (or use the fact that it is true in type $A$) we will no longer be able to see all invariants at all vertices.
Theorem 3.7. For $D_{n,q}$ with $0 \leq \nu < N - 1$ and all $1 \leq \nu + 1$, $W_i$ is generated as a $C[x,y]^{D_{n,q}}$ module by $(xy)^i$ and $w_2^{\Delta_i}w_3^r$, and so is special. Alternatively we can take as generators $(xy)^i$ and $v_2^{\Delta_i}v_3^r$.

Proof. We restrict ourselves to proving the generators $(xy)^i$ and $w_2^{\Delta_i}w_3^r$ by using $D_1$; the other generators follow immediately from the proof below by making the substitutions $w_2 \mapsto v_2$ and $w_3 \mapsto v_3$ throughout and working with $D_2$ instead.

We firstly verify the case $W_{iN}$ then proceed by induction on decreasing $t$. To prove the $W_{iN}$ case we split into 2 subcases:

Case 1: $\alpha_N = 2$. Let $f \in W_{iN}$ and consider $f w_2^{ci_N}w_3^{di_N}$; its an invariant and so view it as sums of cycles at the vertex $W_{iN}$. These must all leave the vertex, so since there are only 2 arrows out we can write

$$f w_2^{ci_N}w_3^{di_N} = w_2^{ci_N}w_3^{di_N}p_{0,N} + (xy)^{iN-1-iN}p_{N-1,N}$$

where $p_{0,N}$ is a sum of paths from $R$ to $W_{iN}$ and $p_{N-1,N}$ is a sum of paths from $W_{iN-1}$ to $W_{iN}$. Note that $iN-1-iN = 1$ since $\alpha_N = 2$. Viewing everything as polynomials $w_2^{ci_N}w_3^{di_N}$ must divide $p_{N-1,N}$ and so after cancelling the $w_2^{ci_N}w_3^{di_N}$ term we may write

$$f = p_{0,N} + (xy)A$$

for some polynomial $A$. By inspection of the quiver $D_1$ there are only two paths from $R$ to $W_{iN}$ that don’t involve cycles, so after moving all cycles to the end of the path (which we can do since there are all invariants at all vertices) we may write $p_{0,N}$ as

$$p_{0,N} = (xy)^{r_{iN}}B_1 + (w_2^{\Delta_N}w_3^{\Gamma_N})B_2$$

for some polynomials $B_1$ and $B_2$. Thus since $r_{iN} = 1$ we see that

$$f = (xy)(B_1 + A) + (w_2^{\Delta_N}w_3^{\Gamma_N})B_2$$

thus by Lemma 3.3 it follows that $W_{iN}$ is generated by $(xy)^i$ and $w_2^{\Delta_N}w_3^{\Gamma_N}$ and so is special.

Case 2: $\alpha_N > 2$. Let $f \in W_{iN}$. Since $\alpha_N > 2$ there is an extra arrow from $W_{iN}$ to $R$ labelled by $(xy)^{r_{iN}+1}w_2^{ci_{N-1}}w_3^{di_{N-1}}$. Further by Lemma 2.13 and Lemma 2.12

$$iN-1 - iN = r_{iN-1} = r_1 + l_{iN-1} + b_{iN-1}$$

thus since $b_{iN-1} = N$ and $iN = 1$ it follows that $r_1 + l_{iN-1} = (iN-1 - iN) - 1$. Consequently the polynomial $f(xy)^{(iN-1-iN)-1}w_2^{ci_{N-1}}w_3^{di_{N-1}}$ is an invariant and so we can view it as a sum of cycles at the vertex $W_{iN}$. They all must leave the vertex, so we may write

$$f(xy)^{(iN-1-iN)-1}w_2^{ci_{N-1}}w_3^{di_{N-1}} = (xy)^{iN-1-iN}A_1 + (xy)^{(iN-1-iN)-1}w_2^{ci_{N-1}}w_3^{di_{N-1}}p_{0,N} + w_2^{\Delta_N}w_3^{\Gamma_N}B_1$$

where $p_{0,N}$ is a sum of paths from $R$ to $W_{iN}$, $A_1$ is a sum of paths from $W_{iN-1}$ to $W_{iN}$, and $B_1$ is some polynomial; we can do this since all the other arrows leaving $W_{iN}$ are divisible by $w_2^{\Delta_N}w_3^{\Gamma_N}$ by Lemma 2.14. Viewing everything as polynomials, $(xy)^{(iN-1-iN)-1}$ must divide $B_1$ and further $w_2^{ci_{N-1}}w_3^{di_{N-1}}$ must divide $A_1$, thus after cancelling these terms we see

$$f = (xy)A_2 + p_{0,N} + w_2^{\Delta_N}w_3^{\Gamma_N}B_2$$
for some polynomials $A_2$ and $B_2$. By inspection of the quiver $D_1$ there are only two paths from $R$ to $W_{it}$ that don’t involve cycles (one is $xy = (xy)^{\Gamma_1}$, the other is $w_2^\Delta w_3^{\Gamma_1}$), so after moving all cycles to the end of the path (which we can do since there are all invariants at all vertices) we may write $p_{0,N}$ as

$$p_{0,N} = (xy)^{\Gamma_1} C_1 + (w_2^\Delta w_3^{\Gamma_1})C_2$$

for some polynomials $C_1$ and $C_2$. Thus since $r_{iN} = 1$ we see that

$$f = (xy)(C_1 + A_2) + (w_2^\Delta w_3^{\Gamma_1})(B_2 + C_2)$$

and so by Lemma 3.3 it follows that $W_{it}$ is generated by $xy = (xy)^{\Gamma_1}$ and $w_2^\Delta w_3^{\Gamma_1}$, hence is special.

For the induction case suppose we are considering $W_{it}$ with $\nu + 1 \leq t < N$ and we have established the result for $W_{it+1}$. Let $f \in W_{it}$ and consider $fw_2^{c_{i+1}}w_3^{d_{i+1}} \in W_{it+1}$. By inductive hypothesis we may write

$$fw_2^{c_{i+1}}w_3^{d_{i+1}} = (xy)^{\alpha_{i+1}}A_1 + w_2^\Delta w_3^{\Gamma_1+1}B_1$$

for some invariant polynomials $A_1, B_1$. Viewing everything as polynomials we see that $w_2^{c_{i+1}}w_3^{d_{i+1}}$ divides $A_1$ and so after cancelling this factor

$$f = (xy)^{\alpha_{i+1}}A_2 + w_2^\Delta w_3^{\Gamma_1+1}B_1$$

for some polynomial $A_2$. Since $B_1$ is invariant $w_2^\Delta w_3^{\Gamma_1+1}B_1 \in W_{it}$, hence since $f \in W_{it}$ we get $(xy)^{\alpha_{i+1}}A_2 \in W_{it}$. This is turn implies that $A_2w_2^{c_{i+1}}w_3^{d_{i+1}}$ is an invariant, and so we can view it as a sum of cycles at vertex $W_{it+1}$. From here we split into 2 subcases:

**Case 1:** $\alpha_{t+1} = 2$. Then there are only two arrows out of $W_{it+1}$, thus

$$A_2w_2^{c_{i+1}}w_3^{d_{i+1}} = (xy)^{\alpha_{i+1}}p_{t+1} + w_2^{c_{i+1}}w_3^{d_{i+1}}p_{t+2,t+1}$$

where $p_{t+1}$ is a sum of paths from $W_{it}$ to $W_{it+1}$ and $p_{t+2,t+1}$ is a sum of paths from $W_{it+2}$ to $W_{it+1}$. But since $\alpha_{t+1} = 2$ it follows that $c_{t+1} = c_i$ and $d_{t+1} = d_i$, thus viewing everything as polynomials $w_2^{c_{i+1}}w_3^{d_{i+1}}$ divides $p_{t+1}$ and so after cancelling this factor

$$A_2 = (xy)^{\alpha_{i+1}}D_1 + p_{t+2,t+1}$$

for some polynomial $D_1$. Now any path from $W_{it+2}$ to $W_{it+1}$ must either factor through the map $(xy)^{\alpha_{i+1} - \alpha_{t+2}}$ or go via $R$ and end through the composition of maps $w_2^\Delta w_3^{\Gamma_{t+1}}$, thus we may write $p_{t+2,t+1}$ as

$$p_{t+2,t+1} = (xy)^{\alpha_{i+1} - \alpha_{t+2}}E_1 + w_2^\Delta w_3^{\Gamma_{t+1}}E_2$$

for some polynomials $E_1$ and $E_2$. But $i_t - i_{t+1} = i_{t+1} - i_{t+2}$ since $\alpha_{t+1} = 2$, hence

$$A_2 = (xy)^{\alpha_{i+1}}(D_1 + E_1) + w_2^\Delta w_3^{\Gamma_{t+1}}E_2.$$

Consequently

$$f = (xy)^{\alpha_{i+1}}(xy)^{\alpha_{i+1} - \alpha_{t+1}}(D_1 + E_1) + w_2^\Delta w_3^{\Gamma_{t+1}}E_2 + w_2^\Delta w_3^{\Gamma_1}B_1 = (xy)^{\alpha_{i+1}}F_1 + w_2^\Delta w_3^{\Gamma_1}F_2$$

for some polynomials $F_1$ and $F_2$. Hence by Lemma 3.3 it follows that $W_{it}$ is generated by $(xy)^{\alpha_i}$ and $w_2^\Delta w_3^{\Gamma_1}$, thus is special.

**Case 2:** $\alpha_{t+1} > 2$. Here we may write

$$A_2w_2^{c_{i+1}}w_3^{d_{i+1}} = (xy)^{\alpha_{i+1}}p_{t+1} + (xy)^{\alpha_{i+1} - \alpha_{t+1}}w_2^{c_{i+1}}w_3^{d_{i+1}}p_{0,t+1} + w_2^\Delta w_3^{\Gamma_{t+1}}G_1$$

where $p_{t+1}$ is a sum of paths from $W_{it}$ to $W_{it+1}$, $p_{0,t+1}$ is a sum of paths from $R$ to $W_{it+1}$, and $G_1$ is some polynomial; we can do this since by Lemma 2.15 all other paths out of $W_{it+1}$ when viewed as polynomials are divisible by $w_2^\Delta w_3^{\Gamma_{t+1}}$. We are also using the fact that
Since either \( f \leq 0 \) since either \( \leq 0 \) since greater than zero since \( \alpha \) by Lemma 2.15. The extra maps go where they should since \((xy)^i \). Proof. For some polynomials \( A \) cancelling these terms we get \((xy)^i \) for some polynomials \( A \). For \( A \) where \( \overline{\text{c}} \) and so by Lemma 3.3 it follows that \( W_{i_2} \) is special. Alternatively we can take as generators \( \overline{\text{c}} \) with \((xy)^i \) and \( (xy)^i \) is generated as a \((xy)^i \) module by \( \overline{\text{c}} \) and further all the extra arrows from \( W_{i_2} \). Consequently viewing \( \overline{\text{c}} \) and consider the invariant \( \overline{\text{c}} \) and \( \overline{\text{c}} \), except \( \overline{\text{c}} \) and \( \overline{\text{c}} \), which is true by Lemma 2.13 and Lemma 2.12. By inspection of \( D \) we can write \( (xy)^i \) or \( (xy)^i \) and \( (xy)^i \). This establishes the induction step, so the result now follows.

Case 2: \( \nu = N - 1 \).

Theorem 3.8. For \( D_{n,q} \) with \( \nu = N - 1 \), \( W_{i_N} \) is generated as a \( \mathbb{C}[x,y]D_{n,q} \) module by \( xy \) and \( w_2 \), thus is special. Alternatively we can take as generators \( xy \) and \( w_2 \).

Proof. We prove the statement regarding \( xy \) and \( w_2 \); the other statement follows immediately since either \( \nu = N - 1 \) is odd and so \( v_2 = w_2 \), or its even in which case (since \( q = i_{v+1} + \nu(n - q) = 1 + \nu(n - q) \) by Lemma 2.14(ii)) \( v_2 = w_2 - 2(xy)((xy)^{2(n-q)} \frac{t}{2} \) with \((xy)^{2(n-q)} \frac{t}{2} \) an invariant.

If \( \alpha_N = 2 \) then the group is in \( SL(2, \mathbb{C}) \) and so the result is well-known, hence we can assume that \( \alpha_N > 2 \). It is clear that we have the following maps from \( R \) to \( W_{i_N} \), and from \( W_{i_N} \) to \( R \):

Note \((xy)^{-1+r_2}w_2^2w_3^2 \) and also by Lemma 2.14 \(-1 + r_2 = 2r_3 - 2 \) which is greater than zero since \( \alpha_N > 2 \) forces \( r_3 > 1 \). Now if \( \alpha_N \geq 3 \), for every \( t \) with \( 4 \leq t \leq \alpha_N + 1 \) add an extra arrow from \( W_1 \) to \( R \) labelled \((xy)^{r_2}w_2^{t-1+r_2}w_3^{t+1} \), where the equality holds since \( r_t = \alpha_N + 1 - t \) by Lemma 2.12 whilst \( c_t = t - 3 \) and \( d_t = 1 \) by Lemma 2.15. The extra maps go where they should since \((xy)^{r_2}w_2^{t-1+r_2}w_3^{t+1} \) is an invariant for all \( 4 \leq t \leq e = \alpha_N + 1 \) and we have a map \( w_2 \) from \( R \) to \( W_{i_N} \). It is easy to see that at both vertices we have all invariants. Let \( f \in W_{i_N} \) and consider the invariant \( f \). Now the two original arrows from \( W_{i_N} \) to \( R \) both have factor \( xy \) and further all the extra arrows from \( W_{i_N} \) to \( R \) have factor \( xy \), except \( w_2^{\alpha_N-2}w_3 \). Consequently viewing \( f \) as cycles at the vertex \( W_{i_N} \) we can write

\[
fw_2^{\alpha_N-2}w_3 = w_2^{\alpha_N-2}w_3A + (xy)B
\]

where \( A \) is a sum of paths from \( R \) to \( W_{i_N} \) and \( B \) is some polynomial. But there are only two arrows from \( R \) to \( W_{i_N} \) (namely \( xy \) and \( w_2 \)) and so writing \( A \) in terms of them

\[
fw_2^{\alpha_N-2}w_3 = w_2^{\alpha_N-1}w_3A_1 + (xy)(B + w_2^{\alpha_N-2}w_3A_2)
\]

for some polynomials \( A_1, A_2 \). Thus \( w_2^{\alpha_N-2}w_3 \) must divide \( B + w_2^{\alpha_N-2}w_3A_2 \) and so after cancelling these terms we get

\[
f = w_2A_1 + (xy)B_2
\]
for some polynomial \(B_2\). Hence by Lemma 3.3 it follows that \(W_1 = W_{i+N}\) is generated by \(xy\) and \(w_2\) and so is special.

We now search for the remaining two rank 1 specials.

**Definition 3.9.** Define \(W_t\) to be the CM module containing \(x^q + y^q\), and \(W_–\) to be the CM module containing \(x^q - y^q\).

These are well defined since \(x^q \pm y^q\) is a relative invariant for the one-dimensional representations

\[
\begin{array}{c|c|c}
 n - q \text{ odd} & n - q \text{ even} \\
\hline
\psi_{2q} & \mp 1 & \psi_{2q} \\
\tau & \varepsilon_{4q} & \tau \varphi_{4(n-q)} \\
\varphi_{2(n-q)} & \varepsilon_{2(n-q)} & \end{array}
\]

Note also that \(W_+\) and \(W_-\) are different representations, and that they are also distinct from the \(W_t\) defined earlier.

**Lemma 3.10.** For any \(D_{n,q}\) with any \(\nu\), \(W_+\) is generated by the two elements \((xy)^{n-q}(x^q - y^q), x^q + y^q\) whilst \(W_-\) is generated by the two elements \((xy)^{n-q}(x^q + y^q), x^q - y^q\). Hence both are special.

**Proof.** Let \(f \in W_+.\) We firstly claim that \(w_2 = (x^q + y^q)(x^q + (-1)^\nu y^q) \in W_{i+1}\); if \(\nu = N - 1\) then this follows since \(w_2\) generates \(W_{i+1} = W_{i+N}\) by Theorem 3.8 if 0 \(\leq \nu < N - 1\) it follows by inspection of \(D_1\). It follows that \(f(x^q + (-1)^\nu y^q) \in W_{i+1}\). But by combining Theorem 3.7 and Theorem 3.8, for any \(\nu\) we know that \(W_{i+1}\) is generated by \((xy)^{i+1}\) and \(w_2\); hence \(f(x^q + (-1)^\nu y^q)\) and so we may write \(f(x^q + (-1)^\nu y^q) = (xy)^{i+1}C_1 + (x^q + y^q)(x^q + (-1)^\nu y^q)\) for some invariant polynomials \(C_1, C_2\). This means \((x^q + (-1)^\nu y^q)\) must divide \(C_1\) and so by Lemma 2.7 by inspection of the list of generators of the invariant ring we may write

\[
C_1 = (xy)^\nu(x^q - y^q)(x^q + (-1)^\nu y^q)d_1 + (x^q + y^q)(x^q + (-1)^\nu y^q)d_2
\]

for some polynomials \(D_1, D_2\), thus

\[
f = (xy)^{\nu+i}(x^q - y^q)d_1 + (x^q + y^q)(C_2 + (xy)^{i+1}D_2).
\]

Now by Lemma 2.9 \(r_3 + i_{i+1} = n - q\) and so

\[
f = (xy)^{n-q}(x^q - y^q)d_1 + (x^q + y^q)(C_2 + (xy)^{i+1}D_2),
\]

hence by Lemma 3.3 it follows that \(W_+\) is generated by \((xy)^{n-q}(x^q - y^q)\) and \(x^q + y^q\), thus is special. The argument for \(W_-\) is symmetrical.

Summarizing what we have proved:

**Theorem 3.11.** For any \(D_{n,q}\), the following CM modules are special and further they are 2-generated as \(C[x, y]^{D_{n,q}}\)-modules by the following elements:

\[
\begin{align*}
W_+ & : x^q + y^q & (xy)^{n-q}(x^q - y^q) & x^q + y^q & (xy)^{n-q}(x^q - y^q) \\
W_- & : x^q - y^q & (xy)^{n-q}(x^q + y^q) & x^q - y^q & (xy)^{n-q}(x^q + y^q) \\
W_{i+1} & : (xy)^{i+1} & w_2 = w_2^{\Delta_{i+1}}w_3^{\Gamma_{i+1}} & (xy)^{i+1} & v_2 = v_2^{\Delta_{i+1}}v_3^{\Gamma_{i+1}} \\
W_{i+2} & : (xy)^{i+2} & w_2^{\Delta_{i+2}}w_3^{\Gamma_{i+2}} & (xy)^{i+2} & v_2^{\Delta_{i+2}}v_3^{\Gamma_{i+2}} \\
& \vdots & & & \\
W_{i+N} & : (xy)^{i+N} & w_2^{\Delta_N}w_3^{\Gamma_N} & (xy)^{i+N} & v_2^{\Delta_N}v_3^{\Gamma_N}
\end{align*}
\]

where the left column is one such choice of generators, and the right hand column is another choice. Further there are no other non-free rank one specials.

**Proof.** Combine Lemma 3.10, Theorem 3.7 and Theorem 3.8. Since they are distinct and we have \(N + 2 - \nu\) of them, as explained in Section 2 these must be them all.
We can now assign to each of the above specials the corresponding vertex in the minimal resolution. The \( \nu > 0 \) version of the following lemma can be found in [Wem09].

**Lemma 3.12.** Consider \( \mathbb{D}_{n,q} \) with \( \nu = 0 \). Then the specials correspond to the dual graph of the minimal resolution in the following way

\[
\begin{array}{c}
\bullet & -2 \\
\bullet & -\alpha_1 & \cdots & -\alpha_{N-1} & -\alpha_N \\
W_- & W_+ & W_{i_1} & \cdots & W_{i_{N-1}} & W_{i_N}
\end{array}
\]

**Proof.** The assumption \( \nu = 0 \) translates into the condition \( \alpha_1 \geq 3 \). Here the fundamental cycle \( Z_f \) is reduced from which, denoting the exceptional curves by \( \{ E_i \}_{i \in I} \), we can easily calculate

\[
(-Z_f \cdot E_i)_{i \in I} = \begin{array}{c}
\bullet \\
1 & -3+\alpha_1 & -2+\alpha_2 & \cdots & -2+\alpha_{N-1} & -1+\alpha_N
\end{array}
\]

\[
((Z_K - Z_f) \cdot E_i)_{i \in I} = \begin{array}{c}
\bullet \\
1 & -1 & 0 & \cdots & 0 & 1
\end{array}
\]

Now denoting \( R = \mathbb{C}[x,y]^{\mathbb{D}_{n,q}} \) if we consider the quiver of \( \text{End}_R(R \oplus W_+ \oplus W_- \oplus \oplus_{i=1}^N W_i) \) we must be able to see the generators of the specials as compositions of irreducible maps out of \( R \). But by [Wem08, 3.3] using the above intersection theory calculation it follows that we must see the generators of the specials using only compositions of the maps.

Inspecting the list of generators of the specials, it is clear that \( xy \in W_{i_N} \) cannot factor through any of the other specials, thus we must have this as an arrow in the quiver and so consequently \( W_{i_N} \) must correspond to one of the vertices above to which \( R \) connects. The same analysis holds for \( (+) := x^q + y^q \in W_+ \) and \( (-) := x^q - y^q \in W_- \) and so these too must correspond to vertices to which \( R \) connects. Now both \( (+)^2 \) and \( (-)^2 \) belong to \( W_{i_1} \), and it is clear that they factor as \( R \xrightarrow{(+)} W_+ \xrightarrow{ (+)} W_{i_1} \) and \( R \xrightarrow{(-)} W_- \xrightarrow{ (-)} W_{i_1} \) respectively. By inspection of the generators of the specials, both the maps \( W_+ \xrightarrow{ (+)} W_{i_1} \) and \( W_- \xrightarrow{ (-)} W_{i_1} \) are irreducible, hence \( W_{i_1} \) must be a common neighbour to both \( W_+ \) and \( W_- \). This forces the positions of \( W_+ \), \( W_- \) and \( W_{i_1} \) in the dual graph as in the statement, and also forces the position of \( W_{i_N} \) since it must occupy the final vertex which is connected to \( R \). Now the polynomial \( (xy)^{i_1} \) factors as

\[
W_{i_1} \xleftarrow{(xy)^{i_1-i_2}} W_{i_2} \cdots W_{i_{N-1}} \xleftarrow{(xy)^{i_{N-1}-i_N}} W_{i_N} \xrightarrow{xy} R,
\]

forcing the remaining positions. \( \square \)

### 4. The Reconstruction Algebra

In this section we define the reconstruction algebra \( D_{n,q} \) with parameter \( \nu = 0 \); when \( \nu > 0 \) a corresponding definition can be found in [Wem09]. In fact we give two different
presentations of this algebra and prove both are isomorphic to the endomorphism ring of the specials.

Consider, for $N \in \mathbb{N}$ with $N \geq 2$ and for positive integers $\alpha_1 \geq 3$ and $\alpha_i \geq 2$ for all $2 \leq i \leq N$, the labelled Dynkin diagram of type $D$:

$$
\begin{array}{c}
\bullet \\
-2 \quad -a_1 \quad \ldots \quad -a_{N-1} \quad -a_N
\end{array}
$$

We call the left hand vertex the $+$ vertex, the top vertex the $-$ vertex and the vertex corresponding to $a_i$ the $i^{th}$ vertex.

To the labelled Dynkin diagram above we add an extended vertex (called $\star$) and ‘double-up’ as follows:

Now if $\sum_{i=1}^{N} (\alpha_i - 2) \geq 2$ we add extra arrows to the above picture in the following way:

- if $\alpha_1 > 3$ then add an extra $\alpha_1 - 3$ arrows from the 1st vertex to $\star$.
- If $\alpha_i > 2$ with $i \geq 2$ then add an extra $\alpha_i - 2$ arrows from the $i^{th}$ vertex to $\star$.

Label the new arrows (if they exist) by $k_2, \ldots, k_{\sum_{i=1}^{N} (\alpha_i - 2)}$ starting from the 1st vertex and working to the right. Name this new quiver $Q$.

**Example 4.1.** Consider $D_{18,5}$ then $\frac{18}{5} = [4, 3, 2]$ and so the quiver $Q$ is
Example 4.2. Consider $\mathbb{D}_{52,11}$ then $\frac{52}{11} = [5, 4, 3]$ and so the quiver $Q$ is

Now for every $\mathbb{D}_{n,q}$ with $\nu = 0$ denote $k_1 := a_+0$ and $k_1 + \sum_{i=1}^{N}(\alpha_i - 2) := c_{N0}$.

Definition 4.3. For all $1 \leq r \leq 1 + \sum_{i=1}^{N}(\alpha_i - 2)$ define $B_r$ ("the butt") to be the number (or +) of the vertex associated to the tail of the arrow $k_r$.

Notice for all $2 \leq r \leq 1 + \sum_{i=1}^{N}(\alpha_i - 2)$ it is true that $B_r = b_r$ where $b_r$ is the $b$-series of $\frac{n}{q}$ defined in Section 2. However $B_1 \neq b_1$ since $b_1 = \nu + 1 = 1$ whilst $B_1 = +$ by the definition of $k_1$.

Now define $u_+ = 1$ and further for $1 \leq i \leq N$ denote

$$u_i := \max\{j : 2 \leq j \leq e - 2 \text{ with } b_j = i\}$$
$$v_i := \min\{j : 2 \leq j \leq e - 2 \text{ with } b_j = i\}$$

if such things exist (i.e. vertex $i$ has an extra arrow leaving it). Also define $W_1 := +$ and for every $2 \leq i \leq N$ consider the set

$$\mathcal{S}_i = \{\text{vertex } j : 1 \leq j < i \text{ and } j \text{ has an extra arrow leaving it}\}.$$

For $2 \leq i \leq N$ define

$$W_i = \begin{cases} 
+ & \text{if } \mathcal{S}_i \text{ is empty} \\
\text{the maximal number in } \mathcal{S}_i & \text{else}
\end{cases}$$

and so $W_i$ is defined for all $1 \leq i \leq N$. The idea behind it is that $W_i$ records the closest vertex to the left of vertex $i$ which has a $k$ leaving it; since we have defined $k_1 := a_+0$ this is always possible to find. Now define, for all $1 \leq i \leq N$, $V_i = uW_i$. Thus $V_i$ records the number of the largest $k$ to the left of the vertex $i$, where since $k_1 := a_+0$ and $u_+ = 1$ it always exists.

Definition 4.4. For $\frac{n}{q} = [\alpha_1, \ldots, \alpha_N]$ with $\nu = 0$ (i.e. $\alpha_1 \geq 3$) define $D_{n,q}$, the reconstruction algebra of type $D$, to be the path algebra of the quiver $Q$ defined above subject to the relations

1. $c_0+c_1 - c_0-c_1 = 4A_{01}$
2. $c_0+a_+0 = c_0-a_-0$
3. $a_-0c_0- = c_-1a_1-$
4. $a_1+c_+1 = a_1-c_1$
together with the relations defined algorithmically as:

**Step 0:** $c_{+0}a_{0+} = c_{+1}a_{1+}$.

**Step 1:** If $\alpha_1 = 3$ then $c_{12}a_{21} = a_{1+}c_{+1}$.

- If $\alpha > 3$ then $k_2A_{01} = a_{1+}c_{+1}, A_{01}k_2 = c_{0+}a_{0+}$.
- $k_1C_{01} = \nu_{t+1}A_{01}, C_{01}k_1 = A_{01}k_{t+1} \forall 2 \leq t < u_1$.
- $k_u, C_{01} = c_{12}a_{21}$.

**Step i:** If $\alpha = 2$ then $c_{ii+1}a_{i+1} = a_{i-1}c_{i-1}$.

- If $\alpha > 2$ then $k_vA_{0i} = a_{ii-1}c_{i-1}, A_{0i}k_v = C_{0B_VV_i}k_V$.
- $k_iC_{01} = \nu_{t+1}A_{0i}, C_{01}k_i = A_{0i}k_{t+1} \forall v_i \leq t < u_i$.
- $k_u, C_{0i} = c_{ii+1}a_{i+1}$.

**Step N:** If $\alpha_N = 2$ then $c_{Na0}a_{0N} = a_{N-1}c_{N-1}, C_{0B_VN}k_VN = a_{0N}c_{N0}$.

- If $\alpha > 2$ then $k_vA_{0N} = a_{N-1}c_{N-1}, a_{0N}k_v = C_{0B_VN}k_VN$.
- $k_iC_{0N} = \nu_{t+1}A_{0N}, C_{0N}k_i = a_{0N}k_{t+1} \forall v_N \leq t < u_N$.

where $A_{0t} := a_{0n} \ldots a_{i+1}$ for every $1 \leq t \leq N$. The $C$’s are defined as follows: we define $C_{0+} := c_{+0}$, and $C_{0i} := c_{i2} \ldots c_{i+1}$ for all $1 \leq t \leq N$. The only thing that remains to be defined is $C_{01}$, and it is this which changes according to the presentation, namely

$$C_{01} := \begin{cases} 
\frac{c_{0+}c_{+1}}{2} & \text{in moduli presentation} \\
\frac{c_{0+}c_{+1} + c_{0-}c_{-1}}{2} & \text{in symmetric presentation}
\end{cases}.$$

The colour red is used so that in the above definition it is clear which relations can change depending on the presentation.

**Remark 4.5.** We should explain why we give two presentations. The symmetric case is pleasing since it treats the two $(-2)$ horns equally, so that the algebra produced is independent of how we view the dual graph (see Lemma 4.13). On the other hand the moduli presentation treats one of the $(-2)$ horns (namely +) to be ‘better’ as relations go through that vertex and not the − vertex. The moduli presentation makes the explicit geometry easier to write down in Section 5, and is also satisfactory from the viewpoint of Remark 4.6 below. We show in Theorem 4.11 that the two presentations yield isomorphic algebras, but note that the explicit isomorphism is difficult to write down. For the moment denote $D_{n,q}$ for the moduli presentation and $D'_{n,q}$ for the symmetric presentation, as a priori they may be different.

**Remark 4.6.** In the moduli presentation (i.e. $C_{01} = c_{0+}c_{+0}$) the algorithmic relations are precisely the same as those for the reconstruction algebra of type $A$ associated to the data

$$-2 \ldots -\alpha_1 \ldots -\alpha_2 \ldots -\alpha_{N-1} \ldots -\alpha_N$$

Consequently you should think of the moduli presentation of the reconstruction algebra of type $D$ for $\nu = 0$ to simply be a reconstruction algebra of type $A$ with a bit stuck on to compensate for the dihedral horns.

**Remark 4.7.** Since $C_{0+} := c_{+0}$, the two presentations are exactly the same if and only if $\alpha_1 = 3$ and $\alpha_2 = \ldots = \alpha_N = 2$, i.e. the family $D_{2s+1,s}$ of Example 6.13. This corresponds to the ‘base case’ of Weym08, 3.6. Note also that the use of $B$ (instead of the $b$-series $b$) in the above definition is not a typo since $W_N = +$ is certainly possible (e.g. in the family $D_{2s+1,s}$) in which case $V_N = u_+ = 1$; consequently $B_{V_N} = B_1 = +$, which is different to $b_1 = 1$. Thus using the $b$-series $b$ can take us to the wrong vertex.

**Remark 4.8.** Double care must be taken to get the algorithmic relations here from the ones in Weym07. Firstly loc. cit labels the extra arrows $k$ in the other direction and secondly it also begins labelling with a $k_1$ instead of $k_2$ (which we use here). The fact that the direction
of the labelling of the $k$'s has changed is awkward but it doesn't matter due to the duality for reconstruction algebras of type $A$ (see [Wem07, 2.10]).

**Example 4.9.** For the group $D_{18,5}$ the symmetric presentation of the reconstruction algebra is the quiver in Example 4.1 subject to the relations
\[
\begin{align*}
c_0 + c_1 &- c_0 - c_1 = 4(a_{03}a_{32}a_{21}) \\
c_0 + a_1 &+ a_0 = c_0 - a_0 \\
a_1 + c_0 &+ c_1 = c_0 - a_1 \\
a_1 + c_1 &+ a_0 = a_1 - c_0
\end{align*}
\]
\[
\begin{align*}
a_1 + c_1 &+ a_0 = k_2(a_{03}a_{32}a_{21}) \\
k_2 &+ a_0 = k_2(a_{03}a_{32}a_{21}) \\
(\frac{1}{2}c_0 + a_0 &+ c_0 - c_1) = c_1 \\
k_2 &+ a_0 = (\frac{1}{2}(c_0 + a_0 + c_0 - c_1)) = c_1 \\
(\frac{1}{2}(c_0 + a_0 + c_0 - c_1)) &+ c_1 = c_2
\end{align*}
\]

**Example 4.10.** For the group $D_{52,11}$ the moduli presentation of the reconstruction algebra is the quiver in Example 4.2 subject to the relations
\[
\begin{align*}
c_0 + c_1 &- c_0 - c_1 = 4a_{03}a_{32}a_{21} \\
c_0 + a_1 &+ a_0 = c_0 - a_0 \\
a_1 + c_0 &+ c_1 = c_0 - a_1 \\
a_1 + c_1 &+ a_0 = a_1 - c_0
\end{align*}
\]
\[
\begin{align*}
a_1 + c_1 &+ a_0 = k_2(a_{03}a_{32}a_{21}) \\
k_2 &+ a_0 = k_2(a_{03}a_{32}a_{21}) \\
(\frac{1}{2}c_0 + a_0 &+ c_0 - c_1) = c_1 \\
k_2 &+ a_0 = (\frac{1}{2}(c_0 + a_0 + c_0 - c_1)) = c_1 \\
(\frac{1}{2}(c_0 + a_0 + c_0 - c_1)) &+ c_1 = c_2
\end{align*}
\]

The following is the main theorem of this paper.

**Theorem 4.11.** For a group $D_{n,q}$ with parameter $\nu = 0$ (i.e. reduced fundamental cycle), denote $R = C[x,y]^D_{n,\ast}$ and let $T_{n,q} = R \oplus W_+ \oplus W_- \oplus \bigoplus_{i=1}^N W_i$ be the sum of the special CM modules. Then
\[
D_{n,q} \cong \text{End}_R(T_{n,q}) \cong D'_{n,q}.
\]

**Proof.** We prove both statements at the same time, by making different choices for the generators of the specials. Using the intersection theory in the proof of Lemma 3.12 it follows immediately from [Wem07, 3.3] that the quiver of the endomorphism ring of the specials is precisely that of the quiver $Q$ defined above. We first find representatives for the known number of arrows:

As before denote $x^q + y^q = (+)$ and $x^q - y^q = (-)$. We must reach the generators of the specials as paths out of $R$ (i.e. $\ast$). We know from the proof of Lemma 3.12 we may choose $c_{0+} = (+)$, $c_{1+} = (+)$, $c_{0-} = (-)$, $c_{1-} = (-)$, $a_{0N} = xy$ and $a_{1+} = (xy)^{\nu}$ for all $1 \leq t < N$ as representatives. Now the generator $(xy)^{n-q}(+) of W_-$ must be reached through $W_i$. Since $(xy)^{n-q}(+) = (a_{0N} \ldots a_{21})(xy)^{n-2q}(+)$ we may choose $a_{1-} = (xy)^{n-2q}(+) = (xy)^{\nu}(-)$. By symmetry we may choose $a_{1+} = (xy)^{\nu}(-)$.

Now consider the generator $w_2 \Delta_2 w_3^{12}$ of $W_2$. We already have the generator $w_2 = c_0 + c_1$ from $R$ to $W_i$, so it is clear that we may choose $c_{12} = w_2^{c_2} w_3^{d_2}$. If we consider the generator $v_2$ of $W_i$, instead (which we have as $w_2^{c_2} w_3^{d_2}$) and want to obtain the generator $v_2 \Delta_2 w_3^{12}$ of $W_{i2}$, instead choose $c_{12} = v_2^{c_2} v_3^{d_2}$. Continuing like this we can choose $c_{it+1} = w_2^{c_i} w_3^{d_i}$ for all $1 \leq t < N$. 


Now it is also fairly clear that we may choose $c_{N0} = w_2^{c_{iN}} w_3^{d_{iN}} = w_2^{c_{i-1}} w_3^{d_{i-1}}$. To see this firstly note that $w_2^{c_{i-1}} w_3^{d_{i-1}}$ doesn’t factor through $a_{N-1}$ (the only possible map to the non-trivial specials). Secondly note that $w_2^{c_{i-1}} w_3^{d_{i-1}}$ can’t factor as some map from $W_{i_k}$ to $R$ multiplied by a non-scalar invariant else the invariant generator $a_{N0}c_{N0} = (xy)^{r+1} w_2^{c_{i-1}} w_3^{d_{i-1}}$ factors into two non-scalar invariants, contradicting the embedding dimension. A similar argument shows that we may choose $a_{+0} = (xy)^{r}(-)$ and $a_{-0} = (xy)^{r}(+)$. Hence we have labelled all arrows in $Q$ by polynomials, except $k_2, \ldots, k_{t-3}$ (if the $k$ arrows exist). How to do this is obvious by the quiver $D_1$ in Section 3: if the $k$’s exist we can choose $k_t$ labelled by $k_t = (xy)^{r+2} w_2^{c_{i+1}} w_3^{d_{i+1}}$. The argument that these choices don’t factor through other specials via maps of strictly positive lower degrees (and so can be chosen as representatives) is similar to the above - for example if $a_1 \geq 4$ consider $(xy)^{r} w_2^{c_2} w_3^{d_2}$. Firstly it does not factor through maps we have already chosen, since if it does we may write $(xy)^{r} w_2^{c_2} w_3^{d_2} = a_1 - f + a_1 + g + c_1 h = (xy)^r F + w_2^{c_1} w_3^{d_1} h$. But by looking at $xy$ powers we know $(xy)^r$ divides $h$ and so after cancelling factors we may write $w_2^{c_1} w_3^{d_1} = (xy)^{r-1} F$ + $w_2^{c_1} w_3^{d_1} h_1$. After cancelling $w_2^{c_1} w_3^{d_1}$ (which $F$ must be divisible by) $1 = w_2^{c_1} w_3^{d_1} - d_1 h_1 + (xy)^{r-1} F'$ which is impossible since the right hand side cannot have degree zero terms. Secondly, $(xy)^r w_2^{c_2} w_3^{d_2}$ does not factor as a map $W_{i_k} \rightarrow R$ followed by a non-scalar invariant since again this would contradict the embedding dimension. Hence we may choose $k_3 = (xy)^r w_2^{c_3} w_3^{d_3}$ in this case. Continue like this: if $a_1 \geq 5$ we want to choose $k_3 = (xy)^r w_2^{c_3} w_3^{d_3}$. If it factors through maps we have already chosen then we may write $k_3 = a_1 - f + a_1 + g + k_2 j + c_1 h = (xy)^{r} F + c_1 h$, so just repeating the argument above shows that this is impossible. By the above, we have justified that we may choose the following as representatives of all the irreducible maps between the specials

and further in the above picture we also have, if $i \leq N$ that the extra arrows $k_t$ labelled by $k_t = (xy)^{r+2} w_2^{c_{i+1}} w_3^{d_{i+1}}$. The symmetric presentation choices are identical, except everywhere we replace $w_2$ by $y_2$ and $w_3$ by $v_3$.

Denote the relations in Definition 4.4 by $S'$. For the relations part of the proof, below we are really working in the completed case (so we can use [W08, 3.3] and [BIRS, 3.4]) and we prove that the completion of the endomorphism ring of the specials is given as the completion of $CQ$ (denoted $\hat{CQ}$) modulo the closure of the ideal $\langle S' \rangle$ (denoted $\langle \hat{S}' \rangle$). The non-completed version of the theorem then follows by simply taking the associated graded ring of both sides of the isomorphism.

Now denote the kernel of the surjection $\hat{CQ} \rightarrow \text{End}_{C\langle[x,y]\rangle}(T_{N,n}) := A$ by $I$, denote the radical of $\hat{CQ}$ by $J$ and further for $t \in \{*, +, - , 1, \ldots, N\}$ denote by $S_t$ the simple corresponding to the vertex $t$ of $Q$. In Lemma 4.12 below we show that the elements of $S'$ are linearly independent in $I/(IJ +JI)$. Thus we may extend $S'$ to a basis $S$ of $I/(IJ +JI)$. Now by [BIRS, 3.4(a)] $I = \langle S \rangle$ and further by [BIRS, 3.4(b)]

$$\dim C \text{ Ext}_A^2(S_a, S_b) = \#(e_a CQ e_b) \cap S$$
for all \(a, b \in \{\ast, +, -, 1, \ldots, N\}\). But on the other hand using the intersection theory in the proof of Lemma 3.12 it follows immediately from [Wem08 3.3] that

\[
\begin{align*}
\dim \mathbb{C} \operatorname{Ext}^2_\Lambda(S_+, S_+) &= 1 \\
\dim \mathbb{C} \operatorname{Ext}^2_\Lambda(S_-, S_-) &= 1 \\
\dim \mathbb{C} \operatorname{Ext}^2_\Lambda(S_i, S_i) &= \alpha_i - 1 \\
\dim \mathbb{C} \operatorname{Ext}^2_\Lambda(S_i, S_+) &= 1 + \sum_{p=1}^N (\alpha_p - 2) \\
\dim \mathbb{C} \operatorname{Ext}^2_\Lambda(S_+, S_i) &= 1
\end{align*}
\]

for all \(1 \leq i \leq N\), and further all other \(\operatorname{Ext}^2\)s between the simples are zero. By inspection of both the above information and the relations \(S'\) we notice that

\[
\dim \mathbb{C} \operatorname{Ext}^2_\Lambda(S_a, S_b) = \#(e_a \mathcal{C} \mathcal{Q} e_b) \cap S'
\]

for all \(a, b \in \{\ast, +, -, 1, \ldots, N\}\). Hence

\[
\#(e_a \mathcal{C} \mathcal{Q} e_b) \cap S = \#(e_a \mathcal{C} \mathcal{Q} e_b) \cap S'
\]

for all \(a, b \in \{\ast, +, -, 1, \ldots, N\}\), proving that the number of elements in \(S\) and \(S'\) are the same. Hence \(S' = S\) and so \(I = \langle S' \rangle\), as required. \(\square\)

**Lemma 4.12.** With notation from the above proof, the members of \(S'\) are linearly independent in \(I/(IJ + JI)\).

**Proof.** Firstly it is easy to verify that all the members in \(S'\) are satisfied by the chosen representatives of the arrows and so belong to \(I\). To see this, note that the first four relations follow immediately by inspection (independent of presentation), as does the Step 0 relation. For the moduli presentation the remaining algorithmic relations are simply the pattern between the invariants and cycles in \(D_4\) from Proposition [3.3] which is the same as the pattern in Type \(A\). The symmetric pattern is just a small modification of this, namely the pattern between the invariants and cycles in the \(D_3\) version of Proposition [3.3]. Thus in either presentation \(S' \subseteq I\).

In what follows we say that a word \(w\) in the path algebra \(\mathcal{C} \mathcal{Q}\) satisfies condition (A) if

(i) It does not contain some proper subword which is a cycle.

(ii) It does not contain some proper subword which is a path from \(\ast\) to 1.

As stated in the proof of the above theorem, we know from the intersection theory and [Wem08 3.3] that the ideal \(I\) is generated by one relation from \(\ast\) to 1, whereas all other generators are cycles. Consequently if a word \(w\) satisfies (A) then \(w \notin IJ + JI\). It is also clear that \(c_{0+}c_{+1} - c_{0-}c_{-1} - 4A_{01} \notin IJ + JI\).

Now since all members of \(S'\) are either cycles at some vertex or paths from \(\ast\) to 1, to prove that the members of \(S'\) are linearly independent in \(I/(IJ + JI)\) we just need to show that

1. the elements of \(S'\) which are paths from \(\ast\) to 1 are linearly independent in \(e_\ast(I/(IJ + JI))e_1\).

2. for all \(t \in \{\ast, +, -, 1, \ldots, N\}\), the elements of \(S'\) that are cycles at \(t\) are linearly independent in \(e_t(I/(IJ + JI))e_1\).

The first condition is easy, since the only relation in \(S'\) from \(\ast\) to 1 is \(c_{0+}c_{+1} - c_{0-}c_{-1} = 4A_{01}\), and we have already noted that it does not belong to \(IJ + JI\), thus it is non-zero and so linearly independent in \(e_\ast(I/(IJ + JI))e_1\). Note also that there are only three paths of minimal grade from \(\ast\) to 1, namely \(\{c_{0+}c_{+1}, c_{0-}c_{-1}, A_{01}\}\), and by inspection of the polynomials they represent we do not have any other relation from \(\ast\) to 1 of this grade.

For the second condition, we must check \(t\) case by case:

**Case** \(t = +\). Here the only relation in \(S'\) from \(+\) to \(+\) is \(a_{+0}c_{+0} = c_{+1}a_{+1}\) (the Step 0 relation) so it is linearly independent provided it is non-zero, i.e. \(a_{+0}c_{+0} - c_{+1}a_{+1} \notin IJ + JI\). But since \(a_{+0}c_{0+}\) satisfies condition (A) we know that \(a_{+0}c_{0+} \notin IJ + JI\). Thus if \(a_{+0}c_{0+} - c_{+1}a_{+1} \in IJ + JI\) we may write

\[
a_{+0}c_{0+} = c_{+1}a_{+1} + u
\]
in the free algebra $\mathcal{C} \hat{Q}$ for some $u \in IJ + JJ$. But the term $a_{-0}c_{0+}$ does not appear in the right hand side, a contradiction.

Case $t = -$. The only relation in $S'$ from $- \to -$ is $a_{-0}c_{0-} = c_{-1}a_{1-}$ (the third relation). A symmetrical argument to the above shows that it is linearly independent.

Case $t = 1$. If $a_{1} = 3$ then the only members of $S'$ from $1 \to 1$ are $c_{12}a_{21} = a_{1} + c_{+1}$ (the Step 1 relation) and $a_{1} + c_{+1} = a_{1} - c_{-1}$ (the fourth relation). Suppose that

$$
\lambda_{1}(c_{12}a_{21} - a_{1} + c_{+1}) + \lambda_{2}(a_{1} + c_{+1} - a_{1} - c_{-1}) = 0
$$

in $e_{1}(I/(IJ + JJ))e_{1}$. Then we may write

$$
\lambda_{1}c_{12}a_{21} + \lambda_{2}a_{1} + c_{+1} = \lambda_{1}a_{1} + c_{+1} + \lambda_{2}a_{1} - c_{-1} + u
$$

in the free algebra $\mathcal{C} \hat{Q}$ for some $u \in IJ + JJ$. But $c_{12}a_{21}$ satisfies (A) so doesn’t appear anywhere in the right hand side, forcing $\lambda_{1} = 0$. But now since $\lambda_{1} = 0$ and further $a_{1} + c_{+1}$ satisfies (A), it cannot appear on the right hand side, forcing $\lambda_{2} = 0$. This proves the assertion when $a_{1} = 3$ and hence we may assume that $a_{1} > 3$, in which case the only relations in $S'$ from $1 \to 1$ are

$$
a_{1} - c_{-1} = a_{1} + c_{+1}
\quad k_{2}A_{01} = a_{1} + c_{+1}
\quad k_{1}C_{01} = k_{t+1}A_{01} \forall 2 \leq t < u_{1}
\quad k_{u_{1}}C_{01} = c_{12}a_{21}.
$$

(i.e. the fourth relation and some of the Step 1 relations). Now suppose that

$$
\lambda_{1}(a_{1} - c_{-1} - a_{1} + c_{+1}) + \lambda_{2}(a_{1} + c_{+1} - k_{2}A_{01}) + \sum_{p=2}^{u_{1}-1} \lambda_{p+1}(k_{p+1}A_{01} - k_{p}C_{01})
\quad + \lambda_{a_{1} - 1}(c_{12}a_{21} - k_{u_{1}}C_{01}) = 0
$$

in $e_{1}(I/(IJ + JJ))e_{1}$ then

$$
\lambda_{1}a_{1} - c_{-1} + \lambda_{2}a_{1} + c_{+1} + \sum_{p=2}^{u_{1}-1} \lambda_{p+1}k_{p+1}A_{01} + \lambda_{a_{1} - 1}c_{12}a_{21}
\quad = \lambda_{1}a_{1} + c_{+1} + \lambda_{2}k_{2}A_{01} + \sum_{p=2}^{u_{1}-1} \lambda_{p+1}k_{p}C_{01} + \lambda_{a_{1} - 1}k_{u_{1}}C_{01} + u
$$

in the free algebra $\mathcal{C} \hat{Q}$ for some $u \in IJ + JJ$. Now $a_{1} - c_{-1}$ satisfies (A) and so cannot appear on the right hand side; consequently $\lambda_{1} = 0$. This combined with the fact that $a_{1} + c_{+1}$ satisfies (A) implies that $\lambda_{2} = 0$. Similarly $c_{12}a_{21}$ satisfies (A) and so can’t appear on the right hand side, so $\lambda_{a_{1} - 1} = 0$. This leaves

$$
(2) \quad \sum_{p=2}^{u_{1}-1} \lambda_{p+1}k_{p+1}A_{01} = \sum_{p=2}^{u_{1}-1} \lambda_{p+1}k_{p}C_{01} + u
$$

in the free algebra $\mathcal{C} \hat{Q}$ and so

$$
\lambda_{u_{1}}k_{u_{1}}A_{01} \equiv \text{terms starting with } k \text{ of strictly smaller index}
\mod IJ + JJ. But since $k_{u_{1}}A_{01}$ does not have any subwords which are cycles, the only way we can change it mod $IJ + JJ$ is to bracket as $k_{u_{1}}(A_{01})$ and use the relation in $I$ from $\ast$ to $1$. Doing this we get $k_{u_{1}}A_{01} \equiv \frac{1}{4}k_{u_{1}}(c_{0+}c_{+1} - c_{0-}c_{-1})$. This still does not start with a $k$ of strictly lower index, so we must again use relations in $IJ + JJ$ to change the terms. But again the words contain no subwords which are cycles, which means we must either change $c_{0+}c_{+1}$ or $c_{0-}c_{-1}$. But there is only one relation between $c_{0+}c_{+1}$, $c_{0-}c_{-1}$ and $A_{01}$ so no matter what we do we arrive back as

$$
k_{u_{1}}A_{01} \equiv \frac{1}{4}k_{u_{1}}(c_{0+}c_{+1} - c_{0-}c_{-1}) \equiv k_{u_{1}}A_{01}.$$

mod $IJ + JI$. Thus mod $IJ + JI$ it is impossible to transform $k_{u_1}A_{01}$ into an expression
involving $k$ terms with strictly lower index, and so we must have $\lambda_{u_1} = 0$. With this in mind
we may re-arrange (2) to get
\[\lambda_{u_1-1}k_{u_1-1}A_{01} \equiv \text{terms starting with } k \text{ of strictly smaller index}\]
mod $IJ + JI$ and so repeating the above argument gives $\lambda_{u_1-1} = 0$. Continuing in this
fashion we deduce all $\lambda$’s are zero, as required.

Case $t$ for $1 < t \leq N$. If $\alpha_t = 2$ then the only relation in $S'$ from $t$ to $t$ is $a_{tt-1}c_{t-1}t = c_{tt+1}a_{t+1}t$ (the Step $t$ relation). This is linearly independent in $\epsilon_t(I/(IJ + JI))\epsilon_t$ using the
same argument as in the case $t = +$. Hence we may assume $\alpha_t > 2$ in which case the only
relations in $S'$ from $t$ to $t$ are
\[
k_{v_t}A_{0t} = a_{tt-1}c_{t-1}t
k_pC_{0t} = k_{p+1}A_{0t} \forall v_t \leq p < E
k_EC_{0t} = c_{tt+1}a_{t+1}t
\]
(i.e. some of the Step $t$ relations) where $E := \{ u_t \text{ if } t < N \} \cup \{ u_N - 1 \text{ if } t = N \}$ and also we mean $c_{NN+1} = c_{N0}$ and $a_{N+1N} = a_{0N}$ if at any place the subscripts become too large. Now if
\[
\lambda_1(k_{v_t}A_{0t} - a_{tt-1}c_{t-1}t) + \sum_{p=0}^{E-v_t-1} \lambda_{p+2}(k_{p+v_t}C_{0t} - k_{p+v_t+1}A_{0t}) + \lambda_{t-1}(k_EC_{0t} - c_{tt+1}a_{t+1}t) = 0
\]
in $\epsilon_t(I/(IJ + JI))\epsilon_t$ then
\[
\lambda_1a_{tt-1}c_{t-1}t + \sum_{p=0}^{E-v_t-1} \lambda_{p+2}k_{p+v_t+1}A_{0t} + \lambda_{t-1}c_{tt+1}a_{t+1}t =
\lambda_1k_{v_t}A_{0t} + \sum_{p=0}^{E-v_t-1} \lambda_{p+2}k_{p+v_t}C_{0t} + \lambda_{t-1}k_EC_{0t} + u
\]
in the free algebra $\mathbb{C}\hat{Q}$ for some $u \in IJ + JI$. But now all terms on the left hand side satisfy
(A) and so none of them can appear on the right hand side, forcing $\lambda_1 = \ldots = \lambda_{t-1} = 0$ as
required.

Case $t = \star$. The only relations left in $S'$ are those from $\star$ to $\star$, which is the second relation
together with the remaining relations from the algorithm. These are precisely

\[
\begin{align*}
&\text{If } \alpha_1 > 3 \quad \{ \begin{array}{ll}
c_{0+a_+0} = c_{0-a_0} \\
A_{01}k_2 = c_{0+a_0} \\
C_{0t}k_t = A_{01}k_{t+1} \forall 2 \leq t < u_1 \\
&\vdots
\end{array} \\
&\text{If } \alpha_t > 2 \quad \{ \begin{array}{ll}
A_{0t}k_{v_t} = C_{0v}V_t k_{V_t} \\
C_{0t}k_t = A_{0t}k_{t+1} \forall v_t \leq t < u_i \\
&\vdots
\end{array} \\
&\text{If } \alpha_N = 2 \quad \{ \begin{array}{ll}
a_{0N}c_{N0} = C_{0v}V_N k_{V_N} \\
a_{0N}k_{v_N} = C_{0v}V_N k_{V_N} \\
&\vdots
\end{array} \\
&\text{If } \alpha_N > 2 \quad \{ \begin{array}{ll}
a_{0N}k_{t+1} = C_{0N}V_t k_{V_N} \forall v_N \leq t < u_N
\end{array}
\end{align*}
\]

Hence if $\alpha_1 = 3$, $\alpha_2 = \ldots = \alpha_N = 2$ then there are no extra arrows $k$ and so the only relations
at $\star$ are $c_{0+a_+0} = c_{0-a_0}$ and $a_{0NC_{N0}} = c_{0+a_0}$. These are linearly independent by using
the same argument as in the case $t = 1$ with $\alpha_1 = 3$. Thus we may assume that some extra
$k$ arrows exist, in which case the proof is similar to the case $t = 1$ with $\alpha_1 > 3$.

The input to define the reconstruction algebra is a certain labelled Dynkin diagram of
type $D$, where the two ‘horns’ are both $(-2)$ curves. These are geometrically indistinguishable
in the sense that their positions in the dual graph can be swapped and the input for the
reconstruction algebra does not change. Thus the reconstruction algebra should be invariant under this change of labels, which leads us to the following.

**Lemma 4.13.** For $\lambda \in \mathbb{C}^*$ denote $D'_{n,q}(\lambda)$ to be the algebra obtained from the symmetric presentation $D'_{n,q}$ by replacing the number 4 in the first relation by $\lambda$. Similarly define $D_{n,q}(\lambda)$. Then for all $\lambda \in \mathbb{C}^*$

$$D'_{n,q}(\lambda) \cong D'_{n,q} \cong D_{n,q} \cong D_{n,q}(\lambda).$$

In particular the algebra obtained from $D'_{n,q}$ by everywhere swapping $+$ and $-$ is isomorphic to $D'_{n,q}$.

**Proof.** We prove that $D'_{n,q}(\lambda) \cong D'_{n,q}$; the proof that $D_{n,q}(\lambda) \cong D_{n,q}$ is identical. All we must do is change the choices of the labels of the arrows made for $D'_{n,q}$ in the proof of Theorem 4.11 such that the relation $(1^t) c_0 c_{-} = \lambda A_01$ together with all the other original relations (except relation 1) hold. Since the only changes we shall make to the choices of arrows in the proof of Theorem 4.11 is by multiplying them by a non-zero scalar (for an arrow $p$, denote by $\kappa_p$ this non-zero scalar), the argument of Theorem 4.11 again goes through to show that $D'_{n,q}(\lambda) \cong \text{End}_R(T_{n,q})$ and so in particular $D'_{n,q}(\lambda) \cong D'_{n,q}$.

Choose $\kappa_{c_0} = \kappa_{c_{-}} = \kappa_{a_0} = \kappa_{c_{+1}} = \kappa_{a_1} = \kappa_{a_{1-1}} = \kappa_{a_{1+1}} = 1$, $\kappa_{a_0 N} = \frac{1}{\lambda}$ and $\kappa_{a_{NN-1}} = \ldots = \kappa_{a_{21}} = 1$. Then certainly relations $1^t, 2, 3$ and 4 hold, as does the Step 0 relation. What remains is to choose $\kappa_{c_{12}}, \ldots, \kappa_{c_{N-1 N}}, \kappa_{c_{N 0}}$ (and the $\kappa_k$ for the $k$ arrows if they exist) and to verify the remaining relations.

Consider the $b$-series of $\frac{c}{\lambda}$. If $b_2 > 1$ then choosing $\kappa_{c_{12}} = \ldots = \kappa_{c_{k_2-1} k_2} = 1$ it is clear that the step 1 to the step $b_2 - 1$ relations hold. Thus in all cases we can consider the $b_2$ relations, knowing the previous step relations hold.

Now if $\alpha_{b_2} = 2$ then $b_2 = N$ and there are no $k$ arrows, so by choosing $\kappa_{c_{k_0} N} = \frac{1}{\lambda}$ it is clear that the Step $N$ relations hold so we are done. Hence we can assume that $\alpha_{b_2} > 2$, in which case choosing $\kappa_{b_2} = \frac{1}{\lambda}, \ldots, \kappa_{k_{b_2} k_{b_2+1}} = \frac{1}{\lambda}$ the step $b_2$ relations hold.

Thus inductively we consider the Step $t$ relations with $\kappa_{c_{12}}, \ldots, \kappa_{c_{t-1} t}$ and $k_2, \ldots, k_{V_t}$ already chosen such that all relations up to and including Step $t - 1$ are satisfied. If $\alpha_{b_2+1} = 2$ choose $\kappa_{c_{t+1 t}} = \kappa_{c_{t-1 t}}$, then the Step $t$ relations hold. Else choose $\kappa_{b_t} = \kappa_{c_{t-1 t}}(\frac{1}{\lambda}), \ldots, \kappa_{a_t} = \kappa_{c_{t-1 t}}(\frac{1}{\lambda})(u_t - v_t) + 1$, $\kappa_{c_{t+1 t}} = \kappa_{c_{t-1 t}}(\frac{1}{\lambda})(u_t - v_t) + 1$ then the Step $t$ relations hold. This concludes the induction step, hence the result follows.

The final statement is now immediate since by inspection of the symmetric presentation the algebra obtained by everywhere swapping $+$ and $-$ is just $D'_{n,q}(-4)$.

\[ \square \]

5. The moduli space of representations

In this section we use quiver GIT on the reconstruction algebra $D_{n,q}$ to obtain the minimal resolution of $\mathbb{C}^2/D_{n,q}$ and so obtain the slightly stronger statement that the special representations not only give the dual graph, but also the whole space.

As in type $A$, fix for the rest of this paper the moduli presentation of the reconstruction algebra, the dimension vector $\alpha = (1, 1, \ldots, 1)$ and the generic stability condition $\theta = (-N + 2, 1, \ldots, 1)$. Notice that a representation $M$ of dimension vector $\alpha$ is $\theta$-stable if and only if it is generated from vertex $\star$, i.e. for every vertex in the representation $M$ there is a non-zero path in $M$ from $\star$ to that vertex. Again, as in type $A$, there is a huge subtlety here since choosing a different stability condition can give us something singular (in fact, not even normal).

Throughout this section we use the moduli presentation convention that $C_{01} = c_0 c_{+1}$ whereas $C_{0t} = C_{01} c_{12} \ldots c_{t-1 t}$ for any $2 \leq t \leq N$. For $D_{n,q}$ with $Z_f$ reduced, we claim (and
prove in Lemma 5.2 that the moduli space is covered by the following \( N + 3 \) open sets:

\[
\begin{align*}
U_0 & \quad C_{0N} \neq 0, c_{0-} \neq 0 \quad (a_{1-}, c_{N0}, a_{0N}) \\
\vdots \\
U_t & \quad C_{0N-t} \neq 0, c_{0-} \neq 0, A_{0N-t+1} \neq 0 \quad (a_{1-}, c_{N-tN-t+1}, a_{N-t+1}) \\
\vdots \\
U_N & \quad c_{0+} \neq 0, c_{0-} \neq 0, A_{01} \neq 0 \quad (a_{1-}, a_{1+}, c_{+1}) \\
U_+ & \quad c_{0+} \neq 0, A_{01} \neq 0, a_{1-} \neq 0 \quad (c_{0-}, a_{1+}, a_{-0}) \\
U_- & \quad c_{0-} \neq 0, A_{01} \neq 0, a_{1+} \neq 0 \quad (c_{0+}, a_{1-}, a_{+0})
\end{align*}
\]

where in the above we have stated for reference the result of Lemma 5.3, which gives the position of where (if we change basis so that the specified non-zero arrows are actually the identity) the co-ordinates can be read off the quiver. In fact in Lemma 5.3 we prove a little more; we show, also reading off the quiver, that these open sets are given abstractly by the following smooth hypersurfaces in \( \mathbb{C}^3 \):

\[
(0 \leq t \leq N - 1) \quad U_t \quad a(1 - 4b\sum_{i=1}^{N} \eta_i^{(i)} c_{N-t}^{(i)} \theta_i^{(i)}) = b^{N+1} c_t^{N+1}
\]

\[
U_N \quad a(c - 4) = bc \\
U_+ \quad b(a^2 + c + 4) = ac \\
U_- \quad b(a^2 c - 4) = ac
\]

where the \( \eta \) and \( \theta \) are specified combinatorics introduced in the proof of Lemma 5.3. Also by taking an arbitrary stable module \( M \) and simply changing basis appropriately the above open cover translates into ratios of polynomials as:

\[
\begin{align*}
U_0 & \quad \mathbb{C}\frac{(xy)^{n+3}(-)}{(x^3)} \quad (xy)^2 + 2\Delta_{N+1} + \Gamma_{N+1} \quad (xy)^{N+1} \\
\vdots \\
U_t & \quad \mathbb{C}\frac{(xy)^{n+3}(-)}{(x^3)} \quad (xy)^2 + 2\Delta_{N-t+1} + \Gamma_{N-t+1} \quad (xy)^{N-t+1} \\
\vdots \\
U_N & \quad \mathbb{C}\frac{(xy)^{n-3}(+)}{(x^3)} \quad (xy)^2 + \Gamma_{N-t} \quad (xy)^{N-t} \\
U_+ & \quad \mathbb{C}\frac{(xy)^{n-3}(+)}{(x^3)} \quad (xy)^2 + (\gamma)^2 \\
U_- & \quad \mathbb{C}\frac{(xy)^{n-3}(+)}{(x^3)} \quad (xy)^2 + (\gamma)^{-2}
\end{align*}
\]

where \( (+) = x^q + y^q \) and \( (-) = x^q - y^q \).

Note that there is a choice of coordinates in \( U_N \) since for the third coordinate \( c \) we could instead choose \( d = \frac{c_{0+} - c_{0-}}{c_{0+}} = \frac{x^3}{(xy)^3} \) since they differ by \( 4 \). Picking this alternative co-ordinate changes the defining equation to \( ad = b(4 - d) \). Although trivial, it makes the gluing of the affine pieces slightly easier: we shall see in Theorem 5.3 that the gluing data between the open pieces is precisely

\[
U_t \ni (a, b, c) \leftrightarrow (a, c^{-1}, c^{N-t}b) \in U_{t+1}
\]

for \( 0 \leq t \leq N - 2 \), whereas for \( t = N - 1 \) the gluing data is

\[
U_{N-1} \ni (a, b, c) \leftrightarrow (ca, c^{a-1}b, c^{-1}) \in U_N.
\]

The choice of coordinate in \( U_N \) gives the final two glues

\[
U_N \ni (a, b, d) \leftrightarrow (a^{-1}, b, a^2 d) \in U_+ \\
U_N \ni (a, b, c) \leftrightarrow (b^{-1}, a, b^2 c) \in U_-
\]

and note that these two do not change from example to example.

We now proceed to prove these statements. To prove that the open sets mentioned above do indeed form an open cover of the moduli space it is convenient to denote \( C_{01} = c_{0-}c_{-1} \),
to denote $\hat{C}_0t = \hat{C}_{01}c_1\ldots c_{t-1}$ for all $2 \leq t \leq N$, and further also to define the following open sets:

$$V_0 : \quad \hat{C}_{0N} \neq 0, c_{0+} \neq 0$$

$$V_i : \quad \hat{C}_{0N-i} \neq 0, c_{0+} \neq 0, A_{0N-i+1} \neq 0$$

**Lemma 5.1.** For every $0 \leq k \leq N - 1$, the open set $V_k$ is contained inside the union of $U_k$ and $U_N$.

**Proof.** Suppose $M$ belongs to $V_k$, then necessarily

$$0 \neq c_{0-}c_{-1} = c_{0+}c_{+1} - 4A_{01}.$$

If $c_{+1} = 0$ then $A_{01} \neq 0$ and so $M$ is in $U_N$. If $c_{+1} \neq 0$ then $M$ is in $U_k$. \hfill $\Box$

**Lemma 5.2.** The open sets $U_0, \ldots, U_N, U_+, U_-$ completely cover the moduli space.

**Proof.** Suppose $M$ is a stable module; we must show that $M$ belongs to one of the open sets in the statement. Note first that if $c_{0+} = c_{0-} = 0$ then the relation $c_{0+}c_{+1} - c_{0-}c_{-1} = 4A_{01}$ forces $A_{01} = 0$, which is impossible since $M$ must be generated as a module from vertex $\star$. Hence we can assume that either $c_{0+} \neq 0$ or $c_{0-} \neq 0$.

**Case 1:** both $c_{0+} \neq 0$ and $c_{0-} \neq 0$. Now if $a_{0N} = 0$ then the only way a nonzero path can reach vertex $N$ is if either $c_{+1}C_{1N} \neq 0$ or $c_{-1}C_{1N} \neq 0$. In the first case $M$ belongs to $U_0$, whereas in the second case $M$ belongs to $V_0$ thus by Lemma 5.1 either $U_0$ or $U_N$. Hence we may assume $a_{0N} \neq 0$. Now if $a_{N+1} = 0$ then by a similar argument $M$ is either in $U_1$ or $V_1$, hence by Lemma 5.1 either $U_1$ or $U_N$. Thus we can assume that also $a_{N+1} \neq 0$ and so $A_{0N-1} \neq 0$. Continuing in this manner either $M$ is in $U_{N-1}$ or we can assume that $A_{01} \neq 0$, in which case $M$ is in $U_N$.

**Case 2:** $c_{0+} \neq 0$ but $c_{0-} = 0$. Then certainly $a_{0-} \neq 0$ since we have to reach vertex $-$ with a non-zero path. Now if $A_{01} = 0$ then by the non-monomial relation it follows that $c_{0+}c_{+1} = 0$ and so we cannot reach vertex $1$ by a non-zero path. This cannot happen, hence $A_{01} \neq 0$ and so we are in $U_+$.

**Case 3:** $c_{0-} \neq 0$ but $c_{0+} = 0$. In this case we are in $U_-$ by the symmetric argument to Case 2. \hfill $\Box$

**Lemma 5.3.** Each open set $U_0, \ldots, U_N, U_+, U_-$ is just a smooth hypersurface in $\mathbb{C}^3$. More precisely the equation of these open sets as a hypersurface in $\mathbb{C}^3$ with co-ordinates $a, b, c$ are given as follows:

$$(0 \leq i \leq N-1) \quad U_t \quad a(1 - 4b)\Sigma_{i=1}^N \eta_i \sum_{i=1}^N \theta_i^+ = b^{r_i} c^{\theta_i^+}$$

$$U_N \quad a(c - 4) = bc$$

$$U_+ \quad b(a^2c + 4) = ac$$

$$U_- \quad b(a^2c - 4) = ac$$

**Proof.** (i) In $U_0$ change basis so that all the specified non-zero arrows equal the identity. By Remark 4.3 the calculation of [Wemyss 07] 4.2 shows every arrow (except for the moment $c_{0-} = 1, a_{0-}, c_{-1}, a_{1-}$) is determined by a monomial in $c_{0N}$ and $a_{0N}$, and the algorithmic relations play no further role. Define

$$\eta_0^+ = \text{the power of } c_{0N} \text{ in } a_{1+}, \quad \theta_0^+ = \text{the power of } a_{N0} \text{ in } a_{1+},$$

$$(1 \leq i \leq N) \quad \eta_i^{(i)} = \text{the power of } c_{0N} \text{ in } a_{i+1}, \quad \theta_i^{(i)} = \text{the power of } a_{N0} \text{ in } a_{i+1}.$$
where by \( a_{N+1N} \) we mean \( a_{0N} \). From this it is clear that \( A_{01} = \sum_{i=1}^{N} \eta_0^{(i)} \sum_{i=1}^{N} \theta_0^{(i)} \). Now we are left with the variables \( c_{N0}, a_{0N}, a_{-0}, a_{1-} \) and \( c_{-1} \) subject to the four relations

\[
\begin{align*}
a_{+0} &= a_{-0} \\
a_{-0} &= c_{-1}a_{1-} \\
a_{1+} &= a_{1-}c_{-1} \\
1 - c_{-1} &= 4A_{01}
\end{align*}
\]

and so really there are only three variables \( c_{N0}, a_{0N} \) and \( a_{1-} \) subject to the one relation

\[
a_{1-}(1 - 4A_{01}) = a_{+0}.
\]

Hence it suffices to show how to put \( a_{+0} \) and \( A_{01} \) in terms of \( c_{N0} \) and \( a_{0N} \). But by the above this becomes

\[
a_{1-}(1 - 4\sum_{i=1}^{N} \eta_0^{(i)} \sum_{i=1}^{N} \theta_0^{(i)}) = c_{N0} \theta_0^{(i)}.
\]

(ii) The proof of the case \( U_t \) for \( 1 \leq t \leq N-1 \) is identical to the above - after setting the specified non-zero elements to be the identity we are down to the three variables \( c_{N-t+1N-t}, a_{N-t+1N-t} \) and \( a_{1-} \) with only one relation

\[
a_{1-}(1 - 4A_{01}) = a_{+0}
\]

so again it suffices to show how to put \( a_{+0} \) and \( A_{01} \) in terms of \( c_{N-tN-t+1} \) and \( a_{N-t+1N-t} \).

Define

\[
\eta_t^{(i)} = \text{the power of } c_{0N} \text{ in } a_{1+}, \quad \theta_t^{(i)} = \text{the power of } a_{0N} \text{ in } a_{1+}
\]

(1 \leq i \leq N) \( \eta_t^{(i)} = \text{the power of } c_{0N} \text{ in } a_{i+1}, \quad \theta_t^{(i)} = \text{the power of } a_{0N} \text{ in } a_{i+1} \)

then it is clear that the equation is simply

\[
a_{1-}(1 - 4\sum_{i=1}^{N} \eta_0^{(i)} \sum_{i=1}^{N} \theta_0^{(i)}) = c_{N-t+1N-t} \theta_t^{(i)}
\]

(iii) For \( U_N \) set the specified non-zero arrows to be 1 then by Remark 4.6 and the calculation [Wen07] 4.2 every arrow (except for the moment \( c_{0-} = 1, a_{-0}, c_{-1}, a_{1-} \)) is determined by a monomial in \( c_{+1} \) and \( a_{1+} \), and the algorithmic relations play no further role. Thus we are down to \( c_{+1}, a_{1+}, a_{-0}, a_{1-} \) and \( c_{-1} \) subject to the four relations

\[
\begin{align*}
a_{+0} &= a_{-0} \\
a_{-0} &= c_{-1}a_{1-} \\
a_{1+}c_{+1} &= a_{1-}c_{-1} \\
c_{+1} - c_{-1} &= 4.
\end{align*}
\]

But these reduce to the three variables \( a_{1-}, a_{1+} \) and \( c_{+1} \) subject to the relation

\[
a_{1+}c_{+1} = a_{1-}(c_{+1} - 4).
\]

Note that since \( c_{+1} - c_{-1} = 4 \) we actually have a choice of coordinate between \( c_{+1} \) and \( c_{-1} \); making the other choice changes the equation in the obvious way.

(iv) For \( U_+ \) after setting the specified non-zero arrows to be 1, by Remark 4.6 and the calculation [Wen07] 4.2 every arrow (except for the moment \( c_{0-} = 1, a_{-0}, c_{-1}, a_{1-} = 1 \)) is determined by a monomial in \( c_{+1} \) and \( a_{1+} \) and the algorithmic relations play no further role. Thus we are down to \( c_{+1}, a_{1+}, c_{0-}, a_{-0} \) and \( c_{-1} \) subject to the four relations

\[
\begin{align*}
a_{+0} &= c_{0-}a_{-0} \\
a_{-0}c_{0-} &= c_{-1} \\
a_{1+}c_{+1} &= c_{-1} \\
c_{+1} - c_{0-}c_{-1} &= 4
\end{align*}
\]

But now these reduce to the three variables \( c_{0-}, a_{1+} \) and \( a_{-0} \) subject to the relation

\[
a_{-0}c_{0-} = a_{1+}(4 + a_{-0}c_{0-}^2)
\]

(v) \( U_- \) follows from \( U_+ \) by a symmetrical argument.
Remark 5.4. The above proof also justifies the position of the coordinates of the open sets as stated in the beginning of this section. Labelling the arrows in the reconstruction algebra with their corresponding polynomials, for any open set if we simply change bases to set the specified non-zero arrows to be 1 we also obtain easily the ratios of polynomials as stated previously.

Theorem 5.5. Consider $D_{n,q}$ with $n = 0$. With the given dimension vector and stability as above, the moduli of representations of the corresponding reconstruction algebra is precisely the minimal resolution of $C^2/D_{n,q}$.

Proof. The open cover is smooth and irreducible, thus we can restrict our attention to the exceptional locus.

Firstly, it is easy to see that the only gluing data needed are the glues mentioned in the introduction to this section - for example if a stable module $M$ belongs to $U_0$ with coordinates (obtained after setting the specified non-zero arrows to be the identity) just $(a_{1-}, c_{N0}, a_{0N})$ then clearly for it to belong to $U_1$ requires $a_{0N} \neq 0$. Now by just looking at the conditions of the non-zero arrows determining the other open sets, its clear that if $U_0$ glues to any other of the open sets it necessarily also has to satisfy $a_{0N} \neq 0$, thus necessarily this glue is through $U_1$. Continuing in this fashion we see that the $N+2$ stated glues are all that are necessary.

Now it is also easy to see that the open pieces glue (in coordinates) in precisely the way mentioned in the introduction to this section - this follows directly from Type A and in fact we can see this from the proof of the last lemma: to see why, for $0 \leq t \leq N-2$

$$U_t \ni (a,b,c) \leftrightarrow (a,c^{-1},c^{N-t}b) \in U_{t+1}$$

is true, notice that given $(a,b,c) \in U_t$ then by the proof of the above lemma the scalar in the position of $a_{N-tN-t-1}$ is $bc^{N-t-1}$, so changing basis at vertex $N-t$ by dividing every arrow into vertex $t$ by $c$ whilst multiplying every arrow out by $c$ yields the result. The remaining three glues are also done by inspection.

Thus by inspecting our open cover and gluing data we see precisely the dual graph of the minimal resolution. To check that the self-intersection numbers are correct, we use adjunction: for example for the curve $C$ in the glue between $U_N$ (co-ordinates $a,b,c$ subject to $f = (a(c-4) - bc)$ and $U_-$ (co-ordinates $a, b, c$ subject to $f = (b(a^2c - 4) - ac)$) given by

$$U_N \ni (a,b,c) \leftrightarrow (b^{-1},a,b^2c) = (a,b,c) \in U_-$$

we have

$$\frac{da \wedge dc}{\partial f/\partial b} = \frac{da \wedge dc}{a^2c - 4} = \frac{d(b^{-1}) \wedge d(b^2c)}{c - 4} = \frac{db \wedge dc}{c - 4} = \frac{db \wedge dc}{\partial f/\partial a}$$

and so $K_X \cdot C = 0$, thus by adjunction $C^2 = -2$. Continuing in this fashion we see that none of the exceptional curves are $(-1)$-curves, thus our resolution is minimal. \hfill \Box

6. Examples

We begin with the easiest family of examples for which $Z_f$ is reduced.

Example 6.1. Consider the group $D_{2s+1,s}$ with $s \geq 2$. We have

$$\frac{2s+1}{s} = 3 - \frac{s-1}{s} = 3 - \frac{1}{s-1} = \left[3, \frac{2, \ldots, 2}{s-1}\right]$$

and so the dual graph of the minimal resolution is

![Graph]

Now

$$\frac{n}{n-q} = \frac{2s+1}{s+1} = \frac{2s+2}{s+1} - \frac{1}{s+1} = \left[2, s+1\right] = [a_2, a_3]$$
and so the moduli and symmetric presentations are the same, namely

\[(xy)^{2(s+1)} - (xy)^{s+2}(x^{2s} + y^{2s}) - (xy)(x^{2s} - y^{2s}) - (x^{2s} + y^{2s})(x^{2s} - y^{2s})^s\]

since \(r_2 = 2n - 3q = s + 2\), \(r_3 = n - 2q = 1\) and \(a_3 - 1 = s\). Denoting \((+) = x^s + y^s\) and \((-) = x^s - y^s\) then we could replace \(x^{2s} + y^{2s}\) by either \((+)^2\) or \((-)^2\) and still get a generating set. Further the reconstruction algebra labelled by polynomials is

\[
\begin{align*}
U_0 &= C[\frac{xy}{(-)}, (+)^{2s}(+) \cdot \frac{xy}{(s+2)(s+1)}] \\
U_t &= C[\frac{xy}{(-)}, (+)^{2s}(+) \cdot \frac{xy}{(s+2)(s+1)}] \\
U_N &= C[\frac{xy}{(-)}, (+)^{2s}(+) \cdot \frac{xy}{(s+2)(s+1)}]
\end{align*}
\]

where there is a choice of coordinates in \(U_N\) since we could alternatively take \((-)^{2}(xy)^2\) as the third coordinate. With respect the above ordering, for \(0 \leq t \leq N - 2\) the gluing data is

\[U_t \ni (a,b,c) \leftrightarrow (a,c^{-1},c^{2b}) \in U_{t+1}\]

and

\[U_{N-1} \ni (a,b,c) \leftrightarrow (ca,c^2b,c^{-1}) \in U_N.\]

It is the choice of coordinate in \(U_N\) which helps with final gluing:

\[
\begin{align*}
U_N &= C[\frac{xy}{(-)}, (xy)^{s+1}(+) \cdot \frac{(-)^2}{(xy)^y}] \\
U_+ &= C[\frac{xy}{(-)}, (xy)^{s+1}(+) \cdot \frac{(-)^2}{(xy)^y}]
\end{align*}
\]

Example 6.2. Consider the group \(D_{56,15}\) of order 2460.

\[
\frac{56}{15} = [4, 4, 4]
\]
and so the dual graph of the minimal resolution is

Now for the invariants

\[
\frac{56}{56 - 15} = [2, 2, 3, 2, 3, 2, 2]
\]

from which we calculate

| \( c \) | 1 | 0 | 1 | 3 | 5 | 12 | 19 | 26 |
|-------|---|---|---|---|---|----|----|----|
| \( d \) | 0 | 1 | 1 | 2 | 3 | 7 | 11 | 15 |

Hence the invariants are

\[(xy)^{22}, (xy)^{57}w_2, (xy)^{20}w_3, (xy)^{11}w_2w_3, (xy)^{7}w_3^2, (xy)^{3}w_3^3, (xy)^{2}w_2^2w_3, (xy)w_2^1w_3^1, w_2^6w_3^1\]

where \( w_2 = (x^{15} + y^{15})^2 \) and \( w_3 = (x^{15} + y^{15})(x^{15} - y^{15}) \). Further in this example the reconstruction algebra labelled by polynomials for the moduli presentation is

which by Theorem 4.11 can be written as a quiver subject to the 19 relations

Now we may view invariants at each vertex as follows: the invariant \((xy)^{22}\) is viewed at the vertex + as \( \frac{1}{16}(c_{+1}a_{1-} - a_{+0}c_{0-})(a_{-0}c_{0+} - c_{-1}a_{1+}) \) whereas at the vertex − it is viewed as
\[ \frac{1}{10}(a_0c_0 - c_1a_1)(c_1a_1 - a_0c_0). \] At all other vertices \((xy)^{82}\) can be seen as

\[ \star \]

followed by \(\frac{1}{4}(a_1 a_0 - a_1 a_0)\). The remainder of the invariants are viewed as follows: \((xy)^{67}(+)\) or \((xy)^{67}(-)^2\):

\[ \star \]

\((xy)^{26}(+)\)(−):

\[ \star \]

\((xy)^{13}(+)\) or \((xy)^{13}(−)^3\):

\[ \star \]

\((xy)^{10}(+)\) or \((xy)^{10}(−)^4\):

\[ \star \]

\((xy)^{8}(+)\) or \((xy)^{8}(−)^4\):

\[ \star \]

\((xy)^{3}(+)\) or \((xy)^{3}(−)^5\):

\[ \star \]
(xy)^2(+)^{21}(-)^7 or (xy)^2(+)^{29}(-)^9:

\[ \begin{align*}
\bullet & \rightarrow \bullet \\
\bullet & \rightarrow \bullet \\
\bullet & \rightarrow \bullet \\
\bullet & \rightarrow \bullet
\end{align*} \]

(xy)^{49}(-)^{11} or (xy)^{47}(-)^{13}:

\[ \begin{align*}
\bullet & \rightarrow \bullet \\
\bullet & \rightarrow \bullet \\
\bullet & \rightarrow \bullet \\
\bullet & \rightarrow \bullet
\end{align*} \]

(+)^{67}(-)^{15} or (+)^{65}(-)^{17}:

\[ \begin{align*}
\bullet & \rightarrow \bullet \\
\bullet & \rightarrow \bullet \\
\bullet & \rightarrow \bullet \\
\bullet & \rightarrow \bullet
\end{align*} \]

where the different shades of red signify the choice between the two invariants. Now by Section 5 the minimal commutative resolution is covered by the following 6 open sets

\[ U_0 = \mathbb{C}[(xy)^{26}(+)^3, (xy)^{67}(-)^{15}, (xy)^{19}(-)^{2}] \quad a(1 - 4bc^15) = b^7c^{26} \]

\[ U_1 = \mathbb{C}[(xy)^{26}(+)^3, (xy)^{19}(-)^4, (xy)^{16}] \quad a(1 - 4bc^4) = b^2c^7 \]

\[ U_2 = \mathbb{C}[(xy)^{26}(+)^3, (xy)^{19}(-)^4, (xy)^{16}] \quad a(1 - 4c) = bc^2 \]

\[ U_3 = \mathbb{C}[(xy)^{41}(+), (xy)^{41}(-), (xy)^{15}] \quad a(c - 4) = bc \]

\[ U_+ = \mathbb{C}[(xy)^{41}(+), (xy)^{41}(-), (xy)^{15}] \quad b(a^2c + 4) = ac \]

\[ U_- = \mathbb{C}[(xy)^{41}(+), (xy)^{41}(-), (xy)^{15}] \quad b(a^2c - 4) = ac \]

To see why the abstract equation for \( U_0 \) is as stated, denoting \( a = a_1 \), \( b = c_{N_0} \) and \( c = a_{0N} \) the quiver picture for \( U_0 \) looks like

\[ \begin{align*}
\bullet & \rightarrow \bullet \\
\bullet & \rightarrow \bullet \\
\bullet & \rightarrow \bullet \\
\bullet & \rightarrow \bullet
\end{align*} \]

thus \( \eta_0^+ = 7, \sum_{i=1}^3 \eta_0^{(i)} = 4, \theta_0^+ = 26 \) and \( \sum_{i=1}^3 \theta_0^{(i)} = 15 \). The other open sets are computed in a similar fashion. Note again there is a choice of coordinate in \( U_3 = U_N \) above since we can pick \( \frac{(xy)^{2}}{(xy)^{2}} \) as the third coordinate (doing this changes the abstract equation to \( a, b, d \)
subject to \( ad = b(4 - d) \). With respect to the above ordering, the gluing of these open sets is:

\[
\begin{align*}
U_0 \ni (a, b, c) &\iff (a, c^{-1}, c^4 b) \in U_1 \\
U_1 \ni (a, b, c) &\iff (a, c^{-1}, c^4 b) \in U_2 \\
U_2 \ni (a, b, c) &\iff (ca, c^3 b, c^{-1}) \in U_3
\end{align*}
\]

and the choice of coordinate in \( U_N \) gives:

\[
\begin{align*}
U_N &= \mathbb{C}[(x y)^{\alpha^+}, \frac{(x y)^{\alpha^-}}{(x y)^{\alpha^+}}, \frac{(x y)^{\alpha^-}}{(x y)^{\alpha^+}}] \ni (a, b, d) \iff (a^{-1}, b, a^2 d) \in U_+ \\
U_N &= \mathbb{C}[(x y)^{\alpha^+}, \frac{(x y)^{\alpha^-}}{(x y)^{\alpha^+}}, \frac{(x y)^{\alpha^-}}{(x y)^{\alpha^+}}] \ni (a, b, c) \iff (b^{-1}, a, b^2 c) \in U_-
\end{align*}
\]

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