The Realistic Scattering of Puffy Dark Matter

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If dark matter has a finite size, the intrinsic interaction responsible for the structure formation is inevitable from the perspective of dark matter self-scattering. To describe the circumstance in which the binding force realizes the finite size dark protons, we first use the Eikonal approximation to simplify the convoluted scattering between dark protons into the case at the $t = 0$ limit. The Chou-Yang model is then introduced to reduce the number of input parameters to one based on the simplicity and analyticity principle. A new definition of velocity dependence and the corresponding implications on the small cosmological structures from Chou-Yang dark protons are shown clearly. Even though the parameter space is not fully covered, the numerical findings show that the amplitude coefficient can alter the self-scattering cross-section, allowing us to recover the excluded parameter space without using binding force. Finally, we demonstrate that the correct relic density from thermal freeze-out production prefers super heavy dark protons.

Introduction Intensive researches lead to the current standard dark matter, a paradigm of cold, collisionless particles which has achieved remarkable success in describing the universe at large scales: the cosmic microwave background radiation (CMB) and the large structure of the universe [1–3]. However, in the small scale of the universe such as clusters and galaxies, ΛCDM does have some discrepancies or anomalies: “missing satellites” [4, 5], “cusp vs. core” [6, 7], “too big to fail” [8] and “diversity” problems [9]. Although there are several astrophysical solutions to these problems, these discrepancies require the cross-sections per mass unit $\sigma/m$ of the dark matter at about 1 cm$^2$/g from the perspective of particle physics, [10, 11].

In light of this, the self-interacting dark matter (SIDM) scenario can address these anomalies in the small structure elegantly [12–16]. The crucial point of the SIDM is that the collision process between the dark matter in the current epoch of our universe is non-relativistic, where the long-range quantum effect dominates in the collision to enhance the cross-section. As a consequence, rich structures are formed at the galaxy scale. In addition to the requirement of the large cross-section, the properties of the velocity dependence play an important role in the small structure observations. As the different cross-sections per mass unit are required by the different small scale observations such as Milky way [17], galaxies [18] and clusters [19] with different average velocities of the halos. Roughly speaking, a decreased cross-section in the velocity is required for the SIDM. (For the detail, see the following context [20, 21].) As the long-range of the quantum effect is much more complicated than the collisionless particle, the properties of the self-scattering dark matter is of great interest in recent studies. For example, the authors proposed a “puffy” dark matter in Ref. [22] which can realize the required velocity dependence. In this scenario, the SIDM with a finite size that is larger than its Compton wave-length can decrease the cross-sections with the corresponding velocity of the dark matter, even in the situation of a long-range force. The finite-size effect can dominate in such cases and is independent of the inner structure of the dark matter. Some QCD-like theories can realize this puffy dark matter and the direct detection can be relevant to the size of the dark matter [23–26].

If the puffy dark matter exists, the scattering between the dark matter will be of great interest as there must be some inner interaction or structure of the dark matter for which the finite-size must be some residual effect of the intrinsic interaction. For example, the Van der Waals force between the molecular is the residual effect of the interaction between electrons and nuclei. The interactions between the confined quarks determine the interaction between the elastic scattering between hadrons like protons. We can not ignore the residue effect and consider its implication on dark matter self-scattering. Fortunately, we have enough knowledge of ordinary matters. It can give us many hints on the dark sector. The scattering between finite-size dark matter is similar to that between protons. The results are consistent with the scattering theory that includes both Coulomb and Nuclear interaction with only very few input parameters [27], even though the detailed inner structure of the component quarks with strong interactions in the proton are unknown to us. With this triumph, we can apply the same paradigm to dark matter scattering with finite size. In this paper, the Eikonal approximation used in $pp$ elastic collision will be adopted and the detailed studies on the scattering of puffy dark matter are shown in the following.

The scattering theory of a puffy particle Before going into the detailed studies on the puffy particle, we
must dwell on the general scattering quantum theory. In principle, analyticity and unitarity are fundamental elements to our understanding of particle physics. Analyticity requires that the forward scattering amplitudes for the scattering of the fundamental particles come from analytic functions. Further, the unitarity provides the relation between the total cross-section and the imaginary portion of the forward scattering amplitude. Generally, the elastic scattering is the function of the impact parameter that is didactic. We can derive numerous theorems relevant to elastic scattering using this approach. We begin our derivation by following the standard scattering theory. As discussed about in the introduction, the long-range force could be the residual effect of the interaction in the center mass frame is

\[ f_c = \frac{m}{q^2 + m^2_\phi} \quad \text{(1)} \]

for simplicity, in which \( m \) is the mass of the scattering particle, \( \alpha \) is the coupling strength, \( q^2 \) is the transfer momentum, \( m_\phi \) is the mass of the mediator, and \( G(q) \) is the form factor. In the case of \( m_\phi \) approaching zero, the amplitude will become a Coulomb scattering, thus a subscript “c” is denoted. From the theoretical side, such Coulomb interaction (or Yukawa interaction in case of a non-zero mass mediator) can come from some specific U(1) gauge symmetry [28] if it is not ordinary electromagnetic interactions. The Coulomb scattering from mediators becomes sufficiently sizable only at small \( q^2 \) with a suppressed contribution to the transfer cross-section discussed in the following section.

The amplitude \( f_c \) listed above is the Born approximation. As shown in the Refs. [15, 16], the small scale problem can be solved in the quantum resonant regime with the angular momentum \( l \) at the order of 10. Nevertheless, if the particle is puffy or has a finite size i.e. \( G(q) \neq 1 \), additional interactions exist at a smaller range, and the interaction is strong enough to confine the U(1) charge in a specific space. As shown in Ref. [22], the authors propose a QCD-like confining theory to realize the puffy dark matter. The form factor \( G(q) \) also generates the desirable velocity-dependent behavior of self-interaction dark matter. However, the equation (1) does not capture the dark QCD interaction that is responsible for the formation of puffy dark matter. We call it Nuclear interaction to distinguish it from the Coulomb one.

In terms of similarity with the elastic scattering of the protons, it suffices to apply the well-developed method to dark protons with Nucleus interaction. We can develop a geometrical picture-based impact-parameter space. The standard partial-wave expansion of the scattering amplitude in terms of Legendre polynomials from QCD interaction in the center mass frame is

\[ f_n(s, t) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1)P_l(\cos \theta) a_l(k), \quad \text{(2)} \]

Here we use subscript \( n \) to stand for Nucleon interaction. \( s, t \) are the Mandelstam variables, \( k \) is the momentum of the incident particle and \( l \)th partial wave is

\[ a_l(k) = \frac{\exp(2i\delta_l) - 1}{2i}. \quad \text{(3)} \]

\( \delta_l \) is the phase shift of \( l \)th partial wave. Note that the subscript “n” denotes the strong interaction in the short range. As the partial wave goes to infinity,

\[ P_l(\cos \theta) \rightarrow J_0((2l + 1)\sin(\theta/2)), \quad \text{(4)} \]

where \( J_0(t) \) is the zero order Bessel function \( J_0(t) = (1/2\pi) \int d\phi \exp(-it\cos \phi) \). The summation of \( l \) can be converted to an integral

\[ f_n(s, t) = 2k \int db J_0(qb) a(b, s), \quad \text{(5)} \]

in which the relation between the newly defined impact parameter \( b \) and angular momentum is \( bk = l + 1/2 \). Then \( a_l \rightarrow a(b, s) \) becomes a function of the impact parameter. It is perpendicular to the beam direction. We can view it as a distribution of wave sources that produce an interference pattern. We can see that the impact parameter is the Fourier transformation of \( f_n(s, t) \) in two dimension space, thus the inversion of \( f_n(s, t) \) gives

\[ a(b, s) = \frac{1}{4\pi k} \int d^2 q \exp(-iq \cdot b) f_n(s, t). \quad \text{(6)} \]

One can easily check that after removing a \( k \) factor, \( f_n \) only depends on \( t \) which is always assumed in elastic scattering, then \( a \) will only depend on impact parameter \( b \). Note that the above methodology is the so-called Eikonal approximation which is used for the situation that the wavelength \( \lambda_d \) much less than the size of the collision particles in the literature. Thus high-energy collision or relativistic collision is needed in the \( pp \) collision. For the slowly moving puffy dark matter, such approximation would be applicable in the case of that the de Broglie wavelength of the heavy dark matter are much smaller than the size of the dark matter. Together with the appropriate high angular momentum discussed above, the application of Eikonal approximation is adopted here with the following parametrization. Note that, in the original puffy dark matter paper Ref. [22], the required suppression behavior \( 1/v^4 \) of the cross section are in the regime \( mv \gg r_{DM}^{-1} \), which is just the applicable regime for the Eikonal approximation.

By above derivation, we can obtain the realistic scattering cross-section including both the Coulomb scattering in Eq. (1) and Nucleus scattering in Eq. (5). A phase factor \( \alpha \phi(q) \) is introduced to address the interference between Coulomb and strong interactions. The complete differential cross-section [29] is

\[ \frac{d\sigma}{d\Omega} = |f_c(q)e^{i\alpha \phi(q)} + f_n(q)|^2 = \frac{d\sigma_c}{d\Omega} + \frac{d\sigma_{\text{int}}}{d\Omega} + \frac{d\sigma_n}{d\Omega}. \quad \text{(7)} \]
where

\[
\frac{d\sigma_r}{d\Omega} = \frac{m^2 \alpha^2 G^4(q)}{(q^2 + m^2)^2},
\]

\[
\frac{d\sigma_{int}}{d\Omega} = -\frac{m^2 v \alpha \sigma_{tot} G^2(q)}{4\pi (q^2 + m^2)} \times e^{-\frac{4m^2}{Bq} (\rho \cos(\alpha \phi) + \sin(\alpha \phi))},
\]

\[
\frac{d\sigma_{el}}{d\Omega} = \frac{(mv)^2 \sigma_{tot}(1 + \rho^2) e^{-Bq^2}}{4\pi 16\pi}.
\]

Here \( v \) is the relative velocity between the scattering particles, \( \phi(q) \) is the phase that is deeply investigated in the literature, and the result from the Eikonal approach gives \[33-35\]

\[
\phi(q) = -\gamma + \ln\left(\frac{Bq^2}{2}\right) + \ln\left(1 + \frac{8r_0^2}{B}\right) + \ln\left(4q^2 r_0^2 + (4q^2 r_0^2) + 2q^2 r_0^2\right).
\]

where \( \gamma \) is Euler’s constant and \( r_0 \) is the radius of charge distribution of the puffy particle. For the Eq. (10), it is originally from the parametrization of the nuclear elastic cross section \( \frac{d\sigma}{dt} = \left(\frac{dp}{dt}\right)^2 e^{Bt} \). The factor \( e^{Bt} \) is a Gaussian form factor indeed. Note that this parametrization is only an effective approach in the Eikonal approximation, the processes similar to the Deep Inelastic Scattering or Drell-Yan processes in the inelastic scattering of proton need further explorations. Also, note that \( B \) is an adequate parameter inspired from the ordinary hadron collision \[30-32\] as no prior form factor exists for the puffy dark matter.

It is easy to find the Nucleus differential cross-section is the function of \( \sigma_{tot} \), \( \rho \) and \( B \). We can derive the cross-section from the analyticity and unitarity conditions. For example, based on the optical theorem

\[
\sigma_{tot} = \frac{4\pi}{k} \text{Im} f_n(s, 0) = 4 \int d^2b \text{Im} a(b, s).
\]

For the detailed study on the scattering impact-parameter space and the application in \( pp \) scattering, one can see Ref. \[27\]. \( \sigma_{tot} \) is the total cross-section which includes all the elastic, inelastic, and annihilation cross-sections. \( \rho = \text{Re} f_n(0) / \text{Im} f_n(0) \) is the ratio between real and imaginary part of the scattering amplitude. \( B \) is the slope parameter when we expand the cross section with \( t \)

\[
B(s, t) = \frac{d}{dt} \left[ \ln \left( \frac{d\sigma}{dt} \right) \right].
\]

\( B(s, t) \) measures the size of the puffy particle. In the case of \( pp \) scattering, we can extract these parameters from the experiments in Coulomb normalization or the “luminosity-free” method. Note that there is a fourth curvature parameter \( C \) when we expand \( t \) to the sub-leading order, which we neglect in this work as it is the higher-order result. From the theoretical side, all the three parameters and other observable should be predicted or calculated from impact-parameter space. As an example, when \( a(b, s) \) is purely imaginary, the elastic scattering cross-section would be \[27\]

\[
\sigma_{el} = 4 \int d^2b (\text{Im} a)^2,
\]

and \( B \) can be derived from

\[
\sigma_{tot} B = 2 \int d^2b b^2 \text{Im} a.
\]

Thus we can use the amplitude \( a(b, s) \) to derive the scattering cross-section by using Eq. (12), Eq. (14) and Eq. (15).

In the above, we show the sketch map of the quantum scattering theory of a puffy particle. We can see that all the concerns lay on the impact-parameter amplitude \( a(b, s) \). As shown in the Ref. \[27\], pure imaginary \( a(b, s) \) are always adopted for the forward high-energy scattering, there are several profiles such as disk, parabolic form, Gaussian shape and Chou-Yang model \textit{etc.} which can be used in the literature. Naively thinking, the puffy particle shows a \( U(1) \) charge locates in finite-size space. Chou-Yang model proposed in Ref. \[36-38\] postulates that the charge distribution should have a similar profile or be determined by the impact-parameter amplitude. The attenuation of the amplitude accompanying the process of the two puffy particles going through each other is governed by the local opaqueness within each particle. It means that the charge distribution or the electro-magnetic form factor should have the same shape as the transverse distribution of the matter, leaving only the strength of the absorption to be fixed. In the visible sector, we require the total cross-section calculated in the model to agree with the experiments. In the dark sector, three more factors are introduced to account for the complete scattering process. From the point of view of minimality and predictivity, this more realistic model, the Chou-Yang model, can perfectly solve the problem. In the model, the absorption at an impact parameter \( b \) is \( \Omega(b) \), then

\[
a(b, s) = \frac{\exp(2i\delta) - 1}{2i} \equiv \frac{i}{2} [1 - \exp(-\Omega(b))].
\]

If the electro-magnetic form factor is dipole

\[
G(q) = \left(\frac{1}{1 + r_0^2 q^2}\right)^2,
\]

using the Fourier transform of the matter distribution, then a convolution of the dipole form factor shows that \( \Omega \) is analytically

\[
\Omega = \frac{A}{8} x^3 K_3(x),
\]

in which \( x = b/r_0 \), \( A \) is a dimensionless parameter which we call amplitude coefficient and \( K_3 \) is the modified
Bessel function. By choosing a suitable parameter $A$, the resulting cross-section agrees with the experiment results. It indicates that we can use a similar picture on the puffy dark matter. Note that the pure imaginary impact factor $a(b, s)$ is generally used in high-energy scattering [27], the validity of such assumption for the $t = 0$ low energy scattering of puffy dark matter needs further check. The Eikonal approximation in such case is discussed in the following section.

**Puffy dark matter and the implication on the small cosmological scale** We generally require a large self-scattering cross-section for the small-scale anomalies. However, the formation of the structure is in fact dependent on the transfer cross-section via combining the Eq. (7, 12, 15,16)

$$
\sigma_T = \int d\Omega (1 - \cos \theta) \frac{d\sigma}{d\Omega}.
$$

(19)

A significant issue in the solution of the anomalies is the behavior of the velocity dependence, where the radius of the dark matter is substantially larger than the range of the Coulomb force for puffy DM.

As shown in Ref. [22], if the interaction is pure dark $U(1)$ interaction, the transfer cross-section $\sigma_T$ will be of $1/v^4$ dependence at $m\nu_{\text{DM}} \gg 1$. Note that the decrease with the velocity of $\sigma_T$ talked about is on

$$
\sigma_0 = 4\pi(m_{\text{DM}}\alpha\lambda^2)^2
$$

(20)

which is the naively dimensional estimation on the corresponding $U(1)$ cross-section at high energy. The corresponding results are shown in the left panel of Fig. 1 in which the vertical coordinate $\lambda = 1/m_\phi$ is the range of $U(1)$ interaction, the horizontal coordinate $r_{\text{DM}} = 2\sqrt{6}\nu_0$ is the radius of the puffy dark matter. The upper left part of the left plot gives the correct velocity dependence coinciding with that of the SIDM with a point particle [15]. The lower right part shows that ratio $\sigma_T/\sigma_0$ also depends on the radius of the dark matter. Furthermore, in the case of a much smaller $\lambda$ that the long-range Coulomb force disappears, the finite-size effect can still maintain the correct velocity dependence.

However, a flaw or an inconsiderate point for the above results is that we can not ignore the binding force of puffy dark matter. The scattering theory in Ref [27] gives us a clue to these questions. At first, the cross-section at high energy is an electromagnetic part ($\sigma_0$) plus a strong interaction part,

$$
\sigma_A = \sigma_0 + \sigma_n,
$$

(21)

To make the impact factor more realistic, we use the Chou-Yang model. Then according to Eq. (12), $\sigma_n$ can be defined as

$$
\sigma_n = 8\pi r_{\text{DM}}^2 \int \! \! dx \! \! \text{Im}a(b, s).
$$

(22)

It is the total cross-section of the strong interaction. By this definition, the dipole electromagnetic form factor has the same profile as the matter distribution, and $A$ is the amplitude coefficient. Eq. (7) and the corresponding calculations can be used for the differential cross-section used for the transfer cross-section. Compared with only the $U(1)$ interaction, we can see that the calculation here would be more realistic. Nevertheless, only one additional parameter $A$ with definite physical implications could be more predictive.

We show the ratio of $\sigma_T/\sigma_A$ numerically with different $A$ in Fig. 1 in which $A = 0$ case talked above lists in the left panel. From Fig. 1 we can see that after the consideration of the strong interaction, the decrease of the ratio changes a lot in the $\lambda$ and $r_{\text{DM}}$ space. Total space consists of two parts: the upper left semi-ellipse zone indicates the $U(1)$ interaction dominants; the lower right ladder-like zone is the radius effect dominant part. Both parts depend on the ratio of radius and force range in the $A = 0$ case. We can also see that in the case of a larger $A$, the effect of the radius becomes more significant. It can affect the ratio even when the force range $\lambda$ is greater than the radius $r_{\text{DM}}$.

Next, we calculate the velocity-averaged transfer cross-section in the Maxwell-Boltzmann distribution to check the constraints in the small cosmological scale. The case of the best-fit point is in the left panel of Fig. 2 in the different value of $A$. The space 95% C.L. contours of $\sigma_A/m_{\text{DM}}$ and $m_{\text{DM}}\nu_{\text{DM}}$ in the $\lambda < r_{\text{DM}}$ case are shown in the right panel of the Fig. 2. The blue dash line of the right panel shows the edge of the validity of the Eikonal approximation. As mentioned in the previous section, such estimates are valid when the de Broglie wavelength is significantly smaller than the size of the puffy dark matter. Here the condition should be

$$
m_{\text{DM}}\nu_{\text{DM}}v \gg 1.
$$

(23)

The blue (purple) dash line shows $m_{\text{DM}}\nu_{\text{DM}}v = 10$ (= 20), implying that we keep the results in about 10% precision limits. Of course, the right part of the rest excluded region will be more concrete along with the increasing of radius $r_{\text{DM}}$. Compared with the $A = 0$ case, we can see that the strong interaction will enlarge the range of $\sigma_A/m_{\text{DM}}$. Some space that is not satisfied in the pure $U(1)$ case can be available when we considered the inner interactions. The cross-section can be extended.
to 3 times larger in the case of $A = 10$, and even larger in other amplitude coefficients. It means that the effect of the radius is significant in the exploration of dark matter simulations. In some spaces, it will dominate the physics of dark matter. Finally, the dark proton can generate the correct relic density by the annihilation cross-section [43]

$$\sigma v_{\text{rel}}^{\text{DM}} (m_{\text{DM}}) = \sigma v_{\text{rel}}^{\text{QCD}} \times \left( \frac{1 \text{GeV}}{m_{\text{DM}}} \right)^2$$  \hspace{1cm} (24)$$

where $\sigma v_{\text{rel}}^{\text{QCD}}$ is the proton-anti-proton annihilation cross-section in Standard QCD, which comes from the direct measurement in experiment [44]. After performing the thermal average, the requirement of $\langle v_{\text{rel}} \rangle = 3 \times 10^{-29}$ cm$^{3}$/s indicates the rough estimate of dark proton mass is around 150 TeV. However, Unitarity’s upper bound on dark matter elastic scattering prevents it from achieving such a large self-scattering cross-section for $m_{\text{DM}} = 150$ TeV. That strongly suggests looking for an additional production mechanism for dark protons to reduce their mass scale.

**Conclusion** If the dark matter is not a point particle but some composite ones located in a finite size space, the scattering between dark matter will be much more complicated. The Eikonal approximation and the similar parametrization as $pp$ collision are adopted. The sketch map of the calculation of the cross-section and a more realistic realization of the matter and charge distribution, as used by the dipole form factor in the Chou-Yang model, are shown in this work. Simultaneously the Chou-Yang model is also introduced to reduce the number of input parameters to one based on the simplicity and analyticity principle. We should emphasize that though the absorption approaches zero the scattering can restore the pure $U(1)$ case, the inner interaction and the structure can not be ignored if the dark matter is puffy from the physical insight. The method we found is a more appropriate way to our understanding of the nature of the puffy dark matter, and we proposed a new definition of velocity dependence in such cases.

The numerical results show that even in the range of sizable $U(1)$ interaction, the non-vanishing amplitude coefficient $A$ can also affect the scattering cross-section of the puffy dark matter. We find that with the participation of the strong interaction, the space of the cross-section to the mass ratio which is needed in the simulation can be enlarged, giving us a more flexible parameter space to other processes related to dark matter.

Though the scattering of the puffy dark matter is realized in the paper, we should note that the main shortcoming of this treatment is that the Eikonal approximation only covers part of parameters space, which might lose the accuracy in low-velocity dark protons. The detailed research on the elastic scattering will give us further insight into the formation of the finite size, such as the QCD-like model, which will be explored in our future work.

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