Nonlinears in black hole ringdowns

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The gravitational wave strain emitted by a perturbed black hole (BH) ringing down is typically modeled analytically using first-order BH perturbation theory. In this Letter we show that second-order effects are necessary for modeling ringdowns from BH merger simulations. Focusing on the strain’s \((\ell, m) = (4, 4)\) angular harmonic, we show the presence of a quadratic effect across a range of binary BH mass ratios that agrees with theoretical expectations. We find that the quadratic \((4, 4)\) mode amplitude exhibits quadratic scaling with the fundamental \((2, 2)\) mode—its parent mode. The nonlinear mode’s amplitude is comparable to or even larger than that of the linear \((4, 4)\) modes. Therefore correctly modeling ringdown—improving mismatches by an order of magnitude—requires the inclusion of nonlinear effects.

Nonlinearity is responsible for the rich phenomenology of general relativity (GR). While many exact nonlinear solutions are known [1, 2], LIGO observables—gravitational waves (GWs) from merging binary black holes (BHs)—must be predicted by numerical relativity (NR). Analytic perturbation theory has an important role far from the merger: at early times, post-Newtonian (PN) theory; and at late times (ringdown), black hole perturbation theory [3–5], provided that the remnant asymptotes to a perturbed Kerr BH [6, 7]. PN theory has been pushed to high perturbative order [8], but the standard paradigm for modeling ringdown is only linear theory (see [9] for a review). It may then come as a surprise if linear theory needs to include nonlinear effects in ringdown models [10–14], the most nonlinear phase of a BH merger.

The “magic” nature of the Kerr geometry [15] leads to a decoupled, separable wave equation for first-order perturbations (the Teukolsky equation [5]), schematically written as

\[ \mathcal{T}\psi = \mathcal{S}, \]  

where \( \mathcal{S} \) is a source term that vanishes for linear perturbations in vacuum, \( \psi \) is related to the first-order correction to the curvature scalar \( \psi_4 \), and the linear differential Teukolsky operator \( \mathcal{T} \) depends on the dimensionless spin parameter \( \chi \equiv |S|/M^2 \) through the combination \( a = |S|/M \) with \( S \) the BH spin angular momentum and \( M \) the BH mass (throughout we use geometric units \( G = c = 1 \)). The causal Green’s function \( \mathcal{G} \sim \mathcal{T}^{-1} \) has an infinite, but discrete set of complex frequency poles \( \omega_{(\ell,m,n)} \) [16]. This makes GWs during ringdown well-described by a superposition of exponentially damped sinusoids, called quasi-normal modes (QNMs). The real and imaginary parts of \( \omega_{(\ell,m,n)} \) determine the QNM oscillation frequency and decay timescale, respectively. These modes are labeled by two angular harmonic numbers \( (\ell, m) \) and an overtone number \( n \). The combination \( M\omega_{(\ell,m,n)} \) is entirely determined by \( \chi \).

To date, the linear QNM spectrum has been used to analyze current GW detections [17–19], forecast the future detectability of ringdown [20–22], and perform tests of gravity in the strong field regime [23, 24].

Since the sensitivity of GW detectors will increase in the coming years [25–28], there is the potential to observe nonlinear ringdown effects in high signal-to-noise ratio (SNR) events. A few previous works have shown that second-order perturbation effects can be identified in some NR simulations of binary BH mergers [29, 30]. In this Letter we show that quadratic QNMs—the damped sinusoids coming from second-order perturbation theory in GR—are a ubiquitous effect present in simulations across various binary mass ratios and remnant BH spins. In particular, for the angular harmonic \((\ell, m) = (4, 4)\), we find that the quadratic QNM amplitude exhibits the expected quadratic scaling relative to its parent—the fundamental \((2, 2)\) mode. The quadratic amplitude also has a value that is comparable to that of the linear \((4, 4)\) QNMs for every simulation considered, thus highlighting the need to include nonlinear effects in ringdown models of higher harmonics.

**Quadratic QNMs.**—Second-order perturbation theory has been studied for both Schwarzschild and Kerr BHs [31–41]. This involves the same Teukolsky operator as in Eq. (1) acting on the second-order curvature correction, and a complicated source \( \mathcal{S} \) that depends quadratically on the linear perturbations [39, 40, 42]. The second-order solution results from a rather involved integral of this source against the Green’s function \( \mathcal{G} \) [36, 41]. We only need to know that it is quadratic in the linear perturbation...
As the linear with frequency \( \omega \) involves purely linear QNMs, the quadratic QNM coming from the bondi time and \( 0 \) modes \([34, 35, 41]\). The quadratic QNM coming from the (2, 2) mode would have frequency \( \omega \) \( (2, 2) = 2 \omega \) and would decay faster than the linear fundamental mode \( (4, 4, 0) \), but slower than the first linear overtone \( (4, 4, 1) \), regardless of the BH spin \([43]\).

The NR strain at future null infinity contains all of the angular information of the GW and is decomposed as

\[
h_{\mathrm{NR}}(u, \theta, \phi) \equiv \sum_{\ell=2}^{\infty} \sum_{|m| \leq \ell} h_{(\ell,m)}(u) - 2Y_{(\ell,m)}(\theta, \phi),
\]

with \( u \) the Bondi time and \( -2Y_{(\ell,m)} \) the spin-weighted \( s = -2 \) spherical harmonics. We model this data with two different QNM ansätze, valid between times \( u \in [u_0, u_f] \).

The first model, which is typically used in the literature, involves purely linear QNMs,

\[
h_{\mathrm{model,}0}(u) = \sum_{n=0}^{N} A_{(\ell,m,n)}(u) e^{-i\omega_{(\ell,m,n)}(u-u_{\text{peak}})},
\]

where \( A_{(\ell,m,n)}(u) \) is the peak amplitude of the quadratic QNM sourced by the linear \( (2, 2, 0) \) QNM interacting with itself. Note that for the quadratic term we do not account for all of the angular structure, which requires the Green’s function integral of the second-order source term. We emphasize that the two models \( h_{\text{model,}0}(u) \) and \( h_{\text{model,}Q}(u) \) contain the same number of free parameters.

In these ringdown models, we fix the QNM frequencies to the values predicted by GR in vacuum and fit the QNM considered in the model, and \( u_{\text{peak}} \) is the time at which the \( L^2 \) norm of the strain over the two-sphere achieves its maximum value (a proxy for the merger time), which we take to be \( u_{\text{peak}} = 0 \) without loss of generality. Note that here we have suppressed the spheroidal-spherical decomposition (which we include as in Eq. (6) of \([44]\)).

We will use Eq. (4) to model both the \( (2, 2) \) and \( (4, 4) \) modes of the strain \([45]\). When modeling the \( (2, 2) \) mode we use \( N = 1 \) and when modeling the \( (4, 4) \) mode we use \( N = 2 \). While prior works have included more overtones in their models \([10–14, 44]\), we restrict ourselves to no more than two overtones because we find that the amplitudes of higher overtones tend to vary with the model start time \( u_0 \) and hence are not very robust.

The novel QNM model, which includes second-order effects and highlights our main result, only changes how the \( (4, 4) \) mode is described, compared to Eq. (4). It is given by

\[
h_{\text{model,}Q}(u) = \sum_{n=0}^{1} A_{(4,4,n)}(u) e^{-i\omega_{(4,4,n)}(u-u_{\text{peak}})} + A_{(2,2,0) \times (2,2,0)}^{(4,4)}(u) e^{-i\omega_{(2,2,0) \times (2,2,0)}(u-u_{\text{peak}})},
\]

where \( A_{(2,2,0) \times (2,2,0)}^{(4,4)} \) is the peak amplitude of the quadratic QNM sourced by the linear \( (2, 2, 0) \) QNM interacting with itself.
We will fix $A(2,2,0)$ and $A(4,4,0)$ clearly satisfy a quadratic relationship, illustrated by the shaded blue region that is obtained by combining the fitted quadratic curves for $u_0 \in [15M, 30M]$. In this region, we find the ratio $A(2,2,0) \times (2,2,0)/A(2,2,0)$ to range between 0.20 and 0.15 [61]. There is no noticeable difference in the quadratic relationship followed by the 0.7 and 0.5 spin families of waveforms, compared to the variations that are observed in the best-fit $A(4,4,0)$.

We emphasize that this quadratic behavior is unique to the $A(2,2,0) \times (2,2,0)$ mode, as can be seen in the bottom panel of Fig. 1, where we show the best-fit linear amplitude $A(4,4,0)$ as a function of $A(2,2,0)$. These two modes are not related quadratically (for more on their scaling with mass ratio, see [62]), which confirms the distinct physical origin of $A(4,4,0)$ and $A(4,4,0)$. The best-fit amplitudes of $A(4,4,0)$ and $A(2,2,0)$ are nearly constants across these values of $u_0$, which is why the four bottom figures look the same. A key result of Fig. 1 is that $A(2,2,0) \times (2,2,0)$ is comparable to or larger (by a factor of $\sim 4$ in cases with $q \approx 1$) than $A(4,4,0)$ at the time of the peak. Given that the exponential decay rates of $A(2,2,0) \times (2,2,0)$ and $A(4,4,0)$ for a BH with $\chi_f = 0.7$ are $\text{Im}[\omega(2,2,0)] = -0.16$ and $\text{Im}[\omega(4,4,0)] = -0.08$, respectively, even beyond $10M$ after $u_{\text{peak}}$ the quadratic mode will be larger than the linear mode for equal mass ratio binaries [63]. Thus, for large SNR events in which the (4, 4) mode is detectable, the quadratic QNM could be measurable.

**Comparisons.—** Fig. 2 shows the GW150914 simulation (SXS:BBH:0305) and its fitting at $u_0 = 18M$, the time at which the residual in the (4, 4) mode reaches its minimum. The top panel shows the waveform fit with the (4, 4) quadratic model $h_{(4,4)}$, $Q$ as a function of time, where we find that it can fit rather well the amplitude and phase evolution of the numerical waveform at late times. The bottom panel shows the residual of the NR waveform with the linear and quadratic (4, 4) QNM models, $h_{(4,4)}$, and $Q$, and a conservative estimate for the numerical error obtained by comparing the highest and second highest resolution simulations for SXS:BBH:0305. We see that even though the linear and quadratic (4, 4) modes have the same number of free parameters, the residual of $h_{(4,4)}$ is nearly an order of magnitude better, which confirms the importance of including quadratic QNMs. Since, in general, the quadratic mode decays in time slower than the (4, 4, 2) QNM, the quadratic model generally better describes the late time behavior of the waveform. In addition, the best-fit value of $A(4,4,0)$ — which is the...
most important QNM in (4, 4) at late times—differs in the linear and quadratic models, which causes the residuals to be rather different even beyond \( u = 50M \) when we expect the overtones and quadratic mode to be sub-dominant.

In addition to the residuals, we quantify the goodness of fit by our models through the mismatch

\[
\mathcal{M} = 1 - \text{Re} \left[ \frac{\langle h_{\text{NR}} | h_{\text{model}} \rangle}{\sqrt{\langle h_{\text{NR}} | h_{\text{NR}} \rangle \langle h_{\text{model}} | h_{\text{model}} \rangle}} \right].
\]  

The top panel of Fig. 3 shows the mismatch in the (4, 4) mode between the NR waveform and the QNM model as a function of \( u_0 \). The red and black lines show the results for the SXS:BBH:0305 simulation when the (4, 4) mode was modeled with \( h_{\text{model}}^{(4,4),L} \) and \( h_{\text{model}}^{(4,4),Q} \), respectively. As a reference, we also show the numerical error calculated for SXS:BBH:0305 [64]. We see that the numerical error is below the fitted model mismatches for \( u_0 \lesssim 40M \), but causes the mismatch to worsen at late times. We also see that the linear model performs worse than the quadratic model for any \( u_0 \), confirming that the residual difference shown in the bottom panel of Fig. 2 was not a coincidence of the particular fitting time chosen there. At times \( u_0 \approx 20M \), we see that the mismatch is about one order of magnitude better in the quadratic model. We find similar results for all of the simulations analyzed in this Letter [65] (grey thin curves show the mismatch of the \( h_{\text{model}}^{(4,4)} \) in those simulations), although the mismatch difference becomes more modest for simulations with \( q \approx 8 \). When comparing the mismatches to the error, we find that in some simulations they start reaching the numerical error floor at around \( u_0 \approx 30M \), which is why we take \( u_0 \leq 30M \) in our fits. It is also beyond \( u_0 \approx 30M \) that we see no considerable differences between the linear and quadratic model mismatches due to the high numerical error relative to the numerical strain’s amplitude.

In the bottom panel of Fig. 3 we show the best-fit amplitudes of the QNMs in the (4, 4) mode as functions of \( u_0 \). We show the results for SXS:BBH:0305 (thick lines) as well as the rest of the simulations (thin lines). We see that at \( u_0 \gtrsim 10M \) the amplitude of \( A_{(4,4,0)} \) is extremely stable, but the faster the additional QNM decays, the more variations that are seen. Nevertheless, the \( A_{(2,2,0)\times(2,2,0)} \) exhibits only \( \sim 20\% \) variations for \( u_0 \in [15M, 30M] \), whereas \( A_{(4,4,1)} \) varies by \( \sim 70\% \) in the same range. Before and near \( u_0 \approx 10M \) every amplitude shows considerable variations, which is why we use \( u_0 \gtrsim 15M \) in this Letter. This suggests a need to improve the QNM model, either by including more overtones as in [10], modifying the time dependence of the linear [66] and quadratic terms, or considering more nonlinear effects.

Finally we check which frequency is preferred by the (4, 4) mode of the numerical strain. For this, we fix two frequencies to be the linear \( \omega_{(4,4,0)} \) and \( \omega_{(4,4,1)} \) frequencies, and keep one frequency free. We vary the frequency of that third term and fit every amplitude to minimize the residual in Eq. (6). Fig. 4 shows contours of the
Note Added.—While preparing this Letter, we learned that Cheung et al. conducted a similar study, whose results agree with ours [67].
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The \((l, m, n) = (2, 2, 0)\) can excite other quasitonic QNMs with frequency \(\omega = \omega_{(2,2,0)} - \omega_{(2,0,0)}\). These will instead be related to the memory effect, as they are non-oscillatory. From angular selection rules they will be most prominent in the \((2,0)\) mode. While these effects could also prove interesting to study, they are much more well-understood than the quasitonic QNMs in the \((4,4)\) mode, so we reserve their examination for future work [44, 69].

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This is because the binary black hole simulations that we consider are non-precessing and are in quasi-circular orbits, so the \(m < 0\) modes can be recovered from the \(m > 0\) modes via \(h_{(l,m)} = (-1)^{n}h_{(l,-m)}\).

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[58] Besides the amplitudes, we can also check the consistency of the phases of the quadratic (4, 4) QNM and the linear (2, 2, 0) QNM. We find that the phase of $A_{(2,2,0)\times(2,2,0)}^{(4,4)}$ is always within 0.4 radians of 0, for each simulation, for start times in the range $u_0 \in [15M, 30M]$.}

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[60] We also find the peak amplitude $A_{(4,4,1)}$ to be comparable or sometimes larger than $A_{(4,4)}$ (see bottom panel of Fig. 3) but, since $\text{Im}[M_{(4,4,1)}]= -0.25$, this (4, 4, 1) mode decays fast enough that it will be comparable or smaller than the quadratic (4, 4) mode after $u = 10M$.

[61] The numerical error for the other simulations tends to be worse since they were not run with as fine of a resolution, but the errors are nonetheless comparable to that of SXS:BBH:0305.

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