REDUCING Dehn FILLINGS AND SMALL SURFACES

SANGYOP LEE, SEUNGSANG OH, AND MASAKAZU TERAGAITO

Abstract. In this paper we investigate the distances between Dehn fillings on a hyperbolic 3-manifold that yield 3-manifolds containing essential small surfaces including non-orientable surfaces. Especially we study the situations where one filling creates an essential sphere or projective plane, and the other creates an essential sphere, projective plane, annulus, Möbius band, torus or Klein bottle, all 11 pairs of such non-hyperbolic manifolds.

1. Introduction

Let \( M \) be a compact, connected, orientable 3-manifold with a torus boundary component \( \partial_0 M \). Let \( \gamma \) be a slope on \( \partial_0 M \), that is, the isotopy class of an essential simple closed curve on \( \partial_0 M \). The 3-manifold obtained from \( M \) by \( \gamma \)-Dehn filling is defined to be \( M(\gamma) = M \cup V_\gamma \), where \( V_\gamma \) is a solid torus glued to \( M \) along \( \partial_0 M \) in such a way that \( \gamma \) bounds a meridian disk in \( V_\gamma \).

By a small surface we mean one with non-negative Euler characteristic including non-orientable surfaces. Such surfaces play a special role in the theory of 3-dimensional manifolds. Thurston’s geometrization theorem for Haken manifolds \( [Th] \) asserts that a hyperbolic 3-manifold \( M \) with non-empty boundary contains no essential small surfaces. Furthermore, if \( M \) is hyperbolic, then the Dehn filling \( M(\gamma) \) is also hyperbolic for all but finitely many slopes \( [Th] \), and a good deal of attention has been directed towards obtaining a more precise quantification of this statement.

Let us say that a 3-manifold is of type \( S, D, A, \) or \( T \), if it contains an essential orientable small surface which is an essential sphere, disk, annulus or torus, and of type \( P, B, \) or \( K \), if it contains a non-orientable small surface which is a projective plane, Möbius band or Klein bottle, respectively. Especially 3-manifolds of type \( S, D, A \) and \( T \) are called reducible, \( \partial \)-reducible, annular and toroidal, respectively. The distance \( \Delta(\gamma_1, \gamma_2) \) between two slopes on a torus is their minimal geometric intersection number. The bound \( \Delta(X_1, X_2) \) is the least nonnegative number \( m \) such that if \( M \) is a hyperbolic manifold which admits two Dehn fillings \( M(\gamma_1), M(\gamma_2) \) of type \( X_1, X_2 \), respectively, then \( \Delta(\gamma_1, \gamma_2) \leq m \). Surveys of the known bounds of various choices \( (X_1, X_2) \) and the maximal values realized by known examples are given in [G2, W3, FA].

In this paper we consider \( M(\gamma_i), i = 1, 2 \) of only the six types \( S, P, A, B, T, \) or \( K \). Suppose that \( M(\gamma_i) \) contains such a small surface \( F_i \). Then we may assume that \( F_i \) meets the attached solid torus \( V_{n_i} \) in a finite collection of meridian disks, and is chosen so that the number of disks \( n_i \) is minimal among all such surfaces in \( M(\gamma_i) \). The main results of this paper are the followings.

Theorem 1.1. Suppose that \( M \) is hyperbolic. If \( M(\gamma_1) \) and \( M(\gamma_2) \) are of type \( S \) or \( P \), then \( \Delta(\gamma_1, \gamma_2) \leq 1 \).

The original proof of the case \( (S, S) \) is very complicated [CL1]. Our proof is remarkably short, although it is based on the analysis of intersections of two surfaces as well as [CL1]. Recently, Hoffman and Matignon [HM] also gives such a short proof in the almost same line, but ours is still simpler than it.

If a 3-manifold contains a projective plane, then it is either a reducible manifold or the 3-dimensional projective space. Hence type \( P \) breaks down into type \( S \) and CYC, which means the
class of manifolds with finite cyclic fundamental groups. References are: [BZ2] for \( \Delta(S, \text{CYC}) = 1 \); [CGLS] for \( \Delta(\text{CYC}, \text{CYC}) = 1 \). See also [M] [T1] for \( \Delta(P, P) = 1 \).

**Theorem 1.2.** Suppose that \( M \) is hyperbolic. If \( M(\gamma_1) \) is of type \( S \) or \( P \), and \( M(\gamma_2) \) is of type \( A \) or \( B \), then either \( \Delta(\gamma_1, \gamma_2) \leq 1 \), or \( \Delta(\gamma_1, \gamma_2) = 2 \) with \( n_2 = 2 \) when \( M(\gamma_2) \) is of type \( A \) (or \( n_2 = 1 \) when of type \( B \)).

For the case \((S, A)\), Wu [W] showed that \( \Delta(S, A) = 2 \) by using the sutured manifold theory, and he asked whether \( n_2 = 2 \) when \( M(\gamma_1) \) is of type \( S \) and \( M(\gamma_2) \) is of type \( A \) with \( \Delta(\gamma_1, \gamma_2) = 2 \) [W] Question 5.8. Our Theorem 1.2 gives the affirmative answer of this question. Note that type \( B \) breaks down into types \( S, A \) and \( D \).

**Theorem 1.3.** Suppose that \( M \) is hyperbolic. If \( M(\gamma_1) \) is of type \( S \) or \( P \), and \( M(\gamma_2) \) is of type \( T \) or \( K \), then either \( \Delta(\gamma_1, \gamma_2) \leq 2 \), or \( \Delta(\gamma_1, \gamma_2) = 3 \) with \( n_2 = 2 \) when \( M(\gamma_2) \) is of type \( T \) (or \( n_2 = 1 \) when of type \( K \)).

For the case \((S, T)\), Oh [O] and Wu [W2] independently showed that \( \Delta(S, T) = 3 \). (See also [LOT] for its short proof.) Hence Theorem 1.3 gives an improvement of this result. In [JLOT], we have showed the conclusions for two cases \((P, T)\) and \((P, K)\).

If a 3-manifold is of type \( K \), then it is of type \( S, T \) or a Seifert fibered manifold with finite fundamental group (a prism manifold). Indeed, non-orientable cases are necessary to prove orientable cases in our arguments. Such phenomenon is also observed in [GL2] and [GL3].

We should emphasize that all 11 cases can be treated in a unified argument in this paper. (Also, we have a plan to continue the study for the remaining pairs of \( S, P, D, A, B, T, K \).) Our main tool in this paper is a two-cornered cycle, which was introduced by Hoffman [H]. Hoffman showed that the disk bounded by a great \( x \)-cycle contains a pair of specific two-cornered cycles, called a seamly pair, and it can be used to find a new essential sphere and projective plane meeting the attached solid torus in a fewer times than an original surface, leading to a contradiction. We define a slight generalization of a great \( x \)-cycle, called an \( x \)-face, and show that it contains a seamly pair by a simpler argument than that in [H] and that such a pair is useful for the types \( A \) and \( B \) as well.

By virtue of Theorem 1.1 and the fact \( \Delta(S, D) = 0 \) [S] (and hence \( \Delta(P, D) = 0 \)), we can put the following assumption throughout the paper to simplify the arguments greatly:

**Assumption.** If \( M(\gamma_i) \) is of type \( A, B, T \) or \( K \), then we assume that \( M(\gamma_i) \) is neither of type \( S \) nor of type \( D \).

### 2. Graphs of surface intersections

Hereafter \( M \) is a hyperbolic 3-manifold with a torus boundary component \( \partial_0 M \). An orientable surface properly embedded in \( M \) is called essential if it is either (i) incompressible, not boundary parallel and not a sphere, or (ii) a sphere which does not bound a 3-ball. Note that any essential small surface is also boundary incompressible by our assumption in Section 1. In this section we describe how (essential) small surfaces \( \hat{F}_1 \) and \( \hat{F}_2 \), in \( M(\gamma_1) \) and \( M(\gamma_2) \) respectively, give rise to labelled intersection graphs \( G_i \subset \hat{F}_i \) for \( i = 1, 2 \) in a general context.

As in Section 1 let \( \hat{F}_i \) be a small surface in \( M(\gamma_i) \) with \( n_i = |\hat{F}_i \cap V_{\gamma_i}| \) minimal. (Recall that if \( \hat{F}_i \) is orientable then it is assumed to be essential.) Then \( n_i > 0 \). Thus \( F_i = \hat{F}_i \cap M \) is a punctured surface properly embedded in \( M \), each of whose \( n_i \) boundary components has slope \( \gamma_i \) on \( \partial_0 M \).

**Lemma 2.1.** For each of six types, \( F_i \) is incompressible and boundary incompressible in \( M \).

**Proof.** For types \( S, A \) and \( T \), the minimality of \( n_i \) guarantees that \( F_i \) is incompressible and boundary incompressible in \( M \).

For type \( P \), assume that \( D \) is a compressing disk for \( F_i \). Since \( \partial D \) is orientation-preserving on \( F_i \), \( \partial D \) bounds a disk \( D' \) on \( \hat{F}_i \). Since \( \text{Int} D' \) meets \( V_{\gamma_i} \), we can create a new projective plane by replacing \( D' \) with \( D \), which meets \( V_{\gamma_i} \) fewer than \( \hat{F}_i \), contradicting the minimality of \( n_i \). Next, assume that \( E \) is a boundary compressing disk for \( F_i \) with \( \partial E = a \cup b \), where \( a \) is an essential arc in \( F_i \) and \( b \) lies in \( \partial_0 M \). If \( a \) joins distinct components of \( \partial F_i \), then a compressing disk for
\( F_i \) is obtained from two parallel copies of \( E \) and the disk obtained by removing a neighborhood of \( b \) from the annulus in \( \partial_b M \) cobounded by those components of \( \partial F_i \) meeting \( a \). Hence both endpoints of \( a \) lie in the same component, say \( \partial_1 F_i \), of \( \partial F_i \). If \( n_i \geq 2 \), then \( b \) bounds a disk \( D' \) in \( \partial_b M \) together with a subarc of \( \partial_1 F_i \). Then \( E \cup D' \) gives a compressing disk for \( F_i \) in \( M \). Therefore \( n_i = 1 \). Then \( M \) would contain a Möbius band, a contradiction.

For type \( S \), assume that \( D \) is a compressing disk for \( F_i \). If \( \partial D \), which is orientation-preserving on \( F_i \), bounds a disk \( D' \) on \( \hat{F}_i \), a new Möbius band obtained by replacing \( D' \) with \( D \) has fewer \( n_i \), a contradiction. Thus \( \partial D \) is essential, and indeed separating on \( \hat{F}_i \). Compressing \( \hat{F}_i \) along \( D \) would give in \( M(\gamma_i) \) a projective plane \( Q_1 \) and a disk \( Q_2 \) which are disjoint. Note that the core of \( V_{\gamma_i} \) must meet both \( Q_1 \) and \( Q_2 \) because \( M \) is hyperbolic. Let \( \ell \) be a subarc of the core of \( V_{\gamma_i} \) which connects \( Q_1 \) and \( Q_2 \), meeting them only on its endpoints. Attaching a tube along \( \ell \) to \( Q_1 \cup Q_2 \) gives a Möbius band in \( M(\gamma_i) \) which intersects the core of \( V_{\gamma_i} \) in \( n_i - 2 \) points, a contradiction. Thus we have shown that \( F_i \) is incompressible. The incompressibility of \( F_i \) then implies the boundary incompressibility of \( F_i \); otherwise, \( n_i = 1 \) as previous, and furthermore a boundary-compressing disk \( E \) allows us to isotope the core of \( V_{\gamma_i} \) into \( \hat{F}_i \) as an orientation-reversing loop. Then \( M \) would contain an essential annulus, a contradiction.

For type \( K \), assume that \( D \) is a compressing disk for \( F_i \). As previous, \( \partial D \) is orientation-preserving and essential on \( \hat{F}_i \). Compressing \( \hat{F}_i \) along \( D \) would give in \( M(\gamma_i) \) either a non-separating 2-sphere or two disjoint projective planes, according as \( \partial D \) is non-separating or separating on \( \hat{F}_i \). But this contradicts the irreducibility of \( M(\gamma_i) \). The incompressibility of \( F_i \) then implies the boundary incompressibility of \( F_i \), unless \( M \) contains a Möbius band, as above. \( \square \)

We use \( i \) and \( j \) to denote 1 or 2, with the convention that, when both appear, \( \{i, j\} = \{1, 2\} \).

By an isotopy of \( F_1 \), say, we may assume that \( F_1 \) intersects \( F_2 \) transversely. By Lemma 2.1 it can be assumed that no circle component of \( F_1 \cap F_2 \) bounds a disk in \( F_1 \) or \( F_2 \). Let \( G_i \) be the graph in \( F_i \) obtained by taking as the (fat) vertices the disks \( \hat{F}_i - \text{Int} F_i \) and as edges the arc components of \( F_1 \cap F_2 \) in \( \hat{F}_i \). Thus the interior of any disk face of \( G_i \) is disjoint from \( F_j \). We number the components of \( \partial F_i \cap \partial_b M \) as 1, 2, \ldots, \( n_i \) in the order in which they appear on \( \partial_b M \). On occasion we will use 0 instead of \( n_i \) in short. This gives a numbering of the vertices of \( G_i \). Furthermore it induces a labelling of the endpoints of edges in \( G_j \) in the usual way (see [CGLS]). Note that \( G_1 \) and \( G_2 \) have no trivial loops by Lemma 2.1. For the sake of simplicity we will say that \( F_i \) and \( G_i \) are of type \( X \) if \( M(\gamma_i) \) is of type \( X \).

Since \( M \) is hyperbolic, we have the following easy lemma.

**Lemma 2.2.** \( n_i \geq 3 \) for \( G_i \) of type \( S \), and \( n_i \geq 2 \) for \( G_i \) of type \( P \).

Although \( F_i \) of type \( P \), \( B \) or \( K \) is non-orientable, we can establish a parity rule, which plays a crucial role. In fact, this is a natural generalization of the usual parity rule [CGLS]. First, orient all components of \( \partial F_i \) so that they are mutually homologous on \( \partial_b M \). Let \( e \) be an edge of \( G_i \). Since \( e \) is an arc properly embedded in \( F_i \), a regular neighborhood \( D \) of \( e \) in \( F_i \) is a disk in \( F_i \). Then \( \partial D = a \cup b \cup c \cup d \), where \( a \) and \( c \) are arcs in \( \partial F_i \) with induced orientations from \( \partial F_i \). If \( a \) and \( c \) are directed along \( \partial D \), then \( e \) is called positive, otherwise negative. See Figure 1. Then we have the parity rule: an edge \( e \) is positive in \( G_i \) if and only if \( e \) is negative in \( G_j \).

The rest of this section will be devoted to several definitions and well known lemmas. Let \( x \) be a label of \( G_i \). An \( x \)-edge in \( G_i \) is an edge with label \( x \) at one endpoint, and an \( xy \)-edge is an edge with label \( x \) and \( y \) at both endpoints. If an \( x \)-edge has an endpoint at a vertex \( v \) with label \( x \), then it is called an \( x \)-edge at \( v \). Especially, a positive \( xx \)-edge is called a level \( x \)-edge, which means that the vertex \( x \) of the other graph \( G_j \) is incident to a negative loop. In particular, \( G_i \) does not contain a level \( x \)-edge unless \( G_j \) is of type \( P \), \( B \) or \( K \).

An \( x \)-cycle is a cycle of positive \( x \)-edges of \( G_i \) which can be oriented so that the head of each edge has label \( x \). A Scharlemann cycle is an \( x \)-cycle that bounds a disk face of \( G_i \), only when \( n_j \geq 2 \). Each edge of a Scharlemann cycle has the same label pair \( \{x, x + 1\} \), so we refer to such a Scharlemann cycle as an \( (x, x + 1) \)-Scharlemann cycle. The number of edges in a Scharlemann cycle is called its length. In particular, a Scharlemann cycle of length two is called an \( S \)-cycle in short.
Suppose that a Scharlemann cycle $\sigma$ is immediately surrounded by a cycle $\kappa$, that is, each edge of $\kappa$ is immediately parallel to an edge of $\sigma$. Then $\kappa$ will be referred to as an extended Scharlemann cycle, only when $n_j \geq 4$. A generalized $S$-cycle is the triple $\{e_1, e_2, e_3\}$ of mutually parallel positive edges in succession and $e_2$ is a level edge, only when $n_j \geq 3$.

**Lemma 2.3.** Assume that $M(\gamma_j)$ is either of type $S$, $A$ or $T$. If $G_i$ has a Scharlemann cycle then $\hat{F}_j$ must be separating, and so $n_j$ is even. Furthermore, for cases $A$ and $T$, the edges of a Scharlemann cycle do no lie in a disk in $\hat{F}_j$.

**Proof.** Let $E$ be a disk face bounded by a Scharlemann cycle with a label pair, say $\{1, 2\}$, in $G_i$. Let $V_{12}$ be the 1-handle cut from $V_{\gamma_j}$ by the vertices 1 and 2 of $G_i$. (When $n_j = 2$, $V_{12}$ is chosen to meet $\partial E$.) Then tubing $\hat{F}_j$ along $\partial V_{12}$ and compressing along $E$ gives a new surface $R$ in $M(\gamma_j)$, homeomorphic to $\hat{F}_j$, that intersects $V_{\gamma_j}$ fewer times than $\hat{F}_j$. If the original $\hat{F}_j$ is non-separating, then so is $R$. Thus $R$ is essential for each case, a contradiction.

If the edges of the Scharlemann cycle lie in a disk $D$ in an annulus or torus $\hat{F}_j$, then $\text{nhd}(D \cup V_{12} \cup E)$ is a once punctured lens space. By the irreducibility of $M(\gamma_j)$, $M(\gamma_j)$ is a lens space, so neither of type $A$ nor of type $T$. □

**Lemma 2.4.**

1. If $G_j$ is of type $S$, $A$ or $T$, then $G_i$ cannot have a level edge. If $G_j$ is of type $P$ or $B$, then $G_i$ has at most one label of level edges. If $G_j$ is of type $K$, then $G_i$ has at most two labels of level edges.

2. If $G_j$ is of type $P$ or $B$, then $G_i$ has at most one label of level edges. If $G_j$ is of type $A$ or $K$, then any two Scharlemann cycles of $G_i$ have the same label pair.

3. When $G_j$ is of type $S$, $M(\gamma_j)$ contains a projective plane if $G_i$ contains an $S$-cycle.

4. If $G_j$ is of type $S$, $A$ or $T$, then $G_i$ cannot have an extended $S$-cycle.

5. If $G_j$ is of type $P$, $B$ or $K$, then $G_i$ cannot have a generalized $S$-cycle.

**Proof.** (1) Let $e$ be a negative loop based at a vertex $x$ in $G_j$. Then $\text{nhd}(x \cup e)$ is a Möbius band in $\hat{F}_j$. Since only projective plane, Möbius band and Klein bottle can contain at most one, one and two Möbius bands respectively, the conclusions follow.

2. When $G_j$ is of type $P$, $B$ or $K$, assume for contradiction that $G_i$ contains a Scharlemann cycle. Then the construction in the proof of Lemma 2.3 gives a new surface homeomorphic to $\hat{F}_j$ in $M(\gamma_j)$, which meets $V_{\gamma_j}$ fewer times than $\hat{F}_j$. This contradicts the minimality of $n_j$.

When $G_j$ is of type $S$ or $A$, this is [GL1] Theorem 2.4 or [W3] Lemma 5.4(2), respectively.

3. Let $\{e_1, e_2\}$ be an $S$-cycle in $G_i$ with label pair $\{k, k + 1\}$. Let $v_k$ be the $h$-th vertex of $G_j$ for $h = k, k + 1$, and let $H$ be the part of $V_{\gamma_j}$ between $v_k$ and $v_{k+1}$. Let $D$ be the disk face bounded by the $S$-cycle, and let $A$ be the Möbius band obtained by taking $H \cup D$ and shrinking $H$ radially to its core. Since $\partial A$ is isotopic to the curve obtained from $e_1 \cup e_2 \cup v_k \cup v_{k+1}$ by shrinking the vertices to points, $\partial A$ bounds a disk on $\hat{F}_j$. Thus $M(\gamma_j)$ contains a projective plane.

4. These are [W1] Lemma 2.3, [W3] Lemma 5.4(3) and [BZ1] Lemma 2.10, respectively.

5. Let $\{e_1, e_2, e_3\}$ be a generalized $S$-cycle in $G_i$, where $e_2$ is a level edge with label $k$, and $e_1, e_3$ have the same label pair $\{k - 1, k + 1\}$. Let $v_k$ be the $h$-th vertex of $G_j$ for $h = k - 1, k, k + 1$, and let $H$ be the part of $V_{\gamma_j}$ between the vertices $v_{k-1}$ and $v_{k+1}$ containing $v_k$. Then $C = \text{nhd}(v_k \cup e_2)$.
is a Möbius band in \( \hat{F}_j \). Let \( D \) be the disk in \( F_j \) representing the parallelism of \( e_1 \) and \( e_3 \) and containing \( e_2 \). Then we have a Möbius band \( A \) from \( H \cup D \) as before. Note that \( \text{Int} A \) is disjoint from \( \hat{F}_j - C \). Since \( \partial A \) is isotopic to the curve obtained from \( e_1 \cup e_3 \cup v_{k-1} \cup v_{k+1} \) by shrinking \( v_{k \pm 1} \) to points, \( \partial A \) is orientation-preserving on \( \hat{F}_j \).

If \( G_j \) is of type \( \mathcal{P} \), then \( \partial A \) above bounds a disk \( E \) in \( \hat{F}_j - C \). Thus \( A \cup E \) gives a new projective plane which meets \( V_{j \gamma} \) fewer times than \( n_j \), a contradiction.

If \( G_j \) is of type \( \mathcal{B} \), then either \( \partial A \) bounds a disk in \( \hat{F}_j - C \) or \( \partial A \) is parallel to \( \partial \hat{F}_j \). In the former, \( M(\gamma_j) \) contains a projective plane, and hence \( M(\gamma_j) \) is reducible, contradicting the irreducibility of \( M(\gamma_j) \). In the latter, let \( A_1 \) be the annulus between \( \partial A \) and \( \partial \hat{F}_j \). Then \( A \cup A_1 \) is a Möbius band in \( M(\gamma_j) \) which meets \( V_{j \gamma} \) fewer times than \( n_j \).

If \( G_j \) is of type \( \mathcal{K} \), then \( \partial A \) bounds either a disk or a Möbius band \( B \) in \( \hat{F}_j - C \). In the former, \( M(\gamma_j) \) contains a projective plane. By the irreducibility of \( M(\gamma_j) \), \( M(\gamma_j) \) is the lens space \( L(2,1) \).

But it is well known that \( L(2,1) \) does not contain a Klein bottle \( \mathbb{K} \). In the latter, \( A \cup B \) gives a new Klein bottle in \( M(\gamma_j) \) which meets \( V_{j \gamma} \) fewer times than \( n_j \) again.

For simplicity we will call \( x \) an \( \text{sl-label} \) of \( G_i \) (or \( \text{sl-vertex} \) of \( G_j \)) if \( x \) is a label of either a Scharrlemann cycle or a level edge in \( G_i \), according as \( \hat{F}_j \) is orientable or not. Thus Lemma 2.4 implies that \( G_i \) contains at most 2, 1, 2, 1 or 2 \( \text{sl-labels} \) when \( G_j \) is of type \( \mathcal{S} \), \( \mathcal{P} \), \( \mathcal{A} \), \( \mathcal{B} \) or \( \mathcal{K} \), respectively. When \( G_j \) is of type \( \mathcal{T} \), we do not need an upper bound for the number of \( \text{sl-labels} \) of \( G_i \), but will show that at most three labels can be labels of \( S \)-cycles of \( G_i \) unless \( M(\gamma_j) \) contains a Klein bottle. (See Claim 3.1)

**Lemma 2.5.** Let \( F \) be a family of mutually parallel positive edges in \( G_i \), and let \( |F| \) denote the number of edges in \( F \).

1. If \( G_j \) is of type \( \mathcal{S} \) or \( \mathcal{A} \), \( |F| \leq \frac{n_j}{2} + 1 \). Furthermore if \( |F| = \frac{n_j}{2} + 1 \), then the first two or the last two edges of \( F \) form an \( S \)-cycle.
2. If \( G_j \) is of type \( \mathcal{P} \) or \( \mathcal{B} \), \( |F| \leq \frac{n_j + 1}{2} \). Furthermore if \( |F| = \frac{n_j + 1}{2} \), then the first or the last edge of \( F \) is a level edge.
3. If \( G_j \) is of type \( \mathcal{T} \) and \( n_j \geq 3 \), \( |F| \leq \frac{n_j}{2} + 2 \).
4. If \( G_j \) is of type \( \mathcal{K} \) and \( n_j \geq 2 \), \( |F| \leq \frac{n_j}{2} + 1 \).

*Proof.* (1) is \([W2\text{ Lemma 1.5(1) and GW\text{ Lemma 2.5(2)}. (We remark that when \( G_j \) is of type \( \mathcal{A} \) and \( n_j = 1 \), \( |F| \leq 1 \), because a pair of edges cannot be parallel in both graphs \([G1\text{ Lemma 2.1}{]) \text{)} \text{. (3) is \([W2\text{ Lemma 1.4}. \text{ For (2), if } |F| > \frac{n_j + 1}{2} \text{ then } F \text{ would contain either an } S \text{-cycle, a generalized } S \text{-cycle or two level edges with distinct labels, except the case where } G_j \text{ is of type } \mathcal{B} \text{ and } n_j = 1. \text{ In the exceptional case, a pair of edges is parallel in both } G_i \text{ and } G_j, \text{ a contradiction. (4) is similar to (2)).} \]

**Lemma 2.6.** Let \( F \) be a family of mutually parallel negative edges in \( G_i \).

1. When \( G_j \) is of type \( \mathcal{S} \) or \( \mathcal{P} \), \( |F| \leq n_j - 1 \).
2. When \( G_j \) is of type \( \mathcal{A} \), \( \mathcal{B} \) or \( \mathcal{K} \), \( |F| \leq n_j \).

*Proof.* (1) See \([G1\text{ Lemma 2.3}. \text{ (This is essentially proved in } \text{ GW\text{ Section 5]). (2) The argument of the proof of } \text{ GW\text{ Lemma 2.5(3) }} \text{ works well. (Also, see the proof of } \text{ GW\text{ Lemma 4.2}.} \text{)} \]

### 3. \( x \)-FACES AND NON-ORIENTABLE SURFACES

In this section we assume that \( G_j \) is of type \( \mathcal{P} \) or \( \mathcal{B} \) (so we may assume that 1 is the only possible \( \text{sl-vertex} \) of \( G_j \) by Lemma 2.4), and thus all level edges of \( G_i \) are level 1-edges.

A disk face of the subgraph of \( G_i \) consisting of all the vertices and positive \( x \)-edges of \( G_i \) is called an \( x \)-face. Remark that the boundary of an \( x \)-face \( D \) may be not a circle, that is, \( \partial D \) may contain a double edge, and more than two edges of \( \partial D \) may be incident to a vertex on \( \partial D \) (see Figure 2.1 in [HM]). A cycle in \( G_i \) is a \( \text{two-cornered cycle} \) if it is the boundary of a disk face containing only 01-labels, 12-labels and positive edges, only when \( n_j \geq 3 \). Recall that 0 denotes \( n_j \). A two-cornered cycle must contain both kinds of corners, because \( G_i \) cannot contain
a Scharlemann cycle by Lemma 2.3. Also, it contains at least one 02-edge. For convenience, when
the labels appear in anticlockwise order around the boundary of a vertex \(v\) of \(G_i\), given three
distinct labels \(x_1, x_2, x_3\), we say \(x_1 < x_2 < x_3\) if \(x_1, x_2, x_3\) appear in anticlockwise order on some
interval in \(\partial v\) containing \(n_j\) edge endpoints. Thus three expressions \(x_1 < x_2 < x_3, x_2 < x_3 < x_1\)
and \(x_3 < x_1 < x_2\) are equivalent.

**Proposition 3.1.** Suppose that \(n_j \geq 3\). An \(x\)-face, \(x \neq 1\), in \(G_i\) contains a pair of two-cornered
cycles sharing a level 1-edge.

**Proof.** Let \(\Gamma_D\) be the subgraph of \(G_i\) in an \(x\)-face \(D\). There is a possibility that \(\partial D\) is not a circle
as mentioned before. Since we will find a pair of two-cornered cycles within \(D\), we can cut formally
the graph \(G_i \cap D\) along double edges of \(\partial D\) and at vertices to which more than two edges of \(\partial D\)
are incident to so that \(\partial D\) is deformed into a circle. (See also Figure 5.1 in [HM].) Thus we may
assume that \(\partial D\) is a circle. We may assume that the labels appear in anticlockwise order around
the boundary of each vertex.

Suppose that \(D\) has a diagonal edge \(d\) with distinct labels \(\{a, b\}\), which must differ from \(x\) because
\(D\) is an \(x\)-face, as in Figure 2(a). Assume without loss of generality that \(a < x < b\).
Formally construct a new \(x\)-face \(D'\) as follows. Keep all corners and edges of \(\Gamma_D\) to the right of
\(d\) (when \(d\) is directed from \(a\) to \(b\)), discard all corners and edges to the left of \(d\), and then insert
additional edges to the left of \(d\), and parallel to \(d\), until you first reach label \(x\) at one or both ends
of this parallel family of edges, as in Figure 2(b). In particular, these additional edges contain no
edges of two-cornered cycles or Scharlemann cycles of the graph on the new \(x\)-face \(D'\).

![Figure 2. Split along a diagonal edge](image)

Repeat the above process for every diagonal edge which is not a level 1-edge, then get a new
\(x\)-face \(E\) and a graph \(\Gamma_E\) in \(E\). All diagonal edges of \(\Gamma_E\) are level 1-edges, and all (and only)
boundary edges are \(x\)-edges, where the label \(x\) possibly appear on both ends, say level \(x\)-edges.

**Claim 3.2.** \(\Gamma_E\) contains a level 1-edge.

**Proof of Claim 3.2.** Assume that \(\Gamma_E\) contains no level 1-edges, and so no diagonal edges. We first
show that if for some vertex \(v\) of \(\Gamma_E\) two boundary edges are incident to \(v\) with label \(x\), then these
should be level \(x\)-edges. For, \(n_j + 1\) edges are incident to \(v\) in \(\Gamma_E\). If \(n_j\) is even, more than \(\frac{n_j}{2}\)
mutually parallel edges are incident to \(v\), and so one of these edges should be a level edge different
from a level \(x\)-edge (recall that \(\Gamma_E\) cannot contain a Scharlemann cycle), a contradiction. If \(n_j\)
is odd, two families of \(\frac{n_j + 1}{2}\) mutually parallel edges are incident to \(v\) by Lemma 2.3(2), and the
boundary edge of each family should be a level \(x\)-edge by the same reason above.

Consider the cycle \(\sigma\) consisting of boundary \(x\)-edges of \(\Gamma_E\). Assume that \(\sigma\) has an \(x\)-edge which
is not a level \(x\)-edge. So the only one end has label \(x\) at \(v_1\), say. By the fact we just proved, another \(x\)-edge incident to \(v_1\) does not have label \(x\) at \(v_1\). Thus this edge has label \(x\) at the
other end \(v_2\), say. After repeating this process, we are led to show that \(\sigma\) is a great \(x\)-cycle in
the terminology of [CGLS]. By the same argument in the proof of Lemma 2.6.2 of [CGLS], \( \Gamma_E \) contains a Scharlemann cycle, since \( \Gamma_E \) does not contain level edges in its interior, a contradiction.

If all edges of \( \sigma \) are level \( x \)-edges, then we have a great \( x + 1 \)-cycle just inside \( \sigma \). Thus we still find a Scharlemann cycle, a contradiction. (See also [HM] Lemma 5.2.)

Let \( e \) be a level 1-edge. So it does not belong to \( \partial \Gamma_E \). Let \( E_1 \) and \( E_2 \) be the faces of \( \Gamma_E \) adjacent to \( e \). Assume for contradiction that \( \partial E_1 \) (or \( \partial E_2 \)) is not a two-cornered cycle. Note that \( \partial E_1 \) may contain many level 1-edges. Let \( \{a_k, a_k + 1\}, k = 1, \ldots, n \), be the consecutive label pairs of the corners between successive level 1-edges on \( \partial E_1 \), which appear in order around \( \partial E_1 \), when one runs clockwise around \( \partial E_1 \) starting at one end of \( e \), as in Figure 3(a). Then some \( a_k \) is neither 0 nor 1. Since \( a_1 = 1 \) and \( a_n = 0 \), there are indices \( l \) and \( m \) so that \( a_k = 0 \) or 1 when \( 1 \leq k < l \) or \( k = m \), and \( a_k \neq 0, 1 \) when \( l \leq k < m \). Consider the edges of the parallelism class containing each \( \{a_k-1, a_k\} \)-edge, \( l \leq k \leq m \). Note that among these edges a level \( x \)-edge on the boundary of \( \Gamma_E \) is the only possible level edge, and there is no \( x \)-edge except on the boundary. See Figure 3(b). Then we have \( x \leq a_k < a_k-1 + 1 \leq x \), and so \( x \leq a_k \leq a_k-1 < x \). Finally we have \( x \leq a_m \leq \cdots \leq a_{l+1} \leq a_l \leq a_{l-1} < x \). This is impossible because \( a_{l-1}, a_m = 0 \) or 1. Thus we have shown that any face adjacent to a level 1-edge is two-cornered.

In fact, we can see from the argument above that \( a_1 = \cdots = a_i = 1 \) and \( a_{i+1} = \cdots = a_n = 0 \) for some \( i \). That is, when we go along a two-cornered cycle from a level 1-edge in some direction, 12-corners appear successively, 23-corners appear after them, and we reach a level 1-edge (possibly, a different one from the start).

Recall that 1 is the only possible \( sl \)-label of \( G_i \). The next theorem essentially follows from the argument of [HM] Section 7.

**Theorem 3.3.** If \( G_j \) is of type \( \cal{P} \), then \( G_i \) cannot contain an \( x \)-face for a non-\( sl \)-label \( x \) of \( G_i \).

**Proof.** Assume that \( G_i \) contains such an \( x \)-face. If \( n_j = 2 \), then a boundary of each face in the \( x \)-face is a Scharlemann cycle or contains both a level 1-edge and a level 2-edge, contradicting Lemma 2.4(1) and (2). Thus we may assume that \( n_j \geq 3 \). Applying Proposition 3.1, \( G_i \) contains a pair of two-cornered cycles sharing a level 1-edge \( e \).

Let \( \tilde{S} \) be the sphere which is the boundary of a regular neighborhood of \( \tilde{E}_j \) in \( M(\gamma_j) \), and let \( S = \tilde{S} \cap M \). The arc components of \( F_i \cap S \) give rise to a pair of labelled graphs \( (G^S_i, \Sigma) \) as usual, where \( G_S \) is a double cover of \( G_j \). Then \( G^S_i \) contains an \( S \)-cycle \( \sigma \) corresponding to the level 1-edge \( e \). We may assume that \( \sigma \) has label pair \( \{1, 2\} \). Also, two cycles adjacent to \( \sigma \) in \( G^S_i \) contain only two types of corners, 01-corners and 23-corners. Let \( E_1 \) and \( E_2 \) be the faces of \( G^S_i \) bounded by these cycles respectively. The edges of \( \sigma \) separate \( \tilde{S} \) into two disks, and two vertices 0 and 3 of \( G_S \) are contained in the same disk component, say \( D_0 \), because of the existence of 03-edges. Let \( D_1 \) be an expansion of \( D_0 \) containing the fat vertices 1 and 2 in \( \hat{S} \), and regard it as properly

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**Figure 3.** Finding two-cornered cycles
embedded in \(X = \text{nhd}(D_1 \cup V_{01} \cup V_{23} \cup E_1 \cup E_2)\) where \(V_{01}\) and \(V_{23}\) are defined as in the proof of Lemma 2.3. Since \(\partial E_1\) and \(\partial E_2\) are non-separating on the boundary of the genus two handlebody \(\text{nhd}(D_1 \cup V_{01} \cup V_{23})\), \(\partial X\) is either a 2-sphere or the disjoint union of a 2-sphere and a torus. The latter happens only when \(\partial E_1\) and \(\partial E_2\) are parallel on the boundary of \(\text{nhd}(D_1 \cup V_{01} \cup V_{23})\). In this case, \(\partial D_1\) lies in the parallelism annulus. Thus \(\partial D_1\) cuts the 2-sphere component of \(\partial X\) into two disk parts, one of which is parallel to \(D_1\), in any case. Let \(D\) be the other disk part and then extend \(\partial D\) down to \(\hat{F}_j\) along an annulus \(A = \partial D \times I\), using the 1-bundle structure of \(\text{nhd}(\hat{F}_j)\).

Note that the boundary component of \(A\) which lies on \(\hat{F}_j\) bounds a Möbius band \(B\) intersecting the core of \(V_{\gamma_j}\) in a single point. In particular, \(D \cup A \cup B\) is a projective plane in \(M(\gamma_j)\). Since \(|D \cap V_{\gamma_j}| \leq |D_0 \cap V_{\gamma_j}| - 2 = n_j - 3\), \(|A \cap V_{\gamma_j}| = 0\) and \(|B \cap V_{\gamma_j}| = 1\), this projective plane intersects the core of \(V_{\gamma_j}\) in less than \(n_j\) points. This contradicts the minimality of \(n_j\). \(\square\)

**Theorem 3.4.** If \(G_j\) is of type \(B\) with \(n_j \geq 2\), then \(G_i\) cannot contain an \(x\)-face for a non-sl-label \(x\) of \(G_i\).

**Proof.** The same argument in the proof of Theorem 3.3 applies here. Assume that \(G_i\) contains an \(x\)-face for a non-sl-label \(x\). Applying Proposition 3.1, \(G_i\) contains a pair of two-cornered cycles sharing a level 1-edge \(e\). In particular, we can assume that one of the pair contains a single level 1-edge. This is guaranteed by choosing the level 1-edge \(e\) to be outermost in the \(x\)-face. In the same notation as before, however, the edges of \(\sigma\) separate \(\hat{S}\), which is an annulus, into two annuli, and so \(D_1\) is an annulus. Also, we may assume that \(\partial E_1\) contains only one 12-edge.

Let \(W = \text{nhd}(D_1 \cup V_{01} \cup V_{23})\) and let \(X = W \cup \text{nhd}(E_1 \cup E_2)\) as before. We may assume that \(D_1\) is properly embedded in \(W\) and \(X\). Thus \(W\) consists of a solid torus \(\text{nhd}(D_1)\) and the two 1-handles attached there. Now \(\pi_1(W) = \langle x, y, t \rangle\), where taking as base “point” a thick ball neighborhood of a meridian disk of \(\text{nhd}(D_1)\), which contains the 12-edge of \(\partial E_1\) and the four attaching disks of two 1-handles \(V_{01}\) and \(V_{23}\), \(x\) is represented by a core of \(V_{23}\) going from vertex 2 to vertex 3, \(y\) is represented by a core of \(V_{01}\) going from vertex 0 to vertex 1, and \(t\) is represented by the 12-edge of \(\sigma\) on \(\partial E_2\) oriented from vertex 1 to vertex 2. In \(\pi_1(W)\), \([\partial E_2]\) (with a clockwise orientation) contains the sequence \(y_1\), but \([\partial E_1]\) has no such sequence by the fact that \(\partial E_1\) has only one level 12-edge. Hence \(\partial E_1\) and \(\partial E_2\) are not parallel on \(\partial W\). Clearly, both are non-separating on \(\partial W\). Thus the boundary of \(W \cup \text{nhd}(E_1)\) is a genus two surface, but \(\partial X\) is either a torus or the disjoint union of two tori, according to whether \(\partial E_2\) is non-separating on the genus two surface or not. In the former, one component of \(\partial D_1\) is essential on the torus. If not, \(\hat{F}_j\) is compressible, and hence \(M(\gamma_j)\) contains a projective plane, which contradicts the irreducibility of \(M(\gamma_j)\). Therefore the frontier of \(X\) is an annulus. In the latter, the frontier of \(X\) is the disjoint union of an annulus and a torus. Thus in either case, we have an annulus component among the frontier of \(X\). In particular, \(\partial D_1\) divides it into two annuli, one of which is parallel to \(D_1\). The rest of the proof is exactly the same as previous, and we would have a new Möbius band having fewer intersection with \(V_{\gamma_j}\). \(\square\)

4. \(x\)-FACES AND ORIENTABLE SURFACES

In this section we assume that \(G_j\) is of type \(S\) or \(A\). By Lemma 2.3 we may assume that \(G_i\) contains only 12-Scharlemann cycles, if they exist.

An \(x\)-face in \(G_i\) is defined as previous. A cycle in \(G_i\) is a two-cornered cycle, slightly different from previous, if it is the boundary of a disk face containing only 01-corners, 23-corners and positive edges, and additionally it contains at least one edge of a 12-Scharlemann cycle. A two-cornered cycle must contain both types of corners and a 03-edge. A cluster \(C\) is a connected subgraph of \(G_i\) satisfying that

(i) \(C\) consists of 12-Scharlemann cycles and two-cornered cycles,
(ii) every 12-edge of \(C\) belongs to both a Scharlemann cycle and a two-cornered cycle, and
(iii) \(C\) contains no cut vertex.

See Figure 4.

**Proposition 4.1.** Suppose that \(n_j \geq 3\). An \(x\)-face, \(x \neq 1, 2\), in \(G_i\) contains a cluster \(C\).
Proof. Let $\Gamma_D$ be the subgraph of $G_i$ in an $x$-face $D$. As in the previous, we may assume that $\partial D$ is a circle. Also assume that the labels appear anticlockwisely.

For all diagonal edges of $D$ which are not of 12-Scharlemann cycles (also these are neither $x$-edges nor level edges), apply the same argument in the proof of Proposition 3.1. Then we get a new $x$-face $E$ and a graph $\Gamma_E$ in $E$ so that all diagonal edges are of 12-Scharlemann cycles and all (and only) boundary edges are $x$-edges. Furthermore the additional edges contain no edges of Scharlemann cycles or two-cornered cycles of $\Gamma_E$. Remark that a level $x$-edge can appear on the boundary of the graph.

**Claim 4.2.** $\Gamma_E$ contains a 12-Scharlemann cycle, so does $\Gamma_D$.

**Proof of Claim 4.2.** It is clear from the proof of Claim 3.2. □

This means that $\tilde{F}_j$ must be separating and $n_j$ is even by Lemma 2.3. The parity rule guarantees that each edge of $\Gamma_E$ connects vertices with one label even and the other label odd, and so there are no level $x$-edges.

Any 12-edge of a Scharlemann cycle does not belong to $\partial \Gamma_E$. Consider the face $E_1$ of $\Gamma_E$ which is adjacent to the 12-edge and whose boundary is not the Scharlemann cycle. It is possible that $E_1$ contains more than one 12-edges of Scharlemann cycles. Again, let $\{a_k, a_k + 1\}$, $k = 1, \ldots, n$, be the consecutive label pairs of the corners between two consecutive 12-edges of Scharlemann cycles when one runs clockwise around $\partial E_1$. Note that $a_1 = 2$ and $a_n = 0$.

Assume for contradiction that $\partial E_1$ is not a two-cornered cycle. Since some $a_k$ then is neither 0 nor 2, there are indices $l$ and $m$ so that $a_k = 0$ or 2 when $1 \leq k < l$ or $k = m$, and $a_k \neq 0, 2$ when $l \leq k < m$.

Consider the edges of the parallelism class containing each $\{a_k - 1, a_k\}$-edge for $l \leq k \leq m$. Since there are neither Scharlemann cycles nor level edges among these edges, one finds that $x \leq a_k < a_k - 1 + 1 \leq x$, and hence $x \leq a_k \leq a_k - 1 < x$. And so $x \leq a_m \leq a_{m-1} \leq \cdots \leq a_l \leq a_{l-1} < x$. This is impossible because $a_{l-1}, a_m = 0$ or 2 and all $a_k$’s are even by the parity rule. Hence $\partial E_1$ is a two-cornered cycle.

Thus we have shown that any face next to a Scharlemann cycle in $\Gamma_E$ is two-cornered. Let $C$ be the union of all the Scharlemann cycles and all the two-cornered cycles adjacent to each 12-edges of the Scharlemann cycles. If necessary, choose a block of $C$. Then it is a desired cluster in $\Gamma_E$ and so in $\Gamma_D$. □

Furthermore, we can see from the argument that $a_1 = \cdots = a_i = 2$ and $a_{i+1} = \cdots = a_n = 0$ for some $i$. 
Let \( R \) be the twice-punctured sphere obtained from \( \hat{F}_j \) by deleting two fat vertices 1 and 2 (if \( G_j \) is of type \( \mathcal{A} \), then use \( \hat{F}_j \) after capping off two boundary circles by disks). The family of all 12-edges of a Scharlemann cycle in the cluster \( C \) separates \( R \) into disks, and one of those disks contains both vertices 0 and 3 of \( G_j \), because of the existence of 03-edges in \( C \). The two 12-edges bounding such a disk are called \textit{good edges} of \( C \). Thus each Scharlemann cycle in \( C \) has exactly two good edges.

Let \( \Lambda \) be the maximal dual graph of \( C \) whose vertices are dual to Scharlemann cycles and two-cornered cycles containing good edges, and edges are dual to good edges of \( C \) as depicted in Figure 4. Thus in \( \Lambda \), a vertex dual to a Scharlemann cycle has valency 2, and a vertex dual to a two-cornered cycle with a good edge in \( G \) contains both vertices 0 and 3 of \( G \), because of the existence of 03-edges in \( C \). Two good edges bounding such a disk are called \textit{good edges} of \( C \). Thus each Scharlemann cycle in \( C \) has exactly two good edges.

From now we apply the argument in [H, Section 6] to get the following two theorems. Recall Proposition 4.3.

**Theorem 4.4.** If \( G_j \) is of type \( \mathcal{S} \), then \( G_i \) cannot contain an \( x \)-face for a non-sl-label \( x \) of \( G_i \).

**Proof.** Suppose that \( G_i \) contains such an \( x \)-face. Note that \( n_j \geq 3 \) by Lemma 2.2. We continue the preceding argument.

Let \( E \) be the disk bounded by \( \sigma_j \). Then \( L = \text{nhd}( (\hat{F}_j - \text{Int} \ D_g) \cup V_{12} \cup E) \) is a punctured lens space in \( M(\gamma_j) \). Since \( \partial L \) is a reducing sphere in \( M(\gamma_j) \), \( \partial L \cap V_{\gamma_j} = 2(\hat{F}_j - \text{Int} \ D_g) \cap V_{\gamma_j} \geq n_j \) by the minimality of \( n_j \). Thus \( \text{Int} \ D_g \cap V_{\gamma_j} \leq \frac{n_j}{2} - 1 \). Let \( X'_{D} = \text{nhd}(D_g \cup V_{01} \cup V_{23}) \) and \( X'_{F} = \text{nhd}(\hat{F}_j \cup V_{01} \cup V_{23}) \). Then \( X'_{D} \) is a genus two handlebody and \( X'_{F} \) is a once-punctured genus two handlebody. The genus two torus component of \( \partial X'_{F} \) is referred to as the outer boundary of \( X'_{F} \). Let \( E_i \) be the face bounded by the two-cornered cycle \( \sigma_i \) for \( i = 1, 2 \). The point is that all edges of \( \sigma_i \) are contained in \( D_g \). Let \( X_{D} = X'_{D} \cup \text{nhd}(E_1 \cup E_2) \) and \( X_{F} = X'_{F} \cup \text{nhd}(E_1 \cup E_2) \) as in Figure 6. Since \( \partial E_1 \) and \( \partial E_2 \) are non-separating on \( \partial X'_{D} \) and \( \partial X'_{F} \), both \( \partial X_{D} \) and the outer components of \( \partial X_{F} \) are either a 2-sphere or the disjoint union of a 2-sphere and a torus, simultaneously. Note that the latter case occurs only when \( \partial E_1 \) and \( \partial E_2 \) are parallel on the boundary of \( X'_{D} \) or \( X'_{F} \).

First, assume that \( \partial X_{D} \) is a 2-sphere \( S_{D} \), and the outer component of \( \partial X_{F} \) is a 2-sphere \( S_{F} \). If \( X_{D} \) is not a 3-ball, then \( S_{D} \) is a reducing sphere. But \( |S_{D} \cap V_{\gamma_j}| = 2|\text{Int} \ D_g \cap V_{\gamma_j}| \leq n_j - 2 \), contradicting the minimality of \( n_j \). Thus it should be a 3-ball, and so \( X_{F} \) is homeomorphic to \( S^2 \times I \). Thus \( S_{F} \) is isotopic to \( \hat{F}_j \), contradicting the minimality of \( n_j \) again.

Next, assume that \( \partial X_{D} \) is the disjoint union of a 2-sphere and a torus, \( S_{D} \cup T_{D} \) and the outer components of \( \partial X_{F} \) are also the disjoint union of a 2-sphere and a torus, \( S_{F} \cup T_{F} \). Recall that this case occurs whenever \( \partial E_1 \) and \( \partial E_2 \) cobound an annulus in \( \partial X'_{D} \) and \( \partial X'_{F} \). This is possible only
when \( \sigma_2 \) corresponds to an end vertex of \( \Lambda_g \) with the same number of 01-corners and 23-corners as those of \( \sigma_1 \). Let \( D' \) be the intersection of \( \partial X_D \) and the inner sphere component of \( \partial X_F \). By the choice of the seemingly pair, this annulus in \( \partial X'_D \) contains \( D' \), so does \( S_D \). Similarly \( S_F \) contains a pushoff of \( R - D_g \).

If \( S_D \) is non-separating in \( M(\gamma_j) \), then it is a reducing sphere with \( |S_D \cap V_{\gamma_j}| \leq n_j - 2 \), contradicting. Thus \( S_D \) is separating in \( M(\gamma_j) \). Let \( X'_D \) be the manifold bounded by \( S_D \) containing \( T_D \) in \( M(\gamma_j) \). If \( X'_D \) is not a 3-ball, then \( S_D \) is a reducing sphere with less intersection with \( V_{\gamma_j} \). Thus it should be a 3-ball, and so \( S_F \) is isotopic to \( \tilde{F}_j \), contradicting again. This completes the proof.

**Theorem 4.5.** If \( G_j \) is of type \( A \) with \( n_j \geq 3 \), then \( G_i \) cannot contain an \( x \)-face for a non-sl-label \( x \) of \( G_i \).

**Proof.** Suppose that \( G_i \) contains such an \( x \)-face. Again we continue the argument stated before Proposition 4.3. The proof is divided into three cases. Recall that \( R \) is a twice-punctured sphere obtained from \( \tilde{F}_j \) by deleting two vertices 1 and 2 and then capping off \( \partial \tilde{F}_j \) by two disks, and that \( D_g \) lies on \( R \).

First, suppose that \( D_g \) contains \( \partial \tilde{F}_j \). Then the edges of the Scharlemann cycle \( \sigma_g \) lie in a disk in \( \tilde{F}_j \), contradicting Lemma 2.8.

Second, suppose that \( D_g \) does not contain any component of \( \partial \tilde{F}_j \). Then \( D_g \) is contained in \( \tilde{F}_j \). The proof is similar to that of the preceding theorem. Recall that \( E_i \) is the face bounded by the two-cornered cycle \( \sigma_i \) for \( i = 1, 2 \). Let \( X_D = \text{nhd}(D_g \cup V_{01} \cup V_{23} \cup E_1 \cup E_2) \) and \( X_F = \text{nhd}(\tilde{F}_j \cup V_{01} \cup V_{23} \cup E_1 \cup E_2) \). Then \( \partial X_D \) is either a 2-sphere or the disjoint union of a 2-sphere and a torus. Also, the frontier of \( X_F \) has a copy of \( \tilde{F}_j \), and the remaining part is referred to as the outer components. It can be seen that the outer components consist of either a single annulus or the disjoint union of an annulus and a torus by the choice of the seemingly pair. (\( \partial X_D \) is disconnected if and only if the outer components of the frontier of \( X_F \) are disconnected.) In either case, we have an annular component \( A_F \). Then \( A_F \) is obtained from \( \tilde{F}_j \) by replacing \( D_g \) with a proper disk which is a part of the 2-sphere component of \( \partial X_D \). Since \( M(\gamma_j) \) is irreducible by the assumption in Section 11 the 2-sphere component of \( \partial X_D \) bounds a 3-ball. Hence \( A_F \) is isotopic to \( \tilde{F}_j \). This contradicts the minimality of \( n_j \).

Finally suppose that \( D_g \) contains exactly one component of \( \partial \tilde{F}_j \). If there is a disk in \( D_g \) which contains all edges of \( \sigma_1 \) and \( \sigma_2 \) and which does not contain the component of \( \partial \tilde{F}_j \), then the case reduces to the previous one. Otherwise, let \( A_1 \) be the annulus \( D_g \cap \tilde{F}_j \). Consider \( Y = \text{nhd}(\tilde{F}_j - \text{Int} A_1) \cup V_{12} \cup E) \), where \( E \) is the face bounded by \( \sigma_g \). Then the frontier \( Q \) of \( Y \) in \( M(\gamma_j) \) is an essential annulus by Claim on page 439 of [10]. Thus \( |Q \cap V_{\gamma_j}| = 2| (\tilde{F}_j - \text{Int} A_1) \cap V_{\gamma_j}| - 2 \geq n_j \).
by the minimality of \( n_j \). Hence \(|\text{Int} \ A_1 \cap V_{\gamma_j}| \leq \frac{n_j}{2} - 1\). Let \( X_A = \text{nhd}(A_1 \cup V_{01} \cup V_{23} \cup E_1 \cup E_2) \) and \( X_F = \text{nhd}(\tilde{F}_j \cup V_{01} \cup V_{23} \cup E_1 \cup E_2) \) again. Then the frontier of \( X_A \) contains an annulus component \( A_\lambda \) by the same argument as in the proof of Theorem 3.4. (By the choice of the seemingly pair, \( \partial E_1 \) and \( \partial E_2 \) are not parallel on the boundary of \( \text{nhd}(A_1 \cup V_{01} \cup V_{23}) \) and \( \text{nhd}(\tilde{F}_j \cup V_{01} \cup V_{23}) \).) Also the outer components of the frontier of \( X_F \) contain an annulus \( A_F \). If \( A_\lambda \) is essential in \( M(\gamma_j) \), then \(|A_\lambda \cap V_{\gamma_j}| \leq 2|\text{Int} \ A_1 \cap V_{\gamma_j}| \leq n_j - 2\), contradicting the minimality of \( n_j \). Remark that \( A_\lambda \) is incompressible, because the central curve of \( A_\lambda \) is isotopic to the central curve of \( \tilde{F}_j \). Thus \( A_\lambda \) should be boundary parallel. That is, there is a solid torus \( U \) such that \( A_\lambda \) is a longitudinal annulus on \( \partial U \) and \( \partial U - A_\lambda \subset \partial M(\gamma_j) \). Then either \( \tilde{F}_j \) is boundary parallel or one can isotope \( \tilde{F}_j \) through \( U \) to \( A_F \), which intersects \( V_{\gamma_j} \) fewer times than \( \tilde{F}_j \). \( \blacksquare \)

Remark that an extended Scharlemann cycle is one of the simplest form of \( x \)-faces. Thus two theorems above guarantee that if \( G_j \) is of type \( S \) or \( A \), then \( G_j \) cannot contain an extended Scharlemann cycle. We may emphasize that only an extended \( S \) cycle was used in the literatures such as \([GW, W1]\) etc.

5. Extremal block

The reduced graph \( G^+_1 \) of \( G_1 \) is defined to be the graph obtained from \( G_1 \) by amalgamating each family of parallel edges into a single edge. Let \( G^+_1 \) denote the subgraph of \( G_1 \) consisting of all vertices and positive edges of \( G_1 \). When \( G_j \) is of type \( S \), it is obvious that each component of \( G^+_1 \) has a disk support, that is, there is a disk in \( F_1 \) which contains the component in its interior. Even if \( G_j \) is of type \( P \), the same is true, because any orientation-preserving loop in a projective plane is contractible.

**Proposition 5.1.** If \( G_1 \) is of type \( S \) or \( P \) and \( G_2 \) is any of the six types, then each non-\( SL \)-vertex of \( G_1 \) has at least \((\Delta(\gamma_1, \gamma_2) - 1)n_2 + \chi(\tilde{F}_2)\) positive edge endpoints.

**Proof.** Assume that there is a non-\( SL \)-vertex \( x \) of \( G_1 \) which has more than \( n_2 - \chi(\tilde{F}_2) \) negative edges. Then \( G_2 \) contains more than \( n_2 - \chi(\tilde{F}_2) \) positive \( x \)-edges by the parity rule. Thus the subgraph \( \Gamma_x \) of \( G_2 \) consisting of all vertices and positive \( x \)-edges of \( G_2 \) has \( n_2 \) vertices and more than \( n_2 - \chi(\tilde{F}_2) \) edges. Then an Euler characteristic calculation shows that \( \Gamma_x \) contains a disk face, which is an \( x \)-face. This contradicts Theorem 3.3 or 4.1. \( \blacksquare \)

Suppose that \( G_1 \) is of type \( S \) or \( P \), and that \( \Delta(\gamma_1, \gamma_2) \geq 2 \). By Lemmas 2.2 and 2.4 \( G_1 \) has a non-\( SL \)-vertex. Note that each of these vertices has therefore valency at least two in \( G^+_1 \) by Lemmas 2.5 and 2.6. Now take an innermost component \( D_0 \) of \( G^+_1 \) with a disk support \( D_0 \), which means that \( D_0 \cap G^+_1 = \Lambda_0 \). The subgraph \( \Lambda_0 \) has therefore at most one \( SL \)-vertex.

Suppose that \( \Lambda_0 \) is a single vertex. Then only negative edges are incident there. By Proposition 5.4 it is an \( SL \)-vertex. If \( G_1 \) is of type \( S \), the edges of a Scharlemann cycle of \( G_2 \) are incident there. Then one of the disks bounded by these edges contains non-\( SL \)-vertices, since \( n_1 \geq 3 \) by Lemma 2.2. Hence we can choose another innermost component of \( G^+_1 \) with more than one vertex and at most one \( SL \)-vertex. If \( G_1 \) is of type \( P \), a negative loop is incident there. Thus we can also choose another innermost component of \( G^+_1 \) which contains only non-\( SL \)-vertices.

We may therefore assume that \( \Lambda_0 \) has more than one vertex. Then \( \Lambda_0 \) has no cut vertex or at least two blocks with at most one cut vertex. Thus we can choose an innermost component \( \Lambda \) with a disk support \( D \) after splitting \( \Lambda_0 \) at all cut vertices, such that \( \Lambda \) has more than one vertex with at most one cut vertex. In such a case that \( \Lambda \) contains a cut vertex and a distinct \( SL \)-vertex, we can choose another innermost one containing no \( SL \)-vertex.

Such a subgraph \( \Lambda \) of \( G^+_1 \) is called an **extremal block** with a disk support \( D \). A vertex of \( \Lambda \) is called a **ghost vertex** if it is either a cut vertex or an \( SL \)-vertex. We emphasize that \( \Lambda \) has more than one vertex and at most one ghost vertex \( y_0 \). A vertex of \( \Lambda \) is called a **boundary vertex** if there is an arc connecting it to \( \partial D \) whose interior is disjoint from \( \Lambda \), and an **inner vertex** otherwise. Then the preceding argument with Proposition 5.1 proves the following theorem which plays key role in this paper:
Theorem 5.2. Suppose that $G_1$ is of type $S$ or $P$.

(1) If $G_2$ is of type $S$, $P$, $A$ or $B$ with the assumption that $\Delta(\gamma_1, \gamma_2) \geq 2$, then $G_1$ contains an extremal block $\Lambda$ with a disk support $D$ so that each boundary vertex, except $y_0$, has at least $n_2 + \chi(F_2)$ consecutive edge endpoints of $\Lambda$.

(2) If $G_2$ is of type $T$ or $K$ with the assumption that $\Delta(\gamma_1, \gamma_2) \geq 3$ and $n_2 \geq 3$ when $G_2$ is of type $T$, and $n_2 \geq 2$ when $G_2$ is of type $K$, then $G_1$ contains an extremal block $\Lambda$ with a disk support $D$ so that each boundary vertex, except $y_0$, has at least $2n_2$ consecutive edge endpoints of $\Lambda$.

6. $(S, S)$ Case

Proof of Theorem 5.2 Assume for contradiction that $\Delta(\gamma_1, \gamma_2) \geq 2$. Theorem 5.2(1) says that $G_1^+$ contains such an extremal block $\Lambda$ with a disk support that each boundary vertex, except $y_0$, has more than $n_2$ consecutive edge endpoints, so different all $n_2$ labels. We can choose a non-sl-label $x$ (at $y_0$ if it exists), unless $G_2$ is of type $S$ and there is $y_0$ with only two edge endpoints which have the sl-labels. Since $\Lambda$ has at least the same number of $x$-edges as that of vertices, it contains an $x$-face, contradicting Theorem 3.4 or 4.4.

For the exceptional case, consider $\Lambda - y_0$. Since all labels still appear on each vertex of $\Lambda - y_0$, the same argument above leads to a contradiction. \hfill \Box

7. $(S, A)$ Case

Proof of Theorem 5.2 Assume that $\Delta = \Delta(\gamma_1, \gamma_2) \geq 2$. Recall that $M(\gamma_2)$ is irreducible and boundary irreducible.

First, assume that $n_2 \geq 3$ when $G_2$ is of type $A$ (or $n_2 \geq 2$ when of type $B$). Theorem 5.2(1) says that $G_1^+$ contains an extremal block $\Lambda$ with a disk support such that each boundary vertex, except $y_0$, has all different $n_2$ labels. When $G_2$ is of type $B$, choose a non-sl-label $x$ at $y_0$, if it exists. Otherwise, $x$ is any non-sl-label. Then $\Lambda$ has at least the same number of $x$-edges as that of vertices, and therefore $\Lambda$ has an $x$-face, contradicting Theorem 3.4. So $G_2$ is of type $A$.

Claim 7.1. $G_1$ contains a Scharlemann cycle.

Proof of Claim 7.1 Assume not. Choose any label $x$ (at $y_0$ if it exists). Then there is an $x$-edge at any vertex of $\Lambda$. Hence $\Lambda$ contains a great $x$-cycle, and so a Scharlemann cycle [CGLS, Lemma 2.6.2].

Thus we may assume that $G_1$ contains only 12-Scharlemann cycles. Equivalently, sl-labels of $G_1$ are 1 and 2. By Lemma 3.3, $F_2$ is separating and $n_2$ is even.

Claim 7.2. (1) $\Lambda$ has no interior vertices;

(2) $\Lambda$ has a ghost vertex $y_0$ such that only two edges are incident to $y_0$ in $\Lambda$ with sl-labels 1 and 2; and

(3) $n_2 = 4$, and the two edges incident to $y_0$ are indeed a 14-edge and a 23-edge.

Proof of Claim 7.2. If $\Lambda$ has an interior vertex or has no ghost vertex, then let $x$ be any non-sl-label. Then $\Lambda$ has at least the same number of $x$-edges as that of vertices, so contains an $x$-face, contradicting Theorem 2.5. Even if there is a ghost vertex $y_0$, we can choose a non-sl-label $x$ at $y_0$, unless (2) holds. Finally, assume $n_2 \geq 6$. Then we can choose a non-sl-label $x$ such that $y_0$ is not incident to an $x$-edge. Thus $\Lambda - y_0$ contains an $x$-face as above again. Even if $n_2 = 4$, there is still such a label $x$, unless the second conclusion of (3) holds. \hfill \Box

In fact, $\Lambda - y_0$ contains a 12-Scharlemann cycle $\sigma$, because there is a 1-edge at any vertex of $\Lambda$ by (3), and hence $\Lambda - y_0$ contains a great 1-cycle. By Lemma 2.3, the edges of $\sigma$ cuts $F_2$ into two annuli $A_1, A_2$ and some disks.

Claim 7.3. $\Lambda$ contains a 34-edge.
Proof of Claim 7.3. Assume not. Then \( \Lambda \) contains only 12, 14 and 23-edges by the parity rule. Let \( y_0, y_1, \ldots, y_k \) be the vertices of \( \Lambda \), numbered consecutively along \( \partial \Lambda \), where the 14-edge \( e_0 \) at \( y_0 \) is incident to \( y_1 \). (Since \( y_i \) (\( i \neq 0 \)) is incident to at least four edges in \( \Lambda \), we see \( k > 1 \).) Let \( e_1 \) be the 3-edge at \( y_1 \), just before \( e_0 \). (This means that the two endpoints of \( e_0 \) and \( e_1 \) are successive around \( y_1 \).) If \( e_1 \) goes to \( y_k \), then \( e_1 \) is a 34-edge. So assume \( e_1 \) goes to \( y_i \) for \( 1 < i < k \). Thus \( e_1 \) has the label 2 at \( y_i \). Split \( \Lambda \) along \( e_1 \), and let \( \Lambda' \) be the part containing \( y_1, y_2, \ldots, y_i \). Then \( \Lambda' \) has a 3-face, contradicting Theorem 4.5. \( \square \)

Hence the vertices 3 and 4 lie in the same component, and we can see that they lie in \( A_1 \) or \( A_2 \) indeed. Otherwise, the vertex 3 or 4 is incident to more than \( n_1 \) negative edges. Thus, \( G_1 \) contains a 3-face or 4-face, a contradiction. We may assume that both are in \( A_2 \). Let \( \Lambda_1' = \text{cl}(\hat{F}_2 - A_2) \), and \( Y = \text{nhd}(\Lambda_1' \cup V_{12} \cup E) \), where \( E \) is the face bounded by \( \sigma \). Then the frontier of \( Y \) gives an essential annulus (see [W3, Claim on page 430]), which meets \( V_{12} \) in two disks. This contradicts the minimality of \( n_2 \).

If \( n_2 = 1 \) when \( G_2 \) is of type \( A \), \( G_2 \) has only positive edges, and these edges are all parallel. This contradicts Lemma 2.5(1-2).

Finally assume \( n_2 = 2 \) when \( G_2 \) is of type \( A \) (or \( n_2 = 1 \) when of type \( B \)). Assume for contradiction that \( \Delta \geq 3 \). Consider the reduced graph \( \overline{G}_2 \). An Euler characteristic calculation shows that each vertex of \( \overline{G}_2 \) has valency at most 4. By Lemmas 2.5(1-2) and 2.6(1) the valency must be 4, so the graph looks like that in Figure 6 (\( (a) \) for type \( A \) and \( (b) \) for type \( B \)).

Claim 7.4. In Figure 6(a), the two vertices have opposite signs.

Proof of Claim 7.4. Assume not. Then \( G_2 \) has only positive edges. For a non-sl-label \( x \) of \( G_2 \), let \( \Gamma \) be the subgraph of \( G_2 \) consisting of all vertices and all \( x \)-edges of \( G_2 \). Then an Euler characteristic calculation shows that \( \Gamma \) has a disk face, which is an \( x \)-face. This contradicts Theorem 3.3 or 4.4. \( \square \)

Thus \( a \) and \( d \) are the families of positive edges and \( b \) and \( c \) are the families of negative edges. When \( G_1 \) is of type \( P \), this is impossible by Lemmas 2.5 and 2.6. When \( G_1 \) is of type \( S \), \( G_2 \) contains an \( S \)-cycle by Lemma 2.5, so \( M(\gamma_1) \) contains a projective plane by Lemma 2.4(3), a contradiction to the previous result. \( \square \)

Figure 6. Annulus and Möbius band

We now give an example realizing the case \( \Delta = 2 \) in Theorem 1.2. Theorem 2.6 of [EW] shows that there is a hyperbolic manifold \( M \) such that \( M(\gamma_1) = (S^1 \times D^2) \natural L(2, 1) \) and \( M(\gamma_2) \) is the union of \( C(2, 1) \) and \( Q(2p, -2p) \), for any integer \( p \geq 2 \), along a torus, with \( \Delta(\gamma_1, \gamma_2) = 2 \). We denote by \( C(r, s) \) the cable space of type \( (r, s) \), and by \( Q(r, s) \) the Seifert fibered manifold with orbifold a disk with two cone points of index \( r \) and \( s \). Note that \( C(2, 1) \) contains a Möbius band and essential annulus with the boundaries on the outside torus, hitting the attached solid torus once and twice, respectively.
8. \((\mathcal{S}, \mathcal{T})\) case

Proof of Theorem \cite{15}. Assume that \(\Delta = \Delta(\gamma_1, \gamma_2) \geq 3\). Recall that \(M(\gamma_2)\) is irreducible and boundary irreducible.

First, assume that \(n_2 \geq 3\). Theorem \cite{22}(2) says that \(G_1^n\) contains an extremal block \(\Lambda\) with a disk support \(D\) so that each boundary vertex, except \(y_0\), has at least \(2n_2\) consecutive edge endpoints of \(\Lambda\). If \(G_2\) is of type \(K\), then we can choose a non-sl-label \(x\) of \(G_1\) by Lemma \cite{14}. Assume that \(G_2\) is of type \(T\).

Claim 8.1. Either \(G_1\) has a label \(x\), which is not a label of \(S\)-cycles in \(G_1\), or \(M(\gamma_2)\) contains a Klein bottle which meets \(V_{\gamma_2}\) at least two times.

Proof of Claim 8.1. If \(n_2 = 3\), then \(G_1\) does not have a Scharlemann cycle by Lemma \cite{23}. Hence any label is a desired one. Assume \(n_2 \geq 4\). If \(G_1\) has four labels of \(S\)-cycles, then there are two \(S\)-cycles whose label pairs are disjoint. Then \(M(\gamma_2)\) contains a Klein bottle \(R\) by the argument of the proof of \cite{22}(Lemma 3.10). If \(R\) meets \(V_{\gamma_2}\) in a single meridian disk, then take a double covering torus \(\bar{R}\) of \(R\) in \(M(\gamma_2)\). By the assumption \(n_2 \geq 4\), \(\bar{R}\) is compressible. Then \(M(\gamma_2)\) is a Seifert fibered manifold, called a prism manifold, which has finite fundamental group. This contradicts the fact that \(M(\gamma_2)\) is toroidal. Hence \(\bar{R}\) meets \(V_{\gamma_2}\) at least two times.

In the second conclusion of Claim 8.1 if the Klein bottle meets \(V_{\gamma_2}\) just two times, then we jump to the case of type \(K\) where \(n_2 = 2\) below. Thus we have a label \(x\) of \(G_1\) which is not an sl-label when \(G_2\) is of type \(K\), or which is not a label of \(S\)-cycles otherwise.

Let \(\Lambda^x\) be the subgraph of \(\Lambda\) consisting of all vertices and \(x\)-edges. Then each boundary vertex of \(\Lambda^x\), except \(y_0\), has at least two edges attached with label \(x\), which cannot be parallel by Lemma \cite{22}(4). Note that \(\Lambda^x\) may not be connected. Then, apply the argument in Section 5 to the present situation; choose an extremal block \(\Lambda'\) of \(\Lambda^x\) with a disk support \(D'\) in \(D\), which we can define in a similar way.

Let \(v, e\) and \(f\) be the numbers of vertices, edges, and disk faces of \(\Lambda'\), respectively. Also let \(v_i, v_0\) and \(v_g\) be the numbers of interior vertices, boundary vertices and ghost vertices. Hence \(v = v_i + v_0 + v_g = 0\) or 1.

Suppose that \(\Lambda'\) has a bigon. By Lemma \cite{23} it contains either a generalized \(S\)-cycle or an extended \(S\)-cycle. (Recall that \(x\) itself is not a label of \(S\)-cycle.) But this is impossible by Lemma \cite{22}. Thus each face of \(\Lambda'\) is a disk with at least 3 sides. Hence we have \(3f + v_0 \leq 2e\). Since \(\Lambda'\) has only disk faces, combined with \(v - e + f = \chi(D') = 1\), we get \(g \leq 3v_g + 2v_0 - 3\). On the other hand, we have \(2(v_g - v_0) + \Delta v_i \leq e\), because each boundary vertex of \(\Lambda'\), except \(y_0\), has at least two edges attached with label \(x\), and \(x\) is not a label of level edges. These two inequalities give us \(3 \leq 2v_0\), a contradiction.

Assume that \(n_2 = 1\) when \(G_2\) is of type \(T\). Then \(G_2\) has only positive edges. Thus it contains an \(x\)-face for a non-sl-label \(x\) of \(G_2\), contradicting Theorems \cite{16} and \cite{14}.

Assume that \(n_2 = 2\) when \(G_2\) is of type \(K\). This case is done by Theorem \cite{22}(2) and the argument of \cite{LOI}(Section 6), which can be carried over without change.

Finally assume that \(\Delta \geq 4\). An Euler characteristic calculation on \(G_2\) shows that each vertex has valency at most 6. Then the graph looks like a subgraph of the graph shown in Figure 7 ((a) for type \(T\) and (b) for type \(K\)) \cite{11}(Lemma 5.2). Here, \(p_i \geq 0\) denotes the number of edges in each parallelism class.

Claim 8.2. When \(G_2\) is of type \(T\), all non-loop edges of \(G_2\) are negative.

Proof of Claim 8.2. This follows from the same argument as the proof of Claim 8.1. □

Thus \(p_1 \leq \frac{\Delta}{2} + 1\), and \(p_i \leq n_1 - 1\) for \(i = 2, 3, 4, 5\) by Lemmas \cite{24} and \cite{22}. Since \(\Delta n_1 \leq (n_1 + 2) + 4(n_1 - 1) = 5n_1 - 2\), \(\Delta = 4\) and all \(p_i\)'s are non-zero. Without loss of generality, we can assume that \(p_1 + p_2 + p_3 \geq 2n_1\). Since \(2n_1 < p_1 + p_2 + p_3 + 1 \leq \left(\frac{\Delta}{2} + 1\right) + 2(n_1 - 1) + 1 < 3n_1\), we can write that \(p_1 + p_2 + p_3 + 1 = 2n_1 + r\), where \(0 < r < n_1\). Then \(p_1 = 2n_1 + r - (p_2 + p_3 + 1) \geq 2n_1 + r - 2(n_1 - 1) - 1 = r + 1\). Hence \(1 \leq r \leq p_1 - 1\). Thus the loop family corresponding to
have two (non-level) edges with the same label \( r \), and so the family contains an \( S \)-cycle or a generalized \( S \)-cycle. This is impossible when \( G_1 \) is of type \( \mathcal{P} \). When \( G_1 \) is of type \( \mathcal{S} \), \( G_2 \) contains an \( S \)-cycle, so \( M(\gamma_1) \) must be of type \( \mathcal{P} \), a contradiction again. \( \square \)

![Figure 7. Torus and Klein bottle](image)

Finally we give an example realizing the case \( \Delta = 3 \) in Theorem 1.3. Let \( M \) be the manifold obtained from the Whitehead link exterior by Dehn filling one component with slope 6. Then \( M \) is hyperbolic [BZ1, p.286] and \( M(1) = L(2,1)\sharp L(3,1) \). Since \( M \) contains a once-punctured Klein bottle whose boundary has slope 4, \( M(4) \) contains a Klein bottle hitting the attached solid torus \( V \) once. Also, the boundary torus of a neighborhood of the Klein bottle in \( M(4) \) is known to be incompressible. Clearly, this torus meets \( V \) twice. (If \( M(4) \) contains a torus meeting \( V \) once, then such a torus is non-separating in \( M(4) \). This is impossible, because \( H_1(M(4)) \) is finite.)

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School of Mathematics, Korea Institute for Advanced Study, 207-43 Cheongryangri-dong, Dongdaemun-gu, Seoul 130-012, Korea

E-mail address: slee@kias.re.kr

Department of Mathematics, Korea University, 1, Anam-dong, Sungbuk-ku, Seoul 136-701, Korea

E-mail address: soh@math.korea.ac.kr

Department of Mathematics and Mathematics Education, Hiroshima University, Kagamiyama 1-1-1, Higashi-Hiroshima 739-8524, Japan

E-mail address: teragai@hiroshima-u.ac.jp