Bose–Einstein Condensation and strong–correlation behavior of phonons in ion traps

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We show that the dynamics of phonons in a set of trapped ions interacting with lasers is described by a Bose–Hubbard model whose parameters can be externally adjusted. We investigate the possibility of observing several quantum many–body phenomena, including (quasi) Bose–Einstein condensation as well as a superfluid–Mott insulator quantum phase transition.

Systems of ultracold bosons present a rich variety of fascinating phenomena, like Bose–Einstein Condensation (BEC) [1], or the superfluid–Mott insulator (SI) quantum phase transition [2]. At present, there exist very few physical systems in which these effects can be observed. Atomic gases at ultra low temperature constitute a unique system in this context, since their physical parameters can be adjusted using external fields, something which has enabled the observation of BEC [3] or the SI transitions [4], for example. In this paper we show that phonons in a crystal of trapped ions interacting with lasers provides us with another system where all these phenomena can be observed in a very clean way. As in the case of neutral atoms, the physical parameters describing the phonon dynamics can be adjusted using lasers. Furthermore, individual addressing yields novel possibilities for investigating novel physical situations.

In the set up we consider here, phonons are associated to the motion of the trapped ions. Coulomb interaction induces the transmission of phonons from one ion to another, whereas anharmonicities in the trapping potentials give rise to an effective phonon–phonon interaction. Thus, an ion crystal is analogous to an optical lattice [5], whereby the ions play the role of lattice sites and the phonons that of the atoms. An important feature of ion crystals is that, due to energy conservation, phonons cannot be created or annihilated, i.e. the phonon number is conserved. This is in contrast with usual solid state systems, where phonons are subjected to processes that do not conserve their number thus preventing them from reaching, e.g., BEC. Furthermore, the theoretical development [6, 7] and experimental progress [8, 9, 10] experienced by the field of trapped ion quantum computation during the last years can be exploited in the present context to gain access to physical observables which are not reachable in other systems.

Let us consider a set of $N$ trapped ions confined by external electric potentials and which move around their equilibrium positions. The Hamiltonian describing this situation can be written as $H = K + V_0 + V_{\text{Coul}}$, where $K$ describes the kinetic energy, $V_0$ the trapping potential, and $V_{\text{Coul}}$ the Coulomb interaction between the ions. We will assume that: (i) the motion of the ions along one particular direction, say $\mathbf{x}$, is decoupled from the motion along the other directions; (ii) the trapping potential along $\mathbf{x}$ for each ion is practically harmonic with frequency $\omega$, i.e. it is given by $\frac{1}{2}m\omega^2x_i^2$, where $x_i$ denotes the operator corresponding to the displacement of the $i$–th ion from its equilibrium position; (iii) the displacements around the equilibrium position are much smaller than the distances between ions; (iv) the Coulomb energy is small compared to the potential energy, i.e. $\beta := \mathcal{E}^2/(d_0^2m\omega^2) \ll 1$, where $d_0$ denotes the average separation between ions. The first requirement (i) allows us to ignore the motional state of the ions along the $y$ and $z$ directions. Condition (ii) allows us to associate phonons to each of the ions in the usual way [2]: if the vibrational state of the $i$–th ion is given by the $n$–th (Fock) excitation state of the corresponding harmonic potential we will say that the ion has $n$ phonons and denote the corresponding state by $|n\rangle$. The position operator of the ion can be then written in terms of creation and annihilation operators for the phonons, i.e. $x_i \propto \hat{a}_i^\dagger + \hat{a}_i$. Condition (iii) allows us to express the Coulomb interaction between ions $i$ and $j$ as $(\mathcal{E}^2/d_{ij}^2)x_ix_j$, where $d_{ij}$ is the distance between the ions. It is clear that this term will induce hopping from phonons between the ions since it contains terms of the form $a_i^\dagger a_j$. On the other hand, condition (iv) imposes that the phonon number is conserved, since the terms of the form $a_i a_j$ would decrease the energy by $2\mathcal{E}$, something which cannot be compensated by the Coulomb interaction which effectively switches off this process. Finally, the anharmonicities of the trapping potential will be, in lowest order, described by terms of the form $x_i^3$ or $x_i^4$. Again, only energy–conserving terms will be important and thus only those proportional to $a_i^\dagger a_i$ or $a_i^{12}a_i^2$ will survive. The first one will add some small correction to the trapping frequency, whereas the second term can be associated to an effective phonon–phonon interaction.

Thus, we have shown that the dynamics of the phonons in an ion crystal will contain hopping terms between different ions as well as on–site phonon–phonon interactions, and therefore will be described by a Bose–Hubbard model (BHM). Typically, the trapping anharmonicities will be very small. However, they can be enhanced by using off–resonant lasers. For instance, one may induce repulsive phonon–phonon interactions by placing the ions near the maximum of a standing wave, something which will induce an AC–Stark shift proportional to $\cos(kx_i)^2 \approx 1 – (kx_i)^2 + (1/3)(kx_i)^4$, where $k$
is the wave-vector of the laser. By placing them at the minimum, one obtains an attractive phonon–phonon interaction.

There are several physical set–ups which realize a BHM as explained above. In the following we will concentrate in the simplest one, which consists of ions in a linear trap and which gives rise to a 1D system. Let us emphasize, however, that with ions in microtraps or in Penning traps one can realize higher dimensional situations.

In a linear trap, ions are arranged in a Coulomb chain. Phonons moving along the chain cannot be used in the way we described above since for them \( \beta \gtrsim 1 \). However, transverse phonons corresponding to the radial modes fulfill \( \beta \ll 1 \) and thus are perfectly suited for our purposes. The radial phonon dispersion relation has a quadratic dependence on the position of the ions. Thus, phonon confinement can be estimated by means of Eq. (10).

The final Hamiltonian takes the form of a BHM:

\[
H = H_{x0} + \sum_{i=1}^{N} H_{0,i},
\]

where we include in \( H_{x0} \) the corrections from the standing wave. Note that as long as the number of phonons is conserved, \( \omega_x \) in \( H_{x0} \) is a global chemical potential that does not play any role in the description of the system.

We discuss now the properties of the solutions of the non-interacting Hamiltonian, \( H_{x0} \). A quite unexpected result is that the Coulomb interaction induces the confinement of the radial phonons. This is due to the fact that the distance between ions is larger at the sides than at the center of the chain, and is well described by a quadratic dependence on the position of the ions. Thus, the corrections \( \omega_{x,i} \) in Eq. (3) are smaller for the ions placed at the center of the chain, in such a way that the radial phonon field is confined (Fig. 14). The harmonic phonon confinement can be estimated by means of Eq.
Let us consider tines. However we can understand the properties of a superfluid phonon state by adiabatic evolution. Hamil-
ments with linear Paul traps: (Fig. 2). rally confined by an approximate harmonic potential (see inset of Fig. 1). Thus, we get the conclusion that radial phonons in ions in linear Paul traps are natu-
re of the radial collective modes, \( \Omega_{x,q} \), that diagonalize \( H_{x0} \).

\[ \alpha - \gamma \left( \frac{i'}{N} \right)^2, \quad \alpha = 3.4, \quad \gamma = 18, \quad (6) \]

where \( i' = i - N/2 \). One can use Eq. (6) to describe qualitatively the dependence of \( \omega_{x,i} \) with the position. We include only Coulomb interaction between nearest-neighbors in order to get analytical results. The spatial dependent part of the non–interacting boson Hamiltonian is given, in this approximation, by:

\[ H_x/(\beta_x \omega_x) = \sum_{i=1}^{N} \frac{1}{2} \gamma (\frac{i'}{N})^2 a_i^\dagger a_i + \frac{1}{2} (\alpha - \gamma (\frac{i'}{N})^2) (a_i^\dagger a_{i+1} + h.c.). \quad (7) \]

In the limit of many ions and low energies, the continuum limit in this expression describes a one dimensional system of bosons trapped by the frequency \( \omega_x \approx (8/N) \beta_x \omega_x \). The lowest collective modes in the exact spectrum show a linear dispersion that is well described by our estimation for \( \omega_x \) (see inset of Fig. 1). Thus, we get the conclusion that radial phonons in ions in linear Paul traps are naturally confined by an approximate harmonic potential (see Fig. 2).

Our ideas lead to the following two proposals of experiments with linear Paul traps:

(i) Superfluid-Mott insulator transition and creation of a superfluid phonon state by adiabatic evolution. Hamiltonian \( \mathcal{H}_{x0} \) describes a BHM with the peculiarity that hopping terms are positive, with a range that is longer than the usual nearest-neighbor hopping in optical lattices. However we can understand the properties of our system by means of the better known model with nearest-neighbor hopping only \( \mathcal{H}_{x0} \). Let us consider \( t = (1/2) \beta_x \hbar \omega_x \), the characteristic hopping energy. If the total number of phonons \( N_{ph} \) is commensurate with the number of ions \( N \), then, for values \( U \gg t \), the ground state of Hamiltonian \( \mathcal{H}_{x0} \) is a Mott insulator, well described by a product of Fock states of \( N_{ph}/N \) in each ion (note that phonon confinement, \( \hbar \omega_x \) is also of order \( t \), so that condition \( U \gg t \) ensures a uniform phonon density). On the other hand, the ground state for \( U < t \) is a superfluid with all the phonons in the lowest energy level. In Fig. 2 we present the results of an exact numerical diagonalization of the complete phonon Hamiltonian (that is, including also the phonon number non-conserving terms) for the case \( N_{ph}/N = 1 \). The transition from the superfluid to the Mott insulator, with one phonon per site, is evident in the evolution of the phonon density as a function of the interaction \( U \).

The properties of the BHM, allows us to propose an experimental sequence that would lead to the observation of the SI quantum phase transition: (1) The ion chain is cooled to the state with zero radial phonons by laser cooling. (2) Starting with a value \( U \gg t \), the eigenstates of the system are well described by Fock states localized at each ion. The ground state of the phonon system can be created by means of sequences of blue/red-side band transitions, in a method that has been successfully implemented with single trapped ions (see \( \mathcal{H}_{x0} \)). (3) The value of \( U \) is varied adiabatically down to a given value \( U_f \), in such a way that the system remains in the ground state. At a given critical value \( U_f \approx t \), the system undergoes a transition to a phonon superfluid. (4) The measurement of the ground state can be accomplished by the coupling of the transverse phonons to a given internal transition. One could apply, for example, a red sideband pulse with intensity \( g \), for a short time \( t \). Under such conditions, the probability of inducing a transition to the excited internal state is \( \propto \sum_n \sin^2(\sqrt{ng})^2 P(n) \approx \sum_n n g^2 t^2 P(n) \), where \( P(n) \) is the probability of having \( n \) phonons. This method would allow us to measure the mean phonon number. By resolving individually the photoluminescence from each ion, one could observe features of the BEC, or SI transition in the variations of the phonon density along the chain. One could also apply well known methods to determine \( P(n) \), or even the whole quantum tomography of the phonon quantum states \( \mathcal{H}_{x0} \).
(ii) (Quasi) Bose–Einstein Condensation by evaporative laser cooling. We propose an experiment that is akin to the usual BEC of cold atoms in harmonic traps. First, we note that techniques for cooling of trapped ions, like laser cooling \[10, 11\] can only be used to destroy phonons. The existence of the trapping phonon potential in ion traps allows us to propose the combination of laser cooling with the idea of evaporative cooling. A possible experimental sequence would be as follows: (1) Start with a Coulomb chain after usual Doppler cooling, that is, a chain with a given number of phonons per site, and induce a small phonon–phonon interaction \( U \ll t \), so that the system remains in the weak interacting regime. (2) Apply laser cooling at the sides of the Coulomb chain, in such a way that the higher energy phonons on the top of the confinement potential are destroyed (evaporated). (3) The interaction \( U \) induces collisions that thermalize the phonons to a lower temperature. Several cycles of laser cooling / thermalization could be applied until the system is cooled below the critical temperature. Detection of the BEC could be accomplished along the same lines exposed above for the case of the BHM. Note that in the case of Coulomb chains (1D) considered here, (quasi) BEC is possible in finite size systems only.

We have shown that phonons in a system of trapped ions can be manipulated in such a way that they undergo BEC, or a SI transition. The main ingredients of our proposal are: (1) The fact that phonons can have a large energy gap that suppresses processes that do not conserve the number of phonons. (2) Phonon–phonon interactions (anharmonicities) can be induced by placing the ions in a standing–wave. (3) In the particular case of ions in a linear trap, radial phonons would be suitable for this proposal, with the advantage that the Coulomb interaction provides us with an approximately harmonic phonon confinement.

In this work we have exposed only a few applications of this idea, but phonons in trapped ions could be used to study quantum phases with a degree of controllability that is not possible with cold neutral atoms. Individual addressing would allow us to design Hubbard Hamiltonians with local interactions that change at will from site to site. Different directions of the radial modes, or different internal states of the ions could play the role of effective spins \[15\] for the phonons. On the other hand, one could also reach the regime dominated by the repulsive interaction and create, thus, a Tonks-Girardeau gas of phonons \[11\] in a Coulomb chain (this idea was implemented recently with optical lattices \[20\]). In a very promising approach, 2D systems of arrays of microtraps \[11\], or ions in Penning traps \[12\], could be considered, because phonons transverse to the crystal plane satisfy the conditions required by our proposal.

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