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Algebra / Algèbre

On cogrowth function of algebras and its logarithmical gap

Sur la fonction de co-croissance des algèbres et son écart logarithmique

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Abstract. Let $A \cong k\langle X \rangle / I$ be an associative algebra. A finite word over alphabet $X$ is $I$-reducible if its image in $A$ is a $k$-linear combination of length-lexicographically lesser words. An obstruction is a subword-minimal $I$-reducible word. If the number of obstructions is finite then $I$ has a finite Gröbner basis, and the word problem for the algebra is decidable. A cogrowth function is the number of obstructions of length $\leq n$. We show that the cogrowth function of a finitely presented algebra is either bounded or at least logarithmical. We also show that an uniformly recurrent word has at least logarithmical cogrowth.

Résumé. Soit $A \cong k\langle X \rangle / I$ une algèbre associative. Un mot fini sur l’alphabet $X$ est $I$-réductible si son image dans $A$ est une combinaison linéaire $k$ de mots de longueur lexicographiquement moindres. Une obstruction dans un mot minimal $I$-réductible. Si le nombre d’obstructions est fini, alors $I$ a une base finie Gröbner, et le mot problème pour l’algèbre est décidable. Une fonction co-croissance est le nombre d’obstructions de longueur $\leq n$. Nous montrons que la fonction de co-croissance d’une algèbre finement présentée est soit bornée, soit au moins logarithmique. Nous montrons également qu’un mot uniformément récurrent a au moins une co-croissance logarithmique.

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1. Cogrowth of associative algebras

Let $A$ be a finitely generated associative algebra over a field $k$. Then $A \cong k\langle X \rangle / I$, where $k\langle X \rangle$ is a free algebra with generating set $X = \{x_1, \ldots, x_s\}$ and $I$ is a two-sided ideal of relations. Further we assume the generating set is fixed. Let “$<$” be a well-ordering of $X$, $x_1 < \cdots < x_s$. This order can be extended to a linear order on the set $\langle X \rangle$ of monomials of $k\langle X \rangle$, i.e. finite words in alphabet $X$: $u_1 < u_2$ if $|u_1| < |u_2|$ or $|u_1| = |u_2|$ and $u_1 <_{\text{lex}} u_2$. Here $|\cdot|$ denotes the length of a word, i.e. the degree of a monomial, and $<_{\text{lex}}$ is the lexicographical order. For $f \in k\langle X \rangle$ we denote by $\hat{f}$ its leading (with respect to $<$) monomial. An algebra $k\langle X \rangle / I$ is said to be finitely presented if $I$ is a finitely generated ideal.

We call a monomial $w \in \langle X \rangle$ $I$-reducible if $w = f$ for some relation $f \in I$. In the opposite case, we call $w$ $I$-irreducible. Denote the set of all monomials of degree at most $n$ by $\langle X \rangle \leq n$. Let $A_n \subseteq A$ be the image of $\langle X \rangle \leq n$ under the canonical map. The growth $V_A(n)$ is the dimension of the linear span of $A_n$. It is easily shown that $V_A(n)$ is equal to the number of $I$-irreducible monomials in $\langle X \rangle \leq n$.

We call a monomial $w \in \langle X \rangle$ an obstruction for $I$ if $w$ is $I$-reducible, but any proper subword of $w$ is $I$-irreducible. The cogrowth of an algebra $A$ is defined as the function $O_A(n)$, the number of obstructions of length $\leq n$.

The celebrated Bergman gap theorem says that the growth function $V_A(n)$ is either constant, linear of no less than $(n + 1)(n + 2)/2$ [2]. In this section we give a non-trivial bound on the cogrowth function for finitely presented algebras.

**Theorem 1.** Let $A$ be a finitely presented algebra. Then the cogrowth function $O_A(n)$ is either constant or no less than logarithmic: $O_A(n) \geq \log_2(n) - C$. The constant $C$ depends only on the maximal length of a relation.

Recall that a Gröbner basis of an ideal $I$ is a subset $G \subseteq I$ such that for any $f \in I$ there exists $g \in G$ such that the leading monomial of $f$ contains the leading monomial of $g$ as a subword. One of Gröbner bases can be obtained by taking for each obstruction $u$ a relation $f_u \in I$ such that $\hat{f_u} = u$.

If $f$ and $g$ are two elements of $k\langle X \rangle$, $g \in I$ and the word $\hat{g}$ is a subword of $\hat{f}$, then $f$ can be replaced by $f'$ such that $\hat{f'} - f \in I$ and $\hat{f'} < \hat{f}$. This operation is called a reduction.

Let $f$ and $g$ be two elements of $k\langle X \rangle$. If $u_1 u_2 = f$ and $u_2 u_3 = \hat{g}$ for some $u_1, u_2, u_3 \in \langle X \rangle$, then the word $u_1 u_2 u_3$ is called a composition of $f$ and $g$, and the normed element $f u_3 - u_1 g$ is the result of this composition.

**Lemma 2 (Diamond Lemma [3]).** Let two-sided ideal $I$ be generated by a subset $U$ of a free associative algebra $k\langle X \rangle$. Suppose that

(i) there are no $f, g \in U$ such that $\hat{g}$ is a proper subword of $\hat{f}$, and

(ii) for any two elements $f, g \in U$ the result of any their composition can be reduced to 0 after finitely many reductions with elements from $U$.

Then the set $U$ is a Gröbner basis of $I$.

**Example.** Consider the associative algebra $A \cong k\langle x, y \rangle / I$, where $I$ is a two-sided ideal generated by $f = x^2 - yx$. It can be shown that the set $\{xy^i x - y^{i+1} x \mid i \geq 0\}$ is a Gröbner basis of $I$, so $O_A(n) = n - 1$ for $n \geq 2$. A monomial is $I$-irreducible if and only if it contains at most one letter $x$, hence $V_A(n) = (n + 1)(n + 2)/2$.

Theorem 1 directly follows from

**Lemma 3.** Let $A \cong k\langle X \rangle / I$ be a finitely presented algebra and let $N$ be greater than the maximal length of its defining relation. Suppose there are no obstructions of length from the interval $[N, 2N]$. Then $I$ has a finite Gröbner basis.
Proof. Let $S$ be the set of all obstructions in $\langle X \rangle_{\leq N}$. Take for each monomial $w \in S$ a relation $f_w$ such that $f_w = w$. Let us show that this set $\{f_w\mid s \in S\}$ forms a Gröbner basis for $I$. Indeed, $I$ is generated by the set $\{f_w\mid w \in S\}$. The condition (i) of the Diamond Lemma holds automatically because no obstruction can be a proper subword of another obstruction. Let us check the condition (ii).

Let $u, v \in S$ and let $h$ be the result of some composition of $f_u$ and $f_v$. It is clear that the leading monomial of $h$ has length less then $2N$. We start reducing $h$ with elements from $\{f_w\mid w \in S\}$. After finally many steps we obtain either 0 or an element $h'$ such that $h'$ does not contain subwords from $S$. But since there are no obstructions from $[N, 2N]$, the second case is impossible. □

The word problem for a finitely presented algebra, i.e. the question whether a given element $f \in k\langle X \rangle$ lies in $I$, is undecidable in the general case. But if $I$ has a finite Gröbner basis $G$, then $A$ has a decidable word problem. Note also that the problem whether a given element in a finitely presented associative algebra is a zero divisor (or is it nilpotent) is undecidable, even if we are given a finite Gröbner basis [6]. But if the ideal of relations is generated by monomials and has a finite Gröbner basis, the nilpotency problem is algorithmically decidable [2].

2. Cogrowth function for a uniformly recurrent sequence

A monomial algebra is a finitely generated associative algebra whose defining relations are monomials. Let $u$ be a finite word in alphabet $X$ and let $A_u$ be the algebra $k\langle X \rangle/I$, where $I$ is generated by the set of monomials that are not subwords of the periodic sequence $u^\infty$. Such algebras $A_u$ play important role in the study of monomial algebras [2].

Let $W$ be a sequence on alphabet $X$, i.e. a map $X^\mathbb{N}$. A finite word $v$ is an obstruction for $W$ if $v$ is not a subword of $W$ but any proper subword $v'$ of $v$ is a subword of $W$. If $u$ is a finite word, the number of obstructions for $u^\infty$ is always finite. We call this number the colength of the period $u$. We say that the period is defined by the set of obstructions.

In [5], G. R. Chelnokov proved that a sequence of minimal period $n$ cannot be defined by fewer than $\log_2 n + 1$ obstructions. G. R. Chelnokov also gave for infinitely many $n_i$ an example of a binary sequence with minimal period $n_i$ and colength of the period $\log_2 \varphi n_i$, where $\varphi = \frac{\sqrt{5}+1}{2}$. P. A. Lavrov found the precise lower estimation for colength of period.

Theorem 4 (cf. [7]). Let $A = \{a, b\}$ be a binary alphabet. Let $u$ be a word of length $n$ and colength $c$, then $\varphi_c \geq n$, where $\varphi_c$ is the $c$-th Fibonacci number ($\varphi_1 = 1, \varphi_2 = 2, \varphi_3 = 3, \varphi_4 = 5$ etc.).

The case of an arbitrary alphabet was considered in [8] by P. A. Lavrov and independently in [4] by I. I. Bogdanov and G. R. Chelnokov.

3. Cogrowth function for an uniformly recurrent sequence

A sequence of letters $W$ on a finite alphabet is called uniformly recurrent (u.r. for brevity) if for any finite subword $u$ of $W$ there exists a number $C(u, W)$ such that any subword of $W$ having length $C(u, W)$ contains $u$ as a subword. This property can be considered as a generalization of periodicity [9].

For a sequence of letters $W$ denote by $A_W$ the algebra $k\langle X \rangle/I_W$, where $I_W$ is generated by the set of monomials that are not subwords of $W$. A monomial algebra $A$ is called almost simple if each of its proper factor algebras $B = A/I$ is nilpotent. In [2] it was shown that almost simple monomial algebras are algebras of the form $A_W$, where $W$ is an u.r. sequence.

Again, a finite word $u$ is an obstruction for $W$ if it is not a subword of $W$ but any its proper subword is a subword of $W$. The cogrowth function $O_W(n)$ is the number of obstructions with length $\leq n$. 

Alexei Ya. Kanel-Belov, Igor Melnikov and Ivan Mitrofanov
Theorem 5. Let \( W \) be an u.r. non-periodic sequence on a binary alphabet. Then 
\[
\lim_{n \to \infty} O_W(n)/\log_3 n \geq 1.
\]

A factorial language is a set \( \mathcal{U} \) of finite words such that for any \( u \in \mathcal{U} \) all subwords of \( u \) also belong to \( \mathcal{U} \). Denote by \( \mathcal{U}_k \) the words of \( \mathcal{U} \) having length \( k \). A finite word \( u \) is called an obstruction for \( \mathcal{U} \) if \( u \notin \mathcal{U} \), but any proper subword belongs to \( \mathcal{U} \). Denote the factorial language consisting of all subwords of a given sequence \( W \) by \( \mathcal{L}(W) \). To prove Theorem 5 we will assume the contrary and construct an infinite factorial language that is a proper subset of \( \mathcal{L}(W) \).

Let \( \mathcal{U} \) be a factorial language and \( k \) be an integer. The Rauzy graph \( R_k(\mathcal{U}) \) of order \( k \) is the directed graph with vertex set \( \mathcal{U}_k \) and edge set \( \mathcal{U}_{k+1} \). Two vertices \( u_1 \) and \( u_2 \) of \( R_k(\mathcal{U}) \) are connected by an edge \( u_1 \) if and only if \( u_3 \in \mathcal{U} \), \( u_1 \) is a prefix of \( u_3 \), and \( u_2 \) is a suffix of \( u_3 \).

For a sequence \( W \) we denote the graph \( R_k(\mathcal{L}(W)) \) by \( R_k(W) \). Further the word graph will always mean a directed graph, the word path will always mean a directed path in a directed graph. The length \( |p| \) of a path \( p \) is the number of its vertices, i.e. the number of edges plus one. If a path \( p_1 \) starts at the end of a path \( p_2 \), we denote their concatenation by \( p_1 p_2 \). Recall that a directed graph is strongly connected if for every pair of vertices \( \{v_1, v_2\} \) it contains a directed path from \( v_1 \) to \( v_2 \) and a directed path from \( v_2 \) to \( v_1 \). It is clear that any Rauzy graph of an u.r. non-periodic sequence is a strongly connected digraph and is not a cycle.

Given a directed graph \( H \), its directed line graph \( L(H) \) is a directed graph such that each vertex of \( L(H) \) represents an edge of \( H \), and two vertices of \( L(H) \) that represent edges \( e_1 \) and \( e_2 \) of \( H \) are connected by an arrow from \( e_1 \) to \( e_2 \) if and only if the head of \( e_1 \) meets the tail of \( e_2 \). For any \( k > 0 \) there is one-to-one correspondence between paths of length \( k \) in \( L(H) \) and paths of length \( k + 1 \) in \( H \).

Let \( \mathcal{U} \) be a factorial language and let \( m \geq n \). A word \( a_1 \ldots a_m \in \mathcal{U}_m \) corresponds to a path of length \( m - n + 1 \) in \( R_n(\mathcal{U}) \), this path visits vertices \( a_1 \ldots a_n, a_2 \ldots a_{n+1}, \ldots, a_{m-n+1} \ldots a_m \). The graph \( R_n(\mathcal{U}) \) can be considered as a subgraph of \( L^{m-n}(R_n(\mathcal{U})) \). Moreover, the graph \( R_{n+1}(\mathcal{U}) \) is obtained from \( L(R_n(\mathcal{U})) \) by removing edges that correspond to obstructions of length \( n + 1 \).

We call a vertex \( v \) of a directed graph \( H \) a fork if \( v \) has out-degree more than one. Furthermore we assume that all forks have out-degrees exactly 2 (this is the case of a binary alphabet). For a directed graph \( H \) we define its entropy regulator: \( er(H) \) is the minimal integer such that any directed path of length \( er(H) \) in \( H \) contains at least one vertex that is a fork in \( H \).

Proposition 6. Let \( H \) be a strongly connected digraph that is not a cycle. Then \( er(H) < \infty \).

Proof. Assume the contrary. Let \( n \) be the total number of vertices in \( H \). Consider a path of length \( n + 1 \) in \( H \) that does not contain forks. Note that this path visits some vertex \( v \) at least twice. This means that starting from \( v \) it is possible to obtain only vertices of this cycle. Since the graph \( H \) is strongly connected, \( H \) coincides with this cycle.

Lemma 7. Let \( H \) be a strongly connected digraph, \( er(H) = K \). Then \( er(L(H)) = K \).

Proof. The forks of the digraph \( L(H) \) are edges in \( H \) that end at forks. Consider \( K \) vertices forming a path in \( L(H) \). This path corresponds to a path of length \( K + 1 \) in \( H \). Since \( er(H) \leq K \), there exists an edge of this path that ends at a fork.

Lemma 8. Let \( H \) be a strongly connected digraph, \( er(H) = K \), let \( v \) be a fork in \( H \), the edge \( e \) starts at \( v \). Let the digraph \( H^* \) be obtained from \( H \) by removing the edge \( e \). Let \( G \) be a subgraph of \( H^* \) that consists of all vertices and edges reachable from \( v \). Then \( G \) is a strongly connected digraph. Also \( G \) is either a cycle of length at most \( K \), or \( er(G) \leq 2K \).

Proof. First we prove that the digraph \( G \) is strongly connected. Let \( v' \) be an arbitrary vertex of \( G \), then there is a path in \( G \) from \( v \) to \( v' \). Consider a path \( p \) of minimum length from \( v' \) to \( v \) in \( H \). Such a path exists, for otherwise \( H \) is not strongly connected. The path \( p \) does not contain the
edge $e$, for otherwise it could be shortened. This means that $p$ connects $v'$ to $v$ in the digraph $G$. From any vertex of $G$ we can reach the vertex $v$, hence $G$ is strongly connected.

Consider an arbitrary path $p$ of length $2K$ in the digraph $G$, suppose that $p$ does not have forks. Since $er(H) = K$, then in $p$ there are two vertices $v_1$ and $v_2$ which are forks in $H$ and there are no forks in $p$ between $v_1$ and $v_2$. The out-degrees of all vertices except $v$ coincide in $H$ and $G$. If $v_1 \neq v$ or $v_2 \neq v$, then we find a vertex of $p$ that is a fork in $G$. If $v_1 = v_2 = v$, then there is a cycle $C$ in $G$ such that $|C| \leq K$ and $C$ does not contain forks of $G$. Since $G$ is a strongly connected graph, it coincides with this cycle $C$.

**Corollary 9.** Let $W$ be a binary u.r. non-periodic sequence, then for any $n$
\[
er(R_{n-1}(W)) \leq 2^{O_w(n)}.
\]

**Proof.** We prove this by induction on $n$. The base case $n = 0$ is obvious. Let $er(R_{n-1}(W)) = K$ and suppose $W$ has exactly $a$ obstructions of length $n + 1$. These obstructions correspond to paths of length 2 in the graph $R_{n-1}(W)$, i.e. edges of the graph $H := L(R_{n-1}(W))$. From Lemma 7 we have that $er(H) = K$. The graph $R_n(W)$ is obtained from the graph $H$ by removing some edges $e_1, e_2, \ldots, e_a$. Since $W$ is a u.r. sequence, the digraphs $H$ and $H - \{e_1, e_2, \ldots, e_a\}$ are strongly connected. This means that the edges $e_1, \ldots, e_a$ start at different forks of $H$. We also know that $R_n(W)$ is not a cycle. The graph $R_n(W)$ can be obtained by removing edges $e_i$ from $H$ one by one. Applying Lemma 8 $a$ times, we show that $er(R_n(W)) \leq 2^K$, which completes the proof.

**Lemma 10.** Let $H$ be a strongly connected digraph, $er(H) = K$, $k \geq 3K$. Let $u$ be an arbitrary edge of the graph $L^k(H)$. Then the digraph $L^k(H) - u$ contains a strongly connected subgraph $B$ such that $er(B) \leq 3K$.

**Proof.** Consider in $H$ the path $p_u$ of length $k + 2$, corresponding to $u$. Divide first $k$ vertices of $p_u$ into three subpaths of length at least $K$. Since $er(H) = K$, each of these subpaths contains a fork (some of these forks can coincide). Next, we consider three cases.

**Case 1.** Assume that the path $p_u$ visits at least two different forks of $H$. Then $p_u$ contains a subpath of the form $pe$, where $p$ is a path connecting two different forks $v_1$ and $v_2$ (and not containing other forks) and $e$ is an edge starting at $v_2$. It is clear that the length of $p_1$ does not exceed $K + 1$. Lemma 8 implies that there is a strongly connected subgraph $G$ of $H$ such that $G$ contains the vertex $v_2$ but does not contain the edge $e_2$.

If $G$ is not a cycle, then $er(G) \leq 2K$. Hence, the graph $B := L^k(G)$ is a subgraph of $L^k(H)$, and from Lemma 7 we have $er(B) \leq 2K$. It is also clear that the digraph $B$ does not contain the edge $u$.

If $G$ is a cycle, we denote it by $p_1$ and denote its first edge by $e_1$ (we assume that $v_2$ is the first and last vertex of $p_1$). The length of $p_1$ does not exceed $K$. Among the vertices of $p_1$ there are no forks of $H$ besides $v_2$. Therefore, $v_1 \notin p_1$. Call a path $t$ in $H$ good, if $t$ does not contain the subpath $pe$. Let us show that for any good path $s$ in $H$ there are two different paths $s_1$ and $s_2$ starting at the end of $s$ such that $|s_1| = |s_2| = 3K$ and the paths $s_1s_2$ are also good.

It is clear that for any good path we can add an edge such that the new path is also good. There is a path $t_1$, $|t_1| < K$ such that $s_1t_1$ is a good path and ends at some fork $v$. If $v \neq v_2$, then two edges $e_i, e_j$ start at $v$, the paths $st_1e_i$ and $st_2e_j$ are good, and each of them can be prolonged further to a good path of arbitrary length. If $v = v_2$, then the paths $st_1p_1e$ and $st_1p_1e_1$ are good and can be extended.

Consider in $L^k(H)$ a subgraph that consists of all vertices and edges that are good paths in $H$, let $B$ be a strongly connected component of this subgraph. It is clear that $er(B) \leq 3K$ and the digraph $B$ does not contain the edge $u$.

**Case 2.** Assume that the path $p_u$ visits exactly one fork $v_1$ (at least 3 times), but there are forks besides $v_1$ in $H$. There are two edges $e_1$ and $e_2$ that start at $v_1$. Starting with these edges and
moving until forks, we obtain two paths $p_1$ and $p_2$. The edge $e_1$ is the first edge of $p_1$, the edge $e_2$ is the first of $p_2$, and $|p_1|, |p_2| \leq K$. We can assume that $p_1$ is a subpath of $p_u$. Then $p_1$ ends at $v_1$ (and is a cycle) and $p_2$ ends at some fork $v_2 \neq v_1$ (if $v_1 = v_2$, then $v_1$ is the only fork reachable from $v_1$). We complete the proof as in the previous case: $p_1 e_1$ is a subpath of $p_u$. We call a path good if it does not contain $p_1 e_1$. As above, we can show that if $s$ is a good path in $H$, then there are two different paths $s_1$ and $s_2$ such that $|s_1| = |s_2| = 3L$ and the paths $s s_1, s s_2$ are also good.

As above, $B$ will be a strongly connected component in the subgraph of $L^k(H)$ that consists of vertices and edges corresponding to good paths in $H$.

**Case 3.** Assume that there is only one fork $v$ in $H$. Then there are two cycles $p_1$ and $p_2$ of length $\leq K$ that start and end at $v$. Let $e_1$ be the first edge of $p_1$ and let $e_2$ be the first edge of $p_2$. The path $p_u$ contains one of the following subpaths: $p_1 e_1, p_2 e_2, p_1 p_1 e_2$ or $p_2 p_2 e_1$. Denote this path by $t$. Call a path good if it does not contain $t$. A simple check shows that we can complete the proof as in the previous cases.

**Proof of Theorem 5.** Arrange all the obstructions $u_i$ of the u.r. binary sequence $W$ by their length in non-descending order. If $\lim_{k \to \infty} \log_2 |u_k| - k = 0$, then the statement of the Theorem holds. If $\lim_{k \to \infty} \log_2 |u_k| - k > 1$ then the sequence $|u_k|/3^k$ tends to infinity. Hence, there exists $n_0$ such that $|u_{n_0}|/3^{n_0} > 10$ and $|u_{n_0}|/3^{n_0} > |u_{n_0}+k|/3^{n_0}$ for all $n > n_0$. In this situation, $|u_{n_0+k}| > |u_{n_0}| + 4 \cdot 2^{n_0} \cdot 3^k$ for any $k > 0$.

Let $v_i = u_i$ if $1 \leq i \leq n_0$ and let $v_i$ be a subword of $u_i$ of length $|u_i|_n + 4 \cdot 2^{n_0} \cdot 3^{i-n_0}$ if $i > n_0$. Denote by $\mathcal{U}$ the set of all finite binary words that do not contain subwords from $\{v_i\}$. It is clear that $\mathcal{U}$ is a proper subset of $\mathcal{L}(W)$. We get a contradiction with the uniform recurrence of $W$ if we show that the language $\mathcal{U}$ is infinite. The Rauzy graph $R_{\mathcal{U}_{n_0-1}}(\mathcal{U})$ is equal to $R_{\mathcal{L}_{n_0-1}}(W)$, and from Corollary 9 we have $\text{er}(R_{\mathcal{L}_{n_0-1}}(\mathcal{L})) \leq 2^{n_0}$.

By induction on $n$ we show that for all $n \geq n_0$ the graph $R_{\mathcal{U}_{n+1-1}}(\mathcal{U})$ contains a strongly connected subgraph $H_n$ such that $\text{er}(H_n) \leq 3^{n-n_0} \cdot 2^{n_0}$. We already have the base case $n = n_0$. The graph $R_{\mathcal{U}_{n+1-1}}(\mathcal{U})$ is obtained from $L_{\mathcal{U}_{n+1-1}}(\mathcal{L})$ by removing at most one edge. Note that $|v_{n+1}| - |v_n| > 3 \cdot \text{er}(H_n)$, so we can use Lemma 10 for the digraph $H_n$ and $k = |v_{n+1}| - |v_n|$. This completes the inductive step.

All the graphs $R_{\mathcal{U}_{n+1-1}}(\mathcal{U})$ are nonempty and, therefore, the language $\mathcal{U}$ is infinite.

**Corollary 11.** Let $W$ be an u.r. non-periodic sequence on a finite alphabet. Then $\lim_{n \to \infty} |O_W(n)|/\log_2 n \geq 1$.

**Example.** Consider a finite alphabet $\{0, 1\}$ and the sequence of words $u_i$, defined recursively as $u_0 = 0$, $u_1 = 01$, $u_k = u_k-1 u_k-2$ for $k \geq 2$. Since $u_i$ is a prefix of $u_{i+1}$, the sequence $(u_i)$ has a limit, called a Fibonacci word $F = 0100101001001 \ldots$. In Example 25 of [1] the set $\{11, 000, 10101, 00100100, \ldots\}$ of obstructions of $F$ is described. These words have lengths equal to Fibonacci numbers. Since the Fibonacci word is u.r., in Theorem 5 we cannot replace the constant 3 by a number smaller than $\sqrt{5}+1$.2

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