Instability of higher dimensional extreme black holes

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Abstract

We study linearized gravitational perturbations of extreme black hole solutions of the vacuum Einstein equation in any number of dimensions. We find that the equations governing such perturbations can be decoupled at the future event horizon. Using these equations, we show that the transverse derivatives of certain gauge invariant quantities blow up at late time along the horizon if the black hole solution satisfies certain conditions. We find that these conditions are indeed satisfied by many extreme Myers–Perry solutions, including all such solutions in five dimensions.

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1. Introduction

Extreme black holes have theoretical importance in the understanding of quantum theory of gravity. For example, Bekenstein–Hawking entropy of supersymmetric black holes was explained by counting BPS states in the view of string theory [1]. Furthermore, a duality called the Kerr/CFT correspondence between extreme black holes and a two-dimensional conformal field theory was proposed [2], and the entropy of the black holes was reproduced as the statistical entropy of the dual CFT.

Recently, it was shown that extreme Reissner–Nordström (RN) and Kerr black holes are classically unstable against test scalar field perturbations [3–6]. Subsequently, the proof is extended to all other extreme black holes [7]. They showed that the second transverse derivative blows up at the horizons as $\partial^2 r \phi \sim v$, where $\phi$ is the scalar field and we take the ingoing Eddington–Finkelstein coordinates, $(v, r)$. For extreme Kerr black holes, similar instabilities were also found in gravitational and electromagnetic perturbations [7].

The following questions arise: are all extreme black holes unstable against gravitational or electromagnetic perturbations? If not, what is the condition for the instability? In this paper, we address these questions by studying the perturbations of any extreme black holes. We use Geroch–Held–Penrose (GHP) formalism in higher dimensions developed in [8, 9] to study the perturbations. So, in section 2, we give a brief review of the GHP formalism. We introduce gravitational, electromagnetic and scalar field perturbation equations based on the formalism.
They are regarded as higher dimensional analogues of Teukolsky equations although they are not decoupled equations for gravitational and electromagnetic perturbations in general. In section 3, we introduce the most general expression for extreme black holes and express them in the view of the GHP formalism. In section 4, we study the scalar field perturbation. Although the scalar field perturbation on any extreme black holes has been already studied in [7], we revisit the problem using the GHP formalism. We find that all extreme black holes are unstable against scalar field perturbations as shown in [7]. In section 5, we study electromagnetic perturbations. We find that, near the horizon, electromagnetic perturbations satisfy decoupled equations. Using the decoupled equations, we show that the perturbations do not decay along the future event horizon if a certain operator on the horizon has a zero eigenvalue. In section 6, we study gravitational perturbations. In a similar way to electromagnetic perturbations, we can show the non-decay of the gravitational perturbations if a horizon operator has a zero eigenvalue. In addition to that, if the background geometry is algebraically special, the first or second transverse derivative of the perturbation variables blow up along the horizon. The eigenvalues for the horizon operators have been calculated for some extreme black holes. In section 7, we see that there are zero eigenvalues in the horizon operators for all higher dimensional extreme black holes with zero cosmological constant as far as we calculated. The final section is devoted to discussions.

2. Geroch–Held–Penrose formalism in higher dimensions

We study the perturbation of general extreme black holes using the GHP formalism in higher dimensions developed in [8, 9]. In this section, we give a brief review of the GHP formalism. In the formalism, we use a null basis \( \{e_0, e_1, e_i\} = \{\ell, n, m_i\} \) \((i = 2, \ldots, d-1)\) which satisfies

\[
\ell^2 = n^2 = \ell \cdot m_i = n \cdot m_i = 0, \quad \ell \cdot n = 1, \quad m_i \cdot m_j = \delta_{ij}. \tag{2.1}
\]

We define the covariant derivatives of basis vectors as

\[
L_{ab} = \nabla_b e_a, \quad N_{ab} = \nabla_b \rho_a, \quad M_{ab}^i = \nabla_b m_{ia}, \tag{2.2}
\]

and

\[
\rho_i = L_{ij}, \quad \tau_i = L_{i1}, \quad \kappa_i = L_{i0}. \tag{2.3}
\]

The expansion, shear and twist of \( \ell \) are described by \( \rho \equiv \rho_0, \rho_{(ij)} = \rho \delta_{ij}/(d-2) \) and \( \rho_{(ij)} \), respectively. The orthogonal relations (2.1) are invariant under spins, boosts and null rotations defined as follows. Spins are local \( SO(d-2) \) rotations of the spatial basis \( \{m_i\} \):

\[
m_i \rightarrow X_{ij} m_j, \tag{2.4}
\]

where \( X_{ij} \in SO(d-2) \) depends on the spacetime coordinate \( x^\mu \). Boosts are local rescaling of the null basis:

\[
\ell \rightarrow \lambda \ell, \quad n \rightarrow n/\lambda, \tag{2.5}
\]

where \( \lambda \) is any real scalar function. Null rotations about \( \ell \) and \( n \) are

\[
\ell \rightarrow \ell, \quad n \rightarrow n + z^i m_i - z^2 \ell/2, \quad m_i \rightarrow m_i - z_i \ell, \tag{2.6}
\]

and

\[
\ell \rightarrow \ell + z'_i m_i - z'^2 n/2, \quad n \rightarrow n, \quad m_i \rightarrow m_i - z'_i n, \tag{2.7}
\]

where \( z \) and \( z' \) are real functions of \( x^\mu \).

In the GHP formalism, we maintain the covariance with respect to spin and boost transformations. An object \( T_{i_1 \cdots i_s} \) is a GHP scalar of spin \( s \) and boost weight \( b \) if it transforms
by the spins and boosts as \( T_{i_1 i_2} \to X_{i_1 j_1} \cdots X_{i_k j_k} T_{j_1 \cdots j_k} \) and \( T_{i_1 i_2} \to \lambda^b T_{i_1 i_2} \). For example, the quantities \( \rho, \tau, \kappa \) are GHP scalars with \( b = 1, 0, 2 \), respectively. We also define priming operation: \( T_{i_1 i_2} \to T'_{i_1 i_2} \), where \( T'_{i_1 i_2} \) is the object obtained by exchanging \( \ell \) and \( n \) in the definition of \( T_{i_1 i_2} \).

We define GHP scalars obtained from Weyl tensor \( C_{abcd} \) as

\[
\Omega_{ij} = C_{00ij}, \quad \Omega'_{ij} = C_{11ij},
\]

(2.8)

\[
\Psi_{ijk} = C_{0ijk}, \quad \Psi'_{ijk} = C_{1ijk}, \quad \Psi_i = C_{0i00}, \quad \Psi'_i = C_{1i00},
\]

(2.9)

\[
\Phi_{ij} = C_{0i1j}, \quad \Phi'_{ijkl} = C_{ijkl}, \quad \Phi = C_{0101}, \quad \Phi'_i = \Phi_{ijj}, \quad \Phi''_{ij} = \Phi_{ijkl}.
\]

(2.10)

where \( \Omega, \Psi, \Phi, \Psi' \) and \( \Omega' \) have boost weights \( b = 2, 1, 0, -1, -2 \), respectively. The null vector \( \ell \) is called multiple WAND (Weyl-aligned null direction) iff all boost weight +2 and +1 components of the Weyl tensor vanish. The spacetime admitting the multiple WAND is called algebraically special spacetime. We can also obtain GHP scalars from Maxwell field strength \( F_{\mu b} \) as

\[
\varphi_i = F_{0i}, \quad F = F_{01}, \quad F_{ij} = F_{ij}, \quad \varphi'_i = F'_{ii},
\]

(2.11)

where \( \varphi, F \) and \( \varphi' \) have boost weights \( b = 1, 0, -1 \), respectively.

The partial derivatives of GHP scalars, such as \( \ell^\mu \partial_\mu T_{i_1 \cdots i_b} \), \( n^\mu \partial_\mu T_{i_1 \cdots i_b} \), or \( m^\mu \partial_\mu T_{i_1 \cdots i_b} \), are not GHP scalars. It is convenient to define derivative operators which are covariant under spins and boosts as

\[
\partial T_{i_1 \cdots i_b} = \ell^\mu \partial_\mu T_{i_1 \cdots i_b} - b L_{10} T_{i_1 \cdots i_b} + \sum_{r=1}^s M^k T_{i_1 \cdots i_b i_k} T_{i_{k+1} \cdots i_b},
\]

(2.12)

\[
\partial' T'_{i_1 \cdots i_b} = n^\mu \partial_\mu T'_{i_1 \cdots i_b} - b L_{11} T'_{i_1 \cdots i_b} + \sum_{r=1}^s M^k T'_{i_1 \cdots i_b i_k} T_{i_{k+1} \cdots i_b},
\]

(2.13)

\[
\partial T'_{j_1 \cdots j_b} = m^\mu \partial_\mu T'_{j_1 \cdots j_b} - b L_{12} T'_{j_1 \cdots j_b} + \sum_{r=1}^s M^k T'_{j_1 \cdots j_b i_k} T_{i_{k+1} \cdots j_b}.
\]

(2.14)

They are called GHP derivatives. We can check that \( \partial T_{i_1 \cdots i_b} \), \( \partial' T'_{i_1 \cdots i_b} \) and \( \partial T'_{j_1 \cdots j_b} \) are all GHP scalars, with boost weights \( (b + 1, b - 1, b) \) and spins \( (s, s, s + 1) \).

The GHP scalars defined above are not independent because of Ricci equations, \([\nabla_\mu, \nabla_\nu] V_\rho = R_{\rho\mu\nu} V_\sigma \), Bianchi equations, \( \nabla [C_{ijkl}] = 0 \), and Maxwell equations, \( dF = d* F = 0 \). The relations for the GHP scalars in Einstein spacetimes \( R_{\mu\nu} = \Lambda g_{\mu\nu} \) are summarized in appendix C. Since these equations are invariant under the spins and boosts, they are written by GHP scalars and their GHP derivatives.

In the GHP formalism, the Klein–Gordon equation \( (\nabla^2 - \mu^2) \phi = 0 \) is written as

\[
(2\partial' \partial + \partial \partial + \rho \partial + \rho' \partial - 2\tau \partial + \rho \partial' - \mu^2) \phi = 0.
\]

(2.15)

From appropriate linear combinations of equations in appendix C, we can obtain useful equations for studying electromagnetic and gravitational perturbations [9]. They are written as

\[
\left( 2\partial' \partial + \partial \partial + \rho \partial + \rho' \partial - 4\tau \partial + \Phi - \frac{d-3}{d-1} \Lambda \right) \varphi_i + \left( -2\tau \partial_j + 2\tau \partial_j + 2\Phi_\partial + 4\Phi_\partial' \right) \varphi_j
\]

\[
= [\kappa \partial' + \rho \partial + (\partial \rho) + (\partial \rho') + \mu] F + (\rho \partial' + \kappa \partial' + \rho \partial' + \mu) \psi
\]

\[
+ (\kappa \partial + \rho \partial + \kappa \tau + \Omega) \psi' + \rho \partial' \partial + \kappa \partial + \rho \partial + \kappa \tau + \Omega) \psi',
\]

(2.16)
and
\[
\begin{align*}
\left(2\Phi' \rho + \delta_{ij} \rho k + \rho' \rho - 6\tau_k \delta_{ij} + 4\Phi - \frac{2d}{d-1}\Lambda\right)\Omega_{ij} + 4\left(\tau_i \delta_{ij} - \tau_j \delta_{ik} + \Phi_{(ijk)} + 4\Phi'_{(ijk)}\right)\Omega_{ij}
\end{align*}
\]
\[
+2\Phi_{(ijk)}\Omega_{ij} + 4\kappa_1 b'(\Psi_{(ijk)} + \Psi_{(jik)})
\]
\[
= [\rho \hat{\Delta} + \tau \rho + \tau' \rho + \kappa \rho' + (b' \kappa) + (\bar{\Omega} \rho) + \Psi] \Psi
\]
\[
+ \rho^2 \Phi + \kappa \rho \Psi' + (\rho b' \Omega + \rho \rho' + \kappa \kappa') \Omega.
\]

The right-hand sides of these equations are very long so we wrote them schematically. In this paper, we do not need the detailed expressions of the right-hand sides.

3. Extreme black holes in the GHP formalism

We consider general extreme black holes. The metric of the extreme black holes can be written as \([10]\)
\[
ds^2 = L^2(x)[-r^2 F(r, x) dv^2 + 2dvd\varphi] + \gamma_{\mu \rho}(r, x)(dx^\mu - r h^\mu(r, x)dv)(dx^\rho - r h^\rho(r, x)dv),
\]
(3.1)
where functions \((F, \gamma_{\mu \rho}, h^\mu)\) and \(L^2\) are smooth function of \([r, x^a]\) and \([x^a]\), respectively. The horizon of the spacetime is located at \(r = 0\). In the metric, there is a residual coordinate transformation, \(r \to \Gamma(x) r\). We choose the free function \(\Gamma(x)\) so that \(F(r = 0, x) = 1\) is satisfied. In this paper, we focus only on Einstein spacetimes satisfying \(R_{\mu \nu} = \Lambda g_{\mu \nu}\).

We assume that the background metric has \(n\) rotational symmetry generated by \(\partial / \partial \phi^I (I = 1, 2, \ldots, n)\). Then, the metric can be written as
\[
ds^2 = L_2(y)[-r^2 F(r, y) dv^2 + 2dvd\varphi] + \gamma_{AB}(r, y)(dy^A - r h^A(r, y)dv)(dy^B - r h^B(r, y)dv)
\]
\[
+ 2\gamma_{A}(r, y)(dy^A - r h^A(r, y)dv)(d\phi^I - r h^I(r, y)dv)
\]
\[
+ \gamma_{J}(r, y)(d\phi^J - r h^J(r, y)dv)(d\phi^I - r h^I(r, y)dv),
\]
(3.2)
where we broke up the coordinates as \([x^a] = [\phi^I, y^A]\) \((I = 1, \ldots, n, A = 1, \ldots, (d - n - 2))\).

We impose further assumption on metric functions as
\[
h^A(r, y) = O(r), \quad h^I(r, y) = k^I + O(r), \quad \gamma_{A}(r, y) = O(r),
\]
(3.3)
where \(k^I\) are constants. These assumptions are true for a large class of extreme black holes \([11–15]\). Under these assumptions, the near horizon geometry of the metric (3.2) takes the ‘standard’ form
\[
ds^2 = L^2(y)[-R^2 dv^2 + 2dV dR] + \gamma_{\mu \beta}(y)dy^\mu dy^\beta + \gamma_{\mu \nu}(y)(d\phi^I - r k^I dv)(d\phi^J - r k^J dv)
\]
(3.4)
where we took the double scaling limit: \(r = \epsilon R, v = V / \epsilon\) and \(\epsilon \to 0\). The induced metric on the horizon is written as
\[
\tilde{ds}_{HJ}^2 = \tilde{g}_{\mu \nu} dv^\mu dv^\nu = \gamma_{\mu \beta}(y)dy^\beta + \gamma_{\mu \nu}(y)(d\phi^I - r k^I dv)(d\phi^J - r k^J dv)
\]
(3.5)
We take null basis \([e_0, e_1, e_2]\) \(= [\ell, n, m_i]\) in the general extreme black hole metric (3.1) as
\[
\ell = \frac{2}{L} \delta_\ell + \frac{r^2 F}{L} \delta_\tau + \frac{2rh^I}{L} \delta_i, \quad n = \frac{1}{2L} \delta_\tau, \quad m_i = \delta_i,
\]
(3.6)
where \(h^I = h^I \delta^I_\ell\) and \(\delta^I_\ell\) is an appropriate orthogonal basis for \(\gamma_{\mu \beta}\). The null basis (3.7) is regular at the future horizon \(r = 0\). Using the basis, we can obtain GHP variables. The full
expressions of the GHP variables are summarized in appendix A. Here, we focus on \( \rho_{ij} \) and \( \kappa_i \) since they will be important later. They are given as

\[
\kappa_i = \frac{2r^2 F_{ij}}{L}, \quad \rho_{ij} = \frac{2r}{L} h^{\nu}_{\phantom{\nu}j} (\hat{\epsilon}_{i\nu}) + \frac{2r^2}{L} h^{\nu}_{\phantom{\nu}j} (\hat{\epsilon}_{i\nu})_k + \frac{r^2 F}{L} \hat{\epsilon}_{ij} (\hat{\epsilon}_{i\nu}),
\]

where \( j \equiv \hat{\epsilon}_{i\nu} \partial_{\nu} \). We chose the residual gauge freedom so that \( F(r = 0, x) = 1 \) is satisfied. Thus, we obtain \( F_{ij} = O(r) \). Therefore, we have \( \kappa_i = O(r^3) \). From the assumption (3.3), \( h^{\nu}_{\phantom{\nu}j} \big|_{r = 0} \) is constant and we have \( h^{\nu}_{\phantom{\nu}j} = O(r) \). Thus, the first term in \( \rho_{ij} \) is \( O(r^2) \). In the second term, there is a derivative operator, \( h^2 \partial_k = k^l \partial_{kl} + O(r) \). Since the \( \partial / \partial \phi \) is a Killing vector, its operation to background variables vanishes and the second term is also \( O(r) \). The last term is trivially \( O(r^2) \). Therefore, we can conclude that \( \rho_{ij} \) is second order in \( r \). By the similar way, we obtain the near horizon expression of the GHP variables as

\[
\rho_{ij} = O(r^2), \quad \kappa_i = O(r^3), \quad \tau_i = \frac{-(L^2)_{ij} + \kappa_j}{2L^2} + O(r),
\]

\[
\rho'_{ij} = O(1), \quad \kappa'_i = 0, \quad \tau'_i = O(1),
\]

\[
L_{10} = \frac{2r}{L} + O(r^2), \quad L_{11} = 0, \quad L_{12} = \frac{k_i}{2L^2} + O(r),
\]

\[
M'_{i0} = O(r^2), \quad M'_{ij} = O(1), \quad M'_{jk} = O(1).
\]

Components of Weyl tensors with boost weights +2 and +1 are given by Newman–Penrose equations (NP1) and (NP3) as

\[
\Omega_{ij} = -\partial_i \kappa_j - \partial_j \kappa_i - \rho_{ik} \rho_{kj} - \kappa_i \tau_j - \tau_i \kappa_j = O(r^3),
\]

\[
\Psi_{ijk} = 2(\tau_i \rho_{jkl} + \kappa_i \rho_{jkl} - \partial_{[ij} \rho_{kl]} ) = O(r^2).
\]

Now, we consider components of Weyl tensor with boost weight 0. In [8], it was shown that the induced Riemann tensor on a space-like surface which is orthogonal to null vectors \( \ell \) and \( n \) is written as

\[
R^{(d-2)}_{ijkl} = 2 \rho_{[k|l]} \rho'_{ij]} + 2 \rho'_{[k|l]} \rho_{ij]} + \Phi_{ijkl} + \frac{2\Lambda}{d-1} \delta_{[k|l} \delta_{i]}.
\]

Thus, we have

\[
\Phi_{ijkl} = \hat{R}_{ijkl} - \frac{2\Lambda}{d-1} \delta_{[k|l} \delta_{i]} + O(r), \quad \Phi^0_{ij} = -\frac{1}{2} \left( \hat{R}_{ij} - \frac{d-3}{d-2} \Lambda \delta_{ij} \right) + O(r),
\]

\[
\Phi = -\frac{1}{2} \left( \hat{R} - \frac{(d-2)(d-3)}{d-1} \Lambda \right) + O(r),
\]

where \( \hat{R}_{ijkl} \) is the induced Riemann tensor on the horizon (3.6). From the antisymmetric part of equation (NP4), we obtain

\[
\Phi^A_{ij} = \left[ -\frac{1}{4L^2} \partial_{[k} + \frac{1}{4L^2} \partial L^2 \wedge k \right]_{ij} + O(r).
\]

The GHP derivatives are

\[
\partial T_{i_1 \cdots i_s} = \frac{2}{L} (\partial_{i_1} + r(k^l \partial_{kl} - b) T_{i_1 \cdots i_s} + O(r^2)),
\]

\[
\partial' T_{i_1 \cdots i_s} = \frac{1}{2L} \partial_{i_1} T_{i_1 \cdots i_s} + \sum_{r=1}^{s} M^r_{i_1} T_{i_1 \cdots i_r i_{r+1} \cdots i_s},
\]

\[
\partial T_{j_1 \cdots j_s} = \left( \hat{\nabla}_{i} - \frac{b k^i}{2L^2} \right) T_{j_1 \cdots j_s} + O(r),
\]

where \( \hat{\nabla} \) is a covariant derivative with respect to the horizon induced metric (3.6).
4. Scalar field perturbations

4.1. Conserved quantity on the horizon

First, we consider the scalar field perturbation equation (2.15). The instability of the massless scalar field perturbation on any extreme black holes has been already shown in [7]. Here, we revisit the problem including the massive scalar field using the GHP formalism.

Substituting near horizon expressions of GHP variables and derivatives (3.9)–(3.13) into the Klein–Gordon equation (2.15), we have the scalar field equation near the horizon as

\[ \partial_0 [2L(2\partial \phi + \mu \phi)] = \mathcal{A}_0 \phi + \mathcal{O}(r), \]  

(4.1)

where the operator \( \mathcal{A}_0 \) is defined as

\[ \mathcal{A}_0 \phi = -\hat{\nabla}_l (L^2 \hat{\nabla}_l \phi) - ik^2 m \phi + \mu^2 L^2 \phi. \]  

(4.2)

Here we decomposed \( \phi \)-dependence of \( \phi \) by Fourier modes \( e^{im \phi} \), that is, \( \partial \phi = im \phi \).

We will do the same decompositions for electromagnetic and gravitational perturbations. The derivation of the equation is given in appendix B. Hereafter, we focus on axisymmetric perturbations, \( m = 0 \). For axisymmetric perturbations, the operator \( \mathcal{A}_0 \) is self-adjoint, \( (Y_1, \mathcal{A}_0 Y_2) = (\mathcal{A}_0 Y_1, Y_2) \), with respect to an inner product

\[ (Y_1, Y_2) = \int_H Y_1^* Y_2, \]  

(4.3)

where \( \int_H = \int d^{d-2} x \sqrt{g} \) and \( \sqrt{g} \) is the determinant of the horizon induced metric defined in equation (3.6).

We assume that operator \( \mathcal{A}_0 \) has a zero eigenvalue. This is always true for massless case \( \mu = 0 \) since we have \( \mathcal{A}_0 Y = 0 \) when \( Y \) is a constant. For massive scalar field, \( \mathcal{A}_0 \) can also have zero eigenvalues depending on the value of \( \mu^2 \) and background geometries. We will discuss the existence of the zero eigenvalues in section 7.2. We denote the eigenfunction for the zero eigenvalue as \( Y \). Operating \( (Y, \ast) \) to both sides of equation (4.1), we obtain

\[ \frac{d I_0}{dv} = 0, \quad I_0 = \int_H Y^* [2L(2\partial \phi + \mu \phi)], \]  

(4.4)

where we used \( (Y, \mathcal{A}_0 Y) = (\mathcal{A}_0 Y, Y) = 0 \). Hence, \( I_0 \) is a conserved quantity along the horizon. Note that the GHP derivative \( \partial \) contains only the radial derivative \( \partial_r \). (See equation (3.13).)

Thus, the integrand in the \( I_0 \) is written by the linear combination of \( \partial \phi \) and \( \phi \). Therefore, we can conclude that if \( I_0 \neq 0 \) at an initial surface, \( \partial \phi \) and \( \phi \) do not both decay along the future horizon as \( v \to \infty \).

4.2. Instability against scalar field perturbations

We assume that \( \phi \) and its angular derivatives \( \partial / \partial \theta^d \) decay along the horizon. For extreme RN and Kerr black holes, it was shown that \( \phi \) decays along the horizon [4, 5]. So, this assumption seems likely also for any other extreme black holes. Then, at the late time, the conserved quantity \( I_0 \) approaches

\[ I_0 \sim 2 \int_H Y^* \partial \phi, \quad (v \to \infty). \]  

(4.5)

\footnote{For axisymmetric perturbations, this operator \( \mathcal{A}_0 \) coincides with the operator \( \mathcal{O}^{(0)} \) defined in the study of perturbation of near horizon geometries [16]. We will also define operators \( \mathcal{A}_s \) for electromagnetic (\( s = 1 \)) and gravitational (\( s = 2 \)) perturbations by similar ways as the scalar field, which relate to \( \mathcal{O}^{(s)} \) defined in [16] as \( \mathcal{O}^{(1)} = \mathcal{A}_1 \) and \( \mathcal{O}^{(2)} = \mathcal{A}_2 + 2 \) for axisymmetric perturbations.}
Now, we differentiate equation (2.15) by \( r \). Near the horizon, the equation can be written as
\[
\frac{\partial}{\partial r} [2L \partial_r (2L \phi + \rho \phi)] = (\mathcal{A}_0 - 2) \partial_r \phi + \mathcal{D}_0 \phi + \mathcal{O}(r),
\]
where, in the linear operator \( \mathcal{D}_0 \), there is no radial derivative \( \partial_r \). Thus, we have \( \mathcal{D}_0 \phi \to 0 \) as \( v \to \infty \). The derivation of above equation is in appendix B. Operating \( (Y, \ast) \) to both sides of equation (4.6), we obtain
\[
\frac{dJ_0}{dv} \simeq -I_0, \quad (v \to \infty), \quad J_0 = \int_H Y^* [2L \partial_r (2L \phi + \rho \phi)].
\]
Therefore, the quantity \( J_0 \) blows up linearly in time \( v \) as
\[
J_0 \simeq -I_0 v, \quad (v \to \infty).
\]
The integrand in the \( J_0 \) is written by the linear combination of \( \partial_r^2 \phi \), \( \partial_{\ast} \phi \) and \( \phi \). Since we assumed that \( \phi \) decays at the horizon, either \( \partial_r^2 \phi \) or \( \partial_{\ast} \phi \) blows up at the horizon. This implies the instability of the extreme black holes against scalar field perturbations.

5. Electromagnetic perturbations

5.1. Decoupled equation on the horizon

Secondly, we consider electromagnetic perturbations. We consider the Maxwell field as a test field and, thus, it vanishes in the background. It follows that GHP scalars obtained from Maxwell field strength are all invariant under the infinitesimal coordinate transformations and basis transformations (2.4)–(2.7). The number of physical degrees of freedom of the Maxwell field in \( d \)-dimensions is \( d - 2 \). The number of components of \( \phi \) is also \( d - 2 \). Thus, we can expect that \( \phi \) has all physical degrees of freedom of electromagnetic perturbations and it would be nice if we can obtain decouple equations for \( \phi \). The right-hand side of equation (2.16) contains coupling terms between \( \phi \) and other components of the perturbation. So, \( \phi \) does not decouple in general. We can see that, however, all the terms on the right-hand side of equation (2.16) are multiplied by \( \rho, \kappa, \Omega \) or \( \Psi \). From equations (3.9) and (3.10), they are at most \( \mathcal{O}(r^2) \). Thus, the right-hand side is \( \mathcal{O}(r^2) \). Therefore, in equation (2.16) and its radial derivative, the right-hand side is zero at the horizon.

5.2. Conserved quantity on the horizon

We consider the left-hand side of equation (2.16) neglecting the right-hand side. Using near horizon expressions of GHP variables and derivatives (3.9)–(3.13), we obtain
\[
\frac{\partial}{\partial r} [2L (2L \phi_i + \rho \phi_i)] = \mathcal{A}_1 \phi_i + \mathcal{O}(r),
\]
where we define the operator \( \mathcal{A}_1 \) as
\[
\mathcal{A}_1 \phi_i = -\frac{1}{L^2} \hat{\nabla}_j (L^4 \hat{\nabla}_j \phi_i) + \left( 2 + 3i k^i m_l - \frac{5}{4L^2} k^i k^l \right) \phi_i + L^2 \left( \hat{R}_{ij} + \frac{1}{2} \hat{R}_{i} \right) \phi_j
+ \left( -\frac{1}{2} (d k)_{ij} + 2(k - d \hat{L}^2) \hat{\nabla}_i \hat{\nabla}_j - \frac{1}{L^2} (d \hat{L}^2)_{ij} \phi_j \right).
\]
The derivation of the equation is written in appendix B. Hereafter, we focus on the axisymmetric perturbations \( m_l = 0 \). We define an inner product as
\[
(Y^1, Y^2) = \int_H L^2 (Y^1, Y^2).
\]
For axisymmetric perturbations, we have \( (Y^1, \mathcal{A}_1 Y^2) = (\mathcal{A}_1 Y^1, Y^2) \), that is, the operator \( \mathcal{A}_1 \) is self-adjoint. We assume that the operator \( \mathcal{A}_1 \) has zero eigenvalue. Although the existence
of the zero eigenvalue is not obvious, we will see that many extreme black holes satisfy this assumption in section 7.2. We denote the eigenfunction by \( Y_i \). Operating \((Y, *)\) to equation (5.1), we have
\[
\frac{dI_1}{dv} = 0, \quad I_1 = \int_H Y_i^* [2L^3 (2\phi'_i + \rho'_i)], \tag{5.4}
\]
where we used \((Y, A_1\phi) = (A_1Y, \phi) = 0\). Therefore, \(I_1\) is a conserved quantity along the horizon. Thus, if \(I_1 \neq 0\) at an initial surface, \(\partial_i \phi_i\) and \(\phi_i\) do not both decay along the future horizon.

We assume that \(\phi_i\) and its tangential derivatives along the horizon decay as \(v \to \infty\). Now, we differentiate equation (2.16) by \(r\). Then, the right-hand side becomes \(O(r)\) and still vanishes on the horizon. Thus, we have
\[
\partial_i [2L \partial_i (2\phi'_i + \rho'_i)] = A_1 \partial_i \phi_i + D_i \phi_i + O(r), \tag{5.5}
\]
where, in the linear operator \(D_i\), there is no radial derivative \(\partial_r\). Hence, we have \(D_1 \phi_i \to 0\) \((v \to \infty)\). The derivation of the equation is in appendix B. Operating \((Y, *)\) to above equation and taking limit of \(v \to \infty\), we obtain
\[
\frac{dJ_1}{dv} \simeq 0, \quad (v \to \infty), \quad J_1 = \int_H Y_i^* [2L^3 \partial_i (2\phi'_i + \rho'_i)]. \tag{5.6}
\]
Therefore, the quantity \(dJ_1/dv\) tends to be zero at the late time even if we consider initial data with \(I_1 \neq 0\). So, we cannot show the instability of extreme black holes against electromagnetic perturbations by the same way as scalar fields. We may be able to find instability in higher order derivatives \(\partial^n \phi_i\) \((n > 2)\). However, since the coupling terms are \(O(r^2)\) in the equation, we cannot neglect these terms in the limit of \(r \to 0\) when we consider the higher order radial derivatives of equation (2.16). It seems to be a difficult problem to show the instability taking into account the coupling terms.

6. Gravitational perturbations

6.1. Gauge invariant variables on the horizon

Finally, we study gravitational perturbations. We consider perturbation of GHP variables as \(\Omega_{ij} \to \tilde{\Omega}_{ij} + \hat{\Omega}_{ij}, \Psi_{ijk} \to \tilde{\Psi}_{ijk} + \hat{\Psi}_{ijk}, \) etc. Here, variables with tildes represent first-order perturbations. Variables without tildes are background variables. The number of physical degrees of freedom of the gravitational perturbations is \(d(d-3)/2\). On the other hand, the number of components of \(\hat{\Omega}_{ij}\) is also \(d(d-3)/2\). Thus, we can expect that \(\hat{\Omega}_{ij}\) has all physical degrees of freedom of the gravitational perturbations. The perturbation variable \(\hat{\Omega}_{ij}\) is transformed by gauge transformations as follows:

Coordinate transformations \((x^a \to x^a + \xi^a (x))\):
\[
\tilde{\Omega}_{ij} \to \hat{\Omega}_{ij} + \xi^a \partial_a \Omega_{ij}. \tag{6.1}
\]

Spins \((\iota_{ij} \in so(d-2))\):
\[
\tilde{\Omega}_{ij} \to \hat{\Omega}_{ij} + 2\iota_{ijk} \tilde{\Omega}_{kij}. \tag{6.2}
\]

Boosts:
\[
\tilde{\Omega}_{ij} \to \hat{\Omega}_{ij} + 2\alpha \Omega_{ij}. \tag{6.3}
\]

Null rotations:
\[
\tilde{\Omega}_{ij} \to \hat{\Omega}_{ij} - 2\alpha (\Psi_{ij} \delta_{jk} + \Psi_{ijk}). \tag{6.4}
\]
Here, $\xi^\mu$, $t_{ij}$, $\alpha$ and $z_k$ are infinitesimal functions depending on spacetime coordinates. In equation (3.10), we obtained $\Omega_{ij} = O(r^3)$ and $\Psi_{ij} = O(r^2)$. Thus, under spin and boost transformations, we have $\tilde{\Omega}_{ij} \rightarrow \Omega_{ij} + O(r^3)$. On the other hand, under coordinate transformations and null rotations, we have $\tilde{\Omega}_{ij} \rightarrow \tilde{\Omega}_{ij} + O(r^3)$. Therefore, $\tilde{\Omega}_{ij}l_{r=0}$ and $\tilde{\partial}_r \tilde{\Omega}_{ij}l_{r=0}$ are not gauge invariant in general. Thus, even if we could show that $\tilde{\partial}_r^2 \tilde{\Omega}_{ij}l_{r=0}$ blows up along the horizon in a similar way to scalar field perturbations, we cannot determine if the instability is a physical one or just a gauge mode. We will avoid this problem by assuming that the background geometry is algebraically special in section 6.4.

6.2. Decoupled equations on the horizon

We consider the first-order perturbation of equation (2.17). The right-hand side contains a coupling term between $\Omega_{ij}$ and other perturbation variables. We can see that all terms on the right-hand side are $O(r^2)$. For example, we have $(r'\rho \Psi) = \tilde{r}' \rho \Psi + r' \tilde{\rho} \Psi + r' \rho \Psi = O(r^2)$ since $\rho$ and $\Psi$ are the second order in $r$. Thus, in equation (2.17) and its radial derivative, the right-hand side is zero at the horizon. On the left-hand side, there is another coupling term, $4\xi_{\ell}^k \int \left( \Psi_{ij} + \Psi_{i}^j \delta_{jk} \right)$ which can be expanded as

$$[4\xi_{\ell}^k \int \left( \Psi_{ij} + \Psi_{i}^j \delta_{jk} \right)] = 4\tilde{\xi}_{\ell}^k \int \left( \Psi_{ij} + \Psi_{i}^j \delta_{jk} \right) + O(r^3).$$

This coupling term is $O(r)$. (Recall that the GHP derivative $\delta'$ contains the radial derivative $\partial_r$. Hence, we have $\delta' \Psi_{ij} = O(r)$. Such a coupling term is harmless when we construct a conserved quantity at the horizon and we can show the non-decay of the gravitational perturbations by the same way as scalar and electromagnetic perturbations. However, when we prove that the perturbations blow up along the horizon, the coupling term is problematic since we differentiate equation (2.17) by $r$. In section 6.4, we will see that this problem can also be avoided by assuming that the background geometry is algebraically special.

6.3. Conserved quantity on the horizon

The coupling terms in equation (2.17) is $O(r)$ and negligible near the horizon. Thus, near horizon, the equation becomes

$$\tilde{\partial}_r [2L \rho' \tilde{\Omega}_{ij} + \rho \tilde{\partial}_r \tilde{\Omega}_{ij}] = A_2 \tilde{\Omega}_{ij} + O(r),$$

where the operator $A_2$ is defined as

$$A_2 \tilde{\Omega}_{ij} = \frac{1}{L^2} \tilde{\nabla}_k (L^2 \tilde{\nabla}_l \tilde{\Omega}_{ij}) + \left( 4 + 3k^l m_l - \frac{4k^l k_l}{L^2} - 2(d - 4) \nabla - 2(d - 4) \nabla^2 \right) \tilde{\Omega}_{ij}
+ 2L^2 \left( \tilde{R}_{ijkl} + \tilde{\nabla}_k \tilde{\nabla}_l \tilde{\Omega}_{ij} \right)
+ \left[ -(dL)^{ij} + \frac{2}{L^2} (d(L^2) \wedge k)^{ij} + 2(k - d(L^2))_k \tilde{\nabla}_l \right] \tilde{\Omega}_{(ij)}.

The derivation of the equation is written in appendix B. For axisymmetric perturbations $m_l = 0$, the operator $A_2$ is self-adjoint with respect to an inner product

$$(Y^1, Y^2) = \int_H L^2 Y^1_{ij} Y^2_{ij}.$$

Hereafter, we focus on axisymmetric perturbations. We assume that operator $A_2$ has a zero eigenvalue. In section 7.2, we will see that many extreme black holes satisfy this assumption.
We denote the eigenfunction for the zero eigenvalue as \( Y_{ij} \). Operating \((Y, \ast)\) to both sides of equation (6.6), we obtain
\[
\frac{dI_2}{dv} = 0, \quad I_2 = \int_H Y^*_ij[2L^5(2\tilde{\varphi}'\tilde{\Omega}_{ij} + \rho'\tilde{\Omega}_{ij})],
\]
where we used \((Y, A_2\tilde{\Omega}) = (A_2Y, \tilde{\Omega}) = 0\). Therefore, \( I_2 \) is a conserved quantity along the horizon. Thus, if \( I_2 \neq 0 \) at an initial surface, \( \partial_r\tilde{\Omega}_{ij} \) and \( \tilde{\Omega}_{ij} \) do not both decay along the future horizon as \( v \to \infty \). Recall that both of \( \partial_r\tilde{\Omega}_{ij} \) and \( \tilde{\Omega}_{ij} \) are gauge invariant at the horizon.

6.4. Null rotation to a multiple WAND

As explained in sections 6.1 and 6.2, we require conditions \( \Psi_{ijk} = \mathcal{O}(r^3) \) and \( \Omega_{ij} = \mathcal{O}(r^4) \) to show the instability of the gravitational perturbations. To satisfy these conditions, we assume that the background geometry is algebraically special. Then, there is a null rotation (2.7) which transforms the null vector \( \ell \) to a multiple WAND, that is, \( \Psi_{ijk} = \Omega_{ij} = 0 \) in the near horizon geometry (3.5), we have already known the multiple WAND: \( \ell^{NH} = L^{-1}(2\partial_r + r^2\partial_r + 2k^l\partial_l) \). We can expect that in the near horizon limit: \( r = \epsilon R, v = \epsilon^4 \) and \( \epsilon \to 0 \), the multiple WAND in the full geometry coincides with the \( \ell^{NH} \) modulo boost transformations.2 The null vector \( \ell \) defined in equation (3.7) satisfies this condition by itself. (\( \ell \to \epsilon\ell^{NH} \) in the near horizon limit.) Thus, \( Z^\prime \) in the null rotation (2.7) should be \( \mathcal{O}(r^2) \). It follow that the near horizon expressions of GHP variables (3.9) are correct even after the this null rotation.

6.5. Instability against gravitational perturbations

We assume that \( \tilde{\Omega}_{ij} \) and its tangential derivatives along the horizon decay along the horizon. Then, at late time, the conserved quantity \( I_2 \) becomes
\[
I_2 \simeq 2 \int_H L^5Y^*_ij \partial_r\tilde{\Omega}_{ij}, \quad (v \to \infty).
\]
Now, we differentiate equation (2.17) by \( r \). Since we assumed that the background geometry is algebraically special, we can neglect the coupling term in the equation. Near the horizon, the equation can be written as
\[
\partial_r[2L^5\partial_r(2\tilde{\varphi}'\tilde{\Omega}_{ij} + \rho'\tilde{\Omega}_{ij})] = (A_2 + 2)\partial_r\tilde{\Omega}_{ij} + D_2\tilde{\Omega}_{ij} + \mathcal{O}(r),
\]
where, in the linear operator \( D_2 \), there is no radial derivative. Thus, we have \( D_2\tilde{\Omega}_{ij} \to 0 \) \( (v \to \infty) \). The derivation of above equation is in appendix B. Operating \((Y, \ast)\) to both sides of equation (6.11), we obtain
\[
\frac{dJ_2}{dv} \simeq J_2, \quad (v \to \infty), \quad J_2 = \int_H Y^*_ij[2L^5\partial_r(2\tilde{\varphi}'\tilde{\Omega}_{ij} + \rho'\tilde{\Omega}_{ij})]
\]
Therefore, the quantity \( J_2 \) blow up linearly in time \( v \) as
\[
J_2 \simeq I_2 v, \quad (v \to \infty).
\]
Thus, either \( \partial_r^2\tilde{\Omega}_{ij} \) or \( \partial_r\tilde{\Omega}_{ij} \) blows up along the horizon. This implies the instability of the extreme black holes against gravitational perturbations.

\[2\] We checked that this is correct for Kerr–NUT–AdS spacetimes [17]. That is, the multiple WAND in Kerr–NUT–AdS spacetimes found in [18] approaches \( \ell^{NH} \) in the near horizon limit.
7. Unstable extreme black holes

7.1. Summary of our statement

Our statements obtained in this paper are as follows. If the operator $A_s$ has a zero eigenvalue for an axisymmetric perturbation and the horizon conserved quantity $I_s$ is non-zero, $\partial_r \psi_s$ and $\psi_s$ do not both decay along the future horizon as $v \to \infty$, where $\psi_0 = \phi$, $\psi_1 = \varphi$, and $\psi_2 = \tilde{\Omega}_{ij}$. The explicit expressions of $A_s$ are given in equations (4.2), (5.2) and (6.7). The horizon conserved quantities (4.4), (5.4) and (6.9) are written as

$$ I_s = \int_H Y^+ \cdot [2L^{2s+1} + 2\rho' \psi_s + \rho' \psi_s], $$

where $Y$ is the eigenfunction satisfying $A_s Y = 0$. In the proof of this statement, we used assumptions (3.3).

Hereafter, we assume that $\psi_s$ and its tangential derivatives along the horizon decay as $v \to \infty$. (Then, $\partial_r \psi_s$ cannot decay.) For scalar field perturbations, we can show that either $\partial^2_r \phi$ or $\partial_r \phi$ blows up along the horizon. For gravitational perturbations, when the background geometry is algebraically special, either $\partial^2_r \tilde{\Omega}_{ij}$ or $\partial_r \tilde{\Omega}_{ij}$ also blows up along the horizon. For electromagnetic perturbations, we could not find the instability in $\partial^2_r \phi$ or $\partial_r \phi$. (There may be instability in the higher order derivative by $r$.)

7.2. Eigenvalues of $A_s$

The existence of a zero eigenvalue for the horizon operator $A_s$ is crucial for the proof of the instability. Surprisingly, in the study of perturbations of near horizon geometries, the eigenvalues of $A_s$ have been calculated for some extreme black holes: four-dimensional extreme Kerr black holes [19, 20], all five-dimensional black holes with two rotational symmetries for $\Lambda = 0$ [21] and Myers–Perry(–AdS) black holes with equal angular momenta [16, 22]. In this subsection, we investigate the existence of zero eigenvalues of $A_s$ using their results.

For massless scalar field, the existence of the zero eigenvalue is trivial since we have $A_0 Y = 0$ when $Y$ is a constant. For massive case, there is no zero eigenvalue in general. In some cases, however, it has a zero eigenvalue depending on mass and background geometry. For example, in odd-dimensional Myers–Perry–AdS spacetimes with equal angular momenta, it was shown that the eigenvalue $\lambda_0$ is given by [16]

$$ \lambda_0 = 4\kappa^2 \left( \kappa + \frac{N}{r^2_{+}} \right), \quad (\kappa = 0, 1, 2, \ldots), $$

where $r_{+}$ is the horizon radius and $L^2$ is defined in equation (3.1), which is given by a constant in the case of odd-dimensional Myers–Perry–AdS spacetimes with equal angular momenta. The integer $N$ relates to the spacetime dimension as $d = 2N + 3$. From this expression, when the scalar field mass is given by $\mu^2 = -4\kappa (\kappa + N)/r^2_{+}$, the operator $A_0$ has a zero mode. (If the horizon radius $r_{+}$ is sufficiently large, then $\mu^2$ does not violate the Breitenlohner–Freedman (BF) bound.)

For electromagnetic and gravitational perturbations ($s = 1, 2$), the existence of a zero eigenvalue is not obvious. In four-dimensional extreme Kerr geometry, the operator $A_s$ ($s = 1, 2$) does not have zero eigenvalue. However, if we consider perturbation equations for $\Omega_{ij}'$ instead of $\Omega_{ij}$, we can show the instability [7]. (In the four-dimensional Kerr geometry, $\Omega_{ij}'$ satisfies a decoupled equation although it does not decouple in Myers–Perry spacetimes even if we consider the near horizon limit.) For all five-dimensional black holes with two
rotational symmetries, eigenvalues $\lambda_d$ are written as

$$
\begin{align*}
\lambda_1 &= \ell(\ell + 1), \ (\ell + 1)(\ell + 2), \ (\ell + 1)(\ell + 3), \ (\ell + 1)(\ell + 4), \\
\lambda_2 &= (\ell - 1)(\ell + 2), \ \ell(\ell + 3), \ \ell(\ell + 3), \ (\ell + 1)(\ell + 4), \ (\ell + 1)(\ell + 4),
\end{align*}
$$

(7.3)

where $\ell = 0, 1, 2, \ldots$. We can find that $\lambda_1$ and $\lambda_2$ can be zero for $\ell = 0$ and $\ell = 1$, respectively. Thus, in these spacetimes, the gravitational and electromagnetic perturbations do not decay in general. In particular, for five-dimensional Myers–Perry black holes, we can show that gravitational perturbations (either $\delta^2 \Omega_{ij}|_{r=0}$ or $\delta \tilde{\Omega}_{ij}|_{r=0}$) blow up along the horizon since the spacetimes are known to be algebraically special [23].

The horizon induced metrics of Myers–Perry black holes with equal angular momenta can be viewed as Hopf fibration over $CP^N$, where $N$ is the integer part of $(d - 3)/2$. Thus, the eigenfunction of $A_s$ can be decomposed into tensor, vector and scalar harmonics on the base space $CP^N$. All eigenvalues for axisymmetric modes are given in [16, 22]. In scalar modes, we can always find zero eigenvalues. (Tensor and vector modes can also have zero eigenvalues depending on the spacetime dimension $d$.) Thus, extreme Myers–Perry black holes with equal angular momenta in all dimensions are unstable against gravitational perturbations. (Electromagnetic perturbations do not decay at least.) Therefore, as far as we calculated, all vacuum extreme higher dimensional black holes with vanishing cosmological constant have zero eigenvalues in the horizon operator $A_s$. It would be nice if we can show the existence of the zero eigenvalues for general black holes.

8. Discussions

We studied perturbations in general extreme black hole spacetimes in all dimensions. We found a sufficient condition for instability which is summarized in section 7.1. Using the condition, we showed that five-dimensional extreme Myers–Perry black holes are unstable against gravitational perturbations. For $d \geq 6$, we also found gravitational instability in extreme Myers–Perry black holes when they have equal angular momenta.

In the study of perturbations of near horizon geometries [16, 21, 22], they considered dimensional reduction of perturbation equations and obtained effective equations of motion in AdS$_2$. Then, they used a criterion $m^2 < -1/4$ to determine the instability of near horizon geometry, where $m$ is the effective mass in the AdS$_2$, since this implies violation of the BF bound. The effective mass relates to the eigenvalue of $A_s$ as $m^2 = \lambda_0, \lambda_1, \lambda_2$ for scalar, electromagnetic and gravitational perturbations, respectively. We can see that the condition of the instability obtained in this paper ($\lambda_s = 0$) differs from theirs. This discrepancy comes from difference of type of instabilities. The violation of BF bound ($m^2 < -1/4$) is considered as a condition for an exponential grow of the perturbations. This was explicitly shown for scalar field perturbations [16]. On the other hand, our condition $\lambda_s = 0$ gives a power law grow of the perturbations, which is more modest than the exponential one. For $(d \geq 6)$-dimensional extreme Myers–Perry black holes with equal angular momenta, it was shown that the BF bound in the near horizon geometries is violated for gravitational perturbations [16, 22]. In fact, in the case of odd-dimensions, such an instability has been found in the full geometry near the extremality [24]. So, the power law instability found in this paper may not be important for these spacetimes. However, for five-dimensional extreme Myers–Perry black holes with equal angular momenta, there is no violation of the BF bound in the near horizon geometries. In addition to that, from the study of perturbations of full geometries, strong evidence of stability

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3 Note that we need to shift the eigenvalues when we use the results in [16, 21, 22] for gravitational perturbations since the operator $A_s$ relates to $O^{(2)}$ defined by them as $O^{(2)} = A_s + 2$ for axisymmetric perturbations.
for non-extreme black holes has been found in [25]. Thus, the power law instability found in this paper can be important for this spacetime.

In this paper, we considered only vacuum black holes. Thus, our instability condition does not apply to gravitational and electromagnetic perturbations of RN or Kerr–Newman black holes. Further work needs to be done to study their stability in extreme limit. They are solutions of N = 2 supergravity and extreme RN black holes are supersymmetric. For RN black holes, it is known that the perturbation equations are decoupled. Using the decoupled equations, we may be able to find conserved quantities on the horizons and show the instability [26]. For Kerr–Newman black holes, the decoupling of the perturbation equations has not been succeeded. Hence, even for the non-extreme case, their stability has not been studied. As we did in this paper, however, if the coupling terms in the perturbation equations are sufficiently small near the horizons, we can study the instability. It would be interesting to estimate the order of the coupling terms and study the instability of extreme Kerr–Newman black hole. It would make good progress in the understanding of the stability of Kerr–Newman black holes.

One of the most interesting problems on the instability is its final state. We need to solve the time evolution of the instability taking into account the backreaction to specify the final state. For scalar field perturbations of RN black holes, we can find an instability for spherically symmetric modes [4]. Thus, the evolution equations of the instability are given by (1 + 1)-dimensional partial differential equations even if we consider the backreaction. Solving the PDEs and finding the final state would be another direction for future research.

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Appendix A. GHP variables for extreme black holes

We take the null basis as in equation (3.6). The dual one-forms for these vectors are written as
\[ e^0 = e_1 = \frac{L}{2} dv, \quad e^1 = e_0 = -r^2 LF dv + 2L dr, \quad e^i = e^i - rh^i dv. \] (A.1)

Using the Cartan equations \( de_a + \omega_{abc} e^c \wedge e^b = 0 \), we can calculate the spin connections \( \omega_{abc} \). From the definition of \( L_{ab} \) in equation (2.2), we have \( L_{ab} = -\omega_{0ab} \). Thus, GHP variables for the background spacetime are given as
\[
\rho_{ij} = \frac{2r}{L} \delta_{ij} \hat{\epsilon}^{a b \gamma} \hat{e}_{\gamma a} + \frac{2r}{L} h^i j (\hat{\epsilon}_{a b \gamma})_i, \quad \kappa_i = \frac{2r^2 F_j}{L}, \quad \tau_i = -\frac{L_i}{L} + \frac{(r h^a)^{\gamma} \hat{e}_{\gamma a}}{2L^2},
\]
\[
\rho'_{ij} = \frac{1}{2L} (\hat{e}_{a b \gamma})_i \hat{e}^{a b \gamma}_j, \quad \kappa'_i = 0, \quad \tau'_i = -\frac{L_i}{L} - \frac{(r h^a)^{\gamma} \hat{e}_{\gamma a}}{2L^2},
\] (A.2)

and
\[
L_{10} = (r^2 F)^{\gamma} \frac{L}{L^2} + 2 r h^a i j \hat{\epsilon}_{a b \gamma} \frac{L}{L^2}, \quad L_{11} = 0, \quad L_{ij} = \frac{(r h^a)^{\gamma} \hat{e}_{\gamma a}}{2L^2},
\]
\[
M'_{j0} = \frac{2r}{L} h^a i j \hat{\epsilon}_{a b \gamma} \frac{L}{L^2} + \frac{2r}{L} h^b \frac{L}{L^2} \hat{\epsilon}_{a b \gamma} \frac{L}{L^2} \hat{\epsilon}_{a b \gamma}, \quad M'_{i1} = -\hat{\omega}_{i1},
\] (A.3)

where \( \hat{\omega}_{i j} \) is defined by \( \hat{d} \hat{\epsilon}_{i} + \hat{\omega}_{i j} \hat{e}^i \wedge \hat{e}^j = 0. \)
Appendix B. Derivation of near horizon equations

Here, we derive near horizon equations (4.1), (4.6), (5.1), (5.5), (6.6) and (6.11). The equations for scalar, electromagnetic and gravitational perturbations are written in a unified form as

\[(2\dot{\psi} + \rho' \dot{\psi})\psi_s + B_\epsilon \psi_s = 0, \tag{B.1}\]

where \(\psi_0 = \phi, \psi_1 = \psi_i \) and \(\psi_2 = \tilde{\Omega}_{ij}\). The angular operators \(B_\epsilon\) are defined by

\[B_\epsilon \psi_0 = (\tilde{\delta}_i \tilde{\delta}_j - 2 \tau_i \tilde{\delta}_j + \rho \dot{\psi} - \mu^2)\phi, \tag{B.2}\]

\[B_1 \psi_1 = \left( \tilde{\delta}_i \tilde{\delta}_j - 4 \tau_i \tilde{\delta}_j + \Phi - \frac{2d - 3}{d - 1} \right) \psi_i + (-2 \tau_i \tilde{\delta}_j + 2 \tau_j \tilde{\delta}_i + 2 \Phi^5 + 4 \Phi^4 \tilde{\Omega}_{ij}) \psi_j, \tag{B.3}\]

\[B_2 \psi_2 = \left( \tilde{\delta}_i \tilde{\delta}_j - 6 \tau_i \tilde{\delta}_j + 4 \Phi - \frac{2d - 3}{d - 1} \right) \tilde{\Omega}_{ij} + 4(\tau_i \tilde{\delta}_j - \tau_j \tilde{\delta}_i + \Phi^5 + 4 \Phi^4 \tilde{\Omega}_{ij}) \tilde{\Omega}_{ij} + 2 \Phi^4 \tilde{\Omega}_{ij}. \tag{B.4}\]

In \(B_\epsilon\), there is no the radial derivative \(\partial_r\). Up to second order in \(r\), the GHP derivative \(\dot{\psi}\) can be written as

\[\dot{\psi}_s = \frac{2}{L} (\partial_r + r(ik m_l - b)) \psi_s + \frac{r^2}{L} \partial_r \psi_s + r^2 C \psi_s + \mathcal{O}(r^3), \tag{B.5}\]

where \(C\) is a operator in which the radial derivative \(\partial_r\) is not contained. From equation (3.13), we obtain \([\dot{\psi}, r] = 1/(2L)\). Thus, we have

\[(2\dot{\psi} + \rho' \dot{\psi})\psi_s \equiv \partial_r \left( \frac{2}{L} (2\dot{\psi} + \rho' \dot{\psi}) \right) + \frac{2}{L} \left( (ik m_l - b) \psi_s + B_s \right) + C' \psi_s + \mathcal{O}(r^2), \tag{B.6}\]

where \(C' = C + \rho'(ik m_l - b)\). We expand the operator \(B_s\) as \(B_s = B_s^H + rB_s^I + \mathcal{O}(r^2)\). Then, equation (B.1) is written as

\[\partial_r \left[ \frac{2}{L} (2\dot{\psi} + \rho' \dot{\psi}) \right] + \frac{2}{L} (ik m_l - b) \psi_s + B_s^H \psi_s + \frac{2}{L} \left( (ik m_l - b) B_s^I + \frac{1}{L} \partial_r + C'' \right) \psi_s = \mathcal{O}(r^2), \tag{B.7}\]

where \(C'' = C' + LB_s^I/2\). Thus, from equation (B.7), we obtain

\[\partial_r [2L(2\dot{\psi} + \rho' \dot{\psi})] = A_s \psi_s + \mathcal{O}(r), \tag{B.8}\]

where

\[A_s = -2(ik m_l - b) - L^2 B_s^H. \tag{B.9}\]

Equation (B.8) expresses equations (4.1), (5.1) and (6.6). Differentiating equation (B.7) by \(r\), we have

\[\partial_r [2L \partial_r (2\dot{\psi} + \rho' \dot{\psi})] = [A_s + 2(b - 1) - 2ik m_l] \partial_r \psi_s + C''' \psi_s + \mathcal{O}(r), \tag{B.10}\]

where \(C''' = -2L[C'' + 2(ik m_l - b)(\dot{\psi} - \partial_r/(2L))]\). (Note that there is no radial derivative in the operator \(\dot{\psi} - \partial_r/(2L)\).) Setting \(m_l = 0\) in above equation, we obtain equations (4.6), (5.5) and (6.11). The explicit expressions of \(A_s\) can be obtained using near horizon expressions of GHP variables and derivatives (3.9), (3.12) and (3.13).
Appendix C. Useful GHP equations

We summarize the useful GHP equations for Einstein spacetime satisfying $R_{\mu\nu} = \Lambda g_{\mu\nu}$. These equations are firstly derived in [8].

C.1. Newman–Penrose equations

From the Ricci equations, $[\nabla_\mu, \nabla_\nu]V_\rho = R_{\mu\nu\rho\sigma}V^\sigma$, we obtain the following equations:

$$\mathfrak{B}\rho_{ij} - \delta_j k = -\rho_{ik}\rho_{kj} - k_i\tau^l_j - \tau_j k_i - \Omega_{ij},$$  \hspace{1cm} (NP1)

$$\mathfrak{B}\tau_i - \mathfrak{B}\tau_i = \rho_{ij}(-\tau_j + \tau^l_j) - \Psi_{ij},$$  \hspace{1cm} (NP2)

$$2\partial_{[ij]}\rho_{[kl]} = 2\tau_{[ij]}\rho_{[kl]} + 2\kappa_{[ij]}\rho'_{[kl]} - \Psi_{[ij]},$$  \hspace{1cm} (NP3)

$$\mathfrak{B}'\rho_{ij} - \delta_j \tau_i = -\tau_i \tau_j - k_i k_j - \rho_{ik}\rho'_{kj} - \Phi_{ij} - \frac{\Lambda}{d-1} \delta_{ij}.$$  \hspace{1cm} (NP4)

Another four equations can be obtained by applying the prime $'$ of these four.

C.2. Bianchi equations

From Bianchi equations, $\nabla_{[\mu}C_{\nu\sigma\rho]} = 0$, we obtain following equations.

Boost weight +2:

$$\mathfrak{B}\psi_{ij} - 2\partial_{ij}\Omega_{kl} = (2\Phi_{ij}\delta_{kl} - 2\delta_{ij}\Phi_{[kl]} - \Phi_{[ik]}k_l)k_j - 2(\psi_{ij}\delta_{kl} + \psi_{ij}\delta_{ik} + \psi_{ik}\delta_{ij})\rho_{kl} + 2\Omega_{ij}\tau_{kl}.$$  \hspace{1cm} (B1)

Boost weight +1:

$$\mathfrak{B}\Phi_{ij} - \delta_j \psi_i + \mathfrak{B}\Omega_{ij} = -(\psi_{ij}\delta_{kl} - \psi_{ij}\delta_{ik})k_l + (\Phi_{ik} + 2\Phi_{ik}' + \Phi_{ik})\rho_{ij} + \psi_{ij}\delta_{ik} + \psi_{ij}\delta_{ik}\tau_l - \Omega_{ij}\tau_{ik}.$$  \hspace{1cm} (B2)

$$\mathfrak{B}\Phi_{mjkl} + 2\partial_{[im]}\psi_{[jkl]} = -2\psi_{[ij]}\psi_{[kl]} - 2\psi_{[ij]}\psi_{[kl]} + 4\Phi_{ij}\rho_{jk} + 2\Phi_{ijkl}\rho_{[jk]} + 2\Phi_{ijkl}\rho_{[jk]} + 2\Omega_{ijkl}\tau_{[jk]}.$$  \hspace{1cm} (B3)

$$\partial_{[im]}\psi_{[jkl]} = 2\Phi_{ijkl}\rho_{[jk]} - 2\Phi_{ijkl}\rho_{[jk]} + \Phi_{im[jk]}\rho_{[il]} - 2\Omega_{ijkl}\rho_{[jk]}.$$  \hspace{1cm} (B4)

Boost weight 0:

$$\mathfrak{B}\psi_{ij} - 2\partial_{ij}\Phi_{[kl]} = 2(\psi_{ij}\delta_{kl} - \psi_{ij}\delta_{kl})\rho_{[kl]} + (2\Phi_{ij}\delta_{kl} - 2\delta_{ij}\Phi_{[kl]} - \Phi_{[ik]}k_l)\tau_j + 2(\psi_{ij}\delta_{kl} - \psi_{ij}\delta_{kl})\rho_{[kl]} + 2\Omega_{ijkl}\kappa_{kl}.$$  \hspace{1cm} (B5)

$$-2\partial_{[im]}\Phi_{ijkl} = 2\psi_{ijkl}\rho_{[lj]} - 2\psi_{ijkl}\rho_{[lj]} - \psi_{ijkl}\rho_{[lj]} - \psi_{ijkl}\rho_{[lj]}.$$  \hspace{1cm} (B6)

$$\partial_{[im]}\Phi_{[jkl]} = -2\psi_{ijkl}\rho_{[jm]} + \psi_{ijkl}\rho_{[jm]} - 2\psi_{ijkl}\rho_{[jm]} - \psi_{ijkl}\rho_{[jm]} + \psi_{ijkl}\rho_{[jm]} - 2\psi_{ijkl}\rho_{[jm]}.$$  \hspace{1cm} (B7)

Another five equations are obtained by applying the prime operator to above equations.
C.3. Maxwell equations

From Maxwell equations, \( d^\ast F = d \ast F = 0 \), we obtain the following equations:

\[
\partial_i \psi_i + \partial^\ast F = \tau'_j \psi_j + \rho_j F_{ij} - \rho F - \kappa_i \psi'_j \tag{M1}
\]

\[
2\partial_i \psi_j - \partial^\ast F_{ij} = 2\tau'_j \psi_j + 2F \rho_{ij} + 2F_{i[k} \rho_{j]l} + 2\kappa_i \psi'_j \tag{M2}
\]

\[
2\psi_j + \partial_j F_{ij} - \partial^\ast F = (2\rho'_{ij} - \rho' \delta_{ij}) \psi_j - 2F_{ij} \tau_j - 2F \tau_i + (2\rho_{ij} - \rho \delta_{ij}) \psi'_j \tag{M3}
\]

\[
\partial_j F_{jk} = \psi_i \rho'_{jk} + \psi'_i \rho_{jk} \tag{M4}
\]

A further three equations can be obtained by priming above equations.

C.4. Commutators of derivatives

The commutation relations for GHP derivatives are given by

\[
\left[ b, b' \right] T_{ij...ik} = (-\tau_j + \tau'_j) \partial_j T_{ij...ik} + b(\tau_j \psi_j' + \kappa_i \kappa'_j + \Phi) T_{ij...ik} + \sum_{r=1}^{s} (\kappa_i \kappa'_j - \kappa'_i \kappa_j + \tau'_j \tau_j - \tau_j \tau'_j + 2\Phi'_{ij}) T_{ij...ik}, \tag{C1}
\]

\[
\left[ b, \partial_j \right] T_{ij...ik} = -(\kappa_i b' + \tau_j b + \rho_j \partial_j) T_{ij...ik} + b(-\tau_j \psi_j' + \kappa_i \kappa'_j + \Psi_i) T_{ij...ik} + \sum_{r=1}^{s} (\kappa_i \rho'_{ij} - \rho_{ij} \tau'_j + \tau'_j \rho_{ij} - \rho'_{ij} \kappa_i - \Psi_{jk} \partial_j) T_{ij...ik}, \tag{C2}
\]

\[
\left[ \partial_j, \partial_l \right] T_{i...jl} = (2\rho'_{ij} b' + 2b' \rho'_{ij}) T_{i...jl} + b(2\rho_{ij} \rho'_{ij} + 2\Phi'_{ij}) T_{i...jl} + \sum_{r=1}^{s} \left( 2\rho'_{ij} \rho_{ij} + 2\rho'_{ij} \rho_{ij} \Phi_{ijkl} + \frac{2\Lambda}{d-1} \delta_{ijkl} \delta_{ijl} \right) T_{i...jl}. \tag{C3}
\]

The result for \([b', \partial_j] \) can be obtained from the prime operation of \([b, \partial_j] \).

References

[1] Strominger A and Vafa C 1996 Phys. Lett. B 379 99 (arXiv:hep-th/9601029)
[2] Guica M, Hartman T, Song W and Strominger A 2009 Phys. Rev. D 80 124008 (arXiv:0809.4266 [hep-th])
[3] Aretakis S 2011 Commun. Math. Phys. 307 17 (arXiv:1110.2007 [gr-qc])
[4] Aretakis S 2011 Ann. Henri Poincare 12 1491 (arXiv:1110.2009 [gr-qc])
[5] Aretakis S 2012 J. Funct. Anal. 263 2770 (arXiv:1110.2006 [gr-qc])
[6] Aretakis S 2012 arXiv:1206.6598 [gr-qc]
[7] Lucietti J and Reall H S 2012 arXiv:1208.1437 [gr-qc]
[8] Durkee M, Pravda V, Pravdova A and Reall H S 2010 Class. Quantum Grav. 27 215010 (arXiv:1002.4826 [gr-qc])
[9] Durkee M and Reall H S 2011 Class. Quantum Grav. 28 035011 (arXiv:1009.0015 [gr-qc])
[10] Reall H S 2003 Phys. Rev. D 68 024024 (arXiv:hep-th/0211290)
[11] Reall H S 2004 Phys. Rev. D 70 089902 (erratum)
[12] Kunduri H K, Lucietti J and Reall H S 2007 Class. Quantum Grav. 24 4169 (arXiv:0705.4214 [hep-th])
[13] Figuera P, Kunduri H K, Lucietti J and Rangamani M 2008 Phys. Rev. D 78 044042 (arXiv:0803.2998 [hep-th])
[14] Kunduri H K and Lucietti J 2009 J. Math. Phys. 50 082502 (arXiv:0806.2051 [hep-th])
[15] Chow D D K, Cvetic M, Lu H and Pope C N 2009 Phys. Rev. D 79 084018 (arXiv:0812.2918 [hep-th])
[16] Durkee M and Reall H S 2011 Phys. Rev. D 83 104044 (arXiv:1012.4805 [hep-th])
[17] Chen W, Lu H and Pope C N 2006 Class. Quantum Grav. 23 5323 (arXiv:hep-th/0604125)
[18] Hamamoto N, Houri T, Oota T and Yasai Y 2007 J. Phys. A: Math. Theor. 40 F177 (arXiv:hep-th/0611285)
[19] Dias O J C, Reall H S and Santos J E 2009 J. High Energy Phys. JHEP08(2009)101 (arXiv:0906.2380 [hep-th])
[20] Amsel A J, Horowitz G T, Marolf D and Roberts M M 2009 J. High Energy Phys. JHEP09(2009)044 (arXiv:0906.2376 [hep-th])
[21] Murata K 2011 Conformal weights in the Kerr/CFT correspondence J. High Energy Phys. JHEP05(2011)117 (arXiv:1103.5635 [hep-th])
[22] Tanahashi N and Murata K 2012 arXiv:1208.0981 [hep-th]
[23] Pravda V, Pravdova A and Ortaggio M 2007 Class. Quantum Grav. 24 4407 (arXiv:0704.0435 [gr-qc])
[24] Dias O J C, Figueras P, Monteiro R, Reall H S and Santos J E 2010 J. High Energy Phys. JHEP05(2010)076 (arXiv:1001.4527[hep-th])
[25] Murata K and Soda J 2008 Prog. Theor. Phys. 120 561 (arXiv:0803.1371 [hep-th])
[26] Lucietti J, Murata K, Reall H S and Tanahashi N 2012 arXiv:1212.2557 [gr-qc]