Theory for Photon-Assisted Macroscopic Quantum Tunneling in a Stack of Intrinsic Josephson Junctions

Masahiko Machida\textsuperscript{a,c} and Tomio Koyama\textsuperscript{b,c}
\textsuperscript{a}CCSE, Japan Atomic Energy Agency, 6-9-3 Higashi-Ueno, Taito-ku Tokyo 110-0015, Japan
\textsuperscript{b}IMR, Tohoku University, 2-1-1 Katahira Aoba-ku, Sendai 980-8577, Japan
\textsuperscript{c}CREST(JST), 4-1-8 Honcho, Kawaguchi, Saitama 332-0012, Japan

We propose a theory for photon-assisted macroscopic quantum tunneling (MQT) in a stack of capacitively-coupled intrinsic Josephson junctions (IJJs) in which the longitudinal Josephson plasma, i.e., longitudinal collective phase oscillation modes, is excited. The scheme of energy-level quantization in the collective oscillatory states is clarified in the N-junction system. When the MQT occurs from the single-plasmon states excited by microwave irradiation in the multi-photon process to the uniform voltage state, our theory predicts that the escape rate is proportional to $N^2$. This result is consistent with the recent observation in Bi-2212 intrinsic Josephson junctions.

PACS numbers: 74.50.+r, 74.81.Fa, 74.72.Hs, 74.25.Nf

Since the discovery of macroscopic quantum tunneling (MQT) in YBCO grain boundary junctions\textsuperscript{1} and Bi-2212 intrinsic Josephson junctions (IJJs)\textsuperscript{2}, a renewed interest has been aroused on the MQT in the Josephson effect from viewpoints of both basic physics and quantum-device applications. In Bi-2212 IJJs the observed crossover temperature at which MQT appears is about 1K\textsuperscript{2}, which is 1-order higher than that in conventional single-junction systems\textsuperscript{2} and\textsuperscript{4}, and, furthermore, the dissipation effect that prevents a quantum tunneling is very weak in spite that the order parameter of the high-$T_e$ cuprates has the d-wave symmetry with nodes. This is because the Josephson plasma frequency in IJJs is far higher compared with artificially-made S/I/S Josephson junctions\textsuperscript{5} and the transfer-integral along the $c$-direction vanishes for the nodal quasi-particles. These remarkable features in IJJs give great promise for quantum-device applications. In addition, MQT in IJJs is expected to provide a new physics originating from the atomic-scale multi-junction structure.

Recently, Jin et al. observed MQT in the switching events to the uniform voltage state in Bi-2212 IJJs where all the junctions are switched into the voltage state\textsuperscript{6}. The switching rate observed in the multi-photon process is greatly enhanced, depending on the number of stacked junctions, $N$. The observed enhancement in the collective switching is proportional to $N^2$, which suggests that collective motion of the phase differences in IJJs is responsible for the quantum tunneling. In IJJs the longitudinal Josephson plasma is known to exist as the collective motion of the phase differences that propagates in the stacking direction of the junctions\textsuperscript{7}. In this paper we formulate a theory for MQT of the collective longitudinal plasma modes. The scheme of quantum energy levels for the collective motion of the phase differences in capacitively-coupled $N$ intrinsic junctions is clarified. Our theory predicts that the switching rate in the multi-photon process is proportional to $N^2$.

Consider a stack of intrinsic Josephson junctions having tiny in-plane area of $W \sim 1 \mu m^2$. In the absence of an external magnetic field the phase of the order parameter $\varphi_\ell$, $\ell$ being the layer index, can be considered uniform along the in-plane direction. In this case the coupling between junctions in equilibrium state is brought about by the long-range Coulomb interaction between charges induced in the superconducting layers with an atomic-scale layer thickness as discussed in\textsuperscript{8}, that is, the coupling originates from the incomplete charge screening between junctions. The charge and phase dynamics in this case is well described by the Lagrangian\textsuperscript{8},

$$\mathcal{L} = W \sum_{\ell=1}^{s} \left\{ \frac{s}{8\pi \mu^2} \left( A_\ell^0 + \frac{\hbar}{e^*} \varphi_\ell \right)^2 + \frac{e d}{8\pi} E_{\ell,\ell-1}^2 - \frac{E_j}{2} \left[ 1 - \cos(\varphi_\ell - \varphi_{\ell-1}) \right] \right\}, \quad (1)$$

where $A_\ell^0$ is the scalar potential at the $\ell$th superconducting layer, $E_{\ell,\ell-1}$ is the electric field inside the insulating layer with the dielectric constant $\epsilon$ between $\ell$th and $(\ell-1)$th superconducting layers, and $s$ and $d$ are, respectively, the thicknesses of the superconducting and insulating layers. The Josephson coupling energy in Eq.(1) is defined as $E_j = \hbar j_c / |e^*|$, $j_c$ being the Josephson critical current density. The first term including $A_\ell^0$ in Eq.(1) corresponds to the charging energy which leads to the finite charge compressibility\textsuperscript{8}. To see this we note that the canonical momentum which is conjugate to $\varphi_\ell$ is given as

$$p_\ell = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_\ell} = \frac{\hbar}{e^*} \cdot \frac{s}{4\pi \mu^2} \left( A_\ell^0 + \frac{\hbar}{e^*} \varphi_\ell \right) = \hbar s n_\ell. \quad (2)$$

where $n_\ell$ is understood to be the density fluctuation of the Cooper-pairs in the $\ell$th layer. Then, the first term in Eq.(1) increases the free energy by the amount of $\propto n_\ell^2$, which leads to the finite charge compressibility discussed by van der Marel and Tsvetkov\textsuperscript{9}. The effect of the charge compressibility has been observed in optical properties of high-$T_e$ superconductors\textsuperscript{10,11}.
The Lagrangian (1) yields the Hamiltonian of the form\cite{12},
\[
\mathcal{H} = \frac{1}{2} \sum_{\ell \neq m} C_{\ell m}^{-1} Q_\ell Q_m + E_J \sum_\ell \left[ 1 - \cos(\varphi_\ell - \varphi_{\ell-1}) \right],
\] (3)
where $Q_\ell$ is the total charge of the $\ell$th layer, i.e., $Q_\ell = e^* W s\ell$, and $C_{\ell m}^{-1}$ is the inverse capacitance matrix. Note that the condition $d^2 \ll W$ is fulfilled even in the intrinsic Josephson junctions with a small in-plane area of $W \sim 1\mu m^2$, since $d \simeq 1.2 nm$. From this fact one understands that the electric field generated in the junctions is well confined inside the stack of the junctions and its direction is perpendicular to the junctions, and as a result, the 1D Coulomb potential is realized almost completely. Then, the inverse mutual capacitance diverges linearly as\cite{10}

\[
C_{\ell m}^{-1} \rightarrow \frac{4\pi d}{\varepsilon W} \ell, \quad \text{for } \ell \gg m.
\] (4)

This result indicates that the long-range nature of the capacitative coupling between junctions should be correctly incorporated, that is, any truncation of the inverse capacitance matrix in Eq.(3) cannot be accepted in the intrinsic Josephson junctions. As shown in\cite{12}, the Hamiltonian (3) can be transformed into more tractable form by the canonical transformation. We choose $(\theta_\ell, u_\ell)$ defined as

\[
\theta_\ell = \varphi_\ell - \varphi_{\ell-1}, \quad u_\ell = \sum_{n=1}^\infty p_n,
\] (5)
as the canonical variables instead of $(\varphi_\ell, p_\ell)$. It is easy to see that the canonical commutation relation $[\theta_\ell, u_\ell] = i\hbar \delta_\ell m$ holds if $[\varphi_\ell, p_n] = i\hbar \delta_\ell n$. The Hamiltonian (3) is greatly simplified if one uses the canonical variables $(\theta_\ell, u_\ell)$ as\cite{12}

\[
\mathcal{H} = \sum_{\ell=1}^\infty \left\{ E_e \left[ (1 + 2\alpha) u_\ell^2 - 2\alpha u_\ell u_{\ell+1} \right] + E_J \left[ 1 - \cos \theta_\ell \right] \right\},
\] (6)

with $E_e = 2\pi e^2 d^2/Wc$ and $\alpha = \epsilon_p^2/sd$. Note that the Josephson coupling term is diagonal and the coupling between junctions survives only for nearest neighbors. The coupling constant $\alpha$ is identical with the one introduced in the capacitively-coupled classical intrinsic Josephson junctions\cite{13,14}. In Bi-2212 $\alpha$ takes a value of $\sim 0.1 - 0.2$, depending on the doping level\cite{13,14,15}. Let us discuss the small quantum phase oscillations in the $N$-junction system. We consider the case of $N \gg 1$ and neglect the boundary effect in the following calculations. For small phase oscillations one can use the approximation,

\[
1 - \cos \theta_\ell \simeq \frac{1}{2} \theta_\ell^2.
\]

Using the periodic boundary condition, we express $u_\ell$ and $\theta_\ell$ in terms of the Fourier series expansions as

\[
u_\ell = N^{-1/2} \sum_n \hat{u}_n \exp(i2\pi n\ell/N),
\theta_\ell = N^{-1/2} \sum_n \hat{\theta}_n \exp(i2\pi n\ell/N),
\]

where the Fourier components $(\hat{u}_n, \hat{\theta}_n)$ satisfy the commutation relation,$(\hat{\theta}_n, \hat{u}_{n+m}) = i\hbar \delta_{nm}$. The Hamiltonian (6) in the harmonic approximation can be expressed in terms of the Fourier components as

\[
\mathcal{H} = \sum_{n=-n_e}^{n_c} \left\{ \frac{E_e}{\hbar^2} \left[ 1 + 2\alpha \left( 1 - \cos \frac{2\pi n}{N} \right) \right] \hat{u}_n \hat{u}_{-n} + \frac{1}{2} E_J \hat{\theta}_n \hat{\theta}_{-n} \right\},
\] (7)

where $n_c$ is given as $N = 2n_c + 1$. Then, in terms of the creation and annihilation operators $c_n$ and $c_n^\dagger$ defined as $\hat{\theta}_n = (c_n + c_n^\dagger)/(\sqrt{2}K_n)$ and $\hat{u}_n = -i\hbar K_n(c_n - c_n^\dagger)/\sqrt{2}$ with $K_n = (E_J/\omega_n)^{1/2}$, one can diagonalize the Hamiltonian as

\[
\mathcal{H} = \hbar \sum_{n=-n_e}^{n_c} \omega_n (c_n^\dagger c_n + \frac{1}{2}),
\] (8)

with $\omega_n$ being the Josephson-plasma frequency defined as $\omega_{pl} = \sqrt{2E_e E_J}$. This phase oscillation mode is identified with the quantum version of the longitudinal Josephson plasma. The dispersion relation given in Eq.(9) is the same as the classical one\cite{16,18}. The longitudinal Josephson plasma has been observed in the microwave absorption experiments in Bi-2212\cite{17,18}. Its frequency $\omega_{pl}$ is located in the range of a few hundreds GHz at $T = 0K$\cite{15}. As seen from Eq.(9), we have $N$ plasma modes in the $N$-junction system, which is crucially different from the single-junction system, and their frequencies range from $\omega_n = \omega_{pl}$ up to $\omega_n = \pm N/2 = \sqrt{1 + 4\alpha} \omega_{pl}$. In the case of $\alpha \sim 0.1$ (under-doped Bi-2212 case) the relation, $\sqrt{1 + 4\alpha} \omega_{pl} - \omega_{pl} \ll \omega_{pl}$, holds. Then, the lowest single-plasmon state is well above the highest zero-plasmon state (see Fig.1). From this observation one expects the energy scheme as depicted schematically in Fig.1 for the capacitively-coupled Bi-2212 intrinsic Josephson junctions in the quantum regime, if the harmonic approximation for the phase oscillation modes is assumed to be valid up to the single-plasmon state. In this figure we also describe the washboard-like effective potential which confines the collective phase oscillation modes. The tilt of the potential can be generated by a bias current. Note that in the presence of a bias current $I$ the plasma frequency is shifted as $\omega_{pl} \rightarrow \omega_{pl}(I) = \omega_{pl}(1 - I^2)^{1/4}$ with $I = I/J_c$ by the effect of the coupling term $-E_J I \sum \theta_\ell$ if $\omega_{pl}/2\pi \sim 100GHZ$, we have the level spacings $\Delta \omega_{pl} \sim 4.8K$ between zero-plasmon and single-plasmon states and $\Delta \omega_{n\pm 1} - \Delta \omega_{n\pm 2} \sim 0.23K$ between adjacent plasma modes for $\alpha = 0.1$ and $N = 50$. From this estimation the quantum effect originating from the discreteness of the $n$-plasmon states may be expected at $T \sim 1K$. The level splitting between the plasma modes can be seen at temperatures below $\sim 100mK$. 

...
In the real systems composed of $N$ junctions this conservation law is understood to hold approximately. One expects that this anharmonic interaction term becomes important for the excited states, i.e., the single-plasmon states, as the bias current is increased.

Let us investigate the excitations of the plasma mode by microwave irradiation, that is, the transition from the ground state $|0\rangle$ to $k$th single-plasmon state $|\omega_k\rangle$ (see Fig. 2). We assume that the coupling between microwave and the phase differences is generated by the oscillating current induced by the microwave. Then, the interaction with the microwave of frequency $\nu$ is described by the term as

$$V_M(t) = E_J \tilde{I}_M \sin \nu t \sum_{\ell} \theta_\ell \sqrt{N} g_M (c_0 + c_0^\dagger) \sin \nu t,$$  \hspace{1cm} (11)

where $g_M = E_J \tilde{I}_M / (2 \sqrt{2} K_0)$, $\tilde{I}_M$ being the intensity of the normalized oscillating current. From Eq. (11) one understands that the microwave can excite only the uniform plasma mode of $n = 0$, $|\omega_0\rangle$, in the one-photon process. The transition rate $1/\tau(1)$ in this process is easily derived as

$$\frac{1}{\tau(1)} = N \pi g_M^2 \delta(\omega_0 (\tilde{I}) - \nu).$$ \hspace{1cm} (12)

On the other hand, in the multi-photon processes in which the mode-mode coupling is incorporated, the plasma modes with $k \neq 0$ can be excited as shown in the following. Note that Eq. (10) includes the interaction term,

$$V_3^k = -\frac{f_k}{\sqrt{N}} c_k c_m c_m^\dagger,$$ \hspace{1cm} (13)
where $f_k = \gamma E_f \tilde{g}(8K_k K_{m_k})^{-1/2}$ and $m_k$ is the index satisfying the relation $k + 2m_k = 0, \pm N$. Let us study the third-order perturbative processes caused by the interaction $\mathcal{H}_{\text{int}} = V_{3M}(t) + V_{3d}^1 + V_{3d}^2$. The two-photon process is included in the third-order perturbation and the transition, $|0\rangle \rightarrow |\omega_k\rangle$, is possible in the presence of $V_{3d}^2$ as seen in Fig. 2. Note that $V_{3M}(T) \propto \sqrt{N}$ and $V_{3d}^1 \propto (\sqrt{N})^{-1}$. Hence, one understands that the transition matrix element $\langle f | i | \omega_k | U^{(3)}(t, -\infty) | 0 \rangle$ in the two-photon process is proportional to $\sqrt{N}$, whereas $U^{(3)}(t, -\infty)$ is the third-order time-evolution operator given by

$$U^{(3)}(t, -\infty) = \left(-i\frac{\Gamma}{\hbar}\right)^3 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t_2} dt_3 \ e^{i(t_1+t_2+t_3)} \mathcal{H}_{\text{int}}(t_1) \mathcal{H}_{\text{int}}(t_2) \mathcal{H}_{\text{int}}(t_3) \Big|_{t \rightarrow 0},$$

and thus the transition rate, $\tau_k^{-1} = \frac{4\pi}{\hbar} \frac{\pi \langle g_M^2 f \rangle^2}{(\omega_\mu(\tilde{I}) - \nu)(\omega_k + \omega_\mu(\tilde{I}) - \nu)} \delta(\omega_k - 2\nu).$ (15)

Then, if the MQT rate of the single-plasmon state $|\omega_k\rangle$ to the voltage state is given by $\Gamma_k$, the photon-assisted switching rate is obtained as $\tau_k^{-1} \Gamma_k$. In the MQT at temperatures around $\sim 1\text{K}$ the splitting between the plasma modes is smeared out for the systems with $N \sim 50$. Then, we find the total switching rate in the two-photon process $\nu \sim \omega_{pl}/2$ as

$$\Gamma^{(2)} = \sum_{k=-n_c}^{n_c} \frac{1}{\Gamma_k} \propto N^2 \frac{\pi \langle g_M^2 f \rangle^2}{\omega_\mu(\tilde{I})^2} \Gamma_0,$$ (16)

which is proportional to $N^2$. The $N^2$-dependence of the switching rate is also obtained in the third-photon process. Note that the transition, $|0\rangle \rightarrow |\omega_k\rangle$, in the three-photon process is possible in the fifth-order perturbative process with respect to $\mathcal{H}_{\text{int}}$. Then, from Eqs.(10) and (11) we expect the relation, $\langle f | i | \omega_k | U^{(3)}(t, -\infty) | 0 \rangle \propto (\sqrt{N})^3 \times (\sqrt{N})^{-2} = \sqrt{N}$, for the three-photon process in the fifth-order perturbation, which is the same as in the two-photon process. This observation indicates that the total switching rate is also proportional to $N^2$ in the three-photon process. Thus, one may conclude that the MQT rate to the uniform resistive state is proportional to $N^2$, if the MQT happens collectively in the multi-photon process. This result is consistent with the recent experimental result in Bi-2212 intrinsic Josephson junctions.\[8\]

Note that the $N^2$-dependence does not appear in the switching rate by way of the single-photon process. This is because the single-plasmon state of $k \neq 0$ cannot be available in the single-photon process. Thus, our theory predicts that the switching rate in the single-photon process is proportional to $N$. Furthermore, from the result given in Eqs.(15) and (16) it is also predicted that the frequency spectrum of the switching rate, i.e., $\Gamma^{(2)}(\nu)$, splits into $N/2$ peaks having centers at $\nu = \omega_k/n$ at low temperatures, say $T \sim 10\text{mK}$, in the $n$-photon process if the damping originating from the coupling with the quasiparticles or environment is weak enough.

In summary we have formulated MQT for the collective longitudinal Josephson plasma modes in intrinsic Josephson junctions, assuming that the switching events to the uniform voltage state occur from the single-plasmon states. The scheme of energy-level quantization for the collective oscillation modes has been clarified in the case of Bi-2212 IJJs. In the $N$-junction system the single-plasmon states are composed of $N$ different oscillation modes whose frequencies range from $\omega_{pl}$ to $\sqrt{1 + \frac{3}{2} \omega_{pl}}$. These modes can be excited by the microwave irradiation in the multi-photon process. Then, there are $N$ channels in the MQT of the collective phase oscillation modes in the multi-photon process. We have also shown that the transition rate to each single-plasmon state in terms of microwave irradiation is proportional to $N$. Therefore, the photon-assisted switching rate to the uniform voltage state is proportional to $N^2$ in the multi-photon process around $1\text{K}$.

The authors thank S.Sato, K.Nakajima, K.Inomata, T.Hatano, H.Kitano, and A.Maeda for discussions on experimental results, and A. Tanaka, S. Kawabata, M.Kato, M.Hayashi, H.Ebisawa, and T.Ishida for helpful discussions.

[1] T. Bauch et al., Phys. Rev. Lett. 94, 087003 (2005).
[2] K. Inomata et al., Phys. Rev. Lett. 95, 107005, (2005).
[3] R. F. Voss and R. A. Webb, Phys. Rev. Lett. 47 265 (1981).
[4] M. H. Devoret, J. M. Martinis and J. Clarke, Phys. Rev. Lett. 55 1988 (1985).
[5] S. Kawabata et al., Phys. Rev. B 70 132505 (2004).
[6] X. Y. Jin et al., Phys. Rev. Lett. 96 177003 (2006).
[7] T. Koyama and M. Tachiki, Phys. Rev. B 54 16183 (1996).
[8] M. Machida, T. Koyama and M. Tachiki, Phys. Rev. Lett. 83 4618 (1999).
[9] D. van der Marel and A. A. Tsvelkov, Phys. Rev. B 64 024530 (2001).
[10] D. Dulic et al., Phys. Rev. Lett. 86 4144 (2001).
[11] T. Kakeshita et al., Phys. Rev. Lett. 86 4140 (2001).
[12] T. Koyama, J. Phys. Soc. Jpn. 70 2114 (2001).
[13] Ch. Preice et al, in SPIE Conference Proceedings 3484 "Superconducting Superlattice II" p236 (1998).
[14] M. Machida and T. Koyama, Phys. Rev. B 70 024523 (2004).
[15] Y. Matsuda et al., Phys. Rev. Lett. 75 4512 (1995).
[16] K. Kadowaki, T. Wada and I. Kakeya, Physica C, 362 71 (2001).
[17] M. B. Gaifullin et al., Phys. Rev. Lett. 83 3928 (1999).
[18] S. Flach and C. R. Willis, Phys. Rep. 295 181 (1998).
[19] S. Takeno and Peyard, Physica D 92 140 (1996).