Consensus on Matrix-weighted Time-varying Networks

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Abstract

This paper examines the consensus problem on time-varying matrix-weighted undirected networks. First, we introduce the matrix-weighted integral network for the analysis of such networks. Under mild assumptions on the switching pattern of the time-varying network, necessary and/or sufficient conditions for which average consensus can be achieved are then provided in terms of the null space of matrix-valued Laplacian of the corresponding integral network. In particular, for periodic matrix-weighted time-varying networks, necessary and sufficient conditions for reaching average consensus is obtained from an algebraic perspective. Moreover, we show that if the integral network with period \( T > 0 \) has a positive spanning tree over the time span \([0, T)\), average consensus for the node states is achieved. Simulation results are provided to demonstrate the theoretical analysis.

I. INTRODUCTION

Reaching consensus is an important construct in distributed coordination of multi-agent systems [1], [2], [3], [4]. Although the consensus problem has been extensively investigated in the literature, it has often been assumed that the network has scalar-weighted edges; extensions of the scalar weights to matrix-valued weights has become relevant in order to characterize interdependencies among multi-dimensional states of neighboring agents. Recently, a broader category of networks referred to as matrix-weighted networks has been introduced to address such interdependencies [5], [6]. In fact, matrix-weighted networks arise in scenarios such as graph effective resistance examined in the context of distributed control and estimation [7], [8], logical inter-dependencies amongst topics in opinion evolution [9], [10], bearing-based formation control [11], dynamics of an array of coupled LC oscillators [12], as well as consensus and synchronization on matrix-weighted networks [5], [13], [14].

For matrix-weighted networks, network connectivity does not translates to achieving consensus. To this end, properties of weight matrices play an important role in characterizing consensus. For instance, positive definiteness and positive semi-definiteness of weight matrices have been employed to provide consensus conditions in [5]; negative definiteness and negative semi-definiteness of weight matrices are further introduced in [14], [15]. In the meantime, the notion of network connectivity can be further extended for matrix-valued networks. For instance, one can identify edges with positive/negative definite matrices as “strong” connections; whereas an edge weighted by positive/negative semi-definite matrices can be considered as a “weak” connection [16].

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To the best of our knowledge, conditions under which consensus can be achieved for time-varying matrix-weighted networks have not been developed in the literature; this is in contrast with conditions that have been examined for scalar-weighted networks [17], [18], [19], [20], [21], [22], [23], [24], [25]. In this paper, we provide necessary and/or sufficient conditions for achieving consensus on matrix-weighted time-varying networks. Under mild assumptions on the switching pattern for such networks, necessary and/or sufficient conditions for which average consensus is achieved are provided in terms of the null space of the matrix-valued Laplacian of the associated integral networks. In particular, for periodic matrix-weighted time-varying networks with period \( T > 0 \), a necessary and sufficient condition for average consensus is obtained; we further show that from a graph-theoretic perspective, when the integral network over time span \([0, T]\) has a positive spanning tree, then average consensus is achieved. Simulation results are provided to demonstrate the theoretical analysis.

The remainder of this paper is organized as follows. Preliminaries are introduced in §II. The problem formulation is provided in §III, followed by the consensus conditions in §IV and §V, respectively. A simulation example is presented in §VI followed by concluding remarks in § VII.

II. PRELIMINARIES

Let \( \mathbb{R} \), \( \mathbb{N} \) and \( \mathbb{Z}_+ \) be the set of real numbers, natural numbers and positive integers, respectively. For \( n \in \mathbb{Z}_+ \), denote \( \underline{n} = \{1, 2, \ldots, n\} \). A symmetric matrix \( M \in \mathbb{R}^{n \times n} \) is positive definite, denoted by \( M > 0 \), if \( z^\top M z > 0 \) for all \( z \in \mathbb{R}^n \) and \( z \neq 0 \) and is positive semi-definite, denoted by \( M \succeq 0 \), if \( z^\top M z \geq 0 \) for all \( z \in \mathbb{R}^n \). The null space of a matrix \( M \in \mathbb{R}^{n \times n} \) is denoted by \( \text{null}(M) = \{z \in \mathbb{R}^n | Mz = 0\} \).

Lemma 1. [26] Let \( M \in \mathbb{R}^{n \times n} \) be symmetric with eigenvalues \( \lambda_1 \leq \cdots \leq \lambda_n \). Let \( x_{i_1}, \cdots, x_{i_k} \) be mutually orthonormal vectors such that \( M x_{i_p} = \lambda_{i_p} x_{i_p} \), where \( i_p \in \mathbb{Z}_+ \), \( p \in \underline{k} \) and \( 1 \leq i_1 < \cdots < i_k \leq n \). Then

\[
\lambda_{i_k} = \max_{\{x \neq 0, x \in S_k\}} \frac{x^\top M x}{x^\top x},
\]

and

\[
\lambda_{i_1} = \min_{\{x \neq 0, x \in S_k\}} \frac{x^\top M x}{x^\top x},
\]

where \( S_k = \text{span}\{x_{i_1}, \cdots, x_{i_k}\} \).

III. PROBLEM FORMULATION

Consider a multi-agent system consisting of \( n > 1 \) \((n \in \mathbb{Z}_+) \) agents whose interaction network is characterized by a matrix-weighted time-varying graph \( G(t) = (\mathcal{V}, \mathcal{E}(t), A(t)) \), where \( t \) refers to the time index. The node and edge sets of \( G \) are denoted by \( \mathcal{V} = \{1, 2, \ldots, n\} \) and \( \mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V} \), respectively. The weight on the edge \((i, j) \in \mathcal{E}(t)\) is encoded by the symmetric matrix \( A_{ij}(t) \in \mathbb{R}^{d \times d} \) such that \( A_{ij}(t) \geq 0 \) or \( A_{ij}(t) > 0 \), and \( A_{ii}(t) = 0_{d \times d} \) for \((i, j) \notin \mathcal{E}(t) \). Thereby, the matrix-weighted adjacency matrix \( A(t) = [A_{ij}(t)] \in \mathbb{R}^{dn \times dn} \) is a block matrix such that the block located in its \( i \)-th row and the \( j \)-th column is \( A_{ij}(t) \). It is assumed that \( A_{ij}(t) = A_{ji}(t) \) for all \( i \neq j \in \mathcal{V} \) and \( A_{ii}(t) = 0_{d \times d} \) for all \( i \in \mathcal{V} \).

Denote the state of an agent \( i \in \mathcal{V} \) as \( x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{id}(t)]^\top \in \mathbb{R}^d \) evolving according to the protocol,

\[
\dot{x_i}(t) = - \sum_{j \in \mathcal{N}_i(t)} A_{ij}(t)(x_i(t) - x_j(t)) , \quad i \in \mathcal{V},
\]

(1)

where \( \mathcal{N}_i(t) = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}(t)\} \) denotes the neighbor set of agent \( i \in \mathcal{V} \) at time \( t \). Note that protocol (1) degenerates into the scalar-weighted case when \( A_{ij}(t) = a_{ij}(t)I \), where \( a_{ij}(t) \in \mathbb{R} \) and \( I \) denotes the \( d \times d \) identity matrix.
Let $C(t) = \text{diag} \{ C_1(t), C_2(t), \cdots, C_n(t) \} \in \mathbb{R}^{dn}$ be the matrix-valued degree matrix of $G(t)$, where $C_i(t) = \sum_{j \in N_i} A_{ij}(t) \in \mathbb{R}^{d \times d}$. The matrix-valued Laplacian is subsequently defined as $L(t) = C(t) - A(t)$. The dynamics of the overall multi-agent system now admits the form,

$$\dot{x}(t) = -L(t)x(t),$$

where $x(t) = [x_1^T(t), x_2^T(t), \ldots, x_n^T(t)]^T \in \mathbb{R}^{dn}$.

**Definition 2.** Let $x_f = 1_n \otimes (\frac{1}{n} \sum_{i=1}^n x_i(0))$. Then the multi-agent system (2) admits an average consensus solution if $\lim_{t \to \infty} x_i(t) = \lim_{t \to \infty} x_j(t) = x_f$ for all $i, j \in \mathcal{V}$.

This work aims to investigate the necessary and/or sufficient conditions under which the multi-agent system (2) admits an average consensus solution. It is well-known that network connectivity plays a central role in determining consensus for scalar-weighted networks [18]. However, as we shall show subsequently, definiteness of the weight matrices is also a crucial factor in examining consensus for a matrix-weighted networks in addition to its connectivity. First, we shall recall a few facts on network connectivity. In graph theory, network connectivity captures how a pair of nodes in the network can be “connected” by traversing a sequence of consecutive edges called paths. A path of $G(t)$ is a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \ldots, (i_{p-1}, i_p)$, where nodes $i_1, i_2, \ldots, i_p \in \mathcal{V}$ are distinct; in this case we say that node $i_p$ is reachable from $i_1$. The graph $G(t)$ is connected if any two distinct nodes in $G(t)$ are reachable from each other. A tree is a connected graph with $n \geq 2$ nodes and $n - 1$ edges where $n \in \mathbb{Z}_+$. For matrix weighted graphs, we adopt the following terminology. An edge $(i, j) \in \mathcal{E}(t)$ is positive definite or positive semi-definite if the associated weight matrix $A_{ij}(t)$ is positive definite or positive semi-definite, respectively. A positive path in $G(t)$ is a path such that every edge on this path is positive definite. A tree in $G(t)$ is a positive tree if every edge contained in this tree is positive definite. A positive spanning tree of $G(t)$ is a positive tree containing all nodes in $G(t)$.

**IV. Consensus on General Matrix-weighted Time-varying Networks**

In order to analyze multi-agent systems of the form (2), we adopt the following assumption on the matrix-weighted time-varying network [18], [19], [21], [23].

**Assumption 1.** There exists a sequence $\{ t_k | k \in \mathbb{N} \}$ such that $\lim_{k \to \infty} t_k = \infty$ and $\Delta t_k = t_{k+1} - t_k \in [\alpha, \beta]$ for all $k \in \mathbb{N}$, where $\beta > \alpha > 0$, $t_0 = 0$, and $G(t)$ is time-invariant for $t \in [t_k, t_{k+1})$ for all $k \in \mathbb{N}$.

When $L(t) = L$ for all $t \in [0, \infty)$, then (2) encodes the consensus protocol on a time-invariant network. The following observation characterizes the structure of the null space of matrix-valued Laplacian $L$ on time-invariant networks, that in turn, can determine the steady-state of the network (2).

**Lemma 3.** [5] Let $G = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-weighted time-invariant network with matrix-valued Laplacian $L$. Then $L \succeq 0$ and $\text{null}(L) = \text{span} \{ \mathcal{R}, \mathcal{H} \}$, where $\mathcal{R} = \text{range} \{ 1 \otimes I_d \}$ and

$$\mathcal{H} = \{ [v_1^T, v_2^T, \ldots, v_n^T]^T \in \mathbb{R}^{dn} | (v_i - v_j) \in \text{null}(A_{ij}), (i, j) \in \mathcal{E} \}.$$

Note that the null space of a matrix-valued Laplacian is not only determined by the network connectivity, but also by the properties of weight matrices; this is distinct from the scalar-weighted networks. For matrix-weighted time-invariant networks, a condition under which the multi-agent system (2) achieves consensus is provided in the following lemma.
Lemma 4. [5] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-weighted time-invariant network with matrix-valued Laplacian $L$. Then the multi-agent system (2) admits an average consensus if and only if $\text{null}(L) = \mathbb{R}$.

Definition 5. Define the consensus subspace of the multi-agent system (2) as $\mathcal{R} = \text{range}\{1 \otimes I_d\}$.

Lemma 6. [5] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-weighted time-invariant network. If $\mathcal{G}$ has a positive spanning tree $T$, then the network (2) admits an average consensus.

In order to characterize the related properties of the time-varying networks $\mathcal{G}(t)$ over a given time span, we introduce the notion of matrix-weighted integral network; this notion proves crucial in characterizing algebraic and graph-theoretic conditions for reaching consensus on matrix-weighted time-varying networks.

Definition 7. Let $\mathcal{G}(t) = (\mathcal{V}(t), \mathcal{E}(t), A(t))$ be a matrix-weighted time-varying network. Then the matrix-weighted integral network of $\mathcal{G}(t)$ over time span $[\tau_1, \tau_2] \subseteq [0, \infty)$ is defined as $\tilde{\mathcal{G}}_{[\tau_1, \tau_2]} = (\mathcal{V}, \tilde{\mathcal{E}}, \tilde{A})$, where
\[
\tilde{A} = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} A(t) dt
\]
and
\[
\tilde{\mathcal{E}} = \left\{(i, j) \in \mathcal{V} \times \mathcal{V} \mid \int_{\tau_1}^{\tau_2} A_{ij}(t) dt > 0 \text{ or } \int_{\tau_1}^{\tau_2} A_{ij}(t) dt \geq 0\right\}.
\]

According to Definition 7, denote by $\tilde{C}$ as the matrix-weighted degree matrix of $\tilde{\mathcal{G}}_{[\tau_1, \tau_2]}$, that is, $\tilde{C} = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} C(t) dt$. Denote the matrix-valued Laplacian of $\tilde{\mathcal{G}}_{[\tau_1, \tau_2]}$ as $\tilde{L}_{[\tau_1, \tau_2]}$. Thus,
\[
\tilde{L}_{[\tau_1, \tau_2]} = \tilde{C} - \tilde{A} = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} L(t).
\]

According to Assumption 1, we denote $\mathcal{G}(t)$ on dwell time $t \in [t_k, t_{k+1})$ as $\mathcal{G}_{[t_k, t_{k+1})}(t) = \mathcal{G}^k$ and denote the associated matrix-valued Laplacian as $L^k$, where $k \in \mathbb{N}$. The following lemma reveals the connection between the null space of the matrix-valued Laplacian of a sequence of matrix-weighted networks and that of the corresponding integral network.

Lemma 8. Let $\mathcal{G}(t)$ be a matrix-weighted time-varying network satisfying Assumption 1. Then $\text{null}(\tilde{L}_{[t_{k'}, t_{k''})}) = \mathcal{R}$ if and only if
\[
\bigcap_{i\in \mathbb{N} - k'} \text{null}(L^{k'+i-1}) = \mathcal{R},
\]
where $k' < k'' \in \mathbb{N}$.

Proof: (Necessity) From the definition of matrix-valued Laplacian, one has $\mathcal{R} \subseteq \bigcap_{i\in \mathbb{N} - k'} \text{null}(L^{k'+i-1})$. 
Assume that \( \bigcap_{i \in k'' - k'} \text{null}(L^{k'+i-1}) \neq \mathcal{R} \); then there exists an \( \eta \notin \mathcal{R} \) such that \( L^{k'+i-1}\eta = 0 \) for all \( i \in k'' - k' \), which would imply,

\[
\bar{L}_{[t_{k'}, t_{k''})} \eta = \left( \frac{1}{\Delta t_{k'}} \int_{t_{k'}}^{t_{k''}} L(t) dt \right) \eta = \frac{1}{\Delta t_{k'}} \sum_{i=1}^{k''-k'} L^{k'+i-1}(t_{k'}+i - t_{k'+i-1}) \eta = 0,
\]

contradicting the fact that \( \text{null}(\bar{L}_{[t_{k'}, t_{k''})}) = \mathcal{R} \). Therefore, \( \bigcap_{i \in k'' - k'} \text{null}(L^{k'+i-1}) = \mathcal{R} \).

(Sufficiency) Assume that \( \text{null}(\bar{L}_{[t_{k'}, t_{k''})}) \neq \mathcal{R} \); then there exists \( \eta \notin \mathcal{R} \) such that \( \bar{L}_{[t_{k'}, t_{k''})} \eta = 0 \). Hence, \( \eta^\top \bar{L}_{[t_{k'}, t_{k''})} \eta = 0 \), implying that,

\[
\frac{1}{\Delta t_{k'}} \eta^\top \left( \int_{t_{k'}}^{t_{k''}} L(t) dt \right) \eta = \frac{1}{\Delta t_{k'}} \sum_{i=1}^{k''-k'} \eta^\top L^{k'+i-1}(t_{k'}+i - t_{k'+i-1}) \eta = 0.
\]

Due to the fact that \( L^{k'+i-1} \) is positive semi-definite for all \( i \in k'' - k' \), \( \eta^\top L^{k'+i-1} \eta = 0 \), which would imply that \( L^{k'+i-1} \eta = 0 \); this on the other hand, contradicts the premise \( \bigcap_{i \in k'' - k'} \text{null}(L^{k'+i-1}) = \mathcal{R} \). Thus \( \text{null}(\bar{L}_{[t_{k'}, t_{k''})}) = \mathcal{R} \). \( \blacksquare \)

In order to link the state evolution of the multi-agent system (2) and the null space of the integral of matrix-weighted time-varying networks, we need to employ the state transition matrix. Denote \( \Phi(k', k'') = e^{-L^{k''-1} \Delta t_{k''-1}} \ldots e^{-L^{k'} \Delta t_{k'}} \). Then \( x(t_{k''}) = \Phi(k', k'') x(t_{k'}) \), where \( k' < k'' \in \mathbb{N} \). Note that the matrix-valued Laplacian \( L \) has at least \( d \) zero eigenvalues. Let \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_d \) be the eigenvalues of \( L \). Then have \( 0 = \lambda_1 = \ldots = \lambda_d \leq \lambda_{d+1} \leq \ldots \leq \lambda_{d+n} \). Denote by \( \beta_1 \geq \beta_2 \geq \ldots \geq \beta_{d+n} \) as the eigenvalues of \( e^{-L t} \); then \( \beta_i(e^{-L t}) = e^{-\lambda_i(t) t} \), i.e., \( 1 = \beta_1 = \ldots = \beta_d \geq \beta_{d+1} \geq \ldots \geq \beta_{d+n} \). In the meantime, the eigenvector corresponding to the eigenvalue \( \beta_i(e^{-L t}) \) is equal to that corresponding to \( \lambda_i(L) \). Consider the symmetric matrix \( \Phi(k', k'') \Phi(k', k'')^\top \) which has at least \( d \) eigenvalues at 1. Let \( \mu_j \) be the eigenvalues of \( \Phi(k', k'') \Phi(k', k'')^\top \), where \( j \in \mathbb{N}_d \) such that \( \mu_1 = \ldots = \mu_d = 1 \) and \( \mu_{d+1} \geq \mu_{d+2} \geq \ldots \geq \mu_{d+n} \). The following lemma provides the relationship between the null space of the matrix-valued Laplacian of \( \bar{G}_{[t_{k'}, t_{k''})} \) and the eigenvalue \( \mu_{d+1} \) of \( \Phi(k', k'') \Phi(k', k'')^\top \). This relationship will prove useful in the proof of our main theorem.

**Lemma 9.** Let \( G(t) \) be a matrix-weighted time-varying network satisfying Assumption 1. Then \( \text{null}(\bar{L}_{[t_{k'}, t_{k''})}) = \mathcal{R} \) if and only if

\[\mu_{d+1}(\Phi(k', k'')^\top \Phi(k', k'')) < 1,\]

where \( k' < k'' \in \mathbb{N} \).

**Proof:** (Sufficiency) Assume that \( \text{null}(\bar{L}_{[t_{k'}, t_{k''})}) \neq \mathcal{R} \); then according to Lemma 8, there exists an \( \eta \notin \mathcal{R} \) such that \( L^{k'+i-1} \eta = 0 \) for all \( i \in k'' - k' \). Thus one can obtain \( e^{-L^{k'+i-1}} \eta = \eta \) for all \( i \in k'' - k' \) and \( \Phi(k', k'') \eta = \eta \). According to
the Lemma 1, one has
\[
\mu_{d+1}(\Phi(k',k'')^T \Phi(k',k'')) \geq \frac{\eta^T \Phi(k',k'')^T \Phi(k',k'') \eta}{\eta^T \eta} = 1,
\]
contradicting,
\[
\mu_{d+1}(\Phi(k',k'')^T \Phi(k',k'')) < 1.
\]
Therefore \(\text{null}(\tilde{L}_{[t_{k_0},t_{k+1})}) = \mathcal{R}\) holds.

(Necessity) Assume that \(\mu_{d+1}(\Phi(k',k'')^T \Phi(k',k'')) \geq 1\). Again, according to Lemma 1, there exists a \(\eta \notin \mathcal{R}\) and \(\eta \neq 0\) such that
\[
\mu_{d+1}(\Phi(k',k'')^T \Phi(k',k'')) = \frac{\eta^T \Phi(k',k'')^T \Phi(k',k'') \eta}{\eta^T \eta} \geq 1.
\]
Thus,
\[
\| \eta \| \leq \| \Phi(k',k'') \eta \|.
\]
Let \(\eta_{k'} = \eta\) and \(\eta_{k'+i} = e^{-L^{k'+i-1} \Delta t_{k'+i-1}} \eta_{k'+i-1}\) for \(i \in k'' - k'\). Due to the fact \(\lambda_j(e^{-L^{k'+i-1} \Delta t_{k'+i-1}}) \leq 1\) for \(j \in \text{dim}\) and \(\eta \notin \mathcal{R}\), then
\[
\| e^{-L^{k'+i-1} \Delta t_{k'+i-1}} \eta \| \leq \| \eta \|,
\]
which implies that,
\[
\| \eta \| \leq \| \Phi(k',k'') \eta \| = \| \eta_{k'} \| \leq \| \eta_{k''-1} \| \leq \ldots \leq \| \eta_{k'} \| = \| \eta \|.
\]
Hence, \(\| e^{-L^{k'+i-1} \Delta t_{k'+i-1}} \eta_{k'+i-1} \| = \| \eta_{k'+i-1} \|\) for \(i \in k'' - k'\). Then, one can further derive \(L^{k'+i-1} \eta_{k'+i-1} = 0\); thus \(\eta_{k'+i-1} \in \text{ker}(L^{k'+i-1})\). Note that since,
\[
\| \eta_{k'+i} - \eta_{k'+i-1} \| = \| e^{-L^{k'+i-1} \Delta t_{k'+i-1}} \eta_{k'+i-1} - \eta_{k'+i-1} \|
\]
\[
= \| \sum_{t=1}^{\infty} \frac{1}{t!} (-L^{k'+i-1} \Delta t_{k'+i-1})^t \eta_{k'+i-1} \|
\]
\[
= 0,
\]
one can further obtain \(\eta_{k'+i-1} = \eta_{k'+i}\) for \(\forall i \in k'' - k'\), which implies that \(\eta \in \bigcap_{i \in k'' - k'} \text{ker}(L^{k'+i-1}) \) and \(\text{null}(\tilde{L}_{[t_{k_0},t_{k+1})}) \neq \mathcal{R}\). This is a contradiction however. As such \(\mu_{d+1}(\Phi(k',k'')^T \Phi(k',k'')) < 1\).

**Theorem 10.** Let \(\mathcal{G}(t)\) be a matrix-weighted time-varying network satisfying Assumption 1. If the multi-agent network (2) admits an average consensus, then there exists a subsequence of \(\{t_k| k \in \mathbb{N}\}\) denoted by \(\{t_k| l \in \mathbb{N}\}\), such that the null space of the matrix-valued Laplacian of \(\mathcal{G}_{[t_{k_0},t_{k+1})}(t)\) is \(\mathcal{R}\), namely, \(\text{null}(\tilde{L}_{[t_{k_0},t_{k+1})}) = \mathcal{R}\) for all \(l \in \mathbb{N}\), where \(\Delta t_{k_l} = t_{k_{l+1}} - t_{k_l} < \infty\) and \(t_{k_0} = t_0\).
Proof: Assume that there does not exist a subsequence \( \{ t_{k_l} | l \in \mathbb{N} \} \) such that \( \text{null}(\tilde{L}_{[t_{k_l}, t_{k_{l+1}}]}) = \mathcal{R} \) for all \( l \in \mathbb{N} \), which implies that there exists \( k^* \in \mathbb{N} \) such that \( \text{null}(\tilde{L}_{[t_{k^*}, \infty)}) \neq \mathcal{R} \). Then for all \( k \geq k^*, k \in \mathbb{N} \), \( \bigcap_{k \geq k^*, k \in \mathbb{N}} \text{null}(L^k) \neq \mathcal{R} \). Denote \( \eta \notin \mathcal{R} \) and \( \eta \in \bigcap_{k \geq k^*, k \in \mathbb{N}} \text{null}(L^k) \).

Then \( L^k \eta = \mathbf{0} \) for all \( k \geq k^*, k \in \mathbb{N} \). One can choose a suitable \( x(0) \) such that \( x(t_{k^*}) = \eta \); then \( \lim_{t \to \infty} x(t) = \eta \), establishing a contradiction to the fact that the multi-agent network (2) admits an average consensus. Thus, there exists a subsequence \( \{ t_{k_l} | l \in \mathbb{N} \} \) such that \( \text{null}(\tilde{L}_{[t_{k_l}, t_{k_{l+1}}]}) = \mathcal{R} \) for all \( l \in \mathbb{N} \).

Remark 11. Although the existence of a subsequence of \( \{ t_k | k \in \mathbb{N} \} \) denoted by \( \{ t_{k_l} | l \in \mathbb{N} \} \) such that \( \text{null}(\tilde{L}_{[t_{k_l}, t_{k_{l+1}}]}) = \mathcal{R} \), for all \( l \in \mathbb{N} \) is a necessary condition for an average consensus, it is not sufficient. To see this fact, we choose, for instance, the multi-agent system \( \dot{x}(t) = -\frac{1}{T}Lx(t) \), where \( L \) is the matrix-valued Laplacian of a time-invariant matrix-weighted network for which \( \text{null}(L) = \mathcal{R} \). Now consider the underlying matrix-weighted time-varying network corresponding to the Laplacian matrix \( \frac{1}{T}L \). Then for the arbitrary subsequence \( \{ t_{k_l} | l \in \mathbb{N} \} \) of \( \{ t_k | k \in \mathbb{N} \} \), one always has \( \text{null}(\tilde{L}_{[t_{k_l}, t_{k_{l+1}}]}) = \mathcal{R} \) for all \( l \in \mathbb{N} \). However, the solution to the above system is \( x(t) = e^\frac{t}{T}x(0) \), and \( \lim_{t \to \infty} x(t) = e^{-L}x(0) \). Therefore, an average consensus cannot be achieved in this example. Thus, we need additional conditions in order to guarantee average consensus for (2). These observations motivate the following result.

Theorem 12. Let \( \mathcal{G}(t) \) be a matrix-weighted time-varying network satisfying Assumption 1; furthermore, suppose there exists a subsequence of \( \{ t_k | k \in \mathbb{N} \} \), denoted by \( \{ t_{k_l} | l \in \mathbb{N} \} \), such that \( \text{null}(\tilde{L}_{[t_{k_l}, t_{k_{l+1}}]}) = \mathcal{R} \) for all \( l \in \mathbb{N} \), where \( \Delta t_{k_l} = t_{k_{l+1}} - t_{k_l} < \infty \) and \( t_{k_0} = t_0 \). If there exists a scalar \( 0 < q < 1 \) such that \( \mu_{d+1}(\Phi(t_{k_l}, t_{k_{l+1}})^T \Phi(t_{k_l}, t_{k_{l+1}})) \leq q \) for all \( l \in \mathbb{N} \), then the multi-agent network (2) admits an average consensus.

Proof: Let \( \omega(t) = x(t) - x_f \). Then \( \dot{\omega}(t) = -L(t)\omega(t) \). Choose \( \omega(0) \notin \mathcal{R} \) and observe that,

\[
\mu_{d+1}(\Phi(t_{k_0}, t_{k_1})^T \Phi(t_{k_0}, t_{k_1})) \\
\geq \frac{\omega(0)^T(\Phi(t_{k_0}, t_{k_1})^T \Phi(t_{k_0}, t_{k_1}))\omega(0)}{\omega(0)^T\omega(0)} \\
= \frac{\omega(t_{k_1})^T\omega(t_{k_1})}{\omega(0)^T\omega(0)},
\]

implying that,

\[
\| \omega(t_{k_1}) \| \leq \mu_{d+1}(\Phi(t_{k_0}, t_{k_1})^T \Phi(t_{k_0}, t_{k_1}))^{1/2} \| \omega(0) \| .
\]

Therefore,

\[
\| \omega(t_{k_{l+1}}) \| \leq \mu_{d+1}(\Phi(t_{k_l}, t_{k_{l+1}})^T \Phi(t_{k_l}, t_{k_{l+1}}))^{1/2} \| \omega(t_{k_l}) \| \\
\leq \mu_{d+1}(\Phi(t_{k_l}, t_{k_{l+1}})^T \Phi(t_{k_l}, t_{k_{l+1}}))^{1/2} \\
\vdots \\
\leq q^{1/2(d+1)} \| \omega(0) \| .
\]

Let

\[
V(t) = \omega(t)^T\omega(t) = \| \omega(t) \|^2;
\]

then

\[
\dot{V}(t) = 2\omega(t)^T(-L(t))\omega(t) \leq 0.
\]
Thus
\[ \| \omega(t) \| \leq \| \omega(t_{k_{i+1}}) \| \leq q^{\frac{i}{2}(l+1)} \| \omega(0) \|, \]
for \( \forall t \in [t_{k_{i+1}}, \infty) \). Note that \( 0 < q < 1 \), and hence,
\[ \lim_{t \to \infty} \| \omega(t) \| = 0. \]
As such, the multi-agent network (2) achieves average consensus.

V. Consensus on Periodic Matrix-weighted Time-varying Networks

In the subsequent discussion, we consider a special class of time-varying networks, where \( G(t) \) is periodic. The periodic network \( G(t) \) is formally characterized by the following assumption.

Assumption 2. There exists a \( T > 0 \) such that \( G(t + T) = G(t) \) for any \( t \geq 0 \). Moreover, there exists a time sequence \( \{t_k | k \in \mathbb{N}\} \) satisfying \( \Delta t_k = t_{k+1} - t_k > \alpha \) for all \( k \in \mathbb{N} \), where \( \alpha > 0 \), and there exists \( m > 2 \ (m \in \mathbb{N}) \) partitions for each time span \( [lT, (l+1)T) \) for which,
\[ lT = t_{lm} < t_{lm+1} < \cdots < t_{(l+1)m} = (l+1)T, \ l \in \mathbb{N}, \]
and \( G(t) \) is time-invariant for \( t \in [t_k, t_{k+1}) \), where \( k \in \mathbb{N} \).

Under Assumption 2, we now proceed to provide the algebraic and graph-theoretic conditions under which the multi-agent system (2) admits average consensus.

Theorem 13. Let \( G(t) \) be a periodic matrix-weighted time-varying network satisfying Assumption 2. Then the multi-agent network (2) admits average consensus if and only if,
\[ \text{null}(L_{[0,T)}) = \mathcal{R}. \]

Proof: (Necessity) Assume that \( \text{null}(L_{[0,T)}) \neq \mathcal{R} \); then there exists a \( \eta \notin \mathcal{R} \) such that \( L^{-1}T \eta = 0 \) for all \( i \in m \). Let \( x(0) = \eta \). Thereby, we can obtain \( x(t) = \eta \) for all \( t > 0 \), contradicting the fact that the multi-agent network (2) admits average consensus.

(Sufficiency) Let \( \omega(t) = x(t) - x_f \); then we have \( \dot{\omega}(t) = -L(t)\omega(t) \). Denote
\[ \Phi(0, T) = e^{-L^{m-1} \Delta t_{m-1}} \cdots e^{-L^0 \Delta t_0}, \]
and choose \( \omega(0) \notin \mathcal{R} \). Then,
\[ \mu_{d+1}(\Phi(0, T)^\top \Phi(0, T)) \geq \frac{\omega(0)^\top (\Phi(0, T)^\top \Phi(0, T)) \omega(0)}{\omega(0)^\top \omega(0)} \]
\[ = \frac{\omega(T)^\top \omega(T)}{\omega(0)^\top \omega(0)}, \]
implying that,
\[ \omega(T)^\top \omega(T) \leq \mu_{d+1}(\Phi(0, T)^\top \Phi(0, T)) \omega(0)^\top \omega(0). \]
Therefore
\[ \| \omega(T) \| \leq \mu_{d+1}(\Phi(0, T)^\top \Phi(0, T)) \| \omega(0) \|, \]
implying that,
\[ \| \omega(kT) \| \leq \mu_{d+1}(\Phi(0, T)^\top \Phi(0, T))^{\top k} \| \omega(0) \|. \]
Hence, one has
\[ \| \omega(t) \| \leq \| \omega(kT) \| \leq \mu_{d+1}(\Phi(0, T)^T \Phi(0, T))^{\frac{1}{2k}} \| \omega(0) \|, \]
for \( t \in [kT, (k+1)T) \); then \( \lim_{t \to \infty} \| \omega(t) \| = 0 \). Therefore, the multi-agent network (2) admits average consensus. \( \blacksquare \)

Theorem 13 provides an algebraic condition for reaching consensus for periodic matrix-weighted time-varying networks using the structure of the null space of the matrix-valued Laplacian matrix of the corresponding integral network. An analogous graph theoretic condition is as follows.

**Theorem 14.** Let \( G(t) \) be a periodic matrix-weighted time-varying network satisfying Assumption 2. If the integral graph of \( G(t) \) over time span \([0, T)\) has a positive spanning tree, then the multi-agent network (2) admits average consensus.

**Proof:** Let \( \tilde{G}_{[0,T)} \) be the integral network of \( G(t) \) over time span \([0, T)\). If \( \tilde{G}_{[0,T)} \) has a positive spanning tree, from Lemma 4 and Lemma 6, one has \( \text{null}(\tilde{L}_{[0,T)}) = R \), where \( \tilde{L}_{[0,T)} \) is the matrix-valued Laplacian matrix of \( \tilde{G}_{[0,T)} \). Theorem 13, now implies that the multi-agent network (2) admits average consensus. \( \blacksquare \)

### VI. Simulation Results

Consider a sequence of matrix-weighted networks, consisting of (the same) four agents, and the topologies of the networks are as \( G_1, G_2 \) and \( G_3 \), as shown in Figure 1. Note that \( n = 4 \) and \( d = 2 \) in this example.

![Fig. 1. Three matrix-weighted networks \( G_1, G_2, \) and \( G_3 \). Those edges weighted by positive definite matrices are illustrated by solid lines and edges weighted by positive semi-definite matrices are illustrated by dotted lines.](image)

The matrix-valued edge weights for each network are,

\[
A_{12}(G_1) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_{23}(G_1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},
\]

\[
A_{24}(G_2) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_{34}(G_2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\]

and

\[
A_{23}(G_3) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},
\]

respectively. The matrix-valued Laplacian matrices corresponding to above three networks are,
Consider a time sequence $\{t_k | k \in \mathbb{N}\}$ such that $t_k = k\Delta t$ where $\Delta t > 0$. The evolution is initiated from network $G_1$ (i.e., $G(0) = G_1$) with $x_1(0) = [0.6787, 0.7577]^T$, $x_2(0) = [0.7431, 0.3922]^T$, $x_3(0) = [0.6555, 0.1712]^T$ and $x_4(0) = [...]

Fig. 2. Switching sequence amongst networks $G_1$, $G_2$, and $G_3$. 

$L(G_1) = \begin{bmatrix}
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
1 & 2 & -1 & -2 & 0 & 0 & 0 & 0 \\
-1 & -1 & 2 & 2 & 1 & -1 & 0 & 0 \\
-1 & -2 & 2 & 3 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\
0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$

and

$L(G_2) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 2 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 2
\end{bmatrix},$

and

$L(G_3) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 2 & 1 & -2 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & -2 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$

respectively.
[0.7060, 0.0318]\top. The switching among networks $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ satisfies $\{t_k \mid k \in \mathbb{N}\}$,

$$
\mathcal{G}(t) = \begin{cases} 
\mathcal{G}_1, & t \in [t_{6l}, t_{6l+2}), \\
\mathcal{G}_2, & t \in [t_{6l+2}, t_{6l+5}), \\
\mathcal{G}_3, & t \in [t_{6l+5}, t_{6_l+1}),
\end{cases}
$$

where $l \in \mathbb{N}$. The network switching process is demonstrated in Figure 2. Examine the dimension of the null space of $L(\mathcal{G}_1)$, $L(\mathcal{G}_2)$ and $L(\mathcal{G}_3)$, respectively. We have $\text{null}(L(\mathcal{G}_1)) \neq \mathbb{R}$, $\text{null}(L(\mathcal{G}_2)) \neq \mathbb{R}$ and $\text{null}(L(\mathcal{G}_3)) \neq \mathbb{R}$. However, note that from Figure 3, the integral graph of $L(\mathcal{G}_1)$, $L(\mathcal{G}_2)$ and $L(\mathcal{G}_3)$ over time span $[t_{6l}, t_{6_l+1})$, where $l \in \mathbb{N}$, denoted by $\tilde{\mathcal{G}}$, has a positive spanning tree $T(\tilde{\mathcal{G}})$. Therefore, according to Theorem 14, the multi-agent system (2) admits an average consensus solution at [0.6958, 0.3382]\top; see Figure 4.

![Fig. 3. The integral graph $\mathcal{G}(t)$ over time span $[t_{6l}, t_{6_l+1})$ where $l \in \mathbb{N}$ (left) and the associated positive spanning tree $T(\tilde{\mathcal{G}})$ (right).](image)

![Fig. 4. State evolution in the multi-agent system (2).](image)

**VII. Conclusion**

This paper examines consensus problems on matrix-weighted time-varying networks. For such networks, necessary and/or sufficient conditions for reaching average consensus are provided. Furthermore, for matrix-weighted periodic time-varying networks, necessary and sufficient algebraic and graph theoretic conditions are obtained for reaching consensus.
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