Stationary Bernstein-Greene-Kruskal structures in a current carrying relativistic cold plasma

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Nonlinear stationary structures, formed in cold plasma with immobile ions, in the presence of a spatially modulated relativistic electron current beam have been investigated analytically in the collisionless limit. These are cold plasma version of the relativistic BGK waves. The structure profile is governed by the ratio of maximum electrostatic field energy density to the relativistic kinetic energy density of the electron beam, i.e., \( \kappa_R = E_m / (8 \pi n_0 (\gamma_0 - 1) m_0 c^2)^{1/2} \), where \( E_m \) is the maximum electric field associated with the nonlinear structure and \( \gamma_0 \) is the Lorentz factor associated with average beam speed. In the linear limit, i.e., \( \kappa_R \ll 1 / \sqrt{\gamma_0} \), the fluid variables, viz, density, electric field, and velocity vary harmonically in space. In the range \( 0 < \kappa_R \leq 1 / \sqrt{\gamma_0} \), the fluid variables exhibit an-harmonic behavior. For values of \( \kappa_R > 1 / \sqrt{\gamma_0} \), the electric field shows finite discontinuities at specific spatial locations indicating formation of negatively charged planes at these locations. Discontinuity in the electric field momentarily stops the electrons, resulting in the formation of periodic electrostatic (BGK) structures consists of negatively charged planes.

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I. INTRODUCTION

Cold relativistic electron beam can support variety of waves in plasma. Here we study a special class of nonlinear waves called stationary BGK waves\(^1\) in a cold plasma which are excited by a relativistic electron beam. In the non-relativistic regime, and in the absence of a beam, propagating BGK waves in a cold plasma have been derived by Albritton et. al.\(^2\). The BGK mode in this case was obtained from the exact space-time dependent solution\(^3\) of the full nonlinear fluid-Maxwell set of equations. Similarly propagating BGK waves in a cold relativistic plasma in the absence of a relativistic electron beam is simply obtained by transforming the governing equations in such a frame, where the wave is at rest, the so-called wave frame\(^4\). Verma et. al.\(^5\) also constructed such a solution for propagating BGK waves (Akhiezer-Polovin wave\(^6\)) from exact space-time dependent solution\(^7\) of the full relativistic fluid-Maxwell set of equations by choosing a special kind of transformation\(^2\). Wang\(^8\) used similar kind of transformation for relativistic streaming plasmas and obtained a nonlinear dispersion relation in Vlasov-Maxwell framework. In the presence of a beam Psimpolous et. al.\(^9\) obtained the solutions for stationary BGK waves (stationary in lab frame) in current carrying non-relativistic cold plasmas for a wide range of parameter \((\kappa = E_m/(4\pi n_0 m v_0^2)^{1/2})\), where \(E_m\) is maximum amplitude of the electric field, \(v_0\) is average electron beam speed and other symbols have their usual meanings.

In this paper, we study BGK structures in a relativistic electron beam propagating through an immobile homogeneous positive background of ions. Under the influence of applied harmonic perturbation, periodic compression and rarefaction occurs in density, so according to equation of continuity electrons accelerate and retard periodically in space, to maintain the constant flux throughout the system. These periodic departures from charge neutrality induce in turn a longitudinal electric field which produces the necessary force on the electrons so that the whole system is kept in stationary state. It is found that the basic parameter that embodies the nonlinear effects in the system, is a ratio of maximum electrostatic energy density to total relativistic kinetic energy density, \(\kappa_R = E_m/(8\pi n_0 (\gamma_0 - 1)m_0 c^2)^{1/2}\), where \(E_m\) is the maximum amplitude of the electric field, \(\gamma_0\) is the Lorentz factor associated with average beam speed \((v_0)\), \(n_0\) and \(m_0\) are respectively the density and rest mass of the electron and \(c\) being the speed of light.

In the non-relativistic limit\(^9\), it is found that if \(\kappa_{NR} \rightarrow \kappa_{NR}^c = E_m/(4\pi n_0 m v_0^2)^{1/2} = 1\),
electric field gradient becomes infinitely steep, periodically in space; so according to Poisson’s law, electron density also becomes infinitely large. In the case of a relativistic beam and in the presence of relativistically intense wave, the critical parameter $\kappa_R^c$ is modified and is found to depend on the average beam speed $v_0$ as $\kappa_R^c = \frac{1}{\sqrt{2}}$. If $\kappa_R \ll \kappa_R^c$, the fluid variables $v_e(x), n_e(x), \phi(x)$, and $E(x)$ vary harmonically in space in accordance with linear theory. As $\kappa_R$ increases, and approaches $\approx \kappa_R^c$ within the interval $0 \ll \kappa_R < \kappa_R^c$, the above variables gradually become anharmonic in space. In the case of $\kappa_R \geq \kappa_R^c$ it is shown that gradient of electric field becomes infinitely steep periodically at certain singular points which in turn implies discontinuity in electric field and explosive behavior of electron density. This discontinuous electric field implies formation of negatively charged perfectly conducting planes, infinitely extended in the transverse direction. In the limit $\kappa_R \to \infty$ the BGK structure collapses to a 1-D crystal. It is also shown that in this limit, results of nonlinear relativistic theory coincide with the nonlinear non-relativistic theory.

In this paper we derive exact stationary solutions of BGK structures in current carrying cold relativistic fluid-Maxwell system. An exact expression for electrostatic potential, electric field, electron density and electron velocity as a function of position are derived which describe the nonlinear BGK structures. It is also shown that, in an appropriate limit, results of relativistic theory coincide with the non-relativistic results. In section I we give an introduction of the problem. In section IA, we derive linear results and in section IB nonlinear theory is derived and results are described. We end the paper with a brief discussion in section II.

A. Linear Theory

Let us consider an infinitely long 1D system, where a relativistic electron beam of density $n_0$ and velocity $v_0$ is propagating through an immobile homogeneous positive background of ions of density $n_0$. The basic set of governing equations required to study nonlinear stationary BGK structures are

$$\frac{\partial n_e v_e}{\partial x} = 0, \quad (1)$$

$$v_e \frac{\partial p_e}{\partial x} = -eE, \quad (2)$$
\[
\frac{\partial E}{\partial x} = 4\pi e(n_0 - n_e)
\]
(3)

where \(p_e = \gamma m_0 v_e\) is momentum of electrons, \(v_e\) is electron velocity, \(n_e\) is electron density, \(E\) is electric field and other symbols have their usual meaning.

In the linear limit \(\kappa_R \ll 1\) and in the spirit of weakly relativistic flow \(v_0 < c\), fluid variables describing the spatial profile can be obtained using linearized set of steady state fluid equations. The continuity equation is

\[
n_0 \frac{\partial v_e}{\partial x} + v_0 \frac{\partial n_e}{\partial x} = 0,
\]
(4)

the momentum equation is

\[
v_0 \frac{\partial p_e}{\partial x} = e \frac{\partial \phi}{\partial x},
\]
(5)

and the Poisson equation is,

\[
\frac{\partial E}{\partial x} = 4\pi e(n_0 - n_e).
\]
(6)

Using equations (4), (5) and (6), solution of stationary equations in the linear limit can be obtained straightforwardly as

\[
E(X) = \kappa_R \sin X,
\]

\[
\Phi(X) = \kappa_R \beta \sqrt{2\gamma_0(\gamma_0 - 1)} \cos X,
\]
(7)

\[
v_e(X) = \beta \left( 1 + \frac{k_R \sqrt{2(\gamma_0 - 1)}}{\beta \gamma_0^{3/2}} \cos X \right),
\]
(8)

and

\[
n_e(X) = \left( 1 - \frac{k_R \sqrt{2(\gamma_0 - 1)}}{\beta \gamma_0^{3/2}} \cos X \right),
\]
(9)

where \(\beta = v_0/c\), \(X = x/s_R\), \(E \rightarrow E/E_0\), \(E_0 = (8\pi n_0(\gamma_0 - 1)m_0 c^2)^{1/2}\), \(\Phi = e(\phi - \phi_0)/\gamma_0 m_0 c^2\), \(v_e \rightarrow v_e/c\) and \(n_e \rightarrow n_e/n_0\). Here \(\phi_0\) is an arbitrary additive potential and \(k_R = (E_m^2/8\pi \gamma_0 n_0 m_0 c^2)^{1/2}\) is ratio of maximum electrostatic energy density to total kinetic energy density and \(s_R = v_0 \gamma_0^{3/2}/\omega_{pe}\) is the wavelength of stationary waves in the linear limit \(\kappa_R \ll \kappa^c_R\). It is readily seen that in the linear limit fluid variables of stationary waves are harmonic in space. Fig. 1 and 2 show the potential, electric field, velocity and density for
two different average beam speeds $\beta \approx 0.1$ and $\beta \approx 0.9$ respectively and nonlinear parameter $\kappa_R \approx 0.01$. In Fig. 1 and 2 continuous curves are obtained from the linear theory and dashed curves are results of nonlinear theory, which will be discussed in next section.

![Graphs showing potential, electric field, velocity, and density](image)

**FIG. 1.** Fig shows (a) potential (b) electric field (c) velocity and (d) density for the parameters $\beta = 0.1; \kappa_R = 0.01$. Here continuous curves are obtained from linear theory and dashed curves are result of nonlinear theory.

### B. Nonlinear Theory

The set of nonlinear stationary relativistic fluid equations are

$$
\nu_e \frac{\partial p_e}{\partial x} = e \frac{\partial \phi}{\partial x},
$$

(10)
\[ \beta = 0.9, \kappa_R = 0.01 \]

FIG. 2. Fig shows (a) potential (b) electric field (c) velocity and (d) density for the parameters \( \beta = 0.9; \kappa_R = 0.01 \). Here continuous curves are obtained from linear theory and dashed curves are result of nonlinear theory.

\[ \frac{\partial n_e v_e}{\partial x} = 0, \]

\[ \frac{\partial E}{\partial x} = 4\pi e (n_0 - n_e). \]

Now integrating equation (11) and assuming that at \( \phi = \phi_0 \) at \( v = v_0 \), relation between electron velocity and electrostatic potential is obtained as

\[ \frac{m_0 c^2}{\sqrt{1 - v_e^2/c^2}} - \frac{m_0 c^2}{\sqrt{1 - v_0^2/c^2}} = e(\phi(x) - \phi_0). \]
Using equation (11), (12) and (13), gradient of electric field as a function of potential can be written as
\[
\frac{d^2 \Phi}{dX^2} = -\beta^2 \gamma_0^2 \left(1 - \beta \frac{1 + \Phi}{\sqrt{(1 + \Phi)^2 - 1 + \beta^2}}\right),
\]
(14)
or
\[
\frac{d^2 \Phi}{dX^2} = -\frac{dV_1(\Phi)}{d\Phi},
\]
(15)
\[
\frac{d}{d\Phi} \left\{ \frac{1}{2} \left( \frac{d\Phi}{dX} \right)^2 + V_1(\Phi) \right\} = 0,
\]
(16)
or
\[
\frac{1}{2} \left( \frac{d\Phi}{dX} \right)^2 + V_1(\Phi) = \text{constant},
\]
(17)
which is an energy equation. Here \(V_1(\Phi)\) is a Sagdeev potential and given by
\[
V_1(\Phi) = \beta^4 \gamma_0^2 \left(1 + \Phi \frac{1}{\beta^2} - \sqrt{1 + \frac{2\Phi}{\beta^2} + \frac{\Phi^2}{\beta^2}}\right).
\]
(18)
Now putting \(d\Phi/dX = -\beta(2\gamma_0(\gamma_0 - 1))^{1/2} E\) in the equation (17), yields
\[
E^2 + \frac{V_1(\Phi)}{\beta^2 \gamma_0(\gamma_0 - 1)} = \text{constant},
\]
(19)
or
\[
E^2 + V(\Phi) = \text{constant},
\]
(20)
\textit{constant} in equation (20) can be obtained using the condition that at \(\Phi = 0\); \(V(\Phi) = 0\) and \(E = \kappa_R\), then equation (20) becomes
\[
E^2 + V(\Phi) = \kappa_R^2
\]
(21)
Equation (21) gives a family of curves in the phase space \(\Phi - E\) modulated by the parameter \(\kappa_R\) and \(\beta\), where \(V(\Phi)\) is defined as
\[
V(\Phi) = \frac{1 + \gamma_0}{\gamma_0} \left(1 + \Phi \frac{1}{\beta^2} - \sqrt{1 + \frac{2\Phi}{\beta^2} + \frac{\Phi^2}{\beta^2}}\right)
\]
(22)
In Fig. 3 solid blue curve shows variation of Sagdeev potential with the electrostatic potential \(\Phi\) for different average beam speeds \(\beta \approx 0.1\), \(\beta \approx 0.5\), \(\beta \approx 0.9\) and \(\beta \approx 0.99\). It is noticed here that Sagdeev potential becomes undefined at \(\Phi^c = -(\gamma_0 - 1)\gamma_0\) (below this
FIG. 3. In this Fig. continuous line shows Sagdeev potential for different speed ratios (a) $\beta = 0.1$ (b) $\beta = 0.5$ (c) $\beta = 0.9$ and (d) $\beta = 0.99$ and dotted line shows level of pseudo-energy for different value of $\kappa_R$.

value of potential, the square root term becomes imaginary). Substituting this value of $\Phi^c$ in $V(\Phi)$, the critical value of pseudo-energy ($\kappa_R$) below which periodic solutions exist turns out to be $\kappa_R^c = 1/\sqrt{\gamma_0}$. The straight lines in Fig. 3 show different values of $\kappa_R < \kappa_R^c$ for which periodic solutions exist; corresponding to these values of $\kappa_R$ closed orbits are seen in $\Phi - E$ space (Fig. 4).

In Fig. 4 the relation $\Phi - E$ is plotted for different values of the parameters ($\kappa_R$) and relativity ($\beta$) parameters. It is readily noticed by looking at the Fig. 4 that the variation of $\beta$ modulates the shape of phase space curves as well as changes the range of electrostatic potential $\Phi$. It is also noticed that phase space becomes discontinuous after a critical value
FIG. 4. $\Phi - E$ phase space for different nonlinear parameter and ratio of average speed of the beam to speed of light (a) $\beta \approx 0.1$, (b) $\beta \approx 0.5$, (c) $\beta \approx 0.9$, (d) $\beta \approx 0.99$.

of $\kappa_R$ and this critical value as mentioned above is $\kappa_R = \kappa^c_R = 1/\sqrt{\gamma_0}$. It is found that at the $\kappa_R = \kappa^c_R$, gradient of electric field becomes infinite, i.e., $dE/dX \rightarrow \infty$, which is a sign of wave breaking of stationary BGK structures in current carrying plasmas.

The range of electrostatic potential $\Phi$ for $0 \leq \kappa_R \leq \kappa^c_R$ and for $\kappa^c_R \leq \kappa_R < \infty$, can be obtained from equation (21) and are respectively given by equations (23) and (24) below

$$\gamma_0(\gamma_0 - 1) \left( \kappa_R^2 - \kappa_R \beta \sqrt{\kappa_R^2 + \frac{2}{\gamma_0(\gamma_0 - 1)}} \right) \leq \Phi \leq \gamma_0(\gamma_0 - 1) \left( \kappa_R^2 + \kappa_R \beta \sqrt{\kappa_R^2 + \frac{2}{\gamma_0(\gamma_0 - 1)}} \right)$$  \hspace{1cm} 0 \leq \kappa_R \leq \frac{1}{\sqrt{\gamma_0}} \hspace{1cm} (23)
\[-\left(1 - \frac{1}{\gamma_0}\right) \leq \Phi \leq \gamma_0(\gamma_0 - 1) \left(\kappa_R^2 + \kappa_R \beta \sqrt{\kappa_R^2 + \frac{2}{\gamma_0(\gamma_0 - 1)}}\right) \leq \frac{1}{\sqrt{\gamma_0}} \leq \kappa_R < \infty \] (24)

In the range $0 \leq \kappa_R \leq \kappa_c^R$, curves of equation (21) are continuous and $E$ is found to be oscillating in the range $-\kappa_R \leq E \leq \kappa_R$. In the range $\kappa_R \geq \kappa_c^R$, $E$ becomes discontinuous at $\Phi_c = (1 - \gamma_0)/\gamma_0$ and jumps from $E = -\sqrt{\kappa_R^2 - \sqrt{1 - \beta^2}}$ to $E = \sqrt{\kappa_R^2 - \sqrt{1 - \beta^2}}$. This implies that $E(X)$ is discontinuous at the positions $X$ satisfying the condition $\Phi = e(\phi(X) - \phi_0)/\gamma_0 m_0 c^2 = (1 - \gamma_0)/\gamma_0$. The critical electrostatic potential at which its gradient $(E(X))$ becomes discontinuous, is not constant as in non-relativistic regime ($\Phi_{c\text{non-relativistic}} = -1$), rather, relativity brings the dependency of critical electrostatic potential on the average beam speed by the relation $e(\phi(X) - \phi_0)/\gamma_0 m_0 c^2 = (1 - \gamma_0)/\gamma_0$. Figure 5 shows variation of critical nonlinear parameter $\kappa_c^R$ and critical electrostatic potential $\Phi_c$ with respect to ratio of average beam speed to speed of light. Potential varies from $\Phi_c = 0 - (-1)$ for the range $\beta = 0 - 1; \gamma_0 = 1 - \infty$. This implies $E(X)$ is discontinuous at the positions $X$ satisfying $e(\phi(X) - \phi_0)/\gamma_0 m_0 c^2 = 0 - (-1)$.

![Variation of critical nonlinear parameter and critical potential](image)

**FIG. 5.** Variation of (a) critical nonlinear parameter $\kappa_c^R = 1/\sqrt{\gamma_0}$ (b) critical potential $\Phi_c = (1 - \gamma_0)/\gamma_0$ with respect to ratio of average beam speed to the speed of light ($\beta$).

Using $E = \frac{-1}{\beta(2\gamma_0(\gamma_0 - 1))^{1/2}} \frac{d\Phi}{dX}$, and assuming $\Phi = \Phi_u = \gamma_0(\gamma_0 - 1) \left(\kappa_R^2 + \kappa_R \beta \sqrt{\kappa_R^2 + \frac{2}{\gamma_0(\gamma_0 - 1)}}\right)$ at $X = 0$, the energy equation (21) can be integrated to obtain potential as a function of
position as

$$\beta(2\gamma_0(\gamma_0 - 1))^{1/2} \int_0^X dX = - \int_\Phi^{\Phi_u} \frac{d\Phi}{\left(\kappa_R^2 - \frac{1+\gamma_0}{\gamma_0} \left(1 + \frac{\Phi}{\beta^2} - \sqrt{1 + \frac{2\Phi}{\beta^2} + \frac{\Phi^2}{\beta^2}}\right)\right)^{1/2}}. \quad (25)$$

For simplification, we assume $\Phi + 1 = \xi \sqrt{1 - \beta^2}$, then integrand of R.H.S. of equation (25) takes the form

$$\frac{d\Phi}{\left(\kappa_R^2 - \frac{1+\gamma_0}{\gamma_0} \left(1 + \frac{\Phi}{\beta^2} - \sqrt{1 + \frac{2\Phi}{\beta^2} + \frac{\Phi^2}{\beta^2}}\right)\right)^{1/2}} = \frac{\sqrt{1 - \beta^2} d\xi}{\left(\kappa_R^2 - \frac{1+\gamma_0}{\gamma_0} \left(1 + \frac{\xi \sqrt{1 - \beta^2}}{\beta^2} - \frac{\sqrt{1 - \beta^2}}{\beta} \sqrt{\xi^2 - 1}\right)\right)^{1/2}}. \quad (26)$$

For the sake of convenience, we define a new mathematical quantity $\alpha$ as

$$\alpha = (\gamma_0 - 1)\kappa_R^2 + \frac{1}{\gamma_0}, \quad (27)$$

which transforms the equation (26) into

$$\frac{d\Phi}{\left(\kappa_R^2 - \frac{1+\gamma_0}{\gamma_0} \left(1 + \frac{\Phi}{\beta^2} - \sqrt{1 + \frac{2\Phi}{\beta^2} + \frac{\Phi^2}{\beta^2}}\right)\right)^{1/2}} = \frac{\beta}{(1 + \gamma_0)^{1/2}} \frac{d\xi}{\left(\alpha - \xi + \beta \sqrt{\xi^2 - 1}\right)^{1/2}}. \quad (28)$$

In order to further simplify the calculation of the above integral, a new variable transformation is introduced which is defined as

$$\sqrt{\xi^2 - 1} = \chi^2 - \xi, \quad (29)$$

and

$$d\xi = \left(\chi - \frac{1}{\chi^3}\right) d\chi, \quad (30)$$

then the integral (28) changes into

$$- \int_\xi^{\xi_u} \frac{\beta d\xi}{\sqrt{1 + \gamma_0} \left(\alpha - \xi + \beta \sqrt{\xi^2 - 1}\right)^{1/2}} = - \left(\frac{1 + \beta}{1 - \beta}\right)^{1/2} \left(\frac{2}{1 + \gamma_0}\right)^{1/2} \int_{\chi}^{\chi_u} \frac{(\chi^2 - 1/\chi^2) d\chi}{((r^2 - \chi^2)(\chi^2 - s^2))^{1/2}}, \quad (31)$$
where $r^2$ and $s^2$ are function of $X$ and are defined as
\begin{align}
r^2 &= \frac{\alpha + \sqrt{\alpha^2 + \beta^2 - 1}}{1 - \beta}, \quad \text{(32)}
s^2 &= \frac{\alpha - \sqrt{\alpha^2 + \beta^2 - 1}}{1 - \beta}.
\end{align}

It must be noted here that substitution of new variable $\chi(X)$, is merely a mathematical manipulation, and does not imply any restriction on the range of the potential. Now equation (31) is in standard form and can be reduced easily in the form of elliptic integral upon using new substitution
\begin{align}
sin^2 \theta &= \frac{r^2 - \chi^2}{r^2 - s^2} \quad \text{(34)}
\end{align}

this implies
\begin{align}
d\chi &= -\frac{(r^2 - s^2) \sin \theta \cos \theta}{\sqrt{r^2 \cos^2 \theta + s^2 \sin^2 \theta}} \quad \text{(35)}
\end{align}

Thus, the exact solution of equation (25) can be written as
\begin{align}
Xr^3 &= -\frac{(1 + \beta)^{1/4}}{\gamma_0 \beta (1 - \beta)^{1/4}} \left( \frac{r^4(k^2 - 1) + 1}{k^2 - 1} \right) \left( E(\theta, k) - \frac{k^2 \sin 2\theta}{2(k^2 - 1)(1 - k^2 \sin^2 \theta)^{1/2}} \right) + c_1(\Phi) \quad \text{(36)}
\end{align}

where $E(\theta, k)$ is an incomplete elliptic integral of second kind and $c_1(\Phi)$ is the constant of integration that can be obtained using the boundary condition that at position $X = 0$, potential is maximum, which is $\Phi_u = \gamma_0(\gamma_0 - 1) \left( \kappa_R^2 + \kappa_R \beta \sqrt{\kappa_R^2 + (2/\gamma_0(\gamma_0 - 1))} \right)$; then the complete solution becomes
\begin{align}
Xr^3 &= -\frac{(1 + \beta)^{1/4}}{\gamma_0 \beta (1 - \beta)^{1/4}} \left( \frac{r^4(k^2 - 1) + 1}{k^2 - 1} \right) \left( E(\theta_u, k) - E(\theta, k) \right)
\end{align}

\begin{align}
&\quad \quad \quad \quad - \frac{k^2 \sin 2\theta_u}{2(k^2 - 1)(1 - k^2 \sin^2 \theta_u)^{1/2}} + \frac{k^2 \sin 2\theta}{2(k^2 - 1)(1 - k^2 \sin^2 \theta)^{1/2}} \quad \text{(37)}
\end{align}

Here the variables $k$, $\theta_u$ and $\theta$ are defined as
\begin{align}
k^2 &= \frac{r^2 - s^2}{r^2} = \frac{2\sqrt{\alpha^2 + \beta^2 - 1}}{\alpha + \sqrt{\alpha^2 + \beta^2 - 1}},

\sin^2 \theta_u &= \frac{r^2 - \gamma_0(1 + \Phi_u + \sqrt{\beta^4 + 2\Phi_u + \Phi_u^2})}{r^2 - s^2},
\end{align}
\[
\sin^2 \theta = \frac{r^2 - \gamma_0(1 + \Phi + \sqrt{\beta^2 + 2\Phi + \Phi^2})}{r^2 - s^2}.
\]  
(38)

Equation (37) gives implicit relation between potential and position. The potential \( \Phi(X) \) as a function of position \( X \) for different values of \( k_R \) and \( \beta \) can be obtained by numerical solution of equation (37) and (38).

The half wavelength (spatial variation between maxima to minima of the electrostatic potential) of the BGK structures can be obtained for the range \( \kappa_R \leq \kappa'_R \) and \( \kappa_R \geq \kappa'_R \) by putting the minimum values of \( \Phi \) (\( \Phi_l = \gamma_0(\gamma_0 - 1)(k^2_R - \kappa R \beta \sqrt{k^2_R + (2/\gamma_0(\gamma_0 - 1))}) \)
and \( \Phi_l = \Phi^c = (1 - \gamma_0)/\gamma_0 \) respectively) in the equation (37). In the range \( 0 \leq \kappa_R \leq \kappa'_R \) wavelength turns out to be

\[
\lambda = 2\mu s_R,
\]
(39a)

where

\[
\mu = -\frac{(1 + \beta)^{1/4}}{\gamma_0\beta(1 - \beta)^{1/4}} \left[ \left( \frac{r^4(k^2 - 1) + 1}{r^3(k^2 - 1)} \right)(E(\theta_l, k) - E(\theta_u, k)) - \frac{k^2 \sin 2\theta_u}{2r^3(k^2 - 1)(1 - k^2 \sin^2 \theta_u)^{1/2}} + \frac{k^2 \sin 2\theta_l}{2r^3(k^2 - 1)(1 - k^2 \sin^2 \theta_l)^{1/2}} \right]
\]
(39b)

and for the range \( \kappa^c_R \leq \kappa_R < \infty \) it becomes

\[
\lambda = 2\mu^c s_R,
\]
(40a)

where

\[
\mu^c = -\frac{(1 + \beta)^{1/4}}{\gamma_0\beta(1 - \beta)^{1/4}} \left[ \left( \frac{r^4(k^2 - 1) + 1}{r^3(k^2 - 1)} \right)(E(\theta_c, k) - E(\theta_u, k)) - \frac{k^2 \sin 2\theta_u}{2r^3(k^2 - 1)(1 - k^2 \sin^2 \theta_u)^{1/2}} + \frac{k^2 \sin 2\theta_c}{2r^3(k^2 - 1)(1 - k^2 \sin^2 \theta_c)^{1/2}} \right].
\]
(40b)

Here wavelengths \( \mu(\kappa_R, \beta) \) and \( \mu^c(\kappa_R, \beta) \) are explicit functions of nonlinear parameter \( \kappa_R \) and speed \( \beta \). Corresponding non-relativistic expression for wavelength can be found in reference 9.

For the non-relativistic case, in the linear limit \( 0 \leq \kappa_{NR} \leq 1 \), Psimpoulous’ observed that wavelength of the BGK structure is constant and independent of \( \kappa_{NR} \), however, in the limit \( 1 \leq \kappa_{NR} < +\infty \), wavelength becomes a function of \( \kappa_{NR} \) and wavelength increases with increasing nonlinear parameter \( \kappa_{NR} \). In the relativistic regime, it is readily seen that
wavelengths (equation (39b) and (40b)) are not only a function of nonlinear parameter \( \kappa_R \) but also has dependence on ratio of average beam speed to the speed of light \( \beta \) through the variable \( k \); where \( k \) is defined by equation (38). In the relativistic regime, within the range \( 0 \leq \kappa_R \leq \kappa_{cR} \), it is found that wavelength depends on \( \beta \) as well as on \( \kappa_R \) (wavelength turns out to be independent of \( \kappa_{NR} \) in non-relativistic regime as long as \( \kappa_{NR} \) lies within the range \( 0 \leq \kappa_R \leq 1 \)). Figure 6 shows variation of wavelength of the BGK structure with the nonlinear parameter \( \kappa_R \) for two different average beam speeds, i.e., \( \beta = 0.1 \) (6a) and \( \beta = 0.9 \) (6b). In fig 6 for the speed \( \beta = 0.1 \) (Fig. 6a), in the range \( 0 \leq \kappa_R \leq \kappa_{cR} \) (blue color curve), wavelength is almost constant or in other words, in the range \( \beta \ll 1 \) wavelength of relativistic BGK structure turns out to be independent of \( \kappa_R \), a feature which is seen in the non-relativistic case also\(^9\). However, for the speed \( \beta = 0.9 \) (Fig. 6b), wavelength increases with increasing \( \kappa_R \) as shown in Fig. 6b, i.e., wavelength shows strong dependence for large values of \( \beta \). Therefore, dependence of wavelength on average beam speed is purely a relativistic effect. In the highly nonlinear limit \( \kappa_{cR} \leq \kappa_R < \infty \), wavelength for all value of \( \beta \) increases with increasing \( \kappa_R \) (orange curve in Fig. 6). The dashed vertical line in Figs. 6a and 6b separates the regime \( 0 \leq \kappa_R \leq \kappa_{cR} \) and \( \kappa_{cR} \leq \kappa_R < \infty \). The rate of increase of wavelength with \( \kappa_R \) increases with increasing \( \beta \). Fig. 7 shows wavelength as a function of nonlinear parameter \( \kappa_R \) for different value of \( \beta \approx 0.1, 0.9, 0.99 \) and 0.999. It is clearly seen that slope \( (d\lambda/d\kappa_R \text{ and } d\lambda_c/d\kappa_R) \) of wavelength increases with increasing \( \beta \).
FIG. 7. Comparison of wavelengths with a range of nonlinear parameter $\kappa_R$ for different values of $\beta$.

The potential $\Phi(X)$ for two different speeds $\beta \approx 0.1; \beta \approx 0.9$ and for a wide range of nonlinear parameter ($\kappa_R$) is plotted in figure 8 and 9. Maxima and minima of electrostatic potential for both the range of $\kappa_R$, i.e., $0 \leq \kappa_R \leq \kappa^c_R$ and $\kappa^c_R \leq \kappa_R < \infty$, coincide with the range of electrostatic potential (equation (23) and (24) ) obtained using $\Phi - E$ relation. In first case, when $\beta \approx 0.1$ is considered, plot (Fig. 8a ) of potential $\Phi(X)$ is shown for nonlinear parameter $k_R \approx 0.1, 0.3, 0.5, 0.7, 0.9$. As it has already been discussed that for $\beta \approx 0.1$, wavelength of the BGK structure remains independent of $\kappa_R$ ($\beta \ll 1$) as shown in Fig. 6a(blue color curve), therefore, minima of the potential occurs at $X \approx \pi$ for the nonlinear parameter range $0 \leq \kappa_R \leq \kappa^c_R$, that is clearly illustrated in Fig. 8a. In second case when $\beta = 0.9$ is considered (Fig. 8b), wavelength increases with increasing nonlinear parameter (see orange curve in Fig. 6) so the position of minima of the electrostatic potential occurs at $X = \mu$ as can be seen in Fig. 8b. In the limit $\kappa^c_R \geq \kappa_R$, the minima of the electrostatic potential $\Phi(X)$ is manifested at $X = \mu_c \approx 3.18$ for speed ratio $\beta \approx 0.1$, as it is shown in figure 9a, and for the speed ratio $\beta \approx 0.9$, Fig. 9b illustrates that minima of $\Phi(X)$ is manifested at $X = \mu_c \approx 4.59$.

We can derive electric field in terms of position $X$ by solving equation (21) and considering two branches depending on the sign of potential $\Phi$

$$
\Phi > 0; \quad \Phi = \gamma_0(\gamma_0 - 1)(E^2 - \kappa^2_R) \left( 1 + \beta \left( 1 + \frac{2}{\gamma_0(\gamma_0 - 1)(E^2 - \kappa^2_R)} \right)^{1/2} \right)
$$
\[ \Phi < 0; \quad \Phi = \gamma_0(\gamma_0 - 1)(E^2 - \kappa_R^2) \left( 1 - \beta \left( 1 + \frac{2}{\gamma_0(\gamma_0 - 1)(E^2 - \kappa_R^2)} \right)^{1/2} \right) \] (41)

Range of \( E \) can be estimated using equation (21):(i) if \( 0 \leq \kappa_R \leq \kappa^c_R \), we have \( 0 \leq E \leq \kappa_R \) for both branches; (ii) if \( \kappa^c_R \leq \kappa_R < +\infty \) we have \( 0 \leq E \leq \kappa_R \) for \( \Phi > 0 \) and \( \sqrt{\kappa_R^2 - \sqrt{1 - \beta^2}} \leq E \leq \kappa_R \) for \( \Phi < 0 \). We observe that in the linear limit \( \kappa_R \ll \kappa^c_R \), results obtained from nonlinear theory coincide with the harmonic solution obtained from the linear theory. Fig. 1 and 2 show the fluid variables in the linear limit for the speed \( \beta \approx 0.1 \) and

FIG. 8. Plot of electrostatic potential \( \Phi(X) \) for (a) \( \kappa_R \approx 0.1, 0.3, 0.5, 0.7, 0.9 \) and \( \beta \approx 0.1 \), (b) \( \kappa_R \approx 0.1, 0.3, 0.5 \) and \( \beta \approx 0.9 \)

FIG. 9. Plot of electrostatic potential \( \Phi(X) \) for (a) \( \kappa_R \approx 0.7 \) and \( \beta \approx 0.1; \mu \approx 3.18 \), (b) \( \kappa_R \approx 0.7 \) and \( \beta \approx 0.9; \mu \approx 4.59 \)

\[ \beta = 0.1 \] (a)  
\[ \beta = 0.9 \] (b)
\[ \beta \approx 0.9 \] respectively, where continuous curves show results obtained from the linear theory and dashed curves show results obtained from the nonlinear theory in the linear limit. Both continuous and dashed curves clearly coincide on each other for both value of \( \beta \). In the range \( 0 \leq \kappa_R < \kappa_R^c \), we obtain that at \( X \approx \mu \) implies \( E = 0 \) and

\[ \frac{dE}{dX} = \left( \frac{\gamma_0(\gamma_0 + 1)}{2} \right)^{1/2} \left( 1 - \frac{\beta(\Phi + 1)}{(\Phi^2 + 2\Phi + \beta^2)^{1/2}} \right), \]

is always negative if \( \Phi^c < \Phi < 0 \). A gradual steepening of wave form occurs as \( \Phi \to \Phi^c \). If \( \Phi = \Phi^c \), \( \kappa_R \) becomes \( \kappa_R^c \) that implies \( E = 0 \); \( dE/dX = -\infty \) at \( X = \mu_c \). If \( \kappa_R > \kappa_R^c \), \( E \) becomes discontinuous at \( X = \mu_c \) and \( E \) jumps from \( -\sqrt{\kappa_R^2 - \sqrt{1 - \beta^2}} \) to \( \sqrt{\kappa_R^2 - \sqrt{1 - \beta^2}} \). This jump in electric field implies the formation of negatively charged plane at \( X = \mu_c \). The surface charge density \( \rho \) of these planes is defined as

\[ \rho = \Delta E/4\pi = \Delta E = \frac{E_0}{2}(\kappa_R^2 - \sqrt{1 - \beta^2})^{1/2}. \]

Fig. 10a and 10b show spatial variation of electric field for two particular case \( \beta \approx 0.1 \) and 0.9. In first case when \( \beta \approx 0.1 \) in Fig. 10a, gradual steepening of electric field is seen at \( X \approx \pi \) as \( \kappa_R \to \kappa_R^c \); \( \Phi \to \Phi^c \). Similar dynamics follows for the second case when \( \beta \approx 0.9 \), where gradual steepening of \( E \) occurs at \( X = \mu \) as shown in Fig. 10b. When \( \kappa_R > \kappa_R^c \), a discontinuity of \( E \) is seen to be manifested at the position \( X \approx 3.17 \) for \( \beta \approx 0.1 \) and at \( X \approx 4.59 \) for \( \beta \approx 0.9 \) as shown in Fig. 11a and 11b respectively. It is also found that in the range \( \beta \to 0 \) and/or \( \kappa_R \to \infty \) range, electron beam is transformed into a crystal of ”negatively charged plane” with inter-distance \( \lambda_0 = E_m/2\pi n_0 e \) having surface charge density \( \sim E_m/2\pi \), which matches with the results found in non-relativistic regime\(^9\).

The gradual steepening and discontinuity of the electric field modulates the electron speed profile. The \( E - v_e \) phase relation can be constructed using equation (13) and (21) as

\[ E^2 - \kappa_R^2 = \frac{1}{\gamma_0(\gamma_0 - 1)} \left( 1 - \frac{\gamma_0(1 - \beta v_e)}{\sqrt{1 - v_e^2}} \right). \]

Fig. 12 shows \( E - v_e \) phase space modulated by the nonlinear parameter \( \kappa_R \) and ratio of average beam speed to the speed of light \( \beta \). It is readily seen that in the range \( \kappa_R > \kappa_R^c \), \( E - v_e \) phase space becomes discontinuous and \( E \) jumps from \( -\sqrt{\kappa_R^2 - \sqrt{1 - \beta^2}} \) to \( \sqrt{\kappa_R^2 - \sqrt{1 - \beta^2}} \) at the potential satisfying \( \Phi = \Phi^c \).

The fluid velocity as function of position can be obtained using equation (13)

\[ v_e = \pm \left( \frac{\Phi^2 + 2\Phi + \beta^2}{\Phi^2 + 2\Phi + 1} \right)^{1/2}. \]
Equation (45) gives the relation between velocity and self consistent electrostatic potential. Since we strictly exclude the existence of trapped electrons in the system therefore $+ve$ sign of the velocity is taken in the account. As $\kappa_R \to \kappa^c_R; \Phi \to \Phi^c$ at the position satisfying $X = \mu$, numerator of equation (45) tends to zero in the limit $\Phi \to \Phi^c$, thus, a gradual decrement in electron velocity occurs at the position $X = \mu$. If $\kappa_R \geq \kappa^c_R$ then $\Phi = \Phi^c$ at the position $X = \mu_c$, this implies that numerator of the equation (45) becomes zero at that position, in other words velocity becomes zero. This means electrons stop momentarily at
the position $X = \mu_c$ and then continue their motion in $+x$ direction. This short time rest of the electrons, consequently, leads to the accumulation of the charge particles at the position $X = \mu_c$ that is further manifested in density burst or, in other words, in order to maintain the electron current, electron density has to increase at the positions where electrons speed decreases. Fig. 13a and 13b show electron velocity for the speed ratios $\beta \approx 0.1$ and 0.9 respectively, and a gradual decrement of electron velocity can be seen clearly at the position $X = \pi$ for $\beta \approx 0.1$ and at $X = \mu$ for $\beta = 0.9$. Fig. 14a and 14b illustrates that in the limit $\kappa_R \geq \kappa_R^c$, velocity becomes zero at the position satisfying $X = 3.17$ for $\beta = 0.1$ and $X = 4.59$ for $\beta = 0.9$.
The electron density can be written as

\[ n_e(X) = 1 - \left( \frac{2}{\gamma_0(\gamma_0 - 1)} \right)^{1/2} \frac{\partial E}{\partial X}. \]  \hspace{1cm} (46)

Using equation (42), the electron density is therefore may be reduced to a function of electrostatic potential

\[ n_e(X) = \frac{\beta(\Phi + 1)}{(\Phi^2 + 2\Phi + \beta^2)^{1/2}}, \]  \hspace{1cm} (47)

FIG. 13. Plot of electron velocity for the parameters (a) \( \beta \approx 0.1; \kappa R \approx 0.1, 0.3, 0.5, 0.7, 0.9 \) at \( X \approx \pi \)
(b) \( \beta \approx 0.9; \kappa R \approx 0.1, 0.3, 0.5 \) at \( X = \mu \).

FIG. 14. Plot of electron velocity for the parameters (a) \( \beta \approx 0.1; \kappa R \approx 1.1 \) at \( X \approx 3.17 \) and (b) \( \beta \approx 0.9; \kappa R \approx 0.7 \) at \( X \approx 4.59 \).
equation (47) gives implicit relation between electron density and spatial position by eliminating electrostatic potential using equation (37) and (38). As it has already been discussed that in the range $0 \leq \kappa_R < \kappa_R^c$, $\Phi$ approaches to $\Phi^c$ and modulation of the electrostatic potential leads to the steepening of the density which can be clearly seen in the Fig. 15a for $\beta \approx 0.1$ at $X = \pi$ and in Fig. 15b for $\beta \approx 0.9$ at $X = \mu$. When $\kappa_R \geq \kappa_R^c$ then $\Phi = \Phi^c$ and denominator of the equation (47) vanishes. This explosive behavior beyond $\kappa_R^c$ can be clearly seen in Figs. 16a and 16b, where density burst is manifested at $X \approx 3.17$ for $\beta \approx 0.1$ (in Fig. 16a) and at $X \approx 4.59$ for $\beta \approx 0.9$ (in Fig. 16b) respectively.

![Fig. 15](image)

**FIG. 15.** Electron density modulation (a) $\beta \approx 0.1; \kappa_R \approx 0.1, 0.3, 0.5, 0.7, 0.9$ at $X \approx \pi$ (b) $\beta \approx 0.9; \kappa_R \approx 0.1, 0.3, 0.5$ at $X = \mu$.

### II. CONCLUSION

An analytical study is carried out for stationary BGK structures in relativistic current carrying fluid-Maxwell system. It is observed that nonlinear BGK structures is governed by the nonlinear parameter $\kappa_R = E_m/(8\pi n_0 m_0 (\gamma_0 - 1)c^2)^{1/2}$. Critical nonlinear parameter scales with average beam speed $v_0$ as $\kappa_R = 1/\sqrt{\gamma_0}$. Amplitude of nonlinear parameter ($\kappa_R$) embodies the nonlinear effects in the problem. In the linear limit $\kappa_R \ll 1/\sqrt{\gamma_0}$, fluid variables vary harmonically in space and results of nonlinear theory coincides with the results of linear theory in this range. As $\kappa_R \to 1/\sqrt{\gamma_0}$ fluid variables gradually begin to shown anharmonic features. In the nonlinear limit $\kappa_R \geq 1/\sqrt{\gamma_0}$, electric field becomes...
discontinuous at certain singular points in space. Average beam speed decreases at the position of electric field discontinuity, so to keep the current constant, density has to shoots up. This process manifests as a density burst periodically in space. These density burst may approach finite values on inclusive of thermal effects. It is found that in the $\beta \to 0$ and/or $\kappa R \to \infty$ range, electron beam is transformed into a crystal of "negatively charged plane" of inter-distance $\lambda_0 = E_m / 2\pi n_0e$ having surface charge density $\sim E_m / 2\pi$, which matches with the results found in non-relativistic regime$^9$. Study of excitation and stability of these BGK structures using a PIC/fluid code is left for future studies.

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