Equivalence groupoid and group classification of a class of nonlinear wave and elliptic equations

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Enhancing and essentially generalizing the results of [Nonlinear Anal. 70 (2009), 3512–3521] on a class on (1+1)-dimensional nonlinear wave and elliptic equations, we apply several new techniques to classify admissible point transformations within this class up to the equivalence generated by its equivalence group. This gives an exhaustive description of its equivalence groupoid. After extending the algebraic method of group classification to non-normalized classes of differential equations, we solve the complete group classification problem for the class under study up to both usual and general point equivalences. The solution includes the complete preliminary group classification of the class and the construction of singular Lie-symmetry extensions, which are not related to subalgebras of the equivalence algebra. The complete preliminary group classification is based on classifying appropriate subalgebras of the entire infinite-dimensional equivalence algebra whose projections are qualified as maximal extensions of the kernel invariance algebra.

1 Introduction

Group classification is concerned with finding an exhaustive list of inequivalent equations from a class of differential equations containing one or more arbitrary elements. It was originally motivated from theoretical physics, where traditionally those equations admitting the maximal number of symmetries among equations from a given class yield the most promising model describing real-world phenomena.

Mathematically, the problem of group classification has been the subject of intensive investigations for more than 100 years, starting with Sophus Lie’s classification of second-order ordinary differential equations [33]. Since then, numerous novel techniques have been introduced for solving various group classification problems, including direct methods [1, 2, 11, 12, 42, 43], the method of furcate splitting [7, 36] and various flavors of the algebraic method [5, 13, 40, 51]. The latter method has proven so far to be the most powerful, in particular for classes of differential equations containing arbitrary elements that are functions of several arguments.

Among the classes considered in the group classification literature, the most prominent one are classes of (1+1)-dimensional evolution equations, see e.g. [8, 13, 40, 51]. What distinguishes evolutionary equations from the symmetry-perspective is that the time variable typically plays the role of a parameter. In other words, while classifying equations from a typical class of evolution equations quite strong restrictions on the form of transformations among elements of the class can be imposed from the beginning, such as the transformation of the time variable only depending on the time variable, and the transformations being projectable on the space of independent variables [27]. This considerably simplifies the classification procedure.

Also wave equations in (1+1)-dimensions have been studied extensively, such as in [5, 25, 29, 30]. Wave equations play an important role in physics and the mathematical sciences as they model the transport of quantities at finite wave speeds. From the symmetry perspective,
wave equations are challenging since time and space variables enter at equal footing, making it harder to exhaustively study the set of admissible transformations among equations from such classes of equations. It is thus also no coincidence that in the field of invariant discretization, which is concerned with deriving numerical schemes for differential equations possessing the same symmetries as the original, undiscretized equation, mostly evolutionary equations have been considered in the past, see e.g. [10].

See also [44, 45, 46, 58] and references therein for exact solutions of nonlinear wave equations using group-theoretical and related methods.

In the present paper, we exhaustively solve the group classification problem for the class \( \mathcal{W} \) of nonlinear wave and elliptic equations of the form

\[
\frac{du}{dt} = f(x, u)u_{xx} + g(x, u). \tag{1}
\]

We need to explicitly impose two auxiliary inequalities on the arbitrary-element tuple \( \theta = (f, g) \) in order to precisely describe the class \( \mathcal{W} \), which is also referred to as the class (1) in the paper. The auxiliary inequality \( f \neq 0 \) is basic since equations of the form (1) with \( f = 0 \) are not true partial differential equations.\(^1\) We denote by \( \mathcal{W}_{\text{gen}} \) the superclass of equations of the form (1) with \( f \neq 0 \). In order to guarantee nonlinearity of equations from the class \( \mathcal{W} \), the definition of this class should also include the auxiliary inequality

\[
(f_u, g_{uu}) \neq (0, 0).
\]

The subclass \( \mathcal{W}_{\text{lin}} \) of linear equations in \( \mathcal{W}_{\text{gen}} \) is the complement of \( \mathcal{W} \) in \( \mathcal{W}_{\text{gen}}, \mathcal{W} = \mathcal{W}_{\text{gen}} \setminus \mathcal{W}_{\text{lin}} \). The reason why we separate nonlinear and linear equations of the form (1) is that they are not mixed by point transformations (cf. Remark 5) and have different Lie symmetry properties. Although linear wave and elliptic equations with two independent variables were already well investigated within the framework of classical symmetry analysis, see, e.g., [11, 32, 42, 43], we discuss specific transformational and symmetry properties of equations from \( \mathcal{W}_{\text{lin}} \) in Remark 16 below, relating them to equations from \( \mathcal{W} \). The sign of \( f \) is not too essential in the course of group classification of the class \( \mathcal{W} \). In fact, we classify both the subclass of hyperbolic equations for which \( f > 0 \) and the subclass of elliptic equations with \( f < 0 \). Hyperbolic and elliptic equations are also not mixed by point transformations.

Note that the consideration is local and all values are real throughout the paper although the transition to the complex case needs only minor corrections.

Following [1, 25], a so-called partial preliminary group classification problem [5, 13] for the class \( \mathcal{W} \) has been considered in [53]. Specifically, the authors selected a six-dimensional subalgebra \( g_6 \) of the infinite-dimensional equivalence algebra \( g \sim \) of the class \( \mathcal{W} \) and tried to only classify one-dimensional subalgebras of the subalgebra \( g_6 \) up to the equivalence generated by the corresponding six-dimensional subgroup \( G_6 \) of the infinite-dimensional equivalence (pseudo)group \( G \sim \) of the class (1). The \( G_6 \)-equivalence is much weaker than the \( G \sim \)-equivalence. This is why the classification in [53] led to an excessively large list of 24 \( G_6 \)-equivalent simplest classification cases of one-dimensional Lie-symmetry extensions most of which are \( G \sim \)-equivalent to each other and, up to \( G \sim \)-equivalence, fit into the first four cases of Table 1 below. Moreover, a number of classification cases were missed even within the posed partial preliminary group classification problem, not to mention other weaknesses.

We enhance and substantially generalize the results of [53]. The class \( \mathcal{W} \) is neither normalized nor semi-normalized in any sense (the usual, the generalized or the extended ones). It cannot be partitioned into normalized or semi-normalized subclasses which are not related by point transformations. There is no mapping of it by families of point transformations to a class with better transformational properties. This is why Lie symmetries of equations from the class \( \mathcal{W} \) cannot be exhaustively classified by the existing versions of the algebraic method of group

\(^{1}\) Since we work within the local framework, auxiliary inequalities on arbitrary elements are interpreted as satisfied for all values of arguments of arbitrary elements on the relevant domain.
classification, which are explicitly [5, 8, 13, 28, 40, 47, 51] or implicitly [3, 4, 15, 16, 17, 18, 19, 29, 30, 31] based on certain normalization properties of classified classes. (Note that most of the above papers are devoted to group classifications of various classes of single (1+1)-dimensional evolution equations.) On the other hand, the class $\mathcal{W}$ is not convenient to be considered within the framework of the direct method of group classification [1, 2, 11, 12, 42, 43], including its advanced versions like the method of furcate splitting suggested in [36]. The last method is especially efficacious for classes of differential equations with arbitrary elements depending on single arguments [20, 21, 26, 41, 50, 55, 57], although it has also been applied to classes whose arbitrary elements depending on two arguments [7, 36]. Various specific algebraic techniques were suggested for group classification of classes such that sets of certain objects related to Lie symmetries of equations from these classes can be endowed by Lie-algebra structures [7, 35, 49] but this is not applicable for the class $\mathcal{W}$.

This is why to efficiently solve the complete group classification problem for the class $\mathcal{W}$, we develop a new version of the algebraic method of group classification for non-normalized classes of (systems of) differential equations, which is based on classifying admissible transformations of the class under study up to their equivalence generated by the equivalence group of this class. We revisit the general framework of the classification of admissible transformations via modifying its basic notion of equivalent admissible transformations and introducing the notion of generating sets for equivalence groupoids. Several new techniques for classifying admissible transformations of non-normalized classes are also suggested. More specifically, we show that the method of furcate splitting and the algebraic method for computing the complete point or contact symmetry groups of single systems of differential equations [22, 23, 24] and the complete equivalence groups of classes of such systems [6] (including discrete symmetry and equivalence transformations) can be extended to admissible transformations. Unexpected is the opportunity of describing admissible transformations via establishing of a functor between the equivalence groupoids of classes that are not related by families of point transformations. Revisiting the algebraic method of group classification, we introduce the notions of regular and singular Lie-symmetry extensions for a class of differential equations, $\mathcal{L}|_{S}$. Regular Lie-symmetry extensions are associated with subalgebras of the equivalence algebra of the class $\mathcal{L}|_{S}$. They are the extensions that can be constructed by the algebraic method in the course of the complete preliminary group classification [5, 13] of $\mathcal{L}|_{S}$. Singular Lie-symmetry extensions can involve only systems from $\mathcal{L}|_{S}$ being sources of admissible transformations of $\mathcal{L}|_{S}$ that are not generated by equivalence transformations of $\mathcal{L}|_{S}$. As a result, the problem that originated this study turns into a proof-of-concept example for the new methods designed for its solution.

The further organization of the present paper is the following. In the subsequent Section 2, we briefly present a new view on the classification of admissible transformations and use it for revisiting the algebraic method of group classification. The determining equations for admissible transformations of the class $\mathcal{W}$, its equivalence group $\mathcal{G}^\sim$ jointly with its equivalence algebra $g^\sim$ and the determining equations for Lie symmetries of equations from this class are computed in Sections 3, 4 and 5, respectively. The results on the complete group classification of the class $\mathcal{W}$ up to $\mathcal{G}^\sim$-equivalence and on the description of the equivalence groupoid $\mathcal{G}^\sim$ of $\mathcal{W}$ in terms of a generating set of its admissible transformations are collected in Section 6 for convenience of further references. The obtained description of $\mathcal{G}^\sim$ allows us to find a complete set of additional equivalence transformations [50] between listed $\mathcal{G}^\sim$-equivalent Lie-symmetry extensions, which leads to the complete group classification of the class $\mathcal{W}$ up to $\mathcal{G}^\sim$-equivalence as well. A generating set of the equivalence groupoid $\mathcal{G}^\sim$ and an exhaustive list of $\mathcal{G}^\sim$-inequivalent singular Lie symmetry extensions are computed in Section 7 using the variety of techniques mentioned above. In Section 8 we classify appropriate subalgebras of the entire infinite-dimensional equivalence algebra whose projections are qualified as maximal extensions of the kernel invariance algebra of equations from the class $\mathcal{W}$. The complete preliminary group classification of the class $\mathcal{W}$ is finished in Section 9. It gives an exhaustive list of $\mathcal{G}^\sim$-inequivalent regular Lie symmetry
extensions, which is in fact relevant for the \(G^\sim\)-equivalence as well. In Section 10 we analyze our approach to simultaneously classifying admissible transformations and Lie symmetries for equations from the class \(\mathcal{W}\). We also compare the obtained list of Lie-symmetry extensions with similar lists existing in the literature for allied classes of (1+1)-dimensional nonlinear wave and elliptic equations.

2 Classification of admissible transformations and group classification problem

In the present paper we more intensively use various concepts of the groupoid theory than in previous papers on admissible transformations in classes of differential equations and their group classification by the algebraic method, see e.g. [3, 5, 8, 13, 15, 16, 28, 40, 47, 51]. This is why below we relate existing notions and results from group analysis of differential equations to the groupoid-relevant terminology and then present procedures for classifying admissible transformations and Lie symmetries for non-normalized classes of differential equations. The notation in this section differs from other parts of the present paper.

Consider a class \(L|_S = \{L_\theta \mid \theta \in S\}\) of systems of differential equations \(L_\theta\) for unknown functions \(u = (u^1, \ldots, u^m)\) of independent variables \(x = (x_1, \ldots, x_n)\) with the arbitrary-element tuple \(\theta = (\theta^1, \ldots, \theta^k)\) running through a set \(S\). Here \(L_\theta\) denotes a system of differential equations of the form \(L(x, u_\theta, \theta(x, u_\theta)) = 0\) with a fixed tuple \(L\) of \(r\)th order differential functions in \(u\) parameterized by \(\theta\). We use the short-hand notation \(u_\theta\) for the tuple of derivatives of \(u\) with respect to \(x\) up to order \(r\), including \(u\) as the zeroth order derivatives. The set \(S\) is the solution set of an auxiliary system of differential equations and inequalities in \(\theta\), where \(r\)th order jets \((x, u_\theta)\) plays the role of independent variables. \(S(x, u_\theta, \theta(q)) = 0\) and, e.g., \(\Sigma(x, u_\theta, \theta(q)) \neq 0\) with the tuple \(\theta(q)\) constituted by the derivatives of \(\theta\) up to order \(q\) with respect to \((x, u_\theta)\). Up to the gauge equivalence of systems from \(L|_S\) [51], which is usually trivial, the correspondence \(\theta \mapsto L_\theta\) between \(L|_S\) and \(S\) is bijective.

The equivalence groupoid \(G^\sim\) of the class \(L|_S\) is the small category with \(L|_S\) or, equivalently, with \(S\) as the set of objects and with the set of point transformations of \((x, u)\), i.e., of (local) diffeomorphisms in the space with the coordinates \((x, u)\), between pairs of systems from \(L|_S\) as the set of arrows. Specifically, for unknown

\[
G^\sim = \{\mathcal{T} = (\theta, \Phi, \tilde{\theta}) \mid \theta, \tilde{\theta} \in S, \Phi \in \text{Diff}^{\text{loc}}(x,u) : \Phi_*L_\theta = L_{\tilde{\theta}}\},
\]

Elements of \(G^\sim\) are called admissible (point) transformations within the class \(L|_S\).

The definitions of all notions related to groupoids are obvious. Thus, the source and target maps \(s, t: G^\sim \to S\) are defined by \(s(\mathcal{T}) = \theta\) and \(t(\mathcal{T}) = \tilde{\theta}\) for any \(\mathcal{T} = (\theta, \Phi, \tilde{\theta}) \in G^\sim\), which gives rise to the groupoid notation \(G^\sim \rightrightarrows S\), where the symbol “\(\rightrightarrows\)” denotes the pair of the source and target maps. Admissible transformations \(\mathcal{T}\) and \(\mathcal{T}' = (\theta', \Phi', \tilde{\theta}')\) are composable if \(\tilde{\theta}' = \tilde{\theta}\), and then their composition is \(\mathcal{T} \circ \mathcal{T}' = (\theta, \Phi' \circ \Phi, \tilde{\theta}')\), which defines a natural partial multiplication on \(G^\sim\). For any \(\theta \in S\), the unit at \(\theta\) is given by \(\text{id}_{\theta} := (\theta, \text{id}_{(x,u)}, \theta)\), where \(\text{id}_{(x,u)}\) is the identity transformation of \((x, u)\). This defines the object inclusion map \(S \rightrightarrows \theta \mapsto \text{id}_{\theta} \in G^\sim\), i.e., the object set \(S\) can be regarded to coincide with the base groupoid \(S \rightrightarrows S := \{\text{id}_{\theta} \mid \theta \in S\}\). The inverse of \(\mathcal{T}\) is \(\mathcal{T}^{-1} := (\tilde{\theta}, \Phi^{-1}, \theta)\), where \(\Phi^{-1}\) is the inverse of \(\Phi\). All required properties like associativity of the partial multiplication, its consistency with the source and target maps, natural properties of units and inverses are obviously satisfied.

The \(s\)-fibre over \(\theta \in S\), \(s^{-1}(\theta) \subseteq G^\sim\), is the set of possible admissible transformations within \(L|_S\) with source at \(\theta\). Similarly, the \(t\)-fibre over \(\theta \in S\), \(t^{-1}(\theta) \subseteq G^\sim\), is the set of possible admissible transformations within \(L|_S\) with target at \(\theta\). The subset \(G(\theta, \tilde{\theta}) := s^{-1}(\theta) \cap t^{-1}(\tilde{\theta})\) of \(G^\sim\) with \(\theta, \tilde{\theta} \in S\) corresponds to the set of point transformations mapping the system \(L_\theta\) to the system \(L_{\tilde{\theta}}\). The vertex group \(G_{\theta} := G(\theta, \theta) = s^{-1}(\theta) \cap t^{-1}(\theta)\) is associated with the point
symmetry (pseudo)group $G_{\theta}$ of the system $L_{\theta}$, $G_{\theta} = \{ \Phi \in \text{Diff}_{\text{loc}}^{x,u} | (\theta, \Phi, \theta) \in G_{\theta} \}$. The orbit $O_{\theta} := t(s^{-1}(\theta))$ of $\theta$ is the subset of systems in the class $L|_{S}$ that are similar to $L_{\theta}$ with respect to point transformations.

Denote by $\varpi$ and $\varpi'$ the projections from the space with the coordinates $(x, u, \varpi)$ to the spaces with the coordinates $(x, u, \varpi(\varpi'))$, respectively.

The (usual) equivalence group $G_{\sim}$ of the class $L|_{S}$ is the (pseudo)group of point transformations, $T$, in the space with the coordinates $(x, u, \varpi(\varpi'))$, that are projectable to the spaces with the coordinates $(x, u, \varpi(\varpi'))$, with $\varpi' T$ being the standard prolongation of $\varpi, T$ to $r$th order jets $(x, u, \varpi(\varpi'))$ and map the class $L|_{S}$ onto itself. The group $G_{\sim}$ can be considered to act in the space with the coordinates $(x, u, \varpi(\varpi'))$, where $r' < r$, if the arbitrary-element tuple depends only on $(x, u, \varpi(\varpi'))$. The notion of usual equivalence group can be generalized in several ways by weakening the specific restrictions on equivalence transformations, which are their projectability or their locality with respect to arbitrary elements. This gives the notions of generalized equivalence group and extended equivalence group, respectively, or the notions of generalized extended equivalence group if both restrictions are weakened simultaneously [26, 34, 40, 47, 51, 54, 56].

There exist generalizations of the notion of equivalence group in the literature, in which some of the restrictions on equivalence transformations (projectability or locality with respect to arbitrary elements) are weakened.

The action groupoid $G_{\sim}^{G_{\sim}}$ of the equivalence group $G_{\sim}$ of the class $L|_{S}$,

$$G_{\sim}^{G_{\sim}} := \{ (\theta, \varpi, T, \bar{\theta}) | \theta \in S, T \in G_{\sim}, \bar{\theta} = (\varpi, T, \theta) \},$$

is a subgroupoid of the equivalence groupoid $G_{\sim}$ of this class, $G_{\sim}^{G_{\sim}} \subseteq G_{\sim}$, with the same object set $S$. We say that an admissible transformation $T$ in the class $L|_{S}$ is generated by an equivalence transformation of this class if $T \in G_{\sim}^{G_{\sim}}$.

The fundamental groupoid $G_{\sim}^{1}$ of the class $L|_{S}$ is the disjoint union of the vertex groups $G_{\theta}$, $\theta \in S$, $G_{\sim}^{1} := \sqcup_{\theta \in S} G_{\theta}$. Since it has the same object set $S$ and the same vertex groups as $G_{\sim}$ and $T^{-1} G_{\theta} T = G_{\theta}$ for any $T \in G(\theta, \theta)$, it is a normal subgroupoid of the equivalence groupoid $G_{\sim}$, which is also called the fundamental subgroupoid of $G_{\sim}$. In other words, the groupoid $G_{\sim}^{1}$ is constituted by the admissible transformations generated by point symmetry transformations of systems from $L|_{S}$, $G_{\sim}^{1} := \{ (\theta, \Phi, \theta) | \theta \in S, \Phi \in G_{\theta} \}$. The kernel point symmetry group $G_{\sim} := \cap_{\theta \in S} G_{\theta}$ of systems from the class $L|_{S}$, which consists of the common point symmetries of these systems, can be associated with the normal subgroup $G_{\sim}^{\sim}$ of $G_{\sim}$ whose elements are obtained from elements of $G_{\sim}$ by the standard prolongation to $r'$th order jets $(x, u, \varpi(\varpi'))$ and the trivial prolongation to the arbitrary-element tuple $\theta$, $G_{\sim}^{\sim} = \varpi_{*} \bar{G}_{\sim}$. Thus, $G_{\sim}^{\sim}$ is the unfaithful subgroup of $G_{\sim}$ under the action on $L|_{S}$.

The $s$-, the $t$- and the conjugation actions of $G_{\sim}$ on $G_{\sim}$ respectively defined by

$$T = (\theta, \Phi, \bar{\theta}) \mapsto (T_{s} \theta, \Phi \circ (\varpi_{*} T)^{-1} \bar{\theta}), (\theta, (\varpi_{*} T) \circ \Phi, T_{s} \bar{\theta}), (T_{s} \theta, (\varpi_{*} T) \circ \Phi \circ (\varpi_{*} T)^{-1}, T_{s} \bar{\theta})$$

for any $T \in G_{\sim}$ and for any $T = (\theta, \Phi, \bar{\theta}) \in G_{\sim}$, induce several equivalence relations on $G_{\sim}$ (s-$G_{\sim}$-equivalence, t-$G_{\sim}$-equivalence, $G_{\sim}$-conjugation and $G_{\sim}$-equivalence).

**Definition 1.** Admissible transformations $T_{1} = (\theta_{1}, \Phi_{1}, \bar{\theta}_{1})$ and $T_{2} = (\theta_{2}, \Phi_{2}, \bar{\theta}_{2})$ in the class $L|_{S}$ are called conjugate with respect to the equivalence group $G_{\sim}$ of this class if there exists $T \in G_{\sim}$ such that $\theta_{2} = T_{s} \theta_{1}$, $\bar{\theta}_{2} = T_{s} \bar{\theta}_{1}$ and $\Phi_{2} = (\varpi_{*} T) \circ \Phi_{1} \circ (\varpi_{*} T)^{-1}$. Admissible transformations $T_{1}$ and $T_{2}$ are called equivalent with respect to $G_{\sim}$ if there exist $T, \tilde{T} \in G_{\sim}$ such that $\theta_{2} = T_{s} \theta_{1}$, $\bar{\theta}_{2} = \tilde{T}_{s} \bar{\theta}_{1}$ and $\Phi_{2} = (\varpi_{*} T) \circ \Phi_{1} \circ (\varpi_{*} T)^{-1}$.

An admissible transformation in $L|_{S}$ belongs to $G_{\sim}^{G_{\sim}}$ if and only if this admissible transformation is $G_{\sim}$-equivalent in the above sense to the identity admissible transformation with the same source system. Another terminology was used in [51], where the stronger equivalence relation
of $G^\sim$-conjugation of admissible transformations was called by their $G^\sim$-equivalence, i.e., in the present paper we use a weaker notion of $G^\sim$-equivalence of admissible transformations.

Since the fundamental groupoid is a normal subgroupoid of $G^\sim$, the Frobenius product

$$G^\sim \ast G^\sim = \{ T \ast T' | T \in G^\sim, T' \in G^\sim, t(T) = s(T') \}$$

is a subgroupoid of $G^\sim$, which coincides with the image of $G^\sim$ under the $s$-action (resp. the $t$-action) of $G^\sim$ on $G^\sim$.

There are several kinds of classes of differential equations that are convenient for group classification by the algebraic method in different ways \cite{9, 28, 47, 51}.

**Definition 2.** The class $L|_S$ is called normalized if $G^\sim = G^\sim$. It is called semi-normalized if $G^\sim \ast G^\sim = G^\sim$. Depending on the kind of the equivalence group $G^\sim$ (the usual, the generalized, the extended or the extended generalized equivalence group of $L|_S$), we distinguish the (semi-)normalization in the usual, the generalized, the extended, or the extended generalized sense.

**Definition 3.** Let $G^H$ be the action groupoid of a subgroup $H$ of $G^\sim$. Suppose that a family $N_S := \{ \theta_\Phi < G^\sim | \theta \in S \}$ of subgroups of the point symmetry groups $G_\theta$ with the associated subgroups $N_\theta := \{ \theta, \Phi, \theta \} | \theta \in S, \Phi \in N_\theta \}$ of the vertex groups $G_\theta$ satisfies the property $T_\Phi \theta = T_\theta \Phi$ for any $\theta \in S$ and for any $T \in G^H$ with $s(T) = \theta$. Then the Frobenius product $N^T \ast G^H = \{ T \ast T' | T \in N^T, T' \in G^H, t(T) = s(T') \}$, with $N^T := \cup_{\theta \in S} N_{\theta}$ is a subgroupoid of $G^\sim$, which coincides with the image of $N^T$ under the $s$-action (resp. the $t$-action) of $H$ on $G^\sim$. If $N^T \ast G^H = G^\sim$, we call the class $L|_S$ semi-normalized with respect to the subgroup $H$ of $G^\sim$ and the family $N_S$ of subgroups of the point symmetry groups. If additionally $G^H \cap N^T = S \Rightarrow S$, then the class $L|_S$ is called disjointedly semi-normalized with respect to the subgroup $H$ of $G^\sim$ and the family $N_S$ of subgroups of the point symmetry groups.

If $H = G^\sim$ and $N_\theta = \{ \text{id}_{x,u} \}$ for any $\theta \in S$, then a class (disjointedly) semi-normalized with respect to the group $H$ and the family $N_S$ is literally normalized. If $H = G^\sim$ and $N_\theta = G_\theta$ for any $\theta \in S$, then a class semi-normalized with respect to the group $H$ and the family $N_S$ is literally semi-normalized. It is obvious that a normalized class is semi-normalized.

The most powerful method for describing admissible transformations within a class of differential equations is the *direct method*, which is based on the definition of admissible transformations. Applying this method to the class $L|_S$, we consider an arbitrary pair $(\theta, \bar{\theta}) \in S \times S$ and a point transformation in the space with coordinates $(x, u)$ of the most general form $\Phi$: $\bar{x} = X(x, u), \bar{u} = U(x, u)$ with nonzero Jacobian $|\partial(X, U)/\partial(x, u)|$ and assume that $\Phi_{\bar{\theta}}L_{\theta} = L_{\bar{\theta}}$. Expressing the required derivatives of $\bar{u}$ with respect to $\bar{x}$ in terms of derivatives of derivatives of $u$ with respect to $x$ using the chain rule, we substitute the deriving expressions into the system $L_{\bar{\theta}}$, obtaining the system $\Phi_{\bar{\theta}}^{-1}L_{\bar{\theta}}$, which should be identically satisfied by solutions of the system $L_{\theta}$. To take into account the last condition, we fix a ranking of derivatives of $u$ that is consistent with the structure of $L_{\theta}$, substitute the expressions for the leading derivatives of $u$ in view of the system $L_{\theta}$ and its differential consequences into $\Phi_{\bar{\theta}}^{-1}L_{\bar{\theta}}$ and split the resulting system with respect to the involved parametric derivatives of $u$. As a result, we obtain a system that implies both the expression of $\bar{\theta}$ via $(\theta, X, U)$ and the system DE of determining equations for components of $\Phi$. The system DE involves only the arbitrary-element tuple $\theta$ (resp. $\Phi_{\bar{\theta}}^{-1} \bar{\theta}$).

Assuming $\theta$ varying within $S$ and splitting with respect to derivatives of $\theta$ in view of the auxiliary system defining the set $S$ \footnote{This means that we set a ranking among the derivatives of $\theta$ that is consistent with structure of the auxiliary system, solve this system jointly with its differential consequences for the leading derivatives of $\theta$, substitute the derived expressions into DE and split the obtained system with respect to the parametric derivatives of $\theta$.}, we get the system DE of determining equations for the $(x, u)$-components of usual equivalence transformations. After finding the $(x, u)$-components via
the integration of $DE^{-\gamma}$, the $\theta$-component of usual equivalence transformations is obtained from the above expression for $\theta$. As a result, we construct the usual equivalence group $G^{\sim}$.

If the solution sets of $DE$ and $DE^{-\gamma}$ coincide, then $G^{\sim} = G^{\sim^{-\gamma}}$, i.e., the class $L|S$ is normalized, which completes the description of the equivalence groupoid $G^{\sim}$. Otherwise, the class $L|S$ is not normalized, and integrating $DE$, which can be carried out up to $G^{\sim^{-\gamma}}$-equivalence of admissible transformations, is a complicated problem. A number of various techniques can be used to simplify the solution of this problem. Below we present some of them.

**Partition of classes.** Let the set $S$ be represented as a disjoint union of its subsets, $S = \sqcup_{\gamma \in \Gamma} S_{\gamma}$ with some index set $\Gamma$, where each of the subsets $S_{\gamma}$ is singled out from $S$ by additional constraints, which are differential equations or differential inequalities. The partition of $S$ is equivalent to the partition of the class $L|S$ into the subclasses $L|S_{\gamma}$, with $\gamma$ running through $\Gamma$, $L|S = \sqcup_{\gamma \in \Gamma} L|S_{\gamma}$. Denote by $G^{\sim^{-\gamma}}$ and by $G^{\sim_{\gamma}}$ the equivalence group and the equivalence groupoid of the subclass $L|S_{\gamma}$, respectively.

If systems from different subclasses of the partition are not related by point transformations, then the partition of the class $L|S$ induces the partition $G^{\sim^{-\gamma}} = \sqcup_{\gamma \in \Gamma} G^{\sim_{\gamma}}$ of its equivalence groupoid. In general, the structure of the groupoid of a subclass may even be more complicated than that of the entire class. This is why a preliminary analysis of the system $DE$ is needed for an appropriate partition of the class $L|S$, where the structure of $G^{\sim_{\gamma}}$ is simpler than $G^{\sim^{-\gamma}}$ for any $\gamma \in \Gamma$. Then we can study subgroupoids separately and then unite them. The best kind of partitions is given by partitions into normalized subclasses, for which $G^{\sim_{\gamma}} = G^{\sim\gamma}$ and thus $G^{\sim^{-\gamma}} = \sqcup_{\gamma \in \Gamma} G^{\sim\gamma}$ [47, 51].

There are several generalization of the partition technique.

Disjoint subclasses may be related by point transformations, and thus the partition of the class $L|S$ into the subclasses $L|S_{\gamma}$ does not induce the partition of the equivalence groupoid $G^{\sim^{-\gamma}}$ into the equivalence groupoids $G^{\sim\gamma}$. Consider a simple situation, where we have a partition of $L|S$ into normalized subclasses $L|S_{\gamma}$, $\gamma \in \Gamma$, and for fixed $\gamma_{0} \in \Gamma$ and for each $\gamma \in \Gamma$ there exists a point transformation $\Phi_{\gamma}$ that maps $L|S_{\gamma_{0}}$ onto $L|S_{\gamma}$. We can assume that $\Phi_{\gamma_{0}} = id_{(x, \omega)}$. In fact, for any $\gamma \in \Gamma$ the normalization of $L|S_{\gamma}$ follows from the normalization of $L|S$ and the existence of $\Phi_{\gamma}$. Then

$$G^{\sim^{-\gamma}} = \left\{ (\Phi_{\gamma}^{-1})_{*}\theta, \Phi_{\gamma_{0}}^{-1} \circ (\omega \mathcal{T}) \circ \Phi_{\gamma}, (\Phi_{\gamma_{0}} \circ (\omega \mathcal{T})_{*}\theta) \mid \theta \in S_{\gamma_{0}}, \mathcal{T} \in G^{\sim\gamma_{0}}, \gamma, \gamma' \in \Gamma \right\}$$

(2)

This structure is admitted by the groupoid of the class (34), where the parameter $\sigma$ plays the role of $\gamma$, see Remark 19 below.

The condition that the class $L|S$ is a disjoint union of appropriate subclasses can be weakened by allowing a proper intersection of subclasses in the union. Thus, in [54] a class of variable-coefficient reaction–diffusion equations with power nonlinearities was represented as a non-disjoint union of normalized subclasses, and its groupoid was proved to be constituted by the admissible transformations for the action groupoids of the subclasses and the compositions of such composable admissible transformations from the action groupoids of different subclasses with nonempty intersections.

**Construction of generalized/extended/generalized extended equivalence group.** If the class $L|S$ is not normalized in the usual sense, one can try to describe its equivalence groupoid $G^{\sim}$ via finding a generalized counterpart of the usual equivalence group $G^{\sim}$, with respect to which the class $L|S$ is normalized in the corresponding sense [40, 51].

**Mappings between classes.** Suppose that there are a class $L'|S'$ of (systems of) differential equations with the same independent and dependent variables $x$ and $u$ as systems from the class $L|S$ and a family of point transformations $\mathcal{F} = \{ \Psi^{\theta}_{u} \mid \theta \in S \}$ such that $\Psi^{\theta}_{u} L_{\theta} \in L'|S'$ for any $\theta \in S$, and for any $\theta' \in S'$ there exists $\theta \in S$ with $\Psi^{\theta}_{u} L_{\theta} = L_{\theta'}$. Then we say that the family $\mathcal{F}$ generates the mapping $\mathcal{F}_{\theta}$ from the class $L|S$ onto the class $L'|S'$, where $\mathcal{F}_{\theta} := \Psi^{\theta}_{u} L_{\theta}$, or, equivalently, the mapping $\mathcal{F}_{\sigma} : S \to S'$ with $F_{\theta} = \theta'$ if $\Psi^{\theta}_{u} L_{\theta} = L_{\theta'}$; see [51, 56] for the first explicit discussions of mappings between classes. Via $\mathcal{F}_{\sigma}$, the family $\mathcal{F}$ also induces the mapping from the
equivalence groupoid $\mathcal{G}^\sim$ of the class $\mathcal{L}|_S$ to the equivalence groupoid $\mathcal{G}^\sim'$ of the class $\mathcal{L}'|_{S'}$ that is defined by
\[
\mathcal{G}^\sim \ni \mathcal{T} = (\theta, \Phi, \tilde{\theta}) \mapsto (\Psi^\theta_\theta, \Psi^\theta \circ \Phi \circ (\Psi^\theta)^{-1}, \Psi^\theta \tilde{\theta}) \in \mathcal{G}^\sim'.
\]
We will denote this mapping by the same symbol as the corresponding mapping between classes. The mapping $\mathcal{F}_* : \mathcal{G}^\sim \to \mathcal{G}^\sim'$ is in fact a groupoid homomorphism since
\[
\mathcal{F}_*(\mathcal{T}_1 \circ \mathcal{T}_2) = (\mathcal{F}_* \mathcal{T}_1) \circ (\mathcal{F}_* \mathcal{T}_2), \quad \mathcal{F}_*(\text{id}_\theta) = \text{id}_{\mathcal{F}_* \theta}, \quad \mathcal{F}_*(\mathcal{T}^{-1}) = (\mathcal{F}_* \mathcal{T})^{-1}
\]
for any $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{G}^\sim$, any $\theta \in S$ and any $\mathcal{T} \in \mathcal{G}^\sim$. Moreover, this homomorphism is surjective. Indeed, take any $\mathcal{T}' = (\theta', \Phi', \tilde{\theta}') \in \mathcal{G}^\sim'$. By the choice of the family $\mathcal{F}$, there exist $\theta, \tilde{\theta} \in S$ such that $\mathcal{F}_*(\theta) = \theta'$ and $\mathcal{F}_*(\tilde{\theta}) = \tilde{\theta}'$. The triple $\mathcal{T} = (\theta, \Phi, \tilde{\theta})$ with $\Phi = (\Psi^\theta)^{-1} \circ \Phi' \circ \Psi^\theta$ belongs to $\mathcal{G}^\sim$, and $\mathcal{F}_* \mathcal{T} = \mathcal{T}'$.

An important particular case of mappings between classes is constituted by mappings of classes to their subclasses that are generated by subgroups of the corresponding equivalence groups. Let $S'$ be a subset of $S$ that is singled out from $S$ by additional auxiliary differential equations or inequalities with respect to the arbitrary-element tuple $\theta$. Thus, $\mathcal{L}|_{S'}$ is a subclass of the class $\mathcal{L}|_S$, and its equivalence groupoid $\mathcal{G}^\sim|$ is a subgroupoid of the equivalence groupoid $\mathcal{G}^\sim$ of the class $\mathcal{L}|_S$. Suppose that for some subgroup $H$ of $\mathcal{G}^\sim$ each orbit of the action groupoid $\mathcal{G}^H$ intersects $S'$ by a single $\theta'$. Denote by $\Psi^\theta$ the point transformation $\tau_* \mathcal{T}$ with $\mathcal{T} \in H$ such that $\tau_* \theta \in S'$. Then the family $\mathcal{F} = \{\Psi^\theta \mid \theta \in S\}$ satisfies the required conditions to generate the corresponding mapping $\mathcal{F}_* : \mathcal{L}|_S \to \mathcal{L}|_{S'}$ and the corresponding surjective homomorphism $\mathcal{F}_* : \mathcal{G}^\sim \to \mathcal{G}^\sim|$. In practice, such mappings are realized via gauging arbitrary elements by equivalence transformations.

Under an appropriate choice of $\mathcal{F}$, the structure of $\mathcal{L}|_{S'}$ is simpler than the structure of $\mathcal{G}^\sim$. Then after the study of $\mathcal{L}|_{S'}$, we can pull back obtained results with respect to $\mathcal{F}$ and thus get results on $\mathcal{G}^\sim$. For example, how to appropriately partition a class into its subclasses can become evident only after a mapping of this class to another class [39]. The complete group classification of $\mathcal{L}|_S$ up to $\mathcal{G}^\sim$-equivalence can be easily derived from the analogous classification of $\mathcal{L}|_{S'}$ using the pullback by $\mathcal{F}$. In this way, the complete group classifications of the classes of (1+1)-dimensional Kolmogorov and Fokker–Planck equations modulo the general point equivalence were obtained from the classical group classification of the class of linear heat equations with potentials, see Corollaries 7 and 17 in [52]. Using mappings between classes for deriving complete group classifications up to $\mathcal{G}^\sim$-equivalence needs a more delicate consideration [56].

**Conditional equivalence groups.** The equivalence group of a subclass of the class $\mathcal{L}|_S$ is called a conditional equivalence group of this class. A conditional equivalence group $\mathcal{G}^\sim'$ of $\mathcal{L}|_S$ associated with the $\mathcal{L}|_{S'}$ is called maximal if for any subclass of $\mathcal{L}|_S$ properly containing $\mathcal{L}|_{S'}$, its equivalence group does not contain $\mathcal{G}^\sim'$. The equivalence group $\mathcal{G}^\sim$ of the entire class $\mathcal{L}|_S$ acts on subclasses of $\mathcal{L}|_S$ simultaneously with their equivalence groups, and the set of maximal conditional equivalence groups of $\mathcal{L}|_S$ is closed under this action. Hence maximal conditional equivalence groups of $\mathcal{L}|_S$ can be classified modulo $\mathcal{G}^\sim$-equivalence. This classification can be a step in the complete description of $\mathcal{G}^\sim$ [40, 38, 51]. It can be combined with a partition of $\mathcal{L}|_S$ into subclasses that is consistent with the structure of the set of maximal conditional equivalence groups of $\mathcal{L}|_S$. Generalized versions of conditional equivalence groups also can be considered [38].

**Generating set of admissible transformations.** A set $\mathcal{B} = \{\mathcal{T}_\gamma \in \mathcal{G}^\sim \mid \gamma \in \Gamma\}$, where $\Gamma$ is an index set, is called a generating set of admissible transformations for the class $\mathcal{L}|_{S'}$ up to $\mathcal{G}^\sim$-equivalence if any admissible transformation of this class can be represented the composition of a finite number of elements of the set $\mathcal{B} \cup \mathcal{B} \cup \mathcal{G}^\sim$, where $\mathcal{B}$ is the set of inverses of admissible transformations from $\mathcal{B}$, $\mathcal{B} := \{\mathcal{T}^{-1} \mid \mathcal{T} \in \mathcal{B}\}$. To make the set $\mathcal{B}$ as small as possible, it is natural to choose $\mathcal{B}$ as a subset of $\mathcal{G}^\sim \setminus \mathcal{G}^\sim$. Moreover, if a canonical representative in a family
of $G^\sim$-equivalent admissible transformations can be assigned, only this representative should be selected from the family for including in $B$.

**Furcate splitting.** This technique was suggested in [36] as a refinement of the direct method of group classification. Its essence is a special way of handling the system of determining equations for Lie symmetries of systems from the class under study depending on the possible number of independent constraints on values of $\theta$ that are induced by this system. This is why it can be extended to descriptions of other objects that are related to systems from classes of differential equations and are computed via solving certain systems of determining equations, including conservation laws [9], conditional equivalence groups [40] and generating sets of admissible transformations (see footnote 4 below). The method of furcate splitting can further be enhanced with involving algebraic techniques [7, 9].

(Bijective) functors between groupoids. Suppose that we construct an isomorphism between $G^\sim$ and the equivalence groupoid $\tilde{G}^\sim$ of a class $\mathcal{L}|_S$, and the description of $\tilde{G}^\sim$ has been known or it is easier or more convenient to describe the groupoid $\tilde{G}^\sim$ than the original groupoid $G^\sim$. For the latter option, for example, some computation techniques that are relevant for $\tilde{G}^\sim$ might be unapplicable to $G^\sim$. Here it is not necessary for the classes $\mathcal{L}|_S$ and $\tilde{\mathcal{L}}|_S$ to be related by a family of point transformations. Then the description of $\tilde{G}^\sim$ implies the description of $G^\sim$.

The technique involving bijective functors is effectively applied to the study of equivalence groupoids of classes of differential equations for the first time in the present paper. It is still unclear whether the construction of non-bijective functors from $G^\sim$ to $\tilde{G}^\sim$ or conversely is useful as well.

The infinitesimal counterparts of the (pseudo)groups $G^\theta$, $G^\cap$ and $G^\sim$ are Lie algebras $\mathfrak{g}^\theta$, $\mathfrak{g}^\cap$ and $\mathfrak{g}^\sim$ that are constituted by the generators of local one-parameter subgroups of the corresponding groups and which are called the maximal Lie invariance algebras of the systems $\mathcal{L}_\theta$, the kernel invariance algebra of systems from the class $\mathcal{L}|_S$ and equivalence algebra of the class $\mathcal{L}|_S$, respectively. Note that $\mathfrak{g}^\cap = \cap_{\theta \in S} \mathfrak{g}_\theta$.

The (complete) group classification problem for the class $\mathcal{L}|_S$ up to $G^\sim$-equivalence (resp. up to $\tilde{G}^\sim$-equivalence) is to find $\mathfrak{g}^\cap$ and an exhaustive list of $G^\sim$-inequivalent (resp. $\tilde{G}^\sim$-inequivalent) values of $\theta$ jointly with the corresponding algebras $\mathfrak{g}_\theta$ for which $\mathfrak{g}_\theta \neq \mathfrak{g}^\cap$. An admissible transformation from $\tilde{G}^\sim \setminus G^\sim$ between systems from the final group classification list modulo $G^\sim$-equivalence is called an additional equivalence transformation. Supplementing the group classification up to $G^\sim$-equivalence with the complete set of additional equivalence transformations results in the group classification up to.

Any version of the algebraic method of group classification in fact reduces to the classification, modulo $G^\sim$-equivalence, of certain subalgebras contained by the span $\mathfrak{g}_(\lambda) := \langle \mathfrak{g}_\theta, \theta \in S \rangle$. The efficiency of using the algebraic method depends on additional conditions satisfied by the class $\mathcal{L}|_S$, in particular, how consistent with $G^\sim$-equivalence the span $\mathfrak{g}_(\lambda)$ is [5, Section 12].

Normalized classes are the most convenient for group classification by the algebraic method. If the class $\mathcal{L}|_S$ is normalized, then $\mathfrak{g}_(\lambda) \subseteq \mathfrak{g}^\sim$, and the solution of the complete group classification for this class reduces to the classification of appropriate subalgebras of $\mathfrak{g}^\sim$ whose pushforwards by $\varpi$ can be qualified as the maximal Lie invariance algebras of systems from $\mathcal{L}|_S$. Since then $G^\sim$-equivalence coincides with $\tilde{G}^\sim$-equivalence, it is obvious that there are no additional equivalence transformations between Lie-symmetry extensions classified modulo $G^\sim$-equivalence. Moreover, it is inessential which of the two equivalences is used in the course of the classification. The above is also true if the class $\mathcal{L}|_S$ is semi-normalized with respect to a subgroup $H$ of $G^\sim$ and a family $\mathcal{N}_S$ of subgroups of the point symmetry groups, and additionally the subgroup $H$ and the family $\mathcal{N}_S$ are known. In this case, we call the class $\mathcal{L}|_S$ definitely semi-normalized, and looking for $G^\sim$-inequivalent subalgebras of $\mathfrak{g}^\sim$ is substituted in the algebraic method by looking for $H$-inequivalent subalgebras of the infinitesimal counterpart $\mathfrak{h}$ of $H$ [28]. The pure
half-normalization of \( L|\mathcal{S} \) at least guarantees the group classification \( L|\mathcal{S} \) up to \( G^\sim \)-equivalence coincides that up to \( G^\sim \)-equivalence.

If the class \( L|\mathcal{S} \) is not normalized, then some Lie symmetry extensions within this class are not related to subalgebras of its equivalence algebra \( \mathfrak{g}^\sim \).

**Definition 4.** We call the maximal Lie invariance algebra \( \mathfrak{g}_\theta \) of the system \( L_\theta \) from the class \( L|\mathcal{S} \) regular in this class if there exists a subalgebra \( \mathfrak{s} \) of \( \mathfrak{g}^\sim \) such that \( \mathfrak{g}_\theta = \varpi_\ast \mathfrak{s} \), and singular in \( L|\mathcal{S} \) otherwise.

If \( \mathfrak{g}_\theta = \mathfrak{g}^\sim \), then the maximal Lie invariance algebra \( \mathfrak{g}_\theta \) is regular in \( L|\mathcal{S} \) since \( \mathfrak{g}^\sim \subseteq \varpi_\ast \mathfrak{g}^\sim \). If \( \mathfrak{g}_\theta \neq \mathfrak{g}^\sim \), then we say that the pair constituted by the value of the arbitrary-element tuple and the algebra \( \mathfrak{g}_\theta \) presents a regular (resp. singular) Lie-symmetry extension of \( \mathfrak{g}^\sim \) in the class \( L|\mathcal{S} \) if \( \mathfrak{g}_\theta \) is a regular (resp. singular) maximal Lie invariance algebra in this class.

It is obvious that the sets of regular and singular Lie-symmetry extensions are separately invariant with respect to the action of \( G^\sim \) but in general this is not the case for the action of \( \mathcal{G}^\sim \). In other words, regular Lie-symmetry extensions are \( G^\sim \)-inequivalent to singular ones but may be \( \mathcal{G}^\sim \)-equivalent to them, see Remark 11 as an example on this claim. This also means that the Lie-symmetry extension for a system \( L_\theta \) with \( \theta \) satisfying \( s^{-1}(\theta) \neq s^{-1}(\theta) \cap G^\sim \) or, equivalently, \( t^{-1}(\theta) \neq t^{-1}(\theta) \cap G^\sim \) (i.e., for a system being sources or targets of admissible transformations that are not generated by equivalence transformations in \( L|\mathcal{S} \)) may also be regular. This is definitely the case if the set \( s^{-1}(\theta) \cap G^\sim \) contains only discrete admissible transformations, as it appears for the regular Cases 14d and 19d of Table 1 below.

Using Definition 4, we suggest the following procedure of group classification for a non-normalized class \( L|\mathcal{S} \) of differential equations within the framework of the algebraic method.

1. Describe the equivalence groupoid \( \mathcal{G}^\sim \) of the class \( L|\mathcal{S} \) up to \( G^\sim \)-equivalence, i.e., via constructing a generating set \( \mathcal{B} \) of admissible transformations.

2. Classify, modulo \( \mathcal{G}^\sim \)-equivalence, Lie symmetries of systems \( L_\theta \) with \( \theta \) satisfying the condition \( s^{-1}(\theta) \neq s^{-1}(\theta) \cap G^\sim \) or, equivalently, \( t^{-1}(\theta) \neq t^{-1}(\theta) \cap G^\sim \). This leads to the complete list of \( \mathcal{G}^\sim \)-inequivalent Lie-symmetry extensions within the class \( L|\mathcal{S} \) that are singular or regular but related to other Lie-symmetry extensions with elements from \( \mathcal{G}^\sim \setminus G^\sim \). Here both the direct and the algebraic methods of group classification might be applicable.

3. Carry out the (complete) preliminary group classification of the class \( L|\mathcal{S} \). The optimized version of such classification includes the classification of candidates for appropriate subalgebras of the equivalence algebra \( \mathfrak{g}^\sim \) and the construction of systems from the class \( L|\mathcal{S} \) that possess the projections of the above candidates by \( \varpi_\ast \) as their Lie invariance algebras. For each obtained system, we select the candidate that is maximal by inclusion; such a candidate always exists.

4. Merge the lists obtained in steps 2 and 3 and exclude repetitions up to \( \mathcal{G}^\sim \)-equivalence, which leads to the complete list of Lie symmetry extensions within the class \( L|\mathcal{S} \) up to \( \mathcal{G}^\sim \)-equivalence.

5. Extend the part of the list from step 4 that is related to the cases of step 2 by admissible transformations from the set \( \mathcal{B} \) modulo \( \mathcal{G}^\sim \)-equivalence. This gives the complete list of Lie symmetry extensions within the class \( L|\mathcal{S} \) up to \( \mathcal{G}^\sim \)-equivalence. All possible additional equivalence transformations between cases in this list are generated elements of \( \mathcal{B} \) modulo \( \mathcal{G}^\sim \)-equivalence.

The order of steps or even single operations may vary depending on the class of differential equations to be studied.

\(^{3}\)More specifically, the kernel invariance group is naturally embedded into \( \mathfrak{g}^\sim \) via the standard prolongation of its elements to \( u_{(r)} \) in view of the contact structure and the trivial prolongation to the arbitrary elements \( \theta \) [13].
3 Preliminary study of admissible transformations

In order to find the complete point equivalence group $G^\sim$ of the class (1) (including both continuous and discrete equivalence transformations) and the equivalence groupoid $G^\sim$ of this class, it is necessary to apply the direct method of computing point transformations that relate systems of differential equations. We will start our consideration with a preliminary study of admissible transformations of the superclass $\mathcal{W}_{\text{gen}}$, which constitute the groupoid $G^\sim_{\text{gen}}$ of $\mathcal{W}_{\text{gen}}$. This will also give relevant information on the group $G^\sim$ and the groupoid $G^\sim$.

Denote by $\mathcal{L}_\theta$ the equation in the class $\mathcal{W}_{\text{gen}}$ that corresponds to a fixed value of the arbitrary-element tuple $\theta = (f, g)$. An admissible transformation of the class $\mathcal{W}_{\text{gen}}$ is a triple $(\theta, \tilde{\theta}, \Phi)$, where $\theta = (f, g)$ and $\tilde{\theta} = (\tilde{f}, \tilde{g})$ are respectively the source and target arbitrary-element tuples for $T$, and

$$
\Phi: \quad \tilde{t} = T(t, x, u), \quad \tilde{x} = X(t, x, u), \quad \tilde{u} = U(t, x, u)
$$

with nonvanishing Jacobian $J := \partial(T, X, U)/\partial(t, x, u)$ is a point transformation in the space with the coordinates $(t, x, u)$ that maps the equation $\mathcal{L}_\theta$ to the equation $\mathcal{L}_{\tilde{\theta}}$. Therefore, we should directly seek for all point transformations mapping a fixed equation $\mathcal{L}_\theta$ of the form (1) to an equation $\mathcal{L}_{\tilde{\theta}}$ of the same form,

$$
\tilde{u}_{\tilde{t}} = \tilde{f}(\tilde{x}, \tilde{u})\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{g}(\tilde{x}, \tilde{u}).
$$

To carry out the transformation $\Phi$ in practice, it is necessary to find its prolongation to derivatives of $u$ up to order two. For this we act by the total derivative operators $D_t$ and $D_x$, respectively, on the expression $\tilde{u}(\tilde{t}, \tilde{x}) = U(t, x, u)$, assuming $\tilde{t} = T(t, x, u)$ and $\tilde{x} = X(t, x, u)$. This gives

$$
\begin{align*}
\tilde{u}_t D_t T + \tilde{u}_x D_t X - D_t U &= 0, \\
\tilde{u}_t D_x T + \tilde{u}_x D_x X - D_x U &= 0, \\
\tilde{u}_{\tilde{t}}(D_t T)^2 + 2\tilde{u}_{\tilde{t}}(D_t T)(D_t X) + \tilde{u}_{\tilde{x}\tilde{x}}(D_t T) + \tilde{u}_{\tilde{x}\tilde{x}}(D_t X)^2 + \tilde{u}_t D_t^2 T + \tilde{u}_x D_t^2 X - D_t^2 U &= 0, \\
\tilde{u}_{\tilde{t}}(D_x T)^2 + 2\tilde{u}_{\tilde{t}}(D_x T)(D_x X) + \tilde{u}_{\tilde{x}\tilde{x}}(D_x X)^2 + \tilde{u}_t D_x^2 T + \tilde{u}_x D_x^2 X - D_x^2 U &= 0,
\end{align*}
$$

cf. [5]. Solving the last two equations for $u_{tt}$ and $u_{xx}$, respectively, and substituting the derived expressions into (1), we obtain

$$
\begin{align*}
\tilde{u}_{\tilde{t}}(D_t T)^2 + 2\tilde{u}_{\tilde{t}}(D_t T)(D_t X) + \tilde{u}_{\tilde{x}\tilde{x}}(D_t X)^2 + \tilde{u}_t V^{tt} T + \tilde{u}_x V^{tx} X - V^{tt} U \\
= f(\tilde{u}_{\tilde{t}}(D_x T)^2 + 2\tilde{u}_{\tilde{t}}(D_x T)(D_x X) + \tilde{u}_{\tilde{x}\tilde{x}}(D_x X)^2 + \tilde{u}_t V^{xx} T + \tilde{u}_x V^{xx} X - V^{xx} U) \\
- g(\tilde{u}_t T_u + \tilde{u}_x X_u - U_u),
\end{align*}
$$

where we use the notation $V^{tt} := \partial_{tt} + 2u_t \partial_{uu} + u_t^2 \partial_{uu}$ and $V^{xx} := \partial_{xx} + 2u_x \partial_{xu} + u_x^2 \partial_{uu}$. The substitution $\tilde{u}_{\tilde{t}} = \tilde{f}\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{g}$ in view of $\mathcal{L}_{\tilde{\theta}}$ into (4) wherever $u_{\tilde{t}}$ occurs leads to an identity with respect to $u_{\tilde{t}}$ and $u_{\tilde{x}}$. In particular, we can collect the coefficients of $\tilde{u}_{\tilde{x}\tilde{x}}$ in (4), which results in

$$
(T_t + T_u u_t)(X_t + X_u u_t) = f(T_x + T_u u_x)(X_x + X_u u_x).
$$

The equation (5) involves only original quantities (without tilde) and is a polynomial in $u_t$ and $u_x$. Therefore, we can split it with respect to $u_t$ and $u_x$ by collecting the coefficients of different powers of these derivatives. As a result, we get

$$
\begin{align*}
T_u X_u &= 0, \\
T_u X_t + T_t X_u &= 0, \\
T_u X_x + T_x X_u &= 0,
\end{align*}
$$

(6) (7) (8)

11
\[ T_t X_t = f T_u X_u. \] (9)

We multiply the equation (7) by \( T_u \) (resp. \( X_u \)) and, in view of the equation (6), derive that \( T_u X_t = 0 \) (resp. \( T_t X_u = 0 \)). We apply the same trick also to the equation (8) to have the equations \( T_u X_t = 0 \) and (resp. \( X_u T_t = 0 \)). The system \( T_u X_t = 0, T_u X_x = 0, T_u X_u = 0 \) (resp. \( X_u T_t = 0, X_u T_x = 0, X_u T_u = 0 \)) implies that \( T_u = 0 \) (resp. \( X_u = 0 \)) since otherwise the Jacobian \( J \) of the point transformation (3) vanishes. The condition

\[ T_u = X_u = 0 \]

means that any admissible point transformation of the class (1) is fiber-preserving. In view of this condition, the equations (6)–(8) are identically satisfied, and the splitting of (4) with respect to \( \tilde{u}_{\pm \varepsilon} \) gives the equations

\[ \tilde{f} T_t^2 + X_t^2 = f (\tilde{f} T_x^2 + X_x^2), \] (10)

\[ \tilde{g} T_t^2 + \tilde{u}_t T_t + \tilde{u}_{\pm \varepsilon} X_t = (U_t + 2 U_t U_t + U_{uu} u_t^2) \]

\[ = f (\tilde{g} T_x^2 + \tilde{u}_t T_x + \tilde{u}_{\pm \varepsilon} X_x = (U_x + 2 U_x U_x + U_{uu} u_x^2)) + g U_u. \]

Splitting the last equation with respect to \( \tilde{u}_t \) and \( \tilde{u}_{\pm \varepsilon} \) gives the equations \( U_{uu} = 0 \) and

\[ T_t - 2 \frac{U_{uu}}{U_u} T_t = f \left( T_x - 2 \frac{U_{uu}}{U_u} T_x \right), \] (11)

\[ X_t - 2 \frac{U_{uu}}{U_u} X_t = f \left( X_x - 2 \frac{U_{uu}}{U_u} X_x \right), \] (12)

\[ \tilde{g} T_t^2 - U_t + 2 \frac{U_{uu}}{U_u} U_t = f \left( \tilde{g} T_x^2 - U_x + 2 \frac{U_{uu}}{U_u} U_x \right) + g U_u. \] (13)

The equation \( U_{uu} = 0 \) implies the representation \( U = U^1(t, x) u + U^0(t, x) \). The additional condition to keep in mind is the nondegeneracy of \( \Phi \), which in view of the conditions \( T_u = X_u = 0 \) reduces to the inequality \( U_u(T_u X_x - T_x X_u) \neq 0 \), and hence \( T_t X_x - T_x X_t \neq 0 \) and \( U^1 := U_u \neq 0 \).

Note that the equations \( T_u = X_u = U_{uu} = 0 \) for admissible point transformations within the class \( \mathcal{W}_{gen} \) can also be derived using item (c) of Theorem 4.4b in [27].

Rewriting the equation (10) as \( \tilde{f}(T_t^2 - f T_x^2) + X_t^2 - f X_x^2 = 0 \) shows that both the expressions \( T_t^2 - f T_x^2 \) and \( X_t^2 - f X_x^2 \) are nonzero,

\[ T_t^2 - f T_x^2 \neq 0, \quad X_t^2 - f X_x^2 \neq 0. \]

Indeed, otherwise \( f > 0 \), \( T_t = \varepsilon_1 f^{1/2} T_x \neq 0 \), where \( \varepsilon_1 = \pm 1 \), and hence the equation (9) would imply that \( X_t = \varepsilon_2 f^{1/2} X_x \) but this contradicts the nondegeneracy of \( \Phi \). Thus, \( \tilde{f}_u = 0 \) if \( f_u = 0 \). Conversely, supposing \( \tilde{f}_u = 0 \) and using the same argumentation for the inverse of \( T \), we derive that \( f_u = 0 \). Therefore, \( \tilde{f}_u = 0 \) if and only if \( f_u = 0 \).

In view of nonvanishing the expression \( T_t^2 - f T_x^2 \), the equation (13) similarly implies that \( \tilde{f}_u = \tilde{g} u_u = 0 \) if and only if \( f_u = g_{uu} = 0 \).

**Remark 5.** Preserving the constraint \( f_u = g_{uu} = 0 \) by admissible transformations of the class \( \mathcal{W}_{gen} \) means that the equivalence groupoid \( \mathcal{G}_{gen}^\sim \) of the class \( \mathcal{W}_{gen} \) is the disjoint union of the equivalence groupoids \( \mathcal{G}^\sim \) and \( \mathcal{G}_{lin}^\sim \) of the subclasses \( \mathcal{W} \) and \( \mathcal{W}_{lin} \), \( \mathcal{G}_{gen}^\sim = \mathcal{G}^\sim \sqcup \mathcal{G}_{lin}^\sim \). In other words, equations from the class \( \mathcal{W} \) are not related to equations from the class \( \mathcal{W}_{lin} \) by point transformations. This justifies the exclusion of the class \( \mathcal{W}_{lin} \) from the consideration, cf. the introduction.
4 Equivalence group and equivalence algebra

At this point, we continue the consideration by computing the equivalence group of the class (1) as it is needed both for the exhaustive description of admissible transformations and for the analysis of the determining equations for components of Lie symmetry vector fields. In the course of computing equivalence transformations, the arbitrary elements $f$ and $g$ of the class should be varied. We can therefore split the equations (9)–(13) with respect to these arbitrary elements. The equation (9) and the nondegeneracy constraint $T_t X_x - T_x X_t \neq 0$ imply that either $T_t = X_x = 0$ and $T_x X_t \neq 0$ or $T_x = X_t = 0$ and $T_t X_x \neq 0$.

For $T_t = X_x = 0$, the equation (10) is simplified to $X_t^2 = f \tilde{T}_x^2$, or $T_x^2 f = X_t^2 / \tilde{f}$. Differentiating the last equation with respect to $t$ and splitting the arising equation with respect to $\tilde{f}$ and its derivatives implies $X_t = 0$, which contradicts the nondegeneracy condition.

Therefore, we necessarily have $X_t = T_x = 0$ and thus $T = T(t)$, $X = X(x)$, where $X_x \neq 0$ and $T_t \neq 0$. Then the equation (10) reduces to $\tilde{T}_x^2 = f X_t^2$ and the differentiation of this equation with respect to $t$ yields

$$2T_t T_{tt} \tilde{f} + T_t^2 U_t \tilde{f}_u = 0. \tag{14}$$

Since the equation (14) holds for all $\tilde{f}$, we can split it with respect to $\tilde{f}$ and $\tilde{f}_u$ and derive $T_{tt} = 0$ and $U_t = 0$.

The equation (11) is identically satisfied in view of the above equations. The equation (12) reduces to the equation $(U_t^2 / X_x)_x = 0$ and the equation (13) yields the transformation relation for $g$.

Integrating the equations derived in view of the nondegeneracy condition $J \neq 0$, we proved the following theorem:

**Theorem 6.** The equivalence (pseudo)group $G^\sim$ of the class (1) consists of the transformations

$$\tilde{t} = c_1 t + c_0, \quad \tilde{x} = \varphi(x), \quad \tilde{u} = c_2 |\varphi_x|^{1/2} u + \psi(x),$$

$$\tilde{T} = \frac{\varphi_x^2}{c_1^2} f, \quad \tilde{g} = \frac{c_2}{c_1} |\varphi_x|^{1/2} \frac{1}{2|\varphi_x|^{1/2}} \left( c_2 \frac{2 \varphi_{xxx} \varphi_x - 3 \varphi_{xx}^2}{4|\varphi_x|^{3/2}} u + \psi_{xx} - \frac{\varphi_{xx}}{\varphi_x} \psi_x \right) f, \tag{15}$$

where $c_0$, $c_1$ and $c_2$ are arbitrary constants satisfying the condition $c_1 c_2 \neq 0$, and $\varphi$ and $\psi$ run through the set of smooth functions of $x$ with $\varphi_x \neq 0$.

**Corollary 7.** The class (1) admits exactly three discrete equivalence transformations that are independent up to combining with each other and with continuous equivalence transformations of this class. These are involutions alternating signs of variables, $(t, x, u, f, g) \mapsto (-t, x, u, f, g)$, $(t, x, u, f, g) \mapsto (t, -x, u, f, g)$ and $(t, x, u, f, g) \mapsto (t, x, -u, f, -g)$.

In contrast to [53], we have proved that there are no other independent discrete equivalence transformations for the class (1).

Theorem 6 implies that any transformation $\mathcal{T}$ from $G^\sim$ can be represented as the composition

$$\mathcal{T} = P^t(c_0) D^t(c_1) S(\psi) D(\varphi) D^u(c_2)$$

of elementary equivalence transformations, each of which belongs to a family of equivalence transformations parameterized by a single constant or functional parameter,
where the parameters are described in Theorem 6, and

$$\alpha^s(x) := \frac{2\phi_{xxx}\phi_x - 3\phi_{xx}^2}{4|\phi_x|^{3/2}}.$$ 

These transformations are shifts and scalings in $t$, arbitrary transformations in $x$, scalings of $u$ and shifts of $u$ with arbitrary functions of $x$, respectively.

The equivalence algebra $\mathfrak{g}^\sim$ of the class (1) can be easily derived as the set of vector fields that generate local one-parametric subgroups of the equivalence group $G^\sim$. It is spanned by the vector fields

$$P^t = \partial_t, \quad D^t = t\partial_t - 2f\partial_f - 2g\partial_g, \quad D^u = u\partial_u + g\partial_g,$$

$$D(\zeta) = \zeta\partial_x + \frac{1}{2}\zeta_x u\partial_u + 2\zeta_x f\partial_f + \frac{1}{2}(\zeta_x g - \zeta_x x u f)\partial_g,$$

$$Z(\chi) = \chi\partial_u - \chi x f \partial_g,$$

where $\zeta = \zeta(x)$ and $\chi = \chi(x)$ run through the set of smooth functions of $x$. The nonvanishing commutation relations between the these vector fields are exhausted by

$$[P^t, D^t] = P^t, \quad [Z(\chi), D^u] = Z(\chi),$$

$$[D(\zeta^1), D(\zeta^2)] = D(\zeta^{12}_x - \zeta^{21}_x), \quad [D(\zeta), Z(\chi)] = Z(\zeta \chi - \frac{1}{2}\zeta_x \chi).$$

### 5 Determining equations for Lie symmetries

Suppose that a vector field $Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$ belongs to the maximal Lie invariance algebra $\mathfrak{g}_{\text{max}}^\sim$ of an equation $\mathcal{L}$: $L = 0$ from the class (1), i.e., it is the generator of a one-parameter Lie symmetry group of $\mathcal{L}$. The criterion for infinitesimal invariance of $\mathcal{L}$ with respect to $Q$ [11, 37, 43] implies that $Q(2)|_{L=0} = 0$, where the notation $|_{L=0}$ means that the condition $Q(2)L$ is required to hold only on equations from the class (1) and $Q(2)$ is the second prolongation of $Q$. $Q(2) = Q + \eta^t\partial_u + \eta^t\partial_u + \eta^t\partial_d + \eta^t\partial_u + \eta^t\partial_d + \eta^t\partial_{ux}$. The coefficients $\eta^t$, $\eta^x$, $\eta^{tx}$ in $Q(2)$ can be determined from the general prolongation formula for vector fields. In particular,

$$\eta^{tx} = D^t_x(\eta - \tau u_t - \xi u_x) + \tau u_{tx} + \xi u_{xxx},$$

$$\eta^t = D^t_x(\eta - \tau u_t - \xi u_x) + \tau u_{tt} + \xi u_{tx},$$

where $D_t$ and $D_x$ denote the total derivative operators with respect to $t$ and $x$, respectively, which in the present case of variables are given by

$$D_t = \partial_t + u_t\partial_u + u_{tt}\partial_u + u_{tx}\partial_u + \cdots,$$

$$D_x = \partial_x + u_x\partial_u + u_{tx}\partial_u + u_{xx}\partial_u + \cdots.$$

Applying the infinitesimal invariance condition to the class (1) then yields

$$\eta^t - (\xi f_x + \eta f_u)u_{xx} - f\eta^{xx} - \xi g_x - \eta g_u = 0 \quad \text{for} \quad u_{tt} = fu_{xx} + g,$$  \hspace{1cm} (17)

It follows from the equations $T_u = X_u = U_{uu} = 0$ for admissible transformation within the class (1) that the vector field $Q$ is projectable to the space of independent variables and linear in $u$, i.e., $\tau_u = \xi_u = \eta_u = 0$ or $\tau = \tau(t, x)$, $\xi = \xi(t, x)$ and $\eta = \eta^t(t, x)u + \eta^0(t, x)$. Taking into account these restrictions and expanding the infinitesimal invariance condition (17), we obtain the equation

$$D^2_t \eta - \tau_t u_t - \xi_t u_x - 2\tau_t u_{tx} - 2\xi_t u_{xx}$$

$$= f(D^2_x \eta - \tau_{xx} u_t - \xi_{xx} u_x - 2\tau_{xx} u_{tx} - 2\xi_{xx} u_{xx}) + (\xi f_x + \eta f_u)u_{xx} + \xi g_x + \eta g_u,$$  \hspace{1cm} (18)
where we have to substitute \(u_{tt} = fu_{xx} + g\). Collecting the coefficients of the derivatives \(u_{tx}, u_{xx}, u_t\) and \(u_x\) in the above equation results in the system of determining equations

\[
\begin{align*}
\xi_t &= \tau_x f, \\
\tau_{tt} - \tau_{xx} f &= 2\eta_{tu}, \\
\xi_{tt} - \xi_{xx} f &= -2\eta_{xu} f, \\
\xi f_x + \eta_{fx} &= 2(\xi_x - \tau_t)f, \\
\xi g_x + \eta_{gx} &= (\eta_u - 2\tau_t)g - \eta_{xx} f + \eta_{tt}.
\end{align*}
\]

In order to derive the kernel algebra \(g^\triangledown\) of class \((1)\), we further split the determining equations with respect to the arbitrary elements and their derivatives. This immediately gives that

\[
g^\triangledown = \langle \partial_t \rangle.
\]

Consequently, the Lie symmetries admitted by each equation from the class \((1)\) are exhausted by transformations of the form \((t, x, u) \mapsto (t + c_0, x, u)\), where \(c_0\) is an arbitrary constant.

### 6 Results of classifications

For convenience, we collect, in a single table, the Lie-symmetry classification cases derived below and formulate the final result of group classification of the class \((1)\) as a theorem.

**Theorem 8.** All \(G^\sim\)-inequivalent (resp. point-inequivalent) cases of Lie symmetry extensions of the kernel algebra \(g^\triangledown = \langle \partial_t \rangle\) in the class \((1)\) are exhausted by cases presented in Table 1.

In each case of Table 1 we present only vector fields which extend the basis \(\langle \partial_t \rangle\) of \(g^\triangledown\) into a basis of the corresponding Lie invariance algebra. The spans of \(g^\triangledown\) and the vector fields given in cases of Table 1 that are parameterized by functions \(\tilde{f}\) or \(\tilde{g}\) are the maximal Lie invariance algebras of the corresponding equations for the general values of these parameter functions \(\tilde{f}\) and \(\tilde{g}\), but for certain values of \(\tilde{f}\) and \(\tilde{g}\) additional extensions are possible, which are equivalent to other cases of Table 1. Thus, \(\tilde{f} \neq \text{const}\) in Case 4 since otherwise up to \(G^\sim\)-equivalence we obtain Case 5a. There are also constraints for constant parameters that are imposed by the condition of inequivalence of the corresponding extensions or the condition of their maximality.

Depending on the dimension of Lie symmetry extension (one, two, three, four or infinity), we split the cases of Table 1 into groups separated by horizontal lines. Note that all Lie symmetry extensions of maximal and submaximal dimensions (infinity and four) for equations from the class \((1)\) are not associated with subalgebras of the equivalence algebra \(g^\sim\), i.e., they are singular.

The usage of two-level numeration for classification cases listed in Table 1 is justified by the presence of additional equivalences among them. Namely, numbers with the same Arabic numerals and different Roman letters correspond to cases that are \(G^\sim\)-inequivalent but equivalent with respect to additional equivalence transformations. To construct all additional equivalence transformations among \(G^\sim\)-inequivalent classification cases and thus to solve the group classification problem for the class \((1)\) up to \(G^\sim\)-equivalence, we need to classify admissible transformations of this class up to \(G^\sim\)-equivalence. This classification is presented in the following theorem, which is proved in Section 7.

**Theorem 9.** A complete list of \(G^\sim\)-inequivalent non-identity admissible transformations of the class \(W\) that are independent up to inversion and composing with each other is exhausted by the following families of admissible transformations, where \(\epsilon, \epsilon', \epsilon'' = \pm 1\):

\[
T1. \ f_x = g_x = 0, \ f_u \neq 0 \text{ or } f = 1, \ \tilde{f} = 1/f, \ \tilde{g} = -g/f, \ \Phi: \ \tilde{t} = x, \ \tilde{x} = t, \ \tilde{u} = u;
\]
Here \( \varepsilon, \varepsilon' = \pm 1 \mod G^-, \delta \in \{0, 1\} \mod G^- \), \( p, q \) and \( \nu \) are arbitrary constants with \( p \neq 0 \) and \( \nu 
eq 0 \). Additionally, \( p \neq \pm 4 \) in Case 16, and \( q \neq 0, 1 \) in Cases 18a–18c. In Case 1, \( \omega := x - \delta \ln |u|, \mathcal{R}(\Phi) := \Phi_x \delta t + \Phi_t \delta x \). The tuple \((\tau, \xi)\) of smooth functions depending on \(t, x\) runs through the solution set of the system \( \tau = \xi, \xi_t = \varepsilon \tau_x \).
T2. \( f = \varepsilon u^{-4}, \ g = \mu(x)u^{-3} + \sigma u, \ \sigma \in \{-1,0,1\}, \ \tilde{f} = \varepsilon \tilde{u}^{-4}, \ \tilde{g} = \mu(\tilde{x})\tilde{u}^{-3}, \ \mu \) runs through the set of smooth functions of \( x \) with \( \mu_x \neq 0 \);

a. \( \Phi: \ i = t^{-1}, \ \tilde{x} = x, \ \tilde{u} = t^{-1}u \) if \( \sigma = 0; \)
b. \( \Phi: \ i = \frac{1}{2}e^{2t}, \ \tilde{x} = x, \ \tilde{u} = e^t u \) if \( \sigma = 1; \)
c. \( \Phi: \ i = \tan t, \ \tilde{x} = x, \ \tilde{u} = u \cot t \) if \( \sigma = -1. \)

T3. \( f = 1, \ g = e^{-2x}g^2(u), \ \tilde{f} = 1, \ \tilde{g} = g^2(\tilde{u}), \ \Phi: \ i = e^{-x} \sinh t, \ \tilde{x} = e^{-x} \cosh t, \ \tilde{u} = u; \)

T4. \( f = 1, \ g = g^1(x)g^2(u), \ \tilde{f} = 1, \ \tilde{g} = \tilde{x}^{-2}g^2(\tilde{u}), \)

a. \( g^1(x) = x^{-2}, \ \Phi: \ i = \frac{t}{x^2 - t^2}, \ \tilde{x} = \frac{x}{x^2 - t^2}, \ \tilde{u} = u; \)

b. \( g^1(x) = \cos^{-2}x, \ \Phi: \ i = \frac{\cos t}{\sin t + \sin x}, \ \tilde{x} = \frac{\cos x}{\sin t + \sin x}, \ \tilde{u} = u; \)

c. \( g^1(x) = -\cosh^{-2}x, \ \Phi: \ i = e^t \sinh x, \ \tilde{x} = e^t \cosh x, \ \tilde{u} = u; \)

d. \( g^1(x) = \sinh^{-2}x, \ \Phi: \ i = e^t \cosh x, \ \tilde{x} = e^t \sinh x, \ \tilde{u} = u; \)

T5. \( f = -1, \ g = e^{-2x}g^2(u), \ \tilde{f} = -1, \ \tilde{g} = g^2(\tilde{u}), \ \Phi: \ i = e^{\varepsilon} \sin t, \ \tilde{x} = e^{-\varepsilon} \cos t, \ \tilde{u} = u; \)

T6. \( f = -1, \ g = g^1(x)g^2(u), \ \tilde{f} = -1, \ \tilde{g} = \tilde{x}^{-2}g^2(\tilde{u}), \)

a. \( g^1(x) = x^{-2}, \ \Phi: \ i = \frac{t}{x^2 + t^2}, \ \tilde{x} = \frac{x}{x^2 + t^2}, \ \tilde{u} = u; \)

b. \( g^1(x) = \cos^{-2}x, \ \Phi: \ i = e^t \sin x, \ \tilde{x} = e^t \cos x, \ \tilde{u} = u; \)

c. \( g^1(x) = \sinh^{-2}x, \ \Phi: \ i = \frac{\sin t}{\cos t + \cosh x}, \ \tilde{x} = \frac{\sinh x}{\cos t + \cosh x}, \ \tilde{u} = u; \)

T7. \( f = -1, \ g = g^2(u) \cosh^{-2}x, \ \tilde{f} = -1, \ \tilde{g} = g^2(\tilde{u}) \cosh^{-2} \tilde{x}, \)

\( \Phi: \ i = \arctan \frac{\sin \gamma \sinh x + \cos \gamma \sin t}{\cos t}, \ \tilde{x} = \arctanh \frac{\cos \gamma \sinh x - \sin \gamma \sin t}{\cosh x}, \ \tilde{u} = u, \)

\( \gamma \in (0,2\pi); \)

T8. \( a. \ f = \tilde{f} = 1, \ g_x = 0, \ \tilde{g} = g, \ \Phi: \ i = t \cosh \gamma + x \sinh \gamma, \ \tilde{x} = t \sinh \gamma + x \cosh \gamma, \ \tilde{u} = u, \ \gamma \in \mathbb{R}\neq 0; \)

b. \( f = \tilde{f} = -1, \ g_x = 0, \ \tilde{g} = g, \ \Phi: \ i = t \cos \gamma - x \sin \gamma, \ \tilde{x} = t \sin \gamma + x \cos \gamma, \ \tilde{u} = u, \ \gamma \in (0,2\pi); \)

T9. \( f = \varepsilon, \ g = e^{\varepsilon}u, \ \tilde{f} = \varepsilon, \ \tilde{g} = e^{\varepsilon}u, \ \Phi: \ i = T(t,x), \ \tilde{x} = X(t,x), \ \tilde{u} = u + \ln |T_x^2 - \varepsilon T_{xx}^2|, \)

where \( \varepsilon, e' = \pm 1, \varepsilon' = 1 \) if \( \varepsilon = 1, \ (T,X) \) runs through the solution set of the system \( T_t = X_x, \ X_t = eT_x \) with \( (T_t,T_x) \neq (0,0), \) i.e., point symmetry transformations specific for the Liouville equation \( u_{tt} - \varepsilon u_{xx} = e^{\varepsilon}u. \)

Throughout the rest of the paper, we use the notation \( T^N \) for the admissible transformation given in item \( N \) of Theorem 9, where \( N \) is the corresponding (one- or two-level) number.

**Remark 10.** The transformational part \( \Phi \) of admissible transformation \( T_{4b} \) can be represented as

\[ T_{4b}: \ i = \cot \frac{x+t}{2} + \tan \frac{x-t}{2}, \ \tilde{x} = \cot \frac{x+t}{2} - \tan \frac{x-t}{2}, \ \tilde{u} = u; \]

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The transformational parts $\Phi$ of admissible transformations $T4c$ and $T4d$ can be replaced by alternative ones, which are analogous to that of $T4b$,

$\mathbf{T4c}$: $\tilde{t} = \coth \frac{x + t}{2} - \tanh \frac{x - t}{2}$, $\tilde{x} = \coth \frac{x + t}{2} + \tanh \frac{x - t}{2}$, $\tilde{u} = u$;

$\mathbf{T4d}$: $\tilde{t} = \tanh \frac{x + t}{2} - \tanh \frac{x - t}{2}$, $\tilde{x} = \tanh \frac{x + t}{2} + \tanh \frac{x - t}{2}$, $\tilde{u} = u$.

There also exist similar alternatives for transformational parts of other admissible transformations in the class $W$. The counterparts of modified admissible transformations $T4b$–$T4d$ and of admissible transformation $T3$ for linear equations from the class $W_{\text{lin}}$ were presented in Notes 1 and 2 of [59].

As a result, we obtain the following independent additional equivalence transformations among classification cases given in Table 1. (Below we do not indicate the corresponding parameters if they are not changed.)

\begin{align*}
\mathbf{T1:} & \quad (a) \ 4f, g \to 4_{1/2f, -g/2}, \ 5a_g \to 5a_{-g} \text{ if } \varepsilon = 1, \ 11f \to 11_{1/f}, \ 12_{q, \varepsilon, \varepsilon'}(u \to -u) \to 12_{1-q, \varepsilon, -\varepsilon'}, \\
& \quad 13_{p, q, \varepsilon, \varepsilon'} \to 13_{-p, q, -\varepsilon, -\varepsilon'}, \ 14d, e' \to 14a, e' = e', \ 16_{p, \varepsilon} \to 16_{-p, \varepsilon}, \ 18a_{q, \varepsilon, -\varepsilon'}, \ 19d \to 19a, \ 20_{\varepsilon, e'} \to 20_{\varepsilon, -e'}; \\
\mathbf{T2:} & \quad (b) \ 8b \to 8a, \ 14b \to 14a, \ 15b \to 15a, \ 19b \to 19a; \\
& \quad (c) \ 8c \to 8a, \ 14c \to 14a, \ 15c \to 15a, \ 19c \to 19a.
\end{align*}

\begin{align*}
\mathbf{T3:} & \quad 5b \to 5a_{\varepsilon = 1}, \ 18b \to 18a_{\varepsilon = 1}. \quad \mathbf{T4:} \quad (b) \ 6b \to 6a_{\varepsilon = 1}, \ (c) \ 6c \to 6a_{\varepsilon = 1}, \ (d) \ 6d \to 6a_{\varepsilon = 1}. \\
\mathbf{T5:} & \quad 5c \to 5a_{\varepsilon = -1}, \ 18c \to 18a_{\varepsilon = -1}. \quad \mathbf{T6:} \quad (b) \ 6e \to 6a_{\varepsilon = -1}, \ (c) \ 6f \to 6a_{\varepsilon = -1}.
\end{align*}

\textbf{Remark 11.} In Table 1, only Cases 1–4, 9–13, 14d, 16, 17 and 19d present regular Lie symmetry extensions in the class $W$. Therefore, the regular Cases 14d and 19d are $G_{\sim}$-inequivalent but $G_{\sim}$-equivalent to the singular Cases 14a and 19a, respectively.

Consider the subclass $W_c$ of the class $W$ singled out by the additional constraints $f_x = g_x = 0$ for the arbitrary-element tuple $\theta = (f, g)$, i.e., the class of equations of the general form

$$ u_{tt} = f(u)u_{xx} + g(u), \tag{24} $$

where $(f_u, g_u) \neq (0, 0)$. Cases 4, 5a, 11, 12, 13, 14a–14d, 16, 17, 18a, 19a–19d and 20 of Table 1 are related to the subclass $W_c$. The kernel Lie invariance algebra $g_c = \langle \partial_t, \partial_x \rangle$ of equations from the subclass $W_c$ is given by Cases 4, which is the general case within this subclass. It is obvious from Theorem 8 and the above list of additional equivalence transformations that a complete list of $G_{\sim}$-inequivalent Lie-symmetry extensions within the subclass $W_c$, where $G_{\sim}$ is its equivalence groupoid, is exhausted by Cases 5a, 11, 12, 13, 14a, 16, 17, 18a, 19a and 20 of Table 1, where additionally $q > 1/2$ in Case 12, $p > 0$ in Cases 13 and 16, and $\varepsilon' = 1$ in Cases 18a and 20 with $\varepsilon = 1$. The group classification of the subclass $W_c$ up to equivalence generated by its equivalence group $G_{\sim}$ is more delicate. The group $G_{\sim}$ is generated by transformations of the form (15) with $\varphi_{xx} = \psi_x = 0$, which constitute the intersection $G_{\sim}^c \cap G_{\sim}$, and one more (discrete) equivalence transformation $t = x, \tilde{x} = t, \tilde{u} = u, \tilde{f} = 1/f, \tilde{g} = -f/g$ of $W_c$, which generates the family $T1$ of admissible transformations within $W$. This is why some $G_{\sim}$-inequivalent Lie-symmetry extensions can be $G_{\sim}$-equivalent, which occurs for Cases 14a and 14d as well as for Cases 19a and 19d. The converse situation is not possible since the subgroupoid of $G_{\sim}$ generated by $G_{\sim}$ is contained in the subgroupoid generated by $G_{\sim}$. This results in the following assertion.

\textbf{Corollary 12.} A complete list of $G_{\sim}$-inequivalent Lie-symmetry extensions within the subclass $W_c$ is exhausted by Cases 5a, 11, 12, 13, 14a–14c, 16, 17, 18a, 19a–19c and 20 of Table 1, where additionally $q > 1/2$ in Case 12, $p > 0$ in Cases 13 and 16, and $\varepsilon' = 1$ in Cases 18a and 20 with $\varepsilon = 1$. 

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7 Equivalence groupoid and singular Lie symmetry extensions

The equivalence groupoid $\mathcal{G}^\sim$ of the class $\mathcal{W}$ contains admissible transformations that are not generated by elements of $G^\sim$, i.e., this class is not normalized. Nevertheless, we can describe the groupoid $\mathcal{G}^\sim$, classifying admissible transformations within the class $\mathcal{W}$ up to the (weak) $G^\sim$-equivalence; see [51] for posing the general problem on classifying admissible transformations.

Since the class $\mathcal{W}$ is a subclass of $\mathcal{W}_{\text{gen}}$, the transformational part $\Phi$ of any admissible transformation $\mathcal{T} = (\theta, \vartheta, \Phi)$ in the class $\mathcal{W}$ takes the form

$$\tilde{t} = T(t, x), \quad \tilde{x} = X(t, x), \quad \tilde{u} = U = U^1(t, x)u + U^0(t, x)$$

with $T_t X_t - T_x X_t \neq 0$ and $U^1 \neq 0$, and additionally the system (9)–(13) is satisfied. Here $\theta = (f, g)$ and $\vartheta = (\tilde{f}, \tilde{g})$ are respectively the source and target arbitrary-element tuples for $\mathcal{T}$, $\mathcal{L}_{\theta}, \mathcal{L}_{\vartheta} \in \mathcal{W}$.

By $\mathcal{W}_0$ and $\mathcal{W}_1$ we respectively denote the subclasses of $\mathcal{W}$ singled out by the constraints $f_u = 0$ and $f_u \neq 0$. The partition $\mathcal{W} = \mathcal{W}_0 \sqcup \mathcal{W}_1$ of the class $\mathcal{W}$ induces the partition of the equivalence groupoid $\mathcal{G}^\sim$ of this class since $f_u = 0$ if and only if $f_u = 0$, cf. the end of Section 3. In other words, equations in the subclass $\mathcal{W}_0$ are not related by point transformations to equations in the subclass $\mathcal{W}_1$. This claim can be nicely reformulated in terms of equivalence groupoids.

**Proposition 13.** The equivalence groupoid $\mathcal{G}^\sim$ of the class $\mathcal{W}$ is the disjoint union of the equivalence groupoids $\mathcal{G}^\sim_0$ and $\mathcal{G}^\sim_1$ of the subclasses $\mathcal{W}_0$ and $\mathcal{W}_1$, $\mathcal{G}^\sim = \mathcal{G}^\sim_0 \sqcup \mathcal{G}^\sim_1$.

We describe the equivalence groupoids $\mathcal{G}^\sim_0$ and $\mathcal{G}^\sim_1$ separately.

**Lemma 14.** The usual equivalence group of the subclass $\mathcal{W}_0$ coincides with $G^\sim$. $G^\sim$-inequivalent non-identity admissible transformations of the subclass $\mathcal{W}_0$ that are independent up to inversion and composing with each other are exhausted by the restriction of the family $T_1$ to $\mathcal{W}_0$ and by the families $T_3$–$T_9$.

**Lemma 15.** A complete list of $G^\sim$-inequivalent singular Lie symmetry extensions for equations from the class $\mathcal{W}_0$, which are not related to appropriated subalgebras of $\mathfrak{g}^\sim$, is exhausted by Cases 5a–7, 18a–18c and 20 within the subclass of equations with arbitrary elements of the form $f = \varepsilon$ and $g = g^1(x)g^2(u)$.

**Proof.** We will simultaneously prove Lemmas 14 and 15. Let $\mathcal{T} \in \mathcal{G}^\sim_0$, i.e., $f_u = 0$, $g_{uu} \neq 0$, $\tilde{f}_u = 0$ and $\tilde{g}_{u} \neq 0$. We express $X_t^2$ from the equation (10), substitute this expression into the squared equation (9). After factorizing the resulting equation, we obtain the equation $(T_t^2 - f T_x^2)(f X_x^2 - \tilde{f} T_t^2) = 0$, which implies in view of $T_t^2 - f T_x^2 \neq 0$ that $f X_x^2 = \tilde{f} T_t^2$. Then the equation (10) yields $X_t^2 = f T_t^2$. This means that $f$ and $\tilde{f}$ have the same sign and up to $G^\sim$-equivalence we can assume that $f = \tilde{f} = \varepsilon$, where $\varepsilon = \pm 1$, i.e., $X_t^2 = T_t^2$ and $X_x^2 = T_x^2$. More precisely, gauging of $f$ and $\tilde{f}$ can be realized via transformations $\mathcal{D}(\varphi)$ and $\mathcal{D}(\tilde{\varphi})$ of $x$ and $\tilde{x}$, respectively. Taking into account the equation (9) and alternating the sign of $t$ (this transformation belongs to the kernel group of the class (1)), we can set $X_x = T_t$ and $X_t = \varepsilon T_x$. Since these equations give $T_u = \varepsilon T_u$ and $X_t = \varepsilon X_x$, the pair of the equations (11) and (12) reduce to the system of linear homogeneous algebraic equations $T_u U_u - \varepsilon T_{ux} U_{ux} = 0$, $X_t U_u - \varepsilon X_u U_{ux} = 0$ with respect to $U_u$ and $U_{ux}$. The determinant of the associated matrix is nonzero, $\varepsilon (T_t X_x - T_x X_t) \neq 0$. Hence $U_u = U_{ux} = 0$, i.e., $U_u$ is a nonzero constant and using a transformation $\mathcal{D}(c_2)$ we can set $U_u = 1$. In view of the above conditions, the equation (13) takes the form

$$(X_x^2 - \varepsilon X_t^2)\tilde{g} = g + U_t^0 - \varepsilon U_x^0,$$

sequentially acting on the equation (25) by the operators $(X_x^2 - \varepsilon X_t^2)^{-1}\partial_t$ and $\partial_t$, we obtain two differential consequences of (25),

$$X_t \tilde{g}_x + U_t^0 \tilde{g}_x + \frac{(X_x^2 - \varepsilon X_t^2)\tilde{g}}{X_x^2 - \varepsilon X_t^2} \partial_t \tilde{g} = \frac{(U_t^0 - \varepsilon U_x^0)\tilde{g}}{X_x^2 - \varepsilon X_t^2},$$

(26)
\[(X_1)_\bar{t}\bar{g}_x + (U^0_1)_\bar{t}\bar{g}_u + \left( \frac{(X^2_\bar{t} - \varepsilon X^2_\bar{t} \bar{t})}{X^2_\bar{t} - \varepsilon X^2_\bar{t}} \right) \bar{g} = \left( \frac{(U^0_\bar{t} - \varepsilon U^0_\bar{t} \bar{t})}{X^2_\bar{t} - \varepsilon X^2_\bar{t}} \right)_\bar{t}, \quad (27)\]

Studying the consistency of the equations (26) and (27) as first order quasilinear partial differential equations with respect to \( \bar{g} \), we consider different cases depending on whether the matrix of coefficients of the derivatives \( \bar{g}_x \) and \( \bar{g}_u \) in the system of these equations is degenerate or nondegenerate.\(^4\)

1. Suppose first that this matrix is nondegenerate, \( X_1(U^0_1)_\bar{t} - U^0_1(X_1)_t \neq 0 \). We solve the system (26)–(27) as a system of linear algebraic equations with respect to \( \bar{g}_x \) and \( \bar{g}_u \),

\[
\bar{g}_x = \alpha^1 \bar{g} + \alpha^0, \quad \bar{g}_u = \beta^1 \bar{g} + \beta^0. \tag{28}
\]

Here the coefficients \( \alpha^0, \alpha^1, \beta^0 \) and \( \beta^1 \) are functions of \( (\bar{t}, \bar{x}) \) whose explicit expressions in terms of \( X \) and \( U^0 \) are not essential for the further consideration. Differentiating the equations (28) with respect to \( \bar{t} \), we derive the consequences \( \alpha^1 \bar{g} + \alpha^0 = 0 \), \( \beta^1 \bar{g} + \beta^0 = 0 \), which implies in view of \( g_u \neq 0 \) that \( \alpha^1 = \alpha^0 = \beta^0 = 0 \), \( \beta^1 = 0 = \alpha^0 \beta^1 \). Therefore, \( \beta^1 \) is a constant. Since \( \bar{g}_{\bar{a}a} \neq 0 \), the second equation in (28) implies that \( \beta^0 \neq 0 \). Using the equivalence transformation \( \mathcal{D}^u(1/\beta^1) \), we can gauge \( \beta^1 \) to 1. Then the second equation in (28) integrates to \( \bar{g} = g^0(x) e^{\bar{g}} + \bar{g}^1(x) \) with \( \bar{g}^1 = -\beta^0 \). The equation (25) implies that the function \( g \) is of similar form in the initial variables, \( g = g^0(x) \psi + \bar{g}^1(x) \), and \( (X^2_\bar{t} - \varepsilon X^2_\bar{t}) g^0_\bar{tt} = g^0, \ (X^2_\bar{t} - \varepsilon X^2_\bar{t}) \bar{g}^1_\bar{tt} = \bar{g}^1 + U^0_\bar{tt} - \varepsilon U^0_{\bar{tt}} \). Using equivalence transformations of the form \( \mathcal{G}(\phi) \) in both the old and new variables, we set \( g^0 = \bar{e}^0 \) and \( \bar{g}^0 = \bar{e}^0 \) with \( \bar{e}, \bar{e}' = \pm 1 \). Then \( (X^2_\bar{t} - \varepsilon X^2_\bar{t}) e^{\bar{e}} = \bar{e}^0 \), i.e., \( U^0 = -\ln |X^2_\bar{t} - \varepsilon X^2| \) and thus \( U^0_\bar{tt} - \varepsilon U^0_{\bar{tt}} = 0 \) since \( X^2_\bar{t} - \varepsilon X^2 \).

The equation (25) takes the form

\[
\frac{g^1(x)}{X^2_\bar{t} - \varepsilon X^2_\bar{t}} = \bar{g}^1(X). \tag{29}
\]

In other words, in this case it suffices to classify admissible transformations within the subclass \( \mathcal{W}_{00} \) of equations of the form

\[
u_{\bar{u}t} = \varepsilon v_{\bar{t}x} + \bar{e}' e^\psi + \bar{g}^1(x)
\]

up to the subgroup \( G_{00} \) of \( G^\sim \) singled out by the constraints \( \phi_x = \pm c_1, c_2 = |c_1|^{-1/2} \) and \( \psi = 0 \). This reduces to deriving possible \( G_{00} \)-inequivalent expressions for \( X = X(t, x), g^1 = g^1(x) \) and \( \bar{g}^1 = \bar{g}^1(x) \) satisfying the joint system of the equation (29) and the equation \( X^2 - \varepsilon X^2 \bar{t} = X^2_\bar{tt} \). We change the independent variables in this system, \( y = x + it \) and \( z = x - it \), where \( i = 1 \) or \( i = \bar{i} \) if \( \varepsilon = 1 \) or \( \varepsilon = -1 \), respectively, \( i \) is the imaginary unit, \( i^2 = -1 \). Hence \( i^2 = \varepsilon \). In the variables \( y \) and \( z \) the equation \( X^2 = \varepsilon X^2 \bar{tt} \) takes the form \( X_{yz} = 0 \) and its general solution is represented as \( X = Y(y) + Z(z) \), where \( Z(z) \) coincides with the conjugate value of \( Y(y) \) if \( \varepsilon = -1 \). Then the equation (29) can be rewritten as

\[
\frac{1}{4Y_y Z_z} g^1(x) = \bar{g}^1(X), \quad \text{assuming} \quad x = \frac{y + z}{2}, \quad X = Y + Z. \tag{31}
\]

\(^4\)This procedure and the previous partition of \( \mathcal{W} \) into \( \mathcal{W}_0 \) and \( \mathcal{W}_1 \) well fits into the framework of the method of furcate splitting [36, 41, 55, 57]. The further consideration is the first construction of a generating set of admissible transformations using this method. The required computations were carried out in 2011 without involving algebraic techniques and gave the first application of furcate splitting to finding admissible transformations but they were too cumbersome, and the obtained results were not published. Later, the method of furcate splitting was used to describe the equivalence groupoid of the class of general Burgers–Korteweg–de Vries equations with space-dependent coefficients via classifying maximal conditional equivalence groups of this class [40], see also [38]. Therefore, the method of furcate splitting can be extended to admissible transformations in various ways depending on which terms the corresponding equivalence groupoid can be described in.
Excluding the parameter function \( \tilde{g}^1 \) via acting by the operator \( Y_g \partial_x - Z_g \partial_y \) on the equation (31), we reduce this equation, after the expansion and algebraic transformations, to

\[
2g^1(\partial_y + \partial_z)(Y_g^{-1} - Z_g^{-1}) = g_1^1(Y_g^{-1} - Z_g^{-1}).
\]

The last equation integrates to \( Y_g^{-1} - Z_g^{-1} = \varepsilon h^0 h^1 \), where \( h^0 \) is a (real-valued) smooth function of \( t \), \( h^1 := |g^1|^{1/2} \neq 0 \) and thus \( h^1 \) is a (real-valued) smooth function of \( x \). We act on the integration result by the operator \( \partial_y^2 - \varepsilon \partial_x^2 = -4\varepsilon \partial_y \partial_x \) to get \( h_0^0 h^1 = \varepsilon h^0 h^1 x_x \).

If \( h^0 = 0 \), then \( Y_g^{-1} - Z_g^{-1} = \text{const} \in \mathbb{R} \) and thus \( Y_g = Z_g = \text{const} \in \mathbb{R} \), which implies \( X_{xX} = 0 \). Therefore, \( T_x = T_{tt} = 0 \) as well. This means that the admissible transformation \( T \) is induced by an element of \( G^\sim \).

The case \( h^1 = g^1 = 0 \) corresponds to the Liouville equation. The sign of \( \varepsilon' \) is alternated by the corresponding admissible transformation from the family \( T1 \) if \( \varepsilon = 1 \) and cannot be alternated in view of the equation \( (X_x^2 - \varepsilon X_t^2) e^{\nu X} \varepsilon'' = \varepsilon' \) if \( \varepsilon = -1 \). This gives the family \( T9 \) of admissible transformations.

Further we assume that \( h_0^0 h^1 \neq 0 \) and thus \( g^1 \neq 0 \) as well. The separation of variables in the equation \( h_0^0 h^1 = \varepsilon h^0 h^1 x_x \) implies that \( h_0^0 h^0 / h^1 \) is a constant, which can be assumed, modulo scalings from \( G^\sim \) preserving the constraint \( f = \varepsilon \), to take values from the set \( \{-1, 0, 1\} \).

Up to shifts of \( x \) and alternating the sign of \( x \), we have

\[
g^1 \in B := \{ \nu, \nu x^{-2}, \nu \cos^{-2} x, -\nu \cosh^{-2} x, \nu \sinh^{-2} x, \varepsilon'' e^{-2x} | \nu \in \mathbb{R}, \nu \neq 0, \varepsilon'' = \pm 1 \}.
\]

Using the same arguments for the inverse of the admissible transformation \( T \), we obtain that the function \( \tilde{g}^1 \) also belongs to the set \( B \) (up to replacing the argument \( x \) by \( \tilde{x} \)).

We first present a complete set of \( G^\sim \)-inequivalent (independent up to inversion and composing with each other) non-identity admissible transformations for \( g^1 \) running through the set \( B \) and then explain the derivation of this list. It is exhausted by the family \( T8 |_{W_0} \) and the following families:

\( T3' \). \( f = 1, \ g = e^u + \varepsilon'' e^{-2x}, \ \tilde{f} = 1, \ \tilde{g} = e^u + \varepsilon'', \ \Phi: \tilde{t} = e^{-x} \sinh t, \ \tilde{x} = e^{-x} \cosh t, \ \tilde{u} = u + 2x; \)

\( T4' \). \( f = 1, \ g = \varepsilon' e^u + g^1(x), \ \tilde{f} = 1, \ \tilde{g} = \varepsilon' e^u + \nu \tilde{x}^{-2}, \ \nu \in \mathbb{R} \neq 0, \)

\[ a. \ g^1(x) = \nu x^{-2}, \ \varepsilon' = \varepsilon', \ \Phi: \tilde{t} = \frac{t}{x^2 - t^2}, \ \tilde{x} = \frac{x}{x^2 - t^2}, \ \tilde{u} = u + 2 \ln |x^2 - t^2|; \]

\[ b. \ g^1(x) = \nu \cos^{-2} x, \ \varepsilon' = \varepsilon', \ \Phi: \tilde{t} = \frac{\cos t}{\sin t + \sin x}, \ \tilde{x} = \frac{\cos x}{\sin t + \sin x}, \ \tilde{u} = u + 2 \ln |\sin t + \sin x|; \]

\[ c. \ g^1(x) = -\nu \cosh^{-2} x, \ \varepsilon' = -\varepsilon', \ \Phi: \tilde{t} = e^t \sinh x, \ \tilde{x} = e^t \cosh x, \ \tilde{u} = u - 2t; \]

\[ d. \ g^1(x) = \nu \sinh^{-2} x, \ \varepsilon' = \varepsilon', \ \Phi: \tilde{t} = e^t \cosh x, \ \tilde{x} = e^t \sinh x, \ \tilde{u} = u - 2t; \]

\( T5' \). \( f = -1, \ g = \varepsilon' e^u + \varepsilon'' e^{-2x}, \ \tilde{f} = -1, \ \tilde{g} = \varepsilon' e^u + \varepsilon'', \ \Phi: \tilde{t} = e^{-x} \sin t, \ \tilde{x} = e^{-x} \cos t, \ \tilde{u} = u + 2x; \)

\( T6' \). \( f = -1, \ g = \varepsilon' e^u + g^1(x), \ \tilde{f} = -1, \ \tilde{g} = \varepsilon' e^u + \nu \tilde{x}^{-2}, \ \nu \in \mathbb{R} \neq 0, \)

\[ a. \ g^1(x) = \nu x^{-2}, \ \Phi: \tilde{t} = \frac{t}{x^2 + t^2}, \ \tilde{x} = \frac{x}{x^2 + t^2}, \ \tilde{u} = u + 2 \ln |x^2 + t^2|; \]

\[ b. \ g^1(x) = \nu \cos^{-2} x, \ \Phi: \tilde{t} = e^t \sin x, \ \tilde{x} = e^t \cos x, \ \tilde{u} = u - 2t; \]

\[ c. \ g^1(x) = \nu \sinh^{-2} x, \ \Phi: \tilde{t} = \frac{\sin t}{\cos t + \cosh x}, \ \tilde{x} = \frac{\sinh x}{\cos t + \cosh x}, \ \tilde{u} = u + 2 \ln |\cos t + \cosh x|; \]

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\[ T^\prime. \quad f = -1, \quad g = \varepsilon' e^v + \nu \cosh^{-2} x, \quad \tilde{f} = -1, \quad \tilde{g} = \varepsilon' e^{u} + \nu \cosh^{-2} \tilde{x} \text{ with } \nu \in \mathbb{R}_{\neq 0}. \]

\[ \Phi: \quad \tilde{t} = \arctan \left( \frac{\sin \gamma \sinh x + \cos \gamma \sin t}{\cos t} \right), \quad \tilde{x} = \arctanh \left( \frac{\cos \gamma \sinh x - \sin \gamma \sin t}{\cosh x} \right), \]

\[ \tilde{u} = u + \ln \left| \cosh^2 x - (\cos \gamma \sinh x - \sin \gamma \sin t)^2 \right|, \quad \gamma \in (0, 2\pi). \]

The direct way of checking which elements of the set \( B \) are related via admissible transformation is to fix an element \( g^1 \) in \( B \), thus defining \( h^1 := |g^1|^1/2 \neq 0 \), to solve the equation \( h^1 \lambda = \lambda h^1 \) with \( \lambda := \varepsilon h^1_{xx}/h^1 = \text{const.} \), to find \( Y \) and \( Z \) by separating variables \( y \) and \( z \) in the equation \( Y^{-1} - Z^{-1} = \varepsilon h^1 h^1 \) and further integration, and finally to determine \( \tilde{g}^1 \) from (29).

We follow an optimized strategy. In the above way, we find the mappings \( \nu e^{-2x} \mapsto \nu \) by \( T^3' \) if \( f = 1 \) and by \( T^5' \) if \( f = -1, \nu \cosh^{-2} x \mapsto \nu x^{-2} \) by \( T^4'b \) if \( f = 1 \) and by \( T^6'b \) if \( f = -1, -\nu \cosh^{-2} x \mapsto \nu x^{-2} \) by \( T^4'c \) if \( f = 1, \nu \sinh^{-2} x \mapsto \nu x^{-2} \) by \( T^4'd \) if \( f = 1 \) and by \( T^6'c \) if \( f = -1 \). The sign of \( \varepsilon' \) is alternated only in \( T^4'c \). For \( f = -1 \), the value \( g^1 = -\nu \cosh^{-2} x \) is mapped to the value \( g^1 = \nu x^{-2} \) by an admissible point transformation only over the complex field.

The maximal Lie invariance algebras of the equations of the form (30) with values of \((f, g^1)\) that has not been reduced to other ones are

\[
(f, g^1) = (1, \nu): \quad g_\theta = \langle \partial_t, \partial_x, t \partial_x + t \partial_z \rangle,
\]

\[
(f, g^1) = (1, -\nu): \quad g_\theta = \langle \partial_t, \partial_x, t \partial_x - t \partial_z \rangle,
\]

\[
(f, g^1) = (\varepsilon, \nu x^{-2}): \quad g_\theta = \langle \partial_t, t \partial_x + x \partial_x - 2 \partial_u, (t^2 + \varepsilon x^2) \partial_t + 2t x \partial_x - 4t \partial_u \rangle,
\]

\[
(f, g^1) = (1, -\nu \cosh^{-2} x): \quad g_\theta = \langle \partial_t, \mathcal{R}'(\cos t \cosh x), \mathcal{R}'(\sin t \cosh x) \rangle.
\]

where \( \mathcal{R}'(\Phi) := \Phi_x \partial_t + \Phi_t \partial_x - 2 \Phi_{tx} \partial_u \). These invariance algebras are given in Cases 5a, 6a, Table 1 and are associated with Cases 6a and 7 of the same table, respectively. They are respectively realizations of the Poincaré algebra \( p(1, 1) \), the Euclidian algebra \( e(2) \), the real special linear algebra \( sl(2, \mathbb{R}) \) and the orthogonal algebra \( o(3) \), which are not isomorphic to each other. At the same time, systems of differential equations are related by point transformations only if their maximal Lie invariance algebras are isomorphic.

Therefore, we need to classify admissible transformations within the four subclasses of equations of the form (30), where for each of these subclasses the tuple \((f, g^1)\) is of a fixed form in \( \{(1, \nu), (-1, \nu), (\varepsilon, \nu x^{-2}),(1, -\nu \cosh^{-2})\} \), and \( \nu \) runs through \( \mathbb{R}_{\neq 0} \). For this purpose, we apply for the first time an extension of the algebraic method to finding admissible transformations. This method was suggested by Hydon in [22, 23, 24] for computing discrete symmetries and extended to equivalence transformations in [6].

We in detail consider only the first subclass. Let \( \mathcal{L}_\theta \) and \( \mathcal{L}_{\tilde{\theta}} \) be two fixed equations of the form (30) with \( f = \tilde{f} = 1, \tilde{g} = \nu, g^1 = \tilde{\nu} \) and some \( \varepsilon', \varepsilon' = \pm 1 \). These equations have the same maximal Lie invariance algebra, \( g_\theta = g_{\tilde{\theta}} = \langle \partial_t, \partial_x, t \partial_x + x \partial_t \rangle \), which is given in Cases 5a, of Table 1 and is a realization of the Poincaré algebra \( p(1, 1) \). Therefore, the pushforward of vector fields by \( \Phi \) induces an automorphism of \( g_\theta \) associated with an automorphism of \( p(1, 1) \). Recall that the transformation \( \Phi \) is completely defined by its \( t- \) and \( x- \) components. Inner automorphisms of \( p(1, 1) \) corresponds to continuous point transformations generated by vector fields from \( g_\theta \). Such transformations are symmetries of \( \mathcal{L}_\theta \), i.e., they do not change the parameters \( \nu \) and \( \varepsilon' \). Up to shifts of \( t \) and \( x \), which are induced by elements of \( G^\sim \), we obtain the family \( T^8a |_{W_{\nu \omega}} \) of admissible transformations. There are only two outer automorphisms of \( p(1, 1) \) that are independent up to composing to each other and to inner automorphisms.\(^5\) The corresponding transformations are the alternation of the sign of \( t \), which is a discrete symmetry of \( \mathcal{L}_\theta \) induced by \( D^\prime(-1) \), and the permutation of \( t \) and \( x \), which belongs to the family \( T^1 \). There are no point transformation that satisfies the restriction for \( \Phi \) and induce the identity automorphism of \( g_\theta \).

\(^5\)See [14, 48] for necessary facts on automorphisms of low-dimensional Lie algebras.
The other three subclasses are considered in a similar way. Each of the algebras e(2) and \(\mathfrak{sl}(2, \mathbb{R})\) possesses a single independent outer automorphism, which is here related, e.g., to alternating the sign of \(t\). The algebra o(3) admits no outer automorphism but alternating the sign of \(t\) generates the identity automorphism of the corresponding algebra \(\mathfrak{g}_\theta\). Factoring out shift and scaling symmetries of related equations, which are induced by elements of \(G^\sim\), we construct the families \(T8b|_{W_{00}}, T4'a\) and \(T7'\), respectively. The last family consists of the non-identity transformations generated by the Lie symmetry vector field \(\mathcal{R}'(\cos t \cosh x)\) of the equation (30) with \((f, g^1) = (-1, \nu \cosh^{-2})\).

It is obvious that \(G^\sim\)-inequivalent singular cases of \(\mathcal{L}\)-symmetry extensions within the subclass \(W_{00}\) are exhausted by those with \(g^1 \in \mathcal{B}\).

2. Now we suppose that the matrix of coefficients of the derivatives \(\tilde{g}_\xi\) and \(\tilde{g}_\eta\) in the system (26), (27) is degenerate,

\[
X_t(U^0_t)_i - U^0_t(X_t)_i = 0. \tag{32}
\]

If \(X_t = 0\) then \(T_\xi = X_{xx} = T_\eta = 0\) and the equation (25) implies in view of the condition \(\tilde{g}_\xi \neq 0\) that \(U^0_t = 0\), i.e., the admissible transformation \(\mathcal{T}\) is generated by an element of \(G^\sim\). This is why in what follows we assume that \(X_t \neq 0\). Representing the equation (32) in the form \((U^0_t(X_t)_i)_t = 0\), we integrate it by \(\tilde{t}\), which yields \(U^0_t = V^0(X)_X \) for some smooth function \(V^0 = V^0(\tilde{x})\). Then we integrate by \(t\), obtaining \(U^0 = V^1(X) + V^2(x)\), where \(V^1\) is an antiderivative of \(V^0\), \(V^1_\xi(\tilde{x}) = V^0(\tilde{x})\), and \(V^2 = V^2(x)\) is a smooth function of \(x\). Therefore, up to \(G^\sim\)-equivalence (namely, up to composing the transformation \(\mathcal{T}\) with transformations from the subgroup \(\{\mathcal{G}(u)\}\)), we can set \(U^0 = 0\) and thus \(U = u\). We then rewrite the equation (26) as

\[
\frac{\tilde{g}_\xi}{g} = -\frac{1}{X_t} \frac{(X^2_\xi - \varepsilon X^2_t)_\xi}{X^2_\xi - \varepsilon X^2_t}.
\]

The left and right hand sides of the last equations do not depend on \(\tilde{t}\) and \(\tilde{u}\), respectively, and hence they are equal to a function of only \(\tilde{x}\). Solving the equation with respect to \(\tilde{g}\) gives the representation of \(g\) as the product of functions of different arguments, \(\tilde{g} = \tilde{g}^1(\tilde{x})\tilde{g}^2(\tilde{u})\). Since \(\tilde{u} = u\) and \(g = (X^2_\xi - \varepsilon X^2_t)\tilde{g}\), the function \(g\) admits the similar representation \(g = g^1(x)g^2(u)\), where \(g^2(u) = \tilde{g}^2(u)\). As result, we again obtain the equation (29). Therefore, equations of the form

\[
u \varepsilon u_{xx} + g^1(x)g^2(u) \quad \text{with} \quad g^2_\xi g^2_{uuu} \neq (g^2_{uu})^2 \tag{33}
\]

with coinciding values of the parameter function \(g^2\) are related by a point transformation if and only if the equations of the form (30) with the same values of the parameters \(\varepsilon\) and \(g^1\) and some values of \(\varepsilon'\) are related by a point transformation. The inequality \(g^2_\xi g^2_{uuu} \neq (g^2_{uu})^2\), which is equivalent to the linear independence of \(g^2_\xi, g^2\) and 1, is imposed for excluding the intersection of the subclass (33) and the subclass of equations of the form \(u_{\eta} = \varepsilon u_{xx} + g^0(x)e^u + g^1(x)\) with \(g^0 \neq 0\), which are reduced by equivalence transformations to equations from the subclass (30). To properly translate the classification of admissible transformations within the subclass (30) to those within the subclass (33), we take into account the condition \(g^1 \neq 0\) for the subclass (33) and replace the \(u\)-components of all transformational parts by \(\tilde{u} = u\). Since the coefficients \(\nu\) and \(\tilde{\nu}\) coincide, then they can just be absorbed by \(g^2\). In total, this gives the restrictions of the families \(T1, T3^* - T7^*\) to the subclass (33). Here the families \(T3^* - T7^*\) respectively correspond to the families \(T3^* - T7^*\).

To complete the classification of admissible transformations in the subclass \(W_1\), we map each equation from the subclass (30) with \(g^1_\xi \in \mathcal{B}\) by the equivalence transformation \(Z(\ln \tilde{g}^1)\) to the equation \(u_{\eta} = \varepsilon u_{xx} + \tilde{g}^1(\tilde{x})g^2(u)\) with \(g^2(u) = \varepsilon' e^u + \tilde{\nu}\). Here \(\tilde{g}^1\) is of the same form as \(g^1\) but with fixed \(\nu = 1, \tilde{\nu} = 2\varepsilon' e'' + \varepsilon'' = 1\) for \(\tilde{g}^1 = \cosh^{-2} x\) and \(\varepsilon'' = 1\) otherwise. As a result, the families
T3’-T7’ are mapped into the families T3–T7. The completion of the latter families allows us to neglect the auxiliary inequality \( g'^2_{4u} \neq (g'^2_{4w})^2 \) of the subclass (33) for these families.

Up to \( G^- \)-equivalence, singular Lie symmetry extensions in the subclass (33) are possible only for \( g' \in B \), which gives Cases 5a–5c, 6a–6f and 7. Since Case 7 reduces to Case 6a over the complex field, for further extensions it suffices to check only equations with \( g'^1 = 1 \) (Case 5a) and with \( g'^1 = x^{-2} \) (Case 6a). For \( g'^1 = 1 \), we obtain only Case 18a with two \( G^- \)-equivalent Cases 18b and 18c. There are no further Lie-symmetry extensions for \( g'^1 = x^{-2} \).

**Remark 16.** A generating set of admissible point transformations within the class \( W_{lin} \) of linear equations of the form \( (1) \) and the group classification of this class can be easily derived from the computation of their counterparts for the class \( W_0 \). For this purpose, we need to consider the essential subgroupoid \( G_{lin}^{^\ast\ast} \) of the equivalence groupoid \( G_{lin}^{\sim} \) of \( W_{lin} \) and essential Lie invariance algebras of equations from \( W_{lin} \), respectively factoring out transformations and Lie-symmetry vector fields related to the linear superposition of solutions, cf. [52, Section 2] and [28]. Elements of \( W_{lin} \) take the form \( u_{tt} = f(x)u_{xx} + g'^1(x)u + g'^0(x) \). Their kernel point symmetry group is generated by the transformations \( \pi_s \mathcal{P}_t(c_0) \) and \( \pi_s \mathcal{D}(c_2) \), and their kernel invariance algebra is \( g_0^0 = \langle \partial_t, u\partial_u \rangle \). The equivalence group of \( W_{lin} \) coincides with \( G^- \). Using equivalence transformations, we can gauge the parameter functions \( f \) and \( g'^t \) to \( \varepsilon \in \{-1, 0, 1\} \) and 0, respectively. As a result, we map the class \( W_{lin} \) onto its subclass \( W_{lin}' \) of linear wave and elliptic equations with \( x \)-dependent potentials, which are of the form \( u_{tt} = \varepsilon u_{xx} + g'^1(x)u \), cf. [59].

Each equation \( \mathcal{L}_\theta \) from \( W_{lin}' \) admits the (pseudo)group \( G_{lin}^{\sim} \) of points symmetries associated with the linear superposition of solutions, \( G_{lin}^{\sim} = \{ \Phi: i = t, \, \tilde{x} = x, \, \tilde{u} = u + h(t, x) \, | \, h \in \mathcal{L}_\theta \} \), where the notation “\( h \in \mathcal{L}_\theta \)” means that the function \( h \) runs through the solution set of \( \mathcal{L}_\theta \). The corresponding Lie algebra is \( g_{lin}^{\sim} = \langle h(t, x) \partial_u \, | \, h \in \mathcal{L}_\theta \rangle \). Let \( G_{lin}^{^\ast\ast} \) be the subgroupoid of the equivalence groupoid \( G_{lin}' \) of the class \( W_{lin}' \) that is constituted by the admissible transformations related the linear superposition of solutions, i.e., \( G_{lin}^{^\ast\ast} \) is the union of \( G_{lin}^{\sim} \) as subgroups of vertex groups in \( G_{lin}' \) for all \( \theta \) with \( \mathcal{L}_\theta \in W_{lin}' \). The essential equivalence groupoid \( G_{lin}^{^\ast\ast} \) of \( W_{lin}' \), which is the complement of \( G_{lin}^{\sim} \) in \( G_{lin}' \), is naturally isomorphic to the equivalence groupoid of the class of equations of the form (30) with a fixed value of \( \varepsilon' \). Therefore, a generating set for \( G_{lin}^{^\ast\ast} \) and, thus, for the essential equivalence groupoid \( G_{lin}^{^\ast\ast} \) of \( W_{lin}' \), which is defined similarly to \( G_{lin}^{^\ast\ast} \), consists of the counterparts of the families T3–T9 for linear equations. That is, one should substitute \( g^2 = u, \, g'' = \bar{u} \) into T3–T7 and \( g = \varepsilon''u, \, \bar{g} = \varepsilon''\bar{u} \) into T8 and replace \( g, \bar{g} \) and the \( u \)-component of \( \Phi \) in T9 by \( g = \varepsilon''u, \, \bar{g} = \varepsilon''\bar{u} \) and \( \bar{u} = u \). A complete list of \( G^- \)-inequivalent essential Lie-symmetry extensions in the class \( W_{lin} \) (i.e., extensions of \( g_{lin}^{\sim} + g_{lin}^{\sim} \)) are exhausted by Cases 5a–7 of Table 1 with \( \bar{g} = \varepsilon''u \) and the counterpart of Cases 20, where \( g = 0 \) and the extension is spanned by \( \tau \partial_\tau + \xi \partial_\xi \), with the same condition on \( (\tau, \xi) \). Additional equivalence transformations between the classification cases are exhausted by the counterparts of those for Cases 5a–6f.

It now remains to study the equivalence groupoid \( G_{lin}^{\sim} \) of the subclass \( W_1 \), which is singled out from the class (1) by the constraint \( f_u \neq 0 \). By \( W_{lin1} \) and \( W_{lin2} \) we respectively denote the subclasses of \( W_1 \) that is associated with the additional auxiliary condition \( f_x = g_x = 0 \) mod \( G^- \) and that consists of the equations \( G^- \)-equivalent to equations of the form

\[
\varepsilon u^{-4}u_{xx} + \mu(x)u^{-3} + \sigma u,
\]

where \( \mu \) runs through the set of smooth functions of \( x \), \( \varepsilon \) and \( \sigma \) are constants, \( \varepsilon \neq 0 \) and hence \( \varepsilon = \pm 1 \) mod \( G^- \) and \( \sigma \in \{-1, 0, 1\} \) mod \( G^- \).

**Lemma 17.** The usual equivalence group of the subclass \( W_1 \) coincides with \( G^- \). Any admissible transformation in \( W_1 \setminus (W_{lin1} \cup W_{lin2}) \) is generated by a transformation from \( G^- \). \( G^- \)-inequivalent non-identity admissible transformations of the subclass \( W_1 \) that are independent up to inversion and composing with each other are exhausted by the restriction of the family T1 to \( W_{lin1} \) and by the family T2, which acts within \( W_{lin2} \).
Proof. For $T \in G^\sim$, the equation (9) immediately implies that $T_t X_t = T_x X_x = 0$ for admissible transformations within the subclass $\mathcal{W}_1$.

Supposing $T_x \neq 0$, we obtain that $X_x = 0, X_t \neq 0$ and hence $T_t = 0$. Up to $G^\sim$-equivalence of admissible transformations we can assume that $T = x$ and $X = t$ as under the above restrictions the transformation $T$ is represented as the composition of $D(T)$, a transformation permuting $t$ and $x$ and $D(X)$. For $T = x$ and $X = t$, the equations (10), (11) and (12) reduce to the simple equations $\tilde{f} = 1$, and $U_{ux} = U_{ut} = 0$, i.e. $U_u = \text{const}$. By a scaling of $u$, which belong to $G^\sim$, the constant $U_u$ can be set equal to 1. Differentiating the equation $\tilde{f} = 1/f$ with respect to $t$ and then, assuming $(\tilde{t}, \tilde{x}, \tilde{u})$ as basic variables, with respect to $\tilde{t}$, we derive the equation $U_{tx}^0 = 0$. Therefore, $U^0 = \psi(x) + \psi(t)$ and the reduced transformation $T_x = \alpha = \tilde{t}$ and $U_x = \tilde{t}$ can be represented as the composition of the transformations $\tilde{s}(\psi), t \mapsto x$ and $g(\psi)$, where $t \mapsto x$ denotes the transformation which only permutes $t$ and $x$: $\tilde{t} = x, \tilde{x} = t$ and $\tilde{u} = u$. This means that up to $G^\sim$-equivalence of admissible transformations the transformation $T$ coincides with $t \leftrightarrow x$. The corresponding transformation components for the arbitrary elements $f$ and $g$ follow from the equations (10) and (13). They read $\tilde{f} = 1/f$ and $\tilde{g} = -g/f$. Since the left (resp. right) hand sides of these equalities do not depend on $\tilde{t} = x$ (resp. $\tilde{x} = t$), the arbitrary elements of equations from the class (1) which connected by the transformation $t \leftrightarrow x$ satisfy the additional auxiliary constraints $f_x = g_x = 0$ and $f_x^2 = g_x^2 = 0$. In other words, admissible transformations of the case under consideration are generated by transformations from the equivalence group $G^\sim$ of the entire class (1) and the equivalence transformation $t \leftrightarrow x$ of the subclass $\mathcal{U}$ which is singled out from the class (1) by the additional auxiliary constraints $f_x = g_x = 0$ and $f_u \neq 0$. In particular, the equivalence group of the subclass $\mathcal{U}$ consists of the transformations of the form (15) with $\varphi_{xx} = \psi_x = 0$ and the compositions of these transformations with $t \leftrightarrow x$.

Now we consider the case $T_x = 0$ for which $T_t \neq 0, X_t = 0$ and $X_x \neq 0$. Then the equations (10)–(12) reduce to $\tilde{f} T_t^2 = f X_x^2, (U^0_u / T_t)_t = 0, (U^0_x / X_x)_x = 0$. From the first equation we can conclude that $\tilde{f}_u \neq 0$ if and only if $f_u \neq 0$. Solving the other two equations with respect to $U_u$, equating the expressions obtained and separating variables in this equality, we derive that $U^1 := U_u = \kappa \sqrt{|T_t X_x|}$, where $\kappa$ is a nonzero constant. Differentiating the equation $\tilde{f} T_t^2 = f X_x^2$ with respect to $t$ results in the consequence

$$\frac{T_{tt}}{T_t} \left( \tilde{u} - U^0 \tilde{f} \right) + 2U^0_\tilde{f} \tilde{f}_u = 0. \tag{35}$$

If $T_{tt} = 0$, the equation (35) implies that $U^1 / \sqrt{|X_x|} = \text{const}$ and $U^0_t = 0$ and, therefore, the transformation $T$ belongs to the equivalence group $G^\sim$.

Further we assume that $T_u \neq 0$. By fixing a value of $t$, we derive from the equation (35) that the arbitrary element $\tilde{f}$ is a solution of an ordinary differential equation of the general form $(\tilde{u} + \tilde{\beta}(\tilde{x})) \tilde{f}_u + 4 \tilde{f} = 0$, where the variable $\tilde{x}$ plays the role of a parameter and $\tilde{\beta}$ is a smooth function of $\tilde{x}$. This implies that $\tilde{f} = \tilde{\alpha}(\tilde{x})(\tilde{u} + \tilde{\beta}(\tilde{x}))^{-4}$ for some smooth function $\tilde{\alpha} = \tilde{\alpha}(\tilde{x})$. Combining the equation $\tilde{f} T_t^2 = f X_x^2$ with the expression for $\tilde{f}$ yields

$$f = \frac{T_t^2 \alpha(X)}{X_x^2 (U^1 u + U^0 + \beta(X))^4} = \frac{\alpha(x)}{(u + \beta(x))^4},$$

where $\alpha(x) := \sqrt[4]{u}(\kappa_\alpha X_x)^{-4} \alpha(X)$ and $\beta(x) := (\tilde{\beta}(X) + U^0)/U^1$. Furthermore, upon using transformations from the equivalence group $G^\sim$, we can set $\tilde{\beta} = \beta = 0$, which consequently implies that $U^0 = 0$. By means of equivalence transformations, we can also set $\alpha, \tilde{\alpha} \in \{-1, 1\}$ and as the multiplier relating $\alpha$ and $\tilde{\alpha}$ is strictly positive, we have that $\tilde{\alpha} = \alpha =: \varepsilon \in \{-1, 1\}$. Then $X_x$ is a constant and we can set $X = x$ and $\kappa = 1$ using a scaling and a translation of $x$ and a scaling of $u$, which belong to $G^\sim$. Therefore, $U = \omega u$, where $\omega := \sqrt{|T_t|}$ and hence $\omega_t \neq 0$. After taking into account all the conditions derived, we reduce the equation (13) to the form $\omega^3 \tilde{g} + \omega(\omega^{-1})_{tt} u = g$. Differentiating the last equation with respect to $t$ and dividing the result by $\omega^2 \omega_t$, we obtain
\[ \tilde{u}\tilde{g}_u + 3\tilde{g} = 4\tilde{\sigma}\tilde{u}, \]  
where \( \tilde{\sigma} := -\left(\omega(\omega^{-1})_u\right)/(4\omega^2\omega) \) is a constant. The general solution of the equation for \( \tilde{g} \) is \( \tilde{g} = \tilde{\mu}(x)\tilde{u}^{-3} + \tilde{\sigma}\tilde{u} \). The expression for \( g \) is similar: \( g = \mu(x)u^{-3} + \sigma u \), where \( \mu = \tilde{\mu} \) and \( \sigma := \tilde{\sigma}^4 + \omega(\omega^{-1})_u \) is also a constant. We rewrite the relation defining \( \sigma \) as an ordinary differential equation for \( \omega \), \( (\omega^{-1})_u = \sigma \omega^{-1} - \tilde{\sigma} \omega^3 \). Up to scalings from \( G^\sim \) there are only three essentially different values of \( \sigma \) (resp. \( \tilde{\sigma} \)), \( \sigma \in \{-1, 0, 1\} \). Finally, from the class \( (1) \) we single out the subclass of equations of the general form \( (34) \).

For each pair of values of \( \sigma \), the corresponding equations from the subclass \( (34) \) with the same value of the parameter function \( \mu \) are related by a point transformation. This is why within this subclass it suffices to classify admissible transformations with \( \tilde{\sigma} = 0 \). We solve the equation \( (\omega^{-1})_u = \sigma \omega^{-1} \) with respect to \( \omega \) and then construct \( T \) using the relation \( T_t = \omega^2 \mod G^\sim \). We find

\[
T = \begin{cases} 
(a_1 t + a_0)/(a_3 t + a_2) & \text{if } \sigma = 0, \\
(a_1 e^{a_3 t} + a_0)/(a_3 e^{a_3 t} + a_2) & \text{if } \sigma = 1, \\
b_1 \tan(t + b_0) + b_2 & \text{if } \sigma = -1,
\end{cases}
\]

where \( a_0, \ldots, a_3 \) are constants with \( a_1 a_2 - a_0 a_3 \neq 0 \) that are determined up to a common nonvanishing multiplier, and \( b_0, b_1 \) and \( b_2 \) are constants with \( b_1 \neq 0 \).

In the case \( \sigma = 0 \) we obtain a subgroup of the complete point symmetry group of the corresponding equation. This group is obviously isomorphic to \( \text{PGL}(2, \mathbb{R}) \). The condition \( T_t \neq 0 \) is equivalent to \( a_3 \neq 0 \) and we can assume \( a_3 = 1 \) due to the indeterminacy up to a constant multiplier. Then \( a_0 - a_1 a_2 \neq 0 \) and we gauge \( a_2, a_0 \) and \( a_1 \) to 0, 1 and 0 using the \( s \)-action of \( \mathcal{P}^s(a_2) \) and the \( t \)-action of \( \mathcal{P}^t(-a_1) \circ \mathcal{D}^t(c_2) \circ \mathcal{D}^u(c_2) \) with \( c_2 := (a_0 - a_1 a_2)^{-1} \). All the above transformations from the equivalence group \( G^\sim \) induce point symmetries of the equation under consideration. Therefore, we can assume that \( T = t^{-1} \mod G^\sim \), obtaining the family \( T_{2a} \) of admissible transformations.

In the same way we derive that \( T = \frac{1}{2} e^{2t} \mod G^\sim \) and \( T = \tan t \mod G^\sim \) if \( \sigma = 1 \) and \( \sigma = -1 \), which gives the family \( T_{2b} \) and \( T_{2c} \) of admissible transformations, respectively.

We set \( \mu_x \neq 0 \) and \( \tilde{\mu}_x \neq 0 \) for admissible transformations from the family \( T_2 \) since similar admissible transformations with \( \mu_x = 0 \) are \( G^\sim \)-equivalent to admissible transformations from the restriction of the family \( T_1 \) to \( W_{11} \cap W_{12} \). The equations of the form \( (34) \) with \( \mu_x \neq 0 \) are not related to those with \( \mu_x = 0 \) by point transformations.

**Lemma 18.** A complete list of \( G^\sim \)-inequivalent Lie symmetry extensions for equations of the general form \( (34) \) is exhausted by the following cases:

1a–1c. \( \varepsilon \) general \( \mu \): \( \mathfrak{g}_\varepsilon = \mathfrak{g}_\sigma \).

2a–2c. \( \mu = \pm 1 \): \( \mathfrak{g}_\mu = \mathfrak{g}_\sigma + \langle \partial_\varepsilon \rangle \).

3a–3c. \( \mu = \nu x^{-2}, \nu \neq 0 \): \( \mathfrak{g}_\mu = \mathfrak{g}_\sigma + \langle 2x\partial_x - u\partial_u \rangle \).

4a–4c. \( \mu = 0 \): \( \mathfrak{g}_\mu = \mathfrak{g}_\sigma + \langle \partial_\varepsilon, 2x\partial_x - u\partial_u \rangle \)

with

a. \( \sigma = 0 \): \( \mathfrak{g}_0 = \langle \partial_t, 2t\partial_t + u\partial_u, t^2\partial_t + tu\partial_u \rangle \),

b. \( \sigma = 1 \): \( \mathfrak{g}_1 = \langle \partial_t, e^{2t}(\partial_t + u\partial_u), e^{-2t}(\partial_t - u\partial_u) \rangle \),

c. \( \sigma = -1 \): \( \mathfrak{g}_{-1} = \langle \partial_t, \cos(2t)\partial_t - \sin(2t)u\partial_u, \sin(2t)\partial_t + \cos(2t)u\partial_u \rangle \).

The cases \( \sigma = 1 \) and \( \sigma = -1 \) reduce to the case \( \sigma = 0 \) with the same value of the parameter function \( \mu = \mu(x) \) by the additional equivalence transformations \( \tilde{t} = \frac{1}{2} e^{2t}, \tilde{x} = x, \tilde{u} = e^t u \) and \( \tilde{t} = \tan t, \tilde{x} = x, \tilde{u} = u \cos t \), respectively.

**Proof.** It follows from Lemma 17 that it suffices to classify only equations of the form \( (34) \) with \( \sigma = 0 \). Spitting the system of determining equations \( (19) \)–\( (23) \) for a Lie-symmetry vector field \( Q \)
of the equation $\mathcal{L}_\theta$ with $\theta = (f, g) = (\varepsilon u^{-3}, \mu(x)u^{-3})$ with respect to $u$, we derive that the components of $Q$ are of the form $\tau = \tau(t)$, $\xi = 2c_1x + c_0$, $\eta = (\frac{3}{2}\tau t - c_1)u$, where $\tau_{uu} = 0$, and $c_0$ and $c_1$ are constants with $(2c_1x + c_0)\mu_x = -4c_1\mu$. The four cases for $\mu$ from the lemma’s statement arise in the course of analysis of the last equation.

Lemma 20. The equivalence group $\mathcal{G}_{120}$ of equations of the form (34) with $\varepsilon = \pm 1$ and $\sigma \in \{-1, 0, 1\}$ can be represented in the form (2), where the parameter $\sigma$ plays the role of $\gamma$. $\Phi_0$ is the identity transformation of $(t, x, u)$, and $\Phi_1$ and $\Phi_{-1}$ are transformational parts of $T_{2a}$ and $T_{2b}$, respectively. Since this class is a subclass of $\mathcal{W}_{12}$ that is obtained by gauging the arbitrary elements $\mathcal{W}_{12}$ with equivalence transformations of $\mathcal{W}_{12}$, then the equivalence groupoid of $\mathcal{W}_{12}$ is of similar structure. The analogue of the last claim also holds for the intermediate class of equations of the form (34) with $\varepsilon \in \mathbb{R}_{\neq 0}$ and $\sigma \in \mathbb{R}$.

Remark 20. The class $\mathcal{W}_{120}$ of equations of the form (34) with $\varepsilon = \pm 1$ and $\sigma = 0$ is normalized. Its equivalence group $\mathcal{G}_{120}$ consists of the transformations

$$\tilde{t} = T(t) := \frac{a_1t + a_0}{a_3t + a_2}, \quad \tilde{x} = b_1x + b_0, \quad \tilde{u} = \pm \sqrt{|b_1^{-1}T_1|} u, \quad \tilde{\varepsilon} = \varepsilon, \quad \tilde{\mu} = b_1^{-2}\mu,$$

where $a_0, \ldots, a_3$ are arbitrary constants with $a_1a_2 - a_0a_3 \neq 0$ that are defined up to common nonzero multiplier, and $b_0$ and $b_1$ are arbitrary constants with $b_1 \neq 0$. The normal subgroup of $\mathcal{G}_{120}$ associated with the kernel point symmetry group of equations from $\mathcal{W}_{120}$ is singled out from $\mathcal{G}_{120}$ by the constraints $b_0 = 0$ and $b_1 = 1$, and thus the kernel invariance algebra of equations from $\mathcal{W}_{120}$ coincides with $g^0_\gamma$. This is why the complete group classification of the class $\mathcal{W}_{120}$ within the framework of the algebraic method reduces to the classification of subalgebras of the algebra $\langle \partial_x, 2x\partial_x - u\partial_u \rangle$, which is trivial. A complete list of inequivalent subalgebras of this algebra are exhausted by $\{0\}, \langle \partial_x \rangle, \langle 2x\partial_x - u\partial_u \rangle, \langle \partial_x, 2x\partial_x - u\partial_u \rangle$, all of which are appropriate, cf. Lemma 18.

8 Classification of appropriate subalgebras

The equivalence group $G^\gamma$ and the equivalence algebra $g^\sim$ admits related representations in the form of a semi-direct product and a semi-direct sum, $G^\sim = G^\gamma \rtimes G^\sim_{\text{ess}}$ and $g^\sim = \hat{g}^\gamma \triangleleft g^\sim_{\text{ess}}$, respectively. Here $G^\gamma = \langle \mathcal{P}^\gamma(t_0) \rangle$ $| t_0 \in \mathbb{R} \rangle$ is the normal subgroup of $G^\sim$ associated with the kernel group $G^\gamma$ of the class (1), $G^\sim_{\text{ess}}$ is the subgroup of $G^\sim$ that consists of the transformations of the form (15) with $t_0 = 0$ and thus effectively acts on the class (1), $\hat{g}^\gamma = \langle \mathcal{P}^\gamma \rangle$ is the ideal of $g^\sim$ corresponding to the kernel algebra $g^\gamma$ and $g^\sim_{\text{ess}} = \langle \mathcal{D}^\gamma, \mathcal{D}^\gamma(\zeta), \mathcal{Z}(\chi) \rangle$ is a subalgebra of $g^\sim$, which is the “essential” part of $g^\sim$ from the point of view of Lie symmetry extensions within the class (1). Denote by $\pi$ the projection from the space with the coordinates $(t, x, u, f, g)$ to the space with the coordinates $(t, x, u)$, and by $\pi_sQ$ the pushforward of a projectable vector field $Q$ in the space with the coordinates $(t, x, u, f, g)$ by $\pi$. A subalgebra $a$ of $g^\sim$ is called appropriate if its projection $\pi_sQ$ is the maximal Lie invariance algebra $g_0$ of an equation $\mathcal{L}_\theta$ from the class (1). Any appropriate subalgebra $a$ of $g^\sim$ should contain $\hat{g}^\gamma$ as an ideal. Hence it can also be represented in the form of semi-direct sum $a = \hat{g}^\gamma \rtimes s$, where $s$ is a subalgebra of $g^\sim_{\text{ess}}$. We call a subalgebra $s$ of $g^\sim_{\text{ess}}$ appropriate if $s = g^\sim_{\text{ess}} \cap a$ for an appropriate subalgebra $a$ of $g^\sim$. Appropriate subalgebras $a_1$ and $a_2$ of $g^\sim$ are $G^\sim$-equivalent if and only if the corresponding subalgebras $s_1$ and $s_2$ of $g^\sim_{\text{ess}}$ are $G^\sim_{\text{ess}}$-equivalent. As a result, the classification of Lie symmetry extensions induced by subalgebras of $g^\sim$ up to $G^\sim_{\text{ess}}$-equivalence reduces to the classification of appropriate subalgebras of $g^\sim_{\text{ess}}$ up to $G^\sim_{\text{ess}}$-equivalence.
For the latter classification, we need to compute the adjoint action of the group $G_{\text{ess}}$ on the algebra $\mathfrak{g}_{\text{ess}}$. Since this algebra is infinite-dimensional, it is convenient to realize this computation via pushing forward the vector fields $\mathcal{D}^u$, $\mathcal{D}^i$, $\mathcal{D}(\zeta)$ and $\mathcal{Z}(\chi)$, which span $\mathfrak{g}_{\text{ess}}$, by elementary equivalence transformations from $G_{\text{ess}}$, i.e., by $\mathcal{D}'(c_1)$, $\mathcal{S}(\psi)$, $\mathcal{D}(\varphi)$ and $\mathcal{D}^u(c_2)$, cf. Section 4. In other words, the usual transformation rule of vector fields under point transformations will be used [5, 13]. This yields the following non-identity actions:

\begin{align*}
\mathcal{S}_*(\psi)\mathcal{D}^u &= \mathcal{D}^u + \mathcal{Z}(\psi), & \mathcal{D}_*(c_2)\mathcal{Z}(\chi) &= c_2\mathcal{Z}(\chi), \\
\mathcal{S}_*(\psi)\mathcal{D}(\zeta) &= \mathcal{D}(\zeta) + \mathcal{Z}(\zeta\psi - \frac{1}{2}\zeta_x\psi), & \mathcal{D}_*(\varphi)\mathcal{Z}(\chi) &= \mathcal{Z}(\hat{\varphi}_x^{-1/2}\chi(\hat{\varphi})), \\
\mathcal{D}_*(\varphi)\mathcal{D}(\zeta) &= \mathcal{D}(\hat{\varphi}\zeta/\hat{\varphi}_x),
\end{align*}

where $\hat{\varphi} = \varphi(x)$ is the inverse of the function $\varphi$.

All vector fields from $\pi_*\mathfrak{g}_{\text{ess}}$ identically satisfy the determining equations for Lie symmetries of equations from the class (1), except the equations (22) and (23). The latter two equations imply restrictions on appropriate subalgebras of $\mathfrak{g}_{\text{ess}}$.

Lemma 21. $\mathfrak{s} \cap \langle \mathcal{D}^u, \mathcal{Z}(\chi) \rangle = \mathfrak{s} \cap \langle \mathcal{D}^i \rangle = \{0\}$ for any appropriate subalgebra $\mathfrak{s}$.

Proof. Suppose that an appropriate subalgebra $\mathfrak{s}$ of $\mathfrak{g}_{\text{ess}}$ contains an vector field $Q = b\mathcal{D}^u + \mathcal{Z}(\chi)$, where the constant $b$ or the function $\chi = \chi(x)$ does not vanish. Then $\pi_*Q$ is a Lie symmetry vector field for an equation $\mathcal{L}_\theta$ from the class (1). Substituting the components of the vector field $\pi_*Q$ into the determining equations (22) and (23) implies the following conditions for the arbitrary-element tuple $\theta = (f, g)$:

\[(bu + \chi)f_u = 0, \quad (bu + \chi)g_u = bg - \chi_x f.\]

Then $f_u = 0$ and $g_{uu} = 0$ if $b \neq 0$ or $\chi \neq 0$. This contradicts the definition of the class (1).

Analogously, the condition $\pi_*\mathcal{D}^i \in \mathfrak{g}_\theta$ gives the equation $f = 0$, which is also inconsistent with the definition of the class (1).

Therefore, any appropriate subalgebra contains no vector fields of the forms considered. □

Lemma 22. $\dim \left( \mathfrak{s} \cap \langle \mathcal{D}(\zeta), \mathcal{Z}(\chi) \rangle \right) \in \{0, 1, 3\}$ for any appropriate subalgebra $\mathfrak{s}$.

Proof. Suppose that $\mathfrak{s}$ is an appropriate subalgebra of $\mathfrak{g}_{\text{ess}}$ and $\dim \left( \mathfrak{s} \cap \langle \mathcal{D}(\zeta), \mathcal{Z}(\chi) \rangle \right) \geq 2$. This means that the subalgebra $\mathfrak{s}$ contains (at least) two vector fields $Q^i = \mathcal{D}(\zeta^i) + \mathcal{Z}(\chi^i)$, where the functions $\zeta^i$, $i = 1, 2$, should be linearly independent in view of Lemma 21. In other words, the projections $\pi_*Q^i$ of $Q^i$ simultaneously are Lie symmetry vector fields of an equation from the class (1). By $W$ we denote the Wronskian of the functions $\zeta^1$ and $\zeta^2$, $W = \zeta^1\zeta^2 - \zeta^2\zeta^1$. $W \neq 0$ as the functions $\zeta^1$ and $\zeta^2$ are linearly independent.

Plugging the coefficients of $\pi_*Q^i$ into the equation (22) gives two equations with respect to $f$ only,

\[2\zeta^i f_x + (\zeta^i u + 2\chi^i)f_u = 4\zeta^i f. \tag{36}\]

We multiply the equation (36) with $i = 1$ by $\chi^2$ and subtract it from the equation (36) with $i = 2$ multiplied by $\chi^1$. Dividing the resulting equation by $W$, we obtain the ordinary differential equation $(u + \beta)f_u = 4f$, where $\beta = \beta(x) := 2(\zeta^1\zeta^2 - \zeta^2\zeta^1)/W$ and the variable $x$ plays the role of a parameter. It is possible to set $\beta = 0$ by means of an equivalence transformation, $\mathcal{G}(-\beta)$. Indeed, this transformation preserves the form of the vector fields $Q^i$, only changing the values of the functional parameters $\chi^i$. In particular, it does not affect the linear independency of the functions $\zeta^i$. The integration of the above equation for $\beta = 0$ yields that $f = \alpha u^4$, where $\alpha = \alpha(x)$ is a nonvanishing function of $x$. In view of the derived form of $f$, splitting of equations (36) with respect to $u$ leads to $\zeta^4 u_x = 0$ and $\chi^4 \alpha = 0$, i.e., $\alpha_x = 0$ and $\chi^4 = 0$. The constant $\alpha$ can be scaled to $\alpha = \pm 1$ by an equivalence transformation.
In a similar manner, consider the equation (23), taking into account the restrictions set on parameter functions and the form of \( f \). For each \( Q^i \), the equation (23) gives an equation with respect to \( g \),

\[
2\zeta^i g_x + \zeta^i u_g = \zeta^i g - \zeta^i_{xx} \alpha u^5.
\]  

(37)

Again, we multiply the equation (37) with \( i = 1 \) by \( \zeta^2 \) and subtract it from the equation (37) with \( i = 2 \) multiplied by \( \zeta^1 \), divide the resulting equation by \( W \) and thereby obtain that \( u_g = g + \mu^0 u^3 \), where \( \mu^0 = \mu^0(x) := -\alpha(\zeta^1 \zeta^2 + \zeta^2 \zeta^3)/W \) and the variable \( x \) again plays the role of a parameter. Integrating the last equation for \( g \) directly gives \( g = \mu^0 u^3/4 + \mu^1 u \), where \( \mu^1 = \mu^1(x) \) is a smooth function of \( x \). The parameter function \( \mu^1 \) can be set equal to zero by the equivalence transformation \( D(\phi) \), where the function \( \phi = \phi(x) \) is a solution of the equation \( \alpha(x) \phi_x - \phi_x = 0 \). Substituting the derived form of \( g \) into equations (37) and splitting with respect to \( u \), we find that \( \mu^1 = 0, \zeta^1 = 0 \).

Summing up, we have proved that any equation of the class (1) admitting (at least) two linearly independent vector fields \( \pi, \psi \) in fact possesses exactly three linearly independent vector fields of this form and is \( G^{\sim} \)-equivalent to an equation of the form \( u_{tt} = \pm u^4 u_{xx} + \mu^1 u \), where \( \mu^1 \) is a constant which can be scaled to \( \pm 1 \) if it is not zero.

The equation \( u_{tt} = \pm u^4 u_{xx} \), for which \( \mu^1 = 0 \), admits an additional Lie-symmetry extension.

**Corollary 23.** There are only two \( G^{\sim} \)-inequivalent cases of Lie-symmetry extensions in the class (1) where the corresponding Lie invariance algebras contain at least two linearly independent vector fields of the form \( \pi, Q^i \) with \( Q^i = D(\zeta^i) + Z(\chi^i) \),

14d: \( u_{tt} = x u^4 u_{xx} + \varepsilon' u = g^{\text{max}} = g^{\text{max}} \),

19d: \( u_{tt} = x u^4 u_{xx} = g^{\text{max}} = g^{\text{max}} \),

with \( \varepsilon, \varepsilon' = \pm 1 \).

Corollary 23 gives the classification of appropriate subalgebras of \( g_{\sim} \) the dimensions of whose intersections with \( \langle D(\zeta), Z(\chi) \rangle \) is not less than two. Hence we should go on with the computation of inequivalent subalgebras of \( g_{\sim} \) which contain at most one linearly independent vector field of the form \( D(\zeta) + Z(\chi) \), where \( \zeta = \zeta(x) \) is a nonvanishing function. In view of Lemma 21 it is obvious that the dimension of such subalgebras cannot be greater than three. Here we select candidates for such subalgebras using only restrictions on appropriate subalgebras presented in Lemma 21. Since there exist specific restrictions for two- and three-dimensional appropriate subalgebras, we will make an additional selection of appropriate subalgebras from the set of candidates directly in the course of the construction of invariant equations.

The result of the classification is formulated in the subsequent lemmas.

**Lemma 24.** A complete list of \( G_{\sim} \)-inequivalent appropriate one-dimensional subalgebras of \( g_{\sim} \) is given by

\[
\langle 2D^u - qD^t + 2D(\delta) \rangle, \quad \langle D^t - D(2) \rangle, \quad \langle D^t - Z(2) \rangle, \quad \langle D(1) \rangle,
\]

where \( \delta \in \{0, 1\} \) and \( q \) is an arbitrary constant.

**Proof.** The classification of the appropriate one-dimensional subalgebras of \( g_{\sim} \) can be carried out effectively by simplifying a general element of \( g_{\sim} \),

\[
Q = a_1 D^u + a_2 D^t + D(\zeta) + Z(\chi),
\]

using scalings of \( Q \) and push-forwards of elementary transformations from \( G_{\sim} \). For this aim, it is necessary to distinguish multiple cases, subject to which of the constants \( a_i \) or the functions

29
and \( \zeta \) and \( \chi \) are nonzero. Note that in the proofs of this and the next two lemmas, we indicate only kinds of elementary transformations to be used for simplifying but not required values of associated parameters.

For \( a_1 \neq 0 \) we can scale the vector field \( Q \) to achieve \( a_1 = 2 \). Using \( G_\ast(\psi) \) we can set \( \chi = 0 \).

If \( \zeta \neq 0 \), then we set \( \zeta = 2 \) by pushing forward \( Q \) by \( D(\varphi) \).

By denoting \( a_2 = -q \) we obtain the first case from the list (38).

If \( a_1 = 0 \) and \( a_2 \neq 0 \), we set \( a_2 = 1 \) by scaling of \( Q \). For \( \zeta \neq 0 \), we can scale \( \zeta = -2 \) by means of \( D_\ast(\varphi) \) and additionally set \( \chi = 0 \) upon using the pushforward of \( G(\psi) \). If \( \zeta = 0 \), then we have \( \chi \neq 0 \) in view of Lemma 21 and hence we can use \( D_\ast^+(c_2) \) and \( D_\ast(\varphi) \) in order to set \( \chi = -2 \).

This gives the second and the third elements of the list (38), respectively.

In case of \( a_1 = a_2 = 0 \) but \( \zeta \neq 0 \), we can set \( \zeta = 1 \) by \( D_\ast(\varphi) \) and use the push-forward \( G_\ast(\psi) \) to arrive at \( \chi = 0 \), which yields the fourth element of the above list of one-dimensional inequivalent subalgebras.

In view of Lemma 21, the case \( a_1 = a_2 = 0 \) and \( \zeta = 0 \) is not appropriate. \( \square \)

**Remark 25.** In Lemma 24 and in the next two lemmas, we choose such values of parameters in basis elements of appropriate subalgebras among possible ones up to \( G^\sim \)-equivalence that the corresponding equations from the class \( W \) have a simple form.

**Lemma 26.** Up to \( G_\ast^\sim \)-equivalence, any appropriate two-dimensional subalgebra of \( g_\ast^\sim \) which contains at most one linearly independent vector field of the form \( D(\zeta) + Z(\chi) \) belongs to the following list:

\[
\begin{align*}
&D_u(1) = D^u - D(1), \quad D^l(2) = D^l - D(2), \\
&D_a(1) = a^1_D\beta + a^2_D\zeta + a^3_D(\delta, D(1)),
\end{align*}
\]

where \( p, a_1, a_2, a_3 \) and \( \delta \) are constants with \( p \neq 0, (a_1, a_2) \neq (0, 0), (a_2, a_3) \neq (0, 0) \) and \( (a_1, a_3, \delta) \neq (0, 0, 0) \). Due to scaling of the first basis element and \( G_\ast^\sim \)-equivalence we can also assume that one of \( a \)'s equals 1, \( (2a_1 + a_3)\delta = 0 \), and \( \delta \in \{0, 1\} \).

**Proof.** Let \( Q^1 \) and \( Q^2 \) be two arbitrary linearly independent vector fields from \( g_\ast^\sim \) that span a subalgebra \( \mathfrak{s} \) of \( g_\ast^\sim \) satisfying the inequality \( \dim(\mathfrak{s} \cap \langle D(\zeta), Z(\chi) \rangle) \leq 1 \) and the conditions from Lemma 21. We simplify the basis elements \( Q^1 \) and \( Q^2 \) as much as possible by linear combining and simultaneous push-forwards by transformations from \( G_\ast^\sim \). The proof is split into two parts.

First, we consider possible two-dimensional subalgebras of \( g_\ast^\sim \) not containing vector fields of the form \( D(\zeta) + Z(\chi) \). In view of this additional restriction and Lemma 21, basis vector fields of \( \mathfrak{s} \) can be chosen in the form

\[
Q^1 = D^u + D(\zeta^1) + Z(\chi^1), \quad Q^2 = D^l + D(\zeta^2) + Z(\chi^2),
\]

where \( \zeta^1 \neq 0 \) and \( (\zeta^2, \chi^2) \neq (0, 0) \).

If \( \zeta^2 \neq 0 \), then we set \( \zeta^2 = -2 \) and \( \chi^2 = 0 \) successively using \( D_\ast(\varphi) \) and \( G_\ast(\psi) \). Since the subalgebra \( \mathfrak{s} \) is closed with respect to the Lie bracket of vector fields, i.e., \( [Q^1, Q^2] \in \langle Q^1, Q^2 \rangle \), we derive \( [Q^1, Q^2]^2 = 2D(\zeta^2) + 2Z(\chi^2) = 0 \), and hence \( \zeta^1 = 0 \) and \( \chi^1 = 0 \), i.e., \( \zeta^1 \) and \( \chi^1 \) are constants. We re-denote the nonzero constant \( \zeta^1 \) by \( -p \). The pushforward \( G_\ast(\psi) \) does not change \( Q^2 \) and sets \( \chi^1 = 0 \), which leads to the first family of subalgebras in the list (39).

For \( \zeta^2 = 0 \), we use \( D_\ast(\varphi) \) and \( G_\ast(\psi) \) to set \( \zeta^1 = -2x \) and \( \chi^1 = 0 \). The condition \( [Q^1, Q^2] \in \langle Q^1, Q^2 \rangle \) implies \( [Q^1, Q^2] = -2Z(\chi^2) = 0 \). Therefore, \( \chi^2 \) is a nonzero constant, which can be gauged by \( D_\ast^+(c_2) \) to \( -2 \), giving the second family of subalgebras in the list (39).

Now we study the case \( \dim(\mathfrak{s} \cap \langle D(\zeta), Z(\chi) \rangle) = 1 \). Up to linearly combining of the basis elements \( Q^1 \) and \( Q^2 \), we can initially take

\[
Q^1 = a_1 D^u + a_2 D^l + D(\zeta^1) + Z(\chi^1), \quad Q^2 = D(\zeta^2) + Z(\chi^2),
\]
where \((a_1, a_2) \neq (0, 0)\) and \(\zeta^2 \neq 0\). We set \(\zeta^2 = 1\) and \(\chi^2 = 0\) using \(D_*(\varphi)\) and \(G_*(\psi)\). Since \(\mathfrak{s}\) is a Lie algebra, we have that \([Q^2, Q^1] = D(\zeta^1) + Z(\chi^1) = a_3 Q_2\) for some constant \(a_3\). Therefore, \(\zeta^1_x = a_3\) and \(\chi^1_x = 0\). Combining \(Q^1\) with \(Q^2\), we obtain that \(\zeta^1 = a_3 x\) and \(\chi^1 = c = \text{const.}\) Up to \(G_{\text{ess}}\)-equivalence we can assume that (2a1 + a3) = 0. Indeed, acting by \(G_s(2c/(2a_1 + a_3))\) in the case 2a1 + a3 ≠ 0, we set \(c = 0\) in \(Q^2\) and do not change the vector field \(Q^1\). Using push-forwards of the variable \(u\) and alternating its signs, we can scale the constant parameter \(c\) and change its sign. Additionally we can multiply the whole vector field \(Q^1\) by a nonvanishing constant in order to scale one of nonvanishing \(a\)'s to one. The conditions \((a_2, a_3) \neq (0, 0)\) and \((a_1, a_3, c) \neq (0, 0, 0)\) follow from Lemma 21. After denoting \(c\) by \(\delta\), this yields the third case of the list (39) and thereby completes the proof of the lemma. □

Lemma 27. Up to \(G_{\text{ess}}\)-equivalence, any appropriate three-dimensional subalgebra of \(\mathfrak{g}_{\text{ess}}\), which contains at most one linearly independent vector field of the form \(D(\zeta) + Z(\chi)\), has the forms

\[
\langle D^a + p_1 D(x), D^b + p_2 D(x), D(1) \rangle, \quad \langle D^a - 2D(x) + Z(d), D^b - Z(2), D(1) \rangle,
\]

where \(p_1, p_2\) and \(d\) are constants such that \(p_1 p_2 \neq 0\).

Proof. In view of Lemma 21, any appropriate three-dimensional subalgebra of \(\mathfrak{g}_{\text{ess}}\), which contains at most one linearly independent vector field of the form \(D(\zeta) + Z(\chi)\), is spanned by vector fields \(Q^1 = D^a + D(\zeta^1) + Z(\chi^1)\), \(Q^2 = D^b + D(\zeta^2) + Z(\chi^2)\) and \(Q^3 = D(\zeta^3) + Z(\chi^3)\), where \(\zeta^i\) and \(\chi^i\) are smooth functions of \(x\), \(\zeta^1\) and \(\zeta^3\) are linearly independent, and \(\zeta^1, \chi^1\) and \(\zeta^2, \chi^2\) are linearly independent as well. We also have \([Q^i, Q^j] \in (Q^1, Q^2, Q^3), \ i, j = 1, 2, 3\).

Using \(D_*(\varphi)\) and \(G_*(\psi)\) with suitably chosen functions \(\varphi\) and \(\psi\) of \(x\), we set \(\zeta^3 = 1\) and \(\chi^3 = 0\), i.e., we make \(Q^3 = D(1)\). The commutation relations of \(Q^3\) with \(Q^1\) and \(Q^2\) are

\[
\begin{align*}
[Q^3, Q^1] &= D(\zeta^1_x) + Z(\chi^1_x) = p_1 Q^3, \\
[Q^3, Q^2] &= D(\zeta^2_x) + Z(\chi^2_x) = p_2 Q^3
\end{align*}
\]

for some constants \(p_i\), \(i = 1, 2\). These commutation relations imply the conditions \(\zeta^1_x = p_1\) and \(\chi^1_x = 0\). Therefore, up to combining \(Q^i\) with \(Q^3\) we obtain \(\zeta^i = p_i x\) and \(\chi^i = d_i\) for some constants \(d_i\), and \(p_1 \neq 0\) and \((p_2, d_2) \neq (0, 0)\) in view of the above linear independence. Then the commutation relation

\[
[Q^2, Q^1] = \frac{1}{2} Z((p_1 + 2)d_2 - p_2 d_1) = 0
\]

yields \(p_2 d_1 = (p_1 + 2)d_2\). If \(p_1 \neq -2\), we can set \(d_1 = 0\) using \(G_s(2d_1/(p_1 + 2))\) and then \(d_2 = 0\). Analogously, in the case \(p_2 \neq 0\) we can set \(d_2 = 0\) using \(G_s(d_2/p_2)\) and then \(d_1 = 0\). Therefore, up to \(G^-\)-equivalence we have two different cases, \(d_1 = d_2 = 0\) and \((p_1, p_2) = (-2, 0)\). In view of Lemma 21 we obtain \(p_1 p_2 \neq 0\) and \(d_2 \neq 0\) in the first and second cases, respectively. The nonzero value of \(d_2\) can be gauged by \(D^a_*(c_2)\) to any nonzero value, e.g., \(-2\). Re-denoting \(d_2\) by \(d\) completes the proof of the lemma. □

9 Regular Lie symmetry extensions

For each vector field \(Q\) from \(\mathfrak{g}^-\), the substitution of the components of \(\pi_\ast \mathcal{Q}\) into the system (22)–(23) results in the condition on the arbitrary-element tuple \(\theta = (f, g)\) for the equation \(L_\theta\) to be invariant with respect to \(\pi_\ast \mathcal{Q}\). This is why equations from the class (1) that are invariant with respect to the projection \(\pi_\ast \mathfrak{s}\) of an appropriate subalgebra \(\mathfrak{s}\) of \(\mathfrak{g}^-\) can be described by the following way: For each basis element \(\mathcal{Q}\) of \(\mathfrak{s}\), we substitute the components of \(\pi_\ast \mathcal{Q}\) into the equations (22) and (23). Collecting all the equations derived from the entire basis \(\mathfrak{s}\) leads to a system of first-order (quasi)linear partial differential equations in the arbitrary elements \(f\)
and $g$ to be solved. Simultaneously we check whether the projection $\pi_\theta$ is really appropriate one-dimensional subalgebras of $\tilde{\mathfrak{g}}_{\text{ess}}$ and results in a simple uncoupled system of two first-order linear differential equations in $f$ and $g$. The corresponding list of equations from the class (1), which possess one-dimensional Lie symmetry extensions of $\tilde{\mathfrak{g}}^\perp$ related to $\tilde{\mathfrak{g}}^\land$, reads

\begin{enumerate}
  \item $2\mathcal{D}u - q\mathcal{D}t + 2\mathcal{D}(\delta)$: $u_{tt} = |u|^q(\tilde{f}(\omega)u_{xx} + \tilde{g}(\omega)u)$,
  \item $\mathcal{D}l - \mathcal{D}(2)$: $u_{tt} = e^\nu(\tilde{f}(u)u_{xx} + \tilde{g}(u))$,
  \item $\mathcal{D}l - \mathcal{D}(2)$: $u_{tt} = e^u(\tilde{f}(x)u_{xx} + \tilde{g}(x))$,
  \item $\mathcal{D}(1)$: $u_{tt} = \tilde{f}(u)u_{xx} + \tilde{g}(u)$,
\end{enumerate}

where $\omega := x - \delta \ln |u|$, $\delta \in \{0, 1\}$ and $q$ is an arbitrary constant. Here and in what follows, in each case we present only vector fields that extend the basis ($\mathcal{P}$) of the ideal $\tilde{\mathfrak{g}}^\land$ of $\tilde{\mathfrak{g}}^\land$ into a basis of the corresponding subalgebra of $\tilde{\mathfrak{g}}^\land$.

Computations related to two-dimensional extensions are more complicated. We first present the computation result and then give some explanations.

\begin{enumerate}
\item $\mathcal{D}u - \mathcal{D}(p)$, $\mathcal{D}(2)$, $p \neq 0$: $u_{tt} = \pm e^p|u|^pu_{xx} + nu$, \item $\mathcal{D}u - 2\mathcal{D}(x)$, $\mathcal{D}(2)$: $u_{tt} = \pm x^2e^nu_{xx} + ve^n$,
\item $-\mathcal{D}u + 2\mathcal{D}l + 2\mathcal{D}(x)$, $\mathcal{D}(1)$: $u_{tt} = \tilde{f}(u)u_{xx}$, \item $(1 - q)\mathcal{D}u + 2\mathcal{D}l - 2(1 - q)\mathcal{D}(x) - \mathcal{Z}(4)$, $\mathcal{D}(1)$: $u_{tt} = \pm e^n u_{xx} + \varepsilon' e^nu$,
\item $(3 - p + q)\mathcal{D}u + 2(1 - q)\mathcal{D}l + 2(1 + p - q)\mathcal{D}(x)$, $\mathcal{D}(1)$: $u_{tt} = \pm |u|^p u_{xx} + \varepsilon'|u|^q$.
\end{enumerate}

Constraints for constant and functional parameters that are imposed by the maximality condition for the corresponding extensions and their inequivalence are discussed after Theorem 8.

Cases 9 and 10 are associated with the first and second families of subalgebras listed in Lemma 26, respectively. In both the cases, $v$ is an arbitrary constant. Note that an arbitrary nonzero constant multiplier in the expression for the arbitrary element $f$, which arises in the course of integrating the equation for $f$, can always be set to $\pm 1$, e.g., by scaling of $t$.

The third span from Lemma 26 in fact represents a multiparametric series of candidates for appropriate extensions, which is partitioned in the course of the construction of invariant equations into Cases 11–13. Not all values of series parameters give appropriate extensions. Additional constraints for parameters follow from the compatibility conditions of the associated system in the arbitrary elements,

$$
\begin{align*}
  f_x &= 0, & ((a_1 + \frac{1}{2}a_3)u + \delta) f_u &= pf, \\
  g_x &= 0, & ((a_1 + \frac{1}{2}a_3)u + \delta) g_u &= qg.
\end{align*}
$$

with the inequalities $f \neq 0$ and $(f_u, g_u) \neq (0, 0)$ and the requirement that the dimension of extensions should not exceed two. Here we introduce the notation $p = 2(a_3 - a_2)$ and $q = a_1 + \frac{1}{2}a_3 - 2a_2$.

The above partition is carried out in the following way. If $a_3 = -2a_1$ and $\delta = 0$, the inequality $f \neq 0$ implies that $p = 0$, i.e., $a_2 = a_3$. Since $a_1$, $a_2$ and $a_3$ cannot simultaneously be zero, we obtain that $q \neq 0$ and hence $g = 0$. Multiplying the first basis element by $-a_1^{-1}$, we set $a_1 = -1$. This gives Case 11. For $a_3 = -2a_1$ and $\delta = 1$ we have $a_2 = -q/2$, $a_3 = (p - q)/2$ and $a_1 = -(p - q)/4$. The parameter $p$ should be nonzero since otherwise we obtain the Liouville equation whose maximal Lie invariance algebra is infinite-dimensional. We additionally multiply the first basis element by $-4$ and scale $p$ with $\mathcal{D}u(c_2)$ for some $c_2$ to 1 and obtain Case 12. Case 13 corresponds to the condition $a_3 \neq -2a_1$. Scaling the first basis element allows us to set $a_1 + \frac{1}{2}a_3 = 4$. Then $a_2 = 2(1 - q)$, $a_3 = 2(1 + p - q)$ and $a_3 = (3 - p + q)$. In both Cases 12
and 13 the parameter \( \varepsilon' \) is nonzero (otherwise the extension dimension is greater than two) and can be gauged to \( \pm 1 \) by simultaneous scaling of \( t \) and \( x \).

Consider the candidates for three-dimensional appropriate extensions listed in Lemma 27. The compatibility of the associated systems in the arbitrary elements, supplemented with the inequality \( f \neq 0 \), implies \( p_1 = 2(p_2 - 1) \) and \( d = -4 \) for the first and the second span of Lemma 27, respectively. The general solutions of these systems up to \( G^- \)-equivalence are \( (f, g) = (\pm |u|^p, 0) \) and \( (f, g) = (\pm e^u, 0) \). This gives the following cases of Lie symmetry extensions:

16. \[ (p - 4)D^u - 2pD(x), \ (p - 4)D^l - 4D(x), \ D(1), \ p \neq 0, 4: \quad u_{tt} = \pm |u|^p u_{xx}, \]
17. \[ D^u - 2D(x) - Z(4), \ D^l - Z(2), \ D(1): \quad u_{tt} = \pm e^u u_{xx}. \]

Here \( p := 4(p_2 - 1)/p_2 \neq 4 \) since for \( p = 4 \) the corresponding equations admits the Lie-symmetry vector fields \( \pi_x D(x) \) and \( \pi_x D(x^2) \). Equations from the class (1) which are invariant with respect to two linearly independent vector fields of the form \( \pi_x Q^i \), where \( Q^i = (\zeta^i) + Z(x^i) \), are classified in Corollary 23. Therefore, \( G^- \)-inequivalent regular Lie symmetry extensions in the class (1) are exhausted by Cases 1–4, 9–13, 14d, 16, 17 and 19d.

10 Conclusion and discussion

In the present paper, we have carry out the complete group classification of the class \( W \) of \((1+1)\)-dimensional nonlinear wave and elliptic equations up to both \( G^- \) and \( G^- \) -equivalences using the new version of the algebraic method of group classification for non-normalized classes of differential equations. The results of the classification are collected in Theorem 8. The key ingredient of the classification procedure is the construction of a generating set for the equivalence groupoid \( G^- \) of the class \( W \) modulo \( G^- \)-equivalence. This generating set is given in Theorem 9.

In view of the partition \( G^-_{\text{gen}} = G^- \cup G^-_{\text{lin}} \) of the equivalence groupoids \( G^-_{\text{gen}} \) of the superclass \( W_{\text{gen}} \) of all equations of the form (1) with \( f \neq 0 \), cf. Remark 5, we can merge the results on \( W \) with the analogous results from Remark 16 on the class \( W_{\text{lin}} \) of linear equations of the form (1) to those for \( W_{\text{gen}} \). In other words, we have also obtained the complete group classifications of the classes \( W_{\text{lin}} \) and \( W_{\text{gen}} \) and the classification of admissible transformations of these classes.

Let us compare this paper’s results with some similar results existing in the literature for related classes of differential equations. The problem of group classification for the class of semilinear wave equations of the general form

\[ u_{tt} = u_{xx} + g(t, x, u, u_x) \]  

(41)

was solved in [29, 30]. The class (41) was partitioned into four (normalized) subclasses, and each of these subclasses was classified separately. One of these subclasses, which we denote by \( K \), is singled out from the class (41) by the constraint \( g_{ux} = 0 \). The group classification of the subclass \( K \) was carried out in Section 6 of [30] and the major part of classification results was collected in Table 1 therein, see also Section V and Table I in [29]. Cases \( 1, \delta = 1, p = 2, f = 1, 2 f = 1, 5 \alpha_1 = 1, 6 \alpha_1 = 1 \) and \( 18 \alpha_1 = 1 \) of Table 1 in the present paper corresponds to Cases 3, 2, 8, 5 and 9 of Table 1 in [30], whereas the Liouville equation is given as Case 20 in Table 1 of the present paper and as the equation (5.4) in [30], and this exhausts all possible analogous cases. The counterpart of Cases \( 1, \delta = 1, p \neq 2, f = 1 \) of our Table 1 is missed in [29, 30]. In fact, each of Cases 3 and 4 of Table 1 in [30] should contain one more constant parameter, which cannot be removed by equivalence transformations of the subclass \( K \). In [31], Lahno and Spichak classified the semilinear elliptic equations of the rather general form

\[ u_{tt} + u_{xx} = F(t, x, u, u_t, u_x) \]

whose maximal Lie invariance algebras are finite-dimensional and have nonzero Levi factors. Cases 6a and 7 of Table 1 are the restriction of the first cases of Theorems 3.1 and 3.2 of [31] to the class \( W \). There are no other related cases in [31] and the present paper.
More important than the solutions of the above specific classification problems are the development and modification of general concepts and techniques and the combination of them with each other, which have been done in the course of solving these problems in the present paper.

Partition of classes of differential equations into subclasses that induce partitions of the corresponding equivalence groupoids had already regularly been applied in the course of the study of equivalence groupoids [5, 47, 51]. We have made the two partitions of classes, \( W_{\text{gen}} = W \sqcup W_{\text{lin}} \) and \( W = W_0 \sqcup W_1 \), obtaining the partitions of the groupoids

\[
G_\sim \sim = G_\sim \sqcup G_{\text{lin}}_\sim \quad \text{and} \quad G_\sim = G_\sim \sqcup G_1 \sim,
\]

see Remark 5 and Proposition 13. All the above classes and subclasses have the same equivalence group \( G_\sim \). Nevertheless, in contrast to the examples existing in the literature, the subclasses in these two partitions do not have better normalization properties than their superclasses. This is why no kind of normalization can be used for justifying the partitions, which are rather derived via the direct analysis of the determining equations for admissible transformations. Although the structure of the partition components is simpler than the entire groupoid for both the groupoid partition, this becomes clear only after a comprehensive study of admissible transformations.

We have separately constructed generating sets \( B_0 \) and \( B_1 \) of the equivalence groupoids \( G_0 \sim \) and \( G_1 \sim \), which are constituted by the families \( T_1|\gamma_0 \) and \( T_3 \sim T_9 \) and by the families \( T_1|\gamma_1 \) and \( T_2 \) given in Theorem 9, respectively. Due to constructing \( B_0 \) and \( B_1 \) modulo \( G_\sim \)-equivalence, we can and should factor out elements of the action groupoid \( G_\sim \) from admissible transformations before including to these sets, which is realized via gauging arbitrary elements of singled out subclasses whose equivalence groupoids contain elements from \( G_\sim \sim \). In other words, mapping these subclasses onto smaller ones with simpler equivalence groupoids by families of point transformations, we have factored out subsets of equivalence transformations and have simplified the consideration for the corresponding classification cases.

To classify admissible transformations of the class \( W_0 \) in the optimal way, we split the construction of the generating set \( B_0 \) in the simultaneous proof of Lemmas 14 and 15 into two cases depending on the number of independent constraints for arbitrary elements that arise in the course of the classification. In this way, we have extended for the first time the method of furcate splitting to getting generating sets of admissible transformations.

Moreover, we have found a bijective functor between two categories, which are the equivalence groupoids \( G_{00} \sim \) and \( G_{01g2} \sim \) of the subclasses \( W_{00} \sim \) and \( W_{01g2} \) of equations of the forms (30) and (33) with \( g^1 \neq 0 \) and a fixed \( \varepsilon \in \{-1, 1\} \) and with a fixed \( g^2 \) satisfying \( g^2_\delta uuu \neq (g^2_\delta uu)^2 \), respectively. The isomorphism from \( G_{00} \sim \) to \( G_{01g2} \sim \) is given by

\[
\varepsilon \mapsto \tilde{\varepsilon} = \varepsilon, \quad g^1 \mapsto \tilde{g}^1 = g^1, \quad \Phi: \tilde{t} = T, \tilde{x} = X, \tilde{u} = u - \ln |X_x^2 - \varepsilon X_t^2| \quad \mapsto \quad \Phi: \tilde{t} = T, \tilde{x} = X, \tilde{u} = u.
\]

Fixing \( \varepsilon \) and of \( g^2 \) is natural since values of these parameters cannot be changed by admissible transformations in the entire classes \( W_{00} \) and \( W_{01} \) up to gauge equivalence transformations of moving a nonzero constant multiplier between \( g^1 \) and \( g^2 \) within \( W_{01} \), which can be neglected. That is, the partitions of the classes \( W_{00} \) and \( W_{01} \) into the subclasses associated with fixed values of \( \varepsilon \) and of \( g^2 \), \( W_{00} = \sqcup \varepsilon W_{00} \) and \( W_{01} = \sqcup g^2 W_{01g2} \), induce the partition of the corresponding equivalence groupoids,

\[
G_{00} \sim \sim = \sqcup \varepsilon G_{00} \sim \quad \text{and} \quad G_{01} \sim = \sqcup g^2 G_{01g2} \sim.
\]

No equations from \( W_{00} \) \( W_{00} \) are related to equations from \( W_{01g2} \) by point transformations. In other words, the functor from \( G_{00} \sim \) to \( G_{01g2} \sim \) is not underlain by a family of point transformations generating a mapping from \( W_{00} \) onto \( W_{01g2} \), or conversely. Nevertheless, it allows us to easily obtain the equivalence groupoid \( G_{01g2} \sim \) from the equivalence groupoid \( G_{00} \sim \).
A necessary preliminary step for finding the above functor is the proper selection of classes to be related via functor. For the first (degenerate) case in the simultaneous proof of Lemmas 14 and 15, we derive the specific form \( g = g^0(x)e^u + g^1(x) \) for values of \( g \) for source and target equations for admissible transformations that are not generated by elements of \( G^- \). There are two possibilities for gauging parameters in the above form of \( g \), either to \( g^1 = 0 \) or to \( g^0 = \varepsilon' \). The first possibility seems preferable since after gauging we obtain equations of the same general form as those in the class (33). In this way, the study can be reduced to describing the equivalence groupoid of the single class of equations of the form (33), where the auxiliary inequality \( g^2 u^2 \neq (g^2 u^u)^2 \) is neglected. At the same time, the structure of the subgroupoid of the above groupoid that is the equivalence groupoid of the subclass singled out by the constraint \( g^2 u^2 = (g^2 u^u)^2 \) is different from and more complicated than the structure of its complement, and thus this subgroupoid needs a separate consideration. As a result, the preferable gauge is in fact \( g^0 = \varepsilon' \). Although we then have to study two classes of equations of different forms, via excluding the evidently marked out value \( g^1 = 0 \), which corresponds to the Liouville equations giving rise to the family \( \text{T9} \) of admissible transformations, and fixing \( \varepsilon' \) and \( g^2 \) we have partitioned the corresponding equivalence groupoids into naturally isomorphic subgroupoids. Therefore, it suffices to describe only one of them.

It is convenient to construct a generating set for the equivalence groupoid \( \mathcal{G}_{00}^- \) up to the equivalence group of the subclass \( \mathcal{W}_{00}^- \), since then we can apply various algebraic techniques, including an original extension of Hydon’s algebraic method to finding admissible transformations. These techniques are based on knowing the maximal Lie invariance algebras of equations from the subclass \( \mathcal{W}_{00}^- \) whose efficient classification involves a preliminary knowledge on admissible transformations within the subclass \( \mathcal{W}_{00}^- \). This is why we have merged the proofs of Lemmas 14 and 15. Mapping the families \( \text{T3}'-\text{T7}' \) into the families \( \text{T3}-\text{T7} \) and uniting the restrictions of the families \( \text{T3}-\text{T8} \) to \( \mathcal{W}_{01} \) and to the class of equations of the same form \( g_2 u^2 = (g_2 u^u)^2 \) provide us with the presentation of the final results in Theorem 9 in a concise form.

An unexpected by-product of the proper gauging the arbitrary elements for equations from the class \( \mathcal{W}_0 \) with \( g = g^0(x)e^u + g^1(x) \) by transformations from the group \( G^- \) is that this gauging is concordant with the maximal natural gauging of the arbitrary elements within the class \( \mathcal{W}_{0} \) by transformations from the same group, which leads to the subclass \( \mathcal{W}_{00}^- \) of \( \mathcal{W}_{0} \). There exists a canonical isomorphism between the essential equivalence groupoid \( \mathcal{G}_{00}^- \) and the equivalence groupoid of the class of equations of the form (30) with a fixed value of \( \varepsilon' \), and it is the above concordance that makes this existent evident. As a result, the complete group classifications of the class \( \mathcal{W}_{00}^- \) up to \( G^- \) - and \( G^- \) -equivalences and the classification of admissible transformations within this class are carried out in the single Remark 16. This is one more demonstration of efficiency of the functor method in classification problems of group analysis of differential equations. Note that analogously to the previous isomorphism between \( \mathcal{G}_{00}^- \) and \( \mathcal{G}_{01}^- \), this groupoid isomorphism is not induced by families admissible point transformations within the superclass \( \mathcal{W}_{\text{gen}} \).

Necessary preliminaries for the classification of singular Lie symmetry extensions within the subclass \( \mathcal{W}_{01} \) have been given by the classification of admissible transformations within this subclass. As a result, the former classification can easily completed both the direct and the algebraic method. The classification of regular Lie symmetry extensions has been carried out within the framework of the algebraic method of group classification. We have use our optimized version of this method, which involves the classification of candidates for appropriate subalgebras of \( \mathfrak{g}^- \) with taking into account the principal restrictions on the dimensions and structure of each subalgebra and the completion of selecting appropriate subalgebras in the course of constructing the corresponding equations possessing Lie-symmetry extensions.

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\(^6\)This consideration shows that the algebraic method can further be developed to the construction of the complete equivalence groupoids for classes of differential equations via applying the algebraic method to the corresponding equivalence algebroids, which are infinitesimal counterparts of the equivalence groupoids.
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