QUANTUM $G_2$ CATEGORIES HAVE PROPERTY (T)

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Abstract. We show that the rigid $C^*$-tensor category of finite dimensional unitary representations of the quantum group $U_q(g_2)$ for positive $q \neq 1$ has property (T) using Kuperberg’s diagrammatic description of the categories.

1. Introduction

Recently there has been a great deal of interest in approximation and rigidity properties for subfactors and rigid $C^*$-tensor categories. Rigid $C^*$-tensor categories, introduced in their modern form by Longo and Roberts [18], are objects which provide a unifying framework for structures that appear as symmetries in a variety of contexts. Often they are described as encoding “quantum symmetries”. They make a prominent appearance in the theory of compact quantum groups as representation categories [21]. These categories arise as DHR super-selection sectors in algebraic quantum field theory [10]. Rigid $C^*$-tensor categories are also realized as categories of bimodules in the standard invariant of finite index subfactors, a theory initiated by V. Jones [12]. Rigid $C^*$-tensor categories and subfactors can be seen as “group-like” objects, and this is the motivation for defining approximation and rigidity properties in these settings.

Popa introduced the concepts of approximation and rigidity properties for subfactors (see [25], [26], [28], [29]). Popa’s definitions for approximation and rigidity properties can be formulated in terms of the symmetric enveloping inclusion $T \subseteq S$ associated to $N \subseteq M$ (see [25], [29]). He showed that the definitions only depend on the standard invariant of the subfactor, which is the 2-category of $M$-$M$ and $N$-$N$ bimodules that appear as tensor powers of $N L^2(M)_M$ and its dual with respect to the relative tensor product. The standard invariant is a powerful invariant of a subfactor, has been axiomatized by Popa as standard $\lambda$-lattices [27], Ocneanu as paragroups [23], and V. Jones as subfactors planar algebras [13]. Alternatively, the standard invariant can be axiomatized as a rigid $C^*$-tensor category along with a tensor generating $Q$-system (see [20] for details).

Recently, approximation and rigidity properties were translated from Popa’s original definitions into the categorical setting by Popa and Vaes in [30]. This allows for the definitions of property (T), the Haagerup property, and amenability to be defined in a conceptually uniform way for standard invariants and tensor categories without reference to an ambient subfactor. They introduce a representation theory for standard invariants and rigid $C^*$-tensor categories generalizing the unitary representation theory for groups. For standard invariants, their representation theory encodes in a natural way the approximation and rigidity properties of the corresponding subfactor. For categories, it is defined by identifying a class of admissible representations of the fusion algebra.

Soon after the initial work of Popa and Vaes, admissible representations of the fusion algebra were interpreted from different points of view. Neshveyev and Yamashita showed that admissible representations can

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be understood as objects in the Drinfeld center of the ind-category [21]. In [9], the authors show that admissible representations have a natural interpretation in the annular representation theory for planar algebras of V. Jones [15], [14]. The tube algebra of a category, introduced by Ocneanu in [24], is one realization of Jones’ annular categories. The fusion algebra is a corner of the tube algebra, and in [9] it is shown that admissible representations of the fusion algebra are precisely representations which are restrictions of representations of the whole tube algebra. Since the tube algebra is computable in principle, this provides a method for determining admissible representations. This is the approach we take in this paper.

Examples of property (T) subfactors have been somewhat elusive. Strictly speaking this is not true, since there are a plethora of property (T) groups. Until recently, however, the only known examples of subfactors with property (T) came in some way from discrete property (T) groups. In particular, the diagonal subfactors and the Bisch-Haagerup subfactors, when constructed with property (T) groups, produce property (T) subfactors [2]. Arano showed in [1] that the discrete dual of the compact quantum groups $SU_q(N)$ have central property (T) for $N \geq 3$ odd and positive $q \neq 1$, which Popa and Vaes showed is equivalent to the corresponding representation category of $SU_q(N)$ having property (T) (see [30]). From these categories, one can construct subfactors whose even bimodule categories are equivalent to $\text{Rep}(SU_q(N))$. This implies their subfactors have (T). Their result provides the first examples of subfactors not coming in some way from discrete groups which have (T).

The categories $\text{Rep}(U_q(g_2))$ have been described diagrammatically by Kuperberg [16], [17], where $U_q(g_2)$ is the Drinfeld-Jimbo deformation of the Lie algebra $g_2$. These diagrammatic categories, denoted $(G_2)_q$, have been further studied by Morsion, Peters, and Snyder in [19], where they appear as “small” examples of trivalent categories in their classification program. The projection categories of $(G_2)_q$ are equivalent to the rigid $C^*$-tensor categories of unitary representations of $U_q(g_2)$ for positive $q$.

In this note, we show that the categories $(G_2)_q$ have property (T). We use a surprisingly small amount of data from the category. We sketch the proof as follows: first we identify the fusion algebra $(G_2)_q$ as polynomials in two self adjoint variables. The irreducible representations correspond to points in the plane, defined by evaluation of polynomials. Using some general restrictions, we reduce the possibilities to a rectangle, with the trivial representation at a corner. Using the description of minimal idempotents provided by [19], we define a function of the plane $f(\alpha, t)$ with the property that this function must be non-negative at $(\alpha, t)$ for the representation corresponding to that point to be admissible. Then using elementary calculus, we show in a neighborhood of the trivial representation this function is strictly negative. A corollary of a proposition due to Popa and Vaes then implies that $(G_2)_q$ has property (T). We note our calculus arguments break down precisely when $q = 1$, which is to be expected since the classical $G_2$ representation category is amenable.

Although the categories we study are quantum group category, our proof uses only the basic skein theoretic description of the category. We hope this note demonstrates the computational usefulness of the planar algebra/diagrammatic approach to approximation and rigidity theory for categories, which was also demonstrated in [3], where the authors give a direct planar algebraic proof that the categories $TLJ(\delta)$ have the Haagerup property for all $\delta \geq 2$. We also remark that there is a subfactor whose category of even bimodules is $(G_2)_q$, hence this subfactor has property (T).

The outline of the paper is as follows: We begin by describing rigid $C^*$-tensor categories and the tube algebra. We then discuss admissible representations and property (T). Finally we give a short proof that $(G_2)_q$ has property (T). In the appendix, we discuss in greater detail the quantum group $U_q(g_2)$ and describe
how the star $*$-structure from the category of unitary representations of $\text{Rep}(U_q(\mathfrak{g}_2))$ transports to the planar algebra $(G_2)_q$.

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2. Preliminaries

2.1. Rigid $C^*$-Tensor Categories. In this paper we will be concerned with semi-simple, $C^*$-categories with strict tensor functor, simple unit and duals. We also assume that $\mathcal{C}$ has countably many isomorphism classes of simple objects. Often in the literature, this is the definition of a rigid $C^*$-tensor category. We elaborate on the meaning of each of these words.

A $C^*$-category is a $\mathbb{C}$-linear category $\mathcal{C}$, with each morphism space $\text{Mor}(X,Y)$ a Banach space, and a conjugate linear, involutive, contravariant functor $\ast : \mathcal{C} \to \mathcal{C}$ which fixes objects and satisfies for every morphism $f$, $\|f\ast f\| = \|ff\ast\| = \|f\|^2$. We say the category is semi-simple if the category has direct sums, sub-objects, and each $(X,Y)$ is finite dimensional.

A strict tensor functor is a bi-linear functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which is associative and has a distinguished unit $id \in \text{Obj}(\mathcal{C})$ such that $X \otimes id = X = id \otimes X$. The category is rigid if for each $X \in \text{Obj}(\mathcal{C})$, there exists $X \in \text{Obj}(\mathcal{C})$ and morphism $R \in \text{Mor}(id, X \otimes X)$ and $\overline{R} \in \text{Mor}(id, X \otimes X)$ satisfying the so-called conjugate equations:

$$(1_X \otimes \overline{R})(R \otimes 1_X) = 1_X \text{ and } (1_X \otimes R')(\overline{R} \otimes 1_X) = 1_X$$

We say two objects are $X,Y$ are isomorphic if there exists $f \in \text{Mor}(X,Y)$ such that $f\ast f = 1_X$ and $ff\ast = 1_Y$. We call an object $X$ simple if $\text{Mor}(X,X) \cong \mathbb{C}$. We note that for any simple objects $X$ and $Y$, $\text{Mor}(X,Y)$ is either isomorphic to $\mathbb{C}$ or 0. Two simple objects are isomorphic if and only if $\text{Mor}(X,Y) \cong \mathbb{C}$. Isomorphism defines an equivalence relation on the collection of all objects and we denote the equivalence class of an object by $[X]$, and the set of isomorphism classes of simple objects $\text{Irr}(\mathcal{C})$.

The semi-simplicity axiom implies that for any object $X$, $\text{Mor}(X,X)$ is a finite dimensional $C^*$-algebra over $\mathbb{C}$, hence a multi-matrix algebra. It is easy to see that each summand of the matrix algebra corresponds to an equivalence class of simple objects, and the dimension of the matrix algebra corresponding to a simple object $Y$ is the square of the multiplicity with which $Y$ occurs in $X$. In general for a simple object $Y$ and any object $X$, we denote by $N^X_Y$ the natural number describing the multiplicity with which $[Y]$ appears in the simple object decomposition of $X$. If $X$ is equivalent to a subobject of $Y$, we write $X \prec Y$. We often write $X \otimes Y$ simply as $XY$ for objects $X$ and $Y$.

For two simple objects $X$ and $Y$, we have that $[X \otimes Y] \cong \oplus Z N^X_Y[Z]$. These means that the the tensor product of $X$ and $Y$ decomposes as a direct sum of simple objects of which $N^X_Y$ are equivalent to the simple object $Z$. The $N^X_Y$ specify the fusion rules of the tensor category and are a critical piece of data.

The fusion algebra is the complex linear span of isomorphism classes of simple objects $\mathbb{C}[\text{Irr}(\mathcal{C})]$, with multiplication given by linear extension of the fusion rules. This algebra has a $*$-involution defined by $[X]^* = [X] \overline{\ast}$ and extended conjugate-linearly. This algebra is a central object of study in approximation and rigidity theory for rigid $C^*$-tensor categories.
For a more detailed discussion and analysis of the axioms of a rigid $C^*$-tensor category, see the paper of Longo and Roberts [13], where these categories were first defined and studied with this axiomatization.

In a rigid $C^*$-tensor category, we can define the statistical dimension of an object $d(X) = \inf_{(R,\alpha)} ||R|| ||\overline{R}||$, where the infimum is taken over all solutions to the conjugate equations for an object $X$. The function $d(\cdot) : \text{Obj}(C) \to \mathbb{R}_+$ depends on objects only up to unitary isomorphism. It is multiplicative and additive and satisfies $d(X) = d(Y)$ for any dual of $X$. We called solutions to the conjugate equations standard if $||R|| = ||\overline{R}|| = d(X)^{\frac{1}{2}}$, and such solutions are essentially unique. For standard solutions of the conjugate equations, we have a well defined trace $\text{Tr}_X$ on endomorphism spaces $\text{Mor}(X, X)$ given by

$$\text{Tr}_X(f) = R^\ast(1_X \otimes f)R = \overline{R}^\ast(f \otimes 1_X)\overline{R} \in \text{Mor}(id, id) \cong C$$

This trace does not depend on the choice of dual for $X$ or on the choice of standard solutions. We note that $\text{Tr}(1_X) = d(X)$. See [19] for details.

2.2. The Tube Algebra. The tube algebra $\mathcal{AC}$ of a semi-simple category was introduced by Ocneanu in [24]. This algebra has proved useful for computing the Drinfeld center $Z(C)$, since finite dimensional irreducible representations of $\mathcal{AC}$ are in one-to-one correspondence with simple objects of $Z(C)$ (see [11]). In general, arbitrary representations of $\mathcal{AC}$ are in one-to-one correspondence with objects in $Z(\text{ind-C})$ studied by Neshveyev and Yamashita in [21] (see [9] for a proof due to Vaes).

For a rigid $C^*$-tensor category $C$, choose a set $\Lambda$ representative of each equivalence class of simple objects in $\text{Irr}(C)$. This set of objects is indexed by $\text{Irr}(C)$, so that $X_k$ is the object we have chosen corresponding to some equivalence class $k \in \text{Irr}(C)$. We choose the strict tensor identity $id$ and label it with index 0, so that $X_0 = id$.

The tube algebra is defined as the algebraic direct sum

$$\mathcal{AC} := \oplus_{i,j,k \in \text{Irr}(C)} \text{Mor}(X_k \otimes X_i, X_j \otimes X_k)$$

An element $x \in \mathcal{AC}$ is given by a sequence $x^k_{i,j} \in \text{Mor}(X_k \otimes X_i, X_j \otimes X_k)$ with only finitely many terms non-zero. For a simple object $\alpha \in \Lambda$, and $\beta \in \text{Obj}(C)$, $\text{Mor}(\alpha, \beta)$ has a Hilbert space structure with inner product defined by $\eta^\ast \xi = \langle \xi, \eta \rangle_\alpha$. Note that this inner product differs by the tracial inner product by a factor of $d(\alpha)$.

$\mathcal{AC}$ carries the structure of an associative $*$-algebra, defined by

$$(x \cdot y)_{i,j}^k = \sum_{j,m,l \in \text{Irr}(C)} \sum_{V \in \text{onb}(X_k, X_m \otimes X_l)} (1_j \otimes V^\ast)(x_{i,j}^m)(1_m \otimes y_{l,i}^k)(V \otimes 1_i)$$

$$(x^\ast)^{k}_{i,j} = (\overline{R}_k \otimes 1_j \otimes 1_k)(1_k \otimes (x_{j,i}^{\ast\ast} \otimes 1_k)(1_k \otimes 1_i \otimes R_k)$$

where $R_k \in \text{Mor}(id, X_k \otimes X_k)$ and $\overline{R}_k \in \text{Mor}(id, X_k \otimes X_k)$ are standard solutions to the conjugate equations for $X_k$. In the first sum, $\text{onb}$ denotes an orthonormal basis with respect to our inner product, and we may have $\text{onb}(X_k, X_m \otimes X_l) = \emptyset$ if $X_k$ is not a sub-object of $X_m \otimes X_l$. We mention the above compact form for the definition of the tube algebra was borrowed from Stefaan Vaes.

We define the subspaces $\mathcal{AC}_{i,j}^k := \text{Mor}(X_k \otimes X_i, X_j \otimes X_k) \subset \mathcal{AC}$. For arbitrary $\alpha \in \text{Obj}(C)$, we have a natural map $\Psi : \text{Mor}(\alpha \otimes X_i, X_j \otimes \alpha) \to \mathcal{AC}$ given by

$$\Psi(f) = \sum_{k \prec \alpha} \sum_{V \in \text{ONB}(k, \alpha)} (1_j \otimes V^\ast)f(V \otimes 1_i).$$
We will use this map later, in our analysis of $(G_2)_q$. For each $k \in \text{Irr}(\mathcal{C})$, there is a projection $p_m \in \mathcal{AC}_{k,k}^0$ given by $p_m := 1_m \in Mor(id \otimes X_m, X_m \otimes id) \in \mathcal{AC}$. In particular $(p_m)_{i,j}^k = \delta_{k,0}\delta_{i,j}\delta_{j,m}1_m$.

We see that $\mathcal{AC}_{0,0} = p_0\mathcal{AC}p_0$. This is a unital $*$-algebra. Recall the fusion algebra of $\mathcal{C}$ is the complex linear span of isomorphism classes of simple objects $\mathbb{C}[\text{Irr}(\mathcal{C})]$. Multiplication is the linear extension of fusion rules and $*$ is given on basis elements by the duality. From the definition of multiplication in $\mathcal{AC}$, one easily sees the following:

**Proposition 2.1.** The fusion algebra $\mathbb{C}[\text{Irr}(\mathcal{C})]$ is $*$-isomorphic to $\mathcal{AC}_{0,0}$, via the map $[X_m] \rightarrow 1_k \in (X_k \otimes id, id \otimes X_k) \in \mathcal{AC}_{k,k}^0$.

2.3. **Representations and property (T).**

**Definition 2.2.** $\text{Rep}(\mathcal{AC})$ is the category of $*$-homomorphisms $\pi : \mathcal{AC} \rightarrow B(H)$ for some Hilbert space $H$, with bounded intertwiners.

**Definition 2.3.** A $*$-homomorphism $\pi : \mathbb{C}[\text{Irr}(\mathcal{C})] \rightarrow B(H)$ is admissible if there exists a $\hat{\pi} \in \text{Rep}(\mathcal{AC})$ such that $\hat{\pi}|_{\mathcal{AC}_{0,0}}$ is unitarily equivalent to $\pi$.

**Proposition 2.4.** ([12], Section 4.2) Let $\phi : \mathcal{AC}_{0,0} \rightarrow \mathbb{C}$ be a linear functional. The following are equivalent:

1. $\phi$ is a vector state in an admissible representation.
2. $\phi(p_0) = 1$, and $\phi(x^* \cdot x) \geq 0$ for all $x \in \mathcal{AC}_{k,k}$ and $k \in \Lambda$.

We call the collection of functionals satisfying the equivalent conditions of the above proposition affine states, denoted $\Phi_0(\mathcal{AC})$. The word affine comes from the correspondence between representations of the tube algebra and representations of the affine annular category of a planar algebra introduced by V. Jones (see [15], [14], [6], [9]). The idea is that affine states play a similar role in our representation theory to positive definite functions in the representation theory of groups.

Every $\phi \in \ell^\infty(\text{Irr}(\mathcal{C}))$ defines a linear functional $\hat{\phi}$ on the fusion algebra $\mathbb{C}[\text{Irr}(\mathcal{C})]$ given by

$$\hat{\phi}(\sum_k c_k[X_k]) = \sum_k c_kd(X_k)\phi(X_k)$$

**Definition 2.5.** $\phi \in \ell^\infty(\text{Irr}(\mathcal{C}))$ is a cp-multiplier if $\hat{\phi}$ is an affine state.

This is not the original definition of cp-multiplier introduced by Popa and Vaes. For their original definition, see [30] Definition 3.1. Their definition was motivated as follows:

Associated to a subfactor $N \subseteq M$ is a von Neumann algebra inclusion $T \subseteq S$, called the symmetric enveloping inclusion [25]. Popa gave definitions for approximation and rigidity properties in terms of sequences of UCP maps $\phi_i : S \rightarrow S$ which are $T$ bimodule satisfying certain properties (see [28]). It turns out that decomposing $L^2(S)$ as a $T-T$ bimodule shows that such maps are determined by functions on $\text{Irr}(\mathcal{C})$, where $\mathcal{C}$ is the category of “even” bimodules of the subfactor. Their definition of cp-multiplier is a necessary and sufficient condition on a function $\phi \in \ell^\infty(\text{Irr}(\mathcal{C}))$ to produce a $T$-$T$ bimodule UCP map on $S$ in the prescribed way. Furthermore, conditions for approximation and rigidity properties on the sequence of UCP maps can be directly translated into conditions on the corresponding functions in $\ell^\infty(\text{Irr}(\mathcal{C}))$ with out reference to the symmetric enveloping inclusion. This results in definitions for approximation and rigidity properties for the category $\mathcal{C}$ without reference to a subfactor.

That our definition is equivalent to theirs follows from [9], Theorem 6.5. We note this theorem also implies the admissible representations of Popa and Vaes are the same our admissible representations.
DEFINITION 2.6. A rigid C*-tensor category has property (T) if every sequence of cp-multipliers \( \phi_i \) which converges to 1 pointwise on \( \text{Irr}(\mathcal{C}) \) converges uniformly.

There exists a C*-closure, \( C^*_u(\mathcal{C}) \), of the fusion algebra introduced in [30] such that continuous Hilbert space representations of this C*-algebra are in one-to-one correspondence with admissible representations. It can be shown that for a simple object \( X \in \text{Irr}(\mathcal{C}) \), for any admissible representation \( \pi \), \( \| \pi(X) \| \leq d(X) \), see [30] Proposition 4.2 or [9] Corollary 4.13.

This bound permits the definition of a universal admissible representation \( \pi_u \). \( C^*_u(\mathcal{C}) \) is defined by taking the C*-completion of \( \pi_u(\mathcal{C}[\text{Irr}(\mathcal{C})]) \). The norm of an element in the fusion algebra can be written

\[
\| f \|_u = \sup_{\phi \in \Phi_0(\mathcal{A} \mathcal{P})} \phi(x^* x)^{\frac{1}{2}}.
\]

We remark that in fact we can define such a universal representation and C*-algebra for the entire tube algebra \( \mathcal{A} \mathcal{C} \) but we do not need this here, see [9], Proposition 4.12.

As with groups, there are many equivalent characterizations of property (T). We record the following provided by Popa and Vaes:

PROPOSITION 2.7. ([30] Proposition 5.5) \( \mathcal{C} \) has property (T) if and only if there exists a projection \( p \in C^*_u(\mathcal{C}) \) such that \( \alpha p = d(\alpha)p \) for all \( \alpha \in \text{Irr}(\mathcal{C}) \).

The so called trivial representation \( 1_{\mathcal{C}} \) is the one dimensional representation of the fusion algebra spanned by \( v_0 \) such that \( 1_{\mathcal{C}}(X)v_0 = d(X)v_0 \) for all \( X \in \text{Irr}(\mathcal{C}) \). Viewing \( 1_{\mathcal{C}} \) as an affine state, the corresponding cp-multiplier is the constant function 1 in \( C^*(\text{Irr}(\mathcal{C})) \).

For categories with abelian fusion rules (for example, all braided categories), \( C^*_u(\mathcal{C}) \cong C(Z) \) for some compact Hausdorff space. Points in \( Z \) correspond to one-dimensional representations of the fusion algebra, so \( 1_{\mathcal{C}} \in Z \). We have the following easy consequence of the above proposition:

COROLLARY 2.8. If \( \mathcal{C} \) has abelian fusion rules so that \( C^*_u(\mathcal{C}) \cong C(Z) \) for some compact Hausdorff space \( Z \), then \( \mathcal{C} \) has property (T) if and only if the trivial representation \( 1_{\mathcal{C}} \) is isolated in \( Z \).

PROOF. If \( \mathcal{C} \) has abelian fusion rules \( C^*_u(\mathcal{C}) \cong C(Z) \), where \( X \) is the spectrum of \( C^*_u(\mathcal{C}) \). If \( 1_{\mathcal{C}} \) is isolated in the spectrum, then the characteristic function \( \delta_{\{1_{\mathcal{C}}\}} \in C(Z) \cong C^*_u(\mathcal{C}) \) is a projection satisfying the required property. Conversely, if we had such a projection \( p \), then it could be represented by the characteristic function of some clopen set \( Y \subseteq Z \). Since \( \alpha p = d(\alpha)p \), this implies as a function on \( Z \) each \( \alpha|_Y = d(\alpha) = \alpha(1_{\mathcal{C}}) \). Extending by linearity, we see that for an arbitrary element in the fusion algebra \( \beta \), \( \beta|_Y = 1_{\mathcal{C}}(\beta) \). This equality extends to the C*-closure \( C^*_u(\mathcal{C}) \cong C(Z) \). Since the points of \( Y \) are not separated by \( C(Z) \) from \( 1_{\mathcal{C}} \), we have \( Y = \{1_{\mathcal{C}}\} \), hence \( \{1_{\mathcal{C}}\} \) is clopen, hence \( 1_{\mathcal{C}} \) is isolated in \( Z \).

In the next section, we describe the categories \( (G_2)_q \), and use this corollary to show they have property (T).

3. \( (G_2)_q \) categories

There are many ways to describe rigid C*-tensor categories. One of the most useful is the planar algebra approach introduced by V. Jones [13]. The idea is to use formal linear combinations of planar diagrams to represent morphisms in your category. These diagrams satisfy some linear dependences called skein relations. The most famous skein relations are the ones defining the Jones and HOMFLY polynomials.
The \((G_2)_q\) categories we describe are a particularly nice type of planar algebra called a trivalent category. These were introduced in their current form by Morrison, Peters, and Snyder \[19\]. Using dimension restrictions on morphism spaces as a notion of “small”, they were able to classify the “smallest” examples. The \((G_2)_q\) categories appear in their classification list.

Formally, a trivalent category \(\mathcal{C}\) is a non-degenerate, evaluable, pivotal category over \(\mathbb{C}\), a generating object \(X\) with \(\dim \text{Mor}(id, X) = 0, \dim \text{Mor}(id, X \otimes X) = 1, \dim \text{Mor}(id, X \otimes X \otimes X) = 1\), generated by a trivalent vertex for \(X\) (see \[19\] Definition 2.4).

We summarize the basic properties of trivalent categories:

1. Objects in the category can be represented by \(\mathbb{N} \cup \{0\}\), and correspond to tensor powers of a generating object \(X\).
2. \(\text{Mor}(k, m)\) is the complex linear span of isotopy classes of planar trivalent graphs embedded in a rectangle, with \(m\) boundary points on the top of the rectangle, \(k\) boundary points on the bottom, and no boundary points on the sides of the rectangle. These diagrams are subject to skein relations, which are linear dependences among the trivalent graphs which make \(\text{Mor}(k, m)\) finite dimensional. (Note: We consider graphs with no vertices at all, namely line segments attached to the boundaries, as trivalent graphs)
3. \(\text{Mor}(0, 0) \cong \mathbb{C}\). In other words, our skein relations reduce every closed trivalent graph to a scalar multiple of the empty trivalent graph. Identifying the empty graph with \(1 \in \mathbb{C}\), this means we have associated to every closed trivalent graph a complex number.
4. Composition of morphisms is vertical stacking of rectangles.
5. Tensor product on objects is addition of natural numbers, on morphisms it is horizontal stacking of rectangles.
6. Duality is given by rotation by \(\pi\) or \(-\pi\) (these manifestly agree in our setting), and the \(*\)-involution is given by reflection of diagrams across a horizontal line and conjugating coefficients.
7. The linear functional \(\text{Tr} : \text{Mor}(k, k) \rightarrow \mathbb{C}\) given by connecting the top strings of the rectangle to the bottom is non-degenerate.
8. The category is a \(C^*\)-trivalent category if \(\text{Tr}(x^*x) \geq 0\) for every \(x \in \text{Mor}(k, m)\).

From the above data, we can construct a rigid \(C^*\)-tensor category as follows: First, it can be shown that a category satisfying all these conditions has a negligible category ideal, generated by diagrams with \(\text{Tr}(x^*x) = 0\). Quotienting by this produces a trivalent category with condition 8 replaced by \(\text{Tr}(x^*x) > 0\). Next, we take the projection completion. Objects in this category will be projections living in some \(\text{Mor}(k, k)\). For two projections \(P \in \text{Mor}(k, k), Q \in \text{Mor}(m, m)\), \(\text{Mor}(P, Q) = \{f \in \text{Mor}(k, m) : QfP = f\}\). Now we formally add direct sums to the category. The resulting category will have objects direct sums of projections, and morphisms matrices of the morphisms between projections. The result is a rigid \(C^*\) tensor category, which we also call \(\mathcal{C}\).

Notice the duality map we have defined is automatically pivotal. Also the strict tensor identity \(id\) is given by the empty diagram. Another consequence of the definitions is that the generating object \(X\) is symmetrically self-dual (see \[2\] Definition 2.10).

The \((G_2)_q\) trivalent categories were introduced by Kuperberg in \[16\] and \[17\]. Kuperberg showed that these categories are equivalent to the category of finite dimensional representations of the Drinfeld-Jimbo quantum groups \(U_q(g_2)\).
To define a trivalent category, it suffices to specify a set of skein relations. In general it is a difficult problem to determine whether an set of skein relations produces a trivalent category. In particular, one has to verify that your relations are consistent. Otherwise you may end up with $\text{Mor}(0, 0)$ being 0 dimensional. Kuperberg showed the following skein relations are indeed consistent, resulting in a trivalent category. The skein theory we present for $(G_2)_q$ can be found in [19], Definition 5.21. It differs from Kuperberg’s description in 2 ways: The trivalent vertex is normalized, and the $q^2$ here is Kuperbergs $q$.

**Definition 3.1.** $(G_2)_q$ for positive $q \neq 1$ is the trivalent category defined by the following skein relations:

\[
\begin{align*}
\text{Fig 1} &= \delta = q^{10} + q^8 + q^2 + 1 + q^{-2} + q^{-8} + q^{-10} \\
\text{Fig 2} &= 0 \\
\text{Fig 3} &= c \\
\text{Fig 4} &= a \left( \text{Fig 3} + \text{Fig 5} \right) + b \left( \text{Fig 6} + \text{Fig 7} \right) \\
\text{Fig 8} &= f \left( \text{Fig 9} + \text{rotations} \right) + g \left( \text{Fig 10} + \text{rotations} \right)
\end{align*}
\]

Where

\[
\begin{align*}
a &= \frac{q^2 + q^{-2}}{(q + 1 + q^{-1})(q - 1 + q^{-1})(q^4 + q^{-4})} \\
b &= \frac{1}{(q + 1 + q^{-1})(q - 1 + q^{-1})(q^4 + q^{-4})^2} \\
f &= -\frac{1}{(q + 1 + q^{-1})(q - 1 + q^{-1})(q^4 + q^{-4})} \\
g &= -\frac{1}{(q + 1 + q^{-1})^2(q - 1 + q^{-1})^2(q^4 + q^{-4})^2}
\end{align*}
\]

[16], [17], and [19] shows this category is actually spherical. The duality maps $\cup$ and $\cap$ provide standard solutions to the simple object (minimal projection) spanning $\text{Mor}(1, 1)$. This is the object $X$, which tensor generates our category.
Kuperberg showed that this category is isomorphic (not just equivalent) to the spherical category generated by the 7-dimensional fundamental representation (which we also call $X$) of $U_q(g_2)$ (see [17] Theorem 5.1). A single string corresponds to the object $X$ in $\text{Rep}(U_q(g_2))$, hence the natural number $k$ an an object in $(G_2)_q$ corresponds to the object $X^\otimes k$ in $\text{Rep}(U_q(g_2))$. Since $X$ tensor generates $\text{Rep}(U_q(g_2))$, we have the whole category appearing. We will call the minimal projection spanning $\text{Mor}(1,1)X$ to remind us of this correspondence.

In both Kuperberg’s work and Morrison, Peters, and Snyder’s no $*$-structure is considered. However, $U_q(g_2)$ has a natural $*$-structure for positive $q \neq 1$ (along with all Drinfeld-Jimbo quantum groups), and it is shown, for example, in [21], Chapter 2.4, that the category of finite dimensional $*$-representations is a rigid $C^*$-tensor category. Every finite dimensional representation of $U_q(g_2)$ for and $q > 0$ is unitarizable ([5], Chapter 10) and thus the rigid $C^*$-tensor category of unitary representations is monoidally equivalent to $\text{Rep}(U_q(g_2))$. In the appendix, we show that this $*$-structure transports to $(G_2)_q$ as the $*$-structure we have defined by reflecting a diagram across a horizontal line (see Proposition 5.1 in the Appendix). Hence we can consider $(G_2)_q$ as a $C^*$-trivalent category.

To prove $(G_2)_q$ has property (T), we need to know the structure of $\text{Mor}(2,2)$. $\text{Mor}(2,2)$ is a 4-dimensional abelian $C^*$-algebra. To determine the minimal projections, we set

$$\xi := \sqrt{\delta^2 c^4 + 2\delta(c^4 - 2c^3 - c^2 + 4c + 2) + (c^2 - 2c - 1)^2} = \frac{(1 + q^2)^2(1 - q^8 + q^{10} - q^{14} + q^{16})}{q^6 (1 + q^8)}.$$ 

This is manifestly non-zero for $q \neq 1$ and $q > 0$. The following can be found in [19], Proposition 4.16: The minimal idempotents in the finite dimensional algebra $\text{Mor}(2,2)$ are given by

$$\frac{1}{\delta} \begin{array}{c} \sqrt{ } \\ \end{array}, \begin{array}{c} \sqrt{ } \\ \end{array}$$

and the two idempotents

$$y_{\pm} = \frac{-(\delta + 1)c^2 + \xi + 1}{\pm 2\xi} + \frac{\delta(c^2 - 2c - 2) + \xi + c^2 - 2c - 1}{\pm 2\xi} - \frac{\delta(c + 2) + \xi + c^2 + 1}{\pm 2\xi} + \frac{\delta c + \delta + c}{\pm \xi}.$$ 

In our setting, we see that the idempotents are in fact projections, since our basis is self-adjoint. These projections correspond to simple objects in the rigid $C^*$-tensor category underlying $(G_2)_q$.

By Kuperberg’s isomorphism, the fusion algebra of the underlying projection category of $(G_2)_q$ is isomorphic to the fusion algebra of the category $\text{Rep}(U_q(g_2))$ for positive $q \neq 1$, which in turn is isomorphic to the fusion algebra of $\text{Rep}(g_2)$. It well known that for compact, connected, simply connected simple Lie groups $G$, $\text{Rep}(g)$ is isomorphic to the algebra of polynomials in the fundamental representations. For a specific reference for $g_2$, see [8], 23.32. This implies the fusion algebra is the (commutative) polynomial algebra in self-adjoint variables $\mathbb{C}[X,Y]$, where $X$ and $Y$ denote the 7 and 14 dimensional fundamental representations of the quantum group $U_q(g_2)$.

The fusion graph with respect to $X$ is given by
Here, the vertex at the bottom corresponds to the identity, the next highest vertex corresponds to $X$ itself, etc.

3.1. **Property (T) for $(G_2)_q$.** Now we consider the tube algebra $A(G_2)_q$. Recall simple objects in the category correspond to minimal projections in some $\text{Mor}(k,k)$. Let us choose our set of representatives $\Lambda$ so that it contains the empty diagram $id$, the single string $X$, and the two projections $y_+$ and $y_-$. For $x \in \text{Mor}(k,k)$, we let $i : \text{Mor}(k,k) \to \text{Mor}(k \otimes id, id \otimes k)$ be the canonical identification. Then define

$$\Delta(x) := \Psi(i(x)) \in (A(G_2)_q)_{0,0},$$

where $\Psi$ is defined in the discussion of the tube algebra.

Translating the fusion algebra description to our setting, $X$ is represented by the single rectangular vertical strand, while $Y$ is represented by the projection $y_+ \in \text{Mor}(2,2)$. In particular, by our discussion of the tube algebra $(A(G_2)_q)_{0,0} \cong \mathbb{C}[\Delta(X), \Delta(y_+)]$.

Going back to our expression for $y_+$, we see that $\frac{c + \delta + c}{\xi} = \frac{(1 + q^2 + q^4)(1 + q^8)}{q^2(1 + q^2)^2} \neq 0$. Thus

$$\Delta(y_+) = \frac{q^4(1 + q^2)^2}{(1 + q^4 + q^8)(1 + q^{16})} \left( (y_+) - \frac{-2c^2 + 2c + 1}{2\xi} \right) + \frac{\delta(c^2 - 2c - 2)}{2\xi}(\xi^2 + c^2 - 2c - 1) + \frac{\delta(c + 2)c + \xi^2 + 1}{2\xi} \Delta(X).$$

Then since $\Delta(y_+) = \Delta(X)$, as a polynomial in $\Delta(X)$ and $\Delta(y_+)$ we have

$$\Delta(y_+) = \frac{q^4(1 + q^2)^2}{(1 + q^4 + q^8)(1 + q^{16})} \left( (y_+) - \frac{-2c^2 + 2c + 1}{2\xi} \Delta(X) - \frac{\delta(c^2 - 2c - 2)}{2\xi}(\xi^2 + c^2 - 2c - 1) + \frac{\delta(c + 2)c + \xi^2 + 1}{2\xi} \Delta(X) \right).$$

We denote $H := \Delta(y_+)$. Since our polynomial expression for $\Delta(H)$ is linear in $\Delta(y_+)$ and the terms with powers of $\Delta(X)$ contain no $\Delta(y_+)$ terms, we can perform an invertible change of basis and write an arbitrary polynomial in $\Delta(X), \Delta(y_+)$ as a polynomial in $\Delta(X), \Delta(H)$. It is then easy to see that $(A(G_2)_q)_{0,0} \cong \mathbb{C}[\Delta(H), \Delta(X)]$.

Therefore irreducible representations of $(A(G_2)_q)_{0,0}$ are 1-dimensional, and they are defined by assigning numbers to $\Delta(X)$ and $\Delta(H)$. Let us denote by $t$ the value assigned to $\Delta(X)$, and $\alpha$ the value assigned to $\Delta(H)$ in our 1-dimensional representation. Let $\gamma_{\alpha,t} : (A(G_2)_q)_{0,0} \to \mathbb{C}$ denote the 1-dimensional representation viewed as a functional, given by evaluating polynomials in $\mathbb{C}[\Delta(H), \Delta(X)]$ at the point $(\alpha, t)$.

The key point is that while arbitrary values of $\alpha$ and $t$ determine a representation of $(A(G_2)_q)_{0,0}$, not all are affine states. $\gamma_{\alpha,t}$ is admissible if and only if $\gamma_{\alpha,t}(x^* x) \geq 0$ for all $x \in (A(G_2)_q)_{0,k}$ and for all simply objects $X_k \in \Lambda$. 
As a first restriction, for our representation to be admissible, \( t \in \mathbb{R} \) since the object corresponding to a single string is self-dual (our representation must be a \(*\)-representation). Also we must have \( \alpha \geq 0 \), since \( \Delta(H) = t^* \cdot t \), where \( t \in \text{Mor}(X \otimes id, X \otimes X) \subseteq (\mathcal{A}(G_2)_q)_{0,0} \), is given by the trivalent vertex \( t := \begin{array}{c}
abla \\
\\n\end{array} \).

We know also that \( |t| \leq \delta \) by [30] or [9], Corollary 4.13. Thus the one dimensional representations are parameterized by some subset of points in the set \( Z \subseteq \{ (\alpha, t) \subseteq \mathbb{R}^2 : \alpha \geq 0, \ t \in [-\delta, \delta] \} \).

Since the fusion algebra is isomorphic to the polynomial algebra in two self adjoint variables, and irreducible representations correspond to evaluation at points \( Z \subseteq \mathbb{R}^2 \), the weak-* topology on \( Z \) as linear functionals on \( C_\text{univ}(G_2)_q \) agrees with the topology on the plane. The trivial representation corresponds to the point \((0, \delta)\). We will show that for positive \( q \neq 1 \), there is a neighborhood of the point \((0, \delta)\) in the rectangle \( \mathbb{R}^+ \times [-\delta, \delta] \) such that the functional \( \gamma_{\alpha, t} \) is not affine.

To see this, let \( s := \begin{array}{c}
\\y \\
\\\end{array} \in \text{Mor}(X \otimes id, y_- \otimes X) \). We view \( s \in (\mathcal{A}(G_2)_q)_{y_-,-} \subseteq \mathcal{A}(G_2)_q \).

for each pair \((\alpha, t)\), we can define the function \( f(\alpha, t) := \gamma_{\alpha, t}(s^* \cdot s) \) This can be directly computed from the representation of \( y_- \) in terms of our planar algebra basis, and we obtain

\[
f(\alpha, t) = \frac{\delta - (\delta + 1)c^2 - \xi + 1}{-2\xi} + t^2 \frac{\delta(c^2 - 2c - 2) + \xi + c^2 - 2c - 1}{-2\xi} - \alpha \frac{\delta(c + 2)c - \xi + c^2 + 1}{-2\xi} + t \frac{\delta c + \delta + c}{-\xi}.
\]

If the functional corresponding to \((\alpha, t)\) is affine, \( f(\alpha, t) \) must be non-negative by construction.

**Proposition 3.2.** For all positive \( q \neq 1 \), \((G_2)_q\) has property (T).

**Proof.** Since \( y_- \) is a minimal projection not equivalent to \( id \), \( f(0, \delta) = 0 \). This can also be seen by direct computation. Let us pick \( v := (x, y) \in \mathbb{R}^2 \) be a non-zero vector in the fourth quadrant including the axes, i.e. \( x \geq 0 \) and \( y \leq 0 \) (but not both \( 0 \)). We wish to show that \( f(x, \delta + y) < 0 \) for sufficiently small \((x, y)\). This will demonstrate that in a neighborhood of \((0, \delta)\) in \( \mathbb{R}^+ \times [-\delta, \delta] \), the function \( f(\alpha, t) \) will be strictly negative, hence the representation cannot exist.

We see that for positive \( q \neq 1 \),

\[
\frac{\partial f}{\partial q}|_{(\alpha, s)} = \frac{(\delta + 1)(c^2 - c - 1)}{-\xi} - 1 = \frac{(-1 + q^2)^2(1 + q^2 + q^4)}{q^4} > 0
\]

This expression is strictly positive for all \( q \neq 1, q \neq 0 \). We note that for \( q = 1 \), we obtain \( \frac{\partial f}{\partial q} = 0 \) hence our proof breaks down. This is to be expected, since \((G_2)_1 \equiv \text{Rep}(G_2)\) is known to be amenable.

Now, we see that

\[
\frac{\partial f}{\partial x}|_{(\alpha, s)} = \frac{\delta(c + 2)c - \xi + c^2 + 1}{2\xi} = -\frac{1 + q^2 + 2q^4 + q^6 + q^8}{(q + q^3)^2} < 0.
\]

This is always strictly negative (for \( q \neq 0 \)). Therefore we have that the directional derivative \( \frac{\partial f}{\partial x}|_{(0, \delta)} \) is continuous for all \( q \neq 0 \). Therefore we have that the directional derivative \( \frac{\partial f}{\partial x}|_{(0, \delta)} < 0 \) for \( v \) in the prescribed range. Notice that the set of unit vectors in the fourth quadrant \( B \) is a compact set, and \( \frac{\partial f}{\partial x}|_{(0, \delta)} \) is a continuous function of \( v \). Therefore, there exists some \( M < 0 \) such that \( \frac{\partial f}{\partial x} \leq M < 0 \) for \( v \in B \).

Now, it is straightforward to compute
\[
\frac{\partial^2 f}{\partial t^2} |_{(0, \delta)} = \frac{2q^6(1 + q^2 + q^4)}{(1 + q^2)^2(1 + q^4) + q^6 + q^8 + q^{10} + q^{12}} > 0.
\]

It is also easy to see that all other second order partial derivatives are 0, and all higher derivatives are 0.

Let us set \(\lambda := \frac{\partial^2 f}{\partial t^2} |_{(0, \delta)}\). Then by your favorite version of Taylors theorem, we have

\[
f(x, \delta + y) = x \frac{\partial f}{\partial t} |_{(0, \delta)} + y \frac{\partial f}{\partial \delta} |_{(0, \delta)} + x^2 \frac{\partial^2 f}{\partial t^2} |_{(0, \delta)}
\]

for arbitrary \(v = (x, y)\) in the fourth quadrant. Let \(v' = \frac{1}{||v||} v\). Since \(x^2 \leq ||v||^2\), we have

\[
f(x, \delta + y) = ||v|| \frac{\partial f}{\partial v'} |_{(0, \delta)} + x^2 \lambda \leq ||v|| M + ||v||^2 \lambda.
\]

If we set \(\epsilon = \frac{|M|}{2}\), then since \(M < 0\), for \(||v|| < \epsilon\), we see that \(f(x, \delta + y) < 0\). Therefore \((G_2)_q\) has property (T) for positive \(q \neq 1\).

\[\square\]

4. Concluding Remarks

(1) Our result provides another class of examples of subfactors having property (T). To see this, we recall that Popa gave an axiomatization of standard invariants of subfactors as standard \(\lambda\)-lattices [27]. These are towers unital inclusions of finite dimensional \(C^*\) algebras, together with some extra data. From a rigid \(C^*\)-tensor \(\mathcal{C}\) and object \(X\), one can construct a standard \(\lambda\)-lattice by

\[
\mathcal{C} \subset \text{End}(X) \subset \text{End}(X \overline{X}) \subset \text{End}(X \overline{X} \overline{X}) \subset \cdots
\]

Then this tower will be the standard invariant of some subfactor \(N \subset M\) by [27]. The category of \(N\)-\(N\) bimodules will be isomorphic to the subcategory of \(\mathcal{C}\) generated by \(X \overline{X}\). Applying this construction to \((G_2)_q\), since \(X\) appears as a sub-object of \(X \otimes X\), the even bimodules of this subfactor will be a category equivalent to the unitarization of \(\text{Rep}(U_q(\mathfrak{g}_2))\) (as opposed to a proper subcategory). Therefore, as in [20], Theorem 8.1, this subfactor will have property (T).

(2) We hope that methods similar to those presented here will be useful to deduce property (T) (or lack there of) for other quantum group categories which have a nice planar algebra description, particularly the BMW planar algebras, which describe the quantum \(SO(n)\) and \(SP(2n)\) categories.

5. Appendix: \(*\)-structure for \((G_2)_q\)

The main point of this section is to show that the \(*\)-structure we described above for \((G_2)_q\) (reflection of diagrams about a horizontal line) gives a \(C^*\)-trivalent category. To do this from the planar algebra perspective is quite a daunting task, since we would have to explicitly construct all idempotents, show they are self adjoint, and that they have positive trace. Fortunately for us, due to Kuperberg’s isomorphism, \((G_2)_q\) can be realized as the category generated by the fundamental 7-dimensional representation \(X\) of the Hopf \(*\)-algebra \(U_q(\mathfrak{g}_2)\), which carries with it a naturally occurring \(*\)-structure.

To elaborate, we know that quantum groups \(U_q(\mathfrak{g})\) with positive \(q\) have a natural \(*\)-structure and that every finite dimensional representation is unitarizable, which means it is equivalent to a \(*\)-representation of the Hopf algebra (see [5], Chapter 10). To obtain a rigid \(C^*\)-tensor category, we restrict to the finite dimensional...
unitary representations, and since every representation is unitarizable, this category is equivalent to the whole category of finite dimensional representations.

Kuperberg’s equivalence from \( \text{Rep}(U_q(\mathfrak{g}_2)) \) to \((G_2)_q\) used the fact that the fundamental 7-dimensional representation was symmetrically self-dual. But being symmetrically self-dual depends on the specific choice of duality maps and most importantly on the map implementing the equivalence from \( \pi \) to \( \pi^c \) (the standard dual, defined using the Hopf algebra antipode). For us to have a \( C^* \)-planar algebra, we need \( \pi \) to be unitarily symmetrically self-dual. Even if \( \pi \) is unitary, the canonical dual \( \pi^c \) is not necessarily unitary, although we know it is equivalent to a unitary representation. In [21], they explicitly identify the unitary dual, \( \pi \) for all representations of \( U_q(\mathfrak{g}_2) \). Using this information, we explicitly compute the unitary intertwiner \( T \in (\pi, \pi) \) in the matrix representations of \( \pi \) and \( \pi^c \), and show that \( T \) along with the standard choices of duality maps implements a unitary symmetrically self-duality on \( \pi \), so there is no problem.

This allows us to use Kuperberg’s isomorphism, but now our category has a nice \( * \)-structure. Compatibility of the \( * \)-structure with the duality functor (which is immediate in rigid \( C^* \)-categories) forces the \( * \)-structure described in section 2.1 for \((G_2)_q\).

**Remark.** Most of this section is, strictly speaking, unnecessary. We include this in an appendix for the reader unfamiliar with quantum groups. In particular, if you believe \((G_2)_q\) must have some positive definite \( * \)-structure from abstract quantum group theory (particularly [21] Chapter 2.4, since every finite dimensional representation is unitarizable), skip to the last proposition.

We give a brief description of the Drinfeld-Jimbo Hopf \( * \)-algebra \( U_q(\mathfrak{g}_2) \), following [21], Definition 2.4.1. Let \( \alpha_1, \alpha_2 \) the standard choice of simple roots of the \( G_2 \) root system, pictured below:

![Diagram of G2 root system](image)

Let \((a_{ij}) = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\) be the Cartan matrix for the \( G_2 \) root system. Let \( d_1 := 1, d_2 := 3 \), so that \( A_{ij} := d_i a_{i,j} = (\alpha_i, \alpha_j) \) is the inner product matrix, given by

\[
A = \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}
\]

For \( q > 0, q \neq 1 \), let \( q_i = q^{d_i}, i = 1, 2 \). Then \( U_q(\mathfrak{g}_2) \) is defined as the universal unital algebra generated by elements \( E_i, F_i, K_i, K_i^{-1}, i = 1, 2 \), satisfying the relations

\[
K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i \\
K_i E_i K_i^{-1} = q_i^{\alpha_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-\alpha_{ij}} F_j
\]
\[ [E_i, F_j] = \delta_{ij} K_i - K_i^{-1} \]

\[ \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right] E_i^k E_j E^{1-a_{ij} - k} = 0 \]

\[ \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right] F_i^k F_j F^{1-a_{ij} - k} = 0 \]

where \[ \left[ \frac{m}{k} \right]_{q_i} = \frac{[m]_{q_i}!}{[m-k]_{q_i}!}, \quad [m]_{q_i} = [m]_{q_i} [m-1]_{q_i} \ldots [1]_{q_i}, \text{ and } [n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}. \]

This algebra is a Hopf \( \star \)-algebra with coproduct \( \hat{\Delta}_q \) defined by

\[ \hat{\Delta}_q(K_i) = K_i \otimes K_i, \quad \hat{\Delta}_q(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \hat{\Delta}_q(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i \]

and with involution given by \( K_i^* = K_i, \quad E_i^* = F_i K_i, \quad F_i^* = K_i^{-1} E_i. \)

The counit \( \hat{\epsilon}_q \) and the antipode \( \hat{S}_q \) are given by

\[ \hat{\epsilon}_q(K_i) = 1, \quad \hat{\epsilon}_q(E_i) = \hat{\epsilon}_q(F_i) = 0, \]

\[ \hat{S}_q(K_i) = K_i^{-1}, \quad \hat{S}_q(E_i) = -K_i^{-1} E_i, \quad \hat{S}_q(F_i) = -F_i K_i \]

The category of finite dimensional representations of this algebra has the structure of a rigid tensor category, with tensor product defined using the comultiplication, and duality the vector space dual with action induced by the antipode. This is a standard result for Hopf algebras. A unitary representation is a \( \star \)-algebra homomorphism \( \pi : U_q(g) \to B(H) \) for some Hilbert space \( H \). We consider here only finite dimensional unitary representations. This category is a rigid \( C^* \)-tensor category. We refer the reader to [21] Chapter 2.4 for details. In fact, every finite dimensional representation of these Hopf algebras is unitarizable (is equivalent to a unitary representation) [5], Chapter 10 for positive \( q \), so that the category of unitary representations is monoidally equivalent to the category of all finite dimensional representations. A key point in quantum group theory is that representations for positive \( q \) are in one-to-one correspondence with classical representations of the Lie algebras, and have the same dimension (as vector spaces) as the classical representations.

We will give an explicit unitary realization of the fundamental 7-dimensional representation \( X \) of \( U_q(g_2) \). Let \( v_0 \) be the highest weight vector, normalized so that \( \langle v_0, v_0 \rangle = 1 \). Then define \( H \) as the Hilbert space with orthonormal basis

\[ v_0, \quad v_1 := q^{\frac{1}{2}} F_1 v_0, \quad v_2 := q^2 F_2 F_1 v_0, \quad v_3 := q^3 [2]_{q}^1 F_1 F_2 F_1 v_0, \quad v_4 := q^3 [2]_{q}^{-1} F_1 F_2 F_1 v_0, \]

\[ v_5 := q^{\frac{5}{2}} [2]_{q}^{-1} F_2 F_1 F_2 F_1 v_0, \quad v_6 := q^5 [2]_{q}^{-1} F_2 F_1 F_2 F_1 v_0. \]

The action \( \pi \) of \( U_q(g_2) \) on vectors can be worked out from the commutation relations. This yields a \( \star \) representation on which the actions of \( E_1, E_2, F_1, F_2 \) can be worked out explicitly. For example, we have the matrix representations with respect to the above basis given by
Then we have \( \hat{\rho} \) positive roots and thus \( 2\pi \) with respect to our chosen basis), and we define \( W \) is given by the same formula as \( T \). Using the description provided above, it is straightforward to check that we see that \( W \) is given by the same formula, with the standard antipode \( \hat{\rho} \).

Alternatively, the dual representation can be given using the unitary antipode \( \hat{\rho} \), defined by

\[
\hat{\rho}(K_1) = \text{diag}(q, q^{-1}, q^2, 1, q^{-2}, q, q^{-1}) \quad \text{and} \quad \hat{\rho}(K_2) = \text{diag}(1, q^3, q^{-3}, 1, q^3, q^{-3}, 1).
\]

If \( f \in B(H, K) \), then define \( \hat{\rho}(f) \in B(\overline{H}, H) \) by \( \hat{\rho}(f)\xi = \overline{f}\xi \) for \( \xi \in K \). Let \( \rho \) be the half-sum of positive roots and thus \( 2\rho \) the sum of positive roots. Then \( 2\rho = 10\alpha_1 + 6\alpha_2 \), so we define \( K_{2\rho} := K_{10} K_{2}^{6} \).

Then we have \( \hat{\rho}(x) = K_{2\rho}^{-1} x K_{2\rho} \). Then we see that \( \hat{\rho}(K_\rho) \) is a positive invertible operator (diagonal with respect to our chosen basis), and we define \( W := \pi(K_{2\rho}^{-1})^{\frac{1}{2}} \in B(H) \). Then \( j(W) \in B(\overline{H}) \), and we define \( \hat{\rho}(,.) = j(W)\pi(,.)j(W^{-1}) \). This is manifestly equivalent to \( \pi^\circ \) but has the advantage of being unitary.

Alternatively, the dual representation can be given using the unitary antipode \( \hat{\rho} \), defined by

\[
\hat{\rho}(K_i) = K_i^{-1}, \quad \hat{\rho}(E_i) = -q_i K_i^{-1} E_i, \quad \hat{\rho}(F_i) = -q_i^{-1} F_i K_i
\]

for \( x \in U_q(G_2) \).

We note the standard dual \( \pi^\circ \) is given by the same formula, with the standard antipode \( \hat{\rho} \) in place of \( \hat{\rho} \).

Since \( W \) is diagonal in the basis described above, the \( i^{th} \) eigenvalue, \( W_i > 0 \), is clear. By inspection, we see that \( W_i W_{7-i+1} = 1 \). Consider the map \( T : H \to \overline{H} \) given by \( T(v_i) = (-1)^{i+1} v_{7-i+1} \). Using the description provided above, it is straightforward to check that \( T \in \text{Hom}(\pi, \overline{\pi}) \). Furthermore, \( T^* = T^{-1} \), and is given by the same formula as \( T \), with the bars reversed, and we have \( T^* j(W) = W^{-1} T^* \).

Consider the map \( r \in (C, H \otimes \overline{H}) \) given by \( r(1) = \sum_i v_i \otimes \overline{v}_i \) (this map does not depend on basis). It is easy to see that \( r \in (\epsilon, \pi \otimes \overline{\pi}) \). By definition \( j(W) \in (\pi^\circ, \overline{\pi^\circ}) \). Then define \( R := (1 \otimes (T^* j(W))) r = (W \otimes T^*) r \in (\epsilon, \pi \otimes \pi) \). We claim that the pair \( (R, R^\circ) \) provide a standard, symmetrically self-dual solution to the conjugate equations for the object \( \pi \). It is straightforward to check that

\[
(1_H \otimes R^\circ)(R \otimes 1_H) = (R^\circ \otimes 1_H)(1_H \otimes R) = 1_H.
\]

Now, to see symmetric self duality, we note that since \( W \) is self adjoint, \( r^\circ \circ (W^{-1} \otimes j(W)) = r^\circ \), and \( (W^{-1} \otimes j(W)) \circ r = r \). Similarly, if \( r(1) = \sum_j v_j \otimes \overline{v}_j \in B(C, \overline{H} \otimes H) \), then since \( T \) is a self adjoint unitary,
\( \tau^* \circ (T^* \otimes T) = \tau^* \) and \((T \otimes T^*) \circ \tau = \tau \). Using these relations and the fact that \( r, \tau \) themselves solve the duality equations in the category of Hilbert spaces, we have

\[
(1_H \otimes 1_H \otimes R^*) \circ (1_H \otimes R \otimes 1_H) \circ R = R = (R^* \otimes 1_H \otimes 1_H) \circ (1_H \otimes R \otimes 1_H).
\]

We let \( X = (\pi, H) \) described above. We define \( C \) to be the strict \( C^* \)-tensor category generated by the symmetrically self dual object \( X \), with unit object \( id := \epsilon_q \) (the trivial representation of the quantum group given by the counit), and duality maps compositions of \( R \) and \( R^* \) in the obvious fashion. Then objects are given by tensor powers \( X^\otimes n \), where \( n \in \mathbb{Z}_+ \), and morphisms are intertwiners of the corresponding quantum group representations. It is easy to check that the duality maps we have defined induce a pivotal structure on this category (essentially following from the fact that \( r, \tau \) induce a pivotal structure on finite dimensional Hilbert spaces).

**Proposition 5.1.** \((G_2)_q\) is a \( C^* \)-trivalent category.

**Proof.** Neglecting the \( * \) structure, \( C \) is precisely the \( G_2 \) algebraic spider defined by Kuperberg, hence must be isomorphic to \((G_2)_q\). We now have a \( * \)-structure on the morphism spaces, which is positive with respect to the trace (since \( C \) is manifestly a \( C^* \)-category). Note that by construction \( \cup^* = R^* = \cap \).

Using the fusion rules, \( Mor(X, X \otimes X) \) is one dimensional, and is spanned in the planar algebra by a rotationally invariant vertex \( t \in (X, X \otimes X) \). We have \( t \) normalized so that \( \bar{t} \circ t = 1_X \), where \( \bar{t} = (1_X \otimes R^*) \circ (1_X \otimes 1_X \otimes R^* \otimes 1_X) \circ (1_X \otimes t \otimes 1_X \otimes 1_X) \circ (R \otimes 1_X \otimes 1_X) \in (X \otimes X, X) \) is the dual of \( t \). We know that \( t^* = \lambda t \) for some \( \lambda \in \mathbb{C} \). This is a \( C^* \) category so \( t \circ t = \lambda 1_X \) must be positive, hence \( \lambda > 0 \). We also have \( \bar{t}^* = \lambda' t \). But \( t = t^** = \lambda' \lambda^{-1}, \) hence \( \lambda = \lambda^{-1} \). Now, since our category is a pivotal \( C^* \)-tensor category, we have that \( \bar{t} = \bar{\bar{t}} \), hence \( \lambda^{-1} t = t \), but since \( \lambda > 0 \), it must be that \( \lambda = 1 \).

The \( * \) of \( t \) and \( \cap \) determine the \( * \)-structure described above is a \( C^* \)-trivalent category. \( \square \)

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