Topologically Massive Models from Higgs Mechanism

Talk given at ”I Congreso Venezolano de Física, ULA Mérida, 1997”

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Abstract

A Higgs mechanism for Abelian theories over non-trivial background flat connections is proposed. It is found that the mass generated for the spin 1 excitation is the same as the one obtained from the standard Higgs mechanism over trivial backgrounds, however, the dynamical structure of the action for the Higgs scalar is completely different from the usual approach. There is a topological contribution to the mass term of the Higgs field. After functional integration over all backgrounds, it is shown that the action for the massive spin 1 excitation is dual to the Topologically Massive Models in any dimension.
Introduction

The Higgs Mechanism of symmetry breaking, has been used very successfully as a mechanism of generating mass to gauge fields. For small coupling the quantum Hilbert space of a field theory is constructed by expanding around the classical minima of the potential. In the standard application of the Higgs mechanism to a massless gauge field coupled to scalar field the absolute minima of the potential correspond to $\langle \Phi \rangle = v e^{i\alpha}, \ (v \ \text{constant})$ and to a potential $A$ gauge equivalent to zero. In other cases mainly for $SU(2)$ gauge potentials, because of terms of the type $Tr (F_{ij}F_{ij})$ and non-trivial structure of the interacting terms for the scalar fields in the Hamiltonian, classical minima correspond to flat connections together with scalar fields satisfying some restrictions. For example in the Witten’s topological field theory[1] describing Donaldson’s invariants the scalar fields satisfy $D\Phi = D\Psi = [\Phi, \Psi] = 0$. Similar terms for the scalar fields occur for the $N = 2$ SUSY Yang-Mills theory. The moduli space of minima is then very rich and allows interesting relations between topological aspects of the base manifold and physical observable of the resulting field theory. After the successful use of the Higgs mechanism to the construction of the Standard Model, several interesting theories have been formulated which describe spin 1 massive excitations in terms of a gauge invariant field theory. In the Topological Massive[2] theory in 3 dimensions the “topological mass” is introduced via a Chern-Simon term in the Lagrangian, several extensions of this theory to four and higher dimensions have been presented in the literature in terms of antisymmetric fields and $BF$ interacting terms. Also much work has been developed to understand the origin of new mechanism of generating mass to gauge field theories [3]. In [4] it was shown that the quantum contribution from one loop diagrams to the effective action in theories with four fermions coupling in four dimensions exactly reproduces the topological terms of an antisymmetric field action describing the new topological photon mass generation. They appear in a similar way as the Chern-Simon terms are generated by parity violating massive fermions in $2 + 1$ dimensions[5]. All these theories describe massive excitations over trivial fiber bundles.

The new relevant points in the topological mass generating approach are the gauge invariant formulation together with the existence of new topological sectors in these theories [6][7], which may be understood in terms of topological theories. In most of the cases in terms of $BF$ topological theories. The main ingredient in the formalism is the presence of non-trivial flat connections (Bohm Aharonov potentials) which carries into the theory the non-trivial topological aspects of the base manifold. Because of the presence of these non-trivial flat connections and the remark that in the standard Higgs mechanism the minima of the Hamiltonian impose the condition of trivial connections, it is thought that these massive gauge theories are related to a different mass generating mechanism.

Since this new mechanism of photon mass generation is topological in nature we will look for a modification of the variational problem of the standard Lagrangian of massless gauge fields coupled to scalars which may take into account topological aspect of the base manifold. This modification al-
allows the introduction of non-trivial flat connections in the description of the background.

We start from the standard Lagrangian for a massless gauge field coupled to scalars, formulated on a topological non-trivial base manifold. We then show the well known result that the absolute minima of the action under arbitrary variations imposes the condition to the connection to be pure gauge. We then introduce a restricted space of variations and consider stationary points on this new space. We show that there are minima associated to non-trivial flat connections. We then follow the Higgs mechanism over this background and obtain the field equations for the Higgs scalar and for the massive gauge field. The mass of the spin 1 excitation is the same as in the standard Higgs approach while the mass of the Higgs scalar has a “topological” contribution from the nontrivial flat connections on the background. Finally we perform a duality transformation to obtain the TM theory in 3 dim and its generalizations to higher dimensions involving antisymmetric fields and $BF$ interacting terms.

**Higgs Mechanism over Non-Trivial Flat Connections**

We start with the action of a complex scalar field interacting with an Abelian gauge field,

$$S = \left\langle D_\mu \Phi \bar{D}_\mu \Phi^* - V(\Phi) + \mathcal{L}(A) \right\rangle$$  \hspace{1cm} (1)

here $\langle \rangle$ denotes integration on a $d$ dimensional space-time, $D_\mu = \partial_\mu + iA_\mu$, the potential $V(\Phi)$ is given by

$$V(\Phi) = \mu^2 (\Phi^* \Phi) + |\lambda| (\Phi^* \Phi)^2,$$  \hspace{1cm} (2)

and $\mathcal{L}(A)$ stands for the Lagrangian associated to the $U(1)$ connection $A$. We will consider $\mathcal{L}(A)$ to be the Chern-Simons Lagrangian on $d = 3$ or the Maxwell Lagrangian on $d \geq 3$ dimensions. In any case the action is invariant up to total derivatives under the gauge transformation: $\Phi \rightarrow e^{i\xi} \Phi$ and $A \rightarrow A - d\xi$.

We would like to extend the spontaneous symmetry breaking process in such a way as to include nontrivial flat connections (one cannot gauge the background $A$ away). The extension is naturally suggested by the fact that the Hamiltonian shows an effective potential $V(\Phi)_{eff}$ given by:

$$V(\Phi)_{eff} = A_i A_i (\Phi^* \Phi) + \mu^2 (\Phi^* \Phi) + \lambda (\Phi^* \Phi)^2$$  \hspace{1cm} (3)

$i = 1, 2, ..., d - 1$
Meaning that the gauge potential could contribute to the mass of the Higgs boson.

We shall begin our analysis showing that a naive study of a vacuum state with \( A \neq 0 \) leads back to the usual Higgs solution \( A = 0 \). After this, we will define the appropriate modifications that will end up in the solution for nontrivial topologies.

We denote the non-trivial vacuum expectation value \( \langle \Phi \rangle = \nu \), \( \nu \) being a constant parameter, which without losing generality we may take to be real.

The field equations for the action (1) are obtained by taking general variations with respect to \( \Phi, \Phi^* \) and \( A_\mu \). We are interested in solutions for the scalar and vector fields of the form:

\[
\begin{align*}
\Phi_0 &= \nu e^{i\xi} \\
A_\mu &= \hat{A}_\mu \neq 0
\end{align*}
\]  

(4)

where \( \nu \) is a constant.

Unrestricted variations of (1) with respect to \( \Phi \) and \( \Phi^* \) yield,

\[
\begin{align*}
\partial_\mu \partial^\mu \Phi^* - i \partial_\mu A^\mu \Phi^* - 2i A^\mu \partial_\mu \Phi^* - A_\mu A^\mu \Phi^* &+ \mu^2 \Phi^* + 2\lambda (\Phi \Phi^*) \Phi^* = 0 \\
\partial_\mu \partial^\mu \Phi + i \partial_\mu A^\mu \Phi + 2i A^\mu \partial_\mu \Phi - A_\mu A^\mu \Phi &+ \mu^2 \Phi + 2\lambda (\Phi \Phi^*) \Phi = 0
\end{align*}
\]  

(5)

After replacing (4) into (5) we obtain for the background field \( \hat{A}_\mu \).

\[
\partial_\mu \hat{A}_\mu = 0
\]  

(6)

\[
-\hat{A}_\mu \hat{A}_\mu + \mu^2 + 2\lambda \nu^2 = 0
\]  

(7)

Unrestricted variations of (1) with respect to \( A_\mu \) yield,

\[
i \Phi \bar{D}^\mu \Phi^* - i \Phi^* D^\mu \Phi + \frac{\delta \mathcal{L}}{\delta A_\mu} = 0
\]  

(8)

After replacing (4) into (8) we get

\[
2\nu^2 \hat{A}_\mu + \frac{\delta \mathcal{L}}{\delta A_\mu} \bigg|_{\hat{A}_\mu} = 0
\]  

(9)

Since we are looking for the ground state of the theory, we are not only interested in a stationary point of the action but also in having a minimum of the Hamiltonian. If \( \mathcal{L}(A) \) is the Lagrangian of Maxwell theory this further condition implies that \( \hat{A} \) must be a flat connection, we will also consider \( \hat{A} \) to be a flat connection in the case of Chern-Simons theory. Under such
conditions, equation (9) implies that:

\[ \hat{A}_\mu = 0. \]  \hfill (10)

Which is the anticipated usual Higgs solution. In (4)-(10) we used the Minkowski metric, however the results are valid for any (pseudo) Riemannian metric over the base manifold.

Our original interest lies in having a non-trivial background \( \hat{A}_\mu \), from the above study, it is clear that we must modify the approach. We would like to have \( \hat{A}_\mu \) a flat connection without further restrictions, this means that we must avoid the conditions imposed by equations (6) and (10). Indeed, (6) seems like a gauge fixing condition to \( \hat{A} \) which is potentially conflicting with (7).

In order to achieve a reasonable modification of the approach we consider stationary points of the action (1) with variations of the field subjected to certain restrictions, which for the scalar field we take

\[ \frac{\delta \Phi}{\Phi_0} = \frac{\delta \Phi^*}{\Phi_0^*}, \]  \hfill (11)

and

\[ d(\delta \Phi \hat{A}) = 0, \]
\[ \left\langle \delta \Phi \hat{A}^2 \right\rangle = 0, \]  \hfill (12)

where \( d \) is the exterior derivative acting on the 1-form \( \delta \Phi \hat{A} \), while for the vector field we take the global constraints

\[ \left\langle \left[ 2 \nu^2 \delta A^\mu + \frac{\delta L(\delta A)}{\delta A_\mu} \right] \hat{A}_\mu \right\rangle = 0, \]
\[ \left\langle \delta \Phi \left[ 2 \nu^2 \delta A^\mu + \frac{\delta L(\delta A)}{\delta A_\mu} \right] \hat{A}_\mu \right\rangle = 0. \]  \hfill (13)

This set of restrictions on the variations of the fields ensures that the stationary points \( \Phi_0 = \nu \) and \( \hat{A} \neq 0 \) a flat connection are strict minima of the action (1). That is, these configurations are strict minima of the action for variations satisfying those constrains. It seems that such set is the unique one to achieve such goal.

We thus obtain several minima depending on the existence of non-trivial flat connections on the base manifold. That is, depending on the non-trivial topology of the base manifold. The number of such minima is in one to one correspondence to the homomorphisms of the fundamental group of the base manifold into the structure group, \( U(1) \) in our case.

All the results we are considering are valid over any (pseudo) Riemannian
metric over the base manifold.

The perturbative expansion around the vacuum of the complex field $\Phi$ satisfying (11) is:

$$\Phi = e^{i \xi \overline{v}} (v + \eta)$$  \hspace{1cm} (14)

$\xi$ and $\eta$ being real fields, then we have

$$D_\mu \Phi = e^{i \xi \overline{D}_\mu} (v + \eta)$$  \hspace{1cm} (15)

Where

$$\tilde{A}_\mu = A_\mu + \frac{1}{v} \partial_\mu \xi$$
$$\tilde{D}_\mu = \partial_\mu + i \tilde{A}_\mu$$  \hspace{1cm} (16)

and

$$V(\Phi) = \mu^2 (v + \eta)^2 + \lambda (v + \eta)^4$$  \hspace{1cm} (17)

Under a gauge transformation one has

$$\xi \to \xi - \Lambda$$
$$\tilde{A} \to \tilde{A} + d\Lambda$$  \hspace{1cm} (18)

so $\xi$ may be eliminated by a gauge transformation.

We notice that (14) is the general solution to (11).

The action (1) then reduces to

$$S = \left\langle \partial_\mu \eta \partial^\mu \eta + \tilde{A}_\mu \tilde{A}^\mu (v + \eta)^2 - \mu^2 (v + \eta)^2 - \lambda (v + \eta)^4 + \mathcal{L}(\tilde{A}) \right\rangle$$  \hspace{1cm} (19)

Now we consider the decomposition of the $U(1)$ connection over a non-trivial flat background $\hat{A}_\mu$.

We have

$$\tilde{A}_\mu = \hat{A}_\mu + a_\mu$$  \hspace{1cm} (20)

where $a_\mu$ satisfies (13) with $\delta A_\mu = a_\mu$.

(19) then reduces to

$$S = \left\langle \partial_\mu \eta \partial^\mu \eta + \hat{A}_\mu \hat{A}^\mu (v + \eta)^2 - 6 \lambda v^2 \eta^2 + [2v(\hat{A}^2 - \mu^2) - 4v^3 \lambda] \eta \right\rangle$$
$$+ \left\langle 4v \hat{A}_\mu a^\mu \eta + \mathcal{L}(a) + v^2 a^2 + 2v^2 \hat{A}_\mu a^\mu \right\rangle$$  \hspace{1cm} (21)
plus terms independent of $\eta$ and $a$ and higher order terms which we have not
written.

The Higgs field $\eta$ and the massive vector field $a_\mu$ are coupled through the
term $4v\hat{A}_\mu a^\mu \eta$ which depends on the background field $\hat{A}_\mu$. We notice also
the presence of a non standart term $2v^2\hat{A}_\mu a_\mu$. If we consider $\eta$ and $a_\mu$ to be
infinitesimal of the same order with respect to $v$ and $\hat{A}_\mu$ respectively in the
expansions (14) and (20), we obtain from (21) two first order infinitesimal
terms

$$2v[\hat{A}^2 - \mu^2 - 2v^2\lambda]\eta + 2v^2\hat{A}_\mu a_\mu,$$

which have to be annilated in order to have a consistent theory. Using (13),
$2v^2\hat{A}_\mu a_\mu$ may be expressed as a total derivative. Using (13) and (12) the
term $4v\hat{A}_\mu a^\mu \eta$ in the action (21) can also be written as a total derivative.
The term $2v\hat{A}^2\eta$ is eliminated by the restriction (12). We take $v^2 = \frac{\mu^2}{2\lambda}$.

The constrained action is then equivalent to

$$S = \left\langle \partial_\mu \eta \partial^\mu \eta + \left(\hat{A}^2 - 4\lambda v^2\right)\eta^2 \right\rangle + \left\langle L(a) + v^2a^2 + (\theta + \rho\eta) \left(2v^2a_\mu + \frac{\delta L}{\delta a_\mu}\right)\hat{A}^\mu \right\rangle,$$

subject to (12).

$\theta$ and $\rho$ are the Lagrange multipliers associated to (13). Since the constraints
are global ones the Lagrange multipliers are constant. (11) has already been
implemented by using (14)

The field equations for the massive vector field $a_\mu$ are

$$\frac{\delta L}{\delta a_\mu} + 2v^2a^\mu + 2(\theta + \rho\eta)v^2\hat{A}^\mu + \frac{\delta^2 L}{\delta a_\mu \delta a_\nu} ((\theta + \rho\eta)\hat{A}^\nu) = 0 \quad (24)$$

$$\left\langle \left(\frac{\delta L}{\delta a_\mu} + 2v^2a^\mu\right)\hat{A}_\mu \right\rangle = 0,$$

$$\left\langle \eta \left(\frac{\delta L}{\delta a_\mu} + 2v^2a^\mu\right)\hat{A}_\mu \right\rangle = 0. \quad (25)$$

The last term in (24) may be rewritten as

$$*d \left((\theta + \rho\eta)\hat{A}\right) \quad (26)$$

for the case of 3 dim and the Chern-Simon Lagrangean, and

$$*d*d \left((\theta + \rho\eta)\hat{A}\right) \quad (27)$$
for the case of the Maxwell Lagrangian in $D$ dimensions.

In both cases, because of (12), the terms vanish. Using (24), (25), (12) and (13) we get

$$\theta = 0$$
$$\rho = 0$$

and the field equations reduce then to

$$\frac{\delta \mathcal{L}}{\delta a_{\mu}} + 2\nu^2 a^\mu = 0$$

(29)

where $a_{\mu}$ can be reexpressed in terms of the connection $A^\mu$ as

$$a_{\mu} = A_{\mu} - \hat{A}_{\mu}.$$  

(30)

(29) shows that the generated mass for the spin 1 excitation is exactly the same as in the standard Higgs mechanism over a trivial background. The scalar field $\eta$, however, is described by a completely different action

$$\left\langle \partial_\mu \eta \partial^\mu \eta + \left( \hat{A}^2 - 4\nu^2 \right) \eta^2 \right\rangle$$

(31)

subject to (12).

The quadratic terms in this action show a distribution of mass depending on the topology of the base manifold since $\hat{A}$ is a closed one form. Its contribution has the same sign as $4\nu^2$ resulting in an increasement of the mass term. The constraint (12) severely restricts the space of solutions of the corresponding field equation.

**Dual Formulation of the Spin 1 Massive Excitations**

We would like now to analyze the action for the massive spin 1 excitations. We will show that it is the dual formulation of the gauge covariant theories for spin 1 excitations in term of antisymmetric fields, for the case in which $\mathcal{L}$ is the Maxwell Lagrangian in $d$ dimensions and of the Topological Massive theory in 3 dimensions. From the point of view of the mass generating mechanism we started with a fixed background $\hat{A}$, we showed that for each $\hat{A}$ we have a minimum of the restricted variational problem and we performed the perturbative analysis around it. We are now going to functionally integrate over all possible backgrounds. In this sense we will obtain an effective action for the massive spin 1 excitation.
The action for the massive spin 1 field is

\[ \langle L(a) + 2v^2 a_\mu a^\mu \rangle \]  

(32)

where \( a_\mu \) is given by (30),

\[ a_\mu = A_\mu - \hat{A}_\mu \]  

(33)

\( A \) being the independent one form connection and \( \hat{A} \) an independent closed one form over the base manifold. (32) may be rewritten as

\[ \langle L(a) + 2v^2 (A - \hat{A}) \wedge * (A - \hat{A}) \rangle \]  

(34)

Where \( * \) is the Hodge dual operation.

We will now follow the general construction of dual formulations described in [8][9]. We introduce an independent 1-form \( L \) globally defined over the base manifold satisfying

\[ dL = 0 \]  

(35)

and introduce this constraint into the action by using a Lagrange multiplier \( d-2 \) form \( B \). We will assume that (35) is the only constraint on \( L \) and then comment on the modification if a global restriction of the form

\[ \oint L = 2\pi n \]  

(36)

is imposed on \( L \). If there is no condition of type (36) then \( B \) is a \( d-2 \) form globally defined over the base manifold.

We then have

\[ \langle L(A) + 2v^2 (A - L) \wedge * (A - L) + iL \wedge dB \rangle \]  

(37)

We may now integrate \( L \) in the functional integral or equivalently take the field equation

\[ -4v^2 * (A - L) + i dB = 0 \]  

(38)

and eliminate \( L \) in terms of \( A \) and \( B \). We obtain the quantum equivalent action

\[ \langle L(A) + (-1)^d \frac{1}{8v^2} * dB \wedge dB + iA \wedge dB \rangle \]  

(39)

If \( L(A) \) is the Maxwell Lagrangean over a 4-dimentional base manifold, (39) is exactly the gauge invariant action describing a massive spin 1 exci-
tation introduced in [3]. Its generalization to $d$ dimensions and equivalence to the Proca formulation was presented in the latest of [3] and also agrees with (39). If $\mathcal{L}(A)$ is the Chern-Simon lagrangean in 3 dimensions and we functionally integrate on $A$ in (39) we end up with the Topological Massive action[2], with the topological mass generated now by a Higgs mechanism on a non-trivial background.

If we consider a global condition of the form (36) then the correct lagrange multiplier $B$ is locally a $d - 2$ form satisfying the global condition

$$\int_{\Sigma_{D-1}} dB = 2\pi m .$$

(40)

The constrains (35) and (36) are introduced into the unconstrained action (37) with the same term

$$iL \wedge dB$$

(41)

the general construction with global constrains was obtained in [8][9]. The same local results are obtained as before, however the global structure of the field $B$ is different. Condition (40) works as a quantum stabilizer for the different minima. The antisymmetric field $B$ satisfying (40) may have non-trivial transitions over the base manifold associated to a higher order bundle as described in [9]. If $B$ is a globally defined $d - 2$ form then $m = 0$ in (39).

**Conclusions**

We introduced a new variational principle which allow the implementation of the Higgs mechanism for Abelian theories over non-trivial background flat connections. The construction has similarities to the variational problem that allows, for integrable systems, to obtain the different multi-solitonic solutions as minima of constrained optimization problems constructed from conserved quantities.

The resulting actions for the massive spin 1 excitations are gauged invariant and dual to the Topological Massive action in 3 dimensions and to its generalizations on higher dimensions in terms of antisymmetric fields. The approach shows how all these massive gauge actions may be obtained from massless gauge actions through the Higgs mechanism. The mass of the spin 1 excitation is the same as the one obtained from the standard Higgs mechanism over trivial backgrounds. However, the dynamics of the scalar field is completely different from the usual approach. The action for the scalar field has a pure topological contribution from the background. The mass term becomes increased because of this contribution. There is also a constraint on the scalar field which severely restricts its dynamics. When the background is switched to zero the standard formulation is regained.

Finally, the approach shows how topological sectors, in these cases re-
lated to $BF$ topological theories, may be incorporated to the standard Higgs mechanism. It is expected that the same approach will lead to the non-Abelian massive gauge theories in three dimensions. This is a very interesting case since there are two massive non-equivalent gauge invariant actions in three dimensions[7], probably corresponding to the weak and strong coupling regimes of the same theory. It would be interesting to see if the Higgs mechanism relates in any way both models. Another interesting point would be to analyze if the same approach may introduce topological sectors to the low energy effective actions of supersymmetric theories.

Acknowledgement

We would like to thank P. Arias and M. I. Caicedo for interesting discussions. This research is supported by Decanato de Investigaciones de la Universidad simón Bolívar Proyecto USB-DID/G11.

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