Shapes of tight composite knots

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Abstract

We present new computations of tight shapes obtained using the constrained gradient descent code \texttt{ridgerunner} for 544 composite knots with 12 and fewer crossings, expanding our dataset to 943 knots and links. We use the new data set to analyze two outstanding conjectures about tight knots, namely that the ropelengths of composite knots are at least $4\pi - 4$ less than the sums of the prime factors and that the writhes of composite knots are the sums of the writhes of the prime factors. Our numerics support the connect sum conjecture and argue against the additivity of writhe conjecture. We also present data on the number of configurations having straight segments and highly curved kinked regions.

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(Some figures may appear in colour only in the online journal)

1. Introduction

When people are given a piece of rope, they almost instinctively tie it into a knot and pull it tight. But what, exactly, is the structure of that tight knot? The last decade has seen great progress in analyzing tight configurations. Researchers have focused on mathematical knots, that is, closed loops forming different topological knot types, with the rope modeled as a non-self-intersecting tube about a smooth (usually $C^2$ or $C^{1,1}$) space curve. One can then define the ropelength of the curve to be the quotient of its length and the maximal radius (or thickness) of a non-self-intersecting tube about the curve. Alternately, the ropelength is the minimal centerline length of a unit-radius tube without self-intersections forming the given knot type.

Configurations that minimize the ropelength within a given knot type are called tight or ideal. These configurations have been used to predict the relative speed of DNA knots under gel electrophoresis [1], the pitch of double helical DNA [2], the average values of different...
spatial measurements of random knots [3] and the breaking points of knots [4]. They also
provide a model for the structure of a class of subatomic particles known as glueballs [5].
Another way to think about the tight knot problem is to see it as a packing problem akin to
Kepler’s conjecture. In this version, instead of packing individual spheres into a volume, we
are packing an entangled tube into a small volume.

Finding analytical solutions to the tight knot problem is difficult; we know the tight
configurations for only some specialized classes of links [6]. For even the simplest non-trivial
knot, the trefoil 31, there is no analytic solution for the tight configuration. Instead, researchers
rely on computer simulations to approximate tight configurations by polygons minimizing
an appropriately discretized version of ropelength [7, 8]. Such calculations yield upper
bounds for the minimal ropelength of knots and links. Combining computer simulations with
theoretical work, we know that the minimal ropelength of the trefoil is between \(31.32\) [9] and
\(32.74317\) [10].

The quality of computer approximations of tight knots has increased immensely over
time. Originally, simple techniques such as simulated annealing were used to determine
approximately tight configurations [1, 11]. Later, Płeranski wrote the SONO (Shrink On
No Overlaps) software [12–15], implementing a gradient-like algorithm that shortens the
length of the polygon and then pushes pairs of vertices apart when they create self-
intersections in a tube about the polygon. Maddocks’s group has used a different approach,
implementing simulated annealing on biarc curves to minimize the ropelength of smooth curves
directly [16, 17].

Over the past few years, our group has developed the knot-tightening code \texttt{ridgerunner},
which implements a constrained gradient-descent algorithm that minimizes the length of a
polygon subject to a family of constraints which define an embedded tube around the polygon
[18]. The algorithm projects the gradient of the length of the polygon onto the subspace of
motions that preserve the integrity of the unit-radius embedded tube about the polygon; we
prove in [18] that a polygon is ropelength-critical when this projection vanishes. We use this
fact to define a quality measure for approximately tight configurations: the residual of such a
configuration is the fraction of the \((L^2)\) norm of the gradient of length after projection onto the
constraint.

Since tight knot configurations are so useful in the sciences, it is desirable to have a
complete catalog of tight knot shapes. The first step is to assemble a table of knot types.
Knot tabulation has a long history, stretching back to the 19th century knot tables of Tait and
Kirkman [19, 20] and very large tables of knots have been generated by computer [21, 22].
Using such tables, we have computed approximately tight shapes for 379 knots and links with 10
and fewer crossings [18]. However, these tables are incomplete in a certain sense: they only contain ‘prime’ knots. To understand primeness for knots, consider the knot ‘product’
defined by splicing two knots \(K_1\) and \(K_2\) together. This is called the connect sum of the knots
and denoted \(K_1 \# K_2\). We say that \(K\) is prime if \(K = K_1 \# K_2\) only when \(K_1\) or \(K_2\) is an unknotted
loop in analogy to the idea of primality for natural numbers, where \(n\) is prime if and only if
\(n = km\) implies \(k \) or \(m\) is equal to 1. The standard knot tables, and our work in [18] include
only prime knots.

But while prime knots and links are mathematically convenient, there is no reason to
expect that the knots and links which occur in scientific applications will be prime. For this
reason, we have continued our work to tabulate and tighten composite knots. In this paper, we
present the results of a large-scale computation of approximately tight shapes for composite
knots with 12 and fewer crossings, covering 544 knot types. We report the ropelengths of these
shapes in tables A1–A6. We computed these shapes with \texttt{ridgerunner}, generating starting
configurations by splicing together the approximately tight configurations of prime knots from

It is an open problem whether the crossing number of a composite knot is the sum of the crossing numbers of its prime factors; if this is true, our list of composite knots covers all the composites with 12 and fewer crossings. The quality of these computations is measured by their resolution (the number of vertices per unit ropelength) and their residual (the fraction of the tightening force which is not balanced by contact forces on the tube). All of our knots have resolution at least 8, and almost all of our knots have residuals between 0.01 and 0.001. The residual for each shape is reported in the supplementary materials for this paper available at stacks.iop.org/JPhysA/45/225202/mmedia. Together with our earlier work, this brings the total number of computed approximately tight shapes to 943 knot and link types.

Katritch, Olsen, Pieranski, Dubochet and Stasiak made the first computations of tight composite knots, reported in Nature in 1997 [23]. In that paper, they observed several interesting phenomena. First, they noted that the 3D average writhing number of tight configurations appeared to be additive under connect sum: \( \text{Wr}(K_1) + \text{Wr}(K_2) = \text{Wr}(K_1 \# K_2) \). This was a particularly striking observation since the writhe is a shape invariant and not a topological invariant and there is no reason to believe that the tight shapes of \( K_1 \) and \( K_2 \) would be exactly repeated in the tight configuration of \( K_1 \# K_2 \). We find that for the vast majority of knots, this conjectured equation holds to a remarkable degree of accuracy. However, we have found a small number of anomalous composites where the conjecture seems to fail.

Another phenomenon noted in [23] was that the ropelength of the tight configuration of \( K_1 \# K_2 \) was shorter than the sum of the minimum ropelengths of \( K_1 \) and \( K_2 \): intuitively, one could save a certain amount of rope by splicing. They conjectured that the amount of rope saved was at least \( 4\pi - 4 \) (this is true for the links in [6]). Our computations support this conjecture, although we find cases where the amount of rope saved is very close to the conjectured \( 4\pi - 4 \).

2. Methodology

2.1. Tabulation of composite knots

It is a classical theorem of Schubert [24] that every composite knot has a unique decomposition into an unordered list of prime summands. From this perspective, it would seem that tabulating composite knots must be easy: one should take a knot table and form all subsets of the table (allowing repeats). Unfortunately, the situation is not quite this simple. The traditional tables of prime knots are computed only up to symmetry. For each entry in the tables, there are actually one, two or four distinct knot types associated with \( K \), depending on the symmetry properties of \( K \).

It is easiest to see this with respect to chirality, a topic of great familiarity in the sciences. The trefoil knot \( 3_1 \), for example, is chiral in that one cannot deform a ‘right-handed’ trefoil to its mirror image. Thus the mirror image of the trefoil is actually a member of a different knot type, denoted \( 3^\text{m}_1 \) and called the ‘left-handed’ trefoil. On the other hand, the figure-8 knot \( 4_1 \) can be deformed to its mirror image, so it is called amphichiral. Most knots are chiral so they are not topologically equivalent to their mirror images, but some are amphichiral (for example, 20 of the 249 prime knots with 10 or fewer crossings are amphichiral).

A reversible knot type is one that can be deformed to itself but with the opposite orientation along the curve. A symmetric configuration of the trefoil, for instance, can be reversed by rotating it by 180°. Only 36 of the 249 prime knots with 10 or fewer crossings are non-reversible (note that the ratio of non-reversible to reversible knot types increases with crossing number), the simplest of which is the 8_{17} knot. We denote the reverse of a knot type \( K \) by \( K^r \), so the reverse of the 8_{17} knot is 8^r_{17}. From a physical standpoint, the reversibility of a knot
Table 1. The five standard symmetry types for a knot type.

| Class                | Amphichiral | Reversible | Isotopy types | Example(s) |
|----------------------|-------------|------------|---------------|------------|
| No symmetry          | No          | No         | 4             | 9_{22}, 9_{33} |
| (-) Amphichiral symmetry | –          | –          | 2             | 12_{127}  |
| Invertible symmetry  | No          | Yes        | 2             | 3_{1}      |
| (+) Amphichiral symmetry | Yes        | No         | 2             | 8_{17}     |
| Full symmetry        | Yes         | Yes        | 1             | 4_{1}      |

Type could be as important as its chirality, for instance, when the knot represents a flux tube or strand of DNA and thus has a natural orientation.

As a result, there are four obvious classes of knots: chiral/non-reversible (no symmetry), chiral/reversible (invertible symmetry, although Conway [25] calls this reversible symmetry), amphichiral/non-reversible ((+) amphichiral symmetry, although Conway [25] calls this invertible symmetry) and amphichiral/reversible (full symmetry). In addition, there is a class of negative amphichiral knot types which are not equivalent to either their reverses or their mirror images, but are equivalent to the reverse of their mirror images. This symmetry type does not fit neatly into the classification above. Luckily, these knots are rare among knots of low crossing number. A summary of the five different knot symmetries are given in table 1.

For tightening computations on prime knots, these fine distinctions are usually immaterial. For instance, although the trefoil knot (3_{1}) is not isotopic to the mirror trefoil (3_{1}^{m}), we know that any tight configuration of 3_{1} is a rigid reflection of a tight configuration of 3_{1}^{m}. Hence both of these knot types have the same minimum ropelength. We note that other geometric invariants which are sensitive to chirality, such as the average writhing number, will be different for the tight configuration of each knot type.

However, when considering composites, symmetries make a real difference in the shapes of tight knots. The connect sum of two trefoils 3_{1}#3_{1} (called the granny knot) is not only a different knot type from the connect sum 3_{1}#3_{1}^{m} (the square knot) but it also has a different minimum ropelength. On the other hand, the mirror granny knot 3_{1}^{m}#3_{1} is a different knot type than the granny knot 3_{1}#3_{1} but has the same minimum ropelength, while the mirror square knot 3_{1}^{m}#3_{1} has the same knot type, and thus the same minimum ropelength, as the square knot 3_{1}#3_{1}^{m}. It turns out there are various possibilities when one takes the connect sum of two knots depending on their symmetry types, with a further simplification when the summands are related to one another by a symmetry (such as in the case above). These possibilities are summarized in table 2.

We compiled symmetry data for prime knots of 9 and fewer crossings from Henry/Weeks [26] and Kodama/Sakuma [27]. For a given prime knot ‘base’ type, we denoted the mirror, reversal and reverse-mirror of the knot (when they are not isotopic to the base) by the tags m, r or rm. We then ordered the list by crossing number, index in the Rolfsen table of knots [28] and symmetry type, so that \( K < K^m < K^{r} < K^{rm} \). For composite knots involving two factors, we used the calculation summarized in table 2 to enumerate the different knot types possible for the connect sum in terms of the symmetries of the summands. There were a few cases where we had more than two summands; these were checked by hand. In our tables, each composite knot type appears once, labeled with the summands in sorted order. For example, the label 3_{1}#3_{1}^{m}#5_{1} appears in our list, but the labels 3_{1}^{m}#3_{1}#5_{1} and 5_{1}#3_{1}#3_{1}^{m} do not. Mastin [29] provides a general algorithm for enumerating composites with any number of prime factors and determining their symmetry types based on the JSJ-decomposition of composite knots. Tabulating composite links is a considerably more difficult problem, treated in [30].
Table 2. The number of knot types and (in parentheses) the number of possible distinct ropelength values which can be obtained by taking a connect sum of two knots of the given symmetry types, assuming that the two summand knots are not related by a symmetry. For example, the connect sum of a 31 knot (reversible) with a 41 knot (full symmetry) yields two knot types: 31#41 and 3m#41, but only one ropelength value since these knot types are related by a mirror symmetry. The connect sum of a 31 knot with a (+) amphichiral 817 knot yields four possible knot types: 31#817, 31#8r17, 3m#817, 3m#8r17, but again only one ropelength value since the last three types are related to the first by a reverse, mirror or mirror-reverse symmetry, respectively. On the other hand, the sum of a 31 knot with a (reversible) 51 knot yields four knot types: 31#51, 3m1#51, 3m1#5m1 and 3m1#5m1 with two potentially different ropelength values, one for 31#51 and 3m1#5m1 (whose tight configurations are related by a mirror symmetry) and one for 3m1#51 and 31#5m1 (where again the tight configurations are related by a mirror symmetry). The tight configurations of the knots 31#51 and 3m1#51m1 do not seem to be related by a rigid motion and have ropelength values 71.544 and 71.579, respectively.

| #       | None   | (−) Amphichiral | Reversible | (+) Amphichiral | Full |
|---------|--------|-----------------|------------|-----------------|------|
| None    | 16 (4) |                 |            |                 |      |
| (−) Amphichiral | 12 (2) | 9 (2)           |            |                 |      |
| Reversible | 8 (2) | 6 (1)           | 4 (2)      |                 |      |
| (+) Amphichiral | 8 (2) | 6 (1)           | 4 (1)      | 4 (2)           |      |
| Full    | 4 (1)  | 3 (1)           | 2 (1)      | 2 (1)           | 1 (1)|

2.2. Algorithms

We minimized polygonal ropelength using the ridgerunner code described in [18]. Recall that we define the residual of an approximately tight polygon to be the fraction of the gradient of length remaining after projection and that the knot is critical when this residual vanishes. For 520 of the 544 tightened composite knots, the residual values were below 0.01 (i.e. over 99% of the gradient is resolved against the constraints).

Since the minimum length of our composite knots varies considerably over our table, we did not choose the same number of edges for each knot. Instead, we choose to keep constant the 'resolution' of each polygon, defined as the quotient of the number of edges and the ropelength of the curve. All of our final configurations have resolution at least 8 (from several hundred to around a thousand vertices).

For each configuration, we give a ropelength upper bound which is given by carefully numerically approximating the length of a piecewise $C^2$ curve constructed by splicing short circle arcs into our polygons. This bound is a rigorous upper bound on the minimum ropelength of the given knot type, the details of which appear in [18].

2.3. Initial configurations

The ridgerunner algorithm requires an initial configuration of each knot type. Since the software proceeds by constrained gradient descent, it is designed to stop at local minima of the ropelength function. As a practical matter, it is impossible to know at present whether any given ropelength-critical knot is a global minimum over its knot type (rigorous, sharp lower bounds are not known for any knot type, and the configuration space of polygonal curves is far too large to attempt any kind of exhaustive search). However, we tried to reduce the probability of false local minima in our dataset in two ways.

First, splicing two given polygons $P_1$ and $P_2$ together requires a choice of arcs on $P_1$ and $P_2$ to cut out and splice. It is clear that the shape of the resulting tight composite probably depends on these choices. To avoid this problem, we took 'all' connect sums of $P_1$ and $P_2$ using the following algorithm.
(i) Identify all arcs of $P_1$ and $P_2$ lying on their convex hulls, i.e. lying on the ‘outside’ of the configuration.
(ii) For each pair of arcs, choose an edge on each arc.
(iii) Translate and rotate the polygons to align the selected edges, delete the edges and splice the polygons together. Repeat this process for all pairs of arcs to form an ensemble of composite polygons.

For the prime summands, we used the approximately tight configurations from [18]. After splicing the polygons together, we then smoothed each of these connect sums and scaled them up, allowing them to retighten from a position with larger thickness. The winning configurations were selected for further runs at higher resolution.

Second, we explored the configuration space of each of our knots using a new version of the ridgerunner core called ‘mangle mode’. In this form, rather than attempting to minimize the length of a polygon subject to the constraints describing the tube, the software applies a randomly chosen toroidal force field to the knot and resolves the resulting motion against the tube constraints. This has the effect of turning the knots ‘inside out’, while preserving the tube around the knot. In practice, using this method to generate 20 alternate start configurations and minimizing ropelength from each position was an effective method of discovering alternate local minima for ropelength.

2.4. Hardware

We minimized our knots on the ACCRE cluster at Vanderbilt University and on a 72 core cluster at the University of Saint Thomas, running computations for most of the year 2010 as we experimented with different start positions and run parameters for the knots. In total, we ran more than 20 000 configurations, distributed among the 544 composite knot types.

3. Results

We report the ropelengths of our tight shapes in tables A1–A6, and their residuals in the supplementary materials available at stacks.iop.org/JPhysA/45/225202/mmedia. As in [18], our data for these composite knots includes both self-contact sets and measures of the compression force on the contacts. We hope that these conformations will inspire other groups to refine them and further investigate them. All of our data, including the vertices of our approximately tight conformations, are freely available on Cantarella and Rawdon’s web pages. Below we discuss the accuracy of our computations and our analysis of the two conjectures.

3.1. Error bars

Although we know that a polygonal configuration with residual zero is exactly critical [18], there is no theorem bounding the distance between a given polygon and a critical polygon in terms of residual. Thus we cannot hope to give hard bounds on how far our ‘almost-critical’ configurations are from truly critical polygons. However, we have validated the ridgerunner code against the few ropelength-critical configurations which are known theoretically, obtaining errors in ropelength between 0.0017% and 0.02%. To get another measure of the errors in this kind of large-scale search, we ran several hundred initial configurations of the 7\text{1} knot, finally obtaining 141 configurations with resolution 8 which converged to residual less than 0.01. Throwing out a few outliers, we get the data in figure 1.

We think about the ropelength ‘landscape’ for a given knot type as dimpled with a large number of ‘pits’, each of which contains a local minimum for ropelength. Each pit is a basin
Figure 1. We plot ropelength/writhe pairs for 138 configurations of 77 with resolution at least 8 which converged to have residual less than 0.01. The top plot of all the configurations reveals several clusters presumably corresponding to different ropelength local minima for this knot. The bottom plot focuses on the cluster with lowest ropelength. On top, we see that knots in different clusters vary in ropelength by 0.677 (1%) and in writhe by 0.038 (6%). On the bottom, knots within the same cluster have a variation of 0.009 (0.1%) in ropelength and 0.004 (0.6%) in writhe.

of attraction for ridgerunner. We can rank them by the ropelength of the corresponding local minimum, with the best basin containing the global minimizer for ropelength among configurations of this knot type. If we have found the best basin of attraction, a conservative estimate based on the data in figure 1 would be that our ropelength figures are accurate to within 0.2% and our writhe figures are accurate to within 1%. If we are within a few basins of the best basin, we can still estimate that our ropelength is within 1% of the true value while our writhe is within 10% of the true value.
Table 3. A selection of cases where the writhe conjecture is not supported by our data. The knot type, writhe of composite, sum of writhes of summands, absolute percentage difference between these numbers, and the number of start configurations tried for the composite (and in parentheses, the two summands). These are not the only cases where we are unable to verify the conjecture, but we have chosen not to show cases with more than two summands (because these more complicated knots simply may not be fully tightened), where the percentage difference was less than 4%, and where the writhe of the composite was less than 0.1 (because in these cases a small absolute difference between writhes close to zero leads to huge percentage differences).

| Knot   | WR(K₁#K₂) | WR(K₁) + WR(K₂) | Difference (%) | # Start positions |
|--------|------------|-----------------|----------------|-------------------|
| 3₁#9₄₈| 0.523      | 0.625           | 19.412 %       | 13 (16, 25)       |
| 4₁#7₇ | −0.534     | −0.632          | 18.359 %       | 55 (25, 331)      |
| 3₁#8₁₄| 0.556      | 0.634           | 14.127 %       | 50 (16, 26)       |
| 3₁#8₁₁| 0.609      | 0.526           | 13.688 %       | 38 (16, 21)       |
| 3₁#6₂₅| −0.564     | −0.632          | 12.000 %       | 17 (16, 27)       |
| 3₁#8₁₆| −0.576     | −0.632          | 9.610 %        | 73 (16, 23)       |
| 3₁#9₄₅| 1.711      | 1.865           | 8.979 %        | 16 (16, 27)       |
| 3₁#9₂₂| −1.123     | −1.223          | 8.936 %        | 42 (16, 24)       |
| 5₁#7₂ | −0.557     | −0.606          | 8.810 %        | 109 (14, 24)      |
| 3₁#9₄₂| −2.197     | −2.022          | 7.950 %        | 12 (16, 24)       |
| 3₁#9₂₃| 1.197      | 1.110           | 7.236 %        | 92 (16, 23)       |
| 3₁#8₁₉| 0.615      | 0.572           | 6.981 %        | 17 (16, 22)       |
| 3₁#9₁₂| 1.171      | 1.092           | 6.782 %        | 83 (16, 29)       |
| 4₁#8₁₃| −1.116     | −1.189          | 6.542 %        | 105 (25, 21)      |
| 5₁#7₄₄| 1.170      | 1.234           | 5.420 %        | 21 (27, 28)       |
| 3₁#5₂₁| 1.183      | 1.134           | 4.200 %        | 16 (16, 27)       |
| 5₁#7₅₄| 2.771      | 2.886           | 4.130 %        | 98 (27, 27)       |

3.2. Writhe of composite knots

Katritch et al [23] conjectured that the average writhing number of a composite knot should obey the relation \( \text{WR}(K₁#K₂) = \text{WR}(K₁) + \text{WR}(K₂) \). This was a surprising conjecture since the writhing number depends on the entire shape of a curve. Laing and Summers [31] showed that given two knots \( K₁ \) and \( K₂ \) which intersect in an arc, the conformation of \( K₁#K₂ \) given by deleting the common arc has writhes equal to \( \text{WR}(K₁) + \text{WR}(K₂) \), but there is no guarantee that the tight configuration of \( K₁#K₂ \) should be constructed in such a way. Nonetheless in the vast majority of the conformations we computed, it seems that the two prime summands appear almost unchanged in the tight composite and the writhes do obey this sum property to a high degree of numerical accuracy.

However, there are a number of cases suggesting that this conjecture is false. A collection of these are summarized in table 3. Pieranski and Przybyl have graciously spot-checked the writhe and tightening computations in this table using their knot-tightening and writhe computation codes [32]. These checks revealed no significant difference in writhes, although they were able to tighten the knots somewhat more using SONO and Przybyl’s FEM-based knot tightener. If any of these configurations represents the global minimum for the knots in question, we can say that the conjecture is (numerically) disproved. In fact, if any of the first five configurations in table 3 are within a few basins of the global minimizer, our numerics argue strongly that the conjecture is false. We cannot rule out the possibility that all of the configurations in table 3 are in basins very far from the global minimum for their knot types, but given the large number of starting positions we took for these knots, this is unlikely. Therefore, the weight of the numerical evidence is against the conjecture.

Inspecting one of our examples, figure 2 provides an explanation: for these knots, one of the summands pushes the other out of the way when the composite knot tightens, changing
Figure 2. Tight configurations of $3_1$, $9_{48}$ and their connect sum $3_1 \# 9_{48}$. The sum of the writhes of the tight $3_1$ and $9_{48}$ configurations is 0.523 while the writhe of the tight composite $3_1 \# 9_{48}$ is 0.625. The difference is easily explained by looking at the pictures; in the connect sum, the lower left arc of the trefoil is spliced to the upper right arc of the $9_{48}$ shown above. The extra loop of the trefoil pushes the remainder of the knot out of its tight shape, changing the writhe of the overall composite. The other examples in our database look similar; in each case it seems that taking the connect sum and then tightening distorts one or both summands, changing their writhe. Each of these configurations shows kinks and straight segments highlighted.

the writhe of the composite. We believe this behavior is real and will be more pronounced when compositions of more complicated knot types are considered. Unfortunately, we cannot discern a definitive pattern in the pairings of factor knots amongst our examples. Thus, an exhaustive search within composite knots with higher crossing numbers would be necessary, which is computationally infeasible at this point.

3.3. Ropelength of composite knots

Figure 3 shows a scatterplot of the ropelength of each of our composites (x-axis) plotted against the sum of the ropelengths of their prime summands (y-axis). Katritch et al [23] conjectured that the ropelength of a composite should be at least $4\pi - 4$ less than the sum of the ropelength of the summands. This has become informally known as the connect sum conjecture for ropelength. The intuition behind the conjecture is easy to understand. When two pairs of linked rings (each with ropelength $8\pi$) are connect-summed to form a three-link chain, the two rings which have been spliced together shrink to form a stadium curve with ropelength $4\pi + 4$. The difference between the original ropelength of the rings ($8\pi$) and the ropelength of the stadium curve ($4\pi + 4$) is the amount of rope saved in the splicing procedure: $4\pi - 4$. For more complicated knots, it is somewhat surprising that the same amount of rope should be ‘exposed’ to a connect sum. If this conjecture holds for very complicated knots, the principle at work would seem to be very different.

However, in this range of composite knots, our data suggests that the conjecture is quite plausible. In our dataset there are only 37 knots where our best conformation of the composite is very slightly longer than the conjecture predicts. Table A7 shows the complete collection of these cases. The discrepancy in ropelength has a maximum value of about 0.4%, which is well within the ‘nearby basin’ ropelength error estimate of 1% obtained above and very close to the ‘correct basin’ ropelength error estimate of 0.2% obtained above.
Figure 3. This scatterplot shows pairs \((x, y)\) where \(x\) is the minimum ropelength we have found for a composite knot and \(y\) is the sum of the minimum ropelengths we have found for its prime summands. On the plot, knots with two prime summands are plotted with circles, knots with three prime summands are plotted with squares and knots with four prime summands are plotted with diamonds. According to the connect sum conjecture of [23], \(y - x\) is at least \((N - 1)(4\pi - 4)\), where \(N\) is the number of prime summands of the knot. This conjecture is shown by lines \(y - x = 4\pi - 4\), \(y - x = 8\pi - 8\) and \(y - x = 12\pi - 12\). As one can see from the plot, all our data are very close to obeying the bounds predicted by the conjecture.

Table 4. Longest and shortest knots of a given crossing number for prime and composite knots. It is interesting to see that the longest and shortest knots of each crossing number are prime.

| Cr | Rop     | Knots     | Cr | Rop     | Knots     |
|----|---------|-----------|----|---------|-----------|
| 6  | [57.042, 57.073] | 3_1 #31, 3_1 #3_1
| 7  | [65.240, 65.240] | 3_1 #3_1, 3_1 #4_1
| 8  | [71.544, 73.193] | 3_1 #5_1, 3_1 #4_1
| 9  | [79.329, 81.088] | 3_1 #3_1, 3_1 #6_1
| 10 | [85.758, 89.472] | 3_1 #7_1, 3_1 #7_1
| 11 | [83.372, 98.171] | 3_1 #8_1, 3_1 #8_1
| 12 | [90.905, 106.508] | 3_1 #9_1, 4_1 #8_1

3.4. Other observations about tight configurations

Table 4 shows the longest and shortest knots by crossing number for both prime and composite knots. We can see that the range of ropelengths for composite knots is smaller than the range for all knots of the same crossing number. One might expect composite knots generally to be longer than prime knots of the same crossing number since composites are separated in two pieces and have less opportunity to nest together and save rope. But it is mildly surprising that...
Figure 4. Two examples of tight composite knots. The left knot is an approximately tight configuration of the square knot $31\#3$ which shows the expected straight segments joining the two summands highlighted. (The other stripe denotes a ‘kink’ where the knot has maximum curvature.) It is interesting that our tight configuration does not have a perfect geometric mirror symmetry (even though a critical configuration with this symmetry surely exists; see [37] for a discussion of symmetric criticality). The right knot is the approximately tight right granny knot $3\#3$. The ropelengths are 57.09 and 57.05, respectively.

the longest knots of each crossing number are not composite. It will be interesting to see if this effect persists through higher crossing numbers.

The embedded tube constraint is controlled by both tube-to-tube contacts (‘struts’) and an upper bound on the curvature of the core polygon (‘kinks’) [18, 33, 34]. It is a theorem (under some mild regularity assumptions) that no closed ropelength-critical curve can fail to have a strut [35], but kinks seem to be optional. However, in our data, tight configurations with kinks seem to be extremely common, occurring in 506 of our 544 composite configurations.

Another theorem is that sections of a ropelength-critical curve without struts or kinks are straight segments (see [35, 36] for different versions of this theorem). Gonzalez conjectured this phenomenon should occur in all tight composite knots with a mirror symmetry (such as the square knot in figure 4). We find this phenomenon in 496 of the 544 composite knots with crossing number at most 12.

4. Future directions

Our publicly available dataset of tight knots and links, now including tight prime knots to 10 crossings, tight prime links to 9 crossings and (with this paper) tight composite knots to 12 crossings should provide a substantial starting point for physicists, biologists and mathematicians interested in the geometry of knotted configurations. We intend to expand the dataset to assemble tight configurations of all prime and composite knots and links to 12 crossings. This is a substantial undertaking even for prime knots and links, but the hardest part is certainly composite links. The problem is that composite links remain untabulated (it is still a challenging open problem to come up with an algorithm for tabulating composite links; see [30]).

It is certainly interesting that so many of our configurations exhibit ‘kinked’ sections of maximum curvature. An excellent confirmation of this phenomenon would be to rerun our configurations with the curvature constraint removed to see whether we can achieve shorter lengths. In addition, the FEM techniques of Pieranski and Przybyl show great promise in computing at very high resolutions (up to tens of thousands of vertices). We intend to collaborate with this group for our next set of computations, using ridgerunner as a medium-resolution search tool to explore the configuration space of curves of a given knot type and then switching to FEM for the final tightening.
Table A1. Ropelengths of tight knots by knot type, part 1 of 6.

| Knot       | $Rop_p$ | $Rop$ |
|------------|---------|-------|
| $3_1#3_1$ | 57.06   | 57.04 |
| $3_1#3_1^m$ | 57.09   | 57.07 |
| $3_1#3_1^m$ | 57.06   | 57.04 |
| $3_1#4_1$ | 65.26   | 65.24 |
| $3_1#4_1^m$ | 65.26   | 65.24 |
| $3_1#5_1$ | 71.56   | 71.54 |
| $3_1#5_1^m$ | 71.59   | 71.58 |
| $3_1#5_1^m$ | 71.56   | 71.54 |
| $3_1#5_2$ | 72.43   | 72.41 |
| $3_1#5_2^m$ | 72.55   | 72.53 |
| $3_1#5_2^m$ | 72.55   | 72.53 |
| $3_1#5_3$ | 72.43   | 72.41 |
| $3_1#5_3^m$ | 73.20   | 73.19 |
| $3_1#6_1$ | 81.10   | 81.09 |
| $3_1#6_1^m$ | 79.73   | 79.72 |
| $3_1#6_1^m$ | 79.73   | 79.72 |
| $3_1#6_1^m$ | 81.10   | 81.09 |
| $3_1#6_2$ | 80.25   | 80.24 |
| $3_1#6_2^m$ | 80.43   | 80.42 |
| $3_1#6_2^m$ | 80.43   | 80.42 |
| $3_1#6_2^m$ | 80.25   | 80.24 |
| $3_1#6_3$ | 81.02   | 81.00 |
| $3_1#6_3^m$ | 81.02   | 81.00 |
| $3_1#6_3^m$ | 81.02   | 81.00 |
| $3_1#7_1$ | 87.88   | 87.86 |
| $3_1#7_1^m$ | 87.88   | 87.86 |
| $3_1#7_1^m$ | 87.88   | 87.86 |
| $3_1#7_1$ | 84.30   | 84.28 |
| $3_1#7_1^m$ | 84.30   | 84.28 |
| $3_1#7_1^m$ | 84.30   | 84.28 |
| $3_1#7_2$ | 87.88   | 87.86 |
| $3_1#7_2^m$ | 87.88   | 87.86 |
| $3_1#7_2^m$ | 87.88   | 87.86 |
| $3_1#7_3$ | 87.88   | 87.86 |
| $3_1#7_3^m$ | 87.88   | 87.86 |
| $3_1#7_3^m$ | 87.88   | 87.86 |

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Table A2. Ropelengths of tight knots by knot type, part 2 of 6.

| Knot | Rop₁ | Rop₂ | Rop₃ | Knot | Rop₁ | Rop₂ | Rop₃ | Knot | Rop₁ | Rop₂ | Rop₃ |
|------|------|------|------|------|------|------|------|------|------|------|------|
| 3₁#₈₄ | 94.63 | 94.61 | 96.48 | 96.47 | 3₁#₈₂₁ | 87.81 | 87.79 | 3₁#₈₂₁ | 88.01 | 87.99 |
| 3₁#₈₅ | 94.98 | 94.96 | 96.44 | 96.42 | 3₁#₈₅ | 88.01 | 87.99 | 3₁#₈₅ | 87.81 | 87.79 |
| 3₁#₈₆ | 94.98 | 94.96 | 96.02 | 95.99 | 3₁#₈₁₃ | 96.02 | 95.99 | 3₁#₈₁₃ | 96.44 | 96.42 |
| 3₁#₈₇ | 94.63 | 94.61 | 96.64 | 96.62 | 3₁#₈₁₄ | 97.19 | 97.17 | 3₁#₈₁₄ | 97.03 | 97.00 |
| 3₁#₈₈ | 95.20 | 95.18 | 97.19 | 97.17 | 3₁#₈₁₁ | 97.03 | 97.00 | 3₁#₈₁₁ | 96.31 | 96.29 |
| 3₁#₈₉ | 95.62 | 95.59 | 96.33 | 96.31 | 3₁#₈₁₆ | 96.44 | 96.42 | 3₁#₈₁₆ | 96.64 | 96.62 |
| 3₁#₈₀ | 95.64 | 95.62 | 96.33 | 96.31 | 3₁#₈₁₅ | 96.64 | 96.62 | 3₁#₈₁₅ | 96.64 | 96.62 |
| 3₁#₈₁ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₄ | 97.19 | 97.17 | 3₁#₈₁₄ | 97.19 | 97.17 |
| 3₁#₈₂ | 95.48 | 95.47 | 96.64 | 96.62 | 3₁#₈₁₃ | 97.19 | 97.17 | 3₁#₈₁₃ | 97.19 | 97.17 |
| 3₁#₈₃ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁ |_ 97.19 | 97.17 | 3₁#₈₁ | 97.19 | 97.17 |
| 3₁#₈₄ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₃ | 97.19 | 97.17 | 3₁#₈₁₃ | 97.19 | 97.17 |
| 3₁#₈₅ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₄ | 97.19 | 97.17 | 3₁#₈₁₄ | 97.19 | 97.17 |
| 3₁#₈₆ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₅ | 97.19 | 97.17 | 3₁#₈₁₅ | 97.19 | 97.17 |
| 3₁#₈₇ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₆ | 97.19 | 97.17 | 3₁#₈₁₆ | 97.19 | 97.17 |
| 3₁#₈₈ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₇ | 97.19 | 97.17 | 3₁#₈₁₇ | 97.19 | 97.17 |
| 3₁#₈₉ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₈ | 97.19 | 97.17 | 3₁#₈₁₈ | 97.19 | 97.17 |
| 3₁#₈₁₀ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₉ | 97.19 | 97.17 | 3₁#₈₁₉ | 97.19 | 97.17 |
| 3₁#₈₁₁ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₀ | 97.19 | 97.17 | 3₁#₈₁₀ | 97.19 | 97.17 |
| 3₁#₈₁₂ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₁ | 97.19 | 97.17 | 3₁#₈₁₁ | 97.19 | 97.17 |
| 3₁#₈₁₃ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₂ | 97.19 | 97.17 | 3₁#₈₁₂ | 97.19 | 97.17 |
| 3₁#₈₁₄ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₃ | 97.19 | 97.17 | 3₁#₈₁₃ | 97.19 | 97.17 |
| 3₁#₈₁₅ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₄ | 97.19 | 97.17 | 3₁#₈₁₄ | 97.19 | 97.17 |
| 3₁#₈₁₆ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₅ | 97.19 | 97.17 | 3₁#₈₁₅ | 97.19 | 97.17 |
| 3₁#₈₁₇ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₆ | 97.19 | 97.17 | 3₁#₈₁₆ | 97.19 | 97.17 |
| 3₁#₈₁₈ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₇ | 97.19 | 97.17 | 3₁#₈₁₇ | 97.19 | 97.17 |
| 3₁#₈₁₉ | 95.62 | 95.59 | 96.64 | 96.62 | 3₁#₈₁₈ | 97.19 | 97.17 | 3₁#₈₁₈ | 97.19 | 97.17 |
| 3₁#₈₂₀ | 85.55 | 85.53 | 85.95 | 85.94 | 5₁#₆₁ | 95.73 | 95.71 | 5₁#₆₁ | 95.73 | 95.71 |
| 3₁#₈₂₁ | 95.70 | 95.68 | 85.95 | 85.94 | 5₁#₆₂ | 95.55 | 95.53 | 5₁#₆₂ | 95.55 | 95.53 |
| 3₁#₈₂₂ | 95.70 | 95.68 | 85.95 | 85.94 | 5₁#₆₁ | 95.73 | 95.71 | 5₁#₆₁ | 95.73 | 95.71 |
| 3₁#₈₂₃ | 95.70 | 95.68 | 85.95 | 85.94 | 5₁#₆₂ | 95.55 | 95.53 | 5₁#₆₂ | 95.55 | 95.53 |
Table A3. Ropelengths of tight knots by knot type, part 3 of 6.

| Knot | $Rop_p$ | $Rop$ |
|------|---------|-------|
| $5_2^m#6_1$ | 96.36 | 96.34 |
| $5_2^m#6_1^m$ | 95.73 | 95.71 |
| $5_2^m#6_2$ | 96.56 | 96.55 |
| $5_2^m#6_2^m$ | 95.64 | 95.62 |
| $5_2^m#6_3$ | 96.56 | 96.55 |
| $5_2^m#6_3^m$ | 95.64 | 95.62 |
| $5_2^m#6_2$ | 96.56 | 96.55 |
| $5_2^m#6_2^m$ | 95.64 | 95.62 |
| $5_2^m#6_3$ | 96.56 | 96.55 |
| $5_2^m#6_3^m$ | 95.64 | 95.62 |

| Knot | $Rop_p$ | $Rop$ |
|------|---------|-------|
| $3_1#3_1#5_1$ | 93.79 | 93.77 |
| $3_1#3_1#5_1^m$ | 94.12 | 94.10 |
| $3_1#3_1#5_1^m$ | 94.08 | 94.06 |
| $3_1#3_1#5_1^m$ | 94.08 | 94.06 |
| $3_1#3_1#5_1^m$ | 94.12 | 94.10 |
| $3_1#3_1#5_1^m$ | 93.79 | 93.77 |
| $3_1#3_1#5_2$ | 94.81 | 94.79 |
| $3_1#3_1#5_2^m$ | 95.04 | 95.02 |
| $3_1#3_1#5_2^m$ | 95.06 | 95.04 |
| $3_1#3_1#5_2^m$ | 95.06 | 95.04 |
| $3_1#3_1#5_2^m$ | 95.04 | 95.02 |
| $3_1#3_1#5_2^m$ | 94.81 | 94.79 |
| $3_1#3_1#5_1#4_1$ | 96.15 | 96.13 |
| $3_1#3_1#5_1#4_1$ | 96.15 | 96.13 |

Appendix. Ropelength tables

Tables A1–A6 show the polygonal ropelength ($Rop_p$) and ropelength upper bounds ($Rop$) that we have obtained for each of the composite knot types that we have considered. The composite knots are organized by dictionary order on their summands, with the primary order coming from position in Rolfsen’s table [28], with the knot $X^i_j$ being the $j$th example of a
Table A4. Ropelengths of tight knots by knot type, part 4 of 6.

| Knot  | Rop $p$ | Rop $p'$ | Rop $p''$ | Rop $p'''$ |
|-------|---------|----------|-----------|-----------|
| $3_1^m$ #9$_1^m$ | 103.83 | 103.80 |          |          |
| $3_1^m$ #9$_2^m$ | 103.77 | 103.75 |          |          |
| $3_1^m$ #9$_3^m$ | 103.57 | 103.55 |          |          |
| $3_1^m$ #9$_4^m$ | 104.08 | 104.06 |          |          |
| $3_1^m$ #9$_5^m$ | 103.57 | 103.55 |          |          |
| $3_1^m$ #9$_6^m$ | 103.67 | 103.65 |          |          |
| $3_1^m$ #9$_7^m$ | 103.42 | 103.39 |          |          |
| $3_1^m$ #9$_8^m$ | 103.67 | 103.65 |          |          |
| $3_1^m$ #9$_9^m$ | 103.83 | 103.81 |          |          |
| $3_1^m$ #9$_{10}^m$ | 103.84 | 103.82 |          |          |
| $3_1^m$ #9$_{11}^m$ | 103.84 | 103.82 |          |          |
| $3_1^m$ #9$_{12}^m$ | 103.83 | 103.81 |          |          |
| $3_1^m$ #9$_{13}^m$ | 103.97 | 103.94 |          |          |
| $3_1^m$ #9$_{14}^m$ | 104.12 | 104.09 |          |          |
| $3_1^m$ #9$_{15}^m$ | 104.12 | 104.09 |          |          |
| $3_1^m$ #9$_{16}^m$ | 103.97 | 103.94 |          |          |
| $3_1^m$ #9$_{17}^m$ | 104.59 | 104.56 |          |          |
| $3_1^m$ #9$_{18}^m$ | 104.20 | 104.18 |          |          |
| $3_1^m$ #9$_{19}^m$ | 103.97 | 103.94 |          |          |
| $3_1^m$ #9$_{20}^m$ | 103.93 | 103.91 |          |          |
| $3_1^m$ #9$_{21}^m$ | 103.93 | 103.91 |          |          |
| $3_1^m$ #9$_{22}^m$ | 103.93 | 103.91 |          |          |
| $3_1^m$ #9$_{23}^m$ | 104.49 | 104.47 |          |          |
| $3_1^m$ #9$_{24}^m$ | 103.93 | 103.91 |          |          |
| $3_1^m$ #9$_{25}^m$ | 104.49 | 104.47 |          |          |
| $3_1^m$ #9$_{26}^m$ | 104.28 | 104.25 |          |          |

Knot Rop $p$ Rop $p'$ Rop $p''$ Rop $p'''$

15 prime X-crossing link of $y$ components in the table. Knots with the same Rolfsen position are ordered by symmetry type, with the convention $K < K^m < K' < K^{mm}$. To aid the reader in making sense of the table, we insert lines where the crossing number changes and spaces where the base types of the summands change.
The tables of the residual values (measuring the relative quality of each of our minimized configurations) can be found in the supplementary materials available at stacks.iop.org/JPhysA/45/225202/mmedia.

The final table, table A7, gives the complete list of composite knots for which our tightest configuration is slightly larger than the value predicted by the connect sum conjecture of [23].
### Table A6. Ropelengths of tight knots by knot type, part 6 of 6.

| Knot            | Rop<sub>p</sub> | Rop<sub>p</sub> | Knot            | Rop<sub>p</sub> | Rop<sub>p</sub> | Knot            | Rop<sub>p</sub> | Rop<sub>p</sub> |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 5_1#7<sub>4</sub> | 102.29          | 102.27          | 5_1#7<sub>4</sub> | 102.96          | 102.94          | 3_1#3<sub>6</sub> | 102.27          | 102.24          |
| 5_1#7<sub>6</sub> | 101.88          | 101.86          | 5_1#7<sub>5</sub> | 102.39          | 103.36          | 3_1#3<sub>6</sub> | 103.30          | 103.28          |
| 5_1#7<sub>7</sub> | 101.88          | 101.86          | 5_1#7<sub>8</sub> | 103.36          | 103.34          | 3_1#3<sub>6</sub> | 103.09          | 102.06          |
| 5_1#7<sub>5</sub> | 102.29          | 102.27          | 5_1#7<sub>5</sub> | 103.39          | 103.36          | 3_1#3<sub>6</sub> | 102.61          | 102.58          |
| 5_1#7<sub>6</sub> | 102.61          | 102.59          | 5_1#7<sub>6</sub> | 103.57          | 103.55          | 3_1#3<sub>6</sub> | 103.50          | 103.47          |
| 5_1#7<sub>7</sub> | 102.79          | 102.77          | 5_1#7<sub>7</sub> | 103.51          | 103.49          | 3_1#3<sub>6</sub> | 103.48          | 103.46          |
| 5_1#7<sub>8</sub> | 102.61          | 102.59          | 5_1#7<sub>8</sub> | 103.51          | 103.49          | 3_1#3<sub>6</sub> | 103.48          | 103.46          |
| 5_1#7<sub>9</sub> | 102.97          | 102.95          | 5_1#7<sub>9</sub> | 103.57          | 103.55          | 3_1#3<sub>6</sub> | 103.61          | 103.58          |
| 5_1#7<sub>10</sub> | 103.14          | 103.12         | 5_1#7<sub>10</sub> | 103.93          | 103.91          | 3_1#3<sub>6</sub> | 103.69          | 103.67          |
| 5_1#7<sub>11</sub> | 104.03          | 104.00          | 5_1#7<sub>11</sub> | 105.36          | 105.34          | 3_1#3<sub>6</sub> | 104.01          | 103.98          |
| 5_1#7<sub>12</sub> | 104.19          | 104.17          | 5_1#7<sub>12</sub> | 105.16          | 105.14          | 3_1#3<sub>6</sub> | 103.69          | 103.67          |
| 5_1#7<sub>13</sub> | 101.01          | 100.99          | 5_1#7<sub>13</sub> | 102.44          | 102.42          | 3_1#3<sub>6</sub> | 106.20          | 106.18          |
| 5_1#7<sub>14</sub> | 102.22          | 102.21          | 5_1#7<sub>14</sub> | 104.19          | 104.17          | 3_1#3<sub>6</sub> | 103.21          | 103.19          |
| 5_1#7<sub>15</sub> | 102.22          | 102.21          | 5_1#7<sub>15</sub> | 102.53          | 102.51          | 3_1#3<sub>6</sub> | 106.20          | 106.18          |
| 5_1#7<sub>16</sub> | 102.22          | 102.21          | 5_1#7<sub>16</sub> | 103.57          | 103.54          | 3_1#3<sub>6</sub> | 104.16          | 104.14          |
| 5_1#7<sub>17</sub> | 102.66          | 102.53          | 5_1#7<sub>17</sub> | 103.57          | 103.54          | 3_1#3<sub>6</sub> | 101.60          | 101.58          |
| 5_1#7<sub>18</sub> | 103.66          | 103.64          | 5_1#7<sub>18</sub> | 103.17          | 103.14          | 3_1#3<sub>6</sub> | 102.80          | 102.78          |
| 5_1#7<sub>19</sub> | 102.55          | 102.53          | 5_1#7<sub>19</sub> | 103.73          | 103.71          | 3_1#3<sub>6</sub> | 102.24          | 102.22          |
| 5_1#7<sub>20</sub> | 102.22          | 102.21          | 5_1#7<sub>20</sub> | 103.17          | 103.14          | 3_1#3<sub>6</sub> | 102.80          | 102.78          |
| 5_1#7<sub>21</sub> | 102.96          | 102.94          | 5_1#7<sub>21</sub> | 103.80          | 103.77          | 3_1#3<sub>6</sub> | 101.60          | 101.58          |
Table A7. The 37 cases where our best conformation of the composite knot is very slightly larger than the connect sum conjecture predicts. The connect sum conjecture, originally due to Katritch et al [23], states that the minimum ropelength of a 2-summand composite knot should be $4\pi - 4$ less than the sum of the minimum ropelengths of the two summands. The conjecture encodes the intuition that splicing two knots together to make a composite knot should allow one to save a certain amount of rope. The excess ropelength in these composites is quite small, with a maximum value of $0.38\%$ for the $5^m \# 7_1$ knot. We encourage further investigation of the minimum ropelength for these knot types.

| Knot     | Rop  | Target | Diff     | Knot     | Rop  | Target | Diff     |
|----------|------|--------|----------|----------|------|--------|----------|
| $5^m \# 7_1$ | 100.426 | 100.042 | 0.384 %  | $3^m \# 3^m_1$ | 57.041 | 56.921 | 0.211 %  |
| $5^m \# 7_1$ | 100.426 | 100.042 | 0.384 %  | $3^m \# 3_1$ | 57.041 | 56.921 | 0.211 %  |
| $5^m \# 7_1$ | 100.359 | 100.042 | 0.317 %  | $3^m \# 7_1$ | 85.758 | 85.584 | 0.204 %  |
| $5^m \# 7_1$ | 100.359 | 100.042 | 0.317 %  | $3^m \# 7_1$ | 85.758 | 85.584 | 0.204 %  |
| $5^m \# 7_1$ | 86.087 | 85.837 | 0.292 %  | $3^m \# 7_1$ | 88.248 | 88.106 | 0.162 %  |
| $6^m \# 6_{12}$ | 105.140 | 104.845 | 0.281 %  | $3^m \# 7_3$ | 88.248 | 88.106 | 0.162 %  |
| $3^m \# 5_1$ | 71.579 | 71.379 | 0.281 %  | $3^m \# 8_3$ | 95.485 | 95.335 | 0.158 %  |
| $3^m \# 5_1$ | 71.579 | 71.379 | 0.281 %  | $3^m \# 8_3$ | 95.485 | 95.335 | 0.158 %  |
| $3^m \# 5_1$ | 57.073 | 56.921 | 0.268 %  | $3^m \# 9_1$ | 99.880 | 99.723 | 0.157 %  |
| $3^m \# 6_1$ | 81.088 | 80.883 | 0.254 %  | $3^m \# 9_1^m$ | 99.880 | 99.723 | 0.157 %  |
| $3^m \# 6_1$ | 81.088 | 80.883 | 0.254 %  | $3^m \# 9_1^m$ | 104.590 | 104.427 | 0.156 %  |
| $3^m \# 7_1$ | 85.800 | 85.584 | 0.253 %  | $3^m \# 7_3$ | 104.590 | 104.427 | 0.156 %  |
| $3^m \# 7_1$ | 85.800 | 85.584 | 0.253 %  | $3^m \# 9_1$ | 105.892 | 105.825 | 0.064 %  |
| $3^m \# 5_1$ | 71.544 | 71.379 | 0.232 %  | $3^m \# 9_1^m$ | 105.892 | 105.825 | 0.064 %  |
| $3^m \# 5_1$ | 71.544 | 71.379 | 0.232 %  | $3^m \# 9_1^m$ | 105.892 | 105.825 | 0.064 %  |
| $3^m \# 5_1$ | 86.034 | 85.837 | 0.230 %  | $3^m \# 9_1$ | 105.892 | 105.825 | 0.064 %  |
| $5^m \# 5_1$ | 86.034 | 85.837 | 0.230 %  | $3^m \# 9_1$ | 105.892 | 105.825 | 0.064 %  |

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