Minimum 2-vertex-twinless connected spanning subgraph problem

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Abstract

Given a 2-vertex-twinless connected directed graph $G = (V, E)$, the minimum 2-vertex-twinless connected spanning subgraph problem is to find a minimum cardinality edge subset $E_t \subseteq E$ such that the subgraph $(V, E_t)$ is 2-vertex-twinless connected. Let $G^1$ be a minimal 2-vertex-connected subgraph of $G$. In this paper we present a $(2 + a_t/2)$-approximation algorithm for the minimum 2-vertex-twinless connected spanning subgraph problem, where $a_t$ is the number of twinless articulation points in $G^1$.

Keywords: Directed graphs, approximation algorithm, Graph algorithms, twinless articulation point

1. Introduction

Let $G = (V, E)$ be a twinless strongly connected graph. A vertex $v \in V$ is called a twinless articulation point if the subgraph obtained from $G$ by removing the vertex $v$ is not twinless strongly connected. A twinless strongly connected graph $G$ is called $k$-vertex-twinless-connected if $|V| \geq k + 1$ and for each $U \subset V$ with $|U| < k$, the induced subgraph on $V \setminus U$ is twinless strongly connected. Therefore, a twinless strongly connected graph $G$ is 2-vertex-twinless-connected if and only if $|V| \geq 3$ and it has no twinless articulation points. Given a $k$-vertex-twinless-connected graph $G = (V, E)$, the minimum $k$-vertex-twinless-connected spanning subgraph problem (denoted by MKVTCS) consist in finding a subset $E_{kt} \subseteq E$ of minimum size such that the subgraph $(V, E_{kt})$ is $k$-vertex-twinless-connected. The MKVTCS problem is NP-hard for $k \geq 1$. Note that an optimal solution for minimum 2-vertex-connected spanning subgraph (M2VCS) problem is not necessarily an optimal solution for the M2VTCS problem, as illustrated in Figure 1.

In 2000, Cheriyan and Thurimella [1] presented a $(1+1/k)$-approximation algorithm for the minimum $k$-vertex-connected spanning subgraph problem. In 2011, Georgiadis [3] improved the running time of this algorithm when
Figure 1: (a) A 2-vertex-twinless connected graph. (b) An optimal solution for the minimum 2-vertex-connected spanning subgraph problem. (c) An optimal solution for the minimum 2-vertex-twinless connected spanning subgraph problem.
$k = 2$ and gave a linear time 3-approximation algorithm for the M2VCS problem. The concept of twinless strongly connected components was first introduced by Raghavan \[11\] in 2006. Raghavan \[11\] showed that twinless strongly connected components of a directed graph can be found in linear time. Georgiadis \[4\] gave a linear time algorithm for testing whether a directed graph is 2-vertex-connected. Italiano et al. \[6, 5\] proved that all the strong articulation points of a directed graph can be calculated in linear time. In 2019, Jaberi \[7, 8, 9\] studied twinless articulation points and some related problems.

In the following section we describe an approximation algorithm for the M2VTCS problem.

2. Approximation algorithm for the M2VTCS Problem

In this section we present an approximation algorithm for the M2VTCS Problem. The following lemma shows a connection between the size of an optimal solution for M2VTCS Problem

**Lemma 2.1.** Let $G = (V, E)$ be a 2-vertex-twinless connected graph. Let $E_{2v} \subseteq E$ be an optimal solution for the M2VCS problem. Then every optimal solution for the M2VTCS problem is also a feasible solution for the M2VCS problem and has size at least $|E_{2v}|$.

**Proof.** Suppose that $E_{2t}$ is an optimal solution for the M2VTCS problem. Then the subgraph $(V, E_{2t})$ does not contain any twinless articulation points. The edge subset $E_{2t}$ is a feasible solution for the M2VCS problem since the subgraph $(V, E_{2t})$ has no strong articulation points. □

If we contract each twinless strongly connected component of a strongly connected graph $G$ into a single supervertex, we obtain a directed graph, called the TSCC component graph of $G$. Let $G = (V, E)$ be a twinless strongly connected graph and let $G^1 = (V, E^1)$ be a strongly connected subgraph of $G$ such that $G^1$ contains at least two twinless strongly connected components. The following lemma shows how to make $G^1$ twinless strongly connected by adding some edges.

**Lemma 2.2.** Let $G = (V, E)$ be a twinless strongly connected graph, and let $G^1 = (V, E^1)$ be a strongly connected subgraph of $G$ such that $G^1$ is not twinless strongly connected. Let $(v, w) \in E \setminus E^1$ such that $v, w$ are not in the same twinless strongly connected component of $G^1$. If $G^1$ contains $k$ twinless strongly connected components, then the number of twinless strongly connected components in the subgraph $(V, E^1 \cup \{(v, w)\})$ is less than $k$. 

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Proof. Since the subgraph $G^1$ is not twinless strongly connected, there are two distinct twinless strongly connected components $C_1, C_2$ of $G^1$ such that $v \in C_1, w \in C_2$. Suppose that $G^1_{tscc} = (V_{tscc}, E_{tscc})$ is the TSCC component graph of the subgraph $G^1$. By [11], Theorem 1, the underlying graph of $G^1_{tscc}$ is a tree and each edge in this tree corresponds to antiparallel edges in $G^1$. Thus, there exists a simple path $p$ from $w$ to $v$ in $G^1$ such that neither $(v, w)$ nor $(w, v)$ belongs to $p$. The path $p$ together with $(v, w)$ forms a simple cycle in $(V, E^1 \cup \{(v, w)\})$. Therefore, the vertices $v$ and $w$ belong to the same twinless strongly connected component in $(V, E^1 \cup \{(v, w)\})$. By [11], Lemma 1, the vertices of $C_1 \cup C_2$ are in the same twinless strongly connected component of the subgraph $(V, E^1 \cup \{(v, w)\})$.

Algorithm 2.3 shows an approximation algorithm for the minimum 2-vertex-twinless connected spanning subgraph problem.

Algorithm 2.3.
Input: A 2-vertex-twinless connected graph $G = (V, E)$
Output: a 2-vertex-twinless connected subgraph $G_{2t} = (V, E_{2t})$
1 calculate a minimal 2-vertex-connected subgraph $G_{2v} = (V, E_{2v})$ of $G$.
2 $E_{2t} \leftarrow E_{2v}$
3 compute the twinless articulation points of $G_{2v}$.
4 for each twinless articulation point $x \in V$ do
5 \hspace{1em} while $G_{2t} \setminus \{x\}$ is not twinless strongly connected do
6 \hspace{2em} identify the twinless strongly connected components of $G_{2t} \setminus \{x\}$
7 \hspace{2em} find an edge $(v, w) \in E \setminus E_{2t}$ such that $v, w$ are in distinct
twinless strongly connected components of $G_{2t} \setminus \{x\}$.
8 \hspace{2em} add the edge $(v, w)$ to $E_{2t}$.

Lemma 2.4. The output of Algorithm 2.3 is 2-vertex-twinless-connected.
Proof. For every vertex $x \in V$, by Lemma 2.2 the while-loop is able to make the subgraph $G_{2t} \setminus \{x\}$ twinless strongly connected since lines 6–9 decrease the number of twinless strongly connected components at least 1.

Theorem 2.5. Algorithm 2.3 achieves an approximation factor of $(2 + a_t/2)$, where $a_t$ is the number of twinless articulation points in the minimal 2-vertex-connected subgraph $G_{2v}$.
Proof. By Lemma 2.1, each optimal solution for the M2VTCS problem has at least $2n$ edges. The number of edges added in Line 9 to $E_{2t}$ is at most $a_t(n - 1)$. Furthermore, the results of Edmonds [2] and Mader [10] imply that $|E_{2v}| \leq 4n$. Therefore, $|E_{2t}| \leq 4n + a_t(n - 1)$.
Theorem 2.6. Algorithm 2.3 runs in $O(n^2m)$ time.

Proof. The twinless strongly connected components of a directed graph can be found in linear time using Raghavan’s algorithm [11]. Jaberi [7] proved that twinless articulation points of a strongly connected graph $G$ can be computed in $O((n-s_a)m)$ time, where $s_a$ is the number of the strong articulation points of $G$. By Lemma 2.2, the number of iterations of the while-loop is at most $n - 1$ for every vertex $x \in V$. Moreover, the number of iterations of the for-loop is at most $n$. □

The running time of Algorithm 2.3 can be improved to $O(n^3)$ using union-find data structure.

3. Open Problems

The results of [2, 10] imply that every minimal 2-vertex-connected spanning subgraph has at most $4n$ edges. An important question is whether the maximum number of every minimal 2-vertex-twinless-connected spanning subgraph is at most $4n$. If every minimal 2-vertex-twinless-connected spanning subgraph has at most $4n$ edges, then there is a 2-approximation algorithm for the M2VTCS problem.

We also leave as open problem whether there is a better approximation algorithm for the M2VTCS problem.

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