Remarks on the large-$N \mathbb{C}P^{N-1}$ model

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In this paper we consider the $\mathbb{C}P^{N-1}$ model confined to an interval of finite size at finite temperature and chemical potential. We compute, in the large-$N$ approximation, the one-loop effective action of the order parameter associated with the effective mass of the quantum fluctuations. To discuss some generic features of the ground state of the model, we work out a mixed-gradient expansion and obtain an expression for the thermodynamic potential density as a functional of the order parameter, generalizing previous calculations to arbitrarily large order and to the case of small finite density. The technique used here relies on analytic regularization and provides an efficient scheme to extract the coefficients of the expansion. These coefficients are then used to deduce some generic properties of the ground state as a function of external conditions. For vanishing chemical potential and intervals of any size, we show that inhomogeneous phases are not energetically favored, but we find evidence that they may become energetically favored for large enough values of the chemical potential. We also show that there can be no transition to a massless phase for any value of the external conditions and clarify a seemingly important point regarding the regularization of the effective action connected to the appearance of logarithmic divergences and to the Mermin-Wagner-Hoenberg-Coleman theorem.
I. INTRODUCTION

The $\mathbb{C}P^{N-1}$ model is $1 + 1$ dimensional ($d = 1$) field theory, consisting of $N$ complex scalar fields $n_i$ ($i = 1, 2, \cdots, N$) with an action of the form

$$S = \int dx dt |\partial_\mu n_i|^2,$$

and obeying a constraint,

$$|n_i|^2 = r.$$  \hspace{1cm} (2)

Original work on the model dates back at least to Refs. [1]-[4] (see Refs. [5, 6] for textbook introductions), but recent years have seen a resurgence of interest in the properties of its ground state when the model is confined to an interval of finite size $\ell$ ($x \in [0, \ell]$) and fluctuations subjected to boundary conditions or other external forcing, as for example temperature variations. Refs. [7, 8] were the first (to the best of our knowledge) to look into questions related to this confined set-up, and since then a renewed interest and a very active debate have resurfaced (see, for example, Refs. [9]-[28]) leading to various, and sometimes contradictory, claims being made. While the above references refer to large-$N$ calculation, the $\mathbb{C}P^{N-1}$ model has also been the subject of extensive lattice simulations [29]-[40] (see in particular Ref. [38] for a lattice study of the $\mathbb{C}P^{N-1}$ model on $S^1 \times S^1$). Issues being currently debated have to do with whether the model can develop a massless ground state for small enough interval size, how the properties of the ground state depend on the external conditions (i.e., size, boundary conditions and temperature), and how everything fits under the umbrella of the large-$N$ approximation.

Our goal here is to re-examine the story and extend the analysis to the case of finite density. We are motivated by two main objectives. The first one is related to the possibility that inhomogeneous phases, even if energetically disfavored at zero density, may become favored above a critical density. This is known to happen for the Gross-Neveu, Nambu-Jona Lasinio and quark-meson models (see, for example, Refs. [41–45]) and it is quite reasonable to expect a similar situation occurring for the $\mathbb{C}P^{N-1}$ model. This may be interesting since it could lead to new features in the geography of the phase diagram of the model (that is the appearance of crossovers into regions characterized by inhomogeneous phases), even for the case of periodic boundary conditions. The second reason is to inspect whether a transition from a massive to a massless phase may or may not occur and whether there is any clear mechanism to exclude the existence of a massless phase (both possibilities have been entertained in the literature with differing conclusions; see Ref. [25] and references given there). Extending the calculation to finite density gives us an excuse to re-consider this debated matter.

The paper is structured as follows. In section II we introduce the main set-up and notation, and illustrate the calculation of the effective action at finite temperature, density, and size using zeta-regularization. This calculation is essentially a repetition of that of Ref. [28] with two major differences: the first being the inclusion of a chemical potential, and the second being a different regularization that allows to capture the infrared behavior of the model and leads to the appearance of a logarithmic contribution. This is an issue of some importance, since it is this term that eventually prevents a massless ground state to be realized and locks the system into a massive phase. Because of this, in section III we will show how the presence of logarithmic contributions can be understood on rather general grounds using zeta-function regularization. This will be done by exploiting the analytic structure of the zeta-function associated with the problem by means of its Mittag-Leffler representation in general dimensionality. In section IV we will discuss the implications of the calculation for the ground state and make some comments on the phase diagram of the model. Some formulae involving polylogarithmic functions used in the computation are given in appendix A.

II. ONE-LOOP EFFECTIVE ACTION AT FINITE CHEMICAL POTENTIAL

Our starting point will be the following general expression for the one-loop effective action at large-$N$ at finite temperature and chemical potential is

$$S_{\text{eff}}^E = \int_0^{\beta/\ell} d\bar{\tau} \int_0^1 d\bar{x} \left\{ (\nabla \sigma)^2 + \ell^2 M^2 (|\sigma|^2 - r) - \ell^2 \mu^2 |\sigma|^2 \right\} + \delta \Gamma,$$

where a coordinate transformation, $x \rightarrow \bar{x} = x/\ell$ and $\tau \rightarrow \bar{\tau} = \tau/\ell$ ($\tau$ is the Wick-rotated Euclidean time and $\beta = 1/T$ in the expression above represents the inverse temperature), has been performed in order to rescale the interval to one of unit length and assumed a background-field configuration to be $n_k = \sigma \times \delta_{1k}$ with $k = 1, 2, \cdots, N$ with $\delta_{1k}$ being the Kronecker

1 Although technically non-trivial, it is obvious to expect, away from periodic boundary conditions, the ground state to become spatially modulated.
delta (see Ref. [23] for details). The rescaled coordinates are dimensionless and the symbol $\bar{V} (= \ell \nabla)$ indicates differentiation with respect to the rescaled coordinate $\bar{x}$. The quantity $\delta \Gamma$ is the one-loop determinant

$$
\delta \Gamma = \frac{(N-1)}{2} \sum_{\pm} \text{Tr} \log \left( -\hat{\Delta} - \frac{\partial^2}{\partial \tau^2} + \ell^2 M^2 - \ell^2 \mu^2 \pm 2 \ell^2 \mu \frac{\partial}{\partial \tau} \right).
$$

(3)

The constraint (2) is incorporated by means of a Lagrange multiplier $M^2$ (as $\delta S/\delta M^2 = 0$) that operates as an effective mass. The quantities $M^2$ and $\sigma$ are assumed to be time-independent, but otherwise general functions of space. The sum over the functional determinant goes over both $\pm$ signs [45-47]. Similarly to Ref. [49], we have introduced a chemical potential $\mu$ associated with the first component of the complex $n_i$, as this is analogue to a chemical potential associated to a $U(1)$ symmetry of a free complex scalar field. More complex configurations (chemical potentials coupled to other or all conserved charges) can be seen as a combination of several elementary configurations as discussed in Ref. [49]. The limits of $\mu \to 0$ and $M \to 0$ recuperate, respectively, known formal expressions (See, for example, Refs. [28, 46-48]).

Using zeta-regularization, we can express the effective action in terms of the zeta-function (see Refs. [51-53] for textbook derivations)

$$
\zeta(s) = \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} \left( p_k^{(s)} + 2 \pi \eta \beta \pm i \ell \mu \right)^2 \left( s \right)^{-(D-d)}
$$

(4)

at $D - d = s$, as

$$
\delta \Gamma = -\zeta_+ (0).
$$

(5)

The (dimensionless) eigenvalues $p_k^{(s)}$ are defined by

$$
\left( \hat{\Delta}_s + \ell^2 M^2 \right) f_k = \left( p_k^{(s)} \right)^2 f_k
$$

and encode the dependence on $M$, $\ell$ and on the boundary conditions. The operator $\hat{\Delta}_s$ is the regularized version of $\hat{\Delta} = \lim_{s \to 0} \hat{\Delta}_s$ (similarly to what is done in dimensional regularization, here we analytically continue the dimensionality, $d \to D = d + s$, and let $s \to 0$ at the end).

The computation of the derivative of the zeta function can be performed in the usual way by utilizing the Mellin transform,

$$
a^{-s} \Gamma(s) = \int_0^\infty t^{s-1} e^{-at} dt,
$$

(6)

to re-express the zeta function [4] in terms of the (integrated) heat-kernel $\mathcal{K}_s(t)$ (defined below) associated to the operator $\left( \hat{\Delta}_s + \ell^2 M^2 \right)$. Simple calculations give

$$
\zeta_+ (s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1-s}} \mathcal{K}_s(t) \sum_{n=-\infty}^{\infty} e^{-\left( \Omega_n^s \right)^2 t},
$$

(7)

where we have defined

$$
\Omega_n^s = 2 \pi n \ell \beta \pm i \ell \mu
$$

(8)

and

$$
\mathcal{K}_s(t) = \sum_{k} e^{-\left( p_k^{(s)} \right)^2 t},
$$

(9)

where $\mathcal{K}_s(t)$ represents the heat-kernel in $D = d + s$ dimensions (In the present case, $d = 1$ and the regularization parameter $s$ is let to zero at the end of the calculations).

The expression of the zeta function can be re-arranged by using the following identity:

$$
\sum_{n=-\infty}^{\infty} e^{-\left( \Omega_n^s \right)^2 t} = \frac{\beta \ell}{\sqrt{4 \pi t}} \theta_3 \left( \pm \frac{i \beta \mu}{2}, e^{\frac{\beta^2}{4 \pi t}} \right),
$$

(10)

where $\theta$ is a Jacobi theta function [54]. Using Eq. (10) in Eq. (7) we get

$$
\zeta_+ (s) = \frac{1}{\Gamma(s)} \frac{\beta \ell}{\sqrt{4 \pi}} \int_0^\infty \frac{dt}{t^{3/2-s}} \mathcal{K}_s(t) \times \theta_3 \left( \pm \frac{i \beta \mu}{2}, e^{\frac{\beta^2}{4 \pi t}} \right).
$$

(11)
To evaluate the derivative of the zeta function and the effective action, we express the integrated heat kernel in terms of the heat-kernel density, \( \mathcal{K}_s(x, t) = \int dx \mathcal{K}_s(x, t) \) and use the following small-\( t \) expansion

\[
\mathcal{K}_s(x, t) = \frac{1}{(4\pi t)^{d/2}} \sum_{k=0}^{\infty} \tilde{a}_k^{(s)} t^k.
\] (12)

The first four coefficients reduce in the limit \( s \to 0 \) to (see, for example, Ref. [50]):

\[
\begin{align*}
\tilde{a}_0^{(0)} &= 1, \\
\tilde{a}_1^{(0)} &= -\ell^2 M^2, \\
\tilde{a}_2^{(0)} &= \frac{1}{2} \ell^4 M^4 - \frac{1}{6} \Lambda (\ell^2 M^2), \\
\tilde{a}_3^{(0)} &= -\frac{1}{6} \ell^6 M^6 + \frac{1}{12} \left( \overline{\nu} (\ell^2 M^2)^2 \right) + \frac{1}{6} \ell^2 M^2 \tilde{\Delta} (\ell^2 M^2) - \frac{1}{60} \tilde{\Delta}^2 (\ell^2 M^2).
\end{align*}
\]

Since we will focus here on the case of periodic boundary conditions, we set all boundary terms to zero (we will comment on the inclusion of boundary terms later). To carry out the integration over \( t \) conveniently, we express the theta function using the following series representation:

\[
\theta_3(x, y) = 1 + 2 \sum_{n=1}^{\infty} \cos(2nx) y^{n^2},
\] (13)

which allows us to write the zeta-function as follows

\[
\zeta_\pm(s) = \frac{1}{\Gamma(s)} \frac{\beta/\ell}{(4\pi)^{(d+1)/2}} \sum_{k=0}^{\infty} \tilde{a}_k^{(s)} \int_{\Lambda^2} \frac{d t}{(d+3-2k-s)/2} \left( 1 + 2 \sum_{n=1}^{\infty} \cosh(\beta \mu n) e^{-2\mu^2 n^2} \right),
\] (14)

with the limit \( \Lambda \to \infty \) understood. It is a good point to remark that in our dimensionless coordinates, the parameter \( \Lambda \) is also dimensionless. Dimension-full quantities can be re-introduced by transforming back to the original coordinate system, as we shall do later. Assuming \( \Re s \) to be sufficiently negative and proceeding by analytical continuation, the integrals over \( t \) can be performed exactly giving

\[
\zeta_\pm(s) = \frac{\beta/\ell}{(4\pi)^{(d+1)/2} \Gamma(s)} \sum_{k=0}^{k_*} \tilde{a}_k^{(s)} \left( -\frac{\Lambda^{-s-2k+d+1}}{s/2 + k - (1 + d)/2} + 2 \Gamma((1 + d)/2 - k - s/2) \left( \frac{\beta}{2\ell} \right)^{s+2k-d-1} \sum_{n=1}^{\infty} \cosh(\pm \beta \mu n) n^{s+2k-d-1} \right),
\]

where the sum over \( k \) extends to finite but arbitrarily large \( k = k_* \). Defining \( z = \beta \mu \) and

\[
\omega_\pm(a) = \sum_{n=1}^{\infty} \cosh(\pm zn) n^{-1+a},
\] (15)

and noticing that any term with \( k \geq 2 \) is regular in the limit \( s \to 0 \) allows us to write in the limit \( d \to 1 \)

\[
\lim_{s \to 0} \frac{d \zeta_\pm}{ds} = \frac{\beta/\ell}{4\pi} \left\{ \left[ \frac{1}{2} \tilde{a}_1^{(0)} \gamma_E + \log \pi + 2 \log \left( \frac{\beta}{\ell} \right) + 2 \log \Lambda^2 - 4 \omega_\pm'(1) \right] \right\} - \lim_{s \to 0} \frac{d \tilde{a}_1^{(s)}}{ds}.
\]

In arriving at the above expression, we have used the following relations (these and the other relations involving the functions \( \omega_\pm \) used here are derived in appendix)

\[
\begin{align*}
\omega_\pm(3) &= \omega_\pm(5) = 0, \\
\omega_\pm(1) &= -1/2,
\end{align*}
\] (16)

and we have defined the quantity (for \( d \to 1 \))

\[
\Omega_0^\pm = \lim_{s \to 0} \frac{d}{ds} \frac{\tilde{a}_0^{(s)}}{(4\pi)^{s/2} \Gamma(s)} \left[ -\frac{\Lambda^{-s+2}}{s/2 - 1} + \Gamma \left( 1 - \frac{s}{2} \right) \left( \frac{\beta}{2\ell} \right)^{-2+s} \omega_\pm(s - 1) \right].
\]
that is a divergent vacuum energy contribution, independent of \(M^2\) in the limit \(s \to 0\). We can now write

\[
\zeta^+_{\ast}(0) + \zeta^-_{\ast}(0) = \frac{\beta \ell^4}{4\pi} \left[ \delta \Omega_0 + \left( a^{(0)}_1 - \gamma_E + \log \pi + 2 \log \left( \frac{\beta}{\ell} \right) + 2 \log \Lambda^2 - 2\omega'(1) \right) \right. \\
\left. - 2 \lim_{s \to 0} \frac{d a^{(s)}_1}{d s} \right] \\
+ \sum_{k=2}^{K} \frac{(-1)^k}{\Gamma(k) 2^{k-3}} \frac{\beta^{2k-2}}{\ell^{2k-2}} \omega' \left( 2k - 1 \right) a^{(0)}_k \right),
\]

where we have defined

\[
\delta \Omega_0 = \Omega_0^\ast + \Omega_0, \\
\omega(z) = \omega_\ast(z) + \omega(z).
\]

The above results can be combined to arrive at the following expression for the one-loop effective action:

\[
S_w^E = \beta \int_0^\ell dx \left\{ (\nabla \sigma)^2 + M^2 \left[ (\nabla \sigma)^2 - r_\ast \right] - \mu^2 \sigma^2 - \frac{(N-1)}{4\pi} \left[ \delta \Omega_0 - \left( \log \left( \frac{\beta^2}{\ell^2} \right) - 2\omega'(1) \right) \right] M^2 \\
- M^2 \log \left( \ell^2 M^2 \right) + \frac{\beta^2}{4} \omega'(3) M^4 + \frac{\beta^4}{16} \omega'(5) \left[ \frac{1}{12} M^6 + \frac{1}{2} \left( \nabla \left( M^2 \right) \right)^2 \right] + \cdots \right\},
\]

after appropriately reabsorbing terms proportional to \(M^2\) with constant coefficients and divergences into a renormalized coupling \(r_\ast\) and after eliminating total derivatives. For \(\mu \to 0\), formulæ

\[
\omega_{\pm}(a) = \zeta_R(1-a) \quad \text{and} \quad \omega(a) = 2 \zeta_R(1-a),
\]

allow to straightforwardly recover the result of Ref. [28], with the exception of the logarithmic contribution \(M^2 \log \left( \ell^2 M^2 \right)\) present here. This term arises from the contribution \(\lim_{s \to 0} d a^{(s)}_1 / d s\) in formula [18] and originates from the regularization of the differential operator in [6] analytically continued from \(d\) to \(D = d + s\) dimensions. In the present case, \(d = 1\), the only nontrivial contribution comes from \(a_{1+d+\beta/2}^\ast\). This term scales as \(a_{1+d+\beta/2} = (\ell M)^{1+d+s}\) that leads, in the effective action in \(d = 1\), to the logarithmic term, \(M^2 \log \left( \ell^2 M^2 \right)\), in [20]. All higher \((k \geq 2)\) order contributions are regular in the limit \(s \to 0\), while the \(k = 0\) contribution is divergent but \(M^2\) independent, thus only resulting in a constant shift in the energy. The logarithmic contribution is quite important in \(1+1\) dimensions since it is a manifestation of the Mermin-Wagner-Hohenberg-Coleman theorem [53, 57] (or, reversing the logic, in the present set-up the restrictions of the theorem follow from this term that encodes an infrared diverging behavior in the \(M \to 0\) limit). This is readily seen once the constraint \(\delta S_w^E / \delta M^2 = 0\) is implemented: the logarithmic correction yields a singularity, as \(\log M^2\), impeding any solution with \(M^2 = 0\) to be realized and thus excluding any massless phase from the spectrum. Once the Lagrange multiplier is integrated out in the path integral, such a zero mode must then be excluded. This conclusion seems to be perfectly in tune with that of Ref. [25] and the additional term is the missing ingredient that brings to an agreement the results of Refs. [25, 28].

## III. LOGARITHMIC CONTRIBUTIONS AND THE MITTAG-LEFFLER REPRESENTATION

The presence of the logarithmic contribution discussed in the preceding section can be understood on rather general grounds and quite easily in zeta-function regularization.

Here, we limit our consideration to a second order differential operator of Laplace type \(D = g^{\mu\nu} \nabla_\mu \nabla_\nu + E\) in \(D = d + \epsilon\) spatial dimensions, where \(\nabla_\mu\) is a suitable covariant derivative and \(E\) is an endomorphism. The covariant derivative may include gauge potentials or connection due to external fields or spacetime curvature, and our consideration below are valid in general. The case considered in the previous section refers to the one-dimensional Laplacian operator with \(E = M^2\). The one-loop effective action \(\Gamma(D)\) can be formally written as [53]

\[
\Gamma = \sum_\lambda \log (\mu^{-2}\lambda),
\]

where the summation over the eigenvalues \(\lambda\) of \(D\) is understood as a regularized sum. In [22] we have assumed that the eigenvalues have \([\text{mass}^2]\) dimension and introduced an arbitrary (renormalization) constant \(\mu\) to keep the argument of the logarithm dimensionless. Introducing the following zeta-function

\[
\zeta (\epsilon|D) = \sum_\lambda (\mu^{-2}\lambda)^{-\epsilon},
\]

allows one to write the above one-loop determinant as follows

\[
\Gamma \sim \lim_{\epsilon \to 0} \zeta (\epsilon|D),
\]
where the limit is understood in the sense of analytical continuation. Now, it is possible to prove that if the operator $\mathcal{D}$ is positive definite, then the zeta function is amenable of an expansion of the form \[ \zeta(\epsilon|\mathcal{D}) = \frac{1}{\Gamma(\epsilon)} \left[ \sum_{p=0}^{\infty} \frac{a_p(\mathcal{D})}{p - (D+1)/2} + \mathcal{F}(\epsilon) \right] \] (25)

known as Mittag-Leffler expansion (the assumption of a strictly positive operator can be relaxed to a non-negative operator with modified coefficients in the numerator of (25)). In the above expression, the quantities $a_p(\mathcal{D})$ are the heat-kernel coefficients associated to the operator $\mathcal{D}$ and $\mathcal{F}(\epsilon)$ is an entire function. In $D + 1$ spacetime dimensions, $a_{(D+1)/2}(\mathcal{D})$ is the heat-kernel coefficient responsible for the divergences and it scales as \[ a_{(D+1)/2}(\mathcal{D}) \sim E^{(D+1)/2} + \cdots, \] (26)

where in flat space and in absence of external gauge potentials the dots denote mixed-derivative terms that vanish in the limit $E$ constant. The term (26) above is sufficient to deal with the present situation of $d = 1$. In higher dimensionality in the presence of curvature of gauge potentials additional terms (not vanishing in the limit of $E \to$ constant) need to be accounted for, but the argument given here does not change. Then, using (23), (25), (26), the presence of the logarithm becomes apparent:

\[ \Gamma \sim E^{(1+\eta)/2} \log \left( E/\mu^2 \right). \] (27)

In the preceding section we have been concerned with the case of $d = 1$, $E = M^2$ and $\mu = \ell^{-1}$, leading precisely to the $M^2 \log \ell^2 M^2$ term appearing in (20). These results have interesting physical implications for the Casimir effect and will be presented elsewhere.

IV. DISCUSSIONS

With the results of the preceding sections in hands, we can examine some of the features of the ground state of the model.

As we have already mentioned, the presence of the logarithm $M^2 \log \ell^2 M^2$ yields a $\log M^2$ divergence once the constraint, $\delta S^\mu/\delta M^2 = 0$, is implemented, impeding the realization of a massless phase. This result is independent of the external conditions, that is a massless ($M^2 = 0$) phase cannot be realized by increasing the density, the temperature or decreasing (or increasing) the size of the interval. This is nothing but the manifestation of the Mermin-Wagner-Hoenberg-Coleman theorem \[ 55-57 \] that becomes evident in the analytic regularization we have used here. Importantly, this also shows that there is no clash between the restrictions of the theorem and the large-$N$ approximation. This is reminiscent of Refs. \[ 60 \].

To inspect whether spatially modulated phases may become energetically favored, as external conditions are varied, we need to compute the dependence of the coefficients $\delta'(1)$, $\delta'(3)$ and $\delta'(5)$ on the temperature and on the chemical potential. This can be easily done either by using formula (A3) derived in appendix, or by brute force numerical computation, starting from the definition of polylogarithmic functions \[ 54 \]. (Using this second approach will result in an imaginary part for the function $\delta'$ due to lack of choice in performing the analytical continuation in our numerical scheme. The imaginary part is then discarded and the real part compared with the result obtained from formula (A9) that gives a real value.). We have carried out the computation in both ways (numerics were carried out with an accuracy of $10^{-7}$) and compared the results that perfectly agreed. Results are shown in figure 1.

In the present case, since no massless phase can be realized, there is no phase transition. Then sign of the coefficient of the $M^2$ term, then simply dictates the order of the potential for small $M^2$. The coefficient of $M^4$ that, in absence of the logarithm would determine the the order of the transition (and a change from second order for $\delta'(3) > 0$ to first order for $\delta'(3) < 0$), here simply controls the concavity of the potential. The coefficient $\delta'(5)$ in the present situation is instead more meaningful since it is the first term in the expansion (20) multiplying a derivative contribution. In the small $z = \mu \beta$ region, the coefficient is positive, signaling that spatially modulated solutions are not energetically favored. However, for $z \gtrsim z_{\text{crit}} \approx 2.05$, $\delta'(5)$ turns negative. This indicates that the derivative term decreases the free energy, making inhomogeneous solutions favored. Naturally, in such a situation, one needs to check that the next order term (i.e. the coefficient of $M^6$) is positive, thus eliminating the possibility of an instability. We have done so and verified that this is the case (the computation of $\delta'(7)$ is straightforward). Of course, in a truncation scheme like the one used here, these results should not be taken as accurate measure of a critical point, but just as an indication that beyond a critical value of the chemical potential, the ground state may become inhomogeneous. This is basically what happens in other models (See Refs. \[ 41-43 \] for examples) and it is reasonable to expect that the same happens here.

Another point worth noticing is the independence of the coefficients of all powers of $M^2$ in the expansion from the size of the interval (with the exception of the logarithms). While this can be explicitly observed from formula (20) for the $M^4$ and $M^6$ coefficients, a proof that extends to all coefficients is worked out very easily from formula (10) and from the scaling of the coefficients $\bar{a}^\mu_k$. This is nothing but an expression of the large-$N$ volume independence for the $C.P^N-1$ model (see \[ 61 \]).
V. CONCLUSIONS

In this paper we have examined a number of issues, recently debated (see refs. [9–28]), on the features of the ground state of the $\mathbb{C}P^{N-1}$ model at finite temperature, (small) density and size. We have worked out an expansion à la Ginzburg-Landau for the effective action as a functional of the Lagrange multiplier $M^2$, that enforces the constraint on the fundamental fields of the model and operates as an effective mass. Assuming $M^2$ to be in principle spatially varying, the coefficients of the expansion easily allow to determine whether there is any phase transition as temperature, density and size vary. Using analytical continuation based on zeta-function regularization, we have been able to show that a logarithmic term of the form $M^2 \log(\ell^2 M^2)$ occurs in the one-loop effective action. This term yields a divergent contribution once the constraint is implemented, preventing the realization of a massless phase in complete agreement with the Mermin-Wagner-Hoenberg-Coleman theorem. To summarize, our calculations indicate:

- the absence of a massless phase for any value of the external conditions (therefore no phase transition towards a massless phase);
- at vanishing density, derivative terms increase the energy of the ground state, therefore inhomogeneous phases are energetically disfavored;
- at finite density, the coefficient of the first derivative term that appears in the GL-like expansion becomes negative beyond a critical point (while the coefficient of the next order term is positive in this region). This suggests that the ground state would develop spatial inhomogeneities for large enough chemical potential.

In all of the above we have taken periodic boundary conditions. In a sense this is the most interesting case, since it is not obvious that the ground state may develop spatial inhomogeneities (contrary to what happens in presence of boundaries, where the ground state naturally becomes inhomogeneous due to the boundary conditions). The boundary part of the action can be easily worked out for the present set-up following a procedure similar to Ref. [28].

In conclusion, we should remark that the scheme presented here is limited by the validity of the derivative expansion (that is in essence an expansion in powers of $\beta/\ell$) and by the series representation of the function $\varphi$ (that is valid for $\beta \mu \leq 2\pi$). It would certainly be desirable to improve the results of this paper in order to be able to make more precise statements and have an accurate estimation of the critical chemical potential. In principle, this could be done by resumming certain classes of terms in the heat-kernel expansion. Another interesting point concerns the interplay between the restrictions resulting from Mermin-Wagner-Hoenberg-Coleman theorem and the Casimir force. We will report on these in a forthcoming work.

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Appendix A: Series representation of the function $\omega(x)$.  

In order to compute the coefficients $\omega'(1)$, $\omega'(3)$ and $\omega'(5)$, we shall start from the following expression

$$f(a;x) = 2 \sum_{n=1}^\infty \cosh(nx) n^{-1+a} = \text{Li}_{1-a}(e^{-x}) + \text{Li}_{1-a}(e^x),$$

(A1)

where $0 \leq |x| < 2\pi$. Using the expression above and tabulated values of polylogarithmic functions [54], it is easy to verify that

$$\omega_\pm(1) = -1/2.$$  

(A2)

Using the following identity

$$\text{Li}_{1-n}(e^{-x}) + (-1)^n \text{Li}_{1-n}(e^x) = 0,$$

(A3)

with $n \in \mathbb{N}$, it follows that

$$\omega_\pm(3) = \omega_\pm(5) = 0.$$  

(A4)

Formulæ (A2) and (A4) are those used in [16] and [17].

To compute the coefficients $\omega'(p)$, we shall adopt the following series representation of the polylogarithmic function:

$$\text{Li}_{1-a}(e^x) = \Gamma(a)(-x)^{-a} + \sum_{k=0}^\infty \frac{1}{k!} \zeta_R(1-a-k)x^k,$$

(A5)

valid for $a \in \mathbb{C}/\mathbb{N}$ and $|x| \in (0,2\pi)$ [62]. In the domain $1-a \leq 0$ with $a \notin \mathbb{N}$, the representation above is an analytic function and the series converge absolutely for all $|x| \leq 2\pi$. For $a \in \mathbb{N}$, it is possible to extend the domain by analytical continuation. The properties of the above series representations have been discussed in various references (see, for example, [63] and the list of references given there). Using the above relation (A5) and defining

$$x_\pm = \pm |x|$$

(A6)

we can easily arrive at

$$\omega(a) = \omega_+(a) + \omega_-(a)$$

$$= 2\Gamma(a)|x|^{-a}(1 + \cos(\pi a)) + 4 \sum_{k=0}^\infty \frac{1}{(2k)!} \zeta_R(1-a-2k)|x|^{2k},$$

(A7)

where we have have analytically continued $\omega_+(a)$ from the top and $\omega_-(a)$ from the bottom. From the above expression is easy to obtain for the coefficients $\omega'(a)$ the following formula:

$$\omega'(a) = -4 \sum_{k=0}^\infty \frac{1}{(2k)!} \zeta_R'(1-a-2k)|x|^{2k}.$$  

(A8)

The above representation is regular and can be compared against a brute force numerical computation carried out using the definition of the polylogarithmic function. Finally, notice that for $z = 0$ (corresponding to $\mu = 0$) we have

$$\omega'(a) = -4\zeta_R'(1-a).$$

[1] A. M. Polyakov, Interaction of Goldstone Particles in Two-Dimensions. Applications to Ferromagnets and Massive Yang-Mills Fields, Phys. Lett. B59 (1975) 79.
[2] A. M. Polyakov and A. A. Belavin, Metastable States of Two-Dimensional Isotropic Ferromagnets, JETP Lett. 22 (1975) 245.
[3] W. A. Bardeen, B. W. Lee and R. E. Shrock, Phase Transition in the Nonlinear Sigma Model in Two + Epsilon Dimensional Continuum, Phys. Rev. D14 (1976) 985.
[4] E. Brezin and J. Zinn-Justin, Spontaneous Breakdown of Continuous Symmetries Near Two-Dimensions, Phys. Rev. B14 (1976) 3110.
(1993) 2483.

[49] F. Bruckmann and T. Sulejmanpasic, *Nonlinear sigma models at nonzero chemical potential: breaking up instantons and the phase diagram*, Phys. Rev. D **90** (2014) no.10, 105010.

[50] F. Bastianelli, O. Corradini, P. A. G. Pisani and C. Schubert, *Scalar heat kernel with boundary in the worldline formalism*, J. High Energ. Phys. **10** (2008) 095.

[51] E. Elizalde, S. D. Odintsov, A. Romeo, A. Bytsenko, and S. Zerbini, *Zeta Regularization Techniques with Applications*, World Scientific, Singapore, (1994).

[52] I.G. Avramidi, *Heat Kernel and Quantum Gravity*, Springer, 2000.

[53] L.E. Parker, D.J. Toms, *Quantum Field Theory in Curved Spacetime*, Cambridge University Press (2009).

[54] M. Abramowitz and I.A. Stegun (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, 1972.

[55] N.D. Mermin, H. Wagner, *Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models*, Phys. Rev. Lett. 17 1133 (1966).

[56] S. Coleman, *There are no Goldstone bosons in two dimensions*, Comm. Math. Phys. **31** 259 (1973).

[57] P.C., Hohenberg, *Existence of Long-Range Order in One and Two Dimensions*, Phys. Rev. **158** 383 (1968).

[58] K. Kirsten, *Spectral Functions in Mathematics and Physics*, CRC Press, Boca Raton, FL, 2001.

[59] P.B. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah-Singer-Index Theorem*, Publish or Perish Inc., Houston, TX, 1984.

[60] E. Witten, *Chiral Symmetry, the 1/N Expansion, and the SU(N) Thirring Model*, Nucl. Phys. B **145** (1978) 110.

[61] T. Sulejmanpasic, *Global Symmetries, Volume Independence, and Continuity in Quantum Field Theories*, Phys. Rev. Lett. **118** (2017) 011601.

[62] John E. Robinson, *Note on the Bose-Einstein Integral Functions*, Phys. Rev. **83** (1951) 678.

[63] G. Fucci, *On the Hurwitz Zeta Function of Imaginary Second Argument*, J. Math. Phys. **52** (2011) 113501.