Monopole Equations on 8-Manifolds with Spin(7) Holonomy

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Abstract
We construct a consistent set of monopole equations on eight-manifolds with Spin(7) holonomy. These equations are elliptic and admit non-trivial solutions including all the 4-dimensional Seiberg-Witten solutions as a special case.
1. Introduction

In a remarkable paper [1] Seiberg and Witten have shown that diffeomorphism invariants of 4-manifolds can be found essentially by counting the number of solutions of a set of massless, Abelian monopole equations [2],[9]. It is later noted that topological quantum field theories which are extensively studied in this context in 2, 3 and 4 dimensions also exist in higher dimensions [4],[5],[6],[7]. Therefore it is of interest to consider monopole equations in higher dimensions and thus generalizing the 4-dimensional Seiberg-Witten theory.

In fact Seiberg-Witten equations can be constructed on any even dimensional manifold (D=2n) with a spin$^c$-structure [8]. But there are problems. The self-duality of 2-forms plays an eminent role in 4-dimensional theory and we encounter projection maps $\rho^+(F_A) = \rho^+(F_A^\perp) = \rho(F_A^\perp)$ (see the next section). The first projection $\rho^+(F_A)$ is meaningful in any dimension $2n \geq 4$. However, a straightforward generalization of the Seiberg-Witten equations using this projection yields an overdetermined set of equations having no non-trivial solutions even locally [9]. To use the other projections, one needs an appropriately generalized notion of self-dual 2-forms. On the other hand there is no unique definition of self-duality in higher than four dimensions. In a previous paper [10] we reviewed the existing definitions of self-duality and gave an eigenvalue criterion for specifying self-dual 2-forms on any even dimensional manifold. In particular, in $D = 8$ dimensions, there is a linear notion of self-duality defined on 8-manifolds with Spin(7) holonomy [11],[12]. This corresponds to a specific choice of a maximal linear subspace in the set of (non-linear) self-dual 2-forms as defined by our eigenvalue criterion [13].

Eight dimensions is special because in this particular case the set of linear Spin(7) self-duality equations can be solved by making use of octonions [14]. The existence of octonionic instantons which realise the last Hopf fibration $S^{15} \to S^8$ is closely related with the properties of the octonion algebra [15],[16],[17].

Here we use this linear notion of self-duality to construct a consistent set of Abelian monopole equations on 8-manifolds with Spin(7) holonomy. These equations turn out to be elliptic and locally they admit non-trivial solutions which include all 4-dimensional Seiberg-Witten solutions as a special case. But before giving our 8-dimensional monopole equations, we first wish in the next section to give the set up and generalizations of 4-dimensional Seiberg-Witten equations to arbitrary even dimensional manifolds with spin$^c$-structure as proposed by Salamon [8]. This is going to help
us put our monopole equations into their proper context. We also wish to note that any 8-manifold with Spin(7) holonomy is automatically a spin manifold \cite{18},\cite{19} and thus carries a spin$^c$-structure; making the application of the general approach possible. In fact our monopole equations can always be expressed purely in the real realm, but in order to relate them to the 4-dimensional Seiberg-Witten equations, it is preferable to use the spin$^c$-structure and complex spinors.

2. Definitions and notation

A spin$^c$-structure on a $2n$-dimensional real inner-product space $V$ is a pair $(W, \Gamma)$, where $W$ is a $2n$-dimensional complex Hermitian space and $\Gamma: V \to \text{End}(W)$ is a linear map satisfying

$$\Gamma(v)^* = -\Gamma(v), \quad \Gamma(v)^2 = -\|v\|^2$$

for $v \in V$. Globalizing this defines the notion of a spin$^c$-structure $\Gamma: TX \to \text{End}(W)$ on a $2n$-dimensional (oriented) manifold $X$, $W$ being a $2n$-dimensional complex Hermitian vector bundle on $X$. Such a structure exists if and only if $w_2(X)$ has an integral lift. $\Gamma$ extends to an isomorphism between the complex Clifford algebra bundle $C^c(TX)$ and $\text{End}(W)$. There is a natural splitting $W = W^+ \oplus W^-$ into the $\pm i^n$ eigenspaces of $\Gamma(e_{2n}e_{2n-1}\cdots e_1)$ where $e_1, e_2, \cdots, e_{2n}$ is any positively oriented local orthonormal frame of $TX$.

The extension of $\Gamma$ to $C_2(X)$ gives, via the identification of $\Lambda^2(T^*X)$ with $C_2(X)$, a map

$$\rho: \Lambda^2(T^*X) \to \text{End}(W)$$

given by

$$\rho\left(\sum_{i<j} \eta_{ij} e_i^* \wedge e_j^*\right) = \sum_{i<j} \eta_{ij} \Gamma(e_i)\Gamma(e_j).$$

The bundles $W^\pm$ are invariant under $\rho(\eta)$ for $\eta \in \Lambda^2(T^*X)$. Denote $\rho^\pm(\eta) = \rho(\eta)|_{W^\pm}$. The map $\rho$ (and $\rho^\pm$) extends to

$$\rho: \Lambda^2(T^*X) \otimes \mathbb{C} \to \text{End}(W).$$

(If $\eta \in \Lambda^2(T^*X) \otimes \mathbb{C}$ is real-valued then $\rho(\eta)$ is skew-Hermitian and if $\eta$ is imaginary-valued then $\rho(\eta)$ is Hermitian.) A Hermitian connection $\nabla$ on $W$ is called a spin$^c$ connection (compatible with the Levi-Civita connection) if

$$\nabla_w(\Gamma(w)\Phi) = \Gamma(w)\nabla_v\Phi + \Gamma(\nabla_v w)\Phi.$$

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where $\Phi$ is a spinor (section of $W$), $v$ and $w$ are vector fields on $X$ and $\nabla_v w$ is the Levi-Civita connection on $X$. $\nabla$ preserves the subbundles $W^\pm$. There is a principal Spin$^c(2n) = \{ e^{i\theta} x | \theta \in \mathbb{R}, x \in Spin(2n) \} \subset C^c(\mathbb{R}^{2n})$ bundle $P$ on $X$ such that $W$ and $TX$ can be recovered as the associated bundles

$$W = P \times_{Spin^c(2n)} \mathbb{C}^{2^n}, \quad TX = P \times_{Ad} \mathbb{R}^{2n},$$

$Ad$ being the adjoint action of Spin$^c(2n)$ on $\mathbb{R}^{2n}$. We get then a complex line bundle $L_\Gamma = P \times \delta \mathbb{C}$ using the map $\delta : Spin^c(2n) \rightarrow S^1$ given by $\delta(e^{i\theta} x) = e^{2i\theta}$.

There is a one-to-one correspondence between spin$^c$ connections on $W$ and spin$^c(2n)=Lie(Spin^c(2n))=spin(2n)\oplus i\mathbb{R}$-valued connection 1-forms $A \in \Omega^1(P,spin^c(2n))$ on $P$. Now consider the trace-part $A$ of $A'$: $A = \frac{1}{2n}trace(A')$. This is an imaginary valued 1-form $A \in \Omega^1(P,i\mathbb{R})$ which is equivariant and satisfies

$$A_p(p \cdot \xi) = \frac{1}{2n}trace(\xi)$$

for $v \in T_p P, g \in Spin^c(2n), \xi \in spin^c(2n)$ (where $p \cdot \xi$ is the infinitesimal action). Denote the set of imaginary valued 1-forms on $P$ satisfying these two properties by $A(\Gamma)$. There is a one-to-one correspondence between these 1-forms and spin$^c$ connections on $W$. Denote the connection corresponding to $A$ by $\nabla_A$. $A(\Gamma)$ is an affine space with parallel vector space $\Omega^1(X,i\mathbb{R})$. For $A \in A(\Gamma)$, the 1-form $2A \in \Omega^1(P,i\mathbb{R})$ represents a connection on the line bundle $L_\Gamma$. Because of this reason $A$ is called a virtual connection on the virtual line bundle $L^{1/2}_\Gamma$. Let $F_A \in \Omega^2(X,i\mathbb{R})$ denote the curvature of the 1-form $A$. Finally, let $D_A$ denote the Dirac operator corresponding to $A \in A(\Gamma)$,

$$D_A : C^\infty(X,W^+) \rightarrow C^\infty(X,W^-)$$

defined by

$$D_A(\Phi) = \sum_{i=1}^{2n} \Gamma(e_i)\nabla_{A,e_i}(\Phi)$$

where $\Phi \in C^\infty(X,W^+)$ and $e_1, e_2, \ldots, e_{2n}$ is any local orthonormal frame.

The Seiberg-Witten equations can now be expressed as follows. Fix a spin$^c$-structure $\Gamma : TX \rightarrow End(W)$ on $X$ and consider the pair $(A, \Phi) \in A(\Gamma) \times C^\infty(X,W^+)$. The Seiberg-Witten equations read

$$D_A(\Phi) = 0 \quad \rho^+(F_A) = (\Phi\Phi^*)_0$$
where \((\Phi\Phi^*)_0 \in C^\infty(X, End(W^+))\) is defined by \((\Phi\Phi^*)_0(\tau) = \langle \Phi, \tau \rangle \Phi\) for \(\tau \in C^\infty(X, W^+)\) and \((\Phi\Phi^*)_0\) is the traceless part of \((\Phi\Phi^*)\).

3. Seiberg-Witten equations on 4-manifolds

Before going over to 8-manifolds, we first show that the Seiberg-Witten equations on 4-manifolds (Ref.[8], p.232) can be rewritten in a different form. The Dirac equation

\[ D_A(\Phi) = 0 \]  

(1)

can be explicitly written as

\[ \nabla_1 \Phi = I \nabla_2 \Phi + J \nabla_3 \Phi + K \nabla_4 \Phi, \]  

(2)

and

\[ \rho^+(F_A) = (\Phi\Phi^*)_0 \]  

(3)

is equivalent to the set

\[ F_{12} + F_{34} = -1/2 \Phi^* I \Phi, \]  

\[ F_{13} - F_{24} = -1/2 \Phi^* J \Phi, \]  

\[ F_{14} + F_{23} = -1/2 \Phi^* K \Phi, \]  

(4)

where \(\Phi : \mathbb{R}^4 \to \mathbb{C}^2\), \(\nabla_i \Phi = \frac{\partial \Phi}{\partial x_i} + A_i \Phi\),
\[ A = \sum_{i=1}^{4} A_i dx_i \in \Omega^1(\mathbb{R}^4, \mathbb{R}), \]  
\[ F_A = \sum_{i<j} F_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbb{R}^4, \mathbb{R}), \]  

and

\[ I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \]

In the most explicit form, these equations can be written as

\[ \frac{\partial \phi_1}{\partial x_1} + A_1 \phi_1 = i(\frac{\partial \phi_1}{\partial x_2} + A_2 \phi_1) + \frac{\partial \phi_2}{\partial x_3} + A_3 \phi_2 + i(\frac{\partial \phi_2}{\partial x_4} + A_4 \phi_2), \]

\[ \frac{\partial \phi_2}{\partial x_1} + A_1 \phi_2 = -i(\frac{\partial \phi_2}{\partial x_2} + A_2 \phi_2) - (\frac{\partial \phi_1}{\partial x_3} + A_3 \phi_1) + i(\frac{\partial \phi_1}{\partial x_4} + A_4 \phi_1) \]  

(5)

(for \(D_A(\Phi) = 0\) and

\[ F_{12} + F_{34} = -i/2(\phi_1 \bar{\phi}_1 - \phi_2 \bar{\phi}_2), \]  

\[ F_{13} - F_{24} = 1/2(\phi_1 \bar{\phi}_2 - \phi_2 \bar{\phi}_1), \]  

\[ F_{14} + F_{23} = -i/2(\phi_1 \bar{\phi}_2 + \phi_2 \bar{\phi}_1) \]  

(6)
We will reinterpret the second part of these equations in the following way: The 6-dimensional bundle of real-valued 2-forms on $\mathbb{R}^4$ has a 3-dimensional subbundle of self-dual forms with orthogonal basis

$$
\begin{align*}
    f_1 &= dx_1 \wedge dx_2 + dx_3 \wedge dx_4, \\
    f_2 &= dx_1 \wedge dx_3 - dx_2 \wedge dx_4, \\
    f_3 &= dx_1 \wedge dx_4 + dx_2 \wedge dx_3,
\end{align*}
$$

in each fiber with respect to the usual metric. These forms span a 3-dimensional complex subbundle of the bundle of complex-valued 2-forms. The projection of a (global) 2-form $F = \sum F_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbb{R}^4, i\mathbb{R})$ onto this complex subbundle is given by

$$
F^+ = 1/2(F_{12} + F_{34}) f_1 + 1/2(F_{13} - F_{24}) f_2 + 1/2(F_{14} + F_{23}) f_3.
$$

We have $\rho^+(f_1) = 2I, \rho^+(f_2) = 2J, \rho^+(f_3) = 2K$, so that,

$$
\rho^+(F^+) = (F_{12} + F_{34}) I + (F_{13} - F_{24}) J + (F_{14} + F_{23}) K.
$$

On the other hand, the orthogonal projection $(\Phi\Phi^*)^+$ of $\Phi\Phi^*$ onto the subbundle of the positive spinor bundle generated by the (Hermitian-) orthogonal basis $(\rho^+(f_1), \rho^+(f_2), \rho^+(f_3))$ is given by

$$
< 2I, \Phi\Phi^* > = I + \frac{1}{2} < J, \Phi\Phi^* > = J + \frac{1}{2} < K, \Phi\Phi^* > = K.
$$

Since

$$
< I, \Phi\Phi^* > = -\Phi^* I \Phi, \quad < J, \Phi\Phi^* > = -\Phi^* J \Phi, \quad < K, \Phi\Phi^* > = -\Phi^* K \Phi,
$$

this shows that the second part of the Seiberg-Witten equations can be expressed as follows: Given any (global, imaginary-valued) 2-form $F$, the image under the map $\rho^+$ of its self-dual part $F^+$ coincides with the orthogonal projection of $\Phi\Phi^*$ onto the subbundle of the positive spinor bundle which is the image bundle of the complexified subbundle of self-dual 2-forms under the map $\rho^+$, that is,

$$
\rho^+(F^+) = (\Phi\Phi^*)^+.
$$

Indeed, in the present case $(\Phi\Phi^*)^+$ is nothing else than $(\Phi\Phi^*)_0$. In this modified form the Seiberg-Witten equations allow a tempting generalisation.
Suppose we are given a subbundle $S \subset \Lambda^2(T^*X)$. Denote the complexification of $S$ by $S^*$, the projection of an imaginary valued 2-form field $F$ onto $S^*$ by $F^+$ and the projection of $\phi \phi^*$ onto $\rho^+(S^*)$ by $(\phi \phi^*)^+$. Then the equation $\rho^+(F^+) = (\phi \phi^*)^+$ can be taken as a substitute of the 4-dimensional equation (3) in 2n-dimensions. An arbitrary choice of $S$ wouldn’t probably give anything interesting, but stable subbundles with respect to certain structures on $X$ are likely to give useful equations.

4. Monopole equations on 8-manifolds

We now consider 8-manifolds with Spin(7) holonomy. In this case there are two natural choices of $S$ which have already found applications in the existing literature. In the 28-dimensional space of 2-forms $\Omega^2(\mathbb{R}^8, \mathbb{R})$, there are two orthogonal subspaces $S_1$ and $S_2$ (7 and 21 dimensional, respectively) which are Spin(7) $\subset$ SO(8) invariant \cite{11,12}. On an 8-manifold $X$ with Spin(7) holonomy (so that the structure group is reducible to Spin(7)) they give rise to global subbundles (denoted by the same letters) $S_1, S_2 \subset \Lambda^2(T^*X)$ which can play the above mentioned role. We will concentrate on the 7-dimensional subbundle $S_1$ and show that the resulting equations are elliptic, exemplify the local existence of non-trivial solutions and show that they are related to solutions of the 4-dimensional Seiberg-Witten equations. We would like to point out that instead of the widely known CDFN 7-plane, we are working with another 7-plane in $\Omega^2(\mathbb{R}^8, \mathbb{R})$, which is conjugated to the CDFN 7-plane and thus invariant under a conjugated Spin(7) embedding in SO(8). This has the advantage that the 2-forms in this 7-plane can be expressed in an elegant way in terms of 4-dimensional self-dual and anti-self-dual 2-forms. (For a general account we refer to our previous work, Ref.\cite{10}.) We will define this 7-plane below, but before that, for the sake of clarity, we first wish to present the global monopole equations. Let $X$ be an 8-manifold with Spin(7) holonomy and $S$ be any stable subbundle of $\Lambda^2(T^*X)$ and $S^*$ its complexification. Given an imaginary valued global 2-form $F$, let us denote its projection onto $S^*$ by $F^+$ and the projection of any global spinor $\phi$ onto the subbundle $\rho^+(S^*) \subset End(W^+)$ by $\phi^+$. Then the monopole equations read

$$D_A(\phi) = 0,$$

$$\rho^+(F_A^+) = (\phi \phi^*)^+.$$
Now, we define $S_1 \subset \Omega^2(\mathbb{R}^8, \mathbb{R})$ to be the linear space of 2-forms

$$\omega = \sum_{i<j} \omega_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbb{R}^8, \mathbb{R}),$$

which can be expressed in matrix form as

$$\omega = \omega_{12} f + \begin{pmatrix} \omega' & \omega'' \\ \omega'' & -\omega' \end{pmatrix},$$  \hspace{1cm} (15)

where $\omega_{12}$ is a real function, $\omega'$ is the matrix of a 4-dimensional self-dual 2-form, $\omega''$ is the matrix of a 4-dimensional anti-self-dual 2-form and we let $f = -J \otimes id_4$. These 2-forms span a 7-dimensional linear subspace $S_1$ in the 28-dimensional space of 2-forms and the square of any element in this subspace is a scalar matrix. $S_1$ is maximal with respect to this property.

We choose the following orhogonal basis for this maximal linear subspace of self-dual 2-forms:

$$f_1 = dx_1 \wedge dx_5 + dx_2 \wedge dx_6 + dx_3 \wedge dx_7 + dx_4 \wedge dx_8,$$

$$f_2 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 - dx_5 \wedge dx_6 - dx_7 \wedge dx_8,$$

$$f_3 = dx_1 \wedge dx_6 - dx_2 \wedge dx_5 + dx_3 \wedge dx_8 + dx_4 \wedge dx_7,$$

$$f_4 = dx_1 \wedge dx_3 - dx_2 \wedge dx_4 - dx_5 \wedge dx_7 + dx_6 \wedge dx_8,$$

$$f_5 = dx_1 \wedge dx_7 + dx_2 \wedge dx_8 - dx_3 \wedge dx_5 - dx_4 \wedge dx_6,$$

$$f_6 = dx_1 \wedge dx_4 + dx_2 \wedge dx_3 - dx_5 \wedge dx_8 - dx_6 \wedge dx_7,$$

$$f_7 = dx_1 \wedge dx_8 - dx_2 \wedge dx_7 + dx_3 \wedge dx_6 - dx_4 \wedge dx_5.$$

(16)

In matrix notation we set $f_1 = f$, and take

$$f_2 = -iI \otimes a_1, \quad f_3 = -iK \otimes b_1,$$

$$f_4 = -iI \otimes a_2, \quad f_5 = -iK \otimes b_2,$$

$$f_6 = -iI \otimes a_3, \quad f_7 = -iK \otimes b_3.$$

(17)

where $(I, J, K)$ are as given as before and we have

$$a_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, a_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, a_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$
and

\[ b_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \]

At this point it will be instructive to show that the above basis corresponds to a representation of the Clifford algebra \( \text{Cl}_7 \) induced by right multiplications in the algebra of octonions. We adopt the Cayley-Dickson approach and describe a quaternion by a pair of complex numbers so that \( a^* = (x + iy) + j(a + iv) \) where \( i, j, ij = k \) are the imaginary unit quaternions. In a similar way an octonion is described by a pair of quaternions \( (a, b) \). Then the octonionic multiplication rule is

\[ (a, b) \cdot (c, d) = (ac - db, da + bc). \] (18)

If we now represent an octonion \( (a, b) \) by a vector in \( \mathbb{R}^8 \), its right multiplication by imaginary unit octonions correspond to linear transformations on \( \mathbb{R}^8 \). We thus obtain the following correspondences:

\[ (0, 1) \rightarrow f_1, (i, 0) \rightarrow f_2, (j, 0) \rightarrow f_3, (k, 0) \rightarrow f_4, \]
\[ (0, i) \rightarrow f_5, (0, j) \rightarrow f_6, (0, k) \rightarrow f_7. \] (19)

The projection \( F^+ \) of a 2-form \( F_\mathbb{R} = \sum_{i<j} F_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbb{R}^8, i\mathbb{R}) \) onto the complexification of the above self-dual subspace is given by

\[
F^+ = \frac{1}{4}(F_{15} + F_{26} + F_{37} + F_{48})f_1 + \frac{1}{4}(F_{12} + F_{34} - F_{56} - F_{78})f_2 + \frac{1}{4}(F_{16} - F_{25} - F_{38} + F_{47})f_3 + \frac{1}{4}(F_{13} - F_{24} - F_{57} + F_{68})f_4 + \frac{1}{4}(F_{17} + F_{28} - F_{35} - F_{46})f_5 + \frac{1}{4}(F_{14} + F_{23} - F_{58} - F_{67})f_6 + \frac{1}{4}(F_{18} - F_{27} + F_{36} - F_{45})f_7. 
\]

We now fix the constant spin\( ^c \)-structure \( \Gamma : \mathbb{R}^8 \rightarrow \mathbb{C}^{16 \times 16} \) given by

\[
\Gamma(e_i) = \begin{pmatrix} 0 & \gamma(e_i) \\ -\gamma(e_i)^* & 0 \end{pmatrix} 
\] (20)
where \( e_i, i = 1, 2, \ldots , 8 \) is the standard basis for \( \mathbb{R}^8 \) and \( \gamma(e_1) = Id, \quad \gamma(e_i) = f_{i-1} \) for \( i = 2, 3, \ldots , 8 \). We note that this choice is specific to 8 dimensions, because \( 2n = 2^{n-1} \) only for \( n = 4 \). We have \( X = \mathbb{R}^8, W = \mathbb{R}^8 \times \mathbb{C}^16, W^\pm = \mathbb{R}^8 \times \mathbb{C}^8 \) and \( L_\Gamma = L_{\Gamma^{1/2}} = \mathbb{R}^8 \times \mathbb{C} \). Consider the connection 1-form

\[
A = \sum_{i=1}^{8} A_i dx_i \in \Omega^1(\mathbb{R}^8, i\mathbb{R})
\]  

(21)
on the line bundle \( \mathbb{R}^8 \times \mathbb{C} \). Its curvature is given by

\[
F_A = \sum_{i<j} F_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbb{R}^8, i\mathbb{R})
\]  

(22)

where \( F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \). The spin connection \( \nabla = \nabla_A \) on \( W^+ \) is given by

\[
\nabla_i \Phi = \frac{\partial \Phi}{\partial x_i} + A_i \Phi
\]  

(23)

\((i = 1, \ldots , 8)\) where \( \Phi : \mathbb{R}^8 \to \mathbb{C}^8 \). Therefore the map

\[
\rho^+ : \Lambda^2(T^*X) \otimes \mathbb{C} \to \text{End}(W^+)
\]

can be computed for our generators \( f_1 \) to give

\[
\rho^+(f_1) = \gamma(e_1)\gamma(e_5) + \gamma(e_2)\gamma(e_6) + \gamma(e_3)\gamma(e_7) + \gamma(e_4)\gamma(e_8)
\]

\[
\rho^+(f_2) = \gamma(e_1)\gamma(e_2) + \gamma(e_3)\gamma(e_4) - \gamma(e_5)\gamma(e_6) - \gamma(e_7)\gamma(e_8)
\]

\[
\rho^+(f_3) = \gamma(e_1)\gamma(e_6) - \gamma(e_2)\gamma(e_5) + \gamma(e_3)\gamma(e_8) + \gamma(e_4)\gamma(e_7)
\]

\[
\rho^+(f_4) = \gamma(e_1)\gamma(e_3) - \gamma(e_2)\gamma(e_4) - \gamma(e_5)\gamma(e_7) + \gamma(e_6)\gamma(e_8)
\]

\[
\rho^+(f_5) = \gamma(e_1)\gamma(e_7) + \gamma(e_2)\gamma(e_8) - \gamma(e_3)\gamma(e_5) - \gamma(e_4)\gamma(e_6)
\]

\[
\rho^+(f_6) = \gamma(e_1)\gamma(e_4) + \gamma(e_2)\gamma(e_3) - \gamma(e_5)\gamma(e_8) - \gamma(e_6)\gamma(e_7)
\]

\[
\rho^+(f_7) = \gamma(e_1)\gamma(e_8) - \gamma(e_2)\gamma(e_7) + \gamma(e_3)\gamma(e_6) - \gamma(e_4)\gamma(e_5).
\]

Then for a connection \( A = \sum_{i=1}^{8} A_i dx_i \in \Omega^1(\mathbb{R}^8, i\mathbb{R}) \) and a given complex 8-spinor \( \Psi = (\psi_1, \psi_2, \ldots, \psi_8) \in C^\infty(X, W^+) = C^\infty(\mathbb{R}^8, \mathbb{R}^8 \times \mathbb{C}^8) \) we state our 8-dimensional monopole equations as follows:

\[
D_A(\Psi) = 0 , \quad \rho^+(F_A^+) = (\Psi \Psi^*)^+.
\]  

(24)

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Here $(\Psi \Psi^*)^\dagger$ is the orthogonal projection of $\Psi \Psi^*$ onto the spinor subbundle spanned by $\rho^+(f_i), i = 1, 2, ..., 7$. More explicitly, $D_A(\Psi) = 0$ can be expressed as

$$\nabla_1 \Psi = \gamma(e_2)\nabla_2 \Psi + \gamma(e_3)\nabla_3 \Psi + ... + \gamma(e_8)\nabla_8 \Psi$$

(25)

and $\rho^+(F_A^+) = (\Psi \Psi^*)^\dagger$ is equivalent to the equation

$$\rho^+(F_A^+) = \sum_{i=2}^8 < \rho^+(f_i), \Psi \Psi^*> \rho^+(f_i)/|\rho^+(f_i)|^2.$$  

(26)

(26) is equivalent to the set of equations

$$F_{15} + F_{26} + F_{37} + F_{48} = 1/8 < \rho^+(f_1), \Psi \Psi^*>,$$

$$F_{12} + F_{34} - F_{56} - F_{78} = 1/8 < \rho^+(f_2), \Psi \Psi^*>,$$

$$F_{16} - F_{25} - F_{38} + F_{47} = 1/8 < \rho^+(f_3), \Psi \Psi^*>,$$

$$F_{13} - F_{24} - F_{57} + F_{68} = 1/8 < \rho^+(f_4), \Psi \Psi^*>,$$

$$F_{17} + F_{28} - F_{35} - F_{46} = 1/8 < \rho^+(f_5), \Psi \Psi^*>,$$

$$F_{14} + F_{23} - F_{58} - F_{67} = 1/8 < \rho^+(f_6), \Psi \Psi^*>,$$

$$F_{18} - F_{27} + F_{36} - F_{45} = 1/8 < \rho^+(f_7), \Psi \Psi^*>.$$  

or still more explicitly to the equations

$$F_{15} + F_{26} + F_{37} + F_{48} = 1/4(\psi_1 \bar{\psi}_3 - \bar{\psi}_3 \psi_1 - \psi_2 \bar{\psi}_4 + \psi_4 \bar{\psi}_2 - \psi_5 \bar{\psi}_5 + \psi_7 \bar{\psi}_7 - \psi_6 \bar{\psi}_8 + \psi_8 \bar{\psi}_6),$$

$$F_{12} + F_{34} - F_{56} - F_{78} = 1/4(\psi_1 \bar{\psi}_5 - \bar{\psi}_5 \psi_1 - \psi_2 \bar{\psi}_6 + \psi_6 \bar{\psi}_2 + \psi_3 \bar{\psi}_3 - \psi_7 \bar{\psi}_7 + \psi_4 \bar{\psi}_4 - \psi_8 \bar{\psi}_4),$$

$$F_{16} - F_{25} - F_{38} + F_{47} = 1/4(\bar{\psi}_1 \psi_7 - \psi_7 \bar{\psi}_1 + \psi_2 \bar{\psi}_8 - \psi_8 \bar{\psi}_2 - \psi_3 \bar{\psi}_3 + \psi_5 \bar{\psi}_5 + \psi_4 \bar{\psi}_4 - \psi_6 \bar{\psi}_6),$$

$$F_{13} - F_{24} - F_{57} + F_{68} = 1/4(\psi_1 \bar{\psi}_2 - \bar{\psi}_2 \psi_1 + \psi_3 \bar{\psi}_4 - \bar{\psi}_4 \psi_3 + \psi_5 \bar{\psi}_6 - \bar{\psi}_6 \psi_5 - \psi_7 \bar{\psi}_7 + \psi_8 \bar{\psi}_8),$$

$$F_{17} + F_{28} - F_{35} - F_{46} = 1/4(\psi_1 \bar{\psi}_4 - \bar{\psi}_4 \psi_1 + \psi_2 \bar{\psi}_3 - \bar{\psi}_3 \psi_2 - \psi_5 \bar{\psi}_8 + \psi_8 \bar{\psi}_5 + \psi_6 \bar{\psi}_7 - \psi_7 \bar{\psi}_6),$$

$$F_{14} + F_{23} - F_{58} - F_{67} = 1/4(-\bar{\psi}_1 \psi_6 + \bar{\psi}_6 \psi_1 - \psi_2 \bar{\psi}_5 + \psi_5 \bar{\psi}_2 - \psi_3 \bar{\psi}_8 + \psi_8 \bar{\psi}_3 + \psi_4 \bar{\psi}_7 - \psi_7 \bar{\psi}_4),$$

$$F_{18} - F_{27} + F_{36} - F_{45} = 1/4(\psi_1 \bar{\psi}_8 - \psi_8 \bar{\psi}_1 - \psi_2 \bar{\psi}_7 + \psi_7 \bar{\psi}_2 - \psi_3 \bar{\psi}_6 + \psi_6 \bar{\psi}_3 - \psi_4 \bar{\psi}_5 + \psi_5 \bar{\psi}_4).$$
5. Conclusion

We will now show that the system of monopole equations (25)-(26) form an elliptic system. These equations can be written compactly in the form

\[ \langle F, f_i \rangle = \frac{1}{8} \langle \rho^+(f_i), \Psi \Psi^* \rangle, \quad i = 1 \ldots 7, \quad D_A(\Psi) = 0. \]

If in addition we impose the Coulomb gauge condition

\[ \sum_{i=1}^{8} \partial_i A_i = 0, \]

we obtain a system of first order partial differential equations consisting of eight equations for the components of the spinor \( \Psi \) and eight equations for the components of the connection 1-form \( A \). The characteristic determinant of this system \(^{[20]}\) is the product of the characteristic determinants of the equations for \( \Psi \) and \( A \). As the Dirac operator is elliptic \(^{[19]}\), the ellipticity of the present system depends on the characteristic determinant of the system consisting of \( \langle F, f_i \rangle = \frac{1}{8} \langle \rho^+(f_i), \Psi \Psi^* \rangle, \quad i = 1 \ldots 7 \) and the Coulomb gauge condition. In the computation of the characteristic determinant, the fifth row, for instance, is obtained from

\[ F_{15} + F_{26} + F_{37} + F_{48} = \partial_1 A_5 - \partial_5 A_1 + \partial_2 A_6 - \partial_6 A_2 + \partial_3 A_7 - \partial_7 A_3 + \partial_4 A_8 - \partial_8 A_4 \]

by replacing \( \partial_i \) by \( \xi_i \). Thus after a rearrangement of the order of the equations, the characteristic determinant can be obtained as

\[
\begin{vmatrix}
\xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_8 \\
-\xi_2 & \xi_1 & -\xi_4 & \xi_3 & -\xi_5 & \xi_8 & -\xi_7 & \\
-\xi_3 & \xi_4 & \xi_1 & -\xi_2 & \xi_7 & -\xi_8 & -\xi_5 & \xi_6 \\
-\xi_4 & -\xi_3 & -\xi_2 & \xi_1 & \xi_8 & \xi_7 & -\xi_6 & -\xi_5 \\
-\xi_5 & -\xi_6 & -\xi_7 & -\xi_8 & \xi_1 & \xi_2 & \xi_3 & \xi_4 \\
-\xi_6 & \xi_5 & \xi_8 & -\xi_7 & -\xi_2 & \xi_1 & \xi_4 & -\xi_3 \\
-\xi_7 & -\xi_8 & \xi_5 & \xi_6 & -\xi_3 & -\xi_4 & \xi_1 & \xi_2 \\
-\xi_8 & \xi_7 & -\xi_6 & \xi_5 & -\xi_4 & \xi_3 & -\xi_2 & \xi_1 \\
\end{vmatrix}
\]

It is equal to

\[(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 + \xi_6^2 + \xi_7^2 + \xi_8^2)^4.\]

and this proves ellipticity.
Finally we point out that the monopole equations (25)-(26) admit non-trivial solutions. For example, if the pair \((A, \Phi)\) with

\[
A = \sum_{i=1}^{4} A_i(x_1, x_2, x_3, x_4) dx_i
\]

and

\[
\Phi = (\phi_1(x_1, x_2, x_3, x_4), \phi_2(x_1, x_2, x_3, x_4))
\]

is a solution of the 4-dimensional Seiberg-Witten equations, then the pair \((B, \Psi)\) with

\[
B = \sum_{i=1}^{4} A_i(x_1, x_2, x_3, x_4) dx_i
\]

(i.e. the first four components \(B_i\) of \(B\) coincide with \(A_i\), thus not depending on \(x_5, x_6, x_7, x_8\) and the last four components of \(B\) vanish) and

\[
\Psi = (0, 0, \phi_1, 0, 0, i\phi_1, -i\phi_2),
\]

where \(\phi_1\) and \(\phi_2\) depend only on \(x_1, x_2, x_3, x_4\), is a solution of these new 8-dimensional monopole equations. It can directly be verified that \(\Psi\) is harmonic with respect to \(B\) and the second part of the equations is also satisfied.
References

[1] N.Seiberg, E.Witten, Nucl.Phys. B426 (1994) 19
[2] E.Witten, Math.Res.Lett. 1 (1994) 764
[3] R.Flume, L.O’Raifeartaigh, I.Sachs, Brief resume of the Seiberg-Witten theory, hep-th/9611118
[4] S.K.Donaldson, R.P.Thomas, Gauge theory in higher dimensions, Oxford University preprint (1996)
[5] L. Baulieu, H. Kanno, I.M. Singer, Special quantum field theories in eight and other dimensions, hep-th/9704167
[6] B.S. Acharya, M.O’Loughlin, B.Spence, Higher dimensional analogues of Donaldson-Witten Theory, hep-th/9705138
[7] C.M.Hull, Higher dimensional Yang-Mills theories and topological terms, hep-th/9710163
[8] D.Salamon, Spin Geometry and Seiberg-Witten Invariants (April 1996 version) (Book to appear).
[9] A.H.Bilge, T.Dereli, Ş.Koçak, Seiberg-Witten equations on $R^8$ in the Proceedings of 5th Gökova Geometry-Topology Conference Edited by S.Akbulut, T.Önder, R, Stern (TUBITAK, Ankara, 1997) p.87
[10] A.H.Bilge, T.Dereli, Ş.Koçak, J.Math.Phys. 38 (1997) 4804
[11] E.Corrigan, C.Devchand, D.Fairlie, J.Nuyts, Nucl.Phys. B214 (1983) 452
[12] R.S.Ward, Nucl.Phys. B236 (1984) 381
[13] A.H.Bilge, T.Dereli, Ş.Koçak, Lett.Math.Phys. 36 (1996) 301
[14] F.Gürsey, C.-H. Tze On the Role of Division, Jordan and Related Algebras in Particle Physics (World Scientific, 1996)
[15] D.Fairlie, J.Nuyts, J. Phys. A17 (1984) 2867
[16] S.Fubini, H.Nicolai, Phys.Lett. B155 (1985) 369
[17] B.Grossman, T.W. Kephart, J.D. Stasheff, Comm. Math. Phys. 96 (1984) 431 (Erratum: ibid, 100 (1985) 311)
[18] D.D. Joyce, Invent.Math. 123 (1996) 507
[19] H.B. Lawson, M.L. Michelsohn Spin Geometry (Princeton U.P., 1989)
[20] F. John, Partial Differential Equations (Springer-Verlag, 1982)