ON A CLASS OF $h$-FOURIER INTEGRAL OPERATORS

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Abstract. In this paper, we study the $L^2$-boundedness and $L^2$-compactness of a class of $h$-Fourier integral operators. These operators are bounded (respectively compact) if the weight of the amplitude is bounded (respectively tends to 0).

1. Introduction

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space), the integral operators

$$F_h \varphi (x) = \int \int e^{i \varphi (S(x,\theta) - y \theta)} a(x,\theta) \varphi (y) \, dy \, d\theta$$

appear naturally in the expression of the solutions of the semiclassical hyperbolic partial differential equations and in the expression of the $C^\infty$-solution of the associate Cauchy’s problem. Which appear two $C^\infty$-functions, the phase function $\phi (x,y,\theta) = S(x,\theta) - y \theta$ and the amplitude $a$.

Since 1970, many efforts have been made by several authors in order to study these type of operators (see, e.g.,[2, 7, 8, 5, 9]). The first works on Fourier integral operators deal with local properties. On the other hand, K. Asada and D. Fujiwara [2] have studied for the first time a class of Fourier integral operators defined on $\mathbb{R}^n$.

For the $h$-Fourier integral operators, an interesting question is under which conditions on $a$ and $S$ these operators are bounded on $L^2$ or are compact on $L^2$.

It has been proved in [2] by a very elaborated proof and with some hypothesis on the phase function $\phi$ and the amplitude $a$ that all operators of the form:

$$(I (a,\phi) \varphi) (x) = \int \int e^{i \phi (x,\theta,y)} a(x,\theta,y) \varphi (y) \, dy \, d\theta$$

are bounded on $L^2$ where, $x \in \mathbb{R}^n$, $n \in \mathbb{N}^\ast$ and $N \in \mathbb{N}$ (if $N = 0$, $\theta$ doesn’t appear in (1.2)). The technique used there is based on the fact that the operators $I (a,\phi)^\ast I (a,\phi)$ and $I (a,\phi)^\ast I (a,\phi)$ are pseudodifferential and it uses Cañéron-Vaillancourt’s theorem (here $I (a,\phi)^\ast$ is the adjoint of $I (a,\phi)$).

In this work, we apply the same technique of [2] to establish the boundedness and the compactness of the operators (1.1). To this end we give a brief and simple proof for a result of [2] in our framework.

We mainly prove the continuity of the operator $F_h$ on $L^2 (\mathbb{R}^n)$ when the weight of the amplitude $a$ is bounded. Moreover, $F_h$ is compact on $L^2 (\mathbb{R}^n)$ if this weight tends
Theorem 2.1. If $\phi$ satisfies (H1), (H2), (H3) and (H3*), and if $a \in \Gamma_{\mu}$, then

1. For all $\varphi \in S(\mathbb{R}^n)$, $\lim_{\sigma \to +\infty} I(a_{\sigma}, \phi; h)\varphi(x)$ exists for every point $x \in \mathbb{R}^n$ and is independent of the choice of the function $g$. We define

$$I(a, \phi; h)\varphi(x) := \lim_{\sigma \to +\infty} I(a_{\sigma}, \phi; h)\varphi(x)$$

2. A general class of $h$-Fourier integral operators

If $\varphi \in S(\mathbb{R}^n)$, we consider the following transformations

$$I(a, \phi; h)\varphi(x) = \int_{\mathbb{R}^n_x \times \mathbb{R}^n_y} e^{i\phi(x, \theta, y)} a(x, \theta, y)\varphi(y) dy \, d\theta$$

where, $x \in \mathbb{R}^n$, $n \in \mathbb{N}^*$ and $N \in \mathbb{N}$ (if $N = 0$, $\theta$ doesn’t appear in (2.3)).

In general the integral (2.3) is not absolutely convergent, so we use the technique of the oscillatory integral developed by Hörmander. The phase function $\phi$ and the amplitude $a$ are assumed to satisfy the following hypothesis:

(H1) $\phi \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y, \mathbb{R})$ ($\phi$ is a real function)

(H2) For all $(\alpha, \beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n$, there exists $C_{\alpha, \beta, \gamma} > 0$

$$|\partial_y^\alpha \partial_\theta^\beta \partial_x^\gamma \phi(x, \theta, y)| \leq C_{\alpha, \beta, \gamma} \lambda^{2(|\alpha| + |\beta| + |\gamma|)}(x, \theta, y)$$

where $\lambda(x, \theta, y) = (1 + |x|^2 + |\theta|^2 + |y|^2)^{1/2}$ called the weight and

$$(2 - |\alpha| - |\beta| - |\gamma|) = \max(2 - |\alpha| - |\beta| - |\gamma|, 0)$$

(H3) There exist $K_1, K_2 > 0$ such that $\forall (x, \theta, y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$

$$K_1 \lambda(x, \theta, y) \leq \lambda(\partial_y \phi, \partial_\theta \phi, y) \leq K_2 \lambda(x, \theta, y)$$

(H3*) There exist $K_1^*, K_2^* > 0$ such that $\forall (x, \theta, y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$

$$K_1^* \lambda(x, \theta, y) \leq \lambda(\partial_y \phi, \partial_\theta \phi, \partial_x \phi) \leq K_2^* \lambda(x, \theta, y)$$

For any open subset $\Omega$ of $\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$, $\mu \in \mathbb{R}$ and $\rho \in [0, 1]$, we set

$$\Gamma_{\rho}^{\mu}(\Omega) = \left\{ a \in C^\infty(\Omega) : \forall (\alpha, \beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n, \exists C_{\alpha, \beta, \gamma} > 0 : |\partial_y^\alpha \partial_\theta^\beta \partial_x^\gamma a(x, \theta, y)| \leq C_{\alpha, \beta, \gamma} \lambda^{\mu - \rho(|\alpha| + |\beta| + |\gamma|)}(x, \theta, y) \right\}$$

When $\Omega = \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$, we denote $\Gamma_{\rho}^{\mu}(\Omega) = \Gamma_{\rho}^{\mu}$.

To give a meaning to the right hand side of (2.3), we consider $g \in S(\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y)$, $g(0) = 1$. If $a \in \Gamma_{\rho}^{\mu}$, we define

$$a_{\sigma}(x, \theta, y) = g(x/\sigma, \theta/\sigma, y/\sigma)a(x, \theta, y), \quad \sigma > 0.$$
2. \( I(a, \phi; h) \in \mathcal{L}(S(\mathbb{R}^n)) \) and \( I(a, \phi; h) \in \mathcal{L}(S'(\mathbb{R}^n)) \) (here \( \mathcal{L}(E) \) is the space of bounded linear mapping from \( E \) to \( E \) and \( S'(\mathbb{R}^n) \) the space of all distributions with temperate growth on \( \mathbb{R}^n \)).

Proof. see \cite{8} or \cite[Proposition II.2]{12}.

Example 2.2. Let’s give two examples of operators of the form \( \mathcal{F} \) which satisfy \((H1)\) to \((H3)^*\):

1. The Fourier transform \( \mathcal{F}(\psi)(x) = \int e^{-ixy} \psi(y) \, dy, \psi \in \mathcal{S}(\mathbb{R}^n) \),
2. Pseudodifferential operators \( \mathcal{A}\psi(x) = (2\pi)^{-n} \int e^{i(x-y)\theta} a(x, y, \theta) \psi(y) \, dy \, d\theta, \psi \in \mathcal{S}(\mathbb{R}^n), a \in \Gamma^0_0(\mathbb{R}^{3n}) \).

3. Assumptions and Preliminaries

We consider the special form of the phase function

\[
\phi(x, y, \theta) = S(x, \theta) - y\theta
\]

where \( S \) satisfies

(G1) \( S \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_\theta^n, \mathbb{R}) \),
(G2) For each \((\alpha, \beta) \in \mathbb{N}^{2n}\), there exist \( C_{\alpha,\beta} > 0 \), such that

\[
|\partial_x^\alpha \partial_\theta^\beta S(x, \theta)| \leq C_{\alpha,\beta} \lambda(x, \theta)^{2-|\alpha|-|\beta|},
\]

(G3) There exists \( \delta_0 > 0 \) such that

\[
\inf_{x,\theta \in \mathbb{R}^n} |\partial^2 S| \geq \delta_0.
\]

Lemma 3.1 (H1). Let’s assume that \( S \) satisfies \((G1), (G2), (G3)\). Then the function \( \phi(x, y, \theta) = S(x, \theta) - y\theta \) satisfies \((H1), (H2), (H3), \) and \((H3^*)\).

Lemma 3.2 (H1). If \( S \) satisfies \((G1), (G2) \) and \((G3)\), then there exists \( C_2 > 0 \) such that for all \((x, \theta), (x', \theta') \in \mathbb{R}^{2n}\),

\[
|x - x'| + |\theta - \theta'| \leq C_2 \left( |(\partial_\theta S)(x, \theta) - (\partial_\theta S)(x', \theta')| + |\theta - \theta'| \right)
\]

where \( \theta = \theta' \) in \((3.5)\), there exists \( C_3 > 0 \), such that for all \((x, x', \theta) \in \mathbb{R}^{3n}\),

\[
|x - x'| \leq C_3 |(\partial_\theta S)(x, \theta) - (\partial_\theta S)(x', \theta)|.
\]

Proposition 3.3. If \( S \) satisfies \((G1)\) and \((G2)\), then there exists a constant \( \epsilon_0 > 0 \) such that the phase function \( \phi \) given in \((3.4)\) belongs to \( \Gamma_0^1(\Omega_{\phi,\epsilon_0}) \) where

\[
\Omega_{\phi,\epsilon_0} = \{(x, \theta, y) \in \mathbb{R}^{3n}; |\partial_\theta S(x, \theta) - y|^2 < \epsilon_0 (|x|^2 + |y|^2 + |\theta|^2) \}.
\]

Proof. We have to show that: \( \exists \epsilon_0 > 0, \forall \alpha, \beta, \gamma \in \mathbb{N}^n, \forall C_{\alpha,\beta,\gamma} > 0; \)

\[
|\partial_x^\alpha \partial_\theta^\beta \partial_\gamma \phi(x, \theta, y)| \leq C_{\alpha,\beta,\gamma} \lambda(x, \theta, y)^{2-|\alpha|-|\beta|-|\gamma|}, \forall (x, \theta, y) \in \Omega_{\phi,\epsilon_0}.
\]

- If \( |\gamma| = 1 \), then \( |\partial_x^\alpha \partial_\theta^\beta \partial_\gamma \phi(x, \theta, y)| = \left| \partial_x^\alpha \partial_\theta^\beta (-\theta) \right| = \left\{ \begin{array}{ll} 0 & \text{if } |\alpha| \neq 0 \\ \partial_\theta^\beta (-\theta) & \text{if } \alpha = 0 \end{array} \right\} \);
- If \( |\gamma| > 1 \), then \( |\partial_x^\alpha \partial_\theta^\beta \partial_\gamma \phi(x, \theta, y)| = 0.\)
Hence the estimate (3.7) is satisfied.
If \(|\gamma| = 0\), then \(\forall \alpha, \beta \in \mathbb{N}^n; |\alpha| + |\beta| \leq 2, \exists C_{\alpha, \beta} > 0;\)
\[
|\partial_x^\alpha \partial_\theta^\beta \phi(x, \theta, y)| = |\partial_x^\alpha \partial_\theta^\beta S(x, \theta) - \partial_x^\alpha \partial_\theta^\beta (y)\theta| \leq C_{\alpha, \beta} \lambda(x, \theta, y)^{(2 - |\alpha| - |\beta|)}.
\]
If \(|\alpha| + |\beta| > 2\), one has \(\partial_x^\alpha \partial_\theta^\beta \phi(x, \theta, y) = \partial_x^\alpha \partial_\theta^\beta S(x, \theta)\). In \(\Omega_{\phi, \varepsilon, 0}\) we have
\[
|y| = |\partial_\theta S(x, \theta) - y - \partial_\theta S(x, \theta)| \leq \varepsilon \sqrt{2 \left(|x|^2 + |y|^2 + |\theta|^2 \right)} + C_3 \lambda(x, \theta), \quad C_3 > 0.
\]
For \(\varepsilon_0\) sufficiently small, we obtain a constant \(C_4 > 0\) such that
\[
(3.8) \quad |y| \leq C_4 \lambda(x, \theta), \quad \forall (x, \theta, y) \in \Omega_{\phi, \varepsilon, 0}.
\]
This inequality leads to the equivalence
\[
(3.9) \quad \lambda(x, \theta, y) \simeq \lambda(x, \theta) \quad \text{in} \quad \Omega_{\phi, \varepsilon_0}
\]
thus the assumption (G2) and (3.9) give the estimate (3.7).

Using (3.9), we have the following result.

**Proposition 3.4**. If \((x, \theta) \rightarrow a(x, \theta)\) belongs to \(\Gamma_k^m(\mathbb{R}^n \times \mathbb{R}^n)\), then \((x, \theta, y) \rightarrow a(x, \theta)\) belongs to \(\Gamma_k^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \cap \Gamma_k^m(\Omega_{\phi, \varepsilon, 0})\), \(k \in \{0, 1\}\).

### 4. \(L^2\)-boundedness and \(L^2\)-compactness of \(F_h\)

**Theorem 4.1.** Let \(F_h\) be the integral operator of distribution kernel
\[
K(x, y; h) = \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x, \theta, y) - y\theta)} a(x, \theta) d\theta d\theta
\]
where \(d\theta = (2\pi h)^{-n} d\theta\), \(a \in \Gamma_k^m(\mathbb{R}^n \times \mathbb{R}^n)\), \(k = 0, 1\) and \(S\) satisfies (G1), (G2) and (G3). Then \(F_h F_h^*\) and \(F_k^* F_k\) are \(h\)-pseudodifferential operators with symbol in \(\Gamma_k^m(\mathbb{R}^{2n})\), \(k = 0, 1\), given by
\[
\sigma(F_h F_h^*)(x, \partial_x S(x, \theta)) \equiv |a(x, \theta)|^2 |(\det \frac{\partial^2 S}{\partial \theta^2})^{-1}(x, \theta)|
\]
\[
\sigma(F_k^* F_k)(\partial_\theta S(x, \theta), \theta) \equiv |a(x, \theta)|^2 |(\det \frac{\partial^2 S}{\partial \theta^2})^{-1}(x, \theta)|
\]
we denote here \(a \equiv b\) for \(a, b \in \Gamma_k^m(\mathbb{R}^n)\) if \((a - b) \in \Gamma_k^{2p-2}(\mathbb{R}^n)\) and \(\sigma\) stands for the symbol.

**Proof.** For all \(v \in \mathcal{S}(\mathbb{R}^n)\), we have:
\[
(F_h F_h^* v)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x, \theta, y) - S(x, \theta))} a(x, \theta) \overline{\mathcal{F}(\xi)(x, \theta)} d\theta.
\]

The main idea to show that \(F_h F_h^*\) is a \(h\)-pseudodifferential operator, is to use the fact that \((S(x, \theta, y) - S(x, \theta))\) can be expressed by the scalar product \((x - \tilde{x}, \xi(x, \tilde{x}, \theta))\) after considering the change of variables \((x, \theta) \rightarrow (x + \xi(x, \tilde{x}, \theta))\). The distribution kernel of \(F_h F_h^*\) is
\[
K(x, \tilde{x}; h) = \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x, \theta, \tilde{x}) - S(x, \theta))} a(x, \theta) \overline{\mathcal{F}(\xi)(x, \theta)} d\theta.
\]
We obtain from (3.6) that if
\[
|x - \tilde{x}| \geq \frac{\varepsilon}{2} \lambda(x, \tilde{x}, \theta) \quad \text{(where \(\varepsilon > 0\) is sufficiently small)}
\]
Choosing $\omega \in C^\infty(\mathbb{R})$ such that

$$\omega(x) \geq 0, \quad \forall x \in \mathbb{R}$$

$$\omega(x) = 1 \quad \text{if} \quad x \in [-\frac{1}{2}, \frac{1}{2}]$$

and setting

$$b(x, \tilde{x}, \theta) := a(x, \theta) \overline{\pi(x, \theta)} = b_{1,\epsilon}(x, \tilde{x}, \theta) + b_{2,\epsilon}(x, \tilde{x}, \theta)$$

$$b_{1,\epsilon}(x, \tilde{x}, \theta) = \frac{|x - \tilde{x}|}{\epsilon \lambda(x, \tilde{x}, \theta)} b(x, \tilde{x}, \theta)$$

$$b_{2,\epsilon}(x, \tilde{x}, \theta) = [1 - \omega(\frac{|x - \tilde{x}|}{\epsilon \lambda(x, \tilde{x}, \theta)})] b(x, \tilde{x}, \theta).$$

We have $K(x, \tilde{x}; h) = K_{1,\epsilon}(x, \tilde{x}; h) + K_{2,\epsilon}(x, \tilde{x}; h)$, where

$$K_{j,\epsilon}(x, \tilde{x}; h) = \int_{\mathbb{R}^n} e^{\mp(S(x, \theta) - S(\tilde{x}, \theta))} b_{j,\epsilon}(x, \tilde{x}, \theta) \overline{a_h \theta}, \quad j = 1, 2.$$

We will study separately the kernels $K_{1,\epsilon}$ and $K_{2,\epsilon}$.

Proof. For all $h$, we have

$$K_{2,\epsilon}(x, \tilde{x}; h) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n).$$

Indeed, using the oscillatory integral method, there is a linear partial differential operator $L$ of order 1 such that

$$L(e^{\pm(S(x, \theta) - S(\tilde{x}, \theta))}) = e^{\pm(S(x, \theta) - S(\tilde{x}, \theta))}$$

where $L = -ih \left| (\partial_\theta S)(x, \theta) - (\partial_\theta S)(\tilde{x}, \theta) \right|^{-2} \sum_{i=1}^{n} \left[ (\partial_{\theta_i} S)(x, \theta) - (\partial_{\theta_i} S)(\tilde{x}, \theta) \right] \partial_{\theta_i}.$

The transpose operator of $L$ is

$${}^t L = \sum_{l=1}^{n} F_l(x, \tilde{x}, \theta; h) \partial_{\theta_l} + G(x, \tilde{x}, \theta; h)$$

where $F_l(x, \tilde{x}, \theta) \in \Gamma_{0}^{-1}(\Omega_{\epsilon}), \; G(x, \tilde{x}, \theta) \in \Gamma_{0}^{-2}(\Omega_{\epsilon})$

$$\left\{ \begin{array}{l}
F_l(x, \tilde{x}, \theta; h) = ih \left| (\partial_\theta S)(x, \theta) - (\partial_\theta S)(\tilde{x}, \theta) \right|^{-2} \sum_{i=1}^{n} \left[ (\partial_{\theta_i} S)(x, \theta) - (\partial_{\theta_i} S)(\tilde{x}, \theta) \right] \partial_{\theta_i} \\
G(x, \tilde{x}, \theta; h) = ih \sum_{l=1}^{n} \partial_{\theta_l} \left[ (\partial_\theta S)(x, \theta) - (\partial_\theta S)(\tilde{x}, \theta) \right]^{-2} \sum_{i=1}^{n} \left[ (\partial_{\theta_i} S)(x, \theta) - (\partial_{\theta_i} S)(\tilde{x}, \theta) \right] \partial_{\theta_i}
\end{array} \right.$$

$$\Omega_{\epsilon} = \left\{ (x, \tilde{x}, \theta) \in \mathbb{R}^{3n}; \; \left| (\partial_\theta S)(x, \theta) - (\partial_\theta S)(\tilde{x}, \theta) \right| > \frac{\epsilon}{\lambda_{2}} \lambda(x, \tilde{x}, \theta) \right\} .$$

On the other hand we prove by induction on $q$ that

$$({}^t L)^q b_{2,\epsilon}(x, \tilde{x}, \theta) = \sum_{\sum |\gamma| \leq q} g_{\gamma, q}(x, \tilde{x}, \theta) \partial_{\theta}^\gamma b_{2,\epsilon}(x, \tilde{x}, \theta), \; g_{\gamma, q}^{(q)} \in \Gamma_{0}^{-q}(\Omega_{\epsilon}),$$
and so,
\[
K_{2, \varepsilon}(x, \bar{x}) = \int_{\mathbb{R}^n} e^{i \Phi(S(x, \theta) - S(\bar{x}, \theta)) (t^L)^q} b_{2, \varepsilon}(x, \bar{x}, \theta) \, d\theta.
\]

Using Leibnitz’s formula, (G2) and the form \((t^L)^q\), we can choose \(q\) large enough such that
\[
\forall \alpha, \alpha', \beta, \beta' \in \mathbb{N}^n, \exists C_{\alpha, \alpha', \beta, \beta'} > 0, \sup_{x, \bar{x} \in \mathbb{R}^n} \left| x^\alpha \bar{x}^{\alpha'} \partial_x^\beta \partial_{\bar{x}}^\beta' K_{2, \varepsilon}(x, \bar{x}; h) \right| \leq C_{\alpha, \alpha', \beta, \beta'}.
\]

Next, we study \(K_{1, \varepsilon}^1\): this is more difficult and depends on the choice of the parameter \(\varepsilon\). It follows from Taylor’s formula that
\[
S(x, \theta) - S(\bar{x}, \theta) = \langle x - \bar{x}, \xi(x, \bar{x}, \theta) \rangle_{\mathbb{R}^n},
\]
\[
\xi(x, \bar{x}, \theta) = \int_0^1 (\partial_x S)(\bar{x} + t(x - \bar{x}), \theta) \, dt.
\]

We define the vectorial function
\[
\tilde{\xi}_\varepsilon(x, \bar{x}, \theta) = \omega \left( \frac{|x - \bar{x}|}{2 \varepsilon \lambda(x, \bar{x}, \theta)} \right) \xi(x, \bar{x}, \theta) + \left( 1 - \omega \left( \frac{|x - \bar{x}|}{2 \varepsilon \lambda(x, \bar{x}, \theta)} \right) \right) (\partial_x S)(\bar{x}, \theta).
\]

We have
\[
\tilde{\xi}_\varepsilon(x, \bar{x}, \theta) = \xi(x, \bar{x}, \theta) \text{ on supp } b_{1, \varepsilon}.
\]

Moreover, for \(\varepsilon\) sufficiently small,
\[
(4.13) \quad \lambda(x, \theta) \simeq \lambda(\bar{x}, \theta) \simeq \lambda(x, \bar{x}, \theta) \text{ on supp } b_{1, \varepsilon}.
\]

Let us consider the mapping
\[
(4.14) \quad \mathbb{R}^{3n} \ni (x, \bar{x}, \theta) \mapsto \left( x, \bar{x}, \tilde{\xi}_\varepsilon(x, \bar{x}, \theta) \right)
\]
for which Jacobian matrix is
\[
\begin{pmatrix}
I_n \\
0 \\
\partial_x \tilde{\xi}_\varepsilon & \partial_{\bar{x}} \tilde{\xi}_\varepsilon & \partial_{\theta} \tilde{\xi}_\varepsilon
\end{pmatrix}.
\]

We have
\[
\frac{\partial \tilde{\xi}_{i,j}}{\partial \theta_i}(x, \bar{x}, \theta) = \frac{\partial^2 S}{\partial \theta_i \partial x_j}(x, \theta) + \omega \left( \frac{|x - \bar{x}|}{2 \varepsilon \lambda(x, \bar{x}, \theta)} \right) \frac{\partial \xi_j}{\partial \theta_i}(x, \bar{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(x, \theta) + \frac{|x - \bar{x}|}{2 \varepsilon \lambda(x, \bar{x}, \theta)} \lambda^{-1}(x, \bar{x}, \theta) \omega' \left( \frac{|x - \bar{x}|}{2 \varepsilon \lambda(x, \bar{x}, \theta)} \right) \left( \xi_j(x, \bar{x}, \theta) - \frac{\partial S}{\partial x_j}(\bar{x}, \theta) \right).
\]

Thus, we obtain
\[
\left| \frac{\partial \tilde{\xi}_{i,j}}{\partial \theta_i}(x, \bar{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(x, \theta) \right| \leq \omega \left( \frac{|x - \bar{x}|}{2 \varepsilon \lambda(x, \bar{x}, \theta)} \right) \left| \frac{\partial \xi_j}{\partial \theta_i}(x, \bar{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(x, \theta) \right| + \lambda^{-1}(x, \bar{x}, \theta) \omega' \left( \frac{|x - \bar{x}|}{2 \varepsilon \lambda(x, \bar{x}, \theta)} \right) \left| \xi_j(x, \bar{x}, \theta) - \frac{\partial S}{\partial x_j}(\bar{x}, \theta) \right|.
\]

Now it follows from \((G2)\), \((4.13)\) and Taylor’s formula that
\[
\left| \frac{\partial \xi_j}{\partial \theta_i}(x, \bar{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(x, \bar{x}, \theta) \right| \leq \int_0^1 \left| \frac{\partial^2 S}{\partial \theta_i \partial x_j}(x + t(x - \bar{x}), \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\bar{x}, \theta) \right| \, dt
\]
\[
(4.15) \quad \leq C_5 \lambda^{-1}(x, \bar{x}, \theta), \quad C_5 > 0
\]
\[
\left| \xi_j(x, \bar{x}, \theta) - \frac{\partial S}{\partial x_j}(\bar{x}, \theta) \right| \leq \frac{1}{\epsilon} \left| \int_0^1 \frac{\partial S}{\partial x_j}(\bar{x} + t(x - \bar{x}), \theta) - \frac{\partial S}{\partial x_j}(\bar{x}, \theta) \right| dt \leq C_0 |x - \bar{x}|, \quad C_0 > 0.
\]

From (4.15) and (4.16), there exists a positive constant \( C_7 > 0 \), such that

\[
|\frac{\partial \xi_j}{\partial \theta_i}(x, \bar{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\bar{x}, \theta)| \leq C_7 \epsilon, \quad \forall i, j \in \{1, \ldots, n\}.
\]

If \( \epsilon < \frac{\delta}{2C} \), then (4.17) and (G3) yields the estimate

\[
\delta_0/2 \leq -C \epsilon + \delta_0 \leq -C \epsilon + \text{det} \left( \frac{\partial^2 S}{\partial x \partial \theta}(x, \theta) \right) \leq \text{det} \left( \frac{\partial \xi_j}{\partial \theta_i}(x, \bar{x}, \theta) \right), \quad \text{with} \ C > 0.
\]

If \( \epsilon \) is such that (4.19) and (4.18) are true, then the mapping given in (4.14) is a global diffeomorphism of \( \mathbb{R}^{3n} \). Hence there exists a mapping

\[
\theta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ni (x, \bar{x}, \xi) \rightarrow \hat{\theta}(x, \bar{x}, \xi) \in \mathbb{R}^n
\]

such that

\[
\begin{align*}
\tilde{\xi}_c(x, \bar{x}, \theta(x, \bar{x}, \xi)) &= \xi \\
\theta(x, \bar{x}, \xi) &= x \\
\hat{\partial}^a \hat{\theta}(x, \bar{x}, \xi) &= \mathcal{O}(1), \quad \forall \alpha \in \mathbb{N}^{3n} \setminus \{0\}
\end{align*}
\]

If we change the variable \( \xi \) by \( \theta(x, \bar{x}, \xi) \) in \( K_{1,\epsilon}(x, \bar{x}) \), we obtain:

\[
K_{1,\epsilon}(x, \bar{x}) = \int_{\mathbb{R}^n} e^{i<x-\bar{x},\xi>} b_{1,\epsilon}(x, \hat{x}, \theta(x, \bar{x}, \xi)) \left| \det \left( \frac{\partial \theta}{\partial \xi}(x, \bar{x}, \xi) \right) \right| d\xi.
\]

From (4.19) we have, for \( k = 0, 1 \), that \( b_{1,\epsilon}(x, \hat{x}, \theta(x, \bar{x}, \xi)) \left| \det \left( \frac{\partial \theta}{\partial \xi}(x, \bar{x}, \xi) \right) \right| \) belongs to \( \Gamma_k^m(\mathbb{R}^{2n}) \) if \( a \in \Gamma_k^m(\mathbb{R}^{2n}) \).

Applying the stationary phase theorem (c.f. [12],[13]) to (4.20), we obtain the expression of the symbol of the \( h \)-pseudodifferential operator \( F_{h}F_{h}^* \):

\[
\sigma(F_{h}F_{h}^*) = b_{1,\epsilon}(x, \hat{x}, \theta(x, \bar{x}, \xi)) \left| \det \left( \frac{\partial \theta}{\partial \xi}(x, \bar{x}, \xi) \right) \right|_{x=\hat{x}} + R(x, \xi; h)
\]

where \( R(x, \xi; h) \) belongs to \( \Gamma_k^{2m-2}(\mathbb{R}^{2n}) \) if \( a \in \Gamma_k^m(\mathbb{R}^{2n}) \), \( k = 0, 1 \).

For \( \hat{x} = x \), we have \( b_{1,\epsilon}(x, \hat{x}, \theta(x, \bar{x}, \xi)) = |a(x, \theta(x, x, \xi))|^2 \) where \( \theta(x, \bar{x}, \xi) \) is the inverse of the mapping \( \theta \rightarrow \partial_x S(x, \theta) = \xi \). Thus

\[
\sigma(F_{h}F_{h}^*)(x, \partial_x S(x, \theta)) \equiv |a(x, \theta)|^2 \left| \det \left( \frac{\partial^2 S}{\partial \partial \theta}(x, \theta) \right) \right|^{-1}
\]

such that

\[
\begin{align*}
\tilde{\xi}_c(x, \bar{x}, \theta(x, \bar{x}, \xi)) &= \xi \\
\theta(x, \bar{x}, \xi) &= x \\
\hat{\partial}^a \hat{\theta}(x, \bar{x}, \xi) &= \mathcal{O}(1), \quad \forall \alpha \in \mathbb{N}^{3n} \setminus \{0\}
\end{align*}
\]

If we change the variable \( \xi \) by \( \theta(x, \bar{x}, \xi) \) in \( K_{1,\epsilon}(x, \bar{x}) \), we obtain

\[
K_{1,\epsilon}(x, \bar{x}) = \int_{\mathbb{R}^n} e^{i\langle x-\bar{x},\xi \rangle} b_{1,\epsilon}(x, \hat{x}, \theta(x, \bar{x}, \xi)) \left| \det \left( \frac{\partial \theta}{\partial \xi}(x, \bar{x}, \xi) \right) \right| d\xi.
\]
Applying the stationary phase theorem, we obtain the expression of the symbol of the \( h \)-pseudodifferential operator \( F_h F_h^* \), is
\[
\sigma(F_h F_h^*)(x, \partial_x S(x, \theta)) \equiv |a(x, \theta)|^2 \det \frac{\partial^2 S}{\partial \theta \partial x}(x, \theta)|^{-1}.
\]
The distribution kernel of the integral operator \( \mathcal{F}(F_h^* F_h)\mathcal{F}^{-1} \) is
\[
\tilde{K}(\theta, \bar{\theta}) = \int_{\mathbb{R}^n} e^{i(S(x, \theta) - S(x, \bar{\theta}))} \overline{a(x, \theta)} a\left(x, \bar{\theta}\right) dx.
\]
Remark that we can deduce \( \tilde{K}(\theta, \bar{\theta}) \) from \( K(x, \bar{x}) \) by replacing \( x \) by \( x \). On the other hand, all assumptions used here are symmetrical on \( x \) and \( \theta \), therefore \( \mathcal{F}(F^* F)\mathcal{F}^{-1} \) is a nice \( h \)-pseudodifferential operator with symbol
\[
\sigma(\mathcal{F}(F^* F)\mathcal{F}^{-1})(\theta, -\partial_\theta S(x, \theta)) \equiv |a(x, \theta)|^2 \left| \det \frac{\partial^2 S}{\partial x \partial \theta}(x, \theta) \right|^{-1}.
\]
Thus the symbol of \( F^* F \) is given by (c.f. [11])
\[
\sigma(F_h^* F_h)(\partial_\theta S(x, \theta), \theta) \equiv |a(x, \theta)|^2 \left| \det \frac{\partial^2 S}{\partial x \partial \theta}(x, \theta) \right|^{-1}.
\]
\[
\square
\]
**Corollary 4.2.** Let \( F_h \) be the integral operator with the distribution kernel
\[
K(x, y; h) = \int_{\mathbb{R}^n} e^{i(S(x, \theta) - y \theta)} a(x, \theta) d\theta
\]
where \( a \in \Gamma^m_0(\mathbb{R}^{2n}) \) and \( S \) satisfies (G1), (G2) and (G3). Then, we have:
(1) For any \( m \) such that \( m \leq 0 \), \( F_h \) can be extended as a bounded linear mapping on \( L^2(\mathbb{R}^n) \)
(2) For any \( m \) such that \( m < 0 \), \( F_h \) can be extended as a compact operator on \( L^2(\mathbb{R}^n) \).

**Proof.** It follows from theorem [11] that \( F_h^* F_h \) is a \( h \)-pseudodifferential operator with symbol in \( \Gamma^m_0(\mathbb{R}^{2n}) \).

1) If \( m \leq 0 \), the weight \( \lambda^m(x, \theta) \) is bounded, so we can apply the Calde´ ron-Vaillancourt theorem (see [4][12][13]) for \( F_h^* F_h \) and obtain the existence of a positive constant \( \gamma(n) \) and a integer \( k(n) \) such that
\[
\| (F_h^* F_h) u \|_{L^2(\mathbb{R}^n)} \leq \gamma(n) Q_{k(n)}(\sigma(F_h^* F_h)) \| u \|_{L^2(\mathbb{R}^n)}, \quad \forall u \in \mathcal{S}(\mathbb{R}^n)
\]
where
\[
Q_{k(n)}(\sigma(F_h^* F_h)) = \sum_{|\alpha| + |\beta| \leq k(n)} \sup_{(x, \theta) \in \mathbb{R}^{2n}} \left| \frac{\partial^{\alpha} \partial^{\beta}_\theta}{\partial \theta ^{\beta}} \sigma(F_h^* F_h)(\partial_\theta S(x, \theta), \theta) \right|
\]
Hence, we have \( \forall u \in \mathcal{S}(\mathbb{R}^n) \)
\[
\| F_h u \|_{L^2(\mathbb{R}^n)} \leq \| F_h^* F_h \|^{1/2}_{L^2(\mathbb{R}^{2n})} \| u \|_{L^2(\mathbb{R}^n)} \leq \left( \gamma(n) Q_{k(n)}(\sigma(F_h^* F_h)) \right)^{1/2} \| u \|_{L^2(\mathbb{R}^n)}.
\]
Thus \( F_h \) is also a bounded linear operator on \( L^2(\mathbb{R}^n) \).

2) If \( m < 0 \), \( \lim_{|x| + |\theta| \to +\infty} \lambda^m(x, \theta) = 0 \), and the compactness theorem (see [12][13]) show that the operator \( F_h^* F_h \) can be extended as a compact operator on \( L^2(\mathbb{R}^n) \).
Thus, the Fourier integral operator $F_h$ is compact on $L^2(\mathbb{R}^n)$. Indeed, let $(\varphi_j)_{j \in \mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$, then
\[
\left\| F_h^* F_h - \sum_{j=1}^n <\varphi_j, \cdot > F_h^* F_h \varphi_j \right\|_{n \to +\infty} \to 0.
\]
Since $F_h$ is bounded, we have \( \forall \psi \in L^2(\mathbb{R}^n) \)
\[
\left\| F_h \psi - \sum_{j=1}^n <\varphi_j, \psi > F_h \varphi_j \right\|^2 \leq \left\| F_h^* F_h \psi - \sum_{j=1}^n <\varphi_j, \psi > F_h^* F_h \varphi_j \right\| \left\| \psi - \sum_{j=1}^n <\varphi_j, \psi > \varphi_j \right\|
\]
then
\[
\left\| F_h - \sum_{j=1}^n <\varphi_j, \cdot > F_h \varphi_j \right\|_{n \to +\infty} \to 0 \]
\]

Example 4.3. We consider the function given by
\[
S(x, \theta) = \sum_{|\alpha|+|\beta|=2} \sum_{\alpha, \beta \in \mathbb{N}^n} C_{\alpha, \beta} x^\alpha \theta^\beta, \text{ for } (x, \theta) \in \mathbb{R}^{2n}
\]
where $C_{\alpha, \beta}$ are real constants. This function satisfies (G1), (G2) and (G3).

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