Josephson junctions as threshold detectors of full counting statistics: open issues

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Abstract. I study the dynamics of a Josephson junction serving as a threshold detector of fluctuations which is subjected to a general non-equilibrium electronic noise source whose characteristic is to be determined by the junction. This experimental setup was proposed several years ago as a prospective scheme for determining the full counting statistics of an electronic noise source. Despite intensive theoretical as well as experimental research in this direction the promise has not been quite fulfilled yet and I will discuss what the unsolved issues are. First, I review a general theory for the calculation of the exponential part of the non-equilibrium switching rates of the junction and compare its predictions with previous results found in different limiting cases by several authors. I identify several possible weak points in the previous studies and I report a new analytical result for the linear correction to the rate due to the third cumulant of a non-Gaussian noise source in the limit of a very weak junction damping. The various analytical predictions are then compared with the results of the numerical method developed. Finally, I analyze the status of the experimental data thus far made publicly available with respect to the theoretical predictions and discuss briefly the suitability of the present experimental schemes as regards their potential for measuring the whole of the full counting statistics for non-Gaussian noise sources as well as their relation to the available theories.

Keywords: driven diffusive systems (theory), driven diffusive systems (experiment), large deviations in non-equilibrium systems

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1. Introduction

Josephson junctions (JJs) were proposed as threshold detectors of full counting statistics (FCS) by Tobiska and Nazarov [1] and independently by Pekola [2] in 2004. Since then there has been continuing effort to implement the proposed schemes experimentally as well as to improve them and better understand their potential theoretically. The original scheme of Tobiska and Nazarov [1] proposed using the overdamped JJ as the threshold detector. This appears to be problematic since in the overdamped junction when the effective phase particle overcomes the tilted washboard potential barrier it gets immediately retrapped in the adjacent minimum. This results in phase diffusion which, however, does not yield enough sensitivity for detecting the whole FCS. This could in principle be overcome by employing a negative-inductance device which apparently has not appealed to the experimentalists enough for them to actually implement it. Instead they opted for an obvious alternative of using underdamped junctions where, under suitable conditions, once the particle overcomes the first barrier it keeps on sliding down the potential, thus producing finite voltage. Thus, the switching of an underdamped junction between the supercurrent (static phase) and running (finite phase velocity, i.e. finite voltage) state would provide a prime example of a threshold detector. Unfortunately, this innocent-looking change in the setup dramatically changes the level of the difficulties involved in the theoretical analysis. This paper addresses those difficulties in some detail.

The structure of the paper is the following. In section 2 I report the theoretical concept of calculating the non-equilibrium escape rate due to a non-Gaussian noise source whose FCS is to be determined. The general theory based on the WKB-like approximation for the weak noise intensity is further carried out, with an analytical result in the case of linear perturbation theory in the third cumulant for very weak junction damping in section 2.1. In this section I also make a comparison with alternative existing theories. In section 2.2 the full theory is numerically implemented and the numerical results in an experimentally relevant regime are discussed and further compared with various analytical predictions. In section 3 I briefly raise some experimentally relevant questions, such as

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what the effect on the rate asymmetry of the nominally subleading terms entering the rate is, and whether one can actually experimentally leave the linear regime and achieve the measurement of the whole FCS of a noise source. In the last section 4 I summarize what has been achieved in this work and review the remaining open problems.

2. Theoretical calculation of the non-equilibrium escape rate

The Josephson element in an electrical circuit is often modeled as a current biased ($I_b$) resistively ($R$) and capacitively ($C$) shunted ideal JJ with the Josephson current–phase relation $I_J(\varphi) = I_0 \sin \varphi$. The voltage across the junction is determined by the second Josephson relation $V_J = \dot{\varphi} \hbar/2e$ with the time derivative denoted by the dot. Moreover, due to the action of ubiquitous thermal (Gaussian) noise $\xi(t)$ characterized by the temperature $T$ and non-equilibrium electronic noise $\eta(t)$ from the measured device whose FCS is to be determined, the JJ is subjected to stochastic forces and its dynamics is thus described by the following Langevin equation (RCSJ model):

$$\ddot{\varphi} + \frac{1}{RC} \dot{\varphi} + \frac{2e}{C\hbar}(I_0 \sin \varphi - I_b) = \xi(t) + \eta(t).$$

(1)

In a realistic experimental situation the current-bias assumption can be inadequate and one may need to generalize the above model. The general consequences of an imperfect current bias are so called environmental or ‘cascade’ corrections to the measured cumulants of the source FCS which were studied in previous works [3, 4]. They could be straightforwardly included here in the same spirit as in those works, especially [4], but since they appear to be of minor importance in the reported experiments so far I will neglect them. In this study I will consider in detail exclusively the simplest case of the Poissonian shot noise $\eta(t)$ corresponding to the measured device being a tunnel junction. In such a case $\eta(t)$ is just a train of $\delta$-function-like spikes which are separated by an exponentially distributed waiting time with a single parameter (inverse mean waiting time), being the mean (particle) current $I_m/e$ flowing through the tunnel junction. This case is also the only one studied experimentally through such experiments to date. Assuming the temporal width of the pulses composing $\eta(t)$ to be very small compared to a characteristic time of the junction dynamics (which is its plasma frequency $\omega_p = \sqrt{2eI_0/\hbar C}$) one can obtain a master equation (analogous to the Fokker–Planck equation in the case of Gaussian noise only) for the probability density $W(x, v, t)$ in dimensionless units $t\omega_p \rightarrow t, \varphi \rightarrow x, \varphi/\omega_p \rightarrow v$:

$$\frac{\partial W}{\partial t} = -v \frac{\partial W}{\partial x} + Q^{-1} \frac{\partial (vW)}{\partial v} + \left( \sin x - s + \frac{I_m}{I_0} \right) \frac{\partial W}{\partial v} + Q^{-1} k_B T \frac{\partial^2 W}{E_J \partial v^2}$$

$$+ \frac{I_m}{I_0 \lambda} \left[ \exp \left( -\lambda \frac{\partial}{\partial v} \right) - 1 \right] W,$$

(2)

with $s = I_b/I_0$ the rescaled bias current, $Q = RC\omega_p$ the quality factor of the (unbiased\(^1\)) junction and $\lambda = \sqrt{e^2/C E_J}$ with the Josephson energy of the junction proportional to the critical current $E_J = I_0 \hbar/2e$. The last term in the equation can be identified as stemming from the cumulant generating function of the Poissonian

\(^1\) Note that my definition of the quality factor differs from that in [5] where a bias-specific quality factor is used instead.
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process\(^2\) \(F_{\text{Poisson}}(x) = I_m(\exp x - 1)\) which suggests how to deal with non-Poissonian noise sources provided the Markovian approximation is made. Thus, the substitution \(I_m[\exp(-\lambda \partial / \partial v)] - 1 \rightarrow F(-\lambda \partial / \partial v)\) for general noise sources described by the cumulant generating function \(F(x)\) generalizes the particular results shown here for the Poissonian process to arbitrary noise sources as long as the Markovian approximation is justified [1, 5].

In order to calculate the escape rates for the junction from the supercurrent branch (zero-voltage state with a static phase \(\varphi\)) to the running state (finite voltage across the junction with non-zero phase velocity \(\dot{\varphi}\)) in the low noise limit we use the standard technique known as the singular perturbation theory in the mathematical literature [6] or as the WKB method in the physical context [7, 8]. It consists in making the ansatz \(W(x, v, t) = \exp[S(x, v, t)/\theta]\) for the probability density \(W(x, v, t)\) with \(\theta\) being a small parameter related to the noise intensity: \(\theta = k_B T \delta / \tau = k_B T / \tau + Q \lambda I_m / 2 I_0 = k_B T / \tau + eRI_m / 2 E J\). Thus, \(\theta\) is a dimensionless effective temperature of the junction due to the summed effect of the thermal noise and the Gaussian part of the non-equilibrium noise [1, 3, 5, 9, 10]. When this ansatz is put into equation (2) and only the lowest order in \(\theta\) is retained (corresponding to the WKB approximation and justified for small \(\theta \ll 1\)) we obtain the following Hamilton–Jacobi (HJ) equation, i.e. a first-order partial differential equation for \(S(x, v, t)\):

\[
\frac{\partial S}{\partial t} = -v \frac{\partial S}{\partial x} + (\sin x - s) \frac{\partial S}{\partial v} + Q^{-1} v \frac{\partial S}{\partial v} + Q^{-1} \left( \frac{\partial S}{\partial v} \right)^2 + \frac{I_m}{I_0} \sum_{n=3}^{\infty} \frac{1}{n!} \left( \frac{\lambda}{\theta} \right)^{n-1} \left( -\frac{\partial S}{\partial v} \right)^n
\]

\[
= -v \frac{\partial S}{\partial x} + (\sin x - s) \frac{\partial S}{\partial v} + Q^{-1} v \frac{\partial S}{\partial v} + Q^{-1} \left( \frac{\partial S}{\partial v} \right)^2 + \frac{\theta}{I_0 \lambda} \tilde{F}_{\text{Poisson}} \left( -\frac{\lambda \partial S}{\theta \partial v} \right).
\]

For a general noise source the last term (given by the sum) in the preceding equations would be replaced by the corresponding expression \(\tilde{F}(x) = F(x) - F'(0)x - F''(0)x^2/2\), i.e. by the reduced cumulant generating function with the first two moments (mean current and the zero-frequency noise) subtracted (notice that \(F(0) = 0\) by definition).

This Hamilton–Jacobi equation can be solved via the method of characteristics, i.e. one can recast the equation as a dynamical system in a four-dimensional phase space \([x(t), v(t), p(t) \equiv \partial S / \partial x, y(t) \equiv \partial S / \partial v]\) governed by the auxiliary Hamiltonian

\[
\mathcal{H} = vp - (\sin x - s)y - Q^{-1} y(v + y) + \frac{\theta}{I_0 \lambda} \tilde{F} \left( -\frac{\lambda}{\theta} y \right).
\]

The coordinates \(x, v\) and their conjugate momenta \(p, y\) are then evolving according to the following equations of motion:

\[
\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = v, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = y \cos x, \quad \dot{y} = -\frac{\partial \mathcal{H}}{\partial v} = -p + Q^{-1} y, \quad \dot{v} = \frac{\partial \mathcal{H}}{\partial y} = -(\sin x - s) - Q^{-1}(v + 2y) - \frac{1}{I_0} \tilde{F}' \left( -\frac{\lambda}{\theta} y \right).
\]

The exponential part of the non-equilibrium escape rate is in this language determined by the difference in stationary, i.e. time-independent, action between the barrier top and bar top and

\(^2\) Here I assume \(I_m \geq 0\) and the equation (2) holds for the positive polarity of the tunneling current. The opposite polarity would just change the sign in the exponential in equation (2).
the metastable minimum of the tilted washboard potential, i.e. $\theta \log \Gamma \propto S(x_{\text{max}}, 0) - S(x_{\text{min}}, 0)$ [7], which can be either determined by the direct solution of equation (3) or by finding the action along the trajectory of the system (5) connecting in infinite time (corresponding to the stationary solution and zero auxiliary energy $H = 0$) the two fixed points $[x_{\text{min}}, 0, 0, 0]$ and $[x_{\text{max}}, 0, 0, 0]$ [11]. I will demonstrate both methods in the next two sections. At this point I would like to stress that the overdamped analogue of the present problem (with $C \to 0$ in equation (1) when the inertial term $\ddot{\varphi}$ can be neglected) studied in [1] is integrable since equations analogous to (3), (5) are to be solved in a two-dimensional phase space only and with the help of the stationarity constraint $H_{\text{overdamped}}(x, p) = 0$ one easily finds the action in equation (3) through a quadrature for an arbitrary strength of the non-equilibrium noise. Unfortunately, this property does not carry over to the underdamped case where the energy constraint is not sufficient for integrability. Therefore, the underdamped problem is conceptually far more difficult than the originally suggested overdamped model.

2.1. Linear perturbation theory of the rate asymmetry due to a weak third cumulant

In this section I will present a linear perturbation theory of the rate asymmetry which is an alternative to the similar previous approaches of a number of authors [3]–[5]. I will use this limiting case for the illustration of the general method, which will be fully developed in the next section, and, at the same time, for pointing out possible discrepancies in the previous studies. As a by-product I will present a new analytical formula for the rate asymmetry in the very low damping limit $Q \to \infty$.

Following the previous studies we consider the linear correction to the escape rate due to a weak third cumulant. To this end we truncate the sum in the stationary version of the HJ equation (3) to the first order, i.e. we consider the effects of the third cumulant $c_3$ only. Further, we formulate the linear perturbation theory for an arbitrary potential $V(x)$ in which the effective particle moves—the present case is then recovered by the choice of the tilted washboard potential describing the JJ (in dimensionless units) $V(x) = -\cos x - sx$. The resulting HJ then reads

$$0 = -v \frac{\partial S}{\partial x} + V'(x) \frac{\partial S}{\partial v} + Q^{-1} v \frac{\partial S}{\partial v} + Q^{-1} \left( \frac{\partial S}{\partial v} \right)^2 - c_3 \left( \frac{\partial S}{\partial v} \right)^3,$$

with $c_3 = I_m x^2 / 6 I_0 \theta^2$. We solve this equation in the linear order in $c_3$ by linearizing the equation. After inserting $S(x, v) = S_0(x, v) + c_3 S_1(x, v)$ into the equation, using the knowledge of the zeroth-order solution $S_0(x, v) = -v^2 / 2 - V(x)$ corresponding to the Boltzmann factor due to the thermal Gaussian noise, and keeping only the linear terms in $S_1$, one obtains

$$v^3 = v \frac{\partial S_1}{\partial x} + [Q^{-1} v - V'(x)] \frac{\partial S_1}{\partial v}.$$

It is very unlikely that equation (7) could be solved analytically for general $Q$. Ankerhold [5] did find a certain solution to the problem of the rate asymmetry for any $Q$; however, his solution is not a solution of the above equation (7) as I will discuss later on. Indeed, one should not expect finding an explicit analytical solution to equation (7) for arbitrary $Q$ since it is generally known that the action $S(x, v)$ (or the ‘non-equilibrium potential’) develops a dense set of singularities close to the barrier top [12]. This does not
happen only in the integrable cases which is certainly the limit \( Q \to 0 \) corresponding to the one-dimensional spatial diffusion and, we hope, also for \( Q \to \infty \) describing the energy diffusion limit, which is effectively one-dimensional again.

Here, I give an analytic expression for the solution \( S_1(x,v) \) in the limit \( Q \to \infty \) for general potential \( V(x) \), in particular for the tilted washboard potential without resorting to its cubic approximation employed in previous works [5, 4]. We look for the solution of equation (7) with \( Q \to \infty \) in the form \( S_1(x,v) = \phi_0(x) + \phi_2(x)v^2/2 \) and find a closed set of equations

\[
\phi_2(x) = 2, \\
\phi_0(x) = V'(x)\phi_2(x).
\]

The solution \( \phi_2(x) = 2(x - x_0), \phi_0(x) = 2 \int^x dy V'(y) - 2x_0V(x) + C \) contains two arbitrary constants \( C, x_0 \). Moreover, one can add an arbitrary solution of the homogeneous part of equation (7) to this particular solution. Solutions to the homogeneous problem are arbitrary (sufficiently smooth and differentiable) functions of the particle energy \( G(v^2/2 + V(x)) \); thus, the freedom in the particular solution can be absorbed into the homogeneous solution since it just represents a linear function of the energy. The arbitrariness stemming from the mathematical solution must be fixed by physical requirements. First, all physical quantities must be ‘gauge invariant’, meaning that an arbitrary constant shift in the potential \( V(x) \to V(x) + \Delta \) cannot change the physical observables. Those are changes of \( S_1(x,v) \) between different points in the phase space, i.e. not just \( S_1(x,v) \) itself, rather its partial derivatives \( \partial S_1/\partial x \) and \( \partial S_1/\partial v \). The conditions to be satisfied are then \( \partial^2 S_1/\partial x \partial \Delta = 0 \) and \( \partial^2 S_1/\partial v \partial \Delta = 0 \). They lead to the same equation \( G''(x) = 0 \) with the linear function solution. This way, we recover the freedom stemming from the particular solution but the larger freedom of the homogeneous solution has been removed. The remaining uncertainty, being basically just the choice of the origin of integration \( x_0 \), since the constant \( C \) is harmless, is fixed by the requirement that in the vicinity of the potential minimum, where the potential can be approximated as harmonic, all the Gaussian averages (i.e. \( \langle v \rangle, \langle x \rangle, \langle x^2 \rangle, \langle xv \rangle, \langle v^2 \rangle \) of the original Fokker–Planck/master equation must stay intact by the third cumulant. In the harmonic regime, this is a necessary consequence of the linearity of the underlying Langevin equation. This condition implies that the origin of integration must be identical with the potential minimum \( x_0 = x_{\text{min}} \). In total, one finally has

\[
S_1(x,v) = 2 \int_{x_{\text{min}}}^{x} dy (y - x_{\text{min}})V'(y) + (x - x_{\text{min}})v^2 + C,
\]

yielding for the exponential part of the rate asymmetry \( R_G \equiv \Gamma_+ / \Gamma_- \) (the factor of 2 stands for the sum of the two equal contributions to the asymmetry from the two opposite polarities of the measured current) \( R_G(Q \to \infty) = \exp[2c_3(S_1(x_{\text{max}},0) - S_1(x_{\text{min}},0))/\theta] = \exp[2D_1(s)E_2^2 \ll I_{\text{in}}^2 / CI_0(k_0T_{\text{eff}})^3] \) with the function \( D_1(s) \) introduced in [3] reading

\[
D_1(s) \equiv \frac{1}{2}[S_1(x_{\text{max}},0) - S_1(x_{\text{min}},0)] = \frac{1}{2} \int_{x_{\text{min}}}^{x_{\text{max}}} dx (x - x_{\text{min}})V'(x)
\]

\[
= \frac{1}{3} \int_{\arcsin s}^{\arcsin s} dx (x - \arcsin s)(\sin x - s)
\]

\[
= \frac{2}{3} \arccos s \left( \sqrt{1 - s^2} - s \arccos s \right), \quad (9)
\]

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where the second part applies to the particular case of the tilted washboard potential.

The value of $D_1(s)$ at zero is $D_1(0) = \pi/3$ in accordance with [3] while the asymptotics for large bias is $D_1(s \to 1) \approx a(1 - s)^2$ with $a = 8/9$ which is exactly equal to the result by Ankerhold [5] but it actually differs from SJ’s numerical finding $a \approx 0.8$ [3] further supported by an independent study by Grabert [4] with $a \approx 0.79$. Although the difference is not severe, being on the order of $10\%$ only and, therefore, practically most likely irrelevant, from the conceptual point of view it matters because all the works claim to calculate the same quantity for the very same model and, thus, the correct result should be unique. It is not simple to follow and reproduce SJ’s approach but Grabert’s method is very transparent and I have fully recovered his numerical findings. Minor generalization of his approach to the full tilted washboard potential (Grabert uses the usual cubic approximation to the tilted washboard potential for a large bias $s \to 1$) gives $D_1(0) = \pi/3$ (within numerical precision) and $a \approx 0.79$ for $s \to 1$. The discrepancy with the above analytical result is thus not a problem of the numerical precision of works [3, 4] but a conceptual problem. Since Grabert uses the trajectory approach of equation (5) I will put off the discussion of his work to the next subsection where the same formalism is also used.

Ankerhold [5] was looking for the correction $S_1(x, v)$ in the form $S_1(x, v) = \phi_0(x) + \sum_{n=1}^{3} v^n \phi_n(x)/n$ and got certain conditions for the arbitrary functions $\phi_n(x)$ from the leading order solution of the Fokker–Planck equation accounting for the noise with the non-zero third cumulant. When his ansatz is plugged into equation (7) one can easily find that the set of equations obtained for the $\phi_n(x)$ is internally inconsistent (suggesting that the truncation at the third power in $v$ in the ansatz is insufficient) for a general $Q$ and potential $V(x)$. There are several exceptions when the inconsistencies are removed, namely for a strictly harmonic potential $V(x) \propto x^2$ (this potential does not exhibit a barrier, at least not a smooth one approximating the tilted washboard potential), and in the limits either $Q \to 0$ or $Q \to \infty$. This suggests that Ankerhold’s solution [5] could yield correctly the two limiting cases $Q \to 0$, $\infty$ for the cubic approximation to the potential considered. Indeed, in the limit $Q \to \infty$ his solution is equal to mine for $s \to 1$ as already mentioned above.

The opposite limit $Q \to 0$ is simple since that case is integrable for any strength of the third cumulant and the linear response can be easily calculated analytically yielding $R_1(Q \to 0) = \exp[2D_2(s)2eR^2E_m^3 \ll I_m^3 / h(k_B T_{\text{eff}})^3] = \exp[2D_2(s)Q^2E_m^3 \ll I_m^3 / C I_0(k_B T_{\text{eff}})^3]$ with $D_2(s) = [(1 + 2s^2) \arccos s - 3s\sqrt{1 - s^2}] / 6 \approx 8\sqrt{2}/45(1 - s)^{5/2}$ for $s \to 1$ [3, 4]. This is identical to Ankerhold’s solution in the corresponding limit $Q \to 0$, $s \to 1$ (recall the multiplicative correction factor of $2/3$ in the Erratum and Ankerhold’s definition of the bias-dependent quality factor of the JJ $Q(s) = Q(1 - s^2)^{1/4}$). Now, Ankerhold’s solution can be interpreted as a simple formula for interpolation between the two limiting cases reading $R_1(Q) = \exp[2D(s, Q)E_m^3 \ll I_m^3 / C I_0(k_B T_{\text{eff}})^3]$ with the interpolating function $D(s, Q) = D_2(s)Q^2/[1 + Q^2D_2(s)/D_1(s)] = 8/9 Q^2(s)(1 - s^2)^2/[Q^2(s)+5]$. It turns out that

3 Of course, this is not too surprising in view of the above stated general properties of the non-equilibrium action; see the comments below equation (7).

4 This fact is also not surprising since the ansatz used in my solution is just a subset of his form of $S_1(x, v)$. The main difference is that I used the ansatz only in the case where it does solve equation (7) and also the discussion of fixing the freedom in the solution due to the homogeneous part etc (present for any $Q$) seems absent in his work.
Ankerhold’s expression (equation (13) of [5]) is a neat scheme of interpolation between the highly underdamped and overdamped junction limits. It certainly provides a very efficient and quite precise interpolation formula for a finite $Q$. Its detailed comparison with the numerically exact solution will be shown in section 2.2.

2.2. Numerical evaluation of the escape rate in a general situation

Now, we turn to the general case of the calculation of the rate asymmetry for an arbitrary intensity of the non-equilibrium noise acting on the junction. This is achieved by the numerical solution of the effective dynamical system equations (5). As already mentioned the solution consists in finding a trajectory satisfying the equations of motion (5) and connecting in infinite time (corresponding to the zero auxiliary energy $\mathcal{H} = 0$) the two fixed points $[x_{\text{min}}, 0, 0, 0]$ and $[x_{\text{max}}, 0, 0, 0]$ being the (metastable) minimum of the potential and the top of the barrier, respectively. There always exists a classical, ‘relaxation’ solution corresponding to the dissipative but noise-free motion of the effective particle from the barrier top down to the minimum. This solution has $p(t) \equiv 0, y(t) \equiv 0$ and also the associated action is zero. On the other hand we are interested in the other, ‘escape’ solution connecting the two potential extrema via trajectory with non-zero conjugate momenta $p(t), y(t)$. For equilibrium, i.e. Gaussian, noise the two types of trajectories are connected by (generalized) time reversal which forms the basis of the Onsager–Machlup theory and was used by Grabert [4] for his linear response calculations. For general non-equilibrium noise sources, however, the two trajectories are not simply related and one has to calculate the escape trajectory directly by solving the full system (5).

This is exactly done here. The problem is formulated as a boundary value problem (BVP) on an infinite time interval reflecting the stationarity condition of the original escape problem. Obviously, this makes the BVP rather tricky and one has to be cautious in its solution. Once the solution $[x(t), v(t), p(t), y(t)]$ is found, the action difference between the two fixed points is calculated from the definition as

$$\Delta S \equiv S(x_{\text{max}}, 0) - S(x_{\text{min}}, 0) = \int_{-\infty}^{\infty} dt \left[ p(t)\dot{x}(t) + y(t)\dot{v}(t) - \mathcal{H}(x(t), v(t), p(t), y(t)) \right]$$

$$= -\int_{-\infty}^{\infty} dt \left\{ Q^{-1}y(t)^2 + \frac{\theta}{I_0\lambda} \left[ \tilde{F}(\frac{\lambda}{\theta}y(t)) - \tilde{F}'(\frac{\lambda}{\theta}y(t)) \left( -\frac{\lambda}{\theta}y(t) \right) \right] \right\}.$$  

The second line was obtained after using the explicit expressions for $\dot{x}(t), \dot{v}(t)$ from equation (5) and $\mathcal{H}$ (4) and the resulting action is thus expressed solely via the conjugate momentum $y(t)$. The exponential part of the rate asymmetry $\log(\Gamma_+/\Gamma_-)$ is then calculated as the difference in action of two opposite measured current polarities.

Technically the BVP is formulated on a long, but finite time interval estimated by the shooting solution used for the initial guess; for details see below. At the ends of this time interval the trajectory is assumed to be close enough to the respective fixed point that the linear approximation to the equations of motion (5) can be employed. The linearized system is then characterized by the stability analysis which identifies stable/unstable directions and corresponding eigenvalues. The boundary conditions at the ends of the time interval are then formulated with help of the respective linearized system in the spirit of the study [13, section IV], i.e. the two-dimensional unstable (stable) manifold...
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around the minimum (maximum) is identified and the solution is required to lie in them which yields two boundary conditions at each fixed point. Solution of the BVP is then sought for by a relaxation method with the 'bvp4c' built-in solver in Matlab [14].

The solver needs a very good initial guess for the solution to converge at all. When it does, it is very fast and efficient. Without a good initial guess it usually does not converge, especially for very large \( Q \). Thus, the initial guess is the crucial part of the whole solution process. For finding a good initial guess I solve the BVP by the shooting method first (in line with a general BVP strategy [14, 15]), i.e. I look for a solution of equation (5) by solving an initial value problem starting at the unstable manifold around the minimum and searching for such an initial condition which evolves into the other fixed point around the potential barrier. Finding a solution by the shooting method solely is a rather difficult task due to characteristic numerical instabilities involved, around the target fixed point [14, 16]. Indeed, the solution found is typically not precise enough to be of any use for the evaluation of the rates; however, it usually suffices as a good enough initial guess to ensure the convergence of the relaxation method. Moreover, the solution found by the shooting method gives a good estimate for the time needed to join the two linearized manifolds around the respective fixed points, a task which it is not obvious how to accomplish otherwise. After solving the problem at this fixed long time interval I include corrections due to the rest of the infinite time interval at the beginning and the end, respectively, by analytically evaluating the associated quadratic action exactly analogously to the method of [13, section V]. These corrections, although relatively small, are necessary within the precision required by the problem. The last technical detail concerns the handling of the (non-)uniqueness of the solution. Since the system (5) is autonomous it has an infinite number of solutions related by a simple time translation, i.e. if \([x(t), v(t), p(t), y(t)]\) is a solution then for an arbitrary \( \tau \) shift in time \([x(t + \tau), v(t + \tau), p(t + \tau), y(t + \tau)]\) is also a solution. This liberty may confuse the relaxation solver and, thus, it is advisable to fix the solution by explicit breaking of the time invariance [14]. I achieved this ‘locally’ by fixing the phase of the oscillatory solution at the unstable manifold around the potential minimum both in the boundary conditions as well as in the initial value problem. Time shift by the period of the local oscillations remains a symmetry operation but the continuous symmetry with an arbitrary shift is in this way broken. This trick does help to stabilize the solution; nevertheless, despite all the tricks used, the numerical implementation is still not absolutely stable and one can occasionally run into problems of non-convergence, especially with increasing value of \( Q \). This is not so surprising since for large \( Q \) the time span needed to connect the two linearized neighborhoods of the fixed points becomes longer and the solution in between exhibits ever increasing number of oscillations.

Further, due to the linear regime in which the problem is to be solved to describe current experiments the effect of non-Gaussian noise leads only to very small asymmetry effects and this requires rather high precision of the calculations which stretches the method used to its limits. Further improvement of the numerical method is thus an interesting and pressing open issue in the solution of the present problem. During the UPoN2008 conference I became aware of the work by the Lancaster group [13, 17, 18] who have apparently developed a toolbox of methods for tackling even far more difficult problems of escape. While I developed some of the techniques that they use independently, there seem to be some more left to explore. It will be interesting to see whether those
techniques, such as the action plot concept [13], will be successful in improving the present numerics. My naive fast implementation attempts have failed thus far. The main difference of the present problem from most established techniques, including the Lancaster group’s ones, seems to lie in the presence of non-Gaussian noise and the task of actually employing and characterizing its effects on the escape characteristics. It is unclear at this point how seriously this influences the feasibility and/or performance of those techniques developed predominantly for Gaussian problems.

For the moment I can only present numerical results obtained with the BVP method described above, which is ‘not yet quite perfect’. The results are shown in figures 1 and 2 for parameters motivated by the Saclay experiment of Huard et al [10]. Their junction was characterized by the critical current $I_0 = 0.48 \, \mu A$, equivalent to the Josephson energy $E_J/k_B = 11.4 \, K$, quality factor $Q = 22$, and the dimensionless ‘kick’ parameter $\lambda = 0.002$. The corresponding plasma frequency of the unbiased junction was $\omega_{p0} \approx 1 \, GHz$. These experimental parameters correspond exactly to figure 1 while in figure 2 I just modified the value of the quality factor $Q = 4$ ‘by hand’ (corresponding to changing the value of the resistance $R$ while keeping the other parameters constant) to explore a more intermediate regime of $Q$ and see the performance of different theories also there, not only in the high $Q \to \infty$ limit as in figure 1. The experiments were performed for temperatures in the range 20–530 mK. This is reflected by the lowest temperature of $T = 20 \, mK$ used in figure 1 while figure 2 presents results for an intermediate temperature of $T = 200 \, mK$. Compared with the Josephson coupling energy $E_J$ both the reservoir and the effective temperatures are small, on the order of a few per cent, which justifies the usage of the above developed theory of the non-equilibrium action valid for the weak noise only.
Figure 2. The logarithm of the escape rate asymmetry for temperature $T = 200$ mK and quality factor $Q = 4$. For detailed explanation of various quantities and the values of other parameters, which are the same as in figure 1 and correspond to the Saclay experiment [10], see the main text.

A quick inspection of both figures reveals that the logarithm of the asymmetry $\log(\Gamma_+ / \Gamma_-)$ presented there is generally a small quantity with a typical magnitude of about 10%. This is consistent with the usage of the linear response theories (in the third cumulant) employed in previous studies [3]–[5]. The curves are not, however, linear functions of the measured current $I_m$ due to the fact that the current contributes also to the effective temperature (and it is an important contribution) which enters the formula for the asymmetry linear in the third cumulant (proportional to $I_m$ as well for the tunnel junction). Moreover, there is another source of non-linearity in the curve, namely the fact that experimentalists for convenience measure in the range of roughly constant mean escape rate on the order of $\approx 30$ kHz. Thus, for each value of the measured current $I_m$, the (dimensionless) bias current $s(I_m)$ is adjusted in such a way that the mean rate stays constant close to that value. For the present junction with $\omega_p \approx 1$ GHz this implies fixing the dimensionless barrier height to a value of roughly 10.4. In other words, for every $I_m$ the value of $s(I_m)$ is determined from the equation $\Delta U(s(I_m))/k_B T_{\text{eff}}(I_m) \approx -\log(3 \times 10^{-5}) \approx 10.4$ with $\Delta U(s) = 2E_J(\sqrt{1 - s^2} - s \arccos s) \approx 4\sqrt{2}E_J(1-s)^{3/2}/3$ for $s \to 1$ being the barrier height of the tilted washboard potential and $T_{\text{eff}}(I_m) = T + eRI_m/2k_B$ the effective temperature.

Let us first discuss figure 1 with high $Q = 22$. This value of the quality factor was nearly at the edge of stability of my BVP numerics described above. The diagnostic quantities are shown in the plot to exemplify the precision achieved in the calculations. I plot the quantity $S(0) - S_{\text{analytic}}(0)$ to assess the overall precision of the calculation. By $S(0)$ I denote the action calculated numerically for zero measured current $I_m = 0$. This value is known analytically and equals the already discussed experimental value of $\approx 10.4$.
This quantity, in the plot labeled ‘$I_m = 0$’, is shown by plus signs which are essentially overlaid by crosses. The crosses show the quantity $[S(I_m) + S(-I_m)]/2 - S_{\text{anal}}(0) = 2$ which probes the linearity of the calculated action in the third cumulant. In the linear regime, the two polarities contain opposite contributions from the third cumulant which cancel in the sum and the subtracted action for zero third cumulant should nullify this quantity. In the second figure 2 with $Q = 4$ this is indeed the case due to more stable numerics but one can see that those two control quantities are not strictly zero for the high $Q$ case. Their overlap, however, actually confirms the linear response regime. The deviations from zero of $[S(I_m) + S(-I_m)]/2 - S_{\text{anal}}(0)$ are solely due to the imprecision of the mean action without any influence of the third cumulant. This is further confirmed by the essentially regular behavior of the asymmetry $S(I_m) - S(-I_m)$. Moreover, it should be stressed that each point presents an independent calculation. Thus, the values of $I_m$ where the control quantities are zero as expected should be trustworthy regardless of the fact that the next value of $I_m$ may be calculated with insufficient precision. Moreover, the overall precision even in the $Q = 22$ case is not catastrophically bad although it does not allow a fully reliable comparison with the concurrent theories.

The asymmetry $S(I_m) - S(-I_m)$ in figure 1 is compared with four different theories grouped into two sets (within the set they are virtually equal in the $Q \to \infty$ limit). It is (generalized numerical) evaluation à la Grabert [4] together with the result by Sukhorukov and Jordan [3], both of which predict basically $S(I_m) - S(-I_m) \propto 0.79(1 - s)^2$ while the other set is my equation (9) and Ankerhold’s [5] result $S(I_m) - S(-I_m) \propto 8/9(1 - s)^2 \approx 0.89(1 - s)^2$. While the difference in the predictions is only on the order of 10% and, thus, most likely irrelevant for experiments, it is relevant from a purely conceptual point of view which one is actually correct since it should help with the identification of possible misconceptions hidden in the failed approach(es).

From the data presented in figure 1 it is clear that the more promising set is the Ankerhold–Novotný one. Despite the scatter in the data, there are reliable points (where the control quantities turn into zero) which are closer to the $8/9$-curve. The numerical calculation did not use any linear perturbation theory or any approximation at all. The data are purely results of the numerical evaluation of the BVP for general values of the parameters. The discrepancy of the data with the theoretical predictions may be caused by the finite, although rather high, value of $Q$. This can account for the difference between the numerics and $8/9$-curve; however, it is inconsistent with Grabert’s theory which predicts monotonic increase of the asymmetry with increasing $Q$; see figure 4 in [4]. Moreover, ‘Grabert’s curve’ was calculated for $Q = 22$ even though it hardly deviates from its limiting $Q = \infty$ counterpart by SJ. Thus, the only salvation for the two theories [3,4] could come from the numerics being wrong which is in principle possible but does not seem too plausible at this point.

If we now turn to the other figure, figure 2, we see in the first place much better precision of the numerics as revealed by the control quantities being zero. The numerical data are again compared with Grabert’s and Ankerhold’s theories which provide alternatives for finite $Q$. I also show a curve for $Q = \infty$ to demonstrate significant deviations of the results for still relatively high $Q = 4$ from the infinite $Q$ limit. This should be remembered when interpreting experimental data of, e.g., the Helsinki group [19,20] with $Q \approx 2.5$ via $Q = \infty$ theories. Ankerhold’s theory (equation (13) of [5] with the $2/3$-correction from the Erratum) is off the numerical data as well as Grabert’s result, thus clearly demonstrating the merely interpolating status of this theory. The discrepancy is,
however, rather small and, therefore, Ankerhold’s formula seems to provide a very cheap and efficient analytical interpolation scheme for an arbitrary $Q$.

Grabert’s result on the other hand lies exactly on top of the numerical data, in stark contrast to its apparent failure for the high $Q$ case in figure 1. This is somewhat mysterious behavior which certainly deserves better understanding. What could go wrong in Grabert’s reasoning? I have no clear answer to that; however, I do have a conjecture as regards where there could be a problem hidden. Of course, I am fully aware that the problem could in fact be also in my numerics for $Q = 22$ although its correspondence with my analytics represented by equation (9) is encouraging and not quite typical for buggy numerics. Grabert’s approach uses a straightforward perturbation theory at the level of trajectories connecting the fixed points. He argues that within the linear response in the third cumulant the equilibrium (unperturbed) solution is enough for evaluating the correction to the action. In more detail, provided the auxiliary Hamiltonian is split into an equilibrium part and a non-Gaussian perturbation $\mathcal{H} = \mathcal{H}_{\text{equil}} + \mathcal{H}_3$, the correction to the action reads $\Delta S_3 = -\int_{-\infty}^{\infty} dt \mathcal{H}_3(y_{\text{equil}}(t))$ (compare with equation (77) in [4]). This is analogous to the standard first-order perturbation theory in quantum mechanics; the correction to the energy is just the mean value of the Hamiltonian in the unperturbed state. However, one should recall that this formula is only applicable if the unperturbed state is non-degenerate. It is not obvious what the analogous condition for classical trajectories is; nevertheless, one may expect certain subtleties to be involved due to several conditions specific to the current problem. First of all, the BVP is formulated on an infinite time interval, there exists a continuous time shift symmetry, and the unstable/stable manifolds around the respective fixed points are two-dimensional (could this be the ‘degeneracy’?) The above formula for $\Delta S_3$ can be easily derived for finite time interval with fixed boundary conditions; however, cannot the infinite time interval bring about omitted surface terms? I am quite sure these questions can be successfully handled by dynamical system theory experts.

3. Experiment-related issues

To this date (end of June 2008) there are two publicly available experimental results, from the Helsinki group [19, 20] and the Saclay group [10]. The Helsinki experiment finds the asymmetry curve as a function of the measured current through the tunnel junction which has its shape in qualitative agreement with all previously mentioned theories (the $\sim 10\%$ difference between different theories is undetectable at the level of precision of the experiment). However, the quantitative comparison with, e.g., Ankerhold’s theory shows a discrepancy on the order of $\sim 10$ (see the comparison in [20]; recall the correction factor $2/3$ missing in that reference and further account for the finite $Q = 2.5$ contributing another factor of $1/2$). I have not discussed the theory used by the Helsinki group for fitting the experiment since it is conceptually different from all the other theories discussed and I consider it to be semi-phenomenological with the prefactor (calculated in other theories) being adjusted to the experimental outcome, thus lacking real predictive power. The other experiment by the Saclay group has been identified as most likely faulty due to a leak in the measurement circuit which prohibited the reliable determination of the bias current. Such an effect largely overshadows any asymmetry due to the third cumulant and, thus, no quantitatively reliable data are available from this experiment.
Regardless of this unsatisfactory state we may consider possible problems which are likely to be encountered, and maybe have already been encountered in the Helsinki experiment, when trying to compare the experimental outcome with theoretical predictions. The first issue is the one of the actual relevance of the exponential part of the rate asymmetry. Clearly, the experiment measures the rate asymmetry, not a theoretical concept of its exponential part. The rationale behind the dominance of the exponential part of the rate unfortunately does not necessarily carry over to the rate asymmetry, especially in the linear regime. The standard argument behind the dominance of the exponential part of (therm) rates is that the large dimensionless barrier entering the exponent simply dominates the whole expression; moreover, the noise intensity (temperature) enters only the exponential part via the Boltzmann factor while the prefactor (attempt frequency) is temperature independent. Now consider a weak noise with the third cumulant non-zero. This weak noise will supposedly weakly modify the rate. This will in general happen both through the exponent and through the prefactor. In the linear response regime in the third cumulant the correction in the exponent can be safely expanded and the resulting linear correction will add to the linear correction stemming from the prefactor. At this stage there is no a priori difference between these two contributions. Of course, in practice one of them (presumably the prefactor part) can still be negligible. What are the prospects for this to happen? We have seen that in the realistic setup studied in the previous section the asymmetry due to the exponential part of the rate reaches values on the order of $\sim 10\%$ at maximum. The expected correction due to the prefactor is of the form $k_B T_{\text{eff}}/E_J \cdot I_m/I_0$. The first factor, dimensionless temperature, is of the order of $\sim 1\%$ while the other factor, dimensionless measured current, is of the order of $\sim 1$. Thus, in total, we have an effect of the order of $\sim 1\%$ which can be, depending on the actual numerical prefactor, comparable to the exponential part. This somewhat pessimistic scenario can be further supported both by the discrepancy found in the Helsinki experiment as well as by the mismatch between Ankerhold’s theory and direct stochastic simulations performed in connection with the Saclay experiment in [10] (see their figure 7(a)) where a multiplicative factor of 2 difference was found for the dimensionless barrier height $\sim 6$. While this value lies at the border of reliability of the WKB approach and corrections for larger barriers (especially the experimentally relevant one 10.4) may be expected, they are not expected to be of order 100%. Therefore, it seems that the asymmetry stemming from the prefactor may be relevant for experiments. This is rather bad news for theoreticians since the calculation of the prefactor for non-equilibrium rates is an involved task; see the discussion in [7] and references therein.

So we finally come to the question of whether one can achieve a non-linear regime with underdamped JJs. The problem is apparently in the fact that the effective temperature rises with the measured current in such a way that it simply dominates the escape mechanism and corrections due to higher order cumulants are just negligible. This is clearly reflected in the plot in figure 1 where the originally growing (with $I_m$) curve for small $I_m$ eventually bends downwards again for larger $I_m$. While the first part is governed by the third cumulant growing with $I_m$, the declining part corresponds to the case when the contribution of $I_m$ to the effective temperature beats the raising third cumulant. The same effect would be seen in figure 2 for larger values of $I_m$. This behavior could be diminished by weakening the effect of the measured current on the effective temperature; see the expression for the effective temperature. This should be achieved
by decreasing the value of the effective shunt resistance $R$. This, in turn, would imply decreasing quality factor which seems experimentally unacceptable beyond the point when the switching ceases to exist and only phase diffusion is present. The quality factor thus should be maintained at a reasonably high value which can be achieved by increasing capacitance $C$. That in turn will decrease the third-cumulant contribution via the formula preceding equation (9). At this point the problem turns into a bad joke. There may be, however, a parameter window where a subtle compromise can be achieved. This should be seriously considered by carefully examining different parameter dependences and testing experimentally acceptable numbers.

4. Conclusions

In this work I have reviewed in detail the status of the problem of the measurement of full counting statistics for the switching dynamics of an underdamped Josephson junction. I have presented a general theory for the weak noise based on the WKB-like approximation and calculated the rate asymmetry due to a weak third cumulant analytically in the limit of very high quality factor of the junction. This calculation has been critically compared to other theories and their possible shortcomings have been identified and pointed out. Further, I have developed a numerical scheme for solving the boundary value problem determining the exponential part of the non-equilibrium escape rate under general circumstances, i.e. beyond the linear perturbation theory. Using this scheme I have calculated the exponential part of the rate asymmetry for an experimentally relevant set of parameters and compared the findings with various linear theories. Again, this helped with the identification of the status of concurrent theories. Eventually, I have briefly discussed issues related to the interpretation of present and future experiments, in particular the question of the relevance of the rate exponential prefactor for the rate asymmetry and the feasibility of achieving the non-linear regime.

There are plenty of unsolved problems and open issues within this field of research. Starting with the more particular and technical ones, it would be rewarding to fully clarify the status of concurrent theories, in particular that of Grabert which performs amazingly well for intermediate range of $Q$ while seeming to fail for large values of $Q$. Although the discrepancy is not too large, Grabert’s theory is supposed to work in that regime as well and, thus, the discrepancy raises serious questions about its very foundation. On the other hand, it would be very helpful to further develop and fully stabilize (if possible) my numerical scheme used for the solution of the BVP. If this attempt were successful the numerical code could be used for interpreting future experiments routinely since it is very fast and efficient as long as it converges, which unfortunately occasionally does not happen. On a more general level, there should be further study of what the effect of nominally subleading terms in the rate on the rate asymmetry is. It appears that the conventional arguments for the dominance of the exponential part of the rate may not be applicable to the rate asymmetry, especially in the linear regime. And last but not least, a most important question, that of whether one can actually use underdamped Josephson junctions for the measurement of the whole FCS and not just the third cumulant in the linear regime, is still open and waiting for a final answer which, if affirmative, could bring the field of FCS to new milestones.

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Post-acceptance note. After the acceptance of this manuscript with minor corrections I became aware of a comment arXiv:0807.2675 by Sukhorukov and Jordan. Although at this point I am unable to decide whether that comment really settles the above mentioned discrepancy between our theories, I recommend the interested reader to check out their paper for an independent point of view on the issue.

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