Shifted genus expanded $\mathcal{W}_\infty$ algebra and shifted Hurwitz numbers

Quan Zheng*
Mathematics College, Sichuan University
610064, Chengdu, Sichuan, PRC

Abstract

We construct the shifted genus expanded $\mathcal{W}_\infty$ algebra, which is isomorphic to the central subalgebra $\mathcal{A}_\infty$ of infinite symmetric group algebra and to the shifted Schur symmetrical function algebra $\Lambda^*$ defined by A. Y. Okounkov and G. I. Olshanskii. As an application, we get some differential equations for the generating functions of the shifted Hurwitz numbers, thus we can express the generating functions in terms of the shifted genus expanded cut-and-join operators.

Key words: shifted genus expanded $\mathcal{W}_\infty$ algebra; shifted genus expanded cut-and-join operator; shifted Hurwitz number;

Subject Classification: 14N10, 14N35, 05E10

Contents

1 Introduction 2

2 Preliminary: The construction of $\mathcal{A}_\infty$ algebra and its structure constants 3

3 Shifted Hurwitz numbers 6

4 Shifted genus expanded $\mathcal{W}_\infty$ algebra 8

5 Generating functions of the shifted Hurwitz number and its differential equations 12

*E-mail: quanzheng21000163.com. Partially supported by NSFC.
1 Introduction

The classical Hurwitz Enumeration Problem [5] and its generalization [2], [3], [4], [17], etc, have been extensively applied in many mathematical and physical fields, such as the integrability system theory, the modular space theory, the relative Gromov-Witten theory, string theory, referring [1], [10], [14], especially, the investigation of the relationship between the Hurwitz numbers and multi-matrix-model theory have displayed its powerful vitality, referring [15], etc. One of the important geometric tools to deal with Hurwitz numbers is the so-called symplectic surgery: cutting and gluing [7], [8], [9], in the views of algebra and differential equations, which is equivalent to the so-called cut-and-join operators [2], [3], [4], [10], [17]. The standard cut-and-join operators can be used to deal with the almost simple Hurwitz numbers and the almost simple double Hurwitz numbers [2], [3], [4], [10], [17]. A. Mironov, A. Morozov, and S. Natanzon, etc, have defined the generalized cut-and-join operators in terms of the matrix Miwa variable [1], [12], [13]. To distinguish the contributions for the Hurwitz number of the source Riemann surface with the different genus, by observing carefully the symplectic surgery ( [8], [9]) and the gluing formulas of the relative GW-invariants developed by A.M. Li and Y.B. Ruan [9], and by E. Ionel and T. Parker [7], in [18], the author has constructed the genus expanded cut-and-join operators and the corresponding genus expanded differential operator algebra \( W_d \) for any nonnegative integer \( d \). In this paper, following the idea of [6], [14] and [16], we extend algebra \( W_d \) to the shifted genus expanded \( W_\infty \) algebra. Thus we have to construct the shifted genus expanded cut-and-join operators to distinguish the contributions of the source Riemann surfaces with different genus, i.e., we introduce one parameter \( z \) to mark the genus, referring the formula (26). Then after normalizing the shifted genus expanded cut-and-join operators by a factor, we obtain the shifted genus expanded \( W_\infty \) algebra, which is isomorphic to the central subalgebra of infinite symmetric group algebra and to the shifted Schur symmetrical function algebra \( \Lambda^* \) [16], referring Corollary 4.6 and Corollary 4.7.

As an application of the genus expanded cut-and-join operators, we get some differential equations for the generating functions of the shifted Hurwitz numbers for the source Riemann surface with different genus, thus we can express the generating functions in terms of the shifted genus expanded cut-and-join operators.
2 Preliminary: The construction of $\mathcal{A}_\infty$ algebra and its structure constants

First of all, let us introduce some notations. In the paper, we always arrange that if the variable is negative, then the any correlated function or invariance is set to be zero or does NOT appear according to the context. Suppose that $\Delta$ is a Youngian diagram or a partition of nonnegative integer. Denote by $m_i(\Delta)$ the number of rows of length $i$ in the partition $\Delta$, so we usually denote partition $\Delta$ by $1^{m_1(\Delta)}2^{m_2(\Delta)}3^{m_3(\Delta)}4^{m_4(\Delta)}\cdots$. We denote

$$l(\Delta) := \sum_{i \geq 1} m_i(\Delta),$$

$$|\Delta| := \sum_{i \geq 1} i m_i(\Delta),$$

$$||\Delta|| := \prod_{i \geq 1} i^{m_i(\Delta)},$$

For any two partition $\Delta_1, \Delta_2$, we define their sum and difference as

$$\Delta_1 \pm \Delta_2 = 1^{m_1(\Delta_1) \pm m_1(\Delta_2)}2^{m_2(\Delta_1) \pm m_2(\Delta_2)}3^{m_3(\Delta_1) \pm m_3(\Delta_2)}\cdots.$$ 

Let $p = (p_1, p_2, p_3, \cdots)$ be indeterminates, which are called the time-variables, then we denote

$$p_\Delta := \prod_{i \geq 1} p_i^{m_i(\Delta)},$$

and

$$\frac{\partial}{\partial p_\Delta} := \prod_{i \geq 1} \frac{\partial p_i^{m_i(\Delta)}}{\partial p_i^{m_i(\Delta)}}.$$ 

In the section, we recall some basic fact about the construction of $\mathcal{A}_\infty$ algebra and its structure constants following [6]. Let $\mathbb{P}_n$ be a set of $n$ positive integers $\{1, \ldots, n\}$, and $\mathcal{S}_n$ be the group of all permutations of $\mathbb{P}_n$. Fix a subset $E \subset \mathbb{P}_n$ of size $|E| = r$ and denote by $\mathcal{S}_E$ the group of permutations of the subset $E$. A partial permutation of the set $\mathbb{P}_n$ is a pair $(E, f)$ consisting of an arbitrary subset $E \subset \mathbb{P}_n$ and an arbitrary permutation $f \in \mathcal{S}_E$. The set $E$ will be referred to as the support of $(E, f)$. Define the degree of a partial permutation $(E, f) \in \mathcal{P}_n$ as $\deg(E, f) = |E|$. Denote by $\mathcal{P}_n$ the set of all partial permutations of the set $\mathbb{P}_n$. Obviously, the number of elements in $\mathcal{P}_n$ equals

$$\sum_{k=0}^{n} \binom{n}{k} k! = \sum_{k=0}^{n} (n \mid k), \quad (1)$$
where \( \binom{n}{k} \) is the binormal coefficient, and \((n \mid k) = n(n-1)\ldots(n-k+1)\) is the falling factorial power.

Given two partial permutations \((E_1, f_1), (E_2, f_2)\), we define their product as the pair \((E_1 \cup E_2, f_1 f_2)\), where we can naturally regard \(f_1, f_2\) as the permutations of \(\mathcal{S}_{E_1 \cup E_2}\), thus we have the product for \(f_1\) and \(f_2\). With this multiplication, \(\mathcal{P}_n\) becomes a semigroup. The partial permutation \((\emptyset, e)\), where \(e\) is the trivial permutation of the empty set \(\emptyset\), is the unity in \(\mathcal{P}_n\).

Denote by \(\mathcal{B}_n = \mathbb{C}[\mathcal{P}_n]\) the complex semigroup algebra of the semigroup \(\mathcal{P}_n\), which is semi-simple and isomorphic to the direct sum of the group algebras of symmetric groups \([6]\),

\[
\mathcal{B}_n \cong \bigoplus_{F \subseteq \mathcal{P}_n} \mathbb{C}[S_F].
\]

The centre of the algebra \(\mathcal{P}_n\) is of the form

\[
Z(\mathcal{B}_n) \cong \bigoplus_{E \subseteq \mathcal{P}_n} Z(\mathbb{C}[S_E]),
\]

where \(Z(\mathbb{C}[S_E])\) is the centre of the group algebra \(\mathbb{C}[S_E]\).

The symmetric group \(S_n\) acts on the semigroup \(\mathcal{P}_n\) by automorphisms \((E, f) \mapsto (vE, vf v^{-1})\) for \(v \in S_n\). The orbits of this action will be referred to as conjugacy classes in \(\mathcal{P}_n\). It is obvious that two partial permutations are conjugate if and only if the sizes of their supports coincide as well as their cycle types. Thus the conjugacy classes \(A_{\Delta; n} \subset \mathcal{P}_n\) are indexed by partial partitions of \(n\), i.e. by partitions \(\Delta \vdash r\) of any integers \(0 \leq r \leq n\). In particular, \(A_{\emptyset; 0} = \{(\emptyset, e)\}\). The action of the symmetric group \(S_n\) on \(\mathcal{P}_n\) can be continued by linearity to an action of \(S_n\) on the algebra \(\mathcal{B}_n\). Denote by \(\mathcal{A}_n = \mathcal{B}_n^{S_n}\) the subalgebra of invariant elements for this action. Let us identify the conjugacy class \(A_{\Delta; n}\) with the invariant element

\[
A_{\Delta; n} = \sum_{(E, f) \in A_{\Delta; n}} (E, f)
\]

of the algebra \(\mathcal{B}_n\). In particular, if \(|\Delta| > n\), then \(A_{\Delta; n} = 0\).

Given a partial partition \(\Delta \vdash r \leq n\), denote by \([\Delta, 1^{n-r}] = \Delta \cup \{1^{n-r}\}\), i.e., the partition of \(n\) obtained by adding an appropriate number of unities, which is called a shifting of \(\Delta\). Let \(C_\Delta\) be the conjugacy class in the group \(\mathcal{S}_{|\Delta|}\) consisting of permutations of cycle type \(\Delta\), and \(|C_\Delta|\) be the number of the permutations of cycle type \(\Delta\). Let \(C_{\Delta; n}\) be the conjugacy class in the group \(\mathcal{S}_n\) consisting of permutations of cycle type \([\Delta, 1^{n-r}]\), and \(|C_{\Delta; n}|\) be the number of the permutations of cycle type \([\Delta, 1^{n-r}]\). Note that \(|C_\Delta| = |C_{\Delta; n}|\) if \(\Delta\) is the
partition of \( n \). Denote by
\[
\psi : \mathcal{P}_n \to \mathcal{S}_n \\
(E, f) \mapsto f \in \mathcal{S}_E \subseteq \mathcal{S}_n
\]
the homomorphism of forgetting the support of a partial permutation, which obviously can be linearly extended to the algebra homomorphism \( \mathcal{B}_n \to \mathcal{S}_n \). It is clear that
\[
\psi(A_{\Delta;n}) = \left( n - |\Delta| + m_1(\Delta) \right) C_{\Delta;n},
\]
thus we can call \( \psi(A_{\Delta;n}) \) the shifted central element of \( \mathcal{S}_n \).

Assume \( m \leq n \), then we introduce a mapping \( \theta_m : \mathcal{B}_n \to \mathcal{B}_m \) by the formula
\[
\theta_m(E, f) = \begin{cases} (E, f) & \text{if } E \subseteq \mathcal{P}_m \\ 0 & \text{otherwise} \end{cases}
\]
The mapping \( \theta_m \) is a homomorphism of algebras and it commutes with the action of the group \( \mathcal{S}_m \) on \( \mathcal{B}_n \) and on \( \mathcal{B}_m \), so we can denote the restricted mapping \( \theta_m|_{\mathcal{A}_n} : \mathcal{A}_n \to \mathcal{A}_m \) by the same denotation \( \theta_m \).

Denote by \( \mathcal{B}_\infty \) and \( \mathcal{A}_\infty \) the projective limit of the algebras \( \mathcal{B}_n \) and \( \mathcal{A}_n \) with respect to the morphisms \( \theta_n \), respectively. Let \( \mathcal{S}_\infty \) be the infinite symmetric group, i.e. the group of finite permutations of positive integers, thus \( \mathcal{A}_\infty \) can be regarded as the central subalgebra of group algebra \( \mathbb{C}[\mathcal{S}_\infty] \).

Denote by \( \theta_n \) the natural homomorphism \( \theta_n : \mathcal{B}_\infty \to \mathcal{B}_n \) as well as its restriction on \( \mathcal{A}_\infty \) to \( \mathcal{A}_n \). The natural inclusion of algebras \( i_n : \mathcal{B}_n \to \mathcal{B}_\infty \) accords with the projection \( \theta_n \): \( \theta_n \circ i_n = id_{\mathcal{B}_n} \).

Given a partition \( \Delta \vdash r \), let \( A_{\Delta} = \sum (E, f) \), where the sum extends to partial permutations \( (E, f) \in \mathcal{P}_\infty \) such that \( |E| = r \) and \( f \) has cycle type \( \Delta \). The elements \( A_{\Delta} \), where \( \Delta \) runs over all partitions, form a linear basis in \( \mathcal{A}_\infty \). Denote by \( C_{\Delta_1, \Delta_2}^{\Delta_3} \) the structure constants of the algebra \( \mathcal{A}_\infty \) in the basis \( \{ A_{\Delta} \} \),
\[
A_{\Delta_1} A_{\Delta_2} = \sum_{\Delta_3} C_{\Delta_1, \Delta_2}^{\Delta_3} A_{\Delta_3},
\]
Note that \( \theta_n(A_{\Delta}) = A_{\Delta;n} \), where \( A_{\Delta;n} \) is the element of the algebra \( \mathcal{A}_n \). Since \( \theta_n : \mathcal{A}_\infty \to \mathcal{A}_n \) and forgetting map \( \psi \) are homomorphisms, for any \( n \geq 0 \), we have

**Proposition 2.1** (**[6]**, Proposition 6.1., Theorem 7.1.)
\[
A_{\Delta_1;n} A_{\Delta_2;n} = \sum_{\Delta_3} C_{\Delta_1, \Delta_2}^{\Delta_3} A_{\Delta_3;n}, \\
\psi(A_{\Delta_1;n}) \psi(A_{\Delta_2;n}) = \sum_{\Delta_3} C_{\Delta_1, \Delta_2}^{\Delta_3} \psi(A_{\Delta_3;n}).
\]
Example 2.2 ([6], [14])

\[ A(1) A(2) = 2A(2) + A(2,1), \]
\[ A(1) A(1,1) = 2A(1,1) + 3A(1,1,1) \]
\[ A(2) A(2) = A(1^2) + 3A(3) + 2A(2^2) \]  

3 Shifted Hurwitz numbers

Denote by \( \Delta_i = (\delta_i^1, \cdots, \delta_i^l(\Delta_i)) \), \( i = 1, \cdots, k \), a series of partitions with any degrees. Let \( \Sigma^h \) be a compact (maybe disconnected) Riemann surface of genus \( h \), and \( \Sigma^g \) a compact connected Riemann surface of genus \( g \). For a given point set \( \{q_1, \cdots, q_k\} \in \Sigma^g \), which is called the set of branch points, we call a holomorphic map \( f: \Sigma^h \to \Sigma^g \) a ramified covering of \( \Sigma^g \) of degree \( n \geq 0 \) by \( \Sigma^h \) with a ramification type \( (\Delta_1, \cdots, \Delta_k) \), if the preimages of \( f^{-1}(q_i) = \{p_1^i, \cdots, p_i^{l(\Delta_i)}, p_i^{l(\Delta_i)+1}, \cdots, p_i^{l(\Delta_i)+|\Delta_i|}\} \) with orders \( (\delta_i^1, \cdots, \delta_i^{l(\Delta_i)}, 1, \cdots, 1) \)

for \( i = 1, \cdots, k \), respectively, i.e., we have to shift by many one to the orders of ramification at the branch points if it is necessary. Two ramified coverings \( f_1 \) and \( f_2 \) with type \( (\Delta_1, \cdots, \Delta_k) \) are said to be equivalent if there is a homeomorphism \( \pi: \Sigma^h \to \Sigma^h \) such that \( f_1 = f_2 \circ \pi \) and \( \pi \) preserves the preimages and the ramification type of \( f_1 \) and \( f_2 \) at each point \( q_i \in \Sigma^g \). Let \( \mu_{h,n}^g(\Delta_1, \cdots, \Delta_k) \) be the number of equivalent covering of \( \Sigma^g \) by \( \Sigma^h \) with ramification type \( (|\Delta_1|, n-|\Delta_1|, \cdots, |\Delta_k|, n-|\Delta_k|) \), which is the classical Hurwitz number, referring [10]. We call

\[ U_{h,n}^g(\Delta_1, \cdots, \Delta_k) := \prod_{i=1}^k \left( n - |\Delta_i| + m_1(\Delta_i) \right)^{\delta_i^1} \mu_{h,n}^g(\Delta_1, \cdots, \Delta_k) \]  

the shifted Hurwitz number. Note that \( U_{h,n}^g(\Delta_1, \cdots, \Delta_k) \) is nonzero only if the Hurwitz formula

\[ (2 - 2g)n - (2 - 2h) = \sum_{i=1}^k (|\Delta_i| - l(\Delta_i)) \]  

and inequalities

\[ |\Delta_i| \leq n, \text{ for } i = 1, \cdots, k \]  

hold. Moreover to stress that the source Riemannian surface is connected, we call the corresponding shifted Hurwitz number by the connected shifted Hurwitz number, which is denoted by \( CU_{h,n}^g(\Delta_1, \cdots, \Delta_k) \). It is well-known that the disconnected shifted Hurwitz number can be related to the connected shifted...
Hurwitz number by the exponential similar to the classical case, especially, for any given partitions $\Delta_1, \cdots, \Delta_k$ and degree $n$, we have

$$U^h,n_g(\Delta_1, \cdots, \Delta_k) p^{(1)}_{\Delta_1} \cdots p^{(k)}_{\Delta_k} = \sum \prod_{i} \frac{1}{m_i!} [CT^h,n_g(\Delta_1^{(i)}, \cdots, \Delta_k^{(i)}) p^{(1)}_{\Delta_1^{(i)}} \cdots p^{(k)}_{\Delta_k^{(i)}}]^{m_i},$$  \hspace{1cm} (12)

where the sum is over all $((n_1, m_1), (n_2, m_2), \cdots)$ and $((\Delta_1^{(i)}, \cdots, \Delta_k^{(i)})$ obeyed the condition: (1) $\sum_j n_j m_j = n$; (2) $p^{(1)}_{\Delta_1} \cdots p^{(k)}_{\Delta_k} = \prod_i [p^{(1)}_{\Delta_1} \cdots p^{(k)}_{\Delta_k}]^{m_i}$, and where $p^{(1)}_{\Delta_1}, \cdots, p^{(k)}_{\Delta_k}$ are the time-variables, moreover $h_i$ satisfies the following equality:

$$(2 - 2g)n_i - (2 - 2h_i) = \sum_{j=1}^k (|\Delta_j^{(i)}| - l(\Delta_j^{(i)})).$$  \hspace{1cm} (13)

By [5], using the notations of section 2, we have

$$U^h,n_g(\Delta_1, \cdots, \Delta_k) = \frac{1}{n!} \prod_{j=1}^{g} \prod_{a,b \in S_n} [a_j, b_j] \psi(A_{\Delta_1;n}) \cdots \psi(A_{\Delta_k;n})$$  \hspace{1cm} (14)

i.e., $U^h,n_g(\Delta_1, \cdots, \Delta_k)$ equals to $\frac{1}{n!}$ times the coefficient of the identity of the product of $g$-tuple commutators $\prod_{a,b \in S_n} [a, b]$ and $k$-tuple shifted central elements $\psi(A_{\Delta_1;n}), \cdots, \psi(A_{\Delta_k;n})$, thus, according to the theory of representations of the symmetric group $S_n$, which is also equivalent to (referring [1])

$$U^h,n_g(\Delta_1, \cdots, \Delta_k) = \sum_{\lambda \vdash n} \frac{\dim \lambda}{n!} \frac{1}{\dim \lambda} [C_{\Delta_1\Delta_2 \cdots \Delta_k}]_{\chi_\lambda(\Delta_1, 1|\lambda| - |\Delta|)}$$  \hspace{1cm} (15)

which expresses them through the properly normalized characters

$$\phi_{\lambda}(\Delta) \equiv \begin{cases} 0 & \text{for } |\Delta| > |\lambda| \\ \frac{\dim \lambda}{|\lambda|!} C_{\Delta_1\Delta_2 \cdots \Delta_k} [C_{\Delta_1\Delta_2 \cdots \Delta_k}]_{\chi_\lambda(\Delta, 1|\lambda| - |\Delta|)} & \text{for } |\Delta| \leq |\lambda| \end{cases}$$  \hspace{1cm} (16)

where $\lambda$ is a Young diagram of degree $|\lambda|$, and $\dim \lambda$ is the dimension of the irreducible representation of the symmetric group $S_{|\lambda|}$ corresponding to $\lambda$, $\chi_\lambda(\Delta, 1|\lambda| - |\Delta|)$ is the character of a permutation $f \in C_{\Delta, |\lambda|} \subset C[S_{|\lambda|}]$ under the irreducible representation $\lambda$. Let $p = (p_1, p_2, p_3, \cdots)$ be time-variables, then for any partition $\lambda$, it is well known that the normalized characters $\phi_{\lambda}(\Delta)$ are related to the Schur functions $S_{\lambda\{p\}}$ as [11], [12]:

$$S_{\lambda\{p\}} = \sum_{\Gamma'} \frac{\dim \lambda}{|\lambda|!} \phi_{\lambda}(\Gamma') p^{\Gamma'} \delta_{|\lambda|,|\Gamma'|}$$  \hspace{1cm} (17)

or to the shifted Schur functions $S_{\lambda\{p_m + \delta_m,1\}}$ as [12]:

$$S_{\lambda\{p_m + \delta_m,1\}} = \sum_{\Gamma'} \frac{\dim \lambda}{|\lambda|!} \phi_{\lambda}(\Gamma') p^{\Gamma'}$$  \hspace{1cm} (18)
where \( S_{\lambda}\{p_m + \delta_{m,1}\} \) means that \( S_{\lambda} \) is the function of \( p_1 + 1, p_2, p_3, \cdots \).

The following examples is important to determine the lower degree genus expanded cut-and-join operators, see Example 4.2:

**Example 3.1** For any positive integer \( a, b, \) we have

\[
\begin{align*}
CU_{0}^{0,1}((0), (1), (0)) &= 1; \\
CU_{0}^{0,a}((0), (a)) &= \frac{1}{a} \\
CU_{0}^{0,a}((a), (1), (a)) &= 1; \\
CU_{0}^{0,a}((a), (1, 1), (a)) &= \frac{1}{2}(a - 1) \\
CU_{0}^{a,a}((a), (1, 1, 1), (a)) &= \frac{1}{6}(a - 1)(a - 2) \\
CU_{0}^{a,a+b}((a, b), (2), (a + b)) &= 1 - \frac{\delta_{a,b}}{2} \\
CU_{0}^{a,a+b}((a, b), (2, 1), (a + b)) &= (1 - \frac{\delta_{a,b}}{2})(a + b - 2)
\end{align*}
\]

**Remark 3.2** From the above examples, we note that the relative Gromov-Witten invariants even with tangent multiple one is different to the relative Gromov-Witten invariants without tangent conditions.

4 **Shifted genus expanded \( \mathcal{W}_\infty \) algebra**

For every nonnegative integer \( n, \) in [18], we constructed a genus expanded differential algebra \( \mathcal{W}_n, \) which is algebraic isomorphic to the central subalgebra \( Z(\mathbb{C}[S_n]) \) (referring [18], Corollary 3.6). In this section we will construct shifted genus expanded differential algebra \( \mathcal{W}_\infty \) by observing the symplectic surgery and the construction of \( \mathcal{A}_\infty \) algebra in the section 2.

For any partition \( \Delta, \) we call a series of partitions \( \tilde{\Delta} = (\Delta^1, \Delta^2, \cdots) \) a proper Re-partition of \( \Delta \) if \( \Delta = \sum_i \Delta^i \) and \( |\Delta^i| \geq 1, \) for any \( i. \) We denote by \( \mathcal{PP}_\Delta \) the all proper Re-partitions of \( \Delta. \) For any series partition \( (\Gamma'_1, \Gamma'_2, \cdots) \) and \( (\Gamma_1, \Gamma_2, \cdots), \) denote \( \sum_i \Gamma'_i := \Gamma' \) and \( \sum_i \Gamma_i := \Gamma. \) We define \( |\text{Aut}(\Gamma', \tilde{\Delta}, \Gamma)| \) the number of the automorphisms of the triples: \( (\Gamma'_i, \Delta^i, \Gamma_i), i = 1, 2, \cdots \).

At now, we can associate any partition \( \Delta \) a shifted genus expanded cut-and-join differential operator \( W(\Delta, z) \) as follows:

\[
W(\Delta, z) = \sum_{\Delta \in \mathcal{PP}_\Delta} \sum_{\Gamma', \Gamma} \frac{z^{|\Delta| - l(\Delta) + l(\Gamma') - l(\Gamma)} |\text{Aut}(\Gamma', \Delta, \Gamma)|}{|\text{Aut}(\Gamma', \Delta, \Gamma)|} \prod_i [||\Gamma'_i|||\delta_{\Gamma_i}|, |\Gamma'_i|] \frac{\partial}{\partial p_{\Gamma'_i}}
\]
where $$\cdot$$ means that the classical normal ordering product, i.e., all $$p_i$$ will always be set in the front of the all partial operator $$\frac{\partial}{\partial p_j}$$, and genus $$h^i_+ \geq 0$$ is determined by the following:

$$2h^i_+ - 2 = |\Delta^i| - l(\Delta^i) - l(\Gamma^i) - l(\Gamma_i).$$  \hspace{1cm} (27)

**Remark 4.1** In generally, the following equality:

$$U^{h^i_+;|\Gamma^i|}(\Gamma', \Delta, \Gamma) = \sum_{\Delta} \frac{1}{|\text{Aut}(\Gamma', \Delta, \Gamma)|} \prod_{t} \delta_{|\Gamma^i_t|; |\Gamma_t|} \left[CU_0^{h^i_+;|\Gamma^i|}(\Gamma^i_t, \Delta^i, \Gamma_i)\right]$$ \hspace{1cm} (28)

doesn’t hold due to shifting the ramification type, for example:

$$U_0^{-1,7}((4, 3), (2, 1), (4, 2, 1)) = 5\frac{4}{4},$$

but

$$CU_0^{0,4}((4), (1), (4))CU_0^{0,3}((3), (2), (2, 1)) = 1,$$

referring ([12], Section 2.2). We denote by $$U(\Gamma', \Delta, \Gamma)$$ the RHS of the formula (28).

**Example 4.2**

$$W((1), z) = \sum_a a p_a \frac{\partial}{\partial p_a},$$ \hspace{1cm} (29)

$$W((1, 1), z) = \frac{1}{2} \sum_a a(a - 1)p_a \frac{\partial}{\partial p_a} + \frac{1}{2} \sum_{a,b} ab p_a p_b \frac{\partial^2}{\partial p_a \partial p_b},$$ \hspace{1cm} (30)

$$W((1, 1, 1)) = \frac{1}{6} \sum_a a(a - 1)(a - 2)p_a \frac{\partial}{\partial p_a} + \frac{1}{2} \sum_{a,b} ab p_a p_b \frac{\partial^2}{\partial p_a \partial p_b},$$ \hspace{1cm} (31)

$$W((2), z) = \frac{1}{2} \sum_{a,b} (a+b)p_a p_b \frac{\partial}{\partial p_{a+b}} + \frac{1}{2} \sum_{a,b} z^2 ab p_a p_b \frac{\partial^2}{\partial p_{a+b} \partial p_c},$$ \hspace{1cm} (32)

$$W((2, 1), z) = \frac{1}{2} \sum_{a,b} (a+b)(a+b-2)p_a p_b \frac{\partial}{\partial p_{a+b}} + \frac{1}{2} \sum_{a,b,c} (a+b)cp p_a p_b p_c \frac{\partial^2}{\partial p_{a+b} \partial p_c},$$ \hspace{1cm} (33)

\hspace{1cm} 9
Remark 4.3 The formula (32) is the standard cut-and-join operator, referring to [2], [3], [4], [10], [17].

To give out the eigenfunctions of the universal genus expanded cut-and-join operator $W(\Delta, z)$, we define the “genus expanded” Schur functions $S_{\lambda}\{p, z\}$ similar to formula (17) as follows:

$$S_{\lambda}\{p, z\} := \sum_{\Gamma} z^{-|\Gamma'|-t(\Gamma')} \frac{dim\lambda}{|\lambda|!} \phi_{\lambda}(\Gamma') p_{\Gamma'} \delta_{|\lambda|,|\Gamma'|}. \quad (34)$$

Theorem 4.4 For any partitions $\Delta_1, \Delta_2$, as operators on the functions of the time-variables $p = (p_1, p_2, \cdots)$, we have

$$W(\Delta_1, z)W(\Delta_2, z) = \sum_{\Delta_3} z^{(|\Delta_1|+|\Delta_2|)+(|\Delta_2|-|\Delta_1|)+(0)} \hat{C}_{\Delta_1, \Delta_2} W(\Delta_3, z). \quad (35)$$

where $\hat{C}_{\Delta_1, \Delta_2}$ is the structure constants of $A_{\infty}$. Moreover, we have

$$W(\Delta, z)S_{\lambda}\{p, z\} = z^{|\Delta|-t(\Delta)}\phi_{\lambda}(\Delta)S_{\lambda}\{p, z\}. \quad (36)$$

Proof Let us firstly prove the equality (36). For any partition $\Delta'$, we have

$$W(\Delta, z)p_{\Delta'} = \sum_{\Gamma', \Gamma} z^{(|\Delta|-t(\Delta)+|\Delta'|+t(\Gamma')) \delta_{|\Gamma'|,|\Gamma'|}||\Gamma'||U(\Gamma', \Delta, \Gamma)p_{\Gamma} \prod_{i \geq 1} (m_i(\Delta') \mid m_i(\Gamma')) \hat{C}_{\Delta, \Delta'} W(\Delta, z). \quad (37)$$

Assume $\Delta'' := \Gamma + (\Delta' - \Gamma')$, then we have

$$W(\Delta, z)p_{\Delta'} = \sum_{\Delta''} z^{(|\Delta|-t(\Delta)+|\Delta'|+t(\Delta'')) \delta_{|\Gamma'|,|\Gamma'|}||\Gamma'||U(\Gamma', \Delta, \Delta' - (\Delta' - \Gamma')) \prod_{i \geq 1} (m_i(\Delta') \mid m_i(\Gamma')) \hat{C}_{\Delta, \Delta''} W(\Delta, z). \quad (38)$$

The second equality follows from the fact that the shifted Hurwitz number $U^{h+, |\Delta''|}_{0}(\Delta', \Delta, \Delta'')$ can be expressed in the many connected shifted Hurwitz number which especially include the form $CU^0_{0, a}(a, 0, a)$, which corresponds to the divisor $p_a$ that need not to be derived, referring the formula (12). For example, let $\Delta = (2, 1)$, $\Delta' = (4, 3)$, $\Delta'' = (4, 2, 1)$, then we have

$$12U^{1, 7}_{0, 0}((4, 3), (2, 1), (4, 2, 1))$$
$$= 12CU^0_{0, 4}((4), (1), (4))CU^0_{0, 3}((3), (2), (2, 1))$$
$$+ 12CU^0_{0, 4}((4), (0), (4))CU^0_{0, 3}((3), (2, 1), (2, 1)). \quad (39)$$
Thus the equality (36) follows from the formula (34), (38), (15) and the orthogonal relation of the irreducible characteristic of symmetric group ( [18], Lemma 2.1) as following:

\[
W(\Delta, z)S_\lambda\{p, z\} = \sum_{\Delta'} \frac{\dim \lambda}{|\lambda|!} \phi_\lambda(\Delta') \delta_{|\Delta'|,|\lambda|} \sum_{\Delta''} z^{|\Delta|-l(\Delta)-|\Delta''|-l(\Delta'')} p_{\Delta''} |\Delta'| U_0^{h+}|\Delta''|(\Delta', \Delta, \Delta'') \delta_{|\Delta'|,|\Delta''|} \\
= \sum_{\Delta''} z^{|\Delta|-l(\Delta)-|\Delta''|-l(\Delta'')} \sum_{\Delta'} \frac{\dim \lambda}{|\lambda|!} \phi_\lambda(\Delta') \delta_{|\Delta'|,|\lambda|} \times \delta_{|\Delta'|,|\Delta''|} \\
= \sum_{\Delta''} z^{|\Delta|-l(\Delta)-|\Delta''|-l(\Delta'')} \sum_{\Delta'} \frac{\dim \lambda}{|\lambda|!} \phi_\lambda(\Delta) \phi_\mu(\Delta') \delta_{|\Delta'|,|\lambda|} p_{\Delta''} \delta_{|\Delta'|,|\Delta''|} \\
= z^{|\Delta|-l(\Delta)} \phi_\lambda(\Delta) S_\lambda\{p, z\}. \tag{40}
\]

The formula (35) follows from the following equalities:

\[
W(\Delta_1, z)W(\Delta_2, z)S_\lambda\{p, z\} = z^{|\Delta_1|+l(\Delta_1)} \phi_\lambda(\Delta_1)W(\Delta_1, z)S_\lambda\{p, z\} \\
= z^{|\Delta_1|+l(\Delta_1)+|\Delta_2|-l(\Delta_2)} \phi_\lambda(\Delta_1) \phi_\lambda(\Delta_2)S_\lambda\{p, z\} \\
= z^{|\Delta_1|+l(\Delta_1)+|\Delta_2|-l(\Delta_2)} \hat{C}_{\Delta_1, \Delta_2} \phi(\Delta_3)S_\lambda\{p, z\} \\
= z^{|\Delta_1|+l(\Delta_1)+|\Delta_2|-l(\Delta_2)-(|\Delta_1|+|\Delta_2|)} \hat{C}_{\Delta_1, \Delta_2} W(\Delta_3, z)S_\lambda\{p, z\}, \tag{41}
\]

where the third equality comes from ( [12], formula (45)).

**Example 4.5** By Example 4.2, it is easy to check that

\[
W((1), z)W((2), z) = 2W((2), z) + W((2, 1), z), \tag{42}
\]

\[
W((1), z)W((1, 1), z) = 2W((1, 1), z) + 3W((1, 1, 1), z), \tag{43}
\]

which coincide with the Example 2.2 and Theorem 4.4.

**Corollary 4.6** If we normalize the shifted genus expanded cut-and-join operator \(W(\Delta, z)\) by a factor \(z^{-|\Delta|+l(\Delta)}\)

\[
\hat{W}(\Delta, z) := z^{-|\Delta|+l(\Delta)} W(\Delta, z), \tag{44}
\]

then as operators on the space of functions in time-variables \(p = (p_1, p_2, \ldots)\), all shifted genus expanded cut-and-join operators \(\hat{W}(\Delta, z)\) form a commutative associative algebra, denoted by \(W_\infty\), which is called by the *shifted genus expanded algebra*.

\[
\hat{W}(\Delta_1, z)\hat{W}(\Delta_2, z) = \sum_{\Delta_3} \hat{C}_{\Delta_1, \Delta_2}^{\Delta_3} \hat{W}(\Delta_3, z) \tag{45}
\]
i.e., we have an algebraical isomorphism:

\[ W_\infty \cong A_\infty \]

Moreover, \( \hat{W}(\Delta, z) \) have the genus expanded Schur function \( S_\lambda\{p, z\} \) as their eigenfunctions and \( \phi_\lambda(\Delta) \) as the corresponding eigenvalues:

\[ \hat{W}(\Delta, z) S_\lambda\{p, z\} = \phi_\lambda(\Delta) S_\lambda\{p, z\}. \] (47)

**Proof** By straightforward calculation, we omit it.

**Corollary 4.7** The shifted genus expanded \( W_\infty \) algebra is isomorphic to the shifted Schur symmetrical function algebra \( \Lambda^* \) defined by A. Y. Okounkov, and G. I. Olshanskii [16].

**Proof** By ([6], Theorem 9.1), we know that \( A_\infty \) is isomorphic to the shifted Schur symmetrical function algebra \( \Lambda^* \), thus the result follows from Corollary 4.6.

**Remark 4.8** Due to the Corollary 4.7, combined the result of the section 3, we call the genus expanded cut-and-join operator \( W(\Delta, z) \) and its normalization \( \hat{W}(\Delta, z) \) as the _shifted genus expanded cut-and-join operators_.

**Remark 4.9** Note a fact that the structure constants of the shifted genus expanded algebra \( W_\infty \) do not dependent on the parameter \( z \) although the shifted genus expanded cut-and-join operators do, which is because that \( z \) recorded the "lost" genus when we execute symplectic surgery and we can obtain the same results if we execute the symplectic surgery once instead of twice, referring [18], Remark 4.3.

5 Generating functions of the shifted Hurwitz number and its differential equations

To apply the shifted genus expanded cut-and-join operators, we assume that \( \Delta_1, \cdots, \Delta_m \) are any fixed partitions, then we can define a generating function

\[
\Phi_g\{z|(u_1, \Delta_1), \cdots, (u_m, \Delta_m)|p^{(1)}, \cdots, p^{(k)}, p\}
\]

\[
= \sum_{n \geq 0} \sum_{l_1, \cdots, l_m \geq 0} \sum_{\Gamma_1, \cdots, \Gamma_k} \frac{e^{2h-2U_{b,n}^{(1)}|l_1, \cdots, l_m, \Delta_1, \cdots, \Delta_m, \Gamma_1, \cdots, \Gamma_k}}{\prod_{j=1}^{m} u_j \prod_{i=1}^{k} p_i^{(j)} p_{\Gamma}}
\]

\[
= \sum_{n \geq 0} \sum_{\lambda} \sum_{l_1, \cdots, l_m \geq 0} \sum_{\Gamma_1, \cdots, \Gamma_k} \frac{e^{2h-2 dim \lambda \prod_{j=1}^{m} \phi_\lambda\{\Gamma_j\} p_{\Gamma_j} \phi_\lambda(\Gamma) p_{\Gamma}}}{\prod_{j=1}^{m} \prod_{i=1}^{k} \phi_\lambda(\Gamma_j) p_{\Gamma_j} \phi_\lambda(\Gamma) p_{\Gamma}}. \] (48)
where \( z, u_1, \ldots, u_m \) are indeterminate variables, \( p, p^{(1)}, \ldots, p^{(k)} \) are time-variables, and \( 2h - 2 \) is determined by the Hurwitz formula:

\[
(2 - 2g)n - (2 - 2h) = \sum_{j=1}^{m} l_j (|\Delta_j| - l(\Delta_j)) + \sum_{j=1}^{k} (|\Gamma_j| - l(\Gamma_j)) + (|\Gamma| - l(\Gamma)).
\]

Moreover, we have some special initial values [11], [12]:

\[
\Phi_0\{z||p\} = \sum_{\lambda} \sum_{\Delta} z^{-|\Delta| - l(\Delta)} \frac{dim\lambda}{|\lambda|!}^2 \phi(\Delta)p_{\Delta} = \sum_{\lambda} \frac{dim\lambda}{|\lambda|!} S_{\lambda}\{p_m + \delta_m, 1, z\} = \exp \frac{p_1 + 1}{z^2};
\]

\[
\Phi_0\{z||p^{(1)}, p\} = \sum_{\lambda} \sum_{\Delta_1, \Delta_2} z^{2|\lambda| - |\Delta_1| - |\Delta_2| - l(\Delta_1) - l(\Delta_2)} \left( \frac{dim\lambda}{d!} \right)^2 \phi(\Delta_1)\phi(\Delta_2)p_{\Delta_1}p_{\Delta_2} = \sum_{\lambda} z^{2|\lambda|} S_{\lambda}\{p_m + \delta_m, 1, z\} S_{\lambda}\{p_m + \delta_m, 1\} = \exp \left( \sum_{m \geq 1} \frac{(p_m + \delta_m)(p_m + \delta_m - 1)!}{z^2} \right).
\]

where we have to add one to \( p^{(1)}_1 \) and \( p_1 \) due to the shifting, referring the equality (19).

Remark 5.1 In the formula (48), we sum all the degree of the covering map \( n \geq 0 \), which is different from [18] formula (23).

Theorem 5.2 For any \( i \), we have

\[
\frac{\partial \Phi_g\{z|(u_1, \Delta_1), \ldots, (u_m, \Delta_m)|p^{(1)}, \ldots, p^{(k)}, p\}}{\partial u_i} = W(\Delta, z)\Phi_g\{z|(u_1, \Delta_1), \ldots, (u_m, \Delta_m)|p^{(1)}, \ldots, p^{(k)}, p\}.
\]

Proof Obviously, we have

\[
\frac{\partial \Phi_g\{z|(u_1, \Delta_1), \ldots, (u_m, \Delta_n)|p^{(1)}, \ldots, p^{(k)}, p\}}{\partial u_i} = \sum_{n \geq 0} \sum_{l_1, \ldots, l_m \geq 0} \sum_{\Gamma_1, \ldots, \Gamma_k} z^{2h_2n - 2h_1} \mu_{g,n}(\Delta_1, \ldots, \Delta_1, \ldots, \Delta_m, \ldots, \Delta_m, \Gamma_1, \ldots, \Gamma_k, \Gamma)\left(\frac{u_i}{l_i - 1}\right)! \prod_{j=1, j \neq i}^{m} \frac{(u_j)^{l_j}}{l_j!} \prod_{j=1}^{k} \prod_{p} (\frac{p}{l_j})^{l_j} p_.
\]
We can write RHS of equation (52) as
\[
\text{RHS} = \sum_{n \geq 0} \sum_{l_1 \geq 1, \ldots, l_m \geq 0} \sum_{\Gamma_1, \ldots, \Gamma_k, \Gamma'} \mu_{\Gamma'}(\Delta_1, \ldots, \Delta_1, \ldots, \Delta_1, \ldots, \Delta_m, \ldots, \Delta_m, \Gamma_1, \ldots, \Gamma_k, \Gamma') \times u_{\Delta_1}^{h_{\Delta_1}} \times u_{\Delta_2}^{h_{\Delta_2}} \times \cdots \times u_{\Delta_m}^{h_{\Delta_m}} \times \prod_{j=1}^{k} p^{(j)} \big| W(\Delta_i, z) \big| p_{\Gamma'},
\]
where \(\tilde{l}_i\) means that we omit \(l_i\), and \(2h^i - 2\) is also determined by the Hurwitz formula:
\[
(2-2g)n - (2-2h^i) = \sum_{j=1}^{m} l_j(|\Delta_j| - l(\Delta_j)) - (|\Delta_i| - l(\Delta_i)) + \sum_{j=1}^{k} (|\Gamma_j| - l(\Gamma_j)) + (|\Gamma'| - l(\Gamma')), \quad (53)
\]
Then the theorem follows from the following facts:

- **Fact (1):** (referring the formula (38))
  \[
  W(\Delta, z) p_{\Delta'} = \sum_{\Delta'} z^{l(\Delta) - l(\Delta')} p_{\Delta'} \big| \Delta''^0(\Delta', \Delta, \Delta'') \delta_{\Delta', \Delta''}, \quad (54)
  \]
- **Fact (2):** \(CU_0^{h_{\Delta^i}, [\beta]}(\alpha, \Delta^i, \beta) \delta_{|\alpha|, |\beta|} \neq 0\) only if
  \[
  2h^i - 2 = -l(\alpha) + (|\Delta^i| - l(\Delta^i)) - l(\beta). \quad (55)
  \]

Immediately, we have

**Corollary 5.3**
\[
\Phi_{g_\text{c}}(\beta|\Delta_1, \ldots, \Delta_n) | p^{(1)}, \ldots, p^{(k)}, p \} = \prod_{i=1}^{n} \exp(u_i W(\Delta_i, z)) \Phi_{g_\text{c}}(\beta| p^{(1)}, \ldots, p^{(k)}, p) \quad (56)
\]

**Example 5.4** Assume \(\Delta = (2, 1)\), then by the formula (33), (50) and (56), we get the generating function(with any genus):
\[
\Phi_0(\beta|\Delta, (2, 1)) | p = \frac{1}{6} (p_1 + 1)^2 z^6 + \frac{1}{2} u(p_1 + 1) p_2 z^{-4} + \frac{1}{2} u^2 p_3 z^{-2} + \frac{1}{4} u^2 (p_1 + 1)^3 z^{-4} + \frac{3}{4} u^3 (p_1 + 1) p_2 z^{-2} + \cdots.
\]
where we omit the higher terms of \(u\) and \(n\).

**Acknowledgements** The author would like to thank the every members of geometry team of Mathematics College in Sichuan University, and Prof. Yongbin Ruan, Prof. Yuping Tu, Prof. Guohui Zhao, Prof. Qi Zhang for their helpful discussions.
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