Multipartite rational functions

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Multivariable Operator Theory

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Plan

1. Introduction: noncommutative rational functions

2. Multipartite rational functions: construction and universality

3. Amitsur’s theorem on multipartite identities

4. Noncommutative function theory perspective
Nc rational expressions

\(\mathbb{k}\) a field of characteristic 0, \(x = \{x_1, \ldots, x_g\}\) freely noncommuting letters, \(\mathbb{k}<x>\) the free algebra of nc polynomials.

\(\mathcal{R}_{\mathbb{k}}(x)\) nc rational expressions built from \(\mathbb{k}<x>\) using 
+ , \cdot, −¹, (, ).
e.g. \(x_2(1 + x_1x_2^{-1}(x_1 − 3))^{-1}\), \((x_1x_2)^{-1} − x_2^{-1}x_1^{-1}\), \((1 − x_1^{-1}x_1)^{-1}\).
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**Evaluations** of \( r \in \mathcal{R}_{\mathbb{k}}(x) \) on tuples of matrices:

- \( M_n(\mathbb{k})^g \rightarrow M_n(\mathbb{k}) \) for all \( n \in \mathbb{N} \);
- \( \text{dom } r \subseteq \bigcup_n M_n(\mathbb{k})^g \) the domain of \( r \);
- \( r \) is degenerate if \( \text{dom } r = \emptyset \) and nondegenerate otherwise.
Nc rational expressions

\( k \) a field of characteristic 0, \( x = \{x_1, \ldots, x_g\} \) freely noncommuting letters, \( k<x> \) the free algebra of nc polynomials.

\( R_k(x) \) nc rational expressions built from \( k<x> \) using 
\(+, \cdot, -1, (, )\), 
e.g. \( x_2(1 + x_1x_2^{-1}(x_1 - 3))^{-1}, (x_1x_2)^{-1} - x_2^{-1}x_1^{-1}, (1 - x_1^{-1}x_1)^{-1} \).

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Define equivalence relation for nondegenerate expressions: \( r_1 \sim r_2 \) iff \( r_1(X) = r_2(X) \) for all \( X \in \text{dom } r_1 \cap \text{dom } r_2 \).
Classes of nondegenerate expressions are called **nc rational functions** and form a skew field $\mathbb{k}\langle x \rangle$, the **free skew field**.
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This construction is due to Helton, McCullough, Vinnikov. Others:

- evaluations on $\infty$-dim skew fields (Amitsur)
- full matrices over $\mathbb{k}\langle x \rangle$ (Cohn)
- Malcev-Neumann series of a free group (Lewin)
- grading on a free Lie algebra (Lichtman)
- unbounded operators associated to a von Neumann algebra (Linnell)
**Nc function context**

Evaluations of nc rational functions respect direct sums and similarities, so they are nc functions equipped with nc difference-differential calculus (Kaliuzhnyi-Verbovetskyi, Vinnikov).
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Here $f = (f_n)_n$, $f_n : \Omega_n \subseteq M_n(\mathbb{k}) \rightarrow M_n(\mathbb{k})$, is a nc function if $f_{m+n}(X \oplus Y) = f_m(X) \oplus f_n(Y)$ and $f_n(PXP^{-1}) = Pf_n(X)P^{-1}$. 
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If \( f \) is a nc function, then

\[
\begin{pmatrix} X & H \\ 0 & Y \end{pmatrix} = \begin{pmatrix} f(X) & \sum_j \Delta_j(f)(X, Y)H_j \\ 0 & f(Y) \end{pmatrix},
\]

where \( \Delta_j \) are (left) directional nc difference-differential operators

\[
\Delta_j(f)_{m,n} : \Omega_m \times \Omega_n \to \text{Hom}_\mathbb{k}(\mathbb{k}^{m\times n}, \mathbb{k}^{m\times n}).
\]

(higher order nc functions)
Polynomial example

For example, if $f = x_1^2 x_2 x_1$, then the directional nc difference-differential operators of $f$ at $(X_1, X_2; Y_1, Y_2)$ are given by

$$\Delta_1(f)(X_1, X_2; Y_1, Y_2)H = HY_1 Y_2 Y_1 + X_1 HY_2 Y_1 + X_1^2 X_2 H$$

$$\Delta_2(f)(X_1, X_2; Y_1, Y_2)H = X_1^2 HY_1$$
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$$\Delta_2(f)(X_1, X_2; Y_1, Y_2)H = X_1^2 H Y_1$$
$$= X_1^2 \otimes Y_1$$

Hence $\Delta_1, \Delta_2 : \mathbb{k}\langle \mathbf{x} \rangle \to \mathbb{k}\langle \mathbf{x} \rangle \otimes \mathbb{k}\langle \mathbf{y} \rangle$. 
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Applying $\Delta_j$ further: $\mathbb{k}<\mathbf{x}^{(1)}> \otimes \cdots \otimes \mathbb{k}<\mathbf{x}^{(G)}>$. What are higher order nc rational functions?
Universal skew field of fractions

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Fix a ring $R$. A skew field $U$ is a **SFF** of $R$ if $R \subset U$ and $R$ generates $U$ as a skew field.
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Furthermore, \( U \) is a **USFF** of \( R \) if for every matrix \( A \) over \( R \) and a homomorphism \( \phi : R \rightarrow D \) into a skew field \( D \),

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\phi(A) \text{ invertible over } D \quad \Rightarrow \quad A \text{ invertible over } U.
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Furthermore, $U$ is a **USFF** of $R$ if for every matrix $A$ over $R$ and a homomorphism $\phi : R \to D$ into a skew field $D$,

$$\phi(A) \text{ invertible over } D \implies A \text{ invertible over } U.$$ 

This notion is due to Cohn (70s). It is a universal property in the category of skew fields with epimorphisms from $R$; morphisms are specializations (local homomorphisms) between skew fields.
Rings with USFF

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- firs, e.g. $\mathbb{k}\langle x \rangle$; semifirs, e.g. $\mathbb{k}[[x]]$, or nc functions analytic at the origin

(semi) free ideal ring: every (finitely generated) left ideal is a free left module of unique rank

Tensor product of free algebras is not a pseudo-Sylvester domain (apart from trivial cases). Cohn ('97) proved that $\mathbb{k}\langle x \rangle \otimes \mathbb{k}\langle y \rangle$ has the USFF, but was unable to show this for more factors.

Today: $\mathbb{k}\langle x(1) \ldots x(G) \rangle := \mathbb{k}\langle x(1) \rangle \otimes \ldots \otimes \mathbb{k}\langle x(G) \rangle$ admits the USFF $\mathbb{k}(\mathbb{k}\langle x(1) \ldots x(G) \rangle)$ for every $G \in \mathbb{N}$.
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**Today:** \( \mathbb{k}<x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)}> := \mathbb{k}<x^{(1)}> \otimes \cdots \otimes \mathbb{k}<x^{(G)}> \) admits the USFF \( \mathbb{k}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)}\rangle \) for every \( G \in \mathbb{N} \).
For $i = 1, \ldots, G$ let $\mathbf{x}^{(i)} = \{x^{(i)}_1, \ldots, x^{(i)}_{g_i}\}$ be sets of freely noncommuting variables and $\mathbf{x} = \mathbf{x}^{(1)} \cup \cdots \cup \mathbf{x}^{(G)}$. 
For $i = 1, \ldots, G$ let $x^{(i)} = \{x_{1}^{(i)}, \ldots, x_{g_{i}}^{(i)}\}$ be sets of freely noncommuting variables and $x = x^{(1)} \cup \cdots \cup x^{(G)}$.

Given $r \in \mathcal{R}_{k}(x)$ and $X^{(i)} \in M_{n_{i}}(\mathbb{K})^{g_{i}}$ we define **mp-evaluation** of $r$ at $X = (X^{(1)}, \ldots, X^{(G)})$ as

$$r^{\text{mp}}(X) := r \left( X^{(1)} \otimes I \otimes \cdots \otimes I, I \otimes X^{(2)} \otimes \cdots \otimes I, \ldots, I \otimes I \otimes \cdots \otimes X^{(G)} \right)$$

in $M_{n_{1} \cdots n_{G}}(\mathbb{K})$, if all nested inverses exist.

Here $\otimes$ denotes Kronecker’s product; note that $(A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I)$.
Example

For example, let $x = \{x_1, x_2\}$, $y = \{y_1, y_2\}$ and
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r = (x_1 + y_2x_2x_1y_1)^{-1} - y_2^{-1}.
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Then

$$r(X; Y) = (X_1 \otimes I + (I \otimes Y_2)(X_2 \otimes I)(X_1 \otimes I)(I \otimes Y_1))^{-1}$$

$$- (I \otimes Y_2)^{-1}$$
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r(X; Y) = (X_1 \otimes I + (I \otimes Y_2)(X_2 \otimes I)(X_1 \otimes I)(I \otimes Y_1))^{-1} - (I \otimes Y_2)^{-1} = (X_1 \otimes I + X_2 X_1 \otimes Y_2 Y_1)^{-1} - I \otimes Y_2^{-1}.
\]
Multipartite rational functions

Given \( r \in \mathcal{R}_k(x) \) let

\[
\text{dom}^{\text{mp}} r \subseteq \bigcup_{n_1, \ldots, n_G} M_{n_1}(\mathbb{K})^{g_1} \times \cdots \times M_{n_G}(\mathbb{K})^{g_G}
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be its \textbf{mp-domain}.
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On the set of rational expressions with non-empty mp-domains we define equivalence relation $r_1 \sim r_2$ if and only if $r_1^{mp}(X) = r_2^{mp}(X)$ for all $X \in \text{dom}^{mp} r_1 \cap \text{dom}^{mp} r_2$. The equivalence class of $r$ is denoted $[r]$ and called a $\textbf{multipartite rational function}$. 

The set of multipartite rational functions is denoted $\mathbb{R}_k(x^{(1)} \cdots x^{(G)})$ and endowed with the natural ring structure.

Theorem $\mathbb{R}_k(x^{(1)} \cdots x^{(G)})$ is a SFF of $\mathbb{R}_k(x^{(1)} \cdots x^{(G)})$. 

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On the set of rational expressions with non-empty mp-domains we define equivalence relation \( r_1 \sim r_2 \) if and only if \( r_1^{\text{mp}}(X) = r_2^{\text{mp}}(X) \) for all \( X \in \text{dom}^{\text{mp}} r_1 \cap \text{dom}^{\text{mp}} r_2 \). The equivalence class of \( r \) is denoted \( \bar{r} \) and called a multipartite rational function.

The set of multipartite rational functions is denoted \( \mathbb{k} \langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle \) and endowed with the natural ring structure.

Theorem

\( \mathbb{k} \langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle \) is a SFF of \( \mathbb{k} \langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle \).
Basic properties

(1) Let $\mathbf{M} \in M_d(\mathbb{K}[x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)}])$. Then $\mathbf{M}$ is invertible if and only if $\mathbf{M}(X)$ is invertible (as a matrix over $\mathbb{K}$) for some $X \in \text{dom } \mathbf{M}$.
Basic properties

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(2) Let $r \in \mathbb{k}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle$ and $Y \in \text{dom} \, r$ with $Y^{(1)} \in M_d(\mathbb{k}\langle x^{(2)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle)$. Then there exists $S \in M_d(\mathbb{k}\langle x^{(2)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle)$ such that

$$r(Y^{(1)}, X) = S(X)$$

for all $X \in \text{dom} \, S$ such that $(Y^{(1)}, X) \in \text{dom} \, r$. 
Basic properties

(1) Let $M \in M_d(\mathbb{K}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle)$. Then $M$ is invertible if and only if $M(X)$ is invertible (as a matrix over $\mathbb{K}$) for some $X \in \text{dom } M$.

(2) Let $r \in \mathbb{K}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle$ and $Y \in \text{dom } r$ with $Y^{(1)} \in M_d(\mathbb{K})^{G_1}$. Then there exists $S \in M_d(\mathbb{K}\langle x^{(2)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle)$ such that

$$r(Y^{(1)}, X) = S(X)$$

for all $X \in \text{dom } S$ such that $(Y^{(1)}, X) \in \text{dom } r$.

(3)

$$\mathbb{K}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G_1)} \rangle \cap \mathbb{K}\langle x^{(G_0)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle = \mathbb{K}\langle x^{(G_0)} \leftrightarrow \cdots \leftrightarrow x^{(G_1)} \rangle$$

holds in $\mathbb{K}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle$ for $G_0 \leq G_1$. 


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(2) Let $r \in \mathbb{k}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle$ and $Y \in \text{dom } r$ with $Y^{(1)} \in M_d(\mathbb{k})_{G_1}$. Then there exists $S \in M_d(\mathbb{k}\langle x^{(2)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle)$ such that $r(Y^{(1)}, X) = S(X)$ for all $X \in \text{dom } S$ such that $(Y^{(1)}, X) \in \text{dom } r$.

(3) $\mathbb{k}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G_1)} \rangle \cap \mathbb{k}\langle x^{(G_0)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle = \mathbb{k}\langle x^{(G_0)} \leftrightarrow \cdots \leftrightarrow x^{(G_1)} \rangle$ holds in $\mathbb{k}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle$ for $G_0 \leq G_1$.

(4) The centralizer of $\mathbb{k}\langle x^{(1)} \rangle$ in $\mathbb{k}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle$ equals $\mathbb{k}\langle x^{(2)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle$ if $|x_1| > 1$. 

Auxiliary result

Let $D$ be an arbitrary skew field containing $k$. Then $D \otimes k\langle x \rangle$ is a fir (Cohn); in particular, it has the USFF.
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Let $D$ be an arbitrary skew field containing $k$. Then $D \otimes k[x]$ is a fir (Cohn); in particular, it has the USFF.

Proposition

Let $M$ be a $d \times d$ matrix over $D \otimes k[x]$. Then $M$ is invertible over the USFF of $D \otimes k[x]$ if and only if $M(X) \in M_d(D \otimes M_n(k)) \cong M_{dn}(D)$ is invertible for some $X \in M_n(k)$.
Auxiliary result

Let $D$ be an arbitrary skew field containing $\mathbb{k}$. Then $D \otimes \mathbb{k}<x>$ is a fir (Cohn); in particular, it has the USFF.

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Ingredients: Cohn’s theory of USFFs, PI theory, skew field constructions and power series expansions.
Universality

Theorem
\[ \mathbb{k}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle \text{ is the USFF of } \mathbb{k}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle. \]
Universality

Theorem
\[ k \langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle \] is the USFF of \( k \langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle \).

Corollary

Let \( r \in \mathcal{R}_k(x) \). TFAE:

(i) \( r^{mp}(X) = 0 \) for all \( X \in \text{dom}^{mp} r \);

(ii) \( r(X) = 0 \) for all \( X \in \text{dom} r \) such that \( [X^{(i_1)}_{j_1}, X^{(i_2)}_{j_2}] = 0 \) for \( i_1 \neq i_2 \);

(iii) for every skew field \( D \), \( r(a) \in \{0, \text{undef}\} \) for every tuple \( a \in D^{g_1 + \cdots + g_G} \) such that \( [a^{(i_1)}_{j_1}, a^{(i_2)}_{j_2}] = 0 \).
Sketch of the proof

Let $M$ be a $d \times d$ matrix over $\mathbb{k}<x^{(1)}\leftrightarrow \cdots \leftrightarrow x^{(G)}>$ and let
$\phi : \mathbb{k}<x^{(1)}\leftrightarrow \cdots \leftrightarrow x^{(G)}> \rightarrow D$ be a homomorphism into a skew field $D$ such that $\phi(M)$ is invertible over $M$. 
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1. Write $a_j^{(i)} = \phi(x_j^{(i)})$; $M(a^{(1)}, a^{(2)}, \ldots)$ invertible over $D$
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3. proposition: $M(X^{(1)}, a^{(2)}, \ldots) \in M_{dn_1}(D)$ invertible for some $X \in M_{n_1}(\mathbb{K})^{n_1}$
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3. proposition: $M(X^{(1)}, a^{(2)}, \ldots) \in M_{dn_1}(D)$ invertible for some $X \in M_{n_1}(\mathbb{k}^{g_1})$

4. induction: $N = M(X^{(1)}, x^{(2)}, \ldots)$ invertible over $\mathbb{k}<x^{(2)}\leftrightarrow \cdots \leftrightarrow x^{(G)}>$
Sketch of the proof

Let $M$ be a $d \times d$ matrix over $\mathbb{k}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle$ and let
$\phi : \mathbb{k}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle \to D$ be a homomorphism into a skew field $D$ such that $\phi(M)$ is invertible over $M$.

1. Write $a_j^{(i)} = \phi(x_j^{(i)})$; $M(a^{(1)}, a^{(2)}, \ldots)$ invertible over $D$

2. $D \otimes \mathbb{k}\langle x^{(1)} \rangle$ fir: $M(x^{(1)}, a^{(2)}, \ldots)$ invertible over the USFF of $D \otimes \mathbb{k}\langle x^{(1)} \rangle$

3. proposition: $M(X^{(1)}, a^{(2)}, \ldots) \in M_{dn_1}(D)$ invertible for some $X \in M_{n_1}(\mathbb{k})^{g_1}$

4. induction: $N = M(X^{(1)}, x^{(2)}, \ldots)$ invertible over $\mathbb{k}\langle x^{(2)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle$

5. basic property: $N(X^{(2)}, \ldots)$ invertible for some $X^{(i)} \in M_{n_i}(\mathbb{k})^{g_i}$
Sketch of the proof

Let $M$ be a $d \times d$ matrix over $\mathbb{L}_k<x^{(1)}\leftrightarrow\cdots\leftrightarrow x^{(G)}>$ and let
\( \phi: \mathbb{L}_k<x^{(1)}\leftrightarrow\cdots\leftrightarrow x^{(G)}> \to D \) be a homomorphism into a skew field $D$ such that $\phi(M)$ is invertible over $M$.

1. Write $a_j^{(i)} = \phi(x_j^{(i)})$; $M(a^{(1)}, a^{(2)}, \ldots)$ invertible over $D$

2. $D \otimes \mathbb{L}_k<x^{(1)}>$ fir: $M(x^{(1)}, a^{(2)}, \ldots)$ invertible over the USFF of $D \otimes \mathbb{L}_k<x^{(1)}>$

3. proposition: $M(X^{(1)}, a^{(2)}, \ldots) \in M_{dn_1}(D)$ invertible for some $X \in M_{n_1}(\mathbb{L})^{g_1}$

4. induction: $N = M(X^{(1)}, x^{(2)}, \ldots)$ invertible over $\mathbb{L}_k\langle x^{(2)}\leftrightarrow\cdots\leftrightarrow x^{(G)}\rangle$.

5. basic property: $N(X^{(2)}, \ldots)$ invertible for some $X^{(i)} \in M_{n_i}(\mathbb{L})^{g_i}$

6. $M(X^{(1)}, X^{(2)}, \ldots)$ invertible, so $M$ invertible over $\mathbb{L}_k\langle x^{(1)}\leftrightarrow\cdots\leftrightarrow x^{(G)}\rangle$. 
Higher order nc rational functions

Let \( r \in \mathbb{K}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle \). Then

1. \( r \) respects direct sums in the first factor and up to canonical shuffle in other factors; \((A \otimes B \sim B \otimes A)\)

2. \( r \) respects similarities in every factor.

Hence \( r \) is a nc function of order \( G - 1 \).
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Directional nc difference-differential operators

\[
\Delta^{(i)}_j : \mathbb{k}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle \rightarrow \mathbb{k}\langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x'\overset{(i)}{\leftrightarrow} x^{(i)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle
\]

satisfy the usual properties.
Higher order nc rational functions cont’d

Furthermore, diagrams like

$\llbracket \langle x^{(1)} \cup x^{(2)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle \rrbracket \rightarrow \llbracket \langle x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle \rrbracket$

$\downarrow \Delta_j^{(1)}$

$\llbracket \langle x'^{(1)} \cup x'^{(2)} \leftrightarrow x^{(1)} \cup x^{(2)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle \rrbracket \rightarrow \llbracket \langle x'^{(1)} \leftrightarrow x^{(1)} \leftrightarrow \cdots \leftrightarrow x^{(G)} \rangle \rrbracket$

commute, where $\rightarrow$ are specializations (local homomorphisms) between skew fields.
Thank you,
and happy birthday!