Fast Entropy Estimation for Natural Sequences

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It is well known that to estimate the Shannon entropy for symbolic sequences accurately requires a large number of samples. When some aspects of the data are known it is plausible to attempt to use this to more efficiently compute entropy. A number of methods having various assumptions have been proposed which can be used to calculate entropy for small sample sizes. In this paper, we examine this problem and propose a method for estimating the Shannon entropy for a set of ranked symbolic “natural” events. Using a modified Zipf-Mandelbrot-Li law and a new rank-based coincidence counting method, we propose an efficient algorithm which enables the entropy to be estimated with surprising accuracy using only a small number of samples. The algorithm is tested on some natural sequences and shown to yield accurate results with very small amounts of data.

I. INTRODUCTION

Machine learning methods typically rely on forming models based on statistical properties of observed data. An area of importance in this regard is information theoretic methods which involve computing Shannon entropy and mutual information. The idea that the randomness of a message can give a measure of the information it conveys formed the basis of Shannon’s entropy theory which gives a means of assigning a value to the information carried within a message [1], [2]. The way in which symbols occurring consecutively, can be derived as

\[ H_N(X|B) = - \sum_{i,j} p(b_i, x_j) \log_2(p(x_j|b_i)) \]

where \( b_i \in \sum_{N-1} \) is a block of \( N-1 \) symbols, \( x_j \) is an arbitrary symbol following \( b_i \), \( p(b_i, x_j) \) is the probability of the \( N \)-gram \( (b_i, x_j) \), \( p(x_j|b_i) \) is the conditional probability of \( x_j \) occurring after \( b_i \) and is given by \( p(b_i, x_j)/p(b_i) \).

One of the limitations of computing entropy accurately is the dependence on large amounts of data, even more so when computing \( N \)-gram entropy. Estimates of entropy based on letter, word and \( N \)-gram statistics have often relied on large data sets [3], [4]. The reliance on long data sequences to estimate the probability distributions used to calculate entropy and attempts to overcome this in coding schemes is discussed in [5] where they provide an estimate of letter entropy extrapolated for infinite text lengths. A method of estimating the number of samples required to compute entropy was proposed in [6] which showed that a very large number of samples may be required to do this accurately.

Various approaches to estimating entropy over finite sample sizes have been considered. A method of computing the entropy of dynamical systems which corrects for statistical fluctuations of the sample data over finite sample sizes has been proposed in [7]. Estimation techniques using small datasets have been proposed in [8], and an online approach for estimating entropy in limited resource environments was proposed in [9]. Entropy estimation over short symbolic sequences was considered in the context of dynamical time series models based on logistic maps and correlated Markov chains, where an effective shortened sequence length was proposed which accounted for the correlation effect [10]. A novel approach for calculating entropy using the idea of estimating probabilities from a quadratic function of the inverse number of symbol coincidences was proposed in [11]. A limitation of this method was that it assumed equiprobable symbols. The difficulty of estimating entropy due to the heavy tailed distribution of natural sequences has been recognized, where it has been shown that the bias using classical estimators depends the sample size and the characteristics of the heavy-tailed distribution [12]. A Bayesian model approach to inferring the probability distributions has been considered at length in [13] and [14]. A computationally efficient method for calculating entropy based on a James-Stein-type shrinkage estimator was proposed by Hausser and Strimmer in [15].

In this paper, by considering a model for the probability distributions of natural sequence data, we propose an extension to the algorithm in [11] which enables a fast method of estimating entropy using a small number of

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samples. The proposed algorithm is derived in the subsequent sections and simulations are given showing its effectiveness.

II. PROPOSED ALGORITHM FOR ESTIMATING ENTROPY

A. Coincidence Counting For Equiprobable Symbols

To compute Shannon entropy by estimating the symbol probabilities using conventional histogram plug-in methods is effective for small alphabet sizes, however for non equiprobable symbols with a large alphabet size, a very large number of symbols may be required. For a given alphabet size $M$, to estimate the entropy with some degree of accuracy it is normally required to estimate the probabilities of $M$ symbols. Another approach is to adopt a parametric model of the symbol probabilities. In this case, the idea is to form an invertible model $J(M)$ of the relationship between the model parameters and some observable statistical feature of the data. Then, the model is inverted and the statistical features of the actual data are observed which enables the model parameters and hence entropy to be estimated.

The method of coincidence detection is based on the idea that a discrete (or symbolic) random variable $x$ which may take on a finite number $M$ of distinct values $x_i \in \{x_1, \ldots, x_M\}$ with probabilities $p(x_i), i \in [1, M]$. Consider the case where $p(x_i) = p(x_j) \forall i, j \in [1, M]$, that is, the symbols are equiprobable. Hence we may proceed as follows. The probability of drawing any symbol on the first try followed by any other different symbol on the second try, that is, any two non repeating symbols is

$$F(2; M) = \frac{M(M-1)}{M^2}$$

and hence the probability of drawing any two repeating or identical symbols out of the entire set is

$$F(2; M) = 1 - \frac{M(M-1)}{M^2}$$

Extending this to $n$ draws, the probability of drawing any symbol on the first try followed by any other different symbol up to the $n$ th draw up to $n$ symbols is

$$F(n; M) = \frac{M(M-1)\cdots (M-n+1)}{M^n}$$

Therefore, it follows that the probability of drawing any $q_n \in [2, \ldots, n]$ identical symbols (ie one or more repeating symbols in any position) out of the entire set is given by

$$F(n; M) = 1 - \frac{M(M-1)\cdots (M-n+1)}{M^n}$$  (6)

To compute the probability of a first coincidence occurring exactly at the $n$ th symbol for $1 < n < M$, means that it is necessary to compute the probability of drawing no repeating symbols in the entire sequence up to the $(n-1)$ th draw given by $F(n-1; M)$ and consequently drawing any $q_{n-1} \in [2, \ldots, n-1]$ identical symbols is given by $F(n-1; M)$. Hence the required probability is given by (11):

$$f(n; M) = F(n; M) - F(n-1; M)$$  (7)

The expectation of the discrete parameter $n$ and its associated probability $f(n; M)$ is given by:

$$E[n] = J(n; M) = \sum_{n=0}^{M} nf(n; M)$$  (8)

Since $n$ is a function of $M$, define

$$D(M) = E[n].$$  (10)

The innovative approach by [11] is to recognize that an invertible smooth curve can be constructed with $D(M)$ as a function of $M$ by using a sequence of uniform iid random data. Now, since Shannon entropy $H_N(X; M)$ is defined as a function of $M$ and for equiprobable symbols, we have

$$H_0(M) = \log_2(M)$$  (11)

This indicates that if the unknown value of $M$ can be estimated directly from the data, then the entropy can be determined.

A model for estimating $M$ can be obtained by forming an appropriate, eg. polynomial model, inverting the original equation found in [11], as

$$\hat{M}(D) = G(\Theta; D) = \sum_{i=0}^{n_p} \theta_i D^i$$  (13)

and appropriate values for the parameters $\theta_i$ by fitting a curve to an ensemble of data. In [11], setting $n_p = 2$, the values obtained were $\theta_0 = 0.1272, \theta_1 = -0.8493, \theta_2 = 0.6366$. The entropy can then be estimated as

$$\hat{H_0} = \log_2(\hat{M})$$  (14)

Experimentally, this approach was shown to provide good accuracy using only a small number of symbol coincidence distance observations [11]. The limitation however is the assumption of equiprobable symbol probabilities. In the next section we propose a new algorithm which extends this method to the case of non-equiprobable symbols.
B. Coincidence Counting For Non-Equiprobable Symbols

For natural sequences, including natural language, a mechanism to model the non-equiprobable symbolic probabilities is to use a Zipfian law where the probability of information events can generally be ranked into monotonically decreasing order. For natural language, it has been shown that Zipf's law approximates the distribution of probabilities of letter or words across a corpus of sufficient size for the larger probabilities. We do not rely on Zipf's law to provide a universal model of human language or other natural sequences (see for example, the discussions in [17], [18], [19]). Nevertheless, Zipfian laws have been proven to be useful as a means of statistically characterizing the observed behaviour of symbolic sequences of data ([20]) and are useful in forming a model of symbolic information transmission which is organized on the basis of sentences made by words in interaction with each other [21]. Here we adopt the Zipf-Mandelbrot-Li law described in [6], as a model for natural sequences with non-equiprobable distribution of symbols.

In the former case, we have a model defined by \( f(n; M) \) from which a smooth invertible model \( J(n; M) \) is obtained. Thus we can obtain a model \( G(\Theta; D) \) which enables the entropy to be estimated directly from the symbol coincidences. To derive a model for the non-equiprobable case, one approach is to model individual \( D_i \) and assume some form of discrete probability related to each distance.

The method we propose is that following [21, 24] a model \( J'(n; M, r) \) is defined for each symbol, indexed by rank \( r \). Therefore, for any given \( M \), each symbol of a specified rank \( r \) can be treated as being equiprobable. Thus, if the probability can be determined for each symbol in terms of its rank, and this can be related to the overall entropy, then the same approach can be followed as for the equiprobable case.

Consider a reformulation of (6) where:

\[
\hat{F} (n; M) = \frac{M(M - 1) \cdots (M - n + 1)}{M^n} = \frac{M}{M} \cdot \left(1 - \frac{1}{M}\right) \cdot \left(1 - \frac{2}{M}\right) \cdots \left(1 - \frac{1}{M}\right) = \frac{1}{M^n} (1 - P_2) \cdot (1 - P_3) \cdots (1 - P_{n-1}) \tag{15}
\]

using the identity \((M - n + 1)/M = 1 - (n - 1)/M\) and \( P_h \) is the probability of independently drawing \( h - 1 \) identical symbols from a set of \( M \) in \( h - 1 \) draws. In the case of equiprobable symbols, we have

\[
\hat{P}_h(M) = 1 - \frac{h - 1}{M} \tag{16}
\]

Now, for a natural sequence where the probability of occurrence of a given word can be defined in terms of rank, the Zipf-Mandelbrot-Li law provides an expression for the probability to be used in (15) where ([6], [21], [22]):

\[
P(r; M) = \frac{\gamma'}{(r + \beta)\alpha} \tag{17}
\]

and for iid samples, the constants can be computed as (17):

\[
\alpha = \frac{\log_2(M + 1)}{\log_2(M)}, \beta = \frac{M}{M + 1}, \gamma' = \frac{M^{\alpha - 1}}{(M - 1)^{\alpha}} \tag{18}
\]

where

\[
\sum_{i=1}^{M} p(i) = 1, \quad \sum_{i=1}^{M} \frac{\gamma}{(r + \beta)\alpha} = \kappa \tag{20}
\]

This approach provides an equiprobable representation of the symbols by considering a different model for each symbol rank, according to the rank. But moreover, once a model is found for one rank, then the whole model can be identified. Hence, adopting a probabilistic model according to the symbolic rank we define

\[
F(n; r, M) = 1 - \prod_{h=1}^{n} (1 - P_h(r, M)) \tag{21}
\]

where

\[
P_h(r, M) = \frac{h\gamma'}{(r + \beta)\alpha} \tag{22}
\]

Therefore, the same approach as before can be adopted by defining

\[
f(n; r, M) = F(n; r, M) - F(n - 1; r, M) \tag{23}
\]

Hence, we now have \( E_r[n] = J'(n; r, M) \) and

\[
D_r(M) = \sum_{n=0}^{M} nf(n; r, M) \tag{24}
\]

Using a similar approach to the previous equiprobable case, a per symbolic rank model for estimating \( M \) can be obtained by prescribing \( J'(n; r, M) \) in (24), and then inverting this to become

\[
\hat{M}_r(D) = G_r(\Theta; M, D_r) \tag{25}
\]

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Note that although it is technically feasible to derive the exact model \( J'(n; r, M) \) in terms of (17), (24), it is not necessary to do so in practice as is evident by the curve fitting approach proposed in [11] and adopted here.
tropy models for $D(M) = J'(n, r, M)$.

Now, unlike the model proposed initially in [11], natural sequence data consists of a non-equiprobable set of symbols and so we cannot simply use [11] to estimate entropy in a single step as before. However, given an estimate $\hat{M}_r(D)$, from the observed inter-symbol distance, it now becomes possible to apply this parameter to the Zipf-Mandelbrot-Li set of equations in addition to our rank-based probability model of symbol drawings, and obtain an overall estimate for the entire set of symbolic probabilities. While this can be achieved using, for example, $D_1$, clearly it is possible to form an estimate which uses $D_i$ for $i = 1..n$ according to any desired criteria such as least squares or any other norm. Having then estimated $\hat{P}_h(r, M)$, the entropy can then be easily estimated as

$$\hat{H}_1(r, X) = -\sum_{h=1}^{M} \hat{P}_h(r, M) \log_2 (\hat{P}_h(r, M))$$  \hspace{1cm} (26)

which defines the rank $r$ Shannon entropy estimate. In the next section, we demonstrate the performance of the model in various simulations.

III. EXAMPLE RESULTS

A. Synthetic Entropy Model of English Text

In this example, a set of data is simulated using the Zipf-Mandelbrot-Li model with 27 symbols corresponding to the 26 letters and a space. The rank-based entropy estimation algorithm described in the previous section is used to estimate the model by counting the coincidences of the symbols. In the first instance, we simply compute the average symbol distance $D_1$ and then apply this to the inverted model. Note that a different model applies to each rank as shown in Fig. 1. The rank-based entropy models for $D(M) = J'(n, r, M)$ are inverted and

$$\hat{M}_r(D) = G_r(\Theta; M, D_r).$$

Each model is derived from the initial rank-based model which describes the symbolic distance $D_r$ as a function of $M$.

The models are shown in Fig. 2. Here, a power based model is used,

$$\hat{M}_r(D) = aD_r^b + c$$  \hspace{1cm} (27)

where $a = 0.0075, b = 4.2345, c = 4.1385$. In the synthetic simulation results, using only 25 symbol coincidences, where the true entropy is $H_0(27) = 4.261$ by application of the rank-based entropy model described in the previous section, we obtain the estimated entropy of $H_e(27) = 4.266$ indicating the efficacy of the method.

B. Entropy of English Text: Tom Sawyer

In this example, the classic English language text Tom Sawyer was used to test the algorithm. In this case, the rank 1 model was again used, where the highest ranked symbol corresponds to the space character. Commencing at Chapter 2 of the text, the intersymbol distance was estimated as $D_1(50) = 6.03$ which leads to an estimated entropy of $H_e(27) = 4.3$ which is in close agreement to the actual entropy of the text where $H_a(27) = 4.4$. Moreover, the result was obtained by using less than 300 characters or 50 words which is quite remarkable.

IV. CONCLUSION

Shannon entropy is a well known method of measuring the information content in a sequence of probabilistic symbolic events. In this paper, we have proposed a fast algorithm for estimating Shannon Entropy for natural sequences. Using a modified Zipf-Mandelbrot-Li law and a coincidence counting method, we have demonstrated a method which gives extremely fast performance in comparison to other techniques and yet is simple to implement. Examples have been given which show the efficacy
of the proposed methodology. It would be of interest to apply this method to various real-world applications to compare the theoretical results against experimentally obtained results. In terms of information-theoretic analytical tools, it may be of interest to consider just how few samples may be required in order to obtain useful results. In order to make the most use of available data, future work could consider optimal strategies for deriving accurate models from multiple symbol ranks; this could be expected to yield fruitful results especially when there is some ‘noise’ in the data, e.g., some symbols are missing. Another area of interest in future work will be to analyze the bias of the model as considered in [23].

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