A Note on the Asymptotic Limit of the Four Simplex

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Abstract

Recently the asymptotic limit of the Barrett-Crane models has been studied by Barrett and Steele. Here by a direct study, I show that we can extract the bivectors which satisfy the essential Barrett-Crane constraints from the asymptotic limit. Because of this the Schlaffi identity is implied by the asymptotic limit, rather than to be imposed as a constraint.

The asymptotic limit \[7\], \[6\], \[4\] of the Barrett-Crane models \[2\], \[3\] has been recently studied systematically in \[5\]. Here by a direct study, I show that we can extract the bivectors which satisfy the essential Barrett-Crane constraints\(^1\) from the asymptotic limit. Because of this the Schlaffi identity is implied by the asymptotic limit, rather than to be imposed as a constraint as in Ref.\[2\]. Here I focus on the Riemannian Barrett-Crane model \[2\] only, but it can be generalized to the Lorentzian Barrett-Crane model \[3\].

Consider the amplitude of a four-simplex\[2\] with a real scale parameter \(\lambda\),

\[
Z_\lambda = (-1)^{\sum_{k<l} 2J_{kl}} \int_{n_k \in S^3} \prod_{i<j} \frac{\sin(\lambda J_{ij} \theta_{ij})}{\sin(\theta_{ij})} \prod_k d n_k,
\]

\[
= (-1)^{\sum_{k<l} 2J_{kl}} (2i)^{10} \int_{n_k \in S^3} \prod_{i<j} \sum_{\epsilon_{ij} = \pm 1} \frac{\varepsilon_{ij} \exp(i \epsilon_{ij} \lambda (p J_{ij} \theta_{ij}))}{\sin(\theta_{ij})} \prod_k d n_k,
\]

where the \(\theta_{ij}\) is defined by \(n_i . n_j = \cos(\theta_{ij})\). Here the \(\theta_{ij}\) is the angle between \(n_i\) and \(n_j\). The asymptotic limit of \(Z_\lambda(s)\) under \(\lambda \to \infty\) is controlled by

\[
S(\{n_i\}, \{J_{ij}\}, \{q_i\})
\]

\[
= \sum_{i<j} \varepsilon_{ij} J_{ij} \theta_{ij} + \sum_i q_i (n_i . n_i - 1),
\]

where the \(q_i\) are the Lagrange multipliers to impose \(n_i . n_i = 1, \forall i\). My goal now is to find stationary points for this action. The stationary value under the

\(^1\)Here by the essential Barrett-Crane constraints \[2\] I mean all the constraints except those that relate to degeneracy conditions of a four simplex.
variation of \( n_j \)'s are determined by

\[
\sum_{i \neq j} \varepsilon_{ij} J_{ij} \frac{\partial \zeta_{ij}}{\partial n_i} + 2 q_j n_j = 0, \forall j,
\tag{1a}
\]

and \( n_j.n_j = 1, \forall j \) where the \( j \) is a constant in the summation.

\[
\frac{\partial \zeta_{ij}}{\partial n_i} = \frac{n_j}{\sin(\zeta_{ij})},
\tag{2}
\]

where the vector index has been suppressed on both the sides.

Using equation (2) in equation (1a) and taking the wedge product of the equation with \( n_j \) we have,

\[
\sum_{i \neq j} \varepsilon_{ij} J_{ij} \frac{n_i \wedge n_j}{\sin(\zeta_{ij})} = 0, \forall j.
\]

If

\[
E_{ij} = \varepsilon_{ij} J_{ij} \frac{n_i \wedge n_j}{\sin(\zeta_{ij})},
\]

then the last equation can be simplified to

\[
\sum_{i \neq j} E_{ij} = 0, \forall j.
\tag{3}
\]

We now consider the properties of \( E_{ij} \):

- Each \( i \) represents a tetrahedron. There are ten \( E_{ij} \)'s, each one of them is associated with one triangle of the four-simplex.

- The square of \( E_{ij} \):

\[
E_{ij} \cdot E_{ij} = \frac{J_{ij}^2}{\sin^2(\zeta_{ij})} \left( n_i^2 n_j^2 - (n_i \cdot n_j)^2 \right)
= \frac{J_{ij}^2}{\sin^2(\zeta_{ij})} \left( 1 - (\cos(\zeta_{ij})^2 \right)
= J_{ij}^2.
\]

- The wedge product of any two \( E_{ij} \) is zero if they are equal to each other or if their corresponding triangles belong to the same tetrahedron.

- Sum of all the \( E_{ij} \) belonging to the same tetrahedron are zero according to equation (3).

It is clear that these properties contain the first four Barrett-Crane constraints. So we have successfully extracted the bivectors corresponding to the triangles of a general flat four-simplex in Riemannian general relativity and the \( n_i \) are the normal vectors of the tetrahedra. The \( J_{ij} \) are the areas of the triangle as
one would expect. Since we did not impose any non-degeneracy Barrett-Crane conditions, it is not guaranteed that the tetrahedra or the four-simplex have non-zero volumes.

The asymptotic limit of the partition function of the entire simplicial manifold with triangulation $\Delta$ is

$$S(\Delta, \{ n_{is} \in S^3, J_{ij}, \varepsilon_{ijs} \}) = \sum_{i<j,s} \varepsilon_{ijs} J_{ij} \zeta_{ijs},$$

where I have assumed variable $s$ represents the four simplices of $\Delta$ and $i, j$ represents the tetrahedra. The $\varepsilon_{ijs}$ can be interpreted as the orientation of the triangles. Each triangle has a corresponding $J_{ij}$. The $n_{is}$ denote the vector associated with the side of the tetrahedron $i$ facing the inside of a simplex $s$. Now there is one bivector $E_{ijs}$ associated with each side facing inside of a simplex $s$ of a triangle $ij$ defined by

$$E_{ijs} = \varepsilon_{ijs} J_{ij} \frac{n_{js} \wedge n_{js}}{\sin(\zeta_{ijs})}.$$ 

If the $n_{is}$ are chosen such that they satisfy stationary conditions

$$\sum_{i \neq j} E_{ijs} = 0, \forall j, s,$$

and if

$$\theta_{ij} = \sum_s \varepsilon_{ijs} \zeta_{ijs},$$

then

$$S(\Delta, \{ J_{ij}, \varepsilon_{ijs} \}) = \sum_{i<j,s} \varepsilon_{ijs} J_{ij} \zeta_{ijs},$$

$$= \sum_{i<j} J_{ij} \theta_{ij}$$

can be considered to describe the Regge calculus for the Riemannian general relativity. The angle $\theta_{ij}$ are the deficit angles associated with the triangles and the $n_{is}$ are the vector normals associated with the tetrahedra.

References

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