Separation of variables via integral transformations

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Abstract

For a system of linear partial differential equations (LPDEs) we introduce an operator equation for auxiliary operators. These operators are used to construct a kernel of an integral transformation leading the LPDE to the separation of variables (SoV).

The auxiliary operators are found for various types of the SoV including conventional SoV in the scalar second order LPDE and the SoV by the functional Bethe anzatz. The operators are shown to relate to separable variables. This approach is similar to the position-momentum transformation to action angle coordinates in the classical mechanics.

General statements are illustrated by some examples.

\textbf{Key words:} separable systems, linear $r$-matrix algebras, classical Yang-Baxter equation, Lax representation, Liouville integrable systems
1 Introduction

We studied certain possibilities to transform a linear partial differential equation (LPDE) into the form allowing the separation of variables (SoV) by integral transformations. This 'non-coordinate' SoV differs from the conventional SoV realized by a purely coordinate transformation. In the classical mechanics, this approach is similar to the position-momentum canonical transformations in the phase space reducing a Hamilton system to the action-angle variables.

One of the realization of the 'non-coordinate' SoV was found in the form of the functional Bethe-anzatz (FBA)\(^{[1]}\) appeared in the frame of the \(R\)-matrix formalism. The \(R\)-matrix formalism was developed on the background of the quantum inverse scattering method \(^{[2]}\). The FBA-method generates effectively classes of integrable systems, however it is difficult to apply the FBA to a certain equation as we believe.

In this paper we consider a relationship between the FBA-constructions and the conventional method of SoV, that, in particular, highlights some ways of the 'direct' non-coordinate SoV, using no \(R\)-matrix formalism constructions.

In the general sense, the method of separation of variables in a linear partial differential equation with \(n\) arguments \((x) = (x_1)_{1\leq i\leq n} \in \mathbb{R}^n\) is a procedure which brings the LPDE

\[ F\psi(x) = 0 \tag{1} \]

into the form where the solution \(\psi\) can be written

\[ \psi_\lambda(x) = S(x)\prod_{i=1}^n \psi_i(x_i, \lambda). \tag{2} \]

(We discuss the local aspects of the SoV, therefore all the functions are suggested to be smooth). Here \(S(x)\) is a function of \((x)\), and the function \(\psi_i(x_i, \lambda)\) depends only on \(x_i\) and separation parameters \(\lambda \in C^{(n-1)}\). The number of parameters \(\lambda\) provides the 'local completeness' of the solution family \(^{[2]}\), so, that an arbitrary solution of \(^{[3]}\) can be formally expanded into \(^{[2]}\).

Below herein we notice some aspects of the theory, that are necessary for this paper.

The theory of complete and partial separation of variables in a scalar LPDE of the second order of both non-paraboloic and paraboloic types is
developed entirely. The theorem on necessary and sufficient conditions of
the SoV has been proved in [1], where two basic problems in the SoV theory
are solved:

1. The general form of an LPDE of both parabaloic and non-parabaloic
types, admitting SoV in a system of separable (privileged) coordinates
was found. The correspondent form of the system of ordinary differential equations on the functions $\psi_i(x_i, \lambda)$ (separated system) and the
SoV procedure were presented.

2. An invariant criterium of the SoV was obtained for a certain equation
in an arbitrary coordinate system. This criterium provides the algo-
rithm of the transformation from an arbitrary coordinate system to the
privileged system.

The criterium is based on the existence of a complete set of pairvise com-
muting symmetry operators of the first and the second orders which satisfy
the special algebraic conditions [1].

The invariant criterium is important because the SoV is not invariant to
a coordinate transformation.

The theorems in [1] assume that the separation of variables in an LPDE
of the second order (1) can be made by the following transformations:

- a coordinate transformation of the form $x' = x'(x)$ independent on the
  separation parameters $\lambda$,

- a function of the form $\psi'(x) = S(x)\psi(x)$ where $S(x)$ is a function of $x$.

This kind of the SoV we shall call the CSov (conventional SoV).

Rather different approach to the integrability problem has provided the
discovery of the quantum inverse scatting method [2] which initiated after-
wards the method of the $R$- matrix. In many papers (see, for example, [4],
[5], [6], [7]), the relationship between the $R$ - matrix formalism and the SoV
was found. The approach based on this relationship and named the FBA was
suggested in the paper [1]. The FBA applied to the classical Hamilton system
admits a possibility to use a 'non-coordinate' position-momentum canonic-
tal transformation to the privileged variables unlike the CSov method. In terms
of the differential equation, the FBA leads to the SoV after some integral
transformation of the LPDE system. In this approach, integro- differential
operators can play the role of the privileged coordinates \( x \). In this coordinates, the solution can be presented in the multiplicative form (2). The equations on the functions \( \psi_i \) in (2) (the separated system) are non-local in general case.

The study of a quantum system (LPDE system) can be carried out basing on the FBA by the following scheme:

1. to find \( R \)- matrix as a solution of quantum Yang-Baxter equation,
2. to look for the representation of \( R \)-matrix algebra \([1]\) in the class of \((n \times n) - L\) operator matrices.
3. to find a system of LPDEs described by this operator matrices.

The operator matrix \( L \) satisfying all the conditions of the FBA and describing some LPDE solves the problem of the SoV: it generates the form of a privileged coordinates and the form of separated equations. As we believe, the direct way to build up all elements of \( R \)-matrix formalism for a certain quantum system (a system of LPDEs) is not found in general case.

In this paper we consider the common features of the CSoV and the FBA. An approach based on auxiliary operators is proposed. It leads a certain LPDE to the SoV by integral transformations. The correspondence of auxiliary operators to the CSoV and FBA is discussed. General statements are illustrated by several examples including the well known quantum Goryachev-Chaplygin top.

2 Operator equation and its relationship to the CSoV and FBA

We consider a quantum system, which is reduced to a pair of compatible two-dimensional LPDEs of an arbitrary order:

\[
\begin{align*}
H_1(x_1, x_2, \partial_{x_1}, \partial_{x_2})\psi(x_1, x_2) &= \epsilon_1 \psi(x_1, x_2), \\
H_2(x_1, x_2, \partial_{x_1}, \partial_{x_2})\psi(x_1, x_2) &= \epsilon_2 \psi(x_1, x_2), \\
[H_1, H_2] &= 0.
\end{align*}
\]  

(3)

Here and below \( \partial_{x_k} = \frac{\partial}{\partial x_k} \), \( \partial_{x_k x_l} = \frac{\partial^2}{\partial x_k \partial x_l} \), \( k, l, s, \ldots = 1, 2 \), \( \epsilon_k \) - are parameters. Let us define the linear partial differential operators \( U_1 \) and \( U_2 \) of
the orders \( n_1 \) and \( n_2 \) respectively (\( n = 0 \) corresponds to the multiplication operator on a function), by the system:

\[
[U_1, H_1] + [U_2, H_2] = 0,
[U_1, U_2] = 0.
\] (4)

If \( U_1 \) and \( U_2 \) are supposed to be functions, we have the following

**Theorem:** If the operators \( H_1 \) and \( H_2 \) of the second order permit the separation of variables in Eqs. (3) via CSoV, then the solution of Eqs. (4) in a class of functions \( (n_1 = 0, n_2 = 0) \) exists.

**Proof:** Let the variables in Eqs. (3) are separated in a certain coordinates \( x \). Then the matrices \( h_{kl}^1 \) and \( h_{kl}^2 \) of the coefficients at the second order partial derivatives of the operators \( H_1 \) and \( H_2 \) have the form:

\[
h_{kl}^s = \sum_r \Phi^{-1}_{sr} \delta_r^k \delta_r^l,
\] (5)

where \( \delta_r^k \) is the Kronecker delta and

\[
\Phi_{sq} = \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}
\]

is the Stäckel matrix. Lower index at the functions \( f_s \) and \( g_s \) indicates the number of correspondent privileged variables on which the function depends. Notice, that a matrix of the form \( \Phi_{ks} = \Phi_{ks}(x_k) \) is the Stäckel matrix by the definition.

For the operators \( H_1 \) and \( H_2 \) of the second order, Eqs. (3) takes the form

\[
\sum_l \left( h_{kl}^1 U_1, l + h_{kl}^2 U_2, l \right) = 0,
\]

\((U_{s,t} = \partial U_{s}/\partial x_t)\) which, taking into the account Eq.(3) in the privileged variables \( V_1 \) and \( V_2 \), results in

\[
\Phi^{-1}_{1s} U_1, s + \Phi^{-1}_{2s} U_2, s = 0.
\] (6)

The integration of Eq.(5) gives:

\[
U_k = \sum_l \Phi^{-1}_{kl} q_l,
\]

where \( q_1 \), \( q_2 \) are arbitrary functions of \( V_1 \) and \( V_2 \), respectively.

The relationship between the functions \( U_1 \) and \( U_2 \) and the separable variables \( V_1 \) and \( V_2 \) can be obtained from the following
**Theorem: 2** If the operators of the second order \( H_1 \) and \( H_2 \) satisfy the CSoV conditions, and \( V_1 \) and \( V_2 \) are the privileged coordinates then Eq. (4) is fulfilled for \( U_1 = V_1 + V_2 \) and \( U_2 = V_1 V_2 \).

**Proof:** According to the CSoV, the privileged coordinates can be taken in the form:

\[
V_1 = \frac{1}{2}(\text{tr}(h_1 \cdot h_2^{-1}) + \sqrt{\text{tr}(h_1 \cdot h_2^{-1})^2 + 4\det(h_1 \cdot h_2^{-1})}),
\]

\[
V_1 = \frac{1}{2}(\text{tr}(h_1 \cdot h_2^{-1}) - \sqrt{\text{tr}(h_1 \cdot h_2^{-1})^2 + 4\det(h_1 \cdot h_2^{-1})}).
\]

(7)

Let us show, that for \( U_1 \) and \( U_2 \) we can take

\[
U_1 = V_1 + V_2 = \text{tr}(h_1 \cdot h_2^{-1}),
\]

\[
U_2 = V_1 \cdot V_2 = \det(h_1 \cdot h_2^{-1}).
\]

(8)

Substitution of the matrices \( h_1 \) and \( h_2 \) of the form (3) into the expressions for the trace and the determinant yields:

\[
U_1 = \text{tr}(h_1 \cdot h_2^{-1}) = (f_2 g_1 + f_1 g_2)/f_1 f_2,
\]

\[
U_2 = \det(h_1 \cdot h_2^{-1}) = g_1 g_2/f_1 f_2.
\]

(9)

From (9) it follows

\[
(g_2 U_1 - f_2 U_2),_1 = 0,
\]

\[
(g_1 U_1 - f_1 U_2),_2 = 0.
\]

(10)

Substitution of \( U_1 \) and \( U_2 \) from (9) into (10) gives:

\[
(g_2^2/f_2),_1 = 0, \quad (g_1^2/f_1),_2 = 0.
\]

The similar connection between the FBA and the system (4) exists if \( U_1 \) and \( U_2 \) are operators. The separated system can be written after the transformation to the representation of these operators.

We consider the quantum system described by \((2 \times 2)\) \( L \)-matrix associated with yangian \( Y[gl(2)] \):

\[
L(u) = \begin{pmatrix}
A(u) & B(u) \\
C(u) & D(u)
\end{pmatrix}.
\]

Here \( A(u), B(u), C(u) \) and \( D(u) \) are linear differential operators depending on the spectral parameter \( u \). Following to the \( R- \) matrix formalism \( \Box \) we have:

\[
\text{tr} (L(u)) = A(u) + D(u) = u \cdot H_2 + H_1
\]

(11)
with

\[
[A(V_k), V_l] = \mu A(V_k) \delta_{kl},
\]
\[
[D(V_k), V_l] = -\mu D(V_k) \delta_{kl}.
\]  

(12)

Here \( \delta_{kl} \) is the Kronecker delta and \( \mu \) is the parameter of \( R \)-matrix. According to the FBA, operators \( V_s \) become the separable variables after transformation to their representation. From commutators for \( \text{tr} \ (L(V_k)) \) with separable variables \( V_l \) and Eqs.(11), (12), we obtain:

\[
[H_1, V_1 + V_2] + [H_2, V_1 \cdot V_2] = 0.
\]  

(13)

So, it is true the following:

**Theorem: 3** If the system (3) permits SoV via the FBA then the operators \( U_1 \) and \( U_2 \) satisfying (4) exist, where \( U_1 = V_1 + V_2 \) and \( U_2 = V_1 V_2 \). Here \( V_1 \) and \( V_2 \) are the operators correspondent to the separable by the FBA variables.

The existence of the operators \( U_1 \) and \( U_2 \) satisfying (4) can be understood as a necessary condition for the applicability of the FBA method.

Let us consider a special case of Eq.(4) where each term in it equals to zero:

\[
[H_1, U_1] = [H_2, U_2] = 0.
\]  

(14)

Eq. (14) appears in the simplest cases of the SoV. Eq.(14) have the solution in the class of functions if each of the equations in the system (3) includes only one variable or if the SoV in (3) can be obtained by the transformations of variables without linear combination of the equations. If \( H_1 \) and \( H_2 \) in (3) are of the form \( H_1 = H_1(U_1, U_2) \) and \( H_2 = H_2(U_1, U_2) \) then this \( U_1 \) and \( U_2 \) are the solution of (14).

We showed that the various types of the SoV relate to the auxiliary operators, which are the solutions of (4) of the correspondent type. This idea is investigated in the approach below.

### 3 Integral transformations

Let us consider an integral transformation of the system (3) with kernel \( K(x_1, x_2; \lambda_1, \lambda_2) \) defined by the system of differential equations:

\[
U_1(x_1, x_2, \partial_{x_1}, \partial_{x_2})K(x_1, x_2; \lambda_1, \lambda_2) = \lambda_1 K(x_1, x_2; \lambda_1, \lambda_2),
\]
\[
U_2(x_1, x_2, \partial_{x_1}, \partial_{x_2})K(x_1, x_2; \lambda_1, \lambda_2) = \lambda_2 K(x_1, x_2; \lambda_1, \lambda_2).
\]  

(15)
Here $U_1$ and $U_2$ are the solutions of (4).

Let us take $\psi(x_1, x_2)$ in the form:

$$\psi(x_1, x_2) = \int K(x_1, x_2; \lambda_1, \lambda_2) \phi(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2. \quad (16)$$

Inserting (16) into (3) and converting the action of the operators $H_1$ and $H_2$ from $\psi(x_1, x_2)$ to $\phi(\lambda_1, \lambda_2)$ we obtain:

$$R_1(\lambda_1, \lambda_2, \partial_{\lambda_1}, \partial_{\lambda_2}) \phi(\lambda_1, \lambda_2) = \epsilon_1 \phi(\lambda_1, \lambda_2),$$
$$R_2(\lambda_1, \lambda_2, \partial_{\lambda_1}, \partial_{\lambda_2}) \phi(\lambda_1, \lambda_2) = \epsilon_2 \phi(\lambda_1, \lambda_2). \quad (17)$$

Here $R_1$ and $R_2$ are integro-differential operators in the general case. The variables can be separated in the system (17) under the proper choice of $U_1$ and $U_2$. The theorems 1 and 3 of the previous section serve for the arguments in this approach.

As mentioned above, $U_1$ and $U_2$ are the functions in the CSov case. Then the solution of Eq. (15) is the Dirac delta. The transformation to the system (17) is identical.

If $U_1$ and $U_2$ are the operators, the integral transformation can lead to SoV the systems, that are not separable by a coordinate transformation. These examples are discussed in the next section. The problem of the proposed approach is the complexity of the transformation to the system (17), if the kernel (15) is complex, and the proper choice of $U_1$ and $U_2$ from all solutions of (4).

Notice, that after transformation to the suitable $(U_1, U_2)$ representation, the separable variables $V_1$ and $V_2$ can be found from: $U_1 = V_1 + V_2$, $U_2 = V_1 V_2$.

4 Examples

4.1 The system of LPDEs of the second order not permitting CSov

If the system (3) permits the CSov, then the operators $H_1$ and $H_2$ must satisfy some algebraic conditions (3). In the example below, this algebraic condition is broken, but it recovers after integral transformation according the approach of sec. 3. Let the operators $H_1$ and $H_2$ in (3) are:

$$H_1 = \partial_{x_1 x_1} + \partial_{x_2} + \mu (x_2^2 \partial_{x_2} + x_1^2),$$
$$H_2 = x_2 \partial_{x_1 x_2} - \frac{1}{2} x_1 \partial_{x_2} + \frac{1}{2} \mu x_1 x_2^2 \partial_{x_2}, \quad (18)$$
where $\mu$ is a constant. The system does not satisfy the necessary condition of the CSoV, which can be formulated here in the form:

$$\text{tr} \left( h_1 \cdot h_2^{-1} \right)^2 + 4 \det(h_1 \cdot h_2^{-1}) > 0.$$  

Here $h_1$ and $h_2$ are the matrices of the coefficients at the second order partial derivatives of the operators $H_1$ and $H_2$ respectively. Solving the system (4) we obtain, in particular:

$$U_1 = x_1,\quad U_2 = x_2^2 \partial_{x_2}.$$  

The solution of (13) is $K = \exp \left(-\lambda_2/x_2\right)\delta(x_1 - \lambda_1)$. The integral transformation yields to:

$$R_1 = \partial_{\lambda_1 \lambda_1} + \lambda_2 \partial_{\lambda_2 \lambda_2} + \mu(\lambda_1^2 + \lambda_2^2),$$  

$$R_2 = \lambda_2 \partial_{\lambda_1 \lambda_2} - \frac{1}{2}\lambda_1 \lambda_2 \partial_{\lambda_2 \lambda_2} + \frac{1}{2}\mu \lambda_1 \lambda_2.$$  

The system (19) allows the CSoV:

$$T(V_1) \psi(V_1) = 0,\quad T(V_2) \psi(V_2) = 0,\quad T(V) = \partial_{VV} + \mu/4V^2 - \epsilon_1 - \epsilon_2/V.$$  

Here the separable variables $V_k$ are related to $U_k$ as follows:

$$V_1 = U_1 + U_2,\quad V_2 = U_1 \cdot U_2.$$  

### 4.2 Quantum Goryachev-Chaplygin top

In this example we apply the approach to the well known quantum system studied by the FBA method in [6].

Let us consider the quantum analogue of the Hamiltonian system on the Lie algebra $e(3)$. The generators can be taken in the class of differential operators in the space $\mathbb{R}^3$ with the conventional commutators:

$$[J_i, J_j] = -i \sum_k \epsilon_{ijk} J_k,\quad [J_i, x_j] = -i \sum_k \epsilon_{ijk} x_k,\quad [x_i, x_j] = 0.$$  

Here $i, j, k = 1, 2, 3$.  

9
The Casimir operators have the form:

\[
K_1 = x_1^2 + x_2^2 + x_3^2, \quad K_2 = x_1J_1 + x_2J_2 + x_3J_3.
\]  

(21)

The Quantum Goryachev-Chaplygin top is described by the Hamiltonian \( H \) and the additional integral of motion \( G \) [6]:

\[
H = \frac{1}{2}(J_1^2 + J_2^2 + 4J_3^2) - bx_1, \\
G = (2J_3 + p)(J_1^2 + J_2^2 - \frac{1}{4}) + b[x_3, J_1].
\]  

(22)

Here \( b \) is a constants, \([A, B]_+ = AB + BA\). The operators \( H \) and \( G \) commute on the subspace of eigenfunctions \( \psi \) of the Casimir operators matching the eigenstates equal to 1, 0:

\[
K_1(J, x)\psi = 1 \cdot \psi, \quad K_2(J, x)\psi = 0.
\]  

(23)

In this case the system of Eq. (21) and (22) is integrable.

The reduction of \( H \) and \( G \) on the selected integrable orbit is obtained here by the method of noncommutative integration of the LPDE developed in [8]. Written in the variables of the orbit \( \zeta_1, \zeta_2 \), the system has a form:

\[
H = \frac{1}{2}((4 + \tan \zeta_2^2)\partial_{\zeta_1\zeta_1} - \tan \zeta_2\partial_{\zeta_2\zeta_2} + \partial_{\zeta_2\zeta_2}) - b\sin \zeta_1 \cos \zeta_2, \\
G = 2\partial_{\zeta_1}(\tan \zeta_2^2\partial_{\zeta_1\zeta_1} + \partial_{\zeta_2\zeta_2} - \tan \zeta_2\partial_{\zeta_2} - 1/4) - b(2\sin \zeta_1 \sin \zeta_2 \tan \zeta_2\partial_{\zeta_1}\zeta_2 + 2\cos \zeta_1 \sin \zeta_2\partial_{\zeta_2}\zeta_2 + \cos \zeta_1 \cos \zeta_2).
\]  

(24)

This system can be considered as the quantum analogue of the reduced system on the orbit of the coadjoint representation of the Lie algebra \( e(3) \).

In this system the constraints (23) produced by the Casimir operators are eliminated and the condition \([H, G] = 0\) is satisfied.

Solving system (23) for \( H_1 = H \) and \( H_2 = G \) we obtain, in particular:

\[
U_1 = \tan \zeta_2^2\partial_{\zeta_1\zeta_1} - \tan \zeta_2\partial_{\zeta_2\zeta_2} + \partial_{\zeta_2\zeta_2}, \quad U_2 = 2\partial_{\zeta_1}. 
\]  

(25)

According to our approach the separable variables \( V_1 \) and \( V_2 \) are related to \( U_1 \) and \( U_2 \) as follows:

\[
U_1 = V_1 + V_2, \quad U_2 = V_1 \cdot V_2.
\]  

(26)

This result can be interpreted in the frame of the FBA. The separable variables are by the FBA of the form [3]:

10
\[ V_1 = J_3 + \sqrt{J_2^2 + J_3^2 + J_4^2}, \]
\[ V_2 = J_3 - \sqrt{J_2^2 + J_3^2 + J_4^2}. \]  

(27)

After the substitution of (27) in (24) in the orbit variables \( \lambda_1, \lambda_2 \) we obtain (24), which is defined from our approach.

The system (17) is difficult to derive due to the nonanalytic kernel of integral transformation defined by (15).

It is known from the FBA, that the system (22) in terms of the variables \( V_1 \) and \( V_2 \) takes the separable form:

\[ T(V_1)\psi(V_1) = 0, \]
\[ T(V_2)\psi(V_2) = 0, \]
\[ T(V) = V^3 - 2(h + 1/8)V + g - b\sqrt{(V - 1)^2 + 1/4e^{2\theta \nu}} - b\sqrt{(V + 1)^2 + 1/4e^{-2\theta \nu}}. \]

Here \( f \) and \( g \) are the eigenfunctions of \( H \) and \( G \), respectively. In (4.2), \( V_1 \) and \( V_2 \) can be taken from (26). We can interpret the separable variables as the quantum analogue variables of the integrable coadjoint orbit of the Lie algebra \( e(3) \).

### 4.3 Electron dynamics in resonant cyclic accelerator

We apply our approach to a quantum system, which exact solution was found in [9]. The quantum system is determined by the Hamiltonian:

\[ H_1 = \partial_{xx} - 2i\gamma_0\gamma\partial_x - a^2x^2 - \gamma\gamma_0\xi^2 \]
\[ -a^2\frac{\partial^2}{f(z)} + f(z)\partial_{zz}. \]  

(28)

The parameters \( a_3, \gamma, \gamma_0 \) of the Hamiltonian have a definite physical sense, \( f(z) \) is a function. The variable \( z \) can be simply separated. The symmetry operator of the second order can be found as:

\[ H_2 = -2i\gamma_0\gamma\xi\partial_x + \gamma_0\xi^2 - \gamma^2\gamma_0x^2 - \gamma\partial_{\xi\xi}. \]  

(29)

This system does not satisfy the CSO\( V \) conditions. One of the solutions of the equation (4) is:

\[ U_1 = \partial_\xi, \quad U_2 = x. \]  

(30)
Substituting (30) in (15) we obtain, with respect to \( x_1 = x \) and \( x_2 = \xi \), identical transformation for the variable \( x \) and Fourier transformation for \( \xi \). This is similar to the results of [9]. After this transformation, the system (28), (29) takes the form:

\[
\begin{align*}
R_1 &= \partial_{\lambda_1} \lambda_1 - \gamma \gamma_0 \partial_{\lambda_2} \lambda_2 + 2 \gamma \lambda_1 \lambda_2 - a^2 \lambda_1^2, \\
R_2 &= -\gamma_0 \partial_{\lambda_2} \lambda_2 + 2 \gamma \gamma_0 \partial_{\lambda_1} \lambda_2 - \gamma^2 \gamma_0 \lambda_2^2 + \gamma \lambda_2^2.
\end{align*}
\] (31)

The transformation of variables

\[
V_1 = \frac{\gamma}{a_4 \sqrt{a_1}} (\frac{a_2}{\gamma} \lambda_1 - \lambda_2), \\
V_2 = \frac{\gamma}{a_4 \sqrt{a_2}} (\lambda_2 - \frac{a_2}{\gamma} \lambda_1)
\]

Here \( a_1, a_2, a_4 \) are expressed in terms of \( \gamma_0, \gamma, a \). This transformation reduces the system (31) to the separated form:

\[
\begin{align*}
(\partial_{V_1} V_1 - V_1^2) \psi(V_1, V_2) &= \epsilon_1 \psi(V_1, V_2), \\
(\partial_{V_2} V_2 - V_2^2) \psi(V_1, V_2) &= \epsilon_2 \psi(V_1, V_2).
\end{align*}
\]

5 Conclusion

We showed the interrelation between the separable variables of the FBA and the method of complete SoV in the scalar second order equation in terms of auxiliary operators \( U_1 \) and \( U_2 \). The conventional SoV appeared to be the simplest type, if \( U_1 \) and \( U_2 \) are operators of multiplication on the functions. The FBA theory, which generalizes the theory of SoV, produces the operators \( U_1 \) and \( U_2 \) in more complicate form.

We put in the background of our approach the operators \( U_1 \) and \( U_2 \), which are the solutions of Eq.(4) in class of the differential operators. It provides a 'direct' way of the 'non-coordinate' SoV for an LPDE system. It can be applied, for example, to the systems with higher order symmetries. Such a problem was researched in various papers (see, for example [10]).

It is interesting, from our point of view, to study the following items:

- the sufficient condition of the SoV in terms of the auxiliary operators
- the analogue of Eq.(4) in 3 and more dimensions can be the theme of latest development
the structure of the auxiliary operators for the different types of LPDE.

It will be the subject of the further papers.

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