FOURTH ORDER QUASI-COMPACT DIFFERENCE SCHEMES FOR (TEMPERED) SPACE FRACTIONAL DIFFUSION EQUATIONS

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Abstract. The continuous time random walk (CTRW) underlies many fundamental processes in non-equilibrium statistical physics. When the jump length of CTRW obeys a power-law distribution, its corresponding Fokker-Planck equation has space fractional derivative, which characterizes Lévy flights. Sometimes the infinite variance of Lévy flight discourages it as a physical approach; exponentially tempering the power-law jump length of CTRW makes it more ‘physical’ and the tempered space fractional diffusion equation appears. This paper provides the basic strategy of deriving the high order quasi-compact discretizations for space fractional derivative and tempered space fractional derivative. The fourth order quasi-compact discretization for space fractional derivative is applied to solve space fractional diffusion equation and the unconditional stability and convergence of the scheme are theoretically proved and numerically verified. Furthermore, the tempered space fractional diffusion equation is effectively solved by its counterpart of the fourth order quasi-compact scheme; and the convergence orders are verified numerically.

Key words. space fractional derivative, tempered space fractional derivative, shifted Grünwald discretization, quasi-compact difference scheme, numerical stability and convergence.

Subject classifications. 65M06, 65M12, 26A33

1. Introduction In recent years, more and more scientific and engineering problems are involved in fractional calculus. They range from relaxation oscillation phenomena [1] to viscoelasticity [2], and from control theory [3] to transport problem [4]. The fractional diffusion equation has been put forward as a more suitable model to describe ion channel gating dynamics [5] and subdiffusive anomalous transport in an external field [6], which are resulted in from the continuous time random walk (CTRW) in the scaling limit. The CTRW is a mathematical formalization of a path that consists of a succession of random steps including the elements of random waiting time and jump length; and it underlies many fundamental stochastic processes in statistical physics. When the first moment of the distribution of waiting time and the second moment of jump length are finite, the probability density function (PDF) of the particle’s location and time satisfies the classical diffusion equation. However, if the jump length obeys the power-law distribution, the PDF of the particle’s location and time is the solution of space fractional diffusion equation; and the corresponding dynamics is called Lévy flight. Sometimes the jumps of the particles are limited by the finite size of the physical system and the infinite variance of Lévy flight discourages it as a physical approach. So the power-law distribution of the jump length is expected to be truncated [7] or exponentially tempered [8]. For the CTRW with the distribution of the tempered jump length \(|x|^{-(1+\alpha)}e^{-\lambda|x|} [9] , the corresponding PDF of the particles satisfies the tempered space fractional diffusion equation [8].

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It seems that there are less works for the numerical solutions of tempered space fractional diffusion equation [10]. However, for the space fractional diffusion or advection-diffusion equation, much progress has been made for its numerical methods, e.g., [11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Transforming the Riemann-Liouville fractional derivative to Caputo fractional derivative, the space fractional Fokker-Planck equation is solved by the method of lines in [11]. Using the superconvergence of Grünwald discretization at a particular point, a second order finite difference scheme is proposed in [17]. Based on the difference discretization and spline approximation to the Riemann-Liouville fractional derivative, a second order scheme is presented for the three dimensional space fractional partial differential equations in [20]. Currently, the most popular discretization scheme for the space Riemann-Liouville fractional derivative seems to be the weighted and shifted Grünwald (WSGD) operator. The first order WSGD operator is firstly presented and detailedly discussed in [12, 13, 14] and the second order convergence is obtained by using extrapolation method [15, 16]. The second order WSGD operator is given in [18]; and the third order compact WSGD (CWSGD) is presented in [19]. Following the idea of weighting and shifting Grünwald operator, this paper provides the basic strategy of deriving the quasi-compact scheme with any desired convergence orders for space fractional diffusion equation; and it can also be extended to solve the tempered space fractional diffusion equation. The fourth order quasi-compact scheme is detailedly discussed in solving space fractional diffusion equation, including stability and convergence analysis and numerical verification of convergence orders. The fourth order quasi-compact scheme for tempered space fractional diffusion equation is also proposed and effectively used to solve the equation; and the convergence orders are numerically verified.

The outline of this paper is as follows. In Sec. 2, the high order quasi-compact discretizations are presented to approximate space Riemann-Liouville fractional derivative. In Sec. 3, following the obtained quasi-compact discretizations, the high order quasi-compact scheme for the one dimensional space fractional diffusion equation is designed and its stability and convergence analysis are performed. Sec. 4 focuses on the quasi-compact scheme and the corresponding stability and convergence analysis in two dimensional case. The high order quasi-compact discretizations is extended to tempered space fractional derivative in Sec. 5 and the corresponding scheme is derived to solve tempered space fractional diffusion equation. In Sec. 6, numerical experiments are performed to testify the efficiency and verify the convergence orders of the schemes. We conclude the paper with some discussions in the last section.

2. Quasi-compact discretizations for Riemann-Liouville space fractional derivatives

We first introduce some definitions and lemmas, including Riemann-Liouville fractional derivatives and shifted Grünwald-Letnikov formulations.

**DEFINITION 2.1.** [21] If the function \( u(x) \) is defined in the interval \((a, b)\) and regular enough, then the \( \alpha \)-th order left and right Riemann-Liouville fractional derivatives are, respectively, defined as

\[
a D^\alpha_x u(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-s)^{n-\alpha-1} u(s) ds, \quad n-1 < \alpha < n
\]  

(2.1)

and

\[
x D^\alpha_x u(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (s-x)^{n-\alpha-1} u(s) ds, \quad n-1 < \alpha < n
\]  

(2.2)

where \( a \) can be \(-\infty\) and \( b \) can be \(+\infty\).
And the standard left and right Grünwald-Letnikov formulations which can be potentially used to approximate the left and right Riemann-Liouville fractional derivatives are, respectively, given as

\[ aD_x^\alpha u(x) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor \frac{x}{h} \rfloor} g_k^{(\alpha)} u(x-kh) \quad (2.3) \]

and

\[ zD_b^\alpha u(x) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor \frac{x}{h} \rfloor} g_k^{(\alpha)} u(x+kh), \quad (2.4) \]

where the Grünwald weights \( g_k^{(\alpha)} = \frac{\Gamma(k+1)}{\Gamma(-\alpha)\Gamma(k+1)} \) are the coefficients of the power series expansion of \((1-z)^\alpha\). For getting the stable scheme, a shifted Grünwald-Letnikov operator is proposed to approximate the left Riemann-Liouville fractional derivative with first order accuracy [15].

**Lemma 2.2 ([15]).** Let \( 1 < \alpha < 2 \), \( u \in C^{n+3}(\mathbb{R}) \), and \( D^k u(x) \in L^1(\mathbb{R}), \ k = 0, 1, \ldots, n+3 \). For any integer \( p \), define the left shifted Grünwald-Letnikov operator by

\[ \Delta_p^\alpha u(x) := \frac{1}{h^\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} u(x-(k-p)h). \quad (2.5) \]

Then we have

\[ \Delta_p^\alpha u(x) = -\infty D_x^\alpha u(x) + \sum_{l=1}^{n-1} a_{p,l}^\alpha D_x^{\alpha+l} u(x) h^l + O(h^n), \quad (2.6) \]

uniformly in \( x \in \mathbb{R} \), where the weights \( a_{p,l}^\alpha \) are the coefficients of the power series expansion of the function \((1-z)^\alpha e^{pz}\), and the first four terms of the coefficients are \( a_{p,0}^\alpha = 1 \), \( a_{p,1}^\alpha = p - \alpha/2 \), \( a_{p,2}^\alpha = (\alpha + 3\alpha^2 - 12\alpha p + 12p^2)/24 \), and \( a_{p,3}^\alpha = (8p^3 + 2p\alpha - 12p^2\alpha - 6p\alpha^2 - \alpha^3)/48 \).

To approximate the right Riemann-Liouville fractional derivative \( zD_b^\alpha u(x) \), the right shifted Grünwald-Letnikov operator is given by \( \Lambda_p^\alpha f(x) := \frac{1}{h^\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} f(x+(k-p)h) \). In the finite interval \([a,b]\), the shifted Grünwald-Letnikov fractional derivatives are

\[ \tilde{\Delta}_p^\alpha u(x) = \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor \frac{b-x}{h} \rfloor} g_k^{(\alpha)} u(x-(k-p)h) \quad (2.7) \]

and

\[ \tilde{\Lambda}_p^\alpha u(x) = \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor \frac{b-x}{h} \rfloor} g_k^{(\alpha)} u(x+(k-p)h). \quad (2.8) \]

In the remaining analysis of the paper, for a function defined in the bounded interval, we suppose that it has been zero extended to \( \mathbb{R} \) whenever the value of \( u(x) \) outside of the bounded interval is used.
2.1. Fourth order quasi-compact approximation to the Riemann-Liouville fractional derivative

According to the definitions of the shifted Grünwald-Letnikov fractional derivatives, we know that \( p \) can be any integer. In order to ensure that the nodes in (2.7) or (2.8) are within the bounded interval, we need to choose the integer \( p \in \{0, 1, 2, \ldots, n\} \) when approximating non-periodic fractional differential equation in the bounded interval. Inspired by the shifted Grünwald-Letnikov operator and the Taylor expansion, we derive the following fourth order combined quasi-compact approximations.

**Theorem 2.3.** Let \( u(x) \in C^7(R) \) and all the derivatives of \( u(x) \) up to order 7 belong to \( L^1(R) \). Then the following quasi-compact approximation has fourth order accuracy, i.e.,

\[
P_x \Delta^\alpha_x u(x) = \mu_1 \Delta^\alpha_x u(x) + \mu_0 \Delta^\alpha_x u(x) + \mu_{-1} \Delta^\alpha_{-1} u(x) + O(h^4),
\]

where \( P_x = 1 + h^2 b_2^0 \delta^2_x \), called CWSGD operator; \( \delta^2_x \) is the centered difference operator; and the coefficients \( b_2^0, \mu_1, \mu_0, \) and \( \mu_{-1} \) are the functions of \( \alpha \) and

\[
\begin{align*}
\mu_1 &= (1 + \alpha)(2 + \alpha)/12, \\
\mu_0 &= (-2 + \alpha)(2 + \alpha)/6, \\
\mu_{-1} &= (-2 + \alpha)(-1 + \alpha)/12, \\
b_2^0 &= (4 + \alpha - \alpha^2)/24.
\end{align*}
\]

In fact, under the assumptions of the theorem, we know that for any fixed order \( \alpha \) and the coefficients \( \mu_1, \mu_0, \) and \( \mu_{-1} \) the following equalities hold.

\[
\mu_1 \Delta^\alpha_x u(x) + \mu_0 \Delta^\alpha_x u(x) + \mu_{-1} \Delta^\alpha_{-1} u(x)
\]

\[
= -\infty D^\alpha_x u(x) + b_2^0 \infty D^\alpha_{-1} u(x) + O(h^4)
\]

\[
= \left(1 + h^2 b_2^0 \frac{\partial^2}{\partial x^2}\right) -\infty D^\alpha_x u(x) + O(h^4)
\]

\[
= P_x -\infty D^\alpha_x u(x) + O(h^4),
\]

where \( b_2^0 = \mu_1 a_{1,2}^0 + \mu_0 a_{1,2}^0 + \mu_{-1} a_{-1,2}^0 \). Then we get (2.11). Since \( \delta^2_x u(x) = (u(x-h) - 2u(x) + u(x+h))/h^2 = \frac{\partial^2 u(x)}{\partial x^2} + O(h^2) \), we have for any function \( u \),

\[
P_x u = \left(1 + h^2 b_2^0 \frac{\partial^2}{\partial x^2}\right) u + O(h^4).
\]

In a similar way, we can derive the fourth order quasi-compact approximation for the right Riemann-Liouville fractional derivative:

\[
P_{-\infty} D^\alpha_x u(x) = \mu_1 \Delta^\alpha_x u(x) + \mu_0 \Delta^\alpha_x u(x) + \mu_{-1} \Delta^\alpha_{-1} u(x) + O(h^4). \tag{2.12}
\]

For \( u(x) \) defined in a bounded interval, supposing its zero extension to \( R \) satisfies the assumptions of Theorem 2.3, the following approximations hold:

\[
P_x \Delta^\alpha_x u(x) = \mu_1 \Delta^\alpha_x u(x) + \mu_0 \Delta^\alpha_x u(x) + \mu_{-1} \Delta^\alpha_{-1} u(x) + O(h^4) \tag{2.13}
\]
Example 2.4. To numerically verify the above statements.

Theorem 2.5. Let \( u(x) \in C^8(R) \) and all the derivatives of \( u(x) \) up to order 8 belong to \( L^1(R) \). Then the following quasi-compact approximation has fifth order accuracy, i.e.,

\[
P_x D^\alpha u(x) = \mu_1 \Delta^\alpha f(x) + \mu_0 \Delta^\alpha f(x) + \mu_{-1} \Delta^\alpha f(x) + O(h^5),
\]

Now using the CWSGD operator, we solve a two-point boundary value problem to numerically verify the above statements.

Example 2.4. Consider the steady state fractional diffusion problem

\[
0 D^\alpha u(x) = \frac{720 x^{6-\alpha}}{\Gamma(7-\alpha)}, \quad x \in (0,1),
\]

with \( 1 < \alpha < 2 \) and the boundary conditions \( u(0) = 0, u(1) = 1 \). Its exact solution is \( u(x) = x^6 \).

Using the quasi-compact scheme (2.13) to solve Example 2.4 leads to the desired convergence orders; see Table 2.1.

| \( \alpha \) | \( h_\alpha \) | \( \| u - \hat{U} \|_2 \) | rate | \( \| u - \hat{U} \|_\infty \) | rate |
|---|---|---|---|---|---|
| 1.1 | 1/8 | 6.0879e-04 | 1.0551e-03 | 1.0551e-03 |
| | 1/16 | 2.7715e-05 | 4.4572 | 5.1560e-05 | 4.3548 |
| | 1/32 | 1.5024e-06 | 4.2054 | 2.8244e-06 | 4.1905 |
| | 1/64 | 9.0430e-08 | 4.0543 | 1.6385e-07 | 4.1075 |
| | 1/128 | 5.5808e-09 | 4.0183 | 9.5651e-09 | 4.0984 |
| 1.5 | 1/8 | 2.9459e-04 | 3.9380e-04 | 3.9380e-04 |
| | 1/16 | 1.8470e-05 | 3.9955 | 2.4150e-05 | 4.0274 |
| | 1/32 | 1.1590e-06 | 3.9942 | 1.5252e-06 | 3.9850 |
| | 1/64 | 7.2639e-08 | 3.9960 | 9.5671e-08 | 3.9948 |
| | 1/128 | 4.5471e-09 | 3.9977 | 9.9911e-09 | 3.9972 |
| 1.9 | 1/8 | 1.1926e-04 | 1.6198e-04 | 1.6198e-04 |
| | 1/16 | 7.4913e-06 | 3.9927 | 1.0174e-05 | 3.9928 |
| | 1/32 | 4.6919e-07 | 3.9970 | 6.3722e-07 | 3.9970 |
| | 1/64 | 2.9352e-08 | 3.9986 | 3.9899e-08 | 3.9974 |
| | 1/128 | 1.8352e-09 | 3.9994 | 2.4947e-09 | 3.9994 |

Using the quasi-compact scheme (2.13) to solve Example 2.4 leads to the desired convergence orders; see Table 2.1.

2.2. Fifth order quasi-compact approximation to the Riemann-Liouville fractional derivative. In this subsection, we present a fifth order quasi-compact approximation given as follows.

Theorem 2.5. Let \( u(x) \in C^8(R) \) and all the derivatives of \( u(x) \) up to order 8 belong to \( L^1(R) \). Then the following quasi-compact approximation has fifth order accuracy, i.e.,

\[
P_x D^\alpha u(x) = \mu_1 \Delta^\alpha f(x) + \mu_0 \Delta^\alpha f(x) + \mu_{-1} \Delta^\alpha f(x) + O(h^5),
\]
where $P_{x}^{5}\to-\infty D_{x}^{\alpha}u(x) = \gamma_{1}-\infty D_{x}^{\alpha}u(x-h) + -\infty D_{x}^{\alpha}u(x) + \gamma_{2}-\infty D_{x}^{\alpha}u(x+h)$, called 5-CWSGD operator, and

\[
\begin{align*}
\gamma_{1} &= \frac{350+331\alpha-15\alpha^2-75\alpha^3-15\alpha^4}{1724-2\alpha-570\alpha^2-30\alpha^3-30\alpha^4}, \\
\gamma_{2} &= \frac{566-329\alpha-15\alpha^2+15\alpha^3-15\alpha^4}{1724-2\alpha-570\alpha^2-30\alpha^3+30\alpha^4}, \\
\mu_{1} &= \frac{566+329\alpha-15\alpha^2-15\alpha^3-15\alpha^4}{1724-2\alpha-570\alpha^2-30\alpha^3+30\alpha^4}, \\
\mu_{0} &= \frac{862+\alpha-285\alpha^2+15\alpha^3+15\alpha^4}{862-\alpha-285\alpha^2-15\alpha^3+15\alpha^4}, \\
\mu_{-1} &= \frac{350-331\alpha+15\alpha^2+75\alpha^3-15\alpha^4}{1724-2\alpha-570\alpha^2-30\alpha^3-30\alpha^4}.
\end{align*}
\]

(2.16)

The way of deriving (2.15) is similar to the derivation of the fourth order quasi-compact approximation. On one hand, from (2.6), we know for different parameter $p \in \{1,0,-1\}$ there exist three equalities

\[
\Delta_{x}^{p}u(x) = -\infty D_{x}^{\alpha}u(x) + \sum_{k=1}^{4} \frac{1}{k!} -\infty D_{x}^{\alpha+k}u(x)h^{k} + O(h^{5}), \quad p = 1,0,-1. \tag{2.17}
\]

On the other hand, in view of the Taylor expansion we know

\[
-\infty D_{x}^{\alpha}u(x-h) = -\infty D_{x}^{\alpha}u(x) + (-1)^{k} \sum_{k=1}^{4} \frac{1}{k!} -\infty D_{x}^{\alpha+k}u(x)h^{k} + O(h^{5}), \tag{2.18}
\]

\[
-\infty D_{x}^{\alpha}u(x+h) = -\infty D_{x}^{\alpha}u(x) + \sum_{k=1}^{4} \frac{1}{k!} -\infty D_{x}^{\alpha+k}u(x)h^{k} + O(h^{5}).
\]

So in order to get the fifth order quasi-compact approximation, combining (2.17) and (2.18), we need to eliminate the low order terms corresponding to $h^{k}$ ($k = 1,2,3,4$), which can be done by solving the algebraic equation

\[
\begin{align*}
\mu_{1} + \mu_{0} + \mu_{-1} - \gamma_{1} - \gamma_{2} &= 1, \\
\mu_{1}a_{1,1}^{0} + \mu_{0}a_{0,1}^{0} + \mu_{-1}a_{-1,1}^{0} + \gamma_{1} - \gamma_{2} &= 0, \\
\mu_{1}a_{1,2}^{0} + \mu_{0}a_{0,2}^{0} + \mu_{-1}a_{-1,2}^{0} - \gamma_{1}/2 - \gamma_{2}/2 &= 0, \\
\mu_{1}a_{1,3}^{0} + \mu_{0}a_{0,3}^{0} + \mu_{-1}a_{-1,3}^{0} + \gamma_{1}/3! - \gamma_{2}/3! &= 0, \\
\mu_{1}a_{1,4}^{0} + \mu_{0}a_{0,4}^{0} + \mu_{-1}a_{-1,4}^{0} - \gamma_{1}/4! - \gamma_{2}/4! &= 0.
\end{align*}
\]

(2.19)

Eq. (2.10) is the solution of (2.19). Then we get Theorem 2.5. Next we utilize the 5-CWSGD operator to solve Example 2.6, and the numerical results are presented in
Table 2.2 from which the accuracy of the 5-CWSGD operator is verified.

**Example 2.6.** We again consider the steady state fractional diffusion problem simulated in Example 2.3, i.e.,

$$aD_x^αu(x) = \frac{270x^{6-α}}{(7-α)}, \quad x \in (0,1),$$

with $1 < α < 2$ and the boundary conditions $u(0) = 0$, $u(1) = 1$; and the exact solution $u(x) = x^6$.

**Remark 2.7.** As the fifth order quasi-compact scheme is not stable in solving the time-dependent space fractional differential equation, we detailedly discuss the fourth order quasi-compact schemes in Sections 3 and 4.

**3. Quasi-compact scheme for one dimensional space fractional diffusion equation** Based on the fourth order quasi-compact discretization to the Riemann-Liouville space fractional derivative, we develop the Crank-Nicolson quasi-compact scheme of the two-sided space fractional diffusion equations. Here we consider the initial boundary value problem of the space fractional diffusion equation

$$\begin{align*}
\frac{∂u(x,t)}{∂t} &= K_1D_x^αu(x,t) + K_2D_x^αu(x,t) + f(x,t), \quad (x,t) \in (a,b) × (0,T], \\
u(x,0) &= u_0(x), \quad x \in [a,b], \\
u(a,t) &= ϕ_a(t), \quad u(b,t) = ϕ_b(t), \quad t \in [0,T],
\end{align*}
\tag{3.1}$$

where $1 < α ≤ 2$. The diffusion coefficients $K_1$ and $K_2$ are nonnegative constants and they satisfy $K_1^2 + K_2^2 ≠ 0$. If $K_1 ≠ 0$, then $ϕ_a(t) ≡ 0$ and $K_2 ≠ 0$, then $ϕ_b(t) ≡ 0$. In the following analysis of the numerical method, we suppose that (3.1) has an unique and sufficiently smooth solution.

**3.1. CN-CWSGD scheme** The time interval $[0,T]$ is partitioned into a uniform mesh with the step size $τ = T/N$ and the space interval $[a,b]$ into another uniform mesh with the step size $h = (b-a)/M$, where $N,M$ are two positive integers. Then the set of grid points can be denoted by $x_j = a + jh$ ($0 ≤ j ≤ M$) and $t_n = nτ$.
Fourth order quasi-compact difference schemes

\(0 \leq n \leq N\). Let \(u_j^n = u(x_j, t_n)\), \(t_{n+1/2} = (t_n + t_{n+1})/2\), and \(f_j^{n+1/2} = f(x_j, t_{n+1/2})\) for \(0 \leq n \leq N - 1\). The maximum norm and the discrete \(L_2\) norm are defined as

\[
\|u\|_\infty = \max_{1 \leq j \leq M-1} |u_j|, \quad \|u\|^2 = h \sum_{j=1}^{M-1} u_j^2. \tag{3.2}
\]

We use the Crank-Nicolson technique for the time discretization of (3.1) and get

\[
\frac{u_j^{n+1} - u_j^n}{\tau} = \frac{1}{2} \left( K_1(aD_x^n u)_j^n + K_1(aD_x^\alpha u)^{n+1}_j + K_2(xD_y^n u)_j^n + K_2(xD_y^\alpha u)^{n+1}_j \right)
+ f_j^{n+1/2} + O(\tau^2). \tag{3.3}
\]

In space, the fourth order quasi-compact discretizations are used to approximate the Riemann-Liouville fractional derivatives. This implies that

\[
P_x \frac{u_j^{n+1} - u_j^n}{\tau} = \frac{K_1 \tau}{2} L D_h^n u_j^n + \frac{K_2 \tau}{2} R D_h^\alpha u_j^n + \frac{K_1 \tau}{2} L D_h^\alpha u_j^{n+1} + \frac{K_2 \tau}{2} R D_h^\alpha u_j^{n+1}
+ P_x f_j^{n+1/2} + R_j^{n+1/2}, \tag{3.4}
\]

where

\[
L D_h^\alpha u_j^n := \mu_1 \tilde{\Delta}_h^\alpha u_j^n + \mu_0 \tilde{\tilde{\Delta}}_h^\alpha u_j^n + \mu_{-1} \tilde{\Delta}_h^{-\alpha} u_j^n = \frac{1}{h^\alpha} \sum_{k=0}^{j+1} w_k^{(\alpha)} u_{j-k+1}^n,
\]

\[
R D_h^\alpha u_j^n := \mu_1 \tilde{\Delta}_h^\alpha u_j^n + \mu_0 \tilde{\tilde{\Delta}}_h^\alpha u_j^n + \mu_{-1} \tilde{\Delta}_h^{-\alpha} u_j^n = \frac{1}{h^\alpha} \sum_{k=0}^{M-j+1} w_k^{(\alpha)} u_{j+k-1}^n,
\]

the coefficients \(w_k^{(\alpha)} = \mu_1 g_k^{(\alpha)}, \quad w_k^{(\alpha)} = \mu_0 g_k^{(\alpha)} + \mu_1 g_k^{(\alpha)}, \quad \text{and} \quad w_k^{(\alpha)} = \mu_1 g_k^{(\alpha)} + \mu_0 g_k^{(\alpha)} + \mu_{-1} g_{k-1}^{(\alpha)}\), \(k = 2, \ldots, M\) and \(P_j^{n+1/2} \leq C(\tau^2 + h^4)\). Then the above equation can be rewritten as

\[
P_x u_j^{n+1} - \frac{K_1 \tau}{2} L D_h^\alpha u_j^{n+1} - \frac{K_2 \tau}{2} R D_h^\alpha u_j^{n+1}
= P_x u_j^n + \frac{K_1 \tau}{2} L D_h^\alpha u_j^n + \frac{K_2 \tau}{2} R D_h^\alpha u_j^n + \tau P_x f_j^{n+1/2} + \tau R_j^{n+1/2}. \tag{3.5}
\]

Denoting \(U_j^n\) as the numerical approximation of \(u_j^n\), we obtain the Crank-Nicolson quasi-compact scheme for (3.1)

\[
P_x U_j^{n+1} - \frac{K_1 \tau}{2} L D_h^\alpha U_j^{n+1} - \frac{K_2 \tau}{2} R D_h^\alpha U_j^{n+1}
= P_x U_j^n + \frac{K_1 \tau}{2} L D_h^\alpha U_j^n + \frac{K_2 \tau}{2} R D_h^\alpha U_j^n + \tau P_x f_j^{n+1/2}. \tag{3.6}
\]

For convenience, the approximation scheme (3.6) can be written in matrix form

\[
(P_\alpha - B_\alpha) U_j^{n+1} = (P_\alpha + B_\alpha) U_j^n + \tau F^n + H^n, \tag{3.7}
\]
Lemma 3.2. Let $f(x) = \frac{x^2}{2}$, then the square root of $f(x)$ is given by $\sqrt{f(x)} = x$. In particular, if $\sqrt{f(x)}$ is defined as $\sqrt{f(x)} = x$, then we have $f(x) = x^2$.

Theorem 3.3. The matrix $A_\alpha + A_\alpha^T$ is negative definite, and $B_\alpha + B_\alpha^T$ is also negative definite, where $A_\alpha$ is given by (3.8) and $B_\alpha$ defined in (3.7).

3.2. Stability and convergence analysis

In this subsection, we prove that the CN quasi-compact scheme has fourth order accuracy in space and is unconditionally stable. Now we give some important lemmas to be used in the analyses.

**Lemma 3.1.** Let $H$ be a Toeplitz matrix with a generating function $f \in C_{2\pi}$. Let $\lambda_{\text{min}}(H)$ and $\lambda_{\text{max}}(H)$ denote the smallest and largest eigenvalues of $H$, respectively. Then we have

$$f_{\text{min}} \leq \lambda_{\text{min}}(H) \leq \lambda_{\text{max}}(H) \leq f_{\text{max}},$$

where $f_{\text{min}}$ and $f_{\text{max}}$ denote the minimum and maximum values of $f(x)$, respectively.

**Lemma 3.2.** Let $A$ be a positive semi-definite matrix. Then there exists a unique $n$-square positive semi-definite matrix $B$ such that $B^2 = A$. Such a matrix $B$ is called the square root of $A$, denoted by $A^{\frac{1}{2}}$.

**Theorem 3.3.** The matrix $A_\alpha + A_\alpha^T$ is negative definite, and $B_\alpha + B_\alpha^T$ is also negative definite, where $A_\alpha$ is given by (3.8) and $B_\alpha$ defined in (3.7).
Fourth order quasi-compact difference schemes

In fact, the generating function \( f(\alpha, x) \) of \( A + A^T \) satisfies

\[
f(\alpha, x) = f_{A, \alpha}(x) + f_{A^T, \alpha}(x) = \left( \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{-i(k-1)x} + \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{i(k-1)x} \right)
\]

\[
= \mu_1 \left( \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{-i(k-1)\sigma} + \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{i(k-1)\sigma} \right) + \mu_0 \left( \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{-i(k+1)\sigma} + \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{i(k+1)\sigma} \right)
\]

\[
= \mu_1((1 - e^{-i\sigma})^\alpha e^{i\sigma} + (1 - e^{i\sigma})^\alpha e^{-i\sigma}) + \mu_0((1 - e^{-i\sigma})^\alpha + (1 - e^{i\sigma})^\alpha)
\]

\[
+ \mu_{-1}((1 - e^{-i\sigma})^\alpha e^{-i\sigma} + (1 - e^{i\sigma})^\alpha e^{i\sigma})
\]

\[
= (2\sin(\sigma/2)^2 (\mu_1(e^{i(\Delta k - \Delta k + \sigma)} + e^{-i(\Delta k - \Delta k + \sigma)}) + \mu_0(e^{i(\Delta k - \Delta k)} + e^{-i(\Delta k - \Delta k)})
\]

\[
+ \mu_{-1}(e^{i(\Delta k - \Delta k - \sigma)} + e^{-i(\Delta k - \Delta k - \sigma)})
\]

\[
= 2 \left( 2\sin(\sigma/2)^2 \right) (\mu_1 \cos(\alpha \pi/2 - \Delta x/2 + x) + \mu_0 \cos(\alpha \pi/2 - \Delta x/2) + \mu_{-1} \cos(\alpha \pi/2 - \Delta x/2 - x))
\]

\[
(3.10)
\]

where \( f_{A, \alpha}(x) \) and \( f_{A^T, \alpha}(x) \) denote the generating functions of the matrix \( A_\alpha \) and \( A^T_\alpha \), respectively. Since \( f(\alpha; x) \) is a real-valued and even function, it’s reasonable to consider its principal value on \([0, \pi]\). Together with Fig. 3.1, we have that \( f(\alpha; x) \leq 0 \) for \( 1 \leq \alpha \leq 2 \) on \([-\pi, \pi]\). Then from Lemma 3.1, we know the matrix \( A_\alpha + A^T_\alpha \) is negative definite. Rewriting \( B_\alpha + B^T_\alpha \) as \( \frac{\pi}{\alpha x} (K_1(A_\alpha + A^T_\alpha) + K_2(A^T_\alpha + A_\alpha)) \), it can be clearly seen that \( B_\alpha + B^T_\alpha \) is negative definite.

**Theorem 3.4.** The difference scheme (3.6) with \( \alpha \in (1, 2) \) is unconditionally stable.
Since the boundary conditions of error equation (3.11) are obtained in finite precision arithmetic. Since \( \tilde{U}_j^n \) satisfies the discretized equation exactly, round-off error \( \epsilon_j^n \) must also satisfy the discretized equation (22). Thus we obtain the following error equation

\[
P_x \epsilon_j^{n+1} + \frac{K_1 \tau}{2} \ell D_h^n \epsilon_j^{n+1} - \frac{K_2 \tau}{2} R D_h^n \epsilon_j^{n+1} = P_x \epsilon_j^n + \frac{K_1 \tau}{2} \ell D_h^n \epsilon_j^n + \frac{K_2 \tau}{2} R D_h^n \epsilon_j^n.
\]

(3.11)

Since the boundary conditions of error equation (3.11) are \( \epsilon_0^n = \epsilon_j^n = \epsilon_j^{n+1} = 0 \), we zero extend the solution of the problem (3.11) to the whole real line \( R \). So it’s reasonable to replace the symbols \( j + 1 \) and \( M - j + 1 \) in error equation (3.11) with \( \infty \). Now we have

\[
b_2^\sigma \epsilon_{j-1}^{n+1} + (1 - 2b_2^\sigma) \epsilon_j^{n+1} + b_2^\sigma \epsilon_{j+1}^{n+1} - \frac{K_1 \tau}{2h^\alpha} \sum_{k=0}^\infty w_k^{(\alpha)} \epsilon_{j-k+1}^{n+1} - \frac{K_2 \tau}{2h^\alpha} \sum_{k=0}^\infty w_k^{(\alpha)} \epsilon_{j+k-1}^{n+1} = b_2^\sigma \epsilon_{j-1}^n + (1 - 2b_2^\sigma) \epsilon_j^n + b_2^\sigma \epsilon_{j+1}^n + \frac{K_1 \tau}{2h^\alpha} \sum_{k=0}^\infty w_k^{(\alpha)} \epsilon_{j-k+1}^n + \frac{K_2 \tau}{2h^\alpha} \sum_{k=0}^\infty w_k^{(\alpha)} \epsilon_{j+k-1}^n.
\]

(3.12)

Let \( \epsilon^n = v^n e^{ij\sigma} \) be the solution of (3.12), where \( i = \sqrt{-1}, \) \( v^n \) is the amplitude at time level \( n \) and \( \sigma(=2\pi h/k) \) is the phase angle with wavelength \( k \). We just need to prove that the amplification factor \( v(\sigma, \alpha) \) satisfies the relation \( |v(\sigma, \alpha)| \leq 1 \) for all \( \sigma \) in \( [-\pi, \pi] \). In fact, by substituting the expressions of \( \epsilon_j^n = v^n e^{ij\sigma} \) and \( \epsilon_j^{n+1} = v^{n+1} e^{ij\sigma} \) into (3.12), we obtain the amplification factor of the CN quasi-compact scheme

\[
v(\sigma, \alpha) = \frac{1 - 4b_2^\sigma \sin^2 \frac{\sigma}{2} + \frac{K_1 \tau}{2h^\alpha} \sum_{k=0}^\infty w_k^{(\alpha)} e^{-i(k-1)\sigma} + \frac{K_2 \tau}{2h^\alpha} \sum_{k=0}^\infty w_k^{(\alpha)} e^{i(k-1)\sigma}}{1 - 4b_2^\sigma \sin^2 \frac{\sigma}{2} - \frac{K_1 \tau}{2h^\alpha} \sum_{k=0}^\infty w_k^{(\alpha)} e^{-i(k-1)\sigma} - \frac{K_2 \tau}{2h^\alpha} \sum_{k=0}^\infty w_k^{(\alpha)} e^{i(k-1)\sigma}}.
\]

\[
= \frac{Q_1(\sigma, \alpha) + Q_2(\sigma, \alpha)}{Q_1(\sigma, \alpha) - Q_2(\sigma, \alpha)},
\]

where \( Q_1(\sigma, \alpha) = 1 - 4b_2^\sigma \sin^2 \frac{\sigma}{2} \) and \( Q_2(\sigma, \alpha) = \frac{K_1 \tau}{2h^\alpha} \sum_{k=0}^\infty w_k^{(\alpha)} e^{-i(k-1)\sigma} + \frac{K_2 \tau}{2h^\alpha} \sum_{k=0}^\infty w_k^{(\alpha)} e^{i(k-1)\sigma} \).
\[
\frac{K \tau^2}{2h^4} \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{i(k-1)\sigma}.
\]
A straightforward calculation yields
\[
Q_2(\sigma, \alpha)
= \frac{K \tau}{2h^{\alpha}} \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{-i(k-1)\sigma} + \frac{K \tau}{2h^{\alpha}} \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{i(k-1)\sigma}
\]
\[
= \frac{\mu_1 \tau}{2h^{\alpha}} (K_1 e^{-i\sigma} - i\sigma + K_2 (1 - e^{i\sigma}) e^{-i\sigma}) + \frac{\mu_0 \tau}{2h^{\alpha}} (K_1 (1 - e^{i\sigma}) e^{-i\sigma} + K_2 (1 - e^{i\sigma}) e^{i\sigma})
\]
\[
+ K_2 e^{i(\frac{\alpha}{2} - \frac{\alpha \sigma}{2})} + \mu_{-1} (K_1 e^{i(\frac{\alpha}{2} - \frac{\alpha \sigma}{2})} + K_2 e^{-i(\frac{\alpha}{2} - \frac{\alpha \sigma}{2})})
\]\nAs \(Q_1(\sigma, \alpha)\) is real-valued,
\[
|v(\sigma, \alpha)| = \frac{|Q_1 + Q_2|}{|Q_1 - Q_2|} = \sqrt{\frac{(Q_1 + Re(Q_2))^2 + (Im(Q_2))^2}{(Q_1 - Re(Q_2))^2 + (Im(Q_2))^2}},
\]
where \(Re(Q_2)\) and \(Im(Q_2)\) are real part and imaginary part of \(Q_2\), respectively. In order to prove that \(|v(\sigma, \alpha)| \leq 1\), we need to check
\[
Q_1 \cdot Re(Q_2) \leq 0.
\]
Note that \(b_2^2 = (4 + \alpha - \alpha^2)/24 \leq 1/6\) for any \(\alpha \in [1, 2]\). So \(Q_1 = 1 - 4b_2^2 \sin^2(\frac{\alpha}{2}) > 0\).
Form [3.13], we know
\[
Re(Q_2) = \frac{(K_1 + K_2) \tau}{2h^\alpha} (2 \sin(\frac{\alpha}{2}))^\alpha (\mu_1 \cos(\frac{\alpha \pi}{2} - \frac{\alpha \sigma}{2}) + \mu_0 \cos(\frac{\alpha \pi}{2} - \frac{\alpha \sigma}{2}))
\]
\[
+ \mu_{-1} \cos(\frac{\alpha \pi}{2} - \frac{\alpha \sigma}{2} - \sigma)
\]
\[
= \frac{(K_1 + K_2) \tau}{4h^\alpha} f(\alpha; \sigma),
\]
where \(f(\alpha; \sigma)\) is defined by [3.10]. Together with \(K_1 + K_2 > 0\) and Fig. 3.1, we obtain \(Re(Q_2) \leq 0\). Thus \(Q_1 \cdot Re(Q_2) \leq 0\). Then \(|v(\sigma, \alpha)| \leq 1\). So the C-N quasi-compact difference scheme is unconditionally stable. \(\Box\)

**Theorem 3.5.** Let \(u(x_j, t_n)\) be the exact solution of [3.1], and \(U_j^n\) the solution of the given finite difference scheme [3.0]. Then we have
\[
\left\| u(x_j, t_n) - U_j^n \right\| \leq C(\tau^2 + h^4),
\]
for all $1 \leq n \leq N$, where $C$ is a constant independent of $n, \tau$, and $h$.

Proof. Denote $\varepsilon^n_j = u(x_j, t_n) - U^n_j$ and $\varepsilon^n = (\varepsilon^n_1, \varepsilon^n_2, \cdots, \varepsilon^n_{M-1})^T$. According to (3.15) and (3.17), we obtain
\[(P_\alpha - B_\alpha)\varepsilon^{n+1} = (P_\alpha + B_\alpha)\varepsilon^n + \tau R^{n+1/2},\]
where $R^{n+1/2} = (R_1^{n+1/2}, R_2^{n+1/2}, \cdots, R_{M-1}^{n+1/2})^T$. The eigenvalues of $P_\alpha$ are given by
\[\lambda(P_\alpha) = 1 - 4b_2^2 \sin^2(j\pi/M), j = 1, \cdots, M-1.
\]
Since $b_2 \in (1/12, 1/6)$, we have $\lambda(P_\alpha) \in (1/3, 1)$. So the matrix $P_\alpha$ is invertible and positive definite, which means that $P_\alpha^{-1}$ exists and is also positive definite. According to Lemma 3.2, we know that $(P_\alpha^{-1})^{\frac{1}{2}}$ uniquely exists and is positive semi-definite. Multiplying $(P_\alpha^{-1})^{\frac{1}{2}}$ and taking the discrete $L_2$ norm on both sides of (3.14) imply
\[\|((P_\alpha)^{\frac{1}{2}} - (P_\alpha^{-1})^{\frac{1}{2}} B_\alpha)\varepsilon^{n+1}\| \leq \|((P_\alpha)^{\frac{1}{2}} + (P_\alpha^{-1})^{\frac{1}{2}} B_\alpha)\varepsilon^n\| + \tau\|((P_\alpha^{-1})^{\frac{1}{2}} R^{n+1/2})\|.
\]
In view of Theorem 3.3, we know that $B_\alpha + B_\alpha^T$ is a negative definite matrix. Furthermore,
\[[((P_\alpha)^{\frac{1}{2}} - (P_\alpha^{-1})^{\frac{1}{2}} B_\alpha)^T((P_\alpha)^{\frac{1}{2}} - (P_\alpha^{-1})^{\frac{1}{2}} B_\alpha) = P_\alpha - B_\alpha - B_\alpha^T P_\alpha^{-1} B_\alpha \geq P_\alpha + B_\alpha^T P_\alpha^{-1} B_\alpha,
\]
and
\[(((P_\alpha)^{\frac{1}{2}} + (P_\alpha^{-1})^{\frac{1}{2}} B_\alpha)^T((P_\alpha)^{\frac{1}{2}} + (P_\alpha^{-1})^{\frac{1}{2}} B_\alpha) = P_\alpha + B_\alpha + B_\alpha^T P_\alpha^{-1} B_\alpha \leq P_\alpha + B_\alpha^T P_\alpha^{-1} B_\alpha,
\]
where the matrix $A \geq B$ means that $A - B$ is positive semi-definite. Denote
\[E^n = \sqrt{h(\varepsilon^n)^T(P_\alpha + B_\alpha P_\alpha^{-1} B_\alpha)\varepsilon^n}.
\]
Since $B_\alpha^T P_\alpha^{-1} B_\alpha$ is positive definite, we know
\[E^n \geq \sqrt{h(\varepsilon^n)^T P_\alpha \varepsilon^n} \geq \sqrt{\lambda_{\min}(P_\alpha)}\|\varepsilon^n\|,
\]
where $\lambda_{\min}(P_\alpha)$ is the minimum eigenvalue of matrix $P_\alpha$. Together with (3.15) and (3.16), we have
\[E^{n+1} - E^n \leq \tau\|((P_\alpha^{-1})^{\frac{1}{2}} R^{n+1/2})\| = \tau\sqrt{h(R^{n+1/2})^T(P_\alpha^{-1})R^{n+1/2}}\]
\[\leq \tau\sqrt{\lambda_{\max}(P_\alpha^{-1})\|R^{n+1/2}\|} = \frac{\tau}{\sqrt{\lambda_{\min}(P_\alpha)}}\|R^{n+1/2}\|.
\]
Summing up (3.19) from 0 to $n - 1$ leads to
\[E^n \leq \tau \sum_{k=0}^{n-1}\|((P_\alpha^{-1})^{\frac{1}{2}} R^{k+1/2})\| \leq \frac{\tau}{\sqrt{\lambda_{\min}(P_\alpha)}} \sum_{k=0}^{n-1}\|R^{k+1/2}\|.
\]
Combining (3.18) and (3.20) and noticing that $\|R^{k+1/2}_{j}\| \leq C(\tau^2 + h^2)$ for $1 \leq j \leq M - 1$, we obtain
\[\|\varepsilon^n\| \leq \frac{cT}{\lambda_{\min}(P_\alpha)}(\tau^2 + h^2) \leq C(\tau^2 + h^2).
\]
4. Quasi-compact scheme for two dimensional space fractional diffusion equation  
To discuss the quasi-compact scheme in two dimensional case, we consider the following space fractional diffusion equation

\[
\left\{ \begin{array}{l}
\frac{\partial u(x,t)}{\partial t} = K_1^x aD_x^\alpha u(x,t) + K_2^x bD_y^\alpha u(x,t) \\
\quad + K_3^y cD_y^\beta u(x,t) + K_4^y dD_y^\beta u(x,t) + f(x,t), \quad (x,y,t) \in \Omega \times (0,T], \\
u(x,y,0) = u_0(x,y), \quad (x,y) \in \Omega, \\
u(x,y,t) = \phi(x,y,t), \quad (x,y) \in \partial \Omega \times (0,T],
\end{array} \right.
\]

(4.1)

where \(\Omega = (a,b) \times (c,d)\) and the fractional orders \(1 < \alpha, \beta \leq 2\). The diffusion coefficients \(K_j^x\) and \(K_j^y\) \((j = 1, 2)\) are non-negative and satisfy \((K_1^x)^2 + (K_2^x)^2 \neq 0\) \((j = x, y)\). The boundary function \(\phi\) satisfies the following condition, if \(K_1^x \neq 0\), then \(\phi(a,y,t) = 0\); if \(K_1^y \neq 0\), then \(\phi(x,c,t) = 0\); if \(K_2^x \neq 0\), then \(\phi(b,y,t) = 0\); if \(K_2^y \neq 0\), then \(\phi(x,d,t) = 0\).

We assume that the equation (4.1) has a unique and sufficiently smooth solution.

4.1. CN-CWSGD scheme  
Let us denote \(x_j = a + jh_x\), \(y_s = c + sh_y\), and \(t_n = n\tau\) for \(0 \leq j \leq M_x\), \(0 \leq s \leq M_y\), and \(0 \leq n \leq N\), where the space step size \(h_x = (b - a)/M_x\), \(h_y = (d - c)/M_y\) and time step size \(\tau = T/N\). Here we take \(u^n_{j,s} = u(x_j, y_s, t_n)\) and \(f^{n+1/2}_{j,s} = f(x_j, y_s, t^{n+1/2})\). The maximum norm and the discrete \(L_2\) norm are defined as

\[
\|u\|_\infty = \max_{1 \leq i \leq m_x-1} |u_{i+1,j,s}|, \quad \|u\|^2 = \sum_{j=1}^{M_x-1} \sum_{s=1}^{M_y-1} u_{j,s}^2.
\]

(4.2)

We still use the Crank-Nicolson technique for the time discretization of equation (4.1) and get

\[
\begin{align*}
\frac{u_{j,s}^{n+1} - u_{j,s}^n}{\tau} &= \frac{1}{2} \left( K_1^x (aD_x^\alpha u)^{n+1}_{j,s} + K_1^x (aD_x^\alpha u)^{n}_{j,s} \\
&+ K_2^x (bD_y^\alpha u)^{n+1}_{j,s} + K_2^x (bD_y^\alpha u)^{n}_{j,s} \\
&+ K_3^y (cD_y^\beta u)^{n+1}_{j,s} + K_3^y (cD_y^\beta u)^{n}_{j,s} \\
&+ K_4^y (dD_y^\beta u)^{n+1}_{j,s} + K_4^y (dD_y^\beta u)^{n}_{j,s} \\
&+ f^{n+1/2}_{j,s} + O(\tau^2). \right)
\end{align*}
\]

(4.3)

In space, the fourth order quasi-compact discretizations are used to approximate the Riemann-Liouville fractional derivatives. This implies that

\[
(P_x P_y - \frac{K_1^x \tau}{2} P_y L D_x^\alpha h_x - \frac{K_2^y \tau}{2} P_y R D_y^\alpha h_x - \frac{K_3^x \tau}{2} P_x L D_x^\alpha h_y - \frac{K_4^y \tau}{2} P_x R D_y^\alpha h_y) u_{j,s}^{n+1}
\]

\[
= (P_x P_y + \frac{K_1^x \tau}{2} P_y L D_x^\alpha h_x + \frac{K_2^y \tau}{2} P_y R D_y^\alpha h_x + \frac{K_3^x \tau}{2} P_x L D_x^\alpha h_y + \frac{K_4^y \tau}{2} P_x R D_y^\alpha h_y) u_{j,s}^n
\]

\[
+ \tau P_x P_y f^{n+1/2}_{j,s} + \tau R^{n+1/2}_{j,s},
\]

(4.4)

where

\[
R^{n+1/2}_{j,s} \leq C(\tau^2 + h_x^4 + h_y^4).
\]
For convenience, we introduce the following discrete operator which works for two variables \( x, y \),

\[
\delta_x^n u_{j,s} = K_1^x D_{hx}^n u_{j,s} + K_2^x D_{hx}^n u_{j,s}.
\]

Then the equation (4.4) can be rewritten as

\[
(P_x P_y - \frac{\tau}{2} P_y \delta_x^\alpha - \frac{\tau}{2} P_x \delta_y^\beta) u_{j,s}^{n+1} = (P_x P_y + \frac{\tau}{2} P_y \delta_x^\alpha + \frac{\tau}{2} P_x \delta_y^\beta) u_{j,s}^n + \tau P_x P_y f_{j,s}^{n+1/2} + \tau R_{j,s}^{n+1/2}.
\] (4.5)

Adding the splitting term

\[
\frac{\tau^2}{4} \delta_x^\alpha \delta_y^\beta (u_{j,s}^{n+1} - u_{j,s}^n) (= \tau^2 O(\tau^2 + h_x^4 + h_y^4)),
\] (4.6)

to the equation (4.5), we obtain

\[
(P_x - \frac{\tau}{2} \delta_x^\alpha)(P_y - \frac{\tau}{2} \delta_y^\beta) u_{j,s}^{n+1} = (P_x + \frac{\tau}{2} \delta_x^\alpha)(P_y + \frac{\tau}{2} \delta_y^\beta) u_{j,s}^n + \tau P_x P_y f_{j,s}^{n+1/2} + \tau R_{j,s}^{n+1/2}.
\] (4.7)

Thus the quasi-compact finite difference scheme for (4.1) is given by

\[
(P_x - \frac{\tau}{2} \delta_x^\alpha)(P_y - \frac{\tau}{2} \delta_y^\beta) U_{j,s}^{n+1} = (P_x + \frac{\tau}{2} \delta_x^\alpha)(P_y + \frac{\tau}{2} \delta_y^\beta) U_{j,s}^n + \tau P_x P_y f_{j,s}^{n+1/2}.
\] (4.8)

As an efficient way to implementation, we give the following equivalent schemes:

- quasi-compact Douglas-ADI scheme:

\[
(P_x - \frac{\tau}{2} \delta_x^\alpha) U_{j,s}^* = (P_x P_y + \frac{\tau}{2} P_y \delta_x^\alpha + \tau P_x \delta_y^\beta) U_{j,s}^n + \tau P_x P_y f_{j,s}^{n+1/2},
\]

\[
(P_y - \frac{\tau}{2} \delta_y^\beta) U_{j,s}^{n+1} = U_{j,s}^* - \frac{\tau}{2} \delta_y^\beta U_{j,s}^n;
\] (4.9)

- quasi-compact D’yakonov-ADI scheme:

\[
(P_x - \frac{\tau}{2} \delta_x^\alpha) U_{j,s}^* = (P_x + \frac{\tau}{2} \delta_x^\alpha)(P_y + \frac{\tau}{2} \delta_y^\beta) U_{j,s}^n + \tau P_x P_y f_{j,s}^{n+1/2},
\]

\[
(P_y - \frac{\tau}{2} \delta_y^\beta) U_{j,s}^{n+1} = U_{j,s}^*.
\] (4.10)

### 4.2. Stability and convergence analysis

The following stability analysis and accuracy analysis indicate that two dimensional CN quasi-compact scheme has fourth order accuracy in space and is unconditionally stable.

**Lemma 1.** (24) Let \( A, B \) be two positive semi-definite matrices, symbolized \( \geq 0 \), \( B \geq 0 \).

**Lemma 2.** (26) Let \( A \in \mathbb{R}^{n \times n} \) have eigenvalues \( \{ \tilde{\rho}_1 \}_{j=1}^n \) and \( B \in \mathbb{R}^{m \times m} \) have eigenvalues \( \{ \rho_j \}_{j=1}^m \). Then the \( mn \) eigenvalues of \( A \otimes B \) are

\[
\tilde{\rho}_1 \rho_1, \cdots, \tilde{\rho}_1 \rho_m, \tilde{\rho}_2 \rho_1, \cdots, \tilde{\rho}_2 \rho_m, \cdots, \tilde{\rho}_n \rho_1, \cdots, \tilde{\rho}_n \rho_m.
\]

**Lemma 3.** (26) Let \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{r \times s}, C \in \mathbb{R}^{n \times p}, D \in \mathbb{R}^{s \times t} \). Then

\[
(A \otimes B) (C \otimes D) = AC \otimes BD,
\]
where \( \otimes \) denotes the Kronecker product. Moreover, if \( A, B \in \mathbb{R}^{n \times n} \), \( I \) is a unit matrix of order \( n \), then matrices \( I \otimes A \) and \( B \otimes I \) commute.

**Lemma 4.4.** Let \( A \) be a \( m \times n \) matrix and \( B \) a \( p \times q \) matrix. We have that the transposition is distributive over the Kronecker product:

\[
(A \otimes B)^T = A^T \otimes B^T.
\]

**Theorem 4.5.** For any \( 1 < \alpha, \beta < 2 \), the finite difference scheme (4.8) is unconditionally stable.

*Proof.* Define the round-off error as \( \epsilon_{j,s}^n = U_{j,s}^n - \tilde{U}_{j,s}^n \). The error equation is given by

\[
(P_x - \frac{\tau}{2} \delta_x^\alpha)(P_y - \frac{\tau}{2} \delta_y^\beta)\epsilon_{j,s}^{n+1} = (P_x + \frac{\tau}{2} \delta_x^\alpha)(P_y + \frac{\tau}{2} \delta_y^\beta)\epsilon_{j,s}^n.
\]  \hspace{1cm} (4.11)

Since the boundary conditions of the above error equation are homogeneous, we zero extend the solution of the problem (4.11) to the whole real plane \( \mathbb{R} \times \mathbb{R} \). It's reasonable to replace the symbols \( j+1 \) and \( M-j+1 \) in error equation (4.11) with \( \infty \). Now we have

\[
(P_x - \frac{\tau}{2} \delta_x^\alpha')(P_y - \frac{\tau}{2} \delta_y^\beta')\epsilon_{j,s}^{n+1} = (P_x + \frac{\tau}{2} \delta_x^\alpha')(P_y + \frac{\tau}{2} \delta_y^\beta')\epsilon_{j,s}^n,
\]  \hspace{1cm} (4.12)

where

\[
\delta_x^\alpha' \epsilon_{j,s} = \frac{K_x^2}{h_x^\alpha} \sum_{k=0}^{\infty} u_k^{(\alpha)} \epsilon_{j-k+1,s} + \frac{K_x^2}{h_x^\alpha} \sum_{k=0}^{\infty} w_k^{(\alpha)} \epsilon_{j+k-1,s},
\]

which works for two variables \( x,y \). Let \( \epsilon_{j,s}^n = v^n e^{i(\sigma_1 + s\sigma_2)} \), where \( i = \sqrt{-1}, \) \( v^n \) is the amplitude at time level \( n \) and \( \sigma_1 = 2\pi h_x/k_x \) and \( \sigma_2 = 2\pi h_y/k_y \) are the phase angles with wavelength \( k_x \) and \( k_y \), respectively. Next we just need to prove that the amplification factor \( G(\sigma_1,\sigma_2) = v^{n+1}/v^n \) satisfies the relation \( |G(\sigma_1,\sigma_2)| \leq 1 \) for all \( \sigma_1 \) and \( \sigma_2 \) in \([-\pi,\pi]\). In fact, substituting the expressions of \( \epsilon_{j,s}^n \) and \( \epsilon_{j,s}^{n+1} \) into the equation (4.12), we get the amplification factor

\[
G(\sigma_1,\sigma_2) = \frac{(1 - 4b_2^\beta \sin^2 \frac{\pi}{2} \frac{\sigma_2}{2} - \frac{K_x^4}{2h_x^6} \sum_{k=0}^{\infty} w_k^{(\beta)} e^{-i(k-1)\sigma_2}} + \frac{K_x^2}{2h_x^4} \sum_{k=0}^{\infty} w_k^{(\beta)} e^{i(k-1)\sigma_2})}{(1 - 4b_2^\beta \sin^2 \frac{\pi}{2} \frac{\sigma_2}{2} - \frac{K_x^4}{2h_x^6} \sum_{k=0}^{\infty} w_k^{(\beta)} e^{-i(k-1)\sigma_2}} + \frac{K_x^2}{2h_x^4} \sum_{k=0}^{\infty} w_k^{(\beta)} e^{i(k-1)\sigma_2})}
\]

\[
= \frac{Q_1(\sigma_1,\alpha) + Q_2(\sigma_1,\alpha)}{Q_1(\sigma_1,\alpha) - Q_2(\sigma_1,\alpha)} \frac{Q_1(\sigma_2,\beta) + Q_2(\sigma_2,\beta)}{Q_1(\sigma_2,\beta) - Q_2(\sigma_2,\beta)} = v(\sigma_1,\alpha) \cdot v(\sigma_2,\beta),
\]

where \( Q_1(\sigma_1,\alpha) = 1 - 4b_2^\beta \sin^2 \frac{\pi}{2} \frac{\sigma_2}{2} \frac{K_x^4}{2h_x^6} \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{-i(k-1)\sigma_1} \)

\[+ \frac{K_x^2}{2h_x^4} \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{i(k-1)\sigma_1}, \]

which work for two pairs of variables \((\sigma_1,\alpha)\) and \((\sigma_2,\beta)\).
According to the analysis of Theorem 3.3, we know that \( |v(\sigma_1, \alpha)| \leq 1 \) and \( |v(\sigma_2, \beta)| \leq 1 \) hold for any \( \alpha, \beta \in (1, 2) \). Then

\[
|G(\sigma_1, \sigma_2)| = |v(\sigma_1, \alpha)| \cdot |v(\sigma_2, \beta)| \leq 1.
\]

So the C-N quasi-compact scheme is unconditionally stable.

**Theorem 4.6.** Let \( u(x_j, y_k, t_n) \) be the exact solution of equation (4.7), and \( U^n_{j,s} \) the solution of the given finite difference scheme (4.8). Then we have

\[
\|u(x_j, y_k, t_n) - U^n_{j,s}\| \leq C(\tau^2 + h_x^4 + h_y^4),
\]

for all \( 1 \leq n \leq N \), where \( C \) is a constant independent of \( \tau, h_x, \) and \( h_y \).

**Proof.** Denote \( \epsilon^n_{j,s} = u(x_j, y_k, t_n) - U^n_{j,s} \), and

\[
P_{(\alpha)} = I_{\beta} \otimes P_{\alpha}, \quad P_{(\beta)} = P_{\beta} \otimes I_{\alpha},
\]

\[
(P_{(\alpha)} - B_{(\alpha)})(P_{(\beta)} - B_{(\beta)})\epsilon^{n+1} = (P_{\alpha} + B_{(\alpha)})(P_{\beta} + B_{(\beta)})\epsilon^n + \tau R^{n+1/2},
\]

where \( A_{\alpha} \) and \( A_{\beta} \) are defined in (3.8) corresponding to \( \alpha \) and \( \beta \). In view of (4.7)-(4.8), we obtain

\[
(P_{(\alpha)} - B_{(\alpha)})(P_{(\beta)} - B_{(\beta)})\epsilon^{n+1} = (P_{\alpha} + B_{(\alpha)})(P_{\beta} + B_{(\beta)})\epsilon^n + \tau R^{n+1/2},
\]

where

\[
\epsilon = (\epsilon_{1,1}, \epsilon_{2,1}, \cdots, \epsilon_{M_x-1,1}, \epsilon_{1,2}, \cdots, \epsilon_{M_x-1,2}, \epsilon_{1,M_y-1}, \epsilon_{2, M_y-1}, \cdots, \epsilon_{M_x-1, M_y-1})^T.
\]

Multiplying \( (P_{(\alpha)}^{-1})^\frac{1}{2}(P_{(\beta)}^{-1})^\frac{1}{2} \) and taking the discrete \( L_2 \) norm on both sides of equation (4.14) imply

\[
\| (P_{(\alpha)} - B_{(\alpha)})(P_{\beta} - B_{(\beta)})\epsilon^{n+1} \|
\]

\[
\leq \| (P_{(\alpha)}^{-1})^\frac{1}{2}(P_{(\beta)}^{-1})^\frac{1}{2} (P_{\alpha} + B_{(\alpha)})(P_{\beta} + B_{(\beta)})\epsilon^n \| + \tau \| (P_{(\alpha)}^{-1})^\frac{1}{2}(P_{(\beta)}^{-1})^\frac{1}{2} R^{n+1/2} \|.
\]

Using Lemmas 4.3 and 4.4, it is easy to check that the matrix \( (P_{(\beta)}^{-1})^\frac{1}{2} \) can commute with \( (P_{(\alpha)}^{-1})^\frac{1}{2} \) and \( P_{(\alpha)} \pm B_{(\alpha)} \), i.e.,

\[
(P_{(\beta)}^{-1})^\frac{1}{2}(P_{(\alpha)}^{-1})^\frac{1}{2} = (P_{(\alpha)}^{-1})^\frac{1}{2}(P_{(\beta)}^{-1})^\frac{1}{2} = (P_{(\beta)}^{-1})^\frac{1}{2} \otimes (P_{(\alpha)}^{-1})^\frac{1}{2},
\]

\[
(P_{(\beta)}^{-1})^\frac{1}{2}(P_{(\alpha)} \pm B_{(\alpha)}) = (P_{(\alpha)} \pm B_{(\alpha)})(P_{(\beta)}^{-1})^\frac{1}{2} = (P_{(\beta)}^{-1})^\frac{1}{2} \otimes \left( P_{\alpha} \pm K_{\alpha}^\frac{1}{2} A_{\alpha} I_2 \pm K_{\alpha}^\frac{1}{2} A_{\alpha} I_2 \right).
\]

After some similar calculations, we also get that \( P_{(\beta)} - B_{(\beta)} \) commutes with \( P_{(\alpha)} - B_{(\alpha)} \), \( (P_{(\alpha)}^{-1})^\frac{1}{2} \), and \( P_{(\alpha)} - B_{(\alpha)}^{-1} \); and \( P_{(\beta)} + B_{(\beta)} \) commutes with \( P_{(\alpha)} + B_{(\alpha)} \), \( (P_{(\alpha)}^{-1})^\frac{1}{2} \),
and $P(\alpha) + B^T(\alpha)$. In view of Theorem 3.3 we know that $B_\alpha + B^T_\alpha$ and $B_\beta + B^T_\beta$ are negative definite matrices. Together with Lemma 4.2 it yields that $B(\alpha) + B^T(\alpha)$ and $B(\beta) + B^T(\beta)$ are also negative definite matrices. Using Lemma 4.1 there exist

$$((P^{-1}(\alpha))^{1/2}((P^{-1}(\alpha) - B(\alpha))(P(\beta) - B(\beta)))^T((P^{-1}(\alpha))^{1/2}(P^{-1}(\alpha) - B(\alpha))(P(\beta) - B(\beta))$$

$$\geq (P(\beta) + B^T(\beta)B(\beta))(P(\alpha) + B^T(\alpha)P^{-1}(\alpha)B(\alpha)) + (B(\beta) + B^T(\beta)(B(\alpha) + B^T(\alpha))$$

and

$$((P^{-1}(\alpha))^{1/2}(P(\alpha) + B(\alpha))(P(\beta) + B^T(\beta)))^T((P^{-1}(\alpha))^{1/2}(P(\alpha) + B(\alpha))(P(\beta) + B^T(\beta))$$

$$\leq (P(\beta) + B^T(\beta)P^{-1}(\beta)B(\beta))(P(\alpha) + B^T(\alpha)P^{-1}(\alpha)B(\alpha)) + (B(\beta) + B^T(\beta)(B(\alpha) + B^T(\alpha))$$

where the matrix $A \geq B$ means that $A - B$ is positive semi-definite. Denoting $E^n = \sqrt{h(\varepsilon^n)^T((P(\beta) + B^T(\beta))P^{-1}(\beta)B(\beta))(P(\alpha) + B^T(\alpha)P^{-1}(\alpha)B(\alpha)) + (B(\beta) + B^T(\beta)(B(\alpha) + B^T(\alpha))}\varepsilon^n$, we have

$$E^n \geq \sqrt{h(\varepsilon^n)^T(P(\alpha))(P(\beta))\varepsilon^n} \geq \sqrt{\lambda_{\min}(P(\alpha))\lambda_{\min}(P(\beta))}\varepsilon^n,$$

where $\lambda_{\min}(P(\alpha))$ and $\lambda_{\min}(P(\beta))$ are the minimum eigenvalues of matrix $P(\alpha)$ and $P(\beta)$, respectively. Together with (4.16) and (4.17), we have

$$E^{n+1} \leq E^0 + \tau \sum_{k=0}^n \|((P^{-1}(\alpha))^{1/2}(P^{-1}(\beta))^{1/2})^n R^{n+1/2} \| \leq \tau \sum_{k=0}^n \sqrt{\lambda_{\max}(P^{-1}(\alpha))}\|R^{n+1/2}\|$$

$$= \frac{\tau}{\sqrt{\lambda_{\min}(P(\alpha))\lambda_{\min}(P(\beta))}} \sum_{k=0}^n \|R^{n+1/2}\|.$$

Using (4.18) and noticing that $|R_{j,s}^{k+1/2}| \leq c(\tau^2 + h_x^2 + h_y^2)$ for $1 \leq j \leq M_x - 1$ and $1 \leq s \leq M_y - 1$, we obtain

$$\|\varepsilon^n\| \leq \frac{cT}{\lambda_{\min}(P(\alpha))\lambda_{\min}(P(\beta))}(\tau^2 + h_x^2 + h_y^2) \leq C(\tau^2 + h_x^2 + h_y^2).$$

5. Extending quasi-compact discretizations and schemes to tempered space fractional derivative and equation This section focuses on developing the high order quasi-compact schemes of tempered fractional differential equation with Dirichlet boundary condition. We begin with the definitions of $\alpha$-th order left and right Riemann-Liouville tempered fractional derivatives.

**Definition 5.1.** ([10]) If the function $u(x)$ is defined in finite interval $[a, b]$ and regular enough, then for any $\lambda \geq 0$ the $\alpha$-th order left and right Riemann-Liouville tempered fractional derivatives are, respectively, defined as

$$aD_x^{\alpha, \lambda}u(x) = e^{-\lambda x}aD_x^{\alpha}(e^{\lambda x}u(x)) = e^{-\lambda x} \frac{d^n}{dx^n} \int_a^x (x-s)^{n-\alpha-1} e^{\lambda s}u(s)ds$$

$$bD_x^{\alpha, \lambda}u(x) = e^{-\lambda x}bD_x^{\alpha}(e^{\lambda x}u(x)) = e^{-\lambda x} \frac{d^n}{dx^n} \int_x^b (x-s)^{n-\alpha-1} e^{\lambda s}u(s)ds$$

(5.1)
and

\[ x D_b^{\alpha,\lambda} u(x) = e^{\lambda x} D_b^\alpha (e^{-\lambda x} u(x)) = \frac{(-1)^n e^{\lambda x}}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (s-x)^{n-\alpha-1} e^{-\lambda s} u(s) ds, \]  

(5.2)

where \( n-1 < \alpha < n \). Moreover, if \( \lambda = 0 \), then the derivatives \( D_x^\alpha u(x) \) and \( D_b^\alpha u(x) \) reduce to the derivatives \( D_x^\alpha u(x) \) and \( D_b^\alpha u(x) \) defined in Definition 2.2.

For getting the stable scheme, we introduce a shifted Grünwald-Letnikov operator to approximate the left tempered Riemann-Liouville fractional derivative with first order accuracy.

**Lemma 5.2** ([10]). Let \( 1 < \alpha < 2, u \in C^{n+3}(R) \) such that \( D^k u(x) \in L^1(R) \), \( k = 0,1,\cdots,n+3 \). For any integer \( k \) and \( \lambda \geq 0 \) define the left shifted tempered Grünwald-Letnikov operator by

\[ \Delta_p^{\alpha,\lambda} u(x) := \frac{1}{h^n} \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{-(k-p)\lambda h} u(x-(k-p)h). \]  

(5.3)

Then we have

\[ \Delta_p^{\alpha,\lambda} u(x) = -\infty D^\alpha u(x) + \sum_{l=1}^{n-1} a^\alpha_{p,l} \infty D^\alpha \lambda u(x) h^l + O(h^n) \]  

(5.4)

uniformly in \( x \in R \), where the weights \( a^\alpha_{p,l} \) are the same as Lemma 2.2.

To approximate the right Riemann-Liouville tempered fractional derivative \( x D^\alpha u(x) \), the right shifted tempered Grünwald-Letnikov operator is defined as

\[ \Lambda_p^{\alpha,\lambda} f(x) := \frac{1}{h^n} \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{-(k-p)\lambda h} u(x+(k-p)h). \]  

If the function \( u(x) \) is defined on the bounded interval \([a,b]\), then the shifted tempered Grünwald-Letnikov formulae approximating the tempered fractional derivative at point \( x \) are written as

\[ \tilde{\Delta}_p^{\alpha,\lambda} u(x) = \frac{1}{h^n} \sum_{k=0}^{[x-a]} g_k^{(\alpha)} e^{-(k-p)\lambda h} u(x-(k-p)h), \]  

(5.5)

\[ \tilde{\Lambda}_p^{\alpha,\lambda} u(x) = \frac{1}{h^n} \sum_{k=0}^{[x-a]} g_k^{(\alpha)} e^{-(k-p)\lambda h} u(x+(k-p)h). \]

Next we establish some suitable high order finite difference discretizations to approximate the tempered fractional derivative.

**5.1. Quasi-compact discretizations to the tempered Riemann-Liouville space fractional derivative**

Now from the Taylor’s expansions of the shifted tempered Grünwald-Letnikov operator, similar to get the CWSGD operator given in Sec. 2, we derive the fourth and fifth order quasi-compact difference operators for Riemann-Liouville tempered fractional derivative.

**5.1.1. Fourth order quasi-compact approximation to the tempered Riemann-Liouville fractional derivative**

**Theorem 5.3.** Let \( u(x) \in C^7(R) \) and all the derivatives of \( u(x) \) up to order 7 belong to \( L_1(R) \). Then the following quasi-compact approximation has fourth order accuracy, i.e.,

\[ P^\lambda_x \infty D^\alpha_x u(x) = \mu_1 \Delta_1^{\alpha,\lambda} u(x) + \mu_0 \Delta_0^{\alpha,\lambda} u(x) + \mu_{-1} \Delta_{-1}^{\alpha,\lambda} u(x) + O(h^4), \]  

(5.6)
Example 5.4. Next we give an example to verify the efficiency and convergence order of the above and corresponding to the left Riemann-Liouville tempered fractional derivative have fifth order accuracy, the coefficients $b_2$, $\mu_1$, $\mu_0$ and $\mu_{-1}$ are given by (2.10).

Note that, by Lemma 5.2, the following equation holds,

$$
\mu_1 \Delta_1^{\alpha,\lambda} u(x) + \mu_0 \Delta_0^{\alpha,\lambda} u(x) + \mu_{-1} \Delta_{-1}^{\alpha,\lambda} u(x) = -\infty D_x^{\alpha,\lambda} u(x) + b_2^\alpha - \infty D_x^{\alpha+2,\lambda} u(x) h^2 + O(h^4)
$$

$$
= (1 + h^2 b_2^\alpha - \infty D_x^{2,\lambda}) - \infty D_x^{\alpha,\lambda} u(x) + O(h^4)
$$

$$
= (1 + h^2 b_2^\alpha - \infty D_x^{2,\lambda}) - \infty D_x^{\alpha,\lambda} u(x) + O(h^4)
$$

$$
P_x^{\alpha,\lambda} u(x) = P_x^{\alpha,\lambda} u(x) + O(h^4).
$$

Then we get (5.4). Since $\delta_x^2 u = \partial^2 \partial x^\mu u + O(h^2)$, we know for any function $u$

$$
P_x^{\alpha,\lambda} u = (1 + h^2 b_2^\alpha - \infty D_x^{2,\lambda}) u + O(h^4).
$$

In a similar way, we obtain quasi-compact approximation of the right Riemann-Liouville tempered fractional derivative:

$$
P_x^{\alpha,\lambda} D_{x+\infty}^{\alpha,\lambda} u(x) = \mu_1 B_1^{\alpha,\lambda} u(x) + \mu_0 B_0^{\alpha,\lambda} u(x) + \mu_{-1} B_{-1}^{\alpha,\lambda} u(x) + O(h^4).
$$

For $u(x)$ defined on a bounded interval, supposing its zero extension to $R$ satisfies the assumptions of Theorem 5.3 the following approximations hold:

$$
P_x u = P_x^{\alpha,\lambda} u(x) = \mu_1 B_1^{\alpha,\lambda} u(x) + \mu_0 B_0^{\alpha,\lambda} u(x) + O(h^4).
$$

and

$$
P_x D_{x+\infty}^{\alpha,\lambda} u(x) = \mu_1 B_1^{\alpha,\lambda} u(x) + \mu_0 B_0^{\alpha,\lambda} u(x) + \mu_{-1} B_{-1}^{\alpha,\lambda} u(x) + O(h^4).
$$

Next we give an example to verify the efficiency and convergence order of the above statement.

**Example 5.4.** Consider the steady state tempered fractional diffusion problem

$$
oD_x^{\alpha,\lambda} u(x) = \frac{720 e^{-\lambda x x_0^\alpha}}{\Gamma(7 - \alpha)}, \quad x \in (0, 1),
$$

with the boundary conditions $u(0) = 0$ and $u(1) = e^{-\lambda}$, and $\alpha \in (1, 2)$. The exact solution is given by $u(x) = e^{-\lambda x x_0^\alpha}$.

Let us denote $u$ and $U$ as the exact solution and approximate value, respectively. In Table 5.1 we show that the proposed approxiamtion in this subsection has fourth order accuracy in $L_\infty$ norm and $L_2$ norm.

**5.1.2. Fifth order quasi-compact approximation to the tempered Riemann-Liouville fractional derivative**

**Theorem 5.5.** Let $u(x) \in C^8(R)$. Then the quasi-compact approximations corresponding to the left Riemann-Liouville tempered fractional derivative have fifth order accuracy,

$$
P_x^{\alpha,\lambda} D_x^{\alpha,\lambda} u(x) = \mu_1 B_1^{\alpha,\lambda} u(x) + \mu_0 B_0^{\alpha,\lambda} u(x) + \mu_{-1} B_{-1}^{\alpha,\lambda} u(x) + O(h^5),
$$

$$
P_x^{\alpha,\lambda} D_{x+\infty}^{\alpha,\lambda} u(x) = \mu_1 B_1^{\alpha,\lambda} u(x) + \mu_0 B_0^{\alpha,\lambda} u(x) + \mu_{-1} B_{-1}^{\alpha,\lambda} u(x) + O(h^5).
$$
To solve Example 5.4, where U
Example 5.6.

theoretical analysis.

respectively. Obviously, the approximations have fifth order accuracy which verify

results in Table 5.2, where

in this subsection, we numerically solve the Example 5.6 and present the numerical
g

To show the efficiency of the proposed approximation

ha

h

U

where the operator $D_x^{\lambda^5}u(x) = me^{-\lambda h}u(x-h) + u(x) + ne^{\lambda h}u(x+h)$ and the coefficients $m, n, \mu_1, \mu_0$ and $\mu_1$ satisfy (2.10).

Similarly to the discussions in Subsection 2.2, we show three equalities

$$\Delta_p^{\alpha,\lambda}u(x) = -\infty D_x^{\alpha,\lambda}u(x) + \sum_{l=1}^{4} a_{p,l}^{\alpha,\lambda} \infty D_x^{\alpha+l,\lambda}u(x)h^l + O(h^5), \quad p = 1, 0, -1. \quad (5.12)$$

In view of the Taylor expansion we know

$$-\infty D_x^2 e^{\lambda(x-h)} u(x-h) = -\infty D_x^2 e^{\lambda x} u(x) + (-1)^{l} \sum_{l=1}^{4} \frac{1}{l!} -\infty D_x^{\alpha+l} e^{\lambda x} u(x)h^l + O(h^5),$$

$$-\infty D_x^2 e^{\lambda(x+h)} u(x+h) = -\infty D_x^2 e^{\lambda x} u(x) + \sum_{l=1}^{4} \frac{1}{l!} -\infty D_x^{\alpha+l} e^{\lambda x} u(x)h^l + O(h^5).$$

As $e^{\lambda x} -\infty D_x^{\alpha,\lambda}u(x) = -\infty D_x^{\alpha,\lambda}e^{\lambda x} u(x)$, multiplying $e^{-\lambda x}$ in equations of (5.13) we obtain

$$e^{-\lambda h} -\infty D_x^{\alpha,\lambda}u(x-h) = -\infty D_x^{\alpha,\lambda}u(x) + (-1)^{l} \sum_{l=1}^{4} \frac{1}{l!} -\infty D_x^{\alpha+l,\lambda}u(x)h^l + O(h^5),$$

$$e^{-\lambda h} -\infty D_x^{\alpha,\lambda}u(x+h) = -\infty D_x^{\alpha,\lambda}u(x) + \sum_{l=1}^{4} \frac{1}{l!} -\infty D_x^{\alpha+l,\lambda}u(x)h^l + O(h^5).$$

So in order to get the fifth order approximation, combining (5.12) and (5.14), we just need to eliminate the low order terms corresponding to $h^k (k = 1, 2, 3, 4)$. Then we get the equation (5.11). To show the efficiency of the proposed approximation in this subsection, we numerically solve the Example 5.6 and present the numerical results in Table 5.2 where $u$ and $U$ denote the exact solution and approximate value, respectively. Obviously, the approximations have fifth order accuracy which verify the theoretical analysis.

**Example 5.6.** Here we also consider the steady state tempered fractional diffusion

| $\alpha$ | $h_x$ | $\|u - U\|_2$ | rate | $\|u - U\|_\infty$ | rate |
|--------|-------|----------------|------|----------------|------|
| 1.1    | 1/8   | 3.8735e-04    |      | 8.7474e-04    |      |
|        | 1/16  | 1.8576e-05    | 4.3821 | 4.6054e-05    | 4.2195 |
|        | 1/32  | 1.0159e-06    | 4.1926 | 2.6950e-06    | 4.1229 |
|        | 1/64  | 6.0438e-08    | 4.0712 | 1.6005e-07    | 4.0737 |
|        | 1/128 | 3.6901e-09    | 4.0337 | 9.4537e-09    | 4.0815 |
| 1.9    | 1/8   | 6.2019e-05    |      | 8.8032e-05    |      |
|        | 1/16  | 3.8991e-06    | 3.9915 | 5.6382e-06    | 3.9647 |
|        | 1/32  | 2.4425e-07    | 3.9967 | 3.5328e-07    | 3.9964 |
|        | 1/64  | 1.5281e-08    | 3.9985 | 2.2104e-08    | 3.9984 |
|        | 1/128 | 9.5548e-10    | 3.9994 | 1.3822e-09    | 3.9993 |
Fourth order quasi-compact difference schemes

problem

\[ aD^{\alpha \cdot \lambda} u(x) = \frac{720e^{-\lambda x} x^{6-\alpha}}{\Gamma(7-\alpha)}, \quad x \in (0,1) \]

with the boundary conditions \( u(0) = 0 \) and \( u(1) = e^{-\lambda} \), and \( \alpha \in (1,2) \). The exact solution is \( u(x) = e^{-\lambda x} x^6 \).

Table 5.2. Numerical errors and convergence rates in \( L_\infty \) norm and \( L_2 \) norm of scheme (5.11) to solve Example 5.6, where \( U \) denotes the numerical solution, \( h_x \) is space step size and \( \lambda = 1.5 \).

| \( \alpha \) | \( h_x \) | \( \| u - U \|_2 \) | rate | \( \| u - U \|_\infty \) | rate |
| --- | --- | --- | --- | --- | --- |
| 1.1 | 1/8 | 8.2144e-06 | 1.4011e-05 | |
| | 1/16 | 2.4016e-07 | 5.0961 | 4.1068e-07 | 5.0924 |
| | 1/32 | 7.3703e-09 | 5.0261 | 1.2489e-08 | 5.0392 |
| | 1/64 | 2.2851e-10 | 5.0114 | 3.8494e-10 | 5.0199 |
| | 1/128 | 7.1140e-12 | 5.0054 | 1.1945e-11 | 5.0101 |
| 1.5 | 1/8 | 3.1463e-06 | 5.3572e-06 | |
| | 1/16 | 9.7972e-08 | 5.0051 | 1.6423e-07 | 5.0276 |
| | 1/32 | 3.1300e-09 | 4.9681 | 5.1523e-09 | 4.9944 |
| | 1/64 | 9.9444e-11 | 4.9689 | 1.6239e-10 | 4.9876 |
| | 1/128 | 3.1783e-12 | 4.9748 | 5.1120e-12 | 4.9895 |

5.2. Quasi-compact scheme for tempered space fractional diffusion equation In this subsection, we present the numerical scheme of the variant of space fractional diffusion equation whose space fractional derivatives are replaced by the tempered fractional derivatives:

\[
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = K_1 a D^{\alpha \cdot \lambda}_x u(x,t) + K_2 a D^{\alpha \cdot \lambda}_h u(x,t) + f(x,t), \quad (x,t) \in (a,b) \times (0,T], \\
u(x,0) = u_0(x), \quad x \in [a,b], \\
u(a,t) = \phi_a(t), \quad u(b,t) = \phi_b(t), \quad t \in [0,T],
\end{cases}
\]

(5.15)

where \( \lambda \geq 0 \). Utilizing the Crank-Nicolson technique for the time discretization of (5.15) and fourth order quasi-compact discretization in space direction, we get

\[
P_x^{\lambda} u_j^{n+1} - u_j^n = \frac{K_1 \tau}{2} L D^{\alpha \cdot \lambda} h u_j^n + \frac{K_2 \tau}{2} R D^{\alpha \cdot \lambda} h u_j^n + \frac{K_1 \tau}{2} L D^{\alpha \cdot \lambda} h u_j^{n+1} + \frac{K_2 \tau}{2} R D^{\alpha \cdot \lambda} h u_j^{n+1} \\
+ P_x^{\lambda} f(x_j,t_{n+1/2}) + R_j^{n+1/2},
\]

(5.16)

where

\[
L D^{\alpha \cdot \lambda} h u_j^n =: \mu_1 \Delta^{\alpha \cdot \lambda}_1 u_j^n + \mu_0 \Delta^{\alpha \cdot \lambda}_0 u_j^n + \mu_{-1} \Delta^{\alpha \cdot \lambda}_{-1} u_j^n = \frac{1}{h^\alpha} \sum_{k=0}^{j+1} w_k^{(\alpha \cdot \lambda)} u_{j-k+1}^n,
\]

\[
R D^{\alpha \cdot \lambda} h u_j^n =: \mu_1 \Delta^{\alpha \cdot \lambda}_1 u_j^n + \mu_0 \Delta^{\alpha \cdot \lambda}_0 u_j^n + \mu_{-1} \Delta^{\alpha \cdot \lambda}_{-1} u_j^n = \frac{1}{h^\alpha} \sum_{k=0}^{M-j+1} w_k^{(\alpha \cdot \lambda)} u_{j+k-1}^n.
\]
the coefficients \( w_0^{(\alpha, \lambda)} = \mu_1 g_0^{(\alpha)} e^{\lambda k} \), \( w_1^{(\alpha, \lambda)} = \mu_1 g_1^{(\alpha)} + \mu_0 g_0^{(\alpha)} \), and \( w_k^{(\alpha, \lambda)} = (\mu_1 g_k^{(\alpha)} + \mu_0 g_{k-1}^{(\alpha)} + \mu_1 g_{k-2}^{(\alpha)}) e^{-(k-1)\lambda} \), \( k = 2, \ldots, M \) and \( R_j^{n+1/2} \leq C(\tau^2 + h^4) \). Denoting \( U^n_j \) as the numerical approximation of \( u^n_j \), we obtain the Crank-Nicolson quasi-compact scheme for (5.15)

\[
P^\alpha_x U^{n+1}_j = \frac{K_1 \tau}{2} L D^\alpha_D U^{n+1}_j - \frac{K_2 \tau}{2} R D^\alpha_H U^{n+1}_j
\]

\[
= P^\alpha_x U^n_{j,s} + \frac{K_1 \tau}{2} L D^\alpha_D U^n_j + \frac{K_2 \tau}{2} R D^\alpha_H U^n_j + \tau P^\alpha_H f^{n+1/2}_j.
\]

For convenience, the approximation scheme (5.17) may be written in matrix form

\[
(P^\alpha_a - B^\lambda) U^{n+1} = (P^\alpha_a + B^\lambda) U^n + \tau F^n + H^\lambda,
\]

where \((P^\alpha_a)_{j,s} = (P^\alpha_a)_{j,s} e^{(j-s)\lambda} \), \( B^\lambda = \frac{2 \tau}{h^4} (K_1 A^\lambda + K_2 (A^\lambda)^T) \), \( (A^\lambda)_{j,s} = (A_{\lambda})_{j,s} e^{(j-s)\lambda} \), \( U^n = (U^0_1, U^1_2, \ldots, U^n_{M-1})^T \), and \( F^n = (f^{n+1/2}_1, f^{n+1/2}_2, \ldots, f^{n+1/2}_{M-1})^T \).

**Remark 5.7.** Note that when taking \( \lambda = 0 \), the tempered fractional diffusion equation (5.15) reduces to the fractional diffusion equation (3.1) and its scheme (5.17) reduces to (3.6).

### 6. Numerical experiments

For the numerical schemes of the fractional diffusion equation, we present some numerical results in one and two dimension cases to verify the theoretical results including the convergence orders and unconditional stability. For the tempered fractional diffusion equation, the numerical simulations are also performed which show the effectiveness of the proposed scheme; and the desired fourth order convergence is also obtained.

**Example 6.1.** Consider the following tempered space fractional diffusion equation

\[
\frac{\partial u}{\partial t} = a D^\alpha_x u(x) - e^{-t-\lambda x} \left( x^6 + \frac{720x^{6-\alpha}}{\Gamma(7-\alpha)} \right), \quad (x, t) \in (0, 1) \times (0, 1],
\]

with the boundary conditions \( u(0,t) = 0 \) and \( u(1,t) = e^{-t-\lambda} \) and the initial value \( u(x,0) = e^{-\lambda x} x^6 \), \( x \in [0,1] \). The exact solution is \( u(x) = e^{-t-\lambda x} x^6 \).

In Table 6.1, we show that the quasi-compact scheme (5.17) is fourth order convergent in space.

**Example 6.2.** Consider the following space fractional diffusion equation

\[
\frac{\partial u}{\partial t} = a D^\alpha_x u(x) + b D^\alpha_t u(x) + f(x,t), \quad (x, t) \in (0, 1) \times (0, 1].
\]

Then the source term is

\[
f(x,t) = -e^{-t} (x^5 (1-x)^5 - \frac{\Gamma(11)(x^{10-\alpha} + (1-x)^{10-\alpha})}{\Gamma(11-\alpha)})
\]

\[
+ 5\Gamma(10)(x^{9-\alpha} + (1-x)^{9-\alpha})/\Gamma(10-\alpha) - 10\Gamma(9)(x^{8-\alpha} + (1-x)^{8-\alpha})/\Gamma(9-\alpha)
\]

\[
+ 10\Gamma(8)(x^{7-\alpha} + (1-x)^{7-\alpha})/\Gamma(8-\alpha) - 5\Gamma(7)(x^{6-\alpha} + (1-x)^{6-\alpha})/\Gamma(7-\alpha)
\]

\[
+ \Gamma(6)(x^{5-\alpha} + (1-x)^{5-\alpha})/\Gamma(6-\alpha).
\]

The exact solution is given by \( u(x) = e^{-t} x^5 (1-x)^5 \). In the domain \( t \in [0,1] \), the boundary conditions are \( u(0,t) = 0 \) and \( u(1,t) = 0 \). The initial value is \( u(x,0) = x^5 (1-x)^5, x \in [0,1] \).
Example 6.3. The following two dimensional two sided fractional diffusion problem

\[
\frac{\partial u(x,y,t)}{\partial t} = 0D_x^\alpha u(x,y,t) + xD_y^\beta u(x,y,t) + 0D_y^\beta u(x,y,t) + yD_1^\beta u(x,y,t) + f(x,y,t),
\]  
(6.3)
is considered in the domain $\Omega = (0,1)^2$ and $t \in (0,1]$. The source term is

$$f(x,t) = -10^6 e^{-t} \left[ x^5 (1-x)^5 y^5 (1-y)^5 \right.$$

$$- \left( \frac{\Gamma(11)}{\Gamma(11-\alpha)} (x^{10-\alpha} + (1-x)^{10-\alpha}) + \frac{5\Gamma(10)}{\Gamma(10-\alpha)} (x^{9-\alpha} + (1-x)^{9-\alpha}) \right.\right.$$

$$- \frac{10\Gamma(9)}{\Gamma(9-\alpha)} (x^{8-\alpha} + (1-x)^{8-\alpha}) + \frac{10\Gamma(8)}{\Gamma(8-\alpha)} (x^{7-\alpha} + (1-x)^{7-\alpha}) \right.$$

$$- \frac{5\Gamma(7)}{\Gamma(7-\alpha)} (x^{6-\alpha} + (1-x)^{6-\alpha}) + \frac{\Gamma(6)}{\Gamma(6-\alpha)} (x^{5-\alpha} + (1-x)^{5-\alpha}) \right.$$

$$\left. \left. y^5 (1-y)^5 \right] \right.\right.$$

$$- \left( \frac{\Gamma(11)}{\Gamma(11-\beta)} (y^{10-\beta} + (1-y)^{10-\beta}) + \frac{5\Gamma(10)}{\Gamma(10-\beta)} (y^{9-\beta} + (1-y)^{9-\beta}) \right.$$

$$- \frac{10\Gamma(9)}{\Gamma(9-\beta)} (y^{8-\beta} + (1-y)^{8-\beta}) + \frac{10\Gamma(8)}{\Gamma(8-\beta)} (y^{7-\beta} + (1-y)^{7-\beta}) \right.$$

$$- \frac{5\Gamma(7)}{\Gamma(7-\beta)} (y^{6-\beta} + (1-y)^{6-\beta}) + \frac{\Gamma(6)}{\Gamma(6-\beta)} (y^{5-\beta} + (1-y)^{5-\beta}) \left. \right) x^5 (1-x)^5 \right].$$

The exact solution is given by $u(x) = 10^6 e^{-t} x^5 (1-x)^5 y^5 (1-y)^5$. The boundary condition is $u(x,y,t) = 0$ with $(x,y) \in \partial \Omega$ and $t \in [0,1]$. The initial value is $u(x,y,0) = 10^6 x^5 (1-x)^5 y^5 (1-y)^5$ with $(x,y) \in [0,1]^2$.

In Table 6.3, we present the numerical errors $\|u - U\|_2$ and the corresponding convergence orders with space step size $h_x = h_y$, where $U$ is the solution of the quasi-compact difference scheme (4.7) or (4.10). It can be noted that the schemes are fourth order convergent, which is in agreement with the theoretical convergence analysis.

| $M_x$ | $\|u - U\|_2$ | Rate | $\|u - U\|_2$ | Rate |
|-------|----------------|-------|----------------|-------|
| D’yakonov | 8 | 7.2903e-04 | 8.4729e-04 | 16 | 5.3915e-05 | 3.7572 | 5.7210e-05 | 3.8885 |
| | 32 | 3.7385e-06 | 3.8502 | 3.8200e-06 | 3.9046 |
| | 64 | 2.4685e-07 | 3.9207 | 2.4748e-07 | 3.9482 |
| | 128 | 1.5880e-08 | 3.9584 | 1.5763e-08 | 3.9727 |
| Douglas | 8 | 7.2903e-04 | 8.4729e-04 | 16 | 5.3915e-05 | 3.7572 | 5.7210e-05 | 3.8885 |
| | 32 | 3.7385e-06 | 3.8502 | 3.8200e-06 | 3.9046 |
| | 64 | 2.4685e-07 | 3.9207 | 2.4748e-07 | 3.9482 |
| | 128 | 1.5880e-08 | 3.9584 | 1.5763e-08 | 3.9727 |

7. Conclusions The continuous time random walk (CTRW) model is the basic stochastic process in statistical physics. The CTRW model characterizes the Lévy flight if the first moment of the distribution of the waiting time is finite, and the jump length obeys the power law distribution and its second moment is infinite; the
corresponding Fokker-Planck equation of the process is the space fractional diffusion equation. Sometimes because of the limit of space size, the power law distribution of the jump length has to be tempered. The Fokker-Planck equation of the new stochastic process is the tempered space fractional diffusion equation. This paper provides the basic strategy of deriving the quasi-compact high order discretizations for space fractional derivative and tempered space fractional derivative. As concrete examples, fourth order discretizations are detailedly discussed and applied to solve the (tempered) space fractional diffusion equation, and the extensive numerical simulations confirm the effectiveness of the provided schemes. In fact, the strict numerical stability and convergence analysis are also performed for the one and two dimensional space fractional diffusion equations.

REFERENCES

[1] F. Mainardi, *Fractional relaxation-oscillation and fractional diffusion-wave phenomena*, Chaos Solitons Fractals, 7(9), 1461–1477, 1996.
[2] R.L. Bagley, P.J. Torvik, *A theoretical basis for the application of fractional calculus to viscoelasticity*, J. Rheol., 27(3), 201–210, 1983.
[3] B.M. Vinagre, I. Podlubny, A. Hernandez, V. Feliu, *Some approximations of fractional order operators used in control theory and applications*, Fract. Calc. Appl. Anal., 3(3), 231–248, 2000.
[4] R. Metzler, J. Klafter, *The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics*, J. Phys. A: Math. Gen., 37(31), R161, 2004.
[5] I. Goychuk, P. Hänggi, *Fractional diffusion modeling of ion channel gating*, Phys. Rev. E, 70(5), 051915, 2004.
[6] E. Barkai, R. Metzler, J. Klafter, *From continuous time random walks to the fractional Fokker-Planck equation*, Phys. Rev. E, 61(1), 132, 2000.
[7] R.N. Mantegna, H.E. Stanley, *Stochastic process with ultraslow convergence to a Gaussian: the truncated Lévy flight*, Phys. Rev. Lett., 73(22), 2946, 1994.
[8] Á. Cartea, D. del-Castillo-Negrete, *Fluid limit of the continuous-time random walk with general Lévy jump distribution functions*, Phys. Rev. E., 76, 041105, 2007.
[9] A. Chakraborty, M.M. Meerschaert, *Tempered stable laws as random walk limits*, Stat. Probab. Lett., 81(8), 989–997, 2011.
[10] C. Li, W.H. Deng, *High order schemes for the tempered fractional diffusion equations*, [arXiv:1402.0064 [physics.comp-ph]].
[11] F. Liu, V. Anh, I. Turner, *Numerical solution of the space fractional Fokker–Planck equation*, J. Comput. Appl. Math., 166(1), 209–219, 2004.
[12] M.M. Meerschaert, C. Tadjeran, *Finite difference approximations for fractional advection–dispersion flow equations*, J. Comput. Appl. Math., 127(1), 65–77, 2004.
[13] M.M. Meerschaert, C. Tadjeran, *Finite difference approximations for two-sided space-fractional partial differential equations*, Appl. Numer. Math., 56(1), 80–90, 2006.
[14] M.M. Meerschaert, H.P. Scheffler, C. Tadjeran, *Finite difference methods for two-dimensional fractional dispersion equation*, J. Comput. Phys., 211(1), 249–261, 2006.
[15] C. Tadjeran, M.M. Meerschaert, H.P. Scheffler, *A second-order accurate numerical approximation for the fractional diffusion equation*, J. Comput. Phys., 213(1), 205-213, 2006.
[16] C. Tadjeran, M.M. Meerschaert, *A second-order accurate numerical method for the two-dimensional fractional diffusion equation*, J. Comput. Phys., 220(2), 813–823, 2007.
[17] H.M. Nasir, B.L.K. Gunawardana, H.M.N.P. Abeyratna, *A second order finite difference approximation for the fractional diffusion equation*, Int. J. Appl. Math. Stat., 3, 237–243, 2013.
[18] W.Y. Tian, H. Zhou, W.H. Deng, *A class of second order difference approximation for solving space fractional diffusion equations*, Math. Comp., in press [arXiv:1201.5949 [math.NA]].
[19] H. Zhou, W.Y. Tian, W.H. Deng, *Quasi-compact finite difference schemes for space fractional diffusion equations*, J. Sci. Comput., 56(1), 45–66, 2013.
[20] W.H. Deng, M.H. Chen, *Efficient numerical algorithms for three-dimensional fractional partial differential equations*, J. Comput. Math., 32(4), 371–391, 2014.
[21] I. Podlubny, *Fractional differential equations*, Academic Press, New York, 1999.
[22] J.D. Anderson, Computational fluid dynamics, Springer, Singapore, 1995.
[23] R.H.F. Chan, X.Q. Jin, An introduction to iterative Toeplitz solvers, SIAM, 2007.
[24] R. Bhatia, Positive definite matrices, Princeton University Press, 2009.
[25] M. Marcus, H. Minc, A survey of matrix theory and matrix inequalities, Allyn and Bacon, 1964.
[26] A. J. Laub, Matrix analysis for scientists and engineers, SIAM, 2005.