THE PETERSON RECURRENCE FORMULA FOR THE CHROMATIC DISCRIMINANT OF A GRAPH

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Abstract. The absolute value of the coefficient of $q$ in the chromatic polynomial of a graph $G$ is known as the chromatic discriminant of $G$ and is denoted $\alpha(G)$. There is a well known recurrence formula for $\alpha(G)$ that comes from the deletion-contraction rule for the chromatic polynomial. In this paper we prove another recurrence formula for $\alpha(G)$ that comes from the theory of Kac-Moody Lie algebras. We start with a brief survey on many interesting algebraic and combinatorial interpretations of $\alpha(G)$. We use two of these interpretations (in terms of acyclic orientations and spanning trees) to give two bijective proofs for our recurrence formula of $\alpha(G)$.

1. Introduction

Let $G$ be a simple graph and let $\chi(G,q)$ denote its chromatic polynomial. The absolute value of the coefficient of $q$ in $\chi(G,q)$ is known as the chromatic discriminant of the graph $G$ [11, 14] and is denoted $\alpha(G)$. It is an important graph invariant with numerous algebraic and combinatorial interpretations. For instance, letting $q$ denote a fixed vertex of the graph $G$, it is well known that each of the following sets has cardinality $\alpha(G)$:

1. Acyclic orientations of $G$ with unique sink at $q$ [7],
2. Maximum $G$-parking functions relative to $q$ [2],
3. Minimal $q$-critical states [6, Lemmas 14.12.1 and 14.12.2],
4. Spanning trees of $G$ without broken circuits [3],
5. Conjugacy classes of Coxeter elements in the Coxeter group associated to $G$ [4, 12, 13],
6. Multilinear Lyndon heaps on $G$ [9, 10, 16].

In addition, $\alpha(G)$ is also equal to the dimension of the root space corresponding to the sum of all simple roots in the Kac-Moody Lie algebra associated to $G$ [1, 15].

We have the following recurrence formula for $\alpha(G)$ (see for instance [5]) which is an immediate consequence of the well-known deletion-contraction rule for the chromatic polynomial:

$$\alpha(G) = \alpha(G\setminus e) + \alpha(G/e),$$

(1.1)

2010 Mathematics Subject Classification. 05C20, 05C30, 05C31.
Key words and phrases. Chromatic discriminant, Acyclic orientations, Spanning trees.

The author acknowledges partial support under the DST Swarnajayanti fellowship DST/SJF/MSA-02/2014-15 of Amritanshu Prasad.
where \( e \) is any edge of \( G \). Here, \( G \setminus e \) denotes \( G \) with \( e \) deleted and \( G/e \) denotes the simple graph obtained from \( G \) by identifying the two ends of \( e \) (i.e., contracting \( e \) to a single vertex) and removing any multiple edges that result.

Yet another recurrence formula for \( \alpha(G) \) was obtained in [15] using its connection to root multiplicities of Kac-Moody Lie algebras. To state this, we introduce some notation: for a graph \( G \), we let \( V(G) \) and \( E(G) \) denote its vertex and edge sets respectively. We say that the ordered pair \((G_1, G_2)\) is an ordered partition of \( G \), if \( G_1 \) and \( G_2 \) are non-empty subgraphs of \( G \) whose vertex sets form a partition of \( V(G) \), i.e., they are disjoint and their union is \( V(G) \). When we don’t care about the ordering of \( G_1, G_2 \), we call the set \( \{G_1, G_2\} \) an unordered partition of \( G \).

We then have:

**Proposition 1.** [15]

\[
2 \ e(G) \ \alpha(G) = \sum_{(G_1,G_2)} \alpha(G_1) \ \alpha(G_2) \ e(G_1,G_2). \tag{1.2}
\]

Here \( e(G) \) is the total number of edges in \( G \), \( e(G_1,G_2) \) is the number of edges that straddle \( G_1 \) and \( G_2 \), and the sum ranges over ordered partitions of \( G \).

We note that the recurrence formula (1.2) does not seem to follow directly from (1.1). In [15], (1.2) was derived from the Peterson recurrence formula [8] for root multiplicities of Kac-Moody Lie algebras. The goal of this paper is to give a purely combinatorial (bijective) proof of (1.2).

To construct a bijective proof, we need sets whose cardinalities are the left and right hand sides of (1.2). We in fact give two bijective proofs, starting with the interpretations of \( \alpha(G) \) in terms of acyclic orientations and spanning trees.

**Acknowledgements:** The author would like to thank Sankaran Viswanath for many fruitful discussions.

2. **Acyclic orientations with unique fixed sink**

In this section we give a bijective proof of the recurrence formula (1.2) in terms of acyclic orientations.

We recall that an acyclic orientation of \( G \) is an assignment of arrows to its edges such that there are no directed cycles in the resulting directed graph. A sink in an acyclic orientation is a vertex which only has incoming arrows. The set of all acyclic orientations of \( G \) is denoted \( \mathcal{A}(G) \). For a vertex \( q \) of \( G \), the set of all acyclic orientations in which \( q \) is the unique sink is denoted \( \mathcal{A}(G, q) \). It is well-known that the cardinality of \( \mathcal{A}(G, q) \) is independent of \( q \) and equals \( \alpha(G) \) [7]. The following characterization of \( \mathcal{A}(G, q) \) is immediate.

**Lemma 1.** Fix a vertex \( q \) of \( G \) and let \( \lambda \in \mathcal{A}(G) \). Then \( \lambda \in \mathcal{A}(G, q) \) if and only if for every \( p \in V(G) \), there is a directed path in \( \lambda \) from \( p \) to \( q \).

This motivates the following:
Definition 1. Given a vertex \( q \) and an acyclic orientation \( \lambda \) of \( G \), let \( V(\lambda, q) \) denote the set of all vertices \( p \) for which there is a directed path in \( \lambda \) from \( p \) to \( q \). We call this the set of \( q \)-reachable vertices in \( \lambda \).

We record the following simple observation:

Lemma 2. Let \( x \) be a vertex of \( G \).

(a) If \( x \notin V(\lambda, q) \), then \( x \notin V(\lambda, p) \) for all \( p \in V(\lambda, q) \).

(b) In particular, an edge joining \( p \) and \( x \) with \( p \in V(\lambda, q) \) and \( x \notin V(\lambda, q) \) is directed from \( p \) to \( x \) in \( \lambda \).

Our next goal is to construct sets \( A \) and \( B \) whose cardinalities are respectively equal to the left and right hand sides of (1.2) and to exhibit a bijection between them. To this end, we first consider the set \( \tilde{E} \) of oriented edges of \( G \); an element of \( \tilde{E} \) is an edge of \( G \) with an arrow marked on it (in one of two possible ways). Thus \( \tilde{E} \) has cardinality \( 2e(G) \). If \( \vec{e} \) is an element of \( \tilde{E} \) corresponding to an edge joining vertices \( p \) and \( q \) with the arrow pointing from \( p \) to \( q \), we call \( p \) the tail of \( \vec{e} \) and \( q \) its head.

We now define \( A \) to be the set consisting of pairs \((\vec{e}, \lambda) \in \tilde{E} \times A(G) \) such that the head of \( \vec{e} \) is the unique sink of \( \lambda \). For a fixed \( \vec{e} \), there are \( \alpha(G) \) choices for \( \lambda \) since \( \lambda \) ranges over \( A(G, q) \) where \( q \) is the head of \( \vec{e} \). It is now clear that \( A \) has cardinality exactly \( 2e(G)\alpha(G) \).

To define \( B \), we first take an ordered partition \((G_1, G_2)\) of \( G \). Let \( E(G_1, G_2) \) denote the set of edges straddling \( G_1 \) and \( G_2 \). Let \( B(G_1, G_2) \) denote the set of triples \((e, \lambda_1, \lambda_2)\) where \( e \in E(G_1, G_2) \), say \( e \) joins \( p_1 \) and \( p_2 \) with \( p_i \) a vertex of \( G_i \), \( i = 1, 2 \), and \( \lambda_i \) is an acyclic orientation of \( G_i \) with unique sink at \( p_i \), \( i = 1, 2 \). Arguing as before, one concludes that \( B(G_1, G_2) \) has cardinality \( \alpha(G_1)\alpha(G_2)e(G_1, G_2) \). We now let \( B \) denote the disjoint union of the \( B(G_1, G_2) \) over all ordered partitions \((G_1, G_2)\) of \( G \). It clearly has cardinality equal to the right hand side of (1.2).

We now define a map \( \varphi : A \rightarrow B \) which will turn out to be the bijection we seek. Given \((\vec{e}, \lambda) \in A \), let \( p \) and \( q \) denote the tail and head of \( \vec{e} \) respectively. Note that \( \lambda \in A(G, q) \). Let \( V_1 = V(\lambda, p) \) denote the set of \( p \)-reachable vertices in \( \lambda \) (definition 1) and let \( V_2 = V(G)\setminus V_1 \). Observe that \( p \in V_1 \) and \( q \in V_2 \). For \( i = 1, 2 \), let \( G_i \) denote the subgraphs of \( G \) induced by \( V_i \), and let \( \lambda_i \) denote the restriction of \( \lambda \) to \( G_i \).

We claim that \( \lambda_1 \) has a unique sink at \( p \) and \( \lambda_2 \) has a unique sink at \( q \). The first assertion follows simply from Lemma 1. For the second assertion, observe that if \( x \in V_2 \subset V(G) \), then there is a directed path from \( x \) to \( q \) in \( \lambda \). Since \( x \notin V(\lambda, p) \), Lemma 2(a) implies that no vertex of this directed path can lie in \( V_1 \). In other words this directed path is entirely within \( G_2 \), and we are again done by Lemma 1.

Let \( e \) denote the undirected edge joining \( p \) and \( q \). We have thus shown that the triple \((e, \lambda_1, \lambda_2)\) is in \( B(G_1, G_2) \subset B \). We define \( \varphi(\vec{e}, \lambda) = (e, \lambda_1, \lambda_2) \).

To see that \( \varphi \) is a bijection, we describe its inverse map. Let \((G_1, G_2)\) be an ordered partition of \( G \); given a triple \((e, \lambda_1, \lambda_2) \subset B(G_1, G_2) \), we construct an acyclic orientation \( \lambda \) of \( G \) as follows:
on $G_1$ and $G_2$, we define $\lambda$ to coincide with $\lambda_1$ and $\lambda_2$ respectively. It only remains to define an orientation for the straddling edges (this includes $\vec{e}$): we orient all of them pointing from $G_1$ towards $G_2$, i.e., such that their tails are in $G_1$ and their heads in $G_2$. We let $\vec{e}$ denote the edge $e$ with the above orientation.

We claim $(\vec{e}, \lambda) \in A$. First observe that $\lambda$ is in fact acyclic; since $\lambda$ extends $\lambda_i$ for $i = 1, 2$, any directed cycle of $\lambda$ must necessarily involve vertices from both $G_1$ and $G_2$. But this is impossible since all straddling edges point the same way, from $G_1$ towards $G_2$. Let $p, q$ denote the tail and head of $\vec{e}$. It remains to show that $\lambda$ has a unique sink at $q$, or equivalently, by Lemma 1, that there is a directed path in $\lambda$ from any vertex $x$ to $q$. This is clear if $x$ is a vertex of $G_2$. If $x$ is in $G_1$, we have a directed path in $\lambda_1$ from $x$ to $p$. Now the edge $\vec{e}$ is directed from $p$ to $q$; concatenating this directed path with $\vec{e}$ produces a directed path from $x$ to $q$ in $\lambda$ as required. We define the map $\psi : B \to A$ by $\psi(e, \lambda_1, \lambda_2) = (\vec{e}, \lambda)$.

Observe that for the $\lambda$ defined above, the set of $p$-reachable vertices is exactly $V(G_1)$. This is because edges straddling $G_1$ and $G_2$ point away from $G_1$, so no vertex of $G_2$ is $p$-reachable. This implies that $\varphi \circ \psi$ is the identity map on $B$. Further, it readily follows from Lemma 2(b) that $\psi \circ \varphi$ is the identity map on $A$. This establishes that $\varphi$ is a bijection. $\square$

3. Spanning trees without broken circuits

In this section we give another bijective proof of the recurrence formula (1.2), this time using the fact that $\alpha(G)$ counts the number of spanning trees of $G$ without broken circuits.

**Definition 2.** Let $\sigma$ be a total ordering on the set $E(G)$ of edges of $G$. Given a circuit in $G$, it has a unique maximum edge with respect to $\sigma$; the set of edges obtained by deleting this edge from the circuit is called a *broken circuit* relative to $\sigma$. The set of all broken circuits relative to $\sigma$ is denoted $B_G(\sigma)$.

Let $S_G(\sigma)$ be the set of all spanning trees of $G$ that contain no broken circuits relative to $\sigma$. It is well-known that the cardinality of $S_G(\sigma)$ is independent of the choice of $\sigma$, and equals $\alpha(G)$ [3].

Given a total ordering $\sigma$ on $E(G)$, let $\max(\sigma)$ denote the maximum edge in $E(G)$. The following lemma is immediate.

**Lemma 3.** Any spanning tree in $S_G(\sigma)$ contains the edge $\max(\sigma)$.

In the sequel, we will fix for each edge $e$, a total order $\sigma_e$ on $E(G)$ for which $\max(\sigma_e) = e$. We will write $B_G(e)$ and $S_G(e)$ for the sets $B_G(\sigma_e)$ and $S_G(\sigma_e)$ respectively.

We now proceed to prove (1.2) in the following equivalent form:

$$e(G) \alpha(G) = \sum_{\{G_1, G_2\}} \alpha(G_1) \alpha(G_2) e(G_1, G_2).$$

(3.1)

We first define the set $A$ to consist of pairs $(e, T)$ where $e$ is an edge and $T \in S_G(e)$; from the above discussion, $A$ has cardinality $e(G) \alpha(G)$. 

...
Next, we define the set $B$. Given an unordered partition \( \{G_1, G_2\} \) of $G$, define $B(\{G_1, G_2\})$ to be the set of pairs \( (e, \{T_1, T_2\}) \) where $e$ is an edge that straddles $G_1$ and $G_2$ and $T_i \in S_{G_i}(e)$ for $i = 1, 2$. Here, $S_{G_i}(e)$ is the set of spanning trees of $G_i$ which contain no broken circuits relative to the total order $\sigma_e$ restricted to the edges of $G_i$. We let $B$ denote the disjoint union of $B(\{G_1, G_2\})$ as $\{G_1, G_2\}$ ranges over unordered partitions of $G$. Clearly $B$ has cardinality equal to the right hand side of \((3.1)\).

We define maps $\varphi : A \to B$ and $\psi : B \to A$ as follows:

Given $(e, T) \in A$, $e$ occurs in $T$ in view of Lemma 3. Deleting $e$ from the spanning tree $T$ will result in a pair of trees $T_1, T_2$ with vertex sets $V_1$ and $V_2$. Let $G_i$ denote the subgraph induced by $V_i$, $i = 1, 2$; clearly $\{G_1, G_2\}$ is an unordered partition of $G$ and $e$ straddles the $G_i$. Observe that since the total order on $E(G_i)$ is defined as the restriction of the total order $\sigma_e$ on $E(G)$, $T_i$ will contain no broken circuits of $G_i$ for $i = 1, 2$, i.e., $T_i \in S_{G_i}(e)$. We set $\varphi(e, T) = (e, \{T_1, T_2\})$.

For the inverse map $\psi$, let $(e, \{T_1, T_2\}) \in B$. Define $T$ to be the spanning tree of $G$ obtained by adding the edge $e$ to the union of $T_1$ and $T_2$. To prove that $T$ contains no broken circuits relative to $\sigma_e$, observe that any broken circuit of $T$ cannot lie entirely within $T_1$ or $T_2$, and must hence contain the edge $e$. But $e$ is the maximum edge relative to $\sigma_e$, so this cannot be a broken circuit by definition. Thus $(e, T) \in A$, and we define $\psi(e, \{T_1, T_2\}) = (e, T)$.

It is straightforward to check that $\varphi$ and $\psi$ are indeed inverse maps. \( \square \)

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