Effective modules of a layered elastic creep medium with power creep kernels

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Abstract. The paper considers the question of constructing the effective characteristics of a layered composite material when periodically repeating layers consist of an isotropic elastically creeping material. For the elastic composites, such characteristics are well known and can be obtained explicitly. In this paper, the effective characteristics of the layered elastic-creeping composite are obtained explicitly when the creep kernel are the power-law functions. The elements of the compliance matrix are presented as the sum of the terms corresponding to the instantaneous elastic compliance, unlimited and limited as the loading time increases due to the creep of the composite individual phases.

Introduction

Often, when solving the problems in the mechanics of inhomogeneous materials with a periodic structure, the method of asymptotic averaging is used \cite{1, 2}. In addition, to study the strength of such materials, an important role is played by modeling their behavior over the long periods of time \cite{3-6}. Analytical representations for the total averaged tensor of an elastic-creeping layered composite were given in \cite{7, 8} if the share relaxation kernel (and, accordingly, the creep kernel) represents the finite number sum of decreasing exponential functions. If the kernels have an arbitrary form, then the analytical representation for the averaged kernel cannot be obtained, however, the creep and the share relaxation kernel can be constructed as a solution of the one-dimensional Volterra integral equation of the second kind \cite{9-11}. These equations are solved on a computer very quickly and with high accuracy. Rheological relations with Abel kernels were also considered in \cite{12,13}.

In this paper, we give an analytical expression for the averaged kernel in the case when the initial kernels are Abelian type kernels.

The defining relations of creep theory

Let us assume that the composite material consists of isotropic layers perpendicular to the axis $Ox_3$, and both phases of the composite have creep kernel of the form $K(t-\tau) = C(t-\tau)^{-\alpha}$, where $0 < \alpha < 1$. Such functions are often used to model the creep processes.
Let us consider an elastic-creeping medium, which is defined by such defining relations that the deviator $s_y(t)$ and ball tensor $P(t)$ voltages have the form

$$s_y(t) = 2\mu e_y(t) - \int_0^t K_e \cdot (t - \tau)^{-\alpha} s_y(\tau) d\tau$$  \hspace{1cm} (1)

$$P(t) = B\theta(t) - \int_0^t K_v \cdot (t - \tau)^{-\alpha} P(\tau) d\tau$$  \hspace{1cm} (2)

Here $e_y(t)$ - is the deviator and $\theta(t)$ - is the ball strain tensor, $K_e$ and $K_v$ - are shear and bulk creep coefficients, respectively, $\mu$ - is the shear modulus, $B$ - is the volume expansion module, $0 < \alpha < 1$,

$$\theta = \varepsilon_i + \varepsilon_z - \varepsilon_z', \sigma_y = s_y + \frac{1}{3} \sigma_i' \cdot \delta_y', \varepsilon_y = e_y + \frac{1}{3} e_i' \cdot \delta_y.$$  

Such constitutive relations are used, for example, in modeling the creeping properties of soils [14]. After applying the Laplace transform in a variable $t$

$$\tilde{f}(p) = \int_0^\infty f(t) e^{-pt} dt$$  \hspace{1cm} (3)

using the equality, we obtain:

$$\frac{t^{-\alpha}}{\Gamma(1 - \alpha)} = \frac{1}{p^{1 - \alpha}},$$  \hspace{1cm} (4)

the following relations

$$\tilde{s}_y(p) = 2\mu \left(1 + \frac{K_e \cdot \Gamma(1 - \alpha)}{p^{1 - \alpha}}\right)^{-1} \cdot \tilde{e}_y(p)$$  \hspace{1cm} (5)

$$\tilde{P}(p) = B \left(1 + \frac{K_v \cdot \Gamma(1 - \alpha)}{p^{1 - \alpha}}\right)^{-1} \cdot \tilde{\theta}(p).$$  \hspace{1cm} (6)

Hence, we have

$$\tilde{\sigma}_y - \frac{1}{3} \tilde{\sigma}_i' \cdot \delta_y = 2\mu \left(1 + \frac{K_e \cdot \Gamma(1 - \alpha)}{p^{1 - \alpha}}\right)^{-1} \left(\varepsilon_y - \frac{1}{3} \varepsilon_i' \cdot \delta_y\right).$$  \hspace{1cm} (7)

Further, using the equalities (5), (6) and the relation $\tilde{\theta} = \frac{1}{3} \tilde{\sigma}_i'$, we get

$$\tilde{s}_y(p) = \left[\frac{B}{3} \left(1 + \frac{K_e \cdot \Gamma(1 - \alpha)}{p^{1 - \alpha}}\right)^{-1} - \frac{2}{3} \mu \left(1 + \frac{K_e \cdot \Gamma(1 - \alpha)}{p^{1 - \alpha}}\right)^{-1}\right] \tilde{\delta}_y +$$

$$+ \mu \left(1 + \frac{K_v \cdot \Gamma(1 - \alpha)}{p^{1 - \alpha}}\right)^{-1} \left(\delta_\alpha \delta_\beta + \delta_\alpha \delta_\gamma\right).$$  \hspace{1cm} (8)

Thus, the defining relation for Laplace’s images $\tilde{s}_y(p)$ can be possible with a fixed value of the complex parameter $p$ considered as a defining relation for an isotropic medium with the Lame constant $\tilde{\lambda}(p), \tilde{\mu}(p)$, having the form

$$\tilde{\lambda}(p) = B \left(1 + \frac{K_e \cdot \Gamma(1 - \alpha)}{p^{1 - \alpha}}\right)^{-1} - \frac{2}{3} \mu \left(1 + \frac{K_e \cdot \Gamma(1 - \alpha)}{p^{1 - \alpha}}\right)^{-1},$$  \hspace{1cm} (9)
\[ \hat{\mu}(p) = \mu \left( 1 + \frac{K_s \cdot \Gamma(1-\alpha)}{p^{\alpha-1}} \right)^{-1} \]  

Denote \( K_s \cdot \Gamma(1-\alpha) = k, K_s \cdot \Gamma(1-\alpha) = c, p^{\alpha-1} = q \). Since the bulk modulus of elasticity \( B = \frac{3\lambda + 2\mu}{3} \), then

\[ (\lambda + 2\mu)(p) = \left[ \frac{3\lambda + 2\mu}{3(q + k)} + \frac{4\mu}{3(q + c)} \right] q, \quad \hat{\lambda}(p) = \left[ \frac{3\lambda + 2\mu}{3(q + k)} - \frac{2\mu}{3(q + c)} \right] q, \quad \tilde{\lambda}(p) = \frac{\mu q}{q + c}. \]  

(10)

The symmetric matrix of elastic moduli depending on a complex parameter \( p \), will have the form

\[
\tilde{M}(p) = \begin{pmatrix}
(\lambda + 2\mu)(p) & \tilde{\lambda}(p) & 0 & 0 & 0 \\
(\lambda + 2\mu)(p) & \tilde{\lambda}(p) & 0 & 0 & 0 \\
(\lambda + 2\mu)(p) & 0 & 0 & 0 & 0 \\
\tilde{\mu}(p) & 0 & 0 & 0 & 0 \\
\tilde{\mu}(p) & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]  

(11)

If

\[
\tilde{\sigma}(p) = \begin{pmatrix}
\tilde{\sigma}_x(p) \\
\tilde{\sigma}_y(p) \\
\tilde{\sigma}_z(p) \\
\tilde{\sigma}_{xy}(p) \\
\tilde{\sigma}_{xz}(p) \\
\end{pmatrix}, \quad \tilde{\varepsilon}(p) = \begin{pmatrix}
\tilde{\varepsilon}_x(p) \\
\tilde{\varepsilon}_y(p) \\
\tilde{\varepsilon}_z(p) \\
\tilde{\varepsilon}_{xy}(p) \\
\tilde{\varepsilon}_{xz}(p) \\
\end{pmatrix},
\]  

then the state equations have the following form \( \tilde{\sigma}(p) = \tilde{M}(p)\tilde{\varepsilon}(p) \).

\[ \tilde{\sigma}(p) = \tilde{M}(p)\tilde{\varepsilon}(p). \]  

(12)

The composite effective characteristics’ calculation

Let us suppose that the sample under consideration consists of the flat layers parallel to each other and perpendicular to the axis \( Ox_3 \). The relative layer thickness \( h \) and \( 1 - h \), type of materials: the materials that are described by the defining relations (1) - (3), while there are two sets of the elastic and creeping materials’ parameters and the layers of materials with these parameters’ sets are periodically repeated. The construction of effective (averaged) characteristics of an elastic layered composite was obtained earlier and the corresponding formulas are given, for example, in [1,2]. Next, we apply these formulas and obtain an averaged defining relation for the Laplace images, expressing the relation for the stress tensor through the strain tensor, and invert the resulting relation. We introduce the notation

\[
\langle f \rangle = \int_0^1 f(\xi) d\xi.
\]  

(13)

The matrix \( Q \), with the help of which the averaged tensor of displacements through the strain tensor components is expressed, will have the form
\[
Q = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\
  a_{22} & a_{23} & 0 & 0 & 0 & 0 \\
  a_{33} & 0 & 0 & 0 & 0 & 0 \\
  a_{44} & 0 & 0 & 0 & 0 & 0 \\
  a_{55} & 0 & 0 & 0 & 0 & 0 \\
  a_{66} & 0 & 0 & 0 & 0 & 0 
\end{pmatrix}.
\]

The calculation shows that the matrix elements (16) are given by the formulas:

\begin{align}
a_{11} &= \frac{\langle (\lambda + 2\mu)(p) \rangle - \langle \mu^2(p) \rangle}{\langle (\lambda + 2\mu)(p) \rangle + \langle \mu^2(p) \rangle - 2\langle \mu^2(p) \rangle}, \\
a_{12} &= \frac{\langle \mu^2(p) \rangle - \langle \mu(p) \rangle}{\langle (\lambda + 2\mu)(p) \rangle + \langle \mu(p) \rangle - 2\langle \mu^2(p) \rangle}, \\
a_{13} &= \frac{1}{\langle (\lambda + 2\mu)(p) \rangle} + 2\langle \mu^2(p) \rangle^{2}/\langle (\lambda + 2\mu)(p) \rangle - 2\langle \mu(p) \rangle^{2}/\langle (\lambda + 2\mu)(p) \rangle, \\
a_{33} &= a_{55} = \langle \mu^{-1}(p) \rangle, \\
a_{66} &= \langle \mu(p) \rangle^{-1},
\end{align}

where \((\lambda + 2\mu)(p), \mu(p), \mu(p)\) are given by the formulas (9), (10).

By the condition, as we have already indicated, the layered rod is considered, all elastic moduli and creep kernels are the periodic functions of the coordinate \(x = \frac{x_3}{\varepsilon}\) (\(\varepsilon\) - a is a small parameter that determines the thickness of the layer with respect to the thickness of the sample under consideration) and are piecewise constant functions of this variable, i.e., the elastic moduli and creep kernel coefficients have the following form

\[
\lambda(x) = \begin{cases}
\lambda_1, & x \in [0; h] \\
\lambda_2, & x \in [1-h; 1]
\end{cases},
\mu(x) = \begin{cases}
\mu_1, & x \in [0; h] \\
\mu_2, & x \in [1-h; 1]
\end{cases},
k(x) = \begin{cases}
k_1, & x \in [0; h] \\
k_2, & x \in [1-h; 1]
\end{cases},
c(x) = \begin{cases}
c_1, & x \in [0; h] \\
c_2, & x \in [1-h; 1]
\end{cases},
\]

where \(0 < h < 1\).

Let us consider the coefficient \(a_{33}\). Performing the indicated averaging operation in (20) and substituting the expressions \((\lambda + 2\mu)(p), \mu(p), \mu(p)\) of (9), (10), we get
In (23), the notation \(i = 1, 2\):

\[
L_i = \lambda_i + 2\mu_i, N_i = 3\lambda_i c_i + 2\mu_i (c_i + 2k_i), p_i = (3\lambda_i + 2\mu_i) c_i - 2\mu_i k_i, r_i = 3\lambda_i + 2\mu_i,
\]

\[
A = 3h(3\lambda_i + 2\mu_i)\mu_i L_2 + (1-h)(3\lambda_i + 2\mu_i)\mu_i L_1,
\]

\[
B = h(3\lambda_i + 2\mu_i)\mu_i N_2 + (1-h)(3\lambda_i + 2\mu_i)\mu_i N_1,
\]

\[
F = h\lambda_i L_2 + (1-h)\lambda_i L_1, ~ I = hc_i k_i N_2 + (1-h)c_i k_i N_1, ~ J = hp_i N_2 + (1-h)p_i N_1,
\]

\[
S = \frac{h(AN_2 - 3BL_2)(Ap_i - 3\beta\lambda_i) + (1-h)(AN_1 - 3BL_1)(Ap_2 - 3\beta\lambda_2))^2}{3A^2B(AN_1 - 3BL_1)(AN_2 - 3BL_2)}, ~ H_i = \begin{cases} h, & i = 1 \\ 1-h, & i = 2 \end{cases}
\]

\[
C = 3h\mu_i L_2 + (1-h)r_i \mu_i L_1, ~ D = hr_i \mu_i N_2 + (1-h)r_i \mu_i N_1.
\]

Similarly

\[
a_{13} = a_{23} = \frac{3F}{2C} + \frac{J}{6Dq} + \frac{2}{6C^2D} \left( q + \frac{D}{C} \right),
\]

In the equalities (23) and (24), we perform the inverse Laplace transform, taking into consideration that \(q = p^{1-\alpha}, \quad 0 < \alpha < 1\).

\[
a_{33}(t) = \frac{[L_2h + L_1(1-h)]A + 3F^2}{AL_1L_2}, \delta(t) + \frac{9BI + J^2}{3BN_1N_2} \left( 1 - \alpha \right) + S \mathcal{E}_a \left( \frac{B}{A}t \right) + \sum_{i=1}^{2} H_i \mathcal{E}_a \left( \frac{N_i}{3L_i}t \right) \left[ \frac{(N_i - 3c_i L_i)(3k_i L_i - N_i)}{3L_i^2 N_i} - \frac{(L_i N_i - L_i N_2)(\lambda_i N_i - L_i p_i)^2}{L_i^2 N_i (3BL_i - AN_i)} \right],
\]

\[
a_{13}(t) = a_{23}(t) = \left[ \frac{3F}{2C} \mathcal{E}_a(\delta(t)) + \frac{3F}{2C} \left( 1 - \alpha \right) + \frac{J}{6Dt^\alpha} + \frac{S \mathcal{E}_a \left( \frac{D}{C} \right) t^\alpha}{6C^2D} \right],
\]

At (25), (26) \(\mathcal{E}_a(a,t)\) - the function of Yu.N. Rabotnov, the first terms correspond to instantaneous elasticity, the second to “limited” creep, the rest to “limited” creep.

If a constant load is applied to the sample, then elastic deformations and displacements are established instantly, the terms corresponding to “unlimited” creep will lead to the unlimited deformations and displacements’ growth, and the terms corresponding to “limited” creep will lead to the creeping deformations that have finite limits with unlimited increasing time. The last characteristic is associated with the function behavior of Yu.N. Rabotnov at infinity - it decreases so that the its integral from zero to infinity is bounded.

In this paper, the formulas for the three coefficients of the compliance matrix are given; similar formulas can be constructed for other coefficients.

**Summary**
Explicit expressions are obtained for effective (averaged) layered elastic-creeping composite, when the individual layers are isotropic elastic-creeping materials, creep kernels of which are the power functions. In analytical representations, $\mathcal{E}$-functions of Yu.N. Rabotnov are used. The results can be used to model soil displacements and the layered composites’ behavior under long-term loads.

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