Entanglement topological invariants for one-dimensional topological superconductors

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Introduction. – In recent years, entanglement has emerged as a groundbreaking diagnostic to characterize and classify many-body quantum phenomena in- and out-of-equilibrium [1–4]. An archetypal example is the possibility of unambiguously detecting topological order in two-dimensional systems via the topological entanglement entropy (TEE) [5–7]. The latter spots the presence of ‘long-range’ entanglement which is not distillable via local operations; consequently, it defines a genuine entanglement order parameter, that distinguishes phases depending on their quasiparticle content [8]. This insight has been widely employed in the characterization of topologically ordered states in numerical simulations [9–11], and has stimulated the search for experimentally realistic entanglement probes [12–18].

While the definition of the TEE naturally emerges from gauge theories in two-dimensions, the existence of topological invariants based solely on entanglement properties in one-dimensional (1D) topological matter - e.g., in the form of an order parameter - is presently not clear. In 1D, bipartite entanglement of connected partitions does not display informative scaling corrections [2, 3], and even its finer structure - captured by the entanglement spectrum - while providing sharp sine qua non [19–21], is not able to distinguish the topological character of wave functions [1]. At the field theory level, the ultimate reason for this is that, in 1D, connected bipartite entanglement is strongly influenced by ultra-violet contributions due to edges, and is thus not immediately linked to ‘universal’ information.

In this work, we show how entanglement and Rényi entropies of disconnected partitions provide a set of entanglement order parameters for one-dimensional topological superconductors (TSCs) [23, 24]. These order parameters satisfy the following properties: (i) are quantized to 0 or \( \log 2 \) depending on the phase being topologically trivial or not-trivial, and are thus able to detect the single entanglement bit - an ebit that can be distillled from the ground state manifold; (ii) display scaling behavior when approaching quantum phase transitions, and thus allow for the definition of entanglement critical exponents that describe the build-up of non-local quantum correlations across such transitions; (iii) are experimentally measurable in- and out-of-equilibrium utilizing recently introduced [14, 15] and demonstrated [25] techniques based on

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1 One simple example is the equivalence between the entanglement spectra of the ground state of finite Ising and Kitaev chains.

FIG. 1. (Color online) Entanglement topological order parameter and the phase diagram of the interacting Kitaev chain. Panel a): schematics of the partitions A (shaded, green) and B (shaded, blue) considered here. Each site of the chain hosts a spin-less fermion degree of freedom \( a_j \), that can be decomposed into two Majorana fermions. The orange circle magnifies the cut across partitions: deep in the topological phase, neighboring Majorana fermions belonging to different physical sites are coupled (dashed line). The partition cut takes place exactly between the two coupled Majorana fermions. Panel b): quadripartite von Neumann entropy \( S^D \) as a function of \( \mu/t, U/t \), at fixed \( \Delta = 1 \) and \( L_A = L_B = 12 \). Black lines as from Ref. [22]. The colour plot is obtained via interpolation on a 6 x 8 grid.
random measurement methods [26].
Following Ref. [27], we consider the $F$-function between two partitions $A, B$, which compensate for all edges and volume contributions in 1D. These properties are required to avoid non-universal effects: as we discuss below, the significance of our diagnostics relies on an underlying gauge theory description, which naturally calls for quantities that are divergence free in the continuum limit. In order to diagnose the presence of non-local correlations in the system, we choose two partitions with different connectivity, as shown in Fig. 1\(^2\). The resulting disconnected $n$-entropies $S_n^D$ read:

$$S_n^D = S_{A,n} + S_{B,n} - S_{A∪B,n} - S_{A∩B,n},$$

where $S_{F,n}$ is the Rényi entropy of order $n$ of the partition $F$. The case $n=1$, that we denote as $S^D$, corresponds to the von Neumann entropy. $S^D > 0$ because of strong subadditivity. This entropy has been considered in Ref. [29], which pointed out a strong analogy between bosonic symmetry-protected topological phases (SPTs) and error-correcting codes. Here, our focus is instead on fermionic phases where topology is stems from an underlying fundamental symmetry (parity) which cannot be broken by any Hamiltonian perturbation. As we discuss below, this fact plays a crucial role in defining a gauge-theory picture describing the entanglement content of such states: the non-local correlations introduced by the fermionic algebra via the Jordan-Wigner string are responsible for such analogy.

At a qualitative level, the key player in $S_n^D$ is the disconnected partition $B$: all other terms are complementary, and only required to eliminate non-universal boundary and volume terms. In Fig. 1\(^b\), we show the finite-size behavior of $S_n^D$ across the phase diagram of interacting Kitaev chains: this plots illustrates graphically how, even at modest system and partition sizes, $S_n^D$ clearly distinguish topological from trivial phases. In the following, we define as $L_a$ the size of a given partition, and as $L_D$ the distance between the two different parts of $B$.

Model Hamiltonian. – We consider the interacting version of Kitaev p-wave superconductor, defined on an open chain of length $L$, and whose Hamiltonian reads:

$$H = \sum_{j=1}^{L-1} \left[-t(a_j^\dagger a_{j+1} + h.c.) + (\Delta a_j a_{j+1} + h.c.)\right] + 4U \left(n_j - \frac{1}{2}\right)\left(n_{j+1} - \frac{1}{2}\right) - \mu \sum_{j=1}^{L} n_j,$$

where $a_j^\dagger$ ($a_j$) are the creation (annihilation) operators of the spinless fermion on site $j$, $n_j = a_j^\dagger a_j$, $t$ is the hopping amplitude, $\Delta$ is the superconducting amplitude, $U$ is a Hubbard-like interaction, and $\mu$ is the chemical potential. The phase diagram of the model is known [22, 30] and displays a TSC phase, in addition to topologically trivial phases, including a band insulator, a Mott insulator, and an incommensurate charge-density-wave (ICDW) phase. For any state in the Hilbert space, the bipartite properties of a simply connected partition are equivalent to the ones of the XYZ spin chain, which is obtained from Eq. 2 after applying a Jordan-Wigner transformation. As such, they are uninformative about the topological origin of a given phase.

Disconnected entropies. – Given the above, the goal is to find a combination of entropies able to unambiguously capture the influence of non-locality in the ground state properties, and that it is able to identify the amount of information - in this case, a single ebit - that can be stored in the ground state manifold. $S_n^D$ introduced above contains the simplest non-trivial disconnected partition, $S^D_B$: all other terms are introduced in order to compensate from possible volume and edge effects.

For conformal phases, $S_n^D$ is immediately given by conformal field theory [2, 28, 31], and vanishes in the thermodynamics limit. For gapped phases, one has to distinguish between topologically trivial and non-trivial phases. We analyze here the limiting cases.

(i) $t = \Delta = U = 0, \mu > 0$: in the band insulator case, the density matrix of arbitrary partitions has rank 1 both in fermionic and spin systems. This immediately gives $S_n^D = 0$. The same result holds in the Mott insulator phase.

(ii) $t = \Delta = 1 \gg |\mu|, U = 0$: this regime is representative of the TSC phase. Its correspondent in the XYZ model is a ferromagnetic phase, which we analyze first as a representative of a symmetry-broken phase. There, the lowest energy states at any finite size are equal weight superpositions of the two ferromagnetic states, $|\Psi_{\text{XYZ}}\rangle = (|\uparrow\uparrow\uparrow\ldots\rangle \pm |\downarrow\downarrow\downarrow\ldots\rangle)/\sqrt{2}$, separated by a gap $\delta \propto e^{-L}$. For both states, any reduced density matrix of an arbitrary spatial partition is equivalent, and thus $S_n^D = 0$.

For the TSC, the situation is different. While $S_A, S_{A∪B}, S_{A∪B}$ are the same as in the spin model, $S_B$ has a sharply different behavior. In this regime, the ground state is two-fold degenerate (again, up to a gap $\delta \propto e^{-L}$): Each of the two states $|\Psi\rangle_\pm$ can be written as an equal weight superposition of states with a given parity $|\psi\rangle_\pm$, i.e., $|\Psi\rangle_\pm = (1/\sqrt{2}) \sum_{\alpha} |\psi\rangle_\pm$. By taking the proper fermionic trace into account, the entanglement structure of arbitrary partitions is straightforwardly evaluated [32], and one obtains $S_B = 2\log 2$. This returns a disconnected entropy $S^D = \log 2$.

While the behavior of these cartoon wave functions sharply distinguishes the TSC phase with respect to all other phases, the question whether this is a property of a phase, and whether the value of $S^D$ remains quantized requires to go beyond these oversimplified picture. Before presenting numerical results in support of these findings, we now illustrate how the quantization of $S^D$ emerges naturally when utilizing a lattice gauge theory (LGT) description of the Kitaev chain.

\(^2\)Note that, at the field theory level, the entanglement properties of these partitions are known to depend on the full operator content [2, 28].
The starting point is the exact relation [33] between Eq. 2 and a $Z_2$ LGT, that we schematically review. The $Z_2$ LGT describes a 1D gauge theory in a phase with strongly fluctuating gauge fields, with Pauli matrices, $\sigma_{j,j+1}^\alpha$, and hardcore Higgs fields $\varphi_j$, with $n_j = \varphi_j^\dagger \varphi_j$. The gauge invariant Hilbert space is defined as the set of states where the local parity $P_i = (1 - 2n_j)\sigma_{j,j+1}^z$ is fixed to 1 (see Fig. 2). Under open boundary conditions (OBC), we can fix the value of the first gauge field without loss of generality to $\sigma_{0,1} = 1$. The value of the last gauge field $\sigma_{N-1,N}$ equals the total parity of the system due to gauge invariance.

In the gauge theory language, the ground state wave functions $|\Psi\rangle_\pm$ can be described in terms of either fermionic or gauge fields, since, in 1D, those are mutually fixed by Gauss law. In the latter language, the ground states are equal weight superpositions of all possible string states of arbitrary length, and compatible with the boundary conditions: a sample of those are depicted in Fig. 2 for the case $P = 1$. This picture describes a 1D gauge theory in a phase with strongly fluctuating gauge fields, and is strongly reminiscent of the loop description of 2D $Z_2$ LGT [8, 34, 35].

Evaluating entanglement entropies in this phase is straightforward by exploiting gauge invariance:

(i) The entropy of each connected partition is $\log 2$. Let us define $\sigma_{b}^x, \sigma_{b}^y$ as the two boundary spins of the partition. Their product is equal to the parity of the partition: the density matrix of the partition is block-diagonal in this conserved quantum number. If the correlation length is much smaller than the partition length, both positive and negative parities are equally probable and all states count with equal weight. The corresponding von Neumann entropy is thus $\log 2$.

(ii) The entropy of disconnected partitions is $(N_c - 1) \log 2$, where $N_c$ is the number of partitions. Let us define as $\sigma_{L,h}^z, \sigma_{R,h}^z$ the gauge fields at the boundaries of the partition $h$. As long as the length of each partition is larger than the correlation length, each patch is an equal weight superposition of all possible states, under the condition that $P_h = \pm 1$ for partitions with or without an out-coming flux. Fixed the total parity, there are $2^{N_c-1}$ finite, equal values of the corresponding density matrix, which indeed returns an entropy equal to $(N_c - 1) \log 2$.

We emphasize that the gauge theory description enables a simple calculation of the entropies (by replacing fermionic statistics with a $Z_2$ gauge field), and, at the same time provides a simple, compelling physical picture, that might be extended to more exotic types of order.

Numerical results. – We now turn to a numerical investigation of Eq. 2. We used free fermion techniques [36] to investigate the non-interacting case $U = 0$, and density-matrix-renormalization group (DMRG) [37, 38] for $U \neq 0$. Since DMRG does not give immediate access to $S_B$, we performed separate simulations to obtain this quantity, by modifying the lattice connectivity [32]. We kept up to 1200 states after truncation, and performed at least 30 sweeps. Typical discarded weights at the end of the simulation were of order $10^{-8}$.

The phase diagram in Fig. 1 shows how, even at very modest partition sizes, $S^D$ is large and finite only the TSC phase; analogous results hold for the $U = 0$ plane [32]. In Fig. 3, we show the finite-size-scaling behavior of $S^D$ for representative points in the TSC ($\mu/t = 1.0, 1.5$) and topologically trivial ($\mu/t = 4.0$) phase. The asymptotic values are quantized within numerical accuracy of our fits to $\log 2$ and 0, respectively, in agreement with the theoretical discussion above. The inset shows how, in the TSC phase, quantization is approached exponentially fast in system size.

Both results hold not only for $S^D$, but also for its low-order Rényi counterparts. As a representative of these results, we show in Fig. 4 the behavior of $S^D_2$ in the $\Delta - \mu$ plane: again, the value of this entropy clearly distinguish topologically trivial and non-trivial phases.

Universal behavior and entanglement critical exponents. – Since $S^D$ captures universal properties of each phase, it is nat-
ural to wonder whether such quantities can display universal scaling behavior when crossing a quantum phase transition. Here, we focus on the transition between TSC and band insulator, which belongs to the Ising universality class. Fixing $L_A = L_B = L/2, L_{A/B} = 3L/4$ in order to avoid effects due to aspect ratios of the different partitions, we fit $S^D$ as one would for an order parameter around the phase transition using a phenomenological finite-size scaling ansatz:

$$\frac{S^D}{L^\frac{2}{D}} = \lambda \left(L^\frac{2}{D} (\alpha - \alpha_c)\right), \quad (3)$$

where $\alpha = \mu$ or $U$ is the parameter chosen, $a$ and $b$ (a priori different depending on the chosen parameter) take place of the usual critical exponent $\beta$ and $\nu$, and $\lambda(x)$ is a scaling function. We extract these parameters using curve intersections and collapse shown in Figs. 5 for two cuts indicated in Fig. 1 with yellow arrows. The results of the collapse scaling locates correctly the transition point (with errors $10^{-4}$). Most surprisingly, we find that the entanglement critical exponents satisfying $a = b = 1$ irrespectively of where the transition line is crossed, a sharp signature of universal behavior. The quality of the collapse scaling in both non-interacting (b) and interacting (d) case is good already for modest system sizes, further corroborating such universal behavior.

**Topological invariance under coherent dynamics.** – A key signature of topological invariants is the fact that, in the thermodynamic limit, those cannot change under unitary evolution (as long as specific symmetries are not broken explicitly [39–41]). In order to check that $S^D$ is a true topological invariant, we performed an extensive investigation based on quantum quenches starting within the topological phase, and quenching with arbitrary values of the Hamiltonian parameters.

A representative sample of our results is presented in Fig. 6. In panel a), we plot the time evolution of $S^D$ for a quench from an initial value of the pairing amplitude $\Delta = 0.5$ to a final value $\Delta = 1.5$. Different lines correspond to different system sizes. For each size, one can sharply distinguish two regimes. At short times, $S^D$ does not change with time, and exhibits a plateau up to a time $t_p$ that depends on $L_A$. After this timescale, quantization is lost, and the dynamics is dictated by non-universal dynamics. In order to understand whether quantization is a robust feature, we perform a finite size scaling analysis in panel b): our results show that $t_p$ (defined as the time where $S^D = 0.95$) grows approximately linearly with system size, and that its extrapolated value to the thermodynamic limit diverges. This behavior confirms the topological invariant nature of $S^D$.

**FIG. 4.** (Color online) Top panel: $S_2^D$ in the non-interacting phase diagram of the Kitaev chain with $U = 0$. Here, $L = 50, L_A = L_B = 12, L_D = 32$.

**FIG. 5.** Finite size scaling of $S^D$ (in $\log_2$) along the line (I) of the phase diagram of Fig. 1 (i.e. $U = 0, t = |\Delta| = 1$) as a function of $\mu$ and $L_D$ using free fermions technique [36] and along the line (II) of Fig. 1 (i.e. $\mu = 1, t = |\Delta| = 1$) as a function of $L_D$ and $U$ using DMRG. In all plots, $L_A = L_B = 12$. (a) and (c): $S^D$ as a function of (a) $\mu$, (c) $U$ for different size: it is possible to extract the critical value $\mu_c(L_A)$ for (a), $U_c(L_A)$ for (c) by taking the point where all curves intersect, here $\mu_c(6) = 1.978$ for (a), $U_c(6) = -0.314$ for (c). (b) and (d) Scaling of $\lambda(x)$ for different system sizes. The collapse is best realized for $a = b = 1$, values that also minimize the square root of the residual sum of squares. Simulations with more sites (especially using free fermion techniques) only confirm these results.

**FIG. 6.** (Color online) Time evolution of $S^D$. a) Evolution of $S^D$ after a quantum quench from $\Delta_0 = 0.5$ to $\Delta = 1.5$, with $U = 0, \mu = 0, L = 4L_A = 4L_B$. The times at which each curve crosses the dashed line (0.95) defines $t_p$. b) Finite-size scaling of $t_p$ for two values of $\Delta$ for the quenched Hamiltonian. In both cases, the width of the plateau diverges linearly with system size, as expected for topological invariants.
**Experimental measurement and comparison to other diagnostics.** – Two key properties of $S^D$ are that they are informative already for modest partition sizes, and that Rényi entropies can be used. The proposals in Ref. [14] discuss how to perform measurements of Rényi-2 entropies in synthetic quantum systems: in particular, within that setting, the complexity of the measurement is not sensitive to the connectivity of the partition itself, but only to its total size. Given that a large $L_D$ allows to distill the correct information from the wave function, measuring $S^D$ is as complex as measuring its largest partition $\mathcal{A}$. We note that partitions of sizes up to 10 spins have already been probed in experiments [25].

Finally, we comment on the relation between our topological invariant and other diagnostics. Topological invariants such as the many-body Chern number are unrelated to bipartite entanglement properties, as they do not depend solely on the spectrum of density matrices, but also on their eigenfunctions. For specific symmetries, ad hoc topological invariants can be defined [35] (and potentially experimentally measured [42]) utilizing the matrix-product-state classification of SPTPs [43]; these quantities are sensitive to the response of a state to specific (symmetric) operations, and not immediately connected to entanglement. From the theoretical viewpoint, all these diagnostics represent complementary tools, that give access to qualitatively different features characterizing topological matter - response of wave functions under changing boundary conditions, properties with respect to protecting symmetry, and non-local entanglement content of wave functions, respectively.

**Conclusions.** – We have shown how entanglement of disconnected partitions uniquely distinguishes topological superconducting phases in one-dimensional systems. This distinction is naturally interpreted within a lattice gauge theory framework, and leads to key footprints both at the ground state level, and in quantum quenches. The entanglement order parameters display universal scaling behavior when crossing phase transitions, characterized by entanglement critical exponents. Our findings show that modest partition sizes - of the order of what has been already experimentally demonstrated - are sufficient to uniquely characterize topological superconductors via entanglement. It would be intriguing to investigate whether other forms of quantum correlations between disconnected partitions, such as discord [44] or quantum coherences [45], display similar characteristic features, and if entanglement topological invariants can be used to characterize the real time dynamics of interesting topological matter [41].

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[1] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. 80, 517 (2008).
[2] P. Calabrese and J. Cardy, J. Phys. A: Math. Theor. 42, 504005 (2009).
[3] J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys. 82, 277 (2010).
[4] E. Fradkin, Field Theories of Condensed Matter Systems (Cambridge University Press, 2013).
[5] A. Hamma, R. Ioriocciou, and P. Zanardi, Phys. Lett. A 337, 22 (2005).
[6] M. Levin and X.-G. Wen, Phys. Rev. Lett. 96, 110405 (2006).
[7] A. Kitaev and J. Preskill, Phys. Rev. Lett. 96, 110404 (2006).
[8] X.-G. Wen, Science 363, eaal3099 (2019).
[9] S. Depenbrock, I. P. McCulloch, and U. Schollwöck, Phys. Rev. Lett. 109, 067201 (2012).
[10] H.-C. Jiang, Z. Wang, and L. Balents, Nat. Phys. 8, 902 (2012).
[11] S. V. Isakov, M. B. Hastings, and R. G. Melko, Nature Physics 7, 772 (2011).
[12] D. A. Abanin and E. Demler, Phys. Rev. Lett. 109 (2012).
[13] A. J. Daley, H. Pichler, J. Schachenmayer, and P. Zoller, Phys. Rev. Lett. 109, 020505 (2012).
[14] A. Elben, B. Vermersch, M. Dalmon, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. 120, 050406 (2018).
[15] B. Vermersch, A. Elben, M. Dalmon, J. I. Cirac, and P. Zoller, Phys. Rev. A 97, 023604 (2018).
[16] H. Pichler, G. Zhu, A. Seif, P. Zoller, and M. Hafezi, Phys. Rev. X 6, 041033 (2016).
[17] M. Dalmon, B. Vermersch, and P. Zoller, Nat. Phys. 14, 827 (2018).
[18] E. Cornfeld, E. Sela, and M. Goldstein, Phys. Rev. A 99 (2019).
[19] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B 81 (2010).
[20] L. Fidkowski, Phys. Rev. Lett. 104 (2010).
[21] A. M. Turner, F. Pollmann, and E. Berg, Phys. Rev. B 83, 075102 (2011).
[22] H. Katsura, D. Schuricht, and M. Takahashi, Phys. Rev. B 92 (2015).
[23] A. Y. Kitaev, Phys.-Usp. 44, 131 (2001).
[24] C. Beenakker, Ann. Rev. Cond. Matt. Phys. 4, 113 (2013).
[25] T. Brydges, A. Elben, P. Jurcevic, B. Vermersch, C. Maier, B. P. Lanyon, P. Zoller, R. Blatt, and C. F. Roos, Science 364, 260 (2019).
[26] S. J. van Enk and C. W. J. Beenakker, Phys. Rev. Lett. 108, 110503 (2012).
[27] H. Casini and M. Huerta, Phys. Lett. B 600, 142 (2004).
[28] M. Caraglio and F. Gliozzi, JHEP 0811, 076 (2008).
[29] B. Zeng and D. L. Zhou, EPL 113, 56001 (2016).
[30] E. M. Stoudenmire, J. Alicea, O. A. Starykh, and M. P. Fisher, Phys. Rev. B 84 (2011).
[31] S. Furukawa, V. Pasquier, and J. Shiraishi, Phys. Rev. Lett. 102, 170602 (2009).
[32] See the Supplemental Material.
[33] B. M. McCoy and M.-L. Yan, Nucl. Phys. B 215, 278 (1983).
[34] C. Lacroix, P. Mendels, and F. Mila, eds., Introduction to Frustrated Magnetism (Springer Series in Solid-State Sciences Vol. 164, 2010).
[35] B. Zeng, X. Chen, D.-L. Zhou, and X.-G. Wen, Quantum Information Meets Quantum Matter (Springer New York, 2019).
[36] I. Peschel, Jour. Phys. A: Math. Gen. 36, L205 (2003).
[37] S. R. White, Phys. Rev. Lett. 69, 2863 (1992).
[38] U. Schollwöck, Rev. Mod. Phys. 77, 259 (2005).
[39] M. D. Caio, N. R. Cooper, and M. J. Bhaseen, Phys. Rev. Lett. 115, 236403 (2015).
[40] L. D’Alessio and M. Rigol, Nat. Commun. 6, 8336 (2015).
[41] M. McGinley and N. R. Cooper, arXiv:1908.06875.
[42] A. Elben, J. Yu, G. Zhu, M. Hafezi, F. Pollmann, P. Zoller, and B. Vermersch, arXiv:1906.05011 (2019).
[43] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science 338, 1604 (2012).
[44] H. Ollivier and W. H. Zurek, Phys. Revs Lett. 88, 017901 (2001).
[45] I. Frérot and T. Roscilde, Phys. Rev. B 94, 075121 (2016).
Supplemental Materials: Entanglement topological invariants for one-dimensional topological superconductors

I. ADDITIONAL INFORMATION ON THE KITAEV MODEL

In this section, we briefly present the Kitaev model, as well as the method and all the relevant formulae used to derive all the analytical results mentioned in the main text. More specifically, we focus on the regime described in Kitaev’s original paper [S1] whose algebra is simpler while containing important features on the entanglement properties of the whole model when it displays a topological phase.

A. The Kitaev model without interaction

We give here a brief reminder of the Kitaev wire of Kitaev’s seminal paper Ref. [S1] for the unfamiliar reader. The Kitaev wire is a chain of \( L \) spinless fermions with open boundary conditions described by the Hamiltonian:

\[
H = \sum_{j=1}^{L-1} \left( -t \left( a_{j}^\dagger a_{j+1} + a_{j+1}^\dagger a_{j} \right) - \mu \left( a_{j}^\dagger a_{j} - \frac{1}{2} \right) + \Delta a_{j} a_{j+1} + \Delta^* a_{j+1}^\dagger a_{j}^\dagger \right),
\]

where \( t \) is the hopping amplitude, \( \mu \) is the chemical potential, and \( \Delta = |\Delta|e^{i\theta} \) the induced superconducting gap. It is convenient to absorb the complex phase of the latter in a (completely local) redefinition of the local creation and annihilations operators \( a_{j}^\dagger \) and \( a_{j} \) such that:

\[
(a_{j}^\dagger, a_{j}) \rightarrow (e^{-i\theta/2} a_{j}^\dagger, e^{i\theta/2} a_{j}),
\]

and consider Eq. S1 with \( \Delta \) real only. It is then useful to introduce the Majorana fermions operators \( c_{j} \) (for \( j = 1, \ldots, L \)):

\[
c_{2j-1} = a_{j} + a_{j}^\dagger, \quad c_{2j} = \frac{a_{j} - a_{j}^\dagger}{i},
\]

such that:

\[
\{c_{m}, c_{l}\} = 2\delta_{m,l}, \quad c_{m}^\dagger = c_{m},
\]

where \( \delta_{m,l} \) is the Kronecker delta. The Hamiltonian Eq. S1 then becomes:

\[
H = \frac{i}{2} \sum_{j=1}^{L-1} \left( -\mu c_{2j-1} c_{2j} + (t + |\Delta|) c_{2j} c_{2j+1} + (-t + |\Delta|) c_{2j-1} c_{2j+2} \right).
\]

In the special regime of parameters when \( |\Delta| = t > 0 \) and \( \mu = 0 \) (which we call the stereotypical regime), the Hamiltonian Eq. S5 becomes:

\[
H = it \sum_{j=1}^{L-1} c_{2j} c_{2j+1},
\]

where it is important to note that the Majorana operators appearing in each term of the sum are not from the same sites. One can define then new local fermionic creation and annihilation operators on the link such that (for \( j = 1, \ldots, L-1 \) only):

\[
\tilde{a}_{j} = \frac{c_{2j} + ic_{2j+1}}{2}, \quad \tilde{a}_{j}^\dagger = \frac{c_{2j} - ic_{2j+1}}{2},
\]

that only mixes two neighbouring sites. The Hamiltonian Eq. S6 becomes diagonal:

\[
H = 2t \sum_{j=1}^{L-1} \left( \tilde{a}_{j}^\dagger \tilde{a}_{j} - \frac{1}{2} \right),
\]
and has two degenerate ground states, each pairing a Majorana fermion of one edge with a Majorana fermion of the other (cf Fig. S1). Defining the non-local operators:

\[ b = \frac{c_2L + ic_1}{2}, \quad b^\dagger = \frac{c_2L - ic_1}{2}, \]  

the two ground states \(|0\rangle\) and \(|1\rangle\) verify:

\[ \forall j \in [1, L - 1], \quad \hat{a}_j|0\rangle = 0, \quad (S10a) \]

\[ \forall j \in [1, L - 1], \quad \hat{a}_j|1\rangle = 0, \quad (S10b) \]

\[ b|0\rangle = 0, \quad (S10c) \]

\[ b^\dagger|0\rangle = |1\rangle. \quad (S10d) \]

In the case of periodic boundary conditions, \(|0\rangle\) becomes the only ground state.

![Figure S1. The Kitaev chain with \(L\) sites and open boundary conditions. Each site \(i\) (denoted by a dashed circle) can be occupied by one spinless fermion, and can be decomposed into two Majorana fermions (denoted by black dots) in \(2i - 1\) and \(2i\). Associating the Majoranas \(2i\) and \(2i + 1\) allows the construction of a new quasi local fermionic basis denoted with tildes. The ground states in the topological stereotypical regime will see its neighbouring Majorana fermions pairing up, so that, in the tilded basis, each site is unoccupied. Only the two Majorana on the edges do not need to pair up.](image)

**B. The entanglement properties of the topological phase in the stereotypical regime**

To understand the entanglement properties of this topological phase and analytically compute the disconnected entanglement entropy \(S^D\), it is useful to compute any reduced density matrices for the ground states \(|0\rangle\) and \(|1\rangle\) obtained in the stereotypical regime when \(\Delta = t > 0\) and \(\mu = 0\). To do so, it is useful to rewrite these states in the "second quantization" formalism, but in the new tilted basis where:

\[ \forall j \in [1, L - 1], \quad \hat{n}_j = \hat{a}_j^\dagger \hat{a}_j, \quad (S11a) \]

\[ n_b = b^\dagger b, \quad (S11b) \]

where the index \(b\) stands for boundary. In that case, the two ground states can be rewritten as:

\[ |0\rangle = |\tilde{n}_1 = 0, \tilde{n}_2 = 0, \ldots, \tilde{n}_{j-1} = 0, n_b = 0 \rangle, \quad (S12a) \]

\[ |1\rangle = |\tilde{n}_1 = 0, \tilde{n}_2 = 0, \ldots, \tilde{n}_{j-1} = 0, n_b = 1 \rangle, \quad (S12b) \]

which is a quasi local basis in the sense that each \(\tilde{n}_j\) can be expressed in terms of operators acting only on sites \(j\) and \(j + 1\). Hence, for a connected bipartition of the system in \(A\) and \(B\) (see Fig. S2), the expressions of the ground states Eq. S12 would already be separable, if it was not for the edges and for the link cut by the partition (taken on site \(c \in [1, L - 1]\)).

To properly do the partial trace and obtain the reduced density matrix \(\rho_A\), it is best to express the ground states in terms of a local basis for both \(A\) and \(B\). This becomes possible when rewriting the two parts of the open Kitaev wire as two open Kitaev wires connected into a singlet on the link \(c\). Calling \(L_A\) the size of \(A\), and \(L_B\) the size of \(B\), such that \(L_A + L_B = L\), we define
Figure S2. A physical cut can only be done between sites, here, on the link \( c \), partitioning the chain into two subsets: \( A \) and \( B \).

A new fermionic basis, local in \( A \) and \( B \):

\[
\begin{align*}
  a_A &= \frac{1}{2} (c_{2L_A} + ic_1), \\
  a_B &= \frac{1}{2} (c_{2L_A+2L_B} + ic_{2L_A+1}), \\
  \tilde{a}_c &= \frac{1}{2} (c_{2L_A} + ic_{2L_A+1}), \\
  b &= \frac{1}{2} (c_{2L} + ic_1),
\end{align*}
\]

so that \( a_A \) and \( a_B \) (and hermitian conjugate) act as boundary operators for the subchains \( A \) and \( B \) respectively. Hence, in second quantization, and after dropping the redundant mentions of the \( \tilde{n}_j \), \( j \in [1, L - 1] \setminus \{ c = L_A \} \), the two ground states of the full chain are \( |n_b = 0, \tilde{n}_c = 0 \rangle \) and \( |n_b = 1, \tilde{n}_c = 0 \rangle \). The local basis of \( A \) and \( B \) is \( \{ |n_A, n_B \rangle \} \) where \( n_A = a_A^\dagger a_A \) and \( n_B = a_B^\dagger a_B \) take the values 0 or 1. Using Eqs. S10 and Eqs. S13, we find:

\[
\begin{align*}
  (|0\rangle = |n_b = 0, \tilde{n}_c = 0 \rangle) &= -\frac{1}{\sqrt{2}} \left( |n_A = 1, n_B = 0 \rangle - |n_A = 0, n_B = 1 \rangle \right), \\
  |n_b = 1, \tilde{n}_c = 1 \rangle &= \frac{1}{\sqrt{2}} \left( |n_A = 1, n_B = 0 \rangle + |n_A = 0, n_B = 1 \rangle \right), \\
  (|1\rangle = |n_b = 1, \tilde{n}_c = 0 \rangle) &= \frac{1}{\sqrt{2}} \left( |n_A = 0, n_B = 0 \rangle + |n_A = 1, n_B = 1 \rangle \right), \\
  |n_b = 0, \tilde{n}_c = 1 \rangle &= \frac{1}{\sqrt{2}} \left( -|n_A = 0, n_B = 0 \rangle + |n_A = 1, n_B = 1 \rangle \right).
\end{align*}
\]

Tracing over \( B \) from there is immediate, as the only vectors of the basis of \( B \) with possible non zero contributions are \( |n_B = 0 \rangle \) and \( |n_B = 1 \rangle \). In particular:

\[
\rho_A (|n_b = 0, \tilde{n}_c = 0 \rangle \langle n_b = 0, \tilde{n}_c = 0 |) = \frac{1}{2} \left( |n_A = 1 \rangle \langle n_A = 1 | + |n_A = 0 \rangle \langle n_A = 0 | \right),
\]

of entanglement entropy \( S_A = \log 2 \). The same happens for the other ground state.

Notice that the partial trace for fermions can induce a change of sign compared to the bosonic case. For example:

\[
\text{Tr}_B (|n_A = 1, n_B = 1 \rangle \langle n_A = 0, n_B = 1 |) = -|n_A = 1 \rangle \langle n_A = 0 |.
\]

Using Eqs. S14, it is possible to get the expressions of the ground states in the local basis of an arbitrary partition. Additionally, taking a partition where all sites are their own individual subsets leads to the expression of the ground states in the original basis, up to a phase. The general expression of the reduced density matrix for an arbitrary partition of the system is obtained recursively, by considering the partition \( A_1, A_2, \ldots, A_n \) of connected subsets \( A_i \) that are next to each others like in Fig. S3. Calling \( A = A_1 \) and \( B = B_1 = \bigcup_{i=2}^n A_i \), allow use of Eqs. S14 to express the two ground states in the local basis of \( A \) and \( B \) instead of \( A \cup B \). The recurrence follows. Naming \( c_i \) the link between subsets \( A_i \) and \( A_{i+1} \) and constructing the “local boundary operators” \( a_{A_i} \) and \( a_{A_i}^\dagger \) for the subset \( A_i \) and \( a_{B_j} \) and \( a_{B_j}^\dagger \) for \( B_j = \bigcup_{i=j+1}^n A_i \), similarly to Eqs. S14, the recurrence can be
Figure S3. A partition of the chain into $n$ consecutive connected subsets $A_1, A_2, ..., A_n$.

written as:

$$\forall n \geq 2, \quad u_n (A_1, \ldots, A_n) \equiv |n_b = 0, \tilde{n}_{c_1} = 0, \ldots, \tilde{n}_{c_{n-1}} = 0\rangle (\equiv |0\rangle)$$  
(S17a)

$$= \frac{1}{\sqrt{2}} \left( |n_{A_1} = 0, n_{B_1} = 1, \tilde{n}_{c_2} = 0, \ldots, \tilde{n}_{c_{n-1}} = 0\rangle - |n_{A_1} = 1, n_{B_1} = 0, \tilde{n}_{c_2} = 0, \ldots, \tilde{n}_{c_{n-1}} = 0\rangle \right)$$  
(S17b)

$$= \frac{1}{\sqrt{2}} (|n_{A_1} = 0\rangle \otimes u_{n-1} (A_2, \ldots, A_n)$$

$$- |n_{A_1} = 1\rangle \otimes u_{n-1} (A_2, \ldots, A_n)), \quad (S17c)$$

$$\forall n \geq 2, \quad v_n (A_1, \ldots, A_n) \equiv |n_b = 1, \tilde{n}_{c_1} = 0, \ldots, \tilde{n}_{c_{n-1}} = 0\rangle$$  
(S17d)

$$= \frac{1}{\sqrt{2}} \left( |n_{A_1} = 0, n_{B_1} = 0, \tilde{n}_{c_2} = 0, \ldots, \tilde{n}_{c_{n-1}} = 0\rangle (\equiv |1\rangle)$$

$$+ |n_{A_1} = 1, n_{B_1} = 1, \tilde{n}_{c_2} = 0, \ldots, \tilde{n}_{c_{n-1}} = 0\rangle \right)$$  
(S17e)

$$= \frac{1}{\sqrt{2}} (|n_{A_1} = 0\rangle \otimes u_{n-1} (A_2, \ldots, A_n)$$

$$+ |n_{A_1} = 1\rangle \otimes u_{n-1} (A_2, \ldots, A_n)). \quad (S17f)$$

Calling:

$$U_n = u_n + iv_n, \quad (S18a)$$

$$V_n = u_n - iv_n, \quad (S18b)$$

$$|+j\rangle = 1/\sqrt{2} \left( i|n_{A_j} = 0\rangle - |n_{A_j} = 1\rangle \right), \quad (S18c)$$

$$|−j\rangle = 1/\sqrt{2} \left( -i|n_{A_j} = 0\rangle - |n_{A_j} = 1\rangle \right), \quad (S18d)$$

these relations become:

$$U_n (A_1, \ldots, A_n) = |+1\rangle \otimes V_{n-1} (A_2, \ldots, A_n)$$  
(S19a)

$$= \left\{ \begin{array}{ll} |+1\rangle \otimes U_2 (A_{n-1}, A_n) & \text{if } n \text{ is even} \\ |+1\rangle \otimes V_2 (A_{n-1}, A_n) & \text{if } n \text{ is odd} \end{array} \right.$$  
(S19b)

$$V_n (A_1, \ldots, A_n) = |−1\rangle \otimes U_{n-1} (A_2, \ldots, A_n)$$  
(S19c)

$$= \left\{ \begin{array}{ll} |−1\rangle \otimes V_2 (A_{n-1}, A_n) & \text{if } n \text{ is even} \\ |−1\rangle \otimes U_2 (A_{n-1}, A_n) & \text{if } n \text{ is odd} \end{array} \right.$$  
(S19d)

where:

$$u_2 (A_{n-1}, A_n) = \frac{1}{\sqrt{2}} \left( |n_{A_{n-1}} = 0, n_{A_n} = 1\rangle - |n_{A_{n-1}} = 1, n_{A_n} = 0\rangle \right), \quad (S20a)$$

$$v_2 (A_{n-1}, A_n) = \frac{1}{\sqrt{2}} \left( |n_{A_{n-1}} = 0, n_{A_n} = 0\rangle + |n_{A_{n-1}} = 1, n_{A_n} = 1\rangle \right), \quad (S20b)$$

so that:

$$U_2 (A_{n-1}, A_n) = \sqrt{2i}|+n-1 - n\rangle, \quad (S21a)$$

$$V_2 (A_{n-1}, A_n) = \sqrt{2i}|−n-1 + n\rangle. \quad (S21b)$$
Therefore, in a local basis of $A_1, \ldots, A_n$, the ground states become:

$$|n_b = 0\rangle = u_n (A_1, \ldots, A_n), \quad (S22a)$$

$$= \frac{i}{\sqrt{2}} (|+1 -2 \ldots \rangle + |-1 +2 \ldots \rangle), \quad (S22b)$$

$$|n_b = 1\rangle = v_n (A_1, \ldots, A_n), \quad (S22c)$$

$$= \frac{i}{\sqrt{2}} (|+1 -2 \ldots \rangle - |-1 +2 \ldots \rangle). \quad (S22d)$$

These states are not Néel states because they are made out of fermions. It becomes clear in the basis of the subsets $\{\otimes|n_{i_j}\rangle\}_j$ up to the global phase change:

$$|0_{j}\rangle \doteq (-1)^j|0_{A_j}\rangle, \quad \text{and} \quad |1_{j}\rangle \doteq |1_{A_j}\rangle. \quad (S23)$$

In that case:

$$|+1 -2 \ldots \rangle = \left(-\frac{1}{\sqrt{2}}\right)^n \otimes_{i=1}^n [|\tilde{0}_i\rangle + |\tilde{1}_i\rangle] \quad (S24a)$$

$$= \left(-\frac{1}{\sqrt{2}}\right)^n \sum_{\{n_{A_i}\}_{i \in [1,n]}=0,1} |\{n_{A_i}\}_{i \in [1,n]}\rangle \quad (S24b)$$

$$|-1 +2 \ldots \rangle = \left(-\frac{1}{\sqrt{2}}\right)^n \sum_{\{n_{A_i}\}_{i \in [1,n]}=0,1} (-1)^{n-\sum n_{A_i}}|\{n_{A_i}\}_{i \in [1,n]}\rangle \quad (S24c)$$

so that:

$$S_{A_1 \cup A_2 \cup A_3 \cup \ldots} = \left\lfloor \frac{n+1}{2} \right\rfloor \log 2, \quad (S25)$$

where $\lfloor \ldots \rfloor$ is the integer subset. This last equation proves the exact additivity of the entropy in this case, which, in addition to the non-nullity of the contribution of the individual subsets, ensure the non-nullity of $S^D$ for any superposition of the ground states. Indeed, for $n = 4$:

$$S^D = S_{A_1 \cup A_2} + S_{A_2 \cup A_4} - S_{A_2} - S_{A_1 \cup A_2 \cup A_4}, \quad (S26)$$

becomes the net contribution of one connected subset only: $\log 2$. Alternatively, it is the contribution of two cut Bell pairs of Majorana fermions. This result is valid for both the von Neumann and the Rényi entropies.

**C. Comparison between the topological and non-topological phases away from the phase transition**

In the topological superconductor phase, away from any phase transition, both the von Neuman and Rényi entanglement entropies for large enough subsets are non zero and additive. If A is a simply connected subset of a partition of the chain (independent of its position), then, $S_{A} = 2\Gamma$, where $\Gamma$ is the contribution of one (Majorana) Bell pair. Eq. S26 then gives:

$$S^D = 2\Gamma + 2 \times 2\Gamma - 2\Gamma - 2\Gamma = 2\Gamma \quad (S27)$$

In that regard, $S^D$ is not unique: combination such as $S_{A_1 \cup A_3} + S_{A_2 \cup A_4} - S_{A_1 \cup A_4} - S_{A_2 \cup A_3}$ would have also work as detectors, but with less experimental relevance, and a more complicated interpretation in terms of mutual information. Furthermore, if the $\Gamma$ obtained from the von Neumann entropy is non zero, so will the $\Gamma'$ obtained from the Rényi entropy and vice-versa (property of minimum value). In addition, if the von Neumann entanglement entropy does not diverge with the size of the system (as is always the case for 1D gapped systems), neither will the Rényi entropy (consequence of the property of monotonicity). In conclusion, the von Neumann and the Rényi entropy can be used interchangeably for 1D gapped systems. The equality $\Gamma = \Gamma'$ for the Kitaev wire is not a generic feature for other systems.

For the non-topological band insulator phase, each individual entanglement entropy goes to zero for large enough subsets, and so is $S^D$.

For the case of a gapped phase displaying ground state equivalent to e.g. a maximally entangled Néel state, the entanglement entropy becomes non zero for each term, but is not additive, such that (calling $\Gamma$ the contribution of the only Bell pair of spin 1/2):

$$S^D = \Gamma + \Gamma - \Gamma - \Gamma = 0. \quad (S28)$$
For a critical phase with a non zero scaling entanglement entropy, the latter is no more additive in simply connected subsets. More specifically, if e.g. $S_{A_i} = \alpha \log L_{A_i} + o(1)$ where $L_{A_i}$ is the size of the subset $A_i$, and $o(1)$ is a negligible correction when $L_{A_i} \to \infty$, then:

$$S^D = \alpha \log \left( \frac{(L_{A_1} + L_{A_2})(L_{A_2} + L_{A_3})}{L_{A_2}(L_{A_1} + L_{A_2} + L_{A_3})} \right) + o(1),$$

(S29)

which can take any value (above $\log 2$) depending on the ratio of $L_{A_1}$, $L_{A_2}$ and $L_{A_3}$. $S^D$ is then not quantized, and not even well-defined.

Finally, for the case of a rigorously dimerized phase, $S^D$ is well-defined and can be considered additive, but is not translation invariant. More precisely, let us define $\epsilon_{ij}$, where $i, j = 1, 2, 3, 4$, such that $\epsilon_{ij} = 1$ if the cut between $A_i$ and $A_j$ is on a dimer, and $\epsilon_{ij} = 0$ otherwise. Then, if $\Gamma$ is the contribution of one dimer:

$$S^D = \Gamma (\epsilon_{41} + \epsilon_{23} + \epsilon_{12} + \epsilon_{23} + \epsilon_{34} + \epsilon_{41} - \epsilon_{12} - \epsilon_{23} - \epsilon_{34}) = 2\Gamma \epsilon_{41},$$

(S30)

which is always zero if $A_1$ and $A_4$ are subsets at both ends of the open chain. $S^D$ is not translation invariant in the case of periodic boundary conditions.

D. Phase diagram of the Kitaev wire without interaction $(U = 0)$

In addition to the phase diagram of the Kitaev wire with interaction (but for $t = |\Delta| > 0$), we also get the phase diagram when $U = 0$ (and varying $\mu/t > 0$ and $|\Delta|/t > 0$) as displayed Fig. S4. The phase transition is also well captured by $S^D$.

![Phase diagram](image)

Figure S4. $S^D$ for the Kitaev wire without interaction $U = 0$ as one vary $\mu/t > 0$ and $|\Delta|/t > 0$ obtained by the free fermion technique. The figure is extrapolated from an array of data of $25 \times 300$. The two theoretically expected phase transitions occur when $|\Delta| = 0$ and $\mu/t = 2$. Using the definitions of Fig.1a) of the main text, $L = 50$, $L_A = L_B = 12$ and $L_D = 32$.

II. ADDITIONAL INFORMATION ON THE NUMERICAL METHODS

A. Computation using free-fermion correlation functions at equilibrium

The main challenge of the numerical analysis consists in calculating the four von Neumann entanglement entropies that are included in the definition of the disconnected topological entanglement entropy $S^D$ of Eq. S26. We recall that each of these four
quantities is associated with a specific possibly disconnected bipartition of the chain, following the scheme shown in Fig. 1a) of the main text. The starting point for evaluating von Neumann or Rényi entanglement entropy $S_A$ of a generic bipartition $A$ and $B$ (not necessarily simply connected) is the computation of the reduced density matrix $\rho_A = \text{Tr}_B |\psi\rangle \langle \psi|$ where $|\psi\rangle$ is a ground state of the whole system. For the Kitaev wire without interactions, i.e. when $U = 0$, the Hamiltonian Eq. S1 is a free-fermion Hamiltonian that only conserves fermion number parity. It can be diagonalized by a Bogoliubov transformation. The reduced density matrix $\rho_A$ of a generic partition can be interpreted as the thermal density matrix at temperature $T = 1$ for an entanglement Hamiltonian $\mathcal{H}_A$. Then $\rho_A = Z_A^{-1} e^{-\mathcal{H}_A}$ with $Z_A = \text{Tr} [e^{-\mathcal{H}_A}]$ [S2–S4]. It is then possible to compute $\rho_A$ of a generic partition by following the well-established approach of Ref. [S5]. The first step of this very general procedure is the computation of the correlation matrices in the original ground-state: $C_{nm} = \langle \psi | a_n^\dagger a_m | \psi \rangle$ and $F_{nm} = \langle \psi | a_n^\dagger a_m^\dagger | \psi \rangle$ where $n$ and $m$ run over the sites of the subset $A$. At equilibrium, $F_{nm}$ is real, so that the spectrum of the entanglement Hamiltonian can be determined by numerically solving the eigenvalue problem [S5]:

$$
\left(2\hat{C} - 2\hat{F} - 1\right) \left(2\hat{C} + 2\hat{F} - 1\right) \phi_l = \tanh^2 \left(\frac{\epsilon_l}{2}\right) \phi_l.
$$

$\epsilon_l$ are the eigenvalues of the entanglement Hamiltonian with eigenvectors $\phi_l$. Once the spectrum $\epsilon_l$ is obtained, we can easily compute the reduced density matrix $\rho_A$ and the entanglement entropy of any bipartition, connected or not.

We carried out these numerical calculations with arbitrary precision by using mpmath Python library [S6]. Two requirements arise: avoiding numerical precision problems when the eigenvalues of the left side of Eq. S31 are approximately 0 and avoiding divergence problems when they are very close to 1. These issues are fixed by taking a number of digits proportional to the total size of the system $L$: we set mp.dps = $20 \times L$ (number of digits in the Python library). While the approach allows generation of a lot of data with relatively little cost, the interaction case, as well as the bilinear biquadratic model remain inaccessible with this algorithm. Instead, we have to switch to the Density Matrix Product State (DMRG) technique presented Sec. II C.

B. Sudden quenches using free-fermion correlation functions

We check that $S^D$ is a topological invariant by looking at its time evolution after sudden quenches. The system starts in its ground state before the quench, for a given set of parameters of the Hamiltonian. At $t = 0$ we change the value of one of these parameters. We then let the system evolve in time.

The generic interacting case is very challenging to follow because the interacting term induces a time evolution of an extensive number of eigenstates of the spectrum. For the free-fermion case, the same previous numerical approach of Sec. II A still applies with the same efficiency. The matrix $F_{nm}$ is however complex during the time evolution of the system so that Eq. S31 is not valid anymore. We follow instead Ref. [S7] and use the definition of the Majorana fermions of Eq. S2. The relevant $2L \times 2L$ correlation matrix is this time $M_{nm} = \langle c_n c_m \rangle$, rewritten as $M_{nm} = \delta_{nm} + \Gamma_{nm}$ with $n, m$ the (half-)site indices spanning only the relevant subset studied. The eigenvalues of $\Gamma_{nm}$ are $\pm \tanh \epsilon_l / 2$, with $\epsilon_l$ the entanglement energies of the reduced density matrix associated with the subset. The time dependent matrix $\Gamma_{nm}$ is directly linked to the time dependent Hamiltonian analytically, and then diagonalized numerically, similarly to Ref. [S8].

We performed several quenches of different amplitudes inside both the TSC and the band insulator, as well as across the phase transition, as seen in Fig. S5. $S^D$ is plotted as a function of time for different chain lengths $L$ with $L/L_A = 4, L_A = L_B$, using the same definition of Fig. 1a) of the main text for the partitions. a), b) and c) exhibits a quantized finite plateau depending on $L_A$. We define the time-size of the plateau with $t_c$: the time at which $S^D$ deviates from log 2 to 0.95 log 2. $t'_c$ is time of revival, when $S^D$ increases from 0 to 0.05 log 2. We plot $1/t_c$ or $4/t'_c$ in d), e) and f) as a function of $1/L$. The linear extrapolations of the length of the plateau show divergences of both time-size when $L \to \infty$ for each kind of quench: $S^D$ stays quantized in the thermodynamical limit because the drops/revivals are only finite size effects. Thus, $S^D$ does not change under coherent time evolution and behaves as a topological invariant.

We used the same implementation as in Sec. II A with mp.dps = $3 \times L$, enough to get reliable results.

C. Computation using DMRG

For the case of the interacting Kitaev wire with $U \neq 0$ (Eq. 2 of the main text), the numerical study is now carried out with DMRG algorithm formulated in the matrix product state (MPS) language [S9].

The main challenge is again the computation of von Neumann or Rényi entanglement entropies of disconnected bipartitions of the chain in order to determine the behavior of $S^D$ of Eq. S26. For DMRG however, only the standard connected bipartition of the MPS state into two halves is very efficient. This efficiency comes from the intimate structure of an MPS state and the use of left/right-orthogonality condition [S9]. The MPS tensors themselves already give the eigenstates of the reduced density matrix of a block of sites starting from either the left or the right edge of the system. Instead, for generic bipartitions of the
Figure S5. (Color online) Time evolution of $S^D$ after a quantum quench from a) $\Delta_0 = 0.5$ to $\Delta = 1.5$, from b) $\mu_0 = 1.0$ to $\mu = 3.0$ and from c) $\mu_0 = 3.0$ to $\mu = 1.0$ with $U = 0, \mu = 0, L = 4L_A$. Finite-size scaling of $t_c$ or $t'_c$ for two values of d) $\Delta$, e) $\mu_0$ and f) $\mu$ of the quenched Hamiltonian. In all cases, the width of the plateau diverges linearly with system size, as expected for topological invariants. The threshold lines of $S^D$ according to the definition of $t_c$ are depicted as a dashed line.

chain, such as partition $B$, $A \cup B$ or $A \cap B$ of Fig. 1a) of the main text, the calculation of the reduced density matrix is highly non-trivial as it involves several multi-index tensor contractions. As a result, the computational cost scales exponentially with the size of the subset.

We circumvent this numerical problem by reordering the MPS sites appropriately, thus considering a long-range Hamiltonian reproducing the original model. Two rearrangements are necessary, as shown in Fig. S6. (i) is a circular permutation of the original multi-partitions to efficiently calculate $S(\rho_A)$, $S(\rho_{A\cup B})$ and $S(\rho_{A\cap B})$ because all these quantities are now related to right-most bipartitions of the chain. (ii) allows computation of $S(\rho_B)$. In this way, we efficiently obtain the four entanglement entropies composing $S^D$.

In our numerical analysis with DMRG algorithm, we used a bond dimension up to 1200, a truncation error of $10^{-8}$ and at least 30 sweeps to reach the convergence and to ensure stability of our findings.

[S1] A. Y. Kitaev, Phys.-Usp. 44, 131 (2001).
[S2] I. Peschel and M.-C. Chung, Jour. Phys. A: Math. Gen. 32, 8419 (1999).
[S3] M.-C. Chung and I. Peschel, Phys. Rev. B 62, 4191 (2000).
[S4] M.-C. Chung and I. Peschel, Phys. Rev. B 64, 064412 (2001).
[S5] I. Peschel, Journal of Physics A: Mathematical and General 36, L205 (2003).
[S6] Fredrik Johansson and others. mpmath: a Python library for arbitrary-precision floating-point arithmetic, http://mpmath.org/.
[S7] I. Peschel and V. Eisler, Jour. Phys. A: Math. Theo. 42, 504003 (2009).
[S8] M. Fagotti and P. Calabrese, Phys. Rev. A 78, 010306 (2008).
[S9] U. Schollwöck, Ann. of Phys. 326, 96 (2011), january 2011 Special Issue.
Figure S6. Scheme of the partitions. Two “tricks” are used to compute only standard bipartition of the chain: (i) a circular permutation of the partition that leaves the entanglement entropies invariant at the thermodynamical limit and (ii) re-indexing the sites, thus making the Hamiltonian long-range, but making the subset $B$ connected.