Scalar-tensor theory with enhanced gravitational effects.

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Abstract. It is shown that a Brans-Dicke scalar-tensor gravitational theory, which also includes Bekenstein’s kind of interaction between the Maxwell and scalar fields, has a particular kind of solutions with highly enhanced gravitational effects as compared with General Relativity, prone to laboratory tests.

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1. Introduction

Scalar-tensor (ST) gravitational theories are attractive candidates for extensions of General Relativity (GR). Part of their appeal comes from the fact that they are induced naturally in the reduction to four dimensions of string and Kaluza-Klein models[1], resulting mostly in the form of a Brans-Dicke (BD) type of ST theory[2], often involving also non-minimal coupling to matter, leading to weak equivalence principle violations. Also, ST theories are possibly the simplest extension of GR that could accommodate cosmological issues as inflation and universe-expansion acceleration, as well as possible space-time variation of fundamental constants[3]. On the other hand, observational and experimental evidence puts strong limits to the observable effects of a possible scalar field. For example, in the case of a massless scalar the BD parameter $\omega$ is constrained by careful Solar-System experiments to be a large number ($\omega > 5 \times 10^5$)[4]. This has not been favorable for the ST theories, although it was found that some Kaluza-Klein models allow ST theories with arbitrarily high values of $\omega$[5], and a ”least-coupling principle” was proposed in string theories to the same effect[6]. Closely related to the ST theories is the question of the space-time variation of fundamental constants. The proposal of models compatible with all known constraints are basically of the Bekenstein’s type[3], involving the coupling of the electromagnetic field to a scalar field[7],[8]. The resulting violations of the equivalence principle are still below the actual experimental limits, either in the propagation of light[9], or in Eötvös experiments. The purpose of the present work is to show that a ST theory of the BD type, that also includes a direct interaction between the Maxwell and the scalar fields, as in Bekenstein’s theory, has a special kind of solutions where the metric effect of the electromagnetic field is strongly enhanced, as compared to the case of GR. This enhanced effect could be observed with relative ease in laboratory experiments with present technology.

2. ST theory

We will consider a ST theory of the BD type with inclusion of a Bekenstein’s direct interaction of scalar and Maxwell fields, with action given by (SI units are used)

$$S = \frac{c^3}{16\pi G_0} \int \sqrt{-g} \phi R d\Omega - \frac{c^3}{16\pi G_0} \int \sqrt{-g} \frac{\omega(\phi)}{\phi} \nabla^l \phi \nabla^l \phi d\Omega$$

$$- \frac{\varepsilon_0 c}{4} \int \sqrt{-g} \lambda(\phi) F_{lm} F^{lm} d\Omega - \frac{1}{c} \int \sqrt{-g} j^l A_l d\Omega - S_{\text{mat}}. \tag{1}$$

In order to have a non-dimensional scalar field $\phi$ of values around unity, in expression[11] the constant $G_0$ representing Newton gravitational constant is included, $c$ is the velocity of light in vacuum, and $\varepsilon_0$ is the vacuum permittivity. $S_{\text{mat}}$ is the non-electromagnetic action of matter. The other symbols are also conventional, $R$ is the Ricci scalar, and $g$ the determinant of the metric tensor $g_{lm}$. The Brans-Dicke parameters $\omega(\phi)$ is considered a function of $\phi$, as it usually results so in the reduction to four dimensions of multidimensional theories. The function $\lambda(\phi)$ in the term of the
action of the electromagnetic field is of the type appearing in Bekenstein’s theory. The electromagnetic tensor is \( F_{lm} = \nabla_l A_m - \nabla_m A_l \), given in terms of the electromagnetic quadri-vector \( A_m \), with sources given by the quadri-current \( j^l \).

Variation of (1) with respect to \( g^{lm} \) results in \((T^E_{lm} \text{ is the usual electromagnetic energy tensor})\)

\[
\phi \left( R_{lm} - \frac{1}{2} R g_{lm} \right) = \frac{8 \pi G_0}{c^4} \left[ \lambda (\phi) T^E_{lm} + T^{mat}_{lm} \right] - \nabla^k \nabla_k \phi g_{lm} + \nabla_l \nabla_m \phi
+ \frac{\omega (\phi)}{\phi} \left( \nabla_l \phi \nabla_m \phi - \frac{1}{2} \nabla^k \nabla_k \phi g_{lm} \right). \tag{2}
\]

Variation with respect to \( \phi \) gives

\[
\phi R + \left( \frac{d\omega}{d\phi} - \frac{\omega}{\phi} \right) \nabla^l \phi \nabla_l \phi + 2 \omega \nabla^l \phi = \frac{4 \pi G_0 \varepsilon_0}{c^2} \frac{d \lambda}{d \phi} F_{lm} F^{lm},
\]

which can be rewritten, using the contraction of (2) with \( g^{lm} \) to replace \( R \), as

\[
(2 \omega + 3) \nabla^l \phi \nabla_l \phi + \frac{d\omega}{d\phi} \nabla^l \phi \nabla_l \phi = \frac{4 \pi G_0 \varepsilon_0}{c^2} \frac{d \lambda}{d \phi} F_{lm} F^{lm} + \frac{8 \pi G_0}{c^4} T^{mat}, \tag{3}
\]

where it was used that \( T^E = T^E_{lm} g^{lm} = 0 \).

Finally, the non-homogeneous Maxwell equations are obtained by varying (1) with respect to \( A_m \),

\[
\nabla_l \left[ \lambda (\phi) F^l_{lm} \right] = \mu_0 j^m. \tag{4}
\]

with \( \mu_0 \) the vacuum permeability.

Having included \( G_0 \), it is understood that \( \phi \) takes values around \( \phi_0 = 1 \). Eqs. (2)-(4) with \( \lambda (\phi) = 1 \) represent the original Brans-Dicke system if \( \omega \) is taken as constant.

3. Special solutions

A rapid inspection of Eqs. (2) and (3) shows that corrections to the metric generated by \( T^{mat}_{lm} \) alone are in general quantified by \( \delta R \sim G_0 \varepsilon_0 F_{lm} F^{lm}/c^2 \), as in GR. However, for those solutions satisfying \( \nabla^l \nabla_l \phi = 0 \), one has instead \( \delta R \sim \sqrt{G_0 \varepsilon_0 F_{lm} F^{lm} (d\lambda/d\phi)} / (c^2 d\omega/d\phi) / L \), with \( L \) a characteristic length-scale of variation of the fields. In this way, depending on how precisely the condition \( \nabla^l \nabla_l \phi = 0 \) holds, and on the possible value of \( (d\lambda/d\phi) / (d\omega/d\phi) \), the system of equations considered allows solutions with enhanced gravitational effects. We now analyze some cases where this kind of possible solutions of the system (2)-(4) exist, in the weak field limit of the equations.

In the weak field approximation, for variations of \( g_{lm} \) around the values \( \eta_{lm} \) taken as those of flat Minkowsky space with signature \((1,-1,-1,-1)\), so that \( g_{lm} = \eta_{lm} + h_{lm} \), and of \( \phi \) around \( \phi_0 = 1 \), so that \( \phi = 1 + \sigma \), we have

\[
R_{lm} - \frac{1}{2} R \eta_{lm} = \frac{1}{2} \left( -\eta^{ik} \partial_{ik} \bar{h}_{lm} + \partial_{il} \bar{h}^l_i + \partial_{lm} \bar{h}^l + \eta_{lm} \partial_{ik} \bar{h}^{ik} \right),
\]
with
\[ h_{\ell m} \equiv h_{\ell m} - \frac{1}{2} h\eta_{\ell m}, \]
where
\[ h \equiv \eta^{i k} h_{i k} = -\eta^{i k} h_{i k}. \]
The system (2)-(4) can be written to lowest order in the perturbations satisfying
\[ \nabla^l \nabla^l \phi = 0 \text{ (which should hold at least up to order two in the perturbed fields)}, \]
\[ \partial_{i k} \sigma \left( \eta^{i k} + \frac{\eta^{i k} h}{2} \right) + \partial_{l} \sigma \left( \partial_{l} h_{i k} + \eta^{i k} \partial_{l} h \right) = 0. \]

While the equation \( \nabla^l \nabla^l \phi = 0 \) is written up to second order as
\[ \partial_{i k} \sigma \left( \eta^{i k} + h_{i k} \right) + \partial_{l} \sigma \left( \partial_{l} h_{i k} + \eta^{i k} \partial_{l} h \right) = 0. \]

The raising and lowering of indices are effected by the tensors \( \eta^{i k} \) and \( \eta_{i k} \) respectively. Even this much simpler system is rather complex. From a practical point of view it is convenient to restrict the choice of coordinates by requiring the additional four conditions on \( h_{\ell m} \)
\[ \partial_{i} h_{i k} + \eta^{i k} \partial_{i} h = -\partial_{\ell m} \sigma \partial_{\ell m} h_{i k} \partial_{i} \sigma, \]
so that Eq. (8) reduces to
\[ \eta^{i k} \partial_{i k} \sigma = 0. \]

The equations (5)-(7), together with (9) and (10), constitute an overdetermined set of equations which has solutions only for particular cases. As it stands, the system can be solved independently for the electromagnetic field given the sources \( j^m \) in (7), and then determine \( \phi \) using (6) and (10), to finally obtain \( h_{\ell m} \) from Eqs. (5) and (9).

In terms of the modulus of the electric and magnetic vector fields, \( E \) and \( B \), respectively, one has
\[ F_{\ell m} F^{\ell m} = 2 \left( B^2 - E^2/c^2 \right), \]
so it is immediately seen from Eq. (9) that a possible solution exists in the case of static fields outside their sources. In effect, consider for instance the case of a pure electrostatic field, so that, using greek indices for the three spatial coordinates, one has for the electrostatic potential \( \varphi \),
\[ E_{\alpha} = -\partial_{\alpha} \varphi, \]
with
\[ \partial_{\alpha \alpha} \varphi = 0, \]
outside the electric sources. In this static case Eq. (6) is written as
\[ \partial_\alpha \sigma \partial_\alpha \sigma = \frac{8\pi G_0 \varepsilon_0 (d\lambda/d\omega)_{\phi_0}}{c^4} \partial_\alpha \varphi \partial_\alpha \varphi, \]
where we have written \((d\lambda/d\omega)_{\phi_0} \equiv (d\lambda/d\phi)_{\phi_0} / (d\omega/d\phi)_{\phi_0}\). In this way, solutions also satisfying Eq. (10) then exist if \((d\lambda/d\omega)_{\phi_0} > 0\), given by
\[ \partial_\alpha \sigma = \pm K \sqrt{\frac{8\pi G_0 \varepsilon_0 (d\lambda/d\omega)_{\phi_0}}{c^4}} \partial_\alpha \varphi, \]
(11)
where
\[ K \equiv \sqrt{\frac{8\pi G_0 \varepsilon_0 (d\lambda/d\omega)_{\phi_0}}{c^2}}. \]
(12)

If \((d\lambda/d\omega)_{\phi_0} < 0\) one has solutions instead only for a magnetostatic field outside its sources, so that \(B_\alpha = \partial_\alpha \psi\), with \(\partial_\alpha \alpha \psi = 0\), given by
\[ \partial_\alpha \sigma = \pm cK \partial_\alpha \psi. \]

For clarity sake we can assume that \((d\lambda/d\omega)_{\phi_0} > 0\), as all derivations are totally analogous in the case \((d\lambda/d\omega)_{\phi_0} < 0\), with \(c\psi\) taking the place of \(\varphi\).

Gravity effects can be quantified in the weak field approximation as a three-dimensional force per unit mass, given in the static case considered by
\[ f_\beta = -\frac{c^2}{2} \partial_\beta h_{00} = -\frac{c^2}{4} \partial_\beta (\overline{h}_{00} + \overline{h}_{\alpha\alpha}). \]
(13)

From Eqs. (11) and (9) we can write the equations for \(\overline{h}_{00}\) and \(\overline{h}_{\alpha\beta}\) as
\[ \partial_\gamma \overline{h}_{00} = H, \]
(14)
\[ \partial_\gamma \overline{h}_{\alpha\beta} = 2\partial_\alpha\beta \sigma + \partial_\alpha \gamma \overline{h}_{\beta\gamma} + \partial_\beta \gamma \overline{h}_{\alpha\gamma} - \delta_{\alpha\beta} H, \]
(15)
where \(\delta_{\alpha\beta}\) is Kroenecker delta, and
\[ H \equiv \partial_{ik} \overline{h}^k = \partial_\gamma \left( \frac{\partial_{\alpha\beta} \sigma \overline{h}_{\alpha\beta}}{3\partial_{\delta\sigma} \partial_{\delta\sigma} \partial_\gamma \sigma} \right). \]
(16)

In particular, one has \(\partial_\gamma \overline{h}_{\alpha\alpha} = -H\), which indicates that the effective potential \(c^2 \left(\overline{h}_{00} + \overline{h}_{\alpha\alpha}\right) / 4\) is harmonic. An estimation of the expected force for a solution of the type (11) is
\[ |f_\beta| \sim \sqrt{8\pi G_0 \varepsilon_0 (d\lambda/d\omega)_{\phi_0}} E. \]
(17)

Another simple case where the particular solutions considered can exist is for propagating electromagnetic fields in vacuum, not dependent on a Cartesian coordinate, which we denote as \(z\). In effect, solenoidal electric and magnetic fields can be represented in that case as \((e_z\) is the unit vector along the \(z\) direction\)
\[ E(x, y, t) = \nabla \Phi \times e_z, \quad B(x, y, t) = \Psi(x, y, t) e_z, \]
while Faraday and Ampère-Maxwell equations require that
\[ \Psi(x, y, t) = \frac{1}{c} \partial_0 \Phi, \quad \eta^{ik} \partial_{ik} \Phi = 0. \]
In this way, Eq. (6) is written as
\[
\partial^l \sigma \partial_l \sigma = \frac{8 \pi G_0 \varepsilon_0 (d\lambda/d\omega)_{\phi_0}}{c^4} \partial^l \Phi \partial_l \Phi,
\]
with the possible solutions, satisfying (10),
\[
\partial_t \sigma = \pm K \partial_t \Phi. \tag{18}
\]
Again, for \((d\lambda/d\omega)_{\phi_0} < 0\) solutions exist with the role of electric and magnetic fields interchanged.

The specific, non-electromagnetic force on a stationary object is given in this case by
\[
f_\beta = -\frac{c^2}{4} \partial_\beta \left( H_{00} + H_{\alpha\alpha} \right) + c^2 \partial_0 H_{0\beta}.
\tag{19}
\]
A direct operation with Eq. (5) allows to write
\[
\eta^{ik} \partial_{ik} f_\beta = -c^2 \partial_0 \left( H_{00} + H_{\alpha\alpha} \right) + c^2 \partial_0 H_{0\beta},
\]
from which, using (18), a simple estimation gives a result similar in form to (17).

Finally, we consider a case analogous to the previous one, where now the free electromagnetic field has rotational symmetry around the \(z\) direction. Using spherical coordinates in three-dimensional space we can write in this case, for the solenoidal electric and magnetic fields (\(e_\phi\) is the unit vector in the local azimuthal direction)
\[
E(r, \theta, t) = \frac{1}{r \sin \theta} \nabla \Phi \times e_\phi, \quad B(r, \theta, t) = \Psi(r, \theta, t) e_\phi,
\]
and now the Faraday and Ampère-Maxwell equations require that
\[
\Psi(r, \theta, t) = \frac{1}{cr \sin \theta} \partial_0 \Phi,
\]
\[
\partial_{00} \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \Phi}{\partial \theta} \right). \tag{20}
\]
It is important to note that \(\Phi\) does not satisfy a D’Alembert equation like the one to be satisfied by \(\sigma\) (Eq.(10)).

Eq. (6) is now written as
\[
(\partial_0 \sigma)^2 - |\nabla \sigma|^2 = \frac{8 \pi G_0 \varepsilon_0 (d\lambda/d\omega)_{\phi_0}}{c^4 r^2 \sin^2 \theta} \left[(\partial_0 \Phi)^2 - |\nabla \Phi|^2\right]. \tag{21}
\]
Special solutions of the kind considered can be obtained noting that Eq. (21) is satisfied if the following equations hold
\[
\mathbf{n} \partial_0 \sigma - \nabla \sigma = \frac{\gamma K}{r \sin \theta} (\mathbf{m} \partial_0 \Phi - \nabla \Phi),
\]
\[
\mathbf{n} \partial_0 \sigma + \nabla \sigma = \frac{K}{\gamma r \sin \theta} (\mathbf{m} \partial_0 \Phi + \nabla \Phi),
\]
where $\gamma$ is an arbitrary function of $(r, \theta, t)$, and $\mathbf{n}$ and $\mathbf{m}$ are two arbitrary unit vectors. Solving these equations for $\partial_0 \sigma$ and $\nabla \sigma$ one has

$$\partial_0 \sigma = \frac{K}{2\gamma r \sin \theta} \left[ (1 + \gamma^2) \mathbf{m} \cdot \mathbf{n} \partial_0 \Phi + (1 - \gamma^2) \mathbf{n} \cdot \nabla \Phi \right], \quad (22)$$

$$\nabla \sigma = \frac{K}{2\gamma r \sin \theta} \left[ (1 - \gamma^2) \mathbf{m} \partial_0 \Phi + (1 + \gamma^2) \nabla \Phi \right]. \quad (23)$$

One can now write Eq. (10) as

$$\partial_0 \left( \partial_0 \sigma \right) = \nabla \cdot (\nabla \sigma), \quad (24)$$

where $\partial_0 \sigma$ and $\nabla \sigma$ are replaced by their expressions (22) and (23). In this way, solved Eq. (20) for $\Phi$, and given arbitrary expressions for the unit vectors $\mathbf{n}$ and $\mathbf{m}$, Eq. (24) turns out to be an equation for the function $\gamma$, that once solved allows to obtain the function $\sigma$ using (22) and (23). The arbitrariness of the units vectors employed only means that the system of equations considered admits a large class of solutions. The actual solution in a real situation must correspond to unit vectors determined by the solution itself. Considering that the physical problem is determined by the scalar $\Phi$ one could take

$$\mathbf{n} = \mathbf{m} = \frac{\nabla \Phi}{|\nabla \Phi|}.$$

### 3.1. Conclusions

It was shown that a Brans-Dicke type of theory, with interaction of Bekenstein’s type between the Maxwell and the scalar fields, admits a special kind of solutions in which the metric effect of the electromagnetic field is much larger than in general relativity. This kind of solution arises for particular electromagnetic field distributions, most notably in all static cases, and also for propagating fields with Cartesian or axial symmetry, in the regions outside the electromagnetic sources. Of course, even if the theory is realist, the actual realization of this kind of solutions depends on the possibility of satisfying the necessary boundary conditions between the regions outside and inside the sources. As the theory is considered as an effective one, and at the classical level, it is not easy to ascertain the correct boundary conditions with real, microscopic sources. A related point to be noted is that, from Eq. (5), the “forcing” of the metric depends on the second derivative of $\sigma$ or, through (11), on the second derivative of the electrostatic potential. This means that in the case of a homogeneous electric field, the actual solution is determined solely by the boundary conditions. It is thus expected that inhomogenous fields are preferred in order to seek solutions of the type considered.

As for the magnitude of the expected forces, from expression (17) one can estimate that for a field of about 10 kV/m and $\sqrt{(d\lambda/d\omega)}_{\partial_0} \sim 1$ the corresponding specific force is about $1 \mu N/Kg$. Although small and difficult to be separated from the forces of electric origin that necessarily should appear in an experiment, such non-electrical forces can be measured with present technology, as is well documented in [10], [11]. It is
also remarkable that forces of the type and magnitude just discussed have been reported in inhomogeneous electrostatic fields[12].

With the above considerations it is concluded that experiments of relative simplicity are possible to test the possibility of scalar-tensor theories with Maxwell-scalar fields interaction.

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