Spin-tomographic symbols of qudit states and spin observables are studied. Spin observables are associated with the functions on a manifold whose points are labelled by spin projections and sphere $S^2$ coordinates. The star-product kernel for such functions is obtained in explicit form and connected with Fourier transform of characters of $SU(2)$ irreducible representation. The kernels are shown to be in close relation to the Chebyshev polynomials. Using specific properties of these polynomials, we establish the recurrence relation between kernels for different spins. Employing the explicit form of the star-product kernel, a sum rule for Clebsch-Gordan and Racah coefficients is derived. Explicit formulas are obtained for the dual tomographic star-product kernel as well as for intertwining kernels which relate spin-tomographic symbols and dual tomographic symbols.

Keywords: spin tomography, star-product, kernel, quantizer, dequantizer, $SU(2)$-group character, qudit

I. INTRODUCTION

In quantum mechanics, states of a system are usually associated with the density operators. The other possibility is to use different maps of quantum states onto the quasi-probability functions \[1, 2\] or the fair probability-distribution function called tomogram (see, e.g., \[3, 4, 5, 6, 7, 8, 9\]). The latter one is of great interest because it can be measured experimentally \[10, 11, 12, 13, 14\]. Though tomograms are often utilized with the only aim to reconstruct the Wigner function or the density matrix, it should be emphasized that tomograms themselves are a primary notion of quantum states. As far as spin states are concerned, the corresponding tomographic map is elaborated in \[15, 16, 17, 18, 19\]. The examples of other maps of spin states onto functions are discussed in \[20, 21, 22\]. By analogy with the density operator, any other operator (observable) on a Hilbert space can be mapped onto the function called tomographic symbol of the operator. Such a scanning procedure is accomplished with the help of the special dequantizer operator \[9, 23\]. Using the quantizer operator \[9, 23\], one can reconstruct the operator in question, i.e., there exists an inverse map of tomographic symbols onto operators. Within the framework of the spin-tomographic

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star-product procedure [24, 25, 26], one deals with symbols instead of operators. In particular, the symbol of the product of two operators is equal to the star-product of the symbols corresponding to the separate operators. The main feature of the star-product is that it is associative but noncommutative in general. The star-product kernel is easily expressed in terms of the dequantizer and quantizer operators. The explicit formula of the kernel was presented previously with the help of Clebsch-Gordan and Racah coefficients in the work [27] and specified for the low-spin states in [28].

The aim of this work is to obtain a new explicit formula of the spin-tomographic star-product kernel in terms of $SU(2)$ irreducible representation character. As mentioned above, any qudit state can be described by the spin-tomographic probability introduced in [15, 16], where the reconstructed states are expressed in terms of such state characteristics as the Wigner function or the density matrix. In the work [29], the discussion is presented how to use such a tomographic probability-distribution in the other known reconstruction procedure. Both the spin tomogram, which coincides with that introduced in [15, 16], and the inversion formula, which provides the density operator of a spin state by means of its spin tomogram, were given in [29] in the compact exponential forms. On the other hand, it was proved in [28] that the exponential form of the inversion formula, the inversion formula found in [16], and that presented in another form in [27] are all identical on the set of spin tomograms. In view of this equivalency, one can use any form of the inversion formula on an equal footing. This means that any form of the quantizer and dequantizer operators is acceptable to study concrete properties of the star-product representation of spin operators and qudit states. In the present paper, the problem is attacked with the help of exponential representation of the quantizer and dequantizer operators. The $SU(2)$ irreducible representation character is known to be nothing else but the Chebyshev polynomial of a specific argument. In turn, the kernel is shown to be a Fourier transformation of the character. We exploit special properties of the Chebyshev polynomials not only to derive the star-product kernel but also to reveal its peculiarities, for instance, the recurrence relation. Comparing different explicit forms of the star-product kernels, we show that the kernel is not defined unambiguously, but the residual must give zero while integrating with tomographic symbols. We point out that the constructed kernels of the spin-tomographic star-product are given for the functions which depend not only on group element of $SU(2)$ but also on weights (spin projection $m$) of $SU(2)$ irreps.

The paper is organized as follows. In Sec. II, we give a brief review of the scanning and reconstruction procedures which are performed in the spin tomography of qudit states. In Sec. III, the star-product kernel is represented in the form of Fourier transformation of the $SU(2)$ irreps character and the explicit formula for the kernel is derived. In Sec. IV, we compare two different explicit forms of the kernel to obtain a new sum rule for Clebsch-Gordan and Racah coefficients. In Sec. V, we establish the recurrence relation between the star-product kernels for different spins. In Sec. VI conclusions are presented. The ambiguity of the spin-tomographic star-product kernel is illustrated in Appendix A. Generalization of the explicit formulas to other types of kernels is given in Appendix B.
II. SPIN TOMOGRAMS AND TOMOGRAPHIC SYMBOLS OF OPERATORS

Unless specifically stated, qudit states with spin \( j \) are considered. We deal with the angular momentum operators \( \hat{J}_x, \hat{J}_y, \) and \( \hat{J}_z \) and the standard state vectors \( |jm\rangle \) defined as follows:

\[
(\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2)|jm\rangle = j(j + 1)|jm\rangle, \quad \hat{J}_z|jm\rangle = m|jm\rangle,
\]

with the spin projection \( m \) taking the values \( -j, -j + 1, \ldots, j \).

As stated above, the state of a qudit is completely determined by its density operator \( \hat{\rho} \) or alternatively by the following probability-distribution function (called spin tomogram):

\[
w_j(m, n) = \langle jm|\hat{R}^\dagger(n)\hat{\rho}\hat{R}(n)|jm\rangle = \text{Tr}\left(\hat{\rho} \hat{R}(n)|jm\rangle\langle jm|\hat{R}^\dagger(n)\right) = \text{Tr}\left(\hat{\rho}\hat{U}_j(m, n)\right),
\]

where we introduced the dequantizer operator \( \hat{U}_j(m, n) \) and the rotation operator \( \hat{R}(n) \). The vector \( n(\theta, \phi) = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \) determines the axis of quantization (the point on the sphere specified by the longitude \( \phi \in [0, 2\pi] \) and the latitude \( \theta \in [0, \pi] \)). The rotation operator is defined through

\[
\hat{R}(n) = e^{-i(n_\perp \cdot \hat{J})\theta}, \quad n_\perp = (-\sin \phi, \cos \phi, 0).
\]

The tomogram satisfies the following normalization conditions:

\[
\sum_{m=-j}^j w_j(m, n) = 1, \quad \frac{2j+1}{4\pi} \int_0^\pi d\phi \int_0^\pi \sin \theta d\theta \ w_j(m, n(\theta, \phi)) = 1.
\]

Some other features of the spin-tomographic functions are discussed in [30, 31, 32, 33].

Taking into account the relation \( \hat{R}(n)\hat{J}_z\hat{R}^\dagger(n) = (n \cdot \hat{J}) \equiv n_\alpha \hat{J}_\alpha \), the dequantizer operator \( \hat{U}_j(m, n) \) can be written in the following exponential form:

\[
\hat{U}_j(m, n) = \hat{R}(n)|jm\rangle\langle jm|\hat{R}^\dagger(n) = \hat{R}(n)\delta(m - \hat{J}_z)\hat{R}^\dagger(n) = \delta \left(m - \hat{R}(n)\hat{J}_z\hat{R}^\dagger(n)\right)
\]

\[
= \delta \left(m - (n \cdot \hat{J})\right) = \frac{1}{2\pi} \int_0^{2\pi} e^{im\varphi} e^{-i(n \cdot \hat{J})\varphi} d\varphi,
\]

where by \( \delta \) we denote the Kronecker delta-symbol [34].

Given the tomogram [2], one can reconstruct the density operator \( \hat{\rho} \) with the help of the quantizer operator \( \hat{D}_j(m, n) \). The reconstruction procedure reads

\[
\hat{\rho} = \sum_{m=-j}^j \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \ w_j(m, n(\theta, \phi))\hat{D}_j(m, n(\theta, \phi))
\]

or briefly

\[
\hat{\rho} = \int w_j(x)\hat{D}_j(x)dx,
\]
where we denoted
\[ x = (m, n), \quad \int d\mathbf{x} = \sum_{m=-j}^{j} \frac{1}{4\pi} \int d\Omega = \sum_{m=-j}^{j} \frac{1}{4\pi} \int_{0}^{2\pi} d\phi \int_{0}^{\pi} \sin \theta d\theta. \tag{8} \]

Similarly to the case of dequantizer, the quantizer operator \( \hat{D}_j(x) \) can be represented in the exponential form \[ \hat{D}_j(m, n) = 2^{j+1} \pi \int_{0}^{2\pi} e^{i(m+n)\hat{J}\phi} d\phi. \tag{9} \]

Both quantizer and dequantizer are Hermitian operators, with the latter one being non-negative as well.

By analogy with the density operator, any spin operator \( \hat{A} \) acting on a Hilbert space of states (1) is mapped onto the function \( f_{\hat{A}}(x) \) and vice versa. By construction, the relation between the tomographic symbol \( f_{\hat{A}}(x) \) and the operator \( \hat{A} \) reads
\[ f_{\hat{A}}(x) = \text{Tr} \left( \hat{A} \hat{U}_j(x) \right), \quad \hat{A} = \int f_{\hat{A}}(x) \hat{D}_j(x) dx. \tag{10} \]

Besides usual tomographic symbols described above, sometimes it is convenient to use dual tomographic symbols defined through
\[ f^d_{\hat{A}}(x) = \text{Tr} \left( \hat{A} \hat{U}_j(x) \right), \quad \hat{A} = \int f^d_{\hat{A}}(x) \hat{U}_j(x) dx. \tag{11} \]

For instance, the average value of the operator \( \hat{A} \) reads
\[ \text{Tr} \left( \hat{\rho} \hat{A} \right) = \text{Tr} \int w_j(x) \hat{D}_j(x) \hat{A} dx = \int w_j(x) \text{Tr} \left( \hat{A} \hat{D}_j(x) \right) dx = \int w_j(x) f^d_{\hat{A}}(x) dx. \tag{12} \]

This implies that one can calculate mean values of observables by using standard and dual tomographic symbols only. The dual tomographic symbols were anticipated in the work \[ and elaborated in \[. Quantumness of qudit states was demonstrated by means of dual tomographic symbols in \[.

III. KERNEL OF STAR-PRODUCT FOR SPIN TOMOGRAPHIC SYMBOLS

Since operators and tomographic symbols are in strong relation with each other, any operation on \( \hat{A} \) and \( \hat{B} \) corresponds to an adequate operation on functions \( f_{\hat{A}}(x) \) and \( f_{\hat{B}}(x) \).

For instance, the sum of \( \hat{A} \) and \( \hat{B} \) maps to the sum of \( f_{\hat{A}}(x) \) and \( f_{\hat{B}}(x) \). As far as the product \( \hat{A} \hat{B} \) is concerned, the corresponding tomographic symbol \( f_{\hat{A}\hat{B}}(x) \) is called the star-product of the symbols \( f_{\hat{A}}(x) \) and \( f_{\hat{B}}(x) \):
\[ f_{\hat{A}\hat{B}}(x) = f_{\hat{A}}(x) * f_{\hat{B}}(x). \tag{13} \]
By definition, one has

\[ f_{AB}(x_1) = \text{Tr} \left( \hat{A} \hat{B} \hat{U}_j(x_1) \right) = \iiint f_A(x_3)f_B(x_2)K_j(x_3,x_2,x_1)dx_2dx_3, \quad (14) \]

where the function

\[ K_j(x_3,x_2,x_1) = \text{Tr} \left( \hat{D}_j(x_3)\hat{D}_j(x_2)\hat{U}_j(x_1) \right) \quad (15) \]

is called the kernel of the star-product scheme. It is worth noting that this kernel is non-local. The non-locality of kernels of this type can also be illustrated if we consider delta-function on the tomogram set. In fact, definitions (2) and (7) are followed by the relation

\[ w_j(x_1) = \int w_j(x_2)\text{Tr} \left( \hat{D}_j(x_2)\hat{U}_j(x_1) \right) dx_2, \quad (16) \]

which holds true for an arbitrary spin tomogram \( w_j(x) \). This implies that the function

\[ K_j^\delta(x_2,x_1) = \text{Tr} \left( \hat{D}_j(x_2)\hat{U}_j(x_1) \right) \quad (17) \]

can be treated as the kernel of the unity operator on the set of spin tomograms and plays the role of an analogue of the Dirac delta-function. Some examples of the star-product kernels and the delta-functions for the low-spin states are depicted in Fig. 1. It is readily seen that apart from being non-local, the delta-function on the tomogram set is not non-negative either.

By analogy with ordinary tomographic symbols, one can also introduce the dual spin-tomographic star-product

\[ f^d_{AB}(x) = f^d_A(x) \ast f^d_B(x) \quad (18) \]

with the non-local kernel of the form

\[ K_j^d(x_3,x_2,x_1) = \text{Tr} \left( \hat{U}_j(x_3)\hat{U}_j(x_2)\hat{D}_j(x_1) \right). \quad (19) \]

Let us calculate the explicit form of the star-product kernel (15) for qudits with spin \( j \).

Using the exponential representation of the dequantizer (5) and the quantizer (9), we obtain

\[
K_j(x_3,x_2,x_1) = \frac{(2j+1)^2}{(2\pi)^3} \sum_{s_2=-1}^{1} \sum_{s_3=-1}^{1} \frac{1}{(1-3s_2^2)(1-3s_3^2)} \times \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \text{Tr} \left( e^{-i(n_3 \cdot J)\varphi_3}e^{-i(n_2 \cdot J)\varphi_2}e^{-i(n_1 \cdot J)\varphi_1} \right) e^{im_1\varphi_1}e^{i(m_2+s_2)\varphi_2}e^{i(m_3+s_1)\varphi_3}d\varphi_1d\varphi_2d\varphi_3. \quad (20)
\]

From this it follows that the spin-tomographic star-product kernel (15) is nothing else but the Fourier transform of \( SU(2) \) irreducible representation character

\[
\chi(n_3,n_2,n_1,\varphi_3,\varphi_2,\varphi_1) = \text{Tr} \left( e^{-i(n_3 \cdot J)\varphi_3}e^{-i(n_2 \cdot J)\varphi_2}e^{-i(n_1 \cdot J)\varphi_1} \right) = \text{Tr} \left( e^{-i(n \cdot J)\Phi} \right), \quad (21)
\]
FIG. 1: Non-locality of tomographic kernels. Delta-function $K^\delta_j(n_2, n_1(\theta, \phi), m_2, m_1)$ on the set of qubit tomograms (a) and on the set of qutrit tomograms (b). Star-product kernel $K_j(n_3, n_2, n_1(\theta, \phi), m_3, m_2, m_1)$ for qubits: real (c) and image (e) parts; and that for qutrits: real (d) and image (f) parts. In (a) we set $n_2 = (0, -\sqrt{3}/2, 1/2)$, $m_2 = -1/2$, and $m_1 = 1/2$. In (b) $n_2 = (-1/2\sqrt{2}, \sqrt{3}/2\sqrt{2}, 1/\sqrt{2})$, $m_2 = 1$, and $m_1 = 0$. In both (c) and (e) $n_3 = (-1/2, -\sqrt{3}/2, 0)$, $n_2 = (-\sqrt{3}/2\sqrt{2}, -\sqrt{3}/2\sqrt{2}, -1/\sqrt{2})$, and $m_3 = m_2 = m_1 = 1/2$. Finally in both (d) and (f) we use $n_3 = (0, 1, 0)$, $n_2 = (1/2, -1/2, 1/\sqrt{2})$, $m_3 = -1$, $m_2 = 1$, and $m_1 = 0$. 


where \( \Phi = \Phi(n_1, n_2, n_3, \varphi_1, \varphi_2, \varphi_3) \) and \( \mathcal{M} = \mathcal{M}(n_1, n_2, n_3, \varphi_1, \varphi_2, \varphi_3) \) are respectively the angle and axis of the resulting rotation which is equivalent to successive rotations around axis \( n_k \) by angle \( \varphi_k \), \( k = 1, 2, 3 \).

To simplify formulas let us introduce the 3-vector \( \varphi \) with components \( (\varphi_1, \varphi_2, \varphi_3) \) and the 9-vector \( N \) with components constructed from components of three vectors \( n_1, n_2, n_3 \), i.e., \( N = (n_1, n_2, n_3) \). Also, we designate

\[
\int d\varphi = \int d\varphi_1 d\varphi_2 d\varphi_3, \quad m = (m_1, m_2, m_3).
\]

Given the angle \( \Phi \), the character has a rather simple form

\[
\chi(\Phi) = \sum_{m=-j}^{j} e^{im\Phi} = \frac{\sin((2j + 1)\Phi/2)}{\sin(\Phi/2)} = U_{2j}\left(\cos(\Phi/2)\right),
\]

where \( U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta} \) is the Chebyshev polynomial of the second kind of degree \( n \).

Thus, combining (20)–(23), we obtain the integral representation of the star-product kernel

\[
K_j(x_3, x_2, x_1) = \frac{(2j + 1)^2}{(2\pi)^3} \sum_{s_2=-1}^{1} \sum_{s_3=-1}^{1} \frac{1}{(1 - 3s_2^2)(1 - 3s_3^2)} I_j(x_3, x_2, x_1),
\]

where by \( I_j(x_3, x_2, x_1) \) we denote the following integral:

\[
I_j(x_3, x_2, x_1) = \int U_{2j}\left(\frac{\Phi(N, \varphi)}{2}\right) e^{i(m \cdot \varphi)} e^{i\varphi_2} e^{i\varphi_3} d\varphi.
\]

This implies that the kernel of spin-tomographic star-product can be treated as the Fourier transform of the Chebyshev polynomial of a specific argument. In the same way, it can easily be checked that the kernel of the dual spin-tomographic star-product reads

\[
K_j^d(x_3, x_2, x_1) = \frac{2j + 1}{(2\pi)^3} \sum_{s_1=-1}^{1} \frac{1}{1 - 3s_1^2} \int U_{2j}\left(\frac{\Phi(N, \varphi)}{2}\right) e^{i(m \cdot \varphi)} e^{i\varphi_1} d\varphi.
\]

Further, the angle \( \Phi \) does not depend on spin \( j \). Consequently it is possible to calculate it for qubits and then extend the obtained result to other spins. Substituting \( 1/2 \) for \( j \) in (23), we get the character for qubits

\[
\chi_{1/2}(\Phi) = U_1\left(\cos(\Phi/2)\right) = 2 \cos(\Phi/2).
\]

On the other hand, from (21) it follows that

\[
\chi_{1/2}(\Phi) = \text{Tr}\left(e^{-i(n_3 \hat{\sigma}_3)\varphi_3/2} e^{-i(n_2 \hat{\sigma}_2)\varphi_2/2} e^{-i(n_1 \hat{\sigma}_1)\varphi_1/2}\right),
\]

where \( \hat{\sigma} = \left(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\right) \) is a set of the Pauli matrices. Employing the known property of Pauli matrices \( \hat{\sigma}_\alpha \hat{\sigma}_\beta = \delta_{\alpha \beta} \hat{1} + i \varepsilon_{\alpha \beta \gamma} \hat{\sigma}_\gamma \), it is not hard to prove that the relations
\begin{align}
(a \cdot \hat{\sigma})(b \cdot \hat{\sigma}) &= a_\alpha b_\beta \hat{\sigma}_\alpha \hat{\sigma}_\beta = (a \cdot b) \hat{1} + i \left( [a \times b] \cdot \hat{\sigma} \right), \\
e^{-i(n \cdot \hat{\sigma})\varphi/2} &= \hat{1} \cos(\varphi/2) - i(n \cdot \hat{\sigma}) \sin(\varphi/2)
\end{align}

are valid whenever \( n^2 \) is equal to unity. In view of these relations, we finally obtain

\begin{align}
e^{-i(n_2 \cdot \hat{\sigma})\varphi_2/2}e^{-i(n_3 \cdot \hat{\sigma})\varphi_3/2}e^{-i(n_1 \cdot \hat{\sigma})\varphi_1/2} &= \hat{1} \cos \left( \Phi(N, \varphi)/2 \right) - i(N \cdot \hat{\sigma}) \sin \left( \Phi(N, \varphi)/2 \right).
\end{align}

Recall that the angle \( \Phi \) depends on three rotation angles \( \varphi_1, \varphi_2, \varphi_3 \) and three directions \( n_1, n_2, n_3 \). Departing from this notation, one can easily derive the resulting rotation angle

\begin{align}
\cos \left( \Phi(N, \varphi)/2 \right) &= \cos(\varphi_1/2) \cos(\varphi_2/2) \cos(\varphi_3/2) - (n_1 \cdot n_2) \sin(\varphi_1/2) \sin(\varphi_2/2) \cos(\varphi_3/2) \\
&- (n_2 \cdot n_3) \cos(\varphi_1/2) \sin(\varphi_2/2) \sin(\varphi_3/2) - (n_3 \cdot n_1) \sin(\varphi_1/2) \cos(\varphi_2/2) \sin(\varphi_3/2) \\
&+ (n_1 \cdot [n_2 \times n_3]) \sin(\varphi_1/2) \sin(\varphi_2/2) \sin(\varphi_3/2)
\end{align}

and the rotation axis

\begin{align}
N \sin \left( \Phi(N, \varphi)/2 \right) &= n_1 \sin(\varphi_1/2) \cos(\varphi_2/2) \cos(\varphi_3/2) \\
&+ n_2 \cos(\varphi_1/2) \sin(\varphi_2/2) \cos(\varphi_3/2) + n_3 \cos(\varphi_1/2) \cos(\varphi_2/2) \sin(\varphi_3/2) \\
&- \left\{ n_1(n_2 \cdot n_3) - n_2(n_1 \cdot n_3) + n_3(n_1 \cdot n_2) \right\} \sin(\varphi_1/2) \sin(\varphi_2/2) \sin(\varphi_3/2) \\
&- [n_1 \times n_2] \sin(\varphi_1/2) \sin(\varphi_2/2) \cos(\varphi_3/2) - [n_2 \times n_3] \cos(\varphi_1/2) \sin(\varphi_2/2) \sin(\varphi_3/2) \\
&- [n_1 \times n_3] \sin(\varphi_1/2) \cos(\varphi_2/2) \sin(\varphi_3/2).
\end{align}

Now, when \( \cos \left( \Phi(N, \varphi)/2 \right) \) is known, integral \((25)\) can be evaluated. Change of variables \( t_k = -\cot(\varphi_k/2), k = 1, 2, 3 \) results in the integral taking the form

\begin{align}
I_j(x_3, x_2, x_1) &= 8 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dt_1 dt_2 dt_3 \frac{(t_3 - i)^{m_3+s_3-1}(t_2 - i)^{m_2+s_2-1}(t_1 - i)^{m_1-1}}{(t_3 + i)^{m_3+s_3+1}(t_2 + i)^{m_2+s_2+1}(t_1 + i)^{m_1+1}} \\
&\times U_{2j} \left( \frac{t_1 t_2 t_3 - t_1(n_2 \cdot n_3) - t_2(n_3 \cdot n_1) - t_3(n_1 \cdot n_2) - (n_1 \cdot [n_2 \times n_3])}{(t_1 - i)^{1/2}(t_1 + i)^{1/2}(t_2 - i)^{1/2}(t_2 + i)^{1/2}(t_3 - i)^{1/2}(t_3 + i)^{1/2}} \right),
\end{align}

where \( z^{1/2} \), \( z \in \mathbb{C} \) is regarded as a principal branch of the square root function, with the branch cut being along the positive real axis. Since the integrand decreases fast enough as \( |t_k| \to \infty, k = 1, 2, 3 \), one can calculate the integral in question with the help of the residue theorem. Indeed, choosing for each complex variable \( t_k, k = 1, 2, 3 \) the path of integration shown in Fig. [2] we have \( I_j(x_3, x_2, x_1) = (2\pi i)^3 \text{Res}_{t_k=i} \text{Res}_{t_2=i} \text{Res}_{t_3=i} \). In order to calculate the residues one needs to find a coefficient corresponding to the term \((t_1 - i)^{-1}(t_2 - i)^{-1}(t_3 - i)^{-1}\).

Employing the expansion of the Chebyshev polynomial

\begin{align}
U_{2j}(x) &= \sum_{k=0}^{[j]} \frac{(-1)^k(2j - k)!}{k!(2j - 2k)!} (2x)^{2j - 2k},
\end{align}
the integral \((34)\) can be written in the form

\[
I_j(x_3, x_2, x_1) = \sum_{k=0}^{[j]} \frac{(-1)^k (2j - k)! 2^{2j-2k+3}}{k! (2j - 2k)!} \int_{-\infty}^{+\infty} dt_1 dt_2 dt_3 f^{2j-2k}(t_1, t_2, t_3) \times (t_3 + i)^{-j-m_3-s_3+k-1}(t_2 + i)^{-j-m_2-s_2+k-1}(t_1 + i)^{-j-m_1+k-1} \frac{(t_3 - i)^{j-m_3-s_3-k+1}(t_2 - i)^{j-m_2-s_2-k+1}(t_1 - i)^{j-m_1-k+1}}{(t_3 - i)^{j-m_3-s_3-k+1}(t_2 - i)^{j-m_2-s_2-k+1}(t_1 - i)^{j-m_1-k+1}}, \tag{36}
\]

where the \((2j - 2k)\)th power of the function

\[
f(t_1, t_2, t_3) = (t_1 - i)(t_2 - i)(t_3 - i) + i(t_1 - i)(t_2 - i) + i(t_2 - i)(t_3 - i) + i(t_3 - i)(t_1 - i)
- (t_1 - i)\left(1 + (n_2 \cdot n_3)\right) - (t_2 - i)\left(1 + (n_3 \cdot n_1)\right) - (t_3 - i)\left(1 + (n_1 \cdot n_2)\right)
- (n_1 \cdot [n_2 \times n_3]) - i\left(1 + (n_1 \cdot n_2) + (n_2 \cdot n_3) + (n_3 \cdot n_1)\right) \tag{37}
\]

resolves to the following sum:

\[
f^{2j-2k}(t_1, t_2, t_3) = \sum_{p_1 + \cdots + p_s = 2j - 2k} \frac{(2j - 2k)!}{\prod_{l=1}^s p_l!} (t_1 - i)^{p_1+p_2+p_4+p_5}(t_2 - i)^{p_1+p_2+p_3+p_6}(t_3 - i)^{p_1+p_3+p_4+p_7}
\times i^{p_2+p_3+2(p_5+p_6+p_7)+3p_8} \left(1 + (n_2 \cdot n_3)\right)^{p_5} \left(1 + (n_3 \cdot n_1)\right)^{p_6} \left(1 + (n_1 \cdot n_2)\right)^{p_7}
\times \left(1 + (n_1 \cdot n_2) + (n_2 \cdot n_3) + (n_3 \cdot n_1) - i(n_1 \cdot [n_2 \times n_3])\right)^{p_8}. \tag{38}
\]

Moreover, one should extract terms \((t_k - i)\) from the general expression \((t_k + i)^r\). It is obvious that the expansion of \((t + i)^r\) to the powers of \((t - i)\) reads

\[
(t + i)^r = (2i)^r \left(1 + \frac{t - i}{2i}\right)^r = \sum_{q=0}^{\infty} \binom{r}{q} \frac{(t - i)^q}{(2i)^{q-r}}, \tag{39}
\]
where we introduced binomial coefficients according to the rule
\[
\binom{r}{q} = \frac{r(r-1)\ldots(r-q+1)}{q!}.
\]

(40)

Here \(r\) is supposed to be real, \(q\) is an integer, \(\binom{r}{q} = 0\) if \(q < 0\), and \(\binom{r}{0} = 1\).

If we combine (36) with (38) and (39), we can calculate the residues involved and then the integral \(I\). The direct computation yields

\[
I_j(x_3, x_2, x_1) = (2\pi)^3 \sum_{k=0}^{[j]} \sum_{p_1+\ldots+p_8=2j-2k} (-1)^k(2j-k)!2^{j-2k+3} \frac{k! \prod_{l=1}^{8} p_l!}{(1 + (n_2 \cdot n_3))^p_5 (1 + (n_3 \cdot n_1))^p_6 (1 + (n_1 \cdot n_2))^p_7 (1 + (n_1 \cdot [n_2 \times n_3]))}\sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} \sum_{q_3=0}^{\infty} (2i)^{3j+3m_3+m_2+m_1+s_3+s_2-3k+3+q_3+q_2+q_1} \cdot \delta_{j-m_1-k-1}(1-3s_3^2)(1-3s_3^2)2^{p_1+p_5+p_6+p_7+2p_8} k! \prod_{l=1}^{8} p_l!
\]

Finally, substituting the calculated value of \(I\) for the integral in (24), we obtain the explicit form of the spin-tomographic star-product kernel

\[
K_j(x_3, x_2, x_1) = (2j+1)2 \sum_{s_3=-1}^{1} \sum_{s_2=-1}^{1} \sum_{k=0}^{[j]} \sum_{p_1+\ldots+p_8=2j-2k} (-1)^k(2j-k)! \frac{k! \prod_{l=1}^{8} p_l!}{(1-3s_3^2)(1-3s_3^2)2^{p_1+p_5+p_6+p_7+2p_8} k! \prod_{l=1}^{8} p_l!} \cdot \delta_{j-m_1-k-1}(1-3s_3^2)(1-3s_3^2)2^{p_1+p_5+p_6+p_7+2p_8} k! \prod_{l=1}^{8} p_l!
\]

(42)

IV. EQUIVALENCY OF KERNEL REPRESENTATIONS

The problem of the explicit form of the star-product kernel has been attacked from different perspectives. In the work [27], the explicit formula is expressed in terms of Clebsch-Gordan and Racah coefficients. To be more precise, the result is
\[ K_j'(x_3, x_2, x_1) = (-1)^j m_1 - m_2 - m_3 \sum_{L_1=0}^{2j} \sum_{L_2=0}^{2j} \sum_{L_3=0}^{2j} (-1)^{L_1+L_2+L_3} \sqrt{(2L_3 + 1)^3(2L_2 + 1)^3(2L_1 + 1)} \]
\[ \times \langle jm_1; j - m_1|L_10\rangle \langle jm_2; j - m_2|L_20\rangle \langle jm_3; j - m_3|L_30\rangle \}\left\{ \begin{array}{ccc} L_2 & L_3 & L_1 \\ j & j & j \end{array} \right\} \]
\[ \times \sum_{M_1=-L_1}^{L_1} \sum_{M_2=-L_2}^{L_2} \sum_{M_3=-L_3}^{L_3} \left( \begin{array}{ccc} L_2 & L_3 & L_1 \\ M_2 & M_3 & M_1 \end{array} \right) D^{(L_1)}_{0-M_1}(0, \theta_1, -\phi_1) D^{(L_2)}_{0-M_2}(0, \theta_2, -\phi_2) D^{(L_3)}_{0-M_3}(0, \theta_3, -\phi_3), \] (43)

where the Wigner \(D\)-function reads
\[ D^{(j)}_{m'm}(\alpha, \beta, \gamma) = e^{-i m' \alpha} e^{-i m \gamma} \sum_s (-1)^s \sqrt{(j + m)! (j - m)! (j + m')! (j - m')! s!} \sqrt{(j + m - s)! (j + m - s)! (m' - m + s)!} \times \left( \cos \frac{\beta}{2} \right)^{2j + m - m' - 2s} \left( -\sin \frac{\beta}{2} \right)^{m' - m + 2s}. \] (44)

In \cite{28}, using the irreducible tensor operators for the \(SU(2)\) group \cite{39, 40}, the same result is specified for the low-spin states and presented in the form of the expansion to orthogonal summands. In the present work, starting from the exponential representation of the quantizer and dequantizer operators, we managed to obtain another explicit form of the star-product kernel which can also be presented in the form
\[ K_j''(x_3, x_2, x_1) \]
\[ = (2j + 1)^2 \sum_{s_3=1}^{1} \sum_{s_2=1}^{1} \sum_{k=0}^{[j]} \sum_{p_1+\ldots+p_8=2j-2k} (-1)^k (2j - k) \frac{1}{(1 - 3s_3^2)(1 - 3s_2^2)} 2^{-p_1 + p_5 + p_6 + p_7 + 2p_8} k! \prod_{l=1}^{8} p_l! \]
\[ \times \left( 1 + (n(\theta_2, \phi_2) \cdot n(\theta_3, \phi_3)) \right)^{p_8} \left( 1 + (n(\theta_3, \phi_3) \cdot n(\theta_1, \phi_1)) \right)^{p_6} \left( 1 + (n(\theta_1, \phi_1) \cdot n(\theta_2, \phi_2)) \right)^{p_4} \]
\[ \times \left( 1 + (n(\theta_1, \phi_1) \cdot n(\theta_2, \phi_2)) + (n(\theta_2, \phi_2) \cdot n(\theta_3, \phi_3)) + (n(\theta_3, \phi_3) \cdot n(\theta_1, \phi_1)) \right) \]
\[ \times \left( -i (n(\theta_1, \phi_1) \cdot [n(\theta_2, \phi_2) \times n(\theta_3, \phi_3)]) \right)^{p_8} \]
\[ \times \left( \begin{array}{ccc} -j - m_1 + k - 1 \\ j - m_1 - k - p_1 - p_2 - p_4 - p_6 \end{array} \right) \left( \begin{array}{ccc} -j - m_2 - s_2 + k - 1 \\ j - m_2 - s_2 - k - p_1 - p_2 - p_3 - p_6 \end{array} \right) \]
\[ \times \left( \begin{array}{ccc} -j - m_3 - s_3 + k - 1 \\ j - m_3 - s_3 - k - p_1 - p_3 - p_4 - p_7 \end{array} \right). \] (45)

It is obvious that all the different formulas must be equivalent on the set of tomograms. This is followed by a specific sum rule for Clebsch-Gordan and Racah coefficients. Namely
\[ K_j'(x_3, x_2, x_1) \sim K_j''(x_3, x_2, x_1), \] (46)
where the sign \(\sim\) is defined through a biconditional implication of the form
The result is of binomial coefficients \( \binom{43}{j} \):

\[
\mathcal{K}_j'(x_3, x_2, x_1) \sim \mathcal{K}_j''(x_3, x_2, x_1) \iff \left\{ \int \int f_A(x_3) f_B(x_2) \mathcal{K}_j'(x_3, x_2, x_1) dx_2 dx_3 \right. \\
= \left. \int \int f_A(x_3) f_B(x_2) \mathcal{K}_j''(x_3, x_2, x_1) dx_2 dx_3 \right\} \quad \text{for all symbols } f_A(x) \text{ and } f_B(x). \quad (47)
\]

Some of sum rules for Clebsch-Gordan coefficients can be found in \([39, 41, 42]\).

Though there takes place an ambiguity in the star-product kernel, all types of the kernel must be equivalent for calculating the symbol of the product of two given operators. As far as functions \((43)\) and \((45)\) are concerned, in case of qubits \((j = 1/2)\), both formulas turned out to be the same (and consequently equal to that found in \([28]\)). In case of qutrits \((j = 1)\), the kernel \((45)\) contains more terms than the kernel \((43)\) expressed in terms of Clebsch-Gordan coefficients. Actually, all redundant terms give zero while integrating with tomographic symbols. In Appendix 1, we discuss the cause of the deviation between kernels and present the difference \(\Delta_j = \mathcal{K}_j''(x_3, x_2, x_1) - \mathcal{K}_j'(x_3, x_2, x_1)\) for qutrits \((j = 1)\).

V. RECURRENCE RELATION FOR SPIN-TOMOGRAPHIC KERNELS

The Chebyshev polynomials obey the recurrence relation of the form \([37, 38]\)

\[
U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x). \quad (48)
\]

Using this peculiar property of the Chebyshev polynomials, it is easy to prove that there exists a similar recurrence relation for integral \((45)\). In fact, one has

\[
I_{j+1/2}(x_3, x_2, x_1) = 2J_j(x_3, x_2, x_1) - I_{j-1/2}(x_3, x_2, x_1), \quad (49)
\]

where

\[
J_j(x_3, x_2, x_1) = \sum_{k=0}^{[j]} \frac{(-1)^k(2j - k)!2^{2j-k+3}}{k!(2j - 2k)!} \int_{-\infty}^{+\infty} dt_1 dt_2 dt_3 f^{2j-2k+1}(t_1, t_2, t_3) \times \frac{(t_3 + i)^{j - m_3 - s_3 + k - 3/2}(t_2 + i)^{j - m_2 - s_2 + k - 3/2}(t_1 + i)^{j - m_1 - k - 3/2}}{(t_3 - i)^{j - m_3 - s_3 + k + 3/2}(t_2 - i)^{j - m_2 - s_2 + k + 3/2}(t_1 - i)^{j - m_1 + k + 3/2}}. \quad (50)
\]

Employing the explicit form \((37)\) of the function \(f(t_1, t_2, t_3)\), one can calculate the integral involved just in the same way as it was fulfilled before and then utilize the following property of binomial coefficients \([43]\):

\[
\binom{r}{q + 1} = \binom{r + 1}{q + 1} - \binom{r}{q}. \quad (51)
\]

The result is
\[
J_j(x, x, x_1) = \sum_{m'_1, m'_2, m'_3} I_j(N, m') \left[ \delta_{m'_1, m_1 + 1/2} \delta_{m'_2, m_2 + 1/2} \delta_{m'_3, m_3 + 1/2}
+ \frac{1}{2} \sum_{k<l} \sum_{h \neq k, l} \sum_{\nu = -1/2}^{1/2} (-1)^{1/2 + \nu} \delta_{m'_h, m_h + \nu} \delta_{m'_k, m_k + 1/2} \delta_{m'_l, m_l + 1/2}
+ \frac{1}{4} \sum_{k<l} \sum_{h \neq k, l} \sum_{\nu = -1/2}^{1/2} (1 + (n_k \cdot n_l)) \sum_{\nu_k = -1/2}^{1/2} \sum_{\nu_l = -1/2}^{1/2} (-1)^{1 + \nu_k + \nu_l} \delta_{m'_h, m_h + \nu_k} \delta_{m'_k, m_k + \nu_k} \delta_{m'_l, m_l + \nu_l}
+ \frac{1}{8} \left( (n_1 \cdot n_2) + (n_2 \cdot n_3) + (n_3 \cdot n_1) - i (n_1 \cdot [n_2 \times n_3]) \right) \times \sum_{\nu_1 = -1/2}^{1/2} \sum_{\nu_2 = -1/2}^{1/2} \sum_{\nu_3 = -1/2}^{1/2} \left( (-1)^{1/2 + \nu_1 + \nu_2 + \nu_3} \delta_{m'_1, m_1 + \nu_1} \delta_{m'_2, m_2 + \nu_2} \delta_{m'_3, m_3 + \nu_3} \right)
- \left( \frac{2j + 2}{2j} \right)^2 K_{j-1/2}(N, m)
\]

Now if we recall (24), we obtain the recurrence relation for spin-tomographic kernels. To be more precise, the kernel for spin \((j + 1/2)\) is expressed in terms of kernels for spin \(j\) and spin \((j - 1/2)\) as follows:

\[
K_{j+1/2}(x, x, x_1) \equiv K_{j+1/2}(N, m) = 2 \left( \frac{2j + 2}{2j + 1} \right)^2 \sum_{m'_1, m'_2, m'_3} K_j(N, m')
\times \left[ \delta_{m'_1, m_1 + 1/2} \delta_{m'_2, m_2 + 1/2} \delta_{m'_3, m_3 + 1/2} + \frac{1}{2} \sum_{k<l} \sum_{h \neq k, l} \sum_{\nu = -1/2}^{1/2} (-1)^{1/2 + \nu} \delta_{m'_h, m_h + \nu} \delta_{m'_k, m_k + 1/2} \delta_{m'_l, m_l + 1/2}
+ \frac{1}{4} \sum_{k<l} \sum_{h \neq k, l} \sum_{\nu = -1/2}^{1/2} (1 + (n_k \cdot n_l)) \sum_{\nu_k = -1/2}^{1/2} \sum_{\nu_l = -1/2}^{1/2} (-1)^{1 + \nu_k + \nu_l} \delta_{m'_h, m_h + \nu_k} \delta_{m'_k, m_k + \nu_k} \delta_{m'_l, m_l + \nu_l}
+ \frac{1}{8} \left( (n_1 \cdot n_2) + (n_2 \cdot n_3) + (n_3 \cdot n_1) - i (n_1 \cdot [n_2 \times n_3]) \right) \times \sum_{\nu_1 = -1/2}^{1/2} \sum_{\nu_2 = -1/2}^{1/2} \sum_{\nu_3 = -1/2}^{1/2} \left( (-1)^{1/2 + \nu_1 + \nu_2 + \nu_3} \delta_{m'_1, m_1 + \nu_1} \delta_{m'_2, m_2 + \nu_2} \delta_{m'_3, m_3 + \nu_3} \right)
- \left( \frac{2j + 2}{2j} \right)^2 K_{j-1/2}(N, m).
\]

This recurrence formula reveals a special feature of the star-product kernels. Indeed, the star-product kernel for qudits with an arbitrary spin \(j\) can be expressed in terms of the kernel for qubits and that for spins equal to zero.

VI. CONCLUSIONS

To resume we point out the main results of our work. We obtained the explicit form of the star-product kernel for spin tomograms in terms of Fourier transform of the Chebyshev
The explicit form of the recurrence relation for spin-tomographic star-product kernels is another new result of the work. This relation provides a connection of the kernels for qudits \((j \geq 1)\) with two basic kernels for the cases \(j = 0\) and \(j = 1/2\). We clarified the relations between different forms of quantizers and dequantizers used in spin tomography and available in the literature \([15, 16, 27, 29]\). We established that all the different expressions for the quantizers and dequantizers are equivalent on the set of tomographic symbols for the spin operators and spin states. The kernel of the dual tomographic star-product is also expressed in terms of Chebyshev polynomials (see Eq. (26)) and calculated explicitly (see Eq. (B3)). Within the proposed technique, we also managed to obtain explicit expressions for delta-function on the tomogram set. In the work \([44]\), the relation of irreps characters for compact and finite groups with kernels of star-products of the functions on the groups was obtained. In the present work, we found the relation of the characters of \(SU(2)\)-group irreps with the star-product of functions depending on both group element and weight of irreps.

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APPENDIX A: EQUIVALENCY OF STAR-PRODUCT KERNELS

Since there exist some different explicit forms of the spin-tomographic star-product kernel, in this Appendix, we consider the difference \(\Delta_j\) between the kernel \((13)\) expressed in terms of Clebsch-Gordan coefficients and the kernel \((15)\) derived on the basis of the exponential representation of the quantizer and dequantizer operators. In order to illustrate the deviation between these kernels one can specify \(\Delta_j = K''_j(x_3, x_2, x_1) - K'_j(x_3, x_2, x_1)\) for the low-spin states. In case of qubits, the reader will have no difficulty in showing that \(\Delta_j = 1/2 = 0\). As far as qutrits are concerned, the direct computation leads to the following rather difficult result:

\[
\Delta_j = \frac{1}{36} \left( 3(n_2 \cdot n_3)^2 - 1 \right) - i \frac{1}{8} m_1 (n_2 \cdot n_3) (n_1 \cdot [n_2 \times n_3]) \\
+ \frac{1}{8} m_1 m_2 \left( 3(n_2 \cdot n_3)(n_3 \cdot n_1) - (n_1 \cdot n_2) \right) + \frac{1}{8} m_1 m_3 \left( 3(n_1 \cdot n_2)(n_2 \cdot n_3) - (n_3 \cdot n_1) \right) \\
+ \frac{1}{144} (3m_1^2 - 2) \left\{ 5 - 3 \left( (n_1 \cdot n_2)^2 + (n_2 \cdot n_3)^2 + (n_3 \cdot n_1)^2 \right) - 9 \left( n_1 \cdot [n_2 \times n_3] \right)^2 \right\} \\
+ \frac{1}{36} (3m_2^2 - 2) \left( 5 \left( 3(n_1 \cdot n_2)^2 - 1 \right) + 2 \right) + \frac{1}{36} (3m_3^2 - 2) \left( 5 \left( 3(n_1 \cdot n_2)^2 - 1 \right) + 2 \right) \\
- \frac{5}{8} m_1 (3m_2^2 - 2)(n_2 \cdot n_3) (n_1 \cdot [n_2 \times n_3]) - i \frac{5}{8} m_1 (3m_3^2 - 2)(n_2 \cdot n_3) (n_1 \cdot [n_2 \times n_3])
\]
The difference is especially written in a form such that each summand gives zero while being integrated with tomographic symbols. The difference of this type is ascribed to the appearance of redundant terms in the quantizer operator. For instance, in case of qutrits, the exponential representation of the quantizer operator contains two additional terms as compared with the quantizer found in [27, 28]:

$$\Delta \hat{D}_{j=1}(m, n) = \frac{3m^2 - 2}{6} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) + \frac{1}{6} \hat{R}(n) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{array} \right) \hat{R}^\dagger(n).$$  \quad (A2)

Let us remark that the quantizer enables to reconstruct the density operator if the state to-mogram is given. It can be easily checked that the integration of the difference $\Delta \hat{D}_{j=1}(m, n)$ with any spin tomogram $w_{j=1}(m, n)$ gives zero and does not change the density operator $\hat{\rho}$.

**APPENDIX B: GENERALIZATION TO OTHER TOMOGRAPHIC KERNELS**

The developed approach for calculating the spin-tomographic star-product kernel can be generalized to other tomographic kernels. Using the results obtained, one can present universal formulas which yield all desired kernels, in particular, the ordinary star-product kernel as well as the dual one and the expression for delta-function on the set of tomograms.

First we introduce the universal constituent part of the form

$$T_j(x_3, x_2, x_1, s_3, s_2, s_1)$$

$$= \frac{1}{(1 - 3s_3^2)(1 - 3s_2^2)(1 - 3s_1^2)} \sum_{k=0}^{[j]} \sum_{p_1 + \ldots + p_4 = 2j - 2k} \frac{(-1)^k(2j - k)!}{2^{p_1 + p_5 + p_6 + p_7 + 2p_8}} \prod_{t=1}^{8} p_t! \times \left(1 + (n_2 \cdot n_3)\right)^{p_5} \left(1 + (n_3 \cdot n_1)\right)^{p_6} \left(1 + (n_1 \cdot n_2)\right)^{p_7} \times \left(1 + (n_1 \cdot n_2) + (n_2 \cdot n_3) + (n_3 \cdot n_1) - i (n_1 \cdot [n_2 \times n_3])\right)^{p_8} \times \left(-j - m_1 - s_1 + k + 1 \right) \left(j - m_1 - s_1 - k - p_1 - p_2 - p_4 - p_5 \right) \left(-j - m_2 - s_2 + k + 1 \right) \left(j - m_2 - s_2 - k - p_1 - p_2 - p_3 - p_6 \right)$$
The relation between symbols has the form

\[
\left( -j - m_3 - s_3 + k - 1 \right) \times \left( j - m_3 - s_3 - k - p_1 - p_3 - p_4 - p_7 \right). \quad (B1)
\]

Then for the ordinary star-product kernel we have

\[
K_j(x_3, x_2, x_1) = \text{Tr} \left( \hat{D}_j(x_3) \hat{D}_j(x_2) \hat{U}_j(x_1) \right) = (2j+1)^2 \sum_{s_3=-1}^{1} \sum_{s_2=-1}^{1} T_j(x_3, x_2, x_1, s_3, s_2, s_1 = 0), \quad (B2)
\]

while the dual star-product kernel reads

\[
K'^d_j(x_3, x_2, x_1) = \text{Tr} \left( \hat{U}_j(x_3) \hat{U}_j(x_2) \hat{D}_j(x_1) \right) = (2j + 1) \sum_{s_1=-1}^{1} T_j(x_3, x_2, x_1, s_3 = 0, s_2 = 0, s_1). \quad (B3)
\]

Now we present the universal function for kernels which depend on two sets of variables \(x_2\) and \(x_1\):

\[
Q_j(x_2, x_1, s_2, s_1) = \frac{1}{(1 - 3s_2^2)(1 - 3s_1^2)} \sum_{k=0}^{[j]} \sum_{p_1+p_4=2j-2k} (-1)^k (2j-k)! \prod_{l=1}^{4} p_l! \left( 1 + (n_1 \cdot n_2) \right)^{p_4} \times \left( -j - m_1 - s_1 + k - 1 \right) \left( j - m_1 - s_1 - k - p_1 - p_2 \right) \left( -j - m_2 - s_2 + k - 1 \right) \left( j - m_2 - s_2 - k - p_1 - p_3 \right). \quad (B4)
\]

Note that this function can be obtained from (B1) if we leave out the third binomial coefficient, put \(p_2 = p_5 = p_6 = p_8 = 0\), and redesignate \(p_4 \rightarrow p_2\).

From (B4) it is readily seen that the kernel of the unity operator on the set of spin tomograms reads

\[
K^u_j(x_2, x_1) = \text{Tr} \left( \hat{D}_j(x_2) \hat{U}_j(x_1) \right) = (2j + 1) \sum_{s_2=-1}^{1} Q_j(x_2, x_1, s_2, s_1 = 0). \quad (B5)
\]

Let us now consider the transition from the ordinary tomographic symbols to the dual ones. The relation between symbols has the form

\[
f^d_A(x_1) = \int f^d_A(x_2) K^{r-o-d}_j(x_2, x_1) dx_2, \quad (B6)
\]

where the intertwining kernel reads

\[
K^{r-o-d}_j(x_2, x_1) = \text{Tr} \left( \hat{D}_j(x_2) \hat{D}_j(x_1) \right) = (2j + 1)^2 \sum_{s_2=-1}^{1} \sum_{s_1=-1}^{1} Q_j(x_2, x_1, s_2, s_1). \quad (B7)
\]

Similarly, a transition from the dual tomographic symbols to the ordinary ones is defined through

\[
f^d_A(x_1) = \int f^d_A(x_2) K^{d-o}_j(x_2, x_1) dx_2, \quad (B8)
\]
where the intertwining kernel reads

\[
K_{d-o}^j(x_2, x_1) = \text{Tr} \left( \hat{U}_j(x_2) \hat{U}_j(x_1) \right) = Q_j(x_2, x_1, s_2 = 0, s_1 = 0).
\]  

(B9)
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