ON THE CATEGORY OF FINITELY PRESENTED SMOOTH MOD $p$ REPRESENTATIONS OF $GL_2(F)$, $F/\mathbb{Q}_p$ FINITE.

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1. Introduction

Let $F$ be a field of characteristic $p$. If $H$ is a locally profinite group, let $C_F(H)$ be the category of smooth representations of $G$ over $F$.

Definition 1.1. Let $V$ be a smooth $F$-representation of a locally profinite group $H$ with centre $Z$. Then $V$ is:

1. **finitely generated** if for some compact open subgroup $K$ of $H$ there is a surjection
   \[ \text{ind}_{KZ}^G W \to V \]
   for a smooth finite-dimensional $F$-representation $W$ of $KZ$;

2. **finitely presented** if for some compact open subgroup $K$ of $H$ there is an exact sequence
   \[ \text{ind}_{KZ}^G W_1 \to \text{ind}_{KZ}^G W_2 \to V \to 0 \]
   for $W_1$ and $W_2$ smooth finite-dimensional $F$-representations of $KZ$.

Let $F$ be a local field of characteristic 0 with ring of integers $\mathcal{O}_F$ and residue field $k$ of characteristic $p$. Then $F$ is:

Theorem 1.2. The category of finitely presented smooth $F$-representations of $G$ is an abelian subcategory of $C_F(G)$.

This is Corollary 4.3 below. In fact, we prove the same result with $F$ replaced by any finite dimensional division algebra over $\mathbb{Q}_p$. We also work with coefficients in a general complete local noetherian $W(F)$-algebra with residue field $F$, referring to [Eme10] for the definitions and basic properties of smooth representations over such rings.

The theorem is equivalent to the statement that the kernel\footnote{and the cokernel, but this is automatic} of any map between finitely presented smooth representations is itself finitely presented. If $C_F(G)$ were the category of modules over a ring $R$, this would be the statement that $R$ is a coherent ring. Indeed, we will prove the theorem by considering smooth $F$-representations as modules over the amalgamated product

\[ F[[K]] \star_{F[[I]]} F[[K']] \]

where $K' = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} K \begin{pmatrix} 1 & 0 \\ 0 & \pi^{-1} \end{pmatrix}$ for $\pi$ a uniformising element of $D$ and $I = K \cap K'$. Then a result of Åberg [Å82] shows that, under certain conditions, an amalgamated product of coherent rings over a noetherian ring is itself coherent.
Throughout, unless otherwise stated, by ‘module’, ‘noetherian’ or ‘coherent’ we mean ‘left module’, ‘left noetherian’ or ‘left coherent’.

The theorem is motivated by Taylor–Wiles patching and the construction of \( CEG_{+13} \). The authors of \( CEG_{+13} \) construct (for \( G = GL_n(F) \)) a smooth \( R_\infty \)-representation \( M_\infty \) of \( G \) for \( R_\infty \) (almost) a local Galois deformation ring. There are many choices made in this construction and it is of great interest in the \( p \)-adic local Langlands program to know to what extent \( M_\infty \) is independent of these choices. However, it seems to be possible to obtain information directly about \( M_\infty \) only by considering \( M_\infty(\sigma) = \text{Hom}_G(\text{ind}_{KZ}^{\sigma} M_\infty, M_\infty) \) for \( \sigma \) a smooth \( W(F) \)-representation of \( K \). This leads to studying the category generated by such compact inductions.

Finitely presented representations were previously studied by Hu ([Hu12]), Vigneras ([Vig11]), and Schraen ([Sch15]). In particular, [Vig11] Theorem 6 shows that a smooth admissible finitely presented representation of \( GL_2(F) \) has finite length, and that all of its subquotients are also admissible and finitely presented. On the other hand, the main result of [Sch15] says that, if \( F \) is a quadratic extension of \( \mathbb{Q}_p \), then an irreducible supersingular representation of \( GL_2(F) \) admitting a central character is never finitely presented, and so there are a great many representations to which our result does not apply!

I do not know whether Theorem 1.2 holds when \( G = GL_n(F) \) (or any \( p \)-adic Lie group), or in the case when \( F \) has positive characteristic. The methods of this paper do not apply in either case — in the former because \( G \) is not (up to centre) an amalgam of two compact open subgroups, and in the latter because then \( GL_2(O_F) \) is not \( p \)-adic analytic, and its completed group ring is not noetherian.

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2. Smooth representations.

For the rest of this article, let \( F \) be a finite field of characteristic \( p \) and let \( A \) be a complete local noetherian \( W(F) \)-algebra with maximal ideal \( m \) and residue field \( F \). Let \( H \) be a locally profinite group with centre \( Z \).

**Definition 2.1.** (see [Eme10] definition 2.2.1) A smooth \( A \)-representation of \( H \) is an \( A \)-module \( V \) with a left action of \( H \) such that

1. every \( v \in V \) is killed by \( m^i \) for some \( i \);
2. every \( v \in V \) has open stabiliser in \( H \).

The category of smooth \( A \)-representations of \( H \) is denoted \( C_A(H) \).

In other words, \( V = \varprojlim V[m^i] \) and the action is continuous for the discrete topology on \( V \).

In the introduction (Definition 1.1) we gave the definition of a finitely presented smooth \( F \)-representation of \( H \). Here is the definition with coefficients.

**Definition 2.2.** If \( V \) is a smooth \( A \)-representation of \( H \), then \( V \) is finitely generated/presented if \( V[m] \) is finitely generated/presented.

**Remark 2.3.** Any finitely generated smooth \( A \)-representation of \( H \) is \( Z \)-finite in the sense of [Eme10] Definition 2.3.1.

If \( Z \) is compact then we may replace \( KZ \) by \( K \) in Definition 1.1.
3. Amalgamated products of completed group rings.

If $K$ is a profinite group, let

$$A[[K]] = \lim_{J \in K_{\text{open}}} A[K/J]$$

be the completed group ring, a compact topological $A$-algebra that is augmented: there is an $A$-algebra homomorphism $A[[K]] \to A$.

Let $K_1, K_2$ and $I$ be profinite groups equipped with inclusions $f_i : I \hookrightarrow K_i$ of $I$ as a common open subgroup of $K_i$. Then there are maps $f_i : A[[I]] \to A[[K_i]]$ of topological augmented $A$-algebras.

**Definition 3.1.** In the above situation, let $H = K_1 \ast_I K_2$. Define the ring $A \langle H \rangle$ as the amalgamated product

$$A \langle H \rangle = A[[K_1]] *_{A[[I]]} A[[K_2]].$$

**Remark 3.2.** If $K_i$ and $I$ are finite, then $A \langle H \rangle$ is simply the group ring of $H$ over $A$. This is because the functor $G \mapsto A[G]$ from groups to $A$-algebras is a left-adjoint, and so commutes with the colimit $*$. We explain the relation of $A \langle H \rangle$ to the category of smooth representations of $H$. First, we describe the topology on $H$. By [Ser77], Théorème 1, the natural map $I \to H$ is injective.

**Proposition 3.3.** With the colimit topology, $H$ is a locally profinite group with a basis of open neighbourhoods of the identity being given by open neighbourhoods of $I$.

**Proof.** Let $H$ and $H'$ respectively denote $H$ with the colimit topology and the topology for which open subgroups of $I$ are a basis of open neighbourhoods of the identity in $H'$. Let $i : H \to H'$ and $j : H' \to H$ be the identity maps; we have to show that they are both continuous. But $i$ is continuous by the universal property of $H$, and $j$ is continuous because the map $I \to H$ is continuous. $\square$

The maps $A[[K_i]] \to A[[K_i]] \to A \langle H \rangle$ agree on $A[I]$ and so induce a homomorphism $\iota : A[H] \to A \langle H \rangle$. In this way, every left $A \langle H \rangle$-module gives rise to an $A$-representation of $H$. Conversely, we have:

**Lemma 3.4.** Suppose that $V$ is a smooth $A$-representation of $H$. Then there is a left action of $A \langle H \rangle$ on $V$ extending that of $A[H]$.

**Proof.** Since $V$ is smooth, the actions of $A[K_1]$ and $A[K_2]$ extend to actions of $A[[K_1]]$ and $A[[K_2]]$. Since $V$ is a representation of $H$, these actions agree on $A[I]$ and hence on $A[[I]]$. Therefore by the universal property we get an action of $A \langle H \rangle$ on $V$, which clearly extends that of $A[H]$. $\square$

**Lemma 3.5.** The map $\iota$ is injective.

**Proof.** Suppose that $f \in A[H]$. Choose a smooth representation $V$ of $H$ on which $f$ acts non-trivially, for instance $(A/m^n)[H/J]$ for a sufficiently small compact open subgroup $J$ of $G$ and sufficiently large power $m^n$ of the maximal ideal $m$ of $A$. Then by lemma 3.4, the action of $A[H]$ extends to an action of $A \langle H \rangle$ on $V$. Since $\iota(f)$ still acts non-trivially, it must be non-zero in $A \langle H \rangle$. $\square$
Next we relate the notions of finitely generated/presented smooth representations of $H$ to finitely generated/presented $A \langle H \rangle$-modules. Note first that the centre of $H$ is compact, being a closed subgroup of $I$, and so remark 2.3 applies.

**Lemma 3.6.** Suppose that $V$ is a smooth $A$-representation of $H$. Then the following are equivalent:

1. $V$ is finitely generated.
2. $V[m]$ is finitely generated as an $F[H]$-module.
3. $V[m]$ is finitely generated as an $F \langle H \rangle$-module.

**Proof.** We can assume that $A = F$, by definition 2.2.

Firstly, (1) implies (2). For this it suffices to show that, if $K \subset H$ is compact open and $W$ is a finite-dimensional smooth representation of $K$, then $\text{ind}_K^H W$ is finitely generated as an $F[H]$-module. This is true, as it is generated by the (finite-dimensional) subspace of functions supported on $K$.

Clearly (2) implies (3).

To see that (3) implies (1), let $V$ be a smooth $F$-representation of $H$ that is finitely generated as a $A \langle H \rangle$-module. Let $v_1, \ldots, v_n$ be a set of generators. For any $v \in V$, we can write $v = \sum_{j=1}^n f_j v_i$ with $f_i \in A \langle H \rangle$. Each $f_i$ can be written as a finite sum of finite products of elements of $F[[K_1]]$ and $F[[K_2]]$. Since every element of $V$ is fixed by an open subgroup of $I$, and the image of $F[[K_1]]$ in $F[[K_2]]$ is dense, we can replace the $f_i$ by elements of $F[K_1] \ast F[H] F[K_2] = F[K]$ without affecting the vectors $f_i v_i$. Therefore $v \in F[H]v_1 \oplus \ldots \oplus F[H]v_n$. So $v_1, \ldots, v_n$ generates $V$ as a $k[H]$-module, as required.

It is not true that a finitely presented smooth $F$-representation of $H$ will be finitely presented as an $F[H]$-module; this is already false for the representation $\text{ind}_K^H F$, as long as $K$ is infinitely generated. If, however, $H$ (or equivalently, any compact open subgroup of $H$) is a $p$-adic analytic group, then we have the following result of Lazard (see [Eme10] Theorem 2.1.1 for the statement with coefficients).

**Theorem 3.7.** If $H$ is a $p$-adic analytic group, then $A[[K]]$ is noetherian for every compact open subgroup $K$ of $H$.

**Lemma 3.8.** Suppose that $H$ is a $p$-adic analytic group. Let $V$ be a smooth $A$-representation of $H$. Then $V$ is finitely presented if and only if $V[m]$ is finitely presented as an $F \langle H \rangle$-module.

**Proof.** We may suppose that $A = F$, by definition 2.2.

The backwards implication follows from Lemma 3.6. Suppose that $V[m]$ is finitely presented as an $F \langle H \rangle$-module. Then it is finitely generated as an $F$-representation of $H$ by Lemma 3.6, and so there is a surjection

$$\alpha : \text{ind}_K^H W \to V \to 0$$

for some finite dimensional smooth representation $W$ of a compact open subgroup $K \subset H$. The kernel of $\alpha$ is a smooth representation of $H$ that is finitely generated as an $F \langle H \rangle$-module, by [Sta17, Tag 0519] (5). Therefore it is finitely generated as an $F$-representation of $H$, by Lemma 3.6.

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2 Strictly speaking, [Sta17, Tag 0519] is only stated for modules over commutative rings. However, it is still true, with an identical proof, in the non-commutative case.
Suppose now that $V$ is finitely presented. We first reduce to the case $V = \text{ind}_K^H F$: there is a short exact sequence
\[
\text{ind}_K^H W_1 \rightarrow \text{ind}_K^H W_2 \rightarrow V \rightarrow 0
\]
with $W_1$ and $W_2$ finite-dimensional smooth $F$-representations of $K$, and we must show that $V$ is finitely presented as an $F \langle H \rangle$-module. By lemma 3.6 above, $\text{ind}_K^H W_1$ is finitely generated as an $F \langle H \rangle$-module. Therefore, by [Sta17, Tag 0519] (4), it suffices to consider the case $V = \text{ind}_K^H W_2$. There is a subgroup $K' \subset K$, smooth representation $W_3$ of $K'$, and $F$-vector space $F^n$ with trivial $K'$-action, and a short exact sequence
\[
0 \rightarrow \text{ind}_K^H W_3 \rightarrow \text{ind}_K^H F^n \rightarrow \text{ind}_K^H W_2 \rightarrow 0
\]
and so, by the same argument again, we may assume that $V = (\text{ind}_K^H F)^n = (F[H/K])^n$. Shrinking $K'$ further if necessary, we see that it suffices to prove that $\text{ind}_K^H F$ is finitely generated whenever $K$ is an open subgroup of $I$.

The natural map $F[H] \rightarrow F[H/K]$ extends to a map $F \langle H \rangle \rightarrow F[H/K]$ with kernel $J$ for some left ideal $J$ of $F \langle H \rangle$. We have to show that $J$ is finitely generated.

**Claim:** The ideal $J$ is the left ideal of $F \langle H \rangle$ generated by the set
\[
\{ k - 1 : k \in K \}.
\]

Granted the claim, we see that $J$ is generated over $F \langle H \rangle$ by the ideal $J_0$ of $F[[K]]$ generated by all $(k - 1)$. But $F[[K]]$ is noetherian by Theorem 3.7, and so $J_0$ is finitely generated, and so $J$ is finitely generated as required.

**Proof of claim:** For any compact open subgroup $K$ of $I$, jet $J_K$ be the left ideal of $F[H]$ generated by $S = \{ (k - 1) : k \in K \}$ and let $J_{K,i}$ be the left ideal of $F[[K_i]]$ generated by the same set. Then $F[K_i] \rightarrow F[[K_i]]/J_{K,i}$ is surjective. Moreover, $J_{K,i}$ contains some two sided ideal $J_{K',i}$ for a compact open subgroup $K' \subset K$ that is normal in $K_i$. It follows by induction on the number of elements of $F[[K_i]]$ needed to write an element of $F \langle H \rangle$ that the map $F[H] \rightarrow F \langle H \rangle /J_K$ is surjective.

Now fix $K$ and let $J = J_K$. The natural surjection $F[H] \rightarrow F[H/K]$ induces an isomorphism $F[H]/J \cong F[H/K]$. But, since $F[H/K]$ is smooth, this map factors as
\[
F[H]/J \rightarrow F \langle H \rangle /F \langle H \rangle \cdot J \rightarrow F[H/K].
\]
The first map is a surjection, as above, and the composite is an isomorphism, and so both maps are isomorphisms. It follows that $J = F \langle H \rangle \cdot J$ as required.

3.1. **Coherence.** Recall that a ring $R$ is coherent if any of the following equivalent definitions hold:

1. every finitely generated left ideal of $R$ is finitely presented;
2. if $f : M \rightarrow N$ is a map of finitely presented left $R$-modules, then $\ker(f)$ is finitely presented;
3. the category of finitely presented left $R$-modules is an abelian of the category of left $R$-modules.

**Lemma 3.9.** The ring $A \langle H \rangle$ is flat as a (left or right) $A[[K_i]]$ or $A[[I]]$-module.

**Proof.** This follows from [Coh59], Corollaries 1 and 2 to Theorem 4.4, since the rings $A[[K_i]]$ are free $A[[I]]$-modules with $A[[I]]$ embedded as a direct summand.

**Proposition 3.10.** If $A[[K_i]]$ are coherent and $A[[I]]$ is noetherian, then $A \langle H \rangle$ is coherent.
Proof. This immediately follows from [Å82] Theorem 12; the hypotheses of that theorem are satisfied by Lemma 3.9. For the convenience of the reader, we summarise the argument of [Å82] in the case of interest to us. It uses the characterisation of left coherent rings as those for which arbitrary products of right flat modules are flat. Let $R$, $S$ and $T$ be rings such that $S$ and $T$ are $R$-algebras, and $Q = S \ast_R T$ is flat as a right $R$, $S$ or $T$-module; we will take $R = A[[f]]$ and $S = A[[K_1]]$, $T = A[[K_2]]$. Then there is a Mayer–Vietoris sequence for $\text{Tor}_Q^i$ in terms of $\text{Tor}_S^i$, $\text{Tor}_R$ and $\text{Tor}_T$. If $R$ is left noetherian and $S$ and $T$ are left coherent, then take a set $(F_i)_{i \in I}$ of right flat $Q$-modules and compare the Mayer–Vietoris sequence for $\text{Tor}(\prod F_i, M)$ with the product of those for $\text{Tor}(F_i, M)$, for an arbitrary left $Q$-module $M$. Using that $S$ and $T$ are left coherent and that, as $R$ is left noetherian and the $F_i$ are right flat $R$-modules, $(\prod F_i) \otimes_R M \to \prod(F_i \otimes_R M)$ is injective, we get that $\text{Tor}_Q^i(\prod F_i, M) = 0$ for $i > 0$ as required.

Combining with Theorem 3.7 we get:

Corollary 3.11. Suppose that $H$ is a $p$-adic analytic group that is an amalgamated product of two compact open subgroups. Then $A(H)$ is coherent.

Theorem 3.12. Suppose that $H$ is a $p$-adic analytic group that is an amalgamated product of two compact open subgroups. Then the category of finitely-presented smooth $A$-representations of $H$ is an abelian subcategory of $C_A(H)$.

Proof. It suffices to show that the kernel or cokernel of map of finitely-presented smooth $A$-representations of $H$ is also a finitely-presented smooth $A$-representation. This is straightforward for cokernels, and does not require the ring $A(H)$. For kernels, suppose that $f : V \to W$ is a map of finitely-presented smooth $A$-representations of $H$. Then $\ker(f)$ is a smooth $A$-representation of $H$, and by Lemma 3.8 and Corollary 3.11 it is finitely presented as a left $A(H)$-module. By Lemma 3.8 again, it is a finitely-presented $A$-representation of $H$.

4. Applications.

Let $F$ be a local field of characteristic 0 with ring of integers $\mathcal{O}_F$ and residue field $k$ of characteristic $p$, and let $D$ be a division algebra over $F$ with ring of integers $\mathcal{O}_D$. Choose a uniformiser $\pi$ of $D$. Let $G = GL_2(D)$ and let $G' = SL_2(D)$ be the subgroup of elements of reduced norm 1. Let $K_1 = GL_2(\mathcal{O}_D)$ and let $K_1' = SL_2(\mathcal{O}_D) = K' \cap SL_2(D)$. Let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \in G$, and let $K_2 = \alpha K_1 \alpha^{-1}$ and $K_2' = K_2 \cap G'$. Let

$$I = K_1 \cap K_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_1 : c \equiv 0 \mod \pi \right\}$$

and $I' = I \cap G' = K_1' \cap K_2'$.

Theorem 4.1. The category of finitely presented smooth $A$-representations of $G'$ is an abelian subcategory of $C_A(G')$.

Proof. By a theorem of Ihara (Serre [Ser77] Chapter II Corollary 1) we know that $G' = K_1' \ast_{I'} K_2'$. The theorem follows from Theorem 3.12.

Theorem 4.2. The category of finitely presented smooth $A$-representations of $G$ is an abelian subcategory of $C_A(G)$.
Proof. We can assume that \( A = F \). Let \( G^0 \) be the subgroup of \( G \) of elements whose reduced norm is in \( O_F^\times \) and let \( Z \) be the centre of \( G \). Then \( ZG^0 \) has finite index in \( G \). It follows that a smooth \( F \)-representation of \( G \) is finitely presented if and only if its restriction to \( ZG^0 \) is. Let \( f : V_1 \rightarrow V_2 \) be a map of smooth finitely presented \( F \)-representations of \( ZG^0 \). Then as \( V_1 \) and \( V_2 \) are finitely presented representations of \( G^0 Z \), they are (by restriction) finitely presented as representations of \( G^0 \). As \( G^0 = K_1 \ltimes K_2 \) by [Ser77] Chapter II Theorem 3, Theorem 3.12 implies that \( \ker(f) \) is a smooth finitely presented \( F \)-representation of \( G^0 \). But \( \ker(f) \) is also a \( Z \)-finite representation of \( G^0 Z \) by remark 2.3 and [Eme10] Lemma 2.3.2. The result follows, as a smooth \( Z \)-finite \( F \)-representation of \( G^0 Z \) that is finitely presented as a \( G^0 \)-representation is finitely presented as a \( G^0 Z \) representation. \( \square \)

In particular we have:

**Corollary 4.3.** The category of finitely presented smooth \( A \)-representations of \( GL_2(F) \) is an abelian subcategory of \( C_A(GL_2(F)) \).

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