FIELDS AND SYMMETRIES OF 2D STRINGS

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ABSTRACT

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1. Introduction

Studies of strings in lower dimensions have produced extensive results with major new insights. First of all, the fruitful relationship to the 1 dimensional matrix model [1] has provided exact solutions for correlation functions and free energy. The dynamics of a sole field theoretic degree of freedom, the massless tachyon has been well understood. It is given by a simple scalar Lagrangian of the collective field [2] theory. In this field theory one can expand perturbatively in the string coupling constant and study loop effects [3]. This field theory is also exactly integrable possessing an infinite sequence of conserved currents and an exact S–matrix [4-8]. In addition to the tachyon there also appears an infinite sequence of imaginary energy discrete states originating in the inverted matrix oscillator. The S–matrix can also be found using the simple dynamics of the harmonic oscillator as we explain. It is in the matrix model language that a larger space–time symmetry appears naturally and was seen [8] to take the form of a $W_\infty$ algebra. It plays a role of a spectrum generating algebra leading to the infinite sequence of discrete states. The exact solution of the theory can be traced to this symmetry. The same symmetry was also established in the conformal field theory approach [11]. Vertex operators for discrete states close under operator products with structure constants of the $W_\infty$ group. The symmetry charges can be constructed. They act nonlinearly on the tachyon (implying nontrivial Ward identities). This nonlinear identities (which are sufficient to deduce the S–matrix) can be seen to be the nonlinear collective field representations [13]. It is in this sense that one can think of the scalar field theory as a (minimal) realization of the $W_\infty$ structure. An extension of this would include a sequence of additional discrete fields (representing the global degrees of freedom). The problem of understanding these fields, their couplings to the tachyon and the complete gauge invariant theory is nontrivial. We address some aspects of this problem. We first describe in Section 2 the simple correspondence of the scalar field theory and the matrix model. This we explain can be used to deduce exact eigenstates and even the S-matrix directly from the matrix harmonic oscillator. In section 3 we discuss $W_\infty$. In section 4 we describe a possible extension involving coupling of discrete topological fields to the scalar tachyon.
2. Scalar Dynamics

The lowest excitation of string theory, the tachyon becomes massless in two dimensions and represents the only real field theoretic degree of freedom. It is described by the scalar field $\phi(x, t)$ and its conjugate $\pi(x, t)$ with its dynamics completely given by the cubic (collective) field theory

$$H = \int \frac{dx}{2\pi} \left\{ \frac{1}{6} (\alpha_+(x, t)^3 - \alpha_-(x, t)^3) - \frac{1}{2} x^2 (\alpha_+ - \alpha_-) \right\}$$

Here $\alpha_{\pm} = \Pi_{x_{\pm}} \pm \pi \phi$ simply denotes the two chiral components. The ground state is given by the static background $\pi \phi_0 = \sqrt{x^2 - \mu}$ and the corresponding perturbation expansion in $g_{st}^2 = 1/\mu^2$ defines the scattering amplitudes [3]. These were computed [3-6] explicitly and compared with similar amplitudes in conformal field theory [7]. The agreement of S–matrix elements (and loop corrections) implies the exactness of the above Lagrangian.

The fact that the collective Lagrangian is induced from the simple matrix dynamics of an inverted oscillator is responsible for the exact solvability [4,8] of the theory. A set of simple rules gives the transition: the matrix variables are $M(t)$ and $P(t) = \dot{M}$ and the transition to field theory can be summarized by:

$$M \rightarrow x$$

$$P \rightarrow \alpha(x, t)$$

$$\text{Tr} \rightarrow \int \frac{dx}{2\pi} \int d\alpha$$

The matrix model hamiltonian

$$H = \frac{1}{2} \text{Tr} (P^2 - M^2)$$

is then seen to transform into the cubic collective hamiltonian. In addition all commutation relations (or Poisson brackets) of $U(N)$ invariant observables simply go over
from the matrix model to the field theory. More importantly a remarkable physical phenomenon taken place: from linear matrix model dynamics

\[ M(t) = M(0)cht + P(0)sh t \]

a highly nonlinear string theory dynamics is generated as a collective effect.

One can use the above transition rules to directly compute general scattering amplitudes and exhibit higher symmetries. First a one parameter set of exact states \[ [8] \] is generated from the oscillators

\[ \text{Tr} (P \pm M)^n \rightarrow T_n^\pm = \int \frac{dx}{2\pi} \frac{(\alpha \pm x)^{n+1}}{n+1} \]

\[ \{H, T_n^\pm\} = \pm nT_n^\pm \]

They correspond to exact tachyon eigenstates \( (p_o = E = \pm in) \) and represent field theoretic versions of tachyon vertex operator of the conformal field theory

\[ T_p^\pm = e^{ipX^0 + (\pm 2 + |p|)\varphi} \]

We now give a simple derivation of the general tachyon scattering amplitudes \[ [13] \]. It turns out that these can be found directly from the oscillator states whose dynamics is exactly known. Note first that the transition to collective field theory seemingly introduces a degeneracy since one could formally write separate states for the \( \alpha_+ \) and \( \alpha_- \) sector. The two sectors commute since the Poisson brackets are \( \{\alpha_+, \alpha_-\} = 0 \).

For the tachyon one would have two states

\[ \int \frac{dx}{2\pi} \frac{(\alpha_+ \pm x)^{1\pm ik}}{1 \pm ik} \quad \text{and} \quad \int \frac{dx}{2\pi} \frac{(\alpha_- \pm x)^{1\pm ik}}{1 \pm ik} \]

with the same quantum numbers

\[ p_o = k, \quad p_x = -2 \pm ik \]

This would imply doubling which would be physically unacceptable. The resolution of the paradox is found in identifying these states with each other \[ [13] \]. This step can be (and is) thought of as a set of boundary conditions imposed in the theory.
The above identification gives a nonlinear relationship between left and right moving waves. After a shift by the classical background we have

\[ \int_{\infty}^{\infty} d\tau e^{+ik\tau} \frac{\hat{\alpha}^+}{\mu} = - \int_{\infty}^{\infty} \frac{d\tau}{ik \pm 1} e^{-ik\tau} \left[ \left(1 + \frac{\hat{\alpha}^-}{\mu}\right)^{ik+1} - 1 \right] \]

This is a solution to the scattering problem in the form first found following a different route in [6]. Expansion of the above equation generates general tree level N-point scattering amplitudes. What we have seen is that the simple linear dynamics of the matrix oscillator induces through collective phenomena the nonlinear dynamics of string scattering.

3. Symmetry

The matrix model and the scalar field theory exhibit in an obvious way a large symmetry algebra. Its fundamental origin lies in the symplectic structure underlying the collective field theory where \( M \to x \) and \( P \to \alpha(x,t) \) represent canonically conjugate variables. The natural symmetry generators [8]

\[ H^m_m = \text{Tr} (P^m M^m) \]

generate general canonical transformations in the \( M,P \) or equivalently \( x,\alpha \) phase space. They close a \( W_\infty \) algebra

\[ \left[ H^{m_1}_m, H^{n_2}_M \right] = i \left( (m_2 - 1) n_1 - (m_1 - 1) n_2 \right) H^{n_1+n_2}_m \]

For the physical system given by the inverted oscillator one takes the spectrum operators

\[ O_{JM} \equiv \text{Tr} (P + M)^{J+M+1} (P - M)^{J-M+1} \]

These in the field theory become

\[ O_{JM} = \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha (\alpha + x)^{J+M+1} (\alpha - x)^{J-M+1} \]

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and obey the $W_\infty$ commutation relations:

$$[O_{J_1 M_1}, O_{J_2 M_2}] = -4i ((J_1 + 1) M_2 - (J_2 + 1) M_1) O_{J_1 + J_2 M_1 + M_2}$$

The hamiltonian itself is a member of this algebra

$$H = O_{0,0} = \text{Tr} (P^2 - M^2)$$

and this implies [8] that an infinite sequence of discrete states is generated described by [9,10] two integers $J, M (-J \leq M \leq +J)$. In this sense the $W_\infty$ generators give a spectrum generating algebra. The nature of the imaginary energy discrete states is nontrivial, in the present approach they are composite states of the tachyon.

Analogous operators are found [11] in conformal field theory. They are given by the discrete state vertex operators

$$\Psi_{JM} = (H(z))^{J-M} e^{(-1+J)\phi(z)}$$

which with the corresponding ground ring operators $O_{J,M}$ combine into a conserved currents

$$O_{JM}(z, \bar{z}) = \Psi_{J+1,M}(z)O_{J,M}(\bar{z})$$

Returning to the collective approach one sees that the symmetry generators $O_{J??}$ directly represented as nonlinear functions of the tachyon field $\alpha(x, t)$. This immediately implies that the generators act in a very special (nonlinear) way on the tachyon states. Studying the action of discrete vertex operators on the tachyon module, a similar conclusion can be reached in the conformal calculus also. This gives rise to nonlinear Ward identities first described in [12]. These identities are, however, explicitly encoded in the collective representation.

We explore this in some detail [13]. First of all the exact tachyon operators

$$T_n = \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha (\alpha \pm x)^n$$

can be written as extensions of the algebra $T_n = O_{\frac{n}{2}-1,n}$. Of particular relevance is
the subalgebra given by

\[ O_{2N} \equiv O_{N,N} = \int \frac{dx}{2\pi} \int d\alpha (\alpha + x)^{2N+1}(\alpha - x) \]

Since it represents a Virasoro algebra:

\[ [O_{2N}, O_{2N'}] = 4i (N - N') O_{2(N+N')} \]

The Virasoro generators act on the tachyon operators \( T_n \) in the simple way

\[ [O_{2N}, T_n] = 2in T_{n+2N} \]

These commutators can be directly verified from the collective representation. Consequently the tachyon field can be thought to have (space–time) conformal spin 1 (in the next section this notion and its generalization will be pursued further).

To exhibit the higher Ward identities, one goes to the approximation with the vanishing cosmological constant. In conformal field theory it is actually only this limit that is well understood. The background shift for the collective fields now becomes

\[ \alpha_\pm = \pm \sqrt{\tilde{x} - \mu} + \bar{\alpha}_\pm \to \alpha_\pm = \pm x + \bar{\alpha}_\pm \]

and after a change \( x = e^{-\tau} \) the generators become

\[
O_{JM} \approx \frac{2^J}{J-M+2} \int \frac{dt}{2\pi} e^{2Mi\tau} \alpha_+^{J-M+2} + \frac{2^J}{J+M+2} \int \frac{dt}{2\pi} e^{-2Mi\tau} \alpha_-^{J+M+2}
\]

Expanding in terms of linearized tachyon creation–annihilation operators \( \bar{\alpha}_+(k) = \alpha(k), \alpha_-(k) = \beta(k) \) we have the expression

\[
O_{JM} = \frac{2^J}{J-M+2} \int dk_1dk_2\cdots dk_{J-M+2} \alpha(k_1) \cdot \alpha(k_{J-M+2}) \delta \left( \sum k_i + 2M \right) + \frac{2^J}{J+M+2} \int dp_1dp_2\cdots dp_{J+M+2} \beta(p_1) \cdots \beta(p_{J+M+2}) \delta \left( \sum p_i + 2M \right)
\]

These are similar (but not identical) to the representation of the symmetry generators constructed on the linearized tachyon by Klebanov in [12]. Some subtle differences
are explained in [13]. The implications when actioning on tachyon states are all the same, one has

\[ O_{M+N,M} |k_1k_2 \cdots k_{N+1} \rangle = \left( M + \sum_i k_i \right) |k \sum_i k_i + M \rangle \]

This nontrivial identity is sufficient to construct the bulk scattering amplitudes as was done in [12].

We have explained how the scalar collective field representation induces the same Ward identities as the conformal vertex operator calculus. One can in fact invert this and realize that the collective formulation gives the simplest realization of the Ward identities. An advantage of the field theoretic approach is the fact that the nonzero cosmological constant is easily incorporated. It represents a nonzero chemical potential in the field theory. An exact discussion of Ward identities can now also be contemplated. The symmetry generators can be completely defined through normal ordering over the field oscillators as has been done in detail in [3] for the hamiltonian.

To summarize the above discussion we have seen that the \( W_\infty \) symmetry occurs naturally in the canonical framework of the matrix model and collective field theory. These are based on phase space notions \((M, P)\) and \((x, \alpha(x, t))\) respectively and the symmetry is given by general canonical transformations. It is an interesting question to ask what happens to these at the full quantum level. In general canonical transformations are known to be highly nontrivial in quantum mechanics.

4. Extension: Coupling of Topological and Collective Fields

We have seen that the \( W_\infty \) symmetry seems to govern the dynamics of the theory. Through Ward identities and its representation in terms of the tachyon field one has a nonlinear realization giving the complete \( c=1 \) tachyon S–matrix. This is manifest in the collective description where the \( W_\infty \) generators are explicit nonlinear functions of the scalar field. The generators play a role of a spectrum generating algebra giving a sequence of additional discrete states. These appear as composite states [8] of
the collective Hamiltonian. One would then like to introduce separate fields [11] to represent these states. This would allow a study of nontrivial backgrounds where these fields play a role. One such example would be the black hole.

The question of how the discrete higher string fields should be introduced is a nontrivial one. On one hand one has the standard BRST gauge symmetry while in the matrix model one finds a $W_{\infty}$ symmetry. It is suggestive to follow the latter as a fundamental principle, and develop an interacting theory of collective and higher fields in two dimension based on $W_{\infty}$. The physical nature of the two sets of fields is, however, quite distinct: the tachyon is a full dynamical particle with the associated scalar field while the higher modes are global in structure and should be of topological nature. It is useful then to introduce [16] the notion of space–time (as opposed to world sheet) central charges. With this we find that the collective tachyon field can be characterized as carrying the central charge $c=1$ while the infinite sequence of higher fields all carry $c=0$ and are therefore topological. To define the coupling we begin [16] with a quantum $c=1 W_{1+\infty}$ algebra defined for example through conformal fermi fields:

$$B^k(z) = \psi^+(z) \partial_z^k \psi(z) :$$

The algebra of commutators (with central charges corresponding to $c=1$) reads

$$[B^0_n, B^0_m] = n \delta n + m_0$$

$$[B^1_n, B^1_m] = -m B^0_{m-n} + \frac{n(n-1)}{2} \delta_{n+m}$$

$$[B^1_n, B^0_n] = (n-m) B^1_{n+m} - \frac{n^3-n}{6} \delta_{n+m}$$

$$[B^2_n, B^0_m] = m^2 B^0_{n+m} - 2 m B^1_{n+m} + \frac{n(n-1)(2n-1)}{6} \delta_{n+m}$$

$$:$$

The idea is then to realize the algebra in terms of bosonic fields. In the first commutator we recognize the scalar collective field

$$\alpha(z) \equiv B^0(z) \quad \{ \alpha(x), \alpha(x') \} = \partial_x \delta(x-x')$$
One then “solves” the commutators for the next $w_\infty$ generator $B^1$ finding

$$B^1_n =: \frac{1}{2} \alpha_{n-\ell} \alpha_\ell + \frac{n-1}{2} \alpha_n + w_n^1$$

Here the scalar field is seen to be responsible for the total central change and a remnant spin 2 field appears with

$$[w_n^1, w_m^1] = (n - m)w_{n+m}^1$$

It consequently has $c=0$ and commutes with the collective boson $[\alpha_n, w_m^1] = 0$. The strategy is then clear, we continue with the higher commutators and find a sequence of topological objects. In particular for the next generator the commutators are uniquely solved by

$$B^2_n = \frac{1}{3} : \alpha^3 : + 2 \sum w_{n-\ell}^1 \alpha_\ell + w_n^2$$

where $w_n^2$ represents a spin 3 topological field. This generator gives the hamiltonian which now reads

$$H = \int dx \left\{ \frac{1}{12\pi} \alpha^3(x,t) + w_1(x,t)\alpha(x,t) + w_2(x,t) \right\}$$

One has an interaction with the collective scalar with a unique coupling given by $\alpha(x,t)w_1(x,t)$. This is an interaction of the spin 1 (collective) and spin 2 (Virasoro) fields.

The infinite component $W_\infty$ symmetry is likely to play a role. The higher spin fields $\{w_n(x,t), n = 1, 2, \cdots\}$ are found to obey the centerless $w_\infty$ algebra

$$[w_n(x), w_m(y)] = ((n - 1)w_{n+m} + (M - 1)w_{n+m}(y)) \partial_x \delta(x - y)$$

These fields are then topological. There is actually a representation which makes this manifest. Consider an infinite component field defined through a general pseudo
differential operator

\[ K(x, \partial) = 1 + \sum_{j=1}^{\infty} a_j(x, t)\partial^{-j} \]

This gives a representation where the hamiltonian densities are all total divergences [17]:

\[ W_n(x, t) = \text{Res} \left( K^{-1}\partial^n K \right) = \partial_x (\sigma_n) \]

This form also exhibits the topological modes as pure gauges of a large \( W_\infty \) gauge group. \( K \) can be thought of as a group element and \( A = K^{-1}\partial K \) as a gauge field. It is likely that a gauge invariant \( W_\infty \) Lagrangian can be written which after gauge fixing reduces to the above.

One expects that the relevance of higher fields will come for studying other classical backgrounds. The formalism discussed in much of this talk is clearly defined around the flat dilaton vacuum. Based on conformal field theory insights, one expects a black hole to arise as a nontrivial background (see recent suggestions [19-21]).

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