Entanglement Manipulation and Concentration

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We introduce a simple, experimentally realisable, entanglement manipulation protocol for exploring mixed state entanglement. We show that for both non-maximally entangled pure, and mixed polarisation-entangled two qubit states, an increase in the degree of entanglement and purity, which we define as concentration, is achievable.

The increasing interest in quantum information and computing as well as other quantum mechanical dependent operations such as teleportation and cryptography have as their cornerstone a reliance on entanglement. There has been a great deal of discussion of measures and manipulation of entanglement in recent years with respect to purification, concentration, and distillable entanglement especially concerning states subject to environmental noise. It is this noise that takes the initially pure maximally entangled resource and leaves us with, at best, a non maximally entangled state, or at worst a mixed state, both less pure and less entangled. We introduce a simple, experimentally realisable protocol to manipulate and explore both pure and mixed-state entanglement. While the scheme will have limitations, in part due to its simplicity, it will allow experimental investigation of the large Hilbert space associated with mixed states.

The motivation for this scheme comes from focusing ideas and proposals of several groups from the past few years into a simple realisation of mixed state entanglement manipulation. It was proposed that quantum correlations on mixed states could be enhanced by positive operator valued measurements. A more specific example by Gisin considered the manipulation of a $2 \times 2$ system using local filters. The scheme we propose here combines these ideas and uses an arrangement similar to the original Procrustean method which dealt solely with pure states. The primary motivation here is in proposing a scheme that can be easily realised experimentally. With the recent advances in the preparation of nonmaximally entangled pure and mixed polarisation-entangled quantum states we now have a source for which there is a high degree of control over the degree of entanglement and purity of the state. This allows us to consider a wide variety of states and examine what operations can be performed so as to make the state more useful in the context of an entanglement resource.

For the purposes of describing the possible manipulation of a state we will define the following three concepts of distillation, purification and concentration (illustrated schematically in Figure 1) as follows,

- **Distillation**: Increasing the entanglement of a state.
- **Purification**: Increasing the purity of a state (decreasing its entropy). This is not purification with respect to some particular state, for example obtaining a singlet state from a mixed state.
- **Concentration**: Increasing both the entanglement and the purity of a mixed state.

These concepts have been used almost interchangeable in the literature but we will follow our primitive definitions to avoid potential confusion. In this letter it is the concentration of a state that is the main aim for the maintenance or recovery of an entanglement resource.

![Figure 1](image.png)

**FIG. 1.** A schematic representation of the entanglement manipulation processes in terms of the degree of Entanglement and Entropy of a state. We propose this distinction between distillation, purification and concentration in an attempt to clarify terminology in the mixed state domain.
Let us now specify the measures which we will be using to characterise the degree of entanglement and purity of a state. The entanglement and purity of a state can be determined using distinct measures. Here we will restrict our attention to $2 \times 2$ systems and hence will use analytic expressions for The Entanglement of Formation and Entropy as our respective measures. The Entanglement of formation as introduced by Wootters \cite{12} is found by considering that for a general two qubit state, $\rho$, the "spin-flipped state" $\tilde{\rho}$ is given by

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$$ (1)

where $\sigma_y$ is the Pauli operator in the computational basis. We calculate the square root of the eigenvalues $\tilde{\lambda}_i$ of $\rho \tilde{\rho}$, in descending order, to determine the “Concurrence”,

$$C(\rho) = \max\{\tilde{\lambda}_1 - \tilde{\lambda}_2 - \tilde{\lambda}_3 - \tilde{\lambda}_4, 0\}$$ (2)

The Entanglement of Formation (EOF) is then given by

$$E(C(\rho)) = h \left(1 + \sqrt{1 - C(\rho)^2}\right)$$ (3)

where $h$ is the binary entropy function

$$h(x) = -x \log(x) - (1 - x) \log(1 - x)$$ (4)

The entropy of the density matrix $\rho$ (our purity measure) is given by

$$S = -\sum_{i=1}^{4} \lambda_i \log_4 \lambda_i$$ (5)

where $\lambda_i$ are the eigenvalues of $\rho$.

We will now describe our entanglement manipulation protocol and emphasise its simplicity. The experimental arrangement for our protocol is described by the schematic in figure\cite{2}. The aim of our protocol is to manipulate mixed states and enhance their degree of entanglement. Let us consider an initial state composed of two subsystems, A and B, each represented by a general $2 \times 2$ matrix. We will describe the joint state of the system, $AB$, in the polarisation basis, \{\(|VV\rangle, |VH\rangle, |HV\rangle, |HH\rangle\}, as

$$\hat{\rho}_{ABin} = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{12}^* & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{13}^* & \rho_{23}^* & \rho_{33} & \rho_{34} \\ \rho_{14}^* & \rho_{24}^* & \rho_{34}^* & \rho_{44} \end{pmatrix}$$ (6)

with the $\hat{\rho}_{ij}$ satisfying the requirements for a legitimate density matrix. From our source (see figure\cite{2}) we have four polarisation modes (two for A and two for B). These polarisation modes are spatially separated and input onto beam splitters (BS), with independent and variable transmission coefficients. The second input port of each of these beam splitters are assumed to be vacuums. With perfectly efficient photodetectors it would be possible to monitor the second output mode of each of these beamsplitter and use the results to conditionally select the concentrated state we wish to produce. We know that if the detection of a photon is made in any of the second output ports then the preparation process is considered to have failed. Non-detection (with perfectly efficient detectors) at all the second output ports is required to prepare our state and here is the problem with current single photon detection efficiencies. Photon detectors have a finite efficiency and it possible that a photon present at these second output ports will not be detected. Hence we will not get the conditioned state we desire. Instead we will examine the transmitted modes of the beamsplitter and consider the situations where joint coincidences are registered at the photodetectors of the two subsystems A and B, or Alice and Bob if you prefer. While this is a post selective process it has the advantage that poor detection efficiency only decreases the coincidence count rate. As we discard any information present at the second output of the beamsplitters, the protocol we describe is not unitary.
FIG. 2. The schematic model of the BS concentration protocol. The source produces an initial state that can be controlled in terms of the entanglement and purity and is thus able to provide a range of initial states for manipulation. This state is then spatially separated with respect to its polarisation modes and then incident on separate beam splitters, BSs, with a different variable reflectivity. By tuning the variable BS it is possible to obtain the desired output state with the corresponding coincidence detections for A and B.

If we consider that having each mode incident on a BS has the effect of expanding the Hilbert space of the system, then in the expanded Hilbert space we can manipulate the state and then project it back onto the polarisation coincidence basis. The BSs transform each mode in the following way

\[ |V, H\rangle|0\rangle \rightarrow \eta_{v,b}|V, H\rangle|0\rangle + \sqrt{1 - \eta_{v,b}^2}|0\rangle|1\rangle \]  

and hence we obtain an output density matrix for this reduced system of the form

\[
\hat{\rho}_{AB\text{out}} = N \begin{pmatrix}
\rho_{11}^{V} & \rho_{12}^{V} & \rho_{13}^{V} & \rho_{14}^{V} \\
\rho_{12}^{V} & \rho_{22}^{V} & \rho_{23}^{V} & \rho_{24}^{V} \\
\rho_{13}^{V} & \rho_{23}^{V} & \rho_{33}^{V} & \rho_{34}^{V} \\
\rho_{14}^{V} & \rho_{24}^{V} & \rho_{34}^{V} & \rho_{44}^{V}
\end{pmatrix}
\]

where \( N = \frac{1}{\rho_{11}^{V} + \rho_{22}^{V} + \rho_{33}^{V} + \rho_{44}^{V}} \) and the probability of obtaining the desired output state is determined from the trace of the unnormalized BS-transformed density matrix, \( N^{-1} \), and thus is dependent on the transmission coefficients. This is the probability of obtaining the output state once the BS parameters have been determined.

This scheme is more easily understood by considering the behaviour of pure states under the protocol. As such we now illustrate the distillation process with a specific example. We will examine a non-maximally entangled pure state of the form of (11) which allows us to simplify the analysis. With this in mind we can set the vertical and horizontal polarisation transmission coefficients for subsystems A and B. The normalisation is given by

\[ N = \sqrt{\rho_{11}^{V} + \rho_{22}^{V} + \rho_{33}^{V} + \rho_{44}^{V}} \]

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\[ N = \sqrt{\rho_{11}^{V} + \rho_{22}^{V} + \rho_{33}^{V} + \rho_{44}^{V}} \]
The outcomes we are interested in are in the joint coincidence basis of A,B and hence the vacuum state components are removed from consideration leaving an effective output state of the form

$$|\varphi_{out}\rangle_{AB} = N_2[\epsilon_1^2|VV\rangle_{AB} + \epsilon_2 e^{i\phi} |HH\rangle_{AB}]$$

(14)

where the normalisation in this coincidence basis is

$$N_2^2 = ||\epsilon_1|^2 + |\epsilon_2|^2|^{-1}$$

(15)

For maximal entanglement we have the following simple relationship

$$|\epsilon_1|^2 = |\epsilon_2|^2$$

(16)

We observe that the entanglement of the output state is dependent on the transmission coefficients of the BSs. Further, this protocol can always take a non-maximally entangled state and obtain a pure maximally entangled one. This protocol can also incorporate a phase adjuster at either A or B to tune any relative phase difference for the state. If we had considered states of the form of (11) then we would need to consider the tuning parameters independently such that the requirement for a pure maximally entangled state is then

$$|\epsilon_1|\eta_a\eta_b = |\epsilon_2|\eta_b\eta_a$$

(17)

This is where the protocol differs from the Procrustean method of Bennett et al. [4]. We have introduced individual depolarising channels, thus obtaining more degrees of freedom, and so allowing the protocol to be extended to mixed states. It is important to mention again that with perfect single photon detection it is possible to monitor the discarded ports for each of the modes, thus preparing the desired state by conditioned measurements.

Let us now turn our attention to the concentration of mixed states. As an extension to the distillation process we take the density matrix $\hat{\rho}_{ABin}$ to be a mixture of the density matrices of two of the Bell-type states, (10) and (11), one of which, say (10), is maximally entangled, $\epsilon_1 = \epsilon_2 = 1$. The mixing can be controlled by the parameter $\gamma$, that is,

$$\hat{\rho}_{ABin} = \gamma N_2^2 \left( \epsilon_1^2 \begin{array}{cccc} |\epsilon_1|^2 & 0 & 0 & \epsilon_1^* \epsilon_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \epsilon_1^* \epsilon_2 & 0 & 0 & |\epsilon_2|^2 \end{array} \right) + \frac{1 - \gamma}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(18)

This state is one of many of the range of mixed states that can be concentrated and has been chosen to easily show the protocols extension from pure to mixed states, from distillation to concentration. Using the BS protocol illustrated in figure (2) the output state for (18) in the coincidence basis, AB, can be represented as

$$\hat{\rho}_{ABout} = N_3^2 \left( \gamma^{\epsilon_1^2} \epsilon_1^2 \eta_a^2 \eta_b^2 \right) \begin{pmatrix} 0 & 0 & \gamma \epsilon_1^2 \epsilon_2 \eta & \gamma \epsilon_1^2 \epsilon_2 \eta \\ 0 & \eta \epsilon_1^2 \epsilon_2 \eta & 0 & \gamma \epsilon_1^2 \epsilon_2 \eta \\ 0 & \eta \epsilon_1^2 \epsilon_2 \eta & \eta \epsilon_1^2 \epsilon_2 \eta & 0 \\ \gamma \epsilon_1^2 \epsilon_2 \eta & \gamma \epsilon_1^2 \epsilon_2 \eta & \gamma \epsilon_1^2 \epsilon_2 \eta & \gamma \epsilon_1^2 \epsilon_2 \eta \end{pmatrix}$$

(19)

with $\Gamma = \frac{(1-\gamma)}{2}$ and the normalisation $N_3$ given by

$$N_3^2 = [\gamma(|\epsilon_1|^2 \eta_a^2 \eta_b^2 + |\epsilon_2|^2 \eta_a^2 \eta_b^2) + \Gamma(\eta_a^2 \eta_b^2 + \eta_a^2 \eta_b^2)]^{-1}$$

(20)

In figure (3) we display the effect of our protocol for a range of $\gamma$ values with $\epsilon_1 = 1$ and $\epsilon_2 = 0.1$ (the $\gamma$ values are labeled at the peak of each curve). The initial points for the fixed $\gamma$, $\epsilon_1$, and $\epsilon_2$ are displayed as solid dots. These curves represent the behaviour of the Entropy and EOF of the states as the BSs are tuned to optimise both. We see how this class of state can be improved is dependent on the amount mixing. The behaviour of the state is similarly dependent on the degree of entanglement in the pure state components of the mixed state of (18), variations in $\epsilon_{1,2}$, though this is not explicitly shown here.
FIG. 3. The plot of the Entanglement of Formation versus the Entropy displays the concentration characteristics of the idealised schematic of figure (1). We consider states of the form of (18) with \((\xi_1, \xi_2) = (1, 0.1)\), the values for \(\gamma\) are labeled at the peak of each curve. The curves illustrate the entire range of values a state can attain in the (S, EOF) plane as the BS parameters, \(\eta_{va} = \eta_{vb} = \eta_v\), are varied, \(0 < \eta_v < 1\). The initial states are marked as filled black circles. If we consider the state \(\gamma = 0.1\), initially \((S, EOF) = (0.23, 0.84)\), then as we decrease \(\eta_v\) from an initial value of 1 we approach the maximum concentration value \((S, EOF) = (0.075, 0.94)\) at \(\eta_v = 0.32\). If we continue to decrease \(\eta_v\) we then follow the curve back down through our initial point on the plane and from then on the state deteriorates from its original value. We also note that the other curves have similar concentration characteristics for \(\gamma \leq 0.5\). \(\gamma = 0.5\) corresponds to the case where the two Bell-type states are evenly mixed.

The curves in figure (3) represent the range of \((S, EOF)\) values for the output states from our protocol. We take the specific case of \(\gamma = 0.1\) and observe the variation of \((S, EOF)\) as we tune \(\eta_{va} = \eta_{vb} = \eta_v\). From the initial state marked with a black circle at \((S, EOF) = (0.23, 0.84)\) with \(\eta_v = 1\) we then adjust the BSs, moving up the curve, to a state with \((S, EOF) = (0.075, 0.94)\) for \(\eta_v = 0.32\). This constitutes a turning point on the plane and if we continue decreasing \(\eta_v\) we follow the curve back to our initial point in the plane after which the entanglement-entropy properties of the state deteriorate from the original values.

What does the state look like? We observe that with \((\xi_1, \xi_2, \xi_3) = (1.00, 0.10, 0.30)\) and allowing all the light through the horizontal BS (an optimal setting provided \(|\xi_1| > |\xi_2|\) to maximise the output), and tuning the vertical beam splitters transmission to \(\eta_v = 0.32\) we can take an initial state

\[
\hat{\rho}_{ABin} = \begin{pmatrix}
0.297 & 0 & 0 & 0.030 \\
0 & 0.350 & 0.350 & 0 \\
0 & 0.350 & 0.350 & 0 \\
0.030 & 0 & 0 & 0.003
\end{pmatrix}
\]  

(21)

to an output state

\[
\hat{\rho}_{ABout} = \begin{pmatrix}
0.039 & 0 & 0 & 0.039 \\
0 & 0.461 & 0.461 & 0 \\
0 & 0.461 & 0.461 & 0 \\
0.039 & 0 & 0 & 0.039
\end{pmatrix}
\]  

(22)

This output state has an increase in the Entanglement of Formation from EOF = 0.52 to EOF = 0.78, while the entropy of the system has decreased from \(S = 0.30\) to \(S = 0.20\), this result is achieved with a finite probability \(P = 7.6\%\).

There exists a critical point with respect to concentration at \(\gamma = 0.5\) which corresponds to the case where the two pure states of (18) are evenly mixed. For those states with the mixing parameter \(\gamma \leq 0.5\) concentration is possible whilst for those states above this value the entanglement can be increased but this is at the cost of purity. All of these states can be concentrated if we choose to tune another BS, thus highlighting the need for all four BSs. Similarly if we considered a mixture of the pure states of (19) and (11), where both \(\xi_1, \xi_2 \neq 1\), then we find that concentration is still achievable.

Now let us consider the incoherent sum of a pure state and a mixed state and take as an example of this the Werner state, a mixture of the identity and some fraction of a pure state. If the pure state fraction of the Werner state is a non-maximally entangled pure state, then it is possible to increase the entanglement of the state. However this entanglement increase comes at the cost of purity and is bound by the amount of entanglement that would be inherent in a Werner state using a maximally entangled pure state.
In conclusion, we have proposed an entanglement concentration protocol that is experimentally realisable and can produce a finite concentration of Bell pairs from some initially mixed states. The key point here is that whilst this is achievable we are more interested in the entanglement properties then the final form of the state. Indeed with such a simple protocol the range of possible tests with respect to quantum information and entanglement are quite diverse, and whilst this protocol does require some knowledge of the state in determining the tuning parameters and is a non-unitary operation, we believe it should provide a most useful tool in the exploration of mixed state entanglement.

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