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**Title:** Photon storage and routing in quantum dots with spin-orbit coupling

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Photon storage and routing in quantum dots with spin-orbit coupling: supplemental document

This supplementary text contains the energy eigenstates and electric-dipole matrix elements of the quantum dots (QDs), the expressions of the Heisenberg-Langevin equations, the solution of Eq. (4) and expressions of some coefficients in the main text, the equations of motion for the single-particle wavefunctions derived from the Heisenberg-Langevin-Maxwell (HLM) equations, and the physical mechanism of suppressing the dephasing.

1. ENERGY EIGENSTATES AND ELECTRIC-DIPOLE MATRIX ELEMENTS OF THE QDS

The Hamiltonian of an electron in an anisotropic QD without SOC reads

\[
\hat{H}' = \frac{1}{2m} \sum_{j=x,y} \left( \hat{p}_j + eA_j \right)^2 + \frac{m}{2} \left( \omega_x^2 x^2 + \omega_y^2 y^2 \right) + \frac{1}{2} \gamma B \hat{p}_z \hat{\sigma}_z, \tag{S1}
\]

where \( A = \frac{1}{2} B(-y, x, 0) \) is the vector potential chosen to be in symmetric gauge, \( m \) is the effective mass of the electron, \( \omega_x (\omega_y) \) is the trapping frequency in \( x (y) \) direction, \( \gamma \) is Landé g-factor, \( B \) is Bohr magneton, \( B = (0, 0, B) \) is the external (static) magnetic field; the last term is Zeeman energy, with \( \hat{\sigma}_z \) the \( z \)-component of Pauli matrices. It is easy to show that the above Hamiltonian can be written into the form

\[
\begin{align*}
\hat{H}' &= \hat{H}_0 + \hat{H}_L, \tag{S2a} \\
\hat{H}_0 &= \frac{1}{2m} \left( \hat{p}_x^2 + \hat{p}_y^2 \right) + \frac{m}{2} \left( \Omega_x^2 x^2 + \Omega_y^2 y^2 \right) + \frac{1}{2} \gamma B \hat{p}_z \hat{\sigma}_z, \tag{S2b} \\
\hat{H}_L &= \frac{eB}{2m} \hat{L}_z, \tag{S2c}
\end{align*}
\]

where \( \Omega_j = \sqrt{\omega_j^2 + e^2 B^2/(4m^2)} \) \((j = x, y)\) are effective trapping frequencies, and \( \hat{L}_z = \hat{x} \hat{p}_y - \hat{y} \hat{p}_x \) is the \( z \)-component of the orbit angular momentum of the electron.

By defining the creation and annihilation operators \( \hat{a}_x = \frac{1}{\sqrt{2\ell_x}} (x - \ell_x \hat{\sigma}_z) \), \( \hat{a}_y = \frac{1}{\sqrt{2\ell_y}} (y + \ell_y \hat{\sigma}_z) \), and \( \hat{a}_x^\dagger = \frac{1}{\sqrt{2\ell_x}} (x + \ell_x \hat{\sigma}_z) \), \( \hat{a}_y^\dagger = \frac{1}{\sqrt{2\ell_y}} (y - \ell_y \hat{\sigma}_z) \), with \( \ell_j = \sqrt{\hbar/(m\Omega_j)} \), the Hamiltonian \( \hat{H}_0 \) can be expressed as

\[
\hat{H}_0 = \left( \hat{a}_x^\dagger \hat{a}_x + \frac{1}{2} \right) \hbar \Omega_x + \left( \hat{a}_y^\dagger \hat{a}_y + \frac{1}{2} \right) \hbar \Omega_y + \frac{1}{2} \gamma B \hbar \hat{\sigma}_z, \tag{S3}
\]

describing a 2D harmonic oscillator under the action of the external magnetic field \( B \).

The energy eigenvectors of \( \hat{H}_0 \) can be obtained by using the relations \( \hat{a}_x^\dagger |n_x, n_y\rangle = \sqrt{n_x + 1} |n_x + 1, n_y\rangle \), \( \hat{a}_y^\dagger |n_x, n_y\rangle = \sqrt{n_y + 1} |n_x, n_y + 1\rangle \), \( \hat{a}_x |n_x, n_y\rangle = \sqrt{n_x} |n_x - 1, n_y\rangle \), \( \hat{a}_y |n_x, n_y\rangle = \sqrt{n_y} |n_x, n_y - 1\rangle \), where \( |n_x, n_y\rangle = \frac{1}{\sqrt{n_x!n_y!}} (\hat{a}_x^\dagger)^{n_x} (\hat{a}_y^\dagger)^{n_y} |0, 0\rangle \) are Fock states, with \( |0, 0\rangle \) the ground state satisfying \( \hat{a}_x \hat{a}_y |0, 0\rangle = 0 \) and \( \langle 0, 0 | 0, 0 \rangle = 1 \). In position space, the ground state wavefunction of \( \hat{H}_0 \) reads

\[
\Psi_{0,0}(x,y) = \frac{1}{\sqrt{\pi \ell_x \ell_y}} e^{-\frac{x^2}{\ell_x^2} - \frac{y^2}{\ell_y^2}}; \tag{S4}
\]

the eigen wavefunctions of excited states have the form

\[
\Psi_{n_x,n_y}(x,y) = C_{n_x,n_y} e^{-\frac{x^2}{\ell_x^2} - \frac{y^2}{\ell_y^2}} H_{n_x} \left( \frac{x}{\ell_x} \right) H_{n_y} \left( \frac{y}{\ell_y} \right), \tag{S5}
\]
where \( C_{nx,ny} = \frac{1}{\sqrt{2^n x^n y^n}} \) are the normalization constants and \( H_{n(x)} \) are Hermite polynomials. The corresponding eigen energies are given by \( E_{n_x,n_y} = (n_x + 1/2)\hbar \Omega_x + (n_y + 1/2)\hbar \Omega_y + (s/2)g_{E}\hbar B \), where \( s = 1 \) (\( s = -1 \)) denotes spin up (down).

When the QD has a significant Rashba SOC, its Hamiltonian is given by \( \hat{H} = \hat{H}' + \hat{H}_{SOC} \). Here \( \hat{H}_{SOC} = \frac{g_1}{2} (\hat{P}_x \hat{P}_y - \hat{P}_y \hat{P}_x) \) is contributed by the SOC, with \( \hat{P} = \hat{p} + eA \) the kinetic momentum and \( g_1 \) the strength of the SOC. The eigen energies \( E_j \) and eigen states of \( \hat{H} \) can be obtained by an exact diagonalization in the basis of the 2D harmonic oscillator described by \( \hat{P}_0 \) given above. These eigen states satisfy the equation

\[
\hat{H} |\psi_j\rangle = E_j |\psi_j\rangle,
\]

and can be expressed as

\[
|\psi_j\rangle = \sum_{n_x,n_y} \xi_{n_x,n_y}^j |n_x,n_y,\psi\rangle,
\]

where \( \xi_{n_x,n_y}^j \) are expansion coefficients for the \( j \)th eigen state. When the diagonalization is implemented, one can obtain \( \xi_{n_x,n_y}^j \) and \( E_j (j = 1, 2, 3, \ldots) \).

Using the relations \( x = \frac{\xi}{\sqrt{2}} \left( \hat{a}_x^+ + \hat{a}_x \right) \) and \( y = \frac{\xi}{\sqrt{2}} \left( \hat{a}_y^+ + \hat{a}_y \right) \), we can obtain the electric-dipole matrix element related to the eigen states \( \psi_1 \) and \( \psi_2 \), i.e.

\[
P_{jk} = \langle \psi_j | e | \psi_k \rangle = \frac{e}{\sqrt{2}} I_{jk},
\]

where

\[
I_{jk} = \sum_{n_x,n_y} \sum_{m_x,m_y} \left( \xi_{n_x,n_y}^j \right)^* \xi_{m_x,m_y}^k \delta_{n_x,n_y} \delta_{m_x,m_y},
\]

with \( I_{m_x,m_y; n_x,n_y} = \delta_{n_y,m_y} \delta_{n_x,m_x+1} + \delta_{n_y,m_y-1} + \delta_{n_x,m_x} \delta_{n_y,m_y+1} + \delta_{n_x,m_x-1} \delta_{n_y,m_y} \).

## 2. Expressions of the Heisenberg-Langevin Equation

Here we give the explicit expressions of the Heisenberg-Langevin equation (3a) in the manuscript, with the microwave field coupling the two lower levels [1] and [2] taken into account. Note that in this case the Hamiltonian (2) in the manuscript is modified to be \( \hat{H}_{H} \rightarrow \hat{H}_{H} + \hat{H}_M \), where

\[
\hat{H}_M = -\frac{\hbar}{\sqrt{2}} \int d^3r \left( \Omega_M \hat{S}_{21}^z + H.c. \right),
\]

where \( \Omega_M = (e_m \cdot \vec{p}) \mathcal{B}_m / \hbar \) is the half Rabi frequency of the microwave field, with \( \vec{p} \) the magnetic dipole matrix element associated with the lower levels [2] and [1], and \( \mathcal{B}_m \) is the magnetic induction of the microwave field. In the following (also in the main text), we have assumed a zero total relative phase for simplicity, i.e., \( \psi_{j1} - \psi_{j2} - \psi_m = 0 \) (where \( \psi_{j1}, \psi_{j2}, \) and \( \psi_m \) are the initial phase of two probe, two control, and microwave fields, respectively) [1].

Then the expressions of the Heisenberg-Langevin equations (3a) in the manuscript reads [2]

\[
\frac{i}{\hbar} \frac{d}{dt} \hat{S}_{11} = -i \sum_{j=1}^3 \Gamma_{1j} \hat{S}_{1j} + \Omega_M \hat{S}_{21} + g_{p1} \hat{E}_{p1} \hat{S}_{31} + g_{p2} \hat{E}_{p2} \hat{S}_{41},
\]

\[
-\Omega_M \hat{S}_{12} - g_{p1} \hat{S}_{13} \hat{E}_{p1} - g_{p2} \hat{S}_{14} \hat{E}_{p2} - i \hat{F}_{11} = 0,
\]

\[
\frac{i}{\hbar} \frac{d}{dt} \hat{S}_{22} + i \Gamma_{12} \hat{S}_{22} - i \Gamma_{23} \hat{S}_{33} - i \Gamma_{24} \hat{S}_{44} + \Omega_M \hat{S}_{12} + \Omega_{33} \hat{S}_{32} + \Omega_{44} \hat{S}_{42} - \Omega_{22} \hat{S}_{21} - \Omega_{32} \hat{S}_{23} - \Omega_{42} \hat{S}_{24} - i \hat{F}_{22} = 0,
\]

\[
\frac{i}{\hbar} \frac{d}{dt} \hat{S}_{33} + i \Gamma_{13} \hat{S}_{33} + g_{p1} \hat{S}_{13} \hat{E}_{p1} + \Omega_{13} \hat{S}_{23} - g_{p1} \hat{S}_{13} \hat{E}_{p1} - \Omega_{13} \hat{S}_{23} - i \hat{F}_{33} = 0,
\]

\[
\frac{i}{\hbar} \frac{d}{dt} \hat{S}_{44} + i \Gamma_{14} \hat{S}_{44} + g_{p2} \hat{S}_{14} \hat{E}_{p2} + \Omega_{41} \hat{S}_{24} - g_{p2} \hat{S}_{14} \hat{E}_{p2} - \Omega_{41} \hat{S}_{24} - i \hat{F}_{44} = 0,
\]
for diagonal elements, and
\[
\begin{align*}
\left(\frac{i}{\partial t} + d_{21}\right) \hat{S}_{21} + \Omega_M \hat{S}_{11} + \Omega_1 \hat{S}_{31} + \Omega_2 \hat{S}_{41} &- \Omega_M \hat{S}_{22} - g_{p1} \hat{S}_{23} \hat{E}_{p1} - g_{p2} \hat{S}_{24} \hat{E}_{p2} - i\hat{F}_{21} = 0, \\
\left(\frac{i}{\partial t} + d_{31}\right) \hat{S}_{31} + g_{p1} \hat{S}_{23} \hat{E}_{p1} + \Omega_1 \hat{S}_{31} - \Omega_M \hat{S}_{32} - g_{p1} \hat{S}_{23} \hat{E}_{p1} - g_{p2} \hat{S}_{24} \hat{E}_{p2} - i\hat{F}_{31} = 0, \\
\left(\frac{i}{\partial t} + d_{22}\right) \hat{S}_{22} + g_{p1} \hat{S}_{23} \hat{E}_{p1} + \Omega_1 \hat{S}_{22} - \Omega_M \hat{S}_{31} - \Omega_1 \hat{S}_{33} - \Omega_2 \hat{S}_{42} - i\hat{F}_{32} = 0, \\
\left(\frac{i}{\partial t} + d_{32}\right) \hat{S}_{32} + g_{p1} \hat{S}_{23} \hat{E}_{p1} + \Omega_1 \hat{S}_{32} - \Omega_M \hat{S}_{41} - \Omega_3 \hat{S}_{43} - \Omega_2 \hat{S}_{42} - i\hat{F}_{42} = 0, \\
\left(\frac{i}{\partial t} + d_{41}\right) \hat{S}_{41} + g_{p2} \hat{S}_{12} \hat{E}_{p2} + \Omega_2 \hat{S}_{22} - \Omega_M \hat{S}_{42} - g_{p1} \hat{S}_{33} \hat{E}_{p1} - g_{p2} \hat{S}_{44} \hat{E}_{p2} - i\hat{F}_{41} = 0, \\
\left(\frac{i}{\partial t} + d_{42}\right) \hat{S}_{42} + g_{p2} \hat{S}_{12} \hat{E}_{p2} + \Omega_2 \hat{S}_{32} - \Omega_M \hat{S}_{43} - \Omega_4 \hat{S}_{44} - i\hat{F}_{42} = 0, \\
\left(\frac{i}{\partial t} + d_{43}\right) \hat{S}_{43} + g_{p2} \hat{S}_{33} \hat{E}_{p2} + \Omega_2 \hat{S}_{43} - g_{p1} \hat{S}_{33} \hat{E}_{p1} - \Omega_1 \hat{S}_{42} - i\hat{F}_{43} = 0,
\end{align*}
\]  
for non-diagonal elements. Here \(d_{\alpha\beta} = \Delta_\alpha - \Delta_\beta + i\gamma_{\alpha\beta} (\alpha \neq \beta)\) with \(\gamma_{\alpha\beta} = (\Gamma_\alpha + \Gamma_\beta)/2 + \gamma_{\alpha\beta}^{\text{dep}}\), \(\Gamma_\beta = \sum_{\alpha < \beta} \Gamma_{\alpha\beta}\) the spontaneous emission decay rate, and \(\gamma_{\alpha\beta}^{\text{dep}}\) the dephasing rate between the levels \(|\alpha\rangle\) and \(|\beta\rangle\); \(\hat{F}_{\alpha\beta}\) are \(\delta\)-correlated Langevin noise operators, with the two-time correlation function given by
\[
\langle \hat{F}_{\alpha\beta}(r,t)\hat{F}_{\alpha'\beta'}(r',t') \rangle = \frac{\delta(r-r')\delta(t-t')}{N} D_{\alpha\beta,\alpha'\beta'}(r,t),
\]  
where \(D_{\alpha\beta,\alpha'\beta'}\) is the diffusion coefficient, which can be obtained from Eqs. (S11) and (S12) using the generalized fluctuation-dissipation theorem [3], assuming that the QDs are coupled with the vacuum reservoir. It can be shown that the diffusion coefficients \(D_{\alpha\beta,\alpha'\beta'}\) have the following forms
\[
\begin{align*}
D_{21,12} &= \Gamma_{23} \langle \hat{S}_{33} \rangle, \\
D_{31,13} &= 0, \\
D_{41,13} &= 0 (\alpha, \beta = 2, 3; \alpha \neq \beta).
\end{align*}
\]  
Note that, for the discussions of the propagation and the storage and retrieval of single photon wavepackets presented in Sec. 3.1 and Sec. 3.2 in the manuscript, a reduced version of the above equations, i.e., the one for the three-level \(\Lambda\)-type system obtained by taking \(\Delta_4 \to \infty\), is used. In this case, one has \(\hat{E}_{p2} = 0, \hat{S}_{34} = \hat{S}_{a4} = 0, \hat{F}_{a4} = 0 (\alpha = 1, 2, 3, 4)\); however, for the discussion on the routing of single-photon wavepackets presented in Sec. 3.3, the above equations for the four-level double \(\Lambda\)-type system must be employed.

3. SOLUTION OF EQ. (4) AND EXPRESSIONS OF SOME COEFFICIENTS IN THE MAIN TEXT

A. Solution of Eq. (4) in the manuscript

Equation (4) in the manuscript can be exactly solved by using the Fourier transformation
\[
\chi(x, t) = \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}} \tilde{\chi}(x, \omega)e^{-i\omega t},
\]  
where \(\chi\) represents \(\hat{E}_p, \hat{S}_{a21}\), and \(\hat{F}_{a21}\) \((\alpha = 1, 2, 3)\). The solution reads
\[
\begin{align*}
\tilde{\chi}_{21} &= -\frac{\Omega^2}{D} \hat{S}_{p1} \hat{E}_p + i \frac{-(\omega + d_{21})\hat{F}_{21} + \Omega_1 \hat{F}_{31}}{\Omega_2^2 - (\omega + d_{21})(\omega + d_{31})}, \\
\tilde{\chi}_{31} &= \frac{\omega + d_{21}}{D} \hat{S}_{p1} \hat{E}_p + i \frac{\Omega_2 \hat{F}_{21} - (\omega + d_{21})\hat{F}_{31}}{\Omega_2^2 - (\omega + d_{21})(\omega + d_{31})}.
\end{align*}
\]  
Substituting Eq. (S16) into Eq. (3b) in the manuscript under the Fourier transformation and solve the closed equation for \(\hat{E}_p\), one can obtain the solution of \(\hat{E}_p\), given by Eq. (5) in the main context.
B. Expression of $\langle S_{33} \rangle$

To give an explicit expression of the photon number of the probe field described by Eq. (7) in the main text, we have to know $\langle \hat{S}_{33} \rangle$. For this aim, we need to solve the reduced equations of Eqs. (3a)-(3c) in the manuscript obtained under the condition $\Delta_4 \to \infty$, and carry out the calculation to the second-order in $g_p \hat{E}_p$, which results in the following equations

$$
i \left( \frac{\partial}{\partial t} + \Gamma_2 \right) \langle \hat{S}_{22} \rangle - i \Gamma_3 \langle \hat{S}_{33} \rangle - \Omega_c \langle \hat{S}_{32} \rangle + \Omega_c^* \langle \hat{S}_{23} \rangle = 0, \quad (S17a)$$

$$
i \left( \frac{\partial}{\partial t} + \Gamma_3 \right) \langle \hat{S}_{33} \rangle + \Omega_c \langle \hat{S}_{32} \rangle - \Omega_c^* \langle \hat{S}_{23} \rangle + g_p \langle \hat{S}_{13}^\dagger \hat{E}_p \rangle - g_p^* \langle \hat{E}_p^\dagger \hat{S}_{13} \rangle = 0, \quad (S17b)$$

$$\left( i \frac{\partial}{\partial t} + d_{32} \right) \langle \hat{S}_{23} \rangle + \Omega_c \langle \hat{S}_{22} \rangle - \langle \hat{S}_{33} \rangle \rangle + g_p \langle \hat{S}_{12}^\dagger \hat{E}_p \rangle = 0, \quad (S17c)$$

$$\left( i \frac{\partial}{\partial t} + d_{32} \right) \langle \hat{S}_{32} \rangle - \Omega_c^* \langle \hat{S}_{22} \rangle - \langle \hat{S}_{33} \rangle \rangle - g_p^* \langle \hat{E}_p^\dagger \hat{S}_{12} \rangle = 0. \quad (S17d)$$

Here an ensemble average on the equations have been made and hence the Langevin noise terms do not appear because $\langle \hat{F}_{xg} \rangle = 0$.

By solving Eq. (S17) with the help of the solution (5) in the main text of $\hat{E}_p$, and the correlation function for Langevin operators (S13) and (S14), the solution of $\langle \hat{S}_{33} \rangle$ can be expressed by

$$\langle \hat{S}_{33}(x, t) \rangle = \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}} \langle \hat{S}_{33}(x, \omega) \rangle e^{-i\omega t}, \quad (S18a)$$

$$\langle \hat{S}_{33}(x, \omega) \rangle = C_1 \frac{|g_p|^2}{\sqrt{2\pi}} W_1 + C_2 \frac{|g_p|^2}{\sqrt{2\pi}} W_2, \quad (S18b)$$

with

$$W_1 = \frac{|\Omega_c|^2}{\omega + d_{32}} \left( \frac{1}{D} \hat{E}_p^\dagger \hat{E}_p \right) + \frac{|\Omega_c|^2}{\omega + d_{32}^3} \left( \hat{E}_p^\dagger \hat{E}_p + \frac{1}{D} \right), \quad (S19)$$

$$W_2 = \langle \hat{E}_p^\dagger \hat{E}_p \rangle - \langle \hat{E}_p^\dagger \hat{E}_p \rangle, \quad (S20)$$

Here $A \cdot B = \int_{-\infty}^{+\infty} d\omega' A(\omega') \hat{B}(\omega')$ and

$$C_1 = \frac{C_{22} + C_{32}}{C_{22} C_{33} - C_{23} C_{32}}, \quad (S21a)$$

$$C_2 = \frac{C_{22}}{C_{22} C_{33} - C_{23} C_{32}}, \quad (S21b)$$

$$C_{23} = \omega + i\Gamma_2 \omega + d_{23} + \frac{|\Omega_c|^2}{\omega + d_{23}} + \frac{|\Omega_c|^2}{\omega + d_{32}}, \quad (S21c)$$

$$C_{33} = \omega + i\Gamma_3 \omega + d_{33} - \frac{|\Omega_c|^2}{\omega + d_{33}} + \frac{|\Omega_c|^2}{\omega + d_{32}} - \frac{|g_p|^2 L_{33}}{4\pi c} \frac{|\Omega_c|^2}{\omega + d_{32}}, \quad (S21d)$$

$$I_{31}(\omega) = \int_{-\infty}^{+\infty} \frac{1}{D^*(\omega' - \omega) D(\omega')}, \quad (S21e)$$

$$I_{21p}(\omega) = \int_{-\infty}^{+\infty} \frac{\omega' - \omega + d_{31}}{D^*(\omega' - \omega) D(\omega')}, \quad (S21f)$$

$$I_{21i}(\omega) = \int_{-\infty}^{+\infty} \frac{\omega' + d_{31}}{D^*(\omega' - \omega) D(\omega')}, \quad (S21g)$$

Substituting Eq. (5) in the main text into Eq. (S18) and using Eqs. (S13) and (S14) again, we obtain the following equation

$$\frac{\partial}{\partial x} \langle \hat{S}_{33}(x, \omega) \rangle = N(\omega) \langle \hat{S}_{33}(x, \omega) \rangle + \frac{\partial}{\partial x} S_{330}(x, \omega), \quad (S22)$$
under the condition of one-photon input. Here
\[
N(\omega) = \frac{N|g_p|^4|\Omega_c|^2}{2\pi c^2} \left\{ C_1 \left[ \frac{\Omega_c^2}{\omega + d_{32}} \left( \frac{1}{\hbar} \frac{\partial}{\partial \omega} + \Omega_c^2 \right) \right] + C_2 \left[ \frac{\Omega_c^2}{\omega + d_{21}} \left( \frac{1}{\hbar} \frac{\partial}{\partial \omega} + \Omega_c^2 \right) \right] \right\},
\]
(S23a)
\[
S_{330}(x, \omega) = \frac{|g_p|^2}{\sqrt{2\pi c}} P_{330}(x, \omega),
\]
(S23b)
\[
P_{330}(x, \omega) = C_1 \left[ \frac{\Omega_c^2}{\omega + d_{32}} \left( \frac{e^{-iKx}}{D^*} f_0 + f_0 e^{iKx} \right) \right] + C_2 \left[ \frac{\Omega_c^2}{\omega + d_{21}} \left( \frac{f_0^* e^{-iKx}}{D^*} f_0 + f_0^* e^{iKx} \right) \right].
\]
(S23c)
The solution of Eq. (S22) is given by
\[
\langle \hat{S}_{33}(x, \omega) \rangle = S_{330}(x, \omega) + N(\omega) \int_0^x dx' e^{N(\omega)(\omega-x')} S_{330}(x', \omega).
\]
(S24)

C. Expression of \( g^{(2)}(x, t_1, t_2) \)
The normalized second-order coherence function of the probe field is defined by
\[
g^{(2)}(x, t_1, t_2) = \frac{G^{(2)}(x, t_1, t_2)}{R(x, t_1)R(x, t_2)},
\]
with
\[
G^{(2)}(x, t_1, t_2) = |f(x, t)|^2 + G_F(x, t_1, t_2),
\]
(S26a)
\[
G^{(2)}(x, t_1, t_2) = |f(x, t_1)|^2 G_F(x, t_2, t_2) + |f(x, t_2)|^2 G_F(x, t_1, t_1)
+ f^*(x, t_1) f(x, t_2) G_F(x, t_2, t_1) + f^*(x, t_2) f(x, t_1) G_F(x, t_1, t_2)
+ G_F(x, t_1, t_1) G_F(x, t_2, t_2) + G_F(x, t_1, t_2) G_F(x, t_2, t_1),
\]
(S26b)
where
\[
f(x, t) = \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}} \left[ f_0(\omega) e^{iK(\omega)x} \right] e^{-i\omega t},
\]
(S27a)
\[
G_F(x, t_1, t_2) = \frac{4\pi}{\sqrt{2\pi}} \left[ \frac{\Omega_c^4}{|g_p|^4 N L} |\Omega_c|^2 F_{23} \right] \left( -\omega_1 D(\omega_2) \right) e^{-i\omega_1 t_1 - i\omega_2 t_2}
\times \int_0^x dx' P_{33}(x', \omega_1 + \omega_2) e^{i[K(\omega_2) - K(\omega_1)](x'-x)}.
\]
(S27b)

4. EQUATIONS OF MOTION OF THE SINGLE-PARTICLE WAVEFUNCTIONS DERIVED FROM THE HLM EQUATIONS

For the probe fields working at single-photon levels, \( \hat{E}_{p1} \) and \( \hat{E}_{p2} \) can be taken as small quantities and hence the HLM equations can be linearized around the initial state solution (i.e., \( \delta_{q0}^{(0)} = \delta_{x1}(\delta_{x1}) \)). Then the HLM equations (3a)-(3c) in the main text in the absence of the microwave field can be simplified as
\[
\left( i \frac{\partial}{\partial t} + d_{21} \right) \hat{S}_{21} + \Omega_c^2 \hat{S}_{31} + \Omega_c^2 \hat{S}_{41} - i \hat{E}_{21} = 0,
\]
(S28a)
\[
\left( i \frac{\partial}{\partial t} + d_{31} \right) \hat{S}_{31} + g_{p1} \hat{E}_{p1} + \Omega_c \hat{S}_{21} - i \hat{E}_{31} = 0,
\]
(S28b)
\[
\left( i \frac{\partial}{\partial t} + d_{41} \right) \hat{S}_{41} + g_{p2} \hat{E}_{p2} + \Omega_c \hat{S}_{21} - i \hat{E}_{41} = 0,
\]
(S28c)
\[
i \left( \frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial t} \right) \hat{E}_{p1} + \frac{N}{c} \hat{S}_{31} = 0,
\]
(S28d)
\[
i \left( \frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial t} \right) \hat{E}_{p2} + \frac{N}{c} \hat{S}_{41} = 0.
\]
(S28e)
Defining $\hat{\psi}_1 = \hat{E}_{p1}/\sqrt{V}$, $\hat{\psi}_2 = \hat{E}_{p2}/\sqrt{V}$, $\hat{\psi}_3 = \sqrt{N}\hat{S}_{31}$, $\hat{\psi}_4 = \sqrt{N}\hat{S}_{41}$, and $\hat{\psi}_5 = \sqrt{N}\hat{S}_{21}$, Equations (S28a)-(S28e) can be written into the form

$$i\hbar \frac{\partial}{\partial t} \hat{\psi} = (\hat{H} - i\hbar \Gamma) \hat{\psi} + i\hbar \hat{F},$$  

(S29)

where

$$\hat{H} = h \begin{bmatrix}
-ic\frac{\partial}{\partial t} & 0 & -\hat{g}_{p1}\sqrt{N} & 0 & 0 \\
0 & -ic\frac{\partial}{\partial t} & 0 & -\hat{g}_{p2}\sqrt{N} & 0 \\
-\hat{g}_{p1}\sqrt{N} & 0 & -\Delta_3 & 0 & -\Omega_{c1} \\
0 & -\hat{g}_{p2}\sqrt{N} & 0 & -\Delta_4 & -\Omega_{c2} \\
0 & 0 & -\Omega_{c1} & -\Omega_{c2} & -\Delta_2
\end{bmatrix},$$  

(S30a)

$$\Gamma = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \gamma_{31} & 0 & 0 \\
0 & 0 & 0 & \gamma_{41} & 0 \\
0 & 0 & 0 & 0 & \gamma_{21}
\end{bmatrix}, \quad \hat{F} = \sqrt{N} \hat{F}_3,$$  

(S30b)

with $\hat{\psi}$ satisfying the bosonic commutation relations

$$[\hat{\psi}_a(r,t), \hat{\psi}_b^\dagger(r',t)] = \delta_{ab}\delta(r - r').$$  

(S31)

As shown in Sec. 3.1 in the main text, due to the EIT effect the Langemvin noise plays a negligible role, and hence the quantum statistical property of the incident single photons can be well preserved during propagation. The physical reasons for this are the following: (i) Since there is no optical pumping, the noise $\hat{F}$ will not be amplified during the propagation. (ii) The population of the excited state $|3\rangle$, given by $\langle \hat{S}_{33} \rangle$, is negligible under the condition of the EIT [see the quantitative analysis given in Sec. III(A)]. Thus the impact of spontaneous emission from $|3\rangle$ on the photon number of the probe field is negligible. (iii) The energy of the optical transition $|1\rangle \leftrightarrow |3\rangle$, given by $\hbar\omega_{31}$, is much larger than that of thermal noises coming from the coupled reservoirs, given by $k_B T$ (where $k_B$ is the Boltzmann constant and $T$ is the experimental temperature). Typically, one has $\hbar\omega_{31}/(k_B T) \approx 1600 \gg 1$ at $T \approx 10 \text{ mK}$, thereby the incoming thermal noises can be safely considered to be at the vacuum states. As a result, the normally-ordered correlation functions $\langle \hat{F}^\dagger \hat{F} \rangle$ is zero, leading to the absent of the incoherent pump of atoms from their ground state $|1\rangle$ to the excited state $|3\rangle$ [4, 5].

Since the Langenvin noise operators play negligible roles in the EIT-based system, Eq. (S29) is equivalent to the following Schrödinger equation

$$i\hbar \frac{\partial |\Psi(t)\rangle}{\partial t} = \hat{H}_0 |\Psi(t)\rangle,$$  

(S32)

with $\hat{H}_0 = \int d^3r \hat{\psi}^\dagger(\hat{H} - i\hbar \Gamma) \hat{\psi}$ the total Hamiltonian and $|\Psi(t)\rangle$ the state vector of the system. For single-particle excitations, the state vector has the form [6]

$$|\Psi^{1P}(t)\rangle = \int d^3r \left[ \sum_{a=1}^5 \Phi_a(r,t) \hat{\psi}_a^\dagger(r) \right] |0\rangle,$$  

(S33)

where $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5)^T$ (here "$T$" means transposition) is called single-particle wavefunction, with $\Phi_a(r,t)$ being its components ($a = 1, 2, ..., 5$). Substituting (S33) into the Schrödinger Eq. (S32), we obtain the following wave equations

$$i\hbar \frac{\partial \Phi}{\partial t} = (\hat{H} - i\hbar \Gamma) \Phi,$$  

(S34)

with $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5)^T$ (here "$T$" means transposition).

Comparing with the operator equation of motion (S29), Eq. (S34) is more convenient for the calculation of photon dynamics because it is a c-number one. Here we use it to calculate the time evolution of a single-photon wavepacket in two cases:
5. THE PHYSICAL MECHANISM OF SUPPRESSING THE DEPHASING

The microwave field is used to realize a coherent population transfer between the two lower states |1⟩ and |2⟩. In this way, a quantum coherence between these two lower states can be acquired.

![Fig. S1](image.png)

Fig. S1. Wavefunction Φ₅ of the coherence ˆS₂₁ between the two lower quantum states |1⟩ and |2⟩ as a function of t/τ₀, for cases with (solid red line) and without (dashed blue line) the use of microwave field at x = 400 µm. The system parameters are the same as those used in Fig. 3 of the main text. We see that when the microwave field is present the coherence is significantly enhanced; while when the microwave field is absent the coherence is very small. Consequently, the application of the microwave field can increase the quantum coherence between the two lower states, by which the system can acquire high retrieval and routing efficiencies of the probe photon after its storage. In fact, such a technique was already used for realizing an efficient classical light memory in a three-level atomic gas [7].

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- the storage and retrieval of the single-photon wavepacket, where one takes Δ₄ → ∞ and hence ˆE_p₂ = 0, ˆψ₂ = ˆψ₄ = 0, equivalent to a Λ-type three-level system with Φ = (Φ₁, Φ₃, Φ₅)^T.

- the routing of the single-photon wavepacket, where Eq. (S34) for non-zero ˆψₐ (a = 1, 2, 3, 4, 5) must be employed.

Note that Eq. (S34) can be generalized to the case when the microwave field coupling with the two lower states |1⟩ and |2⟩ is present.

To illustrate the role of the microwave field for creating the quantum coherence, for simplicity we consider the three-level configuration of QDs with the quantum states given by |1⟩, |2⟩, and |3⟩; see Fig. S1(a). The timing-sequences of the control and microwave fields are shown in the lower part of the figure. By solving Eqs. (S11) and (S12), we can obtain the wavefunction Φ₅ for the coherence ˆS₂₁ between the two lower quantum states |1⟩ and |2⟩ at time t = T_{on}, which reads

Φ₅(x, T_{on}) = Φ₁(x, T_{off})e^{id_{21}(T_{off}−T_{on})} + \frac{Ω_cΩ_M(x)}{εd_{21}g_p\sqrt{N}} \left[ 1 - e^{id_{21}(T_{Moff}−T_{Mon})} \right],

where Φ₁(x, T_{off}) is the wavefunction of the probe field at time t = T_{off}, T_{off} (T_{on}) is the switching-off (switching-on) time of the control field, T_{Moff} (T_{Mon}) is the switching-off (switching-on) time of the microwave field. The first term of the above expression is the contribution from the input probe photon; the second term is the contribution by the microwave field. We see that the shape, intensity, and time duration of the microwave field plays an important role for the coherence wavefunction Φ₅.

Shown in Fig. S1 is Φ₅ as a function of t/τ₀, for cases with (solid red line) and without (dashed blue line) the use of microwave field at x = 400 µm. The system parameters are the same as those used in Fig. 3 of the main text. We see that when the microwave field is present the coherence is significantly enhanced; while when the microwave field is absent the coherence is very small.

Consequently, the application of the microwave field can increase the quantum coherence between the two lower states, by which the system can acquire high retrieval and routing efficiencies of the probe photon after its storage. In fact, such a technique was already used for realizing an efficient classical light memory in a three-level atomic gas [7].
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