1 Introduction

In [1] a interconnected system consisted of two sub-systems is studied, which leads to a generalized small-gain theorem, which gives the express of the gain of the output of the interconnected system w.r.t external input signals. However, sometimes we are interested in the gain from the input to the output of each sub-system. Such a result will offer more flexibility in control synthesis. For this purpose, in this article we provide a modified version of the Theorem 2.1 in [1].

Facts and Notations

• $|\cdot|$ stands for any vector norm.
• $Id$ denotes the identity function.
• $||u||$ denotes the $ess.sup.\{|u(t)|, t \geq 0\}$ for any measurable function $u : \mathbb{R} \rightarrow \mathbb{R}^m$.
• $u_{[t_1,t_2]}$ denotes the truncation of the function $u$:

$$y = \begin{cases} u(t) & \text{if } t \in [t_1, t_2] \\ 0 & \text{otherwise} \end{cases}$$

and $u_T = u_{[0,T]}$.

• Weak triangular inequality:

$$\gamma(a + b) \leq \gamma \circ (Id + \rho)(a) + \gamma \circ (Id + \rho^{-1})(b)$$

(2)

$\gamma$ and $\rho$ are functions of class $K$ and $K_\infty$, respectively. $a$ and $b$ are non-negative real numbers. In particular, let $\rho = Id$, we get: $\gamma(a + b) \leq \gamma(2a) + \gamma(2b)$.

• $\rho$ is any function of class $K_\infty$, then:

$$[Id - (Id + \rho)^{-1}]^{-1} = Id + \rho^{-1}$$

(3)

2 Definition and Main Results

Input-to-Output Practical Stability

Considering the following control system with $x$ as state, $u$ as input, and $y$ as output:

$$\begin{align*}
\dot{x} &= f(x, u) \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\
y &= h(x, u) \quad y \in \mathbb{R}^p
\end{align*}$$

(4)

$f$ and $g$ are smooth functions.
Definition 2.1 System \( \mathbf{4} \) is said to have the boundedness observability (UO) property if a function \( \alpha^0 \) of class \( K \) and a non-negative constant \( D^0 \) exist such that, for each measurable essentially bounded control \( u(t) \) on \( [0, T) \) with \( 0 < T \leq +\infty \), the solution \( x(t) \) of \( \mathbf{4} \) right maximally defined on \( [0, T') \) satisfies:

\[
|x(t)| \leq \alpha^0(|x(0)|) + ||(u^T, y^T)|| + D^0, \forall t \in [0, T')
\]

Definition 2.2 System \( \mathbf{4} \) is input-to-output practically stable (IopS) if a function \( \beta \) of class \( KL \), a function \( \gamma \) of class \( K \), called a (nonlinear) gain from input to output, and a non-negative constant \( d \) exist such that, for each initial condition \( x(0) \), each measurable essentially bounded control \( u(\cdot) \) on \( [0, \infty) \) and each \( t \) in the right maximal interval of definition of the corresponding solution of \( \mathbf{4} \), we have:

\[
|y(t)| \leq \beta(|x(0)|, t) + \gamma(||u||) + d
\]

when \( \mathbf{4} \) is satisfied with \( d = 0 \), system \( \mathbf{4} \) is said to be input-to-output stable (IOS).

Generalized Small-Gain Theorem

Considering now the following general interconnected system:

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, y_2, u_1), \quad y_1 = h_1(x_1, y_2, u_1) \\
\dot{x}_2 &= f_2(x_2, y_1, u_2), \quad y_2 = h_2(x_2, y_1, u_2)
\end{align*}
\]

where, for \( i = 1, 2 \), \( x_i \in \mathbb{R}^{n_i}, u_i \in \mathbb{R}^{m_i} \) and \( y_i \in \mathbb{R}^{p_i} \). The function \( f_1, f_2, h_1 \), and \( h_2 \) are smooth and a smooth function \( h \) exists such that:

\[
\begin{align*}
y_1 &= h(x_1, x_2, u_1, u_2) \\
y_2 &= h_2(x_2, h_1(x_1, y_2, u_1), u_2).
\end{align*}
\]

Theorem 2.1 Suppose \( \mathbf{7} \) is IopS with \((y_2, u_1) \) (resp. \((y_1, u_2) \)) as input, \( y_1 \) (resp. \( y_2 \)) as output, and \((\beta_1, (\gamma^y_1, \gamma^y_2), d_1) \) (resp. \((\beta_2, (\gamma^y_2, \gamma^y_2), d_2) \)) as triple satisfying \( \mathbf{3} \), namely:

\[
\begin{align*}
|y_1(t)| &\leq \beta_1(|x_1(0)|, t) + \gamma^y_1(||y_2(0)||) + \gamma^y_2(||u_1||) + d_1 \\
|y_2(t)| &\leq \beta_2(|x_2(0)|, t) + \gamma^y_2(||y_1||) + \gamma^y_2(||u_2||) + d_2
\end{align*}
\]

Also, suppose that \( \mathbf{7} \) has the UO property with couple \((\alpha^0_1, D^0_1) \) (resp. \((\alpha^0_2, D^0_2) \)). If two functions \( \rho_1 \) and \( \rho_2 \) of class \( K_\infty \) and a non-negative real number \( s_i \) satisfying:

\[
\begin{align*}
(Id + \rho_2) \circ \gamma^y_2 \circ (Id + \rho_1) \circ \gamma^y_1(s) &\leq s, \\
(Id + \rho_1) \circ \gamma^y_1 \circ (Id + \rho_2) \circ \gamma^y_2(s) &\leq s,
\end{align*}
\]

exist, then system \( \mathbf{7} \) with \( u = (u_1, u_2) \) as input, \( y = (y_1, y_2) \) as output, and \( x = (x_1, x_2) \) as state is IopS and has the UO property with \( D^0 = 0 \) when \( s_i = d_i = D^0(i = 1, 2) \).

More specifically, for each pair of class \( K_\infty \) functions \( (\rho_2, r_2^2) \), functions \( \beta_i^2 \) and \( \gamma_i^2 \) of class \( KL \), a function \( r_2^2 \) of class \( K \), and non-negative constants \( d_i^2 \) (equal to zero when \( s_i = d_i = D^0(i = 1, 2) \)) exist such that the system \( \mathbf{7} \) is IopS with the triple \((\beta_i^2 + \beta_i^2, r_1 + r_2 + r_3^1 + r_3^2, d_i^2 + d_i^2) \) where:

\[
\begin{align*}
r_1(s) &= (Id + \rho_2^2)^{-1} \circ (Id + \rho_3^2)^2 \circ \gamma_1(y_1) \circ (Id + \rho_2^2)^{-1} \circ (Id + \rho_2^2) \circ \gamma_2^y(s) \\
r_2(s) &= (Id + \rho_2^2)^{-1} \circ (Id + \rho_3^2)^2 \circ \gamma_2^y \circ (Id + \rho_2^2)^{-1} \circ (Id + \rho_3^2) \circ \gamma_2^y(s)
\end{align*}
\]

and the (nonlinear) gains from input \( u \) to the output components \( y_1(t) \) and \( y_2(t) \) are \((r_1 + r_3^1) \) and \((r_2 + r_3^2) \), respectively.

Remark 1 Furthermore, if the sub-systems have Input-to-State Stable property, namely:

\[
\begin{align*}
|x_1(t)| &\leq \beta_{x_1}(|x_1(0)|, t) + \gamma_{x_1}^u(||y_2(0)||) + \gamma_{x_1}^u(||u_1||) \\
|x_2(t)| &\leq \beta_{x_2}(|x_2(0)|, t) + \gamma_{x_2}^u(||y_1||) + \gamma_{x_2}^u(||u_2||)
\end{align*}
\]

where \( \beta_{x_1} \) and \( \beta_{x_2} \) are KL functions, and \( \gamma_{x_1}^y, \gamma_{x_1}^u, \gamma_{x_2}^y, \gamma_{x_2}^u \) are \( K \) functions. And the IopS property is restricted to IOS property, then we can deduce that the interconnected system is also ISS and IOS.

2
3 Proof of Theorem

A first fact to be noticed is that \((11)\) implies the existence of a non-negative real number \(d_3\) such that:

\[
\gamma^y_2 \circ (Id + \rho_1) \circ \gamma^y_3(s) \leq (Id + \rho_2)^{-1}(s) + d_3 \\
\gamma^y_3 \circ (Id + \rho_2) \circ \gamma^y_2(s) \leq (Id + \rho_1)^{-1}(s) + d_3 \\
\forall s \geq 0
\]

(15)

with \(d_3 = 0\) when \(s_1 = 0\).

**Step 1 : Existence and Boundedness of Solutions on \([0, \infty)\).** For any pair of measurable essentially bounded controls \((u_1(t), u_2(t))\) defined on \([0, \infty)\), for any initial condition \(x(0)\), by hypothesis of smoothness, a unique solution \(x(t)\) of \((2)\) right maximally defined on \([0, T)\) with \(T > 0\) possibly infinite exists. Also, since \([7]\) are \(IOPs\), for any \(\tau \in [0, T)\) and any

\[0 \leq t_{10} \leq t_{20} \leq t_{11} \leq t_{21} < T - \tau\]

we have, using time invariance and causality,

\[
|y_1(t_{11} + \tau)| \leq \beta_1(|x_1(t_{10} + \tau)|, t_{11} - t_{10}) + \gamma^y_1(\{||y_2(t_{10} + \tau, t_{11} + \tau)||\}) + \gamma^y_1(\{u_1\}) + d_1
\]

(17)

\[
|y_1(t_{21} + \tau)| \leq \beta_2(|x_2(t_{20} + \tau)|, t_{21} - t_{20}) + \gamma^y_2(\{||y_1(t_{20} + \tau, t_{21} + \tau)||\}) + \gamma^y_2(\{u_2\}) + d_2
\]

(18)

For ease of notation, set \(\gamma_i = \gamma^y_i\) and \(v_i = \gamma^y_i(\{u_i\})\). Then pick an arbitrary \(T_0 \in [0, T)\) and let

\[\tau = t_{10} = t_{20} = 0, \ t_{21} = T_0, \ t_{11} \in [0, T_0]\]

(19)

By applying (20) and (23) and using (15), we get successively

\[
\|y_{2T_0}\| \leq \beta_2(|x_2(0)|, 0) + \gamma_2(\beta_1(|x_1(0)|, 0) + \gamma_1(\{y_{2T_0}\}) + v_1 + d_1) + v_2 + d_2
\]

(20)

\[
\leq \beta_2(|x_2(0)|, 0) + \gamma_2 \circ (Id + \rho_1) \circ \gamma_1(\{y_{2T_0}\}) + \gamma_2 \circ (Id + \rho_1) \circ \gamma_1(\{u_1\}) + \gamma_2 \circ (Id + \rho_1) \circ \gamma_1(\{u_2\}) + \gamma_2 \circ (Id + \rho_1) \circ \gamma_1(\{v_1\}) + \gamma_2 \circ (Id + \rho_1) \circ \gamma_1(\{v_2\}) + \gamma_2 \circ (Id + \rho_1) \circ \gamma_1(\{d_1\}) + \gamma_2 \circ (Id + \rho_1) \circ \gamma_1(\{d_2\})
\]

(21)

\[
\leq \beta_2(|x_2(0)|, 0) + (Id + \rho_2)^{-1}(\|y_{2T_0}\|) + d_3
\]

(22)

\[
\leq (Id + \rho_2)^{-1}(\beta_2(|x_2(0)|, 0)) + \gamma_2 \circ (Id + \rho_1) \circ \gamma_1(\{u_1\}) + \gamma_2 \circ (Id + \rho_1) \circ \gamma_1(\{u_2\}) + \gamma_2 \circ (Id + \rho_1) \circ \gamma_1(\{v_1\}) + \gamma_2 \circ (Id + \rho_1) \circ \gamma_1(\{v_2\}) + \gamma_2 \circ (Id + \rho_1) \circ \gamma_1(\{d_1\}) + \gamma_2 \circ (Id + \rho_1) \circ \gamma_1(\{d_2\})
\]

(23)

Since \(T_0\) is arbitrary in \([0, T)\) and the right-hand side of (23) is independent of \(T_0\), \(y_2(t)\) is bounded on \([0, T)\).

By symmetry, the same argument shows that \(y_1(t)\) is bounded on \([0, T)\). Since the \(x_1 - \text{subsystem}\) and \(x_2 - \text{subsystem}\) satisfy the UO property, we conclude that \(x_1(t)\) and \(x_2(t)\) are bounded on \([0, T)\). It follows by contradiction that \(T = +\infty\).

**Step 2 : The IOPs Property.** Continuing from (23), we can establish bounds on the outputs in the following manner. From (2), for any function \(\rho_3\) of class \(K_\infty\), we have

\[
|y_2(t)| \leq (Id + \rho_2)^{-1}(\beta_2(|x_2(0)|, 0) + d_3 + \gamma_2 \circ (Id + \rho_1) \circ (Id + \rho_3)^{-1}(\beta_1(|x_1(0)|, 0))
\]

(24)

\[
+ \gamma_2 \circ (Id + \rho_1) \circ (Id + \rho_3)(v_1 + d_1) + v_2 + d_2
\]

\[
\leq (Id + \rho_2)^{-1}(\beta_2(|x_2(0)|, 0) + \gamma_2 \circ (Id + \rho_1) \circ (Id + \rho_3)^{-1}(\beta_1(|x_1(0)|, 0)) + (Id + \rho_3)(d_3 + \gamma_2 \circ (Id + \rho_1) \circ (Id + \rho_3)(v_1 + d_1) + v_2 + d_2)
\]

(25)

So, by symmetry, we have established:

\[
|y_1(t)| \leq \delta_1(|x(0)|) + \Delta_1, \quad |y_2(t)| \leq \delta_2(|x(0)|) + \Delta_2
\]

(26)

with

\[
\delta_1(s) = (Id + \rho_1)^{-1} \circ (Id + \rho_3)^{-1}(\beta_1(s, 0) + \gamma_1 \circ (Id + \rho_2)^{-1} \circ (Id + \rho_3)^{-1}(\beta_2(s, 0))
\]

\[
\delta_2(s) = (Id + \rho_2)^{-1} \circ (Id + \rho_3)^{-1}(\beta_2(s, 0) + \gamma_2 \circ (Id + \rho_1)^{-1} \circ (Id + \rho_3)^{-1}(\beta_1(s, 0))
\]

\[
\Delta_1 = (Id + \rho_1)^{-1} \circ (Id + \rho_3)(d_3 + \gamma_1 \circ (Id + \rho_2)^{-1} \circ (Id + \rho_3)(v_2 + d_2) + v_1 + d_1)
\]

\[
\Delta_2 = (Id + \rho_2)^{-1} \circ (Id + \rho_3)(d_3 + \gamma_2 \circ (Id + \rho_1)^{-1} \circ (Id + \rho_3)(v_1 + d_1) + v_2 + d_2)
\]

(27)
With these bounds on the outputs we can use the UO property to establish bounds on the states \( x_i \). In particular, let \((a_1^0, D_1^0), i = 1, 2\) be two couples satisfying (3) respectively for the subsystems (7). In this case any solution \( x(t) \) of (7) satisfies, for all \( t \geq 0 \):

\[
|x_1(t)| \leq a_1^0 \left( |x_1(0)| + \left\| (u_{1t}, y_{1t}, y_{1tt}) \right\| \right) + D_1^0 \tag{28}
\]

\[
|x_2(t)| \leq a_2^0 \left( |x_2(0)| + \left\| (u_{2t}, y_{2t}, y_{2tt}) \right\| \right) + D_2^0 \tag{29}
\]

From (26), (28), (29), and (2), we have:

\[
\|x\| \leq (a_1^0 + a_2^0) \circ (2Id + 2\delta_1 + 2\delta_2) \left( |x(0)| \right) + [(a_1^0 + a_2^0)(2|u| + 2\Delta_1 + 2\Delta_2) + D_1^0 + D_2^0]
\]

\[
:= \delta_3(|x(0)|) + \Delta_3 \tag{30}
\]

with \( \delta_i \) and \( \Delta_i (i = 1, 2) \) defined in (27).

With these bounds on the states, inequalities (26) can be completed as follows: Let

\[
t_{10} = \frac{t}{4}, \quad t_{20} = \frac{t}{2}, \quad t_{21} = t, \quad t_{11} \in \left[ \frac{t}{2}, t \right]
\]

and substitute (17) in (18), so that we have, for any \( t \geq 0 \) and \( \tau \geq 0 \),

\[
|y_2(t + \tau)| \leq \beta_2 \left( s_\infty \frac{t}{2} \right) + v_2 + d_2 + \gamma_2 \left( (\left\| y_2(t/4 + \tau, \infty) \right\|) + \beta_1 \left( s_\infty \frac{t}{4} \right) + v_1 + d_1 \right)
\]

Thus, by applying (24) and using (19), we obtain, for all \( t \geq 0 \) and tan \( \geq 0 \),

\[
|y_2(t + \tau)| \leq \left[ \beta_2 \left( s_\infty \frac{t}{2} \right) + \gamma_2 \circ (Id + \rho_1^{-1}) \circ (Id + \rho_3^{-1}) \left( \beta_1 \left( s_\infty \frac{t}{4} \right) \right) \right]

+ (Id + \rho_2)^{-1} \left( \left\| y_2(t/4, \infty) \right\| \right) + \gamma_2 \circ (Id + \rho_1^{-1}) \circ (Id + \rho_3)(v_1 + d_1) + v_2 + d_2 + d_3
\]

Note that the term between brackets in (33) is a function if class KL with respect to \((s_\infty, t)\). Further,

\[
\gamma_2 \circ (Id + \rho_1^{-1}) \circ (Id + \rho_3)(v_1 + d_1) + v_2 + d_2 + d_3 = [(Id + \rho_1^{-1}) \circ (Id + \rho_3)]^{-1}(\Delta_2)
\]

So we apply Lemma A.1 of [1] to (33) with \( z(t) = |y_2(t + \tau)|, \mu = \frac{1}{4}, \lambda = (Id + \rho_3), \) and \( \rho = (Id + \rho_2)^{-1} \).

It follows, using symmetry, that two functions \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \) of class KL exist such that, for all \( t \geq 0 \) and \( \tau \geq 0 \),

\[
|y_1(t + \tau)| \leq \hat{\beta}_1(s_\infty, t) + \Delta_1, \quad |y_2(t + \tau)| \leq \hat{\beta}_2(s_\infty, t) + \Delta_2
\]

Now we define the following function on \( R_+^2 \):

\[
\sigma_1(s, \Delta, t) := \min \{ \hat{\beta}_1(\delta_3(s) + \Delta, t), \delta_1(s) \}
\]

then, for any function \( \alpha \) of class \( K_{\infty} \) and for each \((s, \Delta, t)\), we have:

\[
\sigma_1(s, \Delta, t) \leq \sigma_1(s, \alpha^{-1}(s), t) + \sigma_1(\alpha(\Delta), \Delta, t)

\leq \hat{\beta}_1(\delta_3(s) + \alpha^{-1}(s), t) + \delta_1 \circ \alpha(\Delta)
\]

In view of (30) and (37), (26) and (35) imply, for all \( t \geq 0 \),

\[
|y_1(t)| \leq \sigma_1(|x(0)|, \Delta_3, t) + \Delta_1

\leq \hat{\beta}_1 (\delta_3(|x(0)|)) + \alpha^{-1}(|x(0)|), t) + \delta_1 \circ \alpha(\Delta_3) + \Delta_1
\]

Based on the notation (12), we can get:

\[
\Delta_1 \leq r_1(|u|) + \bar{d}_1
\]

\[
\Delta_2 \leq r_2(|u|) + \bar{d}_2
\]

\[
\Delta_3 \leq (a_1^0 + a_2^0) \circ (4Id + 4r_1 + 4r_2)(|u|) + (a_1^0 + a_2^0) (4\bar{d}_1 + 4\bar{d}_2) + D_1^0 + D_2^0
\]
where
\[
\hat{d}_1 = (Id + \rho_1^{-1}) \circ (Id + \rho_3) \circ (Id + \rho_5^{-1}) d_1 + d_3 + \gamma_1 \circ (Id + \rho_2^{-1}) \circ (Id + \rho_3) \circ (Id + \rho_5^{-1}) (d_2)
\]
\[
\hat{d}_2 = (Id + \rho_2^{-1}) \circ (Id + \rho_3) \circ (Id + \rho_5^{-1}) d_2 + d_3 + \gamma_2 \circ (Id + \rho_1^{-1}) \circ (Id + \rho_3) \circ (Id + \rho_5^{-1}) (d_1)
\]
(42)
(43)

Then, using (2) again, we have
\[
\delta_1 \circ \alpha (\Delta_3) \leq \delta_1 \circ \alpha (2\alpha_1^0 + 2\alpha_2^0) \circ (4Id + 4r_1 + 4r_2) (||u||)
\]
\[
+ \delta_2 \circ \alpha (2\alpha_4^0 + 2\alpha_5^0) (4\hat{d}_1 + 4\hat{d}_2 + 2D_1 + 2D_2)
\]
(44)

Now, giving any function \(r_3^1\) of class \(K_\infty\), we can chose \(\alpha\) such that:
\[
\delta_1 \circ \alpha (2\alpha_1^0 + 2\alpha_2^0) \circ (4Id + 4r_1 + 4r_2) (s) \leq r_3^1 (s), \quad \forall s \geq 0
\]
(45)

(For example, \(\alpha = (Id + \delta_1)^{-1} \circ r_3^1 \circ (Id + (2\alpha_4^0 + 2\alpha_5^0) \circ (4Id + 4r_1 + 4r_2)^{-1})\). This in conjunction with (38), (39), (40) and (45) implies:
\[
|y_1(t)| \leq \beta_1^0 (|x(0)|, t) + (r_1 + r_3^1) (||u||) + d_1'
\]
(46)

Where
\[
\beta_1^0 (s, t) = \hat{\beta}_1 (\delta_3 (s) + \alpha^{-1} (s), t)
\]
\[
d_1' = \hat{d}_1 + \delta_1 \circ \alpha (2\alpha_1^0 + 2\alpha_2^0) (4\hat{d}_1 + 4\hat{d}_2 + 2D_1 + 2D_2)
\]
(47)
(48)

By symmetry, we have:
\[
|y_2(t)| \leq \beta_2^0 (|x(0)|, t) + (r_2 + r_3^1) (||u||) + d_2'
\]
(49)

Where
\[
r_3^2 = \delta_2 \circ \delta_1^{-1} \circ r_3^1
\]
\[
\beta_2^0 (s, t) = \hat{\beta}_2 (\delta_3 (s) + \alpha^{-1} (s), t)
\]
\[
d_2' = \hat{d}_2 + \delta_2 \circ \alpha (2\alpha_4^0 + 2\alpha_5^0) (4\hat{d}_1 + 4\hat{d}_2 + 2D_1 + 2D_2)
\]
(50)
(51)
(52)

Finally, with the fact that \(|y(t)| \leq |y_1(t)| + |y_2(t)|\) we obtain the \(IOPs\) property for system \(7\) with the triple \((\beta'_1 + \beta'_2, r_1 + r_2 + r_3^1 + r_3^2, d_1' + d_2')\). When \(d_i = D_i^0 = 0 (i = 1, 2)\) and \(d_3 = 0 (i.e., s_i = 0)\), we get \(d_1' = d_2' = 0\) implying the \(IOS\) property holds. The \(UO\) property for the interconnection follows from the \(UO\) property for each subsystem.

References

[1] Z.-P.Jiang, A.R.Teel, and L.Praly. Small-gain theorem for iss system and applications. \textit{Math.Control Signals Systems, 7}(95-120), 1994.