OPTIMAL INVESTMENT AND CONSUMPTION IN THE
MARKET WITH JUMP RISK AND CAPITAL GAINS TAX

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Abstract. This paper investigates the problem of dynamic investment and
consumption in a market, where a risky asset evolves with jumps and capital
gains are taxed. In addition, the investor’s behavior of tax evasion is taken into
account, and tax evasion is subject to penalty when it is uncovered by audits.
Using dynamic programming approach, we derive an analytical solution for an
investor with the CRRA utility. We find the following: (1) jumps in the risky
asset do not affect the optimal tax evasion strategy; (2) jump risk lessens the
optimal fraction of wealth in the risky asset; (3) tax evasion can be reduced
by increasing the fine and/or the frequency of tax audits; (4) the effects of the
jumps, audits and penalty on the optimal consumption are determined by the
degree of risk aversion of the investor.

1. Introduction. Since the seminal work by [1], optimal investment and consump-
tion problems have been increasingly attractive. On the basis of the Merton model,
many variants have emerged in the literature, including the variations of preference,
asset price dynamics and objective. A comprehensive survey for this aspect is given
in [2].

Because jumps in asset prices, which result from unexpected large shocks such
as the subprime crisis, corporate scandal and bankruptcy, have been empirically
detected (see, e.g., [3, 4, 5, 6, 7, 8, 9, 10]), an important variant is to assume the
asset prices follow jump-diffusion processes. Of the representative papers in this
direction, the effect of systemic risk on portfolio selection is explored in [11] by
using a multivariate system of jump-diffusion processes where the arrival of jumps
is simultaneous across assets, the effects of events-related jumps in stock prices as
well as volatility is analyzed in [12], and the optimal portfolio selection problem
is studied in jump-diffusion models where jumps are incorporated into both stock
returns and state variables in [13]. Besides, [14] considers a consumption-investment

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problem on infinite time horizon maximizing discounted expected HARA utility for a general incomplete market model. However, consumption and capital gains tax are not considered in their models.

Capital gains tax is a realistic factor and it generally has a higher rate than the rate of transaction cost, thus capital gains tax should have greater influence on investment choice. However, it has not received as much attention as transaction cost. The distinction between capital gains tax and transaction cost has been documented in [15]. In the literature, capital gains tax has been incorporated in both discrete-time models and continuous-time models. Among the discrete-time models, [16] and [17] develop a binomial tree model with capital gains tax to work with multi-step portfolio choice problem, and [18] extends the model to a case with multiple risky assets. In addition, [19] studies optimal tax-timing and asset allocation problem when tax rebates on capital losses are limited. Among the continuous-time models, [20] and [21] incorporate capital gains tax in the classical Merton optimal consumption-investment problem, [22] proposes a continuous-time model to examine how the asymmetric tax structure together with limited tax returns affects the behavior of investors, and [15] develops a continuous-time investment and consumption model with capital gains tax, recursive utility and regime switching to study what factors affect optimal portfolio allocation. In these models, however, analytical solutions are not available.

In reality, due to the high rate of capital gains tax, an investor might conceal a portion of his investment to avoid paying capital gains tax. In a continuous-time optimal consumption and investment model, tax evasion is taken into account in [23], where the investor places his wealth in a riskless asset and a risky asset, and both returns will be taxed. Moreover, it is assumed that the investor could evade taxes on the revenue from the risky asset, but he could not on the revenue from the riskless asset. Thus, it means that the investor will make portfolio decisions on three asset classes: the riskless asset where gains tax cannot be evaded, a non-concealed risky asset and a concealed risky asset. Meanwhile, because taxation is subject to audit, once the behavior of evasion is detected, the investor will have to pay a fine, which is a function of the value of the concealed risky asset. However, [23] assumes that the price of the risky asset evolves continuously. As pointed out above, empirical evidence has repeatedly shown the existence of jumps in asset prices. Therefore, we extend their model by assuming that the risky asset price follows a jump-diffusion process, in which the arrival of jumps is governed by a Poisson process. The purpose of this paper is to examine the effect of jumps on optimal consumption and portfolio choice strategy. Using the dynamic programming method, we derive the optimal consumption and portfolio decisions in various cases. We find that: (i) the optimal fractions of wealth invested in the concealed risky asset and the unconcealed risky asset are both time-invariable; (ii) jumps do not affect the optimal tax evasion strategy and the optimal weight in the unconcealed risky asset decreases in jump arrival intensity and the mean jump size; (iii) audits and punishments for tax evasion are effective ways to reduce an investor’s behavior of tax evasion; (iv) the effects of jumps, audits and penalty intensity on the ratio of optimal consumption to wealth are critically dependent on the investor’s degree of risk aversion. Moreover, different from [23], we find that high liquidity portfolio might be selected to hedge jump risk rather than as a sign of tax evasion. Therefore, high liquidity may be a misleading signal used to target audits.
The remainder of this paper is organized as follows. Section 2 presents the model and basic setup. Section 3 gives an analytical solution to the optimal consumption and portfolio choice problem, and examines the sensitivity of the optimal policy to jump risk and audit. Section 3 provides numerical analyses and Section 4 concludes.

2. Model. We assume that there are two assets in the economy. The first is a riskless asset paying a constant rate of interest \( r \). The second is a risky asset with no dividends whose price \( S(t) \) is driven by a jump-diffusion process. Specifically, the price of the risky asset follows the process

\[
dS(t) = (\mu - \phi \lambda_1)S(t)dt + \sigma S(t)dB(t) + S(t-)d\sum_{i=1}^{N_1(t)} Y_i, \tag{1}
\]

where \( \mu \) is the expected return and \( \sigma \) is the volatility, \( B(t) \) is a standard Brownian motion, \( N_1(t) \) is a Poisson process with constant arrival intensity \( \lambda_1 \), and \( \{Y_i\}_{i=1}^{\infty} \) are independently and identically distributed jump sizes with mean \( \phi \) and they are assumed to have support on \((-1, \infty)\) guaranteeing the positivity of \( S \). The parameters \( \mu, \sigma \) and \( \lambda_1 \) are all assumed to be nonnegative. Besides, \( B(t), N_1(t) \) and \( \{Y_i\}_{i=1}^{\infty} \) are assumed to be independent of each other.

In this paper, we consider capital gains taxation. As in [23], we assume that (i) capital gains are taxed continuously and in a symmetric manner, that is, the investor pays tax if the change in asset price is positive and receives a refund if it is negative\(^1\); and (ii) consumption will not be taxed. The tax rates levied on the payoff of the riskless asset and risky asset are denoted by \( \tau_G \) and \( \tau \), respectively. If we let \( r_G = r(1 - \tau_G) \), then \( r_G \) is the after-tax or tax-adjusted return of the riskless asset. In this economy, the investor might conceal part of his investment in the risky asset to evade tax. However, the investor will be fined if tax evasion is exposed in an audit. We also assume that the arrival of the audit is driven by a Poisson process \( N_2(t) \) with constant intensity \( \lambda_2 \), and the proportion of the market value of the evaded risky asset to be fined is \( \alpha \).

Let \( W(t) \) be the investor’s wealth at time \( t \), which consists of \( w_G(t) \) shares of the riskless asset, \( w(t) \) shares of the non-concealed risky asset and \( w_0(t) \) shares of the concealed risky asset. Assume that the investor consumes \( c(t) \) at time \( t \) and there is no endowment stream. In what follows, for convenience we shall omit the variable \( t \) when no confusion is made. Therefore, we have the dynamics of the investor’s wealth as follows:

\[
dW = r_G(W - wS - w_0S)dt + (1 - \tau)wdS + w_0dS - \alpha w_0SdN_2 - cdt. \tag{2}
\]

Combining (1) and (2), we obtain

\[
dW = \{r_G W + wS[(1 - \tau)(\mu - \phi \lambda_1) - r_G] + w_0S(\mu - \phi \lambda_1 - r_G) - c\}dt \\
+ [(1 - \tau)w + w_0]\sigma dB + [(1 - \tau)w + w_0]S_-d\sum_{i=1}^{N_1} Y_i - \alpha w_0SdN_2, \tag{3}
\]

where \( S_- \) is the abbreviation of \( S(t-) \).

\(^1\)This assumption is also made in [15]. For more explanation, please refer to [23].
3. Optimal dynamic consumption and investment. In this section, we study the optimal consumption and investment problem for an investor who has the power utility $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ where $\gamma \in (0, 1) \cup (1, \infty)$. With a positive initial wealth $W(0)$, the investor’s objective is to maximize the total utility of consumption and the terminal wealth by determining his consumption, the fractions of wealth invested in non-concealed and concealed risky asset. To be specific, the optimal problem for the investor is to achieve

$$\max_{\{w(s), w_0(s), c(s): 0 \leq s \leq T\}} \mathbb{E} \left[ \int_0^T \frac{c^{1-\gamma}}{1-\gamma} e^{-\rho s} ds + \chi \frac{W(T)^{1-\gamma}}{1-\gamma} e^{-\rho T} \right]$$

where the wealth process $W$ is governed by Equation (3) and $\chi$ is a constant relative weight for the terminal wealth. The larger the value of $\chi$, the stronger the investor’s preference for terminal wealth with respect to inter-temporal consumption.

We now turn to solve the optimal consumption and asset allocation problem by using dynamic programming method. Following the standard procedure, we first define the indirect utility function or the value function by

$$J(t, W(t)) = \max_{\{w(s), w_0(s), c(s): t \leq s \leq T\}} \mathbb{E}_t \left[ \int_t^T \frac{c^{1-\gamma}}{1-\gamma} e^{-\rho s} ds + \chi \frac{W(T)^{1-\gamma}}{1-\gamma} e^{-\rho T} \right],$$

where $\mathbb{E}_t$ is the conditional expectation given the available information at time $t$. According to the Bellman’s principle of optimality, we easily obtain the following Hamilton-Jacobi-Bellman (HJB) equation for indirect utility function $J$:

$$0 = J_t + \max_{w, w_0, c} \left\{ \frac{c^{1-\gamma}}{1-\gamma} e^{-\rho t} + [r_G W + w S((1-\tau)(\mu - \phi \lambda_1) - r_G) + w_0 S(\mu - \phi \lambda_1 - r_G) - c] J_W + \frac{1}{2} ((1-\tau) w + w_0)^2 S^2 \sigma^2 J_{WW} + \lambda_1 (\mathbb{E}[J(t, W + Y S((1-\tau) w + w_0))] - J) + \lambda_2 (J(t, W - \alpha w_0 S) - J) \right\}$$

where $J_t$ and $J_W$ denote the partial derivatives of $J(t, W)$ with respect to $t$ and $W$, and $J_{WW}$ denotes the second partial derivative. Obviously, the terminal condition for the indirect utility function is

$$J(T, W) = \chi \frac{W^{1-\gamma}}{1-\gamma} e^{-\rho T}.$$ 

The optimal consumption and asset allocation strategy is summarized in the following proposition.

**Proposition 1 (Optimal policy).** Let $\pi^*_c$ be the ratio of the optimal consumption to wealth. Let $\pi^*$ and $\pi^*_0$ be the optimal fractions of wealth invested in unconcealed risky asset and concealed risky asset, respectively. Then

$$\pi^*_c = \frac{c^*}{W} = \frac{\zeta}{1 + \chi^{1/\gamma} \zeta (\tau - t)^{\gamma - 1}},$$

$$\pi^*_0 = \frac{w^*_0 S}{W} = \frac{1}{\alpha} \left\{ 1 - \left[ \frac{\alpha \lambda_2 (1-\tau)}{r_G \tau} \right]^{1/\gamma} \right\},$$
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\[ \pi^* = \frac{w^* S}{W} = \frac{1}{1 - \tau} \left\{ \mu - \frac{\tau \phi}{1 - \tau} + \lambda_1 \left( \frac{E[Y (1 + Y \theta^*)^{-\gamma}] - \phi}{\gamma \sigma^2} \right) - \pi_0^* \right\} , \]  

(8)

where \( \theta^* \) is the solution to the following non-linear equation

\[ \theta^* = (1 - \tau)\pi^* + \pi_0^* = \mu - \frac{\tau \phi}{1 - \tau} + \lambda_1 \left( \frac{E[Y (1 + Y \theta^*)^{-\gamma}] - \phi}{\gamma \sigma^2} \right) \]  

(9)

and \( \zeta = \frac{M - \rho}{\gamma} \).

\[ M = (1 - \gamma)[r_G(1 - \pi^* - \pi_0^*) + (\mu - \phi \lambda_1)\theta^* - \frac{1}{2} \gamma \sigma^2 \theta^*]^2 \]

\[ + \lambda_1[\delta(1 + Y \theta^*)^{1-\gamma} - 1] + \lambda_2[(1 - \alpha \pi_0^*)^{1-\gamma} - 1]. \]  

(10)

Proposition 1 reveals that the optimal investment weights in the non-concealed, concealed risky asset and riskless asset are time-invariant. Moreover, the optimal consumption is also proportional to wealth, but it varies as time goes by. \( \theta^* \) can be viewed as the tax-adjusted investment weight in the risky asset which includes non-concealed and concealed risky asset. In particular, when \( \lambda_1 = 0 \), the optimal consumption and portfolio strategy reduces to that in [23].

**Proof.** According to Equation (5), the first order conditions for the optimal consumption \( c^* \), declared assets \( w^* \) and undeclared assets \( w_0^* \) are

\[ J_W - (c^*)^{-\gamma} e^{-\rho t} = 0, \]  

(11)

\[ ((1 - \tau)(\mu - \phi \lambda_1) - r_G)J_W + S \sigma^2 (1 - \tau)(w_0 + w(1 - \tau))J_{WW} \]

\[ + \lambda_1(1 - \tau)E[Y J_W(t, W + Y S(w_0 + w(1 - \tau)))] = 0, \]  

(12)

and

\[ (\mu - \phi \lambda_1 - r_G)J_W + (w_0 + w(1 - \tau))S \sigma^2 J_{WW} \]

\[ + \lambda_1 E[Y J_W(t, W + Y S(w_0 + w(1 - \tau))) - \alpha \lambda_2 J_W(t, W - \alpha w_0 S)] = 0. \]  

(13)

We conjecture that the value function is of the form

\[ J(t, W) = \frac{W^{1-\gamma} e^{A(t)}}{1 - \gamma}. \]

Then

\[ J_t = JA', \quad J_W = W^{-\gamma} e^{A}, \quad J_{WW} = -\gamma W^{-\gamma-1} e^{A}. \]  

(14)

As a result, it follows from (7) that the optimal consumption process is

\[ c^* = W e^{-\frac{1}{\gamma}(\rho t + A)}. \]  

(15)

The procedure about how to derive the function \( A(t) \) will be shown later. Besides, Equations (12) and (13) become

\[ [(1 - \tau)w^* + w_0^*]S \sigma^2 = -\frac{J_W}{J_{WW}} \frac{(1 - \tau)(\mu - \phi \lambda_1) - r_G}{1 - \tau} \]

\[ - \frac{\lambda_1 E[Y J_W(t, W + Y S(w_0^* + w^*(1 - \tau)))]}{J_{WW}} \]  

(16)
and
\[
[(1-\tau)w^* + w^*_0]S\sigma^2 = -\frac{(\mu - \phi \lambda_1 - r_G) J_W}{J_{WW}} + \frac{\alpha \lambda_2 J_W(t, W - \alpha w^*_0 S)}{J_{WW}} \\
- \frac{\lambda_1 \mathbb{E}[Y J_W(t, W + Y S(w^*_0 + w^*(1 - \tau)))]}{J_{WW}},
\]
respectively. Subtracting (16) from (17), we have
\[
J_W \frac{r_G \tau}{1 - \tau} = \alpha \lambda_2 J_W(t, W - \alpha w^*_0 S).
\]
If \(\alpha \lambda_2 = 0\), the necessary condition for the existence of solutions to (16) and (17) is \(\tau = 0\). If \(\alpha \lambda_2 \neq 0\), the optimal shares of concealed risky asset is
\[
w^*_0 = \frac{W}{\alpha S} \left\{ 1 - \left[ \frac{\alpha \lambda_2 (1 - \tau)}{r_G \tau} \right]^{1/\gamma} \right\}.
\]
Let \(\kappa^* = (1 - \tau)w^* + w^*_0\). It follows from (16) that
\[
\gamma S \sigma^2 \kappa^* = \frac{(1 - \tau)(\mu - \phi \lambda_1) - r_G}{1 - \tau} + \lambda_1 \mathbb{E} \left[ Y \left( 1 + \frac{Y S \kappa^*}{W} \right)^{-\gamma} \right].
\]
Let \(\theta^* = (1 - \tau)\pi^* + \pi^*_0\). Because
\[
\pi^* = \frac{w^* S}{W}, \quad \pi^*_0 = \frac{w^*_0 S}{W} = \frac{1}{\alpha} \left\{ 1 - \left[ \frac{\alpha \lambda_2 (1 - \tau)}{r_G \tau} \right]^{1/\gamma} \right\},
\]
(20) can be rewritten as
\[
\gamma \sigma^2 \theta^* = \mu - \phi \lambda_1 - \frac{r_G}{1 - \tau} + \lambda_1 \mathbb{E} \left[ Y \left( 1 + Y \theta^* \right)^{-\gamma} \right].
\]
Now we show how to derive the analytical expression of \(A(t)\). According to (5), (14), (15) and (19), we get the ordinary differential equation (ODE)
\[
A' + \gamma e^{-\frac{A+\rho t}{\gamma}} + M = 0
\]
where
\[
M = (1 - \gamma)[r_G(1 - \pi^* - \pi^*_0) + \theta^*(\mu - \phi \lambda_1) - \frac{1}{2} \gamma \theta^* \sigma^2] \\
+ \lambda_1 [\mathbb{E}(1 + Y \theta^*)^{1-\gamma} - 1] + \lambda_2 [(1 - \alpha \pi^*_{0})^{1-\gamma} - 1].
\]
Besides, we have the terminal condition
\[
A(T) = \log \chi - \rho T.
\]
The solution to the ODE (22) satisfies
\[
e^{-\frac{A(t)}{\gamma}} = \frac{(M - \rho)e^{\rho t/\gamma}}{\gamma \left[ e^{\frac{1}{\gamma} \frac{T - (M - \rho)}{\gamma}} (1 + \frac{M - \rho}{\gamma \chi - 1/\gamma}) - 1 \right]}, 0 \leq t \leq T.
\]
For the proof, please refer to Appendix A. From (15), the optimal consumption is
\[
c^*(t) = W e^{-\frac{1}{\gamma} \rho (A(t))} = \frac{W(M - \rho)}{\gamma \left[ e^{\frac{1}{\gamma} \frac{T - (M - \rho)}{\gamma}} (1 + \frac{M - \rho}{\gamma \chi - 1/\gamma}) - 1 \right]}.
\]
Next, some important implications about the optimal consumption and investment are given. For instance, we show how the optimal strategy varies with time and as the price jump intensity of the risky asset changes.

**Corollary 1.** The optimal fractions in the concealed risky asset and the non-concealed risky asset are invariable as time elapses, that is,

$$\frac{\partial \pi^*_0}{\partial t} = \frac{\partial \pi^*_c}{\partial t} = 0.$$

In addition, if $M \leq \rho - \gamma \chi^{-1/\gamma}$, then the ratio of optimal consumption to wealth decreases in time $t \in [0, T]$ and if $M \geq \rho - \gamma \chi^{-1/\gamma}$, then the optimal consumption over the wealth is increasing in time $t$ on intervals $[0, \min\{T, T + \frac{1}{\zeta} \log(1 + \zeta \chi^{1/\gamma})\}]$ and $(\min\{T, T + \frac{1}{\zeta} \log(1 + \zeta \chi^{1/\gamma})\}, T]$.

**Proof.** From Proposition 1, it is clear that $Y \equiv \phi$. Mathematically,

$$\frac{\partial \pi^*_0}{\partial \phi} = \frac{\partial \pi^*_c}{\partial \phi} = 0.$$

In addition, according to (6),

$$\frac{\partial \pi^*_c}{\partial t} = \frac{\zeta^2 (1 + \chi^{1/\gamma} \zeta) e^{(T-t) \zeta}}{[1 + \chi^{1/\gamma} \zeta]^2}.$$

If $M \leq \rho - \gamma \chi^{-1/\gamma}$, then for $t \in [0, T]$, $\frac{\partial \pi^*_c}{\partial t} \leq 0$, namely $\pi^*_c$ is decreasing on $[0, T]$; if $M \geq \rho - \gamma \chi^{-1/\gamma}$, then for $t \in [0, \min\{T, T + \frac{1}{\zeta} \log(1 + \zeta \chi^{1/\gamma})\} \cup (\min\{T, T + \frac{1}{\zeta} \log(1 + \zeta \chi^{1/\gamma})\}, T]$, $\frac{\partial \pi^*_c}{\partial t} \geq 0$, namely $\pi^*_c$ is increasing on $[0, \min\{T, T + \frac{1}{\zeta} \log(1 + \zeta \chi^{1/\gamma})\}]$ and $(\min\{T, T + \frac{1}{\zeta} \log(1 + \zeta \chi^{1/\gamma})\}, T]$. □

**Corollary 2 (Jump effect on optimal policy).** The optimal fraction of wealth placed in the undeclared risky asset is irrelevant to the price jump intensity of the risky asset, and the optimal fraction in the declared risky asset is decreasing in jump intensity. Besides, the ratio of the optimal consumption to wealth is decreasing in jump intensity provided $\gamma > 1$ and $\phi \geq 0$, and it is increasing provided $0 < \gamma < 1$ and $-1 < \phi \leq 0$.

In particular, consider the case where price jump sizes are deterministic, namely $Y \equiv \phi$. Then the optimal fraction in the declared risky asset is increasing in the mean of jump sizes if the mean is negative, and it is non-increasing if the mean is nonnegative. Mathematically,

$$\frac{\partial \pi^*_c}{\partial \phi} > 0 \text{ on } (-1, 0), \quad \frac{\partial \pi^*_c}{\partial \phi} \leq 0 \text{ on } (0, \infty). \quad (26)$$

In addition, the ratio of optimal consumption to wealth is a decreasing function of the jump size on $(-1, 0)$ and is an increasing function on $[0, \infty)$ provided that $0 < \gamma < 1$. Conversely, the ratio is an increasing function of the jump size on $(-1, 0)$ and is a decreasing function on $[0, \infty)$ provided that $\gamma \geq 1$.

**Proof.** Firstly, by (8), it is clear that the optimal investment weight $\pi^*_0$ in the concealed risky asset does not depend on time and the price jumps of risky asset.

Secondly, taking the partial derivative of $\theta^*$ with respect to $\lambda_1$ in (21), we derive

$$\frac{\partial \theta^*}{\partial \lambda_1} = \frac{\mathbb{E} \left[ Y (1 + Y \theta^*)^{-\gamma} \right] - \phi}{\gamma \left( \sigma^2 + \lambda_1 \mathbb{E} \left[ Y^2 (1 + Y \theta^*)^{-\gamma - 1} \right] \right)} \leq 0,$$
since $E \left[ (1 + Y \theta^*)^{-\gamma} \right] - \phi \leq 0$. Because $\pi_0^*$ is irrelevant to jump intensity $\lambda_1$, 

$$ \frac{\partial \pi^*}{\partial \lambda_1} = \frac{\partial \theta^*}{1 - \tau \partial \lambda_1} \leq 0. $$

Thirdly, let $\zeta = \frac{M - \rho}{T}$. Then 

$$ \pi_c^* = \frac{1}{\zeta} \times \left[ \frac{r_G}{1 - \tau} - (\mu - \phi \lambda_1) + \gamma \sigma^2 \theta^* - \lambda_1 \lambda Y (1 + Y \theta^*)^{-\gamma} \right] \frac{\partial \theta^*}{\partial \lambda_1} $$

According to (21), we have 

$$ \partial M = (\gamma - 1) \left[ \frac{r_G}{1 - \tau} - (\mu - \phi \lambda_1) + \gamma \sigma^2 \theta^* - \lambda_1 \lambda Y (1 + Y \theta^*)^{-\gamma} \right] \frac{\partial \theta^*}{\partial \lambda_1} $$

Thus, according to Jensen’s inequality, if $\gamma > 1$ and $\phi \geq 0$, 

$$ \frac{\partial M}{\partial \lambda_1} = E (1 + Y \theta^*)^{1 - \gamma} + (\gamma - 1) \phi \theta^* - 1 $$

$$ > (1 + \phi \theta^*)^{1 - \gamma} + (\gamma - 1) \phi \theta^* - 1 $$

$$ > 1 + (1 - \gamma) \phi \theta^* + (\gamma - 1) \phi \theta^* - 1 $$

$$ = 0 $$

and hence 

$$ \frac{\partial \pi^*_c}{\partial \lambda_1} = \frac{\partial \pi^*_c}{\partial \zeta} \frac{\partial \pi^*_c}{\partial \lambda_1} = \frac{\partial \pi^*_c}{\partial \zeta} \frac{\partial \pi^*_c}{\partial \lambda_1} < 0. $$

Likewise, if $\gamma \in (0, 1)$ and $\phi \in (-1, 0)$, we have $\frac{\partial M}{\partial \lambda_1} < 0$ and $\frac{\partial \pi^*_c}{\partial \theta^*} > 0$.

Finally, assume that jump sizes are deterministic, i.e. $Y \equiv \phi$. In this case, replacing $Y$ by $\phi$ and differentiating $\pi^*$ and $\theta^*$ with respect to $\phi$ in (8) and (9) implies the following result:

$$ \frac{\partial \pi^*}{\partial \phi} = \frac{[(1 + \phi \theta^*)^{-\gamma} - 1] - \gamma \phi \theta^*(1 + \phi \theta^*)^{-\gamma - 1}}{\gamma (1 - \tau) [\sigma^2 / \lambda_1 + \phi^2 (1 + \phi \theta^*)^{-\gamma - 1}]} $$

If $\phi < 0$, then $(1 + \phi \theta^*)^{-\gamma} - 1 > 0$ and $\frac{\partial \pi^*_c}{\partial \phi} > 0$; if $\phi \geq 0$, then $(1 + \phi \theta^*)^{-\gamma} - 1 \leq 0$ and $\frac{\partial \pi^*_c}{\partial \phi} \leq 0$. The equality is attained when $\phi = 0$. Meanwhile, differentiating $M$ with respect to $\phi$ in (10) yields 

$$ \frac{\partial M}{\partial \phi} = (1 - \gamma) \frac{[(1 - \tau) [\mu \lambda_1 - r \sigma^2 \theta^* + \lambda_1 \phi (1 + \phi \theta^*)^{-\gamma} - \gamma \phi \theta^*(1 + \phi \theta^*)^{-\gamma - 1} - r_G \phi \partial \pi^*]}{\partial \phi} $$

$$ + (1 - \gamma) \lambda_1 \theta^* [(1 + \phi \theta^*)^{-\gamma} - 1] $$

$$ = (1 - \gamma) \lambda_1 \theta^* [(1 + \phi \theta^*)^{-\gamma} - 1]. $$

Since 

$$ \frac{\partial \pi^*_c}{\partial \phi} = \frac{\partial \pi^*_c}{\partial \zeta} \frac{\partial \pi^*_c}{\partial \lambda_1} = \frac{\partial \pi^*_c}{\partial \zeta} \frac{\partial \pi^*_c}{\partial \lambda_1} < 0, $$

if $0 < \gamma < 1$, $\frac{\partial \pi^*_c}{\partial \phi} \leq 0$ on $(-1, 0)$ and $\frac{\partial \pi^*_c}{\partial \phi} \geq 0$ on $[0, \infty)$; if $\gamma > 1$, $\frac{\partial \pi^*_c}{\partial \phi} \geq 0$ on $(-1, 0)$ and $\frac{\partial \pi^*_c}{\partial \phi} \leq 0$ on $[0, \infty)$. \hfill \Box
Corollary 2 indicates that the jump intensity of the risky asset will not affect the investment weight in the concealed risky asset, while the weight in the declared risky asset is a decreasing function of jump intensity. Particularly, the investor holds a smaller fraction in the risky asset than he would when the price evolves continuously. This is consistent with the findings in [24] and [12] about the effect of jumps in a risky asset on optimal asset allocation, even when capital gains tax and tax evasion are taken into account. Moreover, the conclusion (26) can be more concisely expressed by $\frac{\partial \pi^*}{\partial \varphi^2} \leq 0$, which means the optimal fraction in the declared risky asset is decreasing in the square of the mean jump size. The rationale for this is that, as shown in [25], the volatility of the return of the risky asset is positively proportional to the jump arrival intensity and the square of the mean jump size; therefore, the more frequent the occurrence of jumps and/or the larger the absolute value of the mean jump size, the more volatile the asset return. Accordingly, the investor tends to be more conservative in hedging larger volatility. In other words, in the presence of jump risk, an investor becomes more conservative to prevent his wealth from becoming negative. [23] argues that high liquidity (large weight in the riskless asset) may be seen as a sign of tax evasion and be used to target audits, but it may be a misleading signal in the presence of jumps because high liquidity can also result from the jumps.

On the other hand, the effect of jump intensity on the optimal consumption depends on the risk aversion coefficient and the mean jump size. If the investor is more risk-averse than a log-utility investor and the averaged jump size is positive, he tends to consume less as the jump intensity increases, and if the investor is less risk-averse than a log-utility investor and the averaged jump size is negative, he tends to consume more as the jump intensity increases. In the special case where the jump sizes are deterministic, for the investor who is more risk-averse than a log-utility one, he consumes less than he would if no jump occurs when the jump is either in the upward direction or in the downward direction; however, for the investor who is less risk-averse than a log-utility one, he consumes more. An implication of this is that the risk aversion coefficient determines how the optimal consumption responses to the jumps.

Corollary 3 (Punishment effect on the optimal policy). Let $\pi^*_G$ be the optimal weight in the riskless asset. The effect of the punishment for tax evasion on the optimal policy is included as follows:

(i)

$$\frac{\partial \pi^*_G}{\partial \lambda_2} < 0, \quad \frac{\partial \pi^*}{\partial \lambda_2} > 0, \quad \frac{\partial \pi^*_c}{\partial \lambda_2} < 0$$

and

$$\frac{\partial \pi^*}{\partial \lambda_2} \leq 0, \text{ if } \gamma > 1, \quad \frac{\partial \pi^*_c}{\partial \lambda_2} \geq 0, \text{ if } 0 < \gamma < 1.$$

(ii)

$$\frac{\partial \pi^*_G}{\partial \alpha} < 0, \quad \frac{\partial \pi^*}{\partial \alpha} > 0, \quad \frac{\partial \pi^*_c}{\partial \alpha} < 0$$

and

$$\frac{\partial \pi^*}{\partial \alpha} \leq 0, \text{ if } \gamma > 1, \quad \frac{\partial \pi^*_c}{\partial \alpha} \geq 0, \text{ if } 0 < \gamma < 1.$$
Proof. (i) From Proposition 1, it is clear that $\frac{\partial \pi_G^a}{\partial \lambda_2} < 0$, $\frac{\partial \pi^*}{\partial \lambda_2} > 0$, and $\frac{\partial \pi^*_G}{\partial \lambda_2} < 0$, since

$$\pi^*_G = 1 - \pi_0^* - \pi^*$$

$$= 1 - \frac{\mu - \frac{\delta(1-\pi_G^a)}{1-\tau} + \lambda_1 \left( E[Y(1 + Y^*)^{-\delta}] \right)}{(1-\tau)d\sigma^2} + \frac{\tau}{1-\tau}\pi^*_0.$$ 

Furthermore,

$$\frac{\partial M}{\partial \lambda_2} = (1-\gamma)\phi + \frac{\lambda_1}{(1-\tau)\partial \lambda_2} + [(1 - \alpha\pi_G^a)^{1-\gamma} - 1] + \lambda_2(1 - \gamma)(1 - \alpha\pi_G^a)^{-\gamma}(-\alpha)\frac{\partial \pi_G^a}{\partial \lambda_2}$$

$$= (1-\gamma)[r_G \frac{\lambda_1}{1-\tau} + \alpha\lambda_2(1 - \alpha\pi_G^a)^{-\gamma}\frac{\partial \pi_G^a}{\partial \lambda_2} + [(1 - \alpha\pi_G^a)^{1-\gamma} - 1]$$

$$= \left[ \frac{\alpha\lambda_2(1-\tau)}{r_G\tau} \right]^{\frac{1}{\gamma}} - 1$$

and $0 \leq \pi_0^* \leq 1$ implies

$$0 \leq (1-\alpha)^{\gamma} \leq \frac{\alpha\lambda_2(1-\tau)}{r_G\tau} \leq 1.$$ 

Therefore, if $\gamma > 1$, then $\partial M/\partial \lambda_2 \geq 0$, and if $0 < \gamma < 1$, then $\partial M/\partial \lambda_2 \leq 0$. From the proof of Proposition 2 we have $\frac{\partial \pi^*_G}{\partial \lambda_2} < 0$, and thereby

$$\frac{\partial \pi^*_G}{\partial \lambda_2} = \frac{\partial \pi^*_G}{\partial \zeta} \frac{\partial \zeta}{\partial \lambda_2} \frac{\partial \lambda_1}{\partial \lambda_1} = \frac{1}{\gamma} \frac{\partial \pi^*_G}{\partial \lambda_1} \leq 0$$

provided that $\gamma > 1$, and $\frac{\partial \pi^*_G}{\partial \lambda_2} \geq 0$ provided that $0 < \gamma < 1$.

(ii) Because

$$\frac{\partial \pi_0^*}{\partial \alpha} = -\frac{1}{\alpha^2} \left[ 1 - \left( \frac{\alpha\lambda_2(1-\tau)}{r_G\tau} \right)^{1/\gamma} \right]$$

$$\leq -\frac{1}{\alpha^2} \left[ 1 - \left( \frac{\alpha\lambda_2(1-\tau)}{r_G\tau} \right)^{1/\gamma} \right]$$

$$\leq \frac{\alpha}{\alpha^2}$$

$$\leq 0,$$

$$\frac{\partial \pi^*_G}{\partial \alpha} = -\frac{1}{1-\tau} \frac{\partial \pi_0^*}{\partial \alpha} \geq 0, \quad \frac{\partial \pi^*_G}{\partial \alpha} = \frac{\tau}{1-\tau} \frac{\partial \pi_0^*}{\partial \alpha} \leq 0.$$ 

Additionally,

$$\frac{\partial M}{\partial \alpha} = r_G(1-\gamma) \left[ \frac{\tau}{1-\tau} \frac{\partial \pi_0^*}{\partial \alpha} - \lambda_2(1 - \gamma)(1 - \alpha\pi_G^a)^{-\gamma}(\pi_0^a + \alpha \frac{\partial \pi_0^*}{\partial \alpha}) \right]$$

$$= r_G(1-\gamma) \left[ \frac{\tau}{1-\tau} \frac{\partial \pi_0^*}{\partial \alpha} - \lambda_2(1 - \gamma)(1 - \alpha\pi_G^a)^{-\gamma}(\pi_0^a + \alpha \frac{\partial \pi_0^*}{\partial \alpha}) \right]$$

$$= -\lambda_2(1-\gamma) \frac{r_G\tau}{\alpha\lambda_2(1-\tau)} \pi_0^a.$$ 

Hence, if $\gamma > 1$,

$$\frac{\partial \pi^*_G}{\partial \alpha} = \frac{\partial \pi^*_G}{\partial \zeta} \frac{\partial \zeta}{\partial \lambda_2} \frac{\partial \lambda_1}{\partial \lambda_1} = \frac{1}{\gamma} \frac{\partial \pi^*_G}{\partial \lambda_1} \leq 0,$$

and if $0 < \gamma < 1$, $\frac{\partial \pi^*_G}{\partial \alpha} > 0$. □
From this corollary, we find that as the frequency of audit and/or the harshness of the punishment for tax evasion increases, the weight in the concealed risky asset will decrease and the weight in the non-concealed risky asset will increase, which completely conforms to intuition. The implication is that strengthening audit and increasing punishment are effective ways to reduce the concealing of assets. In addition, the weight in the riskless asset decreases with audit intensity and punishment harshness, which means that the investor tends to pursue higher return on his portfolio with higher risk as supervision on tax evasion rises. Thus, the impacts of the measures to fight evasion on the optimal investment in the declared risky asset and the riskless asset are converse to the impact of the jumps. However, the effect of audit and punishment on the optimal consumption depends on the degree of risk aversion. Specifically, if the investor is more risk-averse than a log-utility investor, he reduces his consumption as the audit arrival intensity increases or (and) the penalty for tax evasion becomes severe; if the investor is less risk averse than a log-utility investor, he behaves conversely. From the policy point of view, the government can fight tax evasion by increasing fine or the frequency of audit. But these measures to fight evasion will make the investor more willing to take risk, i.e., he will invest less in the riskless asset and more in the risky asset. It leads to more uncertainty of the government revenue from the capital gains tax because the riskless asset has a certain return and the risky asset has the stochastic return, while the government can gain higher expected tax revenue since the risky asset has a higher expected return than the riskless asset. As a result, from the fairness point of view, the government should take strict measures against evasion; from the economic point of view, how strict the measures should be depends on how the government makes tradeoff between the risk and the expected value of tax revenue.

4. Examples and numerical analysis. In the numerical examples, we assume as [26] that the jump size of the risky asset price follows the lognormal distribution, specifically assume that $\log(1 + Y) \sim N(\mu_J, \sigma_J)^2$. The parameters, except those associated with the jumps, take the same values as those in [23] and represent a typical business situation. The values of the parameters in the base case are set as follows: the current time is zero and the remaining time for the investor to make decisions about consumption and investment is twenty years, i.e. $t = 0, T = 20$. The return on the riskless asset is 4%. For the risky asset, the expected return is 8% and the volatility is 0.2. The risk aversion coefficient is 2.5, the discount factor is 0.04 and the relative weight for the terminal wealth is 1. The tax rates on the riskless asset is $\tau_G = 0.27$ and the risky asset is $\tau = 0.235$, respectively. The parameters related to tax policy are audit intensity and punishment which are 0.1 and 0.08. Based on the empirical results in [9], the value of the jump arrival intensity of the risky asset is set as $\lambda_2 = 0.15$, and the log jump mean and the jump volatility are $\mu_J = 0$ and $\sigma_J = 0.5$, respectively. The parameter values are summarized in Table 1.

As shown in Figure 1, the optimal ratio of consumption to wealth is increasing as time elapses, and it increases sharply as time is close to the terminal because all the wealth will be used to consume at terminal. The optimal fractions of wealth in concealed and unconcealed risky asset and the riskless asset are time-invariable. These conclusions conform to Corollary 1. Figure 2 demonstrates the impact of the

Another usual assumption in literature is that $\log(1 + Y)$ follows double exponential distribution, which is proposed by [27].
Table 1. The values of the parameters in the base case

| Parameter                        | Value     | Parameter                        | Value     |
|----------------------------------|-----------|----------------------------------|-----------|
| Current time                     | $t = 0$   | Terminal time                    | $T = 20$  |
| Expected return                  | $\mu = 0.08$ | Volatility                      | $\sigma = 0.2$ |
| Riskless interest rate           | $r = 0.04$ | Risk aversion coefficient       | $\gamma = 2.5$ |
| Discount factor                  | $\rho = 0.04$ | Relative weight                 | $\chi = 1$ |
| Audit intensity                  | $\lambda_2 = 0.1$ | Punishment intensity          | $\alpha = 0.08$ |
| Tax rate on riskless asset       | $\tau_G = 0.27$ | Tax rate on risky asset        | $\tau = 0.235$ |
| Jump intensity                   | $\lambda_1 = 0.15$ | Log jump mean              | $\mu_J = 0$ |
| Jump volatility                  | $\sigma_J = 0.5$ |                                   |           |

Figure 1. Dynamic optimal investment and consumption.

Jump intensity of the risky asset on the optimal policy. The optimal fraction in unconcealed risky asset and the ratio of optimal consumption to wealth are both negatively related to the jump intensity, however, the latter is not sensitive to the change of jump intensity. Additionally, the weight in the riskless asset is increasing in the jump intensity that is because the investor holds the high liquidity asset to hedge the jump risk. As indicated in Figure 3, when the jump mean increases from negative to positive, both the optimal fraction on unconcealed risky and the optimal ratio of consumption to wealth rise first and arrive at the peak at $\phi = 0$, then they decline. However, the weight in the riskless asset behaves conversely. Note that the changing trend of $\pi^*_c$ is not apparent in this figure, but it indeed behaves as expected. For example, as $\phi$ changes from $-0.5$ to $0.5$, $\pi^*_c$ increases from $0.068$ to $0.0697$, and then decreases to $0.0682$. As a result, the optimal consumption is not sensitive to the jumps in the risky asset.

Figure 4 shows that the audit arrival intensity and the punishment intensity have a positive effect on the reduction of concealing asset, and the effect is very significant. Moreover, the sensitivity of optimal tax evasion towards $\lambda_2$ and $\alpha$ are...
nearly the same, but the elasticity of optimal tax evasion towards $\lambda_2$ is higher than the elasticity towards $\alpha$. For the investor with high risk aversion, the optimal ratio of consumption to wealth decreases as the audit intensity and (or) the punishment intensity increases; it does not obviously appear for the small variation of audit intensity and punishment intensity.
5. **Conclusion.** In this paper, we study the optimal consumption and investment problem in a market where the risky asset price has jumps and capital gains are taxed. In our model, the investor has CRRA utility and possibly conceals part of his investment in the risky asset to evade tax. Analytical solutions to this optimal problem are obtained. Our results suggest that the optimal fractions of wealth placed in the unconcealed and concealed risky asset do not vary as time elapses. Furthermore, we find that the investment strategy on the concealed risky asset is not influenced by the jumps of risky asset price, but the fraction in the unconcealed risky asset is negatively related to the jump arrival intensity and the absolute mean.
size of the jumps. Thirdly, as anticipated, the investor will reduce his risk-taking behavior of tax evasion if more frequent audits and/or more strict punishment are implemented. Finally, our analysis reveals that the effects of jumps and government policy on optimal consumption are determined by the degree of risk aversion of the investor. Especially for the investor with a high level of risk aversion, strict government policy on tax evasion will reduce the investor’s consumption; conversely, for the investor with a low level of risk aversion, it will improve.

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Appendix A. The ODE in (22) is

\[ A' + \gamma e^{-\frac{A(t) + \rho t}{\gamma}} + M = 0 \]

with the terminal condition

\[ A(T) = \log \chi - \rho T. \]

Let

\[ Q(t) = \gamma e^{-\frac{A(t) + \rho t}{\gamma}} + M. \] (27)

Then

\[ e^{-\frac{A(t) + \rho t}{\gamma}} = \frac{Q(t) - M}{\gamma} \] (28)

and

\[ A'(t) = -Q(t). \] (29)

Obviously, it follows from (27), (28) and (29) that

\[ Q'(t) = -(A'(t) + \rho) e^{-\frac{A(t) + \rho t}{\gamma}} \]

\[ = -(A'(t) + \rho) \frac{Q(t) - M}{\gamma} \]

\[ = \frac{(Q(t) - M)(Q(t) - M)}{\gamma}. \] (30)

Then

\[ \frac{dt}{dQ(t)} = \frac{\gamma}{\rho - M} \left( \frac{1}{Q(t) - \rho} - \frac{1}{Q(t) - M} \right). \]

Therefore,

\[ \log \frac{Q(t) - \rho}{Q(t) - M} = \frac{t(\rho - M)}{\gamma} + C \] (31)

where \( C \) is a constant, and equivalently

\[ Q(t) = \frac{M - \rho}{\exp\{t(\rho - M)/\gamma + C\} - 1} + M. \] (32)

Combining (28) and the terminal condition in (23), we conclude that

\[ e^{-\frac{A(t)}{\gamma}} = \frac{(M - \rho) e^{pt/\gamma}}{\gamma \left[ e^{\frac{(t - T)(\rho - M)}{\gamma}} (1 + \frac{M - \rho}{\gamma\chi^{(p/M)}}) - 1 \right]} \].
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