Decidable Expansions of Labelled Linear Orderings

Alexis Bès\textsuperscript{1} and Alexander Rabinovich\textsuperscript{2}

\textsuperscript{1} University of Paris-Est Créteil, LACL
Email: bes@u-pec.fr
\textsuperscript{2} Tel-Aviv University, The Blavatnik School of Computer Science
Email: rabinoa@post.tau.ac.il

Dedicated to Yuri Gurevich on the occasion of his seventieth birthday

Abstract. Let $M = (A, <, \overline{P})$ where $(A, <)$ is a linear ordering and $\overline{P}$ denotes a finite sequence of monadic predicates on $A$. We show that if $A$ contains an interval of order type $\omega$ or $-\omega$, and the monadic second-order theory of $M$ is decidable, then there exists a non-trivial expansion $M'$ of $M$ by a monadic predicate such that the monadic second-order theory of $M'$ is still decidable.

Key words: monadic second-order logic, decidability, definability, linear orderings

1 Introduction

In this paper we address definability and decidability issues for monadic second order (shortly: MSO) theories of labelled linear orderings. Elgot and Rabin ask in [9] whether there exist maximal decidable structures, i.e., structures $M$ with a decidable first-order (shortly: FO) theory and such that the FO theory of any expansion of $M$ by a non-definable predicate is undecidable. This question is still open. Let us mention some partial results:

- Soprunov proved in [28] that every structure in which a regular ordering is interpretable is not maximal. A partial ordering $(B, <)$ is said to be regular if for every $a \in B$ there exist distinct elements $b_1, b_2 \in B$ such that $b_1 < a$, $b_2 < a$, and no element $c \in B$ satisfies both $c < b_1$ and $c < b_2$. As a corollary he also proved that there is no maximal decidable countable structure if we replace FO by weak MSO logic.

- In [2], Bès and Cègielski consider a weakening of the Elgot-Rabin question, namely the question of whether all structures $M$ whose FO theory is decidable can be expanded by some constant in such a way that the resulting structure still has a decidable theory. They answer this question negatively by proving that there exists a structure $M$ with a decidable MSO theory and such that any expansion of $M$ by a constant has an undecidable FO theory.
The paper [1] gives a sufficient condition in terms of the Gaifman graph of $M$ which ensures that $M$ is not maximal. The condition is the following: for every natural number $r$ and every finite set $X$ of elements of the base set $|M|$ of $M$ there exists an element $x \in |M|$ such that the Gaifman distance between $x$ and every element of $X$ is greater than $r$.

We investigate the Elgot-Rabin problem for the class of labelled linear orderings, i.e., infinite structures $M = (A, <, P_1, \ldots, P_n)$ where $<$ is a linear ordering over $A$ and the $P_i$’s denote unary predicates. This class is interesting with respect to the above results, since on one hand no regular ordering seems to be FO interpretable in such structures, and on the other hand their associated Gaifman distance is trivial, thus they do not satisfy the criterion given in [1].

In this paper we focus on MSO logic rather than FO. The main result of the paper is that for every labelled linear ordering $M$ such that $(A, <)$ contains an interval of order type $\omega$ or $-\omega$ and the MSO theory of $M$ is decidable, then there exists an expansion $M'$ of $M$ by a monadic predicate which is not MSO-definable in $M$, and such that the MSO theory of $M'$ is still decidable. Hence, $M$ is not maximal. The result holds in particular when $(A, <)$ is order-isomorphic to the order of the naturals $\omega = (\mathbb{N}, <)$, or to the order $\zeta = (\mathbb{Z}, <)$ of the integers, or to any infinite ordinal, or more generally any infinite scattered ordering (recall that an ordering is scattered if it does not contain any dense sub-ordering).

The structure of the proof is the following: we first show that the result holds for $\omega$ and $\zeta$. For the general case, starting from $M$, we use some definable equivalence relation on $A$ to cut $A$ into intervals whose order type is either finite, or of the form $-\omega$, $\omega$, or $\zeta$. We then define the new predicate on each interval (using the constructions given for $\omega$ and $\zeta$), from which we get the definition of $M'$. The reduction from $\text{MSO}(M')$ to $\text{MSO}(M)$ uses Shelah’s composition theorem, which allows to reduce the MSO theory of an ordered sum of structures to the MSO theories of the summands.

The main reason to consider MSO logic rather than FO is that it actually simplifies the task. Nevertheless we discuss some partial results and perspectives for FO logic in the conclusion of the paper.

Let us recall some important decidability results for MSO theories of linear orderings (the case of labelled linear orderings will be discussed later for $\omega$ and $\zeta$). In his seminal paper [4], Büchi proved that languages of $\omega$-words recognizable by automata coincide with languages definable in the MSO theory of $\omega$, from which he deduced decidability of the theory. The result (and the automata method) was then extended to the MSO theory of any countable ordinal [5], to $\omega_1$, and to any ordinal less than $\omega_2$ [6]. Gurevich, Magidor and Shelah prove [13] that decidability of MSO theory of $\omega_2$ is independent of ZFC. Let us mention results for linear orderings beyond ordinals. Using automata, Rabin [19] proved decidability of the MSO theory of the binary tree, from which he deduces decidability of the MSO theory of $\mathbb{Q}$, which in turn implies decidability of the MSO theory of the class of countable linear orderings. Shelah [26] improved model-theoretical techniques that allow him to reprove almost all known decidability results about MSO theories, as well as new decidability results for the case of
linear orderings, and in particular dense orderings. He proved in particular that
the MSO theory of \( \mathbb{R} \) is undecidable. The frontier between decidable and unde-
cidable cases was specified in later papers by Gurevich and Shelah [11, 14, 15];
we refer the reader to the survey [12].

Our result is also clearly related to the problem of building larger and larger
classes of structures with a decidable MSO theory. For an overview of recent
results in this area see [3, 32].

2 Definitions, Notations and Useful Results

2.1 Labelled Linear Orderings

We first recall useful definitions and results about linear orderings. A good ref-
erence on the subject is Rosenstein's book [23].

A linear ordering \( J \) is a total ordering. We denote by \( \omega \) (respectively \( \zeta \)) the
order type of \( \mathbb{N} \) (respectively \( \mathbb{Z} \)). Given a linear ordering \( J \), we denote by \( -J \) the
backwards linear ordering obtained by reversing the ordering relation.

Given two elements \( j, k \) of a linear ordering \( J \), we denote by \([j; k]\) the interval
\([\min(j, k), \max(j, k)]\). An ordering is dense if it contains no pair of consecutive
elements. An ordering is scattered if it contains no dense sub-ordering.

In this paper we consider labelled linear orderings, i.e., linear orderings \((A, <)\)
equipped with a function \( f : A \to \Sigma \) where \( \Sigma \) is a finite nonempty set.

2.2 Logic

Let us briefly recall useful elements of monadic second-order logic, and settle
some notations. For more details about MSO logic see e.g. [12, 31]. Monadic
second-order logic is an extension of first-order logic that allows to quantify over
elements as well as subsets of the domain of the structure. Given a signature
\( L \), one can define the set of (MSO) formulas over \( L \) as well-formed formulas
that can use first-order variable symbols \( x, y, \ldots \) interpreted as elements of the
domain of the structure, monadic second-order variable symbols \( X, Y, \ldots \)
interpreted as subsets of the domain, symbols from \( L \), and a new binary predicate
\( x \in X \) interpreted as “\( x \) belongs to \( X \)”. A sentence is a formula without free
variable. As usual, we often confuse logical symbols with their interpretation.

Given a signature \( L \) and an \( L \)-structure \( M \) with domain \( D \), we say that a re-
lation \( R \subseteq D^m \times (2^D)^n \) is (MSO) definable in \( M \) if a nd only if there exists a
formula over \( L \), say \( \varphi(x_1, \ldots, x_m, X_1, \ldots, X_n) \), which is true in \( M \) if and only if
\((x_1, \ldots, x_m, X_1, \ldots, X_n)\) is interpreted by an \((m+n)\)-tuple of \( R \). Given a struc-
ture \( M \) we denote by \( MSO(M) \) (respectively \( FO(M) \)) the monadic second-order
(respectively first-order) theory of \( M \). We say that \( M \) is maximal if \( MSO(M) \)
is decidable and \( MSO(M') \) is undecidable for every expansion \( M' \) of \( M \) by a
predicate which is not definable in \( M \).

We can identify labelled linear orderings with structures of the form \( M =
(A, <, P_1, \ldots, P_n) \) where \( < \) is a binary relation interpreted as a linear ordering
over \( A \), and the \( P_i \)'s denote unary predicates. We use the notation \( \overline{P} \) as a shortcut for the \( n \)-tuple \((P_1, \ldots, P_n)\). The structure \( M \) can be seen as a word indexed by \( A \) and over the alphabet \( \Sigma_n = \{0, 1\}^n \); this word will be denoted by \( w(M) \). For every interval \( I \) of \( A \) we denote by \( M_I \) the sub-structure of \( M \) with domain \( I \).

2.3 Composition Theorems

In this paper we rely heavily on composition methods, which allow to compute the theory of a sum of structures from the ones of its summands. For an overview of the subject see [3, 12, 16, 30]. In this section we recall useful definitions and results. For the whole section we consider signatures of the form \( L = \{<, P_1, \ldots, P_n\} \) where the \( P_i \)'s denote unary predicate names, and deal only with \( L \)-structures where \(< \) is interpreted as a linear ordering – that is, with labelled linear orderings. Given a formula \( \varphi \) over \( L \), the quantifier depth of \( \varphi \) is denoted by \( qd(\varphi) \). The \( k \)-type of an \( L \)-structure \( M \), which is denoted by \( T^k(M) \), is the set of sentences \( \varphi \) such that \( M \models \varphi \) and \( qd(\varphi) \leq k \). Given two structures \( M \) and \( M' \), the relation \( T^k(M) = T^k(M') \) is an equivalence relation with finitely many classes. Let us list some fundamental and well-known properties of \( k \)-types. The proofs of these facts can be found in several sources, see e.g. [26, 31].

**Proposition 1.**

1. For every \( k \) there are only finitely \( k \)-types over a finite signature \( L \)
2. For each \( k \)-type \( t \) there is a sentence \( \varphi_t \) (called "characteristic sentence") which defines \( t \), i.e., such that \( M \models \varphi_t \) iff \( T^k(M) = t \). For every \( k \), a finite list of characteristic sentences for all the possible \( k \)-types can be computed. (We take the characteristic sentences as the canonical representations of \( k \)-types. Thus, for example, transforming a type into another type means to transform sentences.)
3. Each sentence \( \varphi \) is equivalent to a (finite) disjunction of characteristic sentences; moreover, this disjunction can be computed from \( \varphi \).

As a simple consequence, note that the MSO theory of a structure \( M \) is decidable if and only if the function \( k \mapsto T^k(M) \) is recursive.

The sum of structures corresponds to concatenation; let us recall a general definition.

**Definition 2.** Consider an index structure \( \text{Ind} = (I, <^I) \) where \( <^I \) is a linear ordering. Consider a signature \( L = \{<, P_1, \ldots, P_n\} \), where \( P_i \) are unary predicate names, and a family \( (M_i)_{i \in I} \) of \( L \)-structures \( M_i = (A_i; <^i, P_1^i, \ldots, P_n^i) \) with disjoint domains and such that the interpretation \( <^i \) of \(< \) in each \( M_i \) is a linear ordering. We define the ordered sum of the family \( (M_i)_{i \in I} \) as the \( L \)-structure \( M = (A; <^M, P_1^M, \ldots, P_n^M) \) where

- \( A \) equals the union of the \( A_i \)'s
- \( x <^M y \) holds if and only if \( (x \in A_i \text{ and } y \in A_j \text{ for some } i <^I j) \), or \( (x, y \in A_i \text{ and } x <^i y) \)
for every $x \in A$ and every $k \in \{1, \ldots, n\}$, $P^M_k(x)$ holds if and only if $M_j \models P^A_j(x)$ where $j$ is such that $x \in A_j$.

If the domains of the $M_i$ are not disjoint, replace them with isomorphic chains that have disjoint domains, and proceed as before.

We shall use the notation $M = \sum_{i \in I} M_i$ for the ordered sum of the family $(M_i)_{i \in I}$. If $I = \{1, 2\}$ has two elements, we denote $\sum_{i \in I} M_i$ by $M_1 + M_2$.

We need the following composition theorem on ordered sums:

**Theorem 3.**

(a) The $k$-types of labelled linear orderings $M_0, M_1$ determine the $k$-type of the ordered sum $M_0 + M_1$, which moreover can be computed from the $k$-types of $M_0$ and $M_1$.

(b) If the labelled linear orderings $M_0, M_1, \ldots$ all have the same $k$-type, then this $k$-type determines the $k$-type of $\sum_{i \in \mathbb{N}} M_i$, which moreover can be computed from the $k$-type of $M_0$.

Part (a) of the theorem justifies the notation $s + t$ for the $k$-type of a linear ordering which is the sum of two linear orderings of $k$-types $s$ and $t$, respectively. Similarly, we write $t \times \omega$ for the $k$-type of a sum $\sum_{i \in \mathbb{N}} M_i$ where all $M_i$ are of $k$-type $t$.

### 3 The Case of $\mathbb{N}$

In this section we prove that there is no maximal structure of the form $(\mathbb{N}, <, P)$ with respect to MSO logic. The proof is based upon results from [20]. Let us first briefly review results related to the decidability of the MSO theory of expansions of $(\mathbb{N}, <)$. Büchi [4] proved decidability of $\text{MSO}(\mathbb{N}, <)$ using automata. On the other hand it is known that $\text{MSO}(\mathbb{N}, +)$, and even $\text{MSO}(\mathbb{N}, <, x \mapsto 2x)$, are undecidable [22]. Elgot and Rabin study in [9] the MSO theory of structures of the form $(\mathbb{N}, <, P)$, where $P$ is some unary predicate. They give a sufficient condition on $P$ which ensures decidability of the MSO theory of $(\mathbb{N}, <, P)$. In particular the condition holds when $P$ denotes the set of factorials, or the set of powers of any fixed integer. The frontier between decidability and undecidability of related theories was explored in numerous later papers [7, 10, 25, 24, 21, 20, 27, 29]. Let us also mention that [25] proves the existence of unary predicates $P$ and $Q$ such that both $\text{MSO}(\mathbb{N}, <, P)$ and $\text{MSO}(\mathbb{N}, <, Q)$ are decidable while $\text{MSO}(\mathbb{N}, <, P, Q)$ is undecidable.

Most decidability proofs for $\text{MSO}(\mathbb{N}, <, P)$ are related somehow to the possibility of cutting $\mathbb{N}$ into segments whose $k$–type is ultimately constant, from which one can compute the $k$–type of the whole structure (using Theorem 3). This connection was specified in [20] (see also [21]) using the notion of homogeneous sets.

**Definition 4 (k-homogeneous set).** Let $k \geq 0$. A set $H = \{h_0 < h_1 < \ldots\} \subseteq \mathbb{N}$ is called $k$-homogeneous for $M = (\mathbb{N}, <, P)$, if all sub-structures $M_{[h_i, h_{i+1})}$ for $i < j$ (and hence all sub-structures $M_{(h_i, h_{i+1})}$ for $i \geq 0$) have the same $k$-type.
This notion can be refined as follows.

**Definition 5 (uniformly homogeneous set).** A set \( H = \{ h_0 < h_1 < \ldots \} \subseteq \mathbb{N} \) is called uniformly homogeneous for \( M = (\mathbb{N}, <, P) \) if for each \( k \) the set \( H_k = \{ h_k < h_{k+1} < \ldots \} \) is \( k \)-homogeneous.

The following result [20] settles a tight connection between MSO(\( \mathbb{N}, <, P \)) and uniformly homogeneous sets.

**Theorem 6.** For every \( M = (\mathbb{N}, <, P) \), the MSO theory of \( M \) is decidable if and only if (the sets \( P \) are recursive and there exists a recursive uniformly homogeneous set for \( M \)).

One can use this theorem to show that no structure \( M = (\mathbb{N}, <, P) \) is maximal.

Let us give the main ideas. Starting from \( M \) such that MSO(\( M \)) is decidable, Theorem 6 implies the existence of a recursive uniformly homogeneous set \( H = \{ h_0 < h_1 < \ldots \} \) for \( M \).

Let \( M' \) be an expansion of \( M \) by a monadic predicate \( P_{n+1} \) defined as \( P_{n+1} = \{ h_n! \mid n \in \mathbb{N} \} \).

By definition of \( H \), the structures \( M_{[h_k, h_{k+1})} \) have the same \( k \)-type for all \( j, k \geq 0 \). If we combine this with the fact that successive elements of \( P_{n+1} \) are far away from each other, we can prove that \( P_{n+1} \) is not definable in \( M \). For all \( i, k \geq 0 \) let us define the interval \( I(i, k) = [h_{(k+i)!}, h_{(k+i+1)!}] \). In order to prove that MSO(\( M' \)) is decidable, we exploit the fact that all structures \( M_{I(i, k)} \) have the same \( k \)-type for all \( i, k \geq 0 \), and that only the first element of each interval \( I(i, k) \) belongs to \( P_{n+1} \). This allows to compute easily the \( k \)-type of structures \( M'_{I(i, k)} \) from the one of \( M_{I(i, k)} \), and then the \( k \)-type of the whole structure \( M' \).

This provides a reduction from MSO(\( M' \)) to MSO(\( M \)).

The above construction, which we described for a fixed structure \( M \), can actually be defined uniformly in \( M \). This leads to the following result.

**Proposition 7.** There exists a function \( E \) and two recursive function \( g_1, g_2 \) such that \( E \) maps every structure \( M = (\mathbb{N}, <, P) \) to an expansion \( M' \) of \( M \) by a predicate \( P_{n+1} \) such that

1. \( P_{n+1} \) is not definable in \( M \);
2. \( g_1 \) computes \( T^k(M') \) from \( k \) and \( T^{g_2(k)}(M) \).

Hence MSO(\( M' \)) is recursive in MSO(\( M \)). In particular, if MSO(\( M \)) is decidable, then MSO(\( M' \)) is decidable.

Let us discuss item (2). In the proof of the general result (see Sect. 5), we start from a labelled linear ordering \( M = (A, <, P) \) with a decidable MSO theory and try to expand it while keeping decidability. In some case the (decidable) expansion \( M' \) of \( M \) will be defined by applying the above construction to infinitely many intervals of \( A \) of order type \( \omega \). In order to get a reduction from MSO(\( M' \)) to MSO(\( M \)), we need that the reduction algorithm for such intervals is uniform, which is what item (2) expresses.
4 The Case of \( \mathbb{Z} \)

Decidability of the MSO theory of structures \( M = (\mathbb{Z}, <, \mathcal{P}) \) was studied in particular by Compton [8], Semënov [25, 24], and Perrin and Schupp [18] (see also [17, chapter 9]). These works put in evidence the link between decidability of \( \text{MSO}(M) \) and computability of occurrences and repetitions of finite factors in the word \( w(M) \). Let us state some notations and definitions. A set \( X \) of finite words over a finite alphabet \( \Sigma \) is said to be regular if it is recognizable by some finite automaton. Given a \( \mathbb{Z} \)-word \( w \) and a finite word \( u \), both over the alphabet \( \Sigma \), we say that \( u \) occurs in \( w \) if \( w = uw_1w_2 \) for some words \( w_1 \) and \( w_2 \). We say that \( w \) is recurrent if for every regular language \( X \) of finite words over \( \Sigma \), either no element of \( X \) occurs in \( w \), or in every prefix and every suffix of \( w \) there is an occurrence of some element of \( X \). In particular, in a recurrent word \( w \), every finite word \( u \) either has no occurrence in \( w \), or occurs infinitely often on both sides of \( w \). We say that \( w \) is rich if every finite word occurs infinitely often on both sides of \( w \). Given \( M = (\mathbb{Z}, <, \mathcal{P}) \), we say that \( M \) is recurrent if \( w(M) \) is.

We have the following result.

**Theorem 8.** ([25, 18]) Given \( M = (\mathbb{Z}, <, P_1, \ldots, P_n) \),

1. If \( M \) is not recurrent, then every \( c \in \mathbb{Z} \) is definable in \( M \).
2. If \( M \) is recurrent, then no element is definable in \( M \), and \( \text{MSO}(M) \) is computable relative to an oracle which, given any regular language \( X \) of finite words over \( \Sigma \), tells whether some element of \( X \) occurs in \( w(M) \).

Let \( c \in \mathbb{Z} \), and let \( M_1 \) be defined as \( M = M_{[1, \infty[} \) and \( M_2 \) be defined as \( M_{[c, \infty[} \). Then \( M = M_1 + M_2 \).

Let \( M'_1 \) be the expansion of \( M_1 \) by the empty predicate \( P_{n+1} \) and let \( M'_2 \) be obtained by applying the construction of Proposition 7 to \( M_2 \). Let \( M' = M'_1 + M'_2 \).

Note that the above construction of \( M' \) from \( M \) depends on \( c \). We denote by \( E_c \) the function described above that maps every \( M = (\mathbb{Z}, <, P_1, \ldots, P_n) \) to its expansion \( M' \) by \( P_{n+1} \).

It is easy to show that \( P_{n+1} \) is not definable in \( M \), hence \( M' \) is a non-trivial expansion of \( M \).

We claim that if \( M \) is not recurrent, then \( \text{MSO}(M') \) is recursive in \( \text{MSO}(M) \). Indeed, in this case, by Theorem 8, \( c \) is definable in \( M \). Hence, \( M_1 \) and \( M_2 \) can be interpreted in \( M \), which yields that \( \text{MSO}(M_{1}) \) and \( \text{MSO}(M_{2}) \) are recursive in \( \text{MSO}(M) \). Therefore, \( \text{MSO}(M'_1) \) and \( \text{MSO}(M'_2) \) are recursive in \( \text{MSO}(M) \). Finally, applying Theorem 3(a) we obtain that \( \text{MSO}(M') \) is recursive in \( \text{MSO}(M) \).

Hence, we have the following.

**Proposition 9 (Expansion of non-recurrent structures).** There are two recursive function \( g_1, g_2 \) such that if \( M = (\mathbb{Z}, <, P_1, \ldots, P_n) \) is not recurrent, and \( c \in \mathbb{Z} \) is definable in \( M \) by a formula of quantifier depth \( m \), then \( E_c \) maps \( M \) to an expansion \( M' \) by a predicate \( P_{n+1} \) such that

1. \( P_{n+1} \) is not definable in \( M \);
2. $g_1$ computes $T^k(M')$ from $k$ and $T^{2k'(k+m)}(M)$.

Hence $MSO(M')$ is recursive in $MSO(M)$. In particular, if $MSO(M)$ is decidable, then $MSO(M')$ is decidable.

Remark 10. Let us discuss uniformity issues related to Proposition 7 and Proposition 9. Proposition 7 implies that there is an algorithm which reduces $MSO(M')$ to $MSO(M)$. This reduction algorithm is independent of $M$; it only uses an oracle for $MSO(M)$. Proposition 9 implies a weaker property. Namely, it implies that for every non-recurrent $M$ there is an algorithm which reduces $MSO(M')$ to $MSO(M)$. However, this reduction algorithm depends on $M$.

Consider a recurrent structure $M$ and let $M' = E_c(M)$ for some $c \in \mathbb{Z}$. We claim that it is possible that $MSO(M')$ is not recursive in $MSO(M)$. Indeed, we can prove that there exists a recurrent structure $M$ over $\mathbb{Z}$ such that $MSO(M)$ is decidable, and $MSO(M_{[c',\infty]})$ is undecidable for every $c' \in \mathbb{Z}$. Now let $c'$ be the minimal element of $P_{n+1}$. Observe that $c'$ is definable in $M'$ and therefore, $M_{[c',\infty]}$ can be interpreted in $M'$. Since, $MSO(M_{[c',\infty]})$ is undecidable, we derive that $MSO(M')$ is undecidable. Hence, $E_c$ does not preserves decidability of recurrent structures, and we need a different construction for the recurrent case.

To describe our construction for the recurrent case let us introduce first some notations.

For every word $w$ over the alphabet $\Sigma_{n+1} = \{0,1\}^{n+1}$ which is indexed by some linear ordering $(A,<)$ we denote by $\pi(w)$ the word $w'$ indexed by $A$ and over the alphabet $\Sigma_n = \{0,1\}^n$, which is obtained from $w$ by projection over the $n$ first components of each symbol in $w$. The definition and notation extend to $\pi(X)$ where $X$ is any set of words over the alphabet $\Sigma_{n+1}$. Given $M = (\mathbb{Z},<,\overline{P})$ where $\overline{P}$ is an $n$–tuple of sets, and any expansion $M'$ of $M$ by a predicate $P_{n+1}$, by definition $w(M)$ and $w(M')$ are words over $\Sigma_n$ and $\Sigma_{n+1}$, respectively, and we have $\pi(w(M')) = w(M)$.

**Lemma 11.** If $M = (\mathbb{Z},<,\overline{P})$ is recurrent, then there is an expansion $M'$ of $M$ by a predicate $P_{n+1}$ which has the following property:

\begin{equation}
(*) \text{ for every } u \in \Sigma_n, \text{ if } u \text{ occurs infinitely often on both sides of } w(M), \text{ then the same holds in } w(M') \text{ for every word } u' \in \Sigma_{n+1} \text{ such that } \pi(u') = u.
\end{equation}

The proof of Lemma 11 is similar to the proof of Proposition 2.8 in [1], which roughly shows how to deal with the case when $w(M)$ is rich.

Now $w(M')$ has a finite factor in some regular language $X' \subseteq \Sigma_{n+1}$ iff $w(M)$ has a finite factor in $\pi(X') \subseteq \Sigma_n$. The set $\pi(X')$ is regular, and a sentence which defines $\pi(X')$ is computable from a sentence that defines $X'$, thus we obtain, by Theorem 8(2), that if $MSO(M)$ is decidable then $MSO(M')$ is decidable.

One can show that if $M'$ is any expansion of $M$ which has property $(*)$, then $P_{n+1}$ is not definable in $M$. This implies that no recurrent structure is maximal.

From a more detailed analysis of the proof of Theorem 8(2) we can derive the following proposition.
Proposition 12 (Expansion of recurrent structures). There are two recursive functions $g_1, g_2$ such that if $M = (\mathbb{Z}, <, P)$ is recurrent and $M'$ is an expansion of $M$ which has property (*), then

1. $P_{n+1}$ is not definable in $M$;
2. $g_1$ computes $T^k(M')$ from $k$ and $T^{g_2}(k)(M)$.

Hence $\text{MSO}(M')$ is recursive in $\text{MSO}(M)$. In particular, if $\text{MSO}(M)$ is decidable, then $\text{MSO}(M')$ is decidable.

Remark 13. Proposition 12 implies that there is an algorithm which reduces $\text{MSO}(M')$ to $\text{MSO}(M)$. This reduction algorithm (like the algorithm from Proposition 7) is independent of $M$; it only uses an oracle for $\text{MSO}(M)$.

Proposition 9, Lemma 11 and Proposition 12 imply the following corollary.

Corollary 14. Let $M = (\mathbb{Z}, <, P)$. There exists an expansion $M'$ of $M$ by some unary predicate $P_{n+1}$ such that $P_{n+1}$ is not definable in $M$, and $\text{MSO}(M')$ is recursive in $\text{MSO}(M)$. In particular, if $\text{MSO}(M)$ is decidable, then $\text{MSO}(M')$ is decidable.

5 Main Result

The next theorem is our main result.

Theorem 15. Let $M = (A, <, P_1, \ldots, P_n)$ where $(A, <)$ contains an interval of type $\omega$ or $-\omega$. There exists an expansion $M'$ of $M$ by a relation $P_{n+1}$ such that $P_{n+1}$ is not definable in $M$, and $\text{MSO}(M')$ is recursive in $\text{MSO}(M)$. In particular, if $\text{MSO}(M)$ is decidable, then $\text{MSO}(M')$ is decidable.

As an immediate consequence we obtain the following corollary.

Corollary 16. Let $M = (A, <, P)$. There exists an expansion $M'$ of $M$ by some unary predicate $P_{n+1}$ not definable in $M$ such that $\text{MSO}(M')$ is recursive in $\text{MSO}(M)$. We present a sketch of proof for Theorem 15. Let $M = (A, <, P)$ where $(A, <)$ contains an interval of type $\omega$ or $-\omega$.

Consider the binary relation defined on $A$ by $x \approx y$ iff $[x, y]$ is finite. The relation $\approx$ is a condensation, i.e., an equivalence relation such that every equivalence class is an interval of $A$. Moreover the relation $\approx$ is definable in $M$. If $A_1$ and $A_2$ are $\approx$-equivalence classes, we say that $A_1$ precedes $A_2$ if all elements of $A_1$ are less than all elements of $A_2$. Let $I$ be the linear order of the $\approx$-equivalence classes for $(A, <)$. Then $M = \sum_{i \in I} M_{A_i}$ where the $A_i$'s correspond to equivalence classes of $\approx$. Using the definition of $\approx$ and our assumption on $A$, it is easy to check that the $A_i$'s are either finite, or of order type $\omega$, or $-\omega$, or $\zeta$, and that not all $A_i$’s are finite.

We define the interpretation of the new predicate $P_{n+1}$ in every interval $A_i$. The definition proceeds as follows:
1. if some $A_i$ has order type $\omega$ or $-\omega$, then we apply to each substructure $M_{A_i}$ of order type $\omega$ the construction given in Proposition 7, and add no element of $P_{n+1}$ elsewhere. If there is no $A_i$ of order type $\omega$, we proceed in a similar way with each substructure $M_{A_i}$ of order type $-\omega$, but using the dual of Proposition 7 for $-\omega$.

2. if no $A_i$ has order type $\omega$ or $-\omega$, then at least one $\approx-\approx$ equivalence class $A_i$ has order type $\zeta$. We consider two subcases:

(a) if all $\approx-\approx$ equivalence classes $A_i$ with order type $\zeta$ are such that $w(M_{A_i})$ is recurrent, then we apply to each substructure $M_{A_i}$ of order type $\zeta$ the construction given in Proposition 12. For other $\approx-\approx$ equivalence classes $A_i$ we set $P_{n+1} \cap A_i = \emptyset$.

(b) otherwise there exist $\approx-\approx$ equivalence classes $A_i$ with order type $\zeta$ and such that $w(M_{A_i})$ is not recurrent. Let $\varphi(x)$ be a formula with minimal quantifier depth such that $\varphi(x)$ defines an element in some $M_{A_i}$ where $A_i$ has order type $\zeta$. For every $M_{A_i}$ such that $A_i$ has order type $\zeta$ and $\varphi(x)$ defines an element $c_i$ in $M_{A_i}$, we apply the construction $E_{c_i}$ from Proposition 9 to $M_{A_i}$. For other $\approx-\approx$ equivalence classes $A_i$ we set $P_{n+1} \cap A_i = \emptyset$.

The fact that the set $P_{n+1}$ is not definable in $M$ follows rather easily from the construction, which ensures that there exists some $A_i$ such that the restriction of $P_{n+1}$ to $A_i$ is not definable in the substructure $M_{A_i}$.

Let $M'$ be the expansion of $M$ by the predicate $P_{n+1}$. In order to prove that $MSO(M')$ is recursive in $MSO(M)$, we use Shelah’s composition method [26, Theorem 2.4] (see also [12, 30]) which allows to reduce the MSO theory of a sum of structures to the MSO theories of the components and the MSO theory of the index structure.

Theorem 17 (Composition Theorem [26]). There exists a recursive function $f$ and an algorithm which, given $k, l \in \mathbb{N}$, computes the $k$-type of any sum $M = \sum_{i \in I} M_i$ of labelled linear orderings over a signature $\{<, P_1, \ldots, P_l\}$ from the $f(k, l)$-type of the structure $(I, <, Q_1, \ldots, Q_p)$ where

$$Q_j = \{i \in I : T^k(M_i) = \tau_j\} \quad j = 1, \ldots, p$$

and $\tau_1, \ldots, \tau_p$ is the list of all formally possible $k$-types for the signature $L$.

Let us explain the reduction from $MSO(M')$ to $MSO(M)$. We can apply Theorem 17 to $M' = \sum_{i \in I} M'_{A_i}$, which allows to show that for every $k$, the $k$-type of $M'$ can be computed from $f(k, n+1)$-type of the structure $N' = (I, <, Q'_1, \ldots, Q'_p)$ where the $Q'_i$’s correspond to the $k$-types of structures $M'_{A_i}$ over the signature $\{<, P_1, \ldots, P_{n+1}\}$. Using the definition of $P_{n+1}$ and Propositions 7, 9 and 12, one can prove that the $k$-type of $M'_{A_i}$ can be computed form the $g(k)$-type of $M_{A_i}$ for some recursive function $g$ (note that $g$ depends on $M$, namely whether we used case 1, 2(a) or 2(b) to construct $M'$). This allows to prove that $N'$ is interpretable in the structure $N = (I, <, Q_1, \ldots, Q_q)$ where the $Q_i$’s correspond to the $g(k)$-types of structures $M_{A_i}$ over the signature
\{<, P_1, \ldots, P_n\}. It follows that \(\text{MSO}(N')\) is recursive in \(\text{MSO}(N)\). Now using the fact that the equivalence relation \(\approx\) is definable in \(M\), we can prove that \(N\) is interpretable in \(M\), thus \(\text{MSO}(N)\) is recursive in \(\text{MSO}(M)\).

**Remark 18.** Let us discuss uniformity issues related to Theorem 15.

- The choice to expand “uniformly” all \(\approx\)-equivalence classes is crucial for the reduction from \(\text{MSO}(M')\) to \(\text{MSO}(M)\). For example, if some \(A_i\) has order type \(\omega\) and we choose to expand only \(A_i\) then \(\text{MSO}(M')\) might become undecidable. This is the case for the structure \(M\) considered in [2] (Definition 2.4), which has decidable \(\text{MSO}\) theory, and is such that the \(\text{MSO}\) theory of any expansion of \(M\) by a constant is undecidable. For this structure all \(A_i\)'s have order type \(\omega\). If we consider the structure \(M'\) obtained from \(M\) by an expansion of only one \(A_i\), then \(P_{n+1}\) has a least element, which is definable in \(M'\), thus \(\text{MSO}(M')\) is undecidable.
- The definition of \(P_{n+1}\) in case (2) depends on whether all components \(A_i\) with order type \(\zeta\) are such that \(w(M_{A_i})\) is recurrent, which is not a \(\text{MSO}\) definable property. Thus that the reduction algorithm from \(\text{MSO}(M')\) to \(\text{MSO}(M)\) depends on \(M\).

### 6 Further Results and Open Questions

Let us mention some possible extensions and related open questions.

First of all, most of our results can be easily extended to the case when the signature contains infinitely many unary predicates.

Our results can be extended to the Weak MSO logic. In the case \(M\) is countable this follows from Soprunov result [28]. However, our construction works for labelled orderings of arbitrary cardinality.

An interesting issue is to prove uniform versions of our results in the sense of items (2) in Propositions 7 and 12. A first step would be to generalize Proposition 12 to all structures \((\mathbb{Z}, <, P)\).

One can also ask whether the results of the present paper hold for FO logic. Let us emphasize some difficulties which arise when one tries to adapt the main arguments. A FO version of Theorem 6 (about the recursive homogeneous set) was already proven in [21]. Moreover, using ideas from [25] one can also give a characterization of structures \(M = (\mathbb{Z}, <, P)\) with a decidable FO theory, in terms of occurrences and repetitions of finite words in \(w(M)\). This allows to give a FO version of our non-maximality results for labelled orders over \(\omega\) or \(\zeta\). However for the general case of \((A, <, P)\), two problems arise: (1) the constructions for \(\mathbb{N}\) and \(\mathbb{Z}\) cannot be applied directly since they are not uniform, and (2) the equivalence relation \(\approx\) used in the proof of Theorem 15 to cut \(A\) into small intervals is not FO definable. We currently investigate these issues.

Finally, we also study the case of labelled linear orderings \((A, <, P)\) which do not contain intervals of order types \(\omega\) or \(-\omega\). In this case the construction presented in Sect. 5 does not work since the restriction of \(P_{n+1}\) to each \(A_i\) will be empty, i.e., our new relation is actually empty. In a forthcoming paper we show
that it is possible to overcome this issue for the countable orders, and prove that no infinite countable structure \((A, <, \mathcal{P})\) is maximal.

Acknowledgment This research was facilitated by the ESF project AutoMathA. The second author was partially supported by ESF project Games and EPSRC grant.

References

1. A. Bès and P. Cégielski. Weakly maximal decidable structures. *RAIRO-Theor. Inf. Appl.*, 42(1):137–145, 2008.
2. A. Bès and P. Cégielski. Nonmaximal decidable structures. *Journal of Mathematical Sciences*, 158:615–622, 2009.
3. A. Blumensath, T. Colcombet, and C. Löding. Logical theories and compatible operations. In J. Flum, E. Grädel, and T. Wilke, editors, *Logic and automata: History and Perspectives*, pages 72–106. Amsterdam University Press, 2007.
4. J. R. Büchi. On a decision method in the restricted second-order arithmetic. In *Proc. Int. Congress Logic, Methodology and Philosophy of science, Berkeley 1960*, pages 1–11. Stanford University Press, 1962.
5. J. R. Büchi. Transfinite automata recursions and weak second order theory of ordinals. In *Proc. Int. Congress Logic, Methodology, and Philosophy of Science, Jerusalem 1964*, pages 2–23. HOLLAND, 1965.
6. J. R. Büchi and C. Zaimontz. Deterministic automata and the monadic theory of ordinals \(\omega_2\). *Z. Math. Logik Grundlagen Math.*, 29:313–336, 1983.
7. O. Carton and W. Thomas. The monadic theory of morphic infinite words and generalizations. *Inform. Comput.*, 176:51–76, 2002.
8. K. J. Compton. On rich words. In M. Lothaire, editor, *Combinatorics on words. Progress and perspectives, Proc. Int. Meet., Waterloo/Can. 1982*, volume 17 of *Encyclopedia of Mathematics*, pages 39–61. ADDISON, 1983.
9. C. C. Elgot and M. O. Rabin. Decidability and undecidability of extensions of second (first) order theory of (generalized) successor. *J. Symb. Log.*, 31(2):169–181, 1966.
10. S. Fratani. The theory of successor extended with several predicates. preprint, 2009.
11. Y. Gurevich. Modest theory of short chains. i. *J. Symb. Log.*, 44(4):481–490, 1979.
12. Y. Gurevich. Monadic second-order theories. In J. Barwise and S. Feferman, editors, *Model-Theoretic Logics*, pages 479–506. Springer-Verlag, Perspectives in Mathematical Logic, 1985.
13. Y. Gurevich, M. Magidor, and S. Shelah. The monadic theory of \(\omega_2\). *J. Symb. Log.*, 48(2):387–398, 1983.
14. Y. Gurevich and S. Shelah. Modest theory of short chains. ii. *J. Symb. Log.*, 44(4):491–502, 1979.
15. Y. Gurevich and S. Shelah. Interpreting second-order logic in the monadic theory of order. *J. Symb. Log.*, 48(3):816–828, 1983.
16. J. A. Makowsky. Algorithmic uses of the Feferman-Vaught theorem. *Annals of Pure and Applied Logic*, 120(1–3):159–213, 2004.
17. D. Perrin and J.-E. Pin. *Infinite Words*, volume 141 of *Pure and Applied Mathematics*. Elsevier, 2004. ISBN 0-12-532111-2.
18. D. Perrin and P. E. Schupp. Automata on the integers, recurrence distinguishability, and the equivalence and decidability of monadic theories. In Symposium on Logic in Computer Science (LICS ’86), pages 301–305, Washington, D.C., USA, June 1986. IEEE Computer Society Press.

19. M.O. Rabin. Decidability of second-order theories and automata on infinite trees. Transactions of the American Mathematical Society, 141:1–35, 1969.

20. A. Rabinovich. On decidability of monadic logic of order over the naturals extended by monadic predicates. Inf. Comput, 205(6):870–889, 2007.

21. A. Rabinovich and W. Thomas. Decidable theories of the ordering of natural numbers with unary predicates. In Zoltán Ésik, editor, Computer Science Logic, 20th International Workshop, CSL 2006, 15th Annual Conference of the EACSL, Szeged, Hungary, September 25-29, 2006, Proceedings, volume 4207 of Lecture Notes in Computer Science, pages 562–574. Springer, 2006.

22. R.M. Robinson. Restricted set-theoretical definitions in arithmetic. Proc. Am. Math. Soc., 9:238–242, 1958.

23. J. G. Rosenstein. Linear ordering. Academic Press, New York, 1982.

24. A. L. Semenov. Decidability of monadic theories. In M. P. Chytil and V. Koubek, editors, Proceedings of the 11th Symposium on Mathematical Foundations of Computer Science, volume 176 of LNCS, pages 162–175, Praha, Czechoslovakia, September 1984. Springer.

25. A. L. Semenov. Logical theories of one-place functions on the set of natural numbers. Mathematics of the USSR - Izvestia, 22:587–618, 1984.

26. S. Shelah. The monadic theory of order. Annals of Mathematics, 102:379–419, 1975.

27. D. Siefkes. Decidable extensions of monadic second order successor arithmetic. Automatentheorie und Formale Sprachen, (Tagung, Math. Forschungsinst, Oberwolfach), 1969; (Bibliograph. Inst., Mannheim), pages 441–472, 1970.

28. S. Soprunov. Decidable expansions of structures. Vopr. Kibern., 134:175–179, 1988. (in Russian).

29. W. Thomas. A note on undecidable extensions of monadic second order successor arithmetic. Arch. Math. Logik Grundlagenforsch., 17:43–44, 1975.

30. W. Thomas. Ehrenfeucht games, the composition method, and the monadic theory of ordinal words. In Structures in Logic and Computer Science, A Selection of Essays in Honor of A. Ehrenfeucht, number 1261 in LNCS, pages 118–143. Springer, 1997.

31. W. Thomas. Languages, automata, and logic. In G. Rozenberg and A. Salomaa, editors, Handbook of Formal Languages, volume III, pages 389–455. Springer-Verlag, 1997.

32. W. Thomas. Model transformations in decidability proofs for monadic theories. In Michael Kaminski and Simone Martini, editors, CSL, volume 5213 of Lecture Notes in Computer Science, pages 23–31. Springer, 2008.