A SPECTRAL RECIPROCITY FORMULA AND NON-VANISHING FOR
L-FUNCTIONS ON GL(4) × GL(2)

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Abstract. We develop a reciprocity formula for a spectral sum over central values of L-functions on GL(4) × GL(2). As an application we show that for any self-dual cusp form Π for SL(4, Z), there exists a Maaß form π for SL(2, Z) such that L(1/2, Π × π) ≠ 0. An important ingredient is a "balanced" Voronoi summation formula involving Kloosterman sums on both sides, which can also be thought of as the functional equation of a certain double Dirichlet series involving Kloosterman sums and GL(4) Hecke eigenvalues.

1. Introduction

1.1. Non-vanishing of L-functions. Let 1 ≤ m < n be two positive integers. Given a cuspidal automorphic representation Π on GL(n, Q) \ GL(n, A), does there exist a self-dual cuspidal automorphic representation π on GL(m, Q) \ GL(m, A) such that L(1/2, Π × π) ≠ 0? The case m = 1 (i.e., π corresponds to a quadratic Dirichlet character) is one of the prime applications of the theory of multiple Dirichlet series, through which it is possible to answer this question affirmatively for n ∈ {2, 3} (see [Bu2, HK]). In a different direction, approaches through period integrals can sometimes show nonvanishing when m = n − 1 (see, for example, [GJR, GH]).

The case m = 2 is also analytically rather convenient, since GL(2) representations are always self-dual up to a character twist. Hence an averaging argument is more likely to be successful, and one could try to prove an asymptotic formula (or a lower bound) for

\[ \sum_{\pi} e^{-\left(\frac{t_{\pi}}{T}\right)^2} \frac{L(1/2, \Pi \times \pi)}{L(1, \text{Sym}^2 \pi)}, \]

where Π corresponds to a fixed cusp form for GL(n, Z) and π varies over all Maaß forms for SL(2, Z), t_{\pi} being the spectral parameter of π. For n = 3 the second named author [Li1, Theorem 1.1] established an asymptotic formula with power-saving error term for the more complicated sum

\[ \sum_{\pi \text{ even}} e^{-\left(\frac{t_{\pi}}{T}\right)^2} \frac{L(1/2, \Pi \times \pi)L(1/2, \pi)}{L(1, \text{Sym}^2 \pi)}, \]

over the even Maaß forms. In particular, this implies a non-vanishing result for the central values L(1/2, Π × π) for (n, m) = (3, 2).

For n ≥ 4 and any value of m the problem becomes very hard. In special cases, e.g., an isobaric sum Π = Π' ⊕ 1 with Π' on GL(3), Li’s asymptotic formula for (1.2) produces infinitely many non-vanishing L-values, and for Π = π × π' with π, π' on GL(2) it should be possible to use the techniques of Bernstein and Reznikov [BR1, BR2] to prove the same result. However, for general Π on GL(4), the problem is completely open. This is not particularly surprising in view of lack of progress on related problems (such as subconvexity and counts for zeros on the critical line), which remain open for cuspidal automorphic L-functions on GL(n), n ≥ 4. Indeed, even already on GL(3)

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the state-of-the-art analytic machinery (e.g., spectral summation formulae, multiple Dirichlet series, and period formulae) turns out to be unsuccessful for many problems which have long been settled for GL(2).

In this article we solve the non-vanishing problem in the case \((n, m) = (4, 2)\) when \(\Pi\) is self-dual and unramified at all finite places (i.e., \(\Pi\) is associated to a cusp form on \(GL(4, \mathbb{Z}) \setminus GL(4, \mathbb{R})\)). The most direct approach — asymptotically evaluating the corresponding quantity (1.1) — fails, at least with currently available tools. It quickly leads to a “deadlock”, as we shall describe at the end of the introduction in more detail. Therefore we introduce a different path which we now proceed to describe.

Fix a cuspidal automorphic representation \(\Pi\) on \(GL(4, \mathbb{Q}) \setminus GL(4, \mathbb{A})\), which we assume to be unramified at all finite places. Given a test function \(h\) satisfying \(h(t) \ll (1 + |t|)^{-A}\) for some \(A > 4\), we define the following spectral mean value

\[
M^\pm(h) := \sum_{\pi} e^{(1+1)/2} \frac{L(1/2, \Pi \times \pi)}{L(1, \text{Sym}^2 \pi)} h(t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{L(1/2 + it, \Pi) L(1/2 - it, \Pi)}{|\zeta(1+2it)|^2} h(t) dt,
\]

where \(\pi\) runs over all cuspidal automorphic representations on \(GL(2, \mathbb{Q}) \setminus GL(2, \mathbb{A})\) that are spherical at all finite places, and the parity \(\epsilon_\pi \in \{\pm 1\}\) is defined as the root number of the corresponding \(L\)-function \(L(s, \pi)\). In other words, the spectral average \(M^-(h)\) is twisted by the root number \(\epsilon_\pi\), while \(M^+(h)\) is not. The expression (1.3) is absolutely convergent by the convexity bounds (see (2.10) below).

Our nonvanishing result will be shown as a consequence of Theorem 3, which is a reciprocity formula that essentially relates \(M^-(h)\) to \(M^-(h^-)\) (and two similar terms) for an explicitly given integral transform \(h^-\) of \(h\). The precise shape of the transform is rather complicated; see Section 5 for its exact statement (the integral kernel is essentially a \(4F_3\) hypergeometric function). Before giving the exact formula, we shall first state a quantitative version that comes as a corollary of it.

Let \(D \geq 50\) be a fixed integer and define

\[
h_T(t) := e^{-(t/T)^2} P_T(t), \quad \text{where} \quad P_T(t) := \left( \prod_{n=1}^{D} t^2 + \left( \frac{2n-1}{2} \right)^2 \right)^2 T^{-4D}
\]

for a parameter \(T > 1\) (that will be taken large).

**Theorem 1.** Let \(\Pi\) be a cuspidal automorphic representation on \(GL(4, \mathbb{Q}) \setminus GL(4, \mathbb{A})\) which is unramified at all finite places, and let \(h_T\) be as in (1.4) for some fixed integer \(D \geq 50\). Then

\[
M^-(h_T) \ll T \quad \text{for} \quad T > 1,
\]

where the implied constant depends only on \(\Pi\) and \(D\).

Henceforth we shall fix the cusp form \(\Pi\) and not display the implicit dependence of any constants on it. If the generalized Lindelöf hypothesis is true, then by Weyl’s law the \(\pi\)-sum in (1.3) has roughly \(T^2\) terms that are essentially bounded, so Theorem 1 achieves square root cancellation and its linear exponent of \(T\) is expected to be best possible. Notice that (1.5) is a pure bound that contains no \(\varepsilon\) in the exponent, which is absolutely crucial for our application in Theorem 2.

Of course, due to varying signs Theorem 1 cannot be used to bound the individual terms in the sum (1.3). Nevertheless, if \(\Pi\) is self-dual we can still solve the non-vanishing problem mentioned at the beginning of the paper for \((n, m) = (4, 2)\). In this case we have

\[
L\left(\frac{1}{2} + it, \Pi\right) L\left(\frac{1}{2} - it, \Pi\right) = L\left(\frac{1}{2} + it, \Pi\right) L\left(\frac{1}{2} - it, \Pi\right) = |L\left(\frac{1}{2} + it, \Pi\right)|^2,
\]

where \(\Pi\) denotes the contragredient of \(\Pi\). It follows from the lower bound technique of Rudnick and Soundararajan [RSo] that the Eisenstein term in (1.3) is

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|L(1/2 + it, \Pi)|^2}{|\zeta(1 + 2it)|^2} h_T(t) dt \gg T \log T
\]
we will use a more structural approach to studying the
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(1.9)

obtain the
Voronoi summation formula on GL(4). The dual

(1.8)

T

now describe. In analogy to (1.2.

spectral mean value (1.7)

Voronoi formula that includes Kloosterman sums on both sides. One obtains from this an expression

\[ -c \]

\[ \lambda \]

and since 2

\[ \lambda \]

\[ \sum_{\lambda_{\Pi}} \sum_{\lambda_{\pi}} \frac{\lambda_{\Pi}(n)\lambda_{\pi}(n)}{n^{1/2}} = \sum_{\lambda_{\Pi}} \sum_{\lambda_{\pi}} \frac{\lambda_{\Pi}(n)\lambda_{\pi}(-n)}{n^{1/2}}, \]

where \( \lambda_{\Pi} \) and \( \lambda_{\pi} \) denote the Hecke eigenvalues of \( \Pi \) and \( \pi \) respectively. (Strictly speaking, this sum along with the others in this paragraph should be taken with smooth cutoffs.) Using the “opposite-sign”
Kuznetsov formula gives an expression involving a sum over Kloosterman sums, which is very
roughly of the shape

\[ T \sum_{n \in T^4} \frac{\lambda_{\Pi}(n)}{n^{1/2}} \sum_{c < T} \frac{S(-n, 1, c)}{c}. \]

There are no additional oscillatory terms in this sum because the integral kernel (2.15) features the
K-Bessel function (cf. [BK, Lemma 3.8], for example). This is followed by an application of the
Voronoi summation formula on GL(4). The dual \( n \)-sum will then be essentially of bounded length,
and since 2

\[ T \]

\[ \lambda \]

\[ \sum_{\lambda_{\Pi}} \sum_{\lambda_{\pi}} 1 \frac{c \lambda_{\Pi}(n)}{S(-n, 1, c)}, \]

after which the Kuznetsov formula is applied again – but in reverse. This sequence is not involutory
and yields full square-root cancellation in the \( c \)-sum; it therefore bounds (1.8) by \( O(T) \), at least if there are no exceptional eigenvalues (which is known for the full-level modular group \( SL(2, \mathbb{Z}) \)).

This heuristic reasoning described here is of course insensitive to \( \varepsilon \)-powers. In order to delicately obtain the \( O(T) \) estimate in Theorem 1 we will use a more structural approach to studying the
spectral mean value (1.3).

1.2. Spectral reciprocity. We will derive Theorem 1 as a consequence of an equality which we
now describe. In analogy to (1.3) we also define a corresponding discrete series average

\[ \mathcal{M}^{\text{hol}}(h) := \sum_{\pi} \frac{\mathcal{L}^{1/2}(\Pi \times \pi)}{\mathcal{L}(1, \text{Sym}^2 \pi)} h(k_{\pi}), \]
where the notation $\sum_{\text{hol}}$ indicates the sum is taken over all automorphic representations $\pi$ corresponding to classical holomorphic cusp forms of weight $k_\pi$ for $\text{SL}(2, \mathbb{Z})$. For $\diamondsuit \in \{+, -, \text{hol}\}$ we denote by $\mathcal{M}^\diamondsuit$ the same expression as $\mathcal{M}^\diamondsuit$, but with $\Pi$ replaced by its contragredient $\bar{\Pi}$ instead. Let $(\mu, \beta) \in \mathbb{C}^4 \times (\mathbb{Z}/2\mathbb{Z})^4$ denote the representation parameter of $\Pi$ as in [MS2, §1], which we assume (as we may after tensoring with a central character) satisfies

$$
\mu_1 + \mu_2 + \mu_3 + \mu_4 = 0 \quad \text{and} \quad \beta_1 + \beta_2 + \beta_3 + \beta_4 \equiv 0 \pmod{2}
$$

(see [MS2, (2.2)]). For example, if $\Pi$ is $\text{SO}(4)$-fixed, then $\beta \equiv (0, 0, 0, 0)$ or $(1, 1, 1, 1) \pmod{2}$ depending on whether it corresponds to an even or odd automorphic form on $\text{SL}(4, \mathbb{Z}) \setminus \text{SL}(4, \mathbb{R}) \slash \text{SO}(4)$. Let

$$
\mathcal{E}_0(u) := 2 \cos \left(\frac{\pi}{2} u\right) \quad \text{and} \quad \mathcal{E}_1(u) := 2i \sin \left(\frac{\pi}{2} u\right),
$$

where the index is understood as an element in $\mathbb{Z}/2\mathbb{Z}$, and define

$$
\mathcal{G}^\pm(u) := \prod_{j=1}^{4} \Gamma(u + \mu_j) \left(\mathcal{E}_{\beta_j}(u + \mu_j) \mp \mathcal{E}_{\beta_j+1}(u + \mu_j)\right),
$$

which are holomorphic in $\Re u > \frac{1}{2} - \delta$ for some constant $\delta$ depending only on $\Pi$ (see (2.13)). Moreover, let $C^+(u, r) = \cos(\pi u/2)$, $C^-(u, r) = \cosh(\pi r)$, and

$$
\mathcal{K}^\pm(t, r) := \cosh(\pi t) \int_{\Re u = v} \mathcal{G}^\pm \left(\frac{1 - u}{2}\right) C^\pm(u, r) \times \Gamma(u/2 + it)\Gamma(u/2 - it)\Gamma(u/2 + ir)\Gamma(u/2 - ir) \frac{du}{2\pi i}.
$$

The integrand is holomorphic in $0 < \Re u < 2\delta$ when $t, r \in \mathbb{R}$; the path of integration may be taken to be $\Re u = v = \delta$. We also define

$$
K^{\text{hol}}(t, k) := \mathcal{K}^+(t, \frac{k-1}{2i})
$$

for $k \in 2\mathbb{N} = \{2, 4, 6, \ldots\}$. The integrals defining $\mathcal{K}^\diamondsuit$ for $\diamondsuit \in \{+, -, \text{hol}\}$ are absolutely convergent for $t, r \in \mathbb{R}$ and $k \in 2\mathbb{N}$.

**Theorem 3.** Let $\Pi$ be as in Theorem 1 and suppose that for some constant $C_1 \geq 40$ the test function $h : \{t \in \mathbb{C} : |\Im t| < C_1\} \to \mathbb{C}$

$$
h(t) \ll (1 + |t|)^{-C_1} \quad \text{and} \quad h \left(\pm \left(n - \frac{1}{2}\right)i\right) = 0 \quad \text{for} \quad n \in \mathbb{N} \cap [1, C_1].
$$

(1.5)

is holomorphic, and satisfies $h(t) \ll (1 + |t|)^{-C_1}$ and $h \left(\pm \left(n - \frac{1}{2}\right)i\right) = 0$ for $n \in \mathbb{N} \cap [1, C_1]$.

Then

$$
\mathcal{M}^-(h) = \tilde{\mathcal{M}}^-(h^-) + \tilde{\mathcal{M}}^+(h^+) + \tilde{\mathcal{M}}^{\text{hol}}(h^{\text{hol}}),
$$

(1.6)

where

$$
h^\diamondsuit := \frac{1}{8\pi^2} \int_{-\infty}^{\infty} \mathcal{K}^\diamondsuit(t, r)h(t)t \tanh(\pi t) \frac{dt}{2\pi^2}
$$

is absolutely convergent for each $\diamondsuit \in \{+, -, \text{hol}\}$. Moreover, the spectral sums $\tilde{\mathcal{M}}^-(h^-)$, $\tilde{\mathcal{M}}^+(h^+)$, and $\tilde{\mathcal{M}}^{\text{hol}}(h^{\text{hol}})$ are each absolutely convergent, and

$$
\mathcal{K}^+(t, r) \ll e^{-\frac{\pi}{2}|r|}(1 + |t|)^{-1}
$$

for $t, r \in \mathbb{R}$.

The integral kernels $\mathcal{K}^\diamondsuit(t, r)$ will be further discussed in Section 5, where they are computed explicitly in terms of ${}_{4}F_{3}$ hypergeometric functions. In practical situations, the terms $\tilde{\mathcal{M}}^+(h^+)$ and $\tilde{\mathcal{M}}^{\text{hol}}(h^{\text{hol}})$ are easy to bound and/or small, so that we have a “reciprocity formula”

$$
\mathcal{M}^-(h) \sim \tilde{\mathcal{M}}^-(h^-).
$$

(1.18)
An analysis of the $T$-dependence in Theorem 1 shows that if $h(t)$ is (essentially) supported on $t \ll T$, then $h^-(r)$ is of size $T$ for $r \ll 1$ and very small otherwise. Refining this argument, if $h(t)$ is (essentially) supported on a short segment $(T - M, T + M)$ for $T^{1/3} \ll M \ll T$, then $h^-(r)$ is essentially supported on $r \ll T/M$ and very small otherwise. This is a typical “duality” phenomenon of automorphic summation formulae.

Formula (1.16) is motivated by earlier work of Kuznetsov and Motohashi. If $\Pi$ is replaced by a minimal parabolic Eisenstein series, the spectral mean value $M^\pm(h)$ features a fourth moment of $GL(2)$ $L$-functions in the cuspidal sum, along with the eighth moment of the Riemann zeta-function in the contribution of the continuous spectrum. A similar reciprocity formula to (1.16) in this case was envisaged by Kuznetsov, and completed by Motohashi [Mo2]. As far as we know, this interesting formula has not yet been used in applications.

1.3. Balanced Voronoi summation. One of the tools in the proof of Theorem 3 is the analytic continuation and functional equation of a certain double Dirichlet series, which may be of independent interest. Let $a_{\Pi}(n_1, n_2, n_3) = a_{\Pi}(|n_1|, |n_2|, |n_3|)$ denote the abelian coefficients of $\Pi$ (see Section 2), normalized such that $a_{\Pi}(1, 1, 1) = 1$; they are the Hecke eigenvalues of $\Pi$ when $n_1, n_2, n_3 > 0$. Recall that $a_{\Pi}(n_1, n_2, n_3) = a_{\Pi}(n_3, n_2, n_1) = a_{\Pi}(n_1, n_2, n_3)$. For $\epsilon \in \mathbb{Z}/2\mathbb{Z}$ denote

$$V_{\epsilon}(s, z) := \sum_{n \in \mathbb{Z}\setminus\{0\}} \sum_{m, c > 0} \sgn(n) a_{\Pi}(n, m, 1) S(n, 1, c) \left( \frac{|n|}{|c|^2} \right)^{-s},$$

and write $\tilde{V}_\epsilon$ for the same expression with $a_{\Pi}(n, m, 1) = a_{\Pi}(1, m, n)$ instead of $a_{\Pi}(n, m, 1)$. The multiple sum is absolutely convergent in the range

$$\Re s < -1/4, \quad \Re(s + z) > 1,$$

as follows from the Weil bound on Kloosterman sums and the estimate (2.3) below. The following result is a consequence of the “balanced” Voronoi formula given in Theorem 5, which is itself a consequence of the usual $GL(4)$ Voronoi formula from [MS2] (though packaged in a very different way).

**Theorem 4.** The function $V_{\epsilon}(s, z)$ has holomorphic continuation to the region

$$\{(s, z) \in \mathbb{C}^2 \mid \Re(s + 2z) > 5/4, \Re z > 5/4, \text{ and } \Re s < -1/4\}$$

and satisfies the functional equation

$$(1.20) \quad V_{\epsilon}(s, z) = G_{\epsilon}(1 - s - z) \tilde{V}_\epsilon(1 - s - 2z, z),$$

where

$$(1.21) \quad G_{\epsilon}(s) := \prod_{j=1}^4 \frac{\Gamma(s + \mu_j)}{(2\pi)^{s+\mu_j}} \mathcal{E}_{\epsilon+\beta_j}(s + \mu_j)$$

with $\mathcal{E}$ as in (1.11). Notice that

$$G^\pm(u) = (2\pi)^{4u} (G_0(u) \mp G_1(u))$$

because of (1.10) and (1.12). When integrated against the Mellin transform of a test function, (1.20) assumes the symmetric, equivalent form of a summation formula for sums of abelian coefficients times Kloosterman sums. This formula was first obtained by the second two named authors, and has since been generalized to arbitrary sums of $GL(n)$ Fourier coefficients weighted by hyper-Kloosterman sums in [Zh, MZ].

It is crucial for Theorem 2 that (1.19) includes the precise Kloosterman sum arising from the Kuznetsov formula. Here one cannot simply open up the Kloosterman sum and apply Voronoi summation in its usual form [MS2] as in the subconvexity results of [Sa, Li2], because this entails the loss of a multiplicative term of size $T^{\epsilon}$ owing to appearance of additional divisor sums. Rather,
it is a special feature of the GL(4) Hecke algebra that produces the precise form of (1.20), and allows us to cleanly avoid those extra factors. Without this the \( O(T) \) bound in (1.5) would instead exceed the main term \( T \log T \), and we would not able to deduce our nonvanishing result.

We conclude the introduction with a discussion of whether there is a reciprocity formula for the untwisted spectral average \( \mathcal{M}^+(h) \) in analogy to (1.18). Were we to apply the heuristic analysis of (1.7)–(1.8) without the root number \( \epsilon_n \), the “same-sign” Kuznetsov formula produces roughly

\[
T \sum_{n \leq T^4} \frac{\lambda_{11}(n)}{n^{1/2}} e(\pm 2\sqrt{n}) , \quad \text{where} \quad e(x) := e^{2\pi i x},
\]

instead of (1.1), since the \( c \)-sum is now essentially bounded. However, this comes at the cost of the archimedean phase factor \( e(\pm 2\sqrt{n}) \) from the Bessel function. At this point the Voronoi summation formula applied to the \( n \)-sum is completely self-dual, and so the triad Kuznetsov-Voronoi-Kuznetsov turns out to be essentially involutory and gives no useful information. This is the “deadlock” situation mentioned earlier. The extra oscillation introduced by the root number breaks the self-duality here, in such a strong sense that one can obtain pure bounds that are precise even on a log-scale, as a comparison of (1.5) and (1.6) demonstrates.

As a reflection of this phenomenon, formally imitating the proof of Theorem 3 – but without the root number \( \epsilon_n \) yields serious convergence problems: even the final spectral formula does not converge. This was already observed by Motohashi in the case of Eisenstein series [Mo2, (2.16)]. The problem can be repaired by restricting to test functions \( h \) with spectral mass 0, or more precisely by requiring that

\[
\int_{-\infty}^{\infty} h(t) t^{n} \tanh(\pi t) dt = 0
\]

for all integers \( n \) up to some sufficiently large bound. Alternatively, one can consider the difference \( \mathcal{M}^+(h) - \mathcal{M}^{\text{hol}}(h) \), for which the convergence problems disappear and one can derive a spectral identity. These devices are reminiscent of [Mo2, Section 3]. As we are not aware of any interesting applications of such formulae, we shall not go into further detail here.

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2. Automorphic Toolbox

For the rest of the paper \( \pi \) will denote a cuspidal automorphic representation on \( \text{GL}(2, \mathbb{Q}) \backslash \text{GL}(2, \mathbb{A}) \) associated to a Hecke-Maaß cusp form for \( \text{SL}(2, \mathbb{Z}) \). Let \( t_\pi \geq 9.7 \) denote the spectral parameter of \( \pi \) (see [He, Appendix C]) and let \( \lambda_\pi(n) \) denote its Fourier coefficients, normalized so that \( \lambda_\pi(1) = 1 \). The root number is \( \epsilon_\pi = +1 \) for even Maaß forms and \( -1 \) for odd Maaß forms, so that \( \lambda_\pi(-n) = \epsilon_\pi \lambda_\pi(n) \). The Fourier coefficients of the non-holomorphic Eisenstein series \( E_{it}, t \in \mathbb{R} \), are given explicitly as divisor sums

\[
\lambda_t(n) := \sum_{0 < d | n} d^it(n/d)^{-it}
\]

for \( n \neq 0 \). Let

\[
d_{\text{spect}} := \frac{1}{2\pi^2} t \tanh(\pi t) dt
\]

denote the Plancherel measure for \( \text{SL}(2, \mathbb{R}) \).

As in the introduction, let \( \Pi \) be a cuspidal automorphic representation on \( \text{GL}(4, \mathbb{Q}) \backslash \text{GL}(4, \mathbb{A}) \) which is unramified at all finite places. Its abelian coefficients \( a_{\Pi}(n_1, n_2, n_3) \) are the Hecke eigenvalues of \( \Pi \) when each \( n_i > 0 \), and by definition satisfy \( a_{\Pi}(\sigma_1 n_1, \sigma_2 n_2, \sigma_3 n_3) = a_{\Pi}(n_1, n_2, n_3) \) for any choice of \( \sigma_i = \pm 1 \). They are related to those of \( \Pi \) by

\[
a_{\Pi}(n_1, n_2, n_3) = a_{\Pi}(n_3, n_2, n_1) = a_{\Pi}(n_1, n_2, n_3) = a_{\Pi}(-n_3, n_2, n_1).
\]
The following bound of Serre (reprinted in [BB, appendix]) holds uniformly for all finite places: there exists $\delta > 0$ such that
\begin{equation}
\lambda_{\Pi}(n) := a_{\Pi}(n, 1, 1) \ll n^{1/2 - \delta}.
\end{equation}
We will frequently use the Rankin-Selberg bound
\begin{equation}
\sum_{nm^2 \leq x} |a_{\Pi}(n, m, 1)|^2 \ll x,
\end{equation}
typically in combination with the Cauchy-Schwarz inequality; it is a consequence of the fact that the degree-16 tensor product $L$-function $L(s, \Pi \times \Pi)$ has a simple pole at $s = 1$ and analytic continuation with moderate growth in vertical strips. It is also known [Ki, Proposition 6.2] that the Euler product for the degree-10 $L$-function $L(s, \Pi, \text{Sym}^2)$ is absolutely convergent for $\Re s > 1$. With this background in hand, we now turn to the following lemma (whose proof is, in absence of the Ramanujan conjecture, not completely trivial).

**Lemma 1.** For $x \geq 1$ we have
\[ \sum_{n \leq x} |\lambda_{\Pi}(n)|^2 \gg x, \]
where the implied constant depends on $\Pi$.

**Proof.** We start with a variation of [RSA, Proposition 2.4] for $m = 4$. For a prime $p$ let $\{\alpha_j(p) \mid j = 1, \ldots, 4\}$, denote the Satake parameters of $\Pi$ at $p$, so that
\[ \lambda_{\Pi}(p) = \sum_{j=1}^4 \alpha_j(p), \quad \lambda_{\Pi}(p^2) = \sum_{1 \leq i < j \leq 4} \alpha_i(p)\alpha_j(p), \]
and $\lambda_{\Pi}(p^k)$ is a symmetric polynomial in the $\alpha_j(p)$ of degree $k$. These parameters satisfy $\alpha_1(p) \cdots \alpha_4(p) = 1$ and the unitarity condition $\{\alpha_j(p)^{-1} \mid j = 1, \ldots, 4\} = \{\alpha_j(p) \mid j = 1, \ldots, 4\}$. Therefore there are three possibilities for their size: (a) the Ramanujan conjecture holds at $p$, i.e., $|\alpha_j(p)| = 1$ for $1 \leq j \leq 4$; (b) two of the parameters, say $\alpha_1(p)$, $\alpha_2(p)$, are on the unit circle, while $\alpha_3(p)^{-1} = \alpha_4(p)$ is not; or (c) $\{\alpha_1(p), \alpha_2(p), \alpha_3(p), \alpha_4(p)\} = \{\alpha, \bar{\alpha}^{-1}, \beta, \bar{\beta}^{-1}\}$ (as multisets) for complex numbers $\alpha, \beta$ not on the unit circle. In each case it is easy to see that
\[ \max_j |\alpha_j(p)|^2 \ll 1 + |\lambda_{\Pi}(p)|^2 + |\lambda_{\Pi}(p^2)|. \]
Let $a_{\Pi}$ denote the multiplicative function defined on prime powers $p^k$ by $a_{\Pi}(p^k) = \alpha_1(p)^k + \cdots + \alpha_4(p)^k$ (which coincides with $\lambda_{\Pi}$ on primes); it thus satisfies the bound
\[ |a_{\Pi}(p^k)|^2 \ll k^1 + |\lambda_{\Pi}(p)|^{2k} + |\lambda_{\Pi}(p^2)|^k \]
for $k \in \mathbb{N}$.

The quantity $|\lambda_{\Pi}(p)|^2$ is the Dirichlet series coefficient of $p^{-s}$ in the degree-16 Rankin-Selberg $L$-function $L(s, \Pi \times \Pi)$, while the quantity $\lambda_{\Pi}(p^2)$ is the Dirichlet series coefficient of $p^{-s}$ in the degree-10 symmetric square $L$-function $L(s, \Pi, \text{Sym}^2)$. As noted above, both $L$-functions are uniformly and absolutely convergent in the half plane $\Re s \geq 1 + \varepsilon$, for any $\varepsilon > 0$, and consequently $\sum_{p \leq x} |\lambda_{\Pi}(p)|^2$ and $\sum_{p \leq x} |\lambda_{\Pi}(p^2)|$ are both $O(x^{1+\varepsilon})$. It follows that
\begin{equation}
\sum_{p \leq x} |a_{\Pi}(p^k)|^2 \ll \sum_{p \leq x} (1 + |\lambda_{\Pi}(p)|^{2k} + |\lambda_{\Pi}(p^2)|^k) \ll x + x^{1+\varepsilon}x^{\left(\frac{1}{2} - \delta\right)(2k-2)} + x^{1+\varepsilon}x^{(1-2\delta)(k-1)} \ll x^{k-2\delta(k-1)+\varepsilon}
\end{equation}
for any $\varepsilon > 0$, where we have applied the bound (2.2) to $|\lambda_{\Pi}(p)|^{2k-2}$ and $|\lambda_{\Pi}(p^2)|^{k-1}$.

The implied constant in (2.4) depends on $k$, but for $k \geq 1/\delta$ the bound (2.2) trivially implies
\begin{equation}
\sum_{p \leq x} |a_{\Pi}(p^k)|^2 \ll \sum_{p \leq x} p^{2k(\frac{1}{2} - \delta)} \ll x^{1+k-2k\delta}.
\end{equation}
Combining (2.4) and (2.5) we deduce
\[
\sum_{p^k \leq x} |a_M(p^k)|^2 \leq \varepsilon x^{1-\delta+\varepsilon} \leq x^{1-\delta/2}, \quad \text{uniformly for } k \geq 2
\]
and \(\varepsilon < \delta\). By the “prime number theorem” for Rankin-Selberg \(L\)-functions [LWY, Lemma 5.1] we have
\[
\sum_{n \leq x} \Lambda(n)|a_M(n)|^2 \sim x \quad \text{as } x \to \infty,
\]
where the von Mangoldt function \(\Lambda(n)\) is supported on prime powers and \(\Lambda(p^k) = \log(p)\). By (2.6) the contribution from higher prime powers is negligible in this sum, and we conclude
\[
\sum_{p \leq x} |\lambda_M(p)|^2 = \sum_{p \leq x} |a_M(p)|^2 \sim \frac{x}{\log x} \quad \text{as } x \to \infty
\]
by partial summation.

We now apply Wirsing’s Theorem [Wi, Satz 1] to the multiplicative function
\[
f(n) = \begin{cases} |\lambda_M(n)|^2, & n \text{ squarefree;} \\ 0, & \text{otherwise} \end{cases}
\]
to deduce for large \(x\) that
\[
\sum_{n \leq x} f(n) \sim e^{-\gamma} \frac{x}{\log x} \prod_{p \leq x} \left( 1 + \frac{f(p)}{p} \right) \geq e^{-\gamma} \frac{x}{\log x} \exp \left( \sum_{p \leq x} \frac{|\lambda_M(p)|^2}{p} - \frac{1}{2} \sum_{p \leq x} \frac{|\lambda_M(p)|^4}{p^2} \right).
\]
Here we used the inequality \(\log(1+x) \geq x - x^2/2\) for \(x \geq 0\). By applying (2.7) and (2.2), along with partial summation and the absolute convergence of \(L(s, \Pi \times \Pi)\), we have
\[
\sum_{p \leq x} \frac{|\lambda_M(p)|^2}{p} - \frac{1}{2} \sum_{p \leq x} \frac{|\lambda_M(p)|^4}{p^2} = \log \log x + O(1) + O\left( \sum_{p \leq x} \frac{|\lambda_M(p)|^2}{p^{1+2\delta}} \right) = \log \log x + O(1),
\]
so that
\[
\sum_{n \leq x} |\lambda_M(n)|^2 \geq \sum_{n \leq x} f(n) \gg x
\]
as asserted. \(\square\)

It follows from the Dirichlet series expansions of tensor product \(L\)-functions in [Bu1, §2] that
\[
L(s + it, \Pi)L(s - it, \Pi) = \sum_{n, m \geq 1} \frac{\lambda_\Pi(n)a_M(n, m, 1)}{n^s m^{2s}}
\]
and
\[
L(s, \Pi \times \pi) = \sum_{n, m \geq 1} \frac{\lambda_\pi(n)a_M(n, m, 1)}{n^s m^{2s}},
\]
both of which converge absolutely for \(\Re s > 1\) [JS, Theorem 5.3]. In particular, all \(L\)-functions appearing in (1.3) have Dirichlet series expansions which are absolutely convergent in \(\Re s > 1\). The convexity bound for central \(L\)-values and lower bounds for \(L\)-functions at the edge of the critical strip [HL], [Ti, Section 3.6] imply the estimates
\[
\frac{L(1/2 + it, \Pi)L(1/2 - it, \Pi)}{|\zeta(1 + 2it)|^2} \ll \varepsilon (1 + |t|)^{2+\varepsilon} \min(1, |t|^2) \quad \text{and} \quad \frac{L(1/2, \Pi \times \pi)}{L(1, \text{Sym}^2 \pi)} \ll \varepsilon \left\{ \frac{t^{2+\varepsilon}}{k^{3+\varepsilon}}, \frac{1}{k^{2+\varepsilon}} \right\},
\]
where the extra factor of \(t^2\) for \(t\) small comes from the pole of the Riemann \(\zeta\)-function and the two cases on the right hand side correspond to Maaß forms and holomorphic modular forms, respectively.
Our application of Theorem 4 requires some information about the representation parameter \((\mu, \beta) \in \mathbb{C}^4 \times (\mathbb{Z}/2\mathbb{Z})^4\) appearing in (1.21). For \(\eta \in \mathbb{Z}/2\mathbb{Z}\) let

\[
G_\eta(s) := (2\pi)^{-s} \Gamma(s) E_\eta(s) = \begin{cases} 
\frac{\Gamma_R(s)}{\Gamma_R(1-s)}, & \eta \in 2\mathbb{Z}, \\
\frac{\Gamma_R(s+1)}{\Gamma_R(2-s)}, & \eta \in 2\mathbb{Z} + 1,
\end{cases}
\]

where \(\Gamma_R(s) = \pi^{-s/2} \Gamma(s/2)\). Thus \(G_\eta(s) = \prod_{j \leq 4} G_{\epsilon+j}(s+\mu_j)\) in (1.21). Representation parameters for all cusp forms on \(\text{GL}(n, \mathbb{R})\) are explicitly described in [MS3, (A.1)-(A.2)]. In particular, each \(\mu_j\) either has the form \(s'\) or occurs in a pair \((\mu_j, \mu_j') = (s' - \frac{k-1}{2}, s' + \frac{k-1}{2})\) for some \(s' \in \mathbb{C}\) with \(|\Re s'| < \frac{1}{2}\) and \(k \in \mathbb{Z}_{\geq 2}\) with \(k \equiv \beta_j + \beta_{j'} \pmod{2}\). Although in the latter situation \(G_{\epsilon+j}(s+\mu_j)\) has simple poles at \(s = \frac{k-1}{2} - s' - n, \ n \in \mathbb{Z}_{\geq 0}\), the product

\[
G_{\epsilon+j}(s+\mu_j)G_{\epsilon+j'}(s+\mu_{j'}) = i^k (2\pi)^{1-2s-2s'} \frac{\Gamma(s + \frac{k-1}{2} + s')}{\Gamma(1-s + \frac{k+1}{2} - s')}
\]

is holomorphic in \(|s| \Re s > 0\). Thus

\[
G_0(s), \ G_1(s), \ \text{and} \ G^\pm(s) \ \text{are all holomorphic in} \ \Re s > \frac{1}{2} - \delta
\]

for some sufficiently small \(0 < \delta < 1/2\) depending on \(\Pi\) (which we assume, as we may, is simultaneously valid in (2.2)). Stirling’s formula applied to (2.11) gives the asymptotics \(|G_\eta(s)| \sim \frac{1}{\sqrt{2\pi}}|\Re s|^{-1/2}\) in vertical strips of finite width, hence using (1.10) we bound

\[
G_\epsilon(s) \ll (1 + |s|)^{4|\Re s|},
\]

uniformly for \(s\) away from poles in any vertical strip of finite width.

We conclude this section by stating the Kuznetsov summation formula [Ku] in the two different versions we will apply it. Both involve the integral kernels

\[
\mathcal{J}_{t}^+(x) := \frac{\pi i}{\sinh(\pi t)} (J_{2it}(x) - J_{-2it}(x)),
\]

\[
\mathcal{J}_{t}^{-}(x) := 4 \cosh(\pi t) K_{2it}(x) = \frac{\pi i}{\sinh(\pi t)} (I_{2it}(x) - I_{-2it}(x)),
\]

as well as the “holomorphic” kernel

\[
\mathcal{J}_{k}^{\text{hol}}(x) := \mathcal{J}_{(k-1)/(2i)}^+(x) = 2\pi i^k J_{k-1}(x), \quad k \in 2\mathbb{N}.
\]

Let \(h(t) = O((1 + |t|)^{-3})\) be an even function which is holomorphic in \(|\Im t| \leq 1/2\), and let \(n, m \geq 1\). Then (see, e.g., [BK, Lemma 3.3])

\[
\sum_{\pi} \epsilon_\pi \frac{\lambda_\pi(n) \lambda_\pi(m)}{L(1, \text{Sym}^2 \pi)} h(t_\pi) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\lambda_\pi(n) \lambda_\pi(m)}{|\zeta(1+2it)|^2} h(t) dt = \sum_{c} \frac{S(-n, m, c)}{c} \int_{-\infty}^{\infty} \mathcal{J}_{t}^{-} \left( \frac{4\pi \sqrt{nm}}{c} \right) h(t) dt_{\text{spec}}.
\]

Here \(S(n, m, c)\) is the usual Kloosterman sum, which satisfies the Weil bound \(|S(n, m, c)| \leq c^{1/2}(n, m, c)^{1/2}\tau(c)\), where \(\tau(c)\) is the number of positive divisors of \(c\).
Finally, formula (2.17) can be inverted as follows. Suppose that \( \phi \in C^3((0, \infty)) \) satisfies \( x^j \phi^{(j)}(x) \ll \min(x, x^{-3/2}) \) for \( 0 \leq j \leq 3 \), and let \( n, m \in \mathbb{N} \). Then [Mo1, Theorems 2.3 and 2.5] states that

\[
\sum \frac{S(\pm n, m, c)}{c} \phi \left( \frac{4\pi \sqrt{nm}}{c} \right) = \sum_{\lambda \in \mathbb{Z}} \frac{\lambda_\pi(\pm n) \lambda_\pi(m)}{L(1, \text{Sym}^2 \pi)} \int_0^\infty J_\pi^\pm(x) \phi(x) \frac{dx}{x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\lambda_\pi(n) \lambda_\pi(m)}{|\zeta(1 + 2it)|^2} \int_0^\infty J_t^\pm(x) \phi(x) \frac{dx}{x} dt + \sum_{\pi} \frac{\lambda_\pi(\pm n) \lambda_\pi(m)}{L(1, \text{Sym}^2 \pi)} \int_0^\infty J_{\pi, k}^{\text{hol}}(x) \phi(x) \frac{dx}{x},
\]

(2.18)

where the first \( \pi \)-sum runs over automorphic representations associated to cuspidal Maass forms for \( \text{SL}(2, \mathbb{Z}) \) having spectral parameter \( t_x \), and the last \( \pi \)-sum runs over automorphic representations associated to classical holomorphic cusp forms for \( \text{SL}(2, \mathbb{Z}) \) having weight \( k_\pi \in 2N \) (with the convention \( \lambda_\pi(-n) = 0 \), so that it disappears in the minus sign case).

3. Balanced Voronoi summation and proof of Theorem 4

In this section we prove Theorem 4 using the key relations [MS2, Prop. 3.6] between tempered distributions \( \sigma_{j,N,(k_1,k_2,k_3)} \) and \( \rho_{j,N,(k_1,k_2,k_3)} \) on \( \mathbb{R}^1 \) defined in [MS2, (2.47)], where \( 1 \leq j \leq 3 \), \( N > 0 \), and all subscripts are integers. Theorem 4 will be shown as a consequence of the following “balanced” Voronoi formula, which itself follows from the GL(4,\( \mathbb{Z} \))\( \setminus \text{GL}(4, \mathbb{R}) \) Voronoi formula in [MS2].

**Theorem 5.** For any cuspidal automorphic form \( \Pi \) on \( \text{GL}(4, \mathbb{Z}) \setminus \text{GL}(4, \mathbb{R}) \), define

\[
\mathcal{L}_{N,\epsilon}(s, \Pi) := \sum_{m \mid N} a_\Pi(n, m, 1) S \left( n, 1, \frac{N}{m} \right) m^{-1-2s} |n|^{-\epsilon},
\]

(3.1)

where \( N \) is a positive integer, \( \epsilon \in \mathbb{Z}/2\mathbb{Z} \), and \( \Re s > 1 \) (where the sum converges absolutely because of (2.3)). Then \( \mathcal{L}_{N,\epsilon}(s, \Pi) \) has an analytic continuation to an entire function in \( s \) and satisfies the functional equation

\[
\mathcal{L}_{N,\epsilon}(s, \Pi) = N^{2-4\epsilon} \mathcal{G}_\epsilon(1-s) \mathcal{L}_{N,\epsilon}(1-s, \Pi).
\]

(3.2)

As we mentioned above, this theorem has been generalized to GL(n) in [Zh, MZ], where (3.2) is derived using Dirichlet series methods. We will include a different argument here for \( n = 4 \), based on the machinery of [MS2] used to prove the usual GL(4) Voronoi formula. Before giving the proof, we briefly see how it implies Theorem 4. The sum (3.1) satisfies the bounds

\[
\mathcal{L}_{N,\epsilon}(s, \Pi) \ll \begin{cases} 
N^{1/2+\epsilon}, & \Re s > 1, \\
N^{1/2+2-2\Re s+\epsilon}, & 0 \leq \Re s \leq 1, \\
N^{1/2+4-4\Re s+\epsilon}, & \Re s < 0,
\end{cases}
\]

where the first estimate follows from (2.3) and Weil’s bound for Kloosterman sums, the third from the functional equation (3.2), and the second from convexity (it is easy to see \( \mathcal{L}_{N,\epsilon}(s, \Pi) \) has finite order). It follows that

\[
\mathcal{V}_\epsilon(s, z) = \sum_{N>0} N^{2s-1} \mathcal{L}_{N,\epsilon}(s+z, \Pi)
\]

is entire in \( \{ \Re(s+z) > 0, \Re z > 5/4, \Re s < -1/4 \} \cup \{ \Re(s+z) \leq 0, \Re(s+2z) > 5/4 \} = \{ \Re(s+z) > 5/4, \Re z > 5/4, \Re s < -1/4 \} \), and inherits the functional equation (1.20) from (3.2).

**Proof of Theorem 5.** In what follows we mainly follow the notation conventions of [MS2]. The third statement in [MS2, Prop. 3.6] reads

\[
\sigma_{2,N,(1,0,1)}(x) = \chi_2(Nx) \rho_{1,N,(1,0,1)} \left( \frac{1}{N^2 x} \right),
\]

(3.3)
where \( \chi_j : \mathbb{R}^* \to \mathbb{C}^* \) denotes the homomorphism

\[
\chi_j(x) = |x|^{\mu_j - \mu_{j+1} - 1} \text{sgn}(x)^{\beta_j + \beta_{j+1}}.
\]

By [MS2, Prop. 2.51] we have

\[
\rho_{1,N,(1,0,1)}(y) = \sum_{\ell \pmod{N}} e\left( \frac{\ell}{N} \right) (\mathcal{F} \sigma_{1,N,(0,\ell,1)})(Ny)
\]

(3.4)

and

\[
\sigma_{2,N,(1,0,1)}(r) = \sum_{\ell \pmod{N}} e\left( -\frac{\ell}{N} \right) (\mathcal{F} \rho_{2,N,(1,\ell,0)})(-Nr),
\]

where \( (\mathcal{F} \tau)(x) = \int_{\mathbb{R}} \tau(r) e(-xr) dr \) denotes the Fourier transform of a tempered distribution \( \tau \).

Define the normalized coefficients

(3.5) \( c_{r,d,1} = a_1(r,d,1) d^{-\mu_1 - \mu_2} \text{sgn}(r)^{\beta_1} |r|^{-\mu_1} \) and \( c_{1,d,r} = a_1(1,d,r) d^{-\mu_1 - \mu_2} \text{sgn}(r)^{\beta_1} |r|^{\mu_4} \)

for integers \( d > 0 \) and \( r \neq 0 \) (see [MS2, (2.9)]), and the distributions

\[
\Delta_{L; (N, \ell), 1, \frac{\ell}{N}, \ell} = \sum_{n \neq 0} c_{n,(N,\ell),1} e\left( \frac{n\ell}{N/(N, \ell)} \right) \delta_n
\]

(3.6)

\[
\Delta_{R; (N, \ell), 1, \frac{\ell}{N}, \ell} = \sum_{n \neq 0} c_{1,(N,\ell),n} e\left( -\frac{n\ell}{N/(N, \ell)} \right) \delta_n,
\]

where \( \delta_r \in C^\infty(\mathbb{R}) \) denotes the Dirac \( \delta \)-function supported at \( r \in \mathbb{R} \). In terms of the operators

\[
(\mathcal{T}_{j,a,b}^*)_{(x)} = \text{sgn}(ax)^{\beta_j + \beta_{j+1}} |ax|^{\mu_j - \mu_{j+1} - 1} (\mathcal{F} \sigma)\left( \frac{b}{ax} \right) = \chi_j(ax)(\mathcal{F} \sigma)\left( \frac{b}{ax} \right)
\]

defined for \( j = 1, 2, 3 \), we thus have

\[
\sigma_{2,N,(1,0,1)} = \sum_{\ell \pmod{N}} e\left( \frac{\ell}{N} \right) T_{2,N,1}^* \sigma_{1,N,(0,\ell,1)}
\]

(3.7)

\[
= \sum_{\ell \pmod{N}} e\left( \frac{\ell}{N} \right) T_{2,N,1}^* T_{1,(N,\ell), \frac{\ell}{N}, \ell}^* \Delta_{L; (N, \ell), 1, \frac{\ell}{N}, \ell},
\]

where in the first step we have used (3.3) and the first equation of (3.4), and in the second step we have invoked the first definition in (3.6) and [MS2, (4.10)]. We also have that

(3.8) \( \rho_{2,N,(1,\ell,0)} = T_{3, \frac{\ell}{N}, \ell}^* \Delta_{R; (N, \ell), 1, \frac{\ell}{N}, \ell} \)

as follows from the second formula in [MS2, Prop. 3.6].

Identities (3.4), (3.6), (3.7), and (3.8) are all equalities of tempered distributions which vanish to infinite order at \( x = 0 \) and \( x = \infty \) in the sense described in [MS1] and [MS2, Prop. 3.6]. A distribution \( \tau \) on \( \mathbb{R} \) vanishing to infinite order both at zero and at infinity has an entire signed Mellin transform

\[
(M \tau)(s) = \int_{\mathbb{R}} \tau(x) |x|^{s-1} \text{sgn}(x) dx
\]

(see [MS1, Theorem 4.8]). Thus for precisely the same analytic reasons as in [MS2, (1.11)], \( (M \sigma_{2,N,(1,0,1)})(s) \) is entire. (In fact, the distributional identities here are finite linear combinations of the ones there, so no additional analytic overhead is needed beyond what is provided there.) Likewise, the signed Mellin transforms

\[
(M \Delta_{L; (N, \ell), 1, \frac{\ell}{N}, \ell})(s) = \sum_{n \neq 0} c_{n,(N,\ell),1} e\left( \frac{n\ell}{N/(N, \ell)} \right) \text{sgn}(n)^s |n|^{s-1}
\]
establishes the analytic continuation asserted in the Theorem.

\[ (3.12) \]

On the other hand, applying (3.12) and grouping together terms with a common value \( m \) of \((N, \ell)\). In particular, this establishes the analytic continuation asserted in the Theorem.

When both a tempered distribution \( \tau \) and its Fourier transform \( \mathcal{F}\tau \) vanish to infinite order at both 0 at infinity,

\[ (3.10) \]

where both Mellin transforms are entire and we have used the relationship \( (3.12) \) and (3.13) to the second identity in (3.4) yields that \((M_\ell \sigma_{2,N}(1,0,1))(s)\) equals

\[ (3.13) \]
using (3.8) and (3.9). The functional equation (3.2) now follows from comparing (3.12) to (3.13). □

4. Spectral reciprocity and proof of Theorem 3

In this section we prove Theorem 3. For $s$ with $\Re s \geq 0$ and $h$ as in Theorem 3, define

$$\mathcal{M}^\pm(s; h) := \sum_{\pi} \epsilon_x (1+1/2) L\left(\frac{1}{2} + s, \Pi \times \pi\right) h(t_x) + \frac{1}{2\pi} \int_{-\infty}^{\infty} L\left(\frac{1}{2} + s + it, \Pi\right) L\left(\frac{1}{2} + s - it, \Pi\right) \frac{h(t)}{|\zeta(1+2it)|^2} dt$$

and

$$\mathcal{M}^{\text{hol}}(s; h) := \sum_{\pi} L\left(\frac{1}{2} + s, \Pi \times \pi\right) h(t_x).$$

As before, we write $\tilde{\mathcal{M}}^\diamond(s; h)$ for $\diamond \in \{+,-,\text{hol}\}$ for the same expressions with $\Pi$ replaced by $\Pi$. All of these sums are absolutely convergent and holomorphic in a half plane containing $\{s | \Re s \geq 0\}$; their values at $s = 0$ specialize to the sums $\mathcal{M}^\diamond(h)$ defined in (1.3) and (1.9). If $\Re s > 1/2$, the $L$-functions in question can be replaced by their absolutely convergent Dirichlet series (2.8) and (2.9). With further manipulations in mind, let us temporarily assume $3/4 < \Re s < 1$.

By design, applying the Kuznetsov formula (2.17) yields

$$\mathcal{M}^-(s; h) = \sum_{n,m,c > 0} a_{\Pi}(n,m,1) \zeta(1/2) \zeta(1/2 + c) S(-n,1,c) H\left(\frac{4\pi \sqrt{n}}{c}\right),$$

where

$$H(x) := \int_{-\infty}^{\infty} J_t^-(x) h(t) dt = \frac{2}{\pi^2} \int_{-\infty}^{\infty} K_{2it}(x) \sinh(\pi t) h(t) dt.$$

**Lemma 2.** Let $C_2 \in \mathbb{N}$. There exists $C_1 > 0$ (depending on $C_2$) such that

$$x^j \frac{d^j}{dx^j} H(x) \ll_{C_2} C_1 \min(x^{-C_2}, x^{-C_2})$$

for $0 \leq j \leq C_2$, for any $h : \{t \in \mathbb{C} : |\Im t| < C_1\} \to \mathbb{C}$ satisfying (1.15). In particular the Mellin transform $\tilde{H}(u) = \int_{0}^{\infty} H(x) x^{u-1} dx$ is holomorphic in $-C_2 < \Re u < C_2$ and is bounded by $\ll (1 + |u|)^{-C_2}$ in this region. Specifically, one can take $C_1 = 2C_2 + 2$.

**Proof.** We shall take $C_1 = 2C_2 + 2$. First let $x \geq 1$. For $0 \leq j \leq C_2$ it follows from (A.1) and (A.3) that

$$x^j \frac{d^j}{dx^j} H(x) \ll_j \int_{0}^{\infty} e^{\min(0,\pi t-x)} (1 + t/\pi)^{j+1/10} (1 + t)^{-C_1} dt \ll_j x^{2C_1-1} \ll x^{-j-C_2}$$

(as can be seen by separately considering the integrals over $0 \leq t \leq \frac{x}{2\pi}$ and $\frac{x}{2\pi} \leq t$).

Now let $x \leq 1$. We first apply (A.4) and then express the $j$-fold derivative of the integral (4.4) as a sum of terms using (A.2). In those terms containing $I_{2it+n}(x)/\cosh(\pi t)$ for $|n| \leq j$ we shift the contour down to $\Im t = -C_1 + \frac{1}{10}$, while in those terms containing $I_{-2it+n}(x)/\cosh(\pi t)$ for $|n| \leq j$ we shift the contour up to $\Im t = C_1 - \frac{1}{10}$; we then estimate each of these shifted integrals trivially using (A.6). Notice that the contour shifts do not cross poles, since by (1.15) the zeros of $h$ cancel the poles of $\cosh(\pi t)^{-1}$. This gives

$$x^j \frac{d^j}{dx^j} H(x) \ll_j x^j \int_{0}^{\infty} \frac{x^{2C_1-j-1/5}}{(1 + t)^{2C_1-j+3/10}} (1 + t)^{-C_1} dt \ll x^{2C_1-1/3} \ll x^{C_2},$$

proving (4.5). The assertions for $\tilde{H}(u)$ follow from (4.5) and integration by parts. □
Now let $C_2 \geq 10$. Applying (4.5) with $j = 0$, we see that (4.3) is absolutely convergent in $3/4 < \Re s < 1$. We now prepare for the second step, the balanced Voronoi formula in the form of Theorem 4. Using Mellin inversion we first write

$$\mathcal{M}^-(s; h) = \int_{\Re u = v} \hat{H}(u) \sum_{n,m,c > 0} a_1(n,m,1) \left( \frac{\pi n}{c} \right)^{1/2 + s} c S(-n,1,c) \left( 4\pi \sqrt{\frac{n}{c}} \right)^{-u} \frac{du}{2\pi i}.$$  

This multiple sum/integral is absolutely convergent if $-C_2 < v < -1/2$ and $\Re(s + \frac{1}{2}) > 1/2$, a region which is nonempty if $3/4 < \Re s < 1$. Using the functional equation of Theorem 4 we continue our calculation as follows:

$$\mathcal{M}^-(s; h) = \int_{\Re u = v} \hat{H}(u)(4\pi)^{-u} \frac{1}{2} \left[ \mathcal{V}_0 \left( \frac{u}{2} + \frac{1}{2} + s \right) - \mathcal{V}_1 \left( \frac{u}{2} + \frac{1}{2} - s \right) \right] \frac{du}{2\pi i}$$

$$= \int_{\Re u = v} \hat{H}(u)(4\pi)^{-u} \frac{1}{2} \left[ \mathcal{G}_0 \left( \frac{1 - u}{2} - s \right) \mathcal{V}_0 \left( \frac{1 - u}{2} + s \right) \right. \left. - \mathcal{G}_1 \left( \frac{1 - u}{2} - s \right) \mathcal{V}_1 \left( \frac{1 - u}{2} + s \right) \right] \frac{du}{2\pi i}$$

$$= \frac{1}{2} \int_{\Re u = -v - 4\Re s} \left( \mathcal{H}_0(u; s) \mathcal{V}_0 \left( \frac{u}{2} + \frac{1}{2} + s \right) - \mathcal{H}_1(u; s) \mathcal{V}_1 \left( \frac{u}{2} + \frac{1}{2} + s \right) \right) \frac{du}{2\pi i},$$

with

$$\mathcal{H}_\eta(u; s) := \hat{H}(-u - 4s)(4\pi)^{u + 4s} \mathcal{G}_\eta \left( \frac{1 - u}{2} + s \right),$$

for $\eta = 0$ or 1. By (2.13) and Lemma 2, this function is holomorphic in

$$\{ (u, s) \in \mathbb{C}^2 \mid -C_2 < \Re(u + 4s) < C_2, \Re(u + 2s) > 0, \text{ and } 3/4 < \Re s < 1 \},$$

and by (2.14) and again Lemma 2 it satisfies the estimate

$$\mathcal{H}_\eta(u; s) \ll (1 + |u|)^{-C_2 + \Re(4s + 2a)}$$

in this region.

Having applied Voronoi summation in (4.6), we now prepare for the third and final step: the application of the Kuznetsov formula in the reverse direction. Denote by

$$\phi_\eta(x; s) = \int_{\Re u = v} \mathcal{H}_\eta(u; s)(4\pi)^u x^{-u} \frac{du}{2\pi i}$$

the (slightly renormalized) inverse Mellin transform, where

$$-2\Re s < v < \frac{1}{2}(C_2 - 2\Re s - \frac{1}{2})$$

in which case the integrand is holomorphic and the integral is absolutely convergent. Likewise, differentiation under the integral sign gives an absolutely convergent, holomorphic integral for $x^j \phi_\eta^{(j)}(x; s)$, $j = 0, 1, 2, \ldots$, in the smaller range

$$-2\Re s < v < \frac{1}{2}(C_2 - j - 1) - 2\Re s.$$  

Shifting the contour shows that $x^j \phi_\eta^{(j)}(x; s) = O(x^{-v})$ for any such $v$. Since $3/4 < \Re s < 1$ and $C_2 - j - 1 - 4\Re s > 2$ for $0 \leq j \leq 3$, this easily implies the $x^j \phi_\eta^{(j)}(x; s) \ll \min(x, x^{-3/2})$ condition for $\phi_\eta(\cdot; s)$ to be admissible in the Kuznetsov formula (2.18). Thus for $v$ satisfying (4.9),

$$\int_{\Re u = v} \mathcal{H}_\eta(u; s) \mathcal{V}_\eta \left( \frac{u}{2} + \frac{1}{2} + s \right) \frac{du}{2\pi i} = \sum_{n,m,c > 0} \sum_{\kappa = \pm 1} \kappa^n a_1(n,m,1) \left( \frac{\pi n}{c} \right)^{1/2 + s} c S(\kappa n,1,c) \phi_\eta \left( 4\pi \sqrt{\frac{n}{c}}, s \right).$$
by (1.19). Inserting into (4.6), we find

\[
\mathcal{M}^-(s; h) = \frac{1}{2} \sum_{n \neq \pm 1, m \neq 0} \frac{a(n, m)}{(nm)^{1/2+\varepsilon}} S(\varepsilon, 1, c) \left( \phi_0 \left( \frac{4\pi \sqrt{n}}{c}; s \right) - \kappa \phi_1 \left( \frac{4\pi \sqrt{n}}{c}; s \right) \right)
\]

as an absolutely convergent expression. Applying (2.18) with the test function \( \phi_0(r; s) - \kappa \phi_1(r; s) \) gives

\[
\mathcal{M}^-(s; h) = \mathcal{M}^+(s; h) + \mathcal{M}^{\text{hol}}(s; h^\text{hol}) + \mathcal{M}^-(s; h^-),
\]

where

\[
h_s^\pm(r) := \frac{1}{2} \int_0^\infty \mathcal{J}_r^\pm(x) (\phi_0(x; s) \pm \phi_1(x; s)) \frac{dx}{x}
\]

for small \( v > 0 \), where the Mellin transform \( \phi_0(r; s) \) is taken with respect to the first variable. In fact, applying (2.14) and (4.7)–(4.8) gives the bounds

\[
\hat{\phi}_0(-u; s) \pm \hat{\phi}_1(-u; s) = \hat{H}(u - 4s)(4\pi)^{-2u+4s} \left( \mathcal{G}_0 \left( \frac{1-u}{2} + s \right) \mp \mathcal{G}_1 \left( \frac{1-u}{2} + s \right) \right)
\]

with an implied constant that depends locally uniformly on \( \Re u \) and throughout the region \( 4\Re s - C_2 < \Re u < 2\Re s + 2\delta \) (in which the left hand side is holomorphic by (2.13)).

The formula (1.16) would follow as the \( s = 0 \) case of (4.10), were it not for the fact that our present analytic continuation of that formula holds only for \( \Re s > 3/4 \). We shall now continue its terms to a right half plane containing \( s = 0 \). An application of (A.7) and Stirling’s formula shows that the integrand in (4.11) is

\[
\ll (1 + |u|)^{-4\Re s - 2|\Re u| - C_2},
\]

with an implied constant that depends locally uniformly on \( \Re u, s, \) and \( r \). Thus (4.11) is valid for \( 0 < v < 2\Re s + 2\delta \). Therefore taking \( v = \delta \) shows that (4.11) provides an absolutely and locally uniformly convergent expression for \( h_s^\pm(r) \) as a holomorphic function in \( \{ s \mid -\delta/2 < \Re s < 1 \} \). In particular, \( h_s^\pm(r) \) is bounded for \( r \leq 1 \), locally uniformly in \( s \) in that range.

We next show that \( h_s^\pm(r) \) decays sufficiently rapidly for the convergence of the sums \( \hat{M}^\pm(s; h_s^\pm) \) in (4.10). Let \( n \) be an odd positive integer and \( r \geq 1 \). If \( C_2 > 4\Re s + n \), we can shift the contour in (4.11) to \( \Re u = -n \), picking up residues at \(-2m \pm 2ir, m = 0, 1, \ldots, (n - 1)/2\), from the poles of \( \tilde{\mathcal{J}}_r^\pm(u) \): they contribute

\[
\ll r^{-1/2-C_2+4\Re s+3m},
\]

with an implied constant that depends on \( n \) and locally uniformly on \( s \). To estimate the remaining integral over \( \Re u = -n \), we use the bound

\[
\tilde{\mathcal{J}}_r^\pm(u) \ll \left( (1 + |3u + 2r|)(1 + |3u - 2r|) \right)^{-(n+1)/2}
\]

which follows from (A.7) and Stirling’s formula; trivially applying this and (4.12) to the contributions from the three ranges \( |3u| < 2r \), \( r < |3u| \leq 4r \), and \( 4r < |3u| \) gives the estimate

\[
\ll r^{4\Re s + n - C_2} + r^{4\Re s + 2n - C_2 - (n+1)/2} \int_r^{4r} (1 + |y - 2r|)^{-(n+1)/2} dy + r^{4\Re s + n - C_2}.
\]

This last integral is bounded in \( r \), so for any fixed \( A > 0 \) we may choose \( C_2 \) sufficiently large (keeping \( n \) fixed) to arrange that \( h_s^\pm(r) = O(|r|^{-A}) \) as \( r \to \infty \). By (2.10), this ensures that the spectral sums
\( \hat{M}(s, h^\pm) \) defined in (4.1) are absolutely and locally uniformly convergent in \(-\delta/2 < \Re s < 1\), as long as \(C_2\) is taken sufficiently large.

The above argument simplifies for

\[
(4.16) \quad h_{s}^\text{hol}(k) = \frac{1}{2} \int_{\Re u = 0} \hat{f}_k^\text{hol}(u) \tilde{H}(u - 4s)(4\pi)^{-2u+4s} \left( G_0 \left( \frac{1-u}{2} + s \right) - G_1 \left( \frac{1-u}{2} + s \right) \right) \frac{du}{2\pi i}.
\]

Indeed, (4.13) remains true for this integrand, from which we again conclude the holomorphic continuation of \(h_{s}^\text{hol}(k)\) to \(-\delta/2 < \Re s < 1\). For the decay in \(k\), let \(n\) be a fixed positive integer. Since \(\hat{f}_k^\text{hol}(u) = i^k 2^u \pi \Gamma(\frac{1}{2}(u + k - 1)) \Gamma(\frac{1}{2}(1 + k - u))^{-1}\) has no poles for \(\Re u > -n\) when \(k \geq n + 2\), we may shift the contour to the line \(\Re u = -n\). Here \(\Gamma(\frac{1}{2}(u + k + 2n + 1)) = p_{n+1}(u + k) \Gamma(\frac{1}{2}(u + k - 1))\), where \(p_{n+1}(x) = 2^{n-1}(x-1)(x+1)\cdots(x+2n-1)\) is a degree \((n+1)\) polynomial. Hence for fixed \(n\) and \(\Re u = -n\) we have

\[
\hat{f}_k^\text{hol}(u) \ll \frac{\Gamma(\frac{1}{2}(u + k - 1))}{\Gamma(\frac{1}{2}(-u + k + 1))} \ll (k + |3u|)^{-n-1},
\]

instead of the earlier bound (4.14). This shows that for any fixed \(A > 0\) we may choose \(C_2\) sufficiently large to arrange that \(h_{s}^\text{hol}(k) = O(k^{-A})\) as \(k \to \infty\), and in particular ensure the absolute and locally uniform convergence of \(\hat{M}^\text{hol}(s, h_{s}^\text{hol})\).

This completes the analytic continuation of (4.10) to \(s = 0\). We obtain the spectral reciprocity formula of Theorem 3 with \(h^\diamond = h_{s}^\text{hol}\) using (4.11)-(4.12), i.e.,

\[
K^+(t, r) = 8\pi^2 \int_{\Re u = 0} \hat{f}_k^+(u) \hat{f}_k^-(u)(4\pi)^{-2u} \left( G_0 \left( \frac{1-u}{2} + s \right) - G_1 \left( \frac{1-u}{2} + s \right) \right) \frac{du}{2\pi i},
\]

(4.17)

(recall (1.22)) and

\[
K^\text{hol}(t, k) = K^+ \left( t, \frac{k - 1}{2t} \right).
\]

By (A.7) this matches the definitions (1.13) and (1.14).

It remains to prove the bound (1.17) for \(K^+\). Recalling definition (1.11), we see that

\[
\prod_{j \leq 4} \mathcal{E}_{\beta_j}(u + \mu_j) - \prod_{j \leq 4} \mathcal{E}_{\beta_{j+1}}(u + \mu_j) \ll e^{\pi |3u|}
\]

instead of the trivial bound \(e^{2\pi |3u|}\), where the second condition in (1.10) is used to match and then cancel the powers of \(i\) in the leading terms on the left hand side. Thus we have the estimate

\[
G^+(u) \ll e^{-\pi |3u|} (1 + |u|)^{4\Re u - 2}
\]

for (1.12) in fixed vertical strips (see (2.14)). Shifting the \(u\)-contour in (4.17) to \(\Re u = -1\), the contribution of the residues at \(u = \pm 2it\) is

\[
\ll e^{\pi |t| - \frac{1}{2}(2|t| + |t-r| + |t+r|)} ((1 + |t|)(1 + |t-r|)(1 + |t+r|))^{-1/2} \ll e^{-\pi \max(|t|, |r|)}
\]

and the contribution of the residues at \(u = \pm 2ir\) is

\[
\ll e^{\pi |t| - \frac{1}{2}(2|r| + |t-r| + |t+r|)} ((1 + |r|)(1 + |t-r|)(1 + |t+r|))^{-1/2} \ll e^{-|r|}(1 + |t|)^{-1}.
\]

The remaining contour integral over \(u = -1 + 2iw\) is

\[
\ll \int_{-\infty}^{\infty} \frac{(1 + |w|)^2}{(1 + |w^2 - t^2|)(1 + |w^2 - r^2|)} e^{\frac{1}{2}(|w^2 - t^2| - |w^2 - r^2| - |w^2 - r^2|)} dw
\]

\[
\ll \int_{-\infty}^{\infty} \frac{(1 + |w|)^2}{(1 + |w^2 - t^2|)(1 + |w^2 - r^2|)} e^{-\frac{1}{2}(|w^2 - t^2| - |w^2 - r^2|)} dw \ll e^{-\frac{|r|}{(1 + |t|^2)(1 + |r|^2)}}.
\]
This completes the proof of Theorem 3.

**Remark 1:** While \( h^-(r) \) can be arranged to decay to any fixed polynomial order, the kernel \( K^-(t, r) \) itself is not rapidly decaying in \( r \); for fixed \( t \) it is of order of magnitude \((1 + |r|)^{-1}\), which by Weyl’s law does not suffice to make the spectral \( r \)-sum absolutely convergent (not even under GRH for the corresponding central \( L \)-values). It is the extra integration over \( t \) together with the regularity assumptions (1.15) on the test function \( h \) that gives adequate decay properties of \( h^- \). On the other hand, the bound (1.17) shows that the decay of \( h^+ \) happens already on the level of \( K^+ \).

**Remark 2:** The numerical value \( C_1 \geq 40 \) in Theorem 3 arises as follows: choosing \( n = 5 \) and \( C_2 = 12 + 4\Re s \), we obtain from (4.15) that \( h^-\Rightarrow(r) < (1 + |r|)^{-5} \), which by Weyl’s law and (2.10) makes the spectral sum absolutely convergent. For \( 0 \leq \Re s \leq 1 \), we conclude that \( C_2 = 16 \) is admissible and hence \( C_1 = 34 \) by Lemma 2.

5. **INTERLUDE: DESCRIPTION OF THE KERNEL**

The proof of Theorem 3 in Section 4 shows that the passage \( h \mapsto h^\circ \) arises from three steps.

**Step 1 [Kuznetsov (2.17)]:** Define

\[
H(x) := \int_{-\infty}^{\infty} \mathcal{J}_t^-(x)h(t)d_{spect}.
\]

**Step 2 [Voronoi (1.20)]:** Let \( \hat{H}(u) \) denote the Mellin transform of \( H \) and define

\[
\mathcal{H}^\pm(u) := \hat{H}(-u)^{4u}g^\pm\left(\frac{1 + u}{2}\right).
\]

**Step 3 [Kuznetsov in reverse direction (2.18)]:** Let \( \phi^\pm(x) \) denote the inverse Mellin transform of \( \mathcal{H}^\pm(u) \). Then

\[
h^\pm(r) = \frac{1}{8\pi^2} \int_0^\infty \mathcal{J}_r^\pm(x)\phi^\pm(x)\frac{dx}{x}
\]

and

\[
h^\text{hol}(k) = \frac{1}{8\pi^2} \int_0^\infty \mathcal{J}_k^\text{hol}(x)\phi^+(x)\frac{dx}{x}.
\]

This follows from (4.4), (4.11), (4.12), (4.16) with \( s = 0 \), and (1.22).

For completeness we give two additional, alternative expressions for \( K^-(t, r) \) (similar expressions hold for \( K^+(t, r) \) in general). Let

\[
g(x) := \int_{\mathcal{C}} \mathcal{G}^-(u)x^{-u}\frac{du}{2\pi i},
\]

where the contour \( \mathcal{C} \) is taken to the right of \( \Re u = 1/2 - \delta \) for \( u \) small (to avoid poles – see (2.13)), and a bounded distance to the left of \( \Re u = 1/4 \) for \( u \) large (to converge absolutely – see (2.14)).

Then starting from (4.17),

\[
K^-(t, r) = \int_{\Re u = \delta} \mathcal{G}^-(\frac{1 - u}{2})\hat{\mathcal{J}}_t^-(u)\hat{\mathcal{J}}_r^- (u)4^{-u}\frac{du}{2\pi i}
\]

\[
= 2 \int_{\Re z = \frac{1}{40}} \mathcal{G}^-(z)\hat{\mathcal{J}}_t^- (1 - 2z)\hat{\mathcal{J}}_r^- (1 - 2z)4^{2z-1}\frac{dz}{2\pi i}
\]

\[
= 2 \int_{\mathcal{C}} \int_0^\infty \int_0^\infty \mathcal{J}_t^-(x)\mathcal{J}_r^- (y)(xy)^{-2z}dx\,dy\,\mathcal{G}^-(z)4^{2z-1}\frac{dz}{2\pi i}
\]

\[
= \frac{1}{2} \int_0^\infty \int_0^\infty \mathcal{J}_t^-(x)\mathcal{J}_r^- (y)\frac{(xy)^2}{16}\,dx\,dy,
\]

where all integrals are absolutely convergent (as can be seen from standard bounds on \( \mathcal{J}_t^- \) and \( \hat{\mathcal{J}}_t^- \) along the lines of (A.3) and (A.7)). In practice, one may want to asymptotically evaluate \( g(x) \) as in [Mi, (4.11)] or [Li1, Lemma 6.1] (e.g., using stationary phase).
Alternatively, one can shift the u-contour far to the left and pick up the residues at \(-2n \pm 2it\) and \(-2n \pm 2ir\) for \(n \in \mathbb{Z}_{\geq 0}\), which can be written as a sum of quotients having four \(\Gamma\)-factors in the numerator and three \(\Gamma\)-factors in the denominator. Hence the sum of residues can be evaluated in terms of \(qF_3\) hypergeometric functions [GR, (9.14)]. We now describe this in the special case that the \(\beta_j\) from (1.10) all vanish, the other cases being similar. Let

\[
K(t, r) := 8 \cosh(\pi t) \cosh(\pi r) \Gamma(2it) \Gamma(i(r + t)) \Gamma(i(r - t)) \prod_{j=1}^{4} \Gamma(\frac{1}{2} + \mu_j - ir)
\]

\[
\times \left[ \cos(\pi(\mu_2 + \mu_3)) + \cos(\pi(\mu_2 + \mu_4)) + \cos(\pi(\mu_3 + \mu_4)) - \cosh(2\pi r) \right]
\]

\[
\times 4 F_3 \left( \frac{1}{2} + \mu_1 - ir, \frac{1}{2} + \mu_2 - ir, \frac{1}{2} + \mu_3 - ir, \frac{1}{2} + \mu_4 - ir, 1 \right)
\]

for \(r, t \in \mathbb{R}, rt(t^2 - r^2) \neq 0\). With this notation, we have

\[
\mathcal{K}^{-}(t, r) := K(t, r) + K(t, -r) + K(r, t) + K(r, -t)
\]

if \(rt(t^2 - r^2) \neq 0\) (otherwise, \(\mathcal{K}^{-}(t, r)\) is defined by continuity).

6. ASYMPTOTIC ANALYSIS AND PROOF OF THEOREM 1

In this section we prove Theorem 1. Let \(h_T\) be as in (1.4), with \(T\) very large and \(D \geq 50\) fixed. We recall the definitions (1.3) and (1.9) of \(\mathcal{M}^{\diamond}\) for \(\diamond \in \{+, -, \text{hol}\}\), as well as the reciprocity formula (1.16) (where we use \(\widetilde{\mathcal{M}}^{\diamond}\) to denote the analogous quantities for the dual representation \(\widetilde{\Pi}\)). The bound (1.17) together with (2.10) shows that trivially

\[
|\widetilde{\mathcal{M}}^{\diamond}(h_T^\pm)| \ll \int_{-\infty}^{\infty} h_T(t)(1 + |t|)^{-1} dt \ll T.
\]

By (1.16) and (2.10) it suffices to show

**Lemma 3.** If \(D\) in (1.4) is sufficiently large, then

\[
h_T(r) \ll T \min(|r|^{-1}, |r|^{-5})
\]

for \(r \in \mathbb{R}\) and

\[
h_T^{\text{hol}}(k) \ll T k^{-5}
\]

for integers \(k \geq 2\).

The rest of this section is devoted to the proof of Lemma 3. We follow the three steps of Section 5, beginning with an analysis of (4.4),

\[
H_T(x) = \int_{-\infty}^{\infty} \mathcal{J}_T^\diamond(x) h_T(t) dt = 4 \int_{-\infty}^{\infty} K_{2it}(x) \sinh(\pi t) h_T(t) t \frac{dt}{2\pi^2},
\]

in the \(T\)-aspect. As an analogue of Lemma 2 we obtain the following result whose proof uses similar ideas as [BK, Lemma 3.8].

**Lemma 4.** Let \(j \in \mathbb{Z}_{\geq 0}\) and suppose that \(D \geq \max(T, j)\). Then

\[
x^j \frac{d^j}{dx^j} H_T(x) \ll_{D, j} T \min \left( \left( \frac{x}{T} \right)^{D/2}, \left( \frac{x}{T} \right)^{-D/2} \right).
\]

**Proof.** Fix \(D, j\), and three positive integers \(A_1, A_2, A_3\) (which will be chosen later); all implied constants in the proof may depend on these parameters. We argue separately in three ranges for \(x\).

**Range I: \(x \leq 1\).** We proceed similarly as in the proof of Lemma 2 by applying (A.4) and differentiating under the integral sign using (A.2). In those terms containing \(I_{2it+n}(x)/\cosh(\pi t)\) for \(|n| \leq j\) we shift the contour to \(3t = -D\), while in those terms containing \(I_{-2it+n}(x)/\cosh(\pi t)\) for \(|n| \leq j\)
we shift the contour to \( \Im t = D \). The polynomial \( P_T \) in the definition (1.4) ensures that no poles are crossed during the contour shifts. Now we estimate trivially using (A.6) and obtain

\[
x^j \frac{d}{dx} H_T(x) \ll x^j \int_0^\infty e^{-t^2/T^2} \left( 1 + \frac{t}{T} \right)^{4D} \frac{x^{2D-j}}{(1 + t)^{2D-j+1/2}} dt \ll \frac{x^{2D}}{T^{2D-j-3/2}} \lesssim T \left( \frac{x}{T} \right)^{D/2}
\]

for \( x \leq 1 \) and \( D \geq \max(7, j) \).

Range II: \( 1 \leq x \leq T^{13/12} \). Let

\[
h_{\text{spec}}(t) := h_T(t) \frac{d_{\text{spec}} t}{dt} = \frac{1}{2\pi^2} h_T(t) t \tanh(\pi t),
\]

which has Fourier transform

\[
\hat{h}_{\text{spec}}(v) = \int_0^\infty h_{\text{spec}}(t) e^{-itv} dt = \frac{T^2}{2\pi^2} \int_{-\infty}^\infty e^{-t^2} P_T(tT) \tanh(\pi t) e^{-ivt} dt.
\]

We note that

\[
\frac{d^n}{dt^n} P_T(tT) \ll n T^{-4D+n} (1 + |tT|)^{4D-n} \quad (n \geq 0),
\]

and clearly \( \frac{d^n}{dv} e^{-t^2} t^{j+1} \ll_{n,j} e^{-t^2} (1 + |t|)^{n+j+1} \), so that by the Leibniz rule

\[
\frac{d^n}{dt^n} e^{-t^2} P_T(tT) \tanh(\pi T) t \ll e^{-t^2} (1 + |t|)^{n+1} T^{-4D+n} (1 + |tT|)^{4D-n}
\]

for all integers \( n, j \geq 0 \). Applying \( \frac{d}{dv} \) and then integrating by parts \( A_1 \) times, we have

\[
\hat{h}_{\text{spec}}^{(j)}(v) \ll \frac{T^{2+j}}{(T|v|)^{A_1}} \int_{-\infty}^\infty e^{-t^2} (1 + |t|)^{A_1+j+1} T^{-4D+A_1} (1 + |tT|)^{4D-A_1} dt \ll \frac{T^{2+j}}{(T|v|)^{A_1}}
\]

for any fixed \( j \geq 0 \) and any \( A_1 \leq 4D \). Combining this with the case \( A_1 = 0 \), we deduce

\[
\hat{h}_{\text{spec}}^{(j)}(v) \ll T^{2+j} e^{-D|v|}
\]

for \( j \geq 0 \) and \( A_1 \leq 4D \). Alternatively, we apply \( \frac{d}{dv} \) to (6.2) and then shift the contour to \( \Im t = D \) (if \( v < 0 \)) or \( \Im t = -D \) (if \( v > 0 \)); this crosses no poles because \( h_T \) vanishes at the poles of \( \tanh(\pi t) \) in this region. Therefore

\[
\hat{h}_{\text{spec}}^{(j)}(v) \ll T^{2+j} e^{-D|v|}
\]

(as is also clear from (1.4) and Paley-Wiener theory).

Having the bounds (6.3) and (6.4) available, we shall now derive an alternative expression for \( H_T(x) \) and its derivatives (in (6.6) below) that will be easier to analyze since it features the highly-localized Fourier transform \( \hat{h}_{\text{spec}} \) instead of \( h_T \) itself. Applying (A.8) shows

\[
H_T(x) = \int_{-\infty}^\infty J_T^{-}(x) h_{\text{spec}}(t) dt = \int_{-\infty}^\infty e^{ix \sinh(v/2)} \hat{h}_{\text{spec}}(v) dv.
\]

Since

\[
x \frac{d}{dx} e^{ix \sinh(v/2)} = 2 \tanh(v/2) \frac{d}{dv} e^{ix \sinh(v/2)},
\]

we can combine each differentiation in \( x \) with an integration by parts to obtain

\[
\left( x \frac{d}{dx} \right)^j H_T(x) = \int_{-\infty}^\infty e^{ix \sinh(v/2)} D^j \hat{h}_{\text{spec}}(v) dv,
\]

where

\[
Df(v) := -\frac{d}{dv} (\tanh(v/2) f(v)).
\]
(these integrals all converge absolutely by (6.4)). The \( j \)-fold composition \( D^j \) can be expressed as a finite sum
\[
D^j = \sum_{n=0}^{j} \sum_{a \geq 0} \sum_{b \geq 0} \tilde{\rho}_{a,b,n} \text{sech}(v/2)^a \tanh(v/2)^b \frac{d^n}{dv^n},
\]
with \( \tilde{\rho}_{a,b,n} \in \mathbb{R} \). Since the differential operators \( \frac{d}{dx} \) and \( x \) have commutator \( \frac{d}{dx} x - x \frac{d}{dx} = 1 \), by moving all occurrences of \( \frac{d}{dx} \) as far to the right as possible we obtain the expression
\[
x^j \frac{d^j}{dx^j} H_T(x) = \int_{-\infty}^{\infty} e^{ix \sinh(v/2)} (p_j(D) \tilde{h}_{\text{spec}}(v)) dv,
\]
where \( p_j \) is a polynomial of degree \( j \) and
\[
p_j(D) = \sum_{n=0}^{j} \sum_{a \geq 0} \sum_{b \geq 0} \rho_{a,b,n} \text{sech}(v/2)^a \tanh(v/2)^b \frac{d^n}{dv^n}, \quad \rho_{a,b,n} \in \mathbb{R},
\]
is a finite sum. Using the boundedness of \( \text{sech}(v/2) \) we thus have that
\[
(p_j(D) \tilde{h}_{\text{spec}}(v)) \ll \sum_{c=0}^{j} |\tilde{h}_{\text{spec}}^{(c)}(v) \tanh(v/2)^c| \ll T^2(1 + T|v|)^{-A_1+j}
\]
for \( j + 2 \leq A_1 \leq 4D \) by (6.3), where in the first inequality we used the boundedness of \( \tanh(v/2) \) and in the second inequality we instead used the bound \( \tanh(v/2) \ll v \).

In light of the estimate (6.8), we can truncate the \( v \)-integral (6.6) at \( |v| \leq T^{-3/4} \) at the cost of an error
\[
\ll T^2 \int_{T^{-3/4}}^{\infty} (1 + Tv)^{-A_1+j} dv \ll T^{2+j-A_1-3/4}.
\]
From now on we assume \( |v| \leq T^{-3/4} \) and approximate the integrand in (6.6) by various Taylor expansions in a neighbourhood of \( v = 0 \). Since \( x \leq T^{13/12} \), we have \( |xv^3| \leq T^{-7/6} \), so we can write
\[
e^{ix \sinh(v/2)} = e^{ixv/2} \sum_{\beta=0}^{A_2} \frac{(ix)^\beta (\sinh(v/2) - v/2)^\beta}{\beta!} + O(T^{-7(A_2+1)/6})
\]
for any fixed positive integer \( A_2 \). Now writing
\[
(\sinh(v/2) - v/2)^\beta = \sum_{\alpha = 3\beta}^{3A_2} \tilde{c}_{\alpha,\beta} v^\alpha + O(T^{-3(3A_2+1)/4})
\]
for certain constants \( \tilde{c}_{\alpha,\beta} \), we obtain
\[
e^{ix \sinh(v/2)} = e^{ixv/2} \sum_{\alpha = 0}^{3A_2} \frac{[\alpha/3]}{\alpha!} \sum_{\beta=0}^{\alpha/3} \tilde{c}_{\alpha,\beta} x^\beta v^\alpha + O(T^{-7(A_2+1)/2})
\]
with constants \( \tilde{c}_{\alpha,\beta} \) (recalling that \( x \leq T^{13/12} \)).

Similarly, expanding \( \text{sech}(v/2)^a \tanh(v/2)^b \) about \( v = 0 \), we see from (6.3) and (6.7) that
\[
(p_j(D) \tilde{h}_{\text{spec}}(v)) = \sum_{n=0}^{j} \left[ \left( \sum_{\gamma=n}^{A_3} d_{n,\gamma} \gamma \right) \tilde{h}_{\text{spec}}^{(n)}(v) + O\left(T^{-3(A_3+1)/4}T^{2+n}(1 + T|v|)^{-A_1}\right) \right],
\]
where \( d_{n,\gamma} \) are constants. Recalling (6.9) and substituting (6.10) and (6.11) into the truncated version of (6.6), we conclude that \( x^j H_T^{(j)}(x) \) equals
\[
\sum_{\alpha=0}^{3A_2} \frac{[\alpha/3]}{\alpha!} \sum_{\beta=0}^{\alpha/3} c_{\alpha,\beta} x^\beta \sum_{n=0}^{j} d_{n,\gamma} \int_{-T^{-3/4}}^{T^{-3/4}} e^{ixv/2} v^\alpha \gamma \tilde{h}_{\text{spec}}^{(n)}(v) dv + R_1 + R_2 + O(T^{2+j-A_1-3/4}),
\]
where

\[ R_1 \ll T^{-\frac{7}{4}A_2 - \frac{3}{4}} \sum_{n=0}^{\infty} \int_{T^{-3/4}}^{T^{3/4}} T^{2+n} |v|^3 (1 + T|v|)^{-A_1} dv \ll T^{-\frac{7}{4}A_2 + \frac{1}{4}} \]

accounts for the contribution of the error term in (6.10) and the main term in (6.11) (using (6.3), \(|v| \leq T^{-3/4} \leq 1\), and the assumption \(A_1 \geq j + 2\), and

\[ R_2 \ll T^{-3(A_3+1)/4} T^{2+j} \int_{T^{-3/4}}^{T^{3/4}} (1 + T|v|)^{-A_1} dv \ll T^{-3(A_3+1)/4} T^{1+j} = T^{(1-3A_3+4j)/4}. \]

accounts for the contribution of (6.10) and the error term in (6.11).

We now complete the integral in (6.12) to \((-\infty, \infty)\) at the cost of introducing an error (using (6.3) and \(x \leq T^{13/12}\))

\[ \ll \sum_{\alpha=0}^{3A_2} T^{13\alpha/36} \sum_{n=0}^{A_3} \sum_{\gamma=n}^{\infty} \frac{T^{\alpha+\gamma} T^{2+n} (Tv)^{-A_1} dv}{T^{3/4}}, \]

provided \(A_1 \geq 3A_2 + A_3 + 2\). (Note that this error coincides with the error (6.9) from truncating (6.6).) We now choose \(A_1\) to be as large as possible, and \(A_2, A_3\) to simultaneously nearly get equality in this last condition and to nearly equalize 1 (6.13) and (6.14):

\[ A_1 = 4D, \quad A_2 = \left[ \frac{1}{4} (6D - 2j - 3) \right], \quad \text{and} \quad A_3 = \left[ \frac{2}{21} (14D + 9j - 7) \right]. \]

With these choices the three error terms in (6.12) contribute

\[ \ll T^{\frac{4D}{13} - \frac{1}{2} D + \frac{T}{4} + \frac{T}{4} - D} + T^{\frac{4D}{13} - \frac{1}{2} D} \ll T^{2 + \frac{T}{4} - D} \ll T \min \left( \frac{x}{T} \right)^{-D/2}, \left( \frac{x}{T} \right)^{D/2} \]

for \(1 \leq x \leq T^{13/12}\) and \(\max(2, j) \leq D\).

The main term, i.e., the expression (6.12) but with integration extended over the real line, is by Fourier inversion a linear combination of terms of the form

\[ x^\beta \frac{d^{\alpha+\gamma}}{dx^{\alpha+\gamma}} \left( x^n h_{\text{spec}}(x/2) \right) \ll \frac{x^\beta}{T^{4D}} e^{-x^2/T^2} x^{4D+n+1} \left( \frac{x}{T^2} + \frac{1}{x} \right)^{\alpha+\gamma} \]

with \(\alpha \leq 3A_2\), \(\beta \leq \alpha/3\), \(n \leq j \leq D\), \(n \leq \gamma \leq A_3\) (where we have used that \(x \geq 1\) in obtaining this bound). If \(1 \leq x \leq T\) this is \(\ll (x/T)^{4D}\), while if \(T \leq x \leq T^{13/12}\) this is

\[ \ll e^{-x^2/T^2} x \left( \frac{x}{T} \right)^{4D} \frac{x^\beta + \alpha + n + \gamma}{T^{2(\alpha + \gamma)}} \ll e^{-x^2/T^2} x \left( \frac{x}{T} \right)^{4D} \frac{x^\beta + \alpha + \gamma + n}{T^{2(\alpha + \gamma)}} \]

\[ \ll e^{-x^2/T^2} x \left( \frac{x}{T} \right)^{4D+2\alpha + 2\gamma} \ll Te^{-x^2/T^2} \left( \frac{x}{T} \right)^{4D+6A_2+2A_3+1} \ll Te^{-x^2/T^2} \left( \frac{x}{T} \right)^{12D}, \]

and (6.1) follows from the rapid decay of the exponential.

Range III: \(x \geq T^{13/12}\). We use the rapid decay of the Bessel K-function: by (A.1) and (A.3) and splitting the integral at \(|t| = \frac{T}{2x}\), we have

\[ x^j \frac{d^j}{dx^j} H_T(x) \ll x^j \int_{-\infty}^{\infty} e^{-x^2/T^2} \frac{1}{T} e^{-|t|} \left( \frac{|t| + x}{x} \right)^{j+1/10} |t| dt \]

\[ \ll x^j \left( \frac{x}{T} \right)^{4D} e^{-2x/3x^2} + x^j \int_{|t| > x/(3T)} e^{-x^2/T^2} \left( \frac{|t|}{x} \right)^{4D} |t| dt \]

\[ \ll e^{-x^2/2} + T^{j+3} x^{-1} e^{-x^2/(10T)^2} \ll e^{-x^2/2} + x^{j+2} e^{-x^2/(10T)^2}, \]

\(1\)The expressions given here have simpler fractions than the actual equalizers.
which is certainly dominated by the desired bound $T(x/T)^{-D/2}$. This completes the proof of Lemma 4. \qed

\textbf{Proof of Lemma 3.} As in the proof of Lemma 2, the estimates of Lemma 4 imply that the Mellin transform $\tilde{H}_T(u)$ is holomorphic in $-D/2 < R u < D/2$; moreover taking $j = D$ in (6.1) implies the bound

\[ \tilde{H}_T(u) \ll R u \, T^{1+Ru} (1+|u|)^{-D} \]

in this region (using integration by parts). Therefore $\mathcal{H}_T^\pm(u) = \tilde{H}_T(-u) 4^n G^\pm(\frac{1+u}{2})$ from (5.1) is holomorphic in $-2\delta < R u < D/2$ by (2.13), and satisfies the bound

\[ \mathcal{H}_T^\pm(u) \ll R u \, T^{1-Ru} (1+|u|)^{-D+2Ru} \]

using (2.14). This in turn implies that the inverse Mellin transform $\phi_T^\pm$ of $\mathcal{H}_T^\pm$ satisfies

\[ x^j \frac{d^j}{dx^j} \phi_T^\pm(x) \ll x^{-Ru} \int_\mathbb{R} |\mathcal{H}_T^\pm(v+i|t|)(1+|v+it|)^j|dt \ll T \min \left((xT)^\delta, (xT)^{-D/3}\right) \]

upon choosing $v = -\delta$ or $v = D/3$, provided that $j < \frac{1}{3} D - 1$ (to ensure integrability).

It remains to bound $h_T^\pm(r)$ and $h_T^{\text{hol}}(k)$. The former was shown in (5.2) to equal

\[ h_T^\pm(r) = \frac{1}{8\pi^2} \int_0^\infty J_r^-(x) \phi_T^\pm(x) \frac{dx}{x}, \]

where we recall from (2.15) and (A.4) that

\[ J_r^-(x) = 4 \cosh(\pi r) K_{2ir}(x) = \pi i \frac{I_{2ir}(x) - I_{-2ir}(x)}{\sinh(\pi r)} . \]

If $r \leq 1$ and $x \leq 1$, the $I$-Bessel functions in the numerator are bounded by (A.6), while if $r \leq 1$ and $x > 1$ the estimate (A.3) shows $J_r^-(x)$ is bounded. In either case $J_r^-(x) \ll 1/r$, so that

\[ h_T^\pm(r) \ll \frac{1}{r} \int_0^\infty |\phi_T^\pm(x)| \frac{dx}{x} \ll \frac{T}{r}, \]

where we have used (6.15) with $j = 0$.

For $r \geq 1$ we apply a smooth partition of unity to (6.16) and write $h_T^\pm(r) = I_1(r) + I_2(r)$, where

\[ I_1(r) := \int_0^\infty J_r^-(x) \phi_T^\pm(x) \left( \frac{x}{r^{1/3}} \right) \frac{dx}{x} \quad \text{and} \quad I_2(r) := \int_0^\infty J_r^-(x) \phi_T^\pm(x) \left( 1 - w \left( \frac{x}{r^{1/3}} \right) \right) \frac{dx}{x}, \]

with $w$ a fixed, smooth function on $\mathbb{R}$ such that $w(x) \equiv 1$ on $[0,1]$ and $w(x) \equiv 0$ on $[2,\infty)$. We estimate $I_2(r)$ trivially using (6.15) and (A.3):

\[ I_2(r) \ll T \int_{r^{1/3}}^\infty \left( xT \right)^{-D/3} \left( \frac{r+x}{x} \right)^{1/10} \frac{dx}{x} \ll T^{1-\frac{D}{2}} r^\frac{1}{3} - \frac{\delta}{2} \ll T r^{-5} \]

for $D \geq 50$. For $I_1(r)$ we insert the power series expansion (A.5) to obtain

\[ I_1(r) \ll \sum_{k=0}^\infty \frac{1}{k!} \left| \int_0^\infty \frac{x^{\pm 2ir+k}}{\Gamma(\pm 2ir+k+1) \sinh(\pi r)} \phi_T^\pm(x) \left( \frac{x}{r^{1/3}} \right) \frac{dx}{x} \right| \]

\[ \ll \sum_{k=0}^\infty \frac{1}{k!} \left( \frac{r^{1/3}}{r^{1/3}} \right)^{k+2} \frac{1}{\Gamma(\pm 2ir+k+1) \sinh(\pi r)} \int_0^\infty \frac{x^{\pm 2ir+k}}{x^{1/3}} \phi_T^\pm(x) \left( \frac{x}{r^{1/3}} \right) \frac{dx}{x}, \]

where we have used $\Gamma(\pm 2ir+k+1) = (\pm 2ir+k) \cdots (\pm 2ir+1) \Gamma(\pm 2ir+1)$ and then Stirling’s formula to estimate the denominator. We then apply integration by parts five times, integrating $x^{\pm 2ir+k-1}$ and differentiating $\phi_T^\pm(x) w(x/r^{1/3})$. The latter gives

\[ \sum_{i+j=5} \left( \begin{array}{c} 5 \\ j \end{array} \right) \frac{d^i}{dx^j} \phi_T^\pm(x) \left( \frac{w^{(i)}(x/r^{1/3})}{x^{1/3}} \right). \]
Estimating trivially with (6.15) (which is valid for \( j \leq 5 \) if \( D > 18 \)) and keeping in mind the integral is supported on \( x \leq 2^{1/3} \), we find
\[
\mathcal{I}_1(r) \ll \sum_{k=0}^{\infty} \frac{1}{r^{k+1/2}k!} \int_0^{2^{1/3}} r^{-5}(2r^{1/3})^{2k} \max_{0 \leq j \leq 5} |x^j \frac{d^j}{dx^j} \phi_T(x)| \frac{dx}{x} \ll \frac{T}{r^{5/3}}.
\]

This completes the analysis of \( h_T^{-}(r) \).

We now turn to \( h_T^{\text{hol}}(k) \), which by (5.2) and (2.16) can be written as
\[
h_T^{\text{hol}}(k) = \frac{1}{8\pi^2} \int_0^{\infty} J_k^{\text{hol}}(x) \phi_T(x) \frac{dx}{x} = \frac{i^k}{4\pi} \int_0^{\infty} J_{k-1}(x) \phi_T^+(x) \frac{dx}{x}.
\]

We now employ the bound (A.9) for \( J_{k-1}(x) \) and the bound \( \phi_T^+(x) \ll T \min(1, (xT)^{-D/3}) \) (which follows from (6.15)), obtaining
\[
h_T^{\text{hol}}(k) \ll T \int_0^{(k-1)/4} \left( \frac{2\pi}{k-1} \right)^k \frac{dx}{x} + T \int_{(k-1)/4}^{\infty} (xT)^{-D/3} \frac{dx}{x} \ll T(2^{-k} + (kT)^{-D/3})
\]
uniformly in \( k \). This is clearly majorized by the claimed bound \( Tk^{-5} \), completing the proof. \( \square \)

Theorem 1 now follows as a consequence of Lemma 3.

7. Nonvanishing and Moments of \( L \)-functions

In this section we prove Theorem 2, which essentially amounts to a proof of the following lower bound.

**Lemma 5.** For \( T \) sufficiently large we have
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|L(1/2 + it, \Pi)|^2}{\zeta(1 + 2it)^2} h_T(t) dt \gg T \log T,
\]
where \( h_T \) is defined in (1.4).

**Proof.** We follow the method of Rudnick and Soundararajan [RS]. Let \( w \) be a fixed, non-negative, smooth function with support on \([1/4, 1]\) such that \( w(t) \equiv 1 \) on \([1/2, 3/4]\). Like all compactly supported smooth functions, its Fourier transform satisfies
\[
\hat{w}(y) = \int_{-\infty}^{\infty} w(x)e^{-ixy} dx \ll_B (1 + |y|)^{-B}
\]
for any \( B > 0 \). Let \( T \) be a large parameter and let \( A(t) \) be any continuous function that is not identically zero on \([\frac{3}{4}T, \frac{5}{4}T]\). Then the Cauchy-Schwarz inequality implies
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|L(1/2 + it, \Pi)|^2}{|\zeta(1 + 2it)|^2} h_T(t) dt \gg \int_{T/4}^{T} \frac{|L(1/2 + it, \Pi)|^2}{|\zeta(1 + 2it)|^2} w \left( \frac{t}{T} \right) dt \gg \frac{|I_1|^2}{I_2},
\]
with
\[
I_1 := \int_{T/4}^{T} L(1/2 + it, \Pi) A(t) w \left( \frac{t}{T} \right) dt \quad \text{and} \quad I_2 := \int_{T/4}^{T} |A(t)| |\zeta(1 + 2it)|^2 w \left( \frac{t}{T} \right) dt.
\]

We choose
\[
A(t) := \sum_m \lambda_{\Pi}(m) m^{1/2-it} w_1 \left( \frac{m}{T^{1/8}} \right),
\]
where \( w_1 \) is a smooth, non-negative function with support on \([0, 1]\) such that \( w_1(t) \equiv 1 \) on \([0, 3/4]\), and \( \lambda_{\Pi}(m) \) as defined in (2.2). Its Mellin transform \( \hat{w}_1(s) \) is rapidly decaying and has a simple pole with residue 1 at \( s = 0 \). For \( X > 1 \) and \( t \in \mathbb{R} \) we have
\[
\sum_n \lambda_{\Pi}(n) n^{1/2+it} e^{-n/X} = \int_{\mathbb{R}_{s=1}} L(1/2 + s + it, \Pi) \Gamma(s) X^s ds = L(1/2 + it, \Pi) + O_{\varepsilon}(X^{-1/2}(1 + |t|)^{2+\varepsilon})
\]
after shifting the contour to $\Re s = -1/2$. We conclude

$$I_1 = \int_{T/4}^T \left( \sum_n \frac{\lambda(n)}{n^{1/2+it}} e^{-n/T^s} + O\varepsilon(T^{-2}) \right) \sum_m \frac{\overline{\lambda(m)}}{m^{1/2-it}} w_1 \left( \frac{m}{T^{1/8}} \right) w \left( \frac{t}{T} \right) dt.$$  

The $m$-sum can be crudely bounded by $T^{1/8}$ using (2.2), so that integration over $t$ yields

$$I_1 = T \sum_{n,m} \tilde{w} \left( -T \log \frac{m}{n} \right) \frac{\lambda(n)\lambda(m)}{(nm)^{1/2}} w_1 \left( \frac{m}{T^{1/8}} \right) e^{-n/T^s} + O(T^{-1/2})$$

$$= T \tilde{w}(0) \sum_n \frac{\lambda(n)^2}{n} w_1 \left( \frac{n}{T^{1/8}} \right) + O(T^{-1/2}),$$

where we used the Taylor expansion $e^{-n/T^s} = 1 + O(n/T^8)$ and that

$$T \left| \log \frac{m}{n} \right| \geq T \log \left( 1 + \frac{1}{m} \right) \gg \frac{T}{m} \gg T^{7/8}$$

for $m \neq n \leq T^{1/8}$, so that by (7.1) the off-diagonal contribution is negligible. We conclude from Lemma 1 and partial summation that

$$I_1 \gg T \log T$$

for $T$ sufficiently large.

On the other hand, for $\frac{1}{4}T \leq t \leq T$ and any $Y > 1$ we have

$$\sum_n \frac{1}{n^{1+2it}} \frac{w(1)}{Y} = \int_{\Re s = 2} \zeta(1 + s + 2it) w_1(s) Y^s \frac{ds}{2\pi i} = \zeta(1 + 2it) + O(T^{-1/2} + T^{-100})$$

upon shifting the contour to $\Re s = -1$, where the term $T^{-100}$ comes from the residue at $s = -2it$ and the rapid decay of the Mellin transform $\tilde{w}_1$. Thus

$$\zeta(1 + 2it)^2 = \left( \sum_n \frac{1}{n^{1+2it}} \frac{w(1)}{Y^2} \right)^2 + O(T^{-1/6+\varepsilon}),$$

since the $n$-sum is $O(\log T)$. Inserting this and (7.3) into the definition of $I_2$ shows that

$$I_2 = I_{2,\text{main}} + O(\varepsilon T^{-1/6+\varepsilon} dt),$$

where

$$I_{2,\text{main}} = \int_{T/4}^T \sum_{m_1,m_2} \frac{\lambda(m_1)\lambda(m_2)}{m_1^{1/2-it} m_2^{1/2+it}} \left( \frac{m_1}{T^{1/8}} \right) w_1 \left( \frac{m_2}{T^{1/8}} \right) \sum_{n_1,n_2} \frac{w_1 \left( \frac{n_1}{T^{2/3}} \right) w_1 \left( \frac{n_2}{T^{2/3}} \right)}{n_1^{1+2it} n_2^{1-2it}} w \left( \frac{t}{T} \right) dt$$

$$= T \sum_{m_1,m_2} \tilde{w} \left( T \log \frac{m_2}{m_1} \right) \frac{\lambda(m_1)\lambda(m_2)}{m_1^{1/2+it} m_2^{1/2-it}} \left( \frac{m_1}{T^{1/8}} \right) w_1 \left( \frac{m_2}{T^{1/8}} \right) \sum_{n_1,n_2} \frac{w_1 \left( \frac{n_1}{T^{2/3}} \right) w_1 \left( \frac{n_2}{T^{2/3}} \right)}{n_1^{1+2it} n_2^{1-2it}} w_1 \left( \frac{n_1}{T^{2/3}} \right) w_1 \left( \frac{n_2}{T^{2/3}} \right).$$

Since $A(t) = O(T^{1/16})$ (as follows from partial summation, Cauchy-Schwarz, and (2.3)), the error term in (7.5) is $O(T^{23/24+\varepsilon}) = O(T^{24/25})$.

The test function $w_1$ constrains $1 \leq m_1, m_2 \leq T^{1/8}$ and $1 \leq n_1, n_2 \leq T^{2/3}$. For $m_1 m_2 \neq m_2 m_1$ we have $T \left| \log \frac{m_1}{m_2} \right| \gg T^{5/24}$, so that the off-diagonal contribution to $I_{2,\text{main}}$ is negligible because of (7.1). Every quadruple $(m_1, m_2, n_1, n_2)$ with $m_1 n_1 = m_2 n_2$ must be of the form $(rs, rt, us, ut)$ for some integers $r, s, t, u$, hence enlarging the range of summation we have

$$I_2 \ll T \sum_{r,s,t,u \leq T} \frac{|\lambda(r)\lambda(s)|}{r^{s+t} u^2} \ll T \sum_{r,s,t \leq T} \frac{|\lambda(r)\lambda(s)|}{r^{s+t} u^2}.$$
Fix $0 < \varepsilon < \delta$. Using
\[ \lambda_{\Pi}(rs)s^{-1/2+\varepsilon}\lambda_{\Pi}(rt)t^{-1/2+\varepsilon} \leq |\lambda_{\Pi}(rs)|^2 s^{-1+2\varepsilon} + |\lambda_{\Pi}(rt)|^2 t^{-1+2\varepsilon}, \]
we obtain
\[ I_2 \ll T \sum_{r,s \leq T} \frac{|\lambda_{\Pi}(rs)|^2}{rs^{-2-\varepsilon}}. \]
Write $d = \gcd(r, s)$, $r = r_0df$ and $s = s_0dg$, with $\gcd(r_0s_0, d) = 1$ and $f, g \mid d^\infty$ (i.e., both $f$ and $g$ divide some power of $d$). By the multiplicativity of $\lambda_{\Pi}$, we may factorize $\lambda_{\Pi}(rs) = \lambda_{\Pi}(r_0)\lambda_{\Pi}(s_0)\lambda_{\Pi}(d^2fg)$. Then
\[ I_2 \ll T \sum_{d \leq T} \sum_{f, g \mid d^\infty} \frac{1}{r_0s_0d^{-2-\varepsilon}d^{2\varepsilon}fg^{2-\varepsilon}} \ll T(\log T) \sum_{d \leq T} \frac{1}{d^{1+\delta-\varepsilon}} \sum_{f, g \mid d^\infty} \frac{1}{fg^{2\varepsilon}}, \]
where we used (2.3) and partial summation to bound the $r_0, s_0$-sum and then applied (2.2) in the last step. The $g$-sum is $O(1)$ since $\varepsilon < \delta$. The $f$-sum is bounded by any small power of $d$, as can be seen by factoring it as a product over primes. Thus the $d$-sum is $O(1)$ and we altogether obtain $I_2 \ll T \log T$. Together with (7.4) we can therefore bound the left hand side of (7.2) from below by $\gg T \log T$ as desired. This completes the proof of the lemma.

**Proof of Theorem 2.** Combining (1.3), Theorem 1, and Lemma 5, we have
\[ -\sum_{\pi} \varepsilon_{\pi} \frac{L(1/2, \Pi \times \pi)}{L(1, \text{Sym}^2 \pi)} h_T(t_{\pi}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left|L(1/2 + it, \Pi)\right|^2}{\left|\zeta(1+2it)\right|^2} h(t) dt - M^-(h_T) \gg T \log T \]
for $T$ sufficiently large. Fix $\varepsilon > 0$ and choose $D = 50 + \varepsilon^{-1}$ in (1.4). By (2.10) the contribution of representations $\pi$ with $t_{\pi} < T^{1-\varepsilon}$ or $t_{\pi} \geq T^{1+\varepsilon}$ is $O(1)$, hence there must be a non-zero $L$-value in the interval $T^{1-\varepsilon} \leq t_{\pi} \leq T^{1+\varepsilon}$, provided $T$ is sufficiently large in terms of $\Pi$ and $\varepsilon$. Hence the number of non-vanishing $L$-values with $t_{\pi} < X$ is at least $c_1\varepsilon^{-1}(\log \log X - c_2)$ for some absolute constant $c_1 > 0$ and some constant $c_2 = c_2(\Pi, \varepsilon) > 0$, proving Theorem 2.

**Appendix A. Bessel functions**

In this section we compile some useful facts about the $I_-$, $J_-$, and $K$-Bessel functions for easy reference. We have [GR, 8.486.2 and 11]
\[ K_{\nu}'(x) = -\frac{1}{2}(K_{\nu+1}(x) + K_{\nu-1}(x)), \quad I_{\nu}'(x) = \frac{1}{2}(I_{\nu+1}(x) + I_{\nu-1}(x)) \]
for $x > 0$ and $\nu \in \mathbb{C}$, and hence
\[ K_{2it}^{(j)}(x) = \left(-\frac{1}{2}\right)^j \sum_{n=0}^{j} \binom{j}{n} K_{2it-j+2n}(x) \]
and
\[ I_{2it}^{(j)}(x) = \left(\frac{1}{2}\right)^j \sum_{n=0}^{j} \binom{j}{n} I_{2it-j+2n}(x) \]
for $j \in \mathbb{Z}_{\geq 0}$. A weak version of [HM, Proposition 9] states that
\[ J_1^-(x) = 4 \cosh(\pi t)K_{2it}(x) \ll_{\delta t} e^{\min(0, -x + \pi|\Re(t)|)} \left(1 + \frac{|t| + x}{x}\right)^{2|\Re(t)| + 1/10} \]
for \( x > 0 \) and \( t \in \mathbb{C} \). The Bessel \( K \)-function and the Bessel \( I \)-function are related [GR, 8.485] via

\[
\sinh(\pi t) K_{2\mu}(x) = \frac{\pi i}{4} \frac{I_{2\mu}(x) - I_{-2\mu}(x)}{\cosh(\pi t)}.
\]

It follows from the power series expansion [GR, 8.445] and Stirling’s formula that

\[
I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\nu + k + 1)} \left( \frac{x}{2} \right)^{\nu + 2k}
\]

that

\[
e^{-\pi |t|} I_{2\mu}(x) \ll_\mathfrak{M} \frac{x^{-2\Im t}}{(1 + |t|)^{1/2 - 2\Im t}}
\]

for \( t \in \mathbb{C} \) and \( 0 < x < (1 + |t|)^{1/2} \) (the estimate is trivial when \( t \) is constrained to a compact region; for \( \Re t \) large the bound guarantees the terms in \( (A.5) \) are bounded by a constant multiple of the \( k = 0 \) term).

The Bessel kernels defined in \((2.15)\) and \((2.16)\) have Mellin transforms

\[
\begin{align*}
\mathcal{J}_t^+(s) &= 2^s \Gamma(s/2 + it) \Gamma(s/2 - it) \cos(\pi s/2), \\
\mathcal{J}_t^-(s) &= 2^s \Gamma(s/2 + it) \Gamma(s/2 - it) \cosh(\pi t), \\
\mathcal{J}_t^{\text{hol}}(s) &= 2^s \cos\left(\frac{\pi s}{2}\right) \Gamma\left(s + 1 - k\right) \Gamma\left(s + k - 1\right) \\
&= i^k 2^s \pi \Gamma\left(\frac{1}{2}(s + k - 1)\right) \Gamma\left(\frac{1}{2}(1 + k - s)\right),
\end{align*}
\]

where \( k \geq 2 \) is even. These follow from [GR, 17.43.16 & 18] together with functional equation of the Gamma function.

If \( \phi \) is any even Schwartz function bounded by a constant multiple of \( e^{-2\pi|t|} \), then

\[
\int_{-\infty}^{\infty} \mathcal{J}_t^-(x) \phi(t) dt = 2 \int_{-\infty}^{\infty} e^{ix \sinh(v)} \tilde{\phi}(2v) dv,
\]

where \( \tilde{\phi}(v) = \int_{-\infty}^{\infty} \phi(x) e^{-ivx} dx \). This is Parseval’s equality, with the analytic subtlety that the Fourier transform of the tempered distribution \( \mathcal{J}_t^-(x) \) given in [GR, 8.432.4] is not integrable. To prove this identity, first assume \( \phi \) has compact support and insert the absolutely convergent integral formula \( K_{2\mu}(x) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x \cosh(u) - 2\mu tu} du \) [GR, 8.432.1] into definition \((2.15)\) to rewrite the left hand side of \((A.8)\) as

\[
2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\pi t} e^{-x \cosh(u) - 2\mu tu} \phi(t) dt du.
\]

This double integral is also absolutely convergent, with an integrand that decays doubly-exponentially in \( u \) uniformly in horizontal strips. Shifting the \( u \)-contour down to \( \Im u = -\pi/2 \) and changing the order of integration establishes \((A.8)\). (Note that the bound \((6.4)\) assures the \( O(e^{-2\pi|t|}) \) condition, so that \((A.8)\) applies to \((6.5)\)).

Finally we state the uniform bound

\[
J_k(x) \ll (2\pi/k)^k, \quad k \in \mathbb{N}.
\]

The first bound follows from [Ra, Lemma 4.1] for \( k \geq 15 \) and from the power series expansion for \( 1 \leq k \leq 14 \) and \( x \leq 1 \), while the second bound follows from [GR, 8.411.1].

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