ON FINITE SYMPLECTIC MODULES ARISING FROM SUPERCUSPIDAL REPRESENTATIONS

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ABSTRACT. Let $F$ be a non-Archimedean local field with finite residue field. Let $\mathcal{A}_n^c(F)$ be the collection of isomorphism classes of essentially tame irreducible supercuspidal representations of $GL_n(F)$ studied by Bushnell-Henniart. It is known that we can parameterize $\mathcal{A}_n^c(F)$ by the collection $P_n(F)$ of equivalence classes of admissible pairs $(E, \xi)$ consisting of a tamely ramified extension $E/F$ of degree $n$ and an $F$-admissible character $\xi$ of $E^\times$. We are interested in a finite symplectic module $V = V(\xi)$ arising from the construction of the supercuspidal representation from the character $\xi$. This module $V$ is known to admit an orthogonal decomposition with respect to a symplectic form depending on $\xi$. We work with a fixed ambient module $U$ containing $V$ and show that $U$ decomposes in a way analogous to the root space decomposition of the Lie algebra $gl_n(F)$. We then obtain a complete orthogonal decomposition of the submodule $V$ by restriction. Such decomposition relates the finite symplectic module of a supercuspidal and certain admissible embedding of $L$-groups. This relation provides a different interpretation on the essentially tame local Langlands correspondence.

1. Introduction

Let $F$ be a non-Archimedean local field whose residue field $k_F$ has $q$ elements and of characteristic $p$. We study those irreducible supercuspidal representations $\pi$, or simply supercuspidals, of $GL_n(F)$ which are essentially tame. This means if $f(\pi)$ is the number of unramified characters of $F^\times$ that stabilizes $\pi$ then $p$ does not divide $n^f(\pi)$. Let $\mathcal{A}_n^c(F)$ be the set of isomorphism classes of essentially tame supercuspidals of $GL_n(F)$.

We can classify the representations in $\mathcal{A}_n^c(F)$ by using the collection $P_n(F)$ consisting of the equivalence classes of admissible pairs $(E, \xi)$ such that $E/F$ is tamely ramified of degree $n$ and $\xi$ is an character of $E^\times$ admissible over $F$, in the sense of $[BH05a]$. $[Moy86]$. We regard $E^\times$ as the $F$-point of the torus $T = \text{Res}_{E/F} T_i$ and assume that $T$ is $F$-regularly embedded as an elliptic maximal torus of $G = GL_n$. We hence write $E^\times \subseteq GL_n(F)$. There is a natural bijection (see $[BK93]$ and chapter 2 of $[BH05a]$)

$$\pi : P_n(F) \rightarrow \mathcal{A}_n^c(F), (E, \xi) \mapsto \pi_\xi$$

whose construction is independent of the embedding of $E^\times$ into $GL_n(F)$. We briefly describe the correspondence $\mathcal{I}$ in section 3. Note that when $p \nmid n$ every extension of degree $n$ is tamely ramified and every irreducible supercuspidal is essentially tame. In this case the correspondence $\mathcal{I}$ was studied in a precursor article $[Moy86]$.

We will be more specific on the definitions and properties of admissible characters in section 5. Here we point out some of them that carry us to the main result of this article. Attached to each $F$-admissible character $\xi$ of $E^\times$ is a sequence of subfields $E \supseteq E_0 \supseteq E_1 \cdots \supseteq E_d = F$ and a sequence of increasing positive integers $r_0 < \cdots < r_d$. We call these two sequences the jump data of $\xi$. They are defined by the following conditions. For $j = 0, \ldots, d$,

(i) if $r \geq r_j$, then the restriction $\xi|_{U_{r+1}}$ of $\xi$ to the $(r + 1)$th unit group of $E$ factors through $N_{E/E_{r+1}}$, and
(ii) the field $E_{r+1}$ is the minimum between $E$ and $F$ that satisfies condition (i).

The jump data are the basic ingredients in constructing the supercuspidal $\pi_\xi$ of the admissible character $\xi$ as in $\mathcal{I}$. The admissibility of $\xi$ imposes certain conditions on the fields and the integers in the jump data. These conditions are summarized in Proposition 5.1.
As intermediate steps of (1) we produce several representations of subgroups in \(GL_n(F)\), denoted by 
\((H^1, \theta), (J^1, \eta)\), and \((J, \Lambda_0)\). Their constructions and the roles they play can all be found in [BH05a, BK98]. Each of these representations emerges from a theory in its own right. We give several well-known properties.

(i) The subgroups \(H^1, J^1\) and \(J\) depend only on the jump data of \(\xi\) and the embedding \(E^\times \subseteq GL_n(F)\).
We have inclusions \(U_E^1 \subseteq H^1 \subseteq J^1 \subseteq E^\times J^1 \subseteq J\). The subgroups \(H^1\) and \(J^1\) are compact, and \(J\) is compact modulo the center of \(GL_n(F)\).

(ii) The representations \((\xi|_{U_E^1}, U_E^1), (H^1, \theta), (J^1, \eta)\), and \((J, \Lambda_0)\) are successive extensions. The representation \((H^1, \theta)\) has degree 1, and is called a simple character of \(\xi\).
We are particularly interested in the finite quotient \(V = J^1/H^1\), which is an \(k_F\)-space as well as an \(E^\times\)-module. The space \(V\) equips with a non-degenerate alternating \(F\)-bilinear form \(h_\theta\), which is defined by the character \((H^1, \theta)\) and is \(E^\times\)-invariant. Using the admissibility of \(\xi\), the space \(V\) decomposes into an orthogonal sum \(V = V_0 \oplus \cdots \oplus V_d\). Each component \(V_j\) is invariant by \(E^\times\) and has trivial \(E^\times_{j+1}\)-action (see Corollary 1 in 6.3 of [BH05c]). We call this the coarse decomposition of \(V\). The main result of this work is to study a finer decomposition of the finite symplectic module \(V\).

The approach is to first study an ambient module \(U\) called the standard module in this article. It contains respectively all possible \(V\) as \(\xi\) runs through all admissible characters. The action of \(E^\times\) on \(U\) is induced from a two-step descent. We begin with the conjugate action of \(E^\times\) on the space \(\text{End}_F(E)\) of \(F\)-endomorphisms of \(E\). Such action stabilizes the hereditary \(\sigma_F\)-order (see (1.1) of [BK98])
\[\mathfrak{A} = \{X \in \text{End}_F(E) | Xp_E^k \subseteq p_E^k \text{ for all } k \in \mathbb{Z}\}\]
in \(\text{End}_F(E)\) corresponding to the \(\sigma_F\)-lattice chain \(\{p_E^k | k \in \mathbb{Z}\}\) in \(E\). It also stabilizes the Jacobson radical \(\mathfrak{p}_{\mathfrak{A}} := \{X \in \text{End}_F(E) | Xp_E^k \subseteq p_E^{k+1} \text{ for all } k \in \mathbb{Z}\}\) of \(\mathfrak{A}\). Hence the \(E^\times\)-action descends to the \(k_F\)-vector space \(U := \mathfrak{A}/\mathfrak{p}_{\mathfrak{A}}\).

We now recall that \(E^\times\) is \(F\)-regularly embedded into \(GL_n\). Therefore \(E^\times\) acts on the Lie algebra \(\mathfrak{gl}_n\) by the adjoint action. Analogous to the root space decomposition of \(\mathfrak{gl}_n\) in the absolute case (i.e. when \(F\) is algebraically closed), the space \(\text{End}_F(E) \cong \mathfrak{gl}_n(F)\) decomposes into \(E^\times\)-invariant subspaces parameterized by the \(\Gamma_F\)-orbits of the (absolute) root system \(\Phi = \Phi(G,T)\) as follows. Write \([\lambda] = \Gamma_F\lambda\) the \(\Gamma_F\)-orbit of a root \(\lambda \in \Phi\). There is a decomposition
\[\text{End}_F(E) = E \oplus \bigoplus_{[\lambda] \in \Gamma_F \setminus \Phi} \text{End}_F(E)_{[\lambda]},\]
where each component \(\text{End}_F(E)_{[\lambda]}\) is an \(E^\times\)-invariant space over \(F\). We call this the rational root space decomposition. This decomposition therefore restricts along to way from \(\text{End}_F(E)\) to \(\mathfrak{A}, \mathfrak{p}_{\mathfrak{A}},\) and finally \(U\) according to the descents previously described. We call this decomposition on \(U\) the residual root space decomposition. We give a complete description of such decomposition in Theorem 3.6.

As a \(k_F\)-subspace of \(U\) the space \(V\) has the \(E^\times\)-module structure induced from \(U\), which is equivalent to the one induced from the normalization of the group \(J\). The action of \(E^\times\) on \(V\) factors through \(E^\times \to E^\times/F^\times U_E^1\). If we identify \(\mu_E\) and \(k_E^\times\) in the canonical way, then each root \(\lambda \in \Phi\) is well-defined on \(E^\times/F^\times U_E^1\), namely
\[\lambda(tF^\times U_E^1) \equiv \lambda(t) \mod p_E.\]

We have the following main result.

**Theorem 1.1.** (i) The \(k_E(E^\times/F^\times U_E^1)\)-module \(V\) decomposes into a direct sum
\[V = \bigoplus_{[\lambda] \in \Gamma_F \setminus \Phi} V_{[\lambda]}\]
such that if \(V_{[\lambda]}\) is nontrivial, then it is isomorphic to
(a) a field extension of \(k_E\) as \(k_F\)-vector space and
(b) the \((E^\times/F^\times U_E^1)\)-module with action \(\lambda(t)v = \lambda(t)v\) for all \(v \in V_{[\lambda]}\) and \(t \in E^\times/F^\times U_E^1\).

(ii) As a symplectic space (with respect to the alternating form \(h_\theta\)), the direct sum of \(V\) in (1) is orthogonal with symplectic subspaces of the form
(a) \(V_{[\lambda]} \oplus V_{[-\lambda]}\) if \(\lambda\) and \(-\lambda\) are not in the same \(\Gamma_F\)-orbit and
(b) \(V_{[\lambda]}\) if \(\lambda\) and \(-\lambda\) are in the same \(\Gamma_F\)-orbit and \(\lambda^2 \neq 1\).
Theorem 1.1 summarizes the main properties of the symplectic module $V$, which are established in section 9. We call this direct sum the complete decomposition of $V$. Such decomposition is finer than the coarse decomposition. Indeed we can specify the component $V[\lambda]$ in the coarse decomposition that contains $V[\lambda]$ for each orbit $[\lambda]$. Since $V_j$ has trivial $E|_{E_j}$-action for each $j = 0, \ldots, d$, the following fact is clear.

**Corollary 1.2.** The component $V[\lambda]$ is contained in $V_j$ if and only if $\lambda|_{E|_{E_j}}$ is trivial and $\lambda|_{E}$ is non-trivial.

Originating from this Corollary, the orthogonality of the complete decomposition of $V$ is quite elementary to be proved. This implies that the coarse decomposition is orthogonal, the fact that is originally proved in [BH05c].

The decomposition in Theorem 1.1 is related to a result in a preceding article of the author [Tamb]. Recall that Langlands and Shelstad construct certain admissible embedding of L-groups $L \to G$ using a collection of characters called $\chi$-data, in the sense of [LSS77]. We can regard the admissible embedding by a twisted induction process. The twist here is a character $\mu$ of $E^\times$. By comparing Proposition 4.2 of [Tamb] and Theorem 1.1 in this article, we observe that the twist $\mu$ factorizes in a way analogous to the complete decomposition of $V$. This relation then provides a different interpretation on the essentially tame local Langlands correspondence [BH05a]. We briefly discuss this application in section 7.

Throughout the article we will compute explicitly the components $V[\lambda]$ of $V$ and the $E|_{E}$-action on them. The way is to identify the $\Gamma_F$-orbits of the root system $\Phi$ with the non-trivial double cosets $\Gamma_F \backslash \Gamma_E / \{ \Gamma_E \}$ as in Lemma 6.2. Then we use an explicit description of the Galois group in section 2. Our main results will be phrased in a way that the indexes are the double cosets in place of the $\Gamma_F$-orbits of roots.

Recently the author is told by his advisor Professor James Arthur that some of the setups of this article are similar to those in [Koe77]. Since Koch’s result is to classify the primitive representations of the Galois group $\Gamma_F$, it would be interesting to know if we can compare the results in this article to his and hence study the more general Langlands parameters. The author is indebted to Professor Arthur for pointing out this fact and also his support and guidance.

**Notations.** We fix our notations throughout the article.

Let $H$ be a group acting on a set $X$. For $h \in H$ and $x \in X$, we write $hx$ for the action of $h$ on $x$. The $H$-orbit of $x \in X$ is denoted by $Hx$. The collection of all $H$-orbits of $X$ is denoted by $H \backslash X$. The set of fixed points is denoted by $X^H$.

Given a field $F$, let $\bar{F}$ be an algebraic closure of $F$ and $\Gamma_F$ be the absolute Galois group of $F$. Given a field extension $E/F$, its degree is denoted by $|E/F|$. We write $\Gamma_E = \Gamma_F / \Gamma_E$. We denote induction $\text{Ind}_E^{F}$ by $\text{Ind}_{\Gamma_E}^{\Gamma_F}$ and restriction $\text{Res}_{\Gamma_F}^{\Gamma_E}$ by $\text{Res}_{E/F}$. The valuation of a local field $F$ is denoted by $v_F$.

We denote a choice of a primitive $m$th root of unity by $\zeta_m$, and the subgroup it generates by $\mu_m$.

**2. Galois Groups**

Let $E/F$ be a tame extension of degree $n$, with ramification index $e$ and residue degree $f$. The multiplicative group $F^\times$ decomposes into product of subgroups $(\mathcal{O}_E) \times \mu_E \times U_\mathcal{O}_F$. They are namely the group generated by a prime element, the group of roots of unity, and the 1-unit group. We have similar decomposition for $E^\times$. By [Lan94] II.5 we can always assume our choices of $\varpi_E$ and $\varpi_F$ satisfying

\begin{equation}
\varpi_E^\mu = \zeta_{E/F} \varpi_E \quad \text{for some } \zeta_{E/F} \in \mu_E.
\end{equation}

Let $k_F$ be the residual field of $F$. We may identify $\mu_F$ with $k_F^\times$ in the canonical way. Denote $\Psi_{E/F} = E^\times / F^\times U_\mathcal{O}_F$ and its subgroups $\mu_E \mu_F$ and $\varpi = \text{the subgroup generated by the image of } \varpi_E$ in $\Psi_{E/F}$.

Let $L$ be the Galois closure of $E/F$. Hence $L/E$ is unramified and $L/F$ is also a tame extension. With the choice of $\varpi_E$ and $\varpi_F$ as in (2), we define the following $F$-operators on $L$.

- $(i)$ $\phi : \zeta \mapsto \zeta^\mu$ for all $\zeta \in \mu_L$, $\varpi_E \mapsto \varpi_E$ and $\varpi_F \mapsto \varpi_F$.
- $(ii)$ $\sigma : \zeta \mapsto \zeta^\varpi$ for all $\zeta \in \mu_L$, $\varpi_E \mapsto \varpi_E$.

Here $\zeta_\phi$ is in $\mu_F$ satisfying $(\zeta_\phi \varpi_F)^\varpi_E = \zeta_\phi \varpi_F$ and $\varpi_\zeta$ is a choice of a primitive $e$th root of unity in $\bar{F}^\times$.

More generally we write $\phi : \varpi_E = \zeta_\phi \varpi_E$ such that $\zeta_\phi = \zeta_{\phi}^{1+q+\cdots+q^{e-1}}$ is an $e$th root of $\zeta_\phi^{1+q+\cdots+q^{e-1}}$. Notice that we have an action of $\varphi$ on $\sigma$ by $\varphi : \sigma \mapsto \phi \circ \sigma \circ \phi^{-1} = \sigma^q$. Therefore we can write our Galois group as

\begin{equation}
\Gamma_{L/F} = \langle \sigma \rangle \ltimes \langle \phi \rangle \quad \text{and} \quad \Gamma_{L/E} = \langle \phi^q \rangle \subseteq \langle \phi \rangle.
\end{equation}
Proposition 2.1. (i) We can choose \( \{\sigma^k\phi^i|k = 0, \ldots, e - 1, i = 0, \ldots, f - 1\} \) as coset representatives for the quotient \( \Gamma_E/F = \Gamma_F/\Gamma_E \).
(ii) Let \( q^f\langle \sigma \rangle \) be the set of orbits of \( \sigma \) under the action of \( \phi^f \), i.e. \( \sigma \mapsto \sigma^q \), then the double coset \( \Gamma_E\backslash \Gamma_F/\Gamma_E \) is bijective to the set \( \langle q^f\langle \sigma \rangle \rangle \times \langle \phi \rangle \).

Proof. In general, if we have an abelian group \( B \) acting on a group \( A \) as automorphisms, and \( C \) a subgroup of \( B \), then the canonical maps
\[
(A \times B)/(1 \times C) \to A \times (B/C), \quad (a,b)(1 \times C) \mapsto (a,bC)
\]
and
\[
(1 \times C)/(A \times B)/(1 \times C) \to (C\backslash A) \times (B/C), \quad (1 \times C)(a,b)(1 \times C) \mapsto (C\backslash a,bC)
\]
are bijective. We take \( A \times B = \Gamma_{L/F} \) and \( C = \Gamma_{L/E} \) as in \( \Box \).

Later on we will have explicit computations on certain finite modules parameterized by double cosets \( [g] := \Gamma_Eg\Gamma_E \in \Gamma_E\backslash \Gamma_{L/F}/\Gamma_E \).

We will adopt the notations in Proposition 2.1.

The standard module

Again \( \Gamma_{E/F} \) is tamely ramified. Let \( \mathfrak{A} \) be the hereditary order of \( \text{End}_F(E) \) corresponding to the \( \sigma_F \)-lattice chain \( \{p_k^{\mathfrak{A}}|k \in \mathbb{Z}\} \) in \( E \) and \( \mathfrak{P}_A \) be its Jacobson radical, as introduced in section 1. Let \( \mathfrak{P}_{E/F} = E^\times/F^\times U_1^F \).

In this section we show that the \( \mathfrak{A} \text{-subalgebra} \) \( g \) of \( \mathfrak{G} \text{-submodules} \). To begin with, we consider the following \( \mathfrak{A} \)-actions. For all \( t \in E^\times \), we have
\[
E \otimes E \text{ with } t(x \otimes y) = tx \otimes yt^{-1} \text{ for all } x, y \in E,
\]
\[
\text{End}_F(E) \text{ with } (tT)(v) = tT(t^{-1}v) \text{ for all } T \in \text{End}_F(E), \text{ and}
\]
\[
\bigoplus_{[g] \in \Gamma_E \backslash \Gamma_{L/F}/\Gamma_E} gEE \text{ with } t(gx\langle y\rangle)[g] = (gt^{-1}gx\langle y\rangle)[g] \text{ for all } x\langle y\rangle \in E.
\]

The following fact is well known.

Proposition 3.1. The \( F \)-linear maps
(i) \( E \otimes E \to \text{End}_F(E), \quad x \otimes y \mapsto (v \mapsto tr_{E/F}(yx)v), \) and
(ii) \( E \otimes E \to \bigoplus_{[g] \in \Gamma_E \backslash \Gamma_{L/F}/\Gamma_E} gEE, \quad x \otimes y \mapsto (gxy)[g] \)

are isomorphisms of \( E^\times \)-modules.

Proof. Indeed \( \Box \) is isomorphic by the non-degeneracy of the trace form \( tr_{E/F} \), while \( \Box \) is isomorphic by considering all possible \( F \)-algebra embeddings \( E \otimes E \to \bar{F} \). Notice that the isomorphism \( \Box \) is moreover an \( F \)-algebra one. The \( E^\times \)-invariance of both morphisms are clear. \( \Box \)

We identify \( \text{End}_F(E) \) with \( g(F) = gl_n(F) \) by choosing an \( F \)-basis of \( E \) and its subalgebra \( E \) with a Cartan subalgebra \( g(F)_0 \) of \( gl_n(F) \). Recall that the roots of the elliptic maximal \( F \)-torus \( T = Res_{E/F}G_m \) in the \( F \)-reductive group \( G = GL_n \) are of the form \( \left[ \frac{a}{b} \right] \) with \( g, h \in \Gamma_F \) such that
\[
\left[ \frac{a}{b} \right](t) = g(h^t)^{-1} \text{ for all } t \in E^\times = T(F).
\]
If we can choose a collection of representatives \( \{g_1 = 1, g_2, \ldots, g_n\} \) in \( \Gamma_F \) of \( \Gamma_F/\Gamma_E \) (for example, we can take those in Proposition 2.1), then we can write the root system
\[
\Phi = \Phi(G, T) = \{\left[ \frac{a}{b} \right]|i, j \in \{1, \ldots, n\}, i \neq j\}
\]
Hence the canonical permutation of \( \Gamma_F \) on \( \Gamma_F/\Gamma_E \) induces an \( \Gamma_F \)-action on \( \Phi \).

Lemma 3.2. The set \( \Gamma_F/\Phi \) of \( \Gamma_F \)-orbits of the root system \( \Phi \) is bijective to the set \( (\Gamma_E/\Gamma_F/\Gamma_E)' \) of non-trivial double cosets in \( \Gamma_E/\Gamma_F/\Gamma_E \), by the map \( \Gamma_F/\Phi \to (\Gamma_E/\Gamma_F/\Gamma_E)', \Gamma_F/\left[ \frac{a}{b} \right] \mapsto [g] = \Gamma_Eg\Gamma_E \).

Proof. The set of roots is \( \Gamma_F \)-equivalent to the collection of the off-diagonal elements in \( \Gamma_E/\Gamma_F \times \Gamma_E/\Gamma_F \) with \( \Gamma_F \)-action \( \Phi(g_1\Gamma_E, g_2\Gamma_E) = (gg_1\Gamma_E, gg_2\Gamma_E) \), whose orbits are bijective to the non-trivial double cosets. \( \Box \)
Proposition 3.3. The $F$-Lie algebra $\mathfrak{g}(F) = \mathfrak{gl}_n(F)$ decomposes into $\text{Ad}(E^\times)$-invariant subspaces
\[
\mathfrak{g}(F) = \mathfrak{g}(F)_0 \oplus \bigoplus_{[\lambda] \in \Gamma_F \setminus \Phi} \mathfrak{g}(F)_{[\lambda]}.
\]
This decomposition is compatible with the one of $\text{End}_F(E)$ induced by Proposition 3.1 such that $\mathfrak{g}(F)_0 \cong E$ and $\mathfrak{g}(F)_{[\lambda]} \cong \mathfrak{g}_E$.

Proof. The first assertion can be derived by a simple Galois descent argument from the absolute case. More precisely, for every orbit $[\lambda] := \Gamma_F \lambda \in \Gamma_F \setminus \Phi$ there is a subspace $\mathfrak{g}(F)_{[\lambda]}$ in $\mathfrak{g}(F)$ such that $\mathfrak{g}(F)_{[\lambda]} \otimes_F \bar{F} = \bigoplus_{\mu \in [\lambda]} \mathfrak{g}(\bar{F})_{\mu}$
the direct sum of the root space $\mathfrak{g}(\bar{F})_{\mu}$ for $\mu \in \Phi$. The second assertion is clear by Lemma 3.2.

We call the decomposition of $\text{End}_F(E) \cong \mathfrak{g}(F)$ in Proposition 3.3 the rational root space decomposition. We are going to show that such decomposition descends to the ones of certain $\mathfrak{o}_F$-sublattices of $\mathfrak{g}(F)$. Let $K$ be the maximal unramified extension of $F$ in $E$. Consider the following $\mathfrak{o}_F$-lattices contained in the $F$-vector spaces in (4)
\[
\mathfrak{o}_E \subset \text{End}_F(E), \quad \mathfrak{A} \subset \text{End}_F(E), \quad \bigoplus_{[g] \in \Gamma_F \setminus \Gamma_E} \mathfrak{o}_{gE} \subset \bigoplus_{[g] \in \Gamma_F \setminus \Gamma_E} \mathfrak{g}_E.
\]
They are all $E^\times$-conjugate invariant. Under the identification $\text{End}_F(E) \cong \mathfrak{gl}_n(F)$ and choosing suitable basis, the lattice $\mathfrak{A}$ can be expressed in matrices partitioned into $e \times e$ blocks of size $f \times f$, with entries in $\mathfrak{o}_F$, and is blocked upper triangular mod $p_F$.

Proposition 3.4. The isomorphisms in Proposition 3.1 induce isomorphisms of $\mathfrak{o}_F$-lattices as well as of $E^\times$-modules in (3).

Proof. The $\mathfrak{o}_F$-morphism $\mathfrak{o}_{E \otimes E} \to \mathfrak{A}$ restricted from (1) of Proposition 3.1 is clearly injective and $E^\times$-invariant. To show the surjectivity, we choose an $\mathfrak{o}_F$-basis $\{w_1, \ldots, w_n\}$ of $\mathfrak{o}_E$ such that
\[
\begin{array}{c}
v_E(w_1) = \cdots = v_E(w_1) = 0, \quad v_E(w_{f+1}) = \cdots = v_E(w_{2f}) = 1, \\
v_E(w_{(e-1)f+1}) = \cdots = v_E(w_{ef}) = e-1.
\end{array}
\]
We choose another $\mathfrak{o}_F$-basis $\{w_i^*\}$ of $\mathfrak{o}_E$ dual to $\{w_i\}$ in the sense that
\[
\text{tr}_{E/F}(w_i^* w_j) = 0 \text{ for } i \neq j \text{ and } \text{tr}_{E/F}(w_i^* w_i) = \begin{cases} 1 & \text{for } i = 1, \ldots, f \\ \mathfrak{W}_F & \text{for } i = f+1, \ldots, n. \end{cases}
\]
Then we can readily show that, under the isomorphism $E \otimes E \to \text{End}_F(E) \to \mathfrak{gl}_n(F)$, the element $\sum_{i,j} a_{ij}(w_i^* \otimes w_j)$ in $E \otimes E$ is mapped to the matrix $(A_{ij})$ where $A_{ij} = a_{ij} \text{tr}(w_i^* w_j)$. We check the $F$-valuation of these entries. Suppose that $w_i^*$ and $w_j$ corresponds to the $k$th and $\ell$th block respectively with respect to the chosen bases.

1. If $k = 1$, then we have $v_E(w_i^*) = v_E(w_i) = 0$ and so $a_{ij} \text{tr}(w_i^* w_i) = a_{ij} \in \mathfrak{o}_F$.
2. If $\ell = k \geq 1$, then $a_{ij} \in \mathfrak{o}_F \mathfrak{W}_F^{-1}$, $v_E(w_i^*) = e + 1 - k$ and $v_E(w_j) = \ell - 1$. Hence $a_{ij} \text{tr}(w_i^* w_i) \in \mathfrak{o}_F$.
3. When $k > \ell$, we have $a_{ij} \in \mathfrak{o}_F$ and $a_{ij} \text{tr}(w_i^* w_i) = a_{ij} \mathfrak{W}_F \in \mathfrak{p}_F$.

We have just shown that $\mathfrak{o}_{E \otimes E} \to \mathfrak{A}$ is surjective and therefore isomorphic.

To deal with another isomorphism, we use the standard technique in chapter 4 of [Rei03]. Let $M \subseteq N$ be two free $\mathfrak{o}_F$-modules of the same rank. Suppose the quotient $N/M$ is isomorphic to $\bigoplus_j \mathfrak{o}_F / \mathfrak{p}_F^n$ as $\mathfrak{o}_F$-modules. Define the order ideal of $\mathfrak{o}_F$ to be $\text{ord}_\mathfrak{o}_F(N/M) = \prod_j \mathfrak{p}_F^{n_j}$. We can compute the order alternatively as follows. If $M = \bigoplus_j \mathfrak{o}_F x_j$ and $N = \bigoplus_j \mathfrak{o}_F y_j$ such that $x_j = \sum_i a_{ij} y_j$ for some $a_{ij} \in \mathfrak{o}_F$, then $\text{ord}_\mathfrak{o}_F(N/M) = \mathfrak{O}_F \det(a_{ij})$. Suppose we have a trace form of $W = M \otimes \mathfrak{O}_F F$, i.e. a non-degenerate symmetric $F$-bilinear form $\text{tr} : W \times W \to F$, we define the discriminant ideal of $M$ with respect to the form $\text{tr}$ to be $\Delta(M) = \text{det} \text{tr}(x_j x_j) \mathfrak{O}_F$.
Lemma 3.5. (i) We have the relation $d(M) = \mathfrak{o}_F(N/M)^2d(N)$.

(ii) If $P$ and $Q$ are free $\mathfrak{o}_F$-modules, then

(a) $d(P \oplus Q) = d(P)d(Q)$, and

(b) $d(P \otimes_{\mathfrak{o}_F} Q) = d(P)^{\text{rank}Q}d(Q)^{\text{rank}P}$.

Proof. (i) comes from the identity $\det tr(x_1x_2) = \det(a_{ij})^2 \det tr(y_iy_j)$, (ii) is easy, and (iii) is an elementary calculation of the determinant of tensor product of matrices.

In the case when $E$ is a tame extension of $F$ and $M = \mathfrak{o}_E$, we denote the discriminant ideal $d(\mathfrak{o}_E)$ of $\mathfrak{o}_F$ by $d(E/F)$, which is known to be $p_{E/F}^{(e-1)}$.

Now we show that the second $\mathfrak{o}_F$-morphism $\mathfrak{o}_{E\otimes E} \rightarrow \bigoplus_{[g]} \mathfrak{o}_{\ast E\otimes E}$ is isomorphic. It is clearly injective and $E^\times$-invariant. To show that the map is surjective, we compare the sizes of $\bigoplus_{[g]} \mathfrak{o}_{\ast E\otimes E}$ and the image of $\mathfrak{o}_E \otimes_{\mathfrak{o}_F} \mathfrak{o}_E$. We apply Lemma 3.5(ii) on $M = \mathfrak{o}_E \otimes_{\mathfrak{o}_F} \mathfrak{o}_E, N = \bigoplus_{[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E} \mathfrak{o}_{\ast E\otimes E}$ and $P = Q = \mathfrak{o}_E$. Let $m$ be the $F$-valuation of the ideal

$$d(E/F)^{2n} \prod_{[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E} d(\mathfrak{o}_{E\otimes E}/F)^{-1}.$$ 

In the tame case, we can compute

$$m = 2nf(e-1) - (e-1) \sum_{[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E} f(\mathfrak{o}_{E\otimes E}/F).$$

The sum

$$\sum_{[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E} |\mathfrak{o}_{E\otimes E}/F| = \dim_F(E \otimes_F \mathfrak{o}_E) = n^2$$

equals $f\mathfrak{n}$, as we know that

$$\sum_{[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E} |\mathfrak{o}_{E\otimes E}/F| = \dim_F(E \otimes_F \mathfrak{o}_E) = n^2$$

and each $\mathfrak{o}_{E\otimes E}/F$ has the same ramification degree $e$. Therefore by (i) of Lemma 3.5 the order of

$$\left( \bigoplus_{[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E} \mathfrak{o}_{\ast E\otimes E} \right) / (\mathfrak{o}_E \otimes_{\mathfrak{o}_F} \mathfrak{o}_E)$$

is $q^{m/2} = qf^{2(e-1)/2}$. Using the expression of (i), we can check that the order of $\mathfrak{o}_E \otimes_F (\mathfrak{o}_E \otimes_{\mathfrak{o}_F} \mathfrak{o}_E)$ is also $qf^{2(e-1)/2}$. Therefore the morphism $\mathfrak{o}_{E\otimes E} \rightarrow \bigoplus_{[g]} \mathfrak{o}_{\ast E\otimes E}$ is surjective and hence isomorphic.

By similar argument we can show that the $\mathfrak{o}_F$-sublattices of the lattices in (5)

$$\mathfrak{P}_{E\otimes E} := \mathfrak{p}_E \otimes_{\mathfrak{o}_F} \mathfrak{p}_E \subseteq \mathfrak{o}_{E\otimes E},$$

$\mathfrak{P}_{\mathfrak{A}} \subseteq \mathfrak{A}$, and

$$\bigoplus_{[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E} \mathfrak{p}_E \subseteq \bigoplus_{[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E} \mathfrak{o}_{\ast E\otimes E}$$

are all isomorphic. We therefore have an $\mathfrak{k}_F$-isomorphism

$$\mathfrak{A}/\mathfrak{P}_A \cong \bigoplus_{[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E} \mathfrak{o}_{\ast E\otimes E}/\mathfrak{p}_E \subseteq \mathfrak{o}_{E\otimes E}$$

(6)

$$\mathfrak{A}/\mathfrak{P}_A \cong \bigoplus_{[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E} \mathfrak{o}_{\ast E\otimes E}/\mathfrak{p}_E = \bigoplus_{[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E} k_{\ast E\otimes E}$$

which is moreover $E^\times$-equivalent. The $E^\times$-conjugate action on both sides factor through $E^\times / F^\times U_1^E \cong \Psi_{E/F}$. We call the decomposition of $k_F \Psi_{E/F}$-module in (6) the residual root space decomposition.

Let’s describe the action of $\Psi_{E/F}$ on $\mathfrak{A}/\mathfrak{P}_A$ more precisely. Since $\mathfrak{A}/\mathfrak{P}_A$ is isomorphic to $M^e$ where $M = gl_f(k_F)$, we regard $M^e$ as being embedded into diagonal blocks in $gl_f(k_F)$. We first consider $M$ as
a $k_F\mu$-module. By embedding $k_E \hookrightarrow \gl_p(k_F)$ (the choice of such embedding is irrelevant), we have (from section 7.3 of \textbf{BH10})

$$M \cong \bigoplus_{i \in \mathbb{Z}/f} M_i$$

where $M_i \cong k_E$ as $k_F$-vector spaces and $\zeta \in \mu_E$ acts by $v \mapsto \zeta^{q^i}v$ for all $v \in M_i$. Each of the characters $\zeta \mapsto \zeta^{q^i}$, $i \in \mathbb{Z}/f$, is trivial on $\mu_F$, hence is a character of $\mu$. The $\Psi_{E/F}$-module we are interested in is

$$U = \text{Ind}_{\mu}^{\Psi_{E/F}} M.$$

Clearly $U \cong \mathfrak{A}/\mathfrak{P}_\mathfrak{A}$ as $k_F$-vector spaces.

\textbf{Theorem 3.6.} (i) The $\Psi_{E/F}$-module $U$ decomposes into submodules

$$U \cong \bigoplus_{k \in \mathbb{Z}/f} U_{ki}$$

such that for each component $U_{ki}$ the $\Psi_{E/F}$-action is given by $\lambda_{ki} : \zeta \mapsto \zeta^{q^i}$ for all $\zeta \in \mu_E$ and $

\varpi_E \mapsto \zeta^k \zeta^{q^j}.$

(ii) The decomposition of $U$ is equivalent to the residual root space decomposition for $\mathfrak{A}/\mathfrak{P}_\mathfrak{A}$.

\textbf{Proof:} (i) We have the decomposition $U = \bigoplus_{i \in \mathbb{Z}/f} U_i$ for $U_i = \text{Ind}_{\mu}^{\Psi_{E/F}} M_i$. Each $M_i$ is isomorphic to $k_E$, hence $U_i$ is an $e$-dimensional $k_E$-vector space. By \textbf{18}, we can choose a $k_E$-basis for $U_j$ such that $\varpi_E$ acts as conjugation of the matrix

$$
\begin{pmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & \\
\end{pmatrix}
$$

i.e. $\varpi$ cyclically permutes the components $k_E$, with the first component being permuted to the last followed by an action of $\zeta_{E/F} \in \mu_E$, which is the multiplication of $\zeta_{E/F}^{q^i}$. The eigenvalues of such matrix is $\zeta_k^{s_{q^j}}$ for some fixed $s$th root $\zeta_{q^j}$ of $\zeta_{E/F}^{q^i}$ and some $k = 0, \ldots, e - 1$. Hence those $\zeta_k^{s_{q^j}}$, in the same $\Gamma_{k_E}$-orbit, i.e. those $k \in \mathbb{Z}/e$ in the same $q^j$-orbit, form a simple $\Psi_{E/F}$-module. 

(ii) We use the identification in Proposition 5.2. If $g = s^k \phi^i$, then the action of $E^\times$ on $k_{E/E}$ as a component of $\mathfrak{A}/\mathfrak{P}_\mathfrak{A}$ is induced from the conjugate action on $^gEE$ and hence is given by $s^k \phi^i \zeta \mapsto \zeta^{q^i}$ for all $\zeta \in \mu_E$ and $\zeta^{q^i} \varpi_E \varpi_E^{-1} = \zeta^{q^i} \zeta^k \phi^i$. Hence $k_{E/E} = k_E[\zeta_k^{s_{q^j}}]$, which is just $U_{ki}$.

\hfill \Box

The $k_E\Psi_{E/F}$-module $U$ is called the standard module. Using the notation in Lemma 2.1, we write $U_{ki}$ as $U_{[\gamma \phi^i]}$. This notation is well-defined, i.e. the finite module $U_{[\gamma \phi^i]}$ is independent of the coset representative of $[\gamma \phi^i]$, because if $[\gamma] = [\eta]$ then $s^E\eta$ and $s^E\eta^\prime$ have the same residue field.

\section{Symplectic modules}

In this section we introduce a symplectic structure on certain submodules of the standard module $U = \mathfrak{A}/\mathfrak{P}_\mathfrak{A}$. We first give a brief summary on finite symplectic modules, whose details are referred to \textbf{BH10}. We consider a finite cyclic group $\Gamma$ whose order is not divisible by a prime $p$. We call a finite $\mathbb{F}_p\Gamma$-module symplectic if there is a non-degenerate $\Gamma$-invariant alternating form $h : V \times V \rightarrow \mathbb{F}_p$, i.e.

$$h(\gamma v_1, \gamma v_2) = h(v_1, v_2)$$

for all $\gamma \in \Gamma$, $v_i \in V$.

Let $V_{\lambda} = \mathbb{F}_p[\lambda(\Gamma)]$ be the simple $\mathbb{F}_p\Gamma$-module defined by $\lambda \in \text{Hom}(\Gamma, \mathbb{F}_p^\times)$. Its $\mathbb{F}_p$-linear dual is just $V_{\lambda^{-1}}$.

\textbf{Proposition 4.1.} (i) Any indecomposable symplectic $\mathbb{F}_p\Gamma$-module is one of the following two kinds.

(a) A hyperbolic module is of the form $V_{\lambda} \cong V_{\lambda^2} \oplus V_{\lambda^{-1}}$ such that either $\lambda^2 = 1$ or $V_{\lambda} \not\cong V_{\lambda^{-1}}$.

(b) An anisotropic module is of the form $V_{\lambda} \cong V_{\lambda^2} \oplus V_{\lambda^{-1}}$ such that $\lambda^2 \neq 1$ and $V_{\lambda} \cong V_{\lambda^{-1}}$.

(ii) If $V_{\lambda}$ is anisotropic, then $[\mathbb{F}_p[\lambda(\Gamma)]/\mathbb{F}_p]\cong \ker(N_{\mathbb{F}_p[\lambda(\Gamma)]/\mathbb{F}_p}[\lambda(\Gamma)])$ the kernel of the norm map of the quadratic extension $\mathbb{F}_p[\lambda(\Gamma)]/\mathbb{F}_p[\lambda(\Gamma)]^\pm$.

(iii) The $\Gamma$-isometry class of a symplectic $\mathbb{F}_p\Gamma$-module $(V, h)$ is determined by the underlying $\mathbb{F}_p \Gamma$-module $V$. 

Proof. All the proofs can be found in [BH10] chapter 3. □

We also call a symplectic $\mathbb{F}_p \Gamma$-module hyperbolic (resp. anisotropic) if it is a direct sum of hyperbolic (resp. anisotropic) indecomposable submodules. A special case is that if $V$ is anisotropic, then $V \oplus V$ is hyperbolic. By slightly abusing our terminology, we treat this as a special case of hyperbolic module, namely an even anisotropic module. Therefore by Proposition 4.1 (iii), given a finite symplectic module whose $\Gamma$-action is known, we do not have to know the alternating form exactly.

For each symplectic $\mathbb{F}_p \Gamma$-module $V$ we attach a sign $t^0_\Gamma(V) \in \{\pm 1\}$ and a character $t^1_\Gamma \left( \Gamma \right) : \Gamma \to \{\pm 1\}$, called the $t$-factors of $V$. We choose a generator $\gamma$ of $\Gamma$ and set $t_\Gamma(V) = t^0_\Gamma(V) t^1_\Gamma(V)(\gamma)$. We give the algorithm in [BH10] on computing the $t$-factors.

1. If $\Gamma$ acts on $V$ trivially, then
   
   $t^0_\Gamma(V) = 1$ and $t^1_\Gamma(V) \equiv 1$.

2. Let $V$ be an indecomposable symplectic $\mathbb{F}_p \Gamma$-module, then
   
   (a) If $V = V_\lambda \oplus V_{-\lambda}$ is hyperbolic, then
       
       $t^0_\Gamma(V) = 1$ and $t^1_\Gamma(V) = \text{sgn}_{\lambda(\Gamma)}(V_\lambda)$,
   
   where $\text{sgn}_{\lambda(\Gamma)}(V_\lambda) : \Gamma \to \{\pm 1\}$ such that $\gamma \mapsto \text{sgn}_{\lambda(\gamma)}(V_\lambda)$ is the sign of the multiplicative action of $\lambda(\gamma)$ on $V_\lambda$.
   
   (b) If $V = V_\lambda$ is anisotropic, then
       
       $t^0_\Gamma(V) = -1$ and $t^1_\Gamma(V)(\gamma) = \left( \frac{\gamma}{K} \right)$ for any $\gamma \in \Gamma$,
   
   where $K = \ker(N_{\mathbb{F}_p[\lambda(\Gamma)]/\mathbb{F}_p[\lambda(\Gamma)]^{\pm}})$ for the quadratic extension $\mathbb{F}_p[\lambda(\Gamma)]/\mathbb{F}_p[\lambda(\Gamma)]^{\pm}$ and ($-$) is the Jacobi symbol, i.e. for every finite cyclic group $H$,
   
   $$\left( \frac{x}{H} \right) = \begin{cases} 1, & \text{if } x \in H^2 \\ -1, & \text{otherwise} \end{cases}.$$

3. If $V$ decomposes into an orthogonal sum $V_1 \oplus \cdots \oplus V_m$ of indecomposable symplectic $\mathbb{F}_p \Gamma$-modules, then
   
   $t^1_\Gamma(V) = t^1_\Gamma(V_1) \cdots t^1_\Gamma(V_m)$ for $i = 0, 1$.

If $p = 2$, then the order of $\Gamma$ is odd. In this case $t^1_\Gamma(V)$ is always trivial because all sign characters and Jacobi symbols are trivial.

Remark 4.2. If $V_\lambda = \mathbb{F}_p[\lambda(\Gamma)]$ is anisotropic, then $V \oplus V$ is hyperbolic, or precisely even anisotropic. The $t$-factors are the same whether we consider $V \oplus V$ as hyperbolic or anisotropic. It is clear that $t^0_\Gamma(V \oplus V) = 1$ in both cases, while $t^1_\Gamma(V \oplus V) = \text{sgn}_{\lambda(\Gamma)}V$ in the hyperbolic case and $t^1_\Gamma(V \oplus V) = t^1_\Gamma(V)^2 = 1$ in the anisotropic case. Indeed $\text{sgn}_{\lambda(\Gamma)}V \equiv 1$. It is clear for $p = 2$. If $p$ is odd, then by Proposition 4.1 (iii) we have that $s = [\mathbb{F}_p[\lambda(\Gamma)]/\mathbb{F}_p]$ is even and $\lambda(\Gamma) \leq \mu_{r'/2+1}$. Therefore $[\mathbb{F}_p[\lambda(\Gamma)]^s/\mu_{r'/2+1}] = p^{s/2} - 1$ is even and $\text{sgn}_{\lambda(\Gamma)} V = (\text{sgn}_{\lambda(\Gamma)} \mu_{r'/2+1})^{p^{s/2}-1} = 1$. □

Suppose now $\Gamma$ is a cyclic subgroup of $\Psi_{E/F}$. We study the symplectic $\Gamma$-submodules of the standard $k_E \Psi_{E/F}$-module $U$. The cyclic subgroups $\mu$ and $\varpi$ of $\Psi_{E/F}$ are of our particular interest. For $\Gamma = \mu$ or $\varpi$, we regard $U$ as an $\mathbb{F}_p \Gamma$-module by restriction, and compute the $t$-factors of the symplectic $\mathbb{F}_p \Gamma$-submodule of $U$.

Recall in [Tamb] that $g \in (\Gamma \backslash \Gamma_F/\Gamma_F)$ is symmetric if $g = [g^{-1}]$ and is asymmetric otherwise. If $g$ is asymmetric, then $U_{\pm[g]} = U_{[g]} \oplus U_{[g^{-1}]}$ is a hyperbolic $\mathbb{F}_p \Gamma$-module. Write $[g] = [\sigma^k \phi]$ using the description in Proposition 2.1. If $\Gamma = \mu$ then we have

$t^0_{\mu}(U_{\pm[\sigma^k \phi]}) = 1$ and $t^1_{\mu}(U_{\pm[\sigma^k \phi]}) : \zeta \mapsto \text{sgn}_{\zeta^{s-1}}(U_{[\sigma^k \phi]})$ for all $\zeta \in \mu_E$.

In particular, if $i = f/2$, then both $U_{[\sigma^{k/2} \phi]}$ and $U_{([\sigma^{k/2} \phi]^{-1})}$ contain the $\mathbb{F}_p \mu$-module $\mathbb{F}_p(\sigma^{f/2} - 1(\mu_E)) = k_E$, which is anisotropic indecomposable (see Proposition 19 of [BH10]). Hence $U_{\pm[\sigma^{k/2} \phi]}$ is even anisotropic and

$t^0_{\mu}(U_{\pm[\sigma^{k/2} \phi]}) = 1$ and $t^1_{\mu}(U_{\pm[\sigma^{k/2} \phi]}) \equiv 1$. 

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If $\Gamma = \varpi$, then we have similarly

$$t_{\varpi}^0(U_{[\sigma^k \phi^i]}) = 1 \text{ and } t_{\varpi}^i(U_{[\sigma^k \phi^i]})(\varpi E) = \text{sgn}_{\zeta^k}^t(U_{[\sigma^k \phi^i]}).$$

Before we compute more $t$-factors we give some properties of symmetric $[g]$ and the corresponding submodule $U_{[g]}$.

**Lemma 4.3.** \( (i) \) The double coset $[g] = [\sigma^k \phi^i]$ is symmetric if and only if

(a) $i = 0$ or, when $f$ is even, $i = f/2$, and

(b) $e$ divides $(q^f + 1)k$ in case $i = 0$, and divides $(q^{(2f+1)/2} + 1)k$ in case $i = f/2$, for some $t = 0, \ldots, [L/E] - 1$.

(ii) Let $t = t_k$ be the minimal solution of $t$. Assume $[g] \neq [\sigma^{r/2}]$, then $|U_{[g]}|/k_E$ equals $2t_k$ in case $i = 0$, and equals $2t_k + 1$ in case $i = f/2$.

**Proof.** Since $\phi$ acts as $\sigma \mapsto \sigma^e$, we can show that the inverse of $\sigma^k \phi^i$ in $\Gamma_{L/E}$ is $\sigma^{-k} \phi^{-i}$ where $\bar{q}$ is the multiplicative inverse of $q$ in $(\mathbb{Z}/e)^\times$. Also recall that $\Gamma_{L/E} = \langle \phi^f \rangle$, so the double cosets $\Gamma_E \sigma^k \phi^i \Gamma_E$ and $\Gamma_E (\sigma^k \phi^i)^{-1} \Gamma_E$ are equal if and only if $i \equiv -i \mod f$ and $q^{tf} k \equiv -k \mod e$ for some $t$. This implies \((\dagger)\) by simple calculation. Since $U_{[g]}$ is the field extension of $k_E$ generated by $\zeta_k$, the degree of $U_{[g]}/k_E$ equals the minimal solution $d$ of $e|(q^{tf} - 1)k$. Therefore $d$ must be the one indicated in \((\dagger)\).

We write $U_{\pm[g]}$ the subfield of $U_{[g]}$ such that $|U_{[g]}|/U_{\pm[g]}| = 2$. Notice that $U_{\pm[g]}$ contains $k_E$ if and only if $i = 0$.

**Remark 4.4.** In the exception case when $[g] = [\sigma^{r/2}]$, we have $U_{[\sigma^{r/2}]} = k_E$. The minimal solution in Lemma 4.3 is $t_{\sigma^{r/2}} = 0$. Also notice that $[g]$ corresponds to the root $\lambda$ in $\Phi$ that satisfies $\lambda^2 = 1$. As we will see in Proposition 4.5, not all submodules of $U$ admit symplectic structures. This implies in particular to $U_{[\sigma^{r/2}]}$. We only study $U_{[g]}$ with $[g] \neq [1]$ and, in the case $e$ is even, $[g] \neq [\sigma^{r/2}]$.

Now we compute the $t$-factors for symmetric $[g]$. Suppose $[g] = [\sigma^k]$ that $k \neq 0$ or $e/2$. The group $\Psi_{E/F}$ acts by the character $\lambda_{k0}$ where $\lambda_{k0}|_{\mu} = 1$ and $\lambda_{k0}(\varpi E) = \zeta_k$. Since $\mu_E$ acts trivially, we have

$$t_{\varpi}^0(U_{[\sigma^k]}) = 1 \text{ and } t_{\mu}^i(U_{[\sigma^k]}) \equiv 1.$$

To compute $t_{\varpi}^i(U_{[\sigma^k]})$, $i = 0, 1$, we consider $U_{[\sigma^k]}$ as a $k_E[\zeta_k]$-module. Each simple submodule $F_p[\zeta_k]$ of $U_{[\sigma^k]}$ is anisotropic. We have the following property about its multiplicity.

**Lemma 4.5.** The degree $r = r_{[\sigma^k]} = [k_E[\zeta_k]/F_p[\zeta_k]]$ is odd.

**Proof.** We see that $\lambda_{k0}(\varpi)$ is contained in $\ker(N_{U_{[\sigma^k]}}, U_{[\sigma^k]})$ the group of $(q^tf_k + 1)$-roots in $U_{[\sigma^k]}$. Suppose that $r$ is even. If $F_p[\zeta_k]$ has $Q$ elements, then $U_{[\sigma^k]}$ has $Q^r = q^{2tf_k}$ elements. We have $\lambda_{k0}(\varpi) \subseteq F_p[\zeta_k]^{\times} \cap \ker(N_{U_{[\sigma^k]}}, U_{[\sigma^k]}) = \mu_{Q-1} \cap \mu_{Q^r/2+1} \subseteq \{\pm 1\}$, forcing $k = 0$ or $e/2$. This is a contradiction.

By Lemma 4.5 we have

$$t_{\varpi}^0(U_{[\sigma^k]}) = (-1)^r = -1 \text{ and } t_{\mu}^i(U_{[\sigma^k]} : \varpi E) \mapsto \left(\frac{\zeta_k}{\mu_{p^r+1}}\right)^r = \left(\frac{\zeta_k}{\mu_{p^r+1}}\right),$$

where $[F_p[\zeta_k]/F_p] = 2s$ and $\mu_{p^r+1} = \ker(N_{F_p[\zeta_k]/F_p}[\zeta_k])$ for the quadratic extension $F_p[\zeta_k]/F_p[\zeta_k]$.\(^{\ddagger}\)

Now suppose $[g] = [\sigma^k \phi^i/2]$. We first consider $U_{[\sigma^k \phi^i/2]}$ as an $F_p\mu$-module. Since $\lambda_{k, f/2}(\mu_E) = \ker(N_{k_E/k_E}) = \mu_{q^{f/2}+1}$ and $F_p[\lambda_{k, f/2}(\mu_E)] = k_E$ is anisotropic, we have

$$t_{\mu}^0(U_{[\sigma^k \phi^i/2]}) = (-1)^{2t_k+1} = -1$$

and

$$t_{\mu}^i(U_{[\sigma^k \phi^i/2]} : \zeta) \mapsto \left(\frac{\zeta/q^{f/2-1}}{\mu_{q^{f/2}+1}g_i+1}\right)^{2t_k+1} = \left(\frac{\zeta/q^{f/2-1}}{\mu_{q^{f/2}+1}g_i+1}\right) \text{ for all } \zeta \in \mu_E.$$

For $U_{[\sigma^k \phi^i/2]}$ as an $F_p\varpi$-module, the action of $\varpi E$ is the multiplication of $\lambda_{k, f/2}(\varpi E) = \zeta_k^k \zeta_{\phi^i/2}$. We distinguish this value into the following cases.
We need to specify the notations in (7).

5. Admissible characters

We first fix an additive character \( \psi_F : F \to \mathbb{C}^\times \) with level 0, i.e. \( \psi_F \) is trivial on \( \mathfrak{p}_F \) but not on \( \mathfrak{o}_F \). For any field extension \( E/F \), we write \( \psi_E = \psi_F \circ \mathfrak{t}_E/F \). Given a character \( \xi : E^\times \to \mathbb{C}^\times \), the \( E \)-level \( r_E(\xi) \) of \( \xi \) is the minimum integer \( r \) such that \( \xi|_{\mathfrak{u}_E^{r+1}} \equiv 1 \). We say that \( \xi \) is \textit{tamely ramified}, or just tame, if \( r_E(\xi) = 0 \). It is called \textit{admissible} over \( F \) if it satisfies the following conditions.

(i) If \( \xi = \eta \circ N_{E/K} \) for some character \( \eta \) of \( K^\times \), then \( E = K \).

(ii) If \( \xi|_{U_E^r} = \eta \circ N_{E/K} \) for some character \( \eta \) of \( U_K^r \), then \( E/K \) is unramified.

Clearly if \( \xi \) is admissible over \( F \), then by definition it is admissible over \( K \) for every field \( K \) between \( E \) and \( F \). Such character admits a \textit{Howe-factorization} [Moy86] of the form

\[ \xi = (\xi_{d+1} \circ N_{E/F})(\xi_d \circ N_{E/E_d}) \cdots (\xi_0 \circ N_{E_0/E_0})\xi_1. \]

We need to specify the notations in (7).

(i) We have a decreasing sequence of fields

\[ E = E_1 \supseteq E_0 \supseteq E_1 \cdots \supseteq E_d \supseteq E_{d+1} = F. \]

Each \( \xi_i \) is a character of \( E_i^\times \), and \( \xi_d \) is a character of \( F^\times \).

(ii) Let \( r_i \) be the \( E \)-level of \( \xi_i \), i.e. the \( E \)-level of \( \xi_i \circ N_{E/E_i} \), and \( r_{d+1} \) be the \( E \)-level of \( \xi \). We assume that \( \xi_{d+1} \) is trivial if \( r_{d+1} = 1 \). We call the \( E \)-levels \( r_0 < \cdots < r_d \) the jumps of \( \xi \).

(iii) If \( E_0 = E \), then we replace \( (\xi_0 \circ N_{E_0/E_0})\xi_1 \) by \( \xi_0 \). If \( E_0 \subsetneq E \) we have that \( \xi_1 \) is tame and \( E/E_0 \) is unramified.

We define the wild component of \( \xi \) to be \( \Xi_0 \circ N_{E_0/E_0} \) where \( \Xi_0 = (\xi_{d+1} \circ N_{E_{d+1}/F}) \cdots (\xi_1 \circ N_{E_0/E_1})\xi_0 \) and the tame component of \( \xi \) to be \( \xi_1 = \xi(\Xi_0 \circ N_{E_0/E_0})^{-1} \).

\textbf{Proposition 5.1.} For \( i = 0, \ldots, d \), we have the following.

(i) If \( s_i \) is the \( E_i \)-level of \( \xi_i \), then \( s_i E(E/E_i) = r_i \).

(ii) There is a unique \( \alpha_i \in \langle \mathfrak{w}_{E_i}^e \rangle \times \mu_{E_i} \) such that \( v_{E_i}(\alpha_i) = -s_i \) and \( \xi_i|_{U_{E_i}^{r_i}} (1+x) = \psi_{E_i}(\alpha_i x) \).

(iii) Each character \( \xi_i \) is generic over \( E_{i+1} \), in the sense that \( E_{i+1}[\alpha_i] = E_i \).

(iv) We have the relation \( \gcd(r_i, e(E/E_{i+1})) = e(E/E_i) \).

\textbf{Proof.} (i) comes from an elementary calculation of the image of \( N_{E/E_i} \). (ii) and (iii) can be found in section 2.2 of [Moy86]. For (iv), that \( E_{i+1}[\alpha_i] = E_i \) in (iii) implies that \( \gcd(v_{E_i}(\alpha_i), e(E_i/E_{i+1})) = 1 \). Then (ii) and (iv) imply the desired result. \( \square \)
Let $P(E/F)$ be the set of admissible characters of $E^\times$ over $F$. Two admissible characters $\xi, \xi' \in P(E/F)$ are called equivalent if there is $g \in \Gamma_F$ that $gE = E'$ and $\xi = \xi'$. We denote the equivalence class of $\xi$ by $(E, \xi)$. Let $P_n(F)$ be the set of equivalence classes of admissible characters, i.e.

$$ P_n(F) = \text{equivalence classes of } P(E/F) $$

for $E$ goes through tame extensions over $F$ of degree $n$.

### 6. Supercuspidal representations

We briefly describe how to parameterize essentially tame supercuspidals by admissible characters. The details can be found in [Moy86], [BK93], and are also summarized in [BH93a]. We first introduce certain subgroups of $GL_n(F)$. Let $E \supseteq E_0 \supseteq E_1 \cdots \supseteq E_d \supseteq F$ be a decreasing sequence of fields. Write $B_i = \text{End}_{E_i}(E)$ and define

$$ \mathfrak{B}_i = \{ x \in B_i | xP_E \subseteq P_E \} \text{ for all } k \in \mathbb{Z} \} \text{ and } \mathfrak{B}_i^{j} = \{ x \in B_i | xP_E \subseteq P_E^{k+1} \} \text{ for all } k \in \mathbb{Z} \}
$$

the hereditary orders in $B_i$ corresponding to the $\sigma E_i$-lattice chain $\{ P_E^k | k \in \mathbb{Z} \}$ of the $E_i$-vector space $E$ and its radical. We then define subgroups of $B_i^+$

$$ U_{\mathfrak{B}_i} = \{ x \in B_i^+ | xP_E \subseteq P_E^k \text{ for all } k \in \mathbb{Z} \} \text{ and } U_{\mathfrak{B}_i}^j = 1 + \mathfrak{B}_i^j, \text{ for } j > 0. $$

If $E/E_0$ is unramified, then we can replace $P_E$ by $P_{E_0}$ in (9) and (10). We denote $A = \text{End}_F(E)$ and define $\mathfrak{A}$, $\mathfrak{B}_3$, $\mathfrak{A}_3$, and $U_{\mathfrak{A}_3}$ as in (9) and (10) with $B_i$ replaced by $A$. The multiplication of $E$ identifies $E$ as a subspace of $A$. Choose an isomorphism of $A^\times \cong GL_n(F)$ so that $E^\times$ embeds into $GL_n(F)$ by an $F$-regular morphism. Then $\mathfrak{A}$, $\mathfrak{B}_3$, $\mathfrak{A}_3$, $U_{\mathfrak{A}_3}$, and $U_{\mathfrak{B}_3}$ embed into $GL_n(F)$ accordingly.

If the fields $E_i$ are defined by the Howe-factorization of $\xi \in P(E/F)$ as in (7) with jumps $\{ r_0, \ldots, r_d \}$, we define two numbers $j_i$ and $h_i$ by

$$ j_i = \left\lfloor \frac{r_i + 1}{2} \right\rfloor \leq h_i = \left\lfloor \frac{r_i}{2} \right\rfloor + 1. $$

Here $\lfloor x \rfloor$ is the largest integer not greater than $x$. We construct the subgroups

$$ H^1(\xi) = U_{\mathfrak{B}_3}^{j_0}U_{\mathfrak{B}_3}^{j_0} \cdots U_{\mathfrak{B}_3}^{j_0-1}U_{\mathfrak{A}_3}^{h_0}, $$

$$ J^1(\xi) = U_{\mathfrak{B}_3}^{j_0}U_{\mathfrak{B}_3}^{j_0-1} \cdots U_{\mathfrak{B}_3}^{j_0-1}U_{\mathfrak{A}_3}^{h_0} \subseteq J(\xi) = U_{\mathfrak{B}_3}^{j_0}U_{\mathfrak{B}_3}^{j_0} \cdots U_{\mathfrak{B}_3}^{j_0-1}U_{\mathfrak{A}_3}^{h_0}, \text{ and } $$

$$ J(\xi) = E^xJ(\xi) = E_0^xJ(\xi). $$

We abbreviate these groups by $H^1$, $J^1$, $J$ and $\mathbf{J}$ if the admissible character $\xi$ is fixed. Notice that $H^1$, $J^1$, $J$ are compact subgroups and $\mathbf{J}$ is a compact-mod-center subgroup.

Now we briefly describe the construction in [BH93a] of the supercuspidal representation from an admissible character $\xi$ in five steps.

1. From the Howe factorization (7) of $\xi$ we can define a character $\theta = \theta(\xi)$ on $H^1$. This character depends only on the wild component $\Xi_0 \circ N_{E/E_0}$ of $\xi$. In fact according to the definition of simple characters in [BK93], there can be a number of such characters associated to $\xi$. There is a canonical one $\theta(\xi)$ constructed in [Moy86], which is called the simple character of $\xi$ in this article.

2. By classical theory of Heisenberg representation, we can extend $\theta$ to a unique representation $\eta$ of $J^1$ using the symplectic structure of $V = J^1/H^1$ defined by $\theta$.

3. There is a unique extension $\Lambda_0 = \Lambda(\Xi_0 \circ N_{E/E_0})$ of $\eta$ on $\mathbf{J}$ satisfying the conditions in Lemma 1 and 2 of section 2.3 [BH93a]. The restriction $\Lambda_0|J$ is called $\beta$-extension of $\eta$, in the sense of (5.2.1) [BK93].

4. We still need another representation $\Lambda(\xi_{-1})$ on $\mathbf{J}$, which is defined by the tame component $\xi_{-1}$ of $\xi$. Suppose $\xi_{-1}|_{U_E}$ is the inflation of a character $\xi_{-1}$ of $k_E$. We first apply Green’s parametrization to obtain a unique (up to isomorphism) irreducible cuspidal representation $\lambda_{-1}$ of $GL_{1}(F/k_{E_0}) = J/J^1$, then inflate $\lambda_{-1}$ to a representation $\lambda_{-1}$ on $\mathbf{J}$, and finally multiply $\xi(\varpi_E)$ to obtain $\Lambda(\xi_{-1})$ on $\mathbf{J} = (\varpi_E)\mathbf{J}$.

5. The supercuspidal is defined by $\pi_\xi = C\text{Ind}_{\mathbf{J}}^E(\Lambda(\xi_{-1}) \otimes \Lambda_0)$. 
Remark 6.1. The wild component \( \xi_w \) and tame component \( \xi_t \) of \( \xi \) is defined alternatively in [BH05a]. We briefly explain that these choices produce the same representation on \( J \). By construction we have \( \xi = (\Xi_0 \circ N_{E/E_0})\xi_{-1} = \xi_w \xi_t \). Since \( \xi_w|_{U^1_F} = (\Xi_0 \circ N_{E/E_0})|_{U^1_F} \), they induce the same simple character \( \theta = \theta(\xi) \). Therefore we have isomorphism of the \( \beta \)-extensions \( \Lambda(\xi_w) = \Lambda_0 \otimes \alpha \) where \( \alpha \) is a tamely ramified character \( \alpha \) on \( E^\times U_{B_0}/U^1_{B_0} \) such that \[ \alpha|_{U_{B_0}} = \Xi_0|_{\mu_{E_0}} \otimes \det_{k_{E_0}} \circ (\proj_{J/J_1}) \text{ and } \alpha(\varpi_E) = \xi_w^{-1}(\varpi_E)(\Xi_0 \circ N_{E/E_0})(\varpi_E). \]

Here \( \proj_{J/J_1} \) is the natural projection \( J \to J/J^1 \cong GL_{E/E_0}(k_{E_0}) \). (Compare this to (5.2.2) of [BK93] concerning \( \beta \)-extensions.) On the other hand, it can be checked that \( \Lambda(\xi_t) = \Lambda(\xi_{-1}) \otimes \alpha^{-1} \). Indeed by construction in [BH05a] \( \xi_w \) is trivial on \( \mu_{E_0} \). This implies that the Green’s representations \( \tilde{\lambda}_0 \) and \( \lambda_1 \otimes (\Xi_0 \circ \det_{k_{E_0}}) \) on \( GL_{E/E_0}(k_{E_0}) \) are isomorphic. Therefore \( \Lambda(\xi_t) \otimes \Lambda(\xi_w) = \Lambda(\xi_{-1}) \otimes \Lambda_0 \). With the Howe factorization of \( \xi \) in hand, it is more natural to define our wild and tame component of \( \xi \) as in the five steps in the preceding paragraph.

For each irreducible supercuspidal representation \( \pi \) of \( GL_n(F) \) let \( f(\pi) \) be the number of unramified characters \( \chi \) of \( F^\times \) that \( \chi \otimes \pi \cong \pi \). Here \( \chi \) is regarded as a representation of \( GL_n(F) \) by factoring through the determinant map. We have that \( f(\pi) \) divides \( n \) by considering central characters. Recall that \( \pi \) is essentially tame if \( p \) does not divide \( n/f(\pi) \). Let \( \mathcal{A}_{\pi}^t(F) \) be the set of isomorphism classes of essentially tame irreducible supercuspidals.

Proposition 6.2. The map \( P_n(F) \to \mathcal{A}_{\pi}^t(F) \), \( (E, \xi) \to \pi_\xi \) is a bijection, with \( f(\pi_\xi) = f(\pi)/F \).

Proof. The proof is summarized in chapter 2 of [BH05a].

We analyze the group extensions in step 2 and 3 in the five steps of constructing supercuspidals. Since the group \( J \) normalizes \( J^1 \) and \( H^1 \), it acts on the quotient group \( V = V(\xi) = J^1/H^1 \). We usually regard \( V \) as an \( \mathbb{F}_p \)-vector space. We have a direct sum \( V = V_0 \oplus \cdots \oplus V_d \), where \( V_i = U_{B_{i+1}/B_i} U_{B_{i+1}}/U_{B_i} U_{B_{i+1}} \). By the definitions in [11], the module \( V_i \) is non-trivial if and only if the jump \( r_i \) is even, in which case we have \( V_i \cong B_{i+1}/B_i \times \mathbb{P}_{B_{i+1}} \). We call this sum the coarse decomposition of \( V \).

Proposition 6.3. Let \( H^1, J^1, J, V_i, \text{ and } \theta \) be those previously described.

(i) The commutator subgroup \( [J^1, J^1] \) lies in \( H^1 \).
(ii) The group \( J \) normalizes each component \( V_i \) and the simple character \( \theta \).
(iii) The simple character \( \theta \) induces a non-degenerated alternating \( \mathbb{F}_p \)-bilinear form \( h_\theta : V \times V \to \mathbb{C}^\times \) such that the coarse decomposition is an orthogonal sum.

Proof. Some of the proofs can be found in [BK93] chapter 3, for example (1) is in (3.1.15), the non-degeneracy of the alternating form in (11) is in (3.4), and that \( J \) normalizes \( \theta \) in (11) is from (3.2.3). That \( J \) normalizes each \( V_i \) in (11) is clear by definition. That the coarse decomposition is orthogonal in (11) is from 6.3 of [BH05c].

What are we interested in is the conjugate action of \( E^\times \) on \( V \) restricted form \( J \). By Proposition 6.3(i), the action of \( J \), and hence that of \( E^\times \), preserves the symplectic structure defined by \( \theta \). By Proposition 6.3(ii), the subgroup \( J^1 \) of \( J \) acts trivially on \( V \), so the \( E^\times \)-action factors through \( E^\times /F^\times (E^\times \cap J^1) \cong \Psi_{E/F} \). Hence \( V \) is moreover a finite symplectic \( \mathbb{F}_p \Gamma \)-module for each cyclic subgroup \( \Gamma \) of \( \Psi_{E/F} \). By construction the \( \mathbb{F}_p \Psi_{E/F} \)-module \( V \) is always a submodule of the standard one \( U = \mathfrak{A}/\mathfrak{P}_\mathfrak{A} \). We denote the \( U_{[g]} \)-isotypic component in \( V \) by \( V_{[g]} \), and call the decomposition \[ V = \bigoplus_{[g] \in (\Gamma_{E^\times F}/\Gamma_{E^\times})' \cap \Gamma_{E^\times}} V_{[g]} \]
the complete decomposition of \( V \).

Theorem 6.4. The complete decomposition of \( V \) is an orthogonal sum with respect to the alternating form \( h_\theta \).

Proof. Recall the bijection \( \Gamma_{E^\times F} \to (\Gamma_{E^\times F}/\Gamma_{E^\times})' \in \text{Lemma 3.2} \) and write \( V_{[\lambda]} \) as \( V_{[\lambda]} \) for suitable \( [\lambda] \in \Gamma_{E^\times F} \). For every \( \lambda \) and \( \mu \in \Phi \) such that \( \lambda \neq \mu \) or \( \mu^{-1} \), choose a finite field extension of \( \mathbb{F}_p \), for example \( \mathbb{F}_p(\Psi_{E/F}, \lambda) \), such that \( V_{[\lambda]} \otimes \mathbb{F}_p \mathbb{F}_p(\lambda) \) and \( V_{[\mu]} \otimes \mathbb{F}_p \mathbb{F}_p(\lambda) \) decomposes into eigenspaces of \( \Psi_{E/F} \).
Let \( v \in V_{[\lambda]} \) and \( w \in V_{[\mu]} \). We can assume that \( v \) and \( w \) are respectively contained in certain eigenspaces of \( V_{[\lambda]} \otimes_{\mathbb{F}_p} \mathbb{F}_{\lambda_\mu} \) and \( V_{[\mu]} \otimes_{\mathbb{F}_p} \mathbb{F}_{\lambda_\mu} \). There is a unique \( \Psi_{E/F} \)-invariant alternating bilinear form \( \tilde{h} \) of \( V \otimes_{\mathbb{F}_p} \mathbb{F}_{\lambda_\mu} \) extending \( h = h_\theta \). Therefore \( \tilde{h}(v, w) = \tilde{h}(tv, tw) = \lambda(t) \mu(t) \tilde{h}(v, w) \) for all \( t \in \Psi_{E/F} \). The fact that \( \lambda \neq \mu^{-1} \) implies \( h(v, w) = \tilde{h}(v, w) = 0 \).

For our purpose it is not necessary to know the form \( h_\theta \) exactly. Indeed by Proposition 6.1(iv) the symplectic structure of \( V \) is determined by its underlying \( \mathbb{F}_p \Psi_{E/F} \)-module structure. We conclude this section by proving a promised fact, that not all components of \( U \) appears in the symplectic module \( V \).

**Proposition 6.5.** Let \( E/F \) be a tame extension and \( \xi \) run through all admissible characters in \( P(E/F) \).

(i) If \([g] \in \Gamma_E \backslash \Gamma_{E_0}/\Gamma_E = \Gamma_{E_0}/\Gamma_E \), then \( V_{[g]} \) is always trivial.

(ii) If \( e = e(E/F) \) is even, then \( V_{[\sigma^{e/2}]} \) is always trivial.

**Proof.** The first statement is clear from the definition of \( J^1(\xi) \) and \( H^1(\xi) \) in [12]. For the second statement, let \( E_j \supseteq E_{j+1} \) be the intermediate subfields in \([8]\) such that \( e(E_j/E_{j+1}) \) is even and \( e(E_j/E_{j-1}) \) is odd. By Proposition 6.1(iv) the jump \( r_j \) must be odd, so \( V_j \) is trivial. Since \( \sigma^{e/2} \in \Gamma_{E_{j+1}} - \Gamma_{E_j} \), the component \( V_{[\sigma^{e/2}]} \) is contained in \( V_j \) and hence is also trivial. □

A final remark here is that we can easily replace our indexes from the non-trivial double cosets \( (\Gamma_E \backslash \Gamma_{E_0}/\Gamma_E)' \) to the orbits of roots \( \Gamma_E \backslash \Phi \). We can then prove Theorem 1.1 and Corollary 1.2 in a straightforward manner. Indeed Theorem 1.1(b) is clear from the description of the standard module \( U \) in Theorem 3.6 while (b) is direct from Theorem 6.4 and Proposition 6.5. Corollary 1.2 is just an easy observation from Theorem 6.4.

### 7. An application on local Langlands correspondence

For establishing the essentially tame local Langlands correspondence, we include the following fact. Let \( W_E \) be the Weil group of \( F \). For each irreducible complex semi-simple \( n \)-dimensional representation \( \sigma \) of \( W_E \), let \( f(\sigma) \) be the number of unramified characters of \( W_E \) that \( \chi \otimes \sigma \cong \sigma \). We have that \( f(\sigma) \) divides \( n \) by considering determinants of representations. We call \( \sigma \) essentially tame if \( p \) does not divide \( n/f(\sigma) \). Let \( \mathcal{G}_n(F) \) be the set of equivalence classes of essentially tame irreducible representations of degree \( n \). We have the following result.

**Proposition 7.1.** The canonical map \( \sigma : P_n(F) \to \mathcal{G}_n^e(F), (E, \xi) \mapsto \sigma(\xi) = \text{Ind}_{E/F}^E \xi \) is a bijection, with \( f(\sigma(\xi)) = f(E/F) \).

**Proof.** See the Appendix of [BH05a]. □

Let

\[
\mathcal{L} = \mathcal{L}_n^e : \mathcal{G}_n^e(F) \to \mathcal{A}_n^e(F)
\]

be the essentially tame local Langlands correspondence. The ‘naïve’ correspondence \( \mathcal{G}_n^e(F) \to P_n(F) \) does not satisfy all conditions of the essentially tame local Langlands correspondence. In other words, the composition

\[
\mu : P_n(F) \to \mathcal{G}_n^e(F) \xrightarrow{\xi} \mathcal{A}_n^e(F) \xrightarrow{\pi^{-1}} P_n(F)
\]

does not give the identity map on \( P_n(F) \). In [BH10] it is proved that for each admissible character \( \xi \) of \( E^\times \), there is a tame character \( F \mu_\xi \) of \( E^\times \), called the rectifier of \( \xi \), such that \( F \mu_\xi \) is also admissible and

\[
\mu(E, \xi) = (E, F \mu_\xi \xi).
\]

The rectifier \( F \mu_\xi \) is explicitly described in [BH05a], [BH05b], [BH10], and so is the correspondence \( \mathcal{L} \). Each of its value is a product of \( t \)-factors with a factor of a product of Gauss sums. The \( t \)-factors are those \( t_1(V_j) \) for \( i = 0 \) or 1, the group \( \Gamma = \mu \) or \( \varpi \), and \( V_j \) a component in the coarse decomposition of \( V \). With our complete decomposition of \( V \) in hand, we can further decompose these values. Hence we obtain a product of \( t_1(V_{[g]}) \). It can be shown [Tama] that these factors constitute a tamely ramified character \( \mu_{[g], \xi} \) of \( E^\times \). We can write

\[
F \mu_\xi = \prod_{[g] \in (W_E \backslash W_F/W_E)^t} \mu_{[g], \xi}.
\]

Such factorization is related to a previous result about admissible embedding of \( L \)-groups [Tamb]. We write \( G = \text{GL}_n \) and \( T = \text{Res}_{E/F} \mathbb{G}_m \). Let \( LG = \hat{G} \times W_F \) and \( LT = \hat{T} \times W_F \) be the corresponding \( L \)-groups.
Langlands and Shelstad described in general how to construct admissible embedding $L_T \to L_G$, in the sense of [LS87]. It depends on a choice of characters $\{\chi_\lambda\}$ called $\chi$-data. Here each character $\chi_\lambda$ is defined on the multiplicative group $E^*_\lambda$ for certain field extension $E_\lambda/E$ and the parameter $\lambda$ runs through certain subset in the root system $\Phi = \Phi(G,T)$ that represents the $W_F$-orbits of $\Phi$. These characters satisfy certain conditions of symmetry inherited from a chosen positivity of $\Phi$ and its $W_F$-action. By the identification in Lemma 3.2, we can write our $\chi$-data as $\{\chi_g\}$ where $g$ runs through a set in $W_F$ representing the non-trivial double cosets in $(E/F/W_F/W_E)$.

For each character $\xi$ of $T(F) = E^*$ we can attach a morphism $\tilde{\xi} : W_F \to L_T$ (see section 2 of [Tamb]), called a Langlands parameter of $\xi$. Suppose $\chi = \chi(\chi_\lambda)$ : $L_T \to L_G$ is the admissible embedding defined by $\chi$-data $\{\chi_g\}$. Write $\mu = \mu(\chi_\lambda) = \prod g \chi_g|E^*$. The following is proved in [Tamb].

**Proposition 7.2.** The representation of $W_F$ defined by $\chi \circ \tilde{\xi} : W_F \to L_T \to L_G \to \text{GL}_n(\C)$ is isomorphic to $\text{Ind}_{E/F}(\xi \mu)$.

The following theorem is a relation between $\chi$-admissible embedding and the essentially tame local Langlands correspondence, whose proof is in [Tama].

**Theorem 7.3.** For each $\xi \in P(E/F)$, the factor $\mu_{[\xi]}$ of its rectifier $F \mu_\xi$ is of the form $\chi_{g,\xi}|_{E^*}$ for some canonically chosen $\chi$-data $\{\chi_{g,\xi}\}$.

The values of each $\chi_{g,\xi}$ is a product of $t$-factors of $V[g]$ as mentioned. Comparing Theorem 7.3 to Proposition 7.2, we know that the essentially tame local Langlands correspondence should be described as an induction process twisted by a character constructed by canonical choices of $\chi$-data $\{\chi_{g,\xi}\}$. In other words, we can therefore interpret the essentially tame local Langlands correspondence in a reversed way

$$
\pi_{\xi} \xrightarrow{\xi^{-1}} \sigma_{F \mu_{\xi}^{-1} \xi} = \text{Ind}_{E/F}(F \mu_{\xi}^{-1} \xi) = \chi_{(\chi_{g,\xi})^{-1}} \circ \tilde{\xi},
$$

where $\chi_{(\chi_{g,\xi})^{-1}} : L_T \to L_G$ is the admissible embedding defined by $\{\chi_{g,\xi}\}$. Hence the rectifier enjoys the properties, e.g. the symmetry structure, inherited from that of $\{\chi_g\}$. We will study these properties after proving Theorem 7.3 in [Tama]. In a nutshell, the essentially tame local Langlands correspondence can be expressed in terms of the admissible embeddings of L-groups defined by $\chi$-data.

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