ROTATIONAL HYPERSURFACES FAMILY SATISFYING $L_{n-3}G = AG$ IN THE $n$-DIMENSIONAL EUCLIDEAN SPACE

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ABSTRACT. In this paper, we investigate rotational hypersurfaces family in $n$-dimensional Euclidean space $E^n$. Our focus is on studying the Gauss map $G$ of this family with respect to the operator $L_k$, which acts on functions defined on the hypersurfaces. The operator $L_k$ can be viewed as a modified Laplacian and is known by various names, including the Cheng–Yau operator in certain cases. Specifically, we focus on the scenario where $k = n - 3$ and $n \geq 3$. By applying the operator $L_{n-3}$ to the Gauss map $G$, we establish a classification theorem. This theorem establishes a connection between the $n \times n$ matrix $A$, and the Gauss map $G$ through the equation $L_{n-3}G = AG$.

1. Introduction

Chen [6] introduced the challenge of categorizing finite type surfaces within the three dimensional Euclidean space $E^3$. A Euclidean submanifold is designated as having Chen’s finite type characteristic when its coordinate functions can be expressed as a limited combination of its $\Delta$ Laplacian (or Laplace–Beltrami operator)’s eigenfunctions [3, 6]. Additionally, the concept of finite type can be expanded to encompass smooth functions on a submanifold existing within either a Euclidean space or a pseudo-Euclidean space. Readers can refer to [4, 7, 9, 10, 12, 15, 27, 28, 31, 32] for other studies related to the subject mentioned above.

On the other hand, the extended version of the Laplace–Beltrami operator $L_0 = \Delta$ is referred to as the Cheng–Yau operator $L_1 = \Box$. The general operator is denoted by the $L_k$ operator.

Numerous investigations carried out concerning the previously mentioned topics. For instance, Kim and Turgay [18] studied surfaces with $L_1$-pointwise 1-type Gauss map in $E^4$. Kim-et al. [17] focused on $L_1$ operator and Gauss map of surfaces of revolution in $E^3$. Güler and Turgay [14] studied $L_1$ operator and Gauss map of rotational hypersurfaces in $E^4$. Mohammadpour, Kashani, and Pashaie [26] introduced $L_1$-finite type Euclidean surfaces. Kashani [16] studied $L_1$-finite type (hyper)surfaces in $E^{n+1}$.

Alias and Kashani [2] gave hypersurfaces in space forms satisfying the condition $L_k x = Ax + b$, where $A \in \mathbb{R}^{(n+2) \times (n+2)}$ is a constant matrix, $b \in \mathbb{R}^{n+2}$ is a constant vector. Lucas and Ramirez-Ospina [20, 21] worked hypersurfaces in the Lorentz-Minkowski space satisfying $L_k \psi = A\psi + b.$

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See also [22, 23] for their works. Mohammadpour and Kashani [25] studied quadric hypersurfaces of \( L_r \)-finite type. Pashaie and Kashani [29] considered spacelike hypersurfaces in Riemannian or Lorentzian space forms, and they also [30] gave timelike hypersurfaces in the Lorentzian space forms, satisfying \( \mathbb{L}_k x = Ax + b \). Mohammadpour [24] introduced the hypersurfaces with \( L_r \)-pointwise 1-type Gauss map.

The aim of this paper is to investigate the properties of a family of rotational hypersurfaces in \( \mathbb{E}^n \) and establish a connection between the Gauss map \( G \) and specific geometric objects, such as hyperplanes, right circular hypercones, circular hypercylinders, and hyperspheres, by analyzing the equation \( \mathbb{L}_k G = AG \) for a \( n \times n \) matrix \( A \), \( k = n - 3 \geq 0 \), with integers \( n \), involving \( \mathbb{L}_k \) operators.

For further exploration of the \( \mathbb{L}_k \) operators mentioned previously, readers are encouraged to refer to the comprehensive investigation conducted by Chen et al. [8].

We present the main theorem of this work in this section. We then begin by recalling the fundamental notions of \( n \)-dimensional Euclidean geometry in Section 2. Moving on to Section 3, we review the \( \mathbb{L}_k \) operators. Then, in Section 4, we provide the definition of a rotational hypersurfaces family in Euclidean \( n \)-spaces. In Section 5, we calculate the relationships between the mean curvature and the Gauss–Kronecker curvature of the aforementioned family. The pivotal part of our study lies in Section 6, where we apply the operator \( \mathbb{L}_{n-3} \) to the Gauss map \( G \), leading us to establish a classification theorem. This theorem establishes a connection between the \( n \times n \) matrix \( A \) and the Gauss map \( G \) through the equation \( \mathbb{L}_{n-3} G = AG \) for the family with integers \( n \geq 3 \).

Throughout this research, our main focus revolve around the following theorem, which we prove in Section 6.

**Main Theorem.** Let \( \mathfrak{r} = \mathfrak{r}(r, \theta_1, \theta_2, \ldots, \theta_{n-2}) \) be a family of rotational hypersurfaces in \( \mathbb{E}^n \) given by

\[
\mathfrak{r} = \left( f(r) \prod_{i=1}^{n-2} \cos \theta_i, f(r) \sin \theta_1 \prod_{i=2}^{n-2} \cos \theta_i, \right. \\
\left. f(r) \sin \theta_2 \prod_{i=3}^{n-2} \cos \theta_i, \ldots, f(r) \sin \theta_{n-3} \cos \theta_{n-2}, f(r) \sin \theta_{n-2}, \varphi(r) \right).
\]

Then, the Gauss map \( G \) associated with \( \mathfrak{r} \) satisfies the equation \( \mathbb{L}_{n-3} G = AG \) for a \( n \times n \) matrix \( A \) with integers \( n \geq 3 \), if and only if \( \mathfrak{r} \) is an open part of the following:

1. a hyperplane,
2. a right circular hypercone,
3. circular hypercylinder,
4. a hypersphere.

2. Preliminaries

In this section, we describe notations, basic facts and definitions. Let \( \mathbb{E}^n \) denote the Euclidean \( n \)-space with the canonical Euclidean metric tensor given by \( \mathbb{g} = \langle , \rangle = \sum_{i=1}^{n} dx_i^2 \), where \( (x_1, x_2, \ldots, x_n) \) is a rectangular coordinate system in \( \mathbb{E}^n \).
Consider an $n$-dimensional Riemannian submanifold of the space $\mathbb{E}^n$. We denote Levi-Civita connections of $\mathbb{E}^n$ and $M$ by $\nabla$ and $\tilde{\nabla}$, respectively. We shall use letters $X$, $Y$, $Z$, $W$ (resp., $\xi$, $\eta$) to denote vectors fields tangent (resp., normal) to $M$. The Gauss and Weingarten formulas are given, respectively, by
\[
\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad (1)
\]
\[
\tilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi, \quad (2)
\]
where $h$, $D$ and $A$ are the second fundamental form, the normal connection and the shape operator of $M$, respectively.

For each $\xi \in T_p^\perp M$, the shape operator $A_\xi$ is a symmetric endomorphism of the tangent space $T_p M$ at $p \in M$. The shape operator and the second fundamental form are related by
\[
\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle.
\]
The Gauss and Codazzi equations are given, respectively, by
\[
\langle R(X,Y,Z,W) \rangle = \langle h(Y,Z), h(X,W) \rangle - \langle h(X,Z), h(Y,W) \rangle, \quad (3)
\]
\[
(D_X h)(Y,Z) = (\nabla_Y h)(X,Z), \quad (4)
\]
where $R$, $R^D$ are the curvature tensors associated with connections $\nabla$ and $D$, respectively, and $\nabla h$ is defined by
\[
(\nabla_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).
\]

Now, let $M$ be an oriented hypersurface in the Euclidean space $\mathbb{E}^n$, $A$ its shape operator and $x$ its position vector. We consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ consisting of principal direction of $M$ corresponding from the principal curvature $\kappa_i$ for $i = 1, 2, \ldots, n$. Let the dual basis of this frame field be $\{\theta_1, \theta_2, \ldots, \theta_n\}$. Then, the first structural equation of Cartan is indicated by
\[
d\theta_i = \sum_{j=1}^n \theta_j \wedge \omega_{ij}, \quad i = 1, 2, \ldots, n \quad (5)
\]
where $\omega_{ij}$ denotes the connection forms corresponding to the chosen frame field. We denote the Levi-Civita connection of $M$ and $\mathbb{E}^{n+1}$ by $\nabla$ and $\tilde{\nabla}$, respectively. Then, from the Codazzi equation (3), the following hold
\[
e_i(\kappa_j) = \omega_{ij}(e_j)(\kappa_i - \kappa_j), \quad (6)
\]
\[
\omega_{ij}(e_l)(\kappa_i - \kappa_j) = \omega_{il}(e_j)(\kappa_i - \kappa_l), \quad (7)
\]
for distinct $i, j, l = 1, 2, \ldots, n$. Considering
\[
s_j = \sigma_j(\kappa_1, \kappa_2, \ldots, \kappa_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq n} \kappa_{i_1} \kappa_{i_2} \ldots \kappa_{i_j},
\]
we use the notation
\[
r^j_i = \sigma_j(\kappa_1, \kappa_2, \ldots, \kappa_i-1, \kappa_i+1, \kappa_{i+2}, \ldots, \kappa_n).
\]
By the definition, we have $r^0_1 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. 

On the other hand, we will call the function \( s_k \) as the \( k \)-th mean curvature of \( M \). We would like to note that functions \( H = \frac{1}{n} s_1 \) and \( K = s_n \) are called the mean curvature and Gauss–Kronecker curvature of \( M \), respectively. In particular, \( M \) is said to be \( j \)-minimal if \( s_j \equiv 0 \) on \( M \).

3. \( \mathbb{L}_k \) operators

Let \( M \) be a hypersurface of \( \mathbb{R}^{n+1} \) with \( \kappa_1(x), \ldots, \kappa_n(x) \) as its principal curvatures. Associated with the principal curvatures, there are \( n \) algebraic invariants given by

\[
s_k(x) = \sigma_k(\kappa_1(x), \ldots, \kappa_n(x)), \quad 1 \leq k \leq n,
\]

where \( \sigma_k : \mathbb{R}^n \to \mathbb{R} \) is the elementary symmetric function defined by

\[
\sigma_k(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_k}.
\]

Then the characteristic polynomial of the shape operator \( A \) of \( M \) can be expressed in terms of the \( s_k \)'s as

\[
Q_A(t) = \det(tI - A) = \sum_{k=0}^{n} (-1)^k s_k t^{n-k}, \quad s_0 = 1,
\]

where \( I \) describes the identity map. The \( k \)-th mean curvature \( H_k \) of \( M \) is \( s_k = \binom{n}{k} H_k \), \( 0 \leq k \leq n \).

In fact, \( H_1 \) is the mean curvature, \( H_k \) is intrinsic for even \( k \), and \( H_k \) is extrinsic for odd \( k \).

The Newton transformations \( \mathcal{P}_k : \mathcal{X}(M) \to \mathcal{X}(M) \) for \( k = 1, \ldots, n \) are defined inductively from the shape operator \( A \) by

\[
\mathcal{P}_0 = I, \ldots, \mathcal{P}_k = s_k I - A \circ \mathcal{P}_{k-1} = \binom{n}{k} H_k I - A \circ \mathcal{P}_{k-1}, \tag{9}
\]

where \( I \) denotes the identity map on \( \mathcal{X}(M) \). By applying Cayley–Hamilton’s theorem, we have \( \mathcal{P}_n = 0 \) from \( 8 \). If \( k \) is even, \( \mathcal{P}_k \) does not depend on the chosen orientation, but if \( k \) is odd there is a change of sign in \( \mathcal{P}_k \).

Associated with each Newton transformation \( \mathcal{P}_k \), one has the linear differential operator \( \mathbb{L}_k : \mathcal{F}(M) \to \mathcal{F}(M) \) given by \( \mathbb{L}_k(f) = -\text{Tr}(\mathcal{P}_k \circ \nabla^2 f) \), where \( \nabla^2 f : \mathcal{X}(M) \to \mathcal{X}(M) \) is the self-adjoint linear operator metrically equivalent to the Hessian of \( f \) and given by

\[
\langle \nabla^2 f(X), Y \rangle = \langle \nabla_X(\nabla f), Y \rangle, \quad X, Y \in \mathcal{X}(M).
\]

Note that \( \mathbb{L}_k \) is the linearized operator of the first variation of the \( (k+1) \)-th mean curvature arising from the normal variations of \( M \) for \( k = 2, 3, \ldots, n - 1 \). In particular, \( \mathbb{L}_0 = -\Delta \) is called the Laplace–Beltrami operator (i.e., Laplacian), \( \mathbb{L}_1 = \Box \) is called the Cheng–Yau operator introduced by Cheng and Yau \( \llbracket 11 \rrbracket \).

The second order differential operator \( \mathbb{L}_k \) associated with the Newton transformation \( \mathcal{P}_k \) is given by \( \mathbb{L}_k(f) = \text{div} \left( \mathcal{P}_k (\nabla^2 f) \right) \), \( \llbracket 1 \rrbracket \). From this equation, the following appears

\[
\mathbb{L}_k = \sum_i r_k^i (e_i e_i - \nabla_{e_i} e_i), \tag{10}
\]

and the equation

\[
\mathbb{L}_k x = s_k \mathcal{G} \tag{11}
\]
is obtained in [1], where $G$ is the Gauss map of $M$ which assigns every point of $M$ into the unit normal vector associated with the orientation of $M$. Furthermore, by using (10), one can obtain

$$L_k G = -\nabla s_{k+1} - (s_1 s_{k+1} - (k + 2) s_{k+2}) G.$$  \hspace{1cm} (12)

To thoroughly investigate the $L_k$ operators, we suggest readers to refer to the comprehensive research conducted by Chen et al. in their work [8].

4. Rotational Hypersurfaces in Euclidean $n$-Spaces

We note that the definition of rotational hypersurfaces in Riemannian space forms were defined in [13]. A rotational hypersurface $M \subset \mathbb{E}^n$ generated by a curve $C$ around an axis $\ell$ that does not meet $C$ is obtained by taking the orbit of $C$ under those orthogonal transformations of $\mathbb{E}^n$ that leave $\ell$ pointwise fixed. See [13, Remark 2.3] for details.

Let $C$ be the curve of $\mathbb{E}^n$ parametrized by $\gamma = \gamma(r)$:

$$\gamma(r) = (f(r), 0, \ldots, 0, \varphi(r)).$$  \hspace{1cm} (13)

An orthogonal transformations of $\mathbb{E}^n$ that leaves $\ell$ pointwise fixed has the following $n \times n$ matrix form $Z = Z(\theta_1, \theta_2, \ldots, \theta_{n-2})$:

$$Z = \begin{pmatrix}
    \prod_{i=1}^{n-2} C_i & -S_1 & -C_1 S_2 & -\prod_{i=1}^2 C_i S_3 & \cdots & -\prod_{i=1}^{n-4} C_i S_{n-3} & -\prod_{i=1}^{n-3} C_i S_{n-2} & 0 \\
    S_1 \prod_{i=2}^{n-2} C_i & C_1 & -S_1 S_2 & -\prod_{i=2}^2 C_i S_3 & \cdots & -S_1 \prod_{i=2}^{n-4} C_i S_{n-3} & -S_1 \prod_{i=2}^{n-3} C_i S_{n-2} & 0 \\
    S_2 \prod_{i=3}^{n-2} C_i & 0 & C_2 & -S_2 S_3 & \cdots & -S_2 \prod_{i=3}^{n-4} C_i S_{n-3} & -S_2 \prod_{i=3}^{n-3} C_i S_{n-2} & 0 \\
    S_3 \prod_{i=4}^{n-3} C_i & 0 & 0 & C_3 & \cdots & -S_3 \prod_{i=4}^{n-4} C_i S_{n-3} & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & -S_{n-4} S_{n-3} & \vdots & 0 \\
    0 & 0 & 0 & \cdots & C_{n-3} & -S_{n-3} S_{n-2} & \vdots & 0 \\
    0 & 0 & 0 & 0 & \cdots & 0 & C_{n-2} & 0 \\
    0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 
\end{pmatrix},$$

where $Z \in SO(n)$, $Z \ell = \ell$, $\det Z = 1$, $C_i = \cos \theta_i$, $S_i = \sin \theta_i$, $\theta_i \in \mathbb{R}$, $\ell$ denotes the rotation axis $(0, 0, \ldots, 0, 1)^T$.

Next, we define the parametrization of the rotational hypersurfaces family, generated by a curve $\gamma$, represented by Eq. (13), around axis $\ell$. This parametrization is given by

$$\mathbf{r} = Z \cdot \gamma^T.$$  \hspace{1cm} (15)

In the subsequent sections of this paper, a vector or matrix $(v_1, v_2, \ldots, v_n)$ is considered equivalent to its transpose, denoted as the column vector or column matrix $(v_1, v_2, \ldots, v_n)^T$.

Let $\mathbf{r} = \mathbf{r}(\varphi, \theta_1, \ldots, \theta_{n-2})$ be a parametric representation and isometric immersion of a hypersurfaces family $M$ in the Euclidean space $\mathbb{E}^n$. 
The vector product of \( \mathbf{v}_1 = (v_1^1, v_1^2, \ldots, v_1^n), \ldots, \mathbf{v}_{n-1} = (v_{n-1}^1, v_{n-1}^2, \ldots, v_{n-1}^n) \) in \( \mathbb{E}^n \) is defined by

\[
\mathbf{v}_1 \times \mathbf{v}_2 \times \ldots \times \mathbf{v}_{n-1} = \det \begin{pmatrix}
e_1 & e_2 & \cdots & e_n \\
v_1^1 & v_1^2 & \cdots & v_1^n \\
v_2^1 & v_2^2 & \cdots & v_2^n \\
\vdots & \vdots & \ddots & \vdots \\
v_{n-1}^1 & v_{n-1}^2 & \cdots & v_{n-1}^n
\end{pmatrix}_{n \times n}.
\]

The first and the second fundamental form matrices of hypersurface \( \mathbf{r} \) in \( \mathbb{E}^n \) are described, respectively, by \( \mathbf{I} = (g_{ij})_{(n-1) \times (n-1)} \) and \( \mathbf{II} = (h_{ij})_{(n-1) \times (n-1)} \). Here, \( 1 \leq i, j \leq n - 1 \), \( g_{11} = \langle \mathbf{r}_r, \mathbf{r}_r \rangle, g_{12} = \langle \mathbf{r}_r, \mathbf{r}_\theta \rangle, \ldots, g_{(n-1)(n-1)} = \langle \mathbf{r}_{\theta_{n-2}}, \mathbf{r}_{\theta_{n-2}} \rangle \), \( h_{11} = \langle \mathbf{r}_{rr}, \mathbf{G} \rangle, h_{12} = \langle \mathbf{r}_{r\theta}, \mathbf{G} \rangle, \ldots, h_{(n-1)(n-1)} = \langle \mathbf{r}_{\theta_{n-2}\theta_{n-2}}, \mathbf{G} \rangle \), the partial differentials are \( \mathbf{r}_r = \frac{\partial \mathbf{r}}{\partial r}, \mathbf{r}_{\theta_{1}} = \frac{\partial \mathbf{r}}{\partial \theta_1}, \ldots \). and the Gauss map (i.e., the unit normal vector) of hypersurface \( \mathbf{r} \) is expressed by

\[
\mathbf{G} = \frac{\mathbf{r}_r \times \mathbf{r}_{\theta_1} \times \ldots \times \mathbf{r}_{\theta_{n-2}}}{||\mathbf{r}_r \times \mathbf{r}_{\theta_1} \times \ldots \times \mathbf{r}_{\theta_{n-2}}||}.
\] (16)

\((g_{ij})^{-1}(h_{ij})\) gives the shape operator matrix \( \mathbf{A} \). Then, the formulas of the mean curvature and the Gauss–Kronecker curvature are, respectively, determined by

\[
H = \frac{1}{n-1} \text{tr} (\mathbf{A}), \quad \text{and} \quad K = \det(\mathbf{A}) = \frac{\det \mathbf{II}}{\det \mathbf{I}}.
\] (17)

5. CURVATURES OF THE ROTATIONAL HYPERSURFACES FAMILY IN \( \mathbb{E}^n \)

By considering Eq. (15), the following parametric form of the rotational hypersurfaces family \( \mathbf{r} = \mathbf{r}(r, \theta_1, \theta_2, \ldots, \theta_{n-2}) \) appears

\[
\mathbf{r} = \left( f(r) \prod_{i=1}^{n-2} \cos \theta_i, f(r) \sin \theta_1 \prod_{i=2}^{n-2} \cos \theta_i, f(r) \sin \theta_2 \prod_{i=3}^{n-2} \cos \theta_i, \ldots, f(r) \sin \theta_{n-3} \cos \theta_{n-2}, f(r) \sin \theta_{n-2}, \varphi(r) \right)
= (x_1, x_2, x_3, \ldots, x_{n-2}, x_{n-1}, x_n),
\]

where \( f = f(r), r, \theta_i \in \mathbb{R} \setminus \{0\} \). By taking the first differentials with respect to \( r, \theta_1, \theta_2, \ldots, \theta_{n-2} \) of the rotational hypersurfaces family given by Eq. (18), the following first fundamental form matrix occurs

\[
\mathbf{I} = \text{diag} \left( f^2 + \varphi^2, f^2 \prod_{i=2}^{n-2} \cos^2 \theta_i, f^2 \prod_{i=3}^{n-2} \cos^2 \theta_i, \ldots, f^2 \cos^2 \theta_{n-2}, f^2 \right).
\]

Then,

\[
\det \mathbf{I} = (f^2)^{n-2} (f^2 + \varphi^2) \left( \prod_{i=2}^{n-2} \cos \theta_i \right) \left( \prod_{i=3}^{n-2} \cos^2 \theta_i \right) \ldots \left( \prod_{i=n-2}^{n-2} \cos^2 \theta_i \right).
\]
where $f = f(r)$, $f' = \frac{df}{dr}$, $\varphi = \varphi(r)$, $\varphi' = \frac{d\varphi}{dr}$. By taking the second differentials of $f$ and $\theta_i$ of the rotational hypersurfaces family determined by Eq. (18), the following second fundamental form matrix appears

$$\mathbf{II} = \varepsilon \text{diag} \left( \begin{array}{cccc} f'\varphi'' - f''\varphi' & f\varphi' \prod_{i=1}^{n-2} \cos^2 \theta_i & \ldots & f\varphi' \prod_{i=1}^{n-2} \cos^2 \theta_i; \\ \frac{f\varphi' \prod_{i=1}^{n-2} \cos^2 \theta_i}{(f^2 + \varphi^2)^{1/2}} & \frac{f\varphi' \prod_{i=1}^{n-2} \cos^2 \theta_i}{(f^2 + \varphi^2)^{1/2}} & \ldots & \frac{f\varphi' \prod_{i=1}^{n-2} \cos^2 \theta_i}{(f^2 + \varphi^2)^{1/2}} \end{array} \right).$$

Then,

$$\det \mathbf{II} = \varepsilon \frac{f^{n-2} (f' \varphi'' - f'' \varphi') (\varphi')^{n-2} \left( \prod_{i=1}^{n-2} \cos^2 \theta_i \right)}{(f^2 + \varphi^2)^{n-1/2}}.$$

Here,

$$\varepsilon = \begin{cases} -1 & \text{if } n \text{ odd integer}, \\ 1 & \text{if } n \text{ even integer}, \end{cases} \quad (19)$$

for integers $n \geq 3$. The Gauss map of the rotational hypersurfaces family described by Eq. (18) is determined by

$$G = \frac{\varepsilon}{(f^2 + \varphi^2)^{1/2}} \left( \varphi' \prod_{i=1}^{n-2} C_i, \varphi' \prod_{i=2}^{n} S_i \prod_{i=1}^{n-2} C_i, \varphi' \prod_{i=3}^{n} S_i \ldots, \varphi' \prod_{i=n-2}^{n} S_i, -f' \right).$$

The shape operator of the family determined by Eq. (18) is given by

$$A = \text{diag} \left( \kappa_1, \kappa_2, \ldots, \kappa_{n-1} \right).$$

Here, the principal curvatures of the rotational family defined by Eq. (18) are described by

$$\kappa_1 = \varepsilon \frac{f' \varphi'' - f'' \varphi'}{(f^2 + \varphi^2)^{3/2}}, \quad \kappa_2 = \varepsilon \frac{\varphi'}{(f^2 + \varphi^2)^{1/2}}, \quad \kappa_3 = \ldots = \kappa_{n-1}. \quad (21)$$

Then, the mean curvature is given by

$$H = \varepsilon \frac{f f' \varphi'' + (n - 2) \varphi' \varphi' \varphi' + (n - 2) f f' - f f'' \varphi'}{(n - 1) f (f^2 + \varphi^2)^{3/2}}, \quad (20)$$

and the Gauss–Kronecker curvature is determined by

$$K = \varepsilon \frac{(f' \varphi'' - f'' \varphi') (\varphi')^{n-2}}{f^{n-2}(f^2 + \varphi^2)^{n/2}}. \quad (21)$$

Therefore, the following theorems come into view.

**Theorem 1.** The rotational hypersurfaces family parametrized by

$$\mathbf{x} = \left( f(r) \prod_{i=1}^{n-2} \cos \theta_i, f(r) \sin \theta_1 \prod_{i=2}^{n-2} \cos \theta_i, \ldots, f(r) \sin \theta_{n-2} \prod_{i=3}^{n-2} \cos \theta_i, f(r) \sin \theta_{n-3} \cos \theta_{n-2}, f(r) \sin \theta_{n-2} \varphi(r) \right)$$
is minimal (i.e., \( H = 0 \)) if and only if the following holds

\[ \varphi(r) = \pm \int \frac{f^2(r)f'(r)}{(c_1 f^{2n}(r) - f^4(r))^{1/2}}dr + c_2, \]

where \( c_1, c_2 \in \mathbb{R} \), and \((n - 1)f(f'^2 + \varphi'^2)^{3/2} \neq 0\).

Proof. When considering the rotational hypersurface determined by Eq. (18) under the condition of minimality, i.e., when \( H = 0 \) described by Eq. (20), we arrive at a second-order differential equation \( f f' \varphi'' + (n - 2) \varphi'^3 + ((n - 2) f'^2 - f f'') \varphi' = 0 \). The solutions for \( \varphi(r) \) could not be obtained using classical methods.

However, \( \varphi(r) \) is successfully obtained as indicated in the theorem, using Maple program codes:

\[
\begin{align*}
&\texttt{dsolve(f(r)*diff(f(r),r)*diff(f(r),r,r) + (n - 2) * diff(f(r),r)^3} \\
&\quad + ((n - 2) * diff(f(r),r)^2 - f(r) * diff(f(x),r,r)) * diff(f(r),r) \\
&\quad = 0, \varphi(r));
\end{align*}
\]

\(\square\)

Theorem 2. The rotational hypersurfaces family defined by Eq. (18) has zero Gauss–Kronecker curvature (i.e., \( K = 0 \)) if and only if

\[ \varphi(r) = c_1 \quad \text{or} \quad \varphi(r) = c_1 f(r) + c_2, \]

where \( f^{n-2}(f'^2 + \varphi'^2)^{n/2} \neq 0 \), \( c_1, c_2 \in \mathbb{R} \).

Proof. When \( K = 0 \) described by Eq. (21), the second-order differential equation \( f'((\varphi')^{n-2} \varphi'' - f''(\varphi')^{n-1}) = 0 \) transforms to \( \varphi''/\varphi' = f''/f' \). Then, we obtain solutions \( \varphi(r) \).

We also find solution \( \varphi(r) = f(r)/c_1 + c_2 \) using a Maple software program codes:

\[
\begin{align*}
&\texttt{PDEtools[declare]}(f(r), \varphi(r), \text{prime} = r); \\
&\texttt{dsolve(diff(f(r),r) * diff(f(r),r,r) + (n - 2) * diff(f(r),r,r) \\
&\quad - diff(f(r),r) * diff(f(r),r,r) + diff(f(r),r,r) \cdot (n - 1) \\
&\quad = 0, \varphi(r));}
\end{align*}
\]

\(\square\)

Corollary 1. Substituting \( \varphi = c = \text{const.} \) into the Eqs. described by (20) and (21), we find the curvatures \( H = 0, K = 0 \). Then, the hypersurface is minimal.

Corollary 2. Substituting \( f = c = \text{const.} \) into the Eqs. determined by (20) and (21), we obtain the curvatures \( H = -\epsilon \frac{n - 2}{(n - 1)c}, K = 0 \), where \( c \neq 0 \). Hence, the hypersurface has CMC.
6. Gauss map and $L_{n-3}$ operator in $\mathbb{E}^n$

We consider the parametric curve $\gamma = \gamma(r)$ defined by Eq. (13) as a curve with unit speed. The elements of the adapted frame field $\{e_1, e_2, \ldots, e_{n-1}, G\}$ of the rotational hypersurfaces family $\mathbf{r}$ determined by Eq. (18) are given by

\[
e_1 = r_r = \left( f' \prod_{i=1}^{n-2} \cos \theta_i, f' \sin \theta_1 \prod_{i=2}^{n-2} \cos \theta_i, \right.
\]

\[
\left. f' \sin \theta_2 \prod_{i=3}^{n-2} \cos \theta_i, \ldots, f' \sin \theta_{n-3} \cos \theta_{n-2}, f' \sin \theta_{n-2}, \varphi' \right) ,
\]

\[
e_2 = \frac{r_{\theta_1}}{f} = \left( -\sin \theta_1 \prod_{i=2}^{n-2} \cos \theta_i, \cos \theta_1 \prod_{i=2}^{n-2} \cos \theta_i, 0, \ldots, 0 \right),
\]

\[
n-1 = \frac{r_{\theta_{n-2}}}{f}
\]

\[
(22)
\]

\[
(23)
\]

\[
(24)
\]

\[
(25)
\]

The principal curvatures $\kappa_i$ ($1 \leq i \leq n-1$) of $\mathbf{r}$ with respect to the Gauss map $G$ are determined by

\[
\kappa_1 = \langle A(e_1), e_1 \rangle = -\varepsilon \left( f'' \varphi' - f' \varphi'' \right),
\]

\[
\kappa_j = \langle A(e_j), e_j \rangle = -\varepsilon \frac{\varphi'}{f}, \quad (2 \leq j \leq n-1).
\]

Since $\gamma$ denotes a unit speed curve, there exists a smooth function $R = R(r)$ such that $f' = \cos R$ and $\varphi' = \sin R$. Then, the following is presented.
Lemma 1. Consider an oriented hypersurface $\mathfrak{r}$ in the Euclidean space $\mathbb{R}^n$, with its mean curvature $H$ and Gauss–Kronecker curvature $K$. Then, the Gauss map of $\mathfrak{r}$, for integers $n \geq 3$, satisfies the relation

$$\mathbb{L}_{n-3}G = -\nabla s_{n-2} - (s_1 s_{n-2} - (n-1) s_{n-1})G,$$  \hspace{1cm} (26)

where

$$s_1 = \varepsilon \left( \begin{array}{c} (n-2) \big/ 1 \end{array} \right) \left( \begin{array}{c} x' \big/ (n-2) \end{array} \right),$$

$$s_{n-2} = \varepsilon \left( \begin{array}{c} (n-2) \big/ (n-3) \end{array} \right) \left( \begin{array}{c} x' \big/ (n-2) \end{array} \right),$$

$$s_{n-1} = \varepsilon \left( \begin{array}{c} (n-2) \big/ (n-2) \end{array} \right) \left( \begin{array}{c} x' \big/ (n-2) \end{array} \right).$$  \hspace{1cm} (27)

$s_1 = (n-1) H$, $s_{n-1} = K$, $\varphi = \sin R$, $\gamma(r) = (f(r), 0, \ldots, 0, \varphi(r))$ represents a unit speed curve, and $R = R(r)$ denotes a smooth function, $f' = \cos R$, $\varphi' = \sin R$.

Before proving Lemma 1, let’s provide the following examples.

Example 1. Flat hypersurfaces satisfy $\mathbb{L}_{n-3}G = AG$, where $A$ denotes a $n \times n$ matrix, derived from $\mathbb{L}_{n-3}G = -\nabla s_{n-2} - s_1 s_{n-2}G$.

Example 2. Hyperspheres $\sum_{i=1}^{n} (x_i - p_i)^2 = r^2$ have $G = \frac{1}{r} (x_1 - p_1, \ldots, x_n - p_n)$. Then, hyperspheres hold $\mathbb{L}_{n-3}G = AG$, with $A = \varepsilon \frac{1}{r^2} I_n$, where $I_n$ represents an identity matrix.

Example 3. Minimal hypersurfaces satisfy $\mathbb{L}_{n-3}G = -\nabla s_{n-2} + (n-1) s_{n-1}G$.

Now, let’s prove Lemma 1.

Proof. Taking $f' = \cos R$ and $\varphi' = \sin R$, the gradient $\nabla s_{n-2}$ of $\mathfrak{r}$ is given by

$$\nabla s_{n-2} = s'_{n-2} e_1,$$  \hspace{1cm} (28)

where

$$s'_{n-2}(r) = \varepsilon \frac{1}{f^{n-2}} (n-2) \left( f^2 R'' \sin R + (n-3) f^2 R^2 \cos R \right)$$

$$- (n-4) f R' \sin R \cos R - \sin^2 R \cos R \sin^{n-4} R.$$  \hspace{1cm} (29)

Next, we suppose that the Gauss map $G$ given by Eq. (25) of the rotational hypersurface $\mathfrak{r}$ satisfies the following

$$\mathbb{L}_{n-3}G = AG,$$  \hspace{1cm} (30)

with a $n \times n$ matrix $A = (a_{ij})$. Substituting $f' = \cos R$ and $\varphi' = \sin R$ again, into the Gauss map $G$ determined by Eq. (25), the following holds

$$G = \varepsilon \left( \begin{array}{c} \sin R \prod_{i=1}^{n-2} \cos \theta_i, \sin R \sin \theta_1 \prod_{i=2}^{n-2} \cos \theta_i, \sin R \sin \theta_2 \prod_{i=3}^{n-2} \cos \theta_i, \ldots, \sin R \sin \theta_{n-3} \cos \theta_{n-2}, \sin R \sin \theta_{n-2}, - \cos R \end{array} \right).$$  \hspace{1cm} (31)
Consequently, deducing from Eqs. (27), (30) and (31), we infer the following system

\[
(-s'_{n-2} \cos R - (s_1 s_{n-2} - (n - 1) s_{n-1}) \sin R) \sin \theta_{n-2} = \varepsilon(a_{n(1)} \sin R \cos \theta_1 \ldots \cos \theta_{n-2} + a_{n(1)2} \sin R \sin \theta_1 \cos \theta_2 \ldots \cos \theta_{n-2} \\
+ \ldots + a_{n(n-1)(n-2)} \sin R \sin \theta_{n-3} \cos \theta_{n-2} + a_{n(n-1)1} \sin R \sin \theta_{n-2} - a_{n(n-1)n} \cos R),
\]

(32)

\[
(-s'_{n-2} \cos R - (s_1 s_{n-2} - (n - 1) s_{n-1}) \sin R) \sin \theta_{n-2} = \varepsilon(a_{(n-1)1} \sin R \cos \theta_1 \ldots \cos \theta_{n-2} + a_{(n-1)2} \sin R \sin \theta_1 \cos \theta_2 \ldots \cos \theta_{n-2} \\
+ \ldots + a_{n(n-1)(n-2)} \sin R \sin \theta_{n-3} \cos \theta_{n-2} + a_{n(n-1)1} \sin R \sin \theta_{n-2} - a_{n(n-1)n} \cos R),
\]

(33)

\[
(-s'_{n-2} \cos R - (s_1 s_{n-2} - (n - 1) s_{n-1}) \sin R) \sin \theta_{n-2} = \varepsilon(a_{11} \sin R \cos \theta_1 \ldots \cos \theta_{n-2} + a_{12} \sin R \sin \theta_1 \cos \theta_2 \ldots \cos \theta_{n-2} \\
+ \ldots + a_{n(n-1)(n-2)} \sin R \sin \theta_{n-3} \cos \theta_{n-2} + a_{n(n-1)1} \sin R \sin \theta_{n-2} - a_{n(n-1)n} \cos R).
\]

(34)

\[
(-s'_{n-2} \cos R - (s_1 s_{n-2} - (n - 1) s_{n-1}) \sin R) \sin \theta_{n-2} = \varepsilon(a_{11} \sin R \cos \theta_1 \ldots \cos \theta_{n-2} + a_{12} \sin R \sin \theta_1 \cos \theta_2 \ldots \cos \theta_{n-2} \\
+ \ldots + a_{n(n-1)(n-2)} \sin R \sin \theta_{n-3} \cos \theta_{n-2} + a_{n(n-1)1} \sin R \sin \theta_{n-2} - a_{n(n-1)n} \cos R).
\]

(35)

Next, we suppose that \( J = \{ r \in I \mid R'(r) \neq 0 \} \) is nonempty set. Then, \( R(I) \) contains an interval, and obtain from (32)-(35) that \( a_{11} = a_{22} = \ldots = a_{(n-1)(n-1)} \) and \( a_{ij} = 0 \) when \( i \neq j \). Hence, the following hold \( A = \text{diag}(\eta_n, \ldots, \eta_n, \phi) \),

\[
-s'_{n-2} \cos R - (s_1 s_{n-2} - (n - 1) s_{n-1}) \sin R = \varepsilon \eta \sin R,
\]

(36)

and

\[
-s'_{n-2} \cos R - (s_1 s_{n-2} - (n - 1) s_{n-1}) \sin R = -\varepsilon \phi \cos R.
\]

(37)

Eqs. (36) and (37) are equivalent to the following

\[
s'_{n-2} = \lambda \sin R \cos R,
\]

(38)

\[
s_1 s_{n-2} - (n - 1) s_{n-1} = \varepsilon \left( \lambda \sin^2 R + \phi \right),
\]

(39)

where \( \lambda = \phi - \eta \).

Therefore, the following is provided.

**Lemma 2.** Consider a rotational hypersurface \( \varphi \) given by (18) in Euclidean space \( \mathbb{E}^n \), with the set \( J = \{ r \in I \mid R'(r) \neq 0 \} \). Assume that the Gauss map \( \mathcal{G} \) of \( \varphi \) satisfies \( \mathbb{L}_{n-3} \mathcal{G} = A \mathcal{G} \), where \( A \) denotes \( n \times n \) matrix with integers \( n \geq 3 \). Then, \( A \) takes the form \( \eta I_n \), where \( I_n \) represents the identity matrix.
Proof. We understand from the above findings that $A$ is a $n \times n$ diagonal matrix satisfying
$A = \text{diag}(\eta_1, \ldots, \eta_n, \phi)$ for constants $\eta$ and $\phi$. Then, it follows from (27), (29), (38), and (39), the following appear

$$
(n - 2) \left( f^2 R'' \sin R + (n - 3) f^2 R'^2 \cos R \right) - (n - 4) f R' \sin R \cos R - \sin^2 R \cos R) \sin^{n-4} R = \lambda f^{n-1} \sin R \cos R,
$$

and

$$(n - 2) \left( f^2 R'^2 + \sin^2 R + (n - 3) f R' \sin R \right) \sin^{n-3} R = (\lambda \sin^2 R + \phi) f^{n-1},$$

where $R = R(r)$. Differentiating the both sides of Eq. (41) with respect to $r$, the following occurs

$$-2 (n - 2) f^2 R'' R'' \sin^{n-3} R - (n - 3) (n - 2) f R'' \sin^{n-2} R$$

$$- (n - 3) (n - 2) f^2 R'^3 \cos R \sin^{n-4} R - (n - 2) (n^2 - 5n + 8) f R'^2 \cos R \sin^{n-3} R$$

$$-2 \left( (n - 2) \sin^{n-2} R + \lambda f^{n-1} \sin R \right) R' \cos R - (n - 1) f^{n-2} (\lambda \sin^2 R + \phi) \cos R = 0.$$  

Substituting $f' = \cos R$ into Eq. (40), the following holds

$$R'' = \frac{1}{(n - 2) f^2 \sin^{n-4} R} \left[- (n - 2) (n - 3) f^2 R'^2 \cos R \sin^{n-5} R ight.$$

$$+ (n - 2) (n - 4) f R' \cos R \sin^{n-4} R + \lambda f^{n-1} \cos R + (n - 2) \cos R \sin^{n-3} R \right].$$

Substituting right hand of $R''$ into Eq. (42), and using $f' = \cos R$, again,

$$\left[(n - 3) (n - 2) f^3 R'^3 \sin^{n-4} R - 3 (n - 3) (n - 2) f^2 R'^2 \sin^{n-3} R ight.$$

$$- 4 \lambda f^1 R' \sin R - (n - 2) (n^2 - 5n + 10) f R' \sin^{n-2} R$$

$$+ (-\lambda (n - 3) \sin^2 R - (n - 1) (\lambda \sin^2 R + \phi)) f^{n-1}$$

$$- (n - 3) (n - 2) \sin^{n-1} R \cos R = 0.$$  

Taking $R'^2$ in (41), and substituting it into (44), the following appears

$$R' = \frac{a_1 f^{n-1} + a_2}{a_3 f^n + a_4 f},$$

where

$$\begin{align*}
a_1 &= ((n^2 - 7n + 14) \lambda \sin^3 R + (n^2 - 8n + 17) \phi \sin R) \cos R, \\
a_2 &= -(n - 7) (n - 3) (n - 2) \cos R \sin^n R, \\
a_3 &= -(-\lambda (n + 1) \sin^2 R + (n - 3) \phi) \cos R, \\
a_4 &= (n - 2) ((n - 3)^3 - 4 (n^2 - 6n + 10) \cos R \sin^{n-1} R.
\end{align*}$$

Replace $R'$ by Eq. (41) with that by Eq. (45). Then,

$$t_{3(n-1)} f^{3(n-1)} + t_{2(n-1)} f^{2(n-1)} + t_{n-1} f^{n-1} + t_0 = 0.$$
Here, $t_i = t_i(R)$ represent

$$
t_{3(n-1)} = - \left( \phi + \lambda \sin^2 R \right) \alpha_3^2,
$$
$$
t_{2(n-1)} = (n - 2) \alpha_3^2 \sin^{n-1} R + (n - 3) (n - 2) \alpha_1 \alpha_3 \sin^{n-2} R
+ (n - 2) \alpha_1^2 \sin^{-n-3} R - 2 \left( \phi + \lambda \sin^2 R \right) \alpha_3 \alpha_4,
$$
$$
t_{n-1} = 2 (n - 2) \alpha_3 \alpha_4 \sin^{n-1} R + (n - 3) (n - 2) \left( \alpha_1 \alpha_4 + \alpha_2 \alpha_3 \right) \sin^{n-2} R
+ 2 (n - 2) \alpha_1 \sin^{n-3} R - \left( \phi + \lambda \sin^2 R \right) \alpha_2^2,
$$
$$
t_0 = (n - 2) \alpha_4^2 \sin^{n-1} R + (n - 3) (n - 2) \alpha_2 \alpha_4 \sin^{n-2} R + (n - 2) \alpha_2^2 \sin^{-n-3} R.
$$

Differentiating Eq. (47) with respect to $r$, and using $f' = \cos R$, $\frac{\partial \eta}{\partial R} = \frac{\partial \eta}{\partial R} = t' R'$, $R'(r)$ in Eq. (45), the following holds

$$
k_{3(n-1)} f^{3(n-1)} + k_{2(n-1)} f^{2(n-1)} + k_{n-1} f^{n-1} + k_0 = 0,
$$

where $k_i = k_i(R)$,

$$
k_{3(n-1)} = 3 (n - 1) \alpha_4 t_{3(n-1)} + 2 (n - 1) \alpha_3 t_{2(n-1)} + \alpha_2 t'_{3(n-1)} + \alpha_1 t'_{2(n-1)},
k_{2(n-1)} = 2 (n - 1) \alpha_4 t_{2(n-1)} + (n - 1) \alpha_3 t_{3(n-1)} + \alpha_2 t'_{2(n-1)} + \alpha_1 t'_{0},
k_{n-1} = (n - 1) \alpha_4 t_{(n-1)} + \alpha_2 t'_{(n-1)} + \alpha_1 t'_{0},
k_0 = \alpha_2 t'_{0}.
$$

Compute and rewrite $k_i$ given by Eq. (50) in more clear form

$$
k_{3(n-1)} = \left( \alpha_5 (n - 2) \lambda^3 \sin^{n+5} R + \sum_{i=0}^{8} m_i (\phi, \eta) \sin^i R \right) \cos R,
k_{2(n-1)} = \left( \alpha_6 (n - 2)^2 \lambda^2 \sin^{2n+2} R + \sum_{i=0}^{n+5} n_i (\phi, \eta) \sin^i R \right) \cos R,
k_{n-1} = \left( \alpha_7 (n - 2)^3 \lambda \sin^{3n-1} R + \sum_{i=0}^{2n+2} p_i (\phi, \eta) \sin^i R \right) \cos R,
k_0 = \left( \alpha_8 (n - 2)^4 \sin^{4n-4} R + \sum_{i=0}^{3n-1} q_i (\phi, \eta) \sin^i R \right) \cos R,
$$

Here, $m_i, n_i, p_i, q_i$ denotes the polynomials, respectively, in $\phi$ and $\eta$, and the constants represent

$$
\alpha_5 = 2 n^7 - 44 n^6 + 325 n^5 - 807 n^4 - 796 n^3 + 610 n^2 - 10227 n + 4545,
\alpha_6 = 10 n^8 - 273 n^7 + 3089 n^6 - 18843 n^5 + 67423 n^4 - 140907 n^3
+ 162032 n^2 - 81929 n + 5851,
\alpha_7 = 3 n^9 - 104 n^8 + 1549 n^7 - 13028 n^6 + 68261 n^5 - 23091 n^4
+ 502919 n^3 - 670392 n^2 + 486104 n - 137246,
\alpha_8 = -3 (n - 7) (n - 3) (n - 1) (n^4 - 24 n^3 + 194 n^2 - 624 n + 709).
$$

Eliminating $f^{3(n-1)}$ by Eqs. (47) and (49), the following appears

$$
h_{2(n-1)} f^{2(n-1)} + h_{n-1} f^{n-1} + h_0 = 0,
$$

where

$$
h_{2(n-1)} = t_{2(n-1)} k_{3(n-1)} - k_{2(n-1)} t_{3(n-1)},
h_{n-1} = t_{n-1} k_{3(n-1)} - k_{(n-1)} t_{3(n-1)},
h_0 = t_0 k_{3(n-1)} - k_0 t_{3(n-1)}.
$$
Using $t_i$ by Eq. (48) and $k_i$ by Eq. (51) for $i = 0, n - 1, 2(n - 1), 3(n - 1)$, compute $h_j = h_j(R)$ for $j = 0, n - 1, 2(n - 1)$:

$$h_{2(n-1)} = \left( \varsigma_1 (n - 2)^2 \lambda^3 \sin^{2n+8} R + \sum_{j=0}^{n+1} p_{2(n-1)j} (\phi, \eta) \sin^j R \right) \cos R,$$

$$h_{n-1} = \left( \varsigma_2 (n - 2)^3 \lambda^4 \sin^{3n+5} R + \sum_{j=0}^{2n+8} p_{(n-1)j} (\phi, \eta) \sin^j R \right) \cos R,$$

$$h_0 = \left( \varsigma_3 (n - 2)^4 \lambda^5 \sin^{4n+2} R + \sum_{j=0}^{3n+5} p_{0j} (\phi, \eta) \sin^j R \right) \cos R,$$

(53)

where $p_{ij} (\phi, \eta)$ (for $j = 0, n - 1, 2(n - 1)$) denote polynomials in $\phi$ and $\eta$, and

$$\varsigma_1 = \alpha_5 (2n^4 - 29n^3 + 129n^2 - 219n + 105) + \alpha_6 (n + 1)^2,$$

$$\varsigma_2 = \alpha_5 (3n^5 - 56n^4 + 398n^3 - 1380n^2 + 2367n - 1604) + \alpha_7 (n + 1)^2,$$

$$\varsigma_3 = \alpha_5 (n^4 - 24n^3 + 194n^2 - 624n + 709) + \alpha_8 (n + 1)^2$$

describe some constants. Replacing $f^{2(n-1)}$ by Eq. (47) with $f^{2(n-1)} = -\frac{d_{n-1}}{h_{2(n-1)}} f^{n-1} - \frac{d_0}{h_{2(n-1)}}$ given by Eq. (52), then the following occurs

$$\left( -\frac{t_{2(n-1)h_{n-1}} - t_{n-1}}{h_{2(n-1)}} \right) \frac{f^{n-1}}{f^{n-1} + t_0 - \frac{t_{2(n-1)h_0}}{h_{2(n-1)}}} = 0.$$

Using more transparent notation,

$$d_{n-1} f^{n-1} + d_0 = 0,$$

(54)

where

$$d_{n-1} = (n - 2)^4 (\varsigma_1 \varsigma_4 + \varsigma_2 \varsigma_5) \lambda^6 \sin^{4n+8} R,$$

$$d_0 = (n - 2)^5 (\varsigma_1 \varsigma_6 + \varsigma_3 \varsigma_5) \lambda^5 \sin^{5n+5} R,$$

and

$$\varsigma_4 = 3n^5 - 56n^4 + 398n^3 - 1380n^2 + 2367n - 1604,$$

$$\varsigma_5 = -2n^4 + 29n^3 - 129n^2 + 219n - 105,$$

$$\varsigma_6 = n^4 - 24n^3 + 194n^2 - 624n + 709.$$

Replacing $f^{n-1}$ denoted by Eq. (47) with $f^{n-1} = -d_0/d_{n-1}$ determined by Eq. (54), the following appears

$$t_{3(n-1)} \left( -\frac{d_0}{d_{n-1}} \right)^3 + t_{2(n-1)} \left( -\frac{d_0}{d_{n-1}} \right)^2 + t_{n-1} \left( -\frac{d_0}{d_{n-1}} \right) + t_0 = 0.$$

(55)

Clearly,

$$-t_{3(n-1)} (d_0)^3 + t_{2(n-1)} (d_0)^2 d_{(n-1)} - t_{(n-1)} d_0 (d_{(n-1)})^2 + t_0 (d_{(n-1)})^3 = 0.$$

(56)

Compute each terms of Eq. (56):

$$-t_{3(n-1)} (d_0)^3 = \beta_1 \lambda^{18} \sin^{15n+21} R + o(\sin R),$$

$$t_{2(n-1)} (d_0)^2 d_{(n-1)} = \beta_2 \lambda^{18} \sin^{15n+21} R + o(\sin R),$$

$$-t_{n-1} d_0 (d_{(n-1)})^2 = \beta_3 \lambda^{18} \sin^{15n+21} R + o(\sin R),$$

$$t_0 (d_{(n-1)})^3 = \beta_4 \lambda^{18} \sin^{15n+21} R + o(\sin R),$$

(57)
where
\begin{align}
\beta_1 &= (n-2)^{15} \left( n^2 + 1 \right) (c_1 s_6 + c_3 s_5)^3, \\
\beta_2 &= - (n-2)^{15} c_5 (c_1 s_4 + c_2 s_5) (c_1 s_6 + c_3 s_5)^2, \\
\beta_3 &= - (n-2)^{15} c_4 (c_1 s_4 + c_2 s_5)^2 (c_1 s_6 + c_3 s_5), \\
\beta_4 &= (n-2)^{15} s_6 (c_1 s_4 + c_2 s_5)^3, \\
\end{align}
(58)
and \( o(\sin R) \) denotes the lower degree terms in \( \sin R \). Thus, the following holds
\begin{align}
-t_{3(n-1)} (d_0)^3 + t_{2(n-1)} (d_0)^2 d_{n-1} - t_{(n-1)} d_0 (d_{n-1})^2 + t_0 (d_{n-1})^3 \\
= (\beta_1 + \beta_2 + \beta_3 + \beta_4) \lambda^{18} \sin^{15n+21} R + o(\sin R).
\end{align}
(59)
Since \( R(I) \) contains an interval with \( 56 \), then \( \lambda \) should be 0. Therefore, \( \phi = \eta \), i.e., \( A = \lambda I_n \).
It is the end of proof. \( \square \)

Now, we prove the Main Theorem in Section 1.

Proof. A rotational hypersurfaces family \( \mathfrak{r} \) obtained by rotating the unit speed curve \( \gamma(r) = (f(r), 0, \ldots, 0, \varphi(r)) \), with \( f(u) > 0 \) around axis \( x_n \) which is defined on an interval \( I \). Suppose that the Gauss map \( \mathcal{G} \) of \( \mathfrak{r} \) satisfies \( L_{n-3} \mathcal{G} = \mathcal{A} \mathcal{G} \) with a \( n \times n \) matrix \( \mathcal{A} \). For a function \( R = R(r) \) satisfying \( (f'(r), \varphi'(r)) = (\cos R(r), \sin R(r)) \), let us put \( J = \{ r \in I \mid R'(r) \neq 0 \} \).

Let us examine the following two scenarios: the case when \( J = \emptyset \) and the case when \( J \neq \emptyset \).
Now, let’s observe the outcomes.

(a). Assume that \( J \) is nonempty. From the proof of Lemma 2, \( \lambda = \text{diag}(\eta, \ldots, \eta, \phi) \), where \( \lambda = \phi - \eta \) for constants \( \eta, \phi \). When \( \lambda = 0 \), then \( \eta = \phi \). Thus, it follows from \( 38 \) and \( 39 \) that \( s_{n-2} = c \) is constant, and then the Gauss–Kronecker curvature and the mean curvature satisfy \( K - c H = \phi / (n-1) \).

When \( \phi \neq 0 \), \( H \) and \( K \) are nonzero constants. Then, \( \mathfrak{r} \) is an open part of a hypersphere (see \( 19 \), for 3-space case). Using \( 41 \) and \( 45 \) with \( \lambda = 0 \), and then \( R' \) is constant, and \( f(r) = c \sin R \), where \( c \in \mathbb{R}^+ \). That is, the profile curve \( \gamma \) is an open part of a half circle centered on the rotation axis of \( \mathfrak{r} \). Then, \( \mathfrak{r} \) is an open portion of a round hypersphere.

When \( \phi = 0 \), then \( c H = K \). Assume \( K = 0 \), then \( H = 0 \). Thus, rotational hypersurface is flat and minimal. Therefore, \( \mathfrak{r} \) is an open part of a hyperplane. When \( K \neq 0 \), then \( H \neq 0 \). Hence, \( \mathfrak{r} \) is an open part of a right circular hypercone, or a circular hypercylinder.

(b). Assume that \( J \) is empty. Then, the profile curve \( \gamma \) is a straight line. Therefore, \( \mathfrak{r} \) is an open part of a hyperplane, a right circular hypercone, or a circular hypercylinder. The converse is clear from \( 26 \). \( \square \)

As a consequence, we possess the subsequent characterizations.

**Corollary 3.** Let \( \mathfrak{r} \) be a rotational hypersurface in \( \mathbb{E}^n \). Then, following are equivalent:

(a) \( \mathfrak{r} \) is an open part of a round hypersphere.

(b) The Gauss map \( \mathcal{G} \) of \( \mathfrak{r} \) satisfies \( L_{n-3} \mathcal{G} = \mathcal{A} \mathcal{G} \) for a regular \( n \times n \) matrix \( \mathcal{A} \).

**Corollary 4.** Let \( \mathfrak{r} \) be a rotational hypersurface in \( \mathbb{E}^n \). Then, following are equivalent:

(a) \( \mathfrak{r} \) is an open part of a round right circular hypercone.

(b) The Gauss map \( \mathcal{G} \) of \( \mathfrak{r} \) satisfies \( L_{n-3} \mathcal{G} = \mathcal{A} \mathcal{G} \) for a \( n \times n \) matrix \( \mathcal{A} \).
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