Self-force on a static particle near a black hole

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We study the self-force acting on a static charged point-like particle near a Schwarzschild black hole. We obtain the point-like particle as a limit of a spacetime describing a big neutral black hole with a small charged massive object nearby. The massive object is modeled by a black hole or a naked singularity. In this fully interacting system the massive object is supported above the black hole by a strut. Such a strut has a non-zero tension which corresponds to the external force compensating the gravitational force and the electromagnetic self-force acting on the massive object. We discuss details of the limiting procedure leading to the point-like particle situation. As a result, we obtain the standard gravitational force in the static frame of the Schwarzschild spacetime and the electromagnetic self-force. The electromagnetic-self force differs slightly from the classical results in a domain near the horizon. The difference is due to taking into account an influence of the strut on the electromagnetic field. We also demonstrate that higher order corrections to the gravitational force, a sort of the gravitational self-force, are not uniquely defined and they depend on details of the limiting procedure.

I. INTRODUCTION

The self-force problem is a problem of computing of motion of charged particles in an external gravitational field by taking into account a self-interaction of a particle with its own field. This problem has a long history going down to the classical works \cite{1–4}. Even in flat spacetime it is not quite trivial to take into account radiation-reaction effects and the fact that an electromagnetic mass of a charged particle is not localized at a point. In curved spacetime the task becomes much more involved and subtle. In four-dimensional case there were developed many approaches how to deal with this complicated problem. Three decades ago the study of self-force and self-energy has been mostly related with those of the electric charges \cite{5–12}. Currently most these methods are applied to computation of the gravitational self-force. It is mainly motivated by the study of various processes in the vicinity of black holes or during black-hole collisions.

After the discovery of gravitational waves from binary black-hole mergers, a study of the self-force of compact objects in the black hole background got a new life. It can provide a very effective tool to test general relativity in a strong gravity limit. There are excellent reviews of the topic \cite{13,15}, where one can find description of the contemporary methods, results, and applications of the self-force approaches discussed in the literature.

In four dimensions computation of the self-force in strong gravitational field is technically quite involved, but conceptually it is well understood now. It was a surprise \cite{16,17} to higher-dimensional spacetimes had lead to unexpected ambiguities. It was noted \cite{16–18} that in odd-dimensional spacetimes the standard calculation of the self-force leads to some logarithmic terms depending on an unknown scale parameter. This problem appears even in a flat (Rindler) geometry \cite{17}. There were proposed a few approaches \cite{16,17,38,39} how to fix these unpleasant ambiguities in higher dimensions. But although being physically reasonable, they do not necessarily provide an invariant description of the self-force.

In four dimensions there also remain some subtle problems with the invariant description of the gravitational self-force. In a first-order perturbation of the metric everything is clear: Any compact object moves as a test particle in the certain effective metric satisfying the linearized Einstein equations. Taking into account the self-force effects requires computations up to a second order of perturbations of the geometry, which are non-linear. Nevertheless a similar statement is still valid \cite{40}, but requires a refinement of a test point particle approximation.

This approximation, like in electromagnetism, considers an extended object in the limit when mass, charge, momentum, and size scale to zero in proportional manner \cite{37}. The self-force is given by the quadratic in these parameters terms. This limiting procedure leads to physically satisfactory description of self-force effects for a test particle \cite{29,31,36,41,43}, but gauge invariance of the self-force effects with the necessary accuracy still requires special analysis \cite{15,45}. Even an answer on the question: “What is the position of the test particle?”, may be not gauge invariant. This ambiguity may affect the expressions for the self-energy, the self-force, and the effective equations of motion of the particle. It is important to find out what effects and quantities are robust, which are dependent on the limiting procedure, and what is missing.

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in previous approaches.

One of the ways to test our intuition and computational methods is to apply them to the exactly solvable models. This approach was successfully applied to study a self-force acting on a static charge in Schwarzschild spacetime [510], Kerr–Newman spacetime [12], Schwarzschild–de Sitter spacetime [27], cosmic string spacetime [49], or to probe the spacetime global structure [47]. Let us recall that, for a particle at the static orbit in the Schwarzschild spacetime, the “classical” answer for the magnitude of the electromagnetic self-force in the static frame is

\[ F_{sy} = \frac{q^2 M}{r^3} , \]  

(1.1)

and the self-force has the radial direction pointing away from the black hole, [7, 9, 10].

In this paper we study the effect of a self-force exerted on a static electric charge placed in the vicinity of an uncharged black hole. The idea is to use the exact double black hole solutions of the Einstein-Maxwell equations, where all back-reaction effects are taken into account exactly. Then we take the point-particle limit, when the mass and the charge of one component proportionally scale to zero. In the limit, we obtain the test charged particle in the background geometry of the Schwarzschild black hole. We analyze the force which keeps the system in equilibrium.

The limiting system contains the black hole, the test charge, the electromagnetic field due to the charge, and the agent, which balances the particle at its position: the test strut between the black hole and the particle. The perturbation of the geometry due to the last term is frequently overlooked in some approaches. It is true that the strut is a test object which does not affect the limiting Schwarzschild geometry, however, it influences the geometry in the non-test regime and modifies thus the surrounding electromagnetic field during the limiting procedure.

In our model of the fully interacting system, two black holes are kept in equilibrium by a strut described by a conical defect between them. The geometry and consequently the electromagnetic field are affected by the presence of this conical defect. We compute the limiting self-force on the particle taking into account this effect. As a result, we obtain

\[ F_{sy} = \frac{q^2 M}{r^3} \left( 1 - \frac{M}{r} \right) , \]  

(1.2)

This expression differs from the classical result [11]. The reason for this difference is exactly that we do not calculated the self-force of the particle “alone”, but we also included the influence of the strut which keeps the particle at its position. Both expressions for the self-force agree sufficiently far from the horizon, but they differ near the horizon, when the tension and the linear energy density of the strut are big.

We demonstrate robustness of the electromagnetic self-force, i.e., its independence on the limiting procedure. In the contrast, for the gravitational self-force, we demonstrate that it does depend on the limiting procedure. We show that different physically well motivated conditions on the limiting procedure lead to different values of the gravitational self-force.

The plan of our work is as follows: First, in Section II we describe the system of two black holes and review its geometrical and thermodynamical characteristics. In Section III we briefly remind the concept of the test charge and previously known results for the electromagnetic self-force. In Section IV we describe the limiting procedure to a point test particle, derive the gravitational and electromagnetic self-forces and discuss their properties. In Section V we summarize and discuss our results.

II. DOUBLE BLACK-HOLE SOLUTION

A. Geometry

An asymptotically flat static solution of Einstein-Maxwell equations describing two nonextreme charged black holes in equilibrium was obtained in [48, 52]. Its metric and the electromagnetic vector potential can be written in cylindrical Weyl coordinates as

\[ ds^2 = - f dt^2 + f^{-1} \left[ h^2 (d\rho^2 + dz^2) + \rho^2 d\varphi^2 \right] , \]  

(2.1)

\[ A_t = - \Phi , \quad A_\rho = A_z = A_\varphi = 0. \]  

(2.2)

Here \( f, h \) and \( \Phi \) are the functions of the coordinates \( \rho \) and \( z \) only.

The Weyl coordinates describe the exterior of the black holes. Horizons of both black holes degenerate into two rods localized on the axis \( \rho = 0 \), see Fig.1. The centers of these rods are localized at \( z_H \) and \( z_h \), respectively and their coordinate distance is given by the separation parameter \( R \)

\[ R = |z_h - z_H| . \]  

(2.3)

Without loss of generality, we can set the origin of the \( z \) coordinate such that \( z_H = 0 \). The half-lengths \( \Sigma \) and \( \sigma \) of the rods are given by

\[ \Sigma^2 = M^2 - Q^2 + 2\mu Q , \quad \sigma^2 = m^2 - q^2 - 2\mu q . \]  

(2.4)

where \( M, m \) and \( Q, q \) are masses and charges of the black holes, respectively. Here and below we use several constants:

\[ \mu = \frac{mQ - Mq}{R + M + m} , \quad \nu = R^2 - \Sigma^2 - \sigma^2 + 2\mu^2 , \quad \varpi = Mm - (Q - \mu)(q + \mu) , \quad K_s = 4\Sigma \sigma \left( R^2 - (M + m)^2 + (Q - q - 2\mu)^2 \right) . \]  

(2.5)
Finally, we can write the functions

\[ R_+ = \sqrt{\rho^2 + (z - z_H + \Sigma)^2}, \]
\[ r_\pm = \sqrt{\rho^2 + (z - z_H + \sigma)^2}. \]

Next we define functions \( A, B \) and \( C \)

\[ A = \Sigma \sigma \left[ \nu R_+ + C \right] - (\mu^2 - 2\sigma^2)(R_+ - R_-)(r_+ - r_-), \]
\[ B = 2\Sigma \sigma \left[ (\nu M + 2\mu M)(R_+ + R_-) + (\nu M + 2\mu M)(r_+ + r_-) \right. \]
\[ - 2\Delta \left[ \nu (Q - \mu) - 2\sigma RM - \mu - \mu R \right] \left[ (R_+ - R_-) \right. \]
\[ - 2\Sigma \left[ \nu (Q + \mu) - 2\sigma RM + \mu R \right] \left[ (r_+ - r_-) \right], \]
\[ C = 2\Sigma \sigma \left[ \nu (Q + \mu) + 2\sigma (Q - \mu) \right] \left( R_+ + R_- \right) \]
\[ + \left[ \nu (Q - \mu) + 2\sigma (Q + \mu) \right] \left( r_+ + r_- \right) \]
\[ - 2\Delta \left[ \nu \mu M + 2\sigma (\mu R - Q - \mu R) \right] \left( R_+ - R_- \right) \]
\[ - 2\Sigma \left[ \nu \mu M + 2\sigma (\mu R + Q + \mu R) \right] \left( r_+ - r_- \right). \]

Finally, we can write the functions

\[ f = \frac{A^2 - B^2 + C^2}{(A + B)^2}, \quad h^2 = \frac{A^2 - B^2 + C^2}{K_2 R_+ R_- r_+ r_-}, \]

and the potential for the Maxwell field

\[ \Phi = \frac{C}{A + B}. \]
When following the symmetry axis through the disk, at the coordinate distance $M$ (or $m$ for the other disk) one encounters a naked curvature singularity [48].

B. Physical quantities

The described solution has been thoroughly analyzed in [48, 53]. The most of physically interesting quantities has been calculated and we just list them here. The formulae assume the black hole case. But typically, the quantity defined for one of the black holes remains well defined when the other black hole is changed to a naked singularity. In this case one has to remember that corresponding $\Sigma^2$ or $\sigma^2$ is negative.

The total mass of the system is

$$M = M + m.$$ (2.13)

The areas of horizons of both black holes are

$$A = 4\pi \frac{((R + M + m)(M + \Sigma) - Q(Q + q))^2}{(R + \Sigma)^2 - \sigma^2},$$ (2.14)

$$a = 4\pi \frac{(R + M + m)(m + \sigma) - q(Q + q))^2}{(R + \sigma)^2 - \Sigma^2},$$

the surface gravities are

$$K = \frac{\Sigma ((R + \Sigma)^2 - \sigma^2)}{((R + M + m)(M + \Sigma) - Q(Q + q))^2},$$

$$\kappa = \frac{\sigma ((R + \sigma)^2 - \Sigma^2)}{((R + M + m)(m + \sigma) - q(Q + q))^2},$$ (2.15)

and the electric potentials on the horizons are

$$\Phi = \frac{Q - 2\mu}{M + \Sigma}, \quad \phi = \frac{q + 2\mu}{m + \sigma}. \quad (2.16)$$

The total charges of each black hole are $Q$ and $q$, respectively. It is not a simple task to identify a mass of each black hole separately, since one cannot avoid a non-linear nature of the mutual interaction. But it is argued in [49] that the parameters $M$ and $m$ describe directly the individual masses of black holes. One can also observe a remarkable property that both these parameters satisfy the Smarr relations in the form

$$M = 2TS + \Phi Q, \quad m = 2ts + \phi Q, \quad (2.17)$$

where entropies $S$, $s$ and temperatures $T$, $t$ are defined in the standard way

$$S = \frac{A}{4}, \quad s = \frac{a}{4},$$ (2.18)

$$T = \frac{K}{2\pi}, \quad t = \frac{\kappa}{2\pi}. \quad (2.19)$$

Both black holes (or naked singularities) interact besides the gravitational and electromagnetic interaction also through a strut localized on the axis between them. It can be shown that the axis between black holes is not smooth but contains a conical singularity. Such a singularity represents a thin physical source with an internal energy and a tension. These can be related to the conical defect on the axis [49, 51, 54]. When the angle $\Delta \phi$ around the axis is smaller than the full angle $\Delta \phi = 2\pi - \delta$, with $\delta > 0$, the object on the axis is called the cosmic string. If the angle around the axis is bigger than $2\pi$, then $\delta < 0$, the object represents the strut [57]. The strut has a negative energy density $\varepsilon$ and a positive linear pressure $\tau$, which is called also the tension of the strut. These are related to the angular excess $-\delta > 0$ as $\tau = -\varepsilon = -\frac{\delta}{2\pi} > 0$.

Intuitively, because of the equality between linear energy density and tension, the effective gravitational masses of the string or the strut vanish. As a consequence, an influence on a surrounding spacetime is special: it effectively causes only the conical defect on the axis.

The discussed system contains the strut between the black holes with the tension [51]

$$\tau = \frac{\pi - 2\pi}{\nu - 2\nu} = \frac{Mm - (Q - \mu)(q + \mu)}{R^2 - (M + m)^2 + (Q + q)^2}. \quad (2.20)$$

With the strut one can also associate a conjugate thermodynamical observable called the thermodynamic length $\ell$, see [55]. It has meaning of the strut worldsheet area per unit of the Killing time, $h_{\rho=0}$.

$$\ell = \frac{1}{\Delta t} \int_{\text{strut}} d\mathcal{A} = \frac{1}{\Delta t} \int_t^{t+\Delta t} \int_z h_{\rho=0} dz dt. \quad (2.21)$$

The metric function $h$ is constant on the axis and between the black holes (or naked singularities). It has value

$$h|_{\rho=0} = h_0 \equiv \frac{\nu - 2\varepsilon}{\nu + 2\varepsilon}. \quad (2.22)$$

For the case of two black holes one integrates over the part of the axis between the horizons and the thermodynamic length thus is

$$\ell_{\text{nh}} = (R - \Sigma - \sigma) \frac{\nu - 2\varepsilon}{\nu + 2\varepsilon} \quad (2.23)$$

$$= (R - \Sigma - \sigma) \frac{R^2 - (M + m)^2 + (Q + q)^2}{R^2 - (M - m)^2 + (Q - q - 2\mu)^2}.$$ For the case of a naked singularity of mass $m$ and charge $q$ near the black hole of mass $M$ and charge $Q$ one integrates between the horizon and the singularity, yielding to

$$\ell_{\text{ns}} = (R - \Sigma + m) \frac{\nu - 2\varepsilon}{\nu + 2\varepsilon} \quad (2.24)$$

$$= (R - \Sigma + m) \frac{R^2 - (M + m)^2 + (Q + q)^2}{R^2 - (M - m)^2 + (Q - q - 2\mu)^2}.$$ We can evaluate also the proper length of the strut

$$L = \int_{\text{strut}} (h f^{-1/2})|_{\rho=0} dz \quad (2.25)$$
The expression (2.28) contains complex arguments, and in the case of a small black hole or naked singularity near a big neutral black hole of mass \( M \). This case reduces to

\[
S = 4\pi M^2 \frac{(R + M + m)^2}{(R + M)^2 - \sigma^2},
\]

\[
s = \frac{\pi}{2} \frac{((R + m + m)(m + \sigma) - q^2)^2}{(R + \sigma)^2 - 2M^2},
\]

\[
T = \frac{1}{8\pi M} \frac{(R + M)^2 - \sigma^2}{(R + \sigma)^2 - 2M^2},
\]

\[
t = \frac{1}{2\pi} \left( \frac{(R + \sigma)^2 - 2M^2}{(R + M + m)(m + \sigma) - q^2} \right).
\]

The potentials on horizons are

\[
\Phi = \frac{q}{R + M + m}, \quad \phi = \frac{q}{m + \sigma} \frac{R - M + m}{R + M + m}. \tag{2.33}
\]

The tension of the strut gets the form

\[
\tau = \frac{\zeta}{\nu - 2\kappa} = \frac{M - (Q - \mu)(q + \mu)}{R^2 - (M + m)^2 + (q + q)^2}. \tag{2.34}
\]

The thermodynamic length in the case of a small black hole reduces to

\[
\ell_{bh} = \frac{(R - M - \sigma)(R^2 - (M + m)^2 + q^2)}{R^2 - (m - M)^2 + q^2(R - M + m)^2}, \tag{2.35}
\]

and in the case of a small naked singularity to

\[
\ell_{ns} = \frac{(R - M + m)(R^2 - (M + m)^2 + q^2)}{R^2 - (M + m)^2 + q^2(R - M + m)^2}. \tag{2.36}
\]

The extremality condition (2.31) for a small black hole yields

\[
q^2 = m^2 \frac{R + M + m}{R - M + m}. \tag{2.37}
\]

For the square of charge \( q^2 \) smaller than this critical value the spacetime describes two black holes, if \( q^2 \) is larger, it represents a charged naked singularity above the uncharged black hole.

### C. Neutral black hole

In the following sections we will study a small black hole or naked singularity near a big neutral black hole of mass \( M \). In this case \( Q = 0, \Sigma = M \), and the thermodynamic quantities reduce to

\[
S = 4\pi M^2 \frac{(R + M + m)^2}{(R + M)^2 - \sigma^2},
\]

\[
s = \frac{\pi}{2} \frac{((R + m + m)(m + \sigma) - q^2)^2}{(R + \sigma)^2 - 2M^2},
\]

\[
T = \frac{1}{8\pi M} \frac{(R + M)^2 - \sigma^2}{(R + \sigma)^2 - 2M^2},
\]

\[
t = \frac{1}{2\pi} \left( \frac{(R + \sigma)^2 - 2M^2}{(R + M + m)(m + \sigma) - q^2} \right).
\]

The potentials on horizons are

\[
\Phi = \frac{q}{R + M + m}, \quad \phi = \frac{q}{m + \sigma} \frac{R - M + m}{R + M + m}. \tag{2.33}
\]

The tension of the strut gets the form

\[
\tau = \frac{\zeta}{\nu - 2\kappa} = \frac{M - (Q - \mu)(q + \mu)}{R^2 - (M + m)^2 + (q + q)^2}. \tag{2.34}
\]

The thermodynamic length in the case of a small black hole reduces to

\[
\ell_{bh} = \frac{(R - M - \sigma)(R^2 - (M + m)^2 + q^2)}{R^2 - (m - M)^2 + q^2(R - M + m)^2}, \tag{2.35}
\]

and in the case of a small naked singularity to

\[
\ell_{ns} = \frac{(R - M + m)(R^2 - (M + m)^2 + q^2)}{R^2 - (M + m)^2 + q^2(R - M + m)^2}. \tag{2.36}
\]

The extremality condition (2.31) for a small black hole yields

\[
q^2 = m^2 \frac{R + M + m}{R - M + m}. \tag{2.37}
\]

For the square of charge \( q^2 \) smaller than this critical value the spacetime describes two black holes, if \( q^2 \) is larger, it represents a charged naked singularity above the uncharged black hole.

### D. Schwarzschild geometry

For \( Q = 0, m = 0 \), and \( q = 0 \) the geometry reduces to the Schwarzschild solution of mass \( M \). In the Weyl coordinates it has the form given by the metric functions

\[
f = \frac{R_+ + R_- - 2M}{R_+ R_- + 2M}, \quad h^2 = \frac{(R_+ + R_-)^2 - 4M^2}{4R_+ R_-}. \tag{2.38}
\]
The transformation from the Weyl coordinates $t$, $\rho$, $z$, $\varphi$ to the Schwarzschild spherical coordinates $t$, $r$, $\theta$, $\varphi$ is

\[ \rho = \sqrt{r(r-2M)} \sin \theta, \quad z = (r - M) \cos \theta. \]

In particular, along the semi-axis $\theta = 0$, i.e., $\rho = 0$, $z > 0$, we have

\[ r = z + M. \]

III. THE SELF-FORCE OF A TEST CHARGE

A test charged particle in gravitational field, i.e., in a curved spacetime, creates electromagnetic field in the spacetime. For an extended object such a field interacts with the object itself. Therefore, one can expect, that in the limit of a point particle such an interaction survives in the form of a self-force. The self-force acts on the generically moving point particle already in Minkowski spacetime \cite{13, 14}. This interactions can be understood as a reaction on the field radiated by the particle. The self-force can be evaluated also in the curved spacetime \cite{1, 2, 3} where there exist additional contributions due to scattering of the electromagnetic field on the curvature.

For a static charged particle in the Schwarzschild or Reissner-Nordstrom spacetimes the electromagnetic self-force has been evaluated by various methods, see e.g. \cite{5, 6, 7, 8, 9, 10}. We phrase the results in terms of the external force which is needed to support the particle at the static orbit. The total force $F_{\text{ext}} = F_{\text{ext}} e_r$ needed to support the test particle of a rest mass $m_0$ and a charge $q_0$ floating at the Schwarzschild radius $r$ near the black hole of mass $M$ is

\[ F_{\text{ext}} = \frac{m_0 M}{r^2} \left( 1 - \frac{2M}{r} \right)^{-\frac{1}{2}} - \frac{q_0^2 M}{r^3}, \]

where $e_r$ is the normalized radial vector in the static and locally comoving frame. The first term balances a classical gravitational force in the static frame at radius $r$. The second term equals the additional self-force due to the self-interaction of the charged particle with its own electromagnetic field. The characteristic of the self-force is that it is proportional to a square of the charge, to the mass of the black hole, and it always points away from the black hole.

In principle, there exists a self-floating solution when the self-force balances exactly the gravitational force. However, it occurs only for unphysical values of the involved quantities, namely, for the black hole with the gravitational radius smaller than the classical “radius” of the point particle $\frac{2M}{m}$ and at a distance comparable with this radius, see discussion, e.g., in \cite{9}.

Similarly to the electromagnetic self-force, one could expect that the point particle acts on itself also through the gravitational self-force. To estimate such an interaction is, however, much more difficult task since it involves an evaluation of the backreaction of the singular source on the spacetime geometry, which, due to a non-linear nature of the Einstein equation, is not easily defined problem. However, there exist a wide variety of approaches to this problem in the recent literature (see, e.g., \cite{13, 14} and references therein).

A common feature of various approaches to the gravitational self-force is that it is not as unambiguous as the electromagnetic self-force. It usually depends on details how the self-force is evaluated and how the approximation of the point-like particle is obtained.

IV. SELF-FORCE FROM A LIMIT OF A FULLY BACKREACTING SYSTEM

A. Limiting procedure

In our approach of evaluating the self-force acting on the static point-like particle in the Schwarzschild geometry we start with an exact solution of the Einstein-Maxwell equations representing a big uncharged black hole of a mass $M$ and a small massive object of a mass $m$ and a charge $q$, which can be either a small black hole or a naked singularity, depending on the values of $m$ and $q$. Such a solution has been described in Section III.

This solution contains whole information about the gravitational and electromagnetic interaction between the big black hole and the small massive object, including all kinds of gravitational and electromagnetic “self-interaction”. It describes also the agent which keeps the both objects in a static equilibrium, namely, the strut localized on the axis between the objects. This strut has a linear energy $\varepsilon$ and a linear pressure (tension) $\tau$ along the axis. This pressure exactly corresponds to the external force which is needed to keep the massive object at a constant distance above the black hole.

Next we perform a limit in which the mass and charge of the massive object become small and the massive object changes into a point-like test particle. The strut also becomes a test source, which does not influence anymore the resulting background geometry. However, it has still a tension which corresponds to the external force needed to support the test particle at the static orbit and it balances both the gravitational and electromagnetic interactions.

As we said, we are going to perform the limit in the class of double-black hole spacetimes characterized by parameters $M$, $R$, $m$, and $q$. We know, that for $m = 0$, $q = 0$, the geometry reduces to the Schwarzschild geometry, Fig. 3. It means, that we need to approach values $[M_0, R_0, 0, 0]$ by a curve $[M(\varepsilon), R(\varepsilon), m(\varepsilon), q(\varepsilon)]$ in the parametric space, were $M_0$ and $R_0$ are just limiting values of the mass of the big black hole and of the separation parameter.

However, to identify the position of the test particle in the final Schwarzschild geometry of mass $M_0$, it is necessary to identify also points of manifolds during the
limiting process. It is well known 61 that different identifications can lead to different limiting spacetimes. Indeed, the suitably chosen identification of points can incorporate zooming of some parts of the spacetime and squeezing of others.

In our procedure we identify points by fixing the Weyl coordinates during the limiting process. The points for different values of the spacetime parameters are identified if they have the same Weyl coordinates. Of course, it defines the identification only in the static domain outside the black holes, but it is the domain which we are interested in. We also assume that the big uncharged black hole is localized at \( z = z H = 0 \) during the limiting procedure.

With such an identification, as a result of the limit, the small black hole or naked singularity reduces to a point-like object localized on the axis at \( z = z_b \equiv R_0 \) in the Weyl coordinates. In the Schwarzschild coordinates it corresponds to

\[
r = R_0 + M_0, \tag{4.1}
\]

cf. relation (2.40).

**B. Limit \( m, q \to 0 \) with \( m \sim q \)**

Now we have to stipulate in more detail, how we approach the limiting spacetime. For that we specify an expansion of the parametric curve \([M(\epsilon), R(\epsilon), m(\epsilon), q(\epsilon)]\)

near its limiting value \( \epsilon = 0 \),

\[
\begin{align*}
m(\epsilon) &= \hat{m} \epsilon, \\
q(\epsilon) &= q_1 \epsilon + q_2 \epsilon^2 + \ldots , \\
M(\epsilon) &= M_0 + M_1 \epsilon + \ldots , \\
R(\epsilon) &= R_0 + R_1 \epsilon + \ldots .
\end{align*}
\tag{4.2}
\]

It is essentially the limit in small mass \( m \) and we require that the charge scales to zero as well. We assume that the mass and charge of the massive object approach zero in the same order. Therefore, the massive object can represent both, a black hole or a naked singularity, during the limiting procedure.

By setting coefficients \( q_1, q_2, \ldots \) to zero, we get the case when we shrink a small neutral black hole. The case of a naked singularity with charge \( q \) much larger than mass \( m \) will be explored in the next subsection.

We have kept the higher order coefficients in expansions (4.2) to have a control over details of the limiting procedure. Reason is that we still need to specify, based on physical grounds, how we should perform the limit. It is natural to require that we perform the limit keeping the big hole and its separation from the massive object “unchanged.” However, the spacetime is changing during the limit, so we cannot expect that the big black hole remains completely unchanged. We can choose just a particular characteristic which remains the same in the limiting procedure.

Natural candidate is the mass \( M \) of the black hole. But one could consider also entropy \( S \) (the area) of the black hole, or temperature \( T \) (the horizon surface gravity), or maybe the total mass \( M \) of the system.

For the separation of the massive object from the black hole the situation is even more ambiguous. We can keep the separation parameter \( R \) constant, but it does not have a direct physical meaning—it is a coordinate distance between fictitious centers of the black holes. A more plausible choice for two black holes could be to keep the coordinate distance between the horizons, \( R - \Sigma - \sigma \), constant. For a naked singularity near the black hole one could consider the coordinate separation up to the singularity: \( R + m \) from the black hole “center” or \( R - \Sigma + m \) from the horizon. Moreover, instead the coordinate separation, it would be more natural to use the thermodynamic length \( \ell \) or the proper distance \( L \). All these choices define different limiting curves in the parametric space. Therefore, we have to investigate whether this choice influences the resulting force acting on the test particle.

For that we need to expand the tension (2.34) along the limiting curve. See the Appendix for expansions of some intermediate quantities. Here, we list just a leading term of \( \sigma \) for further references,

\[
\sigma = \sigma_1 + \ldots , \quad \sigma_1 = \sqrt{\hat{m}^2 - q_1^2 \frac{R_0 - M_0}{R_0 + M_0}} \tag{4.3}
\]
The expansion of the tension is

\[ \tau = \frac{\hat{m}M}{r^2} \left( 1 - \frac{2M}{r} \right)^{-\frac{1}{2}} \epsilon \]

\[ + \frac{2\hat{m}^2M^2 + \hat{m} \left( M_1r^2 - 2M(M_1 + R_1)(r - M) \right)}{r^4 \left( 1 - \frac{2M}{r} \right)^2} \epsilon^2 \]

\[ - \frac{q_1^2M}{r^3} \left( 1 - \frac{M}{r} \right)^2 \epsilon^2 + \ldots. \quad (4.4) \]

Here, for readability reasons, we have changed at the end the final mass \( M_0 \) and the separation parameter \( R_0 \) to plain \( M \) and \( R \). The force is expressed in terms of the Schwarzschild coordinate \( r \) of the particle with the help of \( 4.1 \).

We see that in the leading order we have obtained just a term not depending on the charge of the particle. It should be compared with the gravitational force acting on the particle in the static frame. However, first of all we have to identify the rest mass of the particle. The mass \( m \) of the massive object in the limiting procedure has a meaning of the asymptotic mass \( \hat{m} \). For a point particle, the asymptotic mass \( \hat{m} \) is the energy evaluated at the infinity and thus it is related to the rest mass \( m_o \) as

\[ \hat{m} = m_o \sqrt{1 - \frac{2M}{r}}. \quad (4.5) \]

Substituting into the expansion of the tension, we find that the first order term of the external force needed to support the particle is

\[ F_{\text{ext 1}} \equiv \tau_1 = \frac{m_oM}{r^2} \left( 1 - \frac{2M}{r} \right)^{-\frac{1}{2}}, \quad (4.6) \]

which is exactly the force acting against the static gravitational force, cf. the first term in \( 3.1 \).

If we were not sure with the interpretation of the mass parameter \( \hat{m} \), we could reverse the argument. The leading term should reproduce the gravitational force and from that we obtain the relation \( 4.5 \) between \( \hat{m} \) and the rest mass \( m_o \).

Returning back to the expansion \( 4.4 \) of the tension we see, that we obtained the self-force contributions only in the second order. The first term in order \( \epsilon^2 \), the term depending on the mass \( \hat{m} \), is related to the gravitational self-force. The second term proportional to \( q_1^2 \), is related to the electromagnetic self-force.

There is an important difference between these two terms. The electromagnetic self-force does not depend on details of the limiting procedure hidden in coefficients \( M_1 \) and \( R_1 \). On the contrary, the gravitational self-force does. We thus obtained that the external force on the point particle needed to balance the electromagnetic self-force is

\[ F_{\text{ext EM 2}} = -\frac{q_2^2M}{r^3} \left( 1 - \frac{M}{r} \right), \quad (4.7) \]

where, for a symmetry reason\(^3\) we changed \( q_1 \to q_o \). This result does not depend on further details of the limit.

Surprisingly, it is not the same as the standard electromagnetic self-force obtained earlier \( 7.9 \), cf. the second term in \( 3.1 \). It coincides with the classical result for large radius, \( r \gg M \), but it differs closer to the horizon. This difference is due to fact that we have consistently incorporated the backreaction of the agent causing the force, namely, of the strut, on the spacetime. The electromagnetic field of the massive object is influenced by the presence of the strut in the fully interacting system. And this influence modifies the resulting force in the limit. The effect is bigger when the strut is short and its energy density and tension are large. This corresponds exactly to the case when the point particle is close to the horizon.

Finally, we should investigate the gravitational self-force. We have already observed that, in contrast to the electromagnetic self-force, it is not uniquely defined. This could be the main result: the gravitational self-force on the point particle does not have a well defined meaning without a reference to a process how the point particle is obtained. However, to demonstrate this ambiguity, we choose several reasonable limiting processes and show the corresponding self-force.

### Constant mass \( M \) and separation between centers

Mathematically the simplest choice is to assume that the mass \( M(\epsilon) \) and the separation parameter \( R(\epsilon) \) are not changing during the limit. It means

\[ M_1 = 0, \quad R_1 = 0, \quad (4.8) \]

leading to

\[ F_{\text{ext gr 2}} = \frac{2\hat{m}^2M^2}{r^4 \left( 1 - \frac{2M}{r} \right)^2} = \frac{2m_o^2M^2}{r^4 \left( 1 - \frac{2M}{r} \right)}, \quad (4.9) \]

where the last formula is expressed using the rest mass \( m_o \), using \( 4.5 \). The corresponding self-force is thus attractive, i.e., it points out in the opposite direction than the electromagnetic self-force. It also falls down faster with the radius.

\(^3\) We will do this substitution in all final expressions for the self-force. However, in the intermediate calculations, we still have to use, \( M = M_0 + M_1 \epsilon + \ldots, \ R = R_0 + R_1 \epsilon + \ldots \), and to distinguish \( M \), \( R \) and \( M_0, R_0 \). Mostly, it should not cause a confusion and it improves the readability of the final results.

\(^4\) Here, \( q_0 \) does not refer to a “rest” charge similarly to the rest mass \( m_o \), but just indicates that it is an intrinsic characteristic of the test particle.
Constant mass $M$ and separation between horizons

A more natural choice may be to keep the coordinate separation between horizons of two black holes fixed,

\[ R - \Sigma - \sigma = (R_0 - M_0) + (R_1 - M_1 - \sigma_1) \epsilon + \cdots = \text{const}, \quad (4.10) \]

with $\sigma_1$ given by (4.3). Assuming also the constant mass, $M = \text{const}$, we obtain

\[ M_1 = 0, \quad R_1 = \sigma_1, \quad (4.11) \]

and for the force

\[ F_{\text{ext gr} \ 2} = - \frac{2m_o\sqrt{m_o^2 - q_o^2} M}{r^3} + \frac{2m_o(m_o - \sqrt{m_o^2 - q_o^2}) M^2}{r^4(1 - \frac{2M}{r})}. \quad (4.12) \]

We see that the gravitational self-force is influenced by the charge of the particle in this case. It is well defined only for $m_o^2 > q_o^2$, which is related to the fact that we have assumed the existence of both horizons, i.e., that the massive object in the limiting process is a black hole. The first term is dominant for large $r$ and also it remains for an uncharged particle, $q_o = 0$, when

\[ F_{\text{ext gr} \ 2} = - \frac{2m_o^2 M}{r^3}. \quad (4.13) \]

The self-force is repulsive from the black hole in this case.

Constant total mass and separation between centers

Requiring the total mass $M = M + m$ and $R$ constant, we get

\[ M_1 = - \hat{m} \dot{M}, \quad R_1 = 0, \quad (4.14) \]

and for the force we obtain a rather simple expression

\[ F_{\text{ext gr} \ 2} = - \frac{2m_o^2}{r^2}. \quad (4.15) \]

Surprisingly, it does not depend on the mass $M$ of the big black hole and it depends on $r$ by the inverse square law. So it falls down the same as the standard gravitational force \( (4.6) \).

Constant entropy $S$ and thermodynamic length

Assuming that the massive object is a black hole, we can require the entropy of the big black hole $S$ and the thermodynamic length $\ell_{\text{th}}$ to be constant during the limit. Expanding the first expression in (2.32) and (2.35), we obtain

\[ S = 4\pi M_0 + 8\pi M_0 \left( M_1 + \frac{\hat{m} M_0}{R_0 + M_0} \right) \epsilon + \cdots, \quad (4.16) \]

\[ \ell_{\text{th}} = (R_0 - M_0) + \left( R_1 - M_1 - \sigma_1 - \frac{4\hat{m} M_0}{R_0 + M_0} \right) \epsilon + \cdots. \quad (4.17) \]

Requiring the first order terms to vanish, we get

\[ M_1 = - \frac{\hat{m} M_0}{R_0 + M_0}, \quad R_1 = \sigma_1 + \frac{3\hat{m} M_0}{R_0 + M_0}. \quad (4.18) \]

Substituting to the formula \( (4.4) \), we get an unimpressive result

\[ F_{\text{ext gr} \ 2} = - \frac{6m_o^2 M}{r^3} \frac{1 - \frac{4M}{r}}{1 - \frac{2M}{r}} \left( 1 - \frac{M}{r} \right)^2 - \frac{2m_o(m_o - \sqrt{m_o^2 - q_o^2}) M}{r^4(1 - \frac{2M}{r})^2}. \quad (4.19) \]

Assuming that the massive object is a naked singularity, we have to require that the thermodynamic length $\ell_{\text{th}}$ given by (2.36) is constant. Its expansion is

\[ \ell_{\text{th}} = (R_0 - M_0) + \left( R_1 - M_1 + \hat{m} - \frac{4\hat{m} M_0}{R_0 + M_0} \right) \epsilon + \cdots, \quad (4.20) \]

which yields

\[ M_1 = - \frac{\hat{m} M_0}{R_0 + M_0}, \quad R_1 = \hat{m} \frac{2M_0 - R_0}{R_0 + M_0}. \quad (4.21) \]

For the force, we obtain

\[ F_{\text{ext gr} \ 2} = \frac{m_o^2 M}{r^3} \left( 1 - \frac{2M}{r} \right)^2. \quad (4.22) \]

It is worth to note that the gravitational self-force is again attractive in this case. It is also independent of the charge of the particle.

Constant temperature $T$ and thermodynamic length

Similarly to the entropy we can keep constant the temperature (the surface gravity) of the big black hole. Its expansion reads

\[ T = \frac{1}{8\pi M_0} - \frac{1}{8\pi M_0} \left( 2\hat{m} \frac{M_1}{M_0} + \frac{M_1}{M_0} \right) \epsilon + \cdots, \quad (4.23) \]

For the limit of a small black hole we require the thermodynamic length $\ell_{\text{th}}$ constant. It means that the first order terms in expansions (4.23) and (4.17) must vanish, which yields

\[ M_1 = - \frac{2\hat{m} M_0}{R_0 + M_0}, \quad R_1 = \hat{m} \frac{2M_0 - R_0}{R_0 + M_0}. \quad (4.24) \]

The force turns out to be

\[ F_{\text{ext gr} \ 2} = - \frac{4m_o^2 M}{r^3} \frac{1 - \frac{M}{r}}{1 - \frac{2M}{r}} \left( 1 - \frac{M}{r} \right)^2 - \frac{2m_o(m_o - \sqrt{m_o^2 - q_o^2}) M}{r^4(1 - \frac{2M}{r})^2}. \quad (4.25) \]

In the case of the limit of a naked singularity, we require the thermodynamic length $\ell_{\text{th}}$ fixed. From (4.23) and (4.20) then follows

\[ M_1 = - \frac{2\hat{m} M_0}{R_0 + M_0}, \quad R_1 = - \hat{m} \frac{R_0 - M_0}{R_0 + M_0}. \quad (4.26) \]
Surprisingly, all contributions to the gravitational self-force cancel each other in this case,
\[ F_{\text{ext gr}} = 0. \]  
\[(4.27)\]

**Constant mass \( M \) and proper length between horizons**

As the last example we discuss the limit of a small black hole with the mass \( M \) and the proper length between black hole horizons \( L_{\text{BH}} \) fixed. The expansion of the proper length \[(2.26)\] is discussed in Appendix B.

\[ L_{\text{BH}} = \sqrt{R_0^2 - M_0^2} + 2M_0 \arctanh \sqrt{R_0^2 - M_0^2 - M_0 R_0 + M_0} \]
\[ - \left( \frac{4M_0}{\sqrt{R_0^2 - M_0^2}} + (M_1 - R_1) \frac{R_0 + M_0}{R_0 - M_0} \right) \]
\[ + 2 \left( \frac{R_0^2 + 3M_0^2}{R_0^2 - M_0^2} - M_1 \right) \arctanh \sqrt{\frac{R_0^2 - M_0^2}{R_0 + M_0}} \]
\[ + \frac{R_0^2 + 3M_0^2}{R_0^2 - M_0^2} \log \frac{\sigma_1 M_0 \epsilon}{4(R_0^2 - M_0^2)} \] \[(4.28)\]

A new feature here is that the expansion contains logarithmic terms \( \log \epsilon \). This reflects a nonanalytic dependence of the proper length on the expansion parameter. However, one can still require that the linear terms of expansion of \( M \) and \( L_{\text{BH}} \) vanish, yielding
\[ M_1 = 0, \]
\[ R_1 = \frac{4\dot{m} M_0}{R_0 + M_0} + \dot{m} \log \frac{\sigma_1 M_0 \epsilon}{4(R_0^2 - M_0^2)} \]
\[ + \frac{2\dot{m} (R_0^2 + 3M_0^2)}{(R_0 + M_0) \sqrt{R_0^2 - M_0^2}} \arctanh \sqrt{\frac{R_0^2 - M_0^2}{R_0 + M_0}}. \]  
\[(4.29)\]

Substituting to the tension \[(4.4)\] gives a complicated expression for the force
\[ F_{\text{ext gr}} = - \frac{2m_0^2 M}{r^2(1 - 2M/r)} \left( M + 2M \left( 1 - \frac{2M}{r} \right) \right) \]
\[ + 2 \frac{r - M}{\sqrt{1 - 2M/r}} \left( 1 - \frac{2M}{r} + 4M^2/r^2 \right) \arctanh \sqrt{1 - \frac{2M}{r}} \]
\[ + (r - M) \log \frac{m_0^2 - q_0^2 M \epsilon}{4r^2 \sqrt{1 - \frac{2M}{r}}}. \]  
\[(4.30)\]

We derived this expression mainly because it shows that the physically well motivated condition of the fixed proper distance can lead to logarithmic divergences in the self-force. Of course, the self-force itself is of the second order in \( \epsilon \), so the logarithmic term is of the type \( \epsilon^2 \log \epsilon \) which is not a real divergence. But it still documents a broad variety of the behavior of the self-force, depending on the limiting procedure.

A similar analysis can be done in the naked singularity case, using the proper length \( L_{\text{BH}} \) given by \[(2.28)\]. The expansions of the elliptic integrals is even more problematic and the result is not a simple expression. It contains logarithmic terms and it depends on the charge of the particle. Because it does not bring anything qualitatively new, we skip it here.

**C. Limit \( m, q \to 0 \) with \( m \ll q \)**

By discussing various limiting procedures we have clearly demonstrated that the gravitational self-force in this approximation is not uniquely defined. However, it raises the question of the well definiteness of the electromagnetic force, which is of the same order. Can one take the expression \[(4.7)\] seriously if it should be combined with a non-unique expression for the gravitational contribution? One could argue that the electromagnetic self-force is identified by its dependence on the square \( q_0^2 \) of the test charge. However, we have seen that the gravitational self-force can also depend on the charge.

We can, however, modify our approximation by assuming that the mass \( m \) of the massive object is much smaller than its charge \( q \). It implies that the massive object must be modeled by a naked singularity. Although it can raise suspicions, the values of the charge and mass of elementary particles satisfy the condition \( m_0 < |q_0| \). We implement this by changing the expansion \[(4.2)\] as follows
\[ m(\epsilon) = \dot{m} \epsilon^2, \]
\[ q(\epsilon) = q_1 \epsilon + q_2 \epsilon^2 + \ldots, \]
\[ M(\epsilon) = M_0 + M_1 \epsilon + \ldots, \]
\[ R(\epsilon) = R_0 + R_1 \epsilon + \ldots. \]  
\[(4.31)\]

Mass \( m \) thus approaches zero faster than charge \( q \). Not surprisingly, the expansions of the tension \[(4.4)\] changes into
\[ \tau = \frac{\dot{m} M}{r^2} \left( 1 - \frac{2M}{r} \right)^{-1} \epsilon^2 - \frac{q_0^2 M}{r^3} \left( 1 - \frac{2M}{r} \right) \epsilon^2 + \ldots. \]  
\[(4.32)\]

It defines the force needed to support the test point particle at a static position as
\[ F_{\text{ext}} = \frac{m_0 M}{r^2} \left( 1 - \frac{2M}{r} \right)^{-1} \epsilon^2 - \frac{q_0^2 M}{r^3} \left( 1 - \frac{2M}{r} \right), \]  
\[(4.33)\]

where we again introduced the rest mass \( m_0 \) by \[(4.5)\] and symmetric notation for the charge, \( q_0 \equiv q_1 \).

Clearly, the first term compensates the gravitational force in the static frame and the second term is the electromagnetic self-force derived above in \[(4.7)\]. Further corrections corresponding to the gravitational self-force are now of higher orders and we ignore them. In this context it has sense to speak about the electromagnetic self-force alone. The result \[(4.33)\] thus should be compared with the classical result \[(4.1)\]. As discussed above, we have obtained a modification of the self-force near the horizon due to gravitational influence of the strut on the electromagnetic field.
V. SUMMARY

To obtain a better understanding of a nature of the point-like particle approximation we investigate a fully interacting system of a big neutral black hole with an extended charged massive object nearby. The massive object is modeled by a small black hole or a naked singularity, which corresponds at the limit to the particle object is modeled by a small black hole or a naked singularity, which corresponds at the limit to the particle.

When we choose the limiting procedure such that the mass and the charge of the massive object approach zero in the same order, we find out that the leading term of the tension of the strut corresponds to the standard gravitational force \(\sigma\) of the black hole acting on the particle. In the next order we find that the tension also compensates the electromagnetic and gravitational self-forces. The electromagnetic self-force is given by the expression \(\mathbf{f}_e\). It is independent of any further details of the limiting procedure. The gravitational self-force, on the other hand, depends on details of the limiting process. We have demonstrated that by a suitable choice of the limit one can achieve very different results for the self-force: it can be attractive or repulsive, cf. (4.9) vs. (4.22), and it can be independent of the charge, cf. (4.19) vs. (4.22), or it can contain terms logarithmic in the expansion parameter, see (4.30). It is clear that one has to choose very well founded physical reasons how to perform the limiting procedure to obtain a trustworthy result.

If we choose the mass and charge magnitude of the object zero in higher order than the charge, i.e. \(m_0 \ll q_0\), we obtain in the leading order the standard gravitational force and the electromagnetic self-force, together given by formula (4.33). The gravitational self-force is of higher order now and can be ignored.

The electromagnetic self-force \(\mathbf{f}_e\) obtained in our model differs from the classical result \[\mathbf{f}(\mathbf{r})\] in a domain near the horizon. The reason for this difference is that we have taken into account the influence of the strut (the agent supporting the massive object) on the surrounding geometry and thus also on the electromagnetic field. The effect is strong for a short strut with large linear energy density and tension, i.e., exactly when the massive object is near the horizon. As a consequence, our formula for the electromagnetic self-force diverges on the horizon.

When considering the result (4.33), one can easily check that there exists a self-floating solution when the electromagnetic self-force compensates the gravitational force and the strut is not needed (it has vanishing energy and tension). However, as for a similar situation discussed for the classical electromagnetic self-force \[\mathbf{f}_e\], parameters of such a solution are unphysical. It happens for the mass of the black hole and the position of the particle being of the order of the “classical radius” \(\sigma_0 = \sigma_0^2 / m_0\) of the point particle. In this regime quantum effects spoil a validity of the classical theory which we are assuming.

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Appendix A: Expansions of various quantities

A derivation of the tension expansion in the limit \(\epsilon \to 0\) is straightforward, but tedious. First, we list some intermediate expansions for involved quantities.

We start with the expansion of the constant \(\mu\) defined in (2.5),

\[
\mu = \frac{M_0 q_1}{R_0 + M_0} \epsilon + \left( \frac{M_1 q_1 + M_0 q_2}{R_0 + M_0} - \frac{M_{q_1} (\hat{m} + M_1 + R_1)}{(R_0 + M_0)^2} \right) \epsilon^2 + \ldots . \quad (A1)
\]

Next, the half-length \(\sigma\), cf. (2.4), is

\[
\sigma = \frac{\sigma_1}{\epsilon} \frac{q_1 (\hat{m} M_0 + M_0 R_1 - M_1 R_0) + q_2 (R_0^2 - M_0^2)}{(R_0 + M_0)^2} \epsilon^2 + \ldots , \quad (A2)
\]
where
\[
\sigma_1 = \sqrt{\tilde{m}^2 - q_0^2} \frac{R_0 - M_0}{R_0 + M_0} = \sqrt{m^2 - q_0^2} \sqrt{1 - \frac{2M}{r}} .
\]

The last formula is just expressed in terms of the rest mass \(m\), charge \(q_0\), and the Schwarzschild coordinate \(r = R_0 + M_0\).

Finally, for the constants \(\nu\) and \(\varkappa\), cf. (2.22), the expansions are
\[
\nu = (R_0^2 - M_0^2) + 2(R_0 R_1 - M_0 M_1) \epsilon + \ldots ,
\]
\[
\varkappa = \tilde{m} M_0 \epsilon + \left( \tilde{m} M_1 - \frac{q_0^2 M_0 R_0}{(R_0 + M_0)^2} \right) \epsilon^2 + \ldots .
\]

**Appendix B: Proper length between the black hole and a massive object**

The proper length along the symmetry axis is
\[
L = \int_{\text{strut}} (h f^{-1/2}) \big|_{\rho=0} dz .
\]

The metric function \(h\) is constant on the axis and on the strut it takes value \(h_0\) given by (2.22). The metric function \(f\) on the strut takes form
\[
f = \frac{((z - z_H)^2 - \Sigma^2)((z - z_h)^2 - \sigma^2)}{((z - z_H + M)(z_h - z + m) - Qq)^2}.
\]

Substituting to the integral \(\text{[B1]}\) with integral limits given by the horizon position, we get
\[
L = h_0 \int_{z_H - \Sigma}^{z_h - \sigma} \frac{(z - z_H + M)(z_h - z + m) - Qq}{\sqrt{((z - z_H)^2 - \Sigma^2)((z - z_h)^2 - \sigma^2)}} dz .
\]

After some substitutions, manipulations and using integral tables, one can derive the result in terms of elliptic integrals
\[
L_{mb} = \frac{h_0}{\xi} \left( \xi^2 E(k) + 4M \Sigma \Pi(\alpha^2, k) + 4M \sigma \Pi(\Lambda^2, k) \right.
\]
\[+ 2(Mm - Qq - M\sigma - M\Sigma - \Sigma\sigma) K(k) \right) .
\]

where
\[
\xi^2 = R^2 - (\Sigma - \sigma)^2 , \quad \alpha^2 = \frac{R - \Sigma - \sigma}{R + \Sigma - \sigma} ,
\]
\[
k^2 = \frac{R^2 - (\Sigma + \sigma)^2}{R^2 - (\Sigma - \sigma)^2} , \quad \Lambda^2 = \frac{R - \Sigma - \sigma}{R + \Sigma + \sigma} .
\]

In the test charge limit \(\sigma \to 0\) and, hence,
\[
k^2 \to 1 , \quad \Lambda^2 \to 1 .
\]

Expansion at \(k = 1\) of the functions \(E(k)\) and \(K(k)\) does not pose any problems. But the expansion of the elliptic integrals \(\Pi(\alpha^2, k)\) and especially \(\Pi(\Lambda^2, k)\) in this limit is less evident.

First of all we rewrite \(\text{[B4]}\) using the following property of the elliptic integrals (see Eq. (19.7.9) at [56])
\[
\sigma \Pi(\beta^2, k) + \Sigma \Pi(\alpha^2, k) = \frac{1}{2}(R + \Sigma + \sigma) K(k). \tag{B7}
\]

This makes it possible to rewrite \(\text{[B4]}\) in an equivalent non-symmetrical form
\[
L_{mb} = \frac{h_0}{\xi} \left( \xi^2 E(k) - 4M \Sigma (M - m) \Pi(\alpha^2, k) \right.
\]
\[+ 2(MR + Mm - Qq - M\sigma + M(\Sigma - m)) K(k) \right) . \tag{B8}
\]

This form is much better suited for the series expansion at small \(m\) and \(q\). Then we use the following representation
\[
\Pi(\alpha^2, k) = K(k) - \frac{\alpha^2}{\sqrt{1 - \alpha^2 - k^2}} \left( E(k) F(\beta, k) - K(k) E(\beta, k) \right) , \tag{B9}
\]

where \(\sin \beta = \frac{q}{k} \). This identity is valid for all \(0 < k < 1\) and \(0 < \alpha < k\) and is convenient to find the series expansion at \(k = 1\), since the expansions of incomplete elliptic integrals \(E(\beta, k)\) and \(F(\beta, k)\) at \(k = 1\) are well known.

To write down these expansions, it is useful to introduce the quantity \(k' = \sqrt{1 - k^2}\) and compute series at \(k' = 0\). The list of necessary expansions is:
\[
K(k) = - \ln \frac{k'}{4} - \frac{1}{4} \left( \ln \frac{k'}{4} + 1 \right) k'^2
\]
\[\quad - \frac{9}{64} \left( \ln \frac{k'}{4} + \frac{7}{6} \right) k'^4 + \mathcal{O}(k'^6) , \tag{B10}\]

\[
E(k) = 1 - \frac{1}{2} \left( \ln \frac{k'}{4} + \frac{1}{2} \right) k'^2
\]
\[\quad - \frac{3}{16} \left( \ln \frac{k'}{4} + \frac{13}{12} \right) k'^4 + \mathcal{O}(k'^6) , \tag{B11}\]

\[
F(\beta, k) = \arctanh \frac{\alpha}{4} \left( \arctanh \frac{\alpha}{1 - \alpha^2} \right) k'^2
\]
\[\quad + \frac{9}{64} \left( \arctanh \frac{\alpha}{3(1 - \alpha^2)} \right) k'^4 + \mathcal{O}(k'^6) , \tag{B12}\]

\[
E(\beta, k) = \alpha + \frac{1}{2} \arctanh \alpha k'^2
\]
\[\quad + \frac{3}{16} \left( \arctanh \alpha + \frac{\alpha}{1 - \alpha^2} \right) k'^4 + \mathcal{O}(k'^6) . \tag{B13}\]
Using the identity (B9) we get

\[ \Pi(\alpha^2, k) = -\ln \frac{k^2}{4} + \alpha \text{arctanh } \alpha - \frac{1 + (1 + \alpha^2)}{1 - \alpha^2} \ln \frac{k^2}{4} + 2\alpha \text{arctanh } \alpha - \frac{1}{4(1 - \alpha^2)^2} \ln \frac{k^2}{4} + \frac{1}{128(1 - \alpha^2)^2} \left( 6(3 + 6\alpha^2 - \alpha^4) \ln \frac{k^2}{4} + (48\alpha \text{arctanh } \alpha + 21 + 12\alpha^2 - 5\alpha^4) \right) k^4 + O(k^6). \]  

(B14)

Using the expansion (4.2) of the spacetime parameters in formulae (2.22), (B5), substituting these to the series expansions above, and all together to (B8), eventually leads to the result (4.28).

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