Commutators of Lepton Mass Matrices, CP Violation, and Matter Effects in Medium-Baseline Neutrino Experiments

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Abstract

We introduce the commutators of lepton mass matrices to describe the phenomenon of lepton flavor mixing, and establish their relations to the effective Hamiltonians responsible for the propagation of neutrinos. The determinants of those commutators are invariant under matter effects, leading to an instructive relationship between the universal CP-violating parameters in vacuum and in matter. In the scenario of low-energy (100 MeV ≤ E ≤ 1 GeV) and medium-baseline (100 km ≤ L ≤ 400 km) neutrino experiments, we illustrate the features of lepton flavor mixing and CP violation. The terrestrial matter effects on CP- and T-violating asymmetries in ν_e ↔ ν_μ and ¯ν_e ↔ ¯ν_μ neutrino oscillations are also discussed. We demonstrate that a relatively pure signal of leptonic CP violation at the percent level can be established from such medium-baseline experiments with low-energy neutrino beams.

PACS number(s): 14.60.Pq, 13.10.+q, 25.30.Pt
Robust evidence for the long-standing anomalies of solar and atmospheric neutrinos has recently been reported by the Super-Kamiokande Collaboration [1]. It strongly implies that neutrinos are massive and lepton flavors are mixed. The admixture of three different lepton flavors generally involves non-trivial complex phases, leading to the phenomenon of CP violation [2]. Leptonic CP violation can manifest itself in neutrino oscillations. In reality, to measure CP-violating asymmetries needs a new generation of accelerator neutrino experiments with very long baselines [3]. In such long-baseline experiments the earth-induced matter effects, which are likely to deform the neutrino oscillation patterns in vacuum and to fake the genuine CP-violating signals, must be taken into account.

Recently some attention has been paid to an interesting possibility to measure lepton flavor mixing and CP violation in the medium-baseline neutrino experiments with low-energy beams [4–6]. The essential idea is on the one hand to minimize the terrestrial matter effects, which are more significant in the long-baseline neutrino experiments, and on the other hand to obtain the fast knowledge of neutrino mixing and CP violation long before a neutrino factory based on the muon storage ring is really built. Although such low-energy and medium-baseline neutrino experiments may somehow suffer from the problems such as smaller detection cross sections and larger beam opening angles, they can well be realized by choosing the optimum baseline length and beam energy. They are also expected to be complementary to the high-energy and long-baseline neutrino experiments, towards a full determination of lepton flavor mixing and CP-violating parameters.

The purpose of this paper is three-fold. First, the commutators of lepton mass matrices are introduced to describe the phenomenon of lepton flavor mixing, and their relations to the effective Hamiltonian responsible for the propagation of Dirac and Majorana neutrinos are established. An elegant relationship between the universal CP-violating parameters in matter and in vacuum, no matter whether neutrinos are Dirac or Majorana particles, can then be derived from the determinants of those commutators, which are invariant under matter effects. Secondly, we study the features of lepton flavor mixing and CP violation in the scenario of low-energy (100 MeV ≤ E ≤ 1 GeV) and medium-baseline (100 km ≤ L ≤ 400 km) neutrino experiments. The terrestrial matter effects on the elements of the flavor mixing matrix and the rephasing-invariant measure of CP violation are in particular illustrated. Finally we analyze the CP- and T-violating asymmetries in $\nu_e \leftrightarrow \nu_\mu$ and $\nu_e \leftrightarrow \bar{\nu}_\mu$ neutrino oscillations based on a low-energy and medium-baseline experiment. It is demonstrated that a relatively pure signal of leptonic CP violation at the percent level can be established from such accelerator neutrino experiments.

Let us denote the mass matrices of charged leptons and neutrinos in vacuum to be $M_l$ and $M_\nu$, respectively. The phenomenon of lepton flavor mixing arises from the mismatch between the diagonalization of $M_l$ and that of $M_\nu$ in an arbitrary flavor basis. Without loss of generality, one can choose to identify the flavor eigenstates of charged leptons with their mass eigenstates. In this specific basis, where $M_l$ takes the diagonal form $M_l = \text{Diag}(m_e, m_\mu, m_\tau)$, the corresponding lepton flavor mixing matrix $V$ links the neutrino mass eigenstates ($\nu_1, \nu_2, \nu_3$) to the neutrino flavor eigenstates ($\nu_e, \nu_\mu, \nu_\tau$). The effective Hamiltonian responsible for the propagation of neutrinos in vacuum can be written as 

$$H_{\text{eff}} = \frac{1}{2E} \left( V M_\nu^2 V^\dagger \right),$$

(1)
where $\overline{M}_\nu \equiv \text{Diag}\{m_1, m_2, m_3\}$ with $m_i$ being the neutrino mass eigenvalues, and $E \gg m_i$ denotes the neutrino beam energy. When neutrinos travel through a normal material medium (e.g., the earth), which consists of electrons but of no muons or taus, they encounter both charged- and neutral-current interactions with electrons. The neutral-current interactions are universal for $\nu_e$, $\nu_\mu$, and $\nu_\tau$ neutrinos, therefore they lead only to an overall and unobservable phase for neutrino mixing. The charged-current interactions are likely to modify the features of neutrino mixing in vacuum and must be taken into account in all realistic neutrino oscillation experiments. Let us use $M_\nu$ and $V$ to denote the effective neutrino mass matrix and the effective flavor mixing matrix in matter. Then the effective Hamiltonian responsible for the propagation of neutrinos in matter can be written as

$$H_{\text{eff}}^m = \frac{1}{2E} (\mathbf{V} \overline{M}_\nu^2 \mathbf{V}^\dagger),$$

in which $\overline{M}_\nu \equiv \text{Diag}\{m_1, m_2, m_3\}$ with $m_i$ being the effective neutrino mass eigenvalues in matter. The deviation of $H_{\text{eff}}^m$ from $H_{\text{eff}}$, denoted as $\Delta H_{\text{eff}}$, is given by

$$\Delta H_{\text{eff}} = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

in which $A = \sqrt{2} G_F N_e$ measures the charged-current contribution to the $\nu_e e^-$ forward scattering, and $N_e$ is the background density of electrons. Assuming a constant earth density profile (i.e., $N_e = \text{constant}$), which is quite close to the reality for most of the long- and medium-baseline neutrino experiments, one can derive the analytical relationship between $m_i$ and $m_i$ as well as that between $V$ and $V$ from Eqs. (1) and (2). Instead of repeating such calculations, we list the relevant results for $m_i$ and $V$ in Appendix A. Subsequently we concentrate on the commutators of lepton mass matrices and explore how leptonic CP violation in matter is related to that in vacuum. Our discussion in this section is essentially the extension of that in Ref. [9].

An instructive measure of the lepton flavor mixing, i.e., the mismatch between the diagonalization of $M_l$ and that of $M_\nu$ (or $M_\nu$), is the commutators of lepton mass matrices, which can be defined as

$$[M_l M_l^\dagger, M_\nu M_\nu^\dagger] \equiv iX,$$

$$[M_l M_l^\dagger, M_\nu^\dagger M_\nu] \equiv i\overline{X},$$

$$[M_l^\dagger M_l, M_\nu M_\nu^\dagger] \equiv iY,$$

$$[M_l^\dagger M_l, M_\nu^\dagger M_\nu] \equiv i\overline{Y}$$

for neutrinos propagating through vacuum; or as

$$[M_l M_l^\dagger, M_\nu M_\nu^\dagger] \equiv iX,$$

$$[M_l M_l^\dagger, M_\nu^\dagger M_\nu] \equiv i\overline{X},$$

$$[M_l^\dagger M_l, M_\nu M_\nu^\dagger] \equiv iY,$$

$$[M_l^\dagger M_l, M_\nu^\dagger M_\nu] \equiv i\overline{Y}$$

for neutrinos propagating through matter. 

\[ \text{Eq. (4)} \]

\[ \text{Eq. (5)} \]
for neutrinos interacting with matter. In the flavor basis where \( M_l = \overline{M}_l \) holds, we obtain \( X = Y, \overline{X} = \overline{Y} \) and \( X = Y, \overline{X} = \overline{Y} \). It is important to note that
\[
X = Y, \quad X = Y, \quad X = Y, \quad X = Y.
\]
It is important to note that \( M_{\nu} M_{\nu}^\dagger V = 2 E \mathcal{H}_{\text{eff}} \), \( M_{\nu} M_{\nu}^\dagger V = 2 E \mathcal{H}_{\text{eff}}^m \) hold in the chosen basis, no matter whether neutrinos are Dirac or Majorana particles (see Appendix B for a brief proof). In contrast, there is no direct relationship between \( M_{\nu}^\dagger M_{\nu} \) (or \( M_{\nu}^\dagger M_{\nu} \)) and \( \mathcal{H}_{\text{eff}} \) (or \( \mathcal{H}_{\text{eff}}^m \)). We are therefore interested only in the commutators \( X \) and \( X \), which can be rewritten as
\[
X = i \left[ V \overline{M}_{\nu}^2 V^\dagger, \overline{M}_l^2 \right],
\]
\[
X = i \left[ V \overline{M}_{\nu}^2 V^\dagger, \overline{M}_l^2 \right].
\]
(7)

The determinants of \( X \) and \( X \) are related to leptonic CP violation in vacuum and in matter, respectively. To see this point clearly, we carry out a straightforward calculation and arrive at
\[
\text{Det}(X) = 2 J \prod_{\alpha<\beta} \left( m_\alpha^2 - m_\beta^2 \right) \prod_{i<j} \left( m_i^2 - m_j^2 \right),
\]
\[
\text{Det}(X) = 2 J \prod_{\alpha<\beta} \left( m_\alpha^2 - m_\beta^2 \right) \prod_{i<j} \left( m_i^2 - m_j^2 \right),
\]
(8)
where the Greek indices run over \((e, \mu, \tau)\); the Latin indices run over \((1, 2, 3)\); \( J \) and \( J \) are the universal CP-violating parameters of \( V \) and \( V \), respectively. In vacuum \( J \) is defined through \( \text{Im} \left( V_{\alpha i} V_{\beta i} V_{\alpha j}^* V_{\beta j}^* \right) = J \sum_{\gamma,k} \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} \),
(9)
where \((\alpha, \beta, \gamma)\) and \((i, j, k)\) run over \((e, \mu, \tau)\) and \((1, 2, 3)\), respectively. Similarly \( J \) can be defined in terms of the matrix elements of \( V \). We see that the quantities \( \text{Det}(X) \) and \( \text{Det}(X) \) measure leptonic CP violation rephasing-invariantly.

We proceed to find out the relationship between the universal CP-violating parameters \( J \) and \( J \). In view of Eq. (6) as well as Eq. (3), we immediately realize that
\[
X = 2i E \left[ \mathcal{H}_{\text{eff}}^m, \overline{M}_l^2 \right] = X + 2i E \left[ \Delta \mathcal{H}_{\text{eff}}, \overline{M}_l^2 \right] = X.
\]
(10)
This interesting result implies that the commutators of lepton mass matrices are invariant under matter effects. As a consequence, \( \text{Det}(X) = \text{Det}(X) \), leading to an elegant relationship between \( J \) and \( J \):
\[
J \prod_{i<j} \left( m_i^2 - m_j^2 \right) = J \prod_{i<j} \left( m_i^2 - m_j^2 \right).
\]
(11)
\( J \) depends on the matter effect (i.e., the parameter \( A \)) through \( m_i^2 \). Of course \( J = J \) if \( A = 0 \), and \( J = 0 \) if \( J = 0 \).
The results obtained above are valid for neutrinos propagating in vacuum and in matter. As for antineutrinos, the corresponding formulas can straightforwardly be written out from Eqs. (1) - (11) through the replacements $V \rightarrow V^*$ and $A \rightarrow (-A)$.

3 Now we illustrate the features of lepton flavor mixing and CP violation in the scenario of low-energy ($100 \text{ MeV} \leq E \leq 1 \text{ GeV}$) and medium-baseline ($100 \text{ km} \leq L \leq 400 \text{ km}$) neutrino experiments. As the large-angle MSW solution to the solar neutrino problem seems to be favored by the latest Super-Kamiokande data [11], we typically take the mass-squared difference $\Delta m_{\text{sun}}^2 = 5 \cdot 10^{-5} \text{ eV}^2$. We take $\Delta m_{\text{atm}}^2 = 3 \cdot 10^{-3} \text{ eV}^2$ in view of the present data on atmospheric neutrino oscillations. The corresponding pattern of lepton flavor mixing is expected to be nearly bi-maximal; i.e., $|V_{e3}| \ll 1$ as supported by the CHOOZ experiment [12], $|V_{e1}| \sim |V_{e2}| \sim O(1)$, and $|V_{\mu3}| \sim |V_{\tau3}| \sim O(1)$. The approximate decoupling of atmospheric and solar neutrino oscillations implies that one can set

$$
\begin{align*}
\Delta m_{21}^2 &\equiv m_2^2 - m_1^2 \approx \pm \Delta m_{\text{sun}}^2, \\
\Delta m_{31}^2 &\equiv m_3^2 - m_1^2 \approx \pm \Delta m_{\text{atm}}^2.
\end{align*}
$$

Of course $\Delta m_{32}^2 \approx \Delta m_{31}^2$ holds in this approximation. Note that the phenomena of flavor mixing and CP violation in neutrino oscillations can fully be described in terms of four independent parameters of $V$. To satisfy the nearly bi-maximal neutrino mixing with large CP violation, we typically choose $|V_{e1}| = 0.816$, $|V_{e2}| = 0.571$, $|V_{\mu3}| = 0.640$, and $J = \pm 0.020$ as the four independent input parameters in the numerical calculations [3]. In addition, the dependence of the terrestrial matter effect on the neutrino beam energy is given as $A \equiv 2EA = 2.28 \cdot 10^{-4} \text{ eV}^2/\text{GeV}$ [3]. With the help of Eq. (11) and those listed in Appendix A, we are then able to compute the ratios $\Delta m_{31}^2/\Delta m_{31}^2$ (for $i = 2, 3$), $|V_{\alpha i}|/|V_{\alpha i}|$ (for $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$), and $J/J$ as functions of $E$. The relevant results are shown respectively in Figs. 1, 2 and 3, in which $\Delta m_{21}^2 > 0$ and $\Delta m_{31}^2 > 0$ have been assumed. Some comments are in order.

(a) Fig.1 shows that the earth-induced matter effect on $\Delta m_{21}^2$ is significant, but that on $\Delta m_{31}^2$ is negligibly small in the chosen range of $E$ (i.e., $\Delta m_{31}^2 \approx \Delta m_{31}^2$ is an acceptable approximation). In addition, the mass-squared differences of neutrinos are relatively more sensitive to the matter effect than those of antineutrinos.

(b) Fig. 2 shows that the matter effects on $|V_{\mu3}|$ and $|V_{\tau3}|$ are negligible in the low-energy neutrino experiment. The smallest matrix element $|V_{e3}|$ is weakly sensitive to the matter effect; e.g., $|V_{e3}|/|V_{e3}|$ deviates about 7% from unity for $E \approx 1 \text{ GeV}$. In contrast, the other six matrix elements of $V$ are significantly modified by the terrestrial matter effects. For $|V_{e1}|$, $|V_{\mu2}|$ and $|V_{\tau2}|$, the relevant matter effects of neutrinos are more important than those of antineutrinos. For $|V_{e2}|$, $|V_{\mu1}|$ and $|V_{\tau1}|$, the matter effects of neutrinos and antineutrinos are essentially comparable in magnitude.

\footnote{This specific choice corresponds to $\theta_{12} \approx 35^\circ$, $\theta_{23} \approx 40^\circ$, $\theta_{13} \approx 5^\circ$, and $\delta \approx \pm 90^\circ$ in the standard parametrization of $V$ [13], in which $J = \sin \theta_{12} \cos \theta_{12} \sin \theta_{23} \cos \theta_{23} \sin \theta_{13} \cos^2 \theta_{13} \sin \delta$ holds. $J > 0$ and $J < 0$ are associated with the case of neutrino mixing and that of antineutrino mixing, respectively.}
(c) Fig. 3 shows that the magnitude of \( J \) may rapidly decrease with the neutrino beam energy \( E \). This feature implies that the low-energy and medium-baseline neutrino experiments are likely to provide a good chance for measurements of leptonic CP-violating and T-violating asymmetries. Note also that \( J \) undergoes a resonance around \( E \sim 100 \) MeV because of the terrestrial matter effect.

So far we have chosen \( \Delta m_{21}^2 > 0 \) and \( \Delta m_{31}^2 > 0 \) in the numerical calculations. As the spectrum of neutrino masses is unknown, it is possible that \( \Delta m_{21}^2 < 0 \) and (or) \( \Delta m_{31}^2 < 0 \). After a careful analysis of the dependence of \( J \) on positive and negative values of \( \Delta m_{21}^2 \) and \( \Delta m_{31}^2 \), we find the following exact relations:

\[
\begin{align*}
J(+\Delta m_{21}^2, +\Delta m_{31}^2, +A) &= -J(-\Delta m_{21}^2, -\Delta m_{31}^2, -A), \\
J(+\Delta m_{21}^2, -\Delta m_{31}^2, +A) &= -J(-\Delta m_{21}^2, +\Delta m_{31}^2, -A), \\
J(+\Delta m_{21}^2, +\Delta m_{31}^2, -A) &= -J(-\Delta m_{21}^2, -\Delta m_{31}^2, +A), \\
J(+\Delta m_{21}^2, -\Delta m_{31}^2, -A) &= -J(-\Delta m_{21}^2, +\Delta m_{31}^2, +A).
\end{align*}
\]  

(13)

The validity of these equations, which are independent of both the neutrino beam energy and the baseline length, can easily be proved with the help of Eqs. (11), (A1) and (A2). The key point is that \( \Delta m_{21}^2 \) and \( \Delta m_{31}^2 \) keep unchanged in proper arrangements of the signs for \( \Delta m_{21}^2 \), \( \Delta m_{31}^2 \) and \( A \). Furthermore, there exist four approximate relations:

\[
\begin{align*}
J(+\Delta m_{21}^2, +\Delta m_{31}^2, +A) &\approx J(+\Delta m_{21}^2, -\Delta m_{31}^2, +A), \\
J(+\Delta m_{21}^2, +\Delta m_{31}^2, -A) &\approx J(+\Delta m_{21}^2, -\Delta m_{31}^2, -A), \\
J(-\Delta m_{21}^2, +\Delta m_{31}^2, -A) &\approx J(-\Delta m_{21}^2, -\Delta m_{31}^2, -A), \\
J(-\Delta m_{21}^2, -\Delta m_{31}^2, +A) &\approx J(-\Delta m_{21}^2, +\Delta m_{31}^2, +A),
\end{align*}
\]  

(14)

which hold to a good degree of accuracy (with the relative errors \( \leq 15\% \)) for the chosen neutrino beam energy. This result implies that changing the sign of \( \Delta m_{31}^2 \) does not affect the magnitude of \( J \) significantly \(^2\). We expect that those interesting relations in Eqs. (13) and (14) can experimentally be tested in the near future.

In a similar way we have carefully analyzed the behaviors of \( |V_{\alpha i}| \) (for \( \alpha = e, \mu, \tau \) and \( i = 1, 2, 3 \)) with respect to the negative values of \( \Delta m_{21}^2 \) and \( \Delta m_{31}^2 \). Instead of presenting the details of our quantitative results, we only make two qualitative remarks: (1) while \( J \) is not sensitive to the sign of \( \Delta m_{31}^2 \), \( |V_{\alpha i}| \) may dramatically be suppressed or enhanced by changing the sign of \( \Delta m_{31}^2 \); (2) in contrast, the dependence of \( |V_{\alpha i}| \) on the sign of \( \Delta m_{21}^2 \) is less significant.

Finally it is worth mentioning that dramatic changes of the results shown in Figs. 1 – 3 do not happen, even if one allows every parameter to change in a reasonable region around the originally chosen value (e.g., \( \Delta m_{21}^2 \sim (10^{-5} - 10^{-4}) \) eV\(^2\), \( \Delta m_{31}^2 \sim (1 - 6) \cdot 10^{-3} \) eV\(^2\), \( |V_{e1}| \sim 0.7 - 0.9 \), \( |V_{e2}| \sim 0.5 - 0.7 \), \( |V_{\mu3}| \sim 0.5 - 0.8 \), and \( |J| \sim 0.01 - 0.03 \)). This observation

\(^2\)Note that the sign of \( \Delta m_{32}^2 \) (\( = \Delta m_{31}^2 - \Delta m_{21}^2 \)) changes simultaneously with that of \( \Delta m_{31}^2 \), simply because \( |\Delta m_{32}^2| \approx |\Delta m_{31}^2| \gg |\Delta m_{21}^2| \) holds. The sensitivity of \( J/J \) to the signs of \( \Delta m_{21}^2 \) and \( \Delta m_{31}^2 \) has been examined in Ref. \(^4\).
implies that the matter-corrected quantities of lepton flavor mixing and CP violation have quite stable behaviors, as a qualitative consequence of the nearly bi-maximal mixing pattern of $V$ and the rough hierarchy $\Delta m^2_{\text{sun}} < A < \Delta m^2_{\text{atm}}$ in the experimental scenario under consideration.

Let us turn to CP- and T-violating asymmetries in the low-energy and medium-baseline neutrino oscillation experiments. The conversion probability of a neutrino $\nu_\alpha$ to another neutrino $\nu_\beta$ in vacuum is given by

$$P(\nu_\alpha \rightarrow \nu_\beta) = -4 \sum_{i<j} \Re[V_{\alpha i} V_{\beta j}^* V_{\alpha j}^* V_{\beta i}^*] \cdot \sin^2 F_{ij} + 8J \prod_{i<j} \sin F_{ij},$$

(15)

where ($\alpha, \beta$) run over ($e, \mu$), ($\mu, \tau$) or ($\tau, e$), and $F_{ij} \equiv 1.27\Delta m^2_{ij} L/E$ with $L$ being the baseline length (in unit of km) and $E$ being the neutrino beam energy (in unit of GeV). The probabilities of $\nu_\beta \rightarrow \nu_\alpha$ and $\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta$ transitions can straightforwardly be read off from Eq. (15) with the replacement $J \rightarrow -J$. Clearly $P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) = P(\nu_\beta \rightarrow \nu_\alpha)$ holds as a straightforward consequence of CPT invariance. The CP-violating asymmetry between $P(\nu_\alpha \rightarrow \nu_\beta)$ and $P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta)$ amounts to the T-violating asymmetry between $P(\nu_\alpha \rightarrow \nu_\beta)$ and $P(\nu_\beta \rightarrow \nu_\alpha)$:

$$\Delta P = P(\nu_\alpha \rightarrow \nu_\beta) - P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta)$$

$$= P(\nu_\alpha \rightarrow \nu_\beta) - P(\nu_\beta \rightarrow \nu_\alpha)$$

$$= -16J \sin F_{21} \cdot \sin F_{31} \cdot \sin F_{32}.$$  

(16)

Because of $|\Delta m^2_{31}| \approx |\Delta m^2_{32}| \gg |\Delta m^2_{21}|$, a favorable signal of CP or T violation can be obtained only when the condition $|\Delta m^2_{21}| \sim E/L$ is satisfied (i.e., $|\sin F_{21}| \sim \mathcal{O}(1)$ is acquired). Therefore only the large-angle MSW solution to the solar neutrino problem is possible to meet such a prerequisite for the observation of CP violation in realistic neutrino oscillation experiments.

When the terrestrial matter effects are taken into account, the probability of the $\nu_\alpha \rightarrow \nu_\beta$ transition becomes

$$P(\nu_\alpha \rightarrow \nu_\beta) = -4 \sum_{i<j} \Re[V_{\alpha i} V_{\beta j} V_{\alpha j}^* V_{\beta i}^*] \cdot \sin^2 F_{ij} + 8J \prod_{i<j} \sin F_{ij},$$

(17)

in which $F_{ij} \equiv 1.27\Delta m^2_{ij} L/E$. The probability $P(\nu_\beta \rightarrow \nu_\alpha)$ can directly be read off from Eq. (17) with the replacement $J \rightarrow -J$. To obtain the probability $P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta)$, however, both the replacements $J \rightarrow -J$ and $A \rightarrow -A$ need be made for Eq. (17). In this case, $P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta)$ is not equal to $P(\nu_\beta \rightarrow \nu_\alpha)$. Such a false signal of CPT violation measures the matter effect! Similar to Eq. (16), the CP- and T-violating asymmetries in the presence of matter effects can be defined respectively as

$$\Delta \tilde{P} = P(\nu_\alpha \rightarrow \nu_\beta) - P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta),$$

$$\Delta \tilde{P} = P(\nu_\alpha \rightarrow \nu_\beta) - P(\nu_\beta \rightarrow \nu_\alpha),$$

(18)

where the flavor indices ($\alpha, \beta$) run over ($e, \mu$), ($\mu, \tau$) or ($\tau, e$). In general, $\Delta \tilde{P} \neq \Delta P$ because of the matter-induced corrections to $\Delta P$. Note that the overall matter effects residing in
\( \Delta \hat{P} \) is expected to be tiny. Indeed \( \Delta \hat{P} \approx \Delta P \) has numerically been confirmed to be an excellent approximation \([3,15]\). The reason is simply that the matter-induced corrections to \( P(\nu_\alpha \rightarrow \nu_\beta) \) and \( P(\nu_\beta \rightarrow \nu_\alpha) \), which depend on the same parameter \( A \), may essentially cancel each other in the asymmetry \( \Delta \hat{P} \). In contrast, \( P(\nu_\alpha \rightarrow \nu_\beta) \) and \( P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) \) are associated separately with \((+A)\) and \((-A)\), thus there is no large cancellation of matter effects in the asymmetry \( \Delta \hat{P} \).

To illustrate, we perform a numerical analysis of the CP-violating asymmetry \( \Delta \hat{P} \) between \( \nu_e \rightarrow \nu_\mu \) and \( \bar{\nu}_e \rightarrow \bar{\nu}_\mu \) transitions as well as the T-violating asymmetry \( \Delta P \) between \( \nu_e \rightarrow \nu_\mu \) and \( \nu_\mu \rightarrow \nu_e \) transitions. The values of the relevant input parameters are taken the same as before. We first choose \( \Delta m^2_{21} \) and \( \Delta m^2_{31} \) to be positive; i.e., \( \Delta P \approx \Delta P \) is a good approximation. The largest deviation of \( \Delta \hat{P} \) from \( \Delta P \), of the magnitude 0.12% or so, takes place when \( E \) and \( L \) satisfy the rough condition \( L/E \approx 500 \text{ km/GeV} \). This observation is of course dependent on the values of the input parameters. We confirm that \( \Delta \hat{P} \approx \Delta P \) holds to a better degree of accuracy than \( \Delta \hat{P} \approx \Delta P \). Following a similar analysis, we find that the CP-violating asymmetry between \( \nu_\mu \rightarrow \nu_e \) and \( \bar{\nu}_\mu \rightarrow \bar{\nu}_e \) transitions amounts essentially to \((-\Delta P)\). It becomes clear that a relatively pure CP-violating asymmetry at the percent level can be obtained from such a low-energy and medium-baseline neutrino experiment.

The continuous dependence of the CP-violating asymmetry \( \Delta P \) on the baseline length \( L \) is shown in Fig. 5 for five different values of the neutrino beam energy \( E \) (i.e., \( E = 100 \text{ MeV}, 200 \text{ MeV}, 300 \text{ MeV}, 400 \text{ MeV} \) and \( 500 \text{ MeV} \)). Fixing \( E = 100 \text{ MeV} \), for example, we observe that the maximal magnitude of \( \Delta P \) may reach 4% at \( L \approx 200 \text{ km} \) and 6% at \( L \approx 280 \text{ km} \). Note that \( L \) should not be too large in realistic experiments, in order to keep the neutrino beam opening angles as small as possible \([1]\). For a specific experiment, one can select the optimum values of \( E \) and \( L \) after taking its luminosity and other technical details into account.

Finally let us examine the sensitivity of \( \Delta P \) to the signs of \( \Delta m^2_{21} \) and \( \Delta m^2_{31} \). To be explicit, we take \( L = 100 \text{ km} \) and \( 400 \text{ km} \). The dependence of \( \Delta \hat{P} \) on \( E \) is then studied with respect to four possible combinations for the signs of \( \Delta m^2_{21} \) and \( \Delta m^2_{31} \). The numerical results are depicted in Fig. 6. We find that the following relations hold to a good degree of accuracy:

\[
\begin{align*}
\Delta P(+\Delta m^2_{21}, +\Delta m^2_{31}) & \approx -\Delta P(-\Delta m^2_{21}, -\Delta m^2_{31}), \\
\Delta P(+\Delta m^2_{21}, -\Delta m^2_{31}) & \approx -\Delta P(-\Delta m^2_{21}, +\Delta m^2_{31}).
\end{align*}
\]  

(19)

To understand the above equation, we notice from Eq. (16) that \( \Delta P(+\Delta m^2_{21}, \pm \Delta m^2_{31}) \) and \( \Delta P(-\Delta m^2_{21}, \mp \Delta m^2_{31}) \) have the same magnitude but the opposite signs. A slight deviation of \( \Delta P(\pm \Delta m^2_{21}, \pm \Delta m^2_{31}) \) from \( \Delta P(\pm \Delta m^2_{21}, \mp \Delta m^2_{31}) \) arises from the terrestrial matter effects, which are very small in the experimental scenario under consideration. Therefore we arrive at the approximate relations in Eq. (19). Note also that the sensitivity of \( \Delta P \) to the sign of \( \Delta m^2_{31} \) is quite weak, as shown in Fig. 6. This result is certainly consistent with that for \( J \) in Eq. (14); i.e., changing the sign of \( \Delta m^2_{31} \) does not affect \( J \) and \( \Delta P \) significantly.

In this paper we have introduced the commutators of lepton mass matrices to
describe the phenomenon of lepton flavor mixing. Their relations to the effective Hamiltonian responsible for the propagation of neutrinos are independent of the nature of neutrinos (Dirac or Majorana). The determinants of those commutators are invariant under matter effects, leading to an elegant relationship between the universal CP-violating parameters in matter and in vacuum.

We have illustrated the features of lepton flavor mixing and CP violation in the scenario of low-energy ($100 \text{ MeV} \leq E \leq 1 \text{ GeV}$) and medium-baseline ($100 \text{ km} \leq L \leq 400 \text{ km}$) neutrino experiments. In particular, the terrestrial matter effects on the elements of the lepton mixing matrix and on the rephasing-invariant measure of CP violation are systematically analyzed. Some useful relations have been found for the matter-corrected parameters of CP or T violation in respect to different signs of the neutrino mass-squared differences.

We have also presented a detailed analysis of CP- and T-violating asymmetries in neutrino oscillations, based on a medium-baseline experiment with low-energy neutrino beams. The terrestrial matter effects are demonstrated to be insignificant and sometimes even negligible in such an experimental scenario. A relatively pure signal of leptonic CP violation at the percent level can be established from the probability asymmetry between $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$ transitions, or between $\nu_e \to \nu_\mu$ and $\bar{\nu}_e \to \bar{\nu}_\mu$ transitions.

To realize a low-energy and medium-baseline neutrino experiment needs high-intensity and narrow-band neutrino beams (e.g., 10 to 100 times more intense than the neutrino flux in the K2K experiment [5]). Considering conventional neutrino beams produced from the charged pion decay, for example, one may maximize the neutrino flux in the forward direction by restricting the pion beam divergence. The required radial focusing can be provided by a quadrupole channel and (or) magnetic horns [4,18]. The peak pion energy and energy spread within the pion decay channel are determined by the beamline optics, which in turn determine the neutrino spectrum. If the optics are designed to accept a large pion momentum spread, the resultant wide-band beam will contain a large neutrino flux with a broad energy spectrum; and if the optics are designed to accept a smaller pion momentum spread, the resultant narrow-band beam will contain a smaller neutrino flux with a narrower energy spectrum [18]. A detailed study of such technical problems is certainly desirable, but beyond the scope of this work.

We expect that the low-energy, medium-baseline neutrino experiments and the high-energy, long-baseline neutrino experiments may be complementary to each other, in the near future, towards a precise determination of the flavor mixing and CP-violating parameters in the lepton sector.

The author is indebted to A. Romanino for her enlightening comments on this paper.
APPENDIX A

In the assumption of a constant earth density profile and with the help of the effective Hamiltonians given in in Eqs. (1) and (2), the author has recently derived the exact and parametrization-independent formulas for the matter-corrected neutrino masses \( m_i \) and the flavor mixing matrix elements \( V_{\alpha i} \) in Ref. [8]. The main results are briefly summarized as follows.

(1) The neutrino mass eigenvalues \( m_i \) in vacuum and \( m_i \) in matter are related to each other through

\[
\begin{align*}
\mathbf{m}_i^2 &= m_i^2 + \frac{1}{3} x - \frac{1}{3} \sqrt{x^2 - 3y} \left[ z + \sqrt{3 (1 - z^2)} \right], \\
\mathbf{m}_2^2 &= m_2^2 + \frac{1}{3} x - \frac{1}{3} \sqrt{x^2 - 3y} \left[ z - \sqrt{3 (1 - z^2)} \right], \\
\mathbf{m}_3^2 &= m_3^2 + \frac{1}{3} x + \frac{2}{3} z \sqrt{x^2 - 3y},
\end{align*}
\]

(A1)

where \( x, y \) and \( z \) are given by [6]

\[
\begin{align*}
x &= \Delta m_{21}^2 + \Delta m_{31}^2 + A, \\
y &= \Delta m_{21}^2 \Delta m_{31}^2 + A \left[ \Delta m_{21}^2 \left( 1 - |V_{e2}|^2 \right) + \Delta m_{31}^2 \left( 1 - |V_{e3}|^2 \right) \right], \\
z &= \cos \left[ \frac{1}{3} \arccos \frac{2 x^3 - 9 xy + 27 A \Delta m_{21}^2 \Delta m_{31}^2 |V_{e1}|^2}{2 (x^2 - 3y)^{3/2}} \right]
\end{align*}
\]

(A2)

with \( \Delta m_{21}^2 \) and \( \Delta m_{31}^2 \) defined in Eq. (12), and \( A \equiv 2 E A = 2 \sqrt{2} G_F N_e E \). Of course, \( \mathbf{m}_i^2 = m_i^2 \) can be reproduced from Eq. (A1) if \( A = 0 \) is taken. Only the mass-squared differences \( \Delta m_{21}^2 \equiv \mathbf{m}_2^2 - \mathbf{m}_1^2 \) and \( \Delta m_{31}^2 \equiv \mathbf{m}_3^2 - \mathbf{m}_1^2 \) are relevant to the practical neutrino oscillations in matter.

At this point it is worth mentioning that \( m_i^2, |V_{ei}|^2, A \) and \( \mathbf{m}_i^2 \) are correlated with one another via an interesting equation, which has not been noticed in the literature:

\[
A \left( m_i^2 - m_k^2 \right) \left( m_j^2 - m_k^2 \right) |V_{ek}|^2 = \prod_{n=1}^{3} \left( m_n^2 - m_k^2 \right),
\]

(A3)

where \( (i, j, k) \) run over \( (1, 2, 3) \) with \( i \neq j \neq k \). Assuming \( |\Delta m_{21}^2| \gg |\Delta m_{31}^2| \) and taking \( k = 3 \), one can easily reproduce the approximate analytical result for the correlation between \( m_i^2 \) and \( m_i^2 \) obtained in Ref. [7].

(2) The analytical relationship between the elements of \( \mathbf{V} \) in matter and those of \( \mathbf{V} \) in vacuum reads:

\[
\mathbf{V}_{\alpha i} = \frac{N_i}{D_i} V_{\alpha i} + \frac{A}{D_i} V_{\alpha i} \left[ (m_i^2 - m_j^2) V_{ek} V_{ak} + (m_i^2 - m_k^2) V_{ej}^* V_{o\alpha j} \right],
\]

(A4)

in which \( \alpha \) runs over \( (e, \mu, \tau) \) and \( (i, j, k) \) over \( (1, 2, 3) \) with \( i \neq j \neq k \), and

\[
\begin{align*}
N_i &= (m_i^2 - m_j^2) (m_i^2 - m_k^2) - A \left[ (m_i^2 - m_j^2) |V_{ek}|^2 + (m_i^2 - m_k^2) |V_{ej}|^2 \right], \\
D_i^2 &= N_i^2 + A^2 |V_{ei}|^2 \left[ (m_i^2 - m_j^2)^2 |V_{ek}|^2 + (m_i^2 - m_k^2)^2 |V_{ej}|^2 \right].
\end{align*}
\]

(A5)
Obviously $A = 0$ leads to $V_{\alpha i} = V_{\alpha i}$. This exact and compact formula shows clearly how the flavor mixing matrix in vacuum is corrected by the matter effects. Instructive analytical approximations can be made for Eq. (A4), once the spectrum of neutrino masses is experimentally known or theoretically predicted.

The results listed above are valid for neutrinos interacting with matter. As for antineutrinos propagating through matter, the relevant formulas can straightforwardly be obtained from Eqs. (A1) – (A5) through the replacements $V \rightarrow V^*$ and $A \rightarrow -A$.

**APPENDIX B**

This Appendix aims to show why $M_\nu M_\nu^\dagger = V M_\nu^2 V^\dagger$ holds in the flavor basis where the charged lepton mass matrix is diagonal (i.e., $M_l = \overline{M_l}$), no matter whether neutrinos are Dirac or Majorana particles. For simplicity, we focus on the standard charged-current weak interactions, in which only the left-handed leptons take part:

$$- L_{\text{weak}} = \frac{g}{\sqrt{2}} (\nu_e, \nu_\mu, \nu_\tau)_L \gamma^\mu \left(\begin{array}{c} e \\ \mu \\ \tau \end{array}\right)_L W^+_\mu + \text{h.c.} ,$$  \hspace{1cm} (B1)

where the flavor eigenstates of charged leptons are identified with their mass eigenstates. As the lepton flavor mixing matrix $V$ is defined to link the neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$ to the neutrino mass eigenstates $(\nu_1, \nu_2, \nu_3)$, we get

$$- L_{\text{weak}} = \frac{g}{\sqrt{2}} (\nu_1, \nu_2, \nu_3)_L V^\dagger \gamma^\mu \left(\begin{array}{c} e \\ \mu \\ \tau \end{array}\right)_L W^+_\mu + \text{h.c.}$$  \hspace{1cm} (B2)

in the chosen flavor basis.

If neutrinos are Dirac particles, the mass term can be written as

$$- L_{\text{Dirac}} = (\nu_e, \nu_\mu, \nu_\tau)_L M_\nu \left(\begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \end{array}\right)_R + \text{h.c.} ,$$  \hspace{1cm} (B3)

where $M_\nu$ is in general an arbitrary $3 \times 3$ matrix. It is always possible to diagonalize $M_\nu$ by a bi-unitary transformation: $V^\dagger M_\nu V = \overline{M_\nu}$, where $V$ is just the flavor mixing matrix of Dirac neutrinos consistent with Eq. (B2). Obviously $M_\nu M_\nu^\dagger = V M_\nu^2 V^\dagger$ holds.

If neutrinos are Majorana particles, the mass terms turns out to be

$$- L_{\text{Majorana}} = \frac{1}{2} (\nu_e, \nu_\mu, \nu_\tau)_L M_\nu \left(\begin{array}{c} \nu^e_e \\ \nu^e_\mu \\ \nu^e_\tau \end{array}\right)_R + \text{h.c.} ,$$  \hspace{1cm} (B4)

in which $\nu^e \equiv C \nu^T$ with $C$ being the charge-conjugation operator. It is well known that $M_\nu$ must be a symmetric matrix and can be diagonalized by a single unitary transformation:
\[ V^\dagger M_\nu V^* = \overline{M_\nu}, \] where \( V \) is just the flavor mixing matrix of Majorana neutrinos consistent with Eq. (B2) \[^3\]. Once again we arrive at \( M_\nu M_\nu^\dagger = V \overline{M_\nu} V^\dagger \).

A more general neutrino mass Lagrangian involves both Dirac and Majorana terms \[^{19}\]. In this case, one can similarly prove that the mass matrix of light (active) Majorana neutrinos is symmetric and satisfies \( M_\nu M_\nu^\dagger = V \overline{M_\nu} V^\dagger \). Thus \( \mathcal{H}_{\text{eff}} = (M_\nu M_\nu^\dagger)/(2E) \) holds in vacuum, no matter whether neutrinos are Dirac or Majorana particles. In matter we have an analogous relation between \( \mathcal{H}_{\text{eff}}^m \) and \( M_\nu M_\nu^\dagger \), as given in Eq. (6).

---

\[^{3}\]If the Majorana mass matrix \( M_\nu \) is diagonalized by the transformation \( U^\dagger M_\nu U = \overline{M_\nu} \), one will see that it is \( U^* \) (instead of \( U \)) linking the flavor eigenstates to the mass eigenstates of neutrinos. Therefore the flavor mixing matrix turns out to be \( V = U^* \) in the chosen flavor basis.
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FIG. 1. Ratios $\Delta m^2_{21}/\Delta m^2_{21}$ and $\Delta m^2_{31}/\Delta m^2_{31}$ changing with the beam energy $E$ (in unit of GeV) for neutrinos ($\nu$) and antineutrinos ($\overline{\nu}$), in which $\Delta m^2_{21} = 5 \cdot 10^{-5}$ eV$^2$, $\Delta m^2_{31} = 3 \cdot 10^{-3}$ eV$^2$, $|V_{e1}| = 0.816$, and $|V_{e2}| = 0.571$ have typically been input.
FIG. 2. Ratios $|V_{\alpha i}|/|V_{\alpha i}|$ (for $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$) changing with the beam energy $E$ (in unit of GeV) for neutrinos ($\nu$) and antineutrinos ($\bar{\nu}$), in which $\Delta m_{21}^2 = 5 \cdot 10^{-5} \text{ eV}^2$, $\Delta m_{31}^2 = 3 \cdot 10^{-3} \text{ eV}^2$, $|V_{e1}| = 0.816$, $|V_{e2}| = 0.571$, $|V_{\mu3}| = 0.640$, and $J = \pm 0.020$ have typically been input.
FIG. 3. The ratio $J/J$ changing with the beam energy $E$ (in unit of GeV) for neutrinos ($\nu$) and antineutrinos ($\bar{\nu}$), in which $\Delta m^2_{21} = 5 \cdot 10^{-5} \text{ eV}^2$, $\Delta m^2_{31} = 3 \cdot 10^{-3} \text{ eV}^2$, $|V_{e1}| = 0.816$, $|V_{e2}| = 0.571$, $|V_{\mu 3}| = 0.640$, and $J = \pm 0.020$ have typically been input.
FIG. 4. The CP-violating asymmetries between $\nu_e \rightarrow \nu_\mu$ and $\bar{\nu}_e \rightarrow \bar{\nu}_\mu$ transitions in matter (a: $\Delta P$ in unit of $10^{-2}$) and in vacuum (b: $\Delta P$ in unit of $10^{-2}$) changing with the beam energy $E$ (in unit of GeV), in which $\Delta m_{21}^2 = 5 \cdot 10^{-5}$ eV$^2$, $\Delta m_{31}^2 = 3 \cdot 10^{-3}$ eV$^2$, $|V_{e1}| = 0.816$, $|V_{e2}| = 0.571$, $|V_{\mu 3}| = 0.640$, and $J = \pm 0.020$ have typically been input.
FIG. 5. The CP-violating asymmetry $\Delta P$ between $\nu_e \rightarrow \nu_\mu$ and $\bar{\nu}_e \rightarrow \bar{\nu}_\mu$ transitions (in unit of $10^{-2}$) changing with the baseline length $L$ (in unit of km) and the beam energy $E$ (a: 100 MeV; b: 200 MeV; c: 300 MeV; d: 400 MeV; e: 500 MeV). Here $\Delta m_{21}^2 = 5 \cdot 10^{-5} \text{eV}^2$, $\Delta m_{31}^2 = 3 \cdot 10^{-3} \text{eV}^2$, $|V_{e1}| = 0.816$, $|V_{e2}| = 0.571$, $|V_{\mu 3}| = 0.640$, and $J = \pm 0.020$ have typically been input.
FIG. 6. Dependence of the CP-violating asymmetry $\Delta P$ (in unit of $10^{-2}$) on the beam energy $E$ (in unit of GeV) and on the signs of $(\Delta m_{21}^2, \Delta m_{31}^2)$ – 1: $(-, +)$; 2: $(-, -)$; 3: $(+, +)$; 4: $(+, -)$. Here $|\Delta m_{21}^2| = 5 \cdot 10^{-5}$ eV$^2$, $|\Delta m_{31}^2| = 3 \cdot 10^{-3}$ eV$^2$, $|V_{e1}| = 0.816$, $|V_{e2}| = 0.571$, $|V_{\mu3}| = 0.640$, and $J = \pm 0.020$ have typically been input.