Mean-Field Analysis of Antiferromagnetic Three-State Potts Model with Next Nearest Neighbor Interaction

M. Itakura

Department of Pure and Applied Sciences,
University of Tokyo,
Meguro-ku, Komaba 3-8-1, Tokyo 153, Japan

(December 7, 2018)

Abstract

The three-state Potts model with antiferromagnetic nearest-neighbor (n.n.) and ferromagnetic next-nearest-neighbor (n.n.n) interaction is investigated within a mean-field theory. We find that the phase-diagram contains two kind of ordered phases, so-called BSS phase and PSS phase, separated by a discontinuous phase transition line. Order-disorder transition is continuous for the weak n.n.n. interaction and becomes discontinuous transition when the n.n.n. interaction is increased. We show that the multicritical point where the order-disorder transition becomes discontinuous is indeed a tricritical point. PACS numbers: 64.40.Kw, 75.10.Hk
I. INTRODUCTION

Antiferromagnetic three-state Potts model has interesting properties. It is described by the following Hamiltonian:

\[ H = J \sum_{<ij>} \delta(s_i, s_j) \]  

where \(<ij>\) indicates summation over nearest neighbor pairs, and \(s_i = 1, 2, 3\) denotes three-state Potts spin on the \(i\)'th site. In antiferromagnetic case \((J > 0)\), the neighboring two spins prefer to take different values. Typical ground state configuration of the model on the square lattice is depicted in Fig.(1). One can change the state of certain spin in Fig.(1) from “2” to “3”, or vice versa, without any energy cost. One half of all the spins are such “semi-free” spins, therefore ground state is infinitely degenerated. It should be noted that if we divide the lattice in Fig.(1) into two interpenetrating sublattices, there are only “1” spins on one sublattice, while random mixture of “2” spins and “3” spins is present on another sublattice.

For the model on simple cubic lattice, it is known that long-range order is realized at finite temperature [1,2] in spite of the high degeneracy of ground-state which usually leads to the disordered state at all temperature range [3]. Recently this model (on the simple cubic lattice) has been studied intensively concerning two interests; one on the nature of the order-disorder transition and another on the nature of ordered phase.

Order-disorder transition is continuous and the associated critical exponents are estimated by various methods, such as Monte Carlo simulation [4,8] and the coherent-anomaly method [9].

As for the nature of ordered phase, it is known that so-called broken-sublattice-symmetry (BSS) phase is realized at a sufficiently low temperature [10,11], in which one of the sublattices is dominated by one spin state, while the second sublattice is dominated by the mixture of the remaining two spin states.

Recently several different claims have been made about the nature of the ordered phase at the region just below the transition temperature. Lapinskas and Rosengren concluded that
so-called permutationally-symmetric-sublattice (PSS) phase, in which each two sublattices are dominated by one spin state, is realized at a very narrow temperature range just below the transition point, based on cluster-variation method analysis [11,12] and Monte Carlo simulation [13]. On the other hand, Kolesik and Suzuki have performed Monte Carlo simulations and observed a rotationally symmetric phase at a certain range near the transition point, finding no evidence of the PSS phase [14].

Now we consider the effect of next-nearest-neighbor (n.n.n.) ferromagnetic interaction. Let us consider the following Hamiltonian:

\[ H = J \sum_{<i,j>_{nn}} \delta(s_i, s_j) - \gamma J \sum_{<i,j>_{nnn}} \delta(s_i, s_j) \]  

(2)

where the first and the second summation runs over all nearest neighbor and next-nearest neighbor pairs on the simple cubic lattice, respectively. We assume that \( J > 0 \) and \( \gamma \geq 0 \). The lattice consists of two sublattices, which we refer as A and B. The first term in (2) represents the inter-sublattice antiferromagnetic interaction and the second one represents the intra-sublattice ferromagnetic interaction, which resolves the high degeneracy of ground state.

Thus the n.n.n. interaction affects the nature of the ordered phase because the BSS phase costs energy proportional to \( \gamma J \) compared to the PSS phase. This effect for the models on the square lattice has been studied by several methods [15–17].

In three dimensions, we expect another effect produced by a strong n.n.n. interaction; one can consider \( \gamma \to \infty \) limit as two independent systems of the ferromagnetic \( q = 3 \) Potts model, which undergoes the first-order phase transition in three dimensions [18]. Thus the order-disorder transition is expected to become discontinuous as \( \gamma \) becomes large.

Mean-field theory gives qualitatively correct answer for both \( \gamma = 0 \) (AF) and \( \gamma \to \infty \) (F) cases in three or greater dimensions, namely the BSS phase is realized below the continuous phase transition point in the AF case, and discontinuous phase transition occurs in the F case. We investigate the intermediate region using mean-field theory in the following sections.
II. MEAN-FIELD ANALYSIS

To perform a mean-field calculation, we use a method of variational free energy which is equivalent to equation of self-consistency [19]. Let us consider the following mean-field Hamiltonian:

\[ H_0 = \sum_i \sum_{s=1}^3 h_{is} \delta(s_i, s) \]  

(3)

where \( h_{is} \) is a mean-field acting on the \( i \)'th spin. Boltzmann probability factor for the system described by \( H_0 \) is denoted by \( P_0(\{s_i\}) \), which is a product of probability factors of individual spins:

\[ P_0(\{s_i\}) = \Pi_i p_i(s_i) \]  

(4)

\[ p_i(s) = \frac{\exp(-\beta h_{is})}{\exp(-\beta h_{i1}) + \exp(-\beta h_{i2}) + \exp(-\beta h_{i3})} \]  

(5)

We minimize the following variational free energy (divided by the total number of spins \( N \)) with respect to \( h_{is} \):

\[ F_0 = (\langle H \rangle_0 - TS_0)/N \]  

(6)

The symbols \( \langle \cdots \rangle_0 \) and \( S_0 \) denotes respectively the expectation value and the entropy associated with the probability distribution \( P_0(\{s_i\}) \):

\[ \langle H \rangle_0 = J \sum_{<i,j>_{nn}} \sum_s p_i(s)p_j(s) - \gamma J \sum_{<i,j>_{nnn}} \sum_s p_i(s)p_j(s) \]  

(7)

\[ S_0 = -\sum_i \sum_s p_i(s) \ln p_i(s) \]  

(8)

Minimization problem with respect to \( h_{is} \) is equivalent to that with respect to \( p_i(s) \) under following constraints:

\[ p_i(1) + p_i(2) + p_i(3) = 1 \quad , \quad p_i(s) \geq 0 \]  

(9)

Translational symmetry assures that the free energy (6) is minimized when:
\[ p_i(s) = C_{sA} \text{ for all } i \in A \] (10)
\[ p_i(s) = C_{sB} \text{ for all } i \in B \] (11)

Obviously, \( C_{s\alpha} \) coincides with the expectation value of the concentration of \( s \)'th spin state on the sublattice \( \alpha \), calculated with the probability distribution \( P_0\{s_i\} \). They are under following constraints:

\[ C_{1\alpha} + C_{2\alpha} + C_{3\alpha} = 1 \quad , \quad C_{s\alpha} \geq 0 \] (12)

Then \( < H >_0 \) and \( S_0 \) are expressed with \( C_{s\alpha} \):

\[ < H >_0 = \frac{z_1 J N}{2} \sum_s C_{sA} C_{sB} - \frac{z_2 \gamma J N}{2} \sum_s (C_{sA}^2 + C_{sB}^2)/2 \] (13)

\[ S_0 = -N \sum_s (C_{sA} \ln C_{sA} + C_{sB} \ln C_{sB})/2 \] (14)

where \( z_1, z_2 \) denotes the coordination number of n.n. and n.n.n. sites respectively (for the simple cubic lattice, \( z_1 = 6 \) and \( z_2 = 12 \)).

Furthermore, we define two-component sublattice magnetization similar as Ono used [20]:

\[ x_\alpha = (C_{1\alpha} - C_{2\alpha})/\sqrt{3} \] (15)
\[ y_\alpha = (C_{1\alpha} + C_{2\alpha} - 2C_{3\alpha})/3 \] (16)

Then the three quantities \( C_{1\alpha}, C_{2\alpha}, \) and \( C_{3\alpha} \) can be expressed by two quantities \( x_\alpha \) and \( y_\alpha \) owing to the constraint \( C_{1\alpha} + C_{2\alpha} + C_{3\alpha} = 1 \).

\[ C_{1\alpha} = \frac{1}{3} + \frac{\sqrt{3}}{2} x_\alpha + \frac{1}{2} y_\alpha \] (17)
\[ C_{2\alpha} = \frac{1}{3} - \frac{\sqrt{3}}{2} x_\alpha + \frac{1}{2} y_\alpha \] (18)
\[ C_{3\alpha} = \frac{1}{3} - y_\alpha \] (19)

The two-component sublattice magnetization \( (x_\alpha, y_\alpha) \) carry an irreducible, unitary representation of the permutation group of the three spin states \( \{1, 2, 3\} \). Owing to the constraints
\( C_{\alpha} \geq 0 \), the sublattice magnetization \( (x_{\alpha}, y_{\alpha}) \) is restricted to take a value within a regular triangle on the \( x_{\alpha} - y_{\alpha} \) plane (Fig.2). Three vertexes of the triangle correspond to completely ordered state \( (C_{1\alpha} = 1, C_{2\alpha} = 0, C_{3\alpha} = 0 \text{ etc.}) \) and the point \((0,0)\) corresponds to completely disordered state \( (C_{1\alpha} = C_{2\alpha} = C_{3\alpha} = 1/3) \).

We have minimized the free energy \( F_0 \) with respect to \( x_A, y_A, x_B, \) and \( y_B \) numerically using the following iteration:

\[
X_{\alpha}^{(n+1)} = X_{\alpha}^{(n)} - \Delta \left. \frac{\partial F_0}{\partial x_{\alpha}} \right|_{x_{\alpha}=X_{\alpha}^{(n)}, y_{\alpha}=Y_{\alpha}^{(n)}}
\]

\[
Y_{\alpha}^{(n+1)} = Y_{\alpha}^{(n)} - \Delta \left. \frac{\partial F_0}{\partial y_{\alpha}} \right|_{x_{\alpha}=X_{\alpha}^{(n)}, y_{\alpha}=Y_{\alpha}^{(n)}}
\]

(20)

where \( \Delta \) is a small, positive quantity. The iteration (20) is repeated until \( F_0 \) converge to some minima. Several values were used as an initial value \( X_{\alpha}^{(0)} \) and \( Y_{\alpha}^{(0)} \) to find absolute minimum value of \( F_0 \) out of all minima.

Finally we obtained a phase diagram of two parameters \( T/J \) and \( \gamma \) (Fig.(3)). The order-disorder transition is continuous for \( \gamma \leq 3/2 \) and discontinuous for \( \gamma > 3/2 \). There are two kind of ordered phases, BSS phase and PSS phase, separated by a discontinuous transition line. Typical values of the concentrations of the three states and the sublattice magnetizations of each ordered phases are shown in Fig.(4). In the PSS phase, the angle between each sublattice magnetizations is non-trivial value; it is greater than 120° and less than 180°. The non-trivial value of the angle can be understood as follows. The antiferromagnetic n.n. interaction makes two points \((x_A, y_A)\) and \((x_B, y_B)\) “repulsive”, therefore prefers 180°. On the other hand, the energy of the ferromagnetic n.n.n. interaction is minimized when \((x_{\alpha}, y_{\alpha})\) locates on the vertex of the triangle, therefore prefers 120°. Thus we can understand the non-trivial value of the angle as a result of the competition of the two interactions.

Banavar and Wu have studied the same model as (2) with \( q = 3, 4 \) using the mean-field theory [21] and concluded that the PSS phase does not appear. Our result disagree with theirs.
III. EFFECTIVE FREE ENERGY FORM

Since the numerical method (20) becomes less precise near the critical line, we expand the mean-field free energy $F_0$ in powers of the order-parameter to investigate the critical behavior.

Firstly we define ferromagnetic and antiferromagnetic order-parameters:

$$x_F = x_A + x_B, \quad x_{AF} = x_A - x_B$$

$$y_F = y_A + y_B, \quad y_{AF} = y_A - y_B$$

Relevant quantities to the order-disorder transition are $x_{AF}$ and $y_{AF}$, which carry an irreducible representation of a group $P_3$ (permutation of three spin states) $\times P_2$ (permutation of two sublattices), isomorphic to the group $c_{6v}$ (point group of a regular hexagon). Indeed, the allowed range of the antiferromagnetic order-parameter $(x_{AF}, y_{AF})$ is a regular hexagon in the $x_{AF} - y_{AF}$ plane.

Then we introduce polar coordinates of ferromagnetic and antiferromagnetic order-parameter:

$$x_F = R_F \cos \theta_F, \quad y_F = R_F \sin \theta_F$$

$$x_{AF} = R_{AF} \cos \theta_{AF}, \quad y_F = R_{AF} \sin \theta_{AF}$$

The PSS phase and the BSS phase can be distinguished by the direction of the antiferromagnetic order-parameter $\theta_{AF}$; the value of $\theta_{AF}$ expected in the PSS and the BSS phase is $k\pi/3$ and $(k + 1/2)\pi/3$ ($k = 0, 1, 2, 3, 4, 5$), respectively. Thus a quantity $\cos 6\theta_{AF}$ is a relevant quantity to the PSS-BSS phase transition [14]. The value of $\cos 6\theta_{AF}$ is $-1$ in the BSS phase, while $\cos 6\theta_{AF} = 1$ in the PSS phase.

Now we use $R_{AF}, \theta_{AF}, R_F,$ and $\theta_F$ as independent variables of the free energy $F_0$ and trace out $R_F$ and $\theta_F$ in order to obtain an effective free energy form which is expressed by $R_{AF}$ and $\theta_{AF}$ only.

$$F_{AF}(R_{AF}, \theta_{AF}) \equiv \min_{R_F, \theta_F} F_0(R_{AF}, \theta_{AF}, R_F, \theta_F) = F_0(R_{AF}, \theta_{AF}, \tilde{R}_F, \tilde{\theta}_F)$$

(23)
where $\tilde{R}_F$ and $\tilde{\theta}_F$ gives minimum value of $F_0$ for fixed $R_{AF}$ and $\theta_{AF}$. Location of minima $\tilde{R}_F$ and $\tilde{\theta}_F$ is determined by solving the following equations:

$$\frac{\partial F_0}{\partial R_F} \bigg|_{(R_F, \theta_F) = (\tilde{R}_F, \tilde{\theta}_F)} = 0$$  \hspace{1cm} (24)

$$\frac{\partial F_0}{\partial \theta_F} \bigg|_{(R_F, \theta_F) = (\tilde{R}_F, \tilde{\theta}_F)} = 0$$  \hspace{1cm} (25)

Since we cannot solve (24) and (25) explicitly, we expand $\tilde{R}_F$ and $\tilde{\theta}_F$ in powers of $R_{AF}$ which is small around the critical line. Firstly we expand (24) and (25) in powers of both $R_F$ and $R_{AF}$. Lowest order terms follow:

$$\frac{\partial F_0}{\partial R_F} = \left(\frac{9(2J - 4\gamma J + T)}{8} R_F - \frac{27T \sin(2\theta_{AF} + \theta_F)}{64} R_{AF}^2\right) + O(R_{AF}^2) + O(R_F R_{AF}^2)$$  \hspace{1cm} (26)

$$\frac{\partial F_0}{\partial \theta_F} = R_F \left( -\frac{27T \cos(2\theta_{AF} + \theta_F) R_{AF}^2}{64} \right) + R_F \left( O(R_{AF}^4) + O(R_F^2) + O(R_{AF}^2 R_F) \right)$$  \hspace{1cm} (27)

Equations (26) and (27) indicate that $\tilde{R}_F \sim O(R_{AF}^2)$ and $\cos(2\theta_{AF} + \tilde{\theta}_F) \sim O(R_{AF}^2)$, so we assume that $\tilde{R}_F$ and $\tilde{\theta}_F$ can be expanded as below:

$$\tilde{R}_F = c_2 R_{AF}^2 + c_4 R_{AF}^4 + c_6 R_{AF}^6 \cdots$$  \hspace{1cm} (28)

$$\tilde{\theta}_F = \frac{\pi}{2} - 2\theta_{AF} + d_2 R_{AF}^2 + d_4 R_{AF}^4 + d_6 R_{AF}^6 \cdots$$  \hspace{1cm} (29)

The coefficients can be determined by letting (28) and (29) into equations (24) and (25), leading to the following result:

$$c_2 = \frac{3\tau}{8}$$  \hspace{1cm} (30)

$$d_2 = \frac{9}{64} (-2 + 4\tau - \tau^2) \sin(6\theta_{AF})$$  \hspace{1cm} (31)

$$c_4 = \frac{27}{64} (\tau - \tau^2) + \frac{27}{512} (-2\tau + 4\tau^2 - \tau^3) \cos(6\theta_{AF})$$  \hspace{1cm} (32)

$$d_4 = \frac{81}{256} (-1 + 4\tau - 4\tau^2 + \tau^3) \sin(6\theta_{AF}) \hspace{1cm} (33)$$

$$\vdots$$
where $t = T/J$ and $\tau = t/(2 - 4\gamma + t)$.

Finally the effective free energy form is obtained by letting (28) and (29) into (23). Note that only the terms which is invariant under the transformations of $c_{6v}$ are present in $F_{AF}$, and they can be written as $R_{AF}^{6n+2m} \cos(6n\theta_{AF})$, $(n, m \geq 0)$. We have calculated the effective free energy $F_{AF}$ up to sixth order term as below:

$$F_{AF} = A_0 + A_2 R_{AF}^2 + A_4 R_{AF}^4 + (A_6 + B_6 \cos 6\theta_{AF}) R_{AF}^6 + O(R_{AF}^8)$$

where

$$A_0 = J - 2\gamma J - T \log 3$$

$$A_2 = \frac{9}{16} J(-2 - 4\gamma + t)$$

$$A_4 = \frac{81}{1024} T(2 - \tau)$$

$$A_6 = \frac{243}{8192} T(4 - 6\tau + 3\tau^2)$$

$$B_6 = \frac{243}{163840} T(-8 + 30\tau - 30\tau^2 + 5\tau^3)$$

The critical line is the region $A_2 = 0$ and $A_4 \geq 0$, which corresponds to the region $t = 2 + 4\gamma$, $\gamma \leq 3/2$. The order-disorder transition is discontinuous for $\gamma > 3/2$ and $(\gamma, t) = (3/2, 8)$ is a tricritical point where $A_2$ and $A_4$ vanish simultaneously [22].

The six-fold anisotropy term $B_6 R_{AF}^6 \cos 6\theta_{AF}$ is the origin of the BSS and the PSS phase; positive $B_6$ corresponds to the BSS phase and negative $B_6$ to the PSS phase. Presence of higher order anisotropic terms such as $R_{AF}^{12} \cos 12\theta_{AF}$ allows a occurrence of minima of $F_{AF}$ at $\theta_{AF} \neq k\pi/6$, which corresponds to neither the PSS nor the BSS phase. However, the positions of minima remain at $\theta_{AF} = k\pi/6$ if the coefficients of such higher order anisotropic terms are sufficiently small. The following example helps understanding above issue: a positions of minima and maxima of a function $f(\theta) = \cos \theta + a \cos 2\theta$ remain at $\theta = k\pi$, as long as $|a| \leq 1/4$.

On the critical line $t = 2 + 4\gamma$, $B_6$ is expressed as a function of $\gamma$:

$$B_6 = 243(2 + 4\gamma)(1 + 30\gamma - 180\gamma^2 + 40\gamma^3)/1310720$$

(40)
It vanishes at $\gamma \sim 0.2032$ on the critical line, where the BSS-PSS transition line merges into the critical line.

Note that higher order term is needed to investigate off-critical regions, where the numerical method \( \text{(20)} \) works well. So we do not calculate higher order terms.

**IV. CONCLUSION**

In this paper we have shown that a mean-field theory analysis indicates the presence of a tricritical point in the three-state Potts model with the antiferromagnetic n.n. interaction and the ferromagnetic n.n.n. one. It is believed that mean-field theory is qualitatively correct above the upper critical dimension, where spatial fluctuations of order-parameter at the critical region become negligible. Since the upper critical dimension of the tricritical phenomena is three \( [22] \), we expect the model on the simple cubic lattice to posses a tricritical point. Owing to the simpleness of the Hamiltonian, this model must serve to develop numerical methods for studies of a tricritical phenomena, such as finite-size scaling in Monte Carlo simulations.

As for the nature of the ordered phase, we have shown that the PSS phase is realized at the strong n.n.n. coupling region, as a result of the competition of the two kind of interactions, while the BSS phase is realized at the weak n.n.n. coupling region.

However, it should be noted that mean-field type analysis like this work may not give collect information, as pointed in Ref. \([14]\), about the six-fold anisotropy term which becomes relatively small compared to the order-parameter fluctuation near the critical line. So further study, such as Monte Carlo simulations, may be needed to clarify the nature of the ordered phase just below the critical line.

**Acknowledgment**

The author is indebted to S. Hikami for reading the manuscript. This work was supported by a Grant-in-Aid for Scientific Research by the Ministry of Education, Science and Culture.
REFERENCES

[1] J. S. Wang, R. H. Swendsen and R. Kotecky, Phys. Rev. Lett. 63, 109 (1989).
[2] J. S. Wang, R. H. Swendsen and R. Kotecky, Phys. Rev. B42, 2465 (1990).
[3] G. H. Wannier, Phys. Rev. 79, 357 (1950).
[4] J. R. Banavar, G. S. Grest and D. Jasnow, Phys. Rev. Lett. 45, 1424 (1980).
[5] Y. Ueno, G. Sun and I. Ono, J. Phys. Soc. Jpn. 58, 1162 (1989).
[6] Y. Ueno, K. Kasono, Phys. Rev. B48, 16471 (1993).
[7] A. P. Gottlob and M. Hasenbusch, Physica A210, 217 (1994).
[8] A. P. Gottlob and M. Hasenbusch, J. Stat. Phys. 77, 919 (1994).
[9] M. Kolesik and M. Suzuki, Physica A216, 469 (1995).
[10] Y. Okabe and M. Kikuchi, in Computational Approaches in Condensed-Matter Physics, edited by S. Miyashita, M. Imada and H. Takayama, (Springer, Berlin, 1992), p. 193.
[11] S. Lapinskas and A. Rosengren, Phys. Rev. B49, 15190 (1994).
[12] A. Rosengren and S. Lapinskas, Phys. Rev. Lett. 71, 165 (1993).
[13] P. J. Kundrotas, S. Lapinskas and A. Rosengren, Phys. Rev. B52, 9166 (1995).
[14] M. Kolesik, M. Suzuki, cond-mat@xxx.lanl.gov No. 9504091
[15] G. S. Grest and J. R. Banavar, Phys. Rev. Lett. 46, 1458 (1981).
[16] M. P. M. den Nijs, M. P. Nightingale, and M. Schick, Phys. Rev. B26, 2490 (1982).
[17] I. Ono, J. Phys. Soc. Jpn. 53, 4102 (1984).
[18] M. Fukugita and M. Okawa, Phys. Rev. Lett. 63, 13 (1989).
[19] J. J. Binney, N. J. Dowrick, A. J. Fisher and M. E. J. Newman, The Theory of Critical
Figure Captions

Fig.1. Typical ground state configuration of antiferromagnetic three-state Potts model on a square lattice. Two different symbols (∇ and ●) show the two interpenetrating sublattices.

Fig.2. A range of sublattice magnetization ($x_\alpha, y_\alpha$). Three vertexes 1, 2, and 3 correspond to the completely ordered states in which all the spins in the sublattice take the same value 1, 2, and 3, respectively.

Fig.3. Schematic phase diagram of two parameters $T/J$ and $\gamma$. The solid line indicates a continuous transition and the dashed line indicates a discontinuous one. The continuous order-disorder transition line ends at $\gamma = 1.5$. The BSS–PSS phase boundary merges into the order-disorder transition line at $\gamma \sim 0.2032$.

Fig.4. Typical values of the concentrations of the three states (upper) and the sublattice magnetizations (lower) of each ordered phases.
Fig. 1. Typical ground state configuration of antiferromagnetic three-state Potts model on a square lattice. Two different symbols (□ and ○) show the two interpenetrating sublattices.

Fig. 2. A range of sublattice magnetization \((x_\alpha, y_\alpha)\). Three vertexes 1, 2, and 3 correspond to the completely ordered states in which all the spins in the sublattice take the same value 1, 2, and 3, respectively.
Fig. 3. Schematic phase diagram of two parameters $T/J$ and $\gamma$. The solid line indicates a continuous transition and the dashed line indicates a discontinuous one. The continuous order-disorder transition line ends at $\gamma = 1.5$. The BSS–PSS phase boundary merges into the order-disorder transition line at $\gamma \sim 0.2032$.

Fig. 4. Typical values of the concentrations of the three states (upper) and the sublattice magnetizations (lower) of each ordered phases.