Equivalence of Geometric Engineering and Hanany-Witten via Fractional Branes

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Abstract

We present an explicit relation between the Hanany-Witten and Geometric Engineering approaches of realising gauge theories in string theory. The last piece in the puzzle is a T-duality relating arbitrary Hanany-Witten setups and fractional branes.
1 Introduction

During recent years it has become clear how supersymmetric quantum field theories can be
derived from string theory in the limit where gravitational effects decouple from the gauge
interactions, i.e. in the limit $M_{\text{Planck}} \to \infty$. In this context there exist two competing
scenarios, namely first the ‘Geometric Engineering’ approach [1, 2, 3] and second the
Hanany-Witten [4] brane set up.

Within Geometric Engineering one considers type IIA/B string compactification on a
complex Calabi-Yau space. For example type IIA/B compactification on a Calabi-Yau
three-fold leads to $\mathcal{N} = 2$ supersymmetric field theories in four dimensions. Perturbative
as well non-perturbative effects in $\mathcal{N} = 2$ field theory can be explained by the geometry
of the underlying Calabi-Yau spaces. Non-Abelian gauge fields together with massless
matter fields arise due to the wrapping of D-branes around internal Calabi-Yau cycles; as
soon as these cycles shrink to zero size the corresponding fields become massless. Since
one assumes that the Calabi-Yau 3-fold is a $K3$-fibration over a sphere, the resulting gauge
groups are immediately determined by the singular $K3$-fibers, which are classified accord-
ing to the simple ADE algebras. In the field theory limit all the interesting information is
encoded in the local geometry around the singularity, which means that the field theory
limit is performed by replacing the compact $K3$ fiber by a non-compact ALE space that
allows for the same ADE singularity classification as $K3$. The non-perturbative effective
action a la Seiberg and Witten is given in terms of the rational instanton numbers of
the Calabi-Yau space, the type IIA superstring is compactified on. It is very beautiful
that these rational instanton numbers can be explicitly computed by the mirror map to
the corresponding type IIB model. Restricting oneself only to the properties of the local
Seiberg-Witten geometry around the singular cycles, the full non-perturbative solution of
the string derived $\mathcal{N} = 2$ gauge theory is provided by the local mirror map acting on the
ALE space fibered over a sphere.

On the other hand, in the Hanany-Witten approach supersymmetric field theories arise
as the world-volume theories of certain brane configurations, which fill the non-compact,
say four-dimensional, spacetime. As soon as parallel D-branes approach each other, the
corresponding non-Abelian gauge bosons become light. In this framework, the solution of
the non-perturbative dynamics is provided by embedding the type IIA brane configura-
tions into M-theory, i.e. lifting all branes to their 11-dimensional parent configurations.
Then all singularities are smeared out, and the Seiberg-Witten curves are rediscovered in
a very nice way.

Since M-theory provides a unified description of all superstring theories and their interac-
tions, one expects on general grounds that the Geometric Engineering approach and the Hanany-Witten brane set up are equivalent to each other and are connected by some duality symmetries. In fact, a vanishing cycle ($A_1$ singularity) is T-dual to two approaching branes, as was already discussed in [5]. In this paper we will explicitly show how this unification between the two ways to derive $\mathcal{N} = 2$ supersymmetric field theories from strings is realized. Concretely, we will demonstrate that the Hanany-Witten brane configurations are connected to fractional D-branes (which arise when considering D-branes on orbifolds [6, 7]) via T-duality. Starting from the Geometric Engineering side, the IIA string on a Calabi-Yau space is also related to the fractional brane configurations by combined S- and T-duality transformations. Therefore, the fractional branes are half way between the Calabi-Yau compactifications with branes wrapped around the internal cycles, and the world-volume gauge theories with spacetime filling branes.

In Section 2 we will review the various approaches to the subject. In Section 3 we will propose the duality relation which allows us to connect the various approaches and treat them in a unified way. Even though this construction is general, we will focus on the approaches yielding $\mathcal{N} = 2$ supersymmetry in 4 dimensions. The connection will be explained in Section 4. In Section 5 we discuss generalizations to the $\mathcal{N} = 1$ case and in Section 6 we conclude.

# 2 The pieces of the puzzle

There have been various approaches to embedding field theory in string theory. The most prominent ones are the Geometric Engineering approach, the Hanany-Witten approach and using Branes as Probes. We will briefly review certain aspects of these 3 approaches before we move on to show that they all fit nicely into a unified picture.

## 2.1 Geometric Engineering

### 2.1.1 The setup: IIA on K3 fibered CY

Four-dimensional $\mathcal{N} = 2$ supersymmetric string vacua are obtained by compactifying the heterotic string on $K3 \times T^2$ or via string-string duality [8, 9] by compactifying the type IIA/B superstring on a Calabi-Yau three-fold (for a recent review see [10]). The $\mathcal{N} = 2$ heterotic/type II string duality can be explained, at least in a heuristic way, from the six-dimensional string-string duality between the heterotic string on $T^4$ and the type IIA
string on $K3$. Using an adiabatic argument and applying the six-dimensional duality fiberwise, the resulting space on the type IIA side is a $K3$ fibered Calabi-Yau space $X^3$ with base $P^1$ [11, 12]; the various ways to perform this $K3$-fibration results in the variety of different type II $\mathcal{N} = 2$ vacua. Later on we will allow for more general two-dimensional bases, namely intersecting $P^1$'s.

The massless spectrum is determined by the cohomology of the three-fold, namely by the two Hodge numbers $h^{(1,1)}$ and $h^{(2,1)}$. As it is known, the type IIA/B compactifications are related by mirror symmetry, which means that the type IIA superstring on the Calabi-Yau space $X^3$ is equivalently described by compactifying the type IIB on the mirror manifold $\tilde{X}^3$ which has exchanged Hodge numbers: $\tilde{h}^{(1,1)} = h^{(2,1)}$, $\tilde{h}^{(2,1)} = h^{(1,1)}$. To be specific, the massless spectrum of the type IIA superstring contains $h^{(1,1)} + 1$ $U(1)$ $\mathcal{N} = 2$ vector multiplets. Their $h^{(1,1)}$ complex scalar moduli in the NS-NS sector correspond to the deformations of the Kähler form $J$ of $X^3$ plus the internal $B_{MN}$ fields; the $h^{(1,1)}$ $U(1)$ R-R vectors originate from the ten-dimensional 3-form gauge potential $A_{MNP}$ with two indices in the internal space. So the Abelian gauge symmetry including the graviphoton is given by $U(1)^{h^{(1,1)}+1}$. Finally there are $h^{(2,1)} + 1$ massless $\mathcal{N} = 2$ hypermultiplets. $h^{(2,1)}$ of them correspond to the complex structure deformations of $X^3$. The additional hypermultiplet contains together with the NS-NS axion field $a$ the four-dimensional dilaton $\phi_{IIA}^4$ plus two more R-R scalar fields. Since the dilaton belongs to a hypermultiplet in the four-dimensional type II vacua it follows that the type II vector couplings (gauge couplings plus moduli space metric) do not depend on the type II dilaton and are purely classical in string perturbation theory. Moreover, in type IIB compactifications, the vector multiplets correspond to complex structure moduli, and hence their interactions are not even affected by world-sheet instantons. This fact allows to calculate exactly all IIB vector couplings in terms of pure geometry.

To discuss the non-Abelian gauge symmetries in the type IIA string, we have to find the charged BPS states which couple to Abelian R-R gauge bosons [13]. These cannot be found within the perturbative type II spectrum, but they are given by the non-perturbative D2 branes which can be wrapped around the 22 homology two-cycles of $K3$. The masses of the BPS particles are proportional to the area of the $K3$ two-cycles. Therefore massless non-Abelian gauge bosons arise at those points in the $K3$ moduli space where the $K3$ degenerates, and the two-cycles shrink to zero sizes. However not all $K3$ cycles can lead to massless gauge bosons since we have to take into account the global structure due to the $K3$ fibration [14]. This means that not the whole intersection lattice $\Lambda$ of $K3$ can lead to a non-Abelian gauge group enhancement, but only the monodromy invariant part of $\Lambda$. This is the so-called Picard lattice $\Lambda_\rho$, whose rank is
given by the Picard number $\rho$. Moreover for the gauge bosons to be massless, the size of the base $P^1$, over which the $K3$ is fibered, must be infinite. This limit corresponds on the dual heterotic side to the perturbative limit with zero string coupling constant. For finite size of $P^1$ the gauge bosons will be in general massive, and instead one deals with massless hypermultiplets at some points in the vector moduli space. This situation precisely reflects the non-perturbative $\mathcal{N} = 2$ dynamics described by Seiberg and Witten. Non-perturbatively, the classical singularities will split, and the quantum moduli space contains special loci of massless BPS hypermultiplets. Therefore, in the type IIA string, the whole Seiberg Witten solution is described entirely in geometric terms.

In summary, concerning the gauge group, it arises due to the ADE singularities of $K3$. In particular, if one deals with a genus $g$ curve with ADE singularity one expects to have ADE gauge symmetry with $g$ adjoint hyper multiplets [15, 16, 17]. Hence the gauge theory is asymptotically free for $g \leq 1$; for $g = 1$ one obtains the $\mathcal{N} = 4$ spectrum. Other matter representations are obtained from different kind of singularities. In particular if there is an ADE singularity over a surface and this singularity is enhanced to a higher one at some points along the surface, then there will be some matter localized at those points. For example, if the $K3$ fiber possesses an $A_{n-1}$ singularity which is enhanced to $A_n$ at $k$ points on the surface, we get an $SU(n) \times U(1)$ gauge group with $k$ hyper multiplets in the fundamental of $SU(n)$ and charged under the $U(1)$ [18]. Other examples with various matter representations have been discussed in [18].

### 2.1.2 Decoupling Gravity and ALE spaces

Now we want to decouple gravitational and stringy effects to keep only all effects due to the gauge dynamics. So we take the limit $\alpha' \to 0$ for describing the low energy physics. In this limit only the local neighborhood of the singularity is relevant. This will be independent of the details of the chosen Calabi-Yau containing the singularity in this limit. This means that the local geometry is described by a non-compact ALE space [1]. This ALE space singles out the relevant shrinking 2-cycles around which the D2 branes are wrapped. Information from more distant parts of $K3$, like effects from 2-branes wrapping other 2-cycles, are suppressed by powers of $\alpha'$.

As an illustrative example [2] consider the geometric engineering of a pure $SU(2)$ gauge symmetry which related to an $A_1$ singularity in $K3$. Locally, one needs a vanishing 2-sphere $P^1$, around which the D-branes, being the $W^{\pm}$ bosons, are wrapped. This $P^1_f$ has to be fibered over the base $P^1_b$ in order to have $\mathcal{N} = 2$ supersymmetry in four dimensions. The different ways to perform this fibration are encoded by an integer $n$,
and the corresponding fiber bundles are the Hirzebruch surfaces $F_n$. The mass (in string units) of the $W^\pm$ bosons corresponds to the area of the fiber $P^1_f$, whereas the area of the base $P^1_b$ is proportional to $1/g^2$ ($g^2$ is the four-dimensional gauge coupling at the string scale). Now let us perform the field theory limit which means that we send the string scale to infinity. The running of the gauge coupling implies that $g^2$ should go to zero in this limit; thus the Kähler class of the base $P^1_b$ must go to zero: $t^b \to \infty$. Second, in the field theory limit the gauge boson masses should go to zero, i.e. $t_f \to 0$. In fact these two limits are related by the running coupling constant,

$$
\frac{1}{g^2} \sim \log \frac{M_W}{\Lambda},
$$

and the local geometry is derived from the following double scaling limit:

$$
t^b \sim \log t_f \to \infty.
$$

Clearly, this picture can be easily generalized to engineer higher ADE gauge group. In this case there is not only one shrinking $P_f$ in the fiber, but several such that the fiber acquires a local ADE singularity. For the close comparison with the Hanany-Witten set up, it is also interesting how product gauge groups can be geometrically engineered [3]. For concreteness consider the group $SU(n) \times SU(m)$ with a hypermultiplet in the bi-fundamental representation. To realize this we have an $A_{n-1}$ singularity over $P^1_b$ and an $A_{m-1}$ singularity over another $P^1_b$. The two $P^1_b$’s intersect at a point where the singularity jumps to $A_{m+n-1}$. This can be seen in 6 dimensions as symmetry breaking of $SU(n+m)$ to $SU(n) \times SU(m) \times U(1)$ by the vevs of some scalars in the Cartan subalgebra of $SU(n+m)$. It is straightforward to generalize this procedure to an arbitrary product of $SU$ groups with matter in bi-fundamentals. One associates to each gauge group a base $P^1_b$ over which there is the corresponding $SU$ singularity; to each pair of gauge groups connected by a bi-fundamental representation one associates an intersection of the base $P^1_b$’s, where over the intersection point the singularity is enhanced to $SU(n+m)$.

### 2.1.3 The Solution: Local Mirror Symmetry

So far our discussing was mainly limited to the classical aspects of the gauge theory derived from Calabi-Yau compactifications. As it is well known, the classical moduli space of $\mathcal{N} = 2$ supersymmetric Yang-Mills theories is corrected by quantum effects. Consider the $\mathcal{N} = 2$ prepotential $F(A)$ of the pure $SU(2)$ gauge theory, spontaneously broken to $U(1)$ by the vev $a$ of the scalar component of the Abelian $\mathcal{N} = 2$ vector multiplet
$A$ [19, 20]:

$$F(A) = \frac{1}{2} \tau_0 A^2 + \frac{i}{\pi} A^2 \log \left( \frac{A}{\Lambda} \right)^2 + \frac{1}{2\pi i} A^2 \sum_{l=1}^{\infty} c_l \left( \frac{\Lambda}{A} \right)^4. \quad (3)$$

Here $\tau_0 = \frac{1}{g^2}$ is the classical gauge coupling, the logarithmic term describes the one-loop correction due to the running of the gauge coupling and the last term collects all non-perturbative contributions from the instantons with instanton numbers $c_l$. The solution to the problem of computing these instanton numbers in field theory was provided by Seiberg and Witten [19, 20] by introducing an auxiliary Riemann surface, the Seiberg-Witten curve, whose period integrals define the period $a$ and the dual period $a_D$ in terms of a gauge invariant order parameter $u$.

As already mentioned, the corresponding prepotential in type IIA Calabi-Yau compactifications, which depends on the Kähler-class moduli $t_A$ ($A = 1, \ldots, N_V = h_{1,1}$), is purely classical in string perturbation theory and receives only corrections due to world-sheet instantons. It has the following structure [21]:

$$F^{II} = -\frac{1}{6} C_{ABC} t_A t_B t_C - \frac{\chi(3)}{2(2\pi)^3} + \frac{1}{(2\pi)^3} \sum_{d_1, \ldots, d_h} n_{d_1, \ldots, d_h} Li_3(e^{i \sum_A d_A t_A}), \quad (4)$$

where we work inside the Kähler cone $\Re t_A \geq 0$. The polynomial part of the type-IIA prepotential is given in terms of the classical intersection numbers $C_{ABC}$ and the Euler number $\chi$, whereas the coefficients $n_{d_1, \ldots, d_h}$ of the exponential terms denote the rational instanton numbers of genus 0 and multi degree $d_A$. The rational instantons are related to non-perturbative effects on the world-sheet, namely they count the embeddings of the genus 0 world-sheet into the Calabi-Yau space; their contributions disappear in the large radius limit $\alpha'/R \to 0$.\(^1\)

The nice way to compute the world-sheet instanton numbers is to perform the mirror map to the type IIB superstring compactified on the mirror Calabi-Yau space. In this way all the necessary non-perturbative information can be obtained by purely geometric considerations on the mirror Calabi-Yau, namely by computing the corresponding Picard-Fuchs equations. In the type IIB picture one is considering vanishing 3-cycles of the mirror Calabi-Yau threefold (corresponding to conifold singularities). The relevant BPS states correspond to type IIB D3 branes wrapped around those 3-cycles. With respect to the $K3$ fibration structure, the type IIB 3-branes are wrapped around the vanishing 2-cycles of $K3$ leaving in this way not massless particles, but non-critical anti-self-dual strings on the base. The tension of this string depends on the size of the relevant 2-cycle.

\(^1\)Several checks had been made that the type II prepotential agrees with the prepotential of the dual heterotic compactification on $K3 \times T^2$ [8, 22, 23, 24].
In order to extract the field theory instanton numbers \( c_l \) and also the logarithmic piece in eq.(3) from the string prepotential eq.(4) one has to perform the double scaling limit (2) which replaces the \( K3 \) by the local ALE space. Therefore this procedure is called the local mirror map. In fact, the local mirror map simplifies the rederivation of the Seiberg-Witten action enormously since only the data in the neighborhood of the ALE singularity is relevant. To understand this we briefly discuss the local mirror map for the pure \( SU(2) \) gauge symmetry [2]. As explained in the last section all the local information we need is encoded in the fibration of \( P^1_f \) over \( P^1_b \). Therefore the prepotential is a function of the moduli \( t_f \) and \( t_b \). \( n_{d_b,d_f} \) denote the number of world-sheet instantons wrapping \( d_b \) times around the base \( P^1_b \) and \( d_f \) times around the fiber \( P^1_f \). First, in the perturbative limit only the instanton numbers \( n_{0,d_f} \) contribute, and, using the limit (2), the string prepotential eq.(4) precisely leads to the logarithmic term in eq.(3). Second, and even more interesting, summing over all rational instantons \( n_{d_b,d_f} \) with \( d_b \neq 0 \) and performing again the limit (2) the Seiberg-Witten instanton numbers \( c_l \) should be derived as explained in [2]. In fact, one should get the same result for all Hirzebruch surfaces \( F_n \).

Having obtained the field theory instanton numbers from string theory via the local mirror map, it is highly suggestive that the field theory Seiberg-Witten curve can be directly derived from the string Calabi-Yau geometry. How this works was explained in [1]. We briefly outline the main arguments for this beautiful picture. Consider first the Seiberg-Witten curve in \( \mathcal{N} = 2 \) supersymmetric Yang-Mills gauge theory based on some gauge group \( G \). The Seiberg-Witten curve can be considered as a fibration of the corresponding weight diagram (points) over \( P^1 \). In string theory, the 0-dimensional local fibers will be replaced by appropriate ALE spaces with corresponding vanishing 2-cycles. Indeed, the vanishing 2-cycles of the ALE space behave exactly like the root vectors of the corresponding ADE groups. This means that after the field theory limit \( \alpha' \to 0 \), the 3-branes wrapped around the type IIB Calabi-Yau 3-cycles are equivalent to the anti-self-dual strings, wrapped around the Seiberg-Witten curve. So in the type IIB picture, the role of the vanishing 1-cycles of the Seiberg-Witten curve is played by the vanishing 3-cycles of the Calabi-Yau space. In this way, the Seiberg-Witten curve has not only an auxiliary function, but gets a real physical interpretation in string theory. This solution is sketched in the following diagram.
2.2 The Hanany Witten setup

Here we will review the method first used by Hanany and Witten to study 3 dimensional gauge theories [4] and since then generalised to other dimensions. In this approach to studying gauge theories, D-branes are placed in flat space. In this way all manifest properties of the gauge theory are encoded in the positions and shapes of the branes. Here we will review the aspects of this construction relevant to this paper. In particular we will consider the setup appropriate for studying 4 dimensional $\mathcal{N} = 2$ $SU(N)$ gauge theories.  

2.2.1 The setup

To study $\mathcal{N} = 2$ $SU(N)$ gauge theories in 4 dimensions Witten [25] considered the following construction of branes in IIA string theory. $N$ parallel D4 branes filling 01236 spacetime were bounded in the 6 direction by 2 NS5 branes filling 012345 spacetime. This produced an effective 4 dimensional world-volume for the D4 branes and it can easily be checked that $\mathcal{N} = 2$ supersymmetry is preserved in 4 dimensions for this setup. Matter can be added to the gauge theory by adding semi-infinite D4 branes filling 01236 and ending “outside” the NS5 branes. Another method to produce matter is to add D6 branes filling 0123789 between the 2 NS5 branes in the 6 direction. These two constructions are in fact related since moving a D6 brane through one of the NS5 branes creates a D4 brane stretched between the NS5 brane and the D6 brane which can then be moved away to infinity.

Classically the gauge coupling is given by the distance between the NS5 branes in the 6 direction, $\Delta$:

$$g^{-2} = \Delta/g_s l_s$$  \hspace{1cm} (5)

$^{2}$The gauge group is U(N) in low dimensions ($d < 4$) but in 4 or more dimensions the U(1) factor is frozen out [25].
where $g_s$ and $l_s$ are the IIA string coupling and length respectively.

2.2.2 Corrections: Bending and instantons

To understand quantum effects in the gauge theory we must consider the consistency of the brane configuration more carefully. In IIA string theory the end points of the D4 branes on the NS5 branes are singular. However an approximate description is that the D4 branes exert a force on the NS5 branes causing them to bend. Considering only the directions not in common to both branes, this leads to a logarithmic bending of the NS5 branes into the 6 direction along the 45 plane. This logarithmic variation of the separation of the NS5 branes is interpreted in the field theory as the logarithmic running of the gauge coupling. In this way the shape of the branes incorporates the 1-loop effect in the field theory.

In $\mathcal{N} = 2$ field theory there are no higher loop effects. However there are still non-perturbative effects due to instantons. These instantons can be seen directly in the brane picture. It is known that a D(p-4) brane within a Dp brane satisfies the 4 dimensional YM instanton equations [26]. So D0 branes are instantons within D4 branes. To interpret these D0 branes as instantons in the 0123 spacetime we should consider Euclidean D0 branes whose world-line is in the 6 direction so that they are contained within the D4 branes between the NS5 branes [27, 28, 29]. So now the field theory objects and quantum effects have been mapped to D-branes and their properties. The problem is now to solve the theory by including all these effects. This can be done by “lifting” the IIA configuration to M-theory [25].

2.2.3 The Solution: Lifting to M-theory

The advantage of considering the above configuration of branes in M-theory is that the D4 branes and NS5 branes are in fact the same object, the M5 brane. The intersection of D4 and NS5 branes in IIA is singular but this is smoothed out in M-theory and in fact it is possible to consider all the D4 branes and NS5 branes as a single M5 brane with complicated world-volume. However the conditions for preserving $\mathcal{N} = 2$ supersymmetry restrict the embedding of the M5 brane world-volume and it is possible to find the function describing this embedding explicitly. Clearly this must incorporate the field theory 1-loop effects through the shape of the M5 brane. However the non-perturbative instantons are also automatically included since D0 branes are simply Kaluza-Klein momentum modes of compactified M-theory. (Here we consider “momentum” with the 6 direction being
Euclidean time, i.e. variation in $x^{11}$ wrt. $x^6$.) The embedding is in fact precisely given by the Seiberg-Witten curve (times the trivial embedding of $R^4$ in the 0123 directions.) The Seiberg-Witten differential can also be naturally identified in this approach [30]. This relation between the IIA setup with separate classical, 1-loop and non-perturbative effects, and the M-theory setup incorporating all effects is shown in the following figure.

![Figure 2: Solving HW setup via M-theory](image)

### 2.2.4 Other Gauge Theories

There are several ways to construct generalisations of the construction given above. Of relevance to us are product gauge groups. These are easily described by adding more NS5 branes. The number of D4 branes between 2 consecutive NS5 branes determines an $SU(N)$ factor of the product gauge group. D6 branes give matter in the fundamental representation of one of the $SU(N)$ factors depending on which NS5 branes they are between. There is also matter in the bi-fundamental of “neighboring” $SU(N)$ factors, coming from strings within each NS5 brane stretching between D4 branes ending from either side.

The 6 direction can be compactified and this leads to “elliptic” models. It is these models which we are particularly interested in here since we will show that a T-duality can be performed on the 6 direction. From the field theory viewpoint these are just product gauge groups as described above where there is in addition bi-fundamental matter transforming under the “first” and “last” $SU(N)$ factors. \(^3\)

$\mathcal{N} = 1$ gauge theories can also be constructed with these gauge groups. This is done by considering some of the NS5 branes to have world-volume 012345 and others to have world-volume 012389 [31, 32, 33, 34, 35, 36]. Despite the reduction of the number of supersymmetries it is still possible to understand many features of the field theory from the branes. This includes a simple interpretation of Seiberg duality [37] as an interchange of the NS5 branes [31, 32].

\(^3\)There is an additional $U(1)$ factor which is not frozen out in the brane picture in this case [25]. However, it decouples since all fields are neutral.
Gauge groups $SO$ and $USp$ can also be studied by including appropriate orientifolds in the brane construction. In this paper we will only consider SU gauge groups for simplicity. However, our analysis should also extend to these cases.

2.3 Branes as Probes

A lot has been learned by putting Dp branes in a nontrivial gravitational background and studying the p+1 dimensional field theory on the brane as a probe of the background geometry. We will focus here on the approach of studying branes in an orbifold background [6, 38]. A variant of this approach has been chosen by [39, 40, 41, 42, 43] where the probe moves in the background of a higher dimensional brane. Some of these cases can be solved exactly by going to F-theory.

2.3.1 D-branes at orbifolds

We want to consider $N$ Dp branes on top of an orbifold singularity, where the orbifold acts in the transverse spacetime. The formalism for this procedure was developed in [6, 44]. In the following we will discuss the example of D3 branes at an orbifold, since this leads to 4d physics. It has received a lot of attention recently, since it has a dual description in terms of supergravity on an AdS space [45]. Many exciting proposals have been made about this system, e.g. it is conjectured to lead to theories that are finite [46, 47] even without supersymmetry (at least for large $N$) and possess certain S-duality properties. The discussion generalizes in the obvious way to other Dp branes.

Hence consider $N$ D3 branes living in the 0123 space at an orbifold fixed point, where the orbifold group $\Gamma$ acts on the “internal” 456789 space. The world-volume theory in the unorbifolded case is $\mathcal{N} = 4 SU(N)$ SYM in 4d. $\Gamma$ has to be a discrete subgroup of the $SO(6)$ acting on the internal space. In the $\mathcal{N} = 4$ setup this $SO(6)$ is the $SU(4)$ $R$ symmetry. Generically all SUSY is broken. If $\Gamma \subset SU(3)$ $\mathcal{N} = 1$ SUSY is preserved and for $\Gamma \subset SU(2)$ $\mathcal{N} = 2$ is preserved.

The $\mathcal{N} = 4$ theory has the following matter content:

- the $U(N)$ vector
- 4 adjoint gauginos
- 6 adjoint scalars
where the scalars and fermions transform under the $SO(6)$ internal (R).

Besides the obvious action of $\Gamma$ on the scalars and fermions it also acts on the vectors. The $U(N)$ gauge group is generated by the Chan Paton factors living at the ends of the open strings connecting the D3 branes. We therefore have to specify the action of $\Gamma$ on the Chan Paton factors. If we want to describe D3 branes which are free to move away from the orbifold fixed point we should restrict ourselves to embed the orbifold group into the Chan Paton factors via the regular representation $R$ of $\Gamma$, that is the $|\Gamma|$ dimensional representation. Note that this representation is reducible and decomposes in terms of the irreducible representations $r_i$ as $R = \oplus_i \text{dim}(r_i)r_i$. Doing so we take into account the D3 branes and all its mirrors, as is required if we want to have branes that are free to move away from the fixed point (and than no longer coincide with their mirrors). Other representations can be considered as well (at least at a classical level) and lead to fractional branes [6, 7, 48, 49]. We will have more to say about these in what follows. For the moment let’s restrict ourselves to the regular representation.

Even though the orbifolded space is singular, string theory physics on the orbifolded space is smooth. String theory automatically provides additional modes in the twisted sectors that resolve the singularity. In the case of an orbifold these twisted sectors are taken care of by including also the $|\Gamma|$ mirror images of each original D3 brane.

So after all we start with an $\mathcal{N} = 4$ $U(|\Gamma|N)$ SYM and just project out degrees of freedom not invariant under $\Gamma$ where $\Gamma$ acts:

- on the $SU(4)$ internal symmetry acting on the scalars and fermions given in the obvious way by embedding of the geometric action in internal spacetime
- on the $U(|\Gamma|N)$ vector index according to $N$ copies of the $|\Gamma|$ dimensional regular representation.

With this we get the following actions on the fields $(i, j$ labelling the columns of length $N$ of vector indices transforming under the same irreducible representation $r_i$ of $\Gamma$, $a = 1, \ldots, 6$ a vector and $\alpha = 1, \ldots, 4$ a spinor index of the internal $SO(6)$ symmetry):

**Vectors:**

$$A^i_j \rightarrow r_i \times \bar{r}_j A^i_j$$

leaving a $\prod_i U(\text{dim}(r_i)N)$ gauge group, since $r_i \times \bar{r}_j$ contains the identity only for $i = j$.

**Scalars:**

$$\Phi^a_i \rightarrow \delta^a_i r_k \times \bar{r}_j \Phi^a_j$$
where \( a_{ik}^6 \) denotes the Clebsch Gordan coefficient in the decomposition of \( r_i \times 6 = \oplus_k a_{ik}^6 r_k \), where \( 6 \) denotes the 6 dimensional representation of the scalars under the R symmetry. We hence obtain \( a_{ik}^6 \) scalars transforming as bi-fundamentals under the \( i \)th and \( k \)th gauge group. For \( i = k \) this is interpreted as an adjoint. From the definition of \( a_{ik}^6 \) we can read off:

\[
dim(r_i) \cdot 6 = \sum_k a_{ik}^6 \dim(r_k)
\]

**Fermions:** The surviving fermions can be determined in the same manner as the surviving scalars, just replacing \( a_{ik}^6 \) with \( a_{ik}^4 \) and \( \bar{a}_{ik}^1 \). If \( \Gamma \subset SU(3) \) we get \( 6 \rightarrow 3 + \bar{3}, 4 \rightarrow 3 + 1 \) and \( \bar{4} \rightarrow \bar{3} + 1 \). This tells us that for every scalar we also have a left and a right handed fermion and in addition we also have an adjoint left and right handed fermion in every gauge group, verifying that the spectrum is indeed \( \mathcal{N} = 1 \) supersymmetric. All potentials are derived from the unique \( \mathcal{N} = 4 \) superpotential.

**Examples:** Consider for example the case of a \( \mathbb{Z}_n \) orbifold group generated by \( \alpha = e^{2\pi i/n} \) acting on the 3 complex dimensional internal space as

\[
\begin{align*}
z_1 & \rightarrow \alpha^{a_1} z_1 \\
z_2 & \rightarrow \alpha^{a_2} z_2 \\
z_3 & \rightarrow \alpha^{a_3} z_3
\end{align*}
\]

For \( a_1 + a_2 + a_3 = 0 \) (mod \( n \)) we have \( \mathbb{Z}_n \subset SU(3) \), for \( a_1 = 1, a_2 = -1 \) and \( a_3 = 0 \) \( \mathbb{Z}_n \subset SU(2) \). One can see easily that these cases indeed yield \( \mathcal{N} = 1,2 \) supersymmetric spectra.

\(|\mathbb{Z}_n| = n \) and the \( n \) irreducible representations are all one dimensional (\( \mathbb{Z}_n \) is Abelian). The \( i \)th representation is given by representing \( \alpha \) as \( \alpha^i \) where \( i \) runs from 0 to \( n - 1 \). According to the general rules the gauge group is hence just \( U(N)^n \). To determine the scalars and fermions we have to read off \( a_{ik}^6 \) and \( a_{ik}^4 \) from the geometric action. To do this note that \( r_i \times r_j = r_{i+j} \) and that the reducible 6 and 4 dimensional representations decompose according to the embedding of \( \mathbb{Z}_n \) in the spacetime \( SO(6) \) as:

\[
\begin{align*}
6 & \rightarrow r_{a_1} + r_{a_2} + r_{a_3} + r_{-a_1} + r_{-a_2} + r_{-a_3} \\
4 & \rightarrow r_{(a_1+a_2+a_3)/2} + r_{(a_1-a_2-a_3)/2} + r_{(-a_1+a_2-a_3)/2} + r_{(-a_1-a_2+a_3)/2} \\
\bar{4} & \rightarrow r_{-(a_1+a_2+a_3)/2} + r_{-(a_1-a_2-a_3)/2} + r_{(-a_1+a_2-a_3)/2} + r_{-(a_1-a_2+a_3)/2}
\end{align*}
\]
All the sums should be understood as mod $n$. The half-integer numbers are due to the fact that to introduce spinors we should really go to the double cover of $SO(6)$, namely $SU(4)$. This additional $Z_2$ action may not commute with the $Z_n$. We see that the matter content consists of scalars which are bi-fundamentals under the $i$th and the $(i + a_\mu)$th gauge group ($\mu = 1, 2, 3$) and similarly for the fermions.

For $a_1 + a_2 + a_3 = 0 \pmod{n}$ we see that we get one adjoint fermion and one for every scalar $((a_1 - a_2 - a_3)/2 = a_1$ and so on) yielding the $\mathcal{N} = 1$ spectrum. For the $\mathcal{N} = 2$ choice we get an adjoint multiplet and the other chiral multiplets can be paired up into hyper multiplets, as required. The $\mathcal{N} = 2$ superpotential is provided automatically, since all interactions are derived from the unique $\mathcal{N} = 4$ potential.

2.3.2 Moving onto the Coulomb branch: Fractional Branes

So far we have identified the gauge theory of the $N$ D3 branes living at the orbifold. Their Higgs Branch can be easily identified as moving the branes away from the singularity, breaking the $U(N)^k$ gauge group to its diagonal $U(N)$ subgroup. This is just the gauge theory on the $N$ D3 branes. At energies below the Higgs scale (given by the mass of open strings stretching between the D3 branes and its distant images), we effectively see the full $\mathcal{N} = 4$ supersymmetry. The metric on the Higgs branch is just the metric on the singular orbifold space. If we turn on FI terms in the gauge theory, the Higgs branch becomes a smooth ALE space. Turning on these FI terms therefore corresponds in spacetime to blowing up the orbifold singularities.

But note that for vanishing FI terms this theory also has a Coulomb branch, corresponding to turning on vevs for the adjoint scalars in the vector multiplets, generically breaking each of the $U(N)$ subgroups to its maximal torus. How can we interpret this branch in spacetime? The answer was given in [7] (see also [6]): on the Coulomb branch, a single D3 brane splits into $|\Gamma|$ fractional branes of equal mass $\frac{ma_4}{|\Gamma|}$. These fractional branes can be interpreted as D5 branes wrapping the vanishing cycles of the orbifold. Therefore they are stuck to the singularity in the orbifolded part of spacetime, but are free to move in the transverse space, giving rise to the Coulomb branch. The example considered in [7] was D0 branes moving on an ALE space. In the orbifold limit these new degrees of freedom supported on the Coulomb branch are precisely those that are necessary for the Matrix description of enhanced gauge symmetry for M-theory on the ALE space.

Looking just at a single fractional brane is achieved by embedding the orbifold group into $\mathbb{Z}_k$. We will now generally use the notation $k$ for the $\mathbb{Z}_k$ orbifold action and $N$ for the number of D-branes.
the Chan Paton factors via a representation other than the regular one. This is consistent with the fact, that only by choosing the regular representation we can describe objects that are free to move away from the orbifold. This identification can be proven by an explicit world-sheet calculation [6] showing that the fractional branes carry charge under the RR fields coupling to D5 branes. This charge vanishes if and only if one chooses the regular representation. So instead of just obtaining fractional branes by splitting D3 branes to move on to the Coulomb branch, one can just add these fractional branes by hand. This is the approach chosen in [50, 51] to explain the wrapped membranes in Matrix theory. We will give a more detailed analysis of the branches, parameters and the quantum behaviour of these theories as we proceed.

3 Duality between Hanany-Witten and fractional branes

3.1 T-duality for fractional branes

Consider again $N$ D3 branes at a transverse $A_{k-1}$ singularity, that is we take the D3 branes to live in the 0123 space and put these on top of an $A_{k-1}$ singularity which is represented by the following $Z_k$ action

$$z_1 = x_8 + ix_9 \rightarrow \alpha z_1, \quad z_2 = x_6 + ix_7 \rightarrow \alpha^{-1} z_2$$

(6)

where $\alpha = e^{2\pi i/k}$. This $Z_k$ is a subgroup of the $SU(2)$ acting on $z_1$, $z_2$ and we therefore have a gauge theory with $\mathcal{N} = 2$ supersymmetry in 4 dimensions. Since the D3 branes can move away from the singularity we should choose to embed the orbifold group in the Chan Paton factors via the regular representation, that is include the D3 brane and all its $k-1$ images. The corresponding gauge group is $U(N)^k$. Upon T-duality in the 6 direction this can be mapped into an elliptic Hanany-Witten setup, that is $k$ NS5 branes living in 012345 directions, with $N$ D4 branes living in 01236 space suspended between every pair of consecutive NS5 branes on the circle. Let us label the NS5 branes with $i = 0, \ldots, k-1$. This kind of duality is by now well known and was for example used to realize various non-trivial fixed points in 6 dimensions via HW setups [52, 53].

Now consider the following setup:
Figure 3: The Hanany-Witten setup: N D4 branes stretching between two of
the NS5 branes

Figure (3) again displays a HW setup very similar to the one above, this time describing
just a single $U(N)$ gauge group. We have just $N$ D4 branes stretching between the $i$th
and the $(i+1)$th NS5 brane. It is natural to assume that the same T-duality in the 6
direction as we used above now translates this into $N$ fractional branes in the sense of
[7], that is $N$ D5 branes wrapping the vanishing homology two-cycle $\sigma_i$ of the $A_{k-1}$
orbifold in the dual picture. Or again in the language of [7], we consider the same orbifold
group, but this time choose to embed the orbifold group into the Chan Paton factors via
the 1d irreducible representation $r_i$ associated with the $i$th node of the extended Dynkin
diagram. Or more generally, if we have $N_i$ D4 branes connecting the $i$th NS5 brane with
the $(i+1)$th NS5 brane the dual setup will again be given by a $Z_k$ orbifold, this time
with the orbifold group embedded in the Chan Paton indices via a general representation
$R$ given by

$$R = \oplus_i N_i r_i. \quad (7)$$

As a first check note that the resulting gauge groups and matter contents agree: both
constructions yield a $\prod_i U(N_i)$ gauge group with hypermultiplets transforming as bi-
fundamentals under neighboring gauge groups. One may wonder whether this is con-
sistent as a quantum theory. Since, as we discussed above, the fractional branes are
charged under the RR gauge fields we need ‘enough’ non-compact space in the transverse
dimensions for this to be consistent. We will have more to say about this when we discuss
the quantum theory. Similarly one can T-dualize any Dp brane stretching in between
NS5 branes as they appear in generalizations of the HW setup to various dimensions to
fractional D(p-1) branes on an orbifold in the same manner.

$^5$where $\sigma_k = -\sum_i \sigma_i$ [7]
3.2 Supporting evidence for the proposed duality

Since the precise metric describing a D4 brane ending on two 5 branes is not known to us we cannot verify our proposal by a direct computation. However there is compelling evidence that it is indeed true.

3.2.1 The 5 brane point of view

Actually a very similar duality has already been proposed in [29, 54]. In both cases it was conjectured that a brane ending on another brane is dual to some object carrying fractional charge by considering the world-volume theory of the brane on which the other brane ends, that is in our case the NS5 brane. Let us briefly review their arguments since they automatically give some evidence for our case, too.

Let us first review the case discussed in [29]. One starts with Euclidean D1 branes living in 36 space ending on D3 branes which live in 0126 space. The D1 can end on the D3 in the 3 direction. This configuration describes the instanton corrections in a HW realization of 3 dimensional gauge theories (after we suspend the D3 branes in between NS5 branes in the 6 direction in the usual fashion). One is interested in how the configuration transforms under T-duality in the 3 direction. If the D1 branes stretch around the circle of radius $R_3$, they simply T-dualize into Euclidean D0 branes stretching along the 6 direction, which are known to yield the instanton corrections to the corresponding gauge theory [27, 28]. Since the D3 branes T-dualize into D4 branes this is now a 4d HW configuration. However the D1 branes can split on the D3 branes just like the D4 branes can split on the NS5 branes in the setup we are interested in. The question of how the system of Euclidean D1 branes ending on the D3 branes behaves under T-duality was answered in [29] with the help of considerations in the world-volume theory of the D4 brane (which is 4d since the D4 brane itself should be thought of as being embedded in a HW setup).

The Euclidean D0 appears in this setup as a usual YM instanton. However it is well known from field theory [55] that the gaugino condensate in the corresponding YM

$$\langle \lambda \lambda \rangle = \Lambda^3 = e^{-8\pi^2 / Ng_Y^2 M}$$  \hspace{1cm} (8)

is not produced by the instantons themselves but by fractional instantons, that is configurations of the form

$$A_\mu = \frac{1}{N} g \partial_\mu g^{-1}.$$  \hspace{1cm} (9)

From the field theory point of view, a usual instanton can shrink to zero size and then split into $N$ fractional instantons. The $4N\nu$ dimensional moduli space of $\nu$ instantons
in 4d is then (after fractionation of all the instantons) just given by the positions of the $N\nu$ fractional instantons. These objects are characterized by the properties that they carry $\frac{1}{N}$th of the mass and charge of a regular instanton. Note that this is precisely the property we are looking for to make contact with our proposal: a D$p$ brane stretching around only $\frac{1}{k}$th of the compact circle T-dualizes into an object that has $1/k$ times the mass and charge of the dual D$(p-1)$ brane: a fractional brane.

This discussion does not directly apply to our setup, since we don’t consider D$p$ branes ending on other D-branes but in our setup they end on NS5 branes. Under T-duality the NS5 branes turn into a geometric background singularity and it is hard to discuss their world-volume point of view. If we however consider a type IIB HW setup (for example the original setup of [4] realizing 3d gauge theories), we can perform an S-duality before we T-dualize, mapping the NS5 branes to D5 branes. This point of view was taken in [54].

### 3.2.2 Deformations and Moduli

It is very nice to see how the various parameters and moduli of the theory are realized as brane motions in the two pictures and how they are mapped to each other. First consider the case of all $N_i = N$, where the duality is well established. This theory has two branches, a Coulomb branch and a Higgs branch. As free parameters we can add FI terms $\xi_i$, for each gauge group factor which are triplets under the $SU(2)_R$ symmetry and totally lift the Coulomb branch. Since we have $k$ gauge group factors we have in addition $k$ gauge coupling constants $g_{2i}$, which at least in the classical theory are additional free parameters. Together with the $\theta$-angles they form the complex coupling constants $\tau_i = \frac{\theta_i}{2\pi} + \frac{4\pi i}{g_{2i}^2}$.

On the orbifold side the Higgs branch just corresponds to moving the D3 branes away from the orbifold point in the ALE space. Therefore the Higgs branch metric coincides with the metric of the transverse ALE space. Since all the $N_i$ are equal in the HW setup all the D4 brane pieces can connect to form $N$ full D4 branes which can then move away from the NS5 branes in the 789 direction. As usual, the 4th real dimension of this branch is given by the $A_6$ component of the gauge field on the D3 branes.

When the D3 branes meet the NS5 branes in 789 space, that is at the origin of the Higgs branch, they can separate into the pieces which are then free to move around 45 space along the NS5 branes giving rise to the Coulomb branch. In the orbifold picture the same process is described by the $N$ D3 branes splitting up into $kN$ fractional branes. These are now localized at the singularity in 6789 space, but are also free to move in the 45 space.
Now consider turning on the FI terms $\xi_i$. For the orbifold this corresponds to resolving
the singularity by blowing up the $i$th vanishing sphere $\sigma_i$. The mass of the states on
the Coulomb branch (which are after all D5 branes wrapping these spheres) become of order $\xi_i^2$. The Coulomb branch is removed as a supersymmetric vacuum. In the dual
picture the FI terms correspond to motion of the NS5 branes in 789 space. This resolves
the singularity since the 5 branes no longer coincide and again the mass of states on the
Coulomb branch (which in this case are D4 brane stretching between the displaced NS5
branes) is of order $\xi_i^2$.

Last but not least we have to discuss the coupling constants. On the HW side they are
simply given in terms of the distances $\Delta_i$ between the NS5 branes in the 6 direction, or
to be more precise
\[ g_{i}^{-2} = \Delta_i / g_s^A l_s \]  \hspace{1cm} (10)
where $g_s^A$ and $l_s$ denote string coupling constant and string length of the underlying type
IIA string theory respectively. Since we are dealing with a theory with compact 6 direction
we have
\[ \sum_i \Delta_i = R_6 \]  \hspace{1cm} (11)
where $R_6$ is the radius of the 6 direction. The easiest case to consider is if all $\Delta_i = \Delta$,
that is the NS5 branes are equally spaced around the circle. In this case we get $g_i^{-2} = g^{-2} = R_6 / g_s^A k l_s$. Applying T-duality to type IIB this means that $g_i^{-2} = 1 / g_s^B k$ since
$g_s^B = g_s^A l_s / R_6$, in accordance with the orbifold analysis [6, 47]. According to our duality
hypothesis the same configuration can be also viewed as $N_i$ fractional branes, that is $N_i$
D5 branes wrapping the $i$th vanishing cycle. According to [47] in this case the coupling
constant is given in terms of fluxes of two-form charge on the vanishing sphere
\[ \tau_i = \int_{\sigma_i} B_{RR} + i \int_{\sigma_i} B_{NS} \]  \hspace{1cm} (12)
At the orbifold the values of the B-fields are fixed to yield again $g_i^{-2} = 1 / g_s^B k$ [7]. Since we
are however dealing with D5 branes wrapping the vanishing sphere, $B_{NS}$ always appears
in the combination $\mathcal{F} = F - B_{NS}$, where $F$ denotes the field strength of the gauge field
on the D5 brane. We therefore expect that the imaginary part of $\tau_i$, that is $\frac{4\pi}{g_i} \tau_i$, is actually
given by $\int_{\sigma_i} \mathcal{F}$. The full gauge coupling is hence given by a constant piece from the fixed
value of the $B_{NS}$ field at the orbifold point plus some modifications due to Wilson lines.
Under T-duality these Wilson lines translate into the positions of the NS5 branes on the
circle, as expected.

It is easy to see that this discussion generalizes without any problems to arbitrary values
of $N_i$. In addition we can introduce D6 branes in the HW setup living in 0123789 space
to give extra matter. These simply T-dualize into D7 branes wrapping the ALE space in the dual picture.

### 3.2.3 Quantum Behaviour

All considerations so far have been purely classical. Now let us discuss the quantum behaviour. Since we are dealing with $\mathcal{N} = 2$ SUSY there are only two kinds of corrections, one-loop and non-perturbative corrections. All higher loop contributions vanish exactly. The one loop beta function gives rise to the usual logarithmic running of the coupling constant. The instanton contributions have been summed up by the Seiberg-Witten solution.

On the HW side it is well known how to incorporate quantum corrections. In the full string theory the D4 brane ending on the NS5 brane actually has a back-reaction on the NS5 brane leading to a logarithmic bending of the NS5 brane. Since the coupling constant is given in terms of distance between the NS5 branes this fact just reflects the 1-loop correction, the running of the gauge coupling. In addition this bending actually freezes out all the $U(1)$ factors yielding just a $\prod SU(N_i)$ gauge group. The FI terms no longer exist. Motions in the 789 space are now associated to baryonic expectation values.

The nonperturbative effects in this case are given by Euclidean D0 branes stretching along the 6 direction [27, 28, 29] using the well known relation that SYM instantons inside a Dp brane are represented by D(p-4) branes [26]. Like the D4 branes themselves these Euclidean D0 branes can of course split on the NS5 branes and become fractional D0 branes [29]. According to the same T duality in the 6 direction we applied to the D4 branes they should become fractional D(-1) branes, that is Euclidean D1 branes wrapping the vanishing cycles. This fits nicely with the picture that this way the instanton corrections on the orbifold side are again given by D(p-4) branes living inside the Dp brane.

On the HW side the instanton corrections can actually be summed up. The D0 branes are just momentum modes along the hidden 11 direction. They are incorporated by lifting the configuration to M-theory [25]. Similarly the instanton contributions on the orbifold side can also be summed up using the tools of geometric engineering. We will have more to say about this connection in the next section.

But how does the 1-loop effect arise in the orbifold picture? Note that the fractional branes carry charge under the appropriate RR fields, as can be born out by a world-sheet computation of the corresponding tadpoles [6]. They represent a charge sitting in the space transverse to the D3 brane and transverse to the orbifold. In our example this
is the 2 dimensional 45 space. In two dimensions the Green’s functions are given by logarithms. Therefore we would expect a theory of a charged object in 2 dimensions to be plagued by divergences. To be more precise we have the following trouble: we want to consider effectively a 3 brane sitting in 6d spacetime. Since the transverse space is only 2d, the corresponding 4-form vector potential $C^{(4)}$ grows logarithmically.

From the D5 brane point of view the 3 brane vector potential couples to the world-volume theory via

$$\int dx^6 \mathcal{F} \wedge C^{(4)}.$$  \hspace{1cm} (13)

After compactification on $\sigma_i$ this leads to a term in the 3 brane action of the form

$$\int_{\sigma_i} \mathcal{F} \cdot \int dx^4 C^{(4)}.$$  \hspace{1cm} (14)

But recall that we identified $\int_{\sigma_i} \mathcal{F}$ as the gauge coupling $\frac{4\pi g_i}{g}$. The growing of $C^{(4)}$ can be absorbed by introducing an effective running coupling constant. The divergences we encounter are just due to the 1-loop running of the gauge coupling! Charge neutrality, which is obtained if we choose the regular representation, corresponds to finiteness \cite{6,47}.

In this way we can see that the correct dependence of the beta function on $N_c$ and $N_f$ is reproduced. This can most easily be seen by observing that the charge clearly depends linearly on the number of fractional branes and is zero when $N_f = 2N_c$, corresponding to the case where the fractional branes can all combine to form a full brane. The relative contributions originate from the (self-)intersection numbers of the spheres $\sigma_i$ given by the (extended) Cartan matrix (here for $A_{k-1}$).

In this way the bending of the branes directly translates into the fall-off of the RR field in transverse space in the orbifold picture. The correspondence also holds in other dimensions (for D5 branes the transverse space is 0 dimensional and we need neutrality reflecting anomaly freedom of the underlying gauge theory, in 5,4,3,2,... dimensions we get linear, logarithmic, $1/r$, $1/r^2$, ... fall off for the field strength). This reflects the appropriate running of the gauge coupling just as in the HW picture. The following diagram illustrates this duality.

![Figure 4: Duality relation between HW setup and fractional branes](image)
4 Connecting the pieces

The duality we proposed above is the missing piece to connect the various approaches to embedding gauge theories into string theory, making their quantum behaviour manifest. We described these approaches above. Here we will show how they are related.

Figure 5: Connection between the various approaches. The non-compact Calabi Yau spaces are ALE spaces fibered over a sphere (or several intersecting spheres). These can be T-dualized via T-duality on the fiber. The last T-duality, connecting to the HW picture is the one discussed in this work.

In Figure (5) the connections are displayed diagrammatically, putting together the pieces of Figures (1), (2) and (4). For the geometric engineering approach of [2] the starting point is IIA string theory on a non-compact Calabi-Yau which is an ALE space fibered over some base (a non-compact version of a K3 fibration). Quantum corrections can be solved by using mirror symmetry to a IIB picture, as reviewed above. A T-duality on
the ALE fiber can be used to map this to a IIA NS5 brane wrapping the Seiberg-Witten curve [1, 2]. While in this approach the solution is obtained via mirror symmetry, this last T-duality leads to the very nice interpretation of the SW curve: it appears as the physical object the 5 brane is wrapping.

Applying the same T-duality, which we have just applied to the IIB solution, directly on the ALE fiber in the IIA setup (which is the starting point for the Geometric Engineering approach), followed by an S-duality in the resulting IIB string theory, it is straightforward to connect this to a setup of D5 branes wrapping the base of the fibration. For the geometric engineering approach to \( \mathcal{N} = 1 \) theories [56, 57] this picture of branes wrapping the base of an ALE fibration is actually the starting point, as we will discuss in the next section. This way we can make direct contact to the language of fractional branes [6, 7, 48] where one discusses the world-volume theory of branes wrapping various cycles.

Using the T-duality proposed in this paper one can translate this into the very intuitive language of a HW setup [4], which finally can be solved for by lifting to M-theory [25]. This way we not only obtain the same solution (the SW curve), but also the same physical interpretation: the SW curve appears directly as the space an M5 brane wraps on.\(^6\)

In the following we will discuss the steps in the above chain of dualities in more detail. There are two important points one should account for. For one thing there are the quantum corrections. Since we are dealing with \( \mathcal{N} = 2 \) there are only two types of these corrections: the 1-loop contribution which gives rise to the logarithmic running of the gauge coupling and the instanton corrections. It is interesting to see how these quantum effects are incorporated in the various pictures and how they are finally solved for.

The other thing one should treat with some care is how the singularity type affects the gauge group. In all the examples we are considering there are always two pieces of information, which we will refer to as the singularity types determining the “gauge group” and the “product structure”. This is easiest to understand in a particular example. Consider \( N \) D3 branes at an \( A_{k-1} \) singularity. This gives rise to an \( SU(N)^k \) gauge group. In this example \( N \) determines the “gauge group” (the size and type of the single gauge group factors) and \( k \) the “product structure” (we have \( k \) \( SU(N) \) factors). Since in the various duality transformations we are using we repeatedly map branes into singularities and vice versa it is quite important to distinguish which ingredient in the picture determines gauge group and product structure. In the cases where the gauge group is determined by a geo-

\(^6\)In the spirit of [58] the difference between IIA and M5 brane is irrelevant, since the radius of the transverse circle does not affect the low-energy SYM on the brane.
metric singularity generalization to D or E type singularities leads to $SO$ or $E_{6,7,8}$ gauge groups, while in the cases where the product structure is determined by a geometric singularity the generalization to D or E type just leads to a more involved product of $SU(N)$ groups determined by the corresponding extended Dynkin diagram. For example in the D3 brane case $N$ D3 branes at a $E_6$ singularity lead to an $SU(N)^3 \times SU(2N)^3 \times SU(3N)$ gauge group.

We will start with the HW setup, since this seems to be the most intuitive approach.

**HW → fractional branes:** This is basically the duality we have discussed in this work. A system of $k$ NS5 branes with $N_i$ D4 branes suspended between the $i$th and the $(i+1)$th NS5 brane is dual to an $A_{k-1}$ orbifold singularity with $N_i$ fractional branes associated to the $i$th shrinking cycle. While in the HW setup the gauge group is determined by the D4 branes and the NS5 branes determine the product structure, the corresponding roles are played by D3 branes and the orbifold type in the dual picture. That is, the vanishing 2-cycles are given by spheres intersecting according to the extended Dynkin diagram. The gauge group can be generalized to D type by including orientifold planes on top of the D-branes. It is not known how to achieve E type gauge groups this way. Using an E type orbifold in the fractional branes only affects the product structure.

The 1-loop quantum effects correspond to bending of the NS5 branes in HW language and get mapped to the logarithmic Green’s functions in the orbifold picture, as discussed in the previous section. Non-perturbative effects are due to Euclidean D0 branes, which are mapped by the same T-duality into Euclidean D1 branes wrapping the vanishing 2-cycles (fractional D-instantons).

**Fractional branes → IIA on non-compact CY:** We can now apply a different T-duality on the setup described by the fractional branes. First we S-dualize, taking the $N_i$ D5 branes into NS5 branes and the Euclidean D1 branes into fundamental strings (that is world-sheet instantons). Performing a T-duality in the overall transverse space (that is 45 space) the NS5 branes turn into an $A_{N_i}$ singularity according to the duality of [5] which we have already used several times. The vanishing 2 cycle, which is still the space built out of the $k$ spheres intersecting according to the extended Dynkin diagram, stays unchanged. The fact that the NS5 branes only wrap parts of this base ($N_i$ NS5 branes on the $i$th sphere $\sigma_i$) translates into the fact that the type of the ALE fibers over the base changes from one sphere to the other. From the IIA point of view this looks like T-duality acting on the fibers as described in [1]. Since now everything is geometric, generalizations to E type are straight forward: the product structure is determined by the intersection pattern of the $k$ spheres, the ADE type of the fiber determines the gauge group. This
way we can even engineer products of exceptional gauge groups.

The non-perturbative effects are now due to world-sheet instantons, as expected. The log-corrections coming from one loop are incorporated in the particular limit one has to choose to decouple gravity [59]. The system can be solved via local mirror symmetry.

5 Generalizations to $\mathcal{N} = 1$

All the approaches discussed in the previous section have been generalized to capture $\mathcal{N} = 1$ supersymmetric gauge dynamics. Even though different approaches are more or less suited for different questions, the above pattern of connections between the various approaches generalizes to this case as well. As a starting point let us take the HW setup, since this case was discussed extensively in the literature [31, 33, 34]. The breaking of the supersymmetry is achieved by rotating some of the NS5 branes into NS5' branes [32], which now live in 012389 space. The theory can still be solved via a lift to M-theory [35, 36].

The first step in the chain of dualities is to translate this into fractional D3 branes sitting at an orbifold singularity. According to the standard lore the system of $k$ NS5 branes and $k'$ NS5' branes T-dualizes into a $\mathbb{Z}_k \times \mathbb{Z}_{k'}$ orbifold action on the 456789 space transverse to the brane given by the combination of the group actions

\begin{align}
  z_1 &= x_8 + ix_9 \rightarrow \alpha z_1, \quad z_2 = x_6 + ix_7 \rightarrow \alpha^{-1} z_2 \\
  z_2 &= x_6 + ix_7 \rightarrow \alpha' z_1, \quad z_3 = x_4 + ix_5 \rightarrow \alpha'^{-1} z_3
\end{align}

This is a particular case of an orbifold group $\Gamma \subset SU(3)$ discussed in Section 2. Again D4 branes stretching all around the circle become D3 branes, while D4 branes ending on the NS5 branes become fractional D3 branes. In this case the fractional D3 branes are D7 branes wrapping a vanishing 4 cycle $S$, which is given by a fibration of $P_1$ over $P_1$ (a Hirzebruch surface), where the $P_1$'s are just the vanishing spheres associated with any one of the two orbifold actions. Instanton corrections are given by fractional D-instantons, that is Euclidean D3 branes wrapping the vanishing 4 cycle.

While the NS5 branes still bend, this bending no longer encodes directly the $\beta$ function. However a different T-duality can be applied to bring the theory into a slightly modified form recently considered by [60], containing D5 branes suspended in rectangles spanned by two types of NS5 branes.\footnote{One applies 2 T-dualities in the 4 and the 7 direction instead of one in the 6 direction to the orbifold described above. This yields the Hanany-Zaffaroni setup (up to some renaming of directions).} In this picture the bending again carries information about
the $\beta$ function: in the case of finite theories there is no bending [61, 62]. In some special cases it is possible to find the coefficient of the beta function in this setup [63]. The orbifold picture leads to a very similar conclusion: if we choose the regular representation charge neutrality still implies finiteness [46]. For non-zero net charge in the orbifold space we encounter divergences which should again be interpreted as being due to the running of the gauge coupling.

Note that this is precisely the setup used as a starting point in the Geometric Engineering approach to $\mathcal{N} = 1$ theories [56, 57]. It can also be reformulated in terms of F-theory, where now the position of the D7 branes is encoded in the singularities of the elliptic fiber over the surface S, which is the compact part of the base of the elliptic fibered CY 4 fold that is used in F-theory compactifications yielding $\mathcal{N} = 1$ in 4d. Since now the 7 branes (which in our language from above determine the gauge group and not the product structure of the gauge group) are translated into geometric singularities, generalizations of this setup to $SO$ and exceptional gauge groups are straight-forward. It would be nice to solve these theories (like the $\mathcal{N} = 2$ ones) via a local mirror symmetry of the 4-fold (which can be thought of as T-duality on a 4 cycle and hence takes IIB (F-theory) back to IIB (F-theory)). Even though there has been some progress in this direction [64, 65], the solution of the problem remains for future work.

6 Conclusions

In this work we have presented a relation between various approaches to obtain field theories as limits of string theory. The last piece in the puzzle was a T-duality between HW setups and fractional branes at orbifolds, which is a generalization of the well known T-duality relating elliptic HW setups to regular branes at orbifolds.

A lot has been learned about latter ones using the recently discovered correspondence of conformal field theory with supergravity on AdS spaces [45, 46]. It would be very interesting to see whether there is also a supergravity description corresponding to the large $N$ limit of non-conformal gauge theories realized in terms of fractional branes. This space will no longer be AdS, since conformal symmetry is lost, but it should encode the logarithmic divergences due to the running of the gauge coupling.
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