BOUNDs FOR $L_p$-DISCREPANCIES OF POINT DISTRIBUTIONS IN COMPACT METRIC MEASURE SPACES

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ABSTRACT. Upper bounds for the $L_p$-discrepancies of point distributions in compact metric measure spaces for $0 < p \leq \infty$ have been established in the paper [6] by Brandolini, Chen, Colzani, Gigante and Travaglini. In the present paper we show that such bounds can be established in a much more general situation under very simple conditions on the volume of metric balls as a function of radii.

1. Introduction

Let $\mathcal{M}$ be a compact metric measure space with a fixed metric $\theta$ and a finite non-negative Borel measure $\mu$, normalized, for convenience, by

$$\mu(\mathcal{M}) = 1, \quad \text{diam } \mathcal{M} = 1,$$

where $\text{diam } \mathcal{E} = \sup\{\theta(y_1, y_2), y_1, y_2 \in \mathcal{E}\}$ denotes the diameter of a set $\mathcal{E} \subseteq \mathcal{M}$.

Since $\mathcal{M}$ is connected and satisfies (1.1), the set of values of $\theta$ coincides with the interval $I = [0, 1]$. We write $B(y, r) = \{x : \theta(x, y) < r\}$ for the ball in $\mathcal{M}$ of radius $r \in I$ centered at $y \in \mathcal{M}$ and of volume $v(y, r) = \mu(B(y, r))$. We can conveniently write $B(y, r) = \emptyset$ and $v(y, r) = 0$ if $r \leq 0$ and $B(y, r) = \mathcal{M}$ and $v(y, r) = 1$ if $r \geq 1$.

The local discrepancy of an $N$-point subset $\mathcal{D}_N \subset \mathcal{M}$ (distribution) in a metric ball $B(y, r)$ is defined by

$$L[B(y, r), \mathcal{D}_N] = \#(B(y, r) \cap \mathcal{D}_N) - N v(y, r)) = \sum_{x \in \mathcal{D}_N} L(y, r, x),$$

where

$$L(y, r, x) = \chi(B(y, r), x) - v(y, r),$$

and $\chi(\mathcal{E}, x)$ denotes the characteristic function of a subset $\mathcal{E} \subset \mathcal{M}$.

For $0 < p < \infty$, the $L_p$-discrepancy is defined by

$$L_p[\mathcal{D}, \mathcal{D}_N] = L_p[\mathcal{M}, \xi, \mathcal{D}_N] = \left(\int_{\mathcal{M} \times I} |L[y, r, \mathcal{D}_N]|^p d\mu(y) d\xi(r)\right)^{1/p},$$

where $\xi$ is a finite (non-negative) measure on $I$ normalized by $\xi(I) = 1$.

For $p = \infty$, we put

$$L_\infty[\mathcal{D}_N] = L_\infty[\mathcal{M}, \mathcal{D}_N] = \sup_{y, r} L[y, r, \mathcal{D}_N],$$

where the supremum is taken over all balls $B(y, r) \subseteq \mathcal{M}$.

2000 Mathematics Subject Classification. 11K38, 52C99.

Key words and phrases. Discrepancies, Point distribution, Metric measure spaces.
We introduce also the following extremal discrepancies

\[
\begin{align*}
\lambda_p[\xi, N] &= \lambda_p[\mathcal{M}, \xi, N] = \inf_{\mathcal{D}_N} \mathcal{L}_p[\xi, \mathcal{D}_N], \\
\lambda_\infty[N] &= \lambda_\infty[\mathcal{M}, N] = \inf_{\mathcal{D}_N} \mathcal{L}_\infty[\mathcal{D}_N],
\end{align*}
\]

(1.6)

where the infimum is taken over all \(N\)-point subsets \(\mathcal{D}_N \subset \mathcal{M}\).

Point distributions on the spheres \(S^d\) have been studied by many authors, see the surveys \([2, 7]\) and references therein. We mention only a few results known at present. First of all, we have the following two-side bounds

\[
N^{\frac{d}{2} - \frac{1}{p}} \lesssim \lambda_2[S^d, \xi, N] \lesssim N^{\frac{d}{2} - \frac{1}{p}},
\]

(1.7)

where the measure \(d\xi(r) = \frac{\pi}{2} \sin(\pi r)dr, \ r \in \mathcal{I}\) (for this measure the geodesic balls \(B(y, r) \subset S^d\) coincide with the spherical caps). The upper bound in (1.7) was proved by Alexander \([1]\) and Stolarsky \([14]\) and the lower bound by Beck \([3]\), see also \([4, \text{Theorem 24A and Corollary 24C}]\). Furthermore, Beck, see \([4, \text{Theorem 24D}]\), established the bound

\[
\lambda_\infty[S^d, N] \lesssim N^{\frac{d}{2} - \frac{1}{d}} (\log N)^{1/2}.
\]

(1.8)

The constants implicit in the symbol \(\lesssim\) are independent of \(N\).

The two-side bounds (1.7) were extended to all compact Riemannian symmetric manifolds of rank one (two-point homogeneous spaces) and arbitrary absolutely continuous measures \(\xi\) on \(\mathcal{I}\), see \([13, \text{Theorem 2.2}]\). Recall that these manifolds are the spheres \(S^d\), the real, complex and quaternionic projective spaces \(\mathbb{P}P^n, \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}\), and the octonionic projective plane \(\mathbb{O}P^2\), see, for example, \([11]\). Recall that any Riemannian manifold can be thought of as a metric measure space with respect to the Riemannian distance and measure (normalized, if needed by (1.1)).

Notice that the upper bound in (1.7) holds for arbitrary compact \(d\)-rectifiable spaces \(\mathcal{M}\) and any measure \(\xi\), see \([12]\). At the same time, the lower bound in (1.7) fails, even for the spheres \(S^d\), if the measure \(\xi\) is singular. The corresponding example can be found in \([5, 12, 13]\). In this example the discrepancy \(\lambda_p[\xi, N]\) is bounded from above by a constant independent of \(N\) and \(p\).

The \(L_p\)-discrepancies for any \(p\) and for general compact metric measure spaces were first estimated in the paper \([6]\) by Brandolini, Chen, Colzani, Gigante and Travaglini. The authors consider point distributions in general sub-regions in \(\mathcal{M}\) and estimate the \(L_p\)-discrepancies in the terms of regularity of the boundary of such sub-regions, see \([6, \text{Theorems 8.1 and 8.5}]\).

Let us discuss in more details the results of the paper \([6]\) related to the \(L_p\)-discrepancies for the metric balls \(B(y, r) \subset \mathcal{M}\). Let \(\mathcal{M}\) be a compact \(d\)-dimensional Riemannian manifold \(\mathcal{M}\), and \(r_\mathcal{M}\) the injectivity radius of \(\mathcal{M}\) (for the definition, see, for example, \([11, p. 142]\)).

It is proved in \([6, \text{Corollary 8.4}]\) that for any \(N\) and each \(0 < p < \infty\) there exists an \(N\)-point subset \(\mathcal{D}_N \subset \mathcal{M}\) such that

\[
\mathcal{L}_p[\mathcal{M}, \xi, \mathcal{D}_N] \lesssim N^{\frac{d}{2} - \frac{1}{p}},
\]

(1.9)

provided that the measure \(\xi\) is concentrated on the sub-interval \([0, r_\mathcal{M}] \subseteq \mathcal{I}\).

Similarly, it is proved in \([6, \text{Corollary 8.6}]\) that for any \(N\) there exists an \(N\)-point subset \(\mathcal{D}_N \subset \mathcal{M}\) such that

\[
\mathcal{L}_\infty[\mathcal{M}, \mathcal{D}_N] \lesssim N^{\frac{d}{2} - \frac{1}{d}} (\log N)^{1/2},
\]

(1.10)
where \( L^p_2[\mathcal{M}, N] \) is defined by (1.5) but the supremum is taken over balls \( B(y, r) \subset \mathcal{M} \) with \( 0 \leq r < r_M \).

Notice that for the spheres \( S^d \), and for some other spaces, the injectivity radius is equal to the diameter, and the bound (1.10) implies Beck’s bound (1.8), while the bounds (1.9) are new for \( S^d \) and \( 2 < p < \infty \).

Naturally, the question arises as to whether the parameter \( r_M \) is really needed in the bounds (1.9) and (1.10).

The occurrence of the injectivity radius in the above results has a purely geometric character and dictated by a local treatment of the discrepancies. For small radii \( 0 \leq r < r_M \) the geodesic balls \( B(y, r) \) are diffeomorphic to the balls in the Euclidean space \( \mathbb{R}^d \), while for large \( r \geq r_M \) the structure of balls \( B(y, r) \) becomes rather complicated.

At the same time, the volume \( v(y, r) \) as a function of radii is quite regular for all \( r \in \mathcal{I} \). Furthermore, \( v(y, r) \) can be estimated very precisely by the volume comparison theorems, well-known in the Riemannian geometry, see [10, Chapter 8] and [11, Chapter 9]. In the present paper we show that the volume function \( v(y, r) \) suffices to treat the discrepancies. This allows us to eliminate \( r_M \) from the bounds (1.9) and (1.10) and estimate the \( L_p \)-discrepancies in a much more general situation.

We specialize metric measure spaces by the following two conditions.

**Condition A.** The volume \( v(y, r) \) satisfies the bounds
\[
\frac{c_1^{-1}}{r^d} \leq v(y, r) \leq c_1 r^d, \quad y \in \mathcal{M}, \ r \in \mathcal{I},
\]
with positive constants \( d \) and \( c_1 \) independent of \( y \in \mathcal{M} \) and \( r \in \mathcal{I} \)

The spaces satisfying the Condition A are known as Ahlfors regular spaces.

In the following, we write consecutively \( c_1, c_2, c_3, \ldots \) for positive constants depending only on \( \mathcal{M} \).

**Condition B.** The volume \( v(y, r) \) as a function of \( r \) is Lipschitz continuous:
\[
|v(y, r_1) - v(y, r_2)| \leq c_2 |r_1 - r_2|, \quad y \in \mathcal{M}, \ r_1, r_2 \in \mathcal{I}.
\]

It is not difficult to give many examples of compact spaces satisfying both Conditions A and B. Particularly, the following is true.

**Proposition 1.1.** Any compact \( d \)-dimensional Riemannian manifold satisfies the Conditions A and B.

The Condition A is well-known for compact Riemannian manifolds, see, for example, [10, Section 9.2], while the Condition B is a little more specific, it can be easily derived from the Bishop–Gromov volume comparison theorem. For completeness, we shall give a short proof of Proposition 1.1 in Appendix in Section 4.

Our first result is the following.

**Theorem 1.1.** Let \( \mathcal{M} \) be a compact connected metric measure space satisfying the Conditions A and B. Then for all \( N \), we have
\[
\lambda_p[\mathcal{M}, \xi, N] \leq c_3 (p + 1)\frac{1}{2} N^{\frac{d}{2}} N^{\frac{d}{p} - \frac{d}{2}}, \quad 0 < p < \infty,
\]
where \( \xi \) is an arbitrary normalized measure on \( \mathcal{I} \).

Particularly, the bound (1.13) holds for any compact Riemannian manifold of dimension \( d \).

The proof of Theorem 1.1 is given in Section 3. In its proof, the well-known random \( N \)-point distributions will be used, see [4, pp. 237–239]. Such random
distributions are constructed in terms of partitions of the space $\mathcal{M}$ into $N$ subsets of equal measure and small diameters. The local discrepancies of such distributions can be written as sums of random independent variables, and the Marcinkiewicz–Zigmund inequality can be applied to obtain the bound (1.13). These arguments in their background are similar to those in [6]. A new observation is that the Conditions A and B are sufficient to prove the bound (1.13) without any additional restrictions.

In (1.13), the dependence on the exponent $p$ is described explicitly. This allows us to obtain upper bounds for the extremal $L_\infty$-discrepancy. For this purpose, we use the following a priori estimate, which is also of interest by itself.

**Proposition 1.2.** Let the assumptions of Theorem 1.1 hold. Then for an arbitrary $N$-point subset $D_N \subset \mathcal{M}$, we have

$$L_\infty[\mathcal{M}, D_N] \leq 2m^{2/p} \lambda_p[\mathcal{M}, \xi_0, D_N] + c_4 N m^{-1/d},$$

(1.14)

where $\xi_0$ is the standard Lebesgue measure on $\mathcal{I}$, while $p > 1$ and integer $m \geq c_5$ are arbitrary parameters. Particularly, we have

$$\lambda_\infty[\mathcal{M}, N] \leq 2m^{2/p} \lambda_p[\mathcal{M}, \xi_0, N] + c_4 N m^{-1/d}.$$  

(1.15)

The proof of Proposition 1.2 is given in Section 2.

Comparing Theorem 1.1 with Propositions 1.2 and 1.1, we arrive at the following.

**Corollary 1.1.** Let the assumptions of Theorem 1.1 hold. Then for all $N$, we have

$$\lambda_\infty[\mathcal{M}, N] \leq c_6 N^{\frac{1}{d} - \frac{1}{d'}} (\log N)^{1/2}.$$  

(1.16)

Particularly, the bound (1.16) holds for any compact Riemannian manifold of dimension $d$.

**Proof.** Putting $m = N^d$ in (1.15) and using (1.13), we obtain

$$\lambda_\infty[N] \leq 2N^{\frac{2d}{d'}} \lambda_p[\xi_0, N] + c_4 \leq 2c_3 N^{\frac{2d}{d'}} (p + 1)^{1/2} N^{\frac{1}{d'}} - \frac{1}{d'} + c_4.$$  

Now, we choose $p = 2d \log N$ (with the log in base 2, say) to obtain

$$\lambda_\infty[N] \leq 2c_3 N^{\frac{1}{d'} - \frac{1}{d'}} (2d \log N + 1)^{1/2} + c_4 \leq c_6 N^{\frac{1}{d'} - \frac{1}{d'}} (\log N)^{1/2},$$

that completes the proof. \qed

The present paper is organized as follows. In Section 2 we describe the necessary facts on partitions of metric measure spaces and prove Proposition 1.2. In Section 3 we describe the construction of random point distributions and prove Theorem 1.1. Finally, in Section 4 we prove Proposition 1.1.

## 2. Partitions of metric spaces. Proof of Proposition 1.2

The following general result is due to Gigante and Leopardi [9, Theorem 2].

**Lemma 2.1.** Let $\mathcal{M}$ be a compact connected metric measure space satisfying the Condition A. Then for all sufficiently large $m > c_7$ there exists a partition $P_\mathcal{M} = \{P_j\}_{j=1}^m$ of $\mathcal{M}$ into $m$ subsets $P_j$ with the following properties

$$\mathcal{M} = \bigcup_{1 \leq j \leq m} P_j, \quad P_j \cap P_i = 0, \quad j \neq i, \quad \mu(P_j) = m^{-1}, \quad 1 \leq j \leq m$$  

(2.1)

and

$$c_8^{-1} m^{-1/d} \leq \operatorname{diam} P_j \leq c_8 m^{-1/d}, \quad 1 \leq j \leq m$$  

(2.2)
Partitions with such properties occur in many fields of geometry and analysis. For special spaces, such as the spheres $S^d$, they have long been in use, see the references in [9].

We wish to give some simple corollaries of Lemma 2.1 needed for the proofs of Theorem 1.1 and Proposition 1.1. We write $\Sigma(y, r) = \{ x : \theta(x, y) = r \}$ for the sphere in $\mathcal{M}$ of radius $r \in I$ centered at $y \in \mathcal{M}$. For a partition $P_m = \{ P_j \}_1^n$ of $\mathcal{M}$, we put
\[
J_m = J_m(y, r) = \{ j : \Sigma(y, r) \cup P_j \neq \emptyset \},
\]
\[
K_m = K_m(y, r) = \# \{ J_m(y, r) \}. \tag{2.3}
\]
Thus, $K_m$ is the number of subsets $P_j \in P_m$ entirely covering the sphere $\Sigma(y, r)$.

**Lemma 2.2.** Let $\mathcal{M}$ be a compact connected metric measure space satisfying the Conditions A and B and let $P_m = \{ P_j \}_1^n$ be the partition of $\mathcal{M}$ from Lemma 2.1. Then, we have
\[
K_m(y, r) \leq c_0 m^{1 - \frac{1}{q}}, \tag{2.4}
\]

**Proof.** Put $\tilde{\Sigma}(y, r) = \bigcup_{j \in J_m} P_j$. In view of (2.1), $\mu(\tilde{\Sigma}(y, r)) = m^{-1} K_m$. From the other hand, in view of (2.2), the union $\tilde{\Sigma}(y, r)$ is a subset in the spherical shell $B(y, r + c_8 m^{-1/d}) \setminus B(y, r - c_8 m^{-1/d})$. By the Condition B, we obtain
\[
K_m \leq m \left( v(y, r + c_8 m^{-1/d}) - v(y, r - c_8 m^{-1/d}) \right) \leq 4 c_2 c_8 m^{1 - \frac{1}{q}},
\]
that completes the proof. $\square$

Introduce the following kernels
\[
\delta^M_m(y, z) = m \sum_{1 \leq j \leq m} \chi(P_j, y) \chi(P_j, z) \quad y, z \in \mathcal{M}, \tag{2.5}
\]
where $P_m = \{ P_j \}_1^n$ is an equal measure partition of $\mathcal{M}$, see (2.1),
\[
\delta^T_m(r, u) = m \sum_{1 \leq i \leq m} \chi(Q_i, y) \chi(Q_i, z) \quad r, u \in I, \tag{2.6}
\]
where $Q_m = \{ Q_j \}_1^n$ is the partition of $I \setminus \{ 0 \}$ into the segments $Q = \left( \frac{1}{m}, \frac{1}{m} \right]$ of equal length $m^{-1}$. We put
\[
\delta_m(y, z; r, u) = \delta^M_m(y, z) \delta^T_m(r, u). \tag{2.7}
\]

The kernel (2.7) is non-negative and one can easily check the following relations
\[
\int_{\mathcal{M} \times I} \delta_m(y, z; r, u) \, d\mu(z) \, du = 1, \tag{2.8}
\]
\[
\left( \int_{\mathcal{M} \times I} \delta_m(y, z; r, u)^q \, d\mu(z) \, du \right)^{1/q} = m^{2/p}, \tag{2.9}
\]
where $1 < q < \infty$, $1 < p < \infty$ and $\frac{1}{q} + \frac{1}{p} = 1$.

For the characteristic function and the volume of a ball $B(y, r)$, we consider the following approximations (piece-wise on the partition $P_m \times I_m$)
\[
\chi_m(B(y, r), x) = \int_{\mathcal{M} \times I} \delta_m(y, z; r, u) \chi(B(z, u), x) \, d\mu(z) \, du, \tag{2.10}
\]
\[
v_m(y, r) = \int_{\mathcal{M} \times I} \delta_m(y, z; r, u) v(z, u) \, d\mu(z) \, du. \tag{2.11}
\]
Replacing in these inequalities
\( r \)
ball
\( B \)
Therefore, the ball both radii
\( r \)
Applying Hölder’s inequality to the integral (2.16) and using (2.9), we obtain
\[ \int \leq \int \]
Integrating (2.12) with respect to
\( \alpha \)
Proof of Proposition 1.2. Substituting (2.12) and (2.13) into (1.2), we obtain
\[ \int \leq \int \]
By the triangle inequality, the ball
\( \varepsilon \)
Proof.
\( \xi \)
where
\( \mu \)
also that the kernel (2.7) does not vanish, if and only if both centers \( y \) and \( z \) belong to the same subset \( \mathcal{P}_j \in P_m \) and both radii \( r \) and \( u \) belong to the same subset \( \mathcal{Q}_i \in Q_m \). In such a situation, from (2.2) and the definition of the partition \( P_m \), we obtain
\[ \varepsilon \]
\[ \varepsilon \]
Therefore, the ball \( B(z, u) \) contains the ball \( B(y, r - \varepsilon_m) \) and is contained in the ball \( B(y, r + \varepsilon_m) \). For the characteristic functions, this means
\[ \varepsilon \]
Substituting these inequalities into (2.10) and using (2.8), we obtain
\[ \varepsilon \]
Replacing in these inequalities \( r \) with \( r - \varepsilon_m \) and next with \( r + \varepsilon_m \), we obtain (2.12). Integrating (2.12) with respect to \( x \in \mathcal{M} \), we obtain (2.13).
Proof of Proposition 1.2. Substituting (2.12) and (2.13) into (1.2), we obtain
\[ \varepsilon \]
where
\( \alpha_m \)
\[ \varepsilon \]
and
\[ \varepsilon \]
From (2.14), we obtain the bound
\[ \varepsilon \]
The quantities (2.15) can be easily estimated by the Condition B
\[ \varepsilon \]
Applying Hölder’s inequality to the integral (2.16) and using (2.9), we obtain
\[ \varepsilon \]
where \( \xi_0 \) is the standard Lebesgue measure on the interval \( \mathcal{I} \).
Notice that the right hand sides in (2.18) and (2.19) are independent of \( y \) and \( r \). Substituting (2.18) and (2.19) into (2.17) and using the definition of \( L_{\infty} \)-discrepancy (1.5), we obtain

\[
L_{\infty}[D_N] \leq 2m^{2/p} L_p[\xi_0, D_N] + 4c_2 c_8 N m^{-1/d}.
\]  

(2.20)

This proves the bound (1.14) with \( c_4 = 4c_2c_8 \) and \( c_5 = c_7 \).

\[ \Box \]

3. Random point distributions. Proof of Theorem 1.1

Random \( N \)-point distributions can be constructed as follows. Suppose that a partition \( P_N = \{P_j\}_1^N \) of the space \( M \) into \( N \) parts \( P_j \subset M \) of equal measure \( N^{-1} \) is given. Introduce the probability space

\[
\Omega_N = \prod_{1 \leq j \leq N} P_j = \{X_N = (x_1, \ldots, x_N) : x_j \in P_i, 1 \leq i \leq N\},
\]

(3.1)

with a probability measure \( \omega_N = \prod_{1 \leq j \leq N} \mu_j \), where \( \mu_j = N \mu | P_j \), and \( \mu | P_j \) denotes the restriction of the measure \( \mu \) to a subset \( P_j \subset M \). We write \( EF[\cdot] \) for the expectation of a random variable \( F[X_N] \), \( X_N \in \Omega_N \):

\[
EF[\cdot] = \int_{\Omega_N} F[X_N] d\omega_N
= N^N \int \cdots \int_{P_1 \times \cdots \times P_N} F(x_1, \ldots, x_N) d\mu(x_1) \cdots d\mu(x_N).
\]

(3.2)

Particularly, if \( F[X_N] = f(x_j) \), where \( j \) is a fixed index and \( f(x), x \in M \), is a summable function, then

\[
EF[\cdot] = N \int_{P_j} f(x) d\mu(x).
\]

(3.3)

Elements \( X_N = (x_1, \ldots, x_N) \in \Omega_N \) can be thought of as random \( N \)-point distributions in the space \( M \), and their discrepancies \( L_p[\xi, X_N] \) as random variables on the probability space \( \Omega_N \). We shall prove the following

**Lemma 3.1.** Let \( M \) be a compact connected metric measure space satisfying the Conditions A and B, and let the probability space \( \Omega_N \) in (3.1) be constructed by the partition \( P_N = \{P_j\}_1^N \) of \( M \) from Lemma 2.1 with \( m = N \). Then, we have

\[
(E | L_p[\xi, |] |^p)^{1/p} \leq c_{10} (p + 1)^{1/2} N^{1/p - \frac{1}{d}}, \quad 0 < p < \infty,
\]

(3.4)

where \( \xi \) is an arbitrary normalized measure on \( \mathcal{I} \).

Theorem 1.1 is a direct corollary of Lemma 3.1.

**Proof of Theorem 1.1.** It follows from (3.4) that for each \( 0 < p < \infty \) there exists an \( N \)-point subset \( X_N^{(p)} \in \Omega_N \) such that

\[
L_p[\xi, X_N^{(p)}] \leq c_{10} (p + 1)^{1/2} N^{1/p - \frac{1}{d}},
\]

and the bound (1.13) follows for \( N > c_7 \) with \( c_3 = c_{10} \), while for \( N \leq c_7 \), we have \( \lambda_p[\xi, N] \leq 2c_7 \). This proves Theorem 1.1.

\[ \Box \]

For the proof of Lemma 3.1 we need the Marcinkiewicz–Zigmund inequality, which can be stated as follows.
Lemma 3.2. Let \( \zeta_j, j \in J, \# \{J\} < \infty \), be a finite collection of real-valued independent random variables on a probability space \( \Omega \) with expectations \( \mathbb{E} \zeta_j = 0, j \in J \). Then, we have
\[
\mathbb{E} \left| \sum_{j \in J} \zeta_j \right|^p \leq 2^p (p + 1)^{p/2} \mathbb{E} \left( \sum_{j \in J} \zeta_j^2 \right)^{p/2}, \quad 1 \leq p < \infty.
\] (3.5)

The proof of Lemma 3.2 can be found in [8, Section 10.3, Theorem 2].

Proof of Lemma 3.1. Introduce the notation
\[
J_m^0 = J_m^0(y, r) = \{ j : \mathcal{P}_j \subset B(y, r) \},
K_m^0 = K_m^0(y, r) = \# \{ J_m^0(y, r) \}. \tag{3.6}
\]

In the notation (2.3) and (3.6) the characteristic function and volume of a ball can be written as
\[
\chi(B(y, r), x) = \sum_{j \in J_m^0} \chi(\mathcal{P}_j, x) + \sum_{j \in J} \chi(B(y, r) \cap \mathcal{P}_j, x),
\]
\[
u(y, r) = N^{-1} K_m^0 + \sum_{j \in J} \mu(B(y, r) \cap \mathcal{P}_j).
\]

With the help of these formulas we can calculate the local discrepancy (1.2) for the random point distribution \( X_N = (x_1, \ldots, x_N) \in \Omega_N \):
\[
L[B(y, r), X_N] = \#(B(y, r) \cap X_N) - N \nu(y, r)
= K_m^0 + \sum_{j \in J} \chi(B(y, r) \cap \mathcal{P}_j, x_j) - K_m^0 - N \sum_{j \in J} \mu(B(y, r) \cap \mathcal{P}_j)
= \sum_{j \in J} \chi(B(y, r) \cap \mathcal{P}_j, x_j) - N \sum_{j \in J} \mu(B(y, r) \cap \mathcal{P}_j),
\]
and we can write
\[
L[B(y, r), X_N] = \sum_{j \in J} \zeta_j[X_N], \tag{3.7}
\]
where
\[
\zeta_j[X_N] = \zeta_j[y, r, X_N] = \chi(B(y, r) \cap \mathcal{P}_j, x_j) - N \mu(B(y, r) \cap \mathcal{P}_j)
\] (3.8)
are random variables on the probability space \( \Omega_N \).

The random variables (3.8) are independent, \( \| \zeta_j[X_N] \| < 1 \) and, in view of (3.3), their expectations \( \mathbb{E} \zeta_j[\cdot] = 0, j \in J_N \). Hence, the Marcinkiewicz–Zigmund inequality (3.5) can be applied to the sum (3.7), and taking the bound (2.4) into account, we obtain
\[
\mathbb{E} \left| \sum_{j \in J} \zeta_j[\cdot] \right|^p \leq 2^p (p + 1)^{p/2} K_m^0
\leq 2^p (p + 1)^{p/2} c_0^{p/2} N^{(\frac{1}{2} - \frac{1}{p})p}, \quad 1 \leq p < \infty.
\] (3.9)

Notice that the right hand side in (3.9) is independent of \( y \) and \( r \). Integrating the inequality (3.9) with respect to the measure \( \mu \times \xi \) on \( \mathcal{M} \times \mathcal{I} \), we obtain
\[
\int_{\mathcal{M} \times \mathcal{I}} \mathbb{E} \left| \sum_{j \in J} \zeta_j[y, r, \cdot] \right|^p \mu(dy) \, d\mu(u)
= \mathbb{E} \left( \mathcal{L}_p[\xi] \right) \leq 2^p (p + 1)^{p/2} c_0^{p/2} N^{(\frac{1}{2} - \frac{1}{p})p}, \quad 1 \leq p < \infty.
\] (3.10)
This proves the bound (3.4) for \( 1 \leq p < \infty \). Since the left hand side in (3.4) is a non-decreasing function of \( p \), the bound (3.4) holds for all \( 0 \leq p < \infty \).

4. Appendix: Proof of Proposition 1.1

In this Section we consider a compact \( d \)-dimensional Riemannian manifold \( M \) with the standard Riemannian geodesic distance \( \theta \) and measure \( \mu \) defined by the corresponding metric tensor on \( M \), see, [11]. Notice that for such \( \theta \) and \( \mu \) the normalization (1.1) fails but this is of no importance for the present discussion, because the choice of normalization has effect only on the constants in the bounds (1.11) and (1.12). We keep the same notation \( v(y, r) \) for the volume of a ball with respect to the metric \( \theta \) and the measure \( \mu \).

The bounds (1.11) for a compact Riemannian manifold are well-known, see, for example, [10]. Recall that local consideration of any Riemannian manifold shows that at each point \( y \in M \) and for small \( r, 0 \leq r < r_M \), one has the asymptotic

\[
v(y, r) = \kappa_d r^d + O(r^{d-1}),
\]

where \( \kappa_d \) is the volume of unit ball in \( \mathbb{R}^d \), see [10, Section 9.2]. This implies the bounds (1.11) for small radii \( r \).

Since \( M \) is compact, the bounds (1.11) can be easily extended to all \( 0 < r \leq \text{diam} \, M \).

In order to prove the bound (1.12), we compare \( v(y, r) \) with the volume \( v_k(r) \) of a geodesic ball in the \( d \)-dimensional simply connected hyperbolic space of constant negative sectional curvature \( -k^2 \). The volume \( v_k(r) \) is independent of the position of its center and is given explicitly by

\[
v_k(r) = \sigma_d \int_0^r \left( \frac{\sinh k u}{k} \right)^{d-1} du, \quad 0 \leq r < \infty,
\]

where \( \sigma_d \) is the \((d-1)\)-dimensional area of the unit sphere in \( \mathbb{R}^d \).

**Lemma 4.1.** For any compact Riemannian manifold \( M \), there exists a constant \( k_M \geq 0 \) depending only on \( M \), such that for all \( k > k_M \) the ratio \( \frac{v(y, r)}{v_k(r)} \) as a function of \( r \) is non-increasing and tends to 1 as \( r \rightarrow 0 \).

Lemma 4.1 is a very special case of the Bishop–Gromov volume comparison theorem, see [10] Section 8.7, Theorem 8.45 and [11] Chapter 9, Lemma 36. The constant \( k_M \) is the smallest \( k_0 \geq 0 \) such that the matrix \( R(y) + k_0^2 (d - 1) I_d \) is not-negative defined for all \( y \in M \), here \( R(y) \) is the Ricci tensor at \( y \in M \) and \( I_d \) is the identity \( d \times d \) matrix.

By Lemma 4.1, for \( 0 < r_1 \leq r_2 \leq \text{diam} \, M \), we have

\[
\frac{v(y, r_2)}{v_k(r_2)} \leq \frac{v(y, r_1)}{v_k(r_1)} \leq 1.
\]

Therefore,

\[
v(y, r_2) - v_k(r_1) \leq \frac{v(y, r_1)}{v_k(r_1)} (v_k(r_2) - v_k(r_1)) \leq v_k(r_2) - v_k(r_1),
\]

and the bound (1.12) follows, since \( v_k(r) \) is smooth and increasing.

The proof of Proposition 1.1 is completed.

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