ON EHRHART POSITIVITY OF TESLER POLYTOPES AND THEIR DEFORMATIONS

YONGGYU LEE AND FU LIU

Abstract. For \( \mathbf{a} \in \mathbb{R}_+^n \), the Tesler polytope \( \text{Tes}_n(\mathbf{a}) \) is the set of upper triangular matrices with non-negative entries whose hook sum vector is \( \mathbf{a} \). Recently, Morales conjectured that \( \text{Tes}_n(1, \ldots, 1) \) and \( \text{Tes}_n(1, 0, \ldots, 0) \) are Ehrhart positive for any positive integer \( n \). In this paper, we consider a certain unimodular copy of \( \text{Tes}_n(\mathbf{a}) \) and show that the majority of the faces of this unimodular copy have positive values under a function constructed by Berline-Vergne. As a consequence, we prove that the 3rd and 4th coefficients of the Ehrhart polynomial of \( \text{Tes}_n(1, \ldots, 1) \) are positive for any \( n \). Using the Reduction Theorem by Castillo and the second author, this result generalizes to any deformations of \( \text{Tes}_n(1, \ldots, 1) \) which includes \( \text{Tes}_n(\mathbf{a}) \) for all \( \mathbf{a} \in \mathbb{R}_+^n \). Furthermore, we give a characterization of which flow polytopes are deformations of \( \text{Tes}_n(1, \ldots, 1) \).

1. Introduction

A subset \( P \) of \( \mathbb{R}^n \) is a polyhedron if \( P \) is the intersection of finitely many half spaces, usually defined by linear inequalities. A polytope is a bounded polyhedron. Equivalently, a polytope \( P \subset \mathbb{R}^n \) may be defined as the convex hull of finitely many points in \( \mathbb{R}^n \). We assume that readers are familiar with the basic concepts related to polytopes, such as face and dimension, presented in [2, 27].

A polytope \( P \subset \mathbb{R}^n \) is called integral if all of its vertices are integer points, i.e., points in \( \mathbb{Z}^n \). In 1962, Ehrhart discovered that for any integral polytope \( P \) of dimension \( d \), the function \( E_P \) which maps any non-negative integer \( t \in \mathbb{Z}_{\geq 0} \) to the number of integer points in \( tP \) (the \( t \)-th dilate of \( P \)) is a polynomial in \( t \) of degree \( d \). We call \( E_P \) the Ehrhart polynomial of \( P \). For each \( 1 \leq i \leq d \), let \( e_i(P) \) be the coefficient of \( t^i \) in \( E_P(t) \), so
\[
E_P(t) = e_d(P)t^d + e_{d-1}(P)t^{d-1} + \cdots + e_0(P).
\]
In [12], Ehrhart showed that for any \( d \)-dimensional polytope \( P \), the leading coefficient \( e_d(P) \) of the Ehrhart polynomial \( E_P(t) \) is the normalized volume of \( P \), the second coefficient \( e_{d-1}(P) \) is one half of the sum of the normalized volumes of the facets of \( P \), and the constant term \( e_0(P) \) is 1. Thus, these three coefficients are always positive. However, the remaining coefficients of \( E_P(t) \) are not always positive. We say an integral polytope \( P \) is Ehrhart positive, if all the coefficients of \( E_P(t) \) are positive. (See [17] for a survey on the problem of Ehrhart positivity.)

An affine transformation from an affine space \( A \subset \mathbb{R}^n \) to an affine space \( B \subset \mathbb{R}^m \) is a unimodular transformation if it induces a bijection from integer points in \( A \) to integer points in \( B \). Two polytopes \( P \subset \mathbb{R}^n \) and \( Q \subset \mathbb{R}^m \) are said to be unimodularly equivalent if there exists a unimodular transformation \( \phi \) from the affine hull of \( P \) to the affine hull of \( Q \) such that \( \phi(P) = Q \). Clearly, if two polytopes are unimodularly equivalent, they have the same Ehrhart polynomial.

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1.1. Tesler polytopes and Morale’s conjecture. Our main object of study in this paper is the family of the Tesler polytopes recently introduced and studied by Mészáros, Morales and Rhoads [19]. Let \( \mathbb{U}(n) \) be the set of \( n \times n \) upper triangular matrices. If we ignore the zeros below the diagonal, then \( \mathbb{U}(n) \cong \mathbb{R}^{\binom{n+1}{2}} \) as a vector space. For any \( M = (m_{i,j}) \in \mathbb{U}(n) \), the hook-sum vector of \( M \) is defined to be

\[
hs(M) := (hs_1(M), hs_2(M), \ldots, hs_n(M)),
\]

where the \( k \)-th entry \( hs_k(M) \) (for each \( 1 \leq k \leq n \)) is called the \( k \)-th hook-sum of \( M \) and is computed as the sum of the entries on the \( k \)-th row minus the sum of the entries on the \( k \)-th column above the diagonal element, that is,

\[
hs_k(M) := (m_{k,k} + m_{k,k+1} + \cdots + m_{k,n}) - (m_{1,k} + m_{2,k} + \cdots + m_{k-1,k}).
\]

It is easy to see that \( hs_k \) and \( hs \) are linear functions on \( \mathbb{U}(n) \). For any \( a = (a_1, \ldots, a_n) \in \mathbb{R}_{\geq 0}^n \), the Tesler polytope of hook sum \( a \), denoted by \( \text{Tes}_n(a) \), is the set of all matrices \( M \in \mathbb{U}(n) \) with non-negative entries satisfying the hook sum condition \( hs(M) = a \):

\[
\text{Tes}_n(a) = \{ M = (m_{i,j}) \in \mathbb{U}(n) \mid hs(M) = a, m_{i,j} \geq 0 \}.
\]

When \( a = 1 := (1, \ldots, 1) \), the integer points of \( \text{Tes}_n(a) \) are called Tesler matrices, which were initially introduced by Tesler, and then rediscovered by Haglund in his work of expressing the diagonal Hilbert series as a weighted sum over these matrices [15]. Consequently, Tesler matrices play an important role in the field of diagonal harmonics [1, 13, 14, 16, 25]. Intrigued by these work, Mészáros, Morales and Rhoads [19] defined and studied the Tesler polytopes of hook sum \( a \) as defined above. We remark that their definition for Tesler polytopes only allows to choose \( a \) from \( \mathbb{Z}_{\geq 0}^n \). However, since important features remain the same, we extend the domain of \( a \) to \( \mathbb{R}_{\geq 0}^n \).

The work in the paper was motivated by a conjecture of Morales’:

**Conjecture 1.** ([20]) Tesler polytopes \( \text{Tes}_n(1) \) and \( \text{Tes}_n(1,0,\ldots,0) \) are both Ehrhart positive for any positive integer \( n \).

As we mentioned above, the integer points of \( \text{Tes}_n(1) \) (known as the Tesler matrices) are important objects in the study of diagonal harmonics. Moreover, it was known [9] that \( \text{Tes}_n(1,0,\ldots,0) \) is unimodularly equivalent to the Chan-Robbins-Yuen (CRY) polytope, which is a face of the Birkhoff polytope and whose volume is the product of the first \( n-2 \) Catalan numbers. (The volume formula was conjectured by Chan, Robbins and Yuen was proved by Zeilberger [26]). Note that the Birkhoff polytope is also a well-studied subject, and in particular computing volumes of Birkhoff polytopes is an extremely hard problem and has attracted a lot of recent research [5, 11, 21]. Therefore, both Tes(1) and Tes_n(1,0,\ldots,0) are fascinating families of polytopes for their interesting combinatorial properties and connection to other fields of mathematics.

1.2. McMullen’s formula and \( \alpha \)-positivity. In this paper, we will use a technique developed by Castillo and the 2nd author [6] to attack Morales’ positivity conjecture and its generalizations. The technique is based on the existence of “McMullen’s formula”. In 1975, Danilov questioned, in the context of toric varieties, whether it is possible to construct a
function $\alpha$ such that for any integral polytope $P \subset \mathbb{R}^n$, the following equation holds

$$|P \cap \mathbb{Z}^n| = \sum_{F : \text{face of } P} \alpha(F, P) \text{nVol}(F),$$

where $\text{nVol}(F)$ is the normalized volume of $F$ and $\alpha(F, P)$ only depends on the normal cone of $P$ at $F$ [10]. McMullen was the first to confirm that it is possible to construct such a function $\alpha$ (in a non-constructive way). Hence, we refer to the above formula as McKMullen’s formula [18]. Since $\alpha$ only depends on normal cones and the normal cones are invariant under dilations, we obtain that

$$e_i(P) = \sum_{F : \text{i-dimensional face of } P} \alpha(F, P) \text{nVol}(F), \text{ for any } 1 \leq i \leq \dim(P).$$

One sees that, as a consequence of the above formula, if $\alpha(F, P) > 0$ for every $i$-dimensional face of $P$, then $e_i(P)$ is positive by the above formula. Moreover, if $\alpha(F, P) > 0$ for every face of $P$, then $P$ is Ehrhart positive. We say a polytope $P$ is $\alpha$-positive if $\alpha(F, P) > 0$ for every face $F$ of $P$. Currently, we know three different constructions for $\alpha$ given by Pommershein and Thomas [22], by Berline and Vergne [4], by Ring and Schürmann [24]. Hence, the choice of the function $\alpha$ for McMullen’s formula is not unique, and thus a specific construction of $\alpha$ has to be chosen before one discusses $\alpha$-positivity. Among known constructions of the function $\alpha$, we will use Berline-Vergne’s construction which we will refer to as the $BV$-$\alpha$ function denoted by $\alpha^{BV}$. Based on discussion above, one sees that Ehrhart positivity can be studied through $\alpha$-positivity or more specifically through $BV$-$\alpha$-positivity, which is the approach we will take. To simplify the calculation of $BV$-$\alpha$ values, we work on the projected Tesler polytopes denoted by $\text{PTes}_n(\alpha)$ which is unimodularly equivalent to $\text{Tes}_n(\alpha)$, and thus has the same Ehrhart polynomial (see [3.1] for the definition of $\text{PTes}_n(\alpha)$). Therefore, if $\text{PTes}_n(\alpha)$ is $BV$-$\alpha$ positive then $\text{Tes}_n(\alpha)$ is Ehrhart positive. We have the following theorem:

**Theorem 1.1.** For all positive integer $n$, all the codimension 2 and 3 faces of $\text{PTes}_n(1)$ have positive $BV$-$\alpha$ values. Therefore, $e_{d-2}(\text{Tes}_n(1))$ and $e_{d-3}(\text{Tes}_n(1))$ are positive, where $d = \binom{n}{2}$ is the dimension of $\text{PTes}_n(1)$.

1.3. Deformations. We say a polytope $Q$ is a deformation of a polytope $P$ if the normal fan of $P$ is a refinement of the normal fan of $Q$. It follows from the Reduction Theorem (Theorem 2.14) by Castillo and the second author that if all the $k$-dimensional faces of polytope $P$ are $BV$-$\alpha$ positive, then the same is true for any deformations of $P$. Therefore, we can generalize our $BV$-$\alpha$ positivity results for codimension 2 and 3 faces of $\text{PTes}_n(1)$ to any deformations of $\text{PTes}_n(1)$:

**Corollary 1.2.** Let $P$ be a $d$-dimensional integral polytope that is a deformation of $\text{PTes}_n(1)$ and let $E_P(t) = e_d(P)t^d + e_{d-1}t^{d-1} + \cdots + 1$ be its Ehrhart polynomial. Then, the followings are true:

1. If $d = \dim(\text{Tes}_n(1))$, then all the codimension 2 and 3 faces of $P$ have positive $BV$-$\alpha$ values. Therefore, $e_{d-2}(P) > 0$ and $e_{d-3}(P) > 0$.
2. If $d = \dim(\text{Tes}_n(1)) - 1$, then all the codimension 2 faces of $P$ have positive $BV$-$\alpha$ values. Therefore, $e_{d-2}(P) > 0$.

Since any invertible affine transformation preserves deformation (Lemma 3.12), we directly obtain:
Corollary 1.3. Let $P$ be any polytope that is a deformation of $\text{Tes}_n(1)$

1. If $\dim(P) = \dim(\text{Tes}_n(1))$, then $e_{d-2}(P) > 0$ and $e_{d-3}(P) > 0$.

2. If $\dim(P) = \dim(\text{Tes}_n(1)) - 1$, then $e_{d-2}(P) > 0$.

It is known that $\text{Tes}_n(a)$ is a deformation of $\text{Tes}_n(1)$ for all $a \in \mathbb{R}^n_{\geq 0}$. Therefore, the positivity result for $\text{Tes}_n(1)$ generalizes to $\text{Tes}_n(a)$ for all $a \in \mathbb{R}^n_{\geq 0}$ including $\text{Tes}_n(1,0,\ldots,0)$, which is the other polytope in Conjecture 1.

Given the results in Corollary 1.3 it is natural to ask which polytopes are deformations of $\text{Tes}_n(1)$. In [19], Mészáros et al showed that for any $a \in \mathbb{Z}^n_{\geq 0}$, the Tesler polytope $\text{Tes}_n(a)$ is unimodularly equivalent to a certain flow polytope (See [11] and references therein). We gave the following characterization of which flow polytopes on a complete graph are deformations of $\text{Tes}_n(1)$:

Theorem 1.4. For any fixed $n$, let $a = (a_1,\ldots,a_n) \in \mathbb{R}^n$ such that Flow$_n(a)$ is non-empty. Suppose $a_1$ is the first positive entry. Then, Flow$_n(a)$ is a deformation of $\text{Tes}_n(1)$ if and only if $a_{t+2},a_{t+3},\ldots,a_n \geq 0$.

Organization of the paper

In section 2 we provide background on polyhedra theory and Tesler polytopes, and a brief description for the construction of BV-$\alpha$ functions. In section 3 we define projected Tesler polytopes and develop a simple method (see section 3.3) that helps to calculate the BV-$\alpha$ values arising from totally unimodular polytopes. Applying this method to $P\text{Tes}_n(1)$, we obtain a proof for Theorem 1.1 by explicitly calculating the $\alpha$-values (Lemmas 3.9 and 3.10). Then, we prove Corollaries 1.2 and 1.3 using the Reduction Theorem (Theorem 2.14). In section 4 we first give a different proof of the known fact that $\text{Tes}_n(a)$ are deformations of $\text{Tes}_n(1)$, and then show that the family of Tesler polytopes already contains all of its deformations up to translations. We finish with a proof for Theorem 1.4.

2. Preliminaries

In this section, we review terminologies related to polytopes/polyhedra that are important to this article.

2.1. Basic definitions in polyhedra theory. In this section, we assume that $V$ is a subspace of $\mathbb{R}^n$, and $\langle \cdot, \cdot \rangle$ is the dot product on $\mathbb{R}^n$. Let $P$ be a polytope in $V \subset \mathbb{R}^n$. We denote by Vert($P$) the set of all vertices of $P$. We say $P$ is a simple polytope, if each of the vertices of $P$ are contained in exactly $\dim(P)$ many facets or equivalently contained in $\dim(P)$ many edges. If a facet $F$ of $P$ contains a face $G$, we call $F$ a supporting facet of $G$. Two vertices $v$ and $w$ of $P$ are said to be adjacent if they are connected by an edge. For an edge $e$ of $P$ connecting two vertices $v$ and $w$, we define the edge direction from $v$ to $w$ to be $d_{v,w} := w - v$. When $P$ is an integral polytope, we define the primitive edge direction from $v$ to $w$ to be $p_{v,w} := a(w - v)$ where $a \in \mathbb{Q}_{>0}$ is chosen so that $p$ is a primitive vector. (Recall that an integer vector $m$ is primitive if the greatest common divisor of its components is 1.) A polyhedron defined by homogeneous linear inequalities is called a (polyhedral) cone. An equivalent definition of a cone is the set of all conic combinations of finitely many points in $V$. A pointed cone is a cone that does not contain a line.
Definition 2.1. Let $K$ be a cone in $\mathbb{R}^n$ that is generated by the primitive vectors $d_1, \ldots, d_l$. We say $K$ is unimodular if \{d_1, \ldots, d_l\} forms a basis of the lattice $\mathbb{Z}^n \cap \text{span}(d_1, \ldots, d_l)$.

Definition 2.2. We say a polytope $P$ in $\mathbb{R}^n$ is totally unimodular if $\text{fcone}(P, v)$ is unimodular for all the vertices $v$ of $P$.

For any subset $S$ of a vector space $V$, let $\text{lin}(S)$ denote the translation of the affine span of $S$ to the origin, and let $V/\text{lin}(S)$ denote the orthogonal complement of $\text{lin}(S)$. For any subset or an element $B$ of $V$, we use $B/\text{lin}(S)$ to denote the canonical projection of $B$ into $V/\text{lin}(S)$.

Definition 2.3. Suppose $P \subseteq V$ is a non-empty polyhedron, and $F$ is a face of $P$.

1. The feasible cone of $P$ at $F$ is:

$$\text{fcone}(F, P) := \{ y \in V : x + \epsilon y \in P \text{ for some } \epsilon > 0 \},$$

where $x$ is any interior point of $F$. (It can be shown that the definition does not depend on the choice of $x$.) The pointed feasible cone of $P$ at $F$ is

$$\text{fcone}^P(F, P) := \text{fcone}(F, P)/\text{lin}(F).$$

2. Given any face $F$ of $P$, the normal cone of $P$ at $F$ with respect to $V$ is:

$$\text{ncone}_V(F, P) := \{ u \in V : \langle u, p_1 \rangle \geq \langle u, p_2 \rangle, \forall p_1 \in F, \forall p_2 \in P \}.$$

The normal fan $\Sigma_P$ of $P$ with respect to $V$ is the collection of all normal cones of $P$. (We omit the subscript $V$ and just write $\text{ncone}(F, P)$ and $\Sigma P$, if $V = \mathbb{R}^n$ and $P$ is full dimensional.)

3. Let $K \subseteq V$ be a cone, and let $W$ be the subspace of $V$ spanned by $K$. The polar cone of $K$ is the cone

$$K^\circ = \{ y \in W : \langle x, y \rangle \leq 0, \forall x \in K \}.$$

Lemma 2.4. [27] The normal fans and the normal cones have the following properties:

1. For any $C \in \Sigma_P$, any face of $C$ is also in $\Sigma_P$.

2. There is an inclusion reversing bijection between the face lattice of $P$ (the set of all the faces of $P$ ordered by inclusion) and $\Sigma_P$ given by $F \mapsto \text{ncone}_V(F, P)$.

3. If $P$ is a full dimensional polytope, $\dim(\text{ncone}(F, P)) = \dim(V) - \dim(F)$ and $\text{ncone}(F, P)$ is pointed for any face $F$ of $P$.

4. $\text{ncone}_V(F, P)^\circ = \text{fcone}^P(F, P)$.

2.2. Prior work on Tesler polytopes by Mészáros, Morales and Rhoads. In this part, we review definitions and results related to Tesler polytope given in [19] that are relevant to this paper.

For any positive integer $n$ and $a \in \mathbb{Z}_{\geq 0}^n$, Mészáros, Morales and Rhoads [19] gave the characterization for the face poset of $\text{Tes}_n(a)$ using the concept of support. Their characterization can be easily generalized to any $a \in \mathbb{R}_{\geq 0}^n$ by the same proof.

Definition 2.5. For any set $\mathcal{A}$ of matrices in $\mathbb{U}(n)$, define the support of $\mathcal{A}$, denoted by $\text{supp}(\mathcal{A})$, to be the matrix $(s_{i,j}) \in \mathbb{U}(n)$,

$$s_{i,j} = \begin{cases} 0, & \text{if the } (i,j)\text{-th entry of every matrix in } \mathcal{A} \text{ is zero,} \\ 1, & \text{otherwise.} \end{cases}$$
For $\mathcal{A} = \{m\}$, containing one matrix $m$, write $\text{supp}(m)$ instead of $\text{supp}(\mathcal{A})$ whenever it is convenient to do so. Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathbb{U}(n)$. We write $\text{supp}(\mathcal{A}) \leq \text{supp}(\mathcal{B})$, if $\text{supp}(\mathcal{A}) = (a_{i,j})$ and $\text{supp}(\mathcal{B}) = (b_{i,j})$ satisfying $a_{i,j} \leq b_{i,j}$ for any $1 \leq i \leq j \leq n$.

For $1 \leq i \leq j \leq n$, let
\begin{equation}
H_{i,j}^n := \{ M = (m_{i,k}) \in \mathbb{U}(n) \mid m_{i,k} = 0 \}.
\end{equation}

We say the intersection of a collection of coordinate planes does not make any zero rows if the intersection is not contained in $H_{i,i}^n \cap H_{i,i+1}^n \cap \cdots \cap H_{i,n}^n$ for any $1 \leq i \leq n$.

**Theorem 2.6** (Lemma 2.4 and Theorem 2.5 in [19]). For any positive integer $n$, the followings are true:

1. Let $a \in \mathbb{R}_{\geq 0}^n$. If $F$ is a codimension $k$ face of $\text{Tes}_n(a)$, then $F$ is of the form $\text{Tes}_n(a) \cap H_{i_1,j_1}^n \cap \cdots \cap H_{i_k,j_k}^n$, where $H_{i_1,j_1}^n \cap \cdots \cap H_{i_k,j_k}^n$ does not make any zero rows. Conversely, any such form is a codimension $k$ face of $\text{Tes}_n(a)$.

2. Let $a \in \mathbb{R}_{\geq 0}^n$. $v \in \text{Tes}_n(a)$ is a vertex if and only if $\text{supp}(v)$ has at most one 1 on each row. In particular, when $a \in \mathbb{R}_{\geq 0}^n$, $v \in \text{Tes}_n(a)$ is a vertex if and only if each row of $v$ has exactly one 1.

Using the above theorem, one can deduce the following when $a \in \mathbb{R}_{\geq 0}^n$. In fact, it is shown to be true for any $a \in \mathbb{R}_{\geq 0}^n$.

**Lemma 2.7** (Theorem 2.5 in [19]). Let $a \in \mathbb{R}_{\geq 0}^n$, and let $F,F'$ be two faces of $\text{Tes}_n(a)$. Then the followings are true:

1. $F = F'$ if and only if $\text{supp}(F) = \text{supp}(F')$.

2. $F \subseteq F'$ if and only if $\text{supp}(F) \leq \text{supp}(F')$.

We also need the following characterization for adjacent vertices of $\text{Tes}_n(a)$ and a specific type of vertex presented in the next two lemmas.

**Lemma 2.8** (Theorem 2.7 in [19]). Let $a \in \mathbb{R}_{\geq 0}^n$. Two vertices $v$ and $w$ of $\text{Tes}_n(a)$ are adjacent if and only if for every $1 \leq k \leq n$, the $k$-th row of $\text{supp}(w)$ can be obtained from the $k$-th of $\text{supp}(v)$ by one of the following operations:

1. Leaving the $k$-th row of $\text{supp}(v)$ unchanged.

2. Changing the unique 1 in $k$-th row of $\text{supp}(v)$ to 0.

3. Changing a single 0 in the $k$-th row to a 1 (if $k$-th row of $\text{supp}(v)$ is a zero row).

4. Moving the unique 1 in the $k$-th row of $\text{supp}(v)$ to a different position in the $k$-th row (this operation must take place in precisely one row of $\text{supp}(v)$).

In particular, when $a \in \mathbb{R}_{\geq 0}^n$, two vertices $v,w$ are adjacent to each other if and only if $\text{supp}(w)$ can be obtained from $\text{supp}(v)$ by moving 1 on a row to a different place on the same row.

**Lemma 2.9** (Lemma 2.4 of [19]). Let $a = (a_1, \ldots, a_n) \in \mathbb{R}_{\geq 0}^n$. The $k$-th row of a vertex $v$ of $\text{Tes}_n(a)$ is a zero row if and only if $a_k = 0$ and the entries of $k$-th column of $v$ above the diagonal (excluding the diagonal entry) are all zero.
Example 2.10. Let \( \alpha = (2, 2, 3, 4) \), and
\[
\mathbf{v} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 4 \\ 3 & 0 & 8 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 2 \\ 5 & 0 & 6 \end{pmatrix}
\]
Then
\[
\text{supp}(\mathbf{v}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \text{supp}(\mathbf{w}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 \end{pmatrix}
\]
Using Theorem 2.6 (1), one can easily check that \( \mathbf{v} \) and \( \mathbf{w} \) are vertices of \( \text{Tes}_4(2, 2, 3, 4) \). Also, they are adjacent vertices, because \( \text{supp}(\mathbf{w}) \) can be obtained by moving 1 on the second row of \( \text{supp}(\mathbf{v}) \) to the right.

2.3. Berline-Vergne’s construction. In this subsection, we provide more details about BV-\( \alpha \) function mentioned in the introduction. In [11], Berline and Vergne associated to every rational affine cone \( c \) an analytic function \( \phi(c) \) on \( \mathbb{R}^n \) which is recursively defined with respect to the dimension of cones. They showed that \( \phi \) is a valuation and \( \phi(c) \) is analytic near the origin. Then they set \( \alpha^{BV}(F, P) \) to be the residue of \( \phi(\text{fcone}^p(F, P)) \) around 0.

One sees that this procedure is complicated. However, when \( \text{fcone}^p(F, P) \) is unimodular, we can obtain a simple formula for \( \alpha^{BV}(F, P) \) when \( \dim(\text{fcone}^p(F, P)) \) is up to 3. Note that it follows immediately from Berline-Vergne’s construction that \( \alpha^{BV}(F, P) = 1 \) when \( \dim(\text{fcone}^p(F, P)) = 0 \), and \( \alpha^{BV}(F, P) = \frac{1}{2} \) when \( \dim(\text{fcone}^p(F, P)) = 1 \) ([2 Example 19.2]). We include formulas for faces of codimensions 2 and 3 below.

Lemma 2.11 (Example 19.3 of [2]). Let \( F \) be a codimension 2 face of a full dimensional polytope \( P \) and \( C = \text{fcone}^p(F, P) = \text{cone}(u_1, u_2) \) where \( u_1 \) and \( u_2 \) form a basis for the orthogonal projection of \( \mathbb{Z}^n \) to \( \mathbb{R}^n / \text{lin}(F) \). Then,
\[
\alpha^{BV}(F, P) = \frac{1}{4} + \frac{1}{12} \left( \frac{\langle u_1, u_2 \rangle}{\langle u_1, u_1 \rangle} + \frac{\langle u_1, u_2 \rangle}{\langle u_2, u_2 \rangle} \right).
\]

Lemma 2.12 (Lemma 3.10 of [6]). Let \( F \) be a codimension 3 face of a full dimensional polytope \( P \) and \( C = \text{fcone}^p(F, P) = \text{cone}(u_1, u_2, u_3) \). Where \( u_1 \) and \( u_2 \) form a basis for the orthogonal projection of \( \mathbb{Z}^n \) to \( \mathbb{R}^n / \text{lin}(F) \). Then,
\[
\alpha^{BV}(F, P) = \frac{1}{8} + \frac{1}{24} \left( \frac{\langle u_1, u_2 \rangle}{\langle u_1, u_1 \rangle} + \frac{\langle u_1, u_2 \rangle}{\langle u_2, u_2 \rangle} + \frac{\langle u_1, u_3 \rangle}{\langle u_1, u_1 \rangle} + \frac{\langle u_1, u_3 \rangle}{\langle u_3, u_3 \rangle} + \frac{\langle u_2, u_3 \rangle}{\langle u_2, u_2 \rangle} + \frac{\langle u_2, u_3 \rangle}{\langle u_3, u_3 \rangle} \right).
\]

Notice that the formulas are in terms of the dot products between the primitive generators of \( \text{fcone}^p(F, P) \).

Remark 2.13. If \( \dim(\text{fcone}^p(F, P)) \geq 4 \), the formula for \( \alpha^{BV}(F, P) \) becomes way more complicated. However, they are still in terms of \( \langle u_i, u_j \rangle \). Therefore, it is important to know the values of \( \langle u_i, u_j \rangle \). In section 3.3 we develop a simple procedure of calculating the dot product values without having to calculate each \( u_i \)'s (calculation of \( u_i \) is not easy because as we project \( \text{fcone}(F, P) \) to a subspace, the lattice structure changes).

It is also possible to obtain the formula even when \( \text{fcone}^p(F, P) \) is not unimodular by decomposing it in to unimodular cones and using the valuation property of \( \Psi \).
benefit of using BV-α is the Reduction Theorem which follows from the valuation property of ψ:

**Theorem 2.14** (Reduction Theorem). Suppose $P$ and $Q$ are two polytopes in $\mathbb{R}^n$. Assume further that $Q$ is a deformation of $P$. Then for any fixed $k$, if $\alpha^{BV}(F,P) > 0$ for every $k$-dimensional face $F$ of $P$, then $\alpha^{BV}(G,Q) > 0$ for every $k$-dimensional face $G$ of $Q$. Therefore, BV-α-positivity of $P$ implies BV-α-positivity of $Q$.

By the above theorem, if we show that all the $k$-th dimensional faces of a polytope $P$ is BV-α positive (which together with (1.1) implies that the corresponding Ehrhart coefficient is positive), then the same is true for any deformation $Q$ of $P$.

3. Positivity of the (projected) Tesler polytope of hook sums $(1, \ldots, 1)$

We prefer working with full dimensional polytopes because facet normal vectors are uniquely determined. One can easily see that Tesler polytopes are not full dimensional in the ambient space, but it is full dimensional in the subspace defined by the hook sum conditions. In this section, we will consider suitable projections that induces unimodular in the ambient space, but it is full dimensional in the subspace defined by the hook sum conditions. For any $a = (a_1, \ldots, a_n) \in \mathbb{R}_{\geq 0}$, let $a$-hook sum space be the subspace of $U(n)$ defined by hook sum conditions:

$$H_n(a) := \{ M \in U(n) \mid hs_k(M) = a_k \text{ for } 1 \leq k \leq n \}.$$  

3.1. **Projection on the Tesler polytopes.** Recall $[n] = \{1, 2, \ldots, n\}$ and $[k,n] = \{k, k+1, \ldots, n\}$.

**Theorem 3.1.** For any $a \in \mathbb{Z}_{\geq 0}^n$ and $j = (j_1, j_2, \ldots, j_{n-1}, j_n) \in [n] \times [n-1] \times [n-1] \times \cdots \times [1]$, define $\psi_j : U(n) \rightarrow U(n-1)$ as

$$\psi_j((x_{i,j})) = (y_{i,j}) \text{ where } y_{i,j} = \begin{cases} x_{i,j} & \text{if } j < j_i, \\ x_{i,j+1} & \text{if } j \geq j_i, \end{cases}$$

($\psi_j$ deletes an entry in a row for each rows where $j$ determines which entries to be deleted). Then $\psi|_{H_n(a)}$ is a unimodular transformation from $H_n(a)$ to $U(n-1)$.

**Proof.** One can easily see that $\psi_j$ is a projection. Thus, it is enough to show that $\psi|_{H_n(a)}$ induces a bijection from the integer points of $H_n(a)$ to that of $U(n-1)$. Let $z = (z_{i,j}) \in U(n-1)$. If $z' = (z'_{i,j}) \in U(n)$ such that $\psi_j(z') = z$, then

$$z'_{i,j} = \begin{cases} z_{i,j} & \text{if } j < j_i, \\ z_{i,j-1} & \text{if } j > j_i \text{ for some } x_{i,j} \in \mathbb{R}, \\ x_{i,j} & \text{if } j = j_i, \end{cases}$$
Since \( z' \in H_n(a) \), one sees that \( z' \) has to satisfy the hook sum conditions \( hs_i(z') = a_i \) for any \( 1 \leq i \leq n \). The first hook sum condition uniquely determines \( x_{1,j} \) in terms of \( z_{i,j} \)'s and \( a_1 \). Since the entries of the first row of \( z' \) are now all in terms of \( z_{i,j} \)'s and \( a_1 \), the second hook sum condition uniquely determines \( x_{2,j} \) in terms of \( z_{i,j} \)'s, \( a_1 \) and \( a_2 \). As we continue this process inductively (row by row), all the \( x_{i,j} \)'s are expressed uniquely in terms of \( z_{i,j} \)'s and \( a_i \)'s. The injectivity and the surjectivity follows from the above construction of \( z' \). Additionally, one sees that if \( z \) was chosen to be an integer point, \( z' \) is also an integer point by the construction of \( z' \). Thus, \( \psi \) is a bijection from \( H_n(a) \) to \( \mathbb{U}(n-1) \) that induces a bijection from integer points of \( H_n(a) \) to integer points of \( \mathbb{U}(n-1) \).

**Example 3.2.** The most natural way to obtain such map would be letting \( j = (1, 2, \ldots, n) \).

Let \( \psi_{\text{diag}} \) be the map in the above example. We call \( \psi_{\text{diag}}(\text{Tes}_n(a)) \) the *projected Tesler polytope of hook sum \( a \) and denote it by \( \text{PTes}_n(a) \).

### 3.2. Faces and the facet normal vectors of the projected Tesler polytopes.

To calculate the BV-\( \alpha \) values of the faces of \( \text{PTes}_n(1) \), we first find the facet normal vectors of \( \text{PTes}_n(1) \).

**Definition 3.3.** For any \( n \),

1. Let \( e_{i,j} \) be the matrix in \( U(n) \) that the entries are all zero except \((i, j)\)-th entry is 1.
2. For \( 1 \leq k \leq n \), define the shifted \( k \)-th hook sum matrix to be

\[
sm_k^n := \sum_{j=k}^{n} e_{k,j}^n - \sum_{i=1}^{k-1} e_{i,k-1}^n \in \mathbb{U}(n).
\]

3. Let \( M = (m_{i,j}) \in \mathbb{U}(n) \) For \( 1 \leq k \leq n \) define shifted \( k \)-th hook sum of \( M \) to be

\[
sh_k^n(M) := (m_{k,k} + m_{k,k+1} + \ldots + m_{k,n}) - (m_{1,k-1} + m_{2,k-1} + \ldots + m_{k-1,k-1}).
\]

We fix a notation for the facets of \( \text{Tes}_n(a) \) and \( \text{PTes}_n(a) \). From now on, for any \( a \in \mathbb{Z}_{>0}^n \), \( 1 \leq i < j \leq n \), we let

\[(3.1) \quad F_{i,j}^\prime(a) := \text{Tes}_n(a) \cap H^n_{i,j} \text{ and } F_{i,j}(a) := \phi_d(\text{Tes}_n(a) \cap H^n_{i,j}).\]

where \( H^n_{i,j} \) is the coordinate plane defined in (2.1) (by Theorem 2.6 (1), any facet of \( \text{Tes}_n(a) \) can be expressed as \( \text{Tes}_n(a) \cap H^n_{i,j} \) and since \( \phi \) is a unimodular transformation, it bijects the facets of \( \text{Tes}_n(a) \) to that of \( \text{PTes}_n(a) \)). We would like to give an example of obtaining the half-space description of \( \text{PTes}_n(a) \).

**Example 3.4.** Let \( a = (4, 5, 2, 2) \). Then

\[
\text{Tes}_4(a) = \left\{ x = (x_{i,j}) \in \mathbb{U}(4) \mid hs_1(x) = 4, \hspace{1em} hs_2(x) = 5, \hspace{1em} hs_3(x) = 2, \hspace{1em} hs_4(x) = 2, \hspace{1em} x_{i,j} \geq 0 \text{ for all } 1 \leq i \leq j \leq 4 \right\}.
\]

We use the following notation to describe the image of \( \psi_{\text{diag}} \) as in Theorem 3.1

\[
\psi_{\text{diag}}(x) = y \in \mathbb{U}(3) \text{ where } y_{i,j} = x_{i,j+1}.
\]
After going through \( \psi_{\text{diag}} \), the diagonal components gets erased. Since all the components are non-negative, the hook-sum conditions become inequalities.

\[
hs_1(x) = x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 4 \implies sh_1(y) = y_{1,1} + y_{1,2} + y_{1,3} \leq 4.
\]
\[
hs_2(x) = x_{2,2} + x_{2,3} + x_{2,4} - x_{1,2} = 5 \implies sh_2(y) = y_{2,2} + y_{2,3} - y_{1,1} \leq 5.
\]
\[
hs_3(x) = x_{3,3} + x_{3,4} - x_{1,3} - x_{2,3} = 2 \implies sh_3(y) = y_{3,3} - y_{1,2} - y_{2,2} \leq 2.
\]
\[
hs_4(x) = x_{4,4} - x_{1,4} - x_{2,4} - x_{3,4} = 2 \implies sh_4(y) = -y_{1,1} - y_{2,2} + y_{3,3} \leq 2.
\]

The last inequality above is redundant, because \(-y_{1,3} - y_{2,3} - y_{3,3} \leq 0\). As a result, we get the description for \( \text{PTes}_4(a) \):

\[
\text{PTes}_4(a) = \left\{ y = (y_{i,j}) \in U(3) \bigg| \begin{array}{c}
sh_1(y) \leq 4, 
sh_2(y) \leq 5, 
sh_3(y) \leq 2, 
y_{i,j} \geq 0 \text{ for all } 1 \leq i \leq j \leq 3
\end{array} \right\}.
\]

One can easily check that all the defining inequalities above are necessary. Thus, the outer normal vectors for the facets are \( sm^1_1, sm^2_2, sm^3_3 \) and \(-e_{1,j}\) for all \( 1 \leq i \leq j \leq 3 \).

It is easy to see that the above method of obtaining the following description of \( \text{PTes}_n(a) \) works for any \( a \in \mathbb{Z}_{>0} \). So, we state the following lemma without the proof.

**Lemma 3.5.** For any \( n \in \mathbb{Z}_{>0} \) and \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{>0} \). Let \( n_{i,j} = \begin{cases} -e^{n-1}_{i,j-1} & \text{if } i < j, \\ sm^{n-1}_k & \text{if } i = j = k. \end{cases} \) Then, \( n_{i,j} \) is an outer facet normal vector of \( F_{i,j} \).

### 3.3. Computing BV-\( \alpha \) values of totally unimodular polytopes.

If an integral polytope \( P \) in \( \mathbb{R}^n \) is totally unimodular (Definition 2.2), then for any face \( F \) of \( P \), the primitive generators of \( \text{fcone}^p(F, P) \) forms a basis for \( \mathbb{Z}^n / \text{lin}(F) \). Thus, we can apply any known BV-\( \alpha \) formula to calculate the BV-\( \alpha \) value of the faces of \( P \). In this section we will apply Lemma 2.11 and Lemma 2.12 to calculate the BV-\( \alpha \) values of all the codimension 2 and 3 faces of \( \text{PTes}_n(1) \). We will present a simple way to obtain the inner product values \( \langle u_i, u_j \rangle \) in Lemma 2.11 and Lemma 2.12 which is important for computing BV-\( \alpha \) values (recall Remark 2.13). We start with the following lemma:

**Lemma 3.6.** Let \( \{n_1, \ldots, n_{k+1}\} \) be the set of linearly independent primitive vectors in \( \mathbb{R}^n \) where the matrix \( A \) formed by taking \( \{n_1, \ldots, n_{k+1}\} \) as rows is totally unimodular. Then, for any \( c = (c_1, \ldots, c_n) \in \mathbb{Z}^n \) there exists an integer solution for the equation \( AX = \cdot c \) where \( X = (x_1, \ldots, x_n) \).

**Lemma 3.7.** Let \( P \) be an integral polytope in \( \mathbb{R}^n \) that is totally unimodular. Let \( S \) be a subspace of \( \mathbb{R}^n \) that is the translation of the affine subspace that \( P \) is full dimensional in. Let \( F \) be a codimension \( k \) face of \( P \) that is the intersection of the facets \( F_1, \ldots, F_k \) and let \( n_i \) be the primitive outer normal vector of \( F_i \) that is in \( S \). Fix a vertex \( v \) of \( F \), let \( v_i \) be a vertex of \( P \) that is adjacent to \( v \) and not in \( F_i \), and \( d_i \) be the primitive edge direction from \( v \) to \( v_i \). If we let \( u_i := d_i / \text{lin}(F_i) \), then the followings are true:

1. \( \langle d_i, n_i \rangle = -1 \).
2. \( \langle u_i, n_i \rangle = -1 \).
3. \( \langle u_i, n_j \rangle = 0 \) for all \( j \neq i \).
4. \( \{u_1, \ldots, u_k\} \) generates \( \text{fcone}^p(F, P) \).
Proof. For (1), let \( H(C) := \{ x \in S \mid \langle n_i, x \rangle = C \} \) be the hyperplane in \( S \) that \( F_i \) is in. Since \( P \) is simple, all the vertices \( v_i \) where \( l \neq i \) are in \( F_i \) which means that the end points of the directions \( d_i \) where \( l \neq i \) are in \( H(0) \). Since \( P \) is totally unimodular, the integer span of \( \{ d_l \mid l \neq i \} \) is precisely all the integer points in \( H(0) \). Also, by Lemma 3.6, there exists an integer point in \( H(-1) \). Therefore, the end point of \( d_i \) should be in \( H(-1) \) (otherwise, \( \{ d_1, \ldots, d_k \} \) doesn’t span \( \mathbb{Z}^n \cap S \)). For (2), since \( u_i \) is a projection of \( d_i \) in to \( \text{lin}(F)^\perp \), \( u_i - d_i \in \text{lin}(F) \). Thus, \( \langle u_i - d_i, n_i \rangle = 0 \) or equivalently, \( \langle u_i, n_i \rangle = -1 \) by (1). For (3), since \( P \) is simple, \( v_i \) is in \( F_j \) for all \( j \neq i \). Thus, \( \langle d_i, n_j \rangle = 0 \) for all \( j \neq i \) which together with the fact that \( \langle u_i - d_i, n_j \rangle = 0 \) for all \( 1 \leq j \leq k \) implies (3). The part (4) follows from Lemma 2.4 (4) and (3).

Since \( u_i \) is in \( \text{lin}(F)^\perp \), \( u_i = a_{i,1}n_1 + \cdots + a_{i,k}n_k \) for some real numbers \( a_{i,1}, \ldots, a_{i,k} \). Let \( c_{i,j} = \langle n_i, n_j \rangle \). We let \( C \) to be the \( n \times n \) matrix with \( (i, j) \)-th entry as \( c_{i,j} \) and \( D \) to be the \( n \times n \) matrix with \( (i, j) \)-th entry as \( a_{i,j} \).

Then we can express the the parts (2) and (3) from the above lemma as,

\[
CD = -I.
\]

Also, if we let \( a_i \) to be the \( i \)-th column vector of \( D \), we have,

\[
\langle u_i, u_j \rangle = a_i^tCa_j = -a_{j,i}.
\]

Let \( M = (m_{i,j}) \) be the inverse matrix of \( C \). Then, (5.1) and (5.2) implies that

\[
\langle u_i, u_j \rangle = m_{i,j}.
\]

3.4. BV-\( \alpha \) on \( \text{PTes}_n(a) \). We have seen in Lemma 3.5 (2) that when \( a \in \mathbb{Z}^n_{\geq 0} \), any face of \( \text{PTes}_n(a) \) that is the intersection of facets \( F_{i,j}'s \) (refer to the equation (4.2)) with \( i < j \) has the facet normal vector as a member of the standard basis up to sign. In this case, we get the BV-\( \alpha \) value for free by Example 3.15 of [7].

Lemma 3.8. Let \( a \in \mathbb{Z}^n_{\geq 0} \) and \( F \) be a codimension \( k \) face of \( \text{PTes}_n(a) \) that is an intersection of the facets \( F_{i_1,j_1}, \ldots, F_{i_k,j_k} \) where \( i_1 < j_1, \ldots, i_k < j_k \). Then, for every \( 1 \leq k \leq n \),

\[
\alpha_{BV}(F) = \frac{1}{2^k}.
\]

By Theorem 2.7 in [19], \( \text{Tes}_n(a) \) is simple when \( a \in \mathbb{Z}^n_{>0} \). Also, one can check that the defining matrix for \( \text{Tes}_n(a) \) is totally unimodular. Thus, \( \text{Tes}_n(a) \) is totally unimodular for any \( a \in \mathbb{Z}^n_{\geq 0} \). Since \( \text{Tes}_n(a) \) and \( \text{PTes}_n(a) \) are unimodularly equivalent, \( \text{PTes}_n(a) \) is also totally unimodular. Therefore, we can apply the procedure described in the previous subsection and use Lemmas 2.11 and 2.12 to calculate the BV-\( \alpha \) values of codimension 2 and codimension 3 faces of \( \text{PTes}_n(1) \).

Since we already have Lemma 3.8, we only consider the case where at least one of the supporting facet for the face of consideration has form \( F_{i,l} \). We calculate the BV-\( \alpha \) values of the faces of \( \text{PTes}_n(1) \) case by case. We will use the following terminologies to describe the cases.

We say that a position \( (i, j) \) of a matrix is on the \( l \)-th hook if either \( i = l \) or \( j = l \). More specifically, we say \( (i, j) \) is on the row of the \( l \)-th hook if \( i = l \) and we say \( (i, j) \) is on the column of the \( l \)-th hook if \( j = l \).
Lemma 3.9. Let $F$ be a codimension 2 face of $\text{PTes}_n(1)$ which is the intersection of facets $F_{i,l}$ and $F_{i,j}$. We consider all possible cases listed below.

1. Suppose only one of the facet normal vectors is a shifted hook sum matrix (i.e., $i \neq j$). Then we have the following:
   
   (i) If $(i, j)$ is on the row of the $l$-th hook, then $\alpha^{\text{BV}}(F, \text{PTes}_n(1)) = \frac{1}{4} + \frac{1}{12} \left( \frac{n}{n-1} \right)$.
   
   (ii) If $(i, j)$ is on the column of the $l$-th hook, then $\alpha^{\text{BV}}(F, \text{PTes}_n(1)) = \frac{1}{6} - \frac{1}{12(n-1)}$.
   
   (iii) If $(i, j)$ is not on the $l$-th hook, then $\alpha^{\text{BV}}(F, \text{PTes}_n(1)) = \frac{1}{4}$.

2. Suppose both of the facet normal vectors are shifted hook sum matrices (i.e., $i = j$).

Then $\alpha^{\text{BV}}(F, \text{PTes}_n(1)) = \frac{1}{4} - \frac{1}{36(n-1)}$.

Hence, all the values are positive for all $n \geq 3$.

Proof. We only provide a proof for case (1)/(i), but one can obtain the BV-$\alpha$ values for the other cases by following the same procedure. For case (1)/(i), by Lemma 3.5, the facet normal vectors are $sm_i^{n-1}$ and $-e_i^{n-1}$. Thus, the matrix $C$ in (3.2) is $\left(\begin{array}{cc} n-1 & -1 \\ -1 & 1 \end{array}\right)$ and the inverse matrix $M$ of $C$ is $\left(\begin{array}{cc} 1 & 1 \\ n-2 & n-1 \\ n-2 & n-2 \end{array}\right)$. Therefore, by (3.4),

$$\langle u_1, u_1 \rangle = \frac{1}{n-2}, \langle u_1, u_2 \rangle = \frac{1}{n-2}, \text{ and } \langle u_2, u_2 \rangle = \frac{n-1}{n-2}.$$

By plugging in the dot product values to the formula in Lemma 2.11, we obtain

$$\alpha^{\text{BV}}(F, \text{PTes}_n(1)) = \frac{1}{4} + \frac{1}{12} \left( \frac{n}{n-1} \right).$$

\[ \square \]

Lemma 3.10. Let $F$ be a codimension 3 face of $\text{PTes}_n(1)$ which is the intersection of facets $F_{i,l}^1, F_{i,j_1}^1, F_{i,j_2}^1$. We consider all possible cases listed below.

1. Suppose only one of the facet normal vectors is shifted hook sum matrix (i.e., $i_1 \neq j_1$ and $i_2 \neq j_2$).

   (i) If both $(i_1, j_1)$ and $(i_2, j_2)$ are not on the $l$-th hook, then $\alpha^{\text{BV}}(F, \text{PTes}_n(1)) = \frac{1}{8}$.
   
   (ii) If one of $(i_2, j_2)$ and $(i_3, j_3)$ is on the row of the $l$-th hook and the other is not on the $l$-th hook, then $\alpha^{\text{BV}}(F, \text{PTes}_n(1)) = \frac{1}{8} + \frac{n}{24(n-1)}$.
   
   (iii) If one of $(i_2, j_2)$ and $(i_3, j_3)$ is on the column of the $l$-th hook and the other is not on the $l$-th hook, then $\alpha^{\text{BV}}(F, \text{PTes}_n(1)) = \frac{1}{12} - \frac{1}{24(n-1)}$.
   
   (iv) If both $(i_2, j_2)$ and $(i_3, j_3)$ are on the row of the $l$-th hook, then $\alpha^{\text{BV}}(F, \text{PTes}_n(1)) = \frac{1}{8} + \frac{n}{12(n-2)}$.
   
   (v) If one of $(i_2, j_2)$ and $(i_3, j_3)$ is on the column of the $l$-th hook and the other is on the row of the $l$-th hook, then $\alpha^{\text{BV}}(F, \text{PTes}_n(1)) = \frac{1}{8} - \frac{1}{12(n-2)}$. 

\[ \square \]
(vi) If both of \((i_2, j_2)\) and \((i_3, j_3)\) are on the column of the \(l\)-th hook, then \(\alpha_{BV}(F, P_{Tes_n}(1)) = \frac{1}{24}\).

(2) Suppose two of the facet normal vectors are shifted hook sum matrices (without loss of generality, assume \(i_2 = j_2\) and let \(m := i_2\)).

(i) If \((i_3, j_3)\) is out of the \(l\)-th and \(m\)-th hook, then \(\alpha_{BV}(F, P_{Tes_n}(1)) = \frac{1}{8} + \frac{1}{12(n - 1)}\).

(ii) If \((i_3, j_3)\) is on the row of either the \(l\)-th hook or the \(m\)-th hook and is not on the hook of the other, then \(\alpha_{BV}(F, P_{Tes_n}(1)) = \frac{1}{8} + \frac{n^2 + n - 3}{24(n^2 - 3n + 2)}\).

(iii) If \((i_3, j_3)\) is on the column of either the \(l\)-th hook or the \(m\)-th hook and is not on the hook of the other, then \(\alpha_{BV}(F, P_{Tes_n}(1)) = \frac{1}{8} - \frac{1}{12(n - 2)}\).

(iv) If \((i_3, j_3)\) is on the column of either the \(l\)-th hook or the \(m\)-th hook and is on the row of the other, then \(\alpha_{BV}(F, P_{Tes_n}(1)) = \frac{1}{8}\).

(3) Suppose all three of the facet normal vectors are shifted hook sum matrices (i.e, \(i_2 = j_2\) and \(i_3 = j_3\)). Then \(\alpha_{BV}(F, P_{Tes_n}(1)) = \frac{1}{8} + \frac{1}{4(n - 2)}\).

Hence, all the values are positive for all \(n \geq 3\).

Proof. We follow the same procedure as in the proof of Lemma 3.9 (1)/(i) to obtain the formulas for \(\alpha_{BV}(F, P_{Tes_n}(1))\). From the formulas for each cases, it is very clear that the values are all positive for \(n \geq 3\). \(\square\)

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Corollary 2.6 in [19], the dimension of \(P_{Tes_n}(1)\) is \(d := \binom{n}{2}\). Thus, when \(n \leq 2\), they are Ehrhart positive. Also, by Lemmas 3.9 and 3.10 and the equation (1.1), we obtain \(e_{d-2}(P_{Tes_n}(1))\) and \(e_{d-3}(P_{Tes_n}(1))\) are positive for all \(n \geq 3\). \(\square\)

3.5. Proof of Corollaries 1.2 and 1.3. Using the Reduction Theorem (Theorem 2.14), we complete the proof of Corollary 1.2 which generalizes the positivity results given in Lemmas 3.9 and 3.10.

Proof of Corollary 1.2. Let \(P\) be any deformation of \(P_{Tes_n}(1)\) with \(\dim(P) = d\). If \(d = \dim(P_{Tes_n}(1))\), then by Lemmas 3.9 and 3.10 and Theorem 2.14, \(\alpha_{BV}(F, P) > 0\) for all codimension 2 and 3 faces \(F\) of \(P\). Therefore, \(e_{d-2}(P), e_{d-3}(P) > 0\) by (1.1). Similarly, if \(d = \dim(P_{Tes_n}(1)) - 1\), Theorem 2.12 and 2.14 implies that \(\alpha_{BV}(F, P) > 0\) for all codimension 3 faces \(F\) of \(P\). Therefore, \(e_{d-3}(P) > 0\) by (1.1). \(\square\)

Recall the definition for deformation in terms of normal fan. However, in the literature, there are multiple ways to define deformations of polytopes. Here, we provide one of them that we use.

Lemma 3.11. [23] Let \(P_0\) and \(Q\) be two polytopes in \(V\). Then \(Q\) is a deformation of \(P_0\) if and only if there exists a surjective map \(\phi\) from the set of the vertices of \(P_0\) to that of \(Q\) and \(r_{i,j} \in \mathbb{R}_{\geq 0}\) such that \(\phi(v_i) - \phi(v_j) = r_{i,j}(v_i - v_j)\) whenever \(v_i\) and \(v_j\) are adjacent vertices.
of $P_0$. In particular, we say that $Q$ is a weak deformation of $P_0$ if all $r_{i,j}$’s are positive and $Q$ is a strong deformation of $P_0$ if some of $r_{i,j}$’s are zero.

Now, Corollary $\text{[1.3]}$ follows directly from Corollary $\text{[1.2]}$ and the following lemma:

**Lemma 3.12.** Let $Q$, $P$ be two polytopes and $\psi$ is an invertible affine transformation from the affine hull of $Q$ to the affine hull of $P$. Then $Q$ is a deformation of $P$ if and only if $\psi(Q)$ is a deformation of $\psi(P)$.

*Proof.* For the forward direction, let $v, w$ be any pair of adjacent vertices of $P$ and $v', w'$ be the corresponding vertices of $Q$. Then by Lemma $\text{[3.1]},$ there exists $r \in \mathbb{R}$ such that $v - w = r(v' - w')$. Since $\psi$ is an affine transformation, $\psi(v) - \psi(w) = r(\psi(v') - \psi(w'))$. Therefore, $\psi(Q)$ is a deformation of $\psi(P)$. The backward direction can be proven by the same argument. □

4. **Tesler polytopes and deformations**

In this section, we explore deformations of $\text{Tes}_n(1)$. More specifically, we give a different proof of the known fact that for all fixed $n$, every tesler polytope $\text{Tes}_n(a)$ is a deformation of $\text{Tes}_n(1)$. Furthermore, we give a characterization for which flow polytope on the complete graph is a deformation of $\text{Tes}_n(1)$. Reduction theorem (Theorem $\text{[2.14]}$) was proved using normal fan definition of a deformation but we use the equivalent definition of deformation since it works better for our case (Lemma $\text{[3.1]}$).

4.1. **For all $a \in \mathbb{R}_{\geq 0}^n$, $\text{Tes}_n(a)$ is a deformation of $\text{Tes}_n(1)$.** To prove our result of this subsection, we will utilize a machinery that was used in the proof of Lemma 2.4 in [19].

**Definition 4.1.** For any $a \in \mathbb{R}_{\geq 0}^n$ and a vertex $v = (v_{i,j}) \in \text{Vert}(\text{Tes}_n(a))$,

1. Define $j_v : \{0, 1, 2, \ldots, n\} \rightarrow \{0, 1, 2, \ldots, n\}$ by

   $j_v(k) = \begin{cases} 
   l & \text{if the } k\text{-th row is not a zero row and } v_{k,l} \neq 0, \\
   0 & \text{if the } k\text{-th row is a zero row or } k = 0.
   \end{cases}$

   (this is well-defined by Theorem $\text{[2.6]}(2)$).

2. Since $v$ is upper triangular, if we keep applying $j_v$, the sequence $\{k, j_v(k), j_v^2(k), \ldots\}$ will stabilize in a finite step. Assume that $q$ is the smallest integer such that $j_v^{q-1}(k) = j_v^q(k)$. Define

   $\text{Dep}_v(k) = \{(k, j_v(k)), (j_v(k), j_v^2(k)), \ldots, (j_v^{q-1}(k), j_v^q(k))\}$.

3. Let $D_v(k) \in \mathbb{U}(n)$ be the matrix recording the positions appearing in $\text{Dep}_v(k)$. More precisely,

   $D_v(k) := (m_{i,j})$ where $m_{i,j} = \begin{cases} 
   1, & \text{if } (i, j) \in \text{Dep}_v(k), \\
   0, & \text{if } (i, j) \notin \text{Dep}_v(k).
   \end{cases}$

   The purpose of this definition is to describe the edge direction from a vertex of $\text{Tes}_n(a)$ to its adjacent vertex.
Example 4.2. Let \( \mathbf{v} \) and \( \mathbf{w} \) be as in Example \([2,10]\). We obtain \( \mathbf{v} \) from \( \mathbf{w} \) by moving 2 on the first row to the second column and adjusting the lower rows to maintain the same hook sum. One sees that \( \text{Dep}_w(1) = \{(1, 3), (3, 3)\} \) is the set of all the positions that are affected by changing \( w_2, 3 \) from 2 to 0 and \( \text{Dep}_v(1) = \{(1, 2), (2, 4), (4, 4)\} \) is the set of all the positions that are affected by the change \( w_1 = 0 \to 2 \) (we have to add 2 to the positions to maintain the same hook sums). Thus, the edge direction from \( \mathbf{w} \) to \( \mathbf{v} \) can be described in terms of \( D_w(1) \) and \( D_v(1) \). Notice that changing the first coordinate of the hook sum vector \( \mathbf{a} \) to any other positive number would only change the length of the edge direction.

\[
\mathbf{v} - \mathbf{w} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 3 & 0 & 0 & 8 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 5 & 0 & 0 & 6 \end{pmatrix} = 2 \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 2(D_v(1) - D_w(1))
\]

We will show that for any \( \mathbf{a} \in \mathbb{R}_{\geq 0}^n \), the edge directions of \( \text{Tes}_n(\mathbf{a}) \) are always of the form as in the above example using the properties of \( D_v(k) \) stated in the Proposition below.

Proposition 4.3. Let \( \mathbf{a} \in \mathbb{R}_{\geq 0}^n \) and \( \mathbf{v} = (v_{i,j}) \in \text{Vert}(\text{Tes}_n(\mathbf{a})) \) such that the \( k \)-th row is a non-zero row. Suppose the non-zero entry in the \( k \)-th row is \( c \) and,

\[
\text{Dep}_v(k) = \{(k, j_v(k)), (j_v(k), j_v^2(k)), \ldots, (j_v^{q-1}(k), j_v^q(k))\}.
\]

Then \( D_v(k) \) has the following properties:

1. The \( k \)-th row has one 1.
2. The \( m \)-th row and column are zero if \( m \neq j_v^l(k) \) for \( 1 \leq l \leq q \).
3. The \( m \)-th row has exactly one 1 and the \( m \)-th column has exactly one 1 above the diagonal if \( m = j_v^l(k) \) for \( 1 \leq l < q \).
4. \( hs(D_v(k)) = e_k \) where \( e_k \) is the \( k \)-th standard basis of \( \mathbb{R}^n \).
5. For any \( 1 \leq l \leq q - 1 \) \( v_{j_v^{l-1}(k), j_v^l(k)} \leq v_{j_v^l(k), j_v^{l+1}(k)} \). Thus, the entries of \( \mathbf{v} - cD_v(k) \) are non-negative.

Proof. (1), (2) and (3) are clear from the definition. For (4), let \( D = D_v(k) \). Then, by (2), \( hs_m(D) = 0 \) if \( m \neq j_v^l(k) \) for some \( 1 \leq l \leq q \). If \( m = j_v^l(k) \) for some \( 1 < l < q \), \( hs_m(D) = 0 \) by (3). Finally, when \( m = k \), since \( k \)-th row has one 1 and the rows above \( k \)-th row are all zero rows, \( hs_k(D) = 1 \). For (5), consider the \( j_v^l(k) \)-th hook sum condition \( hs_{j_v^l(k)}(\mathbf{v}) \). Assume \( a_{j_v^l(k)} = 0 \) and \( v_{j_v^{l-1}(k), j_v^l(k)} > v_{j_v^l(k), j_v^{l+1}(k)} \). Then, since \( v_{j_v^l(k), j_v^{l+1}(k)} \) is the only non-zero element in the row, we get \( hs_{j_v^l(k)}(\mathbf{v}) = a_{j_v^l(k)} < 0 \). Similarly, assume \( a_{j_v^l(k)} > 0 \) and \( v_{j_v^{l-1}(k), j_v^l(k)} \geq v_{j_v^l(k), j_v^{l+1}(k)} \). Then the \( j_v^l(k) \)-th hook sum condition would imply \( a_{j_v^l(k)} \leq 0 \). \( \square \)

Lemma 4.4. For \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_{\geq 0}^n \), let \( \mathbf{v} = (v_{i,j}) \), \( \mathbf{w} = (w_{i,j}) \in \text{Vert}(\text{Tes}_n(\mathbf{a})) \) are adjacent vertices of \( \text{Tes}_n(\mathbf{a}) \) where the rows of \( \text{supp}(\mathbf{v}) \) and \( \text{supp}(\mathbf{w}) \) are the same above the \( k \)-th row and differ in the \( k \)-th row. Then

\[
\mathbf{w} - \mathbf{v} = c(D_w(k) - D_v(k))
\]

where \( c \in \mathbb{R}_{>0} \) is the unique non-zero entry in the \( k \)-th row of \( \mathbf{v} \) (or that of \( \mathbf{w} \)).

Proof. Let \( k \) be as in the above statement. Then, one of the 4 operations in Lemma \([2,8]\) must have occurred in the \( k \)-th row of \( \text{supp}(\mathbf{v}) \) while getting \( \text{supp}(\mathbf{w}) \) from \( \text{supp}(\mathbf{v}) \). Obviously, (a) is impossible. Also, (b) and (c) are impossible, because if we assume that the \( k \)-th row of \( \text{supp}(\mathbf{v}) \) is a zero row but that of \( \text{supp}(\mathbf{w}) \) is not (or vice versa), since the rows above the \( k \)-th row are same, it would imply \( a_k = 0 \) and \( a_k \neq 0 \) by Lemma \([2,9]\). Thus, the operation
(d) occurred and \( v, w \) each have exactly one non-zero entry in the \( k \)-th row, but in different places. Since \( \text{supp}(v) \) and \( \text{supp}(w) \) have the same rows above the \( k \)-th row, by the \( k \)-th hook sum condition, we obtain \( v_{k,j_v(k)} = w_{k,j_w(k)} \). Let

\[
c := v_{k,j_v(k)} = w_{k,j_w(k)}.
\]

Then, by Proposition 4.3 (4), \( c \hs(D_v(k)) = c \hs(D_w(k)) = c e_k \). Thus, if we let \( u := w + cD_v(k) - cD_w(k) \), then

\[
\hs(u) = a - c e_k + c e_k = a.
\]

Also, by Proposition 4.3 (5), the entries of \( w - cD_w(k) \) are non-negative which implies that \( u = w + cD_v(k) - cD_w(k) \in \text{Tes}_n(a) \). We will prove that \( u = v \) in two steps. First, we show that \( u \) is, in fact, a vertex of \( \text{Tes}_n(a) \) by showing that each row of \( u \) has at most one 1. Second, we show that \( \text{supp}(u) \leq \text{supp}(v) \).

We now show \( u \) is a vertex of \( \text{Tes}_n(a) \). Notice that since \( D_v(k) \) and \( w - cD_w(k) \) have non-negative entries, we get \( \text{supp}(u) = \text{supp}(D_v(k) + w - cD_w(k)) = \text{supp}(D_v(k), w - cD_w(k)) \). Also, observe that directly from the definition of \( D_v(k) \), \( \text{supp}(cD_v(k)) = \text{supp}(D_v(k)) \leq \text{supp}(v) \). Similarly, \( \text{supp}(cD_w(k)) \leq \text{supp}(w) \) which implies \( \text{supp}(w - cD_w(k)) \leq \text{supp}(w) \). Therefore,

\[
\text{supp}(u) = \text{supp}(\{D_v(k), w - cD_w(k)\}) \leq \text{supp}(\{v, w\}). \tag{4.1}
\]

By Lemma 2.8, the \( k \)-th row of \( \text{supp}(v, w) \) has two 1’s and at most one 1 in every other rows which implies \( \text{supp}(u) \) has at most one 1 except for the \( k \)-th row. Finally, for the \( k \)-th row of \( u \), since \( v_{k,j_v(k)} = w_{k,j_w(k)} = c \), adding \( cD_v(k) \) and subtracting \( cD_w(k) \) from \( w \) moves the unique 1 in the \( k \)-th row of \( \text{supp}(w) \) to the position \( (k, j_v(k)) \). Therefore \( \text{supp}(u) \) has at most one 1 in each row.

For the second step, it is clear that the rows above the \( k \)-th row of \( u \) are same as those of \( \text{supp}(v) \). Also, we can easily see that the \( k \)-th row of \( v \) and \( u \) are the same. Thus, \( \text{supp}(u) \) is same as \( \text{supp}(v) \) up to the \( k \)-th row. For the lower rows, we have to show the following two statements:

(a) There is no \( l > k \) such that both \( l \)-th row of \( \text{supp}(v) \) and \( \text{supp}(u) \) has one 1 but in a different places.

(b) There is no \( l > k \) such that the \( l \)-th row of \( \text{supp}(u) \) is a non-zero row but that of \( \text{supp}(v) \) is a zero row.

The part (a) directly follow from the inequality (3.2). For (b), assume for the contradiction that we have such \( l \)-s and \( l_0 \) is the smallest of them. Since \( l_0 \)-th row of \( \text{supp}(v) \) is a zero row, by Lemma 2.9 we have \( a_{l_0} = 0 \) and the entries of \( l_0 \)-th column of \( \text{supp}(v) \) above the diagonal are all zero but at least one of the entries of \( l_0 \)-th column of \( \text{supp}(u) \) has to be non-zero. Let \( u_{l_1,l_0} \) be one of the non-zero entry of \( u \) in the \( l_0 \)-th column above the diagonal \( (l_1 < l_0) \). Since \( \text{supp}(v) \) and \( \text{supp}(u) \) are same up to \( k \)-th row, we have \( k < l_1 \). Thus, if the \( l_1 \)-th row of \( \text{supp}(v) \) is a zero row, it contradicts the minimality of \( l_0 \). Hence, the \( l_1 \)-th row of \( \text{supp}(v) \) have exactly one 1 and the position of this 1 cannot be \((l_1, l_0)\)-the position of the 1 in \( l_1 \)-th row of \( \text{supp}(u) \) which is impossible by (a). Therefore, \( \text{supp}(u) \leq \text{supp}(v) \), and by Lemma 2.7 \( \{u\} \subset \{v\} \) which implies \( u = w + cD_v(k) - cD_w(k) = v \). \hfill \Box

We need one more lemma before proving the result of this subsection.
Lemma 4.5. Let \( a_0 \in \mathbb{R}^n_0 \) and \( a \in \mathbb{R}^n_{\geq 0} \). Then, for any \( v = (v_{i,j}) \in \text{Vert}(\text{Tes}_n(a_0)) \), there exists a unique \( v' \in \text{Vert}(\text{Tes}_n(a)) \) such that \( \text{supp}(v') \leq \text{supp}(v) \).

Proof. Let \( v' \) be any point in \( \text{Tes}_n(a) \) such that \( \text{supp}(v') \leq \text{supp}(v) \). Then \( v' \) has to satisfy \( \text{supp}(v') \leq \text{supp}(v) \) and \( \text{hs}(v') = a \).

One can easily see that there is a unique solution to the above two conditions, because \( \text{supp}(v') \leq \text{supp}(v) \) implies that there can be only one possible non-zero entry for each row of \( v' \) (this also implies that \( v' \) is a vertex of \( \text{Tes}_n(a) \) by Lemma 2.6 (2)) and the values of the entries are uniquely determined by the hook sum conditions \( \text{hs}(v') = a \).

Now, we are ready to prove our main result of this section.

Theorem 4.6. Let \( n \) be any fixed positive integer and \( a_0 \in \mathbb{R}^n_{\geq 0} \). Then for any \( a \in \mathbb{R}^n_{\geq 0} \), \( \text{Tes}_n(a) \) is a deformation of \( \text{Tes}_n(a_0) \). In particular, \( \text{Tes}_n(a) \) is a weak deformation of \( \text{Tes}_n(a_0) \) when \( a \in \mathbb{R}^n_{>0} \) and is a strong deformation of \( \text{Tes}_n(a_0) \) when some of the coordinates of \( a \) are zero.

Proof. Since \( \text{Tes}_n(0, \ldots, 0) \) is a point, it is clearly a deformation of any non-empty polytope. Assume that \( a \in \mathbb{R}^n_{>0} \) and \( a \neq (0, \ldots, 0) \). We prove the theorem using Lemma 3.11. To use the lemma, we define a map between the set of the vertices. Let \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n_{\geq 0} \). Define

\[ \phi : \text{Vert}(\text{Tes}_n(a_0)) \longrightarrow \text{Vert}(\text{Tes}_n(a)), \]

by letting \( \phi(v) \) to be the unique \( v' \) satisfying \( \text{supp}(v') \leq \text{supp}(v) \) asserted by Lemma 4.5.

One sees that \( \phi \) is well-defined by Lemma 4.5. The surjectivity of \( \phi \) comes from Lemma 2.6 (2), because for any matrix \( M \) with only 0,1 entries where each row has exactly one 1, there exist \( v \in \text{Tes}_n(a_0) \) such that \( \text{supp}(v) = M \).

Now, we show that \( \text{Tes}_n(a) \) is a deformation of \( \text{Tes}_n(a_0) \). Assume that \( v \) and \( w \) are adjacent vertices of \( \text{Tes}_n(a_0) \) where their support only differ in the \( k \)-th row. Let \( v' = \phi(v) \) and \( w' = \phi(w) \). We need to show that there exists \( r_{w,v} \in \mathbb{R}^n_{\geq 0} \) such that

\[ w' - v' = r_{w,v}(w - v). \]

Case 1: If the \( k \)-th row of \( \text{supp}(v') \) or \( \text{supp}(w') \) is a zero row (without loss of generality, assume that the \( k \)-th row of \( \text{supp}(v') \) is a zero row), then since \( \text{supp}(v) \) and \( \text{supp}(w) \) only differs in the \( k \)-th row, \( \text{supp}(v') \leq \text{supp}(w) \). Thus, \( v' = w' \) by Lemma 4.5. Therefore, we can take \( r_{w,v} = 0 \).

Case 2: If neither the \( k \)-th row of \( \text{supp}(v') \) nor that of \( \text{supp}(w') \) is a zero, since \( \text{supp}(v') \leq \text{supp}(v) \), we have \( j_{w'}(k) = j_v(k) \). Let \( v' = (v'_{i,j}) \). Since \( v'_{k,j_{w'}(k)} \neq 0 \), the \( k \)-th row of \( v' \) is a non-zero row by Lemma 2.9. Since \( \text{supp}(v') \leq \text{supp}(v) \), we get \( j_{w'}(k) = j_v(k) \). By iterating this process, we obtain \( j_{w'}(k) = j_v(k) \) for any integer \( 0 \leq l \). Therefore, \( \text{Dep}_{w'}(k) = \text{Dep}_v(k) \), or equivalently, \( D_{w'}(k) = D_v(k) \). Similarly, we obtain \( D_{w'}(k) = D_v(k) \) by Lemma 4.4.

\[ w - v = c(D_w(k) - D_v(k)) \]

and \( w' - v' = d(D_{w'}(k) - D_{w'}(k)) \)

where \( c \) and \( d \) are the unique non-zero (positive) entries of \( v \) and \( v' \) respectively. Therefore, we can take \( r_{w,v} = \frac{c}{d} \).
In particular, when \( a \in \mathbb{R}^n_+ \), \( r_{v,w} \) is always positive because **Case 1** does not happen. However, when some of the \( a_i \)'s are zero, one can easily see that **Case 1** does happen which means the corresponding \( r_{v,w} \)'s are zero.

\[
\square
\]

4.2. **Flow polytopes on the complete graph that are deformations of the Tesler polytopes.** We first restrict ourselves to the complete graphs with \( n+1 \) vertices \( K_{n+1} \). Flow polytope is defined over (complete) graphs, but in this section, we use another definition that gives a polytope that is unimodularly equivalent to the flow polytope.

**Definition 4.7.** For \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \), the flow polytope on \( K_{n+1} \) with net flow \( a_i \) on vertex \( i \) for \( 1 \leq i \leq n \) is defined by,

\[
\text{Flow}_n(a) := \{ M = (m_{i,j}) \in U(n) \mid hs(M) = a, m_{i,j} \geq 0 \}.
\]

An element of \( \text{Flow}_n(a) \) is called as flow.

One can show that,

\[
\text{Flow}_n(a) \text{ is non-empty if and only if } \sum_{i \leq l} a_i \geq 0 \text{ for all } 1 \leq l \leq n.
\]

In this subsection, we will let \( A_n \) to be the set of all \( a \in \mathbb{R}^n \) that satisfies the above condition. To give a characterization of \( a \in \mathbb{R}^n \) where \( \text{Flow}_n(a) \) is a deformation of Tesler polytope \( \text{Tes}_n(1) \), we first prove that any deformation of Tesler polytope \( \text{Tes}_n(1) \) is a translation of some Tesler polytope (Theorem 4.10).

Let \( P(n) \) be the following set of polytopes

\[
\left\{ P \subset \mathbb{R}^{\binom{n}{2}} \mid P \text{ is a deformation of } \text{Tes}_n(1) \right\}.
\]

By the definition of a deformation, For any \( P \in P(n) \), we can write

\[
P = \{ M = (m_{i,j}) \in U(n) : hs(M) = \beta \text{ and } m_{i,j} \geq -b_{i,j} \text{ for every } i \leq j \}
\]

where \( \beta \in \mathbb{R}^n \) and \( b_{i,j} \in \mathbb{R} \) for any \( i \leq j \). The only difference from Tesler polytopes are that, in this case, we use \( x_{i,j} \geq -b_{i,j} \) for defining inequalities instead of \( x_{i,j} \geq 0 \) which corresponds to translations of facet defining planes. We know that any vertex \( v \) of Tesler polytope \( \text{Tes}_n(1) \) is a matrix that has at most one non-zero element in each rows. So, to find a vertex of \( P \) corresponding to \( v \), we first replace the zero in \((i,j)\)-th position in the matrix \( v \) by \(-b_{i,j}\)'s. Then hook sum conditions determines the values of the undetermined entries. One can deduce from [8], Corollary 2.14 and Proposition 2.16 that the above method of corresponding vertices yields a map from \( \text{Vert}(\text{Tes}_n(1)) \) to \( \text{Vert}(P) \) satisfying the conditions of Lemma 3.11.

**Example 4.8.** With \( P \) as the above remark with \( n = 4 \) and \( \beta = (4,3,8,1) \), let us find the vertex \( v \) of \( P \) corresponding to a vertex \( v' \) of \( \text{Tes}_4(1,1,1,1) \). As explained above, we first change zeros with \(-b_{i,j}\)'s and the hook sum conditions for \( P \) determines \( w, x, y, z \).
By solving the system of equations, we get
\[ v' := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix} \rightarrow v := \begin{bmatrix} -b_{1,1} & w & -b_{1,3} & -b_{1,4} \\ -b_{2,2} & x & -b_{2,4} \\ y & -b_{3,4} & z \end{bmatrix} \rightarrow \begin{cases} h_{s_1}(v) = -b_{1,1} + w - b_{1,3} - b_{1,4} = 4 \\ h_{s_2}(v) = -b_{2,2} + x - b_{2,4} - w = 3 \\ h_{s_3}(v) = y - b_{3,4} - (x - b_{1,3}) = 8 \\ h_{s_4}(v) = z - (-b_{1,4} - b_{2,4} - b_{3,4}) = 1 \end{cases} \]

By solving the system of equations, we get
\[ \begin{align*}
w &= 4 + b_{1,1} + b_{1,3} + b_{1,4} \\
x &= 3 + b_{2,2} + b_{2,4} + w \\
y &= 8 + b_{3,4} + x - b_{1,3} \\
z &= 1 - b_{1,4} - b_{2,4} - b_{3,4} \end{align*} \]

To state the next lemma, we introduce the following notations. Let \( \widetilde{U}(n) \) be the set of \((n - 1) \times n\) matrices \( \widetilde{M} \) obtained by deleting the \( n \)-th row of the matrices \( M \in U(n) \). We define \( h_{s_k}(\widetilde{M}) = h_{s_k}(M) \) for any \( 1 \leq k \leq n - 1 \).

**Lemma 4.9.** Let \( P = \{ M = (m_{i,j}) \in U(n) \mid h_{s}(M) = (\tilde{b}_{1,n}, \ldots, \tilde{b}_{n,n}) \text{ and } m_{i,j} \geq -\tilde{b}_{i,j} \text{ for every } i \leq j \} \)

where \( \tilde{B} := (\tilde{b}_{i,j}) \in \widetilde{U}(n) \). Then \( P \in P(n) \) if and only if \( h_{s_i}(\tilde{B}) \geq 0 \) for all \( 1 \leq i \leq n \).

**Proof.** For the forward direction, assume \( P \in P(n) \). For \( 1 \leq k \leq n - 1 \) let \( d_k \in U(n) \) be \( d_k = (d_{i,j}) \) where \( d_{i,j} = \begin{cases} 1 & \text{if } k \neq i \text{ and } i = j \text{ or } k + 1 \neq i \text{ and } i = j \\ 1 & \text{if } k = i = j - 1 \\ 2 & \text{if } k + 1 = i = j \\ 0 & \text{otherwise} \end{cases} \) Then we see that \( d_k \in \text{Vert}(\text{Tes}_n(1)) \) by Lemma 2.6(2). Let \( \mathbf{w}_k = (w_{i,j}^k) \in U(n) \) be a vertex of \( P \) corresponding to \( d_k \). To calculate the entries of \( \mathbf{w}_k \), replace the zero in \((i,j)\)-th position in \( d_k \) with \( -\tilde{b}_{i,j} \) for all \( 1 \leq i \leq j \leq n \) which determines all the entries of \( \mathbf{w}_k \) except one entry in each row (Let us denote the undetermined entry in \( i \)-th row by \( x_i \)). And then determine \( x_i \)'s by using the hook sum conditions in an iterative manner from the first hook sum condition to the last hook sum condition. After solving the system of equations, (Since \( h_{s_1}(\mathbf{w}_k) = 0 \) only depends on first \( i \) rows, a simple induction argument shows that the system of equations is always solvable and the solution is unique), we obtain
\[ w_{k,k+1} = h_{s_k}(\tilde{B}) - b_{k,k+1}. \]

Let \( I \) be the \( n \) by \( n \) identity matrix (\( I \) is clearly a vertex of \( \text{Tes}_n(1) \)) and \( \mathbf{w} = (w_{i,j}) \in U(n) \) be the corresponding vertex of \( P \). Then,
\[ (d_k - I)_{k,k+1} = 1 \text{ and } (w_k - \mathbf{w})_{k,k+1} = h_{s_k}(\tilde{B}), \]

because \( w_{k,k+1} = -\tilde{b}_{k,k+1} \). By Lemma 3.11, \( h_{s_k}(\tilde{B}) \) must be a non-negative. Also, since \( \mathbf{w} \in P \), we have \( w_{n,n} = h_{s_n}(\tilde{B}) - \tilde{b}_{n,n} \geq -\tilde{b}_{n,n} \). Therefore, we obtain the desired result.

Conversely, if we assume \( h_{s_i}(\tilde{B}) \geq 0 \) for all \( 1 \leq i \leq n \) and let \( B \in U(n) \) be the matrix we obtain by removing the \( n+1 \)-th column of \( \tilde{B} \), one can easily check that the polytope \( P + B \) obtained by shifting \( P \) by the matrix \( B \) is:
\[ P + B = \{ M + B \mid M \in P \} = \text{Tes}_n(h_{s_1}(\tilde{B}), \ldots, h_{s_n}(\tilde{B})), \]
Case 1 : \( f_1 = \begin{bmatrix} a_1 - b_1 & b_1 & 0 \\ a_2 - b_2 & b_1 + b_2 & 0 \\ 0 & a_3 - b_3 & 0 \end{bmatrix} \), \( f_2 = \begin{bmatrix} a_1 - b_1 & 0 & b_1 \\ a_2 - b_2 & b_2 & 0 \\ 0 & a_3 - b_3 & 0 \end{bmatrix} \)

Case 2 : \( f_1 = \begin{bmatrix} 0 & b_1 & 0 & a_1 - b_1 & 0 \\ 0 & b_1 + b_2 & a_2 - b_2 & 0 \\ \ast & \ast & \ast & \ast & \ast \end{bmatrix} \), \( f_2 = \begin{bmatrix} 0 & 0 & b_1 - b_1 & 0 \\ 0 & b_2 & a_2 - b_2 & 0 \\ \ast & \ast & \ast & \ast \end{bmatrix} \)

Figure 1. Case 1 : is when \( k = n = 3 \) and Case 2 : is when \( n = 5 \) and \( k = 3 \).

which is a deformation of Tes\(_n(1)\) by Theorem 4.6 

Now, we are ready to prove the following theorem.

**Theorem 4.10.** Any polytope \( P \in P(n) \) is a translation of some Tesler polytope.

**Proof.** Let \( P = \{ M = (m_{i,j}) \in \mathbb{U}(n) : charged(M) = \beta \text{ and } m_{i,j} \geq c_{i,j} \text{ for every } i \leq j \} \) be a polytope in \( P(n) \) and \( \beta = (\beta_1, \ldots, \beta_n) \). Let \( D = (d_{i,j}) \) be a matrix such that

\[
d_{i,j} = \begin{cases} \beta_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]

\( P - D = \{ M = (m_{i,j}) \in \mathbb{U}(n) : charged(M) = 0 \text{ and } x_{i,j} \geq b_{i,j} \text{ for every } i \leq j \} \)

where \( b_{i,j} = \begin{cases} c_{i,i} - \beta_i & \text{if } i = j \\ c_{i,j} & \text{if } i \neq j \end{cases} \)

because \( charged(D) = \beta \) and \( charged \) is a linear transformation. Since \( P - D \) is just a translation of \( P \), \( P - D \in P(n) \) Thus, if we let \( B := (b_{i,j}) \in \mathbb{U}(n) \) then \( charged_k(B) > 0 \) for \( 1 \leq k \leq n - 1 \) and \( charged_n(B) \geq 0 \) by Lemma 4.9

Thus, \( P - D - B = \{ M = (m_{i,j}) \in \mathbb{U}(n) : charged(M) = (charged_1(-B), \ldots, charged_n(-B)) \text{ and } x_{i,j} \geq 0 \text{ for every } i \leq j \} \)

is a Tesler polytope of hook sum \( (charged_1(-B), \ldots, charged_n(-B)) \) which is a deformation of Tes\(_n(1)\) by Theorem 4.6.

Using the above theorem with the following lemma, we can characterize which flow polytope on a complete graph is a deformation of Tes\(_n(1)\).

**Lemma 4.11.** Let \( n \geq 3 \) and \( a = (a_1, \ldots, a_n) \in A \). Assume that \( a_1 > 0, a_2 \geq 0 \text{ and } a_k \) is the first negative component of \( a \) where \( 3 \leq k \leq n \). Then there exists two flows \( f_1 = (f_{i,j}^1) \) and \( f_2 = (f_{i,j}^2) \) in Flow\(_n(a)\) such that \( f_{i,k}^1 = 0 \) for any \( 1 \leq i \leq k - 2 \) and \( f_{k-1,k}^2 < |a_k| \).

**Proof.** We will provide the result in two cases where in the first case, we will directly construct \( f_1 \) and \( f_2 \) and for the second case, we will construct two points \( p_1, p_2 \) and find a point \( p \) where \( p + p_1 \) and \( p + p_2 \) gives our desired flows. For both of the cases, we use the following fact:

There exist \( b_1, \ldots, b_{k-1} \) such that,

\[
0 < b_1 \leq a_1, \ 0 \leq b_l \leq a_l \text{ for all } 2 \leq l \leq k - 1 \text{ and } \sum_{1 \leq l \leq k-1} b_l + a_k = 0.
\]
The existence of such $b_i$'s is guaranteed since $\sum_{1 \leq k \leq n} a_k \geq 0$, $a_1 > 0$ and $a_2, \ldots, a_{k-1} \geq 0$.

**Case 1:** Suppose $k = n$.
Let $f_1 = (f_{i,j}^1)$ and $f_2 = (f_{i,j}^2)$ where
\[
    f_{i,j}^1 = \begin{cases}
        \sum_{1 \leq l \leq j-1} b_l & \text{if } j = i + 1 \leq k, \\
        a_i - b_i & \text{if } i = j, \\
        0 & \text{otherwise},
    \end{cases} \quad \text{and} \quad f_{i,j}^2 = \begin{cases}
        b_i & \text{if } j = k, \\
        a_i - b_i & \text{if } i = j, \\
        0 & \text{otherwise}.
    \end{cases}
\]

It is clear that $f_1$ and $f_2$ are in $\text{Flow}_n(a)$ and have desired properties.

**Case 2:** Suppose $k < n$.
We first construct $p_1$ and $p_2$. Let $p_1 = (p_{i,j}^1)$ and $p_2 = (p_{i,j}^2)$ where
\[
    p_{i,j}^1 = \begin{cases}
        \sum_{1 \leq l \leq j-1} b_l & \text{if } j = i + 1 \leq k, \\
        a_i - b_i & \text{if } i = j, \\
        0 & \text{otherwise},
    \end{cases} \quad \text{and} \quad p_{i,j}^2 = \begin{cases}
        b_i & \text{if } j = k, \\
        a_i - b_i & \text{if } i = j, \\
        0 & \text{otherwise}.
    \end{cases}
\]

Next, we will find $p$. Since $\sum_{1 \leq l \leq m} a_l$ for all $1 \leq m \leq n$ and $\sum_{1 \leq l \leq k-1} b_l + a_k = 0$, we have,
\[
    \sum_{1 \leq l \leq k-1} a_l - b_l + \sum_{k \leq 1 \leq k} a_1 - (\sum_{1 \leq l \leq k} b_l + a_k) \geq 0 \text{ for all } k + 1 \leq m \leq n.
\]
By the above equation and the fact that $a_l - b_l \geq 0$ for all $1 \leq l \leq k-1$, we have that
\[
    a' := (0, \ldots, 0, \sum_{1 \leq l \leq k} a_i - b_i, a_{k+2}, \ldots, a_n) \in A.
\]

Therefore, $\text{Flow}_n(a')$ is non-empty. Let $p = (p_{i,j})$ be any point in $\text{Flow}_n(a')$. Then, since first $k$ components of $a'$ are zero, we have $p_{i,j} = 0$ for any $i \leq k$. Clearly, $f_1, f_2 \in \text{Flow}_n(a)$ and $f_{i,k} = 0$ for all $1 \leq i \leq k - 2$. Moreover, one can check that $f_{k-1,k} < |a_k|$ follows from the equation (4.2). Therefore, $f_1$ and $f_2$ have the desired properties.

we are now ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** We first show the above statement for the case when $l = 1$ and then show that the other cases can be reduced to this case.

Assume that $l = 1$. For the backward direction, assuming that $a_3, a_4, \ldots, a_n \geq 0$, we will show $\text{Flow}_n(a)$ is a Tesler polytope (up to a shift, which by Theorem 4.10 is a deformation of Tes$_n(1)$). If $a_2 \geq 0$, then $\text{Flow}_n(a) = \text{Tes}_n(a)$. If $a_2 < 0$, then $a_1 \geq |a_2|$ and $m_{1,2} \geq -a_2$ (otherwise the condition $hs_2(M) = a_2$ cannot be satisfied). Thus, if we let $T = (t_{i,j}) \in \mathbb{U}(n)$ where $t_{i,j} = \begin{cases}
        a_{1,2} & (i, j) = (1, 2) \\
        0 & \text{otherwise}
    \end{cases}$ and $a' = (a_1 - |a_2|, a_3, a_4, \ldots, a_n)$, then
\[
    \text{Flow}_n(a) - T = \{M = (m_{i,j}) \in \mathbb{U}(n) \mid \text{hs}(M) = a' \text{ and } m_{i,j} \geq 0 \text{ for all } 1 \leq i \leq j \leq n\},
\]
which is exactly Tes$_n(a_1 - |a_2|, a_3, a_4, \ldots, a_n)$ (notice that the above equation is still true without the assumption that $a_3, \ldots, a_n \geq 0$).

For the forward direction, we assume that $\text{Flow}_n(a)$ is a deformation of Tes$_n(1)$ and therefore, a translation of Tes$_n(a_0)$ for some $a_0 \in \mathbb{R}^n_{\geq 0}$ by Theorem 4.10. Also, for contradiction we assume that at least one of $a_3, a_4, \ldots, a_n$ is negative. It is enough to show for the case when $a_2 \geq 0$ because in the proof of the backward direction, we showed that if $a_2 < 0$, Flow$_n(a)$
is a translation of \( \text{Flow}_n(a_1 - |a_2|, 0, a_3, a_4, \ldots, a_n) \). Let \( T = (t_{i,j}) \in \mathbb{U}(n) \) be the point such that \( \text{Flow}_n(a) + T = \text{Tes}_n(a_0) \). First of all, \( t_{i,j} \leq 0 \) for every \( (i, j) \) because if \( t_{i,j} > 0 \) for some \( (i, j) \), then the \( (i, j) \)-th position of every point in \( \text{Flow}_n(a) + T = \text{Tes}_n(a_0) \) is positive which is impossible by Lemma 2.6 (2) (there exists a vertex of \( \text{Tes}_n(a_0) \) where \( (i, j) \)-th position is 0). Let \( a_k \) be the first negative component of \( a \) \( (3 \leq k \leq n) \). By Lemma 4.11 there exists two flows \( f_1 = (f_{i,j}^1) \) and \( f_2 = (f_{i,j}^2) \) in \( \text{Flow}_n(a) \) such that \( f_{i,k}^1 = 0 \) for all \( 1 \leq i \leq k-2 \) and \( f_{k+1-k}^2 < |a_k| \). Since \( f_1 + T \in \text{Tes}_n(a_0), t_{i,k} = 0 \) for all \( 1 \leq i \leq k-2 \) because otherwise, \( f_1 + T \) would have a negative component which is impossible. Thus, \( h_{s_k}(T) = t_{k,k} + \cdots + t_{k,n} \). Therefore, \( h_{s_k}(T) = 0 \) and \( a_k \geq k-1,k \). Then, the \( (k-1,k) \)-th position of \( f_2 + T \) is negative which is a contradiction.

Now assume \( l > 1 \). Therefore, the backward direction follows from the proof above for \( l = 1 \) case because \( \text{Flow}_n(0, \ldots, 0, a_l, \ldots, a_n) \) and \( \text{Tes}_n(0, \ldots, 1, \ldots, 1) \) are unimodular equivalent to \( \text{Flow}_{n-l+1}(a_l, \ldots, a_n) \) and \( \text{Tes}_{n-l+1}(1, \ldots, 1) \) respectively which the latter is a deformation of \( \text{Tes}_n(1) \) by Theorem 4.6. The forward direction can also be reduced to the \( l = 1 \) case. Similarly as before, \( \text{Flow}_n(a) + T = \text{Tes}_n(a_0) \) for some \( a_0 \in \mathbb{R}_{\geq 0}^n, T \in \mathbb{U}(n) \), where all the entries of \( T \) are either zero or negative. Clearly, the first \( (l-1) \) rows of any point in \( \text{Flow}_n(a) \) is zero, which implies that the first \( (l-1) \) rows of \( T \) is zero (otherwise, \( \text{Tes}_n(a_0) \) have a negative hook sum). Hence, the first \( (l-1) \) rows of any points in \( \text{Tes}_n(a_0) \) is zero and so the first \( l-1 \) entries of \( a_0 \) are zero. Therefore, we can instead work on \( \text{Flow}_{n-l}(a_l+1, \ldots, a_n) + T' = \text{Tes}_{n-l}(a_0') \) where \( T' \in \mathbb{U}(n-l+1) \) is obtained from \( T \) by removing the first \( l-1 \) rows and columns and \( a_0' \in \mathbb{R}_{\geq 0}^{n-l+1} \) is obtained from \( a_0 \) by removing the first \( l-1 \) entries of \( a_0 \) and obtain the same result as in \( l = 1 \) case.

\[\square\]

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