MULTILOOP STRING-LIKE FORMULAS FOR QED

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Abstract

Multiloop gauge-theory amplitudes written in the Feynman-parameter representation are poised to take advantage of two important developments of the last decade: the spinor-helicity technique and the superstring reorganization. The former has been considered in a previous article; the latter will be elaborated in this paper. We show here how to write multiloop string-like formulas in the Feynman-parameter representation for any process in QED, including those involving other non-electromagnetic interactions. The general connection between the Feynman-parameter approach and the superstring/first-quantized approach is discussed. In the special case of a one-loop multi-photon amplitude, these formulas reduce to the ones obtained by the superstring and the first quantized methods. The string-like formulas exhibits a simple gauge structure which makes the Ward-Takahashi identity apparent, and enables the integration-by-parts technique of Bern and Kosower to be applied, so that gauge-invariant parts can be extracted diagram-by-diagram with the seagull vertex neglected.

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1. Introduction

Feynman diagram calculations for gauge theories are always complicated because of the presence of spin and gauge dependence. Spin brings in derivative coupling or Dirac algebra; gauge dependence introduces many gauge non-invariant terms which will be cancelled at the end. For non-abelian theories such as QCD the situation is even worse, for there is the additional complication of color algebra, as well as the presence of three- and four-gluon vertices, and ghosts.

Two techniques have been developed in the last decade to simplify gauge theory calculations for tree and one-loop diagrams. The first is the spinor-helicity technique [1–21], which makes use of the fact that most fermions can be treated as massless at high energies. Being massless, chirality is conserved, a fact which can be exploited effectively to reduce the number of terms in a significant manner. Even photons and gluons benefit from this conservation because a spin-one index can be written as a pair of dotted and undotted spinor indices. Furthermore, one may make use of gauge freedom to choose the polarization vectors wisely to render many more terms zero. Polarized amplitudes benefit enormously from this technique but even unpolarized cross sections for which this technique was first invented are greatly simplified. This technique, however, is applicable only when all the momenta in the problem are linear combinations of massless momenta. This is so for tree diagrams, but not true for loop diagrams where off-shell loop-momenta are present to ruin chirality conservation. Consequently most of the applications of this technique has been confined to tree amplitudes [1–21].

This problem can be circumvented if the off-shell loop-momenta are integrated out before the spinor-helicity technique is applied. One way of doing so is to make use of the connection between a gauge theory and a superstring theory in the infinite tension limit [22–24]. There is a known formula for a one-loop superstring amplitude in which integrations are over the Koba-Niesen variables with internal momenta absent. This formula for a one-loop multi-gluon (or multi-photon) amplitude can also be derived from a first-quantized particle theory interacting with a background gauge field [25], where the particle in the theory is taken to be the one running around the loop. The formula obtained in the first-quantized theory is identical to the superstring formula presumably because the particle theory used in this approach is also reparametrization invariant.

Unfortunately, the generalization of either of these two approaches beyond one loop, and to all possible scattering amplitudes, have not yet been worked out.

A different method to avoid the off-shell momenta is to introduce Feynman parameters to integrate them out [26]. Spinor helicity technique is then applicable in the resulting Feynman-parameter representation, in any process and to any number of loops [27].

The second technique developed in the last decade is independent of the spinor-helicity technique, but works very well in conjunction with it. It can perhaps be labelled as string reorganization [23]. For tree and certain one-loop amplitudes when string formulas are
available, one finds that the individual terms of the string formula in the infinite-tension limit are not identical to the terms obtained from the standard Feynman rules, although the sum of all the terms are necessarily the same for both. Moreover, as a rule, the terms in the string formula are organized in a much neater and a much simpler way, and this beautiful reorganization can greatly simplify calculations. Why the string is so clever is not completely clear; it may have to do with the fact that it treats spacetime and color on an equal footing. I shall refer to field-theoretic formulas reorganized in the string way as string-like formulas.

Because of its neatness and simplicity, and potential simplifications achieved in actual calculations, it would be desirable to develop string-like formulas for all processes in any number of loops. It is not a priori clear that this can be done, and if so in what form the string-like formulas would take, because we only know them for some amplitudes and only to one loop order. Fortunately it turns out that in many cases the string-like formula in one loop is sufficient of a guide for obtaining string reorganization in many loops. For example, in tree and one loop QCD, string [4, 15–17] reorganizes the usual color factors into the Chan-Paton factors [28], which have the advantage of having the corresponding subamplitudes gauge invariant and cyclic-permutation invariant. This reorganization can be generalized to all loops and all amplitudes [27].

Background gauge emerges naturally from the one-loop string calculations of QCD [23] and the first quantized approach [25]. Its use simplifies the calculations considerably in any number of loops as well, especially when a large number of external gluon or photon lines are present [27]. This again illustrates the superiority of string reorganization.

There is yet another important feature in the one-loop string-like formula which is the main subject of this paper. I shall refer to this feature as the gauge characteristic, because it makes explicit the gauge-transformation property and the Ward-Takahashi identity in the Feynman-parameter space. To see its significance take a momentum-space diagram in spinor QED with some external photon lines. A gauge transformation of the polarization vector produces two additional terms given by shrinking one of the two spinor propagators this external photon is attached to. It is this local feature which the Ward-Takahashi identity is depended on. In Feynman-parameter space, this local feature is lost, because rules governing Feynman-parameter amplitudes tend to involve the diagram globally as a whole [26]. Nevertheless, when the amplitude is reorganized into a string-like formula, this local feature reappears once again. The details will be discussed in Sec. 3, but what happens is that in this string-like form, the additional terms arising from the gauge transformation are proportional to derivatives of the integrand wrt some Feynman parameters \( \alpha_r \). The surface terms obtained by integrating these derivatives correspond to diagrams with \( \alpha_r \) set equal to zero, which is equivalent to shrinking and short-circuiting the corresponding propagators. In this way the essential local feature of the momentum-space leading to the Ward-Takahashi identity is restored. This is the gauge characteristic alluded to before.

This gauge characteristic of the string-like formula has been put to good use in the known one-loop string-like formulas [22–25]. To explain it let us define a diagram to be
a standard diagram when none of the external gauge-particle lines are connected to the diagram either through a seagull vertex (in scalar electrodynamics) or through a four-gluon vertex (in QCD). All other diagrams will be called seagull diagrams. Because of the presence of this gauge characteristic in the standard diagrams, it is possible to perform integration by parts (IBP) in the Feynman parameters. At least in simple cases, the gauge-dependent surface terms together add up to cancel contributions from the seagull diagrams. Consequently we never have to worry about the surface terms nor the seagull diagrams. This greatly simplifies the calculation because there are many many seagull diagrams, and because in this way the gauge-invariant parts can be extracted diagram by diagram after discarding the surface terms from suitable IBP’s.

This gauge clarity of the string-like formula and the resulting simplification from the IBP technique makes it important to ask whether string-like formulas can be written for all loops and all processes. We shall leave non-abelian gauge theories aside for future considerations. The purpose of this paper is to show that string-like formulas with the gauge characteristic

can be written for standard diagrams in QED, in the Feynman-parameter representation to any number of loops and for all processes, provided a technical restriction to be discussed in Sec. 3 is obeyed.

In order to derive such multiloop formulas, considerable knowledge of the details of the Feynman-parameter representation and the electric circuit analogy [26, 29–32] is needed. We therefore start out in Sec. 2 with a discussion of these topics. Some but not all of the formulas appearing there are already contained in [26], but the important and subtle roles of external vertices and of level dependence were not sufficiently discussed before. Currents in an electric circuit are the important objects to be considered in momentum-space amplitudes, and voltages are the primary quantities to be dealt with in the configuration-space amplitudes. Voltage at a vertex corresponds to the position of that point in the configuration space. Translational invariance in configuration space corresponds to the impossibility of determining the absolute (rather than the relative) voltage levels, which in turn is the cause of the level-dependence problem. Though there is no direct relationship, it is however interesting to note that in many ways level-dependence in an electric circuit is analogous to gauge-dependence in a gauge theory.

Together with the electric circuit analogy of a Feynman diagram it is also known that there is a particle interpretation of the same, which as we shall see comes very close to being a multiloop generalization of the string approach of Bern and Kosower [22] and the first quantized approach of Strassler [25]. This will also be discussed in Sec. 2.

One feature of the string-like formula is that it treats external vertices somewhat differently than the internal vertices. This can be seen for example in the appearance of the background gauge in QCD [23, 25]. We shall see in Sec. 2 that to some extent this is a general feature, in that the topological structure of Feynman diagrams is such that certain relations which hold for external vertices are no longer true for internal vertices.
The multiloop string-like formula for scalar QED with the gauge characteristic will be derived in Sec. 3. A similar formula for spinor QED will be discussed in Sec. 4. An explicit two-loop example for a photon-meson Compton scattering diagram is given in Sec. 5 to illustrate some of these results.

2. Feynman-parameter Representation

Any $\ell$-loop scattering amplitude with $N$ internal and $n$ external lines is given by a momentum-space integral

$$M(p) = \left[ -\frac{i}{(2\pi)^4} \right]^\ell \int \prod_{a=1}^{\ell} (d^4 k_a) \frac{S_0(q)}{\prod_{r=1}^{N}(q^2_r + m^2_r - i\epsilon)} ,$$

where the internal momenta $q_r$ are understood to be linear combinations of the external \textit{outgoing} momenta $p_i$ and the loop momenta $k_a$. An external momentum $p_i$ will be assigned to every vertex $i$; this allows certain algebraic manipulations that would otherwise be impossible. At the end of the calculation all artificially added $p_i$ will be set equal to zero.

Vertex factors as well as numerators of spinning propagators are incorporated into the \textit{‘primitive spin factor’} $S_0(q)$. We shall work in four dimensions but it is just as easy to develop formulas for an arbitrary dimension. By introducing the Feynman parameters $\alpha_r$, the loop momenta can be integrated out and a Feynman-parameter representation of the amplitude can be obtained [26]:

$$M(p) = \frac{i^N}{(-16\pi^2)^\ell} \int_0^\infty \prod_{r=1}^{N} d\alpha_r \Delta(\alpha)^{-2} \exp[-iD(\alpha, p)]S(J) ,$$

$$D(\alpha, p) = \sum_{r=1}^{N} \alpha_r m^2_r - P(\alpha, p) ,$$

$$P(\alpha, p) = \sum_{r=1}^{N} \alpha_r J^2_r .$$

Eq. (2) is written in the Schwinger proper-time formalism, or the ‘Nambu representation’. One can recover from it the ‘Chisholm representation’ quoted in [27] by making the substitution $\alpha_r = \bar{\alpha}_r \lambda$, with $\sum_r \bar{\alpha}_r = 1$, and carrying out the integration over $\lambda$.

To understand and to describe the quantities $J, P, \Delta$, and $S$ appearing in (2), it is useful to know that a Feynman diagram can be thought of as a passive electric circuit [26, 29–32], in which $\alpha_r$ takes on the role of resistance and $p_i$ become the currents flowing out of the circuit. The quantity $J_r$ is then the current flowing through the $r$th internal line, and $P$ is the power consumed by the circuit. Explicit formulas for these quantities are available [26],

$$\Delta(\alpha) = \sum_{T_1}^{\ell} (\prod \alpha) ,$$

4
\[
P(\alpha, p) = \Delta(\alpha)^{-1} \sum_{T_2}^\ell (\prod \alpha)(\sum p)^2 ,
\]
and
\[
J_r(\alpha, p) = \pm \Delta(\alpha)^{-1} \sum_{T_2(r)}^\ell \alpha_r^{-1}(\prod \alpha)(\sum p) ,
\]
and they mean the following. An \(\ell\)-loop diagram can be made into a connected tree diagram (a ‘1-tree’) by cutting \(\ell\) lines, and into a diagram with two disjoint trees (a ‘2-tree’) by cutting \(\ell + 1\) lines. \(\Delta(\alpha)\) is given by the sum over the set \(T_1\) of all 1-trees so obtained, with the summand consisting of the product of the \(\alpha\)’s of the cut lines. \(P(\alpha, p)\) is given by the sum over the set \(T_2\) of all 2-trees so obtained, with the summand being the product of the \(\alpha\)’s of the cut lines, times the square of the sum of all the external momenta \(p_i\) attached to one of these two trees. Finally, let \(T_2(r)\) be the set of all 2-trees in which line \(r\) is cut, and such that when the line \(r\) is inserted back a 1-tree results. Then \(J_r(\alpha, p)\) is given by the sum of all 2-trees \(T_2(r)\) with the summand equal to the product \(\alpha\)’s of all the cut lines except the \(r\)th, times the sum of all the external momenta \(p_i\) attached to one of these two trees. The sign \(\pm\) in (7) is determined by the orientation of \(J_r\).

We proceed now to describe the ‘modified spin factor’ \(S(J)\) in (2). It is made up of the sum of several terms,
\[
S(J) = \sum_{k \geq 0} S_k(J) ,
\]
of which the first, \(S_0(J)\), is just the numerator factor \(S_0(q)\) in (1) with \(q\) replaced by \(J\). The other terms \(S_k\) are obtained from \(S_0\) by contracting \(k\) pairs of \(J\)’s in all possible ways according to the rule,
\[
J^\mu_r J^\nu_s \rightarrow -\frac{i}{2} g^{\mu\nu} H_{rs}(\alpha) ,
\]
and summing over all the contracted results. The formula for \(H_{rs}\) is
\[
H_{rr}(\alpha) = -\Delta(\alpha)^{-1} \frac{\partial \Delta(\alpha)}{\partial \alpha_r} ,
\]
\[
H_{rs}(\alpha) = \pm \Delta(\alpha)^{-1} \sum_{T_2(rs)} (\alpha_r \alpha_s)^{-1}(\prod \alpha) , \quad (r \neq s) ,
\]
where \(T_2(rs)\) is the set of all 2-trees with both lines \(r\) and \(s\) cut, and such that when either line \(r\) or line \(s\) is inserted back a 1-tree results. The product of \(\alpha\)’s in (10) are over all the cut lines except the \(r\)th and the \(s\)th. If the lines \(r\) and \(s\) both point to the same tree, then the sign in (10) is +1. If they point to different trees, then the sign is −1.

For practical calculations, eq. (2) may not be the best to use because the contractions leading up to \(S_k(J)\) are relatively complicated to compute. It may often be simpler to use the alternate formula [26]
\[
M(p) = \int_0^\infty (\prod_{r=1}^N d\alpha_r) \Delta(\alpha)^{-2} T \left( -\frac{i}{2} \frac{\partial}{\partial p} A\beta \right) \exp[-iD(\alpha, p)] ,
\]
where the function \( T(q) \) is obtained from the function \( S_0(q) \) by the formula

\[
T(q) = \sum_{k \geq 0} T_k(q) ,
\]

(12)

with \( T_0(q) = S_0(q) \), and other \( T_k(q) \) obtained from \( T_0(q) \) by contracting \( k \) pairs of \( q \)'s according to the rule

\[
q_i^\mu q_s^\nu \rightarrow i \beta_r \delta_{rs} g^{\mu\nu} ,
\]

(13)

and summing over all possible contractions. The quantity \( \beta_r \) is the conductance of the line \( r \):

\[
\beta_r = (\alpha_r)^{-1} .
\]

(14)

The contraction rule (13) is far simpler than the contraction rule (9). For example, if each \( q_r \) does not appear more than once in \( S_0(q) \), then no contraction according to (13) is possible because of the \( \delta_{rs} \) factor, so \( T(q) = S_0(q) \), whereas this is not the case according to (9) and \( S(J) \neq S_0(J) \).

In (11), the argument \( q_r \) in \( T(q) \) is replaced by the operator

\[
d_r \equiv -\frac{i}{2} \sum_i \frac{\partial}{\partial p_i} A_{ir} \beta_r ,
\]

(15)

where \( A_{ir} \) is the incidence matrix of the graph, defined to be +1 if line \( r \) points into the vertex \( i \), -1 if the line points out of the vertex, and 0 if line \( r \) is not connected to the vertex \( i \). Momentum conservation at each vertex can be expressed with the help of this matrix to be

\[
p_i = \sum_r A_{ir} J_r .
\]

(16)

If line \( r \) points from vertex \( i \) to vertex \( j \), then we shall also write it as \( r = (ij) \). In that case (15) simply says that each \( q_{(ij)} \) should be replaced by \((-i/2)\beta_{(ij)}[\partial/\partial p_i - \partial/\partial p_j]\).

Let us now look deeper into the various circuit quantities and their relationships. Some of the relations described below already appeared in [26], but the external-vertex relations and the level-dependent relations have not. It is important to understand these formulas, especially the subtle role of level-dependence, for they will be needed in deriving the multiloop string-like formulas in Sec. 3.

Let \( V_i \) be the voltage at vertex \( i \), and

\[
v_r = v_{(ij)} = V_i - V_j = -\sum_i A_{ir} V_i
\]

(17)

be the voltage drop across the resistor \( r \). The current flowing through that line is

\[
J_r = \beta_r v_r .
\]

(18)
Combining (16)–(18), one gets

\[ p = -(A\beta A^t)V \equiv -YV , \]

(19)

where \( p, V \) are \( n \)-dimensional vectors with components \( p_i \) and \( V_i \) respectively, \( A \) is the \( n \times N \) matrix with matrix elements \( A_{ir} \), and \( \beta = \alpha^{-1} \) is a diagonal \( N \times N \) matrix with diagonal matrix elements \( \beta_r = \alpha_r^{-1} \). The power consumed by the network is then

\[ P = -V \cdot p = VYV . \]

(20)

The absolute level of the voltages \( V_i \) are of course never determined; they can all be shifted by a common constant without changing any physical attribute. This is reflected in (19) by the fact that

\[ \sum_j Y_{ij} = 0 , \]

(21)

which follows algebraically from \( \sum_i A_{ir} = 0 \). Being a symmetric matrix, we must also have \( \sum_i Y_{ij} = 0 \), and in (19) this simply expresses conservation of the external currents. The matrix \( Y \) is singular, so (19) cannot be inverted to obtain \( V \) as a function of \( p \), in agreement with the fact that the absolute level of \( V \) cannot be determined. This level-dependence is analogous to a gauge dependence. It is unphysical, it complicates matter, but often we have no choice but to fix a level scheme (analogous to fixing a gauge) to carry out explicit calculations. For example, we must fix a level scheme before the inversion of (19) can be carried out.

In order to invert (19), let us fix a level scheme by choosing \( V_n = 0 \). We shall use a subscript 0 to denote the remaining \( (n-1) \)-dimensional quantities. Then (19) can be inverted to give

\[ V_0 = -Y_0^{-1}p_0 \equiv -Z_0 p_0 . \]

(22)

Incorporating \( V_n = 0 \), we can enlarge this \( (n-1) \)-dimensional relationship into \( n \)-dimensional relationship

\[ V = -Z'p , \]

(23)

\[ Z' = \begin{pmatrix} Z_0 & 0 \\ 0 & 0 \end{pmatrix} . \]

(24)

We shall refer to this level scheme as the primitive level scheme. The chief advantage of this scheme is that an explicit formula for \( Z' \) is available, via eqs. (24), (22), and (19).

The impedance matrix \( Z' \) is a level-dependent quantity. Because of \( p \)-conservation, a change

\[ Z'_{ij} \rightarrow Z_{ij} = Z'_{ij} + a_i + a_j \]

(25)

for any \( a_i \) simply changes the overall level of \( V_i \) by an amount \( \sum_j a_j p_j \) but it does not change the physical attributes of the network. This level change is analogous to a gauge
transformation. Physically measurable quantities such as the current $J_r$ and the power $P$

\[ J = -\beta A^t V = \beta A^t Z' p = \beta A^t Z p , \]

\[ P = -V p = pZ' p = pZ p = J \alpha J , \]

and their level-independence follows easily from

\[ \sum_i p_i = \sum_i A_{ir} = 0 . \]  

The variation of these level-independent quantities wrt a change of $\alpha$ is most easily calculated in the $V_n = 0$ level scheme. Using (19) and (22)–(24), one gets

\[ \frac{\partial P}{\partial \alpha_s} = p \frac{\partial Z'}{\partial \alpha_s} p = (pZ' A \beta)_s (\beta A^t Z' p)_s = J_s^2 , \]

\[ \frac{\partial J_r}{\partial \alpha_s} = -\beta_s \delta_{rs} J_s + [\beta A^t Z' A \beta]_{rs} (\beta A^t Z' p)_s = H_{rs} J_s , \]

in which we have used the definition of $H_{rs}(\alpha)$:

\[ H(\alpha) = G(\alpha) - \beta , \]

\[ G_{rs} = [\beta A^t Z A \beta]_{rs} = \beta_r \beta_s \{Z_{ik} - Z_{jk} - Z_{il} + Z_{kl}\} , \]

for $r = (ij)$ and $s = (kl)$. The level-independence of (29)–(32) can be easily checked using (28). Note that it is this same quantity $H_{rs}(\alpha)$ that appeared in (9) and (10) [26].

Eqs. (26), (27), (31) and (32) allow us to see directly why (2) and (11) are equivalent. To that end, first notice that the only quantity in the scalar integral in (11) depending on $p$ is $P(\alpha, p)$. A single operator (15) operating on $\exp(iP)$ then brings down $(pZ A \beta)_r = J_r$, which corresponds to the replacement $q_r \rightarrow J_r$ used in (2). If $S_0(q) = q^\mu q^\nu$, then $T(d) = d^\mu r d^\nu s + (i\beta_r / 2) \delta_{rs} g^{\mu\nu}$, so operating on $\exp(iP)$, $T(d)$ brings down

\[ J_r J^\nu_s - \frac{i}{2} \{[\beta A^t Z A \beta]_{rs} g^{\mu\nu} - \beta_r \delta_{rs} g^{\mu\nu}\} = J_r J^\nu_s - \frac{i}{2} H_{rs} g^{\mu\nu} , \]

in agreement with (8) and (9). The equivalence for a more complicated $S_0(q)$ can be seen similarly.

One might think that only level-independent quantities need be considered in the Feynman-parameter representation. Indeed both (2) and (11) contain only level-independent quantities. This of course does not mean that certain calculations cannot be carried out more easily in one level scheme than another. In fact, as we shall see later, the gauge characteristic mentioned in the Introduction is borned out only in the particular level scheme described below.
By using the level freedom (25), one can choose \( a_i \) so that

\[
Z_{ii} = 0 \quad \forall i .
\]  

(34)

It is in this zero-diagonal level scheme that the gauge characteristic emerges. It is also in this scheme that the impedance matrix element \( Z_{ij} \) can be computed most easily using a graphical rule. This rule can be derived from the rule (6) for \( \mathbf{P} \), using (27) and (34). If we replace \((\sum p)^2\) in (6) by \(- (\sum p) \cdot (\sum' p)\), where the two sums are respectively sums of outgoing momenta attached to the two disjoint trees, then clearly there are no \( p_i^2 \) term contributing to (6), and the corresponding level scheme is therefore given by (34). The graphical rule in the zero-diagonal scheme is therefore

\[
Z_{ij} = -\frac{1}{2} \sum_{T_{2}^{ij}} \prod_{\alpha}^{l+1} \alpha , \quad (i \neq j),
\]  

(35)

where the sum is over the set of all 2-trees \( T_{2}^{ij} \) in which the vertices \( i \) and \( j \) lie in two different trees.

The zero-diagonal level scheme of (34) treats all indices symmetrically, whereas the primitive level scheme of (24) does not, though the latter has the simplicity of eliminating an irrelevant degree of freedom in a simple manner. In these respects the two level schemes are respectively analogous to the covariant and physical gauges in electrodynamics, in that the former is manifestly covariant and the latter contains no longitudinal photons.

We will now derive three level-dependent relations (eqs. (43), (47), and (48) below), which will be the basis of the string-like formulas of Secs. 3 and 4. In this regard it is important to note that due to external momentum conservation, neither \( \mathbf{J}_r \) nor \( \mathbf{P} \) is a unique function of the \( n \) variables \( p_i \), so the derivatives \( \partial \mathbf{P} / \partial p_i \) and \( \partial \mathbf{J}_r / \partial p_i \), treating each of the \( n \) variables \( p_i \) as independent, is not uniquely defined. However, as can be seen in (26) and (27), this ambiguity is related to the level-dependence of \( Z \). Once a level scheme is defined, such derivatives become meaningful, though the results are obviously level-dependent. Nevertheless, the combination of the derivatives in (15) is still level-independent.

We shall call a vertex with two internal lines (and any number of external lines) an external vertex, and any vertex with more than two internal lines an internal vertex. This terminology is unfortunately somewhat misleading because internal vertices can contain external lines as well. At an external vertex \( a \) we will make the convention that one of the two internal lines points into the vertex \( (A_{ar} = +1) \), and the other points out of the vertex \( (A_{ar} = -1) \). The line upstream of the vertex (with \( A_{ar} = +1 \)) will also be labelled as line \( a'' \), and the line downstream of the vertex will be labelled as line \( a' \). Current conservation at that vertex then takes the form

\[
p_a = J_{a''} - J_{a'} .
\]  

(36)
We shall see that external vertices possess special properties not shared by the internal vertices.

For each external vertex \( a \), define the operator
\[
\partial_a = -\sum_r A_{ar} \frac{\partial}{\partial \alpha_r} = -\frac{\partial}{\partial \alpha_a''} + \frac{\partial}{\partial \alpha_a'}.
\]

(37)

An important property of an external vertex \( a \) is that \( \Delta(\alpha), Z_{ij} \) and \( H_{rs} \) depends on \( \alpha_a' \) only through the combination \( \alpha_a' + \alpha_a'' \), provided \( i \neq a \neq j \) and \( A_{ar} = A_{as} = 0 \), as can be seen from the graphical rules (5), (10), and (35). As a consequence,
\[
\partial_a \Delta(\alpha) = \partial_a H_{rs} = \partial_a Z_{ij} = 0 \quad (38)
\]

provided \( i \neq a \neq j \) and \( A_{ar} = A_{as} = 0 \).

Suppose \( a \) is an external vertex and \( s \) any internal line. Then it follows from (10) that
\[
H_{a's} = H_{a''s}. \quad (39)
\]

To see that suppose first that lines \( a', a'', s \) are all different. Then the claim follows because there is a 1–1 correspondence between 2-trees \( t_2(a's) \) and \( t_2(a''s) \): instead of cutting the line \( a' \) to form a 2-tree, just cut the line \( a'' \). Note that lines \( a' \) and \( a'' \) belong to the same loop so they cannot be cut simultaneously either in \( t_2(a's) \) or in \( t_2(a''s) \). With this 1–1 correspondence, the equality in (39) follows because \( \Delta H_{a's} \) does not contain \( \alpha_a' \) and \( \Delta H_{a''s} \) does not contain \( \alpha_a'' \). Now suppose lines \( a' \) and \( s \) are the same. Then we must show that \( H_{a'a'} = H_{a''a'} \). Since \( \Delta H_{a'a'} = -\partial \Delta / \partial \alpha_a' \), the terms in \( \Delta \) linear in \( \alpha_a' \) is \( -\alpha_a' H_{a'a'} \). We shall now show that it is also given by \( -\alpha_a' H_{a''a'} \), thereby showing that \( H_{a'a'} = H_{a''a'} \). To do so, consider any 2-tree \( t_2(a''a') \) used to compute \( H_{a''a'} \). Since the lines \( a' \) and \( a'' \) are now adjacent to each other, one of the two trees in \( t_2(a''a') \) simply consists of the vertex \( a \), and the other is a tree obtained from the original diagram with the vertex \( a \) removed. There is a 1–1 correspondence between such a tree and a 1-tree of the original diagram when line \( a' \) is cut. Hence the term of \( \Delta \) linear in \( \alpha_a' \) is \( -\alpha_a' H_{a''a'} \). This completes the proof of (39).

Eqs. (38) and (39) are special features of the external vertices not generally shared by internal vertices. We shall refer to them as external-vertex relations.

If \( a, b \) are external vertices, then \( \partial_a P \) can be computed from (29) and (36) to be
\[
\partial_a P = -\sum_r A_{ar} J_r^2 = -J_{a''}^2 + J_{a'}^2 = -p_a \cdot (J_{a''} + J_{a'}) \quad . \quad (40)
\]

Alternatively, using (28) and (38), one gets
\[
\partial_a P = \partial_a (\sum_i p_i Z_{ij} p_j) = 2p_a \cdot \partial_a (\sum_j Z_{aj} p_j) \quad . \quad (41)
\]
Equating (40) and (41), one concludes that
\[ p_a \cdot [J_{a''} + J_{a'} + 2 \partial_a (\sum_j Z_{aj} p_j)] = 0 \] (42)
for every possible \( p \), subject of course only to the restriction \( \sum_i p_i = 0 \). This means that
\[ J_{a''} + J_{a'} = -2 \partial_a (\sum_j Z_{aj} p_j) = 2 \partial_a V_a = -2 \sum_j \dot{Z}_{aj} p_j = \dot{2} V_a . \] (43)

For an external vertex \( a \), it is also useful to define the operator
\[ D_{a}^{\mu} = \partial_a \frac{\partial}{\partial(p_a)_\mu} . \] (44)

Then it is clear from (43) that we can also write
\[ J_{a''} + J_{a'} = -D_a P . \] (45)

Next, employ (29) and (30) to compute \( \partial_a \partial_b P \):
\[ \partial_a \partial_b P = 2 \sum_{r,s} A_{ar} A_{bs} J_r H_{rs} J_s . \] (46)

Since \( a, b \) are external vertices, (40) is true, so using (36) and (39), we get
\[ \partial_a \partial_b P = 2 p_a \cdot p_b H'_{a'b'} . \]

Suppose \( a \neq b \). Then an alternative calculation is to use (38):
\[ \partial_a \partial_b P = \partial_a \partial_b (\sum_{i,j} p_i Z_{ij} p_j) = 2 p_a \cdot p_b \partial_a \partial_b (Z_{ab}) . \]

Again these two expressions must be equal for all momentum configurations, so
\[ H_{a''a'} g^{\mu\nu} = \partial_a \partial_b (Z_{ab}) g^{\mu\nu} = \frac{1}{2} D^{\mu}_{a} D_{b}^{\nu} P \equiv \ddot{Z}_{ab} g^{\mu\nu} \quad (a \neq b) , \] (47)
where (38) has again be used to obtain the last equality. For \( a = b \), (47) is not longer valid. Instead, (39) can be used to relate \( H_{a'a''} \) to \( H_{a'a''} \), which can then be calculated using (47).

There is one more relation for an external vertex \( a \) which we need,
\[ D_{a}^{\mu} J_{s}^{\nu} = -H_{rs} g^{\mu\nu} , \quad (A_{as} = 0) , \] (48)
which is true for \( r = a' \) or \( r = a'' \), and for \( a' \neq s \neq a'' \). To prove it, consider a situation where the primitive spin factor is \( S_0(J) = -(J_{a'} + J_{a''}) \mu^\nu J_s^\nu \). The modified spin factor \( S(J) \) is, according to (9) and (39), given by \( S(J) = S_0(J) + iH_{rs}g^\mu^\nu \). On the other hand, using (11)–(15), and the assumption that \( a' \neq s \neq a'' \), \( S(J) \) is also given by \( \exp(-iP)S_0(d)\exp(iP) \). Using (45), this becomes \( -i\exp(-iP)d^\nu D_a^\mu \exp(iP) = -i\exp(-iP)D_a^\nu J_s^\nu \exp(iP) \), which is equal to \( -i(D_a^\mu J_s^\nu - (J_{a'} + J_{a''}) \mu^\nu J_s^\nu) \). Comparing these two ways of obtaining \( S(J) \), (48) follows.

As it stands, (48) is not true when \( s = a' \) or \( a'' \). For example, using the explicit example to be considered in Sec. 5, one can show explicitly that when \( a = 1 \) and \( s = 2 \), only half of the rhs of (48) is obtained from the lhs.

It is this lack of universal validity of (48) that restricts the validity of the string-like formula, as we shall see in the next section.

Because of the importance of this restriction it should also be pointed out that there is a false derivation of (48) which makes it appear to be valid for all \( a \) and \( s \). The false argument follows from (30), (36), (37), (39), and (44). It is false because strangely enough, (36) is not a valid equation for the present application, where \( D_a \) and hence derivative of external momenta are involved. Eq. (36) is of course valid in the physical case when external momentum conservation (28) is used. On the other hand, since the operation involving \( D_a \) is level-dependent, we must not use this conservation law before the momentum differentiation. In that case the rhs of (36) could be equal to, say, \( -\sum_{b \neq a} p_b \), which is not at all the same thing as the lhs of (36) as far as momentum differentiation is concerned. The correct way to check the validity of (36) is to compute the rhs of (36) using (26), sticking to whatever level scheme one is using and refraining from ever using external momentum conservation before the momentum differentiation. Without using momentum conservation, the currents \( J \) is no longer level-independent, hence the outcome of the rhs of (36) depends on the level scheme one uses. In this way one can show by explicit examples that (36) is generally invalid in the zero-diagonal level scheme. This observation once again shows the subtlety of the level dependence problem.

Let us turn to amplitudes in the configuration space. Since field theories are local in \( x \), interactions should look simpler in the \( x \)-space than in the \( p \)-space. For conceptual reasons it is therefore worthwhile to look at the expressions in the \( x \)-space, although in practical calculations we must return to the momentum space. Fourier-transforming the momentum-space amplitude (1), we get

\[
(2\pi)^4i^4M(p)\delta^4\left(\sum_{i=1}^{n} p_i\right) = \int \left(\prod_{i=1}^{n} d^4 x_i\right) \exp \left[i \sum_{i=1}^{n} p_i \cdot x_i\right] M'(x) \, ,
\]

\[
M'(x) = S_0(-i\partial/\partial y) \prod_{r=1}^{N} \Delta_+(y_r) \, ,
\]
where the scalar propagator is given by

\[
\Delta_+(y_r) = \frac{1}{(2\pi)^4} \int d^4 q_r \frac{\exp[i q_r \cdot y_r]}{m^2_r - q^2_r - i\epsilon} = \frac{1}{16\pi^2} \int_0^\infty d\beta_r \exp \left[-\frac{m^2_r}{\beta_r} + \frac{i}{4} \beta_r y_r^2 \right] = \frac{1}{16\pi^2} \int_0^\infty d\alpha_r \frac{\exp \left[-i\alpha_r m^2_r + i \frac{y_r^2}{4\alpha_r} \right]}{\alpha_r^2}.
\]

Hence

\[
M'(x) = \left(\frac{1}{16\pi^2}\right)^N \int_0^\infty \left(\prod_{r=1}^N d\beta_r\right) S_0 \left(\frac{1}{2} \beta_r y_r\right) \exp \left[-i \sum_r \alpha_r m^2_r + iP\right],
\]

\[
P = \frac{1}{4} \sum_{r=1}^N \beta_r y_r^2.
\]

For a concrete example see eq. (70) below. In (50) and (52), we must substitute \(y_r = x_i - x_j\) for an \(r = (ij)\) and as usual, \(\beta_r = 1/\alpha_r\). In the electric circuit analogy, \(x_i/2\) is really the potential \(V_i\) at vertex \(i\), so \(y_r/2\) is the potential drop \(v_r\) across resistance \(r\) [26, 29–32]. \(P\) is again equal to the power consumed by the network. One can also obtain similar formulas and interpretations when some of the \(x_i\)’s are integrated over. For details, see [26].

A Feynman diagram can also be given a different interpretation [26, 29–32] as a maze in which a particle moves in. The off-shell four-momentum of the particle along an internal line \(r\) is \(J^\mu_r\); the four-distance it travels along \(r\) is \(y^\mu_r = x^\mu_i - x^\mu_j\), and this is accomplished in an amount of ‘proper time’ equal to \(2m\alpha_r\). Incidentally, it is perhaps more appropriate to think of the object travelling around the maze as a quantum mechanical wave rather than a classical particle because it is off-shell, and because it is easier to think of a wave splitting and recombining at vertex junctions than a classical particle. In this interpretation, the integrand in the last expression of (51) is essentially the overlap matrix element \(\langle x_i\tau_i|x_j\tau_j\rangle\) under the Hamiltonian \(m(\dot{x}(\tau)^2 + 1)/2\) where \(\tau = \tau_i - \tau_j = 2m\alpha_r\) is the proper time elapsed.

This interpretation exists for all diagrams and it can perhaps be thought of as the basis for a multiloop generalization of the string approach [22–24] and the first quantized approach [25]. There is however a fundamental difference between a string propagation on a multi-genus worldsheet and this particle propagating in the Feynman-diagram maze. The former satisfies the wave equation everywhere on the worldsheet, whereas the latter can be described by free-particle equation of motion only between vertices. Vertices are singularities where interactions take place, where free-particle equation of motion breaks down. No such singularities are present on the worldsheet on account of string’s conformal invariance [33], but this conformal invariance is lost in the infinite tension limit when the worldsheet collapses into a network of worldlines because the \(\sigma\)-variable along the string disappears. Reparametrization invariance in the \(\tau\) variable can however still be kept as is done in [25],
thereby preserving the string characters and the string-like formulas. As a result of the
difference, although a string amplitude can be written as a path integral of a free-string
over multi-genus worldsheets, a free-particle path integral no longer exists for Feynman
diagrams with internal vertices, unless their associated singularities and interactions can
somehow be incorporated, as is done in the Feynman-parameter representation.

3. Scalar QED

There are two kinds of electromagnetic vertices in scalar electrodynamics: the cubic
vertex of Fig. 1,

\[ C_j = e \varepsilon_j \cdot (q_j'' + q_j') , \]  
(53)

and the quartic (seagull) vertex of Fig. 2,

\[ Q_j = e^2 \varepsilon_j \cdot \varepsilon_{j+1} , \]  
(54)

with \( \varepsilon_j \) being the polarization vector for the \( j \)th photon. We are interested in finding a
string-like formula for standard diagrams, which by definition are diagrams without seagull
vertices.

Fig. 3 represents a one-loop \( n \)-photon standard diagram. All other one-loop standard
diagrams are obtained from it by permuting the external photon lines. The amplitude for
Fig. 3 can be derived from the string or the first quantized approach \([23,25]\) to be

\[ M(p) = - \frac{e^n}{16\pi^2} \int (\prod_{i=1}^{n} dt_i) t_n^{-2} \]

\[ \langle \exp \left\{ -im^2 t_n + i \sum_{i<j} [p_i \cdot p_j G^{ij}_B - (\varepsilon_i \cdot p_j - \varepsilon_j \cdot p_i) \dot{G}^{ij}_B + \varepsilon_i \cdot \varepsilon_j \ddot{G}^{ij}_B] \right\} \}. \]  
(55)

This formula resembles a formula first obtained by Bern and Kosower \([22]\) from string
theory for QCD so it shall be referred to as a string-like formula. The integration region
in (55) is

\[ 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < \infty ; \]  
(56)

the notation \( \langle f(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \rangle \) means that only terms multilinear in all the \( \varepsilon_i \)'s in \( f \) should
be kept. In other words, if \( \theta_i \) are Grassmann variables,

\[ \langle f(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \rangle = \pm \int d\theta_1 d\theta_2 \cdots d\theta_n f(\theta_1 \varepsilon_1, \theta_2 \varepsilon_2, \cdots, \theta_n \varepsilon_n) . \]  
(57)

The function \( G^{ij}_B \) and its derivatives are functions of \( t_{ij} = t_i - t_j \) whose explicit expressions
for \( i < j \) are

\[ G^{ij}_B = t_{ij}(t_n + t_{ij})/t_n , \]  
(58)

\[ \dot{G}^{ij}_B = \frac{\partial G^{ij}_B}{\partial t_i} = (t_n + 2t_{ij})/t_n , \]  
(59)

\[ \ddot{G}^{ij}_B = \frac{\partial^2 G^{ij}_B}{\partial t_i^2} = 2/t_n . \]  
(60)
One can also add in $\delta$-functions to simulate the seagull vertex contributions \[25\] but we shall not do that here.

What is crucial in this formula is that the various terms appearing in the exponent between square brackets are all given by the functions $G_{ij}^B$ and their derivatives. It is this \textit{gauge characteristic} that allows the integration by parts (IBP) technique discussed in the Introduction to be carried out. We shall come back to a discussion of the gauge characteristic after we derive the multiloop string-like formula in scalar QED that exhibits this feature.

As we shall see below, a string-like formula with the gauge characteristic exists for any process to any number of loops, provided the following technical restriction is obeyed. The formula turns out to look very similar to (55) when no derivative couplings are present, \textit{i.e.}, when there are no internal photon lines and when the ‘strong interaction’ between scalar particles contains no derivative couplings. In the more general situation, a string-like formula with the gauge characteristic still exists, but it looks a bit more complicated.

Derivative couplings are present in electromagnetic cubic vertices (53), and perhaps in the ‘strong interactions’ between the scalar particles. All of them contribute to the primitive spin factor $S_0(q)$ in eq. (1). The technical restriction mentioned above is imposed so that eq. (48) can be used. As we shall see below, this means that if $a$ is an \textit{external} vertex, then $S_0(q)$ should \textit{not} contain any $q_a$ or $q_a'$ other than those in $C_a$ of (53). The notations are those used in Sec. 2: $a''$ and $a'$ are respectively the lines pointing into and out of the external vertex $a$. This restriction means for example that if there are internal photon propagators, then these internal vertices should not be adjacent to an external vertex.

To obtain a string-like formula for multiloops, it is necessary to find out first what is the generalization of $G_{ij}^B$. By comparing the terms quadratic in $p$ in the exponents of (55) with (2) and (27), it is clear that if anything works $G_{ij}^B$ has to be $Z_{ij}/2$. However, this alone does not tell us in what level-scheme should $Z_{ij}$ be expressed so as to capture the gauge characteristic. Before we solve this problem there is however a second problem that needs to be considered.

In one loop, $G_{ij}^B$ is a function of the ‘time’ difference $t_{ij} = t_i - t_j$. Time is connected to the Feynman parameters of Fig. 3 by

\[ t_i = \sum_{k=1}^{i} \alpha_k . \tag{61} \]

For multiloop diagrams, $Z_{ij}$ is a rather complicated function of $\alpha_i$, and it is impossible in general to define a ‘time’ parameter $t_i$ so that $Z_{ij}$ is a function only of $t_i - t_j$. This is related to the fact that the particle in the more general diagram has to split and recombine. Given that, the next important question to solve for multiloops is to determine what should replace the time derivatives in (59) and (60). It turns out that the time derivative $\partial/\partial t_a$
should in the general case be replaced by $\partial_a$ defined in (37), and in so doing the gauge characteristic will be preserved provided that $Z_{aj}$ is expressed in the zero-diagonal level scheme of (34).

We shall use the indices $a, b$ to denote cubic electromagnetic vertices (53) with an external photon line, and the indices $i, j$ to label all the vertices. Note that $a, b$ are external vertices in the sense of Sec. 2, viz., two internal lines are connected to each of them. In the simpler case when internal photon lines and derivative couplings between the scalar particles are absent, the multiloop string-like formula for any multiloop standard diagram is

$$M(p) = \frac{i^N - n e^n}{(-16 \pi^2)^l} \int_0^{\infty} \prod_{r=1}^{N} d\alpha_r \Delta(\alpha)^{-2} S_0'' \left\langle \exp \left\{ -i \sum_{r=1}^{N} \alpha_r m_r^2 + i \sum_{i,j} p_i \cdot p_j Z_{ij} - 2 \sum_{a,j} \epsilon_a \cdot p_j \dot{Z}_{aj} + \sum_{a,b} \epsilon_a \cdot \epsilon_b \ddot{Z}_{ab} \right\} \right\rangle,$$

where

$$\dot{Z}_{aj} = \partial_a Z_{aj}, \quad \ddot{Z}_{ab} = \partial_a \partial_b Z_{ab},$$

with $Z_{ii} = 0$ and $\partial_a$ defined in (37). A similar but slightly more complicated formula ((76) below) exists when derivative couplings are present, as long as the technical restriction mentioned earlier is obeyed.

We will proceed now to demonstrate (62). The primitive spin factor $S_0(J)$ for any standard diagram is now given by $S_0(J) = S_0'(J)S_0''$, where

$$S_0'(J) = \prod_a C_a = \prod_a \left[ e^{\epsilon_a \cdot (J_a' + J_a'')} \right] = (-ie)^n \left\langle \exp \left\{ i \sum_a \epsilon_a \cdot (J_a' + J_a'') \right\} \right\rangle$$

is the primitive spin factor for the external cubic electromagnetic vertices, and $S_0''$ is the momentum-independent vertex factors of the rest. Using (43), this becomes

$$S_0'(J) = (-ie)^n \left\langle \exp \left\{ -2i \sum_{a,j} \epsilon_a \cdot p_j \dot{Z}_{aj} \right\} \right\rangle.$$

To compute the modified spin factor $S(J)$, the contraction rule (9) is first used to compute the contraction of a pair of electromagnetic vertices. After using (47), this becomes

$$C_a C_b \rightarrow -2ie^2 \epsilon_a \cdot \epsilon_b H_{ab} = -2ie^2 \epsilon_a \cdot \epsilon_b \ddot{Z}_{ab}.$$

Then (8) is used to sum up all the contractions, giving

$$S(J) = (-ie)^n S_0'' \left\langle \exp \left\{ i \sum_{a,j} [-2\epsilon_a \cdot p_j \dot{Z}_{aj}] + i \sum_{a,b} [\epsilon_a \cdot \epsilon_b \ddot{Z}_{ab}] \right\} \right\rangle.$$
Substituting this into (2), we obtain the string-like formula (62).

The string formula (62) can be simplified by noting that

\[ S(J) \exp(iP) = (-ie)^n \langle \exp \left( -\sum_a \epsilon_a \cdot D_a \right) \rangle S''_0 \exp(iP) , \]

\[ P = \sum_{i,j} p_i Z_{ij} p_j , \]

where \( D_a \) is defined in (44).

From the discussions at the end of Sec. 2, one expects the amplitude to be particularly simple when expressed in the configuration space. Indeed, using (69) and the definition of the configuration-space amplitude (49), one gets

\[ M'(x) = (-ie)^n \langle \exp \left[ -\sum_a \epsilon_a \cdot \partial_a x_a \right] \rangle \prod_{r=1}^N \Delta_+ (y_r) . \]

The string-like formula (62) exhibits a simple gauge transformation property, which is probably the reason why it is inherently important. This is the gauge characteristic we have been talking about. This gauge property is particularly transparent when (62) is written in the form of (68), for then a gauge transformation

\[ \epsilon_a \to \epsilon_a + p_a \lambda_a \]

simply brings on the following change of the integrand of (2):

\[ (-ie)^n \langle \exp \left[ -\sum_a \epsilon_a \cdot D_a \right] \rangle \Delta^{-2} S''_0 \exp(-iD) \to \]

\[ (-ie)^n \langle \exp \left[ -\sum_a (\epsilon_a + \lambda_a p_a) \cdot D_a \right] \rangle \Delta^{-2} S''_0 \exp(-iD) . \]

Note that the first equation in (38) has been used, and we have also used the fact that lines \( a' \) and \( a'' \) have the same mass. Since the gauge parameters \( \lambda_a \) are multiplied by the derivatives \( \partial_a = \partial/\partial \alpha_{a'} - \partial/\partial \alpha_{a''} \), they appear only in surface terms corresponding to diagrams with \( \alpha_{a'} \) or \( \alpha_{a''} \) short-circuited. It is these surface terms that connect the permuted standard diagrams and the seagull diagrams to enable gauge-dependent terms to be cancelled out at the end. In this way the string-like formula makes the Ward-Takahashi identity in Feynman-parameter space almost explicit as the one seen in momentum space, and it is also this same gauge characteristic which enables the IBP technique to be applied.

We turn now to the general case where internal photon lines and ‘strong interaction’ with derivative couplings are allowed to be present in the standard diagram, provided the technical restriction mentioned earlier is met. The primitive spin factor is then given by

\[ S_0(J) = S'_0(J) S''_0(J) , \]
where $S'_0(J)$ is the contribution from the external electromagnetic vertices which is given by (64), and $S''_0(J)$ is the contribution from the rest of the vertex factors which is now momentum dependent. The modified spin factor $S(J)$ is obtained from (74) by binary contractions with the rule of (9). When the technical restriction is met, contractions with the currents at an external vertex $a$ may be accomplished by the operator $D_a$ as in (48), hence we obtain a relation similar to (68), which now reads

$$S(J) \exp(iP) = (-ie)^n \langle \exp \left[ -\sum_a \epsilon_a \cdot D_a \right] \rangle S''(J) \exp(iP),$$

(75)

where $S''(J)$ is the modified spin factor corresponding to the primitive spin factor $S''_0(J)$ of (74). Substituting this into (2), we finally obtain the string-like formula in the general case to be

$$M(p) = \frac{i^{N-n}e^n}{(-16\pi^2)^{N/2}} \int_0^\infty \prod_{r=1}^N d\alpha_r \langle \exp \left[ -\sum_a \epsilon_a \cdot D_a \right] \rangle \cdot \Delta(\alpha)^{-2} S''(J) \exp \left[ -i \sum_{r=1}^N \alpha_r m_r^2 + i \sum_{i,j} p_i \cdot p_j Z_{ij} \right].$$

(76)

Again, the first equality of (38) as well as the fact electromagnetic interaction is diagonal in mass have been used. The gauge transformation property of this is similar to (73). Note that $\exp[ -\sum_a \epsilon_a \cdot D_a]$ as defined in (44) is a translation operator shifting momentum $p_a$ by an amount $-\epsilon_a \cdot \partial_a$. If we carry out this momentum shift in (76), the exponential will return to the form (62), but the momenta $p$ implicitly contained in $S''(J)$ must be so shifted as well, and this shift contains the gauge-dependent quantity $\epsilon_a$. We prefer not to write it in this shifted form for it makes the gauge transformation property more obscure.

### 4. Spinor QED

In spinor electrodynamics, an $n$-photon one-loop amplitude has a string-like formula given in [23,25]. To obtain its multiloop generalization, a Gordon decomposition has to be made to separate the current into a convective part and a spin part.

Consider a fermion propagating in the presence of a background electromagnetic potential $A^\mu(x)$ and a background neutral scalar field $\phi(x)$. Depending on what is required, we can later on replace the background $A^\mu(x)$ by a polarization vector or one end of an internal photon propagator, and the background $\phi(x)$ by an external or internal ‘neutral scalar meson’ coupled to the fermion by ‘strong interaction’. For definiteness we shall assume a Yukawa coupling for the strong interaction, but this point is not crucial for the following discussion. The fermion propagator is $[m - i\gamma(\partial - ieA) + \phi]^{-1} \delta^4(x - y)$; perturbation series are obtained by expanding this in power series of $A$ and $\phi$. Gordon decomposition is accomplished by noting that

$$[m + \phi - i\gamma(\partial - ieA)]^{-1} = [m + \phi + i\gamma(\partial - ieA)] \cdot I,$$

(77)
where
\[
I = \{(m + \phi - i\gamma(\partial - ieA))[m + \phi + i\gamma(\partial - ieA)]\}^{-1}
\]
\[
= \{(m + \phi)^2 - i\gamma(\partial\phi) + (\partial - ieA)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\}^{-1}
\]
\[
= (m^2 + \partial^2)^{-1}\sum_{n=0}^{\infty}[(C + Q + S + M_1 + M_2 + M_3)(m^2 + \partial^2)^{-1}]^n,
\]
(78)

and
\[
\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu],
\]
\[
C = ie(\partial A + A\partial)
\]
\[
Q = e^2A^2
\]
\[
S = \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}
\]
\[
M_1 = -2m\phi
\]
\[
M_2 = -\phi^2
\]
\[
M_3 = i\gamma(\partial\phi).
\]
(79)

C and Q are just the cubic and the seagull electromagnetic vertices for a scalar particle. They give rise to the convective part of the current. On top of these, there is the spin vertex S which gives rise to the magnetic moment of the fermion. The vertices $M_i$ are strong interaction vertices between the fermion and the neutral scalar meson. Note that $M_3$ is both momentum and spin dependent.

In a scattering diagram, a fermion propagator can end in two external fermions, in which case the relevant factor is
\[
\bar{u}(m - i\gamma\partial)[m - i\gamma(\partial - ieA) + \phi]^{-1}(m - i\gamma\partial)u =
\]
\[
\frac{1}{2m}\bar{u}\sum_{n=0}^{\infty}[(C + Q + S + M_1 + M_2 + M_3)(m^2 + \partial^2)^{-1}]^n(m^2 + \partial^2)u,
\]
(80)
or with the wave functions $u, \bar{u}$ replaced by $v, \bar{v}$. Otherwise, it can close on itself in a loop, in which case the proper factor is
\[
\log\{\det[m - i\gamma(\partial - ieA) + \phi]\} = \frac{1}{2}\log\{\det[I]\}
\]
\[
= \frac{1}{2}\text{Tr}\{\log\left[1 + \sum_{n=1}^{\infty}((C + Q + S + M_1 + M_2 + M_3)(m^2 + \partial^2)^{-1})^n\right]\}.
\]
(81)

A constant irrelevant normalization factor has been dropped in the last expression. In other words, other than signs associated with statistics, the electromagnetic interaction of spinor QED differs from that of scalar QED only in having the extra vertex $S$. 

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The rest of the discussion is identical to the scalar case, leading up to the string-like equation (76) for standard diagrams, where \( S''(J) \) is now the modified spin factor for the vertices \( S, M_i \), and the internal \( C \)’s. The technical restriction mentioned in Sec. 3 is automatically fulfilled for the derivatively-coupled vertex \( M_3 \) because it depends only on the momentum of the neutral meson but not of the fermions.

To establish contact with the formula in [22–25], let us specialize to the a one-fermion-loop \( n \)-photon amplitude in the absence of all strong interactions. Then

\[
S''(J) = \prod_c S_c = \text{tr} \left\{ \prod_c [-ie\sigma_{\mu\nu}e_c^\mu P_c^\nu] \right\}. \tag{82}
\]

The trace of a product of \( 2m \) \( \gamma \)-matrices is given by 4 times the sum of all signed contractions, with each signed contraction given by a product of \( m \) factors of \( (\pm g_{\alpha\beta}) \)'s, corresponding to the contraction of \( (\cdots \gamma_\alpha \cdots \gamma_\beta \cdots) \). Using this rule, it is easy to compute the trace

\[
\frac{1}{4} \left( \frac{2}{i} \right)^m \text{tr} \left[ (\epsilon_1 \cdot \sigma \cdot p_1) \cdots (\epsilon_m \cdot \sigma \cdot p_m) \right] \tag{83}
\]
as follows. Take any permutation \( t \in P_m \) of \( m \) objects and express it into cycles: \( t = (t_1t_2\cdots t_k)(\cdots) \cdots \). Then (83) is given by

\[
\eta_t \left[ (\epsilon_{t_1} \cdot p_{t_2})(\epsilon_{t_2} \cdot p_{t_3})\cdots(\epsilon_{t_k} \cdot p_{t_1}) \right][\cdots] \cdots \tag{84}
\]
summed over all permutations \( t \in P_m \), and summed over all possible interchanges between every pair

\[
\epsilon_{t_i} \leftrightarrow p_{t_i}. \tag{85}
\]
\( \eta_t \) is the signature of the permutation, being \( \pm 1 \) for even/odd permutations, and a minus sign is to be associated with each interchange (85).

It is these terms in (85) that give rise to the functions \( G^{ij}_F \) and their associated rules in the formula of [22–25].

5. An Example

To illustrate some of the circuit quantities and relations, let us consider the two-loop Compton amplitude of Fig. 4. The dash lines are photons, and the solid lines are charged scalar mesons. The external vertices are \( a = 1, 3 \), and the ‘internal’ vertices are \( j = 2, 4 \). Using (35), one obtains the impedance matrix in the zero-diagonal scheme to be

\[
\Delta = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) + \alpha_5(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)
\]

\[
Z_{ii} = 0
\]

\[
Z_{ij} = Z_{ji}
\]

\[
-2\Delta Z_{12} = \alpha_2[(\alpha_3 + \alpha_4)(\alpha_1 + \alpha_5) + +\alpha_1\alpha_5]
\]

20
2\Delta Z_{13} = (\alpha_1 \alpha_2 \alpha_3 + \alpha_1 \alpha_2 \alpha_4 + \alpha_1 \alpha_3 \alpha_4 + \alpha_2 \alpha_3 \alpha_4) + \alpha_5 (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3)
-2\Delta Z_{14} = \alpha_1[(\alpha_3 + \alpha_4)(\alpha_2 + \alpha_5) + \alpha_2 \alpha_5]
-2\Delta Z_{23} = \alpha_3[(\alpha_1 + \alpha_2)(\alpha_4 + \alpha_5) + \alpha_4 \alpha_5]
-2\Delta Z_{24} = \alpha_5(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)
-2\Delta Z_{34} = \alpha_4[(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_5) + \alpha_3 \alpha_5)] .

(86)

From these and the definition (37), one can compute the various derivatives \( \dot{Z}_{aj} = \partial_a Z_{aj} \) and \( \ddot{Z}_{ab} = \partial_a \partial_b Z_{ab} \) to be

\[
-2\Delta \dot{Z}_{12} = (a_2 - a_1)(\alpha_4 + \alpha_5) - \alpha_5(-\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4) \\
-2\Delta \dot{Z}_{13} = (a_2 - a_1)(\alpha_3 + \alpha_4) + \alpha_5(-\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4) \\
-2\Delta \dot{Z}_{14} = (a_2 - a_1)(\alpha_3 + \alpha_4) + \alpha_5(-\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \\
-2\Delta \dot{Z}_{31} = (a_4 - \alpha_3)(\alpha_1 + \alpha_2) + \alpha_5(\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4) \\
-2\Delta \dot{Z}_{32} = (a_4 - \alpha_3)(\alpha_1 + \alpha_2) + \alpha_5(\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4) \\
-2\Delta \dot{Z}_{34} = (a_4 - \alpha_3)(\alpha_1 + \alpha_2) + \alpha_5(-\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4) \\
-2\Delta \dot{Z}_{13} = 2\alpha_5 .
\]

(87)

Note that the one loop \( n \)-photon relationship \( \dot{Z}_{13} = -\dot{Z}_{31} \) is no longer valid.

These are all the quantities needed in the string-like formula (62). The string-like formula is obtained by using the external-vertex formulas (38) and (39), as well as the level-dependent relations (43), (47), and (48). Let us look into the explicit form of these relations for

the external vertex \( a = 1 \) in the present example. For \( a = 1 \), we see from Fig. 4 that \( a' = 1 \) and \( a'' = 2 \), so \( \partial_1 = \partial / \partial \alpha_1 - \partial / \partial \alpha_2 \). The quantities relevant to the level-dependent relations are

\[
\Delta(J_1 + J_2) = 2 \dot{V}_1 = (\alpha_1 - \alpha_2)(\alpha_3 + \alpha_4 + \alpha_5)p_1 \\
+ \alpha_5[(\alpha_3 + \alpha_4)(p_4 - p_2) + (\alpha_3 - \alpha_4)p_3]
\]

\[
\Delta(D_1^{\mu} J_{1'}^\nu) = \Delta(D_1^{\mu} J_{2'}^\nu) = \frac{1}{2}(\alpha_3 + \alpha_4 + \alpha_5)g^{\mu\nu} - \frac{1}{2}\Delta H_{11}g^{\mu\nu} = -\frac{1}{2}\Delta H_{12}g^{\mu\nu} \\
\Delta(D_1^{\mu} J_{3'}^\nu) = \Delta(D_1^{\mu} J_{4'}^\nu) = \alpha_5g^{\mu\nu} = -\Delta H_{13}g^{\mu\nu} = -\Delta H_{14}g^{\mu\nu} = -\Delta \ddot{Z}_{13}g^{\mu\nu} \\
\Delta(D_1^{\mu} J_{5'}^\nu) = (\alpha_3 + \alpha_4)g^{\mu\nu} = -\Delta H_{15}g^{\mu\nu} .
\]

(88)

Consequently (43) and (47) are valid, as they should be, and (48) is true when \( a' \neq s \neq a'' \), but is not true when \( s = a' \) or \( a'' \). In the latter cases an extra factor of 1/2 appears. From (86) and (88), it is also easy to check explicitly the external-vertex relations (38) to be correct. We have not written down \( H_{2s} \), but they can be calculated explicitly to see that (39) is indeed valid.

Acknowledgement
I am grateful to Zvi Bern, Matt Strassler, Frank Wilczek, and Tung-Mow Yan for stimulating discussions. This research is supported in part by the Natural Sciences and Engineering Research Council of Canada and the Québec Department of Education.
References

[1] F.A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, W. Troost, and T.T. Wu, *Phys. Lett.* **103B** (1981), 124; *Nucl. Phys.* **B206** (1982), 61; *ibid.* **239** (1984), 382; *ibid.* **239** (1984), 395; *ibid.* **264** (1986), 243; *ibid.* **264** (1986), 265.

[2] P. De Causmaecker, R. Gastmans, W. Troost, and T.T. Wu, *Phys. Lett.* **105B** (1981), 215; *Nucl. Phys.* **B206** (1982), 53.

[3] Z. Xu, D.-H. Zhang, and L. Chang, Tsinghua University Preprints, Beijing, China, TUTP-84/4, TUTP-84/5, TUTP-84/6; *Nucl. Phys.* **B291** (1987), 392.

[4] F.A. Berends and W.T. Giele, *Nucl. Phys.* **B294** (1987), 700; *ibid.* **306** (1988), 759; *ibid.* **313** (1989), 595.

[5] F.A. Berends, W.T. Giele, and H. Kuijf, *Phys. Lett.* **211B** (1988), 91; *ibid.* **232** (1989), 266; *Nucl. Phys.* **B321** (1989), 39; *ibid.* **333** (1990), 120.

[6] J. Gunion and J. Kalinowski, *Phys. Rev.* **D34** (1986), 2119.

[7] J. Gunion and Z. Kunzst, *Phys. Lett.* **159B** (1985), 167; *ibid.* **161** (1985), 333; *ibid.* **176** (1986) 477.

[8] K. Hagiwara and D. Zeppenfeld, *Nucl. Phys.* **B313** (1989), 560.

[9] R. Kleiss and H. Kuijf, *Nucl. Phys.* **B312** (1989), 616.

[10] R. Kleiss and W.J. Stirling, *Nucl. Phys.* **B262** (1985), 235.

[11] D. Kosower, *Nucl. Phys.* **B315** (1989), 391; *ibid.* **335** (1990), 23; *Phys. Lett.* **254B** (1991) 439.

[12] J.G. Körner and P. Sieben, Mainz preprint MZ-TH/90-08.

[13] Z. Kunzst, *Nucl. Phys.* **B271** (1986), 333.

[14] M. Mangano, *Nucl. Phys.* **B309** (1988), 461.

[15] M. Mangano, S. Parke, and Z. Xu, *Nucl. Phys.* **B298** (1988), 653.

[16] M. Mangano and S.J. Parke, *Nucl. Phys.* **B299** (1988), 673; *Phys. Rev.* **D39** (1989), 758; *Phys. Rep.* **200** (1991), 301.

[17] Z. Bern and D.K. Kosower, *Nucl. Phys.* **B362** (1991), 289.

[18] S. Parke and T. Taylor, *Phys. Lett.* **157B** (1985), 81; *Nucl. Phys.* **B269** (1986), 410; *Phys. Rev. Lett.* **B56** (1986), 2459; *Phys. Rev.* **D35** (1987), 313.

[19] C. Dunn and T.-M. Yan, *Nucl. Phys.* **B352** (1989), 402.

[20] G. Mahlon and T.-M. Yan, Cornell preprints CLNS91/1119; 91/1120.

[21] G. Mahlon, Cornell preprint CLNS 92/1154; Cornell University Ph.D. thesis.

[22] Z. Bern and D.K. Kosower, *Phys. Rev. Lett.* **B66** (1991), 1669; *Nucl. Phys.* **B379** (1992) 451; preprint Fermilab-Conf-91/71-T.

[23] Z. Bern and D.C. Dunbar, *Nucl. Phys.* **B379** (1992) 562.

[24] Z. Bern, L. Dixon, and D.A. Kosower, preprint SLAC-PUB-6012 (UCLA/92/TEP/47, CERN-TH.6733/92).

[25] M. Strassler, *Nucl. Phys.* **B385** (1992) 145; SLAC preprint (in preparation).

[26] C.S. Lam and J.P. Lebrun, *Nuovo Cimento* **59A** (1969), 397.

[27] C.S. Lam, preprint McGill/92-32 [hep-ph/9207266].

[28] J. Paton and Chan Hong-Mo, *Nucl. Phys.* **B10** (1969), 519.

[29] J.D. Bjorken, Standford Ph.D. thesis (1958).

[30] J.D. Bjorken and S.D. Drell, ‘Relativistic Quantum Fields’ (McGraw-Hill, 1965).

[31] J. Mathews, *Phys. Rev.* **113** (1959) 381.
[32] S. Coleman and R. Norton, *Nuovo Cimento* 38 (1965) 438.
[33] M.B. Green, J.H. Schwarz, and E. Witten, ‘Superstring Theory’ (Cambridge University Press, 1987).
Figure Captions

Fig. 1: The cubic electromagnetic vertex in scalar QED.

Fig. 2: The seagull electromagnetic vertex in scalar QED.

Fig. 3: A one-loop $n$-photon amplitudes with outgoing external momenta $p_i$, $n = 6$ is shown in the diagram. The numbers around the loop label the internal lines.

Fig. 4: A two-loop Compton scattering amplitude in scalar QED. The $p_i$ are the external outgoing momenta, and the five internal lines are numbered as shown.