The Product $e\pi$ Is Irrational

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Abstract: This note shows that the product $e\pi$ of the natural base $e$ and the circle number $\pi$ is an irrational number.

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1 Introduction

The number $e = 2.718281828459\ldots$ was proved to be irrational by Euler, circa 1744. The proof uses a differential equation to show that the continued fraction

$$e = [2; 2, 1, 2, 1, 4, 1, 6, 1, 8, \ldots]$$

is infinite, see [6], [15, Theorem 3.10]. Later, a simpler proof based on the infinite series

$$e = \sum_{n \geq 0} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

was found by Fourier in 1815. Many versions of the Fourier classical proof are known for $e^r$, where $r \in \mathbb{Q}$ is a rational number, and other numbers, see [13], [1, p. 35], [15]. The number $\pi = 3.141592653589\ldots$ was proved to be irrational by Lambert, circa 1760, see [2, p. 129]. The proof uses the continued fraction of the tangent function $\tan(x)$, the fact that the numbers $\tan(r)$ are irrationals for any nonzero rational number $r \in \mathbb{Q}$, and the value $\arctan(1) = \pi/4$ to indirectly show that the continued fraction

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \ldots]$$

is infinite, see [2], [8], [11]. Later, simpler versions and new proofs were found by several authors, [11], [1, p. 35], [15]. The above short compendium is a glimpse at the vast mathematical literature devoted to the analysis of the numbers $e$ and $\pi$. 
The arithmetic natures of the product $e \cdot \pi = 8.539734222673\ldots$, and of the sum $e + \pi$ are not known. In this note the known information on the continued fractions and the convergents of the two irrational numbers $e$ and $\pi$ are used here to construct an infinite subsequence of rational approximations for the product $e\pi$.

Theorem 1.1. The product $e\pi$ is an irrational number.

The earlier sections cover the basic required background, and the proof of Theorem 1.1 appears in Section 4. An algorithm that illustrates the effectiveness of this result appears in Section 5.

2 Foundation

Except for Theorem 2.4, all the materials covered in this section are standard results in the literature, see [7], [9], [10], [14], [15], et alii.

A real number $\alpha \in \mathbb{R}$ is called rational if $\alpha = a/b$, where $a, b \in \mathbb{Z}$ are integers. Otherwise, the number is irrational. The irrational numbers are further classified as algebraic if $\alpha$ is the root of an irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of degree $\text{deg}(f) > 1$, otherwise it is transcendental.

Theorem 2.1. If a real number $\alpha \in \mathbb{R}$ is a rational number, then there exists a constant $c = c(\alpha)$ such that

$$\frac{c}{q} \leq \left| \frac{\alpha}{q} - \frac{p}{q} \right|$$

holds for any rational fraction $p/q \neq \alpha$. Specifically, $c \geq 1/b$ if $\alpha = a/b$.

This is a statement about the lack of effective or good approximations for any arbitrary rational number $\alpha \in \mathbb{Q}$ by other rational numbers. On the other hand, irrational numbers $\alpha \in \mathbb{R} - \mathbb{Q}$ have effective approximations by rational numbers. If the complementary inequality $|\alpha - p/q| < c/q$ holds for infinitely many rational approximations $p/q$, then it already shows that the real number $\alpha \in \mathbb{R}$ is irrational, so it is sufficient to prove the irrationality of real numbers.

Theorem 2.2 (Dirichlet). Suppose $\alpha \in \mathbb{R}$ is an irrational number. Then there exists an infinite sequence of rational numbers $p_n/q_n$ satisfying

$$0 < \left| \frac{\alpha}{q_n} - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1}^2 q_n^2}$$

for all integers $n \in \mathbb{N}$.

Theorem 2.3. Let $\alpha = [a_0, a_1, a_2, \ldots]$ be the continued fraction of a real number, and let $\{p_n/q_n : n \geq 1\}$ be the sequence of convergents. Then

$$0 < \left| \frac{\alpha - p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2}$$

for all integers $n \in \mathbb{N}$.
This is standard in the literature, the proof appears in [7, Theorem 171], [15, Corollary 3.7], and similar references.

A basic extension of the previous inequalities in Theorem 2.2 and Theorem 2.3 provided here uses a pair of distinct irrational numbers and the corresponding parameters.

**Theorem 2.4.** Let $\alpha = [a_0, a_1, a_2, \ldots]$ and $\beta = [b_0, b_1, b_2, \ldots]$ be distinct continued fractions for two distinct irrational numbers $\alpha$ and $\beta \in \mathbb{R}$ such that $\alpha \beta \neq \pm 1$, respectively. Then

$$0 < \left| \alpha \beta - \frac{p_n u_m}{q_n v_m} \right| < \frac{2\beta}{a_{n+1} q_n^2} + \frac{2\alpha}{b_{m+1} v_m^2},$$

where $\{p_n/q_n : n \geq 1\}$ and $\{u_m/v_m : m \geq 1\}$ are the sequences of convergents respectively.

**Proof.** By Theorem 2.3, there exists a sequence of convergents $\{p_n/q_n : n \in \mathbb{N}\}$ such that

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2},$$

and the corresponding long form is

$$\frac{p_n}{q_n} - \frac{1}{a_{n+1} q_n^2} < \alpha < \frac{p_n}{q_n} + \frac{1}{a_{n+1} q_n^2}. \tag{2.5}$$

Similarly, there exists a sequence of convergents $\{u_m/v_m : m \in \mathbb{N}\}$ such that

$$\left| \beta - \frac{u_m}{v_m} \right| < \frac{1}{b_{m+1} v_m^2},$$

and the corresponding long form is

$$\frac{u_m}{v_m} - \frac{1}{b_{m+1} v_m^2} < \beta < \frac{u_m}{v_m} + \frac{1}{b_{m+1} v_m^2}. \tag{2.6}$$

The product of the last two long forms returns

$$\left( \frac{p_n}{q_n} - \frac{1}{a_{n+1} q_n^2} \right) \left( \frac{u_m}{v_m} - \frac{1}{b_{m+1} v_m^2} \right) < \alpha \beta < \left( \frac{p_n}{q_n} + \frac{1}{a_{n+1} q_n^2} \right) \left( \frac{u_m}{v_m} + \frac{1}{b_{m+1} v_m^2} \right). \tag{2.7}$$

Expanding these expressions produces

$$\frac{p_n u_m}{q_n v_m} - \frac{1}{a_{n+1} q_n^2} \frac{u_m}{v_m} - \frac{1}{b_{m+1} v_m^2} \frac{p_n}{q_n} < \alpha \beta < \frac{p_n u_m}{q_n v_m} + \frac{1}{a_{n+1} b_{m+1} q_n^2 v_m^2}, \tag{2.8}$$

$$\frac{p_n u_m}{q_n v_m} - \frac{1}{a_{n+1} q_n^2} \frac{u_m}{v_m} + \frac{1}{b_{m+1} v_m^2} \frac{p_n}{q_n} < \alpha \beta < \frac{p_n u_m}{q_n v_m} + \frac{1}{a_{n+1} b_{m+1} q_n^2 v_m^2}. \tag{2.9}$$

The second order term $1/(a_{n+1} b_{m+1} q_n^2 v_m^2)$ on the left side and right side is absorbed into the larger first order terms on the right side. Thus, rearranging the inequality yield

$$- \frac{1}{a_{n+1} q_n^2} \frac{u_m}{v_m} - \frac{1}{b_{m+1} v_m^2} \frac{p_n}{q_n} < \alpha \beta - \frac{p_n u_m}{q_n v_m} < \frac{2}{a_{n+1} q_n^2} \frac{u_m}{v_m} + \frac{2}{b_{m+1} v_m^2} \frac{p_n}{q_n}. \tag{2.10}$$

To complete the proof, rewrite it as a standard inequality

$$0 < \left| \alpha \beta - \frac{p_n u_m}{q_n v_m} \right| < \frac{2}{a_{n+1} q_n^2} \frac{u_m}{v_m} + \frac{2}{b_{m+1} v_m^2} \frac{p_n}{q_n}. \tag{2.11}$$
and use the trivial upper bound
\[
\frac{p_n}{q_n} \leq 2\alpha \quad \text{and} \quad \frac{u_m}{v_m} \leq 2\beta,
\]
(2.13)
for all large integers \(n, m \geq 1\), confer (2.6) and (2.8).

For distinct irrationals \(\alpha, \beta \in (0, 1)\), the simpler version
\[
0 < \left| \alpha\beta - \frac{p_n u_m}{q_n v_m} \right| < \frac{2}{a_{n+1} q_n^2} + \frac{2}{b_{m+1} v_m^2}
\]
(2.14)
can be used to streamline the proof of a result such as Theorem 1.1.

**Theorem 2.5** (Euler). The continued fraction \(e = [2, 12, 1, 1, 4, 1, 1, 6, 1, 8, \ldots]\) of the natural base has unbounded quotients and the subsequence of convergents \(p_n/q_n\) satisfies the inequality
\[
\left| e - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2}.
\]
(2.15)
The quotients have the precise form
\[
a_0 = 2, \quad a_{3k} = a_{3k-2} = 1, \quad a_{3k-1} = 2k,
\]
(2.16)
for \(k \geq 1\). The derivation appears in [12], [9, Theorem 2], [15, Theorem 3.10], [5], and other.

### 3 Convergents Correlations

The correlation of a pair of convergents \(\{p_n/q_n : n \geq 1\}\) and \(\{u_m/v_m : n \geq 1\}\) provides information on the distribution of nearly equal values of the continuants \(q_n\) and \(v_m\).

The regular pattern and unbounded properties of the partial quotients \(a_n = a_{3k-1} = 2k\) of the continued fraction of \(e\), see Theorem 2.5, are used here to generate a pair of infinite subsequences of rational approximations \(\{p_{3k-2}/q_{3k-2} : k \geq 1\}\) and \(\{u_{mk}/v_{mk} : k \geq 1\}\), for which the product
\[
\frac{p_{3k-2} u_{mk}}{q_{3k-2} v_{mk}} \rightarrow e\pi \quad \text{as} \ k, m \rightarrow \infty.
\]
(3.1)
Furthermore, the values \(q_{3k-2} \asymp v_{mk}\) are sufficiently correlated. The notation \(f(x) \asymp g(x)\) is defined by \(g(x) \ll f(x) \ll g(x)\).

The recursive relations
\[
p_{-1} = 1, \quad p_0 = a_0, \quad p_n = a_n p_{n-1} + p_{n-2},
\]
\[
q_{-1} = 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2},
\]
(3.2)
for all \(n \geq 1\), see [7], [10], [14], are used to estimate the rate of growth of the subsequences of continuants \(\{q_n : n \geq 1\}\) and \(\{v_m : m \geq 1\}\).
**Lemma 3.1.** Let $e = [a_0, a_1, a_2, \ldots]$ and $\pi = [b_0, b_1, b_2, \ldots]$ be the continued fractions of this pair of irrational numbers. Let $\delta > 0$ and $\varepsilon > 0$ be a pair of arbitrary small numbers. Then, the followings hold.

(i) If $b_m = o(m)$, then the exists a pair of subsequences of convergents $p_{3k-2}/q_{3k-2}$ and $u_{mk}/v_{mk}$ such that

$$(2k)^{1-\varepsilon} q_{3k-2} \ll v_{mk} \ll 2(2k)^{1-\varepsilon} q_{3k-2}. \quad (3.3)$$

(ii) If $b_m = O(m)$, then the exists a pair of subsequences of convergents $p_{3k-2}/q_{3k-2}$ and $u_{mk}/v_{mk}$ such that

$$(2k)^{1-\varepsilon} q_{3k-2} \ll v_{mk} \ll 2(2k)^{1-\varepsilon} q_{3k-2}. \quad (3.4)$$

(iii) If $b_m = O(m^{1+\delta})$, then the exists a pair of subsequences of convergents $p_{3k-2}/q_{3k-2}$ and $u_{mk}/v_{mk}$ such that

$$(m_k)^{1-\varepsilon} v_{mk} \ll q_{nk} \ll 2(m_k)^{1-\varepsilon} v_{mk}. \quad (3.5)$$

**Proof.** Case (i): The partial quotients $b_m = o(m)$ are bounded or unbounded. Make the change of index $n \equiv 1 \mod 3 \rightarrow k = (n+2)/3$ to focus on the subsequence of convergents $p_{3k-2}/q_{3k-2}$ of the number $e$ as $k \rightarrow \infty$, see Lemma ?? for more details. Observe that

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
\multicolumn{12}{c}{k} \\
q_{3k-2} &=& q_{3k-3} + q_{3k-4} &=& q_{3k-2} \\
q_{3k-1} &=& 2kq_{3k-2} + q_{3k-3} &\times& 2^2 k q_{3k-2} \\
q_{3k} &=& q_{3k-1} + q_{3k-2} &\times& 2^3 k q_{3k-2} \\
q_{3(k+1)-2} &=& q_{3(k+1)-3} + q_{3(k+1)-4} &\times& 2^4 k q_{3k-2} \\
q_{3(k+1)-1} &=& 2(k+1)q_{3(k+1)-2} + q_{3(k+1)-3} &\times& 2^5 k (k+1) q_{3k-2} \\
q_{3(k+1)} &=& q_{3(k+1)-1} + q_{3(k+1)} &\times& 2^6 k (k+1) q_{3k-2} \\
q_{3(k+2)-2} &=& q_{3(k+2)-3} + q_{3(k+2)-4} &\times& 2^7 k (k+1) q_{3k-2} \\
q_{3(k+2)-1} &=& 2(k+2)q_{3(k+1)-2} + q_{3(k+2)-3} &\times& 2^8 k (k+1) (k+2) q_{3k-2} \\
\end{array}
\]

This verifies that these numbers has exponential rate of growth in $k$ of the form

$$q_{3(k+t)-1} \asymp (4k)^{t+1} q_{3k-2},$$

for some $t \geq 0$, as $k \rightarrow \infty$. Next, consider the sequence of convergents $u_m/v_m$ of the number $\pi$. By hypothesis, the partial quotients $b_m = o(m)$ are bounded or unbounded. Furthermore, to simplify the notation, assume that $b_{m_k} \asymp m_k^{-\delta}$ for infinitely many integers $m_k = m_{k_0}, m_{k_1}, m_{k_2}, m_{k_3}, \ldots \geq 1$. Then, this implies the existence of a subsequence of convergents $u_{mk}/v_{mk}$ such that

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
\multicolumn{12}{c}{k} \\
v_{mk} &=& b_{m_k} v_{mk-1} + v_{mk-2} &=& v_{mk} \\
v_{mk_1} &=& b_{m_{k_1}} v_{mk_1-1} + v_{mk_1-2} &\times& (2^{k_1} - k m_k^{-\delta}) v_{mk} \\
v_{mk_2} &=& b_{m_{k_2}} v_{mk_2-1} + v_{mk_2-2} &\times& (2^{k_2} - k m_k^{-\delta})^2 v_{mk} \\
v_{mk_3} &=& b_{m_{k_3}} v_{mk_3-1} + v_{mk_3-2} &\times& (2^{k_3} - k m_k^{-\delta})^3 v_{mk} \\
\end{array}
\]
Moreover, the existence of a single value \( v_{m_k} > 1 \), see Tables 1 and 2, such that
\[
(2k)^{1-\varepsilon} q_{3k-2} \ll v_{m_k} \ll 2(2k)^{1-\varepsilon} q_{3k-2},
\]
implies the existence of an infinite subsequence of lower bounds
\[
v_{m_k_s} = (2^{k_s-k} m_k^{1-\delta})^s v_{m_k}
\]
\[
\gg (2^{k_s-k} m_k^{1-\delta})^s (2k)^{1-\varepsilon} q_{3k-2}
\]
\[
\gg (2k)^{1-\varepsilon} q_{3(k+t)-2},
\]
where
\[
q_{3(k+t)-2} = (2^{k_s-k} m_k^{1-\delta})^s q_{3k-2},
\]
and use (3.7) to identify the relation
\[
(2^{k_s-k} m_k^{1-\delta})^s = o((4k)^{t+1})
\]
for some \( s \geq 1 \) depending on \( t \geq 1 \). The corresponding subsequence of upper bounds satisfies
\[
v_{m_k_s} = (2^{k_s-k} m_k^{1-\delta})^s v_{m_k}
\]
\[
\ll 2(2^{k_s-k} m_k^{1-\delta})^s (2k)^{1-\varepsilon} q_{3k-2}
\]
\[
\ll 2(2k)^{1-\varepsilon} q_{3(k+t)-2}.
\]
Combining the last two inequalities yields the required relation
\[
(2k)^{1-\varepsilon} q_{3(k+t)-2} \ll v_{m_k_s} \ll 2(2k)^{1-\varepsilon} q_{3(k+t)-2}
\]
for some \( s, t \geq 1 \) as \( k \to \infty \).

**Case (ii):** The partial quotients \( b_m \) are bounded or unbounded, and \( b_m = O(m) \). The proof for this case is similar to Case (i).

**Case (ii):** The partial quotients \( b_m \) are unbounded, and \( b_m = O(m^{1+\delta}) \). In this case, (3.14) can fail, but since the inequality in Theorem 2.4 is symmetric in \( q_n \) and \( v_m \), the proof is almost the same as Case (i), but the subsequences of convergents are switched to obtain the required relation
\[
(2^{k_t-k} m_k^{1-\delta})^{t+1} v_{m_k_t} \ll q_{m_k+s} \ll 2(2^{k_t-k} m_k^{1-\delta})^{t+1} v_{m_k_t}
\]
for some \( s, t \geq 1 \) as \( k \to \infty \). ■

The distribution of all the continuants \( \{q_n : n \geq 1\} \) associated with a subset of continued fractions of bounded partial quotients is the subject of Zeremba conjecture, see [4] for advanced details. For any continued fraction, the numbers \( \{q_n : n \geq 1\} \) have exponential growth
\[
q_n = a_n q_{n-1} + q_{n-2} \geq \left( (1 + \sqrt{5})/2 \right)^n,
\]
which is very sparse subsequence of integers. The least asymptotic growth occurs for the \((1+\sqrt{5})/2 = [1, 1, 1, \ldots]\). But the combined subset of continuants for a subset of continued fractions of bounded partial fractions has positive density in the sunset of integers \( \mathbb{N} = \{1, 2, 3, \ldots\} \).
4 The Main Result

Proof. (Theorem 1.1) Let $e = [a_0, a_1, a_2, \ldots]$ be the continued fraction of the irrational number $e$. By Theorem 2.3, there exists a sequence of convergents $\{p_n/q_n : n \in \mathbb{N}\}$ such that

$$|e - p_n/q_n| < \frac{1}{a_{n+1}q_n^2}. \quad (4.1)$$

Similarly, let $\pi = [b_0, b_1, b_2, \ldots]$ be the continued fraction of the irrational number $\pi$, and let $\{u_m/v_m : m \in \mathbb{N}\}$ be the sequence of convergents such that

$$|\pi - u_m/v_m| < \frac{1}{b_{m+1}v_m^2}. \quad (4.2)$$

Now, suppose that the product $e\pi = r/s \in \mathbb{Q}$ is a rational number. Then

$$\frac{1}{s} \frac{1}{q_n v_m} \leq \left|e\pi - \frac{p_n u_m}{q_n v_m}\right| < \frac{2\pi}{a_{n+1}q_n^2} + \frac{2e}{b_{m+1}v_m^2}. \quad (4.3)$$

The left side follows from Theorem 2.1, and the right side follows from Theorem 2.4.

Next, use the subsequence of unbounded partial quotients $a_n = a_{3k-1}$ of the continued fraction of $e$ to generate an infinite subsequence of rational approximations

$$\frac{p_{3k-2}u_{m_k}}{q_{3k-2}v_{m_k}} \to e\pi \quad \text{as } k, m_k \to \infty. \quad (4.4)$$

The subsequence of rational approximations is generated by the following algorithm.

Fix an arbitrary small number $\varepsilon > 0$.

1. Input: The integer $n \equiv 1 \mod 3$.
2. Let $k = (n + 2)/3$.
3. Let $a_{n+1} = a_{3k-1} = 2k$, and fix the convergent $p_n/q_n = p_{3k-2}/q_{3k-2}$ of $e$, see Theorem 2.5.
4. Choose a convergent $u_{m_k}/v_{m_k}$ of $\pi$ in the range

$$(2k)^{1-\varepsilon} q_{3k-2} \leq v_{m_k} \leq 2(2k)^{1-\varepsilon} q_{3k-2}. \quad (4.5)$$

5. Output: The quotient $a_{3k-1} = 2k$ and the pair of convergents

$$\frac{p_{3k-2}}{q_{3k-2}} \quad \text{and} \quad \frac{u_{m_k}}{v_{m_k}}. \quad (4.6)$$

Various versions of this algorithm are possible, for example, by modifying the interval in (4.5).

Replacing the subsequence of rational approximations constructed in (4.5) into (4.3) yields

$$\frac{1}{s} \frac{1}{2(2k)^{1-\varepsilon} q_{3k-2}} \leq \left|e\pi - \frac{p_{3k-2}u_{m_k}}{q_{3k-2}v_{m_k}}\right| \leq \frac{2\pi}{2k q_{3k-2}^2} + \frac{2e}{(2k)^2 q_{3k-2}^2} \quad (4.7)$$

$$\leq \frac{\pi}{k q_{3k-2}^2} + \frac{2e}{(2k)^2 q_{3k-2}^2}. \quad (4.8)$$
Multiplying (4.7) by \((2k)^{1-\varepsilon} q_{3k-2}^2\) and using \(b_{m_k+1} \geq 1\) returns
\[
\frac{1}{2s} \leq \frac{2^{1-\varepsilon} \pi}{k^\varepsilon} + \frac{2e}{(2k)^{1-\varepsilon} b_{m_k+1}} \leq \frac{2^{1-\varepsilon} \pi}{k^\varepsilon} + \frac{2e}{(2k)^{1-\varepsilon}}. \tag{4.8}
\]
Since \(e\pi = r/s\) is rational constant, and \(s \geq 1\), it is clear that the inequality (4.8) is a contradiction for infinitely many large rational approximations
\[
\frac{p_{3k-2} u_{m_k}}{q_{3k-2} v_{m_k}} \leq \frac{2^{1-\varepsilon} \pi}{k^\varepsilon} + \frac{2e}{(2k)^{1-\varepsilon}}. \tag{4.9}
\]
as \(k, m_k \to \infty\). Ergo, the product \(e\pi\) is not a rational number.

The structure of the proof, in equations (4.3), and (4.7), is similar to some standard proofs of irrational numbers. Among these well known proofs are the Fourier proof of the irrationality of \(e\), see [1, p. 35], the proofs for \(\zeta(2)\), and \(\zeta(3)\) in [3], et alii.

An algorithm and sample of numerical data is compiled in Section 5 to demonstrate the practicality of this technique.

5 Algorithm And Numerical Data

For each fixed pair of index \((n, m)\), the basic product inequality
\[
0 < \left| e\pi - \frac{p_n u_m}{q_n v_m} \right| \leq \frac{2\pi}{a_n q_n^2} + \frac{2e}{b_m v_m^2} \tag{5.1}
\]
is used to test each rational approximation. It is quite easy to find the pairs \((n, m)\) to construct a subsequence of rational approximations
\[
\frac{p_n u_m}{q_n v_m} \to e\pi \tag{5.2}
\]
as \(n, m \to \infty\). The subsequence of rational approximations is generated by the following algorithm.

Algorithm 1. Fix an arbitrary small number \(\varepsilon > 0\).

1. Input an integer \(n \equiv 1 \mod 3\).
2. Let \(k = (n + 2)/3\).
3. Let \(a_{n+1} = a_{3k-1} = 2k\), and fix the convergent \(p_n/q_n = p_{3k-2}/q_{3k-2}\) of the natural base \(e\).
4. Choose a convergent \(u_{m_k}/v_{m_k}\) of \(\pi\) such that
\[
(2k)^{1-\varepsilon} q_{3k-2}^2 \leq v_{m_k} \leq 2(2k)^{1-\varepsilon} q_{3k-2}. \tag{5.3}
\]
5. If Step 4 fail, then increment \(n \equiv 1 \mod 3\), and repeat Step 2, the existence is proved in Lemma 3.1, see also Remark 5.1.
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| $(n, m)$ | $(19, 10)$ |
|----------|------------|
| $(a_n, b_m)$ | $(12, 1)$ |
| $p_n$ | 13580623 |
| $q_n$ | 4996032 |
| $u_m$ | 5419351 |
| $v_m$ | 1725033 |
| $|e\pi - \frac{p_n}{q_n} - \frac{u_m}{v_m}|$ | $0.00000000000012256862192$ |

Table 1: A Rational Approximation Of The Product $e\pi$.

| $(n, m)$ | $(31, 21)$ |
|----------|------------|
| $(a_n, b_m)$ | $(20, 2)$ |
| $p_n$ | 22526049624551 |
| $q_n$ | 8286870547680 |
| $u_m$ | 3587785776203 |
| $v_m$ | 1142027682075 |
| $|e\pi - \frac{p_n}{q_n} - \frac{u_m}{v_m}|$ | $8.32849575322710174432272 \times 10^{-25}$ |

Table 2: A Rational Approximation Of The Product $e\pi$.

6. Output $a_{3k-1} = 2k$, $p_{3k-2}/q_{3k-2}$ and $u_{m_k}/v_{m_k}$.

The parameters for two small but very accurate approximations are listed in the Tables 1 and 2. These examples demonstrate the practicality of the algorithm.

Remark 5.1. Step 5 makes the algorithm independent of the rate of growth of the partial quotients of the number $\pi$. Various versions of this algorithm are possible, for example, by modifying the interval in (5.3).
References

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