Polynomial approximation of piecewise analytic functions on quasi-smooth arcs

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Abstract

For a function \( f \) that is piecewise analytic on a quasi-smooth arc \( \mathcal{L} \) and any \( 0 < \sigma < 1 \) we construct a sequence of polynomials that converge at a rate \( e^{-n^{\sigma}} \) at each point of analyticity of \( f \) and are close to the best polynomial approximants on the whole \( \mathcal{L} \). Moreover, we give examples when such polynomials can be constructed for \( \sigma = 1 \).

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1 Introduction and main results

Let \( \mathcal{L} \) be a quasi-smooth arc on the complex plane \( \mathbb{C} \), that is, for any \( z, \zeta \in \mathcal{L} \) the length \( |\mathcal{L}(z, \zeta)| \) of the subarc \( \mathcal{L}(z, \zeta) \) of \( \mathcal{L} \) between points \( z, \zeta \) satisfies

\[
|\mathcal{L}(z, \zeta)| \leq c|z - \zeta|
\]

for some \( c = c(\mathcal{L}) \geq 1 \).

Consider a piecewise analytic function \( f \) on \( \mathcal{L} \) belonging to \( C^k(\mathcal{L}) \), \( k \geq 0 \), that means \( f \) is \( k \) times continuously differentiable on \( \mathcal{L} \) and there exist points \( z_2, z_3, ..., z_{m-1} \) such that \( f \) is analytic on \( \mathcal{L}\{z_1, z_2, ..., z_m\} \), \( (z_1, z_m) \) - endpoints of \( \mathcal{L} \), but is not analytic at points \( z_1, z_2, ..., z_m \). We call the \( z_i \) points of singularity of \( f \).

The rate of the best uniform approximation of a function \( f \) by polynomials of degree at most \( n \in \mathbb{N} := \{1, 2, ...\} \) is denoted by

\[
E_n(f) = E_n(f, \mathcal{L}) := \inf_{P_n: \deg P_n \leq n} \|f - P_n\|_{\mathcal{L}}.
\]  (1.1)

Here \( \|\cdot\|_{\mathcal{L}} \) means the supremum norm over \( \mathcal{L} \). Also, let \( p^*_n(f, z) \) be the (unique) polynomial minimizing the uniform norm in (1.1).

It is natural to expect the difference \( f(z) - p^*_n(z) \) to converge faster at points of analyticity of \( f \). But, it turns out, singularities of \( f \) adversely affect the behavior over the whole \( \mathcal{L} \) of a subsequence of the best polynomial approximants \( p^*_n(f, z) \). This so-called "principle of contamination" manifests itself in density of extreme points of \( f - p^*_n \), discussed by A. Kroo and E.B. Saff in [8].
The accumulation of zeros of $p_n^e(f, z)$, showed by H.-P. Blatt and E.B. Saff in [6]. For more details, we refer the reader to [10].

Surprisingly, such behavior of zeros and extreme points need not hold for polynomials of "near-best" approximation, that is for polynomials $P_n$ that satisfy

$$
\| f - P_n \|_L \leq C E_n(f), \quad n = 1, 2, \ldots ,
$$

with a fixed $C > 1$. Hence, it is natural to seek "near-best" polynomials which would converge faster at points $z \in L \setminus \{z_1, z_2, \ldots, z_m\}$.

For the case of $L = [-1, 1]$ and a piecewise analytic function $f$ belonging to $C^k[-1, 1]$, E.B. Saff and V. Totik in [12] have proved that if non-negative numbers $\alpha, \beta$ satisfy $\alpha < 1$ and $\beta \geq \alpha$ or $\alpha = 1$ and $\beta > 1$, then there exist constants $c, C > 0$ and polynomials $P_n, n = 1, 2, \ldots$, such that for every $x \in [-1, 1]$

$$
|f(x) - P_n(x)| \leq C E_n(f)e^{-cn^\alpha d(x)\beta},
$$

(1.2)

where $d(x)$ denotes the distance from $x$ to the nearest singularity of $f$ in $(-1, 1)$.

Accordingly, the question of constructing "near-best" polynomials arises when $[-1, 1]$ is replaced by an arbitrary quasi-smooth arc $L$ in $C$. Polynomial approximation of functions on arcs is an important case of a more general problem of approximation of functions on an arbitrary continuum of the complex plane studied in the works of N.A. Shirokov [13], V.K. Dzjadyk and G.A. Alibeckov [1], V.V. Andrievskii [3] and others (see, for example, [7]).

The behavior of "near-best" polynomials is well studied in the case of approximation on compact sets $K$ with non-empty interior $\text{Int}(K)$. The following results demonstrate how the possible rate of convergence inside $K$ depends on the geometry of $K$. V.V. Maineskul have proved in [9] that if $\Omega := C \setminus K$ satisfies the $\alpha$-wedge condition with $0 < \alpha \leq 1$, then for any $\sigma < \alpha/2$ there exist "near-best" polynomials converging at a rate $e^{-n^\alpha}$ in the interior of $K$. E.B. Saff and V. Totik in [11] show the possibility of geometric convergence of "near-best" polynomials inside $K$ if the boundary of $K$ is an analytic curve. Meantime, N.A. Shirokov and V. Totik in [14] discuss the rate of approximation by "near best" polynomials of a function $f$ given on a compact set $K$ with a generalized external angle smaller than $\pi$ at some point $z_0 \in \partial K$. They showed that if $f$ has a singularity at $z_0$, then geometric convergence inside $K$, where $f$ is analytic, is impossible. Taking into account these results, the most interesting case for us is when singularities of the function $f$ occur at points where the angle between subarcs of $L$ is different from $\pi$. It turns out that for some such arcs there are no restrictions on the rate of convergence of "near-best" polynomials and it can be geometric at points where $f$ is analytic, as opposed to the result for compact sets with non-empty interior. We formulate and prove this assertion in Theorem 2. Furthermore, the general case is given by the following

**Theorem 1.** Let $f$ be a piecewise analytic function on a quasi-smooth arc $L$, i.e. there exist points $z_2, \ldots, z_{m-1} \in L$ such that they divide $L$ into $L^1, L^2, \ldots, L^{m-1}$ and

$$
f(z) = f_i(z), \quad z \in L^i, \quad i = 1, m - 1,
$$

(1.3)
where \( f_i(z) \) are analytic in some neighborhood of \( L_i \), respectively, and satisfy
\[
f^{(r)}_i(z_i) = f^{(r)}_i(z_i), \quad f^{(k)}_i(z_i) \neq f^{(k)}_i(z_i)
\] (1.4)
for \( r = 0, k_i, i = 2, m - 1 \). Then, for any \( 0 < \sigma < 1 \), there exists a sequence \( \{P_n\}_{n=1}^\infty \) of "near-best" polynomial approximants of \( f \) on \( L \), such that
\[
\lim_{n \to \infty} \|f - P_n\| E e^{n\sigma} = 0
\] (1.5)
holds for any compact set \( E \subset L \setminus \{z_2, \ldots, z_{m-1}\} \).

On the complex plane consider lemniscates that are level lines of some complex polynomials. Namely, take \( P(z) = P_N(z) := (z - a_1)(z - a_2)...(z - a_N) \), where \( a_k = R e^{i \frac{2\pi (k - 1)}{N}} \), \( k = 1, N \) and \( R > 0 \) is a fixed number. Then \( |P(z)| = R^N \) is an equation of a lemniscate. Note that the origin is a point of this lemniscate (since \( |P(0)| = R^N \)).

The lemniscate divides the plane into three parts, namely the curve itself, points \( \{z : |P(z)| < R^N\} \) and \( \{z : |P(z)| > R^N\} \). Consider an arc \( L = L' \cup L'' \), where \( L', L'' \) may belong to different petals of the lemniscate, meet at the origin and satisfy \( |P(z)| < R^N, z \in L \setminus \{0\} \). An example for \( N = 4, R = 1 \) you can see below.

In particular, two line segments meeting at the origin at angle \( 0 < \varphi \leq \pi \) satisfy this property: if \( \frac{2\pi}{m+1} < \varphi \leq \frac{2\pi}{m} \) for some integer \( m \), it is enough to take \( R \) to be sufficiently large and \( N = m \).
Let \( f \) be a piecewise analytic function on \( \mathcal{L} \) given by

\[
f(z) = \begin{cases} 
  f_1(z), & \text{if } z \in \mathcal{L}' \\
  f_2(z), & \text{if } z \in \mathcal{L}''
\end{cases}
\]

where \( f_1, f_2 \) are functions, analytic on \( \mathcal{L}' \) and \( \mathcal{L}'' \) correspondingly, satisfying

\[
f_1^{(r)}(0) = f_2^{(r)}(0), \quad r = 0, k, \quad f_1^{(k+1)}(0) \neq f_2^{(k+1)}(0).
\]

With these assumptions we prove the following result

**Theorem 2.** Let \( \mathcal{L} \) and \( f \) be as above. Then there exist a constant \( c > 0 \) and a sequence of "near-best" polynomials \( \{P_n\}_{n=1}^{\infty} \), such that

\[
\lim_{n \to \infty} \|f - P_n\|_E e^{c\text{d}(E)} = 0,
\]

where \( \text{d}(E) > 0 \) for any compact set \( E \subset \mathcal{L} \setminus \{0\} \).

**2 Auxiliary results**

In this section we give some results which allow us to get estimates for the \( E_n(f) \) and are needed for constructing "near-best" polynomials.

For \( a > 0 \) and \( b > 0 \) we will use the notation \( a \lesssim b \) if \( a \leq cb \), with some constant \( c > 0 \). The expression \( a \approxeq b \) means \( a \lesssim b \) and \( b \lesssim a \).

Let \( \mathcal{L} \) be a quasi-smooth arc and \( \Omega := \overline{\mathcal{C}} \setminus \mathcal{L} \). Consider a conformal mapping \( \Phi : \Omega \to \Delta := \{ \omega : |\omega| > 1 \} \), normalized in such a way that \( \Phi(\infty) = \infty, \Phi'(\infty) > 0 \), and denote \( \Psi := \Phi^{-1} \).

By \( \overline{\Omega} \) we denote compactification of the domain \( \Omega \) by prime ends in the Carathéodory sense, and \( \overline{\mathcal{L}} := \overline{\Omega} \setminus \overline{\mathcal{L}} \). For the endpoints \( z_1, z_2 \) of \( \mathcal{L} \) and \( u > 0, j = 1, 2 \), let

\[
\Phi(z_j) := \tau_j; \\
\Delta_1 := \{ \tau : \tau \in \Delta, \arg \tau_1 < \arg \tau < \arg \tau_2 \}; \\
\Delta_2 := \Delta \setminus \Delta_1, \quad \overline{\Omega} := \Psi(\overline{\Delta}_j), \quad \Omega^j := \Psi(\Delta_j); \\
\overline{\mathcal{L}}^j := \overline{\Omega}^j \cap \overline{\mathcal{L}}; \\
\mathcal{L}_u^j := \{ \zeta : \zeta \in \overline{\Omega}^j, |\Phi(\zeta)| = 1 + u \}; \\
\rho_u^j(z) := \text{dist}(z, \mathcal{L}_u^j); \quad \rho_u^j(z) := \max_{j=1,2} \rho_u^j(z).
\]

Let \( z_0 \) be a point of \( \mathcal{L} \), distinct from endpoints of the arc. Then point \( z_0 \) divides \( \mathcal{L} \) into two parts, \( \mathcal{L}' \) and \( \mathcal{L}'' \). Consider the function

\[
f(z) = \begin{cases} 
  f_1(z), & \text{if } z \in \mathcal{L}' \\
  f_2(z), & \text{if } z \in \mathcal{L}''
\end{cases}
\] (2.1)
where \( f_1, f_2 \) are functions, analytic on \( \mathcal{L}' \) and \( \mathcal{L}'' \), i.e. analytic in some neighborhoods of \( \mathcal{L}' \) and \( \mathcal{L}'' \) correspondingly, and satisfying

\[
f_1^{(r)}(z_0) = f_2^{(r)}(z_0), \quad r = 0, k, \quad f_1^{(k+1)}(z_0) \neq f_2^{(k+1)}(z_0). \tag{2.2}
\]

By \( U \) we will denote an open circular neighborhood of the point \( z_0 \), where both \( f_1, f_2 \) are analytic.

Let \( Z^j_0, Z^j_0 \in \tilde{\mathcal{L}} \) be the prime ends, s.t. \( |Z^j_0| = z_0, \ j = 1, 2 \). Set

\[
\tau^j_0 := \Phi(Z^j_0), \quad j = 1, 2.
\]

Points \( \tau^j_1, j = 1, 2 \) we define by

\[
\tau^j_1 = \lambda \tau^j_0,
\]

with \( \lambda > 1 \) such that

\[
\Gamma^1, \Gamma^2 \subset U,
\]

where

\[
\Gamma^j = \Gamma^j_0 := \{ \zeta : 1 < |\Phi(\zeta)| < \lambda, \arg \Phi(\zeta) = \arg \tau^j_0 \}, \quad j = 1, 2. \tag{2.3}
\]

The arcs \( \Gamma^1, \Gamma^2 \) are rectifiable (see \[4\] Chap. 5]), thus, can be oriented in such a way that for all \( z \in \mathcal{L}\backslash \{z_0\} \) function \( f \) can be represented, by the Cauchy formula, as

\[
f(z) = h_1(z) + h_2(z),
\]

where

\[
h_1(z) = \frac{1}{2\pi i} \int_{\Gamma^1 \cup \Gamma^2} \frac{f_1(\zeta) - f_2(\zeta)}{\zeta - z} d\zeta, \tag{2.4}
\]

and \( h_2(z) \) is analytic for all \( z \in \mathcal{L} \), therefore it can be approximated with a geometric rate on \( \mathcal{L} \).

We will make use of the following lemma.

**Lemma 1.** Let \( \mathcal{L} \) be a quasi-smooth arc. Then for any fixed non-negative integer \( k \), a positive integer \( n \) and \( \zeta \in \Gamma^1 \cup \Gamma^2 \) there exists a polynomial kernel \( K_n(\zeta, z) \) of the form

\[
K_n(\zeta, z) = \sum_{j=0}^{n} a_j(\zeta) z^j
\]

with continuous in \( \zeta \) coefficients \( a_j(\zeta) \), \( j = 0, n \), satisfying for \( z \in \mathcal{L} \) and \( \zeta \) with \( |\zeta - z_0| \geq \rho^{*}_{1/n}(z_0) \)

\[
\left| \frac{1}{\zeta - z} - K_n(\zeta, z) \right| \leq c|\rho^{*}_{1/n}(z_0)|^{k+2} |\zeta - z_0|^{-(k+3)}, \tag{2.5}
\]

where \( c = c(\mathcal{L}) > 0 \).

**Proof.** To show (2.5), we repeat word by word the proof for \( k = 0 \), (\[4\] Lemma 5.4]).

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Let $n$ be sufficiently large. For fixed $m$ and $r$ we consider the Dzyadyk polynomial kernel $K_{0,m,r,n}(\zeta, z)$ (see, e.g., \cite{4} Chap. 3). Then, for $r \geq 5$ and $z \in \mathcal{L}$, $\zeta \in \Gamma^j$, $j = 1, 2$,

$$\left| \frac{1}{\zeta - z} - K_{0,m,r,n}(\zeta, z) \right| \leq \frac{1}{|\zeta - z|} |\tilde{\zeta} - \zeta|^{r^m}$$

where $\tilde{\zeta} := \tilde{\zeta}_{1/n} := \Psi((1 + 1/n)\Phi(\zeta))$.

Since

$$|\tilde{\zeta} - \zeta| \leq |\zeta - z| \leq \left| \frac{\rho_{1/n}^1(z_0)}{\zeta - z_0} \right| \leq \left| \frac{\rho_{1/n}^1(z_0)}{\zeta - z_0} \right|^c,$$

it is enough to take $r$ and $m$ such that $rmc \geq k + 2$, and set $K_n(\zeta, z) := K_{0,m,r,|\zeta|n}(\zeta, z)$, where $\varepsilon = \varepsilon(r, m) > 0$ is sufficiently small.

The next theorem is also a generalization of the case $k = 0$ in (2.2) and the proof essentially repeats the proof of \cite{4} Theorem 5.2.

**Theorem 3.** Let $\mathcal{L}$ be a quasi-smooth arc, and let function $f$ be given by (2.1), (2.2). Then

$$c' \left| \rho_{1/n}^1(z_0) \right|^{k+1} \leq E_n(f, \mathcal{L}) \leq c'' \left| \rho_{1/n}^1(z_0) \right|^{k+1},$$

(2.6)

where $c', c''$ don’t depend on $n$.

**Proof.** First, we estimate $E_n(f, \mathcal{L})$ from above.

Without loss of generality, we can assume $z_0 = 0$ and $n$ is sufficiently large. Let $d_n := \rho_{1/n}^1(0)$, $\gamma = \gamma_n := \{ \zeta : \zeta \in \Gamma^1 \cup \Gamma^2, |\zeta| \geq d_n \}$,

$$P_n = \frac{1}{2\pi i} \int_{\gamma} (f_1(\zeta) - f_2(\zeta)) K_n(\zeta, z) d\zeta.$$  

From (2.2), for all $\zeta$ in some neighborhood $U$ of the point $z_0 = 0$

$$f_1(\zeta) = c_0 + c_1 \zeta + \ldots + c_k \zeta^k + c_{k+1} \zeta^{k+1} + \varphi_1(\zeta) \zeta^{k+2}$$

(2.7)

$$f_2(\zeta) = c_0 + c_1 \zeta + \ldots + c_k \zeta^k + \tilde{c}_{k+1} \zeta^{k+1} + \varphi_2(\zeta) \zeta^{k+2},$$

(2.8)

where $c_{k+1} \neq \tilde{c}_{k+1}$ and $\varphi_1(\zeta), \varphi_2(\zeta)$ are functions, analytic in $U$.

Hence, there exists a constant $C$ such that

$$|f_1(\zeta) - f_2(\zeta)| \leq C|\zeta|^{k+1}, \quad \zeta \in U.$$  

(2.9)

By (2.5), (2.9), for all $z \in \mathcal{L}$

$$\left| \frac{1}{2\pi i} \int_{\Gamma^1 \cup \Gamma^2} \frac{f_1(\zeta) - f_2(\zeta)}{\zeta - z} d\zeta - P_n(z) \right|$$

$$\leq \frac{1}{2\pi} \int_{\gamma} |f_1(\zeta) - f_2(\zeta)| \left| \frac{1}{\zeta - z} - K_n(z, \zeta) \right| |d\zeta| + \frac{1}{2\pi} \int_{(\Gamma^1 \cup \Gamma^2) \setminus \gamma} \left| \frac{f_1(\zeta) - f_2(\zeta)}{\zeta - z} \right| |d\zeta|$$
Thus, combining with (2.10), we obtain the estimate from above in (2.6).

Integration by parts of \( \int_{\gamma} \frac{|d\zeta|}{|\zeta|^2} \) yields \( \int_{\gamma} \frac{|d\zeta|}{|\zeta|^2} \approx \frac{1}{\pi^2} \). Since \( \text{dist}(\zeta, \mathcal{L}) \ll |\zeta| \), (see Chap. 5]), and \(|(\Gamma^1 \cup \Gamma^2)\setminus \gamma| \ll d_n \), it implies \( \int_{(\Gamma^1 \cup \Gamma^2)\setminus \gamma} \frac{|c^{k+1}|}{|z - \zeta|} |d\zeta| \ll d_k^{k+1} \).

Thus, combining with (2.10), we obtain the estimate from above in (2.6).

Now, we estimate \( E_n(f, \mathcal{L}) \) from below.

Let \( p_n^* \) be the polynomial of the best approximation, that is

\[
|f(z) - p_n^*(z)| \leq E_n(f), \quad z \in \mathcal{L}
\]

(2.11)

Without loss of generality we can assume that

\[
E_n(f) \leq d_n = \rho_{1/n}(0).
\]

Denote by \( l_3 \subset \Omega^1 \) any arc of a circle \( \{ \zeta : |\zeta| = d_n \} \), separating the prime end \( Z_0^{1} \) from \( \infty \).

Let \( z' \in \mathcal{L}' \) and \( z'' \in \mathcal{L}'' \) be the endpoints of the arc \( l_3 \). Denote

\[
l_1 := \mathcal{L}(0, z'), \quad l_2 := \mathcal{L}(0, z'').
\]

Next, take a point \( z \) so that \( z \in \Gamma, |z| = \varepsilon d_n \) (we’ll choose the constant \( \varepsilon \) later). With a corresponding choice of orientation of arcs \( l_j, j = 1, 2, 3 \)

\[
I := \int_{l_1 \cup l_2} \frac{\hat{f}(\zeta)}{(\zeta - z)^{k+2}} d\zeta = \int_{l_1 \cup l_2} \frac{\hat{f}(\zeta) - \bar{p}_n^*(\zeta)}{(\zeta - z)^{k+2}} d\zeta + \int_{l_3} \bar{p}_n^*(\zeta) d\zeta,
\]

(2.12)

where \( \hat{f}(\zeta) = f(\zeta) - (c_0 + c_1 \zeta + \ldots + c_k \zeta^k) \) and \( \bar{p}_n^*(\zeta) = p_n^* - (c_0 + c_1 \zeta + \ldots + c_k \zeta^k) \).

Notice that \( f(\zeta) - p_n^*(\zeta) = \hat{f}(\zeta) - \bar{p}_n^*(\zeta) \).

In the following estimates we use notations \( a_i, \tilde{a}_i, \tilde{C}, \tilde{C}, C_i \) for constants.

For the left hand side we have

\[
\left| \int_{l_1 \cup l_2} \frac{\hat{f}(\zeta)}{(\zeta - z)^{k+2}} d\zeta \right|
\]

\[
= c_{k+1} \int_{l_1} \frac{\zeta^{k+1}}{(\zeta - z)^{k+2}} d\zeta + \bar{c}_{k+1} \int_{l_2} \frac{\zeta^{k+1}}{(\zeta - z)^{k+2}} d\zeta
\]

\[
+ \int_{l_1} \frac{\varphi_1(\zeta)\zeta^{k+2}}{(\zeta - z)^{k+2}} d\zeta + \int_{l_2} \frac{\varphi_2(\zeta)\zeta^{k+2}}{(\zeta - z)^{k+2}} d\zeta
\]

\[
= c_{k+1} \log \frac{z - z'}{z - z''} + \bar{c}_{k+1} \log \frac{z - z''}{z} + a_1 z^{\varepsilon k+1} + a_2 z^{\varepsilon k} z + \ldots + a_k z^{\varepsilon k+1} + \frac{\bar{a}_1 z^{\varepsilon k+1} + \bar{a}_2 z^{\varepsilon k} z + \ldots + \bar{a}_{k+1} z^{\varepsilon k+1}}{(z'' - z)^{k+1}} + \tilde{C} + \int_{l_1} \frac{\varphi_1(\zeta)\zeta^{k+2}}{(\zeta - z)^{k+2}} d\zeta + \int_{l_2} \frac{\varphi_2(\zeta)\zeta^{k+2}}{(\zeta - z)^{k+2}} d\zeta
\]

\[
\geq \left| (\bar{c}_{k+1} - c_{k+1}) \log \frac{z - z''}{z} + c_{k+1} \log \frac{z - z''}{z} - \frac{C_1 \varepsilon}{(1 - \varepsilon)^{k+1}} \right|
\]

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Next, we estimate the right hand side of (2.12). By (2.11) and by the choice of $z$
\[
\left| \int_{l_1 \cup l_2} \frac{\tilde{f}(\zeta) - \tilde{p}_n^*(\zeta)}{(\zeta - z)^{k+2}} d\zeta \right| \leq C_3 \frac{E_n}{\varepsilon^{k+1} d_n^{k+1}}.
\]
To estimate the integral over $l_3$ notice that by (2.7) and (2.8)
\[
|\tilde{f}(\zeta)| \leq c|\zeta|, \; \zeta \in L
\]
for some constant $c$. Without loss of generality, we assume $c = 1$ (otherwise the arc $l_3$ must be considered with a radius $\frac{d_n}{\varepsilon}$ instead). Since the estimate
\[
|\tilde{p}_n^*(\zeta)| \leq |\tilde{p}_n^*(\zeta) - \tilde{f}(\zeta)| + |\tilde{f}(\zeta)| \leq d_n^{k+1} \left(1 + \frac{\zeta}{d_n}\right)^{k+1}, \; \zeta \in L
\]
holds, [4, Theorem 6.1] implies
\[
|\tilde{p}_n^*(\zeta)| \leq C_4 d_n^{k+1}, \; \zeta \in l_3.
\]
The last inequality yields
\[
\left| \int_{l_3} \frac{\tilde{p}_n^*(\zeta)}{(\zeta - z)^{k+2}} d\zeta \right| \leq \frac{2\pi C_4}{(1 - \varepsilon)^{k+2}}.
\]
Combining the estimates above, for some small but fixed $\varepsilon$ we get
\[
C_3 \frac{E_n}{\varepsilon^{k+1} d_n^{k+1}} \geq |\tilde{c}_{k+1} - c_{k+1}| \log \frac{1 - \varepsilon}{\varepsilon} - \frac{C_1 \varepsilon}{(1 - \varepsilon)^{k+1}} - C_2 - \frac{2\pi C_4}{(1 - \varepsilon)^{k+2}}
\]
\[
\geq \frac{|\tilde{c}_{k+1} - c_{k+1}|}{2} \log \frac{1 - \varepsilon}{\varepsilon}.
\]
Consequently, the estimate from below in (2.6) holds.

With reasoning completely similar, we obtain the following.

**Theorem 4.** Let $L$ be a quasi-smooth arc, and let function $f$ be given by (1.3), (1.4). Then
\[
c' \left[ \rho_{1/n}^*(z_0) \right]^{k+1} \leq E_n(f, L) \leq c'' \left[ \rho_{1/n}^*(z_0) \right]^{k+1},
\]
where $k := \min_{i=2, \ldots, m-1} \{k_i\}$ and $c', c''$ don't depend on $n$. 
3 Proof of Theorem 1

As it was mentioned above, \( f \) can be represented as

\[
f(z) = \sum_{j=2}^{m-1} (h_j^1(z) + h_j^2(z)),
\]

where \( h_j^2(z) \) are analytic functions on \( \mathcal{L} \) and

\[
h_j^1(z) = \frac{1}{2\pi i} \int_{\Gamma_j^1 \cup \Gamma_j^2} \frac{f_j^{-1}(\zeta) - f_j(\zeta)}{\zeta - z} d\zeta,
\]

with \( \Gamma_j^1, \Gamma_j^2 \) being the arcs given by (2.3), that correspond to the point \( z_j \).

Therefore, it’s enough to construct polynomial approximants for \( h_j^1(z) \) only.

To approximate the integral over \( \Gamma_j \), consider a function \( F_j^i : \mathcal{L} \cup \Gamma_j \rightarrow \mathcal{L}^\prime, \) such that \( F_j^i \) is one-to-one and satisfies

\[
|F_j^i(z) - F_j^i(\zeta)| \leq c|z - \zeta|, \quad z, \zeta \in \mathcal{L} \cup \Gamma_j,
\]

\[
F_j^i(z_j) = 0,
\]

\[
F_j^i(\mathcal{L}(z_1, z_j)) = \mathcal{L}^\prime,
\]

\[
F_j^i(\mathcal{L}(z_j, z_m)) = \mathcal{L}^\prime\prime,
\]

\[
F_j^i(\Gamma_j) = \tilde{\Gamma},
\]

where \( z_1, z_m \) are endpoints of \( \mathcal{L} \), \( \mathcal{L}^\prime \) is a line segment in \([0, \infty)\), \( \mathcal{L}^\prime\prime \) is a line segment in the upper half plane that form an angle \( \varphi > 0 \) with \( \mathcal{L}^\prime \), (this angle will be determined below), and \( \tilde{\Gamma} \) – a line segment at an angle \( \frac{\pi}{2} \) to the \( \mathcal{L}^\prime \).

Such a mapping \( F_j^i \in \text{Lip}_1[\mathcal{L} \cup \Gamma_j] \) always exists, and to see this it is enough to note that \( \mathcal{L} \) and \( \Gamma_j \) are quasi-smooth and

\[
\text{dist}(\zeta, \mathcal{L}) \approx |\zeta - z_j|
\]

holds for all \( \zeta \in \Gamma_j \) (see [4] Chap. 5).

By [2] Theorem 4] the function \( F_j^i \) can be approximated by polynomials \( Q_n(z) := Q_{n,j}^i(z) \) with the rate \( \frac{1}{n^\alpha} \), for some \( \alpha > 0 \), that is

\[
|F_j^i(z) - Q_n(z)| \leq C \frac{1}{n^\alpha}, \quad z \in \mathcal{L} \cup \Gamma_j,
\]

where constant \( C \) does not depend on \( z \) and \( n \).

For fixed \( 0 < \sigma < 1 \) take an integer \( k \geq 2 \), such that \( 1 - \sigma > \frac{1}{1+\mu} \). Now, for \( \varphi = \frac{2\pi}{k} \) consider corresponding mapping \( F_j^i \) and approximating polynomials \( Q_n \).
Let
\[
\hat{P}_{i,j}^n(z, \zeta) = 1 - \left( Q_{[n^{\beta}],z}^k(z) - \zeta_0 \right) K_{[n^{\beta}],z}^k(z, \zeta) + \left( Q_{[n^{\beta}],0}^k(z) - \zeta_0 \right) K_{[n^{\beta}],z}^k(0, \zeta) \]

It is not hard to see that \( \hat{P}_{i,j}^n(z, \zeta) \) is a polynomial in \( z \) of degree at most \( n \).

The idea of constructing such a polynomial is motivated by [5, p. 380].

We will show that for some choice of \( \beta \) and \( \zeta_0 \) the term \( \left| \frac{Q_{[n^{\beta}],z}^k(z) - \zeta_0}{Q_{[n^{\beta}],0}^k(z) - \zeta_0} \right| \) is bounded uniformly on \( \mathcal{L} \) by a constant that does not depend on \( n \), and at points of analyticity of \( f \) it can be bounded by \( q \left| \frac{n^{1-\beta}}{n^\alpha} \right| \), for some \( q < 1 \).

For \( n \) sufficiently large the arc \( \mathcal{L} \) can be written as a disjoint union
\[
\mathcal{L} = A_1 \cup A_2 \cup A_3,
\]
where
\[
A_1 := \{ z \in \mathcal{L} : |Q_n(z)| < \frac{C}{n^\alpha \sin \frac{\pi}{4\alpha}} \},
\]
\[
A_2 := \{ z \in \mathcal{L} : |Q_n(z)| \geq \frac{C}{n^\alpha \sin \frac{\pi}{4\alpha}} , \text{dist}(Q_n(z), \mathcal{L}') \leq \frac{C}{n^\alpha} \},
\]
\[
A_3 := \{ z \in \mathcal{L} : |Q_n(z)| \geq \frac{C}{n^\alpha \sin \frac{\pi}{4\alpha}} , \text{dist}(Q_n(z), \mathcal{L}'') \leq \frac{C}{n^\alpha} \},
\]
where $C$ is the constant from (3.1).

Points of $A_1$ satisfy
\[ |Q_n^k(z)| < \frac{C^k}{n^\alpha (\sin \frac{\pi}{4k})^\alpha}. \] (3.5)

For $A_2$ we have
\[ |\sin(\arg Q_n(z))| \leq \frac{C}{n^\alpha |Q_n(z)|} \leq \sin \frac{\pi}{4k}, \]
that implies
\[ -\frac{\pi}{2} < \arg Q_n^k(z) \leq \frac{\pi}{2}. \] (3.6)

Similarly, for $A_3$
\[ |\sin(\frac{2\pi}{k} - \arg Q_n(z))| \leq \frac{C}{n^\alpha |Q_n(z)|} \leq \sin \frac{\pi}{4k}, \]
that yields
\[ -\frac{\pi}{2} < \arg Q_n^k(z) \leq \frac{\pi}{2}. \] (3.7)

$\Gamma_j$ can also be written as a disjoint union
\[ \Gamma_j = B_1 \cup B_2, \]
where
\[ B_1 := \{ \zeta \in \Gamma_j : |Q_n(\zeta)| < \frac{C}{n^\alpha (\sin \frac{\pi}{4k})^\alpha} \}, \] (3.8)
\[ B_2 := \{ \zeta \in \Gamma_j : |Q_n(\zeta)| \geq \frac{C}{n^\alpha (\sin \frac{\pi}{4k})^\alpha} \}. \] (3.9)

Points of $B_1$ satisfy
\[ |Q_n^k(\zeta)| < \frac{C^k}{n^\alpha (\sin \frac{\pi}{4k})^\alpha}. \] (3.10)

For $B_2$ we have
\[ |\sin(\frac{\pi}{k} - \arg Q_n(\zeta))| \leq \frac{C}{n^\alpha |Q_n(\zeta)|} \leq \sin \frac{\pi}{4k}, \]
\[ \pi - \frac{\pi}{4} < \arg Q_n^k(\zeta) \leq \pi + \frac{\pi}{4}. \] (3.11)

Now, if we choose $\zeta_0$ to be a point in $(0, \infty)$ with $\zeta_0 > \max\{|\mathcal{L}|^k, |\mathcal{L}'|^k\}$, then (3.5), (3.6) and (3.7) imply $|Q_n^k(z) - \zeta_0| \leq \zeta_0 + \frac{C^k}{n^\alpha (\sin \frac{\pi}{4k})^\alpha}$, $z \in \mathcal{L}$. Also, by (3.10) and (3.11) the estimate $|Q_n^k(\zeta) - \zeta_0| \geq \zeta_0 - \frac{C^k}{n^\alpha (\sin \frac{\pi}{4k})^\alpha}$ holds for $\zeta \in \Gamma_j$.

According to these observations, we have
\[ \left| \frac{Q_n^k(z) - \zeta_0}{Q_n^k(\zeta) - \zeta_0} \right|^{\frac{\frac{\alpha - \beta}{2\pi}}{\frac{4}{\pi}}} \leq \left( 1 + \frac{C}{n^\alpha \beta \epsilon} \right)^{\frac{\frac{\alpha - \beta}{2\pi}}{\frac{4}{\pi}}} \] (3.12)
where \( \tilde{C} = \frac{2C^h}{\zeta_{\alpha n}^{\alpha + 1} n^{\frac{1}{k^\alpha}} - c^\alpha} \leq C^h \) for \( n \) large enough.

Let \( \beta \) be such that \( 1 - \sigma > \beta > \frac{1}{1+k\alpha} \), so that \( 1 - \beta < \alpha \beta k \) and \( \sigma < 1 - \beta \).

From (3.12) it follows

\[
\left| \frac{Q_{[n^{\alpha}]}(z) - \zeta_0}{Q_{[n^{\alpha}]}(\zeta) - \zeta_0} \right| \left[ \frac{n^{1-\beta}}{2\pi} \right] \leq e^{\zeta_{\alpha \beta n} n^{1-\beta}} \leq \tilde{C},
\]

(3.13)

where \( \tilde{C} \) does not depend on \( n \).

Also, for all points \( z \) of a compact set \( E \subseteq \mathcal{L} \setminus \{z_1, z_2, \ldots, z_m\} \) and \( n \) sufficiently large the estimate

\[
\left| \frac{Q_{[n^{\alpha}]}(z) - \zeta_0}{Q_{[n^{\alpha}]}(\zeta) - \zeta_0} \right| \leq q \left[ \frac{n^{1-\beta}}{2\pi} \right],
\]

(3.14)

holds with some \( q = q(E) < 1 \).

Therefore, if we denote

\[ d_n := \rho_{j_i/\alpha}(z_j), \quad \gamma = \gamma_n := \{ \zeta : \zeta \in \Gamma_j, |\zeta - z_j| \geq d_n \} \]

and consider polynomial

\[
P_{n,j}^i(z) = \frac{1}{2\pi i} \int \frac{(f_{j-1}(\zeta) - f_j(\zeta)) \tilde{P}_{n,j}^i(z, \zeta) d\zeta}{\frown \Gamma_j \setminus \gamma}
\]

\[
+ \frac{1}{2\pi i} \int \frac{(f_{j-1}(\zeta) - f_j(\zeta)) \left( 1 - \left( \frac{Q_{[n^{\alpha}]}(z) - \zeta_0}{Q_{[n^{\alpha}]}(\zeta) - \zeta_0} \right) \left[ \frac{n^{1-\beta}}{2\pi} \right] \right)}{\zeta - z} d\zeta,
\]

by (3.13) and Theorem 4, for all \( z \in \mathcal{L} \) we get

\[
\left| \frac{1}{2\pi i} \int \frac{f_{j-1}(\zeta) - f_j(\zeta)}{\zeta - z} d\zeta - P_{n,j}^i(z) \right|
\]

\[
\leq \frac{1}{2\pi} \int |f_{j-1}(\zeta) - f_j(\zeta)| \left| \frac{Q_{[n^{\alpha}]}(z) - \zeta_0}{Q_{[n^{\alpha}]}(\zeta) - \zeta_0} \right| \left[ \frac{n^{1-\beta}}{2\pi} \right] |1 - \frac{1}{\zeta - z} - K[z_1^{(2)}(z, \zeta)]| d\zeta
\]

\[
+ \frac{1}{2\pi} \int |f_{j-1}(\zeta) - f_j(\zeta)| \left| \frac{Q_{[n^{\alpha}]}(z) - \zeta_0}{Q_{[n^{\alpha}]}(\zeta) - \zeta_0} \right| \left[ \frac{n^{1-\beta}}{2\pi} \right] |d\zeta|
\]

\[
\leq d_k^{\beta+2} \frac{|d\zeta|}{\| \zeta \|^2} + \int_{f_j^{(2)}} \left| \frac{\zeta^{k+1}}{\zeta - z} \right| |d\zeta| \approx E_n(f, \mathcal{L}),
\]

(3.15)

where the last inequality follows by the reasoning, similar to the one we use in (2.10).

If \( z \in E \), by (3.14), (2.10) and Theorem 4 we have

\[
\left| \frac{1}{2\pi i} \int \frac{f_{j-1}(\zeta) - f_j(\zeta)}{\zeta - z} d\zeta - P_{n,j}^i(z) \right|
\]

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4 Proof of Theorem 2

Since changing the \( R \) corresponds to scaling the lemniscate, we can always scale the picture and without loss of generality assume for simplicity \( R = 1 \).

As it was shown above, it's enough to approximate the function

\[
b_1(z) = \frac{1}{2\pi i} \int_{\Gamma_1 \cup \Gamma^2} \frac{f_1(\zeta) - f_2(\zeta)}{\zeta - z} d\zeta.
\]

Here \( \Gamma^1 \) and \( \Gamma^2 \) we choose in such a way that \(|P(\zeta)| > 1\) for all \( \zeta \in (\Gamma^1 \cup \Gamma^2) \setminus \{0\} \).

While the image of \( \Gamma^1 \cup \Gamma^2 \) under the mapping \( P \) belongs to the complement of the unit disc, the image of \( L \) is inside the disc, that yields

\[
\left| \frac{P(z)}{P(\zeta)} \right| \leq 1, \quad z \in L, \quad \zeta \in \Gamma^1 \cup \Gamma^2
\]

Moreover, due to geometry of \( L \) the equality in (4.1) occurs only if \( \zeta = z = 0 \).

Let

\[
\hat{P}_n(z, \zeta) = 1 - \left( \frac{P(z)}{P(\zeta)} \right)^{\frac{1}{n+1}} + \left( \frac{P(z)}{P(\zeta)} \right)^{\frac{n}{n+1}} K_{\hat{z}}(z, \zeta).
\]

One may check that \( \hat{P}_n(z, \zeta) \) is a polynomial in \( z \) of degree at most \( n \).

Let \( d_n := \rho_{1/n}(0), \gamma = \gamma_n := \{ \zeta : \zeta \in \Gamma^1 \cup \Gamma^2, |\zeta| \geq d_n \} \), and consider

\[
P_n(z) = \frac{1}{2\pi i} \int_{\gamma} (f_1(\zeta) - f_2(\zeta)) \hat{P}_n(z, \zeta) d\zeta
\]

\[
+ \frac{1}{2\pi i} \int_{(\Gamma^1 \cup \Gamma^2) \setminus \gamma} (f_1(\zeta) - f_2(\zeta)) \left( \frac{1 - \left( \frac{P(z)}{P(\zeta)} \right)^{\frac{1}{n+1}}}{\zeta - z} \right) d\zeta.
\]

By virtue of Theorem 3, estimates (2.10) and (4.1), for all \( z \in L \)

\[
\left| \frac{1}{2\pi i} \int_{\Gamma^1 \cup \Gamma^2} \frac{f_1(\zeta) - f_2(\zeta)}{\zeta - z} d\zeta - P_n(z) \right|
\]
\[
\leq \frac{1}{2\pi} \int_{\gamma} |f_1(\zeta) - f_2(\zeta)| \left| \frac{P(z)}{P(\zeta)} \right| \left| \frac{1}{\zeta - z} - K_{1/2}(z, \zeta) \right| |d\zeta|
+ \frac{1}{2\pi} \int_{(\Gamma \cup \Gamma^2) \setminus \gamma} \left| \frac{f_1(\zeta) - f_2(\zeta)}{\zeta - z} \right| \left| \frac{P(z)}{P(\zeta)} \right| |d\zeta|
\leq \frac{1}{2\pi} \int_{\gamma} |f_1(\zeta) - f_2(\zeta)| \left| \frac{1}{\zeta - z} - K_{1/2}(z, \zeta) \right| |d\zeta|
+ \frac{1}{2\pi} \int_{(\Gamma \cup \Gamma^2) \setminus \gamma} \left| \frac{f_1(\zeta) - f_2(\zeta)}{\zeta - z} \right| |d\zeta|
\leq d_n^{k+2} \int_{\gamma} \frac{|d\zeta|}{|\zeta|^2} + \int_{(\Gamma \cup \Gamma^2) \setminus \gamma} \left| \frac{\zeta^{k+1}}{\zeta - z} \right| |d\zeta|
\leq [\rho^*_n(0)]^{k+1} \lesssim E_n(f, \mathcal{L}).
\]

If \( E \) is a compact set in \( \mathcal{L} \setminus \{ z_1, 0, z_2 \} \), then for all \( z \in E \)
\[
|P(z)| < q,
\]
for some \( q = q(E) < 1 \).

Let
\[
d(E) := \min_{z \in E} \{ 1 - |P(z)| \}.
\]

By (4.3), \( d(E) > 0 \) for any compact set \( E \subset \mathcal{L} \setminus \{ z_1, 0, z_2 \} \).

Therefore, for all \( z \in E \)
\[
\left| \frac{P(z)}{P(\zeta)} \right| \leq |P(z)| \frac{\pi}{|z|} \leq |1 - d(E)| \frac{\pi}{|z|} \leq e^{-cn d(E)},
\]
where the constant \( c > 0 \) does not depend on \( n \) and \( E \).

Hence, for \( z \in E \)
\[
\left| \frac{1}{2\pi i} \int_{\Gamma \cup \Gamma^2} \frac{f_1(\zeta) - f_2(\zeta)}{\zeta - z} d\zeta - P_n(z) \right|
\leq e^{-cn d(E)} \frac{1}{2\pi} \int_{\gamma} |f_1(\zeta) - f_2(\zeta)| \left| \frac{1}{\zeta - z} - K_{1/2}(z, \zeta) \right| |d\zeta|
+ e^{-cn d(E)} \frac{1}{2\pi} \int_{(\Gamma \cup \Gamma^2) \setminus \gamma} \left| \frac{f_1(\zeta) - f_2(\zeta)}{\zeta - z} \right| |d\zeta|
\leq [\rho^*_n(0)]^{k+1} e^{-cn d(E)} \lesssim E_n(f, \mathcal{L}) e^{-cn d(E)}.
\]

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