MODULAR INVARIANCE AND THE FINITENESS OF SUPERSTRING THEORY

SIMON DAVIS

Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Silver Street, Cambridge CB3 9EW

ABSTRACT. The genus-dependence of multi-loop superstring amplitudes is bounded at large orders in perturbation theory using the super-Schottky group parametrization of supermoduli space. Partial estimates of supermoduli space integrals suggest an exponential dependence on the genus when the integration region is restricted to a single fundamental domain of the super-modular group in the super-Schottky parameter space. Bounds for N-point superstring scattering amplitudes are obtained for arbitrary N and are shown to be consistent with exact results recently obtained for special type II string amplitudes for orbifold or Calabi-Yau compactifications. It is suggested that the generic estimates, which imply the validity of superstring perturbation theory in the weak-coupling limit, might be used to determine scattering amplitudes at strong coupling because of the S-duality of type II and heterotic string theories.
The combination of supersymmetry and the smooth geometry of the interaction region in superstring theory has been shown to lead to finite scattering amplitudes at each order in perturbation theory and vanishing multiloop amplitudes when there are fewer than four external massless states. These results may be contrasted with the divergent amplitudes and even the genus-dependence of regularized amplitudes in bosonic string theory. In particular, a lower bound for the regularized closed bosonic string path integral was found to increase at a factorial rate with respect to the genus. This bound was based on an argument typically used in field theory, the counting of non-isomorphic trivalent graphs, and appeared to be independent of the content of the string theory. It has been suggested that this counting of diagrams, and therefore the growth, could persist in superstring theory [1]. However, the lower bound of [2] was derived from an integral over Teichmüller space rather than moduli space and does not suggest a connection between supersymmetry and large-order string perturbation theory. Similarly, a counting of cells in a triangulation of punctured moduli space yields a factorial growth with respect to the genus [3], but this estimate will be reduced by the introduction of a cut-off in moduli space, excluding those Riemann surfaces with closed geodesics having lengths less than some genus-independent bound.

An analysis of the regularized bosonic string vacuum amplitude in the Schottky parametrization has revealed that the restriction to a single copy of the fundamental region of the modular group reduces the regularized path integral by a genus-dependent factor [4]. Although the regularized integral increases rapidly with respect to the genus, it does not grow at a factorial rate, thus leaving open the possibility of a qualitatively different result for superstrings. Moreover, a connection is established between the genus-dependence of the limits for the Schottky group parameters and the large-order behaviour of the amplitudes, suggesting that the infrared and large-order divergences, which occur in the infinite-genus limit for the bosonic string, might be simultaneously eliminated in the superstring path integral. The techniques of [4] shall be adapted to superstring theory in this paper. It will be demonstrated that, based on a partial estimate of the bounded superstring amplitudes, the maximal dependence on the genus may be exponential, revealing
a higher degree of finiteness in the large-order limit.

Superstring scattering amplitudes are given in general by supermoduli space integrals

\[
\langle V_1(k_1) \ldots V_N(k_N) \rangle_g = \int_{s\mathcal{M}_g} \ d\mu_{sWP} \left( \frac{8\pi s\text{det}^{\hat{\square}_0}}{\text{det}^{\hat{\Psi}_\alpha | \hat{\Psi}_\beta}} \right)^{-5} (s\text{det}^{\hat{P}_1 \hat{P}_1})^\frac{1}{2} 
\langle \langle V_1(k_1) \ldots V_N(k_N) \rangle \rangle_E 
\]

where \( d\mu_{sWP} \) is the super Weil-Petersson measure, \( \hat{\Psi}_\alpha \in \text{Ker} \hat{\square}_0 \), \( \langle \langle \rangle \rangle \) represents the evaluation of the path integral over the scalar position superfields \( X^\mu \), and the integral is restricted to a \((3g - 3|2g - 2)\) complex-dimensional slice of super-Teichmüller space parametrized by the supergeometries \( \{ \hat{E}_A^M \} \). In terms of superghosts,

\[
\langle V_1(k_1) \ldots V_N(k_N) \rangle_g = \int_{s\mathcal{M}_g} \ d^2 m_K \int D(XBC) \langle \langle V_1(k_1) \ldots V_N(k_N) \rangle \rangle \prod_b |\delta(\langle \mu_b | B \rangle)|^2 \prod_k |\langle \mu_k | B \rangle|^2 e^{-I} 
\]

where \( B \) and \( C \) are anti-ghost and ghost superfields of \( U(1) \) weight \( \frac{3}{2} \) and \(-1\) respectively, \( \mu_K, \ K = (k,b) \), are super Beltrami differentials, and \( I = I_m + I_{sgh} \) is the sum of the matter and superghost actions [5]. A formula similar to that given in equation (2) for the scattering amplitude of \( N \) external massless states appears in the twistor-string formalism [6].

The Schottky uniformization of super-Riemann surfaces shall be used to study the superstring measure. The super-Schottky group is generated by \( g \) transformations \( T_n \), \( n=1,\ldots,g \), acting on the super-complex plane with coordinate \( Z = (z, \theta) \)

\[
\frac{T_n(Z) - Z_1n}{T_n(Z) - Z_{2n}} = K_n \frac{Z - Z_1n}{Z - Z_{2n}} \quad |K_n| < 1
\]

where \( Z_{1n} = (\xi_1n, \theta_{1n}) \) and \( Z_{2n} = (\xi_{2n}, \theta_{2n}) \) are attractive and repulsive super-fixed points respectively.

It has been found that the measure on supermoduli space for the Neveu-Schwarz sector of superstring theory, corresponding to the propagation of bosonic states in the loops,
is simpler than that of the Ramond sector \[7\][8][9]. The holomorphic part \[10][11][12]\multiplied by the period-matrix factor is

\[
\frac{1}{dV_{ABC}} \prod_{n=1}^{g} \frac{dK_n}{K_n^{\frac{1}{2}}} \prod_{m,n=1}^{g} \frac{dZ_{1m}dZ_{2n}}{Z_{1n} - Z_{2n}} \left( \frac{1 - K_n}{1 - (-1)^{B_n} K_n^{\frac{1}{2}}} \right)^2 \frac{[\det(I\text{m}\ T)]^{-5}}{ \prod_{\alpha}' \prod_{p=1}^{\infty} \left( \frac{1 - (-1)^{N^B_{\alpha}} K_{\alpha}^{p - \frac{1}{2}}}{1 - K_{\alpha}^{p}} \right)^{10} \prod_{\alpha}' \prod_{p=2}^{\infty} \left( \frac{1 - K_{\alpha}^{p}}{1 - (-1)^{N^B_{\alpha}} K_{\alpha}^{p - \frac{1}{2}}} \right)^2} \tag{4}
\]

where the infinitesimal super-projective invariant volume element is

\[
dV_{ABC} = \frac{dZ_A dZ_B dZ_C}{[(Z_A - Z_B)(Z_C - Z_A)(Z_B - Z_C)]^{\frac{1}{2}}} \cdot \frac{1}{d\Theta_{ABC}}
\]

\[
\Theta_{ABC} = \frac{\theta_A(Z_B - Z_C) + \theta_B(Z_C - Z_A) + \theta_C(Z_A - Z_B) + \theta_A \theta_B \theta_C}{[(Z_A - Z_B)(Z_C - Z_A)(Z_B - Z_C)]^{\frac{1}{2}}}
\tag{5}
\]

and the super-period matrix is

\[
\mathcal{T}_{mn} = \frac{1}{2\pi i} \left[ \ln K_n \delta_{mn} + \sum_{\alpha} \langle m,n \rangle \ln \left[ \frac{Z_{1m} - V_{\alpha} Z_{1n} Z_{2m} - V_{\alpha} Z_{2n}}{Z_{1m} - V_{\alpha} Z_{2m} Z_{2m} - V_{\alpha} Z_{1n}} \right] \right]
\tag{6}
\]

Selecting \(B_n\) to be 0 or 1, depending on the boundary conditions around the \(g\) B-cycles, produces \(2^g\) spin structures associated with the exchange of bosonic states in the \(g\) loops. The number \(N_B^{\alpha}\) equals \(\sum_{n=1}^{g} B_n N^n_{\alpha}\), where \(N^n_{\alpha}\) is the number of times that the generator \(T_n\) or its inverse in the product \(V_{\alpha}\).

Because the measure (4) does not contain the contribution of the Ramond sector, it can only be used to obtain a partial estimate of the amplitude at large genus. However, the restriction to the Neveu-Schwarz sector just necessitates a larger integration region corresponding to the different spin structures, but this would only modify the bound by an exponential function of the genus, as the number of spin structures increases as \(2^{2g}\). The choice of spin structure is defined by the sign of the square root \(K_n^{\frac{1}{2}}\) and \(K_{\alpha}^{\frac{1}{2}}\) in equation (4). While a modular transformation may map the expression (4), with a given choice of signs for the square roots \(K_n^{\frac{1}{2}}\) for all \(n\), into a measure with different signs for the square roots, corresponding to other spin structures, it will be demonstrated later that the integral would be altered only by an exponential factor, from an analysis of sums of \(|K_{\alpha}|\) over elements of the Schottky group [4] and integration regions.
Certain advances have been made in obtaining the Ramond contribution to the full superstring measure. The non-zero mode part of the most general amplitude with an arbitrary number of Ramond loops $g_R$ and $g_{NS}$ Neveu-Schwarz loops [13] is

$$
\prod_{i=1}^{g_R} \prod_{n=2}^{\infty} \left[ \frac{1 \pm (K_i^R)^n}{1 - (K_i^R)^n} \right]^{8} \cdot \left[ \frac{1 \pm K_i^R}{K_i^R} \right]^{10} \prod_{\alpha,NS} \prod_{n=2}^{\infty} \left[ \frac{1 - K_{\alpha}^{-n+\frac{1}{2}}}{1 - K_{\alpha}^{-n}} \right]^{8} \cdot \left[ \frac{1 - K_{\alpha}^{\frac{1}{2}}}{1 - K_{\alpha}} \right]^{10}
$$

(7)

and the ghost zero-mode contribution from one Ramond loop is simply

$$
4 \frac{dK_R}{K_R} \frac{dB_g}{B_g} \frac{d\kappa_R}{\kappa_R} \frac{1}{(1 - \xi_{2g})}
$$

(8)

where $B_g$ is a multiplier variable defined in terms of the fixed points in a subsequent formula and $\kappa_R$ is the fermionic modulus [13].

The overall $OSp(2\mid 1)$ invariance can be used to fix two of the super-fixed points and the even coordinate of a third superfixed point, leaving $3g-3$ even moduli and $2g-2$ odd moduli amongst the super-Schottky group parameters [7]. The integration region is defined to be the fundamental domain of the super-mapping class group in super-Teichmüller space. Equivalently, one may use the intersection of the fundamental region of the super-modular group in the space of positive-definite, symmetric super-period matrices with the set of $T_{mn}$ associated with a super-Riemann surface, which will be contained in the set of $T_{mn}$ such that the ordinary period matrix $\tau_{mn}$, the complex number-valued part of $T_{mn}$, lies in a fundamental domain of the modular group and corresponds to a Riemann surface. This leads to an infinite number of conditions on the multipliers $K_n$ and fixed points $\xi_{1n}$, $\xi_{2n}$ [4] [14].

It has been shown in the analysis of the closed bosonic string that these inequalities may be satisfied by certain categories of isometric circles, $I_{T_n} = \{ z \in \hat{C}||\gamma_n z + \delta_n| = 1 \}$, $T_n z = \frac{\alpha_n z + \beta_n}{\gamma_n z + \delta_n}$ which would then represent a subset of moduli space. In particular, the
following configurations of isometric circles

\[(i)\quad \frac{\epsilon_0}{g^{1-2q'}} \leq |K_n| \leq \frac{\epsilon_0'}{g^{1-2q'}} \quad \delta_0 \leq |\xi_{1n} - \xi_{2n}| \leq \delta_0' \quad 0 \leq q' \leq \frac{1}{2}\]

\[(ii)\quad \frac{\epsilon_0}{g^{1-2q'}} \leq |K_n| \leq \frac{\epsilon_0'}{g^{1-2q'}} \quad \frac{\delta_0}{g^q} \leq |\xi_{1n} - \xi_{2n}| \leq \frac{\delta_0'}{g^q} \quad 0 < q \leq q' < \frac{1}{2}\]

\[(iii)\quad \epsilon_0 \leq |K_n| \leq \epsilon_0' \quad \frac{\delta_0}{\sqrt{g}} \leq |\xi_{1n} - \xi_{2n}| \leq \frac{\delta_0'}{\sqrt{g}}\]

describe a subset of Teichmuller space consistent with a cut-off on the radii of the isometric circles, or equivalently the size of the handles in the intrinsic metric on the surface, \(r_{\text{tr}}^2 = |\gamma_n|^{-2} \geq \frac{1}{g}\) [15].

The absence of divergences in N-point superstring amplitudes at any given finite order in perturbation theory implies that the restriction on the minimum length of closed geodesics should be removed. Effectively closed infinite-genus surfaces, besides completing the domain of string perturbation theory, may have an essential role in the superstring path integral. Configurations of isometric circles arising in the super-Schottky uniformization of these surfaces shall then be included in the supermoduli space integral. The ranges \(\frac{\epsilon_0}{n^{q'}} \leq |K_n| \leq \frac{\epsilon_0'}{n^{q'}}\), \(n = 1, \ldots, g\), \(q' > 1\), and \(\frac{\delta_0}{n^{q''}} \leq |\xi_{1n} - \xi_{2n}| \leq \frac{\delta_0'}{n^{q''}}\), \(n = 1, \ldots, g\), \(q'' > \frac{1}{2}\), therefore will be considered also. While the parameters \(q\), \(q'\), \(q''\) and \(q'''\) are initially chosen to be continuous parameters, to avoid overcounting in the path integral, it is necessary to select a discrete set of values. Specifically, it is sufficient to sum over the values

\[q \leq q'_N = N \frac{\ln \left( \frac{
}{\epsilon_0} \right)}{2 \ln g} \quad N = 0, 1, \ldots, \left[ \frac{\ln g}{\ln \left( \frac{\epsilon_0}{\n} \right)} \right] \quad (9)\]

\[q''_N = 1 + \tilde{N} \frac{\ln \left( \frac{\epsilon_0}{\n} \right)}{\ln n} \quad \tilde{N} = 1, 2, \ldots\]

with the values of \(q'''\) specified later. The values of \(N\) follow from the non-overlapping of the ranges \(\frac{\epsilon_0}{g^{1-2q'}} \leq |K_n| \leq \frac{\epsilon_0'}{g^{1-2q'}}\) for different choices of \(q'_N\) and the covering of the interval \(0 \leq q \leq q'_N < \frac{1}{2}\) [4]. Similarly, non-overlapping of the ranges \(\frac{\epsilon_0}{n^{q''}} \leq |K_n| \leq \frac{\epsilon_0'}{n^{q''}}\) requires that the set \(\{ q''_N \} \) be selected so that

\[\frac{\epsilon_0'}{n^{q''}_{N+1}} = \frac{\epsilon_0}{n^{q''}_N} \quad q''_{N+1} - q''_N = \frac{\ln \left( \frac{\epsilon_0}{\n} \right)}{\ln n} \quad (10)\]
and the last equality is satisfied by the sequence \( \{ q_N'' \} \) in equation (9).

An estimate for the superstring path integral with measure (4) will now be given based on surfaces near the degeneration locus. The use of this measure initially reduces the divergence from \( \int \frac{d|K|}{|K|^2 |\log |K||^3} \) for bosonic strings to \( \int \frac{d|K|}{|K|^2 |\log |K||^5} \) for the Neveu-Schwarz string as \( |K| \to 0 \), reflecting the existence of a tachyon in both cases and a shift in the value of the square of its momentum [16]. It has been noted, however, that an extra factor of \( K^{1/2} \) in the holomorphic part of the measure arises for Ramond fermions circulating in the loops and also when the GSO projection is applied to the Neveu-Schwarz sector. A sum over spin structures, weighted with a phase factor \((-1)^{B_n}\), introduces the factors \( K^{1/2}_n \), \( n = 1, \ldots, g \), since

\[
\frac{1}{[1 - K^{1/2}_n]^2} - \frac{1}{[1 + K^{1/2}_n]^2} = \frac{4K^{1/2}_n}{[1 - K_n]^2} \tag{11}
\]

signalling the absence of a tachyon singularity.

While restriction to the Neveu-Schwarz sector leads to a partial estimate of the amplitude, an estimate of the full amplitude, including a sum over all \( 2^{2g} \) spin structures, can be obtained by considering an integration region consisting of more than one fundamental domain of the modular group.

This may be seen more concretely at one loop. There the fundamental region of the modular group \( SL(2; \mathbb{Z}) \) is mapped onto the adjoining fundamental region by the shift \( \tau \to \tau + 1 \). Labelling the spin structures by the sign of the world-sheet fermions upon one traversal of the A- and B-cycles, there is one odd spin structure \((++)\) and three even spin structures \((+-)\), \((-+)\) and \((-\cdot-\cdot)\) at genus 1. Under the map \( \tau \to \tau + 1 \), \((++) \to (++)\) while \((+-) \to (+-)\), \((-+) \to (-+)\) and \((-\cdot-) \to (-\cdot-)\). Similarly, the modular transformation \( \tau \to -\frac{1}{\tau} \), which maps the original fundamental region to another copy in the upper half plane, leaves \((++)\) invariant, whereas \((+-) \to (-+)\), \((-+) \to (+-)\) and \((-\cdot-) \to (-\cdot-)\). Given one odd spin structure and one even spin structure, all four spin structures may be obtained by using the two transformations \( \tau \to \tau + 1 \), \( \tau \to -\frac{1}{\tau} \) and their product. It may
be recalled that the sum over all four spin structures is required for the vanishing of the one-loop partition function

\[ Z_1(\tau) = \frac{1}{2} \frac{1}{\eta(\tau)^4} [\eta(+) \theta_1^4(0|\tau) + \eta(-) \theta_2^4(0|\tau) + \eta(+) \theta_3^4(0|\tau) + \eta(-) \theta_4^4(0|\tau)] \] (12)

with the choice of phases \( \eta(+) = \pm 1, \eta(-) = -1, \eta(+) = -1 \) and \( \eta(-) = 1 \) [17].

At genus \( g \), there are \( 2^{g-1}(2^g - 1) \) odd and \( 2^{g-1}(2^g + 1) \) even spin structures. Thus, the problem of estimating the full superstring amplitude becomes one of summing over all \( 2^{2g} \) spin structures, beginning with the set of \( 2^g \) spin structures associated with the Neveu-Schwarz sector of the string. Since the number of Dirac zero modes modulo 2 on a Riemann surface is a modular invariant, the sets of odd spin structures and even spin structures each form a representation of the modular group and all \( 2^{2g} \) spin structures arise from modular transformations of one odd and one even spin structure. Thus, if the set of \( 2^g \) spin structures did contain at least one odd and one even spin structure, summation over all \( 2^{2g} \) spin structures can be achieved by enlarging the integration region to contain those regions that are images of the original fundamental region under the corresponding modular transformations.

The choice of signs for spin structures at higher genus can be deduced by noting that the number of Dirac zero modes modulo 2 is additive when two Riemann surfaces are glued [18]. Thus an odd spin structure at genus 2 is composed of an odd spin structure at genus 1 and an even spin structure at genus 1, while an even spin structure at genus 2 consists of a sum of two even spin structures at genus 1 or two odd spin structures at genus 1. The odd spin structures at genus 2 may be listed as \((+++), (+++), (---), (++)\) and \((-+ +)\) while the even spin structures are \((++ +), (+ --), (+ --), (+ --), (+ --), (+ --), (+ --), (+ --)\) and \((-+- +)\). The Neveu-Schwarz sector, defined by anti-periodic boundary conditions around the A-cycles for the world-sheet fermions, corresponds to the spin structures \((-+- +), (+- -), (--- +), \) and \((- - -)\), which are all even. Since the addition of even spin structures is again even, the Neveu-Schwarz sector should always consist of even spin structures. This implies that
modular transformations of the Neveu-Schwarz sector initially can only span the set of even spin structures.

This difficulty of summing over the odd spin structures can be resolved by enlarging the set of $2^g$ spin structures to include an odd spin structure. This can be done as follows. Considering the map from the parallelogram to the annulus, the target space coordinate is $X = e^{2\pi i z} = x + \theta \psi$, under the transformation $z \rightarrow z + 1$, $X \rightarrow X$, but the fermionic coordinates change sign $\theta \rightarrow e^{\pi i} \theta$ and $\psi \rightarrow e^{\pi i} \psi$. Since the annulus can then be mapped to the Schottky plane with two disks removed without changing the sign of the fermions, a change in the transformation of the Grassmann coordinate $\theta$ will induce a change in the sign of the world-sheet fermion leading to an odd spin structure.

While the choice of phase is required for the transformation of amplitudes corresponding to distinct spin structures, the absolute values of the amplitudes are also mapped into each other. Consequently, modular invariance is not only a property of the standard sum over spin structures with the correct phases in the superstring amplitude, but it is also valid for a sum of absolute values of amplitudes associated with different sets of spin structures. Specifically, if $\eta_{(i)}$ denotes the phase associated with the spin structure $(i)$ and $\sigma_r$ is the modular transformation, then the superstring amplitude $A_{N,g} = \int_{F_g} I_{N,g}$, where $F_g$ is the fundamental region of the modular group, is invariant so that

$$\int_{F_g} \sum_{(i)=1}^{2^g} \eta_{(i)} I_{N,g}^{(i)} = \int_{\sigma_r(F_g)} \sum_{(i)=1}^{2^g} \eta_{\sigma_r(i)} I_{N,g}^{\sigma_r(i)}$$

(13)

This sum can be arranged into sets, each consisting of $2^g$ spin structures, so that the sum over spin structures within any individual set introduces a factor of $R_\pi^\pm$, in the same manner as the sum over spin structures (11) in the Neveu-Schwarz sector, implying removal of the tachyon singularity. Denoting the sets as $S_r$, $r = 1, 2, 3, ...$, where $S_1$ represents the Neveu-Schwarz sector, it can be shown that there are $2^g-1$ odd spin structures and $2^g-1$ even spin structures in each set $S_r$, $r = 2, 3, ...$. Recalling that modular transformations acting on the Neveu-Schwarz sector will generate the other even spin structures, but no odd spin structures, it can be seen that $2^{g-1} \cdot 2^g$ even spin structures naturally arise in...
this way, leaving $2^{g-1}$ spin structures. To constitute a set of $2^g$ spin structures, $2^{g-1}$ odd spin structures are needed. An examination of the signs of the associated with the B-cycles reveals that it is this set of $2^{g-1}$ odd and $2^{g-1}$ even spin structures which allows for cancellation of the tachyon divergence, rather than the sets of $2^g$ even spin structures, other than the Neveu-Schwarz sector, that belong to the group $\{S_r\}$. Beginning with this set, which may be labelled as $S_2$, modular transformations might then be used to generate the remaining $2^{g-1}(2^g - 2)$ odd spin structures and simultaneously $2^{g-1}(2^g - 2)$ even spin structures, so that the sets $S_r$, $r = 3, 4, \ldots$ contain an equal number of odd and even spin structures. This description of the sets $\{S_r\}$ can be verified at arbitrary genus. This labelling can be done for genus 1, 2 and 3 in particular.

The existence of a modular transformation mapping the set $S_2$ into $S_r$, $r = 3, 4, \ldots$ should follow from the absence of the tachyon singularity for each of the sets. The tachyon divergences in the limit $|K_n| \to 0$ cancel only when the correct choice of spin structures, consistent with modular transformations of the integrand corresponding to $S_2$, is used. One can consider the following sum over spin structures

\[
\int_{F_g} \sum_{(a_{NS}) = 1}^{2^g} \eta(a_{NS}) I_{N,g}^{(a_{NS})} + \int_{F_g} \sum_{(a_e) = 1}^{2^{g-1}} \eta(a_e) I_{N,g}^{(a_e)}
\]

\[
+ \int_{F_g} \sum_{(a_o) = 1}^{2^{g-1}} \eta(a_o) I_{N,g}^{(a_o)} + \sum_{r=3}^{2^g} \int_{F_g} \sum_{(a_e) = 1}^{2^{g-1}} \eta_{\sigma_r(a_e)} I_{N,g}^{\sigma_r(a_e)}
\]

\[
+ \sum_{r=3}^{2^g} \sum_{(a_o) = 1}^{2^{g-1}} \eta_{\sigma_r(a_o)} I_{N,g}^{\sigma_r(a_o)}
\]  (14)

where $S_2 = \{(a_e)\} \cup \{(a_o)\}$, with $\{(a_e)\}$ representing even spin structures and $\{(a_o)\}$ representing odd spin structures, and $S_r = \{\sigma_r(a_e)\} \cup \{\sigma_r(a_o)\}$. The magnitude of this sum can then be bounded since

\[
\left| \int_{F_g} \sum_{(i) = 1}^{2^g} \eta(i) I_{N,g}^{(i)} \right| \leq \sum_{r=1}^{2^g} \left| \int_{F_g} \sum_{(a) \in S_r} \eta(a) I_{N,g}^{(a)} \right|
\]  (15)

Each of the absolute values of the integrals, labelled by $r$, is free of the infrared divergence in the $|K_n| \to 0$ limit, since the sum over the $2^g$ spin structures in $S_r$ removes the
tachyon singularity. This can be verified at genus 1, for example, as the infinities associated with the Neveu-Schwarz spin structures \((-\rightarrow\) and \((\rightarrow\)) in the limit \(\tau \rightarrow i\infty\) cancel

\[
\lim_{\tau \to i\infty} \left[ \frac{\theta_4^4(0|\tau)}{\eta(\tau)^4} - \frac{\theta_4^4(0|\tau)}{\eta(\tau)^4} \right] = 0
\] (16)

It will be shown in the following analysis that the absence of divergences in the limit \(|K_n| \to 0\) is used in the proof of finiteness of the amplitudes in the other degeneration limits. This implies that when the divergences in these limits are eliminated, bounds on the magnitudes of the integrals \(\int_{F_g} \sum_{a \in S_r} \eta(a) I^{(a)}_{N,g}\) will suffice to provide an upper bound on the total superstring amplitude \(\int_{F_g} \sum_{i=1}^{2^{2g}} \eta(i) I^{(i)}_{N,g}\).

Another divergence associated with the coincidence of fixed points \(\xi_{1n}\) and \(\xi_{2n}\) arises in the Neveu-Schwarz sector, even after summing over spin structures. One might therefore wish to use the combination of the Neveu-Schwarz measure with the part of the Ramond contribution in equations (7) and (8) to estimate the genus-dependence of the amplitudes. For example, the ghost zero-mode contribution at one Ramond loop contains a dependence on the multiplier variable \(B_g\) which represents a softening of the divergences in the \(|B_g| \to 0\) limit.

The actual finiteness of the superstring amplitudes demonstrates that use of the entire superstring measure would remove this divergence. The analysis of [6] shows that this is achieved by transferring the fixed-point divergence to an integral associated with the multipliers \(K_n\), which has been shown to be finite in the limit \(|K_n|\) by the above argument. Specifically, the fixed points can be expressed in terms of alternative variables \(B_m\) and \(H_m\) [11] which may be defined in terms of different radii in the representation of the \(N\)-punctured genus-\(g\) surface as the sewing together of \(2g-2+N\) three-punctured spheres (Fig. 1)
Fig. 1. An N-punctured genus-g surface can be constructed using 2g-2+N three-punctured spheres.

In the twistor-string formalism, the amplitude contains the picture-changing operators $F^\pm = [Q, \xi^\pm]$, where Q is a BRST operator derived from the energy-momentum tensor and ghosts and $\xi^\pm$ are spin-0 fermions required in the expression for the bosonized ghosts. The locations of the picture-changing operators are chosen to be arbitrary points on the 2g-2+N spheres. Since a change in the location of $F^\pm$ implies that $[Q, \xi] \to [Q, \xi] + \int dz [Q, \partial_z \xi]$, the contour of Q initially surrounds the curve joining the initial and final positions of the picture-changing operators [19]. In the degeneration limit $H_m \to 0$, where the Riemann surface splits into two components of genus g-1 and 1, the contour of Q can be pulled off the picture-changing operator to surround the three punctures of $S_{1+g+N}$ (Fig. 2).
Fig. 2. The contour of $Q$ is pulled past two of the punctures sewed together to form a handle on $S_{i+g+N}$ to surround the third puncture.

Around each of the sewed punctures is a closed loop $C_i$ or radius $R_i$ and the contribution from each of the Beltrami differentials is

$$\left| \prod_{i=1}^{3g-3+N} \int_{C_i} \frac{y_i b(y_i)}{R_i} dy_i \right|^2$$

Around the first puncture is the contour integral involving the Beltrami differential for $K_n$, while at the third puncture is the operator

$$|c \exp(-\phi^+-\phi^-) \exp(h^+ + h^-) \psi^+ \psi^-|^2$$

where $c$ is a right-moving boson of spin-$\frac{1}{2}$, $\phi^\pm$ are two scalar bosons of screening charge 2 defining the bosonized ghosts, $h^\pm$ are a pair of right-moving spin-0 fermions [6]. Anticommuting $Q$ with the Beltrami differential for $K_n$ produces a derivative with respect to $K_n$. This total derivative does not represent a divergence because the amplitude already has been shown to be finite at the boundary $K_n = 0$. Anticommutation of $Q$ with the operator at the third puncture produces no terms with zero-modes of the spin-0 fermions $\psi^\pm$, and therefore the amplitude is independent of the locations of the picture-changing
operators. Since all of the picture-changing operators may be moved to the third punc-
ture, the zero-modes of $\psi^+$ and $\psi^-$ cancel, implying that the amplitude associated with the
genus 1 component of the Riemann surface vanishes [6]. This leaves the contribution from
the boundary of the $K_i$ region and, in this sense, it represents a transfer from an integral
involving $H_m$ to a finite integral involving $K_n$. Consequently, divergences are absent in
the (Type IIB Green-Schwarz) superstring amplitudes in the degeneration limit $H_m \to 0$.
A similar argument may be used to demonstrate that the amplitudes are also finite in the
$B_m$ limit [6].

This finiteness result, valid at any given order, suggests a genus-dependence for the
superstring amplitudes mentioned in the introduction. As the amplitude is finite in each of
the degeneration limits $K_n, H_m, B_m \to 0$, and the leading-order behaviour of large-order
amplitudes is essentially determined by taking these limits simultaneously, it follows that
the products of the integrals over these multipliers should be bounded by an exponential
function of the genus, $B_K^g \cdot B_H^{g-2} \cdot B_B^{g-1}$, where $B_K, B_H$ and $B_B$ are upper bounds for
each of the $K_n, H_m$ and $B_m$ integrals respectively.

However, any conclusion derived from the boundedness of the amplitudes in each of the
degeneration limits can be improved by a more precise estimate of the genus-dependence
at large orders. Ideally, therefore, one would like to begin the present calculation of the
amplitudes with the entire superstring measure. The argument of [6], transferring the
problem of the fixed-point divergence to the multiplier integral, implies that one can pro-
ceed with the estimate of the amplitudes using the measure for the Neveu-Schwarz sector
and the measure for the Ramond sector, with the appropriate sum over spin structures, and
apply modular transformations to obtain bounds for the sum over all $2^{2g}$ spin structures.
This procedure would be essentially equivalent to setting an upper bound for an ampli-
tude using the hypothesized divergence-free superstring measure in the super-Schottky
paramatization. Moreover, in terms of the sums over spin structures,

\[
\left| \int_{F_g} \sum_{(a) \in S_r} \eta(a) I^{(a)}_{N,g} \right| = \left| \int_{F_g} \sum_{(ae) = 1}^{2g-1} \eta_{\sigma_r(ae)} I^{\sigma_r(ae)}_{N,g} + \int_{F_g} \sum_{(ao) = 1}^{2g-1} \eta_{\sigma_r(ao)} I^{\sigma_r(ao)}_{N,g} \right|
\]

\[
= \left| \int_{\sigma_r^{-1}(F_g)} \sum_{(ae) = 1}^{2g-1} \eta(ae) I^{(ae)}_{N,g} + \int_{\sigma_r^{-1}(F_g)} \sum_{(ao) = 1}^{2g-1} \eta(ao) I^{(ao)}_{N,g} \right| \quad (19)
\]

Since the integral involving the spin structures \( \{\sigma_r(ae)\} \cup \{\sigma_r(ao)\} \) has been converted to an integral involving the spin structures \( \{(ae)\} \cup \{(ao)\} = S_2 \), with a different integration region, it is sufficient to consider just the spin structures in \( S_1 \) and \( S_2 \) and to determine the effect of integrating over a different region in Schottky group parameter space. Upon evaluation of the integrals over the ranges for the parameters \( K_n, \xi_{1n}, \xi_{2n} \) specified earlier, it follows that the shift of the integration region from \( F_g \) to \( \sigma_r^{-1}(F_g) \) will change the bound on the absolute values of the integrals by an exponential factor \( \sigma(r)^g \). Let \( \sigma_{max} = \max_r \sigma(r) \).

Then

\[
\left| \int_{F_g} \sum_{(i) = 1}^{2^g} \eta(i) I_i^{(i)}_{N,g} \right| \leq (k \sigma_{max})^g \sum_{r=1,2} \left| \int_{F_g} \sum_{(a) \in S_r} \eta(a) I^{(a)}_{N,g} \right| \quad (20)
\]

for some positive integer \( k \). As mentioned earlier, the spin structures in \( S_2 \) can be obtained by changing the boundary conditions for world-sheet fermions at one of the A-cycles. Since this seems to be equivalent to changing the boundary condition for a single Grassmann coordinate, the bounds on the integral corresponding to the set \( S_2 \) should differ from those for \( S_1 \) only by an exponential factor of the genus. It therefore follows from equation (20) that the estimates of the amplitudes given below should have the same genus-dependence as those for the full superstring measure.

Combining the holomorphic part of the measure with its complex conjugate gives a multiplier integral with a dominant part \( \int \frac{d|K_n|}{|K_n|(\ln(\frac{1}{|K_n|}))^4} = \frac{1}{4} \left[ \ln \left( \frac{1}{|K_n|} \right) \right]^{-4} \) in the limit \( |K_n| \to 0 \). With the limits in categories (i) and (ii), the integral gives a factor \((1 - 2q') \ln g)^{-5} \ln \left( \frac{\epsilon_0}{\epsilon_0} \right) + O((1 - 2q') \ln g)^{-6} \). For the third category of isometric circles, a factor of \( \frac{1}{4} \left[ \ln \left( \frac{1}{\epsilon_0} \right) \right]^{-4} - \left( \ln \left( \frac{1}{\epsilon_0} \right) \right)^{-4} \) is obtained. The contribution of a configuration of isometric circles with \( N_0 \) circles belonging to either of the first two categories and \( g - N_0 \)
circles in the third category to the multiplier integral is

\[(1 - 2q')^N_0 \ln \left( \frac{\epsilon'_0}{\epsilon_0} \right)^{N_0} (\ln g)^{-N_0} \]

\[
\cdot \left[ \frac{1}{4} \ln \left( \frac{\epsilon'_0}{\epsilon_0} \right)^4 \left( \ln \frac{1}{\epsilon'_0} \right)^4 \left( \ln \frac{1}{\epsilon_0} \right)^4 \left( \ln \frac{1}{\epsilon'_0} \right)^2 \left( \ln \frac{1}{\epsilon_0} \right)^2 \right] g^{-N_0}.
\]

\[(1 - 2q')^{N_0} = \prod_{m=1}^{N_0} (1 - 2q'_m) \tag{21}\]

In this formula, \(q'\) represents a weighted average of the values \(q'_m\) associated with the \(N_0\) isometric circles in the first two categories. There are

\[
\left( N_0 + \frac{(\ln g)^2}{2 \left( \ln \left( \frac{\epsilon'_0}{\epsilon_0} \right)^2 \right)} \right) \frac{1}{N_0!} \left( N_0 + \frac{(\ln g)^2}{2 \left( \ln \left( \frac{\epsilon'_0}{\epsilon_0} \right)^2 \right)} \right) - 1 \frac{1}{n_1! \ldots n_r!}, \quad \text{with } n_1 + \ldots + n_r = N_0,
\]

different partitions of the \(N_0\) circles into the \(\frac{(\ln g)^2}{2 \left( \ln \left( \frac{\epsilon'_0}{\epsilon_0} \right)^2 \right)}\) subcategories labelled by the indices \(q\) and \(q'\), and each partition \(\{n_i\}\) is weighted by a factor \(\frac{1}{n_1! \ldots n_r!}\), since one set of inequalities defining the fundamental domain of the modular group, \((\operatorname{Im} \tau)_{ss} \geq (\operatorname{Im} \tau)_{rr}, \quad s \geq r\), must be satisfied for each subcategory, leading to restrictions on the ranges of the multipliers \([4]\).

Instead of integrating over the super-fixed points, it is useful to use another set of coordinates on supermoduli space \(\{K_n, B_m, H_m, \theta_{1i}, \theta_{2i}\}\) for which the holomorphic part of the integral over the super-fixed points \([11]\) is replaced by

\[
\prod_{m=2}^{g} dB_m \prod_{m=2}^{g-1} dH_m \prod_{i=2}^{g-1} d\theta_{1i} \prod_{i=1}^{g} d\theta_{2i}, \tag{23}\]

where

\[
\xi_{2n} = \prod_{j=2}^{n} B_j \quad \theta_{2n} = \frac{1}{\sqrt{\xi_{2n}}} \theta_{2n} \quad \theta_{2g} = \frac{1}{\sqrt{\xi_{2g}}} \theta_{2g}
\]

\[
\xi_{1n} = \frac{\xi_{2n}}{1 - H_n - \sqrt{H_n} \theta_{1n} \theta_{2n}} \quad \theta_{1n} = \frac{\sqrt{H_n} \xi_{2n} \theta_{1n} + \theta_{2n}}{1 - H_n} \tag{24}\]
The relation involving $H_n$ can be inverted to give

$$H_n = \frac{\xi_1 - \xi_2}{\xi_1} - \frac{(\xi_1 - \xi_2)^{\frac{1}{2}}}{\xi_1} \vartheta_1 \vartheta_2$$

$$|H_n|^{-1} = \left| \frac{\xi_1}{\xi_1 - \xi_2} \right| \left[ 1 + \frac{1}{2} \left( \frac{\xi_1}{\xi_1 - \xi_2} \right)^{\frac{1}{2}} \vartheta_1 \vartheta_2 ight]$$

$$+ \frac{1}{2} \left( \frac{\xi_1}{\xi_1 - \xi_2} \right)^{\frac{1}{4}} \bar{\vartheta}_1 \bar{\vartheta}_2$$

$$+ \frac{1}{4} \left| \frac{\xi_1}{\xi_1 - \xi_2} \right| \vartheta_1 \vartheta_2 \bar{\vartheta}_1 \bar{\vartheta}_2$$

(25)

The dominant contribution to the integral over $|H_n|$ in $N$-point superstring scattering amplitudes with $N \geq 4$ produces a dependence on the fixed point distance of $|\xi_1 - \xi_2|^{-2}$. This may be verified using the super-fixed points $Z_1$ and $Z_2$.

For the parameters associated with the $g - N_0$ remaining isometric circles in the third category, the constraints $(Im \tau)_{ss} \geq (Im \tau)_{rr}$, $s \geq r$, lead to a reduction of the integral by a factor of $(g - N_0)!$. After summing over the possible values of $N_0$, the combination of the integrals over the multipliers $K_n$ and $H_n$ with the weighting factors is bounded above by

$$\sum_{N_0=0}^{g} \sum_{\{n_i\}} \frac{1}{n_1! \ldots n_r!} (1 - 2q)^{N_0} \left[ \ln \frac{\epsilon'_0}{\epsilon_0} \right]^{N_0} (\ln g)^{-5N_0} g^2 \sum_i q_i n_i$$

$$\sum_{n_i = N_0}^{g} \left[ \frac{1}{4} \ln \frac{\epsilon'_0}{\epsilon_0} \left( \ln \frac{1}{\epsilon_0} \right)^{-4} \left( \ln \frac{1}{\epsilon'_0} \right)^{-4} \left( \ln \frac{1}{\epsilon_0} + \ln \frac{1}{\epsilon'_0} \right) \left( \left( \ln \frac{1}{\epsilon_0} \right)^2 + \left( \ln \frac{1}{\epsilon'_0} \right)^2 \right) \right]^{g - N_0}$$

$$\frac{g^{(g - N_0)}}{(g - N_0)!}$$

(26)

The combinatorial factor $\frac{g^2 \sum_i q_i n_i g^{g - N_0}}{n_1! \ldots n_r!(g - N_0)!}$ obtains a maximum value when $r=1$ and $N_0 = \frac{g}{2}$. If all of the $N_0$ isometric circles are in the subcategory defined by the value $q_i = \frac{1}{2} - \frac{\ln \epsilon'_0}{\ln g}$, then the factor is approximately equal to $\left( \frac{2 \epsilon'_0}{\epsilon_0} \right)^g$.

The bound in equation (26) includes a division by a factor of $(g - N_0)!$ as a result of the constraints on the imaginary part of the period matrix. For isometric circles in the
third category, it had been shown in [4] that there is a sequential ordering of the diagonal elements of $Im \tau$ in a local neighbourhood of $J$ circles upon restriction of arguments of the fixed-point intervals $\xi_{1n} - \xi_{2n}$ and multipliers $K_n$, leading to a reduction of the integral by an exponential function of the genus, and then, that the period matrix inequalities for representative circles from each of the $\frac{(g-N_0)}{J}$ local neighbourhoods by a factor of $\frac{(g-N_0)!}{J!}$.

However, as the genus increases, the values of $Im \tau_{nn}$ for the circles in each of the local neighbourhoods overlap, and the inequalities obeyed by the representative circles will no longer be sufficient to imply the inequalities for the entire set of $(g-N_0)$ circles. In the large-genus limit, conditions on the absolute values of the multipliers will again have to be imposed, leading to a further reduction of the integral. Alternatively, one may state that $J$ tends to 1 in the large-genus limit.

The division by $(g-N_0)!$ rather than $\frac{(g-N_0)!}{J!}$ leads to an upper bound depending exponentially on the genus for all $N_0$, as the combinatorial factor is less than

$$g^{N_0} \quad \text{when} \quad N_0 \leq \frac{(\ln g)^2}{2 \left( \ln \frac{\epsilon_0}{\epsilon_0^0} \right)^2}$$

$$g \left( 1 - 2 \frac{\ln \left( \frac{\epsilon_0^0}{\epsilon_0} \right)}{\ln g} \right)^{\ln \left( \frac{\ln g}{\ln \epsilon_0} \right)} \quad \text{when} \quad \frac{(\ln g)^2}{2 \left( \ln \left( \frac{\epsilon_0^0}{\epsilon_0} \right)^2 \right)} \leq \frac{g}{\ln g}$$

$$\left[ 1 + \frac{\ln \lambda}{\lambda} + 1 \right] \cdot e^g \left( \frac{(\ln g)^2}{2 \left( \ln \left( \frac{\epsilon_0^0}{\epsilon_0} \right)^2 \right)} \right)! \quad \text{when} \quad \frac{g}{\ln g} \leq N_0 \leq \frac{g}{\lambda}, \lambda \gg 1$$

$$\left( 2 \cdot e \cdot \epsilon_0^0 \right)^g \quad \text{when} \quad \frac{g}{\lambda} \leq N_0 \leq \frac{g}{2}$$

Each of these upper bounds must be multiplied by the number of partitions (22) for each
value of $N_0$, and since this is less than $(\ln g)^{2N_0}$, the sum (26) is bounded above by

$$\frac{1}{4g} \left[ (\ln \frac{1}{\epsilon_0})^{-4} - (\ln \frac{1}{\epsilon_0})^{-4} \right]^g \left[ \left( 1 + \frac{\ln \lambda}{\lambda} + \frac{1}{\lambda} \right) \cdot e \right]^g \cdot \left( \frac{(\ln g)^2}{2 (\ln \frac{\epsilon_0}{\epsilon_0'})^2} \right)$$

$$\left[ 1 - 4(1 - 2\epsilon') \left( \ln \frac{1}{\epsilon_0} \right)^4 \left( \ln \frac{1}{\epsilon_0'} \right)^4 \cdot \left( \ln \frac{1}{\epsilon_0} + \ln \frac{1}{\epsilon_0} \right)^{-1} \left( \left( \ln \frac{1}{\epsilon_0} \right)^2 + \left( \ln \frac{1}{\epsilon_0} \right)^2 \right)^{-1} (\ln g)^{-3} \right]^{-1}$$

$$\frac{1}{4g} \left[ \left( \ln \frac{1}{\epsilon_0} \right)^{-4} - \left( \ln \frac{1}{\epsilon_0} \right)^{-4} \right]^g \left( \frac{2 \cdot e \cdot \epsilon_0}{\epsilon_0'} \right)^g g(1 - \frac{1}{\lambda})$$

$$\left[ 4(1 - 2\epsilon') \left( \ln \frac{1}{\epsilon_0} \right)^4 \left( \ln \frac{1}{\epsilon_0'} \right)^4 \left( \ln \frac{1}{\epsilon_0} + \ln \frac{1}{\epsilon_0} \right)^{-1} \left( \left( \ln \frac{1}{\epsilon_0} \right)^2 + \left( \ln \frac{1}{\epsilon_0} \right)^2 \right)^{-1} \cdot (\ln g)^3 \right]^{-\frac{1}{g}}$$

(28)

As $1 \gg \frac{1}{\lambda} > 0$, the second term in the bound (28) is a rapidly decreasing function of the genus for large $g$, while the exponential dependence of the first term on the genus is determined by $\epsilon_0'$. This parameter is constrained by modular invariance. One of the conditions defining the fundamental region of the modular group is $|\text{det}(C\tau + D)| \geq 1$ for

$$\left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \text{Sp}(2g; \mathbb{Z})$$. Since

$$|\text{det}(C\tau + D)| = |\text{det}(\text{Im} \, \tau)| \cdot |\text{det}(C - iC(\text{Re} \, \tau)(\text{Im} \, \tau)^{-1} - iD(\text{Im} \, \tau)^{-1}|$$

(29)

the determinant will be greater than one for all C, D when $|\text{det}(\text{Im} \, \tau)| \geq b > 1$ for some number $b$, so that $|\text{det}(C - iC(\text{Re} \, \tau)(\text{Im} \, \tau)^{-1} - iD(\text{Im} \, \tau)^{-1})|$ will be bounded below when $\text{det} C \neq 0$ and equal to $|\text{det} D| |\text{det}(\text{Im} \, \tau)^{-1}| \geq |\text{det}(\text{Im} \, \tau)^{-1}$ when $C=0$. Moreover, the minimum value of $\frac{\text{tr}(\text{Im} \, \tau)}{g}$ is

$$\ln \frac{1}{\epsilon_0} - \frac{1}{g} \sum_{n=1}^{g} s_{nn}$$

$$s_{nn} \text{ is a least upper bound for } \sum_{\alpha}^{(n,n)} \ln \left( \frac{\xi_{1n} - V_{\alpha} \xi_{2n} \xi_{2n} - V_{\alpha} \xi_{1n}}{\xi_{1n} - V_{\alpha} \xi_{1n} \xi_{2n} - V_{\alpha} \xi_{2n}} \right)$$

(30)

implying that the restriction to a single fundamental domain requires

$$\ln \frac{1}{\epsilon_0} \geq b^\frac{1}{g} + \frac{1}{g} \sum_{n=1}^{g} s_{nn}$$

(31)
Thus, \[
\left( \ln \frac{1}{\epsilon_0^n} \right)^{-4} - \left( \ln \frac{1}{\epsilon_n} \right)^{-4} < 1 \]
and the first term in the bound (28) decreases exponentially. From this result, it follows that the sum over genus of the moduli space integrals defined by the class of Riemann surfaces associated with the first three categories of isometric circles is finite. Finiteness of these integrals at large orders in perturbation theory contrasts with the rapid divergence of the regularized vacuum amplitude in closed bosonic string theory [2].

Another class of surfaces that might be included in the superstring path integral are spheres with an infinite number of handles decreasing in size and accumulating to a point. These surfaces may be constructed using infinitely generated groups of Schottky type, joining together isometric circles by projective transformations in the extended complex plane [20]. Labelling the handles by the index \( n \), the decrease in the square of the radii of the isometric circles, \( |\gamma_n|^{-2} \sim \frac{1}{n^{q''}} \) characterizes the surfaces to be included in the path integral. From the leading behaviour of the multiplier integral, the contribution of a configuration of isometric circles with \( q''_N \) given in equation (9) may be estimated.

\[
\int_{|K_n| \sim \frac{1}{n^{q''_N}}} \frac{d|K_n|}{|K_n| \left( \ln \frac{1}{|K_n|} \right)^3} = \frac{1}{4} [q''_N \ln n - \ln \epsilon'_0]^{-4} - \frac{1}{4} [q''_N \ln n - \ln \epsilon_0]^{-4} \\
= \left( q''_N \ln n \right)^{-5} \ln \left( \frac{\epsilon'_0}{\epsilon_0} \right) \\
= \left( \frac{\ln n}{\ln \left( \frac{\epsilon'_0}{\epsilon_0} \right) + \tilde{N}} \right)^{-5} \left[ \ln \left( \frac{\epsilon'_0}{\epsilon_0} \right) \right]^{-4} \tag{32}
\]

While the measure (4) indicates that the product over \( n \) of the integrals (32) be evaluated, it is more useful to first sum over the index \( \tilde{N} \) as this will allow for the inclusion of all possible combinations of limits for the multipliers and fixed points.

Having specified the limits for \( |K_n| \), one may note that the possible values of \( q'''_{\tilde{N}'} \) consistent with non-overlap of the intervals for \( |\xi_{1n} - \xi_{2n}| \) are

\[
q'''_{\tilde{N}'} = \tilde{N}' \frac{\ln \left( \frac{\delta_{\tilde{N}'}^n}{\delta_{\tilde{0}}^n} \right)}{\ln n} \quad \tilde{N}' = 0, 1, 2, \ldots \tag{33}
\]
As the super-fixed-point integral gives $|\xi_1 - \xi_2|^{-2}$, it will grow as $n^{2q'''}$, which will be shown to give rise to a divergence when $q''' > \frac{1}{2}$. However, it is also essential to eliminate the overcounting of surfaces, which would arise when including every value of $q'''$.

In particular, surfaces corresponding to the limits $\frac{\delta_0}{n^{2q'''}} < |\xi_1 - \xi_2| < \frac{\delta'''_0}{n^{2q'''}}$ can be obtained from surfaces corresponding to $\frac{\delta_0}{n^{q''}} < |\xi_1 - \xi_2| < \frac{\delta'''_0}{n^{q'''}}$ by pinching all handles other than those with the index $n^2$ in the sequential labelling, removing the nodes and deforming the surface by flattening the remaining portion of the handle on the sphere. Pinching one handle produces a surface at the boundary of moduli space, or equivalently, at the boundary of a string vertex, denoted by $\partial \mathcal{M}_{g,0}$ and $\partial \mathcal{V}_{g,0}$ respectively at genus $g$ [21].

Removing the two new punctures in the manner using an analytic map [22] transforms the string vertex from $\mathcal{V}_{g-1,2}$ to $\mathcal{V}_{g-1,0}$, which lies in the compactified moduli space $\overline{\mathcal{M}}_{g-1,0}$. At infinite genus, this procedure still produces a surface with an infinite number of handles lying in $\mathcal{M}_\infty$ [the number of ends of an infinite-genus surface is not required here]. Thus the initial integration over the domain in $\overline{\mathcal{M}}_\infty$ associated with the configurations of isometric circles with the range for $|\xi_1 - \xi_2|$ given by $\left[\frac{\delta_0}{n^{q'''}}, \frac{\delta'''_0}{n^{q''}}\right]$, includes integration over the range $\left[\frac{\delta_0}{n^{2q'''}}, \frac{\delta'''_0}{n^{2q'''}}\right]$ up to pinching of handles. The process of pinching the handles is described, however, by the degeneration limits $|K_m| \to 0$, $m \neq n^2, n \in \mathbb{Z}$. It has already been shown that these limits do not lead to any new divergences in the moduli space integral. Integration over the range $\{|\xi_1 - \xi_2| \sim \frac{1}{n^{q'''}}\}$, therefore, only represents a finite addition to integration over the range $\{|\xi_1 - \xi_2| \sim \frac{1}{n^{q''}}\}$.

Since this argument can be repeated an arbitrary number of times, overcounting will be eliminated by restricting the fixed-point integral to the range $\{|\xi_1 - \xi_2| \sim \frac{1}{n^{q'''}}\}$, $x > q''' > 0$, with $x$ being a small positive number. It is therefore necessary to choose

$$\frac{\ln(\ln n)}{\ln \left(\frac{\delta_0'''}{\delta_0}\right)} < \tilde{N}'_{\text{max}} \leq x \frac{\ln n}{\ln \left(\frac{\delta'''_0}{\delta_0}\right)} \quad (34)$$

An upper bound for the integrals over the fixed points would then grow as $n^{2x}$.

The constraints defining the fundamental region of the modular group, $(Im \, \tau)_{ss} \geq (Im \, \tau)_{rr}$, $s \geq r$ generally reduce the integration range of the Schottky group variables.
However, they are only relevant when the intervals $\left[ \frac{\epsilon_0}{n_q}, \frac{\epsilon_0'}{n_q'} \right]$ and $\left[ \frac{\epsilon_0}{n_q}, \frac{\epsilon_0'}{n_q'} \right]$ overlap, implying

$$\frac{n_s}{n_r} \leq \left( \frac{\epsilon_0}{\epsilon_0'} \right)^{\frac{1}{q'}} \quad \text{(35)}$$

Assuming temporarily that the genus is finite and an integer between $\left( \frac{\epsilon_0}{\epsilon_0'} \right)^{\frac{2}{q''}}$ and $\left( \frac{\epsilon_0}{\epsilon_0'} \right)^{\frac{2}{q''}}$, for some integer $y$, the restrictions on each group of multipliers $K_n$ with $n$ lying between the numbers $1$, $\left( \frac{\epsilon_0}{\epsilon_0'} \right)^{\frac{y+1}{q''}}$, $\left( \frac{\epsilon_0}{\epsilon_0'} \right)^{\frac{y+1}{q''}}$, ... leads to a combinatorial factor

$$\frac{1}{\left( \frac{\epsilon_0}{\epsilon_0'} \right)^{\frac{1}{q''}} - 1} \cdot \frac{\left( \frac{\epsilon_0}{\epsilon_0'} \right)^{\frac{y+1}{q''}} - 1}{\left( \frac{\epsilon_0}{\epsilon_0'} \right)^{\frac{y+1}{q''}} - 1} - 1 \quad \text{(36)}$$

Defining the variable

$$t_N = \left( \frac{\epsilon_0}{\epsilon_0'} \right)^{\frac{1}{q''}} \approx \left( \frac{\epsilon_0}{\epsilon_0'} \right)^{\frac{1}{q''}} \frac{n}{\ln \left( \frac{\epsilon_0}{\epsilon_0'} \right)} = n^{\frac{1}{N}} \quad \text{(37)}$$

one finds that the combinatorial factor (37) is less than $\frac{1}{(g!)^N}$. The sum of the fixed-point integrals over all values of $N'$ up to the maximum value in equation (34) grows as

$$\left( \frac{1}{\delta_0^2} - \frac{1}{\delta_0'^2} \right) \sum_{N' = 0}^{N_{max}} n^{2q_{N'}} \approx \frac{1}{\delta_0^2} n^{2x} \quad \text{(38)}$$
Multiplying this sum with the estimate (32) and dividing by the combinatorial factor arising from the action of the modular group gives

\[
\frac{1}{(g!)^{\frac{1}{N_{max}^1-1}}} \left( \ln \frac{e_0'}{e_0} \right)^{-4g} \frac{1}{\delta_0'2}\prod_{n=1}^{g} n^{2x} \zeta \left( 5, \frac{\ln n}{\ln e_0} - 1 \right)
\]

(39)

in the limit \( g \to \infty \) when \( n^{\frac{1}{N_{max}^1-1}} = 2x \). Since it is assumed that \( x < 1 \), the upper limit in equation (34) implies that the last expression in equation (39) will be an adequate bound if \( x \leq \frac{1}{\sqrt{2}} \left( \ln \left( \frac{\delta_0'}{\delta_0} \right) \right)^{\frac{1}{2}} \). As the product in (39) tends to zero as \( g \to \infty \), the integrals corresponding to the range \( |\xi_{1n} - \xi_{2n}| \sim \frac{1}{n^{2g N'}}, \quad q''_N^\prime > \frac{1}{\sqrt{2}} \left( \ln \left( \delta_0'/\delta_0 \right) \right)^{\frac{1}{2}} \) would be the limit of an exponential function of the genus for each value of \( N' \). Thus, the magnitude of that part of the moduli space integral including spheres having \( g \) handles, with a decreasing cross-sectional area in the intrinsic metric, near a single accumulation point, is therefore a countable sum of contributions depending exponentially on the genus as \( g \to \infty \).

For \( q''_N^\prime > \frac{1}{\sqrt{2}} \left( \ln \left( \delta_0'/\delta_0 \right) \right)^{\frac{1}{2}} \), and particularly for \( q''_N^\prime > \frac{1}{2} \), one initially obtains a divergence such as \( (g!)^{2q''_N^\prime-1} \) in the integral, which would be expected when confining attention to the Neveu-Schwarz sector of the superstring. However, this factorial divergence has been avoided in the previous analysis because the behaviour of the integrals is determined by the \( |K_n| \to 0 \) degeneration limit. As noted earlier, this leads to an exponential dependence on the genus when the GSO projection is used, or when the entire superstring spectrum, including the Ramond sector, is included.

These conclusions are consistent with an analysis of the integrals of the multipliers \( B_m \) and \( H_m \). Like the \( K_n \) integrals, a divergence of factorial type would arise using the measure (23) appropriate for the Neveu-Schwarz sector. However, upon use of the full superstring measure, this divergence must disappear, confirming the finiteness properties of superstring amplitudes. It may be recalled that the finiteness of genus-\( g \) superstring amplitudes has been demonstrated by sewing together three-punctured spheres, labelling the radii \( R_j, \ j = 1, \ldots, 3g - 3 + N \) of the sewed punctures as \( K_n, B_m, H_m \) and \( L_i \) (for external legs), and
considering the limit as $|R_j| \to 0$ [6]. For the limits $|B_m|, |H_m| \to 0$, a finite result is obtained after replacing picture-changing operators by commutators of BRST charges with bosonized ghosts and pulling the BRST charge through the Beltrami differential for $K_n$, leaving a finite integral at the boundary $|K_n| = 0$ and an amplitude, involving picture-changing operators at a single puncture, which must vanish by the cancellation of fermion zero modes [6]. By a parallel argument, similar to the one given before equation (34), it follows that a finite, bounded integral is obtained in the $B_m$ or $H_m$ degeneration limit for one connecting tube, and therefore an exponential dependence on the genus should occur upon evaluation of all of the integrals containing $\{B_m\}$ and $\{H_m\}$. Similarly, the inclusion of the contribution of special surfaces, with handles of arbitrarily small thickness, to the superstring path integral, necessitated by the absence of a cut-off on closed geodesic lengths, does not significantly alter the genus-dependence of the bound on the amplitudes.

The bounds obtained above depended essentially on the restriction of the integral over super-Schottky group parameter space to a single fundamental domain of the supermodular group. In this connection, one may note that the super-mapping class group does not involve any discrete transformations in the odd directions [23][24]. The relevant constraints are those listed for the multipliers and ordinary fixed points. Moreover, conditions such as $-\frac{1}{2} \leq (Re \, \tau)_{mn} \leq \frac{1}{2}$ and $(Im \, \tau)_{1n} \geq 0$ reduce the integrals by an exponential function of the genus [15]. Other exponential factors arise from the angular integrations of the arguments of $K_n$ and $\xi_{1n} - \xi_{2n}$, the integrations over $\xi_{2n}$ and the primitive-element products in the measure (4). Although the powers in the primitive-element products in this measure differ from those in the bosonic string measure, the bounds obtained in [4] suggest that the products in superstring measure will also be bounded by an exponential function of the genus. Finally, a sphere with an infinite number of handles may have more than one accumulation point, and generically, an infinite-genus Riemann surface will have a Cantor set of ends [25], and, if these surfaces can be consistently included in the superstring path integral, the counting of these surfaces would affect the estimate (39) again by an exponential factor.

An alternative calculation switching the roles of the magnitudes of the multipliers $|K_n|$
and the distances between the fixed points $|\xi_{1n} - \xi_{2n}|$ so that the values of $\tilde{N}$ are limited while the values of $\tilde{N}'$ are allowed to be arbitrarily large gives a divergent result, but it is not related to the moduli space integral above, because the elimination of handles on the surface, required in placing an upper limit on $\tilde{N}$, involves the degeneration limit $|\xi_{1m} - \xi_{2m}| \to 0$, with $m \neq n^2$ for example, rather than $|K_m| \to 0$, as the magnitude of $|K_n|$ has already been fixed by the choice of $\tilde{N}$. Since the degeneration limit leads to divergences, the pinching procedure relating configurations of isometric circles of different $\tilde{N}$ does not involve a finite change to the integrals. Thus, the bounds with this range for $\tilde{N}$ and $\tilde{N}'$ are not related to the estimates given in equation (39).

It may also be noted that the non-vanishing of the integrals for finite $g$ is not inconsistent with the vanishing of the superstring amplitudes with less than four massless external states. The integral of the complete measure, including the Ramond sector, over the entire integration range, would be required for a direct comparison. Moreover, it is sufficient to prove finiteness, removing the need to introduce a cut-off near the boundary of moduli space, as the BRST contour integral arguments can then be used to prove the non-renormalization theorems [26][27][28]. Similarly, even though the Euler character of moduli space grows asymptotically as $(-1)^g \frac{(2g-1)!}{2^{2g-1}g!}$ [29], the genus-dependence of the volume is affected to a greater extent by the choice of measure. While there is a rapid, nearly factorial growth of the volume following integration of the bosonic string measure over a subset of moduli space [4], the genus-dependence of the corresponding integral is changed significantly by use of the superstring measure (4).

These estimates do depend on the validity of applying the measure in super-Schottky group coordinates to an integral over all of supermoduli space. Integration over the odd moduli in superstring amplitudes is known to produce a density over moduli space which possesses a total derivative ambiguity associated with a change in the choice of the basis of super-Beltrami differentials [30][31][32]. This integration ambiguity is related to the lack of positive semi-definiteness of the superstring measure on a good global slice, which follows from the absence of a global holomorphic section of the universal Teichmüller curve [31]. Moreover, integration over the odd moduli requires that there is a splitting of su-
permoduli space into even and odd coordinates. The global obstruction to the splitting of supermoduli space can be circumvented by removing a divisor $D_g$ of codimension greater than or equal to one from the stable compactification of moduli space, $\bar{\mathcal{M}}_g$. Integration of a measure with the holomorphic factorization property [33][34] is possible, when it is defined over a subset of moduli space and in particular for the ranges of variables considered earlier. The integral over all of moduli space in superstring scattering amplitudes would differ in the large-genus limit, therefore, from the estimates based on these subdomains by a contribution from the divisor. This, in turn, has been related to tadpoles of massless physical states at lower genera [32]. Tadpole diagrams vanish in stable vacua and more generally, if an exponential dependence on the genus of the scattering amplitudes is assumed up to genus $g-1$, the contribution from the boundary of moduli space should also be bounded by a function of the same order, and it follows that this dependence would continue to hold at genus $g$. By induction, the exponential dependence should then be valid for arbitrarily large genus. Finally, there exist other formalisms which avoid the multi-loop ambiguity although they are not directly related to the approach based on a super-Schottky parametrization of supermoduli space. In the light-cone supersheet formalism, the boundary in supermoduli space is determined by the requirement that the bosonic moduli are pure complex numbers without nilpotent parts [35], eliminating the ambiguity arising from integration over Grassmann variables. Finite, unambiguous scattering amplitudes can also be defined in the twistor-string formalism, which makes use of space-time supersymmetry generators that are independent of the bosonized super-reparametrization ghost fields having poles given by total derivatives in moduli space [28], and it has been suggested previously that the exponential dependence on the genus could be derived after considering the various degeneration limits of these amplitudes.

The vanishing of N-point amplitudes for $N < 4$ provides the first indication that the superstring amplitudes do not necessarily grow at a factorial rate with respect to the genus. Similarly, vanishing of superstring amplitudes has been established when $g + N \leq 8$ [36]. Until recently, however, it has been difficult to determine the superstring scattering amplitudes at high genus. The only result that has been established thus far has been
a consequence of a technical argument relating N=2 string theories with topological field theories. This connection implies that there is a relation between special type II string amplitudes in orbifold and Calabi-Yau backgrounds and topological string amplitudes at any given genus [37],[38]. It is therefore of interest to be able to estimate the generic superstring scattering amplitude with an arbitrary number of vertex operators, receiving contributions from all genus. This follows from the bounds in this paper because the N-point g-loop scattering amplitude is typically given by

$$A_{N,g} = \int_{sM_g} d\mu_g \int_{s\Sigma_g} dt_1 d\bar{t}_1 ... dt_N d\bar{t}_N \langle V_1(t_1, \bar{t}_1)...V_N(t_N, \bar{t}_N) \rangle$$

which may be bounded by

$$\left| \int_{sM_g-N(D_g)} d\mu_g \int_{s\Sigma_g} dt_1 d\bar{t}_1 ... dt_N d\bar{t}_N \langle V_1(t_1, \bar{t}_1)...V_N(t_N, \bar{t}_N) \rangle \right|$$

where $N(D_g)$ is a neighbourhood of the compactification divisor and a subset of supermoduli space, which has measure zero in the large genus limit. For the $R = -1$ slice of Teichmuller space,

$$\frac{1}{2\pi} \int_{\Sigma_g} d^2 \xi \sqrt{g} R = 2 - 2g$$

and using a subtraction procedure [39] to remove divergences in the correlation function associated with the coincidence of vertex operators and to bound its magnitude $|\langle V_1(t_1, \bar{t}_1)...V_N(t_N, \bar{t}_N) \rangle| \leq V_N^{max}$, it follows that the upper bound (41) is less than

$$(4\pi(g - 1))^N \cdot V_N^{max} \cdot B_K^g B_H^{g-2} B_B^{g-1}$$

Lower bounds of this type may also be derived provided there is a suitable lower bound that can be used for expressions containing the supermoduli space integral. Since this particular integral, after the sum over spin structures, actually vanishes, the lower bound that should not be determined strictly by the magnitude of this integral; rather, it should allow for the weighting of the spin structures to be altered in the integration of the correlation function over supermoduli space, leading to a non-vanishing amplitude. It may be noted that the bound (43) involved estimates of N-point functions on surfaces which lie at the boundary of moduli space. In the infinite-genus limit, one is led to consider N-point
functions on spheres with an infinite number of handles of decreasing size. One class of
N-point functions has been studied in [20].

The genus-dependence of the special scattering amplitudes can be obtained from the
general estimates (43). Specifically, the genus-g type II string amplitude \( A_{2g,g} \) in an
orbifold or Calabi-Yau background, with 2g vertex operators including 2g-2 graviphotons
and 2 gravitons, is equal to

\[
A_{2g,g} = (g!)^2 F_g
\]

where \( F_g \) is the partition function of a topological string theory. The partition function
for a topological field theory defined over a particular manifold \( M \) typically is given by the
product of the partition function over the submanifolds \( M_i \), such that \( M \) is the topological
sum \( \cup_i M_i \), so that the partition function for a genus-g surface \( Z_g \) would be \( c^g Z_1^g \). For a
topological gravity theory, the genus-g partition function will also involve a contribution
from a supermoduli space integral associated with modular deformations along the collars
joining the tori. Since the magnitude of this supermoduli integral has already been esti-
ated, the genus-dependence of the partition function \( F_g \) should still be exponential, and
indeed, this has been established in several papers [38][40][41]. Consequently, the type II
string amplitude is \( A_{2g,g} = c_1 c_2^g (g!)^2 \), consistent with the bound (44) upon setting \( N \)
equal to 2g.

The results obtained here imply, therefore, there is a higher degree of finiteness in the
superstring theory in the large-genus limit when modular invariance is considered. The
upper bound (28) decreases exponentially with respect to the genus, indicating that there
may exist an exponential bound for the higher-point superstring amplitudes. In this case,
the perturbation series could be made to converge for an appropriate choice of the string
coupling constant or dilaton expectation value. It may be noted that the estimates in this
analysis are based on an exhaustion of moduli space in the large-genus limit. A further
contribution to the moduli space integrals, with a different dependence on the genus,
is conceivable, but the corresponding ranges of the super-Schottky group parameters at
large genus have not been considered relevant in this approach. Furthermore, improved
summability of the perturbation series would be associated generally with stability of the superstring vacuum. This property might be verified independently by applying positive-energy theorems to the corresponding supersymmetric backgrounds, in particular $\mathbb{R}^{10}$.

The estimates given here can also be regarded as consistent with the addition of the exponential non-perturbative effects associated with the joining of boundaries to the worldsheets. The amplitudes corresponding to surfaces with boundaries attached behave as $\exp(-\frac{1}{\kappa_{str}})[42]$. When Dirichlet boundary conditions are applied to all of the coordinates, the boundary can be mapped to a single point and corresponds to a D-instanton. The one-D-instanton amplitude is given by

$$A_1 = \exp(<1>D_2 + \ldots) A_1^{\text{conn}}.$$  \hspace{1cm} (45)

where $<1>D_2$ is the disk amplitude with no vertex operators and has weight $-\frac{1}{\kappa_{str}}$ [42]. It may be noted that string divergences, arising when the vertex operators approach the Dirichlet boundaries, are eliminated by a Fischler-Susskind mechanism, in which amplitudes, associated with different numbers of boundaries cancel [42][43]. Exponential non-perturbative effects of the order of $\exp(-\frac{1}{\kappa^2})$ typically arise in quantum field theories as a result of non-Borel summability of the perturbation series [44]. Here the non-perturbative amplitudes can be regarded as a separate contribution to the sum over surfaces, to be added to the amplitudes corresponding to the closed surfaces. A complete formulation of string theory requires a sum over specific boundary types, and the leading non-perturbative effects result, for example, from the addition of Dirichlet boundaries. Distinct boundary types also occur in the separation of categories of surfaces into $O_G$ and $P_G$ surfaces, studied in the infinite-genus limit of the S-matrix expansion [4][20]. Further progress on a formula for the S-matrix including the combinatorics of boundaries on the string worldsheets has been recently been reported in [45]. The scattering amplitudes defined in this article arise as the logarithm of the first term in the series expansion of the generating functional

$$\ln \mathcal{Z} = S^{(0)} + \ln \left[ \sum_n \frac{1}{n!} \left( \int \prod_{i=1}^n d^D y_i^\mu \right) e^{S^{(n)}}(y_1, \ldots, y_n) \right]$$ \hspace{1cm} (46)

where $S^{(0)}$ is the path integral over connected worldsheets with no boundaries, $S^{(n)}$ is the
functional integral over worldsheets with boundaries fixed at n points $y_i$ and $S^{(n)}_t$ is $S^{(n)}$ without the zero-boundary term.

The exponential bounds for the amplitudes, based only on the sum over closed surfaces, are therefore not inconsistent, in this scheme, with the non-perturbative effects that have been expected to be present in string theory. Further support for the estimates of superstring amplitudes in this article comes from the convergent perturbation expansion for QED with fermions and finite ultraviolet and infrared cut-offs [46]. From the work on bosonic string theory, one sees that the ultraviolet cut-off in this theory is equivalent to the Gross-Periwal cut-off, while the infrared cut-off removes the divergences in the infrared limit, analogous to the effect of supersymmetry on string amplitudes. It is conceivable that this version of the QED model may arise in the low-energy limit of superstring theory, providing an explanation for the dependence of the amplitudes on the loop order.

It has recently been shown using duality in supersymmetric gauge theories that the strong-coupling regime defined by electric variables is related to the weakly-coupled regime based on magnetic monopoles [47][48]. This approach might also be relevant in developing a complete, non-perturbative formulation of QED, free of divergences of the charge-charge coupling at small length scales [49]. A step towards such a theory has been made by showing that the coupling of an electric charge to lines of magnetic flux increases less rapidly, as the length scale decreases, than the coupling of two electric charges [49], thus reminiscent of the phase of supersymmetric gauge theories corresponding to a weakly-interacting system of magnetic monopoles and abelian photons.

These results also have an analogue in superstring theory [50][51][52]. Specifically, there is a transformation in the S-duality group of the type II superstring and heterotic string theories interchanging electric and magnetic fields, while mapping the dilaton field $\Phi$ to $-\Phi$, so that the string coupling constant $\kappa_{str}^2 = <e^{2\Phi}>$ is mapped to its inverse $\kappa_{str}^2 = <e^{-2\Phi}>$ [53]. This suggests it may be feasible to completely determine the S-matrix for the type IIB superstring, since calculations of scattering amplitudes can be done explicitly in the weak coupling limit. Recently, it has been shown also that the strong-coupling limit of
a ten-dimensional type IIA string theory is related to the weak-coupling limit of eleven-dimensional supergravity [54]. One might therefore be able to improve on earlier studies of quantum amplitudes for the eleven-dimensional theory. Moreover, it is known that certain soliton states of both type II superstring theory and heterotic string theory, saturating the Bogomol’nyi bound and forming representations of $SL(2;\mathbb{Z})$, may be identified with extreme black holes [53][55]. The extreme black hole solutions, which possibly could be used to describe stable elementary particles [55][56], also represent background geometries for which the amplitudes could be evaluated. Conclusions about the bounds for the amplitudes may determine certain properties of the states associated with extreme black holes.

Finiteness of superstring amplitudes at any given order of perturbation theory implies that a cut-off does not need to be introduced in moduli space and that ambiguities associated with surface terms arising from total derivatives resulting from changes in the locations of picture-changing operators are eliminated [6][31][57]. In particular, vanishing of the vacuum amplitude at each order in perturbation theory follows from its representation as the integral of a total derivative on the compactified moduli space $\bar{M}_g$. Similarly, the full vacuum amplitude, given by a sum of multi-loop amplitudes, may be regarded as the integral of a total derivative on the universal moduli space $\bar{M}_\infty$.

The results in this paper represent a further demonstration of the finiteness of superstring theory. While previous studies have been concerned with finiteness at any given order, it is shown here that the leading behaviour of particular moduli space integrals appears to depend only exponentially on the genus. These conclusions were derived using
the super-Schottky group parameters as coordinates on supermoduli space and restricting the integration region to a single fundamental domain of the super-modular group. The physical explanation for the removal of the factorial dependence on the genus in superstring amplitudes is based on a connection between infrared and large-order divergences, following from the genus-dependence of the limits for the Schottky group parameters and leading to their simultaneous elimination as a consequence of supersymmetry. The estimates in this paper have been derived with the purpose of bounding superstring amplitudes and reflect the absence of the tachyon in the superstring spectrum, even though the full superstring measure is not used. The exponential dependence of the moduli space integrals here, evaluated for a specific class of surfaces, suggests that a similar dependence on the genus should occur in full superstring theory, and this is supported by a general argument concerning the degeneration limits of superstring amplitudes. The effect of supersymmetry on large-order string perturbation theory is a greater degree of finiteness of the amplitudes, leading to an improvement of the summability of the loop expansion for superstring interactions.

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