Abstract. We show in Bishop’s constructive mathematics—in particular, using countable choice—that weak König’s lemma implies the uniform continuity theorem.

In [1] Hannes Diener proved, as part of the programme of reverse constructive mathematics, that weak König’s lemma

\textbf{WKL}: Every infinite, decidable, binary tree has an infinite path.

implies the uniform continuity theorem

\textbf{UCT}: Every pointwise continuous function \( f : [0, 1] \to \mathbb{R} \) is uniformly continuous.

in Bishop’s constructive mathematics. Diener’s proof relies on several other results in reverse constructive mathematics; we give a short, direct proof.

\textbf{Theorem 1}. Weak König’s lemma implies the uniform continuity theorem.

The idea of our proof is simple: given a continuous function \( f : [0, 1] \to \mathbb{R} \) and some \( \varepsilon > 0 \), we use WKL to focus in on a point where the function exhibits (almost) greatest variation within \( \varepsilon \) and then use continuity at that point to find our modulus \( \delta \) of uniform continuity for \( \varepsilon \). Before proving Theorem 1 we must set up some notation.

For each \( n > 0 \) we let \( S_n = \{0, 2^{-n}, \ldots, 1 - 2^{-n}, 1\} \) and we write \( \mathcal{D} \) for the set \( \cup \{S_n : n \in \omega\} \) of dyadic rationals. We define a one-one function \( g \) from the set \( 2^{<\omega} \) of finite binary sequences to \( \mathcal{D} \) by

\[
g(a) = \sum_{i=0}^{\lfloor a \rfloor - 1} a(i)2^{-(i+1)}
\]

and associate \( x \in S_n \) with the unique finite binary string \( a \) of length \( n \) such that \( g(a) = x \). The \textit{sum} of two finite binary sequences \( a, b \) each of length \( n \) is the binary sequence \( a \oplus b \) of length \( 2n \) given by

\[
a \oplus b(n) = \begin{cases} a(n/2) & \text{if } n \text{ is even} \\ b((n-1)/2) & \text{if } n \text{ is odd.} \end{cases}
\]

Both \( g \) and \( \oplus \) extend to functions on infinite binary sequences and we make no notational distinction between the functions on finite and infinite sequences. We let \( \pi_0, \pi_1 \) be the left and right inverses of \( \oplus \) respectively; that is \( \pi_0(a \oplus b) = a \) and \( \pi_1(a \oplus b) = b \) for all suitable pairs \( a, b \in 2^{<\omega} \cup 2^\omega \).

For \( \alpha \in 2^\omega \) and \( n \in \mathbb{N} \), \( \bar{\alpha}(n) \) denotes the unique binary sequence of length \( n \) that \( \alpha \) extends. The downward closure of subset \( A \) of \( 2^{<\omega} \) is \( A\downarrow = \{a \in 2^{<\omega} : \exists a' \in A (a < a')\} \), where \( a < a' \) if \( a' \) extends \( a \).

For a given function \( f : [0, 1] \to \mathbb{R} \) we define a predicate \( \varphi_f \) on \( \mathbb{R}^+ \times \mathbb{R}^+ \) by

\[
\varphi_f(\varepsilon, \delta) \equiv \exists x, y \in [a, b] (|x - y| < \delta \land |f(x) - f(y)| > \varepsilon);
\]

\( \varphi_f(\varepsilon, \delta) \) holds if we have a witness that \( \delta \) is not a modulus of uniform continuity for \( f, \varepsilon \). Thus to show that \( f : [0, 1] \to \mathbb{R} \) is uniformly continuous we must, given any \( \varepsilon > 0 \), find some \( \delta > 0 \) such
that \( \varphi_f(\varepsilon, \delta) \) is false. The next lemma shows how we can use \( \text{WKL} \) to reduce the truth of \( \varphi_f(\varepsilon, \delta) \) to whether or not some specific \( x, y \in [0, 1] \) are witnesses of \( \varphi_f(\varepsilon, \delta) \).

**Lemma 2.** \( \text{WKL} \vdash \) Let \( f : [0, 1] \rightarrow \mathbb{R} \) be pointwise continuous and let \( \delta, \varepsilon \) be positive real numbers. Then there exist \( x, y \in [a, b] \) such that if \( \delta \) is not a modulus of uniform continuity for \( f \), then \( x, y \) witness this:

\[
\varphi_f(\varepsilon, \delta) \rightarrow |x - y| < \delta \land |f(x) - f(y)| > \varepsilon.
\]

**Proof.** Using countable choice, construct a function \( \gamma : \mathcal{D}^2 \times \mathbb{N} \rightarrow 2 \) such that

\[
\gamma(x, y, n) = 0 \implies |x - y| > \delta - 2^{-n} \lor |f(x) - f(y)| < \varepsilon + 2^{-n},
\]

\[
\gamma(x, y, n) = 1 \implies |x - y| < \delta \land |f(x) - f(y)| > \varepsilon;
\]

further we may assume that \( \gamma \) is non-decreasing in the third argument. So if \( \gamma(x, y, n) = 1 \), then \( x,y \) are witnesses of \( \varphi_f(\varepsilon, \delta) \). Using \( \gamma \), we can construct an increasing binary sequence \( (\lambda_n)_{n \in \mathbb{N}} \) such that

\[
\begin{align*}
\lambda_n = 0 & \implies \forall_{x, y \in S_n} \gamma(x, y, n) = 0, \\
\lambda_n = 1 & \implies \exists_{x, y \in S_n} \gamma(x, y, m) = 1.
\end{align*}
\]

Finally we construct a decidable binary tree \( T \) as follows. If \( \lambda_n = 0 \) we let \( T_n = 2^n \), and if \( \lambda_n = 1 \) we set \( T_n = \{ \sigma \ast 0 : \sigma \in T_{n-1} \land |\sigma| = \text{ht}(T) \} \). If \( \lambda_n = 1 - \lambda_{n-1} \), we let \( x, y \) be the minimal elements of \( S_n \) such that \( \gamma(x, y, n) = 1 \) and we set \( T_n = (2^n \cup \{ x \oplus y \}) \upharpoonright \)—the branch \( x \oplus y \) is the unique branch of \( T_n \) with length \( \text{ht}(T_n) \), and it codes the witnesses \( x, y \) that \( \delta \) is not a modulus of uniform continuity for \( \varepsilon \). Then

\[
T = \bigcup_{n \in \mathbb{N}} T_n
\]

is an infinite decidable tree.

Using \( \text{WKL} \) we can construct an infinite path \( \alpha \) through \( T \). Set \( x = g(\pi_0 \alpha), y = g(\pi_1 \alpha) \). Suppose there exist \( u, v \in [0, 1] \) such that \( |u - v| < \delta \) and \( |f(u) - f(v)| > \varepsilon \). Since \( f \) is pointwise continuous, there must exist such \( u, v \in \mathcal{D} \). Hence \( \gamma(u, v, n) = 1 \) for some \( n \in \mathbb{N} \) such that \( u, v \in S_n \), so \( \lambda_n = 1 \). It now follows from the construction of \( T \) that \( x, y \) have the desired property.

We recall a result of Hajime Ishihara [2]: \( \text{WKL} \) is equivalent to the longest path principle

**LPP:** Let \( T \) be a decidable tree. Then there exists \( \alpha \in 2^{<n} \) such that for all \( n \), if \( \check{\alpha}(n) \notin T \), then \( T \subset 2^{<n} \).

To get a longest path for a decidable tree \( T \) apply \( \text{WKL} \) to the decidable tree

\[
\{ a \in 2^{<\omega} : a \in T \lor \exists_{b \in T} (|b| = \text{ht}(T) \land a > b) \},
\]

where \( |b| \) is the length of \( b \) and \( \text{ht}(T) \) is the height of \( T \).

Here then is our **proof of Theorem** [1]

**Proof.** Let \( f : [0, 1] \rightarrow \mathbb{R} \) be a pointwise continuous function and fix \( \varepsilon > 0 \). We define a function \( J \) taking finite binary sequences to subintervals of \([0, 1]\) inductively: \( J() = [0, 1] \) and if \( J_u = [p, q] \), then \( J_{u \uparrow 0} = [p, (p + q)/2] \) and \( J_{u \uparrow 1} = [(p + q)/2, q] \). By repeated application of the lemma, let \( x_u, y_u \in J_u \) be such that

\[
\varphi_f|_{J_u}(\varepsilon, 2^{-|u|}) \rightarrow |x_u - y_u| < 2^{-|u|} \land |f(x_u) - f(y_u)| > \varepsilon/2,
\]

and using countable choice construct a decidable tree \( T \) such that

\[
\begin{align*}
u \in T & \implies |f(x_u) - f(y_u)| > \varepsilon/2 - 2^{-|u|} \\
u \notin T & \implies |f(x_u) - f(y_u)| < \varepsilon/2.
\end{align*}
\]
Let \( \alpha \) be a longest path of \( T \), and let \( \xi \) be the unique element of 
\[
\bigcap_{n \in \mathbb{N}} J_{\overline{\alpha}(n)}.
\]
Using the continuity of \( f \) at \( \xi \) we can find \( \delta > 0 \) such that \( |f(x) - f(y)| < \varepsilon / 2 \) for all \( x, y \in (\xi - \delta, \xi + \delta) \); let \( n \) be such that \( 2^{-n+1} < \max\{\delta, \varepsilon\} \). If \( u = \overline{\alpha}(n) \in T \), then \( |x_u - y_u| < \delta \) and 
\[
|f(x_u) - f(y_u)| > \varepsilon - 2^{-n} > \varepsilon / 2
\]
contradicting our choice of \( \delta \). Hence \( \overline{\alpha}(n) \notin T \), so \( T \subset 2^{\mathbb{N}} \). It follows from Lemma 2 and the construction of \( T \) that for all \( x, y \in [0, 1] \), if \( |x - y| < 2^{-n} \), then 
\[
|f(x) - f(y)| < \varepsilon.
\]

References

[1] H. Diener, Weak König’s lemma implies the uniform continuity theorem, Computability 2(1), p. 9–13, 2013.
[2] H. Ishihara, ‘Weak König’s Lemma Implies Brouwer’s Fan Theorem: A Direct Proof’, Notre Dame Journal of Formal Logic 47(2), p. 249–252, 2006.