Conservation laws of generalized higher Burgers and linear evolution equations

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Abstract. By the Cole-Hopf transformation, with any linear evolution equation in 1 + 1 dimensions a generalized Burgers equation is associated. We describe local conservation laws of these equations. It turns out that any generalized Burgers equation has only one conservation law, while a linear evolution equation with constant coefficients has an infinite number of \((x,t)\)-independent conservation laws iff the equation involves only odd order terms and, therefore, is bi-Hamiltonian.

It is well known that the classical Burgers equation \(v_t = v_{xx} + 2v_x v\) is obtained from the linear heat equation \(u_t = u_{xx}\) by the Cole-Hopf transformation \(v = u_x/u\), but it is less known that this construction can be applied to an arbitrary linear evolution equation

\[
u_t = \sum_{i=0}^{m} a^i(x,t)u_i,
\]

(1)

where, as usual, \(u_i = \partial^i u/\partial x^i\) and

\[
a^m(x,t) \neq 0.
\]

(2)

Indeed, denote \(v = v_0 = u_1/u\) and \(v_k = D_x^k(v)\), then

\[
v_t = D_tD_x(\ln |u|) = D_x(\frac{u_t}{u}) = D_x\left(\sum_{i=0}^{m} a^i(x,t)(D_x + v)^i(1)\right),
\]

where \(D_x\) and \(D_t\) are the total derivatives, because it is easily seen that

\[
\frac{u_t}{u} = (D_x + v)^i(1).
\]

The equation

\[
v_t = D_x\left(\sum_{i=0}^{m} a^i(x,t)(D_x + v)^i(1)\right)
\]

(3)

is called the generalized Burgers equation associated with linear equation (1). This is an example of nonlinear C-integrable equations in the sense of Calogero [2].

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Example. Consider linear equations with constant coefficients
\[ u_t = \sum_{i=0}^{m} a_i u_i, \quad a_m \neq 0. \]  (4)

Clearly, flows (4) mutually commute for arbitrary \( a_i \in \mathbb{R} \), hence flows (3) with constant coefficients \( a^i(x,t) = a^i \) also commute and form the integrable Burgers hierarchy (see, for example, [4]). It passes the Painlevé test [6], but, as it follows from Theorem 2 below, any generalized Burgers equation has only one local conservation law. Note that this seems to be the only nonlinear integrable hierarchy that allows complete description of conservation laws for all its members.

In the present article we study conservation laws of equations (1), (3). This question is trivial in the case of evolution equations of order 1, and from now on we assume \( m \geq 2 \).

Recall [1, 5] that a smooth function \( \varphi(x,t,u,u_1,...,u_k) \) is called a conserved density of an equation
\[ u_t = F(x,t,u,u_1,...,u_m) \]  (5)
if \( D_t(\varphi) = D_x(\psi(x,t,u,u_1,...,u_l)) \) for some function \( \psi \). Two conserved densities \( \varphi_1, \varphi_2 \) are said to be equivalent if \( \varphi_1 - \varphi_2 = D_x(\psi) \), and conserved densities of the form \( D_x(\psi) \) are called trivial. Equivalence classes of conserved densities are called conservation laws.

Example. The function \( v \) is a nontrivial conserved density for (3).

Let \( \varphi \) be a conserved density for (5), then its characteristic [5] (also called the generating function [1, 8]) is the variational derivative of \( \varphi \)
\[ \xi = \frac{\delta \varphi}{\delta u} = \sum_{i \geq 0} (-1)^i D_x^i \left( \frac{\partial \varphi}{\partial u_i} \right). \]  (6)
The characteristic satisfies the equation
\[ -D_t(\xi) = \sum_{i=0}^{m} (-1)^i D_x^i \left( \frac{\partial F}{\partial u_i} \xi \right) \]  (7)
and is nonzero if and only if the density \( \varphi \) is nontrivial, i.e., equivalent conserved densities have the same characteristic. The homological interpretation of these concepts can be found in [1, 8].

For a function \( h(x,t,u,u_1,...) \) the maximal integer \( k \) such that \( \partial h/\partial u_k \neq 0 \) will be called the order of \( h \) and denoted by \( o(h) \). If \( \partial h/\partial u_k = 0 \) for all \( k > 0 \), we set \( o(h) = 0 \).

Theorem 1. The characteristic \( \xi \) of any conserved density for (1) has the form
\[ \xi = a(x,t) + \sum_i b^i(x,t) u_i \]  (8)
such that the operator \( \sum_i b^i(x,t) D_x^i \) is self-adjoint, i.e.,
\[ \sum_i b^i(x,t) D_x^i(\psi) = \sum_i (-1)^i D_x^i(b^i(x,t) \psi). \]  (9)
for any smooth function \( \psi = \psi(x,t,u,u_1,...) \).
Proof. We must prove
\[ \frac{\partial^2 \xi}{\partial u_i \partial u_j} = 0, \quad \forall i, j \geq 0, \quad (10) \]
then (9) follows from the properties of variational derivatives [1, 5]. If (10) is not true then there exist \( i_0 \geq j_0 \geq 0 \) such that
\[ \frac{\partial^2 \xi}{\partial u_{i_0} \partial u_{j_0}} \neq 0 \quad (11) \]
and
\[ \frac{\partial^2 \xi}{\partial u_i \partial u_j} = 0, \quad \forall i, j : i > i_0, i + j \geq i_0 + j_0. \quad (12) \]
For (1) equation (7) reads
\[ -D_t(\xi) = m \sum_{i=0}^{\infty} (-1)^i D_i x(a_i(x,t) \xi). \quad (13) \]
Differentiating (13) with respect to \( u_{o(\xi)} + m, \) we get
\[ -a^m(x,t) \frac{\partial \xi}{\partial u_{o(\xi)}} = (-1)^m \cdot a^m(x,t) \frac{\partial \xi}{\partial u_{o(\xi)}}. \]
Combining this identity with (2), we see that \( m \) is odd. Now differentiating (13) with respect to \( u_{i_0+m-1}, u_{j_0+1} \) and taking into account (12), one obtains
\[ -a^m(x,t) \frac{\partial^2 \xi}{\partial u_{i_0-1} \partial u_{j_0+1}} = -a^m(x,t) \frac{\partial^2 \xi}{\partial u_{i_0-1} \partial u_{j_0+1}} - m \cdot a^m(x,t) \frac{\partial^2 \xi}{\partial u_{i_0} \partial u_{j_0}}, \]
which contradicts to our assumption (11); hence (10) is true.

Theorem 2. Any conserved density of a generalized Burgers equation (3) is equivalent to \( Cv \) for some \( C \in \mathbb{R}, \) i.e., the space of conservation laws is one-dimensional.

Remark. For the classical Burgers equation this fact is well known and follows from the general observation [1, 5] that for any even order evolution equation there is an upper bound on the order of the characteristics of conservation laws. However, these arguments do not work for generalized Burgers equations of arbitrary order considered in this theorem.

Proof. For a conservation law \( \Omega \) of (3) consider a conserved density \( \varphi \in \Omega \) of minimal order \( k. \) Replacing \( v_k \) by \( D_k^x(u_1/u) \) in \( \varphi, \) we obtain a conserved density \( \tilde{\varphi} \) for (1). By the construction, \( \tilde{\varphi} \) is invariant under the symmetry \( u_k \mapsto \lambda u_k \) of (1), \( k \geq 0, \lambda \in \mathbb{R}. \) Then from (6) for the characteristic \( \xi \) of \( \tilde{\varphi} \) one has
\[ \xi(x,t,\lambda u_1,\ldots,\lambda u_{2k+2}) = \lambda^{-1} \xi(x,t,u_1,\ldots,u_{2k+2}), \quad \forall \lambda \neq 0. \quad (14) \]
Combining (14) with (8), we obtain \( \xi = 0, \) that is,
\[ \tilde{\varphi} = D_x \psi(x,t,u_1,\ldots,u_k) \quad (15) \]
for some function \( \psi. \)

From (15) we have \( \partial^2 \varphi/\partial u_{k+1}^2 = 0, \) therefore, \( \partial^2 \varphi/\partial v_k^2 = 0. \) This implies \( k = 0, \) because if \( k > 0, \) one easily constructs an equivalent conserved density \( \varphi' \) with \( o(\varphi') < k = o(\varphi), \) which contradicts to our assumption that \( \varphi \in \Omega \) is of minimal order. Thus \( \varphi = l(x,t)v + n(x,t), \) \( \tilde{\varphi} = l(x,t)u_1/u + n(x,t), \) and from (15)
we obtain that \( l(x, t) = l(t) \) does not depend on \( x \). Finally, from the condition that \( \varphi \) is a conserved density it follows that \( l(t) \) is actually a constant.

In contrast to generalized Burgers equations, for a linear equation (1) the space of conservation laws may be infinite-dimensional.

**Theorem 3.** For a linear evolution equation with constant coefficients (4) the space \( \mathcal{CL} \) of \((x,t)\)-independent conservation laws is described as follows.

1. If \( a^0 \neq 0 \) then \( \mathcal{CL} = 0 \).
2. If \( a^0 = 0 \), but \( a^{2j} \neq 0 \) for some \( j \in \mathbb{N} \) then \( \mathcal{CL} \) is one-dimensional and generated by the conserved density \( u \).
3. If all the even coefficients \( a^{2j} \) vanish, \( \mathcal{CL} \) is infinite-dimensional and generated by the conserved densities

\[
u, u^2_k, \quad k \geq 0.
\]

**Proof.** Let \( \varphi = \varphi(u, \ldots, u_k) \) be an \((x,t)\)-independent conserved density. From (8) and (9) it follows that the characteristic \( \xi \) of any \((x,t)\)-independent conserved density has the form

\[
\xi = c + \sum_j c^j u_{2j}, \quad c, c^j \in \mathbb{R}.
\]

Equation (13) reads

\[
-\sum_j \sum_{i=0}^m a^i c^j u_{i+2j} = \sum_j \sum_{i=0}^m (-1)^i a^i c^j u_{i+2j} + a^0 c.
\]

Clearly, if there is a nonzero even coefficient \( a^{2j} \neq 0, j \in \mathbb{N} \), then (18) implies \( \xi = c \in \mathbb{R} \), while if \( a^0 \neq 0 \) then \( \xi = 0 \). On the other hand, if \( a^{2j} = 0 \) for all \( j \) then (18) is valid for any constants \( c, c^j \). It is easily seen that if \( a^0 = 0 \) then \( cu \) is a conserved density with the characteristic \( \xi = c \), while if \( a^{2j} = 0 \) for all \( j \) then the function \( \varphi = u^2_k \) is a conserved density with characteristic \( \xi = (-1)^j u_{2j} \). Hence every function (17) satisfying (18) is indeed a characteristic of a linear combination of the above described conserved densities, which thus span the whole space of conservation laws.

Conservation laws (16) of a linear equation

\[
u_t = \sum_{j \geq 0} a^{2j+1} u_{2j+1}, \quad a^{2j+1} \in \mathbb{R},
\]

can be derived as follows. Performing the Galilean transformation

\[
x \mapsto x + a^1 t, \quad t \mapsto t, \quad u \mapsto u,
\]

we obtain an equation of the form (19) with \( a^1 = 0 \). Such an equation possesses a bi-Hamiltonian structure [7]

\[
u_t = D_x \left( \frac{\delta}{\delta u} \sum_j (-1)^j \frac{1}{2} a^{2j+1} u^2_j \right),
\]

\[
u_t = D^3_x \left( \frac{\delta}{\delta u} \sum_{j \geq 0} (-1)^{j-1} \frac{1}{2} a^{2j+1} u^2_{j-1} \right).
\]
It is easily seen that conservation laws (16) are obtained from this by the Lenard scheme [3, 5] and, therefore, are in involution with respect to the both corresponding Poisson brackets. According to [7], using (20) and the Cole-Hopf transformation, one can also construct a nonlocal bi-Hamiltonian structure for the generalized Burgers equation associated with (19).

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