Scale Invariant Low-Energy Effective Action
in $\mathcal{N} = 3$ SYM Theory

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Abstract

Using the harmonic superspace approach we study the problem of low-energy effective action in $\mathcal{N} = 3$ SYM theory. The candidate effective action is a scale and $\gamma_5$-invariant functional in full $\mathcal{N} = 3$ superspace built out of $\mathcal{N} = 3$ off-shell superfield strengths. This action is constructed as $\mathcal{N} = 3$ superfield generalization of $F^4/\phi^4$ component term which is leading in the low-energy effective action and is simultaneously the first nontrivial term in scale invariant Born-Infeld action. All higher-order terms in the scale invariant Born-Infeld action are also shown to admit an off-shell superfield completion in $\mathcal{N} = 3$ harmonic superspace.

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1 Introduction

Field theories with extended supersymmetry play an important role in modern high-energy physics due to their beautiful properties and intertwining relationships with string theory. A vastly known example is $\mathcal{N} = 4$ super Yang-Mills (SYM) theory. Its geometric and quantum aspects were a subject of numerous studies. It was found that the symmetries of this theory (unified into $\mathcal{N} = 4$ superconformal symmetry) are so rich that many its remarkable quantum properties can be proved solely on the symmetry grounds. E.g. the ultraviolet finiteness of quantum $\mathcal{N} = 4$ SYM theory can be attributed to the exact superconformal invariance of the latter. Some low-energy contributions to the quantum effective action in the $\mathcal{N} = 2$ gauge multiplet sector can be exactly found on the basis of smaller scale and $\gamma_5$ invariances [1] (see also review [2]). Further invoking the hidden (on-shell) $\mathcal{N} = 4$ supersymmetry allows one to extend the analysis of [1] to the hypermultiplet sector and to reveal the complete structure of one-loop effective action on the Coulomb branch [3]. Many proofs and computations are essentially simplified with using the off-shell approach of $\mathcal{N} = 2$ harmonic superspace (HSS) [4, 5] which makes manifest two of four underlying Poincaré supersymmetries of the $\mathcal{N} = 4$ SYM theory.

In this paper we undertake some steps towards studying, along similar lines, the $\mathcal{N} = 3$ SYM theory in $\mathcal{N} = 3$ HSS [6]. This superspace is a generalization of $\mathcal{N} = 2$ HSS, and provides a formulation of $\mathcal{N} = 3$ gauge theory in terms of unconstrained off-shell $\mathcal{N} = 3$ superfields. Such a model, like $\mathcal{N} = 4$ SYM, is known to be finite [7] and superconformally invariant [8]. Actually, $\mathcal{N} = 3$ SYM model describes the dynamics of the same multiplet of physical field as the $\mathcal{N} = 4$ one (see e.g. book [9] for a review), so these theories are different only off shell. Thus $\mathcal{N} = 3$ SYM theory in $\mathcal{N} = 3$ HSS can be regarded as a superfield formulation of $\mathcal{N} = 4$ SYM theory with bigger number of off-shell supersymmetries compared to the more familiar $\mathcal{N} = 2$ superfield formulation of $\mathcal{N} = 4$ SYM theory. Recently, it was argued [10] that the $\mathcal{N} = 3$ superfield formulation of $\mathcal{N} = 4$ SYM theory in fact reveals the maximally attainable number of off-shell supersymmetries, and there is no way to continue off shell the entire $\mathcal{N} = 4$ supersymmetry. This shows the urgency and importance of studying quantum properties of the $\mathcal{N} = 3$ SYM model. So far they were not explored at all, and the structure of the quantum effective $\mathcal{N} = 3$ harmonic superfield action of $\mathcal{N} = 3$ SYM theory remains to be disclosed. The language of $\mathcal{N} = 3$ HSS is capable to greatly simplify quantum calculations, e.g. due to the fact that the interaction superfield Lagrangian of $\mathcal{N} = 3$ SYM theory includes only one or two vertices (in the first- or second-order formalisms [6, 5]), which should be contrasted with an infinite tower of vertices in the $\mathcal{N} = 2$ HSS formulation of $\mathcal{N} = 4$ SYM theory. As another evidence in favour of usefulness of the $\mathcal{N} = 3$ HSS approach, it was recently shown [11] that $\mathcal{N} = 3$ SYM theory in harmonic superspace is naturally generated from superstring theory.

It is well known [11, 2, 12] that the leading term in the low-energy effective action of $\mathcal{N} = 4$ SYM model in the sector of $\mathcal{N} = 2$ vector multiplet has the form $\int d^4 x \frac{\bar{F}^a F_a}{(\bar{\varphi} \varphi)}$, where $F^{\alpha \beta}, \bar{F}^{\dot{\alpha} \dot{\beta}}$ are the Abelian field strengths and $\varphi$ is the scalar field belonging to $\mathcal{N} = 2$ vector multiplet. Complete $\mathcal{N} = 4$ supersymmetric generalization of the low-
energy effective action including both vector fields and hypermultiplets in the $\mathcal{N} = 2$ HSS approach was given in [3].\(^1\) The leading bosonic component of this action is given by

$$
\int d^4x \frac{F^2 \bar{F}^2}{(\phi_i \phi^i)^2}
$$

(1.1)

where $\phi^i$ is a complex SU(3) triplet of scalar fields.\(^2\) Since the models of $\mathcal{N} = 3$ and $\mathcal{N} = 4$ SYM are equivalent on-shell, we expect that the term (1.1) is also leading in the effective action of $\mathcal{N} = 3$ SYM model. Therefore it is interesting and important to find $\mathcal{N} = 3$ superfield action which would reproduce the expression (1.1) in the component expansion. One can expect that such an action presents the low-energy quantum effective action of $\mathcal{N} = 3$ SYM model.

An important step in understanding the possible structure of the effective action in $\mathcal{N} = 3$ gauge theory was the construction of $\mathcal{N} = 3$ supersymmetric Born-Infeld (BI) action in $\mathcal{N} = 3$ HSS [14]. However, this BI action contains a dimensionful parameter, hence it is not scale invariant and cannot reproduce scalar fields in the denominator of the expression (1.1). It is natural to assert that the low-energy effective action of $\mathcal{N} = 3$ gauge theory corresponds to a scale invariant generalization of the $\mathcal{N} = 3$ BI theory. In this work we propose a possible form of such a superfield functional which meets the requirements of $\mathcal{N} = 3$ supersymmetry, as well as gauge, scale and $\gamma_5$ invariances, and gives rise to the term (1.1) after going to components.

The paper is organized as follows. In section 2, just to fix our notation, we overview some facts about $\mathcal{N} = 3$ BI and SYM theories in $\mathcal{N} = 3$ HSS. In section 3 we propose a possible HSS form of a scale invariant $\mathcal{N} = 3$ supersymmetric functional reproducing the leading term in the low-energy effective action of $\mathcal{N} = 3$ SYM theory and study the component structure of this functional. In section 4 we present $\mathcal{N} = 3$ supersymmetric scale invariant completion of all higher terms in BI action, starting with $F^8/\phi^{12}$. Section 5 contains concluding remarks and discussions. In the Appendix we collect useful formulas of the $\mathcal{N} = 3$ HSS formalism.

## 2 $\mathcal{N} = 3$ Supersymmetric Yang-Mills and Born-Infeld actions

The $\mathcal{N} = 3$ HSS was introduced in ref. [6] to construct, for the first time, an off-shell superfield formulation of $\mathcal{N} = 3$ SYM model. The basics of the harmonic superspace method are exposed in book [5]. Throughout this paper we follow the conventions of ref. [14]. Some basic formulas of the $\mathcal{N} = 3$ HSS approach are collected in the Appendix.

The classical HSS action of $\mathcal{N} = 3$ SYM model is

$$
S_{\text{SYM}}^{\mathcal{N}=3}[V] = S_2[V] + S_{\text{int}}[V],
$$

\(^1\)Note also recent papers [13] devoted to discussion of general structure of the effective actions within the $\mathcal{N} = 2$ HSS approach.

\(^2\)In [3] these fields were arranged into a real self-dual 6-plet of SU(4).
\[ S_2[V] = -\frac{1}{4} \text{tr} \int d\zeta^3_{(i_1)} du \left[ V_3^2 D_3^1 V_1^1 + \frac{1}{2} (D_2^1 V_3^1)^2 \right], \quad (2.1) \]
\[ S_{\text{int}}[V] = -\frac{1}{4} \text{tr} \int d\zeta^3_{(i_1)} du \left\{ i (D_2^1 V_3^1 - D_3^1 V_2^1) [V_1^1, V_2^2] - \frac{1}{2} [V_1^1, V_3^2] [V_1^1, V_2^2] \right\}. \quad (2.2) \]

The piece \( S_2[V] \) is the free action and \( S_{\text{int}}[V] \) represents the interaction. We will consider only the Abelian case when the interaction is absent.

The prepotentials \( V_1^2, V_3^2 \) are analytic superfields and the integration in (2.2) is performed over analytic superspace (see eq. (A.5) for the definition of the analytic superspace integration measure). Physical bosonic component fields are contained in the prepotentials as \( \Phi \)

\[ V_3^2 = \left[ (\bar{\theta}^1 \theta^2)^2 u_k^2 - (\bar{\theta}^2)^2 u_k^2 \right] \phi^k + \theta_2^2 \bar{\theta}_2^2 A_{\alpha \alpha} - i \theta_2^2 \bar{\theta}_2^2 (\bar{\theta}^2)^2 H_{\alpha \beta} \]

+ spinors and auxiliary fields,

\[ V_2^1 = -\left( \bar{V}_3^2 \right) = \left[ (\theta_2 \theta_3) \bar{u}_2^2 - (\theta_2)^2 \bar{u}_2^2 \right] \phi^k + \theta_2^2 \bar{\theta}_2^2 A_{\alpha \alpha} + i (\theta_2)^2 \bar{\theta}_2^2 \bar{\theta}_2^2 \bar{H}_{\alpha \beta} \]

+ spinors and auxiliary fields.

Here, \( \phi^i, \bar{\phi}_i \) are complex scalar fields, \( A_{\alpha \alpha} \) is a vector gauge field, \( H_{\alpha \beta}, \bar{H}_{\alpha \beta} \) are the auxiliary fields which ensure the correct structure of the gauge field sector of the theory [14].

The component form of the action \( S_2 \) in the SU(3) singlet gauge field sector is \[ S_2 = \int d^4x [V^2 + \bar{V}^2 - 2(\bar{V} \bar{F} + VF) + \frac{1}{2} (F^2 + \bar{F}^2)] \] \quad (2.4) \]

where \[ V_{\alpha \beta} = \frac{1}{4} (H_{\alpha \beta} + F_{\alpha \beta}), \quad \bar{V}_{\dot{\alpha} \dot{\beta}} = \frac{1}{4} (\bar{H}_{\dot{\alpha} \dot{\beta}} + \bar{F}_{\dot{\alpha} \dot{\beta}}), \]
[\[ F^2 = F_{\alpha \beta} F_{\alpha \beta}, \quad V^2 = V_{\alpha \beta} V_{\alpha \beta}, \quad FV = F_{\alpha \beta} V_{\alpha \beta}, \]

\[ F_{\alpha \beta} = \frac{i}{4} (\sigma_{mn})_{\alpha \beta} (\partial_m A_n - \partial_n A_m), \quad \bar{F}_{\dot{\alpha} \dot{\beta}} = -\frac{i}{4} (\bar{\sigma}_{mn})_{\dot{\alpha} \dot{\beta}} (\partial_m A_n - \partial_n A_m). \]

The auxiliary fields \( V_{\alpha \beta}, \bar{V}_{\dot{\alpha} \dot{\beta}} \) can be eliminated by their algebraic classical equations of motion \[ V_{\alpha \beta} = F_{\alpha \beta}, \quad \bar{V}_{\dot{\alpha} \dot{\beta}} = \bar{F}_{\dot{\alpha} \dot{\beta}}. \] \quad (2.6) \]

As a result, the free classical action (2.4) takes the form of the usual Maxwell action \[ S_2 = -\frac{1}{2} \int d^4x (F^2 + \bar{F}^2). \] \quad (2.7) \]

The higher order terms of the superfield \( \mathcal{N} = 3 \) BI-action can be constructed out of \( \mathcal{N} = 3 \) superfield strengths which are expressed through prepotentials as \[ W_{23} = \frac{1}{2} D_{3 \alpha} \bar{D}_3^\dot{\alpha} V_3^2, \quad \bar{W}^{12} = -\frac{1}{2} D^{1 \alpha} D_1^{\dot{\alpha}} V_1^2, \]

\[ W_{12} = D_1^\alpha W_{23}, \quad \bar{W}^{23} = -\bar{D}_1^{\dot{\alpha}} \bar{W}^{12}, \]

\[ W_{13} = -D_1^\alpha W_{23}, \quad \bar{W}^{13} = \bar{D}_1^{\dot{\alpha}} \bar{W}^{12}. \] \quad (2.8) \]
Here $V_1^2$, $V_3^2$ are non-analytic prepotentials which are the solutions of zero-curvature equations \[ D_2^3 V_2^1 = D_3^1 V_1^2, \quad D_2^3 V_3^2 = D_3^3 V_2^3. \] (2.9)

The superfields (2.8) have the following component structure in the sector of physical bosons \[ W_{23} = u_1^1 \phi(x_{A+}) + 4i \theta_2^\alpha \theta_3^\beta V_{\alpha \beta}(x_{A+}) \text{ + spinors and auxiliary fields} \]

\[ \bar{W}_{12} = \bar{u}_1^3 \bar{\phi}(x_{A-}) + 4i \bar{\theta}_1^\dot{\alpha} \bar{\theta}_2^\dot{\beta} \bar{V}^{\dot{\alpha} \dot{\beta}}(x_{A-}) \text{ + spinors and auxiliary fields} \] (2.10)

where $x_{A+}^{\alpha \dot{\alpha}} = x_{A-}^{\alpha \dot{\alpha}} \pm 2i \theta^\alpha \bar{\theta}^{\dot{\alpha}}$.

The $\mathcal{N} = 3$ supersymmetric Born-Infeld action \[ S_{BI}^{\mathcal{N}=3} = S_2 + S_E, \]

\[ S_E = \sum_{n=1}^{\infty} S_{4n} = \frac{1}{32 X^2} \int d\zeta^{(3)\alpha} du (\bar{W}_{12} W_{23})^2 \hat{E}(A/X^4) \] (2.11)

where $X = f^{-1}$ is a coupling constant of the mass dimension 2. By definition, the composite analytic superfield contains the 4-th degree of the auxiliary fields

\[ A = \frac{1}{210} (D^1)(D_3)^2[D^2 \alpha W_{12} D^2 \alpha W_{12} \bar{D}_{23} \bar{W}^{23} \bar{D}_2^{\dot{\alpha}} \bar{W}^{23}] = V^2 \bar{V}^2 + \ldots. \] (2.12)

The superfield function $\hat{E}(A)$ is connected with the self-interaction function of the auxiliary fields in the BI theory $E(a) = E(V^2 \bar{V}^2)$ \[ \hat{E}(a) = \frac{2}{a} E(a) = \frac{4}{a} [2t^2(a) + 3t(a) + 1], \quad t^4 + t^3 - \frac{1}{4} a = 0, \quad t|_{a=0} = -1. \] (2.13)

The series expansion for the function $\hat{E}(a)$ starts with

\[ \hat{E}(a) = 1 - \frac{a}{4} + \frac{3a^2}{16} + \ldots \] (2.14)

where the 1-st term corresponds to the 4-th order interaction $S_4$. In the gauge field sector the action (2.11) yields the standard bosonic BI action \[ S_{BI} = \int d^4 x \left[ -\frac{1}{2} (F^2 + \bar{F}^2) + \frac{1}{2} \frac{F^2 \bar{F}^2}{X^2} - \frac{1}{4} \frac{F^2 \bar{F}^2 (F^2 + \bar{F}^2)}{X^4} \right. \]

\[ + \left. \frac{1}{8} \frac{F^2 \bar{F}^2 (3F^2 \bar{F}^2 + F^4 + \bar{F}^4)}{X^6} + \ldots \right] \]

\[ = X^2 \int d^4 x \left[ \sqrt{-\det(\eta_{mn} + F_{mn}/X)} - 1 \right] \] (2.15)

where $F_{nm} = \partial_n A_m - \partial_m A_n$ is the Maxwell field strength.
We emphasize the fact that the action $S_E$ (2.11), as well as its component expansion (2.15), contains the coupling constant $X$ of mass dimension 2. Therefore this action is not scale invariant.

Let us single out the quartic superfield part in the action (2.11)

$$S_4 = \frac{1}{32} \int d\zeta^{(3)} \frac{(\bar{W}^{12}W_{23})^2}{X^2}. \quad (2.16)$$

This action produces the first nontrivial term of the BI interaction $E(V^2\bar{V}^2)$

$$\frac{1}{2} \int d^4x \frac{V^2\bar{V}^2}{X^2}. \quad (2.17)$$

and makes the corresponding contributions to the BI action (2.15), starting from the $F^2\bar{F}^2$ term (in particular, it uniquely fixes the coefficient before the 6-th order term). Note that the next, 8-th order superfield term in (2.11) produces the unique term $\sim (V^4\bar{V}^4)$ in the lagrangian of tensor auxiliary fields and, nevertheless, affects the whole infinite sequence of terms $F^{4(1+k)}\bar{F}^{4(1+k)}$, ($k \geq 0$) in the effective Maxwell field strength action after eliminating the auxiliary fields by their equations of motion. The correct BI action is restored only after accounting all superfield terms in (2.11).

### 3 Scale and $\gamma_5$-invariant $\mathcal{N} = 3$ superfield low-energy effective action

In this Section we construct a manifestly $\mathcal{N} = 3$ supersymmetric low-energy effective action containing the term $F^4/\phi^4$ in the bosonic sector.

$\mathcal{N} = 3$ SYM theory is known to be a superconformal field theory [8], like the $\mathcal{N} = 4$ SYM one. Moreover, both these models describe the dynamics of the same multiplet of physical fields and therefore are on-shell equivalent [9]. The effective action of $\mathcal{N} = 3$ SYM model should be scale invariant. The mutually commuting transformations of dilatations (scale invariance) and $\gamma_5$-symmetry (R-symmetry) act on the coordinates of harmonic superspace and superfield strengths as follows

$$\delta x^m_A = ax^m_A, \quad \delta \theta^\alpha_I = \frac{1}{2}(a + ib)\theta^\alpha_I, \quad \delta \bar{\theta}^{I\dot{\alpha}} = \frac{1}{2}(a - ib)\bar{\theta}^{I\dot{\alpha}}$$

$$\delta W_{IJ} = -(a + ib)W_{IJ}, \quad \delta \bar{W}^{IJ} = -(a + ib)\bar{W}^{IJ}. \quad (3.1)$$

We expect that a scale and $\gamma_5$-invariant generalization of the action (2.16) should correspond to the low-energy effective action of $\mathcal{N} = 3$ SYM model. In components such an action should reproduce the scale and $\gamma_5$-invariant generalization of (2.17), that is (1.1). Note that exactly this term is leading in the low-energy effective action of $\mathcal{N} = 4$ SYM model [11, 12]. Thus we wish to construct a generalization of the action (2.16) which would bear the scale- and $\gamma_5$-invariances. Note that the effective $\mathcal{N} = 3$ action of ref. [13] is $\gamma_5$-invariant, so we shall discuss its scale invariant generalization.
To pass from (2.17) to the scale invariant component action (1.1), one should replace the dimensionful constant \( X \) by the function of scalar fields \( \phi^i \bar{\phi}^i \). Therefore, to obtain a scale invariant generalization of the superfield action (2.16) we have to replace the constant \( X \) by some superfield expression having the same dimension and containing \( \phi^i \bar{\phi}^i \) as the lowest component. The suitable expression is

\[
\bar{W}^{IJ} W_{IJ} = \bar{W}^{12} W_{12} + \bar{W}^{23} W_{23} + \bar{W}^{13} W_{13}.
\]

(3.2)

Indeed, the component expansion of this superfield starts with the scalars (see [15] for details)

\[
\bar{W}^{IJ} W_{IJ} \Big|_{\theta = \bar{\theta} = 0} = \phi^i \bar{\phi}_i.
\]

(3.3)

It is worth noting that this superfield, as well as the basic analytic superfields \( W_{23}, W_{12}, W_{13} \), are covariant with respect to the whole superconformal \( \mathcal{N} = 3 \) supergroup [5]. However, we do not study the superconformal properties of the effective action here.

The expression (3.2) cannot be naively inserted into the integral in (2.16) in place of the constant \( X \). The point is that the superfield \( \bar{W}^{IJ} W_{IJ} \) is not analytic since the superfield strengths \( \bar{W}^{23}, \bar{W}^{13}, W_{12}, W_{13} \) are not analytic, while the integration in (2.16) goes over the analytic superspace. Therefore we have to rewrite the action (2.16) in full \( \mathcal{N} = 3 \) HSS and then to insert \( \bar{W}^{IJ} W_{IJ} \) into the integral.

The action (2.16) in the full \( \mathcal{N} = 3 \) HSS is written as

\[
S^4 = \frac{1}{32} \int d^4 x d^2 \theta d u \frac{1}{X^2} \left[ \frac{(\bar{D}_1)^2}{4 \square} (W_{23})^2 \right] \left[ \frac{(D_3)^2}{4 \square} (\bar{W}^{12})^2 \right].
\]

(3.4)

To check that the actions (2.16) and (3.4) are actually identical to each other, one should express the integration measure of the full \( \mathcal{N} = 3 \) superspace through the analytic one

\[
d^4 x d^2 \theta = d \zeta (33) \frac{1}{4} (D_1)^2 \frac{1}{4} (D_3)^2,
\]

(3.5)

and then apply the anticommutation relations

\[
\{ D_\alpha^1, \bar{D}_{\dot{\alpha}} \} = \{ D_\alpha^3, \bar{D}_{3\alpha} \} = -2i (\sigma^m)_{\alpha \dot{\alpha}} \partial_m.
\]

(3.6)

Replacing the constant \( X \) by the superfield \( \bar{W}^{IJ} W_{IJ} \) in (3.4), we arrive at the action

\[
S^4_{scale-inv} = \alpha \int d^4 x d^2 \theta d u \frac{1}{(\bar{W}^{IJ} W_{IJ})^2} \left[ \frac{(\bar{D}_1)^2}{4 \square} (W_{23})^2 \right] \left[ \frac{(D_3)^2}{4 \square} (\bar{W}^{12})^2 \right].
\]

(3.7)

where \( \alpha \) is some dimensionless constant. This constant can be fixed from a straightforward calculation of low-energy effective action in the framework of quantum field theory. Besides, if one assumes e.g. that the term (3.7) is a part of supersymmetric BI action, the constant \( \alpha \) can be fixed from the requirement that the (3.7) reproduces the term \( F^4/\phi^4 \) in the scale invariant version of the expression (2.15) with the correct coefficient. Since the action (3.7) includes no any dimensional constants, it is scale invariant. It can be checked to be \( \gamma_5 \)-invariant as well.
Let us study the component structure of the action (3.7). Note that the superfield strengths entering the action contain a multiplet of physical fields as well as an infinite number of auxiliary fields. We are interested in the component structure of the action (3.7) in the sector of scalar and vector physical fields. For this purpose we neglect all the derivatives on scalar fields and Maxwell field strength. Such an approximation is sufficient for retrieving the term $F^4/\phi^4$ while going to components. Therefore we use the following ansatz for the superfield strengths

\[
\begin{align*}
  \hat{W}^{12} &= \tilde{\phi}_3 + \tilde{\omega}^{12}, \\
  \hat{W}^{23} &= \tilde{\phi}_1 + \tilde{\omega}^{23}, \\
  \hat{W}^{13} &= -\tilde{\phi}_2 + \tilde{\omega}^{13},
\end{align*}
\]

(3.8)

where

\[
\begin{align*}
  \tilde{\phi}_I &= u_I^i \phi_i, \\
  \phi^I &= u_I^i \phi_i, \\
  \tilde{\omega}^{IJ} &= 4i\theta^\alpha \bar{\theta}^J \beta \hat{V}_{\alpha \beta}, \\
  \omega^{IJ} &= 4i\theta^\alpha \theta^J \beta \hat{V}_{\alpha \beta}.
\end{align*}
\]

(3.9)

Here $V_{\alpha \beta}, \hat{V}_{\alpha \beta}$ are auxiliary tensor fields which have the same properties as the Maxwell field strengths $F_{\alpha \beta}, \hat{F}_{\alpha \beta}$ (the fields $V_{\alpha \beta}, \hat{V}_{\alpha \beta}$ are eventually expressed through $F_{\alpha \beta}, \hat{F}_{\alpha \beta}$). The symbol “hat” below indicates that we consider only scalar and vector bosonic fields and discard any auxiliary fields, except for the SU(3) singlet tensor ones just defined. With the ansatz (3.8) and (3.9), we have

\[
(\hat{D}_1)^2(\hat{W}^{23})^2 = 4(\theta_1)^2\Box(\hat{W}^{23})^2, \\
(\hat{D}^3)^2(\hat{W}^{12})^2 = 4(\tilde{\phi}_3)^2\Box(\hat{W}^{12})^2.
\]

(3.10)

Therefore, when we consider only the scalar and vector fields, the action (3.7) contains only local terms

\[
\hat{S}_{4\text{scale-inv}} = \alpha \int d^4x d^4\theta d\phi \frac{(\theta_1)^2(\tilde{\phi}_3)^2}{(W^{12}_{11} W^{11}_{12})^2} (\hat{W}^{12} \hat{W}^{23})^2.
\]

(3.11)

To find the component structure of the action (3.11) we need the following expansions:

\[
\begin{align*}
(\hat{W}^{12} \hat{W}^{23})^2 &= (\tilde{\phi}_3 \phi^1)^2 + (\tilde{\phi}_2 \omega_{23})^2 + (\phi^1 \omega^{12})^2 + (\tilde{\omega}^{12} \omega_{23})^2 \\
&+ 2(\tilde{\phi}_3)^2 \phi^1 \omega_{23} + 2(\phi^1)^2 \tilde{\phi}_3 \omega^{12} + 4\phi^1 \tilde{\phi}_3 \tilde{\omega}^{12} \omega_{23} \\
&+ 2(\phi^1 \omega_{23})^2 \omega_{12} + 2(\phi^1)^2 (\tilde{\omega}^{12})^2 \omega_{23},
\end{align*}
\]

(3.12)

\[
\begin{align*}
\frac{(\theta_1)^2(\tilde{\phi}_3)^2}{(W^{11} W^{12})^2} &= \frac{(\theta_1)^2}{(\phi^1 \phi_i)^2} \left[ 1 - 2 \frac{\phi^{12} \omega_3}{\phi^1 \phi_i} \right] + \\
&+ 3 \frac{(\phi^{12} \omega_3)^2}{(\phi^1 \phi_i)^2} + 3 \frac{(\phi^{12} \omega_3)^2}{(\phi^1 \phi_i)^2} \\
&+ 6 \frac{\phi^{12} \phi_i \omega_3}{(\phi^1 \phi_i)^2} - 12 \frac{(\phi^{12} \omega_3)^2 \phi_i \omega_3}{(\phi^1 \phi_i)^3} - 12 \frac{(\phi^{12} \omega_3)^2 \phi_i \omega_3}{(\phi^1 \phi_i)^3} + 30 \frac{(\phi^{12} \omega_3)^2 \phi_i \omega_3}{(\phi^1 \phi_i)^3}.
\end{align*}
\]

(3.13)

Substituting eqs. (3.12), (3.13) into the action (3.11) we obtain

\[
\hat{S}_{4\text{scale-inv}} = \alpha \int d^4x d^4\theta \frac{24}{(\phi^1 \phi_i)^2} \left[ 30 \frac{(\phi^1 \phi_i)^2}{(\phi^1 \phi_i)^2} - \\
&- 24 \frac{\phi^1 \phi_i}{(\phi^1 \phi_i)^2} + 4 \frac{\phi^1 \phi_i}{(\phi^1 \phi_i)^2} + 1 \right].
\]

(3.14)
The integrand in eq. (3.14) corresponds to the product of expressions (3.12) and (3.13) where only the terms with the maximal number of Grassmann variables are considered since all other terms vanish under the integral. The integration over the Grassmann variables in the action (3.14) yields
\[ \int d^{12}\theta(\theta_1)^2(\bar{\theta}^3)^2(\bar{\omega}_{23})^2 = 16V^2\bar{V}^2 \] (3.15)
where we took into account the relation
\[ (\omega_{23}\bar{\omega}^{12})^2 = 16(\theta_3)^2(\bar{\theta}^1)^2(\bar{\theta}^2)^2V^2\bar{V}^2. \] (3.16)
To perform the integration over harmonic variables we apply the following formula
\[ \int du(\phi^i\bar{\phi}_i)^m(\phi^3\bar{\phi}_3)^n = \frac{2m!n!}{(2+m+n)!}(\phi^i\bar{\phi}_i)^{m+n} \] (3.17)
for each term in eq. (3.14). As a result, we obtain
\[ \hat{S}_{4\text{scale-inv}} = \frac{\alpha_0}{2} \int d^4x \frac{V^2\bar{V}^2}{(\phi^i\bar{\phi}_i)^2} \] (3.18)
where \( \alpha_0 = \frac{32}{15}\alpha. \)
Now we should express the auxiliary fields \( V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}} \) through the physical field strengths \( F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}} \) from the action
\[ \hat{S}_2 + \hat{S}_{4\text{scale-inv}} = \int d^4x \left[ V^2 + \bar{V}^2 - 2(V\bar{F} + V\bar{F}) + \frac{1}{2}(F^2 + \bar{F}^2) + \frac{\alpha_0}{2} \frac{V^2\bar{V}^2}{(\phi^i\bar{\phi}_i)^2} \right]. \] (3.19)
The equations of motion for the auxiliary fields are
\[ 2F_{\alpha\beta} = V_{\alpha\beta} \left[ 2 + \frac{\alpha_0}{(\phi^i\bar{\phi}_i)^2} V^2 \right], \quad 2\bar{F}_{\dot{\alpha}\dot{\beta}} = \bar{V}_{\dot{\alpha}\dot{\beta}} \left[ 2 + \frac{\alpha_0}{(\phi^i\bar{\phi}_i)^2} V^2 \right]. \] (3.20)
Eqs. (3.20) define the auxiliary fields \( V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}} \) as functions of \( F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}} \). The solution to these equations can be represented as a series over the Maxwell field strengths:
\[ V_{\alpha\beta} = F_{\alpha\beta} \left[ 1 - \frac{\alpha_0}{2(\phi^i\bar{\phi}_i)^2} F^2 + O(F^3) \right], \quad \bar{V}_{\dot{\alpha}\dot{\beta}} = \bar{F}_{\dot{\alpha}\dot{\beta}} \left[ 1 - \frac{\alpha_0}{2(\phi^i\bar{\phi}_i)^2} F^2 + O(F^3) \right]. \] (3.21)
Substituting the solutions (3.21) into the action (3.18), we find
\[ S_{4\text{scale-inv}} = \frac{\alpha_0}{2} \int d^4x \left[ F^2\bar{F}^2 + \frac{\alpha_0}{2(\phi^i\bar{\phi}_i)^4} (F^2 + \bar{F}^2) + O(F^8) \right]. \] (3.22)
Setting \( \alpha_0 = 1 \), we obtain the terms of 4-th and 6-th order in BI action (2.15) with the correct coefficients.
Note that the action (3.22) contains all the higher-order terms starting with $F^8$. However, as observed in [14, 17], the coefficients in this series are different from those in the field expansion of the BI action (2.15). In the non-scale-invariant case this discrepancy is corrected by the function $\hat{E}(A)$ in the action (2.14).

Let us finish this Section with several further comments concerning the superfield action (3.7).

This action contains the nonlocal operator $\Box^{-1}$. However, the leading low-energy term in the component action is local. This can be explained as follows. Let us rewrite the action (3.7) in the analytic superspace, using the relation (3.5). We will obtain the lagrangian in which the derivatives $D_1^\alpha, \bar{D}_3\dot{\alpha}$ are distributed in all conceivable ways among the factors of the original lagrangian (3.7). Consider one kind of such terms, namely those in which derivatives do not hit $1/(\bar{W}^{12}W_{12})$. Such terms either vanish due to the analyticity of superfield strengths $\bar{W}^{12}, W_{23}$, or result in local expressions, since the identities

$$(D^1)^2(\bar{D}_1)^2(W_{23})^2 = -16\Box(W_{23})^2, \quad (\bar{D}_3)^2(D^3)^2(W^{12})^2 = -16\Box(W^{12})^2$$

allow one to cancel the factors $\Box$ in the denominators of (3.7). Precisely these superfield terms produce the component term $F^4/\phi^4$. Another type of terms corresponds to the situation when the derivatives $D^1_1, \bar{D}_3\dot{\alpha}$ hit $1/(\bar{W}^{12}W_{12})$. It is easy to see that such terms produce the component expressions which contain the space-time derivatives of component fields and therefore these terms should be neglected in the low-energy approximation.

From the very beginning there is a freedom in distributing the derivatives among different factors in the actions (3.4) and (3.7). However, the local part of the action (3.7) actually does not depend on the specific pattern of such a distribution. The particular pattern we have chosen is most convenient for studying the component structure of the action. Let us dwell on this issue in more details. One can start with the action (3.4) in which the derivatives $D^1_1, \bar{D}_3\dot{\alpha}$ are distributed in a different way. Any such action can be cast in the form (3.4) by integrating by parts. But all such actions with diverse distributions of derivatives lead to different functionals of the type (3.7) where the factor $1/(\bar{W}^{12}W_{12})^2$ is inserted. Integrating by parts, we see that all such actions differ only by non-local terms which appear when the derivatives hit the factor $1/(\bar{W}^{12}W_{12})^2$. As explained above, such terms are irrelevant for our consideration.

As follows from eq. (3.22), the action $S_{4}^{scale-inv}$ contains the term $\int d^4x F^4/(\phi^4 \bar{\phi}_4)^2$ in its component expansion. We observe that in this expression the scalar fields appear in a single SU(3) invariant combination. An analogous result was earlier obtained in ref. [3, 16] for the full low-energy effective action of $\mathcal{N} = 4$ SYM in the $\mathcal{N} = 2$ HSS approach.

The pivotal advantage of $\mathcal{N} = 3$ formalism consists in that all scalar fields from the very beginning are included into a single $\mathcal{N} = 3$ multiplet, while in the $\mathcal{N} = 2$ superspace language the scalar fields are distributed between vector multiplet and hypermultiplet.

To summarize, the off-shell action (3.7) is manifestly supersymmetric, gauge invariant and scale invariant. It also bears the invariance under the $\gamma_5$ and SU(3) transformations. Therefore, it can be considered as a candidate for the low-energy effective action in $\mathcal{N} = 3$ SYM model.
4 Scale invariant $\mathcal{N} = 3$ BI action

In the previous section we have demonstrated that the action (3.7) is responsible for the terms of 4-th and 6-th order in the scale invariant BI action. Now we are going to construct $\mathcal{N} = 3$ superfield scale invariant generalization of all other terms in BI action, starting with $F^8$.

A direct construction of scale invariant generalization of the action (2.11) in analytic superspace faces some difficulties. Therefore, in analogy with the action (3.7) we search for such a scale invariant $\mathcal{N} = 3$ supersymmetric BI action in the full $\mathcal{N} = 3$ superspace. One of possible superfield generalizations of higher terms in the scale invariant BI action is provided by the following action

$$S_{\text{scale-inv}}^{G} = \int d^4x d^{12}\theta du \frac{(D^2)^2(\bar{D}_2)^2(W^{IJ}W_{IJ})^2}{(W^{IJ}W_{IJ})^4} G \left( \frac{A}{(W^{IJ}W_{IJ})^4} \right)$$

(4.1)

where $A$ is defined in eq. (2.12) and $G$ is some function which can be represented as a series

$$G(a) = \sum_{n=0}^{\infty} \beta_n a^n$$

(4.2)

with some coefficients $\beta_n$. The action (4.1) is a scale invariant generalization of eq. (2.11) in the sense that (as will be shown below) it reproduces in components all terms in the scale invariant BI action, starting from $F^8/\phi^{12}$, with definite coefficients which can be fixed by choosing $\beta_n$ in the appropriate way. Therefore, this action, taken in a sum with the quadratic $S_2$ and quartic $S_4^{\text{scale-inv}}$ actions, can generate the scale invariant BI action in the bosonic sector.

It should be noticed that the action (4.1) is by no means the unique superfield expression capable to reproduce the corresponding terms of the BI action in the bosonic limit. There is a freedom in distributing derivatives among different factors in eq. (4.1). As we suppose, this freedom can be compensated by the proper choice of function $G(a)$. Here we consider just an example of such an action which is most convenient for studying the component structure. We prove that a manifestly $\mathcal{N} = 3$ supersymmetric scale invariant BI action exists, but do not discuss how unique it is.\footnote{Perhaps, the freedom just mentioned could be fixed by requiring the action to respect the full $\mathcal{N} = 3$ superconformal symmetry.}

When we consider only scalar and vector fields, the superfield strengths are given by the ansatz (3.8). In such an approximation we derive

$$\hat{A} = \frac{1}{210}(D^1)^2(\bar{D}_3)^2[D^{20}\hat{\hat{W}}_{12}D_2^2\hat{\hat{W}}_{12}\hat{W}^{r23}\hat{D}_2^2\hat{\hat{W}}^{23}] = V^2\bar{V}^2$$

(4.3)

$$\frac{(\bar{\theta}^1)^2(\theta_1)^2}{(W^{IJ}W_{IJ})^m} = \frac{(\bar{\theta}^1)^2(\theta_1)^2}{(\phi_0\phi_3)^m+2}(\bar{\omega}^{23}\omega_{23})^2\frac{m(m+1)}{2}[1 - 2(m + 2)\frac{\delta_1^1\delta_3^1}{\phi_0\phi_3}]$$

(4.4)
\[
\frac{(\bar{\theta}^1)^2(\theta_3)^2}{(\bar{W}^I J\bar{W}_{IJ})^m} = \frac{1}{2} (\hat{\phi}_i^3\theta_3)^2 (\bar{\phi}_i^1)^{m+2} (\bar{\omega}{12}\omega_{12})^2 m(m+1) \left[ 1 - 2(m+2)\frac{\delta^3\phi_i}{\phi^i} \right] + \ldots, \tag{4.5}
\]

\[
\frac{(\bar{\theta}^1)^2(\theta_3)^2}{(\bar{W}^I J\bar{W}_{IJ})^m} = \frac{1}{2} (\hat{\phi}_i^3\theta_3)^2 (\bar{\phi}_i^1)^{m+2} (\bar{\omega}{12}\omega_{12})^2 m(m+1) \left[ 1 - 2(m+2)\frac{\delta^3\phi_i}{\phi^i} \right] + \ldots, \tag{4.6}
\]

\[
\frac{(\bar{\theta}^1)^2(\theta_3)^2}{(\bar{W}^I J\bar{W}_{IJ})^m} = \frac{1}{2} (\hat{\phi}_i^3\theta_3)^2 (\bar{\phi}_i^1)^{m+2} (\bar{\omega}{12}\omega_{12})^2 m(m+1) \left[ 1 - 2(m+2)\frac{\delta^3\phi_i}{\phi^i} \right] + \ldots \tag{4.7}
\]

where \( m = 4 + 4n \). Dots in eqs. (4.5)-(4.7) correspond to the terms with fewer number of Grassmann variables. These terms are not important here since they do not contribute to the action (4.1). Inserting the expressions (4.3)-(4.7) into the action (4.1) and performing there the integration over Grassmann and harmonic variables with the help of eqs. (A.5), (3.17), we find

\[
\hat{S}_{G}^{\text{scale-inv}} = \sum_{n=0}^{\infty} \beta_n \int d^4x \frac{V^{2n+4}V^{2n+4}}{(\bar{\phi}_i)^{6+4n}} \tag{4.8}
\]

where \( \beta_n = 2^{12}(1+n)(5+4n)(3+10n+8n^2)\beta_n \).

The next steps are to express the auxiliary fields \( V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}} \) through the field strengths \( F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}} \) from the equations of motion generated by the action

\[
\hat{S}_{BI}^{\text{scale-inv}} = \hat{S}_2 + \hat{S}_{4}^{\text{scale-inv}} + \hat{S}_{G}^{\text{scale-inv}} \tag{4.9}
\]

and to substitute these expressions back into the action (4.9). As a result, we should obtain the series over field strengths \( F^2, \bar{F}^2 \) with coefficients which depend on \( \beta_n \). The coefficients \( \beta_n \) can be found from the requirement that the action (4.9), in the sector of scalar and vector fields, coincides with the field expansion of the scale invariant version of the BI action (2.13).

For example, if we wish to fix the coefficient \( \beta_0 \) we should expand the above action up to the terms \( \sim F^8 \):

\[
\hat{S}_{BI}^{\text{scale-inv}} = \int d^4x \left[ V^2 + \bar{V}^2 - 2(\bar{V} \bar{F} + VF) + \frac{1}{2} (F^2 + \bar{F}^2) + \frac{1}{2} \frac{V^2 \bar{V}^2}{(\bar{\phi}_i)^2} \right] + \beta_0 \frac{V^4 \bar{V}^4}{(\bar{\phi}_i)^6} + O(\frac{V^6 \bar{V}^6}{(\bar{\phi}_i)^{10}}). \tag{4.10}
\]

The corresponding equations of motion for fields \( V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}} \) are

\[
\frac{\delta \hat{S}}{\delta V_{\alpha\beta}} = 2 V_{\alpha\beta} - 2\bar{F}_{\alpha\beta} + \frac{V_{\alpha\beta} V^2}{(\bar{\phi}_i)^2} + 4\beta_0 V_{\alpha\beta} \frac{V^2 \bar{V}^2}{(\bar{\phi}_i)^6} + \ldots = 0, \tag{4.11}
\]

\[
\frac{\delta \hat{S}}{\delta \bar{V}_{\dot{\alpha}\dot{\beta}}} = 2 \bar{V}_{\dot{\alpha}\dot{\beta}} - 2\bar{F}_{\dot{\alpha}\dot{\beta}} + \frac{\bar{V}_{\dot{\alpha}\dot{\beta}} V^2}{(\bar{\phi}_i)^2} + 4\beta_0 \bar{V}_{\dot{\alpha}\dot{\beta}} \frac{V^2 \bar{V}^2}{(\bar{\phi}_i)^6} + \ldots = 0. \tag{4.12}
\]
The solutions to eqs. (4.11), (4.12) are given by the following series

\[ V_{\alpha\beta} = F_{\alpha\beta} \left( 1 - \frac{1}{2} \frac{F^2}{(\phi^i \phi_i)^2} + \frac{1}{2} \frac{F^2 F^2}{2 (\phi^i \phi_i)^4} - \frac{1}{8} \frac{F^4 F^2 + (4 + 16 \bar{\beta}_0) F^2 F^4}{(\phi^i \phi_i)^6} + \ldots \right) \] (4.13)

\[ \bar{V}_{\dot{\alpha}\dot{\beta}} = \bar{F}_{\dot{\alpha}\dot{\beta}} \left( 1 - \frac{1}{2} \frac{F^2}{(\phi^i \phi_i)^2} + \frac{1}{2} \frac{F^2 F^2}{2 (\phi^i \phi_i)^4} - \frac{1}{8} \frac{F^4 F^2 + (4 + 16 \bar{\beta}_0) F^2 F^4}{(\phi^i \phi_i)^6} + \ldots \right) \] (4.14)

Substituting the expressions (4.13), (4.14) back into the action (4.10), we find that it coincides with the BI action (2.15) up to the \( F^8 \) order, provided that \( \bar{\beta}_0 = -\frac{1}{8} \) and scalar fields are substituted for the constant \( X \), \( X \rightarrow \phi^i \phi_i \). The other coefficients \( \bar{\beta}_n \) can be found in a similar way. Note that it is just the technical problem to find the exact values for all these coefficients.

As a result, complete scale invariant \( \mathcal{N} = 3 \) BI action reads

\[ S_{\text{scale-inv}}^{\text{BI}} = S_2 + S_4^{\text{scale-inv}} + S_G^{\text{scale-inv}} \] (4.15)

where \( S_2 \) is the free quadratic action given by (2.1), \( S_4^{\text{scale-inv}} \) is the non-local action (3.7) corresponding to the \( F^4 / \phi^4 \) term (as well as yielding the \( 6 \)-th order term) and \( S_G^{\text{scale-inv}} \) given by (4.1) is responsible for all higher-order terms with correct coefficients.

To conclude this section, we briefly discuss the relationship between the scale dependent BI action (2.11) and the scale invariant BI action (4.1). The first action is written as an integral over the analytic subspace of \( \mathcal{N} = 3 \) superspace while the second one is given by an integral over the full \( \mathcal{N} = 3 \) superspace. However, the action (2.11) can also be rewritten as an integral over the full \( \mathcal{N} = 3 \) superspace. The basic difference between them consists in the presence of dimensional constant \( X \) in eq. (2.11), while such a constant is absent in the scale invariant action (4.1). The above component analysis shows that both these actions have the same component form in the bosonic sector, provided one replaces the constant \( X \) by \( \phi^i \phi_i \) and picks up the proper functions \( \bar{E}(a) \) in (2.11) and \( G(a) \) in (4.1).

5 Summary

In this paper we analyzed the possible off-shell structure of low-energy effective action of \( \mathcal{N} = 3 \) SYM model written in \( \mathcal{N} = 3 \) harmonic superspace. This action was obtained as \( \mathcal{N} = 3 \) superfield generalization of the term \( F^4 / \phi^4 \) which is leading in the low-energy effective action. This superfield action is written as a functional built out of the superfield strengths in full \( \mathcal{N} = 3 \) superspace. This functional is manifestly supersymmetric, gauge invariant, scale and \( \gamma_5 \)-invariant and corresponds to a scale invariant generalization of 4-th order term in the \( \mathcal{N} = 3 \) supersymmetric BI action. We also constructed a scale invariant \( \mathcal{N} = 3 \) superfield completion of all higher terms in the BI action that might be helpful for studying higher-order terms in the candidate effective action of \( \mathcal{N} = 3 \) SYM model.

Note that the expression (3.7) is off-shell \( \mathcal{N} = 3 \) supersymmetric action. The superfield strengths \( \bar{W}^{IJ}, W_{IJ} \) entering the action (3.7) are unconstrained superfields. These
superfields contain the scalar fields, gauge field strengths and fermions, all being combined into a single $\mathcal{N}=3$ multiplet. This means, in particular, that the on-shell $\mathcal{N}=2$ vector multiplet and hypermultiplet superfields from which the complete low-energy $\mathcal{N}=4$ SYM effective action is composed in the $\mathcal{N}=2$ HSS [3] are now unified within a single off-shell $\mathcal{N}=3$ gauge superfield. Therefore, the action (3.7) can be regarded as an $\mathcal{N}=3$ superfield form of the complete $\mathcal{N}=4$ supersymmetric low-energy effective action found in [3].

In conclusion, let us point out once more that the effective action (3.7) was found solely by employing the symmetries of the model and the requirement that it produces the $F^4/\phi^4$ term in components. This action was determined up to an arbitrary numerical coefficient. We would like to emphasize an analogy with the work [1] where the on-shell low-energy effective action for $\mathcal{N}=4$ SYM theory in the sector of $\mathcal{N}=2$ gauge superfield was found merely on the symmetry grounds up to an arbitrary coefficient. This coefficient has been calculated later in the papers [12] using the quantum field theory considerations. At present, $\mathcal{N}=4$ SYM low-energy effective action is studied in many details both in the $\mathcal{N}=2$ gauge field sector (see e.g. [18]) and in the hypermultiplet sector [3, 16]. The important role in direct calculations of the effective action in supersymmetric models is played by a superfield background field method (see e.g. [9, 19] for $\mathcal{N}=1$ background field method and [20] for $\mathcal{N}=2$ background field method). It allows one to preserve manifest supersymmetry and gauge invariance at any step of quantum consideration. It is clear that the direct way of evaluating an effective action in $\mathcal{N}=3$ supersymmetric gauge theory should be based upon a quantum $\mathcal{N}=3$ formalism. The quantization procedure for this theory in $\mathcal{N}=3$ HSS was developed in [17]. The problem of working out the appropriate $\mathcal{N}=3$ background field method becomes now very important.

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A Appendix

The $\mathcal{N}=3$ HSS [8, 5] is defined as the superspace with coordinates \( \{Z, u\} \), where \( Z = \{x^{\alpha \dot{\alpha}}, \theta^a, \bar{\theta}^{\dot{a}}\}^4 \) is a set of standard $\mathcal{N}=3$ coordinates and $u$ are the harmonics.

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\(^4\)We denote by small Greek symbols the SL(2,$\mathbb{C}$) spinor indices, $\alpha, \dot{\alpha}, \ldots = 1, 2$; the small Latin letters are SU(3) indices, $i, j, \ldots = 1, 2, 3$. 
parameterizing the coset $SU(3)/U(1) \times U(1)$. We consider the harmonics $u^I_i$ and their conjugate $\bar{u}^i_I$ ($I = 1, 2, 3$) as $SU(3)$ matrices

$$u^I_i \bar{u}^j_I = \delta^I_j, \quad u^I_i \bar{u}^j_I = \delta^j_i, \quad \varepsilon^{ijk} u^1_i u^2_j u^3_k = 1. \quad (A.1)$$

The harmonic superspace $\{Z, u\}$ contains the so-called analytic subspace with the coordinates $\{\zeta_A, u\} = \{x^\alpha_A, \theta^\alpha_2, \theta^\alpha_3, \bar{\theta}^\alpha_1, \bar{\theta}^\alpha_2, u\}$ where

$$x^\alpha_A = x^{\alpha A} - 2i(\theta^\alpha_1 \bar{\theta}^{1\alpha} - \theta^\alpha_2 \bar{\theta}^{2\alpha}), \quad \theta^\alpha_i = \theta^\alpha_i u^1_i, \quad \bar{\theta}^{1\alpha} = \bar{\theta}^{1\alpha} u^1_i. \quad (A.2)$$

The analytic superspace plays an important role in harmonic superspace approach since it is closed under supersymmetry and all $\mathcal{N} = 3$ actions can be written in analytic coordinates.

The harmonic superspace is equipped with Grassmann covariant derivatives $D^I_\alpha$, $\bar{D}_{I\dot{\alpha}}$ and harmonic covariant ones $D^I_j$ which form the $su(3)$ algebra (see [[8, 5]] for details). The manifest expressions for these derivatives in the coordinates (A.2) are

$$D^1_\alpha = \frac{\partial}{\partial \theta^\alpha_1}, \quad D_{1\dot{\alpha}} = -\frac{\partial}{\partial \theta^{1\dot{\alpha}}} - 2i\theta^\alpha_1 (\sigma^m)_{\alpha\dot{\alpha}} \partial_m, \quad \bar{D}^2_\alpha = \frac{\partial}{\partial \theta^\alpha_2}, \quad \bar{D}_{2\dot{\alpha}} = -\frac{\partial}{\partial \theta^{2\dot{\alpha}}} - i\theta^\alpha_2 (\sigma^m)_{\alpha\dot{\alpha}} \partial_m, \quad (A.3)$$

$$D^3_\alpha = \frac{\partial}{\partial \theta^\alpha_3}, \quad D_{3\dot{\alpha}} = -\frac{\partial}{\partial \theta^{3\dot{\alpha}}}, \quad \bar{D}^1_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha_1}, \quad \bar{D}^2_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha_2}, \quad \bar{D}^3_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha_3}, \quad (A.4)$$

where $\partial^I_i = u^I_i \frac{\partial}{\partial u^I_i} - \bar{u}^i_I \frac{\partial}{\partial \bar{u}^i_I}$.

The Grassmann and harmonic measures of integration over the $\mathcal{N} = 3$ analytic harmonic superspace are normalized so that

$$\int d^{12} \theta (\theta^1_1)^2 (\theta^2_2)^2 (\theta^3_3)^2 (\bar{\theta}^{1\alpha})^2 (\bar{\theta}^{2\alpha})^2 (\bar{\theta}^{3\alpha})^2 = 1,$$

$$\int d\zeta^{(33)} (\theta^1_1)^2 (\theta^2_2)^2 (\bar{\theta}^{1\alpha})^2 (\bar{\theta}^{2\alpha})^2 = 1,$$

$$\int du = 1. \quad (A.5)$$
We use the following conventions for the relations between spinor and vector indices

\[ x_{\alpha\dot{\alpha}} = (\sigma^m)_{\alpha\dot{\alpha}} x_m, \quad x_m = \frac{1}{2}(\tilde{\sigma}_m)^{\dot{\alpha}\alpha} x_{\alpha\dot{\alpha}}. \quad (A.6) \]

Here \((\sigma^m)_{\alpha\dot{\alpha}} = (1, \vec{\sigma})_{\alpha\dot{\alpha}}\) are Pauli matrices and \((\tilde{\sigma}^m)^{\dot{\alpha}\alpha} = \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} (\sigma^m)_{\beta\dot{\beta}}\). The corresponding relations for antisymmetric rank 2 tensor are

\[ F_{mn} = \frac{i}{2} F^{\alpha\beta}(\sigma_{mn})_{\alpha\beta} - \frac{i}{2} F^{\dot{\alpha}\dot{\beta}}(\tilde{\sigma}_{mn})^{\dot{\alpha}\dot{\beta}}; \]
\[ F_{\alpha\beta} = \frac{i}{4}(\sigma_{mn})_{\alpha\beta} F^{mn}, \quad F_{\dot{\alpha}\dot{\beta}} = -\frac{i}{4}(\tilde{\sigma}_{mn})^{\dot{\alpha}\dot{\beta}} F^{mn} \quad (A.7) \]

where

\[ (\sigma_{mn})^{\alpha}_{\beta} = \frac{i}{2}(\sigma_m \overline{\sigma}_n - \sigma_n \overline{\sigma}_m)^{\alpha}_{\beta}, \quad (\tilde{\sigma}_{mn})^{\dot{\alpha}}_{\dot{\beta}} = \frac{i}{2}(\tilde{\sigma}_m \sigma_n - \tilde{\sigma}_n \sigma_m)^{\dot{\alpha}}_{\dot{\beta}}. \]

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