On the off-shell superfield Lagrangian formulation of $4D, \mathcal{N}=1$ supersymmetric infinite spin theory

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Abstract

We develop a complete off-shell Lagrangian description of the free $4D, \mathcal{N}=1$ supersymmetric theory of infinite spin. Bosonic and fermionic fields are formulated in terms of spin-tensor fields with dotted and undotted indices. The corresponding Lagrangians for bosonic and fermionic infinite spin fields entering into the on-shell supersymmetric model are derived within the BRST method. Lagrangian for this supersymmetric model is written in terms of the complex infinite spin bosonic field and infinite spin fermionic Weyl field subject to supersymmetry transformations. The fields involved into the on-shell supersymmetric Lagrangian can be considered as components of six infinite spin chiral and antichiral multiplets. These multiplets are extended to the corresponding infinite spin chiral and antichiral superfields so that two chiral and antichiral superfields contain among the components the basic fields of an infinite spin supermultiplet and extra four chiral and antichiral superfields containing only the auxiliary fields needed for the Lagrangian formulation. The superfield Lagrangian is constructed in terms of these six chiral and antichiral superfields, and we show that the component form of this superfield Lagrangian exactly coincides with the previously found component supersymmetric Lagrangian after eliminating the component fields added to construct (anti)chiral superfields.

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1 Introduction

The study of various aspects of massless fields corresponding to the infinite spin irreducible representations of the Poincaré group \([1–3]\) attracts attention due to their remarkable relations with higher spin theory and string theory \([4–34]\). One of such interesting aspects is the problem of constructing a supersymmetric infinite spin theory and the corresponding Lagrangian formulation \([23–34]\).

As is well known, \(\mathcal{N} = 1\) supersymmetric field theories can be defined in two ways. First, the theory is formulated in terms of ordinary bosonic and fermionic fields subject to supersymmetry transformations. These fields are called components. In this formulation, supersymmetry is nonmanifest; moreover, if the Lagrangian does not contain special non-dynamical auxiliary fields, the supersymmetry algebra is only on-shell closed. To avoid ambiguities, it is worth emphasizing that the term of the auxiliary fields is used in supersymmetric higher spin theories in two completely different meanings. Firstly, these are the fields that ensure the closure of the supersymmetry algebra. Secondly, this term is used for non-dynamic fields that provide the Lagrangian formulation in higher spin field theory. Since infinite spin theories are similar in some respects to higher spin theories, they can also be characterized by the auxiliary fields irrespective of supersymmetry. Another way to formulate supersymmetric theories is based on the use of unconstrained superfields (see, e.g., \([35]\)), which automatically provide manifest supersymmetry and a closed superalgebra \([1]\).

In the present paper, we solve the problem of constructing the manifest off-shell Lagrangian formulation of \(4D, \mathcal{N} = 1\) supersymmetric free infinite spin fields. Unlike all previous papers, we give a complete superfield description and derive the Lagrangian in terms of six scalar chiral and antichiral superfields. Two of these superfields contain among the components the basic fields corresponding to an infinite spin supermultiplet \([33]\) and the other superfields are exclusively auxiliary in sense of the higher spin theories. It should be noted that we work within the BRST approach (see application of this approach to infinite spin theories in \([25, 26]\), where all the fields are understood as components of the Fock space vectors. Therefore, more precisely, when we talk about superfields, we mean the corresponding vectors of the Fock space with superfield components.

The description of \(4D, \mathcal{N} = 1\) supersymmetric infinite spin theories was studied in the component approach in recent papers \([28–30]\), where supersymmetry transformations were found for one complex scalar field (vector of Fock space in our formulation) and the Dirac field (the Fock space vector in our formulation), in addition, in the paper \([30]\) off-shell supersymmetry transformations (up to gauge transformations) were given for component fields. Unlike the previous works on supersymmetric infinite spin theories we use Lagrangians for bosonic and fermionic fields which were obtained using the BRST method \([25, 26]\). It was noted in \([26]\) that Lagrangian for fermionic infinite spin obtained therein contains twice as many degrees of freedom as Metsaev’s Lagrangian \([12]\). Therefore we split Lagrangian obtained in \([26]\) in terms of Dirac spinors into two parts which are written in terms of Weyl spinors, each of which will describe the fermionic field of an infinite spin. Thus we consider the Lagrangian system of one dynamical complex scalar infinite spin field and one dynamical Weyl infinite spin field (both are the vectors of the Fock space in our formulation). On the basis of these fields, by adding some new additional fields needed for composing (anti)chiral superfields, we construct a superfield Lagrangian formulation for supersymmetric infinite spin theory. Our final result is the first completely superfield Lagrangian formulation for the \(4D, \mathcal{N} = 1\) infinite spin field theory.

\(^1\)Certainly, the constrained superfields or light-cone superfields are useful as well, although they can lead to violation of manifest symmetries.
The paper is organized as follows. In the next section, we recall the Lagrangian construction for fermionic and bosonic continuous spin fields constructed in terms of spin-tensor fields with dotted and undotted indices within the BRST approach suggested in [25, 26]. In section 3, we first consider the sum of Lagrangians of one complex bosonic field and one fermionic Weyl field which are the vectors of the Fock space and find for such system the supersymmetry transformations in component form. After this, we discuss construction of superfields from these component fields by adding some new auxiliary fields needed for composing (anti)chiral superfields. In section 4, by using the resulting superfields, we construct a superfield Lagrangian and present gauge transformation for the superfields. Then we show that after elimination of the redundant fields the superfield Lagrangian indeed reproduces the component Lagrangian given section 3. In Summary, we discuss the results obtained in the paper.

2 BRST Lagrangian formulation of infinite spin fields

The description of fields of infinite spin was presented in [25] for the case of integer spins and in [26] for half-integer spins. Here we summarize the results obtained there for the subsequent formulation of the supersymmetric generalization.

2.1 Generalized infinite spin Fock space

Physical states are defined in the generalized Fock space. For this reason, we introduce the operators

\[ a_\alpha, \quad b_\beta, \] (2.1)

which are Weyl spinors and satisfy the algebra

\[ [a_\alpha, b_\beta] = \delta_\beta^\alpha. \] (2.2)

Hermitian conjugation yield the operators

\[ \bar{a}_\dot{\alpha} = (a_\alpha)^\dagger, \quad \bar{b}_\dot{\beta} = (b_\beta)^\dagger, \] (2.3)

with the commutation relation

\[ [\bar{b}_\dot{\beta}, \bar{a}_\dot{\alpha}] = \delta_\dot{\beta}^\dot{\alpha}. \] (2.4)

Below we use the notation

\[ a_{\alpha(s)} := a_{\alpha_1} \ldots a_{\alpha_s}, \quad \bar{a}_{\dot{\alpha}(s)} := \bar{a}_{\dot{\alpha_1}} \ldots \bar{a}_{\dot{\alpha_s}}, \quad b^{\alpha(s)} := b^{\alpha_1} \ldots b^{\alpha_s}, \quad \bar{b}^{\dot{\beta}(s)} := \bar{b}^{\dot{\beta_1}} \ldots \bar{b}^{\dot{\beta_s}}. \] (2.5)

Following (2.2) and (2.3), we consider the operators \( a_\alpha \) and \( \bar{b}^{\dot{\beta}} \) as annihilation operators and define the “vacuum” state

\[ |0\rangle, \quad \langle 0| = (|0\rangle)^\dagger, \quad \langle 0|0\rangle = 1 \] (2.6)

by the relations

\[ a_\alpha |0\rangle = \bar{b}^{\dot{\beta}} |0\rangle = 0, \quad \langle 0|\bar{a}_{\dot{\alpha}} = \langle 0|b^{\alpha} = 0. \] (2.7)

The Fock space is formed by two types of vectors. Vectors of the first type have the form

\[ |\varphi\rangle = \sum_{s=0}^\infty |\varphi_{s}\rangle, \quad |\varphi_{s}\rangle := \frac{1}{s!} \varphi_{\alpha(s)}^{\beta(s)}(x) b^{\alpha(s)} \bar{a}_{\dot{\beta}(s)} |0\rangle. \] (2.8)
Then the conjugate vector to (2.8) is written as follows:

$$\langle \bar{\varphi} | = \sum_{s=0}^{\infty} \langle \bar{\varphi}_s | , \quad \langle \bar{\varphi}_s | := \frac{1}{s!} \langle 0 | b^{\dagger(s)} a_{\beta(s)} \bar{\varphi}^{\beta(s)}_{\dot{\alpha}(s)}(x).$$  \hfill (2.9)

Expansions (2.8) and (2.9) contain an equal number of operators with undotted and dotted indices. All component fields $\varphi_{\alpha(s)}^{\beta(s)}(x)$ and $\bar{\varphi}^{\beta(s)}_{\dot{\alpha}(s)}(x)$ have bosonic statistics, i.e. these fields are $c$-number.

The Fock space, which we consider, also contains the vectors with external Dirac index $A = 1, 2, 3, 4$:

$$| \Upsilon_A \rangle = \sum_{s=0}^{\infty} | \Upsilon_{A,s} \rangle , \quad | \Upsilon_{A,s} \rangle := \frac{1}{s!} \langle 0 | b^{\dagger(s)} a_{\alpha(s)} \bar{\Upsilon}^{\beta(s)}_{\dot{\beta}(s)}(x) \hfill (2.10)$$

The Dirac conjugate vector to (2.10) is written as follows:

$$\langle \bar{\Upsilon}^A | = (| \Upsilon_B \rangle )^\dagger (\gamma_0) B^A = \sum_{s=0}^{\infty} \langle \bar{\Upsilon}^A_s | , \quad \langle \bar{\Upsilon}^A_s | := \frac{1}{s!} \langle 0 | b^{\dagger(s)} a_{\beta(s)} \bar{\Upsilon}^{\alpha(s)}_{\dot{\alpha}(s)}(x).$$  \hfill (2.11)

Expansions (2.10) and (2.11) contain an equal number of operators with undotted and dotted indices, like expressions (2.8) and (2.9). However, the component fields $\Upsilon_{A\alpha(s)}^{\beta(s)}(x)$ and $\bar{\Upsilon}^{\alpha(s)}_{\dot{\beta}(s)}(x)$ are Grassmann variables and obey fermionic statistics.

It is natural that for the bosonic creation and annihilation operators the vectors (2.8), (2.9) are bosonic while the vectors (2.10), (2.11) with external Dirac index $A$ are fermionic.

We introduce the operators

$$l_0 := \partial^2 = \Box , \quad l_1 := i a^{\alpha} b^{\dot{\beta}} \partial_{\alpha \dot{\beta}} , \quad l_1^+ := i b^{\alpha} \bar{a}^{\dot{\beta}} \partial_{\alpha \dot{\beta}}$$  \hfill (2.12)

in our Fock space. The nonzero commutator of the above operators is

$$[l_1^+, l_1] = K l_0 ,$$  \hfill (2.13)

where

$$K := b^{\alpha} a_{\alpha} + \bar{a}_{\dot{\alpha}} \bar{b}^{\dot{\alpha}} + 2$$  \hfill (2.14)

is a number operator and all other commutators vanish: $[l_1^+, l_0] = 0 = [l_1, l_0]$.  

2.2 Infinite integer spin fields

In [25], it was shown that the BRST description of infinite integer spin representations is carried out by making use of the triplet of vectors

$$| \phi \rangle , \quad | \phi_1 \rangle , \quad | \phi_2 \rangle ,$$  \hfill (2.15)

each of which has the form (2.8).

The equations of motion for the states (2.15) are

$$\Box | \phi \rangle + (l_1^+ - \mu) | \phi_1 \rangle = 0 , \quad K | \phi_1 \rangle - (l_1 - \mu) | \phi \rangle + (l_1^+ - \mu) | \phi_2 \rangle = 0 , \quad \Box | \phi_2 \rangle + (l_1 - \mu) | \phi_1 \rangle = 0$$  \hfill (2.16)
and as it was shown in \[25\], these equations of motion follow from the Lagrangian

\[
\mathcal{L}_\phi = \langle \hat{\phi} \hat{\square} \phi \rangle - \langle \hat{\phi}_1 | K | \phi_1 \rangle - \langle \hat{\phi}_2 \hat{\square} | \phi_2 \rangle + \langle \hat{\phi}_1 | (l_1^+ - \mu) | \phi_1 \rangle + \langle \hat{\phi}_1 | (l_1^+ - \mu) | \phi \rangle - \langle \hat{\phi}_1 | (l_1^+ - \mu) | \phi_2 \rangle - \langle \hat{\phi}_2 | (l_1^+ - \mu) | \phi_1 \rangle.
\]

(2.17)

The dimensionful nonzero parameter $\mu$ in \[2.16\] and \[2.17\] defines the value of the four-order Casimir operators $W^2$ for the massless infinite spin representations of the Poincaré group \[13\]. The Lagrangian \[2.17\] is invariant under transformations

\[
\delta | \phi \rangle = (l_1^+ - \mu) | \lambda \rangle, \quad \delta | \phi_1 \rangle = - | \square | \lambda \rangle, \quad \delta | \phi_2 \rangle = (l_1 - \mu) | \lambda \rangle,
\]

(2.18)

where the local parameter $| \lambda \rangle$ has the form \[2.8\].

The Lagrangian $\mathcal{L}_\phi$ \[2.17\] for complex bosonic infinite spin fields $| \phi \rangle$, $| \phi_1 \rangle$, $| \phi_2 \rangle$ will be used bellow as a bosonic part of the supersymmetric Lagrangian.

### 2.3 Infinite half-integer spin fields

The infinite half-integer spin representation is described in the BRST approach by the triplet \[26\]

\[
| \Psi_A \rangle, \quad | \Psi_{1A} \rangle, \quad | \Psi_{2A} \rangle,
\]

where all vectors have the form \[2.10\]. Three independent equations of motion have the form

\[
i \hat{\theta} | \Psi \rangle + (l_1^+ - \mu) | \Psi_1 \rangle = 0,
K \hat{\theta} | \Psi_1 \rangle - (l_1 - \mu) | \Psi \rangle + (l_1^+ - \mu) | \Psi_2 \rangle = 0,
i \hat{\theta} | \Psi_2 \rangle + (l_1 - \mu) | \Psi_1 \rangle = 0.
\]

(2.20)

where $\hat{\theta}_A^B = \partial_m (\gamma^m)_A^B$. Equations \[2.20\] are Lagrangian equations which can be obtained from the following Lagrangian:

\[
\mathcal{L}_\Psi = \langle \bar{\Psi} | i \hat{\theta} | \Psi \rangle - \langle \bar{\Psi}_1 | K \hat{\theta} | \Psi_1 \rangle - \langle \bar{\Psi}_2 | i \hat{\theta} | \Psi_2 \rangle + \langle \bar{\Psi}_1 | (l_1^+ - \mu) | \Psi_1 \rangle + \langle \bar{\Psi}_1 | (l_1^+ - \mu) | \Psi \rangle - \langle \bar{\Psi}_2 | (l_1^+ - \mu) | \Psi_1 \rangle - \langle \bar{\Psi}_1 | (l_1^+ - \mu) | \Psi_2 \rangle.
\]

(2.21)

Lagrangian \[2.21\] is invariant under gauge transformations

\[
\delta | \Psi \rangle = (l_1^+ - \mu) | \Pi \rangle, \quad \delta | \Psi_1 \rangle = -i \hat{\theta} | \Pi \rangle, \quad \delta | \Psi_2 \rangle = (l_1 - \mu) | \Pi \rangle,
\]

(2.22)

where we omit the external Dirac index, i.e. the local parameter $| \Pi \rangle = | \Pi_A \rangle$ has additional index $A$ and is represented in the form \[2.10\].

In \[26\], it was noted that Lagrangian \[2.21\] contains twice as many degrees of freedom as Metsaev’s Lagrangian \[12\]. Now we split Lagrangian \[2.21\] into two parts, each of which will describe the fermionic field of a infinite spin. For this purpose we decompose the Dirac spinors \[2.19\] in terms of the sum of two Weyl spinors in the following form:

\[
| \Psi_A \rangle = \begin{pmatrix} | \psi_\alpha \rangle \\ | \bar{\chi}^\alpha \rangle \end{pmatrix}, \quad | \Psi_{1A} \rangle = \begin{pmatrix} | \chi_{1\alpha} \rangle \\ | \bar{\psi}_1^\alpha \rangle \end{pmatrix}, \quad | \Psi_{2A} \rangle = \begin{pmatrix} | \psi_{2\alpha} \rangle \\ | \bar{\chi}_2^\alpha \rangle \end{pmatrix},
\]

(2.23)

and

\[
\langle \bar{\Psi}_A \rangle = \langle \bar{\chi}^\alpha, \bar{\psi}_\alpha \rangle, \quad \langle \bar{\Psi}_1 \rangle = \langle \bar{\psi}_1^\alpha, \bar{\chi}_{1\alpha} \rangle, \quad \langle \bar{\Psi}_2 \rangle = \langle \bar{\chi}_2^\alpha, \bar{\psi}_{2\alpha} \rangle.
\]

(2.24)
where
\[
(\psi^\dagger_\alpha) = \langle \tilde{\psi}_\alpha |, \quad (\tilde{\psi}^\dagger_\alpha) = \langle \chi^\dagger_\alpha |,
\]
\[
(\psi^\dagger_1\alpha) = \langle \tilde{\psi}_{1\alpha} |, \quad (\tilde{\psi}^\dagger_1\alpha) = \langle \chi^\dagger_1 |,
\]
\[
(\psi^\dagger_2\alpha) = \langle \tilde{\psi}_{2\alpha} |, \quad (\tilde{\psi}^\dagger_2\alpha) = \langle \chi^\dagger_2 |.
\]

(2.25)

In terms of the Weyl spinors Lagrangian (2.21) takes the form
\[
\mathcal{L}_\psi = \mathcal{L}_\psi + \mathcal{L}_\chi
\]
where the chiral parts of $\mathcal{L}_\psi$ are
\[
\mathcal{L}_\psi = \langle \tilde{\psi}_\alpha | i\partial^\alpha \psi_\alpha \rangle - \langle \psi^\alpha_\dagger | K i\partial_{\alpha\dagger} \tilde{\psi}^\alpha_1 \rangle - \langle \tilde{\psi}_{2\alpha} | i\partial^\alpha \psi_{2\alpha} \rangle
\]
\[
+ \langle \tilde{\psi}_\alpha | (\psi^\dagger_\dagger + 1 - \mu) | \psi_\alpha \rangle + \langle \psi^\alpha_\dagger | (\psi^\dagger_\dagger + 1 - \mu) | \tilde{\psi}^\alpha_1 \rangle + \langle \psi^\alpha_\dagger | (\psi^\dagger_\dagger + 1 - \mu) | \psi_\alpha \rangle ,
\]
\[
\mathcal{L}_\chi = \langle \tilde{\chi}_\dagger \alpha | i\partial^\alpha \chi_\alpha \rangle - \langle \chi^\dagger_\dagger | K i\partial_{\alpha\dagger} \tilde{\chi}^\dagger_1 \rangle - \langle \tilde{\chi}_{2\alpha} | i\partial^\alpha \chi_{2\alpha} \rangle
\]
\[
- \langle \tilde{\chi}_\dagger \alpha | (\psi^\dagger_\dagger + 1 - \mu) | \chi_\alpha \rangle - \langle \chi^\dagger_\dagger | (\psi^\dagger_\dagger + 1 - \mu) | \tilde{\chi}^\dagger_1 \rangle - \langle \chi^\dagger_\dagger | (\psi^\dagger_\dagger + 1 - \mu) | \chi_\alpha \rangle .
\]

Gauge transformations (2.22) in terms of the Weyl spinors are
\[
\delta | \psi_\alpha \rangle = (\psi^\dagger_\dagger + 1 - \mu) | \rho_\alpha \rangle , \quad \delta | \tilde{\psi}^\dagger_\alpha \rangle = -i \partial^\alpha | \rho_\alpha \rangle , \quad \delta | \psi_{2\alpha} \rangle = (l_1 \mu) | \rho_\alpha \rangle ,
\]
\[
\delta | \chi_\alpha \rangle = (\psi^\dagger_\dagger + 1 - \mu) | \bar{\rho}_\alpha \rangle , \quad \delta | \tilde{\chi}^\dagger_\alpha \rangle = -i \partial_{\alpha\dagger} | \bar{\rho}_\alpha \rangle , \quad \delta | \chi_{2\alpha} \rangle = (l_1 \mu) | \bar{\rho}_\alpha \rangle ,
\]
where $| \lambda_\alpha \rangle$ and $| \bar{\rho}_\alpha \rangle$ are the Weyl components of the Dirac ket-vector of the gauge parameter:
\[
| \Pi_\alpha \rangle = \begin{pmatrix} | \lambda_\alpha \rangle \\ | \bar{\rho}_\alpha \rangle \end{pmatrix} .
\]

We see that the Lagrangian (2.21) and gauge transformations (2.22) split into two independent parts, each of which contains Weyl halves of the Dirac spinors. Each of the chiral parts $\mathcal{L}_\psi$ and $\mathcal{L}_\chi$ of the Lagrangian $\mathcal{L}_\psi$ describes the infinite spin multiplet with half of the fermion states of the infinite spin multiplet defined by the Lagrangian $\mathcal{L}_\psi$, which was proposed in [26].

In what follows, we will consider the chiral part $\mathcal{L}_\psi$ (2.21) as constituent part of a supersymmetric system.

### 3 Superfield description

Let us consider the system with the Lagrangian
\[
\mathcal{L} = \mathcal{L}_\phi - \frac{1}{2} \mathcal{L}_\psi ,
\]

This Lagrangian is invariant under the following supersymmetry variations
\[
\delta | \phi \rangle = \epsilon^\beta | \psi_\beta \rangle , \quad \delta \langle \tilde{\phi} | = \langle \psi_{\beta} | \bar{\epsilon}^\beta ,
\]
\[
\delta | \psi_\alpha \rangle = 2i \epsilon^\beta \partial_{\alpha\beta} | \phi \rangle , \quad \delta \langle \tilde{\psi}_\alpha | = -2i \partial_{\alpha\beta} \langle \tilde{\phi} | \epsilon^\beta ,
\]
\[
\delta | \phi_1 \rangle = i \epsilon^\beta \partial_{\alpha\beta} | \tilde{\psi}^\dagger_\beta \rangle , \quad \delta \langle \tilde{\phi}_1 | = -i \partial_{\alpha\beta} \langle \psi^\dagger_\beta | \bar{\epsilon}^\beta ,
\]
\[
\delta | \tilde{\psi}^\dagger_1 \rangle = 2 \epsilon^\alpha | \phi_1 \rangle , \quad \delta \langle \psi^\dagger_1 | = 2 \langle \tilde{\phi}_1 | \epsilon^\alpha ,
\]
\[
\delta | \phi_2 \rangle = \epsilon^\beta | \psi_{2\beta} \rangle , \quad \delta \langle \tilde{\phi}_2 | = \langle \psi_{2\beta} | \bar{\epsilon}^\beta ,
\]
\[
\delta | \psi_{2\alpha} \rangle = 2i \epsilon^\beta \partial_{\alpha\beta} | \phi_2 \rangle , \quad \delta \langle \tilde{\psi}_{2\alpha} | = -2i \partial_{\alpha\beta} \langle \tilde{\phi}_2 | \epsilon^\beta .
\]

(3.2)
One can show that the algebra of these transformations is not closed, which is typical for supersymmetry transformations without auxiliary fields (on-shell supersymmetry).

Now we turn attention that the supersymmetry transformations (3.2) allow one to treat the above vectors as chiral multiplets of $\mathcal{N} = 1$ supersymmetry. More precisely, the pairs of the ket-vectors
\[
\left( |\phi\rangle , |\psi_\alpha\rangle \right), \quad \left( |\phi_\beta\rangle , |\psi_{2\alpha}\rangle \right)
\]
form chiral multiplets, while the other pair of the ket-states forms antichiral multiplet
\[
\left( |\bar{\psi}\rangle , |\phi_1\rangle \right).
\]
Conjugated bra-states have opposite chirality: pairs of bra-states
\[
\left( \langle \bar{\phi} | , \langle \bar{\psi}_\alpha | \right), \quad \left( \langle \bar{\phi}_\beta | , \langle \bar{\psi}_{2\alpha} | \right)
\]
are antichiral while a pair of bra-states
\[
\left( \langle \psi_1^\alpha | , \langle \bar{\phi}_1 | \right)
\]
is chiral. For the off-shell supersymmetric description, we will add auxiliary components and construct the corresponding (anti)chiral superfields.

Off-shell $\mathcal{N} = 1$ supertranslations are realized in $\mathcal{N} = 1$ superspace with supercoordinates $(x^m, \theta^\alpha, \bar{\theta}^\dot{\alpha})$ as follows:
\[
\delta x^m = i \theta \sigma^m \epsilon - i \epsilon \sigma^m \bar{\theta}, \quad \delta \theta^\alpha = \epsilon^\alpha, \quad \delta \bar{\theta}^{\dot{\alpha}} = \bar{\epsilon}^{\dot{\alpha}}.
\]  

The corresponding supersymmetry generators and covariant derivatives have the form
\[
Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i \bar{\partial}_{\alpha \beta} \bar{\theta}^{\dot{\beta}}, \quad \bar{Q}_{\dot{\alpha}} = - \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i \theta^\alpha \partial_{\alpha \dot{\beta}},
\]
\[
D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i \bar{\partial}_{\alpha \beta} \bar{\theta}^{\dot{\beta}}, \quad \bar{D}_{\dot{\alpha}} = - \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i \theta^\alpha \partial_{\alpha \dot{\beta}},
\]

Now we add three additional fields
\[
|f(x_L)\rangle, \quad |e(x_R)\rangle, \quad |g(x_L)\rangle
\]
to three pairs of states (3.3), (3.4) to form three (anti)chiral superfield ket-vectors
\[
|\Phi(x_L, \theta)\rangle = |\phi(x_L)\rangle + \theta^\alpha |\psi_\alpha(x_L)\rangle + \theta^\alpha \theta_\alpha |f(x_L)\rangle,
\]
\[
|\bar{S}_1(x_R, \bar{\theta})\rangle = |\bar{e}(x_R)\rangle + \bar{\theta}_\dot{\alpha} |\bar{\psi}_\dot{\alpha}(x_R)\rangle + \bar{\theta}_\dot{\alpha} \bar{\theta}^{\dot{\alpha}} |\bar{\phi}_1(x_R)\rangle,
\]
\[
|S_2(x_L, \theta)\rangle = |\phi_\beta(x_L)\rangle + \theta^\alpha |\psi_{2\alpha}(x_L)\rangle + \theta^\alpha \theta_\alpha |g(x_L)\rangle,
\]
where
\[
x_L^m = x^m + i \theta^m \bar{\theta}, \quad x_R^m = x^m - i \theta^m \bar{\theta}.
\]

Conjugated states (3.5), (3.6) form three (anti)chiral superfield bra-vectors
\[
\langle \bar{\Phi}(x_R, \bar{\theta}) | = \langle \bar{e}(x_R) | + \bar{\psi}_\dot{\alpha}(x_R) |\bar{\theta}^{\dot{\alpha}} + \langle \bar{f}(x_R) |\bar{\theta}_\dot{\alpha} \bar{\theta}^{\dot{\alpha}},
\]
\[
\langle S_1(x_L, \theta) | = \langle \bar{e}(x_L) | + \psi_\alpha(x_L) |\theta_\alpha + \langle \bar{\phi}_1(x_L) |\theta^\alpha \theta_\alpha,
\]
\[
\langle \bar{S}_2(x_R, \bar{\theta}) | = \langle \bar{\phi}_2(x_R) | + \bar{\psi}_{2\alpha}(x_R) |\bar{\theta}^{\dot{\alpha}} + \langle \bar{g}(x_R) |\bar{\theta}_\dot{\alpha} \bar{\theta}^{\dot{\alpha}}.
\]
Taking into account the general supertranslation formula \( \delta F(x, \theta, \bar{\theta}) = (\epsilon^a Q_a + \bar{\epsilon}^a \bar{Q}^a) F \), the superfield representations (3.11) lead to the following variations of the component states:

\[
\begin{align*}
\delta |\phi\rangle &= \epsilon^\beta |\psi_\beta\rangle, \\
\delta |\psi_\alpha\rangle &= 2i\epsilon^\beta \partial_{\alpha\beta}|\phi\rangle + 2\epsilon_\alpha |f\rangle, \\
\delta |f\rangle &= i\epsilon^\beta \partial_{\alpha\beta}|\psi_\alpha\rangle, \\
\delta |e\rangle &= \epsilon^\beta |\bar{\psi}_1^{\dot{\alpha}}\rangle, \\
\delta |\bar{\psi}_1^{\dot{\alpha}}\rangle &= 2i\epsilon_{\dot{\beta}} \partial^{\dot{\alpha}\dot{\beta}}|e\rangle + 2\bar{\epsilon}^{\dot{\alpha}} |\phi_1\rangle, \\
\delta |\phi_1\rangle &= i\epsilon^\beta \partial_{\alpha\beta}|\bar{\psi}_1^{\dot{\alpha}}\rangle, \\
\delta |\phi_2\rangle &= i\epsilon^\beta |\psi_{2\beta}\rangle, \\
\delta |\psi_{2\beta}\rangle &= 2i\epsilon^\beta \partial_{\alpha\beta}|\phi_2\rangle + 2\epsilon_\alpha |g\rangle, \\
\delta |g\rangle &= i\epsilon^\beta \partial_{\alpha\beta}|\psi_{2\alpha}\rangle,
\end{align*}
\]

(3.14)

As it will be shown later, the field \(|e\rangle\) can be removed by means of its gauge transformation (4.4) and then the states \(|f\rangle, |g\rangle\) vanish due to their equations of motion: \(|f\rangle = 0, |g\rangle = 0\). After this, supersymmetry transformations (3.14), (3.15) pass to the on-shell invariance (3.2) of the Lagrangian (3.1).

4 Superfield Lagrangian

Let us consider the superfield Lagrangian

\[
\mathcal{L}_{\text{super}} = \int d^2\theta d^2\bar{\theta} \left( \langle \bar{\Phi}|\Phi\rangle - \langle S_1|K|\bar{S}_1\rangle - \langle \bar{S}_2|S_2\rangle \right)
\]

(4.1)

\[
+ \int d^2\theta \left( \langle S_1|(l_1 - \mu)|\Phi\rangle - \langle S_1|(l_1^+ - \mu)|S_2\rangle \right)
\]

\[
+ \int d^2\bar{\theta} \left( \langle \bar{\Phi}|(l_1^+ - \mu)|\bar{S}_1\rangle - \langle \bar{S}_2|(l_1 - \mu)|\bar{S}_1\rangle \right),
\]

where

\[
d^2\theta = -\frac{1}{4} d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta}, \quad d^2\bar{\theta} = -\frac{1}{4} d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}}, \quad \int \theta^\alpha \theta_{\dot{\alpha}} d^2\theta = 1, \quad \int \theta^\alpha \theta_{\dot{\alpha}} d^2\bar{\theta} = 1.
\]

The Lagrangian (4.1) is invariant under the gauge transformations

\[
\delta |\Phi\rangle = (l_1^+ - \mu)|\Lambda\rangle, \quad \delta |\bar{S}_1\rangle = \frac{1}{4} D^2|\Lambda\rangle, \quad \delta |S_2\rangle = (l_1 - \mu)|\Lambda\rangle,
\]

(4.2)

where \(|\Lambda\rangle\) is the chiral superfield

\[
|\Lambda(x_L, \theta)\rangle = |\lambda(x_L)\rangle + \theta^\alpha |\pi_\alpha(x_L)\rangle + \theta^\alpha \theta_{\dot{\alpha}} |\epsilon(x_L)\rangle.
\]

(4.3)

The gauge transformation (4.2) in the component form gives (2.18), (2.29) and for the additional fields

\[
\delta |f\rangle = (l_1^+ - \mu)|\epsilon\rangle, \quad \delta |e\rangle = -|\epsilon\rangle, \quad \delta |g\rangle = (l_1 - \mu)|\epsilon\rangle.
\]

(4.4)

After Grassmannian integration the Lagrangian (4.1) acquires the following component form

\[
\mathcal{L}_{\text{super}} = \mathcal{L}_\phi - \frac{1}{2} \mathcal{L}_\psi + \mathcal{L}_{\text{add}},
\]

(4.5)
where the first two terms \( \mathcal{L}_\phi - \frac{1}{2} \mathcal{L}_\psi \) give exactly the Lagrangian (3.1) while the last term has the form

\[
\mathcal{L}_{\text{add}} = \langle \bar{f} | f \rangle - \langle \bar{e} | K | e \rangle - \langle \bar{g} | g \rangle + \langle \bar{e} | (l_1 - \mu) | f \rangle + \langle \bar{f} | (l_1^+ - \mu) | e \rangle - \langle \bar{e} | (l_1^+ - \mu) | g \rangle - \langle \bar{g} | (l_1 - \mu) | e \rangle.
\]

We see that in (4.5) the Lagrangian \( \mathcal{L}_{\text{add}} \) for the additional fields is uncoupled from dynamical Lagrangian (3.1). Moreover, we can remove the state \( |e\rangle \) by the gauge-fixing condition \( |e\rangle = 0 \) for the gauge symmetry (4.4). Then the additional states \( |f\rangle, |g\rangle \) are eliminated with the help of their equations of motion \( |f\rangle = 0, |g\rangle = 0 \). As a result of this, we obtain Lagrangian (3.1) after eliminating some of the auxiliary and gauge states in the system described by Lagrangian (4.5), which is equivalent to the superfield Lagrangian (4.1).

Thus we have shown that Lagrangian (4.1) indeed is the superfield Lagrangian for the supersymmetric infinite spin system.

5 Summary and outlook

Let us summarize the results. We have presented the superfield formulation of the \( \mathcal{N} = 1 \) supersymmetric generalization of the 4D infinite spin theory. To construct a supersymmetric theory of infinite spin, we have used our Lagrangians for bosonic [25] and fermionic [26] infinite spin fields constructed in terms of spin-tensor fields with dotted and undotted indices on the basis of the BRST approach. At the same time, we showed that, in contrast to [26], one can consider a fermionic infinite spin multiplet whose component fields cooperate into a generating field with external Weyl index. This field of infinite spin describes half as many degrees of freedom as the Dirac field of infinite spin from [26]. First, we consider the sum of Lagrangians (2.17) and (2.27), respectively, for the triplet of complex bosonic fields and for the triplet of fermionic Weyl fields. Such content of the fields is due to the desire to compose a chiral superfield. Then we first find supersymmetry transformation (3.2) for such a sum of the Lagrangians in component form and then we add several auxiliary fields (3.10) and construct the superfield Lagrangian (4.1) for infinite spin in terms of three pairs chiral and anti-chiral superfields (3.11). The important point here is that we need supersymmetric generalization (4.2) of the BRST-like gauge invariance, so that it preserves the chirality, or antichirality, of the corresponding superfield states. Thus, in the present paper, we have obtained the superfield Lagrangian formulation for the 4D, \( \mathcal{N} = 1 \) supersymmetric infinite spin theory.

As a continuation of this research, it would be interesting to construct supersymmetric infinite spin systems with extended \( \mathcal{N} > 1 \) supersymmetry. In addition, within the framework of the BRST-like approach we plan to construct field descriptions of infinite spin systems in higher space-time dimensions. In particular, we plan to construct such a description in six dimensions based on our recent works [36,37].

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