Sketching and Neural Networks

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Abstract

High-dimensional sparse data present computational and statistical challenges for supervised
learning. We propose compact linear sketches for reducing the dimensionality of the input,
followed by a single layer neural network. We show that any sparse polynomial function can
be computed, on nearly all sparse binary vectors, by a single layer neural network that takes
a compact sketch of the vector as input. Consequently, when a set of sparse binary vectors is
approximately separable using a sparse polynomial, there exists a single-layer neural network
that takes a short sketch as input and correctly classifies nearly all the points. Previous work
has proposed using sketches to reduce dimensionality while preserving the hypothesis class.
However, the sketch size has an exponential dependence on the degree in the case of polynomial
classifiers. In stark contrast, our approach of using improper learning, using a larger hypothesis
class allows the sketch size to have a logarithmic dependence on the degree. Even in the linear
case, our approach allows us to improve on the pesky $O(1/\gamma^2)$ dependence of random projections.
We empirically show that our approach leads to more compact neural networks than related
methods such as feature hashing at equal or better performance.

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1 Introduction

In many supervised learning problems, input data are high-dimensional and sparse. The high dimensionality may be inherent in the task, for example a large vocabulary in a language model, or the result of creating hybrid conjunction features. Applying standard supervised learning techniques to such datasets poses statistical and computational challenges, as high dimensional inputs lead to models with a very large number of parameters. For example, a linear classifier for \(d\)-dimensional inputs has \(d\) weights and a linear multiclass predictor for \(d\)-dimensional vectors has \(d\) weights per class. In the case of a neural network with \(p\) nodes in the first hidden layer, we get \(dp\) parameters from this layer alone. Such large models can often lead to slow training and inference, and may require larger datasets to ensure low generalization error.

One way to reduce model size is to project the data into a lower-dimensional space prior to learning. Some of the proposed methods for reducing dimensionality are random projections, hashing, and principal component analysis (PCA). These methods typically attempt to project the data in a way that preserves the hypothesis class. While effective, there are inherent limitations to these approaches. For example, if the data consists of linearly separable unit vectors, Arriaga and Vempala [AV06] show that projecting data into \(O(1/\gamma^2)\) dimensions suffices to preserve linear separability, if the original data had a margin \(\gamma\). However, this may be too large for a small margin \(\gamma\). Our work is motivated by the question: could fewer dimensions suffice? Unfortunately, it can be shown that \(\Omega(1/\gamma^2)\) dimensions are needed in order to preserve linear separability, even if one can use arbitrary embeddings (see Section 8). It would appear therefore that the answer is a resounding no.

In this work, we show that using improper learning, allowing for a slightly larger hypothesis class allows us to get a positive answer. In the simplest case, we show that for linearly separable \(k\)-sparse inputs, one can create a \(O(k \log \delta)\)-dimensional sketch of each input and guarantee that a neural network with a single hidden layer taking the sketches as input, can correctly classify \(1-\delta\) fraction of the inputs. We show that a simple non-linear function can “decode” every binary feature of the input from the sketch. Moreover, this function can be implemented by applying a simple non-linearity to a linear function. In the case of 0-1 inputs, one can use a rectified linear unit (ReLU) that is commonly used in neural networks. Our results also extend to sparse polynomial functions.

In addition, we show that this can be achieved with sketching matrices that are sparse. Thus the sketch is very efficient to compute and does not increase the number of non-zero values in the input by much. This result is in contrast to the usual dense Gaussian projections. In fact, our sketches are simple linear projections. Using a non-linear “decoding” operation allows us to improve on previous work.

We present empirical evidence that our approach leads to more compact neural networks than existing methods such as feature hashing and Gaussian random projections. We leave open the question of how such sketches affect the difficulty of learning such networks.

In summary, our contributions are:

- We show that on sparse binary data, any linear function is computable on most inputs by a small single-layer neural network on a compact linear sketch of the data.
- We show that the same result holds also for sparse polynomial functions with only a logarithmic dependence on the degree of the polynomial, as opposed to the exponential dependence
of previous work. To our knowledge, this is the first technique that provably works with such compact sketches.

- We empirically demonstrate that on synthetic and real datasets, our approach leads to smaller models, and in many cases, better accuracy. The practical message stemming from our work is that a compact sketch built from multiple hash-sketches, in combination with a neural network, is sufficient to learn a small accurate model.

2 Related Work

Random Projections. Random Gaussian projections are by now a classical tool in dimensionality reduction. For general vectors, the Johnson-Lindenstrauss Lemma [JL84] implies that random Gaussian projection into \( O \left( \log(1/\delta) / \varepsilon^2 \right) \) dimensions preserves the inner product between a pair of unit vectors up to an additive factor \( \varepsilon \), with probability \( 1 - \delta \). A long line of work has sought sparser projection matrices with similar guarantees; see [Ach03, AC09, Mat08, DKS10, BOR10, KN14, CW13].

Sketching. Research in streaming and sketching algorithms has addressed related questions. Alon et al. [AMS99] showed a simple hashing based algorithm to get unbiased estimators for the Euclidean norm in the streaming setting. Charikar et al. [CCF04] showed an algorithm for the heavy-hitters problem based on the Count Sketch. Most relevant to our works is the Count-min sketch of Cormode and Muthukrishnan [CM05a, CM05b].

Projections in Learning. Random projections have been used in machine learning at least since the work of Arriaga and Vempala [AV06]. For fast estimation of a certain class of kernel functions, sampling has been proposed as a dimensionality reduction technique in [Kon07] and [RR07]. Shi et al. [SPD+09] propose using a count-min sketch to reduce dimensionality while approximately preserving inner products for sparse vectors. Weinberger et al. [WDA+09] use the count-sketch to get an unbiased estimator for the inner product of sparse vectors and prove strong concentration bounds. Previously, Ganchev and Dredze [GD08] had shown empirically that hashing is effective in reducing model size without significantly impacting performance. Hashing had also been used in Vowpal Wabbit [LLS07]. Talukdar and Cohen [TC14] also use the count-min sketch in graph-based semi-supervised learning. Pham and Pagh [PP13] showed that a count sketch of a tensor power of a vector could be quickly computed without explicitly computing the tensor power, and applied it to fast sketching for polynomial kernels.

Compressive Sensing. Our works is also related to the field of Compressive Sensing. For \( k \)-sparse vectors, results in this area, see e.g. [Don06, CT06], imply that a \( k \)-sparse vector \( x \in \mathbb{R}^d \) can be reconstructed, w.h.p. from a projection of dimension \( O(k \ln \frac{d}{k}) \). However, to our knowledge, all known decoding algorithms with these parameters involve sequential adaptive decisions, and are not implementable by a low depth neural network. Recent work by Mousavi et al. [MPB15] empirically explores using a deep network for decoding in compressive sensing and also considers learnt non-linear encodings to adapt to the distribution of inputs.
Parameter Reduction in Deep Learning. Our work can be viewed as a method for reducing the number of parameters in neural networks. Neural networks have become ubiquitous in many machine learning applications, including speech recognition, computer vision, and language processing tasks (see [HDY+12, KSH+12, SEZ+13, VTB+14] for a few notable examples). These successes have in part been enabled by recent advances in scaling up deep networks, leading to models with millions of parameters [DCM+12, KSH+12]. However, a drawback of such large models is that they are very slow to train, and difficult to deploy on mobile and embedded devices with memory and power constraints. Denil et al. [DSD+13] demonstrated significant redundancies in the parameterization of several deep learning architectures. They reduce the number of parameters by training low-rank decompositions of weight matrices. Cheng et al. [CYF+15] impose circulant matrix structure on fully connected layers. Ba and Caruana [BC14] train shallow networks to predict the log-outputs of a large deep network, and Hinton et al. [HVD15] train a small network to match smoothed predictions of a complex deep network or an ensemble of such models. Collines and Kohli [CK14] encourage zero-weight connections using sparsity-inducing priors, while others such as [LDS+89, HSW93, HPTD15] use techniques for pruning weights. HashedNets [CWT+15] enforce parameter sharing between random groups of network parameters. In contrast to these methods, sketching only involves applying a sparse, linear projection to the inputs, and does not require a specialized learning procedure or network architecture.

3 Notation

For a vector $x \in \mathbb{R}^d$, the support of $x$ is denoted $S(x) = \{i : x_i \neq 0\}$. The $p$-norm of $x$ is denoted $\|x\|_p = (\sum_i |x_i|^p)^{\frac{1}{p}}$ and $\|x\|_0 = |S(x)|$. We denote by

$$\text{head}_k(x) = \arg\min_y \|x - y\|_1$$

the closest vector to $x$ whose support size is $k$ and the residue $\text{tail}_k(x) = x - \text{head}_k(x)$. Let $B_{d,k}$ represent the set of all $k$-sparse binary vectors

$$B_{d,k} = \{x \in \{0, 1\}^d : \|x\|_0 \leq k\}.$$

Let $\mathbb{R}_{d,k,c}^+$ represent the set

$$\mathbb{R}_{d,k,c}^+ = \{x \in \mathbb{R}^d : \|\text{tail}_k(x)\|_1 \leq c\},$$

and let $\mathbb{R}_{d,k,c}$ represent the set

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We examine datasets where feature vectors are sparse and come from $B_{d,k}$. Alternatively, the feature can be “near-sparse” and come from $\mathbb{R}_{d,k,c}^+$ or $\mathbb{R}_{d,k,c}$, for some parameters $k, d, c$, with $k$ typically being much smaller than $d$. Let $\mathcal{H}_{d,s}$ denote the set

$$\mathcal{H}_{d,s} = \{w \in \mathbb{R}^d : \|w\|_0 \leq s\}.$$

We denote by $N_{n_1}(f)$ the family of feed-forward neural networks with one hidden layer containing $n_1$ nodes with $f$ as the non-linear function applied at each hidden unit, and a linear function at the output layer. Similarly, $N_{n_1,n_2}(f)$ designates neural networks with two hidden layers with $n_1$ and $n_2$ nodes respectively.
We will use the following family of randomized sketching algorithms based on the count-min sketch. Given a parameter $m$, and a hash function $h : [d] \rightarrow [m]$, the sketch $Sk_h(x)$ is a vector $y$ where

$$y_l = \sum_{i : h(i) = l} x_i.$$ 

We will use several such sub-sketches to get our eventual sketch. It will be notationally convenient to represent the sketch as a matrix with the sub-sketches as columns. Given an ordered set of hash functions $h_1, \ldots, h_t \triangleq h_1 : t$, the sketch $Sk_{h_1 : t}(x)$ is defined as a matrix $Y$, where the $j$'th column $y_j = Sk_{h_j}(x)$.

When $x \in B_{d,k}$, we will use a boolean version where the sum is replaced by an OR. Thus $BSk_h$ is a vector $y$ with

$$y_l = \bigvee_{i : h(i) = l} x_i.$$ 

The sketch $BSk_{h_1 : t}(x)$ is defined analogously as a matrix $Y$ with $y_j = BSk_{h_j}(x)$. Thus a sketch $BSk_{h_1 : t}(x)$ is a matrix $[y_1, \ldots, y_t]$ where the $j$'th column is $y_j \in \mathbb{R}^m$. We define the following decoding procedures:

$$\text{Dec}(y, i ; h) \triangleq y_{h(i)},$$

$$\text{DecMin}(Y, i ; h_1 : t) \triangleq \min_{j \in [t]} (\text{Dec}(y_j, i ; h_j)).$$

In the Boolean case, we can similarly define,

$$\text{DecAnd}(Y, i ; h_1 : t) \triangleq \bigwedge_{j \in [t]} (\text{Dec}(y_j, i ; h_j)).$$

When it is clear from context, we omit the hash functions $h_j$ from the arguments in $\text{Dec}$, $\text{DecMin}$, and $\text{DecAnd}$.

The following theorem summarizes the important property of these sketches. To remind the reader, a set of hash functions $h_1 : t$ from $[d]$ to $[m]$ is pairwise independent if for all $i \neq j$ and $a, b \in [m]$, $\Pr[h_i = a \land h_j = b] = \frac{1}{m^2}$. Such hash families can be easily constructed (see e.g. [MU05]), using $O(\log m)$ random bits. Moreover, each hash can be evaluated using $O(1)$ arithmetic operations over $O(\log d)$-sized words.

**Theorem 4.1.** Let $x \in B_{d,k}$ and for $j \in [t]$ let $h_j : [d] \rightarrow [m]$ be drawn uniformly and independently from a pairwise independent distribution with $m = ek$. Then for any $i$,

$$\Pr[\text{DecMin}(Sk_{h_1 : t}(x), i) \neq x_i] \leq e^{-t},$$

$$\Pr[\text{DecAnd}(BSk_{h_1 : t}(x), i) \neq x_i] \leq e^{-t}.$$ 

**Proof.** Fix a vector $x \in B_{d,k}$. For a specific $i$ and $h$, the decoding $\text{Dec}(y, i ; h)$ is equal to $y_{h(i)}$. Let us denote the set of collision indices of $i$ as

$$\mathcal{E}(i) \triangleq \{ i' \neq i : h(i') = h(i) \}.$$
Then, we can rewrite \( y_{h(i)} \) as \( x_i + \sum_{i' \in \mathcal{E}(i)} x_{i'} \). By pairwise independence of \( h \),

\[
E \left[ \sum_{i' \in \mathcal{E}(i)} x_{i'} \right] = \sum_{i' : x_{i'} = 1} \Pr[h(i') = h(i)] \leq \frac{k}{m} = \frac{1}{e},
\]

since the sum is over at most \( k \) terms, and each term is \( \frac{1}{m} \). Thus by Markov’s inequality, it follows that for any \( j \in [t] \),

\[
\Pr[\text{Dec}(S_{k_h}(\mathbf{x}), i; h_j) \neq x_i] \leq \frac{1}{e}.
\]

Moreover, it is easy to see that \( \text{DecMin}(S_{k_{h,j}}(\mathbf{x}), i) \) equals \( x_i \) unless for each \( j \in t \),

\[
\text{Dec}(S_{k_h}(\mathbf{x}), i; h_j) \neq x_i.
\]

Since the \( h_j \)'s are drawn independently, it follows that

\[
\Pr[\text{DecMin}(S_{k_{h,j}}(\mathbf{x}), i) \neq x_i] \leq e^{-t}.
\]

The argument for \( BSk \) is analogous, with \( + \) replaced by \( \lor \) and \( \text{min} \) replaced by \( \land \).

**Corollary 4.2.** Let \( \mathbf{w} \in \mathcal{H}_{d,s} \) and \( \mathbf{x} \in B_{d,k} \). For \( t = \log(s/\delta) \), and \( m = ek \), if \( h_1, \ldots, h_t \) are drawn uniformly and independently from a pairwise independent distribution, then

\[
\Pr \left[ \sum_{i} w_i \text{DecMin}(S_{k_{h_{1:}\mathbf{x}}}(\mathbf{x}), i) \neq \mathbf{w}^\top \mathbf{x} \right] \leq \delta,
\]

\[
\Pr \left[ \sum_{i} w_i \text{DecAnd}(BSk_{h_{1:}\mathbf{x}}(\mathbf{x}), i) \neq \mathbf{w}^\top \mathbf{x} \right] \leq \delta.
\]

In Appendix A we prove the following extensions of these results to vectors \( \mathbf{x} \) in \( \mathbb{R}^{+}_{d,k,c} \) or in \( \mathbb{R}_{d,k,c} \). Here \( \text{DecMed}(\cdot, \cdot) \) implements the median of the individual decodings.

**Corollary 4.3.** Let \( \mathbf{w} \in \mathcal{H}_{d,s} \) and \( \mathbf{x} \in \mathbb{R}^{+}_{d,k,c} \). For \( t = \log(s/\delta) \), and \( m = e(k + \frac{1}{\varepsilon}) \), if \( h_1, \ldots, h_t \) are drawn uniformly and independently from a pairwise independent distribution, then

\[
\Pr \left[ \sum_{i} w_i \text{DecMin}(S_{k_{h_{1:}\mathbf{x}}}(\mathbf{x}), i) - \mathbf{w}^\top \mathbf{x} \geq \varepsilon c \| \mathbf{w} \|_1 \right] \leq \delta
\]

**Corollary 4.4.** Let \( \mathbf{w} \in \mathcal{H}_{d,s} \) and \( \mathbf{x} \in \mathbb{R}_{d,k,c} \). For \( t = \log(s/\delta) \), and \( m = 4e^2(k + \frac{1}{\varepsilon}) \), if \( h_1, \ldots, h_t \) are drawn uniformly and independently from a pairwise independent distribution, then

\[
\Pr \left[ \sum_{i} w_i \text{DecMed}(S_{k_{h_{1:}\mathbf{x}}}(\mathbf{x}), i) - \mathbf{w}^\top \mathbf{x} \geq \varepsilon c \| \mathbf{w} \|_1 \right] \leq \delta
\]
Figure 1: Schematic description of neural-network sketching. The sparse vector $\mathbf{x}$ is sketched to a sketch using $t = 3$ hash functions, with $m = 8$. The shaded squares correspond to 1’s. This sketching step is random and not learned. The sketch is then used as an input to the single-layer neural network that is trained to predict $\mathbf{w}^\top \mathbf{x}$ for a sparse weight vector $\mathbf{w}$. Note that each node in the hidden layer of a candidate network corresponds to a coordinate in the support of $\mathbf{w}$. We have labelled two nodes “24” and “29” corresponding to decoding for $x_{24}$ and $x_{29}$ and shown the non-zero weight edges coming into them.

5 Sparse Linear Functions

Let $\mathbf{w} \in \mathcal{H}_{d,s}$ and $\mathbf{x} \in B_{d,k}$. As before we denote the sketch matrix by $Y = BSk_{h_1,\ldots,h_t}(\mathbf{x})$ for $m,t$ satisfying the conditions of Corollary 4.2. We will argue that there exists a network in $\mathcal{N}_s(\text{Relu})$ that takes $Y$ as input and outputs $\mathbf{w}^\top \mathbf{x}$ with high probability (over the randomness of the sketching process). Each hidden unit in our network corresponds to an index of a non-zero weight $i \in \mathcal{S}(\mathbf{w})$ and implements $\text{DecAnd}(Y, i)$. The top-layer weight for the hidden unit corresponding to $i$ is simply $w_i$.

It remains to show that the first layer of the neural network can implement $\text{DecAnd}(Y, i)$. Indeed, implementing the $\text{And}$ of $t$ bits can be done using nearly any non-linearity. For example, we can use a $\text{Relu}(a) = \max\{0, a\}$ with a fixed bias of $t - 1$: for any set $T$ and binary matrix $Y$ we have,

$$\bigwedge_{(l,j) \in T} Y_{lj} = \text{Relu} \left( \sum_{l,j} 1 \left[ (l,j) \in T \right] Y_{lj} - (|T| - 1) \right).$$

Using Corollary 4.2, we have the following theorem:

**Theorem 5.1.** For every $\mathbf{w} \in \mathcal{H}_{d,s}$ there exists a set of weights for a network $N \in \mathcal{N}_s(\text{Relu})$ such that for each $\mathbf{x} \in B_{d,k},$

$$Pr_{h_1,t}[N(BSk_{h_1,\ldots,h_t}(\mathbf{x})) = \mathbf{w}^\top \mathbf{x}] \geq 1 - \delta,$$
as long as \( m = ek \) and \( t = \log(s/\delta) \). Moreover, the weights coming into each node in the hidden layer are in \( \{0,1\} \) with at most \( t \) non-zeros.

The final property implies that when using \( w \) as a linear classifier, we get small generalization error as long as the number of examples is at least \( \Omega(s(1 + t \log mt)) \). This can be proved, e.g., using standard compression arguments: each such model can be represented using only \( st \log(mt) \) bits in addition to the representation size of \( w \). Similar bounds hold when we use \( \ell_1 \) bounds on the weight coming into each unit. Note that even for \( s = d \) (i.e. \( w \) is unrestricted), we get non-trivial input compression.

For comparison, we prove the following result for Gaussian projections in the appendix \[3\] In this case, the model weights in our construction are not sparse.

**Theorem 5.2.** For every \( w \in \mathcal{H}_{d,s} \) there exists a set of weights for a network \( N \in \mathcal{N}_s(\text{Relu}) \) such that for each \( x \in B_{d,k} \),

\[
Pr_{h_1 \ldots t}[N(Gx)] = w^\top x \geq 1 - \delta,
\]

as long as \( G \) is a random \( m \times d \) Gaussian matrix, with \( m \geq 4k \log(s/\delta) \).

To implement \( \text{DecMin}(\cdot) \), we need to use a slightly non-conventional non-linearity. For a weight vector \( z \) and input vector \( x \), a min gate implements \( \min_{i:z_i \neq 0} z_i x_i \). Then, using Corollary \[4.3\] we get the following theorem.

**Theorem 5.3.** For every \( w \in \mathcal{H}_{d,s} \) there exists a set of weights for a network \( N \in \mathcal{N}_s(\text{min}) \) such that for each \( x \in \mathbb{R}^{d,k,c}_{+} \) the following holds,

\[
Pr_{h_1 \ldots t}[|N(\text{Sk}_{h_1 \ldots t}(x)) - w^\top x| \geq \varepsilon \|w\|_1] \leq \delta.
\]

as long as \( m = e(k + \frac{1}{2}) \) and \( t = \log(s/\delta) \). Moreover, the weights coming into each node in the hidden layer are binary with \( t \) non-zeros.

For real vectors \( x \), the non-linearity needs to implement a median. Nonetheless, an analogous result still holds.

**6 Representing Polynomials**

In the boolean case, Theorem \[5.1\] extends immediately to polynomials. Suppose that the decoding \( \text{DecAnd}(B\text{Sk}_{h_1 \ldots t}(x), i) \) gives us \( x_i \) and similarly \( \text{DecAnd}(B\text{Sk}_{h_1 \ldots t}(x), j) \) gives us \( x_j \). Then \( x_i \land x_j \) is equal to the And of the two decodings. Since each decoding itself is an And of \( t \) locations in \( B\text{Sk}_{h_1 \ldots t}(x) \), the over all decoding for \( x_i \land x_j \) is an And of at most \( 2t \) locations in the sketch. More generally, for any set \( A \) of indices, the conjunction of the variables in the set \( A \) satisfies

\[
\bigwedge_{i \in A} x_i = \bigwedge_{(i,j) \in T_A} Y_{ij} \text{ where } T_A = \bigcup_{i \in A} T_i.
\]

Since an And can be implemented by a ReLU, the following result holds.
Theorem 6.1. Given \( w \in \mathbb{R}^s \), and sets \( A_1, \ldots, A_s \subseteq [d] \), let \( g : \{0,1\}^d \to \mathbb{R} \) denote the polynomial function

\[
g(x) = \sum_{j=1}^{s} w_j \prod_{i \in A_j} x_i = \sum_{j=1}^{s} w_j \bigwedge_{i \in A_j} x_i.
\]

Then there exists a set of weights for a network \( N \in \mathcal{N}_s(\text{Relu}) \) such that for each \( x \in B_{d,k} \),

\[
Pr_{h_{1,t}}[N(Sk_{h_{1,t}}(x)) = g(x)] \geq 1 - \delta,
\]
as long as \( m = ek \) and \( t = \log(|\cup_{j \in [s]} A_j|/\delta) \). Moreover, the weights coming into each node in the hidden layer are in \( \{0,1\} \) with at most \( t \cdot \left( \sum_{j \in [s]} |A_j| \right) \) non-zeroes overall. In particular, when \( g \) is a degree-\( p \) polynomial, we can set \( t = \log(ps/\delta) \), and each hidden unit has at most \( pt \) non-zero weights.

This is a setting where we get a significant advantage over proper learning. To our knowledge, there is no analog of this result for Gaussian projections. Classical sketching approaches would use a sketch of \( x \otimes p \), which is a \( k^s \)-sparse vector over binary vectors of dimension \( d^s \). Known sketching techniques such as \([PP13]\) would construct a sketch of size \( \Omega(k^s) \). Practical techniques such as Vowpal Wabbit also construct cross features by explicitly building them and have this exponential dependence. In stark contrast, neural networks allow us to get away with a logarithmic dependence on \( p \).

7 Deterministic Sketching

A natural question that arises is whether the parameters above can improved. We show that if we allow large scalars in the sketches, one can construct a deterministic \((2k + 1)\)-dimensional sketch from which a shallow network can reconstruct any monomial. We will also show a lower bound of \( k \) on the required dimensionality.

For every \( x \in B_{d,k} \) define a degree \( 2k \) univariate real polynomial by,

\[
p_x(z) = 1 - (k + 1) \prod_{\{i|x_i=1\}} (z - i)^2.
\]

It is easy to verify that this construction satisfies the following.

Claim 7.1. Suppose that \( x \in B_{d,k} \), and let \( p_x(\cdot) \) be defined as above. If \( x_j = 1 \), then \( p_x(j) = 1 \). If \( x_j = 0 \), then \( p_x(j) \leq -k \).

Let the coefficients of \( p_x(z) \) be \( a_i(x) \) so that

\[
p_x(z) \overset{d}{=} \sum_{i=0}^{2k} a_i(x) z^i.
\]

Define the deterministic sketch \( DS_{k_{d,k}} : B_{d,k} \to \mathbb{R}^{2k+1} \) as follows,

\[
DS_{k_{d,k}}(x) = (a_{x,0}, \ldots, a_{x,2k})
\]
For a non-empty subset $A \subset [d]$ and $y \in \mathbb{R}^{2k+1}$ define

$$DecPoly_{d,k}(y, A) = \frac{\sum_{j \in A} \sum_{i=0}^{2k} y_i j^i}{|A|}. \quad (2)$$

**Theorem 7.2.** For every $x \in B_{d,k}$ and a non-empty set $A \subset [d]$ we have

$$\prod_{j \in A} x_j = \text{Relu}(DecPoly_{d,k}(DSk_{d,k}(x), A)).$$

**Proof.** We have that

$$DecPoly_{d,k}(DSk_{d,k}(x), A) = \frac{\sum_{j \in A} \sum_{i=0}^{2k} a_{x,j} j^i}{|A|} = \frac{\sum_{j \in A} p_x(j)}{|A|}.$$

In words, the decoding is the average value of $p_x(j)$ over the indices $j \in A$. Now first suppose that $\prod_{j \in A} x_j = 1$. Then for each $j \in A$ we have $x_j = 1$ so that by Claim 7.1 $p_x(j) = 1$. Thus the average $DecPoly_{d,k}(DSk_{d,k}(x), A) = 1$.

On the other hand, if $\prod_{j \in A} x_j = 0$ then for some $j \in A$, say $j^*$, $x_j^* = 0$. In this case, Claim 7.1 implies that $p_x(j^*) \leq -k$. For every other $j$, $p_x(j) \leq 1$, and each $p_x(j)$ is non-negative only when $x_j = 1$, which happens for at most $k$ indices $j$. Thus the sum over non-negative $p_x(j)$ can be no larger than $k$. Adding $p_x(j^*)$ gives us zero, and any additional $j$’s can only further reduce the sum. Thus the average in non-positive and hence the Relu is zero, as claimed.

The last theorem shows that $B_{d,k}$ can be sketched in $\mathbb{R}^q$, where $q = 2k+1$, such that arbitrary products of variables be decoded by applying a linear function followed by a ReLU. It is natural to ask what is the smallest dimension $q$ for which such a sketch exists. The following theorem shows that $q$ must be at least $k$. In fact, this is true even if we only require to decode single variables.

**Theorem 7.3.** Let $Sk : B_{d,k} \rightarrow \mathbb{R}^q$ be a mapping such that for every $i \in [d]$ there is $w_i \in \mathbb{R}^q$ satisfying $x_i = \text{Relu}(\langle w_i, Sk(x) \rangle)$ for each $x \in B_{d,k}$, then $q$ is at least $k$.

**Proof.** Denote $X = \{w_1, \ldots, w_d\}$ and let $\mathcal{H} \subset \{0, 1\}^X$ be the function class consisting of all functions of the form $h_x(w_i) = \text{sign}(\langle w_i, Sk(x) \rangle)$ for $x \in B_{d,k}$. On one hand, $\mathcal{H}$ is a sub-class of the class of linear separators over $X \subset \mathbb{R}^d$, hence $\text{VC}(\mathcal{H}) \leq q$. On the other hand, we claim that $\text{VC}(\mathcal{H}) \geq k$ which establishes the proof. In order to prove the claim it suffices to show that the set $A = \{w_1, \ldots, w_k\}$ is shattered. Let $B \subset A$ let $x$ be the indicator vector of $B$. We claim that the restriction of $h_x$ to $A$ is the indicator function of $B$. Indeed, we have that,

$$h_x(w_i) = \text{sign}(\langle w_i, Sk(x) \rangle) = \text{sign}(\text{Relu}(\langle w_i, Sk(x) \rangle))) = x_i = 1[i \in B]$$
We would like to note that both the encoding and the decoding of deterministic sketched can be computed efficiently, and the dimension of the sketch is smaller than the dimension of a random sketch. We get the following corollaries.

**Corollary 7.4.** For every \( w \in \mathcal{H}_{d,s} \) there exists a set of weights for a network \( N \in \mathcal{N}_s(\text{Relu}) \) such that for each \( x \in B_{d,k} \), \( N(\text{DSk}_{d,k}(x)) = w^\top x \).

**Corollary 7.5.** Given \( w \in \mathbb{R}^s \), and sets \( A_1, \ldots, A_s \subseteq [d] \), let \( g : \{0,1\}^d \to \mathbb{R} \) denote the polynomial function

\[
g(x) = \sum_{j=1}^s w_j \prod_{i \in A_j} x_i = \sum_{j=1}^s w_j \bigwedge_{i \in A_j} x_i.
\]

For any such \( g \), there exists a set of weights for a network \( N \in \mathcal{N}_s(\text{Relu}) \) such that for each \( x \in B_{d,k} \), \( N(\text{DSk}_{d,k}(x)) = g(x) \).

Known lower bounds for compressed sensing \([BIPW10]\) imply that any linear sketch has size at least \( \Omega(k \log \frac{d}{k}) \) to allow stable recovery. We leave open the question of whether one can get the compactness and decoding properties of our (non-linear) sketch while ensuring stability.

### 8 Lower Bound for Proper Learning

We now show that if one does not expand the hypothesis class, then even in the simplest of settings of linear classifiers over 1-sparse vectors, the required dimensionality of the projection is much larger than the dimension needed for improper learning. The result is likely folklore and we present a short proof for completeness using concrete constants in the theorem and its proof below.

**Theorem 8.1.** Suppose that there exists a distribution over maps \( \phi : B_{d,1} \to \mathbb{R}^q \) and \( \psi : B_{d,s} \to \mathbb{R}^q \) such that for any \( x \in B_{d,1}, w \in B_{d,s} \),

\[
\Pr \left[ \text{sgn} \left( w^\top x - \frac{1}{2} \right) = \text{sgn} \left( \psi(w)^\top \phi(x) \right) \right] \geq \frac{9}{10},
\]

where the probability is taken over sampling \( \phi \) and \( \psi \) from the distribution. Then \( q \) is \( \Omega(s) \).

**Proof.** If the error is zero, a lower bound on \( q \) would follow from standard VC dimension arguments. Concretely, the hypothesis class consisting of \( h_w(x) = \{e_i^\top x : w_i = 1\} \) for all \( w \in B_{d,s} \) shatters the set \( \{e_1, \ldots, e_s\} \). If \( \text{sgn}(\psi(w)^\top \phi(e_i)) = \text{sgn}(w^\top x - \frac{1}{2}) \) for each \( w \) and \( e_i \), then the points \( \phi(e_i), i \in [s] \) are shattered by \( h_{\psi(w)}(\cdot) \) where \( w \in B_{d,s} \), which is a subclass of linear separators in \( \mathbb{R}^q \). Since linear separators in \( \mathbb{R}^q \) have VC dimension \( q \), the largest shattered set is no larger, and thus \( q \geq s \).

To handle errors, we will use the Sauer-Shelah lemma and show that the set \( \{\phi(e_i) : i \in [s]\} \) has many partitions. To do so, sample \( \phi, \psi \) from the distribution promised above and consider the set of points \( A = \{\phi(e_1), \phi(e_2), \ldots, \phi(e_s)\} \). Let \( W = \{w_1, w_2, \ldots, w_\kappa\} \) be a set of \( \kappa \) vectors in \( B_{d,s} \) such that,

(a) \( S(w_i) \subseteq [s] \)

(b) Distance property holds: \(|S(w_i) \triangle S(w_j)| \geq \frac{s}{4} \) for \( i \neq j \).
Let Theorem 9.1. The proof of the improvement is elementary and provided for completeness. Let $H = \{ h_{\psi(w)} : w \in W \}$ be the linear separators defined by $\psi(w_i)$ for $i \in W$. For brevity, we denote $h_{\psi(w_j)}$ by $h_j$. We will argue that $H$ induces many distinct subsets of $A$.

Let $A_j = \{ y \in A : h_j(x) = 1 \} = \{ x \in A : sgn(\psi(w_j)^\top x) = 1 \}$. Let $E_j \subseteq A$ be the positions where the embeddings $\phi, \psi$ fail, that is,

$$E_j = \{ \phi(e_i) : i \in [s], sgn(\psi(w_j)^\top e_i - \frac{1}{2}) \neq sgn(\psi(w_j)^\top \phi(e_i)) \}.$$ 

Thus $A_j = S(w_j) \triangle E_j$. By assumption, $E[|E_j|] \leq \frac{\delta}{10}$ for each $j$, where the expectation is taken over the choice of $\phi, \psi$. Thus $\sum_j E[|E_j|] \leq \delta \kappa/10$. Renumber the $h_j$’s in increasing order of $|E_j|$ so that $|E_1| \leq |E_2| \leq \ldots \leq |E_n|$. Due to the $E_j$’s being non-empty, not all $A_j$’s are necessarily distinct. Call a $j \in [\kappa]$ lost if $A_j = A_{j'}$ for some $j' \leq j$.

By definition, $A_j = S(w_j) \triangle E_j$. If $A_j = A_{j'}$, then the distance property implies that $E_j \triangle E_{j'} \geq \frac{\delta}{q}$. Since the $E_j$’s are increasing in size, it follows that for any lost $j$, $|E_j| \geq \frac{\kappa}{5}$. Thus in expectation at most $4\kappa/5$ of the $j$’s are lost. It follows that there is a choice of $\phi, \psi$ in the distribution for which $H$ induces $\kappa/5$ distinct subsets of $A$. Since the VC dimension of $H$ is at most $q$, the Sauer-Shelah lemma says that

$$\sum_{t \leq q} \binom{s}{t} \geq \frac{\kappa}{5} = \frac{2^{cs}}{5}.$$ 

This implies that $q \geq c's$ for some absolute constant $c'$.

Note that for the setting of the above example, once we scale the $w_j$’s to be unit vectors, the margin is $\Theta(\frac{1}{\sqrt{s}})$. Standard results then imply that projecting to $\frac{1}{\gamma}s = \Theta(s)$ dimension suffices, so the above bound is tight.

For this setting, Theorem 5.1 implies that a sketch of size $O(\log(s/\delta))$ suffices to correctly classify $1-\delta$ fraction of the examples if one allows improper learning as we do.

## 9 Neural Nets on Boolean Inputs

In this short section show that for boolean inputs (irrespective of sparsity), any polynomial with $s$ monomials can be represented by a neural network with one hidden layer of $s$ hidden units. Our result is a simple improvement of Barron’s theorem \cite{Bar93, Bar94}, for the special case of sparse polynomial functions on 0-1 vectors. In contrast, Barron’s theorem, which works for arbitrary inputs, would require a neural network of size $d \cdot s \cdot p^{O(p)}$ to learn an $s$-sparse degree-$p$ polynomial. The proof of the improvement is elementary and provided for completeness.

**Theorem 9.1.** Let $x \in \{0,1\}^d$, and let $g : \{0,1\}^d \to \mathbb{R}$ denote the polynomial function

$$g(x) = \sum_{j=1}^{s} w_j \prod_{i \in A_j} x_i = \sum_{j=1}^{s} w_j \bigwedge_{i \in A_j} x_i.$$

Then there exists a set of weights for a network $N \in \mathcal{N}(\text{Relu})$ such that for each $x \in \{0,1\}^d$, $N(x) = g(x)$. Moreover, the weights coming into each node in the hidden layer are in $\{0,1\}$. 


Proof. The $j$th hidden unit implements $h_j = \prod_{i \in A_j} x_i$. As before, for boolean inputs, one can compute $h_j$ as $\text{Relu}(\sum_{i \in A_j} x_i - |A_j| + 1)$. The output node computes $\sum_j w_j h_j$ where $h_j$ is the output of $j$th hidden unit.

10 Experiments with synthetic data

In this section, we evaluate sketches on synthetically generated datasets for the task of polynomial regression. In all the experiments here, we assume input dimension $d = 10^4$, input sparsity $k = 50$, hypothesis support $s = 300$, and $n = 2 \times 10^5$ examples. We assume that only a subset of features $I \subseteq [d]$ are relevant for the regression task, with $|I| = 50$. To generate an hypothesis, we select $s$ subsets of relevant features $A_1, \ldots, A_s \subset I$ each of cardinality at most 3, and generate the corresponding weight vector $w$ by drawing corresponding $s$ non-zero entries from the standard Gaussian distribution. We generate binary feature vectors $x \in B_{d,k}$ as a mixture of relevant and other features. Concretely, for each example we draw 12 feature indices uniformly at random from $I$, and the remaining indices from $[d]$. We generate target outputs as $g(x) + z$, where $g(x)$ is in the form of the polynomial given in Theorem 6.1, and $z$ is additive Gaussian noise with standard deviation 0.05. In all experiments, we train on 90% of the examples and evaluate the average squared error on the rest.

We first examine the effect of the sketching parameters $m$ and $t$ on the regression error. We generated sparse linear regression data using the above described settings, with all feature subsets in $\mathcal{A}$ having cardinality 1. We sketched the inputs using several values of $m$ (hash size) and $t$ (number of hash functions). We then trained neural networks in $\mathcal{N}_t(\text{Relu})$ on the regression task. The results are shown in Figure 2. As expected, increasing the number of hash functions $t$ leads to better performance. Using hash functions of size $m$ less than the input sparsity $k$ leads to poor results, while increasing hash size beyond $m = ek$ (in this case, $m = ek \approx 136$) for reasonable $t$ yields only modest improvements.

We next examine the effect of the decoding layer on regression performance. We generated 10
sparse linear regression datasets and 10 sparse polynomial regression datasets. The feature subsets in $\mathcal{A}$ had cardinality of 2 and 3. We then trained linear models and one-layer neural networks in $\mathcal{N}_t(\text{Relu})$ on original features and sketched features with $m = 200$ for several values for $t$. The results are shown in Figure 3. In the case of linear data, the neural network yields notably better performance than a linear model. This suggests that linear classifiers are not well-preserved after projections, as the $\Omega(1/\gamma^2)$ projection size required for linear separability can be large. Applying a neural network to sketched data allows us to use smaller projections.

In the case of polynomial regression, neural networks applied to sketches succeed in learning a small model and achieve significantly lower error than a network applied to the original features for $t \geq 6$. This suggests that reducing the input size, and consequently the number of model parameters, can lead to better generalization. The linear model is a bad fit, showing that our function $g(x)$ is not well-approximated by a linear function. Previous work on hashing and projections would imply using significantly larger sketches for this setting.

To conclude this section, we compared sketches to Gaussian random projections. We generated sparse linear and polynomial regression datasets with the same settings as before, and reduce the dimensionality of the inputs to 1000, 2000 and 3000 using Gaussian random projections and sketches with $t \in \{1, 2, 6\}$. We report the squared error averaged across examples and five datasets of one-layer neural networks in Table 1. The results demonstrate that sketches with $t > 1$ yield lower error than Gaussian projections. Note also that Gaussian projections are dense and hence much slower to train.

Figure 3: Effect of the decoding layer on performance on linear and polynomial regression data.
11 Experiments with language processing tasks

Linear and low degree sparse polynomials are often used for classification. Our results imply that if we have linear or a sparse polynomial with classification accuracy $1 - \varepsilon$ over some set of examples in $\mathcal{B}_{d,k} \times \{0, 1\}$, then neural networks constructed to compute the linear or polynomial function attain accuracy of at least $1 - \varepsilon - \delta$ over the same examples. Moreover, the number of parameters in the new network is relatively small by enforcing sparsity or $\ell_1$ bounds for the weights into the hidden layers. We thus get generalization bounds with negligible degradation with respect to non-sketched predictor. In this section, we evaluate sketches on the language processing classification tasks described below.

**Entity type tagging.** Entity type tagging is the task of assigning one or more labels (such as *person*, *location*, *organization*, *event*) to mentions of entities in text. We perform type tagging on a corpus of new documents containing 110K mentions annotated with 88 labels (on average, 1.7 labels per mention). Features for each mention include surrounding words, syntactic and lexical patterns, leading to a very large dictionary. Similarly to previous work, we map each string feature to a 32 bit integer, and then further reduce dimensionality using hashing or sketches. See [GLG+14] for more details on features and labels for this task.

**Reuters-news topic classification.** The Reuters RCV1 data set consists of a collection of approximately 800,000 text articles, each of which is assigned multiple labels. There are 4 high-level categories: Economics, Commerce, Medical, and Government (ECAT, CCAT, MCAT, GCAT), and multiple more specific categories. We focus on training binary classifiers for each of the four major categories. The input features we use are binary unigram features. Post word-stemming, we get data of approximately 113,000 dimensions. The feature vectors are very sparse, however, and most examples have fewer than 120 non-zero features.

**AG news topic classification.** We perform topic classification on 680K articles from AG news corpus labeled with one of 8 news categories: Business, Entertainment, Health, Sci/Tech, Sports, Europe, U.S., World. For each document, we extract binary word indicator features from the title and description; in total, there are 210K unique features, and on average, 23 non-zero features per document.

Table 1: Comparison of sketches and Gaussian random projections on the sparse linear regression task (top) and sparse polynomial regression task (bottom). See text for details.
Figure 4: F1 score vs. number of non-zero parameters in the first layer for entity type tagging. Each color corresponds to a different sketch size and markers indicate the number of subsketches $t$.

**Experimental setup.** In all experiments, we use two-layer feed-forward networks with ReLU activations and 100 hidden units in each layer. We use a softmax output for multiclass classification and multiple binary logistic outputs for multilabel tasks. We experimented with input sizes of 1000, 2000, 5000, and 10,000 and reduced the dimensionality of the original features using sketches with $t \in \{1, 2, 4, 6, 8, 10, 12, 14\}$ blocks. In addition, we experimented with networks trained on the original features. We encouraged parameter sparsity in the first layer using $\ell_1$-norm regularization and learn parameters using the proximal stochastic gradient method. As before, we trained on 90% of the examples and evaluated on the remaining 10%. We report accuracy values for multiclass classification, and F1 score for multilabel tasks, with true positive, false positive, and false negative counts accumulated across all labels.

**Results.** Since one motivation for our work is reducing the number of parameters in neural network models, we plot the performance metrics versus the number of non-zero parameters in the first layer of the network. The results are shown in Figures 4, 5, and 6 for different sketching configurations and settings of the $\ell_1$-norm regularization parameters ($\lambda_1$). On the entity type tagging task, we compared sketches to a single hash function of size 500,000 as the number of the original features is too large. In this case, sketching allows us to both improve performance and reduce the number of parameters. On the Reuters task, sketches achieve similar performance to the original features with fewer parameters. On AG news, sketching results in more compact models at a modest drop in accuracy. In almost all cases, multiple hash functions yield higher accuracy than a single hash function for similar model size.
12 Conclusions

We have presented a simple sketching algorithm for sparse boolean inputs, which succeeds in significantly reducing the dimensionality of inputs. A single-layer neural network on the sketch can provably model any sparse linear or polynomial function of the original input. For \( k \)-sparse vectors in \( \{0, 1\}^d \), our sketch of size \( O(k \log s/\delta) \) allows computing any \( s \)-sparse linear or polynomial function on a \( 1-\delta \) fraction of the inputs. The hidden constants are small, and our sketch is sparsity preserving. Previous work required sketches of size at least \( \Omega(s) \) in the linear case and size at least \( k^p \) for preserving degree-\( p \) polynomials.

Our results can be viewed as showing a compressed sensing scheme for 0-1 vectors, where the decoding algorithm is a depth-1 neural network. Our scheme requires \( O(k \log d) \) measurements, and we leave open the question of whether this can be improved to \( O(k \log \frac{d}{\delta}) \) in a stable way.

We demonstrated empirically that our sketches work well for both linear and polynomial regression, and that using a neural network does improve over a direct linear regression. We show that on real datasets, our methods lead to smaller models with similar or better accuracy for multiclass and multilabel classification problems. In addition, the compact sketches lead to fewer trainable parameters and faster training.

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Figure 6: AG news topic classification accuracy vs. number of non-zero parameters in the first layer. Each color corresponds to a different sketch size $t_m$, and markers indicate the number of subsketches $t$.

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The results of Section 4 extend naturally to positive and to general real-valued $x$.

**Theorem A.1.** Let $x \in \mathbb{R}^{d,k,c}_+$ and let $h_1, \ldots, h_t$ be drawn uniformly and independently from a pairwise independent distribution, for $m = e(k + \frac{1}{\varepsilon})$. Then for any $i$,

$$\Pr[DecMin(Sk_{h_1,\ldots,h_t}(x), i) \notin [x_i, x_i + \varepsilon c]] \leq \exp(-t).$$

**Proof.** Fix a vector $x \in \mathbb{R}^{d,k,c}_+$. To remind the reader, for a specific $i$ and $h$, $Dec(y, i; h) \triangleq y_{h(i)}$, and we defined the set of collision indices $E(i) \triangleq \{i' \neq i : h(i') = h(i)\}$. Therefore, $y_{h(i)}$ is equal to $x_i + \sum_{i' \in E(i)} x_{i'} \geq x_i$. We next show that

$$\Pr[Dec(Sk_h(x), i; h) > x_i + \varepsilon c] \leq 1/e.$$  

We can rewrite

$$Dec(Sk_h(x), i; h) - x_i = \sum_{i' \in S(head_k(x)) \cap E(i)} x_{i'} + \sum_{i' \in S(tail_k(x)) \cap E(i)} x_{i'}.$$  

By definition of $tail_k(\cdot)$, the expectation of the second term is $c/m$. Using Markov’s inequality, we get that

$$\Pr\left[\sum_{i' \in S(tail_k(x)) \cap E(i)} x_{i'} > \varepsilon c\right] \leq \frac{1}{\varepsilon m}.$$  

To bound the first term, note that

$$\Pr\left[\sum_{i' \in S(head_k(x)) \cap E(i)} x_{i'} \neq 0\right] \leq \Pr\left[\sum_{i' \in S(head_k(x)) \cap E(i)} 1 \neq 0\right] \leq E\left[\sum_{i' \in S(head_k(x)) \cap E(i)} 1\right] \leq \frac{k}{m}.$$  

Recall that $m = e(k + 1/\varepsilon)$, then, using the union bound establishes (3). The rest of the proof is identical to Theorem 4.1. \qed
**Corollary A.2.** Let \( w \in \mathcal{H}_{d,s} \) and \( x \in \mathbb{R}^+_{d,k,e} \). For \( t = \log(s/\delta) \), and \( m = e(k + \frac{1}{\varepsilon}) \), if \( h_1, \ldots, h_t \) are drawn uniformly and independently from a pairwise independent distribution, then

\[
\Pr \left[ \left| \sum_i w_i \cdot \text{DecMin}(Sk_{h_1:t}(x), i) - w^\top x \right| \geq \varepsilon \| w \|_1 \right] \leq \delta.
\]

The proof in the case that \( x \) may not be non-negative is only slightly more complicated. Let us define the following additional decoding procedure,

\[
\text{DecMed}(Y, i; h_{1:t}) \overset{def}{=} \text{Median Dec}(y_j, i; h_j).
\]

**Theorem A.3.** Let \( x \in \mathbb{R}^+_{d,k,e} \), and let \( h_1, \ldots, h_t \) be drawn uniformly and independently from a pairwise independent distribution, for \( m = 4e^2(k + 2/\varepsilon) \). Then for any \( i \),

\[
\Pr \left[ \text{DecMed}(Sk_{h_{1:t}}(x), i) \notin [x_i - \varepsilon c, x_i + \varepsilon c] \right] \leq e^{-t}.
\]

**Proof.** As before, fix a vector \( x \in \mathbb{R}^+_{d,k,e} \) and a specific \( i \) and \( h \). We once again write

\[
\text{Dec}(Sk_h(x), i; h) - x_i = \sum_{i' \in S(\text{head}_k(x)) \cap E(i)} x_{i'} + \sum_{i' \in S(\text{tail}_k(x)) \cap E(i)} x_{i'}.
\]

By the same argument as the proof of Theorem A.1, the first term is nonzero with probability \( k/m \). The second term has expectation in \([-c/s, c/s]\). Once again by Markov’s inequality,

\[
\Pr \left[ \sum_{i' \in S(\text{tail}_k(x)) \cap E(i)} x_{i'} > \varepsilon c \right] \leq \frac{1}{\varepsilon m} \quad \text{and} \quad \Pr \left[ \sum_{i' \in S(\text{tail}_k(x)) \cap E(i)} x_{i'} < -\varepsilon c \right] \leq \frac{1}{\varepsilon m}.
\]

Recalling that \( m = 4e^2(k + 2/\varepsilon) \), a union bound establishes that for any \( j \),

\[
\Pr \left[ |\text{Dec}(Sk_{h_j}(x), h_j, i) - x_i| > \varepsilon c \right] \leq \frac{1}{4e^2}.
\]

Let \( X_j \) be indicator for the event that \( |\text{Dec}(Sk_{h_j}(x), h_j, i) - x_i| > \varepsilon c \). Thus \( X_1, \ldots, X_t \) are binomial random variables with \( \Pr[X_j = 1] \leq \frac{1}{4e^2} \). Then by Chernoff’s bounds

\[
\Pr \left[ \text{DecMin}(Sk_{h_{1:t}}(x), i) \notin [x_i - \varepsilon c, x_i + \varepsilon c] \right] 
\leq \Pr \left[ \sum_j X_j > t/2 \right]
\leq \exp \left( - \left( \frac{1}{2} \ln \frac{1/2}{1/4e^2} + \frac{1}{2} \ln \frac{1/2}{1 - 1/4e^2} \right) t \right)
\leq \exp \left( - \left( \frac{1}{2} \ln 2e^2 + \frac{1}{2} \ln \frac{1}{2} \right) t \right)
\leq \exp(-t).
\]

\( \square \)

**Corollary A.4.** Let \( w \in \mathcal{H}_{d,s} \) and \( x \in \mathbb{R}^+_{d,k,e} \). For \( t = \log(s/\delta) \), and \( m = 4e^2(k + \frac{1}{\varepsilon}) \), if \( h_1, \ldots, h_t \) are drawn uniformly and independently from a pairwise independent distribution, then

\[
\Pr \left[ \sum_i w_i \cdot \text{DecMed}(Sk_{h_{1:t}}(x), i) - w^\top x \right] \geq \varepsilon \| w \|_1 \right] \leq \delta.
\]
B Gaussian Projection

In this section we describe and analyze a simple decoding algorithm for Gaussian Projections.

**Theorem B.1.** Let $x \in \mathbb{R}^d$, and let $G$ be a random Gaussian matrix in $\mathbb{R}^{d' \times d}$. Then for any $i$, there exists a linear function $f_i$ such that

$$E_G \left[ (f_i(Gx) - x_i)^2 \right] \leq \frac{\|x - x_ie_i\|_2^2}{d'}$$

**Proof.** Recall that $G \in \mathbb{R}^{d' \times d}$ is a random Gaussian matrix where each entry is chosen i.i.d. from $N(0, 1/d')$. For any $i$, conditioned on $G_{ji} = g_{ji}$, we have that the random variable $Y_j|G_{ji} = g_{ji}$ is distributed according to the following distribution,

$$(Y_j|G_{ji} = g_{ji}) \sim g_{ji}x_i + \sum_{i' \neq i} G_{ji'}x_{i'} \sim g_{ji}x_i + N(0, \|x - x_ie_i\|_2^2/d') .$$

Consider a linear estimator for $x_i$:

$$\hat{x}_i \overset{def}{=} \sum_j \alpha_{ji}(y_j/g_{ji}) / \sum_j \alpha_{ji} ,$$

for some non-negative $\alpha_{ji}$’s. It is easy to verify that for any vector of $\alpha$’s, the expectation of $\hat{x}_i$, when taken over the random choices of $G_{ji'}$ for $i' \neq i$, is $x_i$. Moreover, the variance of $\hat{x}_i$ is

$$\frac{\|x - x_ie_i\|_2^2}{d'} \sum_j (\alpha_{ji}/g_{ji})^2 / (\sum_j \alpha_{ji})^2 .$$

Minimizing the variance w.r.t $\alpha_{ji}$’s gives us $\alpha_{ji} \propto g_{ji}^2$. Indeed, the partial derivatives are,

$$\frac{\partial}{\partial \alpha_{ji}} \left( \sum_j (\alpha_{ji}/g_{ji})^2 - \lambda \sum_j \alpha_{ji} \right) = \frac{2\alpha_{ji}}{g_{ji}^2} - \lambda ,$$

which is zero at $\alpha_{ji} = \lambda g_{ji}^2/2$. This choice of $\alpha_{ji}$’s translates to

$$E[(\hat{x}_i - x_i)^2] = \frac{\|x - x_ie_i\|_2^2}{d'} / \sum_j g_{ji}^2 ,$$

which in expectation, now taken over the choices of $g_{ji}$’s, is at most $\|x - x_ie_i\|_2^2/d'$. Thus, the claim follows.

For comparison, if $x \in B_{d,k}$, then the expected error in estimating $x_i$ is $\sqrt{\frac{k-1}{d'}}$, so that taking $d' = (k - 1) \log \frac{1}{\delta} / \varepsilon^3$ suffices to get a error $\varepsilon$ estimate of any fixed bit with probability $1 - \delta$. Setting $\varepsilon = \frac{1}{2}$, we can recover $x_i$ with probability $1 - \delta$ for $x \in B_{d,k}$ with $d = 4(k - 1) \log \frac{1}{\delta}$. This implies Theorem 5.2. However, note that the decoding layer is now densely connected to the input layer. Moreover, for a $k$-sparse vectors $x$ that are is necessarily binary, the error grows with the 2-norm of the vector $x$, and can be arbitrarily larger than that for the sparse sketch. Note that $Gx$ still contains sufficient information to recover $x$ with error depending only on $\|x - head_k(x)\|_2$. To our knowledge, all the known decoders are adaptive algorithms, and we leave open the question of whether bounds depending on the 2-norm of the residual $(x - head_k(x))$ are achievable by neural networks of small depth and complexity.