Research on Maneuvering Trajectory Tracking Accuracy of Target Missile Based on Feedback Linearization

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Abstract. Aiming at the trajectory tracking accuracy of missile target, a trajectory tracking guidance law is designed based on feedback linearization theory and linear quadratic optimal control theory, and the tracking accuracy of the designed trajectory tracking guidance law under different maneuvering trajectories is tested. The motion equations of missiles are typical non-linear motion models, so feedback linearization is introduced as an exact linearization method, and the definition and necessary and sufficient conditions of feedback linearization theory are listed. Firstly, the non-linear equations of the longitudinal motion of target missile are established, and the feedback linearization method is used to linearize the non-linear model accurately. Then, the linearized model is designed by using the linear quadratic optimal control theory, and a target trajectory tracking guidance law is obtained. Finally, given gust disturbance and different maneuvering trajectories, the designed trajectory tracking guidance law is simulated, and the tracking accuracy of the guidance law is studied by changing the control coefficient. The results show that the trajectory tracking guidance law based on feedback linearization has excellent trajectory tracking ability, good tracking accuracy and strong anti-jamming ability under different maneuvering trajectories.

1. Introduction

With the rapid development of precision guided weapons, modern warfare puts forward higher requirements for the performance and technical indicators of air defense weapon systems. As a dynamic physical simulator for evaluating the operational effectiveness of air defense weapon system, missile target must have the ability to simulate the shape, size, trajectory and infrared characteristics of the incoming target. The core of the trajectory is to design the trajectory tracking guidance law for the missile target to fly in accordance with the predetermined trajectory [1-5]. Different from missile tracking guidance law for moving target, trajectory tracking guidance law requires missile target to track reference trajectory in a given time domain, which requires a high precision tracking guidance law.

Missile motion is a typical non-linear system. The trajectory tracking problem of missile can be converted into the control problem of non-linear system. At present, scholars at home and abroad have carried out a series of research on trajectory tracking guidance law, and achieved a lot of research results. Current trajectory tracking guidance law has two main design directions: one is to linearize the
non-linear model, and then to design trajectory tracking instructions by using linear control theory. The traditional idea of approximate linearization is first-order approximation, which is mainly applicable to the cases where the working point range is not large, such as least squares method, Taylor series expansion, Jacobian matrix and Fourier series. After linearizing the model, the guidance law is designed by using linear control theory. At present, many scholars design ballistic tracking guidance law based on this method. In reference [6], Zhang Dayuan linearizes the particle model with time and missile coordinates as independent variables for ballistic tracking of air defense missiles, and designs ballistic guidance laws to achieve ballistic tracking using LQR theory. In reference [7] [8], Qing Lu and Zhilei Ge introduced the theory of linear quadratic regulator into the solution of guidance law for determining standard trajectory, linearized the solidification coefficient method of missile model, and obtained guidance instructions satisfying mission requirements. Secondly, the trajectory tracking guidance law is designed directly by using non-linear control theory, such as sliding mode variable structure control, control, which is adopted by many scholars. In document [9], Yang Rongjun studied the three-dimensional sliding mode trajectory tracking control system with attitude angle as control command. In document [9], Zhang Dayuan linearized the model by the solidification coefficient method, and then designed the trajectory tracking guidance law by using sliding mode variable structure control theory. The guidance law obtained has good robustness.

In order to improve the flight performance of target missile, the flight trajectory of missile is optimized by using modern control theory according to the given technical index in order to obtain the benchmark optimal trajectory. Because the traditional model linearization neglects the high order term in the operation, the accumulative error makes the model precision decrease greatly, so the accuracy of linearization is not high. The system using sliding mode variable structure control also has the situation of inaccurate model linearization. Although it has the characteristics of fast response, due to the existence of chattering problem, the requirement of the control system is very high. It is not conducive to the realization of the project. Therefore, in order to improve the accuracy of the linearized model, an exact linearization method, feedback linearization method, is introduced. The linearized model uses the optimal control theory widely used in engineering to design the trajectory tracking guidance instructions. In the literature [10-13], the feedback linearization method is used to design the missile guidance law, and the feedback linearization theory is introduced into the trajectory tracking guidance law. Law provides a theoretical basis.

In this paper, based on the application background of anti-tank missile to guided missile target, the longitudinal channel motion model of missile is established, the state equation of missile motion is listed, the feasibility is analyzed by using the criterion of feedback linearization theory, the accurate linearization of the non-linear model is made, the trajectory tracking guidance law is designed by using linear quadratic optimal control, and finally, under the condition of external wind disturbance, the trajectory tracking guidance law is designed. Simulink is used to simulate the obtained guidance law, and compared with the small perturbation method and the solidification coefficient method.

2. Missile motion equations
The motion of missile can be decoupled into longitudinal motion and lateral motion by decoupling. In order to simplify the discussion of trajectory tracking guidance law, the motion of missile in longitudinal channel can be considered separately. Because of the effect of gravity, under the same trajectory, the required overload of longitudinal channel is larger than that of lateral motion, so the study of longitudinal channel is more representative. Fig. 1 gives the force analysis of missile in two-dimensional inertial coordinate system. In the figure, G is the gravity of the missile, X is the air resistance of the missile, Y is the lift of the missile, V is the speed of the missile, $\alpha$ is the angle of attack of the missile, $\theta$ is the trajectory inclination of the missile.
Then the equations of missile longitudinal motion are:

\[
\begin{align*}
\frac{dv}{dt} &= P \cos \alpha - X - mg \sin \theta \\
\frac{d\theta}{dt} &= P \sin \alpha + Y - mg \cos \theta \\
\frac{dx}{dt} &= v \cos \theta \\
\frac{dy}{dt} &= v \sin \theta \\
\frac{dm}{dt} &= -m,
\end{align*}
\]

(1)

In the formula: \( m \) is missile mass, \( m_s \) is missile unit time flow, \( v \) is missile flight speed, \( P \) is engine axial thrust, \( \alpha \) is missile angle of attack, \( \theta \) is missile trajectory inclination, \( g \) is gravity acceleration, \( X \) and \( Y \) are resistance and lift of missile respectively, where:

\[
\begin{align*}
X &= C_x q S \\
Y &= C_y \alpha q S
\end{align*}
\]

(2)

In the formula, \( q \) is the dynamic pressure, \( S \) is the cross-sectional area of the projectile body. \( C_x \) and \( C_y \) can be obtained by looking up the angle of attack \( \alpha \) and Mach number \( Ma \) as shown in Figure 2 and Figure 3.

**Figure 1.** Force analysis of missile longitudinal channel.

**Figure 2.** Missile aerodynamic parameters \( C_x \).

**Figure 3.** Missile aerodynamic parameters \( C_y \).
In the trajectory design of target missile’s flight plan, because the thrust action stage of the engine is very short and uncontrollable, the flight speed of the missile in the middle guidance stage will not change much. At the same time, in order to simulate the flight trajectory of different aircraft, the height of the missile needs to be changed accordingly. From \( \dot{y} = v \sin \theta \), it can be seen that the change of the trajectory inclination angle \( \theta \) causes the change of the height, so the trajectory inclination angle \( \theta \) and the height of the missile \( h \) are taken as the system state variables. In order to ensure the stability of missile attitude, the absolute value of limiting missile angle of attack is less than \( 20^\circ \). In addition, because the mass change of missile is only related to engine, the fifth formula in formula (1) can be deleted to obtain the missile motion equation:

\[
\begin{bmatrix}
\dot{y} \\
\dot{\theta}
\end{bmatrix} = \begin{bmatrix}
v \sin \theta \\ -\frac{g \cos \theta}{v}
\end{bmatrix} + \begin{bmatrix}
0 \\ \frac{P + C_{p}^a qS}{mv}
\end{bmatrix} \alpha
\]  

(3)

In order to track the trajectory of the scheduled scheme, the deviation of Missile Altitude and trajectory inclination is taken as the state variable.

\[
x = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
y - y_b \\ \theta - \theta_b
\end{bmatrix}
\]  

(4)

The derivation of formula (4) and the substitution of formula (3) are obtained as follows:

\[
\begin{aligned}
\dot{x} &= \begin{bmatrix}
\dot{y} - \dot{y}_b \\
\dot{\theta} - \dot{\theta}_b
\end{bmatrix} = \begin{bmatrix}
a_1 \sin \theta - a_1 \sin \theta_b \\ a_2 \cos \theta - a_2 \cos \theta_b
\end{bmatrix} + \begin{bmatrix}
0 \\ a_3
\end{bmatrix} \Delta \alpha \\
a_1 &= v_b \\
a_2 &= -\frac{g}{v_b} \\
a_3 &= \frac{P + C_{p}^a qS}{mv_b}
\end{aligned}
\]  

(5)

In the formula (5), the ballistic inclination of the reference trajectory at the corresponding time is simplified by the sum-difference product formula with the change of the angle of attack of the missile as the control variable, and the following non-linear model for guidance law design is obtained:

\[
\dot{x} = f(x) + g(x)u
\]

\[
f(x) = \begin{bmatrix}
2a_1 \cos(x_2 + \theta_b) \sin \frac{x_2}{2} \\
-2a_2 \sin(x_2^2 + \theta_b) \sin \frac{x_2}{2}
\end{bmatrix}
\]

\[
g(x) = \begin{bmatrix}
0 \\
a_3
\end{bmatrix}
\]

(6)

3. Feedback linearization theory

Feedback linearization control refers to the application of state feedback to a non-linear system to make the closed-loop system linear, or if the closed-loop system is still non-linear, a local coordinate system \( (\phi, u) \) or non-abnormal state transformation \( z = \phi(u) \) defined on \( U \) can still be found, so that the non-linear system has the form of a linear control system in the new coordinates.
3.1. Definition and judgment theorem of feedback linearization

Feedback linearization is a method of precise linearization of nonlinear systems. It eliminates the nonlinear part of the system through state feedback and obtains a linear state equation. Its basic principles are as follows:

Nonlinear system

\[ \dot{x} = f(x) + g(x)u \]  

The problem of exact linearization of state space is defined as: given a point \( \dot{x}^* \), if possible, finding a neighborhood \( U \) of \( \dot{x}^* \) and a feedback defined on \( U \).

\[ u = \alpha(x) + \beta(x)v \]

And coordinate transformation defined above to make the corresponding closed-loop system

\[ \dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x)v \]

It is linear and controllable. That is to say, to satisfy the conditions

\[ \text{rank}(B, AB, \ldots A^{n-1}B) = n \]

A suitable matrix \( A \in \mathbb{R}^n \times \mathbb{R}^n \) and vector \( B \in \mathbb{R}^n \) of

\[ \left[ \frac{\partial \Phi}{\partial x} (f(x) + g(x)\alpha(x)) \right]_{x=x^*} = Az \]

\[ \left[ \frac{\partial \phi}{\partial x} g(x)\beta(x) \right]_{x=x^*} = B \]

two formulas are established.

For formula (7), \( x = (x_1, x_2, \ldots, x_n)^T \in D \quad (D \in \mathbb{R}^n) \), \( f(x) \) and \( g(x) \) are value maps defined on \( D \) open sets \( \mathbb{R}^n \), and \( u \) are control variables. Mapping \( f: D \rightarrow \mathbb{R}^n \) is called a vector field defined on \( D \) and can be expressed as column vectors:

\[ f(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T \]

Let real-valued function \( h: D \rightarrow \mathbb{R} \) and a vector field \( f: D \rightarrow \mathbb{R}^n \), the Lie derivative of \( h \) along \( f \) be defined as

\[ L_f h(x) = \frac{\partial h}{\partial x} f(x) = \sum_{i=1}^n \frac{\partial h}{\partial x_i} f_i(x) \]

For two vector fields in a non-linear equation \( f: D \rightarrow \mathbb{R}^n \) and \( g: D \rightarrow \mathbb{R}^n \), a new vector field \([f, g]\) is constructed:

\[ [f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) \]

Lie brackets for \( f \) and \( g \), where \( \frac{\partial f}{\partial x} \) and \( \frac{\partial g}{\partial x} \) are Jacobi matrices for \( f \) and \( g \), and higher-order Lie brackets are marked with the following

\[ \text{ad}^0_j g(x) = g(x) \]

\[ \text{ad}^k_j g(x) = [f, \text{ad}^{k-1}_j g](x) \]

In formula: \( k \) is the order of Lie product, and \( k=0 \) is the zero order Lie product.
The nonlinear system shown in equation (7) can be written directly as follows:

$$\dot{x} = Ax + B\gamma^{-1}(x)(u - s(x))$$  \hspace{1cm} (17)

Formula $A$ is $n\times n$ matrix, $B$ is $n \times p$ matrix, $s(x): \mathbb{R}^n \rightarrow \mathbb{R}^p$ and $\gamma(x): \mathbb{R}^n \rightarrow \mathbb{R}^{n \times p}$ are defined above $D$, and $\gamma(x)$ is not singular for all $x \in D$.

Or the coordinate transformation $z = \phi(x)$ causes Formula (7) to have the following form under $z$:

$$\begin{align*}
\dot{z} &= Az + B\gamma^{-1}(x)(u - s(x)) \\
x &= \Phi^{-1}(z)
\end{align*}$$  \hspace{1cm} (18)

The non-linear system shown in formula (7) can be linearized into a linear system by coordinate transformation and state feedback linearization by taking control variables:

$$u = s(x) + \gamma(x)V$$  \hspace{1cm} (19)

Substituting Formula (19) into Formula (18) obtains:

$$\dot{z} = Az + BV$$  \hspace{1cm} (20)

Formula $V$ is a feedback control vector, which can be designed by using linear system control theory. According to [13], the necessary and sufficient conditions for linearization of input states of a nonlinear system shown in Formula (7) are as follows:

1. For all $x = (x_1, x_2, \cdots, x_n)^T \in D$ ($D \subset \mathbb{R}^n$), Matrix $G(x) = (g(x) \text{ad}_{f}^1 g(x) \cdots \text{ad}_{f}^{n-1} g(x))^T$ full rank;
2. The distribution shown below is coincident: $\Delta = \text{span}\{g(x) \text{ad}_{f}^1 g(x) \text{ad}_{f}^2 g(x) \cdots \text{ad}_{f}^{n-2} g(x)\}$

3.2. Solution of Conversion Matrix $\Phi(x)$

For the non-linear system shown in equation (7), suppose $\exists z = \Phi(x)$ that:

$$\dot{z} = \frac{\partial \Phi}{\partial x} \dot{x} = \frac{\partial \Phi}{\partial x} (f(x) + g(x)u)$$  \hspace{1cm} (21)

Formula (21) and Formula (18) are equal on both sides of $\dot{z}$.

$$\begin{align*}
\frac{\partial \Phi}{\partial x} f(x) &= A\Phi(x) - B\gamma^{-1}(x)s(x) \\
\frac{\partial \Phi}{\partial x} g(x) &= B\gamma^{-1}(x)
\end{align*}$$  \hspace{1cm} (22)

The above processes are reversible, so formula (22) is a necessary and sufficient condition for $z = \Phi(x)$ to satisfy formula (21). The transformation matrix $z = \Phi(x)$ is not unique. The transformation matrix $A$ and $B$ can be taken as Brunovsky controllable standard form, let

$$z = \Phi(x) = \left(\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)\right)^T$$  \hspace{1cm} (23)

Formula $\Phi_i(x)$ is a $1 \times n$ dimensional row vector, then Formula (22) can be converted equivalently to
4. Design of ballistic tracking guidance law
Next, the feedback linearization theory is used to linearize the non-linear model (6), design the trajectory tracking guidance law, and use the linear quadratic optimal control theory to calculate the control quantity of the system.

4.1. Definition and judgment theorem of feedback linearization
From the formulas (6), (15) and (16)
\[
\text{ad}_f^1 g(x) = [f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) = \begin{bmatrix} a_i a_s \cos(x_2 + \theta_b) \\ -a_i a_s \sin(x_2 + \theta_b) \end{bmatrix}
\]
(25)
Therefore,
\[
G(x) = (g(x) \text{ ad}_f^1 g(x))^T = \begin{bmatrix} 0 \\ a_3 a_i \cos(x_2 + \theta_b) \\ -a_3 a_i \sin(x_2 + \theta_b) \end{bmatrix}
\]
(26)
Formula (26) shows that as long as \( a_3 = \frac{P + C_s^a S}{m v_b} \neq 0 \), \( G(x) \) has full rank. Because the system dimension \( n = 2 \) and the distribution \( \Delta = \text{span} \{ f(x) \} \) is coincident, the system shown in equation (6) satisfies the condition of feedback linearization.

4.2. Solution transformation \( z = \Phi(x) \)
According to the formula in Section 3.2, formula (6) is substituted for formula (24) to obtain:
\[
\begin{align*}
\frac{\partial \Phi_1(x)}{\partial x} f(x) &= \Phi_2(x) \\
\frac{\partial \Phi_1(x)}{\partial x} g(x) &= 0 \\
\frac{\partial \Phi_2(x)}{\partial x} f(x) &= -\gamma^{-1}(x)s(x) \\
\frac{\partial \Phi_2(x)}{\partial x} g(x) &= \gamma^{-1}(x)
\end{align*}
\]
(27)
Among them: \( z = \Phi(x) = (\Phi_1(x) \quad \Phi_2(x))^T \).
Substitute $g(x) = \begin{bmatrix} 0 \\ a_3 \end{bmatrix}$ into formula (21) and get 
\[
\frac{\partial \Phi_i(x)}{\partial x} g(x) = \frac{\partial \Phi_i(x)}{\partial x_1} x_1 + \frac{\partial \Phi_i(x)}{\partial x_2} a_3 = 0 ,
\]
because $a_3 \neq 0$, in order to satisfy this formula, then 
\[
\frac{\partial \Phi_i(x)}{\partial x_2} = 0 ,
\]
that is $\Phi_i(x)$ does not contain the component of $x_2$. Then from Formula (27) Formula 1, the $\Phi_2(x)$ expression is
\[
\Phi_2(x) = 2a_1 \frac{\partial \Phi_i(x)}{\partial x_1} \cos(x_2 + \theta_0) \sin \frac{x_2}{2}
\]
(28)

By substituting Formula (28) for Formula (21) and Formula 4, the following results can be obtained:
\[
-\gamma^{-1}(x) = a_1 a_3 \frac{\partial \Phi_i(x)}{\partial x_1} \cos(x_2 + \theta_0) \neq 0
\]
(29)

Therefore, $\frac{\partial \Phi_i(x)}{\partial x_1} \neq 0$, let's suppose that $\Phi_i(x) = x_i$ is substituted for formula (28) and formula (29) respectively.
\[
\begin{cases}
\Phi_2(x) = 2a_1 \cos(x_2 + \theta_0) \sin \frac{x_2}{2} \\
\gamma^{-1}(x) = -a_1 a_3 \cos(x_2 + \theta_0)
\end{cases}
\]
(30)

From formula (30), we know that the Lie derivative of $\Phi_2(x)$ to $f$ is
\[
\frac{\partial \Phi_2(x)}{\partial x} f(x) = 2a_1 a_3 \cos(x_2 + \theta_0) \sin \frac{x_2}{2} \sin \frac{x_2}{2}
\]
(31)

By substituting Formula (30) and Formula (31) into Formula (27), the following results can be obtained:
\[
s(x) = \frac{2a_3}{a_1} \sin(x_2 + \theta_0) \sin \frac{x_2}{2}
\]
(32)

By substituting formula (30) and formula (32) into formula (19), the expression of $u$ is obtained as follows
\[
u = \frac{2a_3}{a_1} \sin(x_2 + \theta_0) \sin \frac{x_2}{2} + \frac{1}{a_1 a_3 \cos(x_2 + \theta_0)} V
\]
(33)

Then the linearized system is
\[
\begin{cases}
\dot{z} = Az + BV \\
z = \begin{bmatrix} x_1 \\ 2a_1 \cos(x_2 + \theta_0) \sin \frac{x_2}{2} \end{bmatrix}
\end{cases}
\]
(34)

4.3. Design of guidance instructions using linear quadratic optimal control
Because the target feeding section of target missile is the mid-guidance section and there is no need to restrict the height and trajectory inclination of target missile at the end, the linear system shown in Formula (34) can be regarded as a constant system with infinite terminal time
Using the linear quadratic optimal control design, the quadratic performance index is taken as follows.

\[ J = \frac{1}{2} \int_{t_0}^{t_f} (z^TQz + V^TRV)dt \]  \hspace{1cm} (35)

Formula \( Q \) and \( R \) are weight matrices, and \( Q \) semi-positive definite, \( R \) positive definite. According to the theory of linear quadratic form, there exists the most controllable variable \( V = -Kz \), so that \( J \) is the smallest and the feedback coefficient \( K \) is satisfied.

\[ K = R^{-1}B^TP \]  \hspace{1cm} (36)

In the formula: \( P \) satisfies the following matrix Riccati algebraic equation:

\[ PA + A^TP - PBR^1B^TP + Q = O \]  \hspace{1cm} (37)

At this time, the control law is linear. The trajectory tracking instruction form is obtained by substituting formula (30), formula (32), formula (36) and formula (37) into formula (33).

\[ u = u_0 + \frac{2a_2}{a} \sin (\frac{x_2}{2} + \theta_2) \sin (\frac{x_2}{2}) - \frac{K_1x_1 + K_2 \cdot 2a_1 \cos (\frac{x_2}{2}) \sin (\frac{x_2}{2})}{a_ia_2 \cos (x_2 + \theta_2)} \]  \hspace{1cm} (38)

Among them: \( u_0 \) is the control quantity of the base trajectory and the feedback coefficient \( K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \). So far, the trajectory tracking guidance law has been designed, the workflow of trajectory correction is as follows:

![Workflow of trajectory correction](image)

**Figure 4.** Workflow of trajectory correction

### 5. Missile motion equations

MATLAB/Simulink simulation platform is one of the important tools for dynamic system modeling, simulation and analysis. It has the advantages of simple structure, easy implementation and close to reality. It is widely used in complex simulation and design in various engineering and technical fields. In order to obtain more comprehensive and intuitive trajectory characteristics of the target missile in flight, and further verify the effectiveness of the attitude control system and trajectory tracking control algorithm of the target missile designed above, the trajectory simulation model of the target missile is established based on the simulation platform of MATLAB/Simulink.
The particle motion model of missile is established by Simulink, and random wind disturbance is introduced into the model. The effectiveness of the feedback linearization trajectory tracking guidance law is verified by taking the trajectory of the disturbed missile as the reference trajectory.

According to document [14], the missile is often affected by gust in flight, and the result of gust is that the missile will have additional angle of attack, which will affect the lift and drag of the missile, and make the trajectory of the missile deviate. The disturbance model of random wind is given below.

\[
U_U \text{ represents the vertical gust velocity, } W \text{ represents the horizontal gust velocity, } \rho \text{ represents the air density at the flight altitude of the missile, generally } W = 2U . U_0 \text{ is the vertical gust velocity, } W_0 \text{ is the horizontal gust velocity, } \rho_0 \text{ is the ground air density, } V \text{ is the flight speed of the missile. The calculation formula of the angle of attack is given directly.}
\]

\[
\Delta \alpha = \Delta \alpha_1 + \Delta \alpha_2 = \arctan \left( \frac{U}{V \cos \theta} \right) + \arctan \left( \frac{W}{V \sin \theta} \right)
\]

In order to compare the effectiveness of feedback linearization model horizontally, two kinds of linearization models based on small perturbation method and curving coefficient method are introduced. The control models are based on linear quadratic optimal control method.

Linearization model of missile longitudinal small perturbation method [6]:

\[
\begin{bmatrix}
\dot{y} \\
\dot{\theta}
\end{bmatrix} = A \begin{bmatrix}
y \\
\theta
\end{bmatrix} + B \Delta n_y
\]

(40)

\[
A = \begin{bmatrix}
0 & \frac{v \cos(\theta / 57.3)}{57.3} \\
\frac{57.3(F_y^r - G^r \cos \theta)}{mv} & \frac{g \sin \theta}{v}
\end{bmatrix}
B = \begin{bmatrix}
0 \\
g/v
\end{bmatrix}
\]

(41)

In the formula, \( F_y^r \) is the derivative of lift to longitudinal position variable and \( G^r \) is the derivative of gravity to longitudinal position variable.

Linearized model of missile solidification coefficient method [6]:

\[
\begin{bmatrix}
\dot{y} \\
\dot{\theta}
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} \begin{bmatrix}
y \\
\theta
\end{bmatrix} + \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} \Delta n_y
\]

(42)
\[
\begin{aligned}
    a_{11} &= 0 \\
    a_{12} &= \frac{1}{57.3\cos(\theta / 57.3)} \\
    a_{21} &= -\frac{57.3C_s S}{2m\cos(\theta / 57.3)} \cdot \rho \frac{\ddot{h}_p}{h_p} + g_o \frac{2 \times 57.3R^2}{(R_c + y)^3} \\
    a_{22} &= \frac{n \cdot g \sin(\theta / 57.3)}{V^2 \cos^2(\theta / 57.3)} \\
    b_1 &= 0 \\
    b_2 &= \frac{57.3g}{V^2 \cos(\theta / 57.3)}
\end{aligned}
\]

(43)

The two linearized models can also be used to design guidance instructions by the linear quadratic optimal control method. The tracking guidance laws designed by the feedback linearization method are compared and analyzed with the above two methods.

In order to verify the feasibility of trajectory tracking guidance law, the flight trajectory of missile with uncontrolled flight and gust disturbance is taken as the standard trajectory, and the reference trajectory control is 0. The simulation results are as follows:

Figure 5. Ballistic curve of missile

Figure 5 shows that the trajectory tracking guidance law based on three linearization methods can achieve the trajectory tracking of target missile. The trajectory tracking guidance law based on feedback linearization method has the highest tracking accuracy, can quickly eliminate the initial trajectory error and maintain the error at a low level. Based on the linearized model of small perturbation method, under the same control parameters, The tracking accuracy is poor, while the trajectory tracking guidance law based on small perturbation method fluctuates greatly under the same conditions, and the flight stability is the worst.
As shown in Figure 6 and Figure 7, the trajectory tracking guidance law based on feedback linearization method can suppress the error best and eliminate the error quickly for the longitudinal height and inclination of the controlled variables. For the lateral position and flight speed of the uncontrolled variables, the tracking guidance law based on feedback linearization also has a good tracking effect, because the anti-tank missile has a low range near velocity and a reference projectile. The deviation between trajectory and uncontrolled trajectory is small, so it is impossible to get the advantages and disadvantages of uncontrolled trajectory only from the chart.

In Figure 8, the change of missile angle of attack reflects the change of control force required by missile. The trajectory tracking guidance law based on feedback linearization design has a larger control force at the beginning of interruption, but then maintains a stable level. This change law is also suitable for the actual situation of control force provided by missile, that is, when the takeoff engine works, it can be overloaded heavily and take off. After the operation of the engine, the engine can only maintain the basic speed of the missile, and the available overload is very small.

Because of the random wind interference of Gauss white noise, the missile's control force changes too fast, which does not accord with the actual situation of missile target missile flying. The benchmark trajectory of target missile realizing horizontal flight and serpentine maneuver under given control force is given below. Based on the flight data of the benchmark trajectory, the trajectory tracking guidance law designed by feedback linearization method is validated to the missile's flight trajectory and station. The impact of control is needed.
Table 1. Control force parameters of horizontal flight reference trajectory.

| Flight time t (s) | 0~10 | 10~13 | 13~16 | 16~30 |
|-------------------|------|-------|-------|-------|
| Equivalent control angle of attack $\alpha$ (degree) | 0.5  | 4     | 4.5   | 4.6   |

Table 2. Control force parameters of Serpentine maneuvering reference trajectory.

| Flight time t (s) | 0~10 | 10~18 | 18~20 | 20~22 | 22~24 | 24~26 | 26~28 | 28~30 |
|-------------------|------|-------|-------|-------|-------|-------|-------|-------|
| Equivalent control angle of attack $\alpha$ (degree) | 0.5  | 2     | 8     | 6     | -4    | 4     | 3     | 2     |

In order to highlight the contrast, four groups of feedback coefficients $K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$ are selected as $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 10 \end{bmatrix}$, $\begin{bmatrix} 10 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 100 \\ 100 \end{bmatrix}$, and named as $K-1$, $K-2$, $K-3$ and $K-4$ respectively. Among them, $K-1$ is the optimal feedback coefficient for uncontrolled trajectory simulation. Figure 9 to 12 are the simulation results.

Figure 9. Modified ballistic curve of horizontal flight of missile.
Figure 10. Change of trajectory control force in horizontal flight.

Figure 11. Modified trajectory curve of missile snake maneuver.
From the trajectory simulation results of Figure 9 and Figure 11, it is not difficult to see that the missile can achieve trajectory tracking under four feedback coefficients. When the angle of attack $\alpha$ is 0.5° in the initial stage, the four feedback coefficients can achieve trajectory tracking with high precision. But with the increase of the disturbance in the middle and rear stages, the tracking accuracy gradually increases from $K_1$ to $K_4$. The modification of the trajectory inclination angle by $K_2$ can be obtained by experiment and theoretical analysis. Positive effect is better than $K_1$’s correction effect on height, and when $K_1$ is too large, excessive control will occur, which will destroy the stability of missile flight.

It can be seen from the changes of missile control forces in Figure 10 and Figure 12 that there are great differences in missile control forces under different feedback coefficients. By comparing with Table 1, it is not difficult to see that $K_4$ in Fig. 10 can track the disturbance angle of attack caused by standard trajectory more accurately. Only when the control forces change, there will be a step-by-step correspondence, but the change is about 1°, and the requirement for control system is lower, while adopting the control system. The tracking guidance law designed with the feedback coefficient shown in $K_1$ requires 12°’s required angle of attack in the initial stage, which puts forward higher requirements for the attitude control system of the missile and easily causes the instability of the missile.

6. Conclusion
Aiming at the tracking accuracy of target projectile in trajectory design, this paper establishes the nonlinear equations of target projectile’s longitudinal motion. According to the definition and necessary and sufficient conditions of input state feedback linearization, the linear equations of target projectile's longitudinal motion based on feedback linearization are derived in detail, and the trajectory tracking instructions of target projectile are designed by using linear quadratic optimal control theory. Through trajectory tracking of uncontrolled flying missile and peaceful flying missile, the feasibility and reliability of the designed guidance law are verified. By analyzing the function of feedback coefficient, a trajectory tracking guidance law with low overload requirement is obtained, which solves the design problem of trajectory tracking guidance law involved in the transformation of anti-tank missile to target missile.
At present, many scholars have studied the guidance law of missile attacking target, and the applied theory is relatively new. However, the research results of the guidance law of ballistic tracking are relatively few, and the applied theory is relatively basic. Feedback linearization is a common method of precise model linearization, but it has not been widely used in control system design because of its harsh use conditions and complex calculation process. The trajectory tracking guidance law designed in this paper can complete trajectory tracking well through detailed deduction and combining with the optimal control theory. Aiming at the phenomenon that the current trajectory tracking guidance law research is parabolic trajectory, the more realistic parabolic trajectory is innovatively used to simulate and verify, and good results are achieved. At present, the design of trajectory tracking guidance law is mainly parabola or level flight trajectory. There are few studies on high-maneuvering tactical missiles that need to penetrate. In order to achieve high-maneuvering trajectory tracking, besides high-precision trajectory tracking, it is more necessary to control the missile to complete these maneuvering tasks with overload. Therefore, filter theory and high-order are needed. Advanced theories such as sliding mode control solve current problems.

References
[1] S. T. Tang, Research on trajectory tracking guidance law of air defense missile, Modern defense technology, vol.31, no. 4, pp. 13-16, 2003.
[2] B. H. Song, S. T. Tang, Research on Tracking Guidance Law and Ballistic Simulation of Anti-Ballistic Missile for Air Defense Missile, Modern Defense Technology, vol. 33, no. 4, pp. 25-28, 2005.
[3] C. J. Liu, Z. Y. Liu, Z. H. Liu. Air Defense Weapon Targets. Beijing: Aviation Industry Press, 1997:1-2.
[4] J. L. Bi, Z. Y. Liu, Application and Development Trend of Precision Guided Weapons in Modern Warfare, Tactical Missile Technology, vol. 6, pp. 1-4, 2004.
[5] J. Yun, T. Zhang, G. Y. Guo, The Impact of High and New Technology on the Development of Missile Weapons, Aerial Missiles, vol.12, pp. 1-7, 2002.
[6] D. Y. Zhang, H. M. Lei, L. Wu, Design of trajectory tracking guidance law based on LQR, Solid rocket technology, vol.37, no. 6, pp. 763-768, 2014.
[7] Q. Lu, J. Zhou, LQR tracking guidance law for hypersonic vehicle, 2017 29th Chinese Control And Decision Conference (CCDC), pp. 1436-1442, 2017.
[8] Z. L. Ge, Y. N. Wang, M. B. Lv, Three-dimensional Trajectory Tracking Guidance Law Based on Linear Quadratic Regulator, Journal of Physics: Conference Series, vol. 1039, no. 1, pp. 12-42, 2018.
[9] R. J. Yang, Y. Ye, D. H. Yan, Y. Wang, Application of Sliding Mode Control in Ballistic Tracking of Manufactured Missile Munitions, Journal of Ballistics, 2015,27(04): 24-29.
[10] D. Y. Zhang, H. M. Lei, L. Wu, L. Zhao, J. Li, Design of Ballistic Tracking Guidance Law Based on Sliding Mode Variable Structure, Systems Engineering and Electronic Technology, 2014, 36 (04): 721-727.
[11] X. J. Zhang, M. Y. LIU, Y. LI, impact angle control over composite guidance law based on feedback linearization and finite time control, Journal of Systems Engineering and Electronics, vol. 29, no. 5, pp. 1036-1045, 2018.
[12] A. David, M. Amiram, Guidance Laws Based on Optimal Feedback Linearization Pseudo control with Time-to-Go Estimation, Journal of Guidance, Control and Dynamics, vol. 37, no. 4, pp. 1298-1304, 2014.
[13] Y. Gao, Linear Feedback Guidance for Low-Thrust Many-Revolution Earth-Orbit Transfers, Journal of Spacecraft and Rockets, vol. 46, no. 6, pp. 1320-1325, 2009
[14] D. P. Li, “Nonlinear Control System,” Xi'an: Northwest Polytechnic University Press, pp. 180-195, 2009.
[15] X. G. Li, Q. Fang, “Flight Dynamics of Winged Missiles,” Xi'an: Northwest Polytechnic University Press, pp. 65-66, 2005.