CR-twistor spaces over manifolds with $G_2$- and $\text{Spin}(7)$-structures

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Abstract
In 1984 LeBrun constructed a CR-twistor space over an arbitrary conformal Riemannian 3-manifold and proved that the CR-structure is formally integrable. This twistor construction has been generalized by Rossi in 1985 for $m$-dimensional Riemannian manifolds endowed with a $(m - 1)$-fold vector cross product (VCP). In 2011 Verbitsky generalized LeBrun’s construction of twistor-spaces to 7-manifolds endowed with a $G_2$-structure. In this paper we unify and generalize LeBrun’s, Rossi’s and Verbitsky’s construction of a CR-twistor space to the case where a Riemannian manifold $(M, g)$ has a VCP structure. We show that the formal integrability of the CR-structure is expressed in terms of a torsion tensor on the twistor space, which is a Grassmannian bundle over $(M, g)$. If the VCP structure on $(M, g)$ is generated by a $G_2$- or $\text{Spin}(7)$-structure, then the vertical component of the torsion tensor vanishes if and only if $(M, g)$ has constant curvature, and the horizontal component vanishes if and only if $(M, g)$ is a torsion-free $G_2$ or $\text{Spin}(7)$-manifold. Finally we discuss some open problems.

Keywords  Vector cross product · Formally integrable CR-structure · Torsion-free $G_2$ -and $\text{Spin}(7)$-structure · Metric of constant curvature · Frölicher–Nijenhuis bracket · Invariant algebraic curvature

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1 Introduction

1.1 Motivations and prior works

In his papers [4, 18], motivated by Calabi’s work on almost complex structures on \( S^6 \), Gray introduced the notion of a vector cross product (VCP for short) structure. By definition, a \( r \)-fold VCP structure \( \chi \) on an Euclidean vector space \((V, \langle \cdot, \cdot \rangle)\) is a multilinear alternating map

\[
\chi : \bigwedge^r V \to V
\]
such that

\[
\langle \chi(v_1, \ldots, v_r), v_i \rangle = 0 \quad \text{for} \quad 1 \leq i \leq r,
\]

\[
\langle \chi(v_1, \ldots, v_r), \chi(v_1, \ldots, v_r) \rangle = \|v_1 \wedge \cdots \wedge v_r\|^2,
\]

where \( \|\cdot\| \) is the induced metric on \( \wedge^r V \). For a \( r \)-fold VCP \( \chi \) on \( V \), the associated VCP-form \( \varphi_\chi : \bigwedge^{r+1} V \to \mathbb{R} \) is defined as

\[
\varphi_\chi(v_1, \ldots, v_{r+1}) = \langle \chi(v_1, \ldots, v_r), v_{r+1} \rangle.
\]  

[18, (4.1)]. As a matter of notation, once a \( r \)-fold VCP is fixed one often writes \( v_1 \times v_2 \times \cdots \times v_r \) for \( \chi(v_1, \ldots, v_r) \).

Remark 1.1.2 (1) The Brown–Gray classification [4] asserts that a \( r \)-fold VCP structure exists on \( \mathbb{R}^m \) if and only if one of the following possibilities holds

(i) \( r = 1 \) and \( m \) is even;

(ii) \( r = m - 1 \);

(iii) \( r = 2 \) and \( m = 7 \);

(iv) \( r = 3 \) and \( m = 8 \).

(2) A \((m - 1)\)-fold VCP structure \( \chi \) on \((\mathbb{R}^m, \langle \cdot, \cdot \rangle)\) is defined uniquely by a given orientation on \( \mathbb{R}^m \).
(3) For \( m = 7, r = 2 \) the VCP form \( \varphi_\chi \) is called the \textit{associative 3-form}. Its stabilizer in \( \text{GL}(\mathbb{R}^7) \) is the exceptional group \( G_2 \subset \text{SO}(7) \).

For \( m = 8, r = 3 \) the VCP form \( \varphi_\chi \) is called the \textit{Cayley 4-form}. Its stabilizer in \( \text{GL}(\mathbb{R}^8) \) is the subgroup \( \text{Spin}(7) \subset \text{SO}(8) \). The VCP structures \((\chi, g)\) in these cases are in a 1-1 correspondence with their VCP-forms \( \varphi_\chi \). Given a VCP form \( \varphi \) on a 7-manifold an explicit formula for \( g_\varphi \) is given in [23, §7.1]. Similarly, given a VCP form \( \varphi \) on a 8-manifold, a formula for \( g_\varphi \) can be obtained using the relation \( \varphi^2 = 8\text{vol}_{g_\varphi} \) and Hitchin’s method, see similar results in [35, §3].

One has an immediate notion of a \( r \)-fold VCP on a Riemannian manifold \((M, g)\) as a smooth \( TM \)-valued \( r \)-form \( \chi \in \Omega^r(M, TM) \) such that \( \chi(x) \) is a \( r \)-fold VCP on \( T_xM \) for all \( x \in M \). The corresponding VCP-form will therefore be an element in \( \Omega^{r+1}(M) \).

\textbf{Remark 1.1.3} A VCP form \( \varphi_\chi \) is parallel w.r.t. the Levi-Civita connection \( \nabla^{LC} \) iff either \((M^m, g)\) is an orientable Riemannian manifold and \( r = m - 1 \); or \( m = 2n \), \((M^{2n}, g)\) is a Kähler manifold and \( r = 1 \); or \( m = 7 \) and \((M^7, g)\) is a torsion-free \( G_2 \)-manifold and \( r = 2 \); or \( m = 8 \) and \((M^8, g)\) is a torsion-free \( \text{Spin}(7) \)-manifold and \( r = 3 \). This result singles out Kähler manifolds, torsion-free \( G_2 \)-and \( \text{Spin}(7) \)-manifolds as important classes of Riemannian manifolds with special holonomy [24]. Not unrelated, these classes play a prominent role in calibrated geometry, string theory and M-theory, and F-theory [7, 20, 25].

\textbf{Remark 1.1.4} The VCPs in dimensions 3, 7, 8 can be expressed in terms of algebraic operations on normed algebras. Denote by \( \text{Im} \) the imaginary part of the octonion algebra \( \mathbb{O} \). Harvey and Lawson noticed that, identifying \( \mathbb{R}^7 \) with \( \text{Im} \mathbb{O} \), the associative 3-form \( \varphi_\chi \) on \( \text{Im} \mathbb{O} \) has the following form [21, (1.1), p. 113]:

\[
\varphi_\chi(x, y, z) = (x, yz).
\]  

(1.1.5)

Hence the twofold VCP \( \chi \) on \( \text{Im} \mathbb{O} \) is defined as follows [21, Definition B.1, p. 145]

\[
y \times z = \text{Im} (yz).
\]  

(1.1.6)

The restriction of this twofold VCP to \( \text{Im} \mathbb{H} \subset \text{Im} \mathbb{O} \) coincides with the twofold VCP on \( \mathbb{R}^3 \) [21, p. 145].

The threefold VCP on \( \mathbb{R}^8 = \mathbb{O} \) can be expressed as follows [21, Definition B.3, p. 145]:

\[
u \times u \times w = \frac{1}{2} (((u\bar{v})w - (w\bar{v})u).
\]  

(1.1.7)

The relation between complex structures and VCP structures has been manifested also via CR-twistor spaces over manifolds endowed with a VCP structure. In 1984 LeBrun constructed a CR-twistor space over an arbitrary conformal Riemannian 3-manifold [30]. LeBrun proved that the CR-twistor space of a conformal Riemannian 3-manifold is a CR-manifold, i.e. the CR-structure is integrable. This twistor construction has been generalized by Rossi in 1985 for \( m \)-dimensional Riemannian manifolds endowed with a \((m - 1)\)-fold VCP [37] and utilized further by LeBrun for his proof of the formal integrability of the almost complex structure on the higher dimensional loop space over a Riemannian manifold \((M^m, g)\) endowed with a \((m - 1)\)-fold VCP [31], following a similar proof by Lempert for the weak integrability of the almost complex structure on the loop space over a Riemannian 3-manifold [32]. In 2011 Verbitsky generalized LeBrun’s construction of twistor-spaces to 7-manifolds endowed with the VCP 3-forms \( \varphi \) [41], which subsequently has been used by him for his proof of the formal integrability of the almost complex structure on the loop space over a holonomy \( G_2 \)-manifold [40].
1.2 Our main results

As a first result in this paper, we unify and generalize LeBrun’s, Rossi’s construction of a CR-twistor space over a conformal Riemannian manifold in dimension 3 and in arbitrary dimension, respectively, as well as Verbitsky’s construction of a CR-twistor space over a $G_2$-manifold to the case when the underlying Riemannian manifold $(M, g)$ has a VCP structure, see Definition 1.2.12. In order to state the result we need fixing notation.

**Notation 1.2.1** Let $(M, g)$ be an oriented Riemannian manifold.

- We denote by $G \setminus^+(r-1, M)$ the Grassmannian of oriented $(r-1)$-planes in $TM$, which we shall identify with decomposable unit $(r-1)$-vectors in $\bigwedge^{r-1}TM$. When no confusion is possible we will denote $G \setminus^+(r-1, M)$ simply by $G$. We denote by $\pi : \bigwedge^{r-1}TM \to M$ the natural projection, which also induces the natural projection $\pi : G \to M$. For any point $v \in G$, the fiber of $\pi : G \to M$ through $v$ is naturally identified with the Grassmannian $G \setminus^+(r-1, T_{\pi(v)}M)$ of oriented $(r-1)$-planes in $T_{\pi(v)}M$.

- For $v \in G$ we denote by $E_v \subseteq T_{\pi(v)}M$ the oriented $(r-1)$-plane associated to $v$ and by $E_v^\perp$ its orthogonal complement in $T_{\pi(v)}M$.

The Riemannian metric $g$ induces a natural Riemannian metric on the vector bundle $\bigwedge^{r-1}TM \xrightarrow{\pi} M$ and so endows $\bigwedge^{r-1}TM$ with the corresponding Levi-Civita connection $\nabla^{LC}$. This induces, for any $v \in \bigwedge^{r-1}TM$, a direct sum decomposition

$$T_v(\bigwedge^{r-1}TM) = \bigwedge^{r-1}T_{\pi(v)}M \oplus T_v^{\text{hor}}(\bigwedge^{r-1}TM), \quad (1.2.2)$$

where

$$T_v^{\text{hor}}(\bigwedge^{r-1}TM) \cong T_{\pi(v)}M$$

is the horizontal distribution in $T \bigwedge^{r-1}TM$ w.r.t. $\nabla^{LC}$. Since $G \setminus^+(r-1, TM)$ is a fiber sub-bundle of the vector bundle $\bigwedge^{r-1}TM \xrightarrow{\pi} M$, for $v \in G$ the orthogonal decomposition (1.2.2) induces the decomposition

$$T_vG = T_v^{\text{vert}}G \oplus T_v^{\text{hor}}G, \quad (1.2.3)$$

where

$$T_v^{\text{vert}}G = T_vG \setminus^+(r-1, T_{\pi(v)}M) \quad (1.2.4)$$

and

$$T_v^{\text{hor}}G = T_v^{\text{hor}}(\bigwedge^{r-1}TM) \cong T_{\pi(v)}M. \quad (1.2.5)$$

**Notation 1.2.6** Let $(M^m, g)$ be an $m$-dimensional Riemannian manifold. We denote by $B$ the rank $m - r + 1$ distribution on $G$ defined at a point $v$ of $G$ by

$$B_v := \left\{ w \in T_v^{\text{hor}}G | d\pi_v(w) \in E_v^\perp \subset T_{\pi(v)}M \right\} \subseteq T_v^{\text{hor}}G. \quad (1.2.7)$$

An $r$-fold VCP structure $\chi$ on $(M, g)$ endows the vector spaces $E_v^\perp$ with a complex structure $J_{E_v^\perp}$ defined by

$$J_{E_v^\perp}(z) = \chi(v \wedge z), \quad (1.2.8)$$
for \( z \in E_v^\perp \), see [12, Lemma 3.1], [33, p. 146], [18, Theorem 2.6]. Since \( d\pi_v : B_v \to E_v^\perp \) is an isometry, the complex structure \( J_{E_v^\perp} \) induces a complex structure \( J_{g,\chi} \) on \( B_v \). It is defined by the equation
\[
d\pi_v(J_{g,\chi}(w)) = J_{E_v^\perp}(d\pi_v(w)).
\] (1.2.9)

**Definition 1.2.10** (cf. [9, Definitions 1.1, 1.2, p. 3]) An *almost CR-structure* on a manifold \( N \) is a pair \((B, J_B)\) consisting of a distribution \( B \subseteq TN \) and of an almost complex structure \( J_B \) on \( B \). The triple \((N, B, J_B)\) is called an *almost CR-manifold*. An almost CR-structure \((B, J_B)\) on a manifold \( N \) is said to be *formally integrable* if the complex distribution \( B^{1,0} \subseteq B \otimes \mathbb{C} \) is involutive, i.e., \([B^{1,0}, B^{1,0}] \subseteq B^{1,0}\). If \((B, J_B)\) is integrable, then the almost CR-manifold \((N, B, J_B)\) is called a *CR-manifold*.

**Remark 1.2.11** The condition that the almost CR-structure \((B, J_B)\) is formally integrable can be stated completely in terms of sections of the real vector bundle \( B \) and by \( \Gamma(B) \) the space of smooth sections of \( B \).

1. For any \( X, Y \in \Gamma(B) \) one has \([J_B X, J_B Y] - [X, Y] \in \Gamma(B)\);
2. For any \( X, Y \in \Gamma(B) \) one has
\[
\Pi_B([J_B X, J_B Y] - [X, Y]) - J_B \circ \Pi_B([X, J_B Y] + [J_B X, Y]) = 0.
\]

In literature [5, p. 128], [9, p. 4] the condition (2) is replaced by the following condition
\[
[J_B X, J_B Y] - [X, Y] - J_B([X, J_B Y] + [J_B X, Y]) = 0,
\]
which has meaning only if the condition (1) holds. Clearly the conditions (1) and (2) are equivalent to the condition (1) and the classical condition stated above.

We shall call the condition (1) the *first CR-integrability condition*, and the condition (2) the *second CR-integrability condition*. In Cartan geometry, the condition (1) is also called the *partial integrability* of a CR-structure [8, p. 443].

Now we associate to each VCP-structure on a Riemannian manifold \((M, g)\) an almost CR-manifold as follows.

**Definition 1.2.12** Let \((M, g, \chi)\) be a Riemannian manifold endowed with a VCP structure \(\chi\). The almost CR-manifold \((\mathbb{G}, B, J_{g,\chi})\) consisting of the manifold \(\mathbb{G}\) together with the almost CR-structure given by the distribution \(B\) and the almost complex structure \(J_{g,\chi}\) on \(B\) defined in Eqs. (1.2.7) and (1.2.9) will be called the *CR-twistor space over* \((M, g, \chi)\).

**Example 1.2.13** (1) Let \((\chi, g)\) be a onefold VCP on a smooth manifold \(M^{2n}\). This is equivalent an Hermitian almost complex structure on \(M\). In this case one has \(\mathbb{G} = M^{2n}\), the horizontal distribution \(B\) is identified with the tangent bundle to \(M^{2n}\) and the almost complex structure \(J_{g,\chi}\) is identified with the almost complex structure on \(M\).

(2) Let \((\chi, g)\) be a \((m - 1)\)-fold VCP on an oriented manifold \(M^m\). Then the distribution \(B\) on \(\mathbb{G}\) is 2-dimensional and the CR-twistor structure \((\mathbb{G}, B, J_{g,\chi})\) coincides with the one constructed by LeBrun [30] and extended by Rossi [37].

(3) Let \(\chi\) be a twofold VCP on \((M^7, g)\). Then the CR-twistor structure on \(\mathbb{G}\) coincides with the one constructed by Verbitsky [41].
endowed with a VCP structure. Let $\nabla^L_C$ denote the Levi-Civita connection on $(M, g)$. We say that the VCP $\chi$ is parallel if $\nabla^L_C \chi = 0$.

In this paper we prove the following

**Theorem 1.2.14** (Main Theorem) Let $\chi \in \Omega^{r+1}(M, TM)$ be a VCP structure on a Riemannian manifold $(M, g)$ and $(\mathbb{G}, B, J_{\mathbb{G}}, \chi)$ the associated CR-twistor space. Then there exists a tensor $T \in \Gamma(\wedge^2 B^* \otimes TG)$ on the total space $\mathbb{G}$ such that

1. The first CR-integrability (1)) holds if and only if for any $v \in \mathbb{G}$ and $X, Y \in B(v)$ we have $T^\text{vert}(X, Y) = 0 \in T^\text{vert}\mathbb{G}$.
2. If $(r, m) = (1, 2n)$ or $(m - 1, m)$ then $T^\text{vert} = 0$ for any $(M, g, \chi)$.
3. If $(r, m) = (2, 7)$ or $(3, 8)$ then $T^\text{vert} = 0$ if and only if $(M, g, \chi)$ has constant curvature.
4. The second CR-integrability (2)) holds, if and only if for any $v \in \mathbb{G}$ and $X, Y \in B(v)$ we have $T^\text{hor}(X, Y) = 0 \in T^\text{hor}\mathbb{G}$.
5. If $(r, m) = (1, 2n)$, then $T^\text{hor} = 0$ for $(M, g, \chi)$ if and only if $\chi$ is integrable.
6. If $(r, m) = (m - 1, m)$, then $T^\text{hor} = 0$ for any $(M, g, \chi)$.
7. If $(r, m) = (2, 7)$ or $(r, m) = (3, 8)$, then $T^\text{hor} = 0$ if and only if $\chi$ is parallel.

**Remark 1.2.15** Parts (2&5) and (2&6) of the main theorem above combined, i.e., without decomposing the CR integrability condition into two independent conditions, are classical and we are including them only for completeness. In particular, by combining Example 1.2.13 (1) with Example 2.2.10(1) we recover that a Riemannian manifold $(M^{2n}, g)$ endowed with a onefold VCP $\chi$ is a CR-manifold if and only if the almost complex structure on $M$ induced by $\chi$ is integrable. Part (6) is due to LeBrun, who proved that the CR-twistor space over a Riemannian manifold $(M^m, g)$ with a $(m - 1)$ fold VCP $\chi$ is always a CR-manifold [30], [31]. Note that in this case $\chi$ is always parallel, see [18, Proposition 4.5]. Part (7) for the case (2, 7) is due to Verbitsky [41]. Unfortunately his proof uses a wrong argument, see Remark 2.2.9. Finally, it was also known that the CR-twistor space over any flat Riemannian manifold $(M, g)$ endowed with parallel VCP is a CR-manifold.

### 1.3 Organization of our paper

In the second section we study the first condition (1)) for the formal integrability of the CR-twistor space over a Riemannian manifold $(M, g)$ endowed with a VCP-structure $\chi$. First, using a geometric characterization of the distribution $B$ (Lemma 2.1.7), we express the condition (1)) for the CR-twistor space over $(M, g, \chi)$ in terms of the vertical components of the Lie brackets $[J_B X, J_B Y]$ and $[X, Y]$, where $X, Y \in \Gamma(B)$, with respect to the decomposition (1.2.3) (Corollary 2.1.13). Using this, we prove that the first CR-integrability condition (1)) for the CR-twistor space over $(M, g, \chi)$ holds if and only if the curvature $R(g)$ of the underlying Riemannian manifold $(M, g)$ is a solution of an infinite system of linear equations (Proposition 2.2.8). Next, we study the first CR-integrability condition for the (2, 7) case using Proposition 2.2.8 and computer algebra. Using these results and ad hoc methods in Sect.3, we prove assertions (1) and (3) of Theorem 1.2.14. In Sect.4 we study the second condition (2)) for the formal integrability of the CR-twistor space over $(M, g, \chi)$ using the formalism of the Frölicher–Nijenhuis bracket (Proposition 4.1.1). Then we give the proof of Theorem 1.2.14 (4, 7). Finally we discuss our results and some open questions.

### 1.4 Notation and conventions

- We keep notation in the introduction.
• For a vector bundle $E$ over a manifold $M$ and a smooth section $\alpha \in \Gamma(E)$, we also write $\alpha_x$ for the value $\alpha(x)$ to avoid possibly ugly notation like $\alpha(x)(v)$ occurring, e.g., when $E$ is the endomorphism bundle of $TM$ and $v$ is a tangent vector at $x$.

• If $\xi$ is an element in a vector space $V$ with inner product $\langle , \rangle$, we denote by $\xi^\sharp$ the element on $V^*$ defined by $\xi^\sharp(v) = \langle \xi, v \rangle$ for all $v \in V$.

• Given a $G$-action on a space $X$, for $x \in X$, we denote by $\text{Stab}_G(x)$ the stabilizer of $x$ in $G$.

• We consider in this paper the Killing metric on Lie algebra $\mathfrak{so}(\mathbb{R}^n)$ and any of its Lie subalgebra defined as follows $(X, Y) = -\frac{1}{2} Tr(XY)$.

• Let $(V, \langle , \rangle )$ be an Euclidean vector space. We denote by $\mathcal{AC}(V)$ the vector subspace of $\wedge^2 V^* \otimes \mathfrak{so}(V)$ consisting of elements $R \in \wedge^2 V^* \otimes \mathfrak{so}(V)$ that satisfy the algebraic Bianchi identity, i.e.,

$$R(w_1, w_2)w_3 + R(w_2, w_3)w_1 + R(w_3, w_1)w_2 = 0,$$

The elements of $\mathcal{AC}(V)$ are called algebraic curvature (operators) on $V$. It is known that $\dim \mathcal{AC}(V) = \frac{1}{12} (\dim V)^2(\dim V^2 - 1)$ [17, Corollary 1.8.4, p. 45].

• It is known that the image $R^\text{Id}$ of the operator $\text{Id}: \wedge^2 V^* \to \wedge^2 V^*$ in $\wedge^2 V^* \otimes \mathfrak{so}(V)$ via the identification $\wedge^2 V^*$ with $\mathfrak{so}(V)$ is an algebraic curvature of constant sectional curvature, see, e.g., [17, Lemma 1.6.4, p. 31]. It is immediate to see that

$$R^\text{Id}(w_1, w_2)w_3 := \langle w_2, w_3 \rangle w_1 - \langle w_1, w_3 \rangle w_2,$$ (1.4.1)

for any $w_1, w_2, w_3$, see [17, p. 31]. By the Schur lemma if $(M, g)$ is a connected Riemannian manifold of dimension at least 3, then the Riemannian curvature tensor of $M$ is of the form $R = \lambda(x) R^\text{Id}$ at any point $x \in M$ if and only if $(M, g)$ has constant curvature [28, Theorem 2.2, p. 202].

• Let $\text{Der}(\Omega^*(M))$ be the graded Lie algebra of graded derivations of $\Omega^*(M)$. For $K \in \Omega^*(M, TM)$ we denote by $\iota_K$ and by $\mathcal{L}_K = [d, \iota_K]$ the contraction with $K$ and corresponding the Lie derivative, respectively. It is known that $\mathcal{L} : \Omega^*(M, TM) \to \text{Der}(\Omega^*(M))$ is injective, and moreover [14, 15]

$$\mathcal{L}(\Omega^*(M, TM)) = \{D \in \text{Der}(\Omega^*(M)) | [D, d] = 0\}.$$

Hence $\mathcal{L}(\Omega^*(M, TM))$ is closed under the graded Lie bracket $[,]$ on $\text{Der}(\Omega^*(M))$ and one then defines the Frolicher–Nijenhuis bracket $[,]^{FN}$ on $\Omega^*(M, TM)$ as the pull-back of the graded Lie bracket on $\text{Der}(\Omega^*(M))$ via the linear embedding $\mathcal{L}$, i.e.,

$$\mathcal{L}_{[K, L]}^{FN} := [\mathcal{L}_K, \mathcal{L}_L].$$

2 A reformulation of the first condition for the formal integrability of CR-twistor spaces ($G$, $B$.J$_{g, \chi}$)

In this section we reformulate the first condition for the integrability of CR-twistor spaces ($G$, $B$, $J_{g, \chi}$) in terms of a system of linear equations for the curvature tensor of $(M, g)$. This will in particular imply that the first integrability condition is automatically satisfied in the case $(r, m) = (1, 2n)$ or $(r, m) = (m - 1, m)$. The proof goes in two steps. First we express the first integrability condition as the condition $[J_{g, \chi} X, J_{g, \chi} Y]^{\text{vert}} = [X, Y]^{\text{vert}}$ for any $X, Y \in \Gamma(B)$ (Corollary 2.1.13). Then we translate this in a system of conditions on the curvature tensor of $(M, g)$ (Proposition 2.2.8).
2.1 The equation $[J_g^\chi X, J_g^\chi Y]^\text{vert} = [X, Y]^\text{vert}$

Let $(M, g, \chi)$ be a Riemannian manifold endowed with a $r$-fold VCP structure and $(B, J_g, \chi)$ the almost CR-structure on $\mathbb{G}$. Let $E^*$ be the dual bundle of the tautological bundle $E$ over $\mathbb{G}$. At every point $v$ in $\mathbb{G}$, the Riemannian metric $g$ induces a natural isomorphism $E_v = T_{\pi(v)}M / E^*_v$, where $\pi : \mathbb{G} \to M$ is the projection to the base. This gives a natural identification $E^* = \text{Ann}(E^\perp)$, where $\text{Ann}(E^\perp)$ is the vector bundle over $\mathbb{G}$ whose fiber over $v$ consists of all elements of $T^*_vM$ that annihilate $E^*_v$.

**Remark 2.1.1** Since $E^*$ is a subbundle of $\pi^*T^*M$ via the identification $E^* = \text{Ann}(E^\perp)$, any $\theta \in \Gamma(E^*)$ defines a map of fiber bundles over $M$,

$$\hat{\theta} : \mathbb{G} \to T^*M,$$

(2.1.2)
mapping a point $v \in \mathbb{G}$ to the element $\theta_v$ seen as an element in $T^*_vM$.

The Riemannian metric $g$ induces a natural Riemannian metric on the vector bundle $T^*M \to M$ and so endows $T^*M$ with the associated Levi-Civita connection $\nabla^{LC}$ and the corresponding splitting of the tangent bundle of the total space of $T^*M$ into a vertical and a horizontal subbundle. The same applies to the bundle $\pi^*T^*M$ and to its subbundle $E^*$.

**Notation 2.1.3** For $v \in \mathbb{G}$ we let

$$\Gamma_{\text{hor}(v)}(E^*) := \{ \theta \in \Gamma(E^*) : d\hat{\theta}(T^*_v\mathbb{G}) \subset T^*_v\pi^*\pi M \}. \quad (2.1.4)$$

In other words, $\Gamma_{\text{hor}(v)}(E^*)$ consists of elements in $\Gamma(E^*)$ that are “horizontal” at $v$. Using parallel transport in the Grassmann bundle $\mathbb{G} \to M$ and in the total space of the vector bundle $E^*$ on $\mathbb{G}$ seen as a fiber bundle over $M$ one easily shows that every vector $\xi \in E^*_v$ can be extended to a section of $E^*$ that is horizontal at $v$. For later reference, we state this fact as the following lemma.

**Lemma 2.1.5** For any $\xi \in E^*_v$ there exists an element $\theta \in \Gamma_{\text{hor}(v)}(E^*)$ such that $\theta_v = \xi$.

**Notation 2.1.6** We write

$$\eta : \Gamma(E^*) \to \Omega^1(\mathbb{G}), \quad \theta \mapsto \eta(\theta)$$

for the map sending a smooth section $\theta$ of $E^*$ to the 1-form $\eta[\theta]$ given by

$$\eta[\theta](w) = \theta_v(d\pi_v(w)),$$

for any $w \in T_v\mathbb{G}$. In other words, $\eta[\theta]$ is the section of $T^*\mathbb{G} \to \mathbb{G}$ given by the composition

$$\mathbb{G} \overset{\theta}{\to} E^* \hookrightarrow \pi^*T^*M \overset{(d\pi)^*}{\longrightarrow} T^*\mathbb{G}.$$

**Lemma 2.1.7** Let us consider the subbundle $B \oplus \perp T^\text{vert} \mathbb{G}$ of $T\mathbb{G}$. For any $v \in \mathbb{G}$ we have

$$B_v \oplus \perp T^\text{vert} \mathbb{G} = \bigcap_{\theta \in \Gamma_{\text{hor}(v)}(E^*)} \ker(\eta(\theta)_v).$$

**Proof** Let $w = w \vert_B + w \vert_{\text{vert}} \in T_v\mathbb{G}$, where $w \vert_B \in B_v$ and $w \vert_{\text{vert}} \in T^\text{vert} \mathbb{G}$. Then for every $\theta \in \Gamma_{\text{hor}(v)}(E^*)$ we have

$$\eta[\theta](w) = \theta_v(d\pi_v(w \vert_B)) = 0. \quad (2.1.8)$$
Namely, by definition of $B_v$, Eq. (1.2.7), the vector $d\pi_v(w_B)$ is in $E^\perp_v$ and so it is annihilated by $\theta_v \in E^\perp_v = \text{Ann}(E^\perp_v)$. Vice versa, let $w \in T_v\mathbb{G}$ be such that $\eta[\theta](w) = 0$ for every $\theta \in \Gamma_{\text{hor}(v)}(E^*)$. Let us write $w = w_{\text{hor}} + w_{\text{vert}}$, with $w_{\text{hor/vert}} \in T_{\text{hor/vert}}(\mathbb{G})$. Let $\xi \in E^*_v$. By Lemma 2.1.5, there exists $\theta \in \Gamma_{\text{hor}(v)}(E^*)$ such that $\theta_v = \xi$, and so

$$\xi(d\pi_v(w_{\text{hor}})) = \xi(d\pi_v(w)) = \theta_v(d\pi_v(w)) = \eta[\theta](w) = 0.$$ 

Therefore,

$$d\pi_v(w_{\text{hor}}) \in \bigcap_{\xi \in \text{Ann}(E^\perp_v)} \ker(\xi) = E^\perp_v,$$

and so $w_{\text{hor}} \in B_v$. This completes the proof of Lemma 2.1.7. \qed

Lemma 2.1.9 Let $v \in \mathbb{G}$. For any $\theta \in \Gamma_{\text{hor}(v)}(E^*)$ one has

$$(d\eta[\theta])_v | \wedge^2 T_{\text{hor}}(\mathbb{G}) = 0$$

Proof The bundle $E^* = \text{Ann}(E^\perp)$ over $\mathbb{G}$ is a subbundle of the bundle $\pi^*T^*M$ and therefore a section $\theta$ of $\Gamma(E^*)$ is a section of $\pi^*T^*M$. We have a commutative diagram

![Diagram](image-url)

where $\hat{\theta}: \mathbb{G} \rightarrow T^*M$ is the map defined in Remark 2.1.1, and so

$$\eta[\theta] = \hat{\theta}^*(\Theta_{\text{Liou}};M),$$

(2.1.10)

where $\Theta_{\text{Liou}};M$ is the Liouville 1-form on $T^*M$. Let $\omega$ be the canonical symplectic form on $T^*M$. From (2.1.10) we obtain

$$d\eta[\theta] = \hat{\theta}^*(\omega) = \omega \circ (d\hat{\theta} \wedge d\hat{\theta}).$$

(2.1.11)

Since $\theta \in \Gamma_{\text{hor}(v)}(E^*)$, the differential $d\hat{\theta}$ maps the horizontal space $T_{\text{hor}}^v\mathbb{G}$ to $T_{\text{hor}}^\theta(v) T^*M$. It is well known that the restriction of the canonical 2-form $\omega$ to $\wedge^2 T_{\text{hor}}T^*M$ identically vanishes. This concludes the proof of Lemma 2.1.9. \qed

Lemma 2.1.12 For any $X, Y \in \Gamma(B)$ we have

$$[X, Y] \in \Gamma(B \oplus T^\perp \mathbb{G}).$$

Proof Let $v$ be a point in $\mathbb{G}$. We have to show that $[X, Y]_v \in B_v \oplus T^\text{vert}_v(\mathbb{G})$. By Lemma 2.1.7 this is equivalent to showing that for every $\theta \in \Gamma_{\text{hor}(v)}(E^*)$ we have $\eta[\theta]_v([X, Y]_v) = 0$. By the Cartan formula,

$$\eta[\theta]_v([X, Y]_v) = -(d\eta[\theta])_v(X_v, Y_v) + X_v(\eta[\theta](Y)) - Y_v(\eta[\theta](X)).$$

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By definition of $B_v$, we have $B_v \subset T_{v}^\text{hor}\mathbb{G}$, and so $(d\eta[\theta])_v(X_v, Y_v) = 0$, by Lemma 2.1.9. By definition of $\Gamma_{\text{hor}(v)}(E^*)$, $\theta$ is in particular an element of $\Gamma(E^*) = \Gamma(\text{Ann}(E^*))$. Therefore, for any point $v'$ in $\mathbb{G}$ we have

$$\eta[\theta]_{v'}(X_{v'}) = \theta_{v'}(d\pi_{v'}(X_{v'})) = 0,$$

since $X \in \Gamma(B)$ and so $d\pi_{v'}(X_{v'}) \in E^*_v$, by the defining Eq. (1.2.7). This means that $\eta[\theta](X)$ identically vanish on $\mathbb{G}$. By the same argument, also $\eta[\theta](Y) \equiv 0$, and so we have $\eta[\theta]_v([X, Y]) = 0$. 

**Corollary 2.1.13** Let $J_{g,\chi}$ the complex structure on $B$ induced by the VCP $\chi$ (Eq. (1.2.9)). For any $X, Y \in \Gamma(B)$ we have

$$[J_{g,\chi} X, J_{g,\chi} Y] - [X, Y] \in \Gamma(B) \text{ if and only if } [J_{g,\chi} X, J_{g,\chi} Y]^\text{vert} = [X, Y]^\text{vert}.$$

**Proof** By Lemma 2.1.12, we have $[X, Y] \in \Gamma(B \oplus T^\text{vert}\mathbb{G})$. Since $J_{g,\chi}$ is an vector bundle endomorphism of $B$, we also have $J_{g,\chi} X, J_{g,\chi} Y \in \Gamma(B)$ and so by Lemma 2.1.12 again, $[J_{g,\chi} X, J_{g,\chi} Y] \in \Gamma(B \oplus T^\text{vert}\mathbb{G})$. This gives $[J_{g,\chi} X, J_{g,\chi} Y] - [X, Y] \in \Gamma(B \oplus T^\text{vert}\mathbb{G})$ and so $[J_{g,\chi} X, J_{g,\chi} Y] - [X, Y] \in \Gamma(B)$ if and only if $([J_{g,\chi} X, J_{g,\chi} Y] - [X, Y])^\text{vert} = 0$. 

**2.2 A curvature reformulation of the first CR-integrability condition**

By Corollary 2.1.13, the first CR-integrability condition ((1)) is equivalent to

$$[J_{g,\chi} X, J_{g,\chi} Y]^\text{vert} = [X, Y]^\text{vert}$$

for any $X, Y \in \Gamma(B)$. This latter condition can be conveniently expressed in terms of the curvature operator $R$ of the Levi-Civita connection on $(M, g)$. If $X, Y$ are horizontal vector fields on $\mathbb{G}$ and $v \in \mathbb{G}$, we have $R_{\pi(v)}(d\pi_v(X_v), d\pi_v(Y_v)) \in \mathfrak{so}(T_{\pi(v)}M) \text{ and so a corresponding } \text{SO}(\text{dim } M)\text{-invariant vertical vector field } \tau_{\pi(v)}(d\pi_v(X_v), d\pi_v(Y_v)) \text{ on the fiber of } \mathbb{G} \to M \text{ through } v$. Evaluating this vector field at the point $v$ we obtain a vertical tangent vector $\tau_{\pi(v)}(d\pi_v(X_v), d\pi_v(Y_v)) \big|_v \in T_v^\text{vert}\mathbb{G}$. It is a standard fact, that can be easily derived from see, e.g., [3, p. 290] or [28, p. 89] by noticing that $\left[ -, - \right]^\text{vert} : T^\text{hor}\mathbb{G} \oplus T^\text{hor}\mathbb{G} \to T^\text{vert}\mathbb{G}$ is a tensor, that

$$[X, Y]^\text{vert} = -\tau_{\pi(v)}(d\pi_v(X_v), d\pi_v(Y_v)) \big|_v. \tag{2.2.1}$$

We identify $T^\text{vert}_v\mathbb{G}$ with $\text{Hom}(E_v, E_v^\perp)$ and define a linear embedding $\epsilon : \text{Hom}(E_v, E_v^\perp) \to \mathfrak{so}(E_v \oplus E_v^\perp)$ by extending the following relations linearly for $\xi \in E_v, \, w \in E_v^\perp$, and $X \in E_v \oplus E_v^\perp$:

$$\epsilon(\xi^\perp \otimes w)(X) := \xi(X) \cdot w - \langle w, X \rangle \cdot \xi, \tag{2.2.2}$$

where $\xi^\perp \in E_v^\perp$ is dual to $\xi$ w.r.t. $g \big|_{E_v}$. We shall use the shorthand notation $\xi^\perp \otimes w$ for $\epsilon(\xi^\perp \otimes w)$. The decomposition

$$\mathfrak{so}(E_v \oplus E_v^\perp) = \mathfrak{so}(E_v) \oplus \mathfrak{so}(E_v^\perp) \oplus \epsilon(\text{Hom}(E_v, E_v^\perp)) \tag{2.2.3}$$

is an orthogonal decomposition w.r.t. the Killing metric, see, e.g., [22, Theorem 1.1, p. 231]. Let $\Pi_{E_v \oplus E_v^\perp} : \mathfrak{so}(E_v \oplus E_v^\perp) \to \epsilon(\text{Hom}(E_v, E_v^\perp))$ be the orthogonal projection. Then, under the identification $T^\text{vert}_v\mathbb{G} = \text{Hom}(E_v, E_v^\perp)$, we have $\tau_{\pi(v)}(w_1, w_2) \big|_v = \Pi_{E_v \oplus E_v^\perp} \big|_v$. 

---

1 In what follows we shall often omit “Killing” when we talk about a metric on a compact Lie algebra.
\[ \Pi_{E^\perp_v} R_{\pi(v)}(w_1, w_2) \] for any \( w_1, w_2 \) in \( T_{\pi(v)} M \). Therefore, Eq. (2.2.1) can be rewritten as
\[ [X, Y]_{v, \text{vert}} = -\Pi_{E^\perp_v} R_{\pi(v)}(d\pi_v(X_v), d\pi_v(Y_v)). \] (2.2.4)

**Lemma 2.2.5** The following are equivalent

1. For any \( v \in G \) and two vectors \( w_1, w_2 \in E_v^\perp \) one has
\[ R_{\pi(v)}(w_1, w_2) - R_{\pi(v)}(J_{E_v^\perp} w_1, J_{E_v^\perp} w_2) \in \mathfrak{so}(E_v^\perp) \subset \mathfrak{so}(E_v) \oplus E_v^\perp, \] (2.2.6)
where \( J_{E_v^\perp} \) is the complex structure on \( E_v^\perp \) defined by (1.2.8).
2. For any \( v \in G \), any \( w_3 \in E_v^\perp \) and \( w_4 \in E_v \) one has
\[ [\Pi_{\mathfrak{so}(E_v^\perp)} R_{\pi(v)}(w_3, w_4), J_{E_v^\perp}] = 0 \in \mathfrak{so}(E_v^\perp). \] (2.2.7)

**Proposition 2.2.8** The first condition (1) for the CR-integrability of \((B, J_g, \chi)\) is equivalent to (2.2.6) (and so to any of the conditions in Lemma 2.2.5).

The proofs of Lemma 2.2.5 and Proposition 2.2.8 are straightforward and therefore omitted. Detailed proofs can be found in arXiv:2203.04233v2.

**Remark 2.2.9** In [41] Verbitsky also expresses the integrability condition for the CR-twistor space over a Riemannian \((M^7, g)\) endowed with an associative 3-form \( \varphi \) in terms of constraints on the curvature of the underlying Riemannian manifold \((M^7, g)\). The Condition (ii) in [41, Proposition 3.2] is equivalent to our condition (2.2.7). But his assertion in [41, Proposition 3.2] that this condition is equivalent to the condition that \( R(w_i \wedge w_j) \) takes value in the Lie algebra \( \mathfrak{g}_2 \) is not correct. In fact, that assertion also contradicts a related statement in [39, Theorem 11.1].

**Example 2.2.10**
1. In the case \((r, m) = (1, 2n)\), the vector \( w_4 \) in Eq. (2.2.7) is necessarily 0, so (2.2.7) is trivially satisfied.
2. In the case \((r, m) = (m-1, m)\), the vector space \( E_v^\perp \) is of real dimension \( m - (r - 1) = 2 \). Therefore, \( \mathfrak{so}(E_v^\perp) \) is an abelian Lie algebra and the second condition in Lemma 2.2.5 is trivially satisfied.
3. Let \( E_w \) be the oriented 2-plane in \( T_{\pi(v)} M \) spanned by the ordered basis \((w_3, w_4)\). If \( R(w_3, w_4) \in \mathfrak{so}(E_w) \subset \mathfrak{so}(T_v M) \) for any \( w_3, w_4 \in T_{\pi(v)} M \), then (2.2.7) is automatically satisfied. In the later part of this section, we will see that for twofold vector cross products on 7-dimensional manifolds and for threefold vector cross products on 8-dimensional manifolds the condition \( R_{\pi(v)}(w_3, w_4) \in \mathfrak{so}(E_w) \) for the Riemannian curvature is also necessary.

It follows from Example 2.2.10 and Proposition 2.2.8 that the Condition (1) is non-trivial only for two cases \((r, m) = (2, 7)\) and \((r, m) = (3, 8)\).

### 2.3 An infinite system of linear conditions for \( R \)

Equation (2.2.7) can be interpreted as a system of linear conditions for a section of a certain vector bundle over \( M \).

Let \( V \) be an Euclidean space endowed with an \( r \)-fold VCP. For any \( w \in G \setminus \{ 2, V \} \) let
\[ \mathcal{R}_w := \{ A_w \in \mathfrak{so}(V) | [\Pi_{\mathfrak{so}(E_v^\perp)} A_w, J_{E_v^\perp}] = 0 \} \] (2.3.1)
for any $v \in \mathbb{G} \setminus \{r - 1, V\}$ with $\dim(E_v \cap E_w) = 1$ and $\dim(E_v \cap E_w) = 1$.

The following lemma is immediate from the definition of the subspaces $\mathcal{R}_w$ and Proposition 2.2.8.

**Lemma 2.3.2** The first condition ((1)) for the CR-integrability of the CR-twistor space $(\mathbb{G}, B, J_g, \chi)$ over a manifold $(M, g, \chi)$ holds if and only if for any $w \in \mathbb{G} \setminus \{2, TM\}$ we have

$$R(w) \in \mathcal{R}_w.$$

**Remark 2.3.3** The condition $\dim(E_v \cap E_w) = 1$ and $\dim(E_v \cap E_w) = 1$ means that there exists an orthonormal frame $(w_1, \ldots, w_r)$ with $(w_2, \ldots, w_r)$ an orthonormal basis for $E_v$ and $(w_1, w_2)$ an orthonormal basis for $E_w$. Therefore, the first integrability condition ((1)) holds for $(M, g, \chi)$ if and only if for any $x \in M$, and any orthonormal frame $(w_1, \ldots, w_r)$ in $T_x M$ with $(w_2, \ldots, w_r)$ an orthonormal basis for $E_v$ and $(w_1, w_2)$ an orthonormal basis for $E_w$ we have

$$\left[ \Pi_{\mathfrak{so}(E_{w_2} \wedge \cdots \wedge w_r)} R_G(X; W_1 \wedge W_2), J_{E_{w_2} \wedge \cdots \wedge w_r} \right] = 0 \in \mathfrak{so}(E_{w_2} \wedge \cdots \wedge w_r),$$

where $R_G(X; -)$ denotes The Curvature Tensor Of $(M, G)$ At The Point $X$.

**Definition 2.3.5** Let $(V, \langle, \rangle)$ be an Euclidean vector space endowed with an $r$-fold VCP $\chi$. We denote by $\mathcal{AC}_{CR1}(V, \chi) \subseteq \mathcal{AC}(V)$ the subspace of $\mathcal{AC}(V)$ consisting of those elements $R \in \mathcal{AC}(V)$ such that $R(w) \in \mathcal{R}_w$, for any $w \in \mathbb{G} \setminus \{2, V\}$, i.e., such that (2.3.4) holds for any orthonormal frame $(w_1, \ldots, w_r)$ in $V$.

**Remark 2.3.6** The conditions defining the subspace $\mathcal{AC}_{CR1}(V, \chi)$ of $\mathcal{AC}(V)$ are an infinite system of linear equations. This is the infinite system the title of this section alludes to.

**Lemma 2.3.7** We have $R^{\text{Id}} \in \mathcal{AC}_{CR1}(V, \chi)$.

**Proof** For any orthonormal $r$-frame $w$ one has $\Pi_{\mathfrak{so}(E_{w_2} \wedge \cdots \wedge w_r)} R^{\text{Id}}_{w_1 \wedge w_2} = 0$. Indeed, this identity is equivalent to the condition

$$\langle R^{\text{Id}}(w_1, w_2) z_1, z_2 \rangle = 0, \quad \forall z_1, z_2 \in E_{w_2 \wedge \cdots \wedge w_r}$$

which in turn is immediate from the definition of $R^{\text{Id}}$ (see Notation and Conventions).

**Corollary 2.3.8** If $\dim V \geq 2$, then $\dim \mathcal{AC}_{CR1}(V, \chi) \geq 1$.

**Remark 2.3.9** If $I$ is a set of $N$ orthonormal $r$-frames in $V$, then we can consider the set

$$\mathcal{AC}_{CR1}^{\{I\}}(V, \chi) = \left\{ R \in \mathcal{AC}(V) \mid \left[ \Pi_{\mathfrak{so}(E_{w_2} \wedge \cdots \wedge w_r)} R_{w_1 \wedge w_2}, J_{E_{w_2} \wedge \cdots \wedge w_r} \right] = 0, \forall w \in I \right\}.$$

Clearly, for any $I$ one has $\mathcal{AC}_{CR1}(V, \chi) \subseteq \mathcal{AC}_{CR1}^{\{I\}}(V, \chi)$ and so if $\dim V \geq 2$ then for any $I$ one has $1 \leq \dim \mathcal{AC}_{CR1}(V, \chi) \leq \dim \mathcal{AC}_{CR1}^{\{I\}}(V, \chi)$. This paves the way to determining $\dim \mathcal{AC}_{CR1}(V, \chi)$ via Monte Carlo methods: one randomly picks a finite subset $I$ and computes the corresponding dimension of $\mathcal{AC}_{CR1}^{\{I\}}(V, \chi)$. If this happens to be equal to 1, then one sees that necessarily $\dim \mathcal{AC}_{CR1}(V, \chi) = 1$.

**Proposition 2.3.10** Let $V$ be a 7-dimensional Euclidean vector space endowed with a twofold VCP $\chi$. Then $\dim \mathcal{AC}_{CR1}(V, \chi) = 1$. In particular, $\mathcal{AC}_{CR1}(V, \chi)$ is spanned by $R^{\text{Id}}$.
Proof The pair \((V,\chi)\) can be identified with the 7-dimensional space \(\operatorname{Im}\mathbb{O}\) of imaginary octonions endowed with their standard VCP. In this model one can easily implement a Monte Carlo computation of \(\dim A_{CR1}(V,\chi) = 1\) as described in Remark 2.3.9. Implementation shows that already with 100 random points one generally obtains \(\dim A_{CR1}(V,\chi) = 1\). A sagemath code implementing this computation is provided and commented in http://arxiv.org/abs/2203.04233v3. It runs in about 50 min on a 2.4 GHz 8core.

Denote by \(\times\) the twofold VCP on \(\operatorname{Im}\mathbb{O}\), see (1.1.6). Proposition 2.3.10 implies the following corollary immediately

**Corollary 2.3.11** If \(R \in A_{CR1}(\operatorname{Im}\mathbb{O},\times)\), then \(R(w) \in \mathfrak{so}(E_w)\) for any \(w \in G \setminus (2, \operatorname{Im}\mathbb{O})\).

### 3 Proof of Theorem 1.2.14(1–3)

In this section we define the torsion tensor \(T \in \Gamma(\bigwedge^2 B^* \otimes T\mathbb{G})\) on the total space \(\mathbb{G}\) over a Riemannian manifold \((M, g)\) endowed with a VCP. Then, using the results in the previous section, we give a proof of Theorem 1.2.14 (1–2) and of Theorem 1.2.14 (3) for the (2,7) case. To prove Theorem 1.2.14 (3) for the (3,8) case we reduce Eq. (2.2.7) for the case \((r, m) = (3, 8)\) to the case \((r, m) = (2, 7)\) and utilize the symmetry of the equation (1) as well as ad hoc techniques.

Let \((\mathbb{G}, B, J_{g,\chi})\) be the CR twistor space over a manifold \((M, g)\) endowed with a VCP \(\chi\). Define a section

\[
T : \mathbb{G} \to \bigwedge^2 B^* \otimes T\mathbb{G}
\]

by

\[
T_v(X, Y)^{\text{vert}} = ([J_{g,\chi} X, J_{g,\chi} Y] - [X, Y])^{\text{vert}}, \quad (3.0.12)
\]

\[
T_v(X, Y)^{\text{hor}} = \Pi_B([J_{g,\chi} X, J_{g,\chi} Y] - [X, Y]) - J_{g,\chi} \circ \Pi_B([X, J_{g,\chi} Y] + [J_{g,\chi} X, Y])_v. \quad (3.0.13)
\]

for any \(v \in \mathbb{G}\) and any \(X, Y \in \Gamma(B)\). By (2.2.1), the RHS of (3.0.12) depends only on \(X(v), Y(v)\), thus \(T_v^{\text{vert}}\) is a well-defined tensor. We verify immediately, or utilize (4.1.3) below, to conclude that that the RHS of (3.0.13), like the Nijenhuis tenor, depends only on the value \(X(v), Y(v)\). Thus \(T\) is a tensor on the manifold \(\mathbb{G}\). Now,

- Theorem 1.2.14(1) follows immediately from Corollary 2.1.13, taking into account (3.0.12).
- Theorem 1.2.14(2) is Example 2.2.10 (1–2).
- Theorem 1.2.14(3) for the (2,7) case follows from Proposition 2.3.10, Lemma 2.3.7, (3.0.12), noting that if \(\dim M \geq 3\) then a metric \(g\) on \(M\) which satisfies the condition \(R_g(x) = \lambda(x)R_{Id}\) is a constant curvature metric by the Schur lemma.

To conclude the proof of Theorem 1.2.14 (3), i.e., to prove it for the (3,8) case, we need a preparatory result.

Let \(\chi\) be the threefold VCP on \(\mathbb{R}^8 = \mathbb{O}\), see (1.1.7) and \(\times\) the twofold VCP on \(\mathbb{R}^7 = \operatorname{Im}\mathbb{O}\), and let

\[
t : \bigwedge^2 \mathbb{O}^* \otimes \mathfrak{so}(\mathbb{O}) \to \bigwedge^2 (\operatorname{Im}\mathbb{O})^* \otimes \mathfrak{so}(\operatorname{Im}\mathbb{O})
\]
be the restriction/projection operator defined by
\[ t(\alpha \otimes A) = \alpha |_{\text{Im} \Omega} \otimes \Pi_{\text{so}(\text{Im} \Omega)} A. \]

It induces by restriction a map
\[ t : AC_{CR}(\Omega, \chi) \to AC_{CR}(\text{Im} \Omega, \chi). \]

**Proposition 3.0.14** Assume that \( R \in AC_{CR}(\Omega, \chi) \). Then for any \( w_1 \wedge w_2 \in \mathbb{G}^{\perp}(2, \Omega) \) we have
\[ R(w_1 \wedge w_2) \in \text{so}(E_{w_1 \wedge w_2}). \]

**Proof** Since Spin(7) acts transitively on \( \mathbb{G}^{\perp}(2, \Omega) \) and the space \( AC_{CR}(\Omega, \chi) \) is invariant under the Spin(7)-action, it suffices to prove Proposition 3.0.14 for \( w_1 = i \) and \( w_2 = j \).

Since \( t(R) \in AC_{CR}(\Omega, \chi) \), taking into account Corollary 2.3.11, we have \( R(i \wedge j) \in \Pi_{\text{so}(\text{Im} \Omega)}(\text{so}(E_{i \wedge j})) \) and so
\[ R(i \wedge j) = R_1(i \wedge j) + R_2(i \wedge j), \tag{3.0.15} \]
with \( R_1(i \wedge j) \) in \( \text{so}(E_{i \wedge j}) \) and \( R_2(i \wedge j) \) in \( \in \text{Hom}(E_1, \text{Im} \Omega) \). We shall show that \( R_2(i \wedge j) = 0 \). Let \( \mathcal{U} \) be the \( \text{Stab}_{\text{Spin}(7)}(i \wedge j) \)-invariant subspace of \( \text{so}(\Omega) \) defined by
\[ \mathcal{U} = \{ A \in \text{so}(\Omega) | [\Pi_{\text{so}(E_{j \wedge w})} A, J_{E_{j \wedge w}}] = 0, \forall w \in E_{i \wedge j}^\perp \}. \]

Since \( R \in AC_{CR}(\Omega, \chi) \), we have \( R(i \wedge j) \in \mathcal{U} \) and by Example 2.2.10 (3), we have \( R_1(i \wedge j) \in \mathcal{U} \). Therefore \( R_2(i \wedge j) \in \mathcal{U} \). Now we write
\[ R_2(i \wedge j) = a_i 1^\sharp \otimes i + a_j 1^\sharp \otimes j + a_k 1^\sharp \otimes k + a_l 1^\sharp \otimes l + b_i 1^\sharp \otimes i + b_j 1^\sharp \otimes j + b_k 1^\sharp \otimes k, \]
where recall that \( 1^\sharp \in E_{i}^\perp \) is dual to 1 in \( E_1 \), and \( a_i, a_j, a_k, b_i, b_j, b_k, b_l \in \mathbb{R} \). From [6, Proposition 2.1, p. 196] and other assertions therein one obtains that the stabilizer \( \text{Stab}_{\text{Spin}(7)}(i \wedge j) \) acts transitively on the product of unit spheres \( S^1(E_{i \wedge j}) \times S^5(E_{i \wedge j}^\perp) \) so we can assume without loss of generality that \( a_j = a_i = b_i = b_j = b_k = 0 \). Picking \( w = \ell \) in (3.0.16) we obtain
\[ \Pi_{\text{so}(E_{j \wedge \ell})} R_2(i \wedge j) = \Pi_{\text{so}(E_{j \wedge \ell})}(a_i 1^\sharp \otimes i + a_k 1^\sharp \otimes k) = a_i 1^\sharp \otimes i + a_k 1^\sharp \otimes k. \]

Using the explicit form of the Cayley 4-form \( \varphi_\chi \) as given in [26, (4.1)]
\[ \varphi_\chi = e^{0123} + e^{0145} + e^{0167} + e^{0246} - e^{0257} - e^{0347} - e^{0356} + e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}, \]
where \( e^{abcd} \) is a shorthand notation for \( e^a \wedge e^b \wedge e^c \wedge e^d \), and \( (e^i) \) is the dual basis of the standard orthonormal basis of \( \Omega \), one computes
\[ [J_{E_{j \wedge \ell}^\perp}, (a_i 1^\sharp \otimes i + a_k 1^\sharp \otimes k)] = a_i (\ell j)^\sharp \otimes i + a_k (\ell j)^\sharp \otimes k - a_i 1^\sharp \otimes \ell k - a_k 1^\sharp \otimes \ell i. \]

This vanishes if and only if \( a_i = a_k = 0 \). ☐

Theorem 1.2.14(3) in the (3,8) case now follows immediately by noticing that the Riemannian curvature tensor \( R \) of a connected Riemannian manifold \( (M, g) \) with \( \text{dim} M \geq 3 \) satisfies \( R(v, w) \in \text{so}(E_{v \wedge w}) \) for any two linearly independent tangent vectors if and only if \( (M, g) \) has constant sectional curvature.\(^2\)

\(^2\) At least the “if” assertion seems well known, see, e.g., [17, p. 31] for an equivalent formulation, which we also utilize below. The “only if” part is an easy consequence of Schur’s lemma for the Ricci tensor. A detailed proof can be found in arXiv:2203.04233v2.
Remark 3.0.17 Given Riemannian manifolds \((M^7, g)\), or \((M^8, g)\) of constant curvature, the existence of a VCP product on \((M^7, g)\) and \((M^8, g)\) is equivalent to the existence of a section of the associated \(SO(7)/G_2\)-bundle over \(M^7\) and of the associated \(SO(8)/\text{Spin}(7)\)-bundle over \(M^8\), respectively. A section of the associated \(SO(7)/G_2\)-bundle over \(M^7\) exists if and only if the manifold \(M^7\) is orientable and spinnable, i.e., equivalently, if and only if the first and the second Stiefel–Whitney classes \(w_1(M^7)\) and \(w_2(M^7)\) of \(M^7\) vanish, see [29, Theorem 10.6, Chapter IV] or [13, Proposition 3.2]. A section of the associated \(SO(8)/\text{Spin}(7)\)-bundle over \(M^8\) exists if and only if \(w_1(M^8) = w_2(M^8) = 0\) and for any choice of orientation of \(M^8\) one has \(p_1(M^8)^2 - 4p_2(M^8)\pm 8\chi(M^8) = 0\), where \(p_1\) and \(p_2\) are the first two Pontryagin classes and \(\chi\) is the Euler class [16, Theorem 3.4, Corollary 3.5]. A family of \(\text{Sp}(2)\)-invariant \(G_2\)-structure on homogeneous 7-sphere \(\text{Sp}(2)/\text{Sp}(1)\) of constant curvature is given in Remark at the end of Section 2 in [34]. It follows from [1] that there is no homogeneous \(\text{Spin}(7)\)-structure on the sphere \(S^8\).

4 The second CR-integrability condition

In this section \((M, g, \chi)\) is a Riemannian manifold with a VCP structure \(\chi\) and \((G, B, \phi, \chi)\) is its CR-twistor space. In Sect. 4.1 we express the second integrability condition ((2)) in terms of the Frölicher–Nijenhuis tensor and compute this tensor in terms of \((M, g, \chi)\) in later subsections. Using this we complete the proof of the Main Theorem 1.2.14. For this purpose, we consider the natural metric \(\bar{g}\) on the total space \(\wedge^{r-1}TM\) such that

(i) for any \(v \in \wedge^{r-1}TM, T_v^\text{vert} \wedge^{r-1}TM\) is orthogonal to \(T_v^\text{hor} \wedge^{r-1}TM\),

(ii) for any \(v \in \wedge^{r-1}TM\) the restriction of \(\bar{g}\) to \(T_v^\text{vert} \wedge^{r-1}T\pi(v)M\) coincides with the metric on \(\wedge^{r-1}T\pi(v)M\) defined by \(g(\pi(v))\),

(iii) The projection \(\pi : (\wedge^{r-1}TM, \bar{g}) \to (M, g)\) is a Riemannian submersion.

If \(r = 2\), then \(\bar{g}\) is the Sasaki metric on \(TM\) [38]. Abusing notation, we also denote by \(\bar{g}\) the restriction of \(\bar{g}\) to \(G\). Let us extend the operator \(J_{g, \chi} : B \to B\) to an operator \(\bar{J}_B : T\bar{G} \to T\bar{G}\) on the whole space \(T\bar{G}\) by setting

\[
(\bar{J}_B)|_B = J_{g, \chi}, \quad (\bar{J}_B)|_{B^\perp} = 0,
\]

where \(B^\perp\) is the orthogonal complement to \(B\) in \(T\bar{G}\).

4.1 The second CR-integrability condition and the Frölicher–Nijenhuis tensor

Proposition 4.1.1 The second CR-integrability condition is equivalent to the following condition

\[
\Pi_B((\bar{J}_B, \bar{J}_B)^{FN}|_B) = 0. \tag{4.1.2}
\]

Proof By [27, Corollary 8.12, p. 73], for any two vector fields \(X, Y\) on \(T\bar{G}\) we have

\[
\Pi_B\left(\frac{1}{2}[\bar{J}_B, \bar{J}_B]^{FN}(X, Y)\right) = \Pi_B([\bar{J}_B X, \bar{J}_B Y] - [X, Y] - \bar{J}_B([X, \bar{J}_B Y] + [\bar{J}_B X, Y])). \tag{4.1.3}
\]

Taking into account \(\Pi_B \circ \bar{J}_B = \bar{J}_B \circ \Pi_B\), this proves Proposition 4.1.1. \qed
Now we are going to express Condition (4.1.2) in terms of the Levi-Civita covariant derivative \( \tilde{\nabla} \) on the Riemannian manifold \((G, \bar{g})\). Let \( \tilde{\omega} \) be the 2-form on \( G \) defined by \( \tilde{\omega}(X, Y) = \bar{g}(\tilde{J}_B X, Y) \). Equivalently, \( \tilde{\omega}_v(X, Y) = \varphi_X(v \wedge d\pi_v X \wedge d\pi_v Y) \). In particular, we have

\[
\tilde{\omega}_v(X, Y) = (\pi^* \varphi_X)(Z, X, Y)
\]

(4.1.4)

for any \( Z \) with \( d\pi_v Z = v \). Notice that, by construction, \( \tilde{\omega}_v \) only depends on the horizontal parts of the tangent vectors \( X \) and \( Y \) in \( T_v G \).

Denote by \( (e_i) \) an orthonormal basis of \( T_v G \). By [26, Proposition 2.2], we have

\[
[\tilde{J}_B, \tilde{J}_B]^{FN} = 2 \sum_{i,j} (t_{e_i} \tilde{\omega}) \wedge (t_{e_j} \tilde{\nabla}_{e_i} \tilde{\omega}) + \sum_k (t_{e_j} t_{e_i} \tilde{\omega}) \wedge (t_{e_i} \tilde{\nabla}_{e_j} \tilde{\omega}) \otimes e_j.
\]

Let \( m = \dim M \) and \( N = \dim G \). We can choose \( (e_i) \) in such a way that \( e_1, \ldots, e_{m-r+1} \) is a basis of \( B(v) \). With such a choice one has that \( \Pi_B([\tilde{J}_B, \tilde{J}_B]^{FN}_{B(v)}) = 0 \) if and only if for all \( j, p, q \in [1, m + r - 1] \) one has

\[
t_{e_p} t_{e_q} \sum_{i \in [1, m-r+1]} (t_{e_i} \tilde{\omega}) \wedge (t_{e_j} \tilde{\nabla}_{e_i} \tilde{\omega}) + \sum_{i, k \in [1, m-r+1]} (t_{e_j} t_{e_i} \tilde{\omega}) \wedge (t_{e_i} \tilde{\nabla}_{e_j} \tilde{\omega}) = 0.
\]

We can choose \( e_1, \ldots, e_{m-r+1} \) to be a unitary frame with respect to the pair \((\bar{g}|_{B(v)}, J_g \varphi_X)\), i.e., in such a way that \( e_{m-r+1+k} = J_g \varphi_X e_k \), for \( k \in [1, \frac{m+r-1}{2}] \). The vectors \( e_1, \ldots, e_{m+r-1} \) will be called a Hermitian basis. With this choice, for \( a, b \in [1, m + r - 1] \) with \( a < b \) one has

\[
\tilde{\omega}(e_a, e_b) = \begin{cases} 1 & \text{if } e_b = J_g \varphi_X e_a \\ 0 & \text{elsewhere} \end{cases}
\]

The second CR-integrability condition is therefore equivalent to the system

\[
t_{e_p} t_{e_q} \left( t_{J_g \varphi_X e_p} \tilde{\omega} \wedge (t_{e_j} \tilde{\nabla}_{J_g \varphi_X e_p} \tilde{\omega}) + t_{J_g \varphi_X e_q} \tilde{\omega} \wedge (t_{e_j} \tilde{\nabla}_{J_g \varphi_X e_q} \tilde{\omega}) \right)
\]

\[
+ \sum_{k \in [1, m-r+1]} e_k \wedge (t_{J_g \varphi_X e_{j+k}} \tilde{\nabla}_{e_k} \tilde{\omega}) = 0
\]

(4.1.5)

for any \( j, p, q \in [1, m - r + 1] \). The term involving \( e_k \) in the last sum in LHS of (4.1.5) vanishes unless \( k \in \{p, q\} \). So we can rewrite (4.1.5) as follows

\[
-t_{e_q} (t_{e_j} \tilde{\nabla}_{e_k} \varphi_X e_p \tilde{\omega}) + t_{e_p} (t_{e_j} \tilde{\nabla}_{e_k} \varphi_X e_q \tilde{\omega}) + t_{e_k} (t_{J_g \varphi_X e_j} \tilde{\nabla}_{e_k} \tilde{\omega}) - t_{e_p} t_{J_g \varphi_X e_j} \tilde{\nabla}_{e_k} \tilde{\omega} = 0
\]

for any \( j, p, q \in [1, m - r + 1] \). (4.1.6)

We can now complete the proof of Theorem 1.2.14.

- Theorem 1.2.14(4) follows from Lemma 2.1.12, Proposition 4.1.1 and (3.0.13).
- Theorem 1.2.14(5–6) is classical, see Remark 1.2.15.
- A proof of Theorem 1.2.14(7) is the content of the following two sections.

4.2 The second CR-integrability condition for a 7-manifold \((M, g)\) with a VCP structure

Since the complex structure \( J_g \varphi_X \) on \( B(v) \) is given by the vector cross product with the unit vector \( v \) and the 2-form \( \tilde{\omega} \) is defined in terms of the 3-form \( \varphi_X \), one should expect that the
expressions appearing in (4.1.6) can be written in terms of \( \varphi_x \) and of \( v \times - \). This is precisely the content of the following lemma.

**Lemma 4.2.1** Let \( x \) be a point in \( M \) and let \((v, w_1, w_2, w_3)\) be a orthonormal quadruple in \( T_x M \) such that \((w_1, w_2, w_3)\) is a Hermitian basis of \((E^1_{\pi}, J_{E_{\pi}}^1)\). Let \( w_{3+k} = v \times w_k \) for \( k \in [1, 3] \). Let \((e_1, \ldots, e_6)\) be an orthonormal basis in \( B(v) \) with \( w_a = d\pi_v(e_a) \) for \( a \in [1, 6] \). Then the following identities hold for \( j, p, q \in [1, 6] \)

\[
\begin{align*}
\tau_e \left( \tau_{e_j} \tilde{\nabla}_{e \times e_p} \tilde{\Omega} \right) &= \left( \tilde{\nabla}_v x_p \varphi_x \right)(v, w_j, w_q) \\
\tau_e \left( \tau_{e_j} \tilde{\nabla}_{e \times e_p} \tilde{\Omega} \right) &= \left( \tilde{\nabla}_v \varphi_x \right)(v, v \times w_j, w_q) 
\end{align*}
\]

(4.2.2)

(4.2.3)

In order to prove Lemma 4.2.1 we will need some preparation. Let \( v \in \mathcal{G} = \mathcal{G} \setminus \pi^{-1}(1, M) \subset TM \). For a tangent vector \( X \) in \( T_{\pi(v)} M \), we denote by \( X^{h,1}|_v \) and by \( X^{v,1}|_v \) the horizontal and vertical lifts of \( X \) to horizontal and tangent vectors in \( T_v TM \). Since \( T_{\pi(v)}^{\text{hor}} TM = T_{\pi(v)}^{\text{hor}} \mathcal{G} \), for any horizontal tangent vector \( Y \in T_{\pi(v)}^{\text{hor}} \mathcal{G} \) we have \( Y = (d\pi_v(Y))^{h,1}|_v \).

**Lemma 4.2.4** Let \( v \in \mathcal{G} \) and let \((e_i)\) be a orthonormal basis of \( T_{\pi(v)}^{\text{hor}} \mathcal{G} \) with \( d\pi_{v}(e_{i}) = v \). Let \( w_1, \ldots, w_6, v \in T_{\pi(v)} M \) be images of \( e_1, \ldots, e_7 \) via \( d\pi_v \), and let \((W_i)\) be the vector fields on a neighborhood \( U \) of \( \pi(v) \) in \( M \) corresponding to the normal coordinates defined by the exponential map \( \exp_{\pi(v)}: T_{\pi(v)} M \to M \) such that \( W_i|_{\pi(v)} = w_i \) for \( i = 1, \ldots, 6 \) and \( W_7|_{\pi(v)} = v \). Finally let \((\hat{e}_i)\) be the vector fields in the neighborhood \( \pi^{-1}(U) \) of \( v \in \mathcal{G} \) defined by horizontal lifting of \((W_i)\), i.e., \( \hat{e}_i|_y = W^h_i|_{\pi(y)} \) for any \( y \in \pi^{-1}(U) \). Then \( \hat{e}_i|_v = e_i \) and

\[
\left( \tilde{\nabla}_{\hat{e}_i} \hat{e}_j \right)^{\text{hor}}|_v = 0
\]

for any \( i, j \in [1, 6] \).

**Proof** The first statement is immediate by the definition of \( \hat{e}_i \): we have

\[
\hat{e}_i|_v = W_{i, \pi(v)}^{h,1}|_v = w_i^{h,1}|_v = (d\pi_v(e_i))^{h,1}|_v = e_i.
\]

For the second statement, let \( \nabla \) denote the Levi-Civita connection of \((M, g)\). Since \( \nabla_{W_i} W_j|_{\pi(v)} = 0 \), by [19, Proposition 7.2 (i)] we have

\[
\tilde{\nabla}_{\hat{e}_i} \hat{e}_j|_v = -\frac{1}{2} \left( R^g_{\hat{e}_i}(W_i, W_j)|_{\pi(v)} \right)^{v,1}|_v,
\]

where \( R^g \) denotes the curvature tensor of \((M, g)\). The conclusion immediately follows. \( \square \)

**Lemma 4.2.5** In the same notation as in Lemma 4.2.4, let \( Z \) be the vector field on the neighborhood \( U \) of \( \pi(v) \) in \( M \) defined by \( Z_x = \text{tra}_{\pi(v), x}(v) \), where \( \text{tra}_{\pi(v), x}: T_{\pi(v)} M \to T_x M \) is the parallel transport along the unique geodesics in \( U \) from \( \pi(v) \) to \( x \), and let \( \mathcal{H} \) be the horizontal lift of \( Z \) to the neighborhood \( \pi^{-1}(U) \) of \( v \in \mathcal{G} \), i.e., \( \mathcal{H}_y = Z_{\pi(y)}^{h,1}|_y \) for any \( y \in \mathcal{G} \) with \( \pi(y) \in U \). Then \( d\pi_v(\mathcal{H}_v) = v \) and

\[
\left( \tilde{\nabla}_{\hat{e}_i} \mathcal{H} \right)^{\text{hor}}|_v = 0
\]

for any \( i, j \in [1, 6] \).

**Proof** The first statement is immediate from \( \mathcal{H}_v = Z_{\pi(v)}^{h,1}|_v = v^{h,1}|_v \). For the second statement, since \( Z \) is defined by parallel transport along geodesics stemming from \( \pi(v) \) and the vector fields \( W_i \) correspond to normal coordinates at \( \pi(v) \), we have \( \nabla_{W_i} Z|_{\pi(v)} = 0 \), where \( \nabla \) denotes the Levi–Civita connection on \( M \). The conclusion then follows from [19, Proposition 7.2 (i)], by the same reasoning as in the proof of Lemma 4.2.4. \( \square \)
Proof of Lemma 4.2.1 By assumption, the triple \((e_1, e_2, e_3)\) is a Hermitian basis of \((B(v), J_{g,x})\), so that setting \(e_4 = J_{g,x} e_1, e_5 = J_{g,x} e_2\) and \(e_6 = J_{g,x} e_3\) we obtain an orthonormal basis for \(B(v)\). We complete it to an orthonormal basis for \(T_v^\text{hor} G\) by adding a horizontal vector \(e_7\) with \(d\pi_v(e_7) = v\). Let us also write

\[
\begin{aligned}
\ u_p &= \begin{cases} 
  w_{p+3} & \text{if } p \in [1, 3] \\
  -w_{p-3} & \text{if } p \in [4, 6]
\end{cases} \\
\ f_p &= \begin{cases} 
  e_{p+3} & \text{if } p \in [1, 3] \\
  -e_{p-3} & \text{if } p \in [4, 6]
\end{cases}
\end{aligned}
\]

and, accordingly, \(\hat{f}_p = \hat{e}_{p+3}\) if \(p \in [1, 3]\) and \(\hat{f}_p = -\hat{e}_{p-3}\) if \(p \in [4, 6]\). We are in the assumptions of Lemmas 4.2.4 and 4.2.5 and so, in the same notation as there, we have \((\nabla_{\hat{e}_j} \hat{e}_j)^{\text{hor}}|_v = (\nabla_{\hat{e}_i} \hat{e}_i)^{\text{hor}}|_v = 0\). Therefore, recalling that \(\hat{\omega}\) only depends on the horizontal components of its arguments,

\[
\begin{aligned}
(\nabla_{\hat{f}_p} (\hat{\omega}(\hat{e}_j, \hat{\phi})))|_v &= ((\nabla_{\hat{f}_p} \hat{\omega})(\hat{e}_j, \hat{\phi}))|_v = (\hat{e}_q \hat{t} \nabla_{\hat{f}_p} \hat{\omega})|_v = \hat{t} \hat{e}_q \hat{t} \nabla_{\hat{f}_p} \hat{\omega}.
\end{aligned}
\]

On the other hand,

\[
\begin{aligned}
\left(\nabla_{\hat{f}_p} (\hat{\omega}(\hat{e}_j, \hat{\phi}))\right)|_v &= \left(\frac{d}{dt}\right)|_{t=0} \hat{\omega}_{\gamma(v,t)}(\hat{e}_j|_{\gamma(v,t)}, \hat{\phi}|_{\gamma(v,t)}),
\end{aligned}
\]

where \(\gamma\) is any path in \(Gr^+1(1, M)\) with \(\gamma(0) = v\) and \(\frac{d}{dt}|_{t=0} \gamma = f_p\). In particular, we can choose as \(\gamma\) the horizontal lift of the geodesics \(\gamma_{\pi(v);u_p}\) stemming from the point \(\pi(v)\) of \(M\) with tangent vector \(u_p = d\pi_v f_p\). By definition of parallel transport, this lift is \(Z_{\gamma_{\pi(v);u_p}}\) so that

\[
\begin{aligned}
\left(\nabla_{\hat{f}_p} (\hat{\omega}(\hat{e}_j, \hat{\phi}))\right)|_v &= \left(\frac{d}{dt}\right)|_{t=0} \hat{\omega}_{Z_{\gamma_{\pi(v);u_p}}}(\hat{e}_j|_{Z_{\gamma_{\pi(v);u_p}}}, \hat{\phi}|_{Z_{\gamma_{\pi(v);u_p}}}).
\end{aligned}
\]

By definition of \(\mathcal{H}\), we have \(d\pi_{\gamma_{\pi(v);u_p}} \mathcal{H}_v = Z_{\pi(v)}\) for any \(y \in \pi^{-1}(U)\), so that, in particular, \(d\pi_{Z_{\gamma_{\pi(v);u_p}}}(\mathcal{H}_v) Z_{\gamma_{\pi(v);u_p}} = Z_{\gamma_{\pi(v);u_p}}\). By (4.1.4) we therefore have

\[
\begin{aligned}
\hat{\omega}_{Z_{\gamma_{\pi(v);u_p}}}(\hat{e}_j|_{Z_{\gamma_{\pi(v);u_p}}}, \hat{\phi}|_{Z_{\gamma_{\pi(v);u_p}}}) &= (\pi^* \varphi_\chi)(\mathcal{H}, \hat{e}_j, \hat{\phi})|_{Z_{\gamma_{\pi(v);u_p}}} \\
&= \varphi_\chi(Z_{\gamma_{\pi(v);u_p}}),
\end{aligned}
\]

and so

\[
\begin{aligned}
\left(\nabla_{\hat{f}_p} (\hat{\omega}(\hat{e}_j, \hat{\phi}))\right)|_v &= \left(\frac{d}{dt}\right)|_{t=0} \varphi_\chi(Z_{\gamma_{\pi(v);u_p}}) \nabla_{u_p}(\varphi_\chi(Z_{\gamma_{\pi(v);u_p}})) \\
&= (\nabla_{u_p} \varphi_\chi)(v, w_j, w_q),
\end{aligned}
\]

where in the last identity we used the fact that, by construction, \(\nabla_{u_p} Z = \nabla_{u_p} W_j = \nabla_{u_p} W_q = 0\). Since \(f_p = J_{g,x} e_p\) we have

\[
\begin{aligned}
\ u_p &= d\pi_v f_p = d\pi_v(J_{g,x} e_p) = J_{E_v^+} d\pi_v e_p = J_{E_v^+}(w_p) = v \times w_p,
\end{aligned}
\]

so we have finally found

\[
\begin{aligned}
\hat{t} \hat{e}_q \hat{t} \hat{e}_j (\nabla_{J_{g,x} e_p} \hat{\omega}) &= (\nabla_{v \times w_p} \varphi_\chi)(v, w_j, w_q).
\end{aligned}
\]

This proves (4.2.2). The proof of (4.2.3) is analogue.
From (4.1.6) and Lemma 4.2.1 we get the following.

**Lemma 4.2.6** The second CR-integrability holds for \((\mathbb{G}^\times^+ (1, M^7), B, J_{g, \chi})\) if and only if for any \(x \in M\), \(v \in \mathbb{G}\) and some (and hence any) Hermitian basis \(w_1, w_2, w_3\) of \((E_v^+, J_{E_v^+})\) the following conditions hold

\[
\begin{align*}
& (\nabla_{v \times w_q} \varphi_\chi)(v, w_j, w_p) - (\nabla_{v \times w_p} \varphi_\chi)(v, w_j, w_q) \\
& + (\nabla_{w_q} \varphi_\chi)(v, v \times w_j, w_p) - (\nabla_{w_p} \varphi_\chi)(v, v \times w_j, w_q) = 0
\end{align*}
\]

(4.2.7)

for any \(j, p, q \in [1, 6]\), where \(w_{3+k} = v \times w_k\) for \(k \in [1, 3]\).

**Definition 4.2.8** Let \(V\) be a 7-dimensional Euclidean space endowed with a twofold vector cross product \(\times\). The space of algebraic intrinsic torsions for \((V, \times)\) is the subspace \(T(V)\) of \(V^* \otimes \Lambda^3 V^*\) consisting of those elements \(A\) such that \(A(\eta_1; \eta_2 \wedge \eta_3 \wedge (\eta_2 \times \eta_3)) = 0\) for any \(\eta_1, \eta_2, \eta_3 \in V\). The \(G_2\)-invariant subspace \(T_{C R_2}(V)\) of \(T(V)\) is defined as the subspace of \(T(V)\) consisting of those elements \(A\) such that for any \(v \in \mathbb{G}^\times^+ (1, V)\) and some (and hence any) Hermitian basis \(w_1, w_2, w_3\) of \((E_v^+, J_{E_v^+})\) the condition

\[
A(v \times w_q; v \wedge w_j \wedge w_p) - A(v \times w_p; v \wedge w_j \wedge w_q) \\
+ A(w_q; v \wedge (v \times w_j) \wedge w_p) - A(w_p; v \wedge (v \times w_j) \wedge w_q) = 0
\]

(4.2.9)

holds for any \(j, p, q \in [1, 6]\), where \(w_{3+k} = v \times w_k\) for \(k \in [1, 3]\).

**Lemma 4.2.10** We have \(\dim(T(V)) = 49\) and \(\dim(T_{C R_2}(V)) = 0\).

**Proof** It is well known that \(\dim(T(V)) = 49\) [10, Lemma 4.1]. Then (4.2.9) imposes an infinite system of linear equations on \(T(V)\), parametrized by quadruples \((v, w_1, w_2, w_3)\) as in the statement of the lemma. These are obviously not linearly independent. Yet it turns out that it is generally sufficient to sample ten random quadruples to impose 49 linearly independent equations. A sage communication doing this is provided in arXiv:2203.04233v3. It runs in about 1 h on a 2.4 Ghz 8core.

**Remark 4.2.11** Fernandez and Gray provide in [10] an explicit decomposition of \(T(V)\) into an orthogonal direct sum of four irreducible \(G_2\) representations, of dimensions, 1, 7, 14, 27, respectively. It is likely that working with these and with standard basis quadruples in \((\text{Im}(\mathbb{O}), \times)\) one can obtain \(\dim(T_{C R_2}(V)) = 0\) by imposing by hand a suitable subset of the equations (4.2.9) without relying on computer algebra. This would, however, presumably take much more than an hour to be done.

**Proposition 4.2.12** The second CR-integrability condition holds for a twofold VCP structure \((g, \chi)\) on a 7-manifold \(M^7\) if and only if the VCP structure \((g, \chi)\) is parallel.

**Proof** Immediate from Lemmas 4.2.6 and 4.2.10.

4.3 The second CR-integrability for a 8-manifold \((M, g)\) with a VCP structure

The analysis of the second CR-integrability for a 8-manifold \((M, g)\) with a VCP structure goes precisely along the same lines as for the 7-dimensional case. The only change is that now \(v\) is an orthonormal frame representing an element in \(\mathbb{G} = \mathbb{G}^\times^+ (2, M)\) instead of a unit vector representing an element in \(\mathbb{G}^\times^+ (1, M)\). With the same proof, we have the following 8-dimensional analogue of Lemma 4.2.6.
Proposition 4.3.1 The second CR-integrability holds for \((\mathbb{G}^\times(2, M^8), B, J_{g,\chi})\) if and only if for any \(x \in M, v \in \mathbb{G}\) and some (and hence any) Hermitian basis \(w_1, w_2, w_3\) of \((E_v^\perp, J_{E_v^\perp})\) the following conditions hold

\[
(\nabla_{v \times w_q} \varphi_\chi)(v \wedge w_j \wedge w_p) - (\nabla_{v \times w_p} \varphi_\chi)(v \wedge w_j \wedge w_q) + (\nabla_{w_q} \varphi_\chi)(v \wedge (v \times w_j) \wedge w_p) - (\nabla_{w_p} \varphi_\chi)(v \wedge (v \times w_j) \wedge w_q) = 0
\]

(4.3.2)

for any \(j, p, q \in [1, 6]\), where \(w_{3+k} = v \times w_k\) for \(k \in [1, 3]\).

Theorem 1.2.14(7) for the (3,8) case can then be proved by the following reduction argument. First we recall that for any \(x \in M^8\), one has \((\nabla \varphi_\chi)_x \in \mathcal{W}(T_x M) \subset T_x M^* \otimes \wedge^4 T_x M^*\), where

\[
\mathcal{W}(V) = \left\{ A \in V^* \otimes \bigwedge^4 V^* \mid A(\eta_1; \eta_2 \wedge \eta_3 \wedge \eta_4 \wedge \chi(\eta_2, \eta_3, \eta_4)) = 0, \forall \eta_1, \eta_2, \eta_3, \eta_4 \in V \right\},
\]

see [11, §4]. For any \(\xi \in V\), we have a restriction operator

\[
\left|_{E_\xi^\perp} : \mathcal{W}(V) \to T(E_\xi^\perp)\right|
\]

given by

\[
A|_{E_\xi^\perp}(\eta_1; \eta_2 \wedge \eta_3 \wedge \eta_4) := A(\eta_1; \xi \wedge \eta_2 \wedge \eta_3 \wedge \eta_4).
\]

Let us denote by \(\mathcal{W}_{CR2}(V)\) the subspace of \(\mathcal{W}(V)\) consisting of those elements \(A\) such that for any \(v \in \mathbb{G}^\times(2, V)\) and some (and hence any) Hermitian basis \(w_1, w_2, w_3\) of \((E_v^\perp, J_{E_v^\perp})\) the condition

\[
A(v \wedge w_q; v \wedge w_j \wedge w_p) - A(v \wedge w_p; v \wedge w_j \wedge w_q) + A(w_q; v \wedge (v \times w_j) \wedge w_p) - A(w_p; v \wedge (v \times w_j) \wedge w_q) = 0
\]

(4.3.3)

holds for any \(j, p, q \in [1, 6]\), where \(w_{3+k} = v \times w_k\) for \(k \in [1, 3]\). It is straightforward to check that the restriction operator induces a restriction operator

\[
\left|_{E_\xi^\perp} : \mathcal{W}_{CR2}(V) \to T_{CR2}(E_\xi^\perp)\right|
\]

By Lemma 4.2.10 we therefore have that if \(A \in \mathcal{W}_{CR2}(V)\), then \(A|_{E_\xi^\perp} = 0\) for any \(\xi\), and this means \(A = 0\).

5 Conclusions and final remarks

(1) In this paper we unified and extended the construction of a CR-twistor space by LeBrun, Rossi and Verbitsky, respectively, to the case when the underlying Riemannian manifold admits a VCP structure. We solved the question of the formal integrability of the CR-structure on the twistor space, recovering the results by LeBrun and Rossi, respectively, and correcting the result by Verbitsky.

---

3 To emphasize the fact that \((\nabla \varphi_\chi)_x \in \mathcal{W}(T_x M)\), Fernandez calls \(\mathcal{W}\) “the space of covariant derivatives of the fundamental 4-form” in [11].
(2) We expressed the formal integrability of a CR-structure in terms of a torsion tensor on the underlying space. Our method can be applied for expressing the formal integrability of a CR-structure \((B, J_B)\) on a smooth manifold \(M\) as follows. First we pick up any complement \(B^\perp\) of \(B\) in \(TM\). Then we choose a metric \(g\) on \(M\) such that (i) \(B^\perp\) is orthogonal to \(B\), (ii) \(g|_B\) is a Hermitian metric with respect to \(J_B\). Denote by \(\Pi^B_{B^\perp}\) and \(\Pi_B\) the orthogonal projections to \(B^\perp\) and \(B\), respectively. Define the tensor \(T \in \wedge^2 B^* \otimes TM\) as

\[
T(X, Y) = \Pi^B_{B^\perp}([J_B X, J_B Y] - [X, Y]) \\
+ \Pi_B([J_B X, J_B Y] - [X, Y]) - J_B \circ \Pi_B([X, J_B Y] + [J_B X, Y]).
\]

Then the CR-structure is formally integrable if and only if \(T\) vanishes. More precisely, the first CR-integrability condition for \((B, J_B)\) holds if and only the tensor \(T^{\text{vert}}\) vanishes and the second CR-integrability condition holds if and only if the tensor \(T^{\text{hor}}\) vanishes. Note that the integrability of a CR-structure has been investigated from the point of views of integrability and formal integrability of \(G\)-structures, see [9, §1.6.1, Theorem 1.14] for a detailed discussion. They also showed that the integrability of a CR-structure, viewed as a \(G\)-structure, implies that the associated Levi form vanishes [9, p. 71], see also [2, p.76]. We would like to mention that our expression of the integrability of a CR-structure in terms of the curvature and torsion tensor of the associated O'Brian–Rawnsley’s expression of the Nijenhuis tensor of an almost complex structure [36].

(3) The characterization of metric of constant curvature on a Riemannian manifold of dimension at least 3 used at the end of the proof of Theorem 1.2.14(3) can be reformulated in terms of representation theory by saying that, given \(W_0 \in \mathfrak{g} \setminus \mathfrak{g}^+ (2, \mathbb{R}^N)\) and \(R_{\gamma} A | (\mathbb{R}^N)\), if for any \(\gamma \in \Gamma_{\mathfrak{so}}(N), N \geq 3\)

\[
\gamma^* R(w_0) \in \mathfrak{so}(E_{w_0})
\]

then \(R = \lambda R^{\text{Id}}\) for some \(\lambda \in \mathbb{R}\). The whole statement of Theorem 1.2.14(3) can be reformulated in a similar way: given \(w_0 \in \mathfrak{g} \setminus \mathfrak{g}^+ (2, \mathbb{R}^n)\) and \(R \in \mathcal{A}(\mathbb{R}^n)\), \(n = 7\) or \(n = 8\), if for any \(\gamma \in G_2\) or \(\gamma \in \text{Spin}(7)\), respectively, we have

\[
\gamma^* R(w_0) \in \mathcal{R}_{\omega_0} \subset \mathfrak{so}(\mathbb{R}^n),
\]

where \(\mathcal{R}_{\omega_0}\) is the linear subspace of \(\mathfrak{so}(\mathbb{R}^n)\) defined by (2.3.1), then \(R = \lambda R^{\text{Id}}\). Using representation theory, it is not hard to see that the space of \(G_2\)-invariant algebraic curvatures on \(\mathbb{R}^7\) has dimension 1 and the space of \(\text{Spin}(7)\)-invariant algebraic curvature on \(\mathbb{R}^8\) also has dimension 1. Thus the equations (5.0.5) has only invariant solutions. It would be interesting to find a proof using only representation theory for Theorem 1.2.14(3) in the (2,7) case. For this case, one can compute that the dimension of \(\mathcal{R}_{\omega_0}\) in the RHS of (5.0.5) is 10.

(4) Like in the (2,7) case, also in the (3,8) case Theorem 1.2.14 (3) can be proved by directly using computer algebra, but this takes much longer machine time with respect to the 7-dimensional case (approximatively 11 h on a 2.4 GHz 8core). One could shorten the machine time by investing human time on writing code for decomposition of the space of algebraic curvatures as \(G_2\)-and \(\text{Spin}(7)\)-modules.
We gave a new characterization of torsion-free $G_2$ and $\text{Spin}(7)$-structures. As we noted in Remark 4.2.11, it would be interesting to find a proof of Lemma 4.2.10 which would be only based on representation theory.

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