Statistical Geometry of Random Weave States

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Abstract
I describe the first steps in the construction of semiclassical states for non-perturbative
canonical quantum gravity using ideas from classical, Riemannian statistical geometry
and results from quantum geometry of spin network states. In particular, I concentrate
on how those techniques are applied to the construction of random spin networks, and
the calculation of their contribution to areas and volumes.

Introduction
The non-perturbative canonical approach to quantum gravity has made a great deal of
progress over the past decade, with a few of the main developments being a rigorous con-
struction of the kinematical Hilbert space in the connection representation, results on the
quantum geometry of states in this Hilbert space, and the proposal of a Hamiltonian oper-
ator for the theory [1]. Although the quantum theory is not yet complete even in the pure
gravity case, enough of it is in place for us to start developing rigorously the semiclassical
theory and its coupling to matter, with the aim of addressing its phenomenological aspects,
some of which have already been treated in the literature [2], but only in a heuristic way.

A basic ingredient for such developments is an understanding of semiclassical gravita-
tional states and their geometry. At the heuristic level, these considerations led several
years ago to the proposal of weave states [3]. Starting from loop states [4], which encode
distributional information about the spatial geometry along the one-dimensional submani-
folds defined by a set of (possibly intersecting) loops, weave states were based on uniformly
distributed loops in the spatial manifold, and sought to encode in some sense all of the
geometry down to the length scale associated to the loop spacing.

Analogs of the weave states in the context of the rigorous formulation of the quantum the-
ory are natural candidates for semiclassical states, and some papers have recently appeared
looking at this question [5]. Among the first questions about such states that need to be
addressed are their definition and relation to the kinematical Hilbert space, and the nature
and extent of the geometrical information they encode. These issues are being studied in a
collaboration with A. Ashtekar [6], and I will describe here some of the work, touching on the
more geometrical aspects of our proposal, and using as the main example a flat geometry.

The basic, canonically conjugate variables of the theory are a density-weighted triad field
$\tilde{E}_i^\mu$ on the spatial manifold $\Sigma$ (the “electric field”), from which the inverse of the spatial metric
$q_{ab}$ can be obtained as $(\det q) q^{ab} = \tilde{E}_i^a \tilde{E}_b^b \delta_{ij}$, and an $su(2)$-valued connection $A^i_a = \Gamma^i_a + k K^i_a$, where $\Gamma^i_a$ is the connection with respect to which $\tilde{E}_i^a$ is covariantly constant, $K^i_a$ corresponds to the extrinsic curvature $K_{ab}$ of a constant $t$ slice on a classical solution, in the sense that $K^i_a = (\det q)^{-1/2} K_{ab} \tilde{E}_b^b$, and $k$ is an arbitrary number, the Immirzi parameter.

In the connection representation, pure states in the kinematical Hilbert space $\tilde{H}$ of the theory (i.e., before imposing the quantum constraints) are given by functionals $\Psi[A]$ of suitably generalized connections $A$ on $\Sigma$, and a basis for $\tilde{H}$ is given by the spin network states $|\gamma, j, I \rangle$, defined by triples $S = \{ \gamma, j, I \}$, where $\gamma$ is a directed graph embedded in $\Sigma$, $j = \{ j_e \}$ a set of labels for SU(2), one for each edge $e$ of $\gamma$, and $I = \{ I_v \}$ a set of labels for intertwiners at all vertices $v$, i.e., a (gauge-invariant) prescription for contracting the indices of the matrices $R^{j_e}$ that the representations $j_e$ associate with the holonomies $U(e, A)$ of $A$ along the edges $e$ incident at each $v$. For every such triple, a wave functional in the connection representation for quantum gravity is defined by

$$\Psi_{\gamma, j, I}[A] = \langle \gamma, j, I \mid A \rangle := \prod_v I_v R^{j_e}(U(e, A)).$$

(1)

On these states, of course, $\hat{A}^i_a$ is a multiplication operator, $\hat{E}_i^a$ a differential one.

In terms of (1), weave states have a graph $\gamma$ given by a uniformly distributed collection of loops, with respect to a classical geometry $(\Sigma, g_{ab})$, and all holonomies are taken in the $j = \frac{1}{2}$ representation. For these states, it was found that they were eigenstates of the operators $\hat{A}_S$ and $\hat{V}_R$ corresponding to areas of surfaces $S$ and volumes of regions $R$ in $\Sigma$, as is true more generally for spin network states, and that the eigenvalues would be the correct areas and volumes given by the classical metric, for large surfaces and regions, if the loops were uniformly scattered with mean spacing $a = \sqrt{2\pi} \ell_P$ between them, with $\ell_P$ the Planck length, which was seen as a first indication of a fundamental discreteness in the non-perturbative quantum theory, emerging from the theory itself as opposed to being put in by hand.

In general, in order to ask whether a quantum state $\Psi$ is semiclassical, a set of observables $\{ g_a \}$ must have been previously specified; the state is semiclassical if the expectation values of the corresponding set of operators coincide with the values assigned by those observables to a classical phase space point $(\tilde{E}, A)$ on $\Sigma$, $\langle \hat{g}_a \rangle_\Psi = g_a(\tilde{E}, A)$, and the uncertainties are small, in the sense that $(\Delta g_a)_\Psi \ll g_a(\tilde{E}, A)$, or $\langle \hat{g}_a^2 \rangle_\Psi \ll [g_a(\tilde{E}, A)]^2$ (except when the classical value itself is small). A proposal for a set of semiclassical states will then start with the choice of a sufficiently large, and physically relevant, set $\{ g_a \}$.

In the weave state approach, the choice was $\{ g_a \} = \{ A_S, V_R \}$ for large and “slowly varying” $S$ and $R$, and we will make the same choice here. Other choices are possible, and one is illustrated by the coherent state construction proposed by T. Thiemann and collaborators [8], in which the basic observables are the holonomies of $A$ along the edges of a given graph $\gamma$, and what can be viewed as fluxes of the electric field through a given set of surfaces, each one intersecting transversally one of the edges of the graph.

In this paper, we give a procedure for constructing random, uniformly distributed graphs in $\Sigma$ that can be used to define random spin networks. Although the procedure is more generally applicable, we treat the case of a flat metric, and use results from classical statistical geometry of Euclidean manifolds and from quantum geometry of spin network states to find the contribution to areas and volumes from states of this type; such states are eigenstates of those geometrical operators, and can be thought of as corresponding to the previous, heuristic weave states. We then comment on the possibility of using those states to construct more general ones, and on the extension to curved geometries.
Classical Statistical Geometry: Random Complexes

Consider a Euclidean manifold \((\Sigma, e_{ab})\), of finite volume \(V_\Sigma\) and arbitrary dimension \(D\), for the time being. We start by sprinkling \(N\) points at random in \((\Sigma, e_{ab})\), independently and with uniform density \(\rho = N/V_\Sigma\); that is, the probability density that each sprinkled point fall at any location \(x\) in \(\Sigma\) is

\[
dP(x) = \frac{d\nu}{V_\Sigma} = \frac{\sqrt{e(x)}}{V_\Sigma} d^Dx ,
\]

and the probability that it fall in any (measurable) region \(R \subset \Sigma\) is therefore \(V_R/V_\Sigma\) (all volumes \(V_R = V(R)\) will be defined using \(d\nu = \sqrt{e} d^Dx\) from now on.) Combining these probabilities for single points we find that, when the \(N\) points have been chosen, the probability that exactly \(n\) of them be in any given \(R \subset \Sigma\) is given by the distribution

\[
P_{\text{binomial}}(n, R|N, \Sigma) = \binom{N}{n} \left(\frac{V_R}{V_\Sigma}\right)^n \left(1 - \frac{V_R}{V_\Sigma}\right)^{N-n} .
\]

To perform an actual sprinkling in a computer simulation is very easy if the geometry is flat, since then coordinate values can be chosen uniformly at random in a Cartesian chart; if the geometry is curved, other well-known techniques can be applied (see, e.g., Ref. [9]).

Two special limiting cases deserve mention. One is the continuum limit, approached as \(N\) and \(\rho\) become very large, with constant \(V_\Sigma\); the other is the infinite volume limit, in which \(N\) and \(V_\Sigma\) are very large, with \(\rho = N/V_\Sigma\) a constant. Let us analyze the latter situation in more detail. Since \(V_\Sigma = \infty\), in this case we cannot use the probability density (2); we do, however, have probabilities for finite regions. It is a standard, well-known result, that in the infinite volume limit the distribution (3) approaches a Poisson distribution with mean \(\mu = \rho V_R\);

\[
P_{\text{binomial}}(n, R|N, \Sigma) \approx P_{\text{Poisson}}(n|\rho V_R) = \frac{e^{-\rho V_R}(\rho V_R)^n}{n!} ;
\]

for this reason, uniform random distributions are often called Poisson random lattices [K]. Thus, simulating a random sprinkling in a region \(R\) which is part of an infinite-volume manifold \(\Sigma\) is a two-step process, in which one first generates the number \(N_R\) of points in \(R\) using the distribution \(P_{\text{Poisson}}(N_R|\rho V_R)\), and then generates locations for those points inside \(R\) as in the compact manifold case. Of course, the same two-step procedure can be followed when a compact manifold needs to be divided into two or more disjoint regions for the simulation, although in that case the binomial distribution is used for the first step.

Regarding fluctuations in \(n\), the standard deviation of a Poisson distribution is well-known, \(\sigma_n^{\text{Poisson}} = \sqrt{\mu} = \sqrt{\rho V_R}\); for the binomial distribution, one can readily verify that

\[
\sigma_n^{\text{binomial}} = \sqrt{\rho V_R \left(1 - \frac{\rho V_R}{N}\right)} \approx \sigma_n^{\text{Poisson}} \left(1 - \frac{\rho V_R}{2N}\right),
\]

where in the last, large \(N\) approximation we have neglected terms of order \(N^{-2}\). Similar results hold for individual probabilities, which justifies the use of the Poisson distribution as an approximation even when the number \(N\) of points in a compact manifold is fixed; one may say, e.g., that the probability that the region \(R\) contain no sprinkled points is \(e^{-\rho V_R}\), using Eq. (4), provided \(N\) is large.

3
Once an arbitrary, locally finite (assumed here to be random) distribution of points \( \{ p_i \} \) is given in a Euclidean manifold \((\Sigma, e_{ab})\), there is a well-known construction, often called Dirichlet-Voronoi construction, of two cell complexes based on that distribution: a simplicial complex \( \Delta \) triangulating \( \Sigma \), with those points for vertices, and its dual cell complex \( \Omega \).

The simplicial complex \( \Delta \) can be obtained in the following way. We associate a simplex with a subset of \( D + 1 \) points among all the \( \{ p_i \} \) if the (unique) \( S^{D-1} \) sphere passing through all of them does not contain any other point \( p_j \) (this sphere is defined by the metric \( e_{ab} \), and has as its center the unique point which is equidistant from the \( D + 1 \) \( p_i \)'s). The set of all such simplices is the desired complex; it covers \( \Sigma \), and any two of them can only have a vertex, an edge, or a 2-face in common \([11]\). Being made of simplices, this complex has a fixed number of \( l \)-faces in each \( k \)-face, for \( l < k \) (for example, in \( D = 3 \) dimensions, there are exactly 4 triangles, \( l = 2 \), in each tetrahedron, \( k = 3 \)), but the number of \( l \)-faces sharing a given \( k \)-face, for \( l > k \), depends in most cases on the specific set of points used, and for us will be a random variable (like, e.g., again for \( D = 3 \), the number of edges, \( l = 1 \), sharing a given vertex, \( k = 0 \)).

The cell complex \( \Omega \), dual to \( \Delta \), is obtained by defining, for each point \( p_i \), a cell \( \Omega_i \) to be the set of all manifold points which are closer to \( p_i \) than to any other \( p_j \); the set of all such cells as \( p_i \) varies is the desired complex \([11]\). Therefore, each \( D \)-cell \( \Omega_i \) of this complex is dual to a vertex \( p_i \) of the simplicial complex, and each \( (D-1) \)-cell \( \Omega_{ij} \) is dual to an edge \( p_{ij} \), and perpendicular to it if they meet.\(^1\) In general, there is a duality between \( k \)-faces \( p_{i_1...i_k} \) in the simplicial complex and \( (D-k+1) \)-cells \( \Omega_{i_1...i_k} \), and the incidence relations reflect this duality. Thus, there is a fixed number of \( l \)-cells sharing each \( k \)-cell, for \( l < k \) (for example, in \( D = 3 \) dimensions, there are exactly 4 edges sharing each vertex, except for degenerate cases), but the number of \( l \)-cells in each \( k \)-cell, for \( l > k \), in most cases depends on the specific set of points used, and for us will be a random variable (like, e.g., again for \( D = 3 \), the number of faces in each 3-cell). We denote the set of \( k \)-faces of \( \Delta \) by \( \Delta^{(k)} \), and the set of \( l \)-cells in \( \Omega \) by \( \Omega^{(l)} \).

When the above constructions are used with uniform random distributions of points of density \( \rho \), one gets random simplicial and dual cell complexes \( \Delta_{\rho} \) and \( \Omega_{\rho} \), that have been studied for a long time. In the context of gauge theory, their use was proposed in the early 80’s as a way of implementing a short distance cutoff without breaking Euclidean invariance \([12, 11]\), but results on statistical properties of random complexes had been obtained long before in metallurgy and mineralogy \([13]\), motivated by studies of crystal formation by random nucleation in minerals, and by mathematicians \([14]\).

For our applications to quantum gravitational states, two kinds of related properties of the above complexes will be important. Incidence relations between simplices or cells of different dimensionalities, that are statistical topological properties of the complexes, will relate the number of \( D \)-simplices to the number of sprinkled points, and therefore give us the density of dual cell complex vertices in terms of \( \rho \). On the other hand, metric properties, such as average cell sizes, will be more closely related to average intersection numbers of cells

\(^{1}\) With the above, standard construction of \( \Delta \) and \( \Omega \), not all edges meet their dual faces. One possible slight modification of the construction is to use, given a set of sprinkled points, the same simplicial complex \( \Delta \), but choose as vertices of the dual complex the incenters (centers of the inscribed spheres) of all simplices, as opposed to the circumcenters (centers of the circumscribed spheres), which is what the standard procedure amounts to. The topology of the resulting dual complex \( \Omega' \) is the same as that of \( \Omega \), but in the pair \( (\Delta, \Omega') \) dual elements always intersect, although in general not perpendicularly.
of various dimensionalities with given subsets of $\Sigma$, such as the number of intersections per unit area between a given surface and graph edges. All properties of this type are known for two- and three-dimensional Euclidean space; we will discuss here those properties we will need later for our quantum statistical geometry results.

To calculate incidence relations, we consider the (finite) simplicial complex $\Delta_\rho$. Denote by $N_k$ the total number of $k$-faces of $\Delta_\rho$; in particular, $N_0 = N$ is the number of points. Two relations among the $N_k$’s can be used in any dimension. One is the expression for the Euler characteristic of a simplicial complex,

$$\sum_{k=0}^D (-1)^k N_k = \chi(\Delta_\rho),$$

where the dependence on the global topology of $\Sigma$ can be eliminated by dividing the equation by $N$, and then taking the large $N$ limit, in which $\chi(\Delta_\rho)/N \to 0$, and

$$1 - \frac{N_1}{N_0} + \frac{N_2}{N_0} - \cdots + (-1)^D \frac{N_D}{N_0} = 0 . \quad (6)$$

The other relation is the fact that each $D$-simplex has $D + 1$ faces of codimension 1, each one shared by two $D$-simplexes, so that

$$N_{D-1} = \frac{D + 1}{2} N_D . \quad (7)$$

In $D = 2$ dimensions, these two equations would be sufficient for determining the average number of edges and 2-simplices per unit volume, which in the uniform sprinkling case are the mean densities of the corresponding simplices, constant throughout $\Sigma$; in $D = 3$ dimensions, however, we need more equations. A detailed calculation, where one integrates explicitly over the probabilities of finding points at various locations (see, e.g., Ref. [10]), shows that each cell has on average $96\pi^2/35$ vertices; since each of those vertices is shared by four cells, and there are $\rho$ cells per unit volume, we find the density of cell complex vertices to be

$$\frac{N_3}{V_\Sigma} = \frac{1}{4} \frac{96\pi^2}{35} \rho = \frac{24\pi^2}{35} \rho . \quad (8)$$

To calculate the metric properties of random complexes, we must make extensive use of the probabilities associated to the point distribution. In $D = 3$ dimensions, we are interested in the mean number of edges intersected by a flat surface per unit area, for which we can use the following argument. As has long been known (see, e.g., references in [13]), from a generalization of the Buffon needle method of calculating $\pi$ by random tosses of a stick on a series of parallel lines, when a randomly oriented set of lines of arbitrary (possibly disconnected) shape and total length $L$, contained in a region of volume $V$, is cut by a surface of area $A$, the number of intersections between the lines and the surface per unit area is very simply related to the line length per unit volume, $N_{\text{int}}/A = \frac{1}{2} L/V$. The problem of finding the mean number of intersected edges is thus reduced to that of finding the mean total edge length per unit volume. A known calculation gives that the mean edge length per 3-cell is $(4\pi)^{5/3}(3^{1/3}/5) \Gamma\left(\frac{3}{4}\right) \rho^{-1/3}$, from which $L/V$ can be found multiplying by the mean number of cells per unit volume, $\rho$, and dividing by the number of cells sharing each edge, 3. Putting all of this together then gives

$$\frac{N_{\text{int}}}{A} = \frac{(4\pi)^{5/3} 3^{1/3}}{30} \Gamma\left(\frac{4}{3}\right) \rho^{2/3} \approx 2.917 \rho^{2/3} . \quad (9)$$

This result does not depend on the shape of the surface (in particular, $S$ does not have to be flat), but other moments of the probability distribution for $N_{\text{int}}$, such as its width, do.
Quantum Statistical Geometry: Areas and Volumes

In this section, we start discussing how to use the random cell complexes introduced above to construct quantum gravity states. The amount of structure we use from each pair of complexes is a choice we make in each approach, depending largely on the set \( \{g_\alpha\} \) of observables we want to reproduce. Since our observables here are areas and volumes, results from quantum geometry that we will summarize below allow us to make the simplest possible choice: We use only the 1-skeleton, or set \( \gamma \), of edges of the dual cell complex obtained with a randomly generated set of points \( x = \{x_i\} \), as a random spin network graph. The main advantage of this choice, as opposed to using the simplicial complex \( \Delta_{\rho} \), is the fact that almost every vertex is exactly \((D + 1)\)-valent in \( D \) dimensions; notice that, given that the edges are geodesic segments, the degree of differentiability of the graph depends on the geometry, and in the present case the graph is piecewise analytic.

In this paper, we will also make other simplifying choices, namely those of using only one such graph at a time for the states we consider, and of defining a spin network state \( \Psi_{\gamma,j,I} \) with the same spin label \( j \) at each edge and the same intertwiner \( I \) at each vertex; we can then write our states as \( \Psi_{\gamma,j,I} \). These further choices are made purely for illustrative purposes here, and will be changed in a more complete treatment; a few comments on this point will be included in the final section, and see Ref. [6] for details. This amounts to assuming that any single random graph \( \gamma \) provides a good sampling of the underlying manifold, and we end up with a two-parameter family of states for each such graph. One question we can address is then what constraints are placed on the parameters by the requirement that the area and volume operators have the right values on these states.

Consider a state \( \Psi_{\gamma,j,I} \) of the type just described. Given a surface \( S \) in \( \Sigma \), any given vertex of the random graph \( \gamma \) will fall on \( S \) with probability zero, and we can consider all intersections of \( S \) and \( \gamma \) to be single, transversal edges. In this case the spin network is an eigenstate of the area operator with eigenvalue \[ A_S = 8\pi k \ell_P^2 \sum_{\alpha} \sqrt{j_\alpha(j_\alpha + 1)}, \] where \( j_\alpha \) is the half-integer label for the \( \alpha \)-th edge crossing \( S \). If all the \( j_\alpha \)'s are equal to a given \( j \), the area eigenvalue for a given spin network becomes \( 8\pi k \ell_P^2 N_{\text{int}} \sqrt{j(j + 1)} \), where \( N_{\text{int}} \) is the number of intersections between \( S \) and the graph, so from Eq. (9) we obtain that on average classical and quantum areas agree if \( \rho \) and \( j \) satisfy

\[ \frac{(4\pi)^{5/3} 3^{1/3}}{30} \Gamma\left(\frac{4}{3}\right) \rho^{2/3} = \left(8\pi k \sqrt{j(j + 1)} \ell_P^2\right)^{-1}. \] (11)

Given a region \( R \subset \Sigma \), spin network states are eigenvectors of the volume operator \( \hat{V}_R \), and the corresponding eigenvalues receive a contribution from each vertex of the graph,

\[ \hat{V}_R \Psi_{\gamma,j,I} = \kappa_0 (8\pi k)^{3/2} \ell_P^3 \sum_v \sqrt{|\hat{q}_v|} \Psi_{\gamma,j,I}, \] (12)

where the constant \( \kappa_0 \) is an undetermined factor arising from a regularization ambiguity for the volume operator, and \( \hat{q}_v \) is an operator corresponding to the determinant of the spatial metric at \( v \), whose eigenvalues are determined by the \( j_\alpha \)'s of all edges incident at that vertex and the intertwiner \( I_v \). While a closed formula like the one for area eigenvalues, Eq. (10), is not available for volumes, the calculations simplify in the case of four-valent vertices.
When the four $j$'s are equal, the number of independent intertwiners at $v$, and thus of eigenvalues of $\hat{q}_v$, is $j + 1$; when $j$ is even, one of those eigenvalues is zero, while the others, and all eigenvalues when $j$ is odd, come in pairs of opposite signs and therefore give doubly degenerate volume eigenvalues. The volume eigenvalues have been calculated for the first few values of $j$ \[1\]; for example, the only eigenvalue for $j = 1$, and the non-zero ones for $j = 2$, are

$$
\lambda_1 = \frac{3^{1/4}}{4} \kappa_0 k^{3/2} \ell_p^3, \quad \lambda_2 = \frac{3^{1/4}}{2} \kappa_0 k^{3/2} \ell_p^3.
$$

(13)

In general, we can write the eigenvalues for vertices of our type in the form

$$
\lambda_{j,I} = f(j, I) \kappa_0 k^{3/2} \ell_p^3,
$$

(14)

so from Eq. (8) we obtain that on average classical and quantum volumes agree if

$$
\frac{24\pi^2}{35} \rho = (f(j, I) \kappa_0 k^{3/2} \ell_p^3)^{-1}.
$$

(15)

If the parameters $k$ and $\kappa_0$ of the theory are known, equations (11) and (15) are constraints on the parameters characterizing the random spin network states. Notice, however, that more constraints on the latter parameters are expected to arise from matching the manifold curvature in more general cases, in which we consider non-flat manifolds $(\Sigma, q_{ab})$, and we can also think of those two equations as fixing $\kappa_0$ once values for $j$ and $I$ have been determined,

$$
\left(\sqrt{j(j+1)} \frac{(4\pi)^{8/3} 3^{1/3}}{15} \Gamma(\frac{4}{3})\right)^{3/2} = \frac{24\pi^2}{35} f(j, I) \kappa_0.
$$

(16)

Concluding Remarks

An obvious way to generalize the states used in this paper is to relax the assumption that all $j$'s and all $I$'s on a given graph $\gamma$ be equal, and define a state

$$
|\gamma, C\rangle = \sum_{j,I} C_{j,I} |\gamma, j, I\rangle,
$$

(17)

for some set of coefficients $C_{j,I}$ to be determined. But the more interesting modification to our states comes from the observation that, given $(\Sigma, e_{ab})$, the sprinkling process does not give a unique set of points in $\Sigma$, but rather a probability density for $N$-point distributions,

$$
dP_\rho(x_1, ..., x_N) = N! \prod_{i=1}^N \sqrt{e(x_i)} V_\Sigma d^p x_i,
$$

(18)

parametrized by the density $\rho$, or $N = \rho V_\Sigma$, and therefore the dual cell complex construction also gives us a probability density on the set of graphs embedded in $\Sigma$ depending on the parameter $\rho$, rather than a single graph. We can then use this probability density to integrate over random graphs, and obtain either pure states resulting from their superposition, or mixed states of the form

$$
\Psi_\epsilon(\rho, C) = \int dP_\rho(x_1, ..., x_N) |\gamma_{x}, C\rangle \langle \gamma_{x}, C|,
$$

(19)
where the subscript \( e \) makes the dependence on the metric \( e_{ab} \) explicit. Superposing graphs in this way may seem like a complication, but on the other hand it has the advantage that, while a single \( |\gamma, j, I, \rangle \) would only have an approximate Euclidean invariance, the integral is exactly invariant (despite having an ultraviolet cutoff scale), as well as covariant with respect to the action of diffeomorphisms.

In the previous section, we have only discussed (for the simpler states \( \Psi_{\gamma, j, I} \)) the expectation value of areas and volumes, and seen how they give rise to conditions on the parameters the states depend on. The uncertainties that states of the form \( |\gamma, j, I, \rangle \) associate with those observables, or others we may use to identify semiclassical states, will have a quantum contribution and a classical, statistical one; imposing that they be smaller than the desired tolerance will introduce further conditions, and restrictions on the length scales defined by the geometry and the subsets of \( \Sigma \) we consider.

In addition, it is useful to keep in mind alternatives to some other choices we made; although they were the simplest ones, there is no guarantee that once we understand the dynamical aspects better they will appear as the best ones. Specifically, in a curved manifold, uniform distributions of points are not the only possible covariantly defined ones; a uniform distribution in a Riemannian manifold \( (\Sigma, q_{ab}) \) is equivalent to using the density \( dv = \sqrt{q} d^{D}x \) as a measure \( d\mu \) on \( \Sigma \), while alternative ones can be defined using any scalar constructed from the metric, such as the Ricci scalar, \( d\mu = R dv \), or any other curvature scalar. Possibly related to this is the fact that a better understanding of the theory may show that, in order for a spin network state to have a fully consistent semiclassical interpretation (including, e.g., the fact of giving rise to distributions for the spatial geometry and its time derivative which are both peaked around classical values), the weave we use must be constructed using correlations in the sprinkling process, that were not used here.

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