On singularities of dynamic response functions in the massless regime of the XXZ spin-1/2 chain

Karol K. Kozlowski
Univ Lyon, ENS de Lyon, Univ Claude Bernard Lyon 1, CNRS, Laboratoire de Physique, F-69342 Lyon, France

Abstract

This work extracts, by means of an exact analysis, the singular behaviour of the dynamical response functions -the Fourier transforms of dynamical two-point functions- in the vicinity of the various excitation thresholds in the massless regime of the XXZ spin-1/2 chain. The analysis yields the edge exponents and associated amplitudes which describe the local behaviour of the response function near a threshold. The singular behaviour is derived starting from first principle considerations: the method of analysis does not rely, at any stage, on some hypothetical correspondence with a field theory or other phenomenological approaches. The analysis builds on the massless form factor expansion for the response functions of the XXZ chain obtained recently by the author. It confirms the non-linear Luttinger based predictions relative to the power-law behaviour and of the associated edge exponents which arise in the vicinity of the dispersion relation of one massive excitation (hole, particle or bound state). In addition, the present analysis shows that, due to the lack of strict convexity of the particles dispersion relation and due to the presence of slow velocity branches of the bound states, there exist excitation thresholds with a different structure of edge exponents. These origin from multi-particle/hole/bound state excitations maximising the energy at fixed momentum.

Contents

1 An outline of the problem and main results 3
  1.1 The XXZ chain .................................................. 3
  1.1.1 Singularities of response functions ....................... 4
  1.2 The main achievement of the work ........................... 4
  1.3 The principal theorems ......................................... 5
  1.4 Outline of the paper ............................................ 9
  1.5 Some history of the analysis of dynamic response functions 9
    1.5.1 Heuristic approaches ..................................... 9
    1.5.2 Exact approaches .......................................... 11
    1.5.3 Numerical and Bethe Ansatz based approaches .......... 12

1e-mail: karol.kozlowski@ens-lyon.fr
## 1.5.4 The restricted sum approach

### 2 Main results

2.1 The setting and some generalities on the model

2.2 The behaviour of the longitudinal dynamic response function in the two hole excitations in $\mathcal{F}(c)(k, \omega)$

2.3 The series representation for the dynamic response functions

### 3 The edge singular behaviour of dynamic response functions

3.1 The one free rapidity sector

3.1.1 The one-hole contributions

3.1.2 The one $r$-string contributions

3.2 The multi-hole/$r$-string excitation sector

3.2.1 Excitations built up from holes and, possibly, particles

3.2.2 Excitations built up from holes and a fixed $r$-string species

### 4 Conclusion

### A Main notations

### B Auxiliary theorems

### C Observables in the infinite XXZ chain

C.1 Solutions to linear integral equations

C.2 The velocity of individual excitations

### D Asymptotics of a one-dimensional $\beta$-like integral

D.1 The structural theorem in the one-dimensional case

D.2 Auxiliary lemmata

### E Asymptotics of multi-dimensional $\beta$-like integral

E.1 General assumptions

E.2 The structural theorem in the multidimensional setting

### F Auxiliary results

F.1 A regularity lemma

F.2 Local rectification of $\tilde{z}_\nu$

F.3 Factorisation of the maps $\tilde{z}_\nu$

F.4 Local expansion of a Vandermonde determinant

F.5 Asymptotic behaviour of a local integral

F.5.1 The integral associated with the $|u'_1(k_0)| < v$ regime

F.5.2 The integral associated with the $|u'_1(k_0)| > v$ regime

### G Asymptotic behaviour of a model integral

G.1 Reduction of the model integral into regular and singular parts

G.2 Asymptotic behaviour of the model integral

G.3 Asymptotics of auxiliary functions
1 An outline of the problem and main results

1.1 The XXZ chain

Due to the substantial progress which took place in experimental condensed matter physics, one-dimensional models of quantum many body physics evolved from a status of purely theoretical toy-models of many body physics to concrete compounds exhibiting a genuine one-dimensional behaviour. Even more remarkably, there exist a plethora of compounds whose properties are grasped, within a very good precision, by one-dimensional quantum integrable Hamiltonians. The most prominent example is probably given by the XXZ spin-1/2 chain in an external longitudinal magnetic field. The Hamiltonian of the model takes the form

\[ H = J \sum_{a=1}^{L} \left( \sigma_a^x \sigma_{a+1}^x + \sigma_a^y \sigma_{a+1}^y + \Delta \sigma_a^z \sigma_{a+1}^z \right) - \frac{\hbar}{2} \sum_{a=1}^{L} \sigma_a^z. \]  

Here \( J > 0 \) represents the so-called exchange interaction, \( \Delta \) is the anisotropy parameter, \( \hbar > 0 \) the external magnetic field and \( L \in 2\mathbb{N} \) corresponds to the number of sites. \( H \) acts on the Hilbert space \( \mathcal{H}_{XXZ} = \mathbb{C}^{2L} \) with \( \mathbb{C} \) the complex field, \( \sigma^w, \ w = x,y,z \), are the Pauli matrices and the operator \( \sigma_a^w \) acts as the Pauli matrix \( \sigma^w \) on \( \mathcal{H}_a \) and as the identity on all the other spaces, \( \forall a \).

\[ \sigma_a^w = \text{id} \otimes \cdots \otimes \text{id} \otimes \sigma^w \otimes \text{id} \otimes \cdots \otimes \text{id}. \]  

Finally, the model is subject to periodic boundary conditions, \( \forall a. \sigma_a^y = \sigma_a^y \).

For example, Crystals such as KCuF₃ [60] or Cu(C₄H₄N₂)(NO₃)₂ [61] have been identified to be well-grasped by the isotropic XXX Hamiltonian, \( \forall a. \) the Hamiltonian \( H \) given in (1.1) when \( \Delta \) is set to 1. In its turn, the behaviour of CsCoCl₃ has been found to be well-captured [33] by the XXZ antiferromagnetic Heisenberg chain with \( \Delta = 10 \) while certain aspects of the behaviour of the spin-ladder compound \((\text{C}_5\text{H}_2\text{N})_2\text{CuBr}_4\) are well-described [13] by an effective XXZ Hamiltonian with \( \Delta = 1/2 \).

Most experiments on the above and many other effectively one-dimensional materials measure the Fourier transforms of two-point correlation functions - the so-called dynamic response functions (DRF)- and typically rely on techniques such as Bragg [67] or photoemission spectroscopy or inelastic neutron scattering [52] [55]. In fact, most experiments take place at rather low temperatures, what effectively means that they measure, with good accuracy, the zero-temperature DRF. In the case of the XXZ chain, the zero temperature DRF, \( \mathcal{S}^{(y)}(k, \omega) \), take the form

\[ \mathcal{S}^{(y)}(k, \omega) = \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} \langle \langle \sigma_m^y(t) \sigma_{m+1}^y(0) \rangle \rangle_c \cdot e^{i\omega t} \, dt. \]  

Above, \( \dagger \) stands for the Hermitian conjugation, and the integrand refers to the presumably existing \( \mathcal{S}^{(y)}(k, \omega) \) infinite volume limit of the connected dynamical two-point function at zero temperature

\[ \langle \langle \sigma_m^y(t) \sigma_{m+1}^y(0) \rangle \rangle_c = \lim_{L \to +\infty} \left\langle \left( \Omega, (\sigma_m^y(t) \sigma_{m+1}^y(0)), \Omega \right) - \left| \Omega, (\sigma_m^y, \Omega) \right|^2 \right\rangle. \]  

The connectedness of the correlator allows one to regularise the convergence of the transforms at infinity.

The results of [44] [50] [69] put together entail the existence of the limit at \( t = 0 \) in that they provide a rigorous derivation of a well-defined multiple integral representation for the reduced density matrix of the chain. An appropriate trace thereof allows one to compute \( \langle \langle \sigma_m^y(0) \sigma_{m+1}^y(0) \rangle \rangle_c \). Note that the existence of the limit at \( t = 0 \) also follows from the general theory developed in [63]. It is also fairly easy to see that the limit [13] exists for extracted subsequences in \( L \).
Here $\Omega$ stands for the model’s ground state while the time and space evolution of a spin operator takes the form

$$\sigma_{m+1}(t) = e^{-iHt} \cdot \sigma_1 \cdot e^{-iHt} \cdot e^{imP},$$

(1.5)

where $P$ is the momentum operator and, hence, $e^{iP}$ the translation operator by one-site.

### 1.1.1 Singularities of response functions

Taken that dynamic response functions are natural experimental observables, there is a clear demand to build effective and reliable theoretical tools allowing for their study, at least in some limiting regimes, and providing a satisfactory explanation of the experimental observations. Typically, dynamic response functions in one-dimensional models are observed to exhibit a singular structure in the momentum $k$ – frequency $\omega$ plane. Namely, at fixed momentum $k$, they exhibit a power-law behaviour $(\delta\omega)^\mu$ in $\delta\omega = \omega - \mathcal{E}(k)$, this in the vicinity of certain curves $(k, \mathcal{E}(k))$. The curves $k \mapsto \mathcal{E}(k)$ correspond to dispersion relations of the excitations that are at the root of generating the given non-analytic behaviour. The edge exponent $\mu$ governing a given singularity may be positive or negative. The range of possible values of the edge exponent $\mu$ strongly depends on whether the model is in a massive or massless phase and, in the latter case, on the universality class governing the massless regime. In fact, the singular structure of the DRF, and in particular the form taken by the edge exponents is deeply connected with the critical exponents driving the long-distance and large-time power-law decay of the real space correlators. In the massive case, one expects, unless some non-generic accident happens, that this decay is driven by Gaussian saddle-points, be it in one or several dimensions. Thus, in the massive case, the edge exponents are expected to be of the form $-1/2 + n$, $n \geq 0$ an integer, the typical behaviour being either a square root divergence or a square root cusp in the vicinity of the dispersion curves $k \mapsto \mathcal{E}(k)$. The situation appears to be much richer in a massless model precisely due to the existence of infinitely many zero energy excitations. The latter generate a non-trivial tower of critical exponents which give rise to edge exponents $\mu$ that, generically, exhibit a dependence on the momentum $k$, can be positive or negative and which are, generically, non-rational.

Ideally, one would like to have at one’s disposal tools allowing one to unravel the mentioned singularity structure of the DRFs for a generic, not necessarily integrable, one-dimensional model at zero temperature. The approach should also provide accurate and explicit enough predictions.

### 1.2 The main achievement of the work

A reasonable path for achieving the goal described above appears to start by devising exact tools allowing one to fully describe the singularity structure of the DRF in at least some instances of quantum integrable models; indeed, then, one can hope to rely on the exact solvability of the model which provides one with numerous additional algebraic properties allowing one to simplify the calculations. As will be discussed below, such calculations could have been carried out, at least in part, for some examples of quantum integrable models. However, what would be really useful for the purpose of unravelling a larger picture would be to construct tools and a framework of analysis allowing one to stay as close as possible to objects and pictures usually used in condensed matter physics. The success of such an approach could then allow, by extrapolating the features responsible for the emergence of singularities in an integrable model, to devise an exact phenomenological approach allowing one to grasp the universal part of the structure of DRFs, at least, in certain classes of non-integrable models. By exact phenomenological approach, I mean one being able to produce an exact and analysable to the end expression for the DRF in which the building blocks will be given by specific to the model -but not explicit- functions and such that the part responsible for the singular behaviour of the DRF is captured by a universal structure common to all models belonging to the universality class of interest. In conjunction with the representations that were obtained in my previous work \[51\], this is precisely the program that is achieved in this work.
By starting from the massless form factor\footnote{I remind that a “form factor” refers to a matrix element of some local operator taken between two Eigenstates of the model’s Hamiltonian. Such objects are well-defined in finite volume $L$ as it is the case for the XXZ Hamiltonian (1.1). See \cite{43} where finite-size determinant representation for these objects have been obtained} based representation, which I obtained in [51] for the zero temperature DRF of the massless XXZ spin-1/2 chain, I develop a method of rigorous analysis of the behaviour of each multiple integral present in the series. While the construction of the series that was carried out in [51] relies on a certain amount of hypotheses that are yet to be proven to hold, the analysis of each multiple integral carried out in this work is rigorous.

This allows me to extract the singular behaviour of the DFRs for the XXZ chain and hence determine, through an exact approach, the value of edge exponents $\mu$, singularities curves $k \mapsto E(k)$ and amplitudes characterising the singularities in the $(k, \omega)$ plane. Doing so, allows me to:

i) test and confirm the predictions, issuing form the existing heuristic methods, in respect to the structure of the subset of the singularities associated with one particle/hole/bound state excitations;

ii) fully analyse the effect of multi-particle/hole/bound state processes in the generation of the excitation thresholds. These thresholds take origin in that the velocity of the excitations is not monotonously increasing and, more importantly, in that particles, holes or bound states may share same values of their velocities. Such multi-particle thresholds were, so far, mostly unaccounted for within the existing heuristic methods and not all of the effects present at such thresholds were fully grasped.

I stress that this is the first \textit{ab initio} calculation of the singularities of the response functions in the XXZ spin-1/2 chain, an interacting integrable model containing bound states.

An important point is that the approach developed in the present work is universal in the sense given earlier. I argued in [51] that the massless form factor series expansions of the zero temperature DRF that I obtained for the massless regime of the XXZ spin-1/2 chain has, in fact, a universal form that should be shared by all models belonging to the Luttinger liquid universality class. I refer to that paper for a more precise discussion of that fact. As a consequence, the techniques of analysis -up to trivial modifications- developed in this work will allow one to grasp the singular structure of DFRs in models belonging to the Luttinger liquid universality class. Of course, these will then only be phenomenological results since, for a general model, one does not have an explicit access to form factor densities of local operators or to dispersion relations of the elementary excitations. However, such an approach is not so uncommon in physics and, more importantly, the approximations made to get the result are genuinely constructive and do not rely on this or that heuristics which, in concrete situations, might turn out to be complicated to verify or even simply to have an intuition of. Furthermore, the data (form factor densities, dispersion relations) on which the phenomenological approach builds can, in principle, be computed perturbatively in the vicinity of a free theory, at least on formal grounds [65].

### 1.3 The principal theorems

On technical grounds, the main achievement of this work are the two theorems given below. These results allow one to grasp the small parameter asymptotic expansion of a class of integrals which, upon specialisation, correspond to the one arising in the series expansion for the dynamic response functions obtained in [51]. In order to state these theorems, I first need to introduce a specific class of smooth functions. This definition involves smooth functions on closed set, see Definition B.4 for a precise characterisation of the concept.
**Definition 1.1.** Given $K$ a compact subset $\mathbb{R}^n$ for some $n \in \mathbb{N}^*$, a function $\mathcal{G}$ on $K \times \mathbb{R}^+ \times \mathbb{R}^+$ is said to be in the smooth class of $K$ associated with functions $d_+$ and constant $\tau \in ]0; 1]$, if there exists a decomposition

$$
\mathcal{G}(x, u, v) = d_+(x) d_-(x) \mathcal{G}^{(1)}(x) + d_-(x) \mathcal{G}^{(2)}(x, u) \cdot [u]^{1-\tau} + d_+(x) \mathcal{G}^{(3)}(x, v) \cdot [v]^{1-\tau} + \mathcal{G}^{(4)}(x, u, v) \cdot [u v]^{1-\tau},
$$

where $\mathcal{G}^{(1)}$ is smooth on $K$, $\mathcal{G}^{(2)}, \mathcal{G}^{(3)}$ are smooth and bounded on $K \times \mathbb{R}^+$, $\mathcal{G}^{(4)}$ is smooth and bounded on $K \times \mathbb{R}^+ \times \mathbb{R}^+$.

These functions are such that, for any $(s, \ell, \epsilon) \in \mathbb{N}^n \times \mathbb{N} \times \mathbb{N}$, $s \in [1; 4]$, and $\epsilon > 0$

$$
H_i \prod_{a=1}^n \partial^{\ell_a}_u \cdot \partial^{\ell_a}_v \left\{ \mathcal{G}^{(s)}(x, u)[u]^{1-\tau} \right\} = O\left([u]^{1-\tau-\ell_s} \right) \text{ uniformly in } x \in K, u \in ]0; \epsilon^{-1}] \text{ and for } s = 2, 3;
$$

$H_{ii} \prod_{a=1}^n \partial^{\ell_a}_u \cdot \partial^{\ell_a}_v \cdot \partial^{\ell_v}_v \left\{ \mathcal{G}^{(4)}(x, u, v)[u v]^{1-\tau} \right\} = O\left([u]^{1-\tau-\ell_s} [v]^{1-\tau-\ell_s} \right) \text{ uniformly in } x \in K, (u, v) \in ]0; \epsilon^{-1}]^2.

Finally, if $n \geq 2$, the functions $\mathcal{G}^{(s)}$, $s \in [1; 4]$, along with any of their partial derivatives, all vanish on $\partial K$, $\partial K \times \mathbb{R}^+$, $\partial K \times \mathbb{R}^+ \times \mathbb{R}^+$.

The first theorem deals with the case of one-dimensional integrals.

**Theorem 1.2.** Let $a < b$ be two reals. Let $3_\pm(\lambda)$ be two real-holomorphic functions in a neighbourhood of the interval $\mathcal{I} = [a; b]$, such that

- all the zeroes of $3_\pm$ on $\mathcal{I}$ are simple;
- $3_+$ and $3_-$ admit a unique common zero $\lambda_0 \in \text{Int}(\mathcal{I})$ that, furthermore, is such that $3_+(\lambda_0) \neq 3_-(\lambda_0)$.

Let $\Delta_\nu$ be real analytic on $\text{Int}(\mathcal{I})$ and such that $\Delta_\nu \geq 0$. Let $\mathcal{G}$ be in the smooth class of $\mathcal{I}$ associated with the functions $\Delta_\pm$ and with a constant $\tau$. Then, for $x \neq 0$ and small enough,

$$
\lambda \mapsto \mathcal{G}\left(\lambda, \overline{3_+(\lambda)}, \overline{3_-(\lambda)}\right) \cdot \prod_{\nu=\pm} \left\{ \Xi(\overline{3_\nu(\lambda)}) \cdot \left[\overline{3_\nu(\lambda)}\right]^{\Delta_\nu(\lambda)-1} \right\} \in L^1(\mathcal{I})
$$

(1.7)

where $\overline{3_\pm(\lambda)} = 3_\pm(\lambda) + x$. Let $I(x)$ denote the integral

$$
I(x) = \int_{\mathcal{I}} \mathcal{G}\left(\lambda, \overline{3_+(\lambda)}, \overline{3_-(\lambda)}\right) \cdot \prod_{\nu=\pm} \left\{ \Xi(\overline{3_\nu(\lambda)}) \cdot \left[\overline{3_\nu(\lambda)}\right]^{\Delta_\nu(\lambda)-1} \right\} \cdot d\lambda.
$$

(1.8)

Assume that $\delta_\pm = \Delta_\pm(\lambda_0) > 0$.

**a)** If $3_+(\lambda_0) \cdot 3_-(\lambda_0) < 0$, then $I(x)$ admits the $x \to 0$ asymptotic expansion

$$
I(x) = \Xi\left(3_+(\lambda_0) \cdot \overline{x}\right) \cdot \left\{ \frac{\mathcal{G}^{(1)}(\lambda_0) \cdot \delta_+ \delta_- \cdot |x|^\delta_+ + \delta_- - 1}{|3_+(\lambda_0)|^\delta_+ \cdot |3_-(\lambda_0)|^\delta_-} \cdot \frac{\Gamma(\delta_+) \cdot \Gamma(\delta_-)}{\Gamma(\delta_+ + \delta_-)} + O\left(|x|^\delta_+ + \delta_--1\right) \right\} + f_<(x)
$$

where

$$
x = x \cdot [3_+(\lambda_0) - 3_-(\lambda_0)],
$$

(1.9)
\( G^{(1)} \) is as appearing in (1.6) and \( f_c \) is a smooth function of \( x \). Furthermore, if \( z_\pm \) have no zeroes on \( J \) other than \( \lambda_0 \), then \( f_c = 0 \).

b) If \( z_\prime_+(\lambda_0) \cdot z_\prime_-,(\lambda_0) > 0 \), then \( I(x) \) admits the \( x \to 0 \) asymptotic expansion

\[
I(x) = \frac{G^{(1)}(\lambda_0) \cdot \delta_+ \cdot \delta_-}{|z_\prime_+(\lambda_0)|^{\delta_+} \cdot |z_\prime_-,(\lambda_0)|^{\delta_-}} \cdot \Gamma(\delta_+) \cdot \Gamma(\delta_-) \cdot \Gamma(1 - \delta_+ - \delta_-) \\
\times \left\{ \Xi(x) \pi \sin \left[ \pi \delta_+ \right] + \Xi(-x) \pi \sin \left[ \pi \delta_- \right] \right\} + O(|x|^{\delta_+ + \delta_- - \tau}) + f_c(x) \quad (1.11)
\]

where \( \Xi \) and \( \delta_\pm \) are as above.

\[
p = -\text{sgn}[z_\prime_+(\lambda_0)] \cdot \text{sgn}[z_\prime_+(\lambda_0) - z_\prime_-,(\lambda_0)] \quad (1.12)
\]

and \( f_c \) is a smooth function of \( x \).

The second theorem, deals with a multi-dimensional analogue of the integral given in (D.2). Its statement demands to introduce a few notations and objects. One assumes to be given:

- a strictly positive real \( v > 0 \);
- a choice of signs \( \zeta_r \in \{ \pm \} \);
- a collection of compact intervals \( J_r, r = 1, \ldots, \ell \);
- smooth functions \( u_r \) on \( J_r \) such that \( u_r' \) is strictly monotonous on \( J_r \), and such that \( u_r'(k) \neq \pm v \) for \( k \in \text{Int}(J_r) \).

The intervals \( J_r \) are such that they partition as

\[
J_r = J_r^{(\text{in})} \sqcup J_r^{(\text{out})} \quad \text{with} \quad J_1 = J_1^{(\text{in})} \quad (1.14)
\]

so that

\[
u_r'(\text{Int}(J_r^{(\text{out})})) \cap u_r'(\text{Int}(J_1^{(\text{in})})) = \emptyset \quad \text{and} \quad u_r'(\text{Int}(J_r^{(\text{in})})) = u_1'(\text{Int}(J_1^{(\text{in})})). \quad (1.15)
\]

The above ensures that there exist homeomorphisms

\[
t_r : J_1^{(\text{in})} \to J_r^{(\text{in})} \quad \text{such that} \quad u_1'(k) = u_r'(t_r(k)). \quad (1.16)
\]

One defines the macroscopic "momentum" and "energy" as

\[
\mathcal{P}(k) = \sum_{r=1}^{\ell} n_r \zeta_r t_r(k) \quad \text{and} \quad \mathcal{E}(k) = \sum_{r=1}^{\ell} n_r \zeta_r u_r(t_r(k)) , \quad k \in J_1. \quad (1.17)
\]

It is assumed that \( k \mapsto \mathcal{P}(k) \) is strictly monotonous on \( \text{Int}(J_1) \).
Theorem 1.3. Let \( \mathcal{I}_{\text{tot}} = \mathcal{I}_1^\ell \times \cdots \times \mathcal{I}_n^\ell \) and \( \Delta_\pm \) be smooth positive functions on \( \mathcal{I}_{\text{tot}} \) admitting smooth square roots on \( \mathcal{I}_{\text{tot}} \). Let \( \mathcal{G} \) be in the smooth class of \( \mathcal{I}_{\text{tot}} \) associated with the functions \( \Delta_\pm \) and a constant \( \tau \in [0; 1[ \), c.f. Definition 1.1.

Finally, let

\[
\zeta(p) = E_0 - \sum_{r=1}^{\ell} \sum_{a=1}^{n_r} \zeta_r(p_a^{(r)}) + \nu \nu \left\{ P_0 - \sum_{r=1}^{\ell} \sum_{a=1}^{n_r} \zeta_r(p_a^{(r)}) \right\}, \quad \nu \in \{\pm\},
\]

with \( \zeta_r \in \{\pm\} \) and where \( (P_0, E_0) \in \mathbb{R}^2 \).

Let \( I(x) \) correspond to the multiple integral

\[
I(x) = \prod_{r=1}^{\ell} \left\{ \int_{\mathcal{I}_r} \mathcal{G}_{\text{tot}}(p) \right\} \text{ with } p = (p^{(1)}, \ldots, p^{(\ell)}) \in \mathbb{R}^{n_r},
\]

where

\[
\mathcal{G}_{\text{tot}}(p) = \mathcal{G}(p, \zeta_+(p) + x, \zeta_-(p) + x) \cdot \prod_{r=1}^{\ell} \left\{ \Xi(\zeta_r(p) + x) \cdot \left[ \zeta_r(p) + x \right]^{\Delta_r(p) - 1} \right\} \cdot \prod_{r=1}^{\ell} \prod_{a=1}^{n_r} \left( p_a^{(r)} - p_b^{(r)} \right)^2.
\]

The type of \( x \to 0 \) asymptotic expansion of \( I(x) \) depends on the value of \( (P_0, E_0) \).

a) The regular case.

If the two conditions given below hold

\[
(P_0, E_0) \notin \left\{(P(k), E(k)) : k \in \mathcal{I}_1\right\}
\]

and

\[
\min_{\alpha \in \mathcal{I}_1} \left| E_0 - E(\alpha) + \nu \nu (P_0 - P(\alpha)) \right| > 0
\]

then \( \mathcal{G}_{\text{tot}} \in L^1(\mathcal{I}_{\text{tot}}) \) and \( I(x) \) is smooth in \( x \) for \( |x| \) small enough.

b) The singular case.

Let \( k_0 \in \text{Int}(\mathcal{I}_1) \)

\[
\Delta_r^{(0)} = \Delta_r(t(k_0)) \quad \text{and} \quad \vartheta = \frac{1}{2} \sum_{r=1}^{\ell} n_r^2 - \frac{3}{2} + \Delta_r^{(0)} + \Delta_r^{(0)},
\]

with

\[
t(k_0) = (t_1(k_0), \ldots, t_n(k_0)) \in \mathbb{R}^{n_r} \quad \text{with} \quad t_r(k_0) = (t_r(k_0), \ldots, t_r(k_0)) \in \mathbb{R}^{n_r}.
\]

If

\[
(P_0, E_0) = (P(k_0), E(k_0)), \quad \vartheta \notin \mathbb{N}, \quad \text{and} \quad \Delta_r^{(0)} > 0
\]
then \( \hat{S}_{\text{tot}} \in L^1(\mathcal{F}_{\text{tot}}) \) and \( I(x) \) admits the \( x \to 0^+ \) asymptotic expansion:

\[
I(x) = \frac{\Delta_{\ell}^{(0)} \Delta_{\ell}^{(-1)} G^{(1)}(t(k_0)) \cdot (2\omega)^{\Delta_{\ell}^{(0)} - 1} \cdot \Gamma(\Delta_{\ell}^{(0)}) \Gamma(\Delta_{\ell}^{(-1)})}{\sqrt{P''(k_0)} \cdot \prod_{r=\pm} |\nu - \nu r'(k_0)|^{\Delta_{\ell}^{(0)}}} \cdot \prod_{r=1}^{\ell} \left\{ \frac{G(2 + n_r) \cdot (2\pi)^{n_r - \delta_{r,1}}}{\left| \nu r'(t_r(k_0)) \right|^{\frac{1}{2} (n_r^2 - \delta_{r,1})}} \right\} 
\times |x|^{\theta} \cdot \left\{ \Xi(x) \frac{\sin \left[ \pi \nu_+ \right]}{\pi} + \Xi(-x) \frac{\sin \left[ \pi \nu_- \right]}{\pi} \right\} + \tau(x) + O(|x|^{\theta + 1 - r}).
\]

Above \( \tau(x) \) is smooth in \( x \), for \( |x| \) small enough. Finally,

\[
\nu_\pm = \frac{1}{2} \sum_{\varepsilon_r = \pm} \frac{\epsilon_r^2}{\varepsilon_r} - \frac{1}{4} + \sum_{\varepsilon_r = \pm, |\nu - \nu r'(k_0)| > 0} \Delta_{\nu}^{(0)}
\]

where \( s = -\text{sgn}\left( \frac{\nu r'(k_0)}{\nu r''(k_0)} \right) \) and \( \epsilon_r = -\zeta \text{sgn}\left( \nu r''(t_r(k_0)) \right) \).

### 1.4 Outline of the paper

This paper is organised as follows. This is the Introduction. Sub-section 1.5 to come reviews the various developments that took place in the analysis of the dynamic response functions of one-dimensional models. Section 2 contains a short review of the structure of excitations in the model followed by a discussion of the obtained results in the simple case of the singular structure of the at most two-particle/hole contribution to the longitudinal DRF \( \mathcal{F}^{(1)}(k, \omega) \). Finally, this section closes on the description of the series of multiple integrals representation for \( \mathcal{F}^{(1)}(k, \omega) \) derived in [51]. In particular, I discuss the various properties enjoyed by the integrands of the multiple integrals building up the series. The singular behaviour of each multiple integral, \( \nu_\pm \). summands arising in the series, is then extracted, for the most typical excitations, in Section 3. All the technical details necessary for obtaining these results are relegated to several appendices. Appendix A lists the main notations contained in this work. Appendix B recalls the statements of four theorems, the Morse lemma, the Weierstrass and the Malgrange preparation theorems as well as the Whitney extension theorem. All these will be used in the core of the analysis developed in Appendices D and E. Appendix C recalls the properties of interest of certain observables in the XXZ chain. Sub-appendix C.1 recalls the linear integral equation based description of the observables in the XXZ chain. Sub-appendix C.2 discusses the properties of the velocity of the particles and hole excitations that play an important role in the analysis. Appendices D and E are devoted to a detailed analysis of the asymptotic behaviour of auxiliary integrals whose understanding is necessary for obtaining the per se singular behaviour of the DRF studied in Section 3. Appendix D is devoted to the analysis of the asymptotics of a generalisation of one-dimensional Euler \( \beta \)-integrals while Appendix E carries such an analysis relatively to a multi-dimensional generalisation of one-dimensional \( \beta \) integrals. The rigorous analysis developed in Appendices D and E constitutes the main technical achievement of this work. Finally, Appendices F and G develop several technical results that are needed so as to carry out the analysis developed in Appendix E.

### 1.5 Some history of the analysis of dynamic response functions

#### 1.5.1 Heuristic approaches

There is clearly little hope, for a generic one-dimensional model, to extract the singular structure of DRFs by means of direct, \( ab \text{ inicio} \), calculations. Still, over the years, there emerged various approximation techniques allowing one to analyse certain features of such a singular behaviour. In the massive case, the singularities of the
DRFs appear to be controlled by Van Hove singularities and this completely catches the aforementioned behaviour. A whole lot more attention was dedicated to the massless case where one expects a much richer behaviour and where no such simple explanation exists.

To start with, one can argue that the equal-time long-distance asymptotics of the correlators in a massless model should be grasped by putting the model in correspondence with a Luttinger liquid or, more generally, with a conformal field theory (CFT). Mappings of this kind are built by looking at the momentum and energy of the low-lying excited states above the ground state of the model from where one can read-off the scaling dimensions of the operators on the CFT side which give access to the critical exponents arising in the equal-time long-distance asymptotic behaviour of the zero temperature correlation functions in the original model. In its turn this allows one to argue the behaviour of the Fourier transforms in the vicinity of the point $(k, \omega) = (0, 0)$.

The situation becomes much more involved if one would like to grasp, at least qualitatively, the behaviour of DRFs in the whole $(k, \omega)$ plane. Indeed, then, it becomes necessary to take into account certain of the non-linearities in the spectrum of the model’s excitations. A first phenomenological description of the DRF’s singularities in the $(k, \omega)$ plane was argued by Beck, Bonner and MÃijller in 1979. The approach was substantially developed one year later by these authors and Thomas for the XXX Heisenberg spin-1/2 chain at zero magnetic field. These authors also proposed heuristic reasons based on selection rules so as to predict some of the features of the DRF in the presence of a non-zero magnetic field. A substantial progress towards the setting of an operative phenomenological approach occurred, however, only in the mid ’00. In 2006, Glazmann, Kamenev, Khodas and Pustilnik managed to take into account the non-linearities in the dispersion relation of one-dimensional spinless fermions and argued, in the case of the density structure factor, the presence of a singular behaviour along single particle $k \mapsto \varepsilon_p(k)$ or hole $k \mapsto \varepsilon_h(k)$ excitations thresholds characterised by a non-trivial, \textit{viz.} differing from a half-integer, edge exponents $\mu$. Next year, the authors generalised their approach so as to encompass other DRFs and computed perturbatively the edge exponents arising in the $\delta$-function Bose gas in . More explicit results appeared later on. Pustilnik built on the expressions for the exact form factors in the Calogero-Sutherland model so as to unravel the singular behaviour of the density structure factor in that model. Further, building on the explicit expressions for the spectrum of excitations provided by the Bethe Ansatz, Glazmann and Imambekov proposed closed expressions for the edge exponents arising in the $\delta$-function Bose gas, while Cheianov and Pustilnik argued the expression for the edge exponents associated with the lower threshold -corresponding to one hole excitations- in the massless regime of the XXZ spin-1/2 chain. Such kinds of predictions for the edge exponents were generalised, in 2008-09 by Affleck, Pereira and White, to various other thresholds present in the XXZ chain. In , Glazmann and Imambekov advocated the manifestation of various universal behaviours in the amplitudes appearing in front of the power-law behaviour $(\delta \omega)^\mu$ of the DRF, hence providing a firm ground to the so-called non-linear Luttinger liquid theory supposed to govern the edge singular behaviour of dynamic response functions in massless models. I refer to the review and references therein for a broader discussion of that approach. Similarly to the case of the edge exponents, Caux, Imambekov, Panfil and Shashi, by building on recent techniques pioneered in and allowing one to study the large-volume behaviour of form factors of local operators in quantum integrable models, argued the expressions of the amplitudes in front of the singular power-law behaviour of the DRF in the case of the XXZ spin-1/2 chain, the $\delta$-function Bose gas and the Calogero-Sutherland model. One should also mention that, more recently, the lower thresholds present in DRFs of the spin-1/2 XXX Heisenberg chain where analysed, by Campbell, Carmelo, Machado and Sacramento, within the pseudofermion dynamical theory and by taking the Bethe Ansatz issued input for the energies.

\footnote{The latter corresponds to $\mathcal{S}^{(1)}(k, \omega)$ in the case of the XXZ chain, \textit{c.f.} \cite{13}.}
1.5.2 Exact approaches

The heuristic approaches described above appear quite powerful. It is necessary to check and test the limits of applicability of the mentioned methods versus results stemming from exact, ab initio, calculations of DRF and the extraction of their singularities, carried out on quantum integrable models. Obtaining such exact results constituted a hard and long-standing problem, despite that numerous techniques of exact computations of correlation functions have been developed after the invention of the algebraic Bethe Ansatz \[28\] on the one hand and of the vertex operator approach \[28\] on the other hand.

First results relative to DRFs appeared for free fermion equivalent models. The density structure factor, viz.
the longitudinal Fourier transform \( \mathcal{S}^{(1)}(\mathbf{k}, \omega) \), of the XX chain was computed in a closed form by Beck, Bonner, Müller and Thomas \[22\] in 1981. The case of transverse response functions was much harder, even for the XX chain, due to the much more involved structure of the transverse correlators. An analysis of the power-law divergences in \( \omega \) for the transverse frequency Fourier transform \( \int_0^{2\pi} \mathcal{S}^{(1)}(\mathbf{k}, \omega) \cdot d\mathbf{k} \) of the XX chain was achieved in 1984 by Müller and Shrock \[59\] by exploiting the connection between the associated two-point function and Painlevé’ transcendents.

The development of the vertex operator approach \[28\] in the mid ’90s allowed for a substantial progress in the computation of the correlation functions in an interacting, viz. away from the free fermion point, quantum integrable model, namely for the XXZ chain in its massive regime, i.e. the Hamiltonian (1.1) for \( \Delta > 1 \), this in presence of a zero external magnetic field \( h = 0 \). In 1995, Jimbo and Miwa \[37\] obtained \( 2n \)-fold multiple integral representations for the form factors of local operators of the chain taken between the ground state and an excited state containing \( 2n \)-spinon excitations. Although initially obtained for the XXZ chain at \( \Delta > 1 \), these integral representation admitted a regular \( \Delta \to 1^+ \) limit, hence yielding the corresponding expressions for the XXX Heisenberg chain. The construction of integral representations for the form factors opened the possibility to estimate the DRF of the XXZ chain at \( \Delta \geq 1 \) by taking explicitly the space and time Fourier transforms of the form factor series. Doing so allowed Bougourzi, Karbach and Müller \[17\] to obtain, in 1998, the two-spinon sector contribution \( \mathcal{S}^{(1)}_2(\mathbf{k}, \omega) \) to the transverse dynamic response function \( \mathcal{S}^{(1)}(\mathbf{k}, \omega) \) in the massive regime of the XXZ chain. This analysis was revisited and corrected by Caux, Mosrel and Perez-Castillo \[36\] in 2008 what allowed them to explain the presence of an asymmetry in these DRF. Relatively to singularities, the bottom line of these investigations is that \( \mathcal{S}^{(1)}_2(\mathbf{k}, \omega) \) exhibits square root cusps or singularities along two-spinon excitations thresholds -as expected from a DRF of a massive model-. The two-spinon contribution to \( \mathcal{S}^{(1)}(\mathbf{k}, \omega) \) in the case of the XXX chain was computed by Bougourzi, Couture, Kacic in 1996. Building on these results, Bougourzi, Fledjerjohan, Karbach, Müller and Mütter \[16\] have shown in 1997 that the two-spinon sector saturates ca. 73% of the total intensity of \( \mathcal{S}^{(1)}(\mathbf{k}, \omega) \). They carried as well a thorough analysis of the singularity structure of this DRF, showing the presence of a square root cusp behaviour on the upper two-spinon treshold and a square root divergence on the lower-threshold (plus a logarithmic behaviour). Although the complexity of the integral representations for the higher than 2 spinon sector form factors makes the computations more involved, Abada, Bougourzi, SiLakhal \[1\] and, later, Caux and Hagemans \[22\] still managed to deal with the four spinon contributions to the XXX DRFs. Finally, in 2012, Caux, Konno, Sorrel and Weston \[25\] managed to compute explicitly the two-spinon contribution to the XXZ chain directly in the massless regime and at \( h = 0 \) by using earlier results of Lashkichev and Pugai \[53\] and later rewritings thereof. Here, much in the spirit of the results for the XXX case, the DRF were obtained by first starting from the integral representations for the two-spinon form factors in a massive model (the XYZ chain in this case) and then by taking an appropriate massless scaling limit thereof. Again, the analysis unraveled the presence of square root cusps or divergences, depending on the spinon tresholds. However, due to the much more involved structure on the XYZ chain side, no result exists so far for the higher than two spinon contribution to the form factor series in the massless regime of the XXZ chain at \( h = 0 \).
1.5.3 Numerical and Bethe Ansatz based approaches

All the exact results mentioned so far were obtained in a zero external magnetic field. The obtention of exact results in the presence of a non-zero magnetic field turned out to be much more involved. Nonetheless, it was possible to estimate the response functions numerically. First numerical plots of the longitudinal and transverse DRF in the XXX chain at $h \neq 0$ were obtained by Karbach and Müller [38] in 2000 and then by Biegel, Karbach and Müller [9] in 2002. The plots were obtained by means of a brute force numerical evaluation of the matrix elements of local operators which, in their turn, were computed by using the coordinate Bethe Ansatz representation for the Eigenfunctions of the chain. A qualitative and quantitative step forward of the numerical approach was enabled by the construction of determinant representations for the form factors of local operators in the XXZ chain by Kitanine, Maillet and Terras [43] in 1998. Using such representations which remarkably simplified the numerics, Biegel, Karbach and Müller [10] obtained in 2002 plots of the longitudinal and transverse response functions at fixed momentum $k \in \{\pi, \frac{\pi}{2}\}$ for the XXX chain at finite magnetic field and by distinguishing the contributions of various classes of excitations. Again, for the same values of the momentum, the two-spinon contribution to the longitudinal response function, at various values of the anisotropy $1 > \Delta > 0$ of the XXZ chain, was evaluated numerically by Biegel, Karbach and Müller [11] in 2003. Then, Sato, Shiroishi and Takahashi [64] obtained in 2004 plots at fixed momentum $k = \pi/2$ and energy-momentum plots of the two-spinon contribution to the longitudinal response function at half-saturation field in the massless regime of the XXZ chain, this for various values of the anisotropy. In 2005, Caux and Maillet [26] and then Caux, Maillet and Hagemans [23] obtained $(k, \omega)$ plots of the multi-particle and bound state contribution to the longitudinal $S^{(\omega)}(k, \omega)$ and transverse $S^{(\omega)}(k, \omega)$ DRF. Similar numerics related to the $S^{(\omega)}(k, \omega)$ response function for the XXX chain were carried out by Kohno [45] in 2009, for various values of $h$. In particular, this work has shown that, for the $S^{(\omega)}(k, \omega)$ DRF, the two and three string bound states carry a certain non-negligible part of the spectral weight, as opposed to the $S^{(\omega)}(k, \omega)$ and $S^{(\omega)}(k, \omega)$ response functions where most of the spectral weight is carried by particle-hole excitations. A similar type of numerical analysis was performed in 2006 for the DRFs of the $\delta$-function Bose gas by Caux and Calabrese [20] and in 2007 by Caux, Calabrese and Slavnov [21].

1.5.4 The restricted sum approach

A breakthrough in the exact analysis of certain regimes of form factor expansions of two-point functions in the massless regime of the XXZ chain was achieved by Kitanine, Maillet, Slavnov, Terras and myself [40] in 2011. In that work, we proposed a way to sum up the expansion of XXZ’s static two-point correlation functions over the so-called critical form factor. By heuristically arguing that only such form factors should contribute to the leading order of the large-distance asymptotic behaviour of the two-point functions in this chain, we have been able to compute the amplitude and critical exponent of the leading term associated to every harmonic arising in the long-distance behaviour. Owing to the sole presence of particle-hole excitations in the $\delta$-function Bose gas, we have extended [42] in 2012 the above analysis so as to encompass the case of dynamic two-point functions of that model. We managed to extract, on the basis of first principle arguments, the leading long-time and large-distance asymptotic behaviour of two-point functions while also providing the leading amplitude and critical exponent of every oscillating harmonic (oscillating term at a given frequency and momentum) arising in the asymptotics. The method of analysis we employed also allowed us to investigate the singularity structure of the edge exponents for the dynamic response functions hence confirming, through an ab initio analysis, the predictions stemming from the non-linear Luttinger liquid approach. Although successful for that particular case, the analysis left several open questions. In itself, the method used in [40, 42] only allows one to argue the various asymptotic regimes of the correlators (be it the long-distance/time or the edge singular behaviour of DRFs). In particular, it does not provide

†Expectation values of local operators taken between the ground state and the low-lying excited states exhibiting a conformal structure of their energies.
one with a way to write down a closed form for a massless form factor expansion in the thermodynamic limit and invokes certain heuristics in the handlings of the asymptotic analysis. Furthermore, the applicability of the method to the case of integrable models containing bound states was open. These points were recently solved by myself in [51]. There I managed to circumvent the various problems associated with defining form factor series expansions for massless models and constructed an explicit form factor expansion representation for the dynamical two-point functions in the massless regime of the XXZ spin-1/2 chain at non-zero magnetic field. This representation was enough to take the Fourier transforms explicitly and led to a series of multiple integral representation for the DRFs of the model. The series will be starting point for the analysis carried out in the present work.

The main goal of this paper is to provide a thorough analysis of the edge singularities in the dynamic response functions of the XXZ chain at finite magnetic field and throughout the massless regime, this on the basis of first-principle based calculation: the work starts from the series of multiple integrals representation for the DRFs obtained, on the level of the microscopic model, in [51]. It then carries out rigorously -the well-definedness and some of the properties of the representation obtained in [51] being taken for granted- only those approximations that are consistent with the limiting regimes considered. As a consequence, the analysis carried out in this work does not relies, at any point of our calculations, upon some conjectural or heuristically argued correspondence with a simplified effective model such as a CFT, a Luttinger liquid or its non-linear generalisation.

Furthermore, although obtained for the massless regime of the XXZ chain, taken the "universal" nature of the massless form factor expansion based representation for the DRFs and that the analysis developed in this work solely uses this universal structure, the results will hold -provided one accepts the validity of the phenomenological form of massless form factor expansions advocated in [51]- for any massless one-dimensional quantum Hamiltonian belonging to the Luttinger liquid universality class.

2 Main results

2.1 The setting and some generalities on the model

I shall focus on the so-called massless anti-ferromagnetic regime at positive magnetic field which corresponds to $-1 < \Delta < 1$ and $h_c > h > 0$, where the critical field $h_c$ takes the form $h_c = 4J(1 + \Delta)$. $h_c$ is the saturation field above which the model becomes ferromagnetic. Then, it appears convenient to parametrise the anisotropy $\Delta$ introduced in (1.1) as

$$\Delta = \cos(\zeta) \quad \text{with} \quad \zeta \in [0; \pi[. \quad (2.1)$$

In the thermodynamic limit, the Bethe Ansatz analysis ensures that, for this range of parameters, the excited states above the ground state are built from a pile up of elementary dressed excitations of different types: holes and $r$-strings. For given value of $\zeta$, only certain values of $r$ are possible for the $r$-strings and it is convenient to collect these in the set $\mathbb{N} = \{r_1, \ldots, r_{|\mathbb{N}|}\}$. The set $\mathbb{N}$ is finite when $\zeta / \pi$ is rational and infinite otherwise [68]. Furthermore, independently of the value of $\zeta$, there always exists 1-strings excitations (viz $r_1 = 1$). The 1-string excitations correspond to so-called particle excitations. Among all possible $r$-string excitations, only the particles -i.e. 1-strings- may generate massless excitation, i.e. carrying a zero energy. A given excited state will be made up of $n_h \in \mathbb{N}$ holes, $n_{r_1} \in \mathbb{N}$ $r_1$-strings and left/right Fermi boundary Umklapp excitations with deficiencies $\ell_\pm \in \mathbb{Z}$. These integers satisfy to the constraint

$$n_h = \sum_{r \in \mathbb{N}} r n_r + \sum_{r_1 \in \mathbb{N}} \ell_{r_1}. \quad (2.2)$$

It is convenient to collect the integers labelling the number of excitations of each kind into a single vector

$$n = (\ell_+, \ell_-; n_h, n_{r_1}, \ldots, n_{r_{|\mathbb{N}|}}). \quad (2.3)$$
Owing to the constraint \((2.2)\), there are only finitely many non-zero entries in \(n\).

\(n\) being fixed, the \(n_h\) holes will carry momenta \(t_1, \ldots, t_{n_h}\) which take values in \(\mathcal{R}_h = [-p_F; p_F]\), \(p_F \in [0; \pi/2]\) being the Fermi momentum, and the \(n_r\) r-strings will carry momenta \(k_1^{(r)}, \ldots, k_{n_r}^{(r)}\) taking values in \(\mathcal{I}_r = [p_+^{(r)}; p_+^{(r)}]\). I refer to Appendix C.1 for more precise definitions of these intervals. It appears convenient to gather the momenta carried by the various elementary excitations into the single vector

\[
\mathbf{R} = (\ell_+; \ell_-; t, k_1^{(r)}, \ldots, k_{n_r}^{(r)}) \quad \text{with} \quad t \in \mathcal{R}_h^{n_h} \quad \text{and} \quad k^{(r)} \in \mathcal{I}_r^{n_r}. \tag{2.4}
\]

This notation should be understood as follows. If \(n_h = 0\), resp. \(n_r = 0\) with \(r \in \mathfrak{R}\), then the associated vectors \(t\), resp. \(k^{(r)}\), are to be read as 0, meaning that there is simply no component of the hole or of this r-string momenta in \(\mathbf{R}\), since there are no excitations of this type in the given excited state. The use such a notation allows one to keep the precise track, on the level of the vector \(\mathbf{R}\), of the types of excitations which are present and those which are absent. I stress that formally \(\mathbf{R}\) may contain infinitely many components with such a conventions, but only finitely many of them correspond to non-empty sets since, for fixed \(\ell_\pm\) and \(n_h\), there is only a finite number of integers \(n_r\) that are non-zero. Hence \(\mathbf{R}\) makes sense as an inductive limit. Furthermore, effectively speaking, \(\mathbf{R}\) is built up from vector momenta \(t\), resp. \(k^{(r)}\) with \(r \in \mathfrak{R}\), such that \(n_h \neq 0\), resp. \(n_r \neq 0\).

A given excited state in a sector of relative spin \(s_\gamma\) above the ground state and associated with a vector momentum \(\mathbf{R}\) has a total excitation momentum

\[
\mathcal{P}(\mathbf{R}) = \sum_{r \in \mathfrak{R}} \sum_{a=1}^{n_r} k_a^{(r)} + p_F \sum_{\ell = \pm} \nu \ell_\nu + \pi s_\gamma - \sum_{a=1}^{n_h} t_a \tag{2.5}
\]

and carries a total excitation energy

\[
\mathcal{E}(\mathbf{R}) = \sum_{r \in \mathfrak{R}} \sum_{a=1}^{n_r} \epsilon_a(k_a^{(r)}) - \sum_{a=1}^{n_h} \epsilon_1(t_a). \tag{2.6}
\]

The functions \(\epsilon_a\) correspond to the dispersion relation of the various excitations, \(\epsilon_r\) for the r-strings, \(-\epsilon_1\) for the holes. They are defined as solutions to linear integral equations, see Appendix C.1 equations (C.13)-(C.14) for more details. Again, by convention, sums that are subordinate to \(n_r = 0\) or \(n_h = 0\) are simply understood to be absent.

The velocity of a given r-string excitation with momentum \(k\) is defined as \(v_r(k) = \epsilon_r'(k)\). Moreover, \(v_1(k)\) gives the velocity, depending on the domain where \(k\) evolves, of 1-strings (particles) if \(k \in \mathcal{I}_1\) or holes if \(k \in \mathcal{R}_h\). Particles, holes, and more generally strings, may share the same value of their velocities. In particular, one can prove, c.f. Proposition C.1 in Appendix C.2 that for certain regimes of the model’s parameters, that there exists an interval \([K_m; K_M] \subset \{p_+^{(1)}; p_+^{(1)}\}\) and a diffeomorphism

\[
t : [K_m; K_M] \to \mathcal{I}_h \quad \text{such that} \quad v_1(k) = v_1(t(k)). \tag{2.7}
\]

It is conjectured that this property holds for any regime of the parameters and this is backed by an extensive numerical analysis.

### 2.2 The behaviour of the longitudinal dynamic response function in the two hole excitations in \(\mathcal{I}_h^{(2)}(k, \omega)\)

The excitations thresholds giving rise to singularities of the longitudinal dynamic response function \(\mathcal{I}_h^{(2)}(k, \omega)\) and built up from excitations containing at most two holes and/or two 1-strings are depicted in Figure 1. The curves \(C_h^{(a)}\), \(a = 1, 2\), resp. \(C_p^{(b)}\), \(b = 1, \ldots, 4\), correspond to one hole, resp. particle, excitation above the ground state.
Figure 1: Singularity curves issued from the sectors involving up to two particles, two-holes and no \( r \)-strings with \( r \geq 2 \) for \( \Delta = 0.57 \) and in presence of a magnetic field \( h \) which fixes the per site magnetisation \( m = 1 - 2D \) such that \( D = 0.21 \). Continuous curves correspond to one massive hole or particle excitation. Dotted curves correspond to a collective, coordinated, multi-particle-hole excitation. This excitation is such that all particles and holes building it up have the same velocity.

The curves \( C^{(a)}_{p-h} \), \( a = 1, 2 \) correspond to a joint particle-hole excitation where the particle and hole both have the same velocity. Finally, the curve \( C_{2p-h} \) is built up from a two particle - one hole excitation, all having the same velocity. All the particles or holes building up the excitations in the curves depicted in Figure 1 are massive - viz. carry a finite excitation energy- with the exception of the \( \omega = 0 \) line and of the junctures between the curves that are drawn in continuous and dotted lines. The present approach is unable to analyse the singularity structure at these points. Below, \( 0 < \tau < 1 \) is arbitrary and can be taken as small as necessary.

More precisely, the results established in Section 3 entail that

- \( C^{(1)}_h \) is realised as a one hole excitation with \( \ell_+ = 1, \ell_- = 0 \). It takes the parametric form

\[
(P_0, E_0) = (p_F - t_0, -v_1(t_0)) \quad \text{with} \quad t_0 \in ] - p_F ; p_F [.
\]

Along this curve, the response function behaves as

\[
\mathcal{F}^{(z)}(P_0, E_0 + \delta \omega) = \mathcal{F}^{(z)}_{\text{reg}}(\delta \omega) + \mathcal{A}^{(h)}(\delta \omega) + \mathcal{A}^{(h)} \cdot \Xi(\delta \omega) + O((\delta \omega)^{\Delta^{(h)} + 1 - \rho}) .
\]

\( \mathcal{F}^{(z)}_{\text{reg}}(\delta \omega) \) is smooth in \( \delta \omega \) while the critical exponent takes the form \( \Delta^{(h)} = \delta_+^{(h)} + \delta_-^{(h)} - 1 \). \( \delta_\pm^{(h)} \) are expressed in terms of the dressed phase, c.f. (C.18), as

\[
\delta_+^{(h)} = (\varphi_1(p_F, t_0) - \varphi_1(p_F, -p_F) - 1)^2, \quad \delta_-^{(h)} = (\varphi_1(-p_F, t_0) - \varphi_1(-p_F, p_F))^2.
\]

Finally, the amplitude \( \mathcal{A}^{(h)} \) is closely related to the properly renormalised in the volume form factor squared \( \mathcal{F}^{(z)}(R_0^{(h)}) \) of the operator \( \sigma^z \) taken between the ground state and the excited state associated with \( C^{(1)}_p \):

\[
\mathcal{A}^{(h)} = \frac{(2\pi)^2 \cdot \mathcal{F}^{(z)}(R_0^{(h)})}{\Gamma(\delta_+^{(h)} + \delta_-^{(h)}) \cdot [v_F + v_1(t_0)]^{\delta_+^{(h)}} \cdot [v_F - v_1(t_0)]^{\delta_-^{(h)}}}.
\]
The precise definition of $\mathcal{F}^{(c)}(\boldsymbol{R}_0^{(h)})$ is given in (2.35) and $\mathcal{R}_0^{(h)} = \left( \ell_+ = 1, \ell_- = 0; t = t_0, 0, \ldots \right)$. All building blocks of $\mathcal{A}^{(h)}$ other than the renormalised form factor $\mathcal{F}^{(c)}(\mathcal{R}_0^{(h)})$ correspond to the universal part of the amplitude associated with this hole excitation branch. Finally, $v_F = v_1(q)$ is the velocity of the excitations on the right Fermi boundary.

- $C_p^{(1)}$ is realised as a one particle excitation with $\ell_+ = -1, \ell_- = 0$. It takes the parametric form

$$
(P_0, E_0) = (k_0 - p_F, e_1(k_0)) \quad \text{with} \quad k_0 \in ]p_F; K_m[ .
$$

Along this curve, the response function behaves as

$$
\mathcal{F}^{(c)}(P_0, E_0 + \delta \omega) = \mathcal{F}^{(c)}_{\text{p,reg}}(\delta \omega) + \mathcal{A}^{(p)} \cdot |\delta \omega|^\Delta^{(p)} \left\{ \Xi(\delta \omega)\frac{\sin[\pi \delta^{(p)}_+]}{\pi} + \Xi(-\delta \omega)\frac{\sin[\pi \delta^{(p)}_-]}{\pi} \right\}
+ O\left((\delta \omega)^{\Delta^{(p)}+1-\tau}\right) .
$$

(2.13)

$\mathcal{F}^{(c)}_{\text{p,reg}}(\delta \omega)$ is smooth in $\delta \omega$. The critical exponent takes the form $\Delta^{(p)} = \delta^{(p)}_+ + \delta^{(p)}_- - 1$ and $\delta^{(p)}_\pm$ are expressed in terms of the dressed phase, c.f. (C.18), as

$$
\delta^{(p)}_+ = \left(1 - \varphi_1(p_F, k_0) + \varphi_1(p_F, p_F)\right)^2 , \quad \delta^{(p)}_- = \left(\varphi_1(-p_F, p_F) - \varphi_1(-p_F, k_0)\right)^2 .
$$

(2.14)

The amplitude $\mathcal{A}^{(p)}$ takes the form

$$
\mathcal{A}^{(p)} = \frac{(2\pi)^2 \cdot \Gamma\left(1 - \delta^{(p)}_+ - \delta^{(p)}_-ight)}{|v_F + v_1(k_0)|^{\delta^{(p)}_+} \cdot |v_F - v_1(k_0)|^{\delta^{(p)}_-}} \cdot \mathcal{F}^{(c)}(\mathcal{R}_0^{(p)}) .
$$

(2.15)

$\mathcal{F}^{(c)}(\mathcal{R}_0^{(p)})$ has the same interpretation as given above, is defined in (2.35) and is parameterised by the vector momentum $\mathcal{R}_0^{(p)} = \left( \ell_+ = -1, \ell_- = 0; t = 0, k^{(1)} = k_0, 0, \ldots \right)$. All the other building blocks of $\mathcal{A}^{(p)}$ correspond to the universal part of this particle branch amplitude.

- $C_p^{(1)}$ is realised as an excitation with $\ell_+ = 0, \ell_- = 0$, and containing a particle and a hole, both having the same velocity. It takes the parametric form

$$
(P_0, E_0) = \left(k_0 - t(k_0), e_1(k_0) - e_1(t(k_0))\right) \quad \text{with} \quad k_0 \in ]K_m; K_M[ .
$$

(2.16)

and where $t$ has been introduced in (2.7). Along this curve, the response function behaves as

$$
\mathcal{F}^{(c)}(P_0, E_0 + \delta \omega) = \mathcal{F}^{(c)}_{\text{ph,reg}}(\delta \omega) + \mathcal{A}^{(ph)} \cdot |\delta \omega|^\Delta^{(ph)} \left\{ \Xi(\delta \omega)\frac{\cos[\pi \Delta^{(ph)}]}{\pi} + \Xi(-\delta \omega)\frac{1}{\pi} \right\}
+ O\left((\delta \omega)^{\Delta^{(ph)}+1-\tau}\right) .
$$

(2.17)

$\mathcal{F}^{(c)}_{\text{ph,reg}}(\delta \omega)$ is smooth in $\delta \omega$. The critical exponent takes the form $\Delta^{(ph)} = \delta^{(ph)}_+ + \delta^{(ph)}_- - 1/2$ and $\delta^{(ph)}_\pm$ are expressed in terms of the dressed phase, c.f. (C.18), as

$$
\delta^{(ph)}_+ = \left(\varphi_1(p_F, t(k_0)) - \varphi_1(P_F, k_0)\right)^2 , \quad \delta^{(ph)}_- = \left(\varphi_1(-p_F, t(k_0)) - \varphi_1(-p_F, k_0)\right)^2 .
$$

(2.18)

Finally, the amplitude $\mathcal{A}^{(ph)}$ takes the form

$$
\mathcal{A}^{(ph)} = \frac{(2\pi)^2}{\sqrt{1 - V(k_0)}} \cdot \left(\frac{2\pi}{v_1'(t(k_0))}\right)^{\tau} \cdot \frac{\Gamma\left(-\Delta^{(ph)}\right)}{|v_F + v_1(k_0)|^{\delta^{(ph)}_+} \cdot |v_F - v_1(k_0)|^{\delta^{(ph)}_-}} \cdot \mathcal{F}^{(c)}(\mathcal{R}_0^{(ph)}) .
$$

(2.19)
\( \mathcal{S}^{(c)}(\mathcal{R}_0^{(ph)}) \) has the same interpretation and \( \mathcal{R}_0^{(ph)} = \left( \ell_+ = 0, \ell_- = 0; t = t(k_0), k^{(1)} = k_0, 0, \ldots \right) \). All the other building blocks of \( \mathcal{A}^{(ph)} \) correspond to the universal part of this equal velocity particle-hole branch amplitude.

- \( C_p^{(4)} \) is realised as a one particle excitation with \( \ell_+ = 0, \ell_- = -1 \). It takes the parametric form

\[
(P_0, E_0) = (k_0 + p_F, \epsilon_1(k_0)) \quad \text{with} \quad k_0 \in \mathbb{R}; 2\pi - 3p_F.
\]

Along this curve, the response function behaves as

\[
\mathcal{S}^{(c)}(P_0, E_0 + \delta \omega) = \mathcal{S}_{p_{\text{reg}}}^{(c)}(\delta \omega) + \mathcal{A}^{(p)} \cdot |\delta \omega|^{\Delta^{(p)}} \left\{ \Xi(\delta \omega) \frac{\sin[\pi \delta^{(p)}_+]}{\pi} + \Xi(-\delta \omega) \frac{\sin[\pi \delta^{(p)}_-]}{\pi} \right\} + O((\delta \omega)^{\Delta^{(p)}+1+\epsilon}).
\]

\( \mathcal{S}_{p_{\text{reg}}}^{(c)}(\delta \omega) \) is smooth in \( \delta \omega \). The critical exponent takes the form \( \Delta^{(p)} = \delta^{(p)}_+ + \delta^{(p)}_- - 1 \) and \( \delta^{(p)}_\pm \) are expressed in terms of the dressed phase, c.f. (2.18), as

\[
\delta^{(p)}_+ = \left( \varphi_1(p_F, p_F) - \varphi_1(p_F, k_0) + 1_F(k_0) \right) \sin(\pi - 2\zeta)Z(p_F),
\]

\[
\delta^{(p)}_- = \left( -1 + \varphi_1(-p_F, p_F) - \varphi_1(-p_F, k_0) + 1_F(k_0) \right) \sin(\pi - 2\zeta)Z(p_F).
\]

Here, \( L_\pm = \{2\pi - 2p_F \sin(\pi - 2\zeta - (\pi - 2\zeta) F_0^{(p)}), 2\pi - p_F - 2p_F \sin(\pi - 2\zeta) \} \). Finally, the amplitude \( \mathcal{A}^{(p)} \) takes the same form as in (2.15), with the constants appropriately substituted.

- \( C_{p_{2h}} \) is realised as an excitation with \( \ell_+ = 1, \ell_- = 0 \) that consists of one particle and two holes, all having the same velocity. It takes the parametric form

\[
(P_0, E_0) = \left( k_0 - 2t(k_0) + p_F, \epsilon_1(k_0) - 2\epsilon_1(t(k_0)) \right) \quad \text{with} \quad k_0 \in \mathbb{R}; K_M + 3p_F.
\]

and \( t \) as in (2.7). For this parameterisation, \( P_0 \) increases on the interval \([K_m - p_F; K_M + 3p_F]\). Along this curve, the response function has the singular structure

\[
\mathcal{S}^{(c)}(P_0, E_0 + \delta \omega) = \mathcal{S}_{p_{2h_{\text{reg}}}}^{(c)}(\delta \omega) + \mathcal{A}^{(p_{2h})} \cdot |\delta \omega|^{\Delta^{(p_{2h})}} \times \frac{\sin[\pi \Delta^{(p_{2h})}]}{\pi} + O((\delta \omega)^{\Delta^{(p_{2h})+1+\epsilon}}).
\]

The critical exponent takes the form \( \Delta^{(p_{2h})} = \delta^{(p_{2h})}_+ + \delta^{(p_{2h})}_- + 1 \) and \( \delta^{(p_{2h})}_\pm \) are expressed in terms of the dressed phase, c.f. (2.18), as

\[
\delta^{(p_{2h})}_+ = \left( -1 + 2\varphi_1(p_F, t(k_0)) - \varphi_1(p_F, k_0) - \varphi_1(p_F, p_F) \right)^2,
\]

\[
\delta^{(p_{2h})}_- = \left( 2\varphi_1(-p_F, t(k_0)) - \varphi_1(-p_F, k_0) - \varphi_1(-p_F, p_F) \right)^2.
\]

Finally,

\[
\mathcal{A}^{(p_{2h})} = \frac{(2\pi)^3}{\sqrt{1 - 2t^2(k_0)}} \left( \frac{1}{v_F(t(k_0))} \right)^2 \frac{\Gamma(-\Delta^{(p_{2h})})}{|v_F + v_1(t(k_0))|^{|\Delta^{(p_{2h})}|}; |v_F - v_1(t(k_0))|^{|\Delta^{(p_{2h})}|}; \mathcal{S}_{p_{2h}}^{(c)}(\mathcal{R}_0^{p_{2h}})}.
\]

\( \mathcal{S}^{(c)}(\mathcal{R}_0^{p_{2h}}) \) has the same interpretation and \( \mathcal{R}_0^{(p_{2h})} = \left( \ell_+ = 1, \ell_- = 0; t = (t(k_0), t(k_0)), k^{(1)} = k_0, 0, \ldots \right) \). All the other building blocks of \( \mathcal{A}^{(p_{2h})} \) correspond to the universal part of this equal velocity one particle two hole branch amplitude.
The curves appearing in Fig. 1 are symmetric in respect to the \( k = \pi \) axis. This symmetry also applies relatively to the behaviour along these curves. Thus, the cases that were not listed above can be inferred by this symmetry operation. Also, one should observe that certain curves are realised as \( 2p_F \) or \( 2(\pi - p_F) \) translations of other curves. This is reminiscent of the possibility, in the model, to realise zero energy excitations carrying a non-zero discrete momentum which is an integer multiple of \( 2p_F \). \( C_{p}^{(2)} \) is deduced from \( C_{p}^{(1)} \) by adding a particle on the right end of the Fermi zone and a hole on the left end what corresponds to \((\ell_+, \ell_-) = (-1, 0) \leftrightarrow (\ell'_+, \ell'_-) = (0, 1)\). This, however, changes the values of the critical exponents.

The excitation thresholds \( C_{h}^{(a)}, a = 1, 2 \) and \( C_{p}^{(b)}, b = 1, \ldots, 4 \), along with the associated universal structure of the singular behaviour have been argued in the literature by means of heuristic approaches: the non-linear Luttinger liquid [27] in what concerns \( C_{h}^{(a)} \), [2, 3] relatively to \( C_{h}^{(a)}, C_{p}^{(b)} \) and the pseudofermion dynamic theory [18] relatively to \( C_{h}^{(a)} \). The present analysis does confirm these predictions on the basis of rigorous considerations.

The thresholds corresponding to the curves \( C_{ph} \) and \( C_{p2h} \) have never been discussed within the aforementioned approaches. These excitation thresholds are characterised by a different structure of edge exponents as clearly appears in [2.18] and [2.26]-[2.27]. On physical grounds, these thresholds issue from the presence of excited states built up from various excitations (particles, holes and/or \( r \)-strings), all having equal velocities. The singular structure of the dynamic response functions in the vicinity of multi-hole/\( r \)-string excitations, \( r \in \mathcal{N}_{ul} \) is discussed in Theorem [3.4]. Finally, although it is not detailed in the body of the paper, the structure of the behaviour of dynamic response functions in the vicinity of equal velocity multi-particle/hole/\( r \)-string thresholds can be readily worked out by appropriately adjusting the results of the main theorem established in this paper, Theorem [1.5]. One should mention, that solely the work [3] considered the thresholds generated by a joint multi-particle massive excitation. In [3], the authors argued heuristically the expression for the edge exponents in the case of an equal velocity excitation built up from two holes and one two-string. They also asserted that the singularity is only one-sided. They did not discuss the form of the amplitude though. The present analysis recovers all these features and provides much more thorough information on the amplitude. The presence of one sided singularities does not hold, however, for generic \( r \)-string excitations.

### 2.3 The series representation for the dynamic response functions

Under certain assumptions, I have derived in [51] a series of multiple integral representation for the dynamic response functions of the XXZ spin-1/2 chain in the massless regime \(-1 < \Delta < 1\) and at finite magnetic field \( h_c > h > 0 \). The derivation of the representation relied on the assumption that it is licit to exchange certain limits with summations, that the remainders were uniformly summable and that the resulting series was convergent. The rest of the handling were rigorous. I shall not discuss here further the rigour of the obtained series. In the present work, I shall take for granted the existence and well-definiteness of the series of multiple integrals representing the DRF. The justification of the exchange of limits procedures used in its derivations along with the convergence of the series is left for future investigations and will quite probably demand to invent new mathematical tools adapted for dealing with such questions.

The present work carries out a rigorous analysis of the singularity structure of each summands in the series representing \( \mathcal{S}^{(\gamma)}(k, \omega) \). Developing a technique allowing one for a rigorous analysis of a class of multiple integrals containing, upon specialisations, the integrals of interest constitutes the main achievement of this work. The series of multiple integrals obtained in [51] takes the form

\[
\mathcal{S}^{(\gamma)}(k, \omega) = \sum_{\hat{\Delta} \in \hat{\mathcal{S}}} \mathcal{S}^{(\gamma)}_{\hat{\Delta}}(k, \omega)
\]  

(2.29)

where the summation runs through all the allowed choices of hole, \( r \)-string and Umklapp integers, all gathered in
a single vector \( n \), as in (2.3), while
\[
\mathcal{Z} = \left\{ (\ell_+, \ell_-; n_h, n_r, \ldots, n_r) : \ell_{\pm} \in \mathbb{Z}, n_h, n_r \in \mathbb{N} \right\} \text{ and } n_h = \sum_{r \in \mathbb{R}} n_r + \sum_{\nu=1}^{\ell_-}. \tag{2.30}
\]

A given summand \( \mathcal{S}_n^{(\gamma)}(k, \omega) \) represents the contribution to the dynamic response function of all the excited states whose number of excitations of each type is equal to the corresponding entry of the vector \( n \). It is given by the multidimensional integral
\[
\mathcal{S}_n^{(\gamma)}(k, \omega) = \int_{\mathcal{J}_n^{(\gamma)}} \mathcal{D}_{\mathcal{R}} \prod_{r \in \mathbb{R}} \left\{ \int \mathcal{D}_{\mathcal{R}} k^{(r)} \right\} \cdot \mathcal{F}^{(\gamma)}(\mathcal{R})
\times \sum_{\nu \in \mathbb{R}} \prod_{\nu \neq \pm} \left\{ \Xi(\mathfrak{R}; s) \cdot \left[ \mathfrak{I}_n(\mathfrak{R}; s) \right]^{A_r(\mathfrak{R})-1} \right\} \cdot \left( 1 + \tau(\mathfrak{R}; s) \right). \tag{2.31}
\]

Just as earlier on, by convention, if a hole \( n_h \) or an \( r \)-string \( n_r \), integer is zero, then the associated integration, and \textit{a fortiori} integration variables, are simply absent. The integration variables are collected in the vector \( \mathcal{R} \) that was introduced in (2.4). In the definition of this vector, it should be understood that, if \( n_h = 0 \), than the corresponding vector \( k^{(r)} \) is simply absent. Thus, due to the summation constraint in (2.30), for each \( n \), there are only finitely many \( k^{(r)} \) vectors present in \( \mathcal{R} \), \textit{c.f.} the discussion which followed after (2.4).

I now describe, in detail, the different building blocks of the multiple integral.

- **The integration domain and the regulator \( \epsilon > 0 \)**

The integration variables run through the slightly deformed domains
\[
\mathcal{J}_h^{(\epsilon)} = [-p_F + \epsilon ; p_F - \epsilon], \quad \mathcal{J}_1^{(\epsilon)} = [p_+^{(1)} + \epsilon ; p_-^{(1)} - \epsilon] \tag{2.32}
\]
with \( p_\pm^{(1)} \) being parameterised in terms of \( \zeta \), introduced in (2.1), as \( p_+^{(1)} = 2\pi - p_F - 2p_F \text{sgn}(\pi - 2\zeta) \), \( p_-^{(1)} = p_F \). More generally, \( \mathcal{J}_r^{(\epsilon)} = \mathcal{J}_r = [p_+^{(r)} ; p_-^{(r)}] \) for any \( r \geq 2 \) and the explicit form for \( p_\pm^{(r)} \) can be inferred from the content of Appendix C.1.

Recall that massless excitations are realised by particles and/or holes whose momenta collapse, in the thermodynamic limit, on the left and right endpoint of the Fermi zone, \textit{i.e.} the points \( \pm p_F \) for the holes and the points \( p_\pm^{(r)} \) for the particles. In their turn, the massive excitations carry a finite excitation energy in the thermodynamic limit. Thus, massive particles and/or holes have their momenta located uniformly away from the endpoints of the Fermi zone.

The integral representation (2.31) involves a small but otherwise arbitrary parameter \( \epsilon > 0 \). The latter was introduced in [5] as a regulator defining a separating scale between the massive and massless particle and hole type excitations in the model. The matter is that the contributions of the massless modes cannot be summed by means of a Lebesgue-measure based integral and demand a very different treatment. Their leading effect is already taken into account and manifests itself in the dependence on the functions \( \mathfrak{I}_n(\mathfrak{R}; s) \), \textit{c.f.} (2.43). The \( \epsilon \)-dependence appears explicitly on the level of the domain of integration (2.32), while the rest of \( \epsilon \)-dependence is contained in the remainder \( \tau(\mathfrak{R}; s) \), \textit{c.f.} the later discussion. The whole series (2.29) does not depend on the regulator \( \epsilon \). One cannot take the \( \epsilon \to 0^+ \) individually in each multiple integral due to the presence of non-integrable singularities in the integrals and a non-uniformness in of the control on the remainder in the \( \epsilon \to 0^+ \) limit. However, one can always consider \( \epsilon \) to be as small as necessary for the purpose of the analysis, as long as it remains fixed.
• The integrand

The integrand in (2.31) is built up from two contributions: the static part $F(\mathbf{R})$, and the dynamic part built up from the functions $\gamma_\mathbf{R}(s)$ and $\Delta_\mathbf{R}(s)$. Note that it is precisely the dynamic part that introduces singularities in the integrand and, as such, is the one responsible for the existence of an edge singular behaviour of the DRFs.

- The dynamic part

The function

$$\gamma_\mathbf{R}(s) = \omega - E(\mathbf{R}) + \nu v_F (k - P(\mathbf{R}) + 2\pi s)$$

(2.33)

is the only building block of the integrand that depends on the momentum $k$ and the energy $\omega$. Its expression involves the relative excitation momentum $P(\mathbf{R})$ and relative excitation energy $E(\mathbf{R})$ which were defined, resp., in (2.5) and (2.6). It also involves $v_F = v_1(q)$, the velocity of the excitations on the Fermi boundary.

The exponents $\Delta_\mathbf{R}(\mathbf{R}) \geq 0$ are smooth functions of $\mathbf{R}$. Their explicit expression can be found in equation (C.19) of Appendix C.1 and just above it.

The dynamic part is summed up over $s$ in (2.31). This summation is, in fact, finite. Indeed, for fixed $n \in \mathbb{Z}$, the functions $P(\mathbf{R})$ and $E(\mathbf{R})$ are bounded on the integration domain from below and above. Thus, for $(k, \omega)$ belonging to any compact subset of $\mathbb{R}^2$, there will exist finitely many $s \in \mathbb{Z}$ such that both $\gamma_\mathbf{R}(s) > 0$. In fact, the summation over $s$ in (2.31) simply translates the fact that the spectral function is a $2\pi$ periodic function of $k$, owing to the discrete nature of the XXZ chain.

- The static part

The static part is a smooth function of $\mathbf{R}$, at least when the latter ranges through the integration domain given in (2.31). It is expressed as

$$F^{(\gamma)}(\mathbf{R}) = \frac{(2\pi)^2 \cdot F^{(\gamma)}(\mathbf{R}) \cdot [2\nu v_F]^{-\Delta_+ - \Delta_0 + 1}}{n_h ! \cdot \prod_{r \in R} n_r ! \cdot \Gamma(\Delta_+ \mathbf{R}) \cdot \Gamma(\Delta_- \mathbf{R})}.$$  

(2.34)

$F^{(\gamma)}(\mathbf{R})$ corresponds to the properly renormalised in the volume, thermodynamic limit of the form factor squared of the spin operator $\sigma_1^\mathbf{R}$ taken between the ground state $\Omega$ and the state $\Upsilon_\mathbf{R}$ which is the Eigenstate of $\mathcal{H}$ satisfying the constraints:

i) in the thermodynamic limit, $\Upsilon_\mathbf{R}$ is parameterised in terms of elementary excitations whose momenta are gathered in the vector $\mathbf{R}$;

ii) $\Upsilon_\mathbf{R}$ has the lowest possible, compatible with i), relative excitation energy above the ground state in finite volume $L$.

This properly normalised form factor reads

$$F^{(\gamma)}(\mathbf{R}) = \lim_{L \to +\infty} \left\{ \left( \frac{L}{2\pi} \right)^{\tau(\mathbf{R})} \left| \langle \Upsilon_\mathbf{R}, \sigma_1^\mathbf{R} \Omega \rangle \right|^2 \right\} \quad \text{with} \quad \tau(\mathbf{R}) = \Delta_+ (\mathbf{R}) + \Delta_- (\mathbf{R}) + n_h + \sum_{r \in \mathbb{R}} n_r.$$  

(2.35)

The explicit expression for $F^{(\gamma)}(\mathbf{R})$ can be found in [49].

- The remainder

Finally, $\tau(\mathbf{R}; s)$ is a remainder term. It is controlled as

$$\tau(\mathbf{R}; s) = O\left( \sum_{r \in \mathbb{R}} |\gamma_\mathbf{R}(\mathbf{R}; s)|^{1-\tau} \right)$$

(2.36)
and this estimation is uniform throughout the integration domain. The parameter \(1/2 > \tau > 0\) is arbitrary provided that it is taken small enough. The control on the remainder is also differentiable in respect to the parameters \((\omega, k)\), in the sense of Definition A.1. However, the control on the remainder is not uniform in respect to \(\epsilon \to 0^+\).

In fact, one expects that the optimal control on \(r(\mathbf{R}; s)\) is provided by the sharper bound

\[
\tau(\mathbf{R}; s) = O\left( \sum_{\nu = \pm} \left| \hat{\nu}_0(\mathbf{R}; s) \ln |\hat{\nu}_0(\mathbf{R}; s)| \right| \right). \tag{2.37}
\]

\(\otimes\) An additional property of the integrand

As argued in [47], the series (2.29) taken as a whole has a built-in mechanism which enforces the complete cancellation between the contributions of an immediate vicinity of the boundaries of integration

\[
\partial \left| \left( \mathcal{J}_h^{(e)} \right)^{n_h} \mathcal{J}(\mathbf{h}) \right|_{\text{reR}} \left( \mathcal{J}_r^{(e)} \right)^{n_r} \tag{2.38}
\]

arising in each of the multiple integrals (2.31). This cancellation property effectively results in that the form factor density \(F^{(\gamma)}(\mathbf{R})\) can be considered as a function vanishing smoothly on the boundary (2.38).

3 The edge singular behaviour of dynamic response functions

This section gathers various theorems capturing the singular behaviour of the dynamic response function issuing from various excitation sectors in the model’s spectrum. The statements follows from an application of the general theorems proven throughout Appendix D and E. All the theorems stated below, take as a hypothesis the smooth vanishing of the integrands on the boundary of integration which was discussed above. The precise and rigorous establishing of this property, beyond the arguments given in [47], is left for further study. Also, some of these rely on the properties stated in Conjecture C.2, which can be proven in certain cases of the coupling constants \(\Delta\) and \(h\), c.f. Appendix C.2.

3.1 The one free rapidity sector

In this subsection, I extract the singular behaviour of the dynamical response functions associated with one massive excitation, namely an excitation consisting either of one hole or one particle far from the Fermi boundaries, or one \(r\)-string with \(r \in \mathbb{R} \setminus \{1\}\). Such an excitation can be accompanied by any value of the left or right Umklapp integers \(\ell_\pm\) that are compatible with the constraint (2.2).

3.1.1 The one-hole contributions

For the present purpose, it is convenient to parameterise the momentum-energy \((k, \omega)\) combination as

\[
k = P_0 \text{ where } P_0 = \pi s + p_F \sum_{\nu = \pm} \nu \ell_\nu - t_0 - 2\pi s_0 \quad \text{and} \quad \omega = \delta \omega + E_0 \tag{3.1}
\]

where \(t_0 \in [-\pi; \pi]\), \(s_0 \in \mathbb{Z}\), and \(\ell_\pm\) are subject to the constraint \(\sum_{\nu = \pm} \ell_\nu = 1\).

The one-hole DRF takes the form

\[
\mathcal{J}^{(\gamma)}(P_0, E_0 + \delta \omega) = \int \mathcal{J}^{(\gamma)}(\mathbf{R}) \cdot \sum_{n_h} \sum_{n_r} \left\{ \Xi(\delta \omega + \nu_\nu(h)(t; s)) \cdot \left[ \delta \omega + \nu_\nu(h)(t; s) \right]^{\Delta_h(\mathbf{R}) - 1} \right\} 
\]

\[
\times \left( 1 + \tau(\mathbf{R}^{(h)}; s) \right) \tag{3.2}
\]
Here, I have set
\[ n_h = (\ell_+, \ell_-; n_h = 1, 0, \ldots) \quad \text{and} \quad \mathbf{R}^{(h)} = (\ell_+, \ell_-; t = t, 0, \ldots). \] (3.3)

As a consequence, there are no \( k^{(r)} \) vectors present in \( \mathbf{R}^{(h)}. \) The vector \( \mathbf{R}^{(h)} \) only involves the momentum \( t \) of one hole excitation and the Umklapp integers. The \( \eta^{(h)}_0 \) functions appearing in (3.2) take the form
\[ \eta^{(h)}_0(t; s) = \epsilon_1(t) + \mathcal{E}_0 + \nu \mathcal{F} [t - t_0 + 2\pi(s - s_0)]. \] (3.4)

Finally, the remainder satisfies
\[ \tau(\mathbf{R}^{(h)}; s) = \mathcal{O} \left( \sum_{\nu=\pm} \left| \delta_\nu + \eta^{(h)}_0(t; s) \right|^{1-\tau} \right), \] (3.5)

and the control is differentiable in the sense of Definition [A.1].

**Theorem 3.1.** Assume that

i) \( t_0 \notin [-p_F; p_F], \) i.e. does not belong to the range of available momenta for a hole, in which case \( \mathcal{E}_0 \) can take any value;

ii) \( t_0 \in [-p_F; p_F] \) is within the range of momenta of a hole excitation and that the subsidiary condition holds \( \mathcal{E}_0 = -\epsilon_1(t_0). \)

Then \( \mathcal{J}^{(y)}_{\mathcal{R}^0} (\mathcal{P}_0, \mathcal{E}_0 + \delta\omega) \) is smooth in \( \delta\omega \) belonging to a neighbourhood of \( 0. \)

Assume that
\[ t_0 \in [-p_F; p_F], \quad \mathcal{E}_0 = -\epsilon_1(t_0) \quad \text{and} \quad \delta^{(h)}_\nu = \Delta_\nu(\mathbf{R}^{(h)}_0) > 0 \] (3.6)

where \( \mathbf{R}^{(h)}_0 = (\ell_+, \ell_-; t = t_0, 0, \ldots). \)

Then, one has the \( \delta\omega \to 0 \) asymptotic expansion
\[ \mathcal{J}^{(y)}_{\mathcal{R}^0} (\mathcal{P}_0, \mathcal{E}_0 + \delta\omega) = \frac{\Xi(\delta\omega) \cdot (\delta\omega)^{\delta^{(h)}_+ + \delta^{(h)}_- - 1}}{\Gamma(\delta^{(h)}_+ + \delta^{(h)}_-)} \left( \frac{(2\pi)^2 \cdot \mathcal{F}^{(y)}(\mathbf{R}^{(h)}_0)}{[\mathcal{V}_F + v_1(t_0)]^{\delta^{(h)}_+} \cdot [\mathcal{V}_F - v_1(t_0)]^{\delta^{(h)}_-}} \right)^{\delta^{(h)}_0} + \mathcal{O} \left( |\delta\omega|^{\delta^{(h)}_+ + \delta^{(h)}_- - 1} \right) + \mathcal{J}^{(y)}_{\text{reg}} (\delta\omega). \] (3.7)

The function \( \mathcal{J}^{(y)}_{\text{reg}} (\delta\omega) \) appearing above is smooth in the neighbourhood of the origin.

I recall that \( v_1 \) appearing in (3.7) is defined as
\[ v_1(t) = \epsilon'_1(t). \] (3.8)

Further, one should observe that since \( \Delta_\nu \) is an analytic function of the rapidity \( t_0 \) and that \( \Delta_\nu \geq 0 \) by construction, the constraint of the theorem is always satisfied for a generic choice of parameters.

**Proof —**

Consider the contribution to \( \mathcal{J}^{(y)}_{\mathcal{R}^0} (\mathcal{P}_0, \mathcal{E}_0 + \delta\omega) \) stemming from the integrals in (3.2) associated with picking \( s \neq s_0. \) Then, the functions \( \eta^{(h)}_0(t; s) \) cannot share a common zero on \( \mathcal{J}^{(0)}_h. \) Assume the contrary. Then, denoting this zero as \( t' \in [-p_F; p_F] \) one would have that
\[ 0 = \eta^{(h)}_0(t'; s) - \eta^{(h)}_0(t'; s) = 2 \mathcal{V}_F (t' - t_0 + 2\pi(s - s_0)). \] (3.9)
However, \(|t' - t_0| \leq p_F + \pi < 2\pi\), hence producing a contradiction. Observe that one has

\[ \partial_t \eta^{(h)}(t; s) = v_1(t) + \nu v_F \neq 0 \quad (3.10) \]

with \(v_1\) as defined in (3.8). As discussed in Appendix C.2 one has \(|v_1(t)| < v_F + |t - p_F|\) what ensures that \(\eta^{(h)}(t; s)\) has at most one zero on \(J_h^{(e)}\) and that the latter is simple. A straightforward application of Lemma D.3 then ensures that integrals subordinate to \(s \neq s_0\) only produce smooth functions of \(\delta \omega\) in the neighbourhood of 0.

It remains to focus on the \(s = s_0\) case. First, consider the situation subordinate to the cases i) and ii). If one is in case i), then due to (3.9) the functions \(\eta^{(h)}(t; s_0)\) cannot share a zero on \(J_h^{(e)}\). In case ii), (3.9) would impose that a common zero \(t'\) necessarily coincides with \(t_0\). The latter would then impose that \(0 = \eta^+(t_0; s) = E_0 + e_1(t_0)\) what leads to a contradiction. Thus, since in both cases the functions \(\eta^{(h)}(t; s)\) do not share a common zero on \(J_h^{(e)}\), one has, by Lemma D.3 that \(J_n^{(e)}(P_0, E_0 + \delta \omega)\) is smooth in \(\delta \omega\) around 0.

Finally, I focus on the last case \(t_0 \in ] - p_F ; p_F[\) with \(E_0 = - e_1(t_0)\). Since \(t_0 \in ] - p_F ; p_F[\), one can invoke the freedom of choosing the regulator \(\epsilon > 0\) so that \(t_0 \in \mathbb{N}(J_h^{(e)}), t_0\) is clearly a common zero to \(t \mapsto \eta^{(h)}(t; s_0)\).

It is the only one on \(J_h^{(e)}\) owing to (3.9). Furthermore, due to (3.10), one has \(\partial_t \eta^+(t_0; s) - \partial_t \eta_-(t_0; s) < 0\) and \(\partial_t \eta^+(t_0; s) - \partial_t \eta^-(t_0; s) = 2\nu v_F \neq 0\).

All is set so as to apply Theorem D.1 and thus, in the \(\delta \omega \to 0\) limit, one indeed gets (3.7).

3.1.2 The one \(r\)-string contributions

For the purpose of discussing the contribution of one \(r\)-string excitations to the DRF, it appears convenient to parametrise the momentum-energy \((k, \omega)\) variables of the response function as

\[ k = P_0 \text{ where } P_0 = \pi s \gamma + p_F \sum_{\ell = \pm} \ell \ell_\ell + k_0^{(r)} - 2\pi s_0 \text{ and } \omega = \delta \omega + E_0. \quad (3.11) \]

Above, \(k_0^{(r)} \in J_h\), while the Umklapp integers are subject to the constraints \(\sum_{\ell = \pm} \ell_\ell = -r\) and \(s_0 \in \mathbb{Z}\).

The associated one \(r\)-string, \(r \in \mathfrak{H}\), DRF takes the form

\[ J_n^{(r)}(P_0, E_0 + \delta \omega) = \int_{J_h} d\mathbb{k}^{(r)} \mathbb{F}^{(r)}(\mathbb{R}^{(r)}) \cdot \prod_{\ell = \pm} \{ \Xi(\delta \omega + \eta^{(r)}(k^{(r)}; s)) \cdot [\delta \omega + \eta^{(r)}(k^{(r)}; s)]^\Delta(\mathbb{R}^{(r)}) \} \]

\[ \times (1 + \tau(\mathbb{R}^{(r)}; s)) \quad (3.12) \]

Here,

\[
\begin{align*}
\mathbb{R}^{(r)} &= (\ell_\ell ; n, h = 0, n_1 = 0, \ldots, 0, n_r = 1, 0, \ldots) \\
\mathbb{n}^{(r)} &= (\ell_\ell, \ell_r = 0, k^{(r)} = 0, \ldots, 0, k^{(r)} = k^{(r)}, \ldots) \quad (3.13)
\end{align*}
\]

where the notation means that the only rapidity that is present in \(\mathbb{R}^{(r)}\) is the rapidity \(k^{(r)}\) of one \(r\)-string while, Umklapp integers being set apart, the only non-zero integer in \(\mathbb{n}^{(r)}\) is the one counting the \(r\)-string excitations, and it is set to one.

Also (3.12) involves the functions

\[ \eta^{(r)}(k^{(r)}; s) = E_0 - e_r(k^{(r)}) + \nu v_F [k_0^{(r)} - k^{(r)} + 2\pi(s - s_0)] \]. \quad (3.14)
Finally, the remainder satisfies
\[ v(\mathcal{R}^{(r)}; s) = O\left( \sum_{n \geq s} |\delta \omega + v_0(\mathcal{k}^{(r)}; s)|^{1-r} \right), \]  
and the control on \( v(\mathcal{R}^{(r)}; s) \) it is differentiable in the sense of Definition [A.1]

**Theorem 3.2.** Let \( k_0^{(r)}(s) = k_0^{(r)} + 2\pi (s - s_0) \). Assume that

i) \( k_0^{(r)}(s) \not\in \mathcal{S} \), this for any \( s \), in which case \( \mathcal{E}_0 \) can take any value;

ii) \( k_0^{(r)}(s) \in \mathcal{S} \), at least for one \( s \) and that, for any such \( s \), one has \( \mathcal{E}_0 \neq \varepsilon_0(\mathcal{k}_0^{(r)}(s)) \).

Then \( \mathcal{S}_n^{(\gamma, \gamma)}(\mathcal{P}_0, \mathcal{E}_0 + \delta \omega) \) is smooth in \( \delta \omega \) belonging to a neighbourhood of 0.

Assume that

- \( k_0^{(r)}(s) \in \text{Int}(\mathcal{S}) \) for at least for one \( s \),
- \( \mathcal{E}_0 = \varepsilon_0(\mathcal{k}_0^{(r)}(s) + 2\pi (s - s_0)) \) for the same value of \( s \),
- \( \delta^{(r)}(s) = \Delta_{s}(\mathcal{R}_0^{(r)}(s)) > 0 \), where
\[ \mathcal{R}_0^{(r)}(s) = \left( \ell_\ast, \ell_{\ast-}; t = 0, \mathcal{k}^{(1)}(s) = 0, \ldots, \mathcal{k}^{(r)} = \mathcal{k}_0^{(r)}(s), \mathcal{0}, \ldots \right). \] 

\[ (3.16) \]

\[ \text{Case 1 :} \quad \text{If } |v_r(\mathcal{k}^{(r)}_0(s))| > v_F \]

then, agreeing upon
\[ \eta(s) = -\text{sgn}\left[ v_r(\mathcal{k}^{(r)}_0(s)) \right] \] 

one has the asymptotic expansion
\[ \mathcal{S}_n^{(\gamma)}(\mathcal{P}_0, \mathcal{E}_0 + \delta \omega) = \sum_{s : \mathcal{E}_0 = \varepsilon_0(\mathcal{k}_0^{(r)}(s))} \frac{(2\pi)^2 \cdot \mathcal{F}^{(\gamma)}(\mathcal{R}_0^{(r)}(s)) \cdot \Gamma\left( 1 - \delta^{(r)}(s) - \delta^{(s)}(s) \right)}{v_F + v_r(\mathcal{k}_0^{(r)}(s))^{\delta^{(s)}(s)}} \cdot \frac{|\delta \omega|^{\delta^{(r)}(s) + \delta^{(s)}(s) - 1}}{v_F - v_r(\mathcal{k}_0^{(r)}(s))^{\delta^{(s)}(s)}} \]
\[ \times \left\{ \Xi(\delta \omega) \cdot \frac{\sin \left[ \pi \delta^{(r)}(s) \right]}{\pi} + \Xi(-\delta \omega) \cdot \frac{\sin \left[ \pi \delta^{(s)}(s) \right]}{\pi} \right\} + \mathcal{S}_{\nu_{\text{reg}}^{(\gamma)}(\delta \omega)} + O\left( |\delta \omega|^{\delta^{(r)}(s) + \delta^{(s)}(s) - 1} \right). \] 

\[ (3.18) \]

\( \mathcal{S}_{\nu_{\text{reg}}^{(\gamma)}(\delta \omega)} \) is smooth in the neighbourhood of the origin, and
\[ v_r(t) = \varepsilon'_r(t). \] 

\[ (3.19) \]

\[ \text{Case 2 :} \quad \text{If } |v_r(\mathcal{k}^{(r)}_0(s))| < v_F \]

then, under the same conventions as above
\[ \mathcal{S}_n^{(\gamma)}(\mathcal{P}_0, \mathcal{E}_0 + \delta \omega) = \sum_{s : \mathcal{E}_0 = \varepsilon_0(\mathcal{k}_0^{(r)}(s))} \frac{(2\pi)^2 \cdot \mathcal{F}^{(\gamma)}(\mathcal{R}_0^{(r)}(s)) \cdot \Gamma\left( \delta^{(r)}(s) + \delta^{(s)}(s) \right)}{v_F + v_r(\mathcal{k}_0^{(r)}(s))^{\delta^{(s)}(s)}} \cdot \frac{|\delta \omega|^{\delta^{(r)}(s) + \delta^{(s)}(s) - 1} \Xi(\delta \omega)}{v_F - v_r(\mathcal{k}_0^{(r)}(s))^{\delta^{(s)}(s)}} \]
\[ + \mathcal{S}_{\nu_{\text{reg}}^{(\gamma)}(\delta \omega)} + O\left( |\delta \omega|^{\delta^{(r)}(s) + \delta^{(s)}(s) - 1} \right). \] 

\[ (3.20) \]
Proof —

The analysis is quite similar to the one-hole excitation case. Cases i) and ii) are dealt with by means of Lemma 1.2.

It remains to focus on the case where \( k^{(r)}_0(s) \in \text{Int}(\mathcal{S}_r) \). Again, if \( r = 1 \), then one adjusts \( \epsilon > 0 \) so that \( k^{(r)}_0(s) \in \text{Int}(\mathcal{S}_r) \). By hypothesis, one has that \( \mathcal{E}_0 = c_r(k^{(r)}_0(s)) \), for at least one \( s \). Then, one treats each integral subordinate to a value of \( s \) separately. Only integrals subordinate to values of \( s \) such that \( k^{(r)}_0(s) \in \text{Int}(\mathcal{S}_r) \) and \( \mathcal{E}_0 = c_r(k^{(r)}_0(s)) \) will give rise to a non-smooth behaviour when \( \delta \omega \to 0 \). For any such value of \( s \), just as for the one-hole case, one concludes that \( k^{(r)}_0(s) \) is the only simultaneous zero of \( \eta^{(r)}_\pm(c_\pm s) \) on \( \mathcal{S}_r \) and that

\[
\partial_{k^{(r)}} \eta^{(r)}_\pm(k^{(r)}; s) = -[\nu_r(k^{(r)}) + v \nu_F] .
\]

(3.21)

A priori, and this is supported by numerical investigations, c.f. Appendix C.2 \( |\nu_r| \) may or may not be smaller than \( v_F \), namely depending on the choice of the anisotropy \( \xi \), the values of the magnetic field \( h \) and hence the endpoint of the Fermi zone-, the value of \( r \in \mathbb{R} \) and, finally, the value of \( k^{(r)}_0(s) \) both situations may occur, namely

\[
|\nu_r(k^{(r)}_0(s))| < v_F \quad \text{or} \quad |\nu_r(k^{(r)}_0(s))| > v_F .
\]

(3.22)

In case 1 listed in the statement, viz. \( |\nu_r(k^{(r)}_0(s))| > v_F \), one observes that

\[
\partial_{k^{(r)}} \nu^{(r)}_+ (k^{(r)}_0(s); s) \cdot \partial_{k^{(r)}} \nu^{(r)}_- (k^{(r)}_0(s); s) > 0
\]

(3.23)

and that

\[
-\text{sgn} \left[ \partial_{k^{(r)}} \nu^{(r)}_+ (k^{(r)}_0(s); s) \cdot [\partial_{k^{(r)}} \nu^{(r)}_- (k^{(r)}_0(s); s) - \partial_{k^{(r)}} \nu^{(r)}_+ (k^{(r)}_0(s); s)] \right] = -\text{sgn} \left[ \nu_r(k^{(r)}_0(s)) \right] .
\]

(3.24)

This is all that is needed so as to apply Theorem 1.1 to the situation of interest.

Finally, when \( |\nu_r(k^{(r)}_0(s))| < v_F \) the analysis parallels the one exposed for the contribution of the one hole excitation sector. The details are left to the interested reader.

3.2 The multi-hole/r-string excitation sector

Below, I will consider the contribution to the DRF issuing form the sector containing multiple hole and multiple r-strings all having the same value of \( r \in \mathbb{R} \). Although it will be not discussed here, the case of multiple hole and various numbers r-strings can be treated analogously and leads to a similar structure of singularities. Likewise, one may derive the behaviour in the sector built up only from multi-particle excitations. Such results may be easily extracted from the main structural theorem governing the asymptotics of the class of multiple integrals of interest to the analysis of dynamic response functions, Theorem 1.3 which is established in the Appendix.

3.2.1 Excitations built up from holes and, possibly, particles

For the purpose of the present section, it is convenient to parametrise the momentum-energy \((k, \omega)\) combination as

\[
k = \mathcal{P}_0 \quad \text{where} \quad \mathcal{P}_0 = \pi s_y + p_F \sum_{\nu=\pm} v \ell_\nu + q_0 - 2 \pi s_0 \quad \text{and} \quad \omega = \delta \omega + \mathcal{E}_0
\]

(3.25)
where \( \ell_\pm \) are subject to the constraint \( \sum_{\nu=\pm} \ell_\nu = n_h - n_p \). The integers \( n_p, n_h \) are assumed to satisfy
\[
n_h \geq 1 \quad \text{and} \quad n_p + n_h \geq 2.
\] (3.26)

In this case of interest, the contribution of these types of excitations to the dynamical response function takes the form
\[
\mathcal{J}_{n_p}^{(\gamma)}(\mathcal{P}_0, \mathcal{E}_0 + \delta \omega) = \int d^n t \sum_{k} \left( \mathcal{J}_{n_p}^{(\gamma)}(\mathcal{R}^{(h)p}) \right)
\times \sum_{k \in \mathbb{Z}} \left\{ \mathcal{E}(\delta \omega + \nu_n^{(h)p}(\mathcal{R}^{(h)p}); s) \cdot \left[ \mathcal{E}(\delta \omega + \nu_n^{(h)p}(\mathcal{R}^{(h)p}); s) \right] \right\} \cdot (1 + \nu(\mathcal{R}^{(h)p}); s) .
\] (3.27)

Here, I have set
\[
n_{n_p} = (\ell_+, \ell_-; n_h, n_1 = n_p, 0, \ldots) \quad \text{and} \quad \mathcal{R}^{(h)p} = (\ell_+, \ell_-; t, k^{(1)} = k, 0, \ldots)
\] (3.28)

and agree upon
\[
\nu_n^{(h)p}(\mathcal{R}^{(h)p}); s) = \mathcal{E}_0 - \sum_{a=1}^{n_p} e_1(k_a) + \sum_{a=1}^{n_h} e_1(t_a) + \nu \nu F \left( q_0 - \sum_{a=1}^{n_p} k_a + \sum_{a=1}^{n_h} t_a + 2\pi(s - s_0) \right).
\] (3.29)

Also, recall, c.f. Appendix C.2, that for a reduced range of the model’s parameters and as a conjecture more generally, there exists a strictly decreasing diffeomorphism \( t : [K_m; K_M] \rightarrow [-p_F ; p_F] \) such that \( \nu_1(k) = \nu_1(t(k)) \). Set \( p = t^{-1} \) for its inverse. Further, given \( t \in [-p_F ; p_F] \), let
\[
\mathcal{P}(t) = n_p b(t) - n_h t \quad \text{and} \quad \mathcal{E}(t) = n_p e_1(p(t)) - n_h e_1(t) .
\] (3.30)

Finally, the remainder satisfies
\[
\nu(\mathcal{R}^{(h)p}); s) = O\left( \sum_{\nu=\pm} \left| \delta \omega + \nu_n^{(h)p}(\mathcal{R}^{(h)p}); s) \right|^{1-\gamma} \right),
\] (3.31)

and the control on \( \nu(\mathcal{R}^{(h)p}); s) \) is differentiable in the sense of Definition A.1.

**Theorem 3.3.** Let \( a_0(s) = a_0 + 2\pi(s - s_0) \) and assume that \( n_p + n_h \geq 2 \) with \( n_h \geq 1 \). Also, assume that Conjecture C.2 holds if the set of parameters of the model does not enter into the specifications of Theorem C.7.

If, for any \( s \in \mathbb{Z}\),
\[
(a_0(s), \mathcal{E}_0) \notin (\mathcal{P}(t), \mathcal{E}(t)) \quad t \in [-p_F; p_F]
\] (3.32)

then \( \mathcal{J}_{n_p}^{(\gamma)}(\mathcal{P}_0, \mathcal{E}_0 + \delta \omega) \) is a smooth function in \( \delta \omega \) belonging to a neighbourhood of the origin.

Assume that, for at least one \( s \in \mathbb{Z}\)
\[
(a_0(s), \mathcal{E}_0) = (\mathcal{P}(t_0(s)), \mathcal{E}(t_0(s))) \quad \text{for a} \quad t_0(s) \in [-p_F; p_F]
\] (3.33)

and that for any such \( s \) it holds \( \delta^{(h)p}_n(s) > 0 \) where
\[
\delta^{(h)p}_n(s) = \Delta_n(\mathcal{R}^{(h)p}_0(s)) \quad \text{with} \quad \mathcal{R}^{(h)p}_0 = (\ell_+, \ell_-; t_0(s), p(t_0(s)), \emptyset, \ldots).
\] (3.34)
There \( t_0(s) = (t_0(s), \ldots, t_0(s)) \in \mathbb{R}^{n_h} \) and \( \mathbf{v}(t_0(s)) = (\mathbf{v}(t_0(s)), \ldots, \mathbf{v}(t_0(s))) \in \mathbb{R}^{n_p} \).

Then, the multi particle-hole spectral function has the \( \delta \omega \to 0 \) asymptotic expansion

\[
\mathcal{J}_{n_p}(\mathcal{P}_0, E_0 + \delta \omega) = \mathcal{J}_{n_p, \text{reg}}(\delta \omega) + \sum_{s : \exists s_0(s)} \frac{\mathcal{J}^{(\gamma)}(\mathbf{B}_0^{(hp)}(s))}{\sqrt{\mathcal{P}(t_0(s))}} \cdot \left( \frac{-1}{\mathbf{v}_1'(t_0(s))} \right)^{\frac{n}{2}} \cdot \left( \frac{1}{\mathbf{v}_1(t_0(s))} \right)^{\frac{n-1}{2}} \\
\times G(n_p + 1) G(n_h + 1) \cdot \left( \sqrt{2\pi} \right)^{3+np+n_h} \cdot \Gamma \left( -6_{+}^{(hp)}(s) - 6_{-}^{(hp)}(s) - \frac{n^2 + n_h^2 - 3}{2} \right) \\
\times \left[ \mathbf{v}_F + \mathbf{v}_1(t_0(s)) \right]^{6_{+}^{(hp)}(s)} \left[ \mathbf{v}_F - \mathbf{v}_1(t_0(s)) \right]^{6_{-}^{(hp)}(s)} \\
\left\{ \Xi(\delta \omega) \frac{1}{\pi} \sin \left( \pi \theta_{+}^{(hp)}(s) + \theta_{-}^{(hp)}(s) \right) \right\} + \Xi(-\delta \omega) \frac{1}{\pi} \sin \left( \pi \theta_{+}^{(hp)} + \pi \theta_{-}^{(hp)} \right) \right) + O \left( \delta \omega^{6_{+}^{(hp)}(s) + 6_{-}^{(hp)}(s) + \frac{n^2 + n_h^2 - 1}{2}} \right).
\]

(3.35)

\( \mathcal{J}_{n_p, \text{reg}}(\delta \omega) \) is smooth in the neighbourhood of the origin.

Finally, the summation over \( s \) in (3.35) runs through all solutions \( t_0(s) \) to (3.33). This summation contains at most two terms and, for generic parameters, it only contains one term.

**Proof —**

This is a direct consequence of Theorem[13] In order to identify quantities with the notations of that theorem, one should identify the quantities given in Section[13] of the appendix:

\( \ell = 2, \quad \mathcal{J}_1 = \mathcal{J}_h^{(e)}, \quad \mathcal{J}_2 = \mathcal{J}_1^{(e)}, \quad (n_1, n_2) = (n_h, n_p) \)

(3.36)

in what concerns the intervals. Further,

\( (\mathbf{p}^{(1)}, \mathbf{p}^{(2)}) = (\mathbf{t}, \mathbf{k}), \quad (u_1, u_2) = (e_1, e_1), \quad (\zeta_1, \zeta_2) = (-1, 1). \)

(3.37)

From there one infers that one has

\[ \mathcal{P}'(t) = -(n_h - n_p \mathbf{v}'(t)) < 0 \quad \text{on } J - p_F : p_F \]

(3.38)

since \( \mathbf{v} \) is strictly decreasing. Also, it follows directly from the definition of \( \mathbf{v} \) that

\[ \mathcal{E}'(t) = v_1(t) : \mathcal{P}'(t) \]

(3.39)

so that \( t \mapsto \mathcal{E}(t) \) is strictly increasing on \( J - p_F : 0 \) and strictly decreasing on \( 0 : p_F \). Furthermore, one has that

\[ \partial_t \left[ \mathcal{E}_0 - \mathcal{E}(t) \pm v_1(q_0(s) - \mathcal{P}(t)) \right] = -\mathcal{P}'(t) \cdot \left( v_1(t) \pm v_F \right) \neq 0 \]

(3.40)

on \( J - p_F : p_F \).

I first focus on the regular case, namely when, for any \( s \in \mathbb{Z}, \)

\[ (q_0(s), \mathcal{E}_0) \notin \left\{ (\mathcal{P}(t), \mathcal{E}(t)) : t \in [-p_F : p_F] \right\}. \]

(3.41)

Then observe that (3.40) implies that \( t \mapsto \mathcal{E}_0 - \mathcal{E}(t) \pm v_F(q_0(s) - \mathcal{P}(t)) \) are both strictly monotonous on \( \mathcal{J}_h^{(e)} \).

Thus, should one of these functions vanish on \( \partial \mathcal{J}_h^{(e)} \), it is enough to slightly change \( \epsilon > 0 \), which is a free parameter in the problem (as long as it is small and strictly positive) so as to have a non-vanishing function. Thus,
automatically, condition (1.22) stated in Theorem 1.3 is satisfied. Then, the results of that theorem ensures the smooth behaviour of $\delta \omega \mapsto \mathcal{S}_{\nu}(\mathcal{P}_{\nu}, \mathcal{E}_0 + \delta \omega)$ around 0.

In the case when there exist at least one $s \in \mathbb{Z}$ such that $(q_0(s), \mathcal{E}_0) = \left(\mathcal{P}(t_0(s)), \mathcal{E}(t_0(s))\right)$ for a $t_0(s) \in ] - p_F : p_F [$, one needs to identify additional constants. First, however, one fixes $\epsilon$ such that $t_0(s) \in \text{Int}(\mathcal{F}^{(r)}_{h})$ for any $s$ compatible with the mentioned constraint. One should also observe that the variations of $t \mapsto \mathcal{E}(t)$ and $t \mapsto \mathcal{P}(t)$ on $] - p_F : p_F [\text{ entail that there may at most exist two different } s \text{ such that the previous equality holds. It is also evident that, in the generic case, only one such } s \text{ will exist.}

Since $u'_1(t) = v'_1(t) > 0$ on $] - p_F ; p_F [$, while $u''_1(t) = v''_1(t) < 0$ on $] - p_F ; p_F [$, it follows that

\[
\begin{align*}
\varepsilon_1 &= -\zeta_1 \text{sgn}(u''_1(t_0(s))) = \text{sgn}(v''_1(t_0(s))) = 1 \\
\varepsilon_2 &= -\zeta_2 \text{sgn}(u'_1(t_0(s))) = -\text{sgn}(v'_1(t_0(s))) = -1
\end{align*}
\]

and $s = -\text{sgn}\left(\frac{\mathcal{P}'(t_0)}{u''_1(t_0(s))}\right) > 0$. All parameters being identified, it remains to apply the results of Theorem 1.3 to each $s \in \mathbb{Z}$ such that a $t_0(s)$ exists.

Finally, the $n_h \geq 2$ and $n_p = 0$ case is treated along much the same lines. The results boil down to (3.35) with $n_p$ being set to 0.

### 3.2.2 Excitations built up from holes and a fixed $r$-string species

Below, $r \in \mathfrak{R}_q$ is assumed to be fixed. For the purpose of the present section, it is convenient to parameterise the momentum-energy $(k, \omega)$ combination as

\[
k = \mathcal{P}_0 \text{ where } \mathcal{P}_0 = \pi s_{y} + p_F \sum_{\nu = \pm} v \ell_{\nu} + q_0 - 2\pi s_0 \text{ and } \omega = \delta \omega + \mathcal{E}_0
\]

(3.43)

where $\ell_{\pm}$ are subject to the constraint $\sum_{\nu = \pm} \ell_{\nu} = n_{st} - n_{st}$. The integers $n_{st}, n_h$ are assumed to satisfy

\[
n_h \geq 1 \text{ and } n_{st} + n_h \geq 2.
\]

(3.44)

In this case of interest, the contribution of these types of excitations to the dynamical response function takes the form

\[
\mathcal{S}_{\nu}(\mathcal{P}_{\nu}, \mathcal{E}_0 + \delta \omega) = \int \mathcal{D}^{\nu} t \int \mathcal{D}^{\nu} k \mathcal{T}^{(\gamma)}(\mathfrak{S}^{(hr)})
\]

\[
\times \sum_{s \in \mathbb{Z}} \prod_{\nu = \pm} \left[ \Xi(\delta \omega + \psi^{(hr)}_{\nu}(\mathfrak{S}^{(hr)}; s)) \cdot \left[ \delta \omega + \psi^{(hr)}_{\nu}(\mathfrak{S}^{(hr)}; s) \right]^{\Delta_{\nu}(\mathfrak{S}^{(hr)}) - 1} \right] \cdot (1 + \tau(\mathfrak{S}^{(hr)}; s)).
\]

(3.45)

Here, I have set

\[
n_{hr} = (\ell_{+}, \ell_{-}; n_{h}, 0, \ldots, 0, n_{r} = n_{st}, 0, \ldots) \text{ and } \mathfrak{S}^{(hr)} = (\ell_{+}, \ell_{-}; t, 0, \ldots, 0, k^{(r)} = k, 0, \ldots)
\]

(3.46)

and agree upon

\[
\psi^{(hr)}_{\nu}(\mathfrak{S}^{(hr)}; s) = \mathcal{E}_0 - \sum_{a = 1}^{n_{a}} \epsilon_{r}(k_{a}) + \sum_{a = 1}^{n_{a}} \epsilon_{1}(t_{a}) + \nu v_{F}(q_0 - \sum_{a = 1}^{n_{a}} k_{a} + \sum_{a = 1}^{n_{a}} t_{a} + 2\pi(s - s_{0})
\]

(3.47)
Also, recall, c.f. Appendix [C.2] that for a reduced range of the model’s parameters and as a conjecture more generally, there exists a diffeomorphism \( \delta^{(r)} : [-p_F : p_F] \rightarrow [k_{m}^{(r)} ; k_{M}^{(r)}] \) such that \( v_{1}(t) = v_{r}(\delta^{(r)}(t)) \). Further, given \( t \in [-p_F ; p_F] \), let

\[
\mathcal{P}(t) = n_{st} \delta^{(r)}(t) - n_{ht} \quad \text{and} \quad \mathcal{E}(t) = n_{st} c_{r}(\delta^{(r)}(t)) - n_{ht} c_{t}(t) .
\]

Finally, the remainder satisfies

\[
\nu(\mathcal{R}^{(hr)}; s) = O\left( \sum_{t \in \mathcal{E}} \left| \delta_\omega + n_{r}(\mathcal{R}_{0}^{(hr)}; s) \right|^{1-r} \right)
\]

and the control on \( \nu(\mathcal{R}^{(hr)}; s) \) is differentiable in the sense of Definition [A.1].

**Theorem 3.4.** Let \( a_0(s) = a_0 + 2\pi(s - s_0) \) and assume that \( n_{st} + n_{ht} \geq 2 \) with \( n_{ht} \geq 1 \). Also, assume that Conjecture [C.2] holds if the set of parameters of the model does not enter into the specifications of Theorem [C.7]. Finally, assume that \( t \mapsto \mathcal{P}(t) \) is a diffeomorphism on \([-p_F : p_F]\).

If, for any \( s \in \mathbb{Z} \),

\[
(a_{0}(s), \mathcal{E}_{0}) \notin \big\{(\mathcal{P}(t), \mathcal{E}(t)) : t \in [-p_F ; p_F]\big\},
\]

then, for \( \epsilon > 0 \) small enough, \( \mathcal{F}^{(\gamma)}(\mathcal{P}_{0}, \mathcal{E}_{0} + \delta_\omega) \) is a smooth function in \( \delta_\omega \) belonging to a neighbourhood of the origin.

Assume that, for at least one \( s \in \mathbb{Z} \),

\[
(a_{0}(s), \mathcal{E}_{0}) = (\mathcal{P}(t_{0}(s)), \mathcal{E}(t_{0}(s))) \quad \text{for a } t_{0}(s) \in [-p_F ; p_F],
\]

and that, for any such \( s \) it holds \( \delta_{v}^{(hr)}(s) \) with

\[
\delta_{v}^{(hr)}(s) = \Delta_{v}(\mathcal{R}_{0}^{(hr)}(s)) \quad \text{and} \quad \mathcal{R}_{0}^{(hr)} = \left\{ \ell_{+}, \ell_{-}; t_{0}(s), 0, \ldots, 0, \delta^{(r)}(t_{0}(s)) \right\},
\]

Then, the multi \( r \)-string-hole spectral function has the \( \delta_\omega \rightarrow 0 \) asymptotic expansion

\[
\mathcal{F}^{(\gamma)}(\mathcal{P}_{0}, \mathcal{E}_{0} + \delta_\omega) = \mathcal{F}^{(\gamma)}(\mathcal{P}_{0}, \mathcal{E}_{0}) + \sum_{s : \mathcal{E}(s)} \frac{\mathcal{F}^{(\gamma)}(\mathcal{R}_{0}^{(hr)}(s))}{\sqrt{\mathcal{P}(t_{0}(s))}} \cdot \left( \frac{1}{n_{r}^{(hr)}(t_{0}(s))} \right) \cdot \left( \nu_{1}^{(hr)}(t_{0}(s)) \right) \cdot \left( -\nu_{t}^{(hr)}(t_{0}(s)) \right) \]

\[
\times G(n_{st} + 1) G(n_{ht} + 1) \cdot \frac{\left( \sqrt{2\pi} \right)^{n_{st} + n_{ht}} \Gamma \left( -\delta_{+}^{(hr)}(s) - \delta_{-}^{(hr)}(s) - \frac{n_{st} + n_{ht} - 3}{2} \right)}{\left| \mathcal{V}_{F} + \nu_{1}(t_{0}(s)) \right|^{\delta_{+}^{(hr)}(s)} \left| \mathcal{V}_{F} - \nu_{t}(t_{0}(s)) \right|^{\delta_{-}^{(hr)}(s)}} \cdot \left[ \Xi(\delta_\omega) \frac{1}{\pi} \sin \left( \pi \nu_{+}^{(hr)}(s) \right) + \Xi(-\delta_\omega) \frac{1}{\pi} \sin \left( \pi \nu_{-}^{(hr)}(s) \right) \right] + O\left( \delta_\omega^{n_{st} + n_{ht} + 1 - 3} \right).
\]

\( \mathcal{F}^{(\gamma)}(\mathcal{P}_{0}, \mathcal{E}_{0} + \delta_\omega) \) is smooth in the neighbourhood of the origin. Further, one has

\[
\nu_{+}^{(hr)}(s) = \delta_{+}^{(hr)}(s) + \delta_{-}^{(hr)}(s) + \frac{1}{2} \delta_{-\epsilon r} n_{st} + \frac{1}{2} s_{r} + \frac{1}{2} \frac{1}{2} \]

where

\[
\sigma_{r} = \operatorname{sgn} \left[ \nu_{1}^{(hr)}(t_{0}(s)) \right] \quad \text{and} \quad s_{r} = -\operatorname{sgn} \left[ \mathcal{P}(t_{0}(s)) \right].
\]

Finally, the summation over \( s \) in (3.52) runs through all solutions \( t_{0}(s) \) to (3.51). This summation contains at most two terms and, for generic parameters, it only contains one term.

The proof follows closely the case of the multi-hole multi-particle sector, so that I omit the details.
4 Conclusion

This work developed a technique allowing one to extract, on rigorous grounds, the asymptotic behaviour in certain parameters of a family of multiple integrals. These results are detailed in Sections \[ D \] and \[ E \] of the appendix. The multiple integrals studied in these sections, upon specialisation, contain the multiple integrals which define the coefficients of the series giving the massless form factor expansion issued representation for the DRF in the XXZ chain that was derived in \[ 51 \]. Hence, the analysis I developed allowed, upon relying on additional properties that were argued in \[ 47 \] and \[ 51 \], to give a precise characterisation of the singular behaviour in the \((k, \omega)\) plane of the series’ coefficients. In doing so, this work provides a test and confirmation of the predictions, issuing from the non-linear Luttinger liquid approach, for some of the singularities of the DRF, namely those issuing from a one massive excitation process. Furthermore, the work showed the existence of other singularity lines: the ones issuing from multi-particle/hole/r-string excitations and which correspond to configurations of the various momenta that maximise the multi-excitation energy at fixed momentum. Such multi-species singularity curves generate structurally different edge exponents and universality constants. The edge exponents associated with one such "mixed" excitation were discussed in \[ 3 \], but all the other cases were not considered in the literature. Furthermore, the work \[ 3 \] only focused on the exponents and so did not provide any expression for the universal part of the amplitude. Thus, the expression for the universal part of the amplitude is new.

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Appendix

A Main notations

Sets

- Given a set \( A \), \( \text{Int}(A) \) stands for its interior, \( \overline{A} \) for its closure and \( \partial A \) for its boundary.

- Given a finite set \( A \), \( |A| \) stands for its cardinal.

- \( \mathbb{N} = \{0, 1, 2, \ldots\} \), \( \mathbb{R}^+ = ]0; +\infty[ \), \( \mathbb{R}^* = \mathbb{R} \setminus \{0\} \).

- \( \llbracket 1 ; n \rrbracket = \{1, \ldots, n\} \) and \( \mathfrak{S}_n \) stands for the permutation group of \( \llbracket 1 ; n \rrbracket \).

- \( \amalg \) refers to the disjoint union of sets.

- \( \delta_{a,b} \) stands for the Kronecker symbol: \( \delta_{a,b} = 1 \) if \( a = b \) and \( \delta_{a,b} = 0 \) otherwise.

- Given \( \ell \) integers \( n_1, \ldots, n_\ell \), it is understood that

\[
\overline{n}_\ell = \sum_{r=1}^{\ell} n_r . \quad (A.1)
\]
Vectors and related objects

• $^t$ stands for the transposition of a matrix or vector, depending on the context.
• $I_n$ stands for the identity matrix on $\mathbb{R}^n$, and it will sometimes also be denoted as id.
• Vectors are denoted in bold, viz. $x \in \mathbb{R}^n$ corresponds to the vector $x = (x_1, \ldots, x_n)$. The dimensionality of the vector is always undercurrent by the context.
• If the vector space has a natural Cartesian product structure $\prod_{r=1}^\ell \mathbb{R}^{n_r}$, then any vector $x$ is represented as $x = (x^{(1)}, \ldots, x^{(\ell)})$ with $x^{(r)} = (x^{(r)}_1, \ldots, x^{(r)}_{n_r}) \in \mathbb{R}^{n_r}$.

(A.2)

• Vectors with omitted coordinates are denoted as:
  $x^{(r)}_a = (x^{(r)}_1, \ldots, x^{(r)}_{a-1}, x^{(r)}_{a+1}, \ldots, x^{(r)}_{n_r})$ and $x_{[r,a]} = (x^{(r)}_1, \ldots, x^{(r)}_a, \ldots, x^{(r)}_{\ell})$.

(A.3)

• Given $\alpha, \beta \in \prod_{r=1}^\ell \mathbb{N}^{n_r}$, it is understood that
  $|\alpha| = \sum_{r=1}^\ell \sum_{a=1}^{n_r} \alpha^{(r)}_a$ and $\alpha \geq \beta \iff \forall (r,a) \quad \alpha^{(r)}_a \geq \beta^{(r)}_a$.

(A.4)

• Given $\alpha \in \prod_{r=1}^\ell \mathbb{N}^{n_r}$ and $x \in \prod_{r=1}^\ell \mathbb{R}^{n_r}$, one has
  $x^\alpha = \prod_{r=1}^\ell \prod_{a=1}^{n_r} [x^{(r)}_a]^{\alpha^{(r)}_a}$.

(A.5)

Functions

• Given a set $A$, $1_A$ stands for the indicator function of $A$.
• $\Xi$ refers to the Heaviside step function, viz. $\Xi = 1_{\mathbb{R}^+}$.
• $\Gamma$ refers to the Gamma function which allows one to express the Euler $\beta$-integral as
  $\int_0^1 t^{x-1} (1-t)^{y-1} \cdot dt = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$.

(A.6)

• $G$ stands for the Barnes [4, 5] function.
• The Gaudin-Mehta integral is expressed in terms of the Barnes function as:
  $\int_{\mathbb{R}^n} dy \ e^{-(y,y)} \prod_{a < b}^{n} (y_a - y_b)^2 = \left(\frac{1}{2}\right)^{\frac{n^2}{2}} (2\pi)^{\frac{n}{2}} G(2 + n)$.

(A.7)
• Given $S \subset \mathbb{R}^n$ measurable and a function $f : S \to \mathbb{R}$,

$$||f||_{L^\infty(S)} = \sup \{ |f(x)| : x \in S \}.$$

(A.8)

• Given $g : U \times V \to W$, with $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^o$ the totally even part of a function in respect to a set of variables is defined as

$$[g(z, v)]_{z \text{-even}} = \frac{1}{2^n} \sum_{a=1}^{n} g(\epsilon z_a, v) \quad \text{with} \quad z^{(e)} = (\epsilon_1 z_1, \ldots, \epsilon_d z_d).$$

(A.9)

• Given a smooth function $f : U \to V$ between two open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$, $D^k_x f$ denotes the $k$th-order differential of $f$ at the point $x \in U$. When $k = 1$, it is simply denoted as $D_x f$.

Definition A.1. Given smooth functions $f, g$ on an open neighbourhood of a point $y \in \mathbb{R}^n$, one says that an $O$-remainder relation $f = O(g)$ when $x \to y$ is differentiable if, for each $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$ there exists a smooth function $\psi_\ell$ in the vicinity of $y$ and a constant $C_\ell > 0$ such that

$$\prod_{a=1}^{n} \partial^{\ell_a} x_a \cdot f(x) \leq C_\ell \cdot \prod_{a=1}^{n} \partial^{\ell_a} x_a \cdot [\psi_\ell g](x)$$

(A.10)
on some open neighbourhood of $y$.

Note that the use of $\psi_\ell$ in this definition allows one to encompass a situation when $g$ does not depend explicitly on some of the variables.

B Auxiliary theorems

The proof of theorems [B.1] [B.2] and [B.3] can be found in [34]. The proof of Theorem [B.5] can be found in [12].

Theorem B.1. Morse Lemma

Let $f : U \to \mathbb{R}$ be a smooth function on an open set $U \subset \mathbb{R}^n$. Let $p \in U$ be a non-degenerate critical point of $f$. Let $M$ be the matrix associated with the bilinear form $D^2_x f$:

$$(v, \mathbb{M}w) = D^2_x f(v, w).$$

(B.1)

Then, there exists an open neighbourhood $V_0$ of $0 \in \mathbb{R}^n$ and a smooth diffeomorphism onto $g : V_0 \mapsto U_0 \subset U$ such that

• $0 \in V_0$ and $g(0) = p \in U_0$;

• $f \circ g(x) = (x, \mathbb{M}x)$ on $V_0$.

Here $(\cdot, \cdot)$ is the canonical scalar product on $\mathbb{R}^n$.

Theorem B.2. Weierstrass preparation theorem

Let $f$ be a holomorphic function on an open set $V \subset \mathbb{C}^n$. Let $y \in V$ and $d \in \mathbb{N}$ be such that

$$(\partial^k_x f)(y) = 0 \quad \text{for} \quad k = 0, \ldots, d - 1, \quad \text{and} \quad (\partial^d_x f)(y) \neq 0.$$ 

(B.2)

Then there exists
• open neighbourhoods $U_0 \subset \mathbb{C}^{n-1}$ of $y_{[n]} = (y_1, \ldots, y_{n-1})$ and $W_0 \subset \mathbb{C}$ of $y_n$ such that $V_0 = U_0 \times W_0 \subset V$
• a holomorphic, non-vanishing, function $h$ on $V_0$
• a Weierstrass polynomial

$$W(z) = (z_n - y_n)^d + \sum_{k=0}^{d-1} (z_n - y_n)^k a_k(z_{[n]}) \quad \text{with} \quad z_{[n]} = (z_1, \ldots, z_{n-1}),$$

(B.3)

and $a_k, k \in \mathbb{Z}; d - 1 \mathbb{Z}$, being holomorphic functions on $U_0$ satisfying $a_k(y_{[n]}) = 0$;
such that one has the factorisation

$$f = W \cdot h \quad \text{on} \quad V_0 = U_0 \times W_0.$$  (B.4)

Theorem B.3. Malgrange preparation theorem

Let $f$ be a smooth function on an open set $V \subset \mathbb{R}^n$. Let $y \in V$ and $d \in \mathbb{N}$ be such that

$$(\partial_k^d f)(y) = 0 \quad \text{for} \quad k = 0, \ldots, d - 1, \quad \text{and} \quad (\partial_n^d f)(y) \neq 0.$$  (B.5)

Then there exists

• open neighbourhoods $U_0 \subset \mathbb{R}^{n-1}$ of $y_{[n]} = (y_1, \ldots, y_{n-1})$ and $W_0 \subset \mathbb{R}$ of $y_n$ such that $V_0 = U_0 \times W_0 \subset V$
• a smooth, non-vanishing, function $h$ on $V_0$
• a Weierstrass polynomial

$$W(x) = (x_n - y_n)^d + \sum_{k=0}^{d-1} (x_n - y_n)^k a_k(x_{[n]}) \quad \text{with} \quad x_{[n]} = (x_1, \ldots, x_{n-1}),$$

(B.6)

and the $a_k$'s all being smooth on $U_0$ and such that $a_k(y_{[n]}) = 0$;
such that one has the factorisation

$$f = W \cdot h \quad \text{on} \quad V_0 = U_0 \times W_0.$$  (B.7)

Definition B.4. Let $F$ be a closed set in $\mathbb{R}^n$ such that $F = \text{Int}(F)$. A function $f$ is said to be a smooth function on $F$ if

• $f$ is smooth on $\text{Int}(F)$;
• for any $k \in \mathbb{N}^n, f^{(k)} \equiv \prod_{a=1}^{N} \partial_{x_a}^k f$ extends continuously to $F$;
• for any $a \in \partial F, f$ admits an all order Taylor series expansion, viz. for any $m \geq 0$ it holds

$$f(x) = \sum_{k \in \mathbb{N}^n : |k| \leq m} f^{(k)}(a)(x - a)^k + R_m^n[f](x) \quad \text{with} \quad R_m^n[f](x) = o(||x - a||^m),$$

(B.8)

This definition of smoothness can be stated, in greater generality, in the language of jets where it translates itself in the jet associated to a given function being a Whitney field.

Theorem B.5. Whitney extension theorem

Let $U \subset \mathbb{R}^n$ be open and $X \subset U$ be closed in $\mathbb{R}^n$. Any $f$ smooth on $X$ admits a smooth extension into a function $f_e$ to $U$, with $f_e^{(k)} = f^{(k)}$ on $X$.  

33
C Observables in the infinite XXZ chain

C.1 Solutions to linear integral equations

The observables describing the thermodynamic limit of the spin-1/2 XXZ chain are characterised by means of a collection of functions solving linear integral equations. These equations are driven by an operator $K_{\eta,Q}$ on $L^2([−Q; Q])$ characterised by the integral kernel

$$K(\lambda, \mu) = \frac{\sin(2\eta)}{2\pi \sinh(\lambda + i\eta) \sinh(\lambda - i\eta)}.$$  \hspace{1cm} (C.1)

To introduce all of the functions of interest to this work, one starts by defining the $Q$-dependent dressed energy which allows one to construct the Fermi zone of the model. It is defined as the solution to the linear integral equation

$$\varepsilon(\lambda | Q) + \int_{-Q}^{Q} K(\lambda - \mu | \zeta) \varepsilon(\lambda | Q) \cdot d\mu = h - 4\pi J \sin(\zeta) K(\lambda | \frac{1}{2}\zeta).$$  \hspace{1cm} (C.2)

Note that the unique solvability of (C.2) follows from $K_{\zeta,Q}$ having its spectral radius $< 1$.

The endpoint of the Fermi zone is defined as the unique positive solution $q$ to $\varepsilon(q | q) = 0$. Then, the function $\varepsilon_1(\lambda) = rh - 4\pi J \sin(\zeta) K(\lambda | \frac{1}{2}\zeta) - \int_{-Q}^{Q} K_r(\lambda - \mu)\varepsilon_1(\mu) \cdot d\mu$ \hspace{1cm} (C.3)

with

$$K_r(\lambda) = K(\lambda | \frac{1}{2}\zeta(r + 1)) + K(\lambda | \frac{1}{2}\zeta(r - 1))$$  \hspace{1cm} (C.4)

correspond to the dressed energies of the $r$-bound state excitations. For any $0 < \zeta < \pi/2$ and under some additional constraints for $\pi/2 < \zeta < \pi$, one can show that $\varepsilon_r(\lambda + i\delta\pi/2) > c_r > 0$ for any $\lambda \in \mathbb{R}$, and $\delta \in \{0, 1\}$. However, this lower bound should hold throughout the whole massless regime $0 < \zeta < \pi$, irrespectively of some additional constraints. This property has been checked to hold by numerical study of the solutions to (C.3), c.f. [48].

In order to introduce the dressed momenta of the $r$-bound states and of the particle-hole excitations, I first need to define the $r$-bound state bare phases $\theta_r$:

$$\theta_r(\lambda) = 2\pi \int_{\Gamma_\lambda} K_r(\mu - 0^+) \cdot d\mu \quad \text{for } r \geq 2 \quad \text{and} \quad \theta_1(\lambda) = \theta(\lambda | \zeta)$$  \hspace{1cm} (C.5)

with

$$\theta(\lambda | \eta) = 2\pi \int_{\Gamma_\lambda} K(\mu - 0^+ | \eta) \cdot d\mu.$$  \hspace{1cm} (C.6)

The contour of integration corresponds to the union of two segments $\Gamma_\lambda = [0; i\Im(\lambda)] \cup [i\Im(\lambda); \lambda]$ and the $-0^+$ prescription indicates that the poles of the integrand at $\pm i\eta + i\pi\mathbb{Z}$ should be avoided from the left.
Then, the function

$$p_0(\lambda) = \theta(\lambda + \frac{\pi}{2} \zeta) - \int_{-q}^{q} \theta_2(\lambda - \mu)p'(\mu) \frac{d\mu}{2\pi}$$

$$+ \pi \ell_r(\zeta) - p_F m_r(\zeta) - 2p_F \sum_{\sigma=\pm} \left(1 - \delta_{\sigma,\lambda} \right) \text{sgn} \left(1 - \frac{2}{\pi} \cdot \frac{i r + \sigma \pi}{2} \zeta \right) \cdot |A_{\rho,\sigma}(\lambda)|,$$  (C.7)

extended by $i\pi$-periodicity to $C$, corresponds to the dressed momentum of the $r$-bound states. Above, I have introduced

$$\ell_r(\zeta) = 1 - r + 2\lfloor \frac{\pi}{2} \zeta \rfloor \quad \text{and} \quad m_r(\zeta) = 2 - r - \delta_{r,1} + 2 \sum_{\sigma=\pm} \left\lfloor \frac{i r + \sigma \pi}{2} \right\rfloor.$$  (C.8)

Furthermore, I agree upon

$$\zeta = \eta - \pi \lfloor \frac{\eta}{\pi} \rfloor \quad \text{and} \quad A_{\rho,\sigma} = \left\{ \lambda \in \mathbb{C} : \frac{\eta}{\pi} \geq |\Im(\lambda)| \geq \min\left(\frac{\pi}{2}, \pi - \frac{\pi}{2}\zeta\right) \right\}.$$  (C.9)

In order to obtain $p_0$, one should first solve the linear integro-differential equation for $p_1$ and then use $p_1$ to define $p_0$ by (C.7). $p_1$ corresponds to the dressed momentum of the particle-hole excitations and $p_F = p_1(q)$ corresponds to the Fermi momentum. 1-strings have their rapidities $\lambda \in [R \setminus [-q;q]] \cup [R+\pi/2]$ while $r \geq 2$ strings, $r \in \mathbb{N} \setminus \{1\}$, have their rapidities $\lambda \in \mathbb{R} + i\delta_\pi/2$ for a $\delta_\pi = 0$ or 1, depending on the value of $r$ and $\zeta$. See, e.g. [68] for more details on the string parities.

One can show [48] under similar conditions on $\zeta$ as for the dressed energy that, for any $\lambda \in \mathbb{R}$,

$$\left| p_1'(\lambda + i\delta_\pi \frac{\zeta}{2} \right| > 0 \quad \text{when} \quad r \not\in \mathbb{N} \setminus \{1\} \quad \text{and} \quad \min\left(p_1'(\lambda) = -p_1'(\lambda + i\frac{\pi}{2}) \right) > 0.$$  (C.10)

Again, a numerical investigation indicates that (C.10) does, in fact, hold irrespectively of the value of $\zeta$.

It is convenient to introduce a piecewise shifted deformation of $p_1$:

$$\tilde{p}_1(\lambda) = p_1(\lambda) - 2p_F \text{sgn}(\pi - 2\zeta)1_{-\infty,2\pi\zeta} - q(\lambda) + 2\pi1_{-\infty,2\pi\zeta} - q(\lambda + i\frac{\pi}{2})$$ \quad (C.11)

which is a diffeomorphism from the oriented concatenation of sets

$$[q:+\infty] \cup [\mathbb{R} + i\frac{\pi}{2}] \cup -\infty, -q]$$ \quad onto \quad \{p_F;2\pi - p_F - 2p_F \text{sgn}(\pi - 2\zeta)\}.$$ \quad (C.12)

The image of $[-\mathbb{R} + i\frac{\pi}{2}] \cup [\mathbb{R} \setminus [-q;q]]$ under $\tilde{p}_1$ defines the range $\mathcal{F}_1 = \{p_1^{(-1)}; p_1^{(1)}\}$ with $p_1^{(-1)} = p_F$ and $p_1^{(1)} = 2\pi - p_F - 2p_F \text{sgn}(\pi - 2\zeta)$, where the particles’ momenta evolve. Likewise, the image of $\mathbb{R} + i\delta_\pi/2$ under $p_r$ defines the range $\mathcal{F}_r = \{p_\rho^{(-1)}; p_\rho^{(1)}\}$ where the $r$-string momenta evolve. $p_\rho^{(r)}$ can be readily computed by taking the $\lambda - i\delta_\pi/2 \to \pm\infty$ limits in (C.7). However, since their explicit values do not play a role, we do not provide them here.

The dressed energies of the excitations in the momentum representation are defined as:

$$e_1(k) = e_1 \circ \tilde{p}_1^{-1}(k) \quad \text{for} \quad k \in \mathcal{F}_1$$

$$e_r(k) = e_r \circ \tilde{p}_r^{-1}(k) \quad \text{for} \quad k \in \mathcal{F}_r \quad \text{and} \quad r \not\in \mathbb{N} \setminus \{1\}.$$ \quad (C.13)

The $r$-bound dressed phase is defined as the solution to

$$\phi_r(\lambda, \mu) = \frac{1}{2\pi} \theta_2(\lambda - \mu) - \int_{-q}^{q} K(\lambda - v) \phi_r(v, \mu) \cdot dv + \frac{m_r(\zeta)}{2}$$ \quad (C.15)
and the dressed charge solves

$$Z(\lambda) + \int_{-q}^{q} K(\lambda - \mu)Z(\mu) \, d\mu = 1 \quad . \quad (C.16)$$

The dressed charge is related to the dressed phase by the below identities [46]:

$$\phi_1(\lambda, q) - \phi_1(\lambda, -q) + 1 = Z(\lambda) \quad \text{and} \quad 1 + \phi_1(q, q) - \phi_1(-q, q) = \frac{1}{Z(q)} \quad . \quad (C.17)$$

Similarly to the dressed energy in the momentum representation, it is convenient to introduce the momentum representation of the r-bound dressed phase:

$$\varphi_r(s, k) = \phi_1(\hat{p}_r^{-1}(s), \hat{p}_r^{-1}(k)) \quad \text{for} \quad s \in [-p_F; 2\pi - p_F - 2p_F \text{sgn}(\pi - 2\zeta)] \quad \text{and} \quad k \in \mathcal{S} \quad . \quad (C.18)$$

Here, one should understand that $\hat{p}_r = p_r$ if $r \geq 2$. Also, one sets $Z = Z \circ \hat{p}_1^{-1}$.

Then, the exponents $\Delta_r(\mathcal{R})$ governing the dynamic part of the DRF are expressed as $\Delta_r(\mathcal{R}) = \vartheta_{\nu}^2(\mathcal{R})$ where

$$\vartheta_{\nu}(\mathcal{R}) = - v_1 \ell_{\nu} + \frac{1}{2} \xi_{\nu} Z(p_F) + \sum_{a=1}^{n_a} \varphi_1(u p_F, t_a) - \sum_{r \in \mathcal{R}} \sum_{a=1}^{n_a} \varphi_r(u p_F, k_a^{(r)})$$

$$- \sum_{\nu' \in \{\pm\}} \ell_{\nu'} \varphi_1(u p_F, u' p_F) + \text{sgn}(\pi - 2\zeta) \cdot n_1(\mathcal{R})Z(p_F) \quad . \quad (C.19)$$

Here $n_1(\mathcal{R}) = \#\{k_a^{(1)} \in \mathcal{R} : \hat{p}_1^{-1}(k_a^{(1)}) \in ]-\infty ; -q[\}$. 

### C.2 The velocity of individual excitations

The velocity $v_r$ of an $r$-string excitation if $r \in \mathcal{R}_a$ and of a particle/hole excitation if $r = 1$ is defined by

$$v_r(k) = c_1^r(k) \quad . \quad \text{In particular} \quad v_F = c_1^F(p_F) \quad . \quad (C.20)$$

is the Fermi velocity, namely the velocity of a particle or of a hole excitation located directly on the right edge of the Fermi zone $[-p_F; p_F]$ in the momentum representation. $v_1$ is defined, originally, on

$$[-p_F; 2\pi - p_F - 2p_F \text{sgn}(\pi - 2\zeta)] \quad (C.21)$$

and it is easy to see that it extends to a $2\pi - 2p_F \text{sgn}(\pi - 2\zeta)$ periodic function on $\mathbb{R}$.

Furthermore, $v_1$ enjoys the symmetry

$$v_1(k) = - v_1(2\pi - 2p_F \text{sgn}(\pi - 2\zeta) - k) \quad . \quad (C.22)$$

These properties follow easily from its definition.

Also $v_1$ is a continuous function on $\mathbb{R}$ that is piecewise smooth. The points where smoothness may fail correspond to the two momenta

$$\hat{p}_1^{-1}(\pm \infty) = \hat{p}_1^{-1}(\pm \infty + i\pi/2).$$

One can easily prove for $p_F$ small enough, or for $\zeta$ belonging to a sufficiently small open neighbourhood of $\pi/2$, the below proposition characterising some of the properties of $v_1$. 

36
Proposition C.1. There exists $p_F^{(0)}$ and $\delta \xi^{(0)}$ such that, if either of the two bounds holds
\[ 0 \leq p_F < p_F^{(0)} \quad \text{or} \quad |\xi - \pi/2| < \delta \xi^{(0)}, \] (C.23)
then
- $|v_1| < v_F$ on $]-p_F ; p_F[$;
- there exists $P_m \in ]p_F : \pi - 2p_F \text{sgn}(\pi - 2\xi)[$ such that $v_1$ is strictly increasing on $]-p_F ; P_m[\cup ]P_M : 2\pi - p_F - 2p_F \text{sgn}(\pi - 2\xi)[$
  \[ \text{with } P_M = 2\pi - 2p_F \text{sgn}(\pi - 2\xi) - P_m, \text{ and strictly decreasing on } ]P_m ; P_M[; \]
- there exists an interval $]K_m ; K_M[\subset ]p_F : 2\pi - p_F - 2p_F \text{sgn}(\pi - 2\xi)[$ with $K_M = 2\pi - 2p_F \text{sgn}(\pi - 2\xi) - K_m$ (C.25)
such that
\[ |v_1(k)| < v_F \text{ for } k \in ]K_m ; K_M[ \] (C.26)
and
\[ |v_1(k)| > v_F \text{ for } k \in ]p_F ; 2\pi - p_F - 2p_F \text{sgn}(\pi - 2\xi)[ \setminus [K_m ; K_M]; \] (C.27)
- there exists a strictly decreasing homeomorphism $t : ]K_m ; K_M[ \mapsto ]-p_F ; p_F[$ such that $t(k)$ is the unique solution to $v_1(k) = v_1(t(k))$ (C.28)
with $k \in ]K_m ; K_M[$ and $t(k) \in ]-p_F ; p_F[$. The map $t$ is smooth on $]K_m ; K_M[$;
- there exists a strictly decreasing homeomorphisms $v_L, v_R$
\[ v_L : ]p_F ; P_m[ \mapsto ]P_m ; K_M[ , \quad v_R : ]P_M ; 2\pi - p_F - 2p_F \text{sgn}(\pi - 2\xi)[ \mapsto ]K_M ; P_M[, \] (C.29)
such that $v_L(k)$, resp. $v_R(k)$, is the unique solution to $v_1(k) = v_1(v_L(k))$, resp. $v_1(k) = v_1(v_R(k))$, on their respective range. The maps $v_{R,L}$ are smooth on the interior of their domains.
- $k \mapsto v_r$ is a diffeomorphism from $]r : ]-p_F^{(r)} ; p_F^{(r)}[ \text{ onto } ]-v^{(r)} ; v^{(r)}[$ with $v^{(r)} > v_F$. Furthermore, there exists $K_m^{(r)} , K_M^{(r)} \in ]r$ and a diffeomorphism $b^{(r)} : ]-p_F ; p_F[ \rightarrow ]K_m^{(r)} ; K_M^{(r)}[$ such that $v_1(t) = v_1(b^{(r)}(t))$. (C.30)

In fact, one can check by means of numerical analysis (c.f. Fig.2) that the properties listed in Proposition C.1 above hold true for any $p_F \in ]0 ; \pi/2[ \text{ and } \xi \in ]0 ; \pi[$. Thus the conjecture:

Conjecture C.2. The conclusions of Proposition C.1 hold true irrespectively of the values of $\xi \in ]0 ; \pi[ \text{ or } p_F \in ]0 ; \pi/2[.$
Theorem D.1. Let $a < b$ be two reals. Let $z_\pm(\lambda)$ be two real-holomorphic functions in a neighbourhood of the interval $\mathcal{J} = [a; b]$, such that

- all the zeroes of $z_\pm$ on $\mathcal{J}$ are simple;
- $z_+$ and $z_-$ admit a unique common zero $\lambda_0 \in \text{Int}(\mathcal{J})$ that, furthermore, is such that $z'_+(\lambda_0) \neq z'_-(\lambda_0)$.

Let $\Delta_0$ be real analytic on $\text{Int}(\mathcal{J})$ and such that $\Delta_0 \geq 0$. Let $\mathcal{G}$ be in the smooth class of $\mathcal{J}$ associated with the functions $\Delta_\pm$ and with a constant $\tau$, c.f. Definition[7.7] Then, for $x \neq 0$ and small enough,

$$\lambda \mapsto \mathcal{G}(\lambda, \hat{z}_+(\lambda), \hat{z}_-(\lambda)) \cdot \prod_{\nu = \pm} \left\{ \Xi(\hat{z}_\nu(\lambda)) \cdot \left[ \hat{z}_\nu(\lambda) \right]^{\Delta_\nu(\lambda) - 1} \right\} \in L^1(\mathcal{J}) \quad (D.1)$$

where $\hat{z}_\pm(\lambda) = z_\pm(\lambda) + x$. Let $I(x)$ denote the integral

$$I(x) = \int_{\mathcal{J}} \mathcal{G}(\lambda, \hat{z}_+(\lambda), \hat{z}_-(\lambda)) \cdot \prod_{\nu = \pm} \left\{ \Xi(\hat{z}_\nu(\lambda)) \cdot \left[ \hat{z}_\nu(\lambda) \right]^{\Delta_\nu(\lambda) - 1} \right\} \, \text{d}\lambda \quad (D.2)$$

Assume that $\delta_\pm = \Delta_\pm(\lambda_0) > 0$.

**a)** If $z'_+(\lambda_0) \cdot z'_-(\lambda_0) < 0$, then $I(x)$ admits the $x \to 0$ asymptotic expansion

$$I(x) = \Xi(z'_+(\lambda_0) \cdot x) \cdot \left\{ \frac{\mathcal{G}'(\lambda_0) \cdot \delta_+ \delta_- \cdot |x|^{\delta_+ + \delta_- - 1}}{|z'_+(\lambda_0)|^{\delta_+} \cdot |z'_-(\lambda_0)|^{\delta_-}} \cdot \frac{\Gamma(\delta_+) \cdot \Gamma(\delta_-)}{\Gamma(\delta_+ + \delta_-)} + O(|x|^{\delta_+ + \delta_- - \tau}) \right\} + f_<(x) \quad (D.3)$$

where

$$\hat{x} = x \cdot \left[ z'_+(\lambda_0) - z'_-(\lambda_0) \right] \quad (D.4)$$
\( \mathcal{G}(1) \) is as appearing in (1.6) and \( f_\gamma \) is a smooth function of \( x \). Furthermore, if \( \hat{a}_\pm \) have no zeroes on \( \mathcal{F} \) other than \( \lambda_0 \), then \( f_\gamma = 0 \).

**b)** If \( \hat{a}_\pm(\lambda_0) \cdot \hat{a}_\pm'(\lambda_0) > 0 \), then \( I(x) \) admits the \( x \to 0 \) asymptotic expansion

\[
I(x) = \frac{\mathcal{G}(1)(\lambda_0) \cdot \delta_+ \cdot |\hat{X}|^{\delta_+ - 1}}{|\hat{a}_\pm(\lambda_0)|^{\delta_-} \cdot |\hat{a}_\pm'(\lambda_0)|^\delta} \cdot \Gamma(\delta_+) \cdot \Gamma(\delta_-) \cdot \Gamma(1 - \delta_+ - \delta_-) \times \left\{ \Xi(x)^{\frac{1}{\pi}} \sin[\pi \delta_+] + \Xi(-x)^{\frac{1}{\pi}} \sin[\pi \delta_-] \right\} + O(|x|^\delta_+ - \delta_-) + f_\gamma(x) \quad (D.5)
\]

where \( \hat{X} \) and \( \delta_\pm \) are as above,

\[
p = -\text{sgn}[^{\hat{a}_\pm(\lambda_0)}] \cdot \text{sgn}[^{\hat{a}_\pm(\lambda_0) - \hat{a}_\pm'(\lambda_0)}] \quad (D.6)
\]

and \( f_\gamma \) is a smooth function of \( x \).

**Proof**

The hypothesis on \( \hat{a}_\pm(\lambda) \) ensure that these functions have a holomorphic inverse in a neighbourhood of any of their zeroes. As a consequence, any zero of \( \hat{a}_\pm \) is holomorphic in \( x \) small enough. Thus, the integral can be decomposed as \( I(x) = \sum_{k=1}^n I_k(x) \), where

\[
I_k(x) = \int_{a_k(x)}^{b_k(x)} \mathcal{G} \left( \lambda, \hat{\lambda}_+, (\lambda) \right) \cdot \prod_{\nu = \pm} \left\lfloor \hat{3}_{\nu}(\lambda) \right\rfloor^{\Delta_\nu(\lambda) - 1} \cdot d\lambda . \quad (D.7)
\]

The endpoints \( a_k(x) \), \( k \geq 2 \), and \( b_k(x) \), \( k \leq n - 1 \), all correspond to a zero of \( \hat{3}_+ \) or \( \hat{3}_- \). Furthermore, if \( a_1(x) = a \) and/or \( b_n(x) = b \), then these also correspond to a zero of \( \hat{3}_+ \) or \( \hat{3}_- \). However, it may be that \( a_1(x) = a \) and/or \( b_n(x) = b \) where I remind that \( \mathcal{F} = [a; b] \). Then \( a_1(x) = a \) and/or \( b_n(x) = b \) may or may not correspond to zeroes of \( \hat{3}_\nu \).

The fact that \( \mathcal{F} \) belongs to the smooth class of \( \hat{3}_\nu \) with functions \( \Delta_\nu \) and a constant \( \tau \) ensures that the integrals \( I_k(x) \) are well-defined. Indeed, problems with the \( L^1 \)-nature of its integrand could, in principle, arise if some zero of \( \hat{3}_\nu \) coincides with a zero of \( \Delta_\nu \). However, observe that due to the smooth class property and the hypotheses stated above (also c.f. equation (D.2)), the zeroes of \( \hat{3}_+ \) and \( \hat{3}_- \) are all distinct and simple, at least provided that \( x \) is small enough. Furthermore, one has the decomposition

\[
\mathcal{G} \left( \lambda, \hat{\lambda}_+, (\lambda) \right) \cdot \prod_{\nu = \pm} \left\lfloor \hat{3}_{\nu}(\lambda) \right\rfloor^{\Delta_\nu(\lambda) - 1} = \mathcal{G}(1)(\lambda) \cdot \prod_{\nu = \pm} \left\lfloor \Delta_\nu(\lambda) \right\rfloor^{\Delta_\nu(\lambda) - 1}
\]

\[
+ \mathcal{G}^{(2)} \left( \lambda, \hat{\lambda}_+, (\lambda) \right) \cdot \Delta_\nu(\lambda) \left[ \hat{3}_{\nu}(\lambda) \right]^{\Delta_\nu(\lambda) - 1} \cdot \left[ \hat{3}_{\nu}(\lambda) \right]^{\Delta_\nu(\lambda) - \tau} + \mathcal{G}^{(3)} \left( \lambda, \hat{\lambda}_+, (\lambda) \right) \cdot \Delta_\nu(\lambda) \left[ \hat{3}_{\nu}(\lambda) \right]^{\Delta_\nu(\lambda) - 1} \cdot \left[ \hat{3}_{\nu}(\lambda) \right]^{\Delta_\nu(\lambda) - \tau}
\]

\[
+ \mathcal{G}^{(4)} \left( \lambda, \hat{\lambda}_+, (\lambda) \right) \cdot \Delta_\nu(\lambda) \left[ \hat{3}_{\nu}(\lambda) \right] \cdot \prod_{\nu = \pm} \left\lfloor \hat{3}_{\nu}(\lambda) \right\rfloor^{\Delta_\nu(\lambda) - 1} . \quad (D.8)
\]

By the above, \( \hat{3}_\nu \) vanishes linearly at its zeroes. \( \Delta_\nu \) being holomorphic, it vanishes at least linearly at its zeroes. These two properties ensure the \( L^1 \)-nature of the integrand in (D.7).

In the following, I denote by \( \mu_\pm(x) \) the zeroes of \( \hat{3}_\pm(\lambda) \) such that \( \mu_\pm(0) = \lambda_0 \). If neither \( a_k(x) \) nor \( b_k(x) \) coincides with \( \mu_\pm(x) \), then the endpoint \( a_k(0) \), resp. \( b_k(0) \), is at most a simple zero of one of the functions \( \hat{3}_\pm \), but not of both. The latter is a direct consequence of the assumed properties of the functions \( \hat{3}_\pm \). Hence, \( I_k(x) \) corresponds
to the class of integrals studied in Lemma D.3 and, as such, is smooth in $x$ small enough. Its contribution is thus included in one of the functions $f_c(x)$ or $f_s(x)$, depending on the case of interest.

It thus remains to focus on the integral containing, as one of its endpoints, the zero $\mu_+(x)$. For convenience, denote this integral by $\mathcal{J}_{b_0}(x)$.

As already stated, the zeroes $\mu_\pm(x)$ are analytic functions of $x$, at least for $x$ small enough. Furthermore, it is readily checked that

$$\mu_\pm(x) = \lambda_0 - \frac{x}{\hat{h}_\pm(\lambda_0)} + O(x^2).$$

(D.9)

This ensures that $\mu_+(x) \neq \mu_-(x)$ for $x$ small enough. Being holomorphic, $\hat{h}_\pm(\lambda)$ admits the factorisation:

$$\hat{h}_\pm(\lambda) = (\lambda - \mu_+(x)) \cdot h_\nu(\lambda, x) \quad \text{with} \quad h_\nu(\mu_+(x), x) = \hat{h}_\nu\left(\mu_+(x)\right).$$

(D.10)

By the Weierstrass preparation theorem, cf. Theorem B.2, $h_\nu$ is holomorphic in $\lambda$ and $x$, at least for $|x|$ small enough.

In order to proceed further, one has to distinguish between the cases a) and b) outlined in the statement of the theorem.

- **Case a)**: $\hat{h}_+(\lambda_0) \cdot \hat{h}_-(\lambda_0) < 0$

  Let $\varsigma = \text{sgn}[\hat{h}_+(\lambda_0)]$. Then,

  $$\hat{h}_+(\lambda_0) > 0 \quad \text{on} \quad \varsigma[\mu_+(x) ; \nu_+(x)] \quad \text{while} \quad \hat{h}_-(\lambda_0) > 0 \quad \text{on} \quad \varsigma[\nu_-(x) ; \mu_-(x)]$$

  (D.11)

  where $\nu_\pm(x)$ is the closest zero of $\hat{h}_\pm(\lambda)$ to $\mu_\pm(x)$ such that the function satisfies to the above properties. The $\varsigma$ prefactor in front of the intervals means that the interval is always oriented from the smallest to the largest element.

  One can convince oneself that

  $$\nu_\pm(x) = \mu_\pm(x) \pm \varsigma \delta \nu_\pm(x) \quad \text{where} \quad \delta \nu_\pm(x) > C$$

  (D.12)

  for some $x$-independent constant $C > 0$.

  The above means that the neighbourhood of $\lambda_0$ will produce non-vanishing contributions to $\mathcal{J}_{b_0}(x)$ only if $\mu_+(x) < \mu_-(x)$. Provided this inequality holds, the integration in $\mathcal{J}_{b_0}(x)$ runs through the interval $[\mu_+(x) ; \mu_-(x)]$.

  Since one has

  $$\mu_-(x) - \mu_+(x) = \frac{\hat{h}_+(\lambda_0) - \hat{\nu}_+(\lambda_0)}{-\hat{h}_-(\lambda_0) \cdot \hat{\nu}_-(\lambda_0)} \left(1 + O(x)\right),$$

  (D.13)

  the condition $\mu_+(x) < \mu_-(x)$ can be recast, for $|x|$ small enough, as $\hat{h}_+(\lambda_0) \cdot \hat{\nu}_+(x) > 0$ where $\hat{\nu}_+$ is as defined in (D.4). Thence, upon inserting the factorisation (D.10) into $\mathcal{J}_{b_0}(x)$, the integral can be recast, for $|x|$ small enough, as

  $$\mathcal{J}_{b_0}(x) = \Xi(\hat{\nu}_+(\lambda_0) \cdot \hat{\varsigma}) \cdot \int_{\mu_+(x)}^{\mu_-(x)} \mathcal{H}(\lambda) \cdot \prod_{\nu = \pm} \left\{\nu \varsigma[\lambda - \mu_\nu(x)]\right\}^{\hat{h}_\nu(\lambda_0) - 1} \cdot d\lambda$$

  (D.14)

  with

  $$\mathcal{H}(\lambda) = \mathcal{G}\left(\lambda, \hat{h}_+(\lambda), \hat{h}_-(\lambda)\right) \cdot \prod_{\nu = \pm} \left\{\nu \varsigma h_\nu(\lambda, x)\right\}^{\hat{h}_\nu(\lambda_0) - 1}.$$  

  (D.15)

---

In case there are no more zeroes, one should simply take $\nu_+^{(c)}(x) = b$ and $\nu_-^{(c)}(x) = a$ or $\nu_+^{(s)}(x) = a$ or $\nu_-^{(s)}(x) = b$ depending on the situation, where I remind that $\mathcal{G} = [a ; b]$. 

40
The representation (D.8), the properties of the functions $\mathcal{G}^{(k)}$ and the fact that the $\mathcal{G}$ independent-part of the integrand has constant sign, all lead together to the decomposition

$$\mathcal{J}_\delta(x) = \mathcal{J}[H, \Delta_+, \Delta_-](x) + \sum_{\nu = \pm} \mathcal{O}(\mathcal{J}[1, \Delta_+ + (1 - \tau)\delta_{\nu, +}, \Delta_- + (1 - \tau)\delta_{\nu, -}](x))$$  \hfill (D.16)

where $\delta_{a,b}$ stands for the Kronecker symbol. Here

$$\mathcal{J}[H, \Delta_+, \Delta_-](x) = \mathcal{E}\left(\delta'_+(\lambda_0) \cdot \mathcal{X}\right) \cdot \int_{\mu_{\nu}(x)}^{\mu_{\nu}(x)} H(\lambda) \cdot \prod_{\nu = \pm} \left\{ \nu \chi[\lambda - \mu_{\nu}(x)] \right\}^{\Delta_+(\lambda) - 1} \cdot d\lambda$$  \hfill (D.17)

with

$$H(\lambda) = \Delta_+(\lambda) \Delta_-(\lambda) \cdot \mathcal{G}^{(1)}(\lambda) \cdot \prod_{\nu = \pm} \left\{ \nu \chi h_{\nu}(\lambda, x) \right\}^{\Delta_+(\lambda) - 1}.$$  \hfill (D.18)

Then, the change of variables

$$t = \frac{\lambda - \mu_{\nu}(x)}{\mu_{\nu}(x) - \mu_{\nu}(x)}$$  \hfill (D.19)

recasts the integral as

$$\mathcal{J}[H, \Delta_+, \Delta_-](x) = \mathcal{E}\left(\delta'_+(\lambda_0) \cdot \mathcal{X}\right) \cdot \int_0^1 \tilde{H}(t) \cdot (1 - t) \tilde{\chi}'(t) \cdot \tilde{H}(t) \cdot dt$$  \hfill (D.20)

where

$$\tilde{H}(t) = H\left(\mu_{\nu}(x) + t \cdot [\mu_{-\nu}(x) - \mu_{\nu}(x)]\right) \cdot \left(\mu_{-\nu}(x) - \mu_{\nu}(x)\right)^{\tilde{\Delta}_-(t) + \tilde{\Delta}_+(t) - 1}$$  \hfill (D.21)

and

$$\tilde{\Delta}_{\nu}(t) = \Delta_{\nu}\left(\mu_{\nu}(x) + t \cdot [\mu_{-\nu}(x) - \mu_{\nu}(x)]\right).$$  \hfill (D.22)

Being smooth, all functions have an expansions in $x$ that is uniform in $t \in [0; 1]$. This fact ensures that the leading asymptotic expansion of the integral is obtained by setting the argument $t$ of all functions to 0, leading to

$$\mathcal{J}[H, \Delta_+, \Delta_-](x) = \mathcal{E}\left(\delta'_+(\lambda_0) \cdot \mathcal{X}\right) \cdot \tilde{H}(0) \cdot \frac{\Gamma\left(\tilde{\Delta}_+(0)\right) \cdot \Gamma\left(\tilde{\Delta}_-(0)\right)}{\Gamma\left(\tilde{\Delta}_+(0) + \tilde{\Delta}_-(0)\right)} \cdot (1 + \mathcal{O}(x \ln x)).$$  \hfill (D.23)

Note that the $\mathcal{O}(x \ln x)$ remainder issues from the expansion of the exponents in $\tilde{H}(t)$. One can simplify the formula further. One has $\tilde{\Delta}_{\nu}(0) = \delta_{\nu} + O(x)$ with $\delta_{\pm}$ as in (D.4) as well as

$$\tilde{H}(0) = \delta_{\pm} \delta_- \cdot \mathcal{G}^{(1)}(\lambda_0) \cdot |\mathcal{X}|^{\delta_+ + \delta_- - 1} + \mathcal{O}(x^{\delta_+ + \delta_- \ln |\mathcal{X}|})$$  \hfill (D.24)

what allows one to conclude regarding to (D.3).

• **Case b):** $\delta'_+(\lambda_0) \cdot \delta'_-(\lambda_0) > 0$
Still agreeing upon ζ = sgn[ζ′(λ₀)], and keeping the same definition of ν⁺^{(c)}(x), one gets that
\[ \lambda \in (λ₀) > 0 \text{ on } ζ[μ⁺(x) : ν⁺^{(c)}(x)], \]  
(D.25)
Here, as earlier, the ζ prefactor indicates that the interval is oriented from its smallest to its largest element. It is easy to convince oneself that, in the present case of interest,
\[ ν⁺^{(c)}(x) = μ⁺(x) + ζδν⁺^{(c)}(x) \quad \text{where} \quad δν⁺^{(c)}(x) > C \]  
(D.26)
for some x-independent constant C > 0. Thus, after imposing the positivity constraints and using that |x| is small, the integral \( J_{λ₀}(x) \) runs through \( ζ[b_ζ ; c_ζ] \) where
\[ b_ζ = ζ \max(ζμ⁺(x), ζμ⁻(x)) \quad \text{and} \quad c_ζ = ζ \min(ζμ⁺^{(c)}(x), ζμ⁻^{(c)}(x)). \]  
(D.27)
The integral of interest can then be decomposed as \( J_{λ₀}(x) = J_{λ₀}^{(1)}(x) + J_{λ₀}^{(2)}(x) \)
\[ J_{λ₀}^{(1)}(x) = ζ \int_{b_ζ}^{b_ζ + ζδ} \mathcal{J}(λ, λ⁺(λ), λ⁻(λ)) \prod_{ν=±} [λ ν(λ)]^{C(λ)} \cdot dλ, \]  
(D.28)
and
\[ J_{λ₀}^{(2)}(x) = ζ \int_{b_ζ}^{c_ζ} \mathcal{J}(λ, λ⁺(λ), λ⁻(λ)) \prod_{ν=±} [λ ν(λ)]^{C(λ)} \cdot dλ, \]  
(D.29)
where \( δ > 0 \) is taken small enough.
\( J_{λ₀}^{(2)}(x) \) is a smooth function of \( x \). This can be seen as follows. If \( c_ζ ∈ \partial J \) and if the endpoints of \( J \) are not zeroes of \( λ⁺(λ) \), then the integrand in \( J_{λ₀}^{(2)}(x) \) can be expanded into powers of \( x \) owing to
\[ \inf\{b_ζ(λ) : λ ∈ [b_ζ + ζδ ; c_ζ]\} > C’, \]  
(D.30)
for some \( C’ > 0 \). This entails the claim. Otherwise, \( c_ζ \) coincides with a zero \( λ⁺ \). One treats the part of the integral corresponding to an integration over a domain uniformly away from \( c_ζ \), exactly as in the first case. Then, the neighbourhood of \( c_ζ \) can be treated as in Lemma D.3 and the claim follows.

It thus remains to focus on \( J_{λ₀}^{(1)}(x) \). Recalling the definition (D.6) of \( p \), one can readily check that, for \( x \) small enough,
\[ b_ζ = μ_{−sgn(1)p}(x) \quad \text{and} \quad ζ \min(ζμ⁺(x), ζμ⁻(x)) = μ_{sgn(1)p}(x). \]  
(D.31)
After the change of variables \( λ = b_ζ + ζt \), by using the factorisation (D.10) and setting
\[ a_ζ = ζ(μ_{−sgn(1)p}(x) − μ_{sgn(1)p}(x)) ≥ 0, \]  
(D.32)
one gets that
\[ J_{λ₀}^{(1)}(x) = ζ \int_{0}^{δ} \mathcal{H}(t, tu(t), (t + a_ζ)v(t)) \cdot F(t) \cdot (t + a_ζ)^{B(t)} \cdot dt. \]  
(D.33)
Above, we have set
\[ A(t) = \Delta_{-\sgn(t)}(b_\zeta + \varsigma t) - 1, \quad B(t) = \Delta_{\sgn(t)}(b_\zeta + \varsigma t) - 1 \] (D.34)
and
\[ u(t) = \varsigma h_{-\sgn(t)}(b_\zeta + \varsigma t, x), \quad v(t) = \varsigma h_{\sgn(t)}(b_\zeta + \varsigma t, x). \] (D.35)
Finally,
\[ \mathcal{H}(t, x, y) = \mathcal{G}(b_\zeta + \varsigma t, x, y) \cdot \prod_{\nu = \pm} \{\varsigma h_\nu(b_\zeta + \varsigma t, x)^{\Delta_\nu(b_\zeta + \varsigma t) - 1}. \] (D.36)
The properties of \( \mathcal{G} \) entail that
\[ \mathcal{H}(t, x, y) = H(t) + O(x^{1-\tau} + y^{1-\tau}) \] (D.37)
with a differentiable remainder in the sense of Definition [A.10] and where
\[ H(t) = \left( \Delta_\varsigma \Delta_- \mathcal{G}(1)(b_\zeta + \varsigma t) \cdot \prod_{\nu = \pm} \{\varsigma h_\nu(b_\zeta + \varsigma t, x)^{\Delta_\nu(b_\zeta + \varsigma t) - 1}. \right. \] (D.38)
One is now in position to apply the result of Lemma D.2 given below. This yields that
\[ \mathcal{J}_{\mathcal{G}}(x) = -\frac{H(0)}{\pi}(a_\zeta)^{1+\nu(0)+B(0)} \sin[\pi B(0)] \cdot \Gamma(1 + A(0))\Gamma(1 + B(0))\Gamma(-1 - A(0) - B(0)) \]
\[ + O(a_\zeta^{2+\nu(0)+B(0)} \ln a_\zeta) + r(a_\zeta), \] (D.39)
where \( r \) is some smooth function. At this stage, it remains to use that
\[ \Delta_\nu(b_\zeta) = \delta_\nu + O(x) \quad \text{and} \quad a_\zeta = |\mathcal{X}| \cdot \left[ s_0'(\lambda_0) \cdot s_0''(\lambda_0) \right]^{-1} \cdot (1 + O(x)) \] (D.40)
with \( \mathcal{X} \) as given in (D.34), so as to conclude.

\section*{D.2 Auxiliary lemmata}

\textbf{Lemma D.2.} Let \( 1 > \delta > 0 \) be fixed and \( f(t), A(t), B(t) \) be smooth real valued functions on \([0; \delta]\) admitting the expansion around zero
\[ f(t) = f_0 + O(t), \quad A(t) = a_0 + O(t), \quad B(t) = b_0 + O(t), \] (D.41)
where \( a_0 > -1 \) and \( b_0 > -1 \) are such that \( a_0 + b_0 \notin \mathbb{N} \). Further, let \( \mathcal{F} \) be smooth on \([0; \delta] \times \mathbb{R}^+ \times \mathbb{R}^+ \) and such that, for \( x, y \) bounded
\[ \mathcal{F}(t, x, y) = f(t) + O(x^\alpha + y^\alpha) \quad \text{for some} \quad 0 < \alpha < 1 \] (D.42)
with a differentiable remainder, c.f. Definition [A.7] Let \( u, v \) be smooth on \([0; \delta]\) and such that \( u(t), v(t) > 0 \). Then, the integral
\[ \mathcal{J}[\mathcal{F}, A, B](x) = \int_0^\delta \mathcal{F}(t, u(t), (t + x) v(t)) \cdot f^{A(t)} \cdot (t + x)^{B(t)} \cdot dt \] (D.43)
has the $x \to 0^+$ asymptotic expansion
\[
\mathcal{J}[\mathcal{F}, A, B](x) = -f_0 \frac{\sin(\pi b_0)}{\pi} \Gamma(1 + a_0) \Gamma(1 + b_0) \Gamma \left( -1 - a_0 - b_0 \right) \cdot x^{1+a_0+b_0} + r(x) + O(x^{1+a_0+b_0+\alpha}). \quad (D.44)
\]
where the function $r$ is smooth in $x$, and does depend on $\delta, A, B$.

Proof —
Observe that $x \mapsto \mathcal{J}[\mathcal{F}, A, B](x)$ is smooth in $x \in \mathbb{R}^+$ and that the $x$ derivatives are obtained by differentiating under the integral. Let $n \in \mathbb{N}$ be such that
\[-2 < 1 + a_0 + b_0 - n < -1. \quad (D.45)
\]
Observe that the hypotheses on the differentiability of the remainder ensures that
\[
\partial_x^n \mathcal{J}[\mathcal{F}, A, B](t; u(t), (t + x) v(t)) = \tilde{f}(t) (t + x) \tilde{B}(t) + O((t + x) \tilde{B}(t)) + O(f(t) (t + x) \tilde{B}(t)) \quad (D.46)
\]
with
\[
\tilde{f}(t) = f(t) \cdot \frac{\Gamma(B(t) + 1)}{\Gamma(1 + B(t) - n)} \quad \text{and} \quad \tilde{B}(t) = B(t) - n. \quad (D.47)
\]
Furthermore, being smooth functions on $[0; \delta]$, one has that
\[
\tilde{f}(t) = \sum_{k=0}^p \tilde{f}_k t^k + O(t^{p+1}) \quad A(t) = \sum_{k=0}^p a_k t^k + O(t^{p+1}) \quad \tilde{B}(t) = \sum_{k=0}^p b_k t^k + O(t^{p+1}). \quad (D.48)
\]
From there and the fact that $u(t), v(t) > c$ for some $c > 0$, one readily deduces that
\[
\tilde{f}(t) \cdot t^{A(t)} \cdot (t + x) \tilde{B}(t) = t^{a_0} \cdot (t + x) \tilde{B}(t) \cdot \left( \tilde{f}_0 + O(t \cdot \max(|\ln t|, |\ln(t + x)|, 1)) \right). \quad (D.49)
\]
Then, since $\ln t, \ln(t + x)$ have constant sign on $[0; \delta]$ provided that $|x|$ is small enough, straightforward bounds lead to
\[
\partial_x^n \mathcal{J}[\mathcal{F}, A, B](x) = \tilde{f}_0 \mathcal{T}(a_0, \tilde{b}_0) + O\left( |\partial_2 \mathcal{T}(a, b)| + |\partial_2 \mathcal{T}(a, b)| + |\mathcal{T}(a, b)| \right)_{a=a_0, \ b=\tilde{b}_0}
+ O\left( |\mathcal{T}(a + a_0, \tilde{b}_0)| + |\mathcal{T}(a_0, \tilde{b}_0 + a)| \right) \quad (D.50)
\]
where
\[
\mathcal{T}(a, b) = \int_0^\delta t^a \cdot (t + x)^b \cdot dt = x^{a+b+1} \int_0^\delta t^a \cdot (t + 1)^b \cdot dt. \quad (D.51)
\]
Note that, if need be, one may always slightly decrease the value of $\alpha$ so that $\alpha + a_0 + b_0 \notin \mathbb{Z}$ while preserving the differentiability of the remainder.
The change of variables $v = t/(t + 1)$ recasts the integral as

$$
\mathcal{T}(a, b) = x^{a+b+1} \int_0^\delta (1 - v)^{a-b-2} \cdot dv.
$$

It remains to expand the model integral

$$
\tilde{\mathcal{T}}(x, y; z) = \int_0^z t^{x-1} \cdot (1-t)^{y-1} \cdot dt \quad , \quad \Re(x) > 0 ,
$$

around $z = 1$. Let $p \in \mathbb{N}$ be such that $\Re(y) + p > 0$. Then, using the expansion for $|t - 1| < 1$

$$
t^{x-1} = [1 + (t - 1)]^{x-1} = \sum_{n=0}^{p-1} C_n(x)(1-t)^n \quad \text{with} \quad C_0(x) = 1 ,
$$

one has that

$$
\tilde{\mathcal{T}}(x, y; z) = \int_0^z \left( t^{x-1} - \sum_{n=0}^{p-1} C_n(x)(1-t)^n \right) \cdot (1-t)^{y-1} \cdot dt + \sum_{n=0}^{p-1} \frac{1 - (1-z)^{y+n}}{y+n} C_n(x) = -\mathcal{T}_0 \quad \text{D.55}
$$

where

$$
\mathcal{T}_0 = \sum_{n=0}^{p-1} \frac{C_n(x)}{y+n} + \int_0^1 \left( t^{x-1} - \sum_{n=0}^{p-1} C_n(x)(1-t)^n \right) (1-t)^{y-1} \cdot dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
$$

has been computed by meromorphic continuation in $y$. The above expansion ensures that there exists a function $h$ that is smooth in $x$ belonging to a neighbourhood of 0, and in $a > -1$ and $b \notin \mathbb{Z}$, such that

$$
\mathcal{T}(a, b) = -x^{a+b+1} \frac{\sin[\pi b]}{\pi} \Gamma(a+1)\Gamma(b+1)\Gamma(-a-b-1) + \frac{(\delta + x)^{a+b+1}}{a+b+1} + x \cdot h(x) . \quad \text{D.57}
$$

Owing to the choice of the integer $n$ in (D.45), all integrals $\mathcal{T}(a, b)$ appearing in (D.50) diverge in the $x \to 0$ limit. Thence, upon using the relation between $f_0, b_0$ and their un-tilded counterparts, one gets

$$
\partial_x^p \left[ \mathcal{F}[\mathcal{F}, A, B] \right] = (-1)^{p+1} x^{a_0+b_0+1-n} f_0^\pi \frac{\sin[\pi b_0]}{\pi} \Gamma(a_0+1)\Gamma(b_0+1)\Gamma(n-a_0-b_0-1)
$$

$$
+ O(x^{a_0+b_0+1+a-n}) . \quad \text{D.58}
$$

Then, $n$-fold integration in respect to $x$ entails the claim.

**Lemma D.3.** Let $\xi(x)$ be two real-holomorphic functions in a neighbourhood of $[a; b]$, $a < b$, such that

- $\xi > 0$ on $[a; b]$.

45
Let $\Delta_0 \geq 0$ be smooth on $[a(x);b(x)]$ uniformly in $x$ small enough. Let $G$ be in the smooth class of $[a(x);b(x)]$ with functions $\Delta_0$ and constant $\tau$.

Then, the integral

$$
\mathcal{J}(x) = \int_{a(x)}^{b(x)} \mathcal{G}(\lambda, \Delta_0(\lambda)) \prod_{\nu=\pm} \left[ \frac{\partial}{\partial \lambda} (\Delta_0(\lambda))^{\Delta_0(\lambda)-1} \right] \cdot d\lambda
$$

(D.59)

is a smooth function of $x$ small enough. In particular, it admits a Taylor series expansion around $x = 0$.

**Proof** —

To start with, consider the simpler situation when $\Delta_0 > 0$ on $[a(x);b(x)]$.

First consider the case when $a$ and $b$ are both a zero of one of the functions $\Delta_0$. Then, let $\epsilon_\ell, \epsilon_r \in \{\pm 1\}$ be such that $\Delta_0(a) = 0$, $\Delta_0(b) = 0$. In such a case, for any $\eta > 0$ and small enough, the hypotheses of the lemma ensure that there exists a constant $c > 0$ such that

$$
\begin{cases}
\Delta_0(\lambda) > c \text{ on } [a ; a + \eta] \\
\Delta_0(\lambda) > c \text{ on } [b - \eta ; b]
\end{cases}
$$

and

$$
\Delta_0(\lambda) > c \text{ on } [a + \eta ; b - \eta] .
$$

(D.60)

Since $a$, resp. $b$, is a simple zero of $\Delta_0(\lambda)$, resp. $\Delta_0(\lambda)$, the function is a local biholomorphism in the neighbourhood of that point. Hence, the zeroes $a(x)$ and $b(x)$ are analytic in $x$ small enough and one has the factorisation

$$
\Delta_0(\lambda) = (\lambda - a(x)) \cdot h_\ell(\lambda, x) \quad \text{ and } \quad \Delta_0(\lambda) = (b(x) - \lambda) \cdot h_r(\lambda, x)
$$

(D.61)

with $h_{\ell/r}(\lambda, x) > 0$ and analytic in $\lambda$ and $x$ by the Weierstrass preparation theorem [B.2]. Finally, the inverse $(\Delta_0(\lambda))^{-1}$ takes the explicit form $(\Delta_0(\lambda))^{-1} = \Delta_0(\lambda) - (\lambda - a(x)) \cdot h_\ell(\lambda, x)$

Hence, picking some $\eta > 0$ small enough, one can decompose the original integral as

$$
\mathcal{J}(x) = \mathcal{J}_\ell(x) + \mathcal{J}_r(x) + \mathcal{J}_c(x)
$$

with

$$
\mathcal{J}_c(x) = \int_{\Delta_0(\lambda) = c} G_c(\lambda, x) \cdot d\lambda
$$

(D.62)

and

$$
\mathcal{J}_r(x) = \int_{a(x)}^{b(x)} G_r(\lambda, x) \cdot \frac{\partial}{\partial \lambda} (\Delta_0(\lambda))^{\Delta_0(\lambda)-1} \cdot \Delta_0(\lambda) \cdot d\lambda ,
$$

$$
\mathcal{J}_\ell(x) = \int_{a(x)}^{b(x)} G_\ell(\lambda, x) \cdot \frac{\partial}{\partial \lambda} (\Delta_0(\lambda))^{\Delta_0(\lambda)-1} \cdot \Delta_0(\lambda) \cdot d\lambda .
$$

(D.63)

Above, I have set

$$
G_c(\lambda, x) = \mathcal{G}(\lambda, \Delta_0(\lambda)) \cdot \prod_{\nu=\pm} \left[ \frac{\partial}{\partial \lambda} (\Delta_0(\lambda))^{\Delta_0(\lambda)-1} \right] \cdot \Delta_0(\lambda) \cdot d\lambda .
$$

(D.64)
and

\[ G_{\ell,r}(\lambda, x) = \mathcal{F}(\lambda, \tilde{\tau}_{\pm}(\lambda), \tilde{\tau}_{-}(\lambda)) \cdot [\tilde{\tau}_{\pm-\ell/\lambda}(\lambda)]^{A_{-\ell/\lambda}(1)-1} \cdot \frac{1}{\tilde{\tau}_{\ell/\lambda}(\lambda)}. \]  \hfill (D.65)

The lower bounds (D.60), the smoothness of \( G_x(\lambda; x) \) and the fact that the integration runs through a compact, all together ensure that \( \mathcal{F}_x(x) \) is smooth in \( x \). Furthermore, a change of variables recasts \( \mathcal{F}_x(x) \), with \( a \in \{ \ell, r \} \) as

\[ \mathcal{F}_a(x) = \zeta_a \int_0^\eta G_a(\tau_a^{-1}(s-x), x) \cdot s^{\Delta_A(1)-(s-x)-1} \, ds \]  \hfill (D.66)

with \( \zeta_\ell = 1 \) and \( \zeta_r = -1 \). The same arguments as for \( \mathcal{F}_x(x) \) then allow one to conclude.

The remaining cases of possible values of \( z_\pm(a) \) and \( z_\pm(b) \) can be treated quite similarly.

It remains to discuss the situation when one allows for \( \Delta_v \) to vanish. The latter case remains unchanged relatively to \( \mathcal{F}_x(x) \). As for \( \mathcal{F}_x(x), a \in \{ \ell, r \} \), by the properties of a smooth class function, one may recast

\[ G_a(\tau_a^{-1}(s-x), x) = \Delta_v \circ \tau_a^{-1}(s-x) \cdot G_a^{(1)}(s-x) + G_a^{(2)}(s-x) \cdot s^{1-r} \]  \hfill (D.67)

with \( G_a^{(1)}, G_a^{(2)} \) smooth. Thus,

\[ \mathcal{F}_a(x) = \zeta_a \int_0^\eta G_a^{(2)}(s-x) \cdot s^{\Delta_A(1)-(s-x)-1} \cdot ds - \zeta_a \int_0^\eta \partial_s \{ G_a^{(2)}(s-x) \cdot s^{\Delta_A(1)-(s-x)} \}_{s=\eta} \cdot ds + \zeta_a G_a^{(2)}(\eta-x) - \zeta_a \int_0^\eta \eta^{\Delta_A(1)-(\eta-x)} - \zeta_a G_a^{(2)}(\eta-x) \cdot s^{\Delta_A(1)-(s-x)} \mid_{s=\eta}. \]  \hfill (D.68)

Note that the last term issuing from the integration by parts is present only if \( \Delta_v \circ \tau_a^{-1}(-x) = 0 \) and in that case, the contribution is also smooth in \( x \). Smoothness of all the other terms is clear.

E Asymptotics of multi-dimensional β-like integral

E.1 General assumptions

It is convenient to introduce a few notations and objects that will be used throughout this section. One assumes to be given:

- a strictly positive real \( v > 0 \);
- a collection of compact intervals \( \mathcal{I}_r, r = 1, \ldots, \ell \);
- smooth functions \( u_r \) on \( \mathcal{I}_r \) such that \( u'_r \) is strictly monotonous on \( \mathcal{I}_r \), and such that

\[ u'_r(k) \neq \pm v \quad \text{for} \quad k \in \text{Int}(\mathcal{I}_r). \]  \hfill (E.1)

Taken the physical interpretation that is discussed in the core of the paper,

- \( k \mapsto u_r(k) \) corresponds to the momentum-energy dispersion curve associated with a single particle excitation of "type" \( r \);
• $u'_r$ corresponds to the velocity of this excitation;

• $\mathcal{I}_r$ is the domain, in momentum space, where the dispersion curve $k \mapsto u_r(k)$ is strictly convex or concave, viz., where $u''_r$ has constant sign in its interior.

Given $n_r \in \mathbb{N}^*$, $r = 1, \ldots, \ell$, define the compact subset $\mathcal{I}_{\text{tot}}$ of $\mathbb{R}^{n_\ell}$ with $\mathbf{n}_\ell = \sum_{r=1}^{\ell} n_r$, as

$$\mathcal{I}_{\text{tot}} = \bigcap_{r=1}^{\ell} \mathcal{I}_r^{n_r}. \tag{E.2}$$

It is assumed that the intervals $\mathcal{I}_r$ partition as

$$\mathcal{I}_r = \mathcal{I}_r^{(\text{in})} \sqcup \mathcal{I}_r^{(\text{out})} \tag{E.3}$$

with $\mathcal{I}_1 = \mathcal{I}_1^{(\text{in})}$, i.e. $\mathcal{I}_1^{(\text{out})} = \emptyset$. The partition is such that

$$u'_1(\text{Int}(\mathcal{I}_1^{(\text{out})})) \cap u'_1(\text{Int}(\mathcal{I}_1^{(\text{in})})) = \emptyset \quad \text{and} \quad u'_1(\text{Int}(\mathcal{I}_1^{(\text{in})})) = u'_1(\text{Int}(\mathcal{I}_1^{(\text{in})})). \tag{E.4}$$

The hypothesis of strict monotonicity of $u'_1$ ensures that all the sets $u'_1(\text{Int}(\mathcal{I}_1^{(\text{in})}))$ are in one-to-one correspondence. More precisely, there exist homeomorphisms

$$t_r : \mathcal{I}_1^{(\text{in})} \rightarrow \mathcal{I}_r^{(\text{in})} \quad \text{such that} \quad u'_1(k) = u'_r(t_r(k)). \tag{E.5}$$

The hypotheses on $u'_r$ ensure that $t_r$ is a smooth diffeomorphism from $\text{Int}(\mathcal{I}_1^{(\text{in})})$ onto $\text{Int}(\mathcal{I}_r^{(\text{in})})$. It will appear useful, sometimes, to denote $t_1(k) = k$. The partitioning (E.3) splits the momentum range of type $r$ excitations into an interval $\mathcal{I}_r^{(\text{in})}$ associated with momenta of type $r$ excitations having a velocity that is also shared by type "1" excitations, and an interval $\mathcal{I}_r^{(\text{out})}$ whose associated velocities never coincide with those of type "1" excitations.

Given a choice of signs $\zeta_r \in \{\pm 1\}$, one defines the associated macroscopic "momentum" and "energy"

$$\mathcal{P}(k) = \sum_{r=1}^{\ell} n_r \zeta_r t_r(k) \quad \text{and} \quad \mathcal{E}(k) = \sum_{r=1}^{\ell} n_r \zeta_r u_r(t_r(k)) , \quad k \in \mathcal{I}_1, \tag{E.6}$$

of an agglomeration of equal velocity particles of different types. It is assumed in the following that $k \mapsto \mathcal{P}(k)$ is strictly monotonous on $\text{Int}(\mathcal{I}_1)$, i.e. that

$$k \mapsto \mathcal{P}'(k) = \sum_{r=1}^{\ell} \zeta_r n_r t'_r(k) \tag{E.7}$$

does not vanish on $\text{Int}(\mathcal{I}_1^{(\text{in})})$.

Finally, it is convenient to represent vectors in block form relatively to the Cartesian product structure of $\mathcal{I}_{\text{tot}}$, c.f. (E.2),

$$\mathbf{p} = (p^{(1)}, \ldots, p^{(\ell)}) \quad \text{with} \quad p^{(r)} = (p_1^{(r)}, \ldots, p_n^{(r)}) \in \mathbb{R}^{n_r}. \tag{E.8}$$

Also, given a vector $\mathbf{p}$ as above, it will be useful to introduce a special notation for a related vector where some of the components of $\mathbf{p}$ have been dropped:

$$p_{[u]}^{(r)} = (p_1^{(r)}, \ldots, p_{u-1}^{(r)}, p_{u+1}^{(r)}, \ldots, p_n^{(r)}) \quad \text{and} \quad p_{[r,a]} = (p^{(1)}, \ldots, p_{[u]}^{(r)}, \ldots, p^{(\ell)}). \tag{E.9}$$
In the following, \( k_0 \in \text{Int}(\mathcal{F}_1) \) will single out a point in \( \text{Int}(\mathcal{F}_1) \). Analogously to the above way of writing vectors, one denotes
\[
t(k_0) = (t_1(k_0), \ldots, t_r(k_0)) \in \mathbb{R}^r \quad \text{with} \quad t_r(k_0) = (t_r(k_0), \ldots, t_r(k_0)) \in \mathbb{R}^n
\] (E.10)
as well as
\[
t_{[\ell n]}(k_0) = (t_1(k_0), \ldots, t_{[\ell n]}(k_0)) \in \mathbb{R}^{r-1} \quad \text{with} \quad t_{[\ell n]}(k_0) = (t_r(k_0), \ldots, t_r(k_0)) \in \mathbb{R}^{n-1}
\] (E.11)
and where all the other \( t_r(k_0) \)'s are as given in (E.10).

Finally, the set of all possible labels \( (r, a) \) arising in the coordinates of \( p \) given in (E.8) is denoted as:
\[
\mathcal{M} = \left\{ (r, a) : r \in [1 \ldots \ell] \text{ and } a \in [1 \ldots n_r] \right\}.
\] (E.12)

Sometimes, the notation
\[
\mathcal{M}_{[\ell n]} = \mathcal{M} \setminus \{(\ell, n_r)\}
\] (E.13)
will be used.

It is easily seen that properties \( Hi) \) – \( Hi(i) \) of a function \( \mathcal{G} \) on \( K \times \mathbb{R}^+ \times \mathbb{R}^+ \) that is in the smooth class of \( K \) and associated with functions \( d_\pm \) and a constant \( \tau \), c.f. Definition 1.1, entail that, for any \( (s, \ell, \ell, \cdot) \) as above and for fixed \( \epsilon > 0 \), it holds
\[
\text{H1) (} x, u, v \text{)} \mapsto \prod_{a=1}^n \partial^{\ell u}_a \cdot \partial^{\ell v}_u \mathcal{G}^{(s)}(x, u, v) \text{ is bounded on } K \times [\epsilon; \epsilon^{-1}] \times [0; \epsilon^{-1}];
\]
\[
\text{H2) (} x, u, v \text{)} \mapsto \prod_{a=1}^n \partial^{\ell u}_a \cdot \partial^v_u \mathcal{G}^{(s)}(x, u, v) \text{ is bounded on } K \times [0; \epsilon^{-1}] \times [\epsilon; \epsilon^{-1}];
\]
\[
\text{H3) (} x, u, v \text{)} \mapsto \prod_{a=1}^n \partial^{\ell u}_a \cdot \mathcal{G}^{(s)}(x, u, v) \text{ is bounded on } K \times [0; \epsilon^{-1}]^2.
\]

Note that depending on the values of \( s \in [1 \ldots \ell] \), the \( u \) or \( v \) variables may or may not be effectively present in the above equations, viz. one should understand in the formulae above \( \mathcal{G}^{(s)}(x, u, v) = \mathcal{G}^{(s)}(x) \) etc.

Furthermore, when \( n \geq 2 \), all the functions appearing in \( H1) \) – \( H3) \) vanish upon replacing \( K \leftrightarrow \partial K \).

### E.2 The structural theorem in the multidimensional setting

**Theorem E.1.** Let \( \mathcal{F}_0 \) be as defined in (E.3), and \( \Delta_\pm \) be smooth positive functions on \( \mathcal{F}_0 \) admitting smooth square roots on \( \mathcal{F}_0 \). Let \( \mathcal{G} \) be in the smooth class of \( \mathcal{F}_0 \) associated with the functions \( \Delta_\pm \) and a constant \( \tau \in ]0; 1[ \), according to Definition 1.1.

Finally, let
\[
\zeta_v(p) = \mathcal{E}_0 - \sum_{(r, a) \in \mathcal{M}} \zeta_r u_r(p^{(r)}_a) + v v \mathcal{P}_0 - \sum_{(r, a) \in \mathcal{M}} \zeta_r p^{(r)}_a \, , \quad v \in \{\pm\}
\]
(E.14)

with \( \zeta_\cdot \in \{\pm1\} \) as given in (E.6) and where \( (\mathcal{P}_0, \mathcal{E}_0) \in \mathbb{R}^2 \).

Let \( I(x) \) be given by the multiple integral
\[
I(x) = \int_{\mathcal{F}_0} dp \, \mathcal{G}_0(p)
\]
(E.15)
Proof — some improvement of the method of analysis.

Let \( k_0 \) if the two conditions given below hold

\[
\text{min } _{\alpha \in \mathcal{P}_1} \left| \mathcal{E}_0 - \mathcal{E}(\alpha) + \nu \mathcal{P}_0 - \mathcal{P}(\alpha) \right| > 0
\]

then \( \mathcal{G}_0 \in L^1(\mathcal{I}_0) \) and \( I(x) \) is smooth in \( x \), for \( |x| \) small enough.

b) The singular case.

Let \( k_0 \in \text{Int}(\mathcal{I}_1) \) and recalling \( t(k_0) \) as defined in (E.10), let

\[
\Delta_0 = \Delta_0(t(k_0)) \quad \text{and} \quad \vartheta = \frac{1}{2} \sum_{r=1}^{\ell} n_r^2 - \frac{3}{2} + \Delta_+ + \Delta_-
\]

If

\[
(\mathcal{P}_0, \mathcal{E}_0) = (\mathcal{P}(k_0), \mathcal{E}(k_0)), \quad \vartheta \notin \mathbb{N}, \quad \text{and} \quad \Delta_\pm > 0
\]

then \( \mathcal{G}_0 \in L^1(\mathcal{I}_0) \) and \( I(x) \) admits the \( x \to 0^+ \) asymptotic expansion:

\[
I(x) = \frac{\Delta_+^{(0)} \Delta_-^{(0)} G(1)(t(k_0)) \cdot (2\nu)^{\Delta_+^{(0)} + \Delta_-^{(0)} - 1}}{\sqrt{\mathcal{P}^0(k_0)} \cdot \prod_{r=1}^{\ell} \left| n_r^{(0)} \right| \Gamma(\Delta_+^{(0)}) \Gamma(\Delta_-^{(0)}) \Gamma(-\vartheta) \cdot \prod_{r=1}^{\ell} \left\{ \frac{G_2 + n_r \cdot (2\pi)^{-\frac{1}{2}}}{|u^{(r)}(t_k)|^{\frac{1}{2}} - \delta_{r,1}} \right\}^{\left| n_r^{(0)} \right|} \times \left| x \right|^{\vartheta} \left\{ \frac{\Xi(x)}{\pi} \frac{\sin \frac{\pi \nu_+}{\pi}}{\sin \frac{\pi \nu_-}{\pi}} + \Xi(-x) \frac{\sin \frac{\pi \nu_-}{\pi}}{\sin \frac{\pi \nu_+}{\pi}} \right\} + \tau(x) + O(|x|^{\vartheta + 1 - \tau})
\]

Above \( \tau(x) \) is smooth in \( x \), for \( |x| \) small enough. Finally,

\[
\nu_+ = \frac{1}{2} \sum_{r=1}^{\ell} n_r^2 - \frac{1}{4} + \sum_{\text{even}, n_r^{(0)}} \Delta_0
\]

where \( s = -\text{sgn}(\nu^{(r)}(k_0)) \) and \( \varepsilon_r = -\zeta_s \text{sgn}(u^{(r)}(t_k)) \).

It is to be expected that the conditions \( \Delta_+^{(0)} > 0 \) are only technical and can be relaxed down to \( \Delta_+^{(0)} \geq 0 \), upon some improvement of the method of analysis.

Proof —

50
• $L^1(\mathcal{I}_{\text{tot}})$ character

The integrand is smooth with the exception of the points where $z_o(p) + x = 0$. Thus, to conclude on its $L^1(\mathcal{I}_{\text{tot}})$ integrability it is only necessary to focus on its local behaviour in the vicinity of these points. The local behaviour of the integrand around these points, after an appropriate change of variables that rectifies this behaviour, is thoroughly investigated in the core of the proof. It is the integrations over such vicinity that generate the non-smooth behaviour in $x$. These integrals reduce to the "local" integrals described in (E.121) and (E.151) whose study can be reduced to reasoning on one-dimensional integrals by means of appropriate changes of variables. On the level of these representations, it is easy to see that the local $L^1$ character, in virtue of $\mathcal{G}$ being in the smooth class of $\Delta_{\kappa}$, reduces to $\Delta_{\kappa} \geq 0$.

• A preliminary decomposition into totally collinear and non-collinear parts

The first step of the analysis consists in decomposing the integral into those parts which may, under certain conditions on $(\mathcal{P}_0, \mathcal{E}_0)$, generate a non-smooth behaviour in $x$ and those parts which will always, independently of the value of $(\mathcal{P}_0, \mathcal{E}_0)$, produce a smooth behaviour. This is achieved by decomposing the integration domain into portions where one can directly apply Lemma E.11 hence guaranteeing smoothness in $x$ of their contribution, and those portions which require further study.

Given $\eta > 0$ small enough, one has the below decomposition of $\mathcal{I}_{\text{tot}}$

$$\mathcal{I}_{\text{tot}} = \mathcal{D}_\eta^{(\perp)} \sqcup \mathcal{D}_\eta^{(\parallel)}$$  \hspace{1cm} (E.24)

where

$$\mathcal{D}_\eta^{(\perp)} = \{ p \in \mathcal{I}_{\text{tot}} : \exists (r, a) \neq (1, 1) \text{ such that } |u'_1(p_1^{(1)}) - u'_1(p_a^{(r)})| \geq \eta \}$$  \hspace{1cm} (E.25)

contains vectors $p$ where at least one variable is associated with a different velocity than the one carried by the first component $p_1^{(1)}$ of $p$ and

$$\mathcal{D}_\eta^{(\parallel)} = \{ p \in \mathcal{I}_{\text{tot}} : \forall (r, a) \in \mathcal{M}, |u'_1(p_1^{(1)}) - u'_1(p_a^{(r)})| < \eta \}$$  \hspace{1cm} (E.26)

contains vectors all of whose components have almost equal velocities. Let $\varphi^{(\parallel)}$ be smooth and such that

$$0 \leq \varphi^{(\parallel)} \leq 1 \quad , \quad \varphi^{(\parallel)} = 1 \text{ on } \mathcal{D}_\eta^{(\parallel)} \text{ and } \varphi^{(\parallel)} = 0 \text{ on } \mathcal{D}_\eta^{(\perp)}.$$ \hspace{1cm} (E.27)

Then set $\varphi^{(\perp)} = 1 - \varphi^{(\parallel)}$. This entails that $\varphi^{(\perp)} \neq 0$ only on $\mathcal{D}_\eta^{(\perp)}$ so that one has a partition of unity on $\mathcal{I}_{\text{tot}}$:

$$\varphi^{(\parallel)} + \varphi^{(\perp)} = 1$$

which induces the decomposition of $I(x) = I^{(\parallel)}(x) + I^{(\perp)}(x)$ with

$$I^{(\parallel)}(x) = \int_{\mathcal{D}_\eta^{(\parallel)}} dp \mathcal{G}_{\text{tot}}^{(\parallel)}(p) \quad \text{ and } \quad I^{(\perp)}(x) = \int_{\mathcal{D}_\eta^{(\perp)}} dp \mathcal{G}_{\text{tot}}^{(\perp)}(p).$$  \hspace{1cm} (E.28)

Above and in the following, we agree upon

$$\mathcal{G}_{\text{tot}}^{(\perp)}(p) = \varphi^{(\perp)}(p) \cdot \mathcal{G}_{\text{tot}}(p) \quad \text{ and } \quad \mathcal{G}_{\text{tot}}^{(\parallel)}(p) = \varphi^{(\parallel)}(p) \cdot \mathcal{G}_{\text{tot}}(p).$$  \hspace{1cm} (E.29)
• The integral $I^{(\perp)}$

I establish below that $I^{(\perp)}(x)$ solely generates a smooth behaviour in $x$ small enough.

Given $h \in \mathcal{D}_{\eta/2}^{(\perp)}$, by definition, there exists $(r, a) \neq (1, 1)$ such that $|u_1'(h_1^{(r)}) - u_1'(h_a^{(r)})| \geq \eta/2$. Then, the map

$$f_{(r,a)}(p) = (p_1^{(1)}, \ldots, p_a^{(r)}, \ldots, p^f(z), z_+(p), z_-(p))$$

(E.30)

satisfies

$$\det [D_p f_{(r,a)}] = 2v \xi_1 \xi_r \cdot (-1)^{m_{r,a}} \cdot [u_1'(p_a^{(r)}) - u_1'(p_1^{(1)})] \quad \text{with} \quad m_{r,a} = a + \sum_{b=1}^{r-1} n_b.$$ 

(E.31)

Hence, for all $h \in \mathcal{D}_{\eta/2}^{(\perp)}$,

$$\left| \det [D_h f_{(r,a)}] \right| \geq \nu \eta.$$ 

(E.32)

One is thus in position to apply Lemma F.1 so as to conclude that $x \mapsto I^{(\perp)}(x)$ is smooth in $|x|$ small enough.

As a consequence, it only remains to focus on the $x \to 0$ behaviour of $I^{(\perp)}(x)$.

• Behaviour of $I^{(\perp)}$ in the regular case

This corresponds to case a) appearing in the statement of the theorem. Since $\mathcal{F}_1$ is compact and $k \mapsto (\mathcal{P}(k), \mathcal{E}(k))$ is continuous, where $\mathcal{P}(k), \mathcal{E}(k)$ are as defined in (E.6), hypotheses (E.18) and (E.19) entail that there exists $\varrho > 0$ such that

$$\inf_{k \in \mathcal{F}_1} \left\{ d\left((\mathcal{P}_0, \mathcal{E}_0), (\mathcal{P}(k), \mathcal{E}(k))\right) \right\} > \varrho \quad \text{and} \quad \min_{\alpha \in (0, \mathcal{F}_1)} \left| \mathcal{E}_0 - \mathcal{E}(\alpha) + \nu \mathcal{P}_0 - \mathcal{P}(\alpha) \right| > \varrho.$$ 

(E.33)

It is useful to recast $z_0(p)$ as:

$$z_0(p) = \mathcal{Z}_\nu(p_1^{(1)}) + \delta_{z_0}(p)$$

(E.34)

where

$$\mathcal{Z}_\nu(k) = \mathcal{E}_0 - \mathcal{E}(k) + \nu \{ \mathcal{P}_0 - \mathcal{P}(k) \}$$

(E.35)

and

$$\delta_{z_0}(p) = - \sum_{(r,a) \in \mathcal{M}} \xi_r \omega_{\nu}(p_a^{(r)}; t_\nu(p_1^{(1)})).$$

(E.36)

Here, I have introduced

$$\omega_{\nu}(k; p) = u_\nu(k) - u_\nu(p) + \nu \nu(k - p).$$

(E.37)

One has that $\mathcal{Z}_\nu$ is smooth on $\text{Int}(\mathcal{F}_1)$ and

$$\mathcal{Z}_\nu'(k) = - \left\{ \sum_{r=1}^{\ell} n_r \xi_r t_\nu'(k) \right\} \cdot (u_1'(k) + \nu \nu).$$

(E.38)
Thus owing to hypothesis (E.1) and (E.7), $Z'_p$ does not vanish on $\mathcal{I}$, so that $Z_\nu$ is strictly monotonous on $\mathcal{I}$. This entails that $Z_\nu$ has at most one zero on $\mathcal{I}$.

There are several cases to discuss depending on whether $Z_\nu$ has a zero or not on $\mathcal{I}$.

i) $(P_0, E_0)$ is such that both $Z_\pm$ do not vanish on $\mathcal{I}$.

In such a case, there exists $C_0$, such that $|Z_\pm(k)| \geq 2C_0$, for any $k \in \mathcal{I}$. I now establish that this property entails the non-vanishing of $\delta_0$ on $\mathcal{D}_0$. For that purpose, observe that since $u'_r$ is strictly monotonous on $\mathcal{I}_r$ and continuous, it is continuously invertible on its image. This allows one to recast

$$
\delta_0(p) = - \sum_{(r,a) \in M} \zeta_r \tilde{w}_r^{(r)}(u'_r(p_a^{(r)}); u'_r(t_r(p_1^{(1)}))) = u'_r(p_1^{(1)}) \tag{E.39}
$$

where

$$
\tilde{w}_r^{(r)}(k; p) = u_r \circ (u'_r)^{-1}(k) + u_r \circ (u'_r)^{-1}(p) + v \nu \left( (u'_r)^{-1}(k) - (u'_r)^{-1}(p) \right) \tag{E.40}
$$

Since $u'_r(\mathcal{I}_r) \times \mathcal{I}_1$ is compact, $\tilde{w}_r^{(r)}$ is uniformly continuous on this set. This entails that there exists $s_\eta$, with $s_\eta \to 0^+$ when $\eta \to 0^+$ such that,

uniformly in $p \in D_\eta$ it holds $\delta_0(p) = O(s_\eta)$. \tag{E.41}

Then, by taking $\eta$ small enough in (E.25)-(E.26), one gets that for any $p \in D_\eta$,

$$
|\delta_0(p)| > |Z'_0(p_1^{(1)})| - |\delta_0(p)| > C_0 \tag{E.42}
$$

This lower bound is enough so as to conclude, by derivation under the integral theorems, that $F_0(x)$ is smooth, provided that $|x|$ is small enough.

ii) $(P_0, E_0)$ is such that least one of the two functions $Z_\pm$ vanishes on $\mathcal{I}$.

First of all, by (E.33), $Z_\nu$ cannot vanish on $\partial \mathcal{I}$, and hence, by continuity, on an open neighbourhood thereof. Thus, if a zero exists, it is at a finite distance from the boundary of $\mathcal{I}$. Furthermore, $Z_\pm$ cannot share a common zero on $\text{Int}(\mathcal{I})$. Indeed, if that were the case, then one would have $Z_\pm(k) = 0$ for some $k \in \text{Int}(\mathcal{I})$. This would then entail that

$$
\begin{align*}
0 &= Z_+(k) - Z_-(k) = 2\nu \left[ P_0 - P(k) \right] \\
0 &= Z_+(k) + Z_-(k) = 2\left[ E_0 - E(k) \right]
\end{align*} \tag{E.43}
$$

However, such a vanishing contradicts (E.33).

Denote by $k_+ \in \text{Int}(\mathcal{I})$ the zeroes of $Z_\nu$, if these exists. Let $N_\nu$ be an open neighbourhood of $k_+$ in $\text{Int}(\mathcal{I})$ such that

$$
N_\nu \subset \text{Int}(\mathcal{I}) \quad \text{and} \quad N_+ \cap N_- = \emptyset \tag{E.44}
$$

where the last condition only applies if both zeros exist and can be made possible since $k_+ \neq k_-$ as argued earlier. Then set

$$
K_\nu = \left\{ p \in \mathcal{I} : p_1^{(1)} \in N_\nu \quad \text{and} \quad \forall (r,a) \in M \quad |u'_1(p_1^{(1)}) - u'_r(p_a^{(r)})| < \eta \right\} \tag{E.45}
$$
By construction, $k_0 \not\in \text{pr}_{[1,1]}(\mathcal{D}_\eta^{(f)} \setminus \mathcal{K}_\nu)$, where $\text{pr}_{[1,1]}$ is the projection on the first coordinate. Thus, $\mathcal{Z}_\nu$ does not vanish on $\text{pr}_{[1,1]}(\mathcal{D}_\eta^{(f)} \setminus \mathcal{K}_\nu)$.

Recall that, uniformly on $\mathcal{D}_\eta^{(f)}$, one has $\delta_{3_\nu}(p) = O(s_\eta)$, with $s_\eta \to 0$ as $\eta \to 0^+$. Reducing $\eta$ if necessary, one concludes, as before, that there exists a constant $C > 0$ such that

$$\left| 3_\nu(p) \right| > C \quad \text{for any } \, p \in \mathcal{D}_\eta^{(f)} \setminus \mathcal{K}_\nu.$$ (E.46)

It remains to deal with the behaviour of $3_\nu$ inside of $\mathcal{K}_\nu$. The map

$$f_{[1,1]}^{(\nu)} : \ b (p) \mapsto \ (p^{(1)}, p^{(2)}, \ldots, p^{(\ell)}, 3_\nu(p))$$ (E.47)

satisfies

$$\text{det} \left[ D_p f_{[1,1]}^{(\nu)} \right] = \prod_{\nu = 1}^{\ell} (-1)^{s_{\nu}} \cdot \nu \left( u'(p^{(1)}) \right).$$ (E.48)

Since $u'(k) \neq \pm \nu$ on $\text{Int}(\mathcal{I}_1)$, it follows that $\text{det} \left[ D_p f_{[1,1]}^{(\nu)} \right] \neq 0$ on $\mathcal{K}_\nu$. Upon reducing $\eta$ if necessary, by compactness of $\mathcal{K}_\nu$ and smoothness of $f_{[1,1]}^{(\nu)}$ on an open neighbourhood of $\mathcal{K}_\nu$, there exists:

- points $p_k \in \mathcal{K}_\nu, \ k = 1, \ldots, m_\nu$;
- open neighbourhoods $U_{v:k}$ of $p_k$ forming a finite open cover of $\mathcal{K}_\nu$ such that
  $$\mathcal{K}_\nu \subset \bigcup_{k=1}^{m_\nu} U_{v:k} \subset \mathcal{I}_{\text{tot}};$$ (E.49)
- open sets $V_{v:k}$ and constants $\delta_{v:k} > 0$ satisfying $\delta_{v:k} \pm \delta_{v:k} \neq 0$;

such that

$$f_{[1,1]}^{(\nu)} : \ U_{v:k} \rightarrow \ V_{v:k} \times [3_\nu(p_k) - \delta_{v:k}, \delta_{v:k} + 3_\nu(p_k)]$$ (E.50)

is a diffeomorphism onto and that its inverse $(f_{[1,1]}^{(\nu)})^{-1}$ extends smoothly to a neighbourhood of

$$V_{v:k} \times [3_\nu(p_k) - \delta_{v:k}, \delta_{v:k} + 3_\nu(p_k)].$$ (E.51)

Note that, if both zeroes exists, the neighbourhoods $U_{v} = \bigcup_{k=1}^{m_\nu} U_{v:k}$ can and are chosen such that $U_+ \cap U_- = \emptyset$. Denote by $\{\varphi_{v:k}\}_{k=1}^{m_\nu}$ the partition of unity associated with the open cover $U_{v:k}$.

Below, I only discuss the case when both $\mathcal{Z}_+$ and $\mathcal{Z}_-$ have a zero. All other cases are treated analogously.

By using that the integrand vanishes outside of $\mathcal{D}_{\eta}^{(f)}$, one decomposes the integral as

$$\mathcal{I}^{(f)}(x) = \mathcal{I}_{\text{tot}}^{(f)}(x) + \mathcal{I}_+^{(f)}(x) + \mathcal{I}_-^{(f)}(x).$$ (E.52)

There

$$\mathcal{I}_{\text{tot}}^{(f)}(x) = \int_{\mathcal{D}_{\eta}^{(f)} \setminus \{U_+ \cup U_-\}} \mathcal{G}_{\text{tot}}^{(f)}(p) \cdot dp$$ (E.53)
and
\[ I^{(f)}_{\nu}(x) = \sum_{k=1}^{m_{\nu}} \int_{V_{\nu,k}} dv \int_{\tilde{\nu}(p_{\nu,k})^{q+\epsilon}} dv_{\nu,k}(v, u - x) \cdot [u]^{\tilde{\nu}(u,v)^{-1}}. \]  

(E.54)

Above, I have introduced
\[ \tilde{G}_{\nu,k}(v, u) = \varphi_{\nu,k} \circ \left((f^{(f)})^{-1}(v, u) \cdot G_{\nu}\left((f^{(f)})^{-1}(v, u), u + x, \tilde{\nu}(v, u) + x\right) \right) \cdot \det \left[D_{(v,u)}(f^{(f)})^{-1}\right] \]
\[ \times \xi(x + \tilde{\nu}(v, u)) \cdot \left[\left[\tilde{\nu}(u,v)\right]^{-1}\right] \]  

(E.55)

and used the shorthand notation
\[ G_{\nu}\left(p, u, v\right) = G_{\nu}^{(f)}\left(p, u, v\right) \quad \text{and} \quad G_{\nu}\left(p, u, v\right) = G_{\nu}^{(f)}\left(p, v, u\right) \]  

(E.56)

as well as
\[ \tilde{\Delta}_{\nu}(v, u) = \Delta_{\nu} \circ (f^{(f)})^{-1}(v, u) \quad \text{and} \quad \tilde{\nu}_{\mu}(v, u) = \tilde{\nu}_{\mu} \circ (f^{(f)})^{-1}(v, u). \]  

(E.57)

One can now conclude, individually for each integral.

- The bound (E.46) along with the smoothness of the integrand allows one to conclude that \( I^{(f)}_{\nu}(x) \) are smooth in \( x \) belonging to some open neighbourhood of 0.

- Regarding to \( I^{(f)}_{\nu}(x) \), one should focus on the contribution of each summand \( k \). There are three cases to consider. If \( \tilde{\nu}(p_{\nu}) + \delta_{\nu,k} + x < 0 \), the associated integral simply vanishes for \( x \) small enough and there is nothing more to do. If \( \tilde{\nu}(p_{\nu}) - \delta_{\nu,k} > 0 \), then properties H1) - H3) of a smooth class function as given in Definition 1.1, the fact that the Jacobian determinant in (E.55) never vanishes and has thus a constant sign, the smoothness of the other building blocks and the lower bound (E.46) relatively to \( \tilde{\nu}_{\mu} \) allow one to apply derivation under the integral theorems so as to conclude that the corresponding integral generates a smooth behaviour in \( x \) for \( |x| \) small enough. Finally, if \( \tilde{\nu}(p_{\nu}) + \delta_{\nu,k} > 0 \) and \( \tilde{\nu}(p_{\nu}) - \delta_{\nu,k} < 0 \), then for \( x \) small enough, the corresponding contribution reduces to
\[ I_{\nu,k} = \int_{V_{\nu,k}} dv \int_{0}^{\tilde{\nu}(p_{\nu}) + \delta_{\nu,k} + x} dv_{\nu,k}(v, u - x) \cdot [u]^{\tilde{\nu}(u,v)^{-1}}. \]  

(E.58)

By virtue of the decomposition for smooth class functions on \( \int_{\nu} \) associated with \( \Delta_{\nu} \) and the parameter \( \tau \), one has the decomposition
\[ \tilde{G}_{\nu,k}(v, u - x) = \tilde{G}_{\nu,k}^{(1)}(v, u - x) \cdot [u]^{\tilde{\nu}(u,v)^{-1}} + \tilde{\Delta}_{\nu,k}(v, u - x) \cdot \tilde{G}_{\nu,k}^{(2)}(v, u - x) \]

(E.59)

with \( \tilde{G}_{\nu,k}^{(1)} \) being smooth and bounded in all of their arguments. This allows one for the rewriting
\[ I_{\nu,k} = \int_{V_{\nu,k}} dv \int_{0}^{\tilde{\nu}(p_{\nu}) + \delta_{\nu,k} + x} dv_{\nu,k}(v, u - x) \cdot [u]^{\tilde{\nu}(u,v)^{-1}} - \partial_{u}\left[\tilde{G}_{\nu,k}^{(2)}(v, u - x) \cdot \left[\left[\tilde{\nu}(u,v)\right]^{-1}\right]\right]_{u=\nu} \]
\[ + \int_{V_{\nu,k}} dv \left[\tilde{G}_{\nu,k}^{(2)}(v, \tilde{\nu}(p_{\nu}) + \delta_{\nu,k}) \cdot \left[\tilde{\nu}(p_{\nu}) + \delta_{\nu,k} + x\right]^{\tilde{\nu}(u,v)^{-1}} - \tilde{G}_{\nu,k}^{(2)}(v, -x) \cdot \left[\left[\tilde{\nu}(u,v)\right]^{-1}\right]\right]_{x=0}. \]  

(E.60)
Here, one should note that the terms corresponding to taking the \( s \to 0 \) limit only appear if the exponent \( \lambda^{(\alpha)}_{\nu} \) is vanishing on a set of positive measure. Due to the mentioned properties of the integrand, one may apply derivation under the integral theorems in the above representation so as to infer that the above integral is smooth in \( x \) small enough.

- **Behaviour of \( I(\beta) \) in the singular case**

This corresponds to case \( b) \) appearing in the statement of the theorem and is more tricky to deal with. Extracting the \( x \to 0 \) asymptotics demands several transformations on the integral \( I(\beta)(x) \). I first start with a preliminary decomposition.

Assume that \((\mathcal{P}_0, \mathcal{E}_0)\) takes the form \((E.21)\) for some \( k_0 \in \text{Int}(\mathcal{I}_1) \). Then, one can recast \( z_\nu(p) \) as:

\[
z_\nu(p) = - \sum_{(r,a) \in M} \xi_r w^{(r)}_{\nu}(p^{(r)}_a; t_r(k_0))
\]

(E.61)

where \( w^{(r)}_{\nu} \) is as in \((E.37)\). Then, owing to the proximity of the velocities of the integration variables

\[
u_1'(p^{(1)}_2), \ldots, \nu_1'(p^{(1)}_n) \quad \text{to} \quad \nu_1'(p^{(r)}_1), \ldots, \nu_1'(p^{(r)}_{n_r}) \quad \text{to} \quad \nu_1'(t_r(p^{(1)}_1)), \quad \text{for } r \in \{2, \ell\},
\]

(E.62)

it is convenient to decompose further \( z_\nu(p) \) as

\[
z_\nu(p) = Z_\nu(p^{(1)}_1, k_0) + \delta z_\nu(p) \quad \text{with} \quad Z_\nu(k_1, k_0) = \mathcal{E}(k_0) - \mathcal{E}(k_1) + \nu \mathcal{P}(k_0) - \mathcal{P}(k_1)
\]

(E.63)

and

\[
\delta z_\nu(p) = - \sum_{(r,a) \in M} \xi_r w^{(r)}_{\nu}(p^{(r)}_a; t_r(p^{(1)}_1)).
\]

(E.64)

The previous estimates ensure that \( \delta z_\nu(p) = O(s_{\nu}) \) uniformly on \( D^{(\beta)}_\eta \), c.f. \((E.41)\). In other words, \( Z_\nu \) grasps the dominant part of \( z_\nu(p) \). By the same arguments as earlier on, one gets that

\[
\nu \partial_1 Z_\nu(k, k_0) = -\nu \cdot \mathcal{P}'(k) \cdot (\nu_1'(k) + \nu \nu) \neq 0
\]

(E.65)

so that \( k \mapsto Z_\nu(k, k_0) \) is strictly monotonous on \( \mathcal{I}_1 \). One can then rely on this property so as to split, by means of an appropriate partition of unity, the integral into one over a domain corresponding to a neighbourhood of the point \( t(k_0) = (t_{1}(k_0), \ldots, t_{\ell}(k_0)) \) with \( t_r(k_0) = (t_r(k_0), \ldots, t_r(k_0)) \in \mathbb{R}^n \) which will generate a non-smooth behaviour in \( x \) and an integral over its complement in \( D^{(\beta)}_\eta \) which will only generate a smooth behaviour. However, the steps for achieving such a decomposition depend on the magnitude of \( |u_1'| \) respectively to \( \nu \): one should distinguish between the two possible situations which can arise due to hypothesis \((E.4)\):

\[
|u_1'(k)| < \nu \quad \text{on } \text{Int}(\mathcal{I}_1) \quad \text{or} \quad |u_1'(k)| > \nu \quad \text{on } \text{Int}(\mathcal{I}_1).
\]

(E.66)

- **\( |u_1'(k)| < \nu \text{ on } \text{Int}(\mathcal{I}_1) \)**

Since, \( \delta z_\nu(p) = O(s_{\nu}) \) and since \( k \mapsto Z_\nu(k, k_0) \) is strictly monotonous, the magnitude and sign on \( z_\nu(p) \) will depend on whether one is close to a zero of \( Z_\nu \) or not.

Let

\[
\sigma = \text{sgn} \left( \mathcal{P}'(k) (u_1'(k) + \nu) \right).
\]

(E.67)
Hypothesis (E.7) ensures that $\sigma$ is constant on $\text{Int}(\mathcal{F})$. Taking $\eta$ small enough, the fact that $Z_\nu$ is strictly monotone and that $\delta_{3\nu}(p) = O(s_\nu)$ both ensure that there exists $\rho_\eta > 0$ such that $\rho_\eta \to 0^+$ when $\eta \to 0^+$, and $\gamma_\eta$ strictly increasing in $\eta$, $\gamma_\eta \to 0^+$, so that

\[
\begin{align*}
\n u\sigma_{3\nu}(p) &< -\rho_\eta \quad \text{if} \quad p_1^{(1)} > k_0 + \gamma_\eta \\
\n u\sigma_{3\nu}(p) &> \rho_\eta \quad \text{if} \quad p_1^{(1)} < k_0 - \gamma_\eta
\end{align*}
\]

provided that $p \in D_{\eta}^{(\beta)}$.

(E.68)

The above ensures that, for $\eta$ small enough and $|x| < \rho_\eta$, $x + 3\nu(p)$ will have opposite signs if $|p_1^{(1)} - k_0| \geq \gamma_\eta$. The presence of the Heaviside step function in the integrand then allows one to reduce the integration domain in $I^{(\beta)}(x)$ leading to

\[ I^{(\beta)}(x) = \int_{\mathcal{D}_{\eta}^{(\text{tot})}} dp \, \mathcal{G}_{\text{tot}}^{(\beta)}(p). \]  

(E.69)

Above, I have introduced

\[ \mathcal{D}_{\eta}^{(\text{sg})} = \{ p \in \mathcal{F} \, : \, |p_1^{(1)} - k_0| < \gamma \quad \text{and} \quad \forall (r, a) \in M, \quad |u'_1(p_1^{(1)}) - u'_1(p_a^{(r)})| < \eta \}. \]  

(E.70)

Finally, let $\varphi^{(\text{sg})}$ be smooth on $\prod_{r=1}^{\ell} \mathbb{R}^{r_a}$ and such that

\[ 0 \leq \varphi^{(\text{sg})} \leq 1, \quad \varphi^{(\text{sg})} = 1 \quad \text{on} \quad \mathcal{D}_{\eta}^{(\text{sg})} \quad \text{and} \quad \varphi^{(\text{sg})} = 0 \quad \text{on} \quad \mathcal{D}_{2\eta}^{(\text{out})}. \]

(E.71)

where

\[ \mathcal{D}_{\eta}^{(\text{out})} = \{ p \in \mathcal{F} \, : \, |p_1^{(1)} - k_0| > \gamma \quad \text{and} \quad \forall (r, a) \in M, \quad |u'_1(p_1^{(1)}) - u'_1(p_a^{(r)})| < \eta \}. \]  

(E.72)

Since, by construction, the integrand vanishes on $\mathcal{D}_{2\eta}^{(\beta)} \setminus \mathcal{D}_{\eta}^{(\beta)}$, one may recast $I^{(\beta)}(x)$ in the form

\[ I^{(\beta)}(x) \equiv I_{\text{sg}}^{(\beta)}(x) = \int_{\mathcal{D}_{2\eta}^{(\text{out})}} dp \, \mathcal{G}_{\text{sg}}(p) \quad \text{with} \quad \mathcal{G}_{\text{sg}}(p) = \varphi^{(\text{sg})}(p) \cdot \mathcal{G}_{\text{tot}}^{(\beta)}(p). \]

(E.73)

- $|u'_1(k)| > |v|$ on $\text{Int}(\mathcal{F})$

Define $\sigma$ as in (E.67). Then $\sigma$ is constant on $\text{Int}(\mathcal{F})$ by hypotheses (E.1) and (E.7). Taking $\eta$ small enough, there exists $\rho_\eta > 0$ and $\gamma_\eta > 0$, a strictly decreasing function of $\eta$, such that $\gamma_\eta \to 0^+$ so that

\[
\begin{align*}
\sigma_{3\nu}(p) &< -\rho_\eta \quad \text{if} \quad p_1^{(1)} > k_0 + \gamma_\eta \\
\sigma_{3\nu}(p) &> \rho_\eta \quad \text{if} \quad p_1^{(1)} < k_0 - \gamma_\eta
\end{align*}
\]

provided that $p \in D_{\eta}^{(\beta)}$.

(E.74)

Taken this into account, it appears convenient to introduce $\varphi^{(\text{sg})}$ as in (E.71).

Then, one gets the decomposition of the integral as $I^{(\beta)}(x) = I_{\text{sg}}^{(\beta)}(x) + I_{\text{out}}^{(\beta)}(x)$ where

\[ I_{\text{sg}}^{(\beta)}(x) = \int_{\mathcal{D}_{\eta}^{(\text{sg})}} dp \, \mathcal{G}_{\text{sg}}(p) \quad \text{and} \quad I_{\text{out}}^{(\beta)}(x) = \int_{\mathcal{D}_{\eta}^{(\text{out})}} dp \, (1 - \varphi^{(\text{sg})}(p)) \mathcal{G}_{\text{tot}}(p). \]

(E.75)
There, \( \mathcal{G}_{\text{g}}(p) \) is as appearing in (E.73) and \( D_{2\eta,2\gamma}(\text{out}) \) have been defined in (E.70) and (E.72).

Due to the bound (E.74), one has that
\[
|3_0(p)| > \rho_{\eta} \quad \text{on} \quad D_{2\eta,2\gamma}^{(\text{out})}.
\]

(E.76)

This lower bound allows one to apply derivation under the integral theorems so as to infer that \( T_{\text{out}}^{(f)}(x) \) is smooth in \( x \) belonging to a sufficiently small neighbourhood of 0.

• **Simplified form of \( T_{\text{sg}}^{(f)}(x) \)**

The fact that the integration domain in \( T_{\text{sg}}^{(f)}(x) \) has been reduced \( D_{2\eta,2\gamma}^{(\text{sg})} \) allows one to implement a change of variables which recasts the integral in a simplified form. Doing so is an important step towards the analysis of its \( x \to 0 \) behaviour.

Observe that given any \( p \in D_{2\eta,2\gamma}^{(\text{sg})} \), for any \( (r, a) \in M \), it holds
\[
\left| u'_{r}(p_{a}^{(r)}) - u'_{r}(t_{r}(k_0)) \right| \leq \left| u'_{1}(p_{a}^{(1)}) - u'_{1}(k_0) \right| + \left| u'_{r}(p_{a}^{(r)}) - u'_{1}(p_{a}^{(1)}) \right| < 2\eta + \gamma_{2\gamma} \cdot \| u'_{1} \|_{L^{\infty}(f_{\gamma \eta})}.
\]

(E.77)

Since \( k_0 \in \text{Int}(f_{\gamma}) \), \( t_{r}(k_0) \in \text{Int}(f_{\gamma}) \) and by (E.77), upon reducing \( \eta > 0 \) if need be, it follows that there exists \( \varepsilon > 0 \) such that, for any \( p \in D_{2\eta,2\gamma}^{(\text{sg})} \),
\[
d(u'_{r}(p_{a}^{(r)}), \partial u'_{r}(f_{\gamma})) > \varepsilon,
\]

(E.78)

for the canonical distance \( d \) between points and subsets of \( \mathbb{R} \). This ensures that there exists a compact \( K_{r} \subset \text{Int}(f_{\gamma}) \) containing an open neighbourhood of \( t_{r}(k_0) \), such that both \( p_{a}^{(r)}, t_{r}(p_{a}^{(1)}) \in K_{r} \) uniformly in \( p \in D_{2\eta,2\gamma}^{(\text{sg})} \). Since \( u'_{r} \) is a smooth diffeomorphism on an open neighbourhood of \( K_{r} \), there exist constants \( c_{r}, C_{r} > 0 \) such that
\[
c_{r} \cdot \left| u'_{r}(x) - u'_{r}(y) \right| \leq |x - y| \leq C_{r} \cdot \left| u'_{r}(x) - u'_{r}(y) \right| \quad \text{for any} \quad x, y \in K_{r}.
\]

(E.79)

Recall that \( k_0 \in \text{Int}(f_{\gamma}) \) so that \( t_{r}(k_0) \in \text{Int}(f_{\gamma}) \). Thus, upon taking \( \eta \) small enough, the strict monotonicity of \( u'_{r} \) on \( f_{\gamma} \) and the above bounds ensure that, for any \( p \in D_{2\eta,2\gamma}^{(\text{sg})} \),
\[
\left| t_{r}(k_0) - p_{a}^{(r)} \right| \leq C_{r} \left[ \left| u'_{r}(p_{a}^{(r)}) - u'_{1}(p_{a}^{(1)}) \right| + \left| u'_{r}(k_0) - u'_{1}(p_{a}^{(1)}) \right| \right] \leq 2C_{r}\eta + \frac{C_{r}}{C_{r}}\gamma_{2\gamma} = \eta_{r}.
\]

(E.80)

Therefore, upon denoting \( B_{\eta}(x_{0}) = \{ x \in \mathbb{R} : |x - x_{0}| < \varepsilon \} \) the open ball in \( \mathbb{R} \) of radius \( \varepsilon \) centred at \( x_{0} \), one gets
\[
D_{2\eta,2\gamma}^{(\text{sg})} \subset B_{\eta_{r}}(k_0) \times \prod_{r=1}^{\ell} \left( B_{\eta_{r}}(t_{r}(k_0)) \right)^{\alpha_{r} - \delta_{r,1}}.
\]

(E.81)

with \( \eta_{r} \) as given in (E.80).

Define auxiliary functions on \( \prod_{r=1}^{\ell} \mathbb{R}^{n_{r}} \)
\[
\xi_{r}(x) = - \sum_{(r,a) \in M} \xi_{r} \left( b_{r}(x_{a}^{(r)} + u_{\gamma}x_{a}^{(r)}) \right) \quad \text{and} \quad \left\{ \begin{aligned}
\xi_{r}(x) &= u_{\gamma}^{(r)}(t_{r}(k_0)) \frac{x^{2}}{2} + u'_{r}(k_0) x \\
\xi_{r}^{(0)}(x) &= \frac{u_{\gamma}^{(r)}(t_{r}(k_0))}{u'_{r}(t_{r}(k_0))} x
\end{aligned} \right.,
\]

(E.82)
along with the domain
\[
\mathcal{D}_\eta^{(\text{eff})} = \left\{ x \in \prod_{r=1}^\ell \mathbb{R}^{n_r} : |x^{(1)}_r| \leq C \eta, \forall (r,a) \in \mathcal{M} : \left| f^{(0)}_{r_{(1)}}(x^{(1)}_r) - x^{(r)}_a \right| \leq \frac{\eta}{|u'_r(t_r(k_0))|} \right\}.
\] (E.83)

By the above discussion, \( \mathcal{D}_\eta^{(\text{eff})} \) is an open neighbourhood of the origin.

Then, by virtue of Proposition F.2, there exists \( x_0 > 0 \) and \( \eta' > 0 \) such that there exists:

- smooth functions \( f_0 \) on \( ] - x_0 : x_0] \cap \mathcal{D}_{\eta'}^{(\text{eff})} \) satisfying \( f_0(x ; x ) = 1 + \mathcal{O}(\|x\| + |x|) \),

- a smooth diffeomorphism \( \Psi_k : \mathcal{D}_{\eta'}^{(\text{eff})} \to \Psi_k(\mathcal{D}_{\eta'}^{(\text{eff})}) \) satisfying \( D_\Psi = \text{id} + xN \) with \( \|N\| \leq C \), for some \( x \)-independent \( C > 0 \),

such that
\[
x + \tilde{z}_0 \circ \Psi_k(x) = f_0(x ; x) \cdot \left( x + \tilde{z}_0(x) \right),
\] (E.84)

and, for \( |x| < x_0 \), \( \Psi_k(\mathcal{D}_{\eta'}^{(\text{eff})}) \subset \mathcal{J}_{\text{tot}} \) contains a \( x \)-independent open neighbourhood of \( t(k_0) \in \mathcal{J}_{\text{tot}} \) with \( t(k_0) \) as given in (E.10). Finally,

\[(x, x) \mapsto \Psi_k(x)
\] (E.85)
is smooth on \( ] - x_0 : x_0] \times \mathcal{D}_{\eta'}^{(\text{eff})} \).

Furthermore, by virtue of Proposition F.3 there exists an invertible linear map \( \mathcal{M} \) on \( \prod_{r=1}^\ell \mathbb{R}^{n_r} \) and integers \( m_\pm \in \mathbb{N} \) satisfying \( m_+ + m_- + 1 = \sum_{r=1}^\ell n_r \), such that
\[
\tilde{z}_0(\mathcal{M}x) + x = P_{\nu}(x) \quad \text{with} \quad x = (y, z^+, z^-) \in \mathbb{R} \times \mathbb{R}^{m_+} \times \mathbb{R}^{m_-}
\] (E.86)

and
\[
P_{\nu}(x) = -\frac{u''(k_0)}{2P'(k_0)}y^2 - (u'(k_0) + \nu v)y + \varphi_{\nu}(z)
\] (E.87)
in which
\[
\varphi_{\nu}(z) = x + \sum_{s=1}^{m_+} (z^{(s)})^2 - \sum_{s=1}^{m_-} (z^{(-s)})^2 \quad \text{with} \quad z = (z^+, z^-).
\] (E.88)

To proceed further, it is convenient to introduce the auxiliary function \( F \) which is defined around the origin through its series expansion
\[
F(x) = 1 + \frac{1}{2v} \sum_{k \geq 1} d_k x^k \frac{[v + u'(k_0)]^{2k+1} - [u'(k_0) - v]^{2k+1}}{[v^2 - (u'(k_0))^2]^{2k}}.
\] (E.89)

The sequence \( d_k \) appearing above can be read-off from its generating series
\[
2^{1 - \sqrt{1 - x}} = 1 + \sum_{k \geq 1} d_k x^k.
\] (E.90)
Then, for $\delta_0 > 0$ and small enough, one sets
\[
\mu(z) = \delta_0 \bar{F}(\phi_1(z)) \quad \text{with} \quad \bar{F}(x) = 2v \cdot \frac{F(-2u'(k_0)x)}{|(u'(k_0))^2 - v^2|}. \tag{E.91}
\]

Observe that for any
\[
z = (z^{(+)}, z^{(-)}) \in [-\delta_+^{1/2}; \delta_+^{1/2}]^{m_+} \times [-\delta_-^{1/2}; \delta_-^{1/2}]^{m_-} \tag{E.92}
\]
it holds $\phi(x) = O(\delta_+ + \delta_- + |x|)$. Thus, provided that $\delta_+ > 0$ and $|x|$ are all taken small enough, $\mu(z)$ is well-defined for such $z$’s and, owing to $F(0) = 1$, it holds
\[
\frac{v_0 \delta_0}{|(u'(k_0))^2 - v^2|} \leq \mu(z) \leq \frac{4v_0 \delta_0}{|(u'(k_0))^2 - v^2|}. \tag{E.93}
\]

Enough has now been introduced so as to allow me to define the domain
\[
\mathcal{D} = \left\{ (y, z) : z = (z^{(+)}, z^{(-)}) \in [-\delta_+^{1/2}; \delta_+^{1/2}]^{m_+} \times [-\delta_-^{1/2}; \delta_-^{1/2}]^{m_-} \quad \text{and} \quad y \in [-\mu(z), \mu(z)] \right\}. \tag{E.94}
\]

Then, the inclusion (E.81) ensures that, given $\delta_0 > 0, \delta_+ > 0$ small enough, and upon diminishing, if necessary, the parameter $\eta > 0$, it holds
\[
\mathcal{D}^{(sg)}_{2\eta, \gamma_0} \subset \Psi_\eta(\mathcal{M} \cdot \mathcal{D}) \subset \Psi_\eta(\mathcal{D}^{(eff)}_{\eta}), \tag{E.95}
\]
where $\mathcal{M}$ is as in (E.80), this uniformly in $|x| < x_0$.

Thus, for such a choice of parameters at play, upon using that the integrand vanishes away from $\mathcal{D}^{(sg)}_{2\eta, \gamma_0}$ one gets that
\[
\mathcal{I}^{(f)}_{sg}(x) = \int_{\mathcal{D}^{(sg)}} dp \mathcal{K}_g(p) \quad \text{with} \quad \Psi_\Gamma = \Psi_x \circ \mathcal{M}. \tag{E.96}
\]

The change of variables
\[
p = \Psi_\Gamma(x) \quad \text{with} \quad x = (y, z^{(+)}, z^{(-)}) \in \mathbb{R} \times \mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \tag{E.97}
\]
recasts the integral in the form
\[
\mathcal{I}^{(f)}_{sg}(x) = \int_{\mathcal{D}} dx \mathcal{F}(x) \prod_{\nu = \pm} \left\{ \Xi(P_{\nu}(x)) \cdot [P_{\nu}(x)]^{b_\nu(x) - 1} \right\} \tag{E.98}
\]
where $b_\nu(x) = \Delta_\nu \circ \Psi_\Gamma(x)$ and
\[
\mathcal{F}(x) = V \circ \Psi_\Gamma(x) \cdot \mathcal{J}(\mathcal{H}_\Gamma(x) - \mathcal{J}_{\nu}(x; \mathcal{M} \cdot x)P_{\nu}(x) - \mathcal{J}_{-}(x; \mathcal{M} \cdot x)P_{-}(x)) \times (\phi_{\nu}(\phi_{\nu}(x)))^{b_\nu(x) - 1}. \tag{E.99}
\]

Also, I remind that $V$ has been defined in (E.17).
• \textbf{Properties of the polynomials }$P_{\nu}(x)$

One may put the integral $I_{y_{0}}^{(\nu)}(x)$ into a canonical form by focalising more on the structure of the polynomial $P_{\nu}(x)$. It is easy to see that, uniformly in $x \in \mathcal{D}$ with $\delta_{\pm}$ small enough and $\mathcal{D}$ as defined through $\text{(E.94)}$, it admits the factorisation

\[ P_{\nu}(x) = -\frac{u''_{\nu}(k_{0})}{2P'(k_{0})} \cdot \left( y - y_{+}^{(\nu)} \right) \cdot \left( y - y_{-}^{(\nu)} \right). \]  

(E.100)

Upon setting $\sigma_{\nu} = \text{sgn}(u'_{\nu}(k_{0}) + \nu v)$, one has that

\[ y_{+}^{(\nu)} = \frac{\varphi_{\nu}(z)}{u'_{\nu}(k_{0}) + \nu v} \cdot \left( -\frac{2u''_{\nu}(k_{0}) \varphi_{\nu}(z)}{P'(k_{0}) [u'_{\nu}(k_{0}) + \nu v]} \right), \quad U(x) = 2 \frac{1 - \sqrt{1 - x}}{x}, \]  

(E.101)

where I remind that $z = (z^{(+)}, z^{(-)})$ and $\varphi_{\nu}(z)$ is as introduced in $\text{(E.88)}$. Also, it holds

\begin{align*}
\varphi_{\nu}(z) &= -\frac{\varphi'(k_{0})}{u'_{\nu}(k_{0})} \left( u'_{\nu}(k_{0}) + \nu v \right) \cdot \left( 1 + \frac{2u''_{\nu}(k_{0}) \varphi_{\nu}(z)}{P'(k_{0}) [u'_{\nu}(k_{0}) + \nu v]} \right) \\
&= -\frac{2\varphi'(k_{0})}{u''_{\nu}(k_{0})} \left( u'_{\nu}(k_{0}) + \nu v \right) \cdot \left( 1 + \frac{2u''_{\nu}(k_{0}) \varphi_{\nu}(z)}{P'(k_{0}) [u'_{\nu}(k_{0}) + \nu v]} \right).
\end{align*}

Note that the expressions $\text{(E.101)}$ and $\text{(E.102)}$ entail that $y_{\pm}^{(\nu)}$ are functions of the variable $z$ only through the combination $\varphi_{\nu}(z)$. In the following, unless it will be necessary, this $z$-dependence of $y_{\pm}^{(\nu)}$ will be kept implicit.

Let

\[ s = \text{sgn}(-\frac{u''_{\nu}(k_{0})}{P'(k_{0})}). \]  

(E.103)

One has that $\sigma_{\nu} s y_{+}^{(\nu)} > \sigma_{\nu} s y_{-}^{(\nu)}$ so that

\[
\begin{array}{c|c|c}
\sigma_{\nu} s < 0 & \sigma_{\nu} s > 0 & \sigma_{\nu} s = 0 \\
\hline
sP_{\nu}(x) < 0 & \{ y_{+}^{(\nu)}; y_{-}^{(\nu)} \} & \{ y_{+}^{(\nu)}; y_{-}^{(\nu)} \} \\
\hline
sP_{\nu}(x) > 0 & \{ y_{+}^{(\nu)}; y_{-}^{(\nu)} \} & \{ y_{+}^{(\nu)}; y_{-}^{(\nu)} \}
\end{array}
\]  

(E.104)

gives the domains, in the $y$ variable and for fixed $z$, of positivity and negativity of the polynomials $P_{\nu}(x)$.

To proceed further, one should distinguish between the two cases where $|u'_{1}(k_{0})| < v$ and $|u'_{1}(k_{0})| > v$, since their treatment slightly differs.

\textbf{Joint positivity interval in the }$|u'_{1}(k_{0})| < v$ \textbf{regime}

If $s > 0$, then by $\text{(E.104)}$, one will have

\begin{align*}
P_{+}(x) > 0 \quad \text{on} \quad &[-\infty; y_{-}^{(+)}, \cup \cup y_{+}^{(+)}, +\infty[ \\
\text{with} \quad &\begin{cases} 
\quad y_{+}^{(+)} > 0 \\
\quad y_{-}^{(+)} = O(\delta_{+} + \delta_{-} + |x|)
\end{cases} \\
\text{and} \quad &\begin{cases} 
\quad y_{+}^{(-)} > 0 \\
\quad y_{-}^{(-)} = O(\delta_{+} + \delta_{-} + |x|)
\end{cases}.
\end{align*}

(E.105)

(E.106)

Thus, both polynomials $P_{\pm}(x)$ will be positive on the union of intervals

\[ [-\infty; y_{-}^{(-)}, \cup \cup y_{+}^{(-)}, \cup \cup y_{+}^{(+)}; +\infty[ \]  

(E.107)
where the central interval is present only if the subsidiary condition \( y_{-}^{(+)} - y_{-}^{(-)} > 0 \) holds. In fact, this is the sole interval that will be included in the \( y \) integration domain present in \( D \) [E.94], viz. \([-\mu(z) ; \mu(z)]\), where \( \mu(z) \) given in [E.91] is fixed upon choosing the \( z \) variables and is small enough as in [E.93].

Then, using the local positivity on this interval of the various building blocks present in the factorisation of the polynomials \( P_\nu \), one gets that, for \( x \in D \),

\[
\prod_{\nu=\pm} \left\{ \Xi(P_\nu(x)) \cdot [P_\nu(x)]^{b_\nu(x)-1} \right\} = \Xi\left( y_{-}^{(+)} - y_{+}^{(-)} \right) \cdot 1_{[y_{-}^{(-)} ; y_{+}^{(+)}]}(y) \\
\quad \times \prod_{\nu=\pm} \left\{ \frac{-\mu''(k_\nu)}{2\mu'(k_\nu)} \cdot (y_\nu^{(0)} - y) \right\}^{b_\nu(x)-1} \cdot \prod_{\nu=\pm} \left\{ \nu(y_\nu^{(0)} - y) \right\}^{b_\nu(x)-1} .
\] (E.108)

\( \oplus \) If \( s < 0 \), then by [E.104], one will have

\[
P_+(x) > 0 \quad \text{on} \quad ]y_+^{(+)} ; y_+^{(-)}[ \quad \text{with} \quad \left\{ \begin{array}{l} y_+^{(+)} < 0 \\ y_+^{(-)} = \bigO(\delta_+ + \delta_- + |x|) \end{array} \right. ,
\] (E.109)

\[
P_-(x) > 0 \quad \text{on} \quad ]y_+^{(-)} ; y_+^{(+)}[ \quad \text{with} \quad \left\{ \begin{array}{l} y_+^{(-)} > 0 \\ y_+^{(+)} = \bigO(\delta_+ + \delta_- + |x|) \end{array} \right. .
\] (E.110)

Thus, both polynomials will be simultaneously positive if any only if \( y_{-}^{(+)} - y_{+}^{(-)} > 0 \) and then the interval of joint positivity is

\[
]y_{-}^{(-)} ; y_{+}^{(+)}[ .
\] (E.111)

Upon using the local positivity, on this interval, of the various building blocks present in the factorisation of the polynomials \( P_\nu \), one gets that the factorisation [E.108] also holds in the present case.

- **Joint positivity interval in the \(|u'_1(k_0)| > v\) regime**

In this regime, one has that

\[
\sigma_\nu = \text{sgn}(u'_1(k_0) + \nu v) = \varsigma ,
\] (E.112)

i.e. \( \sigma_\nu \) does not depend on \( \nu \in \{ \pm 1 \} \).

\( \oplus \) If \( s > 0 \), then by [E.104], one will have

\[
P_\pm (x) > 0 \quad \text{on} \quad ] - \infty ; y_\pm^{(\pm)}[ \cup ] y_\pm^{(\pm)} ; +\infty[ .
\] (E.113)

where, for some \( c > 0 \)

\[
y_+^{(\pm)} > c > 0 \quad , \quad y_+^{(\pm)} = \bigO(\delta_+ + \delta_- + |x|) \quad \text{if} \quad \varsigma = +
\]

\[
y_-^{(\pm)} < -c < 0 \quad , \quad y_-^{(\pm)} = \bigO(\delta_+ + \delta_- + |x|) \quad \text{if} \quad \varsigma = - .
\] (E.114)

Thus, the polynomials \( P_\pm (x) \) will be simultaneously positive on the union of intervals

\[
] - \infty ; \min(y_+^{(\pm)}, y_-^{(\pm)})[ \cup ] \max(y_+^{(\pm)}, y_-^{(\pm)}) ; +\infty[ .
\] (E.115)

Indeed, this is a consequence of the fact that the roots \( y_\varsigma^{(\pm)} \) have both "large" absolute value -in respect to \( \mu(z) \) and this uniformly in \( z \) by virtue of [E.93], whereas the roots \( y_\varsigma^{(\pm)} \) are both close to the origin. Thus, taken that the
function $\mu(z)$ \textcolor{red}{[E.91]} delimiting the $y$ integration domain in $D$ \textcolor{red}{[E.94]} is small enough \textcolor{red}{[E.93]}, for any fixed $z$, the interval $[-\mu(z);\mu(z)]$ defining the $y$-integration in \textcolor{red}{[E.98]} will reduce to
\[
J_c(z) = \mu z \left[ \mu z \right] 
\]  \textcolor{red}{(E.116)}
in which the prefactor $\mu$ indicates the orientation of the interval.

Then, using the local positivity, on this interval, of the various building blocks present in the factorisation of the polynomials $P_\nu$, one gets that
\[
\prod_{\nu=\pm} \left( \Xi(P_\nu(x)) \cdot [P_\nu(x)]^{b_\nu(x)-1} \right) = 1_{J_c(z)}(y) \cdot \prod_{\nu=\pm} \left( \frac{-\nu y^{(\nu)}(k_0)}{2\nu^2(y_\nu^{(\nu)} - y)} \right)^{b_\nu(x)-1} \cdot \prod_{\nu=\pm} \left( \frac{\nu(y_\nu^{(\nu)} - y)}{2\nu^2} \right)^{b_\nu(x)-1} . \tag{E.117}
\]
@ If $s < 0$, then by \textcolor{red}{(E.104)}, one will have
\[
P_\nu(x) > 0 \quad \text{on} \quad [y^{(\nu)}_+, y^{(\nu)}_-] \tag{E.118}
\]
where, for some $c > 0$,
\[
y^{(\nu)}_+ < -c < 0 \quad , \quad y^{(\nu)}_- = O(\delta_+ + \delta_- + |x|) \quad \text{if} \quad \nu = +
y^{(\nu)}_+ > c > 0 \quad , \quad y^{(\nu)}_- = O(\delta_+ + \delta_- + |x|) \quad \text{if} \quad \nu = - . \tag{E.119}
\]
Thus, both polynomials will be simultaneously positive only on the interval
\[
\max(y^{(\nu)}_+, y^{(\nu)}_-) ; \min(y^{(\nu)}_+, y^{(\nu)}_-) \]. \tag{E.120}
\]
Since $\mu(z)$ is taken small enough, \textit{c.f.} \textcolor{red}{(E.93)} and in particular such that $0 < \mu(z) < |y^{(\nu)}_\nu|$, $\nu = \pm$, one will have that the presence of Heaviside functions of $P_\nu(x)$ will, effectively, result in a reduction of the $y$-integration domain $[-\mu(z);\mu(z)]$ in $D$ \textcolor{red}{[E.94]} to the interval $J_c(z)$ already introduced in \textcolor{red}{(E.116)}.

Furthermore, using the local positivity on this interval of the various building blocks present in the factorisation of the polynomials $P_\nu$, the factorisation \textcolor{red}{(E.117)} also holds in the present case.

• Canonical form of $I_{sg}^{(\beta)}(x)$ in the $|y^{(\nu)}_\nu| < \nu$ regime

The factorisation \textcolor{red}{(E.108)} entails that, irrespectively of the value of $s$ introduced in \textcolor{red}{(E.103)}, $I_{sg}^{(\beta)}(x)$ as given in \textcolor{red}{(E.98)} now takes the form
\[
I_{sg}^{(\beta)}(x) = \prod_{\nu=\pm} \left\{ \int_{y^{(\nu)}_-}^{y^{(\nu)}_+} \frac{d^{m_\nu}z^{(\nu)}}{\sqrt{\nu^2}} \right\} \int dy \cdot \Xi(y^{(\nu)}_+ - y^{(\nu)}_-) \cdot \mathcal{F}^{(1)}(x) \cdot \prod_{\nu=\pm} \left( \frac{-\nu y^{(\nu)}(k_0)}{2\nu^2(y^{(\nu)}_\nu - y)} \right)^{b_\nu(x)-1} , \tag{E.121}
\]
where $x$ is parameterised in terms of the integration variables as in \textcolor{red}{(E.86)} and, for short,
\[
\mathcal{F}^{(1)}(x) = \mathcal{F}(x) \cdot \prod_{\nu=\pm} \left( \frac{-\nu y^{(\nu)}(k_0)}{2\nu^2(y^{(\nu)}_\nu - y)} \right)^{b_\nu(x)-1} . \tag{E.122}
\]
Note that the integration domain for the $z^{(\nu)}$ variables is symmetric. Hence, only the totally even part of the integrand in respect to these variables does contribute to the value of $I_{sg}^{(\beta)}(x)$. Hence, one has
\[
I_{sg}^{(\beta)}(x) = \prod_{\nu=\pm} \left\{ \int_{y^{(\nu)}_-}^{y^{(\nu)}_+} \frac{d^{m_\nu}z^{(\nu)}}{\sqrt{\nu^2}} \right\} \int dy \cdot \Xi(y^{(\nu)}_+ - y^{(\nu)}_-) \cdot \left[ \mathcal{F}^{(1)}(x) \cdot \prod_{\nu=\pm} \left( \frac{\nu(y^{(\nu)}_\nu - y)}{2\nu^2} \right)^{b_\nu(x)-1} \right]_{z^{(\nu)}_+ z^{(\nu)}_-} \text{even} , \tag{E.123}
\]
63
where \([ \cdot ]_{\text{even}}\) stands for the totally even part of a function in respect to the mentioned variables

\[
\left[ g(z, \nu) \right]_{\text{even}} = \frac{1}{2^d} \sum_{a=1}^{d} g(\zeta^{(a)}), \nu \quad \text{with} \quad \zeta^{(a)} = (\epsilon_1 z_1, \ldots, \epsilon_d z_d).
\]  
(E.124)

At this stage, one observes that

\[
y^{(+)}_+ - y^{(-)}_+ = \frac{2\nu \varphi(z)}{\nu^2 - (u_1'(k_0))^2} F\left(\frac{-2\nu''(k_0)}{\nu'(k_0)} \varphi(z)\right)
\]  
(E.125)

where \(F\) is as defined in (E.89). Note that the series defining \(F\) is convergent since \(\varphi(z) = O(\delta_+ + \delta_- + |x|)\) and \(\delta_+ , |x|\) are all taken small enough. Furthermore, since \(F(0) = 1\), the estimate on \(\varphi(z)\) ensures that the \(F\)-dependent term in (E.125) will be bounded from below by a strictly positive constant, this throughout the whole integration domain \(D\).

Further, setting,

\[
t = (t_0, z) \quad \text{with} \quad \left\{ \begin{array}{ll} z = (z^{(+)}, z^{(-)}) \in [-\sqrt{\delta_0}, \sqrt{\delta_+}]^m \times [-\sqrt{\delta_-}, \sqrt{\delta_-}]^m \\
t_0 = y^{(-)}_+ + t \cdot [y^{(+)}_+ - y^{(-)}_+] \end{array} \right.
\]  
(E.126)

entails that

\[
P_{\nu}(t) = \varphi(z) \cdot F_{\nu}\left(\varphi(z), t_0\right) \cdot \left\{ \begin{array}{ll} (1 - t) & \nu = + \\
t & \nu = - \end{array} \right.
\]  
(E.127)

with

\[
F_{\nu}\left(\varphi(z), t_0\right) = \frac{-\nu u''(k_0)}{\nu'(k_0)(\nu^2 - (u_1'(k_0))^2)} F\left(\frac{-2\nu''(k_0)}{\nu'(k_0)} \varphi(z)\right) \cdot \left[ y^{(+)}_+ - y^{(-)}_+ - t \cdot [y^{(+)}_+ - y^{(-)}_+] \right],
\]  
(E.128)

and \(t_0\) as in (E.126). Note that \(F_{\nu}\) is indeed a sole function of \(\varphi(z)\) and \(t_0\) since the roots \(y^{(\nu)}_\pm\) only depend on \(\varphi(z)\), c.f. (E.101) and (E.102).

Thus, the change of variables

\[
y = y^{(+)}_+ + t \cdot [y^{(+)}_+ - y^{(-)}_+]
\]  
(E.129)

recasts \(I_{sg}^{(\beta)}(t)\) in the form

\[
I_{sg}^{(\beta)}(t) = \prod_{\nu \in \mathbb{N}} \left\{ \int_{-\sqrt{\delta_0}}^{\sqrt{\delta_0}} d\nu \zeta(\nu) \right\} \int_0^1 \mathrm{d}r \left[ (1 - r)^{b_1(t) - 1} r^{b_1(t) - 1} \mathcal{F}_{(t)} \left( \varphi(z) \right) \cdot \left[ b_1(t)^{b_1(t) + b_1(t) - 1} \right] \right]_{\text{even}}
\]  
(E.130)

where \(t\) is as in (E.126) and

\[
\mathcal{F}_{(t)} = \mathcal{F}(t) \cdot \frac{2\nu}{\nu^2 - (u_1'(k_0))^2} F\left(\frac{-2\nu''(k_0)}{\nu'(k_0)} \varphi(z)\right) \cdot \prod_{\nu \in \mathbb{N}} \left[ F_{\nu}\left(\varphi(z), t_0\right)^{b_1(t) - 1} \right].
\]  
(E.131)

At this stage, one decomposes the integral into domains where a square root change of variables is well defined:

\[
\left[-\sqrt{\delta_0}, \sqrt{\delta_0}\right]^{m_\nu} = \bigcup_{e^{(\nu)}(\epsilon_{\nu})} \left\{ \prod_{a=1}^{m_\nu} \left( \epsilon_a^{(\nu)}(\epsilon_{\nu}) \sqrt{\delta_{\nu}} \right) \right\} \quad \text{with} \quad \epsilon^{(\nu)} = (\epsilon^{(\nu)}_1, \ldots, \epsilon^{(\nu)}_{m_\nu}).
\]  
(E.132)
in which the sign prefactor in front of each interval indicates its orientation. Then, in each of the sets building up the partition, one sets
\[ \zeta^{(t)}_a = \epsilon^{(t)}_a \cdot [\nu^{(t)}_a]^{1/2}, \quad a = 1, \ldots, m \nu. \tag{E.133} \]

This yields
\[ \mathcal{I}_{\mathcal{S}_q}^{(t)}(x) = \sum_{e^{(t)}_a \in [4 \nu]} \mathcal{I}_e \left[ \mathcal{F}^{(3)}, d_+ - 1, d_- - 1 \right] \tag{E.134} \]

where the building block integral is defined as
\[ \mathcal{I}_e \left[ F, A, B \right] = \prod_{i = \pm} \left\{ \int_0^{\delta r} \frac{d^n [u^{(t)}]}{\sqrt{\omega^{(t)}_a}} \int_0^1 dt \left[ F(u^{(t)}), (1 - t)\varphi_\chi(u^{(t)}_a), t\varphi_\chi(u^{(t)}_r) \right] \times (1 - t)^{A(u^{(t)})} \cdot t^{B(u^{(t)})} \cdot \Xi[\varphi_\chi(u^{(t)}_r)] \cdot [\varphi_\chi(u^{(t)}_r)]^{A(u^{(t)}) + B(u^{(t))} + 1} \right\} u^{(t)}_{\text{even}}. \tag{E.135} \]

Above, it is undercurrent
\[ u^{(t)} = (u^{(t)}_0, u^{(t)}_r) \quad \text{with} \quad u^{(t)}_r = (u^{(t)}_r^{(+)}, u^{(t)}_r^{(-)}), \quad u^{(t)}_0 = y^{(+)}(\varphi_\chi(u^{(t)}_r)) + i \cdot \left[ y^{(+)}(\varphi_\chi(u^{(t)}_r)) - y^{(-)}(\varphi_\chi(u^{(t)}_r)) \right] \tag{E.136} \]

and, finally,
\[ u^{(t)}_{\text{even}} = \left( \epsilon^{(t)}_1 [w^{(t)}_1]^{1/2}, \ldots, \epsilon^{(t)}_m [w^{(t)}_m]^{1/2} \right). \tag{E.137} \]

Here, I have made explicit the fact that the functions \( y^{(s)}_\pi \) only depend on the \( u^{(t)}_r \) integration variables through the function \( \varphi_\chi(u^{(t)}_r) \). In fact, after this change of variables, it holds
\[ \varphi_\chi(u^{(t)}_r) = x + m \sum_{a=1}^{m} w^{(+)}_a - m \sum_{a=1}^{m} w^{(-)}_a. \tag{E.138} \]

The integrand appearing in \( \mathcal{I}_{\mathcal{S}_q}^{(t)} \) takes the form
\[ \mathcal{F}^{(3)}(u^{(t)}, (1 - t)\varphi_\chi(u^{(t)}_a), t\varphi_\chi(u^{(t)}_r)) = \mathcal{F}(\Psi^{(t)}(u^{(t)}), (1 - t)\cdot \tilde{r}^{(t)}_+(u^{(t)}), t\cdot \tilde{r}^{(t)}_-(u^{(t)})) \times \frac{2^{2 - 2\nu} \cdot \nu}{\sqrt{2} - (u^{(t)}_r)^2} \mathcal{F} \left( \frac{u^{(t)}_r}{\sqrt{2}}, \varphi_\chi(u^{(t)}_r) \right) \cdot \left( V_{\varphi^{(t)}_\chi}(\varphi^{(t)}_\chi) \right) \Psi^{(t)}(u^{(t)}) \times \det D^{(t)} \cdot \prod_{i = \pm} \tilde{r}^{(t)}(u^{(t)}_r)^{b_0(u^{(t)}) - 1}, \tag{E.139} \]

where
\[ \tilde{r}^{(t)}(u^{(t)}_r) = \tilde{r}^{(t)}(x; M \cdot u^{(t)}_r) \cdot F_\nu(\varphi_\chi(u^{(t)}_r), u^{(t)}_r), \tag{E.140} \]

and I remind that \( \tilde{\Pi} = \sum_{r=1}^{\ell} n_r. \)
Here, one observes that
\[ b_\nu(t) = \Delta_\nu(t(k_0)) + O\left(\sqrt{\delta_+} + \sqrt{\delta_-} + |x|\right). \] (E.141)

By hypothesis one has \( \Delta_\nu(t(k_0)) > 0 \) so that reducing \( \delta_+ \) and \( |x| \) if need be, one gets that \( b_\nu > 0 \) throughout the integration domain.

The expansion (E.134) of \( J_{\text{sg}}^{(\ell)}(x) \) decomposes this integral into a sum of elementary integrals (E.135) whose \( \varphi \to 0 \) asymptotic behaviour is analysed in Lemma F.5. Also, the \( L^1 \)-nature of the integrand is part of the conclusions of that lemma. Moreover, upon invoking Lemma F.4 so as to access to the small \( |\mathbf{u}(\epsilon)| \) expansion of \( \mathscr{J}(3) \), Lemma F.5, specialised to the function \( \mathscr{J}(3) \), ensures that there exists a smooth function \( \varphi \to \mathcal{R}_\epsilon(x) \) around \( x = 0 \) such that
\[ J_{\text{eff}}[\mathscr{J}^{(3)}, b_\nu - 1, b_- - 1] = \Delta_\nu(0) \mathcal{G}^{(1)}(t(k_0)) J_{\text{eff}}[\mathcal{R}_\epsilon(0), \Delta_\nu(0), \Delta_-(0)] + O(|x|^{\theta+1-r}) + \mathcal{R}_\epsilon(x) \] (E.142)
where \( \Delta_\nu(0) = \Delta_\nu(t(k_0)), \theta \) is as defined in (E.20), \( \mathcal{G}^{(1)} \) corresponds to the first term of the expansion of \( \mathcal{J} \) as given in (E.6), and
\[ \mathcal{G}^{(1)} = \exp\left\{-\left(\mathbf{M}(\mathbf{u}(\epsilon), \mathbf{D} \cdot \mathbf{M}(\mathbf{u}(\epsilon))\right)\right\} \cdot \left(\mathbf{D}(\mathbf{u}(\epsilon), \mathbf{u}(\epsilon))\right) \cdot \prod_{i=\pm} \left(\frac{2-\pi_v \gamma}{\nu^2 - (\mathbf{u}_i(t(k_0)))^2}\right) \cdot \prod_{i=\pm} \left(\frac{2-\pi_v \gamma}{\nu^2 - (\mathbf{u}_i(t(k_0)))^2}\right) \right\} \] (E.143)

Above, \( \mathbf{M} \) is as introduced in (E.86) while the diagonal matrix \( \mathbf{D} \) defining the Gaussian weight reads
\[ \mathbf{D} = \begin{pmatrix} |u_1'(t(k_0))|I_{|n_1|} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & |u_\nu'(t(k_0))|I_{|n_\nu|} \end{pmatrix} \] (E.144)

In particular, \( \mathcal{G}^{(1)} \) is \( x,y \) independent. Furthermore, \( \varphi^{(\text{sg})} \) and \( \varphi^{(\text{sg})} \) are as appearing in (G.12) of Proposition G.1. Namely, they are smooth on \( \prod_{r=1}^{\nu} \mathbb{R}^{n_r} \) and such that
\[ 0 \leq \varphi^{(\text{sg})} \leq 1 \quad \varphi^{(\text{sg})} = 1 \text{ on } \mathcal{D}^{(\ell)}_{q'} \quad \varphi^{(\text{sg})} = 0 \text{ on } \prod_{r=1}^{\nu} \mathbb{R}^{n_r} \setminus \mathcal{D}^{(\ell)}_{q'} \]
\[ 0 \leq \varphi^{(\text{sg})} \leq 1 \quad \varphi^{(\text{sg})} = 1 \text{ on } \mathcal{D}^{(\ell)}_{q} \quad \varphi^{(\text{sg})} = 0 \text{ on } \prod_{r=1}^{\nu} \mathbb{R}^{n_r} \setminus \mathcal{D}^{(\ell)}_{q} \] (E.145)

The domain \( \mathcal{D}^{(\ell)}_{q'} \), resp. \( \mathcal{D}^{(\ell)}_{q' \alpha} \), is as defined in (G.9), resp. (G.13).

Thus, by performing backwards, on the level of \( J_{\text{eff}}[\mathcal{R}_\epsilon(0), \Delta_\nu(0), \Delta_-(0)] \), all the transformations that were carried out starting from (E.98), one gets that there exists a smooth function \( \varphi \to \mathcal{R}(x) \) around \( x = 0 \) such that
\[ J_{\text{sg}}^{(\ell)}(x) = \Delta_\nu(0) \mathcal{G}^{(1)}(t(k_0)) \cdot J_{\text{sg}^{\text{eff}}}(x) + O(|x|^{\theta+1-r}) + \mathcal{R}(x) \] (E.146)
in which
\[ J_{\text{sg}^{\text{eff}}}(x) = \int_{\mathcal{D}^{(\ell)}_{q'}} d^n x \ e^{-\langle x, \mathbf{D} x \rangle} \cdot (V \varphi^{(\text{sg})} \varphi^{(\text{sg})})(x) \cdot \prod_{i=\pm} \left(\Xi[x + \tilde{z}_\nu(x)] \cdot [x + \tilde{z}_\nu(x)]^{\Delta_\nu(0)-1} \right) \] (E.147)
Now, by virtue of Proposition [G.1] there exists a smooth function \( \tilde{R}(x) \) such that
\[
I_{\text{sg, eff}}^{(\beta)}(x) = \tilde{R}(x) + \int \frac{1}{\prod_{r=1}^{n} |u_r''(t_r(k_0))|^{b_{\rho,v}}_{\eta}} \prod_{r=1}^{n} \left\{ \Xi[x + z_r(x)] \cdot [x + z_r(x)]^{\Delta(0)-1} \right\} \cdot e^{-\langle \alpha, x \rangle} \cdot V(x) \, dx
\]  
(\text{E.148})
in which \( z_r \) is as defined in (G.4) where the below identification of parameters has been made
\[
\varepsilon_r = -\xi_r \text{sgn}(u_r''(t_r(k_0))) \quad \varepsilon_r = |u_r''(t_r(k_0))|^{-\frac{1}{2}} \quad u = u_r'(k_0).
\]
(\text{E.149})

By tracking the previous transformations backwards, one gets that there exists a smooth function \( \tilde{R} \) in the vicinity of the origin
\[
I(x) = \int \frac{1}{\prod_{r=1}^{n} |u_r''(t_r(k_0))|^{b_{\rho,v}}_{\eta}} \prod_{r=1}^{n} \left\{ \Xi[x + z_r(x)] \cdot [x + z_r(x)]^{\Delta(0)-1} \right\} \cdot e^{-\langle \alpha, x \rangle} \cdot V(x) \, dx
\]  
(\text{E.150})

Then, it remains to apply Proposition [G.2] so as to get the form of \( x \to 0 \) asymptotic expansion of the multiple integral appearing above, what yields the claimed form of \( I(x) \) in the regime \( |u_r'(k_0)| < v \).

- **Canonical form of \( I_{\text{sg}}^{(\beta)}(x) \) in the \( |u_r'(k_0)| > v \) regime**

The factorised form of the products of polynomials \( P_r(x) \) given in (E.117) entails that \( I_{\text{sg}}^{(\beta)}(x) \) takes the form
\[
I_{\text{sg}}^{(\beta)}(x) = \prod_{r=1}^{n} \left\{ \int_{-\sqrt{b_{\rho,v}}}^{\sqrt{b_{\rho,v}}} d\xi \mathcal{F}^{(1)}(x) \cdot \prod_{r=1}^{n} \left\{ \varsigma(y^{(r)}_{-\xi}) - y \right\}^{b_{\rho,v}(x)-1} \right\}
\]
(\text{E.151})

where \( \varsigma \) has been introduced, \( J_{\varsigma}(z) \) in (E.116), the argument \( x \) is expressed in terms of the integration variables \( y, z^{(k)} \) as in (E.86), and, for short, I agree upon
\[
\mathcal{F}^{(1)}(x) = \mathcal{F}(x) \cdot \prod_{r=1}^{n} \left\{ \frac{-s_{y^{(r)}_{+\varsigma}}(z_{+\varsigma}^{(r)})}{2y^{(r)}_{+\varsigma}} \right\}^{b_{\rho,v}(x)-1}.
\]
(\text{E.152})

As earlier on, the symmetry of the \( (z^{(+), z^{(-)})} \) integration domain entails that
\[
I_{\text{sg}}^{(\beta)}(x) = \prod_{r=1}^{n} \left\{ \int_{-\sqrt{b_{\rho,v}}}^{\sqrt{b_{\rho,v}}} d\xi \mathcal{F}^{(1)}(x) \cdot \prod_{r=1}^{n} \left\{ \varsigma(y^{(r)}_{-\varsigma}) - y \right\}^{b_{\rho,v}(x)-1} \right\}_{z-\text{even}}
\]
(\text{E.153})

where the \( z \)-even part of a function is as defined in (A.9).

At this stage, one observes that
\[
\varsigma_{y^{(r)}_{+\varsigma}} - \varsigma_{y^{(r)}_{-\varsigma}} = -\varsigma_{\phi_{\varsigma}(z)} \cdot \tilde{F}(\phi_{\varsigma}(z)) \quad \text{with} \quad \tilde{F}(x) = 2\sqrt{\frac{F(-2s_{y^{(r)}_{+\varsigma}}(z_{+\varsigma}))}{(u_r'(k_0))^2 - v^2}}
\]
(\text{E.154})
and where \( F \) is as defined in (E.89). Just as earlier on, one has that, uniformly on \( \mathcal{D} \), it holds \( \tilde{F}(\psi(z)) > 0 \) so that
\[
p = \operatorname{sgn}(\psi_{-c}^{(+) -} - \psi_{-c}^{(-)}) = -c \operatorname{sgn}(\psi(z)) ,
\]
meaning that
\[
a_p = \psi_{-c}^{(p)} - \psi_{-c}^{(-p)} = |\psi(z)| \cdot \tilde{F}(\psi(z)) \geq 0 .
\]
The change of variables
\[
y = b_p(\psi(z)) - c t \tilde{F}(\psi(z)) \quad \text{with} \quad b_p(\psi(z)) = c \min(\psi_{-c}^{(p)}, \psi_{-c}^{(-p)}) = y_{-c}^{(-p)},
\]
in the \( y \)-integration recasts \( I^{(\ell)}_{x_g} (x) \) in the form
\[
I^{(\ell)}_{x_g} (x) = \sum_{\nu = \pm} I^{(\ell)}_{x_g, \nu} (x)
\]
where the two building blocks read
\[
I^{(\ell)}_{x_g, \nu} (x) = \prod_{\nu = \pm} \left\{ \int_{-\nu \delta}^{\nu \delta} d^{m_{x}} \zeta^{(\nu)}(t) \right\} \int_{0}^{\delta_{x}} dt \left[ t^{1 - \beta(t) - 1} \left[ t + |\psi(z)| \right]^{b_{\nu}(t) - 1} \cdot \Xi[\psi(z), \mathcal{F}(t)] \right]_{x-\text{even}} .
\]
There,
\[
t = (t_0, z) , \quad z = (z^{(+)}, z^{(-)}) \in \mathbb{R}^{m_+} \times \mathbb{R}^{\nu_m} , \quad t_0 = b_p(\psi(z)) - c t \tilde{F}(\psi(z)) .
\]
Also,
\[
\mathcal{F}(t) = \mathcal{F}(t) \cdot \tilde{F}(\psi(z)) \prod_{\nu = \pm} \left[ \tilde{F}_\nu(\psi(z), t) \right]^{b_{\nu}(t) - 1}
\]
and
\[
\tilde{F}_\nu(\psi(z), t) = -\frac{\nu^{(t)}(t_0)}{2^b \nu(t_0)} \tilde{F}(\psi(z)) \cdot \left( y_{-c}^{(\nu)} + t c \tilde{F}(\psi(z)) - b_p \right).
\]
Observe that it holds
\[
\eta\psi(z) = \eta x + \sum_{\nu = \pm} m_{\nu} \sum_{\nu = \pm} \left( \zeta^{(\nu)}(z) \right) \cdot \zeta^{(\nu)}(z)
\]
Thus, denoting
\[
\tilde{m}_\nu = m_{\nu} , \quad \text{and} \quad \tilde{\delta}_\nu = \delta_{\nu} ,
\]
the change of variables \( \zeta^{(\nu)}(z) \leftarrow \zeta^{(\nu)}(z) \) leads to
\[
I^{(\ell)}_{x_g, \nu} (x) = \prod_{\nu = \pm} \left\{ \int_{-\nu \delta}^{\nu \delta} d^{m_{x}} \zeta^{(\nu)}(z) \right\} \int_{0}^{\delta_{x}} dt \left[ t^{1 - \beta(t) - 1} \left[ t + \tilde{F}_\nu(\psi(z)) \right]^{b_{\nu}(t) - 1} \cdot \Xi[\tilde{F}(\psi(z), \mathcal{F}(t)] \right]_{x-\text{even}} .
\]
There
\[ \hat{t} = (t_0, z), \quad z = (z^{(0)}, z^{(-0)}) \in \mathbb{R}^{\tilde{m}_+} \times \mathbb{R}^{\tilde{m}_-}, \quad t_0 = b_p(\hat{\varphi}_x(z)) - \varsigma t \hat{F}(\hat{\varphi}_x(z)), \] (E.166)

and
\[ \hat{\varphi}_x(z) = \eta x + \sum_{j=1}^{\tilde{m}_+} (c^{(e)}_j)^2 - \sum_{j=1}^{\tilde{m}_-} (c^{(-e)}_j)^2. \] (E.167)

At this stage, one decomposes the integral into domains where a square root change of variables is well defined:
\[ \left[-[\delta_0]^2; [\delta_0]^2\right]^{\tilde{m}_u} = \bigcup_{\epsilon^{(e)} \in \{\pm 1\}^{\tilde{m}_u}} \prod_{a=1}^{\tilde{m}_u} \left\{ \ell_{\epsilon^{(e)}}^{(0)}(0; \epsilon_{a}^{(e)} \cdot [\hat{\delta}_a]^2) \right\} \quad \text{with} \quad \epsilon^{(e)} = (\epsilon_1^{(e)}, \ldots, \epsilon_{\tilde{m}_u}^{(e)}) \] (E.168)

in which the sign pre-factor indicates the orientation of the interval. Then, in each of the sets building up the partition, one sets
\[ z_{a}^{(e)} = \epsilon_{a}^{(e)} \cdot [w_{a}^{(e)}]^2 \quad a = 1, \ldots, \tilde{m}_u. \] (E.169)

This yields
\[ I_{\text{sg};0}^{(e)}(x) = \sum_{\epsilon^{(e)} \in \{\pm 1\}^{\tilde{m}_u}} \chi_{\epsilon^{(e)}} \left[ \mathcal{F}^{(3)}, \mathcal{d}_{-p} - 1, \mathcal{d}_p - 1 \right] \] (E.170)

where the building block integral is defined as
\[ \chi_{\epsilon^{(e)}}[\mathcal{F}, A, B] = \prod_{a=1}^{\tilde{m}_u} \left\{ \int_{0}^{\delta_a} \frac{\delta_{a}^{\tilde{m}_u} w_{a}^{(e)}}{\sqrt{w_{a}^{(0)}}} \right\} \int_{0}^{\delta_{a}} \left[ \mathcal{F}(u^{(e)}, t, t + \hat{\varphi}_x(u_{w}^{(e)})) \right. \]
\[ \times \left. [t + \hat{\varphi}_x(u_{w}^{(e)})]^{A(u_{a}^{(e)})} \cdot I^{B(u_{a}^{(e)})} \cdot \mathcal{E}\left[\hat{\varphi}_x(u_{w}^{(e)})\right] \right]^{u_{a}^{(0)-even}}. \] (E.171)

Above, it is undercurrent
\[ u^{(e)} = (u_0^{(e)}, u_{w}^{(e)}) \quad \text{with} \quad u_{w}^{(e)} = (u_{w}^{(0)}, u_{w}^{(-0)}), \quad u_0^{(e)} = b_p(\hat{\varphi}_x(u_{w}^{(e)})) - \varsigma t \hat{F}(\hat{\varphi}_x(u_{w}^{(e)})), \] (E.172)

and, finally,
\[ u_{w}^{(e)} = (\epsilon_1^{(e)} [w_1^{(e)}]^2, \ldots, \epsilon_{\tilde{m}_u}^{(e)} [w_{\tilde{m}_u}^{(e)}]^2). \] (E.173)

Also, after the change of variables, one has that
\[ \hat{\varphi}_x(u_{w}^{(e)}) = \eta x + \sum_{a=1}^{\tilde{m}_u} w_{a}^{(e)} - \sum_{a=1}^{\tilde{m}_u} w_{a}^{(-e)}. \] (E.174)
The integrand appearing in (E.170) takes the form

\[ \mathcal{G}^{(3)}(u^{(\epsilon)}, t, t + \varphi_s(u^{(\epsilon)})) = 2^{1 - \eta} \cdot \tilde{F}(\varphi_s(u^{(\epsilon)})) \cdot \left( \psi^{(f)}(\varphi^{(sg)}(\Psi R(u^{(\epsilon)}))) \cdot |\det D_{u^{(\epsilon)}}\Psi R| \right) \]

\[ \times \prod_{l=\pm} \left\{ \mathcal{G}(\Psi R(u^{(\epsilon)}), t, \varphi_s(u^{(\epsilon)}), [t] + \varphi_s(u^{(\epsilon)}) \cdot [t + \varphi_s(u^{(\epsilon)})]) \right\} \]

\[ \text{if } \nu = +, \]

\[ \times \prod_{l=\pm} \left\{ \mathcal{G}(\Psi R(u^{(\epsilon)}), t, \varphi_s(u^{(\epsilon)}), [t + \varphi_s(u^{(\epsilon)})]) \right\} \]

\[ \text{if } \nu = -, \]

(E.175)

where

\[ \tilde{f}_a(u^{(\epsilon)}, u_0^{(\epsilon)}) = \tilde{f}_a(\Psi R^{(3)}(u^{(\epsilon)}), u_0^{(\epsilon)}) \cdot \widetilde{F}(\varphi_s(u^{(\epsilon)}), u_0^{(\epsilon)}). \]

(E.176)

and I remind that \( \overline{\mathcal{M}} = \sum_{r=1}^{L} R_r. \)

As in the previous case, one gets that \( \mathcal{M} \) > 0 throughout the integration domain. The expansion (E.170) of \( \mathcal{I}^{(3)}(x) \) decomposes this integral into a sum of elementary integrals (E.171) whose \( x \to 0 \) asymptotic behaviour is analysed in Lemma F.6. The conclusions of this lemma, specialised to the function \( \mathcal{G}^{(3)} \), ensure that the integrand is in \( L^1 \) and that there exists a smooth function \( x \to R(x) \) around \( x = 0 \) such that

\[ \chi_{e_0} \left[ \mathcal{G}^{(3)}(\mathcal{D} - 1, \mathcal{D} - 1) \right] = \Delta^{(0)}(t(k_0)) \cdot \chi_{e_0} \left[ \mathcal{G}^{(3)}(\mathcal{D} - 1, \mathcal{D} - 1) \right] + O(|x|^{\theta - 1}) + R(x) \]

(E.177)

where \( \Delta^{(0)} = \Delta_0(t(k_0)), \theta \) is as defined in (E.20) and

\[ \mathcal{G}(\mathcal{D} - 1, \mathcal{D} - 1) = \Delta^{(0)}(t(k_0)), \]

\[ \text{if } \nu = +, \]

\[ \times \prod_{l=\pm} \left\{ \tilde{f}_a(u^{(\epsilon)}, u_0^{(\epsilon)}) \right\}^{R_r - 1}. \]

(E.178)

Above, \( \mathcal{M} \) is as introduced in (E.86) and \( \mathcal{D} \) as in (E.144). Finally, \( \psi^{(e)} \) and \( \psi^{(sg)} \) are as appearing in (G.12), c.f. also (E.145). By performing backwards, on the level of \( \chi_{e_0} \left[ \mathcal{G}^{(3)}(\mathcal{D} - 1, \mathcal{D} - 1) \right] \), all the transformations that were carried out starting from (E.98), one arrives to the conclusions stated in (E.146). From there, one concludes as in the regime \( |u^{(\epsilon)}(t)| < \nu \).

\[ \text{F} \]

Auxiliary results

F.1 A regularity lemma

Given \( x \in \mathbb{R}^n \) with \( n \geq 2 \) and integers \( 1 \leq a < b \leq n \) denote

\[ x_{a,b} = (x_1, \ldots, x_{a-1}, x_{a+1}, \ldots, x_{b-1}, x_{b+1}, \ldots, x_n). \]

(F.1)

Lemma F.1. Let \( K \) be a compact subset of \( \mathbb{R}^n, n \geq 2, \) and let \( z_k, \psi_k \) be smooth functions on \( K. \) Let \( 0 < \tau < 1 \) and let \( \mathcal{G} \) be in the smooth class of \( K \) with functions \( \Delta_k = \left[ \psi_k \right]^2 \) and constant \( \tau, \) c.f. Definition B.4.

Assume that for any \( x \in K \) there exists \( a < b, a, b \in \{1, \ldots, n\}, \) such that the differential \( D_x f_a,b \) of the map

\[ f_{a,b} : x \mapsto (x_{a,b}, z_k(x), \tilde{z}^-(x)), \]

(F.2)
with \( x_{a,b} \) as in (F.1), is invertible. Then, the integral

\[
\mathcal{J}(x) = \int_{\mathbb{R}^d} d^d x \, \mathcal{G}(x, \tilde{\xi}_+(x), \tilde{\xi}_-(x)) \cdot \prod_{\nu = \pm} \left\{ \Xi \left[ \tilde{\xi}_+(x) \right] \cdot \left[ \tilde{\xi}_-(x) \right]^{\Delta_v(x)-1} \right\} \quad \text{with} \quad \tilde{\xi}_r(x) = 3_0(x) + x,
\]

is a smooth function of \( x \), provided that \(|x|\) is small enough.

**Proof.** By virtue of the Whitney extension theorem, \( \tilde{\xi}_\pm \) and \( \psi_\pm \) admit smooth extensions to an open neighbourhood \( U_K \) of \( K \). Thus, so does \( \Delta_v = \psi^2 \) and one obviously has that \( \Delta_v(U_K) \subset [0; +\infty] \). One may also extend \( \mathcal{G} \) to \( U_K \times \mathbb{R}^+ \times \mathbb{R}^+ \) smoothly by setting

\[
\mathcal{G}|_{(U_K \setminus K) \times \mathbb{R}^+ \times \mathbb{R}^+} = 0
\]

where the smoothness of this extension is ensured by the smooth vanishing, to all orders in the derivatives, of \( \mathcal{G} \) on \( \partial K \times \mathbb{R}^+ \times \mathbb{R}^+ \).

It follows from the hypothesis of the lemma that, for any \( x \in K \), there exists integers \( a_x < b_x \), an open, relatively compact, neighbourhood \( U_x \) of \( x \), an open, relatively compact, neighbourhood \( V_x \) of \( x_{a_x,b_x} \) in \( \mathbb{R}^{n-2} \) and \( \eta_x > 0 \) such that

\[
f_{a_x,b_x} : U_x \to f_{a_x,b_x}(U_x) = V_x \times [3_+(x) - \eta_x; 3_+(x) + \eta_x] \times [3_-(x) - \eta_x; 3_-(x) + \eta_x]
\]

is a smooth diffeomorphism onto its image. Furthermore, reducing \( \eta_x \) if necessary, one may always assume that \( 3_v(x) \pm \eta_x \neq 0 \) for both values of \( v \in \{\pm\} \). Finally, the sets can always be chosen such that \( f_{a_x,b_x}^{-1} \) has a smooth extension to an open neighbourhood of \( f_{a_x,b_x}(U_x) \) and such that \( U_x \subset U_K \).

Then, \( \bigcup_{x \in K} U_x \subset U_K \) is an open cover of \( K \) and hence there exists a finite sub-cover \( \bigcup_{k=1}^f U_{x_k} \subset U_K \) with associated diffeomorphisms \( f_{a_{x_k},b_{x_k}} \) mapping \( U_{x_k} \) onto \( f_{a_{x_k},b_{x_k}}(U_{x_k}) \). Let \( \{\varphi_k\}_{k=1}^f \) be the partition of unity subordinate to the cover \( \bigcup_{k=1}^f U_{x_k} \). Then, using that the integrand extends by zero outside of \( K \), one has

\[
\mathcal{J}(x) = \int_{U_K} d^n x \, \mathcal{G}(x, \tilde{\xi}_+(x), \tilde{\xi}_-(x)) \cdot \prod_{\nu = \pm} \left\{ \Xi \left[ \tilde{\xi}_+(x) \right] \cdot \left[ \tilde{\xi}_-(x) \right]^{\Delta_v(x)-1} \right\}
\]

what allows one to decompose the integral as \( \mathcal{J}(x) = \sum_{k=1}^f \mathcal{J}_k(x) \) where

\[
\mathcal{J}_k(x) = \int_{U_{x_k}} d^{n-2} \omega \cdot \prod_{\nu = \pm} \left\{ \int_{x + b_{x_k}(x_k) - \eta_{x_k}}^{x + b_{x_k}(x_k) + \eta_{x_k}} d_3 r \right\} \cdot \mathcal{G}_k(\omega) \cdot \prod_{\nu = \pm} \left\{ \Xi \left[ \tilde{\xi}_+(\omega) \right] \cdot \left[ \tilde{\xi}_-(\omega) \right]^{\Delta_v(\omega)-1} \right\}
\]

with \( \omega = (\omega, \tilde{\xi}_+ - x, \tilde{\xi}_- - x) \) and

\[
\mathcal{G}_k(\omega) = \det[D_{a_{x_k}} f_{a_{x_k},b_{x_k}}^{-1}(\omega)] \cdot \det[D_{a_{x_k}} f_{a_{x_k},b_{x_k}}^{-1}(\omega)] \cdot \mathcal{G}(f_{a_{x_k},b_{x_k}}^{-1}(\omega), \tilde{\xi}_+, \tilde{\xi}_-).
\]

Finally, \( \Delta_v(\omega) = \Delta_v \circ f_{a_{x_k},b_{x_k}}^{-1}(\omega) \). This representation is obtained, by restricting the integration domain to \( U_{x_1} \) due to the presence of \( \varphi_k \) followed by making the change of variables \( f_{a_{x_k},b_{x_k}}^{-1}(\omega) = x \). Finally, one shifts the last two integration variables by \(-x\). Note that \( \mathcal{G}_k \) is smooth since the determinant never vanishes and has thus constant sign.

There are four cases to distinguish depending on whether \( 0 \in [3_0(x_k) - \eta_{x_k}; x + 3_0(x_k) + \eta_{x_k}] \) or not.
If $\eta_{x_k} < 0$ for at least one $\nu \in \{\pm\}$, then $F_k(x)$ vanishes for $|x|$ small enough.

If $\eta_{x_k} > 0$ for $\nu = \pm$, then, for $|x|$ small enough, the integral reduces to

$$F_k(x) = \int_{V_{x_k}} d^{n-2}w \cdot \prod_{\nu = \pm} \left\{ \int_{x + h_{x_k} + \eta_{x_k}} d\nu \cdot \tilde{G}_{\nu}^{(b)}(u_\nu) \cdot \prod_{\nu = \pm} \left[ \tilde{\delta}_{\nu}(u_\nu) \right]^{1 - r} \right\}. \quad (F.9)$$

By construction, the endpoints of integration in the $3_\nu$ variables are uniformly away from 0, for $|x|$ small enough. The hypotheses of the lemma ensure that $\tilde{G}_{\nu}(u_\nu)$ is smooth in $x$ small enough and in $(w, 3_+, 3_-)$ provided that $u_\nu \in f_{3_\nu} \cup b_\nu (U_{x_k})$. Taken that the only singularities of the integrand, which are at $3_\nu = 0$, are uniformly away from the integration domain and taken that the integral runs through a compact set, derivation under the integral -and in respect to endpoints of integration- theorems entail that $F_k(x)$ is a smooth function of $x$, for $|x|$ small enough.

If $\eta_{x_k} < 0$ and $\eta_{x_k} > 0$ for both values of $\nu$, then the integral splits as $F_k(x) = \sum_{b=1}^{4} F_k^{(b)}(x)$ where

$$F_k^{(b)}(x) = \int_{V_{x_k}} d^{n-2}w \cdot \prod_{\nu = \pm} \left\{ \int_{0}^{x + h_{x_k} + \eta_{x_k}} d\nu \cdot \tilde{G}_{\nu}^{(b)}(u_\nu) \cdot \prod_{\nu = \pm} \left[ \tilde{\delta}_{\nu}(u_\nu) \right]^{1 - r} \right\} \cdot \begin{cases} \tilde{\Delta}_{+}(u_\nu) \tilde{\Delta}_{-}(u_\nu) & b = 1 \\ \tilde{\Delta}_{+}(u_\nu) \cdot [3_+]^{1 - r} & b = 2 \\ \tilde{\Delta}_{-}(u_\nu) \cdot [3_-]^{1 - r} & b = 3 \\ \prod_{\nu = \pm} \left[ 3_{\nu} \right]^{1 - r} & b = 4 \end{cases}$$

The function $\tilde{G}_{\nu}^{(b)}$ are obtained from $\tilde{G}_{\nu}$ defined in (F.8) upon substituting $\mu \leftrightarrow \mu^{(b)}$ with $\mu^{(b)}$ arising in the expansion (1.6).

Taken that $\mu^{(4)}$ fulfills property $H3$ stated below of (E.13) and that $\gamma \mapsto \gamma^{\delta - r}$ is integrable on $[0; \epsilon]$ for any $\delta \geq 0$, one readily concludes that $F_k^{(4)}(x)$ produces a smooth function of $x$. The analysis of the remaining integrals demands one more step which I detail for $F_k^{(2)}(x)$, the other cases being tractable in a similar way. In the case of $F_k^{(2)}(x)$, an integration by parts yields:

$$F_k^{(2)}(x) = \int_{V_{x_k}} d^{n-2}w \cdot \prod_{\nu = \pm} \left\{ \int_{0}^{x + h_{x_k} + \eta_{x_k}} d\nu \cdot \tilde{\Delta}_{+}(u_\nu) \cdot \tilde{\delta}_{-}(u_\nu) \cdot [3_+]^{1 - r} \cdot [3_-]^{1 - r} \right\} |_{\nu = \pm}$$

$$+ \int_{V_{x_k}} d^{n-2}w \int_{0}^{x + h_{x_k} + \eta_{x_k}} d\nu \left\{ \tilde{G}_{k}^{(2)}(u_\nu) \cdot [3_+]^{1 - r} \cdot [3_-]^{1 - r} \right\} |_{\nu = \pm} - \left\{ \tilde{G}_{k}^{(2)}(u_\nu) \cdot [3_+]^{1 - r} \cdot [3_-]^{1 - r} \right\} |_{\nu = \pm} \cdot \tilde{\delta}_{-}(u_\nu) \cdot \tilde{\delta}_{-}(u_\nu) \cdot [3_+]^{1 - r} \cdot [3_-]^{1 - r} \right\} |_{\nu = \pm}.$$

Note that the last term occurring in this expression is only present if $\tilde{\Delta}_{-}(u_\nu) |_{\nu = \pm} = 0$ on a set of non-zero measure. Property $H3$ fulfilled by $\mu^{(2)}$, the fact that $\mu^{(3)}$ does not depend on the $\nu$ variables as given in (1.6) and the integrability of the $3_\nu$-related part of the integrand all together ensure that the resulting integrals produce smooth contributions in $x$ for $|x|$ small enough.

The situation is quite similar when only one of the $3_\nu$ changes sign, viz. $3_+(x_k) - \eta_{x_k} < 0$ and $3_+(x_k) + \eta_{x_k} > 0$ but $3_-(x_k) - \eta_{x_k} > 0$ or the analogous situation when $\mp \leftrightarrow \mp$. In such a case, one should decompose
the integral similarly to $\textit{iii}$ and then invoke one of the two properties $H1)$ or $H2)$, c.f. below of (E.13), depending on which of among the two integration domains passes though zero, relative to the boundedness of the function $\mathcal{G}$ and its partial derivatives so as to conclude on the smoothness of the associated integral.

Thus the claim.

\[ \square \]

**F.2 Local rectification of $\mathbf{3}_n$**

**Proposition F.2.** Let the assumptions and notation given in Subsection $E.1$ hold. Given $k_0 \in \text{Int } \mathcal{J}_1$ and given any $\zeta_r \in \{\pm1\}$, let

$$
\mathbf{3}_n(p) = -\sum_{(r,a) \in M} \zeta_r w^{(r)}_a(p_0; \ell_r(k_0)), \quad \nu \in \{\pm\},
$$

where

$$
w^{(r)}_a(k; p) = u_r(k) - u_r(p) + \nu v(k - p) \quad \text{and} \quad \nu \in \mathbb{R}^{+*}. \tag{F.10}
$$

Further, let

$$
\xi_r(x) = -\sum_{(r,a) \in M} \zeta_r \left\{ b_r(x^{(r)}_a) + \nu v x_a^{(r)} \right\} \quad \text{where} \quad b_r(x) = -\zeta_r \xi_r \frac{x^2}{2 \xi_r^2} + u(x),
$$

$\epsilon_r \in \{\pm\}, \xi_r \in \mathbb{R}^{+*}$ and $u$ are such that

$$
\frac{\xi_r}{\xi_r^2} \epsilon_r = \xi_r''(t_r(k_0)) \quad \text{and} \quad u = u'_r(k_0). \tag{F.12}
$$

Finally, let

$$
l^{(0)}_r(x) = \frac{\xi_1 \xi_1}{\xi_r \xi_r} \left( \frac{\xi_r}{\xi_1} \right)^2 x
$$

and for $C > 0$, $\eta > 0$, consider the domain

$$
\mathcal{D}_\eta^{(\text{eff})} = \left\{ x \in \prod_{r} \mathbb{R}^{\eta_r} : |x^{(1)}_1| \leq C \eta, \forall (r,a) \in M : |l^{(0)}_r(x^{(1)}_1) - x_a^{(r)}| \leq \xi_r^2 \eta \right\}. \tag{F.14}
$$

Then, there exists $x_0 > 0$, $\eta' > 0$ and

- smooth functions $t_r$ on $]-x_0; x_0[ \times \mathcal{D}_\eta^{(\text{eff})}$ satisfying $t_r(x; x) = 1 + O(\|x\| + |x|)$,

- a smooth diffeomorphism $\Psi_x : \mathcal{D}_\eta^{(\text{eff})} \rightarrow \mathcal{D}_\eta^{(\text{eff})}$ satisfying $D_0 \Psi = \text{id} + xN$ with $\|N\| \leq C$, for some $x$-independent $C > 0$,

such that

$$
\mathbf{3}_n \circ \Psi_x(x) = t_r(x; x) \cdot \left( x + \zeta_r(x) \right),
$$

and $\Psi_x(\mathcal{D}_\eta^{(\text{eff})}) \subset \mathcal{J}_\text{tot}$ contains a $x$-independent open neighbourhood of $t(k_0)$. Furthermore, the map

$$
(x, x) \mapsto \Psi_x(x) \tag{F.16}
$$

is smooth on $]-x_0, x_0[ \times \mathcal{D}_\eta^{(\text{eff})}$.

**Proof —**
• Canonical form of $\delta_{\nu}$

Let $I_{\ell} \subset \text{Int}(\mathcal{J}_{\ell})$ be a segment such that $t_{\ell}(k_{0}) \in \text{Int}(I_{\ell})$. Since $\partial_{k}w_{\nu}^{(\ell)}(k; t_{\ell}(k_{0})) = w_{\nu}(k) + \nu \nabla w_{\nu}(k) \neq 0$ on $I_{\ell}$, $k \mapsto w_{\nu}^{(\ell)}(k; t_{\ell}(k_{0}))$ is strictly monotone on $I_{\ell}$ and thus admits $t_{\ell}(k_{0})$ as its unique zero on $I_{\ell}$. Furthermore, this also implies that there exists

$$c > 0 \quad \text{such that} \quad [-c ; c] \subset w_{\nu}^{(\ell)}(I_{\ell}; t(k_{0})) .$$

(F.18)

Given $\epsilon > 0$, set

$$B_{\ell}(t_{[\ell,n_{\ell}]}(k_{0})) = \left\{ p_{[\ell,n_{\ell}]} \in \prod_{r=1}^{\ell} \mathcal{J}_{\ell}^{n_{r} - \delta_{\ell}} : \left| p_{a}^{(r)} - t_{\ell}(k_{0}) \right| < \epsilon, \forall (a, r) \in M_{[\ell,n_{\ell}]} \right\} ,$$

(F.19)

where $M_{[\ell,n_{\ell}]} = M \setminus \{ (\ell, n_{\ell}) \}$ has been introduced in (E.13) while $p_{[\ell,n_{\ell}]}$ is as defined in (E.9).

Given the function $z_{\nu}$ of $\sum n_{r}$ variables, it is of use to agree to denote its analogue on $B_{\ell}(t_{[\ell,n_{\ell}]}(k_{0}))$, viz. when the last variable is deleted, as:

$$z_{\nu}^{([\ell,n_{\ell}])}(p_{[\ell,n_{\ell}]}; t_{\ell}(k_{0})) = - \sum_{(r,a) \in M_{[\ell,n_{\ell}]}} \zeta_{r}w_{\nu}^{(r)}(p_{a}^{(r)}; t_{\ell}(k_{0})) , \quad \nu \in \{ \pm \} .$$

(F.20)

Let $t_{[\ell,n_{\ell}]}(k_{0})$ be as defined in (E.10) and let $V_{[\ell,n_{\ell}]}$ be any open neighbourhood of $t_{[\ell,n_{\ell}]}(k_{0})$ such that $V_{[\ell,n_{\ell}]} \subset B_{\ell}(t_{[\ell,n_{\ell}]}(k_{0}))$. Then,

$$x + z_{\nu}^{([\ell,n_{\ell}])}(p_{[\ell,n_{\ell}]}; t_{\ell}(k_{0})) = O(|x| + \epsilon) \quad \text{uniformly on} \ V_{[\ell,n_{\ell}]} .$$

(F.21)

Thus, provided that $|x|$ and $\epsilon$ are taken small enough, one has that, for any

$$p_{[\ell,n_{\ell}]} \in V_{[\ell,n_{\ell}]} , \quad \text{it holds} \quad x + z_{\nu}(p_{[\ell,n_{\ell}]}) \in [-c/2 ; c/2] ,$$

(F.22)

with $c > 0$ as appearing in (F.18). Then, the monotonicity of $k \mapsto w_{\nu}^{(\ell)}(k; t_{\ell}(k_{0}))$ on $I_{\ell}$ ensures that there exists a unique $\mathcal{V}_{\nu}(p_{[\ell,n_{\ell}]}) \in I_{\ell}$ such that

$$x + z_{\nu}(p_{\ell}) = 0 \quad \text{with} \quad p_{\ell} = \left( p_{[\ell,n_{\ell}]} ; \mathcal{V}_{\nu}(p_{[\ell,n_{\ell}]}) \right) \quad \text{for any} \quad p_{[\ell,n_{\ell}]} \in V_{[\ell,n_{\ell}]} .$$

(F.23)

Here, for simplicity, the $x$ dependence of $\mathcal{V}_{\nu}$ has been kept implicit. The function $\mathcal{V}_{\nu}$ takes the explicit form

$$\mathcal{V}_{\nu}(p_{[\ell,n_{\ell}]}) = \left( w_{\nu}^{(\ell)} \right)^{-1}(x + z_{\nu}^{([\ell,n_{\ell}])}(p_{[\ell,n_{\ell}]}; t_{\ell}(k_{0})) \bigg/ \zeta_{\ell} .$$

(F.24)

By construction, the function $\mathcal{V}_{\nu}$ is smooth on $V_{[\ell,n_{\ell}]}$ as a composition of smooth functions.

By the Malgrange preparation Theorem [B,3] applied to the function

$$(x, p) \mapsto x + z_{\nu}(p)$$

(F.25)

of $\bar{n}_{\ell} + 1$ variables, $\bar{n}_{\ell} = \sum_{r=1}^{\ell} n_{r}$, at the point $(0, t(k_{0}))$, one concludes that there exist

• $x_{0} > 0$;
• an open neighbourhood $V'_I \subset V_I$ of $I(k_0)$.

• an open neighbourhood $I'_I$ of $I(k_0)$.

• a smooth, non-vanishing, function $h_I$ on $]-x_0 ; x_0[\times V'_I \times I'_I$.

such that, for $p \in V'_I \times I'_I$ and $|x| < x_0$, it holds

$$x + 3.\nu(p) = \zeta_\nu \cdot \left[ p^{(t)}_\nu - \mathcal{V}_\nu(p(x, p)) \right] \cdot h_I(x, p) \quad \text{with} \quad \zeta_\nu = -\zeta_\nu \cdot \text{sgn}(u'_I(k_0) + \nu \nu).$$

Furthermore, given $p'_\nu$, as in (F.23), partial differentiation of the relation (F.26) in respect to $p^{(t)}_\nu$ allows one to conclude that

$$h_I(x, p'_\nu) = \nu \nu + u'_I(\mathcal{V}_\nu(p'_\nu)) > 0.$$ (F.27)

Let $v = p'_\nu - t'(k_0)|x|$. The explicit expression for $\mathcal{V}_\nu$ given in (F.24) warrants that one has the small $|v|$ expansion

$$\mathcal{V}_\nu(p'_\nu) = \mathcal{V}_{v,0} + L \cdot \mathcal{V}_{v,1} + \mathcal{V}_{v,2}(L, Q) + O(|v|^3).$$ (F.28)

There, I have set

$$L = \frac{1}{\zeta_\nu}(y, v) \quad \text{with} \quad y^t = (y^{(1)}^t, \ldots, (y^{(\ell)}^t)^t) \quad \text{and} \quad (y^{(r)})^t = \zeta_r(1, \ldots, 1) \in \mathbb{R}^{n - \delta, \ell},$$ (F.29)

while

$$Q = \begin{pmatrix} \ddots & 0 & 0 \\
0 & \frac{\zeta_\nu}{\nu + u'_I(\nu _{I}(k_0))} \cdot \mathbb{I}_{n - \delta, \ell} & 0 \\
0 & 0 & \ddots \end{pmatrix}.$$ (F.30)

Above, $\mathbb{I}_n$ denotes the identity matrix on $\mathbb{R}^n$, $(\cdot, \cdot)$ denotes the canonical scalar product on $\mathbb{R}^n$ and $^t$ denotes the transposition. The first three coefficients of the expansion (F.28) are given by

$$\mathcal{V}_{v,0} = (w^{(t)}_\nu)^{-1}(x, t_I(k_0)) = t_I(k_0) + \frac{x}{\nu + u'_I(k_0)} + O(x^2),$$ (F.31)

$$\mathcal{V}_{v,1} = -\nu + u'(k_0) \cdot \frac{1}{\nu + u'_I(\nu _{I}(k_0))} = -1 + \frac{x}{\zeta_\nu \nu + u'_I(\nu _{I}(k_0))^2} + O(x^2)$$ (F.32)

and

$$\mathcal{V}_{v,2}(L, Q) = -\frac{Q}{2(\nu + u'_I(\nu _{I}(k_0)))} - u'_I(\nu _{I}(k_0)) \cdot \frac{L^2 [\nu + u'_I(\nu _{I}(k_0))^2]}{2(\nu + u'_I(\nu _{I}(k_0))^3)} - \frac{Q + L^2 \cdot u''(t_I(k_0))}{2(\nu + u'_I(\nu _{I}(k_0)))} + O(x \cdot |Q| + L^2).$$ (F.33)
These expansions ensure that
\[ \mathcal{V}_{-0} - \mathcal{V}_{+0} = -\frac{2x \cdot \zeta^{-1}}{v^2 - (u_1'(k_0))^2} + O(x^2), \quad \mathcal{V}_{-1} - \mathcal{V}_{+1} = 4x \zeta^{-1} \frac{u''(t_1(k_0)) \cdot u'(k_0)}{[v^2 - (u_1'(k_0))^2]^2} + O(x^2) \quad (F.34) \]
and
\[ \mathcal{V}_{-2}(L, Q) - \mathcal{V}_{+2}(L, Q) = \frac{v}{v^2 - (u_1'(k_0))^2} \left(Q + L \cdot u''(t_1(k_0))\right) + O(1 \cdot [|Q| + L^2]) \quad (F.35) \]

Thus, provided that \(|x|\) is small enough, there exist smooth functions \(U_1, U_2\) on \(V'_{(\ell,n)}\) such that
\[ \mathcal{V}_-(p_{(\ell,n)}) - \mathcal{V}_+(p_{(\ell,n)}) = x \mathcal{U}_1(p_{(\ell,n)}) + \mathcal{U}_2(p_{(\ell,n)}) \quad (F.36) \]
The functions \(U_n\), which may depend on \(x\), are such that
\[ U_1(p_{(\ell,n)}) = \frac{-2v \zeta}{v^2 - (u_1'(k_0))^2} + O(|x| + |v|), \quad U_2(p_{(\ell,n)}) = \frac{-\zeta v}{v^2 - (u_1'(k_0))^2} + O(|v|^2) \quad (F.37) \]
where I remind that \(v = p_{(\ell,n)} - t_{(\ell,n)}(k_0)\), \((\cdot, \cdot)\) is the canonical scalar product on \(\prod_{\ell=1}^n \mathbb{R}^{n_{-\delta,\ell}}\) and the matrix \(M\) takes the form
\[ M = -\zeta \cdot u''(t_1(k_0)) \ y \cdot y^T + \begin{pmatrix} 
\ldots & 0 & 0 \\
0 & -\zeta \cdot u''(t_1(k_0))I_{n_{-\delta,\ell}} & 0 \\
0 & 0 & \ldots 
\end{pmatrix} \quad (F.38) \]
with \(y\) as given by \((F.29)\). Since \(M\) is symmetric, it is diagonalisable and has real eigenvalues. For further utility, one still needs to establish that these are non-vanishing. For that purpose, it is enough to show that \(M\) has a non-zero determinant.

Upon factorising the diagonal part, one gets that
\[ \det[M] = \prod_{r=1}^\ell \left\{ -\zeta u''(t_r(k_0)) \right\}^{n_{-\delta,\ell}} \cdot \det[I + w \cdot y^T] \quad (F.39) \]
with \(y\) as in \((F.29)\),
\[ w^T = ((w^{(1)})^T, \ldots, (w^{(\ell)})^T) \quad \text{and} \quad (w^{(\ell)})^T = \zeta \cdot \frac{u''(t_1(k_0))}{u''(t_1(k_0))} \cdot (1, \ldots, 1) \in \mathbb{R}^{n_{-\delta,\ell}}. \quad (F.40) \]
The determinant can be computed explicitly and, upon using the relation \(u''(t_1(k_0))/u''(t_1(k_0)) = \zeta(k_0)/t_1'(k_0)\), which follows from a differentiation of \(u'(t_1(k)) = u'(t_1(k))\) at \(k = k_0\), one eventually obtains that
\[ \det[M] = -\prod_{r=1}^\ell \left\{ -\zeta u''(t_r(k_0)) \right\}^{n_{-\delta,\ell}} \cdot \frac{\mathcal{P}'(k_0)}{u''(k_0)} \neq 0 \quad (F.41) \]

since, by hypothesis \((F.7)\), \(\mathcal{P}'(k_0) \neq 0\).

The above ensures that
\[ \frac{\mathcal{U}_2(t_{(\ell,n)}(k_0))}{\mathcal{U}_1(t_{(\ell,n)}(k_0))} = 0, \quad D_{t_{(\ell,n)}(k_0)} \left( \frac{\mathcal{U}_2}{\mathcal{U}_1} \right) = 0 \quad \text{and} \quad D^2_{t_{(\ell,n)}(k_0)} \left( \frac{\mathcal{U}_2}{\mathcal{U}_1}(s, s') \right) = [c(x)]^2 \cdot (s, Ms') \quad (F.42) \]
for $s, s' \in \prod_{r=1}^{f} \mathbb{R}^{n_r-\delta_r}$ and where $c(x) = 1 + O(x)$ is smooth in the neighbourhood of $x = 0$.

Thus, by virtue of the Morse lemma, Theorem 11 followed by a dilatation of variables by $c(x)$, one infers that there exists

- an open neighbourhood $V''_{[\ell,n_\ell]} \subset V'_{[\ell,n_\ell]}$ of $l_{(\ell,n_\ell)}(k_0)$,
- an open neighbourhood $W_\phi$ of $0$ in $\prod_{r=1}^{f} \mathbb{R}^{n_r-\delta_r}$,
- a smooth diffeomorphism $\phi_\ell : W_\phi \rightarrow V''_{[\ell,n_\ell]}$, with $\phi_\ell(0) = l_{(\ell,n_\ell)}(k_0)$

such that

\[
\frac{U_\ell(\phi_\ell(\nu))}{U_\ell(\phi_\ell(\nu))} = (\nu, M\nu). \tag{F.43}
\]

In particular, one readily infers from (F.43) that $(D_\phi\phi_\ell)^t \cdot M \cdot D_\phi\phi_\ell = 2M$.

It is clear that the size of all the domains appearing above may be taken to be $x$-independent, at least provided that $|x|$ is small enough, say $|x| < x_0$, and that then

\[
(x, \nu) \mapsto \phi_\ell(\nu), \tag{F.44}
\]

is smooth on $]-x_0; x_0[ \times W_\phi$. Clearly, upon adjusting the parameters, one may take $x_0$ as introduced earlier on.

**Canonical form of $\xi_v$**

The very same reasoning applied to the function $\xi_v(x)$, as defined in (F.12), ensures that there exist

- an open neighbourhood $V^{(0)}_{[\ell,n_\ell]}$ of $0 \in \prod_{r=1}^{f} \mathbb{R}^{n_r-\delta_r}$,
- a segment $I^{(0)}_{\ell}$ containing an open neighborhood of $0 \in \mathbb{R}$,
- a smooth, non-vanishing, function $h_v^{(0)}$ on $]-x_0; x_0[ \times V^{(0)}_{[\ell,n_\ell]} \times I^{(0)}_{\ell}$,
- a smooth function $\mathcal{V}_v^{(0)}$ on $]-x_0; x_0[ \times V^{(0)}_{[\ell,n_\ell]}$,

such that

\[
x + \xi_v(x) = s_{v,\ell} \cdot \left[ x^{(0)}_{\ell} - \mathcal{V}_v^{(0)}(x_{[\ell,n_\ell]}) \right] \cdot h_v^{(0)}(x; \nu) \quad \text{with} \quad s_{v,\ell} = -\zeta_\ell \text{sgn}(u'_0(k_0) + \nu\nu). \tag{F.45}
\]

Here, again, I kept the $x$-dependence of $\mathcal{V}_v^{(0)}$ implicit.

Note that here, $s_{v,\ell}$ is exactly as defined in (F.26) owing to the very choice of the parameters $\varepsilon_\alpha, \xi_\alpha, u$, defining the effective function $\xi_v(x)$. The function $h_v^{(0)}$ enjoys the identity

\[
h_v^{(0)}(x; x_\nu) = |\nu + b'_\ell(\mathcal{V}_v^{(0)}(x_{[\ell,n_\ell]}))| > 0 \quad \text{with} \quad x_\nu = \left( x_{[\ell,n_\ell]}, \mathcal{V}_v^{(0)}(x_{[\ell,n_\ell]}), \right). \tag{F.46}
\]

Furthermore, there exist two smooth functions on $V^{(0)}_{[\ell,n_\ell]}$ such that

\[
\mathcal{V}^{(0)}_+(x_{[\ell,n_\ell]}) - \mathcal{V}^{(0)}_-(x_{[\ell,n_\ell]}) = x_1 U^{(0)}_1(x_{[\ell,n_\ell]}) + U^{(0)}_2(x_{[\ell,n_\ell]}), \tag{F.47}
\]
Following the above reasoning, and re-adjusting the domains \( W_{\phi}, V_{[\ell,n]} \) appearing above if necessary, one eventually concludes that there exists a smooth diffeomorphism \( \phi_x^{(0)} : W_{\phi} \to V_{[\ell,n]}^{(0)} \) such that

\[
\frac{\mathcal{U}_2^{(0)}(\phi_x^{(0)}(v))}{\mathcal{U}_1^{(0)}(\phi_x^{(0)}(v))} = (v, \nu) \quad \text{and} \quad \begin{cases} \frac{D_0\phi_x^{(0)}}{\phi_x^{(0)}(0)} = D_0\phi_x \\
\end{cases}.
\]

Likewise to the previous situation, \((x, v) \mapsto \phi_x^{(0)}(v)\) is smooth on \([-x_0 ; x_0] \times W_{\phi}\).

I stress that the open neighbourhood \( W_{\phi} \) appearing above coincides exactly with the domain of the diffeomorphism \( \phi_x \) introduced earlier on. Also, I should comment relatively to the possibility of choosing \( \phi_x^{(0)} \) such that \( D_0\phi_x^{(0)} = D_0\phi_x \). Just as for the case of \( \phi_x \), one deduces that any Morse function \( \phi_x^{(0)} \) rectifying \( \mathcal{U}_2^{(0)}/\mathcal{U}_1^{(0)} \) to has to satisfy \( (D_0\phi_x^{(0)})^{-1} \cdot \mathcal{M} \cdot D_0\phi_x^{(0)} = \mathcal{M} \). A priori this equation has a space of solutions that is isomorphic to \( SO(p,q) \) where \((p,q)\) is the signature of \( \mathcal{M} \). However, upon looking at the proof of the Morse Lemma, one constructs a Morse function from a given choice of a solution to this equation. Thus, when constructing \( \phi_x^{(0)} \), the latter can always be chosen so that \( D_0\phi_x^{(0)} = D_0\phi_x \).

\* The per-se rectification

With all the ingredients being introduced, one may define the smooth diffeomorphism

\[
\Phi_x : \begin{cases} V_{[\ell,n]}^{(0)} \to V_{[\ell,n]}^{(0)} \\
x_{[\ell,n]} \mapsto \phi_x \circ (\phi_x^{(0)})^{-1}(x_{[\ell,n]})
\end{cases}
\]

which satisfies

\[
\mathcal{U}_2 \circ \Phi_x = \frac{\mathcal{U}_2^{(0)}}{\mathcal{U}_1^{(0)}}.
\]

One is now in position to introduce the smooth map

\[
\Psi_x : \begin{cases} V_{[\ell,n]}^{(0)} \times I_\ell^{(0)} \to V_{[\ell,n]}^{(0)} \\
x \mapsto \Psi_x(x) = \left( \Phi_x(x_{[\ell,n]}), \mathcal{V}_- \circ \Phi_x(x_{[\ell,n]}), \left(\mathcal{V}_x^{(0)}(x_{[\ell,n]}) - \mathcal{V}_-^{(0)}(x_{[\ell,n]}) \right)^{-1} \mathcal{U}_1 \circ \Phi_x(x_{[\ell,n]}) \right)
\end{cases}
\]

\[
\frac{\mathcal{U}_1 \circ \Phi_x(x_{[\ell,n]})}{\mathcal{U}_1^{(0)}(x_{[\ell,n]})} \bigg|_{x_{[\ell,n]}=0} = 1 + O(x).
\]

Furthermore, since \( \mathcal{V}_x^{(0)}(t_{[\ell,n]}(0)) = 0 \), one has

\[
\mathcal{V}_x(t_{[\ell,n]}(0)) = (w_0^{(0)})^{-1} \left( \frac{x}{\zeta_\ell} ; t_\ell(k_0) \right) = t_\ell(k_0) + O(x).
\]
and, similarly, \( V^{(0)}_{\nu}(0) = O(x) \). All of the above together entails that

\[
\Psi_{\nu}(x)_{|x=0} = \left( t_{|x=0}(k_0), t_{|x=0}(k_0) + O(x) \right) = t(k_0) + \left( \frac{\pi}{2}, O(x) \right).
\] (F.55)

Furthermore, denote by \( [\Psi_{\nu}(x)]^{(\ell)}_{|x=0} \) the ultimate scalar entry of \( \Psi_{\nu}(x) \). Then, the expansion (F.28) and an analogous one for \( V^{(0)}_{\nu}(x_{|\ell=\nu}) \), yields for any \( u \in \mathbb{R}^{\nu-1} \) and \( s \in \mathbb{R} \)

\[
D_0 \left( [\Psi_{\nu}(x)]^{(\ell)}_{|x=0} \right) \cdot (u, s) = \zeta_{s}^{\frac{1}{2}} V^{(0)}_{\nu} \left( D_0 \Phi_{\nu} \cdot u, y \right) + \left( s - \zeta_{s}^{\frac{1}{2}} V^{(0)}_{\nu} \left( u, y \right) \right) \frac{U_1 \circ \Phi_{\nu}(0)}{U_1^{(0)}(0)} + \frac{V^{(0)}_{\nu}}{s-o(1)} D_0 \left( \frac{U_1 \circ \Phi_{\nu}}{U_1^{(0)}} \right) u. \quad (F.56)
\]

Since \( V^{(0)}_{\nu} = O(x) \) and \( V^{(0)}_{\nu} \cdot V^{(0)}_{\nu} = O(x^2) \), it holds that there exists a linear form \( \mathcal{L} \) on \( \mathbb{R}^{\nu} \) such that

\[
D_0 \left( [\Psi_{\nu}(x)]^{(\ell)}_{|x=0} \right) \cdot (u, s) = s + x \mathcal{L} \cdot (u, s). \quad (F.57)
\]

Thence, there exists an endomorphism \( N_{\nu} \) on \( \mathbb{R}^{\nu} \), with \( \|N_{\nu}\| \leq C \) for some \( x \)-independent constant, such that \( D_0 \Psi_{\nu} = \text{id} + x N_{\nu} \). Thus, \( \Psi_{\nu} \) is invertible in some open neighbourhood of \( 0 \) which, upon reducing \( V^{(0)}_{\nu} \) and \( l^{(0)}_{\ell} \) if necessary, may be taken to be \( V^{(0)}_{\nu} \cdot l^{(0)}_{\ell} \). Note that the estimates on the differential \( D_0 \Psi_{\nu} = \text{id} + x N_{\nu} \) and (F.55) ensures that \( t(k_0) \in \Psi_{\nu} l^{(0)}_{\ell} \times l^{(0)}_{\ell} \) and that the latter set contains a \( x \)-independent open neighbourhood of \( t(k_0) \) in \( \mathbb{R}^{\nu} \).

All is now in place so as to establish the rectification relation. Observe that there exists a direct relation between the zeroes \( \Psi_{\nu} \) and \( V^{(0)}_{\nu} \):

\[
\nu \circ \Phi_{\nu}(x_{|\ell=\nu}) - V^{(0)}_{\nu} \circ \Phi_{\nu}(x_{|\ell=\nu}) = U_1 \circ \Phi_{\nu}(x_{|\ell=\nu}) \left[ x + \frac{U_2 \circ \Phi_{\nu}(x_{|\ell=\nu})}{U_1 \circ \Phi_{\nu}(x_{|\ell=\nu})} \right]
\]

\[
= U_1 \circ \Phi_{\nu}(x_{|\ell=\nu}) \left[ x + \frac{U_2^{(0)}(x_{|\ell=\nu})}{U_1^{(0)}(x_{|\ell=\nu})} \right] \quad \left[ \nu^{(0)}(x_{|\ell=\nu}) - V^{(0)}_{\nu}(x_{|\ell=\nu}) \right].
\] (F.58)

This identity entails that, for any \( \nu \in \{\pm\} \),

\[
\left[ \Psi_{\nu}(x) \right]_{|x=0}^{(\ell)} - \nu \circ \Phi_{\nu}(x_{|\ell=\nu}) = \left[ x_{|\nu}^{(\ell)} \right] - \nu^{(0)}(x_{|\ell=\nu}) \] \frac{U_1 \circ \Phi_{\nu}(x_{|\ell=\nu})}{U_1^{(0)}(x_{|\ell=\nu})}.
\] (F.59)

Thus, starting from the factorisation (F.26) and applying the equality (F.59) followed by an application of the factorisation (F.45) backwards, one gets that

\[
x + \tilde{z}_{\nu} \circ \Psi_{\nu}(x) = \tilde{f}_{\nu}(x; x) \left[ x + \tilde{z}_{\nu}(x) \right] \quad \text{with} \quad \tilde{f}_{\nu}(x; x) = \frac{U_1 \circ \Phi_{\nu}(x_{|\ell=\nu})}{U_1^{(0)}(x_{|\ell=\nu})} \cdot \frac{h_{\nu}(x; \Psi(x))}{h^{(0)}_{\nu}(x; x)}. \quad (F.60)
\]

Clearly, \( \tilde{f}_{\nu} \) is smooth. I now establish that \( \tilde{f}_{\nu} \) has the claimed form of the expansion.

Putting (F.54) and (F.55) together, one infers that, for any \( \nu \in \{\pm\} \),

\[
\Psi_{\nu}(0) = \left( t_{|\ell=\nu}(k_0), V_{\nu}(t_{|\ell=\nu}(k_0)) \right) + O(x).
\] (F.61)
This, along with $\mathcal{V}_f^{(0)}(0) = O(x)$ and the smoothness of $h_\nu$ and $h_\nu^{(0)}$, allows one to use the expressions (F.27) and (F.46) so as to deduce that

$$\frac{h_\nu(x; \Psi_\nu(x))}{h_\nu^{(0)}(x; x)} \bigg|_{x=0} = h_\nu(x; p_\nu) \left|_{p_\nu(\nu; 0) + O(x)} \right. = \frac{\nu v + u_\nu'(k_0) + O(x)}{\nu v + u_\nu'(0) + O(x)} + O(x)$$

where $p_\nu$, resp. $x_\nu$, are as defined through (F.27), resp. (F.46). Furthermore, I used that $u_\nu'(k_0) = u$. Hence, a similar result for the ratio of $\mathcal{U}_\nu$’s established in (F.53), and smoothness in $x$ all together, entail that one has $f_\nu(x; x) = 1 + O(||x|| + |x|)$.

Recall the definition (F.15) of the domain $D^{(eff)}_{\eta'}$. To complete the proof, it remains to establish that

$$D^{(eff)}_{\eta'} \subset V^{(0)}_{[\eta, \eta]} \times I^{(0)}_{\ell} \quad \text{provided that} \quad 0 < \eta' < \eta_0$$

for some $\eta_0 > 0$ small enough. The map

$$G : \mathbb{R}^m \to \mathbb{R}^m \quad \text{such that} \quad \left[G(x)\right]^{(r)} = t^{(r)}_1 (x^{(1)}_1) - \delta^{(r)}(1 - \delta_{0,1} \delta_{r,1})$$

is obviously continuous, and thus, upon agreeing to denote $B_\nu(0) = \{x \in \mathbb{R} : |x| < \epsilon\}$ the open ball around 0 in $\mathbb{R}$ of radius $\epsilon$, one gets that

$$D^{(eff)}_{\eta'} = G^{-1}\left(B_{C\eta'}(0) \times \prod_{r=1}^{\ell} \left(B_{C^{2}\eta'}^{(r)}(0)\right)^{n_r - \delta_{r,1}}\right)$$

is open as a pre-image of an open set by a continuous function. Since $0 \in D^{(eff)}_{\eta'}$, it is an open neighbourhood of that point in $\mathbb{R}^m$. Since its diameter shrinks to 0 as $\eta' \to 0$, and since $V^{(0)}_{[\eta, \eta]} \times I^{(0)}_{\ell}$ is also an open neighbourhood of $0$ in $\mathbb{R}^m$, the claim follows.

\section*{F.3 Factorisation of the maps $\tilde{z}_\nu$}

\begin{proposition}
Let $\tilde{z}_\nu$ correspond to the below multivariate polynomial on $\prod_{r=1}^{\ell} \mathbb{R}^{n_r}$:

$$\tilde{z}_\nu(x) = -\sum_{(r,a) \in M} \xi_r \left\{ \frac{b_r(x_a^{(r)})}{b_r} + \nu \psi x_a^{(r)} \right\} \quad \text{where} \quad b_r(x) = -\xi_r \epsilon_r \frac{x^2}{2e_r} + u x ,$$

$\epsilon_r \in \{\pm\}$, $\xi_r \in \mathbb{R}^+$ and $(u, v) \in \mathbb{R} \times \mathbb{R}^+$. Then, there exists a linear map $M$ on $\prod_{r=1}^{\ell} \mathbb{R}^{n_r}$ such that:

- $M$ is invertible;

- there exist integers $m_+ \in \mathbb{N}$ satisfying $m_+ + m_- + 1 = \sum_{r=1}^{\ell} n_r$ such that it holds

$$\tilde{z}_\nu(M(y, z)) = \tilde{P}_\nu(y, z) \quad \text{with} \quad z = (z^{(+), (-)}) \in \mathbb{R}^{m_+} \times \mathbb{R}^{m_-}$$

and

$$\tilde{P}_\nu(y, z) = \frac{y^2}{2P_{\text{eff}}} - (u + \nu v) y + \sum_{s=1}^{m_+} (z_s^{(+)})^2 - \sum_{s=1}^{m_-} (z_s^{(-)})^2 , \quad \text{with} \quad P_{\text{eff}} = \sum_{r=1}^{\ell} \epsilon_r n_r \xi_r^2 .$$

\end{proposition}
Proof —

One may recast the polynomial \( \tilde{\zeta}_\nu \) in the form

\[
\tilde{\zeta}_\nu(x) = -(u + \nu v) \tilde{\chi}_\nu + \sum_{(r,a) \in M} \frac{\varepsilon_r}{2 \xi_r^2} (\chi_a^{(r)})^2 \quad \text{with} \quad \tilde{\chi}_\nu = \sum_{(r,a) \in M} \xi_a^{(r)}.
\]

(E.69)

Then, let \( \tilde{M} = \tilde{D} + g \cdot e^t \) with \( e^t = (1, \ldots, 1) \in \mathbb{R}^n \). \( g^t = ((g^{(1)})^t, \ldots, (g^{(f)})^t) \) and where

\[
\left( g^{(r)} \right)^t = \frac{\xi_1 \xi_r}{P_{\text{eff}}} \cdot \xi_r^2 \cdot (1, \ldots, 1) \in \mathbb{R}^n \quad \text{and} \quad \tilde{D} = \begin{pmatrix}
0 & \cdots & \cdots & 0 \\
\vdots & -I_{n-1} & \cdots & \vdots \\
0 & 0 & -\zeta_1 \xi_2 I_{n_2} & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & \cdots & 0 & -\zeta_1 \xi_f I_{n_f}
\end{pmatrix}.
\]

(E.70)

It is straightforward to see that \( |\det(\tilde{M})| = |P_{\text{eff}}|^{-1} \neq 0 \).

Then, a straightforward calculation shows that, given \( y \in \prod_{r=1}^f \mathbb{R}^{n_r} \)

\[
\tilde{\zeta}_\nu(\tilde{M}y) = + \frac{(y^{(1)})^2}{2P_{\text{eff}}} - \zeta_1 (u + \nu v) y^{(1)} + Q(y^{(1,1)})
\]

in which I employed the convention introduced in (E.9), while the quadratic form \( Q \) reads

\[
Q(y^{(1,1)}) = \sum_{(r,a) \in M_{(1,1)}} \frac{\varepsilon_r}{2 \xi_r^2} (y^{(r)})^2 - \frac{1}{2P_{\text{eff}}} \left( \sum_{(r,a) \in M_{(1,1)}} y^{(r)} \right)^2.
\]

(E.72)

Here \( M_{(1,1)} \) is as defined in (E.13). Representing the quadratic form as \( Q(y^{(1,1)}) = (y^{(1,1)}, \mathbb{M}_Q y^{(1,1)}) \), one gets that the matrix \( \mathbb{M}_Q \) is a rank one perturbation of a diagonal matrix:

\[
\mathbb{M}_Q = D_Q - \frac{1}{2P_{\text{eff}}} e \cdot e^t \quad \text{with} \quad e^t = (1, \ldots, 1) \in \mathbb{R}^{n-1}
\]

(E.73)

and where I denoted

\[
D_Q = \begin{pmatrix}
\frac{1}{2} \varepsilon_1 \cdot \xi_1^2 \cdot I_{n_1-1} & 0 & \cdots & 0 \\
0 & \frac{1}{2} \varepsilon_2 \cdot \xi_2^2 I_{n_2} & 0 & \vdots \\
0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \frac{1}{2} \varepsilon_f \cdot \xi_f^2 I_{n_f}
\end{pmatrix}.
\]

(E.74)

The determinant of \( \mathbb{M}_Q \) can thus be computed in a closed form

\[
\det[\mathbb{M}_Q] = \frac{\varepsilon_1 \xi_1^2}{P_{\text{eff}}} \prod_{r=1}^f \left( \frac{\varepsilon_r}{2 \xi_r^2} \right)^{n_r - \delta_{r,1}} \neq 0.
\]

(E.75)

\( \mathbb{M}_Q \) being invertible and symmetric, there exists an orthogonal linear map \( \mathbb{N} \) such that

\[
\mathbb{M}_Q = \mathbb{N} \left( \begin{array}{cc}
I_{m_r} & 0 \\
0 & -I_{m_r}
\end{array} \right) \mathbb{N}^t
\]

(E.76)
in which \((m_+, m_-)\) is the signature of \(\mathbb{R}_Q\). Thus the map
\[
\mathbb{M} = \bar{\mathbb{M}} \cdot \begin{pmatrix} \ell_l & 0 \\ 0 & N \end{pmatrix}
\] (F.77)
does the job.

F.4 Local expansion of a Vandermonde determinant

Recall the notations for norms and partial order on vectors of integers (A.4) and the one for exponents \(x^\alpha\) (A.5) with \(x \in \mathbb{R}^\mathbb{N} = \prod_{r=1}^f \mathbb{R}^{n_r}\) and \(\alpha \in \mathbb{N}^\mathbb{N}\). \(\mathbb{N}\) is as defined in (A.1).

**Lemma F.4.** Let \(\Psi : U \to \Psi(U)\) be a smooth diffeomorphism on a open neighbourhood \(U\) of \(0 \in \mathbb{R}^\mathbb{N}\) such that

- \(\Psi(0) = v \in \mathbb{R}^\mathbb{N}\) with \(v = (v^{(1)}, \ldots, v^{(r)})\), each entry \(v^{(r)} = (v^{(r)}_a, \ldots, v^{(r)}_b) \in \mathbb{R}^n\) having equal coordinates;
- \(D_x\Psi = \text{id} + x\mathbb{N}_v\), with \(\mathbb{N}_v \in \mathcal{L}(\mathbb{R}^\mathbb{N})\) such that \(\|\mathbb{N}_v\| \leq C\), for a \(|x|\)-independent constant \(C\).

Let
\[
V(x) = \prod_{r=1}^f \prod_{a < b} n_r (x_a^{(r)} - x_b^{(r)})^2
\] (F.78)
be a product of Vandermonde determinants relative to each of the \(r\)-coordinates. Then, there exists a smooth map \(Q : U \to \mathbb{R}\) such that \(V(\Psi(x)) = V(x) + Q(x)\). The map \(Q\) has the expansion around \(x = 0\) of the form
\[
Q(x) = \sum_{\alpha_0, \alpha_1 \geq m, \alpha_0 \geq \alpha_1} xC_\alpha x^{\alpha} + \sum_{\alpha_0, \alpha_1 \geq m+1, \alpha_0 \geq \alpha_1} D_\alpha x^{\alpha} + O\left(|x|^m + \|x\|^2\right) \quad \text{with} \quad m = \sum_{r=1}^f n_r(n_r - 1).
\] (F.79)

In the above expansion, the even integer coordinate vectors \(\alpha_0, \alpha_1 \in (2\mathbb{N})^\mathbb{N}\) can be taken arbitrary provided that \(|\alpha_0| \geq m + a\), and \(C_\alpha, D_\alpha \in \mathbb{R}\) are coefficients that are bounded uniformly in \(x\).

**Proof** —

The hypotheses on \(\Psi\) entail that \(\Psi(x) = v + x + x\mathbb{N}_v \cdot x + \delta\Psi(x)\), with \(\delta\Psi(x) = O(|x|^2)\). Then, one can write
\[
\left(\Psi(x)\right)^{(r)}_a = x^{(r)}_a + y^{(r)}_a \quad \text{with} \quad y^{(r)}_a = O\left(|x||x| + \|x\|^2\right)
\] (F.80)
and smooth in \(x\). Then, one has
\[
V(\Psi(x)) = \prod_{r=1}^f \det_{\mathcal{N}_v} \left[(x^{(r)}_a + y^{(r)}_a)^{b-1}\right].
\] (F.81)

Upon expanding the power-law, one gets that
\[
(x^{(r)}_a + y^{(r)}_a)^{b-1} = (x^{(r)}_a)^{b-1} + \sum_{k=1}^{b-1} C_{b-1}^k (x^{(r)}_a)^{b-1-k} (y^{(r)}_a)^k = (x^{(r)}_a)^{b-1} + P^{(r)}_{a,b}(x),
\] (F.82)
where \( C_{b-1}^k \) are binomial coefficients. The smoothness of \( y_a^{(r)} \) and the estimates in \( x \) ensure that \( P_{a,b}^{(r)}(x) \) takes the form:

\[
P_{a,b}^{(r)}(x) = \sum_{\alpha,|\alpha| \geq b-1 \atop \alpha_0 \geq 0} x^\alpha C_{a,a,b}^{(r)} x^\alpha + \sum_{\alpha,|\alpha| \geq b \atop \alpha_0 \geq 0} D_{a,a,b}^{(r)} x^\alpha + O(\|x\|^\alpha_0 + x^{\delta_1})
\]

(F.83)

for some coefficients \( C_{a,a,b}^{(r)}, D_{a,a,b}^{(r)} \) and \( \alpha_0, \alpha_1 \in \mathbb{N} \). Developing the determinant in respect to the sum appearing in each column yields

\[
d \det_n \left[ (x_a^{(r)} + y_a^{(r)})^{b-1} \right] = \det_{n'} \left[ (x_a^{(r)})^{b-1} \right] + R^{(r)}(x)
\]

(F.84)

with

\[
R^{(r)}(x) = \sum_{k=1}^n \sum_{\|1 : n_r\| = \|L\| \atop \|\| = k} \det_{n'} [M_{L,T}] , \quad (M_{L,T})_{ab} = \begin{cases} (x_a^{(r)})^{b-1} & \text{if } a \in L \\ P_{a,b}^{(r)}(x) & \text{if } a \in T \end{cases}.
\]

(F.85)

Above, the sum runs through all partitions \( L \cup T \) of \([1 : n_r]\) such that \( T \) has fixed cardinality \( k \). Upon using the expansion

\[
\det_{n'} [M_{L,T}] = \sum_{\sigma \in \mathbb{S}_n} (-1)^\sigma \prod_{a \in L} (x_a^{(r)})^{\sigma(a)-1} \cdot \prod_{a \in L} P_{a,b}^{(r)}(x)
\]

(F.86)

a direct exponent counting argument entails that there exist constants \( C_{a,L,T}^{(r)} D_{a,L,T}^{(r)} \in \mathbb{R} \) such that

\[
\det_{n'} [M_{L,T}] = \sum_{\alpha,|\alpha| \geq n, \alpha_0 \geq 0} x^\alpha C_{a,L,T}^{(r)} x^\alpha + \sum_{\alpha,|\alpha| \geq n, \alpha_0 \geq 0} D_{a,L,T}^{(r)} x^\alpha + O(\|x\|^\alpha_0 + x^{\delta_1}).
\]

(F.87)

All of this being established, it remains to take the square of the expression in (F.84) and then the product over \( r \) so as to get the claim.

\[\blacksquare\]

F.5 \hspace{1em} Asymptotic behaviour of a local integral

F.5.1 \hspace{1em} The integral associated with the \([u_0'(k_0)] < \upsilon\) regime

Given \( \delta_+ > 0 \) and \( m_\pm \in \mathbb{N}^* \) define

\[
I_{b,m_\pm} = [0; \sqrt{\delta_+}]^{m_+} \times [0; \sqrt{\delta_-}]^{m_-}.
\]

(F.88)

**Lemma F.5.** Let \( \delta_+, \eta > 0 \) be fixed and small enough, \( m_\pm \in \mathbb{N} \). Let \( a, b \) be two smooth functions on \([-2m_+\delta_+ - 2m_-\delta_+ ; 2m_+\delta_+ + 2m_-\delta_-] \) depending, possibly, on an auxiliary parameter \( x \) and such that

\[
a(0) = b(0) = 0 \quad \text{and} \quad |a(s)| + |b(s)| \leq \eta \quad (F.89)
\]

uniformly in \( s \in [-2m_+\delta_+ - 2m_-\delta_+ ; 2m_+\delta_+ + 2m_-\delta_-] \). Let \( A, B > -1 \) be smooth function on \( I_{b,m_\pm} \times [-\eta ; \eta] \) and let \( \mathcal{G} \) be a smooth function on \( I_{b,m_\pm} \times [-\eta ; \eta] \times \mathbb{R}^+ \times \mathbb{R}^+ \) such that

\[
\mathcal{G}(u, x, y) = G(u) + O(|x|^{1-\tau} + |y|^{1-\tau}) \quad \text{with} \quad 0 < \tau < 1,
\]

(F.90)
Let \( \mathcal{W} \) be smooth on \( I_{\delta_0,m_0} \times [-\eta; \eta] \times \mathbb{R}^+ \) and admit the expansion around the origin
\[
\mathcal{W}(u, \kappa) = \sum_{\alpha,|a| \geq m_0 \atop \alpha = (m_0, \beta)} c_{\alpha} \cdot \kappa^{|a|} u^\beta \quad \text{with} \quad m_0 \in 2\mathbb{N} .
\] (F.92)

Consider the integral
\[
\mathcal{J}[\mathcal{G}_{\text{tot}}, A, B](x) = \prod_{v = \pm} \left\{ \int_0^{\delta_v} \frac{d^{m_v} w^{(v)}}{\sqrt{w^{(v)}}} \right\} \int_0^1 dt \left[ \mathcal{G}_{\text{tot}}(u, (1 - t)\varphi_x(u_w), t\varphi_x(u_w)) \right. \\
\left. \quad \times (1 - t)^{A(u) \cdot t^{B(u)} \cdot \mathbb{E}[\varphi_x(u_w)] \cdot (\varphi_x(u_w))^{A(u) + B(u) + 1}} \right]_{u_w \text{-even}}
\] (F.93)

where the even part of a function is as defined in (A.9) and vectors \( u, u_w \) appearing under the integral sign are parameterised in terms of \( w^{(v)} \), \( t \) as
\[
u = \pm \] (F.94)

while
\[
\varphi_x(u_w) = x + \sum_{a=1}^{m_+} w_a^{(+) - } - \sum_{a=1}^{m_-} w_a^{(-)}.
\] (F.95)

Finally, the main building block of the integrand reads
\[
\mathcal{G}_{\text{tot}}(u, x, y) = \mathcal{W}(u, \varphi_x(u_w)) \cdot \mathcal{G}(u, x, y).
\] (F.96)

Then, the integrand belongs to \( L^1((0; \delta_+)^{m_+} \times (0; \delta_-)^{m_-} \times [0; 1]) \) and for any smooth function \( F \) on \( I_{\delta_0,m_0} \times [-\eta; \eta] \) satisfying \( F(0) = 1 \), there exists a smooth function \( R \) around 0 such that one has the \( x \to 0 \) behaviour
\[
\mathcal{J}[\mathcal{G}_{\text{tot}}, A, B](x) = G(0) \mathcal{J}[\mathcal{W}_0 F, A(0), B(0)](x) + R(x) + O(|x|^g)
\] (F.97)

where
\[
g = \frac{1}{2}(m_+ + m_0 + m_-) + 1 + (A + B)(0) + 1 - \tau ,
\] (F.98)

and
\[
\mathcal{W}_0(u, \kappa) = \sum_{\alpha,|a| = m_0 \atop \alpha = (m_0, \beta)} c_{\alpha} \cdot \kappa^{|a|} u^\beta \quad \text{with} \quad m_0 \in 2\mathbb{N} .
\] (F.99)
Proof —

One first implements the change of the \( w^{(v)} \)-integration variables \( w^{(v)} \leftrightarrow s^{(v)} \) with

\[
s_k^{(v)} = \sum_{a=k}^{m_v} w_a^{(v)} \quad \text{i.e.} \quad w_k^{(v)} = s_k^{(v)} - s_{k+1}^{(v)} \quad \text{for } k = 1, \ldots, m_v - 1 \quad \text{and } w_{m_v}^{(v)} = s_{m_v}
\]

whose Jacobian equals to 1. The inequalities \( 0 \leq \upsilon_k \leq \delta_v \) defining the integration domain in the original variables can be recast in terms of an equivalent set of encased inequalities defining the integration domain in the \( s^{(v)} \) variables:

\[
\begin{align*}
0 \leq s_1^{(v)} & \leq m_v \delta_v \\
\frac{s_k^{(v) -}}{\delta_v} \leq s_k^{(v)} \leq \frac{s_k^{(v) +}}{\delta_v} \quad & \text{for } k = 2, \ldots, m_v \\
\end{align*}
\]

This recasts the original integral in the form

\[
\mathcal{J}[\mathcal{G}_{\text{tot}}, A, B](x) = \prod_{v=1}^{m_v} \left\{ \int_0^{m_{\delta_v}} ds_k^{(v)} \prod_{k=2}^{m_v} \int_0^{s_k^{(v)}} ds_k^{(v)} \int_0^{\delta_v} dr \prod_{a=1}^{m_v} \left( \frac{1}{s_a^{(v)} - s_a^{(v) -}} \right) \right. \\
\times \left[ (1 - t)^A(x) \cdot t^B(x) \mathcal{G}_{\text{tot}}(x, (1 - t)\varphi_1(x_a), t\varphi_2(x_a)) \cdot \Xi[\varphi_3(x_a)] \cdot [\varphi_3(x_a)]^A(x)+B(x+1) \right]_{x_a=\text{even}}
\]

where it is understood that \( s_{m_v+1}^{(v)} = 0 \), while \( x = (x_a, \alpha \circ \varphi_1(x_a) + t \beta \circ \varphi_2(x_a)) \), in which

\[
x_s = (x_s^{(+)}, x_s^{(-)}) \quad \text{where} \quad x_s^{(v)} = \left( \frac{s_k^{(v) -} - s_1^{(v)}}{\delta_v}, \ldots, \frac{s_k^{(v) +} - s_k^{(v) -}}{\delta_v}, \frac{s_k^{(v)}}{\delta_v} \right).
\]

Finally, one has \( \varphi_1(x_s) = x + s_s^{(+)} - s_1^{(-)} \).

Recall that the integrand vanishes when, either, \( s_1^{(+)} \geq \delta_+ \) or \( s_1^{(-)} \geq \delta_- \). Thus, one may reduce the \( (s_1^{(+)}, s_1^{(-)}) \) integration from \([0; m_v \delta_v] \times [0; m_v \delta_v] \) to the rectangle \([0; \delta_+] \times [0; \delta_-] \). However, as soon as it holds \( 0 \leq s_1^{(v)} \leq \delta_v \), one can readily check that the endpoints of integration \( s_{k-1}^{(v) -}, s_{k-1}^{(v) +} \) in (F.101) for the variable \( s_k^{(v)} \) reduce to

\[
s_{k-1}^{(v) -} = 0 \quad \text{and} \quad s_{k-1}^{(v) +} = s_{k-1}^{(v)} \quad \text{for} \quad k = 2, \ldots, m_v.
\]

Upon this reduction, one may implement another change of variables \( s^{(v)} \leftrightarrow u^{(v)} \) with

\[
s_a^{(v)} = u_a^{(v)} \ldots u_{a-1}^{(v)} \quad \text{and} \quad \det[D_{u^{(v)}} s^{(v)}] = \prod_{a=2}^{m_v} \left\{ u_1^{(v)} \cdots u_{a-1}^{(v)} \right\} = \prod_{a=1}^{m_v} \left\{ u_a^{(v)} \right\}^{m-a}.
\]

Since

\[
\prod_{a=1}^{m_v} \left( \sqrt{s_a^{(v)} - s_a^{(v) +}} \right) = \prod_{a=2}^{m_v} \left( \sqrt{1 - u_a^{(v)}} \cdot \prod_{a=1}^{m_v} \left\{ u_a^{(v)} \right\}^{m_a + 1 - a} \right)
\]

the integral takes the form

\[
\mathcal{J}[\mathcal{G}_{\text{tot}}, A, B](x) = \prod_{v=1}^{m_v} \left\{ \int_0^{\delta_v} du_1^{(v)} \left[ u_1^{(v)} \right]^{m_v - 2} \prod_{k=2}^{m_v} \int_0^{1} du_k^{(v)} \left( \frac{u_k^{(v)} - m_v - 1 - k}{2} \right) \right. \\
\times \left[ \mathcal{G}_{\text{tot}}(y, (1 - t)\varphi_1(y_u), t\varphi_2(y_u)) \cdot (1 - t)^A(y) \cdot t^B(y) \cdot \Xi[\varphi_3(y_u)] \cdot [\varphi_3(y_u)]^A(y)+B(y+1) \right]_{y_u=\text{even}}.
\]
There, one should identify \( y = (y_u, a \circ \varphi(x_u) + t_b \circ \varphi(x_u)) \) where \( y_u = (y_u^+, y_u^-) \) and
\[
y_u^{(o)} = \left( \sqrt{u_{m_2}^{(o)}(1 - u_2^{(o)}), \ldots, \sqrt{u_1^{(o)} \cdots u_{m_2}^{(o)}}} \right).
\]
(F.108)

Finally, \( \varphi(x_u) = x + u_1^{(+)} - u_1^{(-)} \).

At this stage, one may perform explicitly the reduction of the integration domain due to the presence of the Heaviside function. One has \( \varphi(x_u) \geq 0 \) on
\[
\left\{ \left( u_1^{(+)} , u_1^{(-)} \right) \in [0 ; \delta_+] \times [0 ; \delta_-] : -\min(0, x) \leq u_1^{(+)} \leq \delta_+ \text{ and } 0 \leq u_1^{(-)} \leq \min(u_1^{(+)} + x, \delta_-) \right\}.
\]
(F.109)

Since the integrand vanishes when \( u_1^{(-)} \geq \delta_- \), one may just as well, for fixed \( u_1^{(+)} \), extend the \( u_1^{(-)} \) integration to the segment \( [0 ; u_1^{(+)} + x] \).

This form of the integration domain takes explicitly into account the constraints implied by the presence of the Heaviside function. This reduced form of the integration domain leads naturally to the last change of variables
\[
u_1^{(+)} = z_1^{(+)} , \quad a = 1, \ldots, m_+ , \quad u_1^{(-)} = z_1^{(-)} \cdot \left( z_1^{(+)} + x \right) \quad \text{and} \quad u_0^{(-)} = z_a^{(-)} , \quad a = 2, \ldots, m_-.
\]
(F.110)

The integration variables are then gathered in \( r = (r_z, a \circ \varphi(x_r) + t_b \circ \varphi(x_r)) \), with
\[
r_z = \left( r_z^{(+)} , [z_1^{(+)} + x]^2 \cdot r_z^{(-)} \right)
\]
(F.111) and
\[
r_z^{(o)} = \left( \sqrt{z_1^{(o)}(1 - z_2^{(o)}), \ldots, \sqrt{z_1^{(o)} \cdots z_{m_-}^{(o)}}} \right) \cdots \sqrt{z_1^{(o)} \cdots z_{m_+}^{(o)}}.
\]
(F.112)

All of this recasts the integral in the form
\[
\mathcal{J}[\mathcal{G}_{\text{tot}}, A, B](x) = \int_{\min(0, x)}^{\delta_+} \prod_{i=\pm}^{m_+} \int_0^{1} \prod_{k=2}^{1} \frac{d z_k^{(o)} \left( z_k^{(o)+} \cdot \frac{m_k - 1}{2} \right)}{\sqrt{1 - \frac{z_k^{(o)+}}{2}}} \cdot [z_1^{(+)} + x]^\frac{m_+}{2} : \prod_{i=\pm}^{m_+} \left[ z_i^{(o)+} \right]^\frac{m_i}{2} - 1
\]
\[
\times \int dr \left[ (1 - t)^{A(r)} \cdot t^{B(r)} \right] \cdot \left[ (z_1^{(+)} + x)(1 - z_1^{(-)}) \right]^{A(r) + B(r) + 1} \cdot \mathcal{G}_{\text{tot}}(r, (1 - t) \varphi(x_r), t \varphi(x_r)) \right]_{r_{z, \text{even}}}.
\]
(F.113)
in which \( \varphi(x_z) = (x + z_1^{(+)} \cdot (1 - z_1^{(-)}) \text{.} \) The \( L^1 \) nature of the integrand is manifest on the level of \( \mathcal{F}_{\text{tot}} \).

By composing the various expansions at \( 0 \), it is easy to see that
\[
\mathcal{G}_{\text{tot}}(r, (1 - t) \varphi(x_r), t \varphi(x_r)) = \sum_{\begin{array}{c} k \neq l, p = m_0 \\ 2k + 2l = m_0 \end{array}} C_{k,l,p}(z_2^{(+)} , z_1^{(-)} , t) \cdot (z_1^{(+)+} t \cdot (z_1^{(+)} + x)^{k + 1,A(0) + B(0)}
\]
\[
+ O \left( \sum_{\begin{array}{c} k \neq l, p = m_0 + 1 \\ 2k + 2l = m_0 \end{array}} D_{k,l,p}(z_2^{(+)} , z_1^{(-)} , t) \cdot (z_1^{(+)+} t \cdot (z_1^{(+)} + x)^{k + 1,A(0) + B(0)} \left\{ 1 + \left| \ln \left( z_1^{(+)} + x \right) \right| \right\}
\]
\[
+ O \left( \sum_{\begin{array}{c} k \neq l, p = m_0 \end{array}} \tilde{D}_{k,l,p}(z_2^{(+)} , z_1^{(-)} , t) \cdot (z_1^{(+)+} t \cdot (z_1^{(+)} + x)^{k + 2 - \tau,A(0) + B(0)} \right) \right) .
\]
(F.114)

There, \( z_2^{(+)} = (z_2^{(+)} , \ldots , z_2^{(+)}) \) and the functions \( C_{k,l,p}(z_2^{(+)} , z_1^{(-)} , t) , D_{k,l,p}(z_2^{(+)} , z_1^{(-)} , t) , \tilde{D}_{k,l,p}(z_2^{(+)} , z_1^{(-)} , t) \) are all continuous on \( [0 ; 1]^{m_+ + m_-} \). Finally, the remainders are differentiable.

An application of Lemma \( Ω_{\text{as}}[D, 2] \) relatively to the \( z_1^{(+)} \) integration then leads to the claim.
F.5.2 The integral associated with \(|u'_0(k_0)| > \nu\) regime

**Lemma F.6.** Assume the notations and hypotheses outlined in Lemma F.5 with the exception that \(b\) does not have to vanish at \(0\). Pick \(\delta_0\) small enough and such that \(\delta_0 > 2\delta_+\), and assume further that

\[
\mathcal{G}(u, x, y) = 0 \quad \text{if} \quad s > \eta' \quad \text{where} \quad u = (u^{(s^+), u^{(s^-)}, s), \tag{F.115}
\]

such that

\[
a \circ \varphi_t(v) + \int b \circ \varphi_t(v) > 2\eta' \quad \text{uniformly in} \quad t > \frac{\delta_0}{2}, \quad v \in [0; \delta_+]^{m_+} \times [0; \delta_-]^{m_-}. \tag{F.116}
\]

Consider the integral

\[
\chi[\mathcal{G}_{\text{tot}}, A, B](\tau) = \prod_{\nu = \pm} \left\{ \int_{0}^{\delta_\nu} \frac{d^{m_\nu} w^{(\nu)}}{\sqrt{w^{(\nu)}}} \right\} \int_{0}^{\delta_0} dt \times \left[ \mathcal{G}_{\text{tot}}(u, t, t + \varphi_t(u_w)) t^{A(u)} \cdot [t + \varphi_t(u_w)]^{B(u)} \cdot \Xi[\varphi_t(u_w)] \right]_{u_w = \text{even}} \tag{F.117}
\]

where the vectors \(u, u_w\) appearing under the integral sign are as defined in (F.94), the even part of a function is as defined in (A.9), while \(\varphi_t(u_w)\) has been defined in (F.95).

Then, the integrand belongs to \(L^1([0; \delta_+]^{m_+} \times [0; \delta_-]^{m_-} \times [0; \delta_0])\) and, for any smooth function \(F\) on \(I_{\delta_-, m_-} \times [-\eta; \eta]\) satisfying \(F(0) = 1\), there exists a smooth function \(R\) around 0 such that one has the \(x \to 0\) behaviour

\[
\mathcal{J}[\mathcal{G}_{\text{tot}}, A, B](x) = G(0) \mathcal{J}[W_0 F, A(0), B(0)](x) + R(x) + O(|x|^{F}) \tag{F.118}
\]

where

\[
\varrho = \frac{1}{2}(m_+ + m_0 + m_-) + 1 + (A + B)(0) + 1 - \tau, \tag{F.119}
\]

and

\[
W_0(u, \kappa) = \sum_{\alpha, \lambda = m_0 \atop \alpha = (\alpha_0, \beta)} c_\alpha \cdot \kappa^{\alpha_0} u^\beta \quad \text{with} \quad m_0 \in \mathbb{N}. \tag{F.120}
\]

**Proof** —

The very same chain of transformation that was implemented in the proof of Lemma F.5 and the vanishing condition (F.115) allows one to recast the original integral as

\[
\mathcal{J}[\mathcal{G}_{\text{tot}}, A, B](x) = \int_{-\min(0, x)}^{\delta_+} \int_{0}^{\delta_-} \int_{0}^{\delta_0} dt \prod_{\nu = \pm} \left\{ \int_{0}^{\delta_\nu} \frac{d^{m_\nu} w^{(\nu)}}{\sqrt{w^{(\nu)}}} \right\} \times \left[ z_{11}^{(s^+)} + x \right] \prod_{\nu = \pm} \left\{ \left[ z_{k_1}^{(s^+)} \right]^{m_\nu} \right\} \cdot \phi(t) \cdot \left[ t^{A(r)} \cdot [t + (z_{11}^{(s^+)} + x)(1 - z_{11}^{(s^+)} + x)]^{B(r)} \right] \mathcal{G}_{\text{tot}}(r, t, t + (z_{11}^{(s^+)} + x)(1 - z_{11}^{(s^+)})) \right\rvert_{r_2 = \text{even}} \tag{F.121}
\]

in which \(r, r_2\) are as defined in (F.112), (F.111) and \(\phi \geq 0\) is smooth with compact support on \([0; \delta_0]\) and such that \(\phi|_{[0; \delta_0/2]} = 1\). The \(L^1\) nature of the integrand is already manifest on the level of (F.121).

Then a direct application of Lemma F.7 leads to the claim. ■
Lemma F.7. Let $\delta_0 > 2\delta_+ > 0$, $m_\pm \in \mathbb{N}^*$ and $x \in \mathbb{R}^*$. Consider the integral

$$\mathcal{L}(x) = \int_{-\min(0,x)}^{\delta_+} \int_0^1 \int_0^{\delta_0} dt \phi(t) \cdot [z_+ + X]^{\frac{m_+}{\tau}} \cdot [z]^{\frac{m_-}{\tau} - 1} \times \left[ F^{(Z)} \cdot \left[ t + (z_+ + X)(1 - z_+) \right] \right]^{B^{(Z)}(Z_0, t, t + (z_+ + X)(1 - z_+))} \text{d}z_+ \text{d}z \text{d}t \text{z}_0\text{-even}$$

in which

$$Z = \left( z_0, g_1((z_+ + X)(1 - z_-)) + r g_2((z_+ + X)(1 - z_-)) \right), \quad z_0 = \left( \sqrt{z_+}, \left| z_-(z_+ + x) \right|^2 \right), \quad \text{F.123}$$

$g_1, g_2$ are smooth, such that $g_1(0) = 0, g_2(0) = 1$ while the even part of a function is as given in (A.9). Furthermore, the function $F$ is assumed smooth and has the small argument expansion, with a differentiable remainder:

$$\mathcal{F}(x, y, u, v) = x^{2\alpha} y^{2\tau} \left( F_0 + O(u^{1-\alpha} + v^{1-\alpha} + x + y) \right) \text{ for some } 0 < \tau < 1,$$

and integers $s_\pm$. The functions $A, B$ are smooth, $A, B > -1$, while $\phi \geq 0$ is smooth with compact support on $[0; \delta_0]$ and such that $\phi(0, \delta_0)/2 = 1$

Then, for any function of two variables $G$ such that $G(0) = 1$, there exists a smooth function $\tau$ around $x = 0$ such that

$$\mathcal{L}(x) = F_0 \int_{-\min(0,x)}^{\delta_+} \int_0^1 \int_0^{\delta_0} dt \phi(t) \cdot [z_+ + X]^{\frac{m_+}{\tau}} \cdot [z]^{\frac{m_-}{\tau} - 1} \times \left[ F^{(Z)} \cdot \left[ t + (z_+ + X)(1 - z_+) \right] \right]^{B^{(Z)}(Z_0, t, t + (z_+ + X)(1 - z_+))} \cdot G(z_0) \text{z}_0\text{-even}$$

\[ + \tau(x) + O\left( x^{3+\alpha_0 + \beta_0 + \alpha_+ + \alpha_-} \cdot |\ln x| + x^{3+\alpha_0 + \beta_0 + \alpha_+ + \alpha_- - \tau} \right) \text{.} \quad \text{F.125} \]

with

$$\alpha_+ = s_+ + \frac{m_+}{2} - 1, \quad \alpha_- = s_- + \frac{m_-}{2}. \quad \text{F.126}$$

Proof —

We only discuss the proof in the case of $x > 0$ and small enough in that the case $x < 0$ can be dealt with in much the same way.

For further convenience, set $a_0 = A(0)$ and $b_0 = B(0)$. Let $n$ be such that

$$-2 < 1 + a_0 + b_0 + \alpha_- + \alpha_+ - n < -1.$$

It is obvious from the form of the integrand that $\mathcal{L}(x)$ defines a smooth function of $x$ on $\mathbb{R}^*$ and that its derivative can be obtained directly by carrying the derivative under the integral sign. Then, it holds

$$\frac{\partial}{\partial x} \left[ F^{(Z)} \cdot \left[ t + (z_+ + X)(1 - z_+) \right] \right]^{B^{(Z)}(Z_0, t, t + (z_+ + X)(1 - z_+))} \text{z}_0\text{-even}$$

$$= \sum_{p=0}^n C_n^p \cdot (\alpha_-)_{n-p} (b_0)_p \cdot \frac{z_+^{\alpha_+} \cdot z_-^{\alpha_-} (z_+ + X)^{\alpha_- - n + p} \left[ t + (z_+ + X)(1 - z_-) \right]^{b_0 - p} (1 - z_-)^p \cdot t^p}{\left[ t + (z_+ + X)(1 - z_-) \right]^{1-\tau}}$$

\[ \times \left[ F_0 + O\left( t^{1-\tau} + \left[ t + (z_+ + X)(1 - z_-) \right] \right) \right] \text{F.128} \]
where \((x)_p = x(x-1)\cdots(x-p+1)\) refers to the descending Pochhammer symbol.

Thus, upon setting

\[
\mathcal{T}(a, b, c, d, e, f) = \int_{0}^{\delta_2} dz_+ \int_{0}^{1} dz_- \int_{0}^{\delta_1} dt \phi(t) \cdot t^{a}[t + (z_+ + x)(1 - z_-)]^b z_+^{c} z_-^{d} (x + z_+)^{e}(1 - z_-)^{f}
\]

all together, one gets that

\[
\partial_x^n L(x) = \sum_{p=0}^{n} C_{n}^{p} F_0(b_0)p(\alpha_-)_{n-p} T\left(a_0, b_0 - p, \alpha_+, \alpha_-, \alpha_-- n + p, p\right) + \sum_{\nu \in \{\pm\}} \left|T\left(a_0 + \delta_0\nu, b_0 - p, \alpha_+ + \delta_1\nu, \alpha_-, \delta_0\alpha_--\delta_1\alpha_+, \alpha_-- n + p + \delta_0\alpha_-\right)\right|
\]

At this stage, it remains to focus on \(\mathcal{T}\). There, one implements the change of variables

\[
t = \frac{\bar{\xi} - \nu}{1 - \nu} \quad \text{with} \quad \bar{\xi} = (z_+ + x)(1 - z_-) \quad \text{i.e.} \quad \nu = \frac{t}{t + \bar{\xi}},
\]

leading to

\[
\mathcal{T}(a, b, c, d, e, f) = \int_{0}^{\delta_2} dz_+ \int_{0}^{1} dz_- \bar{\xi}_+^{\nu} \bar{\xi}_-^{\nu} (1 - z_-)^{f+a+b+1} (z_+ + x)^{e+a+b+1} h((z_+ + x)(1 - z_-))
\]

where

\[
h(\bar{\xi}) = \int_{0}^{\delta_1} d\nu \frac{\nu^\mu}{(1 - \nu)^{2+a+b}} \phi\left(\frac{\bar{\xi} - \nu}{1 - \nu}\right).
\]

The fact that \(\phi\) is smooth with compact support on \([0; \delta_0]\) entails that for \(\Re(a) > -1\), and irrespectively of the value of \(\nu\), \(h\) is smooth in the neighbourhood of the origin and \(h(0) = \Gamma(a + 1)\Gamma(-1 - a - b)/\Gamma(-b)\), so that upon making use of hypergeometric like notations, an application of Lemma D.2 yields

\[
\mathcal{T}(a, b, c, d, e, f) = \Gamma\left(\begin{array}{c}
-2 - a - b - c - e, -1 - a - b, f + a + b + 2 \\
-b, f + a + b + d + 3, -1 - a - b - e
\end{array}\right) \times \Gamma(a + 1, d + 1, 1 + e) \cdot x^{3+c+e+a+b+c} + \mathcal{R}(x) + O(x^{3+c+e+a+b+c}),
\]

in which \(\mathcal{R}(x)\) is smooth in \(x\). Upon inserting this expansion into (F.130), one gets

\[
\partial_x^n L(x) = \sum_{p=0}^{n} C_{n}^{p} F_0(b_0)p(\alpha_-)_{n-p} \cdot x^{2+a_0+b_0+a_-+a_+} + \mathcal{R}(x) + O(x^{3+a_0+b_0+a_-+a_+})
\]

Then, the bounds (F.127) followed by a direct integration of the above expansion lead to the claim.
G  Asymptotic behaviour of a model integral

G.1  Reduction of the model integral into regular and singular parts

Recall that

\[ V(x) = \prod_{r=1}^{f} \left| \prod_{a \neq b} (x_{a}^{(r)} - x_{b}^{(r)}) \right|^2. \]  \hspace{1cm} (G.1)

Proposition G.1. Let \( \delta_{0} > 0, \sum_{r=1}^{f} n_{r} \geq 2, \xi_{r} \in \mathbb{R}^{*} \) and \( \varepsilon_{r} \in \{ \pm 1 \} \) be such that

\[ \sum_{r=1}^{f} \varepsilon_{r} \xi_{r}^{2} n_{r} \neq 0. \]  \hspace{1cm} (G.2)

Consider the integral

\[ \mathcal{J}(x) = \int_{D_{n}} e^{-(x,x)}V(x) \prod_{\nu = \pm} \left[ \Xi(x + z_{\nu}(x)) \cdot [x + z_{\nu}(x)]^{\delta_{\nu}-1} \right] \cdot dx \quad \text{over} \quad D_{n} = \prod_{r=1}^{f} \mathbb{R}^{n_{r}} \]  \hspace{1cm} (G.3)

where the functions \( z_{\nu}(x), \nu = \pm \), are quadratic forms

\[ z_{\nu}(x) = \frac{1}{2}(x, Ex) + (x, e) \cdot \frac{u + \nu v}{2v} \quad \text{with} \quad E = \begin{pmatrix} \varepsilon_{1}I_{n_{1}} & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & 0 & \varepsilon_{l}I_{n_{l}} \end{pmatrix}. \]  \hspace{1cm} (G.4)

The parameters \((u, v) \in \mathbb{R} \times \mathbb{R}^{+}\) are such that \( u \neq \pm v \) while, one has

\[ e = (e^{(1)}, \cdots, e^{(f)}) \quad \text{with} \quad e^{(r)} = -2v\xi_{r} \cdot (1, \cdots, 1) \in \mathbb{R}^{n_{r}}. \]  \hspace{1cm} (G.5)

Then, for \(|x| \neq 0\) and small enough,

\[ x \mapsto e^{-(x,x)}V(x) \prod_{\nu = \pm} \left[ \Xi(x + z_{\nu}(x)) \cdot [x + z_{\nu}(x)]^{\delta_{\nu}-1} \right] \in L^1(D_{n}) \]  \hspace{1cm} (G.6)

ensuring that \( \mathcal{J}(x) \) is well-defined.

Furthermore, there exists a smooth function \( S(x) \) in an open neighbourhood of \( x = 0 \) such that it holds

\[ \mathcal{J}(x) = \mathcal{J}_{\text{eff}}(x) + S(x) \]  \hspace{1cm} (G.7)

where

\[ \mathcal{J}_{\text{eff}}(x) = \prod_{r=1}^{f} |\xi_{r}|^{-n_{r}^{2}} \cdot \int_{\mathcal{D}_{\eta}^{(f)}} e^{-(x,x)}(\varphi_{\text{eff}}^{(f)}(x)) (\varphi_{\text{eff}}^{(sg)})(x) \prod_{\nu = \pm} \left[ \Xi(x + z_{\nu}(x)) \cdot [x + z_{\nu}(x)]^{\delta_{\nu}-1} \right] \cdot dx \]  \hspace{1cm} (G.8)

The integral above runs through the domain

\[ \mathcal{D}_{\eta}^{(f)} = \left\{ x \in D_{n} : \left| x_{1}^{(1)} \right| \leq C \eta', \forall (r,a) \in M : \left| t_{r}^{(a)}(x_{1}^{(1)}) - x_{a}^{(r)} \right| \leq \xi_{r}^{2} \eta' \right\} \]  \hspace{1cm} (G.9)
where $C > 0$ is some constant while $t_r^{(0)}(x) = \frac{\xi_1}{\xi_r^0}(\xi_1^2) x$. Furthermore, $\mathcal{M}$ stands for the positive definite matrix

$$
\mathcal{M} = \begin{pmatrix} \xi_1^{-2} I_n & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\cdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \xi_1^{-2} I_n \end{pmatrix},
$$

(G.10)

and the function $\tilde{z}_r$ takes the explicit form

$$
\tilde{z}_r(x) = - \sum_{(r,a) \in \mathcal{M}} \zeta_r \left( b_r(x_{(r)}) + \nu \nu x_{(r)} \right)
$$

where $b_r(x) = -\zeta_r \epsilon \frac{x^2}{2\xi_r^0} + ux$.

(G.11)

The variables $\zeta_r \in \{\pm\}$ are arbitrary.

Finally, one has that $\psi_r^{(sg)} \cdot \psi_r^{(s)}$ are arbitrary smooth functions on $\mathcal{D}_n$, such that $0 \leq \psi_r^{(s)} \leq 1$

$$
\begin{cases}
\psi_r^{(sg)} = 1 & \text{on } \mathcal{D}_{\eta r}^{(\text{eff})} \\
\psi_r^{(s)} = 1 & \text{on } \mathcal{D}_{\mathcal{M}_{\eta r}}^{(\text{eff})}
\end{cases}
$$

and

$$
\begin{cases}
\psi_r^{(sg)} = 0 & \text{on } \mathcal{D}_n \setminus \mathcal{D}_{\eta r}^{(\text{eff})} \\
\psi_r^{(s)} = 0 & \text{on } \mathcal{D}_n \setminus \mathcal{D}_{\mathcal{M}_{\eta r}}^{(\text{eff})}
\end{cases}
$$

(G.12)

with

$$
\mathcal{D}_{\eta r}^{(\text{eff})} = \left\{ x \in \mathcal{D}_n : \forall (r,a) \in \mathcal{M} : \left| t_r^{(0)}(x_{(1)}) - x_{(a)}^{(r)} \right| \leq \tilde{c}_{\eta r}^2 \eta \right\}.
$$

(G.13)

Proof —

To start with, I take for granted that

$$
x \mapsto e^{-(x,x)} V(x) \prod_{i=\pm} \left\{ \Xi(x + \tilde{z}_r(x)) \cdot \left[ x + \tilde{z}_r(x) \right]^{\delta_{i,-1}} \right\} \in L^1(\mathcal{D}_n).
$$

(G.14)

This issue will be dealt with at the end of the proof.

It is convenient to introduce $\tilde{\xi}_r = \zeta_r \xi_r$, with $\zeta_r \in \{\pm\}$ and change variables through a rescaling

$$
y^{(r)} = \tilde{\xi}_r x^{(r)}.
$$

(G.15)

This yields

$$
\mathcal{F}(x) = \int_{\mathcal{D}_n} \mathcal{F}_{\eta 0}(x) \cdot dx,
$$

(G.16)

with

$$
\mathcal{F}_{\eta 0}(x) = \left\{ \left| \mathcal{F}_{1}^{(\eta 0)} \right|^{\nu_r} \cdot e^{-(x,x_{(r)})} V(x) \cdot \prod_{i=\pm} \left\{ \Xi(x + \tilde{z}_r(x)) \cdot \left[ x + \tilde{z}_r(x) \right]^{\delta_{i,-1}} \right\} \right\}
$$

(G.17)

and the positive definite matrix $\mathcal{M}$ is as given in (G.10). Finally, the functions $\tilde{z}_r$ have been introduced in (G.11).

Observe that the functions $b_r$ appearing as building blocks of $\tilde{z}_r$ are such that $b_r'$ is strictly monotonous. Furthermore, it is readily checked that $t_r^{(0)}$, as given above of (G.9), satisfies $b_r'(t_r^{(0)}(x)) = b_r'(x)$ on $\mathbb{R}$.
One has the decomposition \( \mathcal{D}_n = \mathcal{D}^{(\perp; \text{eff})}_{\eta'} \sqcup \mathcal{D}^{(\parallel; \text{eff})}_{\eta'} \), with
\[
\mathcal{D}^{(\perp; \text{eff})}_{\eta'} = \left\{ x \in \mathcal{D}_n : \exists (r,a) \in \mathcal{M}, \ |b'_1(x_1^{(1)}) - b'_2(x_a^{(r)})| > \eta' \right\}
\] (G.18)
and
\[
\mathcal{D}^{(\parallel; \text{eff})}_{\eta'} = \left\{ x \in \mathcal{D}_n : \forall (r,a) \in \mathcal{M}, \ |b'_1(x_1^{(1)}) - b'_2(x_a^{(r)})| \leq \eta' \right\}.
\] (G.19)

Let \( \varphi^{(\parallel)}_{\text{eff}} \) be smooth and such that \( 0 \leq \varphi^{(\parallel)}_{\text{eff}} \leq 1 \)
\[
\varphi^{(\parallel)}_{\text{eff}} = 1 \text{ on } \mathcal{D}^{(\parallel; \text{eff})}_{\eta'} \text{ and } \varphi^{(\parallel)}_{\text{eff}} = 0 \text{ on } \mathcal{D}^{(\perp; \text{eff})}_{\eta'}.
\] (G.20)

This allows one to split the original integral as \( \mathcal{I}(x) = \mathcal{I}^{(\perp)}(x) + \mathcal{I}^{(\parallel)}(x) \), with
\[
\mathcal{I}^{(\perp)}(x) = \int_{\mathcal{D}^{(\perp; \text{eff})}_{\eta'}} \mathcal{F}^{(\perp)}_{\text{tot}}(x) \cdot dx \quad \text{and} \quad \mathcal{I}^{(\parallel)}(x) = \int_{\mathcal{D}^{(\parallel; \text{eff})}_{\eta'}} \mathcal{F}^{(\parallel)}_{\text{tot}}(x) \cdot dx,
\] (G.21)
where it is understood that
\[
\mathcal{F}^{(\perp)}_{\text{tot}}(x) = \left(1 - \varphi^{(\parallel)}_{\text{eff}}(x)\right) \cdot \mathcal{F}_{\text{tot}}(x) \quad \text{and} \quad \mathcal{F}^{(\parallel)}_{\text{tot}}(x) = \varphi^{(\parallel)}_{\text{eff}}(x) \cdot \mathcal{F}_{\text{tot}}(x).
\] (G.22)

**The integral \( \mathcal{I}^{(\perp)}(x) \)**

Pick \( R > 0 \) large enough and let \( \varphi^{(\perp)}_{\text{eff}} \) be a smooth function on \( \mathbb{R} \) satisfying
\[
0 \leq \varphi^{(\perp)}_{\text{eff}} \leq 1, \quad \varphi^{(\perp)}_{\text{eff}}(x) = 1 \text{ for } |x| \leq R \quad \text{and} \quad \varphi^{(\perp)}_{\text{eff}}(x) = 0 \text{ for } |x| \geq R + 1.
\] (G.23)

Thus, by writing \( 1 = 1 - \varphi^{(\parallel)}_{\text{eff}}(x_a^{(r)}) + \varphi^{(\parallel)}_{\text{eff}}(x_a^{(r)}) \), \( \varphi^{(\perp)}_{\text{eff}} \) allows one to build a partition of unity on \( \mathcal{D}_n \) which separates, in each variable, the pieces containing \( \infty \) in some of the variables and those being bounded
\[
1 = \sum_{\mathcal{M}_n \in \mathcal{M}_n; \mathcal{M}_\text{out}} \Phi_{\mathcal{M}_n; \mathcal{M}_\text{out}}(x) \quad \text{with} \quad \Phi_{\mathcal{M}_n; \mathcal{M}_\text{out}}(x) = \prod_{(a,r) \in \mathcal{M}_n} \left\{ \varphi^{(\perp)}_{\text{eff}}(x_a^{(r)}) \right\} \cdot \prod_{(a,r) \in \mathcal{M}_\text{out}} \left\{ 1 - \varphi^{(\perp)}_{\text{eff}}(x_a^{(r)}) \right\}
\] (G.24)
where the sum runs through all partitions of \( \mathcal{M} \) into two disjoint sets \( \mathcal{M}_n \) and \( \mathcal{M}_\text{out} \). This partition of unity leads to the decomposition
\[
\mathcal{I}^{(\perp)}(x) = \sum_{\mathcal{M} = \mathcal{M}_n \sqcup \mathcal{M}_\text{out}} \mathcal{I}^{(\perp)}_{\mathcal{M}_n; \mathcal{M}_\text{out}}(x)
\] (G.25)
where, upon making the change of variables
\[
y_a^{(r)} = (x_a^{(r)})^{-1} \text{ if } (r,a) \in \mathcal{M}_\text{out}, \quad y_a^{(r)} = x_a^{(r)} \text{ if } (r,a) \in \mathcal{M}_n
\] (G.26)
and denoting \( x(y; \mathcal{M}_\text{out}) \) the obtained vector, one has
\[
\mathcal{I}^{(\perp)}_{\mathcal{M}_n; \mathcal{M}_\text{out}}(x) = \int_{\mathcal{D}^{(\perp; \text{eff})}_{\eta'}} \mathcal{F}^{(\perp)}_{\text{tot}} (x(y; \mathcal{M}_\text{out})) \prod_{(a,r) \in \mathcal{M}_\text{out}} \left| y_a^{(r)} \right|^2 \cdot dy.
\] (G.27)
with
\[ D^{(1)}_{R;M_{\text{out}}} = \left\{ y : x(y; M_{\text{out}}) \in D^{(1)}_{\eta} \quad \text{and} \quad |y^{(r)}_a| \leq R + 1 \quad \forall (r, a) \in M_{\text{in}} \right\}, \] (G.28)

Note that \( x(y; M_{\text{out}}) \) does depend on the given choice of the partition.

It is easy to see that the integrand in \( (G.27) \) is smooth and vanishes on \( \partial D^{(1)}_{R;M_{\text{out}}} \). The smoothness at the origin follows from the Gaussian decay of \( \mathcal{F}^{(1)}_{\text{tot}} \) at infinity. Furthermore, for any \( k \in D^{(1)}_{R;M_{\text{out}}} \), there exists \( (r, a) \neq (1, 1) \) such that
\[ |b'_1(x^{(1)}_1(k; M_{\text{out}})) - b'_1(x^{(r)}(k; M_{\text{out}}))| > \eta'. \] (G.29)

Then, set
\[ f_{[r,a]}(y) = (y^{(1)}, y^{(2)}, \ldots, y^{(r-1)}, y^{(r)}, y^{(r+1)}, \ldots, y^{(t)}, \tilde{z}_a(x(y; M_{\text{out}})), \tilde{z}_g(x(y; M_{\text{out}}))) \] (G.30)
so that it holds
\[ \det [D_k f_{[r,a]}] = (-1)^{m_{r,a}} \xi_1 \xi_2 \nu \left( \frac{-1}{(k_1^{(1)})^2} \right)^{M_{\text{out}}(1)} \left( \frac{-1}{(k^{(r)})^2} \right)^{M_{\text{out}}(r)} \left( b'_1(x^{(r)}(k; M_{\text{out}})) - b'_1(x^{(1)}_1(k; M_{\text{out}})) \right), \] (G.31)
where \( m_{r,a} = a + \sum_{b=1}^{r-1} n_b \). The latter ensures the local invertibility of \( f_{[r,a]}(y) \) around \( k \) and thus the applicability of Lemma [2] to the integral of interest. Hence, for any given partition \( M_{\text{in}} \sqcup M_{\text{out}} \) of \( M \), \( \mathcal{F}^{(1)}_{M_{\text{in}};M_{\text{out}}}(x) \) is smooth in \( x \) and thus so is \( \mathcal{F}^{(1)}(x) \) as a finite sum of smooth functions.

**The integral \( \mathcal{F}^{(1)}(x) \)**

For the purpose of further reasoning, it is convenient to define
\[ \mathcal{P}(x) = \frac{x}{\xi_1 \xi_1} \sum_{r=1}^{t} n_r \xi_r \xi_r^2 \quad \text{and} \quad \mathcal{E}(x) = \sum_{r=1}^{t} n_r \xi_r \cdot b'_r(x^{(0)}_r(x)) \] (G.32)
so that, by assumption \( (G.2) \), \( |\mathcal{P}'(x)| \geq c > 0 \) for any \( x \in \mathbb{R} \) and some \( c > 0 \).

To start with, one observes that \( \tilde{z}_v(x) = \sum_v (x^{(1)}_1) + \delta \tilde{z}_v(x) \) with
\[ \tilde{z}_v(x) = -\left( \mathcal{E}(x) + \nu \nu \mathcal{P}(x) \right) \quad \text{and} \quad \delta \tilde{z}_v(x) = -\sum_{(r,a) \in M} \xi_r w^{(r)}_{v} (x^{(r)}_a, t^{(0)}_r (x^{(1)}_1)), \] (G.33)
in which \( w^{(r)}_{v}(x,y) = b_r(x) - b_r(y) + \nu \nu (x-y) \).

Observe that the bound
\[ |b'_1(x^{(1)}_1) - b'_r(x^{(r)}_a)| \leq \eta' \quad \text{is equivalent to} \quad |t^{(0)}_r(x^{(1)}_1) - x^{(r)}_a| \leq \xi_r^2 \eta', \] (G.34)
what ensures that \( w^{(r)}_{v}(x^{(r)}_a, t^{(0)}_r (x^{(1)}_1)) = O(\eta') \) under such bounds and thus, it holds uniformly in \( x \in D^{(1)}_{\eta} \) that \( \delta \tilde{z}_v(x) = O(\eta') \).
It is readily seen that
\[ Z'_v(x) = -\mathcal{D}'(x)\left(b'_1(x) + nv\right) \] (G.35)
so that \( Z'_v \) vanishes at the points \( s_v = \xi_1 e_1 \xi_2^2(u + nv) \). However, one has that for \( v, v' \in [\pm 1] \),
\[ Z_v(s_v) = -\frac{1}{2}(u + v'v)(u + (2v - v')v) \cdot \sum_{r=1}^{\ell} \epsilon_r n_r \xi_r^2 \neq 0 \] (G.36)
owing to the hypotheses of the proposition.

Thus, it follows from the above that \( Z_v \) is strictly monotonous on \( ]-\infty; s_v[ \) and on \( ]s_v; +\infty[ \). Furthermore, one has \( s_+ \neq s_- \). Thus one may introduce the three intervals
\[ I^{(-)} = ]-\infty; \min\{s_v\}[ , \quad I^{(c)} = ]\min\{s_v\}; \max\{s_v\}[ , \quad I^{(+) = ]\max\{s_v\}; +\infty[ . \] (G.37)
In each of these intervals \( x \mapsto Z_k(x) \) are both strictly monotonous. First assume that \( u \neq \pm 3v \). Then, the previous calculations ensure that \( Z_k \neq 0 \) on \( \partial I^{(c)} \), for any \( \tau \in \{c, \pm\} \). Let \( \tau_0 \in \{c, \pm\} \) be such that \( 0 \in I^{(\tau_0)} \). Note that \( \tau_0 \) is well defined since \( s_\pm \neq 0 \). This entails that \( 0 \notin \partial I^{(\tau)} \) for any \( \tau \in \{c, \pm\} \).

Finally, one decomposes
\[ \mathcal{D}^{(\text{eff})}_{\eta'} = \bigcup_{\tau \in \{c, \pm\}} \mathcal{D}^{(\text{eff})}_{\eta' \tau} \quad \text{with} \quad \mathcal{D}^{(\text{eff})}_{\eta' \tau} = \left\{ x \in \mathcal{D}^{(\text{eff})}_{\eta'} : x_1^{(1)} \in I^{(\tau)} \right\} \] (G.38)
what induces the decomposition of the original integral \( \mathcal{J}^{(\eta)}(x) = \sum_{\tau \in \{c, \pm\}} \mathcal{J}^{(\eta)}_{\tau}(x) \) with
\[ \mathcal{J}^{(\eta)}_{\tau}(x) = \int_{\mathcal{D}^{(\text{eff})}_{\eta' \tau}} \mathcal{J}^{(\eta)}_{\tau}(x) \cdot dx . \] (G.39)

• The integral \( \mathcal{J}^{(\eta)}_{\tau_0}(x) \)

The analysis depends on whether \(|u| > v \) or \(|u| < v \).

1) The \(|u| < v \) case.

Let \( \sigma = \text{sgn}(\mathcal{D}'(0) (u + v)) \). Since \( x \mapsto Z_v(x) \) is strictly monotonous on \( I^{(\tau_0)} \), \( 0 \in I^{(\tau_0)} \) it follows that
\[ \text{sgn}\left(Z'_v(x)\right) = \text{sgn}(Z'_v(0)) = -uv \sigma \quad \text{for} \quad x \in I^{(\tau_0)} . \] (G.40)

By using that \( Z_v(0) = 0 \), since \( x \mapsto Z_v(x) \) is strictly monotonous and since \( Z'_v(x) \) vanishes, at most, on \( \partial I^{(\tau_0)} \), one infers that
\[ \sigma v \xi_v(x) < -\eta' \quad \text{if} \quad x_1^{(1)} > C\eta' \]
\[ \sigma v \xi_v(x) > \eta' \quad \text{if} \quad x_1^{(1)} < -C\eta' , \] (G.41)
this uniformly in \( x \in \mathcal{D}^{(\text{eff})}_{\eta' \tau_0} \) and for some constant \( C > 0 \).
The domain $D$ integration domain so that it holds

\[ J^{(\beta)}_{\tau_0}(x) = \int_{\frac{x}{\eta'}} F_{\text{tot}}(x) \cdot dx \quad \text{for } |x| < \eta'/2. \]  

(G.42)

The domain $D^{(\text{eff})}_{\eta'}$ appearing above is as defined in (G.9).

Finally, let $\varphi^{(\text{sg})}$ be smooth on $\bigcap_{r=1}^\ell \mathbb{R}^{\tau_r}$ and such that

\[ 0 \leq \varphi^{(\text{sg})} \leq 1, \quad \varphi^{(\text{sg})} = 1 \text{ on } D^{(\text{eff})}_{\eta'} \text{ and } \varphi^{(\text{sg})} = 0 \text{ on } D_n \setminus D^{(\text{eff})}_{2\eta'}. \]  

(G.43)

Since the integrand vanishes anyway outside of $D^{(\text{eff})}_{\eta'}$, it holds

\[ J^{(\beta)}_{\tau_0}(x) = \int_{\frac{x}{\eta'}} \varphi^{(\text{sg})}(x) F_{\text{tot}}^{(\beta)}(x) \cdot dx, \]  

(G.44)

what corresponds exactly to $J^{(\beta)}_{\text{eff}}(x)$ as given in (G.8).

ii) The $|u| > v$ case.

Keeping the definition for $\sigma$ as above, the same reasonings ensure that, now, sgn$(\mathcal{Z}_u(x)) = -\sigma$. This then leads to

\[ \sigma z_u(x) < -\eta' \quad \text{if} \quad x_1^{(1)} > C\eta' \]
\[ \sigma z_v(x) > \eta' \quad \text{if} \quad x_1^{(1)} < -C\eta'. \]  

(G.45)

this uniformly in $x \in D^{(\beta)}_{\eta';\tau_0}$ and for some constant $C > 0$. One then introduces $\varphi^{(\text{sg})}$ as in (G.43) and using that

\[ J^{(\beta)}_{\tau_0}(x) = \int_{D^{(\beta,\text{eff})}_{2\eta';\tau_0}} F_{\text{tot}}^{(\beta)}(x) \cdot dx \]  

(G.46)

since $F_{\text{tot}}^{(\beta)}$ vanishes on $D^{(\beta,\text{eff})}_{2\eta';\tau_0} \setminus D^{(\beta,\text{eff})}_{\eta';\tau_0}$, one may decompose the integral as

\[ J^{(\beta)}_{\tau_0}(x) = J_{\text{eff}}(x) + J_{\tau_0;\text{out}}(x) \quad \text{with} \quad J_{\tau_0;\text{out}}(x) = \int_{D^{(\beta,\text{out})}_{\eta';\tau_0}} \left(1 - \varphi^{(\text{sg})}(x)\right) F_{\text{tot}}^{(\beta)}(x) \cdot dx \]  

(G.47)

in which $J_{\text{eff}}(x)$ is as given in (G.8), while

\[ D^{(\text{out})}_{\eta';\tau_0} = \left\{ x \in D^{(\beta)}_{\eta';\tau_0} : x_1^{(1)} > C\eta', \forall (r, a) \in M : \left| p^{(0)}_r(x_1^{(1)}) - x^{(r)}_a \right| < 2 \xi^2 \right\}. \]  

(G.48)

The above bounds ensure that if $|x| < \eta'/2$, then $|x + \tilde{z}_u(x)| \geq \eta'/2$ uniformly on $D^{(\text{out})}_{\eta';\tau_0}$. Then, derivation under the integral theorems ensure that $x \mapsto J_{\tau_0;\text{out}}(x)$ is a smooth function of $x$ in an open neighbourhood of $x = 0$.  

95
• The integral $J^{(\beta)}(x)$ with $\tau \neq \tau_0$

When $\tau \neq \tau_0$, by construction, it holds that $d(\ell, 0) > 0$, in which $d(A, x)$ stands for the Euclidian distance from the set $A$ to the point $x$. This property, along with the explicit form for $\mathcal{P}$ and $\mathcal{E}$ given in (G.32) both ensure that it holds

$$(0, 0) \notin \{(\mathcal{P}(x), \mathcal{E}(x)) : x \in \ell\}. \quad \text{(G.49)}$$

Furthermore, the previous arguments ensure that

$$\min_{x \in \ell} \left| \mathcal{E}(x) + v \mathcal{P}(x) \right| > 0. \quad \text{(G.50)}$$

The above properties guarantee that the integral $J^{(\beta)}(x)$ is a particular example of the general class of integrals considered in the section “Behaviour of $F^{(\beta)}$ in the regular case” of the proof of Theorem E.1.

One should simply make the identification $u_\nu \leftrightarrow b_\nu$. Then, that very same analysis ensures that $x \mapsto J^{(\beta)}(x)$ is smooth in $x$.

It remains to comment on the case when $u = 3v_1v_2$, for some $v \in \{\pm 1\}$. Then, by (G.36), for any $v \in \{\pm\}$ it holds that $\mathcal{I}(v, s_{\nu}) \neq 0$. However, one has that $\mathcal{I}_-(s_{\nu}) = 0$ and $\mathcal{I}_+(s_{\nu}) \neq 0$. One should then split the integration domain as

$$\left[-\infty; s_{-\nu}\right] \cup \left[s_{-\nu}; s_{\nu}\right] \cup \left[s_{\nu}; \infty\right] = \left[-\infty; s_{-\nu}\right] \cup \left[s_{-\nu}; s_{\nu}\right] \cup \left[s_{\nu}; \infty\right] \quad \text{(G.51)}$$

where $\epsilon > 0$ is taken small enough and, for simplicity, I assumed that $s_{\nu} > s_{-\nu}$, the other situation being tractable in a similar way. The analysis on all the intervals other that $[s_{\nu} - \epsilon; s_{\nu} + \epsilon]$ goes along the lines described above, while on $[s_{-\nu} - \epsilon; s_{-\nu} + \epsilon]$, one should proceed by implementing a change of variables analogous to (E.47). Reasonings as in (E.54)-(E.60) then allow one to conclude on the smoothness of such contributions.

• The $L^1(\mathcal{D}_n)$ character

It remains to prove the $L^1(\mathcal{D}_n)$ nature of the integrand. Since

$$V(x) \prod_{l \in n} \left\{ \Xi(\lambda + z_{\nu}(x)) \cdot [x + z_{\nu}(x)]^{d_{\nu} - 1} \right\} \quad \text{(G.52)}$$

grows algebraically in $\|x\|$ at infinity, the Gaussian prefactor $e^{-\langle x, x \rangle}$ ensures integrability at $\infty$. Furthermore, if both $\delta_{\nu} \geq 1$, then the integrand is bounded on compact subsets of $\mathcal{D}_n$ what entails its $L^1(\mathcal{D}_n)$ nature. If at least one inequality $0 < \delta_{\nu} < 1$ holds, then integrability issues may arise from a neighbourhood of the points where $x + z_{\nu}(x) = 0$. Moreover, since the integrand is strictly positive, it is enough to prove local integrability. By following the integral reduction steps that are outlined in the last part of the proof of Theorem E.1, one eventually ends up with one-dimensional integrals whose direct inspection shows that the local $L^1$-character boils down to the condition $0 < \delta_{\nu} < 1$.

G.2 Asymptotic behaviour of the model integral

Proposition G.2. Let $V$ be as given in (G.1). Consider the integral

$$\mathcal{J}(x) = \int_{\mathcal{D}_n} e^{-\langle x, x \rangle} V(x) \prod_{l \in n} \left\{ \Xi(\lambda + z_{\nu}(x)) \cdot [x + z_{\nu}(x)]^{d_{\nu} - 1} \right\} \cdot dx \quad \text{over } \mathcal{D}_n = \prod_{r=1}^n [0, \infty). \quad \text{(G.53)}$$

96
Here $\delta_0 > 0$ and it is assumed that $\sum_{r=1}^{\ell} n_r \geq 2$. Further, the functions $z_\nu(x)$ are quadratic forms

$$z_\nu(x) = \frac{1}{2}(x, Ex) + (x, e) \cdot \frac{u + \nu v}{2v} \quad \text{with} \quad E = \begin{pmatrix} \epsilon_1 I_{n_1} & 0 & \cdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$  

(G.54)

There $\epsilon_r \in \{\pm 1\}$, $(\cdot , \cdot)$ is the canonical scalar product on $\mathcal{D}_n$, $(u, v) \in \mathbb{R} \times \mathbb{R}^+$ are such that $u \neq \pm v$ and, given $\xi_r \in \mathbb{R}$ one has

$$e = (e^{(1)}, \ldots , e^{(l)}) \quad \text{with} \quad e^{(r)} = -2v\xi_r \cdot (1, \cdots, 1) \in \mathbb{R}^n.$$  

(G.55)

The parameters in play are such that

$$\sum_{r=1}^{\ell} \epsilon_r \xi_r^2 n_r \neq 0.$$  

(G.56)

Then, there exists a smooth function $S$ in a neighbourhood of 0 such that the $x \to 0$ asymptotic expansion holds:

$$J(x) = |x|^\theta \Gamma(\delta_+)\Gamma(\delta_)\Gamma(-\theta) \cdot \frac{[2\nu]^{\delta_+ + \delta_- - 1}}{\prod_{\nu \in \pm} [\nu - \nu u]^{\delta_\nu}} \cdot \frac{\prod_{r=1}^{\ell} G(2 + n_r) \cdot (2\pi)^{n_r - \delta_r}}{\sqrt{\sum_{r=1}^{\ell} \epsilon_r \xi_r^2 n_r}} \times \left( \Xi(t) \sin[\pi \nu u_+] \right) \left( \Xi(-t) \sin[\pi \nu u_-] \right) \left( 1 + O(\theta) \right) + S(x).$$  

(G.57)

where

$$\theta = \frac{1}{2} \sum_{r=1}^{\ell} n_r^2 - \frac{3}{2} + \delta_+ + \delta_-.$$  

(G.58)

and given $\epsilon \in \{\pm\}$

$$\nu_\epsilon = \frac{1}{2} \sum_{r=1}^{\ell} n_r^2 - \frac{1 + \epsilon\nu_0}{4} + \sum_{\nu = \pm \delta_\nu} \delta_\nu \quad \text{with} \quad \left\{ \begin{array}{l} s \rho = \text{sgn} \left( - \sum_{r=1}^{\ell} \epsilon_r n_r \xi_r^2 \right) \\ \sigma_\nu = \frac{1 - \nu_0 - \nu}{v} \end{array} \right.$$

(G.59)

Proof —

Recall the notation (A.2) for the vector $x$. Recall that, for $\omega \in \mathbb{R}^+$, one has the integral representation

$$\Gamma(\alpha) \int_{\mathcal{C}_\omega} \frac{e^{-it\omega}}{|-t|^\alpha} \, dt = \omega^{\alpha - 1} \Xi(\omega).$$  

(G.60)

where the contour $\mathcal{C}_\omega$ passes slightly above 0 and then goes to infinity in either of the two directions $\Re(t) \to \pm \infty$ along two rays $R_\omega$ which enjoy the property that $\Im(\omega t) \to -\infty$, linearly in $|t|$, when $t \to \infty$ along $R_\omega$.

The maps $\Xi_\nu = x + z_\nu$, are such that

$$D_x \Xi_\nu \cdot h = (x, Eh) + \frac{u + \nu v}{2v},$$  

(G.61)
so that $D_{x} \tilde{z}_{\nu}$ is surjective with the exception of the point $x_{\nu} = -E(1 + \nu \psi)$. Still, as ensured by the assumption (G.56), one has, for $|x| \text{small enough,}$

$$\tilde{z}_{\nu}(x_{\nu}) = x - \frac{1}{2} \left( \frac{u + \nu \psi}{2 \nu} \right) (e, E^{-1}e) \neq 0.$$  \hfill (G.62)

Therefore, $\tilde{z}_{\nu}$ is a submersion in an open neighbourhood of $M_{\nu} = (\tilde{z}_{\nu})^{-1}(0)$. Hence, $M_{\nu}$ is a $\sum_{r=1}^{l} n_{r} - 1$ dimensional sub-manifold of $D_{x}$ and, as such, has Lebesgue measure zero. It follows that upon setting $M = M_{x} \cup M_{\nu}$, one has the representation

$$\frac{\mathcal{I}(x)}{\Gamma(\delta_{x}) \Gamma(\delta_{\nu})} = \int_{D_{x} \setminus M} d\mathbf{x} \ e^{-(x,x) \mathbf{V}(\mathbf{x})} \int_{\phi_{\tilde{z}_{\nu}}(x)} \frac{d\lambda}{2\pi |-i\lambda|^{p_{\tilde{z}_{\nu}}}} \int_{\phi_{\tilde{z}_{\nu}}(x)} \frac{d\mu}{2\pi |-i\mu|^{p_{\tilde{z}_{\nu}}}}. \hfill (G.63)$$

This representation follows by first, replacing $D_{x}$ by $D_{x} \setminus M$ in $\mathcal{I}(x)$ as given by (G.53) and then using the integral representation (G.60) for the products involving the $\tilde{z}_{\nu}$ functions. The form of the $\lambda, \mu$ contours ensures exponential decay at infinity of these integrals. Furthermore, it is easy to check that

$$\left| \int_{\phi_{\tilde{z}_{\nu}}(x)} \frac{d\mu}{2\pi |-i\mu|^{p_{\tilde{z}_{\nu}}}} \right| \leq C |\tilde{z}_{\nu}(x)|^{a \nu - 1} \hfill (G.64)$$

for some constant $C$. By using Proposition[G.1] it is easy to see that

$$x \mapsto e^{-(x,x) \mathbf{V}(\mathbf{x})} \prod_{l=\nu} \left[ |\tilde{z}_{\nu}(x)|^{a \nu - 1} \right] \hfill (G.65)$$

is in $L^{1}(D_{x})$. Hence, one can apply dominated convergence to get

$$\frac{\mathcal{I}(x)}{\Gamma(\delta_{x}) \Gamma(\delta_{\nu})} = \lim_{\tau_{l}, \tau_{r} \to 0} \int_{D_{x}} d\mathbf{x} \ e^{-(x,x) \mathbf{V}(\mathbf{x})} \int_{\phi_{\tilde{z}_{\nu}}(x)} \frac{d\lambda}{2\pi |-i\lambda|^{p_{\tilde{z}_{\nu}}}} \int_{\phi_{\tilde{z}_{\nu}}(x)} \frac{d\mu}{2\pi |-i\mu|^{p_{\tilde{z}_{\nu}}}} \exp \left\{ -\tau_{l} \left( \frac{\lambda + \mu}{2} \right)^{2} - \tau_{r} \left( \frac{\lambda - \mu}{2} \right)^{2} \right\}. \hfill (G.66)$$

Since the integrand under the limit has now Gaussian convergence in $\lambda, \mu \to \infty$, one can deform the contours $\phi_{\tilde{z}_{\nu}(x)} \mapsto \mathbb{R} + i \alpha$ for some $\alpha > 0$ and small enough. Since the new $\lambda, \mu$ contours become $x$ independent and one has a rapid convergence of the integrand at infinity, by Fubini’s theorem, one can swap the order of integration and take the $x$ integration first. Then, using again that $M$ has Lebesgue measure 0, yields

$$\frac{\mathcal{I}(x)}{\Gamma(\delta_{x}) \Gamma(\delta_{\nu})} = \lim_{\tau_{l}, \tau_{r} \to 0} \int_{(\mathbb{R} + i\alpha)^{2}} d\mathbf{x} \ e^{-(x,x) \mathbf{V}(\mathbf{x})} \frac{e^{i\mathbf{z}_{\nu}(x)}}{(2\pi)^{2} |\mathbf{V}(\mathbf{x})|^{p_{\tilde{z}_{\nu}}}} \frac{e^{-i\mathbf{z}_{\nu}(x)}}{|-i\mathbf{V}(\mathbf{x})|^{p_{\tilde{z}_{\nu}}}} \exp \left\{ -\tau_{l} \left( \frac{\lambda + \mu}{2} \right)^{2} - \tau_{r} \left( \frac{\lambda - \mu}{2} \right)^{2} \right\}. \hfill (G.67)$$

Then the change of variables

$$\lambda = t(1 - \nu) + s \quad \text{and} \quad \mu = t(1 + \nu) - s \hfill (G.68)$$

recasts the integral in the form

$$\mathcal{I}(x) = \lim_{\tau_{l}, \tau_{r} \to 0} \int_{\mathbb{R} + i\alpha} dt \int_{\mathbb{R} + i\alpha} d\mathbf{w} e^{-\tau_{l} t^{2} - \tau_{r}(t - s\psi)^{2}}} \chi(t, s) \mathcal{I}_{\mathcal{V}}(t, s) \hfill (G.69)$$
where \( \alpha_u = \frac{u}{v} \),

\[
\chi(t, s) = \prod_{\nu = \pm} \left( \frac{i}{t - \sigma_\nu + vs} \right)^{\delta_\nu} \quad \text{and} \quad \sigma_\nu = 1 - \frac{u}{v}.
\]

while

\[
I_V(t, s) = \frac{\Gamma(\delta_+) \Gamma(\delta_-)}{2\pi^2} \int_{D_u} dx \, V(x) \cdot e^{-ix(x, x)} \cdot e^{-(x, [id + i\varepsilon]x) - 2i\alpha}.
\]

Above, id refers to the identity matrix acting on \( D \).

The \( x \) integral can already be taken explicitly. Indeed, for \( t \in \mathbb{R} + i\alpha \) with \( |\alpha| < 1 \), upon dilating the variables and then shifting them, viz., leading to the substitution

\[
x^{(r)} = \frac{x^{(r)} - is e^{(r)}}{2\sqrt{1 + i\varepsilon t}} \quad y^{(r)} \in \mathbb{R}^n_r,
\]

one obtains

\[
I_V(t, s) = \frac{e^{-\beta_s^2} \Gamma(\delta_+) \Gamma(\delta_-)}{2\pi^2} \cdot e^{-2i\alpha} \cdot \int_{D_x} dy \, V(y)e^{-y(y)}.
\]

where

\[
\beta_s = \sum_{r=1}^{\ell} \sqrt{\varepsilon_r^2 n_r} \cdot \frac{1}{1 + i\varepsilon t}.
\]

The remaining integrals can be taken by means of the Gaudin-Mehta formula \( (A.7) \), leading to

\[
I_V(t, s) = \frac{\Gamma(\delta_+) \Gamma(\delta_-)}{2\pi^2} \prod_{r=1}^{\ell} \left\{ \frac{(2\pi)^{\frac{\alpha}{2}} G(2 + n_r)}{2(1 + i\varepsilon_r t)^{\frac{\alpha}{2} n_r}} \right\} \cdot e^{-2i\alpha}.
\]

Thus, all-in-all, one gets

\[
\mathcal{J}(x) = \frac{\Gamma(\delta_+) \Gamma(\delta_-)}{2\pi^2} \prod_{r=1}^{\ell} \left\{ \frac{(2\pi)^{\frac{\alpha}{2}} G(2 + n_r)}{2^{\frac{\alpha}{2} n_r}} \right\} \cdot \mathcal{K}(x),
\]

where

\[
\mathcal{K}(x) = \lim_{\tau_r, \tau_s \to 0^+} \int_{\mathbb{R} + i\alpha} ds \int_{\mathbb{R} + i\alpha u} dt \, e^{-\tau_r t^2 - \tau_s (s - i\frac{\alpha}{2})^2} \frac{\chi(t, s) e^{-\beta_s^2 - 2i\alpha} \Gamma(\delta_+) \Gamma(\delta_-)}{\prod_{r=1}^{\ell} [1 + i\varepsilon_r t]^{\frac{\alpha}{2} n_r}}.
\]

Since the double limit \( \lim_{\tau_r, \tau_s \to 0^+} \lim_{\tau_r \to 0^+} \) exists, it can be computed in any way, in particular, by taking the successive limits \( \lim_{\tau_r \to 0^+} \lim_{\tau_s \to 0^+} \). For \( (t, s) \in (\mathbb{R} + i\alpha) \times (\mathbb{R} + i\alpha u) \) one has the lower bound \( |itr_\nu + vs| \geq \alpha > 0 \) and thus

\[
|\chi(t, s)| \leq \prod_{\nu = \pm} \alpha^{-\delta_\nu}.
\]
For such \( t, s \), owing to \(|\beta_i| \leq C \) for some \( C > 0 \), one thus has the bound
\[
\left| e^{-t_\ell^2 - \tau_\ell(s - \frac{\beta_i}{\text{Re}})^2} \frac{\chi(t, s) e^{-\beta_i s^2 - 2i\tau s}}{\prod_{r=1}^{\ell} [1 + i\epsilon_r t]^{\frac{1}{n_2}}} \right| \leq W(t, s)
\]
(G.79)
with
\[
W(t, s) = C \cdot e^{-\tau_\ell(\text{Re}(t))^2} e^{-\frac{\beta_i(s)^2}{\text{Re}(s)} + 2\alpha_s \text{Re}(s) \beta_i}.
\]
(G.80)

Observe that \( \text{Re}(\beta_i) > 0 \) for finite \( t \) and
\[
\beta_i = \frac{1}{2} \sum_{r=1}^{\ell} \epsilon_r e_r^2 n_r + \frac{1}{2} \sum_{r=1}^{\ell} \epsilon_r^2 n_r + O(t^{-3}) \quad \text{when} \quad \text{Re}(t) \to \pm \infty.
\]
(G.81)

Thus, for \( \alpha \) small enough, one has the bound
\[
\int_{\mathbb{R} + i\alpha} \int_{\mathbb{R} + i\alpha_u} W(t, s) \leq C \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\tau_\ell(\text{Re}(t))^2} e^{\frac{1}{2} (\beta_i(s))^2} \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\text{Re}(s) \beta_i}.
\]
(G.82)

Then, owing to the asymptotics at large \( \text{Re}(t) \) of \( \beta_i \), observe that for some \( C_0 \)
\[
\left| \frac{\beta_i}{\sqrt{\text{Re}(\beta_i)}} \right| \leq C_0.
\]
(G.83)

Upon the rescaling \( \text{Re}(s) \to \frac{1}{2} \text{Re}(s) \), the above bounds entail that, for some constant \( C' \)
\[
\int_{\mathbb{R} + i\alpha} \int_{\mathbb{R} + i\alpha_u} W(t, s) \leq C' \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\tau_\ell t^2} \cdot (1 + |\ell|) \cdot \int_{\mathbb{R}} e^{-s^2} < +\infty.
\]
(G.84)

Hence, since the bounding function in (G.79) is positive, by Fubini’s theorem, the above estimate ensures that it is in \( L^1((\mathbb{R} + i\alpha) \times (\mathbb{R} + i\alpha_u)) \), so that one can apply dominated convergence so as to take the \( \tau_\ell \to 0^+ \) limit in (G.77), hence yielding
\[
\mathcal{K}(x) = \lim_{\tau_\ell \to 0^+} \int_{\mathbb{R} + i\alpha} \int_{\mathbb{R} + i\alpha_u} e^{-\tau_\ell t^2} e^{-2i\tau x} \cdot \mathcal{F}(t) \quad \text{where} \quad \mathcal{F}(t) = \int_{\mathbb{R} + i\alpha_u} \int_{\mathbb{R}} \chi(t, s) e^{-\beta_i s^2}.
\]
(G.85)

The function \( \mathcal{F}(t) \) is analysed in Lemma G.5 which ensures that there exist three functions \( \mathcal{F}^{(\alpha)}(t) \) such that \( \mathcal{F} = \mathcal{F}^{(1)} + \mathcal{F}^{(2)} + \mathcal{F}^{(3)} \) where
- \( \mathcal{F}^{(1)} \) and \( \mathcal{F}^{(2)} \) are holomorphic on the set
  \[
  S_{\theta_0;A} = \left\{ t \in \mathbb{C} : |\text{Re}(t)| > A \quad \text{and} \quad t = \rho e^{i\theta} \quad \text{with} \quad \rho \in \mathbb{R} \quad \text{and} \quad |\theta| < \theta_0 \right\}
  \]
  (G.86)
where \( A \) is large enough while \( \theta_0 \) is small enough;
- when \( \text{Re}(t) \to +\infty \)
  \[
  \mathcal{F}^{(1)}(t) = \sqrt{\pi} \cdot \chi(t, 0) \cdot \left( \frac{1}{t} \right)^{\frac{1}{2}} \cdot (1 + O\left( \frac{1}{t} \right))
  \]
  (G.87)
with a remainder that is uniform and holomorphic on \( S_{\theta_0;A} \);
there exists $C_1, C_2 > 0$ such that, for $\rho \to \pm \infty$, $\theta > 0$ and $\varepsilon = -\operatorname{sgn}(\sum_{r=1}^{\ell} \xi_r^2 n_r)$,
\[
|\mathcal{F}^{(2)}(\rho e^{-i\theta \operatorname{sgn}(\rho)})| \leq C_1 e^{-C_2 \varepsilon \theta} \quad \text{and} \quad |\mathcal{F}^{(3)}(t)| \leq C_1 e^{-C_2 |\Re(t)|} \quad \text{for} \quad t \in \mathbb{R} + i\alpha.
\] (G.88)

Define the contours $\mathcal{C}^{(1)} = \mathcal{R}_c^{(1)} \cup_{\ell=\pm} \mathcal{R}_{c,\theta}^{(1)}$, where $\mathcal{R}_c^{(1)}$ is a curve joining $-A$ to $A$ in the upper half-plane but having a sufficiently small imaginary part, while $\mathcal{R}_{c,\theta}^{(1)}$ are two rays
\[
\mathcal{R}_c^{(1)} = \left\{ z = \pm A \pm i e^{\mp i\theta} \right\}, \quad \rho \in \mathbb{R}^+
\] (G.89)
go to $\infty$ in the direction $\Re(z) \to \infty$ with a slight angle $\theta > 0$ small enough, so that $\Im(tx) \to +\infty$ linearly in $|t|$ along these rays. The contour $\mathcal{C}^{(1)}$ has a similar structure: $\mathcal{C}^{(2)} = \mathcal{R}_c^{(2)} \cup_{\ell=\pm} \mathcal{R}_{c,\theta}^{(2)}$ with the two rays $\mathcal{R}_{c,\theta}$ given as
\[
\mathcal{R}_{c,\theta} = \left\{ z = \varepsilon A + \varepsilon e^{-i\theta} \right\}, \quad \rho \in \mathbb{R}^+
\] (G.90)
for $\theta > 0$ and small enough. Finally, take $\mathcal{C}^{(3)} = \mathbb{R} + i\alpha$.

Upon
i) inserting the decomposition $\mathcal{F} = \sum_{\alpha=1}^{3} \mathcal{F}^{(\alpha)}$ into $\mathcal{K}(x)$;

ii) splitting the integrations for each piece;

iii) deforming the $t$-integration contour to $\mathcal{C}^{(\alpha)}$ in the integrals associated with $\mathcal{F}^{(\alpha)}$;

one obtains three integrals whose respective integrands decay, uniformly in $\tau$, small enough, exponentially fast to 0 along $\mathcal{C}^{(\alpha)}$. Thus, one can invoke dominated convergence so as to send $\tau \to 0^+$ and obtain that
\[
\mathcal{K}(x) = \sum_{\alpha=1}^{3} \mathcal{K}^{(\alpha)}(x) \quad \text{with} \quad \mathcal{K}^{(\alpha)}(x) = \int_{\mathcal{C}^{(\alpha)}} \frac{e^{-2i\pi x t^{(\alpha)}(t)}}{\prod_{r=1}^{\ell} \left[ 1 + \varepsilon_r t \right]^{\frac{i\varepsilon_r}{2}}}. \quad \text{(G.91)}
\]

Since, the integrands in $\mathcal{K}^{(2)}(x)$ and $\mathcal{K}^{(3)}(x)$ are bounded and decay exponentially fast to 0 at $\infty$, this uniformly in $|x|$ small enough, one can apply derivation under the integral theorems so as to infer that $\mathcal{K}^{(2)} + \mathcal{K}^{(3)}$ is smooth in $x$ around 0.

Hence, it remains to focus on the $x \to 0^+$ behaviour of $\mathcal{K}^{(1)}(x)$ which can be decomposed as
\[
\mathcal{K}^{(1)}(x) = \sum_{c \in \{+,-,0\}} \mathcal{K}_c^{(1)}(x) \quad \text{with} \quad \mathcal{K}_c^{(1)}(x) = \int_{\mathcal{K}_c^{(1)}} \frac{e^{-2i\pi x t^{(1)}(t)}}{\prod_{r=1}^{\ell} \left[ 1 + \varepsilon_r t \right]^{\frac{i\varepsilon_r}{2}}}. \quad \text{(G.92)}
\]

Since the integration in $\mathcal{K}_0^{(1)}$ runs through a compact set and since the integrand in bounded, it follows that $\mathcal{K}_0^{(1)}(x)$ is a smooth function of $x$ by derivation under the integral theorems. It thus remains to estimate $\mathcal{K}_c^{(1)}(x)$. The properties of $\mathcal{F}^{(1)}(t)$ ensure that
\[
\frac{\mathcal{F}^{(1)}(t)}{\prod_{r=1}^{\ell} \left[ 1 + \varepsilon_r t \right]^{\frac{i\varepsilon_r}{2}}} = \varphi_{b,c}(t) + \psi_{b,c}(t) \quad \text{with} \quad \psi_{b,c}(t) = O\left( \frac{\varepsilon_{b,c}(t)}{t} \right), \quad \varsigma = \operatorname{sgn}[\Re(t)]
\] (G.93)
and $\varphi_{b,c}(t) = \gamma_{b,c} \cdot (s_c \cdot t)^{-\frac{\theta-1}{2}}$ with $\theta$ as defined in (G.58). The constant prefactor takes the form
\[
\gamma_{b,c} = \frac{\sqrt{\pi} \cdot e^{-i\xi_a}}{\sqrt{2 \sum_{r=1}^{\ell} n_r \xi_r^{2}}} \cdot \prod_{c=\pm} \frac{e^{i\xi_{b,c} \frac{t}{2}}}{|\tau_r|} \cdot \prod_{r=1}^{\ell} \left[ e^{-i\xi_r n_r^{2} t} \right]. \quad \text{(G.94)}
\]
\( \varsigma \) has been introduced in (G.59), \( \sigma_\nu \) is given by (G.70), while \( s_{\nu,\sigma} = \text{sgn}(\sigma_\nu) \).

This being settled, one splits the integrals as

\begin{equation}
\mathcal{K}_\nu^{(1)}(x) = \mathcal{I}_\nu^{(1)}(x) + \delta \mathcal{I}_\nu^{(1)}(x)
\end{equation}

where

\begin{equation}
\mathcal{I}_\nu^{(1)}(x) = \int_{\mathcal{R}_\nu^{(1)}} e^{-2itx} \varphi_\nu(t) \cdot dt \quad \text{and} \quad \delta \mathcal{I}_\nu^{(1)}(x) = \int_{\mathcal{R}_\nu^{(1)}} e^{-2itx} \psi_\nu(t) \cdot dt.
\end{equation}

The rest depends on how large \( \vartheta \) is. Let \( n \in \mathbb{N} \) and \( 0 \leq \alpha < 1 \) be such that \( \vartheta + 2 = \alpha + n \). First, focus on \( \delta \mathcal{I}_\nu^{(1)}(x) \) and introduce \( \psi_{\nu,0} = \psi_\nu \), and, for \( p \geq 1 \),

\begin{equation}
\psi_{\nu,1}(t) = \int_{\mathcal{R}_\nu^{(1)}} \psi_{\nu,1}(s) \cdot ds
\end{equation}

where the integration runs from \( \infty \), along \( \mathcal{R}_\nu^{(1)} \), up to \( t \in \mathcal{R}_\nu^{(1)} \). Then, integrating by parts \( n \) times, one has

\begin{equation}
\delta \mathcal{I}_\nu^{(1)}(x) = -\nu \sum_{p=0}^{n-1} (2ix)^p \psi_{\nu,p+1}(\nu \alpha) e^{-2i\nu \alpha t} + (2ix)^n \int_{\mathcal{R}_\nu^{(1)}} e^{-2itx} \psi_{\nu,n}(t) \cdot dt.
\end{equation}

One has \( \psi_{\nu,p}(t) = O(|t|^{-\alpha}) \). Upon setting \( s_\nu = \text{sgn}(x) \) and taking \( \vartheta > 0 \) and small enough, one gets

\begin{equation}
\left| \int_{\mathcal{R}_\nu^{(1)}} e^{-2ixt} \psi_{\nu,n}(t) \cdot dt \right| = \left| \int_{0}^{+\infty} e^{-2ix(C+|\sin(\vartheta)|^{-1})} \psi_{\nu,n}(\nu A + upe^{-ix\nu \vartheta} \cdot dp) \right|
\end{equation}

\begin{equation}
\leq C \int_{0}^{+\infty} \left[ \frac{e^{-2|x|\nu A |\sin(\vartheta)|^{-1}}}{A + \cos(\vartheta) \varrho} \right] \cdot \varrho \cdot C |x|^{\alpha-1} \int_{0}^{+\infty} u_\nu(\varrho) \cdot \varrho \cdot (G.99)
\end{equation}

for some constant \( C > 0 \). In the last integral, I have set

\begin{equation}
u_\nu(\varrho) = e^{-2\varrho \sin(\vartheta)} \cdot |x| A + \cos(\vartheta) \varrho|^{-\alpha} \longrightarrow u_\nu(\varrho) \in L^1(\mathbb{R}^+)
\end{equation}

point-wise on \( \mathbb{R}^+ \setminus \{0\} \). Since \( u_\nu(\varrho) \leq u_0(\varrho) \), by dominated convergence, the expansion (G.98) ensures that there exist smooth functions \( g_\nu \) such that

\begin{equation}
\delta \mathcal{I}_\nu^{(1)}(x) = g_\nu(x) + O(|x|^\vartheta+1).
\end{equation}

The integral \( \mathcal{I}_\nu^{(1)}(x) \) can be dealt with analogously by doing \( n-1 \) integration by parts, where \( \vartheta + 1 = n-1 + \alpha \). Namely, one has

\begin{equation}
\int_{\mathcal{R}_\nu^{(1)}} e^{-2iut} \cdot dt = \Gamma \left( \begin{array}{c} \vartheta + 1 \\ \vartheta \end{array} \right) \frac{1}{A^\vartheta} e^{-2iuxA} - 2iux \Gamma \left( \begin{array}{c} \vartheta \vartheta + 1 \\ \vartheta \end{array} \right) \int_{\mathcal{R}_\nu^{(1)}} e^{-2iux} \cdot dt
\end{equation}

\begin{equation}
eq e^{-2iux} \sum_{p=0}^{n-2} \frac{(-2iux)^p}{A^\vartheta-p} \Gamma \left( \begin{array}{c} \vartheta - p \vartheta + 1 \\ \vartheta \end{array} \right) + (2iux)^{n-1} \Gamma \left( \begin{array}{c} \vartheta - n + 2 \vartheta + 1 \\ \vartheta \end{array} \right) \int_{\mathcal{R}_\nu^{(1)}} e^{-2iux} \cdot dt.
\end{equation}

102
All this leads to the representation

\[ \mathcal{J}^{(1)}(x) = \sum_{\nu=\pm} \mathcal{J}^{(1)}_{\nu}(x) = \mathcal{J}_{\text{reg}}^{(1)}(x) + \mathcal{J}_{\text{sing}}^{(1)}(x) \]  

(G.103)

where \( \mathcal{J}_{\text{reg}}^{(1)} \) is smooth and given by

\[ \mathcal{J}_{\text{reg}}^{(1)}(x) = \sum_{p=0}^{n-2} \sum_{\nu=\pm} \gamma_{\nu} \cdot \Gamma \left( \frac{\theta - p}{\theta + 1} \right) \cdot e^{-2i\nu A} \frac{(-2i\nu)^p}{A^{\theta-p}} \]  

(G.104)

while, upon deforming the contours \( R^{(1)} \) to \( \nu A - i\sigma \mathbb{R}^+_v \), where \( \mathbb{R}^+_v \) corresponds to \( \mathbb{R}^+ \) oriented with the sign \( \nu \),

\[ \mathcal{J}_{\text{sing}}^{(1)}(x) = \Gamma \left( \frac{\theta - n + 2}{\theta + 1} \right) \cdot |2x|^{\alpha-1} \cdot \sum_{\nu=\pm} \gamma_{\nu} \cdot (i\nu \sigma)^n e^{2i\nu A} \int_0^{+\infty} \frac{e^{-2i|t|}}{(A + i\sigma)^{\theta}} \cdot dt \]

\[ + \gamma_{+} \cdot (-i\nu \sigma)^n e^{-2i\nu A} \int_0^{+\infty} \frac{e^{-2i|t|}}{(A - i\sigma)^{\theta}} \cdot dt \]

\[ = (-1)^n |2x|^\theta \cdot \Gamma \left( \frac{\theta - n + 2}{\theta + 1} \right) \sum_{\nu=\pm} \gamma_{\nu} \cdot (i\nu \sigma)^n e^{-2i\nu A} \int_0^{+\infty} \frac{e^{-t}}{(t + i\nu 2\pi A)^{\theta}} \cdot dt \]  

(G.105)

By dominated convergence, one has that

\[ \int_0^{+\infty} \frac{e^{-t}}{(t + i\nu 2\pi A)^{\theta}} \cdot dt = \Gamma(1 - \alpha) \cdot (1 + o(1)) \]  

(G.106)

when \( x \to 0 \). With some more efforts, one can even establish that it is a \( O(x) \). Since \( \Gamma(1 - \alpha) = \Gamma(n - 1 - \theta) \), straightforward calculation lead to

\[ \mathcal{J}_{\text{sing}}^{(1)}(x) = 2|2x|^\theta \cdot \sqrt{\pi} \cdot \Gamma(-\theta) \cdot \prod_{\nu=\pm} |\sigma_{\nu}|^{-\delta_{\nu}} \cdot \sum_{\nu=\pm} \left\{ \Xi(\nu \xi) \cdot \sin[\pi \nu \xi] \right\} \cdot (1 + O(x)) . \]  

(G.107)

Above, it is understood that

\[ \nu_{\epsilon} = \frac{1}{2} \sum_{\ell=1}^{\ell} \frac{n_{\ell}^2}{\epsilon_{\ell} \cdot 2^{\ell}} - \frac{1 + \epsilon_{\ell}}{4} + \sum_{\ell=1}^{\ell} \delta_{\nu} . \]  

(G.108)

Finally, the obtained estimates are readily seen to be differentiable. Thus, upon putting all the results together, the asymptotic expansion given in (G.57) follows.
G.3 Asymptotics of auxiliary functions

Lemma G.3. Let $\alpha > 0$, $t \in \mathbb{R} + i\alpha$, $(u, v) \in \mathbb{R} \times \mathbb{R}^+$ be such that $u \neq \pm v$ and set $\alpha_u = \frac{u}{v}$. Let

$$\mathcal{F}_n(t) = \int_{\mathbb{R} + i\alpha_u} ds \chi(t, s) s^n e^{-\beta t s^2}$$

(G.109)

where $\beta_t$ is as defined in (G.74) and $\chi(t, s)$ has been introduced in (G.70). Then, there exist functions $\mathcal{F}_n^{(a)}(t)$, $a = 1, 2, 3$ such that $\mathcal{F}_n = \mathcal{F}_n^{(1)} + \mathcal{F}_n^{(2)} + \mathcal{F}_n^{(3)}$, and enjoying the properties:

- $\mathcal{F}_n^{(1)}$ and $\mathcal{F}_n^{(2)}$ are holomorphic on the set
  $$S_{\theta_0; A} = \{ t \in \mathbb{C} : |\Re(t)| > A \text{ and } t = \rho e^{\imath \theta} \text{ with } \rho \in \mathbb{R} \text{ and } |\theta| < \theta_0 \}$$
  (G.110)

  where $A$ is large enough while $\theta_0$ is small enough;

- when $\Re(t) \to +\infty$
  $$\mathcal{F}_n^{(1)}(t) = \chi(t, 0) \cdot \left(\frac{1}{t}\right)^{\frac{n}{2}} \cdot u_n \cdot \left(\frac{1}{t^{2n}}\right)^{w_n} \cdot \left(1 + O\left(\frac{1}{t}\right)\right)$$
  (G.111)

  for some constant
  $$\begin{cases}
  u_n = \Gamma\left(\frac{n+1}{2}\right) & \text{if } n \in 2\mathbb{N} \\
  u_n \neq 0 & \text{if } n \in 2\mathbb{N} + 1 \quad \text{an integer}
  \end{cases}
  \quad \begin{cases}
  w_n = 0 & \text{if } n \in 2\mathbb{N} \\
  w_n \geq 1 & \text{if } n \in 2\mathbb{N} + 1
  \end{cases}$$
  (G.112)

  and with a remainder that is uniform and holomorphic on $S_{\theta_0; A}$;

- there exists $C_1, C_2 > 0$ such that, for $\rho \to \pm \infty$ and $\theta > 0$,
  $$|\mathcal{F}_n^{(2)}(\rho e^{-\imath \theta \sigma_n(\rho)})| \leq C_1 e^{-C_2 |\Re(t)|} \quad \text{and} \quad |\mathcal{F}_n^{(3)}(t)| \leq C_1 e^{-C_2 |\Re(t)|} \quad \text{for} \quad t \in \mathbb{R} + i\alpha,$$
  (G.113)

  where $\zeta = -\text{sgn}\left(\sum_{r=1}^{l} \epsilon_r \xi_r^2 n_r\right)$.

Proof —

One has to distinguish between the two cases: $|u| > v$ or $|u| < v$. Also, recall the notation $\sigma_\nu = 1 - \nu \frac{u}{v}$.

A) The regime $|u| > v$

Let $s_u = \text{sgn}(u)$, so that, using that $\text{sgn}(\sigma_\nu) = -\nu s_u$ one can recast

$$\chi(t, s) = \prod_{\nu \in \Delta} \left(\frac{iv}{s - is_u|\sigma_\nu|}\right)^{\delta_{\nu}}$$

(G.114)

meaning that, for fixed $t$, the map $s \mapsto \chi(t, s)$ has cuts along $s_u|\sigma_\nu| - i \mathbb{R}^\nu$. Since the integration runs through $\mathbb{R} + i\alpha_u$, and, for $t \in \mathbb{R} + i\alpha$,

$$\Im\left(i\alpha_u - s_u|\sigma_\nu|t\right) = \alpha u$$

(G.115)
the cut along $s_u|\sigma_+| - i\mathbb{R}^+$ lies below the integration line $\mathbb{R} + i\alpha u$ and the one along $s_u|\sigma_-| - i\mathbb{R}^-$ lies above the integration line $\mathbb{R} + i\alpha u$. One can represent $\beta_t = e^{i\theta_0|\beta_t|}$. Since, for $t$ large,

$$\beta_t = \frac{S\mathcal{C}_\beta}{t} + O(r^{-2})$$

one has $\theta_{\beta_t} \sim \frac{\pi}{2} s_u\mathcal{S}$ with

$$\mathcal{S} = -\text{sgn}\left(\sum_{r=1}^{\ell} e_r \xi_r^2 n_r\right)$$

$$C_{\beta_t} = \left|\sum_{r=1}^{\ell} e_r \xi_r^2 v^2 n_r\right|.$$  \hfill \text{(G.116)}

Here, I introduced the shorthand notation $s_u = \text{sgn}(\Re(t))$. Also, for further convenience, it is useful to set

$$s_u = s_u \cdot \mathcal{S}.$$  \hfill \text{(G.117)}

It is then convenient to deform the integration contour towards the curve depicted in Figure 3 in the case when $\Re(t)s_u < 0$ and the one depicted in Figure 4 in the case when $\Re(t)s_u > 0$.

Upon the change of variables $s = \beta_t^{-1/2} s'$ in the integration along $e^{-\frac{1}{2}\theta_0|\beta_t|}$, one decomposes $\mathcal{F}_n$ as

$$\mathcal{F}_n(t) = \left\{ \begin{array}{ll} \frac{1}{\beta_t^{\frac{1}{2} (n+1)}} & \text{if } s_u > 0 \\ \int_{\mathbb{R}} ds e^{-s^2} s'' \chi(t, s, \frac{s}{\sqrt{\beta_t}}) + \int_{\Gamma_{\beta_t}^{+}} ds s'' \chi(t, s) e^{-\beta_t s^2} & \text{if } s_u < 0 \end{array} \right.$$  \hfill \text{(G.118)}

Figure 3: Deformed contours in the case $|u| > v$ and for $\Re(t)s_u < 0$

Figure 4: Deformed contours in the case $|u| > v$ and for $\Re(t)s_u > 0$
The Gaussian integral

The first integral appearing in this decomposition can be analysed by observing that

$$\chi(t, \frac{s}{\sqrt{\beta}}) = \chi(t, 0) \cdot \left\{ 1 - \frac{s}{t \sqrt{\beta}} \sum_{n=\pm} \frac{\nu \delta_n}{\sigma_n} + O\left(\frac{s^2}{t^2 \beta}\right) \right\}$$

uniformly in $|s| \leq |\Re(t)|^{1/2}$. Thus, it is convenient to introduce the intervals

$$I_{in} = ]-\tau; \tau[ \quad \text{and} \quad I_{out} = ]-\infty; -\tau[ \cup \tau; +\infty[ \quad \text{with} \quad \tau = |\Re(t)|^{1/2}.$$

Then, one has

$$\left(\frac{1}{\beta_t}\right)^{\frac{1}{2}(n+1)} \int_{\mathbb{R}} ds \, e^{-s^2} \chi(t, \frac{s}{\sqrt{\beta}}) = \mathcal{H}_{in}(t) + \mathcal{H}_{out}(t)$$

(G.121)

where, for even $n$, one has

$$\mathcal{H}_{in}(t) = \chi(t, 0) \cdot \left(\frac{1}{\beta_t}\right)^{\frac{1}{2}(n+1)} \left\{ \int_{I_{in}} e^{-s^2} \cdot ds + O\left(\int_{I_{in}} \frac{s^{2+n}e^{-s^2}}{\beta_t t^2} \cdot ds \right) \right\}$$

(G.122)

The integrations can be extended to infinity upon adding some $O\left(e^{-\frac{1}{2} s^2}\right)$ corrections, so that, for even $n$, one has

$$\mathcal{H}_{in}(t) = \chi(t, 0) \cdot \left(\frac{1}{\beta_t}\right)^{\frac{1}{2}(n+1)} \left\{ \Gamma\left(\frac{n+1}{2}\right) + O\left(1, \frac{1}{\beta_t t^2}\right) + O\left(e^{-\frac{1}{2} \sqrt{\Re(t)}}\right) \right\}.$$ 

(G.123)

Similar handlings in the case of odd $n$ entail that in such a case

$$\mathcal{H}_{in}(t) = \chi(t, 0) \cdot \left(\frac{1}{\beta_t}\right)^{\frac{1}{2}(n+1)} \left\{ \left(\frac{1}{\sqrt{\beta}}\right)^{w_n} \cdot u_n + O\left(1, \frac{1}{\beta_t t^2}\right) + O\left(e^{-\frac{1}{2} \sqrt{\Re(t)}}\right) \right\}$$

(G.124)

for some integer $w_n \geq 1$ and a coefficient $u_n \neq 0$.

Regarding to the second contribution in (G.121), it can be presented as

$$\mathcal{H}_{out}(t) = \left(\frac{1}{\beta_t}\right)^{\frac{1}{2}(n+1)} \int_{0}^{+\infty} ds \, e^{-(s+\tau)^2} \chi(t, \frac{s+\tau}{\sqrt{\beta}}) + (-1)^n \chi(t, \frac{-s+\tau}{\sqrt{\beta}}).$$

(G.125)

Since, for $s \geq 0$,

$$S(t + \nu \frac{s+\tau}{\sqrt{\beta}}) = \alpha(t + \nu \frac{s+\tau}{\sqrt{\beta}}) = \alpha \frac{t + \nu s + \tau}{\sqrt{\beta}} \sin \left(\theta_0 \frac{\nu s + \tau}{\sqrt{\beta}}\right)$$

(G.126)

which obviously does not vanish for $|t|$ large enough, with $t \in \mathbb{R} + i\alpha$, and is dominated in this regime by the second term, it follows that

$$|\chi(t, \frac{s+\tau}{\sqrt{\beta}})| \leq C \cdot |\beta_t|^{-\frac{\delta + \delta_n}{2}}$$

(G.127)
so that, upon inserting the large-\( t \) behaviour of \( \beta_t \), one has

\[
\left| H_{\text{res}}(t) \right| \leq C' \left| \frac{\beta}{\sqrt{2}} \right|^{\frac{1}{2(n+1)-\delta_1-\delta_2}} \cdot e^{-\sqrt{\Re(t)}} \int_0^{+\infty} dx e^{-x^2-2\pi t \left( \left| \beta \right| \left| \beta \right| + \left| \beta \right| + n \right)} = O\left( e^{-\frac{2}{3} \sqrt{\Re(t)} \right) \quad \text{(G.128)}
\]

Thence, all in all, since \( \chi(t,0) \cdot \beta_t \frac{1}{2(n+1)} \) has at most an algebraic growth in \( t \),

\[
\int_{\mathbb{R}} ds e^{-s^2 \chi(t,0) \cdot \beta_t} = \chi(t,0) \cdot \left( \frac{1}{\beta_t} \right)^{\frac{1}{2(n+1)}} \cdot \left( \frac{1}{\sqrt{\beta_t}} \right)^{w_0} \cdot \left\{ u_n + O\left( \frac{1}{t} \right) \right\} + O\left( e^{-\frac{2}{3} \sqrt{\Re(t)} \right)} \quad \text{(G.129)}
\]

Above, \( u_n \) and \( w_n \) are as appearing in (G.112). Finally, it is readily seen that both remainders are holomorphic in \( t \in S_{\theta_0,A} \) for some \( \theta_0 \) small enough and \( A \) large enough.

**The loop integral contribution**

It now remains to focus on the integral along \( \Gamma^\beta_{\text{in}} \). The contour \( \Gamma^\beta_{\text{in}} \) can be deformed as

\[
\Gamma^\beta_{\text{in}} \leftarrow \left\{ s_0 \tau \sigma_{\nu} - \nu \partial \mathcal{D}_{\text{in}} \right\} \cup \left\{ s_0 \tau \sigma_{\nu} + i \nu \epsilon ; -\nu T_{\nu} \right\} \quad \text{(G.130)}
\]

Here, \( -\nu \partial \mathcal{D}_{\text{in}} \) stands for the circle of radius \( \epsilon \) centred at 0 and oriented \( -\nu \) counterclockwise.

However, in doing so, one has to take into account the discontinuity of the integrand along the line \( s_0 \tau \sigma_{\nu} - i \nu \epsilon \cdot T_{\nu} \), where I have set

\[
T_{\nu} = -\nu \left( \tilde{S}(\nu) - s_0 \tau \sigma_{\nu} \right) \sim C \cdot |\Re(t)| \quad \text{(G.131)}
\]

for some \( C > 0 \) and when \( |\Re(t)| \to +\infty \). This yields that

\[
\int_{\Gamma^\beta_{\text{in}}} ds \chi(t,s) e^{\beta s t} = \sum_{\nu=1}^3 \Psi_{\nu \nu}(t) \quad \text{(G.132)}
\]

where, given \( \eta > 0 \) small enough,

\[
\Psi_{1,\nu}(t) = -\nu \epsilon \int_{\partial \mathcal{D}_{\text{in}}} d\nu \frac{e^{-\beta \left( \epsilon + s_0 \beta \sigma_{\nu} \right)}}{-i \nu \epsilon} \cdot \left( \epsilon \sigma_{\nu} + s_0 \sigma_{\nu} \right) \cdot \left\{ \frac{i}{2t - \nu \epsilon} \right\}^{\delta_{-\nu}} \quad \text{(G.133)}
\]

\[
\Psi_{2,\nu}(t) = \frac{1}{2 \sin[\pi \delta_{\nu}]} \int_{\mathbb{R}} dw h_{\nu}(w,t) \quad \text{and} \quad \Psi_{3,\nu}(t) = \frac{1}{2 \sin[\pi \delta_{\nu}]} \int_{\mathbb{R}} dw h_{\nu}(w,t) \quad \text{(G.134)}
\]

Above, I have introduced

\[
h_{\nu}(w,t) = \left( w + i \nu s_0 \tau \sigma_{\nu} \right) e^{\beta \left( w + i \nu s_0 \tau \sigma_{\nu} \right)} \cdot \frac{e^{\beta \left( w + i \nu s_0 \tau \sigma_{\nu} \right)}}{w^{\delta_{-\nu}}} \cdot \left[ w - 2i \right]^{\delta_{-\nu}} \quad \text{(G.135)}
\]
• The bound on $\Psi_{3,\omega}$($t$)

Using the expansion (G.116), one gets, uniformly in $w \in [\eta|\Re(t)|; T_v]$, and in respect to $|\Re(t)| \to +\infty$

$$\beta_i(w + i\nu s_u|\sigma_v| t)^2 = \frac{i\varsigma C_\beta}{t} \left(w^2 + 2i\nu s_u|\sigma_v| t - \sigma_v^2 t^2\right) + O(1)$$

$$= \frac{i\varsigma C_\beta}{t} \left(\frac{w^2}{t} - \sigma_v^2 t^2\right) - 2C_\beta^2 s_u^2|\sigma_v| w + O(1) \quad \text{(G.136)}$$

Also, given $t \in \mathbb{R} + i\alpha$, one has that for $|\Re(t)| \to +\infty$ and $w \in [\eta|\Re(t)|; T_v]$

$$|w + i\nu s_u|\sigma_v| t|^n \leq C|\Re(t)|^n \quad \text{and} \quad |w^{-\delta_v} \cdot [w - 2it]^{-\delta_{-v}}| \leq C|\Re(t)|^{\delta_v + |\delta_{-v}|} \quad \text{(G.137)}$$

for some constant $C > 0$ and where one uses in the intermediate steps that both $|w|$ and $|w - 2it|$ are bounded from below. Hence, by putting these bounds together, one gets for some constants $C, C', C''$,

$$\Psi_{3,\omega}(t) \leq C|\Re(t)|^{n-\delta_v-\delta_{-v}} \int_{\eta|\Re(t)|}^{T_v} e^{-2C_\beta^2 s_u^2|\sigma_v| w} d\text{w} \leq C' e^{-C''|\Re(t)|} \quad \text{(G.138)}$$

where the last bound follows from (G.131) and holds for $t \in \mathbb{R} + i\alpha$ with $|t|$-large enough.

• The bound on $\Psi_{2,\omega}$($t$)

Assume that $t = \rho e^{i\theta}$ with $\rho \in \mathbb{R}, \rho \to +\infty$ and $|\theta| < \theta_0$, with $\theta_0$ small enough. Then, it is convenient to rescale the integration variable as

$$w_x = (1 - x)\epsilon + \chi \eta s_i(t - i\alpha) \quad \text{(G.139)}$$

so that

$$\Psi_{2,\omega}(t) = 2(-i\nu)^n \sin[\pi\delta_v](\eta s_i(t - i\alpha) - \epsilon) \int_0^1 dx h_i(w_x, t) \quad \text{(G.140)}$$

Furthermore, one has that

$$\beta_i(w_x + i\nu s_u|\sigma_v| t)^2 = -i\varsigma C_\beta \left(|\sigma_v| - i\nu s_u\eta x\right)^2 + O(1) \quad \text{(G.141)}$$

and, for $\eta$ small enough, it holds

$$|\sigma_v| - i\nu s_u\eta x = g_{x,v} e^{-\frac{1}{2} s_i s_u \eta x} \quad \text{for}\ 0 \leq \varphi_x < C\eta \quad \text{and}\ g_{x,v} > C' > 0 \quad \text{(G.142)}$$

uniformly in $x$. Thus, for $u = s_u$, 

$$\beta_i(w_x + i\varsigma s_u|\varsigma_v| t)^2 = |\rho| C_\beta g^2_{x,s_u} \exp\left[i[\theta - (\varphi_x + \frac{1}{2}) s_i s_j\psi]\right] + O(1) \quad \text{(G.143)}$$

In particular, for $\theta = -s_i s_j\psi$ with $\psi > 0$, one gets that

$$\Re[\beta_i(w_x + i\nu s_u|\sigma_v| t)^2] \leq -C'' |\rho| \sin(\psi + \varphi_x) \leq -C(3) |\rho| \psi \quad \text{(G.144)}$$
uniformly in \( x \in [0; 1] \). Furthermore,

\[
\Re(w_x) = (1 - x)e + x\eta|\rho|\cos(\theta)
\]

so that \( |w_x^{\delta_\rho}| \leq C \) \hspace{1cm} (G.145)

since \( |w_x| \) is bounded from below away from 0. Finally, one also has that

\[
\Im(w_x - 2i\tau) = -\alpha x \eta \zeta - 2\rho \cos(\theta) + x\eta \pi \sin(\theta)
\]

so that uniformly in \( x \in [0; 1] \) and for \( |\rho| \) large enough and \( \theta_0 \) small enough \( |\Im(w_x - 2i\tau)| \geq C|\rho| \) hence ensuring that \( |w_x - 2i\tau| \) is bounded from below and thus

\[
|w_x - 2i\tau|^{-\delta_{\rho_0}} \leq C.
\]

The numerator in \( h_{\zeta_0}(w_x, t) \) generates a power-law bound in \( \rho \) so that, all-in-all,

\[
|h_{\zeta_0}(w_x, t)| \leq C' e^{-C|\rho|} \quad \text{for} \quad t = |\rho|e^{-i\delta_{\rho_0}}.
\]

This entails that

\[
\Psi_{2;\zeta_0}(t) \leq C' e^{-C|\rho|} \quad \text{for} \quad t = |\rho|e^{-i\delta_{\rho_0}}.
\]

Also, the above reasonings and estimates ensure that \( \Psi_{2;\zeta_0} \) is holomorphic on \( S_{\theta_0;A} \) with \( A \) large enough and \( \theta_0 \) small enough.

**The bound on \( \Psi_{1;\zeta}(t) \)**

Similar handlings to what has been exposed show that, for \( z = e^{i\phi} \),

\[
-\Re(\beta_{\nu}(\varepsilon z + \zeta_0 i|\sigma|)) = -|\sigma|^2 C|\rho|\sin(\theta) + O(1) \quad \text{with} \quad t = \rho e^{-i\delta_{\rho_0}}.
\]

Furthermore, for \( z = e^{i\phi} \) one has

\[
|\Im(\varepsilon z - 2i\tau)| = |\varepsilon \sin(\phi) - 2\rho \cos(\theta)| \geq C|\rho|
\]

provided that \( \theta \) is small enough and \( |\rho| \) large enough, one readily gets that, for some constants \( C, C' \),

\[
|\Psi_{1;\zeta}(t)| \leq C' e^{-C|\rho|} \quad \text{for} \quad t = \rho e^{-i\delta_{\rho_0}}.
\]

Again, the above also ensures that \( \Psi_{1;\zeta} \) is holomorphic on \( S_{\theta_0;A} \) with \( A \) large enough and \( \theta_0 \) small enough.

Thus, upon putting all the intermediate bounds together, the claim follows.

**B) The regime \(|u| < v\)**

The analysis is quite similar to the previous regime. I thus only highlight the main steps.

In the present case, since \( \text{sgn}(\sigma_\pm) = + \), it is convenient to represent

\[
\chi(t, s) = \prod_{\nu=\pm} \left\{ \frac{|u|}{s + \nu|\alpha|} \right\}^{\delta_\nu}
\]

meaning that, for fixed \( t \), the map \( s \mapsto \chi(t, s) \) has cuts along the lines \(-\nu|\alpha| - vi\mathbb{R}^+\). Since \( |\alpha_0| < \alpha \), the cut along \(-|\alpha| + i\mathbb{R}^+\) is located below the original integration line \( \mathbb{R} + i\alpha \), while the one along \( t|\alpha| + i\mathbb{R}^+\) is located above \( \mathbb{R} + i\alpha \).

The analysis then depends on the sign of \( \zeta \).  

109
In this case, one can deform the contour as in Figures 5 or 6 depending on the sign of \( \Re(t) \), without having to deal with the cuts of \( \chi(t, s) \). This yields

\[
F_n(t) = \left( \frac{1}{\beta t} \right)^{\frac{1}{2}(n+1)} \int_{\mathbb{R}} ds \ s^n e^{-s^2} \chi(t, \frac{s}{\sqrt{\beta t}})
\]

and one can conclude by the previous analysis.

\( \varsigma < 0 \)

In this case, when deforming the contour as in Figures 5 or 6 according to the sign of \( \Re(t) \), one observes that one has to take into account both cuts stemming from \( \chi(t, s) \). This yields

\[
F_n(t) = \left( \frac{1}{\beta t} \right)^{\frac{1}{2}(n+1)} \int_{\mathbb{R}} ds \ s^n e^{-s^2} \chi(t, \frac{s}{\sqrt{\beta t}}) + \sum_{\nu=\pm1} \int_{\Gamma_\nu} ds \ s^n \chi(t, s)e^{-\beta t s^2}.
\]

(G.155)
The two cut-issued integrals are very similar in structure to those studied previously, and, eventually, one ends up with the same conclusions.

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