Sensitivity analysis of hybrid systems with state jumps with application to trajectory tracking

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Abstract—This paper addresses the sensitivity analysis for hybrid systems with discontinuous (jumping) state trajectories. We consider state-triggered jumps in the state evolution, potentially accompanied by mode switching in the control vector field as well. For a given trajectory with state jumps, we show how to construct an approximation of a nearby perturbed trajectory corresponding to a small variation of the initial condition and input. A major complication in the construction of such an approximation is that, in general, the jump times corresponding to a nearby perturbed trajectory are not equal to those of the nominal one. The main contribution of this work is the development of a notion of error to clarify in which sense the approximate trajectory is, at each instant of time, a first-order approximation of the perturbed trajectory. This notion of error naturally finds application in the (local) tracking problem of a time-varying reference trajectory of a hybrid system. To illustrate the possible use of this new error definition in the context of trajectory tracking, we outline how the standard linear trajectory tracking control for nonlinear systems—based on linear quadratic regulator (LQR) theory to compute the optimal feedback gain—could be generalized for hybrid systems.

I. INTRODUCTION

Sensitivity analysis for dynamical systems allows to determine directly the change in a trajectory due to (small) changes in initial conditions and parameters and has proven beneficial in many aspects of the analysis of dynamical systems. In this paper, we pursue such sensitivity analysis for a class of hybrid systems. A hybrid system is a dynamic system that exhibits both continuous and discrete dynamic behaviors [8]. It is this inherent nature of hybrid systems that makes sensitivity analysis for this class of systems harder than for nonlinear dynamics, where sensitivity analysis is well established (see, e.g., [12, Chapters 3]). The theoretical framework provided by hybrid systems with (state-triggered) jumps is suitable to model those systems which, at certain instants of time, are subjected to rapid and abrupt changes. Indeed, in the modeling of such systems, it is frequently convenient and valid to neglect the durations of these rapid changes and to assume that the changes can be represented by instantaneous state jumps. Hybrid systems with state-triggered jumps are, for example, suitable to describe dynamical models in the area of robotics and rigid body mechanics with unilateral contact constraints [14], [3], including the study of the dynamics and control of walking or juggling robots (see, e.g., [27], [23], [19]).

The sensitivity result, presented in this paper, draws from the application of classical sensitivity and perturbation theory of nonlinear systems (see, e.g., [12, Chapters 3 and 10]) combined with the use of the implicit function theorem to compute an estimate of the unknown switching time at which the perturbed trajectory jumps. These same mathematical tools have been combined in the investigation of the sensitivity about a nominal trajectory of piecewise-smooth nonlinear systems without state jumps in [15], where the concept of saltation matrix was introduced. One can also find them in [11], where although state jumps are not considered, it is recognized that part the analysis could be carried on even in the presence of state jumps. In [18, Section 6.4], one finds an interesting discussion regarding the sensitivity of hybrid systems in the context of numerical optimal control of mechanical systems where state jumps are specifically taken into account. This work refers to [26, Section 2.2] that in turn refers to [13] (in German) as the source of a key formula for defining the sensitivity of hybrid systems with state jumps. We will re-establish this key formula for the class of hybrid systems considered in this work. From now on, we will refer to the sensitivity over a jump as the jump gain associated with a discontinuous event and it is given in Equation (27) in Section II. Another interesting discussion regarding sensitivity analysis for hybrid system with state jumps is presented in [6], for hybrid systems with jumps obtained by combining multiple differential algebraic equations (DAEs) with switching conditions and reset maps. There, it is mentioned that the jump gain (27) should be credited in fact credited to the (seemingly forgotten) seminal work of Rozenvasser [24, equation (11)]. For completeness, we also mention that a formula related to (27) can be encountered in the context of discontinuity induced bifurcations for hybrid systems. The interested reader is referred to [4, Section 2.5.1] and the work in [20], where the concept of discontinuity map (for transversal intersections) is introduced.

We leave to historians the settling of the question on who discovered (27) first. Here, we limit ourselves to mentioning that the jump gain (27) is indeed a key result that we, as other researcher before us, rediscovered autonomously. Our goal in this paper is to rigorously define what a first-order approximation about a nominal trajectory of a hybrid system with state jumps is on the basis of the result on the jump gain. By doing so, a novel notion of error, which allows to locally compare a nominal trajectory with a perturbed trajectory, emerges. This error notion, in turn, leads naturally
to obtain a trajectory tracking controller to locally stabilize time-varying trajectories for hybrid systems with state jumps. This is, in our view, the main novelty and contribution of this paper. Since the proper understanding of the jump gain is a key ingredient in such developments, we also include a (re)derivation of this result in this work.

Tracking control for hybrid system with state-triggered jumps is a recent and active field of research. Few results exist to design a controller to make a hybrid system with state-triggered jumps track a given, time-varying, reference trajectory. Recent techniques addressing this control problem both from a theoretical and an experimental viewpoint are provided in [17], [22], [21], [16], [5], and [1]. Our interest lies on the situation, commonly encounter in practice, where the jump times of plant and reference trajectories cannot be assumed to coincide.

Aiming at developing an effective trajectory tracking controllers for hybrid systems with state-triggered jumps, we propose to investigate the effects on a nominal trajectory of variations of the control input and initial conditions. In particular, we detail how to construct a linear approximation of the hybrid system about a nominal trajectory with jumps and then show how to use this approximation to construct a local trajectory tracking controller. To the best of our knowledge, the notion of linear approximation that we introduce in this paper has not be presented before.

The non trivial aspect of the problem is how to construct the approximation of the perturbed trajectory as the sum of the nominal trajectory and a linear term when the perturbed and nominal trajectories jump at different, although close, time instants. This difference in the jump times poses also the problem of defining a proper notion of tracking error.

Mimicking what is done for dynamical systems with no jumps, indeed, the most intuitive definition of tracking error is the difference between the nominal and actual state. However, as illustrated in, e.g., the introduction of [1], this definition has the drawback of exhibiting an unstable behavior in the sense of Lyapunov in spite of the convergence of the perturbed trajectory to the nominal one (away from the jump times).

For this reason, different approaches have been proposed in recent years to redefine the notion of tracking error for hybrid systems. In [7], e.g., the tracking problem has been defined in order to neglect in the analysis the times belonging to infinitesimal intervals about the jumping times. In [1], a novel conceptual definition of the notion of distance between two jumping trajectories has been proposed. Moreover, for a subclass class of hybrid system state-triggered jumps corresponding to mechanical systems with fully elastic impacts, the tracking error distance has been defined as the minimum between the distance of state from the nominal trajectory and the distance of state a mirrored version of nominal trajectory. In [2], a modification of this mirroring approach has been consider to deal with dissipative impacts. The use of a mirror reference trajectory has been proposed recently also in [5] for a tracking problem in polyhedral billiards: the controller in this case may decide to track either the real reference or the mirrored reference, mirrored through the billiard boundary.

We claim that the approximation proposed in this paper allows for a reinterpretation of the mirroring technique in terms of what we will call extended ante- and post-event trajectories. Furthermore, we care to emphasize the fact the concept of extended ante- and post-event trajectories allows to cope with the problem of trajectory tracking for a hybrid-system with state-triggered jumps where the (continuous-time) dynamics before and after the jump event are qualitatively different, where simply mirroring the reference trajectory does not appear to be the best choice.

This paper is organized as follows. In Section II we discuss the jump gain associated with a nominal trajectory of a hybrid system and introduce the notion of error between the nominal and perturbed trajectories of hybrid system with state jumps. This notion of error is used in Section III to propose a linear feedback control law for local trajectory tracking of time-varying reference trajectories with jumps. Conclusions are finally drawn in Section IV.

II. SENSITIVITY ANALYSIS FOR HYBRID SYSTEMS

In this section, we propose a framework for the sensitivity analysis of a jumping solution of a hybrid system with state-triggered jumps. To focus on the complexity of the effect of such a jump on the sensitivity, we limit ourselves to sensitivity analysis about one such a jump and leave the treatise of sensitivity analysis of solutions with multiple jumps for future work.

Consider a (sufficiently) smooth time-varying control vector field

\[ \dot{x}(t) = f^a(x,u,t), \]

with state \( x \in \mathbb{R}^n \) and input \( u \in \mathbb{R}^m \). For reasons that will appear clear shortly, we will refer to \( f^a \) as the ante-event control vector field. For a given initial condition \( x_0 \) at time \( t_0 \) and a integrable signal \( \mu(t) \in \mathbb{R}^n, t \in [t_0,t_1] \), we denote with \( \alpha^a(t), t \in [t_0,\tau], \) the solution of (1) with input

\[ u(t) = \mu(t), \quad t \in [t_0,t_1], \]

up to the occurrence of a triggering event at time \( \tau \in [t_0,t_1] \) defined by the satisfaction of the implicit condition

\[ g(\alpha^a(\tau), \tau) = 0, \]

where \( g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) is a smooth real-valued function We assume that, for all \( t \), the level set \( g(\cdot,t) = 0 \) is a \( n-1 \) dimensional smooth manifold embedded in \( \mathbb{R}^n \), (a sufficient condition being \( \partial g(x,t) / \partial x \neq 0 \) for all \( x \) such that \( g(x,t) = 0 \)). At the event time \( \tau \), the state exhibits a jump according to a smooth impulse map \( \Delta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \)

\[ \alpha^p(\tau) = \alpha^a(\tau) + \Delta(\alpha^a(\tau), \tau), \]

and subsequently it evolves according to the following post-event vector field

\[ \dot{x}(t) = f^p(x,u,t) \]

with initial condition at \( x(\tau) = \alpha^p(\tau) \) and input \( (2) \). We indicate with \( \alpha^p(t), t \in [\tau,t_1] \) the post-event trajectory.
Ante- and post-event trajectories can be glued together by “appending” to ante-event trajectory $\alpha^a$ the post-trajectory $\alpha^p$, obtaining the trajectory

$$\alpha(t) := \begin{cases} 
\alpha^a(t), & t \in [t_0, \tau) \\
\alpha^p(t), & t \in [\tau, t_1]. 
\end{cases} \quad (6)$$

The state trajectory $\alpha$ is (generally) not continuous, due to the state jump caused by the impulse map $\Delta$. Without loss of generality, $\alpha$ is by construction right continuous. Together with the nominal input $\mu(t)$, the state trajectory $\alpha(t)$ forms what we term the nominal state-input trajectory $\xi(t) = (\alpha(t), \mu(t)), t \in [t_0, t_1]$.

We are interested in defining and computing the sensitivity of $\xi(t)$ to small variations of the initial condition $x_0$ and input $\mu(t)$, $t \in [t_0, t_1]$. To this end, we will perturb the initial condition in the direction $z_0 \in \mathbb{R}^n$ and the nominal input curve $\mu$ in the direction $v$, where $v$ denotes an integrable curve $v(t) \in \mathbb{R}^m, t \in [t_0, t_1]$. Hence, the perturbed initial condition and input curve are defined, respectively, as

$$x_\varepsilon(t_0) = x_0 + \varepsilon z_0 \quad (7)$$

$$u_\varepsilon(t) = \mu(t) + \varepsilon v(t), \quad t \in [t_0, t_1], \quad (8)$$

with $\varepsilon \in \mathbb{R}$ typically small. The corresponding state-input trajectory will be denoted $\xi_\varepsilon(t) = (x_\varepsilon(t), u_\varepsilon(t)), t \in [t_0, t_1]$. We aim at defining a notion of sensitivity to predict, for small values of $\varepsilon$, the effect of perturbations to the initial condition and the nominal input on a jumping solution of the hybrid system.

The perturbed trajectory $\xi_\varepsilon = (x_\varepsilon, u_\varepsilon)$ is defined, similarly to the nominal trajectory $\xi = (\alpha, \mu)$, by appending to the perturbed ante-event trajectory $x^a_\varepsilon$ the the perturbed post-event trajectory $x^p_\varepsilon$. One cannot expect, however, that the event time $\tau$ will remain constant as $\varepsilon$ is varied. Indeed, the perturbed event time, that we denote $\tau_\varepsilon$, is implicitly defined by the condition

$$g(x^a_\varepsilon(\tau_\varepsilon), \tau_\varepsilon) = 0. \quad (9)$$

The above equation is satisfied, for $\varepsilon = 0$, by the nominal ante-event trajectory $\alpha^a$ at the nominal event time $\tau$, see (3).

In order to ensure that $\varepsilon \mapsto \tau_\varepsilon$ is, about $\varepsilon = 0$, a continuously differentiable function the following assumption needs to be fulfilled by the nominal state-input trajectory $\xi$.

**Assumption 2.1:** The nominal state-input trajectory satisfies the following transversality condition

$$D_1 g \cdot f^a + D_2 g \cdot 1 \neq 0, \quad (10)$$

where $g$ is evaluated at $(\alpha^a(\tau), \tau)$ and the ante-event vector field $f^a$ at $(\alpha^a(\tau), u(\tau), \tau)$.

The above transversality condition is a common requirement when dealing with hybrid system with state-triggered jumps (see, e.g., [19]). Its role in the sensitivity analysis will become clear in the proof of the proposition presented later on in this section.

For $t = \tau_\varepsilon$, the value of $x^p_\varepsilon$ is computed using the impulse map $\Delta$, similarly as done in [4]. Therefore, we can formulate the following conditions that the perturbed ante- and post-event trajectories have to satisfy:

$$x^a_\varepsilon = x_0 + \varepsilon z_0, \quad t = t_0 \quad (11)$$

$$\dot{x}^a_\varepsilon = f^a(x^a_\varepsilon, u_\varepsilon, t), \quad t \in [t_0, \tau_\varepsilon] \quad (12)$$

$$x^p_\varepsilon = x^a_\varepsilon + \Delta(x^a_\varepsilon, t), \quad t = \tau_\varepsilon \quad (13)$$

$$\dot{x}^p_\varepsilon = f^p(x^p_\varepsilon, u_\varepsilon, t), \quad t \in [\tau_\varepsilon, t_1]. \quad (14)$$

One important observation is that, although naturally defined in the time intervals $[t_0, \tau_\varepsilon]$ and $[\tau_\varepsilon, t_1]$, both the ante- and post-event trajectories $x^a_\varepsilon$ and $x^p_\varepsilon$ can individually be extended over the whole time interval $[t_0, t_1]$ by, respectively, forward and backward time integration starting from $\tau_\varepsilon$. We will denote those extensions as $\bar{x}^a_\varepsilon(t)$ and $\bar{x}^p_\varepsilon(t), t \in [t_0, t_1]$.
These extended ante- and post-event perturbed trajectories satisfy

\[
\ddot{x}_ε^a = x_0 + εz_0, \quad t = t_0, \quad (15)
\]
\[
\dot{x}_ε = f^a(\ddot{x}_ε, u_ε, t), \quad t \in [t_0, t_1], \quad (16)
\]
\[
\ddot{x}_ε^p = \ddot{x}_ε + Δ(\ddot{x}_ε, t), \quad t = τ_ε, \quad (17)
\]
\[
\dot{x}_ε^p = f^p(\ddot{x}_ε^p, u_ε, t), \quad t \in [t_0, t_1], \quad (18)
\]

where the perturbed event time \(τ_ε\) is implicitly defined by the condition

\[
g(\ddot{x}_ε^a(τ_ε), τ_ε) = 0. \quad (19)
\]

This apparently innocuous extension is the cornerstone to understand the sensitivity differential equation.

Figure 1 gives an indication on why in general we cannot expect to be able to write the perturbed trajectory \(x_ε\) as

\[
x_ε(t) = α(t) + εz(t) + o(ε)
\]

with \(z\) the solution to an appropriately defined time-varying linear system. The obstacle is represented by the difference in the event times for \(x_ε\) and \(α\). The following proposition shows how to overcome this difficulty, defining the extended ante- and post-event linearization trajectories \(\ddot{x}_ε^a\) and \(\ddot{x}_ε^p\) and defining the expansion about the extended ante- and post-event trajectories \(\ddot{α}_ε^a\) and \(\ddot{α}_ε^p\) instead of simply about \(α\).

**Proposition 2.1:** Consider a nominal state-control trajectory \(ξ(t) = (α, μ)(t), \ t \in [t_0, t_1]\), of system \(\{1, 2, 3, 4, 5\}\) with nominal event time \(τ \in [t_0, t_1]\) and associated extended ante- and post-event trajectories \(\ddot{α}_ε^a(τ)\) and \(\ddot{α}_ε^p(τ), \ t \in [t_0, t_1]\). Adopt Assumption 2.1. The perturbed state trajectory \(x_ε(t), \ t \in [t_0, t_1]\), corresponding to perturbations in the initial condition and input as in \(\{7\}\), satisfies

\[
x_ε(t) = \left\{ \begin{array}{ll}
\ddot{α}_ε^a(t) + ε\ddot{x}_ε^a(t) + o(ε), & t < τ_ε \\
\ddot{α}_ε^p(t) + ε\ddot{x}_ε^p(t) + o(ε), & t ≥ τ_ε.
\end{array} \right. \quad (20)
\]

where the extended ante- and post-event linearization trajectories \(\ddot{x}_ε^a(τ)\) and \(\ddot{x}_ε^p(τ), \ t \in [t_0, t_1]\), are computed as

\[
\ddot{x}_ε^a = z_0, \quad t = t_0 \quad (21)
\]
\[
\ddot{x}_ε^p = A^p(t)z_ε + B^p(t)v \quad t \in [t_0, t_1] \quad (22)
\]
\[
\ddot{x}_ε^a = A^a(t)z_ε + B^a(t)v \quad t = τ \quad (23)
\]
\[
\ddot{x}_ε^p = A^p(t)z_ε + B^p(t)v \quad t \in [t_0, t_1] \quad (24)
\]

where

\[
A^a(t) = D_1f^a(\ddot{α}_ε^a(t), μ(t), t) \quad (25)
\]
\[
B^a(t) = D_2f^a(\ddot{α}_ε^a(t), μ(t), t) \quad (26)
\]

with \(s = \{a, p\}\) and

\[
H(τ) = \frac{f^+ - f^- - \dot{Δ}}{\dot{g}^-} D_1g^- + D_1Δ^- \quad (27)
\]

where

\[
\alpha^p(τ) = α^a(τ) + Δ(α^a(τ), τ) \quad (28)
\]
\[
f^+ = f^p(α^p(τ), μ(τ), τ), \quad (29)
\]
\[
f^- = f^p(α^a(τ), μ(τ), τ) \quad (30)
\]
\[
\dot{Δ}^- = D_1Δ^- · f^- + D_2Δ^- · 1 \quad (31)
\]
\[
\dot{g}^- = D_1g^- · f^- + D_2g^- · 1 \quad (32)
\]
\[
D_k Δ^- = D_k Δ^a(α^a(τ), τ), \quad k = \{1, 2\} \quad (33)
\]
\[
D_k g^- = D_k g^a(α^a(τ), τ), \quad k = \{1, 2\}. \quad (34)
\]

**Proof:** The extended ante- and post-event perturbed trajectories satisfy \(\{15\} - \{18\}\) where the perturbed event time \(τ_ε\) is implicitly defined by the condition \(\{19\}\). On the basis of Assumption 2.1, the implicit function theorem allows to conclude the existence of a unique event time \(τ_ε\) for small values of \(ε\). Furthermore, as \(\{19\}\) is identically zero for each \(ε\) in a neighbourhood of zero, the derivative of \(\{19\}\) with respect to \(ε\) allows to obtain a linear approximation of \(τ_ε\) as a function of \(ε\), namely to compute \(τ_0^ε := \partialτ_ε/\partialε|_{ε=0}\) and approximate \(τ_ε\) as

\[
τ_ε = τ + ετ_0^ε + o(ε). \quad (35)
\]

The formula to compute \(τ_0^ε\) will be obtained at the end of the proof.

The sensitivity equations for the extended ante- and post-event trajectories \(\ddot{x}_ε^a\) and \(\ddot{x}_ε^p\) are the standard (see, e.g., [12, Chapter 10]) sensitivity equations given by \(\{22\} and \{24\}\). The missing link is how to relate \(\ddot{z}_ε^a\) to \(\ddot{z}_ε^a\), that is, to show that \(\ddot{z}_ε^a\) is indeed reinitialized as in \(\{23\}\) at the nominal event time \(τ\) using the linear map \(H\) given by \(\{27\}\).

Expanding \(\ddot{x}_ε^a(τ_ε)\) in series with respect to \(ε\) results in

\[
\ddot{x}_ε^a(τ_ε) = \ddot{α}_ε^a(τ_ε) + ε\ddot{x}_ε^a(τ_ε) + o(ε)
\]
\[
= α^p(τ) + ε(α^p(τ)τ_0^ε + \ddot{x}_ε^a(τ)) + o(ε)
\]
\[
= α^p(τ) + ε(α^p(τ)τ_0^ε + \ddot{x}_ε^a(τ)) + o(ε). \quad (36)
\]

To obtain the above expression, \(\ddot{α}_ε^a(τ_ε)\) has been approximated by linear extrapolation using the value and the time derivative of \(\ddot{α}_ε^p\) at time \(τ\) and \(τ_ε\) has been approximated using \(\{35\}\). We have then discarded the terms of order higher than one in \(ε\ddot{x}_ε^a(τ_ε)\). A similar expansion can be computed for \(\ddot{x}_ε^a(t_ε)\) obtaining

\[
\ddot{x}_ε^a(τ_ε) = α^a(τ) + ε(α^a(τ)τ_0^ε + \ddot{x}_ε^a(τ)) + o(ε). \quad (37)
\]

Using \(\{37\}\) and \(\{35\}\), \(\Delta(\ddot{x}_ε^a(τ_ε), τ_ε)\) appearing in \(\{17\}\) can be expanded as

\[
\Delta(\ddot{x}_ε^a(τ_ε), τ_ε) = \Delta(α^a(τ), τ) + εD_1Δ(α^a(τ), τ) · (α^a(τ)τ_0^ε + \ddot{x}_ε^a(τ)) + o(ε). \quad (38)
\]

Using \(\{37\}\) and \(\{35\}\), \(\{19\}\) can be expanded as

\[
g(\ddot{x}_ε^a(τ_ε), τ_ε) = ε D_1g(α^a(τ), τ) · (α^a(τ)τ_0^ε + \ddot{x}_ε^a(τ)) + ε D_2g(α^a(τ), τ) · τ_0^ε + o(ε). \quad (39)
\]
As the above expression is identically zero for every $\varepsilon$, we get
\[
\tau'_0 = - \frac{D_1 g(\alpha^a(\tau), \tau) \cdot \dot{z}^a(\tau)}{D_1 g(\alpha^a(\tau), \tau) \cdot \dot{\alpha}^a(\tau) + D_2 g(\alpha^a(\tau), \tau) \cdot 1 - \dot{\gamma}^-}
\]
where $\dot{\gamma}^-$ and $D_1 g^-$ are defined, respectively, as in (32) and (34). The expression for (40) is valid as long as $\dot{\gamma}^-$ is different from zero, that is, if Assumption 2.1 is satisfied. Making use of the series expansions (36), (37), and (38), we can match the first-order terms of (17), obtaining, at the event time $\tau$,
\[
\dot{\alpha}^p r'_0 + \dot{z}^p = \dot{\alpha}^a r'_0 + \dot{z}^a
+ D_1 \Delta(a^a, \tau) \cdot (\dot{\alpha}^a r'_0 + \dot{z}^a) + D_2 \Delta(a^a, \tau) \cdot \tau'_0.
\]
The above expression can be rearranged as
\[
\dot{z}^p = \dot{z}^a - (f^+ - f^- - \dot{\Delta}^-) \tau'_0 + D_1 \Delta^- \cdot \dot{z}^a
\]
where $f^+$, $f^-$, $\dot{\Delta}^-$, and $D_1 \Delta^-$ are defined, respectively, as in (29), (30), (31), and (33). Substituting in the above equation the expression for $\tau'_0$ given in (40), we obtain (23) and in particular the expression for $H$ provided in (27). This concludes the proof of the proposition. □

Remark. Note that the reset map (23) is linear in $\dot{z}^a$ and the reset occurs at the nominal event time $\tau$. □

As mentioned in the introduction, (27) is a (uncommonly) known expression in the context of numerical optimal control [26], [18], and parametric sensitivity for hybrid systems [24], [6]. It is also strictly related to equation (54) that appears in [11] and can also be interpreted as a generalization for piecewise-smooth nonlinear systems with state jumps of the saltation matrix introduced in [15] (in [15], $\Delta$ is identically equal to zero as there is no state jump).

Our contribution lies in the use of extended ante- and post-event trajectories to achieve the $o(\varepsilon)$ approximation presented in (20). To the best of our knowledge, this approximation and the use the extended anti- and post-event trajectories is new.

It is worth mentioning that [11, Appendix A] “suggests a procedure for refining the estimate of the perturbed trajectory” in a neighbourhood of the event time. The need for this refining is due to the difference between the nominal and perturbed trajectory event times (in our notation, $\tau$ and $\tau_e$). The approximation in [11] differs from the one that we propose as it is not an $o(\varepsilon)$ approximation of the nominal trajectory (uniformly in $t$) due to the lack of use of the extended anti- and post-event trajectories.

The approximation (29) is key to address the problem of (local) trajectory tracking for systems with state-triggered jumps, as we discuss in the following section.

III. Trajectory Tracking of a Time-Varying Reference Trajectory

Let $\xi = (\alpha, \mu)$ be a nominal state-input trajectory for a hybrid system characterized by ante- and post-event vector fields $f^a$ and $f^p$. As done in the previous section, $\tau$ will indicate the nominal event time and $\dot{\alpha}^a$ and $\dot{\alpha}^p$ the extended ante- and post-event trajectories, respectively.

Consider the following state feedback control law
\[
u = \begin{cases} 
\mu + K(t)(\dot{\alpha}^a - x), & \text{before event detection} \\
\mu + K(t)(\dot{\alpha}^p - x), & \text{after event detection}
\end{cases}
\]
where $K(t) \in \mathbb{R}^{m \times n}$ is a time-varying matrix gain to be designed. In (43), by event detection we mean the satisfaction of the condition $g(x(t), t) = 0$ for the current value of the state at time $t$ (assumed not to be equal to the nominal event time $\tau$). As mentioned in the example of tracking control for a bouncing ball proposed in [1], event detection is not strictly needed to implement a switching feedback law as (43). Indeed, due to the discontinuity in the nominal state trajectory, an equivalent result is obtained by simply choosing, between $\dot{\alpha}^a - x$ and $\dot{\alpha}^p - x$, the one with minimum norm.

Remark. Strictly speaking, in [1], the nominal trajectory $\alpha(t)$ and its negative version $-\alpha(t)$ are used in place of $\dot{\alpha}^a(t)$ and $\dot{\alpha}^p(t)$. For a bouncing ball impacting without energy loss on a surface (located at position zero), the use of the mirror trajectory $-\alpha(t)$ can be justified within our framework observing that $-\alpha(t)$ corresponds to choosing $\dot{\alpha}^p$ before the nominal event time and to $\dot{\alpha}^a$ after the nominal event time, so that an equivalent switching law to (43) is obtained. When the bouncing is not elastic, $-\alpha(t)$ is no longer a good representative of the extended behavior and needs to be corrected. Indeed, in [2], the case of non-elastic impact is considered and the mirrored nominal trajectory is corrected via the use of the impact restitution coefficient. Again, this can be interpreted as the need to obtain an (estimate of) the extended nominal trajectories $\dot{\alpha}^a(t)$ and $\dot{\alpha}^p(t)$ for properly defining the notion of tracking error to deal with the difference between the nominal and perturbed event times. A similar remark applies for the mirroring technique presented in [5]. □

Our goal in this section is to discuss why (43) is a suitable choice to design a trajectory tracking controller (assuming $\xi$ is of infinite extent) and indicate how the time-varying gain $K$ can be designed. Using (23), we obtain the following closed-loop ante- and post-event vector fields
\[
f^a_{cl}(x, t) := f^a(x, \mu + K(t)(\dot{\alpha}^a(t) - x), t),
\]
\[
f^p_{cl}(x, t) := f^p(x, \mu + K(t)(\dot{\alpha}^p(t) - x), t).
\]
By construction, the resulting hybrid system (with no inputs) has $\alpha$ as nominal trajectory and consequently the nominal switching time remains $\tau$. The sensitivity analysis developed in Section II leads to the following state matrices for the ante- and post-event linearization of (44)-(45):
\[
A^a_{cl}(t) = A^a(t) - B^a(t)K(t),
\]
\[
A^p_{cl}(t) = A^p(t) - B^p(t)K(t).
\]
The input matrices $B^a_{cl}$ and $B^p_{cl}$ are zero as $f^a$ and $f^p$ have no input. Finally, the gain $H$, computed using (27), is by
construction equal to the one associated to the nominal open-loop trajectory $\xi = (\alpha, \mu)$. This concludes the derivation of the extended linearization as discussed in Proposition 2.1 for the closed-loop dynamics (44)–(45).

In virtue of Proposition 2.1 and, in particular, of the approximation (20), we expect to be able to shape the local behavior of the closed-loop response of the hybrid system about the nominal trajectory $\xi$ by choosing suitably the matrix gain $K(t)$ in (43). We clearly expect that this will be related to controllability-like assumptions on the antecedent linearizations associated to the time-varying matrices $A^a$, $B^a$, $A^p$, $B^p$, as well as the jump gain $H$ given in (27). Here we limit ourselves to expose the main idea of using a modification of the standard linear quadratic regulator (LQR) problem in order to compute the gain $K$ and leave to future investigation the task of filling in the gaps by providing a mathematical proof that the approach will in fact be effective and demonstrate numerically the strategy on suitable examples.

Over a finite horizon, we consider the minimization of the quadratic cost functional

$$\frac{1}{2} \int_0^T z^T Q z + v^T R v \, ds + \frac{1}{2} z(T)^T P_T z(T) \tag{48}$$

subject to the jump linear dynamics

$$z = z_0, \quad t = t_0, \tag{49}$$

$$\dot{z} = A^a(t) z + B^a(t) v, \quad t \in [t_0, \tau], \tag{50}$$

$$z^+ = z^- + H z^-, \quad t = \tau, \tag{51}$$

$$\dot{z} = A^p(t) z + B^p(t) v, \quad t \in [\tau, T]. \tag{52}$$

with $Q$, $R$, and $P_T$ being (possibly time-varying) positive definite symmetric matrices.

The solution of the above optimal control problem is given by the closed-loop feedback law

$$v = -R^{-1} B^T P(t) z =: -K(t) z \tag{53}$$

where $P(t)$ is the solution of the following Riccati differential equation with jumps

$$P = P_T, \quad t = T,$$

$$\dot{P} = A^a P + P A^a - P B^a R^{-1}(B^a)^T P + Q, \quad t \in [\tau, T],$$

$$P^- = (I + H)^T P^+ (I + H), \quad t = \tau,$$

$$\dot{P} = A^p P + P A^p - P B^a R^{-1}(B^a)^T P + Q, \quad t \in [t_0, \tau].$$

The proof that (53) is indeed the solution to the optimal control problem (48)–(52) is readily obtained by observing that (48)–(52) is equivalent to the minimization of

$$\frac{1}{2} \int_0^T z^T Q z + v^T R v \, ds + \frac{1}{2} z(\tau)^T P^- z(\tau) \tag{54}$$

subject to the linear dynamics

$$z = z_0, \quad t = t_0, \tag{55}$$

$$\dot{z} = A^a(t) z + B^a(t) v, \quad t \in [t_0, \tau], \tag{56}$$

where $P^- = (I + H)^T P^+ (I + H)$ so that $1/2(z^-)^T P^- z^- = 1/2(z^+)^T P^+ z^+$ is the value function associated to the LQR problem

$$\frac{1}{2} \int_\tau^T z^T Q z + v^T R v \, ds + \frac{1}{2} z(\tau)^T P_T z(\tau) \tag{57}$$

subject to the linear dynamics

$$z = z^+, \quad t = \tau, \tag{58}$$

$$\dot{z} = A^p(t) z + B^p(t) v, \quad t \in [\tau, T]. \tag{59}$$

In case the nominal trajectory $\alpha$ becomes constant (resp., periodic) after the finite horizon time $T$, $P_T$ can be initialized to the corresponding algebraic (resp., periodic) solution of the Riccati differential equation to obtain a gain $K$ defined over the infinite horizon. It is left for future investigation the study of the effect on the optimal gain $K$ of the reset map $P^- = (I + H)^T P^+ (I + H)$, found in the Riccati differential equation with jumps associated to the optimal control problem (48)–(52). Our claim is that, as a consequence of this reset, the gain $K$ will be reduced before the impact event.

IV. CONCLUSIONS

This paper addresses the sensitivity analysis of hybrid systems with discontinuous state trajectories. We developed a novel notion of error to obtain, at each instant of time, a first-order approximation of the change in a trajectory due to small changes in initial conditions and inputs. This notion of error naturally finds application in the local tracking problem of a time-varying reference trajectory of a hybrid system. We outlined how the standard linear trajectory tracking control for nonlinear systems – based on linear quadratic regulator (LQR) theory to compute the optimal feedback gain – can be generalized for hybrid systems. We highlighted the connection between the switching linear feedback law that we propose with the idea of trajectory mirroring recently appeared in the literature.

The notion of error developed in this paper opens the possibility of further developing perturbation analysis in the context of hybrid systems. Our current efforts are directed toward the development of a second-order approximation that will find application in the context of numerical optimal control [25], [10], [9].

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