Fractional exclusion statistics in general systems with interaction

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Abstract. I show that fractional exclusion statistics (FES) is manifested in general interacting systems and I calculate the exclusion statistics parameters in a very general model. I show that the mutual exclusion statistics parameters—when the presence of particles in one Hilbert space influences the dimension of another Hilbert space—are proportional to the dimension of the Hilbert space on which they act. Such a result, although surprising and different from the usual way of understanding the FES, renders this statistics consistent and valid in the thermodynamic limit.

I use two standard methods for calculating the partition function and its maximum for a system of interacting bosons. Surprisingly, each method give a different result and after a more careful analysis I prove that neither of the results corresponds indeed to a relative or absolute maximum. Therefore the zero variation of the partition function at the variation of the particle population from the maximum probability population in each of the methods is obtained due to the specific constraints imposed on the particle variations. This observation questions the basic ways of calculating the most probable particle distributions in systems with interaction.
1. Introduction

Fractional exclusion statistics (FES), introduced by Haldane in Ref. [1] and with the thermodynamic properties calculated mainly by Isakov [2] and Wu [3], has received very much attention since its discovery and has been applied to many models of interacting systems (see for example Refs. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]). Several authors have also discussed the microscopic reason for the manifestation of FES [20, 21, 17, 18, 14, 15, 16, 19, 22].

Iguchi and Sutherland [22] showed that liquids of particles in three dimensions, interacting through long-range forces exhibit the nature of quantum liquids with FES, the characteristics of the FES being determined by the interaction.

Murthy and Shankar [17] analysed a system of fermions in the Colagero-Sutherland model. The system has a constant density of states (DOS) (along the single particle energy axis) and has a total energy of

\[ E(\{n_i\}) = \sum_i \epsilon_i n_i + \frac{V}{2\sigma} N(N - 1), \]

(1)

where \( n_i \) is the population of the single particle state of energy \( \epsilon_i \), \( \sigma = (\epsilon_i - \epsilon_{i-1})^{-1} \) (for any \( i > 0 \)) is the DOS, \( V \) is the mean-field interaction potential, and \( N \) is the total number of particles in the system. By redistributing in an uneven way the interaction energy between the particles of the system and associating to the level \( i \) the quasi-particle energy

\[ \tilde{\epsilon}_i = \epsilon_i + V\sigma^{-1}\sum_{j=0}^{i-1} n_j, \]

(2)

Murthy and Shankar obtained a gas with FES of parameter \( \alpha = 1 + V \).

A model which is similar to that of Murthy and Shankar [17] has been employed also in Refs. [15, 16, 19] to describe anyons on the lowest Landau level, coupled chiral particles on a circle, or interacting bosons in two-dimensions.

In Refs. [23, 24] I showed that the same model, with a slight generalization, can lead to a condensation, which is a first order phase transition.

In this paper I will extend the method of Murthy and Shankar to systems of general DOS and any interaction potential, \( V_{ij} \) (where \( i \) and \( j \) label the single particle states) and I will show that such systems lead to a more general manifestation of FES. If in the Murthy and Shankar model we have only direct exclusion statistics (i.e. the exclusion statistics is manifested only in the subspace where the particles are inserted) of constant parameter, \( \alpha \), in the general case we shall have also mutual statistics (acting from one subspace into another); therefore we shall have more complex parameters, denoted as \( \alpha_{ij} \). I will calculate explicitly the parameters \( \alpha_{ij} \) and I will prove that the mutual parameters (\( \alpha_{ij} \), with \( i \neq j \)) are proportional to the dimension of the Hilbert subspace on which they act, verifying in this way the conjecture put forward in the accompanying paper, Ref. [25].

In section 3 I will calculate the equilibrium particle distribution in a system of interacting bosons by maximizing the grandcanonical partition function under
two standard, but different, assumptions. Doing so, I will obtain two different “equilibrium” particle distributions: one is the typical Bose distribution (19) over some specific quasiparticle energies (20) and the other is a FES distribution (21) over the quasiparticle energies of the generalized Murthy and Shankar model (5). The existence of two, (eventually) very different particle distributions of “maximum” probability in a thermodynamic system is very peculiar and the consequence of it is that the distribution of “absolute” maximum probability is neither the Bose, nor the FES distribution, but it must be something else. I do not attempt to calculate here this maximum.

2. FES in systems with interaction

Let us generalize the model of Murthy and Shankar [17, 15, 16, 19] by writing the total energy as

\[ E = \sum_i \epsilon_i n_i + \frac{1}{2} \sum_{ij} V_{ij} n_i n_j \]  

(3)

and the quasiparticle energies as

\[ \tilde{\epsilon}_i = \epsilon_i + \sum_j V_{ij} n_j + \frac{1}{2} V_{ii} n_i. \]  

(4)

To make the calculations and the physical implications as clear as possible, we assume that we have bosons in the systems. In this way we shall not have to worry about adding a unit to the direct exclusion statistics parameters. I also assume that the system is large enough, so that the spectrum is (quasi)continuous, of the (generally not constant) DOS, \( \sigma(\epsilon) \). Then, assuming that \( V_{ij} \) depends only on the energies of the interacting particles, in Eq. (4) I drop the subscript \( i \) and I use \( \epsilon \) as a variable, to write

\[ \tilde{\epsilon} = \epsilon + \int_0^\epsilon V(\epsilon, \epsilon') \sigma(\epsilon') n(\epsilon') d\epsilon'. \]  

(5)

In Eq. (5) I also ignored the term \( V(\epsilon, \epsilon)n(\epsilon) \). Although this term, for \( \epsilon = 0 \), may cause a first order phase transition [23, 24], here I just want to emphasize the characteristics of the emerging FES and carrying along this term in the calculations would be useless. I assume also that the function \( \tilde{\epsilon}(\epsilon) \) is bijective, so that I can use freely its inverse, \( \epsilon(\tilde{\epsilon}) \). If I denote the density of states along the \( \tilde{\epsilon} \) axis by \( \tilde{\sigma}(\tilde{\epsilon}) \) and the number of particles between the energy levels \( \tilde{\epsilon}_1 \) and \( \tilde{\epsilon}_2 \), by \( N(\tilde{\epsilon}_1, \tilde{\epsilon}_2) \), then we have the relation

\[ N(\tilde{\epsilon}_1, \tilde{\epsilon}_2) \equiv \int_{\tilde{\epsilon}_1}^{\tilde{\epsilon}_2} \tilde{\sigma}(\tilde{\epsilon}) n(\tilde{\epsilon}) d\tilde{\epsilon} = \int_{\epsilon(\tilde{\epsilon}_1)}^{\epsilon(\tilde{\epsilon}_2)} \sigma(\epsilon') n(\epsilon') d\epsilon'. \]

If I split the quasiparticle energy interval \([0, \tilde{\epsilon}]\) into the small sub-intervals \([\tilde{\epsilon}_1, \tilde{\epsilon}_0], \ldots, [\tilde{\epsilon}_M, \tilde{\epsilon}_M-1]\) \(\equiv [\epsilon, \epsilon_{M-1}]\) (but which would still contain large enough numbers of particles and energy levels), I can write the expression (5) as

\[ \tilde{\epsilon} = \epsilon + \sum_{i=0}^{M-1} V[\epsilon, \epsilon(\tilde{\epsilon}_i)]N(\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1}) \]  

(6)

where \( \epsilon \) is implicitly \( \epsilon(\tilde{\epsilon}) \). I also used \( V(\epsilon, \epsilon_i) \approx V(\epsilon, \epsilon_{i-1}) \) for any \( i \), assuming that \( V(\epsilon, \epsilon') \) is continuous in both variables and the intervals \( \delta\epsilon_i \) are small enough. If we
include $I_{M-1}$ particles into the interval $[\bar{\epsilon}_M, \bar{\epsilon}_{M-1}]$, we change the value of $\epsilon(\bar{\epsilon})$ (where $\bar{\epsilon}$ is held fixed), but not the values $\epsilon(\tilde{\epsilon}_j)$ for $j < M$, since these particles affect only the energy levels above them. From (6) we get the equation

$$\tilde{\epsilon} = \epsilon' + V(\epsilon', \epsilon')I_{M-1} + \sum_{i=0}^{M-1} V[\epsilon', \epsilon(\tilde{\epsilon}_i)]N(\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1}).$$

If I denote $\delta \epsilon = \epsilon' - \epsilon$ and I expand $V(\epsilon, \epsilon_i)$ around $\epsilon$, I get from (6) and (7) an equation for $\delta \epsilon$:

$$\delta \epsilon = \frac{-V(\epsilon, \epsilon)I_{M-1}}{1 + V(\epsilon, \epsilon)I_{M-1} + \sum_{i=0}^{M-1} \frac{\partial V[\epsilon, \epsilon(\tilde{\epsilon}_i)]}{\partial \epsilon} N(\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1})}.$$  

(8)

Changing the summation into integral, I get

$$\delta \epsilon = \frac{-V(\epsilon, \epsilon)I_{M-1}}{1 + V(\epsilon, \epsilon)I_{M-1} + \int_0^\epsilon \frac{\partial V[\epsilon, \epsilon']}{\partial \epsilon} \sigma(\epsilon') n(\epsilon') \, d\epsilon'}.$$  

(9)

Therefore the insertion of $I_{M-1}$ particles in the interval $[\bar{\epsilon}_M, \bar{\epsilon}_{M-1}]$ changed the number of states in the interval by $\delta G_{M-1} = \sigma(\epsilon) \delta \epsilon$, giving a direct exclusion statistics parameter

$$\alpha_{\tilde{\epsilon}\epsilon} = \frac{V[\epsilon(\tilde{\epsilon}), \epsilon(\tilde{\epsilon})] \sigma[\epsilon(\tilde{\epsilon})]}{1 + \int_0^\epsilon \frac{\partial V[\epsilon, \epsilon']}{\partial \epsilon} \sigma(\epsilon') n(\epsilon') \, d\epsilon'}. $$  

(10)

Note that, as expected, $\alpha_{\tilde{\epsilon}\epsilon}$ has an expression which is identical to that for $\alpha$ calculated before, if $\partial V[\epsilon, \epsilon(\tilde{\epsilon})]/\partial \epsilon = 0$.

Now let’s calculate the mutual exclusion statistics parameters. For this I introduce $I_i$ particles in the interval $[\tilde{\epsilon}_{i+1}, \tilde{\epsilon}_i]$ ($0 \leq i < M - 1$). But unlike the previous case, by introducing these particles I change not only the value of $\epsilon(\bar{\epsilon})$, but also all the energy levels $\epsilon(\tilde{\epsilon}_j)$, of $j > i$. Taking these into account, I write

$$\tilde{\epsilon} = \epsilon' + V(\epsilon', \epsilon_i)I_i + \sum_{j=0}^{M-1} V[\epsilon', \epsilon(\tilde{\epsilon}_j)]N(\tilde{\epsilon}_j, \tilde{\epsilon}_{j+1}).$$

(11)

To simplify the notations a bit, I will denote by $\epsilon_j$ the value $\epsilon(\tilde{\epsilon}_j)$ in the first configuration of particles (i.e. before introducing $I_i$) and by $\epsilon'_j$ the value $\epsilon(\tilde{\epsilon}_j)$ in the configuration with the $I_i$ particles included. Then Eq. (11) becomes

$$\tilde{\epsilon} = \epsilon' + V(\epsilon', \epsilon_i)I_i + \sum_{j=0}^i V[\epsilon', \epsilon_j]N(\tilde{\epsilon}_j, \tilde{\epsilon}_{j+1}) + \sum_{j=i+1}^{M-1} V[\epsilon', \epsilon'_j]N(\tilde{\epsilon}_j, \tilde{\epsilon}_{j+1}).$$

(12)

Expanding again $V(\epsilon, \epsilon')$ to the linear order in both variables, I get the equation for $\delta \epsilon \equiv \epsilon' - \epsilon$:

$$\delta \epsilon \left[ 1 + I_i \frac{\partial V(\epsilon, \epsilon_i)}{\partial \epsilon} + \sum_{j=0}^{M-1} \frac{\partial V(\epsilon, \epsilon_j)}{\partial \epsilon} N(\tilde{\epsilon}_j, \tilde{\epsilon}_{j+1}) \right] = - I_i V(\epsilon, \epsilon_i) - \sum_{j=1}^{M-1} \frac{\partial V(\epsilon, \epsilon_j)}{\partial \epsilon} N(\tilde{\epsilon}_j, \tilde{\epsilon}_{j+1}) \delta \epsilon_j$$

(13)

The unknown quantities, $\delta \epsilon_j = \epsilon'_j - \epsilon_j$, can be calculated recursively, starting from $j = i$, using first Eq. (9) and then Eq. (13). By doing so, we first notice that $\delta \epsilon_j$ is
proportional to $I_i$, for any $j$. Transforming both summations in Eq. (13) into integrals and introducing the notation

$$f(\tilde{\epsilon}, \tilde{\epsilon}_j) = \frac{\sum_{j=i}^{M-1} \frac{\partial V(\epsilon_j, \epsilon_{j+1})}{\partial \epsilon_j} N(\tilde{\epsilon}_j, \tilde{\epsilon}_{j+1}) \delta \epsilon_j}{I_i} = \int_{\epsilon_i}^{\epsilon_f} \frac{\partial V(\epsilon', \epsilon_j)}{\partial \epsilon'} \sigma(\epsilon') n(\epsilon') [\delta \epsilon(\epsilon', \epsilon_i)] d\epsilon',$$  

(14)

I get the final equation for $\delta \epsilon$,

$$\delta \epsilon(\tilde{\epsilon}, \tilde{\epsilon}_i) = -\frac{V[\epsilon(\tilde{\epsilon}), \epsilon(\tilde{\epsilon}_i)] + f(\tilde{\epsilon}, \tilde{\epsilon}_i)}{1 + \int_0^{\epsilon(\tilde{\epsilon})} \frac{\partial V(\epsilon', \epsilon_j)}{\partial \epsilon'} \sigma(\epsilon') n(\epsilon') d\epsilon'} I_i$$  

(15)

If we plug in Eq. (14) into Eq. (15), the latter becomes an integral equation for $\delta \epsilon(\tilde{\epsilon}, \tilde{\epsilon}_i)$. Having now the expression for $\delta \epsilon(\tilde{\epsilon}, \tilde{\epsilon}_i)$, we can calculate the change of the number of states in the interval $\delta \epsilon_{M-1}$:

$$\delta G_{M-1} = \sigma(\epsilon_M) \delta \epsilon_M - \sigma(\epsilon_{M-1}) \delta \epsilon_{M-1} \approx \left. \frac{d \sigma(\epsilon)}{d \epsilon} \right|_{\epsilon_M} (\epsilon_M - \epsilon_{M-1}) \delta \epsilon_M,$$  

(16)

where we ignored $\delta \epsilon_M - \delta \epsilon_M$, since $\delta \epsilon_M$ is itself a small quantity. Plugging Eq. (15) into (16) I obtain the mutual exclusion statistics parameter,

$$\alpha_{\tilde{\epsilon}_i} = \left. \frac{(\epsilon_M - \epsilon_{M-1}) [V[\epsilon(\tilde{\epsilon}), \epsilon(\tilde{\epsilon}_i)] + f(\tilde{\epsilon}, \tilde{\epsilon}_i)]}{1 + \int_0^{\epsilon(\tilde{\epsilon})} \frac{\partial V(\epsilon', \epsilon_j)}{\partial \epsilon'} \sigma(\epsilon') n(\epsilon') d\epsilon'} \left[ \frac{d \sigma(\epsilon)}{d \epsilon} \right]_{\epsilon(\tilde{\epsilon})} \right|_{\epsilon(\tilde{\epsilon})}$$  

(17)

One can see immediately that if $[d \sigma(\epsilon)/d \epsilon]_{\epsilon(\tilde{\epsilon})} = 0$, as it was in the case of constant density spectrum, $\alpha_{\tilde{\epsilon}_i} = 0$ for any $\tilde{\epsilon}_i$.

Now we observe directly the surprising character of the mutual exclusion statistics, namely that it is proportional to the energy interval on which it acts, $(\epsilon_M - \epsilon_{M-1})$. In Ref. 25 I showed that this characteristics is necessary to ensure the self-consistency of the FES formalism, especially in the thermodynamic limit.

3. Maxima of the partition function

Let us now use this model and calculate the thermodynamic properties of the system. We have a gas of bosons of total energy (3) (we shall use equation (11) is only a particular case of (3)) and I will calculate the partition function and the most probable particle distribution. In the view of FES discussed above, I can do that by at least two methods.

In the method 1 I write the entropy of the Bose gas, typically as

$$S_B = k_B \sum_i [(1 + n_i) \ln(1 + n_i) - n_i \ln n_i]$$  

(18)

where $k_B$ is the Boltzman constant. Then the maximum probability particle population at constant total particle number and total energy is obtained by maximizing the grandcanonical partition function, $Z = \exp[S/k_B - \beta(E - \mu N)]$ (where $\beta = 1/k_B T$) with respect to $\{n_i\}$. So, equating $\partial Z/\partial n_i$ to zero, I get the standard Bose population,

$$n_i^{(1)} = \left[ e^{\beta(\tilde{\epsilon}_i^{(1)} - \mu)} - 1 \right]^{-1}$$  

(19)
with the “new” quasiparticle energies,
\[ \epsilon_i^{(1)} = \frac{\partial E}{\partial n_i} \approx \epsilon_i + \sum_j V_{ij} n_j. \] (20)

Expression (19) is “the most probable” particle distribution in the grandcanonical ensemble.

Now we apply method 2. We split the quasiparticle energy axis, \( \tilde{\epsilon} \) (equation 4), into small energy intervals and we calculate the number of configurations, \( W \), the entropy of the system, \( S = k_B \ln W \), and the logarithm of the grandcanonical partition function, \( \log Z = S/k_B + \beta (\mu N - E) \), by the method given in [25]. This way I get in the quasicontinuous limit the integral equation for the particle population [25]

\[ \beta (\mu - \tilde{\epsilon}) + \ln \left[ \frac{1 + n(\tilde{\epsilon})^{1-\alpha}}{n_{\tilde{\epsilon}}} \right] = \int \sigma(\tilde{\epsilon}') \ln \left[ 1 + n(\tilde{\epsilon}')^{1-\alpha_{\tilde{\epsilon}'}} \right] d\tilde{\epsilon}' \] (21)

with \( \alpha_{\tilde{\epsilon}'} \) calculated in the preceding section. Although the distribution (21) looks complicated and cannot be directly compared to the distribution (19), one can check on concrete examples that they are in general very different.

For clarity, I compare here equations (21) and (19) on one of the simplest cases, namely the system of constant DOS from section 1. Then on one hand we have the population \( n(\epsilon^{(1)}) \), given by equation (19), with \( \epsilon^{(1)}(\tilde{\epsilon}) = \epsilon + NV/\sigma \), and on the other hand from (21) we have

\[ \frac{1 + n(\epsilon)^{1-\alpha}}{n_{\epsilon}} = e^{\beta (\epsilon - \mu)}, \] (22)

with \( \alpha = V \). Clearly, the distribution (19) is different from the distribution (22), and for example if we take \( \alpha = 1 \), we obtain \( n(\epsilon) = e^{-\beta (\epsilon - \mu)} \)–the ratio between the population of particles and the population of holes in a Fermi system [25].

How can we interpret this? By method 1 we get (19) as the most probable population, whereas by method 2 we get a completely different maximum probability population, (21), for the same partition function. Since we cannot have two (thermodynamic) equilibrium distribution, except for some very special cases—for example at a phase transition—one could argue that the partition function has two maxima, but only one is dominant. Unfortunately this argument does not hold either. For the system of constant DOS, the two maxima of the partition function are equal for any \( \alpha \) and any temperature, due to the thermodynamic equivalence of systems of the same, constant DOS [26, 27, 28, 29, 30, 31]. So what is the situation?

The obvious conclusion we can draw from the calculations above is that neither of the two distributions corresponds to an (absolute or relative) maximum. The zeros of the functional derivatives of \( Z \) with respect to \( n \) were obtained because of the constraints imposed on the microscopic configurations of the system. To make this clear, let’s say that we set the system at the “equilibrium population” (19) and vary \( n^{(1)}[\epsilon(\tilde{\epsilon}^{(1)})] \) by \( \delta n[\epsilon(\tilde{\epsilon}^{(1)})] \), keeping all the other level populations, \( n^{(1)}(\epsilon') \), unchanged. Then the variation of \( Z \) is zero and the distribution seems to be of maximum probability. But if we vary \( n[\epsilon(\tilde{\epsilon}^{(1)})] \) with other constrains, like the ones corresponding to method 2—i.e.
keeping in the same time the number of particles in all the other energy intervals $[\tilde{\epsilon}_i, \tilde{\epsilon}_{i-1}]$ unchanged, which, from the point of view of method 1 is equivalent to a simultaneous reordering of the particles on the single particle states—the partition function is going to change. This means that by varying the population like in method 2, we can reach particle distributions for which $Z$ is bigger, i.e. particle distributions of higher probability.

The same line of reasoning can be applied to the particle distribution (21) and to method 2: if we impose the constraints of method 1 on changing the particle population, we see that $Z$ is not stationary, and therefore the particle distribution that we started with does not correspond to a maximum. Therefore we draw the conclusion that neither (19), nor (21) correspond to an absolute maximum of $Z$, no matter what is the relative height of $Z(\{n^{(1)}\})$ and $Z(\{n^{(1)}\})$. This is a very peculiar conclusion, since it proves wrong the basic ways of calculating the equilibrium particle distribution in systems with interaction.

4. Conclusions

In conclusion, in this paper I calculated the FES parameters of a general interacting gas and I showed that the mutual exclusion statistics parameters are proportional to the subspace on which they act. Althought this conclusion is surprising and seems peculiar at the beginning, in Ref. [25] I showed that this property is necessary for the general consistency of the FES formalism, especially for the existence of a thermodynamic limit.

In section 3 I showed two methods for calculating the most probable particle distribution in a system with interaction. In general, these methods give very different results and this raised the question of which distribution corresponds to higher probability, i.e. to the “real” thermodynamic equilibrium distribution. The rigorous answer is neither. If we take the maximum probability distribution corresponding to either of the methods and we vary it infinitesimally, with the constraints corresponding to the other method, we see that the partition function is not stationary and therefore the distribution we started with is not a maximum probability distribution.

What is the true maximum probability distribution and what is its physical relevance remains to be clarified. The fact that the partition function has more than one maximum, depending on the way one counts the configurations, is very interesting and leads to surprising results in many different systems of both, ideal and interacting particles [23, 32, 33, 34, 24].

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