Lines crossing a tetrahedron and the Bloch group

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According to B. Totaro ([7]), there is a hope that the Chow groups of a field \( k \) can be computed using a very small class of affine algebraic varieties (linear spaces in the right coordinates), whereas the current definition uses essentially all algebraic cycles in affine space. In this note we consider a simple modification of \( \text{CH}^2(\text{Spec}(k), 3) \) using only linear subvarieties in affine spaces and show that it maps surjectively to the Bloch group \( B(k) \) for any infinite field \( k \). We also describe the kernel of this map.

The second autor is grateful to Anton Mellit, who taught her the idea of passing from linear subspaces to configurations (Lemma 1 below) and pointed out the K-theoretical meaning of Menelaus’ theorem, and to the organizers of IMPANGA summer school on algebraic geometry for their incredible hospitality and friendly atmosphere.

1 Lines crossing a tetrahedron

Let \( k \) be an arbitrary infinite field. Consider the projective spaces \( \mathbb{P}^n(k) \) with fixed sets of homogenous coordinates \((t_0 : t_1 : \cdots : t_n) \in \mathbb{P}^n(k)\). We call a subspace \( L \subset \mathbb{P}^n(k) \) of codimension \( r \) admissible if

\[
\text{codim}(L \cap \{t_{i_1} = \cdots = t_{i_s} = 0\}) = r + s
\]

for every \( s \) and distinct \( i_1, \ldots, i_s \). (Here \( \text{codim}(X) > n \) means \( X = \emptyset \).) Let

\[
C^r_n = \mathbb{Z}\left[ \text{admissible } L \subset \mathbb{P}^n(k), \ \text{codim}(L) = r \right]
\]

be the free abelian group generated by all admissible subspaces of \( \mathbb{P}^n(k) \) of codimension \( r \). Then for every \( r \) we have a complex

\[
\cdots \xrightarrow{d} C^r_{r+2} \xrightarrow{d} C^r_{r+1} \xrightarrow{d} C^r_r \xrightarrow{d} 0 \xrightarrow{d} \cdots
\]

(we assume that \( C^r_n = 0 \) when \( n < r \)) with the differential

\[
[d[L]] = \sum (-1)^i [L \cap \{t_i = 0\}]
\]

(1)

where every \( \{t_i = 0\} \subset \mathbb{P}^n(k) \) is naturally identified with \( \mathbb{P}^{n-1}(k) \) by throwing away the coordinate \( t_i \). We are interested in the homology groups of these complexes \( H^r_n = H_n(C^*_{\cdot}) \).
For example, one can easily see that $H^1_1 \cong k^*$. Indeed, a hyperplane $(\sum \alpha_i t_i = 0)$ is admissible whenever all the coefficients $\alpha_i$ are nonzero, and if we identify $d$ then the differential $d : C^2_2 \longrightarrow C^1_1$ turns into

$$
\begin{align*}
C^1_1 &\cong \mathbb{Z}[k^*] & \{[\alpha_0 t_0 + \alpha_1 t_1 = 0]\} &\longmapsto \left[\frac{\alpha_1}{\alpha_0}\right] \\
C^2_2 &\cong \mathbb{Z}[k^* \times k^*] & \{[\alpha_0 t_0 + \alpha_1 t_1 + \alpha_2 t_2 = 0]\} &\longmapsto \left[\frac{\alpha_1}{\alpha_0}, \frac{\alpha_2}{\alpha_0}\right]
\end{align*}
$$

(2)

then the differential $d : C^2_2 \longrightarrow C^1_1$ turns into

$$
[(x, y)] \longmapsto [x] - [xy] + [y].
$$

(one can recognize Menelaus’ theorem from plane geometry behind this simple computation). Hence we have

$$
H^1_1 \cong \mathbb{Z}[k^*] / \{[x] - [xy] + [y] : x, y \in k^*\} \cong k^* .
$$

Continuing the identifications of (2), $C^2_2$ turns into the bar complex for the group $k^*$ (with the term of degree 0 thrown away) and therefore

$$
H^1_n = H_n(k^*, \mathbb{Z}) , \quad n \geq 1 .
$$

Now we switch to $r = 2$ and try to compute $H^2_3$. The four hyperplanes $\{t_i = 0\}$ form a tetrahedron $\Delta$ in the 3-dimensional projective space $\mathbb{P}^3(k)$ and the line $\ell$ is admissible if it

1) intersects every face of $\Delta$ transversely, i.e. at one point $P_i = \ell \cap \{t_i = 0\}$;

2) doesn’t intersect edges $\{t_i = t_4 = 0\}$ of $\Delta$, i.e. all four points $P_0, \ldots, P_3 \in \ell$ are different .

Therefore it is natural to associate with $\ell$ a number, the cross-ratio of the four points $P_0, \ldots, P_3$ on $\ell$. Namely, there is a unique way to identify $\ell$ with $\mathbb{P}^1(k)$ so that $P_0, P_3$ and $P_2$ become $0, \infty$ and 1 respectively, and we denote the image of $P_3$ by $\lambda(\ell) \in \mathbb{P}^1(k) \setminus \{0, \infty, 1\} = k^* \setminus \{1\}$. We extend $\lambda$ linearly to a map

$$
\sum n_i [\ell_i] \longmapsto \sum n_i [\lambda(\ell_i)]
$$

\[C^3_3 \longrightarrow \mathbb{Z}[k^* \setminus \{1\}] \]

Theorem 1. Let $\sigma : k^* \otimes k^* \longrightarrow k^* \otimes k^*$ be the involution $\sigma(x \otimes y) = -y \otimes x$.

(i) If $d(\sum n_i [\ell_i]) = 0$ then $\sum n_i \lambda(\ell_i) \otimes (1 - \lambda(\ell_i)) = 0$ in $(k^* \otimes k^*)_\sigma$.

(ii) Let $L \subset \mathbb{P}^1(k)$ be an admissible plane and $\ell_i = L \cap \{t_i = 0\}$, $i = 0, \ldots, 4$. If we denote $x = \lambda(\ell_0)$ and $y = \lambda(\ell_1)$ then

$$
\lambda(\ell_2) = \frac{y}{x}, \quad \lambda(\ell_3) = \frac{1 - x^{-1}}{1 - y^{-1}} \quad \text{and} \quad \lambda(\ell_4) = \frac{1 - x}{1 - y} .
$$

(iii) The map induced by $\lambda$ on homology

$$
\lambda_* : H^2_3 \longrightarrow B(k)
$$

(3)
is surjective, where
\[ B(k) = \frac{\text{Ker} \left( \frac{\mathbb{Z}[k^* \setminus \{1\}] \to (k^* \otimes k^*)_\sigma}{[a] \mapsto a \otimes (1-a)} \right)}{\langle [x] - [y] + \left\lfloor \frac{x}{y} \right\rfloor - \left\lfloor \frac{x}{1-y} \right\rfloor + \left\lfloor \frac{1-x}{y} \right\rfloor, x \neq y \rangle} \]

is the Bloch group of \( k \) \([5]\).

(iv) We have \( H_3^2 \cong H_3(\text{GL}_2(k))/H_3(k^*) \) and the kernel of \( \text{K} = \text{Ker}(H_3^2 \xrightarrow{\lambda} B(k)) \)
fits into the exact sequence
\[ 0 \to \text{Tor}(k^*, k^*) \to K/T(k) \to k^* \otimes K_2(k) \to K_3^M(k)/2 \to 0, \]
where \( \text{Tor}(k^*, k^*) \) is the unique nontrivial extension of \( \text{Tor}(k^*, k^*) \) by \( \mathbb{Z}/2 \), and \( T(k) \) is a 2-torsion abelian group (conjectured to be trivial).

We remark that \( \text{Tor}(k^*, k^*) = \text{Tor}(\mu(k), \mu(k)) \) is a finite abelian group if \( k \) is a finitely-generated field. Furthermore, it is proved in \([5]\) that \( B(k) \) has the following relation to \( K_3(k) \): let \( K_3^\text{ind}(k) \) be the cokernel of the map from Milnor’s K-theory \( K_3^M(k) \to K_3(k) \), then there is an exact sequence
\[ 0 \to \text{Tor}(k^*, k^*) \to K_3^\text{ind}(k) \to B(k) \to 0 \] \((5)\)
In particular, if \( k \) is a number field then as a consequence of \((5)\) and Borel’s theorem \((\text{II})\) we have
\[ \dim B(k) \otimes \mathbb{Q} = r_2, \]
where \( r_2 \) is the number of pairs of complex conjugate embeddings of \( k \) into \( \mathbb{C} \).

**Proof of (i) and (ii).** One can check that the diagram
\[ \begin{array}{ccc}
\mathbb{Z}[k^* \setminus \{1\}] & \xrightarrow{[a] \mapsto a \otimes (1-a)} & (k^* \otimes k^*)_\sigma \\
\text{Tor}(k^*, k^*) & \xrightarrow{\lambda} & K_3^\text{ind}(k) \\
\end{array} \]

is commutative, and therefore (i) follows. It is another tedious computation to check (ii).

In the next section we will prove the remaining claims (iii) and (iv) and also show that
\[ H_n^2 \cong H_n(\text{GL}_2(k), \mathbb{Z})/H_n(k^*, \mathbb{Z}) \quad n \geq 3. \] \((6)\)
2 Complexes of configurations

We say that $n + 1$ vectors $v_0, \ldots, v_n \in k^r$ are in general position if every $\leq r$ of them are linearly independent. Let $C(r, n)$ be the free abelian group generated by $(n + 1)$-tuples of vectors in $k^r$ in general position. For fixed $r$ we have a complex

$$\ldots \xrightarrow{d} C(r, 2) \xrightarrow{d} C(r, 1) \xrightarrow{d} C(r, 0)$$

with the differential

$$d[v_0, \ldots, v_n] = \sum (-1)^i[v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

(7)

The augmented complex $C(r, \bullet) \rightarrow \mathbb{Z} \rightarrow 0$ is acyclic. Indeed, if $v \in k^r$ is such that all $(n + 2)$-tuples $[v, v_0, \ldots, v_n]$ are in general position (such vectors $v$ exist since $k$ is infinite) then

$$\sum n_i[v_0^i, \ldots, v_n^i] = d\left(\sum n_i[v, v_0^i, \ldots, v_n^i]\right).$$

**Lemma 1.** $C_n^* \cong C(r, n)_{GL_r(k)}$ for the diagonal action of $GL_r(k)$ on tuples of vectors. Moreover, the complex $C_n^*$ is isomorphic to the truncated complex $C(r, \bullet)_{GL_r(k), \bullet \geq r}$.

**Proof.** For $n \geq r$ there is a bijective correspondence between subspaces of codimension $r$ in $\mathbb{P}^n(k)$ and $GL_r(k)$-orbits on $(n + 1)$-tuples $[v_0, \ldots, v_n]$ of vectors in $k^r$ satisfying the condition that $v_i$ span $k^r$. It is given by

$$L \subset \mathbb{P}^n \mapsto [v_0, \ldots, v_n], \quad v_i = \text{image of } e_i \text{ in } k^{n+1}/\tilde{L} \cong k^r$$

$$[v_0, \ldots, v_n] \mapsto \tilde{L} = \text{Ker}[v_0, \ldots, v_n]^T \subset k^{n+1}$$

where $\tilde{L}$ is the unique lift of $L$ to a linear subspace in $k^{n+1}$ and $e_0, \ldots, e_n$ is a standard basis in $k^{n+1}$.

An admissible point in $\mathbb{P}^r(k)$ is a point which doesn’t belong to any of the $r + 1$ hyperplanes $\{t_i = 0\}$, and for the corresponding vectors $[v_0, \ldots, v_n]$ it means that every $r$ of them are linearly independent. For $n > r$ a subspace $L$ of codimension $r$ in $\mathbb{P}^n(k)$ is admissible whenever all the intersections $L \cap \{t_i = 0\}$ are admissible in $\mathbb{P}^{n-1}(k)$. Hence it follows by induction that admissible subspaces correspond exactly to $GL_r(k)$-orbits of tuples “in general position”. Obviously, differential (7) is precisely (7) for tuples. □

The tuples of vectors in general position in $k^r$ modulo the diagonal action of $GL_r(k)$ are called configurations, so $C(r, n)_{GL_r(k)}$ is the free abelian group generated by configurations of $n + 1$ vectors in $k^r$.

**Proof of (iii) in Theorem 1.** For brevity we denote $C(2, n)$ by $C_n$ and $GL_2(k)$ by $G$. Since the complex of $G$-modules $C_\bullet$ is quasi-isomorphic to $\mathbb{Z}$ we have the hypercohomology spectral sequence with $E^1_{pq} = H_q(G, C_p) \Rightarrow H_{p+q}(G, \mathbb{Z})$. Since all modules $C_p$ with $p > 0$ are free we have $E^1_{pq} = 0$ for $p, q > 0$ and $E^1_{00} = (C_0)_G$. If $G_1 \subset G$ is the stabilizer of $(0)$ then $E^1_{00} = H_q(G, \mathbb{Z}[G/G_1]) = \ldots$
$H_q(G_1, \mathbb{Z})$ by Shapiro's lemma. We have $k^* \subset G_1$ and $H_q(k^*, \mathbb{Z}) = H_q(G_1, \mathbb{Z})$ (see Section 1 in [5]), so $E_{0q}^1 = H_q(k^*, \mathbb{Z})$. Further, $E_{00}^2 = H_p((C_\bullet)_G)$ and $E_{0q}^2 = H_q(k^*, \mathbb{Z})$. This spectral sequence degenerates on the second term. Indeed, the embedding
\[
  k^* \hookrightarrow G
  \quad \alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}
\]
is split by determinant, and therefore all maps $H_q(k^*, \mathbb{Z}) \to H_q(G, \mathbb{Z})$ are injective. Consequently, $E_{\infty}^{pq} = E_{2pq}$ and for every $n \geq 2$ we have a short exact sequence
\[
  0 \to H_n(k^*, \mathbb{Z}) \to H_n(G, \mathbb{Z}) \to H_n((C_\bullet)_G) \to 0.
\]
It follows from Lemma 4 that
\[
  H_n^2 = H_n((C_\bullet)_G) = H_n(G, \mathbb{Z})/H_n(k^*, \mathbb{Z}), \quad n \geq 3.
\]

Let $D_n$ be the free abelian group generated by $(n+1)$-tuples of distinct points in $\mathbb{P}^1(k)$. Again we have the differential like (7) on $D_\bullet$ and the augmented complex $D_\bullet \to \mathbb{Z} \to 0$ is acyclic. We have a surjective map from $C_\bullet$ to $D_\bullet$ since a non-zero vector in $k^2$ defines a point in $\mathbb{P}^1(k)$ and the group action agrees. The spectral sequence $\bar{E}_{pq}^1 = H_q(G, D_p) \Rightarrow H_{p+q}(G, \mathbb{Z})$ was considered in [5]. In particular, $\bar{E}_{p0}^1 = (D_p)_G$ is the free abelian group generated by $(p-2)$-tuples of different points since $G$-orbit of every $(p+1)$-tuple contains a unique element of the form $(0, \infty, 1, x_1, \ldots, x_{p-2})$, and the differential $d^3 : \bar{E}_{03}^3 \to \bar{E}_{03}^4$ is given by
\[
  [x, y] \mapsto [x] - [y] + \left[ \frac{y}{x} \right] - \left[ \frac{1 - x^{-1}}{1 - y^{-1}} \right] + \left[ \frac{1 - x}{1 - y} \right]. \quad (8)
\]
According to [5], terms $\bar{E}_{pq}^2$ with small indices are
\[
  H_3(k^* \oplus k^*)
\]
\[
  H_2(k^*) \oplus (k^* \otimes k^*)_\sigma \quad (k^* \otimes k^*)^\sigma
\]
\[
  k^* \quad 0 \quad 0
\]
\[
  \mathbb{Z} \quad 0 \quad 0 \quad p(k)
\]
where $p(k)$ is the quotient of $\mathbb{Z}[k^* \smallsetminus \{1\}]$ by all 5-term relations as in right-hand side of (8), and the only non-trivial differential starting from $p(k)$ is
\[
  d^3 : p(k) \to H_2(k^*) \oplus (k^* \otimes k^*)_\sigma = \Lambda^2(k^*) \oplus (k^* \otimes k^*)_\sigma
\]
\[
  [x] \mapsto x \wedge (1-x) - x \otimes (1-x)
\]
Therefore $\tilde{E}_3^{30} = \tilde{E}_3^\infty = B(k)$ and we have a commutative triangle

$$
\begin{array}{c}
H_3(G) \\
\downarrow \\
\tilde{E}_3^\infty = H_3^2 \\
\downarrow \\
\tilde{E}_3^\infty = B(k)
\end{array}
$$

where both maps from $H_3(G)$ are surjective, hence the vertical arrow is also surjective. It remains to check that the vertical arrow coincides with $\lambda_*$. A line $\ell$ in $\mathbb{P}^3(k)$ is given by two linear equations and for an admissible line it is always possible to chose them in the form

$$
\begin{align*}
t_0 + x_1 t_2 + x_2 t_3 &= 0, \\
t_1 + y_1 t_2 + y_2 t_3 &= 0.
\end{align*}
$$

This line corresponds to the tuple of vectors

$$
\left( \frac{1}{0}, \frac{0}{1}, \frac{x_1}{y_1}, \frac{x_2}{y_2} \right)
$$

which can be mapped to the points $0, \infty, 1, \frac{x_1 y_2}{y_1 x_2}$ in $\mathbb{P}^1(k)$, hence the vertical arrow maps it to $[x_1 y_2 y_1 x_2]$ (actually we need to consider a linear combination of lines which vanishes under $d$ but for every line the result is given by this expression).

On the other hand, four points of its intersection with the hyperplanes are

$$
\begin{align*}
P_0 &= (0 : y_1 x_2 - y_2 x_1 : -x_2 : x_1) \\
P_1 &= (y_2 x_1 - y_1 x_2 : 0 : -y_2 : y_1) \\
P_2 &= (-x_2 : -y_2 : 0 : 1) \\
P_3 &= (-x_1 : -y_1 : 1 : 0)
\end{align*}
$$

and if we represent every point on $\ell$ as $\alpha P_0 + \beta P_1$ then the corresponding ratios $\frac{\beta}{\alpha}$ will be $0, \infty, -\frac{x_2}{y_2}, -\frac{x_1}{y_1}$. Hence $\lambda(\ell) = \frac{x_1 y_2}{y_1 x_2}$ again and (iii) follows.

To prove (iv) we first observe that the Hochschild-Serre spectral sequence associated to

$$
1 \longrightarrow \text{SL}_2(k) \longrightarrow \text{GL}_2(k) \xrightarrow{\text{det}} k^* \longrightarrow 1
$$

gives a short exact sequence

$$
1 \longrightarrow H_0 \left( k^*, H_3(\text{SL}_2(k), \mathbb{Z}) \right) \longrightarrow \text{Ker} \left( H_3(\text{GL}_2(k), \mathbb{Z}) \xrightarrow{\text{det}} H_3(k^*, \mathbb{Z}) \right) \longrightarrow H_1 \left( k^*, H_2(\text{SL}_2(k), \mathbb{Z}) \right) \longrightarrow 1.
$$

The first term here maps surjectively to $K_3^{\text{ind}}(k)$ (see the last section of [2]), and the map is conjectured by Suslin to be an isomorphism (see Sah [4]). It is known that its kernel is at worst 2-torsion (see Mirzaii [3]). Thus we let

$$
T(k) := \text{Ker} \left( H_0(k^*, H_3(\text{SL}_2(k), \mathbb{Z})) \longrightarrow K_3^{\text{ind}}(k) \right).
$$

6
By the preceding remarks, this is a 2-torsion abelian group. Since the embedding \( k^* \rightarrow \text{GL}_2(k) \) is split by the determinant, the middle term in (9) is isomorphic to \( H_3^2 \). Then applying the snake lemma to the diagram

\[
\begin{array}{ccccccccc}
0 & \to & T(k) & \to & H_0 \left( k^*, H_3(\text{SL}_2(k), \mathbb{Z}) \right) & \to & K_3^{\text{ind}}(k) & \to & 0 \\
0 & \to & K & \to & H_2^2 & \to & B(k) & \to & 0
\end{array}
\]

gives the short exact sequence

\[
0 \rightarrow \text{Tor}(k^*, k^*) \rightarrow K/T(k) \rightarrow H_1 \left( k^*, H_2(\text{SL}_2(k), \mathbb{Z}) \right) \rightarrow 0.
\]

Finally, it follows from \([2]\) that there is a natural short exact sequence

\[
0 \rightarrow H_1 \left( k^*, H_2(\text{SL}_2(k), \mathbb{Z}) \right) \rightarrow k^* \otimes K^2_2(k) \rightarrow K^M_3(k)/2 \rightarrow 0.
\]

This proves \([4]\). \(\square\)

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