On nonlinear profile decompositions and scattering for a NLS-ODE model

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Abstract

In this paper, we consider a Hamiltonian system combining a nonlinear Schrödinger equation (NLS) and an ordinary differential equation (ODE). This system is a simplified model of the NLS around soliton solutions. Following Nakanishi [34], we show scattering of $L^2$ small $H^1$ radial solutions. The proof is based on Nakanishi’s framework and Fermi Golden Rule estimates on $L^4$ in time norms.

1 Introduction

The analysis of the asymptotic behavior for $t \to +\infty$ of solutions of nonlinear dispersive equations is largely an open problem. The Soliton Resolution Conjecture (SR Conjecture) states that generic solutions of nonlinear dispersive equations in Euclidean spaces in the long time limit resolve into trains of solitons plus a dispersing radiative component. For a review we refer to [35]. While the conjecture itself is unsolved, there is a large literature studying scattering (possibly modulo solitons) for some specific equations and in some subsets of phase space invariant for the dynamics. We emphasize two lines of research.

The first, starting from Buslaev-Perelman [2] and Soffer-Weinstein [38], considers invariant sets which are rather small and devotes attention to the so called meta-stable torii. They vanish after a long time and their anomalously weak instability is governed by purely nonlinear interactions, a phenomenon often called Radiation Damping. The linearized and nonlinear dynamics are completely different, because meta–stable torii do not vanish in the linearized equation.

The second line of research, starting from Kenig-Merle [21], centers around the so-called Concentration Compactness Rigidity method (CCR method) and aims to study large regions of phase space. The main idea is that a solution splits into components well separated from each other. In [21, 31] the method is used to prove the scattering of solutions with norm smaller than some critical and not small value. In [8, 9, 10], devoted to arbitrarily large solutions of energy critical equations, the components are either scattering or are solitons. The proofs could be conditional on the absence of discrete internal modes (in the terminology of [24]) in whose presence typical tools like the so called virial inequalities have not been developed yet when meta–stable torii arise, except in [29], a paper which considers only a single discrete coordinate rather simple in terms of the combinatorial structure of the normal form argument needed in the proof of Radiation Damping. Related to these considerations is the fact that the SR Conjecture is known to fail for systems such as discrete NLS’s [32, 19, 33] exactly because of the way Radiation Damping occurs or fails to occur. So one could envisage that between integrable systems, where the SR Conjecture is essentially known to be true and no internal modes are expected to exist, [23], on one end and some discrete equations on the
other end there might be intermediate cases, which might be typical, where the SR Conjecture is correct but requires an explicit elucidation of Radiation Damping.

The CCR method has been applied also to other settings and, for instance, for wave maps we refer to [30].

Nakanishi’s recent paper [34] puts together the two distinct lines of research described above for a problem which, while featuring an eigenvalue which complicates the CCR method, nonetheless does not have meta-stable torii. Our aim is to initiate a theory of the CCR method for equations which have meta-stable torii. Specifically in this paper we consider the following NLS-ODE model:

\[
\begin{align*}
\dot{\xi} &= -\Delta \xi + |\xi|^2 \xi + |z|^2 z G, \\
\dot{z} &= z + \frac{1}{2} z^2 (G|\xi) + |z|^2 (|G|\xi),
\end{align*}
\]

(1.1)

(1.2)

where \(\xi(t) \in H^1(\mathbb{R}^3; \mathbb{C}), z(t) \in \mathbb{C}, (f|g) := \int_{\mathbb{R}^3} f \bar{g} \, dx\) and \(G(x) \in S(\mathbb{R}^3, \mathbb{C})\) (Schwartz function) is a given radially symmetric function.

Schrödinger equations coupled with ODEs naturally appear in the study of asymptotic stability of solitons of NLS (see, for example [36, 37] or [5] and therein for more recent references). The forcing term \(|z|^2 z G\), which governs the interaction between the PDE part and the ODE part, creates the radiation damping. Moreover, such kind of model (with different interaction terms) appears in the study of particle-field interaction [26] and models of friction [11, 13, 14, 15]. There are also studies with the Schrödinger equation replaced by wave, Klein-Gordon and Dirac equations [28, 17, 18, 27].

The system (1.1)-(1.2) is Hamiltonian with symplectic form

\[
\Omega = (id\xi, d\xi) + idz \wedge d\bar{z}
\]

(1.3)

and Hamiltonian function, for \((f, g) = \text{Re}(f|g)\), given by

\[
E(\xi, z) = \frac{1}{2} \|\nabla \xi\|_{L^2}^2 + \frac{1}{4} \|\xi\|_{L^4}^4 + |z|^2 + \langle |z|^2 z G, \xi \rangle.
\]

(1.4)

Notice that for any \(\vartheta \in \mathbb{R}\) the symplectic form \(\Omega\) and the energy \(E\) are invariant with respect to the diffeomorphism \((\xi, z) \rightarrow (e^{i\vartheta} \xi, e^{i\vartheta} z)\). Then the following quadratic form is an invariant of motion for the system (1.1)-(1.2):

\[
M(\xi, z) = -\frac{1}{2} \left. \frac{d}{d\vartheta} \Omega(e^{i\vartheta} \xi, e^{i\vartheta} z) \right|_{\vartheta=0} = \frac{1}{2} \|\xi\|_{L^2}^2 + |z|^2.
\]

(1.5)

By standard arguments, see [3], and using the conservation of \(E\) and \(M\) it is easy to conclude that the Cauchy problem is globally well posed in \(H^1(\mathbb{R}^3; \mathbb{C}) \times \mathbb{C}\) for the system (1.1)-(1.2). The subspace \(H^1_{ad}(\mathbb{R}^3; \mathbb{C}) \times \mathbb{C}\) is invariant for the flow.

We will assume the following, true for most \(G \in S(\mathbb{R}^3, \mathbb{C})\):

\[
\hat{G} \big|_{\{y \in \mathbb{R}^3 : |y| = 1\}} \neq 0, \text{ where } \hat{G}(y) := \int_{\mathbb{R}^3} e^{-izy} G(x) \, dx.
\]

(1.6)

As mentioned before, our aim is to show scattering in a large region of phase space. The assumption (1.6), ensures that the system (1.1)-(1.2) exhibits radiation damping. That is, even though the system is time reversible and Hamiltonian, there is a flow of mass from the ODE part to the Schrödinger part and \(|z(t)|\) converges to 0 as \(t \rightarrow \pm \infty\).

We will now introduce the precise definition of scattering,
Definition 1.1 (Scattering). Let \((\xi, z)\) be solution of system (1.1)-(1.2). We say \((\xi, z)\) scatters forward (resp. backward) in time if there exists \(\varphi \in H^1(\mathbb{R}^3, \mathbb{C})\) s.t. \(\|\varphi\|_{H^1} + |z(t)| \to 0\) as \(t \to +\infty\) (resp. \(-\infty\)). If \((\xi, z)\) scatters forward and backward in time, we simply say that \((\xi, z)\) scatters.

Our aim of this paper is to show that all radial solutions with small \(M\) scatter. The following is our main result.

Theorem 1.2. Assume \(G \in H^1_{rad}(\mathbb{R}^3, \mathbb{C})\) (space of \(H^1\) functions depending only on \(|x|\), \(G \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})\) and (1.6). Then, there exists \(\delta > 0\) s.t. if \(\xi(0) \in H^1_{rad}(\mathbb{R}^3, \mathbb{C})\) satisfies \(\|\xi(0)\|_{L^2} + |z(0)| \leq \delta\) the solution \((z, \xi)\) of the system (1.1)-(1.2) scatters.

Notice that (1.1) is \(L^2\) supercritical because the nonlinear term \(|\xi|^2\xi\) is \(H^{1/2}\) critical. We emphasize that we are not in the perturbation regime, because we are only assuming the smallness of \(M\) (i.e. the \(L^2\) norm) while the \(H^1\) norm is arbitrary. Theorem 1.2 seems to be the first result of radiation damping in the non-perturbation regime and the first result with the CCR method applied to the situation where meta-stable tori exist. We remark that there exist solutions of (1.1)-(1.2) which do not scatter. For example all negative energy solutions do no scatter. We do not address here the more general problem of whether or not all positive energy solutions of (1.1)-(1.2) scatter when we assume (1.6).

Remark 1.3. The choice of \(\xi(0)\) and \(G\) radial guarantees that \(\xi(t, x)\) is radial in \(x\). This condition is important for the profile decomposition in Section 6 which uses the compactness of \(H^1_{rad} \to L^4\).

Remark 1.4. In the case of system (1.1)-(1.2) without the \(|\xi|^2\xi\) term, if (1.6) is true then the proof and result of Theorem 2.1 hold without the hypothesis (2.3). On the other hand, if \(G(y) \equiv 0\) for all \(y \in \{|y| < 1\}\) then for sufficiently small \(\epsilon > 0\), for

\[
\omega_\epsilon := \frac{3}{2} \epsilon^2 (G((\Delta + 1 - \omega_\epsilon)^{-1} G),
\]

one can show \((e^{3\epsilon e^{-i(1+\omega_\epsilon)it}}(\Delta + 1 - \omega_\epsilon)^{-1} G, e^{i(1+\omega_\epsilon)it})\) is a family of standing wave solutions.

As mentioned above, our work is motivated by Nakanishi [34] which studies

\[
i\dot{u} = (-\Delta + V)u + |u|^2 u \text{ in } H^1_{rad}(\mathbb{R}^3, \mathbb{C}),
\]

where \(V(x) = V(|x|)\), \(-\Delta + V\) restricted in \(H^1_{rad}(\mathbb{R}^3, \mathbb{C})\) has just one strictly negative eigenvalue. In [16] it had been proved that for \(\|u(0)\|_{H^1} \ll 1\) the solution \(u(t)\) of (1.7) can be written as

\[
u(t) = e^{i\vartheta(t)} Q_{\omega_\epsilon} + e^{i\Delta \eta_+ + o_{H^1}}(1)
\]

with \(\vartheta \in C^1([0, \infty), \mathbb{R})\), \(Q_{\omega_\epsilon}\) a nonlinear ground state (possibly \(Q_{\omega_\epsilon} = Q_0 := 0\), \(\eta_+ \in H^1(\mathbb{R}^3, \mathbb{C})\) and \(o_{H^1}(1)^{t \to 1 \infty} 0 \text{ in } H^1(\mathbb{R}^3, \mathbb{C})\). Nakanishi [34] has strengthened the result in [16] easing the condition \(\|u(0)\|_{H^1} \ll 1\) by enlarging the basin of attraction into \(\|u(0)\|_{L^2} \ll 1\) and \(u(0) \in H^1(\mathbb{R}^3, \mathbb{C})\), and by adding also that both \(u(0)\) and \(V\) are radially symmetric.

In [6] we extended the result of [16] analyzing small \(H^1(\mathbb{R}^3, \mathbb{C})\) solutions in the case of \(-\Delta + V\) with generic \(\sigma_p(-\Delta + V)\) proving that, up to scattering and symmetries, a small \(H^1(\mathbb{R}^3, \mathbb{C})\) solutions converges to a small soliton, perhaps to vacuum. It would be natural, following [34], to extend the result in [6] to the case of solutions with small \(L^2(\mathbb{R}^3, \mathbb{C})\) norm but arbitrary \(H^1(\mathbb{R}^3, \mathbb{C})\) norm. This remains an open problem although the arguments presented in this paper come very close to prove this, as we explain below.
We explain now the main features of the proof of this paper. Like in [34] the proof is divided in two parts. In the first we perform a profile decomposition of sequences of solutions and, proceeding by contradiction, we find a "minimal non-scattering solution". In the last part of the proof we derive a contradiction using the same argument of [34].

In most of the literate, when there is no small localized state, see for example [1, 22, 21, 31], an important tool is the existence of nonlinear profiles associated to "concentrating waves" (the latter are the waves $\lambda_n$ of the expansion (6.1)). Key is the existence of wave operators, see p. 50 [39], which allow to associate to any solution of the free linear equation a solution of the nonlinear equation with the same asymptotic behavior (the nonlinear profile) as $t \to +\infty$. However, in the presence of some discrete coordinate the existence of wave operators is a nontrivial problem. In fact, in the context considered by Nakanishi [34], where there is a small localized solution, the uniqueness of the nonlinear profile is unknown and the existence is obtained by weak limit (see [16]). In situations where radiation damping occurs, such as our system (1.1)–(1.2), the situation seems to be the same as in Nakanishi [34]. That is, although there is no small localized solution, we do not have the uniqueness of the final data problem. To overcome this difficulty, Nakanishi’s ingenious idea in [34] was to define the nonlinear profiles from weak limits and to consider two different nonlinear perturbation estimates, one close to the profiles and the other away from them. For this purpose Nakanishi introduced a seminorm, here called Nakanishi’s seminorm, based on $s t := L_t^4 L_x^6$, to measure the difference of a solution of the nonlinear problem from an associated solution of the linearized problem.

In this paper, we follow Nakanishi’s strategy. The difficulty is that we have additionally the forcing term $|z|^2 z G$ in (1.1) which could derail Nakanishi’s strategy. Indeed, in [34] the bootstrap arguments, and with them the whole construction, are based on the fact that there are no meta–stable torii and that the nonlinearity does not contain forcing terms like $|z|^2 z G$, which we instead consider in this paper and which are essential for radiation damping. In general we expect that any nonlinear dispersive equation with meta–stable torii and for which it is necessary to prove radiation damping displays the difficulties we face in this paper.

To find the minimal non-scattering solution we consider sequences $(\xi_n, z_n)$ of solutions of (1.1)–(1.2). Following, Nakanishi, it is natural to try nonlinear profile decompositions

$$\xi_n \sim \sum_{j=0}^{J-1} \xi_j (\cdot - s_n^j) + \Gamma_n^J$$

with $(\xi_j, z_j)$ satisfying (1.1)–(1.2), scattering forward and with the $\xi_j (\cdot - s_n^j)$ localized in temporal regions $(s_n^j - \tau, s_n^j + \tau)$, for some $\tau \gg 1$. Then one has to show that also $\xi_n$ scatters by showing $\xi_n$ has finite $st$ norm in $(0, \infty)$. In this argument the difficulties arise with the remainders $\Gamma_n^J$. Since we expect the nonlinear remainder to exist essentially only in the “gap” region $I_n^{J,\tau} := (s_n^J + \tau, s_n^{J+1} - \tau)$, we divide $\Gamma_n^J$ into $J$ pieces $\Gamma_n^{J,\tau}$. The nonlinear remainder $\Gamma_n^{J,\tau}$ will then be given by the solution of NLS with the forcing term and the nonlinear term restricted on $I_n^{J,\tau}$ and initial data given by $\Gamma_n^{J,\tau} (s_n^J + \tau) = \gamma_n^{J,\tau} (s_n^J + \tau)$, where $\gamma_n^{J,\tau}$ is the remainder of the linear profile decomposition (see (A:7) and (6.4) in particular).

Since the key in Nakanishi’s argument is to show that $\xi_n$ are well approximated by the the nonlinear profiles and the remainders and moreover estimate them by the norm $st = L_t^4 L_x^6$, we need to establish various estimates based on the $st$ norm. In particular, we need a

$$L_t^4 \text{ estimate on the forcing term dependent only on discrete modes}, \quad (1.9)$$

which is obtained here by an elementary manipulation of the basic Fermi Golden rule identity, see
We develop in this paper, combined with some other ingredients involving the Fermi Golden rule, in Theorem 1.5, are somewhat restricted to very special classes of systems. However the method strategy can be applied on a diverse set of problems. The result in Theorem 1.2, or the result to get (1.9).

| (4.11)–(4.12) later. We remark that although the $L^4_t$ based FGR can be obtained easily in this case, the obstruction to extending [34] to the setting of [6] comes from the lack of such estimate.

As in [34], we will estimate the profiles step by step by moving from $(s^0_n + \tau, s^0_n + \tau)$ to $I^t_n = (s^0_n, s^0_n + \tau)$ successively estimating $\xi^t(-s^0_n)$ and $\Gamma^{J,j,\tau}$. The difference will come mainly in the “gap” region $I^t_n$ where the nonlinear remainder $\Gamma^{J,j,\tau}$ cannot be estimated a priori and we have to include their estimates in the iterative procedure. In particular, to be able to estimate $\Gamma^{J,j,\tau}$, we need to bound the forcing term $|z_n|^2 z_n G$. To bound the forcing term we have to use the equation of $\xi_n$ and the (1.9), and we need to go back to the fact that in the region $(s^j - \tau, s^j + \tau)$, $\xi_n$ is well approximated by $\xi^j(-s^j_n)$. In the region $I^t_n$, we will show that $\xi_n$ is well approximated by $\xi^j(-s^j_n)$ and the remainder $\Gamma^{J,j,\tau}$ itself is small in st. To proceed from $I^t_n$ to $(s^j_n + \tau, s^j_n + \tau)$, the key is to show that $\Gamma^{J,j,\tau}$ is negligible in this region. Since $\Gamma^{J,j,\tau}$ has no nonlinearity nor forcing term after $s^j_n + \tau$ (by definition, see (A:7)), one can show this by Duhamel estimates (see Lemma 3.12) provided $(\xi^{j+1}, \tau^{j+1})$ scatters backward, which implies the forcing term $|z_n|^2 z_n G$ will be negligible near $s^j_n + \tau$. By such argument, we can estimate the profiles one by one and in the same time show the profiles are good approximation in each regions.

The discussion we made after (1.8), while framed for system (1.1)–(1.2), is in fact very general and can be reproduced in the framework of [6] or in other settings. The only gap remaining in order to include their estimates in the iterative procedure. In particular, to be able to estimate $\Gamma^{J,j,\tau}$, we need to bound the forcing term $|z_n|^2 z_n G$. To bound the forcing term we have to use the equation of $\xi_n$ and the (1.9), and we need to go back to the fact that in the region $(s^j - \tau, s^j + \tau)$, $\xi_n$ is well approximated by $\xi^j(-s^j_n)$. In the region $I^t_n$, we will show that $\xi_n$ is well approximated by $\xi^j(-s^j_n)$ and the remainder $\Gamma^{J,j,\tau}$ itself is small in st. To proceed from $I^t_n$ to $(s^j_n + \tau, s^j_n + \tau)$, the key is to show that $\Gamma^{J,j,\tau}$ is negligible in this region. Since $\Gamma^{J,j,\tau}$ has no nonlinearity nor forcing term after $s^j_n + \tau$ (by definition, see (A:7)), one can show this by Duhamel estimates (see Lemma 3.12) provided $(\xi^{j+1}, z^{j+1})$ scatters backward, which implies the forcing term $|z_n|^2 z_n G$ will be negligible near $s^j_n + \tau$. By such argument, we can estimate the profiles one by one and in the same time show the profiles are good approximation in each regions.

Another result which can be proved exploiting the present paper involves a problem treated in [7] involving

$$i \dot{u}(t, x) = (-\Delta + V(x) + \lambda)u(t, x) + (1 + \gamma_1 \cos(t))|u(t, x)|^2 u(t, x), \ u(0, x) = u_0(x) \tag{1.10}$$

where $\lambda$ is a constant. Specifically we can prove the following result, which we only state here.

**Theorem 1.5.** Assume that $-\Delta + V$ has exactly one negative eigenvalue given by $-\lambda$ with $0 < \lambda < 1$. Assume $0 < \gamma_1 < 1$. Assume the hypotheses stated in Theorem [7] and the $V$ is a radial Schwartz function. Then, there exists an $\epsilon_0 > 0$ s.t. if $\|u_0\|_{L^2} < \epsilon_0$ and $u_0 \in H^1_{rad}(\mathbb{R}^3, \mathbb{C})$ there exists a $\varphi \in H^1(\mathbb{R}^3, \mathbb{C})$ s.t.

$$\lim_{t \to +\infty} \|u(t) - e^{it\Delta}\varphi\|_{H^1} = 0. \tag{1.11}$$

In [7] the above result was proved with $\|u_0\|_{H^1} < \epsilon_0$. Here the restriction $\gamma_1 < 1$, not present in [7], is added to allow any value of the norm $\|u_0\|_{H^1}$. Notice that the result in [7] was extended by [4] to the case when $-\Delta + V$ has any number of eigenvalues in $(-\lambda, 0)$, but that Theorem 1.5 is stated only when $-\Delta + V$ has exactly one eigenvalue exactly because only in this case we can get (1.9). Indeed, in analogy with (4.11)–(4.12), the desired bound can be obtained in an elementary fashion by considering formula (4.23) (in the case $n = 0$) in [4]

$$\frac{1}{2} \frac{d}{dt} |\xi_0|^2 + \pi |\xi_0|^6 (\delta(-\Delta + V + \lambda - 1) \Phi, \Phi) = \text{Im} \left( D_0 \xi_0 \right)$$

where $\Phi(x)$ is a rapidly decreasing and $C^2$ function and where the r.h.s. is a remainder term. Then multiplying the formula by $|\xi_0|^6$ and proceeding in a fashion similar to (4.11)–(4.12) we get an estimate on $\|\xi_0\|_{L^{12}}$. In the presence of one or more further discrete modes we don’t know yet how to get (1.9).

Provided that we can get an st bound on appropriate discrete components interactions our strategy can be applied on a diverse set of problems. The result in Theorem 1.2, or the result in Theorem 1.5, are somewhat restricted to very special classes of systems. However the method we develop in this paper, combined with some other ingredients involving the Fermi Golden rule,
promises to be relevant in much more general situations. We think that the approach to the Soliton Resolution which is currently taking shape, will need ultimately to face the problems we consider in the present paper, and possibly borrow some of the ideas we present here in the presence of metastable tori.

Finally a few words on the organization of the paper. In section 2, we prepare notations and give a proof of Theorem 1.2 under the more restrictive condition \( \| \xi(0) \|_{H^{1/2}} + |z(0)| \ll 1 \). In section 3 we collect the known linear estimates and introduce Nakanishi’s seminorm. In section 4, we provide the \( L^4 \) in time estimates. In section 5, we prove nonlinear perturbation estimates. In section 6, we give the linear and nonlinear profile decomposition. In section 7, we perform the main iteration argument and in section 8, we show the scattering and complete the proof.

2 Notation and preliminary results

We will use the following standard notation.

- \( L^{2,*}(\mathbb{R}^3, \mathbb{C}) := \{ u \in S'(\mathbb{R}^3, \mathbb{C}), \langle x \rangle^s u \in L^2(\mathbb{R}^3, \mathbb{C}) \} \) with \( S'(\mathbb{R}^3, \mathbb{C}) \) the space of tempered distributions and \( \langle x \rangle = \sqrt{1 + |x|^2} \).

- \( B^s_{p,q}(\mathbb{R}^3, \mathbb{C}) \) is the Besov space formed by the tempered distributions \( f \in S'(\mathbb{R}^3, \mathbb{C}) \) s.t.

\[
\| f \|_{B^s_{p,q}} = \left( \sum_{j \in \mathbb{N}} 2^{jsq} \| \hat{\varphi}_j * f \|_{L^p(\mathbb{R}^3)}^q \right)^{\frac{1}{q}} < +\infty
\]

with \( \hat{\varphi} \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}) \) s.t. \( \sum_{j \in \mathbb{Z}} \hat{\varphi}(2^{-j} \xi) = 1 \) for all \( \xi \in \mathbb{R}^3 \setminus \{0\} \), \( \hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j} \xi) \) for all \( j \in \mathbb{N}^* \) and for all \( \xi \in \mathbb{R}^3 \), and \( \hat{\varphi}_0 = 1 - \sum_{j \in \mathbb{N}^*} \hat{\varphi}_j \).

- We will simplify the notation and write \( L^{2,*} \) for \( L^{2,*}(\mathbb{R}^3, \mathbb{C}) \), \( H^* \) for \( H^*(\mathbb{R}^3, \mathbb{C}) \), \( B^s_{p,q} \) for \( B^s_{p,q}(\mathbb{R}^3, \mathbb{C}) \), \( L^p \) for \( L^p(\mathbb{R}^3, \mathbb{C}) \), \( S \) for \( S(\mathbb{R}^3, \mathbb{C}) \) and \( S' \) for \( S'(\mathbb{R}^3, \mathbb{C}) \).

- Given an interval \( I \subseteq \mathbb{R} \) and a Banach space \( X \) we set \( L^p X(I) := L^p(I, X) \).

- Given an interval \( I \subseteq \mathbb{R} \) we set \( \text{Stz}^s(I) := L^\infty H^s(I) \cap L^2 B^s_{6,2}(I), \text{Stz}^{ss}(I) := L^1 H^s(I) + L^2 B^s_{6/5,2}(I) \) and \( \text{st}(I) := L^2 L^0(I) \).

- We set \( R_+ (1) = \lim_{\epsilon \to 0^+} (\Delta - 1 - i \epsilon)^{-1} \) which for \( \sigma > 1/2 \) exists in the strong sense in the space \( B(L^{2,\sigma}, L^{2,-\sigma}) \) of bounded linear operators \( L^{2,\sigma} \to L^{2,-\sigma} \).

- We write that \( 0 \leq a \ll 1 \) if \( 0 \leq a \leq \epsilon \) for a preassigned and arbitrarily small \( \epsilon > 0 \).

- We write \( a \lesssim b \) if \( a \leq Cb \) for a preassigned \( C > 0 \).

We set

\[
\Gamma = - \text{Im} \beta, \quad \beta = (G|R_+(1)G).
\]

By \( R_+ (1) = P.V. \frac{1}{\Delta - 1} + i \pi \delta (\Delta - 1) \), we have

\[
\Gamma = \pi (G|\delta(\Delta - 1)G) = \pi \int_{|\eta|=1} |\hat{G}(\eta)|^2 \, d\eta \geq 0.
\]

Under the assumption (1.6) we have \( \Gamma > 0 \).

The system (1.1)–(1.2) satisfies the following (easier) analogue of the main result in [16, 6].
Theorem 2.1. Assume (1.6). Then, there exist $\delta > 0$ and $C > 0$ s.t. if $\xi(0) \in H^1$ and
\[ \|\xi(0)\|_{H^{1/2}} + |z(0)| \leq \delta \] (2.3)
we have
\[ \|\xi\|_{S^{3}(\mathbb{R})} + \|z\|_{L^{6}(\mathbb{R})} + \|z\|_{L^{\infty}(\mathbb{R})} \leq C(\|\xi(0)\|_{H^{\theta}} + \|\xi(0)\|_{L^{2}} + |z(0)|), \] (2.4)
for $\theta \in [0, 1]$. In particular, $(\xi, z)$ scatters.

Proof. By the Strichartz estimates (Lemma 3.2), for $\theta \in [0, 1]$, we have
\[ \|\xi\|_{S^{3}(t_0, t_1)} \leq C \left( \|\xi(t_0)\|_{H^{\theta}} + \|\xi\|_{L^{6}(t_0, t_1)} + \|z\|^2_{L^{6}(t_0, t_1)} \right). \] (2.5)
We set
\[ Y = - |z|^2 z R_+(1) G, \quad R_\pm(z) := (-\Delta - z \mp i0)^{-1} \] (2.6)
and $\xi = Y + g$. Then, $g$ satisfies
\[ ig = -\Delta g + |\xi|^2 \xi + R_1, \quad R_1 := -i \gamma Y - \Delta Y + |z|^2 z G. \] (2.7)
Substituting (1.2) and (2.6) into the definition of $R_1$, we have
\[ R_1 = \frac{3}{2} |z|^4 \overline{(G \xi)} R_+(1) G. \] (2.8)

Thus, by Strichartz estimate (Lemma 3.2) and Lemma 3.11 below, for $\sigma > 9/2$, we have
\[ \|g\|_{L^2 \Delta L^2, -\sigma(t_0, t_1)} \leq C \|e^{i\Delta(t-t_0)} g(t_0)\|_{L^2 \Delta L^2, -\sigma(t_0, t_1)} \]
\[ + C \|\xi\|^2_{L^6(t_0, t_1)} \|\xi\|_{L^{\infty}(t_0, t_1)} + C \|z\|_{L^{6}(t_0, t_1)} \|\xi\|_{L^2 L^6(t_0, t_1)}. \]
By Lemma 3.11 below we have
\[ \|e^{i\Delta(t-t_0)} g(t_0)\|_{L^2 \Delta L^2, -\sigma(t_0, t_1)} \leq C' \|\xi(t_0)\|_{L^2} + |z(t_0)|^3 \|e^{i\Delta(t-t_0)} R_+(1) G\|_{L^2 \Delta L^2, -\sigma(t_0, t_1)} \]
\[ \leq C \left( \|\xi(t_0)\|_{L^2} + |z(t_0)|^3 \right). \]

Thus, there exists a fixed constant $C$ s.t.
\[ \|g\|_{L^2 \Delta L^2, -\sigma(t_0, t_1)} \leq C \|\xi(t_0)\|_{L^2} + |z(t_0)|^3 \]
\[ + C \|\xi\|^2_{L^6(t_0, t_1)} \|\xi\|_{L^{\infty}(t_0, t_1)} + C \|z\|_{L^{6}(t_0, t_1)} \|\xi\|_{L^2 L^6(t_0, t_1)}. \] (2.9)

Substituting $\xi = Y + g$ into (1.2), we obtain
\[ i \dot{z} = z + \frac{1}{2} z^2 (G g) + |z|^2 \overline{(G g)} - z |z|^4 \left( \frac{1}{2} (G R_+(1) G) + \overline{(G R_+(1) G)} \right). \] (2.10)

Thus, multiplying $\bar{z}$ and taking the imaginary part we have
\[ \frac{d}{dt} |z(t)|^2 = -i \Gamma \frac{1}{2} |z|^6 + \text{Im} \left( \frac{1}{2} |z|^2 z (G g) + |z|^2 \bar{z} \overline{(G g)} \right), \] (2.11)
where $\Gamma$ is given by (2.1). Thus

$$
\Gamma \|z\|_{L^6(t_0, t)} + 2|z(t)|^2 \leq 2|z(t_0)|^2 + 3\|G\|_{L^6} \|g\|_{L^6} L^2 , - \gamma (t_0, t_1) \|z\|_{L^6(t_0, t)}^3,
$$

for $t_0 < t < t_1$. Taking $\sup_{t_0 < t < t_1}$, we have for fixed constants

$$
\Gamma^{1/2} \|z\|_{L^6(t_0, t_1)} + \|z\|_{L^2(t_0, t_1)} \leq C' \|z(t_0)\| + \Gamma^{-1/2} \|g\|_{L^2} L^2 , - \gamma (t_0, t_1),
$$

$$
\leq C|z(t_0)| + CT^{-1/2} \left(|z(t_0)|^3 + \|\xi(t_0)\| L^2 + (\|\xi\|_{L^2(t_0, t_1)}^2 + \|z\|_{L^2} L^2 , - \gamma (t_0, t_1)) \|\xi\|_{L^6(t_0, t_1)}\right).
$$

Substituting (2.13) into (2.5), we have

$$
\|\xi\|_{L^6(t_0, t_1)} \leq C(\|\xi(t_0)\|_{H^6} + \Gamma^{-1/2} \|z(t_0)\| + \Gamma^{-1} |z(t_0)|^3)
$$

$$
+ C \left(\Gamma^{-1} \|z\|_{L^6(t_0, t_1)}^2 + \gamma^{-1} \|\xi\|_{L^2(t_0, t_1)}^2\right) \|\xi\|_{L^6(t_0, t_1)}.
$$

The estimate (2.4) with $\theta \leq 1/2$ follows from a simple continuity argument combined with the smallness of $\|\xi(t_0)\|_{H^6} + |z(t_0)|$ for $t_0 = 0$. For $\theta \in (1/2, 1]$, (2.4) follows from (2.5) combined with $Stz^{1/2} \rightarrow stz$.

Finally, we show scattering, which is a simple consequence of (2.4). Since

$$
e^{-it_3\Delta} \xi(t_2) - e^{-it_1\Delta} \xi(t_1) = -i \int_{t_1}^{t_2} e^{-is\Delta} \left(|\xi(s)|^2 \xi(s) + z(s) |z(s)|^2 G\right) ds,
$$

it suffices to show

$$
\| \int_{t_1}^{t_2} e^{-is\Delta} \left(|\xi(s)|^2 \xi(s) + z(s) |z(s)|^2 G\right) ds \|_{H^1} \rightarrow 0, \quad \text{as } t \rightarrow +\infty.
$$

From Lemma 3.2 the above integral is bounded by $\|\xi\|_{L^6(t_0, t)}^3 + \|z\|_{L^6(t_0, t)}^3 \rightarrow +\infty$. In addition, since $\|\xi\|_{L^6(t_0, t)}^3 \lesssim \|\xi\|_{L^6(t_0, t)}^3$, we conclude from (2.4) a bound on $\|\xi\|_{L^6(t_0, t)}$. Hence

$$
\|\xi\|_{L^6(t_0, t)}^3 \lesssim \|\xi\|_{L^6(t_0, t)}^3 \lesssim \|\xi\|_{L^6(t_0, t)}^3 \lesssim \|\xi\|_{L^6(t_0, t)}^3.
$$

We obtain $|z(t)| \rightarrow +\infty$. This gives forward scattering, and since it is possible to prove backward scattering by the same argument, the proof of Theorem 2.1 is complete.

**Remark 2.2.** The conclusion about forward scattering of Theorem 2.1 continues to hold if we replace the small energy hypothesis (2.3) with the hypothesis $\|\xi\|_{L^6(0, \infty)} < \infty$. Indeed, by (2.13) we have $z \in L^6(0, \infty)$ and therefore the argument at the end of the proof of Theorem 2.1 can be repeated.

We have the following preliminary result, based uniquely of the conservation of $E$ and $M$.

**Lemma 2.3.** Let $(\xi, z)$ be the solution of (1.1)–(1.2) with $\xi(t_0) \in H^1$. Assume $N_0 \lesssim 1$ for

$$
N_0 := \|\xi(t_0)\|_{H^6} + |z(t_0)|.
$$

Then there exist $C_0 = C(N_0)$ s.t.

$$
\|\xi\|_{L^\infty H^1(\mathbb{R})} + \|z\|_{L^\infty(\mathbb{R})} \leq C_0 (N_1^2 + N_1^2).
$$

**Proof.** From the conservation of $E$ and $M$, see (1.4)–(1.5), we have

$$
\|\xi\|_{L^\infty H^1(t_0, t)} + \|z\|_{L^\infty(t_0, t)} \lesssim N_1 + N_0^2 + N_0^2 + \|z\|_{L^\infty(t_0, t)} \|\xi\|_{L^\infty L^2(t_0, t)}^2
$$

which by $\|\xi\|_{L^\infty(t_0, t)} + \|z\|_{L^\infty L^2(t_0, t)} \leq 2N_0$ due to the conservation of $M$, by $N_0 \lesssim 1$ and by $N_0 \leq N_1$ implies immediately (2.16).
3 Linear estimates

In this section we set some notation and list estimates about the linear Schrödinger equation which are used in subsequent sections. We use material from section 4 of [34]. For \( u \in C(\mathbb{R}; H^1) \), we set

\[ u[t_0](t) := e^{i(t-t_0)\Delta}u(t_0), \quad (3.1) \]

For \( u_0 \in H^1 \), we identify \( u_0 \) with \( u(t) \equiv u_0 \) and define

\[ u_0[t_0](t) = e^{i(t-t_0)\Delta}u_0. \quad (3.3) \]

The solution of

\[ iv = -\Delta v + f, \quad v(t_0) = 0 \]

can be written as

\[ Df[t_0](t) := -i \int_{t_0}^t f[s](t) \, ds. \quad (3.4) \]

We can express as \( u_0[t_0] + Df[t_0] \) the solution of

\[ iv = -\Delta v + f, \quad v(t_0) = u_0. \]

Remark 3.1. We have

- \( u[t_1][t_2](t) = e^{i(t-t_2)\Delta}u[t_1](t_2) = e^{i(t-t_2)\Delta}e^{i(t_2-t_1)\Delta}u(t_1) = e^{i(t-t_1)\Delta}u(t_1) = u[t_1](t). \)
- \( (Df[t_1])[t_2](t) = -i \int_{t_1}^{t_2} f[s](t) \, ds. \)

The following are the classical Strichartz estimates, see Theorem 2.3.3 [3].

**Lemma 3.2 (Strichartz estimates).** There exist constants \( C_0 \) s.t. for any interval \( I \subseteq \mathbb{R} \) with \( t_0 \in I \) and any \( f \)

\[ \|u_0[t_0]\|_{\text{Str}^p(I)} \leq C_0 \|u_0\|_{H^s}, \]

\[ \sup_{t \in \mathbb{R}} \left\| \int_I f[s](t) \, ds \right\|_{H^s} \leq C_0 \|f\|_{\text{Str}^p(I)}, \]

\[ \|Df[t_0]\|_{\text{Str}^p(I)} \leq C_0 \|f\|_{\text{Str}^p(I)}. \]

The following estimates are due to Kato [20], Foschi [12] and Vilela [40].

**Lemma 3.3 (Non-admissible Strichartz).** Let

\[ (p_j, q_j) \in (1, \infty) \times (2, 6) \] \( \{j = 1, 2\} \)

and \( \sigma_j := \frac{2}{p_j} + 3 \left( \frac{1}{q_j} - \frac{1}{2} \right) \) \( (3.5) \)

satisfy

\[ \sigma_0 + \sigma_1 = 0 > \sigma_j - \frac{1}{p_j}, \quad |\sigma_j| \leq 2/3. \quad (3.6) \]

Then there exists a constant \( C \) s.t. for any interval \( I \) with \( t_0 \in I \) and any \( f \)

\[ \|Df[t_0]\|_{L^{p_0}L^{q_0}(I)} \leq C \|f\|_{L^{p_1}L^{q_1}(I)}. \]
We further introduce Nakanishi’s seminorm:
\[
\|u\|_{[T_0,T_1]} := \sup_{T_0 < S < T < T_1} \|u[T] - u[S]\|_{st(T_0,\infty)}.
\] (3.7)

**Remark 3.4.** If we take \(S = T_0\) and \(T = T_1\) then restricting the interval to \((T_0, T_1)\) we have
\[
\|u[T_1] - u[T_0]\|_{st(T_0,\infty)} \geq \|u - u[T_0]\|_{st(T_0, T_1)}.
\]
Similarly, restricting the interval to \((T_1, \infty)\), we have
\[
\|u[T_1] - u[T_0]\|_{st(T_0,\infty)} \geq \|u[T_1] - u[T_0]\|_{st(T_1,\infty)}.
\]
Therefore, we have
\[
\|u\|_{[T_0,T_1]} \geq \max \{\|u - u[T_0]\|_{st(T_0,T_1)}, \|u[T_1] - u[T_0]\|_{st(T_1,\infty)}\}.
\] (3.8)
This inequality will be used frequently.

Nakanishi’s seminorm is dominated by Strichartz’s norm.
\[
\|u\|_{[t_0,t_1]} \leq C\|u\|_{Stz^1(t_0,t_1)}.
\] (3.9)
Indeed, for \(t_0 < s < t < t_1\),
\[
\|u[t] - u[s]\|_{st(t_0,\infty)} \leq \|u - u[s]\|_{st(t_0,t)} + \|u[t] - u[s]\|_{st(t_0,\infty)} \\
\leq C\|u\|_{Stz^1(t_0,t)} + C\|u(s)\|_{H^1} + \|u(t)\|_{H^1} \leq C\|u\|_{Stz^1(t_0,t_1)}.
\]

We have the following, see Lemma 4.2 of [34].

**Lemma 3.5 (Subadditivity).** For \(T_0 < T_1 < T_2\),
\[
\|u\|_{[T_0,T_2]} \leq \|u\|_{[T_0,T_1]} + \|u\|_{[T_1,T_2]}.
\]
For \(\chi(-\infty,T]\) being the characteristic function of \((-\infty,T]\) we have the following elementary lemma.

**Lemma 3.6.** For \(u = u_0[0] + \mathcal{D}f[0]\) we have
\[
u[T] - u[S] = \mathcal{D} \left( \chi(-\infty,T] f \right)[S].
\]

**Remark 3.7.** By the above lemma, we see that if \(u\) is the solution of the inhomogeneous problem, we have
\[
\|u\|_{[T_0,T_1]} = \sup_{T_0 < S < T < T_1} \|\mathcal{D} \chi(-\infty,T] f[S]\|_{st(T_0,\infty)}.
\] (3.10)

**Lemma 3.8.** There is a fixed constant \(C\) s.t. \(\|\mathcal{D}f[0]\|_{[T_0,T_1]} \leq C\|f\|_{Stz^{1/2}(T_0,T_1)}\).

**Proof.** By (3.10), we have for fixed constants \(C'\) and \(C\)
\[
\|\mathcal{D}f[0]\|_{[T_0,T_1]} = \sup_{T_0 < S < T < T_1} \|\mathcal{D} \chi(-\infty,T] f[S]\|_{st(T_0,\infty)} \\
\leq C' \sup_{T_0 < S < T < T_1} \|\mathcal{D} \chi(-\infty,T] f[S]\|_{Stz^{1/2}(T_0,\infty)} \\
\leq C \sup_{T_0 < T < T_1} \|\chi(-\infty,T] f\|_{Stz^{1+1/2}(T_0,\infty)} \leq C\|f\|_{Stz^{1+1/2}(T_0,T_1)},
\]
with the embedding \(Stz^{1/2} \hookrightarrow st\) in the 2nd line and Strichartz estimates (Lemma 3.2) in the 3rd.
Lemma 3.9. Let \((p_1, q_1)\) satisfy \(\frac{1}{p_1} = \sigma_1 = 2/p_1 + 3(1/q_1 - 1/2)\) and \(p_1 < 2\). Then for a fixed \(C\)
\[
\|Df[t_0]\|_{L^{p_1}_r L^{q_1}_s(T_0, T_1)} \leq C\|f\|_{L^{p_1}_r L^{q_1}_s(T_0, T_1)}.
\]

Remark 3.10. Lemma 3.9 is an application of the non-admissible Strichartz (Lemma 3.3) with \((p_0, q_0) = (4, 6)\). In this case \(\sigma_0 = -1/2\) and the condition (3.5) is equivalent to \(\sigma_1 = 1/2\) and \(p_1 < 2\).

Proof. By Lemma 3.6 applied to \(u = Df[t_0]\) in the 1st line and by Lemma 3.3 in 2nd line
\[
\|Df[t_0]\|_{L^{p_1}_r L^{q_1}_s(T_0, T_1)} = \sup_{T_0 < S < T < T_1} \|D\chi_{(-\infty, T)} f[S]\|_{L^4 L^6(T_0, \infty)} \leq C \sup_{T_0 < S < T < T_1} \|\chi_{(-\infty, T)} f\|_{L^{p_1}_r L^{q_1}_s(T_0, T_1)} \leq C\|f\|_{L^{p_1}_r L^{q_1}_s(T_0, T_1)}.
\]

The following is well known, for a reference see [6] Lemma 6.5 (where \(\sigma_0 = 9/2\)).

Lemma 3.11. There exists \(\sigma_0 > 0\) s.t. for any \(\sigma > \sigma_0\) the following facts are true:

1. we have \(R_+(1) \in B(L^{2, \sigma}, L^{2, -\sigma})\);
2. there exists a constant \(C_\sigma\) s.t. for \(v \in L^{2, \sigma}\), we have
\[
\|R_+(1)v[0](t)\|_{L^{2, -\sigma}} \leq C_\sigma (t)^{-3/2} \|v\|_{L^{2, \sigma}} \text{ for all } t \geq 0;
\]
3. for all \(p \geq 1\) there is a constant \(C_{p, \sigma}\) s.t. for \(v \in L^{2, \sigma}\)
\[
\|R_+(1)v[0]\|_{L^p L^{2, -\sigma}(R_+)} \leq C_{p, \sigma} \|v\|_{L^{2, \sigma}}.
\]

We will need the following Duhamel estimates too.

Lemma 3.12. There is a \(C > 0\) s.t. for any \(T > 0\) and any \(f \in L^2 L^{6/5}(-\infty, -T)\) we have
\[
\|\int_{-\infty}^{-T} f[s](t) ds\|_{s(0, \infty)} \leq CT^{-1/4} \|f\|_{L^2 L^{6/5}(-\infty, -T)}.
\]

Proof. By \(L^{6/5}, L^6\) decay estimate we have
\[
\|\int_{-\infty}^{-T} f[s](t) ds\|_{L^6_x} \lesssim \int_{-\infty}^{-T} |t - s|^{-1} \|f(s)\|_{L^{6/5}} ds.
\]
Thus,
\[
\|\int_{-\infty}^{-T} f[s](t) ds\|_{s(0, \infty)} \lesssim \left( \int_{-\infty}^{-T} |t - s|^{-2} ds \right)^{1/2} \|s(0, \infty)\|_{L^2 L^{6/5}(-\infty, -T)} \leq T^{-1/4} \|f\|_{L^2 L^{6/5}(-\infty, -T)}.
\]
4 $L^4$ estimates

In this section, we estimate the solutions of (1.1)–(1.2) in terms of the $L^4$ in time based norms $\|\xi\|_{st}$ and $\|z\|_{L^{12}}$ (which can be thought as the $L^4$ norm of $|z|^2z$). In principle $\|z\|_{L^{12}}^2 \leq \|z\|_{L^6}\|z\|_{L^\infty}$. However, in some situations $\|z\|_{L^{12}}$ is small when $\|z\|_{L^6}$ is not as small as we need.

Lemma 4.1. There exist constants $\mu_0 > 0$, $\mu_{1/2} > 0$ and $C_0$ for any $\theta \in [0, 1]$ s.t. for any solution $(\xi, z)$ of (1.1)–(1.2) in $H^1 \times \mathbb{C}$ with $N_0 \leq \mu_0$ and

$$\|\xi\|_{st(t_0, t_1)} \max\{1, N_{1/2}^3\} \leq \mu_{1/2},$$

for $N_0$ defined by (2.15), we have

$$\|\xi\|_{st(t_0, t_1)} + \|z\|_{L^\infty(t_0, t_1)} + \|z\|_{L^6(t_0, t_1)} \leq C_0 N_0.$$  (4.2)

Proof. By (2.5) we have

$$\|\xi\|_{st(t_0, t_1)} \leq \|\xi(0)\|_{H^\theta} + \|z\|_{L^6(t_0, t_1)}^3.$$  (4.3)

Therefore, it suffices to show

$$\|z\|_{L^\infty(t_0, t_1)} + \|z\|_{L^6(t_0, t_1)} \leq N_0.$$  (4.4)

Proceeding like in the proof of Theorem 2.1, for $\sigma > 9/4$, we write

$$\|g\|_{L^2 L^{2, -\sigma}(t_0, t_1)} \leq C(\|\xi(0)\|_{H^2} + |z(0)|^3) + C\|\xi\|_{L^2(t_0, t_1)}^3 \|\xi\|_{L^\infty(t_0, t_1)} + C\|z\|_{L^\infty(t_0, t_1)}^3 \|z\|_{L^6(t_0, t_1)}^3 \|\xi\|_{st(t_0, t_1)}.$$  (4.5)

Inserting this in the inequality (2.12) we obtain

$$\|z\|_{L^6(t_0, t_1)}^2 + \|z\|_{L^\infty(t_0, t_1)} \leq C|z(0)| + \|\xi(0)\|_{H^2} + C\|z\|_{L^\infty(t_0, t_1)}^3 \|z\|_{L^6(t_0, t_1)}^3 \|\xi\|_{st(t_0, t_1)}.$$  (4.6)

Using this inequality and (4.3), we derive immediately (4.4). □

The following proposition is the main $L^4$ estimate in this section.

Proposition 4.2. There exist constants $\mu_0 > 0$, $\mu_{1/2} > 0$ and $C > 0$ s.t. for any solution $(\xi, z)$ of (1.1)–(1.2) in $H^1 \times \mathbb{C}$ with $N_0 \leq \mu_0$ which satisfies either (4.1) or

$$\left(\|\xi(0)\|_{st(t_0, t_1)} + |z(0)|^2\right) \max\{1, N_{1/2}^3\} \leq \mu_{1/2},$$

we have

$$\|\xi\|_{st(t_0, t_1)} + \|z\|_{L^\infty(t_0, t_1)} + \|z\|_{L^6(t_0, t_1)} \leq C(\|\xi(0)\|_{st(t_0, t_1)} + |z(0)|^2),$$

$$\|\xi(0)\|_{st(t_0, t_1)} \leq C(\|\xi\|_{st(t_0, t_1)} + \|z\|_{L^6(t_0, t_1)}^3).$$

Proof. We first assume (4.1). By nonadmissible Strichartz (Lemma 3.3) with (4.6) and (8/5, 4) for the $|\xi|^2\xi$ term and (4.6) and (4/3, 6) for the $|z|^2G$ term in (1.1) we have

$$\|\xi\|_{st(t_0, t_1)} \leq \|\xi(0)\|_{st(t_0, t_1)} + \|\xi\|_{L^2(t_0, t_1)}^3 + \|z^3G\|_{L^4L^{6/5}(t_0, t_1)}$$

$$\leq \|\xi(0)\|_{st(t_0, t_1)} + \|\xi\|_{L^\infty L^3(t_0, t_1)}^3 \|\xi\|_{L^6L^6(t_0, t_1)}^{3/2} + \|z\|_{L^6(t_0, t_1)}^{3/2}.$$  (4.7)
By $H^{1/2} \hookrightarrow L^3$ and (4.2) we have

$$\|\xi\|_{L^3}^{2} \lesssim \|\xi\|_{st(t_0,t_1)}^{2} \lesssim \left( N_{\xi}^{3/2}\|\xi\|_{st(t_0,t_1)} \right)^{1/2} \|\xi\|_{st(t_0,t_1)}.$$  

By (4.1) we conclude

$$\|\xi\|_{st(t_0,t_1)} \lesssim \|\xi[t_0]\|_{st(t_0,t_1)} + \|z\|_{L^{12}(t_0,t_1)}^{3}.$$  

Interchanging $\xi$ and $\xi[t_0]$ we obtain

$$\|\xi\|_{st(t_0,t_1)} + \|z\|_{L^{12}(t_0,t_1)} \sim \|\xi[t_0]\|_{st(t_0,t_1)} + \|z\|_{L^{12}(t_0,t_1)}^{3}. \tag{4.9}$$  

Consider the $g$ in (2.7). Then, again by the nonadmissible Strichartz (Lemma 3.3) with (4.6) and (8/5, 4) for $\|\xi\|_{L^s}$ and (4.1), (4.3, 6) for $R_1$, we have

$$\|g\|_{L^{4}L^{2,-\sigma}(t_0,t_1)} \lesssim \|g[t_0]\|_{L^{4}L^{2,-\sigma}(t_0,t_1)} + \|\xi\|_{L^{8}L^{4}(t_0,t_1)}^{3/2} + \|R_1\|_{L^{A}L^{6/5}(t_0,t_1)},$$  

Recall that $R_1$ is given by (2.8). Then by $\|\xi\|_{L^{8}L^{4}} \lesssim \|\xi\|_{L^{8}L^{4}}^{3/2} + \|\xi\|_{L^{8}L^{6/5}} \lesssim \|z\|_{L^{\infty}} \|\xi\|_{st}$ and

$$\|g[t_0]\|_{L^{4}L^{2,-\sigma}} \lesssim \|\xi[t_0]\|_{st} + \|\nabla Y[t_0]\|_{L^{4}L^{2,-\sigma}} \lesssim \|\xi[t_0]\|_{st} + \|z[t_0]\|,$$

where in the last inequality we used Lemma 3.11. Then we have

$$\|g\|_{L^{4}L^{2,-\sigma}(t_0,t_1)} \lesssim \|\xi[t_0]\|_{st(t_0,t_1)} + \|z(t_0)\|^{3} + \left( N_{\xi}^{3/2}\|\xi\|_{st(t_0,t_1)}^{1/2} + \|z\|_{L^{\infty}(t_0,t_1)}^{4/3} \right) \|\xi\|_{st(t_0,t_1)}. \tag{4.10}$$  

We now estimate the $L^{12}$ norm of $z$. We multiply (2.11) by $|z(t)|^{6}$ obtaining

$$\frac{1}{4} \frac{d}{dt} |z(t)|^{8} = -\Gamma \frac{1}{2} |z|^{12} + \text{Im} \left( \frac{1}{2} |z|^{8}(G|g|) + |z|^{8} \overline{G|g|} \right). \tag{4.11}$$  

Integrating it from $t_0$ to $t \leq t_1$, we have

$$|z(t)|^{8} + 2\Gamma \|z\|_{L^{12}(t_0,t)}^{12} \lesssim |z(t_0)|^{8} + \|z\|_{L^{12}(t_0,t)}^{12} \|g\|_{L^{4}L^{2,-\sigma}(t_0,t_1)}.$$  

Taking sup$_{t_0 < t < t_1}$ by an elementary argument we obtain

$$\|z\|_{L^{\infty}(t_0,t_1)}^{12} + \|z\|_{L^{12}(t_0,t_1)}^{12} \lesssim |z(t_0)|^{2} + |z| \|g\|_{L^{4}L^{2,-\sigma}(t_0,t_1)}.$$  

From (4.9), (4.10) and (4.12) we obtain

$$\|\xi\|_{st(t_0,t_1)} \lesssim \|z(t_0)\|^{2} + \|z\|_{L^{\infty}(t_0,t_1)}^{4} \|g\|_{L^{4}L^{2,-\sigma}(t_0,t_1)}.$$  

By $\|z\|_{L^{\infty}(t_0,t_1)} \leq \mathcal{N}_0 \ll 1$ and the assumption (4.1) with sufficiently small $\mu_{1/2}$, we obtain

$$\|\xi\|_{st(t_0,t_1)} \lesssim \|z(t_0)\|^{2} + \|z(t_0)\|^{2}.$$  

Here, the smallness of $\mathcal{N}_0$ and $\mu_{1/2}$ depends on $\Gamma$. However, $\Gamma$ is a fixed constant, see (2.2), so there is no harm. Thus

$$\|z\|_{L^{\infty}(t_0,t_1)}^{2} + \|z\|_{L^{12}(t_0,t_1)}^{3} \lesssim |z(t_0)|^{2} + \|z(t_0)\|_{st(t_0,t_1)}.$$  

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Next we assume (4.5). If we take $t > t_0$ sufficiently close to $t_0$ then (4.1) is true. Then from the above argument we obtain

$$\|\xi\|_{\text{st}(t_0,t)} \lesssim (\|\xi[t_0]\|_{\text{st}(t_0,t_1)} + |z(t_0)|^2) \ll \min\{N^{-3}_{1/2}, 1\}.$$ 

Thus, by continuity argument we have (4.6) under assumption (4.5).

Finally we prove (4.7). Under assumption (4.1) we know that we have (4.9) which, in turn, implies (4.7). If instead we start with assumption (4.5) then (4.6) implies (4.1) which, in turn, by the previous sentence implies (4.7).

**Remark 4.3.** The conclusions of Lemma 4.1 continue to hold with Assumption (4.1) replaced by Assumption (4.5) since the latter assumption implies the first one by the argument in Proposition 4.2.

We need estimates on the solution of (1.2) with $z(t)|z(t)|^2G$ replaced by some more general $F(t)$.

**Proposition 4.4.** There exist constants $\mu_{1/2} > 0$ and $C > 0$ s.t. for any $\xi$ satisfying

$$i\xi = -\Delta \xi + |\xi|^2 \xi + F,$$  

where $F \in L^2 L^{6/5}(t_0,t_1) \cap L^2 W^{1,6/5}(t_0,t_1)$, with

either $\|\xi\|_{\text{st}(t_0,t_1)} \leq \mu_{1/2} \min\{1, N^{-3}_{1/2}\}$ or $\|\xi[t_0]\|_{\text{st}(t_0,t_1)} + \|F\|_{L^4 L^{6/5}(t_0,t_1)} \leq \mu_{1/2} \min\{1, N^{-3}_{1/2}\}$,

where

$$N_0 := \|\xi[t_0]\|_{H^\infty} + \|F\|_{L^2 W^{1,6/5}(t_0,t_1)},$$

we have

$$1/C \leq \frac{\|\xi\|_{\text{st}(t_0,t_1)} + \|F\|_{L^4 L^{6/5}(t_0,t_1)}}{\|\xi[t_0]\|_{\text{st}(t_0,t_1)} + \|F\|_{L^4 L^{6/5}(t_0,t_1)}} \leq C$$

and

$$\|\xi\|_{t_0,t_1} \leq C(\sqrt{\mu_{1/2}} \|\xi[t]\|_{\text{st}(t_0,t_1)} + \|F\|_{L^4 L^{6/5}(t_0,t_1)}).$$

**Proof.** Suppose $\|\xi\|_{\text{st}(t_0,t_1)} \ll \min\{N^{-3}_{\theta}, 1\}$. By (2.5) we have

$$\|\xi\|_{\text{st}(t_0,t_1)} \lesssim \|\xi[t]\|_{H^\infty} + \|F\|_{L^2 W^{1,6/5}(t_0,t_1)} + \|\xi[t_0]\|_{\text{st}(t_0,t_1)} \|\xi\|_{\text{st}(t_0,t_1)}.$$  

Then by $\|\xi\|^2_{\text{st}(t_0,t_1)} < 1/2$ we obtain

$$\|\xi\|_{\text{st}(t_0,t_1)} \lesssim N_0.$$  

In particular, under the hypothesis $\|\xi\|_{\text{st}(t_0,t_1)} \ll \min\{1, N^{-3}_{1/2}\}$ we have (4.8) for $\theta = 1/2$. From (4.8) with $|z|^2 z G$ replaced by $F$ we have

$$\|\xi\|_{\text{st}(t_0,t_1)} \lesssim \|\xi[t]\|_{\text{st}(t_0,t_1)} + \|\xi\|^{3/2}_{L^\infty L^3(t_0,t_1)} \|\xi\|^3/2_{\text{st}(t_0,t_1)} + \|F\|_{L^4 L^{6/5}(t_0,t_1)}.$$  

By $\|\xi\|_{\text{st}(t_0,t_1)} \ll N^{-3}_{1/2}$ and (4.8) for $\theta = 1/2$ we obtain (4.15).
The proof of (4.15) under the assumption $\|\xi[t_0]\|_{st(t_0,t_1)} + \|F\|_{L^6L^3(t_0,t_1)} \ll \min\{1,N_{1/2}^3\}$ follows from the previous case by a continuity argument similar to that in the proof of Proposition 4.2. Turning to the proof of (4.16), by (3.10) and the non-admissible Strichartz estimate used in (4.8), we have

$$
\|\xi\|_{st(t_0,t_1)} = \sup_{t_0 < s < t < t_1} \|X(-\infty,t) \left(\xi^2(t) + F\right)[s]\|_{st(t_0,t_2)} \\
\lesssim \left(\mathcal{N}_{1/2}^3 \|\xi\|_{st(t_0,t_1)}^{1/2}\right) \|\xi\|_{st(t_0,t_1)} + \|F\|_{L^6L^3(t_0,t_1)}.
$$

In the course of the proof of Proposition 4.4 we proved also the following lemma.

**Lemma 4.5.** There exist constants $\mu_{1/2} > 0$ and $C > 0$ s.t. for $\xi$ satisfying (4.13) with $F \in L^2W^{1,2}(t_0,t_1)$ and, for $N_\theta$ defined by (4.14), s.t. $\|\xi\|_{st(t_0,t_1)} \leq \mu_{1/2} \min\{1,N_{1/2}^3\}$, then

$$
\|\xi\|_{\text{Str}^\theta(t_0,t_1)} \leq CN_\theta \text{ for all } \theta \in [0,1].
$$

The following is (5.14) Lemma 5.1 in [34].

**Lemma 4.6.** Fix $T > 0$ and suppose $u_n \overset{n \to +\infty}{\rightharpoonup} \varphi$ weakly in $H^1$. Then, we have

$$
\|u_n[0] - \varphi[0]\|_{L^\infty(|t|<T;L^1)} \overset{n \to +\infty}{\longrightarrow} 0.
$$

## 5 Nonlinear perturbation

We first recall that in the proof of Theorem 2.1 we have shown that if $\|\xi\|_{\text{Str}^1(0,\infty)} < \infty$ then $\xi$ scatters forward. In the following, for the $N_\theta = \|\xi(t_0)\|_{H^\theta} + |z(t_0)|$ of (2.15) and under the assumption $N_0 \ll 1$, we show that $\|\xi\|_{st(t_0,\infty)} < \infty$ is a sufficient condition for forward scattering.

**Lemma 5.1.** Let $\xi(t) = (\xi(t),z(t)) \in H^1 \times C$. Then, we have

$$(\xi, z) \text{ scatters forward } \iff \|\xi\|_{st(t_0,\infty)} < \infty.$$ 

A similar statement holds for backward scattering.

**Proof.** We first prove $\Leftarrow$. For $T \gg 1$ we have $\|\xi\|_{st(T,\infty)} \overset{T \to +\infty}{\longrightarrow} 0$. Notice that by Lemma 2.3 we have $\|\xi\|_{L^\infty H^1(t_0,\infty)} \lesssim N_2^2$. Thus for $T$ sufficiently large we can apply Lemma 4.1 and conclude that we have $\|\xi\|_{\text{Str}^1(T,\infty)} < \infty$. Therefore by Remark 2.2 we have scattering forward in time.

We next show $\Rightarrow$. Since $\xi$ scatters, by Strichartz estimates, Lemma 3.2, we have

$$
\|\xi(T)\|_{st(T,\infty)} \leq \|\xi[0]\|_{st(T,\infty)} + \|\xi[t]\|_{st(T,\infty)} + \|\xi[0]\|_{st(T,\infty)} \\
\lesssim \|\xi[T] - \xi[t]\|_{H^1} + \|\xi[t]\|_{st(T,\infty)} \to 0 \quad (T \to \infty).
$$

Thus for $T \gg 1$ we have $\left(\|\xi[T]\|_{st(T,\infty)} + |z(T)|^2\right) \max\{1,\|\xi(T)\|_{H^{1/2}}^2\} \ll 1$, where $\|\xi(T)\|_{H^{1/2}}$ is uniformly bounded in $T$ by Lemma 2.3. Then $\|\xi\|_{st(T,\infty)} < \infty$ by Proposition 4.2.

**Lemma 5.2.** Let $\xi(t) = (\xi(t),z(t)) \in H^1 \times C$. Then there exists a $\mu_0 > 0$ s.t. if $N_0 \leq \mu_0$ and if $\xi(z)$ scatters forward then

$$
\|\xi\|_{T,\infty} + \|\xi\|_{L^2W^{1,6}(T,\infty)} \to 0 \text{ as } T \to +\infty.
$$
Proof. By Lemma 5.1 combined with Lemma 4.1, it is easy to conclude that \( \|\xi\|_{L^2W^{1,q}(T,\infty)} \rightarrow 0 \). We have
\[
\|\xi\|_{T,\infty} = \sup_{T < S < T < \infty} \|\xi[T]\| - \xi[S]\|_{s(t,\infty)} \\
\leq \sup_{T < S < T < \infty} (\|\xi - \xi[S]\|_{s(T,T)} + \|\xi[T]\| - \xi[S]\|_{s(T,\infty)}) \\
\leq \|\xi\|_{s(T,\infty)} + 2\|\xi[S]\|_{s(T,\infty)} + \sup_{T < T_1} \|\xi[T]\|_{s(T,\infty)}.
\]
Since \((\xi, z)\) scatters, we have \(\|\xi\|_{s(T,\infty)} < \infty\). Thus, we have \(\|\xi\|_{s(T,\infty)} \rightarrow 0\). By Proposition 4.2, we have \(\|z\|_{L^2(0,\infty)} < \infty\) and therefore \(\|z\|_{L^2(T,\infty)} \rightarrow 0\). Then, by (4.7) we have \(\|S\|_{s(T,\infty)} \rightarrow 0\) for any \(S > T\) uniformly. Hence we conclude \(\|\xi\|_{T,\infty} \rightarrow 0\). \(\square\)

**Lemma 5.3.** Let \((\xi, z)\) be like in Lemmas 5.1–5.2 satisfying also the conclusions therein. Then
\[
\lim_{t_0 \rightarrow +\infty} \|\xi - \xi[t_0]\|_{s(T,\infty)} = 0.
\] (5.2)

Proof. Proceeding like for the proof of inequality (2.14) in Theorem 2.1 we obtain for \(t_1 \rightarrow +\infty\)
\[
\|\xi - \xi[t_0]\|_{s(T,\infty)} \lesssim |z(t_0)| + (\|z\|_{L^2(T,\infty)} + \|\xi\|_{s(T,\infty)})\|\xi\|_{s(t,\infty)}.
\]
and since \(|z(t_0)| + \|\xi\|_{s(T,\infty)} \rightarrow 0\) we obtain (5.2). \(\square\)

We now prepare the long time perturbation estimate. The following lemmas 5.4 and 5.5 correspond to lemmas 6.3 and 6.4 of Nakanishi [34]. Lemmas 5.4 and 5.5 are used in Claim 7.7 and 7.6 in the proof of Proposition 7.1 respectively.

**Lemma 5.4.** There exist fixed constants \(\mu_0 > 0, \mu_+ > 0\) and \(C > 0\) s.t. for any interval \((t_0, t_1)\) and for any solutions of
\[
\frac{d}{dt}\xi_j = -\Delta \xi_j + |\xi_j|^2 \xi_j + F_j \quad \text{in} \quad (t_0, t_1)
\]
with \(F_j \in L^4L^{6/5}(t_0, t_1) \cap L^2W^{1,4}(t_0, t_1)\) for \(j = 1, 2, \) for \(N_0 \leq \mu_0,\) for
\[
\max_{j=1,2} \left(\|\xi_j[t_0]\|_{s(t_0,t_1)} + \|F_j\|_{L^4L^{6/5}(t_0,t_1)}\right) \leq \delta, \quad \|\xi_1 - \xi_2\|_{s(t_0,t_1)} + \|F_1 - F_2\|_{L^4L^{6/5}(t_0,t_1)} \leq \delta,
\]
where
\[
N_\delta := \max_{j=1,2} \left(\|\xi_j[t_0]\|_{H^s} + \|F_j\|_{L^2W^{1,q}(t_0,t_1)}\right),
\]
and finally for \(0 < \delta \leq \delta \leq \mu_\frac{1}{2} \min(N_1^{-3}, 1),\) we have
\[
\|\xi_1 - \xi_2\|_{s(t_0,t_1)} \leq C\delta^{8/7}\delta^{1/7}N_1^{6/7}.
\]

Proof. First by Proposition 4.4 and Lemma 4.5, we have
\[
\|\xi_j\|_{s(t_0,t_1)} \lesssim \delta, \quad \|\xi_j\|_{s(T,\infty)} \lesssim N_1.
\]

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Now, since
\[
i(\xi_1 - \xi_2) = -\Delta (\xi_1 - \xi_2) + |\xi_1|^2 \xi_1 - |\xi_2|^2 \xi_2 + F_1 - F_2,
\]
for \( t_0 < s < t_1 \), we have
\[
\xi_1 - \xi_2 = (\xi_1 - \xi_2) [s] + D \left( |\xi_1|^2 \xi_1 - |\xi_2|^2 \xi_2 + F_1 - F_2 \right) [s].
\]
Therefore, by nonadmissible Strichartz with
\[
\begin{align*}
p_0 &= 4, q_0 = \frac{24}{7}, \sigma_0 = \frac{2}{4} + 3 \left( \frac{7}{24} - \frac{1}{2} \right) = -\frac{1}{8}, |\sigma_0| < \frac{2}{3}, \sigma_0 - \frac{1}{p_0} = -\frac{1}{8} - \frac{1}{4} < 0, \\
p_1 &= \frac{4}{3}, q_1 = 24, \sigma_1 = \frac{3}{2} + 3 \left( \frac{1}{24} - \frac{1}{2} \right) = \frac{1}{8}, |\sigma_1| < \frac{2}{3}, \sigma_1 - \frac{1}{p_1} = \frac{1}{8} - \frac{3}{4} < 0, \\
p_2 &= 4, q_2 = \frac{24}{9}, \sigma_2 = \frac{2}{4} + 3 \left( \frac{9}{24} - \frac{1}{2} \right) = \frac{1}{8}, |\sigma_2| < \frac{2}{3}, \sigma_2 - \frac{1}{p_2} = \frac{1}{8} - \frac{1}{4} < 0,
\end{align*}
\]
we have
\[
\begin{align*}
\|\xi_1 - \xi_2 - (\xi_1 - \xi_2) [s]\|_{L^4 L^{\frac{24}{7}}(t_0, t_1)} &\lesssim \|(|\xi_1|^2 + |\xi_2|^2)(\xi_1 - \xi_2)\|_{L^4 L^{\frac{24}{7}}(t_0, t_1)} + \|F_1 - F_2\|_{L^4 L^{\frac{24}{7}}(t_0, t_1)} \\
&\lesssim (\|\xi_1\|^2_{L^4(t_0, t_1)} + \|\xi_2\|^2_{L^4(t_0, t_1)})\|\xi_1 - \xi_2\|_{L^4 L^{\frac{24}{7}}(t_0, t_1)} + \|F_1 - F_2\|_{L^4 L^{\frac{24}{7}}(t_0, t_1)} \\
&\lesssim \delta^2 \|\xi_1 - \xi_2\|_{L^4 L^{\frac{24}{7}}(t_0, t_1)} + \delta.
\end{align*}
\]
Thus, we have
\[
\|\xi_1 - \xi_2\|_{L^4 L^{24/7}(t_0, t_1)} \lesssim \|\xi_1 - \xi_2\|_{t_0} \|\xi_1 - \xi_2\|_{L^4 L^{24/7}(t_0, t_1)} + \delta.
\]
Next, by Lemma 3.6, for \( t_0 < s < t < t_1 \),
\[
(\xi_1 - \xi_2)[t] - (\xi_1 - \xi_2)[s] = D\chi_{(-\infty, t]} \left( |\xi_1|^2 \xi_1 - |\xi_2|^2 \xi_2 + F_1 - F_2 \right) [s].
\]
Thus,
\[
\begin{align*}
\|((\xi_1 - \xi_2)[t] - (\xi_1 - \xi_2)[s])\|_{L^4 L^{24/7}(t_0, \infty)} &\lesssim \delta^2 \|\xi_1 - \xi_2\|_{L^4 L^{24/7}(t_0, t_1)} + \delta \\
&\lesssim \delta^2 \|\xi_1 - \xi_2\|_{t_0} \|\xi_1 - \xi_2\|_{L^4 L^{24/7}(t_0, t_1)} + \delta.
\end{align*}
\]
Finally, since \( \|f\|_{S^1} \lesssim \|f\|_{L^4 L^{24/7}} \|f\|_{S^1}^{3/4} \), \( \|f\|_{L^4 L^{24/7}} \lesssim \|f\|_{S^1}^{1/4} \|f\|_{S^1}^{3/4} \), we have
\[
\begin{align*}
\|((\xi_1 - \xi_2)[t] - (\xi_1 - \xi_2)[s])\|_{S^1(t_0, \infty)} &\lesssim \|((\xi_1 - \xi_2)[t] - (\xi_1 - \xi_2)[s])\|_{L^4 L^{24/7}(t_0, \infty)} \chi^{3/7} \\
&\lesssim (\delta^2 \|\xi_1 - \xi_2\|_{t_0} \|\xi_1 - \xi_2\|_{L^4 L^{24/7}(t_0, t_1)} + \delta)^{4/7} \chi^{3/7} \\
&\lesssim (\delta^2 \|\xi_1 - \xi_2\|_{t_0} \|\xi_1 - \xi_2\|_{S^1(t_0, t_1)}^{1/4} \chi^{3/7} + \delta)^{4/7} \chi^{3/7} \\
&\lesssim \delta^8 \chi^{1/7} \chi^{3/7}.
\end{align*}
\]
Therefore, we have the conclusion. \[\Box\]
Lemma 5.5. There exists $\mu_0 > 0$ s.t. for solutions $(\xi_j, z_j)$ of (1.1)–(1.2) s.t. $N_0 \leq \mu_0$, where

$$N_0 := \max_{j=1,2} (\|\xi_j(t_0)\|_{H^s} + |z_j(t_0)|), \quad N_2 := \max_{j=1,2} \left(\|\xi_j\|_{L^4(t_0,t_1)} + \|z_j\|_{L^2(t_0,t_1)}^3\right),$$

then for any $\varepsilon > 0$ there exists $\delta_\varepsilon = \delta_\varepsilon (N_0, N_1, N_2, \varepsilon) > 0$ s.t.

$$\|\langle \xi_1 - \xi_2 \rangle [t_0]\|_{L^4(t_0,t_1)} + \|z_1|z_1|^2 - z_2|z_2|^2\|_{L^4(t_0,t_1)} \leq \delta_\varepsilon$$

implies $\|\xi_1 - \xi_2\|_{[t_0,t_1]} \leq \varepsilon$.

Proof. For $N \gg 1$ determined below, we decompose $(t_0, t_1)$ into subintervals $I_0, I_1, \ldots, I_N$ s.t.

$$\|\xi_1\|_{L^4(I_j)} + \|z_1\|_{L^{12}(I_j)} \leq 2N^{-1/4}N_2 =: \tilde{\delta}.$$ 

Let $I_j = (S_j, S_{j+1})$ with $S_0 = t_0$, $S_{N+1} = t_1$. Now, if $\tilde{\delta} N^3_1 \ll 1$, which is true for $N \gg 1$ sufficiently large, we can apply Proposition 4.2 (4.7) and obtain,

$$\|\xi_1[S_j]\|_{L^4(I_j)} \leq \|\xi_1\|_{L^4(I_j)} + \|z_1\|_{L^{12}(I_j)} \leq \tilde{\delta}.$$ 

Suppose we have

$$\|\langle \xi_1 - \xi_2 \rangle [S_0]\|_{L^4(S_0,t_1)} + \|z_1|z_1|^2 - z_2|z_2|^2\|_{L^4(S_0,t_1)} \leq \delta_0 \leq \tilde{\delta}.$$ 

for some $0 < \delta_0$. Then, using $\|z_1\|_{L^{12}} = \|z_2\|_{L^4}$, we have

$$\|\xi_2[S_0]\|_{L^4(S_0,t_1)} + \|z_2\|_{L^{12}(S_0,t_1)} \lesssim \tilde{\delta}.$$ 

Thus, we can apply Lemma 5.4 and Lemma 2.3 and conclude

$$\|\xi_1 - \xi_2\|_{[S_0,S_1]} \leq CN_1^{12/7} \tilde{\delta}^{8/7} \delta_0^{1/7}.$$ 

Now, set

$$\delta_1 := \delta_0 + CN_1^{12/7} \tilde{\delta}^{8/7} \delta_0^{1/7}.$$ 

By the definition of Nakanishi’s seminorm (3.7) we have

$$\|\langle \xi_1 - \xi_2 \rangle [S_1]\|_{L^4(S_1,t_1)} \leq \|\langle \xi_1 - \xi_2 \rangle [S_0]\|_{L^4(S_0,t_1)} + \|\xi_1 - \xi_2\|_{[S_0,S_1]}.$$ 

Thus, we have

$$\|\langle \xi_1 - \xi_2 \rangle [S_1]\|_{L^4(S_1,t_1)} + \|z_1|z_1|^2 - z_2|z_2|^2\|_{L^4(S_1,t_1)} \leq \delta_1.$$ 

If $\delta_1 \leq \tilde{\delta}$, we can repeat the same argument on $I_1$. Set

$$\delta_{j+1} := \delta_j + CN_1^{12/7} \tilde{\delta}^{8/7} \delta_j^{1/7},$$

inductively. Now, for given $N_1$ and $N_2$, we take $N$ large so that

$$\tilde{\delta} = 2N^{-1/4}N \ll N_1^{-3} \leq N_{1/2}^{-3}.$$
Then, if $\delta_j \leq \tilde{\delta}$, we have
\[
\delta_{j+1} \leq \delta_j^{1/7} (\tilde{\delta}^{6/7} + C N_1^{6/7} \tilde{\delta}^{6/7}) \leq \delta_j^{1/7}.
\]
Thus, if we set $\delta_0$ sufficiently small to satisfy
\[
\delta_{N+1} \leq \delta_0 \leq \min(\varepsilon, \tilde{\delta}),
\]
we have $\delta_j \leq \tilde{\delta} \forall j = 0, 1, \ldots, N+1$ and by Lemma 3.5 we have the following, completing the proof:
\[
\|\xi_1 - \xi_2\|_{(0, t_1]} \leq \sum_{j=0}^{N} \|\xi_1 - \xi_2\|_{[S_j, S_{j+1}]}
\]
\[
\leq \sum_{j=0}^{N} (\delta_{j+1} - \delta_j) = \delta_{N+1} - \delta_0 < \varepsilon.
\]
\[\square\]

6 Linear and Nonlinear Profile Decompositions

We first recall the following result on linear profile decompositions, which is a special case of a more general result in Lemma 5.3 of [34]. See also [22, 31].

**Proposition 6.1.** Let $\{s_n\}_n \subset \mathbb{R}$ and $\{\xi_{0n}\}_n \subset H^1_{rad}$ with $\sup \|\xi_{0n}\|_{H^1} < \infty$. Then, passing to a subsequence, there exists $J^* \in \mathbb{N} \cup \{\infty\}$ and $\{s^i_n\}_n \subset \mathbb{R}$ for each $0 \leq i < J^*$ the following holds.

1. $s^0_n = s_n$ and $s^k_n - s^k_n \to \infty$ or $s^k_n - s^k_n \to -\infty$ as $n \to \infty$ for $j \neq k$.

2. For each $j < J^*$, there exists $\varphi^j \in H^1_{rad}$ s.t. $\xi_{0n}[s_n](s^i_n)^{n-\infty} \varphi^j$ weakly in $H^1$. Further, setting $\lambda^j_n = \varphi^j[s^j_n]$, we have $\lambda^j_n(s^j_n)^{n-\infty} 0$ weakly in $H^1$ for $j \neq k$ and $\varphi^j \neq 0$ for $j > 0$.

3. If for each finite $J \leq J^*$ we define $\gamma^j_n$ from the equality
\[
\xi_{0n}[s_n] = \sum_{j=0}^{J-1} \lambda^j_n + \gamma^j_n,
\]
then we have $\gamma^j_n(s^j_n)^{n-\infty} 0$ weakly in $H^1$ for $j < J$.

4. For all $\theta \in [0, 1]$ we have the Pythagorean formula, for $\|f\|_{\tilde{H}^\theta}^2 := \langle (-\Delta)^\theta f, f \rangle$,
\[
\sum_{0 \leq j < J} \|\lambda^j_n\|_{L^\infty(\mathbb{R})}^2 + \|\gamma^j_n\|_{L^\infty(\mathbb{R})}^2 = \|\xi_{0n}\|_{\tilde{H}^\theta}^2 + o_n(1), \text{ with } o_n(1) \to 0.
\] (6.2)

5. $\|\langle (-\Delta)^\theta \lambda^j_n(t), \lambda^k_n(t) \rangle\|_{L^\infty(\mathbb{R})} \to 0$ ($j \neq k$) and $\|\langle (-\Delta)^\theta \lambda^j_n(t), \gamma^j_n(t) \rangle\|_{L^\infty(\mathbb{R})} \to 0$ ($j < J$).

6. For $0 \leq \theta < 1$,
\[
\lim_{J \to J^*} \limsup_{n \to \infty} \|\gamma^j_n\|_{L^\infty L^4(\mathbb{R}, \text{Stz}^2(\mathbb{R}))} = 0.
\]

In particular, $\lim_{J \to J^*} \limsup_{n \to \infty} \|\gamma^j_n\|_{L^\infty(\mathbb{R})} = 0$. 

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We consider now a sequence of solutions of (1.1)–(1.2). More precisely we consider the following steps (A:1)–(A:7).

(A:1) We consider sequences of solutions \((\xi_n, z_n) \in C^0(\mathbb{R}, H^1_{rad} \times \mathbb{C})\) of (1.1)–(1.2) s.t.
\[ \mathcal{N}_1 < \infty \text{ and } \mathcal{N}_0 \ll 1 \text{ for } \mathcal{N}_0 := \sup_n \|\xi_n(0)\|_{H^s} + |z_n(0)|. \tag{6.3} \]
Notice that \(\mathcal{N}_0 \leq C_\theta \mathcal{N}_1^{1-\theta} \mathcal{N}_1^\theta\) for fixed constants \(C_\theta\). We can apply Proposition 6.1 for \(s_n^0 = 0\) and \(\xi_{0n} := \xi_n(0)\).

(A:2) We fix \(J\) in the decomposition of Proposition 6.1, sufficiently large s.t. \(\limsup_n \|\gamma_n^J\|_{st(\mathbb{R})} \ll \min\{1, \mathcal{N}_1^{-3}\}\) and \(\limsup_n \|\gamma_n^J\|_{L^\infty(\mathbb{R})} \ll \mathcal{N}_0\). We order the profiles in Proposition 6.1 so that there exists \(0 < L \leq J\) s.t. for any \(0 < j < L\) we have \(s_n^j - s_n^{j-1} \xrightarrow{n \to \infty} +\infty\) and for \(L \leq j < J\) we have \(s_n^j \xrightarrow{n \to \infty} -\infty\).

(A:3) We introduce a parameter \(\tau > 1\) and set \(s^j_{\pm,n} = s^j \pm \tau\), but with \(s^{0}_{-n} = 0\) and \(s^{L}_n = \infty\).

(A:4) Reducing to subsequences we can assume that \(z_n(\cdot + s^j_n) \xrightarrow{n \to \infty} z^j\) in \(\mathbb{C}\) and \(\xi_n(\cdot + s^j_n) \xrightarrow{n \to \infty} \xi^j\) weakly in \(H^1\) and uniformly on compact sets.

(A:5) We set \(\lambda^j := \varphi^j[0]\).

(A:6) We set \(\Lambda^j_n := \xi^j(\cdot - s^j_n)\) and \(z^j_n := z^j(\cdot - s^j_n)\).

(A:7) For \(0 \leq j < L\) we denote by \(\Gamma^J_{n,j,\tau}\) the function s.t.
\[ \Gamma^J_{n,j,\tau}(s^j_{+,n}, t) = -\Delta \Gamma^J_{n,j,\tau} + \chi_{[s^j_{-n}, s^j_{+,n}]} \left( \|\Gamma^J_{n,j,\tau}\|_{L^\infty(\mathbb{R})} \right) \]
\[ + \left( |\Gamma^J_{n,j,\tau}|^2 \Gamma^J_{n,j,\tau} + |z_n|^2 z_n G \right) \]
\[ \text{with } \Gamma^J_{n,j,\tau}(s^j_{+,n}) = \gamma^J_{n,j,\tau}(s^j_{+,n}). \tag{6.4} \]

In the case \(j = L - 1\), we replace \([s^j_{-n}, s^j_{+,n}]\) by \([s^{L-1}_{-n}, \infty)\).

**Definition 6.2.** Given a sequence \(X_n(\tau)\) dependent on a large parameter \(\tau \gg 1\) we write \(X_n(\tau) = o_\tau\) if \(\lim_{\tau \to +\infty} \limsup_{n \to +\infty} X_n(\tau) = 0\).

In the sequel we will have various quantities and the relation among them will be
\[ o_\tau \ll \sup_n \|\gamma_n^j\|_{st(\mathbb{R})} \ll \max\{\mathcal{N}_1^{-3}, \mathcal{N}_0\} \ll 1. \]
Notice that \(\mathcal{N}_1^2 \mathcal{N}_0\) may be not small but \(\|\gamma_n^j\|_{st(\mathbb{R})} \mathcal{N}_1^2 \ll 1\).

**Lemma 6.3.** For any \(0 \leq j < L\) and \(T > 0\), we have
\[ \|\gamma_n^j\|_{st([s_n^j - T, s_n^j])} \to 0 \text{ as } n \to 0. \tag{6.5} \]

**Proof.** This is (7.18) of Nakanishi [34].
From $\xi_n(0) = \xi_n = \sum_{j=0}^{l-1} \lambda_n^j(0) + \gamma_n^j(0)$, by $z_n(0) \to z^0$, by the conservation of $M$ and $E$ for (1.1)--(1.2) and by the Pythagorean equality (6.2) we have, for $o(1) \to 0$,

\[
M(\xi_n, z_n) = M(\xi^0, z^0) + \sum_{j=0}^{l-1} 2^{-1} \|\lambda_n^j\|_{L^2} + 2^{-1} \|\gamma_n^j\|_{L^2} + o(1),
\]

\[
E(\xi_n, z_n) = E(\xi^0, z^0) + \sum_{j=0}^{l-1} 2^{-1} \|\nabla \lambda_n^j\|_{L^2} + 2^{-1} \|\nabla \gamma_n^j\|_{L^2} + o(1).
\]

The following lemma is proved, see formulas (7.15) and (7.30), in Sect. 7 [34].

**Lemma 6.4.** We have for the $\lambda_n^j$'s of Proposition 6.1 and for $0 \leq j < L$

\[
\sum_{k=0}^{j-1} \|\lambda_n^k\|_{st(s_n^j, \infty)} + \sum_{k=j+1}^{j-1} \|\lambda_n^k\|_{st(0, s_n^j, n)} \to 0 \text{ and } \|\lambda_n^j\|_{st(R\setminus(s_n^j, s_n^{j+1})]} = o_\tau.
\]

### 7 The main iteration argument

The following analogue of Lemma 7.1 [34] is the main property of profile decompositions.

**Proposition 7.1.** Let $0 < l \leq L$ with the $L$ of (A:2) and assume that the $(\xi^j, z^j)$ in (A:4) scatter forward for all $j < l$. Let $\ell = \min\{l, L-1\}$. Then the following are true:

(i) for $0 \leq j \leq \ell$ we have

\[
\|\xi_n(s_n^j, n) - \gamma_n^j - \sum_{i=j}^{j-1} \lambda_n^i\|_{st(s_n^j, \infty)} = o_\tau;
\]

\[
\|\Lambda_n^{j, j+n} - \lambda_n^{j, n}\|_{st(0, s_n^j, \infty)} = o_\tau;
\]

(ii) for $0 \leq j \leq \ell$ we have

\[
\|\Lambda_n^{j, j+n} - \lambda_n^{j, n}\|_{st(0, s_n^j, \infty)} = o_\tau;
\]

(iii) for $0 \leq j < \ell$ we have

\[
\|\Lambda_n^{j, j+n} - \lambda_n^{j, n}\|_{st(0, s_n^j, \infty)} = o_\tau;
\]

(iv) for $0 \leq j \leq \ell$ we have

\[
\|\xi_n - \Lambda_n^j\|_{st(s_n^j, s_n^j+n)} = o_\tau;
\]

(v) for $0 \leq j < \ell$ we have

\[
\|\xi_n - \Gamma_n^{j, j+n}\|_{st(s_n^j, s_n^j+n)} = o_\tau;
\]

(vi) for $0 \leq j < \ell$,

\[
|z^{j+1}(-\tau)| + \|\xi^{j+1} - \varphi^{j+1}[0]\|_{st((-\infty, -\tau)} \to 0, \text{ as } \tau \to \infty.
\]

(vii) for $0 \leq j < \ell$,

\[
\|\Gamma_n^{j, j+n} - \lambda_n^{j, n}\|_{st(s_n^j, \infty)} = o_\tau.
\]
Remark 7.2. We prove Proposition 7.1 by induction. First we prove (7.1) and (7.2) for \( j = 0 \), which are trivial, and then we prove \((i) \Rightarrow (ii) \Rightarrow \cdots \Rightarrow (vi) \Rightarrow ((i) \text{ for } j+1)\). Therefore, step by step (finite induction), we have the conclusion. However, for (vi), we specify that \( \xi_{j+1} \) scatters backward to \( \varphi_{j+1} \) only after we have (7.1) of \( (i) \) for \( j+1 \).

Proof. The proof of Proposition 7.1 is the consequence of Claims 7.3–7.15.

Claim 7.3. (7.1) and (7.2) are true for \( j = 0 \).

Proof. Claims (7.1) and (7.2) for \( j = 0 \) are true because the l.h.s. are 0 by definition. \( \square \)

Claim 7.4 (Proof of (ii) for \( j \)). Assume (7.1) and (7.2) for a \( j \) with \( j \leq \ell \). Then (7.3) is true for \( j \).

Proof. The claim follows from

\[
\|A^j_n\|_{[0,s^j_{n-1},s^j_{n+1}]} = \|A^j_n - \lambda_n^j\|_{[0,s^j_{n-1},s^j_{n+1}]} \lesssim \|A^j_n - \lambda_n^j\|_{st^{(0,s^j_{n-1},n)}} \lesssim \|A_n^j[s^j_{n-1},n\rangle - \lambda_n^j\|_{st^n(0,\infty)} = o_\tau,
\]

where we have used (3.9) as well as (7.2). \( \square \)

Claim 7.5 (Proof of (iii) for \( j \)). Assume (7.1) and (7.2) for a \( j \) with \( j < \ell \). Then (7.4) is true for \( j \).

Proof. By Lemma 5.2 and the hypothesis that \((\xi^j, z^j)\) is scattering forward for \( 0 \leq j < \ell \), by the definition of \( A_n^j \) in \( (A:6) \) we have

\[
\|A^j_n\|_{[s^j_{n-1},n]} = \|\xi^j\|_{[\tau,\infty]} = o_\tau \text{ for } 0 \leq j < \ell.
\]

Claim 7.6 (Proof of (iv) for \( j \)). Assume (7.1) and (7.2) for a \( j \) with \( j \leq \ell \). Then (7.5) is true for \( j \).

Proof. We have

\[
\|(\xi_n - A_n^j)[s^j_{n-1},n]\|_{st(s^j_{n-1},n,s^j_{n+1},n)} \leq \|\xi_n[s^j_{n-1},n] - \gamma_n^j\|_{st(s^j_{n-1},n,s^j_{n+1},n)} + \|\gamma_n^j\|_{st(s^j_{n-1},n,s^j_{n+1},n)} + \sum_{i=j}^{j-1} \|A_n^j\|_{st(s^j_{n-1},n,s^j_{n+1},n)}
\]

\[
+ \sum_{i=j+1}^{\ell} \|A_n^j\|_{st(s^j_{n-1},n,s^j_{n+1},n)} + \|A_n^j - \lambda_n^j\|_{st(s^j_{n-1},n,s^j_{n+1},n)} = o_\tau,
\]

where we used the following bounds for the terms in the r.h.s.: (7.1) for \( j \) for the 1st; Lemma 6.3 for the 2nd; Lemma 6.4 for the 3rd; (7.2) for the 4th. Therefore by \( \|z_n^j z_n^j - z_n^j z_n^j\|_{L^2(s^j_{n-1},n,s^j_{n+1},n)} \to 0 \), which follows from \( (A:4) \) and \( (A:6) \), we can apply Lemma 5.5 and obtain (7.5) for \( j \).

Claim 7.7 (Proof of (v) for \( j \)). Assume (7.1) and (7.2) and (7.5) for a \( j \) with \( j < \ell \). Then (7.6) is true for \( j \).

Proof. Because of forward scattering of \( \xi^j \) and by (5.1) for \( 0 \leq j < \ell \) we have

\[
\|A_n^j[s^j_{n+1},\infty]\|_{st(s^j_{n+1},\infty)} = \|\xi^j[\tau]\|_{st(\tau,\infty)} \to 0.
\]
We have for $0 \leq j < l$

\[
\|\xi_n[s_{n, j}^+ \cdot n] - \gamma_n^j - \sum_{i=j}^{J-1} \lambda_n^i\|_{\text{st}(s_{n, j}^+ \cdot n, \infty)} \leq \|\xi_n[s_{n, j}^+ \cdot n] - \gamma_n^j - \sum_{i=j}^{J-1} \lambda_n^i\|_{\text{st}(s_{n, j}^+ \cdot n, \infty)}
\]

\[
+ \|\lambda_n^j - \Lambda_n^j[s_{n, j}^+ \cdot n]\|_{\text{st}(s_{n, j}^+ \cdot n, \infty)} + \|\xi_n - \Lambda_n^j[s_{n, j}^+ \cdot n] - (\xi_n - \Lambda_n^j)[s_{n, j}^+ \cdot n]\|_{\text{st}(s_{n, j}^+ \cdot n, \infty)}
\]

\[
+ \|\lambda_n^j[s_{n, j}^+ \cdot n]\|_{\text{st}(s_{n, j}^+ \cdot n, \infty)} + \|\lambda_n^j\|_{\text{st}(s_{n, j}^+ \cdot n, \infty)} = o_{\tau},
\]

where we have used the following bounds for the terms in the r.h.s.: (7.1) for $j$ for the 1st and (7.2) for the 2nd; (3.8) and (7.5) for the 3rd; (7.10) for the 4th; Lemma 6.4 for the last. Therefore by Lemma 6.4 and by (6.4)

\[
\|\xi_n[s_{n, j}^+ \cdot n]\|_{\text{st}(s_{n, j}^+ \cdot n, s_{n, j+1}^+ \cdot n)} = \|\xi_n[s_{n, j}^+ \cdot n] - \gamma_n^j\|_{\text{st}(s_{n, j}^+ \cdot n, s_{n, j+1}^+ \cdot n)} = o_{\tau}.
\]

Thus

\[
\|\xi_n[s_{n, j}^+ \cdot n]\|_{\text{st}(s_{n, j}^+ \cdot n, s_{n, j+1}^+ \cdot n)} \leq \|\gamma_n^j\|_{\text{st}(s_{n, j}^+ \cdot n, s_{n, j+1}^+ \cdot n)} + o_{\tau}.
\]

By (7.13), forward scattering of $(\xi^j, z^j)$ and uniform convergence on compact sets $z_n(\cdot + s_n^j) \to z^j$, picking $J \gg 1$ and $\tau \gg 1$ we have

\[
\|\xi_n[s_{n, j}^+ \cdot n]\|_{\text{st}(s_{n, j}^+ \cdot n, s_{n, j+1}^+ \cdot n)} + |z_n(s_n^j)|^2 \ll \min\{1, N_{n/2}^{-3}\}.
\]

Thus by Proposition 4.2 and (A;2), for $\tau \gg 1$ and $n \gg 1$ we have

\[
\|\xi_n\|_{\text{st}(s_{n, j}^+ \cdot n, s_{n, j+1}^+ \cdot n)} + |z_n(s_n^j)|^2 \ll \|\xi_n\|_{\text{st}(s_{n, j}^+ \cdot n, s_{n, j+1}^+ \cdot n)} + o_{\tau} \ll \min\{1, N_{n/2}^{-3}\}.
\]

Then (7.6) is obtained from Lemma 5.4.

We record that from (7.15) and Lemma 4.1, we have

\[
|z_n|^3_{L^6(s_{n, j}^+ \cdot n, s_{n, j+1}^+ \cdot n)} \leq C_0 N_0.
\]

Claim 7.8 (Partial proof of (vi) for $j$). Assume (7.15) for a $j$ with $j < \ell$. Then, there exists some $h_{\ell+1}^j \in H^1$ s.t. we have

\[
|z_{\ell+1}^j(\tau)| + \|\xi_{\ell+1}^j - h_{\ell+1}^j[0]\|_{\text{st}(\infty, -\infty)} \to 0 \text{ as } \tau \to \infty.
\]

Remark 7.9. To get (7.7) we need to show $h_{\ell+1}^j = \varphi_{\ell+1}^j$. This will done after we show (7.1) for $j+1$.

Proof. By Lemmas 5.1 and 5.3, we only have to show $\|\xi_{\ell+1}^j\|_{\text{st}(\infty, 0)} < \infty$. Thus, it suffices to show that for some $\tau > 0$, we have $\|\xi_{\ell+1}^j\|_{\text{st}(\tau, -\tau)} \leq 1$ for arbitrary $T > \tau$. Since $\xi_n(s_n^{j+1} + t) \to \xi_{\ell+1}^j(t)$, by weak lower semi-continuity and by (7.15), we have

\[
\|\xi_{\ell+1}^j\|_{\text{st}(\tau, -\tau)} \leq \liminf_{n \to \infty} \|\xi_n\|_{\text{st}(s_n^{j+1} - T, s_n^{j+1})} \leq \liminf_{n \to \infty} \|\xi_n\|_{\text{st}(s_n^{j+1}, s_n^{j+1})} \leq 1.
\]

Therefore, we have the conclusion.

The proof of (7.8) follows from Claims 7.10–7.12.

Claim 7.10. Assume (7.15) and (7.16) for a $j$ with $j < \ell$. Then $\|\Gamma_n^{j, \ell, \tau}\|_{L^\infty(L^2(\tau \leftarrow T; (s_{n, j}^+ \cdot n, \infty)))} \lesssim N_0$. 

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Proof. First, (7.15) combined with Lemma 4.5 yield
\[
\left\| \Gamma_n^{J,j,\tau} \right\|_{L^\infty L^2(s_{-n}^1, \infty)} \leq \left\| \Gamma_n^{J,j,\tau} \right\|_{\text{Stz}^0(s_{-n}^1, \infty)} \lesssim N_0.
\]
We next estimate \(\left\| \Gamma_n^{J,j,\tau} \right\|_{L^\infty L^4(s_{-n}^1, \infty)}\). By Duhamel's formula
\[
\Gamma_n^{J,j,\tau}(t) = \gamma_n^J(t) + D \left( 1_{(s_{-n}^1, s_{-n}^1 + 1]} \left( z_n^2 \tau_n^G \right) \right) [s_{-n}^j](t) + D \left( 1_{(s_{-n}^1, s_{-n}^1 + 1]} \left( \left| \Gamma_n^{J,j,\tau} \| \Gamma_n^{J,j,\tau} \| \right| \right) \right) [s_{-n}^j](t).
\]
For the first term in the r.h.s. of (7.18), we have \(\left\| \gamma_n^J \right\|_{L^\infty L^4(\mathbb{R})} \lesssim N_0\) by (A:2). The second term can be bounded by Strichartz's estimates and (7.16). Indeed, by \(\text{Stz}^1 \hookrightarrow L^\infty L^4\) and (7.16), we have
\[
\left\| D \left( 1_{(s_{-n}^1, s_{-n}^1 + 1]} \left( z_n^2 \tau_n^G \right) \right) [s_{-n}^j] \right\|_{L^\infty L^4(s_{-n}^1, \infty)} \lesssim \left\| z_n^2 \tau_n^G \right\|_{L^\infty L^3(s_{-n}^1, \infty)} \lesssim \left\| \tau_n \right\|_{L^6(s_{-n}^1, s_{-n}^1 + 1)} \lesssim N_0.
\]
We handle the last term by bootstrap, that is, we assume \(\left\| \Gamma_n^{J,j,\tau} \right\|_{L^\infty L^4(s_{-n}^1, s_{-n}^1 + T)} \leq C N_0\) for sufficiently large \(C > 0\) (but \(CN_0 \ll 1\), and then we show that we can replace \(C\) by \(C/2\), achieving the desired conclusion by standard arguments. The estimates to accomplish this follow. We write
\[
\left\| D \left( 1_{(s_{-n}^1, s_{-n}^1 + 1]} \left( \left| \Gamma_n^{J,j,\tau} \| \Gamma_n^{J,j,\tau} \| \right| \right) \right) [s_{-n}^j] \right\|_{L^\infty L^4(s_{-n}^1, s_{-n}^1 + T)} \lesssim \left( \int_{s_{-n}^1}^{s_{-n}^1 + 1} \left( 1_{s_{-n}^1, s_{-n}^1 + 1}(s) |t-s|^{-3/4} \left\| \Gamma_n^{J,j,\tau} \right\|_{L^4}^3 ds \right) \right)^{1/p}
\]
\[
\lesssim \left( \int_{s_{-n}^1}^{s_{-n}^1 + 1} 1_{s_{-n}^1, s_{-n}^1 + 1}(s) \left\| \Gamma_n^{J,j,\tau} \right\|_{L^4}^2 ds \right)^{1/p} \lesssim \left\| \Gamma_n^{J,j,\tau} \right\|_{L^\infty L^4}.
\]
Then, taking \(p = 4/3\) \((p' = 4)\), by \(\text{Stz}^0 \hookrightarrow L^8 L^4\), we have
\[
\left\| D \left( 1_{(s_{-n}^1, s_{-n}^1 + 1]} \left( \left| \Gamma_n^{J,j,\tau} \| \Gamma_n^{J,j,\tau} \| \right| \right) \right) [s_{-n}^j] \right\|_{L^\infty L^4(s_{-n}^1, s_{-n}^1 + T)} \lesssim \left( \left\| \Gamma_n^{J,j,\tau} \right\|_{L^\infty L^4(s_{-n}^1, s_{-n}^1 + T)} \right)_2 + \left( \left\| \Gamma_n^{J,j,\tau} \right\|_{\text{Stz}^0(s_{-n}^1, s_{-n}^1 + 1)} \right)_2 \left\| \Gamma_n^{J,j,\tau} \right\|_{L^\infty L^4(s_{-n}^1, s_{-n}^1 + T)}.
\]
Therefore, we have the desired estimate. \(\square\)

Claim 7.11. Assume (7.15), (7.16) and (7.17) for a \(j\) with \(j < \ell\). Set
\[
\left\| f \right\|_w := \sup_{s_{-n}^1 < t < s_{-n}^j} w(t) \| f(t) \|_{L^4 L^\infty} \text{ where}
\]
\[
w(t) = \begin{cases} 1 & t > s_{-n}^j, \\ t - s_{-n}^j - \delta & t \leq s_{-n}^j \end{cases}
\]
for a preassigned \(\delta > 0\). Then, we have
\[
\left\| \Gamma_n^{J,j,\tau} \right\|_w = o_{\tau}, \quad (7.19)
\]
Proof. The proof is similar to that of Claim 7.10 because we estimate the three terms in (7.18). First, we have \( \| \gamma_n^j \|_w \to 0 \) as \( n \to \infty \). Indeed, fix \( \varepsilon > 0 \) arbitrary and take \( T > 0 \) so that \( \langle T \rangle^{-\delta} < \varepsilon \). Then

\[
\| \gamma_n^j \|_w \leq \sup_{s_{+}^j < t \leq s_{-}^{j+1} - T} w(t) \langle \gamma_n^j(t) \rangle_{L^4} + \sup_{s_{+}^j - T < t \leq s_{-}^{j+1}} \| \gamma_n^j(t) \|_{L^4}
\]

where the 1st term can be bounded by \( \langle T \rangle^{-\delta} \| \gamma_n^j \|_{L^\infty L^4} < \varepsilon \) and the 2nd term converges to 0 as \( n \to \infty \) by Lemma 6.3.

We next bound the second term of the r.h.s. of (7.18)

\[
\| \mathcal{D} \left( 1_{[s_{-}^j, s_{+}^{j+1}]} \left( |z|^2 z G \right) \right) \left[ s_{+}^j \right] \|_w \leq \sup_{s_{+}^j < t < s_{-}^{j+1}} w(t) \left( \int_{s_{+}^j}^{t-1} + \int_{t}^{t+1} \right) 1_{[s_{-}^j, s_{+}^{j+1}]}(s) \min(|t - s|^{-3/2}, |t - s|^{-3/4}) |z_n(s)|^3 \, ds. \tag{7.20}
\]

For \( S > 1 \) yet to be determined, we divide the time region in three cases \( t < s_{-}^{j+1} - S, s_{-}^{j+1} - S < t < s_{-}^{j+1} + S \) and \( s_{-}^{j+1} + S < t < s_{+}^{j+1} \). In the 1st case we consider, using the mass invariant (1.5),

\[
\sup_{s_{+}^j, t < s_{-}^{j+1} - S} w(t) \int_{s_{+}^j}^{t-1} 1_{[s_{-}^j, s_{+}^{j+1}]}(s) |t - s|^{-3/2} |z_n(s)|^3 \, ds \lesssim \langle S \rangle^{-3} \| z_n \|_{L^\infty(s_{-}^{j+1}, s_{+}^{j+1})}^3 \lesssim \langle S \rangle^{-\delta} M^3.
\]

In the 2nd case we consider

\[
\begin{align*}
\sup_{s_{-}^{j+1} - S < t < s_{-}^{j+1} + S} & w(t) \int_{s_{+}^j}^{t-1} 1_{[s_{-}^j, s_{+}^{j+1}]}(s)(t - s|^{-3/2} |z_n(s)|^3 \, ds \\
& \leq \sup_{s_{-}^{j+1} - S < t < s_{+}^{j+1}} w(t) \left( \int_{s_{-}^{j+1} - S}^{t-1} + \int_{s_{+}^j}^{t-1} \right) 1_{[s_{-}^j, s_{+}^{j+1}]}(s)(t - s|^{-3/2} |z_n(s)|^3 \, ds \\
& \lesssim \langle S \rangle^{-\delta} \| z_n \|_{L^\infty} + \| z_n \|_{L^\infty(s_{-}^{j+1} - S, s_{+}^{j+1} + S)} \lesssim \langle S \rangle^{-\delta} M^3 + o_r,
\end{align*}
\]

where we have used backward scattering for \( j + 1 \) in the last inequality (that is, it has been proved in Claim 7.8 that \( \| z_{j+1} \|_{L^\infty(-s_{-}, -s_{-} + S)} = o_r \) for fixed \( S \), and, using (A:4) and (A:6), we get \( \| z_n \|_{L^\infty(-s_{-}, -s_{-} + S)} = o_r \). Finally, in 3rd case we consider

\[
\begin{align*}
\sup_{s_{-}^{j+1} + S < t < s_{+}^{j+1}} w(t) \int_{s_{+}^j}^{t-1} 1_{[s_{-}^j, s_{+}^{j+1}]}(s)(t - s|^{-3/2} |z_n(s)|^3 \, ds \lesssim S^{-1/2} \| z_n \|_{L^\infty} \lesssim \langle S \rangle^{-\delta} M^3.
\end{align*}
\]

The term with \( \int_{t-1}^{t} \) in (7.20) can be bounded in similarly. For the case \( t < s_{-}^{j+1} - S \), we can use the smallness of \( w(t) \), for the case \( s_{-}^{j+1} - S < t < s_{-}^{j+1} + S \), we can use the backward scattering and for the case \( s_{-}^{j+1} + S < t < s_{+}^{j+1} \) the integral becomes 0.

Finally, we estimate the third term of (7.18) by bootstrap:

\[
\| \mathcal{D} \left( 1_{[s_{-}^j, s_{+}^{j+1}]} \left( |\Gamma|^2 \right) \right) \left[ s_{+}^j \right] \|_{L^4 L^\infty} \lesssim \| \Gamma_{n}^{j, \tau} \|_{L^\infty L^4(s_{+}^j, \infty)} \int_{s_{+}^j}^{t} \min(|t - s|^{-3/4}, |t - s|^{-3/2}) \| \Gamma_{n}^{j, \tau}(s) \|_{L^2 L^4} \, ds.
\]
The conclusion follows from Claim 7.10 and
\[ w(t) \int_{s_{-}^{j,n}}^{t} \min(|t-s|^{-3/4}, |t-s|^{-3/2}) w(s)^{-1} \, ds \lesssim 1. \]
\[ \square \]

Claim 7.12 (Proof of (vii) for \( j \)). We have (7.8).

Proof. By (7.15) and our choice \( J \gg 1 \) we can apply Proposition 4.4 concluding
\[ \| \Gamma_{n}^{J,j,\tau} \|_{S_{t}^{1}(s_{-}^{j,n}, s_{+}^{j+1,n})} \leq \mu_{1/2} \min \{ 1, \mathcal{N}_{1/2}^{-1} \} \]
for the constant \( \mathcal{N}_{1/2} \) in (6.3), a constant that, thanks to the Pythagorean formula (6.2), serves also as a bound for \( \| \gamma_{n}^{J} \|_{L^{\infty}H^{4}(\mathbb{R})} \). Then, by Lemma 4.5, for the interval \( (s_{-}^{j,n}, s_{+}^{j+1,n}) \) and the standard Strichartz’s estimates of Lemma 3.2 for \( (s_{-}^{j,n}, \infty) \), for a fixed \( C \) and the \( \mathcal{N}_{1} \) in (6.3) we obtain
\[ \| \Gamma_{n}^{J,j,\tau} \|_{S_{t}^{1}(s_{-}^{j,n}, \infty)} \leq C \mathcal{N}_{1}. \] (7.21)

We next claim
\[ \| \Gamma_{n}^{J,j,\tau} \|_{S_{t}^{1}(s_{-}^{j,n}, s_{+}^{j+1,n})} = o_{\tau}. \] (7.22)

Notice that we have
\[ \| \Gamma_{n}^{J,j,\tau} \|_{L^{\infty}(L^{4}+L^{\infty})(s_{-}^{j,n}, s_{+}^{j+1,n})} = o_{\tau}, \] (7.23)
from (7.19) and the definition of \( \| \cdot \|_{w} \). By interpolation \( \| f \|_{S_{t}^{1}} \leq \| f \|_{L^{\infty}L^{3}}^{1/3} \| f \|_{L^{\infty}L^{4}}^{2/3}, \| f \|_{S_{t}^{1}} \leq \| f \|_{L^{\infty}L^{3}}^{1/3} \| f \|_{L^{8/3}L^{8}}^{2/3} \) and \( S_{t}^{1} \hookrightarrow L^{8/3}B_{1,2}^{1/2} \hookrightarrow L^{8/3}L^{4}, L^{8/3}L^{8} \), we have
\[ \| \Gamma_{n}^{J,j,\tau} \|_{S_{t}^{1}(s_{-}^{j,n}, s_{+}^{j+1,n})} \leq \| \Gamma_{n}^{J,j,\tau} \|_{S_{t}^{1}(s_{-}^{j,n}, s_{+}^{j+1,n})}^{2/3} \| \Gamma_{n}^{J,j,\tau} \|_{L^{\infty}(L^{4}+L^{\infty})(s_{-}^{j,n}, s_{+}^{j+1,n})}^{1/3} = o_{\tau}. \] (7.24)

Therefore we have (7.22). By (7.22) and Lemma 6.3, to get (7.8) it suffices to prove
\[ \| \Gamma_{n}^{J,j,\tau} - \gamma_{n}^{J} \|_{S_{t}(s_{-}^{j,n}, \infty)} = o_{\tau}. \]

This last formula follows from (7.18) combined with Lemma 3.12. Indeed, by \( S_{t}^{1} \hookrightarrow L^{6}L^{18/5} \), we have, by (7.16), (7.21) and Lemma 3.12
\[ \| \Gamma_{n}^{J,j,\tau} - \gamma_{n}^{J} \|_{S_{t}(s_{-}^{j,n}, \infty)} \lesssim \tau^{-1/4} \left( \| z_{n} \|_{L^{6}(s_{-}^{j,n}, s_{+}^{j+1,n})} + \| \Gamma_{n}^{J,j,\tau} \|_{S_{t}(s_{-}^{j,n}, s_{+}^{j+1,n})}^{3} \right) \lesssim \tau^{-1/4} \mathcal{N}_{1}(1 + \mathcal{N}_{1}^{2}) = o_{\tau}. \]
\[ \square \]

Claim 7.13 (Proof of (7.1) for \( j + 1 \)). Assume all the formulas in the statement of Proposition 7.1 for \( j \), with \( j \leq \ell \). Then (7.1) is true for \( j + 1 \).
Proof. We have
\[
\|\xi_n[s_{-n}^{j+1}] - \gamma_n - \sum_{i=j+1}^{J-1} \lambda_n^j \|_{st(s_{-n}^{j+1}, \infty)} \leq \|\xi_n - \Gamma_n^{j,j,\tau}[s_{-n}^{j+1}] - \sum_{i=j}^{J-1} \lambda_n^j \|_{st(s_{-n}^{j+1}, \infty)} + o_\tau
\]
\[
\leq \|\xi_n - \Gamma_n^{j,j,\tau}[s_{-n}^{j+1}] - \sum_{i=j}^{J-1} \lambda_n^j \|_{st(s_{-n}^{j+1}, \infty)} + \|\xi_n - \Gamma_n^{j,j,\tau}[s_{-n}^{j,1}] - (\xi_n - \Gamma_n^{j,j,\tau})[s_{-n}^{j,1}]\|_{st(s_{-n}^{j+1}, \infty)}
+ o_\tau = o_\tau. \tag{7.25}
\]
where in the first inequality we have used Lemma 6.4 and (7.8) and in the 2nd inequality we have used (7.11) and (7.6).

Claim 7.14 (Back scattering: completed). Assume all the formulas in the statement of Proposition 7.1 for \(j\), with \(j < \ell\) and assume that (7.1) is true for \(j + 1\). Then (7.7) is true.

Proof. First,
\[
\|\xi_n[s_{-n}^{j+1}] - \lambda_n^{j+1} \|_{st(s_{-n}^{j+1}, s_{+n}^{j+1})} \leq \|\xi_n - \Gamma_n^{j,j,\tau}[s_{-n}^{j+1}] - (\xi_n - \Gamma_n^{j,j,\tau})[s_{-n}^{j,1}]\|_{st(s_{-n}^{j+1}, s_{-n}^{j+1})}
+ \|\Gamma_n^{j,j,\tau}[s_{-n}^{j,1}] - \Gamma_n^{j,j,\tau}[s_{-n}^{j,1}]\|_{st(s_{-n}^{j+1}, s_{-n}^{j+1})} + \|\xi_n - \sum_{i=j+1}^{J-1} \lambda_n^j \|_{st(s_{-n}^{j+1}, s_{-n}^{j+1})}
+ \sum_{i=j+1}^{J-1} \|\lambda_n^j \|_{st(s_{-n}^{j+1}, s_{+n}^{j+1})} = o_\tau. \tag{7.26}
\]
Here, for the 1st term we have used (3.8) and (7.6), for the 2nd term we have used Lemma 6.3 and (7.22). Notice that we have \(\Gamma_n^{j,j,\tau}[s_{-n}^{j+1}] - \Gamma_n^{j,j,\tau}[s_{-n}^{j,1}] = \Gamma_n^{j,j,\tau} - \gamma_n^j\). For the 3rd term we have used (7.11) and for the 4th term we used Lemma 6.4.

Since
\[
\xi_n(s_{n+1}^{j+1} - \tau) - \lambda_n^{j+1}(s_{n+1}^{j+1} - \tau) \rightarrow \xi_{n+1}^{j+1}(-\tau) - \lambda_{n+1}^{j+1}(-\tau)
\]
and
\[
(\xi_{n+1}^{j+1}(-\tau) - \lambda_{n+1}^{j+1}(-\tau))[s_{n+1}^{j+1} - \tau](t) = (\xi_{n+1}^{j+1} - \lambda_{n+1}^{j+1})[\tau](t - s_{n+1}^{j+1}),
\]
we have
\[
\|\xi_{n+1}^{j+1} - \lambda_{n+1}^{j+1}\|_{st([-\infty, 0], 0)} \leq \|\xi_{n+1}^{j+1} - \lambda_{n+1}^{j+1}\|_{st(\tau, -\infty)} \leq \|\xi_{n+1}^{j+1} - \lambda_{n+1}^{j+1}\|_{st(\tau, -\infty)} + \|\xi_n - \lambda_n^{j+1}\|_{st(s_{-n}^{j+1}, s_{+n}^{j+1})} = o_\tau \tag{7.27}
\]
where we bound the term in the 2nd line by Lemma 4.6 and the following term by (7.26), since \(\lambda_n^{j+1}[s_{-n}^{j+1}] = \lambda_n^{j+1}\). Now, recall that we have already proved in Claim 7.8 that there exists \(h_{-n}^{j+1} \in H^1\) s.t.
\[
\lim_{\tau \to -\infty} \|\xi_{-n}^{j+1}[-\tau] - h_{-n}^{j+1}[0]\|_{st([-\infty, 0])} \leq \lim_{\tau \to -\infty} \|\xi_{-n}^{j+1}[-\tau] - e^{-ir\Delta}h_{-n}^{j+1}\|_{H^1} = 0. \tag{7.28}
\]
By (7.27) and by \(\lambda^{j+1}[-\tau] = \lambda^{j+1} = \varphi^{j+1}[0]\), we have \(\lim_{\tau \to -\infty} \|\xi_{-n}^{j+1}[-\tau] - \varphi^{j+1}[0]\|_{st(-\infty, 0)} = 0\). Thus we conclude that \(h_{-n}^{j+1} = \varphi^{j+1}\). This completes the proof of (7.7) for \(j + 1\). \(\square\)
Claim 7.15 (Proof of (i)-2 for $j+1$). Assume all the formulas in the statement of Proposition 7.1 for $j$, with $j < \ell$ and assume that (7.1) and (7.7) are true for $j+1$. Then (7.2) is true for $j+1$.

Proof. Since by $\lambda_n^{j+1} := \varphi^{j+1}[s_n^{j+1}]$ and $\lambda^{j+1} := \varphi^{j+1}[0]$ we have $\lambda_n^{j+1} = \lambda^{j+1}(s_n^{j+1})$,

$$
\begin{align*}
\|\varLambda_n^{j+1}[s_n^{j+1}] - \lambda_n^{j+1}\|_{\mathcal{L}^s(0,\infty)} &= \|\varLambda_n^{j+1}[s_n^{j+1}] - \lambda^{j+1}(s_n^{j+1})\|_{\mathcal{L}^s(0,\infty)} \\
&\leq \|\xi^{j+1}(\cdot - s_n^{j+1}) - \lambda^{j+1}(\cdot - s_n^{j+1})\|_{\mathcal{L}^s(0,s_n^{j+1})} + \|\varLambda_n^{j+1}[s_n^{j+1}] - \lambda^{j+1}(\cdot - s_n^{j+1})\|_{\mathcal{L}^s(s_n^{j+1},\infty)} \\
&\lesssim \|\xi^{j+1} - \lambda^{j+1}\|_{\mathcal{L}^s(-\infty,\infty)} + \|\varLambda_n^{j+1} - \lambda^{j+1}\|_{\mathcal{L}^s(s_n^{j+1},\infty)}.
\end{align*}
$$

where in the last line we use (7.7) for $j+1$ and (7.28), where $e^{-ir}\Delta H_n^{j+1} = e^{-ir}\Delta \varphi^{j+1} =: \lambda^{j+1}(-\tau)$ as shown under (7.28).

The proof of Proposition 7.1 is completed.

Corollary 7.16. Assume $l = L$ in Proposition 7.1. Then there exists a fixed constant $C$ s.t.

$$
\|\xi_n\|_{\mathcal{L}^s(0,\infty)} \leq C.
$$

(7.29)

Proof. First by (7.15), we have

$$
\sum_{j=0}^{L-1} \|\xi_n\|_{\mathcal{L}^s(s_{n,j}^{j+1},s_{n,j}^{j+1})} \leq \|\gamma_n^j\|_{\mathcal{L}^s(\mathbb{R})} + o_\tau \leq 1,
$$

for $n$ and $\tau$ sufficiently large. Next, for $0 \leq j \leq L - 1$, we have

$$
\sum_{j=0}^{L-1} \|\xi_n\|_{\mathcal{L}^s(s_{n,j}^{j+1},s_{n,j}^{j+1})} \leq \sum_{j=0}^{L-1} \|\xi_n - \varLambda_n^j\|_{\mathcal{L}^s(s_{n,j}^{j+1},s_{n,j}^{j+1})} + \|\xi^0\|_{\mathcal{L}^s(0,\infty)} + \sum_{j=1}^{L-1} \|\xi^j\|_{\mathcal{L}^s(\mathbb{R})}.
$$

The last two terms are bounded so it suffices to bound $\|\xi_n - \varLambda_n^j\|_{\mathcal{L}^s(s_{n,j}^{j+1},s_{n,j}^{j+1})}$ for each $j$.

$$
\|\xi_n - \varLambda_n^j\|_{\mathcal{L}^s(s_{n,j}^{j+1},s_{n,j}^{j+1})} \leq \|\xi_n - \varLambda_n^j\|_{\mathcal{L}^s(s_{n,j}^{j+1},s_{n,j}^{j+1})} + \|\xi_n - \varLambda_n^j\|_{\mathcal{L}^s(s_{n,j}^{j+1},s_{n,j}^{j+1})} = o_\tau.
$$

Here, we have used (7.9) for the 1st term and (7.5) for the 2nd term.

As in [34] we can formulate the following result, which can be proved similarly.

Proposition 7.17. Let $(\xi_n, z_n) \in C^0(\mathbb{R}, H^1_{rad} \times \mathbb{C})$ be a sequence of solutions of (1.1)–(1.2) satisfying (6.3). Let

$$
\xi_n[0] = \sum_{j=0}^{J-1} \lambda_n^j + \gamma_n^j
$$

be the linearized profile decomposition of Proposition 6.1 where $J$ is fixed but large enough. Let $\{s_n^j\}$

$$
(\xi^j(t), z^j(t)) := \lim_{n \to \infty} (\xi_n, z_n)(t + s_n^j)
$$

be the weak limit in $H^1_{rad} \times \mathbb{C}$. Assume $(\xi^j, z^j)$ scatters as $t \to \sigma \infty$ for each $j < J$ and $\sigma \in \{+, -\}$ satisfying $\lim_{n \to \infty} \sigma s_n^j \geq 0$. Then $\sup_n \|\xi_n\|_{\mathcal{L}^s(\mathbb{R})} < \infty$. 

28
8 Scattering

For each \( \mu > 0 \) and \( A \in \mathbb{R} \) we denote by \( GS(\mu, A) \) the subset of \( C^0_\mu(\mathbb{R}, \mathbb{C} \times H^1) \) formed by the solutions with \( M \leq \mu \) and \( E \leq A \). Let

\[
ST(\mu, A) = \sup\{\|\xi\|_{st(\mathbb{R}_+)} < \infty : (z, \xi) \in GS(\mu, A)\} \quad \chi = \{(\mu, A) : ST(\mu, A) < \infty\}.
\]

We introduce the partial orders in \( \mathbb{R}^2 \)

\[
(\mu_1, A_1) \leq (\mu_2, A_2) \iff \mu_1 \leq \mu_2 \text{ and } A_1 \leq A_2 \\
(\mu_1, A_1) \ll (\mu_2, A_2) \iff \mu_1 < \mu_2 \text{ and } A_1 < A_2.
\]

By the definition of \( \chi \)

\[
(\mu_1, A_1) \leq (\mu_2, A_2) \text{ and } (\mu_2, A_2) \in \chi \Rightarrow (\mu_1, A_1) \in \chi.
\]

Our goal is to prove that there exists \( \mu_0 > 0 \) s.t. \( (0, \mu_0) \times \mathbb{R} \subseteq \chi \). By Theorem 2.1 we know that there exists \( \delta_0 > 0 \) s.t. \( (0, \delta_0) \times (\mathbb{R}, \delta_0) \subseteq \chi \). Suppose there exists \( (\mu_0, A_0) \in \mathbb{R}^2 \setminus \chi \) with \( \mu_0 \ll 1 \) and write

\[
E_* = \sup\{A < A_0 : (\mu_0, A) \in \chi\}, \quad M_* = \sup\{\mu < \mu_0 : (\mu, E_*) \in \chi\}.
\]

Then by Theorem 2.1

\[
0 < E_* \leq A_0, \quad 0 < M_* \leq \mu_0,
\]

and \( (M_*, E_*) \) is s.t.

\[
(\mu_1, A_1) \ll (M_*, E_*) \ll (\mu_2, A_2) \Rightarrow (\mu_1, A_1) \in \chi \text{ and } (\mu_2, A_2) \notin \chi. \tag{8.1}
\]

Hence there is a sequence \( (M_n, E_n) \xrightarrow{n \to \infty} (M_*, E_*) \) and a sequence of solutions \( (\xi_n, z_n) \in GS(M_n, E_n) \) s.t.

\[
M_n \leq \mu_0 + o(1) \text{ and } \|\xi_n\|_{st(\mathbb{R}_+)} = +\infty \text{ for all } n.
\]

We can apply to the sequence \( (\xi_n, z_n) \) the profile decompositions of Section 6. By weak convergence we have

\[
M(\xi^l, z^l) \leq M_* \text{ and } E(\xi^l, z^l) \leq E_* \tag{8.2}
\]

Since \( \|\xi_n\|_{st(\mathbb{R}_+)} = +\infty \) for all \( n \), by Corollary 7.16, the assumptions of Proposition 7.1 must fail and this means that there must exist \( l < L \) s.t. we have \( \|\xi^l\|_{st(\mathbb{R}_+)} = +\infty \). We choose \( l \) minimal, in the sense that if \( \|\xi^j\|_{st(\mathbb{R}_+)} = +\infty \) then \( j \geq l \). By (8.1) and (8.2) we have

\[
(M_*, E_*) = (M(\xi^l, z^l), E(\xi^l, z^l)).
\]

Then \( \xi_n(\cdot + s_n^l) \xrightarrow{n \to \infty} \xi^l \) strongly in \( H^1 \). If \( l > 0 \), (7.7) implies \( z^l(-s_n^l) \xrightarrow{n \to \infty} 0 \) in \( \mathbb{C} \) and \( \xi^l(-s_n^l) - e^{-is_n^l\Delta} \xi^l \xrightarrow{n \to \infty} 0 \) in \( H^1 \). Since \( \lambda^l_n(0) := e^{-is_n^l\Delta} \phi^l_n \) and \( \lambda^l_n(0) \xrightarrow{n \to \infty} 0 \) in \( L^4 \) we get

\[
E_* = E(\xi^l(-s_n^l), z^l(-s_n^l)) = 2^{-1}\|\nabla \lambda^l_n(0)\|^2_{L^2} + o(1),
\]

from which we read

\[
2^{-1}\|\nabla \lambda^l_n(0)\|^2_{L^2} \geq E_* + o(1). \tag{8.3}
\]

Let \( (\xi, z) \in GS(M_*, E_*) \) with \( \|\xi\|_{st(\mathbb{R}_+)} = +\infty \).
Claim 8.1. The image $(\xi(\mathbb{R}_+), z(\mathbb{R}_+))$ is relatively compact in $H^1_{rad} \times \mathbb{C}$.

Proof. We consider a sequence $0 < t_n \to +\infty$ and we apply the above argument based on Proposition 7.17 to $(\xi_n, z_n):= (\xi(\cdot + t_n), z(\cdot + t_n))$ on $(-t_n, 0]$ and on $[0, \infty)$. Notice that we have $\|\xi_n\|_{L^1(-t_n, 0)} \to \infty$ as $n \to \infty$ and $\|\xi_n\|_{L^1(0, \infty)} = \infty$. If in one of the two cases, we have $l = 0$ then $\xi_n(0) = (\xi(t_n)$ is strongly convergent in $H^1$. If in both cases $l = l_0 > 0$ on $(-t_n, 0]$ and $l = l_1 > 0$ on $[0, \infty)$ then $(\xi^0, z^0)$ scatters and thus $E(\xi^0, z^0) \geq 0$ because if the energy is negative, it cannot scatter. Then using (6.6) and (8.3) we have

$$E_* \geq E(\xi^0, z^0) + 2^{-1} \|\nabla \lambda^0_n\|_{L^2}^2 + 2^{-1} \|\nabla \lambda^1_n\|_{L^2}^2 + o(1) \geq 2E_* + o(1)$$

so that $E_* \lesssim o(1)$, and since here $o(1) \to +\infty$ this implies $E_* = 0$, in contradiction with Theorem 2.1 which implies $E_* > 0$. As a consequence, up to a subsequence, $\xi_n(0) = (\xi(t_n)$ is strongly convergent in $H^1$ for any $t_n \to +\infty$.

We now prove the following claim, which completes the proof of Theorem 1.2.

Claim 8.2. There are no $(\xi, z) \in \text{GS}(M_\ast, E_\ast)$ with $\|\xi\|_{L^1(\mathbb{R}_+)} = +\infty$.

Proof. We proceed by contradiction assuming the existence of such a solution. By Claim 8.1 we know that $(\xi(\mathbb{R}_+)) \subset H^1_{rad}$ is relatively compact. On the other hand we know that

$$\|\nabla \xi\|_{L^2}^2 + \frac{3}{4} \|\xi\|_{L^4}^4 \geq 2C \gtrsim 1 \gg \mu_0 \quad (8.4)$$

because otherwise by Theorem 2.1 we can show that $\|\xi\|_{L^1(\mathbb{R}_+)} < +\infty$.

We now consider the Virial Inequality. We consider a smooth function $f(x) = f(|x|)$ with

$$f(r) = \begin{cases} r & \text{for } r \leq 1 \\ \frac{3}{2} & \text{for } r \geq 2 \end{cases}.$$

Then for $f_R(x) := f(x/R)$ and $f_{JR}(x) = f_J(x/R)$ with

$$f_0 = 1 - \partial_r f, \quad f_1 = \Delta (\partial_r + 1/r)f, \quad f_2 = -3/2 + (\partial_r + 1/r)f$$

we have, see [34],

$$\partial_t \langle R f_R \xi, i \partial_r \xi \rangle = \|\nabla \xi\|_{L^2}^2 + \frac{3}{4} \|\xi\|_{L^4}^4 - \int_{\mathbb{R}^3} \left( 2 \partial_r \xi |^2 f_0 R + R^{-2} |\xi|^2 f_1 R - |\xi|^4 f_{2R} \right) dx$$

$$+ (\|z\|^2 G, R f_R \partial_r \xi).$$

Taking $R \gg 1$ by (8.4) we obtain the following, which contradicts $(R f_R \xi, i \partial_r \xi) \in L^\infty(\mathbb{R}_+)$:

$$\partial_t \langle R f_R \xi, i \partial_r \xi \rangle \geq C > 0.$$

Since its denial has led to a contradiction, it follows that Claim 8.2 is true. \qed

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