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Particle with spin 1 in a magnetic field
on the spherical plane $S_2$

Abstract

There are constructed exact solutions of the quantum-mechanical equation for a spin $S = 1$ particle in 2-dimensional Riemannian space of constant positive curvature, spherical plane, in presence of an external magnetic field, analogue of the homogeneous magnetic field in the Minkowski space. A generalized formula for energy levels describing quantization of the motion of the vector particle in magnetic field on the 2-dimensional space $S_2$ has been found, nonrelativistic and relativistic equations have been solved.

1 Introduction

The quantization of a quantum-mechanical particle in the homogeneous magnetic field belongs to classical problems in physics [1, 2, 3, 4]. In 1985 – 2010, a more general problem in a curved Riemannian background, hyperbolic and spherical planes, was extensively studied [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24], providing us with a new system having intriguing dynamics and symmetry, both on classical and quantum levels.

Extension to 3-dimensional hyperbolic and spherical spaces was performed recently. In [25, 26, 27], exact solutions for a scalar particle in extended problem, particle in external magnetic field on the background of Lobachevsky $H_3$ and Riemann $S_3$ spatial geometries were found. A corresponding system in the frames of classical mechanics was examined in [28, 29, 30]. In the present paper, we consider a quantum-mechanical problem a particle with spin $1/2$ described by the Dirac equation in 3-dimensional Lobachevsky and Riemann space models in presence of the external magnetic field.

In the present paper, we will construct exact solutions for a vector particle described by 10-dimensional Duffin–Kemmer equation in external magnetic field on the background of 2-dimensional spherical space $S_2$.

10-dimensional Duffin–Kemmer equation for a vector particle in a curved space-time has the form [31]

$$\{\beta^\gamma [ \imath h ( e^\alpha_{(c)} \partial_\beta + \frac{1}{2} J^{ab} \gamma_{abc}) + \frac{e}{c} A_c ] - mc \} \Psi = 0 , \quad (1.1)$$

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where $\gamma_{abc}$ stands for Ricci rotation coefficients, $A_a = e^\beta_{(a)} A_\beta$ represent tetrad components of electromagnetic 4-vector $A_\beta$; $J^{ab} = \beta^a\beta^b - \beta^b\beta^a$ are generators of 10-dimensional representation of the Lorentz group. For shortness, we use notation $e/c \Longrightarrow e$, $mc/h \Longrightarrow M$.

In the space $S_3$ we will use the system of cylindric coordinates [32]

$$dS^2 = c^2 dt^2 - \rho^2 [\cos^2 z (dr^2 + \sin^2 r d\phi^2) + dz^2],$$

$z \in [-\pi/2, +\pi/2], \ r \in [0, +\pi], \ \phi \in [0, 2\pi].$ (1.2)

Generalized expression for electromagnetic potential for an homogeneous magnetic field in the curved model $S_3$ is given as follows

$$A_\phi = -2B \sin^2 \frac{r}{2} = B (\cos r - 1).$$ (1.3)

We will consider the above equation in presence of the field in the model $S_3$. Corresponding to cylindric coordinates $x^a = (t, r, \phi, z)$ a tetrad can be chosen as

$$e^\beta_{(a)}(x) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^{-1} z & 0 & 0 \\ 0 & 0 & \cos^{-1} z \sin^{-1} r & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$ (1.4)

Eq. (1.1) takes the form

$$\left\{ i\beta^0 \frac{\partial}{\partial t} + \frac{1}{\cos z} \left( i\beta^1 \frac{\partial}{\partial r} + \beta^2 i \partial_\phi + eB (\cos r - 1) + iJ^{12} \cos r \right) \right. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. + i\beta^3 \frac{\partial}{\partial z} + i \frac{\sin z}{\cos z} (\beta^1 J^{13} + \beta^2 J^{23} - M) \right\} \Psi = 0,$$ (1.5)

To separate the variables in eq. (1.5), we are to employ an explicit form of the Duffin–Kemmer matrices $\beta^a$; it will be most convenient to use so called cyclic representation [34], where the generator $J^{12}$ is of diagonal form (we specify matrices by blocks in accordance with $1 - 3 - 3 - 3$-splitting)

$$\beta^0 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad \beta^i = \begin{vmatrix} 0 & 0 & e_i & 0 \\ 0 & 0 & 0 & \tau_i \\ -e_i^t & 0 & 0 & 0 \\ 0 & \tau_i & 0 & 0 \end{vmatrix},$$ (1.6)

where $e_i, e_i^t, \tau_i$ denote

$$e_1 = \frac{1}{\sqrt{2}}(-i, 0, i), \quad e_2 = \frac{1}{\sqrt{2}}(1, 0, 1), \quad e_3 = (0, i, 0),$$

$$\tau_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}, \quad \tau_2 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{vmatrix}, \quad \tau_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = s_3.$$
The generator $J^{12}$ explicitly reads

$$J^{12} = \beta^1 \beta^2 - \beta^2 \beta^1 = -i \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau_3 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 \\ 0 & 0 & 0 & \tau_3 \end{vmatrix} = -iS_3.$$  \hspace{0.5cm} (1.8)

2 Restriction to 2-dimensional model

Let us restrict ourselves to 2-dimensional case, spherical space $S_2$ (formally it is sufficient in eq. (1.5) to remove dependence on the variable $z$ fixing its value by $z = 0$)

$$\left[ i\beta^0 \frac{\partial}{\partial t} + i\beta^1 \frac{\partial}{\partial r} + \beta^2 \frac{i\partial\phi + eB(\cos r - 1) + iJ^{12}\cos r}{\sin r} - M \right] \Psi = 0. \hspace{0.5cm} (2.1)$$

With the use of substitution

$$\Psi = e^{-iet} e^{im\phi} \begin{vmatrix} \Phi_0(r) \\ \bar{\Phi}(r) \\ E(r) \\ \bar{H}(r) \end{vmatrix}, \hspace{0.5cm} (2.2)$$

eq (2.1) assumes the form (introducing notation $m + B(1 - \cos r) = \nu(r)$)

$$\left[ \epsilon \beta^0 + \beta^1 \frac{\partial}{\partial r} - \beta^2 \frac{\nu(r) - \cos r S_3}{\sin r} - M \right] \begin{vmatrix} \Phi_0(r) \\ \bar{\Phi}(r) \\ E(r) \\ \bar{H}(r) \end{vmatrix} = 0. \hspace{0.5cm} (2.3)$$

After separation of the variables and using the notation

$$\frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial r} + \frac{\nu - \cos r}{\sin r} \right) = \hat{a}_-, \hspace{0.5cm} \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial r} + \frac{\nu + \cos r}{\sin r} \right) = \hat{a}_+, \hspace{0.5cm} \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) = \hat{a},$$

$$\frac{1}{\sqrt{2}} \left( - \frac{\partial}{\partial r} + \frac{\nu - \cos r}{\sin r} \right) = \hat{b}_-, \hspace{0.5cm} \frac{1}{\sqrt{2}} \left( - \frac{\partial}{\partial r} + \frac{\nu + \cos r}{\sin r} \right) = \hat{b}_+, \hspace{0.5cm} \frac{1}{\sqrt{2}} \left( - \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) = \hat{b},$$

we arrive at the radial system

$$-\hat{b}_- E_1 - \hat{\alpha}_+ E_3 = M \Phi_0, \hspace{0.5cm} -\hat{b}_- H_1 + i\hat{\alpha}_+ H_3 + \epsilon E_2 = M \Phi_2, \hspace{0.5cm} (2.4)$$

$$i\hat{\alpha} H_2 + \epsilon E_1 = M \Phi_1, \hspace{0.5cm} -\hat{b} H_2 + \epsilon E_3 = M \Phi_3,$$

$$\hat{a} \Phi_0 - \epsilon \Phi_1 = M E_1, \hspace{0.5cm} -\hat{a} \Phi_2 = M H_1, \hspace{0.5cm} \hat{b} \Phi_0 - \epsilon \Phi_3 = M E_3, \hspace{0.5cm} (2.5)$$

$$i\hat{b} \Phi_2 = M H_3, \hspace{0.5cm} -\epsilon \Phi_2 = M E_2, \hspace{0.5cm} i\hat{b} \Phi_1 - \hat{\alpha}_+ \Phi_3 = M H_2.$$
3 Nonrelativistic approximation

Excluding non-dynamical variables \( \Phi_0, H_1, H_2, H_3 \) with the help of equations

\[
\hat{b}_- \Phi_1 - i \hat{a}_+ \Phi_3 = M H_2, \quad i \hat{b}_2 = M H_3. \tag{3.1}
\]

we get 6 equations (grouping them in pairs)

\[
i \hat{a}_i ( i \hat{b}_- \Phi_1 - i \hat{a}_+ \Phi_3) + i \epsilon M E_1 = M^2 \Phi_1, \\
i \hat{a}_i (- \hat{b}_- E_1 - \hat{a}_+ E_3 - i \epsilon M \Phi_1) = M^2 \epsilon_1, \\
- i \hat{b}_- (-i \hat{a}_2 + i \hat{a}_+ (i \hat{b}_2) + i \epsilon M E_2 = M^2 \Phi_2, \\
- i \epsilon M \Phi_2 = M^2 E_2, \\
- i \hat{b}_2 (i \hat{b}_- \Phi_1 - i \hat{a}_+ \Phi_3) + i \epsilon M E_3 = M^2 \Phi_3, \\
\hat{b}_2 (- \hat{b}_- E_1 - \hat{a}_+ E_3) - i \epsilon M \Phi_3 = M^2 E_3. \tag{3.2a}
\]

Now we introduce big and small constituents

\[
\Phi_1 = \Psi_1 + \psi_1, \quad \Phi_2 = \Psi_2 + \psi_2, \quad \Phi_3 = \Psi_3 + \psi_3, \\
i E_1 = \Psi_1 - \psi_1, \quad i E_2 = \Psi_2 - \psi_2, \quad i E_3 = \Psi_3 - \psi_3,
\]

besides we should separate the rest energy by formal change \( \epsilon \to \epsilon + M \); summing and subtracting equation within each pair (3.2) and ignoring small constituents \( \psi_i \) we arrive at three equations for big components

\[
\left( -2 \hat{a}_- + 2 \epsilon M \right) \psi_1 = 0, \\
\left( - \hat{b}_- \hat{a} - \hat{a}_+ \hat{b} + 2 \epsilon M \right) \psi_2 = 0, \\
\left( -2 \hat{b} \hat{a}_+ + 2 \epsilon M \right) \psi_3 = 0. \tag{3.3}
\]

It is a needed Pauli-like system for the spin 1 particle.

Allowing for \( \nu(r) = m + B (1 - \cos r) \), from (3.3) we get radial equations in the form

\[
\left[ \frac{d^2}{dr^2} + \frac{\cos r}{\sin r} \frac{d}{dr} - B - \frac{1 - 2 [m + B (1 - \cos r)] \cos r}{\sin^2 r} - \frac{[m + B (1 - \cos r)]^2}{\sin^2 r} + 2 \epsilon M \right] \psi_1 = 0, \tag{3.4a}
\]

4
\[
\left[ \frac{d^2}{dr^2} + \frac{\cos r}{\sin r} \frac{d}{dr} - \frac{[m + B(1 - \cos r)]^2}{\sin^2 r} + 2\epsilon M \right] \psi_2 = 0, \tag{3.4b}
\]

\[
\left[ \frac{d^2}{dr^2} + \frac{\cos r}{\sin r} \frac{d}{dr} + B - \frac{1 + 2[m + B(1 - \cos r)]}{\sin^2 r} \cos r - \frac{[m + B(1 - \cos r)]^2}{\sin^2 r} + 2\epsilon M \right] \psi_3 = 0. \tag{3.4c}
\]

The first and the third equations are symmetric with respect to formal change \(m \mapsto -m, B \mapsto -B\).

Let us consider eq. (3.4a). In the new variable \(1 - \cos r = 2y\); and with the use of a substitution \(B_1 = y^{C_1}(1 - y)^{A_1} f_1\) the differential equation assumes the form

\[
y(1-y) \frac{d^2 B_1}{dy^2} + [2C_1 + 1 - (2A_1 + 2C_1 + 2)y] \frac{dB_1}{dy} + \left[ B^2 + B + 2\epsilon M - (A_1 + C_1)(A_1 + C_1 + 1) \right. \]
\[
\left. + \frac{1}{4} \frac{4A_1^2 - (2B + m + 1)^2}{1-y} + \frac{1}{4} \frac{4C_1^2 - (m-1)^2}{y} \right] B_1 = 0. \tag{3.5}
\]

At \(A_1, C_1\) taken according to

\[
A_1 = +\frac{1}{2} \left| 2B + m + 1 \right|, \quad C_1 = +\frac{1}{2} \left| m - 1 \right|, \tag{3.6}
\]

eq. (3.5) is recognized as a hypergeometric equation \[33\] with parameters

\[
\alpha_1 = A_1 + C_1 + \frac{1}{2} - \sqrt{B^2 + B + 2\epsilon M + \frac{1}{4}},
\]

\[
\beta_1 = A_1 + C_1 + \frac{1}{2} + \sqrt{B^2 + B + 2\epsilon M + \frac{1}{4}},
\]

\[
\gamma_1 = 2C_1 + 1, \quad B_1 = y^{C_1}(1 - y)^{A_1} F(\alpha_1, \beta_1, \gamma_1; y). \tag{3.7}
\]

To get polynomials we need to impose restriction \(\alpha_1 = -n\), from this it follows

\[
\psi_1 = y^{C_1}(1 - y)^{A_1} F(\alpha_1, \beta_1, \gamma_1; y),
\]

\[
\sqrt{B^2 + B + 2\epsilon M + \frac{1}{4}} = n + \frac{1}{2} + \frac{\left| 2B + m + 1 \right| + \left| m - 1 \right|}{2}. \tag{3.8}
\]

In similar manner, we construct solution of eq. (3.4b)

\[
\psi_2 = y^{C_2}(1 - y)^{A_2} F(\alpha_2, \beta_2, \gamma_2; y),
\]

\[
A_2 = \pm \frac{1}{2} \left( 2B + m \right), \quad C_2 = \pm \frac{m}{2}, \quad \gamma_2 = 2C_2 + 1,
\]

\[5\]
\[ \alpha_2 = A_2 + C_2 + \frac{1}{2} - \sqrt{B^2 + 2\epsilon M + \frac{1}{4}}, \]

\[ \beta_2 = A_2 + C_2 + \frac{1}{2} + \sqrt{B^2 + 2\epsilon M + \frac{1}{4}}; \]  

(3.9)

from quantization condition \( \alpha_2 = -n \) it follows

\[ \sqrt{B^2 + 2\epsilon M + \frac{1}{4}} = n + \frac{1}{2} + \frac{|2B + m| + |m|}{2}. \]  

(3.10)

Finally, we construct solutions for eq. (3.4c):

\[ \psi_3 = y^{C_3}(1 - y)^{A_3} F(\alpha_3, \beta_3, \gamma_3; y), \]

\[ A_3 = \pm \frac{1}{2}(2B + m - 1), \quad C_3 = \pm \frac{1}{2}(m + 1), \quad \gamma_3 = 2C_3 + 1, \]

\[ \alpha_3 = A_3 + C_3 + \frac{1}{2} - \sqrt{B^2 - B + 2\epsilon M + \frac{1}{4}}, \]

\[ \beta_3 = A_3 + C_3 + \frac{1}{2} + \sqrt{B^2 - B + 2\epsilon M + \frac{1}{4}}; \]  

(3.11)

and requiring \( \alpha_3 = -n \) we obtain

\[ \sqrt{B^2 - B + 2\epsilon M + \frac{1}{4}} = n + \frac{1}{2} + \frac{|2B + m - 1| + |m + 1|}{2}. \]  

(3.12)

4 Solution of radial equations in relativistic case

Let start with eqs. (2.4)–(2.5). Excluding six components \( E_i, H_i \) with the help of (2.5), we derive four second order equations for \( \Phi_a \):

\[ (-\hat{b}_- \hat{a} - \hat{a}_+ \hat{b} + \epsilon^2 - M^2)\Phi_2 = 0, \]

\[ (-\hat{b}_- \hat{a} - \hat{a}_+ \hat{b} - M^2)\Phi_0 + i\epsilon(\hat{b}_- \Phi_1 + \hat{a}_+ \Phi_3) = 0, \]

\[ (-\hat{a} \hat{b}_- + \epsilon^2 - M^2)\Phi_1 + \hat{a} \hat{a}_+ \Phi_3 + i\epsilon \hat{a} \Phi_0 = 0, \]

\[ (-\hat{b} \hat{a}_+ + \epsilon^2 - M^2)\Phi_3 + \hat{b} \hat{a}_- \Phi_0 + i\epsilon \hat{b} \Phi_0 = 0. \]  

(4.1)

Once, it should be noted existence of a simple solution of the system

\[ \Phi_0 = 0, \quad \Phi_1 = 0, \quad \Phi_3 = 0, \quad (-\hat{b}_- \hat{a} - \hat{a}_+ \hat{b} + \epsilon^2 - M^2)\Phi_2 = 0, \]  

(4.2a)

at this from (2.5) it follows

\[ E_1 = 0, \quad E_2 = -i\epsilon M^{-1} \Phi_2, \quad E_3 = 0, \]

\[ H_1 = -iM^{-1} \hat{a} \Phi_2, \quad H_2 = 0, \quad H_3 = iM^{-1} \hat{b} \Phi_2. \]  

(4.2b)
Instead of (4.1) we obtain

\[ (\hat{b}_- \Phi_1 = Z_1, \quad \hat{a}_+ \Phi_3 = Z_3, \]

instead of (4.1) we obtain

\[ (-\hat{b}_- \hat{a}_+ \hat{b} + \epsilon^2 - M^2) \Phi_2 = 0, \]
\[ (-\hat{b}_- \hat{a} - \hat{a}_+ \hat{b} - M^2) \Phi_0 + i\epsilon (Z_1 + Z_3) = 0, \]
\[ (-\hat{b}_- \hat{a} + \epsilon^2 - M^2) Z_1 + \hat{b}_- \hat{a} Z_3 + i\epsilon \hat{b}_- \hat{a} \Phi_0 = 0, \]
\[ (-\hat{a}_+ \hat{b} + \epsilon^2 - M^2) Z_3 + \hat{a}_+ \hat{b} Z_1 + i\epsilon \hat{a}_+ \hat{b} \Phi_0 = 0. \] (4.3)

Instead of \( Z_1, Z_3 \), let us use new variables

\[ Z_1 = \frac{f + g}{2}, \quad Z_3 = \frac{f - g}{2}, \]

then the system assumes the form

\[ (-\hat{b}_- \hat{a} - \hat{a}_+ \hat{b} + \epsilon^2 - M^2) \Phi_2 = 0, \]
\[ (-\hat{b}_- \hat{a} - \hat{a}_+ \hat{b} - M^2) \Phi_0 + i\epsilon f = 0, \]
\[ -\hat{b}_- \hat{a} \frac{f + g}{2} + (\epsilon^2 - M^2) \frac{f + g}{2} + \hat{b}_- \hat{a} \frac{f - g}{2} + i\epsilon \hat{b}_- \hat{a} \Phi_0 = 0, \]
\[ -\hat{a}_+ \hat{b} \frac{f - g}{2} + (\epsilon^2 - M^2) \frac{f - g}{2} + \hat{a}_+ \hat{b} \frac{f + g}{2} + i\epsilon \hat{a}_+ \hat{b} \Phi_0 = 0. \] (4.4)

Summing and subtracting equations 3 and 4, we get

\[ (-\hat{b}_- \hat{a} - \hat{a}_+ \hat{b} + \epsilon^2 - M^2) \Phi_2 = 0, \]
\[ (-\hat{b}_- \hat{a} - \hat{a}_+ \hat{b} - M^2) \Phi_0 + i\epsilon f = 0, \]
\[ (-\hat{b}_- \hat{a} + \hat{a}_+ \hat{b}) g + (\epsilon^2 - M^2) f + i\epsilon (\hat{b}_- \hat{a} + \hat{a}_+ \hat{b}) \Phi_0 = 0, \]
\[ (-\hat{b}_- \hat{a} - \hat{a}_+ \hat{b}) g + (\epsilon^2 - M^2) g + i\epsilon (\hat{b}_- \hat{a} - \hat{a}_+ \hat{b}) \Phi_0 = 0. \] (4.5)

Taking into account identities

\[ -\hat{b}_- \hat{a} - \hat{a}_+ \hat{b} = \Delta_2, \quad -\hat{b}_- \hat{a} + \hat{a}_+ \hat{b} = 2B \] (4.6)

eqs. (4.5) reduce to the form

\[ (\Delta_2 + \epsilon^2 - M^2) \Phi_2 = 0, \] (4.7)
\[ (\Delta_2 - M^2) \Phi_0 + i\epsilon f = 0, \]
\[ 2B g + (\epsilon^2 - M^2) f - i\epsilon \Delta_2 \Phi_0 = 0, \]
\[ \Delta_2 g + (\epsilon^2 - M^2) g - 2i\epsilon B \Phi_0 = 0, \] (4.8)
From the second equation, with the use of expression for $\Delta_2 \Phi_0$ according to the first equation, we derive linear relation between three functions
\[ 2B \ g - M^2 f - i\epsilon M^2 \Phi_0 = 0 . \] (4.9)

With the help of (4.9), let us exclude $f$
\[ f = \frac{2B}{M^2} \ g - i\epsilon \Phi_0 \]

from equations 2 and 3:
\[
\begin{align*}
(\Delta_2 + \epsilon^2 - M^2) \ g &= 2i\epsilon B \Phi_0 , \\
(\Delta_2 + \epsilon^2 - M^2) \Phi_0 &= -\frac{2i\epsilon B}{M^2} \ g .
\end{align*}
\] (4.10)

With notation $\gamma = \epsilon^2/M^2$, the system can be written in a matrix form
\[
(\Delta_2 + \epsilon^2 - M^2) \begin{pmatrix} g \\ \epsilon \Phi_0 \end{pmatrix} = \begin{pmatrix} 0 & 2iB \\ -2iB\gamma & 0 \end{pmatrix} \begin{pmatrix} g \\ \epsilon \Phi_0 \end{pmatrix}
\] (4.11)

or symbolically as
\[
\Delta f = Af \quad \Delta f' = SAS^{-1} f' , \quad f' = Sf .
\]

It remains to find a transformation reducing the matrix $A$ to a diagonal form
\[
SAS^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} , \quad S = \begin{pmatrix} a & d \\ c & b \end{pmatrix} ;
\]

the problem is equivalent to the linear system
\[
\begin{align*}
-\lambda_1 a - 2i\gamma B \ d &= 0 , & 2iB a - \lambda_1 d &= 0 ; \\
-\lambda_2 c - 2i\gamma B \ b &= 0 , & 2iB c - \lambda_2 b &= 0 .
\end{align*}
\]

Its solutions can be chosen in the form
\[
\lambda_1 = +\frac{2\epsilon B}{M} , \quad \lambda_2 = -\frac{2\epsilon B}{M} , \quad S = \begin{pmatrix} \epsilon & +iM \\ \epsilon & -iM \end{pmatrix} ,
\]
\[
S^{-1} = \frac{1}{-2i\epsilon M} \begin{pmatrix} -iM & -iM \\ -\epsilon & -\epsilon \end{pmatrix} .
\] (4.12)

New (primed) function satisfy the following equations
\[
\begin{align*}
1) \quad \left( \Delta_2 + \epsilon^2 - M^2 - \frac{2\epsilon B}{M} \right) g' &= 0 , \tag{4.13a} \\
2) \quad \left( \Delta_2 + \epsilon^2 - M^2 + \frac{2\epsilon B}{M} \right) \Phi_0' &= 0 . \tag{4.13b}
\end{align*}
\]

They independent from each other. There exist two linearly independent ones
\[
\begin{align*}
1) \quad g' \neq 0 , & \quad \Phi_0' = 0 , & \quad 2) \quad g' = 0 , & \quad \Phi_0' \neq 0 .
\end{align*}
\] (4.14)
The initial functions for these two cases assume respectively the form

\begin{align*}
1) \quad g &= \frac{1}{2\epsilon} g', \quad \epsilon \Phi_0 = \frac{1}{2iM} g'; \\
2) \quad g &= \frac{1}{2i\epsilon} \epsilon \Phi'_0, \quad \epsilon \Phi_0 = -\frac{1}{2iM} \epsilon \Phi'_0.
\end{align*}

In each case, eqs. (4.10) have the same form coinciding with (4.13a) and (4.13b) respectively. To obtain explicit solutions for these differential equation, we need not any additional calculations, instead it suffices to perform simple formal changes as pointed below

\begin{align}
2\epsilon M \quad \Longrightarrow \quad \begin{cases}
\frac{\epsilon^2 - M^2 - 2\epsilon B}{M} \\
\frac{\epsilon^2 - M^2}{M} \\
\frac{\epsilon^2 - M^2 + 2\epsilon B}{M}
\end{cases}
\end{align}

(4.15)

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References

[1] Rabi I.I. Das freie Electron in Homogenen Magnetfeld nach der Diraschen Theorie. // Z. Phys. 49, 507 – 511 (1928).
[2] Landau L., Diamagnetismus der Metalle, Ztshr. Phys. 64, 629–637 (1930).
[3] Plesset M.S. Relativistic wave mechanics of the electron deflected by magnetic field. // Phys.Rev. 12, 1728 – 1731 (1931).
[4] L.D. Landau, E.M. Lifshitz. Quantum mechanics. Addison Wesley, Reading, Mass., 1958.
[5] A. Comtet, P.J. Houston. Effective action on the hyperbolic plane in a constant external field. J. Math. Phys. 1985. Vol. 26, No. 1. P. 185 – 191
[6] Alain Comtet. On the Landau levels on the hyperbolic plane. Annals of Physics. 1987. Vol. Volume 173. P. 185 – 209.
[7] H. Aoki. Quantized Hall Effect. Rep. Progr. Phys. 1987. Vol. 50. P. 655 – 730.
[8] C. Groshe. Path integral on the Poincaré upper half plane with a magnetic field and for the Morse potential. Ann. Phys. (N.Y.), 1988. Vol. 187. P. 110 – 134.
[9] J.R. Klauder, E. Onofri. Landau Levels and Geometric Quantization. Int. J. Mod. Phys. 1989. Vol. A4. P. 3939 – 3949.
[10] J.E. Avron, A. Pnueli. Landau Hamiltonians on Symmetric Spaces. Pages 96 – 117 in: Ideas and methods in mathematical analysis, stochastics, and applications. Vol. II. S. Alverio et al., eds. (Cambridge Univ. Press, Cambridge, 1990).
[11] M.S. Plyushchay. The Model of relativistic particle with torsion. Nucl. Phys. 1991. Vol. B362. P. 54 – 72.

[12] M.S. Plyushchay. Relativistic particle with torsion, Majorana equation and fractional spin. Phys. Lett. 1991. Vol. B262. P. 71 – 78.

[13] G.V. Dunne. Hilbert Space for Charged Particles in Perpendicular Magnetic Fields. Ann. Phys. (N.Y.) 1992. Vol. 215. P. 233 – 263.

[14] M.S. Plyushchay. Relativistic particle with torsion and charged particle in a constant electromagnetic field: Identity of evolution. Mod. Phys. Lett. 1995. Vol. A10. P. 1463 – 1469; hep-th/9309147

[15] M. Alimohammadi, A. Shafei Deh Abad. Quantum group symmetry of the quantum Hall effect on the non-flat surfaces. J. Phys. 1996. Vol. A29. P. 559.

[16] M. Alimohammadi, H. Mohseni Sadjadi Laughlin states on the Poincare half-plane and their quantum group symmetry, Jour. Phys. 1996. Vol. A29. P. 5551

[17] E. Onofri. Landau Levels on a torus. Int. J. Theoret. Phys., 2001, Vol. 40, no 2, P. 537 – 549; arXiv:quant-ph/0007055v1 18 Jul 2000

[18] J. Negro, M.A. del Olmo, A. Rodriguez-Marco. Landau quantum systems: an approach based on symmetry. arXiv:quantum-ph/0110152.

[19] J. Gamboa, M. Loewe, F. Mendez, J. C. Rojas The Landau problem and noncommutative quantum mechanics. Mod. Phys. Lett. A. 2001. Vol. 16. P. 2075 – 2078.

[20] S.M. Klishevich, M.S. Plyushchay. Nonlinear holomorphic supersymmetry on Riemann surfaces. Nucl. Phys. 2002. Vol. B 640. P. 481 – 503; hep-th/0202077

[21] N. Drukker, B. Fiol, J. Simón. Gödel-type Universes and the Landau problem. hep-th/0309199. Journal of Cosmology and Astroparticle Physics (JCAP) 0410 (2004) Paper 012

[22] A. Ghanmi, A. Intissar. Magnetic Laplacians of differentia forms of the hyperbolic disk and Landau levels. African Journal Of Mathematical Physics. 2004. Vol. 1. P. 21 – 28.

[23] F. Correa, V. Jakubsky, M.S. Plyushchay. Aharonov-Bohm effect on AdS(2) and nonlinear supersymmetry of reflectionless Poschl-Teller system. Annals Phys. 2009. Vol. 324. P. 1078 – 1094,2009; arXiv:0809.2854.

[24] P.D. Alvarez, J.L. Cortes, P.A. Horvathy, M.S. Plyushchay. Super-extended noncommutative Landau problem and conformal symmetry. JHEP. 2009. 0903:034; arXiv:0901.1021

[25] A.A. Bogush, V.M. Red’kov, G.G. Krylov. Schrödinger particle in magnetic and electric fields in Lobachevsky and Riemann spaces. // Nonlinear Phenomena in Complex Systems. 11, no 4, 403 – 416 (2008).
[26] A.A. Bogush, G.G. Krylov, E.M. Ovsiyuk, V.M. Red’kov. Maxwell electrodynamics in complex form, solutions with cylindric symmetry in the Riemann space. Doklady Natsionalnoi Akademii Nauk Belarusi. 33, 52 – 58 (2009).

[27] A.A. Bogush, V.M. Red’kov, G.G. Krylov. Quantum-mechanical particle in a uniform magnetic field in spherical space $S^3$. Proceedings of the National Academy of Sciences of Belarus. Ser. fiz.-mat. 2, 57 – 63 (2009).

[28] V.V. Kudryashov, Yu.A. Kurochkin, E.M. Ovsiyuk, V.M. Red’kov. Motion caused by magnetic field in Lobachevsky space. AIP Conference Proceedings. Vol. 1205, P. 120 – 126 (2010); Eds. Remo Ruffini and Gregory Vereshchagin. The sun, the stars, the Universe and General relativity. International Conference in Honor of Ya.B. Zeldovich. April 20-23, 2009, Minsk.

[29] V.V. Kudryashov, Yu.A. Kurochkin, E.M. Ovsiyuk, V.M. Red’kov. Motion of a particle in magnetic field in the Lobachevsky space. Doklady Natsionalnoi Akademii Nauk Belarusi. 53, 50–53 (2009).

[30] V.V. Kudryashov, Yu.A. Kurochkin, E.M. Ovsiyuk, V.M. Red’kov. Classical Particle in Presence of Magnetic Field, Hyperbolic Lobachevsky and Spherical Riemann Models. SIGMA 6, 004, 34 pages (2010).

[31] V.M. Red’kov. Fields in Riemannian space and the Lorentz group. Publishing House “Belarusian Science”, Minsk, 2009 (in Russian).

[32] M.N. Olevsky. Three-orthogonal coordinate systems in spaces of constant curvature, in which equation $\Delta_2 U + \lambda U = 0$ permits the full separation of variables. Mathematical collection. 1950. Vol. 27. P. 379 – 426.

[33] H. Bateman, A. Erdélyi. Higher transcendental functions. Vol. I. (New York, McGraw-Hill) 1953.

[34] V.M. Red’kov, E.M. Ovsiyuk. Quantum mechanics in spaces of constant curvature. Nova Science Publishers. Inc. 2011. (in press)