Global Schauder estimates for the p-Laplace system

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$p$-Laplacian system

Find a vector field $\mathbf{u} : \Omega \rightarrow \mathbb{R}^N$ satisfying the following system of partial differential equations

$$\begin{cases} \text{div} (|\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u}) = f = \text{div} \mathbf{F} & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial \Omega. \end{cases}$$

- Assume $\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain and $p \in (1, \infty)$;
- Weak solution $\mathbf{u} \in W_{0}^{1,p}(\Omega)$ exists provided $\mathbf{F} \in L^{p'}(\Omega)$;
- How does the regularity of $\mathbf{F}$ transfers to $\mathbf{u}$?
Classical results

Regularity theory for $p$-harmonic functions

$$\text{div}\ (|\nabla u|^{p-2} \nabla u) = 0 \quad \text{in} \quad \Omega.$$ 

- Ural’tseva (’68), $N = 1$: $u \in C^{1,\alpha}(\Omega)$ for some $\alpha < 1$;
- Uhlenbeck (’77), $N \geq 2$ and $p > 2$: $u \in C^{1,\alpha}(\Omega)$;
- Acerbi-Fusco/DiBenedetto/Manfredi/Tolksdorff (’83-’87), $N \geq 2$ and $p < 2$: $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$;
- Manfredi-Iwaniec (’89), $n = 2$: optimal values for $\alpha$. 
Local maximal regularity theory

For a ball $B$ such that $2B \subseteq \Omega$ and a function space $X$

$$\| |\nabla u|^p - 2 \nabla u\|_{X(B)} \leq c \|F\|_{X(2B)}$$

- $X = L^q$ with $q \geq p'$: Iwaniec ('83), DiBenedetto-Manfredi ('93), Kinnuen-Zhou ('99);
- $X = \text{BMO}$: DiBenedetto-Manfredi ('93), Diening-Kaplický-Schwarzacher ('12);
- $X = C^\alpha$: Diening-Kaplický-Schwarzacher ('12).
Global maximal regularity theory

For a bounded domain $\Omega$ of class $C^\infty$ and a function space $X$

\[ \| \nabla u \|_{X(\Omega)}^{p-2} \| \nabla u \|_{X(\Omega)} \leq c \| F \|_{X(\Omega)} \]

- For smooth domains local estimates are expected to extend;
- $X = L^q$ with $q \geq p'$: Kinnuen-Zhou ('01) for $\partial \Omega \in C^{1,\beta}$, Byun-Wang ('08) for Reifenberg-flat domains;
- $X = C^\alpha$ and $\partial \Omega \in C^{1,\beta}$: Lieberman ('88), Chen-Di Benedetto ('89), non-divergence-form;
- Conjecture 1: $X = C^\alpha$ and $\partial \Omega \in C^{1,\alpha}$;
- Conjecture 2: $X = BMO$ and $\partial \Omega \in C^\infty$. 
Continuous gradients

**Minimal assumption for bounded gradients**

\[
\text{div} \left( \left| \nabla u \right|^p \nabla u \right) = f, \\
f \in L^{n,1} \Rightarrow \nabla u \in C^0.
\]

- Lorentz-space \( L^{n,1} \) with \( L^{n+\epsilon} \subsetneq L^{n,1} \subsetneq L^n \);
- Stein ('81) \( p = 2, f \in L^{n,1} \) is optimal: Cianchi ('92);
- \( p > 1 \) by Cianchi-Maz'ya/Kuusi-Mingione ('14);
- Minimal boundary regularity \( \partial \Omega \in W^2 L^{n-1,1} \): Cianchi-Maz'ya.
**BMO and Campanato spaces (1)**

What happens for $q = \infty$?

\[
\mathcal{M}^\# s f(x) := \sup_{B \ni x} \left( \int_{B} |f - (f)_B|^s \, dy \right)^{\frac{1}{s}},
\]

\[
f \in \text{BMO} :\Leftrightarrow \|\mathcal{M}^\# f\|_{\infty} < \infty.
\]

- We have $L^\infty \subsetneq \text{BMO} \subsetneq \cap_q L^q$;
- Singular integrals are continuous on BMO;
- Campanato spaces $C^\alpha$ as weighted $\text{BMO}_\omega$-spaces:

\[
\mathcal{M}^\#_\omega f(x) := \sup_{B \ni x} \frac{1}{\omega(r_B)} \int_{B} |f - (f)_B| \, dy, \quad \omega(r) = r^\alpha.
\]
Campanato semi-norms defined by

$$\|f\|_{L^{\omega(\cdot)}(\Omega)} = \sup_{x \in \Omega, r > 0} \frac{1}{\omega(r)} \int_{\Omega \cap B_r(x)} |f - \langle f \rangle_{\Omega \cap B_r(x)}| \, dy,$$

where $\omega(r)r^{-\beta_0}$ is almost decreasing.

- $C^{r^\alpha} = L^{r^\alpha}$ for $\alpha > 0$, but in general $C^\omega \subsetneq L^\omega$;
- We say that $\omega$ satisfies the Dini condition if $\int_0^{\infty} \frac{\omega(r)}{r} \, dr < \infty$;
- Scale of function spaces between BMO to $C^\alpha$, separated through Dini condition.
Global Schauder estimate (1)

Theorem (Breit-Cianchi-Diening-Schwarzacher)

Let $\Omega$ be bounded s.t. $\partial \Omega \in W^1 L^{\sigma(\cdot)} \cap C^{0,1}$ for some $\sigma$ s.t.

$$
\sup_{r \in (0,1)} \frac{\sigma(r)}{\omega(r)} \int_r^1 \frac{\omega(\rho)}{\rho} \, d\rho \ll 1,
$$

$$
\Rightarrow \| |\nabla u|^{p-2} \nabla u\|_{L^{\omega(\cdot)}(\Omega)} \leq c \| F \|_{L^{\omega(\cdot)}(\Omega)}.
$$

- $\omega = 1$: BMO estimate for $\sigma(r) \ll |\log(r)|^{-1}$;
- $\omega = |\log(r)|^{-1}$: $L^{|\log(r)|^{-1}}$-estimate for $\sigma(r) \ll |\log(r)|^{-1} |\log(|\log(r)|)|^{-1}$;
- $\omega = r^\alpha$: $C^\alpha$-estimate for $\sigma(r) \ll r^\alpha$. 
Global Schauder estimate (2)

Theorem (Breit-Cianchi-Diening-Schwarzacher)

Let \( \omega \in C^1(0, \infty) \) be any concave parameter function, s.t. \( \int_0^1 \frac{\omega(r)}{r} \, dr = \infty \). Then there exist a parameter function \( \sigma \), \( \Omega \subset \mathbb{R}^2 \), and \( F \in \mathcal{L}^{\omega(\cdot)}(\Omega) \) such that, if \( \Delta u = \text{div} F \), then:

\[
\partial \Omega \in W^1 \mathcal{L}^{\sigma(\cdot)} \cap C^{0,1}, \quad \sup_{r \in (0,1)} \frac{\sigma(r)}{\omega(r)} \int_r^1 \frac{\omega(\rho)}{\rho} \, d\rho < \infty,
\]

but \( \nabla u \notin \mathcal{L}^{\omega(\cdot)}(\Omega) \).

- The smallness is necessary!
- Covers BMO and \( \mathcal{L}^{\log(r)-1} \).
Global Schauder estimate (3)

Theorem (Breit-Cianchi-Diening-Schwarzacher)

Let $\Omega$ be bounded s.t. $\partial \Omega \in W^1 L^\omega(\cdot)$ for some parameter function $\omega$ satisfying $\int_0^{\omega(r)} \frac{\omega(\rho)}{\rho} d\rho < \infty$. Then

$$\| \nabla u \|_{C^0,\omega(\cdot)(\Omega)} \leq C \| F \|_{L^\omega(\cdot)(\Omega)}, \quad \omega(r) = \int_0^r \frac{\omega(\rho)}{\rho} d\rho.$$

- No smallness is needed!
- Covers $L^{\log(r)^{-2}}$ and $C^\alpha$ (where $\omega(r) \sim \bar{\omega}(r) \sim r^\alpha$);
- In particular: $F \in C^\alpha$ and $\partial \Omega \in C^{1,\alpha} \Rightarrow |\nabla u|^{p-2} \nabla u \in C^\alpha$;
- $\| F \|_{L^\omega(\cdot)(\Omega)}$ can be replaced with $\| F \|_{C^{\omega(\cdot)}(\Omega)}$. 

Proof of the estimate

Sharpness
Breit-Cianchi-Diening-Kuusi-Schwarzacher (2018):

$$\mathcal{M}^\#(|\nabla u|^{p-2}\nabla u)(x) \leq c \mathcal{M}^{\#,p'}(F)(x) \text{ for a.e. } x \in \mathbb{R}^n.$$ 

- Lower order term for bounded domains;
- Implies the known maximal regularity estimates;
- New estimates in Lorentz- and Orlicz spaces;
- Also for weighted $\mathcal{M}^\#_\omega$. 
Flattening and reflection

After a change of coordinates in a small half ball

\[ \text{div}_Z(|\nabla u|^{p-2} \nabla u) = \text{div}_Z \left( |\nabla u|^{p-2} \nabla u \right) - |\nabla u J|^{p-2} \nabla u J J^T + F \]

- Method due to Chen-Di Benedetto ('89);
- Local coordinates \( \Psi(x', x_n) = (x', x_n - \psi(x')) \), \( J = \langle \nabla \psi \rangle \nabla \psi^{-1} \);
- Reflect at the boundary and apply local estimates;
- Control error between \( \nabla \bar{u} \) and \( \nabla u J \) by boundary regularity.
There are $c > 0$ and $\theta \in (0, 1)$ s.t.

$$
\begin{align*}
&\frac{1}{\omega(\theta s)} \left( \int_{\Omega \cap B_{\theta s}(x)} |A(\nabla u) - A_{\theta s}|^{\min \{2, p'\}} \, dy \right)^{\frac{1}{\min \{2, p'\}}} \\
&\leq \frac{c \sigma(s)}{\omega(s)} \int_{\Omega \cap B_{s}(x)} |A(\nabla u)| \, dy + \frac{c}{\omega(s)} \left( \int_{\Omega \cap B_{s}(x)} |F - F_0|^{p'q} \, dy \right)^{\frac{1}{p'q}} \\
&+ \frac{1}{2\omega(s)} \left( \int_{\Omega \cap B_{s}(x)} |A(\nabla u) - A_s|^{\min \{2, p'\}} \, dy \right)^{\frac{1}{\min \{2, p'\}}}.
\end{align*}
$$
Boundary pointwise estimate (2)

- Proof by pointwise estimate for flat reflected system + boundary Gehring estimate;
- Claim follows if we control \( \frac{\sigma(s)}{\omega(s)} \int_{\Omega \cap B_s(x)} |A(\nabla u)| \, dy \);
- Control \( \int_{\Omega \cap B_s(x)} |A(\nabla u)| \, dy \) by oscillation + lower order term;
- If \( \int_0^\infty \frac{\omega(r)}{r} \, dr < \infty \) we can prove boundedness of \( \nabla u \).
Introduction to $p$-Laplacian

Main results

Proof of the estimate

Sharpness

The domain

Given $\omega$ set $\sigma = \gamma \omega \left( \int \frac{\omega(\rho)}{\rho} \, d\rho \right)^{-1}$, $\psi(r) = \int_0^r \sigma(\rho) \, d\rho$ and

$$\Omega = \{ \xi = x_1 + ix_2 : |\xi| < \frac{1}{2}, x_2 > -\psi(|x_1|) \} = \{ \xi = i\rho \exp(i\theta) : \rho < \frac{1}{2}, |\theta| < \frac{\pi}{2} + \varphi(\rho) \}.$$

- We have $\partial \Omega \in W^1 L^\sigma$.
- Let $\zeta(\xi)$ denote the conformal map of $\Omega$ onto the half-disc $\mathbb{D}^+ = \{ \zeta : \text{Im}(\zeta) > 0, |\zeta| < 1 \}$, with fixed point $\xi = 0$.
- Aim: describe behaviour of $\zeta$ by tools from complex analysis (Warschawski, ’42).
The conformal mapping

Warschawski (’44): recall $\psi(r) = \int_0^r \sigma(\rho) \, d\rho$ and $\varphi \sim \sigma$

$$|\zeta(\xi)| = c \exp \left( - \pi \int_{\rho}^{1/2} \frac{dr}{r(\pi + 2\varphi(r))} + o(1) \right) \quad \text{as } \xi \to 0$$

- Analyse singularity of $\zeta$ at $\xi = 0$;
- Embedding $W^{1,\omega} \hookrightarrow C^v$, where $v = r \int_r^1 \frac{\omega(\rho)}{\rho} \, d\rho$;
- Let $\zeta(\xi)$ denote the conformal map of $\Omega$ onto the
- Show $\zeta \notin C^v$ s.t. $\nabla \zeta \notin W^{1,\omega}$. 
Conclusion

Warschawski with $\theta = 0$ and $|\xi| = \rho + W^1L^\omega \hookrightarrow C^\gamma$ gives

\[ c^{-1}\rho \left( \int_\rho^1 \frac{\omega(r)}{r} \, dr \right)^{\frac{2\gamma}{\kappa \pi}} \leq |\zeta(\xi)| \leq c\rho \left( \int_\rho^1 \frac{\omega(r)}{r} \, dr \right) \quad \rho \to 0. \]

- No Dini condition $\Rightarrow \int_0^\infty \frac{\omega(r)}{r} \, dr = \infty$;
- Choosing $\gamma > \frac{\kappa \pi}{2}$ gives contradiction;
- $\nabla \zeta \notin W^1L^\omega$. 