Chiral magnetic response to arbitrary axial imbalance

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The response of chiral fermions to time and space dependent axial imbalance & constant magnetic field is analyzed. The axial-vector–vector–vector three-point function is studied using a real-time approach at finite temperature in the linear response approximation. The chiral magnetic conductivity is given analytically for non-interacting fermions. It is pointed out that local charge conservation plays an important role when the axial imbalance is inhomogeneous. Proper regularization is needed which makes the constant axial imbalance limit delicate: for static chiral charge the CME current vanishes. In the homogeneous (but possible time-dependent) limit of the axial imbalance the CME current is determined solely by the chiral anomaly. As a phenomenological consequence, the observability of the charge asymmetry caused by the CME turns out to be a matter of interplay between various scales of the system. Possible plasma instabilities resulted from the gradient corrections to the CME current are also pointed out.

I. INTRODUCTION

Anomaly induced transport phenomena in systems with chiral fermions have attracted wide interests ranging from high energy physics to condensed matter physics. Among them is the chiral magnetic effect (CME) which relates the chiral chemical potential $\mu_5$ and the external magnetic field $B$ to the anomaly induced electric current density $J$ by the simple formula [1]?

$$J = \frac{e^2}{2\pi^2} \mu_5 B.$$  \hspace{1cm} (1)

The predictions of CME include the electric charge asymmetries in the final stage of the relativistic heavy ion collisions (RHIC) [2,5] and the negative magnetoresistance in some Weyl and Dirac semimetals [6–11]. While there are experimental evidences of CME in the context of condensed matter physics, the situation in RHIC is far more complicated. It remains to exclude the noisy backgrounds in order to nail down the real CME signals. In the former case, a strong magnetic field is generated during an off-central collision and the chirality imbalance is induced by the transition among different topological sectors. Therefore, the CME is an important probe of the topological structure of QCD.

For the past decade since the concept of CME was proposed there have been a vast amount of theoretical works done on the subject. For thorough reviews, see Refs. [12–17] and the references therein. For a recent review on the status of CME in RHIC see the relevant parts of Ref. [17]. Considering that the CME supposed to have a macroscopic imprint, hydrodynamic descriptions including the effect of the anomaly have been developed in order to simulate the modified dynamics of the medium [18–20]. The underlying assumption when applied to RHIC is that a net macroscopic chiral charge is generated in the initial stage of collisions and its characteristic time of variation is much longer than the relaxation time required to establish a local thermal equilibrium, so the formula (1) can be applied. Hydrodynamic modeling of anomalous transport in condensed matter systems has been actively investigated as well [21–24].

There are several other ways to approach the transport phenomena starting from the microscopic level. Investigations have been conducted ranging from kinetic theory (Boltzmann equations [25–31] or Wigner functions [32–34]) to field theoretical approaches (Kubo formulas [35–41]) even through holographic models [42–46, 51] for an insight of strongly coupled systems. At local thermal equilibrium all of them lead to the same answer.

Eq. (1) can be derived in a grand canonical ensemble with respect to the axial charges by calculating the parity-odd component of the photon self-energy diagram. In early works this was done without UV regularization [34, 37]. It is important to point out that the regularization keeps the electric (vector) charge conservation $\partial \cdot J = 0$ intact. Otherwise, there is charge generation at the boundary of the system, proportional to gradients of the axial imbalance. Because of the anomaly, the naive chiral charge is not conserved, while a conserved charge is required for thermal ensembles. One way is to add the magnetic helicity to the naive axial charge in order to form the conserved axial charge

$$Q_5 = \int d^3r (\bar{\psi}^\gamma \gamma^5 \psi + \frac{e^2}{4\pi^2} A \cdot B).$$  \hspace{1cm} (2)
and the classical action underlying the ensemble average reads as

\[ S = S_{\text{QED}} + \mu_5 Q_5 = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i D_{\mu} \psi - e A_\mu \psi + \mu_5 \left( \gamma^5 \gamma^\mu \psi \right) \right] \]  

(3)

with the Dirac field \( \psi \), the electromagnetic vector potential \( A_\mu \) and \( B = \nabla \times A \). The formula (1) is restored at thermal equilibrium because of the magnetic helicity.

As the time scale of the chiral charge generation/variation in RHIC may be comparable to or shorter than the thermal relaxation time, the hydrodynamic approximation for CME breaks down and one has to consider a space-time dependent chiral charge density. We shall explore this situation in the present work. Without a first principle treatment of the non-perturbative dynamics of the chiral charges in QCD, a simple-minded assumption amounts to extend the constant chiral chemical potential to a space-time dependent one, which will be relied on in the rest of this paper. Under this assumption, the chiral chemical potential encodes the QCD dynamics of the axial charge creation/annihilation process and corresponds to the temporal component of an axial-vector potential and its contribution to the chiral magnetic current (1) is described by the triangle diagram in Fig. 1. An issue arises with the 2nd to the temporal component of an axial-vector potential and its contribution to the temporal component of a WSM sample can be modeled in terms of an axial-vector potential as well. See for example Ref. [48], Sec. 7 of Ref. [13] or Secs. 2, 5 of Ref. [21] for more details.

The electromagnetic (EM) sector of the underlying quantum field theory can be described by the following Lagrangian:

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \gamma^\mu \left( i \partial_\mu - e A_\mu - \gamma^5 A_{5,\mu} \right) \psi + \text{UV reg.} \]

\[ = \mathcal{L}_{\text{QED}} - \bar{\psi} \gamma^\mu \gamma^5 \psi \cdot A_{5,\mu}. \]  

(4)

It is worthwhile to mention that for \( A_5 = (-\mu_5, \mathbf{0}) \) the coupling to the axial-vector field effectively behaves as an axial chemical potential, i.e. \( A_{5,0} = -\mu_5 \). These are, however, two fundamentally different concepts, since there is no need to impose the constraints of thermal equilibrium in order to talk about an axial-vector potential.

In this section we introduce the linear response approximation (LRA) of a fermionic system coupled to electromagnetic fields as well as in presence of axial-vector potential. Such a model might capture the electromagnetic transport in a quark-gluon plasma of which the gluon sector can have nontrivial topological features, locally violating CP–invariance. This local CP–violation is described through the axial-vector potential \( A_5 \) under our assumption. On the fundamental level of QCD there are no axial gauge fields, however, focusing only on the effective description of the EM sector, there are two contributions to the chiral charge non-conservation. The usual EM one proportional to \( \mathbf{E} \cdot \mathbf{B} \), and the one from the gluonic sector: \( \propto \mathbf{E}_a \cdot B^a \), with \( \mathbf{E}_a \) and \( \mathbf{B}_a \) being the components of chromoelectric and -magnetic fields, respectively. It is the latter contribution that is reflected in the hypothetical axial-vector potential.

Solid state systems might happen to be affected by similar circumstances, such as electronic systems in the bulk of Weyl semimetals (WSM). Their low energy behavior is described by the Dirac equation. Since the spatial inversion and time reversal symmetries can be violated in such systems, depending on the details of the material, one can introduce an axial-vector potential. It was explicitly shown in Ref. [47] that the elastic deformations of a WSM sample can be modeled in terms of an axial-vector potential as well. See for example Ref. [48], Sec. 7 of Ref. [13] or Secs. 2, 5 of Ref. [21] for more details.
with the (axial) vector potential \( A^μ(\mathbf{r}) \). Both Eqs. (5) and (6) are to be understood as convolutions in the space-time arguments for inhomogeneous and time dependent sources. In the special case of zero electric field and zero axial magnetic field, the electric current \( J^μ \) is solely controlled by the response function \( \langle J^μ J^ν \rangle \). The functions \( \langle J J J_s \rangle \) containing two vector and one axial-vector current operators and will be referred to as axial-vector–vector–vector (AVV) response from here on.

From the field theoretic perspective, this function corresponds to the fully dressed triangle diagram (see Fig. 1) whose logarithmic component with respect to the A-vertex is – in the constant field limit – dictated by the chiral anomaly. As we will see in the followings, specific orders of limits are tied to the logarithmic component, but this is not the case generally, as it was already pointed out in Ref. [38].

The contribution of the second term in Eq. (5) of the electric current is given by the following expression:

\[
\left\langle J^ν(\mathbf{r}) \right\rangle = \int d^4 y \int d^4 z A_j(y) A_{5,0}(z) \Gamma^{\text{AVV}}_{AVV}(x-y, x-z) = \\
\int d^4 q_1 \int d^4 q_2 A_j(q_1) A_{5,0}(q_2) \Gamma^{\text{AVV}}_{AVV}(q_1, q_2) e^{i q_1 \cdot q_2},
\]

(7)

We again point out that for any explicit calculation one needs to regularize the theory in order to keep the \( U(1) \) Ward–Takahasi-identity intact. Because of the presence of \( \gamma^5 \), we use the method of Pauli and Villars, i.e. coupling to the system an auxiliary field with asymptotically large mass \( M \), although obeys Fermi-Dirac statistics, contributes with an opposite sign to the loop integrals. So in addition to the usual AVV though obeys Fermi-Dirac statistics, contributes with an opposite sign to the loop integrals. So in addition to the usual AVV current is given by the following expression:

\[
\langle J^ν(\mathbf{r}) \rangle = \frac{e^2}{2} \int d^4 q_1 \int d^4 q_2 A_j(q_1) A_{5,0}(q_2) \Gamma^{\text{AVV}}_{AVV}(q_1, q_2) e^{i q_1 \cdot q_2},
\]

(7)

where \( G^{R,A,C} \) are the retarded, advanced and correlation components of the fermionic propagator in the SK formalism, respectively. The above vertex function is retarded with respect to its index \( \mu \), so the response current follows the perturbations both in \( A \) and \( A_5 \). For a detailed introduction into the formalism see for example Ref. [49]. In the spirit of LRA, all the propagators are to be considered at zero external fields in thermal equilibrium. All components are linked to the fermionic spectral density \( \rho = iG^R + iG^A \), \( G^R(p) = \rho(p) \cdot (1 - 2\pi(p_0)) \), where we suppressed the temperature dependence of the Fermi-Dirac distribution \( \tilde{\rho}(p_0) = n_{FD}(p_0/T) \). For the subsequent calculations, we also introduce \( \tilde{\Gamma}^{\text{AVV}}_{AVV} = \gamma^μ \rho^{μν} - \gamma^ν \rho^{μν} \), where the Pauli-Villars-term is \( n_{FD} = m_A(q_1, q_2) \equiv \tilde{\rho}_{\mu=\lambda=M}(q_1, q_2) \), \( M \) being much larger than any other scales \( q_1, q_2, m \) or \( T \). So practically \( M \) is sent to infinity while other external parameters kept finite. The AVV vertex satisfies the following equations due to the the Ward–Takahasi-identities:

\[
\langle q_1 + q_2 \rangle \tilde{\Gamma}^{\text{AVV}}_{AVV} = q_1 \tilde{\Gamma}^{\text{AVV}}_{AVV} = 0,
\]

(15)

\[
q_2 \tilde{\Gamma}^{\text{AVV}}_{AVV} = i e^{μν} q_1 \cdot q_2 \rho, \frac{e^2}{2\pi^2}.
\]

(16)

The first two equations in Eq. (15) implies \( \partial \cdot J = 0 \) at the level of linear response, whilst Eq. (16) is the anomalous non-conservation of the axial-vector current \( J_5 \). The above properties of the AVV vertex are the consequence of the following identities:

\[
G^{R,A}(p + q) \rho G^{R,A}(p) = - G^{R,A}(p) + G^{R,A}(p),
\]

(17)

\[
G^C(p + q) \rho G^C(p) = - G^C(p) + G^C(p),
\]

(18)

\[
G^{R,A}(p + q) \rho G^C(p) = G^C(p).
\]

(19)

Plugging Eq. (16) into Eq. (6) one obtains the well-known anomalous Ward-identity in the chiral limit:

\[
\langle \partial \cdot J_5 \rangle = \frac{1}{2} \int d^4 q_1 \int d^4 q_2 A_5(q_1) A_5(q_2) \times
\]

\[
i(q_1 + q_2) \tilde{\Gamma}^{\text{AVV}}_{AVV}(q_2, -q_1 - q_2) e^{i q_1 \cdot q_2} =
\]

\[
= \frac{e^2}{2\pi^2} E \cdot B.
\]

(20)
Setting $q_2 = 0$, $\Gamma_{AVV}^{\mu
u}(q_1, q_2)$ itself becomes completely determined by the UV sector of the theory, i.e. ruled by the anomaly. Formally this behavior is caused by cancellation between certain terms in the vertex function which are equal upon a shifting of the loop momentum. The details can be found in Appendix A. For the electric current this means the chiral magnetic effect prevails even for time-dependent but homogeneous $A_{5,0}$ and arbitrary $\mathbf{B}$:

$$J(t, r) = \frac{e^2}{2\pi^2} A_{5,0}(t) \mathbf{B}(t, r) = -\frac{e^2}{2\pi^2} \mu_5(t) \mathbf{B}(t, r). \quad (21)$$

Another implication of the behavior of the AVV vertex in the $q_2 = 0$ limit is the robust form of the chiral charge $Q_5$ for $A_{5,0} = 0$:

$$Q_5 = \frac{1}{2} \int_{-\infty}^{\infty} dq'_0 \int_{-\infty}^{\infty} dq_0 \int d^3 q \tilde{A}_\mu(q_0', -q) \tilde{A}_\nu(q_0, q) \times 
\Gamma_{AVV}^{\mu
u}(q_0, q - q_0' - 0, 0) \ e^{iq_0+q_0'} = \frac{e^2}{4\pi^2} \int d^3 r A_j(t, r) \int d^3 q \frac{\partial}{\partial q^\mu} e^{i q^\mu A_\nu(t, r)} = \frac{e^2}{4\pi^2} \int d^3 r A(t, r) \cdot \mathbf{B}(t, r). \quad (22)$$

So for zero axial-vector potential the one-loop vertex generates the chiral charge equivalent to the magnetic helicity term in Eq. (3). This is not unexpected, since the AVV vertex respects EM gauge invariance due to the PV regularization. As we explained in the introduction one can use the identification $A_{5,0} = -\mu_5$, as a space- and time-dependent extension of the chemical potential $\mu_5$. It is worth mentioning that in this case the chiral magnetic current in Eq. (21) implied by Eq. (16) differs from Eq. (1) by a sign at homogeneous $A_{5,0}$. As the origin of the CME current is not the constant value of the chiral charge but rather the axial vector potential $A_5$. The same issue can be considered in the of the Maxwell-Chern-Simons electrodynamics by using the anomalous Ward identity. See Appendix B for details.

### III. THE CASE OF CONSTANT MAGNETIC FIELD

In this section we work out the electric current response in the presence of constant magnetic field and axial imbalance with arbitrary spacetime dependence. Physically, this approximation is meaningful if there is a separation between the scales of the perturbations in the axial imbalance field $A_{5,0}$ and the magnetic field: the latter has to vary on much larger space- and timescales. In RHIC, this is not the case for the whole lifetime of the QGP, but it can be a good approximation describing the initial state, when $\mathbf{B}$ is still large and relativelyunchanged because of the conductivity of the medium. In Ref. [50], the interested reader can find a detailed analysis of the fluctuation pattern of the magnetic field in RHIC. Due to the intense color fields, a region is formed where the axial imbalance is effectively non-zero. This region, however, is still affected by the fast gluonic dynamics, leaving the imbalance field to change fast as well.

Keeping things simple we suppose $A = (0, A)$ and $A_5 = (A_{5,0}, 0)$, so there are no electric and axial magnetic fields present. It is straightforward to check that $\Gamma_{AVV}^{\mu
u}(q_1 = 0, q_2) \approx 0$ by using the Ward-identity to transform the integrand of the AVV vertex into a full derivative with respect to integration momentum. The finite mass and the PV-term then cancel each other out. The first contributing term in the small-$q_1$ expansion is $\frac{\partial \Gamma_{AVV}^{\mu
u}}{\partial q_{1\mu}}$. Equivalently, one can plug the time-independent $A(y) = \frac{1}{2} \mathbf{B} \times \mathbf{y}$ into Eq. (7) to gain the relation:

$$(J')^i(x) = \int d^4 q_2 \tilde{A}_{5,0}(q_2) \ e^{iq_0+q_0'} \left[ \int_{-\infty}^{\infty} \frac{e^{iq_0} B^j}{2} \frac{\partial \Gamma_{AVV}^{\mu
u}}{\partial q_{1\mu}} \right]_{q_1 = 0}$$

$$= \int_{-\infty}^{\infty} d\tau' \int d^3 q \tilde{A}_{5,0}(t', q) \bar{\sigma}^j_A(t' - t, q) e^{-iq\cdot r}. \quad (24)$$

where the kernel

$$\bar{\sigma}^A_A(t, q) = \int_{-\infty}^{\infty} dq_0 e^{iq_0 t} \frac{1}{2} \frac{e^{iq_0} B}{2} \frac{\partial \Gamma_{AVV}^{\mu
u}}{\partial q_{1\mu}} \left|_{q_1 = 0} \right. \quad (25)$$

is the CME conductivity in the mixed representation of spatial momentum and time, and an explicit formula of it will be derived below. We shall omit the averaging sign $\langle \rangle$ in what follows if we can without causing confusion. Before taking the derivative of Eqs. (8)-(12) with respect to $q_{1\mu}$, we note that the sum of Eqs. (8), (9) and (10) equals to the sum of Eqs. (11), (12) and (13), as can be demonstrated by transposing the matrices under the trace of the former employing the charge conjugation property $C(q^\gamma)C^\dagger = -q^\gamma$ with $C = q^\gamma q^\mu$ followed by transforming the integration momentum $p \rightarrow -p - q_1 - q_2$. The transformation of the integration momentum is legitimate as long as the regulator terms kept in the scene. Consequently,

$$\Gamma_{AVV}^{\mu
u}(q_1, q_2) = \gamma_m^{ij}(q_1, q_2) = \lim_{M \rightarrow \infty} \gamma_m^{ij}(q_1, q_2) \quad (26)$$

where

$\gamma_m^{ij}(q_1, q_2) = \int_p \left\{ -\gamma^0 G^A(p + q) \gamma^0 \gamma^5 G^A(p) \gamma^i G^A(p) \cdot (1 - 2\bar{n}(p_0 + q_0) + \gamma^0 G^R(p + q) \gamma^0 \gamma^5 G^R(p) \gamma^i G^R(p) \cdot (1 - 2\bar{n}(p_0)) + \gamma^i G^R(p + q) \gamma^0 \gamma^5 G^A(p) \gamma^i G^A(p) \cdot (2\bar{n}(p_0) - 2\bar{n}(p_0 + q_0)) \right\}.$

Here we have replaced $G^C(p)$ with $(1 - 2\bar{n}(p_0))(G^R(p) -$
\( G^A(p) \). It follows that

\[
\frac{\partial}{\partial q_{1k}} \rho_{im}^{ij}(q_1 \to 0, q_2 = q) = \\
= e^2 \int_p \mathfrak{tr} \left\{ -\gamma^i \gamma^k G^A(p) \gamma^j \frac{\partial}{\partial p_k} G^A(p) \right\} \\
\cdot (1 - 2\delta(p_0 + q_0)) + \\
+ \gamma^i G^R(p + q) \gamma^0 \gamma^j G^R(p) \gamma^j \frac{\partial}{\partial p_k} G^R(p) \\
\cdot (1 - 2\delta(p_0)) + \\
+ \gamma^i G^R(p + q) \gamma^0 \gamma^j G^A(p) \gamma^j \frac{\partial}{\partial p_k} G^A(p) \\
\cdot (2\delta(p_0) - 2\delta(p_0 + q_0))
\] (27)

and

\[
\mathfrak{F}_A(t, q) = \int_{-\infty}^{\infty} dq_0 e^{iq_0 t} \frac{i}{2} e^{iq_k B_k^i} \left[ \frac{\partial}{\partial q_{1k}} \rho_{im}^{ij}(q_1, q_2) \right]_{q_1 = 0, q_2 = q} - \lim_{M \to \infty} \frac{\partial}{\partial q_{1k}} \rho_{im}^{ij}(q_1, q_2) \bigg|_{q_1 = 0, q_2 = q}.
\] (30)

The first step of evaluation is to determine the tensor structure of the expression. In this general would be tedious because there are three propagators left under the trace. However, we are interested in antisymmetric combinations in \( jk \) only, because of the contraction with the magnetic field. The details of the trace calculation are detailed in Appendix C. The resulting expression turns out to remain fairly compact:

\[
\frac{\partial \rho_{im}^{ij}}{\partial q_{1k}} = -16\pi e^2 \int_p \left( 1 - 2\delta(p_0) \right) \text{sgn}(p_0) \frac{1}{2|\mathbf{p}|} \times \\
\left[ (B^r (m^2 - (p_0 + q_0) p_0) - p^i B \cdot (p + q)) \frac{1}{1 + \frac{p^2}{|\mathbf{p}|}} \frac{\partial}{\partial |\mathbf{p}|} \left( \frac{1}{(p_0 + q_0 + i0^+)^2 - (p + q)^2 - m^2} \right) \delta(p_0^2 - p^2 - m^2) + \\
+ (B^r (m^2 - p_0(p_0 + q_0)) - (p^i + q^i) B \cdot p) \frac{1}{(p_0 + q_0 + i0^+)^2 - (p + q)^2 - m^2} \frac{\partial}{\partial |\mathbf{p}|} \delta(p_0^2 - p^2 - m^2) \right].
\] (31)

Here we performed the trace and regrouped the terms from the three propagator-products AAA, RRR and RAA. This way it is possible to deal with the higher-order poles by recasting part of the expression as a derivative of either \( \rho \) or \( G^R \) — more details of the calculation can be found in Appendix D. It proves to be useful to separate the components of \( \mathbf{B} \) parallel to \( \mathbf{q} \): \( \mathbf{B}_1 = (\mathbf{B} \cdot \mathbf{q}) \mathbf{q} \), and perpendicular to it: \( \mathbf{B}_2 = \mathbf{B} - \mathbf{B}_1 \). In this way part of the directional integration can be performed, leaving us with the following expression, the azimuthal integration still left to be done:

\[
\frac{\partial \rho_{im}^{ij}}{\partial q_{1k}} e^{ik B_k} = \\
= \frac{e^2}{\pi^2} \int_{-\infty}^{\infty} dp_0 \text{sgn}(p_0) \int_0^\infty dp \left[ \frac{\partial}{\partial x} \left( \frac{B_1^x (p_0 q_0 + q px + (x^2 + 1) p^2) + B_1^y (p_0 q_0 + (1 - x^2) + 1) p^2}{(p_0 + q_0 + i0^+)^2 - p^2 - q^2 - 2q px - m^2} \right) + \\
+ \frac{p^2}{(p_0 + q_0 + i0^+)^2 - p^2 - q^2 - 2q px - m^2} \right].
\] (32)

Note that the first term in the above expression is a total derivative with respect to \( x \).

The contributions of the two terms in the integrand of Eq. (31) are calculated separately with the detailed steps laid out in Appendix D. For the massless term, we take the Fourier transform of Eq. (32) with respect to \( q_0 \) first and calculate the rest of the integrals afterwards. All integrations can be carried out analytically for \( m = 0 \) and we obtain that
For the PV term, we scale the loop momentum $p$ by the regulator mass $m = M$ as $p = My$ and take the limit $M \to \infty$. The rest of the integrals can be calculated analytically with the result

$$
\int_{-\infty}^{\infty} dq_0 e^{iq_0t} \frac{\partial \beta_{ij}}{\partial q_{1k}} e^{ijB_1} = e^2 \left( \frac{e}{\pi^2} \right)^2 \delta(t) + \theta(-t) \left[ \left( B_i^0 + \frac{B_i}{2} \right) \frac{\partial^2}{\partial t^2} \left( \frac{\sin(qT)}{qt} \right) + \frac{B_i}{2} \frac{\partial}{\partial t} \left( \frac{\sin(qT)}{qt} \right) \right].
$$

(34)

By combining the two, now we are equipped with the mixed representation of the CME conductivity in case of constant, homogeneous magnetic field.

$$
\bar{\sigma}'_A(t, q) = \frac{1}{2} \int_{-\infty}^{\infty} dq_0 e^{iq_0t} \left( \frac{\partial \beta_{ij}}{\partial q_{1k}} + \frac{\partial \beta_{ij0}}{\partial q_{1k}} \right) e^{ijB_1} = e^2 \left( \frac{e}{\pi^2} \right)^2 \left[ B_i^0 \delta(t) + \theta(-t) \left( q \sin(qT) \left( B_i^0 + \vec{q} \cdot \vec{B} \right) - \frac{\partial}{\partial t} \left( \frac{\sin(qT)}{qt} \right) f(tT) \left( B_i^0 - \vec{q} \cdot \vec{B} \right) \right] ,
$$

(35)

where $f(x)$ comes from the Fermi-Dirac distribution:

$$
f(x) = 4x \int_0^\infty d\gamma_F \sin(2\gamma x) = 1 - \frac{2\pi x}{\sinh(2\pi x)} \to \begin{cases} 0, & x \to 0 \\ 1, & x \to \infty \end{cases}
$$

(36)

The above expression is the main original contribution of this paper. Although we end up with a fairly complicated expression, it is still suitable to investigate the charge transport in special situations as we will see in the next section. About to be proven practical as it is, one might find the frequency-momentum representation and the coordinate representation of the response function more useful in other cases. For more details see Appendices [E] and [F].

### A. Limiting cases

In order to gain some insights into the expression in Eq. (35), let us first analyze its behavior in two limiting cases.

A time-independent $A_{5,0}$ will render the conductivity in the zero frequency limit, which is equivalent to the integral of $\bar{\sigma}'_A(t, q)$ with respect to its time-argument:

$$
\int_{-\infty}^{\infty} d\sigma_A = \frac{e^2}{2\pi^2} \left[ B' + \left( B' + \vec{q} \cdot \vec{B} \right) \right] \frac{1}{2} \int_0^0 dqT \sin(qT) - \left( B' - \vec{q} \cdot \vec{B} \right) \frac{1}{2} \int_0^0 d\tau \frac{\partial}{\partial \tau} \left( \frac{\sin(qT)}{qT} \right) f(tT) = \frac{e^2}{4\pi^2} \left( B' - \vec{q} \cdot \vec{B} \right) \left[ 1 - \int_0^0 d\tau \frac{\partial}{\partial \tau} \left( \frac{\sin(\tau)}{\tau} \right) f(\tau T) \right] \xrightarrow{q \to 0} 0.
$$

(37)

In the static but inhomogeneous limit ($q \neq 0$) we see the conductivity is perpendicular to $q$. On one hand this means local charge conservation is fulfilled. On the other hand it shows that the current has a dipole-like structure, which has consequences regarding the long-time behavior of the charge transport, as we shall see soon.

To approach the limit of constant $A_{5,0}$ by sending $q \to 0$ one observes that the current vanishes, since $f(x \to \infty) = 1$. We note here that this limiting behavior was already reported in Ref. [39] and some aspects were also discussed in Ref. [38]. It is not that surprising that the electric current vanishes for constant axial imbalance. The non-existence of the CME at equilibrium was reported by several authors, see for example Refs. [51]-[55]. In the context of Weyl semimetals, where it is possible to prepare the system in such a way that the introduction of a chiral chemical potential makes sense, there is
a consensus that in equilibrium the CME current vanishes – even for nonzero \( A_{5,0} \), see Refs. [21, 52].

When the static limit is taken firstly the UV-originated anomaly contribution is canceled by the following term: 

\[
\text{ie}^2 \int_p (-2\hat{n}(p_0)) \text{tr} \left\{ \gamma^5 \gamma' \gamma^i \gamma^{\delta G^j (p)} \frac{\partial \gamma^{\delta G^j (p)}}{\partial p_0} \right\}.
\]

Although we do not go into the details here, one can show that interactions will not change this expression, see for example Refs. [54–58]. So the vanishing of the conductivity in the mentioned limit is a general result.

For homogeneous \( A_{5,0} \) configurations, i.e. \( q \to 0 \) only the \( \frac{\varepsilon^2}{2\pi^2} B_0(t) \) term in Eq. [55] survives. This term in the end provides the usual homogeneous current parallel to \( B \), tied to the anomaly. Let us expand Eq. [55] around to the first nontrivial order in \( q \) to learn what happens if the system is pushed away from homogeneity:

\[
\delta J(t, q) = J(t, q) - \frac{\varepsilon^2}{2\pi^2} A_{5,0}(t, q) B \approx
\]

\[
\frac{\varepsilon^2}{4\pi^2} \int_{-\infty}^{0} d\tau A_{5,0}(t + \tau, q) q^2 \left[ \left( 1 + \frac{1}{2} f(\tau T) \right) B + \left( 1 - \frac{1}{2} f(\tau T) \right) (B \cdot \vec{q}) \vec{q} \right] + \mathcal{O}(q^4).
\]

The above expression is still too complicated. We now will assume that there is a clear separation between the internal timescale and temperature, namely, send \( T \) either to 0 or \( \infty \). In both cases we end up with the following expression:

\[
\delta J(t, q) = \frac{\varepsilon^2}{4\pi^2} \left( C_1 q^2 B + C_2 (B \cdot k) q \right) \int_{-\infty}^{0} d\tau A_{5,0}(t + \tau, q) \tau,
\]

(40)

with the constants \( \{ C_1, C_2 \} \) being either \( \{ 1, 1 \} \) for \( T = 0 \) or \( \{ \frac{1}{2}, \frac{3}{2} \} \) for \( T \to \infty \). After Fourier transform one arrives at the differential equation below:

\[
\partial^2 t \delta J(t, r) = \frac{\varepsilon^2}{4\pi^2} \left( C_1 B \frac{\partial}{\partial r} + C_2 (B \cdot \nabla_r) \right) A_{5,0}(t, r) =
\]

\[
- \frac{\varepsilon^2}{4\pi^2} \left( C_1 B \left( \nabla \cdot E_5 \right) + C_2 (B \cdot \nabla) E_5 \right).
\]

(42)

For homogeneous \( A_{5,0} \) the RHS of Eq. (42) is zero, leading no deviation from the CME current in Eq. (11). On the other hand, inhomogeneity can add to the dynamics of the electric current. One should, however, keep in mind that the short-time behavior of the corrections in small \( q \) provided by our weak-coupling calculation might be significantly modified at strong coupling [42].

Straightforward analysis shows that \( \partial^2 t \nabla \times \delta J \) is not zero: This means that inhomogeneous \( A_{5,0} \) can, even without sizable electric field or spatial dependence of \( B \), alter the vorticity of the current field. Depending on the dynamics of \( A_{5,0} \) this might cause instabilities in a laminar charge flow, leading to the formation of vortices. Other studies indicate various instabilities in case of the chiral plasma is affected by dynamical EM fields, see for example Refs. [59–61] or the recent first principle study of Ref. [62] simulating QED plasma. Further theoretical investigation is needed by taking the feedback of axial charge and EM fields into account. We will address this question in a future publication.

IV. CME CONTRIBUTION TO THE CHARGE ASYMMETRY

The main goal of this section is to analyze the amount of transported charge by the CME current in situations that might be relevant for RHIC. In that order we first look into some examples of chiral imbalance patterns, and show that there is no contribution for the net charge asymmetry if we wait long enough. Later we generalize this statement and also discuss its limitations.

A. Long-time behavior after quench

First we consider the scenario when there is a sudden onset of the axial imbalance, corresponding to \( \bar{A}_{5,0}(t, q) = \theta(t) \bar{A}_{5,0}(q) \). The electric current is given by this expression:

\[
\bar{J}(t, q) = \frac{\varepsilon^2}{2\pi^2} \bar{A}_{5,0}(q) \left[ B^i - \frac{1}{2} \left( B^i + \bar{q}'(B \cdot \bar{q}) \right) \right] \int_0^t dq \sin(q\tau + \frac{1}{2} \left( B^i - \bar{q}'(B \cdot \bar{q}) \right) \int_0^\tau \frac{\partial}{\partial \tau} \left( \frac{\sin(q\tau)}{q\tau} \right) f(\tau T),
\]

(43)

which for long times simplifies further to

\[
\bar{J}(t, q) \to \frac{\varepsilon^2}{2\pi^2} \bar{A}_{5,0}(q) \left( B^i - \frac{q'(B \cdot q)}{2q^2} \right) F(q/T),
\]

(44)

where

\[
F(x) = 1 + \int_0^\infty \frac{dy}{xy} \frac{\sin(xy)}{xy} f(y) = 1 - \int_0^\infty \frac{dy}{xy} \frac{\sin(xy)}{xy} f'(y)
\]

(45)

The function \( F(x) \) as plotted in Fig. 2 vanishes for small \( x \) and approaches 1 monotonically for large \( x \). This results in \( F \) acting as an infrared cut-off when the Fourier transform is performed to get the coordinate-space expression for \( \bar{J}(t \to \infty, r) \). The suppression of the small \( q \)–domain makes \( \bar{J} \) to be localized in a region with size controlled by \( 1/T \). In the limiting case \( q \to 0 \) the current is zero – as expected for \( \bar{A}_{5,0} \) when the homogeneity limit is taken after the static limit. The result is the same if for some reason \( T \) supersedes any (inverse) spatial scales, since \( T \to \infty \) renders \( \bar{J} \) to be zero.
through $F(0) = 0$ again. For the other extreme, $T \to 0$, $J^i$ reveals a dipole pattern:

$$
\mathbf{J}(t \to \infty, \mathbf{r})|_{T=0} = \frac{e^2}{16\pi^3} \int d^3 r' \left( -\frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla_{\mathbf{r}'} \times (\mathbf{B} \times \nabla_{\mathbf{r}'}) \right) A_{5,0}(\mathbf{r}') = \\
= \frac{e^2}{16\pi^3} \int d^3 r' \frac{\mathbf{B} - \frac{3(\mathbf{r}' \times \mathbf{B})}{(r')^3} \times \mathbf{A}_{5,0}(\mathbf{r} - \mathbf{r}')}{r'} = \\
= \frac{e^2}{16\pi^3} \nabla \times (\mathbf{B} \times \nabla_{\mathbf{r}}) \int d^3 r' \frac{-\mathbf{A}_{5,0}(\mathbf{r} - \mathbf{r}')}{r'},
$$

(46)

One can show that this current dipole transports zero charge in total through a large enough surface perpendicular to the direction of the magnetic field. In general this is the consequence of the structure $\mathbf{B} - \mathbf{q}(\mathbf{B} \cdot \mathbf{q})$. However, a well-localized source is enough to explain what happens: the “current field lines” are closed because $\nabla \cdot \mathbf{J}(t \to \infty, \mathbf{r}) = 0$ so any of them travels through twice on a large enough surface. We can easily show this with a point-like source using the previous dipole formula and integrating over the surface $S \perp \mathbf{B}$:

$$
\int_S d^2 \mathbf{r} \mathbf{B} \cdot \mathbf{J}(t \to \infty, \mathbf{r})|_{T=0} \propto \int_S d^2 r \mathbf{B} \cdot \mathbf{r} = \\
\int_S d\mathbf{l} \frac{\mathbf{B} \times \mathbf{r}}{r^3} \propto \frac{1}{r} \lim_{r \to 0},
$$

(47)

where we used Stokes’ theorem and in the last step we recognized that $d\mathbf{l}$ and $\mathbf{B} \times \mathbf{r}$ are parallel and both are proportional to $r$. Since this observation is based on the long-time behavior of the current, one has to wait sufficiently long for the net transported charge to vanish.

An interesting side-note can be made at this point. Let us further analyze the long-time behavior of the current by integrating it over a spherical region $S_R$ ($R$ being the radius):

$$
V \cdot \mathbf{J} := \int_{S_R} d^3 r \mathbf{J}(t \to \infty, \mathbf{r}) = \\
= \frac{e^2}{2\pi^2} \frac{1}{2} \int d^3 \mathbf{q} \frac{\sin qR - qR \cos qR}{q^3} F(q/T) \frac{\mathbf{q} \cdot (\mathbf{B} \cdot \mathbf{q})}{q^2} \mathbf{q} A_{5,0}(\mathbf{q})
$$

(48)

with $V$ as the volume of $S_R$ and $\mathbf{J}(t \to \infty, \mathbf{r})$ as the coordinate representation of Eq. (44). Now we consider a source of $A_{5,0}$ centered in space around the origin, whose Fourier transform is $\mathbf{A}_{5,0}(\mathbf{q}) = V \cdot \mathbf{A}_{5,0}$. After some calculation whose details can be found in Appendix G7, we arrive at:

$$
\mathbf{J} = \frac{e^2}{2\pi^2} A_{5,0} \frac{1}{\pi^3} \frac{B}{3} \int_0^\infty dQ F(q/(RT)) \frac{\sin Q - Q \cos Q}{Q} = \\
= \frac{e^2}{2\pi^2} A_{5,0} \frac{1}{3} f(R/T),
$$

(49)

This expression does not carry the dipole structure anymore. Instead, there is a suppression factor of $(1 - f(R/T))/3$: for zero temperature or when the spatial averaging is done within an asymptotically small sphere the result is the 1/3 of the anomaly ruled current. Sending either $R$ or $T$ to large values, however, renders $\mathbf{J}$ to zero. Generally, the expression is monotonically interpolates between these two limiting cases depending on the relative value of $R$ and $T$.

\[FIG. 2: Functions characterizing the response function. f is defined in Eq. (38), whereas F is derived from f in order to give the long-time behavior of the response, see Eq. (45).\]

B. Long-time charge transport parallel to B

Returning to the question of transported electric charge, one can argue that its vanishing behavior for long times is a general feature. This is closely related to the fact that the local charge conservation $\partial \cdot \mathbf{J} = 0$ is an essential property of the system. Due to the CME there is electric current in the direction of the magnetic field. The system does not have any boundary, so it is quite natural that the charge flows back somewhere: since the current tends to be parallel to $\mathbf{B}$ in the presence of chiral imbalance, the back-flow happens away from these regions, where $A_{5,0}$ vanishes. We have already seen this dipole structure at work in the previous subsection. Therefore, taking a large enough surface perpendicular to $\mathbf{B}$, we expect that the net charge through this surface tends to zero as time passes.

In order to put the argument onto more general grounds we analyze the following quantity:

$$
\Delta Q_5 := \int_{-\infty}^{\infty} dt \int_S d^2 r \mathbf{B} \cdot \mathbf{J}(t, \mathbf{r}),
$$

(50)

where the surface $S$ is the plane with the normal vector $\mathbf{B}$. 
Utilizing the conductivity relation in Eq. (35) we can write
\[
\Delta Q = \int d^3q \int d^2r B_0 \delta_A(q_0 = 0, q) \tilde{A}_{5,0}(q_0 = 0, q)e^{-iqr} = \frac{e^2B}{4\pi^2} \int d^3q \int d^2r (1 - (B - q\vec{n})) \times \left[ 1 - \int_{-\infty}^{\infty} d\tau \frac{\partial}{\partial \tau} \left( \frac{\sin \tau}{\tau} \right) \frac{\tau}{q} \right] \tilde{A}_{5,0}(0, q)e^{-iq\tau}. \tag{51}\]

Integrating over the surface \(S\) when its size is large enough, the components of \(q\) parallel to \(S\) are forced to vanish and only the component parallel to \(B\) survives:
\[
\Delta Q = \lim_{\text{area of } S \to \infty} \frac{e^2B}{4\pi^2} \int_0^\infty dqq^2F(q/T)x^1 \sqrt{1 - x^2 - 1} \tilde{A}_{5,0}(0, qx\vec{B})e^{-iqrs} \equiv 0. \tag{52}\]

In conclusion, the net charge transported by the CME current for any chiral imbalance in constant magnetic field vanishes for long enough time when it is measured through a large enough plane perpendicular to \(B\).

It is worth pointing out that the above statement is actually the consequence of a more general feature of the vertex function. Taking an infinitely large plane with the normal vector \(\vec{n}\) the transported charge \(\Delta Q\) can be written most generally as follows:
\[
\Delta Q = \int d^4q \int dqq\vec{n}\tilde{\Gamma}_{AVV}^{ij}(q, q') \tilde{\chi}_j(q) \tilde{A}_{5,0}(q'), \tag{53}\]

with \(q = (q_0, \vec{q})\) and \(q' = (-q_0, -q + q\vec{n})\). Now we recognize that because of \(q + q' = (0, q\vec{n})\), the combination in the integrand of the above equation can be recast as the first identity in Eq. (15): \(q\vec{n}\tilde{\Gamma}_{AVV}^{ij}(q, q') = (q + q')\tilde{\Gamma}_{AVV}^{ij}(q, q') \equiv 0\), rendering \(\Delta Q\) to vanish. Therefore the previous statement on the vanishing of the long-time transported charge \(\Delta Q\) is generalized beyond the weak-coupling approximation.

The transported charge shows ambiguity when a homogeneous time-dependent source is considered: performing the \(q\)-integration puts the conductivity in the homogeneity limit – ruled by the anomaly –, so one gets the standard CME current. On the other hand, performing the \(r\)-integration first is equivalent by taking the static limit first. No matter of the \(q\)-dependence, the transported charge is zero in this case because of the dipolar structure of the integrated conductivity, as we mentioned above. The ambiguity boils down to the fact that we have to deal with the different orders of limits of \(q \to 0\) and \(q_0 \to 0\) when computing the transported charge of a homogeneous source. So in that sense \(\Delta Q\) for homogeneous \(A_{5,0}\) is ill-defined.

In order to give meaningful physical interpretation, let us consider the following different situations. In the case when the observation time is way much longer than the lifetime of the source – no matter its spatial structure –: taking \(q_0 \to 0\) first is justified, so \(\Delta Q\) vanishes.

The opposite order of limits is a good approximation only if the source is homogeneous throughout the whole time-evolution – in that case taking \(q \to 0\) first is justified and the charge transport is given by the naive CME expression in Eq. (11). But this scenario is rather unphysical when the observation time is very long: eventually the source \(A_{5,0}\) has to have boundaries in space and/or time. So to relax the ambiguity, one should abandon the infinite-time integration in \(\Delta Q\) and integrate only over a finite period while \(q \to 0\) is a good approximation.

Let us demonstrate this phenomenon by using a simple toy-model within which the current is induced by the axial imbalance profile:
\[
\overline{A}_{5,0}(t, q) = \frac{n_s R^3}{\pi^2} e^{-\frac{q^2}{\tau^2} - \frac{q\cdot\vec{R}}{\tau}}, \tag{54}\]

characterized by its spatial size \(R\) and its lifetime \(\tau\). The constants \(n_s\) and \(\chi\) are the axial charge density and susceptibility, respectively. The above ansatz is meant only to show how the different scales interplay. One needs to investigate further the actual underlying cause of the axial imbalance in order to give a realistic description of the system at hand. Now we also factor in the finite time window of observation, \(t_{\text{obs}}\):
\[
\Delta Q(t_{\text{obs}}) = \int_{-t_{\text{obs}}/2}^{t_{\text{obs}}/2} d\tau \int d^2r B_0 \cdot J(t, r). \tag{55}\]

The infinite size of the plane \(S\) again simplifies the result, making the finite temperature contribution vanish. The integration can be carried out analytically with the technical details and the lengthy formula of \(\Delta Q\) presented in Appendix G2. As shown there, the scaled charge current \(\Delta Q_t\) with
\[
C = \frac{\Delta q}{\Delta t} BA \text{ and } A \text{ being the size of the plane is a function of the dimensionless observation time } \frac{t_{\text{obs}}}{\Delta q}, \text{ and dimensionless extension of the axial imbalance } \rho = \frac{\Delta q}{t_{\text{obs}}}, \text{ rendering } \Delta Q \text{ to vanish. Therefore the previous statement on the vanishing of the long-time transported charge } \Delta Q \text{ is generalized beyond the weak-coupling approximation.} \]
\[
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We conclude this section by emphasizing again that in order to say any conclusive about a physically realistic situation, one needs to know the actual evolution of the axial imbalance. As we have seen above, depending on the various scales of the
non-equilibrium system, the outcome can be vastly different, interpolating between the two limiting behaviors of the CME conductivity discussed in Sec. IIIA. All of these concerns point us to the need of dynamically more detailed and realistic simulations of the anomalous transport in QGP, in order to capture a real CME signal.

\[ \frac{\Delta Q}{C \tau} \]

**FIG. 3:** Behavior of the transported charge \( \Delta Q \) over the source lifetime \( \tau \). \( I_{\text{obs.}} \) dependence of \( \frac{\Delta Q}{C \tau} \) for fixed \( \tau = 1.0 \) and varying \( R/\tau = \{0.2, 0.5, 1.0, 2.0, 3.0, 5.0, 10.0, 100.0\} \): lower value leads the response to vanish faster and in general reach smaller maximum value. For large enough \( R \) we see the homogeneous limit behavior. \( C = \frac{e^2}{2 \pi \hbar} BA \)

\[ I = -\oint_{\text{boundary}} d^2 S \cdot B \frac{e^2}{2 \pi \hbar} A_{5,0} \]

\[ \frac{I}{C} \]

V. DISCUSSION AND OUTLOOK

In this paper we analyzed the chiral magnetic current in constant magnetic field but with an arbitrary axial charge imbalance. In the non-interacting approximation we derived the explicit form of the real-time response function in Eq. (45), which interpolates between the anomaly ruled CME current like in Eq. (1) and zero current at equilibrium, depending on the spatial pattern and time-dependence of the axial imbalance field \( A_{5,0} = -\mu_5 \). Then we explored the consequences of Eq. (45) for different profiles of \( A_{5,0} \). The observation that the static chiral imbalance results zero response current in the homogeneous limit shows the inherent non-equilibrium nature of the CME, as was already pointed out by others [21, 55]. Coming to the phenomenological implications, we computed the electric current through the plane perpendicular to the constant magnetic field. For a localized axial imbalance, we found that the total electric charge transported through the plane over a long time vanishes because of the dipolar spatial structure of the time-integrated CME conductivity \( \sigma_5(t, r) \), Eq. (37), rendering the CME signal to be captured transient in this case. Using a simple impulse-like profile for \( \mu_5 \), we showed that for an axial imbalance source with large enough spatial extension \( R \), the nonzero transported charges persists for a time scale comparable to \( R \). In case when this characteristic timescale is much larger than the observation time, the magnitude of the charge transport is effectively described by Eq. (1).

It is also worthwhile to mention that we derived corrections to the homogeneous electric current, which carries structures sensitive to vorticity. Further investigation is needed to decide how this might change the collective behavior of the chiral plasma.

An important lesson we learned is the role of the spatial variation in the axial imbalance, reflected in the gradients of \( A_{5,0} \), which has not been sufficiently addressed in previous works. In the case of homogeneous \( A_{5,0} \) – which can even be time-dependent – the UV regularization appears optional since it contributes to the effective action only as a total divergence. Here we emphasized that in the presence of a nonzero \( \nabla A_{5,0} \), UV regularization is necessary to maintain the electric charge conservation, i.e. \( \partial \cdot J = 0 \). Without proper UV regulator, the term causing trouble is \( \partial \cdot J = -\frac{e^2}{2 \pi \hbar} \nabla A_{5,0} \cdot B \). For any realistic system there is a boundary where \( A_{5,0} \) changes – most probably vanishes. If there are EM fields still present around this region, a current

\[ I = -\int_{\text{boundary}} d^2 S \cdot B \frac{e^2}{2 \pi \hbar} A_{5,0} \]

is left to be canceled: this is what the UV-term is responsible for in the presented approach. The issue of EM gauge invariance was of course well-recognized in the literature before, see Secs. 2.2–2.4 of Ref. [13], for example. The tool to maintain it even in the case of axial anomaly is to add the so-called Bardeen counterterms or Chern-Simons term [23, 24]: this is what our fermionic effective action realizes via the PV regulator.

In the weak coupling limit, any time variation of external sources will drive the system out of equilibrium. Therefore the calculation presented in this work approximates the situation where the characteristic time for the variation of the axial imbalance is shorter than the relaxation time to equilibrium, opposite to the condition assumed in the hydrodynamic regime. There, it is still justified to use Eq. (1) – in the zeroth order of the gradient expansion. As the result of the two orders of limits persists to higher orders in the coupling, the qualitative aspect of our results, say the quenching of the charge transport over a long time may be carried over to the strong coupling regime. An important limitation is our simple-minded assumption which models a non-equilibrium axial imbalance by a spacetime dependent axial chemical potential. A more realistic approach without using the notion of chemical potential is to factor in the real time QCD process of the axial charge creation/annihilation inspired by instantons or sphalerons. We hope to be able to report our progress along this line in near future.

ACKNOWLEDGMENTS

D. F. H. and H. C. R. were supported by the Ministry of Science and Technology of China (MSTC) under the “973” Project No. 2015CB856904(4) and by the NSFC under Grant Nos. 11735007 and 11875178. M. H. was supported by NSFC under Grant No. 11847242. J. L. is supported by the NSF Grant No. PHY-1913729.
Appendix A: Anomalous Ward-identity

In this Appendix we show that for homogeneous chiral imbalance the $\bar{F}_{AVV}^{ij}$ part of the AVV vertex is completely determined by the axial anomaly. For this we first move $\gamma^i$ into the front in Eqs. (13). This is done by utilizing the Dirac-structure of the propagators, i.e. $\gamma^i \bar{G} + G \gamma^i = g$, where $g$ is a scalar function. Now we can group the terms either as type $GGG$ or $GGg$:

$$\{ GGG \} =$$

$$= \frac{i e^2}{2} \int_p \{ \gamma^i \gamma^j ((p + q_2) \gamma^0 G^A(p) \gamma^i G^A(p-q_1) +$$

$$+ \gamma^i \gamma^j G^R(p + q_2) \gamma^0 G^C(p) \gamma^i G^A(p-q_1) +$$

$$+ \gamma^i \gamma^j G^R(p + q_2) \gamma^0 G^R(p) \gamma^i G^C(p-q_1) +$$

$$+ \gamma^i \gamma^j G^C(p + q_1) \gamma^i G^A(p) \gamma^0 G^A(p-q_2) +$$

$$+ \gamma^i \gamma^j G^R(p + q_1) \gamma^i G^C(p) \gamma^0 G^A(p-q_2) +$$

$$+ \gamma^i \gamma^j G^R(p + q_1) \gamma^i G^R(p) \gamma^0 G^C(p-q_2) \} .$$

(A1)

In the next step we set $q_2 = 0$ and utilize Eqs. (17) and (19) and arrive at:

$$\{ GGG \} = \frac{i e^2}{2} \cdot \frac{1}{Q_{20}} \times$$

$$\int_p \{ \gamma^i \gamma^j (-G^C(p_0 + q_20, p)) \gamma^i G^A(p - q_1) +$$

$$+ \gamma^i \gamma^j G^R(p_0, p) \gamma^i G^A(p - q_1) +$$

$$+ \gamma^i \gamma^j (-G^R(p_0 + q_20, p)) \gamma^i G^C(p - q_1) +$$

$$+ \gamma^i \gamma^j G^R(p + q_1) \gamma^j (-G^C(p_0 - q_20, p)) +$$

$$+ \gamma^i \gamma^j G^R(p + q_1) \gamma^j G^R(p_0 - q_20, p)) \} .$$

(A7)

For the vertex function we have the difference of these terms and their UV-limit provided by the PV-terms. This difference is finite, since any dangerous UV behavior is canceled. Therefore one can shift the integration variables in Eqs. (A7) and realize that $\{ GGG \} - \{ GGG \}_{P_{V}} = 0$.

For the rest, we are interested only in the chiral limit. Then since $g_{\mu
\nu} = 0$, only the PV-terms contribute. The mass scale $M$ in these terms are larger than any other scale in the system. This allows us to replace the fermionic propagators with the non-interacting ones, which also leaves us with $g_{P_{V}}(p) = \frac{2M}{p^2 - M^2}$. One can then make the observation that $q_2$ enters only with $g$, so only in the denominators. Now for the large $M$ limit it is justified to keep $q_1$ only in the nominator, i.e. where it contributes to the spinor structure:

$$\bar{F}_{AVV}^{0ij}(q_1, q_20, 0) = \frac{i e^2}{2} \int_p \{ \gamma^i \gamma^j g_{P_{V}}(p) \gamma^0 G_{P_{V}}(p) \gamma^i G_{P_{V}}(p - q_1) +$$

$$+ \gamma^i \gamma^j G_{P_{V}}(p + q_1) \gamma^j G_{P_{V}}(p - q_1) - (A \leftrightarrow R) \} (1 - 2\bar{n}(p_0)).$$

(A13)

We can perform the trace and combine the $A$ and $R$ pieces together. Since the only spatial structure is in $\{ \gamma^i \gamma^j \gamma^0 \gamma^j \}$, also the directional integration of $p$ can be done. The result is the following:

$$\bar{F}_{AVV}^{0ij}(q_1, q_20, 0) = - \frac{1}{16\pi^4} \frac{i e^2}{2} \int_{-\infty}^{\infty} dp_0 \int_0^\infty d p^2 \frac{1 - 2\bar{n}(p_0)}{[(p_0 - i0^+)^2 - p^2 - M^2]^3} =$$

$$= \frac{4e^2}{\pi^2} \epsilon^{0ijk} q_{1k} M^2 \int_{-\infty}^{\infty} dp_0 (1 - 2\bar{n}(p_0)) \int_0^\infty d p^2 \frac{1}{8p_0 p} \frac{\partial}{\partial p_0} \frac{\delta (p_0^2 - p^2 - M^2)}{M^2} =$$

$$= - \frac{e^2}{2\pi^2} \epsilon^{0ijk} q_{1k} M^2 \int_{-\infty}^{\infty} dp_0 \frac{\partial}{\partial p_0} \frac{1 - 2\bar{n}(p_0)}{p_0} \int_0^\infty d p \text{sign}(p_0) \delta (p_0^2 - p^2 - M^2) =$$

$$= \frac{e^2}{2\pi^2} \epsilon^{0ijk} q_{1k} M^2 \int_0^\infty d p \left( \frac{2\bar{n}(\sqrt{p^2 + M^2})}{p^2 + M^2} + \frac{1 - 2\bar{n}(\sqrt{p^2 + M^2})}{(p^2 + M^2)^{3/2}} \right) \frac{M}{M^4} \frac{e^2}{2\pi^2} \epsilon^{0ijk} q_{1k} + O(M^{-1}).$$

(A17)
Putting the above expression of Eq. (A17) back into Eq. (7) we arrive at the familiar result: the equilibrium CME current of Eq. (1) – with the substitution of $A_{5,0} = -\mu_s$. Although $A_{5,0}$ depends only on time, the magnetic field still can be arbitrary.

Essentially the same argument leads to the anomalous Ward-identity of the vertex function, shown in Eq. (15): for the contraction of $q_{\mu}^2 T_{A\nu}^{\mu
u}$, one moves the $\gamma^5$ into the front under the trace, then through the same steps as in Eqs. (A7–A12) shows that under the integration the regulated version of the expression vanishes – this time even without the assumption of $q_2 = 0$. So the remaining terms are again the PV ones, leading to $q_{\mu}^2 \epsilon^{\mu\nu\rho\sigma} q_{\nu\rho}$, i.e. Eq. (16).

Appendix B: Relation to the Maxwell-Chern-Simons Electrodynamics

Let us briefly return to the phenomenology of the QCD matter. We already mentioned that the local $\mathbb{CP}$–violation is encoded in $A_5$. According to the anomalous Ward-identity, this contributes to $\theta \cdot J_5$ by a source term $\frac{\epsilon^2}{16\pi^2} E_5 \cdot B_5$. We also know, that the gauge sector of QCD has its contribution to the balance equation as $\frac{N_c}{2} \epsilon_\mu E_\mu \cdot B^\mu$. Now we assume that the gluon sector affects the EM transport through the axial imbalance, but there is no back-reaction. After the dynamics of the gauge fields is integrated out an effective action like in Eq. (4) should emerge. Although we do not know how the quark-gluon vertices map into the axial gauge fields, we assume the matching of the previously mentioned sources for $\theta \cdot J_5$.

Setting the fully dynamical origin of $A_5$ aside, what we know that it originates from the vacuum sectors with non-trivial topology (which can be inhomogeneously distributed in space). A minimal approach to model this is to add a so-called axion term to the original EM Lagrangian:

$$\mathcal{L}_\theta = -\frac{1}{4} F_{\mu
u} F^{\mu\nu} + \overline{\psi} \gamma^\mu (i \partial_\mu - e A_\mu) \psi + \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} \tilde{F}_{\rho\sigma}. \quad (B1)$$

At this point we can impose the anomalous Ward-identity to the system and identify the current in terms of the fermionic fields:

$$\frac{e^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} \tilde{F}_{\rho\sigma} \gamma^5 = \partial_\mu \overline{\psi} \gamma^\mu \gamma^5 \psi, \quad (B2)$$

which after partial integration leads to $\mathcal{L}$, with the axial vector potential $A_{5,\mu} = \partial_\mu \theta$. This special form of $A_5$ renders the electric response to a simple form, solely determined by the anomaly. Using Eq. (7) and Eq. (16), we arrive at the following expressions:

$$J^0 = \frac{e^2}{2\pi^2} \nabla \theta \cdot B, \quad (B3)$$

$$J = \frac{e^2}{2\pi^2} \partial \mu B, \quad (B4)$$

which are the well-known equations of motion of the Maxwell-Chern-Simons electrodynamics in the special case of constant $B$ and vanishing $E$. The above result is in agreement with the analysis of the vertex function which has led to Eq. (21), as $\theta = A_{5,0}$. As we pointed out earlier, the vector current expression differs by its sign from Eq. (1) if one identifies the temporal component of the axial field by $-\mu_s$. One should, however, keep in mind that it is not required for the system to be in thermal equilibrium. This simple form of the electric current is the consequence of the anomalous Ward-identity at the level of the vertex function, i.e. Eq. (16), therefore it is not sensitive to the details of the underlying fermionic dynamics in this case. Although this statement remains true even if dynamical EM fields are present, the axial current is not tied to the anomaly anymore. As can be seen from Eq. (6), the first and third terms vanish if $A_5$ is a pure gradient, the second term is sensitive to the EM-fields only. Therefore the IR behavior of the AVV vertex becomes important for $J_5$. But since $F_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a \equiv 0$ for $A_\mu^a = \partial_\mu \theta$, there is no chiral charge generation. So if $E \cdot B = 0$ and initially $Q_5$ is zero, there is still a CME-like current! This might seem troubling, however, one quickly realizes that $L_\theta$ is actually not the system we are interested in! QCD has a $\theta$-term for the gluonic sector. The effective action for the EM sector, only indicating the $GG$ part of the gluon field strength, looks like this:

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \gamma^\mu (i \partial_\mu - e A_\mu) \psi + \frac{e^2}{32\pi^2} 8 G_{\mu\nu}^a \tilde{G}_{\nu}^{\mu a}. \quad (B5)$$

Now, Eq. (B2) is not the right anomaly relation for QCD. Instead, one has

$$\frac{e^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + \frac{e^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} G_{a,\mu\nu} G_{a,\rho\sigma} = \partial_\mu F^\mu_5 = \partial_\mu \overline{\psi} \gamma^\mu \gamma^5 \psi, \quad (B6)$$

which leaves us not only with $\mathcal{L}$, rather with $\mathcal{L} - \frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$, with the so-called Chern-Simons current $J_{\text{CS}}^\mu = \epsilon^{\mu\nu\rho\sigma} F_{\nu\rho}$. It is straightforward to show, that the same components of $J$ generated from $J_{\text{CS}}$ as in Eqs. (B3–B4), but with opposite sign. That is, the CME-like current vanishes! Our intuition therefore restored, there is no CME with zero $Q_5$.

What we can conclude is that the simplest way of taking topological effects into account, namely by adding the $\theta$-term, is not sufficient. The reason is that a pure gradient axial gauge field does not contribute to the chiral charge balance equation. Nevertheless, a possible $\theta$–term still can cause fluctuations both in the vector and the axial currents.
Appendix C: Trace calculation

In this Appendix we give the details of calculating the trace \( \text{tr} \left\{ \gamma^5 \gamma^i G(p+q) \gamma^0 \gamma^5 G(p) \gamma^i \frac{\partial}{\partial q} G(p) \right\} e^{iB_l} \) for non-interacting fermions. Using the explicit form of the propagator, the trace expression can be written like this:

\[
\mathcal{A} = \text{tr} \left\{ \gamma^i (\slashed{p} + \slashed{q} + \slashed{m}) \gamma^0 \gamma^5 (\slashed{p} + \slashed{m}) \gamma^i \frac{\partial}{\partial q} \frac{p^2 - m^2}{p^2} \right\} e^{iB_l} = \frac{1}{p^2 - m^2} e^{iB_l} \text{tr} \left\{ \gamma^i (\slashed{p} + \slashed{q} + \slashed{m}) \gamma^0 \gamma^5 (\slashed{p} + \slashed{m}) \frac{\gamma_i \gamma_k - \gamma_k \gamma_i}{2} \right\} + (C1)
\]

\[
= \mathcal{A} \quad \text{and} \quad 2 \right) \frac{(p^2 - m^2)^2}{(p^2 - m^2)^2} \text{tr} \left\{ \gamma^i (\slashed{p} + \slashed{q} + \slashed{m}) \gamma^0 \gamma^5 (\slashed{p} + \slashed{m}) \right\} (p \times B)^i (\slashed{p} + \slashed{m}) \) \right\}.
\]

(C2)

The detailed evaluation of term \( I \):

\[
I = 2 \text{tr} \left\{ \gamma^i (\slashed{p} + \slashed{q} + \slashed{m}) \gamma^0 \gamma^5 (\slashed{p} + \slashed{m}) \gamma^i \gamma^5 \right\} B^i = 2 \text{tr} \left\{ \gamma^i (\slashed{p} + \slashed{q} + \slashed{m}) \gamma^0 \gamma_5 (\slashed{p} + \slashed{m}) \right\} B^i = (C3)
\]

\[
= 2i (\slashed{p}^2 - \slashed{m}^2) \text{tr} \left\{ \gamma^0 \gamma^5 \gamma^i \right\} = 2i (\slashed{p}^2 - \slashed{m}^2) \text{tr} \left\{ \gamma^i \right\} = (C4)
\]

\[
= 8i \left[ m^2 \eta^B B^i - B^i \left( p^0 + q \right) \gamma^0 \eta^p \eta^q \right] = 8i \left[ m^2 - (p^0 + q) \right] \left[ p^i \right] B \cdot (p + q) = (C6)
\]

Evaluating term \( II \):

\[
II = (p \times B)^i \text{tr} \left\{ \gamma^i (\slashed{p} + \slashed{q} + \slashed{m}) \gamma^0 \gamma^5 (\slashed{p} + \slashed{m}) \gamma^i \right\} = (C7)
\]

\[
= (p \times B)^i \text{tr} \left\{ \gamma^i (\slashed{p} + \slashed{q} + \slashed{m}) \gamma^0 \gamma^5 (\slashed{p} + \slashed{m}) \right\} = (C8)
\]

\[
= - \left( p^2 - m^2 \right) \text{tr} \left\{ \gamma^i (\slashed{p} + \slashed{q} + \slashed{m}) \gamma^0 \gamma^5 \right\} \left( p \times B \right)^i = (C9)
\]

\[
= 4i (p^2 - m^2) \left[ (p + q) \times B \right] = 4i (p^2 - m^2) \left[ p^i B \cdot (p + q) - B^i (p + q) \cdot p \right].
\]

(C10)

Finally, putting the contributions together we get:

\[
\mathcal{A} = \frac{-8i}{p^2 - m^2} \left[ B^i \left( m^2 - p_0 (p_0 + q_0) \right) - (p^i + q^i) B \cdot p \right].
\]

(C11)

Appendix D: Constant B calculation

Here we give the detailed calculations leading to the conductivity \( \sigma_A(t, q) \) in Eq. (35).

\section{1. Manipulations of \( \frac{\partial \ddot{q}_{ij}}{\partial q_{1k}} \)}

First the trace has been calculated. Then we regrouped the terms of Eqs. (271, 272) in a combinations of AAA + RAA and \( RRR + RAA \). After recasting the higher order pole-contributions as derivatives and also changing integration variables so one can separate \( (1 - 2\ddot{q}(p_0)) \frac{\partial \ddot{q}(p_0)}{\partial q_{1k}} \), the resulting expression is:

\[
\frac{\partial \ddot{q}_{ij}}{\partial q_{1k}} e^{iB_l} = -16\pi e^2 \int_0 \left( 1 - 2\ddot{q}(p_0) \right) \text{sgn}(p_0) \frac{1}{2|p|} \times
\]

\[
\int \left[ \left( B^i (m^2 - p_0 (p_0 + q_0)) - p^i B \cdot p \right) \frac{\partial}{\partial \ddot{q}_{ij}} \delta \left( p_0^2 - p^2 - m^2 \right) + \frac{1}{(p_0 + q_0 + i0^+)^2 - (p + q)^2 - m^2} \right] \frac{\partial}{\partial \ddot{q}_{ij}} \delta \left( p_0^2 - p^2 - m^2 \right) \right] +
\]

\[
- \frac{p^i q^j}{1 + \frac{q^i q^j}{p^i p^j}} \left( B^i (m^2 - p_0 (p_0 + q_0)) - p^i B \cdot (p + q) \right) \frac{\partial}{\partial \ddot{q}_{ij}} \left( \frac{1}{(p_0 + q_0 + i0^+)^2 - (p + q)^2 - m^2} \right) \delta \left( p_0^2 - p^2 - m^2 \right)
\]

(D1)

\[
- \frac{p^i q^j}{1 + \frac{q^i q^j}{p^i p^j}} \left( B^i (m^2 - p_0 (p_0 + q_0)) - p^i B \cdot (p + q) \right) \frac{\partial}{\partial \ddot{q}_{ij}} \left( \frac{1}{(p_0 + q_0 + i0^+)^2 - (p + q)^2 - m^2} \right) \delta \left( p_0^2 - p^2 - m^2 \right)
\]

(D2)

\[
- \frac{p^i q^j}{1 + \frac{q^i q^j}{p^i p^j}} \left( B^i (m^2 - p_0 (p_0 + q_0)) - p^i B \cdot (p + q) \right) \frac{\partial}{\partial \ddot{q}_{ij}} \left( \frac{1}{(p_0 + q_0 + i0^+)^2 - (p + q)^2 - m^2} \right) \delta \left( p_0^2 - p^2 - m^2 \right)
\]

(D3)
Eq. (31) is the direct consequence of the above. We proceed by simplifying the angular integration by separating \( B \) into components parallel and perpendicular to \( \mathbf{q} \):

\[
\frac{\partial \delta_{ij}}{\partial q_{1k}} B_l = \frac{e^2}{\pi^2 B_l \int_0^\infty dp \int_{-1}^1 dx} \left[ \delta (p_0^2 - p^2 - m^2) \right] \delta (p_0^2 - p^2 - m^2) + 2qpx \frac{(m^2 - p_0(p_0 + q_0) - px \delta)}{[(p_0 + q_0 + i0^+)^2 - p^2 - q^2 - 2qpx - m^2]^2} \delta (p_0^2 - p^2 - m^2)
\]

(D4)

Regrouping terms leads to:

\[
\frac{\partial \delta_{ij}}{\partial q_{1k}} e^{ij} B_l = -\frac{e^2}{\pi^2} \int_0^\infty dp \int_{-1}^1 dx \left[ \delta (p_0^2 - p^2 - m^2) \right] \delta (p_0^2 - p^2 - m^2) + 2qpx \frac{(m^2 - p_0(p_0 + q_0) - px \delta)}{[(p_0 + q_0 + i0^+)^2 - p^2 - q^2 - 2qpx - m^2]^2} \delta (p_0^2 - p^2 - m^2)
\]

(D5)

which gives us Eq. (32) after part of the expression is written as a total derivative with respect to \( x \).

2. Derivation of Eq. (33)

The Fourier transformation of Eq. (32) with respect to \( q_0 \) can be calculated readily:

\[
\int_{-\infty}^{\infty} dq_0 e^{iq_0t} \frac{\partial \delta_{ij}}{\partial q_{1k}} B_l = \theta(-t) \frac{e^2}{\pi^2} \int_0^\infty dp \int_{-1}^1 dx \left[ \delta (p_0^2 - p^2 - m^2) \right] \delta (p_0^2 - p^2 - m^2) + 2qpx \frac{(m^2 - p_0(p_0 + q_0) - px \delta)}{[(p_0 + q_0 + i0^+)^2 - p^2 - q^2 - 2qpx - m^2]^2} \delta (p_0^2 - p^2 - m^2)
\]

(D6)

\[
\int_{-\infty}^{\infty} dq_0 e^{iq_0t} \frac{\partial \delta_{ij}}{\partial q_{1k}} B_l = \theta(-t) \frac{e^2}{\pi^2} \int_0^\infty dp \int_{-1}^1 dx \left[ \delta (p_0^2 - p^2 - m^2) \right] \delta (p_0^2 - p^2 - m^2) + 2qpx \frac{(m^2 - p_0(p_0 + q_0) - px \delta)}{[(p_0 + q_0 + i0^+)^2 - p^2 - q^2 - 2qpx - m^2]^2} \delta (p_0^2 - p^2 - m^2)
\]

(D7)

\[
\int_{-\infty}^{\infty} dq_0 e^{iq_0t} \frac{\partial \delta_{ij}}{\partial q_{1k}} B_l = \theta(-t) \frac{e^2}{\pi^2} \int_0^\infty dp \int_{-1}^1 dx \left[ \delta (p_0^2 - p^2 - m^2) \right] \delta (p_0^2 - p^2 - m^2) + 2qpx \frac{(m^2 - p_0(p_0 + q_0) - px \delta)}{[(p_0 + q_0 + i0^+)^2 - p^2 - q^2 - 2qpx - m^2]^2} \delta (p_0^2 - p^2 - m^2)
\]

(D8)
After the $p_0$- and $x$-integrations we get:

$$\# = \int_{-\infty}^{\infty} dq_0 e^{i q_0 t} \frac{\partial H_{M \rightarrow 0}}{\partial q_{1k}} e^{i k} B_l =$$

$$= \theta(-t) \frac{e^2}{\pi^2} \int_0^{\infty} dp \frac{1 - 2 \tilde{n}(\Omega)}{2 \Omega} \times$$

$$\times \left\{ B_{\parallel} \left( 2 \cos(\Omega t) \right) \left[ \frac{(\Omega^2 - 2 p^2 - q p) \sin \left( t \sqrt{(p + q)^2 + m^2} \right) + (\Omega^2 - 2 p^2 + q p) \sin \left( t \sqrt{(p - q)^2 + m^2} \right)}{\sqrt{(p + q)^2 + m^2}} \right] +$$

$$+ 2 \Omega \sin(\Omega t) \left[ \cos \left( t \sqrt{(p + q)^2 + m^2} \right) + \cos \left( t \sqrt{(p - q)^2 + m^2} \right) \right] \right\}$$

$$+ B_{\perp} \left( 2 m^2 \cos(\Omega t) \right) \left[ \frac{\sin \left( t \sqrt{(p + q)^2 + m^2} \right) + \sin \left( t \sqrt{(p - q)^2 + m^2} \right)}{\sqrt{(p + q)^2 + m^2}} \right] +$$

$$+ \left( 2 \Omega \sin(\Omega t) + \frac{2 p \cos(\Omega t)}{q t} \right) \cos \left( t \sqrt{(p + q)^2 + m^2} \right) + \left( 2 \Omega \sin(\Omega t) - \frac{2 p \cos(\Omega t)}{q t} \right) \cos \left( t \sqrt{(p - q)^2 + m^2} \right) \right\},$$

where $\Omega = \sqrt{p^2 + m^2}$. Then we take the chiral limit and collect terms carefully. Only a $B_{\perp}$ contribution remains:

$$\# = \theta(-t) \frac{e^2}{\pi^2} \int_0^{\infty} dp \left( 1 - 2 \tilde{n}(p) \right) \left\{ B_{\parallel} \left[ - \cos(pt) \left( \sin(t(p + q)) + \sin(t(p - q)) \right) + \sin(pt) \left( \cos(t(p + q)) + \cos(t(p - q)) \right) \right] +$$

$$+ B_{\perp} \left[ \left( \sin(pt) + \frac{\cos(pt)}{q t} \right) \cos(t(p + q)) \right] + \left( \sin(pt) - \frac{\cos(pt)}{q t} \right) \cos(t(p - q)) \right] \right\}$$

$$= \theta(-t) \frac{e^2}{\pi^2} B_{\parallel} \int_0^{\infty} dp \left( 1 - 2 \tilde{n}(p) \right) \left[ \sin(pt) 2 \cos(pt) \cos(qt) + \frac{1}{q t} \cos(pt)(-2) \sin(pt) \sin(qt) \right] =$$

$$= \theta(-t) \frac{e^2}{\pi^2} B_{\parallel} \left[ \sin(qt) \frac{\cos(qt)}{q t} \right] T \int_0^{\infty} dy \left( 1 - 2 n_{FD}(y) \right) \sin(2y Tr),$$

resulting in Eq. (33) in the end.

3. Derivation of Eq. (34)

Scaling the loop momentum $p$ of Eq. (32) by $p = M y$ and taking the limit $\rightarrow 0$, we find that:

$$\frac{\partial H_{M \rightarrow \infty}}{\partial q_{1k}} e^{i k} B_l \approx - \frac{e^2}{\pi^2} \int_0^{\infty} dy \left( \delta(y_0 - \sqrt{y^2 + 1}) + \delta(y_0 + \sqrt{y^2 + 1}) \right) \frac{1}{2} \int_{-\infty}^{\infty} dx$$

$$= \left\{ \frac{\partial}{\partial x} \left( B_{\parallel} + B_{\perp} \right) \right\} \frac{q_0}{q_0 - y \omega_0} - i \omega_0 + B_{\perp} \left( \frac{q y x}{y_0} + \frac{1}{q y_0 - \frac{y \omega_0}{y_0} + i \omega_0} \right) \right\} +$$

$$- B_{\parallel} \frac{q_0}{q y_0} \frac{q y_0}{q y_0 - \frac{y \omega_0}{y_0} + i \omega_0} \right\} =: @$$

Then comes the Fourier transform and the leftover integration. Step-by-step it is done as follows:
where the frequency \( \omega \) is the Fourier transform of the charge density \( \sigma \).

After collecting terms, Eq. (34) follows.

Performing a Fourier transformation on Eq. (35) with respect to time, one obtains the frequency-momentum representation of the response function

\[
\int_{-\infty}^{\infty} dq_0 e^{i\omega t} \frac{\partial}{\partial t} \left[ B_0 B_1 \frac{\partial}{\partial x} \left( -\omega^2 \sigma + \frac{1}{2} n \right) \right] = 
\]

\[
\left( \frac{\partial^2}{\partial t^2} + q^2 \right) \left( -i\theta(-t)e^{i\omega t} \right) = \int_{-\infty}^{\infty} dx \frac{\partial}{\partial x} \left[ \left( \frac{\partial^2}{\partial t^2} + q^2 \right) \left( -i\theta(-t)e^{i\omega t} \right) \right]
\]

\( (D25) \)

At this point one has to account for the \( t \)-derivatives in the integrand:

\[
[Dots] = -B_0 \left\{ \left( \frac{1}{2} - y_0^2 \right) \delta(t) + \theta(-t) \left[ \left( \frac{1}{2} - y_0^2 \right) q_0 \sin \left( \frac{y_0 q t}{y_0} \right) - 2q^2 \frac{y_0^2}{y_0^2} \cos \left( \frac{y_0 q t}{y_0^2} \right) \right] \right\} +
\]

\[
-D_1 \left\{ \left( \frac{1}{2} - y_0^2 \right) \delta(t) + \theta(-t) \left[ \left( \frac{1}{2} - y_0^2 \right) q_0 \sin \left( \frac{y_0 q t}{y_0^2} \right) - 2q^2 \frac{y_0^2}{y_0^2} \cos \left( \frac{y_0 q t}{y_0^2} \right) \right] \right\},
\]

\( (D27) \)

which we put back into the Eq. \( (D25) \) to gain the expression

\[
\int_{-\infty}^{\infty} dq_0 e^{i\omega t} \frac{\partial}{\partial t} \left[ B_0 B_1 \frac{\partial}{\partial x} \left( -\omega^2 \sigma + \frac{1}{2} n \right) \right] = 
\]

\[
\left[ B_0 \left( \frac{1}{2} - y_0^2 \right) \delta(t) + \theta(-t) \left[ \left( \frac{1}{2} - y_0^2 \right) q_0 \sin \left( \frac{y_0 q t}{y_0^2} \right) - 2q^2 \frac{y_0^2}{y_0^2} \cos \left( \frac{y_0 q t}{y_0^2} \right) \right] \right] +
\]

\[
-D_1 \left[ \left( \frac{1}{2} - y_0^2 \right) \delta(t) + \theta(-t) \left[ \left( \frac{1}{2} - y_0^2 \right) q_0 \sin \left( \frac{y_0 q t}{y_0^2} \right) - 2q^2 \frac{y_0^2}{y_0^2} \cos \left( \frac{y_0 q t}{y_0^2} \right) \right] \right] =
\]

\[
\int_{-\infty}^{\infty} dq_0 e^{i\omega t} \frac{\partial}{\partial t} \left[ B_0 B_1 \frac{\partial}{\partial x} \left( -\omega^2 \sigma + \frac{1}{2} n \right) \right] = 
\]

\[
\left( \frac{1}{2} - y_0^2 \right) \delta(t) + \theta(-t) \left[ \left( \frac{1}{2} - y_0^2 \right) q_0 \sin \left( \frac{y_0 q t}{y_0^2} \right) - 2q^2 \frac{y_0^2}{y_0^2} \cos \left( \frac{y_0 q t}{y_0^2} \right) \right], \quad \text{(D29)}
\]

After collecting terms, Eq. \( (D30) \) follows.

Appendix E: Response functions in frequency-momentum

Performing a Fourier transformation on Eq. \( (D30) \) with respect to time, one obtains the frequency-momentum representation of the response function

\[
\tilde{\sigma}_A(\omega, q) = \int_{-\infty}^{\infty} dt e^{-i\omega t} \tilde{\sigma}_A(t, q) = \frac{e^2}{4\pi^2} \left[ 2B_0^2 + \frac{q^2}{\omega^2 - q^2} \left[ B_0^2 + \tilde{q}^2 (B \cdot \tilde{q}) \right] \right]
\]

\[
+ \int_0^{\infty} dp_{FD} \frac{p}{T} \left[ 2B_0^2 + \frac{q^2}{\omega^2 - q^2} \left[ B_0^2 + \tilde{q}^2 (B \cdot \tilde{q}) \right] \right]
\]

\( (E1) \)

where the frequency \( \omega \) carries an infinitesimal positive imaginary part and the integral representation of \( f(tT) \), Eq. \( (56) \) is employed. The frequency-momentum representation of the electric current reads as

\[
\tilde{J}(\omega, q) = \tilde{\sigma}_A(\omega, q) \tilde{A}_{E_0}(\omega, q).
\]

\( (E2) \)

It follows from the balance equation \( q \cdot \tilde{J} = 0 \) that

\[
\tilde{n}(\omega, q) = \frac{q \cdot \tilde{J}(\omega, q)}{\omega} = \frac{e^2}{2\pi^2} \tilde{A}_{E_0} \frac{\omega}{\omega^2 - q^2} q \cdot B
\]

\( (E3) \)

where \( \tilde{n} \) is the Fourier transform of the charge density \( n \). Interestingly, the resulting expression is temperature independent.
Appendix F: Response functions in coordinate space

It is possible to perform the inverse Fourier transform of Eq. 35 from the spatial momentum $q$ to position $r$ as well. One needs to carefully treat term-by-term the different $q$-contributions:

$$Bq \sin qt, \quad \vec{q}(\vec{B} \cdot \vec{q}) \sin qt,$$

$$B \frac{\partial}{\partial t} \left( \frac{\sin qt}{qt} \right), \quad \vec{q}(\vec{B} \cdot \vec{q}) \frac{\partial}{\partial t} \left( \frac{\sin qt}{qt} \right).$$

After tedious but straightforward calculation, the following expression emerges:

$$J(t, r) = \int d^3\vec{r}' \int_0^\infty dr' \frac{e^2}{2\pi^2} \left\{ BA_{5,0}(t, r' + r) \frac{r'}{4\pi} \left( -\frac{\partial}{\partial r'} \delta(r') \right) +$$

$$+ \frac{B (r')^2}{24\pi^2} \left[ A_{5,0}(t, r + r') \delta(r') + \partial_{3} A_{5,0}(t - r', r + r') + \partial_1 (A_{5,0}(t - r', r + r') \tilde{f}(-r')) \right] +$$

$$+ \frac{B}{2} \left[ \delta(r) A_{5,0}(t, r + r') - \partial_1 A_{5,0}(t - r', r + r') - \frac{A_{5,0}(t - r', r + r') - A_{5,0}(t, r + r')}{r'} \right] +$$

$$+ \frac{\vec{r}'(\vec{B} \cdot \vec{r}')}{2} \frac{1}{4\pi} \left[ \frac{r'}{r'} \delta(r') A_{5,0}(t, r + r') + r' \partial_{3} A_{5,0}(t - r', r + r') - 3\delta(r') A_{5,0}(t, r + r') + 3\partial_1 A_{5,0}(t - r', r + r') +$$

$$+ 3A_{5,0}(t - r', r + r') - A_{5,0}(t, r + r') - \partial_1 (A_{5,0}(t - r', r + r') \tilde{f}(-r')) \right] \right\}, \quad (F1)$$

where $\tilde{f}(x) = f(xT)$. This can be casted in a more compact form by collecting terms into the following groups:

$$J(t, r) = \frac{e^2}{2\pi^2} \left\{ BA_{5,0}(t, r) - \frac{2}{3} BA_{5,0}(t, r) +$$

$$+ \frac{1}{8\pi} \int d^3\vec{r} \int_0^\infty dr' \left[ (r' \partial_{3} A_{5,0}(t - r', r + r')) (B + \vec{r}'(B \cdot \vec{r}')) \right] +$$

$$- \left( \partial_1 A_{5,0}(t - r', r + r') + \frac{A_{5,0}(t - r', r + r') - A_{5,0}(t, r + r')}{r'} \right) (B - 3\vec{r}'(B \cdot \vec{r}')) +$$

$$+ \partial_1 \left( A_{5,0}(t - r', r + r') \tilde{f}(-r') \right) (B - \vec{r}'(B \cdot \vec{r}')) \right\]. \quad (F5)$$

1. **CME response to point-like perturbation**

The formula given in Eq. 58 can be better understood via an example. For that, let us suppose $A_{5,0}$ is well-localized in space as $A_{5,0}(t, r) = A_{5,0}(t) \delta(3)(r)$. All the remaining integration can be done with the aid of the delta-function. The result is the following expression:

$$J(t, r) = \frac{e^2}{2\pi^2} \left\{ \frac{1}{3} BA_{5,0}(t) \delta^{(3)}(r) +$$

$$+ \frac{1}{2} \left[ \frac{A_{5,0}(t - r)}{r} (B + (B \cdot \vec{r})\vec{r}) +$$

$$- \left( \frac{A_{5,0}(t - r)}{r^2} + \frac{A_{5,0}(t - r) - A_{5,0}(t)}{r^3} \right) (B - 3(B \cdot \vec{r})\vec{r}) +$$

$$+ \frac{A_{5,0}(t - r)f(rT) - A_{5,0}(t - r)T f'(rT)}{r^2} (B - (B \cdot \vec{r})\vec{r}) \right\}. \quad (F9)$$
There is a contribution centered at the origin, the one third of the naive CME current. The other contributions carry various position dependence, depicted in Fig. 4. It is interesting to observe a stationary source, i.e., $A_{5,0}(t) \equiv A_{5,0}$. In this case the only contribution except the delta-term is a finite temperature one: the last term in the 4th line of Eq. (49). Averaging the current over a small region around the origin we get back the expression we already derived previously in Eq. (49): the long-time limit of the averaged current after a quench.

\[ J = \frac{e^2}{2\pi^2} A_{5,0} \frac{1}{\pi} 3 B \int_0^\infty dQ \frac{Q - \cos Q}{Q} \left( 1 + \int_0^\infty d\tau \frac{\cos \frac{\tau Q}{\alpha}}{\alpha} f(\tau) \right) \equiv I(\alpha). \]  

To evaluate the integral $I(\alpha)$, some of the steps need regularization. If needed, we insert a factor of $e^{-\epsilon Q}$ ($\epsilon > 0$) or the same for $\tau$, and send the regulator $\epsilon$ to zero in the end of the calculation. In this way both integration can be done and leading us to the following simple result:

\[ I(\alpha) = \pi \left( 1 + \int_0^\infty d\tau \frac{\cos \frac{\tau Q}{\alpha}}{\alpha} f(\tau) \right) = \]  

\[ = \frac{\pi}{2} + \int_0^\infty dQ \frac{\sin Q}{Q} \int_0^\infty d\tau \frac{1}{\alpha} \sin \left( \frac{\tau Q}{\alpha} \right) f(\tau) \frac{1}{\alpha} \sin \left( \frac{\tau Q}{\alpha} \right) f(\tau) = \]  

\[ = \frac{\pi}{2} - \frac{1}{2} \int_0^\infty dx f(x) \left( \frac{\epsilon}{(x-1)^2 + \epsilon^2} - \frac{\epsilon}{(x+1)^2 + \epsilon^2} \right) \bigg|_{\epsilon \rightarrow 0} = \frac{\pi}{2} (1 - f(\alpha)). \]  

Returning to the average-current expression:

\[ J = \frac{e^2}{2\pi^2} A_{5,0} B \frac{1 - f(\tau)}{3}. \]  

### Appendix G: Other side-cals

#### 1. Averaged longtime current of quenched local source

Here we detail the steps leading to the final form of Eq. (49).

\[ I(\alpha) = \int_0^\infty dQ \left( -Q \frac{\partial}{\partial Q} \frac{\sin Q}{Q} \right) \left( 1 + \int_0^\infty d\tau \frac{\cos \frac{\tau Q}{\alpha}}{\alpha} f(\tau) \right) = \]  

\[ = \frac{\pi}{2} + \int_0^\infty dQ \frac{\sin Q}{Q} \int_0^\infty d\tau \frac{1}{\alpha} \sin \left( \frac{\tau Q}{\alpha} \right) f(\tau) \frac{1}{\alpha} \sin \left( \frac{\tau Q}{\alpha} \right) f(\tau) = \]  

\[ = \frac{\pi}{2} - \frac{1}{2} \int_0^\infty dx f(x) \left( \frac{\epsilon}{(x-1)^2 + \epsilon^2} - \frac{\epsilon}{(x+1)^2 + \epsilon^2} \right) \bigg|_{\epsilon \rightarrow 0} = \frac{\pi}{2} (1 - f(\alpha)). \]  

#### 2. Charge transported by an axial imbalance impulse

In this section we elaborate on the expression of $\Delta Q$ which results in Fig. 4 in the main text at the end of Sec. IV. We
plane perpendicular to $\mathbf{B}$. This results in the following expression, containing only zero temperature contributions:

$$
\Delta Q(t_{\text{obs}},) =
\int_{-t_{\text{obs}}/2}^{t_{\text{obs}}/2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dq q^2 \overline{A}_{5,0}(t + t', q) \left( \delta(t') + \theta(-t') q \sin(qt') \right)
$$

which also depends on only the dimensionless quantities $t_{\text{obs}}/\tau$ and $\rho$. Its behavior for various values of $R/\tau$ is shown in Fig. 3. The small–$\tau_o$ behavior is worth mentioning:

$$
\frac{\Delta Q}{C_t} (\tau_o, \rho, R = R/\tau) \approx \frac{1}{\sqrt{2\pi}} \frac{\rho^2 (\rho^2 - 1)}{(\rho^2 + 1)^2} \tau_o + O(\tau_o),
$$

so for large $\tau$ (small $\rho$) we again get a similar behavior to that caused by the CME relation Eq. (1): the transported charge grows with $\tau_o$.

Our toy model has three scales: the observation time $t_{\text{obs}}$, the spatial size $R$ and the pulse length $\tau$. As we have already seen before, in the case of large $t_{\text{obs}}$, the transported charge goes to zero for any finite $R$ and $\tau$: $\Delta Q(t_{\text{obs}} \rightarrow \infty, \tau, R) = 0$. This is the result we have already got for $\Delta Q$ defined in Eq. (G7). Now we can plug in the impulse-like pattern of $\overline{A}_{5,0}$ given in Eq. (54), which is normalized as

$$
\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dq q \overline{A}_{5,0}(t, q) \frac{4\pi^2}{\chi}.
$$

We can now compute $\Delta Q(t_{\text{obs}})$.

For a finite length observation one can get the homogeneous source limit by sending $R$ to infinity while keeping $t_{\text{obs}}$ and $\tau$ finite:

$$
\frac{\Delta Q(t_{\text{obs}}, \tau, R \rightarrow \infty)}{\tau} = C \left( \frac{\tau \ln \frac{t_{\text{obs}}}{2\sqrt{2\tau}}}{\tau} \right),
$$

as well as the short pulse limit $\tau \rightarrow 0$ with finite $t_{\text{obs}}$ and $R$:

$$
\frac{\Delta Q(t_{\text{obs}}, \tau, R)}{\tau} \bigg|_{\tau \rightarrow 0} = C e^{-\frac{t_{\text{obs}}}{4R}} \left( 1 - \frac{t_{\text{obs}}^2}{4R^2} \right),
$$

with $C = \frac{\rho^2}{2\pi^2} \overline{A}$. Limiting cases are depicted in Fig. 5.

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[1] D. E. Kharzeev, L. D. McLerran, H. J. Warringa, Nucl. Phys. A 803, 227–253 (2008)
[2] B. I. Abelev et al. (STAR Collaboration), Phys. Rev. Lett. 103, 251601 (2009)
[3] B. I. Abelev et al. (STAR Collaboration), Phys. Rev. C 81, 054908 (2010)
[4] L. Adamczyk et al. (STAR Collaboration), Phys. Rev. C 88, 064911 (2013)
[5] STAR collaboration, arXiv:1906.03373
[6] Q. Li, D. E. Kharzeev, C. Zhang, Y. Huang, I. Pletikosa, A. V. Fedorov, R. D. Zhong, J. A. Schneeloch, G. D. Gu, and T. Valla., Nature Physics, 12, 550-554 (2016).
[7] J. Xiong, S. K. Kushwaha, T. Liang, J. W. Krizan, M. Hirschberger, W. Wang, R. J. Cava, and N. P. Ong. Science, 350, 413-416 (2015)
[8] X. Huang, L. Zhao, Y. Long, P. Wang, D. Chen, Z. Yang, H. Liang, M. Xue, H. Weng, Z. Fang, X. Dai, and G. Chen. Physical Review X, 5, 031023 (2015)
[9] F. Arnold et al., Nature Commun., 7, 11615, (2016)
[10] H. et al. Nature Commun., 7, 10301, 2016.
[11] X. Yang, Y. Liu, Z. Wang, Y. Zheng, and Zhu-an Xu, arXiv:1506.03190
[12] D. E. Kharzeev, J. Liao, S. A. Voloshin, G. Wang, Prog. Part. Nucl. Phys 88, 1 (2016)
[13] K. Landsteiner, Acta Phys. Pol. B 47, 2617 (2016)
[14] K. Fukushima and K. Mameda,
FIG. 5: Behavior of the transported charge $\Delta Q$ over the source lifetime $\tau$. (left) $\Delta Q/\tau$ for small $\tau$, the asymptotic curve (with blue) given by Eq. (G12), and $(\tau,R) = \{0.3,2.0\}$. (right) $\Delta Q/\tau^2$ in the homogeneous source limit $R \to \infty$, given by Eq. (G11). For measurement times larger than $\tau$ there is a saturation to the naive CME transported charge value.