The Strong Homotopy Structure of Poisson Reduction

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Abstract

In this paper we propose a reduction scheme for multivector fields phrased in terms of $L_\infty$-morphisms. Using well-know geometric properties of the reduced manifolds we perform a Taylor expansion of multivector fields, which allows us to built up a suitable deformation retract of DGLA’s. We first obtained an explicit formula for the $L_\infty$-Projection and -Inclusion of generic DGLA retracts. We then applied this formula to the deformation retract that we constructed in the case of multivector fields on reduced manifolds. This allows us to obtain the desired reduction $L_\infty$-morphism. Finally, we perform a comparison with other reduction procedures.

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1 Introduction

This paper aims to propose a reduction scheme for multivector fields that is phrased in terms of $L_\infty$-morphisms and adapted to deformation quantization. Deformation quantization has been introduced in [1] by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer and it relies on the idea that the quantization of a Poisson manifold $M$ is described by a formal deformation of the commutative algebra of smooth complex-valued functions $C_\infty(M)$, a so-called star product. The existence and classification of star products on Poisson manifolds has been provided by Kontsevich’s formality theorem [15], whereas the invariant setting of Lie group actions has been treated by Dolgushev, see [9, 10]. In the last years many developments have been done, see e.g. [3, 4, 16].

More explicitly, the formality provides an $L_\infty$-quasi-isomorphism between the differential graded Lie algebra of multivector fields and the multidifferential operators resp. the invariant versions.

One open question and our main motivation is to investigate the compatibility of deformation quantization and phase space reduction in the Poisson setting.

In the classical setting one considers here the Marsden-Weinstein reduction [18]. Suppose that a Lie group $G$ acts by Poisson diffeomorphisms on the Poisson manifold $M$ and that it allows an $Ad^*$-equivariant momentum map $J: M \rightarrow g^*$ with $0 \in g^*$ as value and regular value, where $g$ is the Lie algebra of $G$. Then $C = J^{-1}(\{0\})$ is a closed embedded submanifold of $M$ and the reduced manifold $M_{red} = C/G$ is again a Poisson manifold if the action on $C$ is proper and free. Reduction theory is very important and it is still very active field of research. Among the others, we mention the categorical reformulation performed in [7].

In the setting of deformation quantization a quantum reduction scheme has been introduced in [2], see also [14] for a slightly different formulation, which allows the study of the compatibility between the reduction scheme and the properties of the star product, as in [11]. One crucial ingredient are quantum momentum maps (see [20]) and pairs consisting of star products with compatible quantum momentum maps are called equivariant star products. For symplectic manifolds these equivariant star products have recently been classified and it has been shown that quantization commutes with reduction, see [22, 24]. More precisely, equivariant star products on $M$ are classified by certain elements in the cohomology of the Cartan model for equivariant de Rham cohomology [13] and the characteristic classes of the equivariant star product and the reduced star product are related by pull-backs.
In the more general setting of Poisson manifolds, star products are classified by Maurer-Cartan elements in the DGLA of multivector fields, i.e. by formal Poisson structures. Unfortunately in this case there is no pull-back available and one has to use different techniques. Motivated by the aim of reducing the formality, we want to describe the reductions in terms of $L_{\infty}$-morphisms. In particular, in this paper we construct such a reduction for the classical side, i.e. for the equivariant multivector fields $T_q(M)$, a certain DGLA whose Maurer-Cartan elements are invariant Poisson structures with equivariant momentum maps. Assuming for simplicity $M = C \times g^*$, which always holds locally in suitable situations, we can perform a Taylor expansion around $C$, obtaining a new DGLA $T_{\tau \circ}(C \times g^*)$. On $C \times g^*$ we have the canonical momentum map $J$ given by the projection on $g^*$ and the canonical linear Poisson structure $\pi_{KKS}$ induced by the action Lie algebroid. They give a new DGLA structure on $T_{\tau \circ}(C \times g^*)$ with differential $[\pi_{KKS} - J, \cdot]$ and we show that this DGLA is quasi-isomorphic to the multivector fields on $M_{red}$, as desired. One has an $L_{\infty}$-quasi-isomorphism between these two DGLA’s, see Theorems 5.20

**Theorem** There exists an $L_{\infty}$-quasi-isomorphism $\tilde{T}_{\text{red}} : T_{\tau \circ}(C \times g^*) \to T_{\text{poly}}(M_{red})$.

The morphism $\tilde{T}_{\text{red}}$ is obtained by inverting a certain inclusion $i$ of DGLA’s. In order to give a more explicit formula we look at general deformation retracts: let $(A, d_A)$ and $(B, d_B)$ be two differential graded Lie algebras and assume that we have

$$ (A, d_A) \xrightarrow{i} (B, d_B) \xrightarrow{h} $$

where $i$ and $p$ are quasi-isomorphisms of cochain complexes with homotopy $h$, and where $p \circ i = i \circ d_A$ and $h^2 = h \circ i = p \circ h = 0$. Using for a coalgebra morphism $F : S(B[1]) \to S(A[1])$ the notation

$$ L_{\infty, k+1}(F) = Q^A_{k,2} \circ F^{k+1}_B - F^k_B \circ Q^B_{k+1, k+1}, $$

where $Q^A_{k, k+1}$ and $Q^B_{k+1, k+1}$ are the extensions of the Lie brackets to the symmetric algebras, and extending $h$ in an appropriate way to $H_k$ on $S^k(B[1])$, we prove in Proposition 5.2 and 5.3

**Proposition** Given a deformation retract as in (1.1).

i.) If $i$ is a DGLA morphism, then $P : S^*(B[1]) \to S^*(A[1])$ with structure maps $P^1_k = p$ and $P^1_{k+1} = L_{\infty, k+1}(P) \circ H_{k+1}$ for $k \geq 1$ yields an $L_{\infty}$-quasi-isomorphism that is quasi-inverse to $i$.

ii.) If $p$ is a DGLA morphism, then $I : S^*(A[1]) \to S^*(B[1])$ with structure maps $I^1_k = i$ and $I^1_k = h \circ L_{\infty, k}(I)$ for $k \geq 2$ is an $L_{\infty}$-quasi-isomorphism that is quasi-inverse to $p$.

This allows us to give a more explicit description of $\tilde{T}_{\text{red}}$ and its $L_{\infty}$-quasi-inverse. Moreover, it allows us to globalize the result, compare Theorem 5.1

**Theorem** There exists a curved $L_{\infty}$-morphism

$$ T_{\text{red}} : (T_0(\mathfrak{g}), \lambda, -[J, \cdot], [\cdot, \cdot]) \to (T_{\text{poly}}(M_{\text{red}}), 0, 0, [\cdot, \cdot]), $$

where the curvature $\lambda = e^i \otimes (e_i)_M$ is given by the fundamental vector fields of the G-action.

We call $T_{\text{red}}$ reduction $L_{\infty}$-morphism and we extend the statements to the setting of formal power series in $h$. After rescaling the involved curvatures and differentials appropriately, $T_{\text{red}}$ gives in particular a way to associate formal Maurer-Cartan elements. In $T_0(\mathfrak{g})([h])$ resp. $T_{\tau \circ}(C \times g^*)([h])$ with rescaled structures, formal Maurer-Cartan elements can be interpreted as formal Poisson structures $\pi_h$ with formal momentum map $J_h = J + hJ'$. Thus, we have the following properties:

- The Poisson bracket $\{\cdot, \cdot\}_h$ induced by $\pi_h$ is $G$-invariant,
- The fundamental vector fields are given by $\xi_M = \{\cdot, J_h(\xi)\}_h \in \Gamma^\infty(TM)$, and
- $\{J_h(\xi), J_h(\eta)\}_h = J_h(\{\xi, \eta\})$.  

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Comparing the orders of $\hbar$ directly shows that the lowest order is a well-defined Poisson structure on $M$ and that $J$ is an equivariant momentum map with respect to it, and $T_{\text{red}}$ maps such an object to a formal Poisson structure on $M_{\text{red}}$.

Note that there is also another reduction scheme for such formal Poisson structures with formal momentum maps, obtained by adapting the reduction scheme for star products from [2][14], i.e. using the homological perturbation lemma [3]. Finally, we show in Theorem 1.4.

**Theorem** The reduction of formal equivariant Poisson structures with formal momentum maps via the reduction $L_\infty$-morphism coincides with the reduction of formal Poisson structures via the homological perturbation lemma.

The paper is organized as follows: In Section 2 we recall the basic notions of (curved) $L_\infty$-algebras, $L_\infty$-morphisms and twists. In Section 3 we consider general deformation retracts of DGLA’s and prove the explicit formulas for the extensions of the inclusion resp. projection to $L_\infty$-morphisms needed to describe $T_{\text{red}}$ in Section 4. Here we also construct the reduction scheme for the Taylor expansion, both in the classical and the formal setting. Finally, in Section 5 we construct the global reduction $L_\infty$-morphism and compare the reduction via $T_{\text{red}}$ with the classical Marsden-Weinstein reduction and with the reduction of formal Poisson structures via the homological perturbation lemma as explained in Appendix A.

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## 2 Preliminaries

In this section we recall the notions of (curved) $L_\infty$-algebras, $L_\infty$-morphisms and their twists by Maurer–Cartan elements to fix the notation. Proofs and further details can be found in [9][10][12].

We denote by $V^*$ a graded vector space over a field $\mathbb{K}$ of characteristic 0 and define the **shifted** vector space $V[k]^\ell$ by

$$V[k]^\ell = V^{\ell+k}.$$  

A degree $+1$ coderivation $Q$ on the coaugmented counital conilpotent cocommutative coalgebra $S'(\mathcal{L})$ cofreely generated by the graded vector space $\mathcal{L}[1]^\ell$ over $\mathbb{K}$ is called an $L_\infty$-structure on the graded vector space $\mathcal{L}$ if $Q^2 = 0$. The (universal) coalgebra $S'(\mathcal{L})$ can be realized as the symmetrized deconcatenation coproduct on the space $\bigoplus_{n\geq 0} S^n\mathcal{L}[1]$ where $S^n\mathcal{L}[1]$ is the space of coinvariants for the usual (graded) action of $S_n$ (the symmetric group in $n$ letters) on $\otimes^n(\mathcal{L}[1])$, see e.g. [12]. Any degree $+1$ coderivation $Q$ on $S'(\mathcal{L})$ is uniquely determined by the components

$$Q_n:S^n(\mathcal{L}[1]) \rightarrow \mathcal{L}[2]$$

through the formula

$$Q(\gamma_1 \vee \ldots \vee \gamma_n) = \sum_{k=0}^{n} \sum_{\sigma \in \text{Sh}(k,n-k)} \epsilon(\sigma)Q_k(\gamma_{\sigma(1)} \vee \ldots \vee \gamma_{\sigma(k)}) \vee \gamma_{\sigma(k+1)} \vee \ldots \vee \gamma_{\sigma(n)}.$$  

(2.2)

Here $\text{Sh}(k,n-k)$ denotes the set of $(k, n-k)$ shuffles in $S_n$, $\epsilon(\sigma) = \epsilon(\sigma, \gamma_1, \ldots, \gamma_n)$ is a sign given by the rule $\gamma_{\sigma(1)} \vee \ldots \vee \gamma_{\sigma(n)} = \epsilon(\sigma) \gamma_1 \vee \ldots \vee \gamma_n$ and we use the conventions that $\text{Sh}(n,0) = \text{Sh}(0,n) = \{\text{id}\}$ and that the empty product equals the unit. Note in particular that we also consider a term $Q_0$ and thus we are actually considering curved $L_\infty$-algebras (which will be convenient in the following). Sometimes we also write $Q_k = Q^1_k$ and following [5] we denote by $Q_n^k$ the component of $Q_n^k : S^n L[1] \rightarrow S^n L[2]$ of $Q$. It is given by

$$Q_n^k(x_1 \vee \ldots \vee x_n) = \sum_{\sigma \in \text{Sh}(n+1-i,i,1-i)} \epsilon(\sigma)Q^1_{n+1-i}(x_{\sigma(1)} \vee \ldots \vee x_{\sigma(n+1-i)}) \vee x_{\sigma(n+2-i)} \vee \ldots \vee x_{\sigma(n)},$$

(2.3)

where $Q^1_{n+1-i}$ are the usual structure maps.
Example 2.1 (Curved Lie algebra) A basic example of an \( L_\infty \)-algebra is that of a (curved) Lie algebra \((\mathfrak{L}, R, d, [\cdot, \cdot])\) by setting \( Q_0(1) = -R\), \( Q_1 = -d\), \( Q_2(\gamma \lor \mu) = -(-1)^{|\gamma|} |\gamma, \mu|\) and \( Q_i = 0 \) for all \( i \geq 3 \). Note that we denoted the degree in \( \mathfrak{L}[1] \) by \( |\cdot| \).

Let us consider two \( L_\infty \)-algebras \((\mathfrak{L}, Q)\) and \((\tilde{\mathfrak{L}}, \tilde{Q})\). A degree 0 counital coalgebra morphism

\[
F : S^c(\mathfrak{L}) \to S^c(\tilde{\mathfrak{L}})
\]
such that \( FQ = \tilde{Q}F \) is said to be an \( L_\infty \)-morphism. A coalgebra morphism \( F \) from \( S^c(\mathfrak{L}) \) to \( S^c(\tilde{\mathfrak{L}}) \) such that \( F(1) = 1 \) is uniquely determined by its components (also called Taylor coefficients)

\[
F_n : S^n(\mathfrak{L}[1]) \to \tilde{\mathfrak{L}}[1],
\]
where \( n \geq 1 \). Namely, we set \( F(1) = 1 \) and use the formula

\[
F(\gamma_1 \lor \ldots \lor \gamma_n) = \sum_{p \geq 1} \sum_{k_1, \ldots, k_p \geq 1} \sum_{\sigma \in \text{Sh}(k_1, \ldots, k_p)} \frac{\epsilon(\sigma)}{p!} F_k(\gamma_{\sigma(1)} \lor \ldots \lor \gamma_{\sigma(k_1)}) \lor \ldots \lor F_{k_p}(\gamma_{\sigma(n-k_p+1)} \lor \ldots \lor \gamma_{\sigma(n)}),
\]
where \( \text{Sh}(k_1, \ldots, k_p) \) denotes the set of \((k_1, \ldots, k_p)\)-shuffles in \( S_n \) (again we set \( \text{Sh}(n) = \{ \text{id} \} \)). We also write \( F_k = F_k^1 \) and similarly to [23] we get coefficients \( F_k^1 : S^n L[1] \to S^j L'[1] \) of \( F \) by taking the corresponding terms in [35] Equation (2.15). Note that \( F_k^1 \) depends only on \( F_k \) for \( k \leq n - j + 1 \). Given an \( L_\infty \)-morphism \( F \) of (non-curved) \( L_\infty \)-algebras \((\mathfrak{L}, Q)\) and \((\tilde{\mathfrak{L}}, \tilde{Q})\), we obtain the map of complexes

\[
F_1 : (\mathfrak{L}, Q_1) \to (\tilde{\mathfrak{L}}, \tilde{Q}_1).
\]

In this case the \( L_\infty \)-morphism \( F \) is called an \( L_\infty \)-quasi-isomorphism if \( F_1 \) is a quasi-isomorphism of complexes. Given a dgla \((\mathfrak{L}, d, [\cdot, \cdot])\) and an element \( \pi \in \mathfrak{L}[1]^0 \) we can obtain a curved Lie algebra by defining a new differential \( d + [\pi, \cdot] \) and considering the curvature \( R^\pi = d\pi + \frac{1}{2}[\pi, \pi] \).

In fact the same procedure can be applied to a curved Lie algebra \((\mathfrak{L}, R, d, [\cdot, \cdot])\) to obtain the twisted curved Lie algebra \((\mathfrak{L}, R^\pi, d + [\pi, \cdot], [\cdot, \cdot])\), where

\[
R^\pi := R + d\pi + \frac{1}{2}[\pi, \pi].
\]

The element \( \pi \) is called a Maurer–Cartan element if it satisfies the equation

\[
R + d\pi + \frac{1}{2}[\pi, \pi] = 0.
\]

Finally, it is important to recall that given a dgla morphism, or more generally an \( L_\infty \)-morphism, \( F : \mathfrak{L} \to \tilde{\mathfrak{L}} \), one may associate to any Maurer–Cartan element \( \pi \in \mathfrak{L}[1]^0 \) a Maurer–Cartan element

\[
\pi_F := \sum_{n \geq 1} \frac{1}{n!} F_n(\pi \lor \ldots \lor \pi) \in \tilde{\mathfrak{L}}[1]^0.
\]

In order to make sense of these infinite sums we consider complete filtered \( L_\infty \)-algebras and we demand that Maurer–Cartan elements are in a positive filtration, see [9][12] for details on such filtrations.

3 An Explicit Formula for the \( L_\infty \)-Projection and -Inclusion

From the general theory of \( L_\infty \)-algebras one knows that \( L_\infty \)-quasi-isomorphisms always admit \( L_\infty \)-quasi-inverses. Moreover, it is well-known that given a homotopy retract one can transfer \( L_\infty \)-structures. Explicitly, given two cochain complexes \((A, d_A)\) and \((B, d_B)\) with

\[
(A, d_A) \xymatrix{ \ar[r]^i & \ar[r]_{d_B} & (B, d_B) } \xymatrix{ \ar[r]_j & B } \xymatrix{ \ar[r]^{id} & B } \xymatrix{ \ar[r] & h }
\]
where $h \circ d_B + d_B \circ h = \text{id} - i \circ p$ and where $i$ is a quasi-isomorphism, the homotopy transfer Theorem \cite{17} Section 10.3] states that if there exists an $L_\infty$-structure on $B$, then one can transfer it to $A$ in such a way that $i$ extends to an $L_\infty$-quasi-isomorphism.

Let us consider the special case of deformation retracts for DGLA’s. More explicitly, let $A, B$ be two DGLA’s. A deformation retract of $(A, d_A)$ is given by the diagram

\[
(A, d_A) \xrightarrow{i} (B, d_B) \xleftarrow{h} (A, d_A)
\]

where $i$ and $p$ are quasi-isomorphisms of cochain complexes with homotopy $h$, i.e. $h d_B + d_B h = \text{id}_B - ip$, as well as

\[
p \circ i = \text{id}_A, \quad h^2 = 0, \quad h \circ i = 0 \quad \text{and} \quad p \circ h = 0.
\]

In addition, we assume that $i$ is a DGLA morphism. As already mentioned, the homotopy transfer theorem and the invertibility of $L_\infty$-quasi-isomorphisms imply that $p$ extends to an $L_\infty$-quasi-isomorphism denoted by $P$, see e.g. \cite{17} Prop. 10.3.9]. In the following we give a more explicit description of $P$. The DGLA structures yield the codifferentials $Q_A$ on $S(A[1])$ and $Q_B$ on $S(B[1])$ and the map $h$ extends to a homotopy $H_n$: $S^n(B[1]) \rightarrow S^n(B[1])[-1]$ with respect to $Q^n_{B,n}$: $S^n(B[1]) \rightarrow S^n(B[1])[1]$, see e.g. \cite{17} p. 383 for the construction on the tensor algebra, which adapted to our setting works roughly like: we define the operator

\[
K_n: S^n(B[1]) \rightarrow S^n(B[1])
\]

by

\[
K_n(x_1 \vee \cdots \vee x_n) = \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\sigma \in S_n} \epsilon(\sigma) \frac{ipX_{\sigma(1)} \vee \cdots \vee ipX_{\sigma(i)} \vee X_{\sigma(i+1)} \vee X_{\sigma(n)}}{n-1}.
\]

Note that here we sum over the whole symmetric group and not the shuffles, since in this case the formulas are easier. We extend $-h$ to a coderivation to $S(B[1])$, i.e.

\[
\hat{H}_n(x_1 \vee \cdots \vee x_n) := - \sum_{\sigma \in \text{Sh}(1,n-1)} \epsilon(\sigma) hx_{\sigma(1)} \vee x_{\sigma(2)} \vee \cdots \vee x_{\sigma(n)}
\]

and define

\[
H_n = K_n \circ \hat{H}_n = \hat{H}_n \circ K_n.
\]

Since $i$ and $p$ are chain maps, we have

\[
K_n \circ Q^n_{B,n} = Q^n_{B,n} \circ K_n,
\]

where $Q^n_{B,n}$ is the extension of the differential $Q^1_{B,1} = - d_B$ to $S^n(B[1])$ as coderivation. Hence we have

\[
Q^n_{B,n} H_n + H_n Q^n_{B,n} = (n \cdot \text{id} - ip) \circ K_n,
\]

where $ip$ is extended as a coderivation to $S(B[1])$. A combinatorial and not very enlightning computation shows that finally

\[
Q^n_{B,n} H_n + H_n Q^n_{B,n} = \text{id} - (ip)^\vee n.
\]

Suppose that we have constructed a morphism of coalgebras $P$ with structure maps $P^j_k: S^k(B[1]) \rightarrow A[1]$ that is an $L_\infty$-morphism up to order $k$, i.e.

\[
\sum_{\ell=1}^{m} P^j_{\ell} \circ Q^j_{B,m} = \sum_{\ell=1}^{m} Q^j_{A,\ell} \circ P^\ell_{m}
\]

for all $m \leq k$. Then we have the following statement.
Lemma 3.1 Let $P : S(B[1]) \to S(A[1])$ be an $L_{\infty}$-morphism up to order $k \geq 1$. Then
\begin{equation}
L_{\infty,k+1} = \sum_{\ell = 2}^{k+1} Q_{A,\ell}^1 \circ P_{k+1}^\ell - \sum_{\ell = 1}^{k} P_{k+1}^1 \circ Q_{B,k+1}^\ell = Q_{A,2}^1 \circ P_{k+1}^2 - P_{k+1}^1 \circ Q_{B,k+1}^k
\end{equation}

satisfies
\begin{equation}
L_{\infty,k+1} \circ Q_{B,k+1}^k = -Q_{A,1}^1 \circ L_{\infty,k+1}.
\end{equation}

Proof: The statement follows from a straightforward computation. For convenience we omit the index of the differential:
\begin{align*}
L_{\infty,k+1} Q_{B,k+1}^k &= \sum_{\ell = 2}^{k+1} Q_{A,\ell}^1 (P \circ Q)_{k+1}^\ell - \sum_{\ell = 1}^{k} P_{k+1}^1 Q_{B,k+1}^\ell \\
&= \sum_{\ell = 2}^{k+1} Q_{A,\ell}^1 (P \circ Q)_{k+1}^\ell - \sum_{\ell = 1}^{k} P_{k+1}^1 Q_{B,k+1}^\ell \\
&= -Q_{A,1}^1 (Q \circ P)_{k+1}^1 + \sum_{k} P_{k+1}^1 Q_{B,k+1}^k = -Q_{A,1}^1 L_{\infty,k+1},
\end{align*}

where the last equality follows from $Q_{A,1}^1 Q_{B}^1 = 0$.
\hfill $\square$

This allows us to obtain the $L_{\infty}$-quasi-inverse of $i$, denoted by $P$, in (3.2) recursively:

Proposition 3.2 Defining $P_1^1 = p$ and $P_{k+1}^1 = L_{\infty,k+1} \circ H_{k+1}$ for $k \geq 1$ yields an $L_{\infty}$-quasi-isomorphism $P : S(B[1]) \to S(A[1])$ that is quasi-inverse to $i$.

Proof: We observe $P_{k+1}^1 (ix_1 \lor \cdots \lor ix_{k+1}) = 0$ for all $k \geq 1$ and $x_i \in A$, which directly follows from $h \circ i = 0$ and thus $H_{k+1} \circ i^{(k+1)} = 0$. In addition, one also has for all $k \geq 1$ the identity $L_{\infty,k+1} (ix_1, \cdots, ix_{k+1}) = 0$, which follows from the definition of $L_{\infty,k+1}$ and the fact that $i$ is a morphism of DGLAs. We know that $P$ is an $L_{\infty}$-morphism up to order one. Suppose that we already know that it is an $L_{\infty}$-morphism up to order $k \geq 1$, then this implies
\begin{align*}
P_{k+1}^1 \circ Q_{B,k+1}^k &= L_{\infty,k+1} \circ Q_{B,k+1}^k \\
&= L_{\infty,k+1} - L_{\infty,k+1} \circ Q_{B,k+1}^k \circ H_{k+1} - L_{\infty,k+1} \circ (i \circ p)^{(k+1)} \\
&= L_{\infty,k+1} + Q_{A,1}^1 P_{k+1}^1
\end{align*}

by the above lemma, and therefore
\begin{equation}
P_{k+1}^1 \circ Q_{B,k+1}^k - Q_{A,1}^1 P_{k+1}^1 = L_{\infty,k+1}.
\end{equation}

Hence $P$ is an $L_{\infty}$-morphism up to order $k + 1$ and the statement follows inductively.
\hfill $\square$

Let us now assume that $p : B \to A$ in the deformation retract (3.2) is a DGLA morphism and that $i$ is just a chain map. Then we can analogously give a formula for the extension $I$ of $i$ to an $L_{\infty}$-quasi-isomorphism.

Proposition 3.3 The coalgebra map $I : S^\bullet(A[1]) \to S^\bullet(B[1])$ recursively defined by the maps $I_1^1 = i$ and $I_k^1 = h \circ L_{\infty,k}$ for $k \geq 2$ is an $L_{\infty}$-quasi inverse of $p$. Since $h^2 = 0 = h \circ i$, one even has $I_k^1 = h \circ Q_{B}^1 \circ I_k^1$.

Proof: We proceed by induction: assume that $I$ is an $L_{\infty}$-morphism up to order $k$, then we have
\begin{align*}
I_{k+1}^1 Q_{A,k+1}^k - Q_{B,1}^1 I_{k+1}^1 &= -Q_{B,1}^1 \circ h \circ L_{\infty,k+1} + h \circ L_{\infty,k+1} \circ Q_{A,k+1}^k \\
&= -Q_{B,1}^1 \circ h \circ L_{\infty,k+1} - h \circ Q_{B,1}^k \circ L_{\infty,k+1} \\
&= (\text{id} - i \circ p) L_{\infty,k+1}.
\end{align*}

We used that $Q_{B,1}^1 = -d_B$ and the homotopy equation of $h$. Moreover, since $p$ is a DGLA morphism and $p \circ h = 0$, we have that $p \circ L_{\infty,k+1} = 0$ for $k \geq 0$. Since $I$ is an $L_{\infty}$-morphism up to order one, i.e. a chain map, the claim is proven.
\hfill $\square$
4 Reduction of Multivector Fields

In the following, we want to use the above language and considerations to formulate a reduction scheme for multivector fields. We first introduce a new complex which of multivector fields which contains the data of Hamiltonian actions in the case of Lie group actions \( \Phi: G \times M \to M \).

**Definition 4.1 (Equivariant Multivectors)** The DGLA of equivariant multivector fields is given by the complex \( T^*_g(M) \) defined by

\[
T^k_g(M) = \bigoplus_{2r+j=k} (S^j g^* \otimes \Gamma^\infty(\Lambda^{j+1}TM))^G = \bigoplus_{2r+j=k} (S^j g^* \otimes T^j_{\text{poly}}(M))^G,
\]

together with the trivial differential and the following Lie bracket

\[
[\alpha \otimes X, \beta \otimes Y]_g = \alpha \lor \beta \otimes [X,Y]
\]

for any \( \alpha \otimes X, \beta \otimes Y \in T^*_g(M) \).

Here \([\cdot,\cdot]\) refers to the usual Schouten–Nijenhuis bracket on \( T_{\text{poly}}(M) \). Notice that invariance with respect to the group action means invariance under the transformations \( \text{Ad}_g^* \otimes \Phi_g^* \) for all \( g \in G \). We can equivalently interpret this complex in terms of polynomial maps \( g \to T_{\text{poly}}^1(M) \) which are equivariant with respect to adjoint and push-forward action. Using this point of view, the bracket can be rewritten as

\[
[X,Y]_g(\xi) = [X(\xi),Y(\xi)].
\]

(4.1)

Furthermore, we introduce the canonical linear map

\[
\lambda: g \ni \xi \mapsto \xi_M \in T^0_{\text{poly}} M,
\]

(4.2)

where \( \xi_M \) denotes the fundamental vector field corresponding to the action \( \Phi \). It is easy to see that \( \lambda \) is central and as a consequence we can turn \( T^*_g M \) into a curved Lie algebra with curvature \( \lambda \). Now let \( (M,\pi) \) be a Poisson manifold and denote by \([\cdot,\cdot]\) the corresponding Poisson bracket. Recall that an (equivariant) momentum map for the action \( \Phi \) is a map \( J: g \to \mathcal{C}^\infty(M) \) such that

\[
\xi_M = \{ \cdot, J_\xi \} \quad \text{and} \quad J_{\xi,\eta} = \{ J_\xi, J_\eta \}.
\]

(4.3)

An action \( \Phi \) admitting a momentum map is what we called Hamiltonian. In the following we prove a characterization of Hamiltonian actions in terms of equivariant multivectors.

**Lemma 4.2** The curved Maurer–Cartan elements of \( T^*_g(M) \) are equivalent to pairs \((\pi,J)\), where \( \pi \) is a \( G \)-invariant Poisson structure \( J \) is a momentum map \( J: g \to T_{\text{poly}}^1(M) \).

**Proof:** The curved Maurer–Cartan equation reads

\[
\lambda + \frac{1}{2}[\Pi, \Pi]_g = 0
\]

for \( \Pi \in T^1_g(M) \). If we decompose \( \Pi = \pi - J \in (T^1_{\text{poly}}(M))^G \oplus (g^* \otimes T^{-1}_{\text{poly}}(M))^G \), it is easy to see that the curved Maurer–Cartan equation together with the invariance of the elements is equivalent to the conditions \([\mathbf{L}, \mathbf{M}]\) defining the momentum map. \( \blacksquare \)

As in the Marsden-Weinstein reduction procedure, we fix a constraint surface \( C \subseteq M \), by choosing an equivariant map \( J: M \to g^* \) and setting \( C = J^{-1}(\{0\}) \). Here we always assume that \( 0 \in g^* \) is a regular value of the momentum map, making \( C \) a closed embedded submanifold of \( M \). Note that \( G \) acts canonically of \( C \), since \( J \) is equivariant. From now on we also require the action \( \Phi \) to be proper around \( C \) and free on \( C \).

To implement this choice in our algebraic setting we consider from now on on the curved differential graded Lie algebra

\[
(T^*_g(M), \lambda, -[J, \cdot], [\cdot, \cdot, \cdot]).
\]

(4.4)
Note that this is in fact a curved Lie algebra since \([J,J] = 0 = [\lambda, \cdot, \cdot]\). We have to move to the formal setting in order to see why this curved Lie algebra is actually interesting. Therefore, let us consider the curved Lie algebra
\[
(T^*_g(M[[\hbar]], h\lambda, -[J, \cdot], [\cdot, \cdot], )].
\]
Note that one advantage of the setting of formal power series is that we immediately get a complete filtration by the \(h\)-degrees, i.e. by setting
\[
\mathfrak{F}^kT^*_g(M[[\hbar]]) = h^kT^*_g(M[[\hbar]]).
\]
In particular, if we consider formal Maurer-Cartan elements \(h(\pi - J') \in hT^*_g(M[[\hbar]]), \) then the twisting procedures and infinite sums from Section 2 are all well-defined.

**Lemma 4.3** The formal curved Maurer-Cartan elements of \((T^*_g(M[[\hbar]], h\lambda, -[J, \cdot], [\cdot, \cdot])\) are equivalent to pairs \(h(\pi, J')\), where \(\pi\) is a \(G\)-invariant formal Poisson structure with formal moment map \(J + hJ': \mathfrak{g} \to T^*_{g}(M[[\hbar]])\).

**Proof:** The proof follows directly by Lemma 12 by counting \(h\)-degrees. \(\square\)

The rest of this paper is devoted to the construction of a curved \(L_\infty\)-morphism
\[
\text{red}: (T_g(M[[\hbar]], h\lambda, -[J, \cdot], [\cdot, \cdot])) \to (T_{g,\text{poly}}(M_{\text{red}})[[\hbar]], 0, 0, [\cdot, \cdot])
\]
with \(M_{\text{red}} := C/G\). This morphism is frequently referred to as reduction morphism.

### 4.1 Taylor Series Expansion around \(C\)

The main goal of this section is the study of a partial Taylor series expansion of the multivector field on \(M\) around \(C\). Let us assume \(M = C \times \mathfrak{g}^*\). This is not a strong assumption as we know from [2, Lemma 3] that, if \(G\) acts properly on an open neighbourhood of \(C\) we can always find an \(G\)-invariant open neighbourhood \(M_{\text{nice}} \subseteq M\) of \(C\), such that there exists a \(G\)-equivariant diffeomorphism \(M_{\text{nice}} \cong U_{\text{nice}} \subseteq C \times \mathfrak{g}^*\). Here the Lie group \(G\) acts on \(C \times \mathfrak{g}^*\) as
\[
\Phi_g = \Phi^C_g \times \text{Ad}^{-1}_g,
\]
where \(\Phi^C\) is the induced action on \(C\). Note that in this setting the momentum map on \(U_{\text{nice}}\) is simply given by the projection to \(\mathfrak{g}^*\). The idea of a Taylor expansion uses the fact that we have the isomorphism
\[
T^k_{\text{poly}}(C \times \mathfrak{g}^*) \cong \bigoplus_{i+j=k} \mathcal{C}^\infty(C \times \mathfrak{g}^*) \otimes_{\mathcal{C}^\infty(C)} (\Lambda^i\mathfrak{g}^* \otimes T^j_{\text{poly}}(C)).
\]

First, we define
\[
T^k_g: \mathcal{C}^\infty(C \times \mathfrak{g}^*) \ni f \mapsto \sum_{i \in \mathbb{N}_0} \frac{1}{i!} e^i \otimes t^* \partial_{\alpha i} f \in \prod_i (S^i\mathfrak{g} \otimes \mathcal{C}^\infty(C)),
\]
where \(\alpha, e^i\) are coordinates on \(\mathfrak{g}^*\) and \(t^*\) the restriction to \(C\).

**Lemma 4.4** The map \(T^*_g\) is equivariant, i.e.
\[
T^*_g \circ \Phi_{g, \ast} = (\text{Ad}_g \circ \Phi^C_{g, \ast}) \circ T^*_g.
\]

**Proof:** We just observe that
\[
\Phi^*_g \circ \frac{\partial}{\partial \alpha_i} = (\text{Ad}_g^{-1})^j_i \cdot (\Phi^C_g)^*_j \circ \frac{\partial}{\partial \alpha_j}
\]
for \(\text{Ad}_g e_i = (\text{Ad}_g e_i)^j_i e_j\). Hence we have
\[
T^*_g(\Phi_{g, \ast} f) = \sum_{i \in \mathbb{N}_0} \frac{1}{i!} e^i \otimes t^* \partial_{\alpha i} \Phi_{g, \ast} f = (\text{Ad}_g \circ \Phi^C_{g, \ast}) \circ T^*_g f
\]
by shifting the components \((\text{Ad}_g^{-1})^j_i = (\text{Ad}_g)^j_i\) to the symmetric powers of \(\mathfrak{g}\). \(\square\)
Remark 4.5 It is now clear that this map can be restricted to invariant functions in order to obtain invariant elements in \( \prod_i (S^i g \otimes \mathcal{C}^\infty(C)) \). Moreover, with a slight adaption of the proof of the Borel-Lemma, see e.g. [20, Theorem 1.3], one can show that the map \( T_{g^*} \) is surjective. The more remarkable fact is that the properness of the action ensures that the map

\[
T_{g^*} : \mathcal{C}^\infty(M \times g^*)^G \ni f \mapsto \sum_{l \in \mathbb{N}_0^\infty} \frac{1}{l!} e_l \otimes i^* \frac{\partial}{\partial \alpha_l} f \in \prod_i (S^i g \otimes \mathcal{C}^\infty(C))^G
\]

is surjective. We omit this proof as we do not use it here and it is just an adaption of the corresponding statement in [19].

We extend this map to \( T_{poly}^0(C \times g^*) \) via

\[
T_{g^*} : T_{poly}^k(C \times g^*) \ni (f \otimes \xi \otimes X) \mapsto \sum_{l \in \mathbb{N}_0^\infty} \frac{1}{l!} e_l \otimes \xi \otimes i^* \frac{\partial}{\partial \alpha_l} f \cdot X \in \prod_i (S^i g \otimes \mathcal{A}g^* \otimes T_{poly}(C))
\]

and using Lemma [4], we see that also this map can be restricted to invariant multivector fields:

Definition 4.6 (Taylor Expansion around C) The map

\[
T_{g^*} : (Sg^* \otimes T_{poly}(C \times g^*))^G \longrightarrow T_{tayl}(C \times g^*) := (Sg^* \otimes \prod_{i=0}^\infty (S^i g \otimes \mathcal{A}g^* \otimes T_{poly}(C)))^G
\]

(4.6)

is called Taylor expansion around C.

Having in mind that the vector space \( \prod_{i=0}^\infty (S^i g \otimes \mathcal{A}g^* \otimes T_{poly}(C)) \) is just consisting of Taylor expansions, it is not surprising that it also inherits the structure of a DGLA: for \( P, Q \in \prod_i S^i g \) and \( \xi, \eta \in g^* \), the brackets are given by

\[
[P, Q] = 0, \quad [P \otimes \xi, Q] = P \vee i_\xi(Q),
\]

\[
[P \otimes \xi, Q \otimes \eta] = P \vee i_\xi(Q) \otimes \eta - Q \vee i_\eta(P) \otimes \xi,
\]

and they are extended as a Gerstenhaber bracket with respect to the graded commutative product

\[
(P \otimes \xi) \cdot (Q \otimes \eta) := P \vee Q \otimes \xi \wedge \eta.
\]

We combine it with the usual DGLA structure on \( T_{poly}(C) \) and extend it as in the case of \( T_{g^*}(M) \) trivially to all of \( T_{tayl}(C \times g^*) \). Summarizing, we have a DGLA structure on the Taylor expansion around C with zero differential.

Lemma 4.7 The Taylor expansion

\[
T_{g^*} : T_{g^*}(M) \longrightarrow T_{tayl}(C \times g^*)
\]

(4.7)

is a DGLA morphism.

Proof: This is an easy verification on generators. \( \square \)

As a next step we want to include the curvature \( \lambda \in T_{g^*}^2(M) \) from Section [4] Recall that

\[
\lambda = e^i \otimes (e_i)_M \in T_{g^*}^2(M) = (g^* \otimes T_{poly}^0(M))^G.
\]

Using our assumption that \( M = C \times g^* \) and that \( G \) acts as the product of the action on \( C \) and the coadjoint action, we see that

\[
(e_i)_M = (e_i)_C + \alpha_k f^k_{ji} \frac{\partial}{\partial \alpha_j}, \quad (4.8)
\]
where \((e_i)_C\) denotes the fundamental vector field of the action on \(C\) and where \(f^i_{jk}\) are the structure constants of \(\mathfrak{g}\). This means in particular that
\[
T_{\mathfrak{g}^*}(\lambda) = e^i \otimes 1 \otimes 1 \otimes (e_i)_C + f^i_{jk} e^i \otimes e^k \otimes e^k \otimes 1 + 1 \in T_{\mathfrak{t}_{\nu}}(C \times \mathfrak{g}^*).
\]
With a slight abuse of notation we write \(\lambda\) instead of \(T_{\mathfrak{g}^*}(\lambda)\). The same argument leads to the observation that
\[
T_{\mathfrak{g}^*}(J) = e^i \otimes e_i \otimes 1 \otimes 1,
\]
where we also write \(J\) instead of \(T_{\mathfrak{g}^*}(J)\) in the sequel.

**Corollary 4.8** The map
\[
T_{\mathfrak{g}^*} : (T_{\mathfrak{g}}(M), \lambda, -[J, \cdot], [\cdot, \cdot]) \longrightarrow (T_{\nu}(C \times \mathfrak{g}^*), \lambda, -[J, \cdot], [\cdot, \cdot]) \tag{4.9}
\]
is a morphism of curved Lie algebras.

One main advantage of the Taylor expansion \(T_{\nu}(C \times \mathfrak{g}^*)\) consists in the fact that we have a canonical element
\[
\pi_{\mathbb{KKS}} := 1 \otimes \left( \frac{1}{2} f^i_{jk} e^i \otimes e^k \wedge e^j \otimes 1 - 1 \otimes e^i \otimes (e_i)_C \right),
\]
which is not available in \(T_{\mathfrak{g}}(M)\). Note that \(\pi_{\mathbb{KKS}}\) encodes the action on \(C\) and the Lie algebra structure on \(\mathfrak{g}\).

**Remark 4.9 (Action Lie algebroid)** The bundle \(C \times \mathfrak{g} \to C\) can be equipped with the structure of a Lie algebroid since \(\mathfrak{g}\) acts on \(C\) by the fundamental vector fields. The bracket of this action Lie algebroid is given by
\[
[\xi, \eta]_{C \times \mathfrak{g}}(p) = [\xi(p), \eta(p)] - (L_{\xi_C, \eta})(p) + (L_{\eta_C} \xi)(p) \tag{4.10}
\]
for \(\xi, \eta \in \mathfrak{g}C(C, \mathfrak{g})\). The anchor is given by \(\rho(p, \xi) = -\xi_C|_p\). In particular, one can check that \(\pi_{\mathbb{KKS}}\) is the negative of the linear Poisson structure on its dual \(C \times \mathfrak{g}^*\) in the convention of [21].

The canonical \(\pi_{\mathbb{KKS}}\) is of big importance since it is part of some kind of normal form for every invariant Poisson structure on \(C \times \mathfrak{g}^*\) with moment map \(J\). In the Taylor expansion this becomes more clear in the following lemma:

**Lemma 4.10** Let \(\pi \in \left( \prod_{i=0}^{\infty} \mathbb{S}^i \mathfrak{g} \otimes \Lambda^g \otimes T_{\nu}(C) \right)^G \subseteq T_{\nu}(C \times \mathfrak{g}^*)\) be a curved Maurer–Cartan element, then
\[
\pi = \pi_{\mathbb{KKS}} + \pi_C \tag{4.11}
\]
with \(\pi_C \in \left( \prod_{i=0}^{\infty} \mathbb{S}^i \mathfrak{g} \otimes T_{\nu}(C) \right)^G\).

**Proof:** By (4.8) we have for \(\xi \in \mathfrak{g}\), \(c \in C\) and \(\alpha = \alpha_i e^i \in \mathfrak{g}^*\)
\[
\xi_M|_{(c, \alpha)} = (i(dJ(\xi))\pi)|_{(c, \alpha)} = \xi_C|_c + \xi_{\mathfrak{g}^*}|_\alpha = \xi_C|_c - f^i_{jk} i(e_i) \alpha_i \frac{\partial}{\partial \alpha_j}.
\]
This implies directly
\[
\pi = \pi_C + (e_i)_C \wedge \frac{\partial}{\partial \alpha_i} + \frac{1}{2} \alpha_k f^i_{jk} \frac{\partial}{\partial \alpha_i} \wedge \frac{\partial}{\partial \alpha_j},
\]
where \(\pi_C \in \Gamma^\infty(M, \Lambda^2 T_C)\) is tangent to \(C\), but can possibly depend on all of \(M = C \times \mathfrak{g}^*\). In the Taylor expansion \(\frac{\partial}{\partial \alpha_i}\) corresponds to \(i(e^i)\) and the lemma is shown. \(\square\)
Comparing now the terms in $[\pi, \pi] = 0$ with same $g^*$ and $C$ degrees gives hints concerning the coefficient function of $\pi_C$ that can also depend on $g^*$. In particular, the terms in $\Gamma^\infty(A^3 TC)$ are given by

$$[\pi_C, \pi_C] + 2(e_i)_C \left[ \frac{\partial}{\partial e_i}, \pi_C \right] = 0. \quad (4.12)$$

To conclude this section, we define for later use the operator

$$\partial := \text{id} \otimes i_e(e^i) \otimes \text{id} \otimes (e_i)_C \wedge. \quad (4.13)$$

Note that we assume the Koszul sign rule, i.e. applying $\partial$ to $\xi \otimes P \otimes \alpha \otimes X$ we get a sign $(-1)^{[\alpha]}$. We directly see that $\partial \theta = 0$ and Equation (4.12) can be written as

$$\frac{1}{2}[\pi_C, \pi_C] + (e_i)_C \wedge i_e(e^i)\pi_C = \frac{1}{2}[\pi_C, \pi_C] + \partial\pi_C = 0. \quad (4.14)$$

### 4.2 The Cartan Model of Multivector Fields

In the case of symplectic manifolds, it has been shown in [22] that quantization and reduction commute by exploiting the following diagram

$$(\text{Sp}^* \otimes \Omega(M))^G, d_g) \xrightarrow{\iota} (\text{Sp}^* \otimes \Omega(C))^G, d_g) \xrightarrow{\partial^*} (\Omega^*(M_{\text{red}}), d)$$

for $M \leftarrow C \rightarrow M_{\text{red}}$. Here $p^*$ is a quasi-isomorphism and $(\text{Sp}^* \otimes \Omega(C))^G$ is the so-called Cartan model for equivariant de Rham cohomology [13]. We aim to generalize this result to the setting of Poisson manifolds by using the above observation as a guideline. For this reason we introduce our notion for the Cartan Model of equivariant multivector fields and compute its relation with $T_{\text{poly}}(M_{\text{red}})$ and with the Taylor expansion of multivector fields around $C$ from the previous section. We start with the following observation:

**Proposition 4.11** The cohomology of the DGLA $(T_{\text{sym}}(C \times g^*), -[J, \cdot], \cdot, \cdot)$ is given by the Lie algebra $((\prod_{i=0}^\infty (S^i g \otimes T_{\text{poly}}(C)))^G, \cdot, \cdot)$. Therefore, the canonical inclusion

$$\iota: \left( \prod_{i=0}^\infty (S^i g \otimes T_{\text{poly}}(C)) \right)^G, 0, [\cdot, \cdot] \longrightarrow (T_{\text{sym}}(C \times g^*), -[J, \cdot], \cdot, \cdot) \quad (4.15)$$

becomes a quasi-isomorphism of DGLA’s.

**Proof:** The map $h = i_e(e_i) \otimes \text{id} \otimes e^j \wedge \text{id}$ satisfies

$$-[J, \cdot] \circ h(\xi \otimes P \otimes \alpha \otimes X) - h \circ [J, \cdot](\xi \otimes P \otimes \alpha \otimes X) = (\text{deg}(\alpha) + \text{deg}(\xi))(\xi \otimes P \otimes \alpha \otimes X)$$

and the statement follows. \qed

Note that the cohomology $((\prod_{i=0}^\infty (S^i g \otimes T_{\text{poly}}(C)))^G$ can be equipped with a non-trivial, but canonical differential.

**Proposition 4.12** The differential $\partial$ defined in [13] turns $((\prod_{i=0}^\infty (S^i g \otimes T_{\text{poly}}(C)))^G, \cdot, \cdot)$ into a DGLA.

**Proof:** A straightforward computation shows

$$\partial[\xi \otimes X, \eta \otimes Y] = i_e(e^i)(\xi \vee \eta) \otimes (e_i)_C \wedge [X, Y],$$

$$[\partial(\xi \otimes X), \eta \otimes Y] = (-1)^k i_e(e^i)(\xi) \vee \eta \otimes X \wedge [(e_i)_C, Y] + i_e(e^i)(\xi) \vee \eta \otimes (e_i)_C \wedge [X, Y],$$

$$[\xi \otimes X, \partial(\eta \otimes Y)] = (-1)^{k-1} \xi \wedge i_e(e^i)(\eta) \otimes (e_i)_C \wedge [X, Y] - \xi \vee i_e(e^i)(\eta) \otimes [(e_i)_C, X] \wedge Y,$$

where $X \in T^{k-1}_{\text{poly}}(C)$. Using the $G$-invariance we get

$$\xi \otimes [(e_i)_C, X] = -f^k_{ij} e_k \vee i_e(e^i)\xi \otimes X \quad \text{and} \quad \eta \otimes [(e_i)_C, Y] = -f^k_{ij} e_k \vee i_e(e^i)\eta \otimes Y.$$

Summarizing, this yields

$$\partial[\xi \otimes X, \eta \otimes Y] = [\partial(\xi \otimes X), \eta \otimes Y] + (-1)^{k-1}[\xi \otimes X, \partial(\eta \otimes Y)]$$

and the proposition is shown. \qed

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This motivates the following definition.

**Definition 4.13 (Cartan model)** Let $G$ be a Lie group action on a manifold $C$. The DGLA defined by

$$\left(\prod_{i=0}^{\infty} (S^i g \otimes T_{\text{poly}}(C))^G, \partial, [\cdot, \cdot]\right)$$

is called Cartan model and is denoted by $T_{\text{Cart}}(C)$.

Seen as a module, $(\prod_{i=0}^{\infty} (S^i g \otimes T_{\text{poly}}(C))^G)_{G}$ is the dual of the Cartan model $(S^*g \otimes \Omega(C))^G$ for the equivariant de Rham cohomology [13, 23]. Even the differential $\partial$ is dual to the insertion $i_{\bullet} = e^i \lor \otimes i_{\bullet}((e_i)_C)$ that forms together with the de Rham differential the coboundary operator in the usual Cartan model for equivariant cohomology. In the case of forms the equivariant cohomology of the principal fiber bundle $C$ is isomorphic to the de Rham cohomology of the reduced manifold, whereas in our setting we want to show that we get the multivector fields on $M_{\text{red}}$ as cohomology. Note that we have a canonical DGLA map

$$p: (T_{\text{Cart}}(C), \partial, [\cdot, \cdot]) \longrightarrow (T_{\text{poly}}(M_{\text{red}}), 0, [\cdot, \cdot])$$

which is just given by the projection to the symmetric degree 0 followed by the projection to $M_{\text{red}}$. It is well-defined since invariant multivector fields are projectable.

**Proposition 4.14** The DGLA-map

$$p: \left(\prod_{i=0}^{\infty} (S^i g \otimes T_{\text{poly}}(C))^G, \partial, [\cdot, \cdot]\right) \longrightarrow (T_{\text{poly}}(M_{\text{red}}), 0, [\cdot, \cdot])$$

is a quasi-isomorphism.

**Proof:** Consider the principal bundle $pr: C \rightarrow M_{\text{red}}$ and choose a principal bundle connection $\omega = \omega^i \otimes e_i \in \Omega^1(C) \otimes g$, i.e. an equivariant horizontal lift inducing

$$TC = \text{Ver}(C) \oplus \text{Hor}(C) = \ker T pr \oplus \ker \omega,$$

where $\ker \omega \cong pr^* T M_{\text{red}}$. Then we can construct a homotopy for $\partial$ by $h = e_i \lor \otimes i(\omega^i)$. Since $\omega^i((e_j)_M) = \delta^j_i$, it satisfies

$$h \partial + \partial h = (\deg g + \deg_{\text{ver}}) \text{id}.$$

With the vertical degree we mean the degree in the splitting $\Lambda^k TC = \bigoplus_{i+j=k} \Lambda^i \text{Ver}(C) \otimes \Lambda^j \text{Hor}(C)$.

In other words, the above proposition yields for every principal connection $\omega \in \Omega^1(C) \otimes g$ the following deformation retract

$$T_{\text{poly}}(M_{\text{red}}) \xrightarrow{i} \left(\prod_{i=0}^{\infty} (S^i g \otimes T_{\text{poly}}(C))^G\right)_{G} h$$

where $i$ denotes the horizontal lift with respect to the connection $\omega$ and the homotopy $h$ is given on all homogeneous elements by

$$h(\xi \otimes X) = \begin{cases} e_i \lor \xi \otimes i(\omega^i)X & \text{if $\deg(\xi) + \deg_{\text{ver}}(X) \neq 0$} \\ 0 & \text{else.} \end{cases}$$

Indeed, the algebraic relations of a deformation retract between $i$, $p$ and $h$ are easily seen to be verified. Recall that additionally $p$ is a DGLA morphism, which puts us exactly in the situation of Proposition [33]. So before we continue to put the Cartan model in the context of reduction, we give an explicit formula for a quasi-inverse of $p$.
Proposition 4.15 For a fixed principal fiber connection $\omega \in \Omega^1(C) \otimes g$ with curvature $\Omega \in \Omega^2(C) \otimes g$, one obtains an $L_\infty$-quasi-inverse of $p$

$$i_\infty : S(T_{\text{poly}}(M_{\text{red}})[1]) \longrightarrow S\left( \prod_{i=0}^{\infty}(S^i g \otimes T_{\text{poly}}(C))^G \right) [1]$$  \hspace{1cm} (4.19)

given by

$$i_\infty = e^\Omega \circ (\cdot)^{\text{hor}},$$ \hspace{1cm} (4.20)

where one extends $(\cdot)^{\text{hor}}$ as a coalgebra morphism and $\Omega$ as a coderivation of degree 0. In particular,

$$i_\infty,1(X) = X^{\text{hor}} \quad \text{and} \quad i_\infty,2(X,Y) = (-1)^{|X|}e_i \otimes \Omega^f(X^{\text{hor}},Y^{\text{hor}})$$ \hspace{1cm} (4.21)

for a basis $\{e_i\}_{i \in I}$ of $g$.

**Proof:** Let us fix a principal connection $\omega \in \Omega^1(C) \otimes g$ and denote by $h$ the corresponding homotopy and by $\Omega$ its curvature. Due to that fact that Equation 357 is a deformation retract and $p$ is a DGLA morphism, we are exactly in the situation of Proposition 3.3 and the statement becomes a purely computational issue, so let us start with some book-keeping. Throughout the proof, we will make use of the following equation for $X \in \Gamma^\infty(\Lambda^k TC)$, $Y \in \Gamma^\infty(\Lambda^l TC)$ and $\alpha \in \Omega^k(C)$:

$$d\alpha(X,Y) = [i_a(\alpha)X,Y] - (-1)^{|X|} [X,i_a(\alpha)Y] - i_a(\alpha)[X,Y] \quad \text{(*)}$$

where for the left-hand side, we define

$$d\alpha(X,Y) = (d\alpha)_{ij} i_a(dx^i)X \wedge i_a(dx^j)Y$$

in a coordinate patch. The validity of Equation 358 for one-forms of the type $\alpha = f dg$ follows by the usual Schouten calculus. By $R$-linearity of Equation 359, its validity follows for general 1-forms in every coordinate patch and hence also globally. Let us define, using the curvature $\Omega$, the map

$$\Omega : S^2(\prod_{i}(S^i g \otimes T_{\text{poly}}(C)[1])^G) \longrightarrow (\prod_{i}(S^i g \otimes T_{\text{poly}}(C))^G[1]$$

defined on homogeneous and factorizing elements $P_j \otimes X_j \in \prod_{i}(S^i g \otimes T_{\text{poly}}(C)[1])^G$, $j = 1,2$ by

$$\Omega(P_1 \otimes X_1 \vee P_2 \otimes X_2) = (-1)^{|X_1|} e_i \vee P_1 \vee P_2 \otimes \Omega^f(X_1, X_2).$$

This map is well defined, i.e. in fact graded symmetric, and of degree 0. With a slight abuse of notation we denote also by

$$\Omega : S^\bullet(\prod_{i}(S^i g \otimes T_{\text{poly}}(C)[1])^G) \longrightarrow S^{\bullet-1}(\prod_{i}(S^i g \otimes T_{\text{poly}}(C)[1])^G$$

its extension as a coderivation of degree 0, i.e.

$$\Omega(X_1 \vee \cdots \vee X_k) = \sum_{\sigma \in \text{Sh}(2,k-2)} e(\sigma)\Omega(X_{\sigma(1)} \vee X_{\sigma(2)} \vee X_{\sigma(3)} \vee \cdots \vee X_{\sigma(k)}$$

for $X_j \in \prod_{i}(S^i g \otimes T_{\text{poly}}(C)[1])^G$. Note that for every $k \in \mathbb{N}$ and $X_j \in \prod_{i}(S^i g \otimes T_{\text{poly}}(C)[1])^G$, we have that

$$\Omega^k(X_1 \vee \cdots \vee X_k) = 0$$

since $\Omega$ decreases the symmetric degree by one and hence the expression

$$e^\Omega := \sum_k \frac{1}{k!} \Omega^k$$

$$e^\Omega := \sum_k \frac{1}{k!} \Omega^k$$
is a well defined map. Since $\Omega$ is a coderivation of degree 0, it is even a coalgebra morphism. Its components are given by

$$(e^\Omega)^\ell_k = \frac{1}{(k - \ell)!} \Omega^{k-\ell},$$

which can be seen again by counting symmetric degrees. This shows in particular, that $(e^\Omega \circ (\cdot)^\text{hor}) = (\cdot)^\text{hor}$. We proceed now inductively, so let us assume that $e^\Omega \circ (\cdot)^\text{hor}$ coincides with $i_\infty$ from Proposition 3.3 up to order $k$. For $X_j \in T_{\text{poly}}(M_{\text{red}})[1]$, $j = 1, \ldots, k + 1$, we have

$$i_{\infty, k+1}(X_1 \lor \cdots \lor X_{k+1}) = h \circ Q_2 \circ i_{\infty,k+1}^2(X_1 \lor \cdots \lor X_{k+1})$$

$$= \sum_{j=1}^{k} \sum_{\sigma \in \text{Sh}(i,k+1-k)} \binom{k}{j} \epsilon(\sigma) h \circ Q_2 \left( i_{\infty,j}^1(X_{\sigma(1)} \lor \cdots \lor X_{\sigma(\ell)}) \lor i_{\infty,k+1-i}^1(X_{\sigma(j+1)} \lor \cdots \lor X_{\sigma(k+1)}) \right)$$

Let us now take a look at

$$h \circ Q_2 \left( \Omega^{j-1}(X_{\sigma(1)}^{\text{hor}} \lor \cdots \lor X_{\sigma(j)}^{\text{hor}}) \lor \Omega^{k-j}(X_{\sigma(j+1)}^{\text{hor}} \lor \cdots \lor X_{\sigma(k+1)}^{\text{hor}}) \right)$$

$$= (-1)^{\frac{1}{j}} \left( \sum_{j=1}^{k} \sum_{\sigma \in \text{Sh}(j,k+1-k)} \epsilon(\sigma) h \circ Q_2 \left( \Omega^{j-1}(X_{\sigma(1)}^{\text{hor}} \lor \cdots \lor X_{\sigma(j)}^{\text{hor}}) \lor \Omega^{k-j}(X_{\sigma(j+1)}^{\text{hor}} \lor \cdots \lor X_{\sigma(k+1)}^{\text{hor}}) \right) \right)$$

$$= \frac{1}{k} \Omega \left( \Omega^{j-1}(X_{\sigma(1)}^{\text{hor}} \lor \cdots \lor X_{\sigma(j)}^{\text{hor}}) \lor \Omega^{k-j}(X_{\sigma(j+1)}^{\text{hor}} \lor \cdots \lor X_{\sigma(k+1)}^{\text{hor}}) \right)$$

and hence

$$i_{\infty, k+1}(X_1 \lor \cdots \lor X_{k+1})$$

$$= \sum_{j=1}^{k} \sum_{\sigma \in \text{Sh}(i,k+1-k)} \binom{k}{j} \epsilon(\sigma) \Omega \left( \Omega^{j-1}(X_{\sigma(1)}^{\text{hor}} \lor \cdots \lor X_{\sigma(j)}^{\text{hor}}) \lor \Omega^{k-j}(X_{\sigma(j+1)}^{\text{hor}} \lor \cdots \lor X_{\sigma(k+1)}^{\text{hor}}) \right)$$

$$= \frac{1}{k!} \Omega^k(X_1 \lor \cdots \lor X_{k+1}).$$

The last equality follows from the observation that

$$\Omega^k(X_1 \lor \cdots \lor X_{k+1}) = \Omega(\Omega^{k-1}(X_1 \lor \cdots \lor X_{k+1}))$$

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becomes a DGLA morphism.

This series actually well-defined in $T_{\text{Cart}}(C)$, since we have

$$\Pi = \sum_{k \geq 1} \frac{1}{k!} i_{\infty,k}(\pi^\vee)^k,$$

This series actually well-defined in $T_{\text{Cart}}(C)$, since we have

$$\Pi = \sum_{k \geq 1} \frac{1}{k!} \frac{1}{(k-1)!} \Omega^{k-1}((\pi^\text{hor})^\vee)^k$$

using the explicit for of $i_{\infty}$ as in Proposition [4.15]. But

$$\Omega^{k-1}((\pi^\text{hor})^\vee)^k \in (S^{k-1} \otimes T^1_{\text{poly}}(C))^G,$$

whence $\Pi \in MC(T_{\text{Cart}}(C))$ is well-defined. The identity $p(\Pi) = \pi$ is then clear using again the explicit form.

Remark 4.17 In particular, the above proposition shows not only that if $C$ admits a flat connection, then $i_{\infty}$ has $i_1 = (\cdot)^\text{hor}$ as only structure map, but also how to correct the horizontal lift in order to obtain an $L_\infty$-quasi-isomorphism.

Having seen the importance of the ad-hoc defined differential $\partial$ on $T_{\text{Cart}}(C)$, we lean now again towards $T_{\text{Tay}}(C \times g^*)$ and try to find an extension of the differential $-[J, \cdot]$ in order to make the inclusion $\iota: T_{\text{Cart}}(C) \rightarrow T_{\text{Tay}}(C \times g^*)$ a quasi-isomorphism with respect to $\partial$. As a first step we have:

**Proposition 4.18** The map $[\pi_{\text{KKS}}, \cdot]$ is a well-defined differential on $T_{\text{Tay}}(C \times g^*)$ that is explicitly given by

$$[\pi_{\text{KKS}}, \xi \otimes P \otimes \alpha \otimes X] = \xi \otimes \delta_{\text{cr}}(P \otimes \alpha \otimes X) + \partial(\xi \otimes P \otimes \alpha \otimes X).$$

Moreover, the canonical inclusion

$$\iota: (T_{\text{Cart}}(C), \partial, [\cdot, \cdot]) \longrightarrow (T_{\text{Tay}}(C \times g^*), [\pi_{\text{KKS}} = J, \cdot], [\cdot, \cdot])$$

becomes a DGLA morphism.

**Proof:** Since the bracket does not depend on the $S^1$-part we restrict ourselves to $P \otimes \alpha \otimes X$. Let us compute

$$\left[\frac{1}{2} f^k_{ij} e_k \otimes e^i \wedge e^j, P \otimes \alpha \otimes X\right] = \frac{1}{2} f^k_{ij} (e_k \otimes [e^i \wedge e^j, P \otimes \alpha] \otimes X + [e_k, P \otimes \alpha] \wedge e^i \wedge e^j \otimes X)$$

$$= f^k_{ij} e_k \wedge i_{\alpha}(e^i) P \otimes e^\alpha \wedge \alpha \otimes X - \frac{1}{2} f^k_{ij} e_k \otimes e^i \wedge e^j \wedge i_{\alpha}(e_k) \alpha \otimes X$$

and

$$[-e^i \wedge (e_i)_C, P \otimes \alpha \otimes X] = -P \otimes e^i \wedge \alpha \otimes L_{(e_i)_C} X - \frac{1}{2} [\cdot]^{|X|} |X| i_{\alpha}(e^i) P \otimes \alpha \otimes X \wedge (e_i)_C,$$

where $|X|$ denotes the multivector field degree and $|\alpha|$ the form degree. Putting this together we directly get (4.22). Since $\pi_{\text{KKS}}$ is a Poisson structure, we directly see that it squares to zero. Moreover, $[\pi_{\text{KKS}}, \cdot]$ boils down to $\partial$ when restricted to elements in the image of the canonical inclusion $\iota$, i.e. in $(1 \otimes T_{\text{poly}}^\infty(S^1 g \otimes 1 \otimes T_{\text{poly}}(C)))^G$. 


Alternatively, the identity

\[ [\pi_{\text{KKS}}, J] = \lambda, \]  

implies that the canonical \( \pi_{\text{KKS}} \) defines a curved Maurer-Cartan element in the curved DGLA \((T_{\text{poly}}(C \times \mathfrak{g}^*), \lambda, -[J, \cdot], [\cdot, \cdot])\). Therefore, twisting by \( \pi_{\text{KKS}} \) yields a Lie algebra differential on \( T_{\text{poly}}(C \times \mathfrak{g}^*) \) with curvature zero. The next step is, of course, to check if \( \iota \) is still a quasi-isomorphism.

**Proposition 4.19** The inclusion

\[ \iota: (T_{\text{Cart}}(C), \partial, [\cdot, \cdot]) \rightarrow (T_{\text{poly}}(C \times \mathfrak{g}^*), [\pi_{\text{KKS}} - J, \cdot], [\cdot, \cdot]) \]  

is a quasi-isomorphism of DGLAs.

**Proof:** Let us compute the cohomology of \( T_{\text{poly}}(C \times \mathfrak{g}^*) \) by interpreting it as a double complex. The two differentials are \([-J, \cdot]\) and \([\pi_{\text{KKS}}, \cdot]\) and as bigrading we set

\[ C^{p,q} = (S^q \mathfrak{g}^* \otimes \prod_{i=0}^{\infty} (S^i \mathfrak{g} \otimes (\Lambda^q \mathfrak{g}^* \otimes T_{\text{poly}}(C)))^G) \mathfrak{g}. \]

One can directly see that the differentials are compatible with the bigrading in the sense that

\[ [-J, \cdot]: C^{p,q} \rightarrow C^{p,q+1}, \quad \text{and} \quad [\pi_{\text{KKS}}, \cdot]: C^{p,q} \rightarrow C^{p+1,q}. \]

By Proposition 4.11, the cohomology of \([-J, \cdot]\) is given by \((\prod_{i=0}^{\infty} (S^i \mathfrak{g} \otimes T_{\text{poly}}(C)))^G\), on which the horizontal differential \([\pi_{\text{KKS}}, \cdot]\) is just \(\partial\). Thus \(\iota\) is an isomorphism on the first sheet and thus on the cohomology.

The above results show that the Cartan model is an interwinder of \(T_{\text{poly}}(C \times \mathfrak{g}^*)\) and \(T_{\text{poly}}(M_{\text{red}})\), which can be summarized in the following diagram.

\[ \begin{array}{c}
(T_{\text{poly}}(C \times \mathfrak{g}^*), [\pi_{\text{KKS}} - R - J, \cdot], [\cdot, \cdot]) \\
\downarrow i \\
(T_{\text{Cart}}(C), \partial, [\cdot, \cdot]) \\
\downarrow i_{\infty} \\
(T_{\text{poly}}(M_{\text{red}}), [\cdot, \cdot]).
\end{array} \]

So far we have shown that both \(\iota\) and \(p\) are DGLA morphisms and also quasi-isomorphisms. For convenience, we included the \(L_\infty\)-quasi-inverse \(i_{\infty}\) of \(p\). From this diagram and the fact that every \(L_\infty\)-quasi-isomorphism is quasi-invertible, we have the following:

**Theorem 4.20** There exists an \(L_\infty\)-quasi-isomorphism

\[ (T_{\text{poly}}(C \times \mathfrak{g}^*), [\pi_{\text{KKS}} - J, \cdot], [\cdot, \cdot]) \rightarrow (T_{\text{poly}}(M_{\text{red}}), 0, [\cdot, \cdot]). \]

Note that the KKS Poisson structure is not defined on \(M\), but just in an open neighbourhood of \(C\). Recall that we aim to find a curved \(L_\infty\)-morphism

\[ T_{\text{red}}: (T_\mathfrak{g}(M), \lambda, -[J, \cdot], [\cdot, \cdot]) \rightarrow (T_{\text{poly}}(M_{\text{red}}), 0, 0, [\cdot, \cdot]) \]

and its formal correspondence. To achieve this we proceed in the following way: we construct a (non-curved) quasi-inverse of \(\iota\) in Diagram 4.2 denoted by \(P\) and then twist it by \(-\pi_{\text{KKS}}\) in order to find a curved morphism

\[ P^{-\pi_{\text{KKS}}}: (T_{\text{poly}}(C \times \mathfrak{g}^*), \lambda, -[J, \cdot], [\cdot, \cdot]) \rightarrow (T_{\text{poly}}(M_{\text{red}}), \lambda_{\text{red}}, [\pi_{\text{KKS,red}}, \cdot], [\cdot, \cdot]) \]

for

\[ \lambda_{\text{red}} := \sum_{k \geq 0} \frac{(-1)^k}{k!} \sum_{\pi_{\text{KKS}}} P_{1+k} \lambda \pi_{\text{KKS}}^k \]

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\[ \pi_{KKS, \text{red}} := \sum_{k \geq 0} \frac{(-1)^k}{k!} P_k(\pi_{\text{KKS}}). \]

There are now two issues with this approach:

- Since we did not introduce a complete filtration on the involved DGLAs, we have to check by hand that both of the series actually converge in a suitable sense.
- This is actually not what we want, since our target, i.e. \( T_{\text{poly}}(M_{\text{red}}) \), has to have zero curvature and zero differential.

This two problems are solved in Section 4.4, where we construct a quasi inverse of \( \iota \) such that \( \lambda_{\text{red}} = \pi_{\text{KKS,red}} = 0 \) and we show that the series are well-defined. But at first we need to extend our considerations to the formal setting, where we have a complete filtration by \( \hbar \).

### 4.3 Formal Equivariant Multivector Fields and Their Reduction

We want to consider the formal analogue of the equivariant multivector fields on \( M \) from Eq. (4.4). Since we are only interested in formal Maurer-Cartan elements, we have to rescale the curvature by \( \hbar \), i.e. we consider the curved DGLA

\[ ((\mathcal{S}g^* \otimes T_{\text{poly}}(M))^G)[[\hbar]], h\lambda, [-J, \cdot], [\cdot, \cdot], [\cdot, \cdot]. \]

A formal curved Maurer-Cartan elements \( h(\pi - J') \in h(\mathcal{S}g^* \otimes T_{\text{poly}}(M))^G[[\hbar]] \) corresponds to an invariant formal Poisson structure \( \pi \) with formal momentum map \( J + hJ' \).

The Taylor series expansion discussed in Section 4.1 allows us to interpret the element \( h\pi_{\text{KKS}} \) as a formal curved Maurer-Cartan element. Thus we can perform the twisting procedure, yielding the following flat DGLA:

\[ (T_{\text{Tay}}(C \times g^*)[[\hbar]], h\pi_{\text{KKS}} - J, \cdot], [\cdot, \cdot]). \]

For a formal Maurer-Cartan element \( h(\pi - J') \) one can check that \( \pi_{\text{KKS}} + \pi \) is a \( G \)-invariant formal Poisson structure with formal momentum map \( J + hJ' \) as desired and again \( \pi = \pi_C + O(h) \).

Moreover, the Cartan model for the multivector fields reads in the formal setting:

\[ (T_{\text{Cart}}(C)[[\hbar]], h\partial, [\cdot, \cdot]) \]

and the bracket on \( T_{\text{poly}}(M_{\text{red}})[[\hbar]] \) is simply extended \( h \)-bilinearly. Summarizing, we have the following claim.

**Theorem 4.21** We have built the following diagram

\[
\begin{array}{ccc}
(T_{\text{Tay}}(C \times g^*)[[\hbar]], [h\pi_{\text{KKS}} - J, \cdot], [\cdot, \cdot]) & \overset{\iota}{\longrightarrow} & (T_{\text{Cart}}(C)[[\hbar]], h\partial, [\cdot, \cdot]) \\
\downarrow & & \downarrow \\
(T_{\text{poly}}(M_{\text{red}})[[\hbar]], 0, [\cdot, \cdot])
\end{array}
\]

where both maps are DGLA morphisms and where \( \iota \) is still a quasi-isomorphism of DGLAs.

**Proof:** The proof essentially follows from the above considerations. More explicitly, the inclusion of the Cartan model into \( T_{\text{Tay}}(C \times g^*)[[\hbar]] \) is a quasi-isomorphism of DGLAs since the bracket with \( [-J, \cdot] \) is not scaled by \( h \) and \([h\pi_{\text{KKS}}, \cdot]\) is just \( h\partial \) in the cohomology of \([−J, \cdot]\). In other words, the argument from Proposition 4.19 applies.

Note that here we only use the fact that the \( L_\infty \)-quasi-inverse of \( \iota \) exists. In Section 4.4 we give an explicit formula for this map.
Remark 4.22 (Laurent series) We observe that the map \( p \) in the above theorem is not a quasi-isomorphism due to the scaling problem by \( h \). Concerning the projection from the Cartan model to \( \mathcal{M}_{\text{red}} \) we still have the map \( h \) satisfying \( h \partial h + h \partial = h(\deg_q + \deg_{\text{over}}) \) \( \text{id} \), as in Proposition 4.14. However, since we are not allowed to divide by \( h \), the projection \( \text{pr} \) in the formal setting is no longer a quasi-isomorphism. We remark that, if we consider instead Laurent series in \( h \) in all the complexes, e.g. \( T_{\text{poly}}(\mathcal{M}_{\text{red}})[h^{-1}, h] \), then it remains a quasi-isomorphism.

Moreover, we know from [15] Thm. 4.6 that \( L_\infty \)-quasi-isomorphisms induce bijections on the equivalence classes of formal Maurer-Cartan elements. In our setting this yields:

Corollary 4.23 Every formal Maurer-Cartan element \( h(\pi - J') \) in \( T_{\text{poly}}(\mathcal{C} \times \mathfrak{g}^*)[[h]] \) is equivalent to a formal Maurer-Cartan element \( h \pi_C \in T_{\text{Cart}}(\mathcal{C})[[h]] \subset T_{\text{poly}}(\mathcal{C} \times \mathfrak{g}^*)[[h]] \).

In other words, the above Corollary states that every formal Poisson structure \( \pi_{\text{KKS}} + \pi \) with formal momentum map \( J + hJ' \) is equivalent to a formal Poisson structure \( \pi_{\text{KKS}} + \pi_C \) with undeformed momentum map \( J \). Finally, we can construct an explicit equivalence transformation from a generic Maurer-Cartan element \( h(\pi - J') \) to one with \( J' = 0 \). Set \( X^i_h = hJ^i e_i \) and \( J^2_i = \exp(X^i_h)(J_i) - J_i - hJ^i \). One can recursively define for \( k \geq 1 \)

\[
X^i_h + 1 = -J^i_h e_i := -(\exp(X^i_h) \cdots \exp(X^i_h)(J_i) - J_i - hJ^i) e^i. \tag{4.25}
\]

Proposition 4.24 Let \( h(\pi - J') \) be a formal Maurer-Cartan element in \( T_{\text{poly}}(\mathcal{C} \times \mathfrak{g}^*)[[h]] \). Then

\[
X^\infty_h = \log \left( \lim_{k \to \infty} \exp(X^i_h) \cdots \exp(X^i_h) \right) \tag{4.26}
\]

satisfies \( \exp(X^\infty_h)(J_i) = J_i + hJ^i \) and hence \( h \exp(-X^\infty_h)(\pi_{\text{KKS}} + \pi) - h\pi_{\text{KKS}} \) is a formal Maurer-Cartan element in \( T_{\text{poly}}(\mathcal{C} \times \mathfrak{g}^*)[[h]] \) equivalent to \( h(\pi - J') \).

Proof: Note that \( X^i_h \in \mathcal{O}(h) \) and inductively one gets

\[
J_i + hJ^i + J^k_i = \exp(X^i_h) \exp(X^{i-1}_h) \cdots \exp(X^0_h)(J_i) = \exp(X^i_h)(J_i + hJ^i + J^k_i) = J_i + hJ^i + J^k_i + hX^i_h(J_i) + \mathcal{O}(h^{k+1}).
\]

Hence \( J^k_i \in \mathcal{O}(h^{k+1}) \) as well as \( X^k_h \in \mathcal{O}(h^{k+1}) \). In particular, \( X^\infty_h \) is well-defined and satisfies

\[
\exp(X^\infty_h)(J_i) = J_i + hJ^i + \lim_{k \to \infty} J^k_i = J_i + hJ^i
\]

in the \( h \)-adic topology. The gauge equivalence \( \exp(-X^\infty_h) \) therefore maps \( h(\pi - J') \) to

\[
\exp(-X^\infty_h) \circ h(\pi - J') = \exp(-X^\infty_h)(h\pi_{\text{KKS}} - J + h(\pi - J')) - (h\pi_{\text{KKS}} - J) = h \exp(-X^\infty_h)(\pi_{\text{KKS}} + \pi) - h\pi_{\text{KKS}},
\]

compare [25] Prop. 6.2.34 for a formula of the gauge action.

4.4 \( L_\infty \)-quasi-inverse of \( \iota \)

Finally, we want to find an explicit description of the \( L_\infty \)-quasi-inverse of \( \iota \), i.e. an \( L_\infty \)-quasi-isomorphism

\[
P : (T_{\text{poly}}(\mathcal{C} \times \mathfrak{g}^*), [\pi_{\text{KKS}} - J, \cdot], [\cdot, \cdot]) \to (T_{\text{poly}}(\mathcal{M}_{\text{red}}), 0, [\cdot, \cdot]).
\]

One can check that the homotopy \( h \) of \([-J, \cdot]\) from Proposition 4.14 does not commute with \([\pi_{\text{KKS}}, \cdot]\). The idea is to start with \([-J, \cdot]\) as differential on the Taylor decomposition and zero differential on the Cartan model, construct the \( L_\infty \)-quasi-isomorphism \( P \) in this case, and then investigate the compatibility with \([\pi_{\text{KKS}}, \cdot]\).
Let us focus on the following deformation retract of DGLA’s

\[
(T_{\text{Cart}}(C), 0) \xleftarrow{\iota} (T_{\text{asy}}(C \times g^*), [-J, \cdot]) \xrightarrow{\hbar} (\text{Cart}^\ast(C \times g^*), [-J, \cdot])
\]  

(4.27)

and apply the construction from Section 3. By Proposition 3.22 we have an $L_\infty$-quasi-isomorphism $P$ given by $P_1 = p$ and

\[
P_n = P_1^n = (R_2^n P_2^n - P_1^{n-1} Q_n^{n-1}) \circ H_n,
\]

where $Q$ and $R$ denote the $L_\infty$-structure on $S(T_{\text{asy}}(C \times g^*)[1])$ and on $S(T_{\text{Cart}}(C)[1])$, respectively. Moreover, $H_n$ is the extension of

\[
h(\xi \otimes \alpha \otimes \alpha \otimes X) = \begin{cases} \frac{-1}{\deg_S \alpha + \deg_A \alpha} i_\hbar(\c e) \xi \otimes \text{deg}_S \alpha + \deg_A \alpha & \text{if } \deg_S \alpha + \deg_A \alpha \neq 0 \\ 0 & \text{else} \end{cases}
\]

since $Q_1^1 = [J, \cdot]$, compare Proposition 4.14

**Lemma 4.25** For $n = 2$ one has

\[
P_2(X_1 \vee X_2) = -p((-1)^{|X_1|}[hX_1, X_2] - [X_1, hX_2])
\]

for all homogeneous $X_1, X_2 \in T_{\text{asy}}(C \times g^*)[1]$.

**Proof:** One has $P_2^2 \circ H_2 = 0$. Furthermore, for $Q_2^1(X_1, Y_1) = (-1)^{|X_1|}[hX_1, X_2]$ with $|Y_1|$ denoting the shifted degree in $T_{\text{asy}}(C \times g^*)[1]$ we have with the formula for $H_2$, see [17] p. 383,

\[
P_2(X_1 \vee X_2) = -p \circ Q_2^1 \circ H_2(X_1 \vee X_2)
\]

\[
= \frac{p}{2}((-1)^{|X_1|+1}[hX_1, X_2 + ipX_2] + (-1)^{|X_1|+|X_1|+1}[X_1 + ipX_1, hX_2])
\]

\[
= -p((-1)^{|X_1|}[hX_1, X_2] - [X_1, hX_2]).
\]

The last step is easily seen for homogeneous elements by counting the $g^*$-degrees. In fact, if $X_2 = ipX_2$, then $hX_2 = 0$ and the statement holds. If $ipX_2 = 0$, then $p(hX_1, X_2) = 0$ since the bracket contains at least one $g^*$-component that is annihilated by $p$. The same holds for $1 \leftrightarrow 2$. □

As a next step we want to obtain an $L_\infty$-morphism between $(T_{\text{asy}}(C \times g^*), [\pi_{\text{KKS}}, \cdot])$ and $(T_{\text{Cart}}(C), \partial)$. Let us first observe that $P_n$ contains $n - 1$ brackets and $n - 1$ applications of $h$, increasing the $\Lambda^g$-degrees. This implies that the $P_n$ are non-zero only if all arguments have no $\Lambda^g$-contribution and the sum of the $\Sigma^g$-degrees is $n - 1$. As a consequence, all $n - 1$ brackets consist of pairings between $\Lambda^g$-components coming from $h$ and the $\bigwedge \Sigma^g$-components, whereas the $T_{\text{asy}}(C)$-components are just wedged together. Moreover, the first term in (4.28) does not contribute since the bracket $R_2^1$ is here in $C$-direction and we have

\[
P_n = P_1^n = -P_1^{n-1} \circ Q_n^{n-1} \circ H_n.
\]

Therefore, to prove the compatibility of $P$ with the differentials $[\pi_{\text{KKS}}, \cdot]$ and $\partial$ we only have to show

\[-\partial P_1 = P_1^1 \circ (Q^g)^n_n,
\]

where $(Q^g)^n_n$ is the extension of $-[\pi_{\text{KKS}}, \cdot]$. By the proof of Proposition 4.18 and the above arguments, the only part with a non-trivial contribution is the extension of $-\partial = -\id \otimes i_\hbar(e^s) \otimes \id \otimes (\c e)_{C \wedge}$. □

**Proposition 4.26** The map $P$ from (4.30) is an $L_\infty$-quasi-isomorphism from the Taylor series expansion $(T_{\text{asy}}(C \times g^*), [\pi_{\text{KKS}} - J, \cdot])$ to $(T_{\text{Cart}}(C), \partial)$ and an $L_\infty$-quasi-inverse to the inclusion $\iota$ from Proposition 3.22. The same holds in the formal setting with the rescaled differentials $[h\pi_{\text{KKS}} - J, \cdot]$ and $h \partial$. □
We only have to show the desired equations (4.28) vanish later under \( \partial \). In addition, we know \([h, \partial] = 0\) and thus \( H_n(Q^\ast)_n = -(Q^\ast)_n h_n\). If we prove \((Q^\ast)_n Q^n_{n+1} = -(Q^\ast)_{n+1}(Q^\ast)_{n+1}^\ast\) then \((4.30)\) gives inductively
\[
-\partial P^n_{n+1} = \partial P^n_{n+1} Q^n_{n+1} + h_n Q^n_{n+1} h_n + (Q^\ast)_n h_n.
\]
We only have to show the desired \((Q^\ast)_n Q^n_{n+1} = -(Q^\ast)_{n+1}(Q^\ast)_{n+1}^\ast\) on the image of \(H_{n+1}\) on elements with \(\Lambda\)-degree zero and with sum of \(S\)-degrees \(n\), where \(n \geq 1\). In particular, elements in this image have \(\Lambda\)-degree 1 and \(S\)-degree \(n - 1\). Consider \(X_1 \cdots X_{n+1}\) where \(\text{w.l.o.g.}\ X_1\) has a \(\Lambda\)-contribution, then the bracket has to be with respect to this vector field, the other terms vanish later under \(p\).

Using \((2.3)\) we get as only non-vanishing contribution
\[
Q^n_{n+1}(X_1 \cdots X_{n+1}) = \sum_{i=2}^{n+1} (-1)^{|X_i|} (|X_2| + \cdots + |X_{n+1}|) Q^n_{n+1}(X_1 \vee X_i) \land X_2 \land \cdots \land X_{n+1},
\]
where \(|X_i|\) denotes the degree in \(T_{\text{tay}}(C \times g^\ast)[1]\). A straightforward computation shows again
\[
-\partial Q^n_{n+1}(X_1 \vee X_i) = Q^n_{n+1}(\partial X_1 \vee X_i + (-1)^{|X_i|} X_1 \vee \partial X_i)
\]
and combining these two expressions the desired result follows by a comparison of the signs of all terms involving \(\partial X_j\).

\(\square\)

Remark 4.27 Note that here we can not use the usual twisting procedure since we have no complete filtration compatible to \(P\) such that \(\pi_{\text{KKS}}\) is of degree one. Of course, this is to be expected since the differential on \(T_{\text{cart}}(C)\) is not an inner one.

We can also show that \(P\) is compatible with the curvature, which is easier to show in the formal setting.

Proposition 4.28 The map \(P\) from \((4.30)\) is an \(L_\infty\)-morphism between the curved DGLAs \((T_{\text{tay}}(C \times g^\ast)[h], [\cdot, \cdot], [\cdot, \cdot, \cdot])\) and \((T_{\text{cart}}(C)[h], 0, \partial, [\cdot, \cdot], [\cdot, \cdot, \cdot])\).

\(\text{PROOF:}\) We can twist \(P\) from Proposition 4.26 with \(-h\pi_{\text{KKS}}\) as in \([12, \text{Lemma 2.7}]\). Then we obtain an \(L_\infty\)-morphism \((T_{\text{tay}}(C \times g^\ast)[h], [\cdot, \cdot], [\cdot, \cdot, \cdot])\) to \((T_{\text{cart}}(C), 0, \partial)\). This is clear since the new codifferential on \(S(T_{\text{tay}}(C \times g^\ast)[h])[1]\) is given by
\[
Q^n_0 = Q^n_1 - h\pi_{\text{KKS}} + \frac{1}{2}Q_2(-h\pi_{\text{KKS}}, -h\pi_{\text{KKS}}) = -h\pi_{\text{KKS}} + J, h\pi_{\text{KKS}} = -h\lambda.
\]
\[
Q^n_1(X) = Q^n_1(X) + Q_2(-h\pi_{\text{KKS}}, X) = h\pi_{\text{KKS}} + J + h\pi_{\text{KKS}}, X.
\]
Since \(\pi_{\text{KKS}} - R\) contains a \(\Lambda\)-degree the twisting does not change the \(L_\infty\)-structure on the Cartan model and the twisted morphism is just given by \(P\).

\(\square\)

Note that in this case \(P\) is no longer a quasi-isomorphism, and that the result also holds in the classical setting:

Corollary 4.29 The map \(P\) from \((4.30)\) is also an \(L_\infty\)-morphism between the curved DGLAs \((T_{\text{tay}}(C \times g^\ast), \lambda, [\cdot, \cdot], [\cdot, \cdot, \cdot])\) and \((T_{\text{cart}}(C), 0, \delta)\).

\(\text{PROOF:}\) Since the morphism \(P\) is \(h\)-linear we can compute explicitly that the Taylor coefficients of \(P\) are compatible with the above curved DGLA structures. By the construction of \(P\) we know
\[
R_{2}^{1} P_{n}^{2} = P_{n+1}^{1} Q_{n}^{n} + P_{n-1}^{1} Q_{n}^{n-1},
\]
where \(R_{2}^{1}\) is the bracket on the Cartan model and \(Q_{n}^{1}\) is the extension of \([J, \cdot, \cdot]\). Moreover, we have by Proposition 4.28
\[
h R_{1}^{1} P_{n}^{1} + R_{2}^{1} P_{n}^{2} = P_{n+1}^{1} (h Q_{0} \vee \cdot) + P_{n}^{1} Q_{n}^{n} + P_{n-1}^{1} Q_{n}^{n-1},
\]
\[
21
\]
where $R^1 = -\partial$ and $Q_0 = -\lambda$. This gives
\[ \hbar R^1 P_n = P^1_{n+1}(\hbar Q_0 \vee \cdot) \implies R^1 P_n = P^1_{n+1}(Q_0 \vee \cdot) \]
and the statement is shown.

\begin{remark}
This can also be directly shown for the classical setting. Indeed, we do not have the complete filtration, but by the explicit forms of $P$ and $\pi_{\text{nice}}$ all the appearing series in the twisting procedure are still well-defined.
\end{remark}

5 The Reduction $L_\infty$-Morphism and Reduction of Formal Poisson Structures

Let us now merge together all the results we obtained in the previous sections in order to finalize the construction of the reduction scheme. Given a Lie group action $\Phi: G \times M \to M$ on a general manifold $M$ and an equivariant map $J: M \to \mathfrak{g}^*$ with value and regular value $0$ interpreted as an element $J \in (\mathfrak{g}^* \otimes \mathcal{E}_\infty(M))^G$. In [22] we defined the curved differential graded Lie algebra
\[ (T_\mathfrak{g}(M), \lambda_0 - [J, \cdot], [\cdot, \cdot]), \]
and we want to obtain an $L_\infty$-morphism to $T_{\text{poly}}(M_{\text{red}})$ with zero differential in order to reduce in particular formal Poisson structures.

5.1 The Reduction $L_\infty$-morphism

Under the above assumptions that the action is proper in an open neighbourhood of the constraint surface $C := J^{-1}(\{0\})$, we find an open $G$-invariant neighbourhood $C \subseteq M_{\text{nice}} \cong U_{\text{nice}} \subseteq C \times \mathfrak{g}^*$, such that the momentum map on $U_{\text{nice}}$ is just the projection on the second factor and such the group acts as the product of the action on $C$ and the coadjoint action. This yields the curved DGLA morphism
\[ [U_{\text{nice}} : (T_\mathfrak{g}(M), \lambda_0 - [J, \cdot], [\cdot, \cdot]) \longrightarrow (T_\mathfrak{g}(U_{\text{nice}}), \lambda_{U_{\text{nice}}}, -[J]_{U_{\text{nice}}}, [\cdot, \cdot]) \]
which is just the restriction to the invariant open subset $M_{\text{nice}}$ concatenated with the extension of the $G$-equivariant diffeomorphism to $U_{\text{nice}}$. Moreover, we know from [2] Lemma 3 that $U_{\text{nice}}$ is an open neighbourhood of $C \times \{0\}$ such that $U_{\text{nice}} \cap (\{p\} \times \mathfrak{g}^*)$ is star-shaped around $\{p\}$ for all $p \in C$, hence we also have the Taylor expansion as in Equation (4.6). It is a morphism of curved DGLA’s
\[ T_\mathfrak{g} : (T_\mathfrak{g}(U_{\text{nice}}), \lambda_{U_{\text{nice}}}, -[J]_{U_{\text{nice}}}, [\cdot, \cdot]) \longrightarrow (T_{\text{poly}}(C \times \mathfrak{g}^*), \lambda_0 - [J, \cdot], [\cdot, \cdot]), \]
With Proposition 4.28 and Corollary 4.29 we obtain furthermore a curved $L_\infty$-morphism
\[ P: (T_{\text{poly}}(C \times \mathfrak{g}^*), \lambda_0 - [J, \cdot], [\cdot, \cdot]) \longrightarrow (T_{\text{cart}}(C), 0, \partial, [\cdot, \cdot]), \]
and finally we have the projection $p: (T_{\text{cart}}(C), 0, \partial, [\cdot, \cdot]) \to (T_{\text{poly}}(M_{\text{red}}), 0, 0, [\cdot, \cdot])$ from Equation 4.15 that is a DGLA morphism and hence also a morphism of (curved) $L_\infty$-algebras.

Theorem 5.1 The concatenation of all the above morphism results in a curved $L_\infty$-morphism
\[ T_{\text{red}}: (T_\mathfrak{g}(M), \lambda_0 - [J, \cdot], [\cdot, \cdot]) \longrightarrow (T_{\text{poly}}(M_{\text{red}}), 0, 0, [\cdot, \cdot]), \]
called reduction $L_\infty$-morphism. Considering the setting of formal power series in $\hbar$ we can extend $T_{\text{red}}$ $\hbar$-linearly and obtain
\[ T_{\text{red}}: (T_\mathfrak{g}(M)[[\hbar]], \hbar \lambda_0 - [J, \cdot], [\cdot, \cdot]) \longrightarrow (T_{\text{poly}}(M_{\text{red}})[[\hbar]], 0, 0, [\cdot, \cdot]). \]
5.2 Reduction of Formal Poisson Structures

As mentioned above, a formal curved Maurer-Cartan element \( h(\pi - J') \in hT_\theta(M)[[h]] \) is an invariant formal Poisson structure \( h\pi \) with formal moment map \( J + hJ' \). By \( T_{\text{red}} \) we obtain therefore a formal Maurer-Cartan element

\[
h\pi_{\text{red}} = \sum_{k \geq 1} \frac{1}{k!} T_{\text{red}, k}(h(\pi - J')^\vee k)
\]  

(5.2)

in \( T_{\text{poly}}(M_{\text{red}})[[h]] \) which corresponds to a formal Poisson structure \( \pi_{\text{red}} \) on \( M_{\text{red}} \).

In order to show that this morphism gives indeed a non-trivial reduction scheme for formal Poisson structures we show at first that we recover the Marsden-Weinstein reduction. This classical setting is included in our formulation by considering special curved formal Maurer-Cartan elements \( h\pi \in hT_\theta(M)[[h]] \), where in fact \( \pi \in T_{\text{poly}}(M) \) does not depend on \( h \), i.e. is a classical \( G \)-invariant Poisson structure with momentum map \( \pi \).

\textbf{Proposition 5.2} The reduction procedure of Marsden-Weinstein coincides with the one via \( T_{\text{red}} \) from Theorem 5.2, for Maurer-Cartan elements \( h\pi \in hT_\theta(M)[[h]] \) with \( \pi \in T_{\text{poly}}(M) \).

\textbf{Proof:} By Lemma 4.10 we know that \( h\pi \) takes in the Taylor expansion \( h\pi \) the form \( h\pi_{\text{KKS}} + h\pi_C \), where \( \pi_C = \prod_i \pi_C^i \) with \( \pi_C^i \in S^i g \otimes T_{\text{poly}}(C) \). Then the application of \( p \circ P \) yields a Maurer-Cartan element \( h\pi_{\text{red}} \) in the reduced DGLA \( (T_{\text{poly}}(M_{\text{red}})[[h]], 0, [\cdot, \cdot]) \) via

\[
h\pi_{\text{red}} = \sum_{k \geq 1} \frac{1}{k!} p \circ P_k(h(\pi_{\text{KKS}} + \pi_C), \ldots, h(\pi_{\text{KKS}} + \pi_C)) = p(h\pi_C^0),
\]

so this series is indeed well-defined. This Maurer-Cartan element corresponds to a classical Poisson structure \( \pi_{\text{red}} \) with

\[
p^* \pi_{\text{red}}(d\phi, d\psi) = p^0 \pi_C^0(d\phi^* \phi, d\phi^* \phi) = \pi_C^0((\pi_{\text{KKS}} + \pi_C)(d\text{prolp}^* \phi, d\text{prolp}^* \psi)
\]

for \( \phi, \psi \in \mathcal{E}^\infty(M_{\text{red}}) \), where prol: \( \mathcal{E}^\infty(C) \rightarrow \mathcal{E}^\infty(C) \otimes \prod_i S^i g \) is the canonical prolongation. But this is just the usual reduced Poisson structure from Marsden-Weinstein reduction. \( \square \)

Now we want to show that our construction is indeed a non-trivial extension of the classical Marsden-Weinstein reduction to the formal setting. For simplicity, let us consider for a moment just a part of \( T_{\text{red}} \), namely the map

\[
\tilde{T}_{\text{red}} = p \circ P: (T_{\text{poly}}(C \times g^*)[[h]], h\lambda, -[J, \cdot], [\cdot, \cdot]) \rightarrow (T_{\text{poly}}(M_{\text{red}})[[h]], 0, 0, [\cdot, \cdot]).
\]

\textbf{Lemma 5.3} The induced map at the level of Maurer–Cartan elements

\[
\tilde{T}_{\text{red}}: MC(T_{\text{poly}}(C \times g^*)[[h]]) \rightarrow MC(T_{\text{poly}}(M_{\text{red}})[[h]])
\]

\[
h(\pi - J') \longmapsto \sum_{k \geq 1} \frac{1}{k!} \tilde{T}_{\text{red}, k}((h(\pi - J')^\vee k)
\]  

(5.3)

is a surjection.

\textbf{Proof:} Let \( h\pi_{\text{red}} \in MC(T_{\text{poly}}(M_{\text{red}})[[h]]) \), then we know from Corollary 4.10 that

\[
h\Pi = \sum_{k \geq 1} \frac{1}{k!} \tilde{T}_{\text{red}, k}((h\pi_{\text{red}})^\vee k)
\]

is a well-defined Maurer–Cartan element in \( T_{\text{cart}}(C)[[h]] \) with \( p(h\Pi) = h\pi \). Using Proposition 4.18 we see that \( h(\pi_{\text{KKS}} + \Pi) \in MC(T_{\text{poly}}(C \times g)[[h]]) \) and

\[
\sum_{k \geq 1} \frac{1}{k!} \tilde{T}_{\text{red}, k}((h(\pi_{\text{KKS}} + \Pi)^\vee k) = p(h\Pi) = \pi
\]

as desired. \( \square \)
5.3 Comparison of the Reduction Procedures

We conclude with a comparison of the different reduction procedures. More explicitly, we want to compare the reduction via $T_{\text{red}}$ from Theorem 5.4 with the with the reduction of formal Poisson structures via the homological perturbation lemma, see Appendix A.

In the setting of curved DGLA’s or curved $L_\infty$-algebras it is more tricky to talk about equivalent Maurer-Cartan elements. Thus we switch to the description of our reduction in terms of flat DGLA’s as in Theorem 4.21. Here we need $\pi_{\text{KKS}}$ which is not available in the general setting, so from now on we restrict ourselves to the Taylor expansion $(T_{\text{red}}(C \times \mathfrak{g}^*), h\pi_{\text{KKS}} - J, \cdot, [\cdot, \cdot])$.

Consider formal Poisson structure $\pi_h = \sum_{\ell=0}^{\infty} h^\ell \pi_{\ell} \in \Gamma^\infty(\Lambda^2 TM)[[h]]$ with formal equivariant momentum map $J_h = J + hJ^* : \mathfrak{g} \to \mathfrak{e}^\infty(M)[[h]]$. By Proposition A.3 one gets an induced formal Poisson bracket on $M_{\text{red}} = J^{-1}(\{0\})/G$ via

$$\pi^\ast\{u, v\}_{\text{red}} = \iota^\ast([\text{prol}\pi^\ast u], [\text{prol}\pi^\ast v])_h,$$

where the deformed restriction map is given by

$$\iota^\ast = \iota^\ast(\text{id} + i_u(hJ^*)h_0)^{-1} = \iota^\ast \sum_{k=0}^\infty (-i_u(hJ^*)h_0)^k,$$

(compare Proposition A.3). We directly see that the reduction procedure works analogously for $\pi_h \in T^{1}_{\text{Tay}}(C \times \mathfrak{g}^*)[[h]]$.

Theorem 5.4 The reduction of formal equivariant Poisson structures with formal momentum maps via

$$\tilde{T}_{\text{red}} = p \circ P : (T_{\text{red}}(C \times \mathfrak{g}^*))[h], [h\pi_{\text{KKS}} - J, \cdot, [\cdot, \cdot]] \to (T_{\text{poly}}(M_{\text{red}}))[h], 0, [\cdot, \cdot])$$

coincides with the reduction of formal Poisson structures via the homological perturbation lemma from Proposition A.3.

Proof: We show at first that the reduction procedures coincide on Maurer-Cartan elements of the form $h\pi_C$, i.e. where the quantum momentum map is just the classical momentum map. Note that by Corollary 4.23 every formal Maurer-Cartan element $h(\pi' - J')$ is equivalent to such a $h\pi_C$. Writing again $\pi'_C \in (S^\ast \mathfrak{g} \otimes T_{\text{poly}}(C))[h]$, the reduced Poisson structure via $\tilde{T}_{\text{red}}$ is easy to describe, namely by

$$h\pi_{\text{red}} = \sum_{k=1}^\infty \frac{1}{k!} \tilde{T}_{\text{red},k}(h\pi_C \vee \cdots \vee h\pi_C) = \sum_{k=1}^\infty \frac{h^k}{k!} p \circ P_k(\pi_C, \ldots, \pi_C) = p(h\pi_C^0).$$

In the reduction via the homological perturbation lemma one has $\iota^\ast = \iota^\ast$ and thus the reduced formal Poisson structures coincide by the same reasons as in the classical setting of Proposition 5.2.

The idea is now to use the explicit equivalence from Proposition 4.24. Let $h(\pi - J')$ be a formal Maurer-Cartan element in $T_{\text{red}}(C \times \mathfrak{g}^*)[[h]]$ and $X_h^\infty$ be the equivalence between the formal Maurer-Cartan elements $(\pi_{\text{KKS}} + \pi, J + hJ')$ and $(\pi_{\text{KKS}} + \pi_C, J)$. The reduction via the homological perturbation lemma maps both Poisson structures to the same formal Poisson structure on $M_{\text{red}}$. This follows from Formula A.13 for the equivalence between the reduced Poisson structures since $X_h^\infty$ differentiates only in direction of $\mathfrak{g}^*$. We only have to show that $\tilde{T}_{\text{red}}$ also maps both to the same one. But $X_h^\infty$ induces the following equivalence on the level of the reduced manifold

$$p \circ P^1(X_h^\infty \vee \exp(X_h^\infty) \circ h(\pi - J')) = 0,$$

see e.g. [5] Prop. 4.9], whence both reduced structures are again equal. This proves the theorem. \square

A BRST-Like Reduction of Formal Poisson Structures

In this section we want to recall a reduction scheme for formal Poisson structures similarly to the reduction of star products in [14] resp. to the BRST reduction as formulated in [2]. We recall at first the homological perturbation lemma adapted to our setting, see [6] Thm. 2.4] and [23] Chapter 2.4.

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A.1 Homological Perturbation Lemma

Definition A.1 (Homotopy equivalence data) A homotopy equivalence data (HE data) consists of two chain complexes \((C, d_C)\) and \((D, d_D)\) over a commutative ring \(R\) together with two quasi-isomorphisms

\[
p: C \rightarrow D \quad \text{and} \quad i: D \rightarrow C
\]  

(A.1)

and a chain homotopy

\[
h: D \rightarrow D \quad \text{with} \quad \text{id}_D - pi = d_D h + h d_D
\]  

(A.2)

between \(pi\) and \(\text{id}_D\).

For a shorter notation we will denote such a HE data by

\[
p: (C, d_C) \leftrightarrow (D, d_D): i, h.
\]

Moreover, we say that a graded map \(B: D_\bullet \rightarrow D_{\bullet -1}\) with \((d_D + B)^2 = 0\) is a perturbation of the HE data. The perturbation is called small if \(\text{id}_D + Bh\) is invertible, and the homological perturbation lemma states that in this case the perturbed HE data is again a HE data, see [6, Thm. 2.4] for a proof.

Proposition A.2 (Homological perturbation lemma) Let

\[
p: (C, d_C) \leftrightarrow (D, d_D): i, h
\]

be a HE data and let \(B\) be small perturbation of \(d_D\), then the perturbed data

\[
P: (C, \hat{d}_C) \leftrightarrow (D, \hat{d}_D): I, H
\]  

(A.3)

with

\[
A = (\text{id}_D + Bh)^{-1} B, \quad \hat{d}_D = d_D + B, \quad \hat{d}_C = d_C + iAp,
\]

\[
P = p - hAp, \quad I = i - iAh, \quad H = h - hAh
\]  

(A.4)

is again a HE data.

We will even encounter a simpler situation, namely that the complex \(C\) is concentrated in degree 0 and \(D_n = 0\) for \(n < 0\):

\[
\begin{array}{ccccccc}
0 & \leftarrow & D_0 & \leftarrow & D_1 & \leftarrow & \cdots \\
& & \downarrow{h_0} & & \downarrow{h_1} & & \\
& & \leftarrow & & \leftarrow & & \\
P & & & & & & \\
0 & \leftarrow & C_0 & \leftarrow & 0
\end{array}
\]  

(A.5)

In this case, the perturbed HE data corresponding to a small perturbation \(B\) according to (A.4) is given by

\[
P = p, \quad I = i - i(\text{id}_D + B_1 h_0)^{-1} B_1 h_0, \quad H = h - h(\text{id}_D + Bh)^{-1} Bh
\]

and, using the geometric power series, this can be simplified to

\[
P = p, \quad I = i(\text{id}_D + B_1 h_0)^{-1}, \quad H = h(\text{id}_D + Bh)^{-1}.
\]  

(A.6)

Here we denote by \(B_1: D_1 \rightarrow D_0\) the degree one component of \(B\), analogously for \(h\).
A.2 Formal Koszul Complex

We start with the classical Koszul complex $\Lambda_{\mathfrak{g}} \otimes \mathcal{C}^\infty(M)$ that can be interpreted as the smooth functions on $M$ with values in the complexified Grassmann algebra of $\mathfrak{g}$. The Koszul differential $\partial$ is given by

$$\partial: \Lambda^0 \mathfrak{g} \otimes \mathcal{C}^\infty(M) \longrightarrow \Lambda^1 \mathfrak{g} \otimes \mathcal{C}^\infty(M), \quad a \mapsto i(J_0)a = J_{0,0} i(a^0) a,$$

where $i$ denotes the left insertion and $J_0 = J_{0,0} i^0$ the decomposition of $J_0$ with respect to a basis $\epsilon^1, \ldots, \epsilon^n$ of $\mathfrak{g}^*$. The corresponding dual basis will be denoted by $e_1, \ldots, e_n$ and $\partial^2 = 0$ follows immediately with the commutativity of the pointwise product in $\mathcal{C}^\infty(M)$. The differential $\partial$ is also a derivation with respect to associative and super-commutative product on the Koszul complex, consisting of the $\wedge$-product on $\Lambda_{\mathfrak{g}}$ tensored with the pointwise product on the functions. Moreover, it is invariant with respect to the induced $\mathfrak{g}$-representation

$$\mathfrak{g} \ni \xi \mapsto \rho(\xi) = ad(\xi) \otimes id - id \otimes \mathcal{L}_{\xi, M} \in \text{End}(\Lambda_{\mathfrak{g}} \otimes \mathcal{C}^\infty(M))$$

as we have

$$\partial \rho(\epsilon_a)(x \otimes f) = f_{\epsilon_a}^i \epsilon_a \wedge i(\epsilon^i)x \otimes J_{0,i} f + f_{\epsilon_a}^j i(\epsilon^j)x \otimes J_{0,j} f + i(\epsilon^i)x \otimes J_{0,i} \{J_{0,a}, f\} = \rho(\epsilon_a) \partial(x \otimes f)$$

for all $x \in \Lambda_{\mathfrak{g}}$ and $f \in \mathcal{C}^\infty(M)$.

One can show that the Koszul complex is acyclic in positive degree with homology $\mathcal{C}^\infty(C)$ in order zero, and that one has a $G$-equivariant homotopy

$$h_t: \Lambda_{\mathfrak{g}} \otimes \mathcal{C}^\infty(M) \longrightarrow \Lambda^{t+1}_{\mathfrak{g}} \otimes \mathcal{C}^\infty(M)$$

given on $C \subset M_{\text{nice}} \subset C \times \mathfrak{g}^*$ by

$$h_t(c, \mu) = c_t \otimes \partial h_t(c, \mu) = \int_0^1 \partial h_t(c, \mu) dt, \quad \text{with} \quad h_0 = id_0 - \text{prol} \star \text{ and } \quad h_0 \circ \text{prol} = 0,$$

where $x \in \Lambda_{\mathfrak{g}} \otimes \mathcal{C}^\infty(C \times \mathfrak{g}^*)$ and $(c, \mu) \in C \times \mathfrak{g}^*$, see [2] Lemma 6] and [14] for the notation $M_{\text{nice}}$. In other words, this means that

$$\text{prol}: (\mathcal{C}^\infty(C), 0) \Rightarrow (\Lambda_{\mathfrak{g}} \otimes \mathcal{C}^\infty(M), \partial): \ast^*, h$$

is a HE data of the special type of $\mathfrak{g}$, i.e. we have the following diagram:

$$\begin{array}{ccc}
0 & \xleftarrow{\ast} & \mathcal{C}^\infty(M) \\
\text{prol} \downarrow & & \downarrow \partial h_0 \\
0 & \xleftarrow{\ast^*} & \mathcal{C}^\infty(C)
\end{array}$$

Let now $\pi$ be an invariant formal Poisson structure with formal equivariant momentum map $J_0$. In order to take care of the formal momentum map, we extend the Koszul complex $h$-linearly and gain the HE data

$$\text{prol}: (\mathcal{C}^\infty(C)[[h]], 0) \Rightarrow (\Lambda_{\mathfrak{g}} \otimes \mathcal{C}^\infty(M)[[h]], \partial): \ast^*, h.$$

Since the formal momentum map $J_0$ is a deformation of $J_0$ in the sense that the difference $\delta h - J_0 = J': g \mapsto h \mathcal{C}^\infty(M)[[h]]$ starts in order one of $h$, the formal differential $\partial h = i(J_0) = \partial + B$ with $B = i(J')$ on $\Lambda_{\mathfrak{g}} \otimes \mathcal{C}^\infty(M)[[h]]$ is a small perturbation in the sense of the homological perturbation lemma [A.2]. Indeed, $\partial^2 h = 0$ follows for the same reasons as $\partial^2 = 0$, and $id + Bh$ is invertible as formal power series since $Bh$ starts in order one of $h$. Consequently, the corresponding perturbed HE data

$$\text{prol}: (\mathcal{C}^\infty(C)[[\lambda]], 0) \Rightarrow (\Lambda_{\mathfrak{g}} \otimes \mathcal{C}^\infty(M)[[\lambda]], \partial h): \ast^*, h.$$
is given by
\[ \text{prol} = \prol, \quad \iota^* = \iota^*(\text{id} + B_1 h_0)^{-1}, \quad h = h(\text{id} + B h)^{-1}, \] (A.10)
compare (A.6). In particular, we have \( \iota^* \partial h = 0 \),
\[ \text{id}_{A^1 g \otimes \mathbb{C}^\infty(M)[[h]]} - \text{prol}^* = \partial_h h + h \partial_h \] (A.11)
as well as \( \iota^* \text{prol} = \text{id}_{\mathbb{C}^\infty(\mathcal{C} = [[h]])} \) because of \( h_0 \text{prol} = 0 \). Moreover, \( \partial_h \) is still a \( g \)-equivariant derivation of the algebra structure. Therefore, also \( \iota^* \) and \( h \) are \( g \)-equivariant as all involved maps are.

We denote the image of the deformed Koszul differential by
\[ \mathcal{J}_h = \text{im} \partial_h |_{A^1 g \otimes \mathbb{C}^\infty(M)[[h]]} = \mathcal{J}_1 \iota. \]
Since \( \text{prol}^* \) is a projection with kernel \( \mathcal{J}_h \), compare (A.11), we get with the injectivity of prol
\[ \mathcal{J}_h = \ker \iota^* |_{\mathbb{C}^\infty(M)[[h]]}. \]
As \( \partial_h \) is \( \mathcal{C}^\infty(M)[[h]] \)-linear, \( \mathcal{J}_h \) is an ideal in \( \mathcal{C}^\infty(M)[[h]] \) with respect to the pointwise product. Moreover, \( \mathcal{J}_h \) is a Poisson subalgebra of \( (\mathcal{C}^\infty(M)[[h]], \{ \cdot, \cdot \}) \) because of
\[ \iota^* \{ f, g \}_h = \iota^* \{ f^g g^i \{ J_i, J_j \}_h + f^j J_j \{ J_i, g^i \}_h + J_i g^i \{ f^j, J_j \}_h + J_i J_j \{ f^i, g^j \}_h \} = 0 \]
for \( f = f^i J_i, g = g^j J_j \in \mathcal{J}_h \). As usual, one can consider the Poisson normalizer
\[ \mathbb{B}_h = \{ f \in \mathcal{C}^\infty(M)[[h]] | \{ f, \mathcal{J}_h \} \subset \mathcal{J}_h \}, \]
the biggest Poisson subalgebra containing \( \mathcal{J}_h \) as Poisson ideal. Then we know that the quotient
is a Poisson algebra and we even have the following:

**Proposition A.3** There exists a unique formal Poisson structure \( \pi_\text{red} \) on \( M_\text{red} \) such that
\[ \mathbb{B}_h / \mathcal{J}_h \ni [f] \mapsto \iota^* f \in \pi^* \mathcal{C}^\infty(\mathcal{M}_\text{red})[[h]] \]
is an isomorphism of Poisson algebras with inverse \( \pi^* u \mapsto [\text{prol}^* u] \).

**Proof:** We have for \( u \in \mathcal{C}^\infty(\mathcal{M}_\text{red})[[h]], j = j^k J_k \in \mathcal{J}_h \) and \( f \in \mathbb{B}_h \)
\[ \iota^* \{ \text{prol}^* u, j \}_h = \iota^* \{ j^k \{ \text{prol}^* u, J_k \}_h + J_k \{ \text{prol}^* u, j^k \}_h \} = \iota^* \{ j^k J_k \mathcal{L}_{(c_k)} \text{prol}^* u \} = 0 \]
as well as
\[ \mathcal{L}_{(c_k)} \iota^* f = \iota^* \mathcal{L}_{(c_k)} f = \iota^* \{ f, J_i \}_h = 0, \]
thus the maps are both well-defined. The fact that the maps are mutually inverse is clear since
\[ \iota^* \text{prol} = \text{id} \quad \text{and} \quad \text{id} - \text{prol}^* = \partial_h h \in \mathcal{J}_h. \]
The compatibility with the pointwise product follows from the explicit form \( \iota^* = \iota^* \circ \sum_k (-B_1 h_0)^k \) and the fact that
\[ h_0 (f \text{prol} \phi) = \text{prol} \phi \cdot h_0 f, \]
which directly yields
\[ \iota^* ([f g]) = \iota^* ([f \text{prol}^* g]) = \iota^* f \cdot \iota^* g. \]
The compatibility of prol in the setting \( M = M_\text{nice} \) in the notation of [14] is clear since it is just a pull-back. In addition, we get a unique induced formal Poisson structure on \( M_\text{red} \) via
\[ \pi^* \{ u, v \}_\text{red} = \iota^* \{ [\text{prol}^* u], [\text{prol}^* v] \}_h. \]
Antisymmetry is clear and also the Jacobi identity follows directly, where we omit the sign for the equivalence classes:
\[ \pi^* \{ u, \{ v, w \}_\text{red} \}_h = \iota^* \{ \text{prol}^* u, \text{prol}^* \{ \text{prol}^* v, \text{prol}^* w \}_h \}. \]
Then one has even

\[ X = \pi^* \left( \{ \mathrm{prol} \pi^* u, \mathrm{prol} \pi^* v \}_h, \mathrm{prol} \pi^* w \}_h + \{ \mathrm{prol} \pi^* v, \{ \mathrm{prol} \pi^* u, \mathrm{prol} \pi^* w \}_h \} \right) \]

\[ = \pi^* \left( \{ \{ u, v \}_\text{red}, w \} \right)_\text{red} + \{ v, \{ u, w \}_\text{red} \}_\text{red}. \]

Concerning the Leibniz identity we get

\[ \pi^* \{ u, vw \}_\text{red} = \iota^* \{ \mathrm{prol} \pi^* u, \{ \mathrm{prol} \pi^* v \}_h \} \mathrm{prol} \pi^* w \}_h \]

\[ = \iota^* \{ \{ \mathrm{prol} \pi^* u, \{ \mathrm{prol} \pi^* v \}_h \}_h \] \] (A.12)

\[ \pi^* \{ u, vw \}_\text{red} = \iota^* \{ \{ u, v \}_\text{red}, w \}_\text{red} + \{ v, \{ u, w \}_\text{red} \}_\text{red} \]

since \( \iota^* (f \mathrm{prol} \phi) = \iota^* (f) \phi. \)

Now we want to show that the reduction procedure is compatible with equivalences, i.e. that equivalent formal Poisson structures with formal momentum maps are reduced to equivalent reduced Poisson structures.

**Proposition A.4** Let \( T = \exp(X_h) : (\pi_h, J_h) \rightarrow (\pi'_h, J'_h) \) be an equivalence of formal invariant Poisson structures with momentum maps, i.e. \( X_h \in h\Gamma(TM[[h]]) \) such that

\[ T \pi_h = \pi'_h \quad \text{and} \quad T \circ J_h = J'_h. \] (A.12)

Then one has even \( X_h \in h\Gamma(TM)^G[[h]] \) and

\[ T \circ \pi_{\text{red}} = (\pi^*)^{-1} \circ \iota^* \circ T \circ \pi \quad \text{is an equivalence between the reduced formal Poisson structures } \pi_{\text{red}} \text{ and } \pi'_{\text{red}}. \] (A.13)

**Proof:** The proof is analogue to the case of star products in [23, Lemma 4.3.1]. At first, as in [25, Prop. 6.2.20] one can show that \( T \pi_h = \pi'_h \) is equivalent to

\[ T \{ f, g \}_h = \{ Tf, Tg \}'_h. \]

But then (A.12) implies

\[ \mathcal{L}_{\xi_M} T f = \{ T f, J'_h(\xi) \}'_h = T \{ f, J_h(\xi) \}_h = T \mathcal{L}_{\xi_M} f. \]

In particular, this yields \([\xi_M, X_h] = 0 \) and thus the invariance of \( X_h \). In addition, recall from Proposition A.3 that we have an isomorphism of Poisson algebras

\[ (\seis(M_{\text{red}})[[h]], \pi_{\text{red}}) \cong \frac{\mathcal{B}_h}{\mathcal{J}_h}. \]

By [25, Prop. 6.2.7] we know that \( T \) is an automorphism with respect to the pointwise product, thus we see directly from the definition of the deformed Koszul differential that

\[ T \circ \partial_h = \partial'_h \circ T \quad \Rightarrow \quad T : \mathcal{J}_h \xrightarrow{\cong} \mathcal{J}'_h. \]

Analogously, we have for \( j' \in \mathcal{J}'_h \) with \( j = T^{-1} j' \in \mathcal{J}_h \) and \( f \in \mathcal{B}_h \)

\[ \{ Tf, j' \}'_h = T \{ f, j \}_h \in T \mathcal{J}_h = \mathcal{J}'_h \quad \Rightarrow \quad T : \mathcal{B}_h \xrightarrow{\cong} \mathcal{B}'_h. \]

Thus \( T_{\text{red}} \) establishes an isomorphism of the spaces \( \mathcal{B}_h/\mathcal{J}_h \) and \( \mathcal{B}'_h/\mathcal{J}'_h \). It remains to check the compatibility with the Poisson bracket:

\[ \pi^* T_{\text{red}} \{ u, v \}_\text{red} = \iota^{**} T \mathrm{prol} \pi^* \{ \mathrm{prol} \pi^* u, \mathrm{prol} \pi^* v \}_h \]

\[ = \iota^* \{ \{ \mathrm{prol} \pi^* u, \mathrm{prol} \pi^* v \}_h \}_h \]

since \( T \) maps the kernel of \( \iota^* \) into the kernel of \( \iota^* \). On the other hand, we get

\[ \pi^* \{ T_{\text{red}} u, T_{\text{red}} v \}_\text{red} = \iota^{**} \{ \mathrm{prol} \pi^* T \mathrm{prol} \pi^* u, \mathrm{prol} \pi^* T \mathrm{prol} \pi^* v \}'_h \]
\[ = \iota^*\{T_{\text{prol}}\pi^*u, T_{\text{prol}}\pi^*v\}^\prime_h \]

since we take on the right hand side the bracket in \( B_h^\prime/\beta_h^\prime \) where \([\text{prol}^*f] = [f]\). Thus the compatibility with the brackets is shown. It remains to show that \( T_{\text{red}} = \exp(X_{\text{red}}) \) for some vector field \( X_{\text{red}} \in \mathcal{H}^\infty(TM_{\text{red}})[[h]] \). Since \( T = \exp(X_h) \) we know that \( T_{\text{red}} \) is a formal power series of \( \mathcal{C}[[h]] \)-linear operators starting with \( \text{id} + h(\ldots) \). We can write \( T_{\text{red}} = \exp(hD) \) via

\[
hD = \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s} (T - \text{id})^s.
\]

Again by \[25\] Prop. 6.2.7 it suffices to show \( T_{\text{red}}(uv) = T_{\text{red}}(u)T_{\text{red}}(v) \), which directly implies \( T_{\text{red}} = \exp(X_{\text{red}}) \) for some vector field \( X_{\text{red}} \in \mathcal{H}^\infty(TM_{\text{red}})[[h]] \). But this is clear since each of the involved maps in the definition of \( T_{\text{red}} \) is compatible with the pointwise product: The maps \( \text{prol} \), \( \pi^* \) and \( (\pi^*)^{-1} \) since they resp. their inverses are pull-backs, the map \( T \) since \( T = \exp(X_h) \) and \( \iota^* \) by Proposition A.3.

\[ \square \]

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