Duality Rotations in Nonlinear Electrodynamics and in Extended Supergravity

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Abstract

We review the general theory of duality rotations which, in four dimensions, exchange electric with magnetic fields. Necessary and sufficient conditions in order for a theory to have duality symmetry are established. A nontrivial example is Born-Infeld theory with $n$ abelian gauge fields and with $Sp(2n,\mathbb{R})$ self-duality. We then review duality symmetry in supergravity theories. In the case of $N = 2$ supergravity duality rotations are in general not a symmetry of the theory but a key ingredient in order to formulate the theory itself. This is due to the beautiful relation between the geometry of special Kähler manifolds and duality rotations.
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1 Introduction

It has long been known that the free Maxwell’s equations are invariant under the rotation of the electric field into the magnetic fields; this is also the case if electric and magnetic charges are present. In 1935, Schrödinger [2] showed that the nonlinear electrodynamics of Born and Infeld [1], (then proposed as a new fundamental theory of the electromagnetic field and presently relevant in describing the low energy effective action of $D$-branes in open string theory), has also, quite remarkably, this property. Extended supergravity theories too, as first pointed out in [3, 5] exhibit electric-magnetic duality symmetry. Duality symmetry thus encompasses photons self-interactions, gravity interactions and couplings to spinors (of the magnetic moment type, not minimal couplings).

Shortly after [3–5] the general theory of duality invariance with abelian gauge fields coupled to fermionic and bosonic matter was developed in [6, 7]. Since then the duality symmetry of extended supergravity theories has been extensively investigated [8–11], and examples of Born-Infeld type lagrangians with electric-magnetic duality have been presented, in the case of one abelian gauge field [12–16] and in the case of many abelian gauge fields [17–20]. Their supersymmetric generalizations have been considered in [21, 22] and with different scalar couplings and noncompact duality group in [17, 18, 23–25].

We also mention that duality symmetry can be generalized to arbitrary even dimensions by using antisymmetric tensor fields such that the rank of their field strengths equals half the dimension of space-time, see [26, 27], and [11, 16, 18, 24, 25, 28, 30, 31].

We provide a rigorous formulation of the general theory of four-dimensional electric-magnetic duality in lagrangian field theories where many abelian vector fields are coupled to scalars, fermions and to gravity. When the scalar fields lagrangian is described by a non-linear sigma model with a symmetric space $G/H$ where $G$ is noncompact and $H$ is its maximal compact subgroup, the coupling of the scalars with the vector fields is uniquely determined by a symplectic representation of $G$ (i.e. where the representation
space is equipped with an invariant antisymmetric product). Moreover fermions coupled to the sigma model, which lie in representations of $H$, must also be coupled to vectors through particular Pauli terms as implied by electric-magnetic duality.

This formalism is realized in an elegant way in extended supergravity theories in four dimensions and can be generalized to dyons [32] in $D$-dimensions, which exist when $D$ is even and the dyon is a $p$-brane with $p = D/2 - 2$. In the context of superstring theory or $M$ theory electric-magnetic dualities can arise from many sources, namely $S$-duality, $T$-duality or a combination thereof called $U$-duality [29]. From the point of view of a four dimensional observer such dualities manifest as some global symmetries of the lowest order Euler-Lagrange equations of the underlying four dimensional effective theory.

The study of the relations between the symmetries of higher dimensional theories and their realization in four dimension is rich and fruitful, and duality rotations are an essential ingredient. Seemingly different lagrangians with different elementary dynamical fields can be shown to describe equivalent equation of motions by using duality. An interesting example is provided by the $N = 8, D = 4$ supergravity lagrangian whose duality group is $G = E_{7(7)}$, this is the formulation of Cremmer and Julia [5]. An alternative formulation obtained from dimensional reduction of the $D = 5$ supergravity, exhibits an action that is invariant under a different group of symmetries. These two theories can be related only after a proper duality rotation of electric and magnetic fields which involves a suitable Legendre transformation (a duality rotation that is not a symmetry transformation).

Let us also recall that duality rotation symmetries can be further enhanced to local symmetries (gauging of duality groups). The corresponding gauged supergravities appear as string compactifications in the presence of fluxes and as generalized compactifications of (ungauged) higher dimensional supergravities.

As a main example consider again the $N = 8, D = 4$ supergravity lagrangian of Cremmer and Julia, it is invariant under $SO(8)$ (compact subgroup of $E_{7(7)}$). The gauging of $SO(8)$ corresponds to the gauged $N = 8$ supergravity of De Witt and Nicolai [33]. As shown in [34] the gauging of a different subgroup, that is the natural choice in the equivalent formulation of the theory obtained from dimensional reduction of $D = 5$ supergravity, corresponds to the gauging of a flat group in the sense of Scherk and Schwarz dimensional reduction [35], and gives the massive deformation of the $N = 8$ supergravity as obtained by Cremmer, Scherk and Schwarz [36].

Electric-Magnetic duality is also the underlying symmetry which encompasses the physics of extremal black holes and of the “attractor mechanism” [37–39], for recent reviews on the attractor mechanism see [40–42]. Here the Bekenstein-Hawking entropy-area formula

$$S = \frac{1}{4} A$$
is directly derived by the evaluation of a certain black hole potential $\mathcal{V}_{BH}$ at its attractive critical points [43]

$$S = \pi \mathcal{V}_{BH} \bigg|_C$$

where the critical points $C$ satisfy $\partial \mathcal{V}_{BH} \big|_C = 0$. The potential $\mathcal{V}_{BH}$ is a quadratic invariant of the duality group; it depends on both the matter and the gauge fields configuration. In all extended supersymmetries with $N > 2$, the entropy $S$ can also be computed via a certain duality invariant combination of the magnetic and electric charges $p,q$ of the fields configuration [44, 45]

$$S = \pi \mathcal{J}(p,q).$$

In the remaining part of this introduction we present the structure of the paper by summarizing its different sections.

In Section 2 we give a pedagogical introduction to $U(1)$ duality rotations in nonlinear theories of electromagnetism. The basic aspects of duality symmetry are already present in this simple case with just one abelian gauge field: the hamiltonian is invariant (duality rotations are canonical transformations that commute with the hamiltonian); the lagrangian is not invariant but must transform in a well defined way. The Born-Infeld theory is the main example of duality invariant nonlinear theory.

In Section 3 the general theory is formulated with many abelian gauge fields interacting with bosonic and fermionic matter. Necessary and for the first time sufficient conditions in order for a theory to have duality symmetry are established. The maximal symmetry group in a theory with $n$ abelian gauge fields includes $Sp(2n, \mathbb{R})$. If there are no scalar fields the maximal symmetry group is $U(n)$. The geometry of the symmetry transformations on the scalar fields is that of the coset space $Sp(2n, \mathbb{R})/U(n)$ that we study in detail. The kinetic term for the scalar fields is constructed by using this coset space geometry. In Subsection 3.6 we present the Born-Infeld lagrangian with $n$ abelian gauge fields and $Sp(2n, \mathbb{R})$ duality symmetry [18]. The self-duality of this lagrangian is proven by studying another example: the Born-Infeld lagrangian with $n$ complex gauge fields and $U(n,n)$ duality symmetry. Here $U(n,n)$ is the group of holomorphic duality rotations. We briefly develop the theory of holomorphic duality rotations.

The Born-Infeld lagrangian with $U(n,n)$ self-duality is per se interesting, the scalar fields span the coset space $U(n,n)/U(n) \times U(n)$, in the case $n = 3$ this is the coset space of the scalars of $N = 3$ supergravity with 3 vector multiplets. This Born-Infeld lagrangian is then a natural candidate for the nonlinear generalization of $N = 3$ supergravity.

We close this sections by presenting, in a formulation with auxiliary fields, the supersymmetric version of this Born-Infeld Lagrangian [17, 18]. We also present the form without auxiliary fields of the supersymmetric Born-Infeld Lagrangian with a single
gauge field and a scalar field; this theory is invariant under \( SL(2, \mathbb{R}) \) duality, which reduces to \( U(1) \) duality if the value of the scalar field is suitably fixed. Versions of this theory without the scalar field were presented in [46–48].

In Section 4 we apply the general theory of duality rotation to supergravity theories with \( N > 2 \) supersymmetries. In these supersymmetric theories the duality group is always a subgroup \( G \) of \( Sp(2n, \mathbb{R}) \), where \( G \) is the isometry group of the sigma model \( G/H \) of the scalar fields. Much of the geometry underlying these theories is in the (local) embedding of \( G \) in \( Sp(2n, \mathbb{R}) \). The supersymmetry transformation rules, the structure of the central and matter charges and the duality invariants associated to the entropy and the potential of extremal black holes configurations are all expressed in terms of the embedding of \( G \) in \( Sp(2n, \mathbb{R}) \) [11]. We thus present a unifying formalisms. We also explicitly construct the symplectic bundles (vector bundles with a symplectic product on the fibers) associated to these theories, and prove that they are topologically trivial; this is no more the case for generic \( N = 2 \) supergravities.

In Section 5 we introduce special Kähler geometry as studied in differential geometry, we follow in particular the work of Freed [49], see also [50] (and [51]) and then develop the mathematical definition up to the construction of those explicit flat symplectic sections used in \( N = 2 \) supergravity. We thus see for example that the flat symplectic bundle of a rigid special Kähler manifold \( M \) is just the tangent bundle \( TM \) with symplectic product given by the Kähler form. A similar construction applies in the case of local special geometry (there the flat tangent bundle is not of the Kähler manifold \( M \) but is essentially the tangent bundle of a complex line bundle \( L \rightarrow M \)). This clarifies the global aspects of special geometry and the key role played by duality rotations in the formulation of \( N = 2 \) supergravity with scalar fields taking value in the target space \( M \). Duality rotations are needed for the theory to be globally well defined.

In Section 6 duality rotations in nonlinear electromagnetism are considered on a noncommutative spacetime, \([x^\mu, x^\nu] = i \Theta^{\mu\nu}\). The noncommutativity tensor \( \Theta^{\mu\nu} \) must be light-like. A nontrivial example of nonlinear electrodynamics on commutative spacetime is presented and using Seiberg-Witten map between commutative and noncommutative gauge theories noncommutative \( U(1) \) Yang Mills theory is shown to have duality symmetry. This theory formally is nonabelian, \( \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu] \), its self-duality is in this respect remarkable. One can also enhance the duality group to \( Sp(2, \mathbb{R}) \) and couple this noncommutative theory to axion, dilaton and Higgs fields, these latter via minimal couplings. Duality in noncommutative spacetime allows to relate space-noncommutative magnetic monopoles to space-noncommutative electric monopoles [52, 53].

A special kind of noncommutative spacetime is a lattice space (it can be studied with noncommutative geometry techniques). Duality rotations on a lattice have been
studied in [54].

In Appendix 7 we prove some fundamental properties of the symplectic group $Sp(2n, \mathbb{R})$ and of the coset space $Sp(2n, \mathbb{R})/U(n)$. We also collect for reference some main formulae and definitions.

In Appendix 8 a symmetry property of the trace of a solution of a polynomial matrix equation is proven. This allows the explicit formulation of the Born-Infeld lagrangian with $Sp(2n, \mathbb{R})$ duality symmetry presented in Section 3.7.

2 $U(1)$ gauge theory and duality symmetry

Maxwell theory is the prototype of electric-magnetic duality invariant theories. In vacuum the equations of motion are

\[
\begin{align*}
\partial_\mu F^{\mu\nu} &= 0, \\
\partial_\mu \tilde{F}^{\mu\nu} &= 0,
\end{align*}
\]

where $\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. They are invariant under rotations $(F) \mapsto (\cos \alpha \ -\sin \alpha \ \sin \alpha \ \cos \alpha)(F)$, or using vector notation under rotations $(E_B) \mapsto (\cos \alpha \ -\sin \alpha \ \sin \alpha \ \cos \alpha)(E_B)$. This rotational symmetry, called duality symmetry, and also duality invariance or self-duality, is reflected in the invariance of the hamiltonian $H = \frac{1}{2} (E^2 + B^2)$, notice however that the lagrangian $L = \frac{1}{2} (E^2 - B^2)$ is not invariant. This symmetry is not an internal symmetry because it rotates a tensor into a pseudotensor.

We study this symmetry for more general electromagnetic theories. In this section and the next one conditions on the lagrangians of (nonlinear) electromagnetic theories will be found that guarantee the duality symmetry (self-duality) of the equations of motion.

The key mathematical point that allows to establish criteria for self-duality, thus avoiding the explicit check of the symmetry at the level of the equation of motions, is that the equations of motion (a system of PDEs) can be conveniently split in a set of equations that is of degree 0 (no derivatives on the field strengths $F$), the so-called constitutive relations (see e.g. (2.5), or (2.8)), and another set of degree 1 (see e.g. (2.2), (2.3) or (2.9), (2.10)). Duality rotations act as an obvious symmetry of the set of equations of degree 1, so all what is left is to check that they act as a symmetry on the set of equations of degree 0. It is therefore plausible that this check can be equivalently formulated as a specific transformation property of the lagrangian under duality rotations (and independent from the spacetime dependence $F_{\mu\nu}(x)$ of the fields), indeed both the lagrangian and the equations of motions of degree 0 are functions of the field strength $F$ and not of its derivatives.
2.1 Duality symmetry in nonlinear electromagnetism

Maxwell equations read

\[ \partial_t B = -\nabla \times E, \quad \nabla \cdot B = 0 \] (2.2)
\[ \partial_t D = \nabla \times H, \quad \nabla \cdot D = 0 \] (2.3)

they are complemented by the relations between the electric field \(E\), the magnetic field \(H\), the electric displacement \(D\) and the magnetic induction \(B\). In vacuum we have

\[ D = E, \quad H = B. \] (2.4)

In a nonlinear theory we still have the equations (2.2), (2.3), but the relations \(D = E, \ H = B\) are replaced by the nonlinear constitutive relations

\[ D = D(E,B), \quad H = H(E,B) \] (2.5)

(if we consider a material medium with electric and magnetic properties then these equations are the constitutive relations of the material, and (2.2) and (2.3) are the macroscopic Maxwell equations).

Equations (2.2), (2.3), (2.4) are invariant under the group of general linear transformations

\[ \begin{pmatrix} B' \\ D' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix}, \quad \begin{pmatrix} E' \\ H' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}. \] (2.6)

We study under which conditions also the nonlinear constitutive relations (2.5) are invariant. We find constraints on the relations (2.5) as well as on the transformations (2.6).

We are interested in nonlinear theories that admit a lagrangian formulation so that relativistic covariance of the equations (2.2), (2.3), (2.5) and their inner consistency is automatically ensured. This requirement is fulfilled if the constitutive relations (2.5) are of the form

\[ D = \frac{\partial \mathcal{L}(E,B)}{\partial E}, \quad H = -\frac{\partial \mathcal{L}(E,B)}{\partial B}, \] (2.7)

where \(\mathcal{L}(E,B)\) is a Poincaré invariant function of \(E\) and \(B\). Indeed if we consider \(E\) and \(B\) depending on a gauge potential \(A_\mu\) and vary the lagrangian \(\mathcal{L}(E,B)\) with respect to \(A_\mu\), we recover (2.2), (2.3) and (2.7). This property is most easily shown by using four component notation. We group the constitutive relations (2.7) in the constitutive relation

\[ \check{G}^{\mu\nu} = 2\frac{\partial \mathcal{L}(F)}{\partial F_{\mu\nu}}; \] (2.8)

\[ ^1 \text{a practical convention is to define } \frac{\partial F_{\rho\sigma}}{\partial F_{\mu\nu}} = \delta_\rho^\mu \delta_\sigma^\nu \text{ rather than } \delta_\rho^\mu \delta_\sigma^\nu - \delta_\rho^\nu \delta_\sigma^\mu. \] This explains the factor 2 in (2.8).
we also define $G_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\rho\sigma} \tilde{G}^{\rho\sigma}$, so that $\tilde{G}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}$ ($\epsilon^{0123} = -\epsilon_{0123} = 1$). If we consider the field strength $F_{\mu\nu}$ as a function of a (locally defined) gauge potential $A_\mu$, then equations (2.2) and (2.3) are respectively the Bianchi identities for $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the equations of motion for $\mathcal{L}(F(A))$,

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \ , \quad \partial_\mu \tilde{G}^{\mu\nu} = 0 .$$ (2.9) (2.10)

In our treatment of duality rotations we study the symmetries of the equations (2.9), (2.10) and (2.8). The lagrangian $\mathcal{L}(F)$ is always a function of the field strength $F$; it is not seen as a function of the gauge potential $A_\mu$; accordingly the Bianchi identities for $F$ are considered part of the equations of motions for $F$.

Finally we consider an action $S = \int \mathcal{L} d^4x$ with lagrangian density $\mathcal{L} = \mathcal{L}(F)$ that depends on $F$ but not on its partial derivatives; it also depends on a spacetime metric $g_{\mu\nu}$ that we generally omit writing explicitly and on at least one dimensionful constant in order to allow for nonlinearity in the constitutive relations (2.8) (i.e. (2.5)). We set this dimensionful constant to 1.

The duality rotations (2.6) read

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} .$$ (2.11)

Since by construction equations (2.9) and (2.10) are invariant under (2.11), these duality rotations are a symmetry of the system of equations (2.9), (2.10), (2.8) (or (2.2), (2.3), (2.5)), iff on shell the constitutive relations are invariant in form, i.e., iff the functional dependence of $\tilde{G}'$ from $F'$ is the same as that of $\tilde{G}$ from $F$, i.e. iff

$$\tilde{G}'^{\mu\nu} = 2 \frac{\partial \mathcal{L}(F')}{\partial F'_{\mu\nu}} ,$$ (2.12)

where $F'_{\mu\nu}$ and $G'_{\mu\nu}$ are given in (2.11). This is the condition that constrains the lagrangian $\mathcal{L}(F)$ and the rotation parameters in (2.11). This condition has to hold on shell of (2.8)-(2.10); however (2.12) is not a differential equation and therefore has to hold just using (2.8), i.e., off shell of (2.9) and (2.10) (indeed if it holds for constant field strengths $F$ then it holds for any $F$).

\[\text{Notice that (2.9), (2.10) are also the equation of motions in the presence of a nontrivial metric. Indeed } S = \int \mathcal{L} d^4x = \int L \sqrt{g} d^4x . \text{ The equation of motions are } \partial_\mu (\sqrt{g} F^{*\mu\nu}) = \partial_\mu F^{\mu\nu} = 0 , \partial_\mu (\sqrt{g} G^{*\mu\nu}) = \partial_\mu \tilde{G}^{\mu\nu} = 0 , \text{ where the Hodge dual of a two form } \Omega_{\mu\nu} \text{ is defined by } \Omega_{\mu\nu} = \sqrt{g} \epsilon_{\mu\rho\sigma} \Omega^{\rho\sigma} .\]
In order to study the duality symmetry condition (2.12) let \((A \, B \, C \, D) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon(a \, b \, c \, d) + \ldots\), and consider infinitesimal GL\((2, \mathbb{R})\) rotations \(G \rightarrow G + \epsilon \Delta G, \quad F \rightarrow F + \epsilon \Delta F\),

\[
\Delta \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix},
\]

so that the duality condition reads

\[
\tilde{G} + \Delta \tilde{G} = 2 \frac{\partial \mathcal{L}(F + \Delta F)}{\partial (F + \Delta F)}.
\]

The right hand side simplifies to\(^3\)

\[
\frac{\partial \mathcal{L}(F + \Delta F)}{\partial (F + \Delta F)} = \frac{\partial \mathcal{L}(F + \Delta F)}{\partial F} \frac{\partial F}{\partial (F + \Delta F)} = \frac{\partial \mathcal{L}(F + \Delta F)}{\partial F} - \frac{\partial \mathcal{L}(F)}{\partial F} \frac{\partial (\Delta F)}{\partial F}
\]

then, using (2.13) and (2.8), condition (2.14) reads

\[
c\tilde{F} + d\tilde{G} = 2 \frac{\partial \mathcal{L}(F + \Delta F) - \mathcal{L}(F)}{\partial F} - 2a \frac{\partial \mathcal{L}(F)}{\partial F} - b\tilde{G} \frac{\delta G}{\partial F}.
\]

In order to further simplify this expression we write \(2\tilde{F} = \frac{\partial}{\partial F} FF\tilde{F}\) and we factorize out the partial derivative \(\frac{\partial}{\partial F}\). We thus arrive at the equivalent condition

\[
\mathcal{L}(F + \Delta F) - \mathcal{L}(F) - \frac{c}{4} FF\tilde{F} - \frac{b}{4} G\tilde{G} = (a + d)(\mathcal{L}(F) - \mathcal{L}_{F=0}).
\]

The constant term \((a + d)\mathcal{L}_{F=0}\), nonvanishing for example in D-brane lagrangians, is obtained by observing that when \(F = 0\) also \(G = 0\).

Next use \(\mathcal{L}(F + \Delta F) - \mathcal{L}(F) = \frac{\partial \mathcal{L}(F)}{\partial F} \Delta F = \frac{1}{2} a F\tilde{G} + \frac{1}{2} b G\tilde{G}\) in order to rewrite expression (2.16) as

\[
\frac{b}{4} G\tilde{G} - \frac{c}{4} FF\tilde{F} = (a + d)(\mathcal{L}(F) - \mathcal{L}_{F=0}) - \frac{a}{2} F\tilde{G}.
\]

If we require the nonlinear lagrangian \(\mathcal{L}(F)\) to reduce to the usual Maxwell lagrangian in the weak field limit, \(F^4 << F^2\), i.e., \(\mathcal{L}(F) = \mathcal{L}_{F=0} - 1/4 \int F F d^4 x + O(F^4)\), then \(\tilde{G} = -F + O(F^3)\), and we obtain the constraint (recall that \(\tilde{\tilde{G}} = -G\))

\[
b = -c, \quad a = d,
\]

\(^3\)here and in the following we suppress the spacetime indices so that for example \(FG = F_{\mu\nu} G^{\mu\nu}\); notice that \(F\tilde{G} = \tilde{F} G, \quad \tilde{F} = -F\), and \(\tilde{F}\tilde{G} = -FG\) where \(FG = F_{\mu\nu} G^{\mu\nu}\).
the duality group can be at most \( SO(2) \) rotations times dilatations. Condition (2.17) becomes

\[
\frac{b}{4} (\tilde{G}G + F\tilde{F}) = 2a \left( \mathcal{L}(F) - \mathcal{L}_{F=0} - \frac{1}{2} F \frac{\partial \mathcal{L}}{\partial F} \right).
\] (2.18)

The vanishing of the right hand side holds only if either \( \mathcal{L}(F) - \mathcal{L}_{F=0} \) is quadratic in \( F \) (usual electromagnetism) or \( a = 0 \). We are interested in nonlinear theories; by definition in a nonlinear theory \( \mathcal{L}(F) \) is not quadratic in \( F \). This shows that dilatations alone cannot be a duality symmetry. If we require the duality group to contain at least \( SO(2) \) rotations then

\[
G\tilde{G} + F\tilde{F} = 0,
\] (2.19)

and \( SO(2) \) is the maximal duality group. Relation (2.18) is nontrivially satisfied iff

\[
a = d = 0,
\]

and (2.19) hold.

In conclusion equation (2.19) is a necessary and sufficient condition for a nonlinear electromagnetic theory to be symmetric under \( SO(2) \) duality rotations, and \( SO(2) \subset GL(2, \mathbb{R}) \) is the maximal connected Lie group of duality rotations of pure nonlinear electromagnetism.

This conclusion still holds if we consider a nonlinear lagrangian \( \mathcal{L}(F) \) that in the weak field limit \( F^4 \ll F^2 \) (up to an overall normalization factor) reduces to the most general linear lagrangian

\[
\mathcal{L}(F) = \mathcal{L}_{F=0} - \frac{1}{4} FF + \frac{1}{4} \theta F\tilde{F} + O(F^4).
\]

In this case \( G = \tilde{F} + \theta F + O(F^3) \). We substitute in (2.17) and obtain the two conditions (the coefficients of the scalar \( F^2 \) and of the pseudoscalar \( F\tilde{F} \) have to vanish separately)

\[
c = -b(1 + \theta^2) \quad , \quad d - a = 2\theta b.
\] (2.20)

The most general infinitesimal duality transformation is therefore

\[
\begin{pmatrix}
  a & b \\
  -b(1 + \theta^2) & a + 2\theta b
\end{pmatrix} = \begin{pmatrix}
  a + \theta b & 0 \\
  0 & a + \theta b
\end{pmatrix} + \Theta \begin{pmatrix}
  0 & b \\
  -b & 0
\end{pmatrix} \Theta^{-1} \tag{2.21}
\]

\footnote{This symmetry cannot even extend to \( O(2) \) because already in the case of usual electromagnetism the finite rotation \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) does not satisfy the duality condition (2.12). It is instructive to see the obstruction at the Hamiltonian level. The Hamiltonian itself is invariant under \( D \to D, B \to -B \), but this transformation is not a canonical transformation: the Poisson bracket (2.33) is not invariant.}
where $\Theta = \begin{pmatrix} 1 & 0 \\ \theta & 1 \end{pmatrix}$. We have dilatations and $SO(2)$ rotations, they act on the vector $\begin{pmatrix} F \\ G \end{pmatrix}$ via the conjugate representation given by the matrix $\Theta$. Let’s now remove the weak field limit assumption $F^4 << F^2$. We proceed as before. From (2.12) (or from (2.11)) we immediately obtain that dilatations alone are not a duality symmetry of the nonlinear equations of motion. Then if $SO(2)$ rotations are a duality symmetry we have that they are the maximal duality symmetry group. This happens if

$$G\tilde{G} + (1 + \theta)^2 F\tilde{F} = 2\theta F\tilde{G}. \quad (2.22)$$

Finally we note that the necessary and sufficient conditions for $SO(2)$ duality rotations (2.22) (or (2.19)) can be equivalently expressed as invariance of

$$\mathcal{L}(F) - \frac{1}{4} F\tilde{G}. \quad (2.23)$$

Proof: the variation of expression (2.23) under $F \rightarrow F + \Delta F$ is given by $\mathcal{L}(F + \Delta F) - \mathcal{L}(F) - \frac{1}{4} \Delta F\tilde{G} - \frac{1}{4} F\Delta \tilde{G}$. Use of (2.16) with $a + d = 0$ (no dilatation) shows that this variation vanishes.

### 2.2 Legendre Transformations

In the literature on gauge theories of abelian $p$-form potentials, the term duality transformation denotes a different transformation from the one we have introduced, a Legendre transformation, that is not a symmetry transformation. In this section we relate these two different notions, see [15] for further applications and examples.

Consider a theory of nonlinear electrodynamics ($p = 1$) with lagrangian $\mathcal{L}(F)$. The equations of motion and the Bianchi identity for $F$ can be derived from the Lagrangian $\mathcal{L}(F, F_D)$ defined by

$$\mathcal{L}(F, F_D) = \mathcal{L}(F) - \frac{1}{2} F\tilde{F}_D, \quad F_D^{\mu\nu} = \partial^\mu A_D^\nu - \partial^\nu A_D^\mu, \quad (2.24)$$

where $F$ is now an unconstrained antisymmetric tensor field, $A_D$ a Lagrange multiplier field and $F_D$ its electromagnetic field. [Hint: varying with respect to $A_D$ gives the Bianchi identity for $F$, varying with respect to $F$ gives $G^{\mu\nu} = F_D^{\mu\nu}$ that is equivalent to the initial equations of motion $\partial_\mu \tilde{G}^{\mu\nu} = 0$ because $F_D^{\mu\nu} = \partial^\mu A_D^\nu - \partial^\nu A_D^\mu$ (Poincaré lemma)].

Given the lagrangian (2.24) one can also first consider the equation of motion for $F$,

$$G(F) = F_D, \quad (2.25)$$
that is solved by expressing $F$ as a function of the dual field strength, $F = F(F_D)$. Then inserting this solution into $\mathcal{L}(F, F_D)$, one gets the dual model

$$\mathcal{L}_D(F_D) = \mathcal{L}(F(F_D)) - \frac{1}{2} F(F_D) \cdot \tilde{F}_D .$$

(2.26)

Solutions of the (2.26) equations of motion are, together with (2.25), solutions of the (2.24) equations of motion. Therefore solutions to the (2.26) equations of motion are via (2.25) in 1-1 correspondence with solutions of the $\mathcal{L}(F)$ equations of motion.

One can always perform a Legendre transformation and describe the physical system with the new dynamical variables $A_D$ and the new lagrangian $\mathcal{L}_D$ rather than $A$ and $\mathcal{L}$.

The relation with the duality rotation symmetry (self-duality) of the previous section is that if the system admits duality rotations then the solution $F_D$ of the $\mathcal{L}_D$ equations of motion is also a solution of the $\mathcal{L}$ equations of motion, we have a symmetry because the dual field $F_D$ is a solution of the original system. This is the case because for any solution $\mathcal{L}$ of the self-duality equation, its Legendre transform $\mathcal{L}_D$ satisfies:

$$\mathcal{L}_D(F) = \mathcal{L}(F) .$$

(2.27)

This follows from considering a finite $SO(2)$ duality rotation with angle $\pi/2$. Then $F \rightarrow F' = G(F) = F_D$, and invariance of (2.23), i.e. $\mathcal{L}(F') - \frac{1}{4} F'_\rho G' = \mathcal{L}(F) - \frac{1}{4} FG'$, implies $\mathcal{L}_D(F_D) = \mathcal{L}(F_D)$, i.e., (2.27).

In summary, a Legendre transformation is a duality rotation only if the symmetry condition (2.8) is met. If the self-duality condition (2.8) does not hold, a Legendre transformation leads to a dual formulation of the theory in terms of a dual Lagrangian $\mathcal{L}_D$, not to a symmetry of the theory.

### 2.3 Hamiltonian theory

The symmetric energy momentum tensor of a nonlinear theory of electromagnetism (obtained via Belinfante procedure or by varying with respect to the metric) is given by

$$T^\mu_\nu = \tilde{\mathcal{G}}^{\mu\lambda}F_{\nu\lambda} + \partial_\mu \mathcal{L} .$$

(2.28)

The equations of motion (2.10) and (2.9) imply its conservation, $\partial_\mu T^\mu_\nu = 0$. Invariance of the energy momentum tensor under duality rotations is easily proven by observing that for a generic antisymmetric tensor $F_{\mu\nu}$

$$\tilde{F}^\mu_\nu F_{\nu\lambda} = -\frac{1}{4} \partial_\lambda \tilde{F}^\rho_\sigma F_{\rho\sigma} ,$$

(2.29)

\footnote{\textit{Symmetry of $T^\mu_\nu$ follows immediately by observing that the tensor structure of $\tilde{\mathcal{G}}^{\mu\nu}$ implies $\tilde{\mathcal{G}}^{\mu\nu} = f_s(F)F^{\mu\nu} + f_p(F)\hat{F}^{\mu\nu}$ with scalars $f_s(F)$ and $f_p(F)$ depending on $F$, the metric $\eta = \text{diag}(-1,1,1,1)$ and the completely antisymmetric tensor density $\epsilon_{\mu\nu\rho\sigma}$. (Actually, if the lagrangian is parity even, $f_s$ is a scalar function while $f_p$ is a pseudoscalar function).}
and then by recalling the duality symmetry condition (2.19).

In particular the hamiltonian

\[ \mathcal{H} = T^{00} = D \cdot E - \mathcal{L} \]  

(2.30)

of a theory that has duality rotation symmetry is invariant.

In the hamiltonian formalism duality rotations are canonical transformations, since they leave the hamiltonian invariant they are usual symmetry transformations. We briefly describe the hamiltonian formalism of (nonlinear) electromagnetism by avoiding to introduce the vector potential \( A_\mu \); this is appropriate since duality rotations are formulated independently from the notion of vector potential. Maxwell equations (2.2), (2.3) and the expression of the hamiltonian suggest to consider \( B \) and \( D \) as the analogue of canonical coordinates and momenta \( q \) and \( p \), while \( E \), that enters the lagrangian togheter with \( B \), is the analogue of \( \dot{q} \).

Recalling the constitutive relations in the lagrangian form (2.7) we obtain that the hamiltonian \( \mathcal{H} = \mathcal{H}(D, B) \) is just given by the Legendre transformation (2.30). Moreover \( H = \frac{\partial \mathcal{H}}{\partial B} \) and \( E = \frac{\partial \mathcal{H}}{\partial D} \). The equations of motion are

\[ \partial_t B = -\nabla \times \frac{\delta \mathcal{H}}{\delta D} , \]  

(2.31)

\[ \partial_t D = \nabla \times \frac{\delta \mathcal{H}}{\delta B} . \]  

(2.32)

The remaning equations \( \nabla \cdot B = 0, \nabla \cdot D = 0 \) are constraints that imposed at a given time are satisfied at any other time. The Poisson bracket between two arbitrary functionals \( U, V \) of the canonical variables is

\[ \{U, V\} = \int \frac{\partial U}{\partial D} \cdot \left( \nabla \times \frac{\partial V}{\partial B} \right) - \frac{\partial V}{\partial D} \cdot \left( \nabla \times \frac{\partial U}{\partial B} \right) \, d^3r , \]  

(2.33)

in particular the only nonvanishing parenthesis between the canonical variables \( B \) and \( D \) are \( \{ B^i(r), D^j(r') \} = \varepsilon^{ijk} \partial_k \partial^3(r - r') \). The equations of motion (2.31) and (2.32) assume then the canonical form \( \partial_t B = -\{ B, H \} \), \( \partial_t D = \{ D, H \} \) where \( H = \int \mathcal{H} \, d^3r \) is the hamiltonian (\( \mathcal{H} \) being the hamiltonian density). We see that \( H \) as usual is the generator of time evolution. The consitency and the hidden Poincaré invariance of the present formalism is proven in [55].

In the canonical formalism the generator of duality rotations is the following nonlocal integral [57], [56]

\[ \Lambda = \frac{1}{8\pi} \int \int \frac{D_1 \cdot (\nabla \times D_2) + B_1 \cdot (\nabla \times B_2)}{|r_1 - r_2|} \, d^3r_1 d^3r_2 \]  

(2.34)
where the subscripts indicate that the fields are taken at the points \( r_1 \) and \( r_2 \). We have \( \{ \mathbf{D}, \Lambda \} = \mathbf{B} \) and \( \{ \mathbf{B}, \Lambda \} = - \mathbf{D} \).

Finally we remark that it is straightforward to establish duality symmetry in the Hamiltonian formalism. Indeed there are three independent scalar combinations of the canonical fields \( \mathbf{B} \) and \( \mathbf{D} \), they can be taken to be: \( \mathbf{D}^2 + \mathbf{B}^2, \mathbf{D}^2 - \mathbf{B}^2 \) and \( (\mathbf{D} \times \mathbf{B})^2 \). The last two scalars are duality invariant and therefore any Hamiltonian that depends just on them leads to a theory with duality symmetry. The nontrivial problem in this approach is now to constrain the Hamiltonian so that the theory is Lorentz invariant \([58], [57]\). The condition is again (2.19) i.e., \( \mathbf{D} \cdot \mathbf{H} = \mathbf{E} \cdot \mathbf{B} \).

### 2.4 Born-Infeld Lagrangian

A notable example of a Lagrangian whose equations of motion are invariant under duality rotations is given by the Born-Infeld one \([1]\)

\[
\mathcal{L}_{\text{BI}} = 1 - \sqrt{-\det(\mathbf{\eta} + \mathbf{F})} \quad (2.35)
\]

\[
= 1 - \sqrt{1 + \frac{1}{2} \mathbf{F}^2 - \frac{1}{16} (\mathbf{F} \mathbf{F})^2} \quad (2.36)
\]

\[
= 1 - \sqrt{1 - \mathbf{E}^2 + \mathbf{B}^2 - (\mathbf{E} \cdot \mathbf{B})^2}. \quad (2.37)
\]

In the second line we have simply expanded the 4x4 determinant and expressed the Lagrangian in terms of the only two independent Lorentz invariants associated to the electromagnetic field: \( F^2 \equiv F_{\mu\nu} F^{\mu\nu}, \mathbf{F} \mathbf{F} \equiv F_{\mu\nu} \mathbf{F}^{\mu\nu} \).

The explicit expression of \( G \) is

\[
G_{\mu\nu} = \frac{\mathbf{F}_{\mu\nu} + \frac{1}{4} \mathbf{F} \mathbf{F} F_{\mu\nu}}{\sqrt{1 + \frac{1}{2} \mathbf{F}^2 - \frac{1}{16} (\mathbf{F} \mathbf{F})^2}}, \quad (2.38)
\]

and the duality condition (2.19) is readily seen to hold. The Hamiltonian is

\[
\mathcal{H}_{\text{BI}} = \sqrt{1 + \mathbf{D}^2 + \mathbf{B}^2 + (\mathbf{D} \times \mathbf{B})^2} - 1. \quad (2.39)
\]

Notice that while the \( \mathbf{E} \) and \( \mathbf{B} \) variables are constrained by the reality of the square root in the Lagrangian, the Hamiltonian variables \( \mathbf{D}, \mathbf{B} \) are unconstrained. By using the equations of motion and (2.19) it can be explicitly verified that the generator of duality rotations is time independent, \( \{ \Lambda, H \} = 0 \).
2.5 Extended duality rotations

The duality symmetry of the equations of motion of nonlinear electromagnetism can be extended to $SL(2, \mathbb{R})$. We observe that the definition of duality symmetry we used can be relaxed by allowing the $F$ dependence of $G$ to change by a linear term: $G = 2 \frac{\partial L}{\partial F}$ and $G = 2 \frac{\partial L}{\partial F} + \vartheta F$ together with the Bianchi identities for $F$ give equivalent equations of motions for $F$. Therefore the transformation

$$
\begin{pmatrix}
F' \\
G'
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
\vartheta & 1
\end{pmatrix} \begin{pmatrix}
F \\
G
\end{pmatrix}
$$

is a symmetry of any nonlinear electromagnetism. It corresponds to the lagrangian change $\mathcal{L} \to \mathcal{L} + \frac{1}{4} \vartheta F \dddot{F}$. This symmetry alone does not act on $F$, but it is useful if the nonlinear theory has $SO(2)$ duality symmetry. In this case (2.40) extends duality symmetry from $SO(2)$ to $SL(2, \mathbb{R})$ (i.e. $Sp(2, \mathbb{R})$). Notice however that the $SL(2, \mathbb{R})$ transformed solution, contrary to the $SO(2)$ one, has a different energy and energy momentum tensor (recall (2.28)). On the other hand, as we show in Section 3.6, if the constant $\vartheta$ is promoted to a dynamical field we have invariance of the energy momentum tensor under $SL(2, \mathbb{R})$ duality.

3 General theory of duality rotations

We study in full generality the conditions in order to have theories with duality rotation symmetry. By properly introducing scalar fields (sigma model on coset space) we enhance theories with a compact duality group to theories with an extended noncompact duality group. A Born-Infeld lagrangian with $n$ abelian field strengths and $U(n)$ duality group (or $Sp(2n, \mathbb{R})$ in the presence of scalars) is constructed.

3.1 General nonlinear theory

We consider a theory of $n$ abelian gauge fields possibly coupled to other bosonic and fermionic fields that we denote $\varphi^\alpha$, ($\alpha = 1, \ldots, p$). We assume that the $U(1)$ gauge potentials enter the action $S = S[F, \varphi]$ only through the field strengths $F^\Lambda_{\mu\nu}$ ($\Lambda = 1, \ldots, n$), and that the action does not depend on partial derivatives of the field strengths. Define $\tilde{G}^\mu_{\lambda \nu} = 2 \frac{\partial S}{\partial \tilde{F}^\Lambda_{\mu\nu}}$, i.e.,

$$
\tilde{G}^\mu_{\lambda \nu} = 2 \frac{\delta S[F, \varphi]}{\delta F^\Lambda_{\mu\nu}};
$$

(3.1)
then the Bianchi identities and the equations of motions for $S[F, \varphi]$ are
\begin{align}
\partial_\mu \tilde{F}^{\Lambda \mu\nu} &= 0, \\
\partial_\mu \tilde{G}_\Lambda^{\mu\nu} &= 0, \\
\frac{\delta S[F, \varphi]}{\delta \varphi^\alpha} &= 0.
\end{align}

The field theory is described by the system of equations (3.1)-(3.4). Consider the duality transformations
\begin{equation}
\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix},
\end{equation}
\begin{equation}
\varphi^{\alpha} = \Xi^{\alpha}(\varphi)
\end{equation}
where $(A B C D)$ is a generic constant $L(2n, \mathbb{R})$ matrix and the $\varphi^{\alpha}$ fields transformation in full detail reads $\varphi^{\alpha} = \Xi^{\alpha}(\varphi, (A B C D))$, with no partial derivative of $\varphi$ appearing in $\Xi^{\alpha}$.

These duality rotations are a symmetry of the system of equations (3.1)-(3.4) iff, given $F$, $G$, and $\varphi$ solution of (3.1)-(3.4) then $F'$, $G'$ and $\varphi'$, that by construction satisfy $\partial_\mu \tilde{F}^{\alpha \mu\nu} = 0$ and $\partial_\mu \tilde{G}_\alpha^{\mu\nu} = 0$, satisfy also
\begin{equation}
\tilde{G}^{\mu\nu}_\Lambda = 2 \frac{\delta S[F', \varphi']}{\delta F^{\alpha \mu\nu}},
\end{equation}
\begin{equation}
\frac{\delta S[F', \varphi']}{\delta \varphi^{\alpha}} = 0.
\end{equation}

We study these on shell conditions in the case of infinitesimal $GL(2n, \mathbb{R})$ rotations
\begin{equation}
F \to F' = F + \Delta F, \quad G \to G' = G + \Delta G,
\end{equation}
\begin{equation}
\Delta \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix},
\end{equation}
\begin{equation}
\Delta \varphi^{\alpha} = \xi^{\alpha}(\varphi).
\end{equation}
The right hand side of (3.7) can be rewritten as
\begin{equation}
\frac{\delta S[F', \varphi']}{\delta F^{\alpha \mu\nu}} = \int_y \frac{\delta S[F', \varphi']}{\delta F^{\Sigma}(y)} \frac{\delta F^{\Sigma}(y)}{\delta F^{\mu\nu}}.
\end{equation}

We now invert the matrix
\begin{equation}
\begin{pmatrix} \frac{\delta \varphi'}{\delta F} & \frac{\delta \varphi'}{\delta F'} \\ \frac{\delta \varphi}{\delta F} & \frac{\delta \varphi}{\delta F'} \end{pmatrix},
\end{equation}
recall that $F_1 \tilde{F}_2 = \tilde{F}_1 F_2$ and observe that
\begin{equation}
\int_y \frac{\delta S[F, \varphi]}{\delta F(y)} b \frac{\delta G(y)}{\delta F^{\alpha \mu\nu}} = \frac{1}{4} \frac{\delta}{\delta F^{\alpha \mu\nu}} \int_y \tilde{G} b G + \frac{1}{4} \int_y \tilde{G} (b - b') \frac{\delta G}{\delta F^{\alpha \mu\nu}}.
\end{equation}
We thus obtain
\[
\frac{\delta S'[F', \varphi']}{\delta F^\Lambda} - \frac{\delta S[F', \varphi']}{\delta F^\Lambda} - a^\Lambda \sum_{\Sigma} \frac{\delta S[F', \varphi']}{\delta F^\Sigma} - \frac{1}{4 \delta F^\Lambda} \int_y \tilde{G} b G - \frac{1}{4} \int_y \tilde{G} (b - b^t) \frac{\delta G}{\delta F^\Lambda} = 0. \tag{3.12}
\]

Since the left hand side of (3.7) is \( \tilde{G}_\Lambda + \frac{1}{2} \delta F^\Lambda \int_y \tilde{F} c F + \frac{1}{4} (c - c^t) \Lambda \Sigma \tilde{F}^\Sigma + 2 d^\Lambda \sum_{\Sigma} \frac{\delta S[F, \varphi']}{\delta F^\Sigma} \), we rewrite (3.7) as
\[
\frac{\delta}{\delta F^\Lambda} \left( S[F', \varphi'] - S[F, \varphi] - \frac{1}{4} \int_y (\tilde{F} c F + \tilde{G} b G) \right) = (a^t + d) \Lambda \sum_{\Sigma} \frac{\delta}{\delta F^\Sigma} S[F, \varphi] + \frac{1}{4} (c - c^t) \Lambda \Sigma \tilde{F}^\Sigma + \frac{1}{4} \int_y \tilde{G} (b - b^t) \frac{\delta G}{\delta F^\Lambda}. \tag{3.13}
\]

Since this expression does not contain derivatives of \( F \), the functional variation becomes just a partial derivative, and (3.13) is equivalent to
\[
\frac{\partial}{\partial F^\Lambda} \left( \mathcal{L}(F', \varphi') - \mathcal{L}(F, \varphi) - \frac{1}{4} \tilde{F} c F - \frac{1}{4} \tilde{G} b G \right) = (a^t + d) \Lambda \sum_{\Sigma} \frac{\partial}{\partial F^\Sigma} \mathcal{L}(F, \varphi) + \frac{1}{4} (c - c^t) \Lambda \Sigma \tilde{F}^\Sigma + \frac{1}{4} \int_y \tilde{G} (b - b^t) \frac{\partial G}{\partial F^\Lambda}. \tag{3.14}
\]

Here \( \mathcal{L}(F, \varphi) \) is a shorthand notation for a lagrangian that depends on \( F, \varphi^\alpha, \partial \varphi^\alpha \) and eventually higher partial derivatives of the fields \( \varphi^\alpha \), say up to order \( \ell \). Equation (3.14) has to hold on shell of (3.1)-(3.4). Since this equation has no partial derivative of \( F \) and at most derivatives of \( \varphi^\alpha \) up to order \( \ell \), if it holds on shell of (3.1)-(3.4) then it holds just on shell of (3.1), and of the fermions fields equations, the scalar and vector partial differential equations being of higher order in derivatives of \( F \) or \( \varphi^\alpha \) fields. In particular if no fermion is present (3.14) holds just on shell of (3.1).

Since the left hand side of (3.14) is a derivative with respect to \( F^\Lambda \) so must be the right hand side. This holds if we consider infinitesimal dilatations, parametrized by \( \kappa \in \mathbb{R} \), and infinitesimal \( Sp(2n, \mathbb{R}) \) transformations
\[
a^t + d = \kappa \mathbb{I}, \quad b^t = b, \quad c^t = c. \tag{3.15}
\]

We can then remove the derivative \( \frac{\partial}{\partial F^\Lambda} \) and obtain the equivalent condition
\[
\mathcal{L}(F', \varphi') - \mathcal{L}(F, \varphi) - \kappa \mathcal{L}(F, \varphi) - \frac{1}{4} \tilde{F} c F - \frac{1}{4} \tilde{G} b G = f(\varphi) \tag{3.16}
\]
where \( f(\varphi) \) can contain partial derivatives of \( \varphi \) up to the same order as in the lagrangian.

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We now show that \( f(\varphi) \) in (3.16) is independent from \( \varphi \). Consider the \( \varphi \)-equations of motion (3.8),

\[
\frac{\delta S[F', \varphi']}{\delta \varphi^\alpha} = \int_y \frac{\delta S[F', \varphi']}{\delta \varphi^\beta(y)} \frac{\delta \varphi^\beta(y)}{\delta \varphi^\alpha} + \int_y \frac{\delta S[F, \varphi]}{\delta F(y)} \frac{\delta F(y)}{\delta \varphi^\alpha} \\
= \frac{\delta S[F', \varphi']}{\delta \varphi^\alpha} - \frac{\delta S[F, \varphi]}{\delta \varphi^\beta} \frac{\partial \xi^\beta}{\partial \varphi^\alpha} - \frac{\delta}{4 \delta \varphi^\alpha} \int_y \tilde{G} b G \\
= \frac{\delta S[F, \varphi]}{\delta \varphi^\alpha} - \frac{\delta S[F, \varphi]}{\delta \varphi^\beta} \frac{\partial \xi^\beta}{\partial \varphi^\alpha} + \frac{\delta}{\delta \varphi^\alpha} \left( S[F', \varphi'] - S[F, \varphi] - \frac{1}{4} \int_y \tilde{G} b G \right)
\]

where only first order infinitesimals have been retained, and where techniques similar to those used in the study of (3.11) have been applied. On shell the left hand side has to vanish; since the first two addends on the right hand side are proportional to the \( \varphi \)-equations of motion, this happens iff on shell

\[
\frac{\delta}{\delta \varphi^\alpha} \left( S[F', \varphi'] - S[F, \varphi] - \kappa S[F, \varphi] - \frac{1}{4} \int_y (\tilde{G} b G + \tilde{F} c F) \right) = 0 . \tag{3.17}
\]

Comparison with (3.16) shows that on shell

\[
\frac{\delta}{\delta \varphi^\alpha} f(\varphi) = 0 . \tag{3.18}
\]

In this expression no field strength \( F \) is present and therefore the equations of motion of our interacting system are of no use; equation (3.18) holds also off shell and we conclude that \( f(\varphi) \) is \( \varphi \) independent, it is just a constant depending on the parameters \( a, b, c, d \) (it usually vanishes). We thus have the condition

\[
\mathcal{L}(F', \varphi') - \mathcal{L}(F, \varphi) - \kappa \mathcal{L}(F, \varphi) - \frac{1}{4} \tilde{F} c F - \frac{1}{4} \tilde{G} b G = \text{const}_{a,b,c,d} \tag{3.19}
\]

If we expand \( F' \) in terms of \( F \) and \( G \), we obtain the equivalent condition

\[
\Delta_\varphi \mathcal{L}(F, \varphi) = \frac{1}{4} \tilde{F} c F - \frac{1}{4} \tilde{G} b G + \kappa \mathcal{L}(F, \varphi) - \frac{1}{2} \tilde{G} a F + \text{const}_{a,b,c,d} \tag{3.20}
\]

where \( \Delta_\varphi \mathcal{L}(F, \varphi) = \mathcal{L}(F, \varphi') - \mathcal{L}(F, \varphi) \).

Equation (3.20), where \( \tilde{G}^\mu_\alpha = 2 \partial \mathcal{L}/\partial F^\mu_{\alpha \nu} \), is a necessary and sufficient condition in order to have duality symmetry. This condition is on shell of the fermions equations of motion, in particular if no fermion is present this condition is off shell. In the presence of fermions, equation (3.20) off shell is a sufficient condition for duality symmetry.

The duality symmetry group is

\[
\mathbb{R}^+ \times SL(2n, \mathbb{R}) , \tag{3.21}
\]
the group of dilatations times symplectic transformation; it is the connected Lie group
generated by the Lie algebra (3.15). It is also the maximal group of duality rotations
as the example (or better, the limiting case) studied in the next section shows.

We have considered dynamical fermionic and bosonic fields $\varphi^\alpha$. If a subset $\chi^r$ of
these fields is not dynamical the corresponding equations of motion are of the same
order as those defining $G$, and thus (3.14) and (3.20) hold on shell of all these equations.
Moreover since no $\partial \chi^r$ appears in the lagrangian, the duality transformations for these
fields can include the field strength $F$, i.e., $\chi^r \to \chi^r = \Xi^r(F, \chi)$. In this case there is an
extra addend in (3.11). The necessary and sufficient duality condition (3.20) does not
change.

We also notice that condition (3.19) in the absence of dilatations ($\kappa = 0$), and for
$\text{const}_{a,b,c,d} = 0$ is equivalent to the invariance of
\[ \mathcal{L} = \frac{1}{4} \tilde{F} G . \quad (3.22) \]

### 3.2 The main example and the scalar fields fractional transformations

Consider the Lagrangian
\[ \frac{1}{4} N_2 \Lambda \Sigma F^\Lambda F^\Sigma + \frac{1}{4} N_1 \Lambda \Sigma F^\Lambda \tilde{F}^\Sigma + \mathcal{L}(\phi) \quad (3.23) \]
where the real symmetric matrices $N_1(\phi)$ and $N_2(\phi)$ and the lagrangian $\mathcal{L}(\phi)$ are just
functions of the bosonic fields $\phi^i$, $i = 1, \ldots m$, (and their partial derivatives).

Any nonlinear lagrangian in the limit of vanishing fermionic fields and of weak field
strengths $F^4 << F^2$ reduces to the one in (3.23). A straightforward calculation shows
that this lagrangian has $\mathbb{R} > 0 \times SL(2n, \mathbb{R})$ duality symmetry if the matrices $N_1$ and $N_2$
of the scalar fields transform as
\[ \Delta N_1 = c + d N_1 - N_1 a - N_1 b N_1 + N_2 b N_2 , \quad (3.24) \]
\[ \Delta N_2 = d N_2 - N_2 a - N_1 b N_2 - N_2 b N_1 , \quad (3.25) \]
and
\[ \Delta \mathcal{L}(\phi) = \kappa \mathcal{L}(\phi) . \quad (3.26) \]

If we define
\[ \mathcal{N} = N_1 + i N_2 , \]
i.e., $N_1 = \text{Re} \mathcal{N}$, $N_2 = \text{Im} \mathcal{N}$, the transformations (3.24), (3.25) read
\[ \Delta \mathcal{N} = c + d \mathcal{N} - \mathcal{N} a - \mathcal{N} b \mathcal{N} , \quad (3.27) \]
the finite version is the fractional transformation
\[ N' = (C + DN) (A + BN)^{-1}. \] (3.28)
Under (3.28) the imaginary part of \( N \) transforms as
\[ N_2' = (A + BN)^{-\dagger} N_2 (A + BN)^{-1} \] (3.29)
where \(-\dagger\) is a shorthand notation for the hermitian conjugate of the inverse matrix.

The kinetic term \( \frac{1}{4} N_2 A \Sigma F^A F^\Sigma \) is positive definite if the symmetric matrix \( N_2 \) is negative definite. In Appendix 7.2 we show that the matrices \( N = N_1 + i N_2 \) with \( N_1 \) and \( N_2 \) real and symmetric, and \( N_2 \) positive definite, are the coset space \( \text{Sp}(2n, \mathbb{R}) U(n) \).

A scalar lagrangian that satisfies the variation (3.26) can always be constructed using the geometry of the coset space \( \text{Sp}(2n, \mathbb{R}) U(n) \), see Section 3.4.2.

This example also clarifies the condition (3.15) that we have imposed on the \( GL(2n, \mathbb{R}) \) generators. It is a straightforward calculation to check that the equations (3.3) and
\[ \tilde{G} = N_2 F + N_1 \tilde{F} \] (3.30)
have duality symmetry under \( GL(2n, \mathbb{R}) \) transformations with \( \Delta N \) given in (3.27). However it is easy to see that equation (3.14) implies, for the lagrangian (3.23), that condition (3.15) must hold. The point is that we want the constitutive relations \( G = G[F, \varphi] \) to follow from a lagrangian. Those following from the lagrangian (3.23) are (3.30) with \( N_1 \) and \( N_2 \) necessarily symmetric matrices. Only if the transformed matrices \( N_1' \) and \( N_2' \) are again symmetric can we have \( \tilde{G}' = \frac{\partial \mathcal{L}(F', \varphi')}{\partial F'} \) as in (3.7), (or more generally \( \tilde{G}' = \frac{\partial \mathcal{L}(F', \varphi)}{\partial F'} \)). The constraints \( N_1' = N_1'^t, N_2' = N_2'^t \), reduce the duality group to \( \mathbb{R}^\times \times SL(2n, \mathbb{R}) \).

In conclusion equation (3.20) is a necessary and sufficient condition for a theory of \( n \) abelian gauge fields coupled to bosonic matter to be symmetric under \( \mathbb{R}^\times \times SL(2n, \mathbb{R}) \) duality rotations, and \( \mathbb{R}^\times \times SL(2n, \mathbb{R}) \) is the maximal connected Lie group of duality rotations.

### 3.3 A basic example with fermi fields

Consider the Lagrangian with Pauli coupling
\[
\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \bar{\psi} \phi \psi - \frac{1}{2} \bar{\xi} \phi \xi + \frac{1}{2} \lambda F_{\mu\nu} \bar{\psi} \sigma^{\mu\nu} \xi
\] (3.31)
where \( \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \) and \( \psi, \xi \) are two Majorana spinors. We have
\[
\tilde{G}^{\mu\nu} = 2 \frac{\partial \mathcal{L}_0}{\partial F_{\mu\nu}} = -F^{\mu\nu} + \lambda \bar{\psi} \sigma^{\mu\nu} \xi
\] (3.32)
and the duality condition (3.20) for an infinitesimal \( U(1) \) duality rotation \( \left( \begin{array}{cc} 0 & b \\ -b & 0 \end{array} \right) \) reads
\[
\Delta \psi L_0 + \Delta \xi L_0 = -\frac{b}{4} \lambda \bar{F}_{\mu\nu} \sigma \xi + \frac{b}{4} \lambda^2 \bar{\psi} \sigma^{\mu\nu} \xi \bar{\psi} \sigma_{\mu\nu} \xi .
\] (3.33)

It is natural to assume that the kinetic terms of the fermion fields are invariant under this duality rotation (this is also the case for the scalar lagrangian \( L(\phi) \) in (3.26)), then using \( \gamma_5 \sigma^{\mu\nu} = i \tilde{\sigma}^{\mu\nu} \) we see that the coupling of the fermions with the field strength is reproduced if the fermions rotate according to
\[
\Delta \psi = \frac{i}{2} b \gamma_5 \psi ,
\] (3.34)
\[
\Delta \xi = \frac{i}{2} b \gamma_5 \xi ;
\] (3.35)
we also see that we have to add to the lagrangian \( L_0 \) a new interaction term quartic in the fermion fields. Its coupling is also fixed by duality symmetry to be \(-\lambda^2/8\).

The theory with \( U(1) \) duality symmetry is therefore given by the lagrangian [3]
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \bar{\psi} \phi \psi - \frac{1}{2} \bar{\xi} \phi \xi + \frac{1}{2} \lambda \bar{F}_{\mu\nu} \psi \sigma_{\mu\nu} \xi - \frac{1}{8} \lambda^2 \bar{\psi} \sigma_{\mu\nu} \xi \bar{\psi} \sigma^{\mu\nu} \xi .
\] (3.36)

Notice that fermions transform under the double cover of \( U(1) \) indeed under a rotation of angle \( b = 2\pi \) we have \( \psi \to -\psi, \xi \to -\xi \), this is a typical feature of fermions transformations under duality rotations, they transform under the double cover of the maximal compact subgroup of the duality group. This is so because the interaction with the gauge field is via fermions bilinear terms.

### 3.4 Compact and noncompact duality rotations

#### 3.4.1 Compact duality rotations

The fractional transformation (3.28) is also characteristic of nonlinear theories. The subgroup of \( Sp(2n, \mathbb{R}) \) that leaves invariant a fixed value of the scalar fields \( \mathcal{N} \) is \( U(n) \). This is easily seen by setting \( \mathcal{N} = -i \mathbb{I} \). Then infinitesimally we have relations (3.15) with \( \kappa = 0 \) and \( b = -c, a = -a^t \), i.e. we have the antisymmetric matrix
\[
\begin{pmatrix}
a & b \\
-b & a
\end{pmatrix},
\]
a = -a^t, b = b^t. For finite transformations the \( Sp(2n, \mathbb{R}) \) relations (7.2) are complemented by
\[
A = D , \quad B = -C .
\] (3.37)
Thus $A - iB$ is a unitary matrix (see also (7.8)). $U(n)$ is the maximal compact subgroup of $Sp(2n, \mathbb{R})$, it is the group of orthogonal and symplectic $2n \times 2n$ matrices.

More in general from Section 3.1 we easily conclude that a necessary and sufficient condition for a theory with just $n$ abelian gauge fields to have $U(n)$ duality symmetry is (cf. (3.20))

$$\tilde{F}^\Lambda F^\Sigma + \tilde{G}^\Lambda G^\Sigma = 0$$  \hspace{1cm} (3.38)
$$\tilde{G}^\Lambda F^\Sigma - \tilde{F}^\Sigma G^\Lambda = 0$$  \hspace{1cm} (3.39)

for all $\Lambda, \Sigma$. Moreover since any nonlinear lagrangian in the limit of weak field strengths $F^4 << F^2$ reduces to the one in (3.23) (with a fixed value of $\mathcal{N}$), we conclude that $U(n)$ is the maximal duality group for a theory with only gauge fields.

Condition (3.39) is equivalent to

$$(F^\Sigma \frac{\partial}{\partial F^\Lambda} - F^\Lambda \frac{\partial}{\partial F^\Sigma}) \mathcal{L} = 0 ,$$

i.e. to the invariance of the Lagrangian under $SO(n)$ rotations of the $n$ field strengths $F^\Sigma$. Condition (3.38) concerns on the other hand the invariance of the equations of motion under transformation of the electric field strengths into the magnetic field strengths.

In a theory with just $n$ abelian gauge fields the field strengths appear in the Lagrangian only through the Lorentz invariant combinations

$$\alpha^{\Lambda\Sigma} \equiv \frac{1}{4} F^\Lambda F^\Sigma, \quad \beta^{\Lambda\Sigma} \equiv \frac{1}{4} \tilde{F}^\Lambda F^\Sigma,$$

and equation (3.40), tell us that $\mathcal{L}$ is a scalar under $SO(n)$ rotations; e.g. $\mathcal{L}$ is a sum of traces, or of products of traces, of monomials in $\alpha$ and $\beta$ (we implicitly use the metric $\delta_{\Lambda\Sigma}$ in the $\alpha$ and $\beta$ products).

If we define

$$\mathcal{L}_\alpha \equiv \frac{\partial \mathcal{L}}{\partial \alpha}, \quad \mathcal{L}_\beta \equiv \frac{\partial \mathcal{L}}{\partial \beta},$$

then using the chain rule and the definitions (3.41) we obtain that (3.38) is equivalent to

$$\mathcal{L}_\beta \mathcal{L}_\beta - \mathcal{L}_\alpha \beta \mathcal{L}_\alpha + \mathcal{L}_\alpha \alpha \mathcal{L}_\alpha + \mathcal{L}_\beta \beta \mathcal{L}_\beta + \mathcal{L}_\alpha \alpha \mathcal{L}_\alpha + \beta = 0 .$$

If we define

$$p \equiv -\frac{1}{2}(\alpha + i\beta), \quad q \equiv -\frac{1}{2}(\alpha - i\beta),$$

then (3.43) simplifies and reads

$$p - \mathcal{L}_p p \mathcal{L}_p = q - \mathcal{L}_q q \mathcal{L}_q .$$
Condition (3.43) in the case of a single gauge field was considered in [15] together with other equivalent conditions, in particular \(L_uL_v = 1\), where 
\[
u = \frac{1}{2}(\alpha - (\alpha^2 + \beta^2)^{\frac{1}{2}}),
\]
see also [20].

### 3.4.2 Coupling to scalar fields and noncompact duality rotations

By freezing the values of the scalar fields \(N\) we have obtained a theory with only gauge fields and with \(U(n)\) duality symmetry. Vice versa (following [16] that extends to \(U(n)\) the \(U(1)\) interacting theory discussed in [14, 15]) we show that given a theory invariant under \(U(n)\) duality rotations it is possible to extend it via \(n(n + 1)\) scalar fields \(N\) to a theory invariant under \(Sp(2n, \mathbb{R})\). Let \(L(F)\) be the lagrangian of the theory with \(U(n)\) duality. From (3.19) we see that under a \(U(n)\) duality rotation

\[
L(F') - L(F) = -\frac{1}{4}\tilde{F} b F + \frac{1}{4}\tilde{G} b G .
\]

In particular \(L(F)\) is invariant under the orthogonal subgroup \(SO(n) \subset U(n)\) given by the matrix \(\begin{pmatrix} A & 0 \\ 0 & A^t \end{pmatrix}\). This is the so-called electric subgroup of the duality rotation group \(U(n)\) because it does not mix the electric fields \(F\) with the dual fields \(G\).

Define the new lagrangian

\[
L(F, R, N_1) = L(RF) + \frac{1}{4}\tilde{F} N_1 F
\]

where \(R = (R^A_{\Sigma})_{A,\Sigma=1,...,n}\) is an arbitrary nondegenerate real matrix and \(N_1\) is a real symmetric matrix. Because of the \(O(n)\) symmetry the new lagrangian depends only on the combination

\[
N_2 = -R^t R ,
\]

rather than on \(R\). Thus \(L(F, R, N_1) = L(F, N)\) where \(N = N_1 + iN_2\).

We show that \(L\) satisfies the duality condition (3.20),

\[
(\Delta_F + \Delta_R + \Delta_{N_1})L(F, R, N_1) = \frac{1}{4}\tilde{F} c F + \frac{1}{4}\tilde{G} b G
\]

where as always \(\tilde{G} = 2\frac{\partial L}{\partial F}\), and where \(N_1\) transforms as in (3.24) and

\[
\Delta R = -R(a + bN_1) ,
\]

so that \(N_2 = -R^t R\) transforms as in (3.25). Notice that we could also have chosen the transformation \(\Delta R = \Lambda R - R(a + bN_1)\) with \(\Lambda\) an infinitesimal \(SO(n)\) rotation.

We first immediately check (3.49) in the case of the rotation \(\begin{pmatrix} a^t \\ 0 \\ 0 \\ 0 \end{pmatrix}\), where \(a = -d^t\). Finally we consider the duality rotation \(\begin{pmatrix} b^t \\ 0 \\ 0 \\ 0 \end{pmatrix}\). It is convenient to introduce the notation

\[
F = RF , \quad G = 2\frac{\partial L(F)}{\partial F} .
\]

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We observe that $\mathcal{L}(\mathcal{F})$ satisfies the $U(n)$ duality conditions (3.38), (3.39) with $F \rightarrow \mathcal{F}$, $G \rightarrow \mathcal{G}$. Equation (3.49) holds because of (3.38) and proves $Sp(2n, \mathbb{R})$ duality invariance of the theory with lagrangian $\mathcal{L}$.

We end this subsection with few comments. We notice that (3.39) is equivalent to the invariance of the lagrangian under the infinitesimal $SO(n)$ transformation $R \rightarrow \Lambda R$.

We also observe that under an $Sp(2n, \mathbb{R})$ duality transformation $(a \ b \ c \ d)$, the dressed fields $F$ and $G$ transform via the field dependent rotation
\[
\Delta F = R b R^t G ,
\]
\[
\Delta G = -R b R^t F .
\]

The geometry underlying the construction of $Sp(2n, \mathbb{R})$ duality invariant theories from $U(n)$ ones is that of coset spaces. The scalar fields $N$ parametrize the coset space $Sp(2n, \mathbb{R})/U(n)$ (see proof in ...). We also have $Sp(2n, \mathbb{R})/U(n) = SO(n)\backslash GL^{+}(n) \times \mathbb{R}^{(n+1)/2}$ where $GL^{+}(n)$ is the connected component of $GL(n)$ and the equivalence classes $[R] = \{R' \in GL^{+}(n); R'R^{-1} = e^{A} \in SO(n)\}$ parametrize the coset space $SO(n)\backslash GL^{+}(n)$.

The proof of $Sp(2n, \mathbb{R})$ duality symmetry for the theory described by the lagrangian $\mathcal{L}$ holds also if we add to $\mathcal{L}$ an $Sp(2n, \mathbb{R})$ invariant lagrangian for the fields $N$ like the lagrangian in (3.65). Of course we can also consider initial lagrangians in (3.46) that depend on matter fields invariant under the $U(n)$ rotation, they will be $Sp(2n, \mathbb{R})$ invariant in the corresponding lagrangian $\mathcal{L}$. Moreover, by considering an extra scalar field $\Phi$, we can always extend an $Sp(2n, \mathbb{R})$ duality theory to an $\mathcal{R}^{>0} \times Sp(2n, \mathbb{R})$ one.

### 3.5 Nonlinear sigma models on $G/H$

In this section we briefly consider the geometry of coset spaces $G/H$. This is the geometry underlying the scalar fields and needed to formulate their dynamics [59, 60].

We study in particular the case $G = Sp(2n, \mathbb{R})$, $H = U(n)$ [6] and give a kinetic term for the scalar fields $\mathcal{N}$.

The geometry of the coset space $G/H$ is conveniently described in terms of coset representatives, local sections $L$ of the bundle $G \rightarrow G/H$. A point $\phi$ in $G/H$ is an equivalence class $gH = \{g^{-1}\bar{g} \in H\}$. We denote by $\phi^i$ ($i = 1, 2 \ldots m$) its coordinates (the scalar fields of the theory). The left action of $G$ on $G/H$ is inherited from that of $G$ on $G$, it is given by $gH \mapsto g^i gH$, that we rewrite $\phi \mapsto g^i \phi = \phi^i$. Concerning the coset representatives we then have
\[
g^i L(\phi) = L(\phi^i) h ,
\]
but both the left and the right hand side are representatives of $\phi^i$. The geometry of $G/H$ and the corresponding physics can be constructed in terms of coset representatives.
Of course the construction must be insensitive to the particular representative choice, we have a gauge symmetry with gauge group $H$.

When $H$ is compact the Lie algebra of $G$ splits in the direct sum $G = \mathbb{H} + K$, where

$$[\mathbb{H}, \mathbb{H}] \subset \mathbb{H}, \quad [K, K] \subset \mathbb{H} + K, \quad [\mathbb{H}, K] \subset K. \quad (3.55)$$

The last expression defines the coset space representation of $\mathbb{H}$. The representations of the compact Lie algebra $H$ are equivalent to unitary ones, and therefore there exists a basis $(H^a, K_a)$, where $[H^a, K^b] = C^b_{a \alpha} K^\alpha$ with $C^b_{a \alpha} (a \beta = 1, \ldots, m = \dim G/H$ antihermitean matrices. Since the coset representation is a real representation then these matrices $C^a_{\alpha a}$ belong to the Lie algebra of $SO(m)$.

Given a coset representative $L(\phi)$, the pull back on $G/H$ of the $G$ Lie algebra left invariant 1-form $\Gamma = L^{-1} dL$ is decomposed as

$$\Gamma = L^{-1} dL = P^a(\phi) K_a + \omega^\alpha(\phi) H_\alpha. \quad (3.56)$$

The 1-forms $P^a(\phi) = P^a(\phi)_i d\phi^i$ are therefore vielbain on $G/H$ transforming in the fundamental of $SO(m)$, while $\omega = \omega^\alpha(\phi)_i d\phi^i$ is an $\mathbb{H}$-valued connection 1-form on $G/H$. We can then define the covariant derivative $\nabla P^a = [P, \omega]^a = P^b \otimes -C^a_{\alpha b} \omega^\alpha$.

There is a natural metric on $G/H$,

$$g = \delta_{ab} P^a \otimes P^b, \quad (3.57)$$

(this definition is well given because we have shown that the coset representation is via infinitesimal $SO(m)$ rotations). It is easy to see that the connection $\nabla$ is metric compatible, $\nabla g = 0$.

If the coset is furthermore a symmetric coset we have

$$[K, K] \subset \mathbb{H},$$

then the identity $d\Gamma + \Gamma \wedge \Gamma = 0$, that is (the pull-back on $G/H$ of) the Maurer-Cartan equation, in terms of $P$ and $\omega$ reads

$$R + P \wedge P = 0, \quad (3.58)$$
$$dP + P \wedge \omega + \omega \wedge P = 0. \quad (3.59)$$

This last relation shows that $\omega$ is torsionfree. Since it is metric compatible it is therefore the Riemannian connection on $G/H$. Equation (3.58) then relates the Riemannian curvature to the square of the vielbeins.
By using the connection $\omega$ and the vierbein $P$ we can construct couplings and actions invariant under the rigid $G$ and the local $H$ transformations, i.e. sigma models on the coset space $G/H$.

For example a kinetic term for the scalar fields, which are maps from spacetime to $G/H$, is given by pulling back to spacetime the invariant metric (3.57) and then contracting it with the spacetime metric

$$\mathcal{L}_{\text{kin}}(\phi) = \frac{1}{2} P^a_\mu P^\mu_a = \frac{1}{2} P^a_\mu \phi^i \partial^\mu \phi_j \phi^j.$$  

(3.60)

By construction the lagrangian $\mathcal{L}_{\text{kin}}(\phi)$ is invariant under $G$ and local $H$ transformations; it depends only on the coordinates of the coset space $G/H$.

3.5.1 The case $G = Sp(2n, \mathbb{R})$, $H = U(n)$

A kinetic term for the $Sp(2n, \mathbb{R})$ valued scalar fields is given by (3.60). This lagrangian is invariant under $Sp(2n, \mathbb{R})$ and therefore satisfies the duality condition (3.26) with $G = Sp(2n, \mathbb{R})$ and $\kappa = 0$. We can also write

$$\mathcal{L}_{\text{kin}}(\phi) = \frac{1}{2} P^a_\mu P^\mu_a = \frac{1}{2} \text{Tr}(P_\mu P^\mu) ;$$  

(3.61)

where in the last passage we have considered generators $K_a$ so that $\text{Tr}(K_a K_b) = \delta_{ab}$ (this is doable since $U(n)$ is the maximal compact subgroup of $Sp(2n, \mathbb{R})$).

We now recall the representation of the group $Sp(2n, \mathbb{R})$ and of the associated coset $Sp(2n, \mathbb{R})/U(n)$ in the complex basis discussed in the appendix (and frequently used in the later sections) and we give a more explicit expression for the lagrangian (3.61).

Rather than using the symplectic matrix $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of the fundamental representation of $Sp(2n, \mathbb{R})$, we consider the conjugate matrix $A^{-1} S A$ where $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & i \end{pmatrix}$. In this complex basis the subgroup $U(n) \subset Sp(2n, \mathbb{R})$ is simply given by the block diagonal matrices $\begin{pmatrix} \mathbb{1}_n & \mathbb{0} \\ \mathbb{0} & \mathbb{1}_n \end{pmatrix}$. We also define the $n \times 2n$ matrix

$$\begin{pmatrix} f \\ h \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} A - iB \\ C - iD \end{pmatrix}$$  

(3.62)

and the matrix

$$V = \begin{pmatrix} f & \bar{f} \\ h & \bar{h} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} A.$$  

(3.63)

Then (cf. (7.9), (7.10)),

$$V^{-1} dV = \begin{pmatrix} i(f^\dagger df - h^\dagger dh) & i(f^\dagger dh - h^\dagger df) \\ -i(f^\dagger df - h^\dagger dh) & -i(f^\dagger dh - h^\dagger df) \end{pmatrix} \equiv \begin{pmatrix} \omega & \mathcal{P} \\ \mathcal{P} & \omega \end{pmatrix},$$  

(3.64)
where in the last passage we have defined the $n \times n$ sub-blocks $\omega$ and $P$ corresponding to the $U(n)$ connection and the vielbein of $Sp(2n, \mathbb{R})/U(n)$ in the complex basis, (with slight abuse of notation we use the same letter $\omega$ in this basis too).

We finally obtain the explicit expression

$$
\mathcal{L}_{\text{kin}}(\phi) = \text{Tr}(\bar{P}_\mu P^\mu) = \frac{1}{4} \text{Tr}(N_2^{-1} \partial_\mu \bar{N} N_2^{-1} \partial^\mu N) \tag{3.65}
$$

where $P = P_\mu dx^\mu = P_\mu \partial_\mu \phi^i dx^\mu$, $\bar{N} = N_1 - iN_2$ and $N = N_1 + iN_2 = \text{Re}N + i\text{Im}N$. The matrix of scalars $N$ parametrizes the coset space $Sp(2n, \mathbb{R})/U(n)$ (see Appendix 7.2); in terms of the $f$ and $h$ matrices it is given by (cf. (7.19))

$$
N = fh^{-1}, \quad N_2^{-1} = -2ff^\dagger. \tag{3.66}
$$

Under the symplectic rotation $(A^B, C^B, D^B) \rightarrow (A'^B, C'^B, D'^B)(A^B, C^B, D^B)^{-1}$, (3.28),

Another proof of the invariance of the kinetic term (3.65) under the $Sp(2n, \mathbb{R})$ follows by observing that (3.65) is obtained from the pullback to the spacetime manifold of the metric associated to the $Sp(2n, \mathbb{R})$ Kähler form $\text{Tr}(N_2^{-1} d\bar{N} N_2^{-1} dN)$ (here $d = \partial + \bar{\partial}$ is the exterior derivative). This metric is obtained from the Kähler potential

$$
K = -4 \text{Tr} \log (N - \bar{N}). \tag{3.67}
$$

Under the action of $Sp(2n, \mathbb{R})$, $N$ and $N - \bar{N}$ change as in (3.28), (3.29) and the Kähler potential changes by a Kähler transformation, thus showing the invariance of the metric.

### 3.4.2 The case $G = \mathbb{R}^{>0} \times Sp(2n, \mathbb{R})$, $H = U(n)$

In this case the duality rotation matrix $(a^B, b^B, c^B, d^B)$ belongs to the Lie algebra of $\mathbb{R}^{>0} \times Sp(2n, \mathbb{R})$, as defined in (3.15). In particular infinitesimal dilatations are given by the matrix $\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. The coset space is

$$
\frac{\mathbb{R}^{>0} \times Sp(2n, \mathbb{R})}{U(n)} = \mathbb{R}^{>0} \times \frac{Sp(2n, \mathbb{R})}{U(n)}, \tag{3.68}
$$

there is no action of $U(n)$ on $\mathbb{R}^{>0}$. We consider a real positive scalar field $\Phi = e^\sigma$ invariant under $Sp(2n, \mathbb{R})$ transformations. The fields $\Phi$ and $N$ parametrize the coset space (3.68).

Let’s first consider the main example of Section 3.2. The duality symmetry conditions for the lagrangian (3.23) are (3.24)-(3.26). From equations (3.24), (3.25) (that hold for $(a^B)$ in the Lie algebra of $\mathbb{R}^{>0} \times Sp(2n, \mathbb{R})$) we see that the fields $N$, and henceforth the lagrangian $\mathcal{L}_{\text{kin}}(\phi)$, are invariant under the $\mathbb{R}^{>0}$ action. It follows that the scalar lagrangian

$$
\Phi^2 \mathcal{L}_{\text{kin}}(\phi) + \partial_\mu \Phi \partial^\mu \Phi \tag{3.69}
$$
satisfies the duality condition (3.26). This shows that the lagrangian (3.23) with the scalar kinetic term given by (3.69) has \( \mathbb{R}^0 \times Sp(2n, \mathbb{R}) \) duality symmetry. We see that in the lagrangian (3.23) the scalar \( \Phi \) does not couple to the field strenght \( F \). The coupling of \( \Phi \) to \( F \) is however present in lagrangians where higher powers of \( F \) are present.

More in general expression (3.69) is a scalar kinetic term for lagrangians that satisfy the \( \mathbb{R}^0 \times Sp(2n, \mathbb{R}) \) duality condition (3.20).

### 3.6 Invariance of energy momentum tensor

Duality rotation symmetry is a symmetry of the equations of motion that does not leave invariant the lagrangian. The total change \( \Delta \mathcal{L} \equiv \mathcal{L}(F', \varphi') - \mathcal{L}(F, \varphi) \) of the lagrangian is given in equation (3.19). Even if \( \kappa = 0 \) this variation is not a total derivative because \( F \) and \( G \) are the curl of vector potentials \( A_F \) and \( A_G \) only on shell.

We show however that the variation of the action with respect to a duality rotation invariant parameter \( \lambda \) is invariant under \( Sp(2n, \mathbb{R}) \) rotations if the duality rotation (3.10) of the \( \varphi \) fields is \( \lambda \) independent.

Consider the \( \lambda \)-variation of \( \Delta S[F, \varphi] \equiv S[F', \varphi'] - S[F, \varphi] = \int_y \frac{\partial \mathcal{L}}{\partial F} \Delta F + \Delta \varphi S \),

\[
\frac{\delta}{\delta \lambda} \Delta S = \int_y \frac{\delta}{\delta \lambda} \left( \frac{\partial \mathcal{L}}{\partial F} \right) \Delta F + \int_y \frac{\partial \mathcal{L}}{\partial F} \frac{\delta}{\delta \lambda} \left( \Delta F \right) + \frac{\delta}{\delta \lambda} \left( \Delta \varphi S \right) \\
= \int_y \frac{\partial}{\partial F} \left( \frac{\delta \mathcal{L}}{\delta \lambda} \right) \Delta F + \frac{1}{2} \int_y \tilde{G} \frac{\delta}{\delta \lambda} \left( \Delta F \right) + \Delta \varphi \left( \frac{\delta S}{\delta \lambda} \right) \\
= \Delta \left( \frac{\delta S}{\delta \lambda} \right) + \frac{1}{2} \int y \tilde{G} b G \tag{3.70}
\]

where in the second line we used that \( \frac{\delta}{\delta \lambda} \Delta \varphi = 0 \). Thus \( \Delta \left( \frac{\delta S}{\delta \lambda} \right) = \frac{\delta}{\delta \lambda} \left( \Delta S - \frac{1}{2} \int_y \tilde{G} b G \right) \) and therefore from (3.19) we have,

\[
\Delta \left( \frac{\delta S}{\delta \lambda} \right) = \kappa \frac{\delta S}{\delta \lambda} \tag{3.71}
\]

thus showing invariance of \( \frac{\delta S}{\delta \lambda} \) under \( Sp(2n, \mathbb{R}) \) rotations (\( \kappa = 0 \) rotations).

An important case is when \( \lambda \) is the metric \( g_{\mu \nu} \), this is invariant under duality rotations. This shows that the energy momentum tensor \( \frac{\delta S}{\delta g_{\mu \nu}} \) is invariant under \( Sp(2n, \mathbb{R}) \) duality rotations.

Another instance is when \( \lambda \) is the dimensional parameter typically present in a nonlinear theory. Provided the matter fields are properly rescaled \( \varphi \to \hat{\varphi} = \lambda^s \varphi \), so that they become adimensional and therefore their transformation \( \Delta \hat{\varphi} \), usually nonlinear, does not explicitly involve \( \lambda \), then \( \frac{\delta S}{\delta \lambda} \) is invariant, where it is understood that \( \frac{\partial \hat{\varphi}}{\partial \lambda} = 0 \).
For the action of the Born-Infeld theory coupled to the axion and dilaton fields,
\[ L = \frac{1}{\lambda}(1 - \sqrt{1 - \frac{1}{2}\lambda N_2 F^2 - \frac{1}{16}\lambda^2 N_2(F\bar{F})^2}) \]
we obtain the invariant \( \frac{\partial L}{\partial \lambda} = -\frac{1}{\lambda}(L - \frac{1}{4}F\bar{G}); \)
we already found this invariant in (3.22).

3.7 Generalized Born Infeld theory

In this section we present the Born-Infeld theory with \( n \) abelian gauge fields coupled to \( n(n + 1)/2 \) scalar fields \( N \) and show that is has an \( Sp(2n, \mathbb{R}) \) duality symmetry. If we freeze the scalar fields \( N \) to the value \( N = -i \mathbb{1} \) then the lagrangian has \( U(n) \) duality symmetry and reads

\[ L = \text{Tr}[\mathbb{1} - S_{\alpha,\beta} \sqrt{\mathbb{1} + 2\alpha - \beta^2}] \],

(3.72)

where as defined in (3.41), the components of the \( n \times n \) matrices \( \alpha \) and \( \beta \) are \( \alpha^\Lambda\Sigma = \frac{1}{4}F^\Lambda F^\Sigma, \ \beta^\Lambda\Sigma = \frac{1}{4}\tilde{F}^\Lambda\tilde{F}^\Sigma \). The square root is to be understood in terms of its power series expansion, and the operator \( S_{\alpha,\beta} \) acts by symmetrizing each monomial in the \( \alpha \) and \( \beta \) matrices. A world (monomial) in the letters \( \alpha \) and \( \beta \) is symmetrized by averaging over all permutations of its letters. The normalization of \( S_{\alpha,\beta} \) is such that if \( \alpha \) and \( \beta \) commute then \( S_{\alpha,\beta} \) acts as the identity. Therefore in the case of just one abelian gauge field (3.72) reduces to the usual Born-Infeld lagrangian.

The \( Sp(2n, \mathbb{R}) \) Born-Infeld lagrangian is obtained by coupling the lagrangian (3.72) to the scalar fields \( N \) as described in Subsection 3.4.2 and explicitly considered in (3.109).

Following [18] we prove the duality symmetry of the Born-Infeld theory (3.72) by first showing that a Born-Infeld theory with \( n \) complex abelian gauge fields written in an auxiliary field formulation has \( U(n,n) \) duality symmetry. We then eliminate the auxiliary fields by proving a remarkable property of solutions of matrix equations [19]. Then we can consider real fields.

3.7.1 Duality rotations with complex field strengths

From the general study of duality rotations we know that a theory with \( 2n \) real fields \( F_1^\Lambda \) and \( F_2^\Lambda \) (\( \Lambda = 1, \ldots n \)) has at most \( Sp(4n, \mathbb{R}) \) duality if we consider duality rotations that leave invariant the energy-momentum tensor (and in particular the hamiltonian). We now consider the complex fields

\[ F^\Lambda = F_1^\Lambda + iF_2^\Lambda, \quad \bar{F}^\Lambda = F_1^\Lambda - iF_2^\Lambda, \]

(3.73)

the corresponding dual fields

\[ G = \frac{1}{2}(G_1 + iG_2), \quad \bar{G} = \frac{1}{2}(G_1 - iG_2), \]

(3.74)
and restrict the $Sp(4n, \mathbb{R})$ duality group to the subgroup of holomorphic transformations,

\[ \Delta \left( \begin{array}{c} F \\ G \end{array} \right) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} F \\ G \end{array} \right) \]  
(3.75)

\[ \Delta \left( \begin{array}{c} \bar{F} \\ \bar{G} \end{array} \right) = \left( \begin{array}{cc} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{array} \right) \left( \begin{array}{c} \bar{F} \\ \bar{G} \end{array} \right). \]  
(3.76)

This requirement singles out those matrices, acting on the vector $\left( \begin{array}{c} F_1 \\ F_2 \\ G_1 \\ G_2 \end{array} \right)$, that belong to the Lie algebra of $Sp(4n, \mathbb{R})$ and have the form

\[ \left[ \begin{array}{cc} \mathcal{A}(a & 0) \mathcal{A}^{-1} & \frac{1}{2} \mathcal{A}(b & 0) \mathcal{A}^{-1} \\ 2 \mathcal{A}(c & 0) \mathcal{A}^{-1} & \mathcal{A}(d & 0) \mathcal{A}^{-1} \end{array} \right] \]  
(3.77)

where $\mathcal{A} = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array} \right)$. The matrix (3.77) belongs to $Sp(4n, \mathbb{R})$ iff the $n \times n$ complex matrices $a, b, c, d$ satisfy

\[ a^\dagger = -a, \quad b^\dagger = b, \quad c^\dagger = c. \]  
(3.78)

Matrices $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$, that satisfy (3.78), define the Lie algebra of the real form $U(n, n)$. The group $U(n, n)$ is here the subgroup of $GL(2n, \mathbb{C})$ characterized by the relations

\[ M^\dagger \left( \begin{array}{cc} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{array} \right) M = \left( \begin{array}{cc} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{array} \right). \]  
(3.79)

One can check that (3.79) implies the following relations for the block components of $M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$,

\[ C^\dagger A = A^\dagger C, \quad B^\dagger D = D^\dagger B, \quad D^\dagger A - B^\dagger C = \mathbb{I}. \]  
(3.80)

The Lie algebra relations (3.78) can be obtained from the Lie group relations (3.80) by writing $(\mathcal{A}_G) = (\mathcal{A}_D) + \epsilon(a^b)_{cd}$ with $\epsilon$ infinitesimal. Equation (3.77) gives the embedding of $U(n, n)$ in $Sp(4n, \mathbb{R})$.

The theory of holomorphic duality rotations can be seen as a special case of that of real duality rotations, but (as complex geometry versus real geometry) it deserves also an independent formulation based on the holomorphic variables $\left( \begin{array}{c} F \\ G \end{array} \right)$ and maps $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$.

\footnote{In Appendix 7.1 we define $U(n, n)$ as the group of complex matrices that satisfy the condition $U^\dagger \left( \begin{array}{cc} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{array} \right) U = \left( \begin{array}{cc} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{array} \right)$. The similarity transformation between these two definitions is $M = \mathcal{A} U \mathcal{A}^{-1}$.}
The dual fields in (3.74), or rather the Hodge dual of the dual field strength, \( \tilde{G}^{\mu\nu}_\Lambda = \frac{1}{2} \varepsilon_{\mu\rho\sigma} G^{\rho\sigma}_\Lambda \), is equivalently defined via

\[
\tilde{G}^{\mu\nu}_\Lambda \equiv 2 \frac{\partial \mathcal{L}}{\partial F^{\Lambda \mu\nu}} , \quad \tilde{G}^{\mu\nu}_\Lambda \equiv 2 \frac{\partial \mathcal{L}}{\partial \bar{F}^{\Lambda \mu\nu}} .
\]

Repeating the passages of Section 3.1 we have that the Bianchi identities and equations of motion \( \partial_\mu \tilde{F}^{\Lambda \mu\nu} = 0 \), \( \partial_\mu \tilde{G}^{\mu\nu}_\Lambda = 0 \), \( \frac{\delta S_{F,\bar{F},\varphi}}{\delta \varphi} = 0 \) transform covariantly under the holomorphic infinitesimal transformations (3.75) if the lagrangian satisfies the condition (cf. (3.19))

\[
\mathcal{L}(F + \Delta F, \bar{F} + \Delta \bar{F}, \varphi + \Delta \varphi) - \mathcal{L}(F, \bar{F}, \varphi) - \frac{1}{2} \tilde{F} c \tilde{F} - \frac{1}{2} \tilde{G} b \tilde{G} = \text{const}_{a,b,c,d} \quad (3.82)
\]

Of course we can also consider dilatations \( \kappa \neq 0 \), then in the left hand side of (3.82) we have to add the term \( -\kappa \mathcal{L}(F, \bar{F}, \varphi) \).

The maximal compact subgroup of \( U(n,n) \) is \( U(n) \times U(n) \) and is obtained by requiring (3.80) and

\[
A = D , \quad B = -C .
\]

The corresponding infinitesimal relations are (3.78) and \( a = d , \ b = -c \).

The coset space \( \frac{U(n,n)}{U(n) \times U(n)} \) is the space of all negative definite hermitian matrices \( \mathcal{M} \) of \( U(n,n) \), see for example [18] (the proof is similar to that for \( Sp(2n,\mathbb{R})/U(n) \) in Appendix 7.2). All these matrices are for example of the form \( \mathcal{M} = -g^{-1} \tilde{g}^{-1} \) with \( g \in U(n,n) \). These matrices can be factorized as

\[
\mathcal{M} = \begin{pmatrix} \mathbb{I} & -\mathcal{N}_1 \\ \mathbb{I} & \end{pmatrix} \begin{pmatrix} \mathcal{N}_2 & 0 \\ 0 & \mathcal{N}_2^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ -\mathcal{N}_1^\dagger & \mathbb{I} \end{pmatrix} \\
= \begin{pmatrix} \mathcal{N}_2 + \mathcal{N}_1 \mathcal{N}_2^{-1} \mathcal{N}_1^\dagger & -\mathcal{N}_1 \mathcal{N}_2^{-1} \\ -\mathcal{N}_2^{-1} \mathcal{N}_1^\dagger & \mathcal{N}_2^{-1} \end{pmatrix} \\
= -i \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{N} & \mathcal{N} \mathcal{N}_2^{-1} \mathcal{N}_1^\dagger & -\mathcal{N} \mathcal{N}_2^{-1} \\ -\mathcal{N} \mathcal{N}_2^{-1} \mathcal{N}_1^\dagger & \mathcal{N} \mathcal{N}_2^{-1} \end{pmatrix}
\]

(3.83)

where \( \mathcal{N}_1 \) is hermitian, \( \mathcal{N}_2 \) is hermitian and negative definite, and

\[
\mathcal{N} \equiv \mathcal{N}_1 + i \mathcal{N}_2 .
\]

Since any complex matrix can always be decomposed into hermitian matrices as in (3.84), the only requirement on \( \mathcal{N} \) is that \( \mathcal{N}_2 \) is negative definite.
The left action of \( U(n, n) \) on itself \( g \rightarrow (\begin{pmatrix} A & B \\ C & D \end{pmatrix} g \begin{pmatrix} A & B \\ C & D \end{pmatrix}) \) because \( M = -g^{-1}g^{-1} \). Expression (3.83) then immediately gives the action of \( U(n, n) \) on the parametrization \( \mathcal{N} \) of the coset space,

\[
\mathcal{N} \rightarrow \mathcal{N}' = (C + D\mathcal{N})(A + B\mathcal{N})^{-1},
\]

(3.85)

\[
\mathcal{N}_2 \rightarrow \mathcal{N}_2' = (A + B\mathcal{N})^{-1}\mathcal{N}_2(A + B\mathcal{N})^{-1}.
\]

(3.86)

As in Section 3.4, given a theory depending on \( n \) complex fields \( F^\Lambda \) and invariant under the maximal compact duality group \( U(n) \times U(n) \) it is possible to extend it via the complex scalar fields \( \mathcal{N} \) to a theory invariant under \( U(n, n) \). The new lagrangian is

\[
\mathcal{L}(F, R, \mathcal{N}_1) = \mathcal{L}(RF) + \frac{1}{2} \bar{F}\mathcal{N}_1\bar{F}
\]

(3.87)

where \( R = (R^\Lambda_\Sigma)_{\Lambda, \Sigma=1,...,n} \) is now an arbitrary nondegenerate complex matrix. Because of the \( U(n) \) maximal compact electric subgroup this new lagrangian depends only on the combination

\[
\mathcal{N}_2 = -R^\dagger R,
\]

(3.88)

rather than on \( R \). Thus \( \mathcal{L}(F, R, \mathcal{N}_1) = \mathcal{L}(F, \mathcal{N}) \) where \( \mathcal{N} = \mathcal{N}_1 + i\mathcal{N}_2 \). A transformation for \( R \) compatible with (3.85) is

\[
R' = R(A + B\mathcal{N})^{-1},
\]

(3.89)

whose infinitesimal transformation is \( \Delta R = -R(a + b\mathcal{N}) \).

Conversely, if we are given a Lagrangian \( \mathcal{L} \) with equations of motion invariant under \( U(n, n) \) we can obtain a theory without the scalar field \( \mathcal{N} \) by setting \( \mathcal{N} = -i\mathbb{I} \). Then the duality group is broken to the stability group of \( \mathcal{N} = -i\mathbb{I} \) which is \( U(n) \times U(n) \), the maximal compact subgroup.

Similarly to Section 3.4.1 we define the Lorentz invariant combinations

\[
\alpha^{ab} \equiv \frac{1}{2} F^a F^b, \quad \beta^{ab} \equiv \frac{1}{2} \bar{F}^a F^b.
\]

(3.90)

If we consider lagrangians \( \mathcal{L}(F, \bar{F}) \) that depend only on gauge fields and only through sum of traces (or of products of traces) of monomials in \( \alpha \) and \( \beta \), then the necessary and sufficient condition for \( U(n) \times U(n) \) holomorphic duality symmetry is still (3.43), where now \( \alpha \) and \( \beta \) are as in (3.90).
3.7.2 Born-Infeld with auxiliary fields

A lagrangian that satisfies condition (3.82) is

\[ L = \text{Re} \text{Tr} \left[ i(\mathcal{N} - \lambda)\chi - \frac{i}{2}\lambda\chi^\dagger\mathcal{N}_2\chi - i\lambda(\alpha + i\beta) \right] , \quad (3.91) \]

The auxiliary fields \( \chi \) and \( \lambda \) and the scalar field \( \mathcal{N} \) are \( n \) dimensional complex matrices. We can also add to the lagrangian a duality invariant kinetic term for the scalar field \( \mathcal{N} \), (cf (3.65))

\[ \text{Tr}(\mathcal{N}_2^{-1}\partial_\mu\mathcal{N}^\dagger\mathcal{N}_2^{-1}\partial^\mu\mathcal{N}) . \quad (3.92) \]

In order to prove the duality of (3.91) we first note that the last term in the Lagrangian can be written as

\[-\text{Re} \text{Tr} \left[ i\lambda(\alpha + i\beta) \right] = -\text{Tr}(\lambda_2\alpha + \lambda_1\beta) . \]

If the field \( \lambda \) transforms by fractional transformation and \( \lambda_1, \lambda_2 \) and the gauge fields are real this is the \( U(1)^n \) Maxwell action (3.23), with the gauge fields interacting with the scalar field \( \lambda \). This term by itself has the correct transformation properties under the duality group. Similarly for hermitian \( \alpha, \beta, \lambda_1 \) and \( \lambda_2 \) this term by itself satisfies equation (3.82). It follows that the rest of the Lagrangian must be duality invariant. The duality transformations of the scalar and auxiliary fields are

\[ \lambda' = (C + D\lambda)(A + B\lambda)^{-1} , \quad (3.93) \]
\[ \chi' = (A + B\mathcal{N})\chi(A + B\lambda^\dagger)^\dagger , \quad (3.94) \]

and (3.85). Invariance of \( \text{Tr}[i(\mathcal{N} - \lambda)\chi] \) is easily proven by using (3.80) and by rewriting (3.93) as

\[ \lambda' = (A + B\lambda^\dagger)^{-\dagger}(C + D\lambda^\dagger)^\dagger . \quad (3.95) \]

Invariance of the remaining term which we write as \( \text{Re} \text{Tr} \left[ -\frac{i}{2}\lambda\chi^\dagger\mathcal{N}_2\chi \right] = \text{Tr} \left[ \frac{i}{2}\lambda_2\chi^\dagger\mathcal{N}_2\chi \right] \), is straightforward by using (3.86) and the following transformation obtained from (3.95),

\[ \lambda'_2 = (A + B\lambda^\dagger)^{-\dagger}\lambda_2(A + B\lambda^\dagger)^{-1} . \quad (3.96) \]

3.7.3 Elimination of the Auxiliary Fields

The equation of motion obtained by varying \( \lambda \) gives an equation for \( \chi \),

\[ \chi + \frac{1}{2}\chi^\dagger\mathcal{N}_2\chi + \alpha + i\beta = 0 , \quad (3.97) \]

\(^7\)In [18] we use different notations: \( \mathcal{N} \rightarrow S^\dagger, \lambda \rightarrow \lambda^\dagger, \chi \rightarrow \chi^\dagger, \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \rightarrow \left( \begin{array}{cc} D & C \\ B & A \end{array} \right) \).
using this equation in the Lagrangian (3.91) we obtain
\[ \mathcal{L} = \text{Re Tr} \left( iN \chi \right) \]
\[ = \text{Re Tr} \left( -N_2 \chi \right) + \text{Tr} \left( N_1 \beta \right), \tag{3.98} \]
where \( \chi \) is now a function of \( \alpha, \beta \) and \( N_2 \) that solves (3.97). In the second line we observed that the anti-hermitian part of (3.97) implies \( \chi_2 = -\beta \).

In this subsection we give the explicit expression of \( \mathcal{L} \) in terms of \( \alpha, \beta \) and \( N \).

First notice that (3.97) can be simplified with the following field redefinitions
\[ \hat{\chi} = R \chi R^\dagger, \]
\[ \hat{\alpha} = R \alpha R^\dagger, \]
\[ \hat{\beta} = R \beta R^\dagger, \tag{3.100} \]
where, as in (3.88), \( R^\dagger R = -N_2 \). The equation of motion for \( \chi \) is then equivalent to
\[ \hat{\chi} - \frac{1}{2} \hat{\chi}^\dagger \hat{\chi} + \hat{\alpha} - i \hat{\beta} = 0. \tag{3.101} \]

The anti-hermitian part of (3.101) implies \( \hat{\chi}_2 = -\hat{\beta} \), thus \( \hat{\chi}^\dagger = \hat{\chi} - 2i \beta \). This can be used to eliminate \( \hat{\chi}^\dagger \) from (3.101) and obtain a quadratic equation for \( \hat{\chi} \). If we define \( Q = \frac{1}{2} \hat{\chi} \) this equation reads
\[ Q = q + (p - q) Q + Q^2, \tag{3.102} \]
where
\[ p \equiv -\frac{1}{2} (\alpha + i \beta), \quad q \equiv -\frac{1}{2} (\alpha - i \beta). \]
The lagrangian is then
\[ \mathcal{L} = 2 \text{Re Tr} \ Q + \text{Tr} \ (N_1 \beta). \tag{3.103} \]

If the degree of the matrices is one, we can solve for \( Q \) in the quadratic equation (3.102). Apart from the fact that the gauge fields are complex, the result is the Born-Infeld Lagrangian coupled to the dilaton and axion fields \( N \),
\[ \mathcal{L} = 1 - \sqrt{1 - 2N_2 \alpha + N_2^2 \beta^2 + N_1 \beta}. \tag{3.104} \]

For matrices of higher degree, equation (3.102) can be solved perturbatively,
\[ Q_0 = 0, \quad Q_{k+1} = q + (p - q) Q_k + Q_k^2, \tag{3.105} \]
and by analyzing the first few terms in an expansion similar to (3.105) in [17,18] it was conjectured that
\[ \text{Tr} \ Q = \frac{1}{2} \text{Tr} \left[ \mathbb{I} + q - p - S_{p,q} \sqrt{1 - 2(p + q) + (p - q)^2} \right], \tag{3.106} \]
The right hand side formula is understood this way: first expand the square root as a power series in \( p \) and \( q \) assuming that \( p \) and \( q \) commute. Then solve the ordering ambiguities arising from the noncommutativity of \( p \) and \( q \) by symmetrizing, with the operator \( S_{p,q} \), each monomial in the \( p \) and \( q \) matrices. A world (monomial) in the letters \( p \) and \( q \) is symmetrized by considering the sum of all the permutations of its letters, then normalize the sum by dividing by the number of permutations. This normalization of \( S_{p,q} \) is such that if \( p \) and \( q \) commute then \( S_{p,q} \) acts as the identity. Therefore in the case of just one abelian gauge field (3.72) reduces to the usual Born-Infeld lagrangian.

An explicit formula for the coefficients of the expansion of the trace of \( Q \) is [19, 69]

\[
\text{Tr } Q = \text{Tr} \left[ q + \sum_{r,s \geq 1} \left( \begin{array}{c} r + s - 2 \\ r - 1 \end{array} \right) \left( \begin{array}{c} r + s \\ r \end{array} \right) S(p^r q^s) \right].
\] (3.107)

In Appendix 8, following [19], see also [70] and [71], we prove that the trace of \( Q \) is completely symmetrized in the matrix coefficients \( q \) and \( p - q \). Since this is equivalent to symmetrization in \( q \) and \( p \), (3.106) follows. Since symmetrization in \( p \) and \( q \) is equivalent to symmetrization in \( \hat{\alpha} \) and \( \hat{\beta} \), the Born-Infeld lagrangian also reads

\[
\mathcal{L} = \text{Tr}[\mathbb{1} - S_{\alpha,\beta} \sqrt{\mathbb{1} + 2\hat{\alpha} - \hat{\beta}^2 + N_1\beta}].
\] (3.108)

In [69] the convergence of perturbative matrix solutions of (3.97), are studied. A sufficient condition for the convergence of the sequence (3.105) to a solution of (3.102) is that the norms of \( p - q \) and \( q \) have to satisfy \( (1 - ||p - q||^2 > 4||q||) \). Here \( || \cdot || \) denotes any matrix norm with the Banach algebra property \( ||MM'|| \leq ||M|| ||M'|| \) (e.g. the usual norm). This condition is surely met if the field strengths \( F_{\mu\nu}^\Lambda \) are weak.

If equation (3.102) is written as \( (\mathbb{1} + q - p)Q = q + Q^2 \), then the sequence given by \( Q_0 = 0, \ Q_{k+1} = (\mathbb{1} + q - p)^{-1}q + (\mathbb{1} + q - p)^{-1}Q_k^2 \) converges and is a solution of equation (3.102) if \( ||(\mathbb{1} + q - p)^{-1}|| ||(\mathbb{1} + q - p)^{-1}q|| < 1/4 \). Notice that the matrix \( \mathbb{1} + q - p \) is always invertible, use \( \frac{1}{2}(\mathbb{1} + q - p) + \frac{1}{2}(\mathbb{1} + q - p)^\dagger = \mathbb{1} \), and the same argument as in (7.18). Notice also that if \( p \) and \( q \) commute then \( \sqrt{\mathbb{1} - 2(p + q) + (p - q)^2} = (\mathbb{1} + q - p)\sqrt{\mathbb{1} - 4(\mathbb{1} + q - p)^{-2}q} \) and convergence of the power series expansion of this latter square root holds if \( ||(\mathbb{1} + q - p)^{-2}q|| < 1/4 \).

### 3.7.4 Real field Strengths

We here construct a Born-Infeld theory with \( n \) real field strengths which is duality invariant under the duality group \( Sp(2n, \mathbb{R}) \).

We first study the case without scalar fields, i.e. \( N_1 = 0 \) and \( -N_2 = R = \mathbb{1} \). Consider a Lagrangian \( \mathcal{L} = \mathcal{L}(\alpha, \beta) \) with \( n \) complex gauge fields which describes a theory symmetric under the maximal compact group \( U(N) \times U(N) \) of holomorphic duality rotations. Assume that the Lagrangian is a sum of traces (or of products of
traces) of monomials in $\alpha$ and $\beta$. It follows that this Lagrangian satisfies the self-duality equations (3.43) with $\alpha$ and $\beta$ complex (recall end of Section 3.7.1). This equation remains true in the special case that $\alpha$ and $\beta$ assume real values. That is $L = L(\alpha, \beta)$ satisfies the self-duality equation (3.43) with $\alpha = \alpha^T = \bar{\alpha}$ and $\beta = \beta^T = \bar{\beta}$. We now recall that equation (3.43) is also the self-duality condition for Lagrangians with real gauge fields provided that $\alpha$ and $\beta$ are defined as in (3.41) as functions of field strengths $F^\Lambda$ that are real (cf. the different complex case definition (3.90)). This implies that the theory described by the lagrangian $L(\alpha, \beta)$ that is now function of $n$ real field strengths is self-dual with duality group $U(n)$, the maximal compact subgroup of $Sp(2n, \mathbb{R})$. The duality group can be extended to the full noncompact $Sp(2n, \mathbb{R})$, by introducing the symmetric matrix of scalar fields $N$ via the prescription (3.47).

As a straightforward application we obtain the Born-Infeld Lagrangian with $n$ real gauge fields describing an $Sp(2n, \mathbb{R})$ duality invariant theory

\[ L = \text{Tr} \left[ 1 - S_{\hat{\alpha}, \hat{\beta}} \sqrt{1 + 2\hat{\alpha} - \hat{\beta}^2 + \mathcal{N}_1 \beta} \right], \] (3.109)

where $\hat{\alpha} = R\alpha R^t$, $\hat{\beta} = R\beta R^t$, $\mathcal{N}_2 = -R^t R$, and $\alpha^{\Lambda \Sigma} = \frac{1}{4} F^\Lambda F^{\Sigma}$, $\beta^{\Lambda \Sigma} = \frac{1}{4} \tilde{F}^\Lambda F^{\Sigma}$ as in (3.41).

### 3.7.5 Supersymmetric Theory

In this section we briefly discuss supersymmetric versions of some of the Lagrangians introduced. First we discuss the supersymmetric form of the Lagrangian (3.91). Consider the superfields $V^\Lambda = \frac{1}{\sqrt{2}} (V_1^\Lambda + iV_2^\Lambda)$ and $\tilde{V}^\Lambda = \frac{1}{\sqrt{2}} (V_1^\Lambda - iV_2^\Lambda)$ where $V_1^\Lambda$ and $V_2^\Lambda$ are real vector superfields, and define

\[ W^\Lambda_\alpha = -\frac{1}{4} \bar{D}^2 D_\alpha V^\Lambda, \quad \tilde{W}^\Lambda_\alpha = -\frac{1}{4} \bar{D}^2 D_\alpha \tilde{V}^\Lambda. \]

Both $W^\Lambda$ and $\tilde{W}^\Lambda$ are chiral superfields and can be used to construct a matrix of chiral superfields

\[ M^{\Lambda \Sigma} \equiv W^\Lambda \tilde{W}^\Sigma. \]

The supersymmetric version of the Lagrangian (3.91) is then given by

\[ L = \text{Re} \int d^2\theta \left[ \text{Tr} \left( i(\mathcal{N} - \lambda)\chi - \frac{i}{2} \lambda \bar{D}^2 (\chi^\dagger \mathcal{N}_2 \chi) + i\lambda M \right) \right], \]

where $\mathcal{N}$, $\lambda$ and $\chi$ denote chiral superfields with the same symmetry properties as their corresponding bosonic fields. While the bosonic fields $\mathcal{N}$ and $\lambda$ appearing in (3.91) are the lowest component of the superfields denoted by the same letter, the field $\chi$ in the action (3.91) is the highest component of the superfield $\chi$. A supersymmetric kinetic
term for the scalar field $N$ can be written using the Kähler potential (3.67) as described in [72].

Just as in the bosonic Born-Infeld theory, one would like to eliminate the auxiliary fields. This is an open problem if $n \neq 1$. For $n = 1$ just as in the bosonic case the theory with auxiliary fields also admits both a real and a complex version, i.e. one can also consider a Lagrangian with a single real superfield. Then by integrating out the auxiliary superfields the supersymmetric version of the Born-Infeld lagrangian (3.104) is obtained

$$\mathcal{L} = \int d^4 \theta \frac{N_2^2 W^2 \bar{W}^2}{1 + A + \sqrt{1 + 2A + B^2}} + \text{Re} \left[ \int d^2 \theta \left( \frac{i}{2} N W^2 \right) \right], \quad (3.110)$$

where

$$A = \frac{1}{4} (D^2(N_2 W^2) + \bar{D}^2(N_2 \bar{W}^2)) , \quad B = \frac{1}{4} (D^2(N_2 W^2) - \bar{D}^2(N_2 \bar{W}^2)) .$$

If we only want a $U(1)$ duality invariance we can set $N = -i$ and then the lagrangian (3.110) reduces to the supersymmetric Born-Infeld lagrangian described in [46–48].

In the case of weak fields the first term of (3.110) can be neglected and the Lagrangian is quadratic in the field strengths. Under these conditions the combined requirements of supersymmetry and self duality can be used [73] to constrain the form of the weak coupling limit of the effective Lagrangian from string theory. Self-duality of Born-infeld theories with $N = 2$ supersymmetries is discussed in [24].

4 Dualities in $N > 2$ extended Supergravities

In this section we consider $N > 2$ supergravity theories in $D = 4$; in these theories the graviton is also coupled to gauge fields and scalars. We study the corresponding duality groups, that are subgroups of the symplectic group. It is via the geometry of these subgroups of the symplectic group that we can obtain the scalars kinetic terms, the supersymmetry transformation rules and the structure of the central and matter charges of the theory with their differential equations and their duality invariant combinations $\mathcal{J}_{BH}$ and $\mathcal{J}$ (that for extremal black holes are the effective potential and the entropy).

Four dimensional $N$-extended supergravities contain in the bosonic sector, besides the metric, a number $n$ of vectors and $m$ of (real) scalar fields. The relevant bosonic action is known to have the following general form:

$$S = \frac{1}{4} \int \sqrt{-g} d^4 x \left( -\frac{1}{2} R + \text{Im} N_{\Lambda \Gamma} F_{\mu \nu}^\Lambda F^{\Gamma \mu \nu} + \frac{1}{2 \sqrt{-g}} \text{Re} N_{\Lambda \Gamma} \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu}^\Lambda F_{\rho \sigma} + \right.$$

$$+ \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j \bigg), \quad (4.1)$$

37
where $g_{ij}(\phi)$ ($i,j,\cdots=1,\cdots,m$) is the scalar metric on the $\sigma$-model described by the scalar manifold $M_{\text{scalar}}$ of real dimension $m$ and the vectors kinetic matrix $\mathcal{N}_{\Lambda\Sigma}(\phi)$ is a complex, symmetric, $n \times n$ matrix depending on the scalar fields. The number of vectors and scalars, namely $n$ and $m$, and the geometric properties of the scalar manifold $M_{\text{scalar}}$ depend on the number $N$ of supersymmetries and are summarized in Table [1].

The duality group of these theories is in general not the maximal one $Sp(2n,\mathbb{R})$ because the requirement of supersymmetry constrains the number and the geometry of the scalar fields in the theory. In this section we study the case where the scalar fields manifold is a coset space $G/H$, and we see that the duality group in this case is $G$.

In Section 5 we then study the general $N = 2$ case where the target space is a special Kähler manifold $M$ and thus in general we do not have a coset space. There the $Sp(2n,\mathbb{R})$ transformations are needed in order to globally define the supergravity theory. We do not have a duality symmetry of the theory; $Sp(2n,\mathbb{R})$ is rather a gauge symmetry of the theory, in the sense that only $Sp(2n,\mathbb{R})$ invariant expressions are physical ones.

The case of duality rotations in $N = 1$ supergravity is considered in [9], [74], see also [25]. In this case there is no vector potential in the graviton multiplet hence no scalar central charge in the supersymmetry algebra. Duality symmetry is due to the number of matter vector multiplets in the theory, the coupling to eventual chiral multiplets must be via a kinetic matrix $\mathcal{N}$ holomorphic in the chiral fields. We see that the structure of duality rotations is similar to that of $N = 1$ rigid supersymmetry. For duality rotations in $N = 1$ and $N = 2$ rigid supersymmetry using superfields see the review [24].

### 4.1 Extended supergravities with target space $G/H$

In $N \geq 2$ supergravity theories where the scalars target space is a coset $G/H$, the scalar sector has a Lagrangian invariant under the global $G$ rotations. Since the scalars appear in supersymmetry multiplets the symmetry $G$ should be a symmetry of the whole theory. This is indeed the case and the symmetry on the vector potentials is duality symmetry.

Let's examine the gauge sector of the theory. We recall from Section 3.1 that we have an $Sp(2n,\mathbb{R})$ duality group if the vector $(\vec{F})$ transforms in the fundamental of $Sp(2n,\mathbb{R})$, and the gauge kinetic term $\mathcal{N}$ transforms via fractional transformations, if $(A^B) \in Sp(2n,\mathbb{R})$,

$$\mathcal{N} \rightarrow \mathcal{N'} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}. \quad (4.2)$$

Thus in order to have $G$ duality symmetry, $G$ needs to act on the vector $(\vec{F})$ via symplectic transformations, i.e. via matrices $(A^B)$ in the fundamental of $Sp(2n,\mathbb{R})$. This requires a homomorphism

$$S : G \rightarrow Sp(2n,\mathbb{R}). \quad (4.3)$$

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Different infinitesimal $G$ transformations should correspond to different infinitesimal symplectic rotations so that the induced map $\text{Lie}(G) \to \text{Lie}(Sp(2n, \mathbb{R}))$ is injective, and equivalently the homomorphism $S$ is a local embedding (in general $S$ it is not globally injective, the kernel of $S$ may contain some discrete subgroups of $G$).

Since $U(n)$ is the maximal compact subgroup of $Sp(2n, \mathbb{R})$ and since $H$ is compact, we have that the image of $H$ under this local embedding is in $U(n)$. It follows that we have a $G$-equivariant map

$$\mathcal{N} : G/H \to Sp(2n, \mathbb{R})/U(n) ,$$

(4.4)

explicitly, for all $g \in G$,

$$\mathcal{N}(g\phi) = (C + DN(\phi))(A + BN(\phi))^{-1} ,$$

(4.5)

where with $g\phi$ we denote the action of $G$ on $G/H$, while the action of $G$ on $Sp(2n, \mathbb{R})/U(n)$ is given by fractional transformations. Notice that we have identified $Sp(2n, \mathbb{R})/U(n)$ with the space of complex symmetric matrices $\mathcal{N}$ that have imaginary part $\text{Im}\mathcal{N} = -i(\mathcal{N} - \overline{\mathcal{N}})$ negative definite (see Appendix 7.2).

The $D = 4$ supergravity theories with $N > 2$ have all target space $G/H$, they are characterized by the number $n$ of total vectors, the number $N$ of supersymmetries, and the coset space $G/H$, see Table 1.

In general the isotropy group $H$ is the product

$$H = H_{\text{Aut}} \times H_{\text{matter}}$$

(4.6)

where $H_{\text{Aut}}$ is the automorphism group of the supersymmetry algebra, while $H_{\text{matter}}$ depends on the matter vector multiplets, that are not present in $N > 4$ supergravities.

In Section 3.5 we have described the geometry of the coset space $G/H$ in terms of coset representatives, local sections $L$ of the bundle $G \to G/H$. Under a left action of $G$ they transform as $gL(\phi) = L(\phi')h$, where the $g$ action on $\phi \in G/H$ gives the point $\phi' \in G/H$.

We now recall that duality symmetry is implemented by the symplectic embeddings (1.13) and (1.14) and conclude that the embeddings of the coset representatives $L$ in $Sp(2n, \mathbb{R})$ will play a central role. Recalling (3.62) these embeddings are determined by defining

$$L \to f(L) \quad \text{and} \quad L \to h(L) .$$

In Table 1 the group $S(U(p) \times U(q))$ is the group of block diagonal matrices $(P^0_0 \, 0 \; Q)$ with $P \in U(p)$, $Q \in U(q)$ and $\det P \det Q = 1$. There is a local isomorphism between $S(U(p) \times U(q))$ and the direct product group $U(1) \times SU(p) \times SU(q)$, in particular the corresponding Lie algebras coincide. Globally these groups are not the same, for example $S(U(5) \times U(1)) = U(5) = U(1) \times PSU(5) \neq U(1) \times SU(5)$.
Table 1: Scalar Manifolds of $N > 2$ Extended Supergravities

| $N$ | Duality group $G$ | isotropy $H$ | $M_{\text{scalar}}$ | $n$ | $m$ |
|-----|-------------------|--------------|---------------------|-----|-----|
| 3   | $SU(3, n')$       | $S(U(3) \times U(n'))$ | $SU(3, n')$/$S(U(3) \times U(n'))$ | $3 + n'$ | $6n'$ |
| 4   | $SU(1, 1) \times SO(6, n')$ | $U(1) \times S(O(6) \times O(n'))$ | $SU(1, 1)$/$U(1)$ $\times$ $SO(6, n')$/$S(O(6) \times O(n'))$ | $6 + n'$ | $6n' + 2$ |
| 5   | $SU(5, 1)$       | $S(U(5) \times U(1))$ | $SU(5, 1)$/$S(U(5) \times U(1))$ | 10  | 10  |
| 6   | $SO^*(12)$       | $U(6)$       | $SO^*(12)$/$U(6)$ | 16  | 30  |
| 7, 8| $E_7(7)$         | $SU(8)/Z_2$ | $E_7(7)$/$SU(8)/Z_2$ | 28  | 70  |

In the table, $n$ stands for the number of vectors and $m = \dim M_{\text{scalar}}$ for the number of real scalar fields. In all the cases the duality group $G$ is (locally) embedded in $Sp(2n, \mathbb{R})$. The number $n$ of vector potentials of the theory is given by $n = n_g + n'$ where $n'$ is the number of vectors potentials in the matter multiplet while $n_g$ is the number of graviphotons (i.e. of vector potentials that belong to the graviton multiplet). We recall that $n_g = \frac{N(N-1)}{2}$ if $N \neq 6$; and $n_g = \frac{N(N-1)}{2} + 1 = 16$ if $N = 6$; we also have $n' = 0$ if $N > 4$. The scalar manifold of the $N = 4$ case is usually written as $SO_o(6, n')/SO(6) \times SO(n')$ where $SO_o(6, n')$ is the component of $SO(6, n')$ connected to the identity. The duality group of the $N = 6$ theory is more precisely the double cover of $SO^*(12)$. Spinors fields transform according to $H$ or its double cover.

In the following we see that the matrices $f(L)$ and $h(L)$ determine the scalar kinetic term $\mathcal{N}$, the supersymmetry transformation rules and the structure of the central and matter charges of the theory. We also derive the differential equations that these charges satisfy and consider their positive definite and duality invariant quadratic expression $\mathcal{V}_{BH}$. These relations are similar to the Special Geometry ones of $N = 2$ supergravity.

From the equation of motion

\[dF^\Lambda = 4\pi j_m^\Lambda \quad (4.8)\]
\[dG^\Lambda = 4\pi j_e^\Lambda \quad (4.9)\]

we associate with a field strength 2-form $F$ a magnetic charge $p^\Lambda$ and an electric charge $q^\Lambda$ given respectively by:

\[p^\Lambda = \frac{1}{4\pi} \int_{S^2} F^\Lambda, \quad q^\Lambda = \frac{1}{4\pi} \int_{S^2} G^\Lambda \quad (4.10)\]

where $S^2$ is a spatial two-sphere containing these electric and magnetic charges. These are not the only charges of the theory, in particular we are interested in the central
charges of the supersymmetry algebra and other charges related to the vector multiplets. These latter charges result to be the electric and magnetic charges $p^\Lambda$ and $q^\Lambda$ dressed with the scalar fields of the theory. In particular these dressed charges are invariant under the duality group $G$ and transform under the isotropy subgroup $H = H_{Aut} \times H_{matter}$.

While the index $\Lambda$ is used for the fundamental representation of $Sp(2n; R)$ the index $M$ is used for that of $U(n)$. According to the local embedding

$$H = H_{Aut} \times H_{matter} \rightarrow U(n)$$

the index $M$ is further divided as $M = (AB, \bar{I})$ where $\bar{I}$ refers to $H_{matter}$ and $AB = -BA$ ($A = 1, \ldots, N$) labels the two-times antisymmetric representation of the $R$-symmetry group $H_{Aut}$. We can understand the appearence of this representation of $H_{Aut}$ because this is a typical representation acting on the central charges. The index $\bar{I}$ rather than $I$ is used because the image of $H_{matter}$ in $U(n)$ will be the complex conjugate of the fundamental of $H_{matter}$, this agrees with the property that under Kähler transformations of the $U(1)$ bundle $Sp(2n, \mathbb{R})/SU(n) \rightarrow Sp(2n, \mathbb{R})/U(n)$ the coset representatives of the scalar fields in the gravitational and matter multiplets transform with opposite Kähler weights. This is also what happens in the generic $N = 2$ case (cf. (5.61)).

The dressed graviphotons field strength 2-forms $T_{AB}$ may be identified from the supersymmetry transformation law of the gravitino field in the interacting theory, namely:

$$\delta \psi_A = \nabla \epsilon_A + \alpha T_{AB \mu \nu} \gamma^a \gamma^{\mu \nu} \epsilon^B V_a + \ldots$$

Here $\nabla$ is the covariant derivative in terms of the space-time spin connection and the composite connection of the automorphism group $H_{Aut}$, $\alpha$ is a coefficient fixed by supersymmetry, $V^a$ is the space-time vielbein. Here and in the following the dots denote trilinear fermion terms which are characteristic of any supersymmetric theory but do not play any role in the following discussion. The 2-form field strength $T_{AB}$ is constructed by dressing the bare field strengths $F^\Lambda$ with the image $f(L(\phi))$, $h(L(\phi))$ in $Sp(2n; R)$ of the coset representative $L(\phi)$ of $G/H$. Note that the same field strengths $T_{AB}$ which appear in the gravitino transformation law are also present in the dilatino transformation law in the following way:

$$\delta \chi_{ABC} = \mathcal{P}_{ABCD} \partial_\mu \phi^\ell \gamma^\mu \epsilon^D + \beta T_{[AB \mu \nu} \gamma^{\mu \nu} \epsilon_C]$$

Analogously, when vector multiplets are present, the matter vector field strengths $T_I$ appearing in the transformation laws of the gaugino fields, are linear combinations of the field strengths dressed with a different combination of the scalars:

$$\delta \lambda_{IA} = i \mathcal{P}_{IAB} \partial_\mu \phi^r \gamma^\mu \epsilon^B + \gamma T_{I \mu \nu} \gamma^{\mu \nu} \epsilon_A + \ldots$$
Here $P_{ABCD} = P_{ABCD\ell} d\phi^\ell$ and $P^I_{AB} = P^I_{AB\ell} d\phi^\ell$ are the vielbein of the scalar manifolds spanned by the scalar fields $\phi^\ell = (\phi^\ell, \phi^r)$ of the gravitational and vector multiplets respectively (more precise definitions are given below), and $\beta$ and $\gamma$ are constants fixed by supersymmetry.

According to the transformation of the coset representative $gL(\phi) = L(\phi')h$, under the action of $g \in G$ on $G/H$ we have

$$S(\phi) A \rightarrow S(\phi') A = S(g) S(\phi) S(h^{-1}) A = S(g) S(\phi) A U^{-1}$$

(4.15)

where $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ is unitary and symplectic (cf. (7.5)), $S(g) = (AB \atop CD)$ and $S(h)$ are the embeddings of $g$ and $h$ in the fundamental of $Sp(2n, \mathbb{R})$, while $U = A^{-1} S(h) A$ is the embedding of $h$ in the complex basis of $Sp(2n, \mathbb{R})$. Explicitly $U = (u^0 \atop {0 \bar{u}})$, where $u$ is in the fundamental of $U(n)$ (cf. (7.13) and (7.8)). Therefore the symplectic matrix $V = S A =$

$$\begin{pmatrix} f & \bar{f} \\ h & \bar{h} \end{pmatrix}$$

(4.16)

transforms according to

$$V(\phi) \rightarrow V(\phi') = S(g) V(\phi) \begin{pmatrix} u^{-1} & 0 \\ 0 & \bar{u}^{-1} \end{pmatrix} .$$

(4.17)

The dressed field strengths transform only under a unitary representation of $H$ and, in accordance with (4.17), are given by [11]

$$\begin{pmatrix} T \\ -\bar{T} \end{pmatrix} = -i \bar{V}(\phi)^{-1} \begin{pmatrix} F \\ G \end{pmatrix} ;$$

(4.18)

$$T \rightarrow \bar{u} T .$$

(4.19)

Explicitly, since

$$-i \bar{V}^{-1} = \begin{pmatrix} h^i \bar{f}^i \\ -h^i \bar{f}^i \end{pmatrix}$$

(4.20)

we have

$$T_{\Lambda\Lambda} = h_{\Lambda\Lambda} F^\Lambda - f^\Lambda_{\Lambda\Lambda} G^\Lambda$$

$$\bar{T}_I = \bar{h}_{\Lambda I} F^\Lambda - \bar{f}^\Lambda_{\Lambda I} G^\Lambda$$

(4.21)

where we used the notation $T = (T^M) = (T_M) = (T_{\Lambda\Lambda}, \bar{T}_I),$ $f = (f^\Lambda_M) = (f^\Lambda_{\Lambda\Lambda}, \bar{f}^\Lambda_I),$ $h = (h_{\Lambda M}) = (h_{\Lambda\Lambda}, \bar{h}_{\Lambda I})$.

(4.22)
that emphasizes that (for every value of \( \Lambda \)) the sections \((\tilde{f}_\Lambda, \tilde{h}_\Lambda)\) have Kähler weight opposite to the \((f^A_M, f^A_I)\) ones. This may be seen from the supersymmetry transformation rules of the supergravity fields, in virtue of the fact that gravitinos and fotinos with the same chirality have opposite Kähler weight. Notice that this notation (as in [41]) differs from the one in [11], where \((f^A_M) = (f^A_{AB}, f^A_I), (h^A) = (h^A_{AB}, h^A_I)\).

Consequently the central charges are

\[
Z_{AB} = -\frac{1}{4\pi} \int_{S^2_\infty} T_{AB} = f^A_{AB} q_\Lambda - h^A_{AB} p^\Lambda \tag{4.23}
\]

\[
\tilde{Z}_I = -\frac{1}{4\pi} \int_{S^2_\infty} \tilde{T}_I = \tilde{f}^A_I q_\Lambda - \tilde{h}^A_I p^\Lambda \tag{4.24}
\]

where the integral is considered at spatial infinity and, for spherically symmetric configurations, \(f\) and \(h\) in (4.23), (4.24) are \(f(\phi_\infty)\) and \(h(\phi_\infty)\) with \(\phi_\infty\) the constant value assumed by the scalar fields at spatial infinity.

The integral of the graviphotons \(T_{AB\mu\nu}\) gives the value of the central charges \(Z_{AB}\) of the supersymmetry algebra, while by integrating the matter field strengths \(T_I\) one obtains the so called matter charges \(Z_I\). The charges of these dressed field strength that appear in the supersymmetry transformations of the fermions have a profound meaning and play a key role in the physics of extremal black holes. In particular, recalling (4.17) the quadratic combination (black hole potential)

\[
\mathcal{V}_{BH} := \frac{1}{2} \tilde{Z}^{AB} Z_{AB} + \tilde{Z}^I Z_I \tag{4.25}
\]

(the factor 1/2 is due to our summation convention that treats the \(AB\) indices as independent) is invariant under the symmetry group \(G\). In terms of the charge vector

\[
Q = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}, \tag{4.26}
\]

we have the formula for the potential (also called charges sum rule)

\[
\mathcal{V}_{BH} = \frac{1}{2} \tilde{Z}^{AB} Z_{AB} + \tilde{Z}^I Z_I = -\frac{1}{2} Q^t \mathcal{M}(\mathcal{N}) Q \tag{4.27}
\]

where

\[
\mathcal{M}(\mathcal{N}) = -(i\mathring{V}^{-1})^t i\mathring{V}^{-1} = -(S^{-1})^t S^{-1} \tag{4.28}
\]

is a negative definite matrix, here depending on \(\phi_\infty\). In Appendix 7.2 we show that the set of matrices of the kind \(S S^t\) with \(S \in Sp(2n, \mathbb{R})\) are the coset space \(Sp(2n, \mathbb{R})/U(n)\),
hence the matrices $\mathcal{M}(\mathcal{N})$ parametrize $Sp(2n, \mathbb{R})/U(n)$. Also the matrices $\mathcal{N}$ parametrize $Sp(2n, \mathbb{R})/U(n)$. The relation between $\mathcal{M}(\mathcal{N})$ and $\mathcal{N}$ is
\begin{align*}
\mathcal{M}(\mathcal{N}) = \begin{pmatrix} 1 & -\text{Re} \mathcal{N} & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Im} \mathcal{N} & 0 & 0 \\
0 & \text{Im} \mathcal{N}^{-1} & 0 \\
-\text{Re} \mathcal{N} & 0 & 0 \end{pmatrix}.
\end{align*}
(4.29)
This and further properties of the $\mathcal{M}(\mathcal{N})$ matrix are derived in Appendix 7.2.

For each of the supergravities with target space $G/H$ there is another $G$ invariant expression $\mathcal{I}$ quadratic in the charges [63]; the invariant $\mathcal{I}$ is independent from the scalar fields of the theory and thus depends only on the electric and magnetic charges $p^\Lambda$ and $q_\Lambda$. In extremal black hole configurations $\pi \mathcal{I}$ is the entropy of the black hole. In the $N = 3$ supergravity theory $\mathcal{I}$ is the absolute value of a quadratic combination of the charges, while for $N \geq 4$ it is the square root of the absolute value of a quartic combination of the charges. The positive or negative value of this quadratic combination is related to the different BPS properties of the black hole. It turns out that $\mathcal{I}$ coincides with the potential $\mathcal{I}_{BH}$ computed at its critical point (attractor point) [43, 45, 63]. In the next section we give the explicit expressions of the invariants $\mathcal{I}$. They are obtained by considering among the $H$ invariant combination of the charges those that are also $G$ invariant, i.e. those that do not depend on the scalar fields. This is equivalent to require invariance of $\mathcal{I}$ under the coset space covariant derivative $\nabla$ defined in Section 3.5, see also (4.34).

We now derive some differential relations among the central and matter charges. We recall the symmetric coset space geometry $G/H$ studied in Section 3.5, and in particular relations (3.58), (3.59) that express the Maurer-Cartan equation $d\Gamma + \Gamma \wedge \Gamma = 0$ in terms of the vielbein $P$ and of the Riemannian connection $\omega$. Using the (local) embedding of $G$ in $Sp(2n, \mathbb{R})$ we consider the pull back on $G/H$ of the $Sp(2n, \mathbb{R})$ Lie algebra left invariant one form $V^{-1}dV$ given in (3.64), we have
\begin{align*}
V^{-1}dV &= \begin{pmatrix} i(f\dagger dh - h\dagger df) & i(f\dagger dh - h\dagger df) \\
-i(f\dagger dh - h\dagger df) & -i(f\dagger dh - h\dagger df) \end{pmatrix} = \begin{pmatrix} \omega \P \P \bar{\omega} \\
\P \P \bar{\omega} \end{pmatrix},
\end{align*}
(4.30)
where with slight abuse of notation we use the same letters $V$, $P$ and $\omega$ for the pulled back forms (we also recall that $P$ denotes $P$ in the complex basis). Relation (4.30) equivalently reads
\begin{align*}
dV &= V \begin{pmatrix} \omega \P \P \bar{\omega} \\
\P \P \bar{\omega} \end{pmatrix},
\end{align*}
(4.31)
that is equivalent to the $n \times n$ matrix equations:
\begin{align*}
\nabla f &= \bar{f} \P, \\
\nabla h &= \bar{h} \P,
\end{align*}
(4.32)
(4.33)
where
\[ \nabla f = df - f\omega, \quad \nabla h = dh - h\omega. \] (4.34)
Recalling that \( \mathcal{P} \) is symmetric (cf. (7.30)) we equivalently have \( \nabla f = \mathcal{P} \bar{f} \), \( \nabla h = \mathcal{P} \bar{h} \).
In these equations we can now see \( \omega \) and \( \mathcal{P} \) as our data (vielbein and Riemannian connection) on a manifold \( M \), while \( f \) and \( h \) are the unknowns. By construction these equations are automatically satisfied if \( M = G/H \) and \( G \) is a Lie subgroup of \( Sp(2n, \mathbb{R}) \).
More in general equations (4.32), (4.33) hold (with \( f \) and \( h \) invertible) iff the integrability condition, i.e. the Cartan-Maurer equation, \( d(\bar{\omega} \mathcal{P}) + \omega \mathcal{P} \wedge \bar{\omega} \mathcal{P} = 0 \) holds. With abuse of terminology we sometimes call (4.32), (4.33) the Maurer-Cartan equations.

The differential relations among the charges \( Z_{AB} \) and \( \bar{Z}_I \) follow after rewriting (4.32), (4.33) with \( AB \) and \( I \) indices. The embedded connection \( \omega \) and vielbein \( \mathcal{P} \) are decomposed as follows:
\[ \omega = (\omega_C^D) = \left( \begin{array}{ccc} \omega_{AB} & & 0 \\ & \omega_i^j & \\ 0 & & \omega_j^i \end{array} \right), \]
\[ \mathcal{P} = (\mathcal{P}_M^N) = \left( \begin{array}{ccc} \mathcal{P}^A_B C_D & \mathcal{P}^A_B j_i \\ \mathcal{P}^i_C D_j & \mathcal{P}^i_C j_i \end{array} \right), \]
(4.35)
the subblocks being related to the vielbein of \( G/H \), written in terms of the indices of \( H_{\text{Aut}} \times H_{\text{matter}} \). We used the following indices conventions:
\[ f = (f_M^A), \quad f^{-1} = (f_M^A) = (f_M^A) \quad \text{etc.} \] (4.37)
where in the last passage, since we are in \( U(n) \), we have lowered the index \( M \) with the \( U(n) \) hermitian form \( \eta = (\eta_{MN})_{M,N=1,...,n} = \text{diag}(1,1,...,1) \). Similar conventions hold for the \( AB \) and \( I \) indices, for example \( \bar{f}_I^A = \bar{f}^A_I = \bar{f}^A_I \).

Using further the index decomposition \( M = (AB, I) \), relations (4.32), (4.33) read (the factor 1/2 is due to our summation convention that treats the \( AB \) indices as independent):
\[ \nabla f^A_{AB} = \frac{1}{2} \bar{f}^{ACD} \mathcal{P}_{CDAB} + \bar{f}_A^C \mathcal{P}_{AB}^I, \] (4.38)
\[ \nabla h^A_{AB} = \frac{1}{2} \bar{h}^{ACD} \mathcal{P}_{CDAB} + \bar{h}_A^C \mathcal{P}_{AB}^I, \] (4.39)
\[ \nabla f_A^I = \frac{1}{2} \bar{f}^{ACD} \mathcal{P}_{CDI} + f_A^C \mathcal{P}_{JI}, \] (4.40)
\[ \nabla h_A^I = \frac{1}{2} \bar{h}^{ACD} \mathcal{P}_{CDI} + h_A^C \mathcal{P}_{JI}. \] (4.41)
As we will see, depending on the coset manifold, some of the sub-blocks of (4.36) can be actually zero. For \( N > 4 \) (no matter indices) we have that \( \mathcal{P} \) coincides with the vielbein

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$P_{ABCD}$ of the relevant $G/H$. Using the definition of the charges (21) we then get the differential relations among charges: $\nabla Z_M = \tilde{Z}_N P^N_M$, where $\nabla Z_M = \frac{\partial Z_M}{\partial \phi^i} - Z_N \omega^N_M$, with $\phi_r^i$ the value of the $i$-th coordinate of $\phi_r \in G/H$ and $\phi^\infty = \phi(r = \infty)$. Explicitly, using the $AB$ and $I$ indices,

\[
\nabla Z_{AB} = Z_I P^I_{AB} + \frac{1}{2} \tilde{Z}^{CD} P_{CDAB},
\]

(4.42)

\[
\nabla \tilde{Z}_I = \frac{1}{2} \tilde{Z}^{AB} P_{ABI} + Z^j P_{JI},
\]

(4.43)

The geometry underlying the differential equation (4.31) is that of a flat symplectic vector bundle of rank $2n$, a structure that appears also in the special Kähler manifolds of scalars of $N = 2$ supergravities. Indeed if we are able to find $2n$ linearly independent row vectors $V^\xi = (V^\xi_\zeta)_{\zeta = 1,..,2n}$ then the matrix $V$ in (4.31) is invertible and therefore the connection $(\nabla \frac{\partial}{\partial \phi})$ is flat. If these vectors are mutually symplectic then we have a symplectic frame, the transition functions are constant symplectic matrices, the connection is symplectic.

In the present case we naturally have a flat symplectic bundle,

\[
G \times_H \mathbb{R}^{2n} \rightarrow G/H;
\]

this bundle is the space of all equivalence classes $[g,v] = \{(gh,S(h)^{-1}v), g \in G,v \in \mathbb{R}^{2n}, h \in H\}$. The symplectic structure on $\mathbb{R}^{2n}$ immediately extends to a well defined symplectic structure on the fibers of the bundle. Using the local sections of $G/H$ and the usual basis $\{e_\xi\} = \{e_M, e^M\}$ of $\mathbb{R}^{2n} (e_1$ is the column vector with with 1 as first and only nonvanishing entry, etc.) we obtain immediately the local sections $s_\xi = [L(\phi), e_\xi]$ of $G \times_H \mathbb{R}^{2n} \rightarrow G/H$. Since the action of $H$ on $\mathbb{R}^{2n}$ extends to the action of $G$ on $\mathbb{R}^{2n}$, we can consider the new sections $e_\xi = s_\xi S^{-1}(L(\phi))^{\zeta}_{\xi} = [L(\phi), S^{-1}(L(\phi)) e_\xi]$, that are determined by the column vectors $S^{-1}(L(\phi))^{\zeta}_{\xi} = (S^{-1}(L(\phi))^{\zeta}_{\xi})_{\zeta = 1,..,2n}$. These sections are globally defined and linearly independent. Therefore this bundle is not only flat, it is trivial. If we use the complex local frame $\mathcal{V}_\xi = \{s_\xi A_{\zeta}^\xi\}$ rather than the $\{s_\xi\}$ one (we recall that $\mathcal{A} = \frac{1}{\sqrt{2}}(\frac{1}{i1} \frac{1}{1} i)\), then the global sections $e_\xi$ are determined by the column vectors $V^{-1}(L(\phi))^{\eta}_{\xi} = (V^{-1}(L(\phi))^{\eta}_{\xi})_{\eta = 1,..,2n}$,

\[
e_\xi = \mathcal{V}_n V^{-1\eta}_{\xi}.
\]

(4.44)

The sections $\mathcal{V}_\xi$ too form a symplectic frame (a symplectonormal basis, indeed $V_\xi^\rho \Omega_{\sigma \rho} V_\xi^\sigma = \Omega_{\zeta \xi}$, where $\Omega = (\frac{1}{1} \frac{-1}{1} i)$, and the last $n$ sections are the complex conjugate of the first $n$ ones, $\{\mathcal{V}_\xi\} = \{\mathcal{V}_M, \bar{\mathcal{V}}_M\}$. Of course the column vectors $V_\eta = (V_\xi^\rho)_{\xi = 1,..,2n}$, are the coefficients of the sections $\mathcal{V}_\eta$ with respect to the flat basis $\{e_\xi\}$.

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Also the rows of the $V$ matrix define global flat sections. Let’s consider the dual bundle of the vector bundle $G \times_H \mathbb{R}^{2n} \to G/H$, i.e. the bundle with fiber the dual vector space. If $\{s_\zeta\}$ is a frame of local sections of $G \times_H \mathbb{R}^{2n} \to G/H$, then $\{s^\xi\}$, with $\langle s^\xi, s_\zeta \rangle = \delta^\xi_\zeta$, is the dual frame of local sections of the dual bundle. Concerning the transition functions, if $s'_\zeta = s_\eta S_\zeta^\eta$ then $s'^\xi = S^{-1}{}^\lambda_\xi S^\lambda$. This dual bundle is also a trivial bundle and a trivialization is given by the global symplectic sections $e^\xi = V^\xi V^n$, whose coefficients are the row vectors $V^\xi = (V^\xi_{\zeta=1,...,2n})$ i.e., the rows of the symplectic matrix $V$ defined in (4.16),

\[
\begin{align*}
(V^A)_\zeta=1,...,2n &= (f^A_M, \bar{f}^A_M)_{M=1,...,n}, \\
(V^A_M)_{\zeta=1,...,2n} &= (h_{AM}, \bar{h}^A_M)_{M=1,...,n}.
\end{align*}
\] (4.45)

### 4.2 Specific cases

We now describe in more detail the supergravities of Table 1. The aim is to write down the group theoretical structure of each theory, their symplectic (local) embedding $S : G \to Sp(2n, \mathbb{R})$ and $N : G/H \to Sp(2n, \mathbb{R})/U(n)$, the vector kinetic matrix $N$, the supersymmetric transformation laws, the structure of the central and matter charges, their differential relations originating from the Maurer-Cartan equations (3.58), (3.59), and the invariants $\mathcal{F}_{BH}$ and $\mathcal{J}$. As far as the boson transformation rules are concerned we prefer to write down the supercovariant definition of the field strengths (denoted by a superscript hat), from which the supersymmetry transformation laws are retrieved. As it has been mentioned in previous section it is here that the symplectic sections $(f^A_{AB}, \bar{f}^A_B, \bar{f}^A_{AB}, \bar{f}^A_{\bar{A}B})$ appear as coefficients of the bilinear fermions in the supercovariant field strengths while the analogous symplectic section $(h_{AAB}, \bar{h}^A_{AB}, \bar{h}^A_{AAB}, h^A_{\bar{A}I})$ would appear in the dual magnetic theory. We include in the supercovariant field strengths also the supercovariant vielbein of the $G/H$ manifolds. Again this is equivalent to giving the susy transformation laws of the scalar fields. The dressed field strengths from which the central and matter charges are constructed appear instead in the susy transformation laws of the fermions for which we give the expression up to trilinear fermion terms. We stress that the numerical coefficients in the aforementioned susy transformations and supercovariant field strengths are fixed by supersymmetry (or, equivalently, by Bianchi identities in superspace), but we have not worked out the relevant computations being interested in the general structure rather that in the precise numerical expressions. These numerical factors could also be retrieved by comparing our formulae with those written in the standard literature on supergravity and performing the necessary redefinitions. The same kind of considerations apply to the central and matter charges whose precise normalization has not been fixed.
Throughout this section we denote by $A, B, \ldots$ indices of $SU(N)$, $SU(N) \times U(1)$, being $H_{aut}$ the automorphism group of the $N$–extended supersymmetry algebra. Lower and upper $SU(N)$ indices on the fermion fields are related to their left or right chirality respectively. If some fermion is a $SU(N)$ singlet, chirality is denoted by the usual (L) or (R) suffixes.

Furthermore for any boson field $v$ carrying $SU(N)$ indices we have that lower and upper indices are related by complex conjugation, namely: $(v_{AB \ldots}) = \bar{v}^{AB \ldots}$.

### 4.2.1 The $N = 4$ theory

The field content is given by the

- Gravitational multiplet (vierbein for the graviton, gravitino, graviphoton, dilatino, dilaton): \[
(V^a_\mu, \psi_A^\mu, A^{AB}_\mu, \chi_{ABC}^\mu, \kappa) \quad (A, B = 1, \cdots, 4)
\] (4.46)

frequently the upper half plane parametrization $S = \tilde{\kappa}$ is used for the axion-dilaton field.

- Vector multiplets:

\[
(A_\mu, \lambda^A, 6 \phi)^I \quad (I = 1, \cdots, n)
\] (4.47)

The coset space is the product

\[
G/H = \frac{SU(1, 1)}{U(1)} \times \frac{SO(6, n)}{S(O(6) \times O(n))}
\] (4.48)

We have to embed

\[
Sp(2, \mathbb{R}) \times SO(6, n) \to Sp(2(6 + n), \mathbb{R})
\] (4.49)

We first consider the embedding of $SO(6, n)$,

\[
S : SO(6, n) \to Sp(2(6 + n), \mathbb{R})
\]

\[
L \mapsto S(L) = \begin{pmatrix} L^{t^{-1}} & 0 \\ 0 & L \end{pmatrix}
\] (4.50)

we see that under this embedding $SO(6, n)$ is a symmetry of the action (not only of the equation of motions) that rotates electric fields into electric fields and magentic fields into magnetic fields. The natural embedding of $SU(1, 1) \simeq SL(2, \mathbb{R}) \simeq Sp(2, \mathbb{R})$ into $Sp(2(6 + n), \mathbb{R})$ is the $S$-duality that rotates each electric field in its corresponding magnetic field, we also want the image of $Sp(2, \mathbb{R})$ in $Sp(2(6 + n), \mathbb{R})$ to commute with that of $SO(6, n)$ (since we are looking for a symplectic embedding of all $Sp(2, \mathbb{R}) \times SO(6, n)$) and therefore we have

\[
S : Sp(2, \mathbb{R}) \to Sp(2(6 + n), \mathbb{R})
\]

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto S\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A\mathbb{I} & B\eta \\ C\eta & D\mathbb{I} \end{pmatrix}
\] (4.51)
where $\eta = \text{diag}(1,1,\ldots,-1,-1,\ldots)$ is the $SO(6,n)$ metric.

Concerning the coset representatives, on one hand we denote by $L(t)$ the representative in $SO(6,n)$ of the point $t \in SO(6,n)/S(O(6) \times O(n))$. On the other hand we have that $SU(1,1)/U(1) \simeq Sp(2,\mathbb{R})/U(1)$ is the lower half plane (see appendix) and is spanned by the complex number $\kappa$ with $\text{Im} \kappa < 0$, (frequently the upper half plane parametrization $S = \bar{\kappa}$ is used). A coset representative of $SU(1,1)/U(1)$ is

$$U(\kappa) = \frac{1}{n(\kappa)} \left( \begin{array}{cc} 1 & \frac{i \xi}{i + \kappa} \\ \frac{i + \kappa}{i \xi} & 1 \end{array} \right), \quad n(\kappa) = \sqrt{\frac{-4 \text{Im} \kappa}{1 + |\kappa|^2 - 2 \text{Im} \kappa}} \quad (4.52)$$

(In order to show that the $SU(1,1)$ matrix $U(\kappa)$ projects to $\kappa$ use \((7.13,19)\), that reads $\kappa = hf^{-1}$ with $h$ and $f$ complex numbers). The coset representative $U(\kappa)$ is defined for any $\kappa$ in the lower complex plane and therefore $U(\kappa)$ is a global section of the bundle $SU(1,1) \to SU(1,1)/U(1)$. (The projection $SU(1,1) \to SU(1,1)/U(1)$ can be also obtained by extracting $\kappa$ from $\mathcal{M}(\kappa) = \{(0,1)\}A U U^\dagger A^{-1}(0,1\rangle$, cf. \((7.26)\)).

With the given coset parametrizations the symplectic embedded section \((f^A, h_{\Lambda})\) is

$$f^A = (f_{AB}, f^A) = \frac{1}{n(\kappa)} \left( \frac{2}{1 + i \kappa} L_{\Lambda}^{-1} AB, \frac{2}{1 - i \kappa} L_{\Lambda}^{-1} AB \right)$$

$$h_{\Lambda} = (h_{\Lambda AB}, h_{\Lambda}) = \frac{1}{n(\kappa)} \left( \frac{2 \kappa}{i \kappa + 1} L_{\Lambda AB}, \frac{2 \kappa}{i \kappa - 1} L_{\Lambda AB} \right) \quad (4.53)$$

We now have all ingredients to compute the matrix $N$ in terms of $\kappa$ and $L$. The coset representative in $Sp(2(6+m),\mathbb{R})$ of $(\kappa,L)$ is $S(AU(\kappa)A^{-1})S(L)$, and recalling that $N = hf^{-1}$ and \((3.62)\), we obtain after elementary algebra the kinetic matrix

$$\mathcal{N} = \text{Re}N + i \text{Im}N = \text{Re} \kappa \eta + i \text{Im} \kappa LL^t. \quad (4.54)$$

The supercovariant field strengthes and the vielbein of the coset manifold are:

$$\hat{F}^A = dA^A + \left[ f^A_{AB}(c_1 \bar{\psi}^A \psi^B + c_2 \bar{\psi}^A \gamma_a \chi^{ABC} V^a) + f^A_{AB}(c_3 \bar{\psi}^A \gamma_a \chi_{ABC} V^a + c_4 \chi^{ABC} \gamma_a \chi_{AB} \epsilon_{ABCD} V^a V^b) + h.c. \right]$$

$$\hat{P}^{\Lambda}_{AB} = \mathcal{P}^{\Lambda}_{AB} - (\bar{\psi}_A \lambda_B^I + \epsilon_{ABCD} \bar{\psi}^C \lambda^D) \quad (4.55)$$

where $\mathcal{P} = \mathcal{P}_{\kappa} d_{\kappa}$ and $\mathcal{P}^{\Lambda}_{AB} = \mathcal{P}^{\Lambda}_{AB;\kappa} d\phi^i$ are the vielbein of \(\frac{SU(1,1)}{U(1)}\) and \(\frac{SO(6,n')}{S(O(6) \times O(n'))}\) respectively. The fermion transformation laws are:

$$\delta \psi_A = D \epsilon_A + a_1 T_{AB} \nu \gamma^a \gamma_{\mu} \epsilon B V^a + \cdots \quad (4.59)$$

$$\delta \chi_{ABC} = a_2 \mathcal{P}_{\kappa} \partial_{\mu} \gamma^a \epsilon_{ABCD} + a_3 \bar{T}_{AB} \nu \gamma_{\mu} \epsilon_{C} \cdots \quad (4.60)$$

$$\delta \lambda_{A} = a_4 \mathcal{P}^{\Lambda}_{AB;\kappa} \partial_{\mu} \epsilon B + a_5 T_{\mu} \nu \gamma_{\mu} \epsilon_{A} + \cdots \quad (4.61)$$
Table 2: Group assignments of the fields in $D = 4$, $N = 4$

|       | $V^\alpha_{\mu}$ | $\psi_{A\mu}$ | $A^A_i$ | $\chi_{ABC}$ | $A^I_{IA}$ | $U(\kappa)L^A_{AB}$ | $U(\kappa)L^I_{ij}$ | $R_H$ |
|-------|------------------|---------------|---------|--------------|------------|---------------------|---------------------|------|
| $SU(1,1)$ | 1                | 1             | -       | 1            | 1          | $2 \times 1$        | $2 \times 1$        | -    |
| $SO(6, n')$ | 1                | 1             | $6 + n'$| 1            | 1          | $1 \times (6 + n')$ | $1 \times (6 + n')$ | -    |
| $SO(6)$    | 1                | 4             | 1       | 4            | 1          | $1 \times 6$        | 1                   | 6    |
| $SO(n')$   | 1                | 1             | 1       | $n'$         | 1          | $n'$               | $n'$                |      |
| $U(1)$     | 0                | $\frac{1}{2}$ | 0       | $\frac{1}{2}$| $\frac{1}{2}$| 1                   | 1                   | 0    |

In this and in the following tables, $R_H$ is the representation under which the scalar fields of the linearized theory, or the vielbein $P$ of $G/H$ of the full theory transform (recall text after (3.55) and that $P$ is $P$ in the complex basis). Only the left–handed fermions are quoted, right handed fermions transform in the complex conjugate representation of $H$. Care must be taken in the transformation properties under the $H$ subgroups; indeed according to (4.17) the inverse right rep. of the one listed should really appear, i.e. since we are dealing with unitary rep., the complex conjugate

where the 2–forms $T_{AB}$ and $T_I$ are defined in eq.(4.21). By integration of these two–forms we find the central and matter dyonic charges given in equations (4.23), (4.24).

\begin{align}
\nabla^{SU(4) \times U(1)} Z_{AB} &= Z^I P_{IAB} + \frac{1}{2} \epsilon_{ABCD} Z^{CD} P \\
\nabla^{SO(n')} Z_I &= \frac{1}{2} Z^{AB} P_{IAB} + Z_I \bar{P}
\end{align}

where $\frac{1}{2} \epsilon_{ABCD} Z_{CD} = \bar{Z}_{AB}$. In terms of the kinetic matrix (4.33) the invariant $\mathcal{V}_{BH}$ for the charges is given by, cf. (4.27),

\begin{align}
\mathcal{V}_{BH} &= \frac{1}{2} Z_{AB} \bar{Z}^{AB} + Z_I \bar{Z}^I = -\frac{1}{2} Q^I \mathcal{M}(\mathcal{N}) Q .
\end{align}

The unique $SU(1,1) \times SO(6, n')$ invariant combination of the charges that is independent from the scalar fields is $I_1^2 - I_2 I_2$, so that

\begin{align}
\mathcal{J} &= \sqrt{|I_1^2 - I_2 I_2|} .
\end{align}

Here, $I_1$, $I_2$ and $I_2$ are the three $SO(6, n')$ invariants given by

\begin{align}
I_1 &= \frac{1}{2} Z_{AB} \bar{Z}^{AB} - Z_I \bar{Z}^I , \\
I_2 &= \frac{1}{4} \epsilon_{ABCD} Z_{AB} Z_{CD} - Z_I \bar{Z}^I .
\end{align}
4.2.2 The $N = 3$ theory

In the $N = 3$ case [64] the coset space is:

$$ G/H = \frac{SU(3, n')}{S(U(3) \times U(n'))} \quad (4.67) $$

and the field content is given by:

$$ (V^a_\mu, \psi_{A\mu}, A^{AB}_\mu, \chi(L)) \quad A = 1, 2, 3 \quad (gravitational \ multiplet) \quad (4.68) $$

$$ (A_\mu, \lambda_A, \lambda_{(R)}, 3_z)^I \quad I = 1, \ldots, n' \quad (vector \ multiplets) \quad (4.69) $$

The transformation properties of the fields are given in Table 3. We consider the (local)

Table 3: Transformation properties of fields in $D = 4$, $N = 3$

|          | $V^a_\mu$ | $\psi_{A\mu}$ | $A^{AB}_\mu$ | $\chi(L)$ | $\lambda_A^I$ | $\lambda_{(L)}^I$ | $L^{A}_{AB}$ | $L^{I}_{I}$ | $R_H$ |
|----------|-----------|----------------|--------------|-----------|---------------|-----------------|-------------|----------|-------|
| $SU(3, n')$ | 1         | 1             | $3 + n'$     | 1         | 1             | 1               | $3 + n'$    | $3 + n'$ | -     |
| $SU(3)$   | 1         | 3             | 1            | 1         | 3             | 1               | 3           | 1         | 3     |
| $SU(n')$  | 1         | 1             | 1            | $n'$      | 1             | $n'$            | 1           | $n'$     | $n'$  |
| $U(1)$    | 0         | $\frac{a}{2}$| 0            | $3 + \frac{n'}{2}$ | $3 + \frac{n'}{2}$ | $-3(1 + \frac{n'}{2})$ | $n'$       | $-3$     | $3 + n'$ |

embedding of $SU(3, n')$ in $Sp(3 + n', \mathbb{R})$ defined by the following dependence of the matrices $f$ and $h$ in terms of the $G/H$ coset representative $L$,

$$ f^\Lambda_\Sigma = \frac{1}{\sqrt{2}} (L^\Lambda_{AB}, \bar{L}^\Lambda_I) \quad (4.70) $$

$$ h_{\Lambda\Sigma} = -i (\eta f \eta)_{\Lambda\Sigma} \quad \eta = \left( \begin{array}{cc} \mathbb{I}_{3 \times 3} & 0 \\ 0 & -\mathbb{I}_{n' \times n'} \end{array} \right) \quad (4.71) $$

where $AB$ are antisymmetric $SU(3)$ indices, $I$ is an index of $SU(n')$ and $\bar{L}^\Lambda_I$ denotes the complex conjugate of the coset representative. We have:

$$ \mathcal{N}_{\Lambda\Sigma} = (hf^{-1})_{\Lambda\Sigma} = -i (\eta f \eta f^{-1})_{\Lambda\Sigma} \quad (4.72) $$

The supercovariant field strengths and the supercovariant scalar vielbein are:

$$ \hat{F}^\Lambda = dA^\Lambda + \left[ i f^\Lambda_I \chi^A_{\gamma} \psi^A V^\gamma - \frac{1}{2} f^\Lambda_{AB} \bar{\psi}^A \psi^B + i f^\Lambda_{AB} \bar{\chi}_{(R)} \gamma_\alpha \psi^C \epsilon^{ABC} V^\alpha + h.c. \right] $$

$$ \hat{P}_I^A = \mathcal{P}_I^A - \bar{L}^\Lambda_{B} \psi^C \epsilon^{ABC} - \bar{\chi}_{I(R)} \psi^A \quad (4.73) $$
where the only nonvanishing entries of the vierbein $\mathcal{P}$ are

$$\mathcal{P}^A_I = \frac{1}{2} \epsilon^{ABC} \mathcal{P}_{IBC} = \mathcal{P}^A_i dz^i$$  \hspace{1cm} (4.74)

$z^i$ being the (complex) coordinates of $G/H$. The chiral fermions transformation laws are given by:

$$\delta \psi_A = D \epsilon_A + 2i T_{AB\mu\nu} \gamma^a \gamma^\mu V_a \epsilon^B + \ldots$$  \hspace{1cm} (4.75)

$$\delta \chi_{(L)} = \frac{1}{2} T_{AB\mu\nu} \gamma^\mu \epsilon_C \epsilon^{ABC} + \ldots$$  \hspace{1cm} (4.76)

$$\delta \lambda_{IA} = -i \mathcal{P}^A_i \partial_\mu z^i \gamma^\mu \epsilon_A + \ldots$$  \hspace{1cm} (4.77)

$$\delta \lambda_{(L)} = i \mathcal{P}^A_i \partial_\mu z^i \gamma^\mu \epsilon_A + \ldots$$  \hspace{1cm} (4.78)

where $T_{AB}$ and $T_I$ have the general form given in equation (4.21). From the general form of the equations (4.32), (4.33) for $f$ and $h$ we find:

$$\nabla f^A_{AB} = f^I_{AB} \mathcal{P}^I_{AB}$$  \hspace{1cm} (4.79)

$$\nabla h^A_{AB} = h^I_{AB} \mathcal{P}^I_{AB}$$  \hspace{1cm} (4.80)

$$\nabla f^A_I = \frac{1}{2} f^{ACD} \mathcal{P}_{CDI}$$  \hspace{1cm} (4.81)

$$\nabla h^A_I = \frac{1}{2} h^{ACD} \mathcal{P}_{CDI}$$  \hspace{1cm} (4.82)

According to the general study of Section 4.1, using (4.23), (4.24) one finds

$$\nabla^{(H)} Z_{AB} = \bar{Z}^I \mathcal{P}^I_{AB} \epsilon_{ABC}$$  \hspace{1cm} (4.83)

$$\nabla^{(H)} Z_I = \frac{1}{2} \bar{Z}^{AB} \mathcal{P}^I_{AB} \epsilon_{ABC}$$  \hspace{1cm} (4.84)

and the formula for the potential, cf. (4.27),

$$\mathcal{V}_{BH} = \frac{1}{2} Z^{AB} \bar{Z}_{AB} + Z^I \bar{Z}_I = \frac{1}{2} Q^I \mathcal{M}(\mathcal{N}) Q$$  \hspace{1cm} (4.85)

where the matrix $\mathcal{M}(\mathcal{N})$ has the same form as in equation (4.29) in terms of the kinetic matrix $\mathcal{N}$ of equation (4.72), and $Q$ is the charge vector $Q = \frac{1}{2} \mathcal{M}(\mathcal{N}) Q$.

The $G = SU(3, n')$ invariant is $Z^A \bar{Z}_A - Z_I \bar{Z}^I$ (one can check that $\partial_i(Z^A \bar{Z}_A - Z_I \bar{Z}^I) = \nabla^{(H)}(Z^A \bar{Z}_A - Z_I \bar{Z}^I) = 0$) so that

$$\mathcal{S} = |Z^A \bar{Z}_A - Z_I \bar{Z}^I|.$$  \hspace{1cm} (4.86)
4.2.3 The $N=5$ theory

For $N > 4$ the only available supermultiplet is the gravitational one, so that $H_{\text{matter}} = 1$. The coset manifold of the scalars of the $N=5$ theory \cite{33} is:

$$G/H = \frac{SU(5, 1)}{U(5)}$$  \hspace{1cm} (4.87)

The field content and the group assignments are displayed in Table 4.

| $V^a$ | $\psi_A$ | $\chi_{ABC}, \chi_L$ | $A^{\Lambda \Sigma}$ | $L'_A$ | $R_H$ |
|------|-----------|------------------------|----------------------|--------|--------|
| $SU(5, 1)$ | 1 1 1 | - | 6 | - |
| $SU(5)$ | 1 5 | (10, 1) | 1 5 | 5 |
| $U(1)$ | 0 $\frac{1}{2}$ | $(\frac{3}{2}, -\frac{1}{2})$ | 0 1 2 |

In Table 4 the indices $x, y, \ldots = 1, \ldots, 6$ and $A, B, C, \ldots = 1, \ldots, 5$ are indices of the fundamental representations of $SU(5, 1)$ and $SU(5)$, respectively. $L'_A$ denotes as usual the coset representative in the fundamental representation of $SU(5, 1)$. The antisymmetric couple $\Lambda \Sigma, \Lambda, \Sigma = 1, \ldots, 5$, enumerates the ten vector potentials. The local embedding of $SU(5, 1)$ into the Gaillard-Zumino group $Usp(10, 10)$ is given in terms of the three-times antisymmetric representation of $SU(5, 1)$, this is a 20 dimensional complex representation, we denote by $t^{xyz}$ a generic element. This representation is reducible to a complex 10 dimensional one by imposing the self-duality condition

$$\bar{t}^{\bar{y} \bar{z}} = \frac{1}{3!} \epsilon^{\bar{y} \bar{z} \bar{w}} t^{uvw}$$ \hspace{1cm} (4.88)

here indices are raised with the $SU(5, 1)$ hermitian structure $\eta = \text{diag}(1, 1, 1, 1, 1, -1)$. The self duality condition (4.88) is compatible with the $SU(5, 1)$ action (on $\bar{t}^{\bar{y} \bar{z}}$ acts the complex conjugate of the three-times antisymmetric of $SU(5, 1)$). Due to the self-duality condition we can decompose $t^{xyz}$ as follows:

$$t^{xyz} = \begin{pmatrix} t^{\Lambda \Sigma \Xi} \\ \bar{t}^{\bar{\Lambda} \bar{\Sigma} \bar{\Xi}} \end{pmatrix}$$ \hspace{1cm} (4.89)

where $(\Lambda, \Sigma, \cdots = 1, \cdots, 5)$. In the following we set $t^{\Lambda \Sigma} \equiv t^{\Lambda \Sigma 6}$, $\bar{t}^{\bar{\Lambda} \bar{\Sigma}} \equiv \bar{t}^{\bar{\Lambda} \bar{\Sigma} \bar{6}}$, $\bar{t}_{\Lambda \Sigma} \equiv \bar{t}_{\Lambda \Sigma \Xi} = -\bar{t}_{\Lambda \Sigma}$. The symplectic structure in this complex basis is given by the matrix
\[
\begin{pmatrix}
0 & -\mathbb{I} \\
\mathbb{I} & 0
\end{pmatrix},
\]

\[
\langle t, \ell \rangle := \frac{1}{2} (t^{\Lambda \Sigma}, \overline{t}^{\bar{\Lambda} \bar{\Sigma}}) \begin{pmatrix}
0 & -\delta_{\Lambda \Sigma} \ell^{\bar{\Pi}} \\
\delta_{\bar{\Lambda} \bar{\Sigma}} \ell^{\Pi} & 0
\end{pmatrix} \begin{pmatrix}
\ell^{\Pi} \\
\ell^{\bar{\Pi}}
\end{pmatrix} \tag{4.90}
\]

\[
= \frac{1}{2} t^{\Lambda \Sigma} \ell_{\Lambda \Sigma} - \frac{1}{2} \overline{t}^{\bar{\Lambda} \bar{\Sigma}} \ell^{\bar{\Lambda} \bar{\Sigma}}
\]

\[
= \frac{1}{3!3!} t^{xyz} \varepsilon_{xystuv} \ell^{uvw} \tag{4.91}
\]

this last equality implies that the \(SU(5, 1)\) action preserves the symplectic structure. We have thus embedded \(SU(5, 1)\) into \(Sp(20, \mathbb{R})\) (in the complex basis).

The 20 dimensional real vector \((F^{\Lambda \Sigma}, G_{\Lambda \Sigma})\) transforms under the 20 of \(SU(5, 1)\), as well as, for fixed \(AB\), each of the 20 dimensional vectors \((f_{AB}^{\Lambda \Sigma})\) of the embedding matrix:

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix}
f + ih & \bar{f} + i\bar{h} \\
f - ih & \bar{f} - i\bar{h}
\end{pmatrix}. \tag{4.92}
\]

The supercovariant field strengths and vielbein are:

\[
\hat{F}^{\Lambda \Sigma} = dA^{\Lambda \Sigma} + [f^{\Lambda \Sigma}_{AB}(a_1 \bar{\psi}^A \psi^B + a_2 \bar{\psi}^C \gamma_a \chi^{ABC} V^a) + \text{h.c.}] \tag{4.93}
\]

\[
\hat{P}_{ABCD} = \mathcal{P}_{ABCD} - \bar{\chi}_{[ABCD]} - \epsilon_{ABCD\bar{E}} \tilde{X}^{(R)}_{\bar{E}} \psi^F \tag{4.94}
\]

where \(\mathcal{P}_{ABCD} = \epsilon_{ABCDF} \mathcal{P}^F\) is the complex vielbein, completely antisymmetric in \(SU(5)\) indices and \(\bar{\mathcal{P}}_{ABCD} = \bar{\mathcal{P}}_{ABCD}\).

The fermion transformation laws are:

\[
\delta \psi_A = D \varepsilon_A + a_3 T_{AB \mu \nu} \bar{\gamma}^a \gamma^\mu \gamma^\nu V_a + \ldots \tag{4.95}
\]

\[
\delta \chi_{ABC} = a_4 P_{ABCD} \bar{i} \partial \psi_{[ABC]} \bar{\gamma}^\mu \gamma^\nu \epsilon_B + \ldots \tag{4.96}
\]

\[
\delta \chi_{(L)} = a_6 P_{ABCD} \bar{i} \partial \psi_{ABCD} \bar{\gamma}^\mu \gamma^\nu \epsilon_{ABCDEF} + \ldots \tag{4.97}
\]

where:

\[
T_{AB} = \frac{1}{2} (h_{\Lambda \Sigma AB} F^{\Lambda \Sigma} - f_{AB}^{\Lambda \Sigma} G_{\Lambda \Sigma}) \tag{4.98}
\]

\[
\mathcal{N}_{\Lambda \Sigma \Delta \Pi} = \frac{1}{2} h_{\Lambda \Sigma AB} (f^{-1})^{AB \Delta \Pi}. \tag{4.99}
\]

With a by now familiar procedure one finds the following (complex) central charges:

\[
Z_{AB} = i V(\phi_\infty)^{-1} Q \tag{4.100}
\]

---

\(^9\)Strictly speaking we have immersed \(SU(5, 1)\) into \(Sp(20, \mathbb{R})\), in fact this map is a local embedding but fails to be injective, indeed the three \(SU(5, 1)\) elements \(\sqrt{7} \mathbb{I}\) are all mapped into the identity element of \(Sp(20, \mathbb{R})\).
where the charge vector is
\[
Q = \begin{pmatrix}
p^{\Lambda\Sigma} \\
q_{\Lambda\Sigma}
\end{pmatrix} = \left(\frac{1}{4\pi} \int_{S^2} F^{\Lambda\Sigma} \right) \left(\frac{1}{4\pi} \int_{S^2} G_{\Lambda\Sigma}\right)
\]
(4.101)
and \(\phi_{\infty}\) is the constant value assumed by the scalar fields at spatial infinity. From the equations (Maurer-Cartan equations)
\[
\nabla^{(U(5))} f_{AB}^{\Lambda\Sigma} = \frac{1}{2} \overline{f}_{CD}^{\Lambda\Sigma} P_{ABCD}
\]
(4.102)
and the analogous one for \(h\) we find:
\[
\nabla^{(U(5))} Z_{AB} = \frac{1}{2} \overline{Z}_{CD} P_{ABCD}.
\]
(4.103)
Finally, the formula for the potential is, cf. (4.27),
\[
\mathcal{V}_{BH} = \frac{1}{2} \overline{Z}_{AB} Z_{AB} = -\frac{1}{2} Q^i M(N) Q
\]
(4.104)
where the matrix \(M(N)\) has exactly the same form as in equation (4.29), and \(N\) is given in (4.99).

For \(SU(5,1)\) there are only two \(U(5)\) quartic invariants. In terms of the matrix \(A^B_A = Z_{AC} \overline{Z}^{CB}\) they are:
\[
\text{Tr} A = Z_{AB} \overline{Z}^{BA}, \quad \text{Tr}(A^2) = Z_{AB} \overline{Z}^{BC} Z_{CD} \overline{Z}^{DA}.
\]
(4.105)
The \(SU(5,1)\) invariant expression is
\[
\mathcal{J} = \frac{1}{2} \sqrt{4\text{Tr}(A^2) - (\text{Tr}A)^2}.
\]
(4.106)

4.2.4 The \(N = 6\) theory
The scalar manifold of the \(N = 6\) theory has the coset structure [65]:
\[
G/H = \frac{SO^*(12)}{U(6)}
\]
(4.107)
We recall that \(SO^*(2n)\) is the real form of \(O(2n,\mathbb{C})\) defined by the relation:
\[
L^\dagger C L = C, \quad C = \begin{pmatrix}
0 & -\mathbb{I} \\
\mathbb{I} & 0
\end{pmatrix}
\]
(4.108)
Table 5: Transformation properties of fields in $D = 4$, $N = 6$

|   | $V^a$ | $\psi_A$ | $\chi_{ABC}, \chi_A$ | $A^\Lambda$ | $S^\alpha_r$ | $R_H$ |
|---|-------|----------|-----------------------|-------------|-------------|------|
| $SO^\ast(12)$ | 1 | 1 | 1 | - | 32 | - |
| $SU(6)$ | 1 | 6 | $(20 + 6)$ | 1 | $(15, 1) + (15, 1)$ | 15 |
| $U(1)$ | 0 | $\frac{1}{2}$ | $(\frac{2}{7}, -\frac{2}{7})$ | 0 | $(1, -3) + (-1, 3)$ | 2 |

The field content and transformation properties are given in Table 5, where $A, B, C = 1, \ldots, 6$ are $SU(6)$ indices in the fundamental representation and $\Lambda = 1, \ldots, 16$. The 32 spinor representation of $SO^\ast(12)$ can be given in terms of a $Sp(32, \mathbb{R})$ matrix, which in the complex basis we denote by $S^\alpha_r (\alpha, r = 1, \ldots, 32)$. It is the double cover of $SO^\ast(12)$ that embeds in $Sp(32, \mathbb{R})$ and therefore the duality group is this spin group. Employing the usual notation we may set:

$$S^\alpha_r = \frac{1}{\sqrt{2}} \left( f^A_M + ih_{AM} \bar{f}^A_M + ih_{AM} \right) \tag{4.109}$$

where $\Lambda, M = 1, \ldots, 16$. With respect to $SU(6)$, the sixteen symplectic vectors $(f^A_M, h_{AM})$, $(M = 1, \ldots, 16)$ are reducible into the antisymmetric 15 dimensional representation plus a singlet of $SU(6)$:

$$(f^A_M, h_{AM}) \rightarrow (f^A_{AB}, h_{AAB}) + (\bar{f}^A, \bar{h}_{A}) \tag{4.110}$$

It is precisely the existence of a $SU(6)$ singlet which allows for the Special Geometry structure of $SO^\ast(12)$ (cf. (5.71), (5.72))\textsuperscript{10}. Note that the element $S^\alpha_r$ has no definite $U(1)$ weight since the submatrices $f^A_{AB}, \bar{f}^A$ have the weights 1 and $-3$ respectively. The vielbein matrix is

$$\mathcal{P} = \left( \begin{array}{ccc} \mathcal{P}_{ABCD} & \mathcal{P}_{AB} & 0 \\ \mathcal{P}_{CD} & 0 & 0 \end{array} \right) \tag{4.111}$$

where

$$\mathcal{P}_{AB} = \frac{1}{4!} \epsilon_{ABCDEF} \mathcal{P}_{CDEF}; \quad \bar{\mathcal{P}}^{AB} = \bar{\mathcal{P}}_{AB}. \tag{4.112}$$

The supercovariant field strengths and the coset manifold vielbein have the following expression:

$$\hat{F}^A = dA^A + \left[ f^A_{AB}(a_1 \bar{\psi}^A \psi^B + a_2 \bar{\psi}^C \gamma_a \chi^{ABC} V^a) + a_3 f^A \bar{\psi}^C \gamma_a \chi^{AC} V^a + h.c. \right] \tag{4.113}$$

$$\hat{\mathcal{P}}_{ABCD} = \mathcal{P}_{ABCD} - \chi_{[ABC} \psi_{D]} - \epsilon_{ABCDEF} \chi^{E} \psi^{F} \tag{4.114}$$

\textsuperscript{10}Due to its Special Geometry structure the coset space $SO^\ast(12)_{U(6)}$ is also the scalar manifold of an $N = 2$ supergravity. The two supergravity theories have the same bosonic fields however the fermion sector is different.
The fermion transformation laws are:

\[
\delta \psi_A = D \epsilon_A + b_1 T_{AB} \mu \gamma^a \gamma^{\mu} \epsilon^B V_a + \cdots \\
\delta \chi_{ABC} = 2 \mathcal{P}_{ABCD} \partial_z \gamma^a \gamma^D + b_3 T_{AB} \gamma^{ab} \epsilon^C + \cdots \\
\delta \chi_A = b_4 \mathcal{P}^{ABCDEF} \partial_z \gamma^a \gamma^f \epsilon_{ABCDEF} + b_5 T_{ab} \gamma^{ab} \epsilon_A + \cdots 
\]

where according to the general definition (4.21):

\[
T_{AB} = h_{AB} F^A - f_{AB} G_A \\
\bar{T} = \bar{h}_A F^A - \bar{f}_A G_A 
\]

With the usual procedure we have the following complex dyonic central charges:

\[
Z_{AB} = h_{AB} p^A - f_{AB} q_A \\
\bar{Z} = \bar{h}_A p^A - \bar{f}_A q_A
\]

in the \(15\) (recall (4.19)) and singlet representation of \(SU(6)\) respectively. Notice that although we have 16 graviphotons, only 15 central charges are present in the supersymmetry algebra. The singlet charge plays a role analogous to a “matter” charge (hence our notation \(\bar{Z}, \bar{f}_A, \bar{h}_A\)). The charges differential relations are

\[
\nabla^{(U(6))} Z_{AB} = \frac{1}{2} \bar{Z}^{CD} \mathcal{P}_{ABCD} + \frac{1}{4!} Z^{EF} \epsilon_{ABCDEF} \mathcal{P}_{CDEF} \\
\nabla^{(U(1))} \bar{Z} = \frac{1}{2!4!} \bar{Z}^{AB} \epsilon_{ABCDEF} \mathcal{P}_{CDEF}
\]

and the formula for the potential reads, cf. (4.27),

\[
\mathcal{V}_{BH} = \frac{1}{2} \bar{Z}^{AB} Z_{AB} + \bar{Z} Z = -\frac{1}{2} Q^t \mathcal{M} (N) Q .
\]

The quartic \(U(6)\) invariants are

\[
I_1 = (Tr A)^2 \\
I_2 = Tr (A^2) \\
I_3 = \frac{1}{243!} Re (\epsilon_{ABCDEF} Z_{AB} Z_{CD} Z_{EF} Z) \\
I_4 = (Tr A)\bar{Z} Z \\
I_5 = Z^2 \bar{Z}^2
\]

where \(A^B_A = Z_{AC} Z^{CB}\). The unique \(SO^*(12)\) invariant is

\[
\mathcal{I} = \frac{1}{2} \sqrt{|4I_2 - I_1 + 32I_3 + 4I_4 + 4I_5|} .
\]
4.2.5 The $N = 8$ theory

In the $N = 8$ case [5] the coset manifold is:

$$G/H = \frac{E_{7(7)}}{SU(8)/\mathbb{Z}_2}.$$  \hfill (4.130)

The field content and group assignments are given in Table 6.

|          | $V^a$ | $\psi_A$ | $A^{\Lambda\Sigma}$ | $\chi_{ABC}$ | $S^\alpha_r$ | $R_H$ |
|----------|-------|----------|----------------------|--------------|-------------|--------|
| $E_{7(7)}$ | 1     | 1        | -                    | 1            | 56          | -      |
| $SU(8)$  | 1     | 8        | 1                    | 56           | 28 + 28     | 70     |

The embedding in $Sp(56, \mathbb{R})$ is automatically realized because the 56 defining representation of $E_{7(7)}$ is a real symplectic representation. The components of the $f$ and $h$ matrices and their complex conjugates are

$$f^{\Lambda\Sigma}_{AB}, \quad h_{\Lambda\Sigma AB}, \quad \bar{f}^{\Lambda\Sigma}_{AB}, \quad \bar{h}^{\Lambda\Sigma AB},$$  \hfill (4.131)

here $\Lambda\Sigma, AB$ are couples of antisymmetric indices, with $\Lambda, \Sigma, A, B$ running from 1 to 8. The 70 under which the vielbein of $G/H$ transform is obtained from the four times antisymmetric of $SU(8)$ by imposing the self duality condition

$$\bar{t}^{\bar{A}\bar{B}\bar{C}\bar{D}} = \frac{1}{4!} \epsilon_{\bar{A}\bar{B}\bar{C}\bar{D}}^{A'B'C'D'} t^{A'B'C'D'}.$$  \hfill (4.132)

The supercovariant field strengths and coset manifold vielbein are:

$$\hat{F}^{\Lambda\Sigma} = dA^{\Lambda\Sigma} + [f^{\Lambda\Sigma}_{AB}(a_1 \bar{\psi}^A \psi^B + a_2 \bar{\chi}^{ABC} \gamma^C \psi^B V^a) + \text{h.c.}]$$ \hfill (4.133)

$$\hat{P}_{ABCD} = P_{ABCD} - \chi_{[ABC]} \psi^D + \text{h.c.}$$ \hfill (4.134)

where

$$P_{ABCD} = \frac{1}{4!} \epsilon_{ABCDEFGH} P^{EFGH} \equiv (L^{-1} \nabla^{SU(8)} L)_{ABCD} = P_{ABCD} d\phi^i (\phi^i \text{ coordinates of } G/H).$$ \hfill (4.132)

In the complex basis the vielbein $P_{ABCD}$ of $G/H$ are $28 \times 28$ matrices completely antisymmetric and self dual as in (4.132). The fermion transformation laws are given by:

$$\delta \psi_A = D\epsilon_A + a_3 T_{AB} \gamma^a \gamma^{\mu\nu} \epsilon^B V^\alpha + \cdots$$ \hfill (4.135)

$$\delta \chi_{ABC} = a_4 P_{ABCD} \partial_\alpha \phi^\gamma \gamma^a \epsilon^D + a_5 T_{[AB} \gamma^{\mu\nu} \epsilon^C] + \cdots$$ \hfill (4.136)

where:

$$T_{AB} = \frac{1}{2} (h_{\Lambda\Sigma AB} F^{\Lambda\Sigma} - f^{\Lambda\Sigma}_{AB} G_{\Lambda\Sigma}).$$ \hfill (4.137)
with:
\[ N_{\Lambda \Sigma \Gamma \Delta} = \frac{1}{2} h_{\Lambda \Sigma \Lambda B} (f^{-1})^{AB}_{\Gamma \Delta}. \] (4.138)

With the usual manipulations we obtain the central charges:
\[ Z_{AB} = \frac{1}{2} (h_{\Lambda \Sigma AB} p^{\Lambda \Sigma} - f^{\Lambda \Sigma}_{AB} q^{\Lambda \Sigma}), \] (4.139)
the differential relations:
\[ \nabla^{SU(8)} Z_{AB} = \frac{1}{2} Z^{CD} p_{ABCD}, \] (4.140)
and the formula for the potential, cf. (4.27),
\[ \mathcal{V}_{BH} = \frac{1}{2} \bar{Z}^{AB} Z_{AB} = -\frac{1}{2} Q^t \mathcal{M}(N) Q \] (4.141)
where the matrix \( \mathcal{M}(N) \) is given in equation (4.29), and \( \mathcal{N} \) in (4.138).

For \( N = 8 \) the \( SU(8) \) invariants are
\[ I_1 = (TrA)^2 \] (4.142)
\[ I_2 = Tr(A^2) \] (4.143)
\[ I_3 = PfZ = \frac{1}{2^4 4!} \varepsilon^{ABCD}_{EFGH} Z_{AB} Z_{CD} Z_{EF} Z_{GH} \] (4.144)
where \( PfZ \) denotes the Pfaffian of the antisymmetric matrix \( (Z_{AB})_{A,B=1,...,8} \), and where \( A_A^B = Z_{AC} \bar{Z}^{CB} \). One finds the following \( E_7(7) \) invariant \[44]:
\[ \mathcal{S} = \frac{1}{2} \sqrt{|4Tr(A^2) - (TrA)^2 + 32Re(PfZ)|} \] (4.145)

For a very recent study of \( E_7(7) \) duality rotations and of the corresponding conserved charges see [66].

### 4.2.6 Electric subgroups and the \( D = 4 \) and \( N = 8 \) theory.

A duality rotation is really a strong-weak duality if there is a rotation between electric and magnetic fields, more precisely if some of the rotated field strengths \( F'_{IA} \) depend on the initial dual fields \( G^\Sigma \), i.e. if the submatrix \( B \neq 0 \) in the symplectic matrix \( (A^B)_C^D \). Only in this case the gauge kinetic term may transform non-linearly, via a fractional transformation. On the other hand, under infinitesimal duality rotations \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \), with \( b = 0 \), the lagrangian changes by a total derivative so that (in the absence of instantons) these transformations are symmetries of the action, not just of the equation of motion. Furthermore if \( c = 0 \) the lagrangian itself is invariant.
We call electric any subgroup $G_e$ of the duality group $G$ with the property that it (locally) embeds in the symplectic group via matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $B = 0$. The parameter space of true strong-weak duality rotations is $G/G_e$.

The electric subgroup of $Sp(2n, \mathbb{R})$ is the subgroup of all matrices of the kind
\[
\begin{pmatrix} A & 0 \\ C & A^{-1} \end{pmatrix};
\]
we denote it by $Sp_e(2n, \mathbb{R})$. It is the electric subgroup because any other electric subgroup is included in $Sp_e(2n, \mathbb{R})$. This subgroup is maximal in $Sp(2n, \mathbb{R})$ (see for example the appendices in [50, 68]). In particular if an action is invariant under infinitesimal $Sp_e(2n, \mathbb{R})$ transformations, and if the equations of motion admit also a $\pi/2$ duality rotation symmetry $F^\Lambda \rightarrow G^\Lambda, G^\Lambda \rightarrow -F^\Lambda$ for one or more indices $\Lambda$ (no transformation on the other indices) then the theory has $Sp(2n, \mathbb{R})$ duality.

It is easy to generalize the results of Section 2.2 and prove that duality symmetry under these $\pi/2$ rotations is equivalent to the following invariance property of the lagrangian under the Legendre transformation associated to $F^\Lambda$,
\[
\mathcal{L}_D(F, N') = \mathcal{L}(F, N),
\]
where $N' = (C + DN)(A + BN)^{-1}$ are the transformed scalar fields, the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ implementing the $\pi/2$ rotation $F^\Lambda \rightarrow G^\Lambda, G^\Lambda \rightarrow -F^\Lambda$. We conclude that $Sp(2n, \mathbb{R})$ duality symmetry holds if there is $Sp_e(2n, \mathbb{R})$ symmetry and if the lagrangian satisfies (4.147).

When the duality group $G$ is not $Sp(2n, \mathbb{R})$ then there may exist different maximal electric subgroups of $G$, say $G_e$ and $G'_e$. Consider now a theory with $G$ duality symmetry, the electric subgroup $G_e$ hints at the existence of an action $S = \int \mathcal{L}$ invariant under the Lie algebra $\text{Lie}(G_e)$ and under Legendre transformation that are $\pi/2$ duality rotation in $G$. Similarly $G'_e$ leads to a different action $S' = \int \mathcal{L}'$ that is invariant under $\text{Lie}(G'_e)$ and under Legendre transformations that are $\pi/2$ duality rotation in $G$. The equations of motion of both actions have $G$ duality symmetry. They are equivalent if $\mathcal{L}$ and $\mathcal{L}'$ are related by a Legendre transformation. Since $\mathcal{L}'(F, N') \neq \mathcal{L}(F, N)$, this Legendre transformation cannot be a duality symmetry, it is a $\pi/2$ rotation $F^\Lambda \rightarrow G^\Lambda, G^\Lambda \rightarrow -F^\Lambda$ that is not in $G$, this is possible since $G \neq Sp(2n, \mathbb{R})$.

As an example consider the $G_e = SL(8, \mathbb{R})$ symmetry of the $N = 8, D = 4$ supergravity lagrangian whose duality group is $G = E_{7(7)}$ this is the formulation of Cremmer-Julia. An alternative formulation, obtained from dimensional reduction of the $D = 5$ supergravity, exhibits an electric group $G'_e = [E_{6(6)} \times SO(1, 1)] \ltimes T_{27}$ where the nonsemisimple group $G'_e$ is realized as a lower triangular subgroup of $E_{7(7)}$ in its fundamental (symplectic) 56 dimensional representation. $G_e$ and $G'_e$ are both maximal subgroups of $E_{7(7)}$. 

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The corresponding lagrangians can be related only after a proper duality rotation of electric and magnetic fields which involves a suitable Legendre transformation.

A way to construct new supergravity theories is to promote a compact rigid electric subgroup symmetry to a local symmetry, thus constructing gauged supergravity models (see for a recent review [67], and references therein). Inequivalent choices of electric subgroups give different gauged supergravities. Consider again $D = 4$, $N = 8$ supergravity.

The maximal compact subgroups of $G_e = SL(8, \mathbb{R})$ and of $G'_e = [E_{6, (6)} \times SO(1, 1)] \rtimes T_{27}$ are $SO(8)$ and $Sp(8) = U(16) \cap Sp(16, \mathbb{C})$ respectively. The gauging of $SO(8)$ corresponds to the gauged $N = 8$ supergravity of De Witt and Nicolai [33]. As shown in [34] the gauging of the nonsemisimple group $U(1) \rtimes T_{27} \subset G'_e$ corresponds to the gauging of a flat group in the sense of Scherk and Schwarz dimensional reduction [35], and gives the massive deformation of the $N = 8$ supergravity as obtained by Cremmer, Scherk and Schwarz [36].

5 Special Geometry and $N = 2$ Supergravity

In the case of $N = 2$ supergravity the requirements imposed by supersymmetry on the scalar manifold $M_{\text{scalar}}$ of the theory dictate that it should be the following direct product: $M_{\text{scalar}} = M \times M^Q$ where $M$ is a special Kähler manifold of complex dimension $n$ and $M^Q$ a quaternionic manifold of real dimension $4n_H$, here $n$ and $n_H$ are respectively the number of vector multiplets and hypermultiplets contained in the theory. The direct product structure imposed by supersymmetry precisely reflects the fact that the quaternionic and special Kähler scalars belong to different supermultiplets. We do not discuss the hypermultiplets any further and refer to [77] for the full structure of $N=2$ supergravity. Since we are concerned with duality rotations we here concentrate our attention to an $N = 2$ supergravity where the graviton multiplet, containing besides the graviton $g_{\mu \nu}$ also a graviphoton $A^0_{\mu}$, is coupled to $n'$ vector multiplets. Such a theory has a bosonic action of type (4.1) where the number of (real) gauge fields is $n = 1 + n'$ and the number of (real) scalar fields is $2n'$. Compatibility of their couplings with local $N = 2$ supersymmetry lead to the formulation of special Kähler geometry [75], [76].

The formalism we have developed so far for the $D = 4$, $N > 2$ theories is completely determined by the (local) embedding of the coset representative of the scalar manifold $M = G/H$ in $Sp(2n, \mathbb{R})$. It leads to a flat -actually a trivial- symplectic bundle with local symplectic sections $\mathcal{V}_n$, determined by the symplectic matrix $V$, or equivalently by the matrices $f$ and $h$. We want now to show that these matrices, the differential relations among charges and their quadratic invariant $y_{BH}$ (4.27) are also central for the description of $N = 2$ matter-coupled supergravity. This follows essentially from the fact that, though the scalar manifold $M$ of the $N = 2$ theory is not in general a
coset manifold, nevertheless, as for the $N > 2$ theories, we have a flat symplectic bundle
associated to $M$, with symplectic sections $\mathcal{V}_\eta$. While the formalism is very similar there
is a difference, the bundle is not a trivial bundle anymore, and it is in virtue of duality
rotations that the theory can be globally defined on $M$.

In the next section we study the geometry of the scalar manifold $M$ and in detail its
associated flat symplectic bundle. Then in Section 5.2 we see how, in analogy with $N > 2$
supergravities, the flat symplectic bundle geometry of $M$ enters the supersymmetry
transformations laws of $N = 2$ supergravity and the differential relations among the
matter and central charges.

5.1 Special Geometry

There are two kinds of special geometries: rigid and local. While rigid special Kähler
manifolds are the target space of the scalar fields present in the vector multiplets of $N = 2$
Yang Mills theories, the (local) special Kähler manifolds, in the mathematical literature
called projective special Kähler manifolds, describe the target space of the scalar fields
in the vector multiplets of $N = 2$ supergravity (that has local supersymmetry). In
order to describe the structure of a (local or projective) special Kähler manifold it is
instructive to recall that of rigid Kähler manifold.

5.1.1 Rigid Special Geometry

In short a rigid special Kähler manifold is a Kähler manifold $M$ that has a flat connection
on its tangent bundle. This connection must then be compatible with the symplectic
and complex structure of $M$.

More precisely, following [49], see also [50], a rigid special Kähler structure on a
Kähler manifold $M$ with Kähler form $K$ is a connection $\nabla$ that is real, flat, torsionfree,
compatible with the symplectic structure $\omega$:

$$\nabla \omega = 0$$

and compatible with the almost complex structure $J$ of $M$:

$$d_{\nabla} J = 0$$

where $d_{\nabla} : \Omega^1(TM) \to \Omega^2(TM)$ is the covariant exterior derivative on vector-valued
forms. Explicitly, if $J = J^\xi_\zeta \partial_\zeta$ where $J^\xi_\zeta$ are 1-forms, and $\nabla \partial_\zeta = A^\xi_\zeta \partial_\zeta$, with $A^\xi_\zeta$ 1-
forms, then $d_{\nabla} J = dJ^\xi_\zeta \partial_\zeta - J^\xi_\zeta \wedge A^\xi_\zeta \partial_\zeta = (dJ^\xi + A^\xi_\zeta \wedge J^\xi) \partial_\zeta$. Notice that the torsionfree
condition can be similarly written $d_{\nabla} I = 0$, where $I$ is the identity map in $TM$, locally
$I = dx^\xi \otimes \partial_\xi$. The two conditions $d_{\nabla} J = 0, d_{\nabla} I = 0$ for the real connection $\nabla$ can be
written in the complexified tangent bundle simply as

$$d_{\nabla} \pi^{1,0} = 0$$

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where \( \pi^{1,0} \) is the projection onto the \((1, 0)\) part of the complexified tangent bundle; locally \( \pi^{1,0} = dz^i \otimes \overline{\partial} \).

The flatness condition is equivalent to require the existence of a covering of \( M \) with local frames \( \{ e_\xi \} \) that are covariantly constant, \( \nabla e_\xi = 0 \). The corresponding transition functions of the real tangent bundle \( TM \) are therefore constant invertible matrices; compatibility with the symplectic structure, equation (5.1), further implies that these matrices belong to the fundamental of \( Sp(2n, \mathbb{R}) \), where \( 2n \) is the real dimension of \( M \) (each frame \( \{ e_\xi \} \) can be chosen to have mutually symplectic vectors \( e_\xi \)).

Flatness of \( \nabla \) (i.e., the vanishing of the curvature \( R_\nabla \) or equivalently \( d^2 \nabla = 0 \)) implies that (5.3) is equivalent to the existence of a local complex vector field \( \xi \) that satisfies

\[
\nabla \xi = \pi^{1,0} \tag{5.4}
\]

[hint: in a flat reference frame \( d \nabla = d \), and Poincaré lemma for \( d \) implies that any \( d \)-closed section is also \( d \)-exact]. Studying the components of this vector field (with respect to a flat Darboux coordinate system) we obtain the existence of local holomorphic coordinates on \( M \), called special coordinates, their transition functions are constant \( Sp(2n, \mathbb{R}) \) matrices, so that the holomorphic tangent bundle \( TM \) is a flat symplectic holomorphic one. Corresponding to these special coordinates we have a holomorphic function \( \tilde{F} \), the holomorphic prepotential. In terms of this data the Kähler potential and the Kähler form read

\[
K = \frac{1}{2} \text{Im}(\frac{\partial \tilde{F}}{\partial z^i} dz^i \wedge d\overline{z}^j), \tag{5.5}
\]

\[
K = i \partial \bar{\partial} K = \frac{i}{2} \text{Im}(\frac{\partial^2 \tilde{F}}{\partial z^i \partial \overline{z}^j}) dz^i \wedge d\overline{z}^j = \frac{i}{2} \text{Im}(\tau_{ij}) dz^i \wedge d\overline{z}^j, \tag{5.6}
\]

where \( z^i \) are special coordinates, and \( \tau_{ij} = \frac{\partial^2 \tilde{F}}{\partial z^i \partial \overline{z}^j} \).

An equivalent way of characterizing rigid special Kähler manifolds is via a holomorphic symmetric 3-tensor \( C \). This tensor measures the difference between the symplectic connection \( \nabla \) and the Levi-Civita connection \( D \), whose connection coefficients we here denote \( \gamma_{ij}^k \) and \( \bar{\gamma}_{\bar{i}\bar{j}}^k \).

Define

\[
\mathcal{P}_\mathbb{R} = \nabla - D.
\]

The nonvanishing components of \( \mathcal{P}_\mathbb{R} \) are

\[
A_{ij}^k - \gamma_{ij}^k, \quad \bar{A}_{i\bar{j}}^k, \quad A_{i\bar{j}}^k - \bar{\gamma}_{\bar{i}\bar{j}}^k, \quad A_{i\bar{j}}^k, \tag{5.7}
\]

this is so because the components \( A \) of the connection \( \nabla \) are constrained by condition (5.3). Since \( D \) and \( \nabla \) are real and torsionfree we further have that the lower indices in (5.7) are symmetric, and the reality conditions \( A_{ij}^k - \gamma_{ij}^k = A_{i\bar{j}}^k - \bar{\gamma}_{\bar{i}\bar{j}}^k, \quad \bar{A}_{i\bar{j}}^k = A_{i\bar{j}}^k \). Since

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both $D$ and $\nabla$ are symplectic we have that for any vector $u \in T_mM$, $(\mathcal{P}_\mathbb{R})_u : T_mM \to T_mM$ is a generator of a symplectic transformation,

$$
\begin{align*}
    u(K(v, w)) &= D_u(K(v, w)) = K(D_uv, w) + K(v, D_ww) \\
    u(K(v, w)) &= \nabla_u(K(v, w)) = K(\nabla_uv, w) + K(v, \nabla_ww) \\
    0 &= K((\mathcal{P}_\mathbb{R})_uv, w) + K(v, (\mathcal{P}_\mathbb{R})_uw).
\end{align*}
$$

If we set $u = \partial_k$, $v = \partial_i$, $w = \bar{\partial}_j$, and use that $K$ is a $(1,1)$-form, we obtain

$$
A^k_{ij} - \gamma^k_{ij} = 0.
$$

Then the components of

$$
\mathcal{P}_\mathbb{R} = \mathcal{P} + \overline{\mathcal{P}}
$$

are just $A^k_{ij}$ and $A^k_{ij}$. This leads to define the tensor

$$
C_{ijk} = -ig^{i\ell}A^\ell_{jk}.
$$

Setting $u = \partial_k$, $v = \partial_i$, $w = \partial_j$ in (5.8) we obtain that $C_{ijk}$ is totally symmetric in its indices. Since $D_j\pi^{(1,0)} = 0$ we easily compute, recalling (5.4),

$$
C_{ijk} = -\langle \nabla_i \xi, \nabla_j \nabla_k \xi \rangle,
$$

where $\rho = \mathcal{P}_\mathbb{R}$. In local coordinates we have

$$
\begin{align*}
    d\mathcal{P} + d\mathcal{P} = 0 \\
    d\mathcal{P} + d\mathcal{P} = 0
\end{align*}
$$

where $R = d^2\rho$ is the Levi-Civita curvature and $d\mathcal{P}$ is the exterior covariant derivative action on the 1-form $\mathcal{P}$ with values in $T_\mathbb{C}M \otimes T_\mathbb{C}^*M$ (where $T_\mathbb{C}^*M$ is the complexified cotangent bundle). Now in (5.13), the term $R + d\mathcal{P} + d\mathcal{P} \in \Omega^{(1,1)}(M, End(T_\mathbb{C}M, T_\mathbb{C}M))$, i.e., this term maps $T^{(1,0)}M$ into $(1,1)$-forms valued in $T^{(1,0)}M$ (or $T^{(0,1)}M$). On the other hand $\mathcal{P} \in \Omega(End(T_\mathbb{C}M, T_\mathbb{C}^*M))$, in particular it maps $T^{(1,0)}M$ vectors into forms valued in $T^{(0,1)}M$, and annihilates $T^{(0,1)}M$ vectors (hence $\mathcal{P} \wedge \mathcal{P} = 0$). Similar properties hold for the complex conjugate $\overline{\mathcal{P}}$, with $T^{(1,0)}M$ replaced by $T^{(0,1)}M$, and for $d_\mathcal{P}$ and $d_\mathcal{P}$. It follows that equation (5.13) is equivalent to two independent equations,

$$
\begin{align*}
    R + \mathcal{P} + \overline{\mathcal{P}} + \mathcal{P} \wedge \overline{\mathcal{P}} = 0 \\
    d_\mathcal{P} = 0.
\end{align*}
$$

Since the covariant derivative of the metric vanishes, this last equation is equivalent to $d_\mathcal{P} = 0$. In local coordinates we have

$$
\begin{align*}
    dC_{ij} - \gamma^k_{i\ell} \wedge C_{k\ell} - \gamma^k_{j\ell} \wedge C_{k\ell} = 0.
\end{align*}
$$
where $C_{ij} = C_{ik}dz^k$. This equation splits in the condition
\[
\bar{\partial}C = 0 ,
\]  
so that $C$ is holomorphic, and the condition $\partial_D C = 0$, that can be equivalently written
\[
D_i C_j = D_j C_i
\]
where $C_i$ is the matrix $C_i = (C_{ki\ell})_{k,\ell=1,...,n}$, i.e., $C_i \in \Omega^0(M, T^{*,1,0}) \otimes T^{*,1,0}M$, so that $D_i$ is the covariant derivative on functions valued in $T^{*,1,0}M \otimes T^{*,1,0}M$.

The local coordinates expression of (5.13) is
\[
R_{ijk\ell} = -\overline{C}_{i\bar{k}\bar{s}} g^{\bar{s}p} C_{pj\ell}.
\]

In conclusion a rigid special Kähler structure on $M$ implies the existence of a holomorphic symmetric 3-tensor (cubic form) $C$ that satisfies (5.13) and (5.17).

Viceversa if a Kähler manifold $M$ admits a symmetric holomorphic 3-tensor $C$ that satisfies (5.13) and (5.17), then $M$ is a special Kähler manifold. Indeed the contraction of $C$ with the metric gives $P$, so that we can define $\nabla = D - P_R$. The symmetry of $C$ implies that $d\nabla \pi_1,0 = 0$ so that $\nabla$ is torsionfree and compatible with the complex structure, $d\nabla J = 0$. The symmetry of $C$ also implies (5.8) so that $\nabla$ is symplectic. Finally (5.13) and (5.17) imply that $\nabla$ is flat.

In special coordinates the holomorphic 3-tensor $C$ is simply given by $C_{ijk} = \frac{1}{4!} \frac{\partial^3}{\partial z^i \partial z^j \partial z^k}$.

5.1.2 Local Special Geometry

We have recalled that to a rigid special Kähler manifold of dimension $n$ there is canonically associated a holomorphic $n$ dimensional flat symplectic vector bundle. On the other hand, to a projective (or local) special Kähler manifold $M$, of dimension $n'$ there is canonically associated a holomorphic $n = n' + 1$ dimensional flat symplectic vector bundle. The increase by one unit of the rank of the vector bundle with respect to the dimension of the manifold is due to the graviton multiplet. The mathematical description involves the $n = n' + 1$ dimensional manifold $L$, total space of a line bundle over $M$.

**Kähler-Hodge manifolds and their associated principal bundles $\tilde{M} \rightarrow M$**

Consider a Kähler-Hodge manifold, i.e. a triple $(M, L, K)$, where $M$ is Kähler with integral Kähler form $K$, so that it defines a class $[K] \in H^2(M, \mathbb{Z})$, and
\[
L \xrightarrow{\pi} M
\]
is a holomorphic hermitian line bundle with first Chern class equal to $[K]$, and with curvature equal to $-2\pi i K$ (recall that on a hermitian holomorphic vector bundle there is a unique connection compatible with the hermitian holomorphic structure).
Consider the complex manifold $\tilde{M}$, that is $L$ without the zero section of $L \xrightarrow{\pi} M$. The manifold $\tilde{M}$ is a principal bundle over $M$, with structure group $\mathbb{C}^\times$ (complex numbers minus the zero); the action of $\mathbb{C}^\times$ on $\tilde{M}$ is holomorphic. The hermitian connection canonically associated to $L \to M$ induces a connection on $\tilde{M}$ so that in $T\tilde{M}$ we have the subspaces of horizontal and vertical tangent vectors.

Another property of the manifold $\tilde{M}$ is that it has a canonical hermitian line bundle $\pi^*L \to \tilde{M}$; it is the pullback to $\tilde{M}$ of $L \to M$, so that the fiber on the point $\tilde{m} \in \tilde{M}$ is just the fiber of $L$ on the point $m = \pi(\tilde{m}) \in M$,

\[
\begin{array}{ccc}
\pi^*L & \longrightarrow & L \\
\downarrow & & \downarrow \\
\tilde{M} & \xrightarrow{\pi} & M
\end{array}
\] (5.19)

Explicitly $\pi^*L = \{ (\tilde{m}, \ell) ; \pi(\ell) = \pi(\tilde{m}) \}$. The line bundle $\pi^*L$ is trivial indeed we have the globally defined nonzero holomorphic section

\[
\begin{align*}
\Omega : & \tilde{M} \to \pi^*L \\
\tilde{m} & \mapsto (\tilde{m}, \tilde{m}) \\
(m, \lambda) & \mapsto (m, \lambda, \lambda)
\end{align*}
\] (5.20)

In the last line we used a local trivialization of $\tilde{M} \to M$ (and henceforth of $L \to M$) given by a local section $s$, say $\tilde{m} = \lambda s(m) \sim (m, \lambda)$. This induces a local trivialization $\tilde{s} = \pi^*s$ of the line bundle $\pi^*L \to \tilde{M}$. Explicitly $\tilde{s}$ associates to $\tilde{m}$ the point $s(m)$ of $L$, so that a generic element $\tilde{\ell} = \sigma \tilde{s}(\tilde{m}) \in \tilde{L}$ is described by the triple $(m, \lambda, \sigma)$, and in particular

\[
\Omega(\tilde{m}) = \Omega(\lambda s(m)) = \lambda \tilde{s}(\tilde{m}) \sim (m, \lambda, \lambda)
\] (5.21)

It can be shown that $\tilde{M}$ is a pseudo-Kähler manifold (i.e. a Kähler manifold where the metric has pseudo-Riemannian signature). The Kähler form is

\[
\tilde{K} = \frac{i}{2\pi} \partial \bar{\partial} |\Omega|^2,
\] (5.22)

where $|\Omega|^2$ is the evaluation on $\Omega$ of the hermitian structure of $\pi^*(L)$ (this latter is trivially inherited from the hermitian structure of $L$). With respect to the corresponding Kähler metric, horizontal and vertical vectors are orthogonal, moreover the Kähler metric is negative definite along vertical vectors, and positive definite along horizontal vectors, where $\tilde{K}|_{\text{hor}} = |\Omega|^2 \pi^* K$\textsuperscript{11}. Thus $(\tilde{M}, \tilde{K})$ has Lorentzian signature.

\textsuperscript{11}Hint: in the coordinates $(z^i, \lambda)$, associated to the local trivialization $\tilde{m} = \lambda s(m) \sim (m, \lambda)$ induced by a section $s$ of $L$, we have $|\Omega|^2 = \lambda \bar{\lambda} |s|^2$. Moreover horizontal vectors read $u = w^i \partial_i - w^a a_i \partial_{x^a}$, where the local connection 1-form on $M$ is $a = a_i dz^i = |s|^{-2} \partial |s|^2$. The pseudo-Kähler form reads $-2\pi i \tilde{K} = \lambda \lambda \partial_i \partial_j |s|^2 dz^i \wedge d\bar{z}^j + |s|^2 d\lambda \wedge d\bar{\lambda} + \lambda \lambda \partial_i |s|^2 dz^i \wedge d\bar{\lambda} + \lambda \lambda \partial_j |s|^2 d\lambda \wedge d\bar{z}^j$. 

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Concerning the pullback $\pi^*K$ on $\tilde{M}$ of the Kähler form $K$ on $M$; while $K$ is in general only closed, $\pi^*K$ is exact,

$$\pi^*K = \frac{i}{2\pi} \bar{\partial} \partial \log |\Omega|^2.$$  

(5.23)

This last formula easily follows by pulling back the usual local curvature formula for the hermitian connection $K = \frac{i}{2\pi} \bar{\partial} \partial \log |s|^2$ and by observing that $\pi^*\log |s|^2 = \log |\tilde{s}|^2 = \log |\Omega|^2 - \log \lambda - \log \bar{\lambda}$.

In conclusion, one can canonically associate to a Kähler-Hodge manifold $(M, L, K)$ a pseudo-Kähler manifold $(\tilde{M}, \tilde{K})$ that carries a free and holomorphic $\mathbb{C}^\times$ action, and a line bundle $\pi^*L \to \tilde{M}$ that has a canonical global holomorphic section $\Omega$.

The bundle $\tilde{L}$ can be naturally identified as the holomorphic subbundle of $T\tilde{M}$ given by the vertical vectors of $\tilde{M}$ with respect to the holomorphic $\mathbb{C}^\times$ action. The global holomorphic section $\Omega$ corresponds to the vertical vector field that gives the infinitesimal $\mathbb{C}^\times$ action. Under this identification we have

$$\tilde{K}(\Omega, \Omega) = -\frac{i}{2\pi} |\Omega|^2.$$  

(5.24)

This equation shows that under the identification $T\tilde{M}|_{\text{vert}} \simeq L$ the corresponding hermitian structures are mapped one into minus the other.

**Special Kähler manifolds**

Following [49], $(M, L, K)$ is special Kähler if $(\tilde{M}, \tilde{K})$ is rigid special Kähler and if $\Omega$ is compatible with the symplectic connection $\tilde{\nabla}$.

A (projective or local) **special Kähler manifold** is a Kähler-Hodge manifold $(M, L, K)$ such that the associated pseudo-Kähler manifold $(\tilde{M}, \tilde{K})$ has a rigid special pseudo-Kähler structure $\tilde{\nabla}$ which satisfies

$$\tilde{\nabla}\Omega = \pi^{(1,0)}.$$  

(5.25)

Notice that (5.25) is equivalent to the condition $\tilde{\nabla}_u \Omega = u$ for any $u \in T^{(1,0)}\tilde{M}$. As shown in [50], since $\tilde{\nabla}$ is torsionfree and flat, then condition (5.25) implies the $\mathbb{C}^\times$ invariance of $\tilde{\nabla}$, i.e. $dR_b(\tilde{\nabla}_u \Omega) = \tilde{\nabla}_{dR_b u} dR_b \Omega$ where $R_b$ denotes the action of $b \in \mathbb{C}^\times$. Notice also that equation (5.25) is the global version of eq. (5.4).

For ease of notation in the following we denote the flat torsionfree symplectic connection $\tilde{\nabla}$ on $\tilde{M}$ simply by $\tilde{\nabla}$.

We now construct a flat symplectic $2n = 2n' + 2$ dimensional bundle $\mathcal{H}$ on $M$ that is frequently used in the literature in order to characterize projective special Kähler
manifolds. We introduce a new $\mathbb{C}^\times$ action on $T\tilde{M}$. On $\tilde{M}$ it is the usual one $R_b\tilde{m} = \tilde{m}b = b\tilde{m}$, where $b \in \mathbb{C}^\times$, while on vectors we have
\[ v_{\tilde{m}} \mapsto b^{-1}dR_b v_{\tilde{m}}. \] 
(5.26)

From now on by $\mathbb{C}^\times$ action we understand the new above defined one. Thus for example since $b^{-1}dR_b \Omega_{\tilde{m}} = b^{-1}\Omega_{\tilde{m}b}$, then $\Omega$ is not invariant under (5.20). On the other hand the local section (vertical vector field) $\tilde{s}$, obtained from a local section $s$ of $L$, satisfies $b^{-1}dR_b \tilde{s}_{\tilde{m}} = \tilde{s}_{b\tilde{m}}$ (or $b^{-1}R_b \tilde{s} = \tilde{s}$) and is therefore $\mathbb{C}^\times$ invariant. A $\mathbb{C}^\times$ invariant frame associated with local coordinates $z^i$ of $M$ and with the local section $s$ of $L$ is $(\lambda^{-1} \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \lambda})$; it is given by the coordinates $(X^i, X^0) = (\lambda z^i, \lambda)$, they are $\mathbb{C}^\times$ invariant $(b^{-1}R_b^* X = X)$ and therefore are homogeneous (projective) coordinates of $M$.

We define the $2n = 2n' + 2$ dimensional real vector bundle on $M$ $(\dim_{\mathbb{R}} M = 2n')$,
\[ \mathcal{H} \rightarrow M \] 
(5.27)

by identifying its local sections with the $\mathbb{C}^\times$ invariant sections of $T\tilde{M}$. In other words $\mathcal{H}$ is the quotient of $T\tilde{M}$ via the $\mathbb{C}^\times$ action (5.26). A point $(m, h) \in \mathcal{H}$ is the equivalence class $[(\tilde{m}, v_{\tilde{m}})]$ where $(\tilde{m}, v_{\tilde{m}}) \sim (\tilde{m}', u_{\tilde{m}'})$ if $m' = mb$ and $b^{-1}dR_b v_{\tilde{m}} = u_{\tilde{m}'}$. Under this quotient $\pi^* L \subset T\tilde{M}$ becomes $L$, while the subbundle $T\tilde{M}|_{\text{hor}}$ of horizontal vectors becomes $L \otimes T\tilde{M}$.

Since the $\mathbb{C}^\times$ action is holomorphic, then $\mathcal{H}$ is a holomorphic vector bundle on $M$ of rank $n' + 1$. Since $\tilde{K}$ is a $\mathbb{C}^\times$ invariant 2-form the symplectic structure of $T\tilde{M}$ goes to the quotient $\mathcal{H}$: indeed $K(u, v)$ is a homogeneous function on $M$ if $u$ and $v$ are $\mathbb{C}^\times$ invariant vector fields of $T\tilde{M}$. Similarly also the flat symplectic connection $\tilde{\nabla}$ induces a flat symplectic connection on $\mathcal{H}$ (see for example [50]). The inclusion $L \subset \mathcal{H}$ implies that
\[ L^{-1} \otimes \mathcal{H} \rightarrow M \] 
(5.29)

has a nonvanishing global holomorphic section.

In the following we work in $T\tilde{M}$, but we choose $\mathbb{C}^\times$ invariant tensors and therefore our results immediately apply to the bundle $\mathcal{H}$. Let’s consider a $\mathbb{C}^\times$ invariant flat local symplectic framing of $T\tilde{M}$, that we denote by $\{e\xi\} = \{e_\Lambda, f^\Lambda\}$, $\xi = 1, \ldots 2n$, $\Lambda = 1, \ldots n$. The framing is flat because $\tilde{\nabla} e_\Lambda = 0, \tilde{\nabla} f^\Lambda = 0$, and it is symplectic because in this

\[ \text{Hint: denote by } \tilde{e}_m|_{\tilde{m}} \text{ the horizontal lift in } T_{\tilde{m}}\tilde{M} \text{ of the vector } v_m \subset T_m M. \text{ Then the map } L \otimes TM \rightarrow (TM|_{\text{hor}})/\mathbb{C}^\times \text{ action defined by } (\ell_m \otimes v_m) \mapsto [(\ell_m, \tilde{e}_m|_{\ell_m})] \text{ if } \ell_m \neq 0, \text{ and by } 0 \mapsto 0 \text{ is well defined, linear and injective.} \]
basis the symplectic matrix is in canonical form: the components \( \tilde{K}(e_\Lambda, e_\Sigma), \tilde{K}(e_\Lambda, f^\Sigma), \tilde{K}(f^\Lambda, e_\Sigma), \tilde{K}(f^\Lambda, f^\Sigma) \) read
\[
\begin{pmatrix}
0 & -\mathbb{I} \\
\mathbb{I} & 0
\end{pmatrix}
\]  
(5.30)

With respect to the \( \{e_\Lambda, f^\Lambda\} \) frame, the global section \( \Omega \) has local components \( \Omega = \Omega^\xi \xi^e \Lambda e_\Lambda + F^\Lambda \Lambda f^\Lambda \). We also denote by \( \Omega \) this column vector of coefficients,
\[
\Omega = (\Omega^\xi) = (X^\Lambda F^\Lambda).
\]  
(5.31)

The local functions \( X^\Lambda, F^\Lambda \) on \( \tilde{M} \) are holomorphic, indeed (5.25) implies that \( \nabla \Omega \) is a \((1, 0)\)-form valued in \( \tilde{T} \tilde{M} \), since \( \nabla (\Omega^\xi e_\xi) = d\Omega^\xi e_\xi = \partial\Omega^\xi e_\xi + \partial\Omega^\xi e_\xi \), we obtain \( \bar{\partial}\Omega^\xi = 0 \).

In conclusion \((X^\Lambda, F^\Lambda)\) are local components of the global symplectic section \( \Omega \) of the tangent bundle \( \tilde{T} \tilde{M} \).

Each entry \( X^\Lambda, F^\Lambda \) is also a local holomorphic section of the line bundle \( L^{-1} \to M \). Indeed from the transformation properties of \( \Omega \) under the \( \mathbb{C}^\times \) action \( \tilde{m} \mapsto R_{e^{-f(m)}}(\tilde{m}) = e^{-f(m)}\tilde{m} \) (or under a change of local trivialization \( s'(m) = e^{f(m)}s(m) \)) we have
\[
\begin{pmatrix}
X^\Lambda \\
F^\Lambda
\end{pmatrix}' = e^{-f(m)} \begin{pmatrix}
X^\Lambda \\
F^\Lambda
\end{pmatrix},
\]  
(5.32)

therefore for each invertible \( \Omega^\xi \) we have that \( \Omega^{-1}(s)s \) is a section of \( L \to M \) or equivalently each \( X^\Lambda \) and each \( F^\Lambda \) are the coefficients of sections of \( L^{-1} \to M \).

In conclusion \((X^\Lambda, F^\Lambda)\) are local components of the global symplectic section \( \Omega \) of the tangent bundle \( \tilde{T} \tilde{M} \). Each entry is also a local holomorphic section of the line bundle \( L^{-1} \to M \). Under change of local trivialization of \( \tilde{T} \tilde{M} \) we have
\[
\begin{pmatrix}
X^\Lambda \\
F^\Lambda
\end{pmatrix}' = S \begin{pmatrix}
X^\Lambda \\
F^\Lambda
\end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix}
X^\Lambda \\
F^\Lambda
\end{pmatrix},
\]  
(5.33)

where \( S = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \) is a constant symplectic matrix. We can also consider a change of coordinates on \( M \), say \( z \to z' \). Provided we keep fixed the frame of \( \tilde{T} \tilde{M} \) and the trivialization of \( L \) we then have that \( X^\Lambda \) and \( F^\Lambda \) behave like local functions on \( M \), \( X^\Lambda(z) = X'^\Lambda(z') \), \( F^\Lambda(z) = F'^\Lambda(z') \) (here \( X^\Lambda(z) = X^\Lambda(s(z)) \) etc.).

It can be shown [50] that from the set of \( 2n \) elements \( \{X^\Lambda, F^\Lambda\} \) one can always choose a subset of \( n \) elements that form a local coordinate system on \( \tilde{M} \). Contrary to the Kähler case (where the metric is Riemanninan) in this pseudo-Kähler case in general neither \( \{X^\Lambda\} \) nor \( \{F^\Lambda\} \) are coordinates systems on \( \tilde{M} \). The frame \( \{e_\Lambda, f^\Lambda\} \) is determined up to a symplectic transformation, if using this freedom we have that the \( \{X^\Lambda\} \) are coordinates
functions then the \( \{X^\Lambda\} \) are named special coordinates. The sections \( F_\Lambda \) can then be seen as functions of the \( X^\Lambda \) and are obtained via a prepotential \( \Phi \),

\[
F_\Lambda = \frac{\partial \Phi}{\partial X^\Lambda}. \tag{5.34}
\]

Recalling (5.23) and (5.24) we have

\[
\pi^* K = \frac{i}{2\pi} \partial \partial \log i \langle \Omega, \overline{\Omega} \rangle \tag{5.35}
\]

and for the corresponding “Kähler” potential \( K \) we have\(^{13}\)

\[
K = -\log i \langle \Omega, \overline{\Omega} \rangle \tag{5.36}
\]

in these formulae we used the standard notation

\[
\langle \Omega, \overline{\Omega} \rangle = \tilde{K}(\Omega, \overline{\Omega}).
\]

Using the components \( (X^\Lambda, F_\Lambda) \) expression (5.36) reads

\[
K = -\log \left[ i(X, F) \begin{pmatrix} 0 & -\Pi \\ \Pi & 0 \end{pmatrix} \left( \begin{pmatrix} \tilde{X} \\ \tilde{F} \end{pmatrix} \right) \right] = -\log \left[ i(F_\Lambda \bar{X}^\Lambda - X^\Lambda \bar{F}_\Lambda) \right]. \tag{5.37}
\]

By considering local sections of the bundle \( \tilde{M} \to M \), we can then pull back the potential \( K \) to local Kähler potentials on \( M \).

Under the action of \( e^{-f(m)} \in \mathbb{C}^\times \) on \( \tilde{M} \) (or equivalently under change of trivialization of \( \tilde{M} \to M \)) we have

\[
K' = K + f + \bar{f} \tag{5.38}
\]

thus showing that \( e^{-K} \) defines a global nonvanishing section of the bundle \( L \otimes \overline{L} \to M \), in particular this bundle is trivial. Explicitly this global section is \( e^{K(s)}[s, \bar{s}] \) where \( s \) is any local section of \( \tilde{M} \to M \) and \([s, \bar{s}] = \{(s\lambda, \lambda^{-1}\bar{s}) : \lambda \in \mathbb{C}^\times\} \) is the corresponding local section of \( L \otimes \overline{L} \).

**Symplectic Sections and Matrices from local coordinates frames on \( M \)**

Let’s examine few more properties of special Kähler manifolds and introduce those symplectic vectors that we have seen characterizing the geometry of the supergravity scalar fields. Consider a vector \( u \in T^{(1,0)}_m M \), this can be lifted to a horizontal vector \( \hat{u} \in T^{(1,0)}_{\tilde{m}} \tilde{M} \). Because of (5.25) the covariant derivative \( \nabla_{\hat{u}} \Omega \) is again a vector in \( T^{(1,0)}_{\tilde{m}} \tilde{M} \), then

\[
\langle \Omega, \nabla_{\hat{u}} \Omega \rangle = 0, \quad \langle \overline{\Omega}, \nabla_{\hat{u}} \Omega \rangle = 0; \tag{5.39}
\]

\(^{13}\)As usual when \( K \) is integral \( K = \int_{2\pi} g_{ij} dz^i \wedge d\bar{z}^j = \frac{i}{2\pi} \partial_i \partial_j K d z^i \wedge d \bar{z}^j = \frac{i}{2\pi} \partial \bar{\partial} K \).

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the first relation holds because $\tilde{K} = \langle , \rangle$ is a $(1,1)$-form, the second relation holds because horizontal and vertical vectors are orthogonal under $\tilde{K}$ (recall paragraph after (5.22)).

Subordinate to a holomorphic coordinate system $\{z^i\}$ of $M$, and a local section $s$ of $L \to M$ we have the local coordinates $(z^i, \lambda)$ on $\tilde{M}$. The corresponding vector fields are $(\partial_i, \frac{\partial}{\partial \lambda})$. A more natural frame on $\tilde{M}$ is given by considering the vertical vector field associated to the action of $\mathbb{C} \times$ on $\tilde{M}$, $\hat{\partial}_0 \equiv \Omega = \lambda \frac{\partial}{\partial \lambda}$, (5.40)

and the horizontal lift $\hat{\partial}_i$ of the vector fields $\partial_i$ on $M$

$$\hat{\partial}_i = \partial_i - |s|^{-2} \partial_i |s|^2 \lambda \frac{\partial}{\partial \lambda} = \partial_i + \partial_i \lambda \kappa \lambda \frac{\partial}{\partial \lambda} . \quad (5.41)$$

In (5.41), $|s|^2 = h(s,s)$ is the hermitian form of $L \to M$. All these vector fields have degree 1 and are independent from the section $s$ of $L \to M$.

We define

$$\nabla_i \equiv \nabla_{\hat{\partial}_i} . \quad (5.42)$$

The new sections $\nabla_i \Omega$ are exactly the horizontal vector fields $\hat{\partial}_i$, indeed from (5.25) we obtain

$$\nabla_i \Omega = \hat{\partial}_i , \quad \nabla_0 \Omega = \hat{\partial}_0 = \Omega . \quad (5.43)$$

Similarly

$$\nabla_i \bar{\Omega} = 0 , \quad \nabla_0 \bar{\Omega} = 0 . \quad (5.44)$$

Recalling (5.39) we obtain

$$\langle \Omega, \nabla_i \Omega \rangle = 0 \quad (5.45)$$

$$\langle \nabla_i \Omega, \nabla_j \Omega \rangle = 0 \quad (5.46)$$

$$\langle \Omega, \nabla_i \bar{\Omega} \rangle = 0 . \quad (5.47)$$

Notice also that $\langle \Omega, \bar{\Omega} \rangle$ is invariant under horizontal vector fields,

$$\hat{\partial}_i \langle \Omega, \bar{\Omega} \rangle = \nabla_i \langle \Omega, \bar{\Omega} \rangle = \langle \nabla_i \Omega, \bar{\Omega} \rangle + \langle \Omega, \nabla_i \bar{\Omega} \rangle = 0 \quad (5.48)$$

where in the last passage we used (5.39) and (5.44). Similarly $\nabla_i \langle \Omega, \bar{\Omega} \rangle = 0$.

The metric associated to the Kähler form (5.22) on $\tilde{M}$ is block diagonal in the $\hat{\partial}_0, \hat{\partial}_i$ basis, (see paragraph following (5.22)),

$$\begin{pmatrix} \tilde{g}_{00} & 0 \\ 0 & \tilde{g}_{ij} \end{pmatrix} = \begin{pmatrix} -\lambda \bar{\lambda} |s|^2 & 0 \\ 0 & \lambda \bar{\lambda} |s|^2 g_{ij} \circ \pi \end{pmatrix} = \begin{pmatrix} -|\Omega|^2 & 0 \\ 0 & |\Omega|^2 g_{ij} \circ \pi \end{pmatrix} . \quad (5.49)$$
Because of (5.48) the associated Levi-Civita connection coefficients of $\tilde{M}$ in the $\partial_\xi$ basis of horizontal vectors coincide with those of $M$ in the $\frac{\partial}{\partial\xi}$ basis,

$$\tilde{\Gamma}_{ij}^k = \tilde{g}^{kt}\partial_t\tilde{g}_{jk} = g^{kt}\partial_t g_{jk} = \Gamma_{ij}^k. \quad (5.50)$$

In terms of the symplectic frame $\{e_\xi\} = \{e_\lambda, f_\lambda\}$, that is flat, we have $\nabla\Omega = \nabla(\Omega^\xi e_\xi) = d(\Omega^\xi)e_\xi$, and $\nabla_i\Omega = \partial_i(\Omega^\xi)e_\xi = \partial_i\Omega^\xi + \partial_i\mathcal{K}\Omega^\xi$, i.e.,

$$\nabla_i\left(\frac{X^\lambda}{F_\lambda}\right) = \partial_i\left(\frac{X^\lambda}{F_\lambda}\right) + \partial_i\mathcal{K}\left(\frac{X^\lambda}{F_\lambda}\right). \quad (5.51)$$

Recalling the interpretation of $X^\lambda$ or $F_\lambda$ as coefficients of local sections of $L^{-1} \rightarrow M$, we read in equation (5.51) the covariant derivative of $L^{-1} \rightarrow M$.

It is also convenient to normalize $\Omega$ and thus consider the (nonholomorphic) nonvanishing global vector field on $\tilde{M}$ given by

$$\mathcal{V} = e^{\mathcal{K}/2}\Omega. \quad (5.52)$$

From (5.48) the covariant derivatives of $\mathcal{V}$ are

$$\nabla_i\mathcal{V} = e^{\mathcal{K}/2}\nabla_i\Omega, \quad \nabla_i\mathcal{V} = e^{\mathcal{K}/2}\nabla_i\Omega = 0,$$

$$\nabla_i\mathcal{V} = e^{\mathcal{K}/2}\nabla_i\tilde{\Omega}, \quad \nabla_i\mathcal{V} = e^{\mathcal{K}/2}\nabla_i\tilde{\Omega} = 0.$$

Explicitly we have\(^{14}\)

$$\begin{align*}
\nabla_i\mathcal{V} &= (\partial_i\mathcal{V}^\xi + \frac{1}{2} \partial_i\mathcal{K}\mathcal{V}^\xi) e_\xi, & \nabla_i\mathcal{V} &= (\partial_i\mathcal{V}^\xi - \frac{1}{2} \partial_i\mathcal{K}\mathcal{V}^\xi) e_\xi = 0 \quad (5.53) \\
\nabla_i\tilde{\mathcal{V}} &= (\partial_i\tilde{\mathcal{V}}^\xi + \frac{1}{2} \partial_i\mathcal{K}\tilde{\mathcal{V}}^\xi) e_\xi, & \nabla_i\tilde{\mathcal{V}} &= (\partial_i\tilde{\mathcal{V}}^\xi - \frac{1}{2} \partial_i\mathcal{K}\tilde{\mathcal{V}}^\xi) e_\xi = 0. \quad (5.54)
\end{align*}$$

Each coefficient $\mathcal{V}^\xi$ of $\mathcal{V}$ with respect to the $\mathbb{C}^\times$ invariant basis $e_\xi$ is also a coefficient of a local section of the bundle $L^{-1/2} \otimes \tilde{L}^{1/2} \rightarrow M$. This bundle has connection $\frac{1}{2} \partial_i\mathcal{K} - \frac{1}{2} \partial_i\mathcal{K}$. Equation (5.53) can be interpreted as the covariant derivative of these line bundle local sections.

From (5.36), and (5.45)-(5.47) we have

\(^{14}\)we find also instructive to obtain the covariant derivative of the section $\mathcal{V}$ via this straightforward calculation that uses $\lambda_{\partial_\mathcal{K}} = -1$,

$$\begin{align*}
\nabla_i\mathcal{V} &= \nabla_i(e^{\mathcal{K}/2}\Omega^\xi e_\xi) = \partial_i(e^{\mathcal{K}/2}\Omega^\xi)e_\xi = \partial_i(e^{\mathcal{K}/2}\Omega^\xi)e_\xi + \partial_i\mathcal{K}\lambda \frac{\partial}{\partial\lambda}(e^{\mathcal{K}/2}\Omega^\xi)e_\xi = (\partial_i\mathcal{V}^\xi + \frac{1}{2} \partial_i\mathcal{K}\mathcal{V}^\xi) e_\xi. 
\end{align*}$$
\[ \langle V, V \rangle = -i , \quad \langle V, \nabla_i V \rangle = 0 , \quad \langle \nabla_i V, \nabla_j V \rangle = 0 , \quad \langle V, \nabla_i V \rangle = 0 . \] (5.55)

From (5.49), or also from \[ [\nabla_j, \bar{\nabla}] = -\partial_j \bar{\partial} K = -g_{ji} \] and \[ \langle \bar{\nabla}_{\bar{I}} \nabla_i V, V \rangle + \langle \nabla_i V, \bar{\nabla}_{\bar{I}} V \rangle = 0, \] we have

\[ \langle \nabla_j V, \bar{\nabla}_{\bar{I}} V \rangle = i g_{ji} . \] (5.59)

The index \( M \) mixes holomorphic and antiholomorphic indices in order to compensate for the Lorentzian signature of the metric \((-1, 0, 0)\) in (5.55), (5.59).

Explicitly the column vectors of the components of the sections \( V_M = V^\xi_M e_\xi \) are

\[ (V^\xi) = \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} e^{K/2} \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} , \quad (\bar{\nabla}_{\bar{I}} V^\xi) = \begin{pmatrix} \bar{\nabla}_{\bar{I}} L^\Lambda \\ \bar{\nabla}_{\bar{I}} M^\Lambda \end{pmatrix} , \] (5.63)

and they can be organized in a \( 2n \times n \) matrix

\[ (V^\xi_M) = (V, \bar{\nabla}_{\bar{I}} V^\xi) = \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} \begin{pmatrix} \bar{\nabla}_{\bar{I}} L^\Lambda \\ \bar{\nabla}_{\bar{I}} M^\Lambda \end{pmatrix} = \begin{pmatrix} f^\Lambda_M \\ h_{AM} \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix} . \] (5.64)

In the last passage we have denoted by \( f \) (respectively \( h \)) the \( n \times n \) matrix of entries \( f^\Lambda_M \) (respectively \( h_{AM} \)).

The \( N = 2 \) special geometry relations (5.62) are equivalent to

\[ (f^\dagger, h^\dagger) \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} = i \mathbb{I} \quad \text{i.e.} \quad -f^\dagger h + h^\dagger f = i \mathbb{I} \] (5.65)
and
\[(f^t, h^t) \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} = 0 \quad \text{i.e.} \quad -f^t h + h^t f = 0 \quad (5.66)\]

These two relations are equivalent to require the real matrix
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \sqrt{2} \begin{pmatrix} \text{Re} f & -\text{Im} f \\ \text{Re} h & -\text{Im} h \end{pmatrix} \quad (5.67)
\]

to be symplectic. Vice versa any symplectic matrix \(\begin{pmatrix} A B \\ C D \end{pmatrix}\) leads to relations (5.65), (5.66) by defining \(\begin{pmatrix} f \\ h \end{pmatrix} = \frac{1}{\sqrt{2}}(A-iB)\). The matrix
\[
V = \begin{pmatrix} f & \bar{f} \\ h & \bar{h} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} A, \quad (5.68)
\]

where \(A = \frac{1}{\sqrt{2}}(1-i1)\), rotates the flat real symplectic frame \(\{e_\xi\} = \{e^\Lambda, f_\Lambda\}\) in the frame \(\{\mathcal{V}_M, \bar{\mathcal{V}}_{\bar{M}}\}\) that up to a rotation by \(A^{-1} = A^\dagger\) is also real and symplectic (but not flat). This \(\{\mathcal{V}_M, \bar{\mathcal{V}}_{\bar{M}}\}\) frame comes from a local coordinate frame on \(\tilde{M}\), indeed \(\tilde{\mathcal{V}}_{\tilde{M}} = (e^{K/2} \Omega, e^{K/2} e_j^\lambda \partial_j)\). The symplectic connection 1-form in this frame is simply \(\Gamma = V^{-1} dV\), indeed \(\nabla e_\xi = 0\) is equivalent to
\[
dV = V \Gamma. \quad (5.69)
\]

We can write \(\Gamma = \begin{pmatrix} \omega & \bar{P} \\ \bar{P} & \bar{\omega} \end{pmatrix},\) and see this equation as a condition on the Levi-Civita connection \(\omega\) and the tensor \(P\) of \(\tilde{M}\). The block decomposition \(\begin{pmatrix} \omega & \bar{P} \\ \bar{P} & \bar{\omega} \end{pmatrix}\) follows by recalling that \(\tilde{M}\) is in particular a rigid special Kähler manifold. The difference \(\mathcal{P}_R = \nabla - D\) between the flat symplectic connection and the Levi-Civita connection is given by the holomorphic symmetric three form \(C\) (c.f. (5.11))
\[
C = -\langle \nabla \Omega, \nabla \nabla \Omega \rangle. \quad (5.70)
\]

The properties of \(C\) previously discussed in the rigid case apply also to this projective special geometry case.

### 5.2 The \(N = 2\) theory

From the previous section we see that the \(N = 2\) supergravity theories and the higher \(N\) theories have a similar flat symplectic structure. The formalism is the same, indeed since the antisymmetric of the \(U(2)\) authomorphism group of the \(N = 2\) supersymmetry algebra is a singlet we have
\[
f^A_{AB} = f^A_0 \epsilon_{AB}, \quad h_{AAB} = h_{A0} \epsilon_{AB} \quad (5.71)
\]

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where \( f^A_0, h_{A0} \) are the components of the global section \( \mathcal{V} \), therefore from (5.64) we have as in (1.22),

\[
\begin{align*}
f &= (f^M_A) = (f^A_{AB}, \bar{f}^A_I) , \\
h &= (h_{AM}) = (h_{AAB}, \bar{h}_{AI}) ,
\end{align*}
\]

as it should be, the sections \( (\bar{f}^A_I, \bar{h}^A_I) \) have Kähler weight opposite to the \( (f^A_{AB}, h^A_{AB}) \) sections.

The difference between the \( N = 2 \) cases and the \( N > 2 \) cases is that the scalar manifold \( M \) of the \( N = 2 \) case is not in general a coset manifold. The flat symplectic bundle is therefore not in general a trivial bundle. The gauge kinetic term \( \mathcal{N}_{\Lambda \Sigma} = h_{AM} f^{-1}_M \) depends on the choice of the flat symplectic frame \( \{ e_\xi \} = \{ e_\Lambda, f^A_\Lambda \} \). This latter can be defined only locally on \( \tilde{M} \) (and therefore on \( M \)). In another region we have a different frame \( \{ e'_\xi \} = \{ e'_\Lambda, f'^A_\Lambda \} \) and therefore a different gauge kinetic term \( \mathcal{N}'_{\Lambda \Sigma} \). In the common overlapping region the two formulations should give the same theory, this is indeed the case because the corresponding equations of motion are related by a duality rotation. As a consequence the notion of electric or magnetic charge depends on the flat frame chosen. In this sense the notion of electric and magnetic charge is not a fundamental one. The symplectic group is a gauge group (where just constant gauge transformations are allowed) and only gauge invariant quantities are physical.

A related aspect of the comparison between the \( N = 2 \) theory and the \( N > 2 \) theories is that the special Kähler structure determines the presence of a new geometric quantity, the holomorphic cubic form \( C \), which physically corresponds to the anomalous magnetic moments of the \( N = 2 \) theory. When the special Kähler manifold \( M \) is itself a coset manifold [78], then the anomalous magnetic moments \( C_{ijk} \) are expressible in terms of the vielbein of \( G/H \), this is for example the case of the \( N = 2 \) theories with scalar manifold \( G/H = SU(1,1) \times SU(6,2) / O(6) \times O(2) \) [78].

To complete the analogy between the \( N = 2 \) theory with \( n' \) vector multiplets and the higher \( N \) theories in \( D = 4 \), we also give the supersymmetry transformation laws, the central and matter charges, the differential relations among them and the formula for the potential \( \mathcal{V}_{BH} \).

The supercovariant electric field strength \( \hat{F}^A \) is

\[
\hat{F}^A = F^A + f^A \psi^A \psi^B \epsilon_{AB} - i f^A \hat{\lambda}^B_a \psi^B \epsilon^a + h.c. \quad (5.73)
\]

The transformation laws for the chiral gravitino \( \psi_A \) and gaugino \( \lambda^i_A \) fields are:

\[
\delta \psi_A = \nabla_\mu \epsilon_A + \epsilon_{AB} T^A_{\mu \nu} \gamma^\nu \epsilon^B + \cdots , \quad (5.74)
\]

\[
\delta \lambda^i_A = i \partial_\mu \gamma^i \gamma^\mu \epsilon_A + i \frac{1}{2} T^i_{\mu \nu} \gamma^\mu \gamma^\nu \epsilon^B \epsilon^A + \cdots , \quad (5.75)
\]
where:
\[ T = h_\Lambda F^\Lambda - f^A G_\Lambda, \]  
\[ T_i = T_i^j e^j_i, \] with \( T_i = h_{A i} F^A - f_i^A G_\Lambda, \)

are respectively the graviphoton and the matter vectors. In (5.74), (5.75) the position of the \( SU(2) \) automorphism index \( A (A, B = 1, 2) \) is related to chirality, namely \((\psi_A, \lambda^A)\) are chiral, \((\psi^A, \lambda^A)\) antichiral.

In order to define the symplectic invariant charges let us recall the definition of the magnetic and electric charges (the moduli independent charges) in (4.10). The central charges and the matter charges are then defined as the integrals over a sphere at spatial infinity of the dressed graviphoton and matter vectors (4.21), they are given in (4.23), (4.24):
\[ (Z_M) = (Z, \bar{Z}_{\bar{I}}) = iV(\phi_\infty)^{-1}Q \]

where \( \phi_\infty \) is the value of the scalar fields at spatial infinity. Because of (5.61) we get immediately:
\[ \nabla I Z = Z_I. \]

This relation can also be written \( \nabla I Z_{AB} = Z_I \epsilon_{AB}, \) and considering the vielbein 1-form \( P^I \) dual to the frame \( e_I \) introduced in (5.60) and setting \( \nabla \equiv P^I \nabla_I \) we obtain \( \nabla Z_{AB} = Z_I P^I \epsilon_{AB}. \)

The positive definite quadratic invariant \( \mathcal{V}_{BH} \) in terms of the charges \( Z \) and \( Z_I \) reads
\[ \mathcal{V}_{BH} = \frac{1}{2} Z \bar{Z} + Z_I \bar{Z}^I = -\frac{1}{2} Q^I \mathcal{M}(\mathcal{N}) Q. \]

Equation (5.80) is obtained by using exactly the same procedure as in (4.27). Invariance of \( \mathcal{V}_{BH} \) implies that it is a well defined positive function on \( M. \)

6 Duality rotations in Noncommutative Spacetime

Field theories on noncommutative spaces have received renewed interest since their relevance in describing Dp-branes effective actions (see [79] and references therein). Noncommutativity in this context is due to a nonvanishing NS background two form on the Dp-brane. First space-like (magnetic) backgrounds \((B_{ij} \neq 0)\) were considered, then NCYM theories also with time noncommutativity \((B_{0i} \neq 0)\) have been studied [82]. The NCYM theories that can be obtained from open strings in the decoupling limit \( \alpha' \to 0 \) are those with \( B \) space-like or light-like (e.g. \( B_{0i} = -B_{1i} \)), these were also considered the only theories without unitarity problems [83], however by applying a proper perturbative setup it was shown that also time-space noncommutative field theories can be unitary [84].
Following [79], gauge theory on a Dp-brane with constant two-form $B$ can be described via a commutative Lagrangian and field strength $\mathcal{L}(F + B)$ or via a noncommutative one $\hat{\mathcal{L}}(\hat{F})$, where $\hat{F}_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - i [A_{\mu} \star A_{\nu}]$. Here $\star$ is the star product, on coordinates $[x^{\mu} \star x^{\nu}] = x^{\mu} x^{\nu} - x^{\nu} \star x^{\mu} = i \Theta^{\mu\nu}$, where $\Theta$ depends on $B$ and the metric on the Dp-brane. The commutative and the noncommutative descriptions are complementary and are related by Seiberg-Witten map (SW map) [79], [80, 81]. In the $\alpha' \to 0$ limit [79] the exact effective electromagnetic theory on a Dp-brane is noncommutative electromagnetism (NCEM), this is equivalent, via SW map, to a nonlinear commutative $U(1)$ gauge theory.

In this section we consider a D3-brane action in the slowly varying field approximation, we give an explicit expression of this nonlinear $U(1)$ theory and we show that it is self-dual when $B$ (or $\Theta$) is light-like. Via SW map solutions of $U(1)$ nonlinear electromagnetism are mapped into solutions of NCEM, so that duality rotations are also a symmetry of NCEM, i.e., NCEM is self-dual [85], [52]. When $\Theta$ is space-like we do not have self-duality and the S-dual of space-like NCYM is a noncommutative open string theory decoupled from closed strings [87]. Related work appeared in [88–90]. We mention that self-duality of NCEM was initially studied in [86] to first order in $\Theta$. On one hand it is per se interesting to provide new examples of self-dual nonlinear electromagnetism, as the one we give with the lagrangian (6.12). On the other hand this lagrangian is via Seiberg-Witten map, and for slowly varying fields, just NCEM. Formally NCEM resembles $U(N)$ YM on commutative space, and on tori with rational $\Theta$ the two theories are $T$-dual [91]. Self-duality of NCEM then hints to a possible duality symmetry property of the equations of motion of $U(N)$ YM.

**Self-Duality of the D3-brane action**

Consider the D3-brane effective action in a IIB supergravity background with constant axion, dilaton NS and RR two-forms. The background two-forms can be gauged away in the bulk and we are left with the field strength $\mathcal{F} = F + B$ on the D3-brane. Here $B$ is defined as the constant part of $\mathcal{F}$, or $B = \mathcal{F}|_{\text{spatial}}$ since $F$ vanish at spatial infinity. For slowly varying fields the Lagrangian, in Einstein frame is essentially the Born-Infeld action with axion and dilaton. We set for simplicity $N = -i \mathbb{I}$ and $g_s = 1$, where $g_s$ is the string coupling constant. The lagrangian is then $\mathcal{L} = \frac{1}{\alpha'} \sqrt{-\det(g + \alpha' \mathcal{F})}$. The explicit expression of $\mathcal{G}$, is obtained from the definition $\mathcal{G} := \frac{\partial \mathcal{L}}{\partial F}$ and is (cf. (2.38))

$$
\mathcal{G}_{\mu\nu} = \frac{\mathcal{F}^*_{\mu\nu} + \frac{\alpha'^2}{4} \mathcal{F} \mathcal{F}^* \mathcal{F}_{\mu\nu}}{\sqrt{1 + \frac{\alpha'^2}{2} \mathcal{F}^2 - \frac{\alpha'^4}{16} (\mathcal{F} \mathcal{F}^*)^2}}.
$$

(6.1)

Here $\mathcal{F}^*_{\mu\nu} = \sqrt{g} \epsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\rho\sigma}$, cf. footnote 2, Section 2.1. One can then consider a duality
rotation by an angle $\gamma$ and extract how $B$ (the constant part of $F$) transforms

$$B'_\mu \nu = \cos \gamma B_{\mu \nu} - \sin \gamma \frac{B^*_{\mu \nu} + \frac{\alpha'^2}{4} B B^* B_{\mu \nu}}{\sqrt{1 + \frac{\alpha'^2}{2} B^2 - \frac{\alpha'^4}{16} (BB^*)^2}}. \quad (6.2)$$

**Open/closed strings and light-like noncommutativity**

The open and closed string parameters are related by (see [79], the expressions for $G$ and $\Theta$ first appeared in [92])

$$\frac{1}{g + \alpha' B} = G^{-1} + \frac{\Theta}{\alpha'}$$

$$g^{-1} = (G^{-1} - \Theta/\alpha') G (G^{-1} + \Theta/\alpha') = G^{-1} - \alpha'^{-2} \Theta G \Theta$$

$$\alpha' B = -(G^{-1} - \Theta/\alpha') \Theta/\alpha' (G^{-1} + \Theta/\alpha')$$

$$G_s = g_s \sqrt{\frac{\det G}{\det (g + \alpha' B)}} = g_s \sqrt{\det G \det (G^{-1} + \Theta/\alpha') = g_s \sqrt{\det g^{-1} \det (g + \alpha' B)}}$$

The decoupling limit $\alpha' \to 0$ with $G_s, G, \Theta$ nonzero and finite [79] leads to a well defined field theory only if $B$ is space-like or light-like. Looking at the closed and open string coupling constants it is easy to see why one needs this space-like or light-like condition on $B$ in performing this limit. Consider the coupling constants ratio $G_s/g_s$, that expanding the 4x4 determinant reads (here $B^2 = B_{\mu \nu} B^\rho \sigma g^{\mu \rho} g^{\nu \sigma}$, $\Theta^2 = \Theta^{\mu \nu} \Theta^{\rho \sigma} G_{\mu \rho} G_{\nu \sigma}$ and so on)

$$\frac{G_s}{g_s} = \sqrt{1 + \frac{\alpha'^{-2}}{2} \Theta^2 - \frac{\alpha'^{-4}}{16} (\Theta \Theta^*)^2} = \sqrt{1 + \frac{\alpha'^2}{2} B^2 - \frac{\alpha'^4}{16} (BB^*)^2} \quad (6.3)$$

Both $G_s$ and $g_s$ must be positive; since $G$ and $\Theta$ are by definition finite for $\alpha' \to 0$ this implies $\Theta \Theta^* = 0$ and $\Theta^2 \geq 0$. Now $\Theta \Theta^* = 0 \iff \det \Theta = 0 \iff \det B = 0 \iff BB^* = 0$. In this case from (6.3) we also have $\Theta^2 = \alpha'^4 B^2$. In conclusion in order for the $\alpha' \to 0$ limit defined by keeping $G_s, G, \Theta$ nonzero and finite [79], to be well defined we need

$$B^2 \geq 0, \quad BB^* = 0 \quad \text{i.e.} \quad \Theta^2 \geq 0, \quad \Theta \Theta^* = 0 \quad (6.4)$$

This is the condition for $B$ (and $\Theta$) to be space-like or light-like. Indeed with Minkowski metric and in three vector notation (6.4) reads $B^2 - E^2 \geq 0$ and $E \perp B$.

If we now require the $\alpha' \to 0$ limit to be compatible with duality rotations, we immediately see that we have to consider only the light-like case $B^2 = BB^* = 0$. Indeed under $U(1)$ rotations the electric and magnetic fields mix up, in particular under a $\pi/2$ rotation (6.2) a space-like $B$ becomes time-like.
In the light-like case \( \det(g + \alpha'B) = \det(g) \), relations (6.3) simplify considerably. The open and closed string coupling constants coincide, since we set \( g_s = 1 \) we have \( G_s = g_s = 1 \), this also implies \( \det(G) = \det(g) \) so that the hodge dual field \( F^* \) with the \( g \) metric equals the one with the \( G \) metric. Use of the relations

\[
\Omega^*_\mu\nu \Omega^{\ast \rho\nu} - \Omega_{\mu\rho} \Omega^{\ast \rho
u} = \frac{1}{2} \Omega^2 \delta^\nu_\mu, \quad \Omega_{\mu\rho} \Omega^{\ast \rho \nu} = \Omega^*_\mu \Omega^{\ast \rho \nu} = -\frac{1}{4} \Omega \Omega^* \delta^\nu_\mu \quad (6.5)
\]
valid for any antisymmetric tensor \( \Omega \), shows that any two-tensor at least cubic in \( \Theta \) (or \( B \)) vanishes. It follows that \( g^{-1}G \Theta = \Theta \) and that the raising or lowering of the \( \Theta \) and \( B \) indices is independent from the metric used. We also have

\[
B'_{\mu\nu} = -\alpha'^{-2} \Theta_{\mu\nu} \quad (6.6)
\]

**Self-duality of NCBI and NCEM**

We now study duality rotations for noncommutative Born-Infeld (NCBI) theory and its zero slope limit that is NCEM. The relation between the NCBI and the BI Lagrangians is [79]

\[
\hat{\mathcal{L}}_{BI}(F, G, \Theta, G_s) = \mathcal{L}_{BI}(F + B, g) + O(\partial F) + \text{tot.der.} \quad (6.7)
\]
where \( O(\partial F) \) stands for higher order derivative corrections, \( \hat{F} \) is the noncommutative \( U(1) \) field strength and we have set \( g_s = 1 \). The NCBI Lagrangian is

\[
\hat{\mathcal{L}}_{BI}(\hat{F}, G, \Theta, G_s) = -\frac{1}{\alpha'^2 G_s} \sqrt{-\det(G + \alpha'\hat{F})} + O(\partial \hat{F}) \quad (6.8)
\]

In the slowly varying field approximation the action of duality rotations on \( \hat{\mathcal{L}}_{BI} \) is derived from self-duality of \( \mathcal{L}_{BI} \). If \( \hat{F} \) is a solution of the \( \hat{\mathcal{L}}_{BI}^{G_s,G,\Theta} \) EOM then \( \hat{F}' \) obtained via

\[
\hat{F} \xrightarrow{\text{SW map}} \mathcal{F} \xrightarrow{\text{duality rot.}} \mathcal{F}' \xrightarrow{\text{SW map}} \hat{F}'
\]
is a solution of the \( \hat{\mathcal{L}}_{BI}^{G',\Theta'} \) EOM where \( G'_s, G', \Theta' \) are obtained using (6.3) from \( g'_s, B' \) and \( g'_s = g_s = 1 \).

In the light-like case we have \( G_s = g_s = 1 \), the \( B \) rotation (6.2) simplifies to

\[
B'_{\mu\nu} = \cos\gamma B_{\mu\nu} - \sin\gamma B^*_{\mu\nu} \quad (6.9)
\]

Using (6.9) the \( U(1) \) duality action on the open string variables is

\[
G' = G \quad , \quad \Theta'^{\mu\nu} = \cos\gamma \Theta^{\mu\nu} - \sin\gamma \Theta^*^{\mu\nu} \quad (6.10)
\]

For \( \Theta \) light-like, solutions \( \hat{F} \) of \( \hat{\mathcal{L}}^{G,\Theta} \) are mapped into solutions \( \hat{F}' \) of \( \hat{\mathcal{L}}^{G',\Theta'} \). Thus we can map solutions of \( \hat{\mathcal{L}}^{G,\Theta} \) into solutions of \( \hat{\mathcal{L}}^{G',\Theta'} \), therefore the theory described by \( \hat{\mathcal{L}}^{G,\Theta} \) has \( U(1) \) duality rotation symmetry.
In order to show self-duality of NCEM we consider the zero slope limit of (6.7) and verify that the resulting lagrangian on the r.h.s. of (6.7) is self-dual. We rewrite $L_{BI}$ in terms of the open string parameters $G, \Theta$

$$L_{BI} = \frac{-1}{\alpha'2} \sqrt{-\det(g + \alpha' F)} = \frac{-\sqrt{G}}{\alpha^2} \sqrt{ \frac{\det(g + \alpha' B + \alpha' F)}{\det(g + \alpha' B)}}$$

$$= \frac{-1}{\alpha'2} \sqrt{-\det(G + \alpha' F + G\Theta F)}.$$  \hspace{1cm} (6.11)

The determinant in the last line can be evaluated as sum of products of traces (Newton-Leverrier formula). Each trace can then be rewritten in terms of the six basic Lorentz invariants $F^2, FF^*, F\Theta, F\Theta^*, \Theta^2 = \Theta\Theta^* = 0$, explicitly

$$\det G^{-1} \det(G + \alpha' F + G\Theta F) = (1 - \frac{1}{2}\Theta F)^2 + \alpha'^2[\frac{1}{2}F^2 + \frac{1}{2}\Theta F^* FF^*] - \alpha'^4(\frac{1}{4}FF^*)^2$$

Finally we take the $\alpha' \to 0$ limit of (6.11), by dropping the infinite constant and total derivatives the resulting Lagrangian is $\sqrt{G}$ times

$$\left(-\frac{1}{4}F^2 - \frac{1}{8}\Theta F^* FF^*ight) \hspace{1cm} (6.12)$$

We thus have an expression for NCEM in terms of $F, \Theta$ and $G$ (of course $G_{\mu\nu}$ can be taken $\eta_{\mu\nu}$), $\hat{L}_{EM} = \sqrt{G}L_{EM}$

$$\hat{L}_{EM} \equiv -\frac{1}{4}\hat{F}\hat{F} = \frac{-\frac{1}{4}F^2 - \frac{1}{8}\Theta F^* FF^*}{1 - \frac{1}{2}\Theta F} + O(\partial F) + \text{tot. der.} \hspace{1cm} (6.13)$$

The Lagrangian (6.12) satisfies the self-duality condition (3.20) with $\varphi = \Theta, \kappa = 0, a = d = 0, c = -b$ and therefore NCEM is self-dual under the $U(1)$ duality rotations (6.10) and $F' = \cos \gamma F - \sin \gamma G$. The change in $\Theta \to \Theta'$, that is not a dynamical field, can be cancelled by a rotation in space so that therefore we can map solution of the EOM of (6.13) into solutions of the EOM of (6.13) with the same value of $\Theta$.

This duality can be enhanced to $Sp(2, \mathbb{R})$ by considering also axion and dilaton fields; also Higgs fields can be coupled, the coupling is minimal in the noncommutative theory. Using this duality one can relate space-noncommutative magnetic monopoles with a string (D1-string D3-brane configuration) to space-noncommutative electric monopoles (possibly an F-string ending on a D3-brane) [52, 53].
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7 Appendix: Symplectic group and transformations

7.1 Symplectic group \((A, B, C, D \text{ and } f, h \text{ and } V \text{ matrices})\)

The symplectic group \(Sp(2n, \mathbb{R})\) is the group of real \(2n \times 2n\) matrices that satisfy

\[
S^t \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} S = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}
\]

(7.1)

Setting \(S = \begin{pmatrix} A & C \\ B & D \end{pmatrix}\) we explicitly have

\[
A^t C - C^t A = 0 \ , \ B^t D - D^t B = 0 \ , \ A^t D - C^t B = 1 .
\]

(7.2)

Since the transpose of a symplectic matrix is again symplectic we equivalently have

\[
AB^t - BA^t = 0 \ , \ CD^t - DC^t = 0 \ , \ AD^t - BC^t = 1 .
\]

(7.3)

In particular \(A^t C, B^t D, CA^{-1}, BD^{-1}, A^{-1} B, D^{-1} C, AB^t, DC^t\) are symmetric matrices (in case they exist).

If \(D\) is invertible we have the factorization

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathbb{I} & BD^{-1} \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} D^{-1} & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ D^{-1} C & \mathbb{I} \end{pmatrix}
\]

(7.4)

where \(A = D^{-1} + BD^{-1} C\) follows from \(BD^{-1} = D^{-1} B^t\).
The complex basis
It is often convenient to consider the complex basis \( \frac{1}{\sqrt{2}} (F + iG) \) rather than \( (F) \). The transition from the real to the complex basis is given by the symplectic and unitary matrix \( A^{-1} \), where
\[
A = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} \mathbb{1} & \mathbb{1} \\ -i & i \end{array} \right), \quad A^{-1} = A^\dagger.
\] (7.5)

A symplectic matrix \( S \), belonging to the fundamental representation of \( Sp(2n, \mathbb{R}) \), in the complex basis reads
\[
U = A^{-1} S A .
\] (7.6)

There is a 1-1 correspondence between matrices \( U \) as in (7.6) and complex \( 2n \times 2n \) matrices belonging to \( U(n, n) \cap Sp(2n, \mathbb{C}) \),
\[
U^\dagger \left( \begin{array}{cc} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{array} \right) U = \left( \begin{array}{cc} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{array} \right), \quad U^\dagger \left( \begin{array}{cc} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{array} \right) U = \left( \begin{array}{cc} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{array} \right) .
\] (7.7)

Equations (7.7) define a representation of \( Sp(2n, \mathbb{R}) \) on the complex vector space \( \mathbb{C}^{2n} \). It is the direct sum of the representations \( \left( \psi \bar{\psi} \right) \) and \( \left( \psi - \bar{\psi} \right) \), these are real representations of real dimension \( 2n \). (The representation \( \left( \psi \bar{\psi} \right) \) is the vector space of all linear combinations, with coefficients in \( \mathbb{R} \), of vectors of the kind \( \left( \psi \bar{\psi} \right) \)).

The maximal compact subgroup of \( U(n, n) \) is \( U(n) \times U(n) \); because of the second relation in (7.7) the maximal compact subgroup of \( Sp(2n, \mathbb{R}) \) is \( U(n) \). The usual embedding of \( U(n) \) into the complex and the fundamental representations of \( Sp(2n, \mathbb{R}) \) are respectively
\[
\left( \begin{array}{c} u \ 0 \\ 0 \ 0 \end{array} \right) , \quad \left( \begin{array}{cc} \text{Re} \ u & -\text{Im} \ u \\ \text{Im} \ u & \text{Re} \ u \end{array} \right) ,
\] (7.8)
where \( u \) belongs to the fundamental of \( U(n) \).

The \( f \) and \( h \) matrices
The \( f \) and \( h \) matrices are \( n \times n \) complex matrices that satisfy the two conditions
\[
(f^\dagger, h^\dagger) \left( \begin{array}{cc} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{array} \right) \left( \begin{array}{c} f \\ h \end{array} \right) = i \mathbb{1} \quad \text{i.e.} \quad -f^\dagger h + h^\dagger f = i \mathbb{1}
\] (7.9)
and
\[
(f^\dagger, h^\dagger) \left( \begin{array}{cc} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{array} \right) \left( \begin{array}{c} f \\ h \end{array} \right) = 0 \quad \text{i.e.} \quad -f^\dagger h + h^\dagger f = 0
\] (7.10)

These two relations are equivalent to require the real matrix
\[
\left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \sqrt{2} \left( \begin{array}{cc} \text{Re} \ f & -\text{Im} \ f \\ \text{Re} \ h & -\text{Im} \ h \end{array} \right)
\] (7.11)
to be in the fundamental representation of $Sp(2n, \mathbb{R})$. Vice versa any symplectic matrix 
$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ leads to relations (7.9), (7.10) by defining

$$
\begin{pmatrix} f \\ h \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} A - iB \\ C - iD \end{pmatrix}.
$$

(7.12)

In terms of the $f$ and $h$ matrices we have

$$
U = A^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} A = \frac{1}{\sqrt{2}} \begin{pmatrix} f + ih & \bar{f} + i\bar{h} \\ f - ih & \bar{f} - i\bar{h} \end{pmatrix}.
$$

(7.13)

The $V$ matrix and its symplectic vectors

The matrix

$$
V = \begin{pmatrix} A & B \\ C & D \end{pmatrix} A = \begin{pmatrix} f & \bar{f} \\ h & \bar{h} \end{pmatrix}
$$

(7.14)

transforms from the left via the fundamental representation of $Sp(2n, \mathbb{R})$ and from the right via the complex representation of $Sp(2n, \mathbb{R})$. Since $A$ is a symplectic matrix we have that $V$ is a symplectic matrix, $V^t \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} V = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, hence also its transpose $V^t$, $V(\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}) V^t = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ . The columns of the $V$ matrix are therefore mutually symplectic vectors; also the rows are mutually symplectic vectors. Explicitly if $V^\xi$ is the vector with components given by the $\xi$-th row of $V$, then $V^\xi \Omega^\sigma V^\zeta = \Omega^{\xi\zeta}$, where $\Omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

### 7.2 The coset space $Sp(2n, \mathbb{R})/U(n)$ ($M$ and $N$ matrices)

All positive definite symmetric and symplectic matrices $S$ are of the form

$$
S = gg^t, \quad g \in Sp(2n, \mathbb{R}).
$$

(7.15)

Indeed consider the factorization (7.4) (since $S$ is positive definite also its restriction to an $n$ dimensional subspace is positive definite, therefore $D$ is invertible). The factorization (7.15) is obtained for example by considering the symplectic matrix

$$
g = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \sqrt{D^{-1}} & 0 \\ 0 & \sqrt{D} \end{pmatrix},
$$

(7.16)

where the matrix $\sqrt{D}$ is the unique positive definite square root of the symmetric and positive definite matrix $D$. (Notice that the same proof shows that any symmetric and symplectic matrix $\begin{pmatrix} A & B \\ B^t & D \end{pmatrix}$ with invertible and positive definite matrix $D$ is of the form $gg^t$ and therefore is positive definite).
We can now show that the coset space \( Sp(2n, \mathbb{R})/U(n) \) is the space of all positive definite symmetric and symplectic matrices. The maximal compact subgroup of \( Sp(2n, \mathbb{R}) \) is \( H := \{ g \in Sp(2n, \mathbb{R}); gg^t = \mathbb{I} \} \), and we have seen in (7.8) that it is \( U(n) \).

We then denote by \( gH \) the elements of \( Sp(2n, \mathbb{R})/U(n) \), where \( H = U(n) \), and consider the map

\[
\sigma : \frac{Sp(2n, \mathbb{R})}{U(n)} \rightarrow \{ S \in Sp(2n, \mathbb{R}); S = S^t \ and \ S \ \text{positive definite} \}
\]

\[
gH \mapsto gg^t
\]

(7.17)

This map is well defined because it does not depend on the representative \( g \in Sp(2n, \mathbb{R}) \) of the equivalence class \( gH \). Formula (7.15) shows that this map is surjective. Injectivity is also easily proven: if \( gg^t = g'g'^t \) then \( g^{-1}g(g^{-1}g)^t = 1 \), so that \( u = g^{-1}g \) is an element of \( Sp(2n, \mathbb{R}) \) that satisfies \( uu^t = 1 \). Therefore \( u = g^{-1}g \) belongs to the maximal compact subgroup \( H = U(n) \), hence \( g \) and \( g' \) belong to the same coset.

The \( \mathcal{M} \) and \( \mathcal{N} \) matrices

Notice that the \( n \times n \) matrices \( f = (f^\Lambda_a)_{\Lambda,a=1,...,n} \), are invertible. Indeed if the columns of \( f \) were linearly dependent, say \( f^\Lambda_a \psi^a = 0 \), i.e. \( f\psi = 0 \), with a nonzero vector \( \psi \), then sandwiching (7.9) between \( \psi^\dagger \) and \( \psi \) we would obtain

\[
-(f\psi)^\dagger h \psi + \psi^\dagger h^\dagger f\psi = i\psi^\dagger \psi \neq 0
\]

(7.18)

that is absurd. Similarly also the matrix \( h = (h_{\Lambda a}) \) is invertible. We can then define the invertible \( n \times n \) matrix

\[
\mathcal{N} = hf^{-1}
\]

(7.19)

that is symmetric (cf. (7.10)) and that has negative definite imaginary part (cf. (7.9))

\[
\mathcal{N} = \mathcal{N}^t , \quad \text{Im} \mathcal{N} = -\frac{i}{2}(\mathcal{N} - \mathcal{N}^t) = -\frac{1}{2}(ff^\dagger)^{-1},
\]

(7.20)

(while \( \mathcal{N}^{-1} \) has positive definite imaginary part \( \mathcal{N}^{-1} - \mathcal{N}^{-t} = i(hh^\dagger)^{-1} \)). Any symmetric matrix with negative definite imaginary part is of the form (7.19) for some \( (f, h) \) satisfying (7.9) and (7.10) (just consider any \( f \) that satisfies (7.20)). There is also a 1-1- correspondence between symmetric complex matrices \( \mathcal{N} \) with negative definite imaginary part and symmetric negative definite matrices \( \mathcal{M} \) of \( Sp(2n, \mathbb{R}) \). Given \( \mathcal{N} \) we
consider

\[
\mathcal{M}(\mathcal{N}) = \begin{pmatrix} \mathbb{1} & -\text{Re}\mathcal{N} \\ \text{Im}\mathcal{N} & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \text{Im}\mathcal{N}^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -\text{Re}\mathcal{N} \end{pmatrix} \\
= \begin{pmatrix} \text{Im}\mathcal{N} + \text{Re}\mathcal{N}\text{Im}\mathcal{N}^{-1} \text{Re}\mathcal{N} & -\text{Re}\mathcal{N}\text{Im}\mathcal{N}^{-1} \\ -\text{Im}\mathcal{N}^{-1} \text{Re}\mathcal{N} & \text{Im}\mathcal{N}^{-1} \end{pmatrix} \\
= -i \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{N}\text{Im}\mathcal{N}^{-1}\mathcal{N}^\dagger & -\mathcal{N}\text{Im}\mathcal{N}^{-1} \\ -\text{Im}\mathcal{N}^{-1}\mathcal{N}^\dagger & \text{Im}\mathcal{N}^{-1} \end{pmatrix} \\
= -i \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} - 2 \begin{pmatrix} h^\dagger & -hf^\dagger \\ -fh^\dagger & ff^\dagger \end{pmatrix} \\
= -i \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} - 2 \begin{pmatrix} -h \\ f \end{pmatrix} \begin{pmatrix} -h^\dagger \\ f^\dagger \end{pmatrix} \\
= -2 \text{Re} \left[ \begin{pmatrix} -h \\ f \end{pmatrix} \begin{pmatrix} -h^\dagger \\ f^\dagger \end{pmatrix} \right] 
\] (7.21)

Since symmetric negative definite matrices \( \mathcal{M} \) of \( Sp(2n, \mathbb{R}) \) parametrize the coset space \( Sp(2n, \mathbb{R})/U(n) \), the matrices \( \mathcal{N} \) too parametrize this coset space.

Under symplectic rotations (5.33) we have

\[
\begin{pmatrix} f \\ h \end{pmatrix} \rightarrow \begin{pmatrix} f \\ h \end{pmatrix}' = S \begin{pmatrix} f \\ h \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} \] (7.22)

and

\[
\mathcal{N} \rightarrow \mathcal{N}' = (C + D\mathcal{N})(A + B\mathcal{N})^{-1} \] (7.23)

The transformation of the imaginary part of \( \mathcal{N} \) is (recall (7.20))

\[
\text{Im}\mathcal{N} \rightarrow \text{Im}\mathcal{N}' = (A + B\mathcal{N})^{-1}\text{Im}\mathcal{N}(A + B\mathcal{N})^{-1} \] (7.24)

The transformation of the corresponding matrix \( \mathcal{M}(\mathcal{N}) \) is

\[
\mathcal{M}(\mathcal{N}) \rightarrow \mathcal{M}(\mathcal{N}') = S^\dagger \mathcal{M}(\mathcal{N}) S^{-1} , \] (7.25)

this last relation easily follows from (7.21) and from \( \begin{pmatrix} -h \\ f \end{pmatrix} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} \).

The relation between the negative definite symmetric matrix \( \mathcal{M} \) defined in (7.21) and \( S \) defined in (7.15) can be obtained from their transformation properties under \( Sp(2n, \mathbb{R}) \),

\[
\mathcal{M} = -S^{-1} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} S \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \] (7.26)

We also have \( \mathcal{M} = -V^{-\dagger}V^{-1} \). 

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7.3 Lie algebra of \( Sp(2n, \mathbb{R}) \) and \( U(n) \) \((a, b, c, d \text{ matrices})\)

If we write \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( \epsilon \) infinitesimal we obtain that the \( 2n \times 2n \) matrix

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

belongs to the Lie algebra of \( Sp(2n, \mathbb{R}) \) if \( a, b, c, d \) are real \( n \times n \) matrices that satisfy the relations

\[
a^t = -d, \quad b^t = b, \quad c^t = c.
\]

The Lie algebra of \( U(n) \) in this fundamental representation of \( Sp(2n, \mathbb{R}) \) is given by the matrices

\[
\begin{pmatrix} a & b \\ -b & a \end{pmatrix}
\]

with \( b = b^t, \quad a = -a^t \).

In the complex basis \((7.6)\) the Lie algebra of \( Sp(2n, \mathbb{R}) \) is given by the \( 2n \times 2n \) matrices

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

where \( a \) and \( b \) are complex \( n \times n \) matrices that satisfy the relations

\[
a^\dagger = -a, \quad b^t = b.
\]

The Lie algebra of \( U(n) \) in this complex basis is given by the matrices \( \begin{pmatrix} a & 0 \\ 0 & -\bar{a} \end{pmatrix} \) with \( a^\dagger = -a \).

8 Appendix: Unilateral Matrix Equations

The remarkable symmetry property of the trace of the solution of the matrix equation (3.102) holds for more general matrix equations. This trace property and the structure of the solution itself are studied in [18], and with a different method in [70]; see also [71] for a unified approach based on the generalized Bezout theorem, and [69] for convergence of perturbative solutions of matrix equations and a new form of the noncommutative Lagrange inversion formula.

In this appendix we prove the symmetry property of the trace of certain solutions (and their powers) of unilateral matrix equations. These are \( N \)th order matrix equations for the variable \( X \) with matrix coefficients \( A_i \) which are all on one side, e.g. on the left

\[
X = A_0 + A_1X + A_2X^2 + \ldots + A_NX^N.
\]

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The matrices are all square and of arbitrary degree. We may equally consider the \( A_i \)'s as generators of an associative algebra, and \( X \) an element of this algebra which satisfies the above equation. We consider the formal solution of (8.1) obtained as the limit of the sequence \( X_0 = 0, \ X_{k+1} = A_0 + A_1 X_k + A_2 X_k^2 + \ldots + A_N X_k^N \). It is convenient to assign to every matrix a dimension \( d \) such that \( d(X) = -1 \). Using (8.1), the dimension of the matrix \( A_i \) is given by \( d(A_i) = i - 1 \).

First note that we can rewrite equation (8.1) as

\[
1 - \sum_{i=0}^{N} A_i = 1 - X - \sum_{k=1}^{N} A_k (1 - X^k)
\]

The right hand side factorizes

\[
1 - \sum_{i=0}^{N} A_i = (1 - \sum_{k=1}^{N} \sum_{m=0}^{k-1} A_k X^m) (1 - X).
\]

Under the trace we can use the fundamental property of the logarithm, even for non-commutative objects, and obtain

\[
\text{Tr} \log(1 - \sum_{i=0}^{N} A_i) = \text{Tr} \log(1 - \sum_{k=1}^{N} \sum_{m=0}^{k-1} A_k X^m) + \text{Tr} \log(1 - X).
\]

Using \( d(A_k) = k - 1 \) and \( d(X) = -1 \) we have \( d(A_k X^m) = k - m - 1 \) and we see that all the words in the argument of the first logarithm on the right hand side have semi-positive dimension. Since all the words in the expansion of the second term have negative dimension we obtain

\[
\text{Tr} \log(1 - X) = \text{Tr} \log(1 - \sum_{i=0}^{N} A_i) \bigg|_{d<0}.
\]

On the right hand side of (8.2), one must expand the logarithm and restrict the sum to words of negative dimension. Since \( d(X^r) = -r \) by extracting the dimension \( d = -r \) terms from the right hand side of (8.2) we obtain

\[
\text{Tr} \phi^r = \sum_{\{a_i\}} \frac{\left(\sum_{i=0}^{N} a_i - 1\right)!}{a_0! a_1! \ldots a_N!} \text{Tr} \mathcal{S} (A_0^{a_0} A_1^{a_1} \ldots A_N^{a_N}).
\]

The relevant point is that all the terms in the expansion of \( \text{Tr} \log(1 - \sum_{i=0}^{N} A_i) \) are automatically symmetrized, this explains the symmetrization operator \( \mathcal{S} \) in the \( A_0, A_1, \ldots A_N \) matrix coefficients.
If the coefficient $A_N$ is unity, we have the following identity for the symmetrization operators of $N+1$ and of $N$ coefficients (words)

$$S(A_0 a_0 A_1 a_1 \ldots A_N a_N) |_{A_N=1} = S(A_0 a_0 A_1 a_1 \ldots A_N a_N) .$$

This is obviously true up to normalization; the normalization can be checked in the commutative case.

The trace of the solution of (3.102) can now be obtained from (8.3) by considering $r = 1$ and $N = 2$ and by setting $A_2$ to unity.

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