Weight filtration of the limit mixed Hodge structure at infinity for tame polynomials

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Introduction

Let \( X = \mathbb{C}^n \) (\( n \geq 2 \)), and \( S = \mathbb{C} \). Let \( f : X \to S \) be a cohomologically tame polynomial map in the sense of [Sab3], i.e. there is a relative compactification \( \overline{f} : \overline{X} \to S \) of \( f \) such that \( \overline{f} \) is proper and the support of \( \varphi_{\overline{f}} \cdot j_* \mathbb{Q}_X \) is contained in the affine space \( X \) (hence discrete) for any \( c \in \mathbb{C} \), where \( j : X \hookrightarrow \overline{X} \) denotes the inclusion.

With the above notation and assumption, the weight filtration

\[
W(0_m) \text{ filtration shifted by } H
\]

Proposition 1. We thus get the following well-known assertion (see also Appendix of [MT]):

\[
\text{Similarly for the non-unipotent part the weight filtration}
\]

\[
W(0_m) \text{ filtration shifted by } H
\]

\[
\text{for tame polynomials}
\]

\[
\text{H}_s := H^m(X_s, \mathbb{Q}), \quad H^c_s := H^m_c(X_s, \mathbb{Q})
\]

They have the weight filtration \( W \), see [De2]. We have by [DS], Th. 0.3

\[
\{ \text{Gr}^W_{m+k} H_s \}_{s \in S'}, \{ \text{Gr}^W_{m-k} H^c_s \}_{s \in S'} \text{ are constant on } S' \text{ if } k \neq 0.
\]

In fact, the argument in loc. cit. implies that \( \{ H_s / \text{Gr}_m H_s \}_{s \in S'} \) and \( \{ \text{Gr}_m H^c_s \}_{s \in S'} \) are constant, see also (1.1) below.

Let \( H_{\infty} \) be the limit mixed Hodge structure of \( H_s \) for \( s \to \infty \), and similarly for \( H^c_{\infty} \), see [St1], [St2], [SZ]. Set \( N := (2\pi i)^{-1} \log T_u \) with \( T_u \) the unipotent part of the monodromy at infinity. This is an endomorphism of type \((-1, -1)\) of \( H_{\infty}, H^c_{\infty} \). Let \( L \) be the filtration on \( H_{\infty}, H^c_{\infty} \) induced by the weight filtration \( W \) respectively on \( H_s, H^c_s \) for \( s \in S' \). Then the weight filtration \( W \) on \( H_{\infty}, H^c_{\infty} \) coincides with the relative monodromy filtration of \( (L, N) \), see [SZ]. In particular, \( W \) on \( \text{Gr}_m H_{\infty}, \text{Gr}_m H^c_{\infty} \) coincides with the monodromy filtration shifted by \( m \) (i.e. with center \( m \)).

Let \( H_{\infty,1}, H^c_{\infty,1} \) respectively denote the unipotent monodromy part of \( H_{\infty}, H^c_{\infty} \), and similarly for the non-unipotent part \( H_{\infty,\neq 1}, H^c_{\infty,\neq 1} \). By (0.1) we have

\[
H_{\infty,\neq 1} = \text{Gr}_m H_{\infty, \neq 1}, \quad H^c_{\infty, \neq 1} = \text{Gr}_m H^c_{\infty, \neq 1}.
\]

We thus get the following well-known assertion (see also Appendix of [MT]):

**Proposition 1.** With the above notation and assumption, the weight filtration \( W \) on \( H_{\infty, \neq 1}, H^c_{\infty, \neq 1} \) coincides with the monodromy filtration shifted by \( m \).
In this paper we give three new proofs of the following.

**Theorem 1** (C. Sabbah). *With the above notation and assumption, the weight filtration \( W \) on \( H_{\infty,1} \), \( H_{\infty,1}^c \) coincides with the monodromy filtration shifted by \( m+1 \) and \( m-1 \) respectively, and we have the isomorphisms of mixed Hodge structures for \( k \geq 1 \):

\[
\text{Gr}_{m+k}^W H_s \cong \text{Gr}_{m+k}^W \text{Coker}(N|H_{\infty,1}), \quad \text{Gr}_{m-k}^W H_s^c \cong \text{Gr}_{m-k}^W \text{Ker}(N|H_{\infty,1}^c),
\]

where \( \text{Coker}(N|H_{\infty,1}) \) is a quotient of \( H_{\infty,1}(-1) \) and \( \text{Ker}(N|H_{\infty,1}^c) \subset H_{\infty,1}^c \).

Note that the assertions on \( H_{\infty,1} \) and \( H_{\infty,1}^c \) are dual of each other. The last assertion of Theorem 1 means that the primitive part of the graded pieces of the monodromy filtration on \( H_{\infty,1} \) is given by \( \text{Gr}_{m+k}^W H_s \) for \( k \geq 1 \), and the coprimitive part for \( H_{\infty,1}^c \) by \( \text{Gr}_{m-k}^W H_s^c \).

Theorem 1 was first obtained by C. Sabbah as a corollary of [Sab 3], Th. 13.1 where he uses a theory of Fourier transformation, Brieskorn lattices, and spectra at infinity, which was developed by him (see also [Sab1], [Sab2]). Recently another proof has been given also by him in Appendix of [MT] without using Brieskorn lattices or spectra at infinity, but using Fourier transformation where irregular \( D \)-modules inevitably appear.

It does not seem, however, that the above theory is absolutely indispensable for the proof of Theorem 1. In fact, the theorem was almost proved in [DS] where the following was shown (see also [Di1], 4.3–5):

**Theorem 2** ([DS], Th. 0.3). *With the above notation and assumption, let \( \nu_k \) and \( \nu'_k \) denote the number of Jordan blocks of size \( k \) for the monodromy on \( H_{\infty,1} \) and \( \text{Gr}_m^L H_{\infty,1} \) respectively. Let \( s \in S' \). Then

\[
\nu_k = \dim \text{Gr}_{m+k}^W H_s, \quad \nu'_k = \nu_{k+1} \quad \text{for any } k \geq 1.
\]

We give the first proof of Theorem 1 in this paper by showing that Theorem 2 implies Theorem 1 using some lemma of linear algebra, see (1.3–4) below.

The second proof of Theorem 1 in this paper uses a geometric argument together with duality, and is quite different from (and perhaps more intuitive than) the one in the proof of Th. 0.3 in [DS]. It is finally reduced to the following:

**Proposition 2.** *Let \( \iota_s : H_s^c \rightarrow H_s \) be the natural morphism of mixed Hodge structures for \( s \in S' \). Then it induces an isomorphism of Hodge structures

\[
\text{Gr}_m^W \iota_s : \text{Gr}_m^W H_s^c \xrightarrow{\sim} \text{Gr}_m^W H_s \quad \text{for } s \in S'.
\]

We give two proofs of Proposition 2 in this paper. One proof uses the semisimplicity of pure Hodge modules together with a certain property of the mixed Hodge module \( H^0 f_* (Q_{h,X}[n]) \) coming from the cohomologically tame condition. (For \( Q_{h,X} \), see (1.1) below.) Another proof uses the positivity of the polarization on the primitive cohomology of a compact Kähler manifold together with Hironaka’s resolution of singularities.

The third proof of Theorem 1 in this paper is given as a corollary of Theorem 3 below, which holds for any pure Hodge module \( \mathcal{M} \) of weight \( n \) on \( S \) without a constant direct factor, and was proved by C. Sabbah in Appendix of [MT]. The proof in this paper uses the notion of representative functor and the universal extension \( \widetilde{\mathcal{M}} \) of a pure Hodge module.
\( \mathcal{M} \) by a constant mixed Hodge module (see (3.1) below), but Fourier transformation is not used. Note that \( \tilde{\mathcal{M}} \) was defined in loc. cit. by using a sheaf-theoretic operation explicitly.

**Theorem 3** (C. Sabbah). Let \( \mathcal{M} \) be a pure Hodge module of weight \( n \) on \( S \) having no constant direct factor, i.e. \( H^{-1}(S, \mathcal{M}) = 0 \). Let \( \tilde{\mathcal{M}} \) be the universal extension of \( \mathcal{M} \) by a constant mixed Hodge module on \( S \). Set \( H_{1} := \psi_{1/t,1} \mathcal{M} \). Then the weight filtration on \( H_{1} \) coincides with the monodromy filtration shifted by \( n \), and the \( N \)-primitive part \( \text{Gr}_{W}^{W} H_{1} \) is given by \( \text{Gr}_{W}^{k} H^{0}(S, \mathcal{M}) \) for any \( k \) where they vanish unless \( k \geq n \).

Note that we have by the property of the universal extension
\[
\tilde{\mathcal{M}} / \mathcal{M} = a_{X}^{*} H^{0}(S, \mathcal{M})[1], \quad H^{-1}(S, \tilde{\mathcal{M}} / \mathcal{M}) = H^{0}(S, \mathcal{M}).
\]

The proof of Theorem 3 is reduced to the comparison between the global universal extension on \( S \) and the local one on a neighborhood of \( \infty \in \mathbb{P}^{1} \) for the underlying perverse sheaves, where some argument is similar to the one in the proof of [DS], Th. 0.3.

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In Section 1 we explain some basics on cohomologically tame polynomials, and give the proof of Theorem 1 using Theorem 2 after showing Lemma (1.3). In Section 2 we give two proofs of Proposition 2, and then a geometric proof of Theorem 1 after showing Lemma (2.3). In Section 3 we explain the notion of a universal extension by a constant sheaf, and then prove Theorem 3 which implies Theorem 1. In Appendix we give some remarks about the limit mixed Hodge structure and the spectrum.

### 1. Cohomologically tame polynomials

In this section we explain some basics on cohomologically tame polynomials, and give the proof of Theorem 1 using Theorem 2 after showing Lemma (1.3).

**1.1. Some basics on cohomologically tame polynomials.** With the notation in the introduction, let \( j : X \hookrightarrow \overline{X} \) be the inclusion. Note that \( Rj_{!} Q_{X}[n] \) is a perverse sheaf since \( j \) is an affine open immersion, see [BBD]. The intersection complex \( IC_{X} Q \) is a subobject of the perverse sheaf \( Rj_{!} Q_{X}[n] \) (see loc. cit.) and the vanishing cycle functor \( \varphi_{f-} \) (see De2]) is an exact functor (up to a shift). So we get the first inclusion of

\[
\text{supp } \varphi_{f-} IC_{X} Q \subset \text{supp } \varphi_{f-} Rj_{!} Q_{X}[n] = \text{supp } \varphi_{f-} Rj_{!} Q_{X}[n].
\]

For the last isomorphism, we have the relation \( D \circ Rj_{*} = Rj_{!} \circ D \) and the compatibility of \( \varphi_{f-} \) with the dualizing functor \( D \), i.e. \( D \circ \varphi = \varphi \circ D \) where a Tate twist may appear depending on the eigenvalue of the monodromy, see e.g. [Sai1], 5.2.3.

Assume \( f \) is a cohomologically tame polynomial in the sense of [Sab3]. Then the above supports are contained in the affine space \( X \), and is discrete. Since the restrictions of the above perverse sheaves to \( X \) are the same, and \( \varphi \) commutes with the direct images by proper morphisms, we see that the direct image by \( f_{*} \) of the mapping cones of the canonical morphisms

\[
Rj_{!} Q_{X}[n] \to Rj_{*} Q_{X}[n], \quad Rj_{!} Q_{X}[n] \to IC_{X} Q, \quad IC_{X} Q \to Rj_{!} Q_{X}[n],
\]

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are direct sums of constant sheaves on $S$.

Let $Q_{h,X}$ denote the object in $D^b\text{MHM}(X)$ (the bounded derived category of mixed Hodge modules on $X$) which is uniquely characterized by the following two conditions: Its underlying $Q$-complex is $Q_X$, and its $0$-th cohomology $H^0(X, Q_{h,X}) := H^0(a_X)_* Q_{h,X}$ has weight $0$, see [Sa13], 4.4.2.

Replacing $Q_X$ with $Q_{h,X}$ in (1.1.2), the above assertion holds in $D^b\text{MHM}(S)$. Indeed, any admissible variation of mixed Hodge structure $M$ on $S$ is a constant variation by using the mixed Hodge structure on $H^0(S, M)$. This means that $M = a_S^* H$ for $H \in \text{MHS}$ (the category of graded-polarizable mixed Hodge $Q$-structures in [De2]), where $a_S : S \to pt$ is the canonical morphism and $\text{MHM}(pt)$ is naturally identified with $\text{MHS}$. Moreover, we have for $i > 1$

$$(1.1.3) \quad \text{Ext}^i_{\text{MHM}(S)}(a_S^* H, a_S^* H') = \text{Ext}^i_{\text{MHS}}(H, (a_S)_* a_S^* H') = \text{Ext}^i_{\text{MHS}}(H, H') = 0,$$

since $\text{Ext}^i = 0$ ($i > 1$) in $\text{MHS}$ by a well-known corollary of a theorem of Carlson [Ca]. (Indeed, the latter implies the right-exactness of the functor $\text{Ext}^1_{\text{MHS}}(Q, \cdot)$.) So the desired decomposition follows by using the canonical filtration $\tau_{\leq k}$ (see [De2]) on the complex.

1.2. Remark. For $f : X \to S$ as in (1.1), we have the Leray spectral sequence of mixed Hodge structures

$$(1.2.1) \quad E_2^{i,j} = H^i(S, H^j f_*(Q_{h,Z}[n])) \Rightarrow H^{i+j+n}(X, Q).$$

Here $E_2^{i,j} = 0$ for $i \notin [-1, 0]$ since $S = C$. So (1.2.1) degenerates at $E_2$, and we get

$$(1.2.2) \quad H^i(S, H^j f_*(Q_{h,Z}[n])) = 0 \quad \text{for } (i, j) \neq (-1, 1 - n),$$

since $X = C^n$ and $H^j f_*(Q_{h,Z}[n]) = 0$ for $j \leq -n$ (using the classical $t$-structure).

If $f$ is cohomologically tame, then (1.2.2) implies

$$(1.2.3) \quad H^j f_*(Q_{h,Z}[n]) = 0 \quad \text{unless } j = 1 - n \text{ or } 0,$$

using the exactness of $\varphi$ (up to a shift) together with the commutativity of $\varphi$ and the direct image under proper morphisms as in (1.1).

1.3. Lemma. Let $V_\bullet$ be a finite dimensional graded $Q$-vector space with an action of $N$ of degree $-2$, i.e. $N(V_k) \subset V_{k-2}$. Let $V'_\bullet$ be a graded vector subspace stable by $N$. Set $V''_\bullet := V_\bullet / V'_\bullet$. Let $m$ be an integer. Assume the action of $N$ on $V''_\bullet$ vanishes, and

$$N^k : V'_{m+k} \sim V'_{m-k} \quad \text{for any } k \geq 1.$$

Set $C'_k := \text{Coker}(N : V'_{m+k+2} \to V'_{m+k})$ so that $N$ induces $\delta_k : V''_{m+k+2} \to C'_{m+k}$. Let $\nu_k$ be the number of Jordan blocks of size $k$ for the action of $N$ on $V_\bullet$. Then

$$\nu_{k+1} = \begin{cases} \dim \text{Coker} \delta_0 + \sum_{j=1} \dim \text{Ker} \delta_j & \text{if } k = 0, \\ \dim \text{Coker} \delta_k + \dim \text{Im} \delta_{k-1} & \text{if } k \geq 1. \end{cases}$$

Proof. Let $0 V'_{m+k}$ be the primitive part defined by $\text{Ker} N^{k+1} \subset V'_{m+k}$ for $k \geq 0$. We have the primitive decomposition

$$(1.3.1) \quad V'_\bullet = \bigoplus_{k \geq 0} \left( \bigoplus_{j=1}^k N^j_0 V'_{m+k} \right) \quad \text{with } \quad 0 V'_{m+k} \sim C'_{m+k}.$$
Set
\[ n_k = \dim \text{Im} \delta_k. \]

For each \( k \geq 0 \), there are bases \( \{v_{k,j}'\}_j \) of \( V'_{m+k} (= C'_{m+k}) \) and \( \{v''_{k,j}\}_j \) of \( V''_{m+k+2} \) together with lifts \( v_{k,j} \) of \( v''_{k,j} \) in \( V_{m+k+2} \) such that

\[
Nv_{k,j} = \begin{cases} v_{k,j}' & \text{if } 1 \leq j \leq n_k, \\ 0 & \text{otherwise.} \end{cases}
\]

Indeed, by the definition of \( \delta_k \), the assertion is trivial if we consider the equality modulo \( NV_k \), i.e. if we add the term \(+Nu_{k,j}\) for some \( u_{k,j} \in V'_{m+k+2} \) on the right-hand side of (1.3.2). Then we can replace the lift \( v_{k,j} \) of \( v''_{k,j} \) with \( v_{k,j} - u_{k,j} \), and (1.3.2) is proved.

The assertion of Lemma (1.3) now follows from (1.3.1) and (1.3.2).

1.4. Proof of Theorem 1 using Theorem 2. We show the assertion for \( H_{\infty,1} \) since this implies the assertion for \( H_{\infty,1}^c \) by duality. We can replace \( H_{\infty,1} \) with the graded pieces \( \text{Gr}^W_k H_{\infty,1} \) in order to define \( \nu_k, \nu'_k \), since \( W \) is strictly compatible with \( N^k \) for any \( k \geq 1 \). We then apply Lemma (1.3) to

\[
V_k = \text{Gr}^W_k H_{\infty,1}, \quad V'_k = \text{Gr}^W_k \text{Gr}^L_k H_{\infty,1}, \quad V''_k = \begin{cases} \text{Gr}^L_k H_{\infty,1} & \text{if } k > m, \\ 0 & \text{if } k \leq m. \end{cases}
\]

Here \( \text{Gr}^W_j \text{Gr}^L_k H_{\infty,1} = 0 \) for \( j \neq k \) and \( k \neq m \), since \( N = 0 \) on \( H_{\infty,1}/L_m H_{\infty,1} \).

Using the primitive decomposition (1.3.1), we get

\[
(1.4.1) \quad \nu'_{k+1} = \dim C'_{m+k} \quad \text{for } k \geq 0.
\]

Then Theorem 2 together with Lemma (1.3) imply the isomorphism

\[
(1.4.2) \quad \delta_k : V''_{m+k+2} \overset{\sim}{\longrightarrow} C'_{m+k} \quad \text{for } k \geq 0.
\]

Indeed, by (1.4.1) the surjectivity of \( \delta_k \) is equivalent to

\[
(1.4.3) \quad \nu'_{k+1} = \dim \text{Im} \delta_k \quad \text{for } k \geq 0,
\]

and we have by Theorem 2 and Lemma (1.3)

\[
\nu'_{k+1} = \nu_{k+2} = \dim \text{Coker} \delta_{k+1} + \dim \text{Im} \delta_k \quad \text{for } k \geq 0.
\]

So (1.4.3) follows by decreasing induction on \( k \geq 0 \). By Lemma (1.3) together with the surjectivity of \( \delta_0 \) we get

\[
\nu_1 = \dim V''_{m+1} + \sum_{k \geq 0} \dim \text{Ker} \delta_k,
\]

since \( \delta_{-1} \) vanishes. We have moreover \( \nu_1 = \dim V''_{m+1} \) by Theorem 2. So the injectivity of \( \delta_k \) \((k \geq 0)\) follows. Thus (1.4.2) is proved.

Then the primitive decomposition of \( V'_{m+1} \) can be lifted to that of \( V_{m+1+1} \) by the argument as in the proof of Lemma (1.3). We have the last assertion of Theorem 1 since the \( \delta_k \) \((k \geq 0)\) induce isomorphisms of mixed Hodge structures. We thus get the first proof of Theorem 1 in this paper.
2. Geometric proof of Theorem 1

In this section we give two proofs of Proposition 2, and then a geometric proof of Theorem 1 after showing Lemma (2.3).

2.1. One proof of Proposition 2. Consider the following morphisms of mixed Hodge modules on $S$:

\[ (2.1.1) \quad \mathcal{M}_! := H^0 j^!(\mathcal{Q}_{h,X}[n]) \xrightarrow{u^\ast} H^0 j^! \mathcal{Q}_{h,X} \xrightarrow{v^\ast} \mathcal{M}_* := H^0 j^!(\mathcal{Q}_{h,X}[n]). \]

These are induced by the canonical morphisms of mixed Hodge modules on $X$ whose underlying morphisms are as in (1.1.2):

\[ (2.1.2) \quad j^! \mathcal{Q}_{h,X}[n] \xrightarrow{u} IC_X \mathcal{Q}_h \xrightarrow{v} j^* \mathcal{Q}_{h,X}[n]. \]

By (1.1) the kernel and cokernel of $u'$ and $v'$ are constant mixed Hodge modules on $S$.

By the formalism of mixed Hodge modules (see e.g. [Sai3], 2.2 6) we have

\[ (2.1.3) \quad Gr_{n+k}^W (j^! \mathcal{Q}_{h,X}[n]) = Gr_{n-k}^W (j^* \mathcal{Q}_{h,X}[n]) = 0 \text{ if } k > 0, \]

and moreover

\[ (2.1.4) \quad Gr_n^W (j^! \mathcal{Q}_{h,X}[n]) = Gr_n^W (j^* \mathcal{Q}_{h,X}[n]) = IC_X \mathcal{Q}_h. \]

(Indeed, for the last assertion, we use $\text{Hom}(\mathcal{M}', j_* \mathcal{Q}_{h,X}) = \text{Hom}(j^* \mathcal{M}', \mathcal{Q}_{h,X}) = 0$ for any mixed Hodge module $\mathcal{M}'$ supported on $X \setminus X$, and similarly for the dual assertion.) Note that the weight filtration $W$ on $\mathcal{M}_!$ and $\mathcal{M}_*$ are induced by the weight filtration $W$ on $(j^! \mathcal{Q}_{h,X}[n])$ and $(j^* \mathcal{Q}_{h,X}[n])$ respectively.

By (1.1) the kernel and the cokernel of the canonical morphism

\[ (2.1.5) \quad Gr_n^W \mathcal{M}_! \rightarrow Gr_n^W \mathcal{M}_* \]

are constant Hodge modules on $S$. If the cokernel is nonzero, then there is a nontrivial constant Hodge module contained in $\mathcal{M}_*$ using the semisimplicity of $Gr_n^W \mathcal{M}_*$. However, this contradicts the property that $H^{-1}(S, \mathcal{M}_*) = 0$ which follows from the condition that $X = \mathbb{C}^n$ by using the Leray spectral sequence, see Remark (1.2). So we get the surjectivity. For the injectivity we apply the dual argument. Restricting over a general $s \in S$, we then get the desired isomorphism.

2.2. Another proof of Proposition 2. Set $Y = X_s$. More generally, let $Y$ be a smooth variety which is the complement of an ample effective divisor $E$ on a projective variety $\overline{Y}$. Under this assumption, we show the bijectivity of the canonical morphism

\[ (2.2.1) \quad Gr_n^W H^m_{\text{c}}(Y) \rightarrow Gr_n^W H^m(Y). \]

We have a smooth projective compactification $\overline{Y}$ of $Y$ such that $D := \overline{Y} \setminus Y$ is a divisor with simple normal crossings. This is obtained by using Hironaka’s resolution $\sigma: (\overline{Y}, D) \rightarrow (\overline{Y}, E)$ which is a projective morphism. Let $D_\sigma$ be a relatively ample divisor for $\sigma$. We may replace $D_\sigma$ with $D_\sigma - \sigma^* \sigma_* D_\sigma$ so that its support is contained in $D$. Then $k \sigma^* E + D_\sigma$
is an ample divisor on $\tilde{Y}$ for $k \gg 0$. Since its support is contained in $D$, it is a linear combination of the irreducible components $D_i$ of $D$.

By Deligne’s construction of mixed Hodge structure on $H^\ast(\tilde{Y})$ (see [De2]) together with duality, we have

\begin{equation}
Gr^W_m H^m_c(Y) = \text{Ker}(H^m(\tilde{Y}) \rightarrow \bigoplus_i H^m(D_i)),
\end{equation}

\begin{equation}
Gr^W_m H^m(Y) = \text{Coker}(\bigoplus_i H^{m-2}(D_i)(-1) \rightarrow H^m(\tilde{Y})).
\end{equation}

So the assertion is equivalent to the non-degeneracy of the restriction of the natural pairing on the middle cohomology $H^m(\tilde{Y})$ to the kernel of the morphism $H^m(\tilde{Y}) \rightarrow \bigoplus_i H^m(D_i)$. By Hodge theory, it is enough to show that this kernel is contained in the primitive part with respect to the above ample divisor. But it is clear since the action of the cohomology class of each $D_i$ is given by composing the restriction morphism $H^\ast(\tilde{Y}) \rightarrow H^\ast(D_i)$ with its dual. So the assertion follows. This finishes another proof of Proposition 2.

For the geometric proof of Theorem 1 in this section, we also need the following.

**Lemma 2.3.** Let $H$ be a mixed Hodge structure, and $L$ an increasing filtration on $H$. Let $N$ be a nilpotent endomorphism of type $(-1,-1)$ of $H$ preserving the filtration $L$. Assume the relative monodromy filtration $W$ for $(L,N)$ exists, and $W$ coincides with the weight filtration of the mixed Hodge structure $H$. Let $m$ be an integer such that $H = L_m H$. Assume the action of $N$ on $L_{m-1} H$ vanishes, and

\begin{equation}
\dim Gr^W_{m-k}(\text{Ker} N) = \dim Gr^W_{m+k}(\text{Coker} N) \quad (k \geq 1),
\end{equation}

\begin{equation}
Gr^W_{m-k}(\text{Ker} N) = Gr^W_{m+k}(\text{Coker} N) = 0 \quad (k \leq 0),
\end{equation}

where $\text{Coker} N$ is a quotient of $H(-1)$, and $\text{Ker} N \subset H$. Then $W$ coincides with the monodromy filtration with center $m-1$.

**Proof.** Set $H' := L_{m-1} H$, $H'' := Gr^L_m H$. The hypothesis on the action of $N$ on $H'$ implies that

\begin{equation}
Gr^W_k Gr^L_i H' = 0 \quad (k \neq i), \text{ i.e. } W = L \text{ on } H'.
\end{equation}

Set $H_k := Gr^W_k H$, and similarly for $H'_k, H''_k$. Set

$$K_k := \text{Ker}(N : H_k \rightarrow H_{k-2}(-1)), \quad C_k := \text{Coker}(N : H_k \rightarrow H_{k-2}(-1)),$$

and similarly for $K''_k, C'_k$. Note that $Gr^W_k$ commutes with Ker and Coker by the strict compatibility of the weight filtration $W$.

Applying the snake lemma to the action of $N$ on $0 \rightarrow H' \rightarrow H \rightarrow H'' \rightarrow 0$, we get the following long exact sequence for any $k \in \mathbb{Z}$

\begin{equation}
0 \rightarrow H'_k \rightarrow K_k \rightarrow K''_k \xrightarrow{\partial} H'_{k-2}(-1) \rightarrow C_k \rightarrow C''_k \rightarrow 0.
\end{equation}

Here $H'_k = K_k = 0$ for $k \geq m$, and $C_k = 0$ for $k \leq m$ by hypothesis.

We show the following isomorphisms by decreasing induction on $k \geq 0$:

\begin{equation}
H'_{m-k} \xrightarrow{\sim} K_{m-k}, \quad \partial : K''_{m-k} \xrightarrow{\sim} H'_{m-k-2}(-1).
\end{equation}
Here it is enough to show that \( \dim K''_{m-k} = \dim H'_{m-k-2} \), using the long exact sequence (2.3.3) since the surjectivity of \( \partial \) follows from the vanishing of \( C_{k-2} \) for \( k \leq m \).

For \( k \gg 0 \), the assertion trivially holds since all the terms are zero. Assume the isomorphisms hold with \( k \) replaced by \( k + 2 \). We have the following equalities for \( k \geq 0 \):

\[
\dim K''_{m-k} = \dim C''_{m+k+2} = \dim C_{m+k+2} = \dim K_{m-k-2} = \dim H'_{m-k-2}.
\]

Indeed, the first equality follows from the property of the monodromy filtration on \( H'' \), the second from the long exact sequence (2.3.3) together with the hypothesis that \( H'_{k-2} = 0 \) for \( k \geq m + 2 \), the third from the hypothesis (2.3.1) of the lemma, and the last from the inductive hypothesis. So the two isomorphisms in (2.3.4) hold for \( k \geq 0 \).

By (2.3.4) the primitive decomposition of \( \bigoplus_k H''_k \) with center \( m \) can be lifted to the primitive decomposition of \( \bigoplus_k H_k \) with center \( m - 1 \) under the surjection \( H \to H'' \), since \( K'' \) is the coprimitive part of \( H''(k \leq m) \). This finishes the proof of Lemma (2.3).

### 2.4. Proof of Theorem 1.

We show the assertion for \( H^c_\infty \) since that for \( H_\infty \) follows from this using duality. Consider first the following canonical morphisms

\[
\mathbf{R} \Gamma_c(S, f_! Q_{h,X}) \xrightarrow{\alpha} \mathbf{R} \Gamma(S, f_! Q_{h,X}) \xrightarrow{\beta} \mathbf{R} \Gamma(S, f_* Q_{h,X}).
\]

Set

\[
\gamma = \beta \circ \alpha : \mathbf{R} \Gamma_c(S, f_! Q_{h,X}) \to \mathbf{R} \Gamma(S, f_* Q_{h,X}).
\]

By the octahedral axiom of the derived category, we get a distinguished triangle

\[
(2.4.1) \quad C(\alpha) \to C(\gamma) \to C(\beta) \xrightarrow{+1}.
\]

By (1.1) the following mapping cone is a direct sum of constant sheaves on \( S \):

\[
C(f_! Q_{h,X} \to f_* Q_{h,X}) = C(\mathbf{R} \bar{f}_* j_! Q_{h,X} \to \mathbf{R} \bar{f}_* j_* Q_{h,X}),
\]

and this holds in \( D^b \text{MHM}(S) \). Moreover, the stalk at \( s \in S' \) of the mapping cone is given by the cohomology of the mapping cone

\[
C(\mathbf{R} \Gamma_c(X_s, \mathbf{Q}) \to \mathbf{R} \Gamma(X_s, \mathbf{Q})) \quad (s \in S'),
\]

using the generic base change by the inclusion \( \{s\} \hookrightarrow S \).

We then get the following isomorphisms in the derived category of graded-polarizable mixed Hodge structures \( D^b \text{MHS} \):

\[
C(\alpha) = C(N : \psi_{1/1,1} f_! Q_{h,X} \to \psi_{1/1,1} f_! Q_{h,X}(-1))[-1],
\]

\[
(2.4.2) \quad C(\beta) = C(\mathbf{R} \Gamma_c(X_s, \mathbf{Q}) \to \mathbf{R} \Gamma(X_s, \mathbf{Q})) \quad (s \in S'),
\]

\[
C(\gamma) = \mathbf{Q} \oplus \mathbf{Q}(-n)[3 - 2n].
\]

Indeed, the first isomorphism follows from

\[
C(j'_! \mathcal{M} \to j'_* \mathcal{M}) = C(N : \psi_{1/1,1} \mathcal{M} \to \psi_{1/1,1} \mathcal{M}(-1))[-1],
\]
for any mixed Hodge module $M$ on $S$ where $j': S \hookrightarrow \overline{S} := \mathbb{P}^1$ is the inclusion, see [Sai3], 2.24. (In this paper the nearby and vanishing cycle functors $\psi, \varphi$ for mixed Hodge modules are compatible with those for the underlying $\mathbb{Q}$-complexes without any shift of complexes, and do not preserve mixed Hodge modules.) The second isomorphism of (2.4.2) follows from the above argument on the mapping cone, and the last isomorphism of (2.4.2) from

$$R\Gamma_c(X, \mathbb{Q}) = R\Gamma_c(S, f_! \mathbb{Q}_{h,x}), \quad R\Gamma(X, \mathbb{Q}) = R\Gamma(S, f_* \mathbb{Q}_{h,x}).$$

Set $m = n - 1$. With the notation in the main theorem, we have the decompositions

$$Rc\Gamma(X_s, \mathbb{Q}) \cong H^c_{s}[-m] \oplus \mathbb{Q}(-m)[-2m], \quad R\Gamma(X_s, \mathbb{Q}) \cong H_{s}[-m] \oplus \mathbb{Q},$$

using the vanishing of $\text{Ext}^i (i > 1)$ in MHS together with the filtration $\tau_{\leq k}$ as above. We also have

$$\psi_1/\iota_1 f_! \mathbb{Q}_{h,x} \cong H^c_{\infty,1}[-m] \oplus \mathbb{Q}(-m)[-2m].$$

Let $\iota_s: H^c_s \to H_s$ denote the canonical morphism. The distinguished triangle (2.4.1) is then equivalent to the isomorphism in $D^b\text{MHS}$:

$$(2.4.3) \quad C(N : H^c_{\infty,1} \to H^c_{\infty,1}(-1)) \cong C(\iota_s : H^c_s \to H_s).$$

Note that $\text{Ker} \iota_s$ and $\text{Coker} \iota_s$ for $s \in S'$ are extended to constant variations of mixed Hodge structures over $S$ by (1.1).

By duality we have

$$(2.4.4) \quad D(\text{Gr}^W_{m-k} H^c_s) = (\text{Gr}^W_{m+k} H_s)(m) \quad \text{for } k \geq 0.$$

Since $X_s$ is smooth affine, we have

$$\text{Gr}^W_{m-k} H^c_s = \text{Gr}^W_{m+k} H_s = 0 \quad \text{for } k < 0.$$

This implies that $\text{Gr}^W_{m+k} \iota_s = 0 (k \neq 0)$, and $\text{Gr}^W_{m} \iota_s$ is an isomorphism by Proposition 2. Combining these with the isomorphism (2.4.3) in $D^b\text{MHS}$, we get

$$(2.4.5) \quad \text{Gr}^W_{m+k} (\text{Ker} N) \cong \begin{cases} \text{Gr}^W_{m+k} H^c_s & \text{if } k < 0, \\ 0 & \text{if } k \geq 0, \end{cases}$$

$$\text{Gr}^W_{m+k} (\text{Coker} N) \cong \begin{cases} \text{Gr}^W_{m+k} H_s & \text{if } k > 0, \\ 0 & \text{if } k \leq 0, \end{cases}$$

where $\text{Coker} N$ is a quotient of $H^c_{\infty,1}(-1)$. Using also the above duality (2.4.4), we thus get

$$D(\text{Gr}^W_{m-k} (\text{Ker} N)) = (\text{Gr}^W_{m+k} (\text{Coker} N))(m) \quad \text{for } k > 0.$$

Then, applying Lemma (2.3) to $H = H^c_{\infty,1}$ where $L_k H$ is identified with $W_k H^c_s$ for $s \in S'$, we get the second proof of Theorem 1 in this paper.
3. Universal extensions by constant sheaves

In this section we explain the notion of a universal extension by a constant sheaf, and then prove Theorem 3 which implies Theorem 1.

3.1. Universal extensions by constant sheaves over affine line. Let $\mathcal{M}$ be any pure Hodge module of weight $n$ on $S = \mathbb{C}$ having no constant direct factors, i.e. $H^{-1}(S, \mathcal{M}) = 0$. Note that $H^j(S, \mathcal{M}) = 0$ for $j > 0$ since $S$ is affine.

Consider the functor

$$E_M(H) := \text{Ext}^1_{\text{MHM}(S)}(H_S[1], \mathcal{M})$$

for $H \in \text{MHS}$, where $H_S := a^*_S H$ with $a_S : S \to \text{pt}$ the projection. Set

$$H_M := H^0(S, \mathcal{M}), \quad (H_M)_S := a^*_S H_M.$$

Using the adjunction for $a_S : S \to \text{pt}$, we get the functorial isomorphism

$$(3.1.1) \quad E_M(H) \sim \text{Ext}^1_{\text{MHS}}(H[1], (a_S)_* \mathcal{M}) = \text{Hom}_{\text{MHS}}(H, H_M).$$

Here the first isomorphism is given by taking the direct image of $u : H_S[1] \to \mathcal{M}$ by $a_S$, and then composing it with the canonical morphism $H[1] \to (a_S)_* a^*_S H[1]$. The second isomorphism follows from the vanishing of $H^j(S, \mathcal{M})$ for $j \neq 0$.

By (3.1.1), $E_M$ is represented by $H_M$. This means that there is the universal extension $\mathcal{M}$ of $\mathcal{M}$ by a constant mixed Hodge module on $S$ corresponding to the identity on $H_M$, and we have the short exact sequence in $\text{MHM}(S)$:

$$(3.1.2) \quad 0 \to \mathcal{M} \to \mathcal{M} \to (H_M)_S[1] \to 0.$$

Moreover, for any morphism $v : H \to H_M$ in $\text{MHS}$, the corresponding extension class is given by the pull-back of (3.1.2) by $v_S : H_S[1] \to (H_M)_S[1]$.

Lemma 3.2. Let $H_1 := \psi_{1/t,1} \mathcal{M}$, $\mathcal{H}_1 := \psi_{1/t,1} \mathcal{M}$ so that we have the exact sequence in $\text{MHS}$:

$$(3.2.1) \quad 0 \to H_1 \to \mathcal{H}_1 \to H^0(S, \mathcal{M}) \to 0.$$

By the action of $N := (2\pi i)^{-1} \log T_u$ together with the diagram of the snake lemma, we have the morphism

$$(3.2.2) \quad \partial'' : H^0(S, \mathcal{M}) \to \text{Coker}(N|H_1),$$

where $\text{Coker}(N|H_1)$ is a quotient of $H_1(-1)$. Assume the following condition holds:

$$(C) \quad \partial'' \text{ is surjective and } \text{Ker} \partial'' = W_n H^0(S, \mathcal{M}).$$

Then the weight filtration $W$ on $H_1$ coincides with the monodromy filtration shifted by $n$. 

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Proof. This follows from the primitive decomposition as in the proof of Lemma (1.3).

3.3. Proof of Theorem 3. By Lemma (3.2) above we have to prove condition (C). It is enough to show this condition for the underlying perverse sheaves. Let \( \mathcal{F} \) be the underlying \( \mathbb{Q} \)-perverse sheaf of \( \mathcal{M} \). Set

\[
E_{\mathcal{F}}(V) := \text{Ext}^1_{\text{Perv}(S)}(V_S[1], \mathcal{F}) \quad \text{for} \quad V \in M^f(\mathbb{Q}),
\]

where \( M^f(\mathbb{Q}) \) denotes the category of finite dimensional \( \mathbb{Q} \)-vector spaces. By a similar argument, this functor is also represented by \( H^0(S, \mathcal{F}) = H^0(S, \mathcal{M})_{\mathbb{Q}} \).

Let \( \Delta \) be a sufficiently small open disk in \( \mathbb{P}^1 \) with center \( \infty \) such that \( \mathcal{F}_{\Delta^*} \) is a local system up to a shift where \( \Delta^* := \Delta \setminus \{ \infty \} \). Define for \( V \in M^f(\mathbb{Q}) \)

\[
E_{\mathcal{F}_{\Delta^*}}(V) := \text{Ext}^1_{\text{Perv}(\Delta^*)}(V_{\Delta^*}[1], \mathcal{F}_{\Delta^*}) = \text{Hom}_{\mathbb{Q}}(V, H^0(\Delta^*, \mathcal{F}_{\Delta^*})).
\]

Set \( H_{1, \mathbb{Q}} := \psi_{1/t, 1} \mathcal{F}[-1] \). Then \( E_{\mathcal{F}_{\Delta^*}} \) is represented by

\[
H^0(\Delta^*, \mathcal{F}_{\Delta^*}) = \text{Coker}(N : H_{1, \mathbb{Q}} \to H_{1, \mathbb{Q}}(-1)).
\]

We have the canonical functor morphism

\[
E_{\mathcal{F}} \to E_{\mathcal{F}_{\Delta^*}},
\]

which corresponds to the canonical morphism

(3.3.1) \( H^0(S, \mathcal{F}) = H^0(\mathcal{S}, Rj_* \mathcal{F}) \to H^0(\Delta^*, \mathcal{F}_{\Delta^*}) = H^0(\Delta, (Rj_* \mathcal{F})|_{\Delta}) \),

where \( j : S \hookrightarrow \mathcal{S} = \mathbb{P}^1 \). We have to calculate the morphism (3.3.1).

Let \( W \) be the weight filtration on \( Rj_* \mathcal{F} \). By the construction in [Sai3], 2.11 (see also [SZ]), we have

\[
\text{Gr}_k^W H^0(S, \mathcal{F}) = H^0(\mathcal{S}, \text{Gr}_k^W (Rj_* \mathcal{F})),
\]

with

(3.3.2) \[
\text{Gr}_k^W (Rj_* \mathcal{F}) = \begin{cases} 
0 & \text{if } k < n, \\
j_* \mathcal{F} & \text{if } k = n, \\
_i^* \text{Gr}_k^W \text{Coker}(N|H_{1, \mathbb{Q}}) & \text{if } k > n,
\end{cases}
\]

since \( H^1(\mathcal{S}, j_* \mathcal{F}) = 0 \). Here \( N : H_{1, \mathbb{Q}} \to H_{1, \mathbb{Q}}(-1) \) is as above, and \( i : \{ \infty \} \hookrightarrow \mathcal{S} \).

Let \( j' : \Delta^* \hookrightarrow \Delta \) so that

\[
j_{\Delta^*} \mathcal{F}|_{\Delta} = j'_{\ast}(\mathcal{F}_{\Delta^*}).
\]

By the local classification of perverse sheaves on \( \Delta \) (see e.g. [BdM], [BrMa]), we have

(3.3.3) \[
\text{Ext}^1_{\text{Perv}((\Delta)}(V_{\Delta}[1], j'_{\ast}(\mathcal{F}_{\Delta^*})) = 0,
\]

and furthermore

(3.3.4) \[
E_{\mathcal{F}_{\Delta^*}}(V) = \text{Ext}^1_{\text{Perv}(\Delta)}(V_{\Delta}[1], Rj'_{\ast}(\mathcal{F}_{\Delta^*}))
\]

\[
= \text{Ext}^1_{\text{Perv}(\Delta)}(V_{\Delta}[1], i'_* \text{Coker}(N|H_{1, \mathbb{Q}}))
\]

\[
= \text{Hom}_{\mathbb{Q}}(V, \text{Coker}(N|H_{1, \mathbb{Q}})),
\]
where $i' : \{\infty\} \hookrightarrow \Delta$. Let $\tilde{F}$ and $\tilde{F}|_{\Delta^*}$ respectively be the universal extensions of $F$ and $F_{\Delta^*}$ by constant perverse sheaves so that we have the short exact sequences in $\text{Perv}(S)$ and $\text{Perv}(\Delta^*)$:

$$0 \to F \to \tilde{F} \to H^0(S,F)_S[1] \to 0,$$
$$0 \to F_{\Delta^*} \to \tilde{F}|_{\Delta^*} \to \text{Coker}(N|H_1,\mathbf{Q})_{\Delta^*}[1] \to 0.$$ 

By (3.3.3–4) we have the following commutative diagram of exact sequences in $\text{Perv}(\Delta^*)$:

$$\begin{array}{ccc}
0 & \to & H^0(\mathbf{S}, j_*F)_{\Delta^*}[1] \\
\downarrow & & \downarrow \\
0 & \to & H^0(\mathbf{S}, j_*F)_{\Delta^*}[1] \\
\downarrow & & \downarrow \\
0 & \to & \text{Coker}(N|H_1,\mathbf{Q})_{\Delta^*}[1] & \to & 0
\end{array}$$}

(3.5)

$$\begin{array}{ccc}
0 & \to & F_{\Delta^*} \\
\downarrow & & \downarrow \\
0 & \to & F|_{\Delta^*} \\
\downarrow & & \downarrow \\
0 & \to & \text{Coker}(N|H_1,\mathbf{Q})_{\Delta^*}[1] & \to & 0
\end{array}$$

where $\text{Gr}_{i*} H^0(S,F) = H^0(\mathbf{S}, j_*F)$. Indeed, the assertion is equivalent to that the quotient of the middle row by the top row is isomorphic to the bottom row. By (3.3.3–4) this follows from the fact that the restriction to $\Delta$ induces the isomorphism of extension classes:

$$\text{Ext}^1_{\text{Perv}(\mathbf{S})}(V_{\mathbf{S}}[1], i_*\text{Coker}(N|H_1,\mathbf{Q})) \simeq \text{Ext}^1_{\text{Perv}(\Delta)}(V_{\Delta}[1], i'_*\text{Coker}(N|H_1,\mathbf{Q})).$$

Note that the morphism $\partial''$ in Lemma (3.2) is functorially defined for any short exact sequences on $\Delta^*$ whose last term is constant, and it is bijective in the case of the short exact sequence associated to the local universal extension $\tilde{F}|_{\Delta^*}$. So the assertion follows from the above commutative diagram of short exact sequences.

3.4. **Further property of the universal extension.** With the notation of (3.1), let $\delta : S \hookrightarrow S \times S$ be the diagonal, and $q_i : S \times S \to S$ the $i$-th projection ($i = 1, 2$). By [Sai3], 4.4.2, the inverse of the first isomorphism of (3.1.1) is given by taking the pull-back of $v : H[1] \to (a_S)_*\mathcal{M}$ by $a_S$ and then composing it with the functorial morphism:

$$a_S^*(a_S)_*\mathcal{M} = (q_2)_*q_1^*\mathcal{M} \to (q_2)_*\delta_*\delta^*q_1^*\mathcal{M} = \mathcal{M}. $$

Let $j_S$ denote the inclusion of the complement of $\delta(S)$ in $S \times S$. Then we have the distinguished triangle in $D^b\text{MHM}(S)$

$$(q_2)_*(j_S)_*j_S^*q_1^*\mathcal{M} \to (q_2)_*q_1^*\mathcal{M} \to \mathcal{M} \to \mathcal{M} \xrightarrow{+1},$$

and it gives the short exact sequence (3.1.2) in $\text{MHM}(S)$ together with the isomorphism

$$\widetilde{\mathcal{M}} = (q_2)_*(j_S)_*j_S^*q_1^*\mathcal{M}[1].$$

This is essentially the same as the definition of $\widetilde{\mathcal{M}}$ by C. Sabbah in Appendix of [MT].

Consider the long exact sequence associated to (3.1.2):

$$0 \to H^{-1}(S,\widetilde{\mathcal{M}}) \to H_{\mathcal{M}} \xrightarrow{\partial'} H^0(S,\mathcal{M}) \to H^0(S,\widetilde{\mathcal{M}}) \to 0,$$
where $H^{-1}(S, \mathcal{M}) = 0$ by hypothesis and $H^j(S, \mathcal{M}) = H^j(S, \widetilde{\mathcal{M}}) = 0$ for $j > 0$ since $S$ is affine. Here $\partial'$ is the identity (up to a sign) by the definition the first isomorphism in (3.1.1). So we get

\[(3.4.2) \quad H^j(S, \widetilde{\mathcal{M}}) = 0 \quad \text{for any } j \in \mathbb{Z}.\]

Conversely, if there is a short exact sequence

\[(3.4.3) \quad 0 \to \mathcal{M} \to \mathcal{M}' \to H'_S[1] \to 0,\]

with $H' \in \text{MHS}$ and $\mathcal{M}' \in \text{MHM}(S)$ satisfying the vanishing condition as in (3.4.2), then $\mathcal{M}'$ is identified with the universal extension $\mathcal{M}$ of $\mathcal{M}$ by a constant mixed Hodge module on $S$. Indeed, this follows by applying the functor on the right-hand side of (3.4.1) to the short exact sequence (3.4.3), since $\mathcal{M}' = \widetilde{\mathcal{M}}$ by the vanishing condition on $H^j(S, \mathcal{M}')$ and the right-hand side of (3.4.1) vanishes for a constant Hodge module on $S$.

Since the vanishing condition as in (3.4.2) is satisfied for $\mathcal{M}' = H^0 f_*(\mathbb{Q}_{h,X}[n])$ (using the Leray spectral sequence as in Remark (1.2)), we get

\[(3.4.4) \quad \widetilde{\mathcal{M}} = H^0 f_*(\mathbb{Q}_{h,X}[n]) \quad \text{with} \quad \mathcal{M} = \text{Gr}_{t_n}^W H^0 f_*(\mathbb{Q}_{h,X}[n]).\]

So Theorem 3 implies the third proof of Theorem 1 in this paper.

\section*{Appendix}

In this Appendix, we give some remarks about the limit mixed Hodge structure and the spectrum.

\subsection*{A.1. Limit mixed Hodge structures.} In [SZ], the limit mixed Hodge structure was constructed in the unipotent monodromy case. For the non-unipotent case, we can combine it with [St2] as follows. Here we describe the limit of the mixed Hodge structure on the cohomology with compact supports using the Cech-type construction, since this seems to be the easiest way to explain the relation with the theory of motivic Milnor fibers [DL]. For the usual cohomology (i.e. without compact supports), we can use the commutativity of the dualizing functor $D$ and the passage to the limit mixed Hodge structure, i.e. $D \circ \psi_t = \psi_t \circ D$ (up to a Tate twist). Of course, we can also use the two weight filtrations on the logarithmic complex associated with the divisor with $V$-normal crossings [St2] as in [SZ].

Let $f : X \to \Delta$ be a projective morphism of complex manifolds where $\Delta$ is an open disk. We may assume that $f$ is smooth over $\Delta^*$ (shrinking $\Delta$ if necessary). Set $Y := f^{-1}(0)$. Let $D$ be a divisor on $X$ which is flat over $\Delta$, i.e. all the irreducible components $D_j$ of $D$ are dominant over $\Delta$. Assume $D \cup Y$ is a divisor with simple normal crossings. Set

\[U := X \setminus D, \quad f' := f|_U : U \to \Delta, \quad D_j := \bigcap_{j \in J} D_j \quad (\text{where } D_0 = X).\]

Let $Y_i$ be the irreducible components of $Y \subset X$ with $m_i$ the multiplicity of $Y$ along the generic point of $Y_i$. Set $m = \text{LCM}(m_i)$. Let $\tilde{f} : \tilde{X} \to \tilde{\Delta}$ be the normalization of the base change of $f : X \to \Delta$ by the ramified $m$-fold covering $\pi_\Delta : \tilde{\Delta} \to \Delta$ which is finite étale over $\Delta^*$, where $\tilde{\Delta}$ is an open disk. Let $\pi : \tilde{X} \to X$ be the canonical morphism. Set

\[\tilde{U} := \pi^{-1}(U), \quad \tilde{Y} := \pi^{-1}(Y), \quad \tilde{D} := \pi^{-1}(D), \quad \tilde{D}_j := \pi^{-1}(D_j).\]
Then $\tilde{X}$ is a $V$-manifold, and $\tilde{Y} \cup \tilde{D}$ is a divisor with $V$-normal crossings on $\tilde{X}$. Let $\tilde{j} : \tilde{U} \hookrightarrow \tilde{X}$ be the natural inclusion. There is a natural quasi-isomorphism

\begin{equation} \label{eq:A.1.1}
\tilde{j}_! \mathbb{Q}_\tilde{U} \xrightarrow{\sim} \mathcal{K}^\bullet_{\tilde{X}} \quad \text{with} \quad \mathcal{K}^p_{\tilde{X}} := \bigoplus_{|J|=p} \mathbb{Q}_{\tilde{D}_J},
\end{equation}

where the differential of $\mathcal{K}^\bullet_{\tilde{X}}$ is defined in the same way as a Cech complex as is well known.

Consider the complex

$$
\psi_{\tilde{j}} \mathcal{K}^\bullet_{\tilde{X}}.
$$

This naturally underlies a cohomological mixed Hodge complex such that its restriction to $\psi_{\tilde{j}} \mathbb{Q}_{\tilde{D}_J}$ coincides with the one defined in [St2] using the complex of logarithmic forms $\tilde{\Omega}_J^\bullet$ ($\log(\tilde{Y} \cap \tilde{D}_J)$) together with the Hodge filtration $F$ and the weight filtration $W$ on it. Indeed, we have canonical morphisms for $J \subset J'$

$$
\tilde{\Omega}^\bullet_{\tilde{D}_J} ((\log(\tilde{Y} \cap \tilde{D}_J))|_{\tilde{D}_J'} \rightarrow \tilde{\Omega}^\bullet_{\tilde{D}_J'} ((\log(\tilde{Y} \cap \tilde{D}_J'))),
$$

and this is a filtered quasi-isomorphism for $W$ (forgetting the filtration $F$), since $D \cup Y$ is a divisor with normal crossings.

There is a spectral sequence of mixed Hodge structures

\begin{equation} \label{eq:A.1.2}
\infty E_1^{p,q} = \bigoplus_{|J|=p} H^q(\tilde{D}_{\tilde{J},\infty}, \mathbb{Q}) \Rightarrow H_c^{p+q}(\tilde{U}_{\infty}, \mathbb{Q}),
\end{equation}

which is induced by the truncations $\tau_{\geq k}$ on $\mathcal{K}^\bullet_{\tilde{X}}$ for $k \in \mathbb{Z}$, and degenerates at $E_2$. (This is the dual of the spectral sequence in [SZ], 5.7.) Indeed, it is the ‘limit’ by $\tilde{t} \to 0$ of the weight spectral sequence

$$
\iota E_1^{p,q} = \bigoplus_{|J|=p} H^q(\tilde{D}_{\tilde{J},\tilde{t}}, \mathbb{Q}) \Rightarrow H_c^{p+q}(\tilde{U}_t, \mathbb{Q}),
$$

where $\tilde{X}_{\tilde{t}} := \tilde{f}^{-1}(\tilde{t})$ and $\tilde{D}_{\tilde{J},\tilde{t}} := \tilde{D}_{\tilde{J}} \cap \tilde{X}_{\tilde{t}}$ for $\tilde{t} \in \tilde{\Delta}^\ast$. These spectral sequences are the dual of the spectral sequences in [SZ] in the unipotent monodromy case.

Note that (A.1.2) is compatible with the actions of the semisimple part $T_s$ and the nilpotent part $N := (2\pi i)^{-1} \log T_u$ of the monodromy $T$.

**A.2. The relation with motivic nearby fibers.** With the above notation, let $E = Y \cup D$ with $E_i$ the irreducible components of $E$. We may assume $E_i = Y_i$ for $i \leq r$ and $E_i = D_{i-r}$ for $i > r$, where $r$ is the number of the irreducible components of $Y$. For $I$ with $\min(I) \leq r$ (i.e. $E_I \subset Y$), define

\begin{equation} \label{eq:A.2.1}
E_I = \bigcap_{i \in I} E_i, \quad E_I^0 = \bigcap_{i \in I} E_i \setminus \bigcap_{i \in I} E_i, \quad \tilde{E}_I = \pi^{-1}(E_I), \quad \tilde{E}_I^0 = \pi^{-1}(E_I^0).
\end{equation}

Note that $\tilde{E}_I^0 \to \tilde{E}_I^0$ is a cyclic étale covering. Set $I' := I \cap [1, r]$. Let $L$ denote $1(-1)$ as a Chow motive where $1 = [pt]$, and $(-1)$ is the Tate twist, see e.g. [Mu], [Sch]. By [DL], [Lo] and [MT], [Ra], the motivic nearby fibers for the morphisms $f : X \to \Delta$ and $f' : U \to \Delta$ can be given respectively by

\begin{equation} \label{eq:A.2.2}
\sum_{\min(I) \leq r} [(\tilde{E}_I^0, T_s)](1 - L)^{|I'|-1}, \quad \sum_{\max(I) \leq r} [(\tilde{E}_I^0, T_s)](1 - L)^{|I|-1}.
\end{equation}
These belong to the Grothendieck group of Chow motives (with $\mathbb{Q}$-coefficients) endowed with an action of $T_s$ of finite order by using equivariant resolutions of $(\tilde{E}_I, \tilde{E}_I \setminus \tilde{E}_I^o)$, see [DL]. Here $T_s$ denotes the semi-simple part of the monodromy, and is given by the automorphism $\gamma$ of $\tilde{X}$ over $X$ induced by the base change of the automorphism of $\Delta$ defined by $\bar{t} \mapsto \zeta_m \bar{t}$ with $\zeta_m := \exp(2\pi i / m)$. The action of $T_s$ on $L$ is the identity. We denote the images of the two terms of (A.2.2) in the Grothendieck group of mixed Hodge structures with an action of finite order respectively by

\begin{equation}
\sum_{\min(I) \leq r} [(H^*(\tilde{E}_I), T_s) | (1 - L)^{|I'| - 1}], \quad \sum_{\max(I) \leq r} [(H^*(\tilde{E}_I), T_s) | (1 - L)^{|I| - 1}],
\end{equation}

where $H^*(\tilde{E}_I^o)$ is a complex of mixed Hodge structures with zero differential, $L$ means here the class of $\mathbb{Q}(-1)$ with trivial action of the monodromy, and $T_s$ is given by $(\gamma^*)^{-1}$, see (A.4.2) below. Then these respectively coincide in the notation of (A.1.1) with

\begin{equation}
[(H^*(\tilde{X}_\infty), T_s)], \quad [(H^*(\tilde{U}_\infty), T_s)].
\end{equation}

Indeed, this follows from the construction of Steenbrink [St2] together with the long exact sequence of mixed Hodge structures

\begin{equation}
\rightarrow H^j_c(Z') \rightarrow H^j_c(Z) \rightarrow H^j_c(Z \setminus Z') \rightarrow H^j_c(Z') \rightarrow,
\end{equation}

for any open immersions of complex algebraic varieties $Z' \hookrightarrow Z$, which is compatible with the action of automorphisms of varieties. (Here (A.2.5) can be proved by using mixed Hodge modules or the mapping cone construction in [De4] together with the diagram of the octahedral axiom of derived categories.) The dual exact sequence of (A.2.5) for Borel-Moore homology is well known in the theory of cycle maps of higher algebraic cycles.

Note that we get cohomology with compact supports in (A.2.4), and this is quite different from the case of motivic Milnor fibers in [DL].

**A.3. Remarks.** (i) In case $E_I$ is simply connected, the cyclic étale covering $\tilde{E}_I^o \rightarrow E_I^o$ can be determined by the multiplicities $m_j$ of $Y$ along the irreducible components $Y_j$, intersecting $E_I$. For example, assume $\max(I) \leq r$ and

\begin{equation}
E_I = \cap_{i \in I} Y_i = \mathbb{P}^1, \quad E_I^o = E_I \setminus (E_{i'} \cup E_{i''}) = \mathbb{C}^*,
\end{equation}

with $i' \leq r$ (i.e. $E_{i'} = Y_{i'}$). Then the covering degree of $\tilde{E}_I^o \rightarrow E_I^o$ and the number of connected components of $\tilde{E}_I^o$ are given respectively by

\begin{equation}
\text{GCD}(m_i \mid i \in I), \quad \text{GCD}(m_i \mid j \in I \cup \{i'\}).
\end{equation}

This may simplify some argument in [MT].

(ii) In [DL], the semisimple part of the monodromy $T_s$ acts as an automorphism of Chow motives. This seems to be useful for the proof of the independence of the motivic Milnor fiber by the resolutions of singularities. For instance, we have $[\mathbb{P}^1] = 1 \oplus 1(-1)$ with $\text{End}(1) = \text{End}(1(-1)) = \mathbb{Q}$ in the category of Chow motives. This is a special case of the Chow-Künneth decomposition, see e.g. [Mu], [Sch].

**A.4. Geometric monodromy and local system monodromy.** Let $f : X \rightarrow S$ be a continuous map of topological spaces which is locally topologically trivial over $S$. We
assume that the $H_j(X_s)$ and $H^j(X_s)$ with $\mathbb{Q}$-coefficients are finite dimensional for any $j$. Let $s \in S$, and $\gamma \in \pi_1(S, s)$. Let $\rho : Y \to [0, 1]$ be the base change of $f$ by the loop $\gamma$. Choosing a trivialization over $[0, 1]$, we get the geometric monodromy

$$\gamma_\# : X_s = Y_0 \simeq Y_1 = X_s,$$

where the middle homeomorphism is induced by the trivialization. We have the induced action of the geometric monodromy on homology and cohomology (with $\mathbb{Q}$-coefficients):

$$\gamma_* \in \text{Aut}(H_j(X_s)), \quad \gamma^* \in \text{Aut}(H^j(X_s)),$$

such that

$$(A.4.1) \quad \gamma^* = t \gamma_*,$$

where $t$ means the transpose.

On the other hand, we have the local system monodromies

$$\gamma_h \in \text{Aut}(H_j(X_s)), \quad \gamma_c \in \text{Aut}(H^j(X_s)),$$

which are defined by using the following (trivial) local systems of homology and cohomology groups over $[0, 1]$:  

$$\{H_j(Y_u)\}_{u \in [0, 1]}, \quad \{H^j(Y_u)\}_{u \in [0, 1]}.$$

The latter can be identified with the constant sheaf $R^j\rho_* \mathbb{Q}_Y$, see also [De3], XIV, 1.1.2. Note that $\gamma_c$ coincides with the monodromy associated to the nearby cycle functor if $f$ is a Milnor fibration.

The relations between the above monodromies are given by

$$(A.4.2) \quad \gamma_* = \gamma_h, \quad \gamma^* = \gamma_c^{-1}.$$

Indeed, the first assertion easily follows from the definition (using simplicial chains for example). We then get the second equality since

$$\gamma^* = t \gamma_* = t \gamma_h = \gamma_c^{-1},$$

where the last equality follows from

$$(A.4.3) \quad \langle \gamma_c u, \gamma_h v \rangle = \langle u, v \rangle \quad \text{for} \quad u \in H^j(X_s), v \in H_j(X_s).$$

Here $\langle u, v \rangle$ denotes the canonical pairing between cohomology and homology, and it can be extended to a canonical pairing between the local systems so that (A.4.3) follows.

It does not seem that (A.4.2) has been clarified explicitly in the literature. In fact, it does not seem to cause big problems at least in the local monodromy case since it is quasi-unipotent (except possibly for the definition of spectrum as in [MT]).

**A.5. Example.** Let $f$ be a homogeneous polynomial of $n$ variables with degree $d$, having an isolated singularity at the origin. Set $X = \mathbb{C}^n \setminus f^{-1}(0)$ and $S = \mathbb{C}^*$. Here $f$ also denotes the morphism $X \to S$ induced by $f$. Let $\gamma$ be a generator of $\pi_1(S, s) \simeq \mathbb{Z}$ going around the origin counter-clockwise. Then $\gamma_\#$ is induced by the automorphism

$$(A.5.1) \quad \gamma_\# : (x_1, \ldots, x_n) \mapsto (\zeta_d x_1, \ldots, \zeta_d x_n),$$
where \( \zeta_d := \exp(2\pi i/d) \), and \( x_1, \ldots, x_n \) are the coordinates of \( \mathbb{C}^n \). The action of \( \gamma_\# \) is extended to an automorphism of \( \mathbb{C}[x_1, \ldots, x_n] \) over \( \mathbb{C} \) such that

\[
(A.5.2) \quad \gamma_\#^* x_i = \zeta_d x_i.
\]

This can be checked for instance by \( \gamma_\#^*(x_i - \zeta_d c_i) = \zeta_d (x_i - c_i) \).

Set \( \omega = dx_1 \wedge \cdots \wedge dx_n \), and

\[
(A.5.3) \quad H_f'' := \Omega^n_{X,0}/df \wedge d\Omega^{n-2}_{X,0}.
\]

This is called the Brieskorn lattice. Let \( g \in \mathbb{C}[x_1, \ldots, x_n] \) be a monomial of degree \( k \). After Brieskorn, it is well known (and easy to show) that

\[
(A.5.4) \quad \partial_t t(g\omega) = \frac{k+n}{d} g\omega \quad \text{in} \quad H_f'',
\]

see e.g. the proof of Prop. 3.3 in [Sai2] for an argument in a slightly more general case.

In the homogeneous polynomial case, we have moreover the well-known relation

\[
(A.5.5) \quad \gamma_c = \exp(-2\pi i (\text{Res} t\partial_t)),
\]

under the canonical isomorphism

\[
(A.5.6) \quad H^{n-1}(X_1, \mathbb{C}) = H_f''/\Gamma_f''.
\]

where \( s = 1 \). The isomorphism can be defined by using a basis \((\omega_1, \ldots, \omega_\mu)\) of \( H_f'' \) such that \( \partial_t t\omega_i = \alpha_i \omega_i \), and taking the restriction to \( X_1 \) after dividing the \( \omega_i \) by \( df \). Indeed, the assertion is well known if the Brieskorn lattice is replaced by the Deligne extension [De1].

In this case, the inverse isomorphism is given by

\[
u \mapsto \exp(-\frac{\log t}{2\pi i} \log \gamma_c) u,
\]

for \( u \in H^{n-1}(X_1) \) which is identified with a multivalued section, and we have

\[
\partial_t t \exp(-\frac{\log t}{2\pi i} \log \gamma_c) u = -\frac{\log \gamma_c}{2\pi i} \exp(-\frac{\log t}{2\pi i} \log \gamma_c) u,
\]

where the eigenvalues of \( -\frac{1}{2\pi i} \log \gamma_c \) are chosen corresponding to the Deligne extension. This can be extended to the Brieskorn lattice case easily in the homogeneous polynomial case. So (A.5.5) follows.

By (A.5.4) and (A.5.5), the action of \( \gamma_c \) on \( (g/df)|_{X_1} \in H^{n-1}(X_1, \mathbb{C}) \) is given by the multiplication by

\[
(A.5.7) \quad \exp(-2\pi i (k + n)/d).
\]

On the other hand, (A.5.2) implies that the action of the geometric monodromy (A.5.1) on \( (g/df)|_{X_1} \) is given by the multiplication by

\[
(A.5.8) \quad \exp(2\pi i (k + n)/d).
\]

This is the inverse of (A.5.7).
A.6. Brieskorn lattices and mixed Hodge structures. The Brieskorn lattice $H''_f$ in (A.5.3) is defined for any holomorphic function on a complex manifold $X$ having an isolated singularity at $0 \in f^{-1}(0)$. It is a free $\mathbb{C}[[\partial_t^{-1}]]$-module of rank $\mu$, and is contained in the Gauss-Manin system $\mathcal{G}_f$ which is the localization of $H''_f$ by $\partial_t^{-1}$, i.e. $\mathcal{G}_f = H''_f[\partial_t]$. The latter has the Hodge filtration defined by $F_p^{\mathcal{G}_f} := \partial_t^{n-1-p}H''_f$ for $p \in \mathbb{Z}$, and also the filtration $V$ of Kashiwara and Malgrange such that $\partial_t t - \lambda$ is nilpotent on $Gr^\alpha V\mathcal{G}_f$. By an argument similar to the proof of (A.5.6), there are isomorphisms

\begin{equation}
H^{n-1}(X_{f,0}, \mathbb{C})_\lambda = Gr^\alpha V\mathcal{G}_f \quad \text{for} \quad \lambda = \exp(-2\pi i \alpha),
\end{equation}

where $X_{f,0}$ is the Milnor fiber, and $V_\lambda$ denotes the $\lambda$-eigenspace for any vector space $V$ with the action of the local system monodromy $T$. We have moreover

\begin{equation}
F^{n-1-q}H^{n-1}(X_{f,0}, \mathbb{C})_\lambda = Gr^\alpha V\mathcal{G}_f \quad \text{for} \quad q < \alpha \leq q + 1, \quad \lambda = \exp(-2\pi i \alpha),
\end{equation}

where $F$ is the Hodge filtration of the mixed Hodge structure $[St2]$, see $[SS]$, $[Va]$, etc. This is closely related with the definition of the spectrum in (A.7.5) below. In the case of Example (A.5), it is related with $[St3]$ by (A.5.4).

A.7. Spectrum. Let $H$ be a mixed Hodge structure with a semisimple action $T$ of finite order. Set $H_{C,\lambda} := \text{Ker}(T - \lambda) \subset H_{C}$. We define the spectrum $\text{Sp}'(H, T)$ as in $[Sai4]$ (and $[DL]$) by

\begin{equation}
\text{Sp}'(H, T) := \sum_{\alpha \in \mathbb{Q}} n'_{\alpha} t^\alpha,
\end{equation}

with $n'_{\alpha} = \dim_{\mathbb{C}} Gr^p_{\mathcal{G}_f} H_{C,\lambda}$ for $p = [\alpha]$, $\lambda = \exp(2\pi i \alpha)$.

For a holomorphic function $f$ on a complex manifold $X$ of dimension $n$ and $x \in f^{-1}(0)$, we first define $\text{Sp}'(f, x)$ by

\begin{equation}
\text{Sp}'(f, x) := \sum_{j} (-1)^{n-1-j} \text{Sp}'(\tilde{H}^j(X_{f,x}), T_x),
\end{equation}

where $\tilde{H}^j(X_{f,x})$ is the reduced Milnor cohomology endowed with the canonical mixed Hodge structure, and $T_x$ is the semisimple part of the local system monodromy $T$. There are canonical isomorphisms

\begin{equation}
\tilde{H}^j(X_{f,x}) = H^j i_x^* \mathcal{G}_f \mathcal{Q}_X,
\end{equation}

where $i_x : \{x\} \hookrightarrow X$ is the inclusion, see $[De3]$. They can be used to define the mixed Hodge structure on the left-hand side. Note that $T$ is equal to the inverse of the cohomological Milnor monodromy by (A.4), and this is closely related with Example (A.5) by (A.6).

Let $\iota$ denote the involution of $\mathbb{Z}[t^{1/m}, t^{-1/m}]$ over $\mathbb{Z}$ defined by

\begin{equation}
\iota(t^\alpha) = t^{-\alpha}.
\end{equation}

The spectrum $\text{Sp}(f, x)$ is then defined by

\begin{equation}
\text{Sp}(f, x) := t^n \iota(\text{Sp}'(f, x)).
\end{equation}
This spectrum $\text{Sp}(f, x)$ coincides with the one in [St2] for the isolated singularity case (using the complex conjugate of (A.7.1) together with the symmetry (A.7.6) below). It coincides with the one in [St4] up to the multiplication by $t$. Indeed, the above definition of $\text{Sp}(f, x)$ can be rewritten as $\text{Sp}(f, x) := \sum_{\alpha \in \mathbb{Q}} n_\alpha t^n$ with

$$n_\alpha = \sum_j (-1)^{n-1-j} \dim \text{Gr}^{n-1-q}_F \tilde{H}^j(X_{f,x}, \mathbb{C}) \lambda$$

for $q < \alpha \leq q + 1$, $\lambda = \exp(-2\pi i \alpha)$,

and this is used in loc. cit. (up to the multiplication by $t$). In the isolated singularity case, the formula (A.7.5) is closely related with (A.6.2) and also with the calculations in (A.5).

If $f$ has an isolated singularity at $x$, we have the symmetry of mixed Hodge numbers by [St2] so that

$$\text{Sp}(f, x) = \text{Sp}'(f, x).$$

In case $f$ is a weighted-homogeneous polynomial of weights $(w_1, \ldots, w_n)$ (i.e. $f$ is a linear combination of monomials $x_1^{m_1} \cdots x_n^{m_n}$ with $\sum_i w_i m_i = 1$) and has an isolated singularity at 0, it is well known that

$$\text{Sp}(f, 0) = \prod_{i=1}^n \left( \frac{t - t^{w_i}}{t^{w_i} - 1} \right).$$

This follows from [St3], see also [St2] and the proof of Proposition 5.2 in [Di1]. In the case of homogeneous polynomials (i.e. $w_i = 1/d$ for any $i$), this follows also from the calculation in Example (A.5) using (A.6).

For a polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}$, we can define the spectrum at infinity (see also [Sab1]) by

$$\text{Sp}'(f, \infty) := \sum_j (-1)^{n-1-j} \text{Sp}'(\tilde{H}^j(X_\infty), T_s),$$
$$\text{Sp}(f, \infty) := t^n \text{Sp}'(f, \infty),$$

where $\tilde{H}^j(X_\infty)$ is the limit mixed Hodge structure of $\tilde{H}^j(X_t)$ at infinity (of $\mathbb{C}$), and $T_s$ is the semisimple part of the local system monodromy $T$ associated with a sufficiently large loop around the origin which goes counter-clockwise from the origin (and clockwise from $\infty \in \mathbb{P}^1$). This definition is compatible with the one in the weighted-homogeneous isolated singularity case in (A.7.7). In the cohomologically tame case, we have the symmetry by [Sab3] (i.e. Theorem 1 in this paper) so that (A.7.6) holds, and the definition (A.7.8) seems to coincide with the one in [MT] (where the cohomology with compact supports is used) if the local system monodromy is used there.
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