BELLMAN FUNCTIONS AND DIMENSION FREE
L\textsuperscript{p}–ESTIMATES FOR THE RIESZ TRANSFORMS IN BESSEL
SETTINGS

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Abstract. In this article we prove dimension free \( L^p \)–boundedness of Riesz transforms associated with a Bessel differential operator. We obtain explicit estimates of the \( L^p \)–norms for the Bessel–Riesz transforms in terms of \( p \), establishing a linear behaviour with respect to \( p \). We use the Bellman function technique to prove a bilinear dimension free inequality involving Poisson semi-groups defined through this Bessel operator.

1. Introduction and main results

For every \( j = 1, \ldots, d \), the classical Riesz transform \( R_j f \) of \( f \in L^p(\mathbb{R}^d), 1 \leq p < \infty \), is defined by

\[
R_j f(x) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \lim_{\epsilon \to 0^+} \int_{|x-y| > \epsilon} \frac{x_j - y_j}{|x-y|^{d+1}} f(y) dy, \quad \text{a.e. } x = (x_1, \ldots, x_d) \in \mathbb{R}^d.
\]

As it is well-known, \( R_j \) is bounded on \( L^p(\mathbb{R}^d) \) for every \( 1 < p < \infty \) and \( j = 1, \ldots, d \) with constant independent of dimension. Stein [39, Theorem, p. 71] proved something stronger than that: for every \( 1 < p < \infty \), there exists \( A_p > 0 \) such that

\[
\|R f\|_{L^p(\mathbb{R}^d)} \leq A_p \|f\|_{L^p(\mathbb{R}^d)} \text{ for } f \in L^p(\mathbb{R}^d), \text{ where } R f = \left( \sum_{j=1}^d |R_j f|^2 \right)^{1/2}.
\]

By using transference methods, Duoandikoetxea and Rubio de Francia [15] gave another proof of Stein’s result.

Let \( \mathcal{H} \) denote the Hilbert transform on \( \mathbb{R} \). Gokhberg and Krupnik [17] and Pichorides [35] proved that

\[
\| \mathcal{H} \|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} = \begin{cases} \tan\left(\frac{\pi}{2p}\right), & 1 < p \leq 2, \\ \cot\left(\frac{\pi}{2p}\right), & p \geq 2. \end{cases}
\]

Iwaniev and Martin [22] extended this result to higher dimensions proving that, for every \( j = 1, \ldots, d \), \( \|R_j\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} = \|\mathcal{H}\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)}, 1 < p < \infty \). This result was also showed by Bañuelos and Wang [3] using probabilistic methods.

This phenomenon of dimension free estimates has been observed for Riesz transforms appearing in other settings. For instance, Coulhon, Muller and Zienkiewicz

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worked with Riesz transforms on the Heisenberg group; Lust-Piquard considered discrete Riesz transforms on a product of Abelian groups and Urban and Zienkiewicz proved these estimates for Riesz transforms associated with Schrödinger operators.

In recent years, finding dimension free $L^p$-estimates for Riesz transforms associated with orthogonal systems has been on the rise considerably. In the case of Hermite polynomials which are associated with the Ornstein–Uhlenbeck differential operator and the Gaussian measure, this kind of results were obtained by Pisier and Gutiérrez (see also and for proofs using probabilistic methods). Harboure, de Rosa, Segovia and Torrea proved $L^p$-dimension free boundedness for Riesz transforms related to the harmonic oscillator and the Hermite functions (see for another proof). In different settings associated with Laguerre operators, the results have been obtained by Nowak and Stempak and Wróbel. A general result that applies to different orthogonal systems was proved by Forzani, Sasso and Scotto. In [, , , , and ] a procedure based upon the boundedness of Littlewood-Paley functions is used and it is inspired by the Stein’s method of proof developed in [].

Recently, Dragičević and Volberg in a series of papers (,, , and ) have developed another approach, different from any previous ones, where they use a technique involving Bellman functions. These Bellman functions allow them to obtain dimension free bilinear Littlewood-Paley estimates. Then, as a consequence, dimension free $L^p$-estimates for Riesz transforms are also obtained. Dragičević and Volberg have studied classical and Gaussian (Hermite polynomial) context in []). Riesz transforms associated with the harmonic oscillator and more general Schrödinger operators were analyzed in [], respectively. In [], bilinear Littlewood-Paley estimates related to elliptic operators were established.

By using the Bellman function techniques it is possible to get dimension free $L^p$-estimates with better constants than the ones given by other procedures. Furthermore, it is proved that these constants are linear in $\max\{p, p/(p - 1)\}$ for $1 < p < \infty$.

Bellman functions have also been utilized in the boundedness of Riesz transforms appearing in other settings, such as: Bakry context ( and ), Hodge–Laguerre operators (), discrete orthogonal systems ( and ), weighted Riesz transforms () and Beurling operator (). In this paper we obtain dimension free $L^p$-estimates for Riesz transforms in this Bessel setting by using Bellman functions techniques.

Muckenhoupt and Stein started with the study of harmonic analysis associated to Bessel operators. They considered, for $\alpha \geq 0$, the Bessel operator

$$B_\alpha = -x^{2\alpha} \frac{d}{dx} x^{-2\alpha} \frac{d}{dx} = -\frac{d^2}{dx^2} + \frac{2\alpha}{x} \frac{d}{dx},$$

and the Hankel transformation $h_\alpha$ defined, for every $f \in L^1((0, \infty), x^{2\alpha} dx)$, by

$$h_\alpha f(x) = \int_0^\infty \phi_\alpha^*(y) f(y) x^{2\alpha} dy, \quad x \in (0, \infty),$$

where $\phi_\alpha^*(x) = (xy)^{-\alpha-1/2}J_{\alpha-1/2}(xy)$ are the eigenfunctions of $B_\alpha$, and $J_\mu$ represents the Bessel function of first kind and order $\mu$. Notice that $\phi_\alpha^*(x) = \phi_\alpha^*(y)$, $x, y \in (0, \infty)$, and $\phi^\alpha \in L^p((0, \infty), x^{2\alpha} dx)$ if and only if $\frac{2\alpha + 1}{\alpha} < p \leq \infty$, when $\alpha > 0$, and $p = \infty$, when $\alpha = 0$ (see [] (5.16.1), p. 134).
Since the function $z^{-\mu}J_\mu(z)$ is bounded on $(0, \infty)$ for $\mu \geq -1/2$ (see [24] (5.16.1), p. 134), $h_\alpha$ defines a bounded operator from $L^1((0, \infty), x^{2\alpha}dx)$ to $L^\infty((0, \infty), x^{2\alpha}dx) = L^\infty((0, \infty), dx)$. Also, $h_\alpha$ can be extended from $L^1((0, \infty), x^{2\alpha}dx) \cap L^2((0, \infty), x^{2\alpha}dx)$ to $L^2((0, \infty), x^{2\alpha}dx)$ as an isometry on $L^2((0, \infty), x^{2\alpha}dx)$ and $h_\alpha^{-1} = h_\alpha$ (21). Let us note that both the Hankel transformation and the Bessel operator are connected by

$$h_\alpha(B_\alpha f)(x) = x^2 h_\alpha(f)(x), \quad x \in (0, \infty) \quad (1.1)$$

for every $f \in C_c^\infty(0, \infty)$, the space of smooth functions with compact support in $(0, \infty)$.

Muckenhoupt and Stein (29) introduced the Riesz transform $R_\alpha$ associated with $B_\alpha$ as follows. For every $f \in L^2((0, \infty), x^{2\alpha}dx)$,

$$R_\alpha f(x) = -x h_{\alpha+1}(\frac{1}{y}h_\alpha f(y))(x).$$

Since $h_\alpha$ is bounded on $L^2((0, \infty), x^{2\alpha}dx)$, it is clear that $R_\alpha$ is also a bounded operator on $L^2((0, \infty), x^{2\alpha}dx)$. As in the classical case, $R_\alpha$ can be extended from $L^2((0, \infty), x^{2\alpha}dx) \cap L^p((0, \infty), x^{2\alpha}dx)$ to $L^p((0, \infty), x^{2\alpha}dx)$ as an $L^p$–bounded operator, for every $1 < p < \infty$, and from $L^1((0, \infty), x^{2\alpha}dx)$ into $L^{1,\infty}((0, \infty), x^{2\alpha}dx)$ (24). On the other hand, $L^p$-weighted inequalities for $R_\alpha$ were established in [2] and, more recently, in [6].

We define $A_{\alpha,c}(0, \infty)$ as the space consisting of all those functions $\phi \in C_0^\infty(0, \infty) \cap L^1((0, \infty), x^{2\alpha}dx)$ such that $h_\alpha(\phi) \in C_0^\infty(0, \infty)$. Since $h_\alpha$ is an isometry in $L^2((0, \infty), x^{2\alpha}dx)$, the space $A_{\alpha,c}(0, \infty)$ is a dense subspace of $L^2((0, \infty), x^{2\alpha}dx)$.

Motivated by (1.1) we define, for every $f \in A_{\alpha,c}(0, \infty)$, $B_\alpha^{-1/2}f$ by

$$B_\alpha^{-1/2}f = h_\alpha \left( \frac{1}{y}h_\alpha f(y) \right).$$

Then, we have that for every $f \in A_{\alpha,c}(0, \infty)$

$$R_\alpha f(x) = \frac{d}{dx}B_\alpha^{-1/2}f(x), \quad x \in (0, \infty). \quad (1.2)$$

Equality (1.2) says that $R_\alpha$ is the Riesz transform associated with $B_\alpha$ in the sense of Stein (38). Furthermore, $R_\alpha$ can be seen as a principal value integral operator. For every $f \in L^p((0, \infty), x^{2\alpha}dx)$, $1 \leq p < \infty$, we can write

$$R_\alpha f(x) = \lim_{\epsilon \to 0^+} \int_{|x-y| > \epsilon} R_\alpha(x,y)f(y)y^{2\alpha}dy, \quad a.e. x \in (0, \infty),$$

where

$$R_\alpha(x,y) = \int_0^\infty \partial_x P_\alpha(x,y)dt, \quad x,y \in (0, \infty), x \neq y,$$

and $P_\alpha$ denotes the Poisson kernel associated to $B_\alpha$ (see Section 2 for definitions). The Bessel–Riesz transform $R_\alpha$ turns out to be a Calderón–Zygmund operator in the homogeneous type space $((0, \infty), x^{2\alpha}dx, \rho)$ where $\rho$ represents the Euclidean metric on $(0, \infty)$ (see [5]).

Bessel–Riesz transforms in higher dimensions were considered in [4]. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}_{\geq 0}^d := [0, \infty)^d$, let us define the $d$-dimensional
Bessel differential operator as

$$B_\alpha = \sum_{i=1}^{d} B_{\alpha,i}x_i = \sum_{i=1}^{d} \left( -\frac{\partial^2}{\partial x_i^2} + \frac{2\alpha}{x_i} \frac{\partial}{\partial x_i} \right).$$

The Hankel transform $h_\alpha(f)$ of $f$ is defined, for $x \in \mathbb{R}_+^d$, by

$$h_\alpha(f)(x) = \int_{\mathbb{R}_+^d} \phi_y^\alpha(x)f(y)y^{2\alpha}dy,$$

where

$$\phi_y^\alpha(x) = \prod_{j=1}^{d} (x_jy_j)^{-\alpha_j+1/2}J_{\alpha_j-1/2}(x_jy_j), \quad x = (x_1, \ldots, x_d), \quad y = (y_1, \ldots, y_d) \in \mathbb{R}_+^d,$$

$y^{2\alpha} = \prod_{j=1}^{d} y_j^{2\alpha_j}$ and $f \in L^1(\mathbb{R}_+^d, x^{2\alpha}dx)$. For every $y \in \mathbb{R}_+^d$, the function $\phi_y^\alpha$ is an eigenfunction of $B_\alpha$, being $B_\alpha \phi_y^\alpha = |y|^2 \phi_y^\alpha$. The Hankel transform $h_\alpha$ can be extended from $L^1(\mathbb{R}_+^d, x^{2\alpha}dx) \cap L^2(\mathbb{R}_+^d, x^{2\alpha}dx)$ to $L^2(\mathbb{R}_+^d, x^{2\alpha}dx)$ as an isometry on $L^2(\mathbb{R}_+^d, x^{2\alpha}dx)$ and $h_\alpha^{-1} = h_\alpha$, as in the one-dimensional case.

Let $i = 1, \ldots, d$ be given. For every $f \in L^2(\mathbb{R}_+^d, x^{2\alpha}dx)$, the $i$-th Bessel-Riesz transform $R_{\alpha,i}f$ is defined by

$$R_{\alpha,i}f(x) = -x_i h_{\alpha+e_i} \left( \frac{1}{|y|} h_\alpha(f) \right)(x), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d,$$

where $e_i$ is the $i$-th unit vector on $\mathbb{R}_+^d$. The operator $R_{\alpha,i}$ is bounded on $L^2(\mathbb{R}_+^d, x^{2\alpha}dx)$.

By $S(\mathbb{R}_+^d)$ we represent the space consisting of all those functions $\phi \in C^\infty(\mathbb{R}_+^d)$ such that, for every $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$,

$$\gamma_{m,k}(\phi) =: \sup_{x \in \mathbb{R}_+^d} |x|^m \left| \frac{\partial^{k_1+\ldots+k_d}}{\partial x_1 \ldots \partial x_d} \phi(x) \right| < \infty.$$

$S(\mathbb{R}_+^d)$ is endowed with the topology generated by the system $\{\gamma_{m,k}\}_{m \in \mathbb{N}_0, k \in \mathbb{N}_0^d}$ of seminorms. The Hankel transformation $h_\alpha$ is an automorphism in $S(\mathbb{R}_+^d)$ (1.1). For every $\phi \in S(\mathbb{R}_+^d)$, we have that

$$h_\alpha(B_\alpha \phi)(x) = |x|^2 h_\alpha(\phi)(x), \quad x \in \mathbb{R}_+^d. \quad (1.3)$$

We denote by $A_{\alpha,c}(\mathbb{R}_+^d)$ the tensor product space

$$A_{\alpha,c}(\mathbb{R}_+^d) = A_{\alpha,1,c}(\mathbb{R}_+) \otimes \ldots \otimes A_{\alpha,d,c}(\mathbb{R}_+).$$

$A_{\alpha,c}(\mathbb{R}_+^d)$ is dense in $L^2(\mathbb{R}_+^d, x^{2\alpha}dx)$. By taking into account (1.3) we define, for every $\phi \in A_{\alpha,c}(\mathbb{R}_+^d)$,

$$B_\alpha^{-1/2}(\phi) = h_\alpha \left( \frac{1}{|y|} h_\alpha(\phi) \right)$$

Note that if $\phi \in A_{\alpha,c}(\mathbb{R}_+^d)$, $0 \notin \text{supp}(h_\alpha(\phi))$ and then $B_\alpha^{-1/2} \phi \in S(\mathbb{R}_+^d)$. According to [23] (5.37), p. 103, we get

$$R_{\alpha,i} \phi = \partial_{x_i} B_\alpha^{-1/2}(\phi), \quad \phi \in A_{\alpha,c}(\mathbb{R}_+^d).$$

$R_{\alpha,i}$ is a Riesz transform associated with $B_\alpha$ in the sense of Stein [33].

In [4] Theorem 1.4 it was proved that $R_{\alpha,i}$ can be extended from $L^2(\mathbb{R}_+^d, x^{2\alpha}dx) \cap L^p(\mathbb{R}_+^d, x^{2\alpha}dx)$ to $L^p(\mathbb{R}_+^d, x^{2\alpha}dx)$ as an $L^p$-bounded operator, for every $1 < p < \infty,$
and from \( L^1(\mathbb{R}^d_+, x^{2\alpha} dx) \) into \( L^{1,\infty}(\mathbb{R}^d_+, x^{2\alpha} dx) \). The proof of [4, Theorem 1.4] is very laborious. The underlying set \( \mathbb{R}^d_+ \) is divided into many regions, one of them is close to the diagonal of the \( i \)-th variable and the others far from this diagonal set. Then, the \( i \)-th Riesz transform is decomposed in some products of Hardy type operators and a local classical Riesz transform. The \( L^p \)-boundedness properties of \( R_{\alpha,i} \) are deduced from the corresponding properties of the operators appearing in the decomposition. This procedure does not allow them to obtain dimension free \( L^p \)-estimates for \( R_{\alpha,i} \) since many iterations of operators are involved.

We define

\[
R_\alpha = \left( \sum_{i=1}^d |R_{\alpha,i}|^2 \right)^{1/2}.
\]

By using Bellman functions technique we prove the following results.

**Theorem 1.1.** For every \( d \in \mathbb{N}, \alpha \in \mathbb{R}^d_{\geq 0}, 1 < p < \infty \) and \( f \in L^p(\mathbb{R}^d_+, x^{2\alpha} dx) \), we have

\[
\|R_\alpha f\|_{L^p(\mathbb{R}^d_+, x^{2\alpha} dx)} \leq 48(p^* - 1) \|f\|_{L^p(\mathbb{R}^d_+, x^{2\alpha} dx)}, \tag{1.4}
\]

where \( p^* = \max\{p, p/(p-1)\} \).

For every \( k \in \mathbb{N} \), we define the set

\[
C_{\alpha,k} = \{ \text{compositions of } k \text{ operators among } R_{\alpha,1}, \ldots, R_{\alpha,d} \}.
\]

**Theorem 1.2.** For every \( d, k \in \mathbb{N}, \alpha \in \mathbb{R}^d_{\geq 0}, 1 < p < \infty \) and \( f \in L^p(\mathbb{R}^d_+, x^{2\alpha} dx) \)

\[
\left\| \left( \sum_{R \in C_{\alpha,k}} |Rf|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d_+, x^{2\alpha} dx)} \leq 48^k(p^* - 1)^k \|f\|_{L^p(\mathbb{R}^d_+, x^{2\alpha} dx)}.
\]

It should be mentioned that these theorems are not covered by the general results in [4].

From Theorem 1.1 it is immediate to see that, for every \( i = 1, \ldots, d \),

\[
\|R_{\alpha,i} f\|_{L^p(\mathbb{R}^d_+, x^{2\alpha} dx)} \leq 48(p^* - 1) \|f\|_{L^p(\mathbb{R}^d_+, x^{2\alpha} dx)}, \tag{1.5}
\]

for every \( f \in L^p(\mathbb{R}^d_+, x^{2\alpha} dx) \) and \( 1 < p < \infty \). Dimension free estimates like (1.5) were established in [43, Theorem 4.1 (i)] but there the constants are not specified.

This article is organized as follows. In Section 2 we prove some properties related to Bessel–Poisson kernels and integrals. Proofs of Theorems 1.1 and 1.2 are presented in Section 3.

Throughout this paper, \( c \) and \( C \) will denote positive constants that may change from one line to another.

### 2. Definitions and properties of Bessel–Poisson operators

Let \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d_{\geq 0} \) and set \( \mathbb{R}^d_+ := (1, 1, \ldots, 1) \). From now on, by \( x_j \) and \( y_j \) we denote the \( j \)-th coordinate of \( x \) and \( y \), respectively. Let us write symbols for several operations

\[
x y = (x_1y_1, \ldots, x_d y_d),
\]

\[
\alpha - 1/2 = \left( \alpha_1 - \frac{1}{2}, \alpha_2 - \frac{1}{2}, \ldots, \alpha_d - \frac{1}{2} \right)
\]
\[ x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d} = \prod_{j=1}^{d} x_j^{\alpha_j}. \]

Given \( \nu \in (-1/2, \infty)^d \), we define
\[
J_\mu(x) = \prod_{j=1}^{d} J_{\nu_j}(x_j), \quad I_\mu(x) = \prod_{j=1}^{d} I_{\nu_j}(x_j),
\]
where, for every \( \mu > -1/2 \), \( J_\mu \) and \( I_\mu \) are the Bessel function and the modified Bessel function, respectively, of first kind and order \( \mu \).

Let us define the \( d \)-dimensional Bessel operator as
\[
\Delta_{\alpha,x} = -\sum_{j=1}^{d} \delta_j^* \delta_j = \sum_{j=1}^{d} x_j^{-2\alpha_j} \frac{\partial}{\partial x_j} \left(x_j^{2\alpha_j} \frac{\partial}{\partial x_j}\right) = \sum_{j=1}^{d} \left( \frac{\partial^2}{\partial x_j^2} - \frac{2\alpha_j}{x_j} \frac{\partial}{\partial x_j} \right),
\]
with
\[
\delta_j^* = -x_j^{-2\alpha_j} \frac{\partial}{\partial x_j}(x_j^{2\alpha_j}), \quad \delta_j = \frac{\partial}{\partial x_j} =: \partial_{x_j},
\]
then \( B_\alpha = -\Delta_{\alpha,x} \). Let us remark that the commutator
\[
[\delta_j, \delta_j^*] = \delta_j \delta_j^* - \delta_j^* \delta_j = \frac{2\alpha_j}{x_j^2} \geq 0.
\]

In this setting the Hankel transformation \( h_\alpha \) plays the same role as the Fourier transformation in the Laplacian context.

We define the heat semigroup \( \{T^\alpha_t\}_{t>0} \) and the Poisson semigroup \( \{P^\alpha_t\}_{t>0} \) associated with the Bessel operator \( B_\alpha \) as follows. For every \( f \in L^2(\mathbb{R}_+^d, x^{2\alpha}dx) \),
\[
T^\alpha_t f = h_\alpha(e^{-|\cdot|^2t}h_\alpha f), \quad t > 0,
\]
and
\[
P^\alpha_t f = h_\alpha(e^{-|\cdot|^2t}h_\alpha f), \quad t > 0.
\]
Therefore, for every \( t > 0 \), we have
\[
T^\alpha_t f(x) = \int_{\mathbb{R}_+^d} e^{-|z|^2t} \phi^\alpha_z(x) h_\alpha f(z) z^{2\alpha} dz
\]
\[
= \int_{\mathbb{R}_+^d} \left( \int_{\mathbb{R}_+^d} e^{-|z|^2t} \phi^\alpha_z(x) \phi^\alpha_y(y) z^{2\alpha} dz \right) f(y) y^{2\alpha} dy
\]
\[
= \int_{\mathbb{R}_+^d} W^\alpha_t(x,y) f(y) y^{2\alpha} dy, \quad x \in \mathbb{R}_+^d,
\]
with
\[
W^\alpha_t(x,y) = (xy)^{-\alpha+1/2} \int_{\mathbb{R}_+^d} e^{-|z|^2t} J_{\alpha-\frac{1}{2}}(xz) J_{\alpha-\frac{1}{2}}(yz) z^{2\alpha} dz
\]
\[
= \frac{e^{-|xy|^2t/(2t)}}{(2\pi t)^{d/2}} \frac{1}{(xy)^{-\alpha+1/2}} I_{\alpha-\frac{1}{2}} \left( \frac{xy}{2t} \right), \quad x, y \in \mathbb{R}_+^d,
\]
(see [12] p. 395 for the one-dimensional case), where \( W^\alpha_t(x,y) > 0 \) for every \( t > 0 \) and every \( x, y \in \mathbb{R}_+^d \). Thus, \( T^\alpha_t \) turns out to be a positive operator, i.e., for every \( 0 \leq f \in L^2(\mathbb{R}_+^d, x^{2\alpha}dx) \), \( T^\alpha_t f \geq 0 \).
By using the subordination formula, we get the following expression for the Poisson semigroup

\[ P_t^\alpha f(x) = \frac{t}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2}{2u}} T_u^\alpha f(x) du \]

\[ = \int_0^\infty \left( \frac{t}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2}{2u}} W_u^\alpha(x,y) du \right) f(y) y^{2\alpha} dy \]

\[ := \int_0^\infty P_t^\alpha(x,y) f(y) y^{2\alpha} dy, \quad x \in \mathbb{R}^d_+ \text{ and } t > 0, \]

where, again, \( P_t^\alpha(x,y) \) is a positive kernel, so \( P_t^\alpha \) is a positive operator as well. In Section 2.1 we shall give several estimates for the heat and Poisson operators and kernels.

Also, for every \( f \in S(\mathbb{R}^d_+) \), we have that

\[ \left( \frac{\partial}{\partial t} - B_\alpha \right) T_t^\alpha f(x) = 0, \quad x \in \mathbb{R}^d_+ \text{ and } t > 0, \]

and,

\[ L_\alpha P_t^\alpha f(x) = 0, \quad x \in \mathbb{R}^d_+ \text{ and } t > 0, \]

where

\[ L_\alpha := \frac{\partial^2}{\partial t^2} - B_\alpha. \]

Let \( i = 1, \ldots, d \). In order to define the \( i \)-th conjugate diffusion and conjugate Poisson semigroups, we shall consider the conjugate operator of \( B_\alpha \), given by \( B^i_{\alpha,x} = -\Delta^i_{\alpha,x} \), being

\[ \Delta^i_{\alpha,x} := -B_\alpha - [\delta_i, \delta^*_i] = \Delta_{\alpha,x} - \frac{2\alpha_i}{x_i^2}. \]

In this case we have

\[ \mathbb{E}^i_{\alpha,x}(-\partial_x \phi^\alpha_{\alpha}(x)) = |y|^2 (-\partial_x \phi^\alpha_{\alpha}(x)), \quad x, y \in \mathbb{R}^d_+. \]

Then, the diffusion and the Poisson semigroups associated with this new operator are defined, respectively, by

\[ \mathbb{T}^\alpha_{\alpha} g(x) = \int_{\mathbb{R}^d_+} e^{-|y|^2 t} (-\partial_x \phi^\alpha_{\alpha}(x)) h_{\alpha + e_i} \left( \frac{g}{\cdot i} \right) (y) y^{2\alpha} dy, \quad x \in \mathbb{R}^d_+ \text{ and } t > 0, \]

and

\[ \mathbb{P}^\alpha_{\alpha} g(x) = \int_{\mathbb{R}^d_+} e^{-|y|^2 t} (-\partial_x \phi^\alpha_{\alpha}(x)) h_{\alpha + e_i} \left( \frac{g}{\cdot i} \right) (y) y^{2\alpha} dy, \quad x \in \mathbb{R}^d_+ \text{ and } t > 0, \]

with \( g \in L^2(\mathbb{R}^d_+, x^{2\alpha} dx) \) and \( \frac{g}{\cdot i} (z) := \frac{g(z_i)}{z_i}, \quad z = (z_1, \ldots, z_d) \in \mathbb{R}^d_+ \). Thus, for every \( g \in S(\mathbb{R}^d_+) \),

\[ \left( \frac{\partial}{\partial t} - \Delta^i_{\alpha,x} \right) \mathbb{T}^\alpha_{\alpha} g(x) = 0, \quad x \in \mathbb{R}^d_+ \text{ and } t > 0, \]

and, if we define

\[ L^i_\alpha = \frac{\partial^2}{\partial t^2} + \Delta^i_{\alpha,x}, \]

we also have

\[ L^i_\alpha \mathbb{P}^\alpha_{\alpha} g(x) = 0, \quad x \in \mathbb{R}^d_+ \text{ and } t > 0, \]
or, equivalently,
\[ L_\alpha P^\alpha_t g(x) = \frac{2\alpha_t}{x^2} P^\alpha_t g(x), \quad x \in \mathbb{R}^d_+ \text{ and } t > 0. \]

From the fact that
\[ \frac{\partial}{\partial x_i} \phi^\alpha_{xy}(x) = -x_i y_i^\alpha \phi^\alpha_{xy}(x), \quad x, y \in \mathbb{R}^d_+, \]
we can rewrite the heat and Poisson semigroups by means of Hankel transforms in the following way
\[ T^\alpha_t g = x_i h_{\alpha+e_i} \left( e^{-|t|^2} h_{\alpha+e_i} \left( \frac{y}{x_i} \right) \right), \quad t > 0, \]
\[ P^\alpha_t g = x_i h_{\alpha+e_i} \left( e^{-|t|^2} h_{\alpha+e_i} \left( \frac{y}{x_i} \right) \right), \quad t > 0, \]
for every \( g \in L^2(\mathbb{R}^d_+, x^{2\alpha} dx). \)

Hence we can rewrite the heat and Poisson semigroups by means of Hankel transforms in the following way
\[ T^\alpha_t g(x) = x_i T_\alpha^{t+e_i} \left( \frac{y}{x_i} \right) (x). \]
Hence
\[ P^\alpha_t g(x) = x_i P_\alpha^{t+e_i} \left( \frac{y}{x_i} \right) (x). \] (2.1)

From these observations we conclude that both \( T^\alpha_t \) and \( P^\alpha_t \) are positive operators. Besides, since we know that for every \( \mu \geq -1/2, I_{\mu+1}(z) < I_\mu(z) \) for all \( z > 0 \), (see [30] and [37]) then for \( \mu \geq 0, u, z > 0 \) we have
\[ u_z W_t^{\mu+1}(u, z) \leq W_t^\mu(u, z), \quad u, z, t > 0. \]

Hence if \( 0 \leq g \in L^2(\mathbb{R}^d_+, x^{2\alpha} dx) \), then
\[ T^\alpha_t g \leq T_\alpha^{t} g, \quad t > 0, \]
and also
\[ P^\alpha_t g \leq P_\alpha^{t} g, \quad t > 0. \] (2.2)

2.1. **Estimates for Bessel–Poisson operators.** In this section we establish some properties for the Poisson kernels and the operators \( P^\alpha_t \) and \( P^\alpha_t \) that will be useful in the proof of our main result.

In the sequel, given \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d_+ \), we will use the notation
\[ \tilde{x}^i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \in \mathbb{R}^{d-1}_+ \]
and we will understand \( x = (x_i, \tilde{x}^i) \) and \( (x, t) = (x_i, \tilde{x}^i, t), \ t > 0 \). Similarly, when \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^{d}_{\geq 0}, \tilde{\alpha}^i := (\alpha_1, \ldots, \alpha_{i-1}, 0, \alpha_{i+1}, \ldots, \alpha_d) \in \mathbb{R}^{d-1}_{\geq 0} \)
and \( \alpha = (\alpha_i, \tilde{\alpha}^i). \)

**Proposition 2.1.** Let \( \alpha \in \mathbb{R}^d_{\geq 0} \) and \( f \in C_\infty(\mathbb{R}^d_+) \). Then,
\[ (P.1) \lim_{t \to +\infty} P_\alpha^t f(x) = 0, \text{ for every } x \in \mathbb{R}^d_+, \]
\[ (P.2) \lim_{t \to 0^+} t \partial_t P_\alpha^t f(x) = 0, \text{ for every } x \in \mathbb{R}^d_+, \]
\[ (P.3) \lim_{t \to +\infty} t \partial_t P_\alpha^t f(x) = 0, \text{ for every } x \in \mathbb{R}^d_+, \]
\[ (P.4) \lim_{x \to 0^+} \partial_{x_i} P_\alpha^t f(x, \tilde{x}^i) = 0, \text{ for every } i = 1, \ldots, d, \ t > 0, \ \tilde{x}^i \in \mathbb{R}^{d-1}_+. \]
\((P.5)\) For every \(x, y \in \mathbb{R}^d_+\) and \(t > 0\),
\[0 \leq P^\alpha_t(x, y) \leq C \frac{t}{(t^2 + |x - y|^2)^{\alpha + \frac{d+1}{2}}}.
\]

\((P.6)\) For every \(i = 1, \ldots, d\),
\[|\partial_{x_i} P^\alpha_t(x, y)| \leq C \frac{t(x_i + y_i)}{(t^2 + |x - y|^2)^{\alpha + \frac{d+1}{2}}}, \ x, y \in \mathbb{R}^d_+ \text{ and } t > 0.
\]

**Proof.** Since \(h_\alpha(f) \in S(\mathbb{R}^d_+)\), by taking into account that, for every \(\mu \geq -1/2\), the function \(z^{-\mu} J_\mu(z)\) is bounded on \((0, \infty)\), we can differentiate under the integral sign to get
\[t \partial_{x_i} P^\alpha_t(x) = -\int_{\mathbb{R}^d_+} |y| |t e^{-|y|^2} \phi_\alpha''(x) h_\alpha(f)(y) y^2 \alpha dy
\]
\[= -h_\alpha(|t | e^{-|t|} h_\alpha(f)(\cdot)(x), \ x \in \mathbb{R}^d_+ \text{ and } t > 0,
\]
and, since \(\frac{d}{dz}(z^{-\mu} J_\mu(z)) = -z^{-\mu} J_{\mu+1}(z), \ z > 0\) and \(\mu \geq -1/2\) ([23] (5.1), p. 103), for every \(i = 1, \ldots, d, \ x \in \mathbb{R}^d_+ \text{ and } t > 0,
\]
\[\partial_{x_i} P^\alpha_t(x) = -x_i \int_{\mathbb{R}^d_+} e^{-|y|^2} \phi_\alpha''(x_i, y_i) \alpha \alpha -1/2 J_{\alpha+1/2}(x_i y_i) y^2 \alpha h_\alpha(f)(y) y^2 \alpha dy.
\]

By using the Dominated Convergence Theorem we can prove properties \([P.1]\).

Let us prove \([P.5]\). Recall that, for \(\mu \geq -1/2\), the modified Bessel functions of first kind \(I_\mu\) verify
\[I_\mu(z) \leq Cz^\mu, \ z \in (0, 1), \quad (2.3)
\]
and
\[I_\mu(z) = \frac{e^z}{\sqrt{2\pi z}} \left(1 + O \left(\frac{1}{z}\right)\right), \ z \geq 1 \quad (2.4)
\]
(see [23] (5.11.8), p. 123).

By using the above one-dimensional estimates \([23]\) and \([24]\) on each variable, we have
\[W^\alpha_t(x, y) \leq C_\alpha \frac{e^{-|x-y|^2/4}}{(2t)^{\alpha + \frac{d}{2}}}, \ x, y \in \mathbb{R}^d_+ \text{ and } t > 0. \quad (2.5)
\]

Then, the subordination formula leads to
\[P^\alpha_t(x, y) \leq \frac{t}{\sqrt{2\pi}} \int_0^\infty e^{-u^2/2} |W^\alpha_u(x, y)| du
\]
\[\leq C \int_0^\infty e^{-u^2/2} \frac{u^{\alpha + \frac{d}{2}}}{\alpha + \frac{d}{2}} du
\]
\[\leq C \frac{t}{(t^2 + |x - y|^2)^{\alpha + \frac{d+1}{2}}}, \ x, y \in \mathbb{R}_d^+ \text{ and } t > 0.
\]

Let \(i = 1, \ldots, d\). In order to prove \([P.6]\) we need estimates for \(\partial_{x_i} W^\alpha_t(x_i, y_i), \ x_i, y_i > 0\), and for \(W^\alpha_t(x, y), \ x, y \in \mathbb{R}^d_+, \ i = 1, \ldots, d, \text{ with } t > 0.
\]

Similarly to \([2.5]\) we get
\[W^\alpha_t(x^i, y^i) \leq C_\alpha \frac{e^{-|x^i-y^i|^2}}{|x^i-y^i|^{\alpha + \frac{d+1}{2}}}, \ x^i, y^i \in \mathbb{R}^{d-1}_+ \text{ and } t > 0. \quad (2.6)
\]
Since $\frac{d}{dz} [z^{-\mu} I_{\mu}(z)] = z^{-\mu} I_{\mu+1}(z)$ for every $z > 0$ and $\mu \geq -1/2$ ([23] (5.7.9), p. 110), we have

$$\partial_x W^{\alpha \beta}_t(x,y) = \partial_x W^{\alpha \beta}_t(x_i,y_i) W^{\gamma \delta}_t(\vec{x}^i, \vec{y}^j)$$

$$= e^{-\frac{x^2+y^2}{4t}} \left( \frac{x_i y_i}{2t} \right)^{-\alpha_i + \frac{1}{2}} \left[ -\frac{x_i}{2t} I_{\alpha_i - \frac{1}{2}} \left( \frac{x_i y_i}{2t} \right) + \frac{y_i}{2t} I_{\alpha_i + \frac{1}{2}} \left( \frac{x_i y_i}{2t} \right) \right] \times W^{\gamma \delta}_t(\vec{x}^i, \vec{y}^j).$$

By using (2.3) we get, for every $t, x_i, y_i > 0$, being $x_i y_i < 2t$,

$$|\partial_x W^{\alpha \beta}_t(x_i, y_i)| \leq C e^{-\frac{x^2+y^2}{4t}} \left( \frac{x_i y_i}{2t} \right)^{-\alpha_i + \frac{1}{2}} \left[ \frac{x_i}{2t} \left( \frac{x_i y_i}{2t} \right)^{\alpha_i} + \frac{y_i}{2t} \left( \frac{x_i y_i}{2t} \right)^{\alpha_i + 1} \right] e^{\frac{x_i y_i}{16t}}$$

$$\leq C x_i + y_i e^{-\frac{x^2+y^2}{4t}} t^{\alpha_i + 1/2}.$$

Now, if $t, x_i, y_i > 0$ with $x_i y_i > 2t$, using (2.4) we have

$$|\partial_x W^{\alpha \beta}_t(x_i, y_i)| = \left| e^{-\frac{x^2+y^2}{4t}} \left( \frac{x_i y_i}{2t} \right)^{-\alpha_i} \frac{e^{\frac{x_i y_i}{2t}}}{\sqrt{2\pi}} \right| \times \left| \frac{-x_i}{2t} \left( 1 + O \left( \frac{2t}{x_i y_i} \right) \right) + \frac{y_i}{2t} \left( 1 + O \left( \frac{2t}{x_i y_i} \right) \right) \right|$$

$$\leq C e^{-\frac{x^2-y^2}{16t}} \left( \frac{x_i y_i}{t} \right)^{-\alpha_i} x_i + y_i \leq C x_i + y_i e^{-\frac{(x_i - y_i)^2}{4t}} t^{\alpha_i + 1}.$$

Therefore, combining the above estimate and (2.0), we deduce

$$|\partial_x W^{\alpha \beta}_t(x,y)| \leq C_{t^{1/2}} x_i + y_i e^{-\frac{|x-y|^2}{8t}} t^{\alpha_i + \frac{1}{2}}. \quad (2.7)$$

From (2.7), we obtain, for every $x, y \in \mathbb{R}_+^d$ and $t > 0$,

$$|\partial_x P^{\alpha \beta}_t(x,y)| \leq \frac{t}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{u^2}{16t}} |\partial_x W^{\alpha \beta}_u(x,y)| du$$

$$\leq C t \int_0^\infty e^{-\frac{u^2}{4|x|^2}} du$$

$$\leq C t \frac{x_i + y_i}{\sqrt{2}\sqrt{|x-y|^2}}.$$ 

\[\Box\]

**Proposition 2.2.** Let $\alpha \in \mathbb{R}_+^d$ and $g \in C_0^\infty(\mathbb{R}_+^d)$. Then, for every $i = 1, \ldots, d$,

- (P.1) $\lim_{t \to +\infty} t^{\alpha_i + 1} g(x) = 0$, for every $x \in \mathbb{R}_+^d$;
- (P.2) $\lim_{t \to 0^+} t^{\alpha_i + 1} g(x) = 0$, for every $x \in \mathbb{R}_+^d$;
- (P.3) $\lim_{t \to +\infty} t^{\alpha_i + 1} g(x) = 0$, for every $x \in \mathbb{R}_+^d$.

**Proof.** We can proceed as in the proof of the corresponding items of Proposition 2.1. \[\Box\]
3. Proofs of Theorems 1.1 and 1.2

In this section we present the proofs of Theorems 1.1 and 1.2. First, we shall prove some auxiliary results.

Lemma 3.1. Let $\alpha \in \mathbb{R}_{\geq 0}^d$ and $f, g \in L^2(\mathbb{R}_+, x^{2\alpha} \, dx)$. Then,

$$\int_{\mathbb{R}_+^d} R_{\alpha,f}(x)g(x)x^{2\alpha} \, dx = -4 \int_0^\infty \int_{\mathbb{R}_+^d} \partial_x P_t^{\alpha}(f)\partial_t \mathbb{P}_t^{\alpha,i}(g)(x)x^{2\alpha} \, dx \, dt,$$

(3.1)
for every $i = 1, \ldots, d$.

Proof. Let $i = 1, \ldots, d$. According to [23, (5.3.7), p. 103] we have that

$$\partial_x P_t^{\alpha}(f)(x) = -x_i h_{\alpha+e_i} \left( e^{-|y|t} h_{\alpha}(f)(y) \right)(x), \quad x \in \mathbb{R}_+^d \text{ and } t > 0. \quad (3.2)$$

Also, we get

$$\partial_t \mathbb{P}_t^{\alpha,i}(g)(x) = -x_i h_{\alpha+e_i} \left( |y| e^{-|y|t} h_{\alpha+e_i}(g)(y) \right)(x), \quad x \in \mathbb{R}_+^d \text{ and } t > 0. \quad (3.3)$$

Note that differentiations under the integral sign to obtain (3.2) and (3.3) are justified because the function $z^{-\mu} J_{\mu}(z)$ is bounded on $(0, \infty)$ for every $\mu \geq -1/2$ ([23, (5.16.1), p. 134]).

From (3.2) and (3.3) we can write

$$\int_0^\infty \int_{\mathbb{R}_+^d} \partial_x P_t^{\alpha}(f)(x)\partial_t \mathbb{P}_t^{\alpha,i}(g)(x)x^{2\alpha} \, dx \, dt$$

$$= \int_0^\infty \int_{\mathbb{R}_+^d} \left( x_i h_{\alpha+e_i}(e^{-|y|t} h_{\alpha}(f)(x)x_i h_{\alpha+e_i} \left( |y| e^{-|y|t} h_{\alpha+e_i}(g)(y) \right) \right)(x)x^{2\alpha} \, dx \, dt$$

$$= \int_0^\infty \int_{\mathbb{R}_+^d} \left( |y| e^{-|y|t} h_{\alpha+e_i}(x_i h_{\alpha+e_i}(g)(y) \right)(x)x^{2\alpha} \, dx \, dt.$$

Since the Hankel transform $h_{\alpha+e_i}$ is self-adjoint on $L^2(\mathbb{R}_+, y^{2(\alpha+e_i)} \, dy)$ and coincides with its inverse, we obtain

$$\int_0^\infty \int_{\mathbb{R}_+^d} \partial_x P_t^{\alpha}(f)(x)\partial_t \mathbb{P}_t^{\alpha,i}(g)(x)x^{2\alpha} \, dx \, dt$$

$$= \int_0^\infty \int_{\mathbb{R}_+^d} |y| e^{-|y|t} h_{\alpha}(f)(y)h_{\alpha+e_i} \left( g(\cdot) \right)(y)y^{2(\alpha+e_i)} \, dy \, dt.$$

Notice that we can interchange the order of integration above since $h_{\alpha}$ and $h_{\alpha+e_i}$ are isometries in $L^2(\mathbb{R}_+, y^{2(\alpha+e_i)} \, dy)$ and $L^2(\mathbb{R}_+, y^{2(\alpha+e_i)} \, dy)$, respectively. Indeed, we have that

$$\int_0^\infty \int_{\mathbb{R}_+^d} |y| e^{-|y|t} h_{\alpha}(f)(y)h_{\alpha+e_i} \left( g(\cdot) \right)(y)y^{2(\alpha+e_i)} \, dy \, dt$$

$$\leq \frac{1}{4} \int_{\mathbb{R}_+^d} \left| h_{\alpha}(f)(y) \right| h_{\alpha+e_i} \left( g(\cdot) \right)(y)y^{2(\alpha+e_i)} \, dy$$

$$\leq \frac{1}{4} \left( \int_{\mathbb{R}_+^d} \left( \frac{y_i}{|y|} |h_{\alpha}(f)(y)| \right)^2 y^{2\alpha} \, dy \right)^{1/2} \left( \int_{\mathbb{R}_+^d} \left| y_i h_{\alpha+e_i} \left( \frac{g(\cdot)}{y} \right)(y) \right|^2 y^{2\alpha} \, dy \right)^{1/2}$$
been used in different settings ([7], [11], [12], [14], [43], and [44]).

A function introduced by Nazarov and Treil ([30]) and some modifications of it have been used in different settings ([7], [11], [12], [14], [43], and [44]).

By using Lemma 3.1 and Hölder’s inequality we can prove that, for every \(1 < p < \infty\), such that

\[ \| \mathcal{F}(f) \|_{L^p(\mathbb{R}^d_+)} \leq C_p \| f \|_p. \]

Thus, we get

\[
\int_0^\infty \int_{\mathbb{R}^d_+} \partial_x \mathcal{P}_t^\alpha(f)(x) \partial_t \mathcal{P}_t^\alpha(i)(y)(x) x^{2\alpha} dx dt \\
= \int_{\mathbb{R}^d_+} \int_0^\infty |y|^2 e^{-|y|^2 t} dt \frac{h_\alpha f(y)}{|y|} h_{\alpha+\epsilon_i}(\frac{g}{|x|}) (y) y^{2(\alpha+\epsilon_i)} dy \\
= \frac{1}{4} \int_{\mathbb{R}^d_+} h_{\alpha+\epsilon_i}(\frac{h_\alpha f}{|y|}) (y) y^{2(\alpha+\epsilon_i)} dy \\
= \frac{1}{4} \int_{\mathbb{R}^d_+} y_i h_{\alpha+\epsilon_i}(\frac{h_\alpha f}{|y|}) (y) y^{2\alpha} dy \\
= -\frac{1}{4} \int_{\mathbb{R}^d_+} R_{\alpha,i} f(y) (y) y^{2\alpha} dy. \quad \square
\]

**Remark 3.2.** Let \(\alpha \in \mathbb{R}^d_{\geq 0}\). We define Littlewood-Paley functions associated with Bessel-Poisson semigroups as follows: for every \(f \in L^p(\mathbb{R}^d_+, x^{2\alpha} dx), \ 1 < p < \infty\) and \(i = 1, \ldots, d\),

\[ G_{\alpha,0}^i(f)(x) = \left( \int_0^\infty t |\partial_{\alpha} \mathcal{P}_t^\alpha(f)(x)|^2 dt \right)^{1/2}, \ x \in \mathbb{R}^d_+, \]

and

\[ G_{\alpha,i}^i(f)(x) = \left( \int_0^\infty t |\partial_x \mathcal{P}_t^\alpha(f)(x)|^2 dt \right)^{1/2}, \ x \in \mathbb{R}^d_+. \]

The heat semigroup \(\{T_t^\alpha\}_{t>0}\) is a diffusion semigroup in the Stein’s sense ([38, p. 65]). Then, \(\{P_t^\alpha\}_{t>0}\) and \(\{P_t^{\alpha,i}\}_{t>0}\) are also Stein diffusion semigroups. According to [38, Theorem 10] (see also [27]), for every \(1 < p < \infty\) and \(i = 1, \ldots, d\), there exists \(C_p > 0\) depending only on \(p\), such that

\[ \| G_{\alpha,0}^i(f) \|_p \leq C_p \| f \|_p, \ f \in L^p(\mathbb{R}^d_+, x^{2\alpha} dx). \]

By proceeding as in the proof of [31, Theorem 6] we can see that for every \(1 < p < \infty\) and \(i = 1, \ldots, d\), there exists \(C_p > 0\) depending only on \(p\), such that

\[ \| G_{\alpha,i}^i(f) \|_p \leq C_p \| f \|_p, \ f \in L^p(\mathbb{R}^d_+, x^{2\alpha} dx). \]

By using Lemma 3.1 and Hölder’s inequality we can prove that, for every \(1 < p < \infty\) and \(i = 1, \ldots, d\), there exists \(C_p > 0\) depending only on \(p\), such that

\[ \| R_{\alpha,i} f \|_p \leq C_p \| f \|_p, \ f \in L^p(\mathbb{R}^d_+, x^{2\alpha} dx). \]

Our next argument, that uses Bellman functions, allows us to improve the last boundedness inequality getting an explicit expression for the constant \(C_p\).

A fundamental ingredient for the proof of our main result is the Bellman function. A function introduced by Nazarov and Treil ([30]) and some modifications of it have been used in different settings ([7], [11], [12], [14], [43], and [44]).
Assume that $p \geq 2$. Note that $p^* = p$. We define as usual $p^* = p/(p-1)$ and we define $\gamma(p) = \frac{p'(p'-1)}{8}$. As in [14] we consider the function $\beta_p$, defined by

$$
\beta_p(s_1, s_2) = s_1^p + s_2^p + \gamma \left\{ \begin{array}{ll} s_2^p - s_1^p, & s_1 \leq s_2, \\ \frac{p}{2} s_1^p + \left( \frac{p}{p-1} \right) s_2^p, & s_1 > s_2, \end{array} \right. \quad s_1, s_2 \geq 0.
$$

For $m = (m_1, m_2) \in \mathbb{N}^2$, the Nazarov-Treil Bellman function $B_p = B_{p,m} : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to [0, \infty)$ is defined by

$$
B_p(\zeta, \eta) = \frac{1}{2} \beta_p(|\zeta|, |\eta|),
$$

with $\zeta \in \mathbb{R}^{m_1}$ and $\eta \in \mathbb{R}^{m_2}$. Let us remark that $B_p \in C^1(\mathbb{R}^{m_1+m_2})$ and $B_p \in C^2(\mathbb{R}^{m_1+m_2} \setminus \{(\zeta, \eta) : \eta = 0 \text{ or } |\zeta|^p = |\eta|^q\})$. In order to get rid of this lack of smoothness we convolve this function with a smooth non-negative radial approximation to the identity $\{\psi_\kappa\}$ where $\kappa > 0$. We call it

$$
B_{\kappa,p} = B_p * \psi_\kappa.
$$

Since both $B_p$ and $\psi_\kappa$ are bi-radial then $B_{\kappa,p}$ is bi-radial and therefore there exists $\beta_{\kappa,p} : [0, \infty]^2 \to [0, \infty)$ such that $B_{\kappa,p}(\zeta, \eta) = \frac{1}{2} \beta_{\kappa,p}(|\zeta|, |\eta|)$. The properties of the functions $\beta_{\kappa,p}$ and $B_{\kappa,p}$ that we need are stated in [14] Proposition 4. We list them below for the sake of completeness.

**Proposition 3.3.** Let $\kappa \in (0, 1)$. Then for $r, s > 0$, we have

$$
\begin{align*}
(B.1) & \quad 0 \leq \beta_{\kappa,p}(r, s) \leq (1 + \gamma(p))(r + \kappa) + (s + \kappa)^{p'}, \\
(B.2) & \quad 0 \leq \partial_r \beta_{\kappa,p}(r, s) \leq C_p \max\{(r + \kappa)^{p-1}, s + \kappa\} \quad \text{and} \quad 0 \leq \partial_s \beta_{\kappa,p}(r, s) \leq C_p(s + \kappa)^{p'-1}, \quad \text{with } C_p \text{ being a positive constant.}
\end{align*}
$$

Moreover $B_{\kappa,p} \in C^\infty(\mathbb{R}^{m_1+m_2})$, and for any $((\zeta, \eta)) \in \mathbb{R}^{m_1+m_2}$ there exists a positive $\tau_{\kappa} = \tau_{\kappa}((|\zeta|, |\eta|))$ such that for $\omega = (\omega_1, \omega_2) \in \mathbb{R}^{m_1+m_2}$ we have

$$
(B.3) \quad \omega \text{Hess}(B_{\kappa,p})(\zeta, \eta) \omega^T \geq \frac{\gamma(p)}{\tau_{\kappa}((|\zeta|, |\eta|))} \tau_{\kappa}((|\omega_1|^2 + |\omega_2|^2)), \quad \text{where Hess}(B_{\kappa,p}) \text{ denotes the Hessian matrix of } B_{\kappa,p}.
$$

Suppose now that $f, g_i \in C_\infty^\infty(\mathbb{R}_+^d)$ and $g_i \geq 0, i = 1, \ldots, d$. Let $g = (g_1, \ldots, g_d)$. We define the function $b_{\kappa,p}(x, t)$ as follows

$$
b_{\kappa,p}(x, t) = B_{\kappa,p}(u(x, t)), \quad x \in \mathbb{R}_+^d \quad \text{and } t > 0,
$$

where $u(x, t) := (P_t^\alpha(f)(x), P_t^\alpha(g_1)(x))$ and $P_t^\alpha g(x) := (P_t^{\alpha,1}(g_1)(x), \ldots, P_t^{\alpha,d}(g_d)(x))$. Note that in this case $m_1 = 1$ and $m_2 = d$.

The following lemma allows us to obtain a crucial connection between the function $b_{\kappa,p}$ and the right hand side of (3.1).

**Lemma 3.4.** Let $\alpha \in \mathbb{R}_{\geq 0}$, and $\kappa \in (0, 1)$. Assume that $f, g_i \in C_\infty^\infty(\mathbb{R}_+^d)$, $g_i \geq 0$, and $i = 1, \ldots, d$. Then, we have the following pointwise equality

$$
\mathcal{L} \alpha b_{\kappa,p}(x, t) = (\partial_t u) \text{Hess}(B_{\kappa,p}(u(x, t)))(\partial_t u)^T + \sum_{j=1}^d (\partial_{x_j} u) \text{Hess}(B_{\kappa,p}(u(x, t)))(\partial_{x_j} u)^T + \sum_{j=1}^d \partial_{\eta_j} B_{\kappa,p}(u(x, t)) \frac{2\alpha_j}{x_j} P_t^{\alpha,j}(g_1)(x),
$$

with $u(x, t) := (P_t^\alpha(f)(x), P_t^\alpha(g_1)(x))$. The properties of the function $\beta_{\kappa,p}$ and $B_{\kappa,p}$ that we need are stated in [14] Proposition 4. We list them below for the sake of completeness.
for every \( x \in \mathbb{R}^d_+ \) and \( t > 0 \).

Moreover,
\[
\mathcal{L}_\alpha b_{\kappa,p}(x,t) \geq \gamma(p)|P^\alpha_t(f)(x)|_\tau |\mathbb{P}^\alpha_t(g)(x)|_\tau \quad , \quad x \in \mathbb{R}^d_+ \text{ and } t > 0.
\]

Here (see \cite{44} (3.4)) we define
\[
|P^\alpha_t(f)(x)|_\tau := \left( |\partial_t P^\alpha_t(f)(x)|^2 + \sum_{j=1}^d |\partial x_j P^\alpha_t(f)(x)|^2 \right)^{1/2}, \quad x \in \mathbb{R}^d_+ \text{ and } t > 0,
\]
and \(|\mathbb{P}^\alpha_t(g)(x)|_\tau := \left( \sum_{i=1}^d |\mathbb{P}^\alpha_t(g_i)(x)|^2 \right)^{1/2}, \quad x \in \mathbb{R}^d_+ \text{ and } t > 0.
\]

**Proof.** The first part of this lemma can be proved following the same lines as in \cite{44} Proposition 5. We use that \( \mathcal{L}_\alpha P^\alpha_t(f)(x) = 0 \) and, for every \( i = 1, \ldots, d \),
\[
\mathcal{L}_\alpha \mathbb{P}^{\alpha,i}_t(g_i)(x) = \frac{2\alpha_i}{x^2} \mathbb{P}^{\alpha,i}_t(g_i)(x),
\]
\( x \in \mathbb{R}^d_+ \) and \( t > 0 \).

By applying now the Bellman function properties \([(B.3)]\) on each term involving the Hessian, \([(B.2)]\) and taking into account that \( (B.3) \) is non-negative (since the semigroup and the function are non-negative), we get, for every \( \kappa \in (0,1) \),
\[
\mathcal{L}_\alpha b_{\kappa,p}(x,t) \geq \frac{\gamma(p)}{2} \left[ \tau_\kappa |\partial_t P^\alpha_t(f)(x)|^2 + \tau_\kappa^{-1} \sum_{i=1}^d |\partial x_i \mathbb{P}^{\alpha,i}_t(g_i)(x)|^2 \right.
\]
\[
\left. + \sum_{j=1}^d \left( \tau_\kappa |\partial x_j P^\alpha_t(f)(x)|^2 + \tau_\kappa^{-1} \sum_{i=1}^d |\partial x_j \mathbb{P}^{\alpha,i}_t(g_i)(x)|^2 \right) \right]
\]
\[
\geq \frac{\gamma(p)}{2} \left( \tau_\kappa |P^\alpha_t(f)(x)|^2_\tau + \tau_\kappa^{-1} |\mathbb{P}^\alpha_t(g)(x)|^2_\tau \right), \quad x \in \mathbb{R}^d_+ \text{ and } t > 0,
\]
being \( \tau_\kappa = \tau_\kappa(x,t) > 0 \).

Then, since \( as + b/s \geq 2\sqrt{ab} \) for every \( s, a, b \in \mathbb{R}_+ \), we get
\[
\mathcal{L}_\alpha b_{\kappa,p}(x,t) \geq \gamma(p)|P^\alpha_t(f)(x)|_\tau |\mathbb{P}^\alpha_t(g)(x)|_\tau.
\]

\( \Box \)

In the sequel, we shall use the following notation: given \( F = (F_1, \ldots, F_d) \) with \( F_i, i = 1, \ldots, d \), defined on \( \mathbb{R}^d_+ \), we denote as usual
\[
|F| = \left( \sum_{i=1}^d |F_i|^2 \right)^{1/2}
\]
and, for \( 1 < p < \infty \), we will write
\[
\|F\|_p := \left( \left( \sum_{i=1}^d |F_i|^2 \right)^{1/2} \right)_p.
\]

We are now in position to prove (1.4). Let \( f, g_i \in C^\infty_c(\mathbb{R}^d_+) \), \( g_i \geq 0, i = 1, \ldots, d \). We deduce from Lemma \cite{54} that
\[
\left| \sum_{i=1}^d \int_{\mathbb{R}^d_+} R_{\alpha,i}(f)(x)g_i(x)x^{2\alpha}dx \right|
\]
\[
\begin{align*}
&= 4 \left| \int_0^\infty \int_{\mathbb{R}_+^d} t \sum_{i=1}^d \partial_{x_i} P_t^{\alpha}(f)(x) \partial_{i} P_t^{\alpha}(g_i)(x) x^{2\alpha} \, dx \, dt \right| \\
&\leq 4 \int_0^\infty \int_{\mathbb{R}_+^d} t \left( \sum_{i=1}^d |\partial_{x_i} P_t^{\alpha}(f)(x)|^2 \right)^{1/2} \left( \sum_{i=1}^d |\partial_{i} P_t^{\alpha}(g_i)(x)|^2 \right)^{1/2} x^{2\alpha} \, dx \, dt \\
&\leq 4 \int_0^\infty \int_{\mathbb{R}_+^d} |P_t^{\alpha} f(x)| \cdot |P_t^{\alpha} g(x)| \cdot x^{2\alpha} \, dx \, dt \\
&\leq 4 \lim_{\epsilon \to 0^+} \lim_{m \to \infty} \int_0^\infty \int_{\frac{1}{m} \mathbb{R}^d} |P_t^{\alpha} f(x)| \cdot |P_t^{\alpha} g(x)| \cdot x^{2\alpha} \, dx \, dt \\
&\leq 4 \lim_{\epsilon \to 0^+} \lim_{m \to \infty} \int_0^\infty \int_{\frac{1}{m} \mathbb{R}^d} L_\alpha b_{\kappa_m, p}(x, t) x^{2\alpha} \, dx \, dt,
\end{align*}
\]
provided that \( \kappa_m \in (0, 1) \), for every \( m \in \mathbb{N} \). We shall prove now that the right-hand side of the above inequality is bounded by a constant times \( ||f||_p + ||g||_p \), with constant independent of dimension.

Let \( \epsilon > 0 \) and \( m \in \mathbb{N} \). We take \( \kappa_m \in (0, 1) \) such that
\[
\kappa_m^{p-1/2} m^{2(\alpha)+d} \leq 1,
\]
which implies, since \( p \geq 2 \), \( \kappa_m^{p-1/2} m^{2(\alpha)+d} \leq 1 \) and \( \lim_{m \to \infty} \kappa_m = 0 \). Let \( 0 < a < b \) be given in such a way that \( \text{supp}(f) \) and \( \text{supp}(g_i), i = 1, \ldots, d \) are contained in \( [a, b]^d \). Recalling the definition of \( L_\alpha \), we have to estimate
\[
I_{m, \epsilon} := \int_0^\infty \int_{\frac{1}{m} \mathbb{R}^d} \partial_t^2 b_{\kappa_m, p}(x, t) x^{2\alpha} \, dx \, dt,
\]
and, for every \( \ell = 1, \ldots, d \),
\[
J_{m, \epsilon, \ell} := \int_0^\infty \int_{\frac{1}{m} \mathbb{R}^d} x^{2\alpha} \partial_x (x^{2\alpha} \partial_{x_\ell} b_{\kappa_m, p}(x, t)) x^{2\alpha} \, dx \, dt.
\]

**Estimation of \( I_{m, \epsilon} \):** By using integration by parts we get
\[
I_{m, \epsilon} = \int_0^\infty \partial_t \left( \int_{\frac{1}{m} \mathbb{R}^d} \partial_t b_{\kappa_m, p}(x, t) x^{2\alpha} \, dx \right) e^{-\epsilon t} \, dt \\
= \lim_{T \to \infty} T e^{-\epsilon T} \int_{\frac{1}{m} \mathbb{R}^d} \partial_t b_{\kappa_m, p}(x, T) x^{2\alpha} \, dx \\
- \lim_{T \to 0^+} t e^{-\epsilon t} \int_{\frac{1}{m} \mathbb{R}^d} \partial_t b_{\kappa_m, p}(x, t) x^{2\alpha} \, dx \\
- \int_0^\infty \int_{\frac{1}{m} \mathbb{R}^d} \partial_t b_{\kappa_m, p}(x, t) x^{2\alpha} \, dx e^{-\epsilon t} (1 - \epsilon t) \, dt.
\]
The third term can be rewritten in the following way
\[
- \int_0^\infty \int_{\frac{1}{m} \mathbb{R}^d} \partial_t b_{\kappa_m, p}(x, t) x^{2\alpha} \, dx e^{-\epsilon t} (1 - \epsilon t) \, dt \\
= - \int_0^\infty \partial_t \left( \int_{\frac{1}{m} \mathbb{R}^d} b_{\kappa_m, p}(x, t) x^{2\alpha} \, dx \right) e^{-\epsilon t} (1 - \epsilon t) \, dt.
\]
Let us prove that

\[ b \]

where we have used again integration by parts and the fact that \( b \) is non-negative.

The function \( P_t^\alpha f(x) \) is continuous in \((x, t) \in (0, \infty) \times [0, \infty) \) and \( \lim_{t \to 0^+} P_t^\alpha f(x) = f(x), \ x \in (0, \infty). \) Also, we have that the function \( P_t^{\alpha,i} g_i(x) \) is continuous in \((x, t) \in (0, \infty) \times [0, \infty) \) and \( \lim_{t \to 0^+} P_t^{\alpha,i} g_i(x) = \lim_{t \to 0^+} x_i P_t^{\alpha+\epsilon_i}(g_i/\epsilon_i)(x) = g_i(x), \ x \in (0, \infty) \) for each \( i = 1, \ldots, d. \) We obtain then

\[
\lim_{t \to 0^+} \int_{[1/m]^d} b_{k_m,p}(x, t) x^{2\alpha} dx = \int_{[1/m]^d} b_{k_m,p}(x, 0) x^{2\alpha} dx
\]

\[
= \int_{[1/m]^d} B_{k_m,p}(f(x), g(x)) x^{2\alpha} dx.
\]

Therefore,

\[
I_{m,\epsilon} \leq \lim_{T \to \infty} T e^{-\epsilon T} \int_{[1/m]^d} \partial_t b_{k_m,p}(x, T) x^{2\alpha} dx
\]

\[
- \lim_{t \to 0^+} t e^{-\epsilon t} \int_{[1/m]^d} \partial_t b_{k_m,p}(x, t) x^{2\alpha} dx
\]

\[
+ \lim_{T \to \infty} e^{-\epsilon T}(T - 1) \int_{[1/m]^d} b_{k_m,p}(x, T) x^{2\alpha} dx
\]

\[
+ \int_{[1/m]^d} B_{k_m,p}(f(x), g(x)) x^{2\alpha} dx
\]

\[
+ \int_0^\infty \int_{[1/m]^d} b_{k_m,p}(x, t) x^{2\alpha} dx^2 t e^{-\epsilon t} dt =: \sum_{j=1}^5 I_{j, m, \epsilon}.
\]

Let us prove that \( I_{1, m, \epsilon} = I_{2, m, \epsilon} = I_{3, m, \epsilon} = 0. \) Since \( \int_{\mathbb{R}^d_+} P_t^\alpha(x, y) g^{2\alpha} dy = 1 \) for every \( x \in \mathbb{R}^d_+ \) and \( t > 0, |\partial_t P_t^\alpha(x, y)| \leq C P_t^{\alpha/2}(x, y), \ x, y \in \mathbb{R}^d_+ \) and \( t > 0, \) we deduce that \( |P_t^\alpha(f)(x)| + |\partial_t P_t^\alpha(f)(x)| \leq C\|f\|_\infty, \) for every \( x \in \mathbb{R}^d_+ \) and \( t > 0. \) Also, by \( (2.1) \) and \( (2.2), \) it follows that \( |P_t^{\alpha,i}(g_i)(x)| + |\partial_t P_t^{\alpha,i}(g_i)(x)| \leq C\|g_i\|_\infty, \) for every \( i = 1, \ldots, d. \) Then, from \( (B.2) \) we have

\[
t |\partial_t b_{k_m,p}(x, t)| \leq C \left( |P_t^\alpha f(x)|^{p-1} + |P_t^{\alpha,i} g(x)| + |P_t^\alpha g(x)|^{p'-1} + K_{m-1}^{p'-1} \right)
\]

\[
\times (|\partial_t P_t^\alpha f(x)| + |t \partial_t P_t^\alpha g(x)|))
\]
\[ \leq C \left( \| f \|_\infty^{-1} + \left( \sum_{i=1}^{d} \| g_i \|_\infty^2 \right)^{1/2} + \left( \sum_{i=1}^{d} \| g_i \|_\infty^2 \right)^{\frac{\epsilon - 1}{\epsilon}} + \kappa_m^{p'-1} \right) \]
\[ \times \left( \| f \|_\infty + \left( \sum_{i=1}^{d} \| g_i \|_\infty^2 \right)^{1/2} \right). \]

for \( x \in \mathbb{R}_d^d \) and \( t > 0 \).

Then, from \((\text{P.2})\) \((\text{P.3})\) and \((\text{P.2})\) we deduce that
\[ t \partial_t b_{\kappa_m,p}(x,t) \to 0, \]
whenever \( t \to \infty \) or \( t \to 0^+ \).

Thus, by applying the Dominated Convergence Theorem we get that \( I_{1,m,\epsilon}^1 = I_{1,m,\epsilon}^2 = 0 \).

Let us take care of \( I_{3,m,\epsilon}^3 \). In this case, by using \((\text{B.1})\) for \( T > \epsilon^{-1} \), we have
\[ (\epsilon T - 1)e^{-\epsilon T} \int [\frac{1}{m}]^d b_{\kappa_m,p}(x,T)x^{2\alpha} \, dx \]
\[ \leq Ct e^{-\epsilon T} \int [\frac{1}{m}]^d \left[ (|P_T^p f(x)| + \kappa_m)^p + (|P_T^p g(x)| + \kappa_m)^p \right] x^{2\alpha} \, dx \]
\[ \leq Ct e^{-\epsilon T} \left( \| f \|_p^p + \| g \|_{p'}^p + m^{2\alpha+d} (\kappa_m^p + \kappa_m^{p'}) \right) \to 0, \]
as \( T \to \infty \). Hence, \( I_{3,m,\epsilon}^3 = 0 \) which yields that
\[ I_{3,m,\epsilon}^3 \leq I_{4,m,\epsilon}^1 + I_{5,m,\epsilon}^5 \]
\[ = \int [\frac{1}{m}]^d B_{\kappa_m,p}(f(x), g(x))x^{2\alpha} \, dx + \int_0^\infty \int [\frac{1}{m}]^d b_{\kappa_m,p}(x,s/\epsilon)x^{2\alpha} \, dx \, ds. \]

Following the ideas in \([44, \text{p. 15}]\), from \((\text{B.1})\) and the choice of \( \kappa_m \), we have that
\[ I_{4,m}^4 \leq \frac{1 + \gamma(p)}{2} \left( (1 + \epsilon)^p \| f \|_p^p + (1 + \epsilon)^p \| g \|_{p'}^p \right) \]
\[ + (1 + \epsilon^{-1})^p m^{2\alpha+d} (1 + \epsilon^{-1})^p \kappa_m ^{p'} \]
\[ \leq \frac{1 + \gamma(p)}{2} \left( (1 + \epsilon)^p \| f \|_p^p + (1 + \epsilon)^p \| g \|_{p'}^p + \kappa_m^{1/2} \right) \left( (1 + \epsilon^{-1})^p + (1 + \epsilon^{-1})^p \right). \]

Then, since \( \lim_{m \to \infty} \kappa_m = 0 \), we obtain
\[ \limsup_{m \to \infty} I_{4,m}^4 \leq \frac{1 + \gamma(p)}{2} \left( \| f \|_p^p + \| g \|_{p'}^p \right). \]

In order to estimate \( I_{5,m,\epsilon}^5 \), we apply \((\text{B.1})\) \((\text{2.2})\) and the properties on \( \kappa_m \), to get
\[ I_{5,m,\epsilon}^5 \leq (1 + \gamma(p)) \left[ \int_0^\infty \int [\frac{1}{m}]^d |P_{s\epsilon}^\alpha f(x)|^p x^{2\alpha} \, dx \, ds + \kappa_m^{p} m^{2\alpha+d} \right]. \]
\[ +2^{p-1} \left( \int_0^\infty \int_0^\infty \int_{|x-y|\leq 1/m} |P_{s/e}^\alpha f(x)|^p |x|^{s} \, dx \right) \left( \int_0^\infty \int_0^\infty \int_{|x-y|\leq 1/m} |P_{s/e}^\alpha g(x)|^p |x|^{s} \, dx \right) \]

\[ \leq (1 + \gamma(p))2^{p-1} \int_0^\infty \int_0^\infty \int_{|x-y|\leq 1/m} \left( |P_{s/e}^\alpha f(x)|^p + |P_{s/e}^\alpha g(x)|^p \right) |x|^{s} \, dx \]

\[ + (1 + \gamma(p))2^{p-1} \kappa_m \cdot \]

with \(|P_{s/e}^\alpha g(x)| = \left( \sum_{i=1}^d |P_{s/e}^\alpha g(x)|^q \right)^{1/2}\) and \(|P_{s/e}^\alpha f(x)| = \left( \sum_{i=1}^d |P_{s/e}^\alpha f(x)|^q \right)^{1/2}\).

Thus

\[ \limsup_{m \to \infty} I_{m, \epsilon}^5 \leq (1 + \gamma(p))2^{p-1} \int_0^\infty \int_0^\infty \int_{|x-y|\leq 1/m} \left( |P_{s/e}^\alpha f(x)|^p + |P_{s/e}^\alpha g(x)|^p \right) |x|^{s} \, dx \]

If we define

\[ H_e(x, s) = \left( |P_{s/e}^\alpha f(x)|^p + |P_{s/e}^\alpha g(x)|^p \right) se^{-s}, \quad x \in \mathbb{R}^d_+ \text{ and } s > 0, \]

we observe that \(\lim_{\epsilon \to 0^+} H_e(x, s) = 0\) for every \(x \in \mathbb{R}^d_+\) and \(s > 0\), by virtue of (P.1).

On the other hand, since the maximal operator \(P_t^\alpha f = \sup_{t>0} |P_t^\alpha f|\) is bounded on \(L^q(\mathbb{R}^d_+, x^{2\alpha} dx)\) for every \(1 < q \leq \infty\) by virtue of Proposition 6.2 and the fact that \(\|P_t^\alpha f\|_q \leq \|T_t^\alpha f\|_q \leq C\|f\|_q\), for \(1 < q \leq \infty\), we get that

\[ H_e(x, s) \leq \left( |P_{s/e}^\alpha f(x)|^p + \sum_{i=1}^d |P_{s/e}^\alpha g_i(x)|^p \right) se^{-s} \in L^1(\mathbb{R}^d_+, x^{2\alpha} dx ds). \]

Finally, by applying the Dominated Convergence Theorem we obtain that

\[ \limsup_{\epsilon \to 0^+} \limsup_{m \to \infty} I_{m, \epsilon}^5 = 0. \]

Therefore

\[ \limsup_{\epsilon \to 0^+} \limsup_{m \to \infty} \int_0^\infty \int_{|x-y|\leq 1/m} \partial_x^\alpha b_{\kappa_m, p}(x, t)x^{2\alpha} dx e^{-\epsilon t} dt \leq \frac{1 + \gamma(p)}{2} \left( \|f\|_p + \|g\|_p \right). \]

**Estimation of** \(J_{m, \epsilon, t}^\ell\): Let \(\ell = 1, \ldots, d\). We shall see that

\[ \limsup_{\epsilon \to 0^+} \limsup_{m \to \infty} J_{m, \epsilon, t}^\ell \leq 0. \]

Actually, we will show that for every \(\epsilon > 0\),

\[ \limsup_{m \to \infty} J_{m, \epsilon, t} \leq 0. \quad (3.5) \]

In order to do so, we integrate by parts on the \(\ell\)-th variable to get

\[ J_{m, \epsilon, t}^\ell = \int_0^\infty \int_{|x-y|\leq 1/m} m^{2\alpha/\ell} \partial_x^\alpha b_{\kappa_m, p}(m, \tilde{x}^\ell, t)(\tilde{x}^\ell)^{2\alpha/\ell} d\tilde{x}^\ell e^{-\epsilon t} dt \]

\[ - \int_0^\infty \int_{|x-y|\leq 1/m} (1/m)^{2\alpha/\ell} \partial_x^\alpha b_{\kappa_m, p}(1/m, \tilde{x}^\ell, t)(\tilde{x}^\ell)^{2\alpha/\ell} d\tilde{x}^\ell e^{-\epsilon t} dt \]

\[ =: J_{m, \epsilon, t}^1 - J_{m, \epsilon, t}^2. \]

Then, we will see that

\[ \limsup_{m \to \infty} J_{m, \epsilon, t}^1 \leq \liminf_{m \to \infty} J_{m, \epsilon, t}^2 \leq 0. \]
Let us first remark that from (B.2), (2.2), and the boundedness of $P_t^\alpha$ on $L^\infty(\mathbb{R}^d_+)$, we get, for every $x \in \mathbb{R}^d_+$ and $t > 0$,

$x^{2\alpha_i} |\partial_{x_i} b_{\kappa, \beta}(x, t)|$

$$\leq C_p \left( \left| (P_t^\alpha f(x)) + \kappa_m \right|^{p-1} + \left| \mathbb{P}_t^\alpha g(x) \right| + \kappa_m \right) x^{2\alpha_i} |\partial_{x_i} P_t^\alpha f(x)|$$

$$+ C_p \left( \left| P_t^\alpha g(x) \right| + \kappa_m \right)^{p'-1} x^{2\alpha_i} |\partial_{x_i} P_t^\alpha g(x)|$$

$$\leq C_p \left( \left| (P_t^\alpha f(x)) + \kappa_m \right|^{p-1} + \left| P_t^\alpha g(x) \right| + \kappa_m \right) x^{2\alpha_i} |\partial_{x_i} P_t^\alpha f(x)|$$

$$+ C_p \left( \left| P_t^\alpha g(x) \right| + \kappa_m \right)^{p'-1} x^{2\alpha_i} |\partial_{x_i} P_t^\alpha g(x)|$$

$$\leq C_p \left( \left\| f \right\|_\infty + \kappa_m \right)^{p-1} + \left( \sum_{i=1}^d \left\| g_i \right\|_\infty^2 \right)^{1/2} + \kappa_m \right) x^{2\alpha_i} |\partial_{x_i} P_t^\alpha f(x)|$$

$$+ C_p \left( \sum_{i=1}^d \left\| g_i \right\|_\infty^2 \right)^{1/2} + \kappa_m \right)^{p'-1} x^{2\alpha_i} |\partial_{x_i} P_t^\alpha g(x)|.$$

In order to estimate $J^m_{\alpha, \beta, \ell}$, we consider the functions

$A(x, t) = x^{2\alpha_i} |\partial_{x_i} P_t^\alpha f(x)|, x \in \mathbb{R}^d_+$ and $t > 0,$

and, for $i = 1, \ldots, d,$

$B_i(x, t) = x^{2\alpha_i} |\partial_{x_i} P_t^\alpha g_i(x)|, x \in \mathbb{R}^d_+$ and $t > 0.$

According to (P.6) we get

$A(x, t) \leq C x^{2\alpha_i} \int_{[a, b]^d} \frac{t(x + 1)}{(t^2 + |x - \gamma|^2)^{1/2} + \frac{\alpha_i + 2}{2}} y^{2\alpha} dy$

$$\leq C t x^{2\alpha_i + 1} \int_{[a, b]^d} \frac{1}{(x^2 + |\hat{x} - \hat{\gamma}|^2)^{1/2} + \frac{\alpha_i + 2}{2}} dy$$

$$\leq C t \int_{[a, b]^d} \frac{1}{(1 + |\hat{x} - \hat{\gamma}|^2)^{1/2} + \frac{\alpha_i + 2}{2}} dy,$$

when $t > 0, x \in \mathbb{R}^d_+$ and $x_\ell > 2b,$ and

$A(x, t) \leq C x^{2\alpha_i} \int_{[a, b]^d} \frac{t(x + 1)}{(t^2 + |x - \gamma|^2)^{1/2} + \frac{\alpha_i + 2}{2}} y^{2\alpha} dy$

$$\leq C t x^{2\alpha_i} \int_{[a, b]^d} \frac{1}{(1 + |\hat{x} - \hat{\gamma}|^2)^{1/2} + \frac{\alpha_i + 2}{2}} dy,$$

when $t > 0, x \in \mathbb{R}^d_+$ and $x_\ell < a/2.$

By (2.1) we get

$\partial_{x_i} P_t^{\alpha_i}(g)(\cdot) = \delta_{i, \ell} P_t^{\alpha_i+e_i}(\frac{g}{a_i})(\cdot) + x_i \partial_{x_i} P_t^{\alpha_i+e_i}(\frac{g}{a_i})(\cdot),$ (3.6)

for $t > 0$ and $x \in \mathbb{R}^d_.$ Here, $\delta_{i, \ell} = 0,$ when $i \neq \ell,$ and $\delta_{i, \ell} = 1,$ when $i = \ell.$ Then, by using (P.5) and (P.6) and proceeding as above, we obtain

$B_i(x, t) \leq C x^{2\alpha_i} \left( \int_{[a, b]^d} \frac{t}{(t^2 + |x - \gamma|^2)^{1/2} + \frac{\alpha_i + 2}{2}} y^{2\alpha} dy \right).$
for every

\( t > 0, \ x \in \mathbb{R}^d_+ \) and \( x_\ell > 2b \), and

\[
B_\ell(x, t) \leq C t x_\ell^{2\alpha}(1 + x_\ell) \int_{[a, b]^d} \frac{1}{(1 + |\tilde{x}^\ell - \tilde{y}^\ell|^\alpha + \frac{m}{m^\ell})} dy,
\]

where \( t > 0, \ x \in \mathbb{R}^d_+ \) and \( x_\ell < a/2 \).

By the case \( x_\ell > 2b \) in the estimates of \( A(x, t) \) and \( B_\ell(x, t), i = 1, \ldots, d \), we get, for every \( \tilde{x}^\ell \in [1/m, m]^{d-1} \) and \( t > 0 \), that

\[
m^{2\alpha} |\partial_{x_\ell} b_{\kappa_m,p}(x, \tilde{x}^\ell, t)| \leq C(f, g, b) \frac{t}{m} \int_{[a, b]^d} \frac{1}{(1 + |\tilde{x}^\ell - \tilde{y}^\ell|^\alpha + \frac{m}{m^\ell})} dy.
\]

Then,

\[
m^{2\alpha} |\partial_{x_\ell} b_{\kappa_m,p}(x, \tilde{x}^\ell, t)| \rightarrow 0,
\]
as \( m \rightarrow \infty \). Also, the right-hand side of the above inequality belongs to \( L^1(\mathbb{R}_{x_\ell} \times (0, \infty), (\tilde{x}^\ell)^{2\alpha} dx^\ell e^{-ct} dt) \). Therefore \( \limsup_{m \rightarrow \infty} J_{m, \epsilon, \ell}^1 = 0 \).

We shall see now that \( \liminf_{m \rightarrow \infty} J_{m, \epsilon, \ell}^2 \geq 0 \). From \( \text{[8.10]} \) and the fact that \( P_\ell^{\alpha + \epsilon} \) is positive, we know that

\[
\partial_{x_\ell} \| P_\ell^{\alpha} (g_i) (x) \| \geq x_\ell \partial_{x_\ell} P_\ell^{\alpha + \epsilon} \left( \frac{g_i}{\lambda_1} \right) (x),
\]

for every \( i = 1, \ldots, d \), \( x \in \mathbb{R}^d_+ \) and \( t > 0 \).

On the other hand, \( \text{[B.2]} \) implies that \( \partial_{x_\ell} B_{\kappa_m,p}(u(x, t)) \geq 0 \), for every \( i = 1, \ldots, d \). This, together with the above inequality, yields

\[
x_\ell^{2\alpha} \partial_{x_\ell} b_{\kappa_m,p}(x, t) = x_\ell^{2\alpha} \partial_{x_\ell} B_{\kappa_m,p}(u(x, t)) \partial_{x_\ell} P_\ell^{\alpha} f(x)
\]

\[
+ x_\ell^{2\alpha} \sum_{i=1}^d \partial_{x_\ell} B_{\kappa_m,p}(u(x, t)) \partial_{x_\ell} P_\ell^{\alpha, i} (g_i)(x)
\]

\[
\geq \partial_{x_\ell} B_{\kappa_m,p}(u(x, t)) x_\ell^{2\alpha} \partial_{x_\ell} P_\ell^{\alpha} f(x)
\]

\[
+ \sum_{i=1}^d \partial_{x_\ell} B_{\kappa_m,p}(u(x, t)) x_\ell x_\ell^{2\alpha} \partial_{x_\ell} P_\ell^{\alpha + \epsilon} \left( \frac{g_i}{\lambda_1} \right) (x), \ x \in \mathbb{R}^d_+ \text{ and } t > 0.
\]

Thus,

\[
J_{m, \epsilon, \ell}^2 \geq \int_0^{\infty} \left( \frac{1}{m} \right)^{2\alpha} \int_{[1/m, m]}^{d-1} \partial_{x_\ell} B_{\kappa_m,p}(u(1/m, \tilde{x}^\ell, t))
\]

\[
\times \partial_{x_\ell} P_\ell^{\alpha} f(1/m, \tilde{x}^\ell) (\tilde{x}^\ell)^{2\alpha} d\tilde{x}^\ell e^{-ct} dt
\]

\[
+ \sum_{i=1}^d \int_0^{\infty} \left( \frac{1}{m} \right)^{2\alpha} \partial_{x_\ell} B_{\kappa_m,p}(u(1/m, \tilde{x}^\ell, t))
\]

\[
\times x_\ell \partial_{x_\ell} P_\ell^{\alpha + \epsilon} \left( \frac{g_i}{\lambda_1} \right) (1/m, \tilde{x}^\ell) (\tilde{x}^\ell)^{2\alpha} d\tilde{x}^\ell e^{-ct} dt
\]
\[
=: J_{m,e,t}^3
\]

From property \([PA]\) we get that the integrands of \(J_{m,e,t}^3\) converge pointwise to 0 as \(m \to \infty\). On the other hand, in order to obtain an integrable function independent of \(m\) we proceed to bound the integrands as we have done before, but now considering the estimates of \(A\) and \(B_i, i = 1, \ldots, d\), on the range \(x_i < a/2\). Then, by the Dominated Convergence Theorem, we get that \(\lim_{m \to \infty} J_{m,e,t}^3 = 0\).

And so \(\liminf_{m \to \infty} J_{m,e,t}^2 \geq 0\).

Bringing things together we have proved that

\[
\left| \sum_{i=1}^{d} \int_{\mathbb{R}_+^d} R_{\alpha,i} f(x)g_i(x)x^{2\alpha} \, dx \right| \leq 2 \frac{1 + \gamma(p)}{\gamma(p)} \left( \|f\|_p^p + \|g\|_{p'}^p \right).
\]

Let us take \(\lambda > 0\). By applying the above inequality to \(\lambda f\) and \(\frac{g_i}{x}\), \(i = 1, \ldots, d\), we have

\[
\left| \sum_{i=1}^{d} \int_{\mathbb{R}_+^d} R_{\alpha,i} f(x)g_i(x)x^{2\alpha} \, dx \right| \leq 2 \frac{1 + \gamma(p)}{\gamma(p)} \left( \lambda^p \|f\|_p^p + \frac{1}{\lambda^{p'}} \|g\|_{p'}^p \right),
\]

and minimizing on \(\lambda\) yields

\[
\left| \sum_{i=1}^{d} \int_{\mathbb{R}_+^d} R_{\alpha,i} f(x)g_i(x)x^{2\alpha} \, dx \right| \leq 2 \frac{1 + \gamma(p)}{\gamma(p)} \left( \left( \frac{p}{p'} \right)^{1/p} + \left( \frac{p'}{p} \right)^{1/p'} \right) \|f\|_p \|g\|_{p'}^p.
\]

Approximation arguments allow us to see that the last inequality also holds when \(0 < g_i \in L^p(\mathbb{R}_+^d, x^{2\alpha} \, dx), i = 1, \ldots, d\) and \(f \in C_c^\infty(\mathbb{R}_+^d)\). Therefore, if \(f \in C_c^\infty(\mathbb{R}_+^d)\) and \(g_i \in L^p(\mathbb{R}_+^d, x^{2\alpha} \, dx)\) for every \(i = 1, \ldots, d\), by considering \(g_i = g_i^+ - g_i^-\), where \(g_i^+ = \max\{g_i, 0\}\), we deduce

\[
\|R_{\alpha} f\|_{L^p(\mathbb{R}_+^d, x^{2\alpha} \, dx)} \leq 8 D_p \|f\|_{L^p(\mathbb{R}_+^d, x^{2\alpha} \, dx)}.
\]

Finally, as it was proved in [14] p. 761], the constant \(D_p\) is smaller than \(6(p^* - 1)\), which let us conclude with the proof in this case.

In order to prove the result when \(1 < p < 2\) we can proceed in a similar way by interchanging the roles between \(p\) and \(p'\) and between \(P_\alpha^p(f)\) and \(P_\alpha^{p'}(g)\). More precisely, we need to consider \(b_{n,p}(x,t) = B_{n,p}(P_\alpha^p(g)(x), P_\alpha^{p'}(f)(x))\), where \(B_{n,p}\) is the Bellman function defined on \(\mathbb{R}^d \times \mathbb{R}\), that is, \(m_1 = d\) and \(m_2 = 1\). This function satisfies Proposition 3.3 with \(p\) replaced by \(p'\) and viceversa.

Theorem 1.2 follows the same lines of [12] Corollary 2], by using above arguments and proceeding as in the proof of [12] Proposition 3].

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