The tempered space-fractional Cattaneo equation

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Abstract

We consider the time-fractional Cattaneo equation involving the tempered Caputo space-fractional derivative. There is an increasing interest in the recent literature for the applications of the fractional-type Cattaneo equations to heat transfer models. Our main aim is to discuss the role played by a fractional tempered operator in this framework. We show that the fundamental solution coincides with the probability law of a time-changed Brownian motion, obtained by means of a tempered stable subordinator. We find the characteristic function of this process and we explain the main differences with previous stochastic treatments of the time-fractional Cattaneo equation. We also provide the solution of a Dirichlet problem for the tempered fractional Cattaneo equation by means of the H-Fox function.

Keywords: Cattaneo equation, tempered fractional derivative, stochastic processes

1. Introduction

The recent experimental observation of wavelike thermal transport in graphite at temperatures above 100 K \cite{13}, which confirms the just derived theoretical
prediction \cite{7}, solves the long-standing challenge to establish the existence, in
certain materials, of phonon hydrodynamics and second sound phenomenon at
relatively high temperatures \cite{19,16}. In fact, the occurrence of second sound
was previously limited to a handful of materials at low temperatures and there-
fore the scientific and practical significance of this phenomenon was limited.
This new experimental evidence indeed potentially indicate an important role
of second sound in microscale transient heat transport in two-dimensional and
layered materials in a wide temperature range.

A wavelike thermal transport implies a phonon hydrodynamics regime that
is intermediate between ballistic and diffusive regimes, and it is properly de-
scribed by a generalization of Fourier’s law into the viscous heat equation (or
damped wave equation) \cite{30}. Thus, second sound in solids occurs when the local
temperature follows an hyperbolic equation analogue to the telegrapher’s equa-
tion in electromagnetism and to Cattaneo’s equation in conduction problems
\cite{12} that reads

\[
\frac{\partial^2 f}{\partial t^2} + \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.
\]  

(1)

In this respect, we remind that when the same hyperbolic equation governs the
pressure or the density then we have the first sound. In general, for second
sound phenomena the damping term dominates the inertial term and we have
the diffusion equation, while for the first sound phenomena the opposite is true
and we have the wave equation.

This experimental ascertainment at relative high temperature of the second
sound in graphene, and in general in solids, motivated us to a mathematical
investigation of non-local extensions of the viscous heat equation (1) in the
spirit of taking into account the already widely established evidences of non-
local effects both in diffusive, see, e.g., \cite{15}, and viscous, see, e.g. \cite{22}, systems.
Since from the physical point of view, it is more appropriate to speak about
telegrapher’s equation when we speak of applications to electromagnetism, while
it is common to use the name Cattaneo equation in problems regarding heat
conduction and anomalous transport processes, then hereinafter we refer to the
Cattaneo equation, which, we remind, still calls for a deep mathematical analysis both in the classical formulation \[31, 5\] and in the non-local extension \[9, 2\]. 

The Cattaneo equation plays indeed a relevant role in many different physical contexts. In particular, in random motion and heat propagation models with finite front velocity \[14\] ranging from the micro-scale of the run-and-tumble bacterial dynamics \[1\] to the the regional-scale of wildland fire propagation \[25, 4\]. The space- and time-fractional counterpart of this equation has gained a relevant interest both for physical applications and stochastic models related to continuous-time persistent random walks \[10, 23\]. Non-local generalization of the Cattaneo equation through the fractional calculus has been already investigated, e.g., \[6, 26, 27, 8, 10, 23\]. Moreover, some recent papers have been devoted to the analysis of the generalizations of the Fourier law leading to time and space-fractional Cattaneo equations in the framework of heat transfer models, e.g., \[28, 29\]. There is a wide engineering literature about the applications of the Cattaneo law (and its experimental validation) in thermoelectricity and heat conduction in solids, e.g., \[21\]. 

Here, the novel contribution with respect to the literature, lays on the analysis of the role of the tempering of fractional derivative when applied for generalizing the space-derivative. We remind that, indeed, the tempered fractional diffusion equation has been already investigated, e.g., \[18, 20\].

The rest of the paper is organized as follows. First, we provide in Section 2 the preliminaries notions on the tempered derivatives and the related processes, and later in Section 3 the main results are reported.

2. Preliminaries on tempered fractional derivatives and related stochastic processes

The shifted fractional derivative has been used in the physical literature for mathematical models of wave propagation in porous media \[11\] and in probability in relation with the Tempered Stable Subordinator (TSS). The shifted
fractional derivative is defined as
\[
\left(\lambda + \frac{d}{dx}\right)^\alpha f(x) = e^{-\lambda x} D_x^\alpha [e^{\lambda x} f(x)], \quad \alpha \in (0, 1), \quad \lambda \geq 0, \quad x \geq 0,
\]
where \(D_x^\alpha\) denotes the space fractional derivative in the sense of Caputo of order \(\alpha \in (0, 1)\), i.e.,
\[
D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha} \frac{\partial f}{\partial \xi} d\xi.
\]
Thus, the Laplace transform of the shifted fractional derivative is given by
\[
\mathcal{L}\left\{\left(\lambda + \frac{d}{dx}\right)^\alpha f(x)\right\}(s) = (s + \lambda)^\alpha \tilde{f}(s) - (s + \lambda)^{\alpha-1} f(0^+),
\]
where we denoted by \(s\) the Laplace parameter and \(\tilde{f}(s)\) the Laplace transform of the function \(f(x)\), i.e.,
\[
\int_0^\infty e^{-sx} f(x) \, dx = \tilde{f}(s).
\]
In probability the transition density \(f_{\lambda,\alpha}(x,t)\) of the TSS can be introduced by "tempering" the transition density of the \(\alpha\)-stable subordinator \(h_\alpha(x,t)\) as follows
\[
f_{\lambda,\alpha}(x,t) := e^{-\lambda x + \lambda^\alpha t} h_\alpha(x,t).
\]
Indeed, it is possible to prove that the density of the TSS satisfies the following tempered-fractional equation
\[
\frac{\partial}{\partial t} f_{\lambda,\alpha}(x,t) = \lambda^\alpha f_{\lambda,\alpha}(x,t) - \left(\lambda + \frac{\partial}{\partial x}\right)^\alpha f_{\lambda,\alpha}(x,t) = -\frac{\partial^{\lambda,\alpha}}{\partial x^{\lambda,\alpha}} f_{\lambda,\alpha}(x,t),
\]
where
\[
\frac{\partial^{\lambda,\alpha}}{\partial x^{\lambda,\alpha}} f(x,t) := \left(\lambda + \frac{\partial}{\partial x}\right)^\alpha f(x,t) - \lambda^\alpha f(x,t),
\]
under the conditions
\[
f_{\lambda,\alpha}(x,0) = \delta(x), \quad f_{\lambda,\alpha}(0,t) = 0.
\]
We here briefly recall the notion of fractional tempered stable (TS) process. We denote by \(\mathcal{L}_\alpha(t):= \inf\{s: H_\alpha(s) > t\}, t \geq 0, \alpha \in (0,1),\) the inverse of the \(\alpha\)-stable subordinator \(H_\alpha(t)\), then
Definition 2.1. Let $\mathcal{L}_\nu(t)$, with $t \geq 0$, be the inverse of the stable subordinator, then the fractional TS process is defined as

$$T_{\lambda,\alpha}^\nu(t) := T_{\lambda,\alpha}(\mathcal{L}_\nu(t)), \quad t \geq 0, \quad \lambda \geq 0, \quad \nu, \alpha \in (0, 1),$$

where $\mathcal{L}_\nu$ is independent of the tempered stable subordinator (TSS) $\mathcal{T}_{\lambda,\nu}$.

The density of the fractional TS process $T_{\lambda,\alpha}^\nu(t)$ satisfies the tempered fractional equation \[3, \text{Theorem 6}\]

$$D^\nu_t f = \left[ \lambda^\alpha - \left( \lambda + \frac{\partial}{\partial x} \right)^\alpha \right] f, \quad \lambda \geq 0, \quad \nu, \alpha \in (0, 1),$$

under the initial-boundary conditions

$$\begin{cases} f(x, 0) = \delta(x), \\ f(0, t) = 0. \end{cases}$$

Finally, we recall also that the density of the time-changed Brownian motion

$$X_{\lambda,\alpha}^\nu(t) := B(T_{\lambda,\alpha}^\nu(t)), \quad \lambda \geq 0, \quad \nu, \alpha \in (0, 1),$$

coincides with the solution of the tempered equation \[3\]

$$D^\nu_t g = \left[ \lambda^\alpha - \left( \lambda - \frac{\partial^2}{\partial x^2} \right)^\alpha \right] g, \quad -\infty < x < +\infty.$$

3. The tempered space-fractional Cattaneo-type equation

Let $\mathcal{L}^\beta(t)$, with $t > 0$, be the inverse process of the sum of two independent positively skewed stable subordinators $H_{1}^{2\beta}$ and $H_{2}^{\beta}$, that is

$$\mathcal{L}^\beta(t) := \inf \left\{ s \geq 0, \ H_{1}^{2\beta}(s) + (2k)^{1/\beta}H_{2}^{\beta}(s) \geq t \right\}, \quad t, k > 0, \quad \beta \in (0, 1/2).$$

We recall that the Laplace transform with respect to $t$ of the law $l_\beta(x, t)$ of the process $\mathcal{L}^\beta(t)$ is given by \[8\]

$$\tilde{l}_\beta(x, s) = (s^{2\beta - 1} + 2ks^{\beta - 1})e^{-x^2s^{2\beta} - 2ks^\beta},$$

and satisfies the fractional equation

$$D^\beta_t u + 2kD^\beta_1 u = -\frac{\partial u}{\partial x}.$$
Theorem 3.1. The solution of the tempered fractional equation

\[ D_t^{2\beta} f + 2kD_t^\beta f = \left[ \lambda^\alpha - \left( \lambda - \frac{\partial^2}{\partial x^2} \right)^\alpha \right] f, \quad -\infty < x < +\infty, \tag{17} \]

with \( \beta \in (0, 1/2) \) and \( \alpha \in (0, 1) \), under the conditions \( u(x,0) = \delta(x) \) and \( u(0,t) = 0 \), coincides with the probability law of the process \( W(t) := B(T_{\lambda,\alpha}(L^\beta(t))), \quad t > 0. \tag{18} \)

Moreover, the fundamental solution of equation \ref{eq:tempered_fractional_equation} has the following Fourier transform with respect to \( x \)

\[ \hat{u}(\xi,t) = \frac{1}{2} \left[ \left( 1 + \frac{k}{\sqrt{k^2 - \theta(\xi)}} \right) E_{\beta,1}(r_1 t^\beta) + \left( 1 - \frac{k}{\sqrt{k^2 - \theta(\xi)}} \right) E_{\beta,1}(r_2 t^\beta) \right], \tag{19} \]

with

\[
\theta(\xi) = (\lambda + \|\xi\|^2)^\alpha - \lambda^\alpha, \\
r_1 = -k + \sqrt{k^2 - \theta(\xi)}, \\
r_2 = -k - \sqrt{k^2 - \theta(\xi)},
\]

and where

\[ E_{\beta,\gamma}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\beta k + \gamma)}, \]

is the Mittag-Leffler function, for \( \beta > 0 \) and \( \gamma \in \mathbb{C} \).

Proof. The probability law of the process \( W(t) \) is given by

\[ w(x,t) = \int_0^\infty v(x,\mu) l_\beta(\mu, t) \, d\mu, \tag{20} \]

where \( v(x,t) \) is the density of the tempered stable subordinator whose Fourier transform is equal to

\[ \hat{v}(\xi,t) = e^{-t\theta(\xi)}. \tag{21} \]

Therefore, the Fourier transform of \( w(x,t) \) is given by

\[ \hat{w}(\xi,t) = \int_0^\infty e^{-t\theta(\xi)} l_\beta(\mu, t) \, d\mu. \tag{22} \]
Since the Laplace transform with respect to \( t \) of \( l_\beta(x,t) \) is given by

\[
\tilde{l}_\beta(x,s) = (s^{2\beta-1} + 2ks^{\beta-1}) e^{-xs^{2/\beta}-2ks^{\beta}},
\]

then the Fourier–Laplace transform of the probability law of the process \( W(t) \) is given by

\[
\hat{\tilde{w}}(\xi,s) = \int_0^\infty e^{-st} dt \int_0^\infty e^{-\mu t} l_{\beta}(\mu, t) d\mu = \frac{s^{2\beta-1} + 2ks^{\beta-1}}{s^{2\beta} + 2ks^{\beta} + \theta(\xi)}.
\]  

If we compare it with the Fourier–Laplace transform of the fundamental solution of the equation (17), we observe that they are equal and the claimed result holds.

Regarding the inverse time-Laplace transform of (24), we report that it can be obtained through an algebraic manipulations [8, pp. 1021–1022] and by recalling the Laplace transform formulas for the Mittag–Leffler functions.

We observe that the process \( X(t) := B(T_{\lambda,\alpha}(t)) \) is indeed a Lévy process with Lévy exponent

\[
\psi(\xi) = -\frac{1}{t} \ln \mathbb{E}[e^{i\xi X(t)}]
= -\frac{1}{t} \ln \mathbb{E}[\mathbb{E}[e^{i\xi X(T(t))}| T(t)]] = -\ln \mathbb{E}e^{-\xi^2 T(t)/2}
= -\frac{1}{t} \ln \left[ \int_0^\infty e^{-\xi^2 x^2} f_{\lambda,\alpha}(x,t) dx \right],
\]

and by using equation (5) we have that

\[
\psi(\xi) = -\frac{1}{t} \ln \left[ \int_0^\infty e^{\lambda x^2 - \lambda x - \frac{\xi^2}{2}} x^{\alpha} \right] = \left[ \frac{\xi^2}{2} + \lambda \right] - \lambda^\alpha.
\]  

Since the Laplace exponent of the sum of stable subordinators \( H_{1,2}^{2\beta}(t) \) is given by

\[
\phi(s) = s^{2\beta} + 2ks^{\beta},
\]

then the Fourier–Laplace transform of the probability law of the process \( W(t) \) is given by

\[
\hat{\tilde{w}}(\xi,s) = \frac{s^{2\beta-1} + 2ks^{\beta-1}}{s^{2\beta} + 2ks^{\beta} + \theta(\xi)}.
\]  

If we compare it with the Fourier–Laplace transform of the fundamental solution of the equation (17), we observe that they are equal and the claimed result holds.
then we have that the inverse process $\mathcal{L}^\beta(t)$ has Lévy exponent given by

$$
\mathbb{L}\left\{ E\mathcal{L}^\beta(t); s \right\} = \frac{1}{s \phi(s)} = \frac{1}{s^{2\beta+1} + 2ks^{\beta+1}},
$$
(27)

whose inverse Laplace transform, namely $U(t)$, is given by

$$
U(t) := t^{2\beta} E\mathcal{L}^{\beta,2\beta+1}(-2kt^\beta).
$$
(28)

We recall now [17, Theorem 2.1] that if we consider the time-changed Lévy process $X(Y(t))$, where $X(t)$ is an homogeneous Lévy process and $Y(t)$ a non-decreasing process independent of $X$, then we have that

$$
\mathbb{E}X(Y(t)) = U(t)\mathbb{E}X(1),
$$
(29)

and

$$
\text{Var}X(Y(t)) = \mathbb{E}[X(1)]^2 \text{Var}[Y(t)] + U(t)\text{Var}[X(1)],
$$
(30)

where $U(t) = \mathbb{E}Y(t)$. In our case, since the mean value of $X(t)$ is null for all $t > 0$, we have that

$$
\mathbb{E}X(\mathcal{L}^\beta(t)) = 0, \quad \text{for all} \quad t > 0,
$$
(31)

and

$$
\text{Var}X(\mathcal{L}^\beta(t)) = t^{2\beta} E\mathcal{L}^{\beta,2\beta+1}(-2kt^\beta)\text{Var}[X(1)].
$$
(32)

We observe that

$$
\text{Var}[X(1)] = \mathbb{E}\left[ \mathbb{E}[B(T_{\lambda,\alpha}(1))^2|T_{\lambda,\alpha}(1)] \right] = \mathbb{E}\left[ T(1)^2 \right]
$$

$$
= \int_0^{+\infty} z^2 e^{-\lambda z + \lambda \alpha} h_{\alpha}(z, 1) \, dz = e^{\lambda \alpha} \frac{d^2}{d\lambda^2} \int_0^{+\infty} e^{-\lambda z} h_{\alpha}(z, 1) \, dz
$$

$$
= e^{\lambda \alpha} \frac{d^2}{d\lambda^2} e^{-\lambda \alpha} = \alpha \lambda^{\alpha-2} [1 - \alpha + \alpha \lambda^{\alpha}]
$$
(33)

and we conclude that

$$
\text{Var}X(\mathcal{L}^\beta(t)) = \alpha \lambda^{\alpha-2} [1 - \alpha + \alpha \lambda^{\alpha}] t^{2\beta} E\mathcal{L}^{\beta,2\beta+1}(-2kt^\beta).
$$
(34)

We recover, for $\lambda = 0$, the result for the space-time fractional (non-tempered) Cattaneo process [8, Theorem 4.1]

$$
\text{Var}X(\mathcal{L}^\beta(t)) = \alpha [1 - \alpha] t^{2\beta} E\mathcal{L}^{\beta,2\beta+1}(-2kt^\beta).
$$
(35)
Remark 3.2. We observe that the probabilistic interpretation given by the time-changed process \((15)\) works only for \(\beta \in (0, 1/2)\), that is a sort of multi-term time-fractional diffusion equation with space-tempered derivatives. On the other hand, the analytical representation of the solution is correct also for \(1/2 < \beta < 1\), under the additive constraint \(\partial_t u \big|_{t=0} = 0\). Moreover, we recover the Fourier transform of the fundamental solution that was originally found by Beghin and Orsingher [27].

We can also consider the space-Laplace transform of the solution for the more general case \(\alpha \in (0, 1)\), even if in this case we loose the probabilistic representation that is valid only in the case \(\alpha \in (0, 1/2)\).

For the particular case \(\beta = 1\), we have the following

Proposition 3.3. The space-Laplace transform of the solution for the fractional problem \((17)\), under the conditions \(u(x, 0) = \delta(x)\) and \(\partial_t u(x, t) \big|_{t=0} = 0\) is given by

\[
\tilde{u}(s, t) = \frac{e^{-kt}}{2} \left[ \left( 1 + \frac{k}{\sqrt{k^2 - \psi(s)}} \right) e^{t\sqrt{k^2 - \psi(s)}} \right. \\
+ \left. \left( 1 - \frac{k}{\sqrt{k^2 - \psi(s)}} \right) e^{-t\sqrt{k^2 - \psi(s)}} \right]
\]

where

\[
\psi(s) = (s + \lambda)^\alpha - \lambda^\alpha.
\]

Proof. We take the space-Laplace transform and, by using \((4)\), we have that

\[
\frac{\partial^2 \tilde{u}}{\partial t^2} + 2k \frac{\partial \tilde{u}}{\partial t} = (s + \lambda)^\alpha \tilde{u} - \lambda^\alpha \tilde{u} = \psi(s)\tilde{u},
\]

whose solution, under the given conditions, is given by \((36)\). \(\square\)

To conclude, we consider the following Dirichlet problem

\[
\begin{cases}
\frac{\partial^2 u}{\partial x^2} + 2k \frac{\partial u}{\partial t} = \frac{\partial^{\lambda,\alpha} u}{\partial x^{\lambda,\alpha}}, & x \geq 0, \\
u(x, 0) = 0, & u(0, t) = \phi(t), \\
\frac{\partial u}{\partial t} \big|_{t=0} = 0.
\end{cases}
\]

We have the following
Proposition 3.4. The time-Laplace transform of the solution for the Dirichlet problem (39) is given by
\[ \tilde{u}(x,s) = \tilde{\phi}(s)e^{-\lambda x} E_{\alpha,1} \left[ -(s^2 + 2ks + \lambda^\alpha)x^\alpha \right]. \] (40)

Proof. By taking the time-Laplace transform of (39), we have that
\[ s^2 \tilde{u} + 2ks \tilde{u} = e^{-\lambda x} D^\alpha_x [e^{\lambda x} \tilde{u}] - \lambda^\alpha \tilde{u}. \] (41)
Therefore, we have that
\[ e^{-\lambda x} D^\alpha_x [e^{\lambda x} \tilde{u}] = (s^2 + 2ks + \lambda^\alpha) \tilde{u}, \] (42)
whose solution, according to the boundary condition and by recalling that the one-parameter Mittag–Leffler function is an eigenfunction of the Caputo fractional derivative $D^\alpha_x$, is given by (40).

We consider now the special case when $k = \lambda^\alpha/2$, then (40) can be rewritten as
\[ \tilde{u}(x,s) = \tilde{\phi}(s)e^{-\lambda x} E_{\alpha,1} \left[ -(s + \lambda^\alpha/2)^\alpha x^\alpha \right], \] (43)
and the solution $u(x,t)$ can be explicitly derived.

We start by considering that
\[ \int_0^{+\infty} e^{-\eta x} E_{\alpha,1}(-\theta^2 x^\alpha)dx = \frac{\eta^{\alpha-1}}{\eta^\alpha + \theta^2}, \] (44)
and we observe that its inverse Laplace transform with respect to $\theta$ is given by
\[ \mathcal{L}^{-1} \left\{ \frac{\eta^{\alpha-1}}{\eta^\alpha + \theta^2}t \right\} = \eta^{\alpha-1} t E_{2,2}(-\eta^\alpha t^2). \] (45)
Now we invert the Laplace transform with respect to $\eta$, by considering the following representation of the Mittag–Leffler function as H function [24, formula (1.136)]:
\[ E_{\alpha,\beta}(x) = H^{1,1}_{1,2} \left[ -x \left| \begin{array}{cc} (0,1) \\ (0,1) & (1-\beta,\alpha) \end{array} \right. \right]. \] (46)
We then apply the inverse transformation [24, formula (2.21)] (after checking that the conditions are satisfied for $\sigma = \alpha$ and $\rho = 1 - \alpha$), as follows

\[
L^{-1} \{ \eta^{\alpha-1} t E_{2,2}(-\eta^{\alpha} t^2); x \} = t x^{-\alpha} H_{2,2}^{1,1} \left[ \frac{t^2}{x^\alpha} \right] \begin{bmatrix} (0, 1) & (1 - \alpha, \alpha) \\ (0, 1) & (-1, 2) \end{bmatrix} = [24, formula (1.60)]
\]

\[
= \frac{1}{t} H_{2,2}^{1,1} \left[ \frac{t^2}{x^\alpha} \right] \begin{bmatrix} (1, 1) & (1, \alpha) \\ (1, 1) & (1, 2) \end{bmatrix} = [24, formula (1.58)]
\]

\[
= \frac{1}{t} H_{2,2}^{1,1} \left[ \frac{x^\alpha}{t^2} \right] \begin{bmatrix} (0, 1) & (0, 2) \\ (0, 1) & (0, \alpha) \end{bmatrix} = \frac{1}{t} \frac{1}{2\pi i} \int_L \left( \frac{x^\alpha}{t^2} \right)^{-w} \frac{\Gamma(w)\Gamma(1-w)}{\Gamma(2w)\Gamma(1-\alpha w)} dw
\]

\[
= \text{[by the duplication property of the Gamma function]}
\]

\[
= 2\sqrt{\frac{\pi}{t}} \frac{1}{2\pi i} \int_L \left( \frac{x^\alpha}{4t^2} \right)^{-w} \frac{\Gamma(1-w)}{\Gamma(1/2 + w)\Gamma(1-\alpha w)} dw
\]

\[
= 2\sqrt{\frac{\pi}{t}} H_{2,1}^{1,1} \left[ \frac{x^\alpha}{4t^2} \right] \begin{bmatrix} (0, 1) & (1/2, 1) \\ (0, 1) & (0, \alpha) \end{bmatrix},
\]

where $L$ is the loop beginning and ending at $+\infty$, denoted by $L_{+\infty}$ [24 point ii), p. 3], since, in this case, $\mu = \alpha - 2 > 0$.

As a consequence of the previous steps we can write the solution $u(x,t)$ as follows

\[
u(x,t) = 2\sqrt{\pi} e^{-\lambda x} \int_0^t \phi(t-z) e^{-\lambda^{\alpha/2} z} \frac{1}{t^2} \frac{H_{2,1}^{1,1} \left[ \frac{x^\alpha}{4z^2} \right]}{z} \begin{bmatrix} (0, 1) & (1/2, 1) \\ (0, 1) & (0, \alpha) \end{bmatrix} dz. (47)
\]

4. Conclusions

In this paper we have considered the generalized Cattaneo equation involving the space-fractional tempered derivative. We have shown that the fundamental solution coincides with the probability law of a time-changed Brownian motion related to the fractional tempered stable process [3]. This is a Lévy process and we provide a full characterization of it. In particular, the aim of this paper is to
show the role of the tempered fractional operator in the context of the studies about the time-fractional Cattaneo equation that has been widely studied in the literature both engineering [28] and probabilistic [27]. From the analytical point of view, we have found the Fourier transform of the fundamental solution for this new generalization of the Cattaneo equation by means of Mittag–Leffler functions. In the second part of this paper we have also considered the Dirichlet problem for the tempered fractional Cattaneo equation. In this case we have found the solution by means of H-functions. This is the first step for further investigations on the applications of tempered fractional Cattaneo equations for heat transfer problems. In view of the relevant impact of the generalized fractional Cattaneo in the recent literature, it is interesting to understand the role and utility of the tempered derivatives in real models as, for example, in thermoelasticity.

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