SSD – Software for Systems with Delays:
Reproducible Examples and Benchmarks on
Model Reduction and $H_2$ Norm Computation

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Abstract: We present SSD – Software for Systems with Delays, a de novo MATLAB package for the analysis and model reduction of retarded time delay systems (RTDS). Underneath, our delay system object bridges RTDS representation and Linear Fractional Transformation (LFT) representation of MATLAB. This allows seamless use of many available visualizations of MATLAB. In addition, we implemented a set of key functionalities such as $H_2$ norm and system gramian computations, balanced realization and reduction by direct integral definitions and utilizing sparse computation. As a theoretical contribution, we extend the frequency-limited balanced reduction to delay systems first time, propose a computational algorithm and give its implementation. We collected two set of benchmark problems on $H_2$ norm computation and model reduction. SSD is publicly available in GitHub*. Our reproducible paper and two benchmarks collections are shared as executable notebooks.

Keywords: H2 norm, frequency-limited balanced reduction, software, sparse computation.

1. INTRODUCTION

Software packages are essential and practical tools for analysis and design of control systems. Considerable research effort is devoted to extend classical and modern control techniques to accommodate delays. Without being exhaustive and focusing only publicly available ones, we can group software packages as:

- Solution of delay differential equations, Hairer et al. (1995); Enright et al. (1997); Shampine et al. (2001).
- Stability and bifurcation analyses, Engelborghs et al. (2002); Breda et al. (2009); Wu et al. (2012); Avanessov et al. (2013); Vythidal et al. (2014); Breda et al. (2016).
- System norms and other metrics, Michiels et al. (2010); Gumussoy et al. (2010); Appeltans et al. (2019).
- Controller design:
  - Stabilization, Michiels (2011b).
  - Pole placement, Boussaada et al. (2021).
  - PID design, Gumussoy et al. (2012); Appeltans et al. (2022).
  - $H_\infty$ control, Gumussoy et al. (2011).

This paper gives a guided tour to a new MATLAB package, Software for Delay Systems (SSD) focusing on model reduction and $H_2$ norm computation for retarded time delay systems (RTDSs). Our main contributions are

- allowing easy-to-access MATLAB’s time and frequency domain visualizations by bridging RTDS and MATLAB’s LFT representation,
- publicly available implementation of model reduction and $H_2$ norm computation using their direct definitions of the integral form and utilizing sparse computation,
- as a theoretical contribution, extension of the frequency-limited balanced reduction first time for RTDS and a computational algorithm via integral expressions for delay systems,
- collecting two sets of benchmark problems on model reduction and $H_2$ norm computation and sharing them as two executable notebooks,
- making our paper completely executable and reproducible to facilitate reproducible research.

Our goals are two folds: First, to introduce publicly available functionalities as a baseline approach for comparison with the advanced techniques in model reduction and $H_2$ norm computation. Second, to facilitate the analysis of delay systems having small to mid-size state dimensions by SSD’s easy-to-use interface.

The paper is organized as follows. First we define the delay system representation as an RTDS and show its use in the next section. We illustrate how time and frequency domain visualizations are used in Section 3. The model reduction functionalities and $H_2$ norm computation are overviewed in Section 4. We summarized the collected benchmark problems in Section 5. We give details on the computational aspects of our MATLAB package in Section 6. We end our paper with concluding remarks and some future directions.

2. DELAY SYSTEM DEFINITION

SSD constructs a retarded time-delay system as
\[
\begin{align*}
\dot{x}(t) &= \sum_{i=0}^{m_A} A_i x(t-h_i^A) + \sum_{i=0}^{m_B} B_i u(t-h_i^B), \\
y(t) &= \sum_{i=0}^{m_C} C_i x(t-h_i^C) + \sum_{i=0}^{m_D} D_i u(t-h_i^D).
\end{align*}
\]

The system matrices are \( A_i \in \mathbb{R}^{n \times n} \), \( B_i \in \mathbb{R}^{n \times n_u} \), \( C_i \in \mathbb{R}^{n_u \times n_y} \) and \( D_i \in \mathbb{R}^{n_y \times n_u} \). The number of time delays are \( m_A \), \( m_B \) for states and inputs in the state equation and \( m_C \), \( m_D \) for states and inputs in the output equation. By convention, \( h_i^A = h_i^B = h_i^C = h_i^D = 0 \). All delays are non-negative real numbers.

The ssd object is constructed using the function \( \text{ssd} \) as follows:
\[
\text{sys} = \text{ssd}(A, hA, B, hB, C, hC, D, hD);
\]

where \( A, B, C, \) and \( D \) matrices are 3-dimensional matrices and \( hA, hB, hC, \) and \( hD \) are unique, increasing non-negative vectors as shown below in Figure 1.

\[
\begin{align*}
A, & \quad B, & \quad C, & \quad D \text{ matrices} \\
hA, & \quad hB, & \quad hC, & \quad hD \text{ matrices}
\end{align*}
\]

Code: \( A = \text{cat}(3,A0,A1,...,AN) \)

Code: \( hA = [0,hA1,...,hAN] \)

Fig. 1. System matrices and delays in ssd object.

We borrow the following delay system from Jarlebring et al. (2013) as a motivational example,
\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} -2 & -1 \\ -3/2 & -1/2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1/2 \\ 1 & 0 \end{bmatrix} x(t-1) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 2 & 0.2 \end{bmatrix} x(t).
\end{align*}
\]

We first define the system matrices as follows,
\[
\begin{align*}
A0 &= [-2, -1, -3/2, -1/2]; \\
A1 &= [0, 1/2, 1, 0]; \\
A &= \text{cat}(3,A0,A1); \\
B &= [1; -1]; \\
C &= [2, 0.2];
\end{align*}
\]

Then we create the ssd object by
\[
\text{sys} = \text{ssd}(A, [0, 1], B, 0, C, 0)
\]

\[
\text{sys} = \text{ssd}(A, hA, B, hB, C, hC, D, hD);
\]

Note that there is no \( D \) matrix in the above example. Therefore, the matrix and its delay argument are not included. Alternatively, one can set them to empty vector, \([\ ]\) or enter zero matrix with appropriate dimensions and zero delay.

The interface of \text{ssd} is a natural extension of state-space object, \text{ss}(A,B,C,D) of MATLAB by introducing extra delays per system matrix. While the ssd object keeps the delay system structure, it also constructs the MATLAB’s LFT-based ss object with delays behind the scenes to leverage pre-defined functionalities of MATLAB as we illustrate in the next section.

3. TIME / FREQUENCY DOMAIN VISUALIZATIONS

Time and frequency domain plots gives additional insights for the analysis and design of delay systems. The ssd object interfaces with MATLAB plotting functionality and enables frequency domain plots, \text{bode}, \text{bodemag}, \text{sigma}, \text{nyquist} and the time domain plot, \text{step} as in MATLAB including multiple system support.

Continuing with the same system matrices of previous example, \( \text{sys} \), we define two additional systems, one with a state delay of 0.5 and the other one with no delay.
\[
\begin{align*}
\text{sys1} &= \text{ssd}(A, [0, 0.5], B, 0, C, 0); \\
\text{sys2} &= \text{ssd}(\text{sum}(A, 3), 0, B, 0, C, 0); \\
\text{sys2}.\text{name} &= \text{’no delay’}; \\
\text{step} &\text{(sys, ‘r’, sys1, ‘g-’, sys2, ‘b--’)};
\end{align*}
\]

Fig. 2. Step responses of multiple systems.

Fig. 3. Analytical and numerical \( H_2 \) norm values.

Note that \( \text{sys2} \) is a standard state-space system with no delay. As seen in Figure 2, SSD shows the system names and allows the user to modify the names as needed.

4. \( H_2 \) NORM AND MODEL REDUCTION

SSD provides a set of functionality on \( H_2 \) norm computation and model reduction. In next two sections, we introduce the basic definitions, outline our direct computational approach and illustrate their use with the example codes.

4.1 \( H_2 \) Norm Computation

Assuming the exponential stability of the delay system, its \( H_2 \) norm is defined in frequency domain as,
\[
\|G\|_2 := \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} \left( G(j\omega)^* G(j\omega) \right) \, dw}
\]
where \( G(s) = \mathcal{L}(s^{-1}A(s)^{-1}B(s) \text{ with the auxiliary terms, } A(s) := \sum_{i=0}^{m_A} A_i e^{-j\omega h^i}, B(s) := \sum_{i=0}^{m_B} B_i e^{-j\omega h^i}, \text{ and } \mathcal{C}(s) := \sum_{i=0}^{m_C} C_i e^{-j\omega h^i}. \) By definition, \( D \) matrix is zero for a well-defined \( H_2 \) norm.

We borrow the hot shower problem from Jarlebring et al. (2011) (included in benchmarks_h2norm.mlx at GitHub as the first benchmark). The system equations are
\[
\dot{x} = -ax(t - h) + bu(t), \quad y(t) = cx(t).
\]
This example admits a closed form for the \( H_2 \) norm as, \( ||G||^2_2 = \frac{c^2}{a} \frac{\cos(ah)}{1 - \sin(ah)}. \) Set the system parameters as \( a = b = c = 1 \), then the following code computes the analytical and approximate \( H_2 \) norms.

```matlab
>> a = 1; b = 1; c = 1;
>> h = 0.1:.1:1;
>> h2 = ((c*b)^2/2/a)*((cos(a*h))./(1-sin(a*h)));
>> for ct=1:length(h)
    >> sys = ssd(cat(3,0,-a),[0 h(ct)],1,0,1,0);
    >> happrox(ct) = h2norm(sys);
    >> end
    >> plot(h,sqrt(h2),'o',h,happrox);
```

Figure 3 shows the comparison. The values are same within numerical tolerances.

SSD utilizes sparse computation inside the integrals for \( H_2 \) norm computation (2). For delay systems with large dimensions, enabling sparse option inside ssdoptions can speed up the computation considerably. Computing \( H_2 \) norm without and with the sparse option enabled for our example system \( sys4 \) of order 1000 (included in our benchmark problems) results in 50 times speed up.

```matlab
>> tic; h2 = h2norm(sys4); time=toc
time = 400.7489
```

```matlab
>> opt = ssdoptions('sparse',true);
```

```matlab
```

where \( sysb \) is a balanced realization of \( sys \) with the same system matrices given in Jarlebring et al. (2013), p. 162. One can see that the state contributions by plotting the \textbf{info} object. Figure 4 shows that the first transformed state’s contribution is major.

Note that we define the gramians in the sense of position balancing, Jarlebring et al. (2013), p.160. For convenience, we drop the term position in our paper.

The function gram computes the system gramians. It calculates the system gramians by the direct definitions (3) and (4) and the integral function of MATLAB. For our example system, \( sys \), the system gramians are

```matlab
>> [Wc,Wo] = gram(sys,'co')
```

\[
Wc =
\begin{bmatrix}
0.9273 & -1.7426 \\
-1.7426 & 3.6292
\end{bmatrix}
\]

\[
Wo =
\begin{bmatrix}
1.2674 & -0.4129 \\
-0.4129 & 0.3674
\end{bmatrix}
\]

Based on the energy contributions of the states, the balanced reduction eliminates the states of the balanced realization up to a desired order defined by the user. The function balred performs this reduction. Figure 4 suggests that the first-order model may approximate the \textbf{sys} well. We reduce the system and compare the step responses of the original and the reduced systems by

```matlab
>> sysr = balred(sys,1);
```

```
```

Figure 5 shows that the first-order model captures most of the dynamics as expected.
As a theoretical contribution, to the best of the author’s knowledge, SSD extends the frequency-limited balanced reduction to delay systems first time where the system gramians are computed over the frequency intervals of interest (see Gawronski et al. (1990) for standard case). The resulting reduced system approximates the dynamics better over the desired frequency range with less number of states.

Let \( \text{sys3} \) represent the delay system Example 3 in Jarlebring et al. (2011) (included as the third benchmark in \textit{benchmarks_h2norm.mlx}). The frequency interval is defined in \texttt{FreqInt} field of \texttt{ssdoptions} object.

\begin{verbatim}
>> sys3r = balred(sys3,2);
>> opt = ssdoptions(’FreqInt’,[20 Inf]);
>> sys3rf = balred(sys3,2,opt);
>> sigma(sys3,sys3r,sys3rf);
\end{verbatim}

The zoomed version of the sigma plot, Figure 6, shows that the frequency-limited balanced reduction approximates better compared to the standard reduction over the focused frequency range.

As a final remark, the object \texttt{info} stores the gramian related information and the full balancing transformation matrices. The function \texttt{balred} computes the reduced orders very fast when \texttt{info} is provided since it does not recompute gramian information, i.e.,

\begin{verbatim}
>> tic; [sys3r,info3] = balred(sys3,2); time=toc
>> tic; sys3r1 = balred(sys3,4,info3); time=toc

\texttt{time} = 3.1990
\texttt{time} = 0.0041
\end{verbatim}

5. BENCHMARK PROBLEMS

We surveyed the literature and collected two sets of benchmark problems for the model reduction and the \( \mathcal{H}_2 \) norm computation. We prepared two executable notebooks in which we define the benchmark problems and show how to use SSD on them in GitHub repository.

5.1 Model Reduction

There are five benchmark problems provided in the executable notebook, \textit{benchmarks_model_reduction.mlx}. We used the model reduction functionality of SSD on the selected benchmarks:

1. **The heated rod (HR2) Example in Michiels (2011a):** A 100\textsuperscript{th}-order model with a single delay derived from the discretization of heat equations describing the temperature in a rod controlled with distributed delayed feedback.

2. **Mass-spring (MS) system in Saadvandi et al. (2012):** A 1000\textsuperscript{th}-order coupled mass-spring system with dampers and feedback controls with delays.

3. **Platoon of eight vehicles (P8V) in Scarciotti et al. (2014):** A 23\textsuperscript{rd}-order model describing the problem of controlling a group of vehicles tightly spaced following a leader, all moving in longitudinal direction.

4. **Example 1 in Lordejani et al. (2020):** A 6\textsuperscript{th}-order synthetic model with one state and one output delay.

5. **Second-order system with proportional damping example (SOSPD) in Jiang et al. (2019):** A 2000\textsuperscript{th}-order model with a single state delay.

Table 1 shows the state, output, input dimensions and the number of delays for all system matrices. Since the maximum delay may affect the complexity of the method, this information is included in the table.

The examples in Saadvandi et al. (2012) and Jiang et al. (2019) are second-order mechanical systems and the orders of their first-order equivalent models are twice of their dimensions. The benchmark problems are single-input-single-output (SISO) systems and mostly single state delay in the state equation is considered.

| Ex. | Dimensions \( (n_u,n_y,n_x) \) | \# Delays \( (m_A,m_B,m_C,m_D) \) | Max. Delay |
|-----|----------------------------------|----------------------------------|-----------|
| HR2 | \( (100,1,1) \)                 | \( (1,0,0,0) \)                 | 1         |
| MS  | \( (1000^5,1,1) \)              | \( (1,0,0,0) \)                 | 2         |
| P8V | \( (23,1,1) \)                  | \( (1,0,0,0) \)                 | 0.005     |
| Ex.1| \( (6,1,1) \)                   | \( (1,0,1) \)                   | 1.6       |
| SOSPD| \( (2000^5,1,1) \)          | \( (1,1,0,0) \)                 | 1         |

Table 1. Model reduction benchmark problems

A snapshot of the heated rod example from our notebook can be seen in Figure 7.

5.2 \( \mathcal{H}_2 \) Norm Computation

There are six benchmark problems for \( \mathcal{H}_2 \) norm computation collected in our notebook, \textit{benchmarks_h2norm.mlx}:

1. **The hot shower (HS) problem in Jarlebring et al. (2011), Ex.1:** A first-order state-space model whose \( \mathcal{H}_2 \) norm can be computed analytically. Its delay can be set by the user.

2. **Example 2.b in Jarlebring et al. (2011), Ex.2b:** A 3\textsuperscript{rd}-order synthetic model with two state delays.
(3) Example 3 in Jarlebring et al. (2011), Ex.3: A 9th-order model with two state-delays obtained from discretization of partial differential equations with delays.

(4) The heat exchanger (HE) example in Michiels et al. (2019): A 5th-order model with seven delays for a heat exchanger for which the controller based on a combination of static state feedback and proportional integral control.

(5) The heated rod (HR2) Example in Peeters et al. (2013): A 100th-order model with one single delay, same as the previous example in the model reduction benchmarks.

(6) The heated rod (HR4) Example in Michiels et al. (2019): A 10000th-order model with a single delay, same as the previous example with larger size.

| Ref. | Dimensions $(n_n, n_u)$ | # Delays $(m_A, m_B, m_C, m_D)$ | Max. Delay |
|------|-------------------------|---------------------------------|-----------|
| HS   | (1, 1, 1)               | (1, 0, 0, 0)                    | User-defined |
| Ex.2b| (3, 1, 1)               | (2, 0, 0, 0)                    | 1         |
| Ex.3 | (9, 1, 1)               | (2, 0, 0, 0)                    | 3         |
| HE   | (5, 1, 5)               | (7, 0, 0, 0)                    | 40        |
| HR2  | (100, 1, 1)             | (1, 0, 0, 0)                    | 1         |
| HR4  | (10000, 1, 1)           | (1, 0, 0, 0)                    | 1         |

Table 2. $H_2$ benchmark problems

Table 2 summarizes the dimensions and the number of delays for all system matrices. In general, the examples in the literature have one or two state delays and no delays at input and output matrices excluding $D$ matrices which must be zero for well-posedness of the norm. Most examples are SISO systems. Finally, we share a snapshot from the executable notebook for the hot shower example in Figure 8.

6. COMPUTATIONAL ASPECTS

For experimental\(^1\) testing of the computational complexity of balanced reduction and $H_2$ norm computation, we randomly generate delay systems with the following properties:

- Input and output sizes are 3, $n_y = n_u = 3$,
- There are 3 delays, $m_A = m_B = m_C = 3$,
- All delays are uniformly selected from $[0, 1)$,
- All matrices are dense whose elements uniformly selected from $[-3, 3]$ and in addition, $3I$ subtracted from $A_0$ matrix to make it stable,
- $D$ matrices are set to zero since they are not needed for $H_2$ norm computation and balanced reduction.

We validated that the generated delay systems have rich dynamics. We measured the computation time of $h2norm$ and $gram$ (computing both gramians) functions since the latter function is the main computational bottleneck of balreal and balred. The number of states varies as $1 – 10$, $10 – 100$ by 10 increments and $100 – 500$ for by 100 increments (The upper bound is 400 for model reduction). For each step, we created 5 delay systems, computed the average of computational times and the standard deviation.

\(^1\) All experiments are performed on HP Z6 G4 Workstation with Intel Xeon Silver 4114 2.2GHz CPU, 64GB RAM.

Figure 9 (left) shows that the $H_2$ norm computation time is in the order of seconds up to 100 states, that of minutes up to 300 states and around 8 minutes for 500 states. Figure 9 (right) shows that the $gram$($sys$, ‘co’) computation time is under a minute up to 50 states, under 10 minutes up to 200 states and almost 2 hours for 400 states. In the light of these experiments, we see that SSD can handle small to medium size state dimensions.

For large-scale delay systems with sparse matrices, the sparse option of ssdoptions can be used to speed up the computation. As a large-scale benchmark, we computed the $H_2$ norm of HR4 example with 10,000 states from Section 5.2. We set the absolute tolerance a little high $10^{-3}$ to make the relative error active and set the relative error to $10^{-4}$ as reported in Michiels et al. (2019) and get similar order of time magnitude, around 6 seconds, compared to their results 4.8 seconds obtained by their special large-scale approach.

```
>> opt = ssdoptions('Sparse',true,...
    'RelTol',1e-4,'AbsTol',1e-3);
>> tic
>> h2 = h2norm(sys_hr4,opt)
>> time=toc
```

```
h2 = 0.4357
```

```
time = 5.9651
```

As a final remark, the function integral over $[0, \infty]$ interval may be computationally expensive or less accurate when the integrand has highly oscillatory behavior for a large part of the interval. In our experiments, it shows satisfactory performance.

7. CONCLUDING REMARKS

We took a guided tour of a new MATLAB package, SSD, on several examples. SSD seamlessly integrates with the time and frequency domain visualization capabilities of MATLAB while offering new features on model reduction and $H_2$ norm computations built on a frequently-used RTDS delay system representation. The list of SSD functions is given in Table 3.

Our computational analysis shows that SSD is suitable for small to mid-size state dimensions. We hope that SSD’s delay system definition of matrices and delays, easy-to-use interface, and provided functionalities will facilitate the analysis of delay systems and be used as a baseline for advanced techniques.
All the examples in this paper are reproducible by our executable notebook, introduction.mlx. We share two executable notebooks, benchmarks_model_reduction.mlx and benchmarks_h2norm.mlx with a collection of benchmark problems on model reduction and $H_2$ norm computation. SSD and three notebooks are available at the GitHub repo

https://github.com/gumussoysuat/ssd.

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| Focus | Function names |
|-------|----------------|
| Visualizations | bode, bodemag, sigma, nyquist, step |
| Norm | h2norm
| Model Reduction | gram1, balreal, balred |
| Options | ssdoptions |

† Sparse computation is available.