Symmetry protection of critical phases and global anomaly in 1 + 1 dimensions

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We derive a selection rule among the (1 + 1)-dimensional SU(2) Wess-Zumino-Witten theories, based on the global anomaly of the discrete $\mathbb{Z}_2$ symmetry found by Gepner and Witten. In the presence of both the SU(2) and $\mathbb{Z}_2$ symmetries, an RG flow is possible between level-$k$ and level-$k'$ Wess-Zumino-Witten theories only if $k \equiv k' \mod 2$. This classifies the Lorentz-invariant, SU(2)-symmetric critical behaviors into two “symmetry-protected” categories, corresponding to even and odd levels.

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Introduction.— Classification of quantum phases is a central problem in condensed matter and statistical physics. They can be first classified into gapped and gapless phases. Gapped quantum phases are relatively easier to be handled theoretically. In fact, there has been a significant progress in further classification of gapped phases. In particular, symmetry protected topological (SPT) phases \cite{1,2} have become an important concept in the classification. That is, even when two states have no long-range entanglement and are indistinguishable in terms of any local observables, in the presence of a certain symmetry, they could still belong to distinct phases separated by a quantum phase transition.

In contrast, classification of gapless quantum phases remains very much open. Symmetries are naturally expected to play an important role also in the classification of gapless quantum phases. Interestingly, symmetry protection of gapless quantum phases has been better understood for the gapless edge states of SPT phases. In fact, the symmetry protection of an SPT phase in the bulk is often conveniently analyzed in terms of symmetry protection of the gapless edge state. As a simple and well-known example, the helical edge state of a topological insulator remains gapless even in the presence of impurities, as long as the system is time-reversal invariant \cite{3}. This indeed leads to the stability of the topological insulator as an SPT phase in the presence of the time-reversal symmetry. This approach is recently refined and generalized in terms of anomaly \cite{4,5}, so that it can be applied for interacting systems. In short, the edge state of an SPT phase exhibits an anomaly with respect to the relevant symmetry, which implies the “ingappability” of the edge state in presence of the symmetry \cite{6}. This also motivates us to question if there is a mechanism of symmetry protection of the universality class of bulk gapless critical phases.

In this Letter, we argue that there is a protection of bulk gapless critical phases by discrete symmetry. This symmetry protection is quite analogous to that of the well-known (gapped) SPT phases; here we show that the concept can be generalized to bulk \textit{gapless critical} phases. We demonstrate this for the SU(2)-symmetric quantum antiferromagnetic chains and their effective field theory, SU(2) Wess-Zumino-Witten (WZW) theory as an example. The SU(2) WZW theory is characterized by a natural number $k$, which is called level. Hereafter we denote the level-$k$ SU(2) WZW theory as SU(2)$_k$ WZW theory. The SU(2)$_k$ WZW theories with $k = 1, 2, \ldots$ are thought to be a complete classification of the universality classes of critical points in 1+1 dimensions with the Lorentz and SU(2) symmetry. We can also identify the level $k = 0$ with the gapped state with a unique “disordered” ground state. In the presence of the SU(2) and a certain discrete $\mathbb{Z}_2$ symmetry of the WZW theory, which corresponds to the translation symmetry of the spin chain, we find that a renormalization-group (RG) flow is possible between SU(2)$_k$ and SU(2)$_{k'}$ WZW theories only if $k \equiv k' \mod 2$. That is, the gapless critical phases in 1+1 dimension with the SU(2), the $\mathbb{Z}_2$, and the Lorentz symmetries are classified into the two “symmetry-protected” categories: one corresponds to even levels and the other to odd levels. In terms of spin chain models, as long as the SU(2) spin-rotation and the lattice translation symmetries are unbroken (either explicitly or spontaneously), a spin chain with $S \in \mathbb{Z}$ can only realize SU(2)$_k$ WZW theory with an even $k$, while one with $S \in \mathbb{Z} + 1/2$ can only realize SU(2)$_k$ WZW theory with an odd $k$.

As we will discuss in detail later, the present result includes, as special cases, the earlier semi-classical ($k \to \infty$) analysis \cite{7} and the Lieb-Schultz-Mattis theorem \cite{8–10} applied to SU(2)-symmetric one-dimensional systems. However, the present result is much stronger than the Lieb-Schultz-Mattis theorem, in restricting the possible universality class of the gapless critical phase.

Model.— The standard Heisenberg antiferromagnetic (HAFM) chain is defined by the Hamiltonian $\mathcal{H}_\text{HAFM} = J_1 \sum_j S_j \cdot S_{j+1}$ with $J_1 > 0$, which possesses an SU(2) symmetry of the global spin rotation, the lattice translation symmetry, and the lattice inversion symmetry. We can also consider various generalizations of this model by
including next-nearest-neighbor interaction, biquadratic interaction, and so on.

In order to explore the possible quantum critical behaviors of quantum spin chains, non-Abelian bosonization is useful. In the non-Abelian bosonization of quantum antiferromagnetic chains, first the spin-S spin chain is represented in terms of fermions with 2S “colors.” The resulting effective field theory is SU(2) WZW theory defined by the action

$$S_{SU(2)_k} = -\frac{1}{\lambda} \int_{S^2} dx_1 dx_2 \text{Tr}[(g^{-1} \partial_\mu g)(g^{-1} \partial_\mu g)] + k \Gamma_{wz},$$

with the field $g(x_1, x_2) \in SU(2)$, the coupling constant $\lambda > 0$ and the Wess-Zumino term $k \Gamma_{wz}$. We consider the space-imaginary time compactified as the two-dimensional sphere $S^2$. The Wess-Zumino term is then given by

$$k \Gamma_{wz} = \frac{ik}{12\pi} \int_{S^2} \text{Tr}(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg)$$

(2)

where $dg = \partial_\mu g dx_\mu$. In the Wess-Zumino term, the field $g$ originally defined on $S^2$ is extended to $g(x_0, x_1, x_2)$ on the northern hemisphere $S^2_N$ of the three-dimensional sphere $S^3$. The hemisphere $S^2_N$ has the spacetime manifold $S^2$ as a boundary, $\partial S^2 = S^1$. The Wess-Zumino term is a topological term, which is unaffected by the infinitesimal variation $\delta S^2$. Nevertheless, $\Gamma_{wz}$ can take values different by integral multiples of $2\pi$, corresponding to topologically inequivalent extensions. To define the partition function consistently, the level $k$ is quantized to be $k \in \mathbb{Z}$. Exact features of the SU(2)$_k$ WZW theory are known, thanks to its infinite dimensional symmetry governed by Kac-Moody algebra. Each value of $k$ represents different critical behaviors.

For the effective theory for a quantum antiferromagnetic chain with spin $S$, the level is given as $k = 2S$. Generically the effective theory of the spin-S chain contains various perturbations to the WZW theory. The general principle is that, all the possible perturbations allowed by the symmetries should be present, unless parameters in the Hamiltonian are fine-tuned. The special model called Takhtajan-Babujian model is exactly solvable for any $S$ by Bethe Ansatz; the exact solution shows that the system is indeed described by the SU(2)$_{2S}$ WZW theory. The Takhtajan-Babujian model corresponds to the special multicritical point where the parameters are fine-tuned so that all the relevant perturbations vanish. Away from the Takhtajan-Babujian point, there are usually relevant perturbations which drive the system away from the original SU(2)$_{2S}$ WZW fixed point, under RG.

In order to discuss the RG flow, we need to identify symmetries of the system and their representation in the field theory. In this Letter, we limit ourselves to models with the global SU(2) symmetry of spin rotation. Furthermore, we consider the models which are invariant under the translation by one site, $T_1: S_j \rightarrow S_{j+1}$. The lattice translation symmetry $T_1$ is represented by the $\mathbb{Z}_2$ under $g \rightarrow -g$, in the WZW theory.

**Modular invariance.**—We are naturally motivated to promote the $g \rightarrow -g$ symmetry to the gauge symmetry as well as the SU(2) rotational symmetry. In some cases the resultant gauged theory turns out to suffer from the anomaly. To see the anomaly of the theory, it is convenient to consider the system on a torus. The torus can be defined in terms of complex coordinates $z$ and $\bar{z}$ with the identifications $z \sim z + 2\pi$ and $\bar{z} \sim \bar{z} + 2\pi$. The resulting effective field theory is SU(2) WZW theory corresponding to periodic and anti-periodic boundary conditions. This requirement, which is called modular invariance, is represented by the $\mathbb{Z}_2$ antiperiodic function $Z_{SU(2)}(\tau, \bar{\tau}) = \sum_{j,j'} \chi_{j,j'} X_{j,j'} \bar{\chi}_{j,j'}$.

$$Z(\tau, \bar{\tau}) = \sum_{j,j'} \chi_{j,j'} X_{j,j'} \bar{\chi}_{j,j'}$$

(3)

where $X_{j,j'}$ are nonnegative integers and $\chi_{j,j'}$ are Kac-Moody characters as functions of the modulus, corresponding to holomorphic and anti-holomorphic parts. The characters are labelled by the spin $j = 0, 1/2, \ldots, k/2$.

The same torus can be represented by different modular parameters, which are related by modular transformations generated by $\mathcal{T}: \tau \rightarrow \tau + 1$ and $\mathcal{S}: \tau \rightarrow -1/\tau$. The modular transformations of the Kac-Moody characters read $\mathcal{T} \chi_j(\tau) = e^{i\pi(\Delta_j - \frac{1}{2})} \chi_j(\tau)$ and $\mathcal{S} \chi_j(\tau) = \sum_{j'} S_{j,j'} \chi_{j'}(\tau)$, where $c = 3k/(2 + k)$, $\Delta_j = j(j + 1)/(2 + k)$ and $S_{j,j'} = \sqrt{2/(k + 2)} \sin[(2j + 1)(2j' + 1)/(k + 2)]$. The character of the antiholomorphic part is similarly transformed. Since the modular transformations do not change the underlying torus, a physically sensible partition function should be invariant under these transformations. This requirement, which is called modular invariance, in fact leads to quite a powerful constraint on possible consistent conformal field theories (CFT). In the present context, $X_{j,j'}$ are strongly constrained by the modular invariance so as to be non-negative integer.

The simplest possible modular invariant partition function for the SU(2)$_k$ WZW theory is $Z_{SU(2)}(\tau, \bar{\tau}) = \sum_{j,j'} |\chi_{j,j'}|^2$, which corresponds to $X_{j,j'} = \delta_{j,j'}$. This indeed can be interpreted as the partition function of the SU(2)$_k$ WZW theory on the torus with periodic boundary conditions. The modular invariance of the partition function $Z_{SU(2)}$ can be easily verified with an explicit calculation.

However, this is not the only possible modular invariant partition function. Since the SU(2)$_k$ WZW theory also possesses the discrete $\mathbb{Z}_2$ symmetry $g \rightarrow -g$, we can consider “gauging” the $\mathbb{Z}_2$ symmetry by identifying $g \sim -g$. This procedure is known as orbifold construction. The periodic boundary condition for the $\mathbb{Z}_2$ orbifold of the SU(2)$_k$ WZW theory corresponds to periodic and
antiperiodic boundary conditions of the original SU(2)k WZW theory, because of the identification \( g \sim -g \). The \( \mathbb{Z}_2 \) orbifold partition function thus reads \[ Z_+ (\tau, \bar{\tau}) = (1 + S + T \mathcal{S}) Z_+^{\text{proj}} (\tau, \bar{\tau}) - Z_{SU(2)} (\tau, \bar{\tau}). \] Here \( Z_+^{\text{proj}} \) is the projected partition function

\[ Z_+^{\text{proj}} = \text{Tr} (P_+ e^{-2\pi \tau \mathcal{H} - i2\pi \tau \mathcal{P}}), \]

where \( P_+ \) is the projection operator to the subspace even under \( g \to -g \) and projects out the conformal towers with \( j \in \mathbb{Z} + 1/2 \), that is, \( Z_+^{\text{proj}} = \sum_{j \in \mathbb{Z}} |\chi_j|^2 \). It appears that the partition function \[ \text{(4)} \] is modular invariant by construction. However, it can be checked by explicit calculations that it is not always the case. The \( \mathbb{Z}_2 \) orbifold of the SU(2)k WZW theory is modular invariant only if \( k \) is even; it is modular non-invariant if \( k \) is odd. Indeed \( Z_+ \) for odd \( k \) has non-trivial coefficients \( X_{j,j'} = [1 + e^{i2\pi (\Delta - \Delta_{j,j'})}] / 2 \notin \mathbb{Z} \) for \( j \neq j' \). This is an example of local anomaly in quantum field theory.

A physical interpretation of this fact was also given \[ \text{(12)} \]: the \( \mathbb{Z}_2 \) orbifold of the SU(2)k WZW theory can be regarded as an SO(3) WZW theory, because SU(2) modulo \( g \sim -g \) is SO(3). As in the case of the SU(2) WZW theory, the requirement that the probability amplitude is independent on the extension of \( g \) to \( S^3 \) leads to quantization of the Wess-Zumino term. Since the volume of SO(3) is half of that of SU(2), the \( k \) must be even.

Consequences of the global anomaly.— While the global anomaly of the SU(2)k WZW theory with an odd \( k \) has been known for many years in string theory, its implications in condensed-matter and statistical physics were never elucidated. Now we shall argue that there are indeed very profound consequences. First, the anomaly for the SU(2)k WZW with an odd \( k \) corresponds to a field-theory version of Lieb-Schultz-Mattis theorem. Naively, since the theory has the discrete \( \mathbb{Z}_2 \) symmetry \( g \to -g \), we expect that we can consider projection on the subspace of the entire Hilbert space which is symmetric under \( g \to -g \). However the anomaly for odd \( k \) precisely means that there is no consistent CFT defined within the symmetric subspace. Likewise the antisymmetric sector is also inconsistent CFT. The odd-\( k \) SU(2)k WZW theory is inconsistent unless both symmetric and antisymmetric sectors are included. This is analogous to the chiral edge state of two-dimensional systems that is modular non-invariant but becomes modular invariant when it is considered together with the other edge state with opposite chirality.

This observation is also related to the “gappability” of the theory. In general, a CFT has relevant operators in its spectrum. Once the theory is perturbed by a relevant operator, generically the theory would become massive; the excited states would be separated from the ground state by a non-vanishing mass gap. Usually the ground state in such a system is unique. If this is the case, because of the \( \mathbb{Z}_2 \) symmetry of the theory, the unique ground state is either symmetric or antisymmetric with respect to the \( \mathbb{Z}_2 \) symmetry. Then, in order to describe the low-energy physics, we can consider a projection onto symmetric or antisymmetric sector of the Hilbert space. However, for odd \( k \), the global anomaly of the SU(2)k WZW theory means that such a projection does not yield a consistent quantum field theory. Therefore, in order to open a mass gap, the global anomaly for odd \( k \) requires that the ground states below the gap exist in both the symmetric and antisymmetric sectors and are doubly degenerate. This signals the spontaneous breaking of the \( g \to -g \) symmetry \[ \text{(17)} \].

The implications of the above statement in the context of spin chains is as follows. As discussed earlier, non-Abelian bosonization of a spin-S HAFM chain yields the SU(2)k WZW theory with \( k = 2S \) with perturbations allowed by the symmetries. Thus, the global anomaly is present when the spin \( S \) is a half-odd-integer. Since the \( \mathbb{Z}_2 \) symmetry of \( g \to -g \) corresponds to the lattice translation symmetry, when the spin-S chain acquires a gap, the lattice translation symmetry must be spontaneously broken so that there are two degenerate ground states. This is precisely the statement of the Lieb-Schultz-Mattis theorem \[ \text{(10)} \] on the spin-S HAFM chain.

Furthermore, we can derive stronger results which restrict possible universality classes of gapless critical phase. Let us consider RG flows from the SU(2)k WZW theory induced by perturbations allowed within the SU(2) and \( g \to -g \) symmetries. If \( k \) is even, the \( \mathbb{Z}_2 \) orbifold of the SU(2)k WZW theory is modular invariant. Thus the \( \mathbb{Z}_2 \) orbifold must be a consistent field theory along the RG flow starting from the SU(2)k WZW fixed point. If the RG flow reaches another fixed point, it should be SU(2)k′ WZW with a different \( k' \). Zamolodchikov’s c-theorem dictates that the central charge of the infrared fixed point \[ \text{(18)} \], the SU(2)k′ WZW theory, should be smaller than that of the ultraviolet fixed point, the SU(2)k WZW theory. Thus \( k' < k \). In addition, the consistency of the \( \mathbb{Z}_2 \) orbifold requires the modular invariance of the SU(2)k WZW theory. Thus \( k' \) must also be even.

Likewise, we can consider RG flows from the SU(2)k WZW theory with an odd \( k \) to the SU(2)k′ WZW theory, under the SU(2) and \( g \to -g \) symmetries. Again \( k' < k \) because of the c-theorem. Moreover, if \( k' \) is even, we can further perturb the system to obtain a massive (gapped) field theory with a unique ground state (corresponding to \( k = 0 \)). This contradicts with the gappability of the SU(2)k WZW theory discussed above. Thus \( k' \) must be also odd.

Therefore, we obtain the following statement.

When there is an RG flow from the SU(2)k WZW theory to the SU(2)k′ theory, if the SU(2) symmetry and the \( \mathbb{Z}_2 \) symmetry \( g \to -g \) are respected, \( k' < k \) and \( k' \equiv k \mod 2 \).
mod 2.

In terms of spin chains, this implies

The critical behavior of a general spin-S HAFM chain is described by the SU(2)k WZW theory with k ≡ 2S mod 2, as long as the Hamiltonian possesses the SU(2) spin rotation symmetry and the lattice translation symmetry.

In other words, critical phenomena in 1+1 dimensions with SU(2) symmetry and $Z_2$ symmetry of $g → −g$ are grouped into two symmetry-protected classes: one consists of the SU(2)k WZW theories with even k and the other with odd k. In the presence of the SU(2) spin rotation symmetry and the lattice translation symmetry, general HAFM chains with integer spin $S$ can only realize the former, while those with half-odd-integer spin $S$ can only realize the latter.

As discussed earlier, identifying the SU(2)0 WZW theory as a gapped phase with a unique ground state, the above includes Lieb-Schultz-Mattis theorem as a special case. Affleck and Haldane argued, based on the large-k semiclassical analysis, the SU(2)k WZW theory with the leading perturbation allowed under the $g → −g$ symmetry, $(trg)^2$, can be mapped to the O(3) nonlinear sigma model. Here the Wess-Zumino term gives rise to the topological term of the O(3) nonlinear sigma model, at large-$g$, (tr$g$)$^2$, can be mapped to the O(3) nonlinear sigma model. Here the Wess-Zumino term gives rise to the topological term of the O(3) nonlinear sigma model, at large-$g$, (tr$g$)$^2$, can be mapped to the O(3) nonlinear sigma model, the SU(2) spin chain context, the SU(2) symmetry, $S_z$ inversion symmetry does not.

Another interesting example is the translation invariant spin-S model

$$\mathcal{H}_{J_1,J_3} = \sum_j [J_1 S_j S_{j+1} + J_3 ((S_{j-1} S_j) (S_j S_{j+1}) + \text{H.c.})].$$

This model has the quantum critical point $J_{3c} > 0$ described by the SU(2)$_{2S}$ WZW theory, even though the model is not at the integrable Takhtajan-Babujian point. Again this is consistent with our selection rule.

We stress that our selection rule is protected by the symmetry under the lattice translation by one site. In fact, it can be removed by breaking the translation symmetry explicitly. For example, in the extended model

$$\mathcal{H}_{J_1,J_3,\delta} = \mathcal{H}_{J_1,J_3} - J_1 \delta \sum_j (-1)^j S_j \cdot S_{j+1}$$

The model (7) breaks the one-site translation symmetry explicitly, when $\delta \neq 0$. When $S = 1$, the model (7) exhibits a critical line of $c = 1$ connected to the multicritical point with $c = 3/2$. This means that the SU(2)$_2$ WZW theory can flow to the SU(2)$_1$ one in the absence of the translation symmetry. A similar RG flow from the level 2 to the level 1 is found in the $S = 1$ bilinear-biquadratic chain with the bond alternation. The bond alternation breaks the lattice translation and the site-centered inversion symmetries, but keeps the time reversal and the bond-centered inversion symmetries. This is consistent with our analysis that either the lattice translation or the site-centered inversion symmetry protects the two categories, while the time reversal nor the site-centered inversion symmetry does not.

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See Supplemental Material for derivations of the partition function of the $\mathbb{Z}_2$ orbifold in the conformal field theory. To this end the relation (4) in the main text is useful. The partition function of the original $SU(2)_k$ WZW theory is simply given in the diagonal form,

$$Z_{SU(2)}(\tau, \bar{\tau}) = \sum_{j=0}^{k-1} |\chi_j(\tau)|^2.$$  \hfill (8)

The highest weight state $|j, j\rangle$ of the conformal tower yielding the term $|\chi_j(\tau)|^2$ in Eq. (8) is transformed by $g \to -g$ as follows.

$$|j, j\rangle \to (-1)^{2j}|j, j\rangle.$$  \hfill (9)

Here we implicitly assumed that the state $|0, 0\rangle$ is transformed as $|0, 0\rangle \to |0, 0\rangle$ because $|0, 0\rangle$ corresponds to the identity operator and it is even with respect to the $g \to -g$ transformation.

The conformal tower is composed of the highest weight state and descendant states. The latter are generated from the former by multiplying generators of the Kac-Moody and the $SU(2)$ current algebras. Since these generators are odd under $g \to -g$, every state generated from $|j, j\rangle$ acquires the nontrivial phase $(-1)^{2j}$ by the $g \to -g$ transformation.

Now we are ready to derive the projected partition function $Z^\text{proj}_+$. The definition (5) in the main text immediately gives

$$Z^\text{proj}_+ = \sum_{j=0}^{k-1} |\chi_j|^2.$$  \hfill (10)

The index $j$ runs over integers only. We apply $\mathcal{S}$ and $\mathcal{T}\mathcal{S}$ transformations on $Z^\text{proj}_+$:

$$SZ^\text{proj}_+(\tau) = \sum_{j=0}^{k-1} |S_{j, 0}\chi_0 + S_{j, \frac{k}{2}}\chi_{\frac{k}{2}} + \cdots + S_{j, \frac{3k}{2}}\chi_{\frac{3k}{2}}|^2,$$  \hfill (11)

$$TSZ^\text{proj}_+(\tau) = \sum_{j=0}^{k-1} |S_{j, 0}\chi_0 + S_{j, \frac{k}{2}}e^{i2\pi\Delta_1/2}\chi_{\frac{k}{2}} + \cdots + S_{j, \frac{3k}{2}}e^{i2\pi\Delta_1/2}\chi_{\frac{3k}{2}}|^2.$$  \hfill (12)

Combining them with the formula (4) in the main text, we obtain the coefficient $X_{j, j'}$ of Eq. (3) in the main text.
For \( j = 0, 1, 2, \ldots, \frac{k}{2} \),
\[
X_{j,j} = 2(S_0^2 + S_{1,j}^2 + \cdots + S_{\frac{k}{2},j}^2).
\] (13)

For \( j = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{k}{2} \),
\[
X_{j,j} = 2(S_0^2 + S_{1,j}^2 + \cdots + S_{\frac{k}{2},j}^2) - 1.
\] (14)

For \( j \neq j' \),
\[
X_{j,j'} = (S_0 S_{j'} + S_{1,j} S_{j'} + \cdots + S_{\frac{k}{2},j} S_{\frac{k}{2},j'})
\times (1 + e^{i2\pi(\Delta_j - \Delta_{j'})}).
\] (15)

The diagonal coefficients \( (13) \) and \( (14) \) are simple. The summation \( 2(S_0^2 + S_{1,j}^2 + \cdots + S_{\frac{k}{2},j}^2) \) becomes
\[
2\left( \frac{k}{2} + 1 \right) - \sum_{n=0}^{k/2-1} \cos \left( \frac{\pi(2j+1)(2n+1)}{2 + k} \right)
= \frac{2}{2 + k} \left[ \frac{k+1}{2} - \sum_{n=0}^{k-1} \cos \left( \frac{\pi(2j+1)(2n+1)}{2 + k} \right) \right]
\] (16)

with \( \theta = 2\pi(2j+1)/(2 + k) \). The identity \( \sin(\theta + k + 1) = \sin(2\pi(2j+1) - \theta) = - \sin(\theta) \) leads to
\[
X_{j,j} = 1, \quad (j = 0, 1, 2, \ldots, \frac{k}{2})
\] (17)

and
\[
X_{j,j} = 0, \quad (j = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{k}{2}).
\] (18)

The off-diagonal term is similarly derived.
\[
S_0 S_{j'} + S_{1,j} S_{j'} + \cdots + S_{\frac{k}{2},j} S_{\frac{k}{2},j'}
= \frac{1}{2 + k} \left[ \frac{k}{2} - \sum_{n=0}^{k-1} \cos \left( \frac{\pi(2j - 2j')(2n+1)}{2 + k} \right) \right].
\] (19)

When \( j + j' \neq \frac{k}{2} \), the summation in the last line leads to
\[
S_0 S_{j'} + S_{1,j} S_{j'} + \cdots + S_{\frac{k}{2},j} S_{\frac{k}{2},j'}
= \frac{1}{2 + k} \left[ \frac{\sin(\theta_+(k+1))}{2 \sin(\theta_+)} + \frac{\sin(\theta_-(k+1))}{2 \sin(\theta_-)} \right]
= \frac{1}{2(2 + k)} (-1)^{2j-2j'} [(-1)^{j'}-1]
= 0,
\] (20)

where \( \theta_+ = \pi(2j + 2j' + 2)/(2 + k) \) and \( \theta_- = \pi(2j - 2j')/(2 + k) \). On the other hand, when \( j + j' = \frac{k}{2} \), the summation does not vanish.
\[
S_0 S_{j'} + S_{1,j} S_{j'} + \cdots + S_{\frac{k}{2},j} S_{\frac{k}{2},j'}
= \frac{1}{2 + k} \left( \frac{k+1}{2} - \frac{1}{2} (-1)^{4j-k+1} \right)
= \frac{1}{2}.
\] (21)

Here \( 4j - k + 1 \) is an even integer because \( k \) is odd. In the end, the nonzero coefficients \( X_{j,j'} \) are
\[
X_{j,j'} = \begin{cases} 
1, & (j = 0, 1, 2, \ldots, \frac{k}{2}) \\
\frac{1}{2} \left( 1 + e^{i2\pi(\Delta_j - \Delta_{j'})} \right), & (j + j' = \frac{k}{2}).
\end{cases}
\] (22)

resulting in the partition function,
\[
Z_+ = \sum_{j \in \mathbb{Z}^+} |\chi_j + \chi_{\frac{k}{2}-j}|^2 + 2|\chi_{\frac{k}{2}}|^2.
\] (25)

Note that the term \( |\chi_{\frac{k}{2}}|^2 \) comes from the first two lines of Eq. (24).

When \( k \) is divisible by 4, the coefficient \( 22 \) becomes
\[
X_{j,j'} = \begin{cases} 
1, & (j = 0, 1, 2, \ldots, \frac{k}{2}) \\
\frac{1}{2} [1 + (-1)^{2j}], & (j + j' = \frac{k}{2});
\end{cases}
\] (24)

resulting in the partition function,
\[
Z_+ = \sum_{j \in \mathbb{Z}} |\chi_j + \chi_{\frac{k}{2}-j}|^2 + 2|\chi_{\frac{k}{2}}|^2.
\] (25)

That is, the partition function is given by
\[
Z_+ = \sum_{j \in \mathbb{Z}} |\chi_j|^2 + |\chi_{\frac{k}{2}}|^2 + \sum_{j=1/2}^{k/2} \left( \chi_j \bar{\chi}_{\frac{k}{2}-j} + \chi_{\frac{k}{2}-j} \bar{\chi}_j \right).
\] (27)
When $k$ is odd, the coefficient is not real:

$$X_{j,j'} = \begin{cases} 
1, & \left( j = 0, 1, 2, \ldots, \frac{k-1}{2} \right) \\
\frac{1}{2} \left[ 1 + (-1)^{2j} e^{-i\pi k/2} \right], & \left( j + j' = \frac{k}{2} \right) \\
0 & \text{otherwise}
\end{cases}.$$  

The partition function is then anomalous,

$$Z_+ = \sum_{\substack{j=0 \atop j \in \mathbb{Z}}} \left| \chi_j \right|^2 + \sum_{\substack{j=0 \atop j \in \mathbb{Z}}} \left( \frac{1 + (-1)^{2j} e^{-i\pi k/4}}{2} \chi_j \chi_{\frac{k}{2} - j} \right) + \frac{1 + (-1)^{2j} e^{i\pi k/4}}{2} \chi_{\frac{k}{2} - j}.$$  

Modular invariance

The modular noninvariance of the odd-level $\mathbb{Z}_2$ orbifold is obvious because, for instance, $\chi_j \chi_{\frac{k}{2} - j}$ is transformed as

$$\mathcal{T}(\chi_j \chi_{\frac{k}{2} - j}) = (-1)^j e^{-i\pi k/4} \chi_j \chi_{\frac{k}{2} - j}.$$  

It is also straightforward to confirm the modular invariance of the even-level $\mathbb{Z}_2$ orbifolds (25) and (27). According to a complete classification of all the possible modular invariants in the $SU(2)$ WZW theory, the partition function (25) for $k = 4n$ is the modular invariant of the type $D_{2n+2}$ and one (27) for $k = 4n - 2$ is that of the type $D_{2n+1}$.