Instability of group formation through indirect reciprocity under imperfect information and implementation error

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(Dated: February 19, 2018)

Abstract

Indirect reciprocity is a key mechanism behind the evolution of cooperation. Oishi et al. analytically showed the formation of two exclusive groups under the KANDORI assessment rule in the case of perfect information and no implementation error, regardless of the population size $N$. Here, we numerically show the formation of many exclusive groups under the JUDGING assessment rule in the same case. Introducing degrees of exclusive groups, we numerically examine the stability of the group formation under imperfect information and implementation error.
I. INTRODUCTION

Indirect reciprocity is a key mechanism driving the evolution of cooperation.\cite{1} One feature of indirect reciprocity is that helpful acts are returned, not by the recipient as in direct reciprocation, but by third parties: players collect information about other player’s behavior, and determine their actions by using this information. This behavior requires the following two modules: (a) an assessment rule for the acts of others as good or bad, and (b) an action rule specifying how to act toward others based on that assessment.

Assessment rules are classified into three types, depending on what information they use. A first-order assessment rule only takes into account whether a donor $X$ helps a recipient $Y$. A second-order assessment rule takes also into account the image of the recipient $Y$. A third-order assessment rule additionally takes into account the image of the donor $X$.

Assuming binary assessments, Ohtsuki and Iwasa\cite{2,3} showed that among the resulting possible strategies, only eight lead to a stable regime of mutual cooperation under public and perfect information about others. These are called the leading eight.

On the other hand, Uchida\cite{4} examined the effects of private and imperfect information as well as those of implementation errors. Although Uchida concluded that private information leads to the collapse of the sterner (KANDORI\cite{5}) assessment rule, the results presented here are somewhat different.

Oishi et. al.\cite{6} showed the emergence of two exclusive groups in the JUDGING assessment rule when information is perfect and private. In this study, we numerically examine the stability of the exclusive groups under imperfect information and implementation errors. Furthermore, we numerically show the emergence of many exclusive groups in the third-order (JUDGING) assessment rule.

II. MODEL AND METHODS

We consider the donation game defined as follows: there are $N$ players, each with their own opinion as to whether each of the other players is good ($G$) or bad ($B$) (image matrix). $\beta_{ij}(t) \in G, B$ represents player $i$’s opinion of player $j$ at a round $t$. Each element of the image matrix at the initial round $t = 1$, is $G$ with probability $p$ or $B$ with probability $1 - p$. The probability $p$ is called the initial trust probability and $0 < p < 1$. 

The game is repeated over a large number of rounds. In each round, one player is randomly chosen as a donor and another as a recipient. It is the donor’s choice whether to cooperate or defect. We assume the decision follows an action rule. If the donor cooperates, the payoff of the donor, \(-c\), is less than 0 and that of the recipient, \(b\), is greater than 0. If the donor defects, the payoffs of both players are 0. We assume \(b \geq c\).

All players observe which player is the donor, which is the recipient, and what the donor does to the recipient. Then, all players independently revise their own opinion on the donor based on the observation and their assessment rule.

Here we use only one action rule, namely that the donor only cooperates with recipients whom the donor regards as good. On the other hand, we mainly study two assessment rules, which are called the KANDORI and JUDGING ones. In Table 1, we show definitions of the assessment rules used in this paper.

We consider two types of noises, following Uchida’s paper. We investigate the effect of imperfect information, in which each interaction is only observed by a fraction, \(q < 1\), of the population. We also study the effect of implementation error \(\epsilon\), which is the probability that an intended help is not actually given.

| pre-assessment to the donor | good | bad | good | bad |
|-----------------------------|------|-----|------|-----|
| cooperate(C) or defect(D)   | C    | D   | C    | D   |
| All C                       | G    | G   | G    | G   |
| All D                       | B    | B   | B    | B   |
| SCORING                     | G    | B   | G    | B   |
| TYPE 1                      | G    | B   | G    | G   |
| KANDORI                     | G    | B   | B    | G   |
| STANDING                    | G    | B   | G    | B   |
| JUDGING                     | G    | B   | B    | B   |
III. RESULTS USING THE KANDORI ASSESSMENT RULE

First, we show our results in the case of the KANDORI assessment rule.

A. The case of \( q = 1 \) and \( \epsilon = 0 \)

We first consider the case of perfect information, \( q = 1 \), and no implementation error, \( \epsilon = 0 \). Oishi et al. \[6\] analytically investigated this case and showed the formation of two exclusive groups with ratios of \( p \) and \( 1 - p \), irrespective of the population size, \( N \).

B. The case of \( q < 1 \) and \( \epsilon = 0 \)

In this case, the results depend upon the initial state of the image matrix, \( \beta_{ij}(t = 1) \), and the number of players, \( N \). We consider two initial states of an image matrix for some \( p \) \((0 < p < 1)\), namely 1) a random initial state that is randomly set to \( G \) with a probability \( p \) or \( B \) with a probability \( 1 - p \), and 2) a uniform initial state where two exclusive groups with ratios \( p \) and \( 1 - p \) are initially formed. In the latter case, we easily confirm that imperfection of the information, \( 1 - q \), does not destroy the two exclusive groups.

In the former case, we can see that, after a large number of rounds and sorting (see Appendix A), two imperfect exclusive groups still appear for small \( N \) and small \( 1 - q \), as seen in Fig. 1. Here, we introduce the degree of the two exclusive groups, \( DEG_1 \), as follows:

\[
DEG_1 = \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1}^{N} \text{sign}(\beta_{ij})\beta_{1i}\beta_{ji}.
\]  

(1)

\( DEG_1 \) is equal to one when the two exclusive groups are perfectly formed, and becomes zero when they are destroyed. In Fig. 2, the results of \( DEG_1 \) versus \( q \) are shown for \( N = 100, 200, \) and 400. For larger numbers of players, we can see that smaller values of the incompleteness of the information, \( 1 - q \), disturbs the formation of the two exclusive groups from a random initial state.

C. The case of \( q = 1 \) and \( \epsilon > 0 \)

After a large number of rounds, two groups appear. Players within the same group have the same opinions; those in different groups have opposite opinions. The two groups are not...
FIG. 1. Image matrix $\beta_{ij}$ at the 20,000th round starting with a random initial state, where $N=100$, $q=0.99$, $p=0.8$, and $\epsilon = 0$.

FIG. 2. $DEG_1$ versus $q$ for $N = 100, 200$ and $400$ after a large number of rounds from a random initial state, where $p = 0.7$ and $\epsilon = 0$ and the error bars show a standard deviation of 10 samples with different random numbers.

necessarily exclusive. In Fig. 3, we show that $DEG_1$ linearly decreases with $\epsilon$, irrespective of the population size $N$.

D. The case of $q < 1$ and $\epsilon > 0$

In contrast with the case of $q < 1$ and $\epsilon = 0$, there is no dependence upon different initial states with the same $p$ after a large number of rounds. In Fig. 4, the results of $DEG_1$ versus $q$ are shown for $N = 100, 200$, and $400$. We obtain results similar to the case where $q < 1$ and $\epsilon = 0$ for a random initial state.
IV. RESULTS USING THE JUDGING ASSESSMENT RULE

We show results using the JUDGING assessment rule.

A. The case of $q = 1$ and $\epsilon = 0$

Many exclusive groups are formed after a large number of rounds from a random initial state at some $p$, where $0 < p < 1$, as shown in Fig. 5. The sizes of these groups and their frequencies depend upon $p$. Fig. 6 shows relations between group sizes and frequencies. The larger the value of $p$, the greater the ease with which large groups are formed. When $p$ is small, it shows a behavior close to a power-law distribution. This distribution is almost the same for different population sizes $N$, as shown in Fig. 7.

Because $DEG1$ is only applicable to two exclusive groups, we introduce the following...
FIG. 5. Image matrix $\beta_{ij}$ at the 60,000th round from a random initial state, where $N=100$, $p=0.7$, $q=1$, and $\epsilon=0$.

FIG. 6. Relations between group size and frequency for $p=0.3$, 0.5, 0.7, and 0.9, where $N=100$, $q=1$, and $\epsilon=0$. The frequency is a sum over 100 samples.

function for the degree of the exclusive groups for the JUDGING assessment rule ($DEG2$):

\[
DEG2 = \frac{\sum_{i=2}^{N} \sum_{j=1}^{i-1} (\beta_{ij} + 1)(\beta_{ji} + 1)/4}{\sum_{i=2}^{N} \sum_{j=1}^{i-1} (\beta_{ij} + 1)/2} \tag{2}
\]

$DEG2$ is one when many exclusive groups are perfectly formed, and zero when they are destroyed."
B. The case of $q < 1$ and $\epsilon = 0$

In this case, many smaller exclusive groups are formed from a random initial state (see Fig. 8), although $DEG2$ still remains one. Once the formation of exclusive groups has been completed, imperfection of information, $1 - q$, does not change the state (i.e. the state is stationary). This behavior results in a decrease in good assessments. Fig. 9 shows the fraction of good assessments as a function of $q$.

C. The case of $q = 1$ and $\epsilon > 0$

The degree of exclusive groups ($DEG2$) decreases almost linearly with $\epsilon$, similar to the case of the KANDORI assessment rule. This does not necessarily mean the collapse of the group. As seen in Fig. 10, there are some white horizontal lines in addition to squares representing the exclusive groups. This indicates that there are players to assess the players in the different group as good: i.e., the exclusivity of the group formed in this case is not complete.

D. The case of $q < 1$ and $\epsilon > 0$

The degree of the exclusive groups ($DEG2$) decreases and groups are destroyed. We also examine the stability against mutants with different assessment rules. We consider the world
FIG. 8. Image matrix $\beta_{ij}$ at the 60,000th round from a random initial state, where $N=100$, $p=0.7$, $q=0.7$, and $\epsilon=0$.

FIG. 9. Fraction of good assessments in the image matrix $\beta_{ij}$ at the 60,000th round, starting with a random initial state, where $N=100$, $p=0.7$, $q=0.7$, and $\epsilon=0$. Error bars show the standard deviation of 10 samples.

of players with two or three different assessment rules in Table 1, and compare the average total payoffs. One example is shown in Fig. 11. The results show that, with the exception of the STANDING mutants, the JUDGING players have a larger average payoff than that of the other mutants, although $DEG2$ decreases in the case of $q < 1$ and $\epsilon > 0$. 

9
FIG. 10. Image matrix $\beta_{ij}$ at the 60,000th round starting with a random initial state, where $N=100$, $p=0.9$, $q=1$, and $\epsilon=0.1$.

FIG. 11. Average total payoff at the 60,000th round in the world of players with two assessment rules (90 JUDGING players and 10 mutants), where $N=100$, $p=0.7$, $q=0.9$, $\epsilon=0.1$, $b=10$, and $c=5$. Error bars show the standard deviation of 10 samples.

V. SUMMARY AND DISCUSSION

We summarize the results of this study as follows:

1) In contrast with the claims of Uchida[4], imperfect information does not completely destroy two exclusive groups in the KANDORI assessment rule. For a small population size of $N$ and small imperfect information $1-q$, two incomplete exclusive groups are formed.
2) Using the JUDGING assessment rule with $q = 1$ and $\epsilon = 0$, many exclusive groups appear from a random initial state with a trust probability $p$, where $0 < p < 1$. Distributions of the group sizes of exclusive groups show nearly power-law behaviors when $p$ is small and do not depend upon population size $N$. Incomplete information $1 - q$ does not decrease the degree of exclusive groups ($D_{EG2}$), but makes their group sizes smaller, resulting in an increased number of bad assessments.

3) If $b$ is sufficiently larger than $c$, JUDGING players have larger average payoffs than mutants (other than STANDING mutants) with different assessment rules, as shown in Table 1, even in the case where $q < 1$ and $\epsilon > 0$.

The model investigated here has considerable limitations. Introducing more realistic effects to the model may overcome the negative points of the JUDGING players (namely smaller average payoffs than those of the STANDING players and an increased number of bad assessments due to the incomplete information). One of the limitations is that players have a very short memory: they determine their opinions of another player according to their last observation of that player’s actions. In real life, images are likely to be based on a longer memory. In fact, we observe that introducing a longer memory to the model makes the two exclusive groups of KANDORI players more robust under incompleteness of information.

The other limitation is that the images are binary in this model, whereas the moral world is not just black or white. Tanabe et al. investigated the indirect reciprocity with trinary reputations and found that this model allows cooperation under the SCORING assessment rule under some mild conditions. We may expect an increase in good assessments for the JUDGING players under the imperfect information by changing the binary images into trinary ones.

**ACKNOWLEDGMENTS**

We are thankful for the fruitful discussions with Professors F. Matsubara and N. Suzuki. Part of the results in this research was obtained using supercomputing resources at Cyber-science Center, Tohoku University.
Appendix A: Sorting image matrices

We sort image matrices to make their structure easier to see. First, matrices are sorted based on the image of the first row. The columns are also sorted in the same order. At this time, a white square is formed in the upper left corner. In the case of the KANDORI assessment rule, a white square is also formed in the lower right corner, and sorting is completed. However, in the case of the JUDGING assessment rule, the lower right area has not been sorted. Therefore, the lower right area is sorted in the same way. This process is repeated until sorting becomes impossible. Finally, many white squares are formed on the diagonal of the image matrix. In other words, the players are divided into many exclusive groups.

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