Electron-Magnon Scattering in Anomalous Hall Effect

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We study the role played by electron-magnon scattering in the anomalous Hall effect. We find that it has important contributions distinct from other scattering processes like impurities scattering and phonon scattering. As a demonstration, we calculate the Hall conductivity for a two dimensional Dirac model. The result indicates that as system control parameter varies, the competition between magnon scattering and other types of scattering changes the Hall conductivity drastically. In particular, the side jump contribution could acquire a strong temperature dependence.

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The anomalous Hall effect (AHE), in which a transverse voltage is induced by a longitudinal current flow in ferromagnetic materials, is one of the most intriguing effects in physics. While it has been widely used experimentally as a standard technique for the characterization of ferromagnets, the theoretical study of the AHE proves to be complicated and is a subject full of controversial issues and conflicting results [1]. In recent years, an important connection has been established between the AHE and the Berry phase of Bloch electrons [2, 3, 4, 5]. This triggers revived interest in this subject and is followed by extensive researches both theoretically and experimentally [6]. It is now generally accepted that the AHE is due to spin-orbit coupling in ferromagnets, and apart from an intrinsic contribution which is scattering independent, there are also important extrinsic contributions to the AHE. Especially, there is a peculiar side jump contribution that arises from scattering, but does not depend on the scattering strength.

Up to now, most of the theoretical studies of the AHE only take into account the impurity scattering, despite that many experimental measurements are performed at finite temperatures hence other scattering processes like phonon scattering and magnon scattering should also be relevant. It has been shown that the phonon scattering produces similar contributions as impurity scattering [7], which seems to imply that the side jump contribution should only weakly depend on temperature. However, recent experiment by Tian et al. shows that the side jump does have strong temperature dependence [8].

In this paper, we show that the magnon scattering, which has been largely overlooked so far, has distinct contributions to the AHE from both impurity scattering and phonon scattering. The underlying reason for this difference is that the magnon scattering involves spin-flip, whereas the impurity scattering and the phonon scattering are both spin independent. The extrinsic contribution to the AHE turns out to depend sensitively on the type of spin structure of the scattering process, which has also been noticed in the recent study of the magnetic impurity scattering [9, 10]. Therefore, the anomalous Hall conductivity would change drastically as two or more types of scattering compete. In particular, the side jump contribution will depend on the relative scattering strength.

We begin with a description of the electron-magnon scattering process in ferromagnetic systems. The exchange coupling between conduction electron and local magnetic order parameter can be written as

$$\hat{H}_{\text{int}} = -J \int d\mathbf{r} \left[ \hat{\sigma}(\mathbf{r}) \cdot \mathbf{S}(\mathbf{r}) \right]$$

where $J$ is the exchange coupling constant, $\hat{\sigma}$ is the vector of Pauli matrices for conduction electron spin, $\mathbf{S}$ is the local spin, $\hat{\sigma}_x \equiv \hat{\sigma}_x \pm i\hat{\sigma}_y$, $S_\pm \equiv S_x \pm iS_y$, and hat means the quantity is a matrix in spin space. The last term above describes the exchange splitting which should be included in the non-interacting part of the Hamiltonian, whereas the first two terms describe the electron-magnon scattering. Using Holstein-Primakoff representation, for temperatures below Curie point when the number of local spin flips is much smaller than the total spin, we can write the interaction Hamiltonian as

$$\hat{H}_{\text{int}} = -J\sqrt{2S} \frac{1}{V} \sum_{k,q} \left( c^\dagger_{k+q} c_{k} a^\dagger_{-q} + c^\dagger_{k+q} c_{k} a_{-q} \right),$$

where $V$ is the system volume, $c^\dagger$ ($c$) and $a^\dagger$ ($a$) are the electron and magnon creation (annihilation) operators respectively. From Eq.[2], it is clear that the magnon scattering flips spin, unlike impurity scattering and phonon scattering which are spin independent.

When the energy of magnons involved in the scattering is much lower than the Fermi energy and the electron energy spectrum (and density of states) varies smoothly around Fermi surface, we can approximate the scattering process as quasi-elastic. In typical ferromagnetic materials like Fe or Co, the magnon has a large effective mass...
about $10^{-29} \sim 10^{-28} \text{kg}$ \cite{11}. The quasi-elastic treatment will be a good approximation if the electron effective mass at Fermi level is much smaller than that of the magnon. In this case, the electron sees an effective scattering potential

$$
\hat{V}_m(q) = \frac{1}{\sqrt{2}} V^m_\alpha(q) (\hat{\sigma}_+ + \hat{\sigma}_-),
$$

where $V^m_\alpha(q) = -J \sqrt{n_m(q)/2}$ is the orbital part of the scattering potential, $n_m(q)$ is the distribution function of magnon, and the spin part $\hat{\sigma}_\pm$ is off-diagonal representing spin-flip processes.

To demonstrate that the magnon scattering gives distinct contributions to the AHE, we calculate the Hall conductivity of the two dimensional (2D) Dirac model. The AHE from impurity scattering in this model has been studied previously by Sinitsyn et al. \cite{12}. We choose this model not only because of its simplicity to demonstrate our ideas, but also because it describes low energy physics of interesting systems such as graphene \cite{13}, kagome lattice \cite{14} and surface states of topological insulator \cite{15} (though it should be noticed that for graphene and kagome lattice the ‘spin’ refers to the sublattice degrees of freedom rather than electron spin as discussed here). Therefore the results presented here will also be important in understanding transport properties of these systems.

The AHE occurs in 2D Dirac model when a suitable symmetry breaking mechanism is introduced. Our model Hamiltonian reads (we set $\hbar = 1$ in the following)

$$
\mathcal{H} = v (k_x \hat{\sigma}_x + k_y \hat{\sigma}_y) + \Delta \hat{\sigma}_z,
$$

where the last term is the symmetry breaking term which introduces a gap of $2\Delta$. We assume that this model describes low energy physics near the Fermi surface of certain 2D ferromagnetic system, hence the term $\Delta \hat{\sigma}_z$ represents the exchange splitting which corresponds to the last term in Eq.\cite{1}.

The eigenstates of the system are given by $\psi^c,v(k) = (1/\sqrt{V}) e^{i k \cdot r} |u^c,v(k)\rangle$ with energy eigenvalues $\varepsilon^c,v(k) = \pm \sqrt{(vk)^2 + \Delta^2}$, where $c$ and $v$ stand for conduction and valence band respectively, and $|u^c,v(k)\rangle$ is the spin part of the eigenstate which can be written as

$$
|u^c(k)\rangle = \left( \begin{array}{c} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i \phi} \end{array} \right), \quad |u^v(k)\rangle = \left( \begin{array}{c} \sin \frac{\theta}{2} \\ -\cos \frac{\theta}{2} e^{i \phi} \end{array} \right),
$$

with $\theta$ and $\phi$ being the spherical angles of the vector $(vk_x, vk_y, \Delta)$ such that $\cos \theta = \Delta/\sqrt{(vk)^2 + \Delta^2}$ and $\tan \phi = k_y/k_x$. Due to spin-orbit coupling, the eigen-spinors are $k$-dependent.

We evaluate the Hall conductivity by using the Kubo-Streda formalism \cite{14, 17}. In this approach, the Hall conductivity can be separated into two parts, $\sigma_{xy} = \sigma_{xy}^I + \sigma_{xy}^H$, where $\sigma_{xy}^I$ is a Fermi surface contribution, and $\sigma_{xy}^H$ is a Fermi sea contribution for which we only need to retain the scattering-free component in the weak disorder limit \cite{12}. All the important scattering effects are contained in $\sigma_{xy}^I$, which takes the form

$$
\sigma_{xy}^I = \frac{e^2}{2\pi^2} \text{Tr} \left( \hat{v}_x \hat{G}^R(\varepsilon_F) \hat{v}_y \hat{G}^A(\varepsilon_F) \right),
$$

where $\hat{G}^R$ and $\hat{G}^A$ are the retarded and advanced Green’s functions respectively, $\hat{v}_x,y = \varepsilon \hat{\sigma}_x,y$ are the velocity operators, $\varepsilon_F$ is the Fermi energy, the trace is taken over both momentum and spin spaces and the bracket means statistical average over disorder configurations. In the following calculation, we take the Fermi energy to be in the conduction band.

The intrinsic contribution results from the Berry curvatures of spin-orbit coupled bands and is independent of scattering. In Kubo-Streda formalism, the intrinsic contribution comes from the scattering-free components of $\sigma_{xy}^I$ and $\sigma_{xy}^H$. It can be decomposed into two parts $\sigma_{xy}^I = \sigma_{xy}^{I(v)} + \sigma_{xy}^{I(c)}$, where $\sigma_{xy}^{I(v)}$ is the contribution from all the completely occupied valence bands below the Fermi surface and $\sigma_{xy}^{I(c)}$ is the contribution from the partially filled conduction band where the Fermi surface lies in. The contribution from completely filled bands $\sigma_{xy}^{I(v)}$ has a topologically quantized value $N e^2/(2\pi)$ with $N$ being an integer known as the first Chern number. The calculation of $N$ goes beyond any low energy effective model since it involves the entire Fermi sea. On the contrary, the contribution $\sigma_{xy}^{I(c)}$ from the partially filled conduction band can be regarded as a Fermi surface property \cite{13}. For Dirac model, we have

$$
\sigma_{xy}^{I(c)} = \frac{e^2}{4\pi} (1 - \cos \theta_F),
$$

where $\theta_F$ is the spherical angle $\theta$ evaluated on the Fermi surface when $k = k_F$.

The extrinsic contribution consists of the side jump and the skew scattering. In the semiclassical picture, the side jump arises from the coordinate shift of a wavepacket during the scattering process, and the skew scattering appears due to the asymmetry of scattering rate for higher order scattering processes \cite{1}. It has been clarified recently that the skew scattering contribution as defined in the semiclassical picture actually contains two different parts: a conventional skew scattering part with $n^{-1}$ dependence and an intrinsic skew scattering part with $n^0$ dependence with $n$ being the disorder density \cite{14}. According to its parametric dependence, the intrinsic skew scattering can be included as part of the side jump. In Kubo-Streda formalism, the various contributions listed above have been identified with different sets of Feynman diagrams in the self-consistent non-crossing approximation \cite{12}.

Let’s first consider a clean system with only magnon scattering. In the quasi-elastic approximation, the disorder lines in Feynman diagrams do not carry energy
arguments. Since the population of magnon bath is conserved in steady state, each disorder line must have a pair of \( \sigma_+ \) and \( \sigma_- \) at the two ends, corresponding to magnon emission and absorption processes. Therefore the conventional skew scattering which involves third order scattering events must vanish. Furthermore, due to the angular average at velocity vertices, the intrinsic skew scattering also vanishes, leaving only the side jump contribution,

\[
\sigma_{xy}^{\text{ext}} = \frac{e^2}{4\pi} \cos \theta_F, \tag{8}
\]

which cancels with the part of intrinsic contribution that depends on Fermi energy, such that the final result becomes a constant value \( e^2/(4\pi) \) and is the same as the intrinsic contribution for a completely filled conduction band, i.e. in the limit \( \theta_F \to \pi/2 \).

Next we shall include the impurity scattering as well and investigate the competition between magnon scattering and impurity scattering in the AHE. For simplicity, we consider the short range impurity as been studied in Ref. [12]. The result of the total Hall conductivity is (including only \( \sigma_{xy}^{\text{int}(c)} \) for intrinsic contribution)

\[
\sigma_{xy} = \frac{e^2}{4\pi} (1 - \cos \theta_F) - \frac{e^2}{\pi} \left( \frac{\sin^2 \theta_F \cos \theta_F (1 - \zeta)}{1 + 3 \cos^2 \theta_F} + 4 \sin^2 \theta_F \zeta \right) - \frac{e^2}{\pi} \left( \frac{\sin^4 \theta_F \cos \theta_F (\frac{1}{2} - \zeta + 2\eta)}{(1 + 3 \cos^2 \theta_F) + 4 \sin^2 \theta_F \zeta} \right)^2, \tag{9}
\]

where \( \zeta = \tau^{-1}_m / \tau^{-1}_i \) is the ratio of magnon scattering rate to impurity scattering rate with

\[
\tau^{-1}_{m,i} = 2\pi \int \frac{d^2 \mathbf{k}'}{(2\pi)^2} \left| V_m^o(\mathbf{k}', \mathbf{k}) \right|^2 \left[ 1 - \cos (\phi - \phi') \right] \delta (\varepsilon_F - \varepsilon_{k'},) \tag{10}
\]

and \( \eta = \tau^{-2}_k / \tau^{-2}_i \) represents the conventional skew scattering contribution from impurities. For standard white noise impurity model, the third order correlation of scattering potential is zero, so \( \eta \) vanishes identically. It will be nonzero if a non-Gaussian part \( V_1 \) of the impurity potential is included [12], then \( \tau^{-2}_k = n_i \varepsilon_i^3 V_1^2 / (4\pi v^4) \).

Observe that when the magnon scattering is dominant over impurity scattering, i.e. for very large \( \zeta \), we recover the result of Eqs. (7,8) with a constant value \( e^2/(4\pi) \). In the opposite limit, when impurity scattering dominates, \( \zeta \to 0 \), we retain the previous result by Sinitsyn et al. [12]. From our result Eq. (2), it is clear that as \( \zeta \) varies, which results from the competition between different scattering mechanisms, the extrinsic contributions to the Hall conductivity varies drastically and can have a sign change.

It is of fundamental importance to experimentally separate contributions from different mechanisms, especially the intrinsic contribution [8]. Let us collect the terms of \( \sigma_{xy} \) that are of order \( n^0 \), denoted as \( \sigma_{xy}^0 \). These include the intrinsic contribution and the side jump (including intrinsic skew scattering). For soft magnon modes, the ratio \( \zeta \) does not sensitively depend on the Fermi energy. In Fig[1] we plot \( \sigma_{xy}^0 \) as a function of Fermi energy for different values of \( \zeta \). As \( \zeta \) increases, the curve of Hall conductivity is shifting upward from the impurity scattering dominated situation and approaching the limiting value \( e^2/4\pi \) for the magnon scattering dominated case. This competition behavior is more clearly observed in Fig[2] where \( \sigma_{xy}^0 \) is plotted at fixed Fermi level as a function of \( \zeta \). As \( \zeta \) increases, \( \sigma_{xy}^0 \) increases monotonically. For impurity scattering dominated case, \( \sigma_{xy}^0 \) takes negative value for Fermi energies below \( \varepsilon_F \approx 7.3\Delta \). Hence in this energy range, there is a sign change of \( \sigma_{xy}^0 \) as \( \zeta \) increases, i.e. when magnon scattering gradually takes dominant place.

As observed from this model calculation, the magnon scattering indeed plays a quite different role as compared with impurity scattering. This difference comes from their different structures in spin space. Magnon scattering flips spin hence its the matrix element is off-diagonal in spin space while both impurity scattering and phonon scattering is proportional to the identity in spin space. As a result, the extrinsic contributions for each type of scattering involve different combinations of structure factors (such as \( \cos \theta_F \) and \( \sin \theta_F \) in our example), which
leads to the competition behavior. Magnetic impurity scattering has yet another spin structure different from the above three, as being proportional to $\sigma_z$ if the average magnetization is along $z$-direction. Previous studies indeed show that the magnetic impurity scattering behaves differently from the normal impurity scattering for the AHE [10]. The above analysis suggests that we could classify various scattering processes according to their structures in spin space which is the major factor that determines their contributions to the AHE [21]. The competition between different classes could change the Hall conductivity dramatically as system control parameter such as temperature varies.

Finally, we point out that the above discussion is not limited to ferromagnetic systems with electron spin degrees of freedom. Any quasi-particle index which has two degrees of freedom can be generally referred to as ‘spin’ (or pseudospin). Anomalous Hall effect will arise if the system has ‘spin’-orbit coupling as well as ‘spin’ splitting. For example, in a biparticle lattice such as graphene, the sublattice degree of freedom can be treated as pseudospin. Anomalous Hall transport occurs in graphene when there is sublattice symmetry breaking in the system [22]. As another example, for bilayer systems, it is the layer index that plays the role of pseudospin and the pseudospin splitting can be realized by imposing a bias between the two layers. In general, our result indicates that a careful analysis of various scattering processes according to its pseudospin structure is indispensable in the study of AHE for these systems.

In summary, we have shown that the electron-magnon scattering plays an important role in the anomalous Hall effect. The competition between magnon scattering and other scatterings changes the anomalous Hall conductivity drastically as system control parameters are varied.

As a result, the side jump contribution can have strong temperature dependence.

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