Distributed Decision-Making over Adaptive Networks

Sheng-Yuan Tu, Student Member, IEEE and Ali H. Sayed, Fellow, IEEE

Abstract

In distributed processing, agents generally collect data generated by the same underlying unknown model (represented by a vector of parameters) and then solve an estimation or inference task cooperatively. In this paper, we consider the situation in which the data observed by the agents may have arisen from two different models. Agents do not know beforehand which model accounts for their data and the data of their neighbors. The objective for the network is for all agents to reach agreement on which model to track and to estimate this model cooperatively. In these situations, where agents are subject to data from unknown different sources, conventional distributed estimation strategies would lead to biased estimates relative to any of the underlying models. We first show how to modify existing strategies to guarantee unbiasedness. We then develop a classification scheme for the agents to identify the models that generated the data, and propose a procedure by which the entire network can be made to converge towards the same model through a collaborative decision-making process. The resulting algorithm is applied to model fish foraging behavior in the presence of two food sources.

Index Terms

Adaptive networks, diffusion adaptation, classification, decision-making, biological networks.

I. INTRODUCTION

Self-organization is a remarkable property of biological networks [2], [3], where various forms of complex behavior are evident and result from decentralized interactions among agents with limited

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capabilities. One example of sophisticated behavior is the group decision-making process by animals [4]. For example, it is common for biological networks to encounter situations where agents need to decide between multiple options, such as fish deciding between following one food source or another [5], and bees or ants deciding between moving towards a new hive or another [6], [7]. Although multiple options may be available, the agents are still able to reach agreement in a decentralized manner and move towards a common destination (e.g., [8]).

In previous works, we proposed and studied several diffusion strategies [9]–[14] that allow agents to adapt and learn through a process of in-network collaboration and learning. References [13], [14] provide overviews of diffusion techniques and their application to distributed adaptation, learning, and optimization over networks. Examples of further applications and studies appear, e.g., in [15]–[20]. Diffusion networks consist of a collection of adaptive agents that are able to respond to excitations in real-time. Compared with the class of consensus strategies [21]–[27], diffusion networks have been shown to remain stable irrespective of the network topology, while consensus networks can become unstable even when each agent is individually stable [28]. Diffusion strategies have also been shown to lead to improved convergence rate and superior mean-square-error performance [14], [28]. For these reasons, we focus in the remainder of this paper on the use of diffusion strategies for decentralized decision-making.

Motivated by the behavior of biological networks, we study distributed decision-making over networks where agents are subject to data arising from two different models. The agents do not know beforehand which model accounts for their data and the data of their neighbors. The objective of the network is for all agents to reach agreement on one model and to estimate and track this common model cooperatively. The task of reaching agreement over a network of agents subjected to different models is more challenging than earlier works on inference under a single data model. The difficulty is due to various reasons. First, traditional (consensus and diffusion) strategies will converge to a biased solution (see Eq. (14)). We therefore need a mechanism to compensate for the bias. Second, each agent now needs to distinguish between which model each of its neighbors is collecting data from (this is called the observed model) and which model the network is evolving to (this is called the desired model). In other words, in addition to the learning and adaptation process for tracking, the agents should be equipped with a classification scheme to distinguish between the observed and desired models. The agents also need to be endowed with a decision process to agree among themselves on a common (desired) model to track. Moreover, the classification scheme and the decision-making process will need to be implemented in a fully distributed manner and in real-time, alongside the adaptation process.
There have been useful prior works in the literature on formations over multi-agent networks \cite{29}–\cite{35} and opinion formation over social networks \cite{36}–\cite{38} using, for example, consensus strategies. These earlier works are mainly interested in having the agents reach an average consensus state, whereas in our problem formulation agents will need to reach one of the models and not the average of both models. Another difference between this work and the earlier efforts is our focus on combining real-time classification, decision-making, and adaptation into a single integrated framework running at each agent. To do so, we need to show how the distributed strategy should be modified to remove the bias that would arise due to the multiplicity of models — without this step, the combined decision-making and adaptation scheme will not perform as required. In addition, in our formulation, the agents need to continuously adjust their decisions and their estimates because the models are allowed to change over time. In this way, reaching a static consensus is not the objective of the network. Instead, the agents need to continuously adjust and track in a dynamic environment where decisions and estimates evolve with time as necessary. Diffusion strategies endow networks with such tracking abilities — see, e.g., Sec. VII of \cite{39}, where it is shown how well these strategies track as a function of the level of non-stationarity in the underlying models.

II. DIFFUSION STRATEGY

Consider a collection of \(N\) agents (or nodes) distributed over a geographic region. The set of neighbors (i.e. neighborhood) of node \(k\) is denoted by \(N_k\); the number of nodes in \(N_k\) is denoted by \(n_k\). At every time instant, \(i\), each node \(k\) is able to observe realizations \(\{d_k(i), u_{k,i}\}\) of a scalar random process \(d_k(i)\) and a \(1 \times M\) row random regressor \(u_{k,i}\) with a positive-definite covariance matrix, \(R_{u,k} = \mathbb{E}u^T_{k,i}u_{k,i} > 0\). The regressors \(\{u_{k,i}\}\) are assumed to be temporally white and spatially independent, i.e., \(\mathbb{E}u^T_{k,i}u_{l,j} = R_{u,k}\delta_{kl}\delta_{ij}\) in terms of the Kronecker delta function. Note that we are denoting random quantities by boldface letters and their realizations or deterministic quantities by normal letters. The data \(\{d_k(i), u_{k,i}\}\) collected at node \(k\) are assumed to originate from one of two unknown column vectors \(\{w_0^k, w_1^k\}\) of size \(M\) in the following manner. We denote the generic observed model by \(z_k^o \in \{w_0^k, w_1^k\}\); node \(k\) does not know beforehand the observed model. The data at node \(k\) are related to its observed model \(z_k^o\) via a linear regression model of the form:

\[
d_k(i) = u_{k,i}z_k^o + v_k(i)
\]

where \(v_k(i)\) is measurement noise with variance \(\sigma^2_{v,k}\) and assumed to be temporally white and spatially independent. The noise \(v_k(i)\) is assumed to be independent of \(u_{l,j}\) for all \(\{k, l, i, j\}\). All random processes are zero mean.
The objective of the network is to have all agents converge to an estimate for one of the models. For example, if the models happen to represent the location of food sources \([13], [40]\), then this agreement will make the agents move towards one particular food source in lieu of the other source. More specifically, let \(w_{k,i}\) denote the estimator for \(z_k^q\) at node \(k\) at time \(i\). The network would like to reach an agreement on a common \(q\), such that

\[
w_{k,i} \to w_k^q \quad \text{for } q = 0 \text{ or } q = 1 \quad \text{and for all } k \text{ as } i \to \infty
\]  

(2)

where convergence is in some desirable sense (such as the mean-square-error sense).

Several adaptive diffusion strategies for distributed estimation under a common model scenario were proposed and studied in \([9]–[13]\), following the developments in \([41]–[45]\) — overviews of these results appear in \([13], [14]\). One such scheme is the adaptive-then-combine (ATC) diffusion strategy \([11], [45]\). It operates as follows. We select an \(N \times N\) matrix \(A\) with nonnegative entries \(\{a_{l,k}\}\) satisfying:

\[
1_N^T A = 1_N^T \quad \text{and} \quad a_{l,k} = 0 \text{ if } l \notin N_k
\]  

(3)

where \(1_N\) is the vector of size \(N\) with all entries equal to one. The entry \(a_{l,k}\) denotes the weight that node \(k\) assigns to data arriving from node \(l\) (see Fig. 1). The ATC diffusion strategy updates \(w_{k,i-1}\) to \(w_{k,i}\) as follows:

\[
\psi_{k,i} = w_{k,i-1} + \mu_k \cdot u_{k,i}^T [d_k(i) - u_{k,i} w_{k,i-1}]
\]  

(4)

\[
w_{k,i} = \sum_{l \in N_k} a_{l,k} \psi_{l,i}
\]  

(5)

where \(\mu_k\) is the constant positive step-size used by node \(k\). The first step (4) involves local adaptation, where node \(k\) uses its own data \(\{d_k(i), u_{k,i}\}\) to update the weight estimate at node \(k\) from \(w_{k,i-1}\) to an intermediate value \(\psi_{k,i}\). The second step (5) is a combination step where the intermediate estimates \(\{\psi_{l,i}\}\) from the neighborhood of node \(k\) are combined through the weights \(\{a_{l,k}\}\) to obtain the updated weight estimate \(w_{k,i}\). Such diffusion strategies have found applications in several domains including distributed optimization, adaptation, learning, and the modeling of biological networks — see, e.g., \([13], [14], [40]\) and the references therein. Diffusion strategies were also used in some recent works \([46]–[49]\) albeit with diminishing step-sizes (\(\mu_k(i) \to 0\)) to enforce consensus among nodes. However, decaying step-sizes disable adaptation once they approach zero. Constant step-sizes are used in (4)-(5) to enable continuous adaptation and learning, which is critical for the application under study in this work.

When the data arriving at the nodes could have risen from one model or another, the distributed strategy (4)-(5) will not be able to achieve agreement as in (2) and the resulting weight estimates will
tend towards a biased value. We first explain how this degradation arises and subsequently explain how it can be remedied.

**Assumption 1** (Strongly connected network). The network topology is assumed to be strongly connected so that the corresponding combination matrix $A$ is primitive, i.e., there exists an integer power $j > 0$ such that $[A^j]_{l,k} > 0$ for all $l$ and $k$.

As explained in [13], Assumption 1 amounts to requiring the network to be connected (where a path with nonzero weights exists between any two nodes), and for at least one node to have a non-trivial self-loop (i.e., $a_{k,k} > 0$ for at least one $k$). We conclude from the Perron-Frobenius Theorem [50], [51] that every primitive left-stochastic matrix $A$ has a unique eigenvalue at one while all other eigenvalues are strictly less than one in magnitude. Moreover, if we denote the right-eigenvector that is associated with the eigenvalue at one by $c$ and normalize its entries to add up to one then it holds that:

$$Ac = c, \quad 1^T_N c = 1, \quad 0 < c_k < 1. \quad (6)$$

Let us assume for the time being that the agents in the network have agreed on converging towards one of the models (but they do not know beforehand which model it will be). We denote the desired model generically by $w_q^\circ$. In Section IV, we explain how this agreement process can be attained. Here we explain that even when agreement is present, the diffusion strategy (4)-(5) leads to biased estimates unless it is modified in a proper way. To see this, we introduce the following error vectors for any node $k$:

$$\tilde{w}_{k,i} \triangleq w_q^\circ - w_{k,i} \quad \text{and} \quad \tilde{z}_k^\circ \triangleq w_q^\circ - z_k^\circ. \quad (7)$$
Then, using model (1), we obtain that the update vector in (4) becomes

\[ h_{k,i} \triangleq u_{k,i}^T [d_k(i) - u_{k,i} w_{k,i-1}] \]
\[ = u_{k,i}^T u_{k,i} w_{k,i-1} - u_{k,i} u_{k,i} \tilde{z}^o_k + u_{k,i} v_k(i). \] (8)

We collect all error vectors across the network into block vectors: \( \tilde{w}_i \triangleq \text{col}\{ \tilde{w}_{k,i} \} \) and \( \tilde{z}^o \triangleq \text{col}\{ \tilde{z}^o_k \} \). We also collect the step-sizes into a block diagonal matrix and introduce the extended combination matrix:

\[ M = \text{diag}\{ \mu_k I_M \} \quad \text{and} \quad A \triangleq A \otimes I_M \] (9)

where \( I_M \) denotes the identity matrix of size \( M \). In (9), the notation \( \text{diag}\{ \cdot \} \) constructs a diagonal matrix from its arguments and the symbol \( \otimes \) denotes the Kronecker product of two matrices. Moreover, the notation \( \text{col}\{ \cdot \} \) denotes the vector that is obtained by stacking its arguments on top of each other. Then, starting from (4)-(5) and using relation (8), we can verify that the global error vector \( \tilde{w}_i \) of the network evolves over time according to the recursion:

\[ \tilde{w}_i = B_i \tilde{w}_{i-1} + y_i \] (10)

where the matrix \( B_i \) and the vector \( y_i \) are defined in Table II with \( R_i \triangleq \text{diag}\{ u_{k,i}^T u_{k,i} \} \) and \( s_i \triangleq \text{col}\{ u_{k,i}^T v_k(i) \} \). Note that the matrix \( B_i \) is a random matrix due to the randomness of the regressors \( \{ u_{k,i} \} \). Since the regressors are temporally white and spatially independent, then \( B_i \) is independent of \( \tilde{w}_{i-1} \). In addition, since \( u_{k,i} \) is independent of \( v_k(i) \), the vector \( s_i \) in \( y_i \) has zero mean. Then, from (10), the mean of \( \tilde{w}_i \) evolves over time according to the recursion:

\[ \mathbb{E} \tilde{w}_i = B \cdot \mathbb{E} \tilde{w}_{i-1} + y \] (11)

where \( B \triangleq \mathbb{E} B_i \) and \( y \triangleq \mathbb{E} y_i \) are defined in Table II with \( R \triangleq \mathbb{E} R_i = \text{diag}\{ R_{u,k} \} \). It can be easily verified that a necessary and sufficient condition to ensure the convergence of \( \mathbb{E} \tilde{w}_i \) in (11) to zero is

\[ \rho(B) < 1 \quad \text{and} \quad y = 0 \] (12)

where \( \rho(\cdot) \) denotes the spectral radius of its argument. It was verified in [13], [28] that a sufficient condition to ensure \( \rho(B) < 1 \) is to select the site-sizes \( \{ \mu_k \} \) such that

\[ 0 < \mu_k < \frac{2}{\rho(R_{u,k})} \] (13)

for all \( k \). This conclusion is independent of \( A \). However, for the second condition in (12), we note that in general, the vector \( y = A^T M R \tilde{z}^o \) cannot be zero no matter how the nodes select the combination matrix \( A \). When this happens, the weight estimate will be biased. Let us consider the example with three
The error vector evolves according to the recursion \( \tilde{w}_i = B_i \tilde{w}_{i-1} + y_i \), where the variables \( \{B_i, y_i\} \) and their respective means are listed below for the conventional and modified diffusion strategies.

|                | Diffusion (4)-(5) | Modified diffusion (15)-(16) |
|----------------|-------------------|------------------------------|
| \( B_i \)     | \( A^T(I_{NM} - M_R) \) | \( A^T_1(I_{NM} - M_R) + A^T_2 \) |
| \( B \triangleq \mathbb{E}[B_i] \) | \( A^T(I_{NM} - M_R) \) | \( A^T_1(I_{NM} - M_R) + A^T_2 \) |
| \( y_i \)     | \( A^T M R \tilde{z} - A^T M s_i \) | \( A^T_1 M R \tilde{z} - A^T_2 M s_i \) |
| \( y \triangleq \mathbb{E}[y_i] \) | \( A^T M R \tilde{z} \) | \( A^T_1 M R \tilde{z} \) |

Fig. 2. A three-node network. Node 1 observes data from \( w_0^\circ \) while nodes 2 and 3 observe data from \( w_1^\circ \).

Nodes in Fig. 2 where node 1 observes data from model \( w_0^\circ \), while nodes 2 and 3 observe data from another model \( w_1^\circ \). The matrix \( A \) in this case is shown in Fig. 2 with the parameters \( \{a, b, c, d\} \) lying in the interval \([0, 1]\) and \( b + c \leq 1 \). We assume that the step-sizes and regression covariance matrices are the same, i.e., \( \mu_k = \mu \) and \( R_{u,k} = R_u \) for all \( k \). If the desired model of the network is \( w_q^\circ = w_0^\circ \), then the third block of \( y \) becomes \( \mu R_u(w_0^\circ - w_1^\circ) \), which can never become zero no matter what the parameters \( \{a, b, c, d\} \) are. More generally, using results on the limiting behavior of the estimation errors \( \{\tilde{w}_{k,i}\} \) from [52], we can characterize the limiting point of the diffusion strategy (4)-(5) as follows.

**Lemma 1.** For the diffusion strategy (4)-(5) with \( \mu_k = \mu \) and \( R_{u,k} = R_u \) for all \( k \) and for sufficiently small step-sizes, all weight estimators \( \{w_{k,i}\} \) converge to a limit point \( w^\circ \) in the mean-square sense, i.e., \( \mathbb{E}[\|w^\circ - w_{k,i}\|^2] \) is bounded and of the order of \( \mu \), where \( w^\circ \) is given by

\[
w^\circ = \sum_{k=1}^{N} c_k z_k^\circ
\]

where the vector \( c \) is defined in (6).

**Proof:** The result follows from Eq. (25) in [52] by noting that the variable \( s_k(w^\circ) \) used in [52] is given by \( R_u(z_k^\circ - w^\circ) \). □
Thus, when the agents collect data from different models, the estimates using the diffusion strategy (4)-(5) converge to a convex combination of these models given by (14), which is different from any of the individual models because $c_k > 0$ for all $k$. A similar conclusion holds for the case of non-uniform step-sizes $\{\mu_k\}$ and covariance matrices $\{R_{u,k}\}$.

III. MODIFIED DIFFUSION STRATEGY

To deal with the problem of bias, we now show how to modify the diffusion strategy (4)-(5). We observe from the example in Fig. 2 that the third entry of the vector $\mathbf{y}$ cannot be zero because the neighbor of node 3 observes data arising from a model that is different from the desired model. Note from (8) that the bias term arises from the gradient direction used in computing the intermediate estimates in (4). These observations suggest that to ensure unbiased mean convergence, a node should not combine intermediate estimates from neighbors whose observed model is different from the desired model. For this reason, we shall replace the intermediate estimates from these neighbors by their previous estimates $\{\mathbf{w}_{k,i-1}\}$ in the combination step (5). Specifically, we shall adjust the diffusion strategy (4)-(5) as follows:

$$\psi_{k,i} = \mathbf{w}_{k,i-1} + \mu_k \cdot \mathbf{u}_{k,i}^T [d_k(i) - \mathbf{u}_{k,i} \mathbf{w}_{k,i-1}]$$  \hspace{1cm} (15)

$$\mathbf{w}_{k,i} = \sum_{l \in \mathcal{N}_k} \left( a_{l,k}^{(1)} \psi_{l,i} + a_{l,k}^{(2)} \mathbf{w}_{l,i-1} \right)$$  \hspace{1cm} (16)

where the $\{a_{l,k}^{(1)}\}$ and $\{a_{l,k}^{(2)}\}$ are two sets of nonnegative scalars and their respective combination matrices $A_1$ and $A_2$ satisfy

$$A_1 + A_2 = A$$  \hspace{1cm} (17)

with $A$ being the original left-stochastic matrix in (3). Note that step (15) is the same as step (4). However, in the second step (16), nodes aggregate the $\{\psi_{l,i}, \mathbf{w}_{l,i-1}\}$ from their neighborhood. With such adjustment, we will verify that by properly selecting $\{a_{l,k}^{(1)}, a_{l,k}^{(2)}\}$, unbiased mean convergence can be guaranteed. The choice of which entries of $A$ go into $A_1$ or $A_2$ will depend on which of the neighbors of node $k$ are observing data arising from a model that agrees with the desired model for node $k$.

A. Construction of Matrices $A_1$ and $A_2$

To construct the matrices $\{A_1, A_2\}$ we associate two vectors with the network, $f$ and $g$. Both vectors are of size $N$. The vector $f$ is fixed and its $k$th entry, $f(k)$, is set to $f(k) = 0$ when the observed model for node $k$ is $w_0^k$; otherwise, it is set to $f(k) = 1$. On the other hand, the vector $g_i$ is evolving with time;
its \( k \)th entry is set to \( g_i(k) = 0 \) when the desired model for node \( k \) is \( w^0_\circ \); otherwise, it is set equal to \( g_i(k) = 1 \). Then, we shall set the entries of \( A_1 \) and \( A_2 \) according to the following rules:

\[
\begin{align*}
\alpha_{l,k,i}^{(1)} &= \begin{cases} 
 \alpha_{l,k}, & \text{if } l \in \mathcal{N}_k \text{ and } f(l) = g_i(k) \\
 0, & \text{otherwise}
\end{cases} \\
\alpha_{l,k,i}^{(2)} &= \begin{cases} 
 \alpha_{l,k}, & \text{if } l \in \mathcal{N}_k \text{ and } f(l) \neq g_i(k) \\
 0, & \text{otherwise}
\end{cases}
\end{align*}
\]

That is, nodes that observe data arising from the same model that node \( k \) wishes to converge to will be reinforced and their intermediate estimates \( \{\psi_{l,i}\} \) will be used (their combination weights are collected into matrix \( A_1 \)). On the other hand, nodes that observe data arising from a different model than the objective for node \( k \) will be de-emphasized and their prior estimates \( \{w_{l,i-1}\} \) will be used in the combination step \((16)\) (their combination weights are collected into matrix \( A_2 \)). Note that the scalars \( \{\alpha_{l,k,i}^{(1)}, \alpha_{l,k,i}^{(2)}\} \) in \((18)-(19)\) are now indexed with time due to their dependence on \( g_i(k) \).

**B. Mean-Error Analysis**

It is important to note that to construct the combination weights from \((18)-(19)\), each node \( k \) needs to know what are the observed models influencing its neighbors (i.e., \( f(l) \) for \( l \in \mathcal{N}_k \)); it also needs to know how to update its objective in \( g_i(k) \) so that the \( \{g_i(l)\} \) converge to the same value. In the next two sections, we will describe a distributed decision-making procedure by which the nodes are able to achieve agreement on \( \{g_i(k)\} \). We will also develop a classification scheme to estimate \( \{f(l)\} \) using available data. More importantly, the convergence of the vectors \( \{f, g_i\} \) will occur before the convergence of the adaptation process to estimate the agreed-upon model. Therefore, let us assume for the time being that the nodes know the \( \{f(l)\} \) of their neighbors and have achieved agreement on the desired model, which we are denoting by \( w^0_\circ \), so that (see Eq. \((24)\) in Theorem \(2\))

\[
g_k(1) = g_i(2) = \cdots = g_i(N) = q, \quad \text{for all } i.
\]

Using relation \((8)\) and the modified diffusion strategy \((15)-(16)\), the recursion for the global error vector \( \tilde{w}_i \) is again given by \((10)\) with the matrix \( \mathcal{B}_i \) and the vector \( y_i \) defined in Table \(1\) and the combination matrices \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) defined in a manner similar to \( \mathcal{A} \) in \((9)\). We therefore get the same mean recursion as \((11)\) with the matrix \( \mathcal{B} \) and the vector \( y \) defined in Table \(1\). The following result establishes asymptotic mean convergence for the modified diffusion strategy \((15)-(16)\).
Fig. 3. Decision-making process (a) node $k$ receives the desired models from its neighbors (b) node $k$ updates its desired model using (22)-(23) with 20 neighbors.

**Theorem 1.** Under condition (20), the modified diffusion strategy (15)-(16) converges in the mean if the matrices $A_1$ and $A_2$ are constructed according to (18)-(19) and the step-sizes $\{\mu_k\}$ satisfy condition (13) for those nodes whose observed model is the same as the desired model $w_q^i$ for the network.

**Proof:** See Appendix A.

We conclude from the argument in Appendix A that the net effect of the construction (18)-(19) is the following. Let $w_q^i$ denote the desired model that the network wishes to converge to. We denote by $N_q$ the subset of nodes that receive data arising from the same model. The remaining nodes belong to the set $N_q^c$. Nodes that belong to the set $N_q$ run the traditional diffusion strategy (4)-(5) using the combination matrix $A$ and their step-sizes are required to satisfy (13). The remaining nodes in $N_q^c$ set their step-sizes to zero and run only step (5) of the diffusion strategy. These nodes do not perform the adaptive update (4) and therefore their estimates satisfy $\psi_{k,i} = w_{k,i-1}$ for all $k \in N_q^c$.

**IV. DISTRIBUTED DECISION-MAKING**

The decision-making process is motivated by the process used by animal groups to reach agreement, and which is known as quorum response [4], [6], [7]. The procedure is illustrated in Fig. 3 and described as follows. At time $i$, every node $k$ has its previous desired model $g_{i-1}(k)$, now modeled as a random variable since it will be constructed from data realizations that are subject to randomness. Node $k$ exchanges $g_{i-1}(k)$ with its neighbors and constructs the set

$$N_{k,i-1}^q \equiv \{ l \mid l \in N_k, g_{i-1}(l) = g_{i-1}(k) \}.$$  

That is, the set $N_{k,i-1}^q$ contains the subset of nodes that are in the neighborhood of $k$ and have the same desired model as node $k$ at time $i-1$. This set changes over time. Let $n_q^i(i-1)$ denote the number of nodes in $N_{k,i-1}^q$. Since at least one node (node $k$) belongs to $N_{k,i-1}^q$, we have that $n_q^i(i-1) \geq 1$. Then,
one way for node \( k \) to participate in the quorum response is to update its desired model \( g_i(k) \) according to the rule:

\[
g_i(k) = \begin{cases} 
g_{i-1}(k), & \text{with probability } q_{k,i-1} \\
1 - g_{i-1}(k), & \text{with probability } 1 - q_{k,i-1}
\end{cases}
\]  

(22)

where the probability measure is computed as:

\[
q_{k,i-1} = \frac{n_k^g(i-1)^K}{n_k^g(i-1)^K + [n_k - n_k^g(i-1)]^K}
\]  

(23)

and the exponent \( K \) is a positive constant (e.g., \( K = 4 \)). That is, node \( k \) determines its desired model in a probabilistic manner, and the probability that node \( k \) maintains its desired target is proportional to the \( K \)th power of the number of neighbors having the same desired model (see Fig. 3(b)). Using the above stochastic formulation, we are able to establish agreement on the desired model among the nodes.

**Theorem 2.** For a connected network starting from an arbitrary initial selection for the desired models vector \( g_i \) at time \( i = -1 \), and applying the update rule (21)-(23), then all nodes eventually achieve agreement on some desired model, i.e.,

\[
g_i(1) = g_i(2) = \ldots = g_i(N), \quad \text{as } i \to \infty.
\]

(24)

**Proof:** See Appendix B.

Although rule (21)-(23) ensures agreement on the decision vector, this construction is still not a distributed solution for one subtle reason: nodes need to agree on which index (0 or 1) to use to refer to either model \( \{w_0^\circ, w_1^\circ\} \). This task would in principle require the nodes to share some global information. We circumvent this difficulty and develop a distributed solution as follows. Moving forward, we now associate with each node \( k \) two local vectors \( \{f_k, g_{k,i}\} \); these vectors will play the role of local estimates for the network vectors \( \{f, g_i\} \). Each node will then assign the index value of one to its observed model, i.e., each node \( k \) sets \( f_k(k) = 1 \). Then, for every \( l \in N_k \), the entry \( f_k(l) \) is set to one if it represents the same model as the one observed by node \( k \); otherwise, \( f_k(l) \) is set to zero. The question remains about how node \( k \) knows whether its neighbors have the same observed model as its own (this is discussed in the next section). Here we comment first on how node \( k \) adjusts the entries of its vector \( g_{k,i-1} \). Indeed, node \( k \) knows its desired model value \( g_{k,i-1}(k) \) from time \( i - 1 \). To assign the remaining neighborhood entries in the vector \( g_{k,i-1} \), the nodes in the neighborhood of node \( k \) first exchange their desired model indices with node \( k \), that is, they send the information \( \{g_{l,i-1}(l), l \in N_k\} \) to node \( k \). However, since \( g_{l,i-1}(l) \) from node \( l \) is set relative to its \( f_i(l) \), node \( k \) needs to set \( g_{k,i-1}(l) \) based on the value of \( f_k(l) \).
Specifically, node $k$ will set $g_{k,i-1}(l)$ according to the rule:

$$g_{k,i-1}(l) = \begin{cases} 
g_{l,i-1}(l), & \text{if } f_k(l) = f_k(k) \\
1 - g_{l,i-1}(l), & \text{otherwise} 
\end{cases}$$

That is, if node $l$ has the same observed model as node $k$, then node $k$ simply assigns the value of $g_{l,i-1}(l)$ to $g_{k,i-1}(l)$.

In this way, computations that depend on the network vectors $\{f, g_i\}$ will be replaced by computations using the local vectors $\{f_k, g_{k,i}\}$. That is, the quantities $\{f(l), g_i(l)\}$ in (18)-(19) and (21)-(23) are now replaced by $\{f_k(l), g_{k,i}(l)\}$. We verify in the following that using the network vectors $\{f, g_i\}$ is equivalent to using the local vectors $\{f_k, g_{k,i}\}$.

**Lemma 2.** It holds that

$$f(l) \oplus g_i(k) = f_k(l) \oplus g_{k,i}(k)$$

$$g_i(l) \oplus g_i(k) = g_{k,i}(l) \oplus g_{k,i}(k)$$

where the symbol $\oplus$ denotes the exclusive-OR operation.

**Proof:** Since the values of $\{f_k(l), g_{l,i}(l), g_{k,i}(l)\}$ are set relative to $f_k(k)$, it holds that

$$f(k) \oplus f(l) = f_k(k) \oplus f_k(l)$$

$$f(k) \oplus g_i(k) = f_k(k) \oplus g_{k,i}(k)$$

$$f(k) \oplus g_i(l) = f_k(k) \oplus g_{k,i}(l)$$

Then relations (26) and (27) hold in view of the fact:

$$(a \oplus b) \oplus (a \oplus e) = b \oplus e$$

for any $a$, $b$, and $e \in \{0, 1\}$.

With these replacements, node $k$ still needs to set the entries $\{f_k(l)\}$ that correspond to its neighbors, i.e., it needs to differentiate between their underlying models and whether their data arise from the same model as node $k$ or not. We propose next a procedure to determine $f_k$ at node $k$ using the available estimates $\{w_{l,i-1}, \psi_{l,i}\}$ for $l \in \mathcal{N}_k$.

### V. Model Classification Scheme

To determine the vector $f_k$, we introduce the belief vector $b_{k,i}$, whose $l$th entry, $b_{k,i}(l)$, will be a measure of the belief by node $k$ that node $l$ has the same observed model. The value of $b_{k,i}(l)$ lies in
the range $[0, 1]$. The higher the value of $b_{k,i}(l)$ is, the more confidence node $k$ has that node $l$ is subject to the same model as its own model. In the proposed construction, the vector $b_{k,i}$ will be changing over time according to the estimates $\{w_{l,i-1}, \psi_{l,i}\}$. Node $k$ will be adjusting $b_{k,i}(l)$ according to the rule:

$$b_{k,i}(l) = \begin{cases} \alpha b_{k,i-1}(l) + (1 - \alpha), & \text{to increase belief} \\ \alpha b_{k,i-1}(l), & \text{to decrease belief} \end{cases}$$

(32)

for some positive scalar $\alpha \in (0, 1)$, e.g., $\alpha = 0.95$. That is, node $k$ increases the belief by combining in a convex manner the previous belief with the value one. Node $k$ then estimates $f_k(l)$ according to the rule:

$$\hat{f}_{k,i}(l) = \begin{cases} 1, \quad \text{if } b_{k,i}(l) \geq 0.5 \\ 0, \quad \text{otherwise} \end{cases}$$

(33)

where $\hat{f}_{k,i}(l)$ denotes the estimate for $f_k(l)$ at time $i$ and is now a random variable since it will be computed from data realizations. Note that the value of $\hat{f}_{k,i}(l)$ may change over time due to $b_{k,i}(l)$.

Since all nodes have similar processing abilities, it is reasonable to consider the following scenario.

**Assumption 2 (Homogeneous agents).** All nodes in the network use the same step-size, $\mu_k = \mu$, and they observe data arising from the same covariance distribution so that $R_{u,k} = R_u$ for all $k$.

Agents still need to know whether to increase or decrease the belief in (32). We now suggest a procedure that allows the nodes to estimate the vectors $\{f_k\}$ by focusing on their behavior in the far-field regime when their weight estimates are usually far from their observed models (see (37) for a more specific description). The far-field regime generally occurs during the initial stages of adaptation and, therefore, the vectors $\{f_k\}$ can be determined quickly during these initial iterations.

To begin with, we refer to the update vector from (35), which can be written as follows for node $l$:

$$h_{l,i} = \mu^{-1}(\psi_{l,i} - w_{l,i-1}) = u_{l,i}^T u_{l,i}(z_i^0 - w_{l,i-1}) + u_{l,i}^T v_{l_i}(i).$$

(34)

Taking expectation of both sides conditioned on $w_{l,i-1} = w_{l,i-1}$, we have that

$$\tilde{h}_{l,i} \triangleq \mathbb{E}[h_{l,i} | w_{l,i-1} = w_{l,i-1}] = R_u(z_i^0 - w_{l,i-1}).$$

(35)

That is, the expected update direction given the previous estimate, $w_{l,i-1}$, is a scaled vector pointing from $w_{l,i-1}$ towards $z_i^0$ with scaling matrix $R_u$. Note that since $R_u$ is positive-definite, then the term $\tilde{h}_{l,i}$ lies in the same half plane of the vector $z_i^0 - w_{l,i-1}$, i.e., $\tilde{h}_{l,i}^T (z_i^0 - w_{l,i-1}) > 0$. Therefore, the update
Fig. 4. Illustration of the vectors $\tilde{h}_{k,i}$ and $\tilde{h}_{l,i}$ (a) when both nodes are in far-field and have the same observed model or (b) different observed models.

vector provides useful information about the observed model at node $l$. For example, this term tells us how close the estimate at node $l$ is to its observed model. When the magnitude of $\tilde{h}_{l,i}$ is large, or the estimate at node $l$ is far from its observed model $z_l^0$, then we say that node $l$ is in a far-field regime. On the other hand, when the magnitude of $\tilde{h}_{l,i}$ is small, then the estimate $w_{l,i-1}$ is close to $z_l^0$ and we say that the node is operating in a near-field regime. The vector $\tilde{h}_{l,i}$ can be estimated by the first-order recursion:

$$\hat{h}_{l,i} = (1 - \nu)\hat{h}_{l,i-1} + \nu \mu^{-1}(\psi_{l,i} - w_{l,i-1})$$

(36)

where $\hat{h}_{l,i}$ denotes the estimate for $\tilde{h}_{l,i}$ and $\nu$ is a positive step-size. Note that since the value of $\tilde{h}_{l,i}$ varies with $w_{l,i-1}$, which is updated using the step-size $\mu$, then the value of $\nu$ should be set large enough compared to $\mu$ (e.g., $\mu = 0.005$ and $\nu = 0.05$ are used in our simulations) so that recursion (36) can track variations in $\tilde{h}_{l,i}$ over time. Moreover, since node $k$ has access to the $\{w_{l,i-1}, \psi_{l,i}\}$ if node $l$ is in its neighborhood, node $k$ can compute $\hat{h}_{l,i}$ on its own using (36). In the following, we describe how node $k$ updates the belief $b_{k,i}(l)$ using $\{\hat{h}_{k,i}, \hat{h}_{l,i}\}$.

During the initial stage of adaptation, nodes $k$ and $l$ are generally away from their respective observed models and both nodes are therefore in the far-field. This state is characterized by the conditions

$$\|\hat{h}_{k,i}\| > \eta \quad \text{and} \quad \|\hat{h}_{l,i}\| > \eta$$

(37)

for some threshold $\eta$. If both nodes have the same observed model, then the estimates $\hat{h}_{k,i}$ and $\hat{h}_{l,i}$ are expected to have similar direction towards the observed model (see Fig. 4(a)). Node $k$ will increase the belief $b_{k,i}(l)$ using (32) if

$$\hat{h}_{k,i}^T \hat{h}_{l,i} > 0.$$  

(38)

Otherwise, node $k$ will decrease the belief $b_{k,i}(l)$. That is, when both nodes are in the far-field, then node $k$ increases its belief that node $l$ shares the same observed model when the vectors $\hat{h}_{k,i}$ and $\hat{h}_{l,i}$
lie in the same quadrant. Note that it is possible for node \( k \) to increase \( b_{k,i}(l) \) even when nodes \( k \) and \( l \) have distinct models. This is because it is difficult to differentiate between the models during the initial stages of adaptation. This situation is handled by the evolving network dynamics as follows. If node \( k \) considers that the data from node \( l \) originate from the same model, then node \( k \) will use the intermediate estimate \( \psi_{l,i} \) from node \( l \) in (16). Eventually, from Lemma 1, the estimates at these nodes get close to a convex combination of the underlying models, which would then enable node \( k \) to distinguish between the two models and to decrease the value of \( b_{k,i}(l) \). Clearly, for proper resolution, the distance between the models needs to be large enough so that the agents can resolve them. When the models are very close to each other so that resolution is difficult, the estimates at the agents will converge towards a convex combination of the models (which will be also close to the models). Therefore, the belief \( b_{k,i}(l) \) is updated according to the following rule:

\[
b_{k,i}(l) = \begin{cases} 
\alpha b_{k,i-1}(l) + (1 - \alpha), & \text{if } E_1 \\
\alpha b_{k,i-1}(l), & \text{if } E_1^c
\end{cases}
\]  

(39)

where \( E_1 \) and \( E_1^c \) are the two events described by:

\[
E_1 : \|\hat{h}_{k,i}\| > \eta, \|\hat{h}_{l,i}\| > \eta, \text{ and } \hat{h}_{k,i}^T \hat{h}_{l,i} > 0
\]  

(40)

\[
E_1^c : \|\hat{h}_{k,i}\| > \eta, \|\hat{h}_{l,i}\| > \eta, \text{ and } \hat{h}_{k,i}^T \hat{h}_{l,i} \leq 0.
\]  

(41)

Note that node \( k \) updates the belief \( b_{k,i}(l) \) only when both nodes \( k \) and \( l \) are in the far-field.

VI. DIFFUSION STRATEGY WITH DECISION-MAKING

Combining the modified diffusion strategy (15)-(16), the combination weights (18)-(19), the decision-making process (21)-(23), and the classification scheme (33) and (39) with \( \{f(l), g_i(l)\} \) replaced by \( \{\hat{f}_{k,i}(l), \hat{g}_{k,i}(l)\} \), we arrive at the listing shown in the table. It is seen from the algorithm that the adaptation and combination steps of diffusion, which correspond to steps 1) and 8), are now separated by several steps. The purpose of these intermediate steps is to select the combination weights properly to carry out the aggregation required by step 8). Note that to implement the algorithm, nodes need to exchange the quantities \( \{w_{k,i-1}, \psi_{k,i}, g_{k,i-1}(k)\} \) with their neighbors. We summarize the computational complexity and the amount of scalar exchanges of the conventional and modified diffusion strategies in Table II. Note that the modified strategy still requires in the order of \( O(M) \) computations per iteration. Nevertheless, the modified diffusion strategy requires about \( 2n_k M \) more additions and multiplications than conventional diffusion. This is because of the need to compute the terms \( \{\hat{h}_{l,i}\} \) in step 2). If the...
TABLE II
Comparison of the number of multiplications and additions per iteration, as well as the number of scalars that are exchanged for each iteration of the algorithms at every node \( k \). In the table, the symbol \( n_k \) denotes the degree of node \( k \), i.e., the size of its neighborhood \( N_k \).

| Multiplications | Diffusion (4)-(5) | Modified diffusion |
|-----------------|------------------|--------------------|
| \( (n_k + 2)M \) | \( (3n_k + 2)M + n_k - 1 \) |
| \( (n_k + 1)M \) | \( (3n_k + 1)M + n_k - 1 \) |
| Scalar exchanges | \( n_k M \) | \( n_k (2M + 1) \) |

nodes can afford to exchange extra information, then instead of every node connected to node \( l \) computing the term \( \hat{h}_{l,i} \) in step 2), this term can be computed locally by node \( l \) and shared with its neighbors. This reveals a useful trade-off between complexity and information exchange.

Due to the dependency among the steps of the algorithm, the analysis of its behavior becomes challenging. However, by examining the various steps, some useful observations stand out. Specifically, it is observed that the convergence of the algorithm occurs in three phases as follows (see also Sec. VIII):

1) Convergence of the classification scheme: The first phase of convergence happens during the initial stages of adaptation. It is natural to expect that during this stage, all weight estimates are generally away from their respective models and the nodes operate in the far-field regime. Then, the nodes use steps 2)-5) to determine the observed models \( \{\hat{f}_{k,i}(l)\} \) of their neighbors. We explain later in Eq. (77) in Theorem 3 that this construction is able to identify the observed models with high probability. In other words, the classification scheme is able to converge reasonably well and fast during the initial stages of adaptation.

2) Convergence of the decision-making process: The second phase of convergence happens right after the convergence of the classification scheme, once the \( \{\hat{f}_{k,i}(l)\} \) have converged. Because the nodes now have correct information about their neighbor’s observed models, they use steps 5)-6) to determine their own desired models \( \{g_{k,i}(l)\} \). The convergence of this step is ensured by Eq. (24) in Theorem 2.

3) Convergence of the diffusion strategy: After the classification and decision-making processes converge, the estimates \( \{\hat{f}_{k,i}(l), g_{k,i}(l)\} \) remain largely invariant and the combination weights in step 7) therefore remain fixed for all practical purposes. Then, the diffusion strategy becomes unbiased and converges in the mean according to Theorem 1. Moreover, when the estimates are close to steady-state, those nodes whose observed models are the same as the desired model enter the near-
field regime and they stop updating their belief vectors (this will be justified by the future result (75)).

VII. PERFORMANCE OF CLASSIFICATION PROCEDURE

It is clear that the success of the diffusion strategy and decision-making process depends on the reliability of the classification scheme in (33) and (39). In this section, we examine the probability of error for the classification scheme under some simplifying conditions to facilitate the analysis. This is a challenging task to pursue due to the stochastic nature of the classification and decision-making process, and due to the coupling among the agents. Our purpose in this section is to gain some insights into this process through a first-order approximate analysis.

Now, there are two types of error. When nodes $k$ and $l$ are subject to the same observed model (i.e., $z_k = z_l^o$ and $f_k(l) = 1$), then one probability of error is defined as:

$$P_{e,1} = \Pr (\hat{f}_{k,i}(l) = 0 \mid f_k(l) = 1)$$

$$= \Pr (b_{k,i}(l) < 0.5 \mid z_k = z_l^o)$$

(42)

where we used rule (33). The second type of probability of error occurs when both nodes have different observed models (i.e., when $z_k^o \neq z_l^o$ and $f_k(l) = 0$) and refers to the case:

$$P_{e,0} = \Pr (\hat{f}_{k,i}(l) = 1 \mid f_k(l) = 0)$$

$$= \Pr (b_{k,i}(l) > 0.5 \mid z_k^o \neq z_l^o).$$

(43)

To evaluate the error probabilities in (42)-(43), we examine the probability distribution of the belief variable $b_{k,i}$. Note from (39) that the belief variable can be expressed as:

$$b_{k,i}(l) = \alpha b_{k,i-1}(l) + (1 - \alpha) \xi_{k,i}(l)$$

(44)

where $\xi_{k,i}(l)$ is a Bernoulli random variable with

$$\xi_{k,i}(l) = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1 - p \end{cases}.$$

(45)

The value of $p$ depends on whether the nodes have the same observed models or not. When $z_k^o = z_l^o$, the belief $b_{k,i}(l)$ is supposed to be increased and the probability of detection, $P_d$, characterizes the probability that $b_{k,i}(l)$ is increased, i.e.,

$$P_d = \Pr (\xi_{k,i}(l) = 1 \mid z_k^o = z_l^o).$$

(46)
In this case, the probability $p$ in (45) will be replaced by $P_d$. On the other hand, when $z_k^0 \neq z_l^0$, the probability of false alarm, $P_f$, characterizes the probability that the belief $b_{k,i}(l)$ is increased when it is supposed to be decreased, i.e.,

$$P_f = \Pr(\xi_{k,i}(l) = 1 \ | \ z_k^0 \neq z_l^0)$$

(47)

and we replace $p$ in (45) by $P_f$. We will show later (see Lemma 4) how to evaluate the two probabilities $P_d$ and $P_f$. In the sequel we denote them generically by $p$.

Expanding (44), we obtain

$$b_{k,i}(l) = \alpha^{i+1} b_{k,-1}(l) + (1 - \alpha) \sum_{j=0}^{i} \alpha^j \xi_{k,i-j}(l).$$

(48)

Although it is generally not true, we simplify the analysis by assuming that the $\{\xi_{k,i}(l)\}$ in (45) are independent and identically distributed (i.i.d.) random variables. This assumption is motivated by conditions (40)-(41) and by the fact that the type of model that is observed by node $k$ is assumed to be independent from the type of model that is observed by node $l$. The assumption is also motivated by the fact that the regression data and noise across all nodes are assumed to be temporally white and independent over space. Now, since $\alpha$ is a design parameter that is smaller than one, after a few iterations, say, $C$ iterations, the influence of the initial condition in (48) becomes small and can be ignored. In addition, the distribution of $b_{k,i}(l)$ can be approximated by the distribution of the following random variable, which takes the form of a random geometric series:

$$\zeta_k(l) \triangleq (1 - \alpha) \sum_{j=0}^{C} \alpha^j \xi_{k,j}(l)$$

(49)

where we replaced the index $i - j$ in (48) by $j$ because the $\{\xi_{k,i}(l)\}$ are assumed to be i.i.d. There have been several useful works on the distribution function of random geometric sequences and series [53]–[55]. However, it is generally untractable to express the distribution function in closed form. We instead resort to the following two inequalities to establish bounds for the error probabilities (42)-(43).

First, for any two generic events $E_1$ and $E_2$, if $E_1$ implies $E_2$, then the probability of event $E_1$ is less than the probability of event $E_2$ [56], i.e.,

$$\Pr(E_1) \leq \Pr(E_2) \quad \text{if} \quad E_1 \subseteq E_2.$$  

(50)

The second inequality is the Markov inequality [56], i.e., for any nonnegative random variable $x$ and positive scalar $\delta$, it holds that

$$\Pr(x \geq \delta) = \Pr(x^2 \geq \delta^2) \leq \frac{\mathbb{E}x^2}{\delta^2}.$$  

(51)
To apply the Markov inequality (51), we need the second-order moment of $\zeta_k(l)$ in (49), which is difficult to evaluate because the $\{\xi_{k,j}(l)\}$ are not zero mean. To circumvent this difficulty, we introduce the change of variable:

$$\xi_{k,j}^o(l) \triangleq \frac{\xi_{k,j}(l) - p}{\sqrt{p(1-p)}}. \quad (52)$$

It can be verified that the $\{\xi_{k,j}^o(l)\}$ are i.i.d. with zero mean and unit variance. Then, we can write (49) as

$$\zeta_k(l) = p(1 - \alpha^{C+1}) + \sqrt{p(1-p)}\zeta_k^o(l) \quad (53)$$

where the variable $\zeta_k^o(l)$ is defined by

$$\zeta_k^o(l) \triangleq (1 - \alpha^2) \sum_{j=0}^{C} \alpha^j \xi_{k,j}^o(l) \quad (54)$$

and its mean is zero and its variance is given by

$$\mathbb{E}(\zeta_k^o(l))^2 = \frac{1 - \alpha}{1 + \alpha} \left( 1 - \alpha^{2(C+1)} \right) \approx \frac{1 - \alpha}{1 + \alpha}. \quad (55)$$

Then, from (42) and (53) and replacing the probability $p$ by $P_d$ and for $C$ large enough so that $1 - \alpha^{C+1} \approx 1$, we obtain that

$$P_{e,1} \approx \Pr(\zeta_k(l) < 0.5 \mid z_k^o = z_k^o)$$

$$= \Pr \left( \zeta_k^o(l) < \frac{- \left( P_d - 0.5 \right)}{\sqrt{P_d(1-P_d)}} \mid z_k^o = z_k^o \right)$$

$$\leq \Pr \left( |\zeta_k^o(l)| > \frac{P_d - 0.5}{\sqrt{P_d(1-P_d)}} \mid z_k^o = z_k^o \right)$$

$$\leq \frac{1 - \alpha}{1 + \alpha} \cdot \frac{P_d(1-P_d)}{(P_d - 0.5)^2} \quad (56)$$

where we used (50) and the Markov inequality (51) in the last two inequalities. Note that in (56), we assume the value of $P_d$ to be greater than 0.5. Indeed, as we will argue in Lemma 4, the value of $P_d$ is close to one. Similarly, replacing the probability $p$ by $P_f$ and assuming that $P_f < 0.5$, we obtain from (43) and (53) that

$$P_{e,0} \leq \frac{1 - \alpha}{1 + \alpha} \cdot \frac{P_f(1-P_f)}{(0.5 - P_f)^2} \quad (57)$$

To evaluate the upper bounds in (56)-(57), we need the probabilities of detection and false alarm in (46)-(47). Since the update of $b_{k,i}(l)$ in (39) depends on $\{\hat{h}_{k,i}, \hat{h}_{l,i}\}$, we need to rely on the statistical properties of these latter quantities. In the following, we first examine the statistics of $\hat{h}_{k,i}$ constructed via (56) and then evaluate $P_d$ and $P_f$ defined by (46)-(47).
A. Statistics of $h_{k,i}$

We first discuss some assumptions that lead to an approximate model for the evolution of $h_{k,i}$ in (72) further ahead. As we mentioned following (36), since the step-sizes $\{\mu, \nu\}$ satisfy $\mu \ll \nu$, the variation of $w_{k,i-1}$ can be assumed to be much slower than the variation of $h_{k,i}$. For this reason, the analysis in this section will be conditioned on $w_{k,i-1} = w_{k,i-1}$, as we did in (35), and we introduce the following assumption.

**Assumption 3** (Small step-size). The step-sizes $\{\mu, \nu\}$ are sufficiently small, i.e.,

$$0 < \mu \ll \nu \ll 1$$

so that $w_{k,i} \approx w_{k,i-1}$ for all $k$.

In addition, since the update vector from (35) depends on the covariance matrix $R_u$, we assume $R_u$ is well-conditioned so that the following is justified.

**Assumption 4** (Regression model). The regression covariance matrix $R_u$ is well-conditioned such that it holds that

$$\begin{align*}
&\text{if } \|z^0_k - w_{k,i-1}\| \gg 1, \text{ then } \|h_{k,i}\| \gg \eta \\
&\text{if } \|z^0_k - w_{k,i-1}\| \ll 1, \text{ then } \|h_{k,i}\| \ll \eta.
\end{align*}$$

Moreover, the fourth-order moment of the regression data $\{u_{k,i}\}$ is assumed to be bounded such that

$$\nu \tau \ll 1$$

where the scalar $\tau$ is a bound for

$$\frac{\mathbb{E}\|u^T_{k,i} u_{k,i}(z^0_k - w_{k,i-1}) - h_{k,i}\|^2}{\|h_{k,i}\|^2} \leq \tau$$

and its value measures the randomness in variables involving fourth-order products of entries of $u_{k,i}$.

Note that condition (62) can be rewritten as

$$\left(\frac{z^0_k - w_{k,i-1}}{z^0_k - w_{k,i-1}}\right)^T \mathbb{E}(u^T_{k,i} u_{k,i} u^T_{k,i} u_{k,i} - R_u^2)(z^0_k - w_{k,i-1}) \leq \tau$$

which shows that (62) corresponds to a condition on the fourth-order moment of the regression data. Combining conditions (58) and (61), we obtain the following constraint on the step-sizes $\{\mu, \nu\}$:

$$0 \ll \mu \ll \nu \ll \min\{1, 1/\tau\}.$$
To explain more clearly what conditions (59)-(60) entail, we obtain from (35) that \( \| \hat{h}_{k,i} \| ^2 \) can be written as the weighted square Euclidean norm:

\[
\| \hat{h}_{k,i} \| ^2 = \| z^0_k - w_{k,i-1} \| _{R_u}^2. \tag{65}
\]

We apply the Rayleigh-Ritz characterization of eigenvalues \[50\] to conclude that

\[
\lambda_{\text{min}}(R_u) \cdot \| z^0_k - w_{k,i-1} \| \leq \| \hat{h}_{k,i} \| \leq \lambda_{\text{max}}(R_u) \cdot \| z^0_k - w_{k,i-1} \|
\]

where \( \lambda_{\text{min}}(R_u) \) and \( \lambda_{\text{max}}(R_u) \) denote the minimum and maximum eigenvalues of \( R_u \). Then, condition (59) indicates that whenever node \( k \) is operating in the far-field regime, i.e., whenever \( \| z^0_k - w_{k,i-1} \| \gg 1 \), then we would like

\[
\lambda_{\text{min}}(R_u) \cdot \| z^0_k - w_{k,i-1} \| \gg \eta. \tag{66}
\]

Likewise, whenever \( \| z^0_k - w_{k,i-1} \| \ll 1 \), then

\[
\lambda_{\text{max}}(R_u) \cdot \| z^0_k - w_{k,i-1} \| \ll \eta. \tag{67}
\]

Therefore, the scalars \( \lambda_{\text{min}}(R_u)/\eta \) and \( \lambda_{\text{max}}(R_u)/\eta \) cannot be too small or too large, i.e., the matrix \( R_u \) should be well-conditioned.

We are now ready to model the average update vector \( \hat{h}_{k,i} \). From Assumption [3] since the estimate \( w_{k,i-1} \) remains approximately constant during repeated updates of \( \hat{h}_{k,i} \), we first remove the time index in \( w_{k,i-1} \) and examine the statistics of \( \hat{h}_{k,i} \) under the condition \( w_{k,i-1} = w_k \). From (34) and (36), the expected value of \( \hat{h}_{k,i} \) given \( w_{k,i-1} = w_k \) converges to

\[
\lim_ {i \to \infty} \mathbb{E} \hat{h}_{k,i} = R_u (z^0_k - w_k) \triangleq \bar{h}_k. \tag{68}
\]

We can also obtain from (34) and (36) that the limiting second-order moment of \( \hat{h}_{k,i} \), which is denoted by \( \sigma^2_{\hat{h},k} \), satisfies:

\[
\sigma^2_{\hat{h},k} \triangleq \lim_{i \to \infty} \mathbb{E} \| \hat{h}_{k,i} - \bar{h}_k \|^2 = (1 - \nu)^2 \sigma^2_{h,k} + \nu^2 \sigma^2_{\hat{h},k} \tag{69}
\]

where \( \sigma^2_{h,k} \triangleq \mathbb{E} \| h_{k,i} - \bar{h}_k \|^2 \) is given by

\[
\sigma^2_{h,k} = \mathbb{E} \| u^T_{k,i} u_{k,i} (z^0_k - w_k) - \bar{h}_k \|^2 + \sigma^2_{v,k} \text{Tr}(R_u). \tag{70}
\]

Note that the cross term on the right-hand side of (69) is zero because the terms \( \hat{h}_{k,i-1} - \bar{h}_k \) and \( h_{k,i-1} - \bar{h}_k \) are independent under the constant \( w_k \) condition. Note also that \( h_{k,i} - \bar{h}_k \) has zero mean. Then, from (69) and Assumption [3] the variance \( \sigma^2_{\hat{h},k} \) is given by

\[
\sigma^2_{\hat{h},k} = \frac{\nu}{2 - \nu} \sigma^2_{h,k} \approx \frac{\nu}{2} \sigma^2_{\hat{h},k}. \tag{71}
\]
Since $w_{k,i-1}$ remains approximately constant, the average update vector $\hat{h}_{k,i}$ has mean and second-order moment close to expressions (68) and (71). We then arrive at the following approximate model for $\hat{h}_{k,i}$.

**Assumption 5** (Model for $\hat{h}_{k,i}$). The estimate $\hat{h}_{k,i}$ is modeled as:

$$\hat{h}_{k,i} = \bar{h}_{k,i} + n_{k,i}$$  \tag{72}

where $n_{k,i}$ is a random perturbation process with zero mean and

$$\mathbb{E}[\|n_{k,i}\|^2] \leq \frac{\nu[\tau\|\bar{h}_{k,i}\|^2 + \sigma^2_{v,k}\text{Tr}(R_u)]}{2}$$  \tag{73}

with the scalar $\tau$ defined by (61).

Note that since the perturbation $n_{k,i}$ is from the randomness of the regressor and noise processes $\{u_{k,i}, v_k(i)\}$, then it is reasonable to assume that the $\{n_{k,i}\}$ are independent of each other.

Before we proceed to the probability of detection (46) and the probability of false alarm (47), we note that the update of the belief $b_{k,i}(l)$ happens only when both nodes $k$ and $l$ are in the far-field regime, which is determined by the magnitudes of $\hat{h}_{k,i}$ and $\hat{h}_{l,i}$ being greater than the threshold $\eta$. The following result approximates the probability that a node is classified to be in the far-field.

**Lemma 3.** Under Assumptions 3-5, it holds that

$$\Pr(\|\hat{h}_{k,i}\| > \eta \mid \|z^0_k - w_{k,i-1}\| \gg 1) \geq 1 - \frac{\nu\tau}{2}$$  \tag{74}

$$\Pr(\|\hat{h}_{k,i}\| > \eta \mid \|z^0_k - w_{k,i-1}\| \ll 1) \leq \frac{\nu\sigma^2_{v,k}\text{Tr}(R_u)}{2\eta^2}.$$  \tag{75}

**Proof:** See Appendix C.

From Assumptions 3-4, the probability in (74) is close to one and the probability in (75) is close to zero. Therefore, this approximate analysis suggests that during the initial stages of adaptation, the magnitude of $\{\|\hat{h}_{k,i}\|\}$ successfully determines that the nodes are in the far-field state and they update the belief using rule (39). When the estimates approach steady-state, the nodes whose observed models are the same as the desired model satisfy the condition $\|z^0_k - w_{k,i-1}\| \ll 1$ and, therefore, they stop updating their belief vectors in view of (75). On the other hand, when both nodes $k$ and $l$ have observed models that are different from the desired model (and, therefore, their estimates are away from their observed models), they will continue to update their beliefs. The proof in Appendix D then establishes the following bounds on $P_d$ and $P_f$. 
Lemma 4. Under Assumptions \[3\] and during the far-field regime \[59\], the probabilities of detection and false alarm defined by \((46)-(47)\) are approximately bounded by

\[
P_d \geq 1 - \frac{16\nu \tau}{\pi^2} \quad \text{and} \quad P_f \leq \frac{16\nu \tau}{\pi^2}.
\]

(76)

The above result establishes that the probability of detection is close to one and the probability of false alarm is close to zero in view of \(\nu \tau \ll 1\). That is, with high probability, node \(k\) will correctly adjust the value of \(b_{k,j}(l)\). We then arrive at the following bound for error probabilities in \((42)-(43)\).

Theorem 3. Under Assumptions \[3\] and in the far-field regime \[59\], the error probabilities \(\{P_{e,1}, P_{e,0}\}\) are approximately upper bounded by

\[
P_u = 1 - \frac{\alpha}{1 + \alpha} \cdot \frac{16\nu \tau}{\pi^2} \cdot \frac{1 - 16\nu \tau/\pi^2}{(1/2 - 16\nu \tau/\pi^2)^2} = O(\nu).
\]

(77)

Proof: Let the function \(f(p)\) be defined as \(p(1-p)/(p-0.5)^2\). It can be verified that the function \(f(p)\) is strictly increasing when \(p \in [0, 0.5]\) and strictly decreasing when \(p \in (0.5, 1]\). From Lemma \[4\] we conclude that \(P_d > 0.5\) and \(P_f < 0.5\). Therefore, an upper bound for \(P_{e,1}\) can be obtained by replacing \(P_d\) in \((56)\) by the lower bound in \((76)\). Similar arguments apply to the upper bound for \(P_{e,0}\).

This result reveals that the \(\{P_{e,1}, P_{e,0}\}\) are upper bounded by the order of \(\nu\). In addition, the upper bound \(P_u\) also depends on the value of \(\alpha\) used to update the belief in \((39)\). We observe that the larger the value of \(\alpha\), the smaller the values of the error probabilities. In simulations, we choose \(\nu = 0.05\) and \(\alpha = 0.95\), which will give the upper bound in \((77)\) the value \(P_u \approx 0.008\tau < \nu \tau\). This implies that the classification scheme \[33\] identifies the observed models with high probability.

VIII. RATES OF CONVERGENCE

There are two rates of convergence to consider for adaptive networks running a decision-making process of the form described in the earlier sections. First, we need to analyze the rate at which the nodes reach an agreement on a desired model (which corresponds to the speed of the decision-making process). Second, we analyze the rate at which the estimates by the nodes converge to the desired model (which corresponds to the speed of the diffusion adaptation).

A. Convergence Rate of Decision-Making Process

From the proof of Theorem \[2\] (see Appendix \[B\]), the decision-making process can be modeled as a Markov chain with \(N + 1\) states \(\{\chi_i\}\) corresponding to the number of nodes whose desired vectors are...
$w_i^n$. The Markov chain has two absorbing states \( \{0, N\} \) and its transition probability matrix \( P \) can be written as:

\[
P = \begin{pmatrix}
1 & 0 & 0 \\
b & Q & c \\
0 & 0 & 1
\end{pmatrix}
\] (78)

where the matrix \( Q \) of size \((N - 1) \times (N - 1)\) is the transition matrix among the transient states \(\{1, 2, \cdots, N - 1\}\), and the vectors \(\{b, c\}\) of size \(N - 1\) are the transition probabilities from the transient states to the absorbing states. The convergence rate of the decision-making process is then determined by the rate at which, starting at any arbitrary transient state, the Markov chain converges to one of the absorbing states. The argument that follows is meant to show that the rate of convergence of the decision making process improves with the parameter \(K\) used in (23); the larger the value of \(K\) the faster is the convergence.

To arrive at this conclusion, we first remark that to assess the rate of convergence, we need to compute the \(j\)th power of \(P\) from (78) to find that

\[
P^j = \begin{pmatrix}
1 & 0 & 0 \\
b & Q^j & c \\
0 & 0 & 1
\end{pmatrix}
\] (79)

where \(\{b, c\}\) are two \(N \times 1\) vectors. Let the Markov chain start from any arbitrary initial state distribution, \(y\), of the form

\[
y^T = \begin{bmatrix} 0 \ y_Q^T \ 0 \end{bmatrix}
\] (80)

where \(y_Q\) is a vector of size \(N - 1\) and its entries add up to one, i.e., \(y_Q^T \mathbb{1}_{N-1} = 1\). We shall select \(y_Q\) in a manner that enables us to determine how the convergence rate depends on \(K\). Thus, note that the state distribution after \(j\) transitions becomes

\[
y^T P^j = \begin{bmatrix} y_Q^T b \ y_Q^T Q^j \ y_Q^T c \end{bmatrix}
\] (81)

Therefore, the convergence rate is measured by the rate at which the matrix \(Q^j\) converges to zero, which is determined by the spectral radius of \(Q\). Since \(Q\) is the sub-matrix of the transition probability matrix, all entries of \(Q\) are nonnegative, then by the Perron-Frobenius Theorem [50], the vector \(y_Q\) can be selected to be the left eigenvector of \(Q\) corresponding to the eigenvalue \(\rho(Q)\), i.e., \(y_Q^T Q = \rho(Q) y_Q^T\). Moreover, from (111), the matrix \(Q\) is primitive and, therefore, all entries of \(y_Q\) are positive. Furthermore, since the transition probability matrix \(P\) is right-stochastic (i.e., \(P \mathbb{1}_{N+1} = \mathbb{1}_{N+1}\)), from (78) it holds that

\[
b + c + Q \mathbb{1}_{N-1} = \mathbb{1}_{N-1}.
\] (82)
Pre-multiplying the vector $y_Q$ on both sides of (82), we obtain that the convergence rate of the decision-making process can be determined by

$$\rho(Q) = y_Q^T Q \mathbb{1}_{N-1} = 1 - y_Q^T (b + c). \quad (83)$$

We now determine the value of the vector sum $b + c$. We note from (110) that the transition probabilities $\{p_{n,m}\}$ in $Q$ are determined by the probability $q_{k,i-1}$ from (23), so is the spectral radius of $Q$. We further note from (23) that there is a single parameter $K$ dictating the value of $q_{k,i-1}$. In the following, we examine the dependence of the convergence rate $\rho(Q)$ on the parameter $K$. It is generally challenging to develop the relation because the transition probability $p_{n,m}$ needs to be computed in a compounded way where we need to evaluate the summation of the products of $\{q_{k,i-1}\}$. Nevertheless, some useful insights can be obtained by means of the following approximate argument. Suppose the network size is sufficiently large and that the nodes are uniformly distributed in the spatial domain so that each of the nodes in the network has approximately the same number of neighbors collecting data from model $w_0^1$; likewise, each of the nodes in the network has approximately the same number of neighbors collecting data from the other model. Suppose that there are $\chi_{i-1} = n$ out of $N$ nodes with desired model $w_0^1$, then, on average, node $k$ with $n_k$ neighbors will have $n_k n/N$ neighbors whose desired model is $w_0^1$ and have $n_k (1 - n/N)$ neighbors whose desired model is $w_0^2$. Then, from rule (22)-(23), node $k$ chooses $w_0^1$ as its desired model with probability

$$q_n \triangleq \frac{(n_k n/N)^K}{(n_k n/N)^K + (n_k (N - n)/N)^K} = \frac{n^K}{n^K + (N - n)^K} \quad (84)$$

which is independent of the node index $k$ and is denoted by $q_n$. Then, the second summation in (110) can be evaluated in a way that there are $m$ out of $N$ nodes choosing $w_0^1$ as their desired model and the remaining $N - m$ nodes choosing $w_0^2$, which is equal to

$$\binom{N}{m} q_n^m (1 - q_n)^{N-m}. \quad (85)$$

Note that the probability in (85) depends on $g_{i-1}$ only through its sum, which is equal to $n$. Therefore, the transition probability $p_{n,m}$ in (110) has the same form as (85). To evaluate the spectral radius of $Q$ from (83), we need the value of $p_{n,0} + p_{n,N}$ (i.e., the $n$th entry of $b + c$), which is given by:

$$p_{n,0} + p_{n,N} = \frac{n^{NK} + (N - n)^{NK}}{(n^K + (N - n)^K)^N}. \quad (86)$$

The following result establishes a monotonicity property of the sum in (86).
**Lemma 5.** Let \( f(x) \) be a function of the form
\[
f(x) = \frac{a^N x + b^N x}{(a^x + b^x)^N}
\] (87)
for some positive scalars \( \{a, b, N\} \) with \( N > 1 \). Then, \( f(x) \) is a non-decreasing function, i.e.,
\[
f'(x) \geq 0
\] (88)
with equality if, and only if, \( a = b \).

**Proof:** The proof follows from evaluating \( f'(x) \).

Since the spectral radius of \( Q \) depends on the value of \( K \) in (22), we will index the quantities with the parameter \( K \). For example, we denote the spectral radius of \( Q \) by \( \rho[Q(K)] \). The following result relates the convergence rate of the decision-making process to the parameter \( K \).

**Theorem 4.** The spectral radius \( \rho[Q(K)] \) is a strictly decreasing function of \( K \) for \( N > 2 \), i.e.,
\[
\rho[Q(K+1)] < \rho[Q(K)].
\] (89)

**Proof:** From (83), the spectral radius \( \rho[Q(K)] \) is given by:
\[
\rho[Q(K)] = 1 - \sum_{n=1}^{N-1} y_{Q,n}[p_{n,0}(K) + p_{n,N}(K)]
\] (90)
where \( y_{Q,n} \) is the \( n \)th entry of \( y_Q \) and the sum inside the brackets is shown in (86). From Lemma 5, we have that
\[
p_{n,0}(K+1) + p_{n,N}(K+1) \geq p_{n,0}(K) + p_{n,N}(K)
\] (91)
with equality if, and only if, \( n = N/2 \). Therefore, if \( N > 2 \), there exists \( n \in 1, 2, \ldots, N - 1 \) such that strict inequality holds in (91). Moreover, since the matrix \( Q \) is primitive, the \( \{y_{Q,n}\} \) are positive and we arrive at (89).

We therefore conclude that to improve the convergence rate of the decision-making process, the nodes should use larger values of \( K \). Nevertheless, it may not be beneficial for the network to seek fast convergence during the decision making process because the network (e.g., a fish school) may converge to a bad model (e.g., a food source of poor quality). There exists a trade-off between exploration and exploitation, as in the case of multi-armed bandit problem [57]. Such trade-off can be taken into account by introducing some weighting scalar \( \beta_k(i-1) \) that measures the quality of the desired model of node \( k \) at time \( i - 1 \) relative to the other model. The higher values of \( \beta_k(i-1) \), the better the quality of the
model and the higher probability that node $k$ will maintain its desired model. Therefore, node $k$ adjusts the probability $q_{k,i-1}$ from (23) to

$$q_{k,i-1} = \frac{\left[\beta_k(i-1)n_k^q(i-1)\right]^K}{\left[\beta_k(i-1)n_k^q(i-1)\right]^K + \left[n_k - n_k^q(i-1)\right]^K}. \tag{92}$$

B. Convergence Rate of Diffusion Adaptation

Using the arguments in Section VI, we assume in the following that the nodes have achieved agreement on the desired model, say, $w^q$ as in (20). We know from the proof of Theorem 1 (see Appendix A) that a modified diffusion network is equivalent to a network with a mixture of informed and uninformed nodes, as studied in [58]. That is, nodes whose observed model is identical to its desired model ($f(l) = q$) are informed; otherwise they are uninformed. The convergence rate of the learning process specifies the rate at which the mean-square error converges to steady-state. Using the results of [58], we can deduce that the convergence rate, denoted by $r$, of the modified diffusion strategy (15)-(16) is given by:

$$r = \left[\rho(B)\right]^2 \tag{93}$$

where $B$ is defined in Table I. Note that the value of $r$ depends on the combination matrix $A$. Under Assumptions 2-3, it was shown that the convergence rate is bounded by [58]:

$$(1 - \mu\lambda_{\min}(R_u))^2 \leq r < 1. \tag{94}$$

To improve the convergence rate, it is desirable for the nodes to select their combination weights so that the network has lower value of $r$. It was shown in [58] that for any connected network, the convergence rate (93) can achieve the lower bound in (94) (namely, the network is able to converge to steady-state at the fastest rate) by selecting the combination matrix $A$ according to the following rules:

1) If there are informed nodes (i.e., nodes with positive step-sizes) in the neighborhood of node $k$, then it will assign positive combination weights to those nodes only.

2) Otherwise, node $k$ will assign positive combination weights to neighbors that are closer (with shorter path) to informed nodes.

However, there are two issues with this construction. First, it is difficult to construct the weights in a distributed manner because rule 2) requires spatial distribution of informed nodes. Second, the constructed combination matrix is not primitive (i.e., Assumption 1 does not hold) because there are no links from uninformed nodes to informed nodes. Therefore, Theorem 1 would not apply here. In the following, we first propose a way to select combination weights that approximate rule 2) and then show that the approximate weights ensure mean convergence.
Let $N^f_k$ denote the set of nodes that are in the neighborhood of $k$ and whose observed model is the same as the desired model $w^q_0$ (i.e., they are informed neighbors)

$$N^f_k = \{ l \mid l \in N_k, f(l) = q \}. \quad (95)$$

Also, let $n^f_k$ denote the number of nodes in the set $N^f_k$. The selection of combination weights is specified based on three types of nodes: informed nodes ($f(k) = q$), uninformed nodes with informed neighbors ($f(k) \neq q$ and $n^f_k \neq 0$), and uninformed nodes without informed neighbors ($f(k) \neq q$ and $n^f_k = 0$). The first two types correspond to rule 1) and their weights can satisfy rule 1) by setting

$$a_{l,k} = \begin{cases} 1/n^f_k, & \text{if } l \in N^f_k \\ 0, & \text{otherwise} \end{cases}. \quad (96)$$

That is, node $k$ places uniform weights on the informed neighbors and zero weights on the others. The last type of nodes corresponds to rule 2). Since these nodes do not know the distribution of informed nodes, a convenient choice for the approximate weights they can select is for them to place zero weights on themselves and uniform weights on the others, i.e.,

$$a_{l,k} = \begin{cases} 1/(n_k - 1), & \text{if } l \in N_k \text{ and } l \neq k \\ 0, & \text{otherwise} \end{cases}. \quad (97)$$

Note that the weights from (96)-(97) can be set in a fully distributed manner and in real-time. To show the mean convergence of the modified diffusion strategy using the combination matrix $A$ constructed from (96)-(97), we resort to Theorem 1 from [58]. It states that the strategy converges in the mean for sufficiently small step-sizes if for any node $k$, there exists an informed node $l$ and an integer power $j$ such that

$$[A^j]_{l,k} > 0. \quad (98)$$

Condition (98) is clearly satisfied for the first two types of nodes. For any node belonging to the last type, since the network is connected and from (97), there exists a path with nonzero weight from a node of the second type (uninformed with informed neighbors) to itself. In addition, there exist direct links from informed nodes to the nodes of the second type, condition (98) is also satisfied. This implies that the modified diffusion strategy using the combination weights from (96)-(97) converges in the mean.

**IX. Simulation Results**

We consider a network with 40 nodes randomly connected. The model vectors are set to $w^q_0 = [5; -5; 5; 5]$ and $w^q_1 = [5; 5; -5; 5]$ (i.e. $M = 4$). Assume that the first 20 nodes (nodes 1 through
observe data originating from model $w_0^q$, while the remaining nodes observe data originating from model $w_1^q$. The regression covariance matrix $R_u$ is diagonal with each diagonal entry generated uniformly from $[1, 2]$. The noise variance at each node is generated uniformly from $[-35, -5]$ dB. The step-sizes are set to $\mu = 0.005$, $\nu = 0.05$, and $\alpha = 0.95$. The threshold $\eta$ is set to $\eta = 1$. The network employs the decision-making process with $K = 4$ in (25) and the uniform combination rule: $a_{l,k} = 1/n_k$ if $l \in \mathcal{N}_k$.

In Fig. 5, we illustrate the network mean-square deviation (MSD) with respect to the two model vectors over time, i.e.,

$$\text{MSD}_q(i) = \frac{1}{N} \sum_{k=1}^{N} E\|w_q^i - w_{k,i}\|^2$$

for $q = 0$ and $q = 1$. We compare the conventional ATC diffusion strategy (4)-(5) and the modified ATC diffusion strategy (15)-(16) with decision-making. We observe the bifurcation in MSD curves of the modified ATC diffusion strategy. Specifically, the MSD curve relative to the model $w_0^q$ converges to 23 dB, while the MSD relative to $w_1^q$ converges to -50 dB. This illustrates that the nodes using the modified ATC diffusion are able to agree on a model and to converge to steady-state (to model $w_1^q$ in this case). We also show in Fig. 5 the evolution of the beliefs $\{b_{k,i}(l)\}$ for a particular node using the update rule (39). The node has two neighbors observing data that originate from the same model and two neighbors observing data from a different model. We observe that, at the initial stage of adaptation, all beliefs increase. Nevertheless, as time evolves, the node is able to differentiate between the two models and the beliefs for the latter two neighbors decrease. Note that the belief converges to one if a node has the same observed model; otherwise, it converges to zero. This indicates that the classification scheme successfully identifies the observed models of neighboring nodes. On the other hand, for the conventional diffusion strategy, the nodes also converge because the MSD curves in Fig. 5 remain flat. However, the MSD values are large (about 18 dB). This implies that the nodes converge to a common vector that does not coincide with either of the model vectors.

We also show the dependence of the convergence rate on the parameter $K$. We compare two modified diffusion strategies using decision-making with $K = 1$ and $K = 4$ in (23). The network MSD curves for these two strategies are shown in Fig. 7. We observe that the MSD curves relative to the model $w_0^q$ decrease at the same rate and converge to the same steady-state value. However, there is about 75 shift in time between these curves: the MSD with $K = 4$ is 75 time steps ahead of the MSD curve with $K = 1$. As the analytical result revealed, the decision-making processes adopting larger values of parameter $K$ achieve agreement at faster rate. We also consider the effect of the combination weights on the convergence rate of the adaptation strategies. Figure 8 illustrates the modified diffusion strategies.
Fig. 5. Transient network MSD over a network using the conventional diffusion strategy (4)-(5) and using the modified diffusion strategy (15)-(16). The network with decision-making converges to the model $w_0^1$ while the network without decision making converges to a vector that is not identical to either of the model vectors.

![Graph showing transient network MSD](image)

Fig. 6. Evolution of beliefs using (39) at a particular node. The node has four neighbors; two of them collect data from the same model while the other two collect data from a different model.

![Graph showing belief evolution](image)

with different combination weights: one with the uniform combination rule and the other one with the combination rule in (96)-(97). We observe that the diffusion strategy using the proposed rule converges at faster rate with some degradation in steady-state MSD. Note that the trade-off between convergence rate and MSD is also indicated in [58].

We apply the results of this paper to model the fish schooling behavior in the presence of two food sources (located at $w_0^0$ and $w_1^0$). It is observed in nature that fish move in a harmonious manner so that
they align their motion and keep a safe distance from each other [29], [32], [59]–[61]. We apply the motion control mechanism from [40] to model mobile agents. Let $x_{k,i}$ denote the location vector of node $k$ at time $i$. Every node $k$ adjusts its location vector according to the rule:

$$x_{k,i+1} = x_{k,i} + \Delta t \cdot v_{k,i+1}$$  \hspace{1cm} (100)$$

where $\Delta t$ is a positive time step and $v_{k,i+1}$ is the velocity vector at node $k$, which is set according to
the rule:

\[ v_{k,i+1} = \lambda \frac{w_{k,i} - x_{k,i}}{\|w_{k,i} - x_{k,i}\|} + \beta \sum_{l \in N_k} c_l k v_{k,i} + \gamma \delta_{k,i} \]  

(101)

where \( \{\lambda, \beta, \gamma\} \) are nonnegative scalars and \( \delta_{k,i} \) helps the nodes keep a certain distance \( d_s \) to each other and is given by

\[ \delta_{k,i} = \frac{1}{n_k - 1} \sum_{l \in N_k \setminus \{k\}} \left( \frac{\|x_{l,i} - x_{k,i}\| - d_s}{\|x_{l,i} - x_{k,i}\|} \right). \]

The nodes employ the diffusion strategy to estimate the location of food sources. This is achieved as follows. We showed in [40] that the distance, \( d_k^o(i) \), between the target located at \( w^o \) and a node \( k \) located at \( x_{k,i} \) can be expressed as the inner product (see Fig. [9]):

\[ d_k^o(i) = u_{k,i}^o(w^o - x_{k,i}) \]

where \( u_{k,i}^o \) denotes the unit direction vector pointing to \( w^o \) from \( x_{k,i} \). However, the nodes observe a noisy distance \( d_k(i) \) and a noisy direction \( u_{k,i} \) to the target, which can be related to \( w^o \) as follows (the same form as (1)):

\[ \hat{d}_k(i) \triangleq d_k(i) + u_{k,i} x_{k,i} = u_{k,i} w^o + v_k(i) \]  

(102)

where \( v_k(i) \) is the scalar noise term and its variance is proportional to the distance of node \( k \) to the target, i.e.,

\[ \sigma^2_{v,k,i} = \kappa \|w^o - x_{k,i}\|^2 \]  

(103)

with \( \kappa = 0.01 \). In simulation, there are two targets located at \( w^o_0 = [10, 10] \) and \( w^o_1 = [-10, 10] \). The nodes then apply Algorithm in Section [VI] to achieve agreement on a desired target. The simulation results are illustrated in Fig. [10]. The parameters used in (100)-(101) are set to \((\Delta t, \lambda, \beta, \gamma, d_s) = (0.1, 0.3, 0.7, 1, 3)\).

Initially, there are 40 nodes uniformly distributed in a \( 20 \times 20 \) square area around the origin. There are 20 nodes collecting data that originate from target \( w^o_0 \) and the remaining 20 nodes collecting data arising from the other target \( w^o_1 \). In Fig. [10], nodes that would like to move towards \( w^o_0 \) are shown as blue dots and nodes that would like to move towards \( w^o_1 \) are shown as red circles. We observe that the node achieve agreement on a desired target and get to the target (at \( w^o_1 = [40, -40] \) in this case).

X. CONCLUDING REMARKS

In the presence of distinct models observed by the nodes in a network, conventional distributed estimation strategies will lead to biased solutions. In this paper, we proposed a modified strategy to address this issue. To do so, we allow the nodes to exchange not only intermediate estimates, but also previous estimates. We also developed a classification scheme and a decision-making procedure for the nodes to identify the underlying models that generate data and to achieve agreement among the nodes on the desired objective. It is useful to comment on some features of the proposed framework.
Fig. 9. Distance and direction of the target $w^o$ from node $k$ at location $x_k$. The unit direction vector $u_k^o$ points towards $w^o$.

We focused in this work on the case where nodes need to choose between two models. Extension of the techniques to multiple models require additional analysis. The case of two models is not a serious limitation especially since many hypothesis testing problems tend to be formulated as deciding between two choices. In addition, it is natural to expect that convergence of the decision process will occur towards one model or the other in a probabilistic manner since the outcome is influenced by the fraction of nodes that sense data from one model or another. Interestingly, though, the decision-making process and the estimation task are largely independent of each other. This is because there are two tasks that the nodes need to accomplish. First, they need to decide which of the two models to follow and, second, they need to estimate the model. To solve the first task, agents do not need to know the exact model values. An arbitrary node $k$ only needs to know whether a neighboring node $l$ is observing data from the same
model or from a different model regardless of the model values. This property enables the initial decision process to converge faster and to be largely independent of the estimation task.

APPENDIX A

PROOF OF THEOREM 1

Without loss of generality, let \( w_0^0 \) be the desired model for the network (i.e., \( q = 0 \) in (20)) and assume there are \( N_0 \) nodes with indices \( \{1, 2, \ldots, N_0\} \) observing data arising from the model \( w_0^0 \), while the remaining \( N - N_0 \) nodes observe data arising from model \( w_1^0 \). Then, we obtain from (7), (18), and (19) that

\[
\tilde{z}^0_k = \begin{cases} 
0, & \text{if } k \leq N_0 \\
 w_0^0 - w_1^0, & \text{if } k > N_0 
\end{cases}
\]  

(104)

\[
a^{(1)}_{l,k} = 0 \text{ if } l > N_0 \quad \text{and} \quad a^{(2)}_{l,k} = 0 \text{ if } l \leq N_0.
\]

(105)

Since the matrix \( \mathcal{M} \mathcal{R} \) is block diagonal, we conclude that

\[
y = 0 \quad \text{and} \quad \mathcal{B} = A^T (I_{NM} - \mathcal{M}_e \mathcal{R})
\]

(106)

where \( \mathcal{M}_e \) is an \( N \times N \) block diagonal matrix of the form

\[
\mathcal{M}_e \triangleq \text{diag}\{\mu_1 I_M, \cdots, \mu_{N_0} I_M, 0, \cdots, 0\}.
\]

(107)

That is, its mean recursion in (11) is equivalent to the mean recursion of a network running the traditional diffusion strategy (4)-(5) with \( N_0 \) nodes (nodes 1 to \( N_0 \)) using positive step-sizes and \( N - N_0 \) nodes (nodes \( N_0 + 1 \) to \( N \)) having zero step-sizes. Then, according to Theorem 1 of [58] and under the assumption that the matrix \( A \) is primitive, if the step-sizes \( \{\mu_1, \mu_2, \cdots, \mu_{N_0}\} \) are set to satisfy (13), then the spectral radius of \( \mathcal{B} \) will be strictly less than one.

APPENDIX B

PROOF OF THEOREM 2

For a given vector \( g_{i-1} \), we denote by \( \chi_{i-1} \) the number of nodes whose desired model is \( w_1^0 \) at time \( i - 1 \), i.e.,

\[
\chi_{i-1} \triangleq \sum_{k=1}^{N} g_{i-1}(k).
\]

(108)

From (21)-(23), the vector \( g_i \) depends only on \( g_{i-1} \). Thus, the value of \( \chi_i \) depends only on \( \chi_{i-1} \). Therefore, the evolution of \( \chi_i \) forms a Markov chain with \( N + 1 \) states corresponding to the values
\{0, 1, 2, \ldots, N\} for \(\chi_i\). To compute the transition probability, \(p_{n,m}\), from state \(\chi_{i-1} = n\) to state \(\chi_i = m\), let us denote by \(G_n\) the set of vectors \(g = \{g(1), g(2), \ldots, g(N)\}\) whose entries are either 1 or 0 and add up to \(n\), i.e.,
\[
G_n = \left\{ g \mid \sum_{k=1}^{N} g(k) = n \right\}.
\] (109)

Then, the \(p_{n,m}\) can be written as:
\[
p_{n,m} = \sum_{g_{i-1} \in G_n} \Pr(g_{i-1}) \sum_{g_i \in G_m} \prod_{l=1}^{N} \Pr(g_i(l) \mid g_{i-1}(l))
\] (110)
where \(\Pr(g_{i-1})\) is a priori probability and where the probability \(\Pr(g_i(l) \mid g_{i-1}(l))\) is determined by (23). Note that for a static network, the transition probability \(p_{n,m}\) is independent of \(i\), i.e., the Markov chain is homogeneous \[62\].

Now we assume that \(\chi_{i-1} = n \neq 0, N\). Since the network is connected, for any \(g_{i-1} \in G_n\) at least one node (say, node \(k\)) has desired model \(w_0^k\) and has a neighbor with distinct desired model \(w_0^k\) so that \(n_k^g(i-1) < n_k\) and \(1 - q_{k,i-1} > 0\) from (23). Since \(q_{l,i-1} > 0\) for all \(l\), we obtain from (110) that
\[
p_{n,n-1} \geq \sum_{g_{i-1} \in G_n} \Pr(g_{i-1})(1 - q_{k,i-1}) \prod_{l \neq k} q_{l,i-1} > 0
\] (111)
for \(n \neq 0, N\). When \(n = 0\) or \(n = N\), we have that \(p_{0,0} = p_{N,N} = 1\). This indicates that the Markov chain has two absorbing states: \(\chi_i = 0\) (or, \(g_i(1) = g_i(2) = \cdots = g_i(N) = 0\)) and \(\chi_i = N\) (or, \(g_i(1) = g_i(2) = \cdots = g_i(N) = 1\)), and for any state \(\chi_i\) different from 0 and \(N\), there is a nonzero probability traveling from an arbitrary state \(\chi_i\) to state 0 and state \(N\). Therefore, no matter which state the Markov chain starts from, it converges to state 0 or state \(N\) \[62\] p.26, i.e., all nodes reach agreement on the desired model.
**APPENDIX C**

**PROOF OF LEMMA 3**

Let $C_1$ denote the far-field condition: $\|z_k^0 - w_{k,i-1}\| \gg 1$. We obtain from Assumption 4 and (72) that

$$\Pr(\|\hat{h}_{k,i}\| > \eta \mid C_1) \geq \Pr(\|\tilde{h}_{k,i}\| - \|n_{k,i}\| > \eta \mid C_1)$$

$$= 1 - \Pr(\|n_{k,i}\| \geq \|\tilde{h}_{k,i}\| - \eta \mid C_1)$$

$$\geq 1 - \frac{\mathbb{E}\|n_{k,i}\|^2}{(\|h_{k,i}\| - \eta)^2}$$

$$\geq 1 - \frac{\nu[\tau \|h_{k,i}\|^2 + \sigma_{v,k}^2 \text{Tr}(R_u)\|n_{k,i}\|^2]}{2(\|h_{k,i}\| - \eta)^2}$$

(112)

where step (a) follows from the triangle inequality of norms and (50), step (b) is by the Markov inequality (51) and Assumption 4, and step (c) is by (73). Moreover, under conditions $C_1$ and (59), we can ignore the term $\eta$ in the denominator of (112). In addition, from condition $C_1$ and (65), and since the variance $\sigma_{v,k}^2$ is generally small, we may ignore the term $\nu \sigma_{v,k}^2 \text{Tr}(R_u)$ in (112) and obtain (74). Similar arguments apply to (75).

**APPENDIX D**

**PROOF OF LEMMA 4**

Under condition (59) and from (39), the probability $P_d$ in (46) becomes

$$P_d = \Pr(\|\hat{h}_{k,i}\| > \eta, \|\hat{h}_{l,i}\| > \eta, \hat{h}_{k,i}^T \hat{h}_{l,i} > 0 \mid z_k^0 = z_l^0)$$

$$\approx \Pr(\hat{h}_{k,i}^T \hat{h}_{l,i} > 0 \mid z_k^0 = z_l^0)$$

(113)

where we used the fact that $\hat{h}_{k,i}$ and $\hat{h}_{l,i}$ are independent, as well as the result of Lemma 3 which ensures that $\|\hat{h}_{k,i}\| > \eta$ with high probability (likewise, for the norm of $\hat{h}_{l,i}$). Note that the event $\hat{h}_{k,i}^T \hat{h}_{l,i} > 0$ is equivalent to the fact that the angle between these two vectors is less than $\pi/2$. Let $\theta_{k,i}$ denote the angle between the vectors $\hat{h}_{k,i}$ and $\hat{h}_{l,i}$ due to the noise $n_{k,i}$ (see Fig. 11(a)). The value of $\theta_{k,i}$ is positive if the vector $\hat{h}_{k,i}$ rotates counter-clockwise relative to $\hat{h}_{k,i}$; otherwise, its value is negative. Then, we have that the angle $\theta_{k,i}$ is upper bounded by (see Fig. 11(a)):

$$|\theta_{k,i}| \leq \sin^{-1}\left(\frac{\|n_{k,i}\|}{\|\hat{h}_{k,i}\|}\right) \approx \frac{\|n_{k,i}\|}{\|h_{k,i}\|}$$

(114)

That is, the maximum value of $\theta_{k,i}$ occurs when the vectors $\hat{h}_{k,i}$ and $n_{k,i}$ are perpendicular. The approximation in (114) is from (59) and (73) so that it holds that

$$\mathbb{E}\|n_{k,i}\|^2 / \|h_{k,i}\|^2 \leq \frac{\nu[\tau \|h_{k,i}\|^2 + \sigma_{v,k}^2 \text{Tr}(R_u)]}{2\|h_{k,i}\|^2} \approx \frac{\nu \tau}{2}$$

(115)
Since all nodes start from the same initial estimate (i.e., \( w_{k,-1} = 0 \) for all \( k \)), the estimates \( \{w_{k,i-1}\} \) are close to each other during the initial stages of adaptation and it is reasonable to assume that \( \|w_{k,i-1} - w_{l,i-1}\| \ll \|z^0_k - w_{k,i-1}\| \). Therefore, we arrive at the approximation \( \bar{h}_{k,i} \approx \bar{h}_{k,i} \) for computing \( P_d \). This implies that the vectors \( \hat{h}_{k,i} \) and \( \hat{h}_{l,i} \) can be modeled as starting approximately at the same location \( w_{k,i-1} \) but having deviated by angles \( \theta_{k,i} \) and \( \theta_{l,i} \), respectively (see Fig. 11(b)). Therefore, the angle between \( \hat{h}_{k,i} \) and \( \hat{h}_{l,i} \) is equal to \( |\theta_{k,i} - \theta_{l,i}| \). From (113), we obtain that

\[
P_d \approx \Pr \left( \theta_{k,i} - \theta_{l,i} < \frac{\pi}{2} \mid z^0_k = z^0_l \right) \geq \Pr \left( |\theta_{k,i} + |\theta_{l,i}| < \frac{\pi}{2} \mid z^0_k = z^0_l \right) \geq \Pr \left( \frac{\|n_{k,i}\|}{\|\bar{h}_{k,i}\|} + \frac{\|n_{l,i}\|}{\|\bar{h}_{k,i}\|} < \frac{\pi}{2} \right) = 1 - \Pr(\|n_{k,i}\| + \|n_{l,i}\| \geq \pi\|\bar{h}_{k,i}\|/2) \tag{116}
\]

where step (a) is by the triangle inequality of norms and (50) and step (b) is by (114). To evaluate the probability in (116), we resort to the following fact. For any two random variables \( x \) and \( y \) and for any constant \( \eta \), it holds from (50) that

\[
\Pr(x + y > \eta) \leq \Pr(x > \eta/2) + \Pr(y > \eta/2). \tag{117}
\]

This leads to

\[
P_d \geq 1 - \Pr(\|n_{k,i}\| > \pi\|\bar{h}_{k,i}\|/4) - \Pr(\|n_{l,i}\| > \pi\|\bar{h}_{k,i}\|/4). \tag{118}
\]

We then arrive at (76) because

\[
P_d \geq 1 - \frac{16(\mathbb{E}\|n_{k,i}\|^2 + \mathbb{E}\|n_{l,i}\|^2)}{\pi^2\|\bar{h}_{k,i}\|^2} \geq 1 - \frac{16\nu\tau}{\pi^2} \tag{119}
\]

where we used the Markov inequality (51) and (115). Similar arguments apply to \( P_f \) when \( z^0_k \neq z^0_l \) by noting that the vectors \( \bar{h}_{k,i} \) and \( \bar{h}_{l,i} \) can again be modeled as starting approximately at the same location \( w_{k,i-1} \), but pointing towards different directions: \( \bar{h}_{k,i} \) towards \( z_k \) and \( \bar{h}_{l,i} \) towards \( z_l \), and the angle between these two vectors now assumes a value close to \( \pi \) according to Lemma 1.

REFERENCES

[1] S. Y. Tu and A. H. Sayed, “Adaptive decision making over complex networks,” Proc. Asilomar Conference on Signals, Systems, and Computers, pp. 525–530, Pacific Grove, CA, Nov. 2012.

[2] S. Camazine, J. L. Deneubourg, N. R. Franks, J. Sneyd, G. Theraulaz, and E. Bonabeau, Self-Organization in Biological Systems. Princeton University Press, 2003.

[3] I. D. Couzin, “Collective cognition in animal groups,” Trends in Cognitive Sciences, vol. 13, pp. 36–43, Jan. 2009.
Fig. 11. Illustration of (a) the angle $\theta k,i$ between $\hat{h}_{k,i}$ and $\tilde{h}_{k,i}$ due to the noise $n_{k,i}$ and (b) the angle between $\hat{h}_{k,i}$ and $\hat{h}_{l,i}$ when $z k = z l$.

[4] D. J. T. Sumpter and S. C. Pratt, “Quorum responses and consensus decision making,” *Phil. Trans. R. Soc. B*, vol. 364, pp. 743–753, Dec. 2009.

[5] I. D. Couzin, C. C. Ioannou, G. Demirel, T. Gross, C. J. Torney, A. Hartnett, L. Conradt, S. A. Levin, and N. E. Leonard, “Uninformed individuals promote democratic consensus in animal groups,” *Science*, vol. 334, pp. 1578–1580, Dec. 2011.

[6] N. F. Britton, N. R. Franks, S. C. Pratt, and T. D. Seeley, “Deciding on a new home: How do honeybees agree?” *Proc. R. Soc. Lond. B*, vol. 269, pp. 1383–1388, May 2002.

[7] S. C. Pratt, E. B. Mallon, D. J. T. Sumpter, and N. R. Franks, “Quorum sensing, recruitment, and collective decision-making during colony emigration by the ant Leptothorax albipennis,” *Behav. Ecol. Sociobiol.*, vol. 52, pp. 117–127, May 2002.

[8] M. Beekman, R. L. Fathke, and T. D. Seeley, “How does an informed minority of scouts guide a honey bee swarm as it flies to its new home?” *Animal Behavior*, vol. 71, pp. 161–171, 2006.

[9] C. G. Lopes and A. H. Sayed, “Diffusion least-mean squares over adaptive networks: Formulation and performance analysis,” *IEEE Trans. on Signal Processing*, vol. 56, no. 7, pp. 3122–3136, Jul. 2008.

[10] F. S. Cattivelli, C. G. Lopes, and A. H. Sayed, “Diffusion recursive least-squares for distributed estimation over adaptive networks,” *IEEE Trans. on Signal Processing*, vol. 56, no. 5, pp. 1865–1877, May 2008.

[11] F. S. Cattivelli and A. H. Sayed, “Diffusion LMS strategies for distributed estimation,” *IEEE Trans. on Signal Processing*, vol. 58, no. 3, pp. 1035–1048, Mar. 2010.

[12] J. Chen and A. H. Sayed, “Diffusion adaptation strategies for distributed optimization and learning over networks,” *IEEE Trans. on Signal Processing*, vol. 60, no. 8, pp. 4289–4305, Aug. 2012.

[13] A. H. Sayed, “Diffusion adaptation over networks,” in *Academic Press Library in Signal Processing*, vol. 3, R. Chellapa and S. Theodoridis, editors, pp. 323-454, Elsevier, 2014. Also available online at http://arxiv.org/abs/1205.4220, May 2012.

[14] A. H. Sayed, S. Y. Tu, J. Chen, X. Zhao, and Z. Towfic, “Diffusion strategies for adaptation and learning over networks,” *IEEE Signal Processing Magazine*, vol. 30, no. 3, pp. 155–171, May 2013.

[15] L. Li and J. A. Chambers, “Distributed adaptive estimation based on the APA algorithm over diffusion networks with changing topology,” *Proc. IEEE SSP Workshop*, pp. 757–760, Cardiff, Wales, Sep. 2009.

[16] S. Chouvardas, K. Slavakis, and S. Theodoridis, “Adaptive robust distributed learning in diffusion sensor networks,” *IEEE Trans. on Signal Processing*, vol. 59, no. 10, pp. 4692–4707, Oct. 2011.

[17] P. D. Lorenzo, S. Barbarossa, and A. H. Sayed, “Bio-inspired decentralized radio access based on swarming mechanisms over adaptive networks,” *IEEE Trans. on Signal Processing*, no. 12, pp. 3183–3197, Jun. 2013.

[18] Y. Xia, D. P. Mandic, and A. H. Sayed, “An adaptive diffusion augmented CLMS algorithm for distributed filtering of non-circular complex signals,” *IEEE Signal Processing Letters*, no. 11, pp. 659–662, Nov. 2011.
[19] N. Takahashi and I. Yamada, “Link probability control for probabilistic diffusion least-mean squares over resource-constrained networks,” Proc. IEEE ICASSP, pp. 3518–3521, Dallas, TX, Mar. 2010.

[20] Y. K. S. Chouvardas, K. Slavakis and S. Theodoridis, “A sparsity promoting adaptive algorithm for distributed learning,” IEEE Trans. on Signal Processing, vol. 60, no. 10, pp. 5412–5425, Oct. 2012.

[21] J. N. Tsitsiklis, J. N. Bertsekas, and M. Athans, “Distributed asynchronous deterministic and stochastic gradient optimization algorithms,” IEEE Trans. on Autom. Control, vol. 31, no. 9, pp. 803–812, Sep. 1986.

[22] A. Nedic and A. Ozdaglar, “Distributed subgradient methods for multi-agent optimization,” IEEE Trans. on Autom. Control, vol. 54, no. 1, pp. 48–61, Jan. 2009.

[23] I. Schizas, G. Mateos, and G. Giannakis, “Distributed LMS for consensus-based in-network adaptive processing,” IEEE Trans. on Signal Processing, vol. 57, no. 6, pp. 2365–2382, Jun. 2009.

[24] G. Mateos, I. D. Schizas, and G. B. Giannakis, “Performance analysis of the consensus-based distributed LMS algorithm,” EURASIP Journal on Advances in Signal Processing, Nov. 2009, doi:10.1155/2009/981030.

[25] A. G. Dimakis, S. Kar, J. M. F. Moura, M. G. Rabbat, and A. Scaglione, “Gossip algorithms for distributed signal processing,” Proc. IEEE, vol. 98, no. 11, pp. 1847–1864, Nov. 2010.

[26] S. Kar and J. M. F. Moura, “Convergence rate analysis of distributed gossip (linear parameter) estimation: Fundamental limits and tradeoffs,” IEEE J. Selected Topics in Signal Processing, vol. 5, no. 5, pp. 674–690, Aug. 2011.

[27] ——, “Distributed parameter estimation in sensor networks: Nonlinear observation models and imperfect communication,” IEEE Trans. on Info. Theory, vol. 58, no. 6, pp. 3575–3605, Jun. 2012.

[28] S. Y. Tu and A. H. Sayed, “Diffusion strategies outperform consensus strategies for distributed estimation over adaptive networks,” IEEE Trans. on Signal Processing, vol. 60, no. 12, pp. 6217–6234, Dec. 2012.

[29] A. Jadbabaie, J. Lin, and A. S. Morse, “Coordination of groups of mobile autonomous agents using nearest neighbor rules,” IEEE Trans. on Automatic Control, vol. 48, no. 6, pp. 988–1001, Jun. 2003.

[30] R. Olfati-Saber and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” IEEE Trans. on Automatic Control, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[31] J. A. Fax and R. M. Murray, “Information flow and cooperative control of vehicle formations,” IEEE Trans. on Autom. Control, vol. 49, no. 9, pp. 1465–1476, Sep. 2004.

[32] R. Olfati-Saber, “Flocking for multi-agent dynamic systems: algorithms and theory,” IEEE Trans. on Automatic Control, vol. 51, no. 3, pp. 401–420, Mar. 2006.

[33] M. E. Yildiz, A. Scaglione, and A. Ozdaglar, “Asymmetric information diffusion via gossiping on static and dynamic networks,” IEEE Proc. of CDC, pp. 7467–7472, Atlanta, GA, Dec. 2010.

[34] U. A. Khan, S. Kar, and J. M. F. Moura, “Higher dimensional consensus: Learning in large-scale networks,” IEEE Trans. on Signal Processing, vol. 58, no. 5, pp. 2836–2849, May 2010.

[35] P. Forero, A. Cano, and G. B. Giannakis, “Consensus-based distributed support vector machines,” Journal of Machine Learning Research, vol. 11, pp. 1663–1707, 2010.

[36] C. Castellano, S. Fortunato, and V. Loreto, “Statistical physics of social dynamics,” Rev. Mod. Phys., vol. 81, pp. 591–646, 2009.

[37] D. Acemoglu and A. Ozdaglar, “Opinion dynamics and learning in social networks,” Dynamic Games and Applications, vol. 1, no. 1, pp. 3–49, 2010.

[38] A. Jadbabaie, P. Molavi, A. Sandroni, and A. Tahbaz-Salehi, “Non-Bayesian social learning,” Games and Economic Behavior, vol. 76, no. 1, pp. 210–225, Sep. 2012.
[39] X. Zhao, S. Y. Tu, and A. H. Sayed, “Diffusion adaptation over networks under imperfect information exchange and non-stationary data,” IEEE Trans. on Signal Processing, vol. 60, no. 7, pp. 3460–3475, Jul. 2012.

[40] S. Y. Tu and A. H. Sayed, “Mobile adaptive networks,” IEEE J. Selected Topics on Signal Processing, vol. 5, no. 4, pp. 649–664, Aug. 2011.

[41] C. G. Lopes and A. H. Sayed, “Distributed processing over adaptive networks,” Proc. Adaptive Sensor Array Processing Workshop, pp. 1–5, MIT Lincoln Laboratory, MA, Jun. 2006.

[42] A. H. Sayed and C. G. Lopes, “Adaptive processing over distributed networks,” IEICE Trans. on Fundamentals of Electronics, Communications and Computer Sciences, vol. E90-A, no. 8, pp. 1504–1510, 2007.

[43] C. G. Lopes and A. H. Sayed, “Diffusion least-mean-squares over adaptive networks,” Proc. IEEE ICASSP, pp. 917–920, Honolulu, Hawaii, Apr. 2007.

[44] F. S. Cattivelli, C. G. Lopes, and A. H. Sayed, “A diffusion RLS scheme for distributed estimation over adaptive networks,” Proc. IEEE Workshop on Signal Process. Advances Wireless Comm. (SPAWC), pp. 1–5, Helsinki, Finland, Jun. 2007.

[45] F. S. Cattivelli and A. H. Sayed, “Diffusion LMS algorithms with information exchange,” Proc. Asilomar Conference on Signals, Systems and Computers, pp. 251–255, Pacific Grove, CA, Nov. 2008.

[46] S. Ram, A. Nedic, and V. V. Veeravalli, “Distributed stochastic subgradient projection algorithms for convex optimization,” Journal of Optimization Theory and Applications, vol. 147, no. 3, pp. 516–545, 2010.

[47] P. Bianchi, G. Fort, W. Hachem, and J. Jakubowicz, “Convergence of a distributed parameter estimator for sensor networks with local averaging of the estimates,” Proc. IEEE ICASSP, pp. 3764–3767, Prague, Czech, May 2011.

[48] K. Srivastava and A. Nedic, “Distributed asynchronous constrained stochastic optimization,” IEEE J. Selected Topics on Signal Processing, vol. 5, no. 4, pp. 772–790, Aug. 2011.

[49] S. S. Stankovic, M. S. Stankovic, and D. M. Stipanovic, “Decentralized parameter estimation by consensus based stochastic approximation,” IEEE Trans. on Autom. Control, vol. 56, no. 3, pp. 531–543, Mar. 2011.

[50] R. Horn and C. R. Johnson, Matrix Analysis. Cambridge University Press, 1985.

[51] A. Berman and R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, 1994.

[52] J. Chen and A. H. Sayed, “On the limiting behavior of distributed optimization strategies,” Proc. Allerton Conference on Communication, Control, and Computing, pp. 1535–1542, Allerton, IL, Oct. 2012.

[53] J. F. S. Hill and M. A. Blanco, “Random geometric series and intersymbol interference,” IEEE Trans. on Information Theory, vol. 19, no. 3, pp. 326–335, May 1973.

[54] P. J. Smith, “The distribution functions of certain random geometric series concerning intersymbol interference,” IEEE Trans. on Information Theory, vol. 37, no. 6, pp. 1657–1662, Nov. 1991.

[55] A. Bovier and P. Picco, “A law of the iterated logarithm for random geometric series,” The Annals of Probability, vol. 21, no. 1, pp. 168–184, 1993.

[56] A. Papoulis and S. U. Pillai, Probability, Random Variables, and Stochastic Processes. McGraw-Hill, 2002.

[57] J. C. Gittins, Multi-Armed Bandit Allocation Indices. John Wiley and Sons, New York, NY, 1989.

[58] S. Y. Tu and A. H. Sayed, “On the influence of informed agents on learning and adaptation over networks,” IEEE Trans. on Signal Processing, vol. 61, no. 6, pp. 1339–1356, Mar. 2013.

[59] V. Gazi and K. M. Passino, “Stability analysis of social foraging swarms,” IEEE Trans. on Systems, Man, and Cybernetics-Part B: Cybernetics, vol. 34, pp. 539–557, Feb. 2004.

[60] P. D. Lorenzo and S. Barbarossa, “A bio-inspired swarming algorithm for decentralized access in cognitive radio,” IEEE Trans. on Signal Processing, vol. 59, no. 12, pp. 6160–6174, Dec. 2013.
[61] M. M. Zavlanos, A. Ribeiro, and G. J. Pappas, “Distributed control of mobility & routing in networks of robots,” Proc. IEEE SPAWC, pp. 236–240, Jun. 2011.

[62] G. F. Lawler, Introduction to Stochastic Processes. Chapman & Hall/CRC, 2006.
Algorithm (Diffusion strategy with decision-making)

For each node \( k \), initialize \( \mathbf{w}_{k,-1} = 0 \), \( \mathbf{h}_{k,-1} = 0 \), \( b_{k,-1}(l) = 0.5 \), and \( \mathbf{g}_{k,-1}(k) = 1 \).

for \( i \geq 0 \) and \( k = 1 \) to \( N \) do

1) Perform an adaptation step using the local data \( \{ \mathbf{d}_{k}(i), \mathbf{u}_{k,i} \} \):

\[
\psi_{k,i} = \mathbf{w}_{k,i-1} + \mu \mathbf{u}_{k,i}^T \mathbf{d}_{k}(i) - \mathbf{u}_{k,i} \mathbf{w}_{k,i-1}.
\]

2) Exchange the vectors \( \{ \psi_{k,i}, \mathbf{w}_{k,i-1} \} \) with neighbors and update the average update vectors \( \{ \mathbf{h}_{l,i} \} \) for \( l \in \mathcal{N}_k \):

\[
\mathbf{h}_{l,i} = (1 - \nu)\mathbf{h}_{l,i-1} + \nu \mu^{-1} (\psi_{l,i} - \mathbf{w}_{l,i-1}).
\]

3) Update the beliefs \( \{ b_{k,i}(l) \} \) for \( l \in \mathcal{N}_k \setminus \{ k \} \):

\[
b_{k,i}(l) = \begin{cases} 
\alpha b_{k,i-1}(l) + (1 - \alpha), & \text{if } E_1 \\
\alpha b_{k,i-1}(l), & \text{if } E_1^c
\end{cases}
\]

where \( E_1 \) and \( E_1^c \) are defined in (40)–(41).

4) Identify the observed models \( \{ \hat{f}_{k,i}(l) \} \) for \( l \in \mathcal{N}_k \setminus \{ k \} \):

\[
\hat{f}_{k,i}(l) = \begin{cases} 
1, & \text{if } b_{k,i}(l) \geq 0.5 \\
0, & \text{otherwise}
\end{cases}
\]

5) Collect the desired models \( \{ g_{k,i-1}(l) \} \) for \( l \in \mathcal{N}_k \setminus \{ k \} \) and construct the set \( \mathcal{N}^p_{k,i-1} \) as follows:

\[
g_{k,i-1}(l) = \begin{cases} 
g_{l,i-1}(l), & \text{if } \hat{f}_{k,i}(l) = 1 \\
1 - g_{l,i-1}(l), & \text{otherwise}
\end{cases}
\]

\[
\mathcal{N}^p_{k,i-1} = \{ l \mid l \in \mathcal{N}_k, g_{k,i-1}(l) = g_{k,i-1}(k) \}.
\]

6) Update the desired model \( \mathbf{g}_{k,i}(k) \):

\[
\mathbf{g}_{k,i}(k) = \begin{cases} 
g_{k,i-1}(k), & \text{w.p. } q_{k,i-1} \\
1 - g_{k,i-1}(k), & \text{w.p. } 1 - q_{k,i-1}
\end{cases}
\]

where the probability \( q_{k,i-1} \) is defined in (23).

7) Adjust the combination weights \( \{ a_{1,k,i}^{(1)} \} \) and \( \{ a_{1,k,i}^{(2)} \} \):

\[
a_{1,k,i}^{(1)} = \begin{cases} 
a_{l,k}, & \text{if } l \in \mathcal{N}_k \text{ and } \hat{f}_{k,i}(l) = \mathbf{g}_{k,i}(k) \\
0, & \text{otherwise}
\end{cases}
\]

\[
a_{1,k,i}^{(2)} = \begin{cases} 
a_{l,k}, & \text{if } l \in \mathcal{N}_k \text{ and } \hat{f}_{k,i}(l) \neq \mathbf{g}_{k,i}(k) \\
0, & \text{otherwise}
\end{cases}
\]

8) Perform the combination step:

\[
\mathbf{w}_{k,i} = \sum_{l \in \mathcal{N}_k} \left( a_{1,k,i}^{(1)} \psi_{l,i} + a_{1,k,i}^{(2)} \mathbf{w}_{l,i-1} \right).
\]

end for