DISCRIMINANT LOCI OF AMPLE 
AND SPANNED LINE BUNDLES

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Abstract. Let \((X, L, V)\) be a triplet where \(X\) is an irreducible smooth complex projective variety, \(L\) is an ample and spanned line bundle on \(X\) and \(V \subseteq H^0(X, L)\) spans \(L\). The discriminant locus \(\mathcal{D}(X, V) \subseteq |V|\) is the algebraic subset of singular elements of \(|V|\). We study the components of \(\mathcal{D}(X, V)\) in connection with the jumping sets of \((X, V)\), generalizing the classical biduality theorem. We also deal with the degree of the discriminant (codegree of \((X, L, V)\)) giving some bounds on it and classifying curves and surfaces of codegree 2 and 3. We exclude the possibility for the codegree to be 1. Significant examples are provided.

Introduction

Let \(X\) be an irreducible smooth complex projective variety of dimension \(n\). Take \(L\) an ample line bundle on \(X\) and a linear system \(|V| \subseteq |H^0(X, L)|\) with \(\dim(|V|) = N\) and \(V\) spanning \(L\). We define the discriminant locus \(\mathcal{D}(X, V)\) of the triplet \((X, L, V)\) as the algebraic subset of \(|V|\) parameterizing the singular elements of \(|V|\). In the particular case in which \(\phi_V\) is an embedding, from now on the classical setting, the discriminant locus is just the dual variety \(\phi_V(X) \subset \mathbb{P}^N\), an irreducible subvariety of \(\mathbb{P}^N\). A nice survey on results on duality can be found in [T]. When \(\phi_V\) is not an embedding some considerations on the morphism \(\phi_V\) enter into the picture. In fact the main ingredients to build \(\mathcal{D}(X, V)\) are the jumping sets (and their images by \(\phi_V\)), measuring the deviation of \(\phi_V\) from being an immersion, see [LPS1]. Inspired by the classical setting, different problems on the discriminant locus can be faced. In our previous paper on this subject [LM1] (see also in [LPS1]) we have focused on the dimension of the discriminant locus. By the Bertini theorem \(\dim(\mathcal{D}(X, V)) < N\). Hence it can be written as \(\dim(\mathcal{D}(X, V)) = N - 1 - k\), where \(k \geq 0\) is called the (discriminant) defect of \((X, L, V)\). Some bounds on \(k\) and classification results in the extremal cases (where \(k\) is maximal) are provided in [LM1]. These results deeply rely on the geometry of \(\phi_V(X) \subset \mathbb{P}^N\) since \(\phi_V(X) \subset \mathcal{D}(X, V)\). We have also studied this problem dropping the hypothesis that \(L\) is ample in [LM2].
When $\phi_V$ is an immersion $\phi_V(X)^\vee = D(X,V)$. The locus where $\phi_V$ is not an immersion, consisting of the jumping sets, is important to study the discriminant locus in more general settings. In [LPS1], among other things, $D(X,V)$ is written as a union of algebraic subsets built with the jumping sets (see (0.3)). These sets are related with the Chern classes of the first jet bundle of $L$. This approach is continued in [LPS2] where a partial study of this decomposition of the discriminant locus (in the particular case in which $\phi_V$ is generically one–to–one) is given. Some considerations on the singular locus of a general $D \in D(X,V)$ are also presented. In the current paper we follow this line of research started in [LPS1] and developed in [LPS2], [LM1], [LM2]. Our main goal is to find appropriate generalizations of theorems holding in the classical setting to the more general setting of an ample line bundle $L$ spanned by $V$.

A main theorem in the classical setting is the so called biduality theorem. For $X \subset \mathbb{P}^N$ an irreducible complex projective variety, $X^{\vee\vee} = X$ via the canonical identification between $\mathbb{P}^N$ and $\mathbb{P}^{\mathbb{P}^N\vee\vee}$. In Section 1 we present a natural generalization of this theorem. In fact we prove that any irreducible component of $D(X,V)$ is the dual of the image of a component of a jumping set, see (1.3). Moreover, the dual of any irreducible component of $D(X,V)$ is contained in $\phi_V(X)$ as proved in (1.4). These results help to understand the relation between the decomposition in (0.3) and the irreducible components of the discriminant. Significant examples are provided.

Another basic fact in the classical setting is the irreducibility of the dual variety of an irreducible complex projective variety. In the non-classical setting this is no longer true. But if $\phi_V$ is just an immersion, then $D(X,V)$ is still irreducible. In Section 2 we show that, for curves, the facts of $\phi_V$ being an immersion and the irreducibility of $D(X,V)$ are equivalent. This is not true in higher dimension. We can construct examples of surfaces for which the discriminant locus is irreducible and any possible configuration of the decomposition in (0.3) is achieved, $\phi_V$ not being, in particular, an immersion. The most relevant consequence of irreducibility of the discriminant locus is the emptiness of the biggest jumping set, presented in (2.7).

Last problem we are concerned with is that of the degree of the discriminant locus called, according to [Z1], codegree of $(X,L,V)$ (denoted codeg$(X,V)$). In the classical setting this invariant is the class of $\phi_V(X) \subset \mathbb{P}^N$ when dim$(D(X,V)) = N - 1$. In [LPS1] it is shown that the Chern classes of the first jet bundle are related with the singular locus of elements in general linear subsystems of appropriate dimension of $|V|$. Using this identification we get an expression of the top Chern class of the first jet bundle involving the degrees of the maximal dimensional components of the discriminant. This expression and some consequences of it lead to a complete classification of curves and surfaces of codegree less than or equal to three. Let us recall that in the classical setting a complete classification of smooth projective varieties of codegree $\leq 3$ is provided in [Z1], [Z2, Thm. 5.2]. We prove that there are no triplets $(X,L,V)$ with codegree one and establish the complete list of curves and surfaces of codegree two (see (4.4) and (6.11)) and three (see (4.4)
and (7.5)). All cases in the lists are effective and examples are provided.

The final section is devoted to three further possible developments of the theory. As a first thing we introduce the concept of tame codegree for triplets \((X, L, V)\) for which the general element in \(\mathcal{D}(X, V)\) is singular in just one point and the singularity is quadratic and ordinary. This occurs in the classical setting, but not only in this case. We classify (see (8.1.4)) surfaces of tame codegree less than or equal to eight. The second point is concerned with the study of the subvariety of the discriminant made of the reducible or non-reduced elements in \(|V|\). The third one deals with two important facts holding in the classical case for positive defect varieties but not yet explored in the ample and spanned case: the parity theorem (the dimension and the defect have the same parity) and the linearity of the singular locus of a general element in the discriminant.

0. Background material

We work over the complex field and we use standard notation in algebraic geometry. In particular, if \(X\) is a projective manifold, \(K_X\) will denote the canonical bundle of \(X\). We say that a line bundle on \(X\) is spanned by a vector space \(V\) of sections if \(V\) generates \(L\) at every point of \(X\). By a little abuse of notation line bundles and divisors are used with little (or no) distinction. The symbol \(\equiv\) denotes numerical equivalence. We use the word scroll along the paper in the classical sense, i.e., the projectivized of an ample vector bundle with the polarization given by the tautological line bundle. We fix our setting as follows.

\((0.0)\) Let \((X, L, V)\) be a triplet where: \(X\) is an irreducible smooth projective variety of dimension \(n\), \(L\) is an ample and spanned line bundle on \(X\) and \(V \subseteq H^0(X, L)\) spans \(L\) at every point of \(X\). By a little abuse of notation line bundles and divisors are used with little (or no) distinction. The symbol \(\equiv\) denotes numerical equivalence. We use the word scroll along the paper in the classical sense, i.e., the projectivized of an ample vector bundle with the polarization given by the tautological line bundle. We fix our setting as follows.

The discriminant locus \(\mathcal{D}(X, V)\) of the triplet \((X, L, V)\) parameterizes the singular elements of \(|V|\). More precisely, taking the incidence correspondence

\[
\mathcal{Y} := \{(x, [s]) \in X \times |V| : j_1(s)(x) = 0\} \xrightarrow{p_1} X
\]

\(\mathcal{D}(X, V) \subseteq \mathbb{P}^N\),

where \(j_1(s)\) denotes the first jet of the section \(s \in V\), \(\mathcal{D}(X, V)\) is the image of \(\mathcal{Y}\) via the second projection of \(X \times |V|\). Thus \(\mathcal{D}(X, V)\) is an algebraic subset in \(|V| = \mathbb{P}^{N\vee}\). By the Bertini Theorem \(\dim(\mathcal{D}(X, V)) < N\). Hence we can write \(\dim(\mathcal{D}(X, V)) = N - 1 - k\), where \(k \geq 0\) is called the defect of \((X, L, V)\). It is important to point out the following fact.

\((0.2)\) We always look at the discriminant locus \(\mathcal{D}(X, V) \subseteq |V|\) as an algebraic set with its reduced structure.

If \(\phi_V(X) \neq \mathbb{P}^N\) then the dual variety \(\phi_V(X)^\vee\) is a non-empty irreducible subvariety of \(\mathcal{D}(X, V)\). Furthermore, if \(\phi_V\) is an immersion then \(\phi_V(X)^\vee = \mathcal{D}(X, V)\)
Anyway, points in \( D(X, V) \setminus \phi_V(X)^\vee \) are coming from points on \( X \) where the differential of \( \phi_V \) is not injective. In this context it is natural to define the jumping sets \( J_i = J_i(V) \) (1 ≤ i ≤ n) as in [LPS1, (1.1)], i.e., \( J_i = \{ x \in X : \text{rk}(d\phi_V(x)) \leq n - i \} \). As in [LPS2, (0.3.1)] \( X_i \) stands for \( J_i \setminus J_{i+1} \), with the convention that \( J_0 = X \) and \( J_{n+1} = \emptyset \). This allows to define \( D_i(X, V) \subseteq D(X, V) \) as \( p_2 \circ p_1^{-1}(X_i) \), that is, the Zariski closure in \( \mathbb{P}^{n\vee} \) of the locus of elements of \( |V| \) singular at points of \( X_i \), so that:

\[
D(X, V) = \cup_{i=0}^{n} D_i(X, V).
\]

When no confusion arises we will write \( D \) (respectively \( D_i \)) instead of \( D(X, V) \) (respectively \( D_i(X, V) \)). For further use we end this section with the following easy consequence of the second Bertini theorem [H, III Ex. 11.3, p. 280].

**0.4 Remark.** For \((X, L, V)\) as in (0.0), if \( \dim(X) \geq 2 \) then any element of \( |V| \) is connected. So if \( D \in |V| \) is reducible then \( D \in D(X, V) \).

1. **On the Components of the Discriminant**

Let us study some properties of \( D_i(X, V) \) (0 ≤ i ≤ n) and relate them with the geometry of \( \phi_V(X) \subseteq \mathbb{P}^N \). A first basic fact is the following.

**1.1** \( D_0(X, V) \) is always irreducible because, if non-empty, it is the dual variety of \( \phi_V(X) \subseteq \mathbb{P}^N \).

This in fact does not mean that \( D_0 \) (when non-empty) is always an irreducible component of \( D \), as is shown, for example, in [LM1, Example 0.2]. Let us recall this example for further references.

**1.1.0 Example.** Let \( S \) be a Del Pezzo surface with \( K_S^2 = 1 \) and let \( L = -2K_S \). We know that \( L \) is ample and spanned and \( \phi_L : S \to \Gamma \subseteq \mathbb{P}^3 \) is a double cover of the quadric cone \( \Gamma \), branched at the vertex \( v \) and along the smooth curve \( B \) cut out on \( \Gamma \) by a transverse cubic surface. We have \( D(S, L) = D_0 \cup D_1 \cup D_2 \), where \( D_0 = \Gamma^\vee \) is a conic, the dual of \( \Gamma \), \( D_1 = B^\vee \) is the dual of \( B \) and \( D_2 = v^\vee \) is a plane. Recalling that \( B \) is a sextic of genus 4 we thus get \( \deg(D_1) = 2(\deg(B) + g(B) - 1) = 18 \). Furthermore we can note that \( D_0 \subseteq D_1 \cap D_2 \). Actually, \( \Gamma^\vee \subseteq v^\vee \), since any plane tangent to \( \Gamma \) must contain \( v \); moreover, \( \Gamma^\vee \subseteq B^\vee \) since any plane tangent to \( \Gamma \) is tangent to it along a generator \( \ell \), hence it is also tangent to \( B \) at the points where \( \ell \) meets \( B \) (note that they are three distinct points for the general \( \ell \)). On the other hand, note that \( B^\vee \cap v^\vee \) is a hyperplane section of \( B^\vee \), so it has degree 18. Let \( \ell_i \) be a generator of \( \Gamma \) tangent to \( B \). The line parameterizing the pencil of planes through \( \ell_i \) is contained in \( B^\vee \cap v^\vee \). Actually any plane in such pencil cuts \( \Gamma \) along \( \ell_i + \ell_i' \), where \( \ell_i' \) is another generator. So, this plane is in \( v^\vee \) (since containing \( \ell_i \) it contains the vertex \( v \)); moreover it is in \( B^\vee \) (since \( \ell_i \) is a line tangent to \( B \)). Any such generator \( \ell_i \) does correspond to a branch point of the morphism \( p : B \to \gamma \), where \( \gamma = \mathbb{P}^1 \) is a directrix of \( \Gamma \). Since \( p \) has degree 3 and \( B \) has genus 4, by Riemann–Hurwitz formula we get that this number is 12. All this shows that the
intersection \( D_1 \cap D_2 \) is given (scheme theoretically) by \( 3D_0 \) plus 12 lines all tangent to \( D_0 \).

Assertion (1.1) is not true for \( D_i(X, V) \) when \( i > 0 \), as shown by the following examples.

(1.1.1) Examples.

(a) Consider the canonical system of a smooth hyperelliptic curve of genus \( g \geq 2 \). The discriminant locus consists of the union of \( D_0 \) and \( D_1 \). [LPS1, (1.8)]. In fact, \( D_0 = C^v \) is the dual variety of the corresponding rational normal curve \( C \subset \mathbb{P}^{g-1} \), \( D_1 \) is reducible, being the union of \( 2g + 2 \) linear spaces of dimension \( g - 2 \), and \( D_0 \setminus D_1 \neq \emptyset \neq D_1 \setminus D_0 \).

(b) Take \( r > 0 \) triplets as in (0.0), say \( (X_1, L_1, V_1), \ldots, (X_r, L_r, V_r) \), with the corresponding morphisms \( \phi_{V_i} : X_i \to \mathbb{P}^{N_i} \). Consider the product morphism:

\[
X = X_1 \times \cdots \times X_r \xrightarrow{\phi_{V_1} \times \cdots \times \phi_{V_r}} \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_r}
\]

and compose with the Segre embedding to obtain \( F : X \to \mathbb{P}^N \). For the triplet \( (X, L = F^* \mathcal{O}_{\mathbb{P}^N}(1), V = F^* H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))) \) it is straightforward to check the following fact: \((x_1, \ldots, x_r) \in J_i(X, V)\) if and only if \( x_j \in J_{i_j}(X, V) \) for \( 1 \leq j \leq r \) and \( \Sigma_{j=1}^{r}(i_j) \geq i \). Let us comment some particular cases.

(b.1) Take \( r = 2 \), \( X_1 = C \), a smooth curve of genus \( g \) and \( |V_1| \) a base-point free pencil of degree \( d \) defining a \( d \)-to-1 map \( \phi_{V_1} : C \to \mathbb{P}^1 \). Choose \( X_2 = \mathbb{P}^{n-1} \), \( L_2 = \mathcal{O}_{\mathbb{P}^{n-1}}(1) \) and \( V_2 = H^0(X_2, L_2) \). For the triplet \( (X, F^* \mathcal{O}_{\mathbb{P}^N}(1), V = F^* H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))) \), \( D(X, V) \) is the union of \( D_0(X, V) = (\mathbb{P}^1 \times \mathbb{P}^{n-1})^v \) (that is, \( \mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}^v \)) and \( D_1(X, V) \), which is the union of \( s = 2g - 2 + 2d \) linear spaces of dimension \( N - 1 - (n - 1) = 2n - 1 - n = n - 1 \). Therefore, \( D_0(X, V) = \mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1} \), \( D_1(X, V) = \cup_{i=1}^{s} \mathbb{P}^{n-1}_i \subset D_0(X, V) \), and so \( D(X, V) = D_0(X, V) \).

(b.2) Now take \( r = 2 \) and two triplets \( (C_1, L_1, V_1) \) and \( (C_2, L_2, V_2) \) where \( C_1 \) and \( C_2 \) are smooth curves and, for \( i = 1, 2 \), \( |V_i| \) is a pencil of degree \( d_i \). Consider the corresponding morphisms \( \phi_{V_i} : C_i \to \mathbb{P}^1 \) and their ramification loci \( R_1 = \{c_1, \ldots, c_{s_1}\} \subset C_1 \) and \( R_2 = \{d_1, \ldots, d_{s_2}\} \subset C_2 \). For the triplet \( (X = C_1 \times C_2, F^* \mathcal{O}_{\mathbb{P}^3}(1), V = F^* H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))) \) we have: \( D_0(X, V) = (\mathbb{P}^1 \times \mathbb{P}^1)^v \) is an union of lines; \( J_1 = \{(c, d) \in C_1 \times C_2 : c \in R_1 \text{ or } d \in R_2\} \) and \( D_1(X, V) \subset D_0(X, V) \) is a union of planes. Note that \( D_2(X, V) \) is reducible and \( D(X, V) = D_0(X, V) \cup D_2(X, V) \).

(b.3) Let us recall here [LPS2, Example 4.2.4]. Consider \( C' \subset \mathbb{P}^2 \) an irreducible curve of degree \( \geq 4 \) whose singular locus is just a cusp. Call \( \nu : C' \to C' \) the desingularization. Take \( X_1 = C, \phi_{V_i} \) the composition of the desingularization with the inclusion \( C' \subset \mathbb{P}^2 \) and \( (X_2, L_2) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \). In this situation one can prove that \( D = D_0 \) since \( D_1 \) is a linear space of dimension three contained in \( D_0 \).

(c) Take a surface \( \Sigma \subset \mathbb{P}^N \) having only an even set of nodes as singularities. One can take the double cover \( \pi : S \to \Sigma \), branched exactly at the nodes. Here, “even”
just means the following: consider the blowing–up $Y \to \Sigma$ at the nodes, let $C_i$ be the $(-2)$-curve corresponding to the node $p_i$ ($i = 1, \ldots , \mu$), and let $\Delta = \sum_{i=1}^{\mu} C_i$. The set of nodes of $\Sigma$ is even if $\Delta \in 2\text{Pic}(Y)$. Under this condition, we can consider the smooth surface $X$, double cover of $Y$ branched along $\Delta$. Then the preimages on $X$ of the $C_i$’s are $(-1)$-curves, and by contracting them we finally get the smooth surface $S$ and the required double cover. Now let $L := \pi^*\Omega_{\Sigma}(1)$ and $V = \pi^*W$, where $|W|$ is the trace of $|\Omega_{\mathbb{P}^N}(1)|$ on $\Sigma$. Then for our $(S, L)$, $J_2$ consists of $\mu$ points ($\mu$ being the number of nodes of $\Sigma$). Moreover $J_1 \setminus J_2 = \emptyset$. So, $D_2$ consists of $\mu$ hyperplanes, $D_1$ is empty and, of course, $D_0$ is the dual of $\Sigma$.

This example is effective. Let $S = JC$ be the jacobian of a smooth curve $C$ of genus 2 and call $C$ again the image of the curve embedded in $JC$ via the usual Abel–Jacobi map. Note that $C$ is the theta divisor up to a translation, hence it is an ample divisor. Set $L := [2C]$. Then the ample line bundle $L$ is also spanned, as Reider’s theorem immediately shows; furthermore $L^2 = 8$ and $h^0(L) = \chi(L) = L^2/2 = 4$. Moreover, $\phi_L : S \to \mathbb{P}^3$ is a morphism of degree 2 onto the Kummer quartic surface $\Sigma$ having 16 nodes as singular locus [GH, pp. 785–786]. This morphism of degree 2 has exactly these 16 points as branch locus, as can be checked by a local computation. Then for this triplet $(S, L, H^0(S, L))$, $J_2$ consists of the preimages of these 16 points, while $J_1 \setminus J_2 = \emptyset$. Correspondingly, $D_2$ consists of 16 planes in $\mathbb{P}^{3V} = |V|$, and $D_1$ is empty. Note also that $D_0 = (\Sigma)^\vee = \Sigma \subset \mathbb{P}^3$, [GH, p. 784].

The following propositions generalize the fact that $\phi_V(X)^\vee = D_0(X, V)$. Concretely, any irreducible component of the discriminant locus is proved to be the dual variety of the image by $\phi_V$ of an irreducible component of a jumping set.

(1.2) Proposition. Let $(X, L, V)$ be a triplet as in (0.0) such that $\dim(X_i) = n-i$ and consider the irreducible components of maximal dimension of $X_i$, that is, $Y_{ij} \subseteq X_i$ ($1 \leq j \leq s_i$) such that $\dim(Y_{ij}) = n-i$. Then $\bigcup_{j=1}^{s_i} \phi_V(Y_{ij})^\vee \subseteq \mathcal{D}_i(X, V) \subseteq \mathcal{D}(X, V)$.

Proof. Take a general point $y \in Y_{ij}$. Since $y \in J_i \setminus J_{i+1}$ then $\text{rk}(d\phi_V(y)) = n-i$. Hence the kernel $K := \ker(d\phi_V(y))$ is a subspace of dimension $i$ of the Zariski tangent space $T_{X, y}$. The other hand $\dim(Y_{ij}) = n-i$. If $\dim(K \cap T_{Y_{ij}, y}) > 0$ then $\phi_V|_{Y_{ij}}$ is not finite, a contradiction. As a consequence of the previous discussion one can choose local coordinates $z_1, \ldots , z_n$ around $y$ such that: (i) $\partial s/\partial z_h = 0$ for $1 \leq h \leq i$ and for all $s \in V$; and (ii) $z_{i+1}, \ldots , z_n$ are local parameters for $Y_{ij}$. In this setting the vanishing of the derivatives with respect to $z_{i+1}, \ldots , z_n$ just means that the corresponding hyperplane in $\mathbb{P}^N$ is tangent to $\phi_V(Y_{ij})$ at $\phi_V(y)$.

Let us note that it can be $\dim(X_i) < n-i$ (for example in special projections of smooth projective varieties). We can refer to [LPS2, Example 4.2.5] where a surface for which $\dim(X_1) = 0$ and $X_2 = \emptyset$ is provided. In fact it will be a consequence of the next proposition that for [LPS2, Example 4.2.5] $\mathcal{D}_1 \subset \mathcal{D}_0$ and $\mathcal{D} = \mathcal{D}_0$.

(1.3) Proposition. Let $(X, L, V)$ be a triplet as in (0.0) and $\mathcal{D}'$ an irreducible
component of $D(X, V)$. Then there exists an index $i$ ($0 \leq i \leq n$) and an irreducible component $Y_{ij} \subseteq \mathfrak{X}_i$ such that $\dim(Y_{ij}) = n - i$ and $D' = \phi_V(Y_{ij})^\vee$.

**Proof.** Consider the following incidence correspondence

$$
\mathcal{Y}_{D'} := \{(x, \lbrack s \rbrack) \in X \times D' : j_1(s)(x) = 0\} \quad \xrightarrow{p_1} \quad X
$$

Let $\dim(D') = N - 1 - k'$. By [LPS2, Lemma (0.6)] the dimension of the generic fibre of $p_2$ is $k' = N - 1 - \dim(D')$ and so $\dim(\mathcal{Y}_{D'}) = N - 1$. Take a $(N - 1)$–dimensional irreducible component $\mathcal{Y}^0 \subseteq \mathcal{Y}_{D'}$ such that $p_2(\mathcal{Y}^0) = D'$. Let $i$ be the maximum integer such that $p_1(\mathcal{Y}^0) \subseteq \mathfrak{X}_i$, and consider a general $p \in p_1(\mathcal{Y}^0) \cap X_i$. As $p \in X_i$ it holds that $|V - 2p| \subseteq D(X, V)$ is a linear space $T_p$ of dimension $N - 1 - (n - i)$. Since $T_p \cap D' \neq \emptyset$ then $T_p \subseteq D'$. Whence the dimension of the general fibre of $p_1$ is $N - 1 - (n - i)$. In particular, $\dim(X_i) = n - i$ and there exists an irreducible component $Y_{ij} \subseteq \mathfrak{X}_i$ such that $Y_{ij} = p_1(\mathcal{Y}^0)$. Just by definition of dual variety, $\phi_V(Y_{ij})^\vee = D'$. \hfill \Box

(1.3.1) With the same notation as in the proof of (1.3) we have maps $D' \xleftarrow{p_2} \mathcal{Y}^0 \xrightarrow{p_1} X$ where $\mathcal{Y}^0$ is characterized by the following properties: irreducibility, $\dim(\mathcal{Y}^0) = N - 1$ and $p_2(\mathcal{Y}^0) = D'$. Moreover, by classical biduality theorem, any other $(N - 1)$–dimensional irreducible component $\mathcal{Y}^1 \subseteq \mathcal{Y}_{D'}$ such that $p_2(\mathcal{Y}^1) = D'$ verifies $\phi_V(p_1(\mathcal{Y}^1)) = \phi_V(Y_{ij})$.

As a consequence we have the following statement, analogous to the classical **biduality theorem**, offering in particular some control on the linear components of $D(X, V)$.

**1.4) Biduality Theorem.** Let $(X, L, V)$ be a triplet as in (0.0). Then $(D')^\vee \subseteq \phi_V(X)$ for any irreducible component $D' \subseteq D(X, V)$.

**Proof.** From (1.3) and with the same notation as there it holds that $D' = \phi_V(Y_{ij})^\vee$. Then our result is a consequence of the classical biduality theorem, that is, $(D')^\vee = (\phi_V(Y_{ij})^\vee)^\vee = \phi_V(Y_{ij}) \subseteq \phi_V(\mathfrak{X}_i) \subseteq \phi_V(X)$. \hfill \Box

(1.4.1) Let $(X, L, V)$ be a triplet as in (0.0). Suppose there exists a linear irreducible component $\mathbb{P}^{N-1-r} \subseteq D(X, V)$. By biduality $\phi_V(X)$ contains a linear space $T$ of dimension $r$ and by (1.3) there exists $Y \subset J_{n-r}$ such that $T = \phi_V(Y)$. It is of particular interest the fact that if $D(X, V)$ contains a hyperplane, then $J_n \neq \emptyset$. In fact, any hyperplane in $D(X, V)$ defines a point in $J_n$. Note that the converse is obvious, because $|V - x| \subset D(X, V)$ if $x \in J_n$. So we have the following

**1.4.2) Corollary.** Let $(X, L, V)$ be a triplet as in (0.0). Then $D(X, V)$ contains a hyperplane if and only if $J_n \neq \emptyset$. 


(1.5) Lemma. Let \((X, L, V)\) be a triplet as in (0.0) and let \(\dim(D(X, V)) = N - 1 - k\). Then \(J_i = \emptyset\) for \(i > n - k\). Moreover, if \(\dim(X_{n-k}) = k\) then any maximal dimensional component of \(\phi_V(X_{n-k})\) is linear.

Proof. Let us suppose there exists \(i > n - k\) for which \(X_i \neq \emptyset\). For any \(p \in X_i\) we get \(|V - 2p| = \mathbb{P}^{N-1-(n-i)} \subseteq D(X, V)\) a contradiction. The last assertion is just a consequence of (1.3). \(\square\)

Let \((X, L, V)\) and \((Y, M, W)\) be two triplets as in (0.0) such that \(\dim(V) = \dim(W) = N + 1\). In the classical case, that is, \(\phi_V\) and \(\phi_W\) embeddings, the biduality theorem states that if \(D(X, V) = D(Y, W) \subset \mathbb{P}^N\) (that is, there exists a linear transformation of \(\mathbb{P}^N\) sending isomorphically \(D(X, V)\) to \(D(Y, W)\)) then \((X, L) = (Y, M)\). It is natural to ask to what extent this theorem is true when \(\phi_V\) or \(\phi_W\) are not embeddings. Next examples show that it cannot be true in the same terms and the right hypotheses to impose.

(1.6) Examples.

(a) Choose \(X_1\) a smooth elliptic curve and \(L_1\) giving a \(g_1^1 = |V_1|\) on \(X_1\). Take \((X_2, L_2, V_2) = (\mathbb{P}^1, O_{\mathbb{P}^1}(1), H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1)))\). As in (b) of (1.1.1) we have \((X, L)\) such that \(\phi_V(X) \subset \mathbb{P}^3\) is a smooth quadric and the branch locus of \(\phi_V\) is made of four disjoint lines on \(\phi_V(X)\). Hence \(D(X, V)\) is a smooth quadric in \(\mathbb{P}^3\). For \((Y = Q, L = O_Q(1), V = H^0(Y, L))\) a smooth quadric with its corresponding embedding in \(\mathbb{P}^3\) we have \(D(X, V) = D(Y, W)\), \(\phi_V(X) = \phi_W(Y) \subset \mathbb{P}^3\) but \(X\) and \(Y\) are not isomorphic.

(b) Choose \(X_1\) a smooth conic such that \((X_1, L_1, V_1 = H^0(X_1, L_1))\) defines the embedding \(\phi_{V_1}(X_1) \subset \mathbb{P}^2\). Take \(X_2\) a double cover of the plane, \(f : X_2 \rightarrow \mathbb{P}^2\), branched along \(\phi_{V_1}(X_1, L_2 = f^*O_{\mathbb{P}^2}(1), V_2 = f^*H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(1)))\). Then \(\phi_{V_1}(X_1) \subset \mathbb{P}^2\) is a smooth conic and \(\phi_{V_2}(X_2) = \mathbb{P}^2\). In fact \(\phi_{V_1}(X_1)\) and \(\phi_{V_2}(X_2)\) are not isomorphic but \(D(X_1, H^0(X_1, L_1)) = D(X_2, V_2) = \phi_{V_1}(X_1)^\vee\) a smooth conic.

(c) Consider two smooth plane curves \(C_1, C_2 \subset \mathbb{P}^2\), not isomorphic. Let \(f_1 : X_1 \rightarrow C_1 \times \mathbb{P}^2\) and \(f_2 : X_2 \rightarrow C_2 \times \mathbb{P}^2\) be cyclic covers, both branched along \(C_1 \times C_2\). Let \(F_i\) be the composition of the Segre embedding \(\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8\) with \(f_i\). Then we get triplets \((X, L_i = F_i^*O_{\mathbb{P}^8}(1), V_i = F_i^*H^0(\mathbb{P}^8, O_{\mathbb{P}^8}(1)))\), \(i = 1, 2\). Whence \(D(X_1, V_1) = D_0 \cup D_1\) where \(D_0 = (C_1 \times \mathbb{P}^2)^\vee\) and \(D_1 = (C_1 \times C_2)^\vee\). We know \(\dim(D_0) = 6\) (since \(D_0\) is the dual of a three-dimensional scroll over a curve). We claim that \(D_0 \subset D_1\). In fact a general element of \(D_0\) corresponds to a hyperplane \(H \in (C_1 \times \mathbb{P}^2)^\vee\) which is tangent to \(C_1 \times \mathbb{P}^2\) along a line contained in a fiber \(f\). Since this line is meeting \(f \cap (\mathbb{P}^2 \times C_2) = C_2\) then \(H \in (C_1 \times C_2)^\vee\). Hence \(D(X_1, V_1) = (C_1 \times C_2)^\vee\). In the same way we see that \(D(X_2, V_2) = (C_1 \times C_2)^\vee\). Note however that \(\phi_{V_1}(X_1) = C_1 \times \mathbb{P}^2\) is not isomorphic to \(\phi_{V_2}(X_2) = \mathbb{P}^2 \times C_2\).

(1.7) Proposition. Let \((X, L, V)\) and \((Y, M, W)\) be two triplets as in (0.0) such that \(\dim(X) = n = \dim(Y)\) and \(\dim(V) = \dim(W) = N + 1\). If \(D_0(X, V)\) is an irreducible component of \(D(Y, W)\) then \(\phi_V(X) = \phi_W(Y) \subset \mathbb{P}^N\).
Proof. Since $D_0(X, V) = \phi_V(X)^\vee$ is a component of $D(Y, W)$ then, by (1.3), there exists $Y_{ij} \subset Y$ such that $\phi_V(X)^\vee = \phi_V(Y_{ij})^\vee$. By the classical biduality theorem $\phi_V(X) = \phi_V(Y_{ij})$. This gives $\dim(\phi_V(Y_{ij})) = \dim(Y)$ and so $Y_{ij} = Y$. □

2. IRREDUCIBILITY OF THE DISCRIMINANT LOCUS

In this section we study some consequences of the irreducibility of the discriminant locus. A general fact is the following.

(2.1) If $\phi_V$ is an immersion then $D(X, V) = D_0$ and so it is irreducible, see (1.1).

The converse of (2.1) is also true for curves. We need the following lemma.

(2.2) Lemma. Let $(X, L, V)$ be as in (0.0) such that $D(X, V)$ is irreducible and $\mathcal{J}_n \neq \emptyset$, then:

(2.2.1) $\mathcal{J}_n$ is a finite set and $\phi_V(x) = \phi_V(y)$ for any $x, y \in \mathcal{J}_n$;

(2.2.2) $D(X, V)$ is a hyperplane of $|V|$ and $\phi_V(X)$ is a cone whose vertex contains $\phi_V(\mathcal{J}_n)$.

Proof. By [LPS1, Theorem 1.2] $\dim(\mathcal{J}_n) = 0$. For any $x \in \mathcal{J}_n$ one has $\mathbb{P}^{N-1} = |V - x| \subseteq D(X, V)$. Since $D(X, V)$ is irreducible then $|V - x| = |V - y|$ for any $x, y \in \mathcal{J}_n$ and (2.2.1) follows. Moreover $D(X, V) = D_n = |V - x|$ for any $x \in \mathcal{J}_n$. Since $\phi_V(X)^\vee = D_0 \subseteq D(X, V)$ then it is either empty or non-empty and degenerate (in the sense that it is contained in a hyperplane of $\mathbb{P}^N$). If empty then $\phi_V(X) = \mathbb{P}^N$ and $N = n$. So, $\phi_V(X) = \mathbb{P}^n$ is a linear cone. If non-empty and degenerate then $\phi_V(X) \subset \mathbb{P}^N$ is a cone whose vertex contains $\phi_V(x)$ for any $x \in \mathcal{J}_n$. □

We will see in (3.5) that (2.2.2) cannot occur.

(2.3) Remark. For $C$ a smooth irreducible curve, it is not possible to construct a finite morphism $\pi : C \to \mathbb{P}^1$ of degree $d \geq 2$ with a single branch point $p \in \mathbb{P}^1$. Let us call $m$ the number of distinct points in $\pi^{-1}(p)$. The claim is just a consequence of the Riemann–Hurwitz formula: $2g(C) - 2 = -2d + d - m$.

We can prove the following result for curves.

(2.4) Proposition. Let $(C, L, V)$ be as in (0.0) with $\dim(C) = 1$. Then $D(C, V)$ is irreducible if and only if $\phi_V$ is an immersion.

Proof. In view of (2.1) it is enough to prove the only if part. So let us assume $D(C, V)$ to be irreducible. If $\mathcal{J}_1 \neq \emptyset$ then $\phi_V(C) = \mathbb{P}^1$ by (2.2.2). The contradiction comes from (2.2.1) because $\mathcal{J}_1 \neq \emptyset$ implies that one can construct a map as in (2.3). □

This statement is not true for higher dimension. In the examples (b.1) and (b.3) of (1.1.1) (see [LPS2, Example 4.2.4]) $D(X, V)$ is not only irreducible but
equal to $\phi_V(X)^V$ not being $\phi_V$ an immersion. Let us record a list of examples of surfaces with irreducible discriminant locus showing different behaviors of the $D_i'$s. Since scrolls are of particular interest, first consider the following result, that is important also for the next sections. We follow the usual notation of [Ha, V 2].

(2.5) Lemma. Let $(S, L, V)$ be a triplet as in (0.0) such that $\dim(S) = 2$ and $(S, L)$ is a scroll over a smooth curve $B$. Let $C_0$ and $f$ be a fundamental section and a fibre respectively. Then

(2.5.1) $L \equiv C_0 + bf$, with $b > 0$.
(2.5.2) $J_2 = \emptyset$.
(2.5.3) If $\dim(V) = 3$ there is a surjection $i: B \to D(S, V)$. In particular $i$ is an isomorphism if $b = 1$.

Proof. Write $S = \mathbb{P}(E)$, where $E$ is a vector bundle of rank 2 on $B$. We can assume that $E$ is normalized in the sense of [Ha, p. 373] and that $C_0$ is the tautological section on $S$. Of course $L \equiv C_0 + bf$ for some integer $b$, since $(S, L)$ is a scroll. Let $\pi: S \to B$ be the scroll projection. Since $L$ is ample and the general element in $|L|$ is irreducible then $b \geq 0$ by [Ha, V Prop. 2.20 and 2.21]. It is not hard to see that $b = 0$ implies that $L$ is not globally generated and this proves (2.5.1). Alternatively an argument based on the non emptiness of the discriminant locus can be given. Take $D \in \mathcal{D}(S, V)$, $\mathcal{D}(S, V) \neq \emptyset$ by [LPS, Thm. 2.8)]. Choose $x \in \operatorname{Sing}(D)$, and let $f_{\pi(x)} = \pi^*\mathcal{O}_B(\pi(x))$ be the fibre of $S$ containing $x$. Then

(2.5.4) $D = f_{\pi(x)} + R$

for some effective divisor $R \equiv C_0 + (b - 1)f$ containing $x$. Otherwise $1 = Lf = Df_{\pi(x)} \geq \operatorname{mult}_x(D)\operatorname{mult}_x(f_{\pi(x)}) \geq 2$, a contradiction. Suppose that $b \leq 0$. This would mean that $0 < h^0(R) = h^0(E \otimes \mathcal{L})$ where $\mathcal{L}$ is a line bundle on $B$ with $\deg(\mathcal{L}) < 0$, contradicting the assumption that $E$ is normalized made at the beginning.

To prove (2.5.2), assume that $x \notin J_2$. Then $|V - x| = |V - 2x|$. As we have seen before any $D \in |V - 2x|$ is as in (2.5.4). Hence $|V - 2x| = f_{\pi(x)} + |V - f_{\pi(x)} - x|$. Then $f_{\pi(x)}$ would be contained in the base locus of $|V - x|$. But $\operatorname{Bs}(|V - x|) = \phi_V^{-1}(\phi_V(x))$ must be a finite set since $L$ is ample and spanned by $V$. This gives a contradiction.

Now assume that $\dim(V) = 3$, so that $\phi_V(S) = \mathbb{P}^2$. For every $p \in B$ let $x, y$ be any two distinct points lying on the fibre $f_p = \pi^{-1}(p)$. So $|V - x - y|$ consists of a single element $D_p$, and $D_p = f_p + R_p$ for an effective $R_p$ such that $R_p \equiv C_0 + (b - 1)f$. In particular, $D_p$ has a (exactly one) singular point on $f_p$. Then the mapping $i(p) = D_p$ defines a morphism $i: B \to \mathcal{D}(S, V)$. Now pick an element $D \in \mathcal{D}(S, V)$ and let $x$ be a singular point of $D$. By (2.5.4) it holds that $D = i(\pi(x))$. Then $i$ is surjective. If $b = 1$ then $i$ is also injective. If $i(p) = i(q)$ with $p \neq q$, then $D_p = D_q$; in particular $D_q - f_p - f_q \equiv C_0 - f$ would be effective, a contradiction. $\square$
In the previous lemma we have shown that for any \( N = \dim(|V|) \) we have the following maps:

\[
B := \{(b, D) \in B \times \mathcal{D}(S, V) : \pi^{-1}(b) \subset D\} \xrightarrow{\pi_1} B
\]

where any fibre of \( \pi_1 \) is a linear space of dimension \( N - 2 \). If (2.5.3) holds, \( \pi_1 \) is an isomorphism since \( N = 2 \) and \( i = \pi_2 \circ \pi_1^{-1} \).

We are going to show several examples of surfaces whose discriminant locus is irreducible. The list shows that any possible relation between \( D_0, D_1 \) and \( D \) can occur. In fact (2.4) is no longer true when the dimension is bigger than one. In the following examples \( D_2 = \emptyset \), being \( D \) irreducible. This is a general fact, as it will be proved later, see (2.7).

(2.6) Examples.

(a) Take \( C \subset \mathbb{P}^3 \) an irreducible non-degenerate smooth curve. As in [LM, Example 3.2] consider the conormal variety \( X = \{(c, H) : T_{C,c} \subset H\} \subset C \times C^\vee \) and the corresponding projections \( \pi_1 \) and \( \pi_2 \). The triplet

\[
(X, \pi_2^*\mathcal{O}_{\mathbb{P}^3}(1), \pi_2^*H^0(\mathbb{P}^3^\vee, \mathcal{O}_{\mathbb{P}^3}(1))
\]

is as in (0.0). Recall that \( X \) is a \( \mathbb{P}^1 \)-bundle over \( C \). A local computation shows that \( J_1 \) consists of a section plus the fibres over the hyperflexes of \( C \). In fact the section corresponds to \( \{(c, \text{Osc}_c^2(C)) : c \in U\} \subset C \times C^\vee \), where \( \text{Osc}_c^2(C) \) stands for the second osculating projective linear space to \( C \) at \( c \in C \) and \( U \subset C \) is the open subset of points of \( C \) where the osculating plane is defined. One can check that \( D \) consists of the tangent developable \( TC \) of \( C \) plus the lines corresponding to the hyperflexes. Then \( D_0 = (C^\vee)^\vee \subset D = D_1 \).

(b) Cyclic coverings of \( \mathbb{P}^2 \) give rise to: \( \emptyset = D_0 \subset D_1 = D = B^\vee \) where \( B \) is the branch locus of the covering.

(c) In the example (b.3) of (1.1.1), see [LPS2, Example 4.2.4], we have \( \emptyset \neq D_1 \subset D_0 = D \). If \( \phi_V \) is an immersion then \( \emptyset = D_1 \subset D_0 = D \).

(d) Let \( B \) be a smooth elliptic curve and \( p \in B \). Consider the rank two vector bundle \( \mathcal{E} \) defined as the non-trivial extension: \( 0 \to \mathcal{O}_B \to \mathcal{E} \to \mathcal{O}_B(p) \to 0 \). Set \( S = \mathbb{P}(\mathcal{E}), L \equiv C_0 + f \) and \( V = H^0(S, L) \). One can check that \( h^0(S, L) = 3 \) and \( L \) is ample and spanned. By (2.5) there is an isomorphism between \( B \) and \( \mathcal{D}(S, V) \). Then \( \phi_V : S \to \mathbb{P}^2 \) is a degree 3 map whose branch locus is the dual of the smooth plane cubic \( \mathcal{D}(X, V) \). In this case \( \emptyset = D_0 \subset D_1 = D \).

The remainder of this section is devoted to prove that irreducibility of the discriminant locus implies emptiness of the maximal jumping set. As (2.6) shows in the case of surfaces, this is in principle the only consequence of irreducibility of the discriminant locus one can expect in general.
(2.7) Lemma. Let \((X, L, V)\) be as in (0.0). If \(\mathcal{D}(X, V)\) is irreducible then \(\mathcal{J}_n = \emptyset\).

Proof. If \(\mathcal{J}_n \neq \emptyset\) and \(\mathcal{D}(X, V)\) is irreducible then \(\mathcal{D}(X, V) = \mathbb{P}^{N-1} = |V - x|\) for any \(x \in \mathcal{J}_n\). Consider a general \(W \subseteq V\) such that \(\dim(W) = n + 1\) and note that \(W\) spans \(L\). Then \(\mathcal{D}(X, W) = |W - x|\) and \(\phi_W(X) = \mathbb{P}^n\). In particular there exists \(p \in \phi_W(X)\) such that \(p = \phi_W(x)\) for any \(x \in \mathcal{J}_n\). Moreover, for this choice of \(W\), \(\mathcal{J}_1 = \mathcal{J}_1(W)\) is a divisor on \(X\) and for any component \(\mathcal{J}_1^i \subseteq \mathcal{J}_1\) we have \(\phi_W(\mathcal{J}_1^i)^\vee \subseteq \mathcal{D}(X, W)\). That is, \(\phi_W(\mathcal{J}_1^i)^\vee\) is contained in the hyperplane of \(\mathbb{P}^N\) of hyperplanes of \(\mathbb{P}^N\) through \(p\). Then

\((2.7.1)\) \(\phi_W(\mathcal{J}_1)\) is a union of cones with vertex containing \(p\).

Choose a general line \(T \subset \mathbb{P}^n\) through \(p\). Since \(T \cap \phi_W(\mathcal{J}_1) = \{p\}\) then, by Bertini type theorems, \(\phi_W^{-1}(T)\) is a curve whose singular locus is contained in \(\phi_W^{-1}(p)\). In fact, consider a general hyperplane \(H \subset \mathbb{P}^n\) and define \(f : X \setminus \mathcal{J}_1 \to H \setminus \phi_W(\mathcal{J}_1)\) as \(f(z) = \langle \phi_W(z), \phi_W(p) \rangle \cap H\). Hence \(T \setminus \phi_W^{-1}(p)\) is smooth because it is a general fibre of \(f\). Then, consider any component \(\Gamma_i \subset \phi_W^{-1}(T)\) and let \(\mu_i : \Gamma_i \to \mathbb{P}^1\) be its desingularization. The morphism \(\phi_W \circ \mu : \gamma_i \to \mathbb{P}^1\) has only one branch point, so, by (2.3), it is an isomorphism. This says that \(\Gamma_i\) is a smooth rational curve such that \(\text{Ld}_i = 1\). This implies in particular that \(X\) is swept out by lines, that is, there exists a family \(\mathcal{T}\) of smooth rational curves of \(L\)-degree 1, sweeping out \(X\). Moreover, since \(\phi_V^{-1}(p)\) is finite, there exists \(x \in \phi_V^{-1}(p)\) such that \(x \in \ell\) for \(\ell\) general in \(\mathcal{T}\). By [LPS2, Lemma 3.1] the normal bundle \(N_{\ell/X}\) splits as \(O_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus O_{\mathbb{P}^1}(a_{n-1})\) with \(0 \leq a_1 \leq \cdots \leq a_{n-1}\). Let us suppose that \(a_1 = \cdots = a_t = 0\), \(a_{t+1} > 0\). Since \(\phi_W(X) = \mathbb{P}^n\) and \(x \in \ell\) for \(\ell\) general we conclude that the irreducible component \(\mathcal{T}_x\) through \(\ell\) of the Hilbert scheme of rational curves of \(L\)-degree 1 on \(X\) through \(x\) has dimension greater than or equal to \(n - 1\). Since \(h^1(\ell, N_{\ell/X}(-1)) = 0\) then \(\mathcal{T}_x\) is smooth; moreover \(h^0(\ell, N_{\ell/X}(-1)) = a_{t+1} + \cdots + a_{n-1} \geq n - 1\). Then \(h^0(\ell, N_{\ell/X}) \geq n - 1 + n - 1 = 2n - 2\). This implies \((X, L) = (\mathbb{P}^n, O_{\mathbb{P}^n}(1))\) by [LP2, Thm. 1.4], a contradiction. \(\square\)

3. Codegree

The definition of codegree can be established as in the classical case, see [Z1].

(3.1) Definition. For \((X, L, V)\) as in (0.0) define its codegree, say \(\text{codeg}(X, V)\), as the degree of \(\mathcal{D}(X, V) \subset \mathbb{P}^{N\vee}\).

As said in (0.2), \(\mathcal{D}(X, V)\) is an algebraic subset of \(|V|\) with its reduced structure; hence \(\text{codeg}(X, V) = \Sigma_{\dim(D) = \max} \deg(D^i)\), the sum ranging over all irreducible components of maximal dimension.

As in the classical case we can relate the codegree of \((X, L, V)\) with the Chern classes of the jet bundle \(J_1(L)\). Suppose \(\dim(\mathcal{D}(X, V)) = N - 1\) and consider the maximal dimensional components of its discriminant locus, say \(D^1, \ldots, D^s \subseteq \mathcal{D}(X, V)\). Let \(d_i = \deg(D^i)\). For \(D \in D^i, 1 \leq i \leq s\), with isolated singularities take \(s \in V\) the section defining \(D\) and \(x \in \text{Sing}(D)\). Define the 0–cycle \(z_x = \mu_x(D)x\),
where $\mu_x(D)$ is the Milnor number of $x$ as an isolated singular point of $D$, and $z(D) = \Sigma_{x \in \text{Sing}(D)} z_x$. Then we can prove the following.

(3.2) **Theorem.** Let $(X, L, V)$ be as in (0.0) with $\dim(D) = N - 1$ and consider a general pencil $|W| \subseteq |V|$. With the notation of the previous paragraph, let $|W| \cap D^i = \{D_j^i\}$, $(j = 1, \ldots, d_i)$ and consider the 0–cycles $z(D_j^i)$. Then $c_n(J_1(L)) = \Sigma_{i=1}^{s} \Sigma_{j=1}^{d_i} z(D_j^i)$.

**Proof.** From [LPS1, Cor. 2.6] we know that $c_n(J_1(L))$ is represented by $j_1(W)^{-1}(0)$. For any $D_j^i \in |W| \cap D^i$ and for any $x \in \text{Sing}(D_j^i)$ we can take $s \in V$ the section defining $D_j^i$ and $t \in V$ not vanishing at $x$ such that $W = \langle s, t \rangle$. Now we can use the notation of [LPS2, Prop. 1.1], i.e., there exist local coordinates $x_1, \ldots, x_n$ at $x \in X$ such that $s = \Sigma_{j=1}^{n} a_{ij} x_i x_j + \text{h.o.t.}$ (higher order terms) and $j_1(t) = (1, 0, \ldots, 0)$ since $t$ does not vanish at $x$. Hence $j_1(t) \wedge j_1(s) = \left( \frac{\partial s}{\partial x_1}, \ldots, \frac{\partial s}{\partial x_n}, 0, \ldots, 0 \right)$. This shows that the zero sub-scheme of $j_1(W)^{-1}(0)$ supported at $x$ is defined by the Jacobian ideal $\left( \frac{\partial s}{\partial x_1}, \ldots, \frac{\partial s}{\partial x_n} \right)$. Hence this 0–cycle has to be $z_x = \mu_x(D_j^i) x$, where $\mu_x(D_j^i)$ is the Milnor number of $x$ as an isolated singular point of $D_j^i$. This gives the equality of 0–cycles of the statement. $\square$

From now on $c_n(J_1(L))$ will stand for the degree of the corresponding 0–cycle. With the previous notation take a general $D \in D^l$ where $1 \leq l \leq s$. Let us describe more explicitly the singularities of $D$. By (1.3) (with the notation there) and the classical biduality theorem, $D$ defines an element of $\phi_V(Y_{i,j,l})^\vee$ whose contact locus is a single point $q \in \phi_V(Y_{i,j,l})$. Then we claim that the singular locus of $D$ is confined to $\phi_V^{-1}(q) = \{x_1, \ldots, x_m\}$, that is,

\begin{equation}
(3.2.1) \quad \text{Sing}(D) \subseteq \phi_V^{-1}(q).
\end{equation}

Since $D$ corresponds to an element of $\phi_V(Y_{i,j,l})^\vee$ there exists $x \in \phi_V^{-1}(q)$ such that $D \in \langle |V| - 2y \rangle$. Consider $y \neq x$ such that $D \in \langle |V| - 2y \rangle$. Since $D \in \langle |V| - 2y \rangle \cap D^l$, $|V| - 2y \subset D^l$. By (1.3) $y \in X_j, j \geq i$. We have then an irreducible component $Y_{j,k} \subset \overline{X}_j$ such that $\dim(Y_{j,k}) = n - j$ and $D^l = \phi_V(Y_{i,k,l})^\vee = \phi_V(Y_{j,k})^\vee$. If $j > i$, by the classical biduality theorem, we get the contradiction $\phi_V(Y_{i,k,l}) = \phi_V(Y_{j,k})$. Hence $y \in \overline{X}_i$ and $\dim(|V| - 2y|) = N - 1 - (n - i_l)$. If $y$ is a smooth point of $X_{i_l}$ then there exists $Y_{i,k,l} \subset \overline{X}_i$ with $\dim(Y_{i,k,l}) = n - i_l$ such that $\phi_V(Y_{i,k,l})^\vee = \phi_V(Y_{i,k,l})^\vee$. Then $\phi_V(Y_{i,k,l}) = \phi_V(Y_{i,k,l})$ by the the classical biduality theorem, as pointed out in (1.3.1). In particular $\phi_V(x) = \phi_V(y)$ because the contact locus of a general element on $\phi_V(Y_{i,k,l})$ just is a point. This proves the claim unless $y$ is a singular point of $\overline{X}_i$ and $\dim(|V| - 2y|) = N - 1 - (n - i_l)$. If this occurs, since $\dim(\text{Sing}(\overline{X}_i)) < n - i_l$, we get the contradiction $N - 1 = \dim(D^l) = N - 1 - (n - i_l) + \dim(\text{Sing}(\overline{X}_i)) < N - 1$.

By the previous discussion there exists an index $k, 1 \leq k \leq m$, such that, after reordering if necessary, $\text{Sing}(D) = \{x_1, \ldots, x_k\} \subset X_i \cap \phi_V^{-1}(q)$ and $z(D) = \mu_{x_1}(D)x_1 + \cdots + \mu_{x_k}(D)x_k$. By semicontinuity the degree of this 0–cycle is constant at the general point of $D^l$. Let $m_l = \deg(z(D))$ for $D \in D^l$ general. Hence, by
(3.2) and recalling that \( \text{codeg}(X, V) = d_1 + \cdots + d_s \), we get

\begin{equation}
(3.2.2) \quad c_n(J_1(L)) = m_1d_1 + \cdots + m_sd_s \geq \text{codeg}(X, V). \tag{3.2.2}
\end{equation}

In particular, if the discriminant locus has just one maximal dimensional irreducible component then there exists a positive integer \( m \) such that

\begin{equation}
(3.2.3) \quad c_n(J_1(L)) = m \text{ codeg}(X, V). \tag{3.2.3}
\end{equation}

Classical results on Milnor numbers can be applied, for instance, see [DJP, Thm. 3.4.29, p. 122]:

\begin{equation}
(3.2.4) \quad z_x = x \text{ if and only if } x \text{ is an isolated non-degenerate quadratic singularity of } D_j. \tag{3.2.4}
\end{equation}

In the classical case, see for example [BS, Rmk. 1.6.11, p. 33], when the dual variety is a hypersurface the general singular hyperplane section has only an isolated non-degenerate quadratic singularity, so that \( c_n(J_1(L)) = \text{codeg}(X, V) \), being \( m = 1 \) in (3.2.3).

Let us give a bound when the dual of the image of \( X \) by \( \phi_V \) is the only maximal dimensional irreducible component of the discriminant locus.

\begin{equation}
(3.2.5) \quad \text{Lemma. Let } (X, L, V) \text{ be as in (0.0) with } \dim(D) = N - 1 - k. \text{ If } \phi_V(X)^{\circ} \text{ is the only } (N - 1 - k)\text{-dimensional irreducible component of } D \text{ then } c_n(J_1(L)) \leq \frac{\text{codeg}(X, V) L^n}{\deg(\phi_V(X))}. \tag{3.2.5}
\end{equation}

Proof. If \( k > 0 \) then \( c_n(J_1(L)) = 0 \) and the assertion is obvious. If \( k = 0 \) then (3.2.4) and the fact that the general element of \( D(X, V) \) has only isolated non-degenerate quadratic singularities lead to the inequality \( m \leq \deg(\phi_V) = L^n/\deg(\phi_V(X)) \) in (3.2.3) and the assertion follows. \( \square \)

Let \( (X, L, V) \) be as in (0.0) with \( J_2 = \emptyset \) and \( \phi_V(J_1) \) not contained in the singular locus of \( \phi_V(X) \). As in [BDL, Lemma 1 (3)], for general \( y \in \phi_V(J_1) \) and \( x \in \phi_V^{-1}(y) \) we can choose local coordinates \( x_1, \ldots, x_n \) on \( X \) centered at \( x \) and \( y_1, \ldots, y_n \) on \( \phi_V(X) \) centered at \( y \) to write \( y_1 = x_1^k, y_2 = x_2, \ldots, y_n = x_n \), the branch locus locally being defined by \( y_1 = 0 \). Then, an element \( D \in D^1 \) singular at \( x \) is defined by \( s(x_1^k, \ldots, x_n) \) where \( s(y_1, \ldots, y_n) \) defines the corresponding hyperplane section through \( y \). Recall that \( y = (0, \ldots, 0) \) is a smooth point of \( s(y_1, \ldots, y_n) = 0 \) but \( s(x_1^k, x_2, \ldots, x_n) = 0 \) is singular at \( x = (0, \ldots, 0) \). Hence \( s(y_1, \ldots, y_n) = y_1 + \text{h.o.t.} \).

In fact, \( s(x_1^k, x_2, \ldots, x_n) = x_1^k + \text{h.o.t.} \). In particular, suppose \( n = N = 2 \) and \( J_2 = \emptyset \). Since \( s(y_1, y_2) = 0 \) is the tangent line to the branch locus at the (general) point \( y = (0, 0) \) then \( s(y_1, y_2) = y_1 + \alpha y_2^2 + \text{h.o.t., where } \alpha \in \mathbb{C} - \{0\} \). Hence:

\begin{equation}
(3.2.6) \quad s(x_1^k, x_2) = x_1^k + \alpha x_2^2 + \text{h.o.t.} \quad \text{and so } \mu_x(D) = k - 1,
\end{equation}
relating the index of ramification of $\phi_V$ at $x$ and the Milnor number of the singularity $x$. Of course $k \leq \deg \phi_V \leq L^2$; then, specializing (3.2) in this particular setting, we obtain the bound:

$$(3.2.7) \quad c_2(J_1(L)) \leq \text{codeg}(X, V)(L^2 - 1),$$

which is sharp as example (b) in (3.3) will show.

If $\dim(D(X, V)) = N - 1 - k$, $k > 0$, then to get an expression as in (3.2) we have to consider $\zeta(D^j_i)$, the $k$-cycle corresponding to the singular locus of a general point $D^j_i \in D^i$, $\dim(D^i) = N - 1 - k$, counted with its appropriate number. Generalizations of the Milnor number are naturally considered, see [A]. Let us produce some examples when $k = 0$.

### (3.3) Examples.

**a)** Consider the example (1.1.0): $S$ is a a Del Pezzo surface with $K_S^2 = 1$ and $L = -2K_S$. We know that $\phi_L : S \rightarrow \Gamma \subset \mathbb{P}^3$ is a double cover of the quadric cone $\Gamma$, branched at the vertex $v$ and along the smooth curve $B$ cut out on $\Gamma$ by a transverse cubic surface. As explained in (1.1.0) the maximal dimensional components of $D(S, L)$ are $D_1 = B^\vee$, of degree 18, and the plane $D_2 = v^\vee$. Note that $D_0 = \gamma^\vee$ is a conic, the dual of $\gamma$, contained in $D_1 \cap D_2$. Hence $\text{codeg}(S, L) = \deg(D_1) + \deg(D_2) = 19$. On the other hand, since $S$ is obtained by blowing-up $\mathbb{P}^2$ at 8 points, the Euler–Poincaré characteristic of $S$ is $e(S) = 11$. Moreover, $L^2 = 4$ and genus formula shows that $g(L) = 2$. Thus $c_2(J_1(L)) = 11 + 4 + 4 = 19$. Whence $c_2(J_1(L)) = \text{codeg}(X, V)$ which implies that the general element in $D_1$ as well as that in $D_2$ has a single isolated non-degenerate quadratic singularity.

**b)** Let $\pi : S \rightarrow \mathbb{P}^2$ be a cyclic cover of degree $d$ branched along a smooth curve $\Delta \in |\mathcal{O}_{\mathbb{P}^2}(bd)|$. Let $L := \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. We have $K_S = \pi^*\mathcal{O}_{\mathbb{P}^2}(b(d - 1) - 3)$ by the ramification formula, and, for $i > 0$ $h^i(\mathcal{O}_S) = h^i(\mathcal{O}_{\mathbb{P}^2}) + h^i(\mathcal{O}_{\mathbb{P}^2}(-b)) + \cdots + h^i(\mathcal{O}_{\mathbb{P}^2}(-b(d - 1)))$ by the projection formula. Thus $q(S) = 0$, while $p_g(S) = \sum_{i=1}^{d-1} \binom{i(b-1)}{2}$. Therefore $12\chi(\mathcal{O}_S) = b^2(d - 1)(d)(2d - 1) - 9bd(d - 1) + 12d$. Since $K_S^2 = d((d - 1) - 3)^2$, Noether’s formula and [BS, Lemma 1.6.4] give

$$c_2(J_1(L)) = 12\chi(\mathcal{O}_S) - K_S^2 + 2K_SL + 3L^2 = (d - 1)bd(bd - 1).$$

Since $\text{class}(\Delta) = \deg(\Delta^\vee) = bd(bd - 1)$ we have $c_2(J_1(L)) = (d - 1)\text{codeg}(S, L)$ and therefore the general element of $D$ has only a single isolated non-degenerate singularity whose Milnor number is $d - 1$.

**c)** Let $(X, L, V)$ be as in $(0, 0)$ and suppose furthermore that $L$ is very ample and $(X, L) \not\in (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Let $h^0(L) = M + 1$. Then $D(X, L) = \phi_L(X)^\vee \subset \mathbb{P}^{M^\vee}$. Suppose $D(X, L)$ is a hypersurface. Then $c_n(J_1(L))$ represents its degree. Take $V$ general and let $\dim(V) = N + 1$. Note that $n \leq N \leq M$, the first inequality following from the fact that $V$ spans $L$. Geometrically, $\phi_V(X)$ is the general projection of $\phi_L(X) \subset \mathbb{P}^M$ into $\mathbb{P}^N$ and in the dual projective space $D(X, V)$ can be regarded as a general linear section of $D(X, L)$. In fact $D(X, V) = D(X, L) \cap$
Therefore $\mathcal{D}(X, V)$ is an irreducible (reduced) hypersurface of $|V|$, since $V$ was chosen general. Moreover it has the same degree as $\mathcal{D}(X, L)$, hence $\deg(\mathcal{D}(X, V)) = c_n(J_1(L))$, which exactly means that the general element of $\mathcal{D}(X, V)$ has an isolated non-degenerate quadratic singularity. Moreover, if $n = N$ then $\mathcal{D}^0 = \emptyset$ and $\mathcal{D}(X, V)$ is birational to $J_1$. The birational map is given by $p_2 \circ p_1^{-1}$ recalling (0.1).

(d) Let us consider some special projections of embedded projective varieties. Let $X = \mathbb{P}^n$, $L = \mathcal{O}_{\mathbb{P}^n}(m)$ and $V = \langle x_0^m, \ldots, x_n^m \rangle$. Then $V \subset H^0(X, L)$ spans $L$ and $\phi_L : X \to \mathbb{P}^n$ is a finite morphism of degree $m^n$. Take a general pencil $\{D_t\}_{t \in \mathbb{P}^1}$ in $|V|$. For every $t = (t_0 : t_1) \in \mathbb{P}^1$, the hypersurface $D_t$ has equation $a_0(t)x_0^m + \cdots + a_n(t)x_n^m = 0$, where $a_i(t) = a_{i,0}t_0 + a_{i,1}t_1$ for $i = 0, \ldots, n$. Then $D_t$ is singular at $(x_0 : \cdots : x_n)$ if and only if $a_0(t)x_0^{m-1} = \cdots = a_n(t)x_n^{m-1} = 0$. We thus see, the pencil being general, that there are exactly $n + 1$ singular hypersurfaces, say $D_0, \ldots, D_n$, defined by $t$ satisfying $a_i(t) = 0$ for $i = 0, \ldots, n$ respectively. Each of them is singular at one of the vertices of the homogeneous coordinate system having a point of multiplicity $m$ there. E.g., $D_0$ has equation $a_1(\tau)x_1^m + \cdots + a_n(\tau)x_n^m = 0$, where $\tau$ satisfies $a_0(\tau) = 0$. In particular, this shows that $\text{codeg}(X, V) = n + 1$. Moreover, $\mathcal{D}(X, V)$ consists of $n + 1$ hyperplanes. To see this, let $p$ be any of the points $(1 : 0 : \cdots : 0), \ldots, (0 : 0 : \cdots : 1)$. Then each hyperplane $|V - p|$ lies in $\mathcal{D}(X, V)$ (and in fact $|V - p| = |V - mp|$ for each of them). Actually, let $a = (a_0 : \cdots : a_n)$; then the hypersurface $D_a$ has equation: $a_0x_0^m + a_1x_1^m + \cdots + a_nx_n^m = 0$. Let $p = (1 : 0 : \cdots : 0)$. Then $D_a \in |V - p|$ if and only if $a_0x_0^m = 0$, i.e., if $a_0 = 0$. Thus $|V - p|$ consists exactly of the hypersurfaces of equation $a_1x_1^m + \cdots + a_nx_n^m = 0$, which have $p$ as a singular point of multiplicity $m$. Let us prove also the following

\textbf{(3.3.1) Proposition.} For $(\mathbb{P}^n, L = \mathcal{O}_{\mathbb{P}^n}(m), \langle x_0^m, \ldots, x_n^m \rangle)$, $c_n(J_1(L)) = (m - 1)^n\text{codeg}(X, V)$.

\textit{Proof.} Let $p = (1 : 0 : \cdots : 0)$ and $D \in |V - p|$. Let $y_1, \ldots, y_n$ be local coordinates of $X$ at $p$. Arguing as after (3.1) and noting that $j_1(s) = (s, ma_1y_1^{m-1}, \ldots, ma_ny_n^{m-1})$, where $s$ defines $D$, we get $\mu_p(D) = (m - 1)^n$. Since the same computation can be done for any point of the set $\{(1 : 0 : \cdots : 0), \ldots, (0 : 0 : \cdots : 1)\}$ we get the final assertion. $\Box$

The previous computation can be done in a different way. Let $X$ be any projective manifold of dimension $n$, let $L$ be a line bundle on $X$ and consider the exact sequence: $0 \to \Omega_X^1 \otimes L \to J_1(L) \to L \to 0$. We have (e.g., see [BS, Lemma 1.6.4]) $c_n(J_1(L)) = \sum_{i=0}^{n} (n+1-i)c_i(\Omega_X^1)L^{n-i} = \sum_{i=0}^{n} (n+1-i)c_i(\Omega_X^1)L^{n-i}$. So, for $X = \mathbb{P}^n$, letting $h = \mathcal{O}_{\mathbb{P}^n}(1)$ and recalling that the total Chern class of $\Omega_X^1$ is $(1 - h)^{n+1} \mod h^{n+1}$, we get $c_n(J_1(L)) = \sum_{i=0}^{n} (n+1-i)(-1)^i \binom{n+1}{i} h^i L^{n-i}$. An immediate check shows that $(n+1-i) \binom{n+1}{i} = (n+1) \binom{n}{i}$. Hence, for $L = \mathcal{O}_{\mathbb{P}^n}(m)$, we get

\begin{equation}
(3.3.2) \quad c_n(J_1(L)) = (n + 1) \sum_{i=0}^{n} \binom{n}{i} m^{n-i}(-1)^i = (n + 1)(m - 1)^n.
\end{equation}
This example is nice also from the point of view of the biduality theorem. Let $p_i = (0 : \cdots : 0 : 1 : 0 \cdots : 0)$ be the $i$th vertex of the coordinate system, $i = 0, \ldots, n$. The first jumping set $J_1$ is the union of the coordinate hyperplanes $x_i = 0$, and it is easy to realize that each of the further jumping sets is the union of all coordinate linear subspaces of the appropriate dimension; in particular, $J_n$ consists of the $n+1$ points $p_0, \ldots, p_n$. Now, let $D \in \mathcal{D}(X, V)$ be defined by the section $s = \sum_{i=0}^{n} a_i x_i^m \in V$ and suppose that the singular locus of $D$ includes a point $p$, distinct from the $p_i$'s. Then, up to reordering the coordinates, $p = (0 : \cdots : 0 : y_s : \cdots : y_n)$, with $y_j \neq 0$ for $j \geq s$, and

$$a_s = \cdots = a_n = 0.$$  

Therefore $D$ lies on the intersection of $n - s + 1$ of the $n + 1$ hyperplanes of $\mathbb{P}^n$, constituting $\mathcal{D}(X, V)$. On the other hand, according to our choice, $p$ lies on an irreducible component $Y$ (a linear space) of the jumping set $J_s$ (since it is required that $s$ coordinates vanish). Note that here $J_s$ is the same as $X_s$ (the closure of $J_s \setminus J_{s+1}$, as defined in Section 0). Moreover, $\phi_V(Y)^\vee$ is the $\mathbb{P}^{s-1} \subset |V|$ defined by (3.3.3), which is contained in $\mathcal{D}_s(X, V) \subset \mathcal{D}(X, V)$. This discussion illustrates (1.2) very well. Moreover, letting $s = n$ we get a significant example also for (1.3): the index named $i$ is $n$ and the corresponding $Y$ is simply the point $p_n$.

Let us now focus on low codegree triplets. By [LPS1, Th. 2.8] the only $(X, L, V)$ as in (0.0) such that $\text{codeg}(X, V) = 0$, which means $\mathcal{D}(X, V) = \emptyset$, is $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)))$. It becomes then natural, as in the classical case, to face the problem of classifying low codegree triplets $(X, L, V)$. Let us first show some examples.

(3.4) Examples.

(a) When considering smooth projective varieties $X \subseteq \mathbb{P}^N$, the only codegree 1 varieties are (degenerate) linear spaces $\mathbb{P}^n \subset \mathbb{P}^N$ with $n < N$, and the only codegree 2 varieties are quadrics. That is, there are no triplets $(X, L, V)$ as in (0.0) with $\phi_V$ an embedding and $\text{codeg}(X, V) = 1$ and the only example with $\text{codeg}(X, V) = 2$ is $(Q, \mathcal{O}_Q(1), H^0(Q, \mathcal{O}_Q(1)))$ where $Q \subset \mathbb{P}^{n+1}$ is a smooth quadric. If $\text{codeg}(X, V) = 3$ and $\phi_V$ is an embedding a complete classification can be found in [Z1], [Z2, IV.5].

(b) Take a cyclic covering $f : X \rightarrow \mathbb{P}^n$, branched along a smooth quadric $Q \subset \mathbb{P}^n$. For the triplet $(X, L = f^*\mathcal{O}_{\mathbb{P}^n}(1), V = f^*H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)))$ the discriminant locus is a smooth quadric, $\mathcal{D}(X, V) = Q^\vee$. In fact, $(X, L)$ is as in (a) above, but $V$ is a codimension 1 general linear subspace of $H^0(X, L)$.

(c) For the example (b.1) of (1.1.1) $\mathcal{D}(X, V) = \mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$ and so $\text{codeg}(X, V) = n$.

(d) Consider now $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. If $V = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ then $\mathcal{D}(\mathbb{P}^2, V)$ is the dual variety of the Veronese surface $S \subset \mathbb{P}^5$, that is the cubic symmetroid $S^\vee \subset \mathbb{P}^5$, hence the codegree is 3. In fact, taking coordinates $x_0, \ldots, x_5$ in $\mathbb{P}^5$, $S^\vee$ is defined by $rk(M) \leq 2$ being $M = \begin{pmatrix} x_0 & x_3 & x_4 \\ x_3 & x_1 & x_5 \\ x_4 & x_5 & x_2 \end{pmatrix}$ and its singular locus is a Veronese
surface defined by \( rk(M) = 1 \).

Let \( V \subset H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(2)) \) be a subspace such that \(|V|\) is a base-point free web of conics and consider the morphism \( \phi_V : \mathbb{P}^2 \to \mathbb{P}^3 \). Since \( \Sigma := \phi_V(\mathbb{P}^2) \) is non-degenerate, there are only two possibilities: either \( \Sigma \) is a quartic surface and \( \phi_V \) is birational, or \( \Sigma \) is a quadric surface and \( \phi_V \) has degree 2. Note that in the latter case \( \Sigma \) must be a quadric cone. Actually, \( \mathbb{P}^2 \) cannot be a branched double cover of a smooth quadric surface; otherwise, the ramification formula would imply that \( 9 = K^2_{\mathbb{P}^2} \in 2\mathbb{Z} \), a contradiction. Here are examples of both cases:

(d.1) Let \((u : x : y)\) be homogeneous coordinates in \( \mathbb{P}^2 \) and let \( V_1 = \langle u^2 + x^2 + y^2, xy, uy, ux \rangle \). Then \(|V_1|\) is a base-point free web of conics and \( \phi_{V_1} : \mathbb{P}^2 \to \mathbb{P}^3 \) is a birational morphism onto the roman Steiner quartic surface \( \Sigma \) of equation \( y_1^2y_2^2 + y_1^2y_3^2 + y_2^2y_3^2 - y_0y_1y_2y_3 = 0 \), where \((y_0 : y_1 : y_2 : y_3)\) are the corresponding homogeneous coordinates in \( \mathbb{P}^3 \). Note that \( \mathcal{D}(\mathbb{P}^2, V_1) \) is the irreducible cubic surface defined by: \( \lambda^3 - \lambda(\mu^2 + \nu^2 + \epsilon^2) + 2\mu\nu\epsilon = 0 \), where \((\lambda : \mu : \nu : \epsilon)\) are the dual homogenous coordinates in \( \mathbb{P}^{3\vee} \).

(d.2) Let \( V_2 = \langle u^2, x^2, xy, y^2 \rangle \). Then \(|V_2|\) is a base-point free web of conics and the morphism \( \phi_{V_2} : \mathbb{P}^2 \to \mathbb{P}^3 \) is two-to-one onto the quadric cone \( \Sigma \) of equation \( y_1y_3 - y_2^2 = 0 \). Note that \( \mathcal{D}(\mathbb{P}^2, V_2) \) is defined by the equation \( \lambda(\mu\nu - \nu^2) = 0 \), hence it is reducible into a plane plus a quadric cone.

Now let us choose \( V \subset H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(2)) \) such that \(|V|\) is a base-point free net of conics. By [W]:

(d.3) either \( \mathcal{D}(\mathbb{P}^2, V) \subset \mathbb{P}^2 \) is irreducible and there exists a suitable choice of homogeneous coordinates \( x, y, z \) in the plane (suggested in [W]) so that

\[
(3.4.1) \quad V = \langle 2xz + y^2, 2yz, -x^2 - 2gy^2 + cz^2 + 2gxz \rangle \subset H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(2)),
\]

where \( g, c \) are complex parameters, or

(d.4) \( \mathcal{D}(\mathbb{P}^2, V) \subset \mathbb{P}^2 \) is reducible and for a suitable choice of homogeneous coordinates, either \( V = \langle x^2, y^2, z^2 \rangle \) or \( V = \langle x^2, y^2, z^2 + 2xy \rangle \).

Then, the possibilities are the following:

(d.3.1) If \( c \neq -9g^2 \) in (3.4.1) then \( \phi_V : \mathbb{P}^2 \to \mathbb{P}^2 \) is a degree four map branched along a sextic \( C \subset \mathbb{P}^2 \) with 9 cusps (and no other singularities). Hence \( \mathcal{D}(\mathbb{P}^2, V) = \mathcal{D}_1(\mathbb{P}^2, V) = C^\vee \subset \mathbb{P}^2 \) is a smooth plane cubic.

(d.3.2) If \( c = -9g^2 \neq 0 \) in (3.4.1) then \( \phi_V : \mathbb{P}^2 \to \mathbb{P}^2 \) is a degree four map branched along a quartic curve \( C \subset \mathbb{P}^2 \) with three cusps (and no other singularities). Hence \( \mathcal{D}(\mathbb{P}^2, V) = \mathcal{D}_1(\mathbb{P}^2, V) = C^\vee \) is a nodal plane cubic.

Note that for \( c = g = 0 \) \(|V|\) is not base point free, \( Bs|V| = \{(0 : 0 : 1)\} \).

(d.4.1) If \( V = \langle x^2, y^2, z^2 + 2xy \rangle \) then \( \phi_V \) is a degree four map branched along the union of a smooth conic \( Q \) and two lines tangent to \( Q \). Hence \( \mathcal{D}_1(\mathbb{P}^2, V) = Q^\vee \), \( \mathcal{D}_2(\mathbb{P}^2, V) = \ell \) and \( \mathcal{D}(\mathbb{P}^2, V) = \mathcal{D}_1(\mathbb{P}^2, V) \cup \mathcal{D}_2(\mathbb{P}^2, V) \) where \( Q^\vee \) is a smooth conic and \( \ell \) is a line transverse to \( Q^\vee \).
(d.4.2) If \( V = \langle x^2, y^2, z^2 \rangle \) then \( \phi_V : \mathbb{P}^2 \to \mathbb{P}^2 \) is a degree four map branched along three general lines. Hence \( D(\mathbb{P}^2, V) = \ell_1 \cup \ell_2 \cup \ell_3 \subset \mathbb{P}^2 \), being \( \ell_1, \ell_2 \) and \( \ell_3 \) the dual trilateral to the branch locus. Hence \( D(\mathbb{P}^2, V) = D_2(\mathbb{P}^2, V) = \{ \ell_1, \ell_2, \ell_3 \} \) and \( D_1(\mathbb{P}^2, V) = \{ (\ell_1 \cap \ell_2), (\ell_2 \cap \ell_3), (\ell_1 \cap \ell_3) \} \).

A corollary of the study of nets of conics according to [W] is the following. For a suitable 3-dimensional vector subspace \( V \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \) it may happen that the corresponding plane section of the cubic symmetroid either is a cuspidal curve, or contains a double line, or is the union of a smooth conic and one of its tangent lines. In all these cases, however, \(|V|\) is not base-point free.

We can exclude the possibility for the codegree to be one.

(3.5) **Theorem.** Let \( (X, L, V) \) be as in (0.0). Then \( \text{codeg}(X, V) \geq 2 \) except for \( (X, L, V) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))) \) (for which \( \text{codeg}(X, V) = 0 \)).

**Proof.** By assumption \( \text{codeg}(X, V) = 0 \) if and only if \( D(X, V) = \emptyset \), which implies \( \dim(D(X, V')) = -1 \). We conclude by [LPS, Thm. 2.8]. Let \( \text{codeg}(X, V) = 1 \) then \( D(X, V) = T \cup D' \) where \( T = \mathbb{P}^N-1-k \) and \( D' \) is the union of the irreducible components of the discriminant locus of dimension \( < N - 1 - k \). By (1.4.1) there exists \( Y \subseteq J_{n-k}(X, V) \) such that \( \phi_V(Y) = \mathbb{P}^k \), \( T = \phi_V(Y)^{\vee} \) and \( J_{n-k+i} = \emptyset \) for \( i > 0 \). If \( D' \neq \emptyset \) consider \( D^0 \) the union of the maximal dimensional components of \( D' \). By (1.3) \( D^0 = \phi_V(Y_1)^{\vee} \cup \cdots \cup \phi_V(Y_s)^{\vee} \). Suppose \( \dim(D^0) = N - 1 - k_0 \). Take general sections \( s_1, \ldots, s_{k_0+1} \in V \) and let \( M = |\langle s_1, \ldots, s_{k_0+1} \rangle| \subseteq |V| \). Since \( \dim(M \cap D^0) = 0 \) the classical biduality theorem implies that the image by \( \phi_V \) of the singular locus of any element corresponding to a point \( p \in M \cap D^0 \) is a linear space of dimension \( k_0 \), say \( \mathbb{P}^{k_0}_p \). Then, by [LPS1, Thm. 2.4], \( \phi_V((j_1(s_1) \wedge \cdots \wedge j_1(s_{k_0+1}))^{-1}(0)) \) is a finite union of linear spaces of dimension \( k_0 \). By [LPS1, 2.3.2] \( \phi_V(J_{n-k_0}) \) is the intersection of these linear spaces. Since \( \phi_V(Y) = \mathbb{P}^k \subseteq \phi_V(J_{n-k}) \subseteq \phi_V(J_{n-k_0}) \) it follows that all the contact loci of \( \phi_V(Y_i) \) (\( 1 \leq i \leq s \)) are meeting along \( \phi_V(Y) \). This in particular means that the intersection of all the projective tangent spaces to \( \phi_V(Y_i) \subset \mathbb{P}^N \) is not empty. So, \( \phi_V(Y_i) \) is a cone [R, Prop. 1.2.6] whose vertex contains \( \phi_V(Y) \), contradicting the non-emptyness of \( D' \). Hence \( D' = \emptyset \) so that \( D \) is irreducible. This implies \( \phi_V(X) \) is a cone whose vertex contains \( \phi_V(Y) \) and \( \phi_V(J_1) \) is a union of cones with vertex containing \( \phi_V(Y) \) as in (2.7.1). In this situation the general line in \( \mathbb{P}^n \) only meets \( \phi_V(J_1) \) at one point. This leads to a contradiction as in the proof of (2.7). \( \square \)

4 Low codegree curves

In this section we classify curves of codegree less than or equal to three.

(4.1) **Remark.** For \( (X, L, V) \) as in (0.0) \( \phi_V(X)^{\vee} \) cannot be a cone.

In fact, if \( \phi_V(X)^{\vee} \) is a cone, then \( \phi_V(X)^{\vee\vee} = \phi_V(X) \) is degenerate, contradicting the assumptions of (0.0). We collect two basic facts on curves in the following remark. Proofs are straightforward.
If deg(C, V) = 1 then \( \phi_V(C) = \mathbb{P}^1 \) and \(|\phi_V(J_1)| = 1\) by (4.2), contradicting (2.3). This gives a different proof of (3.5) in the case of curves.

If deg(C, V) = 2 then either \( \phi_V(C) \) is a quadric and \( J_1 = \emptyset \), or \( \phi_V(C) = \mathbb{P}^1 \) and \( \phi_V(J_1) = \{p_1, p_2\} \). If the former holds then \( \phi_V(C) \) is smooth by (4.1), and so, by biduality, \( \phi_V(C) \) is a smooth conic. Since \( J_1 = \emptyset \) then \( (C, L, V) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2), H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))) \). If the latter holds then, arguing as in (2.3), from the Riemann–Hurwitz formula we obtain \( 2g(C) - 2 = -m_1 - m_2 \) being \( m_1 = |\phi_V^{-1}(p_1)| \) (respectively \( m_2 = |\phi_V^{-1}(p_2)| \)). Then \( g(C) = 0 \) and \( m_1 = m_2 = 1 \). In fact, there exists an integer \( r > 1 \) such that \( C = \mathbb{P}^1 \), \( L = \mathcal{O}_{\mathbb{P}^1}(r) \) and \( V \) has to be chosen in the following way: since the complete linear system embeds \( \mathbb{P}^1 \) in \( \mathbb{P}^r \) as a rational normal curve, we have to project it from \( T = \mathbb{P}^{r-2} \) in such a way that \( |\phi_V(J_1)| = 2 \). Then \( T \) is the intersection of two linear spaces of dimension \( r - 1 \) that are \((r-1)\)-osculating to the rational normal curve. This concludes the codegree 2 case.

If deg(C, V) = 3 then either

(4.3.1) \( \phi_V(C) \) is a cubic and \( J_1 = \emptyset \), that is, \( \phi_V \) is an immersion, or

(4.3.2) \( \phi_V(C) \) is a conic and there exist \( p \in \phi_V(C) \) such that \( \phi_V(J_1) = p \), or

(4.3.3) \( \phi_V(C) = \mathbb{P}^1 \) and there exist three distinct points \( p_1, p_2, p_3 \in \phi_V(C) \) such that \( \phi_V(J_1) = \{p_1, p_2, p_3\} \).

If (4.3.1) holds, \( \phi_V \) is an immersion and then \( \mathcal{D}(X, V) = \phi_V(C)^\vee \). By (3.2.5) \( c_1(J_1(L)) \leq 3\text{deg}(\phi_V) \). Since \( \phi_V(C)^\vee \) is a cubic, \( \text{deg}(\phi_V(C)) \geq 3 \). The previous bounds and the fact that \( c_1(J_1(L)) = 2g(C) - 2 + 2\text{deg}(L) = 2g(C) - 2 + 2\text{deg}(\phi_V)\text{deg}(\phi_V(C)) \) leads to a contradiction. Hence this case does not occur. If (4.3.2) holds then \( 2g(C) - 2 = -d - m \), where \( m = |\phi_V^{-1}(p)| \), by the Riemann–Hurwitz formula. This gives \( g(C) = 0 \) and \( d = m = 1 \), a contradiction. If (4.3.3) holds then, just as before, Riemann–Hurwitz formula says \( 2g(C) - 2 = d - (m_1 - m_2 - m_3) \) where \( m_i = |\phi_V^{-1}(p_i)| \) for \( i = 1, 2, 3 \). Whence:

(4.4) Theorem. Let \( (X, L, V) \) as in (0.0) with \( \text{dim}(X) = 1 \) and \( \text{deg}(X, V) \leq 3 \). Then, either

(4.4.1) \( (X, L, V) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2), H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))) \), \( \text{deg}(X, V) = 2 \), or

(4.4.2) \( (X, L, V) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r), V) \) with \( r \geq 2 \) and \( V \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r)) \) is such that \( \phi_V \) is the projection of \( \phi_L(\mathbb{P}^1) \subset \mathbb{P}^r \) from the intersection of two \((r-1)\)-dimensional linear spaces of \( \mathbb{P}^r \) that are \((r-1)\)-osculating to \( \phi_L(\mathbb{P}^1) \); \( \text{deg}(X, V) = 2 \), or

(4.4.3) \( \phi_V(X) = \mathbb{P}^1 \), \( \phi_V(J_1) = \{p_1, p_2, p_3\} \) and \( \text{deg}(X, V) = 3 \).

(4.5) Examples (showing that the list of (4.4) is effective).
(a) Consider a degree r rational normal curve $C \subset \mathbb{P}^r$ and for every $k \leq r - 1$ let $\text{Osc}_c^k(C)$ be the $k$-th osculating space to $C$ at $p$. Take two general points $p_1, p_2 \in C$ and consider $M = \text{Osc}_{p_1}^{r-1}(C) \cap \text{Osc}_{p_2}^{r-2}(C) = \mathbb{P}^{r-3}$. Let $T$ be a general $\mathbb{P}^{r-2}$ in $\text{Osc}_{p_1}^{r-1}(C)$ containing $M$. Then the projection from $T$ ramifies in $p_1$ with ramification index $r$ and in $p_2$ with ramification index $r - 1$. Since $-2 = -2r + (r - 1 + r - 2 + 1)$ then the projection from $T$ ramifies in a third point $p_3$ with ramification index 2 and we are in case (4.4.3).

(b) With the notation of (4.3.3) let us construct an example with $g(C) = 1$, $d = 3$ and $m_1 = m_2 = m_3 = 1$. Take the projection of a smooth plane cubic $C \subset \mathbb{P}^2$ from a point $x \in \mathbb{P}^2 \setminus C$ onto $\mathbb{P}^1$. In order to have codegree three we have to choose $x$ in the intersection of three tangent lines to $C$ at flexes of $C$. For example consider the cubic defined by the equation $x_0^3 - x_1x_2^2 + x_1^2x_2 = 0$ and project from $(1 : 0 : 0)$ which is on the intersection of the tangent lines to the cubic at the three flexes $(0 : 0 : 1), (0 : 1 : 0)$ and $(0 : 1 : 1)$.

5 Low codegree surfaces: general facts

Consider now a triplet $(S, L, V)$ as in (0.0), $S$ being a surface. Recall that $\dim(\mathcal{D}(S, V)) = N - 1$, see [LPS1, Thm. 2.8]. Suppose that $\phi_V(S)^\vee$ is the only $(N - 1)$-dimensional irreducible component of $\mathcal{D}(X, V)$. If $\text{codeg}(S, V) < \deg(\phi_V(S))$ then (3.2.5) combined with [LPS1, Prop. A.1] gives the bound $L^2 - 1 \leq c_2(J_1(L)) \leq \text{codeg}(S, V)\frac{L^2}{\deg(\phi_V(S))} < L^2$. It follows that the first inequality has to be an equality, and so [LPS1, Prop. A.1] implies that $(S, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. In particular, $(S, V)$ is one of the pairs discussed in (d) of (3.4). It turns out that, apart from this case, the inequality $\text{codeg}(S, V) < \deg(\phi_V(S))$ cannot be true. Moreover, if equality holds then $(S, L)$ is a scroll by [LPS1, Prop. A1]. So this proves the following

(5.1) Corollary. Let $(S, L, V)$ be as in (0.0), where $\dim(S) = 2$. If $\phi_V(S)^\vee$ is the only $(N - 1)$-dimensional irreducible component of $\mathcal{D}(X, V)$ then, either

(5.1.1) $(S, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, or

(5.1.2) $\text{codeg}(S, V) \geq \deg(\phi_V(S))$.

Moreover if equality holds in (5.1.2) and we are not in (5.1.1) then $(S, L)$ is a scroll.

We can regard (5.1) as a natural extension of classical results of Marchionna [M] and Gallarati [G] to the ample and spanned setting. Let us recall here that for a triplet $(S, L, V)$ as in (0.0) with $\dim(S) = 2$, $L$ very ample and $\phi_V$ an embedding, it is usual to use the term class to refer to $\text{codeg}(S, V)$. Marchionna proved that the class of a surface is greater than or equal to its degree minus one and equality holds when $(S, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t)), t = 1, 2$, see [M]. Moreover Gallarati showed that the class is equal to the degree if and only if $(S, L)$ is a scroll, see [G]. The example (d) of (2.6) is interesting in connection with (5.1). In fact, for the elliptic scroll of invariant $e = -1$ when considering $L = C_0 + f$, $V = H^0(S, L)$ we obtain $\text{codeg}(S, V) = 3$ and $\phi_V(S) = \mathbb{P}^2$. On the other hand, when considering $L = C_0 + 2f$, $V = H^0(S, L)$
then $\phi_V$ is an embedding and $\phi_V(S) \subset \mathbb{P}^4$ is the quintic elliptic scroll. Whence $\text{codeg}(S, V) = 5$.

As said in the introduction, the geometry of $\phi_V(S) \subseteq \mathbb{P}^N$ is an important tool in the study of $\mathcal{D}(S, V)$. In particular $\phi_V(S)^\vee \subseteq \mathcal{D}(X, L)$ is a relevant part of the discriminant. Let us comment some consequences of $\phi_V(S)^\vee$ to be small. We will recall the following definition and notation:

**Definition.** Let $Y \subset \mathbb{P}^M$ a projective variety. The tangent developable to $Y$ is denoted by $TY$ an is defined as the closure in $\mathbb{P}^M$ of the union of the embedded projective tangent spaces to $Y$ at its smooth points.

(5.2) **Proposition.** Let $(S, L, V)$ be a triplet as in (0.0) with $\dim(S) = 2$ and $\dim(\phi_V(S)^\vee) < N - 1$. Then:

(5.2.1) either $\phi_V(S) \subseteq \mathbb{P}^N$ is a cone, or

(5.2.2) there exists a curve $C \subset \mathcal{J}_1$ such that $\deg(\phi_V(S)) \leq \deg(\phi_V(C)^\vee) \leq \text{codeg}(S, V)$ and $\phi_V(S) = T\phi_V(C)$.

In particular if (5.2.1) does not hold then $\text{codeg}(S, V) > 3$.

**Proof.** Suppose $\phi_V(S) \neq \mathbb{P}^2$, if not we are in case (5.2.1). Since the dual of $\phi_V(S) \subset \mathbb{P}^N$ is not a hypersurface then the general tangent hyperplane is tangent to $\phi_V(S)$ along a line. In particular $\phi_V(S) \subset \mathbb{P}^N$ is swept out by lines. Since $\phi_V(S) \neq \mathbb{P}^2$ there is a finite number of lines through the general point of $\phi_V(S)$. Consider $x \in \phi_V(S)$ general. The general tangent hyperplane $H$ to $\phi_V(S)$ at $x$ is tangent along a line $\ell_H$ through $x$. Since there is a finite set of lines on $\phi_V(S)$ through $x$ it holds that $\ell_H = \ell$ for the general $H$ containing $T_{\phi_V(S), x}$. This says that $T_{\phi_V(S), x} = T_{\phi_V(S), y}$ for a general $y \in \ell$. In particular $\phi_V(S) \subset \mathbb{P}^N$ is a developable surface. Then, by [FP, Thm. 2.2.8], either $\phi_V(S) \subset \mathbb{P}^N$ is a cone, and so (5.2.1) holds, or it is the tangent developable to a curve $E \subset \phi_V(S)$. Suppose $\phi_V(S) = TE$. Then the general line in $\phi_V(S)$ is the tangent line to $E$ at a smooth point $e \in E$, say $T_{E,e}$. The general hyperplane $H$ containing this line cuts out $\phi_V(S)$ along a reducible curve by degree reasons, and so its corresponding element $D \in |V|$ is reducible, hence singular by (0.4). This implies $E^\vee \subset \mathcal{D}(S, V)$. In particular, it is a non-linear $(N - 1)$–dimensional irreducible component of $\mathcal{D}(S, V)$ and we conclude the existence of a curve $C \subset \mathcal{J}_1$ by (1.3).

Consider now a general line $R$ contained in $|V|$ corresponding to the hyperplanes in $\mathbb{P}^N$ containing a fixed $T = \mathbb{P}^{N-2}$. Since $E^\vee$ is an irreducible component of the discriminant then there exist points $e_1, \ldots, e_r \in E$, $r = \deg(E^\vee) \leq \text{codeg}(S, V)$, such that $\dim((T_{E,e_i}, T)) = N - 1$ for $1 \leq i \leq r$. This is equivalent to saying that $T \cap T_{E,e_i} = x_i \neq \emptyset$ for $1 \leq i \leq r$. In particular $\{x_1, \ldots, x_r\} \subset T \cap \phi_V(S)$. We claim that this is an equality. Indeed, if there exists $x \in (T \cap \phi_V(S)) \setminus \{x_1, \ldots, x_r\}$ it holds that there exists an analytic arc $\{e(t)\} \subset E$ such that $x \in \ell \subset \phi_V(S)$, being $\ell$ the limit of the tangent lines $T_{E,e(t)}$. Let us remark that if $\ell = T_{E,e_i}$ for some $i$ and $x \in \ell = T_{E,e_i} \setminus \{x_1, \ldots, x_r\}$ then $|\ell \cap T| \geq 2$ and so $\ell \subset T \cap \phi_V(S)$, contradicting the general choice of $T$. Then $\ell \cap T = \{x\} \neq \emptyset$ and the set of hyperplanes containing $\ell$
is contained in $E^\vee$, a contradiction with $\deg(E^\vee) = r$. Since developable quadrics and cubics have to be cones, see [E, pp. 32–33], then $\text{codeg}(X, V) > 3$. □

For low codegree we can study the possibility for any maximal dimensional component of the discriminant to be linear.

**Lemma 5.3.** Let $(S, L, V)$ be as in (0.0) and $\dim(S) = 2$. If any maximal dimensional component of $\mathcal{D}$ is linear then $\phi_V(S) = \mathbb{P}^2$, $\phi_V(J_1)$ is a union of at least three non-collinear lines and $\phi_V(p)$ is contained in a line of $\phi_V(J_1)$ for any $p \in J_2$. Moreover $\text{codeg}(S, V) \geq 3$ and if equality holds then $(S, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r))$.

**Proof.** Since $\phi_V(S)^\vee \subseteq \mathcal{D}(S, V)$, (5.2) applies. Then either $\phi_V(S) \subseteq \mathbb{P}^N$ is a cone or $\phi_V(S)$ is a developable surface (different from a cone). In the second case by (5.2.2) there exists a curve $C \subseteq J_1$ such that $T\phi_V(C) = \phi_V(S)$, in particular $\phi_V(C) \subset \mathbb{P}^N$ is non-degenerate, and $\phi_V(C)^\vee$ is a component of $\mathcal{D}(S, V)$. This is a contradiction because the dual of any non-degenerate curve is a non-linear hypersurface. Hence $\phi_V(S) \subseteq \mathbb{P}^N$ is a cone.

If $\phi_V(S) \subseteq \mathbb{P}^N$ is a not linear cone then the vertex is a point, say $v$. If $\phi_V(J_1)$ contains an irreducible curve of degree $\geq 2$ then its dual is a non-linear component of dimension $N - 1$ of $\mathcal{D}$, a contradiction. Then $\phi_V(J_1)$ is a union of lines through $v$. This gives $(S, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ by exactly the same argument as after (2.7.1). In fact, the preimage of a general line $\ell \subset \phi_V(S)$ through $v$ is a curve whose singular locus is contained in $\phi_V^{-1}(v)$. Then for the normalization $\gamma$ of any of its irreducible components, $\phi_V$ defines a map from $\gamma$ onto $\mathbb{P}^1$ branched only at $v$. By (2.3) any irreducible component of $\phi_V^{-1}(\ell)$ is then isomorphic to $\mathbb{P}^1$ via $\phi_V$. Hence $(S, L)$ is swept out by lines and we conclude by [LP2, Thm. 1.4]. Hence $\phi_V(S) = \mathbb{P}^2$.

Since $\phi_V(S) = \mathbb{P}^2$, $J_1$ is a union of curves and so $\phi_V(J_1)$ is a union of lines, being any maximal dimensional component of the discriminant. Moreover, since $J_2 \subset J_1$, any $p \in J_2$ is contained in a curve of $J_1$, hence $\phi_V(p)$ is contained in a line of $\phi_V(J_1)$. By exactly the same argument of the previous paragraph, $\phi_V(J_1)$ cannot be a set of lines through a point, then $\phi_V(J_1)$ contains at least three non-collinear lines $\ell_1, \ell_2, \ell_3$. Then we can write $\phi_V(J_1) = \ell_1 \cup \cdots \cup \ell_s$, where $s \geq 3$ and $\ell_i$ is a line $(1 \leq i \leq s)$. Let $R$ be the ramification divisor of $\phi_V$, then $R \in |K_S + 3L|$. We can write $R = R_1 + \cdots + R_s$ where $\phi_V(R_i) = \ell_i$. Moreover, $R_i = \sum_{j=1}^{s_i} \alpha_{ij} R_{ij}$ where each $R_{ij}$ is an irreducible curve and $\alpha_{ij} \geq 1$. Since $\phi_V$ ramifies along $R$ then for $1 \leq i \leq s$ there exists a divisor $H_i = \phi_V^{-1}(\ell_i) \in |V|$ and a divisor $E_i \geq 0$ such that $H_i = \sum_{j=1}^{s_i} (\alpha_{ij} + 1) R_{ij} + E_i$. We claim that

(5.3.1) for any $1 \leq i \leq s$ there exists $j \neq i$ such that $R_i R_j > 0$.

If (5.3.1) holds then for $x_{ij} \in R_i \cap R_j$ we get $|V - R_i|, |V - R_j| \subset |V - 2x_{ij}|$ and $|V - R_i| \neq |V - R_j|$. Whence $x_{ij} \in J_2$ and $\text{codeg}(S, V) \geq 3$. Let us prove (5.3.1). We argue with $R_1$ and the same argument holds when $i \neq 1$. Suppose $R_i R_j = 0$ for any $j \neq 1$. Since $L$ is ample then $0 < LR_{11} = H_j R_{11}$ and so

(5.3.2) $R_{11} E_j > 0$ for any $j \neq 1$.
This in particular implies $E_i > 0$ for $1 \leq i \leq s$. Moreover, since $H_1$ is ample, its support is connected and then

\begin{equation}
(5.3.3) \quad R_{11}(R_{12} + \cdots + R_{1s_1} + E_1) > 0.
\end{equation}

Now we have:

\begin{align*}
-(K_S + R_{11})R_{11} &= -(R_{11} + R - 3L)R_{11} = -(R_{11} + R - H_1 - H_2 - H_3)R_{11} = \\
&= -(R_{11} + R - H_1 + R_2 - H_2 + R_3 - H_3 + \Sigma j \geq 4 R_j)R_{11} = \\
&= (R_{12} + \cdots + R_{1s_1} + E_1)R_{11} + (R_{21} + \cdots + R_{2s_2} + E_2)R_{11} + \\
&\quad + (R_{31} + \cdots + R_{3s_3} + E_3)R_{11}.
\end{align*}

Note that the first summand in the final expression is $\geq 1$ by (5.3.3) and the same inequality holds for the other two summands by (5.3.2). Thus, by adjunction formula we get:

\[ -2 \leq 2g(R_{11}) - 2 = (K_S + R_{11})R_{11} \leq -3, \]

a contradiction. This proves (5.3.1). Now suppose that $\text{codeg}(S, V) = 3$ (i.e., $s = 3$ in the previous notation) and $E_1 > 0$. Let $C_1$ be an irreducible component of $E_1$. Recall that $E_1$ has no non-reduced components (otherwise they would be part of $R$). Then

\begin{align*}
-(K_S + C_1)C_1 &= -(R - 3L + C_1)C_1 = (R_{11} + \cdots + R_{1s_1} + (E_1 - C_1))C_1 + \\
&\quad + (R_{21} + \cdots + R_{2s_2} + E_2)C_1 + (R_{31} + \cdots + R_{3s_3} + E_3)C_1 \geq 3,
\end{align*}

each of the three summands being $\geq 1$: the first one by the connectedness of $H_1$ and the remaining two by the the ampleness of $L$. By adjunction formula this gives a contradiction again. This shows that $E_1 = 0$ and the same argument gives $E_2 = E_3 = 0$. Hence

\[ K_S = -(R_{11} + \cdots + R_{1s_1}) - (R_{21} + \cdots + R_{2s_2}) - (R_{31} + \cdots + R_{3s_3}). \]

We claim that $s_1 = s_2 = s_3 = 1$. We have: $-(K_S + R_{11})R_{11} = (R_{12} + \cdots + R_{1s_1})R_{11} + (R_{21} + \cdots + R_{2s_2})R_{11} + (R_{31} + \cdots + R_{3s_3})R_{11}$. If $s_1 > 1$ then by the connectedness of $H_1$ and the ampleness of $L$ we have the contradiction $(K_S + R_{11})R_{11} \leq -3$. The same argument works with $s_2$ and $s_3$.

Since $s_1 = s_2 = s_3 = 1$, $H_i = (\alpha_i + 1)R_i$, $1 \leq i \leq 3$. Then $R_{i1}$ is an ample divisor for $1 \leq i \leq 3$. Moreover $K_S = -R_{11} - R_{21} - R_{31}$ and so $S$ is a Del Pezzo surface with $-K_S$ being the sum of three ample divisors. Hence $S = \mathbb{P}^2$, $R_{11}, R_{21}, R_{31} \in |O_{\mathbb{P}^2}(1)|$ and $L = O_{\mathbb{P}^2}(\alpha_{11} + 1)$. □

Let us observe that (5.3) gives a different proof of (3.5) in the case of surfaces. Moreover, recall that for any $V \subseteq H^0(S, L)$, $\mathcal{D}(S, V) = \mathcal{D}(S, H^0(S, L)) \cap |V|$ (possibly set-theoretically). We know that $\text{codeg}(\mathbb{P}^2, H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(2))) = 3$ and $\text{codeg}(\mathbb{P}^2, H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(r))) > 3$ for $r \geq 3$. Hence, in the previous discussion, either $\alpha_{11} = 1$, $(S, L) = (\mathbb{P}^2, O_{\mathbb{P}^2}(2))$ and $V \subseteq H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(2))$ is as in (d.4.2) of (3.4) or special projections of the $r$-Veronese embedding of $\mathbb{P}^2$ have to be considered: $V = \langle x_r^0, x_r^1, x_r^2 \rangle$ provides an example of codegree 3 for any $r$. 

\[ 24 \]
(5.4) **Proposition.** Let \((S, L, V)\) be as in \((0.0)\), \(\dim(S) = 2\). If \(\text{codeg}(S, V) \leq 3\) and \(\mathcal{J}_2 = \emptyset\) then \(S\) is a ruled surface.

**Proof.** Choose a general vector subspace \(V' \subseteq V\) such that \(\dim(V') = 3\). Then \(\phi_{V'}(S) = \mathbb{P}^2\) and \(\text{codeg}(S, V') \leq 3\). Moreover \(\mathcal{J}_2(V') = \emptyset\) because, if not, \(\mathcal{D}(X, V')\) has a linear component of maximal dimension and so \(\mathcal{D}(X, V)\) has a linear component of maximal dimension contradicting \(\mathcal{J}_2(V) = \emptyset\). Hence, by the usual expression of \(c_2(J_1(L))\) in terms of the invariants of \(S\), see for example [LPS1, A.1.1], and (3.2.7) we have:

\[
(5.4.1) \quad c_2(J_1(L)) = e(S) + 2K_S L + 3L^2 \leq \text{codeg}(S, V')(L^2 - 1),
\]

where \(e(S)\) is the topological Euler–Poincaré characteristic of \(S\). In particular \(e(S) + 2K_S L \leq -3\). Hence either \(e(S) < 0\) or \(2K_S L < 0\) and we are done either by Castelnuovo–De Franchis theorem or by Enriques theorem, see [B]. □

6 **Codegree 2 surfaces**

Now we deal with codegree two surfaces. Let us consider \((S, L, V)\) as in \((0.0)\) with \(\dim(S) = 2\) and \(\text{codeg}(S, V) = 2\). By [LPS1, Thm. 2.8], \(\dim(\mathcal{D}(S, V)) = N - 1\) and by (5.3), \(\mathcal{D}(S, V)\) has just one maximal dimensional irreducible component, say \(\mathcal{D}^0\). Whence \(\mathcal{J}_2 = \emptyset\) and \(\mathcal{D}^0\) is either an irreducible quadric cone, or a smooth quadric. We can exclude the first possibility: If \(\phi_V(S)\) is a cone then \(\phi_V(S)\) would be degenerate and so \(\phi_V(S) = \mathbb{P}^2\). Whence \(\mathcal{D}^0\) cannot be an irreducible quadric cone. To deal with the second possibility note that if \(\mathcal{D}^0\) is a smooth quadric then, by biduality, \(\mathcal{D}^0 \subset \phi_V(S)\). Hence, either \(\phi_V(S) \subset \mathbb{P}^3\) is a smooth quadric or \(\phi_V(S) = \mathbb{P}^2\) and there is a smooth conic in \(\phi_V(J_1)\). We will need the following general fact.

(6.1) **Lemma.** Let \((S, L, V)\) be as in \((0.0)\) with \(\dim(S) = 2\) and \(N \geq 3\). If there exists an \((N - 1)\)-dimensional component \(\mathcal{D}^0 \subseteq \mathcal{D}\) which is a smooth quadric and any other irreducible components of \(\mathcal{D}\) is linear then either \((S, L)\) is a scroll or \(|\phi_V(\mathcal{J}_2)| \geq 2\).

**Proof.** By hypothesis \(N \geq 3\) so \(\phi_V(S) \subset \mathbb{P}^3\) is a smooth quadric. Since the other components of \(\mathcal{D}\) are linear, \(\phi_V(J_1)\) is a union of lines. Let us observe that if \((S, L)\) is not a scroll then the ramification divisor \(R \in |K_S + 2L|\) of \(\phi_V\) is an ample and effective divisor, see [LP1, Thm. 2.5]. Consider \(\ell_i \in \mathcal{T}_i\) \((i = 1, 2)\) a general line in the ruling \(\mathcal{T}_i\) of \(\phi_V(S)\). Then \(C_i = \phi_V^{-1}(\ell_i)\) is a smooth curve. Let us observe that \(C_1 + C_2 \in |V|\), \(C_1^2 = C_2^2 = 0\) and \(C_1C_2 = L^2/2\). Since \((S, L)\) is not a scroll that for any irreducible component \(C\) of \(C_1\) or \(C_2\) the branch locus of \(\phi_V|_C : C \to \mathbb{P}^1\) can be neither empty nor a point (see (2.3)). Then for any component \(C\) of \(C_1\) or \(C_2\) the restriction \(\phi_V|_C : C \to \mathbb{P}^1\) branches in at least two points. This means that \(R\) has at least four components, say \(R_1, R_2, R_3\) and \(R_4\), such that \(R_1C_1 > 0\), \(R_2C_1 > 0\), \(R_3C_2 > 0\) and \(R_4C_2 > 0\). Since any component of \(R\) maps onto a line on \(\phi_V(S)\) we have \(R_i^2 \leq 0\), \(R_1R_2 = 0\) and \(R_3R_4 = 0\). Moreover \(\phi_V(R_1) \neq \phi_V(R_2)\) and,
since $R$ is ample, $RR_4 > 0$. Hence there exist two components $R_1'$ and $R_2'$ of $R$ such that $R_1R_1' > 0$ and $R_2R_2' > 0$. Now take $p \in R_1 \cap R_1'$ and respectively $q \in R_2 \cap R_2'$. We claim that $p, q \in J_2$ and this proves the lemma because $\phi_V(p) \neq \phi_V(q)$. Since $p \in R_1 \subset J_1$ then $|V - R_1| = |V - 2p|$ and equivalently $|V - R_1'| = |V - 2p|$. Then $|V - 2p|$ contains two different lines and so $|V - 2p| = |V - p|$, that is, $p \in J_2$. The same argument can be applied to $q$. □

Let us come back to the codegree two case. Suppose $\phi_V(S) \subset \mathbb{P}^3$ to be a smooth quadric. By (6.1) $(S, L)$ is a scroll over a smooth curve $B$ and $L \equiv C_0 + bf$. We have $L|_f = O_{\mathbb{P}^2}(1)$ for any fibre $f$. In particular $\phi_V(f)$ is a line on $\phi_V(S)$ and all lines image by $\phi_V$ of fibres of the scroll are on the same ruling $T_1$ of the quadric $\phi_V(S)$. Consider two general lines $\ell_1$ and $\ell_2$ on the other ruling $T_2$ of $\phi_V(S)$. It holds that $\phi^{-1}_V(\ell_1) = C_1$ and $\phi^{-1}_V(\ell_2) = C_2$, with $C_1$ and $C_2$ smooth irreducible curves which are sections of the scroll. Moreover $C_1 \cap C_2 = \emptyset$. Thus, by [Ha, Ex. V.2.2], we have $S = \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ splits as $\mathcal{E} = \mathcal{O}_B \oplus \mathcal{L}$ with $\mathcal{L} \in \text{Pic}(C)$ and $\deg(\mathcal{L}) \leq 0$. Consider $\ell$ a general line in the ruling $T_1$. Since $\phi^{-1}_V(\ell_1 \cup \ell)$ is linearly equivalent to $\phi^{-1}_V(\ell_2 \cup \ell)$ then $C_1$ and $C_2$ are linearly equivalent. In particular this says that $S = B \times \mathbb{P}^1$ and we exactly are in case (b.1) of (1.1.1) with $n = 2$.

By what is said just before (6.1) it remains to consider the case when $\phi_V(S) = \mathbb{P}^2$ and $\phi_V(J_1)$ contains a smooth conic $C$. In this situation $\mathcal{D}(S, V) = C^\vee \cup \mathcal{D}^1 \cup \cdots \cup \mathcal{D}^s$ where, for $1 \leq i \leq s$, each $\mathcal{D}^i$ is a point in $|V|$ corresponding to a line in $\phi_V(J_1)$.

Suppose for the moment that $(S, L)$ is a scroll. Take a general $\ell_1 \in C^\vee$ corresponding to $D_1 \in \mathcal{D}(S, V)$. By (2.5.4) for any $x \in \text{Sing}(D_1)$ it holds that $D_1 = f_{\pi(x)} + R$ with $R$ an effective divisor smooth at $x$. In particular $\mu_x(D_1) = 1$ and $c_2(J_1(L)) = L^2$ is an even number by (3.2.3). Then, by the same argument as in (2.5.4), we obtain $D_1 = M + f_1 + \cdots + f_{L^2/2}$. Since $L^2 = D_1^2$ then $M^2 = 0$. The same construction can be done for another general line $\ell_2(\neq \ell_1) \in C^\vee$. In particular $D_2 = M' + f'_1 + \cdots + f'_{L^2/2}$. Now $\phi_V(M) = \ell_1 \neq \ell_2 = \phi_V(M')$ and so $M \neq M'$. Since $MM' = 0$ and $M, M'$ are irreducible, we have constructed a one dimensional family of pairwise disjoint sections. Moving the point $\ell$ on $C^\vee$ we get a rational parametrization of these sections $M$. So $M$ is linearly equivalent to $M'$ showing that we again obtain a product of a smooth curve cross $\mathbb{P}^1$ as in (b.1) of (1.1.1) with $n = 2$.

Suppose now that $(S, L)$ is not a scroll. In this case (3.2.7) reads as $e(S) + 2K_S L + 3L^2 \leq 2L^2 - 2$, where $e(S)$ is the topological Euler–Poincaré characteristic of $S$. In particular this gives:

\begin{equation}
(6.2) \quad e(S) + 2K_S L \leq -L^2 - 2.
\end{equation}

Since $(S, L)$ is neither a scroll nor $(\mathbb{P}^2, O_{\mathbb{P}^2}(r))$ with $r = 1, 2$ then $(K_S + L)^2 \geq 0$ by [LP1, 2.1], or equivalently:

\begin{equation}
(6.3) \quad 2K_S L \geq -K_S^2 - L^2.
\end{equation}

Substituting (6.3) in (6.2) we have:

\begin{equation}
(6.4) \quad -L^2 - 2 \geq e(S) + 2K_S L \geq e(S) - K_S^2 - L^2.
\end{equation}

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In particular

\[(6.5)\]
\[e(S) - K_S^2 \leq -2.\]

By (2.5.4) \(S\) is ruled, then either \((S, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r)), r \geq 3,\) or \(S \neq \mathbb{P}^2, e(S) = 4(1 - q) + s\) \((s\) is a nonnegative integer), \(K_S^2 = 8(1 - q) - s\). By Bezout’s theorem (6.6) the sum of the Milnor numbers of the singularities of a plane curve of degree \(r\) with isolated singularities is less than or equal to \((r - 1)^2\).

If \((S, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r))\) then by (6.6), (3.2.7) and (3.3.2) we get the contradiction \(3(r - 1)^2 = c_2(J_1(L)) \leq 2(r - 1)^2\).

Let us observe the following general fact: take a general \(D \in \mathcal{D}\) which corresponds to a line tangent to \(C\) at \(y\). Then, using (3.2) and (3.2.6) we get:

\[(6.7)\] If \(c_2(J_1(L)) = 2(L^2 - 1)\) then \(\phi_V^{-1}(y) = \{x\} \subset \text{Sing}(D)\) and \(\mu_x(D) = L^2 - 1\).

\[(6.8)\] If \(c_2(J_1(L)) = 2(L^2 - 2)\) then \(\phi_V^{-1}(y) = \{x, x'\} \subset \text{Sing}(D)\) and \(\mu_x(D) + \mu_{x'}(D) = L^2 - 2\) (possibly \(x = x'\) and \(\mu_x(D) = L^2 - 2\)).

If \((S, L) \neq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r))\) then (6.4) gives

\[-L^2 - 2 \geq 4(1 - q) + s + 2K_SL \geq 4(1 - q) + s - K_S^2 - L^2 \geq -4(1 - q) + s - L^2.\]

In particular \(s + 2 \leq 4(1 - q)\), that is, \(q = 0\) and \(s \leq 2\). Hence, by Noether formula, we have \(K_S^2 = 8 - s\). Then either \(q = 0, s = 2, K_S^2 = 6, e(S) = 6,\) not compatible with (6.5); or \(q = 0, s = 1, K_S^2 = 7, e(S) = 5;\) or \(q = 0, s = 0, K_S^2 = 8, e(S) = 4.\)

If \(q = 0, s = 1, K_S^2 = 7, e(S) = 5\) then equality holds in (6.4), hence in both (6.2) and (6.3). By [LP1,2.1] equality \((K_S + L)^2 = 0\) implies that either \(S\) is a Del Pezzo surface, \(L \equiv -K_S, c_2(J_1(L)) = 12,\) (excluded by (6.6) because we are dealing with plane cubic curves), or

\[(6.9)\] \((S, L)\) is a rational conic bundle, more precisely, \(S\) is a blowing up at \(s = 1\) point of a \(\mathbb{P}^1\) bundle over \(\mathbb{P}^1\) of invariant \(e \geq 0.\) Denote by \(E\) the exceptional divisor and \(C_0\) and \(f,\) as an abuse of notation, the proper transforms of the corresponding \(C_0\) and \(f\) on the \(\mathbb{P}^1\) bundle. Hence \(L = 2C_0 + bf - E, L|_f = \mathcal{O}_{\mathbb{P}^1}(2), c_2(J_1(L)) = 2(L^2 - 1)\) and \(L^2 = 4(b - e) - 1.\)

If \(q = 0, s = 0, K_S^2 = 8, e(S) = 4\) then \(S\) is a rational \(\mathbb{P}^1\)-bundle of invariant \(e \geq 0\) and \((S, L)\) is not a scroll. Whence \(L = aC_0 + bf, a \geq 2\) and \(b > ae.\) By (5.4.1) we have

\[(6.9.1)\]
\[4 + 2ae - 4a - 4b + 6ab - 3a^2e \leq 4ab - 2a^2e - 2,\]

or equivalently \(b(2a - 4) - a^2e + 2ae - 4a \leq -6.\) Since \(a \geq 2\) and \(b \geq ae + 1\) then \((ae + 1)(2a - 4) - a^2e + 2ae - 4a \leq -6.\) Then \(a(e(a - 2)) \leq -2\) which gives that either \(e = 0,\) or \(e = 1,\) \(a = 2, 3\) or \(e \geq 2\) and \(a = 2.\) If \(a = 2\) then we get (6.10) else we get the following. In the case \(e = 0\) we can suppose \(0 < a \leq b\) and the inequality (6.9.1) implies that \(a = 3 = b.\) If \(e = 1\) and \(a = 3\) then \(c_2(J_1(L))\) is
odd, contradicting (3.2.3). If \( e = 0, a = b = 3 \) then the ramification divisor’s class is \( R = K_S + 3L = 7C_0 + 7f \) and \( c_2(J_1(L)) = 2(L^2 - 1) = 34 \). By (6.7), moving the singular point, there exists an effective divisor \( F \) on \( S \) such that \( R - 17F \geq 0 \). This clearly gives a contradiction.

(6.10) If \( a = 2 \) then \( S \) is a \( \mathbb{P}^1 \) bundle over \( \mathbb{P}^1 \), \( L|_f = \mathcal{O}_{\mathbb{P}^1}(2) \) for any fibre \( f \) of \( S \), \( c_2(J_1(L)) = 2(L^2 - 2) \) and \( L = 2C_0 + bf \). We just need to face (6.9) and (6.10).

Let us observe in (6.10) that ampleness is equivalent to very ampleness, hence \( \phi_V(S) \) is the projection of \( \phi_L(S) \subset \mathbb{P}^{h^0(S,L)-1} \) from a codimension three linear space \( T \) such that \( T \cap \phi_L(S) = \emptyset \). A similar situation occurs in (6.9). We can blow down the exceptional divisor to obtain \( S' \). Consider the line bundle \( L' = 2C_0 + bf \) that is, in fact, very ample. Take \( V' \subset H^0(S',L') \) defining a linear system with just one base point. Then \( \phi_V \) is the morphism resolving the indeterminacy of the rational map defined by \( |V'| \). Hence we are projecting \( \phi_{L'}(S') \subset \mathbb{P}^{h^0(S',L')-1} \) from a codimension 3 linear space \( T \) meeting \( \phi_{L'}(S') \) in one point.

First let us deal with the case (6.10). In this case \( L^2 = 4(b-e) \) and \( c_2(J_1(L)) = 2(L^2 - 2) > L^2 \) since \( (S,L) \) is not a scroll, hence

\[(6.10.1) \quad L^2 = 4(b-e) \geq 8.\]

Consider a general \( D \in |V| \). By Riemann–Hurwitz formula the ramification divisor \( R_D \) of \( \phi_{V|D} \) verifies \( \deg(R_D) = L(L + K_S) + 2L^2 = 10(b-e) - 4 \). On the other hand there exist effective divisors \( F_i = a_iC_0 + b_if > 0 \) (\( 1 \leq i \leq s \)) such that \( |V - 2F_i| = D^i \) and \( \phi_V(F_i) = \ell_i \). Then there exist integers \( \alpha_i \geq 1 \) and a divisor \( G > 0 \) such that the ramification divisor \( R \) of \( \phi_V \) verifies \( R = \alpha_1F_1 + \cdots + \alpha_sF_s + G \) and \( \phi_V(G) = C \). Moreover, by (6.8), there exist two divisors \( G_1 = A_1C_0 + B_1f > 0, G_2 = A_2C_0 + B_2f > 0 \) (maybe equal) and two integers \( z_1, z_2 \geq 0, z_1 + z_2 = L^2 - 2 \) such that

\[(6.10.2) \quad R = \alpha_1F_1 + \cdots + \alpha_sF_s + z_1G_1 + z_2G_2.\]

Since any ramification point of \( \phi_{V|D} \) is a ramification point of \( \phi_V \) we get

\[10(b-e) - 4 = \deg(R_D) = RL \leq \alpha_1 + \cdots + \alpha_s + L^2 - 2 = \alpha_1 + \cdots + \alpha_s + 4(b-e) - 2,\]

which implies

\[(6.10.3) \quad \alpha_1 + \cdots + \alpha_s \geq 6(b-e) - 2.\]

By the ramification formula \( R = K_S + 3L = 4C_0 + (3b - 2 - e)f \) then:

\[
\begin{align*}
\alpha_1a_1 + \cdots + \alpha_sa_s + z_1A_1 + z_2A_2 &= 4 \\
\alpha_1b_1 + \cdots + \alpha sb_s + z_1B_1 + z_2B_2 &= 3b - e - 2.
\end{align*}
\]

Note that the effectiveness of \( F_i \) and \( G_i \) implies that \( a_i + b_i \geq 1 \) and \( A_i + B_i \geq 1 \). Adding both equalities and using (6.10.3) we get \( 7b \leq 9e + 6 \). This is a contradiction because \( b \geq 2e + 1 \) by the ampleness of \( L \).
If (6.9) holds we can argue in exactly the same way. In fact $L^2 = 4(b - e) - 1$ and $c_2(J_1(L)) = 2(L^2 - 1)$ then the expression for the ramification divisor is

\[(6.10.4) \quad R = \alpha_1 F_1 + \cdots + \alpha_s F_s + z_1 G = K_S + 3L = 4C_0 + (3b - 2 - e)f - 2E\]

with $z_1 = L^2 - 1$, $F_i = a_i C_0 + b_i f + c_i E$ and $G = AC_0 + B f + C E$. The formula now gives $\deg(R_D) = 10(b - e) - 6$. Whence the analogue of (6.10.3) is $\alpha_1 + \cdots + \alpha_s \geq 6(b - e) - 4$. Since $G > 0$ and $F_i > 0$ ($1 \leq i \leq s$), $A + B + C \geq 1$ and $a_i + b_i + c_i \geq 1$. Then, by (6.10.4) and the previous inequality we get $7b \leq 9e + 6$, a contradiction.

Summing up the discussion in the codegree two case we have proved the following

**Theorem.** Let $(X, L, V)$ be as in (0.0) with $\dim(X) = 2$ and $\text{codeg}(X, V) = 2$. Then $X = C \times \mathbb{P}^1$ for a smooth curve $C$, $|V|_{C}$ is a $g_3^1$ on $C$ defining a degree $d$ morphism $f : C \to \mathbb{P}^1$, and $L = F^*O_{Q}(1)$ where $F = s \circ (f \times \text{Id})$ is the composition of $f \times \text{Id} : C \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ with the Segre embedding $s : \mathbb{P}^1 \times \mathbb{P}^1 \to Q \subset \mathbb{P}^3$.

### 7 The codegree 3 case.

Let $(S, L, V)$ be a triplet as in (0.0) such that $\dim(S) = 2$ and $\text{codeg}(S, V) = 3$. Let $D^0$ be the union of the $(N - 1)$-dimensional irreducible components of $D(S, V)$. We have already considered the case of $D^0$ being the union of three distinct hyperplanes in (5.3). Hence one of the following holds:

1. $D^0 = Q \cup H$, where $Q$ is a smooth quadric hypersurface and $H = \mathbb{P}^N - 1$,
2. $D^0 = \phi_V(S)^\vee$,
3. $D^0 = Q \cup H$, where $Q$ is an irreducible quadric cone and $H = \mathbb{P}^N - 1$,
4. $D^0 = \phi_V(C)^\vee$ for $C \subset J_1$ an irreducible curve.

Consider (7.1) Since $Q^\vee \subset \phi_V(S)$ then either $\phi_V(S) \subset \mathbb{P}^3$ is a smooth quadric hypersurface or $\phi_V(S) = \mathbb{P}^2$ and $Q$ is a smooth conic. If $\phi_V(S)$ is a smooth quadric hypersurface then, by (6.1), either $(S, L)$ is a scroll, so that $J_2 = \emptyset$, or $|\phi_V(J_1)| \geq 2$. Hence this case does not occur. It remains to consider $\phi_V(S) = \mathbb{P}^2$ and $\phi_V(J_1) = C \cup \ell_1 \cup \cdots \cup \ell_s$, where $C$ is a smooth conic and $\ell_i$ is a line for $1 \leq i \leq s$. This case is effective as shown in (d.4.1) of (3.4).

Now consider (7.2). Since $\dim(\phi_V(S)^\vee) = N - 1$, $\phi_V(S)$ cannot be a cone. By (5.1), either $(S, L) = (\mathbb{P}^2, O_{\mathbb{P}^2}(2))$ (discussed in (d) of (3.4)) or $\deg(\phi_V(S)) \leq 3$ and $(S, L)$ is a scroll.

When $(S, L)$ is a scroll and $\deg(\phi_V(S)) = 3$ then $N = 3, 4$. Recall that $\phi_V(S)$ is not a cone. If $N = 4$ then $\phi_V(S) \subset \mathbb{P}^4$ is the rational normal scroll $\mathbb{P}(O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}) \subset \mathbb{P}^4$, [XXX], which has a (single) section that is a line, say $C$. If $N = 3$ then $\phi_V(S)$ is a (finite and birational) projection of $\mathbb{P}(O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}) \subset \mathbb{P}^4$ from a point $p \in \mathbb{P}^4 \setminus \mathbb{P}(O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1})$. In view of the classification of cubic ruled surfaces [E, Ch. I, §37, 38, 48, 49] there are two types:
(7.2.1) $p$ lies on the plane spanned by a smooth conic $Q \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}) \subset \mathbb{P}^4$. Then the image by the projection from $p$ has two directrix lines $C_1$ and $C_2$ (lines meeting each line of the ruling), being $C_1$ the projection of $C$ and $C_2$ the projection of $Q$ (a double line).

(7.2.2) $p$ lies on the plane plane spanned by a general fiber $f$ and $C$. Then the image by the projection has just one directrix line $C_1$ that is the projection of $C$ (or of $f$, so that a double line).

Let us consider first $N = 4$. Take a general point $p \in \phi_V(S)$ and a general hyperplane section $H$ singular at $p$. Then $H$ decomposes as the fibre through $p$, say $f_p$ and a conic $Q_p$ cutting $f_p$ just at $p$. Since $c_2(J_1(L)) = L^2 = \text{codeg}(X,V)$ and $\deg(\phi_V(S)) = 3$ then, by (3.2.4), $\phi_V^*(H)$ is singular at exactly the $L^2/3$ points constituting $\phi_V^{-1}(p)$. Then, by (2.5.4), $\phi_V^*(H) = M + f_1 + \cdots + f_{L^2/3}$ where $M$ is a smooth curve and $f_i$ is a fibre of $S$ such that $\phi_V(f_i) = f_p$. In particular $\phi_V^*(f_p) = f_1 + \cdots + f_{L^2/3}$. Take now $H$ cut out by the hyperplane containing two general fibres $f_q$ and $f_p$. Then $H = C + f_p + f_q$. By the previous arguments $\phi_V^*(H) = D + f_1 + \cdots + f_{L^2/3} + g_1 + \cdots + g_{L^2/3}$ where $D = \phi_V^*(C)$ is a curve such that $D^2 = -L^2/3$ and $\phi_V^*(f_q) = g_1 + \cdots + g_{L^2/3}$. Now $DM = D(D + g_1 + \cdots + g_{L^2/3}) = 0$. This implies that $E$ is decomposable [Ha, Exercise 2.2 p. 383]. Assume that $E$ is normalized, in the usual sense [Ha, p. 373]; then $E = \mathcal{O}_B \oplus \mathcal{L}$, where $\mathcal{L} \in \text{Pic}(B)$ is such that $e := -\deg\mathcal{L} \geq 0$ [Ha, Theorem 2.12 p. 376]. Let $C_0$ be, as usual, a tautological section. We can write $D \equiv C_0 + af$ for some integer $a$ and $M \equiv C_0 + bf$, where $b = a + \frac{L^2}{3}$. Moreover, $0 = DM = -e + a + b$. As $D$ is irreducible, we know from [Ha, Proposition 2.20 p. 382] that either $D = C_0$ or $a \geq e$. However, in the latter case we get $e = a + b = 2a + \frac{L^2}{3} \geq 2e + \frac{L^2}{3}$, giving $e + \frac{L^2}{3} \leq 0$, a contradiction. Therefore $D = C_0$, hence $a = 0$ and then $\frac{L^2}{3} = b = e$. In particular, $\phi_V|_D : D \to B$ is an isomorphism and $L \equiv C_0 + 2\frac{L^2}{3}f$.

If $N = 3$, as said before, there exists a finite and birational morphism $\pi_p : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}) \to \phi_V(S)$ (the projection from $p$) which is the normalization morphism. Then, by the universal property of the normalization, $\phi_V : S \to \phi_V(S)$ factors through $\pi_p$. Now we can get the same conclusion as in the previous paragraph with the following warning. For the general line $f_p$ on $\phi_V(S)$, we can consider $\phi_V^*(f_p) = \pi^*(f_p) = f_1 + \cdots + f_{L^2/3}$ and $M$ is defined exactly as before. To construct $D$ we do the following. In (7.2.1) just consider a hyperplane section $H$ containing $C_1$ and $f_p$, then there exists another line of the ruling, say $f_q$ and $H = C_1 + f_p + f_q$ and proceed as before. In (7.2.1) take a hyperplane section containing $C_1$ and $f_p$. Then $\phi_V^*(H) = \pi^*(C + f + f_p)$ and proceed as before.

If $\deg(\phi_V(S)) = 2$ then either $\phi_V(S)$ is a quadric cone or $\phi_V(S) \subset \mathbb{P}^3$ is a smooth quadric. If the former occurs then we get the contradiction that $D^0$ is degenerate. The latter contradicts $\deg(\phi_V(S)^\vee) = 3$.

Next consider (7.3). If $D^0$ contains an irreducible quadric cone $Q$ then by (1.4) $Q^\vee \subset \phi_V(S)$. Hence $Q^\vee$ is a smooth plane conic. Moreover $\dim(\phi_V(S))^\vee < N - 1$, and so, by (5.2), $N = 3$ and $\phi_V(S)$ is a quadric cone.
Finally we deal with (7.4). Since dim(ϕ_V(S)) < N − 1 and J_1 = ∅ then, by (5.2), ϕ_V(S) = \mathbb{P}^2.

Summing up the discussion on the codegree three case we get:

(7.5) **Theorem.** Let (X, L, V) as in (0.0) with dim(X) = 2 and codeg(X, V) = 3. Then, either

\begin{enumerate}
\item[(7.5.1)] (X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r)), or
\item[(7.5.2)] X = \mathbb{P}_B(\mathcal{O}_B \oplus \mathcal{O}_B(p^*\mathcal{O}_{\mathbb{P}^1}(-1))) \xrightarrow{\pi} B, where B is a smooth curve, p : B \to \mathbb{P}^1 is a surjective morphism and L ≡ C_0 + \pi^*p^*(\mathcal{O}_{\mathbb{P}^1}(2)); N = 3, 4 and ϕ_V(X) is a cubic ruled surface, or
\item[(7.5.3)] N = 3, ϕ_V(X) \subset \mathbb{P}^3 is an irreducible quadric cone, ϕ_V(J_1) = C \cup ℓ_1 \cup \cdots \cup ℓ_s where C is a smooth plane conic and ℓ_i is a line for 1 ≤ i ≤ s, or
\item[(7.5.4)] ϕ_V(X) = \mathbb{P}^2 and there exists an irreducible curve C \subseteq J_1 such that ϕ_V(C)\} is one of the maximal dimensional components of D.
\end{enumerate}

In (3.4.d) we have studied (7.5.1) when r = 2; (3.3.d) with n = 2 also provides examples of (7.5.1) for any r ≥ 3. Let us observe that the general hyperplane section of the Segre variety \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5 is the only example as in (7.5.2) in the classical setting, [Z2, p. 93]. We end this section with examples corresponding to (7.5.2), (7.5.3) and (7.5.4). Then all situations described in (7.2) are effective.

(7.6) **Examples**

(a) Let B be an elliptic curve, and let \rho' : B \to \mathbb{P}^1 be the morphism defined by a line bundle of degree 2 on B. Let p : S \to \mathbb{F}_1 be the double cover branched along the fibres of \mathbb{F}_1 corresponding to the four branch points of \rho'. Then S is a \mathbb{P}^1 bundle over B. Moreover its invariant is 2. To see this denote by γ_0 the (-1)-section of \mathbb{F}_1 and note that C_0 = p^{-1}(γ_0) = p^*γ_0 is the section of minimal self-intersection on S. Set L := p^*[γ_0+2φ], where φ is a fibre of \mathbb{F}_1, and V := p^*H^0(\mathbb{F}_1,[γ_0+2φ]). Let f be the general fibre of S. As p^*φ consists of two fibres of S, we have L ≡ C_0 + 4f. Note that L is ample by [Ha, Proposition 2.20 p. 382]; moreover V spans L by construction, L^2 = 6 and φ_V|_{C_0} = p. This gives an example of (7.5.2) with g(B) > 0.

(b) Let ρ : \mathbb{F}_2 \to \Gamma \subset \mathbb{P}^3 be the minimal desingularization of the quadric cone. Let C_0 and f be the minimal section and a fibre of \mathbb{F}_2. Note that ρ^*(\mathcal{O}_\Gamma(1)) = [C_0 + 2f]. Let C \in [C_0 + 2f] be a smooth curve (the pull-back of a general hyperplane section γ of Γ). Then Δ := C_0 + C is a smooth divisor in the linear system |2B|, where B = [C_0 + f]. Thus there exists a smooth surface X and a morphism ρ : X \to \mathbb{F}_2 of degree 2 branched along Δ. Let E := ρ^{-1}(C_0); thus ρ^*(C_0) = 2E, since C_0 is in Δ. Moreover, 4E^2 = (2E)^2 = (ρ^*C_0)^2 = 2C_0^2 = -4. So E is a (-1)-curve inside X, and we can contract it, obtaining a smooth surface S. Let μ : X \to S be the contraction and set x = μ(E). Then we get a commutative
Therefore

$$\pi : S \to \Gamma$$

is the induced double cover. Note that $\pi$ is branched along

$$\gamma = \nu(C)$$

and at vertex $v$ of $\Gamma$, and $v = \pi(x)$. Put

$$L := \pi^* \mathcal{O}_\Gamma(1) \quad \text{and} \quad V = H^0(S, L).$$

Then $\phi_V = \pi$. We have

$$\mathcal{J}_2(S, V) = \{ x \},$$

while

$$\mathcal{J}_1(S, V) \setminus \mathcal{J}_2(S, V) = \pi^{-1}(\gamma).$$

It follows that $\mathcal{D}(S, V)$ consists of: $\mathcal{D}_0$, the dual of $\Gamma$, which is a smooth conic; $\mathcal{D}_1$, the dual of $\gamma$, which is a quadric cone (because $\gamma$ is a plane curve) containing $\mathcal{D}_0$; and

$$\mathcal{D}_2$$

the plane of $\mathbb{P}^3\setminus \mathcal{V}$ parameterizing the planes through the vertex $v$. Therefore $\text{codeg}(S, V) = \deg \mathcal{D}_1 + \deg \mathcal{D}_2 = 3$. This gives an example as in (7.5.3).

It deserves to explore the example above a little bit more, to recognize a situation early described in (3.4). Note that the ruling projection $\mathbb{P}_2 \to \mathbb{P}^1$ induces a fibration $X \to \mathbb{P}^1$, whose general fibre $F := \rho^*(f)$ is a $\mathbb{P}^1$, being a double cover of $f$ branched at $\Delta \cap f$. Hence $X$ is rational, and so is $S$. By the ramification formula we have

$$\text{(7.6.1)} \quad K_X = \rho^*(K_{\mathbb{P}_2} + \mathcal{B}) = \rho^*(-2C_0 - 4f + C_0 + f) = -\rho^*(C_0 + 3f).$$

It thus follows that $K_X^2 = 2(C_0 + 3f)^2 = 8$, and so $K_2^2 = K_X^2 + 1 = 9$. Therefore $S = \mathbb{P}^2$. From the commutativity of the diagram above, recalling (7.6.1) and the fact that $K_X = \mu^*K_S + E$, we also see that

$$\mu^*(\nu^* \mathcal{O}_\Gamma(1)) = \rho^*(\nu^* \mathcal{O}_\Gamma(1)) = \rho^*(C_0 + 2f) = (2E + 2F) = \frac{2}{3}(2E + 3F + E) = \frac{2}{3}(-K_X + E) = \frac{2}{3}\mu^*(-K_S).$$

Therefore $\nu^* \mathcal{O}_\Gamma(1) = \frac{2}{3}(-K_S) = \mathcal{O}_{\mathbb{P}_2}(2)$. This shows that $(S, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}_2}(2))$, and we fall in case (d.2) of (3.4).

(c) In (7.5.4), as $N = 2$, either $\phi_V(C)$ is a smooth plane conic and $|\phi_V(\mathcal{J}_2)| = 1$ or $\phi_V(C)^\vee = \mathcal{D}$, $\mathcal{J}_2 = \emptyset$ and so $S$ is a ruled surface by (5.4). Note that for a general codimension one subvector space $V' \subset V$ of any example as in (7.5.3) we get the former situation. On the other hand something else can be said thanks to Plücker formulas. If $\phi_V(C)$ has ordinary singularities then either

$\phi_V(C)$ is a sextic with nine cusps (and no other singularities) and $\phi_V(C)^\vee$ is a smooth plane cubic (this example is effective as shown in (d) of (2.6), and in (d.3.1) of (3.4)), or

$\phi_V(C)$ is a quartic with three cusps (and no other singularities) and $\phi_V(C)^\vee$ is a nodal cubic (this example is effective as shown in (d.3.2) of (3.4)), or

$\phi_V(C)$ and so $\phi_V(C)^\vee$ are cuspidal cubics. Let us put an example of this last situation. Take $X = \mathbb{P}^1 \times \mathbb{P}^2$ and $L$ the line bundle defining the Segre embedding in $\mathbb{P}^5$. Consider $V \subset H^0(X, L)$ defining a general base-point free linear system with
dim(⌈V⌉) = 3. Since $X^\vee = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^{5\vee}$ we have that $X^\vee \cap |V| = C_0 \subset \mathbb{P}^3$ is a twisted cubic. Call $TC_0 \subset \mathbb{P}^3$ the tangent variety to $C_0 \subset \mathbb{P}^3$, that is, the union of its tangent lines. Consider $H \in TC_0 \setminus C_0$. In particular $H \notin X^\vee$. Then $X \cap H = S \subset \mathbb{P}^4$ is a smooth cubic scroll. Let us observe that the restriction of $|V|$ to $S$ is a base-point free linear system of dimension 2. In fact one can suppose that $V = \langle s_0, s_1, s_2, s_3 \rangle$, where $s_0$ defines $H$. Then, when restricting to $S$, $V|_S = \langle s_1|_S, s_2|_S, s_3|_S \rangle$. If there exists $x \in Bs|V|_S$ then $s_0, \ldots, s_3$ vanish at $x$, contradicting that $|V|$ is base-point free. It is classically well known that the projection $\pi_H$ from $H$ gives the identification $S^\vee = D(S, L|_S) = \pi_H(X^\vee) = \pi_H(D(X, L))$; then $D(S, V|_S) = \pi_H(D(X, L)) \cap |V|_S = \pi_H(D(X, L) \cap |V|) = \pi_H(C_0)$ a cuspidal curve.

**8 Final remarks**

In this section we present some problems which we consider of interest.

(8.1) As pointed out after (5.1), in the classical setting it is possible to classify surfaces for which the difference

\[(8.1.1) c_2(J_1(L)) - L^2 = \text{class}(S) - \deg(S)\]

is small. In the ample and spanned case, triplets $(S, L, V)$ for which the right hand term of (8.1.1) is less than or equal to zero are listed in [LPS 1, Prop. A.1]. In line with this we have stated (5.1) where surfaces for which $\text{codeg}(S, V) - \deg(\phi_V(S)) \leq 0$ are considered. In this context it has sense the following definition:

(8.1.2) **Definition.** Let $(X, L, V)$ be a triplet as in (0.0). We say that $(X, V)$ has tame codegree if $\text{codeg}(X, V) = c_n(J_1(L))$.

Pairs in examples (a) and (c) of (3.3) have tame codegree while for (b) (and $d \geq 2$ in (3.3) we have $\text{codeg}(X, V) < c_n(J_1(L))$. In the classical setting, i. e., when $\phi_V$ gives an embedding, having tame codegree simply means that $D(X, V)$ is a hypersurface, because in that case $c_n(J_1(L)) = \deg(D(X, V))$. More generally, in the ample and spanned setting, having tame codegree means that the general element in $D$ is singular in a single point and the singularity is just a non-degenerate quadratic singularity, see (3.2.4). Let us show another example.

(8.1.3) Let $S = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} B$, where $\mathcal{E}$ is the rank-2 vector bundle over a smooth curve $B$ of genus 1 of (2.6.d), defined by a non-split exact sequence

$$0 \to \mathcal{O}_B \to \mathcal{E} \to \mathcal{O}_B(p) \to 0 \quad (p \in B),$$

and $L = 2\xi$, where $\xi$ is the tautological line bundle of $\mathcal{E}$. Clearly $L$ is ample since $\xi$ is so. Moreover, $L$ is spanned as Reider’s Theorem immediately shows. Note that $h^0(L) = h^0(S^2\mathcal{E}) = \deg(\mathcal{E}) = 3$. So $(S, L, V = H^0(S, L))$ is as in (0.0). Let us show that $(S, V)$ has tame codegree. We can regard $S$ as the twofold symmetric product of the base elliptic curve $B$. Hence $L$ lifts to $B \times B$ via the natural double cover.
p : B × B → S as the line bundle \( p_i^*O_B(x+y) \otimes p_i^*O_B(x+y) \), where \( p_i : B × B → B \) is the projection onto the \( i \)-th factor, and \( x, y ∈ B \). On the other hand, since \( B \) is an elliptic curve, for any two points \( x, y ∈ B \) the linear series \( |x+y| \) is a \( g^2_2 \). So our \( L \) is like that appearing in [BDL, Ex. 9], where the ramification locus of \( φ \) is described. In fact the branch locus of the 4–to–1 map \( φ \) is the union of a smooth conic and four of its tangent lines. For any \( b \in B \) we will denote \( f_b = π^*O_B(b) \) and \( O_S(C_0) = ξ \). Note that \( L^2 = 4 \), hence \( c_2(J_1(L)) = 8 \). Observe that \( φ \) embeds \( f_p \) as a smooth plane conic, say \( γ \), and gives a 2–to–1 map from \( C_0 \) onto a line. For any \( t ∈ B \) note that \( h^0(S, O_S(C_0+f_p−f_t)) = h^0(S, O_S(C_0+f_t−f_p)) = 1 \). Then we can choose \( Γ_t ∈ |C_0+f_p−f_t| \) and \( Γ'_t ∈ |C_0+f_t−f_p| \) such that \( Γ_t + Γ'_t ∈ |2C_0| = |L| \) and so \( φ(Γ_t) = φ(Γ'_t) \). Moreover one can check by [BDL, Ex. 9] that \( φ(Γ_t + Γ'_t) \) meets \( γ \) in just one point. This gives \( γ^0 ⊆ D(S, V) \). Moreover there exist \( p_i ∈ B, i = 1, 2, 3 \), such that \( O_B(2p_i) = O_B(2p) \). Call \( p_0 = p \). This produces four non–reduced elements \( 2Γ_{p_i} ∈ |2C_0| \). Hence the six lines \( ⟨2Γ_{p_1}, 2Γ_{p_2}⟩ ⊂ |V| \) are contained in \( D(S, V) \). This gives \( \text{codeg}(X, V) ≥ 8 \) and in fact an equality by (3.2.2).

In the following paragraphs we classify surfaces as in (0.0) having tame codegree \( ≤ 8 \). The argument relies on some rough inequalities. In fact, a more careful analysis would permit to discuss also higher values. We confine to codegree \( ≤ 8 \) because 8 is the smallest value giving rise to the nice example discussed above.

So, let \( n = 2 \) and set \( S = X \). Recall that \( c_2(J_1(L)) − L^2 = e(S) + 4(g − 1) \), where \( e(S) \) is the topological Euler–Poincaré characteristic of \( S \) and \( g = g(L) \). Suppose that \( (S, L) \) is neither \( (\mathbb{P}^2, O(e)) \), \( e = 1, 2 \), nor a scroll. Then \( c_2(J_1(L)) − L^2 > 0 \) [LPS1, Prop. A.1]. If \( S \) is not (birationally) ruled, then \( e(S) ≥ 0 \) by the Castelnuovo–De Franchis theorem. Moreover, \( g ≥ 2 \), with equality if and only if \( S \) is the K3 double plane [LP3, Thm. 3.1], in which case, however, \( e(S) = 24 \). Thus \( e(S) + 4(g − 1) ≥ 8 \) if \( S \) is non-ruled. In particular \( c_2(J_1(L)) > 8 \). Now suppose that \( S \) is ruled. Due to our assumptions on \( (S, L) \) we know that \( g ≥ 1 \), and equality occurs if and only if \( S \) is a Del Pezzo surface and \( L = −K_S \). For such surfaces we have \( e(S) = 12 − L^2 \) by Noether’s formula. Hence \( c_2(J_1(L)) = 12 \). Assume that \( g ≥ 2 \). If \( S ∼= \mathbb{P}^2 \), then \( g ≥ 3 \) by Clebsch formula, hence \( e(S) + 4(g − 1) ≥ 3 + 8 = 11 \). So, \( c_2(J_1(L)) > 12 \). On the other hand, if \( S ∼= \mathbb{P}^2 \), then there exists a birational morphism \( η : S → S_0 \), where \( S_0 \) is a \( \mathbb{P}^1 \)-bundle over a smooth curve of genus \( q = h^1(O_S) \). Thus \( e(S) = e(S_0) + s = 4(1 − q) + s \), where \( s \) is the number of blowing–ups \( η \) factors though. So we have \( e(S) + 4(g − 1) = 4(1 − q) + s + 4(g − 1) ≥ 4(g − q) \).

As \( (S, L) \) is not a scroll, we know that \( K_S + L \) is nef, hence

\[
0 ≤ (K_S + L)^2 = K_S^2 + 2(K_S + L)L − L^2 ≤ 8(1 − q) + 4(g − 1) − L^2 < 4(1 + g − 2q).
\]

This says that \( g ≥ 2q \). All cases with \( g ≤ 1 \) being already considered, we conclude that \( g − q ≥ 1 \) equality occurring only for \( g = 2 \). Since \( L \) is ample and spanned, taking into account [LP3, Theorem 3.1] we see that \( 2 = g = q + 1 \) only for the pair \( (S, L) \) in (8.1.3)

So, apart from the pair in (8.1.3) we have \( g − q ≥ 2 \), and then \( e(S) + 4(g − 1) ≥ 8 \). In particular, \( c_2(J_1(L)) > 8 \). The discussion above proves the following
(8.1.4) Proposition. Let \((S, L, V)\) be as in (0.0), with \(\dim S = 2\) and suppose that \((S, V)\) has tame codegree \(\leq 8\). If \((S, L)\) is neither \(\left(\mathbb{P}^2, \mathcal{O}(e)\right)\), \(e = 1, 2\), nor a scroll, then \(\text{codeg}(S, V) = 8\) and \((S, L, V)\) is as in (8.1.3).

Another question is the following:

(8.2) In view of (0.4), for \((X, L, V)\) as in (0.0) with \(\dim(X) \geq 2\) we can define \(R(X, V) = \{D \in |V| : D \text{ is reducible or non-reduced}\} \subseteq D(X, V)\). The following conjecture is quite similar to [BDL, Conjecture 1]:

(8.2.1) Conjecture. Let \((X, L, V)\) be a triplet as in (0.0) such that \(\dim(X) \geq 2\), \(N > n\). Then \(R(X, V) = D(X, V)\) if and only if \((X, L)\) is either \(\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)\right)\) or a scroll over a curve.

In fact, in [BDL, Conjecture 1] the requirement on the discriminant locus is replaced with the following condition: for all \(x \in X\), (i) \(|V - 2x| \neq \emptyset\) and (ii) any \(D \in |V - 2x|\) is reducible or non-reduced. Let us note that (i) is equivalent to \(\dim(|V|) \geq n + 1\) and we cannot drop out this hypothesis, allowing \(\phi_V(X) = \mathbb{P}^N\), as example (d) in (3.3) shows. A first evidence for this conjecture is that it is true in the classical case, [BDL, Prop. 7].

Finally we comment on the following problem.

(8.3) A relevant result in the classical setting is the so-called Landman’s parity theorem. The precise statement is as follows: for \((X, L, V)\) as in (0.0) with \(\phi_V\) an embedding the difference between the defect and the dimension of \(X\) is an even number. This result is not known to be true or false in the ample and spanned case. First proof of this theorem comes from Landman [L], [K, II (22)] and is essentially topological. In fact since the codimension of the discriminant is bigger than one one can construct a pencil in \(|V|\) not cutting \(D\), i.e., all its elements are smooth. A consequence of the existence of these pencils is that the equalities of Betti numbers of sections of \(X\) provided by the Lefschetz theorem go further for positive defect varieties. This is also true in the ample and spanned case. Last part of the proof relies on the fact that the singular locus of a general element of \(D\) is very well known and provides a vanishing cycle and a monodromy relation giving the parity result. This last part cannot be applied to the ample and spanned case. Another proof of Landman’s parity theorem can be found in [Ei]. In the classical setting, when the discriminant is not a hypersurface, the singular locus of a general element in \(D\) is a linear space \(\mathcal{T}\) of dimension \(k > 0\) and any of its points is a non-degenerate quadratic singularity. Then, the second fundamental form gives a symmetric isomorphism between the normal bundle \(N_{\mathcal{T}/X}\) and the twist of its dual \(N_{\mathcal{T}/X}^\vee(1)\). The symmetry of the isomorphism and basic considerations on normal bundles have several relevant consequences like parity theorem (among others). This puts in relation (8.3) with [LPS1, Conjecture 2.11], the conjecture stating that the the singular locus of a general element in \(D\) is a disjoint union of linear spaces \(\mathcal{T}\) of dimension \(k\); in particular \(X\) is swept out by these linear spaces (for \(\mathcal{T}\) to be linear we mean isomorphic to \(\mathbb{P}^k\) and \(TL^{n-k} = 1\)). Also [A, Prop. 2.5] shows
that [LPS1, Conjecture 2.11] implies parity theorem.

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