Special functions from quantum canonical transformations

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Abstract

Quantum canonical transformations are used to derive the integral representations and Kummer solutions of the confluent hypergeometric and hypergeometric equations. Integral representations of the solutions of the non-periodic three body Toda equation are also found. The derivation of these representations motivate the form of a two-dimensional generalized hypergeometric equation which contains the non-periodic Toda equation as a special case and whose solutions may be obtained by quantum canonical transformation.

1 Introduction

The confluent hypergeometric and hypergeometric equations underlie many of the special functions of mathematical physics and, by extension, many of the exactly soluble problems of quantum mechanics in one dimension. The solutions of these

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equations have of course been well understood for more than a hundred years (see e.g. [1]). This paper returns to derive the integral representations and Kummer solutions of the confluent hypergeometric and hypergeometric equations using a new approach whose motivation comes from physics. The purpose of this is two-fold: to give an elegant and economical derivation of time-honored results and to illustrate the use of some basic tools which have application to solving more difficult problems. As an illustration, simple derivations of two integral representations of the wavefunctions for the (non-periodic) three-body Toda equation are given. This motivates a 2-dimensional generalized hypergeometric equation which includes the non-periodic Toda equation as a special case.

The origin of the method to be used lies in quantum mechanics. Given a linear differential equation, it can be interpreted as a time-independent Schrodinger equation at a given energy, where the dependent variable is taken to be the wavefunction and the differential operator to be the Hamiltonian in the coordinate representation. (For convenience, the discussion will be given assuming the differential equation is in terms of a single variable; the generalization to many variables is straightforward.) The differential operator \( \text{cum} \) Hamiltonian can be abstracted from the coordinate representation and viewed directly as a function of non-commuting variables \( x \) and \( \partial \) which satisfy the commutation relation \( [x, \partial] = -1 \). The coordinate representation consists in taking \( \partial = d/dx \). We use \( \partial \) instead of the physicists’ \( p \) to avoid unnecessary factors of \( i \); expressions in terms of \( p \) are simply obtained by substituting \( \partial = ip \).

The method consists in making a quantum canonical transformation from the given Hamiltonian to another one having known solutions. A quantum canonical transformation is a change of variables \( (x, \partial) \rightarrow (x'(x, \partial), \partial'(x, \partial)) \) which preserves the algebraic relation \( [x, \partial] = -1 = [x', \partial'] \). It is so called by analogy to a classical canonical transformation which is a change of the phase space variables which preserves the Poisson bracket. Quantum canonical transformation are produced by acting with a function \( C(x, \partial) \),

\[
q \mapsto q' = CqC^{-1}, \quad \partial \mapsto \partial' = C\partial C^{-1}. \tag{1}
\]

Under this transformation, any function of \( x \) and \( \partial \) transforms

\[
H(x, \partial) \mapsto H'(x, \partial) = CH(x, \partial)C^{-1} = H(CxC^{-1}, C\partial C^{-1}). \tag{2}
\]

A solution \( \psi \) of \( H\psi = 0 \) is obtained from a solution \( \psi' \) of \( CHC^{-1}\psi' = H'\psi' = 0 \) by

\[
\psi = C^{-1}\psi'. \tag{3}
\]

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Care must be taken if the kernel of $C$ as an operator in the coordinate representation is non-trivial as not all $\psi$ obtained in this way will be solutions of $H$ (see \[2\]).

2 Quantum canonical transformations

To be more precise about the quantum canonical transformations, one must address the nature of functions $C(x, \partial)$ and their representation as operators. By introducing $x^{-1}$ and $\partial^{-1}$ as the algebraic inverses of $x$ and $\partial$ ($xx^{-1} = x^{-1}x = 1, \partial\partial^{-1} = \partial^{-1}\partial = 1$), one can define an algebra $U$ consisting of functions of the variables $x$ and $\partial$ and their inverses, consistent with the commutation relation $[x, \partial] = 1$. The algebra is constructed so that every element $C \in U$ has an algebraic inverse $C^{-1} \in U$, and so that elements transform under a quantum canonical transformation by (2). Details are given in Ref. \[2\]. The precise class of functions has not been identified with a named class, but it essentially consists of those functions and their algebraic inverses which do not involve distributions in themselves or any of their derivatives (the functions may have poles, branch points or essential singularities). Every linear differential operator with suitably smooth coefficients corresponds to an element of this algebra. Very likely it is possible to generalize the analysis to treat operators whose coefficients are not smooth, but this has not been investigated.

The functions $C(x, \partial)$ and their inverses are well-defined as elements of the algebra $U$. Their realization as operators in the coordinate representation is more subtle. (For convenience, the same notation will be used for the elements of the algebra and their realization as operators.) Care must be taken because the operators corresponding to $C$ and $C^{-1}$ are not strictly inverses because their kernels may be non-vanishing. One can define them as inverses when restricted to act on suitable subspaces with the kernels projected out\[2\], but it is practical to think of them as being inverses up to arbitrary linear combinations of elements in their respective kernels. For example, the inverse of differentiation, $\partial = d/dx$, is indefinite integration, $\partial^{-1} = \int x dx$, up to an arbitrary additive constant (an element of $\ker \partial$).

One might be concerned about possible problems in defining the operator realization of the quantum canonical transformations. The property that makes the quantum canonical transformations useful is that they can be decomposed into a product of elementary transformations, each of which has a well-defined action on wavefunctions\[2\]. There are three elementary canonical transformations:
1) similarity transformations, \( C = \exp(\int x f(q) dq) \),

\[
x \mapsto x, \quad \partial \mapsto \partial - f(x) : \quad C\psi(x) = \exp(\int x f(q) dq)\psi(x),
\]

(4)

2) point canonical transformations, \( P_{f(x)} \),

\[
x \mapsto f(x), \quad \partial \mapsto \frac{1}{f'(x)}\partial : \quad P_{f(x)}\psi(x) = \psi(f(x)),
\]

(5)

and 3) interchange, \( I \),

\[
x \mapsto -i\partial, \quad \partial \mapsto -ix : \quad I\psi(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dy e^{ixy}\psi(y).
\]

(6)

The functions in each transformation are allowed to be complex. In the differential equation context, these correspond to similarity transformations, change of variables, and Fourier transform. By composing them, one obtains ordered functions of operators whose behavior on wavefunctions is well-defined. In particular,

\[
I \exp(\int_{i}^{x} f(q) dq) I^{-1} = \exp(\int_{\partial} f(q) dq)
\]

and

\[
IP_{f(ix)} I^{-1} = P_{f(\partial)}
\]

give the analogous similarity \((x, \partial) \mapsto (x + f(\partial), \partial)\) and point canonical transformations \((x, \partial) \mapsto (f'(\partial)^{-1}x, f(\partial))\) involving \(\partial\). These are the basic tools for making quantum canonical transformations. They encode in an efficient way the details of performing the standard manipulations of differential equations.

As an illustration, consider the transformation generated by \(t^{x\partial}\) which will be used later. This can be decomposed as

\[
t^{x\partial} = \exp(\ln t x\partial) = P_{\ln x} \exp(\ln t \partial) P_{e^x}.
\]

(7)

The middle transformation is recognized as a translation while the outer ones are point canonical transformations. Their collective action on the algebraic element \(x\) is then

\[
t^{x\partial} e^{x\partial} = P_{\ln x} \exp(\ln t \partial) P_{e^x} x P_{\ln x} \exp(-\ln t \partial) P_{e^x}
\]

(8)
and their action on a wavefunction is also to rescale $x$

$$t^{x\partial} \psi(x) = P_{lnx} \exp(ln t \partial) P_{e^x} \psi(x) = \psi(tx).$$

(9)

Alternatively, one could simply have observed that since

$$x^{-1}t^{x\partial}x = t^{x^{-1}x\partial x} = t^{x\partial + 1},$$

one has

$$t^{x\partial}xt^{-x\partial} = xt^{x\partial + 1}t^{-x\partial} = xt.$$ 

For simple manipulations like this, it is often not necessary to explicitly decompose the transformation into elementary canonical transformations, though it may be done.

There is one main difference between the use of quantum canonical transformations for physics and for simply solving differential equations. In physics, one often wants to find a transformation which takes the entire spectrum of an operator into that of another. In particular, one would like to find a canonical transformation from an interacting to a free theory. If one is interested instead in an equation like the hypergeometric equation, one is not concerned with the spectrum of the operator, but rather with a particular solution. The significance of this is that one does not need to transform to an operator for which one can find the general solution, only to one for which a single solution is easily identified. This will generate a single solution of the operator one is interested in. Other solutions can be obtained from other canonical transformations, especially from those which act as symmetries of one’s operator, taking it into another operator of the same form but with different parameters.

3 Confluent Hypergeometric Function

The differential operator for the confluent hypergeometric equation can be represented by the Hamiltonian

$$H = x\partial^2 + (b - x)\partial - a.$$  

(10)

The confluent hypergeometric functions are the solutions $\psi$ of

$$H\psi = 0.$$  

(11)

The Hamiltonian is easily canonically transformed to an equation soluble on inspection after it is rewritten as

$$H = (b + x\partial)\partial - (a + x\partial).$$  

(12)
The quantum canonical transformation produced by

\[ C = \frac{\Gamma(b + x\partial)}{\Gamma(a + x\partial)}. \]  

(13)

transforms the derivative

\[ \partial \mapsto C\partial C^{-1} = \frac{a + x\partial}{b + x\partial}\partial. \]  

(14)

This is easily verified by observing

\[ \partial \Gamma(a + x\partial)\partial^{-1} = \Gamma(a + \partial x) = \Gamma(a + x\partial + 1). \]

Since \( C \) commutes with \( b + x\partial \) and \( a + x\partial \), it transforms the Hamiltonian to

\[ H^{(a)} = CHC^{-1} = (a + x\partial)(\partial - 1). \]  

(15)

One of the solutions of \( H^{(a)}\psi^{(a)} = 0 \) is clearly \( \psi^{(a)} = e^x \). The complementary solution is easily found but won’t be needed. (In a problem for which one was interested in the complete spectrum \( H\psi = E\psi \), one wouldn’t be satisfied with this form. One would look for further transformations leading to, say, a free Hamiltonian \( H' = \partial^2 \).)

The solution of the confluent hypergeometric equation which corresponds to \( \psi^{(a)} = e^x \) is

\[ \psi = C^{-1}\psi^{(a)} = \frac{\Gamma(a + x\partial)}{\Gamma(b + x\partial)}e^x. \]  

(16)

Expanding the exponential in a power series in \( x \) and allowing the operators to act, one finds the familiar power series representation of the confluent hypergeometric function

\[ \psi = \sum_{n=0}^{\infty} \frac{\Gamma(a + n)x^n}{\Gamma(b + n)n!} = \frac{\Gamma(a)}{\Gamma(b)}M(a, b, x). \]  

(17)

One can obtain an integral representation of (17) by recognizing that \( C^{-1} \) times \( \Gamma(b - a) \) is a beta function,

\[ \frac{\Gamma(a + x\partial)\Gamma(b - a)}{\Gamma(b + x\partial)} = \int_0^1 dt t^{a+x\partial-1}(1 - t)^{b-a-1} \]  

(18)
\((\Re(b) > \Re(a) > 0)\). Applying this to \(e^x\), and using (9), one finds
\[
\frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)} M(a, b, x) = \int_0^1 dt \, t^{a-1} (1 - t)^{b-a-1} e^t x. \tag{19}
\]
This is a familiar integral representation from which others may be obtained by change of variables.

A Barnes-type contour integral representation is also easily obtained by expressing \(e^x\) as
\[
e^x = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \, \Gamma(-s)(-x)^s. \tag{20}
\]
Applying \(C^{-1}\) to this, one has
\[
\frac{\Gamma(a)}{\Gamma(b)} M(a, b, x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \, \frac{\Gamma(-s)\Gamma(a+s)}{\Gamma(b+s)} (-x)^s. \tag{21}
\]
One must choose the contour of integration so that it separates the poles of \(\Gamma(-s)\) and \(\Gamma(a+s)\). Again, this is the standard result.

An expression for the other familiar confluent hypergeometric function \(U(a, b, x)\) can be found from a second canonical transformation. The transformation
\[
C'' = \frac{1}{\Gamma(a + x\partial)\Gamma(1 - b - x\partial)} \tag{22}
\]
transforms the momentum
\[
\partial \mapsto -\frac{a + x\partial}{b + x\partial} \partial. \tag{23}
\]
The transformed Hamiltonian is
\[
H'(a) = (a + x\partial)(-\partial - 1). \tag{24}
\]
This has the immediate solution \(e^{-x}\). The confluent hypergeometric function corresponding to \(e^{-x}\) is
\[
\Gamma(a)\Gamma(1 + a - b) U(a, b, x) = \Gamma(a + x\partial)\Gamma(1 - b - x\partial) e^{-x}. \tag{25}
\]
One recognizes another instance of the beta function,
\[
\frac{\Gamma(a + x\partial)\Gamma(1 - b - x\partial)}{\Gamma(1 + a - b)} = \int_0^\infty dt \, t^{a+x\partial-1} (1 + t)^{b-a-1}. \tag{26}
\]
Applying this to \( e^{-x} \) gives

\[
\Gamma(a)U(a, b, x) = \int_0^\infty dt \, t^{a-1}(1 + t)^{b-a-1}e^{-xt}.
\] (27)

Additional forms of solutions can be found by applying symmetry transformations of the equation. These give the Kummer relations between solutions. A symmetry is a canonical transformation which takes an operator into another operator of the same form but with different parameters. Since we are interested here in equations of the form \( H\psi = 0 \), the term “symmetry” will be broadened to include transformations which produce an operator of the form of \( H \) up to an overall factor, which can be divided out. Let \( H(a, b) \) denote the confluent hypergeometric operator. A symmetry \( C \) gives

\[
CH(a, b)C^{-1} = FH(a', b')
\] (28)

where \( F \) is an overall factor. Given a solution \( \psi' \) of \( H(a', b') \), \( \psi = C^{-1}\psi' \) is a solution of \( H(a, b) \) (assuming here that the kernels of \( F \) and \( C \) vanish; a more general discussion is possible[2]).

The simplest symmetries are those obtained by similarity transformation. Since the coefficient of the highest derivative term is \( x \), the transformation \( \partial \mapsto \partial + \alpha x \),

\[
(29)
\]

will produce a linear derivative term of correct form. The transformed Hamiltonian is

\[
H(a) = x\partial^2 + (2\alpha + b - x)\partial + \frac{\alpha^2 - \alpha(1 - b)}{x} - (a + \alpha).
\] (30)

Taking \( \alpha = 0 \) or \( 1 - b \) cancels the undesired \( 1/x \) potential term, and leaves an operator of confluent hypergeometric form. The symmetry is

\[
C_1H(a, b)C_1^{-1} = H(a + \alpha, b + 2\alpha),
\] (31)

where \( \alpha = 0, 1 - b \). The transformation with \( \alpha = 0 \) is trivial, but the other produces a solution of \( H(a, b) \)

\[
\psi = C_1^{-1}\psi^{(a)} = x^{1-b}M(a - b + 1, 2 - b, x).
\] (32)

Since this solution has different behavior than \((17)\) at the regular singular point \( x = 0 \), it is an independent solution.
A second symmetry transformation is obtained from the transformation $C_2 = e^{-\beta x}$,

$$\partial \mapsto \partial + \beta. \quad (33)$$

This leads to

$$H^{(a)} = x\partial^2 + (b - (1 - 2\beta)x)\partial + b\beta - a + (\beta^2 - \beta)x.$$ \quad (34)

Requiring the coefficient of $x$ to vanish gives $\beta = 1$. The transformed Hamiltonian is

$$H^{(a)} = x\partial^2 + (b + x)\partial + b - a.$$ \quad (35)

This is not of confluent hypergeometric form because the sign of $x$ in the linear term is wrong. A second transformation $S = \exp(i\pi x\partial)$

$$x \mapsto -x, \quad \partial \mapsto -\partial \quad (36)$$

takes $H^{(a)}$ to $H^{(b)}$ in canonical form \((10)\) up to an overall sign. This sign is not important because the equation to be satisfied is $H(a,b)\psi = 0$ and an overall factor can be divided out. The symmetry here is

$$SC_2H(a,b)C_2^{-1}S^{-1} = -H(b-a,b). \quad (37)$$

A solution of \((10)\) is then given by

$$\psi = C_2^{-1}S^{-1}\psi^{(b)} = e^xM(b-a,b,-x). \quad (38)$$

Combining these two transformations gives a third transformation,

$$C_1SC_2H(a,b)C_2^{-1}S^{-1}C_1^{-1} = -H(b-a+\alpha,b+2\alpha), \quad (39)$$

where $\alpha = 1 - b$. This leads to the solution

$$\psi = x^{1-b}e^xM(1-a,2-b,-x). \quad (40)$$

Inspection of the behavior near the regular singular point $x = 0$ shows that

$$M(a,b,x) = e^xM(b-a,b,-x), \quad (41)$$

$$x^{1-b}M(a-b+1,2-b,x) = x^{1-b}e^xM(1-a,2-b,-x).$$

These are the Kummer transformations \([3]\). The above symmetry transformations may also be applied to $U(a,b,x)$ to find similar relations.

9
4 Hypergeometric Equation

A similar story can be repeated for the hypergeometric equation. The Hamiltonian corresponding to the hypergeometric equation is

\[ H = x(1 - x)\partial^2 + (c - (a + b + 1)x)\partial - ab. \] (42)

This can be rewritten as

\[ H = (c + x\partial)\partial - (a + x\partial)(b + x\partial). \] (43)

The canonical transformation

\[ C = \frac{\Gamma(c + x\partial)}{\Gamma(a + x\partial)\Gamma(b + x\partial)} \] (44)

transforms the derivative

\[ \partial \mapsto C\partial C^{-1} = \frac{(a + x\partial)(b + x\partial)}{c + x\partial} \partial. \] (45)

The transformed Hamiltonian is

\[ H^{(a)} = (a + x\partial)(b + x\partial)(\partial - 1). \] (46)

This has \( \psi^{(a)} = e^x \) as a solution. The solution of \( H \) corresponding to \( \psi^{(a)} \) is the familiar hypergeometric function

\[ \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a, b; c; x) = C^{-1}e^x \] (47)

\[ = \frac{\Gamma(a + x\partial)\Gamma(b + x\partial)}{\Gamma(c + x\partial)} e^x \]

\[ = \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)} \frac{x^n}{n!}. \]

Using (20), one obtains a Mellin-Barnes integral representation

\[ \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a, b; c; x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a + s)\Gamma(b + s)\Gamma(-s)}{\Gamma(c + s)} (-x)^s ds, \] (48)
where the contour separates the poles of $\Gamma(a+s)\Gamma(b+s)$ from those of $\Gamma(-s)$, $a, b, c$ are not negative integers, and $a - b$ is not an integer.

Another integral representation is found by recognizing that $C^{-1}$ times $\Gamma(c-b)$ contains a beta function. Thus, one finds

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, b; c; x) = \frac{\Gamma(b + x\partial) \Gamma(c-b)}{\Gamma(c+x\partial)} \frac{\Gamma(a+x\partial)}{\Gamma(a)} e^x$$

(49)

$$= \int_0^1 dt t^{b+x\partial-1}(1-t)^{c-b-1} \frac{\Gamma(a+x\partial)}{\Gamma(a)} e^x$$

($\Re(c) > \Re(b) > 0$). This can be further simplified by expanding $e^x$ in a power series and allowing $\Gamma(a+x\partial)$ to act on it. This gives

$$\frac{\Gamma(a+x\partial)}{\Gamma(a)} e^x = \sum_{n=0}^\infty \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} x^n.$$  

(50)

Using the gamma function inversion formula, $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$, this gives

$$\frac{\Gamma(a+x\partial)}{\Gamma(a)} e^x = \sum_{n=0}^\infty \frac{\Gamma(1-a)}{\Gamma(1-a-n)\Gamma(n+1)} (-1)^n x^n = (1-x)^{-a}.$$  

(51)

Inserting this in (49), and letting $t^{x\partial}$ act on $(1-x)^{-a}$, one obtains the familiar integral representation

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, b; c; x) = \int_0^1 dt t^{b-1}(1-t)^{c-b-1} (1-tx)^{-a}.$$  

(52)

The set of Kummer’s 24 solutions of the hypergeometric equation can be obtained by symmetry transformations which take the hypergeometric equation into itself. Let $H(a, b, c)$ represent the hypergeometric operator (42). Since a quadratic function of the coordinate multiplies the $\partial^2$ term, the most general shift of the derivative which produces a linear term of the correct form is generated by $C_1 = x^{-a}(1-x)^{-b}$,

$$\partial \mapsto C_1 \partial C_1^{-1} = \partial + \frac{\alpha}{x} - \frac{\beta}{1-x}.$$  

(53)

This produces a new Hamiltonian

$$H^{(a)} = x(1-x)\partial^2 + (c + 2\alpha - (a + b + 1 + 2\alpha + 2\beta)x)\partial$$

$$+ \frac{\alpha^2 - (1-c)\alpha}{x} + \frac{\beta^2 + (a + b - c)\beta}{1-x} - (\alpha + \beta + a)(\alpha + \beta + b).$$  

(54)
This is in hypergeometric form if the undesired potential terms are cancelled by taking
\[
\begin{align*}
\alpha^2 - (1 - c)\alpha &= 0 \\
\beta^2 + (a + b - c)\beta &= 0.
\end{align*}
\] (55)

The solutions of the conditions are \(\alpha = 0, 1 - c\) and \(\beta = 0, c - a - b\), and these give three transformed hypergeometric functions. (Note that these are the exponents of the regular singular points at 0 and 1.) The symmetry transformation is
\[
C_1 H(a, b, c) C_1^{-1} = H(a + \alpha + \beta, b + \alpha + \beta, c + 2\alpha).
\] (56)

A solution of the original hypergeometric equation is given in terms of the transformed one as
\[
\psi = C_1^{-1} \psi^{(a)} = x^\alpha (1 - x)^\beta F(\alpha + \beta + a, \alpha + \beta + b; c + 2\alpha; x)
\] (57)

For the four choices of \((\alpha, \beta)\), one finds
\[
\begin{align*}
\psi^{(1)} &= F(a, b; c; x) \\
\psi^{(2)} &= x^{1-c}F(a - c + 1, b - c + 1; 2 - c; x) \\
\psi^{(3)} &= (1 - x)^{c-a-b}F(c - a, c - b; c; x) \\
\psi^{(4)} &= x^{1-c}(1 - x)^{c-a-b}F(1 - b, 1 - a; 2 - c; x).
\end{align*}
\]

Inspecting the behavior at the regular singular point \(x = 0\), one finds that \(\psi^{(1)} = \psi^{(3)}\) and \(\psi^{(2)} = \psi^{(4)}\). For \(c \neq 1\), \(\psi^{(1)}\) and \(\psi^{(2)}\) are independent solutions.

In addition to similarity transformations, one can make point canonical transformations. The transformation \(P_{1/x}\) takes
\[
x \mapsto 1/x, \quad \partial \mapsto -x^2 \partial.
\] (59)

This transforms the Hamiltonian to
\[
H^{(a)} = -x[x(1 - x)\partial^2 + (1 - a - b - (2 - c)x)\partial + \frac{ab}{x}].
\] (60)

This is not in canonical form because of the term in \(1/x\). This can be cancelled by making the similarity transformation \(C_1\) (53). The transformed Hamiltonian is
\[
H^{(b)} = -x[x(1 - x)\partial^2 + (1 - a - b + 2\alpha - (2\alpha + 2\beta + 2 - c)x)\partial + \frac{\alpha^2 - \alpha(a + b) + ab}{x} + \frac{\beta^2 + \beta(a + b - c)}{1 - x} \\
- (\alpha + \beta)(1 - c + \alpha + \beta)].
\] (61)
Taking \( \alpha = a, b \) and \( \beta = 0, c - a - b \) cancels the undesired potentials and returns this to the hypergeometric form. (Note that \( \alpha \) is the exponent of the regular singular point at \( \infty \).) One has

\[
C_1 P_{1/x} H(a, b, c) P_{1/x} C_1^{-1} = -x H(\alpha + \beta, 1 - c + \alpha + \beta, 1 - a - b + 2\alpha).
\] (62)

The overall factor of \( x \) is irrelevant since the equation is \( H^{(b)} \psi^{(b)} = 0 \), and it may be divided out. The solutions of the original hypergeometric are equation

\[
\psi = P_{1/x} C_1^{-1} \psi^{(b)} = x^{-\alpha-\beta}(x - 1)^{\beta} F(\alpha + \beta, 1 - c + \alpha + \beta; 1 - a - b + 2\alpha; 1/x) \tag{63}
\]

For the different choices of \((\alpha, \beta)\), one finds

\[
\begin{align*}
\psi^{(5)} &= x^{-a} F(a, 1 - c + a; 1 + a - b; 1/x), \\
\psi^{(6)} &= x^{-b} F(b, 1 - c + b; 1 + b - a; 1/x), \\
\psi^{(7)} &= x^{b-c}(x - 1)^{c-a-b} F(c - b, 1 - b; 1 + a - b; 1/x), \\
\psi^{(8)} &= x^{a-c}(x - 1)^{c-a-b} F(c - a, 1 - a; 1 + b - a; 1/x).
\end{align*}
\] (64)

Checking behavior at the regular singular point \( x = \infty \), one finds \( \psi^{(5)} = \psi^{(7)} \) and \( \psi^{(6)} = \psi^{(8)} \).

A second point canonical transformation \( P_{1-x} \) also leads to new solutions. This transforms

\[
x \mapsto 1 - x, \partial \mapsto -\partial.
\] (65)

The transformed Hamiltonian is

\[
H^{(a)} = x(1 - x) \partial^2 + (1 + a + b - c - (a + b + 1)x) \partial - ab.
\] (66)

This is in hypergeometric form. Applying the transformation \( C_1 \), with \( \alpha = 0, c - a - b, \beta = 0, 1 - c \) (because \( c \) in the original hypergeometric equation has been replaced by \( 1 + a + b - c \)), one obtains

\[
\begin{align*}
H^{(b)} &= x(1 - x) \partial^2 + (1 + a + b - c + 2\alpha - (2\alpha + 2\beta + a + b + 1)x) \partial \\
&\quad - (\alpha + \beta + a)(\alpha + \beta + b).
\end{align*}
\] (67)

One has

\[
C_1 P_{1-x} H(a, b, c) P_{1-x} C_1^{-1} = H(a + \alpha + \beta, b + a + \beta, 1 + a + b - c + 2\alpha),
\] (68)
with $\alpha = 0, c - a - b, \beta = 0, 1 - c$. The solutions to the original hypergeometric equation are

$$\psi = P_{1-x}C_1^{-1}\psi^{(b)} = x^\beta (1 - x)\alpha F(\alpha + \beta + a, \alpha + \beta + b; 1 + a + b - c + 2\alpha; 1 - x). \quad (69)$$

For the different choices of $(\alpha, \beta)$, these are

$$\psi^{(9)} = F(a, b; 1 + a + b - c; 1 - x),$$
$$\psi^{(10)} = (1 - x)^{c-a-b}F(c - b, c - a; 1 + c - a - b; 1 - x),$$
$$\psi^{(11)} = x^{1-c}F(a - c + 1, b - c + 1; 1 + a + b - c; 1 - x),$$
$$\psi^{(12)} = x^{1-c}(1 - x)^{c-a-b}F(1 - b, 1 - a; 1 + c - a - b; 1 - x).$$

Checking behavior at the regular singular point $x = 1$, one finds $\psi^{(9)} = \psi^{(11)}$ and $\psi^{(10)} = \psi^{(12)}$.

The previous two transformations can be composed to lead to further solutions. First, one can apply $C_1P_{1/x}C_1P_{1-x}$. One must remember that $C_1$ depends on two parameters $\alpha, \beta$ that are not reflected in the notation. These parameters are determined by the form of the hypergeometric function that $C_1$ acts upon, and are in general different for each factor of $C_1$. Using (68) and (62), one finds

$$C_1P_{1/x}C_1P_{1-x}H(a, b, c)P_{1-x}C_1^{-1}P_{1/x}C_1^{-1} =$$

$$= C_1P_{1/x}H(a + \alpha_1 + \beta_1, b + \alpha_1 + \beta_1, 1 + a + b - c + 2\alpha_1)P_{1/x}C_1^{-1}$$

$$= -xH(\alpha_2 + \beta_2, c - a - b - 2\alpha_1 + \alpha_2 + \beta_2, 1 - a - b - 2\alpha_1 - 2\beta_1 + 2\alpha_2),$$

where

$$\alpha_1 = 0, \quad c - a - b,$$
$$\beta_1 = 0, \quad 1 - c,$$
$$\alpha_2 = \alpha_1 + \beta_1 + a, \quad \alpha_1 + \beta_1 + b,$$
$$\beta_2 = 0, \quad 1 - c - 2\beta_1.$$  

There are only four distinct forms of solution, and without loss of generality one can take $(\alpha_1, \beta_1) = (0, 0)$. This gives

$$\psi^{(13)} = (1 - x)^{-a}F(a, c - b; 1 + a - b; \frac{1}{1 - x}),$$
$$\psi^{(14)} = (1 - x)^{-b}F(b, c - a; 1 + b - a; \frac{1}{1 - x}).$$
\[ \psi^{(15)} = (-1)^{-a} x^{1-c} (x-1)^{c-a-1} F(1+a-c, 1-b; 1+a-b; \frac{1}{1-x}), \]
\[ \psi^{(16)} = (-1)^{-b} x^{1-c} (x-1)^{c-b-1} F(1+b-c, 1-a; 1+b-a; \frac{1}{1-x}). \]

A second transformation is induced by \( C_2 = C_1 p_{1-x} C_1 p_{1/x} \). Without loss of generality, one can take \((\alpha_2, \beta_2) = (0, 0)\), this makes the symmetry transformation

\[ C_2 H(a, b, c) C_2^{-1} = (x - 1) H(\alpha_1 + \beta_1, 1 - c + \alpha_1 + \beta_1, 1 + a + b - c + 2\beta_1), \quad (74) \]

where \( \alpha_1 = a, b, \beta_1 = 0, c - a - b \). The solutions of the hypergeometric equation are

\[ \psi^{(17)} = x^{-a} F(a, a - c + 1; a + b - c + 1; 1 - \frac{1}{x}), \quad (75) \]
\[ \psi^{(18)} = x^{-b} F(b, b - c + 1; a + b - c + 1; 1 - \frac{1}{x}), \]
\[ \psi^{(19)} = x^{b-c}(x-1)^{c-a-b} F(c-b, 1-b; c-a-b+1; 1 - \frac{1}{x}), \]
\[ \psi^{(20)} = x^{a-c}(x-1)^{c-a-b} F(c-a, 1-a; c-a-b+1; 1 - \frac{1}{x}). \]

Finally, the transformation induced by

\[ C_3 = C_1 p_{1-x} C_1 p_{1/x} C_1 p_{1-x}, \]

taking \((\alpha_1, \beta_1) = (0, 0)\) and \((\alpha_2, \beta_2) = (a, 0)\), makes gives the symmetry

\[ C_3 H(a, b, c) C_3^{-1} = (x - 1) H(a + \alpha_3 + \beta_3, c - b + \alpha_3 + \beta_3, c + 2\alpha_3), \quad (76) \]

where \( \alpha_3 = 0, 1 - c, \beta_3 = 0, b - a \). This gives the hypergeometric solutions

\[ \psi^{(21)} = (1 - x)^{-a} F(a, c - b; c; \frac{x}{x - 1}), \quad (77) \]
\[ \psi^{(22)} = (1-x)^{-c} x^{1-c}(1-x)^{c-a-1} F(a-c+1, 1-b; 2-c; \frac{x}{x - 1}), \]
\[ \psi^{(23)} = (1 - x)^{-b} F(b, c - a; c; \frac{x}{x - 1}), \]
\[ \psi^{(24)} = (-1)^{1-c} x^{1-c}(1-x)^{c-b-1} F(b-c+1, 1-a; 2-c; \frac{x}{x - 1}). \]

This completes the construction of the 24 Kummer solutions of the hypergeometric equation.
5 Three-body Toda Quantum Mechanics

As an application of canonical transformations to a more difficult problem, consider the nonperiodic three-body Toda potential. The Schrödinger operator is

$$H = -\frac{1}{2} (\partial_1^2 + \partial_2^2 + \partial_3^2) + e^{q_1 - q_2} + e^{q_2 - q_3}. \quad (78)$$

This operator does not separate in any of the standard orthogonal coordinate systems, yet integral representations for its eigenfunctions can be constructed. The n-body Toda potential, generalizing (78) in the obvious way, is physically interesting as an exactly solvable nonlinear oscillator which may be obtained from the Korteweg-deVries equation by an appropriate discretization[5].

Two integral representations of the three-body Toda wavefunctions are quoted in the review by Olshanetsky and Perelomov [6], and both can be economically derived using canonical transformations. The result of Vinogradov and Takhtadjan [7] was originally found using methods from number theory. A second form was obtained in [8] by constructing power series solutions for the wavefunctions and then by simply stating the associated integral representations. The derivations of these integral representations by canonical transformation are analogous to the solutions of the hypergeometric equations, in that a sequence of transformations, one involving the Gamma function, is used to transform the equation to a form where a solution is obvious by inspection. An alternative solution in which the Toda problem is transformed to a free one with vanishing potential is discussed in [9].

The result of Vinogradov and Takhtadjan [7] for the 3-body Toda wavefunction is the most easily obtained and so we shall begin with it. The 3-body Toda Hamiltonian (78) can be transformed into center-of-mass coordinates with separated interaction potential by the transformation

$$P_{\text{C.O.M.}} \quad q_1 \mapsto \frac{4}{3} q_1 - \frac{2}{3} q_2 + \frac{1}{3} q_3, \quad \partial_1 \mapsto \frac{1}{3} \partial_1 + \frac{1}{3} \partial_3$$
$$q_2 \mapsto -\frac{2}{3} q_1 + \frac{2}{3} q_2 + \frac{1}{3} q_3, \quad \partial_2 \mapsto -\frac{1}{3} \partial_1 + \frac{1}{3} \partial_2 + \frac{1}{3} \partial_3$$
$$q_3 \mapsto -\frac{2}{3} q_1 - \frac{3}{3} q_2 + \frac{2}{3} q_3, \quad \partial_3 \mapsto -\frac{1}{3} \partial_2 + \frac{1}{3} \partial_3. \quad (79)$$

The resulting Hamiltonian takes the form

$$H = -\frac{1}{4} (\partial_1^2 + \partial_2^2 + \partial_1 \partial_2) - \frac{3}{2} \partial_3^2 + e^{2q_1} + e^{2q_2}. \quad (81)$$
The following sequence of transformations then reduces the Hamiltonian to a form for which an eigenfunction can be found by inspection.

\[
\begin{align*}
\text{[VT1]} & \quad \exp(-\ln 2(\partial_1 + \partial_2)) : \\
& \quad \{ q_1 \mapsto q_1 - \ln 2, \quad \partial_1 \mapsto \partial_1 \\
& \quad \{ q_2 \mapsto q_2 - \ln 2, \quad \partial_2 \mapsto \partial_2 \\
\text{[VT2]} & \quad \exp(-ik(q_1 - q_2)) : \\
& \quad \{ q_1 \mapsto q_1, \quad \partial_1 \mapsto \partial_1 + ik \\
& \quad \{ q_2 \mapsto q_2, \quad \partial_2 \mapsto \partial_2 - ik \\
\text{[VT3]} & \quad \frac{\Gamma(-\frac{\beta_1+\beta_2}{2})}{\Gamma(-\frac{\beta_1+3\beta}{2})\Gamma(-\frac{\beta_2-3\beta}{2})} : \\
& \quad \{ e^{2q_1} \mapsto \frac{\partial_1+3i\beta_1}{\partial_1+\partial_2} e^{2q_1}, \quad \partial_1 \mapsto \partial_1 \\
& \quad \{ e^{2q_2} \mapsto \frac{\partial_2-3i\beta_2}{\partial_1+\partial_2} e^{2q_2}, \quad \partial_2 \mapsto \partial_2 \end{align*}
\]

Suppressing the free-particle kinetic term \(-\frac{\beta^2}{2}\) describing the motion of the center of mass, the Hamiltonian transforms as follows:

\[
H^a = -\frac{1}{4}(\partial_1^2 + \partial_2^2 - \partial_1\partial_2) + e^{2q_1} + e^{2q_2}
\]

\[
\text{[VT1]} \quad \mapsto -\frac{1}{4}(\partial_1^2 + \partial_2^2 - \partial_1\partial_2 - e^{2q_1} - e^{2q_2})
\]

\[
\text{[VT2]} \quad \mapsto -\frac{1}{4}(\partial_1^2 + 3ik\partial_1 + \partial_2^2 - 3ik\partial_2 - \partial_1\partial_2 - e^{2q_1} - e^{2q_2}) + \frac{3k^2}{4}
\]

\[
\text{[VT3]} \quad \mapsto -\frac{1}{4(\partial_1+\partial_2)}[(\partial_1 + 3i\beta_1)(\partial_1^2 - e^{2q_1}) + (\partial_2 - 3i\beta_2)(\partial_2^2 - e^{2q_2})] + \frac{3k^2}{4}.
\]

Inspection shows that this final Hamiltonian has \(B_\nu(e^{q_1})C_\nu(e^{q_2})\) as an eigenfunction where \(B_\nu, C_\nu\) stand for modified Bessel functions, each either \(I_\nu, K_\nu\) or a linear combination of them. The eigenvalue of this eigenfunction is \(\frac{1}{4}(-\nu^2 + 3k^2)\).

The eigenfunction of the Toda Hamiltonian \((78)\) (with zero center-of-mass momentum) corresponding to this eigenfunction is

\[
\psi_{k,\nu}(q_1, q_2) = N_{k,\nu} P_{C.O.M.}^{-1} \exp((\partial_1 + \partial_2) \ln 2) \exp(ik(q_1 - q_2)) \exp \left( \frac{1}{2}(\partial_1 + \partial_2) \ln(\frac{\partial_1+3i\beta_1}{\partial_1+\partial_2} ) \right) B_\nu(e^{q_1})C_\nu(e^{q_2}).
\]

Recognizing the product of Gamma functions as a Beta function, and using a familiar integral representation for the Beta function, one finds the result

\[
\psi_{k,\nu} = N_{k,\nu} P_{C.O.M.}^{-1} e^{ik(q_1 - q_2)} \int_0^\infty dt \left( t^{-1} - \frac{1}{2}(\partial_2 - 3i\beta_2)(1 + t) \frac{1}{2}(\partial_1 + \partial_2) B_\nu(2e^{q_1})C_\nu(2e^{q_2}) \right. \\
= N_{k,\nu} e^{\frac{1}{2}ik(q_1 - q_2 + q_3)} \int_0^\infty dt \left( t^{-1} + \frac{3}{2}ik \right) \left( B_\nu(2\sqrt{(1 + t)e^{\frac{1}{2}(q_1 - q_2)})}C_\nu(2\sqrt{(1 + t^{-1})e^{\frac{1}{2}(q_2 - q_3)})} \right). 
\]
The physically interesting solution is the wavefunction which dies in the classically forbidden region, and this is found by taking $B_\nu$ and $C_\nu$ to be $K_\nu$. With the identifications $k = -\frac{1}{2}i(t-s)$, $\nu = -1 + \frac{1}{2}(3s+t)$, this agrees with the result of Vinogradov and Takhtadjan, as quoted in [1].

The key idea is to begin by factoring out the asymptotic plane-wave behaviour. Denote the momentum of this asymptotic plane wave by $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ in the coordinate system where the Toda Hamiltonian takes the form (78). Also let $a = \lambda_1 - \lambda_2$, $b = \lambda_2 - \lambda_3$. The sequence of transformations to a theory for which an eigenfunction can be recognized by inspection is

$$H = -\frac{1}{2}(\partial_1^2 + \partial_2^2 + \partial_3^2) + e^{q_1-q_2} + e^{q_2-q_3}$$

$$e^{-i\lambda \cdot q} : \quad \mapsto -\frac{1}{2}((\partial_1 + i\lambda_1)^2 + (\partial_2 + i\lambda_2)^2 + (\partial_3 + i\lambda_3)^2) + e^{q_1-q_2} + e^{q_2-q_3}$$

$$P_{C.O.M.} : \quad \mapsto -\frac{1}{4}(\partial_1^2 + \partial_2^2 - \partial_1 \partial_2 + 2ia\partial_1 + 2ib\partial_2 - 4e^{2q_1} - 4e^{2q_3})$$

where $P_{C.O.M.}$ as in (79). Supposing the eigenvalue $\frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$ and terms involving $\partial_3$ which describe the center of mass motion, the transformations continue as

$$e^{-\ln 2(\partial_1 + \partial_2)} : \quad \mapsto -\frac{1}{4}(\partial_1^2 + \partial_2^2 - \partial_1 \partial_2 + 2ia\partial_1 + 2ib\partial_2 - e^{2q_1} - e^{2q_3})$$

$$\frac{\Gamma(1 + i\alpha + \frac{1}{2}i\partial_3)}{\Gamma(-ib - \frac{1}{2}i\partial_2)\Gamma(1 + i(a+b) + \frac{1}{2}i\partial_1 + \frac{1}{2}i\partial_2)} : \quad \mapsto -\frac{1}{4(2i(a+b) + \partial_1 + \partial_2)}[(\partial_1 + 2ia)(\partial_2^2 + 2i(a+b)\partial_2 - e^{2q_1})]$$

$$+ (\partial_2 + 2ib)(\partial_2^2 + 2i(a+b)\partial_2 + e^{2q_2})$$

$$e^{i(a+b)q_1 + q_2} : \quad \mapsto -\frac{1}{4(\partial_1 + \partial_2)}[(\partial_1 + i(a-b))(\partial_2^2 + e^{2q_1} + (a+b)^2)$$

$$+ (\partial_2 + i(b-a))(\partial_2^2 + e^{2q_2} + (a+b)^2)]$$

The final Hamiltonian in this sequence has $K_{i(a+b)}(q_1)H_{i(a+b)}^{(1)}(q_2)$ as an eigenfunction with eigenvalue $\frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$. This eigenfunction will lead to a solution which is asymptotically free in the classically allowed region with asymptotic momentum $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and which dies in the classically forbidden region. Other solutions to the equation can be obtained by using other products of Bessel functions. This corresponds to the Toda solution of Bruschi et al. Assembling the full transformation, this solution is

$$\psi_\lambda(q) = N_\lambda e^{i\lambda \cdot q} P_{C.O.M}^{-1} e^{\ln 2(\partial_1 + \partial_2)} \frac{\Gamma(-ib - \frac{1}{2}i\partial_2)\Gamma(1 + i(a+b) + \frac{1}{2}i\partial_1 + \frac{1}{2}i\partial_2)}{\Gamma(1 + ia + \frac{1}{2}i\partial_1)}$$

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\[ e^{-i(a+b)(q_1+q_2)}K_{i(a+b)}(e^{q_1})H_{i(a+b)}^{(1)}(e^{q_2}). \] (88)

The product of Gamma functions is once again a Beta function and has the integral representation (with parameters at the limits of convergence)

\[
\Gamma(-ib - \frac{1}{2} \partial_2)\Gamma(1 + i(a + b) + \frac{1}{2} \partial_1 + \frac{1}{2} \partial_2) = \Gamma(1 + ia + \frac{1}{2} \partial_1) \]

\[= e^{-b\pi + \frac{1}{2} \pi \partial_2} \int_0^\infty d\bar{z}_3 \left(1 - \bar{z}_3\right)^{-1-ia-\frac{1}{2} \partial_1} \bar{z}_3^{-1-ib-\frac{1}{2} \partial_2} . \] (89)

The translation operator \( \exp(i\pi \partial_2/2) \) acts to transform \( H_{i(a+b)}^{(1)}(e^{q_2}) \) into \( K_{i(a+b)}(e^{q_2}) \) times some constant factors. Letting the \( \exp[\ln 2(\partial_1 + \partial_2)] \) factor act, one finds

\[
\psi_{\lambda}(q) = \frac{-2iN_{\lambda}}{\pi} e^{i\lambda q} P_{-1}^{1} e^{-ib_2 - i2(a+b)} \int_0^\infty d\bar{z}_3 \left(1 - \bar{z}_3\right)^{-1-ia-\frac{1}{2} \partial_1} \bar{z}_3^{-1-ib-\frac{1}{2} \partial_2} \] (90)

The integral representation for the modified Bessel function

\[
\int_0^\infty dx x^{\nu-1} \exp \left( i\mu(x - \frac{e^{2q}}{x}) \right) = 2e^{\nu q} e^{i\nu\pi/2} K_{-\nu}(2\mu e^{q}), \] (91)

\((\text{Im}(\mu) > 0, \text{Im}(e^{2q}\mu) < 0)\) for \( \nu = -i(a + b) \), with \( \mu = 1 \) and \( e^q \) appropriately displaced into the complex plane for convergence, can be used to replace the Bessel functions. Acting with the translation operators appearing in the Beta function, the result is

\[
\psi_{\lambda}(q) = \frac{-iN_{\lambda}}{2\pi} e^{-b\pi - i2(a+b)} e^{i\lambda q} \int_0^\infty d\bar{z}_3 \left(1 - \bar{z}_3\right)^{-ia-1} \bar{z}_3^{-ib-1} \]

\[
\int_0^\infty dz_1 \int_0^\infty dz_2 \left(z_1 z_2\right)^{-i(a+b)-1} e^{i(z_1 + z_2)} \exp \left(-i\frac{e^{q_1+q_2}}{z_1}, i\frac{e^{q_1+q_2}}{z_3}\right). \] (92)

Finally, absorbing the constant factors into the normalization and making the change of variables \( \bar{z}_3 = z_3/(z_1 z_2) \), one reaches the form of Bruschi et al.

\[
\psi_{\lambda}(q) = N_{\lambda} e^{i\lambda q} \int_0^\infty \int_0^\infty \int_0^\infty dz_1 dz_2 dz_3 \left(z_1 z_2 - z_3\right)^{i(\lambda_2 - \lambda_1)-1} \left(z_3 - \lambda_2\right)^{-1} \]

\[e^{i(z_1 + z_2)} \exp \left(-i\frac{z_1}{z_1 z_2 - z_3} e^{q_1+q_2} - i\frac{z_2}{z_3} e^{q_2+q_1}\right). \] (93)
Note that this corrects some typos in the formula quoted in [3], most importantly concerning the range of integration. Note also that this is an eigenfunction of (78) and not of the Hamiltonian of the form discussed in Section 12 of [3].

The power of the quantum canonical transformations is evident in the simplicity of these derivations.

6 2-d Generalized Hypergeometric Functions

The example of the 3-body Toda problem suggests a nontrivial generalization of the hypergeometric equation to two dimensions. The center of mass form of the Toda problem (81) has the form of two Bessel operators in \( q_1 \) and \( q_2 \) with a cross term \( \partial_1 \partial_2 \). The Gamma function transformation [VT3] in (82) removes the cross term and effectively separates the problem so that a product of two Bessel functions is a solution. In the resulting full solution, the Gamma function transformation acts as a convolution of the Bessel functions. Since a Bessel function is essentially a special form of hypergeometric function, it is natural to consider the function obtained by similarly convolving two (generalized) hypergeometric functions.

The characteristic feature of generalized hypergeometric functions is the Gamma function structure of their series expansions

\[
_{r}F_{s}(a_1, \ldots, a_r; b_1, \ldots, b_s; x) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{r} \Gamma(a_j + n)}{\prod_{j=1}^{s} \Gamma(a_j)} \frac{\prod_{k=1}^{s} \Gamma(b_k + n)}{\prod_{k=1}^{r} \Gamma(b_k)} \frac{x^n}{n!} \]  

\[
(94)
\]

\[ _{r}F_{s}(a_1, \ldots, a_r; b_1, \ldots, b_s; x) = \frac{\prod_{j=1}^{r} \Gamma(a_j + x \partial)}{\prod_{j=1}^{s} \Gamma(a_j)} \frac{\prod_{k=1}^{s} \Gamma(b_k + x \partial)}{\prod_{k=1}^{r} \Gamma(b_k + x \partial)} e^x. \]

\[ _{r}F_{s} \] is a solution of the operator

\[ H = \prod_{k=1}^{s} (b_k + x \partial) \partial - \prod_{j=1}^{r} (a_j + x \partial), \]  

\[ (95) \]

The canonical transformation

\[ C = \frac{\prod_{k=1}^{s} \Gamma(b_k + x \partial)}{\prod_{j=1}^{r} \Gamma(a_j + x \partial)} \]  

\[ (96) \]

transforms \( H \) to

\[ H^a = \prod_{j=1}^{r} (a_j + x \partial) (\partial - 1), \]  

\[ (97) \]
which has $e^x$ as a solution.

In a two-dimensional form, the intent is to couple two hypergeometric equations with a cross term. Define

\[
A_1 = \prod_{j_1=1}^{r_1} (a_{j_1}^{(1)} + x_1 \partial_1),
\]

\[
A_2 = \prod_{j_2=1}^{r_2} (a_{j_2}^{(2)} + x_2 \partial_2).
\]

A natural equation is

\[
H = A_1 \partial_1 + A_2 \partial_2 - \left[ (x_1 \partial_1)^2 + (x_2 \partial_2)^2 - x_1 \partial_1 x_2 \partial_2 \right] + \alpha (x_1 \partial_1 - x_2 \partial_2) + \beta.
\]

By only making transformations which depend on $x_1 \partial_1$ and $x_2 \partial_2$, only the isolated $\partial_1$ and $\partial_2$ factors will transform. The transformation

\[
C_1 = \Gamma(x_1 \partial_1 + x_2 \partial_2)
\]

takes

\[
\partial_1 \mapsto \frac{1}{x_1 \partial_1 + x_2 \partial_2} \partial_1, \quad \partial_2 \mapsto \frac{1}{x_1 \partial_1 + x_2 \partial_2} \partial_2.
\]

When the factor $x_1 \partial_1 + x_2 \partial_2$ is multiplied into the expression in square brackets, the cross term is cancelled and a separated expression remains. The result is

\[
H^\alpha = \frac{1}{x_1 \partial_1 + x_2 \partial_2} \left( A_1 \partial_1 - (b_1 + x_1 \partial_1)(b_2 + x_1 \partial_1)x_1 \partial_1 \right)
\]

\[
+ A_2 \partial_2 - (-b_1 + x_2 \partial_2)(-b_2 + x_2 \partial_2)x_2 \partial_2,
\]

where $\alpha = b_1 + b_2$, $\beta = b_1 b_2$. This has a product of two generalized hypergeometric functions as a solution, and the desired result has been obtained.

An operator of the form occurring in the three-body Toda equation can be obtained from (99) by a few simple transformations which indicate other natural forms of this 2-dimensional hypergeometric equation: A Fourier transform leads to an operator where the isolated $\partial_1$, $\partial_2$ become $x_1$ and $x_2$. A point transformation $x_1 \mapsto x_1^k$, $x_2 \mapsto x_2^k$ has the mild effect $x_1 \partial_1 \mapsto k^{-1} x_1 \partial_1$, $x_2 \partial_2 \mapsto k^{-1} x_2 \partial_2$, so one can consider powers of the isolated factors different than 1. Finally taking $k = 2$ and $A_1 = 1 = A_2$,
an exponential point transformation $x_1 \mapsto e^{x_1}, \ x_2 \mapsto e^{x_2}$ results in an operator of the form that the three-body Toda equation takes after [VT2] in (83).

A superficial difference in the treatment of Toda is that the Bessel functions are eigenfunctions with nonzero eigenvalue of the Bessel operators and not simply the zero-eigenvalue solutions, as the hypergeometric functions are here. This is possible because the factors multiplying the Bessel operators conspire to cancel the leading factor $(x_1 \partial_1 + x_2 \partial_2)^{-1}$ when they both act on the same function. Appropriate $A_1$ and $A_2$ would allow a similar freedom in (102). Or, one could use further Gamma function canonical transformations to cancel $A_1$ and $A_2$ and substitute operators of appropriate form. Using this freedom can be a powerful way to obtain more explicit forms of solutions to the equation than found by simply using the immediate solutions in terms of generalized hypergeometric functions.

As well, one can make the observation that the “eigenvalue” is implicit in the definition of the hypergeometric operators. By changing the constant $\beta$ in (99), one gets solutions of the remaining operator with different eigenvalue. The Bessel operator itself cannot be put in hypergeometric form until the eigenvalue is specified, and alone it is in this sense incomplete.

It is not difficult to further generalize (99) by multiplying the term in brackets by a product of polynomials in $x_1 \partial_1, x_2 \partial_2$. The remaining terms must also be appropriately modified so that the operator which results after the transformation (100) has a separable product as a solution. Defining

$$B_1 = \prod_{k_1=1}^{s_1} (b_{k_1}^{1} + x_1 \partial_1),$$

$$B_2 = \prod_{k_2=1}^{s_2} (b_{k_2}^{2} + x_2 \partial_2),$$

the appropriate generalization is

$$H = B_2 A_1 \partial_1 + B_1 A_2 \partial_2 - B_1 B_2 [(x_1 \partial_1)^2 + (x_2 \partial_2)^2 - x_1 \partial_1 x_2 \partial_2 + \alpha(x_1 \partial_1 - x_2 \partial_2) + \beta].$$

After making the transformation induced by (100), this becomes

$$H^\alpha = \frac{1}{x_1 \partial_1 + x_2 \partial_2} \left( B_2 [A_1 \partial_1 - B_1 (b_1 + x_1 \partial_1)(b_2 + x_1 \partial_1)x_1 \partial_1] + B_1 [A_2 \partial_2 - B_2 (-b_1 + x_2 \partial_2)(-b_2 + x_2 \partial_2)x_2 \partial_2] \right).$$
It is clear that the essential step in solving (99) is the transformation $\Gamma(x_1 \partial_1 + x_2 \partial_2)$. In the series expansion of the final solution, if the powers of $x_1$ and $x_2$ are indexed by $n_1$ and $n_2$, respectively, this produces a factor of $1/\Gamma(n_1 + n_2)$. It is not hard to see that this kind of trick could be fruitfully applied to uncouple other equations and to produce coupled factors in the series expansions of other functions.

7 Conclusion

Quantum canonical transformations have been used to give a simple and economical derivation of the integral representations and Kummer solutions of the confluent hypergeometric and hypergeometric equations. It is straightforward to extend this discussion to derive the differentiation formulae relating hypergeometric functions of different indices. The transformations to the specific forms of the equations for the many classical special functions are also easily handled with canonical transformations. The quantum canonical transformations are seen to provide a useful systematization of techniques for solving differential equations.

Two integral representations for the physically interesting solutions of the non-periodic three-body Toda equation were also found using quantum canonical transformations. These motivated a nontrivial two-dimensional generalized hypergeometric equation of which the non-periodic three-body Toda equation is a special case. The methods described here can be applied to solve other nonseparable partial differential equations which do not separate in the standard orthogonal coordinate systems.

References

[1] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis, 4th ed.*, (Cambridge Univ. Press: Cambridge, 1902, 1927).

[2] A. Anderson, Phys. Lett. **B305** (1993), 67; preprint Imperial-TP-92-93-31/hep-th-9305054, to appear in Ann. Phys. (1993).

[3] M. Abramowitz and I.A. Stegun, eds. *Handbook of Mathematical Functions* (U.S. Government Printing Office: Washington, D.C., 1964). p. 504.

[4] *Ibid.* p. 556.

[5] M. Toda, *Nonlinear Waves and Solitons*, (KTK Scientific, Tokyo, 1989).
[6] M.A. Olshanetsky and A.M. Perelomov, Phys. Rep. 94 (1983) 313.

[7] A.I. Vinogradov and A.A. Takhtadjan, “Theory of the Eisenstein series for the group SL(3,R) and its applications to a binary problem, I” Notes at the LOMI seminars, 76 (1978) 5.

[8] M. Bruschi, D. Levi, M.A. Olshanetsky, A.M. Perelomov and O. Ragnisco, Phys. Lett. A88 (1982) 7.

[9] A. Anderson, B. Nilsson, C. Pope, and K.S. Stelle, Imperial/TP/93-94/13, hep-th/9401004, to appear in Nucl. Phys. B (1994).