GRADIENT ESTIMATES FOR POSITIVE SOLUTIONS
OF THE LAPLACIAN WITH DRIFT

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Abstract. Let $M$ be a complete Riemannian manifold of dimension $n$ without boundary and with Ricci curvature bounded below by $-K$, where $K \geq 0$. If $b$ is a vector field such that $\|b\| \leq \gamma$ and $\nabla b \leq K_*$ on $M$, for some nonnegative constants $\gamma$ and $K_*$, then we show that any positive $C^\infty(M)$ solution of the equation $\Delta u(x) + (b(x)\nabla u(x)) = 0$ satisfies the estimate

$$\frac{\|\nabla u\|^2}{u^2} \leq \frac{n(K + K_*)}{w} + \frac{\gamma^2}{w(1 - w)},$$

on $M$, for all $w \in (0, 1)$. In particular, for the case when $K = K_* = 0$, this estimate is advantageous for small values of $\|b\|$ and when $b \equiv 0$ it recovers the celebrated Liouville theorem of Yau (Comm. Pure Appl. Math. 28 (1975), 201–228).

1. Introduction

In this paper we investigate the behaviour of positive $C^\infty(M)$ solutions of the equation

$$\Delta u(x) + (b(x)\nabla u(x)) = 0 \tag{1.1}$$
on $M$, where $M$ is an $n$-dimensional complete Riemannian manifold without boundary.

We require smoothness of the manifold, uniform bound on the norm of the vector field $b$ as well as lower bounds on the tensor fields of the Ricci curvature and $\nabla b$ where

$$\nabla b(X, Y) = (\nabla_X b)Y, \quad \forall X, Y \in \mathfrak{X}(M), \tag{1.2}$$

where $\mathfrak{X}(M)$ denotes the Lie algebra of vectors fields on $M$ and $\nabla_X b$ the associated (Levi-Civita) Riemannian covariant derivative of $b$ with respect to $X$.

Our main result is a gradient estimate for positive $C^\infty(M)$ solutions of equation (1.1), namely,

$$\frac{\|\nabla u\|^2}{u^2} \leq \frac{n(K + K_*)}{w} + \frac{\gamma^2}{w(1 - w)} \tag{1.3},$$

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on $M$, for any $w \in (0, 1)$, where the Ricci curvature is bounded below by $-K$, $\nabla b \leq K_*$ and $\|b\| \leq \gamma$ for some nonnegative constants $K$, $K_*$ and $\gamma$.

For the particular case when $K = K_* = 0$ inequality (1.3) yields

\begin{equation}
\frac{\|\nabla u\|^2}{u^2} \leq 4\gamma^2.
\end{equation}

Note that this simple estimation is independent of the dimension of $M$ and for the case when $b \equiv 0$ it recovers the Liouville theorem of Yau [10]. The proof of (1.3), and thus of (1.4), is essentially along the lines of Li and Yau [7] and Davies [3, Chap. 5].

This method, originated first in Yau [10] and Cheng and Yau [2], has been developed by several authors (cf. [3], [6], [7], [8], and [9], amongst others). More specifically, for the case when $b = \nabla \phi$, and $\phi \in C^\infty(M)$, a gradient estimate for any positive $C^\infty(M)$ solution of (1.1) has been obtained by Setti in [9].

In order to start, however, we need an extension of the Bochner-Lichnerowicz-Weitzenböck formula for the operator $L^b = \Delta + (b|\nabla|)$. This remarkable fact is proved as an independent lemma. It is known for drift vectors $b = \nabla \phi$ of gradient form; see e.g. the monograph of Deuschel and Stroock (cf. [4], §6.2).

2. Gradient estimates revisited

In the derivation of the main results, a central role will be played by the next formula.

**Lemma 2.1** (Bochner-Lichnerowicz-Weitzenböck formula for $L^b$). Let $M$ be a Riemannian manifold and assume that $f \in C^\infty(M)$. Then,

\begin{equation}
L^b(\|\nabla f\|^2) = 2\|\text{Hess}(f)\|_{\text{H.S.}}^2 + 2(\nabla f|\nabla (L^b f)) + 2(\text{Ric} - \nabla b)|\nabla f, \nabla f|,
\end{equation}

where $\|\text{Hess}(f)\|_{\text{H.S.}}$ denotes the Hilbert-Schmidt norm of $\text{Hess}(f)$ (cf. [4, p. 262]), $\text{Ric}$ denotes the Ricci curvature and $\nabla b$ denotes the tensor field given by (1.2).

**Proof.** Applying the well-known Bochner-Lichnerowicz-Weitzenböck formula for the Laplacian, one obtains

\begin{equation}
L^b(\|\nabla f\|^2) = 2\|\text{Hess}(f)\|_{\text{H.S.}}^2 + 2(\nabla f|\nabla (\Delta f)) + 2\text{Ric}(\nabla f, \nabla f) + (b|\nabla(\|\nabla f\|^2)).
\end{equation}

So, in order to prove (2.1) first we establish that

\begin{equation}
\nabla b(\nabla f, \nabla f) = (\nabla((b|\nabla f))|\nabla f) - \frac{1}{2}(b|\nabla(\|\nabla f\|^2)).
\end{equation}

Now, one has

\[
\nabla b(\nabla f, \nabla f) = (\nabla_{\nabla f} b|\nabla f) = \nabla_{\nabla f}((b|\nabla f)) - (b|\nabla\nabla f|\nabla f)
\]

\[
= -(b|\nabla\nabla f|\nabla f) + \nabla f((b|\nabla f)) = -(b|\nabla\nabla f|\nabla f) + (\nabla((b|\nabla f))|\nabla f).
\]

Thus, all that remains is to show that

\begin{equation}
\nabla\nabla f|\nabla f = \frac{1}{2}\nabla(\|\nabla f\|^2).
\end{equation}

In order to check (2.3) let any $Z \in \mathfrak{X}(M)$; then

\[
(\nabla(\|\nabla f\|^2), Z) = Z(\|\nabla f\|^2) = Z((\nabla f|\nabla f))
\]

\[
= 2(\nabla Z|\nabla f|\nabla f) = 2((\nabla\nabla f Z + [Z, \nabla f])|\nabla f)
\]

\[
= 2(\nabla\nabla f Z|\nabla f) + 2([Z, \nabla f]|\nabla f)
\]
boundary. Let $B$ on $M$ (2.4) and so which agrees with the Bochner-Lichnerowicz-Weitzenböck formula derived in [4, p. 262] and [5, p. 32].

Theorem 2.1. Let (1.2) given by

$$
L(\nabla (\| \nabla f \|^2)) = 2 (\nabla_{X,Y} \nabla f) + 2 (\nabla (\| \nabla f \|^2) \nabla),
$$

and so

$$
(\nabla (\| \nabla f \|^2))Z = 2 (\nabla_{X,Y} f) \nabla f)Z, \quad \forall Z \in \mathfrak{X}(M),
$$

from which (2.3) follows.

Remark 2.1. For the case when $b = -\nabla U$, $U \in C^\infty(M)$, denoting $L^U = L^{-\nabla U}$ and taking into account that $\text{Hess}(U)(X,Y) = (\nabla_X (\nabla U))Y)$, for all $X,Y \in \mathfrak{X}(M)$ (cf. [4, p. 261]), formula (2.1) is written as

$$
L^U(\| \nabla f \|^2) = 2 \| \nabla f \|^2_{M^u} + 2 (\nabla f \nabla (L^U f)) + 2 (\text{Ric} + \text{Hess}(U))(\nabla f, \nabla f),
$$

which agrees with the Bochner-Lichnerowicz-Weitzenböck formula derived in [4, p. 262] and [5, p. 32].

Now, formula (2.1) enables us to prove the next local gradient estimate.

Theorem 2.1. Let $M$ be a complete Riemannian manifold of dimension $n$ without boundary. Let $B_p(2R)$ be a geodesic ball of radius $2R$ around $p \in M$ and denote by $-K(2R)$, with $K(2R) \geq 0$, a lower bound on $B_p(2R)$ of the Ricci curvature. Set $b$ a vector field on $M$ and denote by $\gamma(2R)$ and $K_*(2R)$ some nonnegative constants satisfying $\| b \| \leq \gamma(2R)$ and $\nabla b \leq K_*(2R)$ on $B_p(2R)$, where $\nabla b$ is the tensor field given by (1.2). If $u(x,t)$ is a positive $C^\infty$ solution of the equation

$$
\Delta u(x,t) + (b(x)|\nabla u(x,t)) = \frac{\partial u(x,t)}{\partial t},
$$

on $M \times [0,\infty)$, then for any $\alpha > 1$ and any $w \in (0,1)$, the estimate

$$
\| \nabla u \|^2 (x,t) - \alpha \frac{u_t(x,t)}{u(x,t)} 
\leq \frac{n\alpha^2}{2w} + \frac{n^2}{2w} \left\{ \frac{2\epsilon}{R^2} + \frac{(n-1)(1+R\sqrt{K})}{R^2} + \frac{\nu}{R^2} + \frac{K + K_*}{2(\alpha - 1)} \right\}
$$

holds on $B_p(R) \times (0,\infty)$, where $\epsilon > 0$ and $\nu > 0$ are some constants.

Proof. Observe that the function $f(x,t) = \log u(x,t)$ satisfies the equation

$$
L^f u + \| \nabla f \|^2 = f_t.
$$

Now, using formula (2.1), it follows that

$$
F(x,t) = t \{ \| \nabla f \|^2 (x,t) - \alpha f_t(x,t) \},
$$
satisfies the estimate
\[ L^b F - F_t + 2(\nabla f \mid \nabla F) + t^{-1}F \]
\[ = t \left[ L^b (f) - \alpha L^b f - 2(\nabla f \mid \nabla f_t) + \alpha f_t \right. \]
\[ + 2(\nabla f \mid \nabla (\|f\|^2)) \left. - 2\alpha (\nabla f \mid \nabla f_t) \right] \]
\[ \geq t \left[ \frac{2(\Delta f)^2}{n} - 2(K + K_*) \|f\|^2 \right] \]
\[ = t \left[ \frac{2}{n} \|f\|^2 + (b \mid \nabla f) - f_t^2 - 2(K + K_*) \|\nabla f\|^2 \right] \]
on \( B_p(2R) \times (0, \infty) \), where we have used the inequalities \( \|\text{Hess} \ f\|_{\text{H.S.}} \geq (\Delta f)^2/n \)
and \( (\text{Ric} - \nabla b) \geq -(K + K_*) \).

Let \( \psi \) be a \( C^\infty(\mathbb{R}) \) function such that
\[ \psi(r) = \begin{cases} 1 & \text{if } r \in (-\infty, 1], \\ 0 & \text{if } r \in [2, \infty), \end{cases} \]
and \( 0 \leq \psi(r) \leq 1, \ \forall r \in \mathbb{R} \).

Denote by \( \epsilon > 0 \) and \( \nu > 0 \) some constants with
\[ 0 \geq \psi^{-1/2}(r) \frac{d}{dr} \psi(r) \geq -\epsilon \]
and
\[ \frac{d^2}{dr^2} \psi(r) \geq -\nu. \]

Now we set \( \phi(x) = \psi \left( \frac{d(p, x)}{R} \right) \), where \( d(p, x) \) is the distance between \( p \) and \( x \).

Using an argument of Calabi [1] (see also Cheng and Yau [2] and Setti [9]), we can assume without loss of generality that the function \( \phi \), with support in \( B_p(2R) \), is of class \( C^2 \).

Let \( (a, s) \) be the point in \( B_p(2R) \times [0, t] \) at which \( \phi F \) takes its maximum value, and assume that this value is positive (otherwise the proof is trivial). Then at \( (a, s) \) one has
\[ \nabla(\phi F) = 0, \quad \Delta(\phi F) \leq 0, \quad F_t \geq 0. \]

Therefore at \( (a, s) \) one has
\[ \phi \Delta F + F \Delta \phi - 2F \|\nabla \phi\|^2 \phi^{-1} \leq 0. \]

This inequality together with the estimates
\[ \|\nabla \phi\|^2 \leq \frac{c^2 \phi}{R^2}, \tag{2.6} \]
and
\[ \Delta \phi \geq -\frac{(n-1)(1+R\sqrt{K})\epsilon^2}{R^2} - \frac{\nu}{R^2} \tag{2.7} \]
(cf. [1]) yields
\[ \phi \Delta F \leq \left( \frac{2c^2}{R^2} + \frac{(n-1)(1+R\sqrt{K})\epsilon}{R^2} + \frac{\nu}{R^2} \right) F, \text{ at } (a, s). \tag{2.8} \]

Inequalities (2.6) and (2.7) at \( (a, s) \) imply that
\[ \phi \Delta F - (b \nabla \phi) F - \phi F_t - 2(\nabla f \nabla \phi) F + s^{-1} \phi F \]
(2.9) \[ \geq \left\{ \frac{2}{n} [ \| \nabla f \|^2 + (b | \nabla f) - f_t]^2 - 2(K + K_*) \| \nabla f \|^2 \right\} s \phi. \]

From (2.8) the left-hand side of (2.9) satisfies
\[ \phi \Delta F - (b \, | \nabla \phi) F - \phi F_t - 2(\nabla f \, | \nabla \phi) F + s^{-1} \phi F \]
\[ \leq \left( \frac{2 \epsilon^2}{R^2} + \frac{(n-1)(1 + R \sqrt{K}) \epsilon}{R^2} + \frac{\nu}{R^2} \right) F - (b \, | \nabla \phi) F - 2(\nabla f \, | \nabla \phi) F + s^{-1} \phi F. \]

Denoting \( \mu = \| \nabla f \|^2 (a, s) \), using (2.9) and the last inequality, we obtain
\[ \left\{ \frac{2}{n} \left( \mu - \frac{\mu s - 1}{\alpha s} \right)^2 F^2 - \frac{4(\mu \phi)^{1/2} \gamma (\mu - \frac{\mu s - 1}{\alpha s})}{n} \right\} s \phi \geq 2(K + K_*) s \mu \phi F. \]

Multiplying this inequality by \( \phi F \) and since \( \phi \leq 1 \), we obtain
\[ \frac{2(1 + (\alpha - 1) \mu s)^2(\phi F)^2}{\alpha^2 n} - 2 \left\{ \frac{2s^2 \mu^{1/2}(1 + (\alpha - 1) \mu s)}{\alpha n} + \frac{\epsilon s \mu^{1/2}}{R} \right\} (\phi F)^{3/2} \]
\[ \geq \left\{ \frac{2 \epsilon^2}{R^2} + \frac{(n-1)(1 + R \sqrt{K}) \epsilon}{\epsilon} + \frac{\nu}{R^2} \right\} s + \frac{\gamma \epsilon s}{R} + 2(K + K_*) \mu s^2 \}
\[ (\phi F) \leq 0. \]

On the other hand, for any \( w \in (0, 1) \) we have
\[ -2 \left\{ \frac{2s^2 \mu^{1/2}(1 + (\alpha - 1) \mu s)}{\alpha n} + \frac{\epsilon s \mu^{1/2}}{R} \right\} (\phi F)^{3/2} \]
\[ \geq -\frac{2(1 - w)(1 + (\alpha - 1) \mu s)^2(\phi F)^2}{\alpha^2 n} \]
\[ \left( 2(1 - w)(1 + (\alpha - 1) \mu s)^2 \right) \left[ \frac{2s^2 \mu^{1/2}(1 + (\alpha - 1) \mu s)}{n} + \frac{\epsilon s \alpha \mu^{1/2}}{R} \right]^2 (\phi F). \]

From (2.11) inequality (2.10) becomes
\[ A_1 \lambda^2 - 2A_2 \lambda \leq 0, \]
where
\[ \lambda = \phi F, \quad A_1 = \frac{2w}{\alpha^2 n}(1 + (\alpha - 1) \mu s)^2, \]
and
\[ 2A_2 = \left( \frac{2 \epsilon^2}{R^2} + \frac{(n-1)(1 + R \sqrt{K}) \epsilon}{R^2} + \frac{\nu}{R^2} \right) s + \frac{\gamma \epsilon s}{R} + 2(K + K_*) \mu s^2 \]
\[ + \frac{n}{2(1 - w)(1 + (\alpha - 1) \mu s)^2} \left[ \frac{2s^2 \mu^{1/2}(1 + (\alpha - 1) \mu s)}{n} + \frac{\epsilon s \alpha \mu^{1/2}}{R} \right]^2. \]
As in [3, Lemma 5.3.3], we use the estimate
\[
\frac{\mu s^2}{(1 + (\alpha - 1)\mu s)^2} \leq \frac{s}{4(\alpha - 1)},
\]
and so we obtain
\[
\frac{2A_2}{A_1} \leq \frac{n\alpha^2}{2w} + \frac{n\alpha^2 s}{2w} \left\{ \frac{2\gamma^2}{R^2} + \frac{(n - 1)(1 + R\sqrt{K})\epsilon}{R^2} + \frac{\nu}{R^2} \right\} + \frac{K + K_*}{2(\alpha - 1)} + \frac{\gamma\epsilon}{R} + \frac{n}{8(1 - w)(\alpha - 1)} \left( \frac{2\gamma}{n} + \frac{\epsilon\alpha}{R} \right)^2.
\]
(2.12)

Now, since \( \lambda \leq \frac{2A_2}{A_1} \), \( s \in [0, t] \) and using (2.12), estimate (2.5) holds.

From Theorem 2.1 one obtains the next global gradient estimate

**Corollary 2.1.** Let \( M \) be a complete Riemannian manifold of dimension \( n \) without boundary and assume that the Ricci curvature of \( M \) is bounded from below by \( -K \) with \( K \geq 0 \). Also we suppose that the vector field \( b \) satisfies \( ||b|| \leq \gamma \) and that the tensor field \( \nabla b \), given by (1.2), is bounded from above by \( K_* \), for some nonnegative constants \( \gamma \) and \( K_* \). If \( u(x) \) is a positive \( C^\infty(M) \) solution of equation (1.1), then

\[
\|\nabla u\|^2 u^2 \leq n(K + K_*) + \frac{\alpha^2 \gamma^2}{4w(\alpha - 1)} + \frac{\alpha^2 \gamma^2}{4w(1 - w)(\alpha - 1)},
\]

on \( M \). Setting \( \alpha = 2 \) (which minimizes the right-hand side of (2.13)), the result holds.

**Proof.** Letting \( R \to \infty \) and \( t \to \infty \) in (2.5) one has

\[
\|\nabla u\|^2 u^2 \leq \frac{n\alpha^2(K + K_*)}{w} + \frac{\alpha^2 \gamma^2}{w(1 - w)},
\]

on \( M \). Setting \( \alpha = 2 \) (which minimizes the right-hand side of (2.13)), the result holds. \( \square \)

**Remark 2.2.** If \( u(x) \) is a positive \( C^\infty(M) \) solution of \( \Delta u(x) + (b(x)||\nabla u(x)|| \leq 0 \), and assuming that \( \text{Ric} \geq 0, \nabla b \leq 0 \) and \( ||b|| \leq \gamma \), for some \( \gamma \geq 0 \), it follows from Corollary 2.1 above that

\[
\|\nabla u\|^2 u^2 \leq \frac{\gamma^2}{w(1 - w)},
\]

for any \( w \in (0, 1) \). Setting \( w = 1/2 \) (which minimizes the right-hand side of (2.14)) one obtains

\[
\|\nabla u\|^2 u^2 \leq 4\gamma^2.
\]

**Remark 2.3.** Let \( M = \mathbb{R} \) be the one-dimensional Euclidean space with its standard Riemannian metric. It is a complete Riemannian manifold without boundary and with Ricci curvature identically zero. In this setting consider the equation

\[
u''(x) + bu'(x) = 0,
\]

where \( b \) is a real constant. It is clear that \( u(x) = e^{-bx} \) is a positive \( C^\infty(\mathbb{R}) \) solution of (2.15), such that \( \frac{||\nabla u||^2}{u^2} = b^2 \). This case is contemplated by Corollary 2.1 with \( K = K_* = 0 \) and \( \gamma = |b| \), which establishes that \( \frac{||\nabla u||^2}{u^2} \leq 4b^2 \).
On the other hand, the equation
\begin{equation}
 u''(x) - (1 + e^x)u'(x) = 0
 \end{equation}
has the function \( u(x) = e^{e^x} \) as a positive \( C^\infty(\mathbb{R}) \) solution such that \( \|\nabla u\|^2 \) is unbounded. Note that the function \( b(x) = -(1 + e^x) \) satisfies \( b' \leq 0 \) and \( b \) is unbounded. Thus, we see that the assumption of the boundedness of \( \|b\| \) is needed for the kind of results obtained here.

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