Algorithm implementation and numerical analysis for the two-dimensional tempered fractional Laplacian

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Abstract
Tempered fractional Laplacian is the generator of the tempered isotropic Lévy process. This paper provides the finite difference discretization for the two-dimensional tempered fractional Laplacian by using the weighted trapezoidal rule and the bilinear interpolation. Then it is used to solve the tempered fractional Poisson equation with homogeneous Dirichlet boundary condition and the error estimate is also derived. Numerical experiments verify the predicted convergence rates and effectiveness of the schemes.

Keywords Two-dimensional tempered fractional Laplacian · The weighted trapezoidal rule · Bilinear interpolation · Error estimate

Mathematics Subject Classification 35R11 · 65M06

1 Introduction
Anomalous diffusion refers to the movements of particles whose trajectories’ second moment is a nonlinear function of the time \( t \) [20], being widely observed in the natural world [17] and having many applications in various fields, such as physical systems [14], stochastic dynamics [5], finance [19], image processing [7] and so on. The fractional Laplacian \( \Delta^\beta/2 \) is the fundamental non-local operator for modelling anomalous diffusion.
lous dynamics, introduced as the infinitesimal generator of a $\beta$-stable Lévy process \cite{3,12,22}, being the scaling limit of the Lévy flight. The extremely long jumps make the second and all higher order moments of the Lévy flight diverge, which sometimes fails to well model some practical physical processes. To overcome this, a trivial idea is to introduce a parameter $\lambda$ (a sufficiently small number) to exponentially temper the isotropic power law measure of the jump length; the new processes generate the tempered fractional Laplacian $(\Delta + \lambda)^{\beta} u$, being physically introduced and mathematically defined in \cite{9} with its definition

\[
(\Delta + \lambda)^{\beta} u(x) = -c_{n,\beta} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{e^{\lambda|x-y|}|x-y|^{n+\beta}} dy \quad \text{for } \beta \in (0, 2),
\]

where

\[
c_{n,\beta} = \begin{cases} 
\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}|\Gamma(-\beta)|} & \text{for } \lambda > 0 \text{ and } \beta \neq 1, \\
\frac{\beta \Gamma\left(\frac{n+\beta}{2}\right)}{2^{1-\beta} \pi^{n/2} \Gamma(1-\beta/2)} & \text{for } \lambda = 0 \text{ or } \beta = 1,
\end{cases}
\]

P.V. denotes the principal value integral, $n$ stands for the dimension of the space, and $x, y \in \mathbb{R}^n$. Here $\Gamma(t) = \int_0^{\infty} s^{t-1}e^{-s} ds$ is the Gamma function. Evidently, when $\lambda = 0$, the expression (1.1) reduces to the fractional Laplacian in the singular integral form \cite{18,23},

\[
(\Delta)^{\beta} u(x) = -c_{n,\beta} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+\beta}} dy.
\]

The main challenge about numerical approximations of (1.1) and (1.2) comes from their non-locality and singularity, especially in higher-dimensional case. Currently, fractional Laplacian (1.2) is the trend and a hot topic in both mathematical and numerical fields. For example, \cite{2} introduces the finite element approximation for the $n$-dimensional Dirichlet homogeneous problem about fractional Laplacian and \cite{1} presents the code employed for implementation in two dimensions; \cite{16} provides a finite difference-quadrature approach and gives its convergence proof; \cite{15} proposes several finite difference discretizations and tackles the non-locality, singularity and flat tails in practical implementations; \cite{11} provides a weighted trapezoidal rule for the fractional Laplacian and gives the additional insights into the convergence behaviour of the method by the extensive numerical examples. To our knowledge, using the finite difference method to discretize (1.2) is mostly for the one-dimensional case. For the tempered fractional Laplacian (1.1), the existing numerical methods at present are also mainly analyzed in one-dimensional case. Among them, \cite{25} presents a Riesz basis Galerkin method for the tempered fractional Laplacian and gives the well-posedness proof of the Galerkin weak formulation and convergence analysis; \cite{24} proposes a finite difference scheme and gives the convergence analysis for the tempered fractional Laplacian.

In this paper, we extend the discretization provided in \cite{24} to a two-dimensional case, but it can’t be got by taking a tensor product of two one-dimensional cases directly. Here we derive a finite difference scheme for (1.1) in two dimensions, based on the weighted trapezoidal rule combined with the bilinear interpolation. Furthermore,
we apply the discretization to solve the two-dimensional tempered fractional Poisson equation with homogeneous Dirichlet boundary condition \[9\], i.e.,

\[
\begin{align*}
- (\Delta + \lambda)^{\beta} u(x) &= f(x) \quad \text{for } x \in \Omega, \\
u(x) &= 0 \quad \text{for } x \in \mathbb{R}^2 \setminus \Omega,
\end{align*}
\]

(1.3)

where \(\beta \in (0, 2)\) and \(\Omega\) is a bounded domain in \(\mathbb{R}^2\). The accuracy of the scheme is proved to be \(O(h^{2-\beta})\) for \(u \in C^2(\mathbb{R}^2)\), where \(h\) denotes the mesh size.

It’s worth mentioning that the coefficient matrix generated by discretizing (1.1) in two dimensions needs a huge memory requirement, so we choose a suitable region to discrete (1.1) to get a symmetric block Toeplitz matrix with Toeplitz block according to [11], which reduces the memory requirement effectively. On the other hand, because the coefficient matrix is full, designing an efficient iteration scheme for solving (1.3) makes more sense. Here, we use Conjugate Gradient iterator to solve (1.3); and in iterative process, we calculate the \(B \mathbf{v}\) (\(B\) denotes the coefficient matrix and \(\mathbf{v}\) is a vector) by fast Fourier transform [8] to reduce the computational complexity. Our algorithm has a memory requirement of \(O(N^2)\) and a computational cost of \(O(N^2 \log N^2)\) instead of a memory requirement of \(O(N^4)\) and a computational cost of \(O(N^6)\) per iteration, where \(N^2\) is the order of the coefficient matrix. Next, to verify the convergence rates, numerical experiments are performed for the equation with known exact solution. For the unknown source term, we give an algorithm to approximate it, which changes the unbounded integration region into a bounded one by polar coordinate transformation in some special cases. For the details, see “Appendix A”.

The rest of this paper is organized as follows. In Sect. 2, we propose a discretization scheme for the two-dimensional tempered fractional Laplacian by using the weighted trapezoidal rule combined with the bilinear interpolation, and give its corresponding error estimate. In Sect. 3, we solve the tempered fractional Poisson equation with homogeneous Dirichlet boundary condition by the presented scheme and provide the error estimates of the solution. In Sect. 4, through numerical experiments for the equation with/without known solution, we verify the convergence rates and show the effectiveness of the schemes. Finally, we conclude the paper with some discussions.

2 Numerical approximation of the tempered fractional Laplacian and its corresponding error estimate

This section provides the approximation of the two-dimensional tempered fractional Laplacian by using the weighted trapezoidal rule and the bilinear interpolation on a bounded domain \(\Omega = (-l, l) \times (-l, l)\) with extended homogeneous Dirichlet boundary condition: \(u(x, y) \equiv 0\) for \((x, y) \in \Omega^c\). Afterwards, we analyze the error of the approximation.

Let us introduce the inner product and norms that will be used in the paper. Define the discrete \(l_2\) inner product, \(l_2\) norm and \(l_\infty\) norm as
\[(V, W) = h^2 \sum_{i=1}^{M} v_i w_i,\]
\[\|V\| = \sqrt{(V, V)},\]
\[\|V\|_\infty = \max_{1 \leq i \leq M} |v_i|,\]

where \(V = \{v_i\}_{i=1}^{M}, W = \{w_i\}_{i=1}^{M}\) and \(V, W \in \mathbb{R}^M\), denote
\[\|v\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |v(x)|\]
as the continuous \(L^\infty\) norm.

2.1 Numerical approximation

According to (1.1), the tempered fractional Laplacian in two dimensions can be written as
\[-(\Delta + \lambda)^{\beta} u(x, y) = -c_{2, \beta} P.V. \int \int_{\mathbb{R}^2} u(x + \xi, y + \eta) - u(x, y) e^{\lambda \sqrt{\xi^2 + \eta^2}} \left( \frac{\sqrt{\xi^2 + \eta^2}}{2 + \beta} \right)^{2+\beta} d\xi d\eta, \quad (2.1)\]
which can be symmetrized as
\[-(\Delta + \lambda)^{\beta} u(x, y) = -\frac{c_{2, \beta}}{2} \int \int_{\mathbb{R}^2} u(x + \xi, y + \eta) - 2u(x, y) + u(x - \xi, y - \eta) e^{\lambda \sqrt{\xi^2 + \eta^2}} \left( \frac{\sqrt{\xi^2 + \eta^2}}{2 + \beta} \right)^{2+\beta} d\xi d\eta \quad (2.2)\]
with
\[g(x, y, \xi, \eta) = u(x + \xi, y + \eta) + u(x - \xi, y + \eta) + u(x - \xi, y - \eta) + u(x + \xi, y - \eta) - 4u(x, y).\]

By the symmetry of the integral domain and integrand, (2.2) can be reformulated as
\[-(\Delta + \lambda)^{\beta} u(x, y) = -c_{2, \beta} \int_0^\infty \int_0^\infty g(x, y, \xi, \eta) e^{\lambda \sqrt{\xi^2 + \eta^2}} \left( \frac{\sqrt{\xi^2 + \eta^2}}{2 + \beta} \right)^{2+\beta} d\eta d\xi. \quad (2.3)\]
If we denote
\[\phi_{\gamma}(\xi, \eta) = \frac{g(x, y, \xi, \eta)}{e^{\lambda \sqrt{\xi^2 + \eta^2}} \left( \frac{\sqrt{\xi^2 + \eta^2}}{2 + \beta} \right)^{2+\beta}}, \quad (2.4)\]
where \( \gamma \in (\beta, 2) \), then (2.3) becomes

\[
- (\Delta + \lambda)^{\beta} u(x, y) = -c_{2, \beta} \int_0^\infty \int_0^\infty \frac{\phi_\gamma(\xi, \eta)}{(\sqrt{\xi^2 + \eta^2})^{-\gamma+2+\beta}} d\eta d\xi. \tag{2.5}
\]

Now, we just need to discretize the tempered fractional Laplacian in \([0, \infty) \times [0, \infty)\) instead of \(\mathbb{R} \times \mathbb{R}\). To keep the discretization scheme of \(- (\Delta + \lambda)^{\beta} u(x, y)\) uniform for any \((x, y) \in (-l, l) \times (-l, l)\) and satisfy \(u(x + \xi, y + \eta) = 0\) for any \((\xi, \eta) \notin (-L, L) \times (-L, L)\) and \((x, y) \in (-l, l) \times (-l, l)\), we take \(L = 2l\). Thus,

\[
- (\Delta + \lambda)^{\beta} u(x, y) = -c_{2, \beta} \left( \int_0^L \int_0^L \frac{\phi_\gamma(\xi, \eta)}{(\sqrt{\xi^2 + \eta^2})^{-\gamma+2+\beta}} d\eta d\xi 
- \frac{4}{\nu} \int_0^L \int_0^\infty \frac{u(x, y)}{e^{\nu \sqrt{\xi^2 + \eta^2}} (\sqrt{\xi^2 + \eta^2})^{2+\beta}} d\eta d\xi 
- \frac{4}{\nu} \int_0^\infty \int_0^L \frac{u(x, y)}{e^{\nu \sqrt{\xi^2 + \eta^2}} (\sqrt{\xi^2 + \eta^2})^{2+\beta}} d\eta d\xi 
- \frac{4}{\nu} \int_0^\infty \int_0^\infty \frac{u(x, y)}{e^{\nu \sqrt{\xi^2 + \eta^2}} (\sqrt{\xi^2 + \eta^2})^{2+\beta}} d\eta d\xi \right). \tag{2.6}
\]

For ease of presentation, we denote

\[
G^\infty = \int_0^L \int_0^\infty \frac{1}{e^{\nu \sqrt{\xi^2 + \eta^2}} (\sqrt{\xi^2 + \eta^2})^{2+\beta}} d\eta d\xi 
+ \int_0^\infty \int_0^L \frac{1}{e^{\nu \sqrt{\xi^2 + \eta^2}} (\sqrt{\xi^2 + \eta^2})^{2+\beta}} d\eta d\xi \tag{2.7}
+ \int_0^\infty \int_0^\infty \frac{1}{e^{\nu \sqrt{\xi^2 + \eta^2}} (\sqrt{\xi^2 + \eta^2})^{2+\beta}} d\eta d\xi.
\]

Let the mesh sizes \(h_1 = L/N_1, h_2 = L/N_2\); denote grid points \(\xi_i = ih_1, \eta_j = jh_2\), for \(0 \leq i \leq N_1, 0 \leq j \leq N_2\); for convenience, we set \(N_1 = N_2 = N\) and \(h_1 = h_2 = h\) in the following. Then, the first integral in (2.6) can be formulated as

\[
\int_0^L \int_0^L \phi_\gamma(\xi, \eta)(\xi^2 + \eta^2)^{\frac{\gamma-2-\beta}{2}} d\eta d\xi = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \phi_\gamma(\xi, \eta)(\xi^2 + \eta^2)^{\frac{\gamma-2-\beta}{2}} d\eta d\xi. \tag{2.8}
\]
For (2.8), when \((i, j) = (0, 0)\), it is easy to see that the integration is weak singular, so we approximate the integral by the weighted trapezoidal rule. For different \(\gamma\), we use different integral nodes to approximate it, namely,

\[
\int_{\xi_0}^{\xi_1} \int_{\eta_0}^{\eta_1} \phi_\gamma(\xi, \eta)(\xi^2 + \eta^2)^{\frac{\gamma-2-\beta}{2}} d\eta d\xi
\]

\[
\approx \begin{cases} 
\frac{1}{4} \left( \lim_{(\xi, \eta) \to (0, 0)} \phi_\gamma(\xi, \eta) + \phi_\gamma(\xi_0, \eta_1) + \phi_\gamma(\xi_1, \eta_1) + \phi_\gamma(\xi_1, \eta_0) \right) G_{0,0}, & \gamma \in (\beta, 2); \\
\frac{1}{3} \left( \phi_\gamma(\xi_0, \eta_1) + \phi_\gamma(\xi_1, \eta_1) + \phi_\gamma(\xi_1, \eta_0) \right) G_{0,0}, & \gamma = 2.
\end{cases}
\]

(2.9)

where

\[
G_{0,0} = \int_{\xi_0}^{\xi_1} \int_{\eta_0}^{\eta_1} (\xi^2 + \eta^2)^{\frac{\gamma-2-\beta}{2}} d\eta d\xi.
\]

To deal with the singularity, \(G_{0,0}\) can be calculated in polar coordinates, i.e. taking \(\xi = r \cos(\theta)\) and \(\eta = r \sin(\theta)\), one has

\[
G_{0,0} = \frac{\pi}{2(\gamma - \beta)} h^{\gamma-\beta} + \int_{\xi_0}^{\xi_1} \int_{\eta_0}^{\eta_1} (\xi^2 + \eta^2)^{\frac{\gamma-2-\beta}{2}} d\eta d\xi.
\]

(2.10)

So we only need to use the built-in function ‘integral2.m’ in MATLAB to calculate the second term in (2.10) with a tolerance of \(10^{-13}\).

Assuming \(u\) is smooth enough, for \(\gamma \in (\beta, 2)\), we have \(\lim_{(\xi, \eta) \to (0, 0)} \phi_\gamma(\xi, \eta) = 0\). Then, (2.9) can be rewritten as

\[
\int_{\xi_0}^{\xi_1} \int_{\eta_0}^{\eta_1} \phi_\gamma(\xi, \eta)(\xi^2 + \eta^2)^{\frac{\gamma-2-\beta}{2}} d\eta d\xi = k_\gamma \frac{1}{4} \left( \phi_\gamma(\xi_0, \eta_1) + \phi_\gamma(\xi_1, \eta_1) + \phi_\gamma(\xi_1, \eta_0) \right) G_{0,0},
\]

where

\[
k_\gamma = \begin{cases} 
1 & \gamma \in (\beta, 2), \\
\frac{4}{3} & \gamma = 2.
\end{cases}
\]
For another part of (2.8), when \((i, j) \neq (0, 0)\), we deal with the integration by the bilinear interpolation. Before discretizing it, we define the following functions

\[
G_{i,j} = \frac{1}{h^2} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} (\xi^2 + \eta^2)^{\gamma-2-\beta} \frac{d\eta d\xi}{2},
\]

\[
G_{i,j}^\xi = \frac{1}{h^2} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \xi (\xi^2 + \eta^2)^{\gamma-2-\beta} \frac{d\eta d\xi}{2},
\]

\[
G_{i,j}^\eta = \frac{1}{h^2} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \eta (\xi^2 + \eta^2)^{\gamma-2-\beta} \frac{d\eta d\xi}{2},
\]

\[
G_{i,j}^{\xi\eta} = \frac{1}{h^2} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \xi \eta (\xi^2 + \eta^2)^{\gamma-2-\beta} \frac{d\eta d\xi}{2}.
\]

(2.11)

It is easy to see that \(G_{i,j}\) depend on the mesh size \(h\). To get \(G_{i,j}\) for different \(h\) conveniently, taking \(\xi = ph\) and \(\eta = qh\), we rewrite it as

\[
G_{i,j} = \frac{1}{h^{2+\beta-\gamma}} g_{i,j} \quad \text{with } i, j \in \mathbb{Z} \text{ and } (i, j) \neq (0, 0),
\]

where

\[
g_{i,j} = \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \left(\sqrt{p^2 + q^2}\right)^{\gamma-2-\beta} dp dq \quad \text{with } i, j \in \mathbb{Z} \text{ and } (i, j) \neq (0, 0).
\]

Here, \(g_{i,j}\) is got by the built-in function ‘integral2.m’ in MATLAB with a tolerance of \(10^{-13}\). And the same skill can be used to calculate \(G_{i,j}^\xi, G_{i,j}^\eta\) and \(G_{i,j}^{\xi\eta}\) when \((i, j) \neq (0, 0)\).

Further denote \(I_{i,j}\) as the interpolation integration about \(\phi_\gamma(\xi, \eta)\) in \([\xi_i, \xi_{i+1}] \times [\eta_j, \eta_{j+1}]\), i.e.,

\[
I_{i,j} = \phi_\gamma(\xi_i, \eta_j)(G_{i,j}^{\xi\eta} - \xi_{i+1}G_{i,j}^\eta - \eta_{j+1}G_{i,j}^\xi + \xi_{i+1}\eta_{j+1}G_{i,j})
\]

\[
- \phi_\gamma(\xi_{i+1}, \eta_j)(G_{i,j}^{\xi\eta} - \xi_iG_{i,j}^\eta - \eta_{j+1}G_{i,j}^\xi + \xi_i\eta_{j+1}G_{i,j})
\]

\[
- \phi_\gamma(\xi_i, \eta_{j+1})(G_{i,j}^{\xi\eta} - \xi_{i+1}G_{i,j}^\eta - \eta_jG_{i,j}^\xi + \xi_{i+1}\eta_jG_{i,j})
\]

\[
+ \phi_\gamma(\xi_{i+1}, \eta_{j+1})(G_{i,j}^{\xi\eta} - \xi_iG_{i,j}^\eta - \eta_jG_{i,j}^\xi + \xi_i\eta_jG_{i,j});
\]

and let

\[
W_{i,j}^1 = G_{i-1,j}^{\xi\eta} - \xi_{i+1}G_{i,j}^\eta - \eta_{j+1}G_{i,j}^\xi + \xi_{i+1}\eta_{j+1}G_{i,j},
\]

\[
W_{i,j}^2 = -(G_{i-1,j}^{\xi\eta} - \xi_{i-1}G_{i,j}^\eta - \eta_{j+1}G_{i,j}^\xi + \xi_{i-1}\eta_{j+1}G_{i-1,j}),
\]

\[
W_{i,j}^3 = -(G_{i-1,j-1}^{\xi\eta} - \xi_{i+1}G_{i,j-1}^\eta - \eta_{j-1}G_{i,j-1}^\xi + \xi_{i+1}\eta_{j-1}G_{i-1,j-1}),
\]

\[
W_{i,j}^4 = G_{i-1,j-1}^{\xi\eta} - \xi_{i-1}G_{i-1,j}^\eta - \eta_{j-1}G_{i-1,j}^\xi + \xi_{i-1}\eta_{j-1}G_{i-1,j-1}.
\]

(2.12)
Then, \( I_{i,j} \) can be rewritten as

\[
I_{i,j} = \phi_\gamma (\xi_i, \eta_j) W_{i,j}^1 + \phi_\gamma (\xi_{i+1}, \eta_j) W_{i+1,j}^2 + \phi_\gamma (\xi_i, \eta_{j+1}) W_{i,j+1}^3 + \phi_\gamma (\xi_{i+1}, \eta_{j+1}) W_{i+1,j+1}^4
\]

(2.13)

and (2.8) becomes

\[
\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \phi_\gamma (\xi, \eta) (\xi^2 + \eta^2)^{-\frac{\gamma-2-\beta}{2}} d\eta d\xi
\]

\[
\approx \frac{k_\gamma}{4} \left( \phi_\gamma (\xi_0, \eta_1) + \phi_\gamma (\xi_1, \eta_1) + \phi_\gamma (\xi_1, \eta_0) \right) G_{0,0}
\]

(2.14)

Combining (2.13) with (2.14), we derive

\[
\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \phi_\gamma (\xi, \eta) (\xi^2 + \eta^2)^{-\frac{\gamma-2-\beta}{2}} d\eta d\xi
\]

\[
\approx \left( \frac{k_\gamma}{4} G_{0,0} + W_{1,1}^1 + W_{1,1}^2 + W_{1,1}^3 \right) \phi_\gamma (\xi_1, \eta_1)
\]

\[
+ \left( \frac{k_\gamma}{4} G_{0,0} + W_{1,0}^1 \right) \phi_\gamma (\xi_1, \eta_0)
\]

\[
+ \left( \frac{k_\gamma}{4} G_{0,0} + W_{0,1}^1 \right) \phi_\gamma (\xi_0, \eta_1)
\]

\[
+ \sum_{i=2}^{N-1} \left( W_{i,0}^1 + W_{i,0}^2 \right) \phi_\gamma (\xi_i, \eta_0)
\]

\[
+ \sum_{j=2}^{N-1} \left( W_{0,j}^1 + W_{0,j}^3 \right) \phi_\gamma (\xi_0, \eta_j)
\]

\[
+ \sum_{i=1}^{N-1} \left( W_{i,N}^3 + W_{i,N}^4 \right) \phi_\gamma (\xi_i, \eta_N)
\]

\[
+ \sum_{j=1}^{N-1} \left( W_{N,j}^2 + W_{N,j}^4 \right) \phi_\gamma (\xi_N, \eta_j)
\]

\[
+ W_{0,0}^3 \phi_\gamma (\xi_0, \eta_0) + W_{0,0}^2 \phi_\gamma (\xi_0, \eta_0) + W_{0,0}^4 \phi_\gamma (\xi_0, \eta_0)
\]

\[
+ \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \left( W_{i,j}^1 + W_{i,j}^2 + W_{i,j}^3 + W_{i,j}^4 \right) \phi_\gamma (\xi_i, \eta_j).
\]

For the second part of (2.6), namely \( G^\infty \), when \( \lambda = 0 \), we calculate it in the polar coordinates, so the integration in two dimensions can be translated into a bounded integration in one dimension; when \( \lambda \neq 0 \), \( G^\infty \) can be calculated by the built-in function ‘integral2.m’ in MATLAB with a tolerance of \( 10^{-13} \).

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Denote \( u_{p,q} = u(-l + ph, -l + qh) \) \((p, q \in \mathbb{Z})\). Then we can get the discretization scheme

\[
- (\Delta + \lambda)^{\beta/2}_h u_{p,q} = \sum_{i=-N}^{i=N} \sum_{j=-N}^{j=N} w_{i,j} u_{p-i,q-j},
\]

(2.15)

where \( w_{i,j} \) are defined in “Appendix B”.

For convenience, we write the matrix form of the scheme (2.15) as

\[
- (\Delta + \lambda)^{\beta}_h \mathbf{U} = \mathbf{B}\mathbf{U},
\]

(2.16)

where

\[
\mathbf{U} = (u_{1,1}, u_{1,2}, \ldots, u_{1,N-1}, u_{2,1}, \ldots, u_{2,N-1}, \ldots, u_{N-1,1,N-1})^T,
\]

(2.17)

and

\[
\mathbf{B} = \begin{bmatrix}
w_{1-1,1-1} & w_{1-1,1-2} & \cdots & w_{1-1,1-N} & w_{1-1,2-1} & \cdots & w_{1-1,2-N} & \cdots & w_{1-1,N-1,1} & \cdots & w_{1-1,N-1,2-N} & \cdots & w_{1-1,N-1,N-1} \\
w_{2-1,1-1} & w_{2-1,1-2} & \cdots & w_{2-1,1-N} & w_{2-1,2-1} & \cdots & w_{2-1,2-N} & \cdots & w_{2-1,N-1,1} & \cdots & w_{2-1,N-1,2-N} & \cdots & w_{2-1,N-1,N-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
w_{1-N,1-1} & w_{1-N,1-2} & \cdots & w_{1-N,1-N} & w_{1-N,2-1} & \cdots & w_{1-N,2-N} & \cdots & w_{1-N,N-1,1} & \cdots & w_{1-N,N-1,2-N} & \cdots & w_{1-N,N-1,N-1}
\end{bmatrix}.
\]

Here, \( u_{p,q} \) is the value of the exact solution \( u \) at the mesh point \((-l + ph, -l + qh)\).

Denote the numerical solution of (1.3) at point \((-l + ph, -l + qh)\) as \( u_{p,q}^h \), and the source term \( F \) at point \((-l + ph, -l + qh)\) as \( f_{p,q} \) \((p, q \in \mathbb{Z})\). Then (1.3) can also be written as

\[
\mathbf{B} \mathbf{U}_h = \mathbf{F},
\]

(2.18)

where

\[
\mathbf{U}_h = (u_{1,1}^h, u_{1,2}^h, \ldots, u_{1,N-1}^h, u_{2,1}^h, \ldots, u_{2,N-1}^h, \ldots, u_{N-1,1,N-1}^h)^T,
\]

(2.19)

and

\[
\mathbf{F} = (f_{1,1}, f_{1,2}, \ldots, f_{1,N-1}, f_{2,1}, \ldots, f_{2,N-1}, \ldots, f_{N-1,1,N-1})^T.
\]

**Remark 2.1** In this subsection, we introduce \( \phi_\gamma \) to transform (2.1) into a weighted integral, where the term \((\sqrt{\xi^2 + \eta^2})^{\gamma-2-\beta}\) in (2.5) is referred to as the weight function, which not only deals with the singularity of (1.1) by converting it into a weighted integral but also reduces the difficulty caused by the term \(e^{-\lambda |x-y|}\) of computing weight in (B.1) numerically. To avoid the singularity of the weight function, we choose \( \gamma \in (\beta, 2] \). By this way, we can take numerical approximation to (2.1) effectively. Moreover, we find that the numerical errors decrease as the parameter \( \gamma \) increases and the convergence orders are better than our theoretical ones when \( \gamma = 2 \) by numerical experiments in Sect. 4 and the main reason is that the extra regularity of the exact solution in our examples can not be exploited in our theoretical analysis.
When $\Omega$ is not a square/symmetric domain, one can choose a larger square domain $\mathbb{D}$ such that $\Omega \subset \mathbb{D}$ and apply our numerical scheme in domain $\mathbb{D}$. Then the numerical solution in $\Omega$ can be obtained.

2.2 Structure of the coefficient matrix $B$

Definition 2.1 ([8]) The symmetric $N \times N$ matrix $T$ is called the symmetric Toeplitz matrix if its entries are constant along each diagonal, i.e.,

$$T = \begin{bmatrix} t_0 & t_1 & \cdots & t_{N-2} & t_{N-1} \\ t_1 & t_0 & \cdots & t_{N-3} & t_{N-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{N-2} & t_{N-1} & \cdots & t_1 & t_0 \end{bmatrix}.$$  

And the symmetric $N^2 \times N^2$ matrix $H$ is called the symmetric block Toeplitz matrix with Toeplitz block, which has following structure

$$H = \begin{bmatrix} T_0 & T_1 & \cdots & T_{n-2} & T_{n-1} \\ T_1 & T_0 & \cdots & T_{n-3} & T_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{n-2} & T_{n-1} & \cdots & T_1 & T_0 \end{bmatrix},$$

where each $T_i$ is a symmetric Toeplitz matrix.

Since a symmetric Toeplitz matrix $T$ is determined by its first column and each block of $H$ is a symmetric Toeplitz matrix, we can store $H$ by a $N \times N$ matrix to reduce the memory requirement [8]. In our scheme (2.16), it is easy to verify that the coefficient matrix $B$ is a symmetric block Toeplitz matrix with Toeplitz block according to (B.1) in “Appendix B”, so we store $B$ by a $N \times N$ matrix to reduce the memory requirement from $O(N^4)$ to $O(N^2)$. When solving $BU_h = F$, the fast Fourier transform is used in the iterative process and the computational cost of calculating $BV$ ($V \in \mathbb{R}^{N^2}$ is a vector) can be reduced to $O(N^2 \log N^2)$.

2.3 Error estimate for the discretized tempered fractional Laplacian

Lemma 2.1 Let $\beta \in (0, 2)$, $\xi > 0$ and $\eta > 0$. If $u(x, y) \in C^2(\mathbb{R}^2)$, and the derivatives $D^\alpha \phi_\gamma$ ($\alpha$ is a multi-index and $|\alpha| \leq 2$) exist for any $\gamma \in (\beta, 2]$, then for $(x, y) \in \Omega$, there are

$$|\phi_\gamma| \leq C \left(\xi^2 + \eta^2\right)^{1-\frac{\gamma}{2}},$$

$$\left|\frac{\partial^2 \phi_\gamma}{\partial \xi^2}\right| \leq C \left((\xi^2 + \eta^2)^{-\frac{\gamma}{2}} + (\xi^2 + \eta^2)^{\frac{1}{2}-\frac{\gamma}{2}} + (\xi^2 + \eta^2)^{1-\frac{\gamma}{2}}\right),$$

$$\left|\frac{\partial^2 \phi_\gamma}{\partial \eta^2}\right| \leq C \left((\xi^2 + \eta^2)^{-\frac{\gamma}{2}} + (\xi^2 + \eta^2)^{\frac{1}{2}-\frac{\gamma}{2}} + (\xi^2 + \eta^2)^{1-\frac{\gamma}{2}}\right)$$

$(2.20)$
with $C$ being a positive constant.

**Proof** Using Taylor’s formula, we obtain

$$\left| \phi_\gamma (\xi, \eta) \right| \leq \left| \frac{g(x, y, \xi, \eta)}{(\xi^2 + \eta^2)^{\frac{\gamma}{2}}} \right| + \left| \frac{(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y})^2 u}{2!(\xi^2 + \eta^2)^{\frac{\gamma}{2}}} \right| \times \left| (x_1^*, y_1^*) \right| \times \left| (-\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y})^2 u \right| \times \left| (x_2^*, y_2^*) \right|

$$

$$+ \left| \frac{(\xi \frac{\partial}{\partial x} - \eta \frac{\partial}{\partial y})^2 u}{2!(\xi^2 + \eta^2)^{\frac{\gamma}{2}}} \right| \times \left| (x_3^*, y_3^*) \right| \times \left| (-\xi \frac{\partial}{\partial x} - \eta \frac{\partial}{\partial y})^2 u \right| \times \left| (x_4^*, y_4^*) \right|

$$

$$\leq C \left| \frac{\xi^2 + \eta^2}{(\xi^2 + \eta^2)^{\frac{\gamma}{2}}} \right| + C \left| \frac{\xi^2 + \eta^2}{(\xi^2 + \eta^2)^{\frac{1}{2} + \frac{\gamma}{2}}} \right| \leq C (\xi^2 + \eta^2)^{1-\frac{\gamma}{2}}.$$

where

$$(x_1^*, y_1^*) \in [x, x + \xi] \times [y, y + \eta], \ (x_2^*, y_2^*) \in [x - \xi, x] \times [y, y + \eta],$$

$$(x_3^*, y_3^*) \in [x, x + \xi] \times [y - \eta, y], \ (x_4^*, y_4^*) \in [x - \xi, x] \times [y - \eta, y].$$

For $\left| \frac{\partial^2 \phi_\gamma}{\partial \xi^2} \right|$, we have

$$\left| \frac{\partial^2 \phi_\gamma}{\partial \xi^2} \right| \leq C \left| \frac{g^{(2,0)}(x, y, \xi, \eta)}{(\xi^2 + \eta^2)^{\frac{\gamma}{2}}} \right| + C \left| \frac{g^{(1,0)}(x, y, \xi, \eta)}{(\xi^2 + \eta^2)^{\frac{1}{2} + \frac{\gamma}{2}}} \right| \xi^2$$

$$+ C \left| \frac{g^{(1,0)}(x, y, \xi, \eta)}{(\xi^2 + \eta^2)^{\frac{1}{2} + \frac{\gamma}{2}}} \right| + C \left| \frac{g(x, y, \xi, \eta)}{(\xi^2 + \eta^2)^{\frac{1}{2} + \frac{\gamma}{2}}} \right| \xi^2$$

$$+ C \left| \frac{g(x, y, \xi, \eta)}{(\xi^2 + \eta^2)^{\frac{3}{2} + \frac{\gamma}{2}}} \right| + C \left| \frac{g(x, y, \xi, \eta)}{(\xi^2 + \eta^2)^{\frac{3}{2} + \frac{\gamma}{2}}} \right| \xi^2$$

$$\leq C \left( (\xi^2 + \eta^2)^{-\frac{\gamma}{2}} + (\xi^2 + \eta^2)^{\frac{1}{2} - \frac{\gamma}{2}} + (\xi^2 + \eta^2)^{1 - \frac{\gamma}{2}} \right).$$
The estimate for \( \frac{\partial^2 \phi}{\partial y^2} \) can be similarly obtained as \( \frac{\partial^2 \phi}{\partial x^2} \). Then the desired inequalities (2.20) hold. □

Next, we introduce a lemma about the error of the bilinear interpolation.

**Lemma 2.2** ([21]) Let \( I \) denote bilinear interpolation on the box \( K = [0, h] \times [0, h] \). For \( f \in W^{2,\infty}(K) \) \((W^{k,p}(K) \text{ denotes the Sobolev space})\), the error of bilinear interpolation is bounded by

\[
\| f - I f \|_{L^\infty(\Omega)} \leq Ch^2 \left( \left\| \frac{\partial^2 f}{\partial x^2} \right\|_{L^\infty(\Omega)} + \left\| \frac{\partial^2 f}{\partial y^2} \right\|_{L^\infty(\Omega)} \right),
\]

where \( C \) is a positive constant and independent of \( h \).

**Proof** The proof can be completed by using the tensor-product polynomial approximation given in [6]. We omit the details here. □

**Theorem 2.1** Denote \( (\Delta + \lambda)^\frac{\beta}{h} \) as the finite difference approximation of the tempered fractional Laplacian \( (\Delta + \lambda)^\frac{\beta}{2} \). Suppose that the exact solution \( u(x, y) \in C^2(\mathbb{R}^2) \) is supported on an open set \( \Omega \subset \mathbb{R}^2 \). Then, for any \( \gamma \in (\beta, 2] \), there is

\[
\left\| (\Delta + \lambda)^\frac{\beta}{2} u(x, y) - (\Delta + \lambda)^\frac{\beta}{h} u(x, y) \right\|_{L^\infty(\Omega)} \leq Ch^{2-\beta} \quad \text{for } \beta \in (0, 2)
\]

with \( C \) being a positive constant depending on \( \beta \) and \( \gamma \).

**Proof** From (2.3), (2.6), (2.8) and (2.14), we obtain the error function

\[
e^h_{\beta, \gamma}(x, y) = (\Delta + \lambda)^\frac{\beta}{h} u(x, y) - (\Delta + \lambda)^\frac{\beta}{2} u(x, y)
= \left( \sum_{i=0, j=0; (i, j) \neq (0, 0)} \left( \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \phi_{\gamma}(\xi, \eta)(\xi^2 + \eta^2)^{\frac{\gamma-2-\beta}{2}} d\eta d\xi \right) + I + II \right.
\]

For the first part of (2.21), according to Lemma 2.1, there holds

\[
|I| \leq \int_{\xi_0}^{\xi_1} \int_{\eta_0}^{\eta_1} \left( \phi_{\gamma}(\xi, \eta) + \frac{k_\gamma}{4} (|\phi_{\gamma}(\xi_0, \eta_1)| + |\phi_{\gamma}(\xi_1, \eta_0)| + |\phi_{\gamma}(\xi_1, \eta_1)|) \right)
\cdot (\xi^2 + \eta^2)^{\frac{\gamma-2-\beta}{2}} d\eta d\xi
\leq \int_{\xi_0}^{\xi_1} \int_{\eta_0}^{\eta_1} \left( C(\xi^2 + \eta^2)^{1-\frac{\gamma}{2}} + Ch^{2-\gamma} \right) (\xi^2 + \eta^2)^{\frac{\gamma-2-\beta}{2}} d\eta d\xi,
\]
where we have used the fact \( \xi_0 = \eta_0 = 0 \) and \( \xi_1 = \eta_1 = h \). Taking \( \xi = ph, \eta = qh \), we obtain

\[
|I| \leq Ch^{2-\beta} \int_0^1 \int_0^1 (p^2 + q^2)^{-\frac{\beta}{2}} dqdp
+ Ch^{2-\beta} \int_0^1 \int_0^1 (p^2 + q^2)^{\frac{\gamma - 2 - \beta}{2}} dqdp.
\]

Since \( \beta < \gamma \leq 2 \), we obtain \( -\beta > -2 \) and \( \gamma - 2 - \beta > -2 \). Then it holds

\[
|I| \leq Ch^{2-\beta}.
\]

For the second part of (2.21), according to Lemma 2.2 and (2.13), we have

\[
|II| \leq C \sum_{i=0, j=0; (i, j) \neq (0, 0)} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \left( \left\| \frac{\partial^2 \phi_Y}{\partial \xi^2} \right\|_{L_\infty(\Omega)} + \left\| \frac{\partial^2 \phi_Y}{\partial \eta^2} \right\|_{L_\infty(\Omega)} \right) h^2 (\xi^2 + \eta^2)^{\frac{\gamma - 2 - \beta}{2}} d\eta d\xi.
\]

Denote \( \Omega_{i, j} = [\xi_i, \xi_{i+1}] \times [\eta_j, \eta_{j+1}] \). Lemma 2.1 gives

\[
\left\| \frac{\partial^2 \phi_Y}{\partial \xi^2} \right\|_{L_\infty(\Omega_{i, j})} \leq C \sup_{(\xi, \eta) \in \Omega_{i, j}} \left( (\xi^2 + \eta^2)^{-\frac{\gamma}{2}} + (\xi^2 + \eta^2)^{\frac{1}{2} - \frac{\gamma}{2}} + (\xi^2 + \eta^2)^{1 - \frac{\gamma}{2}} \right),
\]

\[
\left\| \frac{\partial^2 \phi_Y}{\partial \eta^2} \right\|_{L_\infty(\Omega_{i, j})} \leq C \sup_{(\xi, \eta) \in \Omega_{i, j}} \left( (\xi^2 + \eta^2)^{-\frac{\gamma}{2}} + (\xi^2 + \eta^2)^{\frac{1}{2} - \frac{\gamma}{2}} + (\xi^2 + \eta^2)^{1 - \frac{\gamma}{2}} \right).
\]

For any \( (\xi, \eta) \in \Omega_{i, j} \), there exists a constant \( C \) satisfying

\[
\sup_{(\xi, \eta) \in \Omega_{i, j}} \left( (\xi^2 + \eta^2)^{-\frac{\gamma}{2}} \right) \leq C (\xi^2 + \eta^2)^{-\frac{\gamma}{2}},
\]

\[
\sup_{(\xi, \eta) \in \Omega_{i, j}} \left( (\xi^2 + \eta^2)^{\frac{1}{2} - \frac{\gamma}{2}} \right) \leq C (\xi^2 + \eta^2)^{\frac{1}{2} - \frac{\gamma}{2}},
\]

\[
\sup_{(\xi, \eta) \in \Omega_{i, j}} \left( (\xi^2 + \eta^2)^{1 - \frac{\gamma}{2}} \right) \leq C (\xi^2 + \eta^2)^{1 - \frac{\gamma}{2}}.
\]

Thus

\[
|II| \leq C \sum_{i=0, j=0; (i, j) \neq (0, 0)} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \left( h^2 (\xi^2 + \eta^2)^{\frac{\gamma - 2 - \beta}{2}} + (\xi^2 + \eta^2)^{\frac{1}{2} - \frac{\gamma}{2}} + (\xi^2 + \eta^2)^{-\frac{\gamma}{2}} \right) d\eta d\xi.
\]

\[
\leq |II_1| + |II_2| + |II_3|.
\]
Taking $\xi = ph, \eta = qh$, we have

$$|I_1| \leq C \sum_{i=0, j=0; (i, j) \neq (0, 0)} h^{2-\beta} \int_i^{i+1} \int_j^{j+1} (p^2 + q^2)^{-\frac{2-\beta}{2}} dq dp.$$  

And since $-2 - \beta < -2$, it holds

$$|I_1| \leq Ch^{2-\beta}. \tag{2.22}$$

Then, we arrive at

$$|I_3| \leq Ch^2 \sum_{i=0, j=0; (i, j) \neq (0, 0)} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} (\xi^2 + \eta^2)^{-\frac{\beta}{2}} d\eta d\xi.$$ 

Take $\xi = r \cos(\theta), \eta = r \sin(\theta)$. Since $0 < \beta < 2$, we obtain

$$|I_3| \leq Ch^2 \int_0^{\pi} \int_h^{\sqrt{2L}} r^{1-\beta} dr d\theta \leq \begin{cases} Ch^2 & \beta \in (0, 1], \\ Ch^{3-\beta} & \beta \in (1, 2). \end{cases} \tag{2.23}$$

For $|I_2|$, being similar to $|I_1|$ and $|I_3|$, we have

$$|I_2| \leq \begin{cases} Ch^2 & \beta \in (0, 1], \\ Ch^{3-\beta} & \beta \in (1, 2). \end{cases} \tag{2.24}$$

Therefore, (2.22), (2.23) and (2.24) lead to

$$|II| \leq Ch^{2-\beta}.$$ 

So for $u(x, y) \in C^2(\mathbb{R}^2)$, it holds

$$\|e^h_{\beta, y}(x, y)\|_{L_\infty(\Omega)} \leq Ch^{2-\beta}. $$

Then, the proof is completed. \qed

### 3 Error estimates for the tempered fractional Poisson problem

Now, we turn to the convergence proof of the designed scheme for solving the tempered fractional Poisson problem with homogeneous Dirichlet boundary condition (1.3).
Lemma 3.1 ([4]) The spectrum $\lambda(A)$ of the $n$-order matrix $A = [a_{i,j}]$ is enclosed in the union of the discs

$$C_i = \{z \in \mathbb{C}; |z - a_{i,i}| \leq \sum_{i \neq j} |a_{i,j}|, \ 1 \leq i, j \leq n\}$$

and in the union of the discs

$$C'_i = \{z \in \mathbb{C}; |z - a_{i,i}| \leq \sum_{i \neq j} |a_{j,i}|, \ 1 \leq i, j \leq n\}.$$

Proposition 3.1 The weights of approximating the tempered fractional Laplacian defined in (2.15) satisfy

$$\begin{cases} 
\sum_{i=-N}^{N} \sum_{j=-N}^{N} w_{|i|,|j|} > CG_0^\infty > 0; \\
 w_{i,j} < 0, \ (i, j) \neq (0, 0),
\end{cases}$$

where $G_0^\infty$ is defined in (2.7).

Proof According to (B.1) in “Appendix B”, we only need to prove that $W_{i,j}^1$, $W_{i,j}^2$, $W_{i,j}^3$ and $W_{i,j}^4$ are positive. Combining (2.11) and (2.12), we obtain

$$W_{i,j}^1 = \frac{1}{h^2} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} (\xi - \xi_{i+1})(\eta - \eta_{j+1})(\xi^2 + \eta^2)^{\gamma-\beta/2} d\eta d\xi \geq 0.$$ 

The proofs for $W_{i,j}^2$, $W_{i,j}^3$ and $W_{i,j}^4$ are similar to the one for $W_{i,j}^1$. Combining $G_0^\infty > 0$ and $G_{0,0} > 0$, one can get $\sum_{i=-N}^{N} \sum_{j=-N}^{N} w_{|i|,|j|} > CG_0^\infty > 0$ for some constant $C > 0$. For $w_{i,j} < 0 ((i, j) \neq (0, 0))$, one can directly get it from (B.1) in “Appendix B”.

According to Proposition 3.1 and Lemma 3.1, the minimum eigenvalue of matrix $B$ satisfies

$$\lambda_{\text{min}}(B) > CG_0^\infty > 0.$$ 

So $B$ is a strictly diagonally dominant and symmetric positive definite matrix.

Theorem 3.1 Suppose that $u \in C^2(\mathbb{R}^2)$ is the exact solution of the tempered fractional Poisson Eq. (1.3) and $U_h$ is the solution of the finite difference scheme (2.18). Then, there exists some constant $C > 0$ satisfying

$$\|U - U_h\| \leq C \left\| (\Delta + \lambda)^{\beta/2}_h U - (\Delta + \lambda)^{\beta/2}_h U_h \right\|,$$

$$\|U - U_h\|_{\infty} \leq C \left\| (\Delta + \lambda)^{\beta/2}_h U - (\Delta + \lambda)^{\beta/2}_h U_h \right\|_{\infty},$$
Fig. 1 $l_2$ errors and convergence orders for the system with different $\gamma$ when $\beta = 0.5$ and $\lambda = 0$.

Fig. 2 $l_2$ errors and convergence orders for the system with different $\gamma$ when $\beta = 0.5$ and $\lambda = 0.5$.

where $U$ and $U_h$ are defined in (2.17) and (2.19), respectively.

Proof According to the definition of $G^\infty$, making an inner product of (2.18) with $U_h$ and using the Cauchy-Schwarz inequality, we have

$$CG^\infty \|U_h\|^2 \leq (B U_h, U_h) \leq \|F\| \|U_h\|,$$

which leads to

$$\|U_h\|^2 \leq \frac{1}{CG^\infty} \|F\| \|U_h\|.$$

Thus

$$\|U_h\| \leq \frac{1}{CG^\infty} \|F\|.$$

$\square$ Springer
Table 1  Numerical approximation errors and convergence orders for \((\Delta + \lambda)^{\beta/2} (1-x^2)^3 (1-y^2)^3\) with \(\lambda = 0\) and \(\gamma = 1 + \frac{4}{\beta}\)

| \(h\) | \(0.5\) | \(1/16\) | \(1/32\) | \(1/64\) | \(1/128\) | \(1/256\) |
|------|--------|--------|--------|--------|--------|--------|
| \(l_\infty\) | 2.5155E-02 | 9.9477E-03 | 3.7440E-03 | 1.3771E-03 | 4.9991E-04 | 1.7998E-04 |
| Rate | 1.3384 | 1.4098 | 1.4430 | 1.4619 | 1.4738 |
| \(l_2\) | 1.5053E-02 | 6.0283E-03 | 2.2798E-03 | 8.4041E-04 | 3.0548E-04 | 1.1007E-04 |
| Rate | 1.3202 | 1.4029 | 1.4397 | 1.4600 | 1.4727 |
| \(0.8\) | \(l_\infty\) | 9.0498E-02 | 4.2795E-02 | 1.9321E-02 | 8.5654E-03 | 3.7652E-03 |
| Rate | 1.3202 | 1.4029 | 1.4397 | 1.4600 | 1.4727 |
| \(l_2\) | 5.4199E-02 | 2.5992E-02 | 1.1792E-02 | 5.2379E-03 | 2.3045E-03 | 1.0091E-03 |
| Rate | 1.3202 | 1.4029 | 1.4397 | 1.4600 | 1.4727 |
| \(1.2\) | \(l_\infty\) | 4.0132E-01 | 2.4528E-01 | 1.4377E-01 | 8.3182E-02 | 4.7909E-02 |
| Rate | 1.3202 | 1.4029 | 1.4397 | 1.4600 | 1.4727 |
| \(l_2\) | 2.4038E-01 | 1.4921E-01 | 8.7882E-02 | 5.0922E-03 | 2.9344E-03 | 1.6876E-03 |
| Rate | 1.3202 | 1.4029 | 1.4397 | 1.4600 | 1.4727 |
| \(1.5\) | \(l_\infty\) | 1.1249E+00 | 8.4067E-01 | 6.0367E-01 | 4.2877E-01 | 3.0360E-01 |
| Rate | 1.3202 | 1.4029 | 1.4397 | 1.4600 | 1.4727 |
| \(l_2\) | 6.7336E-01 | 5.1163E-01 | 3.6915E-01 | 2.6256E-01 | 1.8598E-01 | 1.3158E-01 |
| Rate | 1.3202 | 1.4029 | 1.4397 | 1.4600 | 1.4727 |

Assuming \(\|U_h\|_\infty = |u_{p,q}^h|\), according to (B.1), we obtain that

\[
\begin{align*}
&u_{p,q}^h \left( \sum_{i=-N}^{i=N} \sum_{j=-N}^{j=N} w_{|i|,|j|} u_{p-i,q-j}^h - 4c_{2,\beta} G_\infty u_{p,q}^h \right) \\
&= u_{p,q}^h \left( \sum_{i=-N, j=-N; (i,j) \neq (0,0)}^{i=N, j=N} w_{|i|,|j|} u_{p-i,q-j}^h + (w_{0,0} - 4c_{2,\beta} G_\infty) u_{p,q}^h \right) \\
&\geq \sum_{i=-N, j=-N; (i,j) \neq (0,0)}^{i=N, j=N} -w_{|i|,|j|} \left( (u_{p,q}^h)^2 - u_{p,q}^h u_{p-i,q-j}^h \right) \\
&\geq 0,
\end{align*}
\]

which implies

\[
CG_\infty \|U_h\|_\infty \leq |f_{p,q}|.
\]
Table 2  Numerical approximation errors and convergence orders for $(\Delta + \lambda)^{\beta/2}(1 - x^2)^3(1 - y^2)^3$ with $\lambda = 0.5$ and $\gamma = 1 + \frac{\beta}{2}$

| $\beta$ | $h$ | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 | 1/256 |
|---------|-----|-----|-----|-----|-----|-----|-----|
| 0.5     | $l_\infty$ | 2.1316E−02 | 9.0524E−03 | 3.5350E−03 | 1.3277E−03 | 4.8808E−04 | 1.7711E−04 |
|         | Rate     | 1.2356  | 1.3566 | 1.4128 | 1.4437 | 1.4624 |
| 0.8     | $l_\infty$ | 7.6879E−02 | 3.9268E−02 | 1.8428E−02 | 8.3409E−03 | 3.7089E−03 | 1.6338E−03 |
|         | Rate     | 0.9693  | 1.0915 | 1.1436 | 1.1692 | 1.1827 |
| 1.2     | $l_\infty$ | 3.3838E−01 | 2.2543E−01 | 1.3772E−01 | 8.1362E−02 | 4.7368E−02 | 2.7388E−02 |
|         | Rate     | 0.5860  | 0.7110 | 0.7593 | 0.7805 | 0.7904 |
| 1.5     | $l_\infty$ | 9.3441E−01 | 7.6900E−01 | 5.7753E−01 | 4.1937E−01 | 3.0023E−01 | 2.1357E−01 |
|         | Rate     | 0.2811  | 0.4131 | 0.4617 | 0.4821 | 0.4914 |

So we have

\[ CG^\infty \| U_h \|_\infty \leq \| F \|_\infty. \]  \hspace{1cm} (3.3)

In addition, by (2.16), we get

\[ B(U - U_h) = (- (\Delta + \lambda)^{\frac{\beta}{2}} U) - (- (\Delta + \lambda)^{\frac{\beta}{2}} U_h). \]  \hspace{1cm} (3.4)

Applying (3.2) and (3.3) to (3.4), the desired results are obtained.

**Theorem 3.2** Suppose $u \in C^2(\mathbb{R}^2)$ is the exact solution of (1.3), and $U_h$ is the solution of the difference scheme (2.18). Then we get

\[ \| U - U_h \| \leq Ch^{2-\beta}, \quad \| U - U_h \|_\infty \leq Ch^{2-\beta}, \]

where $U$ and $U_h$ are defined in (2.17) and (2.19), respectively.
Proof Combining Theorem 2.1 and Theorem 3.1, we get that for $u \in C^2(\mathbb{R}^2)$,

$$\|U - U_h\| \leq Ch^{2-\beta}, \quad \|U - U_h\|_\infty \leq Ch^{2-\beta}.$$ 

Remark 3.1 After modifying the above discretization (2.15), one can also solve tempered fractional Poisson equation with the nonhomogeneous Dirichlet boundary condition, i.e.,

$$- (\Delta + \lambda)^{\beta/2} u(x) = f(x) \quad \text{for } x \in \Omega,$$

$$u(x) = \psi(x) \quad \text{for } x \in \mathbb{R}^2 \setminus \Omega.$$ 

In fact, one just needs to modify (2.6) as

$$- (\Delta + \lambda)^{\beta/2} u(x, y) = -c_{2,\beta} \left( \int_0^L \int_0^L \frac{\phi_{\gamma}(\xi, \eta)}{(\xi^2 + \eta^2)^{\gamma+2+\beta}} d\eta d\xi \right) - (\Delta + \lambda)^{\beta/2} u(x, y)$$
Table 4  Errors and convergence orders of \((\Delta + \lambda)^{\beta/2}(1-x^2)^3(1-y^2)^3 = f\) with \(\lambda = 0.5\) and \(\gamma = 1 + \frac{\beta}{2}\)

| \(\beta\) | \(h\) | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 | 1/256 |
|---|---|---|---|---|---|---|---|
| 0.5 | \(l_{\infty}\) | 4.5025E−02 | 2.3185E−02 | 1.0960E−02 | 4.9802E−03 | 9.0392E−04 | 3.3199E−04 |
| | Rate | 1.2436 | 1.3578 | 1.4140 | 1.4451 | 1.4635 | 1.4635 |
| \(l_2\) | 6.0576E−02 | 4.0453E−02 | 2.5086E−02 | 1.5004E−03 | 8.0636E−04 | 5.1173E−04 |
| | Rate | 0.9748 | 1.0870 | 1.1406 | 1.1679 | 1.1823 | 1.1823 |
| 0.8 | \(l_{\infty}\) | 9.4091E−02 | 6.3722E−02 | 3.9735E−02 | 2.3827E−02 | 1.4004E−02 | 8.1432E−03 |
| | Rate | 0.5825 | 0.6894 | 0.7415 | 0.7687 | 0.7832 | 0.7832 |
| \(l_2\) | 1.5112E−01 | 1.2580E−01 | 9.7124E−02 | 7.2367E−02 | 5.2879E−02 | 3.8195E−02 |
| | Rate | 0.2645 | 0.3732 | 0.4245 | 0.4526 | 0.4693 | 0.4693 |
| 1.2 | \(l_{\infty}\) | 1.0043E−01 | 8.2556E−02 | 6.3448E−02 | 4.7169E−02 | 3.4420E−02 | 2.4840E−02 |
| | Rate | 0.2827 | 0.3798 | 0.4277 | 0.4546 | 0.4706 | 0.4706 |

\[
-4 \int_0^L \int_0^\infty \frac{u(x, y)}{e^{\lambda \sqrt{\xi^2 + \eta^2}}} \left( \frac{\sqrt{\xi^2 + \eta^2}}{\xi^2 + \eta^2} \right)^{2+\beta} d\eta d\xi
-4 \int_0^L \int_0^{\infty} \frac{u(x, y)}{e^{\lambda \sqrt{\xi^2 + \eta^2}}} \left( \frac{\sqrt{\xi^2 + \eta^2}}{\xi^2 + \eta^2} \right)^{2+\beta} d\eta d\xi
-4 \int_0^L \int_0^{\infty} \frac{u(x, y)}{e^{\lambda \sqrt{\xi^2 + \eta^2}}} \left( \frac{\sqrt{\xi^2 + \eta^2}}{\xi^2 + \eta^2} \right)^{2+\beta} d\eta d\xi + B(x, y),
\]

where

\[
B(x, y) = \int_0^L \int_0^\infty \psi(x + \xi, y + \eta) + \psi(x + \xi, y - \eta) + \psi(x - \xi, y + \eta) + \psi(x - \xi, y - \eta) e^{\lambda \sqrt{\xi^2 + \eta^2}} \left( \frac{\sqrt{\xi^2 + \eta^2}}{\xi^2 + \eta^2} \right)^{2+\beta} d\eta d\xi
\]
Table 5 Errors and convergence orders of $(\Delta + \lambda)^{\beta}/2(1-x^2)^3(1-y^2)^3 = f$ with $\lambda = 0$ and $\gamma = 2$

| $h$   | $\beta$ |
|-------|---------|
|       | 1/8     | 1/16    | 1/32    | 1/64    | 1/128   | 1/256   |
|       | 0.5     | 0.5     | 0.5     | 0.5     | 0.5     | 0.5     |
| $l_\infty$ | 9.1029E-04 | 1.9637E-04 | 4.6351E-05 | 1.1347E-05 | 2.8156E-06 | 7.0180E-07 |
| Rate  | 2.2128  | 2.0829  | 2.0302  | 2.0109  | 2.0043  |         |
| $l_2$ | 5.7429E-04 | 1.2021E-04 | 2.7950E-06 | 6.8027E-06 | 1.6843E-06 | 4.1961E-07 |
| Rate  | 2.2562  | 2.1046  | 2.0387  | 2.0140  | 2.0050  |         |
| $0.8$ |         |         |         |         |         |         |
| $l_\infty$ | 1.8685E-03 | 3.9361E-04 | 9.0560E-05 | 2.1800E-05 | 5.3592E-06 | 1.3295E-06 |
| Rate  | 2.2471  | 2.1198  | 2.0545  | 2.0242  | 2.0111  |         |
| $l_2$ | 1.2028E-03 | 2.4720E-04 | 5.5768E-05 | 1.3284E-05 | 3.2495E-06 | 8.0453E-07 |
| Rate  | 2.2827  | 2.1481  | 2.0697  | 2.0314  | 2.0140  |         |
| $1.2$ |         |         |         |         |         |         |
| $l_\infty$ | 3.8160E-03 | 7.8100E-04 | 1.7136E-04 | 3.9489E-05 | 9.3971E-06 | 2.2808E-06 |
| Rate  | 2.2887  | 2.1883  | 2.1175  | 2.0712  | 2.0427  |         |
| $l_2$ | 2.4861E-03 | 5.0566E-04 | 1.0904E-04 | 2.4736E-05 | 5.8223E-06 | 1.4037E-06 |
| Rate  | 2.2977  | 2.1401  | 2.0870  | 2.0523  |         |         |
| $1.5$ |         |         |         |         |         |         |
| $l_\infty$ | 6.3257E-03 | 1.3143E-03 | 2.8396E-04 | 6.3320E-05 | 1.4493E-05 | 3.3880E-06 |
| Rate  | 2.2670  | 2.2105  | 2.1649  | 2.1273  | 2.0968  |         |
| $l_2$ | 4.0732E-03 | 8.5417E-04 | 1.8368E-04 | 4.0561E-05 | 9.1888E-06 | 2.1290E-06 |
| Rate  | 2.2536  | 2.2174  | 2.1790  | 2.1422  | 2.1097  |         |

\[
\int_{-L}^{L} \int_{-L}^{L} \psi(x + \xi, y + \eta) - \psi(x, y) + \psi(x + \xi, y - \eta) + \psi(x - \xi, y + \eta) + \psi(x - \xi, y - \eta) d\eta d\xi
\]

Thus the discretization scheme can be rewritten as

\[
-(\Delta + \lambda)^{\beta/2}_h u_{p,q} = \sum_{i=-N}^{i=N} \sum_{j=-N}^{j=N} w_{|i|,|j|} u_{p-i,q-j} - c_{2,\beta} B(x_p, y_q),
\]

where $w_{i,j}$ and $c_{2,\beta}$ are given in (2.15). As for numerical approximation of $B(x, y)$, denoting $\mathbb{M} = \mathbb{R}^2 \setminus (-L, L)^2$, then $B(x, y)$ can be written as

\[
B(x, y) = \int_{\mathbb{M}} \psi(x + \xi, y + \eta) e^{\lambda \sqrt{\xi^2 + \eta^2}} \left(\sqrt{\xi^2 + \eta^2}\right)^{2+\beta} d\eta d\xi.
\]
Table 6 Errors and convergence orders of \((\Delta + \lambda)^{\beta}/2(1 - x^2)^3(1 - y^2)^3 = f\) with \(\lambda = 0.5\) and \(\gamma = 2\)

| \(h\) | \(\beta\) | \(1/8\) | \(1/16\) | \(1/32\) | \(1/64\) | \(1/128\) | \(1/256\) |
|-------|---------|-------|-------|-------|-------|-------|-------|
| \(l_\infty\) | 0.5 | 3.2465E−03 | 6.9337E−04 | 1.5697E−04 | 3.6753E−05 | 8.7832E−06 | 2.1265E−06 |
| Rate  | 2.2272 | 2.1431 | 2.0946 | 2.0650 | 2.0463 |
| \(l_2\) | 1.9408E−03 | 4.1294E−04 | 9.3343E−05 | 2.1844E−05 | 5.2193E−06 | 1.2634E−06 |
| Rate  | 2.2326 | 2.1453 | 2.0953 | 2.0653 | 2.0466 |
| \(l_\infty\) | 0.8 | 6.3390E−03 | 1.4709E−03 | 3.5361E−04 | 8.6471E−05 | 2.1319E−05 | 5.2764E−06 |
| Rate  | 2.1076 | 2.0564 | 2.0319 | 2.0201 | 2.0145 |
| \(l_2\) | 3.7839E−03 | 8.7736E−04 | 2.1096E−04 | 5.1602E−05 | 1.2724E−05 | 3.1489E−06 |
| Rate  | 2.1086 | 2.0562 | 2.0315 | 2.0199 | 2.0146 |
| \(l_\infty\) | 1.2 | 1.4822E−02 | 4.0632E−03 | 1.1494E−03 | 3.2981E−04 | 9.5204E−05 | 2.7540E−05 |
| Rate  | 1.8670 | 1.8217 | 1.8012 | 1.7925 | 1.7895 |
| \(l_2\) | 8.9250E−03 | 2.4663E−04 | 7.0290E−04 | 2.0284E−04 | 5.8789E−05 | 1.7056E−05 |
| Rate  | 1.8555 | 1.8110 | 1.7930 | 1.7867 | 1.7853 |
| \(l_\infty\) | 1.5 | 2.8458E−02 | 9.2196E−03 | 3.1122E−03 | 1.0731E−03 | 3.7417E−04 | 1.3125E−04 |
| Rate  | 1.6261 | 1.5668 | 1.5361 | 1.5200 | 1.5114 |
| \(l_2\) | 1.7457E−02 | 5.7510E−03 | 1.9678E−03 | 6.8487E−04 | 2.4023E−04 | 8.4579E−05 |
| Rate  | 1.6019 | 1.5472 | 1.5227 | 1.5114 | 1.5060 |

\[
\mathcal{M} = \int \int \int_M \psi(x + \xi, y + \eta) e^{\lambda \sqrt{\xi^2 + \eta^2}} (\sqrt{\xi^2 + \eta^2})^{2+\beta} d\eta d\xi
\]

and the last integral can be approximated by trapezoidal rule or calculated by the built-in function ‘\texttt{integral2.m}’ in MATLAB.

As for the tempered fractional Poisson equation with the Neumann boundary condition [9], spectral method should be a good choice instead of the finite difference.

### 4 Numerical experiments

In this section, extensive numerical experiments are performed, including verifying the predicted convergence rates and showing the effectiveness of the scheme by simulating (1.3) with homogeneous and nonhomogeneous Dirichlet boundary conditions. Without loss of generality, we consider the domain \(\Omega = (-1, 1) \times (-1, 1)\). For convenience, we take \(\gamma = 1 + \beta/2\) and \(\gamma = 2\) in this section and we explain the reasonability of this selection for \(\gamma\) by Figs. 1 and 2. The \(l_\infty\) norm and \(l_2\) norm are used to measure the corresponding errors here.
Table 7 Errors and convergence orders of \((\Delta + \lambda)^{\beta/2} u = 1\) with \(\lambda = 0\) and \(\gamma = 1 + \frac{\beta}{2}\)

| \(\beta\) | \(h\) | \(1/16\) | \(1/32\) | \(1/64\) | \(1/128\) | \(1/256\) |
|---|---|---|---|---|---|---|
| 0.5 | \(l_\infty\) | 5.5241\(\times\)10^{-02} | 4.2765\(\times\)10^{-02} | 3.4306\(\times\)10^{-02} | 2.8808\(\times\)10^{-02} | 2.4210\(\times\)10^{-02} |
|   | Rate | 0.3693 | 0.3180 | 0.2520 | 0.2509 | 0.2509 |
|   | \(l_2\) | 2.8760\(\times\)10^{-02} | 1.7970\(\times\)10^{-02} | 1.0996\(\times\)10^{-02} | 6.6540\(\times\)10^{-03} | 4.0008\(\times\)10^{-03} |
|   | Rate | 0.6784 | 0.7086 | 0.7247 | 0.7339 | 0.7339 |
| 0.8 | \(l_\infty\) | 4.1983\(\times\)10^{-02} | 3.1363\(\times\)10^{-02} | 2.3612\(\times\)10^{-02} | 1.7837\(\times\)10^{-02} | 1.3496\(\times\)10^{-02} |
|   | Rate | 0.4207 | 0.4095 | 0.4046 | 0.4023 | 0.4023 |
|   | \(l_2\) | 2.8257\(\times\)10^{-02} | 1.6889\(\times\)10^{-02} | 9.7784\(\times\)10^{-03} | 5.5543\(\times\)10^{-03} | 3.1158\(\times\)10^{-03} |
|   | Rate | 0.7425 | 0.7884 | 0.8160 | 0.8340 | 0.8340 |
| 1.2 | \(l_\infty\) | 2.5049\(\times\)10^{-02} | 1.5905\(\times\)10^{-02} | 1.0213\(\times\)10^{-02} | 6.6579\(\times\)10^{-03} | 4.3582\(\times\)10^{-03} |
|   | Rate | 0.6553 | 0.6391 | 0.6173 | 0.6113 | 0.6113 |
|   | \(l_2\) | 2.2415\(\times\)10^{-02} | 1.4148\(\times\)10^{-02} | 8.6240\(\times\)10^{-03} | 5.1498\(\times\)10^{-03} | 3.0366\(\times\)10^{-03} |
|   | Rate | 0.6639 | 0.7142 | 0.7438 | 0.7621 | 0.7621 |
| 1.5 | \(l_\infty\) | 2.1610\(\times\)10^{-02} | 1.5623\(\times\)10^{-02} | 1.1253\(\times\)10^{-02} | 8.0821\(\times\)10^{-03} | 5.7890\(\times\)10^{-03} |
|   | Rate | 0.4680 | 0.4733 | 0.4775 | 0.4814 | 0.4814 |
|   | \(l_2\) | 1.5619\(\times\)10^{-02} | 1.1405\(\times\)10^{-02} | 8.2246\(\times\)10^{-03} | 5.8995\(\times\)10^{-03} | 4.2199\(\times\)10^{-03} |
|   | Rate | 0.4536 | 0.4717 | 0.4793 | 0.4834 | 0.4834 |

### 4.1 The performance of approximating tempered fractional Laplacian

This subsection shows the errors and convergence rates of approximating the tempered fractional Laplacian.

**Example 4.1** Compute \((\Delta + \lambda)^{\beta/2} u(x, y)\) with \(u(x, y) = (1-x^2)^3(1-y^2)^3 (u(x, y) \in C^2(\mathbb{R}^2))\).

Table 1 shows the accuracy of computing \((\Delta + \lambda)^{\beta/2} u(x, y)\) with \(\lambda = 0\) and \(\gamma = 1 + \frac{\beta}{2}\), which verifies the results of Theorem 2.1. Table 2 shows the accuracy of computing \((\Delta + \lambda)^{\beta/2} u(x, y)\) with \(\lambda = 0.5\) and \(\gamma = 1 + \frac{\beta}{2}\). We find that for the fixed mesh size \(h\), the numerical errors will increase as the parameter \(\beta\) increases and the convergence rates are \(O(h^{2-\beta})\) for any \(\beta \in (0, 2)\) from Tables 1 and 2. These results are consistent with the theoretical predictions.

Comparing Tables 1 with 2, we find that the convergence rates are independent of \(\lambda\) and the numerical errors will decrease as the parameter \(\lambda\) increases for fixed \(h\) and \(\beta\).
Table 8 Errors and convergence orders of \((\Delta + \lambda)^{\beta/2} u = 1\) with \(\lambda = 0.5\) and \(\gamma = 2\)

| \(\beta\)  | \(h\)     | \(l_\infty\)            | \(\tau\)       | \(l_2\)         | \(\tau\)       | \(l_\infty\)            | \(\tau\)       | \(l_2\)         | \(\tau\)       |
|------------|-----------|--------------------------|-----------------|-----------------|-----------------|--------------------------|-----------------|-----------------|-----------------|
| 0.5        | 1/16      | 1.2879E−01               | 0.4233          | 8.7742E−02      | 0.7332          | 6.2353E−02               | 0.4692          | 4.3717E−02      | 0.7288          |
|            | 1/32      | 9.6038E−02               | 0.3996          | 5.2782E−02      | 0.7968          | 4.5041E−02               | 0.4563          | 2.6380E−02      | 0.8107          |
|            | 1/64      | 7.2802E−02               | 0.3700          | 3.0383E−02      | 0.8279          | 3.2535E−02               | 0.4563          | 1.5039E−02      | 0.8558          |
|            | 1/128     | 5.6334E−02               | 0.3420          | 1.7117E−02      | 0.8809          | 2.3712E−02               | 0.4417          | 8.3099E−03      | 0.8811          |
|            | 1/256     | 4.4445E−02               | 0.3373          | 9.5561E−03      |                | 1.7459E−02               | 0.4417          | 4.5118E−03      |                |
| 0.8        | 1/16      | 6.2353E−02               | 0.4692          | 4.3171E−02      | 0.7288          | 2.8717E−02               | 0.4692          | 1.5039E−02      | 0.8811          |
|            | 1/32      | 4.5041E−02               | 0.4693          | 2.6380E−02      | 0.8107          | 1.5039E−02               | 0.4693          | 8.3099E−03      | 0.8811          |
|            | 1/64      | 3.2535E−02               | 0.4563          | 1.5039E−02      | 0.8558          | 8.3099E−03               | 0.4693          | 4.5118E−03      | 0.8811          |
|            | 1/128     | 2.3712E−02               | 0.4417          | 8.3099E−03      | 0.8811          | 4.5118E−03               | 0.4417          | 4.5118E−03      | 0.8811          |
|            | 1/256     | 1.7459E−02               | 0.4417          | 4.5118E−03      | 0.8811          | 4.5118E−03               | 0.4417          | 4.5118E−03      | 0.8811          |
| 1.2        | 1/16      | 2.1830E−02               | 0.5287          | 1.5544E−02      | 0.6740          | 7.1398E−03               | 0.5287          | 9.7421E−03      | 0.6740          |
|            | 1/32      | 1.5132E−02               | 0.5870          | 9.7421E−03      | 0.7984          | 5.2215E−03               | 0.5870          | 5.6015E−03      | 0.7984          |
|            | 1/64      | 1.0074E−02               | 0.6044          | 5.6015E−03      | 0.8687          | 3.3499E−03               | 0.6044          | 3.0677E−03      | 0.8687          |
|            | 1/128     | 6.6261E−03               | 0.6078          | 3.0677E−03      | 0.9108          | 2.0513E−03               | 0.6078          | 1.6317E−03      | 0.9108          |
|            | 1/256     | 4.3480E−03               | 0.6078          | 1.6317E−03      |                | 1.2336E−03               | 0.6078          | 1.2336E−03      |                |
| 1.5        | 1/16      | 7.1398E−03               | 0.6740          | 4.2726E−03      | 0.4450          | 7.1398E−03               | 0.6740          | 3.1385E−03      | 0.4450          |
|            | 1/32      | 5.2215E−03               | 0.6403          | 3.1385E−03      | 0.6824          | 5.2215E−03               | 0.6403          | 1.9557E−03      | 0.6824          |
|            | 1/64      | 3.3499E−03               | 0.7076          | 1.9557E−03      | 0.8028          | 3.3499E−03               | 0.7076          | 1.1211E−03      | 0.8028          |
|            | 1/128     | 2.0513E−03               | 0.7336          | 1.1211E−03      | 0.8723          | 2.0513E−03               | 0.7336          | 6.1242E−04      | 0.8723          |
|            | 1/256     | 1.2336E−03               | 0.7336          | 6.1242E−04      |                | 1.2336E−03               | 0.7336          | 6.1242E−04      |                |

4.2 The performance of solving the tempered fractional Poisson equation

Example 4.2 We solve (1.3) with different \(\beta\) and \(\gamma\), and the exact solution is taken as \(u(x, y) = (1 - x^2)^3(1 - y^2)^3\), where \(u \in C^2(\mathbb{R}^2)\). The source term \(f(x, y)\) is obtained numerically by the algorithm in “Appendix A”.

Tables 3 and 4 show that the convergence rates are both \(O(h^{2-\beta})\) when \(\lambda = 0\), \(\gamma = 1 + \frac{\beta}{2}\) and \(\lambda = 0.5\), \(\gamma = 1 + \frac{\beta}{2}\). The results show that \(\lambda\) has no effect on the convergence rates when \(\gamma = 1 + \frac{\beta}{2}\).

But when \(\lambda = 0\) and \(\gamma = 2\), the convergence rates shown in Table 5 are higher than our theoretical results and the convergence rates are \(O(h^2)\) for any \(\beta \in (0, 2)\). For \(\lambda > 0\), the convergence rates provided in Table 6 depend on \(\beta\), that is, when \(\beta < 1\), the convergence rates are \(O(h^\beta)\), and the convergence rates are \(O(h^{3-\beta})\) for \(\beta > 1\). This phenomenon indicates that the provided scheme works very well for the equation (1.3) when \(\gamma = 2\).

Next, we give Figs. 1 and 2 to show the influence on the convergence rates about different choices for \(\gamma\). Here, we show the errors and convergence rates with \(\gamma = 0.6, 0.75, 1.25, 1.5\) and 1.8 and the ‘Order line 1.5’ denotes the convergence rate is \(O(h^{1.5})\) and the ‘Order line 2’ denotes the convergence rate is \(O(h^2)\). Figure 1 shows that the convergence rates are almost \(O(h^{2-\beta})\) except for \(\gamma = 2\) when \(\beta = 0.5\) and \(\lambda = 0\);
for the same mesh size $h$, the numerical errors decrease as the parameter $\gamma$ increases. We can get the same results from Fig. 2 when $\beta = 0.5$ and $\lambda = 0.5$. Comparing Figs. 1 with 2, we obtain that $\gamma$ has the same influence on the convergence rates for any $\lambda$ except for $\gamma = 2$, so our choices for $\gamma = 1 + \beta/2$ and $\gamma = 2$ are valid.

**Example 4.3** We consider the model (1.3) in $\Omega$ with the source term $f = 1$. Here

$$\text{rate} = \frac{\ln(e_{2h}/e_h)}{\ln(2)}$$

is utilized to measure the convergence rates [13], where $u_h$ means the numerical solution under mesh size $h$ and $e_h = \|u_{2h} - u_h\|$.

Tables 7 and 8 show the numerical errors and the convergence rates with $\lambda = 0$, $\gamma = 1 + \frac{\beta}{2}$ and $\lambda = 0.5$, $\gamma = 2$, respectively, which indicate the effectiveness of our scheme. The convergence rates are lower than the desired ones because the regularity of the exact solution $u$ is lower than our assumption. These results are similar to those in one dimension [24].

In statistical physics [10], the solution $u$ of Example 4.3 represents the mean first exit time of a particle starting at $(x, y)$ away from the given domain $\Omega$. Figure 3 shows the dynamical behaviors when $\lambda = 0$, 0.5 and $\beta = 0.5$, 0.8, 1.2, 1.5; for any $\lambda$.
Table 9 Errors and convergence orders with $\lambda = 0.1$ and $\gamma = 2$

| $\beta$ | $h$ | $l_\infty$ | $l_2$ | $l_\infty$ | $l_2$ | $l_\infty$ | $l_2$ |
|---------|-----|----------|-------|----------|-------|----------|-------|
|         | 1/16 | 1/32     | 1/64  | 1/128    | 1/256 |
| 0.5     | 9.1102E-04 | 7.3522E-04 | 5.9459E-04 | 4.8380E-04 | 3.9636E-04 | 0.3093 | 0.3063 | 0.2975 | 0.2876 |
| Rate    | 0.7032 | 0.7469   | 0.7687 |          |        |
| 0.8     | 6.1728E-04 | 4.6550E-04 | 3.4949E-04 | 2.6243E-04 | 1.9741E-04 | 0.0787 | 0.1135 | 0.1387 | 0.1487 |
| Rate    | 0.7487 | 0.8007   | 0.8317 |          |        |
| 1.2     | 2.3436E-04 | 1.5745E-04 | 1.0465E-04 | 6.9215E-05 | 4.5685E-05 | 0.5738 | 0.5965 | 0.6094 | 0.6217 |
| Rate    | 0.8094 | 0.8643   | 0.8998 |          |        |
| 1.5     | 7.0869E-05 | 4.3491E-05 | 2.6298E-05 | 1.5764E-05 | 9.4078E-06 | 0.7044 | 0.7257 | 0.7383 | 0.7447 |
| Rate    | 0.8417 | 0.8911   | 0.9246 |          |        |

and $\beta$, the mean first exit time of particles starting near the center are longer than those starting near the boundary of $\Omega$; for any fixed $\lambda$, the mean first exit time will get shorter as $\beta$ increases; the mean first exit time of any fixed starting point will get longer as $\lambda$ increases. These results well conform to the physical phenomena.

Lastly, we use our discretization to solve the following equation [10]. Because the boundary conditions of the solution are discontinuous, designing the efficient algorithm becomes very delicate. We find that our method is valid since the scheme well converges, but the convergence orders aren’t optimal. The main reason is that the regularity of the solution is lower than the assumption. Here, we only solve it roughly by using our presented scheme. We will study the topic carefully and give an efficient algorithm for this equation in the near future study.

**Example 4.4** Consider the following equation with nonhomogeneous boundary condition [10], i.e.

$$
\begin{array}{l}
-(\Delta + \lambda)^{\beta} u(x) = 0 \text{ for } x \in \Omega, \\
u(x) = 1 \text{ for } x \in E,
\end{array}
$$

(4.2)

where $\Omega = [-1, 1] \times [-1, 1]$ and $E = [3, 5] \times [-1, 1]$. 
Algorithm implementation and numerical analysis for the…

Fig. 4 Dependence of escape probability $u$ on $\beta$ and $\lambda$.

Table 9 shows that the numerical errors and convergence orders when $\lambda = 0.1$ and $\gamma = 2$, which verifies the effectiveness of our scheme. The convergence rates are also obtained by (4.1). Because of the low regularity of exact solution, the convergence orders aren’t optimal.

In statistical physics [10], the solution $u$ of (4.2) represents escape probability [10] from a domain $\Omega$ to a subset $E$ of $\Omega^c$. The first four drawings of Fig. 4 show the dynamical behaviors when $\lambda = 0.1$ and $\beta = 0.5, 0.8, 1.2, 1.5$. Following that, we take $\lambda = 0.2, 0.3, 0.4, 0.5$ for $\beta = 0.5$ to show the change of the escape probability, as is shown in the last four drawings. Comparing 4a with 4e, 4f, 4g and 4h, we know that $\lambda$ has a strong effect on the escape probability.

5 Conclusion

This paper provides the finite difference scheme for the two-dimensional tempered fractional Laplacian, being physically introduced and mathematically defined in [9]. The operator is written as the weighted integral of a weak singular function by introducing the function $\phi_\gamma$. The weighted trapezoidal rule is used to approximate the integration of the weak singular part and the bilinear interpolation for the rest of the integration. The detailed error estimates are presented for the designed numeri-
Fig. 5 Division of the integral region for a fixed point \((x, y)\)

Appendix

A Numerically calculating \((\Delta + \lambda)^{\beta/2}\) performed on a given function

According to the equation \(- (\Delta + \lambda)^{\beta/2} u(x, y) = f(x, y)\), we can compute \(- (\Delta + \lambda)^{\beta/2} u(x, y)\) to get the source term \(f(x, y)\). Since the singularity and non-locality of \(- (\Delta + \lambda)^{\beta/2} u(x, y)\), one can’t directly approximate it by the trapezoidal rule. Now we provide the technique to calculate it. For a fixed point \((x, y)\), we denote

\[
\begin{align*}
    r_1 &= \sup_{(\xi, \eta) \in \partial \Omega} \max(|x - \xi|, |y - \eta|), \\
    r_2 &= \inf_{(\xi, \eta) \in \partial \Omega} \sqrt{(x - \xi)^2 + (y - \eta)^2}. 
\end{align*}
\]

Without loss of generality, we set \(\Omega = (-1, 1) \times (-1, 1)\). For any \((x, y) \in \Omega\), we denote \(A_1\) as a square whose length is \(2r_1\) and center point is \((x, y)\) and \(A_2\) as a square whose length is \(2r_2\) and center point is \((x, y)\). To compute the source term \(f(x, y)\), we divide the domain into four parts, i.e., \(\mathbb{R} \times \mathbb{R} = (\mathbb{R} \times \mathbb{R}) \setminus A_1 \cup (A_1 \setminus \Omega) \cup (\Omega \setminus A_2) \cup A_2\), shown in Fig. 5.

For the term

\[
\int \int_{(\mathbb{R} \times \mathbb{R}) \setminus (A_1)} \frac{u(\xi, \eta) - u(x, y)}{e^{\lambda \sqrt{(x-\xi)^2+(y-\eta)^2}}} \left(\sqrt{(x-\xi)^2+(y-\eta)^2}\right)^{2+\beta} d\xi d\eta, \tag{A.1}
\]

since \(\text{supp } u(x, y) \subset \Omega\), (A.1) can be rewritten as

\[
- u(x, y) \int \int_{(\mathbb{R} \times \mathbb{R}) \setminus (A_1)} \frac{1}{e^{\lambda \sqrt{(x-\xi)^2+(y-\eta)^2}}} \left(\sqrt{(x-\xi)^2+(y-\eta)^2}\right)^{2+\beta} d\xi d\eta.
\]
Next, we establish polar coordinates at point \((x, y)\) and let \(x - \xi = r \cos(\theta), y - \eta = r \sin(\theta)\). Then, by simple calculations, we can obtain

\[
\int \left(\mathbb{R} \times \mathbb{R}\right)_{(A_1)} \frac{1}{e^{\lambda \sqrt{(x - \xi)^2 + (y - \eta)^2}} \left(\sqrt{(x - \xi)^2 + (y - \eta)^2}\right)^{2+\beta}} d\xi d\eta
\]

\[
= \int_{0}^{\frac{\pi}{4}} \int_{r_1 \cos(\theta)}^{\frac{\pi}{4}} \frac{1}{r^{1+\beta} e^{\lambda r}} dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{r_1 \cos(\theta)}^{\frac{\pi}{4}} \frac{1}{r^{1+\beta} e^{\lambda r}} dr d\theta
\]

\[
+ \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{r_1 \cos(\theta - \frac{\pi}{2})}^{\frac{3\pi}{4}} \frac{1}{r^{1+\beta} e^{\lambda r}} dr d\theta + \int_{\frac{3\pi}{4}}^{\frac{\pi}{2}} \int_{r_1 \cos(\theta - \frac{\pi}{2})}^{\frac{3\pi}{4}} \frac{1}{r^{1+\beta} e^{\lambda r}} dr d\theta
\]

\[
+ \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \int_{r_1 \cos(\theta - \frac{\pi}{2})}^{\frac{5\pi}{4}} \frac{1}{r^{1+\beta} e^{\lambda r}} dr d\theta + \int_{\frac{5\pi}{4}}^{\frac{\pi}{2}} \int_{r_1 \cos(\theta - \frac{\pi}{2})}^{\frac{5\pi}{4}} \frac{1}{r^{1+\beta} e^{\lambda r}} dr d\theta
\]

\[
= 8 \int_{0}^{\frac{\pi}{4}} \int_{r_1 \cos(\theta)}^{\frac{\pi}{4}} \frac{1}{r^{1+\beta} e^{\lambda r}} dr d\theta.
\]

When \(\lambda = 0\) for (A.2), we have

\[
8 \int_{0}^{\frac{\pi}{4}} \int_{r_1 \cos(\theta)}^{\frac{\pi}{4}} \frac{1}{r^{1+\beta} e^{\lambda r}} dr d\theta = 8 \int_{0}^{\frac{\pi}{4}} \left(\frac{r_1 \cos(\theta)}{r^{1+\beta} e^{\lambda r}}\right) d\theta.
\]

We just approximate it by the trapezoidal rule in a finite interval. When \(\lambda \neq 0\), we approximate (A.2) by the function ‘integral2.m’ in MATLAB.

For the term

\[
\int \int_{A_2} e^{\lambda \sqrt{(x - \xi)^2 + (y - \eta)^2}} \left(\sqrt{(x - \xi)^2 + (y - \eta)^2}\right)^{2+\beta} d\xi d\eta,
\]

using its symmetry leads to

\[
\int \int_{A_2} e^{\lambda \sqrt{(x - \xi)^2 + (y - \eta)^2}} \left(\sqrt{(x - \xi)^2 + (y - \eta)^2}\right)^{2+\beta} d\xi d\eta
\]

\[
= \int_{r_2}^{r_1} \int_{0}^{r_2} u(x + \xi, y + \eta) - u(x, y) \end{align*}
\]

\[
= \int_{0}^{r_2} \int_{0}^{r_2} u(x + \xi, y + \eta) + u(x + \xi, y - \eta) + u(x - \xi, y + \eta) + u(x - \xi, y - \eta) - 4u(x, y) e^{\lambda \sqrt{\xi^2 + \eta^2}} \left(\sqrt{\xi^2 + \eta^2}\right)^{2+\beta} d\xi d\eta.
\]
Because of the weak singularity, we try to compute it in polar coordinates. Let $\xi = r \cos(\theta)$, $\eta = r \sin(\theta)$. Then (A.3) can be rewritten as

$$
\int_{A_2} \frac{u(\xi, \eta) - u(x, y)}{e^{\lambda \sqrt{(x-\xi)^2 + (y-\eta)^2}} \left( \sqrt{(x-\xi)^2 + (y-\eta)^2} \right)^{2+\beta}} \, d\xi \, d\eta
$$

$$
= \int_0^\pi \int_0^{\frac{\pi}{4}} \frac{1}{r^2} (u(x + r \cos(\theta), y + r \sin(\theta)) + u(x - r \cos(\theta), y + r \sin(\theta))
+ u(x + r \cos(\theta), y - r \sin(\theta)) + u(x - r \cos(\theta), y - r \sin(\theta))
- 4u(x, y)) r^{-1-\beta} e^{-\lambda r} \, dr \, d\theta
+ \int_0^\pi \int_0^{\frac{\pi}{4}} (u(x + r \cos(\theta), y + r \sin(\theta)) + u(x - r \cos(\theta), y + r \sin(\theta))
+ u(x + r \cos(\theta), y - r \sin(\theta)) + u(x - r \cos(\theta), y - r \sin(\theta))
- 4u(x, y)) r^{-1-\beta} e^{-\lambda r} \, dr \, d\theta.
$$

(A.4)

In (A.4), for some special function, such as $u(x, y) = (1 - x^2)(1 - y^2)^2$, we can expand it as

$$
(u(x + r \cos(\theta), y + r \sin(\theta)) + u(x - r \cos(\theta), y + r \sin(\theta))
+ u(x + r \cos(\theta), y - r \sin(\theta)) + u(x - r \cos(\theta), y - r \sin(\theta))
- 4u(x, y)) r^{-1-\beta} e^{-\lambda r}
= 4r^{1-\beta} e^{-\lambda r} (r^6 \sin^4(\theta) \cos^4(\theta) + 6r^4 x^2 \sin^4(\theta) \cos^2(\theta) + 6r^4 y^2 \sin^2(\theta) \cos^4(\theta)
- 2r^4 \sin^2(\theta) \cos^4(\theta)
- 2r^4 \sin^4(\theta) \cos^2(\theta) + r^2 x^4 \sin^4(\theta) + 3r^2 x^2 y^2 \sin^2(\theta) \cos^2(\theta) - 2r^2 x^2 \sin^4(\theta)
- 12r^2 x^2 \sin^2(\theta) \cos^2(\theta) + r^2 y^4 \cos^4(\theta) - 2r^2 y^2 \cos^4(\theta) - 12r^2 y^2 \sin^2(\theta) \cos^2(\theta)
+ r^2 \sin^4(\theta) + r^2 \cos^4(\theta) + 4r^2 \sin^2(\theta) \cos^2(\theta)
+ 6x^2 y^2 \sin^2(\theta) - 2x^4 \sin^2(\theta) + 6x^2 y^2 \cos^2(\theta)
- 12x^2 y^2 \sin^2(\theta) - 12x^2 y^2 \cos^2(\theta) + 4x^2 \sin^2(\theta)
+ 6x^2 \cos^2(\theta) - 2y^4 \cos^2(\theta) + 6y^2 \sin^2(\theta)
+ 4y^2 \cos^2(\theta) - 2 \sin^2(\theta) - 2 \cos^2(\theta)).
$$

For (A.4), the inner integration about $r$ can be calculated analytically when $\lambda = 0$, so we approximate the outer integration about $\theta$ by the trapezoidal rule; when $\lambda \neq 0$, we can transform the inner integration about $r$ to a nonsingular numerical integration by virtue of integration by parts.

For another two terms,

$$
\int_{(A_1 \setminus \Omega) \cup (\Omega \setminus A_2)} \frac{u(\xi, \eta) - u(x, y)}{e^{\lambda \sqrt{(x-\xi)^2 + (y-\eta)^2}} \left( \sqrt{(x-\xi)^2 + (y-\eta)^2} \right)^{2+\beta}} \, d\xi \, d\eta,
$$

we get them by the trapezoidal rule directly.
B Weights of approximating tempered fractional Laplacian

\[-4 \left( \frac{G_{0,0} + W_{1,1} + W_{2,1} + W_{3,1}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}} \right) \]
\[+ \frac{G_{0,0} + W_{1,0}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}} + \frac{G_{0,0} + W_{4,1}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}} \]
\[+ \sum_{r=2}^{N-1} \frac{W_{0,r} + W_{2,0}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}} + \sum_{j=2}^{N-1} \frac{W_{0,j} + W_{3,j}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}} \]
\[+ \sum_{r=1}^{N-1} \frac{W_{r,0} + W_{4,N}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}} + \sum_{j=1}^{N-1} \frac{W_{2,j} + W_{4,j}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}} \]
\[+ \sum_{r=1}^{N-1} \sum_{j=1}^{N-1} \frac{W_{r,j} + W_{r,j} + W_{r,j} + W_{r,j}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}} \]
\[+ \frac{W_{0,N}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}} + \frac{G^N}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}}, \quad i = 0, j = 0, \]
\[\frac{G_{0,0} + W_{1,1} + W_{2,1} + W_{3,1}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}}, \quad i = 1, j = 1, \]
\[\frac{G_{0,0} + W_{1,0}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}}, \quad i = 1, j = 0, \]
\[\frac{G_{0,0} + W_{4,1}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}}, \quad i = 0, j = 1, \]
\[\frac{W_{0,0} + W_{2,0}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}}, \quad 1 < i < N, j = 0, \]
\[\frac{W_{0,j} + W_{3,j}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}}, \quad i = 0, 1 < j < N, \]
\[\frac{W_{3,j} + W_{4,j}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}}, \quad 1 < i < N, j = N, \]
\[\frac{W_{2,j} + W_{4,j}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}}, \quad i = N, 1 < j < N, \]
\[\frac{W_{2,j} + W_{4,j}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}}, \quad i = 0, j = N, \]
\[\frac{W_{N,0}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}}, \quad i = N, j = 0, \]
\[\frac{W_{N,j}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}}, \quad i = N, j = N, \]
\[\frac{W_{N,N}}{e^{\sqrt{\lambda + \eta^2}} \sqrt{\xi^2 + \eta^2}}, \quad \text{otherwise.} \]

(B.1)
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