Quantum return probability of a system of \( N \) non-interacting lattice fermions

P L Krapivsky\(^1\), J M Luck\(^2,3\) and K Mallick\(^2\)

\(^1\) Department of Physics, Boston University, Boston, MA 02215, United States of America
\(^2\) Institut de Physique Théorique, Université Paris-Saclay, CEA and CNRS, 91191 Gif-sur-Yvette, France
E-mail: pkrapivsky@gmail.com, jean-marc.luck@ipht.fr and kirone.mallick@ipht.fr

Received 23 October 2017
Accepted for publication 28 December 2017
Published 12 February 2018

Abstract. We consider \( N \) non-interacting fermions performing continuous-time quantum walks on a one-dimensional lattice. The system is launched from a most compact configuration where the fermions occupy neighboring sites. We calculate exactly the quantum return probability (sometimes referred to as the Loschmidt echo) of observing the very same compact state at a later time \( t \). Remarkably, this probability depends on the parity of the fermion number—it decays as a power of time for even \( N \), while for odd \( N \) it exhibits periodic oscillations modulated by a decaying power law. The exponent also slightly depends on the parity of \( N \), and is roughly twice smaller than what it would be in the continuum limit. We also consider the same problem, and obtain similar results, in the presence of an impenetrable wall at the origin constraining the particles to remain on the positive half-line. We derive closed-form expressions for the amplitudes of the power-law decay of the return probability in all cases. The key point in the derivation is the use of Mehta integrals, which are limiting cases of the Selberg integral.

Keywords: mesoscopic systems, quantum transport in one-dimension

\(^3\) Author to whom any correspondence should be addressed.
1. Introduction

The advent of techniques to manipulate cold atoms has led to the experimental realization of low-dimensional quantum gases [1] that, for many decades, were mainly thought of as being toy models for theoreticians [2, 3]. The continuous tuning of the pair interaction between atoms thanks to Feshbach resonances allows one to create many-body systems ranging from free to strongly interacting. This revolution triggered a profusion of studies of the non-equilibrium dynamics of quantum systems [4–6], including quenches, entanglement and thermalization, that resulted in many new ways of exploring the quantum world [7].

Non-interacting tight-binding fermions on a lattice form the simplest of all quantum gases. They can be viewed as continuous-time quantum walkers. Quantum walks were initially introduced in the context of quantum information theory, as a general framework to state quantum algorithms [8–10]. Many fundamental problems that have been studied for classical random walks have natural counterparts in the realm of quantum walks, often leading to results that look unexpected and surprising from a classical point of view. First of all, quantum walkers behave ballistically rather than diffusively [11, 12]. The survival in the presence of traps [13, 14] and the ballistic spreading of bound states [15, 16] provide yet other examples of qualitatively different behavior in the classical and quantum cases.
A foremost question that has been investigated since the earlier days of quantum mechanics concerns the decay of individual quantum states \[17–23\]. In many circumstances the return probability of a quantum system to its very initial state, also referred to as the Loschmidt echo \[24–27\], falls off exponentially in time. Quantum revival therefore corresponds to a rare event, somehow analogously to those described by the laws of large deviations in classical statistical mechanics \[28, 29\]. The return probability has been recently studied for various systems of bosons or fermions confined to one dimension. There, a wide variety of temporal decays can be found, ranging from a power law to a superexponential decay. These recent works include two-particle systems \[30–32\], many-particle systems \[33–36\], and systems with infinitely many particles \[37–39\]. For instance, for free lattice fermions with domain-wall initial condition, the return probability was found to obey a pure Gaussian decay in time \[37, 38\]. More recently, the XXZ quantum spin chain with domain-wall initial condition has been investigated in the whole massless phase, where the anisotropy parameter obeys \(|\Delta| < 1\) \[39\]. There, the decay of the return probability is found to be either Gaussian or exponential, depending on whether the angle parameter \(\gamma\) (defined by \(\Delta = \cos \gamma\)) is commensurate to \(\pi\) or not. This highly discontinuous asymptotic behavior is attributed to integrability.

The aim of the present work is to study the return probability for a system of \(N\) non-interacting fermions hopping on a one-dimensional lattice. The fermions are launched from the most compact state where they occupy \(N\) successive lattice sites. We consider two different settings: an infinite chain (section 2) and a semi-infinite chain bounded by an impenetrable wall (section 3). In the latter situation, the compact initial state lies near the wall. We first derive general expressions for the return probability at arbitrary finite times (sections 2.1 and 3.1). We then perform an asymptotic analysis of the regime of late times (sections 2.3 and 3.2). In both settings the return probability manifests a dependence on the parity of the fermion number, oscillating forever for odd \(N\) and decaying monotonically for even \(N\). In all cases it falls off as a power of time, whose exponent also slightly depends on the parity of \(N\), and is roughly twice smaller than what it would be in the continuum limit. We also derive closed-form expressions for the amplitudes of the power-law decay of the return probability in all these situations. This derivation relies on the usage of Mehta integrals (\[40\] chapter 17). Our results suggest a non-trivial crossover behavior in the scaling regime where time \(t\) and the fermion number \(N\) are both large and proportional to each other (sections 2.4 and 3.3). The determination of the corresponding scaling functions \(F\) and \(F^{(w)}\) remains a challenging open problem. Section 4 contains a brief summary and a discussion of our findings. Two appendices are respectively devoted to the Andr"{e}ief identity (appendix A) and to Barnes’ \(G\)-function (appendix B).

2. Free fermions on the infinite chain

We consider \(N\) non-interacting tight-binding fermions hopping on an infinite chain. At the initial time \((t = 0)\), the fermions are launched from \(N\) consecutive sites, labeled \(n = 1, \ldots, N\) (see figure 1).
We are interested in the return probability \( R_N(t) \), i.e. the probability that the fermions occupy the same sites at a later time \( t \), and especially in the asymptotic decay of this quantity in the regime of long times.

### 2.1. General expressions at finite times

General expressions for the return probability \( R_N(t) \) at arbitrary finite times can be derived as follows. Within the formalism of the second quantization, the Hamiltonian of the system reads

\[
H = \sum_n \left( a_n^\dagger a_{n+1} + a_{n+1}^\dagger a_n \right),
\]

in dimensionless units, with the standard fermionic anti-commutation rules

\[
\{a_m, a_n\} = \{a_m^\dagger, a_n^\dagger\} = 0, \quad \{a_m, a_n^\dagger\} = \delta_{m,n}.
\]

The many-body wavefunction \( |\Psi(t)\rangle \) obeys the time-dependent Schrödinger equation

\[
i \frac{d|\Psi\rangle}{dt} = H|\Psi\rangle,
\]

and so

\[
|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle.
\]

The return probability therefore reads

\[
R_N(t) = |A_N(t)|^2,
\]

with

\[
A_N(t) = \langle \Psi(0)|\Psi(t)\rangle = \langle \Psi(0)|e^{-iHt}|\Psi(0)\rangle.
\]

The above expression can be made explicit by bringing the Hamiltonian \( H \) to the diagonal form

\[
H = \int_0^{2\pi} \frac{dq}{2\pi} \varepsilon_q \hat{a}_q^\dagger \hat{a}_q,
\]

with

\[
\varepsilon_q = 2 \cos q \quad (0 \leq q \leq 2\pi).
\]

The operators

\[
\hat{a}_q = \sum_n e^{iqa_n}, \quad \hat{a}_q^\dagger = \sum_n e^{-iqa_n^\dagger}
\]

obey the anti-commutation relation

\[
\{a_m, a_n\} = \{a_m^\dagger, a_n^\dagger\} = 0, \quad \{a_m, a_n^\dagger\} = \delta_{m,n}.
\]
\{\hat{a}_q, \hat{a}^\dagger_q\} = 2\pi \delta(q - q'). \tag{2.10}

We find after some elementary algebra

\[ A_N(t) = \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{n=1}^{N} \frac{dq_n}{2\pi} e^{-2it \cos q_n} |\langle \Psi(0)|q\rangle|^2, \tag{2.11} \]

where \( q = (q_1, \ldots, q_N) \), and the \( q_n \) are the momenta of the \( N \) fermions. The many-body amplitude \( \langle \Psi(0)|q\rangle \) is given by a normalized Slater determinant. In the present situation, where the fermions are launched from the sites \( m = 1, \ldots, N \), we have

\[ \langle \Psi(0)|q\rangle = \frac{1}{\sqrt{N!}} \det \left( e^{imq_n} \right), \tag{2.12} \]

and therefore

\[ A_N(t) = \frac{1}{N!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{n=1}^{N} \frac{dq_n}{2\pi} e^{-2it \cos q_n} |\det \left( e^{imq_n} \right)|^2. \tag{2.13} \]

The formula (2.12) can be recast as

\[ \langle \Psi(0)|q\rangle = \frac{1}{\sqrt{N!}} \prod_{n=1}^{N} e^{iql_n} \prod_{1 \leq m < n \leq N} \left( e^{iq_n} - e^{iq_m} \right), \tag{2.14} \]

where the last product is a Vandermonde determinant. We are thus left with the following integral expression for the amplitude:

\[ A_N(t) = \frac{1}{N!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{n=1}^{N} \frac{dq_n}{2\pi} e^{-2it \cos q_n} \times \prod_{1 \leq m < n \leq N} \left| e^{iq_n} - e^{iq_m} \right|^2. \tag{2.15} \]

Applying the Andréief identity (A.1) to (2.13), we get the alternative expression

\[ A_N(t) = \det (\psi_{m,n}(t)) = \det (J_{m-n}(2t)), \tag{2.16} \]

where

\[ \psi_{m,n}(t) = \int_0^{2\pi} \frac{dq}{2\pi} e^{-2it \cos q + i(n-m)q} = e^{im-n} J_{n-m}(2t) \tag{2.17} \]

is the one-body wavefunction at site \( n \) at time \( t \) for a particle launched from site \( m \) at time 0, and the \( J_n \) are Bessel functions.

The expressions (2.15) and (2.16) are complementary. The first one is more amenable to some analytical investigations, including the asymptotic analysis of the long-time regime performed in section 2.3. The second one could have been obtained by more elementary means, along the lines of [14–16]. It demonstrates explicitly that the amplitude is properly normalized, i.e. it obeys \( A_N(0) = 1 \). It is also more suited for numerical evaluations and power-series expansions in \( t \).

\(^4\) Throughout this work, determinants are of size \( N \times N \), with indices in the range \( 1 \leq m, n \leq N \).
2.2. The first few values of \( N \)

It is interesting to first look at the first few values of the fermion number \( N \).

- **\( N = 1 \).** For a single particle, we have \( A_1(t) = J_0(2t) \), and so

\[
R_1(t) = J_0^2(2t) = \frac{\cos^2(2t - \pi/4)}{\pi t}. \tag{2.18}
\]

The return probability therefore oscillates forever, becoming exactly zero at infinitely many times \( t_k \approx (k - 1/4)\pi/2 \). These oscillations can be averaged out by replacing the squared cosine by a factor 1/2. We thus obtain the power-law decay

\[
\overline{R}_1(t) \approx \frac{1}{2\pi t} \tag{2.19}
\]

for the mean return probability.

- **\( N = 2 \).** The situation of two fermions is the simplest one where quantum statistics plays a role. We have \( A_2(t) = J_0^2(2t) + J_1^2(2t) \), and so

\[
R_2(t) = (J_0^2(2t) + J_1^2(2t))^2 \approx \frac{1}{t^2}. \tag{2.20}
\]

In this case the return probability falls off monotonically to zero.

- **\( N = 3 \).** We have

\[
A_3(t) = (J_0(2t) + J_2(2t))(J_0^2(2t) + 2J_1^2(2t) - J_0(2t)J_2(2t)) = \frac{2J_1(2t)}{t}(J_0^2(2t) + J_1^2(2t)) - \frac{J_0(2t)J_1^2(2t)}{t^2}, \tag{2.21}
\]

where the first line gives the raw determinantal expression (2.16). The second one, obtained by means of the recursion

\[
J_{n+1}(2t) + J_{n-1}(2t) = \frac{n}{t}J_n(2t), \tag{2.22}
\]

is more suitable to study the late-time behavior of the return probability. Keeping only the first group of terms in the second line of (2.21), we obtain

\[
R_3(t) \approx \frac{4\cos^2(2t - 3\pi/4)}{\pi^3 t^5}. \tag{2.23}
\]

The return probability again oscillates forever, becoming exactly zero at infinitely many times \( t_k \approx (k + 1/4)\pi/2 \). The situation is therefore qualitatively similar to that of a single particle. Averaging again over the oscillations, we obtain

\[
\overline{R}_3(t) \approx \frac{2}{\pi^3 t^5}. \tag{2.24}
\]

2.3. Asymptotic analysis in the long-time regime

In this section we determine the asymptotic decay of the return probability \( R_N(t) \) in the long-time regime, for arbitrary values of the fermion number \( N \). To do so, we
evaluate the multiple integral entering (2.15) by means of the saddle-point approximation. Saddle points are defined by the condition that every momentum \( q_n \) is either 0 or \( \pi \). The vicinity of the most general saddle point can thus be parametrized by choosing \( M \) momenta near 0, of the form \( q_n = x_n \), and the remaining \( N - M \) momenta around \( \pi \), of the form \( q_n = \pi + y_n \). For a fixed \( M \), there are \( \binom{N}{M} \) ways of choosing which momenta are near 0 and near \( \pi \). Expanding in (2.15) the arguments of the exponentials and the products to quadratic order in the variables \( x_n \) and \( y_n \), we obtain the estimate

\[
A_N(t) \approx \sum_{M=0}^{N} \frac{I_{N,M}(t)}{M!(N-M)!},
\]

(2.25)

with

\[
I_{N,M}(t) = 2^{2M(N-M)} e^{2i(N-2M)t} \cdot \prod_{1 \leq m < n \leq M} (x_n - x_m)^2 \cdot \prod_{1 \leq m < n \leq N-M} (y_n - y_m)^2.
\]

The above integrals can be performed exactly. They are indeed given by the analytical continuation to imaginary values of \( \alpha \) of the Mehta integral [40, equation (17.6.7), \( \gamma = 1 \)]

\[
\mathcal{I}_N(a) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{n=1}^{N} dx_n e^{-a x_n^2} \prod_{1 \leq m < n \leq N} (x_n - x_m)^2
\]

\[
= (2\pi)^{N/2}(2a)^{-N^2/2} \prod_{j=1}^{N} j!
\]

\[
= (2\pi)^{N/2}(2a)^{-N^2/2} G(N+2),
\]

(2.27)

where \( G \) is Barnes’ \( G \)-function (see appendix B). We thus obtain

\[
I_{N,M}(t) = 2^{2M(N-M)} (2\pi)^{-N^2/2} e^{i(N-2M)(2t-N\pi/4)} \cdot G(M+2)G(N+2)(2t)^{M(N-M)-N^2/2}.
\]

(2.28)

All the quantities \( I_{N,M}(t) \) which enter the estimate (2.25) thus fall off as power laws in time, albeit with an \( M \)-dependent exponent. The behavior of the return probability at long times is governed by the term \( I_{N,M}(t) \) with the slowest decay. Even and odd values of the fermion number \( N \) yield different behaviors, and have to be dealt with separately. The emerging picture fully corroborates the observations made in section 2.2 for the first few values of \( N \).

- If \( N = 2m \) is even, the slowest decay is reached for \( M = m \), i.e. for equal numbers of momenta near 0 and \( \pi \). We thus obtain the estimate

\[
A_{2m}(t) \approx \frac{2^{m(m-1)}G(m+1)^2}{\pi^m t^m},
\]

(2.29)
showing that the amplitude is positive and decays monotonically, at least for large times. The echo therefore falls off monotonically as a power law

\[ R_{2m}(t) \approx \frac{2^{2m(m-1)}G(m+1)^4}{\pi^{2m}t^{2m^2}}, \tag{2.30} \]

with exponent \( 2m^2 = \frac{1}{2}N^2 \).

- If \( N = 2m + 1 \) is odd, the slowest decay is reached both for \( M = m \) and \( M = m + 1 \). We thus obtain the estimate

\[ A_{2m+1}(t) \approx \frac{2^{m^2}G(m+1)G(m+2)}{\pi^{m+1/2}t^{m^2+m+1/2}} \cos(2t - (2m + 1)\pi/4), \tag{2.31} \]

showing that the amplitude behaves for large times as an oscillatory function modulated by a decaying power law. Averaging over the oscillations, we obtain the following power-law decay for the mean echo:

\[ \overline{R}_{2m+1}(t) \approx \frac{2^{2m^2-1}G(m+1)^2G(m+2)^2}{\pi^{2m+1}t^{2m^2+2m+1}}, \tag{2.32} \]

with exponent \( 2m^2 + 2m + 1 = \frac{1}{2}(N^2 + 1) \).

Figure 2 illustrates the above results, showing log-log plots of the return probability \( R_N(t) \) against time \( t \) for \( N = 1, 2, 3 \) and \( 4 \). Data have been obtained using the expression (2.16). For odd \( N \) (left), the echo oscillates forever. For even \( N \) (right), it falls off monotonically but exhibits damped oscillations, which can be attributed to the first subleading contributions \( I_{N,M}(t) \) for \( N = 2m \) and \( M = m \pm 1 \). The blue straight lines—slightly translated for a better readability—have the predicted slopes 1, 2, 5 and 8.

It is worth comparing the above results for lattice fermions to the corresponding predictions in the continuum limit. Within the present framework, taking the continuum limit just amounts to approximating the dispersion relation (2.8) by a quadratic law of the form

\[ \varepsilon_q \approx 2 - q^2, \tag{2.33} \]
and therefore to restricting the saddle-point approximation to the sector where all momenta are near zero (i.e. \( M = N \)). This yields

\[
R_N^{(\text{cont})}(t) \approx \frac{G(N + 1)^2}{2^{N(N+1)} \pi^N t N^2}.
\]  

(2.34)

We still obtain a power-law decay of the return probability, albeit with no parity effect and with a different—roughly twice larger—decay exponent.

The above decay of the return probability in the continuum can be alternatively derived by the following heuristic reasoning. Suppose the fermions have mass \( m \) and are launched from the interval \( 0 < x < \ell \). In reduced units (\( \hbar = 1 \)), the typical momentum of each particle is \( p \sim 1/\ell \), and so its wavefunction spreads ballistically over a region of size

\[
L(t) \sim \frac{t}{m \ell}.
\]  

(2.35)

The return probability of one single particle therefore scales as \( R_1(t) \sim \ell / L(t) \sim m \ell^2 / t \). The exponent of this decay is in agreement with (2.32) and (2.34). In the same setting, the modulus of the wavefunction of \( N \) non-interacting fermions at late times can be estimated as

\[
|\psi_N(x_1, \ldots, x_N; t)| \sim C_N(t) \prod_{1 \leq m < n \leq N} |x_m - x_n|,
\]  

(2.36)

as long as all the coordinates \( x_n \) are less than \( L(t) \), whereas it falls off very fast at larger distances. Dimensional analysis implies that the normalization of the wavefunction scales as \( C_N(t) \sim L(t)^{-N^2/2} \), and that the return probability scales as

\[
R_N(t) \sim \left( \frac{\ell}{L(t)} \right)^{N^2} \sim \left( \frac{m \ell^2}{t} \right)^{N^2}.
\]  

(2.37)

The exponent of this decay is in agreement with (2.34). Furthermore, comparing the prefactors of the estimates (2.34) and (2.37) at large \( N \) yields the identification \( m \ell^2 \approx N/(2e^{3/2}) \). The scaling \( \ell \sim \sqrt{N} \) of the length of the confining interval is an artifact of the continuum framework.

2.4. Large-\( N \) asymptotics

In the \( N \to \infty \) limit, the amplitude \( A_N(t) \) admits the following remarkably simple expression for all finite times:

\[
A_\infty(t) = \lim_{N \to \infty} \det (J_{m-n}(2t)) = e^{-t^2}.
\]  

(2.38)

This expression has been derived independently in two recent works, devoted to quantum quenches of fermionic chains [38], and to volumes of balls in unitary groups [41]. The return probability therefore reads

\[
R_\infty(t) = e^{-2t^2}.
\]  

(2.39)
Quantum return probability of a system of \( N \) non-interacting lattice fermions

It can be verified, by expanding the result (2.16) as a power series in \( t \) for the first values of \( N \), that the expressions for \( R_N(t) \) and \( R_\infty(t) \) start differing at order \( t^{2N+2} \). This phenomenon has already been noticed in [41].

On the other hand, for large but finite \( N \), the results of section 2.3 can be given more explicit forms. Using the asymptotic expansion (B.5), the expressions (2.30) for \( N \) even and (2.32) for \( N \) odd respectively yield

\[
\begin{align*}
N \text{ even: } & \quad \ln R_N(t) = -\frac{N^2}{2} \left( \ln \frac{t}{N} + \frac{3}{2} \right) \\
& \quad - \frac{1}{3} \ln N + \frac{1}{3} \ln 2 + 4\zeta'(-1) + \cdots, \\
N \text{ odd: } & \quad \ln R_N(t) = -\frac{N^2 + 1}{2} \left( \ln \frac{t}{N} + \frac{3}{2} \right) \\
& \quad - \frac{1}{3} \ln N - \frac{2}{3} \ln 2 + \frac{3}{4} + 4\zeta'(-1) + \cdots,
\end{align*}
\]

where the remainders go to zero for large \( N \). To leading order, both expressions read

\[
\ln R_N(t) \approx -\frac{N^2}{2} \left( \ln \frac{t}{N} + \frac{3}{2} \right). \tag{2.42}
\]

The results (2.39) and (2.42) suggest a scaling law of the form

\[
\ln R_N(t) \approx -\frac{N^2}{2} F(x), \quad x = \frac{t}{N}, \tag{2.43}
\]

all over the regime where \( t \) and \( N \) are large and comparable, with

\[
F(x) \approx \begin{cases} 
4x^2 & (x \ll 1), \\
\ln x + \frac{3}{2} & (x \gg 1).
\end{cases}
\tag{2.44}
\]

The form of the scaling variable \( x \) reflects the ballistic nature of the dynamics of a free tight-binding particle.

3. Free fermions near a wall

In this section we consider the same problem on a semi-infinite chain ending with an impenetrable wall. At time \( t = 0 \), the fermions are launched from the \( N \) sites which are closest to the wall (see figure 3).

We are interested in the temporal decay of the return probability \( R_N^{(w)}(t) \), where the superscript \( (w) \) reminds of the presence of a wall.

3.1. General expressions at finite times

General expressions for the return probability \( R_N^{(w)}(t) \) at arbitrary finite times can be derived along the very lines of section 2.1.
The presence of an impenetrable wall imposes Dirichlet boundary conditions at site \( n = 0 \). With these boundary conditions, the Hamiltonian \( \mathcal{H} \) is brought to the diagonal form

\[
\mathcal{H} = \int_{0}^{\pi} \frac{dq}{\pi} \varepsilon_q \hat{a}_q^{\dagger} \hat{a}_q, \tag{3.1}
\]

with

\[
\varepsilon_q = 2 \cos q \quad (0 \leq q \leq \pi). \tag{3.2}
\]

The operators

\[
\hat{a}_q = \sqrt{2} \sum_n \sin nq a_n, \quad \hat{a}_q^{\dagger} = \sqrt{2} \sum_n \sin nq a_n^{\dagger}
\]

(3.3)

obey the anti-commutation relation

\[
\{\hat{a}_q, \hat{a}_{q'}^{\dagger}\} = \pi \delta(q - q'). \tag{3.4}
\]

The many-body amplitude \( \langle \Psi(0)|q \rangle \) is again given by a normalized Slater determinant. Since the fermions are launched from the sites \( m = 1, \ldots, N \), the latter reads

\[
\langle \Psi(0)|q \rangle = \sqrt{2^N N!} \det (\sin mq_n). \tag{3.5}
\]

The analogue of (2.11) therefore reads

\[
A_N^{(w)}(t) = 2^N \frac{N!}{N!} \int_{0}^{\pi} \cdots \int_{0}^{\pi} \frac{dq_n}{\pi} \prod_{n=1}^{N} e^{-2it \cos q_n} (\det (\sin mq_n))^2. \tag{3.6}
\]

The formula (3.5) can be brought to a form similar to (2.14). We recall that

\[
\sin mq = \sin q U_{m-1}(\cos q), \tag{3.7}
\]

where the \( U_m \) are the Tchebyshev polynomials of the second kind ([42] volume II, chapter X). The \( m \)th polynomial has degree \( m \) and the leading term \( U_m(z) = (2z)^m + \cdots \). We have therefore

\[
\det (\sin mq_n) = \prod_{n=1}^{N} \sin q_n \det (U_{m-1}(q_n)) = 2^{N(N-1)/2} \prod_{n=1}^{N} \sin q_n \times \prod_{1 \leq m < n \leq N} (\cos q_n - \cos q_m), \tag{3.8}
\]
where the last product is again a Vandermonde determinant. We are thus left with the following integral expression for the amplitude:

\[
A_N^{(w)}(t) = 2^{N^2} \frac{1}{N!} \int_0^\pi \cdots \int_0^\pi \prod_{n=1}^N \frac{dq_n}{\pi} \sin^2 q_n e^{-2it \cos q_n} \times \prod_{1 \leq m < n \leq N} (\cos q_n - \cos q_m)^2.
\] (3.9)

Applying the Andréief identity (A.1) to (3.6), we get the alternative expression

\[
A_N^{(w)}(t) = \det \left( \psi_{m,n}^{(w)}(t) \right),
\] (3.10)

where

\[
\psi_{m,n}^{(w)}(t) = \psi_{m,n}(t) - \psi_{-m,n}(t)
= i^{-(n-m)} J_{n-m}(2t) - i^{-(n+m)} J_{n+m}(2t)
\] (3.11)

is the one-body wavefunction at site \( n \) at time \( t \) for a particle located on the semi-infinite chain and launched from site \( m \) at time 0. The expression (3.11) can be recovered by applying the reflection principle (i.e. the method of images) to (2.17).

The complementary expressions (3.9) and (3.10) are the exact analogues on the semi-infinite chain of their counterparts (2.15) and (2.16) on the infinite chain.

### 3.2. Asymptotic analysis in the long-time regime

In this section we determine the asymptotic decay of the return probability \( R_N^{(w)}(t) \). We again apply the saddle-point approximation to the multiple integral in (3.9). Saddle points are still defined by the condition that every momentum \( q_n \) is either 0 or \( \pi \). The vicinity of the most general saddle point will be parametrized by choosing \( M \) momenta near 0, of the form \( q_n = x_n \), and the remaining \( N - M \) momenta around \( \pi \), of the form \( q_n = \pi - y_n \). At variance with the previous situation (section 2.3), the variables \( x_n \) and \( y_n \) are now positive. We thus obtain the estimate

\[
A_N^{(w)}(t) \approx 2^{N^2} \sum_{M=0}^N \frac{I_{N,M}^{(w)}(t)}{M!(N-M)!},
\] (3.12)

with

\[
I_{N,M}^{(w)}(t) = 2^{N-(N-2M)^2} e^{2i(N-2M)t} \times \int_0^\infty \cdots \int_0^\infty \prod_{n=1}^M \frac{dx_n}{\pi} x_n^2 e^{itx_n} \prod_{1 \leq m < n \leq M} (x_n^2 - x_m^2)^2 \times \int_0^\infty \cdots \int_0^\infty \prod_{n=1}^{N-M} \frac{dy_n}{\pi} y_n^2 e^{-ity_n} \prod_{1 \leq m < n \leq N-M} (y_n^2 - y_m^2)^2.
\] (3.13)

The above integrals can still be performed exactly. They are indeed proportional to the analytical continuation to imaginary values of \( a \) of the Mehta integral [40, equation (17.6.6), \( \alpha = 3/2, \gamma = 1 \)]

https://doi.org/10.1088/1742-5468/aaa79a
Quantum return probability of a system of $N$ non-interacting lattice fermions

$$I_N^{(w)}(a) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{n=1}^{N} \frac{dx_n}{x_n^2} e^{-a x_n^2} \prod_{1 \leq m < n \leq N} (x_n^2 - x_m^2)^2$$

$$= a^{-N(2N+1)/2} \prod_{j=1}^{N} j! \Gamma(j + 1/2)$$

$$= \pi^{N/2} (2a)^{-N(2N+1)/2} N! G(2N + 2), \quad (3.14)$$

where $G$ is again Barnes’ $G$-function (see appendix B). We thus obtain

$$I_{N,M}^{(w)}(t) = 2^{-(N-2M)^2} \pi^{-N/2} e^{i(N-2M)(2t-(N+1/2)\pi/4)} \times \sqrt{M!(N-M)! G(2M + 2) G(2N - 2M + 2)} \times (2t)^{2M(N-M)-N(2N+1)/2}. \quad (3.15)$$

The behavior of the return probability at long times is still governed by the term with the slowest decay. Even and odd values of the fermion number $N$ again yield different behaviors.

- If $N = 2m$ is even, the slowest decay is reached for $M = m$, i.e. for equal numbers of momenta near 0 and $\pi$. We thus obtain the estimate

$$A_{2m}^{(w)}(t) \approx \frac{2^{m(2m-1)} G(2m + 2)}{\pi^{m} m! t^{m(2m+1)}} \quad , \quad (3.16)$$

showing that the amplitude is positive and decays monotonically, at least for large times. The echo therefore falls off monotonically as a power law

$$R_{2m}^{(w)}(t) \approx \frac{2^{2m(2m-1)} G(2m + 2)^2}{\pi^{2m} m!^2 t^{2m(2m+1)}} \quad . \quad (3.17)$$

with exponent $2m(2m + 1) = N(N + 1)$.

- If $N = 2m + 1$ is odd, the slowest decay is reached both for $M = m$ and $M = m + 1$. We thus obtain the estimate

$$A_{2m+1}^{(w)}(t) \approx \frac{2^{m(2m+1)} G(2m + 3)}{\pi^{m+1/2} m! t^{m(2m+3m+3/2)}} \cos(2t - (2m + 3/2)\pi/4), \quad (3.18)$$

showing that the amplitude behaves for large times as an oscillatory function modulated by a decaying power law. Averaging over the oscillations, we obtain the following power-law decay for the mean echo:

$$\overline{R}_{2m+1}^{(w)}(t) \approx \frac{2^{2m(2m+1)-1} G(2m + 3)^2}{\pi^{2m+1} m!^2 t^{4m^2+6m+3}} \quad , \quad (3.19)$$

with exponent $4m^2 + 6m + 3 = N^2 + N + 1$.

Figure 4 illustrates the above results, showing log-log plots of the return probability $R_N^{(w)}(t)$ against time $t$ for $N = 1$, 2, 3 and 4. Data have been obtained using the expression (3.10). For odd $N$ (left), the echo oscillates forever. For even $N$ (right), it falls off monotonically but exhibits damped oscillations, which can again be attributed to the first
subleading contributions \( R^{(w)}_{N,M}(t) \) for \( N = 2m \) and \( M = m \pm 1 \). The blue straight lines—slightly translated for a better readability—have the predicted slopes 3, 6, 13 and 20.

It is again worth comparing the above results for lattice fermions to the corresponding prediction in the continuum limit. Restricting the saddle-point approximation to the sector where all momenta are near zero (i.e. \( M = N \)) yields

\[
R_{N,\text{cont}}^{(w)}(t) \approx \frac{G(2N + 2)}{2^{N(2N+1)}\pi^N N! t^{N(2N+1)}}. \tag{3.20}
\]

We again obtain a power-law decay of the return probability, albeit with no parity effect and with a different, larger decay exponent.

The above power-law decay can still be recovered by means of heuristic reasoning. Suppose the fermions are launched from the interval \( 0 < x < \ell \) near the wall. The modulus of the many-body wavefunction at time \( t \) now reads approximately

\[
|\psi^{(w)}_N(x_1, \ldots, x_N; t)| \sim C^{(w)}_N(t) \prod_{n=1}^N x_n \prod_{1 \leq m < n \leq N} |x_m^2 - x_n^2|. \tag{3.21}
\]

Dimensional analysis determines the scaling of the normalization of the wavefunction, \( C^{(w)}_N(t) \sim L(t)^{-N(2N+1)} \), and of the return probability,

\[
R_N^{(w)}(t) \sim \left( \frac{\ell}{L(t)} \right)^{N(2N+1)} \sim \left( \frac{m\ell^2}{t} \right)^{N(2N+1)}. \tag{3.22}
\]

A very similar result can be found in [33]. The decay exponent of the above estimate agrees with (3.20). Furthermore, comparing the prefactors of the estimates (3.20) and (3.22) at large \( N \) yields \( m\ell^2 \approx N/e^{3/2} \). The latter estimate is twice larger than its counterpart on the infinite line, given below (2.37). An intuitive interpretation of this factor two will be given below (3.27).

### 3.3. Large-\( N \) asymptotics

In the \( N \to \infty \) limit, the return probability is expected to be equal to the square root of its counterpart (2.39) on the infinite chain, namely

\[
\text{https://doi.org/10.1088/1742-5468/aaa79a}
\]
Quantum return probability of a system of $N$ non-interacting lattice fermions

$$R^w_{\infty}(t) = e^{-t^2}. \quad (3.23)$$

A heuristic way of showing this goes as follows. On the infinite chain, the compact fermionic state has two ends, and can therefore decay through either end, whereas it has only one right end if confined near a wall. Hence we can expect $R_{\infty}(t) = (R^w_{\infty}(t))^2$. The latter result can be checked by expanding the expression (3.10) as a power series in $t$ for the first values of $N$. Doing so confirms our expectation and shows that $R^w_N(t)$ and $R^w_{\infty}(t)$ again start differing at order $t^{2N+2}$.

On the other hand, using the asymptotic formula (B.5), we can still derive more explicit forms of the above results for large $N$. The expressions (3.17) for $N$ even and (3.19) for $N$ odd respectively yield

$$N \text{ even:} \quad \ln R^w_N(t) = - N(N + 1) \left( \ln \frac{t}{2N + 1} + \frac{3}{2} \right) - \frac{1}{6} \ln N + \ln 2 - \frac{3}{8} + 2\zeta'(-1) + \cdots, \quad (3.24)$$

$$N \text{ odd:} \quad \ln R^w_N(t) = - (N^2 + N + 1) \left( \ln \frac{t}{2N + 1} + \frac{3}{2} \right) - \frac{1}{6} \ln N - 2\ln 2 + \frac{9}{8} + 2\zeta'(-1) + \cdots. \quad (3.25)$$

The argument $t/(2N + 1)$ of the logarithms has been chosen in order to minimize the order of magnitude of the correction terms given in the second lines of the above expressions. To leading order, both results read

$$\ln R^w_N(t) \approx -N^2 \left( \ln \frac{t}{2N} + \frac{3}{2} \right). \quad (3.26)$$

The results (3.23) and (3.26) again suggest a scaling law of the form

$$\ln R^w_N(t) \approx -N^2 F^w(x), \quad x = \frac{t}{2N}. \quad (3.27)$$

The effective fermion number $2N$ entering the scaling variable $x$ can be interpreted as the total distance to be traveled by an excitation entering the compact fermionic state from the right, bouncing at the wall, and exiting from the right. Finally, the scaling function $F^w(x)$ obeys the very same asymptotics (2.44) as $F(x)$ both for $x \ll 1$ and for $x \gg 1$. This observation might be more than a coincidence, in the sense that both scaling functions might be identical.

4. Discussion

We studied the quantum return probability for a system of $N$ non-interacting lattice fermions launched from $N$ consecutive sites, either on the infinite chain or near an impenetrable wall ending a semi-infinite chain.

https://doi.org/10.1088/1742-5468/aaa79a

15
In each case we derived exact expressions for the return probability valid for all fermion numbers $N$ and time $t$. We thus obtained two complementary kinds of expressions, namely integral formulas (see (2.15) and (3.9)), which are the natural outcome of the second-quantized formalism, and determinantal formulas (see (2.16) and (3.10)), which could have been obtained by a more elementary, first-quantized approach as well. We deduced the asymptotic long-time behavior of the return probability by evaluating the integral formulas by means of the saddle-point method. Even and odd values of the fermion number $N$ yield different qualitative behaviors, as well as slightly different expressions for the decay exponents. For even $N$, the echo falls off monotonically as a power law. For odd $N$, it exhibits periodic oscillations modulated by a decaying power law. This qualitative dependence on the parity of the fermion number is a pure lattice effect, which is absent in the continuum limit. The return probability of $N$ particles thus provides yet another example of a situation where quantum dynamics exhibits qualitatively different features on the lattice and in the continuum. The exponents characterizing the temporal decay of the return probability are gathered in table 1.

Our results also yield explicit expressions for the prefactors of the asymptotic power-law decay of the return probability, in both geometries and for all values of the fermion number $N$. The key point in the derivation of these results has been the use of Mehta integrals, which have been extensively used in random matrix theory ([40] chapter 17) and can be viewed as limiting cases of the Selberg integral. Reference [43] provides a historical overview of the Selberg and related integrals, whereas the recent work [44]

Table 1. Exponents of the temporal decay of the return probability on the infinite chain and near a wall. Comparison between the values of the exponents for lattice fermions and the predictions of the continuum limit.

| Infinite chain (lattice) | Infinite chain (continuum) | Near a wall (lattice) | Near a wall (continuum) |
|--------------------------|---------------------------|---------------------|------------------------|
| $N$ even: $\frac{1}{2}N^2$ | $N^2$ | $N$ even: $N(N + 1)$ | $N(2N + 1)$ |
| $N$ odd: $\frac{1}{2}(N^2 + 1)$ | | $N$ odd: $N^2 + N + 1$ | |

Table 2. Asymptotic temporal decay of the return probability on the infinite chain and near a wall, for fermion numbers up to $N = 10$. The exact prefactors are given in factorized form.

| $N$ odd | Infinite chain | Near a wall | $N$ even | Infinite chain | Near a wall |
|---------|---------------|--------------|----------|---------------|-------------|
|         | $R_N(t)$      | $R_N^{(w)}(t)$ |          | $R_N(t)$      | $R_N^{(w)}(t)$ |
|         | (2.32)        | (3.19)       |          | (2.30)        | (3.17)      |
| 1       | $\frac{1}{2\pi t}$ | $\frac{1}{2\pi t}$ | 2        | $\frac{1}{\pi^2 t^2}$ | $\frac{2^4}{\pi^4 t^4}$ |
| 3       | $\frac{2}{\pi^3 t^3}$ | $\frac{2^3}{\pi^3 t^3}$ | 4        | $\frac{2^6}{\pi^6 t^6}$ | $\frac{2^{10}}{\pi^{10} t^{10}}$ |
| 5       | $\frac{2^2}{\pi^5 t^5}$ | $\frac{2^{10}}{\pi^5 t^5}$ | 6        | $\frac{2^{16}}{\pi^{16} t^{16}}$ | $\frac{2^{32} \pi^{32}}{\pi^{32} t^{32}}$ |
| 7       | $\frac{2^3}{\pi^7 t^7}$ | $\frac{2^{12} \pi^{12}}{\pi^{12} t^{12}}$ | 8        | $\frac{2^{32}}{\pi^{32} t^{32}}$ | $\frac{2^{64} \pi^{64}}{\pi^{64} t^{64}}$ |
| 9       | $\frac{2^5}{\pi^9 t^9}$ | $\frac{2^{12} \pi^{12}}{\pi^{12} t^{12}}$ | 10       | $\frac{2^{60} \pi^{60}}{\pi^{60} t^{60}}$ | $\frac{2^{160} \pi^{160}}{\pi^{160} t^{160}}$ |

In each case we derived exact expressions for the return probability valid for all fermion numbers $N$ and time $t$. We thus obtained two complementary kinds of expressions, namely integral formulas (see (2.15) and (3.9)), which are the natural outcome of the second-quantized formalism, and determinantal formulas (see (2.16) and (3.10)), which could have been obtained by a more elementary, first-quantized approach as well. We deduced the asymptotic long-time behavior of the return probability by evaluating the integral formulas by means of the saddle-point method. Even and odd values of the fermion number $N$ yield different qualitative behaviors, as well as slightly different expressions for the decay exponents. For even $N$, the echo falls off monotonically as a power law. For odd $N$, it exhibits periodic oscillations modulated by a decaying power law. This qualitative dependence on the parity of the fermion number is a pure lattice effect, which is absent in the continuum limit. The return probability of $N$ particles thus provides yet another example of a situation where quantum dynamics exhibits qualitatively different features on the lattice and in the continuum. The exponents characterizing the temporal decay of the return probability are gathered in table 1.

Our results also yield explicit expressions for the prefactors of the asymptotic power-law decay of the return probability, in both geometries and for all values of the fermion number $N$. The key point in the derivation of these results has been the use of Mehta integrals, which have been extensively used in random matrix theory ([40] chapter 17) and can be viewed as limiting cases of the Selberg integral. Reference [43] provides a historical overview of the Selberg and related integrals, whereas the recent work [44]
Quantum return probability of a system of $N$ non-interacting lattice fermions and the references therein mention yet other connections between random matrix theory and systems of free fermions. Our expressions for the prefactors of the power-law decays involve as an essential ingredient Barnes’ $G$-function, which is also ubiquitous in random matrix theory. Table 2 gives our predictions in factorized form up to $N = 10$.

The behavior of our results at large fermion numbers led us to hypothesize that the return probability obeys the scaling laws (2.43) and (3.27), with arguments $x = t/N$ and $x = t/(2N)$, in the regime where $t$ and $N$ are both large and comparable. The form of these scaling variables reflects the ballistic nature of the motion of a free tight-binding particle. It would clearly be desirable to investigate this ballistic scaling regime in a more thorough fashion, in order to eventually determine the analytic form of the scaling functions $F$ and $F^{(w)}$, and thus prove or disprove their conjectural equality suggested by the similarity of their behavior at small and large values of their argument. Such an investigation would certainly imply the usage of more advanced techniques. There is indeed no simple way to generalize the saddle-point approach used here in order to encompass the whole ballistic scaling regime.

The framework of the present study could also be extended in several other directions. One could investigate the return probability of $N$ fermions launched from other types of localized states, either on the chain, such as compactly supported but not most compact states, or in the continuum, with arbitrary initial states whose many-body wavefunction is localized in the center-of-mass coordinate. A further direction of research consists in replacing free fermions by other integrable systems of interacting particles in one dimension. Finally, higher-dimensional settings are also worth being envisaged. We intend to return to some of these matters in the future.

Acknowledgments

It is a pleasure to thank Jean-Paul Blaizot, Adolfo del Campo, Bertrand Eynard and Jean-Marie Stéphan for fruitful discussions.

Appendix A. The Andréief identity

The Andréief identity

$$
\int_a^b \cdots \int_a^b \prod_{n=1}^{N} \rho(x_n) \, dx_n \, \det (f_m(x_n)) \, \det (g_m(x_n)) = N! \, \det \left( \int_a^b \rho(x) \, dx \, f_m(x) g_n(x) \right)
$$

relates the integral of the product of the determinants built upon two families of functions to the determinant of their scalar products. This identity comes in many guises in various branches of the mathematical literature, concerning especially orthogonal polynomials and random matrix theory. Although it seems to appear for the first time in print in an article by Andréief in 1883 [45], it is also associated with other names, including Cauchy–Binet, Gram and Heine. Here, it allows us to respectively derive (2.16) and (3.10) from (2.13) and (3.6).
Let us give an elementary proof of the above identity for the sake of completeness. Starting from the left-hand side, let us introduce the Leibniz expansions of the determinants:

$$
\text{det } (f_m(x_n)) = \sum_{\sigma} \text{sgn } \sigma \prod_{n=1}^{N} f_{\sigma_n}(x_n),
$$

$$
\text{det } (g_m(x_n)) = \sum_{\tau} \text{sgn } \tau \prod_{n=1}^{N} g_{\tau_n}(x_n),
$$

(A.2)

where $\sigma$ and $\tau$ are permutations acting on $N$ symbols, and $\text{sgn } \sigma = \pm 1$ and $\text{sgn } \tau = \pm 1$ are their signatures. We thus obtain

$$
I_N = \sum_{\sigma, \tau} \text{sgn } \sigma \text{sgn } \tau \int_{-b}^{b} \cdots \int_{-b}^{b} \prod_{n=1}^{N} \rho(x_n) \, \text{d}x_n \, f_{\sigma_n}(x_n) g_{\tau_n}(x_n).
$$

(A.3)

The integrand is now a product, and so

$$
I_N = \sum_{\sigma, \tau} \text{sgn } \sigma \text{sgn } \tau \prod_{n=1}^{N} \int_{a}^{b} \rho(x) \, \text{d}x \, f_{\sigma_n}(x) g_{\tau_n}(x).
$$

(A.4)

For fixed permutations $\sigma$ and $\tau$, let us change the index from $n$ to $m = \tau_n$. We have then $\sigma_n = \mu_m$, where $\mu = \sigma \cdot \tau^{-1}$, and so $\text{sgn } \mu = \text{sgn } \sigma \text{sgn } \tau$. For fixed $\tau$, the sum over $\sigma$ can be replaced by a sum over $\mu$. The sum over $\tau$ simply yields a factor $N!$. We thus obtain

$$
I_N = N! \sum_{\mu} \text{sgn } \mu \prod_{n=1}^{N} \int_{a}^{b} \rho(x) \, \text{d}x \, f_{\mu_m}(x) g_m(x).
$$

(A.5)

The sum over $\mu$ is nothing but the Leibniz expansion of the determinant given in the right-hand side of (A.1).

On the infinite chain, the expression (2.13) of the amplitude $A_N(t)$ is proportional to the left-hand side of (A.1), with

$$
x_n = q_n, \quad \rho(x_n) = e^{-2it \cos q_n}, \quad f_m(x_n) = e^{imq_n}, \quad g_m(x_n) = e^{-imq_n}.
$$

(A.6)

Applying the identity (A.1) yields (2.16).

On the semi-infinite chain, the expression (3.6) of the amplitude $A^w_N(t)$ is proportional to the left-hand side of (A.1), with the same $\rho(x_n)$ and

$$
f_m(x_n) = g_m(x_n) = \sqrt{2} \sin mq_n.
$$

(A.7)

Applying the identity (A.1) yields (3.10).

**Appendix B. Barnes’ G-function**

Barnes’ G-function shares many common features with Euler’s $\Gamma$-function. This appendix summarizes the main properties of both functions, which can be found in the
Quantum return probability of a system of \( N \) non-interacting lattice fermions Wikipedia article [46] or in the Digital Library of Mathematical Functions ([47] chapter 5.17).

Euler’s \( \Gamma \)-function and Barnes’ \( G \)-function are meromorphic functions in the complex plane obeying the recursion relations
\[
\Gamma(z + 1) = z\Gamma(z), \quad G(z + 1) = G(z)G(z),
\]
with appropriate regularity conditions.

When \( z \) is a positive integer, Euler’s \( \Gamma \)-function becomes the usual factorial:
\[
\Gamma(n + 1) = n!,
\]
whereas Barnes’ \( G \)-function becomes the ‘superfactorial’:
\[
G(n + 2) = \prod_{k=1}^{n} k! = \prod_{\ell=1}^{n} \ell^{n+\ell} = \prod_{1\leq i<j\leq n+1} (j - i).
\]
We have in particular \( \Gamma(1) = \Gamma(2) = 1 \) and \( G(1) = G(2) = G(3) = 1 \). The ‘superfactorial’ numbers \( G(n + 2) \) appear in the OEIS [48] as sequence number A000178, together with many further properties and references.

Euler’s \( \Gamma \)-function and Barnes’ \( G \)-function have the following asymptotic expansions as \( z \to +\infty \):
\[
\ln \Gamma(z + 1) = \left(z + \frac{1}{2}\right) \ln z - z + \ln \sqrt{2\pi} + \cdots, \quad (B.4)
\]
\[
\ln G(z + 1) = \left(\frac{z^2}{2} - \frac{1}{12}\right) \ln z - \frac{3z^2}{4} + z \ln \sqrt{2\pi} + \zeta'(-1) + \cdots, \quad (B.5)
\]
where \( \zeta'(-1) = -0.16542143\ldots \) (\( \zeta \) being Riemann’s \( \zeta \)-function), and the remainders go to zero for large \( z \).

References

[1] Bloch I, Dalibard J and Nascimbène S 2012 Quantum simulations with ultracold quantum gases Nat. Phys. 8 267
[2] Paredes B, Widera A, Murg V, Mandel O, Fölling S, Cirac I, Shlyapnikov G V, Hänsch T W and Bloch I 2004 Tonks-Girardeau gas of ultracold atoms in an optical lattice Nature 429 277
[3] Kinoshita T, Wenger T and Weiss D S 2006 A quantum Newton’s cradle Nature 440 900
[4] Cazalilla M A and Rigol M 2011 Focus on dynamics and thermalization in isolated quantum many-body systems New J. Phys. 12 055006
[5] Polkovnikov A, Sengupta K, Silva A and Vengalatorre M 2011 Nonequilibrium dynamics of closed interacting quantum systems Rev. Mod. Phys. 83 863
[6] Eisert J, Friedsfeld M and Gogolin C 2015 Quantum many-body systems out of equilibrium Nat. Phys. 11 124
[7] Haroche S and Raimond J M 2006 Exploring the Quantum: Atoms, Cavities, and Photons (Oxford: Oxford University Press)
[8] Aharonov Y, Davidovich L and Zagury N 1993 Quantum random walks Phys. Rev. A 48 1687
[9] Ambainis A 2003 Quantum walks and their algorithmic applications Int. J. Quantum Inf. 1 507
[10] Kempe J 2003 Quantum random walks: an introductory overview Contemp. Phys. 44 307
[11] Farhi E and Gutmann S 1998 Quantum computation and decision trees Phys. Rev. A 58 915
[12] de Toro Arias S and Luck J M 1998 Anomalous dynamical scaling and bifractality in the one-dimensional Anderson model J. Phys. A: Math. Gen. 31 7699
[13] Parris P E 1989 One-dimensional trapping kinetics at zero temperature Phys. Rev. Lett. 62 1392
Quantum return probability of a system of N non-interacting lattice fermions

[14] Krapivsky P L, Luck J M and Mallick K 2014 Survival of classical and quantum particles in the presence of traps J. Stat. Phys. 154 1430

[15] Krapivsky P L, Luck J M and Mallick K 2015 Interacting quantum walkers: two-body bosonic and fermionic bound states J. Phys. A: Math. Theor. 48 475301

[16] Krapivsky P L, Luck J M and Mallick K 2016 Quantum centipedes: Collective dynamics of interacting quantum walkers J. Phys. A: Math. Theor. 49 335303

[17] Gammel W 1928 Zur Quantentheorie des Atomkerns Z. Phys. 51 204

[18] Gurney R W and Condon E U 1929 Quantum mechanics and radioactive disintegration Phys. Rev. 33 127

[19] Weisskopf V and Wigner E 1930 Berechnung der natürlichen Linienbreite auf Grund der Diracschen Lichttheorie Z. Phys. 63 54

[20] Weisskopf V and Wigner E 1930 Über die natürliche Linienbreite in der Strahlung des harmonischen Oszillators Z. Phys. 65 18

[21] Khalil L A 1958 Contribution to the decay theory of a quasi-stationary state Sov. Phys.—JETP 6 1053

[22] Winter R G 1961 Contribution to the decay theory of a quasi-stationary state Zh. Eks. Teor. Fiz. 33 1371

[23] Adler S L 2003 Weisskopf-Wigner decay theory for the energy-driven stochastic Schrödinger equation Phys. Rev. B 67 025301

[24] Peres A 1984 Stability of quantum motion in chaotic and regular systems Phys. Rev. A 30 1610

[25] Jalabert R A and Pastawski H M 2001 Environment-independent decoherence rate in classically chaotic systems Phys. Rev. Lett. 86 2490

[26] Manfredi G and Hervieux P A 2006 Loschmidt echo in a system of interacting electrons Phys. Rev. Lett. 97 190404

[27] Gorin T, Prosen T, Seligman T H and Znidaric M 2006 Dynamics of Loschmidt echoes and fidelity decay Phys. Rep. 435 33

[28] Derra B 2007 Non-equilibrium steady states: fluctuations and large deviations of the density and of the current J. Stat. Mech. P07023

[29] Touchette H 2009 The large deviation approach to statistical mechanics Phys. Rep. 478 1

[30] Taniguchi T and Sawada S I 2011 Escape behavior of quantum two-particle systems with Coulomb interactions Phys. Rev. E 83 026208

[31] García-Calderón G and Mendoza-Luna L G 2011 Time evolution of decay of two identical quantum particles Phys. Rev. A 84 032106

[32] Hunn S, Zimmermann K, Hiller M and Buchleitner A 2013 Tunneling decay of two interacting bosons in an asymmetric double-well potential: a spectral approach Phys. Rev. A 87 043626

[33] del Campo A 2011 Long-time behavior of many-particle quantum decay Phys. Rev. A 84 012113

[34] Longhi S and della Valle G 2012 Many-particle quantum decay and trapping: the role of statistics and Fano resonances Phys. Rev. A 86 021212

[35] Marchewka A and Granot E 2015 Role of quantum statistics in multi-particle decay dynamics Ann. Phys. 355 348

[36] del Campo A 2016 Exact quantum decay of an interacting many-particle system: the Calogero-Sutherland model New J. Phys. 18 015014

[37] Stéphan J M and Dubail J 2011 Local quantum quenches in critical one-dimensional systems: entanglement, the Loschmidt echo, and light-cone effects J. Stat. Mech. P08019

[38] Viti J, Stéphan J M, Dubail J and Haque M 2016 Inhomogeneous quenches in a free fermionic chain: exact results Europhys. Lett. 115 40011

[39] Stéphan J M 2017 Return probability after a quench from a domain-wall initial state in the spin-1/2 XXZ chain J. Stat. Mech. P103108

[40] Mehta M 1990 Random Matrices 2nd edn (London: Academic)

[41] Wei L, Pitaevsk R A, Corander J and Tirkkonen O 2017 From random matrix theory to coding theory: volume of a metric ball in unitary groups IEEE Trans. Inf. Theory 63 2814

[42] Erdélyi A (ed) 1953 Higher Transcendental Functions (The Bateman Manuscript Project) (New York: McGraw-Hill)

[43] Forrester P J and Warnaar S O 2008 The importance of the selberg integral Bull. Am. Math. Soc. 45 489

[44] Dean D S, Le Doussal P, Majumdar S N and Schehr G 2016 Non-interacting fermions at finite temperature in a d-dimensional trap: universal correlations Phys. Rev. A 94 060622

[45] Andréjé C 1883 Note sur une relation entre les intégrales définies des produits des fonctions Mém. Soc. Sci. Nat. Math. Bordeaux 2 1

[46] Wikipedia, The Free Encyclopedia https://en.wikipedia.org/wiki/Barnes_G-function

[47] NIST Digital Library of Mathematical Functions http://dlmf.nist.gov

[48] OEIS The On-Line Encyclopedia of Integer Sequences http://oeis.org

https://doi.org/10.1088/1742-5468/aaa79a