BOUNDINESS OF GAUSSIAN BESSEL POTENTIALS AND BESSEL FRACTIONAL DERIVATIVES ON VARIABLE GAUSSIAN BESOV-LIPSCHITZ SPACES.

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ABSTRACT. In this paper we study the regularity properties of the Gaussian Bessel potentials and Gaussian Bessel fractional derivatives on variable Gaussian Besov-Lipschitz spaces $B^{\alpha,p}_{\gamma, q}(\gamma_d)$, that were defined in a previous paper [11], under certain conditions on $p(\cdot)$ and $q(\cdot)$.

1. Introduction and Preliminaries

On $\mathbb{R}^d$ let us consider the Gaussian measure

$$\gamma_d(x) = \frac{e^{-|x|^2}}{\pi^{d/2}} dx, \ x \in \mathbb{R}^d$$

and the Ornstein-Uhlenbeck differential operator

$$L = \frac{1}{2} \Delta_x - \langle x, \nabla_x \rangle.$$ 

Let $\nu = (\nu_1, \ldots, \nu_d)$ be a multi-index such that $\nu_i \geq 0, i = 1, \ldots, d$, let $\nu! = \prod_{i=1}^d \nu_i !, |\nu| = \sum_{i=1}^d \nu_i, \partial_i = \frac{\partial}{\partial x_i}$, for each $1 \leq i \leq d$ and $\partial^\nu = \partial_{x_1}^{\nu_1} \cdots \partial_{x_d}^{\nu_d}$, consider the normalized Hermite polynomials of order $\nu$ in $d$ variables,

$$h_\nu(x) = \frac{1}{(2^{\nu_1!} \nu! \sqrt{\pi})^{1/2}} \prod_{i=1}^d (-1)^{\nu_i} e^{x_i^2} \frac{\partial^{\nu_i}}{\partial x_i^{\nu_i}} (e^{-x_i^2}),$$

it is well known, that the Hermite polynomials are eigenfunctions of the operator $L$,

$$Lh_\nu(x) = -|\nu| h_\nu(x).$$

Given a function $f \in L^1(\gamma_d)$ its $\nu$-Fourier-Hermite coefficient is defined by

$$\hat{f}(\nu) = \langle f, h_\nu \rangle_{\gamma_d} = \int_{\mathbb{R}^d} f(x) h_\nu(x) \gamma_d(dx).$$

Let $C_\nu$ be the closed subspace of $L^2(\gamma_d)$ generated by the linear combinations of $\{h_\nu : |\nu| = n\}$. By the orthogonality of the Hermite polynomials with respect to $\gamma_d$
it is easy to see that \( \{C_n\} \) is an orthogonal decomposition of \( L^2(\gamma_d) \),

\[
L^2(\gamma_d) = \bigoplus_{n=0}^{\infty} C_n,
\]

this decomposition is called the Wiener chaos.

Let \( J_n \) be the orthogonal projection of \( L^2(\gamma_d) \) onto \( C_n \), then if \( f \in L^2(\gamma_d) \)

\[
J_n f = \sum_{|\nu| = n} \hat{f}(\nu) \psi_{\nu},
\]

Let us define the Ornstein-Uhlenbeck semigroup \( \{T_t\}_{t \geq 0} \) as

\[
T_t f(x) = \frac{1}{2\pi d/2 (1 - e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{1}{1 - e^{-2t}} |\nu|^2} f(y) \gamma_d(dy)
\]

\[
= \frac{1}{\pi^{d/2} (1 - e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y-x|^2}{1 - e^{-2t}}} f(y) dy
\]

(1.5)

The family \( \{T_t\}_{t \geq 0} \) is a strongly continuous Markov semigroup on \( L^p(\gamma_d), 1 \leq p \leq \infty \), with infinitesimal generator \( L \). Also, by a change of variable we can write,

\[
T_t f(x) = \int_{\mathbb{R}^d} f(\sqrt{1 - e^{-2t}} u + e^{-t} x) \gamma_d(du).
\]

(1.6)

Now, by Bochner subordination formula, see Stein [13] page 61, we define the Poisson-Hermite semigroup \( \{P_t\}_{t \geq 0} \) as

\[
P_t f(x) = \frac{1}{\sqrt{t}} \int_0^{\infty} e^{-\frac{u}{\sqrt{t}}} T_{t^2/4u} f(x) du
\]

(1.7)

From (1.5) we obtain, after the change of variable \( r = e^{-t^2/4u} \),

\[
P_t f(x) = \frac{1}{2\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^1 \exp\left(\frac{t^2}{4 \log r}\right) \frac{\exp\left(-\frac{|y-x|^2}{1 - r^2}\right)}{(1 - r^2)^{d/2}} dr f(y) dy
\]

\[
= \int_{\mathbb{R}^d} p(t, x, y) f(y) dy,
\]

(1.8)

with

\[
p(t, x, y) = \frac{1}{2\pi^{(d+1)/2}} \int_0^1 \frac{t \exp\left(\frac{t^2}{4 \log r}\right) \frac{\exp\left(-\frac{|y-x|^2}{1 - r^2}\right)}{(1 - r^2)^{d/2}}}{r} dr.
\]

(1.9)

Also by the change of variables \( s = t^2/4u \) we have,

\[
P_t f(x) = \frac{1}{\sqrt{t}} \int_0^{\infty} e^{-\frac{u}{\sqrt{t}}} T_{t^2/4u} f(x) du = \int_0^{\infty} T_s f(x) \mu_t^{(1/2)}(ds),
\]

(1.10)

where the measure

\[
\mu_t^{(1/2)}(ds) = \frac{t}{2\sqrt{\pi}} e^{-s^2/4t} s^{3/2} ds,
\]

(1.11)

is called the one-side stable measure on \((0, \infty)\) of order 1/2.
The family \( \{P_t\}_{t \geq 0} \) is also a strongly continuous semigroup on \( L^p(\gamma_\nu) \), \( 1 \leq p < \infty \), with infinitesimal generator \( -(-L)^{1/2} \). In what follows, often we are going to use the notation
\[
\alpha \to u(x, t) = P_t f(x),
\]
and
\[
\alpha \to u^{(k)}(x, t) = \frac{\partial^k}{\partial t^k} P_t f(x).
\]

Observe that by (1.4) we have that
\[
T_t h_\nu(x) = e^{-t|\nu|} h_\nu(x),
\]
and
\[
P_t h_\nu(x) = e^{-t \sqrt{P} h_\nu(x)},
\]
i.e. the Hermite polynomials are eigenfunctions of \( T_t \) and \( P_t \) for any \( t \geq 0 \).

For completeness, let us get more background on variable Lebesgue spaces with respect to a Borel measure \( \mu \).

A \( \mu \)-measurable function \( p(\cdot) : \Omega \to [1, \infty] \) is an exponent function, the set of all the exponent functions will be denoted by \( \mathcal{P}(\Omega, \mu) \). For \( E \subset \Omega \) we set
\[
p_-(E) = \text{ess inf}_{x \in E} p(x) \quad \text{and} \quad p_+(E) = \text{ess sup}_{x \in E} p(x),
\]
and \( \Omega_\infty = \{ x \in \Omega : p(x) = \infty \} \).

We use the abbreviations \( p_+ = p_+(\Omega) \) and \( p_- = p_-(\Omega) \).

**Definition 1.1.** Let \( E \subset \mathbb{R}^d \). We say that \( \alpha(\cdot) : E \to \mathbb{R} \) is locally log-Hölder continuous, and denote this by \( \alpha(\cdot) \in LH_0(E) \), if there exists a constant \( C_1 > 0 \) such that
\[
|\alpha(x) - \alpha(y)| \leq \frac{C_1}{\log(e + \frac{1}{|x - y|})}
\]
for all \( x, y \in E \). We say that \( \alpha(\cdot) \) is log-Hölder continuous at infinity with base point at \( x_0 \in \mathbb{R}^d \), and denote this by \( \alpha(\cdot) \in LH_\infty(E) \), if there exist constants \( \alpha_\infty \in \mathbb{R} \)
and \( C_2 > 0 \) such that
\[
|\alpha(x) - \alpha_\infty| \leq \frac{C_2}{\log(e + |x - x_0|)}
\]
for all \( x \in E \). We say that \( \alpha(\cdot) \) is log-Hölder continuous, and denote this by \( \alpha(\cdot) \in LH(E) \) if both conditions are satisfied. The maximum, \( \max(C_1, C_2) \) is called the log-Hölder constant of \( \alpha(\cdot) \).

**Definition 1.2.** We say that \( p(\cdot) \in \mathcal{P}_d \) \((E) \), if \( \frac{1}{p(\cdot)} \) is log-Hölder continuous and denote by \( C_{\log(p)} \) or \( C_{\log} \) the log-Hölder constant of \( \frac{1}{p(\cdot)} \).

**Definition 1.3.** For a \( \mu \)-measurable function \( f : \mathbb{R}^d \to \mathbb{R} \), we define the modular
\[
\rho_{p(\cdot), \mu}(f) = \int_{\mathbb{R}^d \setminus \Omega_\infty} |f(x)|^{p(\cdot)} \mu(dx) + \|f\|_{L^\infty(\Omega_\infty, \nu)}
\]
The variable exponent Lebesgue space on $\mathbb{R}^d$, $L^{p(\cdot)}(\mathbb{R}^d, \mu)$ consists on those $\mu$-measurable functions $f$ for which there exists $\lambda > 0$ such that $\rho_{p(\cdot), \mu}(f/\lambda) < \infty$, i.e.

$L^{p(\cdot)}(\mathbb{R}^d, \mu) = \{ f : \mathbb{R}^d \to \mathbb{R}, \mu$-measurable and $\rho_{p(\cdot), \mu}(f/\lambda) < \infty$, for some $\lambda > 0 \}$.

and the norm

\begin{equation}
\|f\|_{p(\cdot), \mu} = \inf \{ \lambda > 0 : \rho_{p(\cdot), \mu}(f/\lambda) \leq 1 \}.
\end{equation}

**Observation 1.1.** When $\mu$ is the Lebesgue measure, we write $\rho_{p(\cdot)}$ and $\|f\|_{p(\cdot)}$ instead of $\rho_{p(\cdot), \mu}$ and $\|f\|_{p(\cdot), \mu}$ respectively.

**Theorem 1.1.** (Norm conjugate formula) Let $\nu$ a complete, $\sigma$-finite measure on $\Omega$. $p(\cdot) \in \mathcal{P}(\Omega, \nu)$, then

\begin{equation}
\frac{1}{2}\|f\|_{p(\cdot), \nu} \leq \|f\|_{p'(\cdot), \nu} \leq 2\|f\|_{p(\cdot), \nu},
\end{equation}

for all $f \nu$-measurable on $\Omega$, where

\[ \|f\|_{p'(\cdot), \nu} = \sup \left\{ \int_{\Omega} |f| \|g\| d\mu : g \in L^{p'(\cdot)}(\Omega, \nu), \|g\|_{p'(\cdot), \nu} \leq 1 \right\}. \]

*Proof.* See Corollary 3.2.14 in [3].

**Theorem 1.2.** (Hölder’s inequality) Let $\nu$ a complete, $\sigma$-finite measure on $\Omega$. $r(\cdot), q(\cdot) \in \mathcal{P}(\Omega, \nu)$, and define $p(\cdot) \in \mathcal{P}(\Omega, \nu)$ by $\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}$ $\nu$ a.e. $x \in \Omega$. Then, for all $f \in L^{q(\cdot)}(\Omega, \nu)$ and $g \in L^{r(\cdot)}(\Omega, \nu)$, $fg \in L^{p(\cdot)}(\Omega, \nu)$ and

\begin{equation}
\|fg\|_{p(\cdot), \nu} \leq 2\|f\|_{q(\cdot), \nu}\|g\|_{r(\cdot), \nu}
\end{equation}

*Proof.* See Lemma 3.2.20 in [3].

**Theorem 1.3.** (Minkowski’s integral inequality for variable Lebesgue spaces) Given $\mu$ and $\nu$ complete $\sigma$-finite measures on $X$ and $Y$ respectively, $p \in \mathcal{P}(X, \mu)$. Let $f : X \times Y \to \mathbb{R}$ measurable with respect to the product measure on $X \times Y$, such that for almost every $y \in Y$, $f(\cdot, y) \in L^{p(\cdot)}(X, \mu)$. Then

\begin{equation}
\left\| \int_Y f(\cdot, y) \nu(y) \right\|_{p(\cdot), \mu} \leq C \int_Y \|f(\cdot, y)\|_{p(\cdot), \mu} \nu(y)
\end{equation}

*Proof.* It is completely analogous to the proof of Corollary 2.38 in [1] by interchanging the Lebesgue measure for complete $\sigma$-finite measures $\mu$ and $\nu$ on $X$ and $Y$ respectively, and by using (1.17), Fubini’s theorem and then (1.16).

In what follows $\mu$ represents the Haar measure $\mu(dt) = \frac{dt}{t}$ on $\mathbb{R}^+$. 
Observation 1.2. For a \( \mu \)-measurable function \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \), \( q(\cdot) \in \mathcal{P}(\mathbb{R}^+, \mu) \), and any \( \lambda > 0 \)
\[
\rho_{q(\cdot), \mu} \left( \frac{f}{\lambda} \right) = \int_0^\infty \frac{|f(t)|^{q(t)} \mu(dt)}{\lambda} = \int_0^\infty \left| \frac{t^{-1/q(t)} f(t)}{\lambda} \right|^{q(t)} \mu(dt)
\]
Thus,
\[
\|f\|_{q(\cdot), \mu} = \|t^{-1/q(\cdot)} f\|_{q(\cdot)}
\]
In the case \( \Omega = \mathbb{R}^+ \), we denote by \( \mathcal{M}_{0,\infty} \) the set of all measurable functions \( p(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) which satisfy the following conditions:
\( i) \) 0 ≤ \( p_- \leq p_+ < \infty \).
\( ii) \) There exists \( p(0) = \lim_{x \to 0} p(x) \) and \( |p(x) - p(0)| \leq A \frac{p'(0)}{p(0)} x, 0 < x \leq 1/2 \).
\( iii) \) There exists \( p(\infty) = \lim_{x \to \infty} p(x) \) and \( |p(x) - p(\infty)| \leq A \frac{p'(0)}{p(0)} x, x > 2 \).
We denote by \( \mathcal{P}_{0,\infty} \) the subset of functions \( p(\cdot) \) such that \( p_- \geq 1 \).

Let \( \alpha(\cdot), \beta(\cdot) \in LH(\mathbb{R}^+), \) bounded with
\[
\alpha(0) < \frac{1}{p'(0)}, \quad \alpha(\infty) < \frac{1}{p'(\infty)}
\]
and
\[
\beta(0) > -\frac{1}{p'(0)}, \quad \beta(\infty) > -\frac{1}{p'(\infty)}
\]

Theorem 1.4. Let \( p(\cdot) \in \mathcal{P}_{0,\infty}, \alpha(\cdot), \beta(\cdot) \in LH(\mathbb{R}^+), \) bounded. Then the Hardy-type inequalities
\[
\left\| x^{\alpha(x)-1} \int_0^x \frac{f(y)}{y^{\alpha(y)} dy} \right\|_{p(\cdot)} \leq C_{\alpha(\cdot), p(\cdot)} \|f\|_{p(\cdot)}
\]
\[
\left\| x^{\beta(x)} \int_x^\infty \frac{f(y)}{y^{\beta(y)+1} dy} \right\|_{p(\cdot)} \leq C_{\beta(\cdot), p(\cdot)} \|f\|_{p(\cdot)}
\]
are valid, if and only if, \( \alpha(\cdot), \beta(\cdot) \) satisfy conditions (1.20) and (1.21)

Proof. For the proof see Theorem 3.1 and Remark 3.2 in [4].

As a consequence, we obtain the Hardy inequalities associated to the exponent \( q(\cdot) \in \mathcal{P}_{0,\infty} \) and the measure \( \mu \).

Corollary 1.1. Let \( q(\cdot) \in \mathcal{P}_{0,\infty} \) and \( r > 0 \), then
\[
\left\| t^r \int_0^t g(y) dy \right\|_{q(\cdot), \mu} \leq C_{r, q(\cdot)} \|y^{r+1} g\|_{q(\cdot), \mu}
\]
and
\[
\left\| t^r \int_t^\infty g(y) dy \right\|_{q(\cdot), \mu} \leq C_{r, q(\cdot)} \|y^{r+1} g\|_{q(\cdot), \mu}
\]
Proof. See [11].

In what follows we also need the classical Hardy’s inequalities, so for completeness we will write then here, see [13] page 272,

\begin{equation}
\int_0^{+\infty} \left( \int_0^x f(y) dy \right)^p x^{-r-1} dx \leq \frac{p}{r} \int_0^{+\infty} \left( y f(y) \right)^p y^{-r-1} dy,
\end{equation}

and

\begin{equation}
\int_0^{+\infty} \left( \int_y^{+\infty} f(y) dy \right)^p x^{-r-1} dx \leq \frac{p}{r} \int_y^{+\infty} \left( y f(y) \right)^p y^{-r-1} dy,
\end{equation}

where $f \geq 0$, $p \geq 1$ and $r > 0$.

We will consider only Lebesgue variable spaces with respect to the Gaussian measure $\gamma_d$, $L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$. The next condition was introduced by E. Dalmasso and R. Scotto in [2].

**Definition 1.4.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d, \gamma_d)$, we say that $p(\cdot) \in \mathcal{P}_{\gamma_d}(\mathbb{R}^d)$ if there exist constants $C_{\gamma_d} > 0$ and $p_{\infty} \geq 1$ such that

\begin{equation}
|p(x) - p_{\infty}| \leq \frac{C_{\gamma_d}}{\|x\|^2},
\end{equation}

for $x \in \mathbb{R}^d$, $x \neq 0$.

**Example 1.1.** Consider $p(x) = p_{\infty} + \frac{A}{(e + \|x\|)^q}$, $x \in \mathbb{R}^d$, for any $p_{\infty} \geq 1, A \geq 0$ and $q \geq 2$ then $p(\cdot) \in \mathcal{P}_{\gamma_d}(\mathbb{R}^d)$.

**Observation 1.3.** If $p(\cdot) \in \mathcal{P}_{\gamma_d}(\mathbb{R}^d)$, then $p(\cdot) \in LH_\infty(\mathbb{R}^d)$.

Additionally, we need some technical results.

**Lemma 1.1.** Given $k \in \mathbb{N}$ and $t > 0$ then $\mu^{(1/2)}_t$ the one-side stable measure on $(0, \infty)$ of order $1/2$ satisfies

\begin{equation}
\int_0^{+\infty} \left| \frac{\partial^k \mu^{(1/2)}_t}{\partial s^k} \right| (ds) \leq \frac{C_k}{t^k}.
\end{equation}

For the proof see inequality (3.21) in [16].

**Lemma 1.2.** Let $p(\cdot) \in \mathcal{P}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$. Suppose that $f \in L^{p(\cdot)}(\gamma_d)$, then for any integer $k$

\[ \left\| \frac{\partial^k}{\partial t^k} P_t f \right\|_{p(\cdot), \gamma_d} \leq C_{p(\cdot)} \left\| \frac{\partial^k}{\partial s^k} P_s f \right\|_{p(\cdot), \gamma_d}, \]

for any $0 < s < t < +\infty$. Moreover,

\begin{equation}
\left\| \frac{\partial^k}{\partial t^k} P_t f \right\|_{p(\cdot), \gamma_d} \leq \frac{C_{k, p(\cdot)}}{t^k} \|f\|_{p(\cdot), \gamma_d}, \quad t > 0.
\end{equation}
For the proof see [11].

One of the main results in [11] was the definition of the variable Gaussian Besov-Lipschitz spaces $B^\alpha_{p,q}(\gamma_d)$, following [13] and [7]. They were defined as follows:

**Definition 1.5.** Let $p(\cdot) \in \mathcal{P}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q(\cdot) \in \mathcal{P}_{0,\infty}$. Let $\alpha \geq 0$, $k$ the smallest integer greater than $\alpha$. The variable Gaussian Besov-Lipschitz space $B^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)$ is defined as the set of functions $f \in L^{p(\cdot)}(\gamma_d)$ such that

\begin{equation}
\| f \|_{B^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)} := \| f \|_{p(\cdot),\gamma_d} + \| (\partial^k_P f) \|_{p(\cdot),\gamma_d} < \infty,
\end{equation}

the norm of $f \in B^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)$ is defined as

\begin{equation}
\| f \|_{B^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)} := \| f \|_{p(\cdot),\gamma_d} + \| (\partial^k_P f) \|_{p(\cdot),\gamma_d} < \infty.
\end{equation}

The variable Gaussian Besov-Lipschitz space $B^\alpha_{p(\cdot),\infty}(\gamma_d)$ is defined as the set of functions $f \in L^{p(\cdot)}(\gamma_d)$ for which there exists a constant $A$ such that

\begin{equation}
\| \partial^k_P f \|_{p(\cdot),\gamma_d} \leq A t^{-k+\alpha}, \forall t > 0
\end{equation}

and then the norm of $f \in B^\alpha_{p(\cdot),\infty}(\gamma_d)$ is defined as

\begin{equation}
\| f \|_{B^\alpha_{p(\cdot),\infty}(\gamma_d)} := \| f \|_{p(\cdot),\gamma_d} + A_k(f),
\end{equation}

where $A_k(f)$ is the smallest constant $A$ in the above inequality.

For more details about the definition of variable Gaussian Besov-Lipschitz spaces, we refer to [11].

Additionally in [11] we obtained some inclusion relations between variable Gaussian Besov-Lipschitz spaces. These results are analogous to Proposition 10, page 153 in [13], see also [10] or Proposition 7.36 in [16].

**Proposition 1.1.** Let $p(\cdot) \in \mathcal{P}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q_1(\cdot), q_2(\cdot) \in \mathcal{P}_{0,\infty}$. The inclusion $B^{\alpha_1}_{p(\cdot),q_1(\cdot)}(\gamma_d) \subset B^{\alpha_2}_{p(\cdot),q_2(\cdot)}(\gamma_d)$ holds if:

\begin{enumerate}
  \item $\alpha_1 > \alpha_2 > 0$ (If $q_1(\cdot)$ and $q_2(\cdot)$ not need to be related), or
  \item If $\alpha_1 = \alpha_2$ and $q_1(t) \leq q_2(t)$ a.e.
\end{enumerate}

Finally, the operators that are going to be considered in this paper are the following:

- The Gaussian Bessel Potential of order $\beta > 0$, $\mathcal{J}_\beta$, is defined formally as

\begin{equation}
\mathcal{J}_\beta = (I + \sqrt{-L})^{-\beta},
\end{equation}
meaning that for the Hermite polynomials we have,

\[ J_{\beta} h_\nu(x) = \frac{1}{(1 + \sqrt{|\nu|})^\beta} h_\nu(x). \]

Again by linearity can be extended to any polynomial and Meyer’s theorem allows us to extend Bessel Potentials to a continuous operator on \( L^p(\gamma_d) \), \( 1 < p < \infty \). It can be proved that the Bessel potentials can be represented as

\[ \mathcal{J}_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} t^{\beta - 1} e^{-t} P_t f(x) \frac{dt}{t}. \]

Moreover, \( \mathcal{J}_\beta \) is bounded on \( L^p(\cdot) \), for \( p(\cdot) \in \mathcal{P}^\infty(\mathbb{R}^d) \) with \( 1 < p_- \leq p_+ < \infty \). For the proof see [9].

- The Gaussian Bessel fractional derivative \( D^\beta \), defined formally for \( \beta > 0 \) as

\[ D^\beta = (I + \sqrt{-L})^\beta, \]

which means that for the Hermite polynomials, we have

\[ D^\beta h_\nu(x) = (1 + \sqrt{|\nu|})^\beta h_\nu(x), \]

Let \( k \) be the smallest integer greater than \( \beta \) i.e. \( k - 1 \leq \beta < k \), then the fractional derivative \( D^\beta \) can be represented as

\[ D^\beta f = \frac{1}{c_\beta^k} \int_0^{\infty} t^{\beta - 1}(e^{-t} P_t - I)^k f dt, \]

where \( c_\beta^k = \int_0^{\infty} u^{\beta - 1}(e^{-u} - 1)^k du. \)

As usual in what follows \( C \) represents a constant that is not necessarily the same in each occurrence.

2. Main results

The main results of the paper are the study of the regularity properties of the Gaussian Bessel potentials and the Gaussian Bessel fractional derivatives on variable Gaussian Besov-Lipschitz spaces.

Let us start considering the regularity properties of the Gaussian Bessel potentials. In the following theorem we consider their action on \( B^\alpha_{p(\cdot),\infty}(\gamma_d) \) spaces, which is analogous to Theorem 4 in [7].

**Theorem 2.1.** Let \( \alpha \geq 0, \beta > 0 \) then for \( p(\cdot) \in \mathcal{P}^\infty(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d) \) with \( 1 < p_- \leq p_+ < \infty \), Then, the Gaussian Bessel potential \( \mathcal{J}_\beta \) is bounded from \( B^\alpha_{p(\cdot),\infty}(\gamma_d) \) into \( B^{\alpha+\beta}_{p(\cdot),\infty}(\gamma_d) \).
Proof. Let \( k > \alpha + \beta \) a fixed integer and \( f \in B^\alpha_{p(\cdot),\infty}(\gamma_d) \), then \( \mathcal{J}_\beta f \in L^p(\gamma_d) \) (see [9]). By using the representation of Bessel potential (1.35) and properties of \( P_t \), we get

\[
P_t(\mathcal{J}_\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{\infty} s^\beta e^{-s} P_{t+s}f(x) \frac{ds}{s},
\]

thus using the dominated convergence theorem and chain rule, we obtain

\[
\frac{\partial^k}{\partial t^k} P_t(\mathcal{J}_\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{\infty} s^\beta e^{-s} u^{(k)}(x, t+s) \frac{ds}{s}.
\]

This implies, using Minkowski’s integral inequality (1.18), that

\[
\left\| \frac{\partial^k}{\partial t^k} P_t(\mathcal{J}_\beta f) \right\|_{p(\cdot),\gamma_d} \leq \frac{C}{\Gamma(\beta)} \int_0^{\infty} s^\beta e^{-s} \left\| u^{(k)}(\cdot, t+s) \right\|_{p(\cdot),\gamma_d} \frac{ds}{s} \leq \frac{C}{\Gamma(\beta)} \int_0^{\infty} s^\beta e^{-s} \left\| u^{(k)}(\cdot, t+s) \right\|_{p(\cdot),\gamma_d} \frac{ds}{s} \leq C \frac{\beta}{\Gamma(\beta)} \beta A_k(f) t^{-k+\alpha} = C \beta A_k(f) t^{-k+\alpha+\beta}.
\]

Now, as \( \beta > 0 \), using Lemma 1.2 (as \( t+s > t \)) and since \( f \in B^\alpha_{p(\cdot),\infty}(\gamma_d) \),

\[
(1) \leq \frac{C}{\Gamma(\beta)} \int_0^{\infty} s^\beta e^{-s} \left\| \frac{\partial^k}{\partial t^k} P_t f \right\|_{p(\cdot),\gamma_d} ds \leq \frac{C}{\Gamma(\beta)} \left\| \frac{\partial^k}{\partial t^k} P_t f \right\|_{p(\cdot),\gamma_d} \int_0^{\infty} s^\beta e^{-s} ds \leq \frac{C}{\Gamma(\beta)} \left\| \frac{\partial^k}{\partial t^k} P_t f \right\|_{p(\cdot),\gamma_d} \int_0^{\infty} s^\beta e^{-s} s^{-k+\alpha} ds \leq \frac{C}{\Gamma(\beta)} \left\| \frac{\partial^k}{\partial t^k} P_t f \right\|_{p(\cdot),\gamma_d} \int_0^{\infty} s^\beta e^{-s} s^{-k+\alpha+\beta} ds \leq C A_k(f) \frac{r^{-k+\alpha+\beta}}{\Gamma(\beta)} = C A_k(f) t^{-k+\alpha+\beta}.
\]

Therefore,

\[
\left\| \frac{\partial^k}{\partial t^k} P_t(\mathcal{J}_\beta f) \right\|_{p(\cdot),\gamma_d} \leq C A_k(f) t^{-k+\alpha+\beta}, \forall t > 0.
\]

Then \( \mathcal{J}_\beta f \in B^{\alpha+\beta}_{p(\cdot),\infty}(\gamma_d) \) and \( A_k(\mathcal{J}_\beta f) \leq A_k(f) \). Thus,

\[
\| \mathcal{J}_\beta f \|_{B^{\alpha+\beta}_{p(\cdot),\infty}} = \| \mathcal{J}_\beta f \|_{p(\cdot),\gamma_d} + A_k(\mathcal{J}_\beta f) \leq C \| f \|_{p(\cdot),\gamma_d} + C A_k(f) \leq C \| f \|_{B^\alpha_{p(\cdot),\infty}}.
\]

Now, in the following theorem we consider the action of Gaussian Bessel potentials on \( B^\alpha_{p(\cdot),p(\cdot)}(\gamma_d) \) spaces. It is analogous to Theorem 2.4 (i) of [10].
\textbf{Theorem 2.2.} Let \( \alpha \geq 0, \beta > 0, p(\cdot) \in \mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d) \) with \( 1 < p_- \leq p_+ < \infty \) and \( q(\cdot) \in \mathcal{P}_{0,\infty} \). Then, the Gaussian Bessel potential \( \mathcal{J}_\beta \) is bounded from \( B^p_{p(\cdot),q(\cdot)}(\gamma_d) \) into \( B^{p_+\beta}_{p(\cdot),q(\cdot)}(\gamma_d) \).

\textit{Proof.} Let \( f \in B^p_{p(\cdot),q(\cdot)}(\gamma_d) \) then \( \mathcal{J}_\beta f \in L^{p(\cdot)}(\gamma_d) \) since \( \mathcal{J}_\beta \) is bounded on \( L^{p(\cdot)}(\gamma_d) \).

Let denote \( u(x,t) = P_t f(x) \) and \( U(x,t) = P_t \mathcal{J}_\beta f(x) \). Using the representation (1.10) of \( P_t \), we have

\[
U(x,t) = \int_0^{+\infty} T_s(\mathcal{J}_\beta f)(x) \mu_t^{1/2}(ds).
\]

Thus, by the semigroup's property of \( P_t \)

\[
U(x,t_1 + t_2) = P_{t_1}(P_{t_2}(\mathcal{J}_\beta f))(x) = \int_0^{+\infty} T_s(P_{t_2}(\mathcal{J}_\beta f))(x) \mu_t^{1/2}(ds).
\]

Now, fix \( k \) and \( l \) integer greater than \( \alpha \) and \( \beta \) respectively. By using the dominated convergence theorem, differentiating \( k \) times respect to \( t_2 \) and \( l \) times respect to \( t_1 \) we get

\[
\frac{\partial^{k+l} U(x,t_1 + t_2)}{\partial t_1^{k} \partial t_2^{l}} = \int_0^{+\infty} T_s(\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathcal{J}_\beta f))(x) \frac{\partial^l}{\partial t_1^l} \mu_t^{1/2}(ds).
\]

Thus, making \( t = t_1 + t_2 \), we get

\[
\frac{\partial^{k+l} U(x,t)}{\partial t^{k+l}} = \int_0^{+\infty} T_s(\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathcal{J}_\beta f))(x) \frac{\partial^l}{\partial t_1^l} \mu_t^{1/2}(ds),
\]

therefore, by using Minkowski’s integral inequality (1.18), the \( L^{p(\cdot)} \)-continuity of \( T_s \) and Lemma 1.1

\[
\left\| \frac{\partial^{k+l} U(x,t)}{\partial t^{k+l}} \right\|_{p(\cdot),\gamma_d} \leq C \int_0^{+\infty} \left\| T_s(\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathcal{J}_\beta f)) \right\|_{p(\cdot),\gamma_d} \left\| \frac{\partial^l}{\partial t_1^l} \mu_t^{1/2}(ds) \right\|.
\]

\[
\leq C \int_0^{+\infty} \left\| \frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathcal{J}_\beta f) \right\|_{p(\cdot),\gamma_d} \left\| \frac{\partial^l}{\partial t_1^l} \mu_t^{1/2}(ds) \right\|
\]

\[
= C \left\| \frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathcal{J}_\beta f) \right\|_{p(\cdot),\gamma_d} \int_0^{+\infty} \left\| \frac{\partial^l}{\partial t_1^l} \mu_t^{1/2}(ds) \right\|
\]

\[
(2.1)
\]

On the other hand, using the representation of the Bessel potential (1.35), we have

\[
P_t(\mathcal{J}_\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta} e^{-s} P_{t+s} f(x) \frac{ds}{s}
\]
Thus,
\[
\frac{\partial^k P_t}{\partial t^k} (\mathcal{J}_\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{\infty} s^{\beta} e^{-s} \frac{\partial^k P_{t+s} f(x)}{\partial t^k} \frac{ds}{s} = \frac{1}{\Gamma(\beta)} \int_0^{\infty} s^{\beta} e^{-s} \frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k} \frac{ds}{s},
\]
and again by Minkowski’s integral inequality (1.18)
\[
\left\| \frac{\partial^k P_t}{\partial t^k} (\mathcal{J}_\beta f) \right\|_{p(\cdot,\gamma_d)} \leq \frac{C}{\Gamma(\beta)} \int_0^{\infty} s^{\beta} e^{-s} \left\| \frac{\partial^k P_{t+s} f}{\partial (t+s)^k} \right\|_{p(\cdot,\gamma_d)} ds.
\]
Now, since the definition of \( B^\alpha_{p(\cdot),q(\cdot)}(\gamma_d) \) is independent of the integer \( k > \alpha \) that we choose, take \( k > \alpha + \beta \) and \( l > \beta \), then \( k + l > \alpha + 2\beta > \alpha + \beta \), this is, \( k + l \) is an integer greater than \( \alpha + \beta \). Now we will show that
\[
\left\| \frac{\partial^{k+l-(\alpha+\beta)} \mathcal{U}(\cdot, t)}{\partial t^{k+l}} \right\|_{p(\cdot,\gamma_d)} < +\infty.
\]
In fact, taking \( t_1 = t_2 = t/2 \) in (2.1), we get
\[
\left\| \frac{\partial^{k+l-(\alpha+\beta)} \mathcal{U}(\cdot, t)}{\partial t^{k+l}} \right\|_{p(\cdot,\gamma_d)} \leq C \left\| \frac{\partial^{k+l-(\alpha+\beta)} \mathcal{U}(\cdot, t)}{\partial t^{k+l}} \right\|_{p(\cdot,\gamma_d)} \left\| \frac{\partial^k P_{t+s} f}{\partial (s + \frac{k}{2})^k} \right\|_{p(\cdot,\gamma_d)} \left( \frac{l}{2} \right)^{-1} \left\| \frac{\partial^k P_{t+s} f}{\partial (t+s)^k} \right\|_{p(\cdot,\gamma_d)} \left( \frac{l}{2} \right)^{-1}
\]
\[
\leq C \frac{\Gamma(\beta)}{\Gamma(\beta)} \left\| \frac{\partial^{k+l-(\alpha+\beta)} \mathcal{U}(\cdot, t)}{\partial t^{k+l}} \right\|_{p(\cdot,\gamma_d)} \left( \int_0^{\infty} s^{\beta} e^{-s} \left\| \frac{\partial^k P_{t+s} f}{\partial (s + \frac{k}{2})^k} \right\|_{p(\cdot,\gamma_d)} ds \right) \left\| \frac{\partial^k P_{t+s} f}{\partial (t+s)^k} \right\|_{p(\cdot,\gamma_d)} \left\| \frac{\partial^k P_{t+s} f}{\partial (t+s)^k} \right\|_{p(\cdot,\gamma_d)} \left( \frac{l}{2} \right)^{-1} \left\| \frac{\partial^k P_{t+s} f}{\partial (t+s)^k} \right\|_{p(\cdot,\gamma_d)} \left( \frac{l}{2} \right)^{-1}
\]
\[
= I + II.
\]
Using Lemma 1.2, the change of variables $u = t/2$ and since $\beta > 0$, we have

$$I = \frac{C}{\Gamma(\beta)} \left| k^{-(\alpha+\beta)} \left( \int_0^{s^2} \frac{\partial^k P_{s+\frac{t}{2}} f}{\partial(s + \frac{t}{2})^k} \bigg| \frac{ds}{s} \bigg| \right) \right|_{q(\gamma_d), \mu}$$

$$\leq \frac{C}{\Gamma(\beta)} \left| k^{-(\alpha+\beta)} \left( \int_0^{s^2} \frac{\partial^k P_{s+\frac{t}{2}} f}{\partial(s + \frac{t}{2})^k} \bigg| \frac{ds}{s} \bigg| \right) \right|_{q(\gamma_d), \mu}$$

$$= \frac{C}{\beta \Gamma(\beta)} k^{-\alpha} \left| \frac{\partial^k P_{s+\frac{t}{2}} f}{\partial(s + \frac{t}{2})^k} \bigg|_{p(\gamma_d), q(\gamma_d)} \right|_{q(\gamma_d), \mu} = C_{k,\alpha,\beta} \left| u^{k-\alpha} \left| \frac{\partial^k P_{s+\frac{t}{2}} f}{\partial(s + \frac{t}{2})^k} \bigg|_{p(\gamma_d), q(\gamma_d)} \right|_{q(\gamma_d), \mu} < +\infty,$$

since $f \in B^\alpha_{p(\gamma_d), q(\gamma_d)}$.

On the other hand, using the Hardy’s inequality (1.25), since $k > \alpha + \beta$ and again by Lemma 1.2, we get

$$II = \frac{C}{\Gamma(\beta)} \left| k^{-(\alpha+\beta)} \left( \int_t^{+\infty} \frac{\partial^k P_{s+\frac{t}{2}} f}{\partial(s + \frac{t}{2})^k} \bigg| \frac{ds}{s} \bigg| \right) \right|_{q(\gamma_d), \mu}$$

$$\leq \frac{C}{\Gamma(\beta)} \left| k^{-(\alpha+\beta)} \left( \int_t^{+\infty} \frac{\partial^k P_{s+\frac{t}{2}} f}{\partial(s + \frac{t}{2})^k} \bigg| \frac{ds}{s} \bigg| \right) \right|_{q(\gamma_d), \mu} \leq C_{k,\alpha,\beta} \left| s^{k-\alpha} \left| \frac{\partial^k P_{s+\frac{t}{2}} f}{\partial(s + \frac{t}{2})^k} \bigg|_{p(\gamma_d), q(\gamma_d)} \right|_{q(\gamma_d), \mu} < +\infty,$$

since $f \in B^\alpha_{p(\gamma_d), q(\gamma_d)}$). This is, $J^\beta f \in B^{\alpha+\beta}_{p(\gamma_d), q(\gamma_d)}$.

Moreover,

$$\|J^\beta f\|_{B^{\alpha+\beta}_{p(\gamma_d), q(\gamma_d)}} \leq C\|f\|_{B^\alpha_{p(\gamma_d), q(\gamma_d)}}.$$

Now, we will study the action of Bessel fractional derivative $\mathcal{D}^\beta$ on variable Gaussian Besov-Lipschitz spaces $B^\alpha_{p(\gamma_d), q(\gamma_d)}$. We will use the representation (1.37) of the Bessel fractional derivative and Hardy’s inequalities.

First, we need to consider the forward differences. Remember for a given function $f$, the $k$-th order forward difference of $f$ starting at $t$ with increment $s$ is defined as,

$$\Delta^k_s f(t) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j f(t + (k - j)s).$$

The forward differences have the following properties (see Appendix 10.9 in [16]) we will need the following technical result

**Lemma 2.1.** For any positive integer $k$

i) $\Delta_s^k (f, t) = \Delta_{s-1}^k (\Delta_s (f, \cdot), t) = \Delta_s^{k-1} (\Delta_s (f, \cdot), t)$
\[ \Delta_s^k(f, t) = \int_{v_1}^{i+s} \cdots \int_{v_{k-2}+s}^{v_{k-1}+s} f^{(k)}(v_k) dv_k dv_{k-1} \cdots dv_2 dv_1. \]

For any positive integer \( k, \)

\[
\frac{\partial}{\partial s} (\Delta_s^k(f, t)) = k \Delta_s^{k-1}(f', t + s),
\]

and for any integer \( j > 0, \)

\[
\frac{\partial}{\partial t^j} (\Delta_s^k(f, t)) = \Delta_s^j(f^{(j)}, t).
\]

Observe that, using the Binomial Theorem and the semigroup property of \([P_t],\)

\[
(P_t - I)^k f(x) = \sum_{j=0}^{k} \binom{k}{j} (-I)^j f(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j P_t^{k-j} f(x)
\]

\[
= \sum_{j=0}^{k} \binom{k}{j} (-1)^j (x, (k - j)t)
\]

(2.4) \[ = \Delta_t^k(u(x, t), 0), \]

where as usual, \( u(x, t) = P_t f(x). \)

Additionally, we will need in what follows the next result,

**Lemma 2.2.** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}_+^d, \gamma_d), \) \( f \in L^p(\gamma_d) \) and \( k, n \in \mathbb{N} \) then

\[ \|\Delta_s^k(u^{(n)}, t)\|_{p(\cdot), \gamma_d} \leq C_{k,p(\cdot)} s^k \|u^{(k+n)}(\cdot, t)\|_{p(\cdot), \gamma_d} \]

**Proof.** From ii) of Lemma 2.1, we have

\[ \Delta_s^k(u^{(n)}(x, t), t) = \int_{v_1}^{v_{k-1}+s} \cdots \int_{v_{k-2}+s}^{v_{k-1}+s} u^{(k+n)}(x, v_k) dv_k dv_{k-1} \cdots dv_2 dv_1, \]

then, using Minkowski’s integral inequality (1.18) and Lemma 1.2 \( k \)-times respectively

\[ \|\Delta_s^k(u^{(n)}, t)\|_{p(\cdot), \gamma_d} \leq C^k \int_{v_1}^{v_{k-1}+s} \cdots \int_{v_{k-2}+s}^{v_{k-1}+s} \|u^{(k+n)}(\cdot, v_k)\|_{p(\cdot), \gamma_d} dv_k dv_{k-1} \cdots dv_2 dv_1 \]

\[ \leq C^k (C_{p(\cdot)} s^k \|u^{(k+n)}(\cdot, t)\|_{p(\cdot), \gamma_d}) = C_{k,p(\cdot)} s^k \left\| \frac{\partial^{k+n}}{\partial t^{k+n}} u(\cdot, t) \right\|_{p(\cdot), \gamma_d}. \]

\[ \Box \]

We are now ready to consider the action Gaussian Bessel fractional derivatives on general \( B^\alpha_{p(\cdot), q(\cdot)}(\gamma_d) \) spaces. The result analogous to Theorem 8 in [7].

**Theorem 2.3.** Let \( 0 < \beta < \alpha, p(\cdot) \in \mathcal{P}_0(\gamma_d) \cap LH_0(\mathbb{R}_+^d) \) with \( 1 < p_- \leq p_+ < \infty \)

and \( q(\cdot) \in \mathcal{P}_{0, \infty}. \) Then, the Gaussian Bessel fractional derivative \( D_\beta \) is bounded from \( B^\alpha_{p(\cdot), q(\cdot)}(\gamma_d) \) into \( B^{\alpha-\beta}_{p(\cdot), q(\cdot)}(\gamma_d). \)
Proof. Let $f \in B^{\alpha}_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$, $k \in \mathbb{N}$ such that $k-1 \leq \beta < k$ and set $v(x,t) = e^{-t}u(x,t)$ then using the classical Hardy’s inequality (1.26), the fundamental theorem of calculus and Lemma 2.1,

$$|D_\beta f(x)| \leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |\Delta^k_s(v(x,\cdot),0))|ds$$

$$\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^t \frac{|\partial}{\partial r} \Delta^k_r(v(x,\cdot),0))|dr ds \leq \frac{k}{\beta c_\beta} \int_0^{+\infty} r^{-\beta} |\Delta^k_{r-1}(v'(x,\cdot),r)|dr$$

and by Minkowski’s integral inequality (1.18) this implies

$$\|D_\beta f\|_{p(\cdot),q_d} \leq \frac{k}{\beta c_\beta} C \int_0^{+\infty} r^{-\beta} \|\Delta^k_{r-1}(v',r)\|_{p(\cdot),q_d} dr.$$ 

Now, using Lemma 2.1 and again Minkowski’s integral inequality (1.18)

$$\|\Delta^k_{r-1}(v',r)\|_{p(\cdot),q_d} \leq C \int_r^{2r} \int_{v_1}^{v_1+r} \cdots \int_{v_{k-2}}^{v_{k-2}+r} \|v^{(k)}(v,v_{k-1})\|_{p(\cdot),q_d} dv_{k-1} dv_{k-2} \cdots dv_2 dv_1,$$

and by Leibnitz’s differentiation rule for the product

$$\|v^{(k)}(v,v_{k-1})\|_{p(\cdot),q_d} = \left\| \sum_{j=0}^{k} \binom{k}{j} (e^{-v_{k-1}})^j u^{(k-j)}(v,v_{k-1}) \right\|_{p(\cdot),q_d}$$

$$\leq \sum_{j=0}^{k} \binom{k}{j} e^{-v_{k-1}} \|u^{(k-j)}(v,v_{k-1})\|_{p(\cdot),q_d}.$$ 

Then, by Lemma 1.2

$$\|\Delta^k_{r-1}(v',r)\|_{p(\cdot),q_d} \leq C \sum_{j=0}^{k} \binom{k}{j} \int_r^{2r} \int_{v_1}^{v_1+r} \cdots \int_{v_{k-2}}^{v_{k-2}+r} e^{-v_{k-1}} \|u^{(k-j)}(v,v_{k-1})\|_{p(\cdot),q_d} dv_{k-1} dv_{k-2} \cdots dv_2 dv_1$$

$$\leq C_{k,p(\cdot)} \sum_{j=0}^{k} \binom{k}{j} e^{-r} \|u^{(k-j)}(v,r)\|_{p(\cdot),q_d}.$$
Therefore, using the $L^{p(.)}$-boundedness of $P_j$ (see [9])

$$
\|D_{\beta} f\|_{p(\cdot), \gamma_d} \leq \frac{k}{\beta c_\beta} C_{k,p(\cdot)} \sum_{j=0}^{k-1} \binom{k}{j} \int_0^{+\infty} r^{k-j-1} e^{-r} \|u^{(k-j)}(\cdot, r)\|_{p(\cdot), \gamma_d} dr
$$

$$
= C_{k,p(\cdot)} \frac{k}{\beta c_\beta} \sum_{j=0}^{k-1} \binom{k}{j} \int_0^{+\infty} r^{k-j-(\beta-j)-1} e^{-r} \left\| \frac{\partial^{k-j}}{\partial r^{k-j}} P_j f \right\|_{p(\cdot), \gamma_d} dr
$$

$$
+ C_{k,p(\cdot)} \frac{k}{\beta c_\beta} \int_0^{+\infty} r^{k-j-1} e^{-r} \|P_j f\|_{p(\cdot), \gamma_d} dr
$$

$$
\leq C_{k,p(\cdot)} \frac{k}{\beta c_\beta} \sum_{j=0}^{k-1} \binom{k}{j} \int_0^{+\infty} r^{k-j-(\beta-j)-1} \left\| \frac{\partial^{k-j}}{\partial r^{k-j}} P_j f \right\|_{p(\cdot), \gamma_d} dr
$$

$$
+ C_{k,p(\cdot)} \frac{k}{\beta c_\beta} \int_0^{+\infty} r^{k-j-1} e^{-r} \|f\|_{p(\cdot), \gamma_d} dr
$$

Thus,

$$
\|D_{\beta} f\|_{p(\cdot), \gamma_d} \leq C_{k,p(\cdot)} \frac{k}{\beta c_\beta} \sum_{j=0}^{k-1} \binom{k}{j} \int_0^{+\infty} r^{k-j-(\beta-j)-1} \left\| \frac{\partial^{k-j}}{\partial r^{k-j}} P_j f \right\|_{p(\cdot), \gamma_d} dr
$$

$$
+ C_{k,p(\cdot)} \frac{k\Gamma(k-\beta)}{\beta c_\beta} \|f\|_{p(\cdot), \gamma_d} < \infty,
$$

since $f \in B^\alpha_{p(\cdot), \gamma_d} \subset B^\beta_{p(\cdot), \gamma_d}$ as $\alpha > \beta - j \geq 0$, for $j \in \{0, \ldots, k-1\}$. Hence, $D_{\beta} f \in L^{p(\cdot)}(\gamma_d)$.

On the other hand,

$$
P_j(e^{-s}P_s - I)^k f(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{-s(k-j)} u(x, t + (k-j)s).
$$

Let $n$ be the smaller integer greater than $\alpha$, i.e. $n - 1 \leq \alpha < n$, we have

$$
\frac{\partial^n}{\partial t^n} P_j(D_{\beta} f)(x) = \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{-s(k-j)} u^{(n)}(x, t + (k-j)s) ds
$$

$$
= \frac{\epsilon_j}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{-(t+s(k-j))} u^{(n)}(x, t + (k-j)s) ds
$$

$$
= \frac{\epsilon_j}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \Delta^k_j(w(x, \cdot), t) ds,
$$
where \( w(x,t) = e^{-t}u^{(n)}(x,t) \). Now using the fundamental theorem of calculus,

\[
\frac{\partial^n}{\partial t^n} P_t(D\beta f)(x) = \frac{e'}{c\beta} \int_0^{+\infty} s^{-\beta-1} \Delta_s^k(w(\cdot),t)ds
\]

\[
= \frac{e'}{c\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s \frac{\partial}{\partial s} \Delta_s^k(w(\cdot),t)dr ds.
\]

Then, using classical Hardy’s inequality (1.26), and Lemma 2.1,

\[
\left| \frac{\partial^n}{\partial t^n} P_t(D\beta f)(x) \right| \leq \frac{e'}{c\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s \left| \frac{\partial}{\partial s} \Delta_s^k(w(\cdot),t) \right| dr ds
\]

\[
\leq \frac{e'}{c\beta^2} \int_0^{+\infty} r \left| \frac{\partial}{\partial r} \Delta_r^k(w(\cdot),t) \right| r^{-\beta-1}dr
\]

\[
= \frac{ke'}{c\beta} \int_0^{+\infty} r^{-\beta} |\Delta_r^{k-1}(w'(\cdot),t+r)|dr
\]

and by Minkowski’s integral inequality (1.18) we get

\[
\left\| \frac{\partial^n}{\partial t^n} P_t(D\beta f) \right\|_{p(\cdot),\gamma_d} \leq C \frac{ke'}{\beta c\beta} \int_0^{+\infty} r^{-\beta} \|\Delta_r^{k-1}(w',t+r)\|_{p(\cdot),\gamma_d} dr.
\]

Now, by analogous argument as above, Lemma 2.1 and again Leibnitz’s differentiation rule for the product, give us

\[
\|\Delta_r^{k-1}(w',t+r)\|_{p(\cdot),\gamma_d} \leq C_{k,p(\cdot)} \sum_{j=0}^{k} \binom{k}{j} r^{-j-\beta} e^{-r} \|u^{(k+n-j)}(\cdot,t+r)\|_{p(\cdot),\gamma_d},
\]

and this implies that

\[
\left\| \frac{\partial^n}{\partial t^n} P_t(D\beta f) \right\|_{p(\cdot),\gamma_d} \leq C_{k,p(\cdot)} \frac{e'}{c\beta^2} \int_0^{+\infty} r^{-\beta} \left( \sum_{j=0}^{k} \binom{k}{j} r^{-j-\beta} e^{-r} \|u^{(k+n-j)}(\cdot,t+r)\|_{p(\cdot),\gamma_d} \right) dr
\]

\[
= C_{k,p(\cdot)} \frac{k}{c\beta^2} \sum_{j=0}^{k} \binom{k}{j} \int_0^{+\infty} r^{-\beta-1} e^{-r} \|u^{(k+n-j)}(\cdot,t+r)\|_{p(\cdot),\gamma_d} dr.
\]

Thus,

\[
\left\| \rho^{\alpha-\beta} \left\| \frac{\partial^n}{\partial t^n} P_t(D\beta f) \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu}
\]

\[
\leq C_{k,p(\cdot)} \frac{k}{c\beta^2} \sum_{j=0}^{k} \binom{k}{j} \left\| \rho^{\alpha-\beta} \int_0^{+\infty} r^{-\beta-1} e^{-r} \|u^{(k+n-j)}(\cdot,t+r)\|_{p(\cdot),\gamma_d} dr \right\|_{q(\cdot),\mu}.
\]
Now, for each $1 \leq j \leq k$, $0 < \alpha - \beta + k - j \leq \alpha$ and by lemma 1.2

$$
\left\| \int_0^{\mu} \int_0^{\infty} r^{\alpha - \beta - 1} e^{-r} \|u^{(k+n-j)}(\cdot, t + r)\|_{\gamma_d}^a \right\|_{q(\cdot), \mu} 
\leq C_{p(\cdot)} \left\| \int_0^{\alpha - \beta} \int_0^{\infty} r^{\alpha - \beta - 1} e^{-r} \|u^{(n+k-j)}(\cdot, t)\|_{\gamma_d}^a \right\|_{q(\cdot),\mu} 
\leq C_{p(\cdot)} \Gamma(k - \beta) \|u^{(n+k-j)}(\cdot, t)\|_{\gamma_d}^a < \infty,
$$
as $f \in B_{p(\cdot),q(\cdot)}^a(\gamma_d) \subset B_{p(\cdot),q(\cdot)}^{a-k+\beta}(\gamma_d)$ for any $1 \leq j \leq k$.

Now, for the case $j = 0$,

$$
\left\| \int_0^{\alpha - \beta} \int_0^{\infty} r^{\alpha - \beta - 1} e^{-r} \|u^{(n+k)}(\cdot, t + r)\|_{\gamma_d}^a \right\|_{q(\cdot), \mu} 
\leq \left\| \int_0^{\alpha - \beta} \int_0^a r^{\alpha - \beta - 1} e^{-r} \|u^{(n+k)}(\cdot, t)\|_{\gamma_d}^a \right\|_{q(\cdot), \mu} 
+ \left\| \int_0^{\alpha - \beta} \int_a^{\infty} r^{\alpha - \beta - 1} e^{-r} \|u^{(n+k)}(\cdot, t + r)\|_{\gamma_d}^a \right\|_{q(\cdot), \mu} 
= (I) + (II).
$$

Using Lemma 1.2, and $k > \beta$,

$$
(I) \leq C_{p(\cdot)} \left\| \int_0^{\alpha - \beta} \int_0^a r^{\alpha - \beta - 1} \|u^{(n+k)}(\cdot, t)\|_{\gamma_d}^a \right\|_{q(\cdot), \mu} 
= C_{p(\cdot)} \left\| \int_0^{\alpha - \beta} \|u^{(n+k)}(\cdot, t)\|_{\gamma_d}^a \int_0^a r^{\alpha - \beta - 1} \right\|_{q(\cdot), \mu} 
= \frac{C_{p(\cdot)}}{k - \beta} \|u^{n+k-a}(\cdot, t)\|_{\gamma_d}^a < \infty,
$$
since $f \in B_{p(\cdot),q(\cdot)}^a(\gamma_d)$ and $n + k > \alpha$.

For the second term, using Lemma 1.2 and Hardy’s inequality (1.25)

$$
(II) \leq C_{p(\cdot)} \left\| \int_0^{\alpha - \beta} \int_a^{\infty} r^{\alpha - \beta - 1} \|u^{(n+k)}(\cdot, t + r)\|_{\gamma_d}^a \right\|_{q(\cdot), \mu} 
\leq C_{p(\cdot)} C_{q(\cdot)} \left\| \int_a^{\alpha - \beta} \|u^{(n+k)}(\cdot, t)\|_{\gamma_d}^a \right\|_{q(\cdot), \mu} < \infty,
$$
since $f \in B_{p(\cdot),q(\cdot)}^a(\gamma_d)$. Therefore, $D_{\beta} f \in B_{p(\cdot),q(\cdot)}^{a-k+\beta}(\gamma_d)$.
Moreover,

\[
\|D_\beta f\|_{B^{\nu-\alpha-\beta}_{p,q}(\cdot)} = \|D_\beta f\|_{B^{\nu}_{p,q}(\cdot)} + \left\| \frac{\partial^n}{\partial r^n} P_r D_\beta f \right\|_{B^{\nu}_{p,q}(\cdot)} \\
\leq C_{k,p,q}^{1} \|f\|_{B^{\nu}_{p,q}(\cdot)} + \frac{k}{\epsilon_{1}^{p_0}} \sum_{j=0}^{k} \left( \frac{k}{j} \right)^{2} C_{k,p,q}^{2} \left\| \frac{\partial^n}{\partial r^n} P_r f \right\|_{B^{\nu}_{p,q}(\cdot)} \\
\leq C_{p,q} \|f\|_{B^{\nu}_{p,q}(\cdot)}
\]

The boundedness of Gaussian Riesz potentials on variable Gaussian Besov-Lipschitz spaces and the regularity of all these operators on variable Gaussian Triebel-Lizorkin spaces, spaces that were also defined in [11], will be considered in a forthcoming paper.

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