Continuous connection of two adjacent pipe parts defined by line, bézier and hermit center curves

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Abstract. A shape of pipe part consist of the longitudinal and cross section boundary curve. Also, it can be defined by only its center curve of the pipe. To build a complete pipe, we can connect continually the pipe parts. This paper discuss about continuous connection of two adjacent pipe parts that are defined by line, Bézier and Hermit curves. The method is as follows, we join continually two adjacent center curves of the pipe parts and then by using the same formulae, we define its coincidence cross section boundary curves of the pipes. Finally, the joining of the longitudinal section boundary curves of the pipe parts can be done. The results of this study show that we can evaluate the joint continuity between two adjacent pipe parts defined by line, Bézier and Hermit curves, using the various forms of its center and cross-longitudinal section boundary curves.

1. Introduction
There are many introduced methods for modeling a shape of pipe. We can use a single-valued tubular patches and a cyclide to model radius and pipe design of tubular geometry [7,8]. By using the geometric continuity condition for algebraic surface we can unify some pieces of pipes at common vertex [1]. By joining two cylinders of revolution with axes in a common plane and different radii we will find a circular surfaces [4]. Also, the shape of pipe can be evaluated by geometric properties of canal surfaces in $E^3$ [6]. The physical model of transitional pipeline parts can be made of materials that cannot be wrinkled or stretched [5]. Then, a modular Pipe-Z parametric design system will give the results for a trefoil, a gure-eight knot and a pentafoil [9]. Different from the earlier methods, this paper discusses about constructing the whole pipe using small parts connection of the pipes. In this discussion, we calculate the continuous connection of two adjacent pipe parts defined by line, Bézier and Hermit curves.

This paper is organized in the following sections. In the first section, we talk about the formulation of pipe parts that depend on its cross, longitudinal and center curves. In the second, we evaluate the continuous connection of two adjacent pipe parts of the line, Bézier and Hermit center curves. Finally, the results will be summarized in the conclusion section.

2. Pipe Parts Formulation
Consider a regular curve $\Gamma(u)$ continuous on the interval $0 \leq u \leq 1$ that can be differentiated twice. Also it can be expressed as a function of the natural parameter $\Gamma(s)$. The tangent unit vector $t$ and the normal unit vector $n$ are orthogonal in the form (Figure 1a)

$$t = \frac{d\Gamma}{ds} = \frac{\Gamma'}{|\Gamma'|}$$

and

$$n = -\frac{k}{|k|}$$

(1a)

(1b)

$$k = \frac{dt}{ds} = \frac{t'}{|\Gamma'|}$$

(1c)
Using the cross product operation of the both vectors \( \mathbf{t} \) and \( \mathbf{n} \), we can define the unit binormal vector \( \mathbf{b} = \mathbf{t} \wedge \mathbf{n} \) such that the triplet \([\mathbf{t}, \mathbf{n}, \mathbf{b}]\) form the Frenet frame of curve [3].

Consider along the direction of parameter \( u \) each points of the curve \( \Gamma(u) \) as the center points of the defined circles of radius \( \mathcal{Y}(u,v) \) in the normal plane \([\mathbf{b}, \mathbf{n}]\) of the curve \( \Gamma(u) \) that are orthogonal to unit tangent vector \( \mathbf{t} \). Using the parametric tubular surface formulae of the curve \( \Gamma(u) \), we can define a part of tubular pipe in the form [2]

\[
\mathbf{T}(u,v) = \mathbf{T}(u) + \mathcal{Y}(u,v)\left[\cos (\varphi) \mathbf{b} + \sin (\varphi) \mathbf{n}\right] \tag{2}
\]

with the real function \( \mathcal{Y}(u,v) = \rho(u), r(v) \) expresses the radius of the pipe, \( \varphi = 2\pi v \) and \( 0 \leq u, v \leq 1 \). In this case, the real function \( \rho(u) \) characterizes the inflate-deflate surfaces form of the pipe patches along its center curve, meanwhile \( r(v) \) defines the cross-section curve form of the pipe. To facilitate the creation of pipe part, we determine the real function \( \rho(u) \) and \( r(v) \) from the Bézier, Hermit and trigonometric curves as follows

\[
\rho_1(u) = P_0(1 - u)^3 + 3P_1(1 - u)^2u + 3P_2(1 - u)u^2 + P_3u^3 \tag{3a}
\]

\[
\rho_2(u) = r_2(0)H_1(u) + r_2(1)H_2(u) + r_2''(0)H_2(u) + r_2''(1)H_4(u) \tag{3b}
\]

\[
\rho_3(u) = a + b \cos (u) + c \sin (u) \tag{3c}
\]

\[
r_1(v) = a \cos (n \varphi) \pm b \sin (n \varphi) \tag{4a}
\]

where

\[
H_1(u) = 2u^3 - 3u^2 + 1; \quad H_2(u) = 2u^3 + 3u^2;
\]

\[
H_3(u) = u^3 - 2u^2 + u; \quad H_4(u) = u^3 - u^2.
\]

with \( a, b, c \) real constants, \( \varphi = 2\pi v \) and \( n \) is the number of defined rose leafs, \( 0 \leq u \leq 1 \) and \( 0 \leq v \leq 1 \). In addition, we define the cross section curve form of the pipe using unify \( n \) curves of the different circle parts to the same origin \( O \) in the form

\[
r_2^i(v) = r_0\left[\cos \left((2i + 1)\frac{\varphi}{n} - \frac{\varphi}{n} \right)\right] \pm \sqrt{\left[r_0^2 \cos^2 \left((2i + 1)\frac{\varphi}{n} - \frac{\varphi}{n}\right) - r_2^2 - r^2 \right]} \tag{4b}
\]

for \( i = 0, 1, ..., n - 1 \) and \( r_0 \) as a ray of polar form, \( \varphi = 2 \pi v \) with \( 0 \leq v \leq 1 \). The first, we determine \( \Gamma(u) \) as a line curve of equation

\[
\mathbf{L}(u) = \mathbf{c} + \lambda \ u \mathbf{l} \tag{5}
\]

with \( \mathbf{c} \) constant vector, \( \mathbf{l} \) unit direction vector, \( \lambda \) positive real constant and \( 0 \leq u \leq 1 \) . Because of the \( \mathbf{L}(u) \) line, we must change the circles of radius \( \mathcal{Y}(u,v) \) in the normal plane \([\mathbf{b}, \mathbf{n}]\) in equation (2) become in plane \([\mathbf{v}_1, \mathbf{v}_2]\) with \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) any unit constant vectors such that \( \mathbf{l} = \mathbf{v}_1 \wedge \mathbf{v}_2 \). The second, we define the center curve \( \Gamma(u) \) in the form cube Bézier curve \( \mathbf{B}(u) \) and Hermit curve \( \mathbf{H}(u) \) with

\[
\mathbf{B}(u) = P_0(1 - u)^3 + 3P_1(1 - u)^2u + P_2(1 - u)u^2 + P_3u^3 \tag{6}
\]

\[
\mathbf{H}(u) = H_0H_1(u) + H_1H_2(u) + H_2H_3(u) + H_3H_4(u) \tag{7}
\]
The vectors $P_0$, $P_1$, $P_2$ and $P_3$ are the control points of the Bézier curve $B(u)$, $H_0 = H(0)$, $H_1 = H(1)$, $H_2 = H'(0)$, $H_3 = H'(1)$ and $0 \leq u \leq 1$.

The implementation of formulae (2-7) can be demonstrated as follows. Let $v_1 = \langle 0,1,0 \rangle$, $v_2 = \langle 0,0,1 \rangle$ and $L(u) = \langle -15,0,0 \rangle + 25u \langle 1,0,0 \rangle$. When we decide $\rho(u) = 2 + \cos (2\pi u) - \sin (2\pi u)$ and $r(v) = 1$ such that $\rho(u) = 2 + \cos (2\pi u) - \sin (2\pi u)$, equation (2) will show Figure 1a. If we determine $\rho(u) = 1.5; r(v) = -2 - \cos (4\pi v) - \cos (8\pi v)$ and $\rho(u) = 2 (1-u)^3 + 12 (1-u)^2. u + 3 (1-u)^2 + 2u^3; r(v) = -2 - \cos (4\pi v) - \cos (8\pi v)$, we will find Figure 1b and 1c, respectively. Figure 1d and 1e show the graph of equation (2) and (6) with control points $[P_0, P_1, P_2, P_3]$.

3. Connection of Two Adjacent Pipe Parts

Consider two parametric tubular surface pieces $T_1(u,v)$ and $T_2(u,v)$ of equation (2). Its center curve and its cross-longitudinal boundary curves are respectively $[L_1(u), L_2(u)]$ of line curve (5) and $[Y_1(u,v), Y_2(u,v)]$ of equations (3-4) in the same direction condition. Because of the center curves of degree one, we can join the both tubular surfaces using tangential continuity, that is $T_{L1}(1,v) = T_{L2}(0,v)$ and $T_{L1}^{u}(1,v) = T_{L2}^{u}(0,v)$ along the interval $0 \leq v \leq 1$. These mean that, the first, at $T_{L1}(1,v)$ and $T_{L2}(0,v)$ it must be

$$[c + \lambda L_{1}]_{L1} = [c]_{L2}$$

$$Y_{L1}^{u}(1,v) = Y_{L2}^{u}(0,v)$$

in plane $[v_1,v_2]$. The second, along the interval $0 \leq v \leq 1$, the tangent vectors $L_1^u(u)$ and $L_2^u(u)$ of its center curves and the first derivation of the tube radius $Y_{L1}^u(u,v)$ and $Y_{L2}^u(u,v)$ have to
\[ \mathbf{L}_1^u (1) = \sigma \mathbf{L}_2^u (0) \text{ or } \mathbf{l}_1 = \sigma \mathbf{l}_2 \]  
\[ \gamma_1^u (1,v) = \gamma_2^u (0,v) \]  
\[ \mathbf{l}_1 = 3 \tau \mathbf{P}_1 \mathbf{P}_2 \]  
\[ \gamma_1^u (1,v) = \gamma_2^u (0,v) \]  
\[ \mathbf{B}(u) = 3[(1-\mathbf{P}_1 \cdot \mathbf{P}_2)(1-u^2) + 2(\mathbf{P}_2 \cdot \mathbf{P}_1)(1-u)u + (\mathbf{P}_3 \cdot \mathbf{P}_2)u^2], \quad \mathbf{B}(0) = 3[(\mathbf{P}_1 \cdot \mathbf{P}_2)] \]

with \( \sigma \) positive real scalar. When the center curve of pipe \( \mathbf{T}_3(u,v) \) is a cubic Bézier curve of equation (6), then \( \mathbf{B}(u) = 3[(\mathbf{P}_1 \cdot \mathbf{P}_2)(1-u^2) + 2(\mathbf{P}_2 \cdot \mathbf{P}_1)(1-u)u + (\mathbf{P}_3 \cdot \mathbf{P}_2)u^2], \quad \mathbf{B}(0) = 3[(\mathbf{P}_1 \cdot \mathbf{P}_2)] \)

The tangential continuity conditions in equation (11) of two pipe parts \( \mathbf{T}_1(1,v) \) and \( \mathbf{T}_2(0,v) \) coincide each other respectively and the second derivation of the direction \( u \) of radius \( \gamma_1^u (u,v) \) and \( \gamma_2^u (u,v) \) at \( \mathbf{T}_1(1,v) \) and \( \mathbf{T}_2(0,v) \) are equal respectively. So, it must be

\[ \mathbf{n}_1(1) = \beta \mathbf{n}_2(0) \]  
\[ \mathbf{b}_1(1) = \gamma \mathbf{b}_2(0) \]  
\[ \gamma_1^{uu}(1,v) = \gamma_2^{uu}(0,v) \]  

with \( \beta \) and \( \gamma \) positive real scalars. The calculation of the unit vectors \( \mathbf{t}, \mathbf{n} \) and \( \mathbf{b} \) is as follows.

\[ \mathbf{B}(u) = < R_x(u), R_y(u), R_z(u) > \]

with

\[ R_x(u) = 3[(\mathbf{P}_{1z} \cdot \mathbf{P}_{2z})(1-u^2) + 2(\mathbf{P}_{2z} \cdot \mathbf{P}_{1z})(1-u)u + (\mathbf{P}_{3z} \cdot \mathbf{P}_{2z})u^2], \]  
\[ R_y(u) = 3[(\mathbf{P}_{1y} \cdot \mathbf{P}_{2y})(1-u^2) + 2(\mathbf{P}_{2y} \cdot \mathbf{P}_{1y})(1-u)u + (\mathbf{P}_{3y} \cdot \mathbf{P}_{2y})u^2], \]  
\[ R_z(u) = 3[(\mathbf{P}_{1z} \cdot \mathbf{P}_{2z})(1-u^2) + 2(\mathbf{P}_{2z} \cdot \mathbf{P}_{1z})(1-u)u + (\mathbf{P}_{3z} \cdot \mathbf{P}_{2z})u^2]. \]
\begin{equation}
\mathbf{B}''(u) = \langle W_x(u), W_y(u), W_z(u) \rangle
\end{equation}

with

\begin{align*}
W_x(u) &= 6((P_{2x} - 2P_{1x}) + (P_{3x} - 2P_{2x} + P_{1x})u] \\
W_y(u) &= 6((P_{2y} - 2P_{1y}) + (P_{3y} - 2P_{2y} + P_{1y})u] \\
W_z(u) &= 6((P_{2z} - 2P_{1z}) + (P_{3z} - 2P_{2z} + P_{1z})u].
\end{align*}

\begin{equation}
\mathbf{H}''(u) = \langle N_x(u), N_y(u), N_z(u) \rangle
\end{equation}

with

\begin{align*}
N_x(u) &= H_{2x}(6u^2 - 6u) + H_{1x}(-6u^2 + 6u) + H_{0x}(3u^2 - 4u + 1) + H_{-1x}(3u^2 - 2u) \\
N_y(u) &= H_{2y}(6u^2 - 6u) + H_{1y}(-6u^2 + 6u) + H_{0y}(3u^2 - 4u + 1) + H_{-1y}(3u^2 - 2u) \\
N_z(u) &= H_{2z}(6u^2 - 6u) + H_{1z}(-6u^2 + 6u) + H_{0z}(3u^2 - 4u + 1) + H_{-1z}(3u^2 - 2u).
\end{align*}

\begin{equation}
\mathbf{H}''(u) = \langle Z_x(u), Z_y(u), Z_z(u) \rangle
\end{equation}

with

\begin{align*}
Z_x(u) &= H_{2x}(12u - 6) + H_{1x}(-12u + 6) + H_{0x}(6u - 4) + H_{-1x}(6u - 2) \\
Z_y(u) &= H_{2y}(12u - 6) + H_{1y}(-12u + 6) + H_{0y}(6u - 4) + H_{-1y}(6u - 2) \\
Z_z(u) &= H_{2z}(12u - 6) + H_{1z}(-12u + 6) + H_{0z}(6u - 4) + H_{-1z}(6u - 2).
\end{align*}

\begin{equation}
k_B = \langle M_x(u), M_y(u), M_z(u) \rangle
\end{equation}

with

\begin{align*}
M_x &= [s^2W_x(u) - R_x(u)W_x(u) + R_y(u)R_z(u)W_x(u) + R_z(u)R_y(u)W_x(u)]/s^4 \\
M_y &= [s^2W_y(u) - R_y(u)W_y(u) + R_x(u)R_z(u)W_y(u) + R_z(u)R_x(u)W_y(u)]/s^4 \\
M_z &= [s^2W_z(u) - R_z(u)W_z(u) + R_x(u)R_y(u)W_z(u) + R_y(u)R_x(u)W_z(u)]/s^4 \\
s &= [R_x^2(u) + R_y^2(u) + R_z^2(u)]^{1/2}.
\end{align*}

\begin{equation}
k_H = \langle S_x(u); S_y(u); S_z(u) \rangle
\end{equation}
with
\[ S_t = [n^2Z(u) - \{N_uZ(u) + N_z(u)Z(u) + N_z(u)N_z(u)Z(u)\}]n^4 \]
\[ S_s = [n^2Z(u) - \{N_uZ(u) + N_z(u)Z(u) + N_z(u)N_z(u)Z(u)\}]n^4 \]
\[ S_c = [n^2Z(u) - \{N_uZ(u) + N_z(u)Z(u) + N_z(u)N_z(u)Z(u)\}]n^4 \]
n = \[N_u^2(u) + N_z^2(u) + N_z^2(u)]^{1/2}.

So, the unit vector tangent, normal and binormal of the Bezier center curve is

\[ t_B = (1/s) < R_z(u), R_y(u), R_x(u) > \] (13a)
\[ n_B = < M_z(u)/s_z, M_y(u)/s_y, M_x(u)/s_x > \] (13b)
\[ b_B = < [M_y(u)R_z(u) - R_y(u)M_z(u)]/s_z, \]
\[ [M_z(u)R_y(u) - R_z(u)M_y(u)]/s_y, \]
\[ [R_y(u)M_x(u) - M_y(u)R_x(u)]/s_x > \] (13c)

with \( s_0 = [M_z^2(u) + M_y^2(u) + M_x^2(u)]^{1/2} \). On the other hand, we have the unit vector tangent, normal and binormal of the Hermit center curve

\[ t_H = 1/n < N_z(u); N_y(u); N_x(u) > \] (14a)
\[ n_H = < S_z(u)n_z, S_y(u)n_y, S_x(u)n_x > \] (14b)
\[ b_H = < [N_z(u)S_y(u) - S_z(u)N_y(u)]/(n_n_z), \]
\[ [S_z(u)N_y(u) - N_z(u)S_y(u)]/(n_n_y), \]
\[ [N_y(u)S_z(u) - S_y(u)N_z(u)]/(n_n_x) > \] (14c)

with \( n_0 = [S_z^2(u) + S_y^2(u) + S_x^2(u)]^{1/2} \).

4. Conclusions
We have formulated the continuous connection of various pipe shapes which its center curves \( \Gamma(u) \) are defined by line, Bézier and Hermit curves. We can conclude that two parametric tubular surface pieces \( T_1(u,v) \) and \( T_2(u,v) \) connect continuously order-1, if they verify the conditions: first, \( T_1(1,v) = T_2(0,v) \). Second, its tangent vectors of center curve \( \Gamma_1(u) \) and \( \Gamma_2(u) \) at the values \( T_1(1,v) \) and \( T_2(0,v) \) and its tangent vectors of longitudinal boundary curve along the cross section boundary curve are equal. They are also in the moving trihedron continuity, if its triplet \( [t_n b] \) of the center curves at \( \Gamma_1(1) \) and \( \Gamma_2(0) \) coincide each other respectively and the second derivation of the direction \( u \) of radius \( F_1(u,v) \) and \( F_2(u,v) \) respectively at \( T_1(1,v) \) and \( T_2(0,v) \) are equal.

The continuous connection of various pipe shapes have been introduced. The interesting thing to discuss ahead is how to build a complete pipe using some parts of pipes in various thickness. Also, the connection have to be varied and continuous.

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