The Periodic Response of Periodically Perturbated Stochastic Systems

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I. INTRODUCTION

In paper [1] a possible way was noticed regarding how a neuron retains or responds to periodic stimulation. This new feature was established as a result of computational study of the deep differential structure of actual neuron spike trains. We observed, that for many cases the sequences of higher order finite differences, taken from periodically stimulated neuron spike trains, can be divided into several subsequences of approximately equal lengths, on some of which the changes in monotony of these differences are strongly periodic. On the other hand, the presence of chaotic dynamics in neural activity is well known. This work investigates the following problem: In which extent the deterministic stochastic systems hold such type of periodic response property, or does our remark from [1] remain valid for living matter only? With this end in view, we introduce a new numerical characteristic of stochastic one dimensional systems, expressing this kind of hidden and incomplete periodicity in a quantitative form. Its computational study on instances of some nonlinear systems is presented. We have chosen three one dimensional maps, most frequently mentioned in nonlinear dynamics: the tent map, logistic map, and Poincare displacement of Chirikov’s standard map. The tent map is the simplest system exhibiting the strong chaotic properties, while the behavior of logistic map is typical for many one dimensional systems having quadratic maximum. The Chirikov map [2-4] has a fundamental role in Hamilton dynamics. We study the behavior of one of its ordinates, so-called [5] Poincare displacement. The Billingsley-Eggleston formula type inequality as well as a theoretical result on sequences of
fractional parts, generated by a simple aperiodic map, are also presented. We recall [6], that namely these sequences are well accepted for the aims of mathematical and computational modeling of randomness.

Note, that in many problems of nonlinear dynamics (e.g. various neuron mathematical models, Duffing equation – will be studied in our subsequent work) namely a one dimensional Poincare displacement is of the main interest. Let us also note, that the research of various two dimensional maps and three dimensional flows (for instance, the orbits of Rössler and Lorentz attractors - see [2], Ch. 1.5.b and Ch. 7.1.b) can be mostly reduced to studying some one dimensional systems. In this regard, we recall Bridges-Rowlands general theoretical scheme, can be found in [2], Ch. 7.3.b.

We give some statements of finite difference analysis [1, 7], on which this paper is based, and explain our approach. The mentioned new notion, the numerical characteristic $\gamma$, is some measure of (in certain sense) minimal periodicity, and hence, can also be treated as a measure of chaosity. The main claim of this work is that this quantity, being a measure of irregularity, is also that characteristic of stochastic system, that is able to change essentially its numerical value when the system undergoes a weak periodic perturbation. This feature of $\gamma$ makes it a new tool when researching the problems of weak signal detection in presence of strong (deterministic) noise. In particular, it can be used in research of stochastic resonance phenomena (see, e.g., [8]).

Accepted approaches in stochastic resonance problems are the use of power spectrum and correlation function. However, in this work we give only the comparisons of new measure with Lyapunov exponent, leaving the consideration of spectral characteristics for the further work. The criticism on Lyapunov exponent one can find, e.g., in [9]. Because of its computation simplicity, the $\gamma$ characteristic is well adapted for research of various applied problems, where the analytic law of system evolution usually remains unknown. In this respect, we note important works by J. Kurths and his colleagues ([10]; see also [11] where the approach from [1, 7] to study the seismic time series is applied) on nontraditional measures and its applications to studying different chaotic time series in medicine and astronomy.
II. THE ABSOLUTE FINITE DIFFERENCES AND A NEW MEASURE OF IRREGULARITY

The differential method, suggested in [1, 7], reduces the research of chaotic properties of the orbits $\bar{X} = (x_i)_{i=1}^{\infty}$ generated by a given one dimensional system to analysis of alternations of the monotone increase and decrease of higher order absolute finite differences

$$\Delta^{(s)} x_i = |\Delta^{(s-1)} x_{i+1} - \Delta^{(s-1)} x_i| \quad (\Delta^{(0)} x_i = x_i; \ i, s = 1, 2, 3, \ldots).$$

In this section we introduce some statements of this approach and briefly describe some computations, explaining the basic notions involved in this work.

Let us have an one dimensional system, generating numerical sequences $\bar{X} = (x_i)_{i=1}^{\infty}$, $0 \leq x_i \leq 1$ from interval $(0, 1)$; we impose no restrictions on the system, and the generating mechanism can be quite arbitrary. Following [1, 7] we consider a special representation of finite orbit $\bar{X}_k = (x_i)_{i=1}^{k}$, which emphasizes its deep differential structure. Indeed, it can be easily obtained, that for $1 \leq s \leq k - 1$ we have

$$\Delta^{(s-1)} x_i = \mu_{k,s-1} + \sum_{p=1}^{i-1} (-1)^{\delta_p^{(s)}} \Delta^{(s)} x_p - \min_{0 \leq i \leq k-s} \left( \sum_{p=1}^{i} (-1)^{\delta_p^{(s)}} \Delta^{(s)} x_p \right) \quad (1)$$

where

$$\delta_p^{(s)} = \begin{cases} 0 & \Delta^{(s)} x_{p+1} \geq \Delta^{(s)} x_p \\ 1 & \Delta^{(s)} x_{p+1} < \Delta^{(s)} x_p \end{cases} \quad \mu_{k,s} = \min\{\Delta^{(s)} x_i : 1 \leq i \leq k-s\},$$

and it is supposed that $\sum_{1}^{0} = 0$. Hence, one can consider that finite orbits with length $k$ are given in some special form:

$$\bar{\xi}_k = (r_1^{(k)}, r_2^{(k)}, \ldots, r_m^{(k)}; \mu_{k,1}, \mu_{k,2}, \ldots, \mu_{k,m}; \rho_{k,1}, \rho_{k,2}, \ldots, \rho_{k,k-m}) \quad (2)$$

where $r_s^{(k)} = 0, \delta_1^{(s)}, \delta_2^{(s)}, \ldots, \delta_{k-s}^{(s)}$ ($1 \leq s \leq m$) and

$$\mu_{k,1}, \mu_{k,2}, \ldots, \mu_{k,m} \quad \text{and} \quad \rho_{k,1}, \rho_{k,2}, \ldots, \rho_{k,k-m} \quad (\rho_{k,i} = \Delta^{(m)} x_i)$$

are some numbers from interval $[0, 1]$. Here $m = m_k$ are some given numbers which tend to $\infty$ as $k \to \infty$ and everywhere below we have chosen $m_k = \lfloor k/2 \rfloor$. It is important to note,
that after applying the recurrent procedure (1) the sequence $X_k$ can be completely recovered by $\zeta_k$. For given orbit $\bar{X}$, we consider the binary sequences

$$\bar{X}^{(n)} = (\delta_1^{(n)}, \delta_2^{(n)}, \ldots, \delta_k^{(n)}, \ldots) \quad (n, k = 1, 2, \ldots).$$

The method from [7] reduces the study of orbits $\bar{X}$ to analysis of some conjugate orbits $\bar{\nu} = (\nu_n)_{n=1}^{\infty}$ which terms are defined as follows:

$$\nu_n = 0, \delta_1^{(n)} \delta_2^{(n)} \delta_3^{(n)} \ldots.$$  

It is shown [1, 7], that provided some rather general conditions, the conjugate orbits are attracted to some Cantor set $\mathcal{A}$ of zero Lebesgue measure.

The main claim is that the periodic response of (weak) periodic perturbation of given stochastic system is localized in $\bar{X}^{(n)}$ and its presence can be detected, when considering these sequences. More exactly, for that purpose it should be studied the asymptotical (as $N \to \infty$) relative volume (i.e. the density in natural series) of the set of all those indeces $i$, for which the changes of binary symbol occur,

$$\delta_i^{(N)} = 1 - \delta_i^{(N)} \quad (1 \leq i \leq N - 1).$$

In order to explain how this statement relates to irregularity notion and how the transition to chaos occurs, let us consider, from such a differential point of view, one of our particular chaotic systems - the logistic map $x \to rx(1-x)$ (it is assumed $0 < x < 1$ and $0 < r < 4$). It is well known, that numerical interval $[0, 4]$ is divided into two subintervals $[0, r_\infty)$ and $[r_\infty, 4]$ ($r_\infty = 3.569 \ldots$), where the orbits $\bar{X}$ of consecutive iterates of this map demonstrate regular periodic and stochastic and aperiodic motion respectively. We observed how frequently, in dependence of the control parameter value $r$, the changes (4) of binary terms from Eq. (3) occur. By this computational way one can find that for each $r \in [0, r_\infty)$ there exists some index $n_r$ such that for all $N \geq n_r$ the finite sequence $\bar{X}_N^{(N)}$ contains the series with the same binary symbol having the lengths tending to $\infty$ as $N \to \infty$. However, when $r$ increasingly approaches to $r_\infty$, these lengths are decreased and for $r = r_\infty$ they become upper bounded.
for all $N$. It can be proved [7, 17], that if they are bounded by a number $K \geq 2$ then for Hausdorff dimension of attractor $\mathcal{A}$ we have

$$\dim(\mathcal{A}) \leq 1 - \frac{1}{(4 \ln 2)(K - 1)}.$$  

Let us now introduce, based on such kind of computational experience, the following characteristic of one dimensional systems: for an orbit $\bar{X} = (x_k)_{k=0}^{\infty}$ of given system we define

$$\gamma = \gamma(\bar{X}) = \lim_{N \to \infty} \frac{\gamma(\bar{X}, N)}{N}. \quad (5)$$

Here, $\gamma(\bar{X}, N)$ denotes the total number of those indeces $1 \leq i \leq N - 1$ for each of which equation (4) holds. Further, we consider the comparisons of $\gamma$ with Lyapunov exponent $\lambda$ (see e.g., [12], Ch. 5 and [2], Ch. 7.2.b): if $x_{k+1} = F(x_k)$, then

$$\lambda = \lambda(\bar{X}) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \ln |\frac{dF(x_k)}{dx_k}|. \quad (6)$$

This quantity, along with Kolmogorov-Sinai entropy (that for one dimensional system coincides with $\lambda$), power spectrum and correlation function is one of the most propagated measures of chaosity. As the Lyapunov exponent, $\gamma$ exhibits a weak dependence on initial value $x_0$.

For the processes, which can be reduced [7] to that ones generating binary sequences, we have established (see [17]) the following Billingsley-Eggleston [13] formula type relation: for Hausdorff dimension of attractor $\mathcal{A}$ we have

$$\dim(\mathcal{A}) \leq -H(\gamma)$$

where $\gamma$ is the system response charactersitic defined by Eq. (5), and

$$H(x) = x \log_2 x + (1 - x) \log_2(1 - x) \quad (0 < x < 1)$$

is the Shannon entropy function.

If $\bar{X}$ is either constant or periodic, then we obviously have
\[ \gamma(X, N) = \gamma N + O(1) \quad (N \to \infty) \quad (7) \]

and \(0 \leq \gamma \leq 1\) is rational. According to next theorem (see [7, 17]) this also represents a sufficient condition of regularity in sense of definition from [1] (given for theoretical neuron spike trains).

**Theorem 1** For the sequence \(X = \{\{\alpha n\}\}_{n=1}^{\infty}\) of fractional parts, where \(0 < \alpha < 1\) is irrational, the next statements are true: (a) the conjugate to \(X\) orbit \(\nu = (\nu_n)_{n=1}^{\infty}\) is a periodic sequence; (b) if entire part of \(1/\alpha\) is of the form \([1/\alpha] = 2^p - 1\) \((p \geq 1)\) then \(\nu_n \equiv 0\) for all enough large indices \(n\).

This implies, that relation (7) holds for sequences of fractional parts as well. It is well known [2, 16], that through use of some canonical transformation, accepted in classical mechanics, an arbitrary integrable Hamilton system in fact is reduced to some billiard system in cube. In its turn, the billiards boundary behavior, due to results from [14] (see also [7, 15]), is reduced to countable set of sequences of fractional parts. It should be also noticed that the Lyapunov exponent of every integrable system is equal to zero [2, Ch. 5.3]. In this respect, Theorem 1 and results from [14], imply that every integrable Hamilton system appears to be a regular system in some generalized sense of above mentioned definition from [1] (see [17] for rigorous formulations).

We note especially, that we are more interested not in numerical value of quantity \(\gamma\) – our main interest is focused on the value of its discrepancy when applying to given system a weak periodic perturbation. The computational analysis, partially presented in next section, shows that \(\gamma\) possesses the following basic properties:

(A) For the most irregular systems \(\gamma\) is positive;

(B) When applying to irregular system any small periodic perturbation, \(\gamma\) is increased;

(C) Different systems have different rate of increase of \(\gamma\).

The second point of Theorem 1 implies that there exist aperiodic (but integrable) systems for which \(\gamma\) is zero. The tent and logistic maps, after applying a stimulation with a small
intensity $10^{-4}$ demonstrate an increase of $\gamma$ up to 25% and 30% respectively, while for $\theta$-ordinate of standard map the increase rate can be more than 45%.

### III. COMPUTATIONAL STUDY OF $\gamma$-CHARACTERISTIC

The introduced above response coefficient $\gamma$ reflects the asymptotical measure of irregularity of a given system’s orbits in respect of changes in monotony of finite-differences, taken from the orbit. In the context of remarks from [1] on neural activity, it can be said that $\gamma$ also expresses the degree of ability of given system “to feel” the stimulation and to respond on it as the living matter does. We study the response properties of three parametric stochastic systems — the tent map $F_T$:

$$F_T^{(t)}(x) = t(1 - 2\left|\frac{1}{2} - x\right|) = \begin{cases} 2tx, & 2x \leq 1 \\ 2t - 2tx, & 2x > 1 \end{cases} \quad (0 < x < 1; \ 0 < t \leq 1),$$

the logistic map $F_L$:

$$F_L^{(r)}(x) = rx(1 - x) \quad (0 < x < 1; \ 0 < r \leq 4),$$

and the Poincare $\theta$-displacement $\bar{X} = (\theta_n)_{n=1}^\infty$ of Chirikov’s standard map $F_S^{(K)} = F_S^{(K)}(I, \theta)$:

$$I_{n+1} = I_n + K \sin \theta_n \quad (0 < I, \theta < 2\pi; \ K > 0)$$

$$\theta_{n+1} = \theta_n + I_{n+1} \mod (2\pi).$$

Here $t$, $r$ and $K$ are control parameters. It is well known that for $0 < t < 1/2$ and $0 < r < r_\infty$ respectively, the tent and logistic maps iterates has regular behavior while in rest part of parameters they mostly exhibit irregular chaotic motion. The $K > 0$ in map $F_S$, so-called stochasticity coefficient, parametrizes transition from local chaos ($K \approx 0$) to global ”stochastic sea” ($K \approx 1$) [2, 4].

We have been considering also actual neuron spike trains, obtained from electrophysiological recordings [18]. However, the data appear to be too short (note that the work [18] followed other aims) in order to determine the values of $\gamma$ with a sufficient accuracy.
Let us now describe the results obtained and to compare the $\gamma$-characteristic and Lyapunov exponent $\lambda$. We note, that (see, e.g., [2, 12, 16])

\[ \lambda_T(t) = \log_2 2t \quad (0 < t \leq 1), \quad \lambda_S(K) = \ln \frac{K}{2} \quad (K \approx 6) \]

while $\lambda_L(r)$ has a complex behavior on numerical interval $3 < r < 4$; here, the second relation where $\lambda_S$ denotes the maximal Lyapunov exponent of standard map, has been obtained by Chirikov for large values of $K$ (see also [12], Ch.5). After computations we found that different systems may have different values of $\gamma$. The computations show (see Fig. 2) that for logistic map $F_L$ and tent map $F_T$ the relation

\[ \gamma(r) = \alpha(r)\lambda^+(r) \quad (\lambda^+ = \max\{0, \lambda\}) \]

holds for all values of control parameter $r$. Here, $0 < \alpha(r) < \infty$ is some continuous function, which zeroes can be situated only in zero points of $\lambda^+(r)$. Indeed, one can see (Fig. 2), that our new measure $0 \leq \gamma \leq 1$ always reaches its local minimal values in small intervals, containing zero points of $\lambda^+$.

More or less exact (with preciseness about $10^{-3}$) computation of $\gamma$ needs quite large number of iterates of given map (tens of thousands). On the other hand, the new measure has that important advantage, that is the simplicity of its computation. It can be easily implemented just over the data $\bar{X}$, without referring to process generation law. In contrary, this cannot be said about Lyapunov exponent for which, in order to get an approximate to its numerical value, it is usually required either a given system’s explicit analytic form (cp. Eq. (6)) or rather complex theoretical constructions ([2], Ch. 5.3). By this reason, as for computation of $\gamma$ we need only the corresponding time series to be available, the $\gamma$ is better adapted for research of various applied nonlinear problems.

We were studying the influence of simplest periodic perturbation on the numerical value of $\gamma$. Namely, let $(s_n)_{n=1}^\infty$ be a periodic sequence of the form

\[ s_n = \begin{cases} 
\epsilon & n/\tau \text{ is integer} \\
0 & n/\tau \text{ is fractional},
\end{cases} \]
i.e. we let $s_n = \epsilon$ for the numbers $n$ of the form $n = \tau, 2\tau, 3\tau, \ldots$ and $s_n = 0$ elsewhere in natural series. Here, $0 < \epsilon < 1$ and natural $\tau \geq 2$ are pre-given. We call such a sequence $(s_n)_{n=1}^\infty$ the perturbation (or stimulation) with intensity $\epsilon$ and period $\tau$. We have been studying the additively perturbed systems

$$\bar{X}_{\epsilon,\tau} = (x'_n)_{n=1}^\infty \text{ where } x'_{n+1} = F(x'_n) + s_n,$$

where $F$ denotes one of three mentioned systems. Such a perturbed system where $F$ is logistic map but assuming that $s_n$ is white noise, was studied earlier in series of works ([19]; see details in [16], Ch. 3). In this work we deal with an inverse statement, considering $F$ as a (deterministic) noise source and $s_n$ as a regular signal. For those actual systems, for which the analytic shape of generating law $F$ remains unknown (for instance, earthquake time series or neuron spike trains; an explanation of quite complex neuron stimulation mechanism can be found in [20]), one can consider $\bar{X}_{\epsilon,\tau}$ as the stimulated system.

Further, for mentioned systems $F$ we have been studying by computational way the properties of the response coefficient

$$\gamma(\epsilon, \tau) = \gamma(\bar{X}_{\epsilon,\tau}).$$

First, for different values of intensity $\epsilon$ we have considered the problem: how large can be the value of function $\gamma(\epsilon, \tau)$ for a given intensity $\epsilon$. Particularly, whether for given level of intensity $\epsilon$ there exist such values of period $\tau$, when the response $\gamma$ is close to its possible maximal value 1? The result obtained, which demonstrate the Figs. 3 and 4, can be formulated in the following form:

for given $\epsilon$ the function $\gamma(\epsilon, \tau)$ has the self-affine structure;

for given $\epsilon$ those values of $\tau$, where the function $\gamma(\epsilon, \tau)$ reaches its maximal (in respect of $\tau$) value, are spreaded on the whole natural series and possess a positive density in this series;

the maximal (in respect of period $\tau$) response depends on stimulation intensity $\epsilon$ in nonpredictable way.
The second statement implies an important conclusion, that for a given $\epsilon$ the "maximal" period can be found in natural series with a positive probability. The Fig. 3 presents the graph of function $\gamma(\epsilon, \tau)$ for perturbed systems $F$ with applied stimulations of intensity $\epsilon = 0.0001$ and periods $\tau = 2, 3, \ldots, 100$, computed with a supercomputer (we used Silicon-Graphics); the control parameter values are: $r = 0.7$ for tent map, $r = 3.7$ for logistic map, $K = 0.6$ for standard map. Comparing graphs on Figs. 1 and 3, one can see that the value of $\gamma$, which for these three non-perturbated systems are approximately equal $0.49$, $0.62$ and $0.35$ (see Fig. 1), under stimulation is increased (for different periods $\tau$) up to $0.62$, $0.80$, and $0.50$ respectively. In this connection, let us note the following: if to apply to perturbed system $F$ another periodic stimulation with a small intensity, it should result a similar increase of $\gamma$ (of already once perturbated system). If to continue this process, it seems possible to construct for a given stochastic deterministic system a (multiperiodic) stimulation of any small positive intensity, which will increase the $\gamma$ up to its possible maximal value $1$.

We have also studied the maximal coefficient

$$\gamma(\epsilon) = \max_{2 \leq \tau < \infty} \gamma(\epsilon, \tau)$$

that gives the dependence of maximal values of response on stimulation intensity $\epsilon$. This function also possesses the self-affine shape and it is probably possible to study the corresponding fractal characteristics. On the other hand, one can see that this function has certain general tendency to monotone decrease with growth of $\epsilon$. This peculiarity can be emphasized in the following way, accepted in function theory and classical mechanics: we consider the averaged function $\mu_s$,

$$\mu_s(\epsilon) = \frac{1}{s} \int_\epsilon^{\epsilon+s} \gamma(t)\delta_s(\epsilon, t)dt$$

where

$$\delta_s(x, y) = \begin{cases} 1 & |x - y| < s \\ 0 & |x - y| \geq s \end{cases}$$

and $s > 0$ is some number. Computations show that for enough small $s > 0$ the $\mu_s(\epsilon)$ is
a monotone decreasing function. This means, that the maximal (averaged) response of the (above considered) nonlinear systems is found in the area of small intensities.
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V. FIGURE CAPTIONS

Fig. 1. The graphs of $\gamma$-characteristic for three different systems $F$, constructed through relation (5), where is taken $N = 30000$ for tent and logistic maps, and $N = 40000$ for standard map. The value of $\gamma$ has been computed for 50 values of control parameters $t$, $r$ and $K$ with the step 0.01 and 0.02 respectively.

Fig. 2. The graphs of $\gamma$-characteristic (solid line) and Lyapunov exponent (dotted line) for logistic map with initial value $x_0 = 0.55$. (a) Computations made for $N = 50000$ and values of control parameter, starting from $r = 3.56$ till $r = 4$ with the constant step is equal 0.005. (b) Computations made for $N = 50000$, starting from $r = 3.7$ till $r = 3.9$ with the step 0.002. (c) The same computations as in (b) with step 0.001.

Fig. 3. The graphs of function $\gamma(\epsilon, \tau)$ for three different systems. It is taken $\epsilon = 0.0001$ and computations made by Eq. (5) and for the same $N$ as in Fig. 1. The systems are: the tent map's iterates for $t = 0.7$ and $x_0 = 0.17$, the logistic map for $r = 3.7$ and $x_0 = 0.317$, and the standard map for $K = 0.6$ and $I_0 = 0.5$, $\theta_0 = 0.2$. 
