ARTIN’S CRITERIA FOR ALGEBRAICITY REVISITED

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ABSTRACT. Using notions of homogeneity we give new proofs of M. Artin’s algebraicity criteria for functors and groupoids. Our methods give a more general result, unifying Artin’s two theorems and clarifying their differences.

INTRODUCTION

Classically, moduli spaces in algebraic geometry are constructed using either projective methods or by forming suitable quotients. In his reshaping of the foundations of algebraic geometry half a century ago, Grothendieck shifted focus to the functor of points and the central question became whether certain functors are representable. Early on, he developed formal geometry and deformation theory, with the intent of using these as the main tools for proving representability. Grothendieck’s proof of the existence of Hilbert and Picard schemes, however, is based on projective methods. It was not until ten years later that Artin completed Grothendieck’s vision in a series of landmark papers. In particular, Artin vastly generalized Grothendieck’s existence result and showed that the Hilbert and Picard schemes exist—as algebraic spaces—in great generality. It also became clear that the correct setting was that of algebraic spaces—not schemes—and algebraic stacks.

In his two eminent papers [Art69b, Art74], M. Artin gave precise criteria for algebraicity of functors and stacks. These criteria were later clarified and simplified by B. Conrad and J. de Jong [CJ02], who replaced Artin approximation with Néron–Popescu desingularization, by H. Flenner [Fle81] using Exal, and the first author [Hal12b] using coherent functors. The criterion in [Hal12b] is very streamlined and elegant and suffices—to the best knowledge of the authors—to deal with all present problems. It does not, however, supersede Artin’s criteria as these are weaker. Another conundrum is the fact that Artin gives two different criteria—the first [Art69b, Thm. 5.3] is for functors and the second [Art74, Thm. 5.3] is for stacks—but neither completely generalizes the other.

The purpose of this paper is to use the ideas of Flenner and the first author to give a new criterion that supersedes all present criteria. We also introduce several new ideas that strengthen the criteria and simplify the proofs of [Art69b, Art74, Fle81]. In positive characteristic, we also identify a subtle issue in Artin’s algebraicity criterion for stacks. With the techniques that we develop, this problem is circumvented. We now state our criterion for algebraicity.

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Main Theorem. Let $S$ be an excellent scheme. Then a category $X$, fibered in groupoids over the category of $S$-schemes, $\text{Sch}/S$, is an algebraic stack, locally of finite presentation over $S$, if and only if it satisfies the following conditions.

1. $X$ is a stack over $(\text{Sch}/S)_{\text{fppf}}$.
2. $X$ is limit preserving (Definition 1.1).
3. $X$ is $\text{Art}^\text{triv}$-homogeneous.
4. $X$ is effective (Definition 9.1).
5a. Automorphisms and deformations are bounded (Conditions 6.1(i) and 6.1(ii)).
5b. Automorphisms, deformations and obstructions are constructible (Condition 6.3).
5c. Automorphisms, deformations and obstructions are Zariski-local (Condition 6.5); or $S$ is Jacobson; or $X$ is $\text{DVR}$-homogeneous (Definition 2.11).

Condition 6.3(iii) (resp. 6.5(iii)) on obstructions can be replaced with either Condition 7.3 or 8.3 (resp. either Condition 7.4, or 8.3). Finally, we may replace $1$ and $3$ with

1'. $X$ is a stack over $(\text{Sch}/S)_{\text{Ét}}$.
3'. $X$ is $\text{Art}^\text{insep}$-homogeneous.

If every residue field of $S$ is perfect, e.g., if $S$ is a $\mathbb{Q}$-scheme or of finite type over $\text{Spec}(\mathbb{Z})$, then $3$ and $3'$ are equivalent.

The $\text{Art}^\text{triv}$-homogeneity (resp. $\text{Art}^\text{insep}$-homogeneity) condition is the following Schlessinger–Rim condition: for any diagram of local artinian $S$-schemes of finite type $[\text{Spec} B \leftarrow \text{Spec} A \hookrightarrow \text{Spec} A']$, where $A' \to A$ is surjective and the residue field extension $B/\mathfrak{m}_B \to A/\mathfrak{m}_A$ is trivial (resp. purely inseparable), the natural functor

$$X(\text{Spec}(A' \times_A B)) \to X(\text{Spec } A') \times_{X(\text{Spec } A)} X(\text{Spec } B)$$

is an equivalence of categories.

The perhaps most striking difference to Artin’s conditions is that our homogeneity condition 3 only involves local artinian schemes and that we do not need any conditions on étale localization of deformation and obstruction theories. If $S$ is Jacobson, e.g., of finite type over a field, then we do not even need compatibility with Zariski localization. There is also no condition on compatibility with completions for automorphisms and deformations. We will do a detailed comparison between our conditions and other versions of Artin’s conditions in Section 10.

All existing algebraicity proofs, including ours, consist of the following four steps:

(i) existence of formally versal deformations;
(ii) algebraization of formally versal deformations;
(iii) openness of formal versality; and
(iv) formal versality implies formal smoothness.

Step 1 was eloquently dealt with by Schlessinger [Sch68, Thm. 2.11] for functors and Rim [SGA7, Exp. VI] for groupoids. This step uses conditions 3 and 5a ($\text{Art}^\text{triv}$-homogeneity and boundedness of tangent spaces). Step 2 begins with the effectivization of formally versal deformations using condition 4. One may then algebraize this family using either Artin’s results [Art69a, Art69b] or B. Conrad and J. de Jong’s result [C.02]. In the latter approach, Artin approximation is replaced with Néron–Popescu desingularization and $S$ is only required to be excellent. This step requires condition 2.
The last two steps are more subtle and it is here that \cite{Art69b, Art74, Fle81, Sta06, Hal12b} and our present treatment diverges—both when it comes to the criteria themselves and the techniques employed. We begin with discussing step (iv).

It is readily seen that our criterion is weaker than Artin’s two criteria \cite{Art69b, Art74} except that, in positive characteristic, we need $X$ to be a stack in the fppf topology, or otherwise strengthen (3). This is similar to \cite[Thm. 5.3]{Art69b} where the functor is assumed to be an fppf-sheaf. In [loc. cit.], Artin uses the fppf sheaf condition and a clever descent argument to deduce that formally universal deformations are formally étale \cite[pp. 50–52]{Art69b}, settling step (iv) for functors. This argument relies on the existence of universal deformations and thus does not extend to stacks with infinite or non-reduced stabilizers.

In his second paper \cite{Art74}, Artin only assumes that the groupoid is an étale stack. His proof of step (iv) for groupoids \cite[Prop. 4.2]{Art74}, however, does not treat inseparable extensions. We do not understand how this problem can be overcome without strengthening the criteria and assuming that either (1) the groupoid is a stack in the fppf topology or (3) requiring homogeneity for inseparable extensions. Flenner does not discuss formal smoothness, and in \cite{Hal12b} formal smoothness is obtained by strengthening the homogeneity condition (3).

With a completely different and simple argument, we show that formal versality and formal smoothness are equivalent. The idea is that with homogeneity, rather than semi-homogeneity, we can use the stack condition (1) to obtain homogeneity for artinian rings with arbitrary residue field extensions (Lemma \ref{homogeneityartinian}). This immediately implies that formal versality and formal smoothness are equivalent (Lemma \ref{equivVersForm}) so we accomplish step (iv) without using obstruction theories.

Finally, Step (iii) uses constructibility, boundedness, and Zariski localization of deformations and obstruction theories (Theorem \ref{construction}). In our treatment, localization is only required when passing to non-closed points of finite type. Such points only exist when $S$ is not Jacobson, e.g., if $S$ is the spectrum of a discrete valuation ring. Our proof is very similar to Flenner’s proof. It may appear that Flenner does not need Zariski localization in his criterion, but this is due to the fact that his conditions are expressed in terms of deformation and obstruction sheaves.

As in Flenner’s proof, openness of versality becomes a matter of simple algebra. It comes down to a criterion for the openness of the vanishing locus of half-exact functors (Theorem \ref{openness}) that easily follows from the Ogus–Bergman Nakayama Lemma for half-exact functors (Theorem \ref{Ogus-Bergman}). Flenner proves a stronger statement that implies the Ogus–Bergman result (Remark \ref{Ogus-Bergman}).

At first, it seems that we need more than Art\textsuperscript{triv}-homogeneity to even make sense of conditions (5a)–(5c). This will turn out to not be the case. Using steps (ii) and (iv), we prove that conditions (1)–(4) guarantee that we have homogeneity for arbitrary integral morphisms (Lemma \ref{homogeneity}). It follows that $\text{Aut}_{X/S}(T, -)$, $\text{Def}_{X/S}(T, -)$ and $\text{Obs}_{X/S}(T, -)$ are additive functors.

Outline. In Section \ref{Outline} we recall the notions of homogeneity, limit preservation and extensions from \cite{Hal12b}. We also introduce homogeneity that only involves artinian rings and show that residue field extensions are harmless for stacks in the fppf topology. In Section \ref{Outline2} we then relate formal versality, formal smoothness and vanishing of Exal.
In Section 3 we study additive functors and their vanishing loci. This is applied in Section 4 where we give conditions on $\text{Exal}$ that assure that the locus of formal versality is open. The results are then assembled in Theorem 4.4.

In Section 5 we repeat the definitions of automorphisms, deformations and minimal obstruction theories from [Hal12b]. In Section 6 we give conditions on $\text{Aut}$, $\text{Def}$ and $\text{Obs}$ that imply the corresponding conditions on $\text{Exal}$ needed in Theorem 4.4. In Section 7 we introduce $n$-step obstruction theories. In Section 8 we formulate the conditions on obstructions without using linear obstruction theories, as in [Art69b]. Finally, in Section 9 we prove the Main Theorem. Comparisons with other criteria are given in Section 10.

**Notation.** We follow standard conventions and notation. In particular, we adhere to the notation of [Hal12b]. Recall that if $T$ is a scheme, then a point $t \in |T|$ is of finite type if $\text{Spec}(\kappa(t)) \to T$ is of finite type. Points of finite type are locally closed. A point of a Jacobson scheme is of finite type if and only if it is closed. If $f: X \to Y$ is of finite type and $x \in |X|$ is of finite type, then $f(x) \in |Y|$ is of finite type.

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1. **Homogeneity, Limit Preservation, and Extensions**

In this section, we review the concept of homogeneity—a generalization of Schlessinger’s Conditions that we attribute to J. Wise [Wis11 §2]—in the formalism of [Hal12b §§1–2]. We will also briefly discuss limit preservation and extensions.

Fix a scheme $S$. An $S$-groupoid is a category $X$, together with a functor $a_X: X \to \text{Sch}/S$ that is fibered in groupoids. A 1-morphism of $S$-groupoids $\Phi: (Y, a_Y) \to (Z, a_Z)$ is a functor between categories $Y$ and $Z$ that commutes strictly over $\text{Sch}/S$. We will typically refer to an $S$-groupoid $(X, a_X)$ as “$X$”.

An $X$-scheme is a pair $(T, \sigma_T)$, where $T$ is an $S$-scheme and $\sigma_T: \text{Sch}/T \to X$ is a 1-morphism of $S$-groupoids. A morphism of $X$-schemes $U \to V$ is a morphism of $S$-schemes $f: U \to V$ (which canonically determines a 1-morphism of $S$-groupoids $\text{Sch}/f: \text{Sch}/U \to \text{Sch}/V$) together with a 2-morphism $\alpha: \sigma_U \Rightarrow \sigma_V \circ \text{Sch}/f$. The collection of all $X$-schemes forms a 1-category, which we denote as $\text{Sch}/X$. It is readily seen that $\text{Sch}/X$ is an $S$-groupoid and that there is a natural equivalence of $S$-groupoids $\text{Sch}/X \to X$. For a 1-morphism of $S$-groupoids $\Phi: Y \to Z$ there is an induced functor $\text{Sch}/\Phi: \text{Sch}/Y \to \text{Sch}/Z$.

We will be interested in the following classes of morphisms of $S$-schemes:

- **Nil** – locally nilpotent closed immersions,
- **Cl** – closed immersions,
- **rNil** – morphisms $X \to Y$ such that there exists $(X_0 \to X) \in \text{Nil}$ with the composition $(X_0 \to X \to Y) \in \text{Nil},$
- **rCl** – morphisms $X \to Y$ such that there exists $(X_0 \to X) \in \text{Nil}$ with the composition $(X_0 \to X \to Y) \in \text{Cl},$
- **Artfin** – morphisms between local artinian schemes of finite type over $S$,
- **Artinsep** – $\text{Artfin}$-morphisms with purely inseparable residue field extensions,
- **Arttriv** – $\text{Artfin}$-morphisms with trivial residue field extensions,
- **Fin** – finite morphisms,
\textbf{Int} – integral morphisms,
\textbf{Aff} – affine morphisms.

We certainly have a containment of classes of morphisms of $S$-schemes:

\[
\begin{align*}
&\text{Nil} \subset \cap \text{Cl} \subset \cup r\text{Nil} \subset \cup r\text{Cl} \subset \cup \text{Int} \subset \cup \text{Aff}, \\
&\text{Art}^{\text{triv}} \subset \text{Art}^{\text{insep}} \subset \text{Art}^{\text{fin}}
\end{align*}
\]

Note that for a morphism $X \to Y$ of locally noetherian $S$-schemes, the properties $r\text{Nil}$ and $r\text{Cl}$ simply mean that $X_{\text{red}} \to Y$ is Nil and Cl respectively.

Let $P \subseteq \text{Aff}$ be a class of morphisms. In [Hal12b, §1] the notion of a $P$-homogeneous 1-morphism of $S$-groupoids $\Phi: Y \to Z$ was defined. We say that an $S$-groupoid is $P$-homogeneous if its structure 1-morphism is. We will not recall the definition in full (it is somewhat lengthy), but we will give an explicit description in Lemma 1.2 for $S$-groupoids that are stacks in the Zariski topology. We will also show that for limit preserving Zariski stacks, it is enough to verify $P$-homogeneity for $S$-schemes of finite type.

**Definition 1.1.** Let $X$ be an $S$-groupoid that is a Zariski stack. We say that $X$ is limit preserving if for any inverse system of affine $S$-schemes $\{\text{Spec } A_i\}_{i \in J}$, with limit $\text{Spec } A$, the natural functor:

\[
\lim_{\to j} X(\text{Spec } A_j) \to X(\text{Spec } A)
\]

is an equivalence of categories [Art74, §1].

The definition just given also agrees with the definition in [Hal12b, §3]. When $X$ is an algebraic stack, then $X$ is limit preserving if and only if $X \to S$ is locally of finite presentation [LM18, Prop. 4.15].

**Lemma 1.2.** Let $S$ be a scheme. Consider a class of morphisms $P \subseteq \text{Aff}$ that is local for the Zariski topology. Let $X$ be an $S$-groupoid that is a stack for the Zariski topology. Then the following conditions are equivalent.

1. $X$ is $P$-homogeneous.
2. For any diagram of affine schemes $[\text{Spec } B \leftarrow \text{Spec } A \xrightarrow{i} \text{Spec } A']$, where $i$ is a nilpotent closed immersion and $\text{Spec } A \to \text{Spec } B$ is $P$, the natural functor:

\[
X(\text{Spec } (A' \times_A B)) \to X(\text{Spec } A') \times_{X(\text{Spec } A)} X(\text{Spec } B)
\]

is an equivalence of categories.

If, in addition, $X$ is limit preserving, and $P \in \{\text{Nil, Cl, rNil, rCl, Int, Aff}\}$, then in (2) it suffices to take $\text{Spec } A$, $\text{Spec } A'$, and $\text{Spec } B$ to be locally of finite presentation over $S$. In particular, $\text{Int}$-homogeneity is equivalent to $\text{Fin}$-homogeneity and, if $S$ is locally noetherian, then $r\text{Cl}$-homogeneity is equivalent to the condition (S1') of [Art73, 2.3].

**Proof.** The first part follows from the definitions. To see the second part, assume that $X$ is limit preserving and that $P \in \{\text{Nil, Cl, rNil, rCl, Int, Aff}\}$. As $X$ is a Zariski stack we may assume that $S = \text{Spec } (R)$ is affine. Let $[\text{Spec } B \leftarrow \text{Spec } A \xrightarrow{i} \text{Spec } A']$ be a diagram as in (2) and let $B' = A' \times_A B$. Then, by Proposition A.2 it
can be written as an inverse limit of diagrams \([\text{Spec } B_\lambda \leftarrow \text{Spec } A_\lambda \rightarrow \text{Spec } A'_\lambda]\) of finite presentation over \(S\) and, furthermore, \(\text{Spec } B'\) is the inverse limit of \(\text{Spec } B'_\lambda\), where \(B'_\lambda = A'_\lambda \times_{A_\lambda} B_\lambda\). The result then follows from our assumption that \(X\) is limit preserving. \(\square\)

By [Wis11 Prop. 2.1], any algebraic stack is \(\text{Aff}\)-homogeneous. It is easily verified, as is done in [loc. cit.], that if the stack is not necessarily algebraic but has representable diagonal, then the functor above is at least fully faithful. Moreover, \(r\text{Cl}\)-homogeneity is equivalent to Artin’s semi-homogeneity condition [Art74 2.2(S1a)] for \(X\), its diagonal \(\Delta_X\), and its double diagonal \(\Delta_{\Delta_X}\).

The main computational tool that \(P\)-homogeneity brings is [Hal12b Lem. 1.4], which we now recall.

**Lemma 1.3.** Let \(S\) be a scheme and let \(P \subset \text{Aff}\) be a class of morphisms. Let \(X\) be a \(P\)-homogeneous \(S\)-groupoid. Consider a diagram of \(X\)-schemes \([V \overset{i}{\leftarrow} T \overset{p}{\rightarrow} T']\), where \(i\) is a locally nilpotent closed immersion and \(p\) is \(P\). Then there exists a cocartesian diagram in the category of \(X\)-schemes:

\[
\begin{array}{ccc}
T & \xrightarrow{i} & T' \\
\downarrow p & & \downarrow p' \\
V & \xrightarrow{i'} & V'
\end{array}
\]

This diagram is also cocartesian in the category of \(S\)-schemes, the morphism \(i'\) is a locally nilpotent closed immersion, \(p'\) is affine, and the induced homomorphism of sheaves:

\[
\mathcal{O}_{V'} \rightarrow i'_* \mathcal{O}_V \times_{p'_* \mathcal{O}_T} p'_* \mathcal{O}_{T'}
\]

is an isomorphism. Moreover if \(P \in \{\text{Nil, Cl, } r\text{Nil, } r\text{Cl, } \text{Fin, } \text{Art}^\text{fin}, \text{Art}^\text{insep}, \text{Art}^\text{triv}\}\), then \(p'\) is \(P\).

**Proof.** Everything except the last claim is [Hal12b Lem. 1.4]. The last claim is trivial except for \(P \in \{\text{Nil, Cl, } \text{Fin, } \text{Int}\}\). In these cases, however, it is well-known—see e.g. [Fer03 5.6 (3)]. \(\square\)

**Remark 1.4.** Let \(S\) be a noetherian scheme. If \([\text{Spec } B \leftarrow \text{Spec } A \rightarrow \text{Spec } A']\) is a diagram of schemes of finite type over a scheme \(S\) such that \(\text{Spec } A \rightarrow \text{Spec } B\) is integral (or equivalently finite) and \(\text{Spec } A \rightarrow \text{Spec } A'\) is a locally nilpotent immersion, then \(\text{Spec } (B \times_A A')\) is of finite type over \(S\). This follows from the fact that \(B \times_A A' \subset B \times A'\) is an integral extension [AM69 Prop. 7.8]. On the other hand, if \(\text{Spec } A \rightarrow \text{Spec } B\) is only affine, then \(\text{Spec } (B \times_A A')\) is typically not of finite type over \(S\). For example, if \(B = k[x], A = k[x, x^{-1}]\) and \(A' = k[x, x^{-1}, y]/y^2\), then \(B' = B \otimes_A A' = k[x, y, xy^{-1}, yx^{-2}, \ldots]/(y, xy^{-1}, \ldots)^2\) which is not of finite type over \(S = \text{Spec}(k)\).

Homogeneity supplies an \(S\)-groupoid with a quantity of linear data, which we now recall from [Hal12b §2]. An \(X\)-extension is a square zero closed immersion of \(X\)-schemes \(i: T \hookrightarrow T'\). The collection of \(X\)-extensions forms a category, which we denote as \(\text{Exal}_X\). There is a natural functor \(\text{Exal}_X \rightarrow \text{Sch}/X\) that takes \((i: T \hookrightarrow T')\) to \(T\).

We denote by \(\text{Exal}_X(T)\) the fiber of the category \(\text{Exal}_X\) over the \(X\)-scheme \(T\)—we call these the \(X\)-extensions of \(T\). There is a natural functor

\[
\text{Exal}_X(T)^\circ : Q\text{Coh}(T), \quad (i: T \hookrightarrow T') \mapsto \ker(i^{-1} \mathcal{O}_{T'} \rightarrow \mathcal{O}_T).
\]
We denote by $\text{Exal}_X(T, I)$ the fiber category of $\text{Exal}_X(T)$ over the quasi-coherent $\mathcal{O}_T$-module $I$—we refer to these as the $X$-extensions of $T$ by $I$.

Let $W$ be a scheme and let $J$ be a quasi-coherent $\mathcal{O}_W$-module. We let $W[J]$ denote the $W$-scheme $\text{Spec}_W(\mathcal{O}_W[J])$ with structure morphism $r_{W,J}: W[J] \to W$. If $W$ is an $X$-scheme, we consider $W[J]$ as an $X$-scheme via $r_{W,J}$. The $X$-extension $W \to W[J]$ is thus trivial in the sense that it admits an $X$-retraction.

By [Hal12b, Prop. 2.3], if the $S$-groupoid $X$ is $\text{Nil}$-homogeneous, then the groupoid $\text{Exal}_X(T, I)$ is a Picard category. Denote the set of isomorphism classes of the category $\text{Exal}_X(T, I)$ by $\text{Exal}_X(T, I)$. Thus, we have additive functors

$$\begin{align*}
\text{Der}_X(T, -): \text{QCoh}(T) &\to \text{Ab}, \quad I \mapsto \text{Aut}_{\text{Exal}_X(T, I)}(T[I]) \\
\text{Exal}_X(T, -): \text{QCoh}(T) &\to \text{Ab}, \quad I \mapsto \text{Exal}_X(T, I).
\end{align*}$$

We now record here the following easy consequences of [Hal12b 2.2–2.5 & 3.4].

**Lemma 1.5.** Let $S$ be a scheme, let $X$ be an $S$-groupoid, and let $T$ be an $X$-scheme.

1. Let $I$ be a quasi-coherent $\mathcal{O}_T$-module. Then $\text{Exal}_X(T, I) = 0$ if and only if every $X$-extension $i: T \hookrightarrow T'$ of $T$ by $I$ admits an $X$-retraction.

2. If $X$ is $\text{rNil}$-homogeneous, then the functor $M \mapsto \text{Exal}_X(T, M)$ is half-exact.

3. Suppose that $X$ is $\text{Nil}$-homogeneous and limit preserving. If $T$ is locally of finite presentation over $S$, then the functor $M \mapsto \text{Exal}_X(T, M)$ preserves direct limits.

4. Let $p: U \to T$ be an affine étale morphism and let $N$ be a quasi-coherent $\mathcal{O}_U$-module. Then there is a natural functor $\psi: \text{Exal}_X(T, p, N) \to \text{Exal}_X(U, N)$. If $(i: T \hookrightarrow T') \in \text{Exal}_X(T, p, N)$ with image $(j: U \hookrightarrow U') \in \text{Exal}_X(U, N)$, then there is a cartesian diagram of $X$-schemes

$$
\begin{array}{ccc}
U & \xrightarrow{i} & U' \\
\downarrow p & & \downarrow p' \\
T & \xrightarrow{i} & T',
\end{array}
$$

which is cocartesian as a diagram of $S$-schemes. If $X$ is $\text{Aff}$-homogeneous, then $\psi$ is an equivalence.

Finally, we give conditions that imply $\text{Art}^{\text{fin}}$-homogeneity.

**Lemma 1.6.** Let $S$ be a scheme and let $X$ be an $S$-groupoid that is $\text{Art}^{\text{triv}}$-homogeneous. Assume that one of the following conditions is satisfied.

1. $X$ is a stack in the fppf topology.
2. $X$ is a stack in the étale topology and $\text{Art}^{\text{insep}}$-homogeneous.
3. $S$ is a $\mathbb{Q}$-scheme and $X$ is a stack in the étale topology.

Then $X$ is $\text{Art}^{\text{fin}}$-homogeneous.

**Proof.** We begin by noting that trivially (3) implies (2). Next, let $[\text{Spec} B \leftarrow \text{Spec} A \to \text{Spec} A']$ be a diagram of local artinian $S$-schemes, with $A' \to A$ a surjection of rings with nilpotent kernel, and $B \to A$ finite so that $\text{Spec} A \to \text{Spec} B$ belongs to $\text{Art}^{\text{fin}}$. Let $\text{Spec} B' = \text{Spec}(A' \times_A B)$ be the pushout of this diagram in the category of $S$-schemes. We have to prove that the functor

$$\varphi: X(\text{Spec}(A' \times_A B)) \to X(\text{Spec} A') \times_{X(\text{Spec} A)} X(\text{Spec} B)$$
is an equivalence. Assume that $X$ is $\text{Art}^{\text{triv}}$-homogeneous (resp. $\text{Art}^{\text{insep}}$-homogeneous).

We first show that $\varphi$ is an equivalence when $A$, $A'$ and $B$ are not necessarily local but the residue field extensions of $\text{Spec}(A) \rightarrow \text{Spec}(B)$ are trivial (resp. purely inseparable). As $\text{Spec } B \leftrightarrow \text{Spec } B'$ is bijective, and $X$ is a Zariski stack, we can work locally on $\text{Spec } B'$ and assume that $\text{Spec } B'$ is local. Then $\text{Spec } B$ is also local and if we let $A = \prod_{i=1}^{n} A_i$ and $A' = \prod_{i=1}^{n} A'_i$ be decompositions such that $A' \rightarrow A_i$ factors through $A'_i$, then $B' = (A'_1 \times_{A_1} B) \times_B (A'_2 \times_{A_2} B) \times_B \cdots \times_B (A'_n \times_{A_n} B)$ is an iterated fiber product of local artinian rings. The equivalence of $\varphi$ in the non-local case thus follows from the local case.

If $X$ is a stack in the fppf (resp. étale) topology, then the equivalence of $\varphi$ is a local question in the fppf (resp. étale) topology on $B'$ since fiber products of rings commute with flat base change. As $\text{Spec } B \rightarrow \text{Spec } B'$ is a nilpotent closed immersion, the scheme $\text{Spec } B'$ is local artinian and the residue fields of $B$ and $B'$ coincide. Choose a finite (resp. finite separable) field extension $K/k_B$ such that the residue fields of $k_A \otimes_{k_B} K$ are trivial (resp. purely inseparable) extensions of $K$.

There is then a local artinian ring $\tilde{B}'$ and a finite flat (resp. finite étale) extension $B' \twoheadrightarrow \tilde{B}'$ with $k_{\tilde{B}'} = K$. Let $A = A \otimes_{B'} \tilde{B}'$, $A' = A' \otimes_{B'} \tilde{B}'$ and $\tilde{B} = B \otimes_{B'} \tilde{B}'$.

Then $\tilde{A}$, $\tilde{A}'$, $\tilde{B}$ are artinian rings such that all residue fields equal $K$ (resp. are purely inseparable extensions of $K$). Thus, equivalence of $\varphi$ follows from the case treated above.

\[ \square \]

2. Formal versality and formal smoothness

In this section we address a subtle point about the relationship between formal versality and formal smoothness. To be precise, we desire sufficient conditions for a family, formally versal at all closed points, to be formally smooth. In the algebraicity criterion for functors [Art69a, Thm. 5.3] a precise statement in this form is not present, but is addressed in [op. cit., Lem. 5.4]. In the algebraicity criterion for groupoids [Art74, Thm. 5.3] the relevant result is precisely stated in [op. cit., Prop. 4.2]. We do not, however, understand the proof.

In the notation of [loc. cit.], to verify formal smoothness, the residue fields of $A$ are not fixed. But the proof of [loc. cit.] relies on [op. cit., Thm. 3.3], which requires that the residue field of $A$ is equal to the residue field of $R$. If the residue field extension is separable, then it is possible to conclude using [op. cit., Prop. 4.3], which uses étale localization of obstruction theories. We do not know how to complete the argument if the residue field extension is inseparable. The essential problem is the verification that formal versality is smooth-local.

We also wish to point out that, in [loc. cit.], the techniques of Artin approximation are used via [op. cit., Prop. 3.3]. In this section we demonstrate that excellence (or related) assumptions are irrelevant with our formulation.

We begin this section with recalling, and refining, some results of [Hal12b, §4].

**Definition 2.1.** Let $S$ be a scheme, let $X$ be an $S$-groupoid, and let $T$ be an $X$-scheme. Consider the following lifting problem: given a square zero closed immersion of $X$-schemes $Z_0 \hookrightarrow Z$ fitting into a commutative diagram of $X$-schemes:

\[
\begin{array}{ccc}
Z_0 & \rightarrow & T \\
\downarrow & & \downarrow \\
Z & \rightarrow & X.
\end{array}
\]
The $X$-scheme $T$ is:

- **formally smooth** – if the lifting problem can always be solved étale-locally on $Z$;
- **formally smooth at $t \in |T|$** – if the lifting problem can always be solved whenever the $X$-scheme $Z$ is local artinian, with closed point $z$, such that $g(z) = t$, and the field extension $\kappa(t) \subset \kappa(z)$ is finite;
- **formally versal at $t \in |T|$** – if the lifting problem can always be solved whenever the $X$-scheme $Z$ is local artinian, with closed point $z$, such that $g(z) = t$ and $\kappa(t) \cong \kappa(z)$.

We certainly have the following implications:

\[
\text{formally smooth} \implies \text{formally smooth at all } t \in |T| \implies \text{formally versal at all } t \in |T|.
\]

It is readily observed that formal smoothness is smooth-local on the source. Without stronger assumptions, it is not obvious to the authors that formal versality is smooth-local on the source. Similarly, formal smoothness at $t$ and formal versality at $t$ are not obviously equivalent. We will see, however, that these subtleties vanish whenever the $S$-groupoid is $\text{Art}^{\text{fin}}$-homogeneous. For formal versality and formal smoothness at a point, it is sufficient that liftings exist when $\kappa(z) \cong g_*(\ker(\mathcal{O}_Z \to \mathcal{O}_{Z_t}))$.

**Lemma 2.2.** Let $S$ be a locally noetherian scheme and let $X$ be a limit preserving $S$-groupoid. Let $T$ be an $X$-scheme that is locally of finite type over $S$ and let $t \in |T|$ be a point such that:

1. $T$ is formally smooth at $t \in |T|$ as an $X$-scheme;
2. the morphism $T \to X$ is representable by algebraic spaces.

Let $W$ be an $X$-scheme. Then the morphism $T \times_X W \to W$ is smooth in a neighborhood of every point over $t$. In particular, if $T$ is formally smooth at every point of finite type, then $T \to X$ is formally smooth.

**Proof.** By a standard limit argument we can assume that $W \to S$ is of finite type. It is then enough to verify that $T \times_X W \to W$ is smooth at closed points in the fiber of $t$ and this follows from [EGA IV.17.14.2]. The last statement follows from the fact that any closed point of $T \times_X W$ maps to a point of finite type of $T$. \qed

There is a tight connection between formal smoothness (resp. formal versality) and $X$-extensions in the affine setting. Most of the next result was proved in [Hal12b Lem. 4.3], which utilized arguments similar to those of [Ele81 Satz 3.2].

**Lemma 2.3.** Let $S$ be a scheme, let $X$ be an $S$-groupoid, and let $T$ be an affine $X$-scheme. Let $t \in |T|$ be a point. Consider the following conditions.

1. The $X$-scheme $T$ is formally smooth at $t$.
2. The $X$-scheme $T$ is formally versal at $t$.
3. $\text{Exal}_X(T, \kappa(t)) = 0$.

Then $1 \implies 2$ and if $X$ is $\text{Art}^{\text{fin}}$-homogeneous and $t$ is of finite type, then $2 \implies 1$. If $X$ is $\text{Cl}$-homogeneous, $T$ is noetherian and $t$ is a closed point, then $2 \implies 3$. If $X$ is $r\text{Cl}$-homogeneous and $t$ is a closed point, then $3 \implies 2$.

Thus, assuming that an $S$-groupoid $X$ is $r\text{Cl}$-homogeneous, we can reformulate formal versality of an affine $X$-scheme $T$ at a closed point $t \in |T|$ in terms of
the triviality of the abelian group \( \text{Exa}_X(T, \kappa(t)) \). Understanding the set of points \( U \subset |T| \) where \( \text{Exa}_X(T, \kappa(u)) = 0 \) for \( u \in |U| \) will be accomplished in the next section.

**Remark 2.4.** If \( X \) is \( \text{Aff} \)-homogeneous and \( \text{Exa}_X(T, -) \equiv 0 \), then \( T \) is formally smooth \([\text{Hal}12b, \text{Lem. 4.3}]\) but we will not use this. If \( \text{Exa}_X \) commutes with Zariski localization, that is, if for any open immersion of affine schemes \( U \subseteq T \) the canonical map \( \text{Exa}_X(T, M) \otimes_{\mathcal{O}_T} \Gamma(\mathcal{O}_U) \rightarrow \text{Exa}_X(U, M|_U) \) is bijective, then the implications \( 2 \implies 3 \) and \( 3 \implies 2 \) also hold for non-closed points. This is essentially what Flenner proves in \([\text{Fle}81, \text{Satz 3.2}]\) as his \( \text{Ex}(T \rightarrow X, M) \) is the sheafification of the presheaf \( U \mapsto \text{Exa}_X(U, M|_U) \).

**Proof of Lemma 2.3.** The implication \( 1 \implies 2 \) follows from the definition. The implications \( 2 \implies 3 \) and \( 3 \implies 2 \) are proved in \([\text{Hal}12b, \text{Lem. 4.3}]\). The implication \( 2 \implies 1 \) follows from a similar argument: assume that \( T \) is formally versal at \( t \) and let \( Z_0 \hookrightarrow Z \) be a square zero closed immersion of local artinian \( X \)-schemes fitting into a commutative diagram

\[
\begin{array}{ccc}
Z_0 & \rightarrow & T \\
\downarrow & & \downarrow \\
Z & \rightarrow & X,
\end{array}
\]

such that the closed point \( z \in |Z_0| \) is mapped to \( t \in |T| \) and \( \kappa(z)/\kappa(t) \) is a finite extension. Let \( W_0 \) be the image of \( Z_0 \rightarrow \text{Spec}(\mathcal{O}_{T,t}) \). Then \( W_0 \) is a local artinian scheme with residue field \( \kappa(t) \). As \( X \) is \( \text{Art}^{\text{fin}} \)-homogeneous, there is a commutative diagram

\[
\begin{array}{ccc}
Z_0 & \rightarrow & W_0 & \rightarrow & T \\
\downarrow & & \downarrow & & \downarrow \\
Z & \rightarrow & W & \rightarrow & X,
\end{array}
\]

where \( W_0 \hookrightarrow W \) is a square zero closed immersion. As \( W_0 \hookrightarrow W \) is a sequence of closed immersions with kernel isomorphic to \( \kappa(t) \), there is a lift \( W \rightarrow T \) and thus a lift \( Z \rightarrow T \). \( \square \)

Combining the two lemmas above we obtain an analogue of \([\text{Art}74, \text{Prop. 4.2}]\).

**Proposition 2.5.** Let \( S \) be a locally noetherian scheme and let \( X \) be a limit preserving and \( \text{Art}^{\text{fin}} \)-homogeneous \( S \)-groupoid. Let \( T \) be an \( X \)-scheme such that

1. \( T \rightarrow S \) is locally of finite type,
2. \( T \rightarrow X \) is formally versal at all points of finite type, and
3. \( T \rightarrow X \) is representable by algebraic spaces.

Then \( T \rightarrow X \) is formally smooth.

We also obtain the following result showing that formal versality is étale-local under mild hypotheses. This improves \([\text{Art}74, \text{Prop. 4.3}]\), which requires the existence of an obstruction theory that is compatible with étale localization.

**Proposition 2.6.** Let \( S \) be a scheme and let \( X \) be an \( \text{Art}^{\text{triv}} \)-homogeneous \( S \)-groupoid that is a stack in the étale topology. Let \( T \) be an \( X \)-scheme and let \((U, u) \rightarrow (T, t) \) be a pointed étale morphism of \( S \)-schemes. Then formal versality at \( t \in |T| \) implies formal versality at \( u \in |U| \).
Proof. Reasoning as in the proof of Lemma 1.6 we see that $X$ is homogeneous with respect to morphisms of artinian rings with separable residue field extensions. Arguing as in the proof of Lemma 2.3(2) ⇒ (1) we thus see that formal versality at $t \in |T|$ implies formal versality at $u \in |U|$. □

Using Lemma 2.3 one can show that Proposition 2.6 admits a partial converse. Indeed, if $u \in |U|$ and $t \in |T|$ are closed, $X$ is rCl-homogeneous, $U$ and $T$ are affine and noetherian, and $T \rightarrow X$ is representable by algebraic spaces, then formal versality at $u \in |U|$ implies formal versality at $t \in |T|$. This will not be used, however.

**Remark 2.7.** Artin remarks [Art74, 4.9] that to verify the criteria for algebraicity, it is enough to find suitable obstruction theories étale-locally. We do not, however, understand the given arguments as [Art74, Prop. 4.3] uses the existence of a global obstruction theory. Since our Proposition 2.6 does not use obstruction theories, it is enough to find obstruction theories étale-locally on $T$ in the Main Theorem. If one replaces semihomogeneity by homogeneity we can thus confirm [Art74, 4.9].

Next, we give a condition that ensures that if an $X$-scheme $T$ is formally versal at all closed points, then it is formally versal at all points of finite type.

**Condition 2.8 (Zariski localization of extensions).** For any open immersion of affine $X$-schemes $p: U \rightarrow T$, locally of finite type over $S$, and any point $u \in |U|$ of finite type, the natural map:

$$\text{Exal}_X(T, \kappa(u)) \rightarrow \text{Exal}_X(U, \kappa(u))$$

is surjective.

Note that Lemma 1.5(4) implies that Condition 2.8 is satisfied whenever the $S$-groupoid $X$ is Aff-homogeneous. It is also satisfied whenever $S$ is Jacobson.

**Lemma 2.9.** Let $X$ be a Zariski $S$-stack and let $p: U \rightarrow T$ be an open immersion of affine $X$-schemes. If $u \in |U|$ is a point that is closed in $T$, then the natural map

$$\text{Exal}_X(T, \kappa(u)) \rightarrow \text{Exal}_X(U, \kappa(u))$$

is an isomorphism. In particular, if $X$ is a Zariski stack and $S$ is Jacobson, then Condition 2.8 is always satisfied.

**Proof.** We construct an inverse by taking an $X$-extension $U \hookrightarrow U'$ of $U$ by $\kappa(u)$ to the gluing of $U'$ and $T \setminus \kappa(u)$ along $U' \setminus \kappa(u) \cong U \setminus \kappa(u)$. If $S$ is Jacobson and $T \rightarrow S$ is locally of finite type, then $T$ is Jacobson and every point of finite type $u \in |U|$ is closed in $T$ so Condition 2.8 holds. □

We now extend the implication $\Rightarrow \Rightarrow 2.3$ of Lemma 2.3 to points of finite type.

**Proposition 2.10.** Fix a scheme $S$ and an rCl-homogeneous $S$-groupoid $X$ satisfying Condition 2.8 (Zariski localization of extensions). Let $T$ be an affine $X$-scheme, locally of finite type over $S$, and let $t \in |T|$ be a point of finite type. If $\text{Exal}_X(T, \kappa(t)) = 0$ then the $X$-scheme $T$ is formally versal at $t$.

**Proof.** Finite type points are locally closed so there exists an open affine neighborhood $U \subseteq T$ of $t$ such that $t \in |U|$ is closed. By Condition 2.8 we have that $\text{Exal}_X(U, \kappa(t)) = \text{Exal}_X(T, \kappa(t)) = 0$ so the $X$-scheme $U$ is formally versal at $t$ by Lemma 2.8. It then follows, from the definition, that the $X$-scheme $T$ also is formally versal at $t$. □
We conclude this section by showing that DVR-homogeneity implies that formal smoothness is stable under generizations. Recall that a geometric discrete valuation ring is a discrete valuation ring $D$ such that Spec($D$) $\to S$ is essentially of finite type and the residue field is of finite type over $S$ [Art69b, p. 38].

**Definition 2.11.** Let $S$ be an excellent scheme. We say that an $S$-groupoid $X$ is DVR-homogeneous if for any diagram of affine $S$-schemes $[\text{Spec } D \leftarrow \text{Spec } K^i \rightarrow \text{Spec } K']$, where $D$ is a geometric discrete valuation ring with fraction field $K$ and $i$ is a nilpotent closed immersion, the natural functor:

$$X(\text{Spec}(K' \times_K D)) \to X(\text{Spec } K') \times_{X(\text{Spec } K)} X(\text{Spec } D)$$

is an equivalence of categories.

Artin’s condition [4a] of [Art69b, Thm. 3.7] implies DVR-semihomogeneity and Artin’s conditions [5′](b) and [4′](a,b) of [Art69b, Thm. 5.3] imply DVR-homogeneity. The following lemma is a generalization of [Art69b, Lem. 3.10] from functors to categories fibered in groupoids.

**Lemma 2.12.** Let $S$ be an excellent scheme and let $X$ be a limit preserving DVR-homogeneous $S$-groupoid. Let $T$ be an $X$-scheme such that

1. $T \to S$ is locally of finite type,
2. $T \to X$ is representable by algebraic spaces, and
3. $T \to X$ is formally smooth at a point $t \in |T|$ of finite type.

Then $T \to X$ is formally smooth at every generization $t' \in |T|$ of $t$.

**Proof.** Consider a diagram of $X$-schemes

$$\begin{array}{ccc}
Z_0 & \xrightarrow{g} & T \\
\downarrow & & \downarrow \\
Z & \to & X
\end{array}$$

where $Z_0 \hookrightarrow Z$ is a closed immersion of local artinian schemes and the image $t' = g(z_0)$ of the closed point $z_0 \in |Z_0|$ is a generization of $t \in T$ and $\kappa(z_0)/\kappa(t')$ is finite. We have to prove that every such diagram admits a lifting as indicated by the dashed arrow.

As $X$ is limit preserving, we can factor $Z \to X$ as $Z \to W \to X$ where $W$ is an $S$-scheme of finite type. Let $h : T \times_X W \to T$ denote the first projection. The pull-back $T \times_X W \to W$ is smooth at every point of the fiber $h^{-1}(t)$ by Lemma 2.14. Let $T_t$ denote the local scheme Spec($\mathcal{O}_{T,t}$). It is enough to prove that $T \times_X W \to W$ is smooth at every point of $h^{-1}(T_t)$.

Let $y \in |T \times_X W|$ be a point of $h^{-1}(T_t)$. It is enough to prove that $Y = \{y\}$ contains a point at which $T \times_X W \to W$ is smooth. By Chevalley’s theorem, $h(Y)$ contains a constructible subset. Thus, there is a point $w \in h(Y) \cap T_t$ such that the closure $W = \{w\}$ in the local scheme $T_t$ is of dimension 1. By Lemma 2.14 it is enough to show that $T \to X$ is formally smooth at $w$. Thus, consider a diagram

$$\begin{array}{ccc}
\text{Spec}(K') & \xrightarrow{g} & T \\
\downarrow & & \downarrow \\
\text{Spec}(K'') & \to & X
\end{array}$$
of $X$-schemes where $K'' \to K'$ is a surjection of local artinian rings such that $g(\eta) = w$ and $\kappa(\eta)/\kappa(w)$ is finite. Let $D \subseteq K = \kappa(\eta)$ be a geometric DVR dominating $\mathcal{O}_{W,t}$ (which exists since $\mathcal{O}_{W,t}$ is excellent). We may then, using DVR-homogeneity, extend the situation to a diagram

$$
\begin{array}{ccc}
\text{Spec}(K') & \longrightarrow & \text{Spec}(D') \\
\downarrow & & \downarrow \\
\text{Spec}(K'') & \longrightarrow & \text{Spec}(D'') \\
\end{array}
\quad
\begin{array}{c}
\longrightarrow \\
\text{Spec}(T) \\
\end{array}
\quad
\begin{array}{c}
\quad \\
\text{Spec}(T) \\
\end{array}
\quad
\begin{array}{c}
\longrightarrow \\
\text{Spec}(X) \\
\end{array}
$$

where $D' = D \times_K K'$ and $D'' = D \times_K K''$ so that $D' \to D$ and $D' \to D$ have nilpotent kernels. Now, by Lemma \[2.2\] the pullback $T \times_X \text{Spec}(D'') \to \text{Spec}(D'')$ is smooth at the image of $\text{Spec}(D')$ so there is a lifting as indicated by the dashed arrow. Thus $T \to X$ is formally smooth at $w$ and hence also at $t'$. \hfill \square

In Lemma \[9.3\] we will show that, under mild hypotheses, DVR-homogeneity actually implies Aff-homogeneity and thus also Condition 2.8.

3. Vanishing loci for additive functors

Let $T$ be a scheme. In this section we will be interested in additive functors $F: \text{QCoh}(T) \to \text{Ab}$. It is readily seen that the collection of all such functors forms an abelian category, with all limits and colimits computed “pointwise”. For example, given additive functors $F, G: \text{QCoh}(T) \to \text{Ab}$ as well as a natural transformation $\varphi: F \to G$, then $\ker \varphi: \text{QCoh}(T) \to \text{Ab}$ is the functor

$$(\ker \varphi)(M) = \ker(F(M) \xrightarrow{\varphi(M)} G(M)).$$

Next, we set $A = \Gamma(\mathcal{O}_T)$. Note that the natural action of $A$ on the abelian category $\text{QCoh}(T)$ induces for every $M \in \text{QCoh}(T)$ an action of $A$ on the abelian group $F(M)$. Thus we see that the functor $F$ is canonically valued in the category $\text{Mod}(A)$. It will be convenient to introduce the following notation: for a quasi-compact and quasi-separated morphism of schemes $g: W \to T$ and a functor $F: \text{QCoh}(T) \to \text{Ab}$, define $F_W: \text{QCoh}(W) \to \text{Ab}$ to be the functor $F_W(N) = F(g_*N)$. If $F$ is additive (resp. preserves direct limits), then the same is true of $F_W$. The vanishing locus of $F$ is the following subset $\text{Hal12a} \ [6.2]$

$$\forall(F) = \{ t \in |T| : F(M) = 0 \quad \forall M \in \text{QCoh}(T), \text{Supp}(M) \subseteq \text{Spec}(\mathcal{O}_{T,t}) \}$$

$$= \{ t \in |T| : F_{\text{Spec}(\mathcal{O}_{T,t})} \equiv 0 \} \quad \text{(if $T$ is quasi-separated).}$$

The main result of this section, Theorem \[3.3\], which gives a criterion for the set $\forall(F)$ to be Zariski open, is essentially due to H. Flenner \[Ple81\] Lem. 4.1. In \textit{loc. cit.}, for an $S$-groupoid $X$ and an affine $X$-scheme $V$, locally of finite type over $S$, a specific result about the vanishing locus of the functor $M \mapsto \text{Exal}_X(V,M)$ is proved. In \textit{op. cit.}, the standing assumptions are that the $S$-groupoid $X$ is semi-homogeneous, thus the functor $M \mapsto \text{Exal}_X(T,M)$ is only set-valued, which complicates matters. Since we are assuming Nil-homogeneity of $X$, the functor $M \mapsto \text{Exal}_X(T,M)$ takes values in abelian groups. As we will see, this simplifies matters considerably.

We now make the following trivial observation.

\textbf{Lemma 3.1.} Let $T$ be a scheme and let $F: \text{QCoh}(T) \to \text{Ab}$ be an additive functor. Then the subset $\forall(F) \subseteq |T|$ is stable under generization.
By Lemma 3.1, we thus see that the subset $\mathcal{V}(F) \subset |T|$ will be Zariski open if we can determine sufficient conditions on the functor $F$ and the scheme $T$ so that the subset $\mathcal{V}(F)$ is (ind)constructible. We make the following definitions.

**Definition 3.2.** Let $T = \text{Spec}(A)$ be an affine scheme and let $F : \text{QCoh}(T) \to \text{Ab}$ be an additive functor.

- The functor $F$ is **bounded** if the scheme $T$ is noetherian and $F(M)$ is finitely generated for any finitely generated $A$-module $M$.
- The functor $F$ is **weakly bounded** if the scheme $T$ is noetherian and for any integral closed subscheme $i : T_0 \hookrightarrow T$, the $\Gamma(\mathcal{O}_{T_0})$-module $F(i_*\mathcal{O}_{T_0})$ is coherent.
- The functor $F$ is GI (resp. GS, resp. GB) if there exists a dense open subset $U \subset |T|$ such that for all points $u \in |U|$ of finite type, the map $F(\mathcal{O}_T) \otimes_A \kappa(u) \to F(\kappa(u))$ is injective (resp. surjective, resp. bijective).
- The functor $F$ is CI (resp. CS, resp. CB) if for any integral closed subscheme $T_0 \hookrightarrow T$, the functor $F_{T_0}$ is GI (resp. GS, resp. GB).

We can now state the main result of this section.

**Theorem 3.3 (Flenner).** Let $T$ be an affine noetherian scheme and let $F : \text{QCoh}(T) \to \text{Ab}$ be a half-exact, additive, and bounded functor that commutes with direct limits. If the functor $F$ is CS, then the subset $\mathcal{V}(F) \subset |T|$ is Zariski open.

Functors of the above type occur frequently in algebraic geometry.

**Example 3.4.** Let $T$ be an affine noetherian scheme and let $Q \in D_{\text{coh}}^{-}(T)$. Then, for all $i \in \mathbb{Z}$, the functors on quasi-coherent $\mathcal{O}_T$-modules given by $M \mapsto \text{Ext}^i_{\mathcal{O}_T}(Q, M)$ and $M \mapsto \text{Tor}^i_{\mathcal{O}_T}(Q, M)$ are additive, bounded, half-exact, commute with direct limits, and CB.

**Example 3.5.** Let $T$ be an affine noetherian scheme and let $p : X \to T$ be a morphism that is projective and flat. Then the functor $M \mapsto \Gamma(X, p^*M)$ is CB. Indeed, one interpretation of the Cohomology and Base Change Theorem asserts that the functor $M \mapsto \Gamma(X, p^*M)$ is of the form given in Example 3.4.

**Example 3.6.** Let $T$ be an affine noetherian scheme. An additive functor $F : \text{QCoh}(T) \to \text{Ab}$, commuting with direct limits, is coherent [Aus66] if there exists a homomorphism $M \to N$ of coherent $\mathcal{O}_T$-modules such that $F(-) = \text{coker}(\text{Hom}_{\mathcal{O}_T}(N, -) \to \text{Hom}_{\mathcal{O}_T}(M, -))$. It is easily seen that a coherent functor is CB and bounded. Indeed, boundedness is obvious and if $i : T_0 \hookrightarrow T$ is an integral closed subscheme, then $F|_{T_0} = \text{coker}(\text{Hom}_{\mathcal{O}_{T_0}}(i^*N, -) \to \text{Hom}_{\mathcal{O}_{T_0}}(i^*M, -))$ and after passing to a dense open subscheme, we may assume that $i^*N$ and $i^*M$ are flat. Then $F|_{T_0}(-) = \text{coker}((i^*N)^Y \to (i^*M)^Y) \otimes_{\mathcal{O}_{T_0}} (-)$ commutes with all tensor products. It is well-known, and easily seen, that the functors of the previous two examples are coherent.

Conversely, let $F : \text{QCoh}(T) \to \text{Ab}$ be a half-exact bounded additive functor that commutes with direct limits and is CS. Then for every integral closed subscheme $T_0 \hookrightarrow T$, there is an open dense subscheme $U_0 \subset T_0$ such that $F|_{U_0}$ is coherent. In particular, for half-exact bounded additive functors that commute with direct limits, CS implies CB.
Lemma 3.10. Let $A$ be an additive functor. Using Theorem 3.7 we infer that $U_A$ is an isomorphism. In particular, for any $U$-coherent $\Gamma(O)$ is closed under generization and its intersection with an irreducible closed subset $T$.

Proof of Theorem 3.3. By [EGA, IV, 1.10.1], the set $\mathcal{V}(F)$ is open if and only if it is closed under generization and its intersection with an irreducible closed subset $T_0 \subset |T|$ contains a non-empty open subset or is empty. By Lemma 3.1, we have witnessed the stability under generization. Thus it remains to address the latter claim.

Let $T_0 \hookrightarrow T$ be an integral closed subscheme. If $|T_0| \cap \mathcal{V}(F) \neq \emptyset$, then the generic point $\eta \in |T_0|$ belongs to $\mathcal{V}(F)$ (Lemma 3.1), thus $F(\kappa(\eta)) = 0$. Since the functor $F$ is, by assumption, CS, there exists a dense open subset $U_0 \subset |T_0|$ such that, $\forall u \in U_0$ of finite type, the map $F_{T_0}(\mathcal{O}_{T_0}) \otimes_{\Gamma(\mathcal{O}_{T_0})} \kappa(u) \to F(\kappa(u))$ is surjective.

As $\kappa(\eta)$ is a quasi-coherent and flat $\mathcal{O}_{T_0}$-module, the natural map $F_{T_0}(\mathcal{O}_{T_0}) \otimes_{\Gamma(\mathcal{O}_{T_0})} \kappa(\eta) \to F(\kappa(\eta))$ is an isomorphism by Proposition 3.9. But $\eta \in \mathcal{V}(F)$, thus the coherent $\Gamma(\mathcal{O}_{T_0})$-module $F_{T_0}(\mathcal{O}_{T_0})$ is torsion. Hence there is a dense open subset $U_0 \subset |T_0|$ with the property that if $u \in U_0$ is of finite type, then $F(\kappa(u)) = 0$. Using Theorem 3.7 we infer that $U_0 \subset \mathcal{V}(F) \cap |T_0|$.

We record for future reference a useful lemma.

Lemma 3.10. Let $T = \text{Spec}(A)$ be an affine noetherian scheme and let $F : \text{QCoh}(T) \to \text{Ab}$ be an additive functor.
(1) If the functor $F$ is half-exact, then $F$ is bounded if and only if $F$ is weakly bounded.
(2) If the functor $F$ is (weakly) bounded, then any additive sub-quotient functor of $F$ is (weakly) bounded.
(3) If $F$ is GS (resp. CS), then so is any additive quotient functor of $F$.
(4) If $F$ is weakly bounded and CI, then so is any additive subfunctor of $F$.
(5) Consider an exact sequence of additive functors $\mathcal{QCoh}(T) \to \mathbf{Ab}$:
$$
\begin{array}{ccc}
H_1 & \longrightarrow & H_2 \\
\downarrow & & \downarrow \\
H_3 & \longrightarrow & H_4
\end{array}
$$
(a) If $H_1$ and $H_3$ are CS and $H_4$ is CI and weakly bounded, then $H_2$ is CS.
(b) If $H_1$ is CS, $H_2$ and $H_4$ are CI, and $H_4$ is weakly bounded, then $H_3$ is CI.

If the scheme $T$ is reduced, then (1), (5a), and (5b) hold with GI and GS instead of CI and CS.

Proof. For claim (1), note that any coherent $\mathcal{O}_T$-module $M$ admits a finite filtration whose successive quotients are of the form $i_*\mathcal{O}_{T_0}$, where $i: T_0 \hookrightarrow T$ is a closed immersion with $T_0$ integral. Induction on the length of the filtration, combined with the half-exactness of the functor $F$, proves the claim. Claims (2) and (3) are trivial. For (1), it is sufficient to prove the claim about GI and we can assume that $T$ is a disjoint union of integral schemes. Fix an additive subfunctor $K \subset F$, then there is an exact sequence of additive functors: $0 \to K \to F \to H \to 0$. By (2) we see that $H$ is weakly bounded and so $H(\mathcal{O}_T)$ is a finitely generated $A$-module. As $A$ is reduced, generic flatness implies that there is a dense open subset $U \subset |T|$ such that $H(\mathcal{O}_T)_u$ is a flat $A$-module $\forall u \in U$. Thus, for all $u \in |U|$ the sequence:
$$
\begin{array}{ccc}
0 & \longrightarrow & K(\mathcal{O}_T) \otimes_A \kappa(u) \\
& \longrightarrow & F(\mathcal{O}_T) \otimes_A \kappa(u) \\
& \longrightarrow & H(\mathcal{O}_T) \otimes_A \kappa(u) \\
& \longrightarrow & 0
\end{array}
$$
is exact. By shrinking $U$, we may further assume that the map $F(\mathcal{O}_T) \otimes_A \kappa(u) \to F(\kappa(u))$ is injective for all points $u \in |U|$ of finite type. We then conclude that $K$ is GI from the commutative diagram:
$$
\begin{array}{ccc}
K(\mathcal{O}_T) \otimes_A \kappa(u) & \subseteq & F(\mathcal{O}_T) \otimes_A \kappa(u) \\
\downarrow & & \downarrow \\
K(\kappa(u)) & \subseteq & F(\kappa(u)).
\end{array}
$$
Claims (5a) and (5b) follow from a similar argument and the 4-Lemmas.

We conclude this section with a criterion for a functor to be GI (and consequently a criterion for a functor to be CI). This will be of use when we express Artin’s criteria for algebraicity without obstruction theories.

**Proposition 3.11.** Let $T = \text{Spec}(A)$ be an affine and integral noetherian scheme with function field $K$. Let $F: \mathcal{QCoh}(T) \to \mathbf{Ab}$ be an additive functor that commutes with direct limits such that $F(\mathcal{O}_T)$ is a finitely generated $A$-module. Then $F$ is GI if and only if the following condition is satisfied:

(†) for any $f \in A$, any free $A_f$-module $M$, and $\omega \in F(M)$ such that for all non-zero maps $\epsilon: M \to K$ we have $\epsilon_*\omega \neq 0$ in $F(K)$, there exists a dense open subset $V_\omega \subset D(f) \subset |T|$ such that for every non-zero map $\gamma: M \to \kappa(v)$, where $v \in V_\omega$ is of finite type, we have $\gamma_*\omega \neq 0$ in $F(\kappa(v))$. 
Proof. Let $M$ be a free $A_f$-module of finite rank and let $M^\vee = \text{Hom}_{A_f}(M,A_f)$. Then the canonical homomorphism $F(A)_f \otimes_{A_f} M \to F(M)$ is an isomorphism (Proposition 3.9), so that there is a one-to-one correspondence between elements $\omega \in F(M)$ and homomorphisms $\varpi: M^\vee \to F(A)_f$. Moreover, $\varpi$ is injective if and only if $\varpi \otimes_A K: M^\vee \otimes_A K \to F(A) \otimes_A K = F(K)$ is injective and this happens exactly when $\epsilon_\omega \neq 0$ in $F(K)$ for every non-zero map $\epsilon: M \to K$.

Let $t \in [T]$ and let $\delta_t: F(A) \otimes_A \kappa(t) \to F(\kappa(t))$ denote the natural map. Then condition (1) can be reformulated as: for any free $A_f$-module $M$ of finite rank and any injective homomorphism $\varpi: M^\vee \to F(A)_f$, there exists a dense open subset $V_\omega \subset D(f)$ such that $\delta_t \circ (\varpi \otimes_A \kappa(t))$ is injective for all points $t \in V_\omega$ of finite type.

To show that (1) implies that $F$ is GI, choose $f \in A \setminus 0$ such that $F(A)_f$ is free, let $M = F(A)_f$ and let $\omega \in F(M)$ correspond to the inverse of the canonical isomorphism $F(A)_f \to M^\vee$. If (1) holds, then there exists an open subset $V$ such that $\delta_t$ is injective for all $t \in V_\omega$, i.e., $F$ is GI.

Conversely, if $F$ is GI, then there is an open subset $V$ such that $\delta_t$ is injective for all $t \in V$ of finite type. Given a finite free $A_f$-module $M$ and $\omega \in F(M)$, we let $V_\omega = V \cap W$ where $W \subset D(f)$ is an open dense subset over which the cokernel of $\varpi$ is flat. If $\varpi$ is injective, it then follows that $\delta_t \circ (\varpi \otimes_A \kappa(t))$ is injective for all $t \in V_\omega$ of finite type, that is, condition (1) holds.

4. Openness of formal versality

As the title suggests, we now address the openness of the formally versal locus. Let $S$ be a scheme. We isolate the following conditions for a Nil-homogeneous $S$-groupoid $X$.

| Condition 4.1 | (Boundedness of extensions). For any affine $X$-scheme $T$, locally of finite type over $S$, the functor $M \mapsto \text{Exal}_X(T,M)$ is bounded. |
| Condition 4.2 | (Constructibility of extensions). For any affine $X$-scheme $T$, locally of finite type over $S$, the functor $M \mapsto \text{Exal}_X(T,M)$ is CS. |

To see that these conditions are plausible, observe the following.

Lemma 4.3. Let $S$ be a locally noetherian scheme, let $X$ be an algebraic $S$-stack, and let $T$ be an affine $X$-scheme. Suppose that both $X$ and $T$ are locally of finite type over $S$. Then the functors $M \mapsto \text{Der}_X(T,M)$ and $M \mapsto \text{Exal}_X(T,M)$ are bounded and CB.

Proof. By [Ols06] Thm. 1.1] there is a complex $L_{T/X} \in D^-_{\text{coh}}(T)$ such that for all quasi-coherent $O_T$-modules $M$, there are natural isomorphisms $\text{Der}_X(T,M) \cong \text{Ext}^0_{O_T}(L_{T/X},M)$ and $\text{Exal}_X(T,M) \cong \text{Ext}^1_{O_T}(L_{T/X},M)$. The result now follows from a consideration of Example 3.4.

In their current form, Conditions 4.1 and 4.2 are difficult to verify. In [Si] this will be rectified. In any case, we can now prove

Theorem 4.4. Let $S$ be a locally noetherian scheme. Let $X$ be an $S$-groupoid satisfying the following conditions:

1. $X$ is limit preserving,
2. $X$ is rCI-homogeneous,
3. Condition 4.1 (boundedness of extensions),
4. Condition 4.2 (constructibility of extensions), and
(5) Condition [2.8] (Zariski localization of extensions)
Let $T$ be an affine $X$-scheme that is locally of finite type over $S$ and let $t \in |T|$ be a closed point. If $T$ is formally versal at $t \in |T|$, then $T$ is formally versal at every point of finite type in a Zariski open neighborhood of $t$. In particular, if $X$ is also $\text{Art}_{\text{fin}}$-homogeneous and $T \to X$ is representable, then $T$ is formally smooth in a Zariski open neighborhood of $t$.

Proof. By Condition [4.1] and Lemma [1.5] the functor $M \mapsto \text{Exal}_X(T, M)$ is bounded, half-exact, and preserves direct limits. Condition [1.2] now implies that the functor $M \mapsto \text{Exal}_X(T, M)$ satisfies the criteria of Theorem [3.6]. Thus, $\mathcal{V}(\text{Exal}_X(T, -)) \subset |T|$ is a Zariski open subset. By Lemma [2.3] and Theorem [3.7] we have that $t \in \mathcal{V}(\text{Exal}_X(T, -))$. So, there exists an open neighborhood $t \in U \subset |T|$ with $\text{Exal}_X(T(t), \kappa(u)) = 0$ for all $u \in U$. By Proposition [2.10], every point $u \in |U|$ of finite type is formally versal. The last assertion follows from Lemma [2.2].

5. Automorphisms, deformations, and obstructions

In this section, we introduce the necessary deformation-theoretic framework that makes it possible to verify Conditions [2.8], [4.1], and [1.2]. To do this, we recall the formulation of deformations and obstructions given in [Hal12b], §6.

Let $S$ be a scheme and let $\Phi: Y \to Z$ be a 1-morphism of $S$-groupoids. Define the category $\text{Def}_\Phi$ to have objects the triples $(T, J, \eta)$, where $T$ is a $Y$-scheme, $J$ is a quasi-coherent $\mathcal{O}_T$-module, and $\eta$ is a $Y$-scheme structure on the trivial $Z$-extension of $T$ by $J$. Graphically, it is the category of completions of the following diagram:

$$
\begin{array}{ccc}
T & \xrightarrow{\eta} & Y \\
\downarrow & & \downarrow \Phi \\
T[J] & \xrightarrow{\emptyset} & Z.
\end{array}
$$

There is a natural functor $\text{Def}_\Phi \to \text{Sch}/Y$ taking $(T, J, \eta)$ to $T$ and we denote the fiber of this functor over the $Y$-scheme $T$ by $\text{Def}_\Phi(T)$. There is also a functor $\text{Def}_\Phi(T) \to \text{QCoh}(T)$ taking $(J, \eta)$ to $J$. We denote the fiber of this functor over a quasi-coherent $\mathcal{O}_T$-module $J$ as $\text{Def}_\Phi(T, J)$. Note that this category is naturally pointed by the trivial $Y$-extension of $T$ by $J$. If the 1-morphism $\Phi$ is fibered in setoids, then the category $\text{Def}_\Phi(T, J)$ is discrete. By [op. cit., Prop. 8.3], if $Y$ and $Z$ are $\text{Nil}$-homogeneous, then the groupoid $\text{Def}_\Phi(T, J)$ is a Picard category. Denote the set of isomorphism classes of $\text{Def}_\Phi(B, J)$ by $\text{Def}_\Phi(B, J)$. Thus we obtain $\Gamma(T, \mathcal{O}_T)$-linear functors:

$$
\begin{align*}
\text{Def}_\Phi(T, -): & \text{QCoh}(T) \to \text{Ab}, & J & \mapsto \text{Def}_\Phi(T, J) \\
\text{Aut}_\Phi(T, -): & \text{QCoh}(T) \to \text{Ab}, & J & \mapsto \text{Aut}_{\text{Def}_\Phi(T, J)}(T[J]).
\end{align*}
$$

The Lemma that follows is an easy consequence of [Hal12b], Lem. 6.2.

Lemma 5.1. Let $S$ be a scheme and let $\Phi: Y \to Z$ be a 1-morphism of $\text{Cl}$-homogeneous $S$-groupoids. Let $i: W \hookrightarrow T$ be a closed immersion of $Y$-schemes and let $N$ be a quasi-coherent $\mathcal{O}_W$-module. Then the natural maps:

$$
\text{Aut}_\Phi(T, i_*N) \to \text{Aut}_\Phi(W, N) \quad \text{and} \quad \text{Def}_\Phi(T, i_*N) \to \text{Def}_\Phi(W, N),
$$

are isomorphisms.
We recall the exact sequence of [op. cit., Prop. 8.5], which is our fundamental computational tool.

**Proposition 5.2.** Let $S$ be a scheme and let $\Phi: Y \to Z$ be a 1-morphism of $\text{Nil}$-homogeneous $S$-groupoids. Let $T$ be a $Y$-scheme and let $J$ be a quasi-coherent $\mathcal{O}_T$-module. Then there is a natural 6-term exact sequence of abelian groups:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Aut}_\Phi(T, J) & \longrightarrow & \text{Der}_Y(T, J) & \longrightarrow & \text{Der}_Z(T, J) \\
&&& \Phi_*(T, J) & \longrightarrow & \text{Exal}_Y(T, J) & \longrightarrow & \text{Exal}_Z(T, J).
\end{array}
$$

We now define $\text{Obs}_\Phi(T, J) = \text{coker}(\text{Exal}_Y(T, J) \to \text{Exal}_Z(T, J))$ so that we obtain an $\Gamma(T, \mathcal{O}_T)$-linear functor

$$\text{Obs}_\Phi(T, -): \text{Qcoh}(T) \to \text{Ab}, \quad J \mapsto \text{Obs}_\Phi(T, J).$$

This is the *minimal obstruction theory* of $\Phi$ in the sense of Section 7.

Recall that if $\Phi$ is $\text{rCl}$-homogeneous, then $\text{Aut}_\Phi(T, -)$ and $\text{Def}_\Phi(T, -)$ are half-exact [Ha12b, Cor. 6.4]. There is no reason to expect that $\text{Obs}_\Phi(T, -)$ is half-exact, however. We have the following analogue of Lemma 5.1 for obstructions.

**Lemma 5.3.** Let $S$ be a scheme $S$ and let $\Phi: Y \to Z$ be a 1-morphism of $\text{Cl}$-homogeneous $S$-groupoids. Let $i: W \hookrightarrow T$ be a closed immersion of $Y$-schemes and let $N$ be a quasi-coherent $\mathcal{O}_W$-module. Then there is a natural map $\text{Obs}_\Phi(W, N) \to \text{Obs}_\Phi(T, i_*N)$, which is injective and functorial in $N$.

Moreover, if $T$ is noetherian and $\text{Obs}_\Phi(T, i_*N)$ is finitely generated, then there exists an infinitesimal neighborhood $W_n$ of $W$ in $T$, i.e., a factorization of $i$ through a locally nilpotent closed immersion $j: W \hookrightarrow W_n$, such that

$$\text{Obs}_\Phi(W_n, j_*N) \to \text{Obs}_\Phi(T, i_*N)$$

is an isomorphism.

**Proof.** Note that for $\text{Def}$ and $\text{Aut}$ there are always, without any homogeneity, natural maps $\text{Aut}_\Phi(T, i_*N) \to \text{Aut}_\Phi(W, N)$ and $\text{Def}_\Phi(T, i_*N) \to \text{Def}_\Phi(W, N)$ and $\text{Cl}$-homogeneity equips these with natural inverses. For $\text{Obs}$ there is no natural map $\text{Obs}_\Phi(T, i_*N) \to \text{Obs}_\Phi(W, N)$, but if $Y$ and $Z$ are $\text{Cl}$-homogeneous, we have natural maps for $\text{Exal}_Y$ and $\text{Exal}_Z$ in the opposite direction and thus a natural map for $\text{Obs}_\Phi$ as stated in the lemma. That this map is injective follows immediately from the $\text{Cl}$-homogeneity of $\Phi$.

Now, given an obstruction $\omega \in \text{Obs}_\Phi(T, i_*N)$, we can realize it as a $Z$-extension $k: T \hookrightarrow T'$ of $T$ by $i_*N$. The ideal sheaf $k_*i_*N \subset \mathcal{O}_{T'}$ is then annihilated by the ideal sheaf $I$ defining the closed immersion $k \circ i: W \hookrightarrow T'$. Thus, by the Artin–Rees lemma, we have that $(k_*i_*N) \cap I^n = 0$ for some $n$. Let $W'_1$ and $W_1$ be the closed subschemes of $T'$ defined by $I^n$ and $I^n + k_*i_*N$. Then the morphisms in the diagram:

$$
\begin{array}{ccc}
W' & \xrightarrow{j_1} & W_1 \\
\downarrow & & \downarrow \\
W'_1 & \xrightarrow{j} & T
\end{array}
$$

are closed immersions and the square is cartesian and cocartesian in the category of $Z$-schemes (because $Z$ is $\text{Cl}$-homogeneous). Let $\omega_1 = [W_1 \hookrightarrow W'_1] \in \text{Obs}_\Phi(W_1, i_*N)$.
Observing $\Omega(W_1, (j_1)_*N)$ be the obstruction, so that $\omega$ is the image of $\omega_1$ along the natural map given by the first part.

We have thus shown that every element $\omega \in \Omega(W, i_*N)$ is in the image of $\Omega(W_1, (j_1)_*N)$ for some infinitesimal neighborhood $j_1 : W \hookrightarrow W_1$, depending on $\omega$. Since $\Omega(W, i_*N)$ is finitely generated and $\mathcal{O}_W$ is noetherian, it follows that there exists an infinitesimal neighborhood $j : W \hookrightarrow W_n$ such that $\Omega(W_n, j_*N) \rightarrow \Omega(W, i_*N)$ is an isomorphism.

\[\square\]

6. Relative conditions

Let $S$ be a locally noetherian scheme. In this section we introduce a number of conditions for a 1-morphism of Nil-homogeneous $S$-groupoids $\Phi : Y \rightarrow Z$. These are the relative versions of the conditions that appear in (5a), (5b) and (5c) of the Main Theorem. For any of the conditions given in this section, a Nil-homogeneous $S$-groupoid $Y$ is said to have that condition, if the structure 1-morphism $X \rightarrow \text{Sch}/S$ has the condition. These conditions are provided in the relative version so that this paper can be more readily seen to subsume the results of [Sta06]. That these conditions are stable under composition follows from the exact sequence of [Hal12b] Prop. 6.9 and Lemma 3.10. Moreover, we can also bootstrap the diagonal using [Hal12b] Prop. 6.9—the conditions for $\text{Aut}_{X/S}$ and $\Omega_{X/S}$ imply the corresponding conditions for $\Omega_{X/S}$ and $\Omega_{X/S}$.

\begin{center}
\textbf{Condition 6.1} (Boundedness of automorphisms, deformations and obstructions). For every affine $Y$-scheme $T$ that is locally of finite type over $S$ and every integral closed subscheme $i : T_0 \hookrightarrow T$,

(i) the $\Gamma(\mathcal{O}_T)$-module $\text{Aut}_\Phi(T_0, \mathcal{O}_{T_0})$ is coherent;

(ii) the $\Gamma(\mathcal{O}_T)$-module $\text{Def}_\Phi(T_0, \mathcal{O}_{T_0})$ is coherent; and

(iii) the $\Gamma(\mathcal{O}_T)$-module $\Omega(T, i_*\mathcal{O}_{T_0})$ is coherent.
\end{center}

We note that Condition 6.1(iii) often is satisfied for trivial reasons. If, for example, the $S$-groupoid $Z$ satisfies Condition 4.1, which is the case when $Z$ is algebraic, then $\Phi$ satisfies Condition 6.1(iii).

\textbf{Lemma 6.2.} Let $S$ be a locally noetherian scheme and let $\Phi : Y \rightarrow Z$ be a 1-morphism of rCl-homogeneous $S$-groupoids satisfying Condition 6.1(ii). If $Z$ satisfies Condition 4.1 (boundedness of extensions), then so does $Y$.

\textbf{Proof.} Let $T = \text{Spec}(R)$ be an affine $Y$-scheme that is locally of finite type over $S$. By Lemma 1.9.2 the functor $M \mapsto \text{Exal}_Y(T, M)$ is half-exact. Thus, by Lemma 3.10, it is sufficient to prove that for any integral closed subscheme $i : T_0 \hookrightarrow T$ the $R$-module $\text{Exal}_Y(T, i_*\mathcal{O}_{T_0})$ is coherent. Now, by Proposition 5.2 there is an exact sequence:

$\text{Def}_\Phi(T, i_*\mathcal{O}_{T_0}) \longrightarrow \text{Exal}_Y(T, i_*\mathcal{O}_{T_0}) \longrightarrow \text{Exal}_Z(T, i_*\mathcal{O}_{T_0}).$

By Condition 4.1 the $R$-module $\text{Exal}_Z(T, i_*\mathcal{O}_{T_0})$ is coherent. By Lemma 5.1 we also have that $\text{Def}_\Phi(T, i_*\mathcal{O}_{T_0}) \cong \text{Def}_\Phi(T_0, \mathcal{O}_{T_0})$, which is a coherent $\Gamma(\mathcal{O}_{T_0})$-module by Condition 6.1(ii). It now follows from the exact sequence that $\text{Exal}_Y(T, i_*\mathcal{O}_{T_0})$ is a coherent $R$-module.

\[\square\]

Similarly, to expand Condition 4.2 (constructibility of extensions), we introduce the following conditions.
We now move on and address Condition 2.8 (Zariski localization of extensions). For every affine and irreducible \( T \) scheme that is locally of finite type over \( S \), with reduction \( i: T_0 \hookrightarrow T \),

(i) the functor \( \text{Aut}_\Phi(T_0, -) : \text{QCoh}(T_0) \to \text{Ab} \) is GB;

(ii) the functor \( \text{Def}_\Phi(T_0, -) : \text{QCoh}(T_0) \to \text{Ab} \) is GB; and

(iii) the functor \( \text{Obs}_\Phi(T, i_*-): \text{QCoh}(T_0) \to \text{Ab} \) is GI.

**Lemma 6.4.** Let \( S \) be a locally noetherian scheme. Let \( \Phi: Y \to Z \) be a 1-morphism of CI-homogeneous \( S \)-groupoids satisfying Conditions 6.1(iii), 6.3(ii) and 6.3(iii). If \( Z \) satisfies Condition 6.3 (constructibility of automorphisms, deformations and obstructions), then so does \( Y \).

**Proof.** Let \( T \) be an affine \( Y \)-scheme that is locally of finite over \( S \). By Proposition 5.2 there is an exact sequence of additive functors \( \text{QCoh}(T) \to \text{Ab} \):

\[
\text{Def}_\Phi(T, -) \to \text{Exal}_Y(T, -) \to \text{Exal}_Z(T, -) \to \text{Obs}_\Phi(T, -) \to 0.
\]

Let \( i: T_0 \hookrightarrow T \) be an integral closed subscheme. By Lemma 5.1 we have that \( \text{Def}_\Phi(T_0, -) = \text{Def}_\Phi(T, i_*(-)) \). Condition 6.3(iii) gives that \( \text{Def}_\Phi(T_0, -) \) is GS, so the functor \( \text{Def}_\Phi(T, -) \) is CS. Condition 4.3 says that \( \text{Exal}_Z(T, -) \) is CS. The remaining two conditions together with Lemma 5.3 imply that \( \text{Obs}_\Phi(T, -) \) is CI and weakly bounded. In fact, for any integral subscheme \( i: T_0 \hookrightarrow T \), there is an infinitesimal neighborhood \( j_n: T_0 \hookrightarrow T_n \) such that \( \text{Obs}_\Phi(T_n, (j_n)_*\mathcal{O}_{T_0}) \cong \text{Obs}_\Phi(T, i_*\mathcal{O}_{T_0}) \) and \( \text{Obs}_\Phi(T_n, \kappa(t)) : \text{Obs}_\Phi(T, \kappa(t)) \) is injective for all \( t \in |T_0| \). It now follows from Lemma 3.10 [5a] that the functor \( \text{Exal}_Y(T, -) \) is CS.

We now move on and address Condition 2.8 (Zariski localization of extensions).

**Condition 6.5** (Zariski localization of automorphisms, deformations and obstructions). For every affine and irreducible \( Y \)-scheme \( T \) that is locally of finite type over \( S \), with reduction \( T_0 \), such that the generic point \( \eta \in |T| \) is of finite type,

(i) the natural map \( \text{Aut}_\Phi(T_0, \kappa(\eta)) \to \text{Aut}_\Phi(U_0, \kappa(\eta)) \) is bijective;

(ii) the natural map \( \text{Def}_\Phi(T_0, \kappa(\eta)) \to \text{Def}_\Phi(U_0, \kappa(\eta)) \) is bijective; and

(iii) the natural map \( \text{Obs}_\Phi(T, \kappa(\eta)) \to \text{Obs}_\Phi(U, \kappa(\eta)) \) is injective

for every non-empty open subset \( U \subset T \) with reduction \( U_0 \).

Note that Condition 6.5 trivially holds when \( S \) is Jacobson since then \( U = T = \{\eta\} \). The proof of the next result is similar, but easier, than the proof of Lemma 6.3, thus is omitted.

**Lemma 6.6.** Let \( S \) be a locally noetherian scheme. Let \( \Phi: Y \to Z \) be a 1-morphism of CI-homogeneous \( S \)-groupoids satisfying Conditions 6.1(iii), 6.5(ii) and 6.5(iii). If \( Z \) satisfies Condition 2.8 (Zariski localization of extensions), then so does \( Y \).

7. Obstruction theories

As in the previous section, we let \( S \) be a locally noetherian scheme and let \( \Phi: Y \to Z \) be a 1-morphism of \( \text{Nil} \)-homogeneous \( S \)-groupoids. We will expand the conditions on obstructions and obtain conditions that are more readily verifiable. We begin with recalling the definition of an \( n \)-step relative obstruction theory given in [12] Defn. 6.6].
An \( n \)-step relative obstruction theory for \( \Phi \), denoted \( \{ o^l(\cdot, \cdot), O^l(\cdot, \cdot) \}_{l=1}^n \), is for each \( Y \)-scheme \( T \), a sequence of additive functors (the obstruction spaces):

\[
O^l(T, -) : \text{QCoh}(T) \to \text{Ab}, \quad J \mapsto O^l(T, J), \quad l = 1, \ldots, n
\]
as well as natural transformations of functors (the obstruction maps):

\[
o^l_1(T, -) : \text{Exal}_Z(T, -) \Rightarrow O^l_1(T, -)
\]

\[
o^l_l(T, -) : \ker o^{l-1}(T, -) \Rightarrow O^l_l(T, -) \quad \text{for } l = 2, \ldots, n,
\]
such that the natural transformation of functors:

\[
\text{Exal}_X(T, -) \Rightarrow \text{Exal}_Z(T, -)
\]

has image \( \ker o^n(T, -) \). Furthermore, we say that the obstruction theory is

- **(weakly) bounded**, if for any affine \( Y \)-scheme \( T \), locally of finite type over \( S \), the obstruction spaces \( M \mapsto O^l(T, M) \) are (weakly) bounded functors;
- **Zariski- (resp. étale-) functorial** if for any open immersion (resp. étale morphism) of affine \( Y \)-schemes \( q : V \to T \), and \( l = 1, \ldots, n \), there is a natural transformation of functors:

\[
C^l_g : O^l(T, g^*_N) \Rightarrow O^l(V, -),
\]

which for any quasi-coherent \( \mathcal{O}_V \)-modules \( N \), make the following diagrams commute:

\[
\begin{align*}
\text{Exal}_X(T, g^*_N) & \longrightarrow O^l(T, g^*_N) & \ker o^{l-1}(T, g^*_N) & \longrightarrow O^l(T, g^*_N) \\
\text{Exal}_X(V, N) & \longrightarrow O^l(V, N) & \ker o^{l-1}(V, N) & \longrightarrow O^l(V, N).
\end{align*}
\]

Here the leftmost map is the map \( \psi \) of Lemma 1.5 (4). We also require for any open immersion (resp. étale morphism) of affine schemes \( h : W \to V \), an isomorphism of functors:

\[
o^l_{g,h} : C^l_h \circ C^l_g \Rightarrow C^l_{gh}.
\]

**Remark 7.1 (Comparison with Artin’s obstruction theories).** An obstruction theory in the sense of \([\text{Art74}, 2.6]\) is a 1-step bounded obstruction theory “that is functorial in the obvious sense”. We take this to mean étale-functorial in the above sense. Obstruction theories are usually half-exact and functorial for any morphism, but Exal is only contravariantly functorial for étale morphisms so the condition above does not make sense for arbitrary morphisms. On the other hand, for Aff-homogeneous stacks, Exal is covariantly functorial for any morphism, cf. \([\text{Hal12b}, \text{Proof of Cor. 2.5}]\). Also note that the minimal obstruction theory \( \text{Obs}_\Phi \) is étale-functorial.

We have the following simple

**Lemma 7.2.** Let \( S \) be a locally noetherian scheme and let \( \Phi : Y \to Z \) be a 1-morphism of \( \text{Nil} \)-homogeneous \( S \)-groupoids. Let \( \{ o^l, O^l \}_{l=1}^n \) be an \( n \)-step relative obstruction theory for \( \Phi \). Let \( \overline{O}^l(T, M) \subset O^l(T, M) \) be the image of \( o^l(T, M) \) for \( l = 1, \ldots, n \). Then \( \{ o^l, \overline{O}^l \}_{l=1}^n \) is an \( n \)-step relative obstruction theory for \( \Phi \). Moreover,
Definition 8.1. Let $\text{Obs}^{l}(T, -) = \text{Exal}_{Z}(T, -)/\ker o^{l}$ and $\text{Obs}^{0}(T, -) = 0$. Then $\text{Obs}^{n}(T, -) = \text{Obs}_{\text{red}}(T, -)$ and we have exact sequences

$$
0 \longrightarrow \tilde{O}^{l}(T, -) \longrightarrow \text{Obs}^{l}(T, -) \longrightarrow \text{Obs}^{l-1}(T, -) \longrightarrow 0
$$

for $l = 1, 2, \ldots, n$. In particular, if the obstruction theory is (weakly) bounded, then so is the minimal obstruction theory $\text{Obs}_{\text{red}}(T, -)$.

We now introduce variants of Conditions 6.3(iii) and 6.5(iii) (constructibility and Zariski localization of obstructions) in terms of an $n$-step relative obstruction theory.

| Condition 7.3 (Constructibility of obstructions II). | There exists a weakly bounded $n$-step relative obstruction theory for $\Phi$, $\{o^{l}(\cdot, -), O^{l}(\cdot, -)\}_{l=1}^{n}$, such that for every affine irreducible $Y$-scheme $T$ that is locally of finite type over $S$, the obstruction spaces $O^{l}(T, -)|_{T_{\text{red}}} : \text{QCoh}(T_{\text{red}}) \to \text{Ab}$ are GI for $l = 1, \ldots, n$. |
| --- | --- |
| Condition 7.4 (Zariski localization of obstructions II). | There exists a functorial, $n$-step relative obstruction theory for $\Phi$, $\{o^{l}(\cdot, -), O^{l}(\cdot, -)\}_{l=1}^{n}$, such that for every affine irreducible $Y$-scheme $T$ that is locally of finite type over $S$ and whose generic point $\eta \in T$ is of finite type, and every open subscheme $U \subset T$, then the canonical maps $O^{l}(T, \kappa(\eta)) \to O^{l}(U, \kappa(\eta))$ are injective for $l = 1, \ldots, n$. |

Lemma 7.5. Let $S$ be a locally noetherian scheme and let $\Phi : Y \to Z$ be a 1-morphism of $\text{Nil}$-homogeneous $S$-groupoids.

1. (Constructibility) $\Phi$ satisfies Conditions 6.1(iii) and 6.3(iii) (boundedness and constructibility of obstructions) if and only if $\Phi$ satisfies 7.3.
2. (Zariski localization) Conditions 6.5(iii) and 7.4 for $\Phi$ are equivalent.

Proof. If $\Phi$ satisfies Conditions 6.1(iii) and 6.3(iii), then the minimal obstruction theory satisfies 7.3. Conversely, assume that we are given an obstruction theory $O^{l}(\cdot, -)$ as in 7.3. Let $T$ be an affine irreducible $Y$-scheme that is locally of finite type over $S$. Then the subfunctors $O^{l}(T, -)|_{T_{\text{red}}} \subset O^{l}(T, -)|_{T_{\text{red}}}$ of Lemma 7.2 are also GI and weakly bounded by Lemma 6.4. Since $\text{Obs}_{\text{red}}(T, -)$ is an iterated extension of the $O^{l}(T, -)$’s, it follows that $\text{Obs}_{\text{red}}(T, -)|_{T_{\text{red}}}$ is GI and weakly bounded by Lemma 6.10(5b) — thus Conditions 6.3(iii) and 6.1(iii) hold.

If Condition 6.5(iii) holds then the minimal obstruction theory satisfies 7.4. That Condition 7.4 implies Condition 6.5(iii) follows from Lemma 7.2.

8. Conditions on obstructions without an obstruction theory

In this section we give conditions without reference to linear obstruction theories, just as in [Art69b, Thm. 5.3 [5’c]] and [Sta06].

Definition 8.1. By a deformation situation for $\Phi : Y \to Z$, we will mean data $(T_{0} \hookrightarrow T \twoheadrightarrow T', M)$, where $T$ is an irreducible affine $Y$-scheme that is locally of finite type over $S$, where $T_{0} = T_{\text{red}}$ is integral, where $M$ is a quasi-coherent $\mathcal{O}_{T}$-module, and where $T \twoheadrightarrow T'$ is an $Z$-extension of $T$ by $M$. We say that the deformation situation is obstructed if the $Z$-extension $T \hookrightarrow T'$ cannot be lifted to a $Y$-extension $T \hookrightarrow T'$.
Let \( \eta_0 = \text{Spec}(K_0) \) denote the generic point of \( T_0 \), let \( \eta = \text{Spec}(\mathcal{O}_{T_0, \eta_0}) \), and let \( \eta' = \text{Spec}(\mathcal{O}_{T', \eta_0}) \). Thus \( \eta \mapsto \eta' \) is a \( Z \)-extension of \( \eta \) by \( M_\eta = M \otimes_{\mathcal{O}_{T_0}} K_0 \).

**Condition 8.2** (Constructibility of obstructions III). Given a deformation situation such that \( M \) is a free \( \mathcal{O}_{T_0} \)-module and such that for every non-zero map \( \epsilon: M_\eta \to K_0 \), the resulting \( Z \)-extension \( \eta \mapsto \eta' \) of \( \eta \) by \( K_0 \) is obstructed, then there exists a dense open subset \( U_0 \subset |T_0| \) such that for all points \( u \in U_0 \) of finite type, and all non-zero maps \( \gamma: M_0 \to \kappa(u) \), the resulting \( Z \)-extension \( T \mapsto T'_\gamma \) of \( T \) by \( \kappa(u) \) is obstructed.

**Condition 8.3** (Zariski localization of obstructions III). For every affine and irreducible \( Y \)-scheme \( T \) that is locally of finite type over \( S \), such that the generic point \( \eta \in |T| \) is of finite type, if a \( Z \)-extension of \( T \) by \( \kappa(\eta) \) is obstructed, then for every affine open neighborhood \( U \subset T \) of \( \eta \), the induced \( Z \)-extension of \( U \) by \( \kappa(\eta) \) is obstructed.

**Lemma 8.4.** Let \( S \) be a locally noetherian scheme and let \( \Phi: Y \to Z \) be a 1-morphism of \( \text{Nil} \)-homogeneous \( S \)-groupoids.

1. (Constructibility) If obstructions satisfy boundedness and Zariski localization, if \( Z \) is \( \text{Aff} \)-homogeneous and if \( Y \) and \( Z \) are limit preserving, then Conditions 6.3(iii) and 8.3 for \( \Phi \) are equivalent.
2. (Zariski localization) Conditions 6.3(iii) and 8.3 for \( \Phi \) are equivalent.

**Proof.** Condition 8.3 is a straightforward expansion of Condition 6.5(iii).

To see that Conditions 6.3(iii) and 8.3 are equivalent we will use Proposition 3.11 with \( F(-) = \text{Obs}_\Phi(T_0, -)|_{T_0} \). Some care is needed, though, as condition [1] of Proposition 3.11 is not quite equivalent to Condition 8.3.

If condition [1] is satisfied for \( F(-) = \text{Obs}_\Phi(T_0, -)|_{T_0} \), i.e., if \( F \) is GI, then it is easily seen that Condition 8.3 holds for \( T \). Indeed, consider a deformation situation as in Condition 8.3 and let \( \omega \in F(M) \) be the corresponding obstruction. Then \( \epsilon, \omega \in F(K_0) \) is non-zero since its image in \( \text{Obs}_\Phi(T_0, K_0) \) is non-zero. Thus, there is an open dense subset \( U_0 \subset |T_0| \) such that \( \gamma \circ \omega \in F(\kappa(u)) \) is non-zero for all \( u \in U_0 \) of finite type and non-zero maps \( \gamma: M_0 \to \kappa(u) \), that is, Condition 8.3 holds.

Conversely, fix a deformation situation \( (T_0 \mapsto T \mapsto T', M) \) and assume that Condition 8.3 holds for all deformation situations that are restrictions of the fixed one along any open dense subset \( W \subset T \). We will prove that \( F_W(-) = \text{Obs}_\Phi(W_0, -)|_{W_0} \) is GI for sufficiently small \( W \). It then follows from the Zariski localization Condition 6.5(iii) and Lemma 5.3 that \( F(-) \) is GI as well.

There are natural maps \( \text{Exal}_Y(T, -) \to \text{Exal}_Y(W, -) \to \text{Exal}_Y(\eta, -) \) and similarly for \( Z \). Since \( Z \) is \( \text{Aff} \)-homogeneous, the maps \( \text{Exal}_Z(T, -) \to \text{Exal}_Z(W, -) \to \text{Exal}_Z(\eta, -) \) are bijective (Lemma 1.2(iii)), so that the induced maps \( \text{Obs}_\Phi(T, -) \to \text{Obs}_\Phi(W, -) \to \text{Obs}_\Phi(\eta, -) \) are surjective. As \( Y \) and \( Z \) are limit preserving, it follows that for sufficiently small \( W \), the homomorphism \( \text{Obs}_\Phi(W, K_0) \to \text{Obs}_\Phi(\eta, K_0) \) of \( K_0 \)-modules is bijective.

It is now easily verified that condition [1] of Proposition 3.11 holds for \( F_W(-) = \text{Obs}_\Phi(W_0, -)|_{W_0} \). Indeed, let \( f, M \) and \( \omega \) be as in condition [1]. Let \( V = D(f) \) and let \( V \mapsto V' \) be an \( Z \)-extension of \( V \) by \( M \) with obstruction \( \omega \). Since \( \text{Obs}_\Phi(W, K_0) \to \text{Obs}_\Phi(\eta, K_0) \) is injective (even bijective), it follows that Condition 8.3 applies for the extension \( V \mapsto V' \). Thus there is some dense open subset \( U_0 \subset V_0 \) such that
for every non-zero map $\gamma : M \to \kappa(u)$, with $u \in U_0$ of finite type, $\gamma_\ast \omega$ is non-zero in $\text{Obs}_\Phi(V, \kappa(u))$. Then, by the Zariski localization Condition 6.5(iii) it follows that $\gamma_\ast \omega$ is non-zero in $\text{Obs}_\Phi(W, \kappa(u))$ so that condition (†) of Proposition 3.11 holds.

**Remark 8.5.** If $S$ is of finite type over a Dedekind domain as in [Art69b] (or Jacobson), then in Condition 8.2 it is enough to consider closed points $u \in U$. Indeed, in the proof of the lemma above, we are free to pass to open dense subsets and every $S$-scheme of finite type has a dense open subscheme which is Jacobson.

9. Proof of Main Theorem

In this section we prove the main theorem. We begin by giving a precise definition of effectivity.

**Definition 9.1.** Let $X$ be a category fibered in groupoids over the category of $S$-schemes. We say that $X$ is effective if for every local noetherian ring $(B, \mathfrak{m})$, such that $B$ is $\mathfrak{m}$-adically complete, with an $S$-scheme structure $\text{Spec} B \to S$ such that the induced morphism $\text{Spec}(B/\mathfrak{m}) \to S$ is locally of finite type, the natural functor:

$$X(\text{Spec} B) \to \lim_{\leftarrow n} X(\text{Spec}(B/\mathfrak{m}^n))$$

is dense and fully faithful. Here dense means that for every object $(\xi_n)_{n \geq 0}$ in the limit and for every $k \geq 0$, there exists an object $\xi \in X(\text{Spec} B)$ such that its image in $X(\text{Spec}(B/\mathfrak{m}^k))$ is isomorphic to $\xi_k$.

If $X$ is an algebraic stack, then the functor $X(\text{Spec} B) \to \lim_{\leftarrow n} X(\text{Spec}(B/\mathfrak{m}^n))$ is an equivalence of categories—thus every algebraic stack is effective.

We now obtain the following algebraicity criterion for groupoids.

**Proposition 9.2.** Let $S$ be an excellent scheme. Then an $S$-groupoid $X$ is an algebraic $S$-stack that is locally of finite presentation over $S$, if and only if it satisfies the following conditions.

1. $X$ is a stack over $(\text{Sch}/S)_{\text{Et}}$.
2. $X$ is limit preserving.
3. $X$ is $\text{Art}^{\text{insep}}$- and $\text{rCl}$-homogeneous.
4. $X$ is effective.
5. The diagonal morphism $\Delta_{X/S} : X \to X \times_S X$ is representable.
6a. Condition 6.1(ii) (boundedness of deformations).
6b. Condition 4.2 (constructibility of extensions).
6c. Condition 2.5 (Zariski localization of extensions)

**Proof.** Just as in the proof of [Hal12b, Cor. 4.6] conditions (2)–(4)—using only $\text{Art}^{\text{triv}}$-homogeneity—together with Condition 6.1(ii) permit the application of [CJ02, Thm. 1.5]. Thus for any pair $(\text{Spec} k \to S, \xi)$, where $k$ is a field, $x$ is a morphism locally of finite type, and $\xi \in X(x)$, there exists a pointed and affine $X$-scheme $(Q_\xi, q)$ such that $Q_\xi$ is locally of finite type over $S$. There is also an isomorphism of $X$-schemes $\text{Spec} \kappa(q) \cong \text{Spec} k$ and $Q_\xi$ is a formally versal $X$-scheme at the closed point $q$.

As $X$ is $\text{rCl}$-homogeneous and satisfies Condition 6.1(ii), Lemma 6.2 implies that $X$ satisfies Condition 4.3 (boundedness of extensions). By Lemma 1.6, $X$ is
Art$^{\text{fin}}$-homogeneous. Using the Conditions 2.8 (Zariski localization), 4.1 (boundedness of extensions), and 4.2 (constructibility of extensions), together with Art$^{\text{fin}}$-homogeneity, it follows from Theorem 4.4 that we are free to assume—by passing to an affine open neighborhood of $q$—that the $X$-scheme $Q_\xi$ is formally smooth.

The remainder of the proof of [Hal12b, Cor. 4.6] applies without change. □

Before we get to the proof of the Main Theorem we must prove the following Lemma.

**Lemma 9.3.** Let $S$ be a locally noetherian scheme and let $X$ be an $S$-groupoid satisfying the following conditions:

1. $X$ is an étale stack.
2. $X$ is limit preserving.
3. The diagonal morphism $\Delta_{X/S} : X \to X \times_S X$ is representable.
4. Let $k$ be a field, let $x : \text{Spec} k \to S$ be a morphism that is locally of finite type, and let $\xi \in X(x)$. Then there exists a pointed affine $X$-scheme $(T_\xi, t)$ with the following properties.
   a. $T_\xi$ is locally of finite type over $S$.
   b. The point $t \in |T_\xi|$ is closed and the $X$-schemes $\xi$ and $\text{Spec} \kappa(t)$ are isomorphic.
   c. The $X$-scheme $T_\xi$ is formally smooth at $t \in |T_\xi|$ (resp. at every generalization of $t$).

Then $X$ is Int$^\text{-}$homogeneous (resp. Aff$^\text{-}$homogeneous).

**Proof.** Since $X$ is a Zariski stack that is also limit preserving, to show that $X$ is Int$^\text{-}$homogeneous, it suffices, by Lemma 4.2 to prove the following assertion: for any diagram of affine $S$-schemes $\text{Spec} A_2 \leftarrow \text{Spec} A_0 \to \text{Spec} A_1$ that are locally of finite type over $S$, where $A_1 \to A_0$ is surjective with nilpotent kernel and $A_0 \to A_2$ is finite (resp. arbitrary), the canonical map

$X(\text{Spec}(A_2 \times_{A_0} A_1)) \to X(\text{Spec} A_2) \times_{X(\text{Spec} A_0)} X(\text{Spec} A_1)$

is an equivalence. Let $A_3 = A_2 \times_{A_0} A_1$ and for $j = 0, \ldots, 3$ let $W_j = \text{Spec} A_j$. Then we must uniquely complete all commutative diagrams:

$$
\begin{array}{c}
W_1 \\
\downarrow i \\
W_0 \\
\downarrow p \\
\downarrow j \\
W_2 \\
\downarrow \pi \\
W_3 \\
\downarrow \rho \\
X
\end{array}
$$

Since $X$ is an étale stack, the problem of constructing a map $W_3 \to X$ is étale-local on $W_3$. Thus it is sufficient to construct for each point $w_3 \in |W_3|$ a smooth morphism of pointed schemes $(U_3, u_3) \to (W_3, w_3)$, together with a unique map $U_3 \to X$ which is compatible with pulling back the square above by $U_3 \to W_3$.

The morphism $W_2 \to W_3$ is a nilpotent closed immersion, so $w_3$ is in the image of a unique point $w_2$ of $W_2$ and $\kappa(w_2) \cong \kappa(w_3)$. The points of $W_2$ which are of finite type over $S$ are dense, thus the same is true of $W_3$. So, we may assume that the morphism $\text{Spec} \kappa(w_3) \to S$ is locally of finite type.

By condition (4), there exists an affine $X$-scheme $T$ that is locally of finite type over $S$, which is formally smooth at a closed point $t \in |T|$, and the $X$-schemes $\text{Spec} \kappa(t)$ and $\text{Spec} \kappa(w_3)$ are isomorphic. Let $W'_j = W_j \times_X T$ for $j = 0, 1, 2$. 




By (3), the morphism $T \to X$ is representable so, by Lemma 2.2, the pull-back $W'_j \to W_j$ is smooth in a neighborhood of the inverse image of $t$ (resp. the inverse image of $T = \text{Spec}(O_{T,t}))$. Let $W''_j \subset W'_j$ be the smooth locus of $W'_j \to W_j$.

Let $Z_2 = p(W_0' \setminus i^{-1}(W^\text{sm}_1))$ and let $W''_2 = W''_j \setminus \overline{Z_2}$ and $W''_0 = p^{-1}(W''_j)$ and $W''_1 = i(W''_j)$ as open subsets of $W''_j$. Then all points above $t$ belong to the $W''_j$.

Indeed, it is enough to check that $Z_2$ does not contain any points above $t$. But $Z_2$ does not contain any points above $t$ (resp. $T$) and since $p$ is finite, $Z_2$ is closed (resp. every point of $Z_2$ is a specialization of a point in $Z_2$). In particular, we have that $w_2$ is in the image of $W''_1$.

By [Ha12b, Lem. A.4], there is a commutative diagram of $S$-schemes:

$$
\begin{array}{ccc}
W''_0 & \longrightarrow & W''_1 \\
\downarrow & & \downarrow \\
W''_2 & \longrightarrow & W''_3 \\
\downarrow & & \downarrow \\
W_0 & \longrightarrow & W_1 \\
\downarrow & & \downarrow \\
W_2 & \longrightarrow & W_3
\end{array}
$$

where all faces of the cube are cartesian, the top and bottom faces are cocartesian, and the map $W''_3 \to W_3$ is flat. By [Ha12b Lem. A.5], the morphism is $W''_3 \to W_3$ is smooth. Since the top square is cocartesian, and there are compatible maps $W''_j \to T$ for $j \neq 3$, there is a uniquely induced map $W''_3 \to T$. Taking the composition of this map with $T \to X$, we obtain a map $W''_3 \to X$ which is compatible with the data. This map is unique because the diagonal of $X$ is representable. As $W''_3 \to W_3$ is a smooth neighborhood of $w_3$, the result follows. \hfill \Box

We are now ready to prove the Main Theorem.

**Proof of Main Theorem.** Repeating the bootstrapping techniques of the proof of [Ha12b Thm. A], it is sufficient to prove the result in the case where the diagonal 1-morphism $\Delta_{X/S} : X \to X \times_S X$ is representable.

As in the first part of the proof of Proposition 9.2, we see that for every point $x : \text{Spec}(k) \to X$ that is of finite type over $S$, there exists an affine $X$-scheme $(Q_\xi, q)$ such that $Q_\xi$ is locally of finite type over $S$, together with an isomorphism of $X$-schemes $\text{Spec} \ k(q) \cong \text{Spec} \ k$, and $Q_\xi$ is a formally versal $X$-scheme at the closed point $q$. Now since $X$ is $\text{Art}^{\text{fm}}$-homogeneous (Lemma 1.0), it follows that $Q_\xi$ is formally smooth at the closed point $q$ by Lemma 2.3. If $X$ is $\text{DVR}$-homogeneous, then $Q_\xi$ is even formally smooth at every generalization of $q$ by Lemma 2.2. Then, by Lemma 9.3, we see that the $S$-groupoid $X$ is $\text{Int}$-homogeneous (and $\text{Aff}$-homogeneous if $X$ is $\text{DVR}$-homogeneous) and thus also $\text{rCl}$-homogeneous.

So, by Proposition 9.2, it remains to show that the hypotheses of the Theorem guarantee that Conditions 1.2 and 2.8 (constructibility and Zariski localization of extensions) hold for $X$. We saw that if $X$ is $\text{DVR}$-homogeneous, then $X$ is $\text{Aff}$-homogeneous and Condition 2.8 holds by Lemma 1.0. Likewise, if $S$ is Jacobson, then Condition 2.8 holds by Lemma 2.9.

Now, by Lemmata 1.3 and 1.5, we have that Conditions 1.1, 1.2 and 2.8 (boundedness, constructibility and Zariski localization of extensions) hold for $S$. Trivially, Condition 6.1(iii) (boundedness of obstructions) then holds for $X$. By
Lemmas 7.5(2), 8.4(2) and 6.6 we see that the hypothesis (5c) implies Condition 2.3. Similarly, by Lemmas 7.5(1), 8.4(1) and 6.4, the hypothesis (5b) implies Condition 4.2. We may thus apply Proposition 9.2 to conclude that $X$ is an algebraic stack that is locally of finite presentation over $S$. □

10. Comparison with other criteria

In this section we compare our algebraicity criterion with Artin’s criteria [Art69b, Art74], Starr’s criterion [Sta06], the criterion of the first author [Hal12b], the criterion in the stacks project [Stacks], and Flenner’s criterion for openness of versality [Fle81].

10.1. Artin’s algebraicity criterion for functors. In [Art69b, Thm. 5.3] Artin assumes [0′] = 1 (fppf stack), [1′] = 2 (limit preserving) and [2′] = 4 (effectivity). Further [4′](b) + [5′](a) is Nil-homogeneity for irreducible schemes, which implies (3). His [4′](a) + (c) is boundedness, Zariski-localization and constructibility of deformations (Conditions 6.1(ii), 6.5(ii) and 6.3(ii)). His [5′](c) is Condition 8.2 (constructibility of obstructions). Finally, [5′](b) together with [4′](a) and [4′](b) implies DVR-homogeneity and hence (5c). Conditions on automorphisms are of course redundant for functors. Condition [3′](a) is only used to assure that the resulting algebraic space is locally separated (resp. separated) and condition [3′](b) guarantees that it is quasi-separated. If one is willing to accept non quasi-separated algebraic spaces, no separation assumptions are necessary.

10.2. Artin’s algebraicity criterion for stacks. Let us begin with correcting two typos in the statement of [Art74, Thm. 5.3]. In (1) the condition should be that (S1′, 2) holds for $F$, not merely (S1, 2), and in (2) the canonical map should be fully faithful with dense image, not merely faithful with dense image. Otherwise it is not possible to bootstrap and deduce algebraicity of the diagonal.

Artin assumes that $X$ is a stack for the étale topology [op. cit., (1.1)], and that $X$ is limit preserving. He assumes (1) that the Schlessinger conditions (S1′, 2) hold and boundedness of automorphisms. In our terminology, (S1′) is rCl-homogeneity, which implies Arttriv-homogeneity, our (3). The other two conditions are exactly boundedness of automorphisms and deformations (5m). Artin’s condition (2) is our (4) (effectivity). Artin’s condition (3) is étale localization and constructibility of automorphisms, deformations and obstructions, and compatibility with completions for automorphisms and deformations. The constructibility condition is slightly stronger than our (5b) and the étale localization condition implies the much weaker (5c). We do not use compatibility with completions. Finally, Artin’s condition (4) implies that the double diagonal of the stack is quasi-compact and this condition can be omitted if we work with stacks without separation conditions. Thus [Art74, Thm. 5.3] follows from our main theorem, except that Artin only assumes that the groupoid is a stack in the étale topology. This is related to the issue when comparing formal versality to formal smoothness mentioned in the introduction and discussed in the beginning of Section 2.

Remark 10.1. That automorphisms and deformations are sufficiently compatible with completions for Artin’s proof to go through actually follows from the other conditions. In fact, let $A$ be a noetherian local ring with maximal ideal $m$, let
\( T = \text{Spec}(A) \) and let \( T \to X \) be given. Then the injectivity of the comparison map

\[ \varphi: \text{Def}_{X/S}(T, M) \otimes_A \hat{A} \to \lim_{\leftarrow n} \text{Def}_{X/S}(T, M/m^n) \]

for a finitely generated \( A \)-module \( M \) follows from the boundedness of \( \text{Def}_{X/S}(T, -) \), see Remark 3.8. If \( T \to X \) is formally versal, then \( \varphi \) is also surjective. Indeed, from (S1) it follows that \( \text{Der}_S(T, M/m^n) \to \text{Def}_{X/S}(T, M/m^n) \) is surjective for all \( n \), so the composition \( \text{Der}_S(T, M) \otimes_A \hat{A} \simeq \lim_{\leftarrow n} \text{Der}_S(T, M/m^n) \to \lim_{\leftarrow n} \text{Def}_{X/S}(T, M/m^n) \), which factors through \( \varphi \), is surjective.

The variant [Sta06, Prop. 1.1], due to Starr, has the same conditions as [Art74, Thm. 5.3] except that it is phrased in a relative setting. From Section 6, it is clear that our conditions can be composed. The salient point is that with \( r\text{Cl} \)-homogeneity (or even with just (S1), i.e., \( r\text{Cl} \)-semihomogeneity, as in [Fle81]), there is always a linear minimal obstruction theory. There is further an exact sequence relating the minimal obstruction theories for the composition of two morphisms [Hal12b, Prop. 6.9]. Thus [Sta06, Prop. 1.1] also follows from our main theorem.

10.3. The criterion [Hal12b] using coherence. There are two differences between [Hal12b, Thm. A] and our main theorem. The first is that Condition (3) is strengthened to \( \text{Aff} \)-homogeneity. As this includes \( \text{DVR} \)-homogeneity, it becomes redundant. Zariski localization also follows immediately from \( \text{Aff} \)-homogeneity without involving \( \text{DVR} \)-homogeneity, see discussion after Condition 2.8. We thus have the following version of our main theorem.

**Theorem 10.2.** Let \( S \) be an excellent scheme. Then a category \( X \) that is fibered in groupoids over the category of \( S \)-schemes, \( \text{Sch}/S \), is an algebraic stack that is locally of finite presentation over \( S \), if and only if it satisfies the following conditions.

\begin{enumerate}
  \item (1') \( X \) is a stack over \( (\text{Sch}/S)_{\text{fri}} \).
  \item (2) \( X \) is limit preserving.
  \item (3'') \( X \) is \( \text{Aff} \)-homogeneous.
  \item (4) \( X \) is effective.
  \item (5a) Automorphisms and deformations are bounded (Conditions 6.1(i) and 6.1(ii)).
  \item (5b) Automorphisms, deformations and obstructions are constructible (Conditions 6.3(i) and 6.3(ii) and either Condition 6.3(iii), 7.3 or 8.2).
\end{enumerate}

The second difference is that (5a) and (5b) are replaced with the condition that \( \text{Aut}_{X/S}(T, -) \), \( \text{Def}_{X/S}(T, -) \), and \( \text{Obs}_{X/S}(T, -) \) are coherent functors. This implies that the functors are bounded and \( CB \) (Example 3.6), hence satisfy (5a) and (5b).

10.4. The criterion in the Stacks project. In the Stacks project, the basic version of Artin’s axiom [Stacks 07XJ07Y5] requires that

\begin{enumerate}
  \item [0] \( X \) is a stack in the étale topology,
  \item [1] \( X \) is limit preserving,
  \item [2] \( X \) is \( \text{Art}^{\text{fin}} \)-homogeneous (this is the Rim–Schlessinger condition RS),
  \item [3] \( \text{Aut}_{X/S}(\text{Spec}(k), k) \) and \( \text{Def}_{X/S}(\text{Spec}(k), k) \) are finite dimensional,
  \item [4] \( X \) is effective, and
  \item [5] \( X, \Delta_X \) and \( \Delta_{\Delta_X} \) satisfy openness of versality.
\end{enumerate}
There is also a criterion for when $X$ satisfies openness of versality \cite{Stacks, 07YU} using naive obstruction theories with finitely generated cohomology groups. This uses the (RS$^*$)-condition which is our Aff-homogeneity \cite{Stacks, 07Y8}. The existence of the naive obstruction theory implies that $\text{Aut}_{X/S}(T, -)$, $\text{Def}_{X/S}(T, -)$, $\text{Obs}_{X/S}(T, -)$ are bounded and $CB$ (Example \ref{ex:cb}), hence satisfy (5a) and (5b).

In \cite{Stacks}, the condition that the base scheme $S$ is excellent is replaced with the condition that its local rings are $G$-rings. In our treatment, excellency enters at two places: in the application of Néron–Popescu desingularization in Proposition \ref{prop:nnr} via \cite{CJ02} and in the context of DVR-homogeneity in Lemma \ref{lem:dvr}. In both cases, excellency can be replaced with the condition that the local rings are $G$-rings without modifying the proofs.

10.5. \textbf{Flenner’s criterion for openness of versality.} Flenner does not give a precise analogue of our main theorem, but his main result \cite[Satz 4.3]{Fle81} is a criterion for the openness of versality. In his criterion he has a limit preserving $S$-groupoid which satisfies (S1)–(S4). The first condition (S1) is identical to Artin’s condition (S1), i.e., rCl-semihomogeneity. The second condition (S2) is boundedness and Zariski localization of deformations. The third condition (S3) is boundedness and Zariski localization of the minimal obstruction theory. Finally (S4) is constructibility of deformations and obstructions. The Zariski localization condition is incorporated in the formulation of (S3) and (S4) which deals with sheaves of deformation and obstructions modules. His (S2)–(S4) are marginally stronger than our conditions, for example, treating arbitrary schemes instead of irreducible schemes. Theorem \cite[Satz 4.3]{Fle81} thus becomes the first part of Theorem \ref{thm:main} in the view of Section \ref{sec:sat} except that we assume rCl-homogeneity instead of rCl-semihomogeneity. This is a pragmatic choice that simplifies matters since $\text{Exal}_X(T, M)$ becomes a module instead of a pointed set. Also, in any algebraicity criterion, we would need homogeneity to deduce that the diagonal is algebraic and, conversely, if the diagonal is algebraic, then semihomogeneity implies homogeneity.

10.6. \textbf{Criterion for local constructibility.} There is a useful criterion for when a sheaf (or a stack) is locally constructible, that is, when it corresponds to an étale algebraic space (or algebraic stack) \cite[VII.7.2]{Art73}:

\textbf{Theorem 10.3.} Let $S$ be an excellent scheme. Then a category $X$ that is fibered in groupoids over $\mathbf{Sch}/S$, is an algebraic stack that is étale over $S$, if and only if it satisfies the following conditions.

1. $X$ is a stack over $(\mathbf{Sch}/S)_{\text{ét}}$.
2. $X$ is limit preserving.
3. $X(B) \to X(B/m)$ is an equivalence of categories for every local noetherian ring $(B, m)$, such that $B$ is $m$-adically complete, with an $S$-scheme structure $\text{Spec } B \to S$ such that the induced morphism $\text{Spec}(B/m) \to S$ is of finite type.

The necessity of the conditions is clear. That the conditions are sufficient can be proven directly as follows. Let $j : (\mathbf{Sch}/S)_{\text{ét}} \to S_{\text{ét}}$ denote the morphism of topoi corresponding to the inclusion of the small étale site into the big étale site. It is enough to prove that $j^{-1} j_* X \to X$ is an equivalence. As $X$ is limit preserving, it is enough to verify that $f^*(X|_{S_{\text{ét}}}) \to X|_{T_{\text{ét}}}$ is an equivalence for every morphism $f : T \to S$ locally of finite type, and this can be checked on stalks at points of finite
type. Therefore, it suffices to prove that $X(B) \to X(B/\mathfrak{m})$ is an equivalence when $B$ is the henselization of $\mathcal{O}_T,t$, for every $t \in [T]$ of finite type. This follows from general Néron–Popescu desingularization and the three conditions.

A proof more in the lines of this paper goes as follows: from [3] it follows that: $X$ is Art$^{\text{fin}}$-homogeneous; $X$ is effective; and $X \to S$ is formally étale at every point of finite type. In particular, $\text{Aut}_{X/S}(T,N) = \text{Def}_{X/S}(T,N) = \text{Obs}_{X/S}(T,N) = 0$ for every $X$-scheme $T$ that is of finite type over $S$ and every quasi-coherent $\mathcal{O}_T$-module $N$ with support that is artinian (use Lemmata 5.1 and 5.3). Thus, $\text{Aut}_{X/S}(T,-) = \text{Def}_{X/S}(T,-) = 0$ by Theorem 3.7. Theorem 10.3 would follow from the main theorem if we also can show that $\text{Obs}_{X/S}(T,-) = 0$. As we do not yet know that $\text{Obs}_{X/S}(T,-)$ is half-exact, it is apparently difficult to deduce that $\text{Obs}_{X/S}(T,-) = 0$ without invoking Popescu desingularization. A more elementary approach, that does not rely on the main theorem, is to note that given an $X$-scheme $T$ that is locally of finite presentation over $S$, and a point $t \in [T]$ of finite type, then $T \to X$ is formally smooth at $t$ if and only $T \to S$ is formally smooth at $t$. Thus, openness of formal smoothness for $T \to X$ follows.

**APPENDIX A. APPROXIMATION OF INTEGRAL MORPHISMS**

In this appendix, we give an approximation result for integral homomorphisms. It is somewhat technical since the properties that we need—surjective and surjective with nilpotent kernel—cannot be deduced for an arbitrary approximation. In fact, the approximation has to be built with these properties in mind.

**Lemma A.1.** Let $A$ be a ring, let $B$ be an $A$-algebra and let $C$ be an $B$-algebra. Assume that $B$ and $C$ are integral $A$-algebras. Then there exists a filtered system $(B_\lambda \to C_\lambda)_\lambda$ of finite and finitely presented $A$-algebras, with direct limit $B \to C$. In addition, if $A \to B$ (resp. $B \to C$, resp. $A \to C$) has one of the properties:

1. surjective,
2. surjective with nilpotent kernel,

then $A \to B_\lambda$ (resp. $B_\lambda \to C_\lambda$, resp. $A \to C_\lambda$) has the corresponding property.

**Proof.** We begin by writing $B = \varinjlim_{\lambda \in \Lambda} B_\lambda^\circ$ and $C = \varinjlim_{\lambda \in \Lambda} C_\lambda^\circ$ as direct limits of finitely generated subalgebras. We may then replace $C_\lambda^\circ$ with the $C$-subalgebra generated by the images of $B_\lambda^\circ$ and $C_\lambda^\circ$ so that we have homomorphisms $B_\lambda^\circ \to C_\lambda^\circ$ for all $\lambda$. If $B \to C$ is surjective, then we let $C_\lambda^\circ$ be the image of $B_\lambda^\circ \to C$. It is now easily verified that if $A \to B$ (resp. $B \to C$, resp. $A \to C$) is surjective or surjective with nilpotent kernel then so is $A \to B_\lambda$ (resp. $B_\lambda \to C_\lambda$, resp. $A \to C_\lambda$).

For every $\lambda$, choose surjections $P_\lambda \to B_\lambda^\circ$ and $Q_\lambda \to C_\lambda^\circ$ where $P_\lambda$ and $Q_\lambda$ are finite and finitely presented $A$-algebras. We may assume that we have homomorphisms $P_\lambda \to Q_\lambda$ compatible with $B_\lambda^\circ \to C_\lambda^\circ$ and if $B \to C$ is surjective, then we take $P_\lambda = Q_\lambda$. For any finite subset $L \subseteq \Lambda$ let $P_L = \bigotimes_{\lambda \in L} P_\lambda$ and $Q_L = \bigotimes_{\lambda \in L} Q_\lambda$, where the tensor products are over $A$.

For fixed $L \subseteq \Lambda$ choose finitely generated ideals $I_L \subseteq \ker(P_L \to B)$ and $I_L Q_L \subseteq J_L \subseteq \ker(Q_L \to C)$ and let $B_L = P_L/I_L$ and $C_L = Q_L/J_L$. If $A \to B$ (resp. $A \to C$) is surjective, then for sufficiently large $I_L$ (resp. $J_L$), we have that $A \to B_L$ (resp. $A \to C_L$) is surjective. If $B \to C$ is surjective, then by construction $P_L = Q_L$ so that $B_L \to C_L$ is surjective. If $B \to C$ has nilpotent kernel, with nilpotency index $n$, then we replace $I_L$ with $I_L + J^n_L$ so that $B_L \to C_L$ has nilpotent kernel.
Consider the set $\Xi$ of pairs $\xi = (I, J, L, J)$ where $L \subseteq \Lambda$ is a finite subset, and $I_L \subseteq \mathcal{P}_L$ and $J_L \subseteq \mathcal{Q}_L$ are finitely generated ideals as in the previous paragraph. Then $(B_L, C_L)_{\xi}$ is a filtered system of finite and finitely presented $A$-algebras with direct limit $(B \to C)$ which satisfies the conditions of the lemma.

Fix a scheme $S$ and consider the category of diagrams $[Y \xrightarrow{i, j} X \xrightarrow{\lambda} X']$ of $S$-schemes. We say that a morphism $[Y_1 \xrightarrow{i_1, j_1} X_1 \xrightarrow{\lambda_1} X_1'] \to [Y_2 \xrightarrow{i_2, j_2} X_2 \xrightarrow{\lambda_2} X_2']$ is affine if the components $Y_1 \to Y_2$, $X_1 \to X_2$ and $X_1' \to X_2'$ are affine. Given an inverse system of diagrams with affine bonding maps the inverse limit then exists and is calculated component by component.

**Proposition A.2.** Let $S$ be an affine scheme and let $P \in \{\text{Nil, Cl, rNil, rCl, Int, Aff}\}$ (cf. Section 1). Let $W = [Y \xrightarrow{i} X \xrightarrow{\lambda} X']$ be a diagram of affine $S$-schemes where $i$ is a nilpotent closed immersion, and $f$ is $P$. Then $W$ is an inverse limit of diagrams $W_\lambda = [Y_\lambda \xrightarrow{i_\lambda} X_\lambda \xrightarrow{\lambda} X'_\lambda]$ of affine finitely presented $S$-schemes where $i_\lambda$ is a nilpotent closed immersion, and $f_\lambda$ is $P$. Moreover, if we let $Y' = Y \amalg_X X'$ and $Y'_\lambda = Y_\lambda \amalg X_\lambda$ denote the push-outs, then $Y' = \lim_{\leftarrow \lambda \in \Lambda} Y'_\lambda$.

**Proof.** We will begin by looking at the induced diagram $[Y \xrightarrow{i} Y' \xrightarrow{\beta} X']$. As $j$ is a nilpotent closed immersion it follows that $g$ has property $P$. The first step will be to write this diagram as an inverse limit of diagrams $[Y_\lambda \xrightarrow{i_\lambda} \amalg X_\lambda \xrightarrow{\lambda} X'_\lambda]$ of finite presentation over $S$ where $j_\lambda$ is a nilpotent closed immersion and $g_\lambda$ has property $P$. To this end, we begin by writing $Y'$ as an inverse limit of finitely presented $S$-schemes $Y'_\alpha$. By Lemma A.1 we may also write $g: X' \to Y'$ (resp. $j: Y \to Y'$) as an inverse limit of finitely presented $P$-morphisms $X'_J \to Y' \to Y' \to Y' \to Y'$. For every pair $(\beta, \gamma)$ there is ([EGA IV.8.10.5]) an $\alpha = \alpha(\beta, \gamma)$, and a cartesian diagram

$$
\begin{array}{ccc}
Y_{\beta} & \xrightarrow{\gamma} & Y' \\
\downarrow & & \downarrow \\
Y_{\alpha \beta \gamma} & \xrightarrow{\lambda} & X'_{\alpha \beta \gamma}
\end{array}
$$

where $X'_{\alpha \beta \gamma} \to Y'_{\alpha}$ is a finitely presented $P$-morphism and $Y_{\alpha \beta \gamma} \to Y'_{\alpha}$ is a nilpotent closed immersion.

For every $\alpha \geq \alpha(\beta, \gamma)$ we also let $[Y_{\alpha \beta \gamma} \to Y'_{\alpha} \xleftarrow{X'_{\alpha \beta \gamma}}]$ denote the pull-back along $Y'_{\alpha} \to Y'_{\alpha(\beta, \gamma)}$. Let $I = \{(\beta, \gamma, \alpha)\}$ be the set of indices such that $\alpha > \alpha(\beta, \gamma)$. For every finite subset $J \subset I$, we let

$$
\begin{align*}
\amalg_J Y_{\alpha \beta \gamma} &= \prod_{(\beta, \gamma, \alpha) \in J} Y_{\alpha \beta \gamma}, \\
J &= \prod_{(\beta, \gamma, \alpha) \in J} Y_{\alpha \beta \gamma}, \quad \text{and} \quad X'_J = \prod_{(\beta, \gamma, \alpha) \in J} X'_{\alpha \beta \gamma}
\end{align*}
$$

where the products are taken over $S$. The finite subsets $J \subset I$ form a partially ordered set under inclusion and the induced morphisms:

$$
\begin{array}{ccc}
\amalg_J Y_{\alpha \beta \gamma} & \to \lim_J Y_J, & Y \to \lim_J Y_J, \\
\amalg_J X'_J & \to \lim_J X'_J & X' \to \lim_J X'_J
\end{array}
$$

are closed immersions. Now, let $K_{Y_J} = \ker(\mathcal{O}_{Y_J} \to (g_J)_* \mathcal{O}_Y)$ and similarly for $K_{X'_J}$. Note that $K_{Y_J} \mathcal{O}_{Y_J} \subseteq K_{Y_J}$ and $K_{X'_J} \mathcal{O}_{X'_J} \subseteq K_{X'_J}$. We then let $\Lambda = \{(J, R_{Y_J}, R_{X'_J})\}$ where $J \subset I$ is a finite subset and $R_{Y_J} \subset K_{Y_J}$, $R_{X'_J} \subset$
$K_{\mathcal{Y}}$ and $R_{X_j} \subset K_{X_j}$ are finitely generated ideals such that $R_{X_j} \mathcal{O}_{X_j} \subset R_{Y_j}$ and $R_{Y_j} \mathcal{O}_{Y_j} \subset R_{X_j}$. For every $\lambda \in \Lambda$ we put

$$\overline{Y}_{\lambda} = \text{Spec}(\mathcal{O}_{\overline{Y}} / R_{\overline{Y}}), \; \; Y_{\lambda} = \text{Spec}(\mathcal{O}_{Y_j} / R_{Y_j}), \; \; \text{and} \; \; X_{\lambda} = \text{Spec}(\mathcal{O}_{X_j} / R_{X_j})$$

Then $[Y \to Y' \leftarrow X'] = \lim_{\lambda} [Y_{\lambda} \to \overline{Y}_{\lambda} \leftarrow X'_{\lambda}]$. Finally, we take $X_{\lambda} = X'_{\lambda} \times_{\overline{Y}_{\lambda}} Y_{\lambda}$ so that $[Y \xleftarrow{i_Y} X \xrightarrow{i_X} X'] = \lim_{\lambda} [Y_{\lambda} \xleftarrow{i_{Y_{\lambda}}} X_{\lambda} \xrightarrow{i_{X_{\lambda}}} X'_{\lambda}]$. Indeed, $X = X' \times_{\overline{Y}} Y$ and inverse limits commute with fiber products.

For the last assertion, we note that all schemes are affine and that there is an exact sequence

$$0 \to \Gamma(\mathcal{O}_{Y'}) \to \Gamma(\mathcal{O}_Y) \times \Gamma(\mathcal{O}_{X'}) \to \Gamma(\mathcal{O}_X) \to 0$$

and similarly for the approximations $Y'_\lambda$ (which can be different from $\overline{Y}_\lambda$). As direct limits are exact it follows that $Y' = \lim_{\lambda} Y'_\lambda$. \hfill \Box

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