A GENERALIZATION OF GOODSTEIN’S THEOREM: 
INTERPOLATION BY POLYNOMIAL FUNCTIONS OF 
DISTRIBUTIVE LATTICES

MIGUEL COUCEIRO AND TAMÁS WALDHAUSER

Abstract. We consider the problem of interpolating functions partially defined over a distributive lattice, by means of lattice polynomial functions. Goodstein’s theorem solves a particular instance of this interpolation problem on a distributive lattice \( L \) with least and greatest elements 0 and 1, resp.: Given a function \( f: \{0,1\}^n \to L \), there exists a lattice polynomial function \( p: L^n \to L \) such that \( p|_{\{0,1\}^n} = f \) if and only if \( f \) is monotone; in this case, the interpolating polynomial \( p \) is unique.

We extend Goodstein’s theorem to a wider class of partial functions \( f: D \to L \) over a distributive lattice \( L \), not necessarily bounded, and where \( D \subseteq L^n \) is allowed to range over cuboids \( D = \{a_1, b_1\} \times \cdots \times \{a_n, b_n\} \) with \( a_i, b_i \in L \) and \( a_i < b_i \), and determine the class of such partial functions which can be interpolated by lattice polynomial functions. In this wider setting, interpolating polynomials are not necessarily unique; we provide explicit descriptions of all possible lattice polynomial functions which interpolate these partial functions, when such an interpolation is available.

1. Introduction

Let \( L \) be a distributive lattice and let \( f: D \to L \) \((D \subseteq L^n)\) be an \( n \)-ary partial function on \( L \). In this paper we are interested in the problem of extending such partial functions to the whole domain \( L^n \) by means of lattice polynomial functions, i.e., functions that can be represented as compositions of the lattice operations \( \land \) and \( \lor \) and constants. More precisely, we aim at determining necessary and sufficient conditions on the partial function \( f \) that guarantee the existence of a lattice polynomial function \( p: L^n \to L \) which interpolates \( f \), that is, \( p|_D = f \).

An instance of this problem was considered by Goodstein [8] in the case when \( L \) is a bounded distributive lattice, and the functions to be interpolated were of the form \( f: \{0,1\}^n \to L \). Goodstein showed that such a function \( f \) can be interpolated by lattice polynomial functions if and only if it is monotone. Furthermore, if such an interpolating polynomial function exists, then it is unique.

The general solution to the above mentioned interpolation problem eludes us. However, we are able to generalize Goodstein’s result by allowing \( L \) to be an arbitrary (possibly unbounded) distributive lattice and considering functions \( f: D \to L \), where \( D = \{a_1, b_1\} \times \cdots \times \{a_n, b_n\} \) with \( a_i, b_i \in L \).
and \( a_i < b_i \). More precisely, we furnish necessary and sufficient conditions for the existence of an interpolating polynomial function. As it will become clear, in this more general setting, uniqueness is not guaranteed, and thus we determine all possible interpolating polynomial functions.

The structure of the paper is as follows. In Section 2 we recall basic background on polynomial functions over distributive lattices (for general background see [7, 9]) and formalize the interpolation problem that we are interested in. In Section 3 we state and prove the characterization of those functions that can be interpolated by polynomial functions and we describe the set of all solutions of the interpolation problem. We discuss variations of the interpolation problem in Section 4 and relate our work to earlier results obtained for finite chains in [11]. Finally, in Section 5 we consider potential applications of our results in mathematical modeling of decision making.

2. Preliminaries

Let \( L \) be a bounded distributive lattice with least element 0 and greatest element 1. It can be shown that a function \( p : L^n \to L \) is a lattice polynomial function if and only if there exist \( c_I \in L, I \subseteq [n] := \{1, \ldots, n\} \), such that \( p \) can be represented in disjunctive normal form (DNF for short) by

\[
p(x) = \bigvee_{I \subseteq [n]} (c_I \land \bigwedge_{i \in I} x_i), \quad \text{where } x = (x_1, \ldots, x_n) \in L^n.
\]

It is easy to verify that taking \( c'_I = \bigvee_{J \subseteq I} c_J \), we also have

\[
p(x) = \bigvee_{I \subseteq [n]} (c'_I \land \bigwedge_{i \in I} x_i),
\]

and hence the coefficients \( c_I \) can be assumed to be monotone in the sense that \( I \subseteq J \) implies \( c_I \leq c_J \). This monotonicity assumption allows us to recover the coefficients of the DNF from certain values of the polynomial function \( p \). Indeed, denoting by \( 1_I \) the characteristic vector of \( I \subseteq [n] \) (i.e., the tuple \( 1_I \in L^n \) whose \( i \)-th component is 1 if \( i \in I \) and 0 if \( i \notin I \)), we then have that \( p(1_I) = c_I \). In the sequel, we will always consider lattice polynomials in DNF, and we will implicitly assume that the coefficients are monotone. These observations contain the essence of Goodstein’s theorem.

**Theorem 1** (Goodstein [8]). Let \( L \) be a bounded distributive lattice, and let \( f \) be a function \( f : \{0, 1\}^n \to L \). There exists a polynomial function \( p \) over \( L \) such that \( p|_{\{0, 1\}^n} = f \) if and only if \( f \) is monotone. In this case \( p \) is uniquely determined, and can be represented by the DNF

\[
p(x) = \bigvee_{I \subseteq [n]} (f(1_I) \land \bigwedge_{i \in I} x_i).
\]

Informally, Goodstein’s theorem asserts that polynomial functions are uniquely determined by their restrictions to the hypercube \( \{0, 1\}^n \), and a
function on the hypercube extends to a polynomial function if and only if it is monotone.

Let us now consider a distributive lattice $L$ without least and greatest elements. (We omit the analogous discussion of the cases where $L$ has one boundary element.) Polynomial functions over $L$ can still be given in DNF of the form (1) by allowing the coefficients $c_I$ to take also the values 0 and 1, which are considered as external boundary elements (see, e.g., [1, 3]). For example, a polynomial function $p(x, y) = a \lor x \lor (b \land x \land y)$ can be rewritten as $p(x, y) = a \lor (1 \land x) \lor (0 \land y) \lor (b \land x \land y)$.

We can still assume monotonicity of the coefficients, and any such system $c_I \in L \cup \{0, 1\}$ of coefficients gives rise to a polynomial function $p$ over $L$, provided that $c_\emptyset \neq 1$ and $c_{[n]} \neq 0$. (The latter two cases correspond to the constant 1 and constant 0 functions.) Just like in the case of bounded lattices, there is a one-to-one correspondence between such DNF’s and polynomial functions, since we can recover the coefficients of the DNF of $p$ from certain values of $p$. To see this, let us choose elements $a < b$ from $L$ to play the role of 0 and 1, and let $e_I$ be the “characteristic vector” of $I \subseteq [n]$ (i.e., the tuple $e_I \in L^n$ whose $i$-th component is $b$ if $i \in I$ and $a$ if $i \notin I$). If $a$ is sufficiently small (less than all non-zero coefficients in the DNF of $p$) and $b$ is sufficiently large (greater than all non-one coefficients in the DNF of $p$), then a routine computation shows that

$$p(e_I) = \begin{cases} 
 c_I & \text{if } c_I \in L, \\
 a & \text{if } c_I = 0, \\
 b & \text{if } c_I = 1. 
\end{cases}$$

Thus we can learn the coefficient $c_I$ from the behavior of the value $p(e_I)$ by letting $a$ decrease and $b$ increase indefinitely, i.e., the polynomial function $p$ is uniquely determined by its values on a sufficiently large cube $\{a, b\}^n$ (for a more detailed discussion, see [1]). As the next example shows, this does not imply that there is only one polynomial function that takes prescribed values on a fixed cube $\{a, b\}^n$.

**Example 2.** Let $L$ be the lattice of open subsets of a topological space $X$, and let $a, b \in L$ with $a \subset b$. Since $L$ is a bounded distributive lattice, every unary polynomial function $p$ over $L$ can be represented by a unique DNF of the form $p(x) = c_0 \cup (c_1 \cap x)$ with $c_0, c_1 \in L, c_0 \subseteq c_1$. It is straightforward to verify that such a polynomial function satisfies $p(a) = p(b) = b$ if and only if

$$b \setminus a \subseteq c_0 \subseteq b \quad \text{and} \quad b \subseteq c_1 \subseteq X.$$

Thus, there may be infinitely many polynomial functions $p$ whose restriction to the “one-dimensional cube” $\{a, b\}$ is constant $b$ (for instance, let $X$ be the real line, and let $a$ and $b$ be open intervals).

Let us go one step further, and choose a “zero” and “one”, possibly different in each coordinate: Let $a_i, b_i \in L$ with $a_i < b_i$ for each $i \in [n]$, and let $\hat{e}_I$ be the “characteristic vector” of $I \subseteq [n]$ (i.e., the tuple $\hat{e}_I \in L^n$ whose $i$-th
Combining these inequalities, we get

\[ f(\hat{e}_I) \land a_k \leq f(\hat{e}_I) \leq f(\hat{e}_{I \setminus \{k\}}) \lor b_k \quad \text{for all } I \subseteq [n], k \in [n]. \]

Observe that the first inequality is trivial if \( k \in I \), and the second inequality is trivial if \( k \notin I \).

Our first lemma shows how to obtain inequalities between \( f(\hat{e}_S) \) and \( f(\hat{e}_T) \) for \( S \subseteq T \) by repeated applications of (\( \star \)).

**Lemma 3.** If the function \( f \) satisfies (\( \star \)), then for all \( S \subseteq T \subseteq [n] \) we have

\[ f(\hat{e}_T) \land \bigwedge_{k \in T \setminus S} a_k \leq f(\hat{e}_S) \quad \text{and} \quad f(\hat{e}_T) \leq f(\hat{e}_S) \lor \bigvee_{k \in T \setminus S} b_k. \]

**Proof.** We only prove the first inequality; the second one follows similarly. Let \( T \setminus S = \{k_1, \ldots, k_r\} \), and let us apply (the first inequality of) condition (\( \star \)) with \( I = S \cup \{k_1, \ldots, k_{m-1}\} \) and \( k = k_m \) for \( m = 1, \ldots, r \):

\[ f(\hat{e}_{S \cup \{k_1\}}) \land a_{k_1} \leq f(\hat{e}_S), \]

\[ f(\hat{e}_{S \cup \{k_1,k_2\}}) \land a_{k_2} \leq f(\hat{e}_{S \cup \{k_1\}}), \]

\[ \vdots \]

\[ f(\hat{e}_{S \cup \{k_1,\ldots,k_{r-1}\}}) \land a_{k_r} \leq f(\hat{e}_{S \cup \{k_1,\ldots,k_{r-1}\}}). \]

Combining these \( r \) inequalities, we get

\[ f(\hat{e}_{S \cup \{k_1,\ldots,k_r\}}) \land a_{k_1} \land \cdots \land a_{k_r} \leq f(\hat{e}_S). \]
Let us now show that (1) is a necessary condition for the existence of a solution of the Interpolation Problem.

**Lemma 4.** If there is a polynomial function \( p \) over \( L \) such that \( p|_D = f \), then \( f \) is monotone and satisfies (1).

**Proof.** Assume that \( p \) is a polynomial function that extends \( f \). Since \( p \) is monotone, \( f \) is also monotone. To show that (1) holds, let us fix \( I \subseteq [n] \) and \( k \in [n] \), and let us assume that \( k \notin I \) (the case \( k \in I \) can be dealt with similarly). Let \( (\hat{e}_I)^c_k \in L^n \) denote the \( n \)-tuple obtained from \( \hat{e}_I \) by replacing its \( k \)-th component by the variable \( x \). We can define a unary polynomial function \( u \) over \( L \) by \( u(x) := p((\hat{e}_I)^c_k) \). Using this notation, (1) takes the form \( u(b_k) \land a_k \leq u(a_k) \). The DNF of \( u \) is of the form \( u(x) = c_0 \lor (c_1 \land x) \), where \( c_0, c_1 \in L \cup \{0, 1\} \). Using distributivity and the fact that \( a_k < b_k \), we can now easily prove the desired inequality:

\[
u(b_k) \land a_k = (c_0 \lor (c_1 \land b_k)) \land a_k = (c_0 \land a_k) \lor (c_1 \land b_k \land a_k) = (c_0 \land a_k) \lor (c_1 \land a_k) \leq c_0 \lor (c_1 \land a_k) = u(a_k) . \]

To find all polynomial functions \( p \) satisfying \( p|_D = f \), we will make use of the Birkhoff-Priestley representation theorem to embed \( L \) into a Boolean algebra \( B \). For the sake of canonicity, we assume that \( L \) generates \( B \); under this assumption \( B \) is uniquely determined up to isomorphism. The boundary elements of \( B \) will be denoted by 0 and 1. This notation will not lead to ambiguity since if \( L \) has a least (resp. greatest) element, then it must coincide with 0 (resp. 1). The complement of an element \( a \in B \) is denoted by \( a' \). Given a function \( f : D \to L \), we define the following two elements in \( B \) for each \( I \subseteq [n] \):

\[
c^-_I := f(\hat{e}_I) \land \bigwedge_{i \notin I} a'_i, \quad c^+_I := f(\hat{e}_I) \lor \bigvee_{i \in I} b'_i.
\]

Observe that \( c^-_I \leq c^+_I \), and if \( f \) is monotone, then \( I \subseteq J \) implies \( c^-_I \leq c^-_J \) and \( c^+_I \leq c^+_J \). Let \( p^- \) and \( p^+ \) be the polynomial functions over \( B \) which are given by these two systems of coefficients. We will see that \( p^- \) and \( p^+ \) are the least and greatest polynomial functions over \( B \) whose restriction to \( D \) coincides with \( f \) (whenever there exists such a polynomial function).

**Lemma 5.** If \( f \) is monotone and satisfies (1), then \( p^+(\hat{e}_J) \leq f(\hat{e}_J) \) for all \( J \subseteq [n] \).

**Proof.** Let us fix \( J \subseteq [n] \) and consider the value of \( p^+ \) at \( \hat{e}_J \):

\[
p^+(\hat{e}_J) = \bigvee_{I \subseteq [n]} (c^+_I \land \bigwedge_{j \in I} (\hat{e}_J)_j) = \bigvee_{I \subseteq [n]} (c^+_I \land \bigwedge_{j \in I \setminus J} a_j \land \bigwedge_{j \in I \cap J} b_j).
\]

It is sufficient to verify that each joinand is at most \( f(\hat{e}_J) \). Taking into account the definition of \( c^+_I \), this amounts to showing that

\[
(\hat{e}_J) \lor \bigvee_{i \in I} b'_i \land \bigwedge_{j \in I \setminus J} a_j \land \bigwedge_{j \in I \cap J} b_j \leq f(\hat{e}_J) \tag{2}
\]
holds for all \( I \subseteq [n] \). Distributing meets over joins, the left hand side of (2) becomes
\[
(f(\hat{e}_I) \land \bigwedge_{j \in I \cap J} a_j \land \bigwedge_{j \in I \setminus J} b_j) \lor \bigvee (b'_i \land \bigwedge_{j \in I \setminus J} a_j \land \bigwedge_{j \in I \cap J} b_j).
\]

Let us examine each joinand of this expression. For each \( i \in I \), the joinand involving \( b'_i \) equals 0, since
\[
b'_i \land \bigwedge_{j \in I \setminus J} a_j \land \bigwedge_{j \in I \cap J} b_j \leq b'_i \land \bigwedge_{j \in I \cap J} b_j = b'_i \land \bigwedge_{j \in I \cap J} b_j = 0.
\]

The joinand of (3) that involves \( f(\hat{e}_I) \) can be estimated using (r) and Lemma 3 (with \( T = I \) and \( S = I \cap J \)):
\[
f(\hat{e}_I) \land \bigwedge_{j \in I \cap J} a_j \land \bigwedge_{j \in I \setminus J} b_j \leq f(\hat{e}_I) \land \bigwedge_{j \in I \setminus (I \cap J)} a_j \leq f(\hat{e}_{I \cap J}).
\]
Since \( f \) is monotone, we have \( f(\hat{e}_{I \cap J}) \leq f(\hat{e}_{I}) \), and this proves (2).
\[\square\]

The following lemma is the dual of Lemma 5 and it can be proved by using the conjunctive normal form of \( p^- \).

**Lemma 6.** If \( f \) is monotone and satisfies (r), then \( p^-(\hat{e}_J) \geq f(\hat{e}_J) \) for all \( J \subseteq [n] \).

The estimates obtained in the previous two lemmas allow us to find all solutions of our interpolation problem over \( B \), whenever a solution exists.

**Theorem 7.** Let \( D = \{\hat{e}_I : I \subseteq [n]\} \) and \( f : D \to L \) be given, as in the Interpolation Problem. Suppose that \( f \) is monotone and satisfies (r), and let \( p \) be an \( n \)-ary polynomial function over \( B \) given by the DNF corresponding to a system of coefficients \( c_I \in B (I \subseteq [n]) \). Then the following three conditions are equivalent:

(i) \( p|_D = f; \)

(ii) for all \( I \subseteq [n] \) the inequalities \( c_I^- \leq c_I \leq c_I^+ \) hold;

(iii) for all \( x \in L^n \) we have \( p^-(x) \leq p(x) \leq p^+(x) \).

**Proof.** Implication (ii) \(\implies\) (iii) is trivial. To prove (ii) \(\implies\) (i), assume that \( p|_D = f \), i.e., \( p(\hat{e}_J) = f(\hat{e}_J) \) for all \( J \subseteq [n] \). Then we can replace \( f(\hat{e}_J) \) by \( p(\hat{e}_J) \) in the definition of \( c_J^- \), and we can compute its value by substituting \( \hat{e}_J \) into the DNF of \( p \):
\[
c_J^- = f(\hat{e}_J) \land \bigwedge_{j \notin J} a'_j = p(\hat{e}_J) \land \bigwedge_{j \notin J} a'_j = \left( \bigvee_{I \subseteq [n]} (c_I \land \bigwedge_{i \in I} (\hat{e}_I)_i) \right) \land \bigwedge_{j \notin J} a'_j
\]
\[
= \bigvee_{I \subseteq [n]} \left( c_I \land \bigwedge_{i \in I \setminus J} a_i \land \bigwedge_{i \in I \cap J} b_i \land \bigwedge_{j \notin J} a'_j \right).
\]
If there exists \( i \in I \setminus J \), then \( a_i \land a'_j = 0 \) appears in the joinand corresponding to \( I \), hence we can omit each of these terms from the join, and keep only
those where $I \setminus J = \emptyset$:

$$c_J = \bigvee_{I \subseteq J} (c_I \land \bigwedge_{i \in I \setminus J} a_i \land \bigwedge_{i \in I \cap J} b_i \land \bigwedge_{j \notin J} a'_j) \leq \bigvee_{I \subseteq J} c_I = c_J.$$  

This proves $c_J^- \leq c_J$. The inequality $c_J \leq c_J^+$ can be proved by a dual argument.

Finally, to prove $(iii) \implies (i)$ let us assume that $p^- \leq p \leq p^+$ holds in the pointwise ordering of functions. Applying Lemma 5 and Lemma 6 we get the following chain of inequalities for every $I \subseteq [n]$:

$$f(\hat{e}_I) \leq p^-(\hat{e}_I) \leq p(\hat{e}_I) \leq p^+(\hat{e}_I) \leq f(\hat{e}_I).$$

This implies $p(\hat{e}_I) = f(\hat{e}_I)$ for all $I \subseteq [n]$, therefore we have $p|D = f$. □

Note that in Lemma 4 we did not make use of the fact that $p$ is a polynomial function over $L$: the proof works also for polynomial functions over $B$. This fact together with Theorem 7 shows that monotonicity and property $(\star)$ of $f$ are necessary and sufficient for the existence of a solution of our interpolation problem over $B$. This observation leads to the following result.

**Theorem 8.** The Interpolation Problem has a solution if and only if $f$ is monotone and satisfies $(\star)$. In this case a polynomial function $p$ over $L$ verifies $p|D = f$ if and only if $c_I^- \leq c_I \leq c_I^+$ holds for the coefficients $c_I$ of the DNF of $p$ for all $I \subseteq [n]$. In particular, $p$ can be chosen as the polynomial function $p_0$ given by the coefficients $c_I = f(\hat{e}_I)$:

$$p_0(x) = \bigvee_{I \subseteq [n]} (f(\hat{e}_I) \land \bigwedge_{i \in I} x_i).$$

**Proof.** The necessity of the conditions has been established in Lemma 4. To prove the sufficiency, we just need to observe that if $f$ is monotone and satisfies $(\star)$, then the polynomial function $p_0$ is a solution of the Interpolation Problem by Theorem 7 as $c_I^- \leq f(\hat{e}_I) \leq c_I^+$ follows immediately from the definition of $c_I^-$ and $c_I^+$. Since $f(\hat{e}_I) \in L$ for all $I \subseteq [n]$, the polynomial function $p_0$ is actually a polynomial function over $L$. The description of the set of all solutions over $L$ also follows from Theorem 7. □

Let us note that if $L$ is bounded and $a_i = 0, b_i = 1$ for all $i \in [n]$, then Theorem 8 reduces to Goodstein’s theorem. Indeed, in this case $(\star)$ holds trivially, hence a solution exists if and only if $f$ is monotone. Moreover, we have $c_I^- = c_I^+ = f(\hat{e}_I)$, hence $p_0$ (which is the same as the polynomial function given in Theorem 7) is the only solution of the Interpolation Problem.

4. Variations

We have seen that monotonicity and property $(\star)$ are necessary and sufficient to guarantee the existence of a solution of the Interpolation Problem. The following example shows that these two conditions are independent, hence neither of them can be dropped.
Example 9. Let $L$ be a distributive lattice, let $a, b, c \in L$ such that $a < b < c$, and let $D = \{a, b\}$. Then the function $f : D \to L$ defined by $f(a) = b$, $f(b) = a$ satisfies (2) but it is not monotone, while the function $g : D \to L$ defined by $g(a) = a$, $g(b) = c$ is monotone but it does not satisfy (2).

Considering polynomial functions over the Boolean algebra $B$ generated by $L$, the Interpolation Problem has a least and a greatest solution, namely $p^{-}$ and $p^{+}$, whenever a solution exists (see Theorem 7). On the other hand, the instance of the Interpolation Problem considered in Example 2 has no least solution over $L$ itself (since usually there is no least open set containing $b \setminus a$), and a dual example shows that in general there is no greatest solution over $L$. However, if $L$ is complete, then extremal solutions exist over $L$.

To describe these, let us introduce the following notation. For an arbitrary $b \in B$, we define the elements $\text{cl}(b)$ and $\text{int}(b)$ of $L$ by

\[ \text{cl}(b) := \bigwedge_{a \in L} a \quad \text{and} \quad \text{int}(b) := \bigvee_{a \in L} a. \]

Completeness of $L$ ensures that these (possibly infinite) meets and joins exist, and one can verify that $\text{cl}$ is a closure operator on $B$ (the closed elements being exactly the elements of $L$), while $\text{int}$ is the dual closure operator on $B$ (also called as “interior operator”).

Theorem 10. If $L$ is a complete distributive lattice, then a polynomial function $p$ over $L$ is a solution of the Interpolation Problem if and only if

\[ \text{cl}(c^{-}_{I}) \leq c_{I} \leq \text{int}(c^{+}_{I}) \]

holds for the coefficients $c_{I}$ of the DNF of $p$, for all $I \subseteq [n]$.

Proof. Theorem 10 follows directly from Theorem 8, since, by the very definition of $\text{cl}$ and $\text{int}$, we have that $c^{-}_{I} \leq c_{I} \leq c^{+}_{I}$ holds for a given $c_{I} \in L$ if and only if $\text{cl}(c^{-}_{I}) \leq c_{I} \leq \text{int}(c^{+}_{I})$. \qed

Now let us consider a general version of the Interpolation Problem, where $D$ is an arbitrary subset of $L^{n}$, not necessarily the set of vertices of a rectangular box. This problem is still open for distributive lattices; however, for finite chains the solution has been given in [11]. That paper deals with Sugeno integrals (cf. Section 5) instead of lattice polynomials; here we reformulate the criterion for the existence of a solution (Theorem 3 in [11]) in the language of lattice theory.

Theorem 11 ([11]). Let $L$ be a finite chain, and let $D$ be an arbitrary subset of $L^{n}$. A function $f : D \to L$ extends to a lattice polynomial function on $L$ if and only if

\[ \forall a, b \in D : f(a) < f(b) \implies \exists i \in [n] : a_i \leq f(a) < f(b) \leq b_i. \]

Let us explore the relationship between Theorem 11 and Theorem 8. Our condition (2) is defined only for sets $D$ of the form $D = \{e_I : I \subseteq [n]\}$, whereas (3) can be interpreted for any set $D \subseteq L^{n}$ for any distributive
Example 12. Let \( L = \{0, 1, a, b\} \) with \( 0 < a, b < 1 \) and \( a, b \) incomparable. Let \( n = 1 \) and \( D = \{0, b\} \), and define \( f: D \to L \) by \( f(0) = b, f(b) = a \) and \( g: D \to L \) by \( g(0) = a, g(b) = 1 \). Then \( f \) trivially satisfies (1), but \( f \) is not monotone, hence it is not the restriction of any polynomial function. On the other hand, \( g \) does not satisfy (1), although it is the restriction of the polynomial function \( p(x) = x \lor a \) to \( D \).

Observe that if \( L \) is a chain, then (1) implies that \( f \) is monotone\(^1\) but this is not true for arbitrary distributive lattices (see the example above). Thus we may want to require that \( f \) is a monotone function satisfying (1). We will prove below that if \( D \) is of “rectangular” shape, then monotonicity of \( f \) and condition (4) are sufficient to ensure that \( f \) extends to a polynomial function (but (4) is not necessary, as we have seen in Example 12).

Proposition 13. Let \( L \) be a distributive lattice and \( D = \{\hat{e}_I: I \subseteq [n]\} \) as in the Interpolation Problem. If \( f: D \to L \) is monotone and satisfies (4), then there exists a polynomial function \( p \) over \( L \) such that \( p|_D = f \).

Proof. Let \( f: D \to L \) be a monotone function satisfying (1). By Theorem 8 we only have to prove that \( f \) also satisfies (2). Let us assume that \( k \notin I \); the other case is similar. Then only \( f(\hat{e}_{I \cup \{k\}}) \land a_k \leq f(\hat{e}_I) \) needs to be verified, as the second inequality of (2) is trivial in this case. Since \( f \) is monotone, we have \( f(\hat{e}_I) \leq f(\hat{e}_{I \cup \{k\}}) \), and if equality holds here, then we are done. On the other hand, if \( f(\hat{e}_I) < f(\hat{e}_{I \cup \{k\}}) \), then (1) implies that there is an \( i \in [n] \) such that

\[
(5) \quad (\hat{e}_I)_i \leq f(\hat{e}_I) < f(\hat{e}_{I \cup \{k\}}) \leq (\hat{e}_{I \cup \{k\}})_i.
\]

This is clearly impossible for \( i \neq k \), since then the \( i \)-th component of \( \hat{e}_I \) and \( \hat{e}_{I \cup \{k\}} \) is the same (namely, \( a_i \)). Thus we must have \( i = k \), and then (5) reads as

\[ a_k \leq f(\hat{e}_I) < f(\hat{e}_{I \cup \{k\}}) \leq b_k. \]

From this we immediately obtain the desired inequality:

\[ f(\hat{e}_{I \cup \{k\}}) \land a_k \leq a_k \leq f(\hat{e}_I). \]

Finally, we give an example that shows that monotonicity and condition (1) together do not guarantee the existence of a solution of the Interpolation Problem if \( L \) is an arbitrary distributive lattice and \( D \) is an arbitrary subset of \( L^n \). Thus it remains as a topic of further research to find an appropriate criterion for the existence of an interpolating lattice polynomial function in this general setting.

\(^1\)Of course, this follows from Theorem 11 but it is also easy to verify directly.
Example 14. Let $L$ be the same lattice as in Example 12, and let $D = \{a, b\}$. Then the function $f: D \to L$ defined by $f(a) = b$, $f(b) = a$ is monotone and satisfies (4), but it is not the restriction of a polynomial function.

5. Application in decision making

The original motivation for considering the Interpolation Problem lies in the following mathematical model of multicriteria decision making. Let us assume that we have a set of alternatives from which we would like to choose the best one (e.g., a house to buy). Several properties of these alternatives could be important in making the decision (e.g., the size, price, etc., of a house), and this very fact can make the decision difficult (for instance, maybe it is not clear whether a cheap and small house is better than a big and expensive one). To overcome this difficulty, the values corresponding to the various properties of each alternative should be combined to a single value, which can then be easily compared.

To formalize this situation, let us assume that there are $n$ criteria along which the alternatives are evaluated, and these take their values in linearly ordered sets $L_1, \ldots, L_n$. These linearly ordered sets could be quantitative scales (e.g., $L_1$ could be the real interval $[40, 200]$, measuring the size of a house in square meters) or qualitative scales (e.g., $L_1$ could be the finite chain $\{\text{very small} < \text{small} < \text{big} < \text{very big}\}$). Thus, to each alternative corresponds a profile $x \in L_1 \times \cdots \times L_n$. Since this product is usually not a linearly ordered set, some alternatives may be incomparable. Therefore, we choose a common scale $L$, and monotone functions $\varphi_i: L_i \to L (i \in [n])$ to translate the values corresponding to the different criteria (which may have different units of measure, e.g., square meters, euros, etc.) to this common scale, and which are then combined into a single value (for each alternative) by a so-called aggregation function $p: L^n \to L$. In this way we obtain a function $U: L_1 \times \cdots \times L_n \to L$ defined by

$$U(x) = p(\varphi_1(x_1), \ldots, \varphi_n(x_n)),$$

and we can choose the alternative that maximizes $U$. The function $U$ is called a global utility function, whereas the maps $\varphi_i$ are called local utility functions. The relevance of such functions is attested by their many applications in decision making, in particular, in representing preference relations [2].

It is common to choose the real interval $[0, 1]$ for $L$, and consider $\varphi_i(x_i)$ as a kind of “score” with respect to the $i$-th criterion. In this case, simple aggregation functions $p$ are for instance the weighted arithmetic means, but there are of course other, more elaborate ways of aggregating the scores such as the so-called Choquet integrals. However, in the qualitative approach, where only the ordering between scores is taken into account (for instance, when $L = \{\text{bad} < \text{OK} < \text{good} < \text{excellent}\}$), such operators are of little use since they rely heavily on the arithmetic structure of the real unit.
interval. In the latter setting, one of the most prominent class of aggregation functions is that of discrete Sugeno integrals, which coincides with the class of idempotent lattice polynomial functions (see [10]).

In [5] and [6] a more general situation was considered: $L$ is an arbitrary finite distributive lattice, the lattice polynomial functions are not assumed to be idempotent, and the local utility functions are not assumed to be monotone (instead they have to satisfy the boundary conditions $\varphi_i(0_i) \leq \varphi_i(x_i) \leq \varphi_i(1_i)$ for all $x_i \in L_i$, where $0_i$ and $1_i$ denote the least and greatest element of $L_i$). The corresponding compositions (6) were called pseudo-polynomial functions, and several axiomatizations were given for this class of functions. Besides axiomatization, another noteworthy problem is the factorization of such functions: given a function $U: L_1 \times \cdots \times L_n \to L$, find all factorizations of $U$ in the form (6). Such a factorization can be useful in real-life applications, when only the function $U$ is available (from empirical observations), and an analysis of the behavior of the local utility functions $\varphi_i$ and of the aggregation function $p$ could give valuable information about the decision maker’s attitude.

Suppose that we have already found the local utility functions $\varphi_i$ (see [5] and [6] for a method to find them), and let $a_i = \varphi_i(0_i), b_i = \varphi_i(1_i)$. If $x \in L_1 \times \cdots \times L_n$ is such that $x_i = 1_i$ if $i \in I$ and $x_i = 0_i$ if $i \notin I$, then $U(x) = p(\hat{e}_I)$. Thus, knowing the global utility function $U$, we have information about $p|_{D}$, and we can use Theorem 8 to find all possible lattice polynomial functions $p$ that can appear in a factorization (6) of $U$. (Of course, one has to take into account the other values of $U$ as well, but this can be done by using the boundary conditions.)

Acknowledgments. The first named author is supported by the internal research project F1R-MTH-PUL-09MRDO of the University of Luxembourg. The second named author acknowledges that the present project is supported by the TAMOP-4.2.1/B-09/1/KONV-2010-0005 program of the National Development Agency of Hungary, by the Hungarian National Foundation for Scientific Research under grants no. K77409 and K83219, by the National Research Fund of Luxembourg, and cofunded under the Marie Curie Actions of the European Commission (FP7-COFUND).

References

[1] Behrisch, M., Couceiro, M., Kearnes, K., Lehtonen, E., Szendrei, Á.: Commuting polynomial operations of distributive lattices. To appear in Order
[2] Bouyssou, D., Dubois, D., Prade, H., Pirlot, M. (eds): Decision-Making Process – Concepts and Methods. ISTE/John Wiley (2009)
[3] Couceiro, M., Lehtonen, E.: Self-commuting lattice polynomial functions on chains. Aequationes Math. 81(3), 263–278 (2011)
[4] Couceiro, M., Marichal, J.-L.: Characterizations of discrete Sugeno integrals as polynomial functions over distributive lattices. Fuzzy Sets and Systems 161(5), 694–707 (2010)
[5] Couceiro, M., Waldhauser, T.: Axiomatizations and factorizations of Sugeno utility functions. Internat. J. Uncertain. Fuzziness Knowledge-Based Systems 19(4), 635–658 (2011)

[6] Couceiro, M., Waldhauser, T.: Pseudo-polynomial functions over finite distributive lattices. In: Liu, W. (ed.) ECSQARU 2011. LNCS (LNAI), vol. 6717, pp. 545–556. Springer (2011)

[7] Davey, B. A., Priestley, H. A.: Introduction to Lattices and Order. Cambridge University Press, New York (2002)

[8] Goodstein, R. L.: The Solution of Equations in a Lattice. Proc. Roy. Soc. Edinburgh Section A, 67, 231–242 (1965/1967)

[9] Grätzer, G.: General Lattice Theory. Birkhäuser Verlag, Berlin (2003)

[10] Marichal, J.-L.: Weighted lattice polynomials. Discrete Mathematics 309(4), 814–820 (2009)

[11] Rico, A., Grabisch, M., Labreuche, Ch., Chateauneuf, A.: Preference modeling on totally ordered sets by the Sugeno integral. Discrete Applied Math. 147(1), 113–124 (2005)

(Miguel Couceiro) Mathematics Research Unit, FSTC, University of Luxembourg, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg, Luxembourg

E-mail address: miguel.couceiro[at]uni.lu

(Tamás Waldhauser) Mathematics Research Unit, FSTC, University of Luxembourg, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg, Luxembourg, and Bolyai Institute, University of Szeged, Aradi vétanúk tere 1, H-6720 Szeged, Hungary

E-mail address: twaldha@math.u-szeged.hu