JACOB’S LADDERS, NONLINEAR INTERACTIONS BETWEEN 
ζ-OCCILLATING SYSTEMS AND CORRESPONDING 
CONSTRAINTS

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Abstract. In this paper we introduce new class of nonlinear interactions of 
ζ-oscillating systems. The main formula is generated by corresponding subset 
of the set of trigonometric functions. Next, the main formula generates certain 
set of two-parts forms. For this set the following holds true: the cube of two-
part form is asymptotically equal to other two-part form – short functional 
algabra.

1. Introduction

1.1. Let us remind that we have obtained in our paper \[8\] certain class of formulae 
linear in different variables of the following type

\[
\prod_{r=1}^{k} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r \right)}{\zeta \left( \frac{1}{2} + i\beta_r \right)} \right|^2, \quad k = 1, \ldots, k_0,
\]

i.e. linear in different oscillating systems. Namely, see formulae in \[8\]:

\[
(1.1)
\]

\[
(1.2)
\]

It is, for example, the formula (7.1):

\[
\cos^2(\alpha_0^{2,2}) \prod_{r=1}^{k_2} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{2,2} \right)}{\zeta \left( \frac{1}{2} + i\beta_r^{2} \right)} \right|^2 +
\]

\[
+ \sin^2(\alpha_0^{1,1}) \prod_{r=1}^{k_1} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{1,1} \right)}{\zeta \left( \frac{1}{2} + i\beta_r^{1} \right)} \right|^2 \sim 1,
\]

\[
L \to \infty, \quad 1 \leq k_1, k_2 \leq k_0, \quad L, k_0 \in \mathbb{N},
\]

where

\[
\begin{align*}
\alpha_0^{1,1} &= \alpha_0^1(U, \mu, L, k_1; \sin^2 t), \\
\beta_r^{1} &= \beta_r(U, \mu, L, k_1), \\
\alpha_0^{2,2} &= \alpha_0^2(U, \mu, L, k_2; \cos^2 t), \\
\beta_r^{2} &= \beta_r(U, \mu, L, k_2).
\end{align*}
\]

Remark 1. We have called the formula (1.2) as the ζ-analogue of the elementary 
trigonometric identity

\[
\cos^2 t + \sin^2 t = 1.
\]

Key words and phrases. Riemann zeta-function.
Next, we have introduced in our paper [8] the following notions in connection with the formula (1.2):

(a) functionally depending $\zeta$-oscillating systems and linearly connected $\zeta$-oscillating systems, (see [8], beginning of section 2.4),
(b) interaction between corresponding $\zeta$-oscillating systems (see [8], Definition 4).

Remark 2. By (a) and (b) we may assume in the context of the paper [8] the following:

interaction = linear interaction,

(see, for example, (1.3), comp. [9], Remark 1).

Remark 3. Moreover, we notice that the $\zeta$-oscillating system itself (comp. (1.1)) is a complicated nonlinear system (comp. [8], (1.7), i.e. the spectral form of the Riemann-Siegel formula).

1.2. Next, let us remind the following notions we have introduced (see [1] – [7]) within the theory of the Riemann zeta-function:

(A) Jacob’s ladders, (see [1], comp. [2]),
(B) $\zeta$-oscillating system, (see [7], (1.1)),
(C) factorization formula, (see [5], (4.3) – (4.18), comp. [7], (2.1) – (2.7)),
(D) metamorphosis of the $\zeta$-oscillating systems:
   (a) first, the notion of metamorphoses of an oscillating multiform [4],
   (b) after that, the notion of metamorphoses of a quotient of two oscillating multiform, [5],
(E) $Z_{\zeta, Q^{2}}$-transformation (or device), [7].

1.3. In this paper, we shall present certain class of nonlinear intera ction formulae, namely:

(a) containing nonlinearities in variables of kind (1.1), (comp. Remark 2),
(b) describing interaction between corresponding $\zeta$-oscillating systems, i.e. ev- ery of these is functionally depending upon others of these $\zeta$-oscillating systems.

The main result is expressed by the following nonlinear formula:

\[
\begin{align*}
\prod_{r=1}^{k_1} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_{r}^{1,k_1} \right)}{\zeta \left( \frac{1}{2} + i\beta_{r}^{1,k_1} \right)} \right|^2 & \sim \\
\sim \frac{\cos(\alpha_{0}^{2,k_2})}{\cos^3(\alpha_{0}^{1,k_1})} \prod_{r=1}^{k_2} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_{r}^{2,k_2} \right)}{\zeta \left( \frac{1}{2} + i\beta_{r}^{2,k_2} \right)} \right|^2 & - \\
- \frac{U^2}{3} \frac{1}{\cos^3(\alpha_{0}^{1,k_1}) \cos^3 U} \times \\
\times \left\{ \cos^2(\alpha_{0}^{3,k_3}) \prod_{r=1}^{k_3} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_{r}^{3,k_3} \right)}{\zeta \left( \frac{1}{2} + i\beta_{r}^{3,k_3} \right)} \right|^2 \right. & - \sin^2(\alpha_{0}^{4,k_4}) \prod_{r=1}^{k_4} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_{r}^{4,k_4} \right)}{\zeta \left( \frac{1}{2} + i\beta_{r}^{4,k_4} \right)} \right|^2 \left. \right\}^3, \\
L \to \infty, 1 \leq k_1, k_2, k_3, k_4 \leq k_0,
\end{align*}
\]

(1.4) (comp. (1.3), (1.4) and (a) in the beginning of this section).
2. Lemmas

2.1. Since
\[ \int \cos^3 t \, dt = \sin t - \frac{1}{3} \sin^3 t, \]
then
\[ \int_{2\pi L}^{2\pi L+U} \cos^3 t \, dt = \sin U - \frac{1}{3} \sin^3 U, \ L \in \mathbb{N}, \]
and
\[ \frac{1}{U} \int_{2\pi L}^{2\pi L+U} \cos^3 t \, dt = \frac{\sin U}{U} - \frac{U^2}{3} \left( \frac{\sin U}{U} \right)^3. \]

Of course,
(2.1)
\[ f_1(t) = \cos^3 t \in \tilde{C}_0[2\pi L, 2\pi L + U], \]
the class of functions \( \tilde{C}_0 \) has been defined in our paper [9], where
\[ t \in [2\pi L, 2\pi L + U], \ U \in \left( 0, \frac{\pi}{2} - \epsilon \right), \]
and \( \epsilon > 0 \) is sufficiently small. Consequently, we obtain for generating of the factorization formula by making use our algorithm (see [8], (3.1) – (3.11)) the following

Lemma 1. For the function (2.1) there are vector-valued functions
\[ (\alpha_1^{1, k_1}, \ldots, \alpha_1^{1, k_1}, \ldots, \alpha_1^{1, k_1}, \beta_1^{1, k_1}, \ldots, \beta_1^{1, k_1}), \ k_1 = 1, \ldots, k_0, \ k_0 \in \mathbb{N} \]
\((k_0 \text{ is arbitrary and fixed})\) such that the following factorization formula holds true:
\[ \prod_{r=1}^{k_1} \left| \frac{\zeta \left( \frac{1}{2} + i \alpha_r^{1, k_1} \right)}{\zeta \left( \frac{1}{2} + i \beta_r^{1, k_1} \right)} \right|^2 \sim \left[ \sin \frac{U}{U} - \frac{U^2}{3} \left( \frac{\sin \frac{U}{U}}{\sin \frac{U}{U}} \right)^3 \right] \frac{1}{\cos^3 \left( \alpha_0^{1, k_1} \right)}, \ L \to \infty, \]
where
\[ \alpha_r^{1, k_1} = \alpha_r(U, L, k_1; f_1), \ r = 0, 1, \ldots, k_1, \]
\[ \beta_r^{1, k_1} = \beta_r(U, L, k_1), \ r = 1, \ldots, k_1, \]
\[ 2\pi L < \alpha_0^{1, k_1} < 2\pi L + U \Rightarrow 0 < \alpha_0^{1, k_1} - 2\pi L < U, \]
\[ 1 \leq k_1 \leq k_0. \]

Remark 4. In the asymptotic formula (2.2) the symbol \( \sim \) stands for (see [8], (3.8))
\[ = \left\{ 1 + O \left( \frac{\ln \ln L}{\ln L} \right) \right\}. \]

2.2. Let
\[ f_2(t) = \cos t, \ t \in [2\pi L, 2\pi L + U], \ U \in \left( 0, \frac{\pi}{2} - \epsilon \right). \]
Since
\[ f_2(t) \in \tilde{C}_0[2\pi L, 2\pi L + U] \]
then we obtain by similar way as in the case of Lemma 1 the following.
Lemma 2. For the function \((2.4)\) there are vector-valued functions
\[
(\alpha_0^{2,k_2}, \alpha_1^{2,k_2}, \ldots, \alpha_k^{2,k_2}, \beta_1^{2,k_2}, \ldots, \beta_k^{2,k_2}), \quad k_2 = 1, \ldots, k_0, \quad k_0 \in \mathbb{N}
\]
\((k_0 \text{ is arbitrary and fixed})\) such that the following factorization formula holds true:
\[
(2.5) \quad \prod_{r=1}^{k_0} \frac{\zeta \left( \frac{1}{2} + i \alpha_r^{2,k_2} \right)}{\zeta \left( \frac{1}{2} + i \beta_r^{2,k_2} \right)} \sim \frac{\sin U}{U} \frac{1}{\cos(\alpha_0^{2,k_2})}, \quad L \to \infty,
\]
where
\[
\alpha_r^{2,k_2} = \alpha_r(U, L, k_2; f_2), \quad r = 0, 1, \ldots, k_2,
\]
\[
\beta_r^{2,k_2} = \beta_r(U, L, k_2), \quad r = 1, \ldots, k_2,
\]
\[
0 < \alpha_0^{2,k_2} - 2\pi L < U,
\]
\[
1 \leq k_2 \leq k_0.
\]

2.3. Next, let us remind (see [8], (8.1) and also (4.2), (4.3), (4.6), (4.7)) the following formula
\[
(2.6) \quad \frac{\cos^2(\alpha_0^{2, k_2})}{\cos(2\mu + U)} \prod_{r=1}^{k_1} \left| \frac{\zeta \left( \frac{1}{2} + i \alpha_r^{2, k_2} \right)}{\zeta \left( \frac{1}{2} + i \beta_r^{2, k_2} \right)} \right|^2 \sim \frac{\sin^2(\alpha_0^{1, k_1})}{\cos(2\mu + U)} \prod_{r=1}^{k_1} \left| \frac{\zeta \left( \frac{1}{2} + i \alpha_r^{1, k_1} \right)}{\zeta \left( \frac{1}{2} + i \beta_r^{1, k_1} \right)} \right|^2 \sim \frac{\sin U}{U}, \quad L \to \infty,
\]
\[
t \in [\pi L + \mu, \pi L + \mu + U].
\]
Now, if we put (in our present context)
\[
\mu = 0, L \to 2L,
\]
\[
\alpha_r^{2, k_2} \to \alpha_r^{3, k_1}, \quad r = 0, 1, \ldots, k_3,
\]
\[
\beta_r^{2} \to \beta_r^{3, k_1}, \quad r = 1, \ldots, k_3,
\]
\[
\alpha_r^{1, k_1} \to \alpha_r^{4, k_4}, \quad r = 0, 1, \ldots, k_4,
\]
\[
\beta_r^{1} \to \beta_r^{4, k_4}, \quad r = 1, \ldots, k_4,
\]
then we obtain from \((2.6)\) in the case
\[
t \in [2\pi L, 2\pi L + U], \quad U \in \left(0, \frac{\pi}{2} - \epsilon\right]
\]
the following
\[
(2.8) \quad \frac{\cos^2(\alpha_0^{3, k_3})}{\cos U} \prod_{r=1}^{k_1} \left| \frac{\zeta \left( \frac{1}{2} + i \alpha_r^{3, k_3} \right)}{\zeta \left( \frac{1}{2} + i \beta_r^{3, k_3} \right)} \right|^2 - \frac{\sin^2(\alpha_0^{4, k_4})}{\cos U} \prod_{r=1}^{k_1} \left| \frac{\zeta \left( \frac{1}{2} + i \alpha_r^{4, k_4} \right)}{\zeta \left( \frac{1}{2} + i \beta_r^{4, k_4} \right)} \right|^2 \sim \frac{\sin U}{U}, \quad L \to \infty,
\]
where

\[ f_3(t) = \cos^2 t \rightarrow \prod_{r=1}^{k_3} \frac{\zeta\left(\frac{1}{2} + i\alpha^{3,k_3}_r\right)}{\zeta\left(\frac{1}{2} + i\beta^{3,k_3}_r\right)} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha^{3,k_3}_r\right)}{\zeta\left(\frac{1}{2} + i\beta^{3,k_3}_r\right)} \right|^2, \]

\[ f_4(t) = \sin^2 t \rightarrow \prod_{r=1}^{k_4} \frac{\zeta\left(\frac{1}{2} + i\alpha^{4,k_4}_r\right)}{\zeta\left(\frac{1}{2} + i\beta^{4,k_4}_r\right)} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha^{4,k_4}_r\right)}{\zeta\left(\frac{1}{2} + i\beta^{4,k_4}_r\right)} \right|^2, \]

\[ t \in [2\pi L, 2\pi L + U], \]

and

\[ \alpha^{3,k_3}_0 = \alpha^{2,2}_0(U, 0, 2L; f_3), \ldots \]
\[ \beta^{3,k_3}_r = \beta^{2}_r(U, 0, 2L, k_3), \]
\[ \alpha^{4,k_4}_0 = \alpha^{1,1}_0(U, 0, 2L; f_4), \ldots \]
\[ \beta^{4}_r \rightarrow \beta^{4}_{r}, \ r = 1, \ldots, k_4, \]
\[ \beta^{4}_{k_4} = \beta^{1}_r(U, 0, 2L, k_4). \]

3. **Theorem**

3.1. Now, we obtain by making use of (2.2), (2.5) and (2.7) the following nonlinear interaction formula.

**Theorem.**

\[
\prod_{r=1}^{k_1} \frac{\zeta\left(\frac{1}{2} + i\alpha^{1,k_1}_r\right)}{\zeta\left(\frac{1}{2} + i\beta^{1}_{k_1}\right)} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha^{1,k_1}_r\right)}{\zeta\left(\frac{1}{2} + i\beta^{1}_{k_1}\right)} \right|^2 \sim \frac{\cos(\alpha^{2,k_2}_0)}{\cos^3(\alpha^{1,k_1}_0)} \prod_{r=1}^{k_2} \frac{\zeta\left(\frac{1}{2} + i\alpha^{2,k_2}_r\right)}{\zeta\left(\frac{1}{2} + i\beta^{2}_{k_2}\right)} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha^{2,k_2}_r\right)}{\zeta\left(\frac{1}{2} + i\beta^{2}_{k_2}\right)} \right|^2 - \frac{U^2}{3 \cos^2(\alpha^{1,k_1}_0) \cos^3 U} \right\}

\[
\times \left\{ \cos^2(\alpha^{3,k_3}_0) \prod_{r=1}^{k_3} \frac{\zeta\left(\frac{1}{2} + i\alpha^{3,k_3}_r\right)}{\zeta\left(\frac{1}{2} + i\beta^{3}_{k_3}\right)} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha^{3,k_3}_r\right)}{\zeta\left(\frac{1}{2} + i\beta^{3}_{k_3}\right)} \right|^2 - \sin^2(\alpha^{4,k_4}_0) \prod_{r=1}^{k_4} \frac{\zeta\left(\frac{1}{2} + i\alpha^{4,k_4}_r\right)}{\zeta\left(\frac{1}{2} + i\beta^{4}_{k_4}\right)} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha^{4,k_4}_r\right)}{\zeta\left(\frac{1}{2} + i\beta^{4}_{k_4}\right)} \right|^2 \right\}^3, \]

\[ L \rightarrow \infty, \]

\[ 1 \leq k_1, k_2, k_3, k_3 \leq k_0. \]

Next, we give following corollaries from the Theorem, (comp. (b) in the beginning of the section 1.3).
Corollary 1.

\[
\left\| \frac{\zeta \left( \frac{1}{2} + i \alpha_r \kappa_r \right)}{\zeta \left( \frac{1}{2} + i \beta_r \kappa_r \right)} \right\|^2 \sim \\
\frac{\cos^3(\alpha_{0,1}^1, \kappa_1)}{\cos(\alpha_{0,2}^{k_2})} \prod_{r=1}^{k_1} \frac{\zeta \left( \frac{1}{2} + i \alpha_r \kappa_r \right)}{\zeta \left( \frac{1}{2} + i \beta_r \kappa_r \right)} \right\|^2 + \frac{U^2}{3} \frac{1}{\cos(\alpha_{0}^{2, k_2}) \cos^3 U} \times \\
\left\{ \cos^2(\alpha_{0}^{3, k_3}) \prod_{r=1}^{k_3} \frac{\zeta \left( \frac{1}{2} + i \alpha_r \kappa_r \right)}{\zeta \left( \frac{1}{2} + i \beta_r \kappa_r \right)} \right\}^2 - \sin^2(\alpha_{0}^{4, k_4}) \prod_{r=1}^{k_4} \frac{\zeta \left( \frac{1}{2} + i \alpha_r \kappa_r \right)}{\zeta \left( \frac{1}{2} + i \beta_r \kappa_r \right)} \right\}^2 \}
\]

\( L \to \infty. \)

Corollary 2.

\[
\left\| \frac{\zeta \left( \frac{1}{2} + i \alpha_r \kappa_r \right)}{\zeta \left( \frac{1}{2} + i \beta_r \kappa_r \right)} \right\|^2 \sim \\
\frac{\sin^2(\alpha_{0}^{4, k_4})}{\cos^2(\alpha_{0}^{3, k_3})} \prod_{r=1}^{k_3} \frac{\zeta \left( \frac{1}{2} + i \alpha_r \kappa_r \right)}{\zeta \left( \frac{1}{2} + i \beta_r \kappa_r \right)} \right\|^2 + \\
\left\{ \frac{3}{U^2} \frac{\cos^3 U \cos(\alpha_{0}^{2, k_2})}{\cos^6(\alpha_{0}^{3, k_3})} \prod_{r=1}^{k_2} \frac{\zeta \left( \frac{1}{2} + i \alpha_r \kappa_r \right)}{\zeta \left( \frac{1}{2} + i \beta_r \kappa_r \right)} \right\}^2 - \\
\left\{ \frac{3}{U^2} \frac{\cos^3 U \cos^3(\alpha_{0}^{1, k_1})}{\cos^6(\alpha_{0}^{3, k_3})} \prod_{r=1}^{k_1} \frac{\zeta \left( \frac{1}{2} + i \alpha_r \kappa_r \right)}{\zeta \left( \frac{1}{2} + i \beta_r \kappa_r \right)} \right\}^2 \}
\]

\( L \to \infty. \)

Corollary 3.

\[
\left\| \frac{\zeta \left( \frac{1}{2} + i \alpha_r \kappa_r \right)}{\zeta \left( \frac{1}{2} + i \beta_r \kappa_r \right)} \right\|^2 \sim \\
\frac{\cos^2(\alpha_{0}^{3, k_3})}{\sin^2(\alpha_{0}^{4, k_4})} \prod_{r=1}^{k_4} \frac{\zeta \left( \frac{1}{2} + i \alpha_r \kappa_r \right)}{\zeta \left( \frac{1}{2} + i \beta_r \kappa_r \right)} \right\|^2 - \\
\left\{ \frac{3}{U^2} \frac{\cos^3 U \cos(\alpha_{0}^{2, k_2})}{\sin^6(\alpha_{0}^{4, k_4})} \prod_{r=1}^{k_2} \frac{\zeta \left( \frac{1}{2} + i \alpha_r \kappa_r \right)}{\zeta \left( \frac{1}{2} + i \beta_r \kappa_r \right)} \right\}^2 - \\
\left\{ \frac{3}{U^2} \frac{\cos^3 U \cos^3(\alpha_{0}^{1, k_1})}{\sin^6(\alpha_{0}^{4, k_4})} \prod_{r=1}^{k_1} \frac{\zeta \left( \frac{1}{2} + i \alpha_r \kappa_r \right)}{\zeta \left( \frac{1}{2} + i \beta_r \kappa_r \right)} \right\}^2 \}
\]

\( L \to \infty. \)
3.2. Let us remind the following correspondences (see (2.1), (2.2); (2.3), (2.4); (2.9)):

\[ f_1(t) = \cos^3 t \rightarrow \prod_{r=1}^{k_1} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_{r,k_1} \right)}{\zeta \left( \frac{1}{2} + i\beta_{r,k_1} \right)} \right|^2 = (k_1, f_1), \]

\[ f_2(t) = \cos t \rightarrow \prod_{r=1}^{k_2} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_{r,k_2} \right)}{\zeta \left( \frac{1}{2} + i\beta_{r,k_2} \right)} \right|^2 = (k_2, f_2), \]

\[ f_3(t) = \cos^2 t \rightarrow \prod_{r=1}^{k_3} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_{r,k_3} \right)}{\zeta \left( \frac{1}{2} + i\beta_{r,k_3} \right)} \right|^2 = (k_3, f_3), \]

\[ f_4(t) = \sin^2 t \rightarrow \prod_{r=1}^{k_4} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_{r,k_4} \right)}{\zeta \left( \frac{1}{2} + i\beta_{r,k_4} \right)} \right|^2 = (k_4, f_4), \]

\[ 1 \leq k_1, k_2, k_3, k_4 \leq k_0. \]

**Remark 5.** First, we see that the formulae (3.1) – (3.4) define the set of nonlinear interactions of the \((k_0)^4\) oscillating systems in (3.5). Namely, this set contains \(4(k_0)^4\) elements of different type. Now, if we use our short notions in (3.5) we may write down the following diagram

\[
\begin{array}{ccc}
(k_1, f_1) & \longleftrightarrow & (k_2, f_2) \\
\uparrow & & \uparrow \\
(k_4, f_4) & \longleftrightarrow & (k_3, f_3)
\end{array}
\]

4. **Short functional algebra**

We shall call each of the following type of functional combinations

\[ \cos^2(\alpha_{0,k_3}^3) \prod_{1}^{k_3} - \sin^2(\alpha_{0,k_4}^4) \prod_{1}^{k_4}, \]

\[ \frac{3}{U^2} \cos^3 U \cos(\alpha_{0,k_2}^2) \prod_{1}^{k_2} - \frac{3}{U^2} \cos^3 U \cos^3(\alpha_{0,k_1}^1) \prod_{1}^{k_1}, \]

of two \(\zeta\)-oscillating systems as the two-parts form. Since

\[ 1 \leq k_1, k_2, k_3, k_4 \leq k_0 \]

then the set of formulae (4.1) (just as similar set (4.2)) contains \((k_0)^2\) two-part forms of different type. In this direction we have (see (3.1)) the following
Corollary 4.

\[
\left\{ \cos^2(\alpha_0^{3,k_3}) \prod_{r=1}^{k_3} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{3,k_3}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^{k_3}\right)} \right|^2 - \\
- \sin^2(\alpha_0^{4,k_4}) \prod_{r=1}^{k_4} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{4,k_4}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^{k_4}\right)} \right|^2 \right\} \sim \]

(4.3)

\[
\sim \frac{3}{U^2} \cos^3 U \cos(\alpha_0^{2,k_2}) \prod_{r=1}^{k_2} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{2,k_2}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^{k_2}\right)} \right|^2 - \\
- \frac{3}{U^2} \cos^3 U \cos^3(\alpha_0^{1,k_1}) \prod_{r=1}^{k_1} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{1,k_1}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^{k_1}\right)} \right|^2,
\]

\[L \to \infty.\]

Remark 6. The following property is expressed by the formula (4.3): the cube of two-parts form (4.1) is asymptotically equal to the two-part form (4.2). Of course, we have also the following formula

\[\sqrt[3]{(4.2)} \sim (4.1).\]

Next, it is true (see (4.3)) that for every fixed pair \((k_3, k_4)\) we have set of \((k_0)^2\) asymptotic formulae of different types for the cube of corresponding two-parts form (4.1). For example, in the case \(k_0 = 10^3\) we have \(10^6\) of these formulae.

Remark 7. Consequently, we may regard the formula (4.3) as a kind of simplification of the well-known school-formula

\[(a - b)^3 = a^3 - 3a^2b + 2ab^2 - b^3\]

in this short functional algebra generated by the formula (3.1). Namely, the right-hand side of (4.3) contains two-parts form (i.e. the two members only in comparison with (4.3)).

5. The iteration formula as a constraint on the corresponding vector-valued functions generated by the operator \(\hat{H}\)

Let us remind that we have defined (see [8], Definition 2 and Definition 5; [9], Definition) new type of vector-valued operator \(\hat{H}\) as follows:

\[\forall f(t) \in \tilde{C}_0[T, T + U] \to \hat{H}f(t) = \]
\[= (\alpha_0, \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k), \quad k = 1, \ldots, k_0,\]

for every fixed \(k\).

Consequently, we have the following set of \(k_0\) vector-valued functions

\[(\alpha_0, \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k), \quad k = 1, \ldots, k_0.\]
Remark 8. That is, we may say that we have defined the following matrix-valued operator $\hat{H}$:

$$
\begin{pmatrix}
\alpha_0 & \alpha_1 & \beta_1 & 0 & \ldots \\
\alpha_0 & \alpha_1 & \alpha_2 & \beta_1 & \beta_2 & 0 & \ldots \\
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \beta_1 & \beta_2 & \beta_3 & 0 & \ldots \\
\vdots \\
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k_0} & \beta_1 & \beta_2 & \ldots & \beta_{k_0}
\end{pmatrix}_{k_0 \times (2k_0+1)}
$$

Next, it is true that every interaction formula:

(a) contains some set of $\zeta$-oscillating systems,
(b) every $\zeta$-oscillating system from that set contains the components of corresponding vector-valued function of type (5.1).

Remark 9. Consequently, we may understand every interaction formula (3.1), for example, as the constraint on the set of corresponding vector-valued functions of type (5.1) which are contained in this formula.

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