RECENT PROGRESSES ON ELLIPTIC TWO-PHASE FREE
BOUNDARY PROBLEMS

DANIELA DE SILVA
Department of Mathematics, Barnard College, Columbia University
New York, NY 10027, USA

FAUSTO FERRARI
Dipartimento di Matematica dell’Università di Bologna
Piazza di Porta S. Donato, 5
40126 Bologna, Italy

SANDRO SALSA*
Dipartimento di Matematica del Politecnico di Milano
Leonardo da Vinci, 32
20133 Milano, Italy

To Luis, with friendship and admiration

Abstract. We provide an overview of some recent results about the regularity
of the solution and the free boundary for so-called two-phase free boundary
problems driven by uniformly elliptic equations.

1. Introduction. In this survey paper we consider free boundary problems gov-
erned by uniformly elliptic equations in which the state variable can assume two
phases and the condition across the free boundary is expressed by an energy balance
involving the fluxes from both sides (so called Bernoulli type problems). Typical
examples come from constraint minimization problems as

$$\min_{u \in g + H^1_0(B_1)} \left\{ a_{ij}(x) u_{x_i} u_{x_j} + q(x) \chi_{\{u>0\}} \right\}$$

over $u \in g + H^1_0(B_1)$, with $q \geq c > 0$ a.e. Here the free boundary condition takes
the form

$$(u^+_\nu)^2 - (u^-_\nu)^2 = q(x).$$

Other examples arise in flame propagation as limits of singular perturbation prob-
lems with forcing term (see [33, 37]), in the Stokes and Prandtl-Batchelor models
in classical hydrodynamics (see [5, 7]), in free transmission problems (see [4].)
Among the several concepts of solutions we use the notion of viscosity solution
introduced by Caffarelli in the seminal papers [8, 9, 10], which seems to be the most

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* Corresponding author: Sandro Salsa.
appropriate to study optimal regularity of both solutions and free boundaries in
great generality. In these papers Caffarelli developed a general strategy to attack the
optimal regularity of both the solution and the free boundary based on: a powerful
monotonicity formula, Harnack principles and families of continuous perturbations.
A different approach, relying on Harnack principles and linearization, has been
introduced by De Silva in [15] for a one phase model motivated by the classical
Stokes problem in hydrodynamics.

The main common underlying idea of both techniques to obtain the regularity
of the free boundary is to set up an iterative improvement of flatness argument in
a neighborhood of a point where one of the two phases enjoys a non-degeneracy
condition (e.g. a linear growth).

In the last two decades there has been a huge development in various directions
of the ideas in [8, 9, 10] and more recently of [15].

Our aim is to describe some of the main results, confining ourselves for simplicity
to isotropic free boundary conditions. At the same time, we shall single out some
questions that are still open, also adding some clues about the typical difficulties
one has to face in trying to get an answer.

2. Elliptic free boundary problems and their viscosity solutions. In a
bounded domain $\Omega \subset \mathbb{R}^n$, we consider the problem

$$
\begin{aligned}
\mathcal{L}_1 u &= f_1 & \text{in } \Omega^+(u) \\
\mathcal{L}_2 u &= f_2 & \text{in } \Omega^-(u) \\
u_+^\nu &= G(u_\nu) & \text{on } F(u) = \partial \Omega^+(u),
\end{aligned}
$$

where

$$
\Omega^+(u) = \{ x \in \Omega : u(x) > 0 \}, \quad \Omega^-(u) = \{ x \in \Omega : u(x) \leq 0 \}.
$$

Here $f_1, f_2$ are bounded on $\Omega$ and continuous in $\Omega^+(u) \cup \Omega^-(u)$, while $u_+^\nu$ and $u_-^\nu$ denote the normal derivatives in the inward direction to $\Omega^+(u)$ and $\Omega^-(u)$ respectively. $F(u)$ is called the free boundary.

$\mathcal{L}_1, \mathcal{L}_2$ are uniformly elliptic operators with ellipticity constants $0 < \lambda < \Lambda$, in divergence or nondivergence form

$$
\mathcal{L} u = \text{div}(A(x) \nabla u) \quad \text{or} \quad \mathcal{L} u = \text{Tr} (A(x) D^2 u) + b(x) \cdot \nabla u,
$$

with $A$ a Hölder continuous matrix and $b$ a vector of continuous bounded coefficients (indeed it is sufficient to assume that $b$ is bounded measurable), or $\mathcal{L}$ are fully nonlinear operators:

$$
\mathcal{L} u = \mathcal{F} (D^2 u) \quad (\mathcal{F}(O) = 0)
$$

where $D^2 u$ is the Hessian matrix of $u$.

We assume the function $G(t) : [0, \infty) \to (0, \infty)$ to be regular and strictly increasing with $G(0) > 0$, $G(t) \to \infty$ as $t \to \infty$. In addition we also require that there exists $N > 0$ such that $t^{-N} G(t)$ is strictly decreasing.

We now recall the definition of weak or viscosity solution. We say that a point
$x_0 \in F(u)$ is regular from the right (resp. left) if there is a ball $B \subset \Omega^+(u)$ (resp. $\Omega^-(u)$), such that $B \cap F(u) = \{ x_0 \}$.

**Definition 2.1.** We say that $u \in C(\Omega)$ is a viscosity solution of f.b.p. (1) if:

i). $\mathcal{L}_1 u = f_1$ in $\Omega^+(u)$ and $\mathcal{L}_2 u = f_2$ in $\Omega^-(u)$ in the usual viscosity sense;

ii). $u$ satisfies the free boundary condition in the following sense:
1. If \( x_0 \in F(u) \) is regular from the right with tangent ball \( B \) then
\[
    u^+(x) \geq \alpha(x - x_0, \nu)^+ + o(|x - x_0|) \quad \text{in } B, \quad \text{with } \alpha \geq 0
\]
and
\[
    u^-(x) \leq \beta(x - x_0, \nu)^- + o(|x - x_0|) \quad \text{in } B^c, \quad \text{with } \beta \geq 0
\]
with equality along every non tangential domain, and \( \alpha \leq G(\beta) \).

2. If \( x_0 \in F(u) \) is regular from the left with tangent ball \( B \), then
\[
    u^-(x) \geq \beta(x - x_0, \nu)^- + o(|x - x_0|) \quad \text{in } B, \quad \text{with } \beta \geq 0
\]
and
\[
    u^+(x) \leq \alpha(x - x_0, \nu)^- + o(|x - x_0|) \quad \text{in } B^c, \quad \text{with } \alpha \geq G(\beta).
\]

Here \( \nu = \nu(x_0) \) denotes the unit normal to \( \partial B \) at \( x_0 \), pointing towards \( \Omega^+(u) \) and \( B^c \) the complementary set of \( B \) in \( \mathbb{R}^n \).

Another way of defining viscosity solutions is through comparison with classical sub/super solutions. For the equivalence of the two definitions we refer to [12]. See also [18]. For completeness, we recall also this definition.

**Definition 2.2.** We say that \( v \in C(\Omega) \) is a \( C^2 \) strict (comparison) subsolution (resp. supersolution) to our f.b.p. in \( \Omega \), if \( v \in C^2(\Omega^+(v)) \cap C^2(\Omega^-(v)) \) and the following conditions are satisfied:
1. \( \mathcal{L}_1 v > f_1 \) (resp. \( < f_1 \)) in \( \Omega^+(v) \) and \( \mathcal{L}_2 v > f_2 \) (resp. \( < f_2 \)) in \( \Omega^-(v) \);
2. If \( x_0 \in F(v) \), then
\[
    v_\nu^+(x_0) > G(v_\nu^-(x_0)) \quad \text{(resp. } v_\nu^+(x_0) < G(v_\nu^-(x_0)) \text{)}, \quad v_\nu^+(x_0) \neq 0.)
\]

Viscosity sub/super solutions are now defined in the usual way. Recall that, given \( u, \varphi \in C(\Omega) \), we say that \( \varphi \) touches \( u \) below (resp. above) at \( x_0 \in \Omega \) if \( u(x_0) = \varphi(x_0) \), and
\[
    u(x) \geq \varphi(x) \quad \text{(resp. } u(x) \leq \varphi(x) \text{)} \quad \text{in a neighborhood } O \text{ of } x_0.
\]
If this inequality is strict in \( O \setminus \{x_0\} \), we say that \( \varphi \) touches \( u \) strictly by below (resp. above).

**Definition 2.3.** Let \( u \) be a continuous function in \( \Omega \). We say that \( u \) is a viscosity solution to our f.b.p. in \( \Omega \), if the following conditions are satisfied:
1. \( \mathcal{L}_1 u = f_1 \) in \( \Omega^+(v) \) and \( \mathcal{L}_2 u = f_2 \) in \( \Omega^-(v) \) in the viscosity sense;
2. Any (strict) comparison subsolution \( v \) (resp. supersolution) cannot touch \( u \) by below (resp. by above) at a point \( x_0 \in F(v) \).

After the aforementioned seminal papers of Caffarelli in the 80’s, the theory for problem (1) can be developed according to by now a well established paradigm:

(a) Existence of solutions (e.g. Perron or variational solutions or solutions obtained as a limit of singular perturbations).

Due to the highly nonlinear nature of the problems, uniqueness is hardly to be expected, see [2].

However, some results on uniqueness are shown in [34].

(b) Weak regularity properties of the f.b., such as finite perimeter and density properties for the positivity set.

(c) Lipschitz continuity of viscosity solutions. Clearly, given the jump on the gradient along the free boundary, this is the optimal regularity for a solution.
(d) Strong regularity properties of the f.b. For instance Lipschitz or “flat” (see below) free boundaries are $C^1$ or better.

(e) Higher regularity: Schauder estimates and analyticity for both solution and f.b.

(f) Classification of global solutions and analysis of nonregular points.

3. Existence of viscosity solutions and weak regularity of the free boundary. We start by describing the existence theory for viscosity solutions and the weak regularity properties of the free boundary. Throughout the section $\mathcal{L}_1 = \mathcal{L}_2$ and this common operator is denoted by $\mathcal{L}$.

In [10] Caffarelli considered the homogeneous case, i.e. $f_i = 0$, $i = 1, 2$, when $\mathcal{L} = \text{div}(A(x) \nabla)$. His main result is the following

**Theorem 3.1.** Given a Lipschitz domain $\Omega$ and $g \in C(\partial \Omega)$, there exists a viscosity solution $u \in C(\overline{\Omega})$ with $u = g$ on $\partial \Omega$. Moreover, $u$ is locally Lipschitz in $\Omega$, $\Omega^+(u)$ is a set of finite perimeter and

$$0 < \alpha_1 \leq \frac{u^+(x)}{\text{dist}(x, F(u))} \leq \alpha_2.$$  \hspace{1cm} (4)

Moreover, for $c, r_0$ universal, $r < r_0$, we have for all $x \in F(u)$,

$$\mathcal{H}^{n-1}(F(u) \cap B_r(x)) \leq cr^{n-1},$$

and, denoting by $F^*(u)$ the reduced free boundary,

$$\mathcal{H}^{n-1}(F^*(u) \cap B_r(x)) \geq cr^{n-1}, \quad \mathcal{H}^{n-1}(F(u) \setminus F^*(u)) = 0. \hspace{1cm} (5)$$

Note the linear growth of the positive part of the solution, expressed by the left inequality in (4) and the density property of $F(u)$ at every point of the reduced boundary.

Caffarelli used a Perron method for constructing a minimal solution given by

$$u(x) = \inf_{v \in S} v(x)$$

where $S$ is a special class of continuous supersolutions $v$ in $\overline{\Omega}$. From a philosophical point of view, this result implies the existence of universal regularity and nondegeneracy properties for these kind of problems. This result has been extended by P.Y. Wang in [42] to the case $\mathcal{L} = F(D^2u)$ with $F$ concave. Existence and weak regularity for $\mathcal{L}u := F(D^2u, Du)$, with $F$ non concave in the Hessian matrix and even for the linear case $\mathcal{L}u := \text{Tr}(A(x) D^2u) + b(x) \cdot \nabla u$ remain open questions.

The main obstruction in extending Caffarelli’s method is the lack of the following monotonicity formula of Alt, Caffarelli and Friedman [3].

**Theorem 3.2.** Let $u = u^+ - u^- \in C(B_1) \cap H^1(\Omega)$, such that $\text{div}(A(x) \nabla u^\pm) \geq 0$ in $B_1$, with $a_{ij}(0) = \delta_{ij}$, $u^+(0) = u^-(0) = 0$. Assume that $A$ is a Hölder continuous matrix of coefficients with exponent $\alpha$. Define the function

$$\Phi(r) \equiv r^{-4} \int_{B_r} \frac{|\nabla u^+|^2}{|x|^{n-2}} \, dx \int_{B_r} \frac{|\nabla u^-|^2}{|x|^{n-2}} \, dx.$$  \hspace{1cm} (6)

Then for, say, $0 < r \leq 1/2$, and some constant $c(n) > 0$, the function $\Phi(r) = c(n) r^{n \alpha}$ is increasing and

$$\Phi(r) \leq c(n) \|u\|_{L^\infty(B_1)}^4.$$
Moreover, if $a_{ij}(x) = \delta_{ij}$, then $\Phi$ is constant if and only if $u^+$ and $u^-$ are linear functions.

Observe that if the supports of $u^+$ and $u^-$ were separated by a smooth surface with normal $\nu$ at $x = 0$ then, by taking the limit as $r \to 0$, we could deduce that

$$(u^+_r(0))^2(u^-_r(0))^2 \leq \Phi(1/2)$$

so that, “morally” $\Phi(r)$ gives a control in average of the product of the normal derivatives of $u$ at the origin. Then, from the free boundary condition one deduces that $u^+_r(0)$ is bounded above.

As a consequence the monotonicity formula plays a key role in proving the Lipschitz continuity of the minimal solution, but it has other consequences, as we will see in next section.

The ACF monotonicity formula has been used in various contexts and extended to nonhomogeneous equations. In particular the following almost monotonicity formula holds: if $L$ is in divergence form and $Lu^+ \geq -1, Lu^- \geq -1$, then for $0 < r \leq r_0$,

$$\Phi(r) \leq c_0(1 + \|u^+\|^2_{L^\infty(B_r)} + \|u^-\|^2_{L^\infty(B_r)})$$

with $r_0$ and $c_0$ universal (see [13, 36] and also [38]). In the nonhomogeneous case, using the almost monotonicity formula, Lipschitz Perron solutions for $f_1, f_2$ bounded and continuous, are constructed in [18] when $L$ is in divergence form and in [39], when $L := F(D^2u)$ with $F$ concave. These papers follow the main guidelines of [10], although the presence of a distributed source requires to face new situations and requires new delicate arguments. Indeed, for instance, the standard technique of “harmonic” replacement cannot be applied in these cases, since no sign condition is posed on the right-hand-sides $f_1$ and $f_2$. This difficulty is bypassed by resorting to special obstacle type problems.

We conclude this section by mentioning a different approach to prove existence for a specific class of free boundary problems in divergence form that can be found for example in [37].

4. Lipschitz continuity and global solutions. The Lipschitz continuity of solutions to (1) is a crucial ingredient in the study of the regularity of the free boundary $F(u)$. Indeed it provides compactness to carry on a blow-up analysis around a point $x_0 \in F(u)$, reducing the problem to the classification of global Lipschitz solution. For instance, if $L_1 = L_2 = -\Delta, f_1 = f_2 = 0$, having at disposal a monotonicity formula, it is possible to classify global solutions $U$ as either purely two-plane functions ($\nu$ a unit direction),

$$U(x) = \alpha(x - x_0, \nu)^+ - \beta(x - x_0, \nu)^- \quad \alpha = G(\beta), \beta > 0$$

or one-phase solutions, in case we have

$$U^- \equiv 0.$$

In particular, if $u$ is a solution to (1) with $0 \in F(u)$, then via a blow-up analysis and flatness results (see Section 5) either $u^-(x) = o(|x|)$ or $F(u)$ is $C^{1,\gamma}$ in a neighborhood of 0 that is, the only types of singular points are the ones that occur in the one-phase setting when the negative phase is identically zero (see Subsection 4.4 for further details on the analysis of singular points of one-phase problems).

On points of the reduced boundary of the minimal Perron solutions from Section 2, one can conclude that in the case when the blow up is a one-phase solution, then
it is in fact a one-plane solution $\alpha(x - x_0, \nu)$, hence $C^{1,\gamma}$ regularity of the reduced boundary follows from flatness results.

For viscosity solutions in general, when the governing equation is a fully nonlinear operator more robust arguments are required.

4.1. Lipschitz continuity. In [22] De Silva and Savin proved the following result under the assumptions that $G$ behaves like $t$ for $t$ large.

**Theorem 4.1.** Let $u$ be a viscosity solution to (1) for $\mathcal{L}_i u = F(D^2 u), f_i = 0, i = 1, 2$, and assume that $G \in C^2([0, \infty))$ and

$$G'(t) \to 1, \quad G''(t) = O(1/t) \quad \text{as} \quad t \to \infty. \quad (6)$$

Then

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C \left(1 + \|u\|_{L^\infty(B_{1})}\right)$$

with $C = C(n, \lambda, \Lambda, G)$ universal.

The heuristic behind the proof is that “big gradients” force the free boundary condition to become a no-jump condition for $\nabla u$ and then interior $C^{1,\alpha}$ estimates for fully nonlinear equations provide gradient estimates.

The dependence on $G$ in the constant above is determined by the rate of convergence in the limit (6). In particular (6) can be relaxed to $G' \in [1-\delta, 1+\delta]$ for large values of $t$. If $F$ is homogeneous of degree 1, then it suffices to require $G(t)/t \to c_0$ as $t \to \infty$ for some constant $c_0$.

We remark that the method of the proof still works in the presence of a non-zero right hand side.

The technique used in the proof of Theorem 4.1 provides also H"older continuity of viscosity solution to (1) for a general class of problems governed by uniformly elliptic linear equations with continuous coefficients $a_{ij}$. Precisely:

**Theorem 4.2.** Let $\mathcal{L}_i u = \text{Tr}(A(x) D^2 u), i = 1, 2$, with $A$ continuous. Assume that

$$\sigma^{-1}t \geq G(t) \geq \sigma t,$$

for $t > M$, large and some $\sigma > 0$. Then $u \in C^{0,\gamma}(B_{1/2})$ for some $\gamma > 0$ depending on $\sigma$, and

$$\|u\|_{C^{0,\gamma}(B_{1/2})} \leq C(\sigma, M)(\|u\|_{L^\infty(B_{1})} + 1).$$

In dimension $n = 2$ these results can be improved significantly (see Section 4.3.)

4.2. Classification of global solutions. In [23] De Silva and Savin proved the following Liouville theorem for global Lipschitz solutions.

**Theorem 4.3.** Let $\mathcal{L}_i u = F(D^2 u), f_i = 0, i = 1, 2$, and $u$ be a globally Lipschitz continuous viscosity solution to (1) in $\mathbb{R}^n$. Assume that

$$F \text{ is concave (or convex) and } F \text{ is homogeneous of degree 1.} \quad (7)$$

Then either $u$ is a two plane-solution

$$u = \alpha (x - x_0, \nu)^+ - \beta (x - x_0, \nu)^- \quad \text{with} \quad \alpha, \beta > 0, \alpha = G(\beta), \quad (8)$$

or

$$u^- \equiv 0, \quad (9)$$

which means that $u$ solves the one-phase problem for $F$. 


As in the case of the Laplacian, the main consequence of Theorem 4.3 is that it reduces the question of the regularity of the free boundary for the two-phase problem to the classification of global blow-up solutions to the one-phase problem. In particular, if $u$ is a solution to (1) with $0 \in \Gamma(u)$, then either $u^-(x) = o(|x|)$ or $F(u)$ is $C^{1,\gamma}$ in a neighborhood of 0.

Theorem 4.3 can be extended to more general operators $\mathcal{F}(D^2u, \nabla u, u)$ that depend also on $\nabla u$ and $u$ if appropriate assumptions are imposed on $\mathcal{F}$. For example this result holds when the problem is governed by quasilinear equations of the type

$$\sum_{i,j=1}^{n} a_{ij} \left( \frac{\nabla u}{|\nabla u|} \right) u_{ij} = 0,$$

with uniformly elliptic coefficients $a_{ij} \in C^1(S^{n-1})$. In [22] Lipschitz continuity of solutions to such a problem is also established.

We roughly outline a formal idea of the proof of Theorem 4.3. Notice that $|\nabla u^+|$ is a subsolution of the linearized equation in $\{u > 0\}$ and then its supremum occurs on the boundary $F(u)$. Assume for the moment that $F(u)$ is of class $C^2$ and that $|\nabla u^+|$ achieves its maximum at a point, say at 0 $\in F(u)$. Suppose the normal to $F(u)$ at 0 (pointing in the positive phase) is $e_n$ and let $\kappa_1, \ldots, \kappa_{n-1}$ be the principal curvatures of $F(u)$ at 0. Then the mixed derivatives vanish, i.e. $u_{in}^+(0) = 0, i < n$,

$$D^2u^+(0) = \text{diag}(\kappa_1 u_{1n}^+(0), \ldots, \kappa_{n-1} u_{n-1,n}^+(0), u_{nn}^+(0))$$

and similarly (in view of the free boundary condition)

$$D^2u^-(0) = \text{diag}(\kappa_1 u_{1n}^-(0), \ldots, -\kappa_{n-1} u_{n-1,n}^-(0), u_{nn}^-(0)).$$

Using that $\mathcal{F}(D^2u^+(0)) = \mathcal{F}(-D^2u^-(0)) = 0$ and that $\mathcal{F}$ is homogenous of degree one we have

$$\mathcal{F} \left( \text{diag} \left( \kappa_1, \ldots, \kappa_{n-1}, \frac{u_{nn}^+(0)}{u_n^+(0)} \right) \right) = \mathcal{F} \left( \text{diag} \left( \kappa_1, \ldots, \kappa_{n-1}, -\frac{u_{nn}^-(0)}{u_n^-(0)} \right) \right) = 0.$$

which by ellipticity of $\mathcal{F}$ gives

$$\frac{u_{nn}^+(0)}{u_n^+(0)} = -\frac{u_{nn}^-(0)}{u_n^-(0)}.$$

On the other hand, by Hopf lemma applied to $u_n^+$, $u_n^-$ we have

$$u_{nn}^+(0), u_{nn}^-(0) < 0,$$

(unless $u_n^+$ and $u_n^-$ are constant) and we reach a contradiction.

The rigorous proof requires somewhat involved and technical arguments. One of the main steps consists in obtaining a weak Evans-Krylov type estimate for a nonlinear transmission problem. The assumptions that $F(u)$ is of class $C^2$ (even if $F$ is concave) and that $|\nabla u^+|$ achieves its maximum at a point (rather than at infinity) cannot be justified. One major difficulty is that $F(u)$ is not known to be better than $C^{1,\alpha}$ even in the perturbative setting.

The idea of proof of Theorem 4.3 is to show a “reversed” improvement of flatness for the solution $u$, which means that if $u$ is sufficiently close to a two plane solution at a small scale then it remains close to the same two-plane solution at all larger scales. The key ingredient in the proof of Theorem 4.3 is the Proposition 1 below.

Since $\mathcal{F}$ is homogenous of degree one, we can multiply $u^+$ and $u^-$ by suitable constants and assume that $G(1) = 1$ and $\|\nabla u\|_{L^\infty} = 1$. Denote by $P_{M,\nu}$ quadratic
approximations of slope 1 to our free boundary problem,

$$P_{M,\nu}(x) := x \cdot \nu + \frac{1}{2} x^T M x,$$

with \(\nu\) a unit direction and \(M \in S^{n \times n}\) such that \(M\nu = 0\) and \(F(M) = 0\).

The condition \(M\nu = 0\) expresses the fact that we require the approximating polynomial to have gradient at most 1 near the origin up to linear order.

Proposition 1 is a dichotomy result for solutions \(u\) which are \(\epsilon\)-perturbations of polynomials \(P_{M,\nu}\) at scale 1. It says that either \(u\) can be approximated by another polynomial \(\bar{P}_{\bar{M},\bar{\nu}}\) in a \(C^{2,\alpha}\) fashion at a smaller scale, or that \(|\nabla u(0)|\) has to be strictly below 1 an amount of order \(\epsilon\).

**Proposition 1 (Nonlinear Dichotomy).** Assume that \(0 \in F(u), G(1) = 1\) and \(|\nabla u| \leq 1\). There exist small universal constants, \(\epsilon_0, \delta_0, r_0, c_0, \alpha_0 > 0\) such that if

$$|u(x) - P_{M,\epsilon_0}(x)| \leq \epsilon$$

in \(B_1\), \(\epsilon \leq \epsilon_0\) (12)

with

$$\|M\| \leq \delta_0 \epsilon^{1/2},$$

then one of the following alternatives holds:

(i) \(|u - P_{\bar{M},\nu}| \leq \epsilon_0^{2+\alpha_0}\) in \(B_{r_0}\) (14)

for some \(\bar{M}, \nu\) with

$$\|M - \bar{M}\| \leq C \epsilon$$

and \(C\) universal;

(ii) \(|\nabla u^+| \leq 1 - c_0 \epsilon\) (15)

We remark that assumption (12) implies that \(F(u) \in C^{1,\alpha}\) and \(u\) is a classical solution, hence \(\nabla u^+ (0)\) is well defined (see Section 2).

In the case of the Laplacian, Proposition 1 can be easily deduced from the stronger \(C^{2,\alpha}\) improvement of flatness estimates established in [18]. In the fully nonlinear case this is not known, however Proposition 1 states that we can obtain a quadratic improvement of flatness as long as the slope of the next approximating polynomial is 1.

**4.3. Different operators.** When \(L_1\) and \(L_2\) are different operators few results are known. In [4], Amaral and Teixeira consider the case of divergence form operators, with \(f_1, f_2 \in L^p, p > n/2\) and \(A_j\) merely measurable. They prove that local minimizers of the associated energy functional have a universal Hölder modulus of continuity.

In [11], Caffarelli, De Silva and Savin obtained Lipschitz continuity and classification of global Lipschitz solutions, that is Theorem 4.1 and Theorem 4.3, for a two-phase problem driven by two different operators with measurable coefficients under very general free boundary conditions \(u_+^\nu = G(u_-, \nu, x)\). We remark that Theorem 4.1 cannot hold in this generality in dimension \(n \geq 3\). Indeed, say for \(n = 3\), it is not difficult to construct two homogeneous functions of degree less than one, that solve two different uniformly elliptic equations in complementary domains in \(\mathbb{R}^3\) and satisfy the free boundary condition for a specific \(G\).

The proof in dimension \(n = 2\) relies on two-dimensional topological arguments which involve intersecting the graph of the solution with a family of planes.
4.4. **Analysis of singular points.** We conclude this section by mentioning a recent result by Engelstein, Spolaor and Velichkov [25] concerning the analysis of singular points for energy minimizing solutions of the Bernoulli one-phase free boundary problem, i.e. the minimization problem for the functional:

\[ J(u) := \int_{\Omega} (|\nabla u|^2 + \chi_{\{u>0\}}) \, dx. \]

In this paper the authors prove uniqueness of blow-ups and \(C^{1,\log}\)-regularity for the free boundary of energy minimizing solutions at points where one blow-up has an isolated singularity. They do this by establishing a (log-)epiperimetric inequality for the Weiss energy for traces close to that of a cone with isolated singularity, whose free-boundary is graphical and smooth over that of the cone in the sphere. With additional assumptions on the cone, they can prove a classical epiperimetric inequality which can be applied to deduce a \(C^{1,\gamma}\) regularity result. These additional assumptions are satisfied by the De Silva-Jerison-type cones [21], which are the only known examples of minimizing cones with isolated singularity.

5. **Regularity of the free boundary: Strong results.**

5.1. **The homogeneous case.** The regularity theory for the Laplace operator in the homogeneous case has been developed by Caffarelli in the two seminal papers [8, 9].

In particular the “Lipschitz implies \(C^{1,\gamma}\)” part is contained in [8] while the flat implies Lipschitz part is shown in [9]. The flatness condition in [9] is stated in terms of \(\varepsilon\)-monotonicity along a cone of directions \(\Gamma(\theta_0, e)\) of axis \(e\) and large opening \(\theta_0\). Precisely, a function \(u\) is said to be \(\varepsilon\)-monotone (\(\varepsilon > 0\) small) along the direction \(\tau\) in the cone \(\Gamma(\theta_0, e)\) if for every \(\varepsilon' \geq \varepsilon\),

\[ u(x + \varepsilon' \tau) \leq u(x). \]

Geometrically, the \(\varepsilon\)-monotonicity of \(u\) can be interpreted as \(\varepsilon\)-closeness of \(F(u)\) to the graph of a Lipschitz function.

In those papers Caffarelli set up a general strategy to attack the regularity of the free boundary. Let us briefly describe the central idea of the proof in [8].

Starting from a Lipschitz graph, one shows that in a neighborhood of \(F(u)\) the level sets of \(u\) are still Lipschitz graphs, locally in the same direction. Then one improves the Lipschitz constant (i.e. the flatness) of the level sets of \(u\) away from the free boundary. Here Harnack inequality applied to directional derivatives of \(u\) plays a major role. Then the task is to carry this interior gain up to the free boundary. To this aim, Caffarelli introduces a powerful method of continuity based on the construction of a continuous family of deformations, constructed as the supremum of a harmonic function over balls of variable radius (supconvolutions). Finally, by rescaling and iterating the last two steps, one obtains a geometric decay of the Lipschitz constant, which amounts to the \(C^{1,\gamma}\) regularity of \(F(u)\).

After 10 years Feldman [28] considered different anisotropic operators with constant coefficients and extends to this case the results in [8].

P. Y. Wang managed to extend the results in [8, 9] to a class of concave fully nonlinear operators of the type \(F(D^2 u)\) (see [40, 41]). One year later Feldman in [29] considered a class of non concave fully nonlinear operators of the type \(F(D^2 u, Du)\). He showed that Lipschitz free boundaries are \(C^{1,\alpha}\) thus extending to this case the results in [8].
The first papers dealing with variable coefficient operators are by Cerutti, Ferrari, Salsa [14] and by Ferrari [26] and Argiolas, Ferrari [6]. They considered respectively, linear elliptic operators in non-divergence form and a rather general class of fully nonlinear operators $F(D^2u, Du, x)$, with Hölder continuity in $x$, including Bellman’s operators. One of the main difficulty in extending the theory to variable coefficients operator is the fact that directional derivatives do not satisfy any reasonable elliptic equation.

A refinement of the techniques in [14] leads to the following results (see [27]), where the drift coefficient is merely bounded measurable, with two different operators on the two phases.

**Theorem 5.1.** Let $\mathcal{L}u = \text{Tr}(A_i(x) D^2u) + b_i(x) \cdot \nabla u$, $i = 1, 2$. Let $u$ be a weak solution of our free boundary problem in $B_1$. Suppose $0 \in F(u)$ and that

i) $A_i \in C^{0, \alpha}(B_1)$, $0 < \alpha \leq 1$, $b_i \in L^\infty(B_1)$.

ii) $0 < \alpha_1 \leq \frac{u^+}(x) \leq \alpha_2$.

iii) $G(0) > 0$.

There exist $0 < \tilde{\theta} < \pi/2$ and $\tau > 0$ such that, if for $0 < \varepsilon < \tau$, $F(u)$ is contained in an $\varepsilon$–neighborhood of a graph of a Lipschitz function $x_n = g(x')$ with

$$\text{Lip}(g) \leq \tan \left( \frac{\pi}{2} - \tilde{\theta} \right)$$

then $F(u)$ is a $C^{1, \gamma}$–graph in $B_{1/2}$.

The same conclusion holds if $B_1^+(u) = \{(x', x_n) : x_n > g(x')\} \cap B_1$ where $f$ is a Lipschitz continuous function.

Condition ii) expresses a linear behavior of $u^+$ at the free boundary while being trapped in a neighborhood of two Lipschitz graph with small Lipschitz constant is another way to express a flatness condition. Thus, flatness plus linear behavior of the positive part imply smoothness.

This theorem has been extended to the same class of fully nonlinear operators considered in [26] by Argiolas and Ferrari in [6]. Note that, in principle, all these results do not need the a priori assumption of Lipschitz continuity of the solution, that comes as a consequence of the regularity of the free boundary.

**5.2. The nonhomogeneous case.** In [15], De Silva introduced a new strategy to investigate inhomogeneous free boundary problems, motivated by a classical one phase problem in hydrodynamic. Based on a Harnack type theorem and linearization, this technique avoids the use of supconvolutions, that in presence of distributed sources produces several complicacies. The method can be very well adapted to nonhomogeneous two-phase problems to prove that flat (see below) or Lipschitz free boundaries of (1) are $C^{1, \gamma}$, when the governing equation is the same in both phases (see [16], [17], [18]). Throughout this section, $\mathcal{L}_1 = \mathcal{L}_2$ and this common operator will be denoted by $\mathcal{L}$. Also, $f_1 = f_2 = f$.

We have (we will always assume that $0 \in F(u)$):

**Theorem 5.2.** Let $\mathcal{L}$ be as in (2), or (3) and $u$ be Lipschitz viscosity solution to (1) in $B_1$. Assume that $f$ is bounded and continuous in $B_1^+(u) \cup B_1^-(u)$. There exists a universal constant $\delta > 0$ such that, if

$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\}, \quad (\delta – \text{flatness})$$

(16)

with $0 \leq \delta \leq \tilde{\delta}$, then $F(u)$ is $C^{1, \gamma}$ in $B_{1/2}$.
Condition (16) expresses that the zero set of $u^+$ is trapped between two parallel hyperplanes at $\delta$–distance from each other for a small $\delta$, which is a kind of flatness ($\delta$–flatness). While this looks like a somewhat strong assumption, it is indeed a natural one since it is satisfied for example by rescaling a solution around a point of the free boundary where there is a normal in some weak sense (regular points), for instance in the measure theoretical one. We have seen that in the homogeneous case $H^{n-1}$–a.e. points on $F(u)$ are of this kind, when $L$ is a divergence form operator or $L = F(D^2u)$. Moreover, starting from a Lipschitz free boundary, $H^{n-1}$–a.e. points on $F(u)$ are regular, by Rademacher Theorem.

When $L$ is linear or if $F$ is (positively) homogeneous of degree one (or when $F_r(M)$ has a limit $F^*(M)$, as $r \to 0$, which is always homogeneous of degree one), we also have:

**Theorem 5.3.** (Lipschitz implies $C^{1,\gamma}$) Let $u$ be a Lipschitz viscosity solution to (1) in $B_1$. Assume that $f$ is bounded and continuous in $B_1^+ \cup B_1^-$. If $F(u)$ is a Lipschitz graph in a neighborhood of $0$, then $F(u)$ is $C^{1,\gamma}$ in a (smaller) neighborhood of $0$.

Theorem 5.3 follows from via a blow-up argument and flatness.

Further extensions can be achieved with small extra effort. Actually, there is no problem in extending our results when $b_j$ and $f$ are merely bounded measurable.

As already pointed out the proof of Theorem 5.2 follows the strategy developed in [15]. The main difficulty in the analysis in this two-phase problem comes from the case when $u^-$ is degenerate, that is very close to zero without being identically zero. In this case the flatness assumption does not guarantee closeness of $u$ to an “optimal” (two-plane or one-plane) configuration. Thus one needs to work only with the positive phase $u^+$ to balance the situation in which $u^+$ highly predominates over $u^-$ and the case in which $u^-$ is not too small with respect to $u^+$.

In particular, the proof is based on a recursive improvement of flatness, obtained via a compactness argument, provided by a geometric type Harnack inequality, which linearizes the problem into a limiting one. The limiting problem turns out to be a transmission problem in the nondegenerate case and a Neumann problem in the other case. The information to set up the iteration towards regularity is precisely stored in the analysis of this problem.

Let us heuristically show how the free boundary condition

$$|\nabla u^+| = G(|\nabla u^-|)$$

linearizes in the nondegenerate case when

$$Lu = \text{Tr} \left( A(x) D^2u \right) + b(x) \cdot \nabla u = f \quad \text{in} \ B_1.$$

In the nondegenerate case, letting $U_\beta(t) = \alpha t^+ - \beta t^-$, $\alpha = G(\beta)$, the flatness condition is equivalent to

$$U_\beta(x_n - \varepsilon) \leq u(x) \leq U_\beta(x_n - \varepsilon)^- \quad \text{in} \ B_1,$$

with $0 < \beta \leq L$, and $L = \text{Lip}(u)$. After rescaling we may assume that

$$|a_{ij}(x) - \delta_{ij}| \leq \varepsilon, \quad |b_j(x)| \leq \varepsilon^2, \quad |f| \leq \varepsilon^2 \min\{\alpha, \beta\}$$
This suggests the renormalization
\[
\tilde{u}_\varepsilon(x) = \begin{cases} 
\frac{u(x) - \alpha x_n}{\alpha \varepsilon}, & x \in B_1^+(u) \cup F(u) \\
\frac{u(x) - \beta x_n}{\beta \varepsilon}, & x \in B_1^-(u)
\end{cases}
\]
or
\[
u(x) = \begin{cases} 
\alpha x_n + \varepsilon \alpha \tilde{u}_\varepsilon(x), & x \in B_1^+(u) \cup F(u) \\
\beta x_n + \varepsilon \beta \tilde{u}_\varepsilon(x), & x \in B_1^-(u).
\end{cases}
\quad (18)
\]
We have
\[
\mathcal{L}\tilde{u}_\varepsilon = \frac{f - \alpha b_n}{\alpha \varepsilon} \sim \varepsilon \quad \text{in } B_1^+(u) \cup B_1^-(u).
\]
On \( F(u) \),
\[
|\nabla u^+| = \alpha |e_n + \varepsilon \nabla \tilde{u}_\varepsilon(x)| \sim \alpha \left(1 + \varepsilon (\tilde{u}_\varepsilon)_{x_n} + \varepsilon^2 |\nabla \tilde{u}_\varepsilon|^2 \right)
\]
and
\[
G(|\nabla u^-|) = G(|b e_n + \varepsilon b \nabla \tilde{u}_\varepsilon|) \sim G \left(b \left(1 + \varepsilon (\tilde{u}_\varepsilon)_{x_n} + \varepsilon^2 |\nabla \tilde{u}_\varepsilon|^2 \right) \right)
\sim G(\beta) + \varepsilon G'(\beta) \left(\beta (\tilde{u}_\varepsilon)_{x_n} + \varepsilon \beta |\nabla \tilde{u}_\varepsilon|^2 \right) + \varepsilon^2.
\]
Letting \( \varepsilon \to 0 \), we get formally for “the limit” \( \tilde{u} \) the following problem:
\[
\Delta \tilde{u} = 0, \quad \text{in } B_{1/2}^+ \cup B_{1/2}^-
\quad (19)
\]
and the transmission condition (linearization of the free boundary condition)
\[
\alpha(\tilde{u}_{x_n})^+ - \beta G'(\beta) (\tilde{u}_{x_n})^- = 0 \quad \text{on } B_{1/2} \cap \{x_n = 0\}
\quad (20)
\]
where \((\tilde{u}_{x_n})^+\) and \((\tilde{u}_{x_n})^-\) denote the \(e_n\)-derivatives of \( \tilde{u} \) restricted to \( \{x_n > 0\} \) and \( \{x_n < 0\} \), respectively.

When \( \mathcal{L}u = F(D^2u) \), the Laplace equation in (19) must be replaced by the fully nonlinear equations
\[
\mathcal{F}^\pm(D^2\tilde{u}) = 0 \quad \text{in } B_{1/2}^\pm
\quad (21)
\]
where \( \mathcal{F}^+(M), \mathcal{F}^-(M) \) are limits (of sequences) of operators of the form
\[
\frac{1}{\alpha \varepsilon} \mathcal{F}^+(\alpha \varepsilon M) \quad \text{and} \quad \frac{1}{\beta \varepsilon} \mathcal{F}^-(\beta \varepsilon M)
\]
respectively.

Thus, at least formally, we have found an asymptotic problem for the limits of the renormalizations \( \tilde{u}_\varepsilon \). The crucial information we were mentioning before is contained in the following regularity result that we state for the fully nonlinear case (see [20] when distributed sources are present). Consider the transmission problem, \( \tilde{a} \neq 0 \)
\[
\begin{align*}
\mathcal{F}^+(D^2\tilde{u}) &= 0 \quad \text{in } B_1^+ \cap \{x_n \neq 0\}, \\
\mathcal{F}^-(D^2\tilde{u}) &= 0 \quad \text{in } B_1^- \cap \{x_n \neq 0\}, \\
\tilde{a} \cdot (\tilde{u}_{x_n})^+ - \tilde{b} \cdot G'(\tilde{b}) \cdot (\tilde{u}_{x_n})^- &= 0 \quad \text{on } B_1 \cap \{x_n = 0\}.
\end{align*}
\quad (22)
\]
Theorem 5.4. Let \( \tilde{u} \) be a viscosity solution to (22) in \( B_1 \) such that \( \|\tilde{u}\|_\infty \leq 1 \). Then \( \tilde{u} \in C^{1,\gamma} (\tilde{B}_1^+) \) and in particular, there exists a universal constant \( \tilde{C} \) such that
\[
|\tilde{u}(x) - \tilde{u}(0) - (\nabla_{\nu} \tilde{u}(0) \cdot x' + \tilde{p} x_n^+ - \tilde{q} x_n^-)| \leq \tilde{C} r^{1+\gamma}, \quad \text{in } B_r
\]
for all \( r \leq 1/2, \) with \( \tilde{a}^2 \tilde{p} - \tilde{b}^2 \tilde{q} = 0 \).

Transferring the estimate (23) to \( \tilde{u}_\varepsilon \) and then reading it in terms of flatness for \( u \) through formulas (18), one deduce that if \( 0 < r \leq r_0 \) for \( r_0 \) universal, and \( 0 < \varepsilon \leq \varepsilon_0 \) for some \( \varepsilon_0 \) depending on \( r \),
\[
U_\beta(x \cdot \nu_1 - r \frac{\varepsilon}{2}) \leq u(x) \leq U_\beta(x \cdot \nu_1 + r \frac{\varepsilon}{2}) \quad \text{in } B_r,
\]
with \( |\nu_1| = 1, \, |\nu_1 - e_n| \leq \tilde{C} \varepsilon \), and \( |\beta - \beta'| \leq \tilde{C} \beta \varepsilon \) for a universal constant \( \tilde{C} \).

Rescaling and iterating (24) one deduces uniform pointwise \( C^{1,\gamma} \) estimates in neighborhood of the origin.

6. Higher regularity.

6.1. Main results. In view of the results in Section 5, under suitable flatness assumptions, the free boundary \( F(u) \) is locally \( C^{1,\gamma} \) and the same conclusion holds if \( F(u) \) is a graph of a Lipschitz function. Therefore \( u \) is a classical solution, i.e. the free boundary condition is satisfied in a pointwise sense. Coming to the question of higher regularity in [19] we proved the following result.

Theorem 6.1. Let \( k \) be a nonnegative integer. Assume that \( f_i \in C^{k,\gamma} (B_1), \, i = 1, 2, \) and \( G \) is \( C^{2+k} \). Then \( F(u) \cap B_{1/2} \) is \( C^{k+2,\gamma} \). If \( f_i \) are \( C^\infty \), \( i = 1, 2 \), or real analytic in \( B_1 \), then \( F(u) \cap B_{1/2} \) is \( C^\infty \) or real analytic, respectively.

Before entering into more technical argument, we describe the state of the art about the higher regularity theory for two-phase free boundary problems. In the seminal paper [30], the authors used a zero order hodograph transformation and a suitable reflection map, to locally reduce a two-phase problem to an elliptic, coercive nonlinear system of equations. The existing literature on the regularity of solutions to nonlinear systems developed in [1, 35] can be applied as long as the solution \( u \) is \( C^{2,\alpha} \) for some \( \alpha > 0 \) up to the free boundary (from either side).

Remark 1. As noted in the recent work [32], in the case when the governing equation in (1) is linear in divergence form, the initial assumption to obtain the above theorem is actually \( u \in C^{3,\alpha} \). It is not completely evident that the general case of linear uniformly elliptic equations with \( \text{Hölder} \) coefficients can also be treated in a similar manner. On the other hand, the case when the leading operator is a fully nonlinear operator definitely requires the solution to have \( \text{Hölder} \) second derivatives (from both sides).

In [19] we consider the case of linear operators \( L_1 = L_2 = \mathcal{L} \) with the purpose to develop a general strategy that would apply to a larger class of problem, to include also fully nonlinear operators. The application to the latter case would be rather straightforward once \( C^{2,\alpha} \) estimates for the limiting problem (22) were available. This remains an open problem, object of future investigations. Thus, the main result in [19] is the following.
Theorem 6.2. Let $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$ be a linear divergence or nondivergence form operator and $u$ be a (Lipschitz) viscosity solution to (1) in $B_1$. There exists a universal constant $\bar{\eta} > 0$ such that, if
\[ \{x_n \leq -\eta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \eta\}, \quad \text{for } 0 \leq \eta \leq \bar{\eta}, \]
then $F(u)$ is $C^{2,\gamma}$ in $B_{1/2}$ for a small $\gamma$ universal, with the $C^{2,\gamma}$ norm bounded by a universal constant.

Other related higher regularity results can be found in [24, 31]. The overall strategy for the proof of Theorem 6.2 follows the ideas described in the previous section. However, reaching the $C^{2,\gamma}$ regularity requires a much more involved process because of the possible degeneracy of the negative part. Indeed this causes a delicate interplay between the two phases. Ultimately the main source of difficulties is due to the presence of a forcing term of general sign in the negative phase. Indeed, if $f_2 \geq 0$, the Hopf maximum principle would imply nondegeneracy (also) on the negative side, making the two-phases of comparable size and considerably simplifying the final iteration procedure. It is worth noticing, that however even in this easier scenario (and in particular in the homogeneous case), if one wants to attain uniform estimates with universal constants, then one must employ the more involved methods developed in [19] for the degenerate case. In the next subsection we outline the general strategy.

6.2. Outline and strategy. In this section, we outline the main strategy in the proof of Theorem 6.2, trying to emphasize the key points, also in comparison to the flatness implies $C^{1,\gamma}$ case. The first thing to do is to reinforce the notion of flatness, tailoring it for the attainment of $C^{2,\gamma}$ regularity. This can be done by introducing a suitable class of functions that we call two-phase and one-phase polynomials. In principle second order polynomials should be enough, but it turns out that we need a small third order perturbation.

Given $\omega \in \mathbb{R}^n$, with $|\omega| = 1$, and let $S_\omega$ be an orthonormal basis containing $\omega$. Let $M \in S^{n \times n}$ satisfy
\[ M\omega = 0 \]
and define
\[ P_{M,\omega}(x) = x \cdot \omega - \frac{1}{2} x^T M x. \]
Set,
\[ V_{M,\omega,a,b}^{\alpha,\beta}(x) = \alpha(1 + a \cdot x)P_{M,\omega}^+(x) - \beta(1 + b \cdot x)P_{M,\omega}^-(x), \quad \alpha > 0, \beta \geq 0, a, b \in \mathbb{R}^n \]
where the superscripts $\pm$ denote as usual the positive/negative part of a function. These are our two-phase polynomials, one-phase if $\beta = 0$. In the particular case when $M = 0, a = b = 0, \omega = e_n$ we obtain the two-plane function:
\[ U_{\beta}(x) = \alpha x_n^+ - \beta x_n^- \]
The unit vector $\omega$ establishes the “direction of flatness”.

We shall need to work with a subclass, strictly related to problem (1), at least at the origin. We denote by $\mathcal{V}_{f_1,G}$ the class of functions of the form $V_{M,\omega,a,b}^{\alpha,\beta}$ for which
\[ 2\alpha a \cdot \omega - \alpha tr M = f_1(0) \]
\[ 2\beta b \cdot \omega - \beta tr M = f_2(0) \quad \text{if } \beta \neq 0, \]
\[ \alpha = G(\beta), \quad \text{if } \beta \neq 0, \]
and, in view of the linearization of the free boundary condition,

$$\alpha a \cdot \omega^+ = \beta G'(\beta) b \cdot \omega^+, \quad \forall \omega^+ \in S_\omega.$$  

When $\beta = 0$, then there is no dependence on $b$ and $a \cdot \omega^+ = 0$. Thus, we drop the dependence on $\beta, b, G$ and $f_2$ in our notation above and we indicate the dependence on $a_\omega := a \cdot \omega$.

We introduce the following definitions.

**Definition 6.3.** Let $V = V_{M,\omega,a,b}^{\alpha,\beta}$. We say that $u$ is $(V, \epsilon, \delta)$ flat in $B_1$ if

$$V(x - \epsilon \omega) \leq u(x) \leq V(x + \epsilon \omega) \quad \text{in } B_1$$

and

$$|a|, |b|, \|M\| \leq \delta \epsilon^{1/2}, \quad |h_n| \leq \delta^2, \quad |h_n||M| \leq \delta^2 \epsilon.$$  

Given $V = V_{M,\omega,a,b}^{\alpha,\beta}$, under the rescaling

$$V_r(x) = \frac{V(rx)}{r}$$

we get

$$V_r = V_{rM,\omega,ra,rb}^{\alpha,\beta}.$$  

**Definition 6.4.** Let $V = V_{M,\omega,a,b}^{\alpha,\beta}$. We say that $u$ is $(V, \epsilon, \delta)$ flat in $B_r$ if the rescaling

$$u_r(x) := \frac{u(rx)}{r}$$

is $(V_r, \overline{\epsilon}, \delta)$ flat in $B_1$.

Notice that if $u$ is $(V, \epsilon, \delta)$ flat in $B_r$ then

$$V(x - \epsilon \omega) \leq u(x) \leq V(x + \epsilon \omega) \quad \text{in } B_r.$$  

The parameter $\epsilon$ measures the level of polynomial approximation and $\delta$ is a flatness parameter (also controlling the $C^{0,\gamma}$ norms of $f_1$ and $f_2$).

Thus, roughly, the purpose is to show that $u$ is $(V_k, \lambda_k^{2+\gamma}, \delta)$ flat in $B_{\lambda_k}$ for $\lambda_k = \eta^k$ and all $k \geq 0$, for some $\delta, \eta$ small and a sequence of $V_k$ converging to a final profile $V_0$. This would give uniform pointwise $C^{2,\gamma}$ regularity both for the solution and the free boundary in $B_{1/2}$.

The starting point is to show that the flatness condition (16) allows us to normalize our solution so that a rescaling $u_r$ of $u$ falls into one of the following cases, with suitable $\lambda, \delta$. This kind of dichotomy parallels in a sense what happens in [16] and related papers.

**Case a.** $u_r$ is $(V, \lambda_k^{2+\gamma}, \delta)$ flat for some $V = V_{0,\epsilon,\alpha,b}^{\alpha,\gamma} \in \mathcal{V}_{f_1,G}$. Moreover, $\beta \delta$ controls the $C^{0,\gamma}$ semi-norm of $f_2$. This case corresponds to a nondegenerate configuration, in which the two phases have comparable size and $u_r$ is trapped between two translations of a genuine two-phase polynomial.

**Case b.** $u_r^+$ is $(V, \lambda_k^{2+\gamma}, \delta)$ flat for some $V = V_{0,\epsilon,\alpha,b}^{\alpha,\gamma} \in \mathcal{V}_{f_1}$, and $u_r^-$ is close to a purely quadratic profile $cx_n^2$. This case corresponds to a degenerate configuration, where the negative phase has either zero slope or a small one (but not negligible) with respect to $u_r^+$, and $u_r^+$ is trapped between two translations of a one-phase polynomial. Note that this situation cannot occur if $f_2 \geq 0$, unless $u^-$ is identically zero.
The initial flatness corresponding to cases a) and b) above improves successively at a smaller scale. This is described by the following “subroutines”, to be implemented in the course of the final iteration towards $C^{2,\gamma}$ regularity.

Two-phase flatness improvement: if $u$ is $(\bar{V}, \bar{\lambda}^{2+\gamma}, \bar{\delta})$ flat for some $V = V^{\alpha,\beta}_{M,\omega,a,b} \in \mathcal{V}_{f,\gamma}$ in $B_{\lambda}$, the $C^{0,\gamma}$ seminorms of $f_1$ and $f_2$ are controlled by $\delta$ and $\beta \delta$, respectively, then, in $B_{3\lambda}$, $u$ enjoys a $C^{2,\gamma}$ flatness improvement, i.e. $u$ is $(\bar{V}, (\eta \lambda)^{2+\gamma}, \bar{\delta})$ flat for some $\bar{V} \in \mathcal{V}_{f,\gamma}$, properly close to $V$.

One-phase flatness improvement: if $u^+ = (V, \lambda^{2+\gamma}, \delta)$ flat for some $V = V^\alpha_{M,\omega,a,b} \in \mathcal{V}_{f_1}$ in $B_{\lambda}$, the $C^{0,\gamma}$ seminorm of $f_1$ is controlled by $\delta$ and $u^+_\nu$ is close to $\alpha$ on $F(u)$, then $u^+$ enjoys a $C^{2,\gamma}$ flatness improvement, with $\bar{V} \in \mathcal{V}_{f_1}$, properly close to $V$.

The achievement of the improvements above relies first of all on a higher order refinement of the geometric Harnack inequality [16]. This gives the necessary compactness to pass to the limit along a sequence of suitable higher order renormalizations of $u$ and obtain a limiting transmission problem (Neumann problem in the one phase-case). From the regularity of the solution of this problem we get the information to improve the two-phase or one-phase approximation for $u$ or $u^+$ respectively, and hence their flatness.

As we have seen, after a suitable rescaling, we face a first dichotomy “degenerate versus nondegenerate” and the iteration can start.

In the latter case the two-phase subroutine can be applied indefinitely to reach pointwise $C^{2,\gamma}$ regularity for some universal $\gamma^*$.

When $u$ falls into the degenerate case a new delicate dichotomy appears. First of all, to run the one-phase subroutine one needs to make sure that the closeness of $u^-\nu$ to a purely quadratic profile makes $u^\pm$ to be a (viscosity) solution of a one-phase free boundary problem, with $u^\pm$ close to an appropriate $a$ on $F(u)$. At this point two alternatives occur at a smaller scale:

- **D1**: either $u^-$ is closer to a purely quadratic profile at a proper $C^{2,\gamma}$ rate and $u^+$ enjoys a $C^{2,\gamma}$ flatness improvement;
- **D2**: or $u^-$ is closer (at a $C^{2,\gamma}$ rate) to a one-phase polynomial profile with a small non-zero slope but $u^+$ only enjoys an “intermediate” $C^2$ flatness improvement.

If D1 occurs indefinitely we are done. If not, we prove that the intermediate improvement in D2 is kept for a while, at smaller and smaller scale. The final and crucial step is to prove that, at a given universally small enough scale, the $C^{2,\gamma}$ one-phase approximation of $u^-$, together with the intermediate $C^2$ flatness improvement of $u^\pm$, is good enough to recover a full $C^{2,\gamma}$ two-phase improvement of $u$ with a universal $\gamma^* < \gamma$.

We emphasize that it is the interplay between the parallel improvements on both sides of the free boundary that makes possible the full two-phase improvement, at the price of a little decrease of the Hölder exponent. This kind of situation has no counterpart in the flatness implies $C^{1,\gamma}$ case of [16].

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E-mail address: desilva@math.columbia.edu
E-mail address: fausto.ferrari@unibo.it
E-mail address: sandro.salsa@polimi.it