EXTENDED HAMILTON-LAGRANGE FORMALISM AND ITS APPLICATION TO FEYNMAN'S PATH INTEGRAL FOR RELATIVISTIC QUANTUM PHYSICS

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With this paper, a consistent and comprehensive treatise on the foundations of the extended Hamilton-Lagrange formalism will be presented. In this formalism, the system’s dynamics is parametrized along a time-like system evolution parameter $s$, and the physical time $t$ is treated as a dependent variable $t(s)$ on equal footing with all other configuration space variables $q^i(s)$. In the action principle, the conventional classical action $L dt$ is then replaced by the generalized action $L_e ds$, with $L$ and $L_e$ denoting the conventional and the extended Lagrangian, respectively. Supposing that both Lagrangians describe the same physical system then provides the correlation of $L$ and $L_e$.

In the existing literature, the discussion is restricted to only those extended Lagrangians $L_e$ that are homogeneous forms of first order in the velocities. As a result, the Legendre transformation of $L_e$ to a corresponding extended Hamiltonian is singular and thus does not provide us with an equivalent extended Hamiltonian $H_e$.

In this paper, it is shown that a class of extended Lagrangians $L_e$ exists that are correlated to corresponding conventional Lagrangians $L$ without being homogeneous functions in the velocities. Then the Legendre transformation of $L_e$ to an extended Hamiltonian $H_e$ exists. With this class of extended Hamiltonians, an extended canonical formalism is presented that is completely analogous to the conventional Hamiltonian formalism. The physical time $t$ and the negative value of the conventional Hamiltonian then constitute an additional pair of conjugate canonical variables. The extended formalism also includes a theory of extended canonical transformations, where the time variable $t(s)$ is also subject to transformation.

In the extended formalism, the system’s dynamics is described as a motion on a hypersurface within an extended phase space of even dimension. It is shown that the hypersurface condition does not embody a constraint as the condition is automatically satisfied on the system path that is given by the solution of the extended set of canonical equations.

It is furthermore demonstrated that the value of the extended Hamiltonian and the parameter $s$ constitute a second additional pair of canonically conjugate variables. In the corresponding quantum system, we thus encounter an additional uncertainty relation.

As a consequence of the formal similarity of conventional and extended Hamilton-Lagrange formalisms, Feynman’s non-relativistic path integral approach can be converted on a general level into a form appropriate for relativistic quantum physics. In the emerging parametrized quantum description, the additional uncertainty relation serves as the means to incorporate the hypersurface condition and hence to finally eliminate
the parametrization.

As the starting point, the non-homogeneous extended Lagrangian $L_e$ of a classical relativistic point particle in an external electromagnetic field will be presented. It will be shown that this extended Lagrangian can be transformed into a corresponding extended Hamiltonian $H_e$ by a regular Legendre transformation. With this $L_e$, it is shown that the generalized path integral approach yields the Klein-Gordon equation as the corresponding quantum description. Moreover, the space-time propagator for a free relativistic particle will be derived. These results can be regarded as the proof of principle of the relativistic generalization of Feynman’s path integral approach to quantum physics.

**Keywords:** Extended Hamilton-Lagrange formalism, relativity, path integral, relativistic quantum physics

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1. Introduction

Even more than hundred years after the emerging of Einstein’s special theory of relativity, the presentation of classical dynamics in terms of the Lagrangian and the Hamiltonian formalisms is still usually based in literature on the Newtonian absolute time as the system evolution parameter

The way to generalize the Hamilton-Lagrange formalism in order to render it compatible with special relativity is obvious and well-established. It consists of introducing a system evolution parameter, $s$, as a new time-like independent variable, and of subsequently treating the physical time $t = t(s)$ as a dependent variable of $s$, in parallel to all configuration space variables $q^i(s)$. This idea has been pursued in numerous publications, only a few of them being cited here.

Despite this unambiguity in the foundations and the huge pile of publications on the matter — dating back to P. Dirac and C. Lanczos — a truly consistent extended Hamilton-Lagrange formalism is still missing. The reason for this is that the discussion in the existing literature is restricted to only those extended Lagrangians that are homogeneous forms of first order in the velocities. In this paper, this class of Lagrangians will be referred to as trivial extended Lagrangians. For the class of trivial extended Lagrangians, corresponding trivial extended Hamiltonians cannot be directly derived by a Legendre transformation as the transformation is singular. Yet, trivial extended Hamiltonians can always be set of on the basis of a given conventional Hamiltonian.

As will be shown in this paper, extended Lagrangians $L_e$ indeed exist for given conventional Lagrangians $L$ that both describe the same physical system and that are no homogeneous forms in the velocities $dq^i/ds$. In other words, the correlation of $L$ and $L_e$ is not unique is the sense that we can find more than one extended Lagrangian $L_e$ that can be reduced to the same conventional Lagrangian $L$. This will be demonstrated for the simple case of the free relativistic point particle.

If for a given conventional Lagrangian $L$ a non-trivial extended Lagrangian $L_e$ can be found, then the Legendre transformation is regular, and hence an equivalent extended Hamiltonian $H_e \neq 0$ can be derived directly. This will be shown for the case of a relativistic particle in an external electromagnetic field, whose
extended Hamiltonian will be derived by Legendre-transforming the corresponding non-homogeneous extended Lagrangian. Remarkably, we thus derive an extended Hamiltonian which coincides with the “super-Hamiltonian” that was postulated earlier by Misner, Thorne, and Wheeler.

For extended Hamiltonians $H_e$, the subsequent extended set of canonical equations is found to perfectly coincide in its form with the conventional one. This also applies for the theory of extended canonical transformations. The trivial extended generating function $F_2$ is shown to generate exactly the subgroup of conventional canonical transformations within the group of extended canonical transformations. This subgroup consists of exactly those extended canonical mappings that transform the time variables identically.

On grounds of the formal similarity of conventional and extended Hamilton-Lagrange formalisms, it is possible to formally convert non-relativistic approaches that are based on conventional Lagrangians into relativistic approaches in terms of extended Lagrangians. This idea is worked out exemplarily for Feynman’s path integral approach to quantum physics.

The paper is organized as follows. We start in Sect. 2.1 with the Lagrangian description and derive from the extended form of the action integral the extended Lagrangian $L_e$, together with its relation to the conventional Lagrangian $L$. It is shown that this relation reduces to the factor $\frac{dt}{ds}$. The extended set of Euler-Lagrange equations then follows from the dependencies of the extended Lagrangian.

In the extended Hamilton-Lagrange description of dynamics, the system’s motion takes place on hypersurfaces in extended phase spaces. In the extended Lagrangian formalism, this space is given by the tangent bundle $T(M \times \mathbb{R})$, whereas in the extended Hamiltonian formalism, the hypersurface lies within the cotangent bundle $T^* (M \times \mathbb{R})$, both cases built over the space-time configuration manifold $M \times \mathbb{R}$. It is proved that the emerging of a hypersurface condition does not imply the system to be constrained as the condition is always satisfied on the system path that is given by the solution of the (unconstrained) extended set of canonical equations. This perception corresponds to the case of a conventional Hamiltonian system with no explicit time dependence, where the system’s motion takes place on a phase-space hypersurface of constant energy. Likewise, the correlation of the dynamical variables that is induced by this hypersurface of constant energy is not considered to be a constraint as for autonomous systems the energy is automatically maintained by any solution of the set of canonical equations. The hypersurface condition thus distinguishes physical from unphysical phase-space locations that cannot represent at any time the system’s state for the given canonical equations and the initial conditions. In this sense, the hypersurface condition is the classical particle analogue of the mass shell condition of quantum field theory.

To provide a simple example, we derive in Sect. 3.1 the non-homogeneous extended Lagrangian $L_e$ for a free relativistic point particle. This Lorentz-invariant Lagrangian $L_e$ has the remarkable feature to be quadratic in the velocities. This contrasts with the conventional Lorentz-invariant Lagrangian $L$ that describes the...
identical dynamics. For this system, the hypersurface condition depicts the constant square of the four-velocity vector.

We show in Sect. 3.4 that the extended Lagrangian \( L_e \) of a relativistic particle in an external electromagnetic field agrees in its form with the corresponding non-relativistic conventional Lagrangian \( L \). The difference between both is that the derivatives in the extended Lagrangian \( L_e \) are being defined with respect to the particle’s proper time, which are converted into derivatives with respect to the Newtonian absolute time in the non-relativistic limit.

In Sect. 2.2, we switch to the extended Hamiltonian description. As the extended Hamiltonian \( H_e \) springs up from a non-homogeneous extended Lagrangian \( L_e \) by means of a regular Legendre transformation, both functions equally contain the total information on the dynamical system in question. The Hamiltonian counterparts of the Lagrangian description, namely, the extended set of canonical equations, the hypersurface condition, and the correlation of the extended Hamiltonian \( H_e \) to the conventional Hamiltonian \( H \) are presented. On this basis, the theory of extended canonical transformations and the extended version of the Hamilton-Jacobi equation are worked out as straightforward generalizations of the conventional theory. As a mapping of the time \( t \) is incorporated in an extended canonical transformation, not only the transformed coordinates emerging from the Hamilton-Jacobi equation are constants, as usual, but also the transformed time \( T \). The extended Hamilton-Jacobi equation may thus be interpreted as defining the mapping of the entire dynamical system into its state at a fixed instant of time, i.e., for instance, into its initial state. In the extended formulation, the Hamilton-Jacobi equation thus reappears in a new perspective.

We furthermore show that the value of the extended Hamiltonian \( H_e \) and the system evolution parameter \( s \) yield an additional pair of canonically conjugate variables. For the corresponding quantum system, we thus encounter an additional uncertainty relation. Based on both the extended Lagrangian \( L_e \) and the additional uncertainty relation, we present in Sect. 2.5 the path integral formalism in a form appropriate for relativistic quantum systems. An extension of Feynman’s approach was worked out earlier\(^8\) for a particular system. Nevertheless, the most general form of the extended path integral formalism that applies for any extended Lagrangian \( L_e \) is presented here for the first time. By consistently treating space and time variables on equal footing, the generalized path integral formalism is shown to apply as well for Lagrangians that explicitly depend on time. In particular, the transition of a wave function is presented here as a space-time integral over a space-time propagator. In this context, we address the physical meaning of the additional integration over \( t \). The uncertainty relation is exhibited as the quantum physics’ means to incorporate the hypersurface condition in order to finally eliminate the parameterization.

On grounds of a generalized understanding of the action principle, Feynman showed that the Schrödinger equation emerges as the non-relativistic quantum description of a dynamical system if the corresponding classical system is described
by the non-relativistic Lagrangian $L$ of a point particle in an external potential. Parallel to this beautiful approach, we derive in Section 3.10 the Klein-Gordon equation as the relativistic quantum description of a system, whose classical counterpart is described by the non-homogeneous extended Lagrangian $L_e$ of a relativistic point particle in an external electromagnetic field. The reason for this to work is twofold. As the extended Lagrangian $L_e$ agrees in its form with the conventional non-relativistic Lagrangian $L$, the generalized path integral formalism can be worked out similarly to the non-relativistic case. Furthermore, as we proceed in our derivation an infinitesimal proper time step $\Delta s$ only and consider the limit $\Delta s \to 0$, the hypersurface condition disappears by virtue of the uncertainty relation.

We finally derive in Sect. 3.11 the space-time propagator for the wave function of a free particle with spin zero from the extended Lagrangian of a free relativistic point particle. The hypersurface condition, as the companion of the classical extended description, is taken into account in the quantum description by integrating over all possible parameterizations of the system’s variables. This integration is now explained in terms of the uncertainty relation. We regard these results as the ultimate confirmation of the relativistic generalization of Feynman’s path integral formalism.

2. Extended Hamilton-Lagrange formalism

2.1. Extended set of Euler-Lagrange equations

The conventional formulation of the principle of least action is based on the action functional $S[q(t)]$, defined by

$$S[q(t)] = \int_{t_a}^{t_b} L(q, \dot{q}, t) \, dt,$$  

with $L(q, \dot{q}, t)$ denoting the system’s conventional Lagrangian, and the vector of configuration space variables $q(t) = (q^1(t), \ldots, q^n(t))$ as a function of time. In this formulation, the independent variable time $t$ plays the role of the Newtonian absolute time. The actual path $(\bar{q}(t), \dot{\bar{q}}(t))$ the physical system “realizes” is given as the extremum of the action $S$, hence for $\delta S = 0$. The path representing this extremum of $S$ is the solution of the set of Euler-Lagrange equations $(i = 1, \ldots, n)$ for the given initial conditions $q_0, \dot{q}_0$,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0.$$  

The reformulation of the least action principle (1) that is eligible for relativistic physics is accomplished by treating the time $t(s) = q^0(s)/c$ — like the vector $q(s)$ of configuration space variables — as a dependent variable of a newly introduced timelike independent variable, $s$. The action functional then writes in
terms of an extended Lagrangian $L_e$

$$S_e[q(s), t(s)] = \int_{s_a}^{s_b} L_e \left( q, \frac{dq}{ds}, t, \frac{dt}{ds} \right) ds \equiv \int_{s_a}^{s_b} L_e \left( q^\mu, \frac{dq^\mu}{ds} \right) ds.$$  \hspace{1cm} (3)

Herein, the index $\mu = 0, \ldots, n$ denotes the entire range of extended configuration space variables. As the action functional (3) has the form of (1), the subsequent Euler-Lagrange equations that determine the particular path $(\bar{q}(s), \bar{t}(s))$ on which the value of the functional (3) takes on an extreme value, adopt the customary form of Eq. (2)

$$\frac{d}{ds} \left( \frac{\partial L_e}{\partial \left( \frac{dq^\mu}{ds} \right)} \right) - \frac{\partial L_e}{\partial q^\mu} = 0.$$  \hspace{1cm} (4)

For the index $\mu = 0$, the Euler-Lagrange equation can be expressed equivalently in terms of $t(s)$ as

$$\frac{d}{ds} \left( \frac{\partial L_e}{\partial \left( \frac{dt}{ds} \right)} \right) - \frac{\partial L_e}{\partial t} = 0.$$  \hspace{1cm} (5)

The equations of motion for both $q(s)$ and $t(s)$ are thus determined by the extended Lagrangian $L_e$. The solution $q(t)$ of the Euler-Lagrange equations that equivalently emerges from the corresponding conventional Lagrangian $L$ may then be constructed by eliminating the evolution parameter $s$.

As the actions, $S$ and $S_e$, are supposed to be alternative characterizations of the same underlying physical system, the action principles $\delta S = 0$ and $\delta S_e = 0$ must hold simultaneously. This means that

$$\delta \int_{s_a}^{s_b} L \frac{dt}{ds} ds = \delta \int_{s_a}^{s_b} L_e ds,$$

which, in turn, is assured if both integrands differ at most by the $s$-derivative of an arbitrary differentiable function $F(q, t)$

$$L \frac{dt}{ds} = L_e + \frac{dF}{ds}.$$  \hspace{1cm} (6)

Functions $F(q, t)$ define a particular class of point transformations of the dynamical variables, namely those ones that preserve the form of the Euler-Lagrange equations. Such a transformation can be applied at any time in the discussion of a given Lagrangian system and should be distinguished from correlating $L_e$ and $L$. We may thus restrict ourselves without loss of generality to those correlations of $L$ and $L_e$, where $F \equiv 0$. In other words, we correlate $L$ and $L_e$ without performing simultaneously a transformation of the dynamical variables. We will discuss this issue in the more general context of extended canonical transformations in Sect. 2.3.

The extended Lagrangian $L_e$ is then related to the conventional Lagrangian, $L$, by

$$L_e \left( q, \frac{dq}{ds}, t, \frac{dt}{ds} \right) = L \left( q, \frac{dq}{dt}, t \right) \frac{dt}{ds}, \quad \frac{dq}{dt} = \frac{dq}{ds} \frac{ds}{dt}.$$  \hspace{1cm} (6)
The derivatives of $L_e$ from Eq. (6) with respect to its arguments can now be expressed in terms of the conventional Lagrangian $L$ as

$$\frac{\partial L_e}{\partial q^\mu} = \frac{\partial L}{\partial q^\mu} \frac{dt}{ds}, \quad \mu = 1, \ldots, n$$

(7)

$$\frac{\partial L_e}{\partial t} = \frac{\partial L}{\partial t} \frac{dt}{ds}$$

(8)

$$\frac{\partial L_e}{\partial \left( \frac{dq^\mu}{ds} \right)} = \frac{\partial L}{\partial \left( \frac{dq^\mu}{ds} \right)} \frac{dt}{ds}, \quad \mu = 1, \ldots, n$$

(9)

$$\frac{\partial L_e}{\partial \left( \frac{dt}{ds} \right)} = L + \sum_{\mu=1}^{n} \frac{\partial L}{\partial \left( \frac{dq^\mu}{dt} \right)} \frac{dq^\mu}{ds} \frac{dt}{ds} = L - \sum_{\mu=1}^{n} \frac{\partial L}{\partial \left( \frac{dq^\mu}{dt} \right)} \frac{dq^\mu}{ds} \left( \frac{ds}{dt} \right)^2 \frac{dt}{ds}$$

$$= L - \sum_{\mu=1}^{n} \frac{\partial L}{\partial \left( \frac{dq^\mu}{dt} \right)} \frac{dq^\mu}{ds}.$$ 

(10)

Equations (9) and (10) yield for the following sum over the extended range $\mu = 0, \ldots, n$ of dynamical variables

$$\sum_{\mu=0}^{n} \frac{\partial L_e}{\partial \left( \frac{dq^\mu}{ds} \right)} \frac{dq^\mu}{ds} = L \frac{dt}{ds} - \sum_{\mu=1}^{n} \frac{\partial L}{\partial \left( \frac{dq^\mu}{dt} \right)} \frac{dq^\mu}{ds} \frac{dt}{ds} + \sum_{\mu=1}^{n} \frac{\partial L}{\partial \left( \frac{dq^\mu}{dt} \right)} \frac{dq^\mu}{ds}$$

$$= L_e.$$ 

The extended Lagrangian $L_e$ thus satisfies the equation

$$L_e = \sum_{\mu=0}^{n} \frac{\partial L_e}{\partial \left( \frac{dq^\mu}{ds} \right)} \frac{dq^\mu}{ds} \begin{cases} \neq 0 & \text{if } L_e \text{ not homogeneous} \\ = 0 & \text{if } L_e \text{ homogeneous} \end{cases}$$

(11)

Regarding the correlation (6) and the pertaining condition (11), two different cases must be distinguished. In the first case, an extended Lagrangian $L_e$ can be set up immediately by multiplying a given conventional Lagrangian $L$ with $dt/ds$ and expressing all velocities $dq/dt$ in terms of $dq/ds$ according to Eq. (6). Such an extended Lagrangian $L_e$ is called a trivial extended Lagrangian as it contains no additional information on the underlying dynamical system. A trivial extended Lagrangian $L_e$ constitutes a homogeneous form of first order in the $n+1$ variables $dq^0/ds, \ldots, dq^n/ds$. This may be seen by replacing all derivatives $dq^\mu/dt$ with $a \cdot dq^\mu/ds$, $a \in \mathbb{R}$ in Eq. (6), which yields

$$L_e \left( q, a \frac{dq}{ds}, t, a \frac{dt}{ds} \right) = L \left( q, \frac{dq}{dt}, t \right) a \frac{dt}{ds}$$

$$= aL_e \left( q, \frac{dq}{ds}, t, \frac{dt}{ds} \right).$$

Consequently, Euler’s theorem on homogeneous functions states that Eq. (11) constitutes an identity\textsuperscript{11}. In that case, we may differentiate the identity with respect
to the velocity $dq^\mu/ds$ to get

$$
\sum_{\mu=0}^{n} \frac{\partial^2 L_e}{\partial (dq^\mu/ds)} \frac{dq^\mu}{ds} \equiv 0.
\tag{12}
$$

This is a homogeneous set of $n$ equations for the velocities $dq^\mu/ds$. It has a non-trivial solution ($dq/ds \neq 0$) only if the coefficient matrix is singular

$$
\det \left( \frac{\partial^2 L_e}{\partial (dq^\mu/ds)} \frac{dq^\mu}{ds} \frac{dq^\nu}{ds} \right) = 0.
\tag{13}
$$

Due to Eq. (13), a corresponding extended Hamiltonian $H_e$ does not follow from a trivial extended Lagrangian $L_e$ as the mediating Legendre transformation is singular.

The Euler-Lagrange equation (5) for $dt/ds$ then reduces to the conventional set of Eqs. (2) for arbitrary $t(s)$, hence, we do not obtain a substantial equation of motion for $t(s)$. Inserting Eq. (10) into Eq. (5), one finds

$$
\sum_{\mu=1}^{n} \frac{dq^\mu}{dt} \left[ \frac{\partial L}{\partial q^\mu} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^\mu} \right) \right]_{=0} = 0.
$$

The parametrization of time $t(s)$ is thus left undetermined — which reflects the fact that a conventional Lagrangian does not provide any information on a parametrization of time and that a trivial extended Lagrangian does not incorporate additional information.

The second case is completely overlooked in literature (cf, for instance\textsuperscript{10,11,23}), namely that extended Lagrangians $L_e$ exist that are related to a given conventional Lagrangian $L$ according to Eq. (6) \textit{without being homogeneous forms} in the $n + 1$ velocities $dq^\mu/ds$. In Sect. 3.1, a simple example will be furnished by setting up such a non-homogeneous extended Lagrangian $L_e$ for the free relativistic point particle. For a non-homogeneous extended Lagrangian $L_e$, the extended set of Euler-Lagrange equations (4) is not redundant and the Legendre transformation to an extended Hamiltonian $H_e$ exists. In that case, Eq. (11) does not represent an identity, which implies that Eq. (12) and, subsequently, Eq. (13) do not hold. Then, Eq. (11), regarded as an \textit{implicit equation}, is always satisfied on the extended system evolution path parametrized by $s$, which is given by the solution of the extended set of Euler-Lagrange equations (4). This can be seen by calculating the
total $s$-derivative of Eq. (11) and inserting the Euler-Lagrange equations (4)

\[
\frac{d}{ds} L_e \left( q^\mu, \frac{dq^\mu}{ds} \right) - \sum_{\mu=0}^{n} \frac{dq^\mu}{ds} \frac{d}{ds} \left( \frac{dq^\mu}{ds} \right) - \sum_{\mu=0}^{n} \frac{\partial L_e}{\partial \left( \frac{dq^\mu}{ds} \right)} \frac{d}{ds} \left( \frac{dq^\mu}{ds} \right) = 0.
\]

For this reason, Eq. (11) actually does not impose a constraint on the system's evolution along $s$ but separates unphysical states that do not satisfy Eq. (11) from the physical states that are solutions of the Euler-Lagrange equations (4). In this respect, Eq. (11) exactly corresponds to the case of the conserved energy function $e(t) = e_0$ of a conventional Lagrangian system $L(q, \dot{q})$ with no explicit time dependence. In that case, the quantity $e(t)$

\[
e(t) = \sum_{\mu=1}^{n} \frac{\partial L}{\partial \dot{q}^\mu} \dot{q}^\mu - L(q, \dot{q}) = e_0
\]

is a constant of motion and hence defines a surface in $TM$ on which the system’s motion takes place. Nevertheless, it is not considered a constraint as the condition (15) is automatically satisfied by means of the conventional Euler-Lagrange equations (2),

\[
\frac{de(t)}{dt} = \sum_{\mu=1}^{n} \left( \dot{q}^\mu \frac{dL}{dt} \frac{\partial L}{\partial \dot{q}^\mu} + \frac{\partial L}{\partial q^\mu} \dot{q}^\mu - \frac{\partial L}{\partial \dot{q}^\mu} \dot{q}^\mu - \frac{\partial L}{\partial \dot{q}^\mu} \dot{q}^\mu \right)
\]

\[
= \sum_{\mu=1}^{n} \dot{q}^\mu \left( \frac{dL}{dt} \frac{\partial L}{\partial q^\mu} \right) = 0.
\]

To summarize, by switching from the conventional variational principle (1) to the extended representation (3), we have introduced an extended Lagrangian $L_e$ that in addition depends on $dt(s)/ds$. Due to the emerging conserved quantity that follows from Eq. (14), the actual number of degrees of freedom is unchanged. In the language of Differential Geometry, the system’s motion along the parameter $s$ now takes place on a hypersurface, defined by Eq. (11), within the tangent bundle $TM_e \equiv T(M \times \mathbb{R})$ over the “spatial-plus-time” configuration manifold $M_e \equiv M \times \mathbb{R}$. This is the basis required for the description of relativistic point particle dynamics, which mandates configuration space coordinates and time to be treated on equal footing in a chart representation. It contrasts with the conventional Lagrangian description in $TM$ over the spatial configuration manifold $M$ for Lagrangians that do not explicitly depend on the system’s parameter $t$, which is commonly identified in applications with Newton’s absolute time.
2.2. Extended set of canonical equations

The Lagrangian formulation of particle dynamics can equivalently be expressed as a Hamiltonian description. The complete information on the given dynamical system is then contained in a Hamiltonian $H$, which carries the same information content as the corresponding Lagrangian $L$. It is defined by the Legendre transformation

$$H(q, p, t) = \sum_{\mu=1}^{n} p_{\mu} \frac{dq^{\mu}}{dt} - L\left(q, \frac{dq}{dt}, t\right),$$  \hspace{1cm} (16)

with the covariant momentum vector components $p_\mu$ being defined by

$$p_\mu = \frac{\partial L}{\partial \left(\frac{dq^{\mu}}{dt}\right)}.$$

Correspondingly, the extended Hamiltonian $H_e$ is defined as the extended Legendre transform of the extended Lagrangian $L_e$ as

$$H_e(q, p, q^0, p_0) = \sum_{\mu=0}^{n} p_{\mu} \frac{dq^{\mu}}{ds} - L_e\left(q', \frac{dq'}{ds}\right),$$  \hspace{1cm} (17)

wherein $q^0(s) = ct(s)$ and $p_0(s)$ denotes the canonical conjugate variable of $q^0(s)$. In order for $H_e$ to take over the complete information on the dynamical system from $L_e$, the Hesse matrix must be non-singular

$$\det \left( \frac{\partial^2 L_e}{\partial \left(\frac{dq'}{ds}\right)^2} \right) \neq 0.$$

We know from Eq. (9) that for $\mu = 1, \ldots, n$ the momentum variable $p_\mu$ is equally obtained from the extended Lagrangian $L_e$,

$$p_\mu = \frac{\partial L_e}{\partial \left(\frac{dq^{\mu}}{ds}\right)}.$$

This fact ensures the Legendre transformations (16) and (17) to be compatible. For the index $\mu = 0$, i.e., for $q^0 = ct$ we must take some care as the derivative of $L_e$ with respect to $dt/ds$ evaluates to

$$\frac{\partial L_e}{\partial \left(\frac{dt}{ds}\right)} = L - \sum_{\mu=1}^{n} \frac{\partial L}{\partial \left(\frac{dq^{\mu}}{dt}\right)} \frac{dq^{\mu}}{dt} = -H(q, p, t).$$

The momentum coordinate $p_0(s)$ that is conjugate to $q^0 = ct(s)$ must therefore be defined as

$$p_0(s) = -\frac{e(s)}{c}, \hspace{1cm} e(s) \neq H(q(s), p(s), t(s)),$$  \hspace{1cm} (19)

with $e(s)$ representing the instantaneous value of the Hamiltonian $H$ at $s$, but not the function $H$ proper as these functions are different. The canonical coordinate
p_0 must be conceived — like all other canonical coordinates — as a function of the independent variable, s, only. Thus, p_0 has solely a derivative with respect to s. In contrast, the Hamiltonian H contains the complete information on the underlying dynamical system — which is provided as the dependence of the value e(s) of H on the individual values of the q^\mu(s), p_\mu(s), and t(s) — and thus has derivatives with respect to all these canonical coordinates. We may express the definition of p_0(s), and e(s), by means of the comprehensible notation

\[ p_0(s) = \frac{\partial L_e}{\partial \left( \frac{dq^0}{ds} \right)}(s) \iff e(s) = -\frac{\partial L_e}{\partial \left( \frac{dt}{ds} \right)}(s). \]  

(20)

According to the extended Legendre transformation (17), the condition (11) translates in the extended Hamiltonian description simply into

\[ H_e(q(s), p(s), t(s), e(s)) = 0. \]  

(21)

This means that the extended Hamiltonian H_e directly defines the hypersurface within the extended phase space the classical particle motion is restricted to. Geometrically, the hypersurface lies in the cotangent bundle T^*M_e ≡ T^*(M × \mathbb{R}) over the same extended configuration manifold M_e ≡ M × \mathbb{R} as in the case of the Lagrangian description. This is exactly the higher-dimensional analogue of the case of an autonomous conventional Hamiltonian system, hence a Hamiltonian with no explicit time dependence, H(q(t), p(t)) = e_0 — where the system’s initial energy e_0 embodies a constant of motion. In that case, the system’s motion again takes place on a hypersurface that is now defined by H(q, p) = e_0 and represents the phase-space surface of constant energy within the cotangent bundle T^*M over the configuration manifold M.

By virtue of the Legendre transformations (16) and (17), the correlation from Eq. (6) of extended and conventional Lagrangians is finally converted into

\[ H_e(q, p, t, e) = \sum_{\mu=1}^{n} p_\mu \frac{dq^\mu}{ds} - e \frac{dt}{ds} - L_e \left( q, \frac{dq}{ds}, t, \frac{dt}{ds} \right) \]

\[ = \sum_{\mu=1}^{n} p_\mu \frac{dq^\mu}{ds} - e \frac{dt}{ds} - L \left( q, \frac{dq}{ds}, t \right) \frac{dt}{ds} \]

\[ = \sum_{\mu=1}^{n} p_\mu \frac{dq^\mu}{ds} - e \frac{dt}{ds} + H(q, p, t) - \sum_{\mu=1}^{n} p_\mu \frac{dq^\mu}{dt} \frac{dt}{ds} \]

\[ = (H(q, p, t) - e) \frac{dt}{ds}. \]  

(22)

The extended Legendre transformation (17) in conjunction with (18) and the extended set of Euler-Lagrange equations (4) immediately yields the extended set of canonical equations (\mu = 0, \ldots, n),

\[ \frac{\partial H_e}{\partial p_\mu} = \frac{dq^\mu}{ds}, \quad \frac{\partial H_e}{\partial q^\mu} = -\frac{\partial L_e}{\partial q^\mu} = -\frac{dp_\mu}{ds}. \]  

(23)
The right-hand sides of these equations follow directly from the Legendre transformation (17) as the Lagrangian \( L_e \) does not depend on the momenta \( p_\mu \) and has, up to the sign, the same space-time dependence as the Hamiltonian \( H_e \). The extended set is characterized by the additional pair of canonical equations for the index \( \mu = 0 \), which reads in terms of \( t(s) \) and \( e(s) \)

\[
\frac{de}{ds} = \frac{\partial H_e}{\partial t}, \quad \frac{dt}{ds} = -\frac{\partial H_e}{\partial e}.
\]

(24)

For the total derivative of \( H_e(q, p, t, e) \) we thus find

\[
\frac{dH_e}{ds} = \frac{\partial H_e}{\partial p_i} \frac{dp_i}{ds} + \frac{\partial H_e}{\partial q^i} \frac{dq^i}{ds} + \frac{\partial H_e}{\partial t} \frac{dt}{ds} + \frac{\partial H_e}{\partial e} \frac{de}{ds}
\]

\[
= \frac{dq^i}{ds} \frac{dp_i}{ds} - \frac{dp_i}{ds} \frac{dq^i}{ds} + \frac{de}{ds} \frac{dt}{ds} - \frac{dt}{ds} \frac{de}{ds} = 0.
\]

Thus, if \( e(0) = e_0 \) is identified with the system’s initial energy \( e_0 = H(q_0, p_0, 0) \) at \( t = 0 \), then the condition \( H_e(q, p, t, e) = 0, \frac{dH_e(q, p, t, e)}{ds} = 0 \) is automatically fulfilled along the system’s trajectory that is given by the solution of the extended set of canonical equations (23).

The extended phase-space variable \( e(s) \) is defined as the particular function of the independent variable, \( s \), that represents the value of the conventional Hamiltonian, \( H \). In accordance with Eqs. (19) and (21), we thus determine \( H \) for any given extended Hamiltonian \( H_e \) by solving \( H_e = 0 \) for \( e \). Then, \( H \) emerges as the right-hand side of the equation \( e = H \).

In the converse case, if the conventional Hamiltonian \( H \) is given and \( H_e \) is set up according to Eq. (22), then the canonical equation for \( \frac{dt}{ds} \) yields an identity, hence allows arbitrary parametrizations of time,

\[
\frac{dt}{ds} = -\frac{\partial H_e}{\partial e} = -\frac{\partial}{\partial e} \left[ (H(q, p, t) - e) \frac{dt}{ds} \right] = \frac{dt}{ds}.
\]

Exactly as in the Lagrangian description, this is not astonishing as a conventional Hamiltonian \( H \) generally does not provide the information for an equation of motion for \( t(s) \), i.e., for a particular parametrization of time \( t \). Furthermore, setting up the extended Hamiltonian \( H_e \) according to Eq. (22) on the basis of a given conventional Hamiltonian \( H \) does not generate additional information on the actual dynamical system.

Corresponding to Eq. (19), we may introduce the variable \( e_e \) as the value of the extended Hamiltonian \( H_e \). We can formally imagine \( H_e \) to be also a function of \( s \) in addition to its dependence of the extended phase-space variables,

\[
e_e \equiv H_e(q, p, t, e, s).
\]

(25)

By virtue of the extended set of canonical equations (23), we find that \( e_e \) is a constant of motion if and only if \( H_e \) does not explicitly depend on \( s \),

\[
e_e(s) = \text{const.} \quad \iff \quad H_e = H_e(q, p, t, e).
\]
In this case, \( s \) can be regarded as a cyclic variable, with \( e_e \) the pertaining constant of motion, and hence its conjugate. Thus, in the same way as \( (e, t) \) constitutes a pair of canonically conjugate variables, so does the pair \( (e_e, s) \), i.e., the value \( e_e \) of the extended Hamiltonian \( H_e \) and the parameterization of the system’s variables in terms of \( s \). In the context of a corresponding quantum description, this additional pair of canonically conjugate variables gives rise to the additional uncertainty relation

\[
\Delta e_e \Delta s \geq \frac{1}{2} \hbar.
\]  

Thus, in a quantum system whose classical limit is described by an extended Hamiltonian \( H_e \), we cannot simultaneously measure exactly both a deviation \( \Delta e_e \) from the hypersurface condition \( \Delta e_e(s) = 0 \) from Eqs. (21), (25) and the actual value of the system evolution parameter \( s \). For the particular extended Hamiltonian \( H_e \) of a relativistic particle in an external electromagnetic field, to be discussed in Sect. 3.5, the condition reflects the relativistic energy-momentum correlation, whereas the parameter \( s \) represents the particle’s proper time. For this particular system, the uncertainty relation (26) thus states the we cannot have simultaneous knowledge on a deviation from the relativistic energy-momentum correlation (66) and the particle’s proper time. The extended Lagrangian \( L_e \) and the uncertainty relation (26) constitute together the cornerstones for deriving the relativistic generalization of Feynman’s path integral approach to non-relativistic quantum physics, to be presented in Sect. 2.5.

To end this section, we remark that the extended Hamiltonian \( H_e \) most frequently found in literature is given by (cf, for instance, Refs. 11,12,14,13,15,20)

\[
H_e(q, p, t, e) = H(q, p, t) - e.
\]  

According to Eqs. (24), the canonical equation for \( dt / ds \) is obtained as

\[
\frac{dt}{ds} = -\frac{\partial H_e}{\partial e} = 1.
\]

Up to arbitrary shifts of the origin of our time scale, we thus identify \( t(s) \) with \( s \). As all other partial derivatives of \( H_e \) coincide with those of \( H \), so do the respective canonical equations. The system description in terms of \( H_e \) from Eq. (27) is thus identical to the conventional description and does not provide any additional information. The extended Hamiltonian (27) thus constitutes the simplest form of a trivial extended Hamiltonian.

### 2.3. Extended canonical transformations

The conventional theory of canonical transformations is built upon the conventional action integral from Eq. (1). In this theory, the Newtonian absolute time \( t \) plays the role of the common independent variable of both original and destination system. Similarly to the conventional theory, we may build the extended theory of canonical equations on the basis of the extended action integral from Eq. (3). With the
time \( t = q^0/c \) and the configuration space variables \( q^i \) treated on equal footing, we are enabled to correlate two Hamiltonian systems, \( H \) and \( H' \), with different time scales, \( t(s) \) and \( T(s) \), hence to canonically map the system’s time \( t \) and its conjugate quantity \( e \) in addition to the mapping of generalized coordinates \( q \) and momenta \( p \). The global timelike evolution parameter \( s \) then plays the role of the common independent variable of both systems, \( H \) and \( H' \). A general mapping of all dependent variables may be formally expressed as

\[
Q^\mu = Q^\mu(q^\nu, p_\nu), \quad P_\mu = P_\mu(q^\nu, p_\nu), \quad \mu = 0, \ldots, n
\]  

(28)

Completely parallel to the conventional theory, the subgroup of general transformations (28) that satisfy the principle \( \delta S_e = 0 \) of the action functional (3) is referred to as “canonical”.

\[
\delta \int_{s_a}^{s_b} L_e(q^\nu, \frac{dq^\nu}{ds}) \, ds = \delta \int_{s_a}^{s_b} L'_e(Q^\nu, \frac{dQ^\nu}{ds}) \, ds.
\]  

(29)

The action integrals may be expressed equivalently in terms of an extended Hamiltonian by means of the Legendre transformation (17). We thus get the following condition for a transformation (28) to be canonical

\[
\delta \int_{s_a}^{s_b} \left[ \sum_{\mu=0}^n p_\mu \frac{dq^\mu}{ds} - H_e(q^\nu, p_\nu) \right] \, ds = \delta \int_{s_a}^{s_b} \left[ \sum_{\mu=0}^n P_\mu \frac{dQ^\mu}{ds} - H'_e(Q^\nu, P_\nu) \right] \, ds.
\]  

(30)

As we are operating with functionals, the conditions (29) and (30) hold if the integrands differ at most by the derivative \( \frac{dF_1}{ds} \) of an arbitrary differentiable function \( F_1(q^\nu, Q^\nu) \)

\[
L_e = L'_e + \frac{dF_1}{ds}
\]  

(31)

\[
\sum_{\mu=0}^n p_\mu \frac{dq^\mu}{ds} - H_e = \sum_{\mu=0}^n P_\mu \frac{dQ^\mu}{ds} - H'_e + \frac{dF_1}{ds}.
\]  

(32)

Because of

\[
\delta \int_{s_a}^{s_b} \frac{dF_1}{ds} \, ds = \delta (F_1|_{s_b}) - \delta (F_1|_{s_a}) = 0,
\]

a term \( \frac{dF_1}{ds} \) does not contribute to the variation of the action functional (3). This means that the particular path \( (\bar{q}(s), \bar{t}(s)) \) on which the action integral takes on an extremum is maintained.

We restrict ourselves to functions \( F_1(q^\nu, Q^\nu) \) of the old and the new extended configuration space variables, hence to a function of those variables, whose derivatives match those of the integrands in Eq. (30). Calculating the \( s \)-derivative of \( F_1 \),

\[
\frac{dF_1}{ds} = \sum_{\mu=0}^n \left[ \frac{\partial F_1}{\partial q^\mu} \frac{dq^\mu}{ds} + \frac{\partial F_1}{\partial Q^\mu} \frac{dQ^\mu}{ds} \right],
\]  

(33)
we then get unique transformation rules by comparing the coefficients of Eq. (33) with those of (32)

\[ p_\mu = \frac{\partial F_1}{\partial q^\mu}, \quad P_\mu = -\frac{\partial F_1}{\partial Q^\mu}, \quad H'_e = H_e. \]  

(34)

\( F_1 \) is referred to as the extended generating function of the — now generalized — canonical transformation. The extended Hamiltonian \( H_e \) has the important property that its value is conserved under extended canonical transformations. This means that the system’s physical evolution is kept being confined to the surface \( H'_e = 0 \), hence that the condition (21) is maintained in the transformed system, as required. Corresponding to the extended set of canonical equations, the additional transformation rule is given for the index \( \mu = 0 \). This transformation rule may be expressed equivalently in terms of \( t(s), e(s), \) and \( T(s), E(s) \) as

\[ e = -\frac{\partial F_1}{\partial t}, \quad E = \frac{\partial F_1}{\partial T}, \]  

(35)

with \( E \), correspondingly to Eq. (19), the value of the transformed Hamiltonian \( H' \)

\[ P_0(s) = -\frac{E(s)}{c}, \quad E(s) \not\equiv H'(Q(s), P(s), T(s)). \]  

(36)

The addressed transformed Hamiltonian \( H' \) is finally obtained from the general correlation of conventional and extended Hamiltonians from Eq. (22), and the transformation rule \( H'_e = H_e \) for the extended Hamiltonian from Eq. (34)

\[ \left[ H'(Q, P, T) - E \right] \frac{dT}{ds} = \left[ H(q, p, t) - e \right] \frac{dt}{ds}. \]

Eliminating the evolution parameter \( s \), we arrive at the following two equivalent transformation rules for the conventional Hamiltonians under extended canonical transformations

\[ \left[ H'(Q, P, T) - E \right] \frac{dT}{dt} = H(q, p, t) - e \]

\[ \left[ H(q, p, t) - e \right] \frac{\partial T}{\partial T} = H'(Q, P, T) - E. \]  

(37)

The transformation rules (37) are generalizations of the rule for conventional canonical transformations as now cases with \( T \neq t \) are included. We will see at the end of this section that the rules (37) merge for the particular case \( T = t \) into the corresponding rules of conventional canonical transformation theory.

By means of the Legendre transformation

\[ F_2(q^\nu, P_\nu) = F_1(q^\nu, Q^\nu) + \sum_{\mu=0}^{n} Q^\mu P_\mu, \quad P_\mu = -\frac{\partial F_1}{\partial Q^\mu}, \]  

(38)

we may express the extended generating function of a generalized canonical transformation equivalently as a function of the original extended configuration space variables \( q^\nu \) and the extended set of transformed canonical momenta \( P_\nu \). As, by
definition, the functions $F_1$ and $F_2$ agree in their dependence on the $q^\mu$, so do the corresponding transformation rules

$$\frac{\partial F_1}{\partial q^\mu} = \frac{\partial F_2}{\partial q^\mu} = p_\mu.$$  

This means that all $q^\mu$ do not take part in the transformation defined by (38). As $F_1$ does not depend on the $P_\nu$, the new transformation rule pertaining to $F_2$ thus follows immediately as

$$\frac{\partial F_2}{\partial P_\nu} = \sum_{\mu=0}^{n} Q^\mu \frac{\partial P_\mu}{\partial P_\nu} = \sum_{\mu=0}^{n} Q^\mu \delta_\mu^\nu = Q^\nu.$$

The new set of transformation rules, which is, of course, equivalent to the previous set from Eq. (34), is thus

$$p_\mu = \frac{\partial F_2}{\partial q^\mu}, \quad Q^\mu = \frac{\partial F_2}{\partial P_\mu}, \quad H'_e = H_e.$$  

(39)

Expressed in terms of the variables $q, p, t, e$, and $Q, P, T, E$ the new set of coordinate transformation rules takes on the more elaborate form

$$p_i = \frac{\partial F_2}{\partial q^i}, \quad Q^i = \frac{\partial F_2}{\partial P^i}, \quad e = -\frac{\partial F_2}{\partial t}, \quad T = -\frac{\partial F_2}{\partial E}.$$  

(40)

Similarly to the conventional theory of canonical transformations, there are two more possibilities to define a generating function of an extended canonical transformation. By means of the Legendre transformation

$$F_3(p_\nu, Q^\nu) = F_1(q^\nu, Q^\nu) - \sum_{\mu=0}^{n} q^\mu p_\mu, \quad p_\mu = -\frac{\partial F_1}{\partial q^\mu},$$

we find in the same manner as above the transformation rules

$$q^\nu = -\frac{\partial F_3}{\partial p_\nu}, \quad P_\mu = -\frac{\partial F_3}{\partial Q^\mu}, \quad H'_e = H_e.$$  

(41)

Finally, applying the Legendre transformation, defined by

$$F_4(p_\nu, P_\nu) = F_3(p_\nu, Q^\nu) + \sum_{\mu=0}^{n} Q^\mu P_\mu, \quad P_\mu = -\frac{\partial F_3}{\partial Q^\mu},$$

the following equivalent version of transformation rules emerges

$$q^\mu = -\frac{\partial F_4}{\partial p_\mu}, \quad Q^\mu = \frac{\partial F_4}{\partial P_\mu}, \quad H'_e = H_e.$$  

Calculating the second derivatives of the generating functions, we conclude that the following correlations for the derivatives of the general mapping from Eq. (28) must hold for the entire set of extended phase-space variables,

$$\frac{\partial Q^\mu}{\partial q^\nu} = \frac{\partial p_\nu}{\partial P_\mu}, \quad \frac{\partial Q^\mu}{\partial p_\nu} = -\frac{\partial q^\nu}{\partial P_\mu}, \quad \frac{\partial P_\mu}{\partial q^\nu} = -\frac{\partial p_\nu}{\partial Q^\mu}, \quad \frac{\partial P_\mu}{\partial p_\nu} = \frac{\partial q^\nu}{\partial Q^\mu}.$$
Exactly if these conditions are fulfilled for all \( \mu, \nu = 0, \ldots, n \), then the extended coordinate transformation (28) is canonical and preserves the form of the extended set of canonical equations (23). Otherwise, we are dealing with a general, non-canonical coordinate transformation that does \emph{not} preserve the form of the canonical equations.

The connection of the extended canonical transformation theory with the conventional one is furnished by the particular extended generating function

\[
F_2(q, P, t, E) = f_2(q, P, t) - tE,
\]

with \( f_2(q, P, t) \) denoting a conventional generating function. According to Eqs. (40), the coordinate transformation rules following from (42) are

\[
\begin{align*}
p_i &= \frac{\partial f_2}{\partial q^i}, \\
Q^i &= \frac{\partial f_2}{\partial P_i}, \\
e &= -\frac{\partial f_2}{\partial t} + E, \\
T &= t.
\end{align*}
\]

With \( \partial T/\partial t = 1 \), the general transformation rule (37) for conventional Hamiltonians now yields the well-known rule for Hamiltonians \( H' \) under conventional canonical transformations,

\[
H'(Q, P, t) = H(q, p, t) + E - e = H(q, p, t) + \frac{\partial f_2}{\partial t}.
\]

Canonical transformations that are defined by extended generating functions of the form of Eq. (42) leave the time variable unchanged and thus define the subgroup of conventional canonical transformations within the general group of extended canonical transformations. Corresponding to the trivial extended Hamiltonian from Eq. (27), we may refer to (42) as the \emph{trivial extended generating function}.

### 2.4. Extended Hamilton-Jacobi equation

In the context of the extended canonical transformation theory, we may derive an extended version of the Hamilton-Jacobi equation. We are looking for a generating function \( F_2(q^\nu, P_\nu) \) of an extended canonical transformation that maps a given extended Hamiltonian \( H_e = 0 \) into a transformed extended Hamiltonian \( H'_e = 0 \) with the property that \emph{all} partial derivatives of \( H'_e(Q^\nu, P_\nu) \) vanish. Hence, according to the extended set of canonical equations (23), the derivatives of all canonical variables \( Q^\mu(s), P_\mu(s) \) with respect to the system’s evolution parameter \( s \) must vanish

\[
\begin{align*}
\frac{\partial H'_e}{\partial P_\mu} = \frac{dQ^\mu}{ds} &= 0, \\
-\frac{\partial H'_e}{\partial Q^\mu} = \frac{dP_\mu}{ds} &= 0, \\
\mu = 0, \ldots, n.
\end{align*}
\]

This means that \emph{all} transformed canonical variables \( Q^\mu, P_\mu \) must be constants of motion. Writing the variables for the index \( \mu = 0 \) separately, we thus have

\[
T = \text{const.}, \quad Q^i = \text{const.}, \quad E = \text{const.}, \quad P_i = \text{const}.
\]

Thus, corresponding to the conventional Hamilton-Jacobi formalism, the vectors of the transformed canonical variables, \( Q \) and \( P \), are constant. Yet, in the extended
formalism, the transformed time $T$ is also a constant. The particular generating function $F_2(q^ν, p^ν)$ that defines transformation rules for the extended set of canonical variables such that Eqs. (43) hold for the transformed variables thus defines a mapping of the entire system into its state at a fixed instant of time, hence — up to trivial shifts in the origin of the time scale — into its initial state at $T = t(0), Q^i = q^i(0), P^i = p^i(0), E = H(q(0), p(0), t(0))$.

We may refer to this particular generating function as the extended Hamiltonian action function $F_2 ≡ S_e(q^ν, p^ν)$. According to the transformation rule $H'_e = H_e$ for extended Hamiltonians from Eq. (34), we obtain the transformed extended Hamiltonian $H'_e ≡ 0$ simply by expressing the original extended Hamiltonian $H_e = 0$ in terms of the transformed variables. This means for the conventional Hamiltonian $H(q, p, t)$ according to Eq. (22) in conjunction with the transformation rules from Eqs. (40),

$$H\left(q^1, \ldots, q^n, \frac{∂S_e}{∂q^1}, \ldots, \frac{∂S_e}{∂q^n}, t\right) + \frac{∂S_e}{∂t} = 0.$$  

(44)

Equation (44) has exactly the form of the conventional Hamilton-Jacobi equation. Yet, it is actually a generalization as the extended action function $S_e$ represents an extended generating function of type $F_2$, as defined by Eq. (38). This means that $S_e$ is also a function of the (constant) transformed energy $E = -P(0)$.

Summarizing, the extended Hamilton-Jacobi equation may be interpreted as defining the mapping of all canonical coordinates $q, p, t,$ and $e$ of the actual system into constants $Q, P, T,$ and $E$. In other words, it defines the mapping of the entire dynamical system from its actual state at time $t$ into its state at a fixed instant of time, $T$, which could be the initial conditions.

2.5. Generalized path integral with extended Lagrangians

In Feynman’s path integral approach to quantum mechanics, the space and time evolution of a wave function $ψ(q, t)$ is formulated in terms of a transition amplitude density $K(b, a)$, also referred to as a kernel, or, a propagator:

$$ψ(q_b, t_b) = \int_{−∞}^{∞} K(q_b, t_b; q_a, t_a) ψ(q_a, t_a) d^3q_a.$$  

(45)

The parameterized kernel $K_σ(b, a)$ for a parameterized action $S_e$ is given by the multiple path integral

$$K_σ(b, a) = \int \int \exp \left\{ i \frac{1}{\hbar} S_e[q(s), t(s)] \right\} D^3q(s) D^3t(s).$$  

(46)
Herein, the integrals are to be taken over all paths that go from \((q_a, t_a)\) at \(s_a\) to \((q_b, t_b)\) at \(s_b\). The justification for integrating over all times is that in relativistic physics we must treat space and time on equal footing. Hence, we must allow the laboratory time \(t\) to take any value — negative and even positive ones — if we regard \(t\) from the viewpoint of a particle with its proper time \(s\). We thus additionally integrate over all histories of the particle. The integration over all futures can then be interpreted as integration over all histories of the anti-particle, whose proper timescale runs backwards in terms of the particle’s proper timescale.

If the time paths and the spatial paths are taken to be independent of each other, hence if we do not incorporate the shell condition (11) into the integration boundaries, we also sum over all particles off the mass shell. The action functional \(S_e\) stands for the \(s\)-integral over the extended Lagrangian \(L_e\), as defined by Eq. (3).

In classical dynamics, the parameterization of space and time variables can be eliminated by means of the shell condition (11). For the corresponding quantum description, the uncertainty principle from Eq. (26) applies. It tells us that an accurate fulfillment of the condition \(\Delta e_e(s) = 0\) is related to a complete uncertainty about the parameterization of the system’s variables phase-space in terms of \(s\). Therefore, in the context of the path integral approach, the condition \(\Delta e_e(s) = 0\) is incorporated by integrating the parameterized kernel \(K_\sigma(b, a)\) over all possible parameterizations \(\sigma = s_b - s_a > 0\) of coordinates \(q(s)\) and time \(t(s)\). The final kernel, hence the transition amplitude density is thus given by

\[
K(b, a) = \int_0^\infty K_\sigma(b, a) \, d\sigma. \tag{47}
\]

This means that all parameterized kernels \(K_\sigma(b, a)\) contribute with equal weight to the total transition amplitude \(K(b, a)\). As an example, we calculate in Sect. 3.11 the explicit form of the space-time propagator for the wave function of a relativistic free particle from the extended Lagrangian \(L_e\) of the pertaining classical system.

For an infinitesimal step \(\delta \epsilon = s_b - s_a\), we may approximate the action functional \(S_e\) from Eq. (3) by

\[
S_{e, \delta \epsilon}[q^\mu(s)] = \delta \epsilon \, L_e \left( \frac{q^\mu_s + q^\mu_a}{2}, \frac{q^\mu_b - q^\mu_a}{\delta \epsilon} \right).
\]

For \(s_b = s_a + \delta \epsilon\), the kernel \(K_\sigma(s_a + \delta \epsilon, s_a)\) from Eq. (46) that yields the transition amplitude density for a particle along this infinitesimal interval \(s_b - s_a\) is accordingly given by

\[
K(b, a) = \frac{1}{M} \exp \left[ \frac{i}{\hbar} \int_{s_a, \delta \epsilon} S_{e, \delta \epsilon} \right].
\]

As we proceed an infinitesimal step \(\delta \epsilon\) only, and then take the limit \(\delta \epsilon \to 0\), the integration (47) over all possible parameterizations of this step must be omitted. For, conversely to the situation discussed beforehand, a small \(\delta \epsilon = \Delta s\) is related to a large uncertainty with respect to satisfying the condition \(\Delta e_e(s) = 0\), so that in the limit \(\delta \epsilon \to 0\) the condition ceases to exist.
The yet to be determined normalization factor $M$ represents the integration measure for one step of the multiple path integral (46). Clearly, this measure must depend on the step size $\delta \epsilon$. The transition of a given wave function $\psi(q^\mu_a)$ at the particle’s proper time $s_a$ to the wave function $\psi(q^\mu_b)$ that is separated by an infinitesimal proper time interval $\delta \epsilon = s_b - s_a$ can now be formulated according to Eq. (45) as

$$\psi(q^\mu_b) = \frac{1}{M} \exp \left[ \frac{i}{\hbar} S_{c, \delta \epsilon} \right] \psi(q^\mu_a) \, dq_a.$$  

(48)

Note that we integrate here over the entire space-time. To serve as test for this approach, we derive in Sect. 3.10 the Klein-Gordon equation on the basis of the extended Lagrangian $L_e$ for a relativistic particle in an external electromagnetic field.

3. Examples of extended Hamilton-Lagrange systems

3.1. Extended Lagrangian for a relativistic free particle

As only expressions of the form $q^2 - c^2 t^2$ are preserved under the Lorentz group, the conventional Lagrangian for a free point particle of mass $m$, given by

$$L^{nr}(q, \frac{dq}{dt}, t) = T - V = \frac{1}{2} m \left( \frac{dq}{dt} \right)^2 - mc^2,$$  

(49)

is obviously not Lorentz-invariant. Yet, in the extended description, a corresponding Lorentz-invariant Lagrangian $L_e$ can be constructed by introducing $s$ as the new independent variable, and by treating the space and time variables, $q(s)$ and $q^\mu = ct(s)$ equally. This is achieved by adding the corresponding derivative of the time variable $t(s)$,

$$L_e \left( q, \frac{dq}{ds}, t, \frac{dt}{ds} \right) = \frac{1}{2} mc^2 \left[ \frac{1}{c^2} \left( \frac{dq}{ds} \right)^2 - \left( \frac{dt}{ds} \right)^2 - 1 \right].$$  

(50)

The constant third term has been defined accordingly to ensure that $L_e$ converges to $L^{nr}$ in the limit $dt/ds \to 1$. Of course, the dynamics following from (49) and (50) are different — which reflects the modification our dynamics encounters if we switch from a non-relativistic to a relativistic description. The Lagrangian (50) is no homogeneous form of first order in the velocities $dq^\mu/ds, \mu = 0, \ldots, 3$. Therefore, we obtain from Eq. (11) the hypersurface condition, also referred to as the mass shell condition:

$$\frac{1}{c^2} \left( \frac{dq}{ds} \right)^2 - \left( \frac{dt}{ds} \right)^2 + 1 = 0 \iff \frac{1}{c^2} \left( \frac{dq}{dt} \right)^2 + \left( \frac{ds}{dt} \right)^2 - 1 = 0.$$  

(51)

We thus encounter the reciprocal value of the relativistic scale factor, $\gamma$,

$$\frac{ds}{dt} = \sqrt{1 - \frac{1}{c^2} \left( \frac{dq}{dt} \right)^2} = \gamma^{-1},$$  

(52)
which shows that in the case of the Lagrangian (50) the system evolution parameter $s$ is physically nothing else than the particle’s proper time. Inserting the condition (51) into the Lagrangian yields the constant value of $L_e$,

$$L_e \big|_{\text{mass shell}} = -mc^2.$$ 

In contrast to the non-relativistic description, the constant rest energy term $-\frac{1}{2}mc^2$ in the extended Lagrangian (50) is essential. Consequently, the extended Lagrangian (50) is no homogeneous form of first order in the velocities $dq^\mu/ds, \mu = 0, \ldots, 3$, the condition (11) is not satisfied identically. Yet, in the derivation of (11), we have assumed that a corresponding conventional Lagrangian $L$ exists, hence a Lagrangian that depends on $dt/ds$ only indirectly via the reparameterization condition

$$\frac{dq}{dt} = \frac{dq}{ds} \frac{dt}{ds}$$

from Eq. (6) applied to its velocities. We must, therefore, make sure that such a corresponding conventional Lagrangian $L$ exists, hence a function $L = L_e \, ds/dt$ that does not depend anymore on $s$. For the extended Lagrangian $L_e$ from Eq. (50), a corresponding conventional Lagrangian $L$ indeed exists. Inserting Eq. (51) into Eq. (50), we find with Eq. (52)

$$L \left( q, \frac{dq}{dt}, t \right) = L_e \left( q, \frac{dq}{ds}, \frac{dt}{ds} \right) \bigg|_{\text{mass shell}} \frac{ds}{dt} = -mc^2 \left( \frac{ds}{dt} \right)^2 = -mc^2 \sqrt{1 - \frac{1}{c^2} \left( \frac{dq}{dt} \right)^2}. \quad (53)$$

We thus encounter the well-known conventional Lagrangian of a relativistic free particle. In contrast to the equivalent extended Lagrangian from Eq. (50), the Lagrangian (53) is not quadratic in the derivatives of the dependent variables, $q(t)$. The loss of the quadratic form originates from the projection of the hypersurface description within the tangent bundle $T(\mathcal{M} \times \mathbb{R})$ to the description within $(T\mathcal{M}) \times \mathbb{R}$. The quadratic form is recovered in the non-relativistic limit by expanding the square root, which yields the Lagrangian $L^\text{nr}$ from Eq. (49).

Denoting by $q^\mu$ the components of the contravariant four-vector of space-time variables $(q^0, \ldots, q^3) = (ct, x, y, z)$, the corresponding covariant vector is then $(q_0, \ldots, q_3) = (-ct, x, y, z)$ for the metric $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$. Adopting the “summation convention,” which means to sum over all quantities with pairs of identical covariant and contravariant indices, the non-homogeneous extended Lagrangian from Eq. (50) can then be rewritten in covariant notation as

$$L_e \left( q^\mu, \frac{dq^\mu}{ds} \right) = \frac{1}{2}m \left( \frac{dq^\alpha}{ds} \frac{dq^\alpha}{ds} - c^2 \right).$$
The hypersurface condition (11) is then expressed as
\[ \frac{dq^\alpha}{ds} \frac{dq_\alpha}{ds} = -c^2, \]
which depicts the constant length of the four-velocity vector.

To summarize, with \( L_e \) from Eq. (50), we have found a \textit{non-trivial} extended Lagrangian \( L_e \), i.e. an extended Lagrangian that is \textit{non-homogeneous} in its velocities and possesses a corresponding conventional Lagrangian \( L = L_e \frac{ds}{dt} \), with \( ds/dt \) determined by Eq. (11) that now embodies an implicit equation rather than an identity. In addition to the equations of motion for \( q(s) \), this \( L_e \) determines uniquely the correlation \( t(s) \) of the laboratory time \( t \) to the particle’s proper time, \( s \).

### 3.2. Trivial extended Lagrangian for a relativistic free particle

Given the conventional Lagrangian (53), we may immediately set up the corresponding \textit{trivial} extended Lagrangian according to Eq. (6) by multiplying \( L \) with \( dt/ds \)
\[ L^{\text{triv}}_e \left( q, \frac{dq}{ds}, t, \frac{dt}{ds} \right) = -mc \sqrt{c^2 \left( \frac{dt}{ds} \right)^2 - \left( \frac{dq}{ds} \right)^2} \]
\[ = -mc \sqrt{-\frac{dq^\alpha dq_\alpha}{ds}}. \]

We easily convince ourselves that the trivial extended Lagrangian satisfies Eq. (11) \textit{identically}
\[ \frac{\partial L^{\text{triv}}_e}{\partial \left( \frac{dq^\mu}{ds} \right)} \frac{dq^\mu}{ds} = mc \sqrt{\frac{dq^\mu dq_\mu}{ds}} \frac{dq^\mu}{ds} \]
\[ = -mc \sqrt{-\frac{dq^\alpha dq_\alpha}{ds}} \]
\[ \equiv L^{\text{triv}}_e \]
and thus fulfills Eq. (12),
\[ \frac{\partial^2 L^{\text{triv}}_e}{\partial \left( \frac{dq^\mu}{ds} \right) \partial \left( \frac{dq_\mu}{ds} \right)} \frac{dq^\mu}{ds} = mc \left( \frac{dq^\nu dq_\mu}{ds} - \delta^\nu_\mu \frac{dq^\beta dq_\beta}{ds} \right) \frac{dq^\mu}{ds} \]
\[ \equiv 0. \]

The subsequent equation of motion for \( t(s) \) does \textit{not} determine a parametrization of time \( t \) but rather allows for arbitrary parametrizations. As a trivial extended
Lagrangian $L_{\text{triv}}$ generally follows by multiplying a given conventional Lagrangian $L$ by $dt/ds$, a formally covariant description is encountered in the sense that space and time variables are then treated on equal footing. Yet, no additional information on the dynamical system is provided by the transition from $L$ to $L_{\text{triv}}$.

### 3.3. Trivial extended Hamiltonian for a relativistic free particle

For a trivial extended Lagrangian, it is not possible to derive the corresponding trivial extended Hamiltonian as the Legendre transformation of a homogeneous extended Lagrangian is singular. This does not mean that a corresponding extended Hamiltonian does not exist, as it is frequently claimed in literature\textsuperscript{23}. To the contrary, for any conventional Lagrangian $L$ that can be Legendre-transformed into a corresponding conventional Hamiltonian $H$, one can always set up $L_e$ according to Eq. (6) and $H_e$ according to Eq. (22). Setting up the extended set of Euler-Lagrange equations for a trivial extended Lagrangian then yields exactly the same description of the given dynamical system as setting up the extended set of canonical equations for the trivial extended Hamiltonian obtained this way.

In order to set up the trivial extended Hamiltonian $H_{\text{triv}}^e$ that corresponds to the trivial extended Lagrangian $L_{\text{triv}}^e$ from Eq. (54) of the free relativistic point particle, one must first Legendre-transform the underlying conventional Lagrangian (53) to the corresponding conventional Hamiltonian according to

$$H(q, p, t) = p\dot{q} - L(q, \dot{q}, t), \quad p = \frac{\partial L}{\partial \dot{q}}.$$  

For the particular Lagrangian (53), one finds

$$p = \frac{m\dot{q}}{\sqrt{1 - \frac{\dot{q}^2}{c^2}}}, \quad H = \frac{mc^2}{\sqrt{1 - \frac{\dot{q}^2}{c^2}}}.$$  

A Hamiltonian must be expressed in terms of the canonical momenta rather than by the velocities, hence $\dot{q}$ must be expressed in terms of $p$.

$$H^2 = \frac{m^2c^4}{1 - \frac{\dot{q}^2}{c^2}}, \quad p^2 = \frac{m^2\dot{q}^2}{1 - \frac{\dot{q}^2}{c^2}} \Rightarrow H^2 - p^2c^2 = \frac{m^2c^4}{1 - \frac{\dot{q}^2}{c^2}} \left(1 - \frac{\dot{q}^2}{c^2}\right) = m^2c^4,$$  

hence

$$H(q, p, t) = \sqrt{p^2c^2 + m^2c^4}.$$  

The corresponding trivial extended Hamiltonian $H_{\text{triv}}^e$ can now be set up according to the general recipe from Eq. (22)

$$H_{\text{triv}}^e(q, p, t, e) = \left(\sqrt{p^2c^2 + m^2c^4} - e\right) \frac{dt}{ds}. \quad (55)$$

In contrast to the Lagrangian description, the factor $dt/ds$ does not represent a conjugate variable but enters into the canonical equations as an external factor. The trivial extended Hamiltonian (55) has exactly the same information content...
on the underlying dynamical system as the trivial extended Lagrangian from (54) and thus yields identical equations of motion. In particular, $H_{triv}$ equally does not determine a parametrization of time, $t = t(s)$, but rather allows for arbitrary parametrizations. This can be seen by setting up the respective canonical equation

$$\frac{dt}{ds} = -\frac{\partial H_{triv}}{\partial e} = -\frac{dt}{ds}.$$  

One thus finds an identity but no substantial canonical equation for $t = t(s)$.

### 3.4. Extended Lagrangian for a relativistic particle in an external electromagnetic field

The non-homogeneous extended Lagrangian $L_{e}$ of a point particle of mass $m$ and charge $\zeta$ in an external electromagnetic field that is described by the potentials $(\phi, A)$ is given by

$$L_{e}(q, \frac{dq}{ds}, t, \frac{dt}{ds}) = \frac{1}{2} mc^2 \left[ \frac{1}{c^2} \left( \frac{dq}{ds} \right)^2 - \left( \frac{dt}{ds} \right)^2 - 1 \right] + \zeta c A(q, t) \frac{dq}{ds} - \zeta \phi(q, t) \frac{dt}{ds}. \quad (56)$$

The associated hypersurface condition (11) for $L_{e}$ coincides with that for the free-particle Lagrangian from Eq. (51) as all terms linear in the velocities drop out

$$\left( \frac{dt}{ds} \right)^2 - \frac{1}{c^2} \left( \frac{dq}{ds} \right)^2 - 1 = 0. \quad (57)$$

Similar to the free particle case from Eq. (53), the extended Lagrangian (56) may be projected into $(T\mathbb{R}) \times \mathbb{R}$ to yield the well-known conventional relativistic Lagrangian $L$

$$L(q, \frac{dq}{dt}, t) = -mc^2 \sqrt{1 - \frac{1}{c^2} \left( \frac{dq}{dt} \right)^2} + \frac{\zeta}{c} A \frac{dq}{dt} - \zeta \phi. \quad (58)$$

Again, the quadratic form of the velocity terms is lost owing to the projection.

For small velocity $\frac{dq}{dt}$, the quadratic form is regained as the square root in (58) may be expanded to yield the conventional non-relativistic Lagrangian for a point particle in an external electromagnetic field,

$$L_{nr}(q, \frac{dq}{dt}, t) = \frac{1}{2} m \left( \frac{dq}{dt} \right)^2 + \frac{\zeta}{c} A \frac{dq}{dt} - \zeta \phi - mc^2. \quad (59)$$

Significantly, this Lagrangian can be derived directly, hence without the detour over the projected Lagrangian (58), from the extended Lagrangian (56) by letting $dt/ds \rightarrow 1$.

It is instructive to review the Lagrangian (56) and its non-relativistic limit (59) in covariant notation. With Einstein’s summation convention and the notation...
$A_0(q^\mu) = -\phi(q^\mu)$ for the particular constant metric $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$, the extended Lagrangian (56) then writes

$$L_e\left(q^\mu, \frac{dq^\mu}{ds}\right) = \frac{1}{2}m \eta_{\alpha\beta} \frac{dq^\alpha}{ds} \frac{dq^\beta}{ds} + \frac{\zeta}{c} A_\alpha \frac{dq^\alpha}{ds} - \frac{1}{2}mc^2. \quad (60)$$

The hypersurface condition (57) is then converted into

$$\eta_{\alpha\beta} \frac{dq^\alpha}{ds} \frac{dq^\beta}{ds} = -c^2. \quad (61)$$

Correspondingly, the non-relativistic Lagrangian (59) has the equivalent representation

$$L^{nr}\left(q^\mu, \frac{dq^\mu}{dt}\right) = \frac{1}{2}m \eta_{\alpha\beta} \frac{dq^\alpha}{dt} \frac{dq^\beta}{dt} + \frac{\zeta}{c} A_\alpha \frac{dq^\alpha}{dt} - \frac{1}{2}mc^2. \quad (62)$$

Note that $(dq^0/dt)(dq_0/dt) = -c^2$, which yields the second half of the rest energy term, so that (62) indeed agrees with (59). Comparing the Lagrangian (62) with the extended Lagrangian from Eq. (60) — and correspondingly the Lagrangians (56) and (59) — we notice that the transition to the non-relativistic description is made by identifying the proper time $s$ with the laboratory time $t = q^0/c$. The remarkable formal similarity of the Lorentz-invariant extended Lagrangian (60) with the non-invariant conventional Lagrangian (62) suggests that approaches based on non-relativistic Lagrangians $L^{nr}$ may be transposed to a relativistic description by (i) introducing the proper time $s$ as the new system evolution parameter, (ii) treating the time $t(s)$ as an additional dependent variable on equal footing with the configuration space variables $q(s)$ — commonly referred to as the “principle of homogeneity in space-time” — and (iii) by replacing the conventional non-relativistic Lagrangian $L^{nr}$ with the corresponding Lorentz-invariant extended Lagrangian $L_e$, similar to the transition from (62) to (60).

### 3.5. Extended Hamiltonian for a relativistic particle in an external electromagnetic field

The extended Hamiltonian counterpart $H_e$ of the non-homogeneous extended Lagrangian (56) for a relativistic point particle in an external electromagnetic field is obtained via the Legendre transformation prescription from Eqs. (17) and (18). The transition to the extended Hamiltonian $H_e$ is easiest calculated by starting from the covariant form (60) of $L_e$ and afterwards converting the results to 3-vector notation. According to Eqs. (18) and (20), the canonical momenta $p_\mu$ are introduced by

$$p_\mu = \frac{\partial L_e}{\partial \left( \frac{dq^\mu}{ds} \right)} = m \eta_{\mu\alpha} \frac{dq^\alpha}{ds} + \frac{\zeta}{c} A_\mu = p_{\mu,\text{kin}} + \frac{\zeta}{c} A_\mu. \quad (63)$$

We notice that the kinetic momentum $p_{\mu,\text{kin}} = m dq_\mu/ds$ differs from the canonical momentum $p_\mu$ in the case of a non-vanishing external potential $A_\mu \neq 0$. 

---

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The condition for the Legendre transform of \( L_e \) to exist is that its Hesse matrix 
\[
\frac{\partial^2 L_e}{\partial \left( \frac{dq^\mu}{ds} \right) \partial \left( \frac{dq^\nu}{ds} \right)}
\]
must be non-singular, hence that the determinant of this matrix does not vanish. For the Lagrangian \( L_e \) from Eq. (60), this is actually the case as
\[
\det \left( \frac{\partial^2 L_e}{\partial \left( \frac{dq^\mu}{ds} \right) \partial \left( \frac{dq^\nu}{ds} \right)} \right) = \det (m \delta^\nu_\mu) = m^4 \neq 0.
\]
This falsifies claims made in literature\(^{23}\) that the Hesse matrix associated with an extended Lagrangian \( L_e \) be generally singular, and that for this reason an extended Hamiltonian \( H_e \) generally could not be obtained by a Legendre transformation of an extended Lagrangian \( L_e \). The necessary condition for an extended Hamiltonian \( H_e \) to emerge from a Legendre transformation of an extended Lagrangian \( L_e \) is that \( L_e \) must not be a homogeneous function of first order in its velocities \( \frac{dq^\mu}{ds} \).

With the Hesse condition being actually satisfied, the extended Hamiltonian \( H_e \) that follows as the Legendre transform (17) of \( L_e \) reads
\[
H_e(q^\mu, p_\mu) = \frac{d q^\alpha}{d s} \left( m \frac{d q_\alpha}{d s} + \frac{\zeta}{c} A_\alpha \right) - \frac{1}{2} m \frac{d q^\alpha}{d s} \frac{d q_\alpha}{d s} - \frac{\zeta}{c} A_\alpha \frac{d q^\alpha}{d s} + \frac{1}{2} mc^2
\]
\[= \frac{1}{2} m \frac{d q^\alpha}{d s} \frac{d q_\alpha}{d s} + \frac{1}{2} mc^2.
\]
As any Hamiltonian must be expressed in terms of the canonical momenta rather than through velocities, \( H_e \) takes on the more elaborate final form according to Eq. (63)
\[
H_e(q^\mu, p_\mu) = \frac{1}{2m} \left( p_\alpha - \frac{\zeta}{c} A_\alpha \right) \left( p^\alpha - \frac{\zeta}{c} A^\alpha \right) + \frac{1}{2} mc^2. \tag{64}
\]
This extended Hamiltonian coincides with the “super-Hamiltonian” that was postulated by Misner, Thorne, and Wheeler\(^{24}\).

In covariant notation, the condition \( H_e = 0 \) thus follows as
\[
\left( p_\alpha - \frac{\zeta}{c} A_\alpha \right) \left( p^\alpha - \frac{\zeta}{c} A^\alpha \right) + m^2 c^2 = 0,
\]
which follows equivalently if the velocities in the hypersurface condition (61) are replaced by the canonical momenta according to Eq. (63). In terms of the conventional 3-vectors for the canonical momentum \( p \) and vector potential \( A \), and the scalars, energy \( e \) and electric potential \( \phi \), the extended Hamiltonian \( H_e \) is equivalently expressed as
\[
H_e(q, p, t, e) = \frac{1}{2m} \left[ \left( p - \frac{\zeta}{c} A(q, t) \right)^2 - \left( \frac{e - \zeta \phi(q, t)}{c} \right)^2 \right] + \frac{1}{2} mc^2, \tag{65}
\]
and the condition \( H_e = 0 \) furnishes the usual relativistic energy relation
\[
\left( e - \zeta \phi(q, t) \right)^2 = c^2 \left( p - \frac{\zeta}{c} A(q, t) \right)^2 + m^2 c^4. \tag{66}
\]
The conventional Hamiltonian $H$ that describes the same dynamics is determined according to Eq. (19) as the particular function, whose value coincides with $e$. Solving $H = 0$ from Eq. (65) for $e$, we directly find $H$ as the left-hand side of the equation $H = e$,

$$H(q, p, t) = \sqrt{c^2 \left( p - \zeta \frac{\alpha}{c} A(q, t) \right)^2 + m^2 c^4 + \zeta \phi(q, t)} = e. \quad (67)$$

The conventional Hamiltonian $H_{nr}$ that describes the particle dynamics in the non-relativistic limit is obtained from the Lorentz-invariant Hamiltonian (67) by expanding the square root

$$H_{nr}(q, p, t) = \frac{1}{2m} \left( p - \zeta \frac{\alpha}{c} A(q, t) \right)^2 + \zeta \phi(q, t) + mc^2. \quad (68)$$

In contrast to the extended Lagrangian description, a direct way to transpose the relativistic extended Hamiltonian from Eq. (65) into the non-relativistic Hamiltonian $H_{nr}$ does not exist. We conclude that the Lagrangian approach is more appropriate if we want to “translate” a given non-relativistic Hamilton-Lagrange system into the corresponding Lorentz-invariant description.

In order to show that the extended Hamiltonian (65) and the well-known conventional Hamiltonian (67) indeed yield the same dynamics, we now set up the extended set of canonical equations (23) for the covariant extended Hamiltonian (64)

$$-\frac{\partial H_e}{\partial q^\mu} = \frac{dp_\mu}{ds} = \frac{\zeta}{mc} \eta^{\alpha \beta} \left( p_\alpha - \zeta \frac{\alpha}{c} A_\alpha \right) \frac{\partial A_\beta}{\partial q^\mu},$$

$$\frac{\partial H_e}{\partial p_\mu} = \frac{dq_\mu}{ds} = \frac{1}{m} \eta^{\mu \alpha} \left( p_\alpha - \zeta \frac{\alpha}{c} A_\alpha \right). \quad (68)$$

In the notation of scalars and 3-vectors, the pair of equations (68) separates into the following equivalent set of four equations

$$\frac{dp_i}{ds} = \frac{\zeta}{mc} \left( p_j - \zeta \frac{\alpha}{c} A_j \right) \frac{\partial A_i}{\partial q^j} - \frac{\zeta}{mc^2} \left( e - \zeta \phi \right) \frac{\partial \phi}{\partial q^i},$$

$$\frac{de}{ds} = -\frac{\zeta}{mc} \left( p_j - \zeta \frac{\alpha}{c} A_j \right) \frac{\partial A_i}{\partial t} + \frac{\zeta}{mc^2} \left( e - \zeta \phi \right) \frac{\partial \phi}{\partial t},$$

$$\frac{dq_i}{ds} = \frac{1}{m} \left( p^j - \zeta \frac{\alpha}{c} A^j \right),$$

$$\frac{dt}{ds} = \frac{1}{mc^2} \left( e - \zeta \phi \right). \quad (69)$$

From the last equation, we deduce the derivative of the inverse function $s = s(t)$ and insert the condition from Eq. (66)

$$\frac{ds}{dt} = \frac{mc^2}{e - \zeta \phi} = \frac{mc^2}{\sqrt{c^2 \left( p - \zeta \frac{\alpha}{c} A(q, t) \right)^2 + m^2 c^4}}. \quad (70)$$
The canonical equations (69) can now be expressed equivalently with the time $t$ as the independent variable

$$
- \frac{dp_i}{dt} = - \frac{dp_i}{ds} \frac{ds}{dt} = - \frac{\zeta c}{\sqrt{c^2(p - \frac{\zeta}{c}A(q,t))^2 + m^2c^4}} \left( p - \frac{\zeta}{c}A \right) \frac{\partial A}{\partial q^i} + \zeta \frac{\partial \phi}{\partial q^i}
$$

$$
\frac{de}{dt} = \frac{de}{ds} \frac{ds}{dt} = - \frac{\zeta c}{\sqrt{c^2(p - \frac{\zeta}{c}A(q,t))^2 + m^2c^4}} \left( p - \frac{\zeta}{c}A \right) \frac{\partial A}{\partial t} + \zeta \frac{\partial \phi}{\partial t}
$$

$$
\frac{dq^i}{dt} = \frac{dq^i}{ds} \frac{ds}{dt} = \frac{c^2}{\sqrt{c^2(p - \frac{\zeta}{c}A(q,t))^2 + m^2c^4}} \left( p^i - \frac{\zeta}{c}A^i \right).
$$

(71)

The right-hand sides of Eqs. (71) are exactly the partial derivatives $\frac{\partial H}{\partial q^i}, \frac{\partial H}{\partial t},$ and $\frac{\partial H}{\partial p_i}$ of the Hamiltonian (67) — and hence its canonical equations, which was to be shown.

The physical meaning of the $\frac{dt}{ds}$ is worked out by casting it to the equivalent form

$$
\frac{dt}{ds} = \sqrt{1 + \frac{(p - \frac{\zeta}{c}A(q,t))^2}{m^2c^2}} = \sqrt{1 + \left( \frac{p_{\text{kin}}(s)}{mc} \right)^2} = \gamma(s),
$$

with $p_{\text{kin}}(s)$ the instantaneous kinetic momentum of the particle. The dimensionless quantity $\frac{dt}{ds}$ thus represents the instantaneous value of the relativistic scale factor $\gamma$.

3.6. Lorentz transformation as an extended canonical transformation

We know that the Lorentz transformation provides the rules according to which a physical system is transformed from one inertial reference system into an other. On the other hand, a mapping of one Hamiltonian into another is constituted by a canonical transformation. Consequently, the Lorentz transformation must be a particular canonical transformation. As the Lorentz transformation always involves a transformation of the time scales $t \mapsto T$, this transformation can only be represented by an extended canonical transformation. Its generating function $F_2$ is given by

$$
F_2(q, p_{\text{kin}}, t, E_{\text{kin}}) = p_{\text{kin}}q - \gamma \left( E_{\text{kin}}t + \beta \left( \frac{p_{\text{kin}}ct - E_{\text{kin}}}{c}q \right) \right) + \frac{\gamma - 1}{\beta^2} (\beta p_{\text{kin}})(\beta q)
$$

(72)

with $\beta = v/c$ the constant vector that delineates the scaled relative velocity $v$ of both reference systems, and $\gamma$ the dimensionless relativistic scale factor $\gamma = 1/\sqrt{1 - \beta^2}$. In order to also cover cases where the particle moves within an
external potential, the index “kin” indicates that the momenta and the energy are to be understood as the “kinetic” quantities, as defined in Eq. (63). The generating function (72) generalizes the free-particle generator presented earlier in Ref. 19. The general transformation rules (40) for extended generating functions of type $F_2$ yield for the particular generator from Eq. (72)

$$p_{\text{kin}} = \frac{\partial F_2}{\partial q} = P_{\text{kin}} + \frac{\gamma \beta}{c} E_{\text{kin}} + \frac{\gamma - 1}{\beta^2} \beta (\beta P_{\text{kin}}), \quad e_{\text{kin}} = - \frac{\partial F_2}{\partial t} = \gamma E_{\text{kin}} + c \gamma \beta P_{\text{kin}},$$

$$Q = \frac{\partial F_2}{\partial P_{\text{kin}}} = q - \gamma \beta ct + \frac{\gamma - 1}{\beta^2} \beta (\beta q), \quad T = - \frac{\partial F_2}{\partial E_{\text{kin}}} = \gamma t - \frac{\gamma}{c} \beta q.$$

In matrix form, the transformation rules for the space-time coordinates, $Q$ and $T$, are

$$\begin{pmatrix} Q \\ cT \end{pmatrix} = \begin{pmatrix} 1 + \left(\frac{\gamma - 1}{\beta^2}\right) \beta & -\gamma \beta \\ -\gamma \beta & \gamma \end{pmatrix} \begin{pmatrix} q \\ ct \end{pmatrix}. \tag{73}$$

The corresponding linear relation for the kinetic momentum vector $p_{\text{kin}}$ and the kinetic energy $e_{\text{kin}}$ is

$$\begin{pmatrix} p_{\text{kin}} \\ e_{\text{kin}/c} \end{pmatrix} = \begin{pmatrix} 1 + \left(\frac{\gamma - 1}{\beta^2}\right) \beta & \gamma \beta \\ \gamma \beta & \gamma \end{pmatrix} \begin{pmatrix} P_{\text{kin}} \\ E_{\text{kin}/c} \end{pmatrix}. \tag{74}$$

If we replace the kinetic momenta with the canonical momenta according to Eq. (63), it is not astonishing to find that the external potentials obey the same transformation rule as the momenta,

$$\begin{pmatrix} A \\ \phi \end{pmatrix} = \begin{pmatrix} 1 + \left(\frac{\gamma - 1}{\beta^2}\right) \beta & \gamma \beta \\ \gamma \beta & \gamma \end{pmatrix} \begin{pmatrix} A' \\ \phi' \end{pmatrix}.$$  

We easily convince ourselves that the transformation (73) preserves the condition (51) that equally applies for a particle in an external potential. Correspondingly, the transformation (74) preserves the conditions (66). As a consequence, we have established the important result that the extended Hamiltonian $H_e$ from Eq. (65) is also preserved under Lorentz transformations

$$H'_e(P, Q, T, E) = H_e(p, q, t, e).$$

This is in agreement with the general canonical transformation rule for extended Hamiltonians from Eq. (34)

According to the subsequent rule for the conventional Hamiltonians, $H$ and $H'$, from Eq. (37), and $\partial T / \partial t = \gamma$, we find

$$\left(H' - E_{\text{kin}}\right) \gamma = H - e_{\text{kin}}. \tag{75}$$

In conjunction with the energy transformation rule from Eq. (74), $e_{\text{kin}} = \gamma E_{\text{kin}} + \beta \gamma P_{\text{kin}c}$, we get from Eq. (75) the transformation rule for a Hamiltonian $H$ under Lorentz transformations

$$H = \gamma \left(H' + \beta c P_{\text{kin}}\right).$$
As expected, the Hamiltonians, $H$ and $H'$, transform equally as their respective values, $e_{\text{kin}}$ and $E_{\text{kin}}$.

3.7. Infinitesimal canonical transformations, generalized Noether theorem

A general infinitesimal extended transformation is generated by

$$F_2(q^{\nu'}, p^{\nu'}) = \sum_{\alpha=0}^{n} q^{\alpha} P_{\alpha} + \delta \epsilon I(q^{\nu'}, p^{\nu'}).$$

(76)

In this generating function, $\delta \epsilon \in \mathbb{R}$ denotes an infinitesimal parameter, whereas the differentiable function $I(q^{\nu'}, p^{\nu'})$ quantifies the deviation of the actual infinitesimal transformation from the identity. We first derive the coordinate transformation rules for the particular generating function (76) according to the general rules (39),

$$p_{\mu} = \frac{\partial F_2}{\partial q^\mu} = P_{\mu} + \delta \epsilon \frac{\partial I}{\partial q^\mu},$$

$$Q^{\mu} = \frac{\partial F_2}{\partial P_{\mu}} = q^{\mu} + \delta \epsilon \frac{\partial I}{\partial P_{\mu}},$$

(77)

$$H'_{e} = H_{e}.$$

To first order in $\delta \epsilon$, the variations $\delta p_{\mu}$, $\delta q^{\mu}$, and $\delta H_{e}$ are obtained from the transformation rules (77) as

$$\delta p_{\mu} \equiv P_{\mu} - p_{\mu} = -\delta \epsilon \frac{\partial I}{\partial q^\mu},$$

$$\delta q^{\mu} \equiv Q^{\mu} - q^{\mu} = \delta \epsilon \frac{\partial I}{\partial P_{\mu}},$$

(78)

$$\delta H_{e} \equiv H'_{e} - H_{e} = 0.$$

Obviously, any function $I(q^{\nu'}, p^{\nu'})$ is invariant under the infinitesimal transformation if it defines,

$$\delta I = \sum_{\alpha=0}^{n} \left( \frac{\partial I}{\partial q^\alpha} \delta q^\alpha + \frac{\partial I}{\partial P_{\alpha}} \delta p_{\alpha} \right) = \delta \epsilon \sum_{\alpha=0}^{n} \left( \frac{\partial I}{\partial q^\alpha} \frac{\partial I}{\partial p_{\alpha}} - \frac{\partial I}{\partial P_{\alpha}} \frac{\partial I}{\partial q^\alpha} \right) \equiv 0.$$

This is not necessarily true for the extended Hamiltonian $H_{e}$. The condition $H_{e} = 0$ from Eq. (21) enters into the extended canonical transformation theory in the way that we must explicitly verify that $H'_{e} = H_{e}$ actually holds under the transformation rules of the canonical variables that are defined by the generating function. Only then the physical motion of the transformed system keeps being confined to the phase-space surface $H'_{e} = 0$, as required for the system to be physical. In the case of the infinitesimal transformation (78), the transformation rule for the extended Hamiltonian $H_{e}$ is satisfied exactly if $\delta H_{e} = 0$ under the infinitesimal variations of
the canonical variables. For the transformation rules (78), the variation of $H_e$ due to the variations $\delta q^\alpha$ and $\delta p_\alpha$ of the canonical variables is given by

$$
\delta H_e = \sum_{\alpha=0}^n \left( \frac{\partial H_e}{\partial q^\alpha} \delta q^\alpha + \frac{\partial H_e}{\partial p_\alpha} \delta p_\alpha \right)
= \delta \epsilon \sum_{\alpha=0}^n \left( \frac{\partial H_e}{\partial q^\alpha} \frac{\partial I}{\partial p_\alpha} - \frac{\partial H_e}{\partial p_\alpha} \frac{\partial I}{\partial q^\alpha} \right)
= \delta \epsilon [H_e, I]_{\text{ext}},
$$

with the last expression defining the extended Poisson bracket. Thus, the canonical transformation rule $\delta H_e = 0$ from Eqs. (78) is actually fulfilled if and only if the characteristic function $I(q^\nu, p_\nu)$ in (76) satisfies

$$
\sum_{\alpha=0}^n \left( \frac{\partial I}{\partial q^\alpha} \frac{\partial H_e}{\partial p_\alpha} - \frac{\partial I}{\partial p_\alpha} \frac{\partial H_e}{\partial q^\alpha} \right) = [I, H_e]_{\text{ext}} = 0.
$$

(79)

Along the system trajectory, the canonical equations (23) apply. As a consequence, the partial derivatives of $H_e$ in (79) may be replaced accordingly to yield

$$
\sum_{\alpha=0}^n \left( \frac{\partial I}{\partial q^\alpha} \, dq^\alpha + \frac{\partial I}{\partial p_\alpha} \, dp_\alpha \right) = \frac{dI}{ds} = 0.
$$

(80)

Thus, $I(q^\nu, p_\nu)$ must “commute” with the extended Hamiltonian $H_e$, hence must be invariant along the system’s phase-space trajectory in order for the transformation (76) to comply with the requirement $\delta H_e = 0$ for an extended canonical transformation. Then and only then the generating function (76) defines an extended canonical transformation and thus ensures the action functional (29) to be preserved. The correlation (80) of a system invariant $I$ to a transformation that preserves the action functional — hence to a canonical transformation — establishes the most general form of Noether’s theorem in the realm of the extended Hamilton-Lagrange formulation of point mechanics,

$$
[I, H_e]_{\text{ext}} = 0 \iff \frac{dI}{ds} = 0 \iff \delta H_e = 0.
$$

(81)

We may rewrite the condition (79) in terms of a conventional Hamiltonian $H$ if we distinguish the space coordinates $q^i$, $i = 1, \ldots, n$ from the time coordinate $t$. With the replacements $q^0 = ct, p_0 = -e/c$, $e$ denoting the instantaneous value of the conventional Hamiltonian $H$, and

$$
\frac{\partial H_e}{\partial t} = \frac{\partial H}{\partial t} \frac{dt}{ds}, \quad \frac{\partial H_e}{\partial c} = -\frac{dt}{ds}, \quad \frac{\partial H_e}{\partial q^i} = \frac{\partial H}{\partial q^i} \frac{dt}{ds}, \quad \frac{\partial H_e}{\partial p_i} = \frac{\partial H}{\partial p_i} \frac{dt}{ds},
$$

according to the correlation (22) of extended and conventional Hamiltonians, we find for $I = I(p, q, t, e)$

$$
\frac{\partial I}{\partial t} + \frac{\partial I}{\partial e} \frac{dt}{ds} + \sum_{i=1}^n \left( \frac{\partial I}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial I}{\partial p_i} \frac{\partial H}{\partial q^i} \right) = 0.
$$

(82)
Due to the conventional canonical equations
\[
\frac{\partial H}{\partial t} = \frac{de}{dt}, \quad \frac{\partial H}{\partial p_i} = \frac{dq^i}{dt}, \quad \frac{\partial H}{\partial q^i} = -\frac{dp_i}{dt},
\]
Eq. (82) is thus equivalent to
\[
\frac{dI}{dt} = 0. \tag{83}
\]
In this notation, the symmetry transformation rules (78) pertaining to the invariant (83) assume the equivalent form
\[
\delta p_i = -\delta e \frac{\partial I}{\partial q^i}, \quad \delta q^i = \delta e \frac{\partial I}{\partial p_i}, \quad \delta e = \delta e \frac{\partial I}{\partial t}, \quad \delta t = -\delta e \frac{\partial I}{\partial e}. \tag{84}
\]
We can always eliminate or induce an \(e\)-dependence of \(I\) by inserting the conventional Hamiltonian according to \(e = H\). A representation \(I = I(p, q, t)\) of the invariant \(I\) does not depend on \(e\), which means that \(\delta t = 0\). Then, the resulting symmetry transformation does not involve a transformation of time. In contrast, if \(I = I(p, q, t, e)\), then the invariant defines a symmetry transformation that includes a transformation of time, \(\delta t \neq 0\). Equivalent representations \(I = I(p, q, t, e)\) and \(I = I(p, q, t)\) of the invariant \(I\) reflect the same underlying system symmetry, yet depicted at different instants of time \(t\).

Summarizing, the set of extended canonical transformations covers all transformations that leave the action functional in the generalized form of Eq. (30) invariant. As each canonical transformation can be defined in terms of an infinitesimal generating function \(F_2\) from Eq. (76), the characteristic function \(I(p, q, t, e)\) that is contained in \(F_2\) then constitutes the corresponding constant of motion. Conversely, each invariant \(I\) of a dynamical system can be inserted into the generating function \(F_2\) of the infinitesimal canonical transformation. The subsequent canonical transformation rules then define the corresponding infinitesimal symmetry transformation of the respective dynamical system. With the extended canonical transformation approach, we thus encounter a generalization of Noether’s theorem in the realm of Hamiltonian point dynamics.

3.7.1. Example: Symmetry generated by the extended Hamiltonian \(H_e\)

A trivial yet important example of an invariant \(I\) is furnished by the extended Hamiltonian \(H_e\) itself
\[
\delta H_e = \delta e [H_e, H_e]_{\text{ext}} = 0, \quad \frac{dH_e}{ds} = 0.
\]
The infinitesimal transformation rules (78) thus define a canonical transformation. With \(\delta e = \delta s\), their explicit form is
\[
\delta p_\mu = -\delta e \frac{\partial H_e}{\partial q^\mu} = \frac{dp_\mu}{ds} \delta s, \quad \delta q^\mu = \delta e \frac{\partial H_e}{\partial p_\mu} = \frac{dq^\mu}{ds} \delta s.
\]
This is obviously the infinitesimal transformation that shifts the extended set of canonical coordinates one step $\delta s$ along the system’s extended phase-space trajectory, which always resides on the surface $H_e(q^\nu, p^\nu) \neq 0$. Thus, the symmetry transformation corresponding to the constant value of $H_e$ is that the system’s symplectic structure is maintained along its evolution parameter, $s$.

### 3.7.2. Example: Symmetry of the time-dependent harmonic oscillator at $\delta t = 0$

The time-dependent harmonic oscillator is a simple one-degree-of-freedom example of a non-autonomous dynamical system, i.e., a system whose Hamiltonian depends explicitly on the independent variable, $t$,

$$H(q, p, t) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2(t) q^2. \quad (85)$$

Herein, $\omega(t)$ denotes the system’s time-dependent circular frequency. The value of the Hamiltonian $H$ is thus not a conserved quantity. The canonical equations and the equation of motion immediately follow as

$$\dot{q} = \frac{\partial H}{\partial p} = p, \quad \dot{p} = -\frac{\partial H}{\partial q} = -\omega^2(t) q, \quad \ddot{q} + \omega^2(t) q = 0.$$

A conserved quantity $I$ for this system is constituted by the quadratic form

$$I = \beta_e(t) p^2 + 2\alpha_e(t) pq + \gamma_e(t) q^2, \quad (86)$$

provided that the time functions $\beta_e(t)$, $\alpha_e(t)$, and $\gamma_e(t)$ satisfy the equations

$$\frac{1}{2}\beta_e \ddot{\beta}_e - \frac{1}{4}\dot{\beta}_e^2 + \omega^2(t) \beta_e^2 = 1, \quad \dot{\beta}_e = -2\alpha_e, \quad \beta_e \gamma_e - \alpha_e^2 = 1. \quad (87)$$

We easily prove the invariance of $I$ directly by calculating its total time derivative and inserting the canonical equations and the conditions (87).

Geometrically, the quadratic form (86) represents an ellipse centered at the origin of the $(q, p)$-phase space with the actual coordinates $q, p$ defining its boundary, which varies its shape but retains its area $\pi I$. Thus, the invariant $I$ represents the conserved area of an ellipse with time-dependent parameters $\beta_e(t)$, $\alpha_e(t)$, and $\gamma_e(t)$ that passes through $(q(t), p(t))$.

The symmetry transformation corresponding to the invariant (86) follows from Eqs. (84)

$$\delta p = -\delta \epsilon \frac{\partial I}{\partial q} = \delta \sigma (\gamma_e q + \alpha_e p), \quad \delta q = \delta \epsilon \frac{\partial I}{\partial p} = -\delta \sigma (\alpha_e q + \beta_e p), \quad \delta t = -\delta \epsilon \frac{\partial I}{\partial \epsilon} = 0,$$

introducing the abbreviation $-2\delta \epsilon \equiv \delta \sigma$. In matrix notation, this infinitesimal canonical transformation of coordinate $q$ and momentum $p$ reads

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{bmatrix} I + \delta \sigma & \alpha_e \\ -\gamma_e & I \end{bmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \quad \delta \sigma = \delta \sigma \begin{pmatrix} -\alpha_e & -\beta_e \\ \gamma_e & \alpha_e \end{pmatrix}, \quad (88)$$

with $I$ denoting the $2 \times 2$ unit matrix. As the coefficients of $\delta \sigma$ do not depend on the canonical variables $q, p$, we may directly set up the pertaining finite transformation.
Equation (88) may be regarded as a Taylor expansion that could be truncated after
the linear term because of very small \( \delta \sigma \). The finite transformation for arbitrary
\( \sigma \in \mathbb{R} \) is then given by the exponential of \( A_{\sigma} \), hence
\[
\begin{pmatrix} Q \\ P \end{pmatrix} = M \begin{pmatrix} q \\ p \end{pmatrix}, \quad M = \exp(A_{\sigma}).
\]
The general scheme for deriving the matrix exponential \( \exp(A) \) for a 2 × 2 matrix
\( A = (a_{ij}) \), \( i, j = 1, 2 \) is expressed in terms of the expression \( D \),
\[
D = \sqrt{\frac{1}{2}(a_{11} - a_{22})^2 + a_{12} a_{21}}.
\]
As
\[
M = \exp\left(\frac{1}{2}(a_{11} + a_{22})\begin{pmatrix} \cosh D + \frac{1}{2}(a_{11} - a_{22})D^{-1} \sinh D & a_{12}D^{-1} \sinh D \\ a_{21}D^{-1} \sinh D & \cosh D - \frac{1}{2}(a_{11} - a_{22})D^{-1} \sinh D \end{pmatrix}\right).
\]
(89)
For the particular matrix \( A_{\sigma} \) from Eq. (88), we find \( a_{11} + a_{22} = 0 \) and \( D = i \sigma \).
Due to the purely imaginary \( D \), the hyperbolic sine and cosine functions in matrix
exponential are thus converted into trigonometric sines and cosines, which finally
yields
\[
\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \cos \sigma - \alpha_{e} \sin \sigma & -\beta_{e} \sin \sigma \\ \gamma_{e} \sin \sigma & \cos \sigma + \alpha_{e} \sin \sigma \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}.
\]
(90)
Note that \( (Q, P) \) and \( (q, p) \) as well as the ellipse parameters \( \alpha_{e}, \beta_{e}, \) and \( \gamma_{e} \) refer to
the same instant of time as the actual symmetry transformation is associated with \( \delta t = 0 \). The inverse transformation is then obtained as
\[
\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \cos \sigma + \alpha_{e} \sin \sigma & \beta_{e} \sin \sigma \\ -\gamma_{e} \sin \sigma & \cos \sigma - \alpha_{e} \sin \sigma \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}.
\]
Inserting \( q \) and \( p \) as functions of \( Q \) and \( P \) into the invariant (86), we find that the
representation of \( I \) retains its form in the transformed variables
\[
I = \beta_{e}(t) P^2 + 2 \alpha_{e}(t) PQ + \gamma_{e}(t) Q^2.
\]
Thus, \( (Q, P) \) and \( (q, p) \) both lie on the same ellipse, but shifted with respect to
each other on the ellipse’s perimeter. The geometric meaning of the one-parameter
symmetry transformation \( M \) from Eq. (90) that is associated with the invariant \( I \)
from Eq. (86) is thus to map any point on this ellipse into another point on the
same ellipse. The free parameter \( \sigma \) of the transformation group then specifies the
particular destination point \( (Q, P) \) with respect to the source point, \( (q, p) \). This can
be seen from the parametric representation of the ellipse (86)
\[
q = \sqrt{\frac{I}{\gamma_{e}}} \left( \cos \phi - \alpha_{e} \sin \phi \right), \quad p = \sqrt{\gamma_{e}} \sin \phi.
\]
(91)
Letting $\phi$ run along the interval $0 \leq \phi \leq 2\pi$, we perform one turn on the ellipse’s perimeter. The symmetry transformation (90) then acts on $(q,p)$ according to

$$
\begin{pmatrix}
Q \\
P
\end{pmatrix} = \begin{pmatrix}
\cos \sigma - \alpha \sin \sigma & -\beta \sin \sigma \\
\gamma \sin \sigma & \cos \sigma + \alpha \sin \sigma
\end{pmatrix} \begin{pmatrix}
\sqrt{I/\gamma_\epsilon} (\cos \phi - \alpha \sin \phi) \\
\sqrt{I/\gamma_\epsilon} \sin \phi
\end{pmatrix}
$$

Thus, $(Q,P)$ is shifted counterclockwise with respect to $(q,p)$ on the ellipse’s perimeter exactly by the phase angle $\sigma$ in the parameter representation (91). This accounts for $\sigma$ being referred to as a “phase advance”. The integral over the closed curve $C$ comprising the shaded region $A_\sigma$ of Fig. 1 measures the enclosed area $A_\sigma = \frac{1}{2} \int_C q dp - pdq = \frac{1}{2} \int_{\phi}^{\phi+\sigma} \left( q \frac{dp}{d\phi} - p \frac{dq}{d\phi} \right) d\phi$

$$
= \frac{1}{2} I \int_{\phi}^{\phi+\sigma} \left( \cos^2 \phi - \alpha \sin \phi \cos \phi + \sin^2 \phi + \alpha \sin \phi \sin \phi \right) d\phi
$$

$$
= \frac{1}{2} I \sigma.
$$

Note that the phase advance $\sigma$ does not depict the polar angle from vectors $(q,p)$ to $(Q,P)$. Instead, $\sigma$ is proportional to the shaded area $A_\sigma$.

3.7.3. Example: Symmetry of the time-dependent harmonic oscillator at $\delta t \neq 0$

Replacing the quadratic $p$-dependence in the invariant (86) of the time-dependent harmonic oscillator (85) according to

$$
\epsilon = \frac{1}{2} p^2 + \frac{1}{2} \omega^2(t) q^2,
$$
we arrive at an equivalent representation of the invariant that now depends on the energy variable, \( e \)

\[
I = 2\dot{\beta}_e(t) e - \dot{\beta}_e(t) pq + \frac{1}{2} \ddot{\beta}_e(t) q^2. \tag{92}
\]

Of course, the function \( \beta_e(t) \) must again satisfy the second-order equation from Eq. (87) in order for \( I \) to actually establish an invariant. The particular infinitesimal rules for the corresponding symmetry transformation from Eq. (84) are

\[
\begin{pmatrix} Q \\ P \end{pmatrix} \bigg|_T = \left[ 1 + \delta \epsilon \right] \begin{pmatrix} q \\ p \end{pmatrix} \bigg|_t, \quad \delta \epsilon = \delta \epsilon \left( \begin{array}{cc} -\dot{\beta}_e(t) & 0 \\ \beta_e(t) & -\dot{\beta}_e(t) \end{array} \right), \quad T = t - 2\delta \epsilon \beta_e(t). \tag{93}
\]

As the coefficients of \( \delta \epsilon \) do not explicitly depend on \( e \), we can set up the matrix exponential \( M = \exp(\delta \epsilon) \) according to the general scheme (89) in order to finally derive the finite symmetry mapping that corresponds to the infinitesimal mapping (93),

\[
M = \begin{pmatrix} \exp(-\delta \epsilon \dot{\beta}_e) & 0 \\ -\left(\delta \epsilon \dot{\beta}_e / \beta_e \right) \sinh(\delta \epsilon \dot{\beta}_e) & \exp(\delta \epsilon \dot{\beta}_e) \end{pmatrix}, \quad \delta \epsilon = -\frac{\delta t}{2\beta_e}.
\]

Here, \( \delta \epsilon \) still denotes an infinitesimal \( e \) interval. The actual one-parameter symmetry transformation (93) is associated with a transformation of time \( t \rightarrow T \). As the coefficients of \( \delta \epsilon \) are time-derivatives of the ellipse function \( \beta_e(t) \) and thus generally depend on time \( t \), we must substitute \( \delta \epsilon = -\delta t / 2\beta_e(t) \) and integrate all terms in \( M \) that are proportional to \( \delta t \) over the finite interval \( T - t \) that corresponds to a finite interval \( \Delta \epsilon = \epsilon_1 - \epsilon_0 \),

\[
m_{11} = \exp(-\delta \epsilon \dot{\beta}_e) \quad \rightarrow \quad m_{11} = \exp \left( \int_t^T \frac{\dot{\beta}_e(\tau)}{2\beta_e(\tau)} d\tau \right) = \frac{1}{m_{22}}.
\]

With the identity \( \sinh \ln x = (x - x^{-1}) / 2 \), the matrix element \( m_{21} \) follows as

\[
m_{21} = -\frac{\delta t \dot{\beta}_e}{\beta_e} \sinh(\delta \epsilon \dot{\beta}_e) \quad \rightarrow \quad m_{21} = -\frac{\dot{\beta}_e(T) - \dot{\beta}_e(t)}{\beta_e(T) - \beta_e(t)} \sinh \left( \frac{\beta_e(t)}{\beta_e(T)} \right) = \frac{\dot{\beta}_e(T) - \dot{\beta}_e(t)}{2\beta_e(T)\beta_e(t)}.
\]

The finite symmetry mapping \((q, p)_t \rightarrow (Q, P)_T\) is thus finally obtained as

\[
\begin{pmatrix} Q \\ P \end{pmatrix} \bigg|_T = \frac{1}{\sqrt{\beta_e(T)\beta_e(t)}} \begin{pmatrix} \beta_e(T) & 0 \\ \alpha_e(t) - \alpha_e(T) & \beta_e(t) \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \bigg|_t \tag{94}
\]

\[
\Delta \sigma = -2\Delta \epsilon = \int_t^T \frac{d\tau}{\beta_e(\tau)}.
\]

The symmetry mapping (94) is referred to as the Floquet transformation.
3.7.4. Example: Rotational symmetry of the Kepler system

The classical Kepler system is a two-body problem with the mutual interaction following an inverse square force law. In Cartesian coordinates, where no distinction between covariant and contravariant coordinates is needed (all indexes lowered), this system is described by a Hamiltonian

$$ H(q, p, t) = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + V(q, t) $$

containing the interaction potential

$$ V(q, t) = -\frac{\mu(t)}{\sqrt{q_1^2 + q_2^2}} = -\frac{\mu(t)}{r}, $$

with $\mu(t) = G[m_1(t) + m_2(t)]$ the possibly time-dependent gravitational coupling strength that is induced by possibly time-dependent masses $m_1$ and $m_2$ of the interacting bodies. As the potential (96) spatially depends on $r$ only, it is obviously invariant with respect to rotations in configuration space $(q_1, q_2)$,

$$ Q_1 = (\cos \epsilon \sin \epsilon) q_1 $$
$$ Q_2 = (-\sin \epsilon \cos \epsilon) q_2 $$

where $\epsilon$ denotes the counterclockwise rotation angle. This symmetry is not affected if we choose $\epsilon \equiv \delta \epsilon$ to be very small. We may then restrict ourselves in Eq. (97) to first-order terms in $\delta \epsilon$ and insert the replacements $\cos \delta \epsilon \approx 1$, $\sin \delta \epsilon \approx \delta \epsilon$. This yields the infinitesimal transformation rules

$$ Q_1 = q_1 + \delta \epsilon q_2, \quad Q_2 = q_2 - \delta \epsilon q_1. $$

This transformation can be regarded as being defined by a generating function of the form of Eq. (76), namely

$$ F_2(q, p, P_1, P_2, t, E) = -tE + q_1P_1 + q_2P_2 + \delta \epsilon (p_1q_2 - p_2q_1). $$

The transformation rules for the canonical momenta, energy, and time emerge from the generating function (99) by applying the general canonical rules from Eqs. (40),

$$ p_1 = \frac{\partial F_2}{\partial \dot{q}_1} = P_1 - \delta \epsilon p_2, \quad p_2 = \frac{\partial F_2}{\partial \dot{q}_2} = P_2 + \delta \epsilon p_1, \quad T = -\frac{\partial F_2}{\partial \dot{E}} = t, \quad e = -\frac{\partial F_2}{\partial \dot{t}} = E. $$

The rules from Eqs. (98) are indeed reproduced as to first order in $\delta \epsilon$, we find the configuration space transformation rules

$$ Q_1 = \frac{\partial F_2}{\partial P_1} = q_1 + \delta \epsilon q_2, \quad Q_2 = \frac{\partial F_2}{\partial P_2} = q_2 - \delta \epsilon q_1. $$

According to Eq. (83), the expression proportional to $\delta \epsilon$ in Eq. (99) must be a constant of motion in order for the infinitesimal generating function $F_2$ to define a canonical transformation, hence to comply with the finite symmetry transformation (97) that preserves the physical system. Thus

$$ I = p_1q_2 - p_2q_1, \quad \frac{dI}{dt} = 0, $$
which establishes the well-known conservation law of angular momentum in — possibly time-dependent — central-force fields. As the transformation rules (97) only depend on the parameter \( \epsilon \) and not on the canonical variables, the transformation is referred to as a global symmetry transformation.

As with any generating function of a canonical transformation, we can derive from Eq. (99) the rules of both the configuration space coordinates and the respective canonical momenta. In matrix form, the infinitesimal rules for the momenta can be rewritten as

\[
\begin{pmatrix}
P_1 \\
P_2
\end{pmatrix} = \begin{pmatrix} 1 + \delta \epsilon \end{pmatrix} \begin{pmatrix} p_1 \\
p_2\end{pmatrix},
\]

\( \delta \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \),

with 1 denoting the 2 × 2 unit matrix. The corresponding finite transformation is then

\[
\begin{pmatrix}
P_1 \\
P_2
\end{pmatrix} = \exp(\delta \epsilon) \begin{pmatrix} p_1 \\
p_2\end{pmatrix}, \quad \exp(\delta \epsilon) = \begin{pmatrix} \cos \epsilon & \sin \epsilon \\ -\sin \epsilon & \cos \epsilon \end{pmatrix},
\]

which coincides with the rules of the configuration space variables from Eq. (97). This reflects the fact that the Hamiltonian (95) is equally invariant under rotations in momentum space.

3.7.5. Example: Symmetry associated with the Runge-Lenz invariant of the time-independent Kepler system

As Noether’s theorem associates the constants of motion of a dynamical system with system symmetries, it can be applied in both directions. In Sect. 3.7.4, the constant of motion was determined for a system symmetry that could be deduced directly from the form of the Hamiltonian. Conversely, if a constant of motion is known to exist, then we can then derive the related system symmetry. For the time-independent Kepler system (95), (96) with \( \mu = \text{const.} \), one component of the Runge-Lenz vector is given by

\[
I_1 = -q_1 p_2^2 + q_2 p_1 p_2 + \mu \frac{q_1}{\sqrt{q_1^2 + q_2^2}}.
\]

(100)

We easily convince ourselves that \( I_1 \) commutes with the Hamiltonian \( H \) from (95) with (96). Along the system’s phase-space trajectory, we then have

\[
[I_1, H] = 0 \iff \frac{dI_1}{dt} = 0.
\]

Using the invariant \( I_1 \) as the characteristic function \( I \) in the generating function (76), the subsequent transformation rules (78) then define the corresponding infinitesimal symmetry transformation that preserves the action functional (29). The so obtained transformation is not particularly enlightening. Yet, a better representation of the symmetry that is associated with the Runge-Lenz invariant can be derived in the extended Hamiltonian formalism. In this context, we may express
the invariant \( I_1 \) equivalently as a function of \( q, p, \) and \( e \), with \( e \) being defined as the \textit{value} of the Hamiltonian \( H \) from Eq. (95),

\[
e = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{\mu}{\sqrt{q_1^2 + q_2^2}}.
\]

The \( \mu \)-dependent term of the invariant \( I_1 \) can thus be replaced by an \( e \)-term according to

\[
\mu \frac{q_1}{\sqrt{q_1^2 + q_2^2}} = \frac{1}{2}q_1p_1^2 + \frac{1}{2}q_1p_2^2 - q_1e,
\]

which yields an equivalent extended phase-space representation of the Runge-Lenz invariant \( I_1 = I_1(q, p, e) \) as a \textit{symmetric} quadratic form in the canonical momenta,

\[
I_1 = \frac{1}{2}q_1p_1^2 + q_2p_1p_2 - \frac{1}{2}q_1p_2^2 - q_1e.
\]  

(101)

As expected, the invariant \( I_1 \) commutes with the Hamiltonian of the time-independent Kepler system (\( \mu = \text{const.} \))

\[
[I_1, H]_{\text{ext}} = p_1(H - e) = 0,
\]

hence establishes an invariant along the system’s phase-space trajectory as \( H = e \) by definition. Due to the \( e \)-dependence of the invariant \( I_1 \), the corresponding symmetry transformation now includes a transformation of time according to rules (84).

Explicitly, the infinitesimal transformation rules are obtained as

\[
\begin{align*}
\delta p_1 &= -\delta \epsilon \frac{\partial I_1}{\partial q_1} = \delta \epsilon \left( \frac{1}{2}p_2^2 - \frac{1}{2}p_1^2 + e \right) \\
\delta p_2 &= -\delta \epsilon \frac{\partial I_1}{\partial q_2} = -\delta \epsilon p_1 p_2 \\
\delta q_1 &= \delta \epsilon \frac{\partial I_1}{\partial p_1} = \delta \epsilon (q_1p_1 + q_2p_2) \\
\delta q_2 &= \delta \epsilon \frac{\partial I_1}{\partial p_2} = \delta \epsilon (p_1q_2 - p_2q_1) \\
\delta e &= \delta \epsilon \frac{\partial I_1}{\partial t} = 0 \\
\delta t &= -\delta \epsilon \frac{\partial I_1}{\partial \epsilon} = \delta \epsilon q_1.
\end{align*}
\]

(102)

The transformation rules for the new configuration space \( Q_1, Q_2 \) variables depend \textit{linearly} on the original ones, \( q_1, q_2 \). We may thus rewrite the infinitesimal configuration space transformation \( Q_i = q_i + \delta q_i, \ i = 1, 2 \) in matrix form as

\[
\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \bigg|_{t+q_i \delta \epsilon} = \left[ 1 + \mathcal{A}_{\delta \epsilon} \right] \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \bigg|_t, \quad \mathcal{A}_{\delta \epsilon}(p_1, p_2) = \delta \epsilon \begin{pmatrix} p_1 \\ p_2 \\ -p_2 \\ -p_1 \end{pmatrix} \bigg|_t.
\]

(103)

with \( 1 \) denoting the \( 2 \times 2 \) unit matrix. The form of the \( 2 \times 2 \) matrix \( \mathcal{A}_{\delta \epsilon} = (a_{ij}) \) from Eq. (103) with \( a_{11} = a_{22} \) and \( a_{12} = -a_{21} \) results from the particular representation (101) of the Runge-Lenz invariant \( I_1 \). With \( \delta \epsilon \) still an \textit{infinitesimal} variation of the parameter \( \epsilon \), the transformation (103) can be expressed equivalently in terms of the matrix exponential \( \exp(\mathcal{A}_{\delta \epsilon}) \). Then, the \textit{infinitesimal} symmetry transformation then takes on the exceptionally simple form

\[
\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \bigg|_{t+q_i \delta \epsilon} = \exp(p_1 \delta \epsilon) \begin{pmatrix} \cos(p_2 \delta \epsilon) & \sin(p_2 \delta \epsilon) \\ -\sin(p_2 \delta \epsilon) & \cos(p_2 \delta \epsilon) \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \bigg|_t.
\]

(104)
The system symmetry that corresponds to the Runge-Lenz invariant from Eq. (101) is thus given by a local scaled rotation of the configuration space variables. In contrast to the example of Sect. 3.7.4, the transformation (104) depends on the actual coordinates \( q_1, p_1, p_2 \). It is, therefore, referred to as a local symmetry transformation.

Owing to the fact that the Hamiltonian (95) with potential (96) is invariant under swappings \( q_1 \leftrightarrow q_2 \) and \( p_1 \leftrightarrow p_2 \), the second component \( I_2 \) of the invariant Runge-Lenz vector is obtained by flipping all indexes of \( I_1 \),

\[
I_2 = \frac{1}{2} q_2 p^2_2 + q_1 p_1 p_2 - \frac{1}{2} q_2 p^2_1 - q_2 \epsilon.
\]

The infinitesimal transformation of the configuration space coordinates follows as

\[
\begin{pmatrix}
Q_1 \\
Q_2
\end{pmatrix}
\bigg|_{t+\delta\epsilon} = \left[ I + \mathcal{E}_{\delta\epsilon} \right] \begin{pmatrix}
q_1 \\
q_2
\end{pmatrix}
\bigg|_t, \quad \mathcal{E}_{\delta\epsilon}(p_1, p_2) = \delta\epsilon \begin{pmatrix}
p_1 p_2 \\
p_1 p_2
\end{pmatrix}.
\]

Again, the transformation can be expressed equivalently in terms of the matrix exponential \( \exp(\mathcal{E}_{\delta\epsilon}) \), where \( \delta\epsilon \) denotes an infinitesimal shift of the symmetry transformation's parameter

\[
\begin{pmatrix}
Q_1 \\
Q_2
\end{pmatrix}
\bigg|_{t+\delta\epsilon} = \exp(p_2 \delta\epsilon \begin{pmatrix}
\cos(p_1 \delta\epsilon) & -\sin(p_1 \delta\epsilon) \\
\sin(p_1 \delta\epsilon) & \cos(p_1 \delta\epsilon)
\end{pmatrix}) \begin{pmatrix}
q_1 \\
q_2
\end{pmatrix}
\bigg|_t.
\]

### 3.8. Extended point transformations, conventional Noether theorem

The derivation of Noether’s theorem in the context of the Lagrangian formalism is restricted to extended point transformations, hence canonical transformations for which the new space-time coordinates only depend on the old space-time coordinates and not on the set of old momentum coordinates. Yet, the extended canonical transformation approach allows to describe more general possible symmetry mappings as the rules (78) are not restricted to point transformations. Consequently, equation (81) in conjunction with the infinitesimal canonical mapping (78) represents a generalized formulation of Noether’s theorem. In order to derive the conventional Noether theorem in the Hamiltonian description, we restrict ourselves to the case of an infinitesimal point transformation, which is defined by a generating function (76) with characteristic function \( I \) that is linear in the momenta \( p_\nu \),

\[
I(q', p_\nu) = - \sum_{\alpha=0}^n \eta^\alpha(q') p_\alpha + f(q'), \quad (105)
\]

hence with functions \( \eta^\mu = \eta^\mu(q') \), \( f = f(q') \) that depend on the space-time coordinates only. With this \( I \), the transformation rules for space and time coordinates follow as \( (\mu, \nu = 0, \ldots, n, i = 1, \ldots, n) \)

\[
\delta q^\mu = -\epsilon \eta^\mu(q'), \quad \delta q^i = -\epsilon \eta^i(q, t), \quad \delta t = -\epsilon \xi(q, t), \quad \xi = q^0/c.
\]
The condition (79) for this transformation to preserve the extended Hamiltonian $H_e$, hence for the function (105) to represent a conserved quantity along the system’s evolution is

$$\sum_{\beta=0}^{n} \left[ \eta^{\beta} \frac{\partial H_e}{\partial q^{\beta}} + \frac{\partial f}{\partial q^{\beta}} \left( \frac{\partial f}{\partial q^{\alpha}} - \sum_{\alpha=0}^{n} p^{\alpha} \frac{\partial \rho^{\alpha}}{\partial q^{\beta}} \right) \right] = 0. \quad (106)$$

Distinguishing the canonical time and energy variables from the canonical space and momentum coordinates, the Noether function (105) has the equivalent representation

$$I(q, p, e, t) = \xi(q, t)e - \sum_{i=1}^{n} \eta^{i}(q, t)p_{i} + f(q, t), \quad (107)$$

which represents a conserved quantity if Eq. (82) is satisfied. In the last step, the energy variable $e$ may be replaced by the conventional Hamiltonian $H$. We thus find the conventional Noether function in the Hamiltonian formulation

$$I(q, p, t) = \xi(q, t)H - \sum_{i=1}^{n} \eta^{i}(q, t)p_{i} + f(q, t), \quad (108)$$

which is an invariant provided that Eq. (82) holds with $\partial I/\partial e = 0$. Due to their different dependence on the canonical variables, the Noether functions (107) and (108) yield different transformation rules from Eqs. (84). However, these rules are compatible as

$$\delta \bar{p}_{i} = \delta p_{i} - \frac{dp_{i}}{dt} \delta t, \quad \delta \bar{q}^{i} = \delta q^{i} - \frac{dq^{i}}{dt} \delta t, \quad \delta \bar{e} = \delta e - \frac{dH}{dt} \delta t, \quad \delta \bar{t} = 0, \quad (109)$$

if the barred quantities denote the variations derived from Eq. (108) and the unbarred those derived from Eq. (107). As the function $I(q, p, t)$ does not depend on the energy variable, $e$, the subsequent transformation rules are associated with an identical time transformation, $T = t$, $\delta \bar{t} = 0$. In contrast, $I(q, p, e, t)$ from Eq. (107) accounts for an infinitesimal time shift transformation $T = t - \epsilon \xi$, $\delta \bar{t} = -\epsilon \xi$. The connection of both equally valid sets of transformation rules is given by Eqs. (109).

With these formulations, we are led to interpreting the conventional Noether theorem in the reverse direction. If we can find functions $f(q, t), \xi(q, t), \text{and} \eta^{i}(q, t)$ such that for a given conventional Hamiltonian $H$ the total time derivative of $I$ vanishes, $dI/dt = 0$, then the invariant $I$ in the forms of Eqs. (107) or (108) defines a corresponding extended canonical point transformation according to Eqs. (84).

### 3.9. Canonical quantization in the extended Hamiltonian formalism

The transition from classical dynamics to the corresponding quantum description is most easily made in terms of the “canonical quantization prescription.” The quantum description of a dynamical system whose classical limit is represented by a Hamiltonian $H$ is accordingly obtained by reinterpreting our dynamical variables
$q^\mu(s)$ and $p_\mu(s)$ as operators $\hat{q}^\mu(s)$ and $\hat{p}_\mu(s)$ that act on a wave function $\psi$. In the configuration space representation, the quantum mechanical operators are

$$\hat{q}^\mu = q^\mu \mathbb{1}, \quad \hat{p}_\mu = -i\hbar \frac{\partial}{\partial q^\mu},$$  

with $\mathbb{1}$ denoting the identity operator. In the extended formalism, an additional pair of operators is given for the index $\mu = 0$. Because of $q^0 \equiv ct$, $p_0 \equiv -e/c$, these operators are expressed equivalently as

$$\hat{t} = t \mathbb{1}, \quad \hat{e} = i\hbar \frac{\partial}{\partial t}.$$  

With $e_e$ denoting the value of the extended Hamiltonian $H_e$, we encountered in Sect. 2.2 another additional pair of canonically conjugate variables, $(e_e, s)$. The corresponding operators are

$$\hat{s} = s \mathbb{1}, \quad \hat{e}_e = i\hbar \frac{\partial}{\partial s}.$$  

For explicitly $s$-dependent extended Hamiltonians $H_e$ and wave functions $\psi(q^\mu, s)$, the classical equation $H_e = e_e$ from Eq. (25) thus translates into the equation of motion for the wave function $\psi(q^\mu, s)$,

$$\hat{H}_e \psi = i\hbar \frac{\partial \psi}{\partial s}.$$  

This equation was postulated earlier by Feynman.$^{25}$ The usual cases with no $s$-dependence of $H_e$ and $\psi$ are then directly obtained from the condition $H_e = 0$ for the classical extended Hamiltonian (21)

$$\hat{H}_e \psi(q^\mu) = 0.$$  

Equation (111) is the relativistic extension of the Schrödinger equation.

For the extended Hamiltonian of a point particle in an external electromagnetic field from Eq. (64), we immediately find the Klein-Gordon equation, inserting Eqs. (110)

$$\left[ (i\hbar \frac{\partial}{\partial q^\alpha} + \frac{\xi}{c} A_\alpha) \left( i\hbar \frac{\partial}{\partial q^\alpha} + \frac{\xi}{c} A^\alpha \right) + m^2 c^2 \right] \psi(q^\mu) = 0.$$  

The non-relativistic limit is encountered by letting $s \to t$. The corresponding extended Hamiltonian $H_e = H - e = 0$ from (27) with $H(q, p, t)$ a conventional non-relativistic Hamiltonian then yields the associated non-relativistic wave equation for $\psi(q^\mu) \equiv \psi(q, t)$:

$$\hat{H} \psi = i\hbar \frac{\partial \psi}{\partial t},$$  

which is referred to as the Schrödinger equation.
3.10. Path integral derivation of the Klein-Gordon equation for a relativistic point particle in an electromagnetic field

Apart from the important additional rest energy term $-\frac{1}{2}mc^2$, the extended Lagrangian (60) for a relativistic classical point particle in an external electromagnetic field agrees with the Lagrangian proposed by Feynman\(^9\) on the basis of a formal reasoning. We have seen that this Lagrangian \(L_c\) is actually not a mere formal construction, but has the physical meaning to describe the same dynamics as the corresponding conventional Lorentz-invariant Lagrangian from Eq. (58). As the extended Lagrangian (60) is thus identified as physically significant, it can be concluded that the path integral erected on this Lagrangian yields the correct quantum description of a relativistic point particle in an external electromagnetic field.

For an infinitesimal proper time step $\epsilon \equiv \Delta s$, the action \(S_{c,\epsilon}\) for the extended Lagrangian (60) writes to first order in $\epsilon$

\[
S_{c,\epsilon} = \epsilon L_c = \frac{1}{2} m \eta_{\alpha\beta} \left(\frac{q^\alpha_{\epsilon} - q^\alpha_0}{\epsilon}\right) \left(\frac{q^\beta_{\epsilon} - q^\beta_0}{\epsilon}\right) + \frac{\zeta}{c} \left(q^\alpha_{\epsilon} - q^\alpha_0\right) A_\alpha(q^\mu_\epsilon) - \frac{1}{2} mc^2 \epsilon. \tag{113}
\]

The potentials \(A_\alpha\) are to be taken at the space-time location \(q^\mu_\epsilon = (q^\mu_0 + q^\mu_\epsilon)/2\). We insert this particular action function into Eq. (48) and perform a transformation of the integration variables \(q^\mu_\epsilon\),

\[
q^\mu_\epsilon - q^\mu_0 = \xi^\mu \quad \Rightarrow \quad d^4 q_\epsilon = d^4 \xi.
\]

The integral (48) has now the equivalent representation

\[
\psi(q^\mu_\epsilon) = \frac{1}{M} \int \exp \left[\frac{i}{\hbar} S_{c,\epsilon}\right] \psi(q^\mu_0 - \xi^\mu) d^4 \xi, \tag{114}
\]

while the action \(S_{c,\epsilon}\) from Eq. (113) takes on the form

\[
S_{c,\epsilon} = \frac{m}{2} \eta_{\alpha\beta} \left(\frac{\xi^\alpha \xi^\beta}{\epsilon}\right) + \frac{\zeta}{c} \xi^\alpha \left[A_\alpha(q^\mu_0) - \frac{1}{2} \xi^\beta \frac{\partial A_\alpha(q^\mu_\epsilon)}{\partial q^\beta}\right] - \epsilon mc^2 \frac{\xi^\alpha \xi^\beta}{2}.
\]

Here, we expressed the potentials \(A_\alpha(q^\nu_\epsilon)\) to first order in terms of their values at \(q^\nu_0\). In the following, we skip the index “\(b\)” in the coordinate vector as all \(q^\nu\) refer to that particular space-time event from this point of our derivation.

In order to match the quadratic terms in \(S_{c,\epsilon}\), the wave function \(\psi(q^\nu_\epsilon - \xi^\mu)\) under the integral (114) must be expanded up to second order in the \(\xi^\mu\),

\[
\psi(q^\mu_\epsilon - \xi^\mu) = \psi(q^\mu_0) - \xi^\alpha \frac{\partial \psi(q^\mu_\epsilon)}{\partial q^\alpha} + \frac{1}{2} \xi^\alpha \xi^\beta \frac{\partial^2 \psi(q^\mu_\epsilon)}{\partial q^\alpha \partial q^\beta} - \ldots
\]

The rest energy term in \(S_{c,\epsilon}\) depends only on \(\epsilon\). It can, therefore, be taken as a factor in front of the integral and expanded up to first order in \(\epsilon\). The total expression (114) for the transition of the wave function \(\psi\) thus follows as

\[
\psi = \frac{1}{M} \left(1 - \frac{imc^2}{2\hbar}\right) \int_{-\infty}^{\infty} \exp \left\{ \frac{i}{\hbar} \left[\frac{m}{2} \eta_{\alpha\beta} \xi^\alpha \xi^\beta + \frac{\zeta}{c} A_\alpha \xi^\alpha - \frac{\zeta}{2c} \frac{\partial A_\alpha}{\partial q^\beta} \xi^\alpha \xi^\beta\right]\right\} \times \left[\psi - \xi^\alpha \frac{\partial \psi}{\partial q^\alpha} + \frac{1}{2} \xi^\alpha \xi^\beta \frac{\partial^2 \psi}{\partial q^\alpha \partial q^\beta}\right] d^4 \xi. \tag{115}
\]
Prior to actually calculating the Gaussian type integrals, we may simplify the integrand in (115) by taking into account that the third term in the exponential function is of order of $\epsilon$ smaller than the first one. We may thus factor out this term and expand it up to first order in $\epsilon$

$$\exp\left[-\frac{i\zeta \partial A_\alpha}{2\hbar c} \xi^\alpha \xi^\beta\right] = 1 - \frac{i\zeta \partial A_\alpha}{2\hbar c} \xi^\alpha \xi^\beta + \ldots$$

Omitting terms of higher order than quadratic in the $\xi^\mu$, the integral becomes

$$\psi = \frac{1}{M} \left(1 - \epsilon \frac{mc^2}{2\hbar}\right) \int_{-\infty}^{\infty} \exp\left\{\frac{i}{\hbar} \left[\frac{m}{2\epsilon} \eta_{\alpha\beta} \xi^\alpha \xi^\beta + \frac{\zeta}{c} A_\alpha \xi^\beta\right]\right\} \times \left[\psi - \xi^\alpha \frac{\partial \psi}{\partial q^\alpha} + \frac{1}{2} \xi^\alpha \xi^\beta \left(\frac{\partial^2 \psi}{\partial q^\alpha \partial q^\beta} - \frac{i\zeta}{\hbar c} \frac{\partial A_\alpha}{\partial q^\beta} \psi\right)\right] d^4 \xi.$$

The integral over the entire space-time can now be solved analytically to yield

$$\psi = \frac{1}{M} \left(\frac{2\pi\hbar}{im}\right)^2 \left(1 - \epsilon \frac{mc^2}{2\hbar}\right) \exp\left\{-\frac{i\zeta^2}{2\hbar mc^2} A^\alpha A_\alpha\right\} \times \left[\psi + \epsilon \frac{\zeta}{mc} A^\alpha \frac{\partial \psi}{\partial q^\alpha} + \frac{\epsilon}{2} \left(\frac{\partial^2 \psi}{\partial q^\alpha \partial q^\beta} - \frac{i\zeta}{\hbar c} \frac{\partial A_\alpha}{\partial q^\beta} \psi\right)\left(\frac{\epsilon\zeta^2}{m^2c^2} A^\alpha A^\beta + \frac{i\hbar}{m} \eta^{\alpha\beta}\right)\right].$$

We may omit the term quadratic in $\epsilon$ that is contained in the rightmost factor and finally expand the exponential function up to first order in $\epsilon$

$$\psi = \frac{1}{M} \left(\frac{2\pi\hbar}{im}\right)^2 \left(1 - \epsilon \frac{mc^2}{2\hbar}\right) \left(1 - \epsilon \frac{mc^2}{2\hbar}\right) \left(1 - \epsilon \frac{i\zeta^2}{2\hbar mc^2} A^\alpha A_\alpha\right) \times \left[\psi + \epsilon \frac{\zeta}{mc} A^\alpha \frac{\partial \psi}{\partial q^\alpha} + \frac{\epsilon}{2m} \left(\frac{\partial^2 \psi}{\partial q^\alpha \partial q^\beta} - \frac{i\zeta}{\hbar c} \frac{\partial A_\alpha}{\partial q^\beta} \psi\right)\right].$$

The normalization factor $M$ is now obvious. As the equation must hold to zero order in $\epsilon$, we directly conclude that $M = (2\pi\hbar/im)^2$. This means, furthermore, that the sum over all terms proportional to $\epsilon$ must vanish. The five terms in (116) that are linear in $\epsilon$ thus establish the equation

$$\frac{m^2c^2}{\hbar^2} \psi = \frac{\partial^2 \psi}{\partial q^\alpha \partial q^\alpha} - \frac{\zeta^2 A^\alpha A_\alpha}{\hbar^2c^2} \psi + \frac{2\zeta A^\alpha}{\hbar c} \frac{\partial \psi}{\partial q^\alpha} + \frac{\zeta}{\hbar c} \frac{\partial A_\alpha}{\partial q^\alpha} \psi.$$

This equation has the equivalent product form

$$\left(\frac{\partial}{\partial q^\alpha} - \frac{i\zeta}{\hbar c} A_\alpha\right) \left(\frac{\partial}{\partial q^\alpha} - \frac{i\zeta}{\hbar c} A_\alpha\right) \psi = \left(\frac{mc}{\hbar}\right)^2 \psi,$$

which constitutes exactly the Klein-Gordon equation for our metric $\eta_{\mu\nu}$. It coincides with the wave equation (112) that emerged from the canonical quantization formalism.

We remark that Feynman$^{25}$ went the procedure developed here in the opposite direction. He started with the Klein-Gordon equation and deduced from analogies with the non-relativistic case a classical Lagrangian similar to that of Eq. (60), but without its rest energy term $-\frac{1}{2}mc^2$. The obtained Lagrangian was not identified as physically significant, i.e., as exactly the extended Lagrangian $L_e$ that describes the corresponding classical system, but rated as “purely formal.”$^9$
3.11. **Space-time kernel for the free relativistic point particle**

The hypersurface condition (51) is to be disregarded setting up the parameterized kernel (46) as virtual particles are to be included. The components of the extended free-particle Lagrangian (50) can then be treated as independent. The corresponding action functional $S$ from Eq. (3) thus splits into a sum of independent action functionals,

$$S_c[q^\alpha(s)] = \frac{1}{2}m \int_{s_a}^{s_b} \left( \frac{dq^\alpha}{ds} \frac{dq_\alpha}{ds} - c^2 \right) ds = \sum_\alpha S[q^\alpha(s)].$$

Hence, the parameterized space-time kernel (46) separates into a product of path integrals. For the free particle, the individual path integrals can be solved analytically.\(^\text{16}\)\(^\text{,7}\) Expressed in terms of $s$ as the independent variable, the result for one degree of freedom $q^k$ is

$$K_s(q^k_b, q^k_a) = \sqrt{\frac{m}{2\pi i\hbar(s_b - s_a)}} \exp \left[ \frac{im}{2\hbar} \left( q^k_b - q^k_a \right)^2 \right].$$

The total parameterized space-time kernel $K_\sigma(b, a)$ is then obtained for $S_c$ from Eq. (118) as

$$K_\sigma(b, a) = -\frac{m^2c}{4\pi^2\hbar^2(s_b - s_a)} \exp \left\{ \frac{im}{2\hbar} \left( \frac{(q^\alpha_b - q^\alpha_a)(q_{\alpha,b} - q_{\alpha,a})}{s_b - s_a} - c^2(s_b - s_a) \right) \right\}.$$  

The term proportional to $(s_b - s_a)$ in the exponential function originates from the rest energy term $-\frac{1}{2}mc^2$ in the extended Lagrangian (50) and, correspondingly, in the action integral (118). The integration over the parameter variable $s$ is worked out by means of a Wick rotation. The parameter interval is then $\sigma = i(s_b - s_a)$. With $\tau$ defined by

$$\tau^2 = \frac{(q^\alpha_b - q^\alpha_a)(q_{\alpha,b} - q_{\alpha,a})}{c^2},$$

the parameterized space-time kernel $K_\sigma(b, a)$ takes on the equivalent form

$$K_\sigma(b, a) = \frac{m^2c}{4\pi^2\hbar^2} \sigma^{-2} \exp \left\{ \frac{-mc^2}{2\hbar} \left( \frac{\tau^2}{\sigma} + \sigma \right) \right\}.$$  

According to Eq. (47), the space-time propagator $K(b, a)$ for a free relativistic wave packet is finally acquired by integrating $K_\sigma(b, a)$ over all parameter intervals $\sigma$

$$K(b, a) = \frac{m^2c}{4\pi^2\hbar^2} \int_0^{\infty} \sigma^{-2} \exp \left\{ \frac{-mc^2}{2\hbar} \left( \frac{\tau^2}{\sigma} + \sigma \right) \right\} d\sigma.$$  

The integral is proportional to the integral representation of the Bessel function $K_1$ of second kind and order one,\(^\text{26}\), that is also referred to as MacDonald function,

$$\int_0^{\infty} \sigma^{-2} \exp \left\{ -\frac{M}{2} \left( \frac{\tau^2}{\sigma} + \sigma \right) \right\} d\sigma = \frac{2}{\tau}K_1(M\tau), \quad M = \frac{mc^2}{\hbar}.$$
For our metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, a positive $\tau^2$ represents a space-like connection of the events $a$ and $b$. The kernel $K(b, a)$ from Eq. (120) is then given by

$$K(b, a) = \frac{m^2 c^2}{2\pi^2 \hbar^2} \frac{1}{|q_b - q_a|} K_1 \left( \frac{mc}{\hbar} |q_b - q_a| \right), \quad \tau^2 > 0. \quad (122)$$

If $\tau^2$ is negative, one encounters a time-like connection of the events $a$ and $b$. The kernel $K(b, a)$ is then expressed in terms of the Hankel function $H^{(1)}_1(x) = -\frac{2}{\pi} K_1(ix)$ as:

$$K(b, a) = \frac{im^2 c^2}{4\pi \hbar^2} \frac{1}{|q_b - q_a|} H^{(1)}_1 \left( \frac{mc}{\hbar} |q_b - q_a| \right), \quad \tau^2 < 0. \quad (123)$$

We may convince ourselves by direct substitution that the kernels (122) and (123) satisfy the zero-potential case ($A_\mu = 0$) of the Klein-Gordon equation (117):

$$\frac{\partial^2}{\partial q^\alpha \partial q_\alpha} K(b, a) = \pm \frac{m^2 c^2}{\hbar^2} K(b, a).$$

As a consequence, so does a free-particle wave function $\psi(q, t)$ if its space-time propagation is calculated according to Eq. (45).

In order to determine the non-relativistic limit $c \to \infty$ of Eq. (122), we consider the asymptotic behavior of $\tau$ and the Bessel function $K_1$:

$$\tau = \sqrt{-(t_b - t_a)^2 + (q_b - q_a)^2/c^2} \quad \overset{c \to \infty}{\approx} \quad i(t_b - t_a)$$

$$\frac{1}{\tau} K_1(M\tau) \quad \overset{c \to \infty}{\approx} \quad \sqrt{\frac{\pi}{2M\tau^3}} \exp(-M\tau)$$

$$\exp \left( -\frac{mc^2}{\hbar} \tau \right) \quad \overset{c \to \infty}{\approx} \quad \exp \left[ \frac{im}{2\hbar} \frac{(q_b - q_a)^2}{t_b - t_a} \right].$$

The nonrelativistic kernel $K(b, a)$ the kernel for three spatial degrees of freedom becomes

$$K_q(b, a) = \left[ \frac{m}{2\pi i \hbar (t_b - t_a)} \right]^{3/2} \exp \left[ \frac{im}{2\hbar} \frac{(q_b - q_a)^2}{t_b - t_a} \right].$$

This kernel generalizes the one-dimensional case (Eq. 119) and satisfies again the Schrödinger equation\textsuperscript{16,7}.

### 4. Conclusions

Starting from the space-time formulation of the action principle, we have demonstrated that the Lagrangian as well as the Hamiltonian description of classical dynamics can consistently be reformulated in order to be compatible with special relativity. In the emerging extended version of the Hamilton-Lagrange formalism, the dynamics is described as a motion on a hypersurface within an extended phase space. With the specific correlations of extended Lagrangian $L_e$ and extended
Hamiltonian $H_e$ to their conventional counterparts $L$ and $H$ given in this paper, the extended formalism retains the form of the long-established conventional Hamilton-Lagrange formalism. The extended Hamilton-Lagrange formalism thus provides an equivalent physical description of dynamical systems that is particularly appropriate for special relativity.

The physical significance of the Lorentz invariant extended Hamiltonian $H_e$ of a point particle in an external electromagnetic field was demonstrated by showing that the subsequent extended set of canonical equations, in conjunction with the condition $H_e = 0$, is equivalent to the set of canonical equations that follows from the well-known conventional Hamiltonian $H$ for this system. It was shown that the condition $H_e = 0$ is automatically satisfied on the system path that is defined by the solution of the canonical equations. For this reason, the hypersurface condition $H_e = 0$ actually does not represent a constraint for the system. The corresponding non-homogeneous extended Lagrangian $L_e$ was shown to be quadratic in its velocity terms, hence similar in its form with the conventional Lagrangian $L$ that describes the non-relativistic limit. This makes the extended formalism particularly suited for analytical approaches that depend on the Lagrangian to be quadratic in the velocities — like Feynman’s path integral formalism. Devising the “quantum version” of the action principle, one of Feynman’s achievements was to derive — by means of his path integral approach to quantum physics — the Schrödinger equation as the quantum description of a physical system whose classical limit is described by the non-relativistic Lagrangian $L$ for a point particle in an external potential. This is generally regarded as the proof of principle for the path integral formalism.

Similar to the extension of the conventional Hamilton-Lagrange formalism in the realm of classical physics, the general form of the relativistic extension of Feynman’s path integral approach is obtained by consistently treating space and time variables on equal footing. We have shown that the hypersurface condition from the classical extended formalism appears in the context of the extended path integral formalism as an additional uncertainty relation.

On the basis of the extended Lagrangian $L_e$ of a classical relativistic point particle in an external electromagnetic field, we could derive the Klein-Gordon equation as the corresponding quantum description by means of the space-time version of the path integral formalism. Correspondingly, we can regard the emerging of the Klein-Gordon equation as the proof of principle of the relativistic generalization of Feynman’s path integral approach that is based on Lorentz invariant extended Lagrangians $L_e$ in conjunction with the additional uncertainty relation.

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