ASYMPTOTIC INVARIANCE AND THE DISCRETISATION OF NONAUTONOMOUS FORWARD ATTRACTING SETS

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Abstract. The $\omega$-limit set $\omega_B$ of a nonautonomous dynamical system generated by a nonautonomous ODE with a positive invariant compact absorbing set $B$ is shown to be asymptotic positive invariant in general and asymptotic negative invariant if, in addition, the vector field is uniformly continuous in time on the absorbing set. This set has been called the forward attracting set of the nonautonomous dynamical system and is related to Vishik’s concept of a uniform attractor. If $\omega_B$ is also assumed to be uniformly attracting, then its upper semi continuity in a parameter and the upper semi continuous convergence of its counterparts under discretisation by the implicit Euler scheme are established.

1. Introduction. Invariant sets play a fundamental role in characterising the behaviour of dynamical systems, in particular their attractors. The weaker concepts of positive and negative invariance are also very useful in describing more easily determined absorbing sets that contain an attractor, in the first case, or in establishing the upper semi continuity of attractors in a parameter and the upper semi continuous convergence of discretised attractors in the second case [8, 7, 16, 17]. This applies for autonomous dynamical systems and also for the more complicated situation of nonautonomous dynamical systems whose attractors consist of an invariant family of time-dependent subsets, $A = \{A_t : t \in \mathbb{R}\}$, which attracts all bounded subsets of the state space in either the pullback or forward sense [2, 3, 9, 10, 15].

In autonomous dynamical systems the attractor corresponds to the $\omega$-limit set $\omega_B$ of an absorbing set $B$. Although this set is defined as the limit as time goes to infinity, it actually exists for all time, since the time variable in an autonomous dynamical system $\pi(t, x)$ is just the elapsed time from the start and not the actual time. The situation is rather different in a nonautonomous dynamical system $\phi(t, t_0, x)$ which depends on both the actual time and starting time and not only their difference. $\omega$-limit sets can be defined here, but depend on the starting time,
i.e., \( \omega_{B,t_0} \) [14]. These are increasing in \( t_0 \) and the closure of their union \( \omega_B \) represents the set of all future limit points of the system. Vishik [18] introduced the concept of a uniform attractor as the smallest set that attracts all bounded sets uniformly in the initial time. It contains all forward \( \omega \)-limit points, hence \( \omega_B \), but can be a larger set as an example in the appendix of [2] shows. However neither this uniform attractor nor \( \omega_B \) need be invariant or even positive invariant, e.g., as the scalar ODE \( \dot{x} = -x + e^{-t} \) with \( \omega_B = \{0\} \) shows.

Bortolan et al [1] introduced the concept of lifted invariance of a set \( A \), by which they mean, in terms of the processes or two-parameter semi-groups considered here, that for any point \( x_0 \) in \( A \) there is some time \( t_0 \) and an entire trajectory contained in \( A \) and taking the value \( x_0 \) at time \( t_0 \). They show that the set \( \omega_B \) of a nonautonomous dynamical system formulated as a skew-product flow is lifted invariant, but their proof assumes that the state space of the driving system has a compact invariant subset, which does not apply for processes. Some related concepts for limiting equations and their associated skew-product flows are given in Chapter 4 and the appendix of LaSalle [14].

Recently, Kloeden & Yang [12] showed \( \omega_B \) is asymptotic positive invariant for nonautonomous difference equations. This concept has been long known in the differential equations literature [6, 13]. In this note it will be shown that \( \omega_B \) is also asymptotic negative invariant provided the vector field of the nonautonomous ODE and its spatial gradient are uniformly continuous in time inside the compact absorbing set \( B \).

These results are then used to show the upper semi continuity dependence of \( \omega_B \) on parameters and the upper semi continuous convergence of its counterparts for the implicit Euler scheme when, in addition, \( \omega_B \) is uniformly attracting.

The results here also hold for more general one-step schemes as well as more general dissipativity assumptions on the ODE. The main issue is to show that the numerical scheme inherits the dissipativity of the ODE and hence has a nonempty \( \omega \)-limit set. This will be discussed briefly in the last section. This property is easily established for the implicit Euler scheme considered here, which allows the focus of the paper to be on the essentially new ideas of asymptotic invariance and the required uniformity assumptions.

2. Dissipative nonautonomous system. Consider a nonautonomous ODE in \( \mathbb{R}^d \)

\[
\frac{dx}{dt} = f(x,t) \quad \text{for all } x \in \mathbb{R}^d, t \geq T^*, \tag{2.1}
\]

where \( f : \mathbb{R}^d \times [T^*, \infty) \to \mathbb{R}^d \) is (at least) continuously differentiable, so there exists a unique solution \( \phi(t,t_0,x_0) \) for the initial value \( x(t_0) = x_0 \).

Also assume that the nonautonomous system (2.1) is dissipative. In particular, assume that the nonautonomous system (2.1) is ultimately bounded in the closed and bounded (hence compact) subset \( B \) in \( \mathbb{R}^d \).

**Assumption 1.** There exists a \( \phi \)-positive invariant compact subset \( B \) in \( \mathbb{R}^d \) such that for any bounded subset \( D \) of \( \mathbb{R}^d \) and every \( t_0 \geq T^* \) there exists a \( T_D \geq 0 \) for which

\[
\phi(t,t_0,x_0) \in B \quad \forall t \geq t_0 + T_D, x_0 \in D.
\]
It follows from this dissipativity assumption and the compactness of the set $B$ that the $\omega$-limit set

$$\omega_{B,t_0} := \bigcap_{t \geq t_0} \bigcup_{s \geq t} \phi(s,t_0,B)$$

is a nonempty compact set of $\mathbb{R}^d$ for each $t_0 \geq T^*$.

In fact, by the positive invariance $\omega_{B,t_0} = \bigcap_{t \geq t_0} \phi(t,t_0,B)$. Note that

$$\lim_{t \to \infty} \text{dist}_{\mathbb{R}^d}(\phi(t,t_0,B), \omega_{B,t_0}) = 0 \quad (2.2)$$

for each $t_0 \geq T^*$ and that $\omega_{B,t_0} \subset \omega_{B,t_0'} \subset B$ for $t_0 \leq t_0'$. (Here $\text{dist}_{\mathbb{R}^d}(\cdot, \cdot)$ is the Hausdorff semi-distance between nonempty compact subsets of $\mathbb{R}^d$).

Hence, the set

$$\omega_B := \bigcup_{t_0 \geq T^*} \omega_{B,t_0} \subset B$$

is nonempty and compact. It contains all of the future limit points of the process starting in the set $B$ at some time $t_0 \geq T^*$. In particular, $\bar{y} \in \omega_B$ if there exist sequences $b_{0,j} \in B$ and $t_{0,j} \leq t_{0,j}'$ with $t_{0,j} \to \infty$ as $j \to \infty$ such that $\phi(t_{0,j}',t_{0,j},b_{0,j}) \to \bar{y}$ as $j \to \infty$.

The set $\omega_B$ characterises the forward asymptotic behaviour of the nonautonomous system. It is closely related to what Vishik [18] (see also [1, 2, 3]) called the uniform attractor, but it may be smaller and does not require the generating ODE (2.1) to be defined for all time or the attraction to be uniform in the initial time [9, 12].

Furthermore, the simple example $\dot{x} = -x + e^{-t}$ with $\omega_B = \{0\}$ shows that the set $\omega_B$ need not be invariant or even positive invariant.

2.1. Asymptotic positive invariance. The set $\omega_B = \{0\}$ in the previous example appears to become more and more invariant the later one starts in the future. This motivated the concept of asymptotic positive invariance in the literature for differential equations [6, 13]. Positive invariance says that a set is mapped into itself at future times, while asymptotic positive invariance says it is mapped almost into itself and closer the later one starts.

**Definition 2.1.** A set $A$ is said to be asymptotic positively invariant if for any monotonic decreasing sequence $\varepsilon_p \to 0$ as $p \to \infty$ there exists a monotonic increasing sequence $T_p \to \infty$ as $p \to \infty$ such that

$$\phi(t_{0},A) \subset B_{\varepsilon_p}(A), \quad t \geq t_0, \quad (2.3)$$

for each $t_0 \geq T_p$, where $B_{\varepsilon_p}(A) := \{x \in \mathbb{R}^d : \text{dist}_{\mathbb{R}^d}(x,A) < \varepsilon\}$.

The following result was proved in [12] for difference equations. The proof, essentially the same, is repeated here in the present context for the reader’s convenience.

**Theorem 2.2.** Let the above assumptions hold. Then $\omega_B$ is asymptotic positively invariant.

**Proof.** Since $\omega_{B,t_0} \subset \omega_B \subset B$ and $\phi(t_{0},\omega_B) \subset \phi(t_{0},B) \subset B$, by the convergence (2.2), for every $\varepsilon > 0$ and $t_0 \geq T^*$ there exists $T_0(t_0,\varepsilon) \in \mathbb{R}^+$ such that

$$\text{dist}_{\mathbb{R}^d}(\phi(t_{0},\omega_B), \omega_B) < \varepsilon \quad \text{for } t \geq t_0 + T_0(t_0,\varepsilon).$$

Suppose for an $\varepsilon_1 > 0$ that there are sequences $t_{0,j} \leq t_j \leq t_{0,j} + T_0(t_{0,j},\varepsilon_1)$ with $t_{0,j} \to \infty$ as $j \to \infty$ such that

$$\text{dist}_{\mathbb{R}^d}(\phi(t_j,t_{0,j},\omega_B), \omega_B) \geq \varepsilon_1, \quad j \in \mathbb{N}.$$
Asymptotic negative invariance.

3. \( \partial \).

Since \( \phi(t_j, t_{0,j}, \omega_B) \) is compact there exists a \( b_j \in \omega_B \subset B \) such that
\[
\text{dist}_{\mathbb{R}^d}(\phi(t_j, t_{0,j}, b_j), \omega_B) = \text{dist}_{\mathbb{R}^d}(\phi(t_j, t_{0,j}, \omega_B), \omega_B) \geq \varepsilon_1, \quad j \in \mathbb{N}.
\]

Define \( y_j := \phi(t_j, t_{0,j}, b_j) \). Since the points \( y_j \in B \), which is compact, there exists a convergent subsequence \( y_{j_k} \to \bar{y} \in B \). Moreover, \( \bar{y} \in \omega_B \) by the definition. However, \( \text{dist}_{\mathbb{R}^d}(y_j, \omega_B) \geq \varepsilon_1 \), so \( \text{dist}_{\mathbb{R}^d} \bar{y}, \omega_B) \geq \varepsilon_1 \), which is a contradiction.

Hence for this \( \varepsilon_1 > 0 \) there exists \( t_1 = t_1(\varepsilon_1) \) large enough such that
\[
\text{dist}_{\mathbb{R}^d}(\phi(t, t_0, \omega_B), \omega_B) < \varepsilon_1 \quad \text{for} \ t \geq t_0 \geq t_1(\varepsilon_1).
\]
The argument can be repeated inductively with \( \varepsilon_{p+1} < \varepsilon_p \) and \( t_{p+1}(\varepsilon_{p+1}) > t_p(\varepsilon_p) \).
It follows that \( \omega_B \) is asymptotic positively invariant.

The set \( \omega_B \) was called the forward attracting set of the nonautonomous system in [12].

3. Asymptotic negative invariance. The concept of negative invariance of a set implies that any point in it can be reached in any even time from another point in it. The set \( \omega_B \) is generally not negatively invariant, but under an additional uniformity assumption it is asymptotic negatively invariant.

Definition 3.1. A set \( A \) is said to be asymptotic negatively invariant if for every \( a \in A, \varepsilon > 0 \) and \( T > 0 \), there exist \( t_\varepsilon \) and \( a_\varepsilon \in A \) such that
\[
\| \varphi(t_\varepsilon, t_\varepsilon - T, a_\varepsilon) - a_\varepsilon \| < \varepsilon.
\]

To show that this property holds the future uniform behaviour of the vector field \( f \) in time, the following condition is needed.

Assumption 2. The mappings \( t \mapsto f(x, t) \) and \( t \mapsto \nabla_x f(x, t) \) are uniformly continuous in \( t \geq T^* \) uniformly in \( x \in B \).

This assumption holds if, for example, \( f \) has the form \( f(x, t) = \bar{f}(x, \phi(t)) \), where \( \phi : [T^*, \infty) \to \mathbb{R}^m \) is uniformly bounded and uniformly continuous, see [11], such as an almost periodic or recurrent function.

By Assumption 2, the vector field \( f \) of the ODE (2.1) is Lipschitz on the compact absorbing set \( B \) uniformly in time, i.e.,
\[
\| f(x, t) - f(y, t) \| \leq L_B \| x - y \|, \quad x, y \in B, t \geq T^*,
\]
with the constant \( L_B = \sup_{x \in B, t \geq T^*} \| \nabla_x f(x, t) \| < \infty \).

Thus for any two solutions \( \varphi(t, \tau, x_0), \varphi(t, \tau, y_0) \) in \( B \)
\[
\| \varphi(t, \tau, x_0) - \varphi(t, \tau, y_0) \| \leq \| x_0 - y_0 \| + \int_{\tau}^{t} \| f(\varphi(s, \tau, x_0), s) - f(\varphi(s, \tau, y_0), s) \| \ ds
\]
\[
\leq \| x_0 - y_0 \| + L_B \int_{\tau}^{t} \| \varphi(s, \tau, x_0) - \varphi(s, \tau, y_0) \| \ ds.
\]
Gronwall’s inequality then gives
\[
\| \varphi(t, \tau, x_0) - \varphi(t, \tau, y_0) \| \leq \| x_0 - y_0 \| e^{L_B(t - \tau)} \leq \| x_0 - y_0 \| e^{L_B T},
\]
where \( 0 \leq t - \tau \leq T \). Note that the bound depends just on the length of the time interval and not on the starting point of the interval.

Theorem 3.2. Let the Assumptions 1 and 2 hold. Then \( \omega_B \) is asymptotic negatively invariant.
Proof. To show this let \( \omega \in \omega_B \), \( \varepsilon > 0 \) and \( T > 0 \) be given. Then there exist sequences \( b_n \in B \) and \( \tau_n < \tau_n^* \) with \( \tau_n \to \infty \) and an integer \( N(\varepsilon) \) such that

\[
\| \phi(t_n, \tau_n, b_n) - \omega \| < \frac{1}{2} \varepsilon, \quad n \geq N(\varepsilon).
\]

Define \( a_n := \phi(t_n - T, \tau_n, b_n) \in B \). Since \( B \) is compact, there exists a convergent subsequence \( a_{n_j} \) such that \( \phi(t_n, \tau_n, b_n) \to \omega_\varepsilon \) as \( n_j \to \infty \). By definition, \( \omega_\varepsilon \in \omega_B \).

From Assumption 2 the process \( \phi \) is continuous in initial conditions uniformly on finite time intervals of the same length, i.e., (3.2). Hence

\[
\| \phi(t_{n_j}, t_{n_j} - T, a_{n_j}) - \phi(t_{n_j}, t_{n_j} - T, \omega_\varepsilon) \| < \frac{1}{2} \varepsilon, \quad n_j \geq N(\varepsilon). \tag{3.3}
\]

By the 2-parameter semi-group property

\[
\phi(t_{n_j}, t_{n_j} - T, a_{n_j}) = \phi(t_{n_j}, t_{n_j} - T, \phi(t_{n_j}, \tau_n, b_n)) = \phi(t_{n_j}, \tau_n, b_n).
\]

Then

\[
\| \omega - \phi(t_{n_j}, t_{n_j} - T, \omega_\varepsilon) \| \leq \| \omega - \phi(t_{n_j}, t_{n_j} - T, a_{n_j}) \| + \| \phi(t_{n_j}, t_{n_j} - T, a_{n_j}) - \phi(t_{n_j}, t_{n_j} - T, \omega_\varepsilon) \|
\]

\[
\leq \| \omega - \phi(t_{n_j}, \tau_n, b_n) \| + \| \phi(t_{n_j}, t_{n_j} - T, a_{n_j}) - \phi(t_{n_j}, t_{n_j} - T, \omega_\varepsilon) \|
\]

\[
< \frac{1}{2} \varepsilon + \frac{1}{2} \varepsilon = \varepsilon.
\]

This is the desired result. \( \square \)

4. Upper semi continuity in a parameter. Now consider a parameterised family of nonautonomous ODEs in \( \mathbb{R}^d \)

\[
\frac{dx}{dt} = f^\nu(x, t) \quad \text{for all } x \in \mathbb{R}^d, t \geq T^* \tag{4.1}
\]

for \( \nu \in [0, \nu^*] \), where each \( f^\nu : \mathbb{R}^d \times [T^*, \infty) \to \mathbb{R}^d \) is (at least) continuously differentiable. Let \( \phi^\nu(t, t_0, x_0) \) denote the unique solution with the initial value \( x(t_0) = x_0 \).

Assumption 1 is strengthened so the ODEs (4.1) are equi-ultimately bounded in the compact absorbing \( B \) uniformly in \( \nu \in [0, \nu^*] \).

Assumption 3. The exists nonempty compact set \( B \) which is \( \phi^\nu \)-positive invariant for each \( \nu \in [0, \nu^*] \) and for any bounded subset \( D \) of \( \mathbb{R}^d \) and \( t_0 \geq T^* \) there exists a \( T_D \geq 0 \) (independent of \( \nu \)) such that

\[
\phi^\nu(t, t_0, x_0) \in B \quad \forall t > t_0 + T_D, x_0 \in D, \nu \in [0, \nu^*].
\]

This holds, for example, if the vector fields of the ODEs (4.1) satisfy a dissipative inequality such as: there exists an \( R^* > 0 \) such that

\[
\langle f^\nu(x, t), x \rangle \leq -1 \quad \text{for all } \|x\| \geq R^*, t \geq T^*, \nu \in [0, \nu^*].
\]

In this case, \( B \) is the closed and bounded ball \( B_0[R^*] \) of radius \( R^* \) about the origin in \( \mathbb{R}^d \).

It will also be assumed that the vector fields of the ODEs (4.1) converge as the parameter \( \nu \to 0 \) uniformly in time on the set \( B \).
Assumption 4. For every $\varepsilon > 0$ and $T > 0$ there exists a $\delta(\varepsilon, T) > 0$ such that
\[
\left\| f^\nu(x, t) - f^0(x, t) \right\| < \varepsilon, \quad t_0 \leq t \leq t_0 + T, x \in B.
\]
for $|\nu| < \delta(\varepsilon, T)$ and any $t_0 \geq T^*$.

The continuous convergence of the solutions of the ODEs (4.1) as the parameter $\nu \to 0$ on the set $B$ uniformly in time then follows.

Lemma 4.1. Suppose that Assumptions 3 and 4 hold. For every $\varepsilon > 0$ and $T > 0$ there exists a $\delta(\varepsilon, T) > 0$ such that
\[
\left\| \varphi^\nu(t, t_0, b) - \varphi^0(t, t_0, b) \right\| < \varepsilon, \quad t_0 \leq t \leq t_0 + T, b \in B.
\]
for $|\nu| < \delta(\varepsilon, T)$ and $t_0 \geq T^*$.

Proof. For the solutions $\varphi^0(t, \tau, x_0)$ and $\varphi^\nu(t, \tau, x_0)$ of the ODEs (2.1) and (4.1) in the set $B$
\[
\left\| \varphi^\nu(t, \tau, x_0) - \varphi^0(t, \tau, x_0) \right\| \leq \int_{\tau}^{t} \left\| f^\nu(\varphi^\nu(s, \tau, x_0), s) - f^\nu(\varphi^0(s, \tau, x_0), s) \right\| \, ds
\leq \int_{\tau}^{t} \left\| f^\nu(\varphi^\nu(s, \tau, x_0), s) - f^\nu(\varphi^0(s, \tau, x_0), s) \right\| \, ds
+ \int_{\tau}^{t} \left\| f^\nu(\varphi^0(s, \tau, x_0), s) - f^0(\varphi^0(s, \tau, x_0), s) \right\| \, ds
\leq L_B \int_{\tau}^{t} \left\| \varphi^\nu(s, \tau, x_0) - \varphi^0(s, \tau, x_0) \right\| \, ds + \varepsilon T,
\]
when $0 \leq t - \tau \leq T$. Here Assumption 2 gives the common Lipschitz constant, while Assumption 4 is used to estimate the integral in the second last line. Then Gronwall’s inequality gives
\[
\left\| \varphi^\nu(t, \tau, x_0) - \varphi^0(t, \tau, x_0) \right\| \leq \varepsilon T e^{L_B(t-\tau)} \leq \varepsilon T e^{L_B T},
\]
when $0 \leq t - \tau \leq T$, i.e., the bound depends just on the length of the time interval and not the starting time. Then the assertion of the Lemma holds with $\delta(\varepsilon, T) = \delta(\varepsilon T^{-1} e^{-L_B T}, T)$. $\square$

Finally, the uniform attraction of the set $\omega_B^0$ for the system $\varphi_0$ is needed for the following result.

Assumption 5. $\omega_B^0$ uniformly attracts the set $B$, i.e., for every $\varepsilon > 0$ there exists a $T(\varepsilon)$, which is independent of $t_0 \geq T^*$, such that
\[
\text{dist}_d(\varphi^0(t_0 + t, t_0, B), \omega_B^0), \quad t \geq T(\varepsilon), t_0 \geq T^*.
\]

Theorem 4.2. Suppose that Assumptions 3, 4 and 5 hold. Then
\[
\text{dist}_d(\omega_B^\nu, \omega_B^0) \to 0 \quad \text{as } \nu \to 0.
\]

Proof. A proof by contradiction will be used.

Suppose for some sequence of parameters $\nu_j \to 0$ that the above limit is not true, i.e., there exists an $\varepsilon_0 > 0$ such that
\[
\text{dist}_d(\omega_B^\nu_j, \omega_B^0) \geq \varepsilon_0, \quad j \in \mathbb{N}.
\]
Since $\omega^\nu_B$ is compact, there exists $\omega_j \in \omega^\nu_B$ such that
\[
\text{dist}_{\mathbb{R}^d} (\omega_j, \omega_B^0) = \text{dist}_{\mathbb{R}^d} (\omega^\nu_B, \omega_B^0) \geq \varepsilon_0, \quad j \in \mathbb{N}.
\] (4.2)
By Assumption 5 there is a $T = T(\varepsilon_0/8)$ such that for any $t_0 \geq T^*$
\[
\text{dist}_{\mathbb{R}^d} (\varphi^0 (t_0 + T, t_0, B), \omega_B^0) < \frac{1}{8} \varepsilon_0
\]
Then use Lemma 4.1 with this $T$ to pick a $\nu_j < \delta(\varepsilon_0/2, T)$ to ensure that
\[
\|\varphi^\nu_j (t_0, t_0 - T, b)) - \varphi^0 (t, t_0 - T, b)\| < \frac{1}{2} \varepsilon_0, \quad b \in B, t_0 \gg 0.
\] (4.3)
Fix such a $\nu_j$ and use the asymptotic negative invariance of $\omega^\nu_B$ to obtain the existence of an $\omega_{j,T} \in \omega^\nu_B$ and a $t_j^T > 0$ so that
\[
\|\varphi^\nu_j (t_j^t, t_j^T, \omega_{j,T}) - \omega_j\| < \frac{1}{8} \varepsilon_0.
\]
Then, with $t_0$ taken as $t_j^T$ above,
\[
\text{dist}_{\mathbb{R}^d} (\omega_j, \omega_B^0) \leq \|\omega_j - \varphi^\nu_j (t_j^t, t_j^T - T, \omega_{j,T})\|
\quad + \|\varphi^\nu_j (t_j^t, t_j^T - T, \omega_{j,T}) - \varphi^0 (t_j^t, t_j^T - T, \omega_{j,T})\|
\quad + \text{dist}_{\mathbb{R}^d} (\varphi^0 (t_j^t, t_j^T - T, \omega_{j,T}), \omega_B^0)
\quad < \frac{1}{8} \varepsilon_0 + \frac{1}{2} \varepsilon_0 + \frac{1}{8} \varepsilon_0 = \frac{3}{4} \varepsilon_0,
\]
which contradicts the assumption (4.2).
\[\square\]

5. Numerical discretization. The ODE (2.1) is assumed to be dissipative.

**Assumption 6.** The vector field of the ODE (2.1) satisfies the dissipative inequality
\[
\langle f(x, t), x \rangle \leq -\|x\|^2 \quad \text{for all } \|x\| \geq R^*, \, t \geq T^*,
\]
for some $R^* > 0$.

Thus the compact ball $B_0[R^*]$ of radius $R^*$ centered at the origin in $\mathbb{R}^d$ is positive definite absorbing set uniformly in the initial time $t_0$ for the ODE (2.1). Thus, the limit set $\omega_B$ exists and is contained in $B = B_0[R^*]$.

It will also be assumed in this section that the vector field and its gradient are uniformly continuous in future time on $B$, i.e., satisfy Assumption 2, and that $\omega_B$ is uniformly attracting, i.e., satisfies Assumption 6.

The ODE (2.1) will be discretised using variable time steps $h_n$. Let $\mathcal{H}$ be the set of infinite sequence $h = \{h_0, h_1, h_2, \cdots \}$ with values in $(0, 1]$ such that $\sum_{n=0}^\infty h_n = \infty$ and let $\Sigma(h, t_0)$ be set of times $t_n = t_0 + \sum_{j=0}^{n-1} h_j$ for a given (but otherwise arbitrary) $t_0 \geq T^*$.

The implicit Euler scheme for the ODE (2.1) and the initial value $x_0$ and $t_0 \geq T^*$ is given by
\[
x_{n+1}^{(h)} = x_n^{(h)} + h_n f\left(x_n^{(h)}, t_{n+1}^{(h)}\right)
\] (5.1)
for a step size sequence $h \in \mathcal{H}$ with $\{t_n\} \in \Sigma(h, t_0)$. The implicit equation (5.1) is uniquely solvable for sufficiently small step sizes [17]. Thus the implicit Euler scheme (5.1) can be written as a discrete time process $\Phi^{(h)}$ defined by $\Phi^{(h)} (m, n, x_n^{(h)}) :=$
Let Assumptions 2 and 6 hold. Then the implicit Euler scheme (5.1) is also dissipative.

**Lemma 5.1.** Let Assumption 6 hold. Then the ball $B_0[R^*]$ is also uniform absorbing and positive invariant for the implicit Euler scheme (5.1) and all time step sequences $h \in \mathcal{F}$.

**Proof.** By the inequality in Assumption 6
\[
\|x_{n+1}^{(h)}\|^2 = \langle x_{n+1}^{(h)}, x_n^{(h)} \rangle + h \| f(x_{n+1}, t_{n+1}) \| = \langle x_{n+1}^{(h)}, x_n^{(h)} \rangle + h \| f(x_{n+1}, t_{n+1}) \| = \|x_n^{(h)}\| \|x_{n+1}^{(h)}\| - h\|x_n^{(h)}\|^2
\]
provided $\|x_n^{(h)}\| \geq R^*$, so
\[
\|x_{n+1}^{(h)}\| \leq \frac{1}{1 + h}\|x_n^{(h)}\|.
\]
Hence, if $\|x_n^{(h)}\| \leq R^*$, then $\|x_{n+1}^{(h)}\| \leq R^*$, so $B_0[R^*]$ is positive invariant. On the other hand, if $\|x_n^{(h)}\| > R^*$, then $\|x_{n+1}^{(h)}\| \leq R^*$, where $n^*$ is the first integer such that $\prod_{j=0}^{n-1} \frac{1 - h\|x_j^{(h)}\|}{1 + h\|x_j^{(h)}\|} \leq R^*$. Thus $B_0[R^*]$ is uniform absorbing.

**Lemma 5.2.** Let Assumptions 2 and 6 hold. Then the implicit Euler scheme (5.1) is continuous in initial conditions on $B$ for time step sequences $h \in \mathcal{F}$ with sup$_n h_n$ sufficiently small.

**Proof.** Fix $t_0 \geq T^*$ and $h \in \mathcal{F}$. Let $x_n^{(h)}$, $y_n^{(h)} \in B$. Then the next iterates $x_{n+1}^{(h)}$, $y_{n+1}^{(h)} \in B$, so
\[
\|x_{n+1}^{(h)} - y_{n+1}^{(h)}\| \leq \|x_n^{(h)} - y_n^{(h)}\| + h\| f(x_{n+1}, t_{n+1}) - f(y_{n+1}, t_{n+1})\| \leq \|x_n^{(h)} - y_n^{(h)}\| + hL_{B}\|x_{n+1}^{(h)} - y_{n+1}^{(h)}\|,
\]
where $L_B$ be the Lipschitz constant of the vector field $f$ on the set $B$, which gives
\[
\|x_{n+1}^{(h)} - y_{n+1}^{(h)}\| \leq \frac{1}{1 - L_B\|h\|}\|x_n^{(h)} - y_n^{(h)}\|
\]
provided $\|h\|\infty := \sup_n h_n < 1/L_B$.

Thus, the discrete time process $\Phi^{(h)}$ corresponding to the implicit Euler scheme has a nonempty compact limit sets $\omega_{B,n}^{(h)} \subset \omega_{B,n+1}^{(h)} \cdots \subset B$ and
\[
\omega_{B,n}^{(h)} := \bigcup_{n \geq 0} \omega_{B,n}^{(h)} \subset B
\]
for all time step sequences $h \in \mathcal{F}$ with $L_B\|h\|\infty < 1$. See also [16].

The proofs of the following theorems are essentially the same as those of their continuous time counterparts, Theorems 2.2 and 3.2.
Theorem 5.3. Let Assumptions 2 and 6 hold. Then \( \omega_B(h) \) is asymptotic positive invariant for the implicit Euler scheme (5.1).

Theorem 5.4. Let Assumptions 2, 5 and 6 hold and let \( L_B \|h\|_\infty < 1 \). Then \( \omega_B(h) \) is asymptotic negative invariant for the implicit Euler scheme (5.1).

The implicit Euler scheme (5.1) is a first order scheme. Under the above uniformity assumptions its global discretisation error depends only on the length of the time interval under consideration and not its starting time.

Lemma 5.5. Let Assumptions 2 and 6 hold and let \( L_B \|h\|_\infty < 1 \). Then the global discretisation error of the implicit Euler scheme (5.1) on the set \( B \) satisfies the uniform estimate

\[
\|\phi(t_n + t_0, t_0, x_0) - x_n^{(h)}\| \leq K_T h
\]

for all \( x_0 \in B, t_0 \geq T^* \) and \( t_n \in [t_0, t_0 + T] \) for any finite \( T > 0 \), where \( h = \|h\|_\infty \) and \( K_T \) does not depend on \( t_0 \geq T^* \).

**Proof.** The estimate (5.2) is obtained [17] by iterating a difference inequality and using the local discretisation error

\[
\|\phi(h + t_0, t_0, x_0) - x_1\| \leq Kh^2
\]

for all \( x_0 \in B, t_0 \geq T^* \) and \( h \in (0,1] \). The local error (5.3) is derived by a Taylor expansion of the ODE solution and the constant \( K = L_B = \sup_{x \in B, t \geq T^*} \|\nabla_x f(x,t)\| \) \( < \infty \). The constant \( K_T \) in (5.2) has the form \( e^{L_B T} \) and is independent of initial time \( t_0 \).

Theorem 5.6. Let Assumptions 2, 5 and 6 hold and let \( L_B \|h\|_\infty < 1 \). Then

\[
\text{dist}_{R^n} \left( \omega_B^{(h)}, \omega_B \right) \to 0 \text{ as } \|h\|_\infty = \sup_n h_n \to 0.
\]

**Proof.** Suppose for some sequence of time step sequences \( h^j \in J \) with \( \|h^j\|_\infty \to 0 \) that this limit is not true. Then there exists \( \varepsilon_0 > 0 \) such that

\[
\text{dist}_{R^n} \left( \omega_B^{(h^j)}, \omega_B \right) \geq \varepsilon_0, \quad j \in \mathbb{N}.
\]

Since \( \omega_B^{(h^j)} \) is compact, there exists \( \omega_j \in \omega_B^{(h^j)} \) such that

\[
\text{dist}_{R^n} (\omega_j, \omega_B) = \text{dist}_{R^n} \left( \omega_B^{(h^j)}, \omega_B \right) \geq \varepsilon_0, \quad j \in \mathbb{N}.
\]

Let \( t_n^j \to \infty \) as \( n \to \infty \) be the time sequence in \( \mathcal{I}(h^j, t_0) \) determined by adding the time steps starting at \( t_0 \) and let \( \Phi^{(h^j)}(n,m,b) \) be the value of the numerical scheme at time \( t_n^j \) starting at \( b \) at time \( t_n^j \), where \( n \geq m \).

First, by Assumption 5 pick \( T_0 = T(\varepsilon_0/4) \) such that for any \( t_0 \geq T^* \)

\[
\text{dist}_{R^n} (\Phi(t_0 + T_0, t_0, B), \omega_B) < \frac{1}{4} \varepsilon_0
\]

Next consider the global discretisation error of the implicit Euler scheme (5.1) on the interval of length \( 2T_0 \), i.e.,

\[
\|\Phi^{(h^j)}(m,n,b) - \Phi(t_m^j, t_n^j, b)\| \leq K_{2T_0} \|h^j\|_\infty, \quad t_m^j \leq t_n^j \leq t_m^j + 2T_0, b \in B.
\]

Recall that by Lemma 5.5 the constant \( K_{2T_0} \) is independent of the starting time \( t_m^j \).
Fix such an \( h^j \). Define \( h_* = \min \{ \varepsilon_0/(4K_{2T_0}), \varepsilon_0 \} \). Pick \( h^j \in \mathcal{H} \) with \( \| h^j \|_\infty \leq h_* \) and \( N \) such that
\[
T_0 \leq N\delta(h^j) \leq N\| h^j \|_\infty \leq 2T_0.
\]
Then \( K_{2T_0}\| h^j \|_\infty^p \leq \varepsilon_0/4 \) and the global discretisation error estimate becomes
\[
\| \varphi(h^j)(m, n, b) - \varphi(t^j_m, t^j_n, b) \| \leq K_{2T_0}\| h^j \|_\infty \leq \varepsilon_0/4, \quad t^j_n \leq t^j_m \leq t^j_n + 2T_0, b \in B,
\]
i.e., so \( m - n \leq N \).

Also, by the asymptotic negative invariance of \( \omega_B^{(h^j)} \), for the above \( \omega_j \in \omega_B^{(h^j)} \), \( N \) and \( \varepsilon_0 \) there exist an \( n(\varepsilon_0) \geq 0 \) and \( \omega_{j,N} \in \omega_B^{(h^j)} \) such that
\[
\| \varphi(h^j)(n(\varepsilon_0), n(\varepsilon_0) - N, \omega_{j,N}) - \omega_j \| \leq \frac{1}{4}\varepsilon_0.
\]

Let \( t^j_{n(\varepsilon_0) - N} \) and \( t^j_{n(\varepsilon_0)} \) be times in the corresponding time sequence in \( \mathbb{T}((h^j), t_0) \). Then, \( t^j_{n(\varepsilon_0)} - t^j_{n(\varepsilon_0) - N} \geq N\delta(h^j) \geq T_0 \), so
\[
\text{dist}_{\mathcal{R}^d} \left( \varphi_0 \left( t^j_{n(\varepsilon_0)}, t^j_{n(\varepsilon_0) - N}, \omega_{j,N} \right), \omega_B \right) \leq \frac{1}{4}\varepsilon_0
\]
by the the assumption that \( \omega_B \) uniformly attracts the set \( B \).

Then, with \( t^j_0 \) and \( t^j_{n(\varepsilon_0)} \) in the global discretisation error replaced by \( t^j_{n(\varepsilon_0)} \) and \( t^j_{n(\varepsilon_0) - N} \), resp., we obtain
\[
\text{dist}_{\mathcal{R}^d} \left( \omega_j, \omega_B \right) \leq \left\| \omega_j - \varphi(h^j)(n(\varepsilon_0), n(\varepsilon_0) - N, \omega_{j,N}) \right\|
+
\left\| \varphi(h^j)(n(\varepsilon_0), n(\varepsilon_0) - N, \omega_{j,N}) - \varphi \left( t^j_{n(\varepsilon_0)}, t^j_{n(\varepsilon_0) - N}, \omega_{j,N} \right) \right\|
+
\text{dist}_{\mathcal{R}^d} \left( \varphi \left( t^j_{n(\varepsilon_0)}, t^j_{n(\varepsilon_0) - N}, \omega_{j,N} \right), \omega_B \right)
< \frac{1}{4}\varepsilon_0 + \frac{1}{4}\varepsilon_0 + \frac{1}{4}\varepsilon_0 = \frac{3}{4}\varepsilon_0,
\]
which contradicts the above assumption (5.4). \( \square \)

6. Higher order numerical schemes. The above results hold for more general one-step schemes as well as more general dissipativity assumption on the ODE (2.1). The main issue is to show that the numerical scheme inherits the dissipativity of the ODE and hence has a nonempty \( \omega \)-limit set. This is not always straightforward [4, 5], but holds here because of the uniformity assumptions. It can be shown similarly to Kloeden & Lorenz [8] using a Lyapunov function to characterise the dissipativity or equi-ultimate boundedness of the ODE system.

Instead of working with the nonautonomous ODE (2.1) one could work with the autonomous inclusion equation
\[
\frac{dx}{dt} \in F(x) \quad \text{for all } x \in \mathbb{R}^d,
\]
where the setvalued mapping \( F \) is defined by \( F(x) := \bigcup_{t \geq T} J(t, x) \) for each \( x \in \mathbb{R}^d \).
Under Assumption 6 the dissipativity of the inclusion (6.1) follow easily:
\[
\langle F(x), x \rangle \leq \sup_{t \geq T^*} \langle f(x, t), x \rangle \leq -\|x\|^2 \quad \text{for all } \|x\| \geq R^*.
\]
Modifications of the arguments in Kloeden & Lorenz [8] with the Lyapunov function
\[
V(x) := \max\{\|x\|^2 - (R^*)^2, 0\}
\]
can then be used to show that one-step scheme is also dissipative.

More generally, when the ODE (2.1) satisfies Assumption 1, there are general
existence theorems of Lyapunov function characterising this equi-ultimate bound-
ess of the system that can be used.

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