Fragmentation of a sheet by propagating, branching and merging cracks

Deepak Dhar

Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

E-mail: ddhar@theory.tifr.res.in

Received 2 February 2015, revised 9 March 2015
Accepted for publication 12 March 2015
Published 8 April 2015

Abstract
We consider a model of the fragmentation of a sheet by cracks that move with a velocity in a preferred direction but which undergo random transverse displacements as they move. There is a non-zero probability of crack-splitting and the split cracks move independently. If two cracks meet, they merge, and move as a single crack. In the steady state, there is non-zero density of cracks and the sheet left behind by the moving cracks is broken into a large number of fragments of different sizes. The evolution operator for this model reduces to the Hamiltonian of a quantum XY spin chain, which is exactly integrable. This allows us to determine the steady state and to also determine the distribution of the sizes of the fragments.

Keywords: fragmentation, cracks, self-organized criticality, coagulation-decoagulation models

(Some figures may appear in colour only in the online journal)

1. Introduction

There is a huge body of work in the literature of engineering and physics that deals with the distribution of the sizes of the fragments that arise when a large piece of a solid is broken into much smaller fragments. This interest has been driven by the applications of this research in many industrial processes, such as mining and milling, in geophysics, and in environmental science [1–4]. Theoretical approaches to modelling fragmentation have generally focussed on the distribution of the sizes of the fragments. It is expected that the general form of the distribution would be independent of material details and may be captured in a simple theoretical model that is based on some general physical principles. In one of the first theoretical attempts in this direction, Mott and Linfoot [5] modelled the fragmentation that is caused by randomly drawn Poisson-distributed horizontal and vertical lines. Grady and Kipp studied different variants of the
Mott construction, as well as their effect on the fragment-size distribution [1]. Similar ideas were also used by Gilvarry [6, 7] to predict the fragment size distribution. These ideas have been studied using the chemical reaction rate equations approach [8]. Several random fuse-network type models for fractures have been developed. In these models the material is seen as a network of coupled elastic elements, which have a random threshold for failure [9, 10]. Molecular-dynamic simulations of the assembly of sub-units have also been used, such as those using the Lennard-Jones interaction, and subjected to high-impact projectile or expansive stress [11–15]. Models assuming a fractal structure of defects have also been studied [16].

In recent years, there has been a lot of concern about climate change, leading to increased melting of the polar ice and a consequent rise of sea-levels. To make quantitative predictions, one needs to understand the fragmentation of the the polar ice cap into much smaller iceberg fragments. This concern has led to a recent study of calving glaciers by Astrom et al [17]. The precise model studied by Astrom et al is rather complicated and has to be solved numerically [18, 19]. In this paper, we propose a simplified model of the stochastic growth of cracks on a two dimensional sheet, which was inspired by these papers. The cracks have a preferred direction of propagation but have a random transverse velocity. There is a finite probability per unit time that the a crack splits into two, which then propagates independently. If two cracks meet, they merge, and evolve subsequently as a single crack. The region left behind by the cracks is broken into fragments of different sizes. An example of the fragments generated in our model is shown in figure 1. These may be compared with the photographs of natural calving glaciers in [20], or with the simulations in [17]. We see that the model does capture the qualitative features of the geometrical structure.

This model was studied earlier by Ben Avraham et al [21, 22] as a model of particles with diffusion, aggregation, and birth. They noted that the detailed balance condition is satisfied, and hence the steady state is easily written down. They also showed that the spacing distribution between consecutive particles satisfies a linear equation, and hence can be determined. The evolution operator for this model reduces to the Hamiltonian of a quantum XY spin model, which is well known to be diagonalizable exactly in terms of free fermions [23–26]. Viewed as a model of fragmentation, geometrical questions about fragment-size distribution and correlations seem quite natural, but these questions have not been discussed.

![Figure 1. Structure of the fragments formed in our model A. The picture shown is obtained from a numerical simulation, size of lattice \( L = 80 \), evolved to depth 150, with the crack-splitting rate \( \lambda = 0.1 \). Note the wide distribution of the fragment sizes.](image-url)
before. A model that is rather similar in spirit was discussed earlier by Inaoka and Takayasu [27], but their detailed model is actually quite different and can only be solved numerically. Our model has the advantage of being analytically tractable and we are also able determine the exact distribution of fragment sizes. We show that in the limit when the splitting rate is small, there is an exact mapping between our model A (which is defined precisely below) and the directed Abelian sandpile model [28, 29], which brings out the self-organized origin of the divergence in the fragment-size distribution for small sizes. In fact, this distribution is the same as that encountered in a rather different model, in which particles aggregate and do not undergo fragmentation, which was also earlier studied by Takayasu [30].

2. Definition of the model

We start with an intact cylinder with no cracks. At the start, we introduce one or more cracks at the left boundary. We can define a discrete-time update rule (model A), or continuous-time update rule (model B).

Model A: We consider the model defined on a square grid, which is drawn on the surface of a semi-infinite cylinder. The sites of the lattice are labelled by integers \((x, t)\), with \(1 \leq x \leq 2L, t \geq 0\), and \((x + t)\) even. The neighbours of site \((x, t)\) are four of the sites \((x \pm 1, t \pm 1)\). The periodic boundary conditions along the cylinder are imposed by identifying \((X + 2L, t) \equiv (x, t)\) for all \(x, t\). This corresponds to the axis of the cylinder being in the direction \((1, 1)\) of the lattice (figure 2).

We consider a discrete-time parallel update, where each crack moves inwards with speed 1. Then, the cracks reach up to distance \(t\) along the cylinder axis, and we may equivalently think of \(t\) as the time coordinate. If there is a crack at site \((x, t)\), then it moves according to the following rule:

(a) With probability \(\lambda\), it splits into two cracks, and these cracks go to \((x + 1, t + 1)\) and \((x - 1, t + 1)\). Clearly, \(0 < \lambda \leq 1\).

(b) If the crack does not split, it moves to any one of the two forward neighbours \((x + 1, t + 1)\) or \((x - 1, t + 1)\) with equal probability.

(c) If two cracks reach a site, then they merge, and move as a single crack.

Model B: Here we consider the time-coordinate \(t\) to be a continuous variable, \(t > 0\), but the spatial coordinate is still discrete, \(0 \leq x \leq 2L\), with periodic boundary conditions.
A configuration at time $t$ is specified by the occupation numbers $\{n_x\}$, where $n_x$ takes values 1 and 0, depending on whether or not there is a crack at $x$ at time $t$.

The configurations $\{n_x\}$ undergo continuous-time Markovian evolution with the following rates:

(a) A crack at position $x$ jumps to site $x'$ with rate 1, if $x'$ is a nearest neighbour of $x$.

(b) An empty neighbour $x'$ of an occupied site $x$ becomes occupied with rate $\lambda$ due to crack-splitting at $x$.

(c) If a crack arrives at a site already occupied by a crack, then it merges with it and the state of the site does not change.

If we represent the configuration by a bit string, e.g. $...001010110001...$, then the transition rates are given by the equations

$$01 \rightarrow 10, \quad \text{with rate 1;} \quad (1)$$

$$01 \rightarrow 11, \quad 10 \rightarrow 11, \quad \text{with rate } \lambda; \quad (2)$$

$$11 \rightarrow 01, \quad 11 \rightarrow 10, \quad \text{with rate 1.}$$

Figure 3 shows a schematic time–history of evolution in this case.

### 3. Calculation of the steady state

In our model, the number of cracks at any given time (equivalently at distance along the axis of the cylinder) is not conserved. So long as there is at least one crack at the start of the configuration $t = 0$, then as the cracks propagate along the cylinder, they may split or merge. For large times, there is a non-trivial steady state of the system with a non-zero density $\rho(\lambda)$ of cracks and a finite mean area per fragment. We now determine this steady state and the asymptotic density.

**Model A**: Let us label the bonds of the lattice by the coordinates of their midpoints. Then, a bond has coordinates $(m + 1/2, n + 1/2)$, where $m$ and $n$ are any integers, with $0 \leq m \leq M - 1$, and $0 \leq n < \infty$. The configuration $\Sigma$ of the cracks is specified by giving the occupation numbers $\{\sigma(m, n)\}$ of all of the bonds, where $\sigma(m, n) = 1$, if the bond
\((m + 1/2, n + 1/2)\) is occupied by a crack, and zero otherwise. The restriction of \(\Sigma\) to the vertical row \(t\) will be denoted by \(\Sigma_t\).

We can think of the set \((\sigma(m, n))\) as the evolution history of a set of \(2M\) Ising spins on a line, undergoing stochastic Markovian dynamics. Here, \(\sigma(m, n)\) is +1 if the \(m\)-th spin is up at time \(t\), and zero otherwise. In this formulation, the Markovian evolution is defined as follows: at odd times, we consider pairs of adjacent spins \([\sigma(2j, t), \sigma(2j + 1, t)]\). These evolve according to the following rules:

(a) \([0, 0]\) remains unchanged.

(b) If \([\sigma, \sigma'] \neq [0, 0]\), then it is reset to \((1, 0), (0, 1)\) or \((1, 1)\) with probabilities \((1 - \lambda)/2, (1 - \lambda)/2, \lambda\), respectively.

At even times \(t\), we apply the same rule but choose the pairs to be \([\sigma(j + 1, t), \sigma(j + 2, t)]\).

Now, we specify the state of the system at time \(t\) by giving a probability measure on the space of spin configurations. Let \(\text{Prob} (\Sigma_t)\) denote the probability that the configuration of Ising spins at time \(t\) will be found to be \(\Sigma_t\). Note that there are \(2^{2L} - 1\) allowed values of \(\Sigma_t\) because the configuration with all spins down is not reached in the steady state if we start with at least one crack present at \(t = 0\). The master equation for the evolution of \(\text{Prob} (\Sigma_t)\) is

\[
\dot{\text{Prob}}(\Sigma_{t+1}) = \mathcal{W}[\Sigma_{t+1}, \Sigma_t] \text{Prob}(\Sigma_t)
\]

where \(\mathcal{W}\) is the Markov transition matrix.

We note that the transition rates here satisfy the detailed balance condition corresponding to the Hamiltonian

\[
\mathcal{H}(\sigma) = -K \sum_{i=1}^{M} \sigma_i
\]

where \(e^K = 2\lambda/(1 - \lambda)\).

There are \(2^{2L}\) states and \(\mathcal{W}\) is a \(4^L \times 4^L\) matrix. There are two steady state eigenvectors: one in which there are no cracks and one in which a configuration with \(n\) cracks has a probability proportional to \(e^{Kn}\).

**Model B:** In this case, the analysis is simpler. We note that probability of transition of the state of two adjacent sites from the state 01 to 11 is \(\lambda\) but the rate of transition from 11 to 01 is 1. Also, there is a rate 1 for the transition 01 to 10, and vice-versa. Then, a detailed balance is clearly satisfied for the Hamiltonian function given by equation (4) with \(\lambda = e^K\).

4. Distribution of fragment sizes

We will first discuss model A. The cracks divide the plane into fragments of different sizes. We note that each fragment is a polygon that has a unique starting point and a unique endpoint, which are the boundary sites with minimum and maximum values of the time-coordinate, respectively. The boundary itself consists of two directed walks. These shapes have been called skew Ferrers diagrams or staircase polygons in the literature.

The left boundary of the fragment is a simple directed biased random walk, where the step occurs to the right with probability \((1 + \lambda)/2\), and to the left with probability \((1 - \lambda)/2\). Similarly, the right boundary is an independent walk with step to the left with probability \((1 + \lambda)/2\) and step to the right occurs with probability \((1 - \lambda)/2\). The fragment ends when these two boundaries meet.

Consider the stochastic evolution of the system. Suppose that there is a tip-splitting event at some site \((x_0, t_0)\), which starts a new fragment. Let \(\text{Prob}(A, \lambda)\) be the probability that this
fragment will have area $A$. For small $A$, these are easily determined by explicit enumeration of possible shapes. For example, consider a fragment starting at $(x_0, t_0)$. We can have $A = 1$, only if the fragment boundary, when at the site $(x_0 + 1, t_0 + 1)$ takes the next step to the left, and the boundary, when at $(x_0 - 1, t_0 + 1)$ takes the next step to the right. Thus we have $\text{Prob}(A = 1, \lambda) = (1 + \lambda)^2/4$. Similarly, $\text{Prob}(A = 2, \lambda) = (1 + \lambda)^2 (1 - \lambda^2)/8$. More generally, if a given fragment shape $S$ has perimeter $2P$, then it occurs with a probability $(1 + \lambda)^2 (1 - \lambda^2)^{P-2}$. 

Thus, the question of finding $\text{Prob}(A)$ for general $A$ is reduced to the finding the number of different staircase polygons of area $A$. This has been studied by Delest and Fedou [31], and by Prellberg[32], and Prellberg and Brak [33]. Let the number of a staircase polygon having $2h$ horizontal bonds and $2v$ vertical bonds, and area $A$ be $C(h, v, A)$. We define the generating function 

\[ G(x, y, A) = \sum_{h,v,A} x^h y^v q^A \]  

Prellberg showed that $G(x, y, A)$ satisfies a functional equation, and used it to determine $G(x, y, q)$ exactly. One gets 

\[ G(x, y, q) = y \left[ \frac{H(q^x, qy, q)}{H(qx, y, q)} - 1 \right] \]  

where 

\[ H(x, y, q) = \sum_{n=0}^{\infty} \frac{(-x)^n q^\binom{n}{2}}{(q; q)_n (y; q)_n} \]  

with 

\[ (r; q)_n = \prod_{m=1}^{n-1} (1 - rq^m) \]  

These explicit formulas are a bit complicated. They simplify in the continuum limit, corresponding to $\lambda \to 0$. The continuum problem of determining the probability distribution of area in a Brownian bridge has been studied in [34, 35], who showed that in this case the distribution function $\text{Prob}(A, \lambda)$ is the Airy distribution. For small $\lambda$, $\text{Prob}(A, \lambda)$ has the scaling form 

\[ \text{Prob}(A, \lambda) \approx \lambda^\frac{4}{3} \left( \frac{\lambda A}{4} \right)^{\frac{1}{3}} \]  

In the limit $\lambda \to 0$ and fixed $A$, $\text{Prob}(A, \lambda)$ tends to a non-zero value. In this case the boundaries of the fragment do unbiased random walk and the corresponding probabilities are exactly as found in the directed abelian sandpile model in 2-dimensions [28]. For this case, simple dimensional arguments already show that $\text{Prob}(A) \sim A^{-4/3}$ [28]. This implies that 

\[ \text{Prob}(A, \lambda \to 0) = \text{Prob}(A)_{\text{ASM}} \]  

where $\text{Prob}(A)_{\text{ASM}}$ is the probability that adding a grain in the steady state of a 2-dimensional directed abelian sandpile model will cause an avalanche with exactly $A$ topplings [28, 29]. It is known that $\text{Prob}(A)_{\text{ASM}} \sim A^{-4/3}$ for large $A$, we must have 

\[ g(x) \sim K x^{-4/3}, \text{ for small } x. \]
For large $x$, the Airy distribution, $g(x)$, decreases as $\exp(-x^{1/2})$ [35]. In the literature, there has been some discussion about whether the size-distribution has an exponential or stretched exponential decay for large sizes [1].

Note the unusual $\lambda^4$ scaling of the prefactor in Equation (11). This arises because the function $g(x)$, which varies as $x^{-4/3}$ for small $x$, is not integrable. Then, if we take the scaling limit

$$\sum_{A=1}^{\infty} \text{Prob}(A, \lambda) \approx \int_{\lambda_{\min}}^{\infty} dx g(x),$$

we have to impose a cut-off of $\lambda_{\min} = O(\lambda^3)$ on the lower limit of the integral. The integral then diverges as $\lambda_{\min}^{-1/2}$, which cancels the extra power of $\lambda$ in equation (11).

For finite but small $\lambda$, the mean density of cracks varies as $\lambda^{1/2}$, and the average longitudinal extent of a fragment scales as $\lambda^{1/2}$. Thus, the mean size of a fragment varies as $\lambda^{1/3}$.

Now, we consider Model B. Let $\text{Prob}(A, m) dA$ be the probability that a segment of width $m$ will generate a fragment with additional area between $A$ and $A + dA$. The Laplace transform of this distribution will be denoted by $\tilde{P}(s, m)$, and is defined by

$$\tilde{P}(s, m) = \int_0^\infty dA \text{Prob}(A, m) e^{-sA}$$

Consider the time evolution of a fragment whose transverse size is $m$ at some time $t = t_0$. In time, the boundaries of this fragment will evolve stochastically, until they merge. The segment of length $m$ can become a segment of length $m \pm 1$ if either end makes a diffusive jump (with rate 2 for each) or if either end creates a new crack which decreases the size of this segment (total rate $2\lambda$). The waiting time $t_1$ before any one of these occurs is an exponentially distributed random variable. The probability that this occurs between time $t_1$ and $t_1 + dt_1$ is given by $\text{Prob}(t_1) dt_1 = (4 + 2\lambda) e^{-(4+2\lambda)t_1} dt_1$. The total area of the fragment is the sum of two independent random variables: the area $A_1 = mt_1$ up to this time and remainder area $A_2$. The remainder has a starting length $m + 1$ with probability $p_+ = 1/(2 + \lambda)$, and $m - 1$, with probability $p_- = (1 + \lambda)/(2 + \lambda)$. This gives

$$\tilde{P}(s, m) = \left[ \int_0^\infty \text{Prob}(t_1) dt_1 e^{-smt_1} \right] \left[ p_+ \tilde{P}(s, m + 1) + p_- \tilde{P}(s, m - 1) \right]$$

which simplifies to

$$(4 + 2\lambda + ms)\tilde{P}(s, m) = 2\tilde{P}(s, m + 1) + (2 + 2\lambda)\tilde{P}(s, m - 1)$$

Write $\tilde{P}(sm) = (1 + \lambda)^{-s/2} Q(s, m)$. Then, $Q(s, m)$ satisfies the equation of the form

$$\left( \frac{4 + 2\lambda + ms}{\sqrt{1 + \lambda}} - 2 \right) Q(s, m) = [Q(s, m + 1) + Q(s, m - 1) - 2Q(s, m)]$$

which is a discrete variant of the Airy equation. In the continuum limit of small $\lambda$, it reduces to the Airy equation and the scaling form of the fragment distribution is the same as studied in [34].

The experimental/simulation data for the distribution of masses of fragments in [17] is fitted to either a simple power law form $m^{-5/3}$ or to a power-law with an exponential cut-off for large sizes of the form $m^{-1.2} \exp(-am^{1/2})$. We see that our theoretical prediction $m^{-4/3} \exp(-b m^{1/2})$ is quite consistent with these results.
5. Ultra-locality of the correlations between fragments

This model has the remarkable property in that two fragments starting at any two sites \((x_1, t_1)\) and \((x_2, t_2)\) are completely uncorrelated unless they are actually touching each other. This is true not only for their area but also their shapes. This lack of correlations between fragments is very surprising in view of the fact that the condition that the arrangement of fragments has to tile the plane imposes very strong geometrical constraints on their shapes. The probability that a fragment starting at \((x_1, t_1)\) will have any given shape \(S\) depends only on \(S\) and is completely independent of the state of other sites at time \(t_1\). If there is another crack that comes near this crack, then at best it can merge with this fragment’s boundary, but this will not influence its subsequent motion.

We have hitherto considered all of the cracks moving into the unfragmented region with unit speed. This assumption is not really necessary. At the start of the simulation, we can assign to each site a random variable by taking one of three possible possible values, which tells us what would happen if a crack reaches that site: that is, will it split into two? Or, will it go left unsplit? Or, will it go right unsplit? We can then start with any configuration of cracks at the boundary on the left, and evolve cracks in any order. The final configuration is independent of the order in which the cracks are updated and the waiting time between these updates.

In particular, we may consider a geometry where a glacier fills the upper half of the plane in figure 2, with the lower half being the sea. Cracks can be initiated only at the boundary of sea and ice with some probability, and once formed, the propagation is along the horizontal direction, as before. If the crack returns to the boundary, the fragment formed breaks off into the sea, and some new boundary is exposed. These boundary conditions better reflect the formation of cracks in calving termini. It is easy to see that this does not change distribution of fragment sizes.

The independence of probability distributions of shapes and sizes of non-adjacent fragments is a kind of ‘ultra-locality’, that is also seen in percolation clusters in the standard bond/site percolation models. The present model may also be of this type, where the shapes of two percolation clusters are perfectly uncorrelated unless they have a common boundary. The present model may also be seen as a model of this type in that the shape of a cluster depends only on the values of random variables within or at the boundary of the cluster. This property is much less obvious if we look at the transfer matrix of the model. Consider the configuration of a particular column \(t = t_0\). We assume that a crack-splitting occurs at a point \((x_0, t_0)\). The probability that the generated fragment with \((x_0, t_0)\) as its left-most point has area \(A\), denoted by \(\text{Prob}(A, \lambda)\) above, would in general be expressible in terms of the eigenvalues and eigenvectors of the transfer matrix, and the matrix elements of some projection operators between eigenvectors. That this probability does not depend on the starting vector, so long as a splitting occurs at \((x_0, t_0)\), requires the transfer matrix to have a very special structure. For example, the transfer matrix \(\mathcal{W}\) can be written as

\[
\mathcal{W} = \mathbb{I}_1 \mathbb{I}_2
\]

where \(\mathbb{I}_1\) and \(\mathbb{I}_2\) are the matrices corresponding to evolution at odd and even times. But \(\mathbb{I}_2\) can be written as a direct product of \(L\ \times \times 4\) matrices, where each matrix gives the transition probabilities from the state of a pair of adjacent sites \(\sigma(2j - 1, t), \sigma(2j, t)\) to the state \(\sigma(2j - 1, t + 1), \sigma(2j, t + 1)\). Each of these is the configuration basis, as is easily seen to be of the form
where $a = (1 - \lambda)/2$, $b = \lambda$. Clearly, the rank of $T_2$, and hence of $W$, is at most $2^L$. Out of the $4^L$ eigenvalues of $L$, at least $4^L - 2^L$ are exactly zero. This occurs when the rate of coagulation ($A A \to 0 A$) is the same as diffusion rate ($0 A \to A 0$), which corresponds to the case $\Delta' = 0$ studied in [25]. The large number of zero eigenvalues provides the mathematical mechanism underlying the ultra-locality in this model.

**Acknowledgments**

I thank S N Majumdar for our discussions and for referring me to [21]. X Viennot for pointing out the work of Delest and Fedou to me and D Vallot for correspondence and help in getting copies of several papers on glaciers. This work is supported in part by the Department of Science and Technology, Government of India via grant DST-SR/S2/JCB-24/2005.

**References**

[1] Grady D E and Kipp M E 1985 *J. Appl. Phys.* **58** 1210
[2] Turcotte D L 1986 *J. Geophys. Res.* **91** 1921
[3] Herrmann H J and Roux S (ed) 1990 *Statistical Models for the Fracture of Disordered Media* (Amsterdam: North Holland)
[4] Astrom J A 2006 *Adv. Phys.* **55** 247
[5] Mott N F and Linfoot E H 1943 Ministry of Supply AC3348
[6] Gilvarry J J 1961 *J. Appl. Phys.* **32** 391
[7] Gilvarry J J and Bergstrom B H 1961 *J. Appl. Phys.* **32** 400
[8] Redner S 1990 *Statistical Models for the Fracture of Disordered Media* (Amsterdam: North Holland) pp 321–48
[9] de Arcangis L and Redner S 1985 *J. Physique Lett.* **46** L585
[10] Nakula S, Nakula P K V V, Simunovic S and Guess F 2006 *Phys. Rev.* E **73** 036109
[11] Anderson J V and Lewis L J 1998 *Phys. Rev.* E **57** R1211
[12] Meakin P 1990 *Statistical Models for the Fracture of Disordered Media* (Amsterdam: North Holland) pp 291–320
[13] Termonia Y, Meakin P and Smith P 1985 *Macromolecules* **18** 2246
[14] Timar G, Kun F, Carmona H A and Herrmann H J 2012 *Phys. Rev.* E **86** 016113
[15] Astrom J A, Holian B L and Timonen J 2000 *Phys. Rev. Lett.* **84** 5061
[16] Perfect E 1997 *Engineering Geol.* **48** 185
[17] Astrom J A et al 2014 *Nat. Geosci.* **7** 874–78
[18] Kekalainen P P, Astrom J A and Timonen J 2007 *Phys. Rev.* E **76** 026112
[19] Astrom J A 2013 *Cryosphere* **7** 1591–602
[20] Chapius A and Tetzlaff T 2014 *J. Glaciology* **60** 622
[21] Ben-Avraham D, Burschka M A and Doering C R 1990 *J. Stat. Phys.* **60** 695
[22] Doering C R and Ben-Avraham D 1988 *Phys. Rev. A* **38** 3035
[23] Alcaraz F, Droz M, Henkel M and Rittenberg V 1994 *Ann. Phys.* **230** 250
[24] Peschel I, Rittenberg V and Schultze U 1994 *Nucl. Phys.* **430** 633
[25] Krebs K, Pfannmueller M, Wehefritz B and Hinrichsen H 1995 *J. Stat. Phys.* **7** 1425
[26] Hinrichsen H 2010 *Adv. Phys.* **49** 815
[27] Inaoka H and Takayasu H 1996 *Physica A* **229** 5
[28] Dhar D and Ramaswamy R 1989 *Phys. Rev. Lett.* **63** 1659
[29] Dhar D 2006 *Physica A* **369** 29
[30] Takayasu H 1989 *Phys. Rev. Lett.* **63** 2563
[31] Delest M P and Fedou J M 1993 *Discrete Math.* **112** 65
[32] Prellberg T 1995 *J. Phys. A: Math. Gen.* **28** 1289
[33] Prellberg T and Brak R 1995 *J. Stat. Phys.* **78** 701
[34] Kearney M J and Majumdar S N 2005 *J. Phys. A: Math. Gen.* **38** 4097
[35] Kearney M J, Majumdar S N and Martin R J 2007 *J. Phys. A: Math. Gen.* **40** F863