Analysis on Symmetric and Locally Symmetric Spaces
(Multiplicities, Cohomology and Zeta functions)

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1 Multiplicities of principal series

Let us first fix some notation. Let $G$ be a semisimple Lie group with finite center and $G \cong KAN$ be the Iwasawa decomposition of $G$, where $K$ is a maximal compact subgroup, $A$ a maximal $\mathbb{R}$-split torus of $G$, and $N$ is a nilpotent subgroup. Correspondingly, we have a decomposition of the Lie algebra:

$$g \cong \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$  

We denote by $P := MAN$ a minimal parabolic subgroup, where $M = Z_K(A)$ is the centraliser of $K$ in $A$. For a finite dimensional representation $(\sigma,V_\sigma)$ of $M$, we define

$$\sigma^\lambda : P \rightarrow GL(V_\sigma)$$

by $m \mapsto a^{\rho - \lambda} \sigma(m)$. 

**Definition 1** The induced representation $\pi^{\sigma,\lambda} := \text{Ind}_{G}^{P}(\sigma,\lambda)$ is called the principal series representation with parameter $(\sigma,\lambda)$. 

$\pi^{\sigma,\lambda}$ acts on the space of sections of regularity $? \in K_{\text{fin}},\omega,\infty,L^{2},-\infty$ or $-\omega$. To be honest, $C^{K_{\text{fin}},\omega}(G/P,V(\sigma,\lambda))$ is only a $(\mathfrak{g},K)$-module. Denote the $C$-linear dual of $\sigma$ by $\sigma^*$. To define the hyperfunction and distribution valued sections, observe that $V(1,-\rho) \rightarrow G/P$ is the bundle of densities, and we have a $G$-equivariant pairing $V(\sigma,\lambda) \otimes V(\sigma^*,-\lambda) \rightarrow V(1,-\rho)$. Combining this pairing with integration, we have a pairing of spaces of sections.

**Definition 2**

$$C^{-\omega}(G/P,V(\sigma,\lambda)) := C^{\omega}(G/P,V(\sigma^*,\lambda))^*$$
$$C^{-\infty}(G/P,V(\sigma,\lambda)) := C^{\infty}(G/P,V(\sigma^*,\lambda))^*.$$  

$C^{t,\omega}(G/P,V(\sigma,\lambda))$ is a $G$-Banach representation. If $\Re(\lambda) = 0$, it is unitary in a natural way.
Example We consider $G = SL(2, \mathbb{R})$. The Iwasawa components are

$$K = \left\{ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \mid \phi \in [0, 2\pi) \right\}$$

$$A = \left\{ \begin{pmatrix} t^{\frac{1}{2}} & 0 \\ 0 & t^{-\frac{1}{2}} \end{pmatrix} \mid t \in (0, \infty) \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

In this case, we have $a \cong \mathbb{R}H$, where $H = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$. Calculating $\rho(H) = \frac{1}{2}$, we obtain that $\rho \cong \frac{1}{2}$. Since $M \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \cong \mathbb{Z}/2\mathbb{Z}$, $\hat{M} = \{1, \theta\}$ consists of two elements. Furthermore, $G/P = K/M \cong S^1$. Choosing $\sigma = 1$, the principal series representations are parametrized by $\lambda \in \mathbb{C}$. For generic $\lambda$, the representation is irreducible.

$D^-$ is called the holomorphic/antiholomorphic discrete series, and $F_{2n+1}$ is the $2n+1$-dimensional irreducible representation of $SL(2, \mathbb{R})$. The unitary principal series representations, the complementary series representations, $D^+_{2n+1}$ and $F_{2n+1}$ are unitary. Note that $\pi^{1+\pm}$ is a non-trivial extension and therefore not unitary.
Let $\Gamma \subset G$ be a cocompact, torsion-free discrete subgroup, $X := G/K,$ 
$Y := \Gamma \backslash X$ and $\Gamma V^{-\omega}_{\pi^{\lambda}} := \{ \phi \in V^{-\omega}_{\pi^{\lambda}} | \pi^{\lambda}(\gamma) \phi = \phi \ \forall \ \gamma \in \Gamma \}$, the space of $\Gamma$-invariant hyperfunction vectors.

Lemma 1 (Frobenius reciprocity)

$$\Gamma V^{-\omega}_{\pi^{\lambda}} \cong \text{Hom}_{G}(V^{-\omega}_{\pi^{\lambda}}, C^\infty(\Gamma \backslash G))$$

$$\phi \mapsto (f \mapsto (g \mapsto \phi(\pi^{\lambda}(g)f)))$$

Since $\Gamma$ is cocompact, we have a decomposition

$$C^\infty(\Gamma \backslash G) \cong \bigoplus_{(\pi, V)} m_{\Gamma}(\pi)V^\infty_{\pi},$$

where $\hat{G}_\pi$ is the unitary dual of $G$, i.e. the set of equivalence classes of irreducible unitary representations of $G$. As a consequence, if $\pi^{\lambda}$ is unitary and $\lambda \neq 0$, then it is irreducible and therefore $m_{\Gamma}(\pi^{\lambda}) = \dim(\Gamma V^{-\omega}_{\pi^{\lambda}})$. Moreover, if $\pi^{\lambda}$ is irreducible and not unitary, then $\Gamma V^{-\omega}_{\pi^{\lambda}} = \{ 0 \}$.

We now return to our example and connect $\Gamma V^{-\omega}_{\pi^{\lambda}}$ with the eigenvalues of the Laplacian on $Y$. As $K$-homogenous bundles, $V(\sigma, \lambda) = G \times_{\sigma} V_{\sigma} = K \times_{M} V_{\sigma}$, and for $\sigma = 1$, $K \times_{M} V_{1} \cong S^1 \times \mathbb{C}$. Denote by $f \in C^\infty(K \times_{M} V_{1})$ the unique $K$-invariant section with $f(1) = 1$. For any choice of $\lambda$, this corresponds to a real-analytic section $f_\lambda$ of $V(1, \lambda)$.

Using the reciprocity homomorphism $i : V^{-\omega}_{\pi^{\lambda}} \rightarrow \text{Hom}_{G}(V^{-\omega}_{\pi^{\lambda}}, C^\infty(G))$ for trivial $\Gamma$, we define the Poisson transformation

$$i.(f_\lambda) : V^{-\omega}_{\pi^{\lambda}} \rightarrow C^\infty(G).$$

Now, any $\phi \in V^{-\omega}_{\pi^{\lambda}}$ defines $i_{\phi}(f_\lambda) \in C^\infty(X)$. The Casimir-operator $\Omega$ of $\mathfrak{g}$ acts on $\pi^{\lambda}$ by a scalar, $\pi^{\lambda}(\Omega) = \mu(\lambda) := \frac{1}{4} - \lambda^2$, while $\Omega|_{C^\infty(X)} = \Delta_X$, the Laplace operator. Therefore, $i_{\phi}(f_\lambda) \in \text{Ker}(\Delta_X - \mu(\lambda))$.

Theorem 1 (Helgasson [9]) For $\lambda \notin \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, ...\}$, the Poisson transformation

$$i.(f_\lambda) : V^{-\omega}_{\pi^{\lambda}} \rightarrow \text{Ker}(\Delta_X - \mu(\lambda))$$

is an isomorphism.

To calculate $H^*(\Gamma, V^{-\omega}_{\pi^{\lambda}})$, we need the following classical theorems.

Theorem 2 Let $M$ be a $C^\omega$-manifold, $E, F \rightarrow M$ $C^\omega$-vector bundles and $A : C^\infty(M, E) \rightarrow C^\infty(M, F)$ an elliptic operator with $C^\omega$-coefficients. If $M$ is non-compact, then $A$ is surjective.

Theorem 3 ([1], Lemma 2.4, 2.6) Let a discrete group $U$ act properly on a space $M$ and let $F$ be a soft or flabby $U$-equivariant sheaf on $M$. Then $H^i(U, F(M)) = 0$ for all $i \geq 1$. 

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Combining Theorem 1 and Theorem 2, we get a short exact sequence for \( \lambda \notin \{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots\} \)

\[
0 \to V_{\pi_1 \lambda}^{-\omega} i(f_{\lambda}) C^\infty(X) \xrightarrow{\Delta_Y - \mu(\lambda)} C^\infty(X) \to 0.
\]

By Theorem 3, this is a \( \Gamma \)-acyclic resolution of \( V_{\pi_1 \lambda}^{-\omega} \), so for \( \lambda \notin \frac{1}{2} - N_0 \), we get

\[
\begin{array}{c|c|c}
1 & H^i(\Gamma, V_{\pi_1 \lambda}^{-\omega}) & 0 \\
0 & \ker(\Delta_Y - \mu(\lambda)) & \text{dim}(H^i(\Gamma, F_{2n+1})) \\
1 & \text{coker}(\Delta_Y - \mu(\lambda)) & \text{dim}(H^i(\Gamma, V_{\pi_1 \lambda}^{-\omega})) \\
\end{array}
\]

Since \((\Delta_Y - \mu(\lambda))\) is an elliptic operator, both spaces are finite dimensional. To compute \( H^i(\Gamma, V_{\pi_1 \lambda}^{-\omega}) \) for \( \lambda \in -\frac{1}{2} - N_0 \), we need \( H^i(\Gamma, V_{\pi_1 \lambda}^{-\omega}) \) for \( i \geq 1 \), \( \lambda \in \frac{1}{2} + N \) as well.

In the case of \( \lambda \in \frac{1}{2} + N \), we have the exact sequence

\[
0 \to F_{2n+1} \to V_{\pi_1 \lambda}^{-\omega} \to D_{2n+1}^{-\omega} \to 0
\]

Since in this case \( \mu(\lambda) = \frac{1}{4} - \lambda^2 \) is negative, \( \Delta_Y - \mu(\lambda) \) is a positive operator on a compact manifold and therefore injective, so \( V_{\pi_1 \lambda}^{-\omega} \) is trivial. Using the long exact sequence

\[
\ldots \to H^i(\Gamma, F_{2n+1}) \to H^i(\Gamma, V_{\pi_1 \lambda}^{-\omega}) \to H^i(\Gamma, D_{2n+1}^{-\omega}) \to \ldots
\]

and the topological result

\[
\begin{array}{c|c|c}
1 & \text{dim}(H^i(\Gamma, F_{2n+1})) & \text{dim}(H^i(\Gamma, F_1)) \\
0 & 0 & 1 \\
1 & (2g-2)(2n+1) & 2g \\
2 & 0 & 1 \\
\end{array}
\]

where \( g \) denotes the genus of \( Y \), we get

\[
\begin{array}{c|c|c|c}
1 & \text{dim}(H^i(\Gamma, F_{2n+1})) & \text{dim}(H^i(\Gamma, V_{\pi_1 \lambda}^{-\omega})) & \text{dim}(H^i(\Gamma, D_{2n+1}^{-\omega})) \\
0 & 0 & 0 & (2g-2)(2n+1) \\
1 & (2g-2)(2n+1) & 0 & 0 \\
2 & 0 & 0 & 0 \\
\end{array}
\]

For \( \lambda = 1/2 \), we have that \( \text{dim}(\ker(\Delta_Y)) = \text{dim}(\text{coker}(\Delta_Y)) = 1 \), so we get

\[
\begin{array}{c|c|c|c}
1 & \text{dim}(H^i(\Gamma, F_1)) & \text{dim}(H^i(\Gamma, V_{\pi_1 1/2}^{-\omega})) & \text{dim}(H^i(\Gamma, D_{1/2}^{-\omega})) \\
0 & 1 & 1 & 2g \\
1 & 2g & 1 & 0 \\
2 & 1 & 0 & 0 \\
\end{array}
\]
Here, the long exact sequence gave us that $\dim(H^1(\Gamma, D_1^{\omega}))$ is either 1 or 2 and since $D_1^{\omega}$ is the sum of two conjugate isomorphic $G$ submodules, it has to be even. For $\lambda \in -1/2 - \mathbb{N}_0$, we have a long exact sequence

$$\ldots \to H^i(\Gamma, D_{2n+1}^{\omega}) \to H^i(\Gamma, V_{\pi,1}^{\omega}) \to H^i(\Gamma, F_{2n+1}) \to \ldots$$

Using the result for positive $\lambda$, we obtain immediately for $\lambda \in -\frac{1}{2} - \mathbb{N}$

$$\begin{array}{|c|c|c|c|}
\hline
1 & \dim(H^i(\Gamma, D_{2n+1}^{\omega})) & \dim(H^i(\Gamma, V_{\pi,1}^{\omega})) & \dim(H^i(\Gamma, F_{2n+1})) \\
\hline
0 & (2n+1)(2g-2) & (2n+1)(2g-2) & 0 \\
1 & 0 & (2n+1)(2g-2) & (2n+1)(2g-2) \\
2 & 0 & 0 & 0 \\
\hline
\end{array}$$

In the case of $\lambda = -1/2$, it is

$$\begin{array}{|c|c|c|}
\hline
1 & \dim(H^i(\Gamma, D_{1}^{\omega})) & \dim(H^i(\Gamma, V_{\pi,1}^{\omega})) \\
\hline
0 & 2g & 2g \\
1 & 2 & 2g+1 \\
2 & 0 & 1 \\
\hline
\end{array}$$

Everything except $\dim(H^0(\Gamma, V_{\pi,1}^{\omega}))$ is determined by the long exact sequence. For $\dim(H^0(\Gamma, V_{\pi,1}^{\omega}))$, we could have $2g$ or $2g + 1$, but all invariants are in the submodule $D_{1}^{\omega}$ because otherwise we would get an embedding of $V_{\pi,1}^{\omega}$ into $L^2(\Gamma \backslash G)$ which is impossible, since $V_{\pi,1}^{\omega}$ is not unitary.

In all these cases, the cohomology groups are finite dimensional and $\chi(\Gamma, V_{\pi,1}^{\omega}) = 0$.

Let us now consider the general case, so let $G_C$ be a connected reductive group over $\mathbb{C}$, $G_R$ a real form, $K \subset G_R$ maximal compact, and $\Gamma \subset G_R$ be cocompact, torsion-free and discrete. Denote by $\text{Mod}(\mathfrak{g}, K)$ the category of $(\mathfrak{g}, K)$ modules and by $\mathcal{HC}(\mathfrak{g}, K) \subset \text{Mod}(\mathfrak{g}, K)$ the subcategory of Harish-Chandra modules. Recall that a module is called Harish-Chandra if it is finitely generated over the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and admissible, i.e. $\forall \gamma \in K, \dim(V(\gamma)) < \infty$, where $V(\gamma)$ is the $\gamma$-isotypical component of $V$. For a $G$ module $V$ denote the $K$-finite submodule by $V_{K-fin} = \{v \in V|\dim(\text{span}(K)v)) < \infty\}$. For $V \in \mathcal{HC}(\mathfrak{g}, K)$ denote by $\hat{V} := \text{Hom}(V, \mathbb{C})_{K-fin}$ the dual in $\mathcal{HC}(\mathfrak{g}, K)$.

**Definition 3** For a Harish-Chandra module $V$, the maximal globalisation of $V$ is $MG(V) := \text{Hom}_{\mathfrak{g}, K}(\hat{V}, C^\infty(G)_{K-fin})$.

Thus, the maximal globalisation $MG(V)$ is a particular continuous representation of $G$ whose $K$-finite part is isomorphic to $V$.

For a Harish-Chandra module $V$, define the hyperfunction vectors of $V$ to be $V^{-\omega} := (\hat{V})^*$, the topological dual of the analytic vectors in any Banach globalisation of the dual representation.
Theorem 4 (Schmid [15]) $MG(V) \cong V^{-\infty}$.

Theorem 5 (Bunke/Olbrich [2], 1.4) For $V \in \mathcal{H}(g, K)$,

$$H^*(\Gamma, MG(V)) \cong \bigoplus_{(\pi,V_\pi) \in \hat{G}_u} m_{\Gamma}(\pi) Ext^*_g(V_\pi, K_{-fin}).$$

For the proof of this theorem, we need the generalisation of Theorem 3

$$H^i(\Gamma, C^\infty(G)_{K_{-fin}}) = 0 \quad \forall i \geq 1$$

and the following generalisation of Theorem 1.

Theorem 6 (Kashiwara-Schmidt [12])

$$Ext^i_{(g,K)}(\tilde{V}, C^\infty(G)_{K_{-fin}}) = \begin{cases} MG(V), & i = 0 \\ 0, & i > 0 \end{cases}$$

Now, let us chose a projective resolution $P \rightarrow \tilde{V} \rightarrow 0$ of $\tilde{V}$ in $Mod(g, K)$. By Theorem 6 $Hom_{(g,K)}(P, C^\infty(G)_{K_{-fin}})$ resolves $MG(V)$ $\Gamma$-acyclically, therefore $Hom_{(g,K)}(P, C^\infty(\Gamma\backslash G)_{K_{-fin}})$ calculates $H^*(\Gamma, MG(V))$. Finally, an analytic argument ([2], Lemma 3.1) shows that only a finite part of the decomposition

$$C^\infty(\Gamma\backslash G)_{K_{-fin}} = \bigoplus_{(\pi,V_\pi) \in \hat{G}_u} m_{\Gamma}(\pi)V^\infty_{\pi,K_{-fin}}$$

contributes to $Hom_{(g,K)}(P, C^\infty(\Gamma\backslash G)_{K_{-fin}})$.

2 The Selberg Zeta function

In this section we will additionally assume that the real rank of $G$ is equal to one. A group element $g \in G$ is called hyperbolic if there exist $m_g \in M$, $a_g \in A$ such that $g$ is conjugate in $G$ to $m_g a_g$ with $a_g^T > 1$. If $g$ is hyperbolic, $a_g$ is unique, and $m_g$ is unique up to conjugation in $M$. Note that if $\gamma \in \Gamma$, $\gamma \neq 1$, then $\gamma$ is hyperbolic.

For a hyperbolic $g \in G$, we define

$$Z(g, \sigma, \lambda) := \prod_{k=0}^{\infty} det(1 - \sigma_{\lambda-2\rho}(m_g a_g) \otimes S^k(Ad(m_g a_g)|\pi)),$$

where $\sigma_{\lambda-2\rho}$ denotes the representation of $P$ as in section 1 and $S^k$ is the $k$'th symmetric power.

In the example of $G = SL(2, \mathbb{R})$, a hyperbolic $g$ is conjugate to $m_g \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix}$, $t > 1$. Therefore, we get

$$Z(g, 1, \lambda) = \prod_{k=0}^{\infty} (1 - t^{-\lambda - k - 1/2}).$$
Definition 4  The conjugacy class \([γ] \in \Gamma\) is called primitive, if \([γ]\) is not of the form \([γ] = [γ′]^n\) for some \(γ′ \in \Gamma\), \(n > 1\).

Definition 5  The Selberg Zeta function is defined for \(\Re(λ) > ρ\) by the converging product

\[
Z(\Gamma, σ, λ) = \prod_{[γ] \in CΓ \text{primitive}} Z(γ, σ, λ).
\]

Theorem 7 (Fried [8])  \(Z(Γ, σ, λ)\) has a meromorphic continuation to \(a^*_c\).

The proof uses Ruelles thermodynamic formalism. The disadvantage of this method of proof is that it doesn’t give any information about the singularities of the continuation or about a functional equation.

We use the Selberg trace formula to calculate the singularities of the Zeta function and to determine the functional equation. To this end, we introduce the logarithmic derivative

\[
L(Γ, σ, λ) := \frac{Z′(Γ, σ, λ)}{Z(Γ, σ, λ)},
\]

considered as a 1-form on \(a^*_c\). Denote by \(X^d\) the dual of the symmetric space \(X\), i.e.

\[
\begin{array}{cccc}
X & H^m & \mathbb{C}H^m & \mathbb{H}H^m & CaH^2 \\
X^d & S^n & \mathbb{C}P^n & \mathbb{H}P^n & CaP^2
\end{array}
\]

In the case of \(n := \dim X\) odd, we assume that either \(σ\) is irreducible and the isomorphism class of \(σ\) is fixed under the action of the Weyl group, or \(σ = σ′ ⊕ σ^w\) with \(w\) the non-trivial element of the Weyl group \(W(\mathfrak{g}, a) \cong \mathbb{Z}/2\mathbb{Z}\) and \(σ′\) irreducible.

The positive root system of \((\mathfrak{g}, a)\) is either of the form \(\{α\}\) or \(\{\frac{α}{2}, α\}\). We call α the long root and identify \(a^*_c \cong \mathbb{C}\) sending \(α\) to 1. Define the root vector \(H_α \in a\) corresponding to a positive root \(α\) by the condition that

\[
λ(H_α) = \frac{⟨λ, α⟩}{⟨α, α⟩} \quad \forall λ \in a^*,
\]

where \(⟨.,.⟩\) denotes an \(Ad(G)\)-invariant inner product on \(\mathfrak{g}\). For \(σ \in \hat{M}\), we define \(ε_σ(σ) \in \{0, \frac{1}{2}\}\) by the condition \(e^{2πir_α(σ)} = σ(exp(2πiH_α))\) and \(ε_σ \in \{0, \frac{1}{2}\}\) by requiring \(ε_σ \equiv |ρ| + ε_α(σ) \mod \mathbb{Z}\).
Theorem 8 (Bunke/Olbrich [6]) There exists a virtual elliptic operator $A_X(\sigma)$ and a corresponding operator $A_{X^d}(\sigma)$, such that in the case of $n$ even

$$L(\Gamma, \sigma, \lambda) = \frac{Tr'((\lambda^2 + A_Y(\sigma))^{-1} - \frac{(-1)^{\frac{n}{2}} vol(Y)\pi P_\sigma(\lambda)}{vol(X^d)\lambda}}{2\lambda} \left\{ \begin{array}{ll}
tan(\pi \lambda) & \text{if } \epsilon_\sigma = \frac{1}{2} \\
-cot(\pi \lambda) & \text{if } \epsilon_\sigma = 0 \end{array} \right. (1)$$

$$- \frac{(-1)^{\frac{n}{2}} vol(Y)}{vol(X^d)} Tr'((\lambda^2 - A_{X^d}(\sigma))^{-1} (2)$$

and in the case of $n$ odd,

$$L(\Gamma, \sigma, \lambda) = \frac{Tr'((\lambda^2 + A_Y(\sigma))^{-1} - \frac{(-1)^{\frac{n+1}{2}} vol(Y)\pi P_\sigma(\lambda)}{vol(X^d)\lambda}}{2\lambda}$$

In the theorem, $P_\sigma$ is some polynomial depending on $\sigma$. Since $(\lambda^2 + A_Y^2(\sigma))^{-1}$ is not trace-class, we have to use a regularised trace $'Tr'$. We refer to [6], 3.2. for a discussion of the regularisation.

From this formula, it follows that $L(\Gamma, \sigma, \lambda)$ is meromorphic. In the case of $n$ even, (2) is odd and (3) is even. The poles of the two terms cancel for $\lambda > 0$, add up for $\lambda < 0$ and $I := 2\lambda((2 + 3))$ is regular at zero ([6], 3.2.3). Hence, the set of poles of $I$ is $\epsilon_\sigma - N$.

For $A$ an operator on some Hilbert space, $'Tr'(\lambda - A)^{-1}$ has first order poles at the eigenvalues of $A$ with $res (2'Tr'(\lambda^2 - A)^{-1})$ equal to the dimension of the corresponding eigenspace, hence

$$res_{-\epsilon_\sigma - k}I = -\frac{(-1)^{\frac{n}{2}} 2vol(Y)dimE_{A_X^d}(\sigma)(-\epsilon_\sigma - k)^2}{vol(X^d)}.$$  

It follows from Hirzebruch proportionality that $\frac{vol(Y)}{vol(X^d)} \in \mathbb{Z}$ ([6],Prop. 3.14), so the residues of $I$ are integral.

(1) has first order poles at $\pm is$ with residue $dimE_{A_Y(\sigma)}(s^2)$, where $s^2$ is a non-zero eigenvalue of $A_Y(\sigma)$ and if zero is an eigenvalue of $A_Y(\sigma)$, there is an additional pole at zero with residue $2dimkerA_Y(\sigma)$.

In the case of $n$ odd, the only poles are those coming from the first term. Note that all the residues of $L$ are integral.
We’ll now come back to the example of \( G = SL(2, \mathbb{R}) \), \( \sigma = 1 \). The above theorem specializes to

\[
\frac{L(\Gamma, \sigma, \lambda)}{2\lambda} := Tr'((\Delta_Y - \frac{1}{4} + \lambda^2)^{-1}) - \frac{2g - 2}{2}Tr'((\Delta_{S^2} + \frac{1}{4} - \lambda^2)^{-1}) + \frac{2g - 2 - 2\pi \tan(\lambda\pi)}{\lambda},
\]

where \( g \) is the genus of \( \Gamma \setminus X \). The eigenvalues of \( \Delta_{S^2} \) are \( \frac{n^2}{4} \), \( n \in \mathbb{N} \) with \( \text{mult}(\frac{n^2}{4}) = 2n + 1 \). Using this, we get the following picture for \( \text{ord}(Z(\Gamma, 1, \lambda)) \).
Theorem 9 (Bunke/Olbrich, conjectured by Patterson [1])

For $\lambda \neq 0$,

$$\text{ord}_\lambda Z(\Gamma, \sigma, \lambda) = -\mathcal{X}'(\Gamma, V_{\pi^\lambda}^{-\omega})$$

where $\mathcal{X}'(\Gamma, V_{\pi^\lambda}^{-\omega}) = \sum_{p=0}^{\infty} (-1)^p \dim H^p(\Gamma, V_{\pi^\lambda}^{-\omega})$.

Proof: In the example, the proof is just by inspection. In the general case, use Theorem 5 to obtain

$$H^p(\Gamma, V_{\pi^\lambda}^{-\omega}) = \bigoplus_{(\pi, V_{\pi}) \in \hat{G}u} m(\pi) \text{Ext}^p_{\mathfrak{g}, K}(V_{\pi^\lambda}^{-\omega}, V_{\pi^\lambda}^{-\omega})$$

$$= \bigoplus_{(\pi, V_{\pi}) \in \hat{G}u} m(\pi) \text{Hom}_{MA}(H^p(n, V_{\pi^\lambda}^{-\omega}) \otimes H_{\pi^n}(n, V_{\pi^\lambda}^{-\omega}), V_{\pi^\lambda})$$

$$= \bigoplus_{(\pi, V_{\pi}) \in \hat{G}u} m(\pi) \text{Hom}_{MA}((H^{n-p}(n, \pi^\lambda) \otimes H^{n-p}(n, \pi^\lambda)), V_{\pi^\lambda})$$

$$= \bigoplus_{(\pi, V_{\pi}) \in \hat{G}u} m(\pi) \left( (H^n(n, \pi^\lambda) \otimes H^{n-p}(n, \pi^\lambda)) \otimes V_{\pi^\lambda} \right)$$

Setting

$$\mathcal{X}(\pi, \sigma, \lambda) := \sum_{p=0}^{\infty} (-1)^p \dim (H^p(n, V_{\pi^\lambda}^{-\omega}) \otimes V_{\pi^\lambda}^{-\omega})_{MA},$$

the following theorem finishes the proof.

Theorem 10 (Juhl, [10], [11])

$$\text{ord}_\mu Z(\Gamma, \sigma, \lambda) = (-1)^{n-1} \sum_{(\pi, V_{\pi}) \in \hat{G}u} m(\pi) \mathcal{X}(\pi, \sigma, \mu).$$

3 Generalisations

There are two obvious directions in which the theory could be generalised. First, one could try to generalise the group $G$. Unfortunately, we do not know of a satisfactory general definition of a Zeta function for groups of higher rank, but see [13], [14], [7] for special cases.

The other direction is to weaken the conditions on $\Gamma$. In [3] and [4], the theory is generalised to the case of subgroups $\Gamma \subset G$ of finite covolume. Examples of infinite covolume (convex co-compact) are considered in [5].

Footnote: For $\lambda = 0$ see [1].
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