SAYD modules over Lie-Hopf algebras

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Abstract

In this paper a general van Est type isomorphism is established. The isomorphism is between the Lie algebra cohomology of a bicrossed sum Lie algebra and the Hopf cyclic cohomology of its Hopf algebra. We first prove a one to one correspondence between stable-anti-Yetter-Drinfeld (SAYD) modules over the total Lie algebra and SAYD modules over the associated Hopf algebra. In contrast to the non-general case done in our previous work, here the van Est isomorphism is found at the first level of a natural spectral sequence, rather than at the level of complexes. It is proved that the Connes-Moscovici Hopf algebras do not admit any finite dimensional SAYD modules except the unique one-dimensional one found by Connes-Moscovici in 1998. This is done by extending our techniques to work with the infinite dimensional Lie algebra of formal vector fields. At the end, the one to one correspondence is applied to construct a highly nontrivial four dimensional SAYD module over the Schwarzian Hopf algebra. We then illustrate the whole theory on this example. Finally explicit representative cocycles of the cohomology classes for this example are calculated.

1 Introduction

Hopf cyclic cohomology was invented by Connes-Moscovici in 1998 [2]. It is now beyond dispute that this cohomology is a fundamental tool in noncommutative geometry. Admitting coefficients is one of the most significant properties of this theory [7, 6, 10]. These coefficients are called stable-anti-Yetter-Drinfeld (SAYD) modules [7].

A “geometric” Hopf algebra is a Hopf algebra associated to (Lie) algebraic group or Lie algebra via certain functors. Such Hopf algebras are defined as representative (smooth) polynomial functions on the object in question or as the universal

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enveloping algebras of the Lie algebra or even as a bicrossed product of such Hopf algebras. The latter procedure is called semi-dualization. The resulting Hopf algebra via semi-dualization is usually neither commutative nor cocommutative [11].

The study of SAYD modules over “geometric” Hopf algebras begins in [15], where we proved that any representation of the Lie algebra induces a SAYD module over the associated Hopf algebra. Therefore those SAYD modules are called induced modules [15]. We also proved that the Hopf cyclic cohomology of the associated Hopf algebra is isomorphic to the Lie algebra cohomology of the Lie algebra with coefficients in the original representation.

In [16], the notion of SAYD modules over Lie algebras was defined and studied. It was observed that the corresponding cyclic complex has been known with different names for different SAYD modules. As the main example we proved that the (truncated) polynomial algebra of a Lie algebra is a SAYD module. The corresponding cyclic complex is identified with the (truncated) Weil algebra [16]. In the same paper we identify the category of SAYD modules over the enveloping algebra of a Lie algebra with those on the Lie algebra.

Let us recall the main result of [16] as follows. For an arbitrary Lie algebra \( \mathfrak{g} \), the comultiplication of \( U(\mathfrak{g}) \) does not use the Lie algebra structure of \( \mathfrak{g} \). This fact has been discouraged attention in comodules over \( U(\mathfrak{g}) \). It is shown that such comodules are in one to one correspondence with the nilpotent modules over the symmetric algebra \( S(\mathfrak{g}^*) \). Using this fundamental fact we can identify AYD modules over \( U(\mathfrak{g}) \) with modules over the semi-direct product Lie algebra \( \tilde{\mathfrak{g}} = \mathfrak{g}^* \rtimes \mathfrak{g} \). Here \( \mathfrak{g}^* = \text{Hom}(\mathfrak{g}, \mathbb{C}) \) is considered to be a commutative Lie algebra and to be acted upon by \( \mathfrak{g} \) via the coadjoint representation. We show that the notion of comodule over Lie algebras make sense. Furthermore, SAYD modules over Lie algebras and the cyclic cohomology of a Lie algebra with coefficients in such modules is defined. It is shown that SAYD modules over \( U(\mathfrak{g}) \) and over \( \mathfrak{g} \) have a one-to-one correspondence and their cyclic homologies are identified.

Let \( \mathfrak{g} = \mathfrak{g}_1 \rtimes \mathfrak{g}_2 \) be a bicrossed sum Lie algebra. Let us denote \( R(\mathfrak{g}_2) \) and \( U(\mathfrak{g}_1) \) by \( \mathcal{F} \) and \( \mathcal{U} \) respectively. Here \( R(\mathfrak{g}_2) \) is the Hopf algebra of all representative functions on \( \mathfrak{g}_2 \), and \( U(\mathfrak{g}_1) \) is the universal enveloping algebra of \( \mathfrak{g}_1 \). A module-comodule over \( \mathcal{H} := \mathcal{F} \triangleright \mathcal{U} \) is naturally a module-comodule over \( \mathcal{U} \) and comodule-module over \( \mathcal{F} \). In [15], we completely determined those module-comodule whose \( \mathcal{U} \)-coaction and \( \mathcal{F} \)-action is trivial. It is proved that such a module-comodule is induced by a module over \( \mathfrak{g} \) if and only if it is a YD module over \( \mathcal{H} \).

Continuing our study in [15, 16], we completely determine SAYD modules over the bicrossed product Hopf algebra \( \mathcal{H} = \mathcal{F} \triangleright \mathcal{U} \). Roughly speaking, we show that SAYD modules over \( \mathcal{H} \) and SAYD module over \( \mathfrak{g} \) are the same. We then take advantage of a spectral sequence in [10] to prove a van Est isomorphism between the Hopf cyclic cohomology of \( \mathcal{H} \) with coefficients in \( \sigma M_\delta = M \otimes \sigma C_\delta \) and the Lie algebra cohomology of \( \mathfrak{g} \) relative to a Levi subalgebra with coefficients in a \( \mathfrak{g} \)-module \( M \).
One of the results of this paper is about the SAYD modules over Connes-Moscovici Hopf algebras $\mathcal{H}_n$. We know that $\mathcal{H}_n$ is the bicrossed product Hopf algebra of $\mathcal{F}(N)$ and $U(gl_n)$ [13]. However, the group $N$ is not of finite type. So we cannot apply our theory freely on $\mathcal{H}_n$. We overcome this problem by carefully analyzing the SAYD modules over $\mathcal{H}_n$ to reduce the case to a finite type problem. As a result, we prove that $\mathcal{H}_n$ has no AYD module except the most natural one, $\mathbb{C}_\delta$, which was found by Connes-Moscovici in [2].

To illustrate our theory in a nontrivial example we introduce a SAYD module over the Schwarzian Hopf algebra $\mathcal{H}_{1S}$ introduced in [2]. By definition, $\mathcal{H}_{1S}$ is a quotient Hopf algebra of $\mathcal{H}_1$ by the Hopf ideal generated by $\delta_2 - \frac{1}{2}\delta_1^2$.

Here $\delta_i$ are generators of $\mathcal{F}(N)$. So the Hopf algebra $\mathcal{H}_{1S}$ is generated by

$$X, Y, \delta_1$$

As we know, $\mathcal{H}_{1S}^{\text{cop}}$ is isomorphic to $R(C) \bowtie U(gl_{1\text{aff}})$. So our theory guarantees that any suitable SAYD module $M$ over $sl_2 = gl_{1\text{aff}} \bowtie \mathbb{C}$ will produce a SAYD module $M_\delta$ over $\mathcal{H}_{1S}^{\text{cop}}$. We take the truncated polynomial algebra $M = S(sl_2)[2]$ as our candidate. The resulting 4-dimensional SAYD module $M_\delta$ is then generated by

$$1, \ R^X, \ R^Y, \ R^Z,$$

with the $\mathcal{H}_{1S}^{\text{cop}}$ action and coaction defined by

| $\triangleright$ | $\delta_1$ | $Y$ | $X$ | $\delta_1$ |
|---|---|---|---|---|
| $1$ | $0$ | $0$ | $R^Z$ | $\delta_1$ |
| $R^X$ | $-R^Y$ | $2R^X$ | $0$ | $Y$ |
| $R^Y$ | $-R^Z$ | $R^Y$ | $0$ | $X$ |
| $R^Z$ | $0$ | $0$ | $0$ | $1$ |

$$\triangleright: \ M_\delta \longrightarrow \mathcal{H}_{1S}^{\text{cop}} \otimes M_\delta$$

$$1 \mapsto 1 \otimes 1 + X \otimes R^X + Y \otimes R^Y$$

$$R^X \mapsto 1 \otimes R^X$$

$$R^Y \mapsto 1 \otimes R^Y + \delta_1 \otimes R^X$$

$$R^Z \mapsto 1 \otimes R^Z + \delta_1 \otimes R^Y + \frac{1}{2}\delta_1^2 \otimes R^X.$$ 

The surprises here are the nontriviality of the action of $\delta_1$ and the appearance of $X$ and $Y$ in the coaction. In other words this is not an induced module [15].

We illustrate our results in this paper on this example. We then apply the machinery developed in [13] by Moscovici and one of the authors to prove that the following two cocycles generates the Hopf cyclic cohomology of $\mathcal{H}_{1S}^{\text{cop}}$ with coefficients in $M_\delta$. 

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\[ c^{\text{odd}} = 1 \otimes \delta_1 + R_Y \otimes X + R_X \otimes \delta_1 X + R_Y \otimes \delta_1 Y + 2R_Z \otimes Y, \]

\[ c^{\text{even}} = 1 \otimes X \otimes Y - 1 \otimes Y \otimes X + 1 \otimes Y \otimes \delta_1 Y - R_X \otimes XY \otimes X \]
\[- R_X \otimes Y^2 \otimes \delta_1 X - R_X \otimes Y \otimes X^2 + R_Y \otimes XY \otimes Y + R_Y \otimes Y^2 \otimes \delta_1 Y \]
\[+ R_Y \otimes X \otimes Y^2 + R_Y \otimes Y \otimes \delta_1 Y^2 - R_Y \otimes Y \otimes X - R_X \otimes XY^2 \otimes \delta_1 \]
\[- \frac{1}{3} R_X \otimes Y^3 \otimes \delta_1^2 + \frac{1}{3} R_Y \otimes Y^3 \otimes \delta_1 - \frac{1}{4} R_X \otimes Y^2 \otimes \delta_1^2 - \frac{1}{2} R_Y \otimes Y^2 \otimes \delta_1. \]

As can be seen by the inspection, the expression of the above cocycles cannot be easily found with bare hands. It is the mentioned machinery in [13] which allows to arrive at this elaborate formulae.

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2 Matched pair of Lie algebras and SAYD modules over double crossed sum Lie algebras

In this section, matched pair of Lie algebras and their bicrossed sum Lie algebras are reviewed. We also recall double crossed product of Hopf algebras from [11]. Next, we provide a brief account of SAYD modules over Lie algebras from [16]. Finally we investigate the relation between SAYD modules over the double crossed sum Lie algebra of a matched pair of Lie algebras and SAYD modules over the individual Lie algebras.

2.1 Matched pair of Lie algebras and mutual pair of Hopf algebras

Let us recall the notion of matched pair of Lie algebras from [11]. A pair of Lie algebras \((\mathfrak{g}_1, \mathfrak{g}_2)\) is called a matched pair if there are linear maps

\[
\alpha : \mathfrak{g}_2 \otimes \mathfrak{g}_1 \to \mathfrak{g}_2, \quad \alpha_X(\zeta) = \zeta \rhd X, \quad \beta : \mathfrak{g}_2 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1, \quad \beta_\zeta(X) = \zeta \lhd X,
\]

satisfying the following conditions,

\[
\begin{align*}
[\zeta, \xi] \rhd X &= \zeta \rhd (\xi \rhd X) - \xi \rhd (\zeta \rhd X), \\
\zeta \lhd [X, Y] &= (\zeta \lhd X) \lhd Y - (\zeta \lhd Y) \lhd X, \\
\zeta \rhd [X, Y] &= [\zeta \rhd X, Y] + [X, \zeta \rhd Y] + (\zeta \rhd X) \rhd Y - (\zeta \rhd Y) \rhd X, \\
[\zeta, \xi] \lhd X &= [\zeta \lhd X, \xi] + [\zeta, \xi \lhd X] + \zeta \lhd (\zeta \rhd X) - \xi \lhd (\zeta \rhd X).
\end{align*}
\]

Given a matched pair of Lie algebras \((\mathfrak{g}_1, \mathfrak{g}_2)\), one defines a double crossed sum Lie algebra \(\mathfrak{g}_1 \rhd \lhd \mathfrak{g}_2\). Its underlying vector space is \(\mathfrak{g}_1 \oplus \mathfrak{g}_2\) and its Lie bracket is defined by:

\[
[X \oplus \zeta, Z \oplus \xi] = ([X, Z] + \zeta \rhd Z - \xi \lhd X) \oplus ([\zeta, \xi] + \zeta \lhd Z - \xi \lhd X).
\]

Both \(\mathfrak{g}_1\) and \(\mathfrak{g}_2\) are Lie subalgebras of \(\mathfrak{g}_1 \rhd \lhd \mathfrak{g}_2\) via obvious inclusions. Conversely, if for a Lie algebra \(\mathfrak{g}\) there are two Lie subalgebras \(\mathfrak{g}_1\) and \(\mathfrak{g}_2\) so that \(\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2\) as vector spaces, then \((\mathfrak{g}_1, \mathfrak{g}_2)\) forms a matched pair of Lie algebras and \(\mathfrak{g} \cong \mathfrak{g}_1 \rhd \lhd \mathfrak{g}_2\) as Lie algebras [11]. In this case, the actions of \(\mathfrak{g}_1\) on \(\mathfrak{g}_2\) and \(\mathfrak{g}_2\) on \(\mathfrak{g}_1\) for \(\zeta \in \mathfrak{g}_2\) and \(X \in \mathfrak{g}_1\) are uniquely determined by

\[
[\zeta, X] = \zeta \rhd X + \zeta \lhd X, \quad \zeta \in \mathfrak{g}_2, \quad X \in \mathfrak{g}_1.
\]
\(V\)-module coalgebra. We call them mutual pair if their actions satisfy the following conditions.

\[
v \triangleright (u^1 u^2) = (v^1) (v^2) (u^1 (1)) (u^2 (2)), \quad 1 \triangleleft u = \varepsilon(u), \quad (2.8)
\]

\[
(v^1 v^2) \triangleleft u = (v^1 (1) (v^2 (1) \triangleright u^1 (1))) (v^2 (2) \triangleleft u^2 (2)), \quad v \triangleright 1 = \varepsilon(v), \quad (2.9)
\]

\[
\sum v^1 (1) \triangleleft u^1 (1) \otimes v^2 (2) \triangleright u^2 (2) = \sum v^2 (2) \triangleleft u^2 (2) \otimes v^1 (1) \triangleright u^1 (1). \quad (2.10)
\]

Having a mutual pair of Hopf algebras, one constructs the double crossed product Hopf algebra \(U \bowtie V\). As a coalgebra, \(U \bowtie V\) is \(U \otimes V\). However, its algebra structure is defined by the rule

\[
(u^1 \bowtie v^1)(u^2 \bowtie v^2) := u^1 (v^1 (1) \triangleright u^2 (1)) \bowtie (v^1 (2) \triangleleft u^2 (2)) v^2, \quad (2.11)
\]

together with \(1 \bowtie 1\) as its unit. The antipode of \(U \bowtie V\) is defined by

\[
S(u \bowtie v) = (1 \bowtie S(v))(S(u) \bowtie 1) = S(v^1 (1)) \bowtie S(u^1 (1)) \bowtie S(v^2 (2)) \bowtie S(u^2 (2)). \quad (2.12)
\]

It is shown in [11] that if \(a = g_1 \bowtie g_2\) is a double crossed sum of Lie algebras, then the enveloping algebras \((U(g_1), U(g_2))\) becomes a mutual pair of Hopf algebras. Moreover, \(U(a)\) and \(U(g_1) \bowtie U(g_2)\) are isomorphic as Hopf algebras.

In terms of the inclusions

\[
i_1 : U(g_1) \to U(g_1 \bowtie g_2) \quad \text{and} \quad i_2 : U(g_2) \to U(g_1 \bowtie g_2), \quad (2.13)
\]

the Hopf algebra isomorphism mentioned above is

\[
\mu \circ (i_1 \otimes i_2) : U(g_1) \bowtie U(g_2) \to U(a). \quad (2.14)
\]

Here \(\mu\) is the multiplication on \(U(g)\). We easily observe that there is a linear map

\[
\Psi : U(g_2) \bowtie U(g_1) \to U(g_1) \bowtie U(g_2), \quad (2.15)
\]

satisfying

\[
\mu \circ (i_2 \otimes i_1) = \mu \circ (i_1 \otimes i_2) \circ \Psi. \quad (2.16)
\]

The mutual actions of \(U(g_1)\) and \(U(g_2)\) are defined as follows

\[
\triangleright := (\text{Id}_{U(g_2)} \otimes \varepsilon) \circ \Psi \quad \text{and} \quad \triangleleft := (\varepsilon \otimes \text{Id}_{U(g_1)}) \circ \Psi. \quad (2.17)
\]

### 2.2 SAYD modules over double crossed sum Lie algebras

We first review the Lie algebra coactions and SAYD modules over Lie algebras. To this end, let us first introduce the notion of comodule over a Lie algebra.
Definition 2.1. [16]. A vector space $M$ is a left comodule over a Lie algebra $g$ if there is a map $\nabla_g : M \to g \otimes M, \quad m \mapsto m_{[-1]} \otimes m_{[0]}$ such that

$$m_{[-2]} \wedge m_{[-1]} \otimes m_{[0]} = 0,$$

(2.18)

where

$$m_{[-2]} \otimes m_{[-1]} \otimes m_{[0]} = m_{[-1]} \otimes (m_{[0]}{_{[-1]}} \otimes (m_{[0]}{_{[0].}})$.

By [16, Proposition 5.2], corepresentations of a Lie algebra $g$ are nothing but the representations of the symmetric algebra $S(g^*)$. The most natural corepresentation of a Lie algebra $g$, with a basis $\{X_1, \ldots, X_N\}$ and dual basis $\{\theta^1, \ldots, \theta^N\}$, is $M = S(g^*)$ via $m \mapsto X_i \otimes m\theta^i$. This is called the Koszul coaction. The corresponding representation on $M = S(g^*)$ coincides with the initial multiplication of the symmetric algebra.

Next, let $\nabla_g : M \to g \otimes M$ be a left $g$-comodule structure on the linear space $M$. If the $g$-coaction is locally conilpotent, i.e., for any $m \in M$ there exists $n \in \mathbb{N}$ such that $\nabla^k_g(m) = 0$, then it is possible to construct a $U(g)$-coaction structure $\nabla_U : M \to U(g) \otimes M$ on $M$, [16, Proposition 5.7]. Conversely, any comodule over $U(g)$ results a locally conilpotent comodule over $g$ via its composition with the canonical projection $\pi : U(g) \to g$ as follows:

$$M \xrightarrow{\nabla_U} U(g) \otimes M \xrightarrow{\nabla_0} g \otimes M \xrightarrow{\pi \otimes \text{Id}} g \otimes M$$

We denote the category of locally conilpotent left $g$-comodules by $g_{\text{conil}}$, and we have $g_{\text{conil}} = U(g)g$, [16, Proposition 5.8].

Definition 2.2. [16]. Let $M$ be a right module and left comodule over a Lie algebra $g$. We call $M$ a right-left AYD module over $g$ if

$$\nabla_g (m \cdot X) = m_{[-1]} \otimes m_{[0]} \cdot X + [m_{[-1]}, X] \otimes m_{[0]}.$$

(2.19)

Moreover, $M$ is called stable if

$$m_{[0]} \cdot m_{[-1]} = 0.$$

(2.20)

Example 2.3. Let $g$ be a Lie algebra with a basis $\{X_1, \ldots, X_N\}$ and a dual basis $\{\theta^1, \ldots, \theta^N\}$, and $M = S(g^*)$ be the symmetric algebra of $g^*$. We consider the following action of $g$ on $S(g^*)$:

$$S(g^*) \otimes g \to S(g^*), \quad m \otimes X \mapsto m \circ X := -L_X(m) + \delta(X)m$$

(2.21)

Here, $L : g \to \text{End}S(g^*)$ is the coadjoint representation of $g$ on $S(g^*)$ and $\delta \in g^*$ is the trace of the adjoint representation of the Lie algebra $g$ on itself. Via the action (2.21) and the Koszul coaction

$$M \to g \otimes M, \quad m \mapsto X_i \otimes m\theta^i,$$

(2.22)

$M = S(g^*)$ is a SAYD module over the Lie algebra $g$. 

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Example 2.4. Let $\mathfrak{g}$ be a Lie algebra and $M = S(\mathfrak{g}^*)_{[2]}$ be a truncation of the symmetric algebra of $\mathfrak{g}^*$. Then by the action (2.21) and the coaction (2.22), $M$ becomes an SAYD module over the Lie algebra $\mathfrak{g}$. Note that in this case the coaction is locally conilpotent.

We recall from [7] the definition of a right-left stable-anti-Yetter-Drinfeld module over a Hopf algebra $\mathcal{H}$. Let $M$ be a right module and left comodule over a Hopf algebra $\mathcal{H}$. We say that it is a stable-anti-Yetter-Drinfeld (SAYD) module over $\mathcal{H}$ if

\[
\nabla(m \cdot h) = S(h_{(3)}) m_{(-1)} h_{(1)} \otimes m_{(0)} \cdot h_{(2)},
\]

(2.23)

\[
m_{(0)} \cdot m_{(-1)} = m,
\]

(2.24)

for any $v \in V$ and $h \in \mathcal{H}$.

According to [16, Proposition 5.10], AYD modules over a Lie algebra $\mathfrak{g}$ with locally conilpotent coaction are in one to one correspondence with AYD modules over the universal enveloping algebra $U(\mathfrak{g})$. In this case, while it is possible to carry the $\mathfrak{g}$-stability to $U(\mathfrak{g})$-stability [16, Lemma 5.11], the converse is not necessarily true [16, Example 5.12].

A family of examples of SAYD modules over a Lie algebra $\mathfrak{g}$ is given by the modules over the Weyl algebra $D(\mathfrak{g})$, [16, Corollary 5.14]. As for finite dimensional examples, it is proven in [16] that there is no non-trivial $\mathfrak{sl}_2$-coaction that makes a simple two dimensional $\mathfrak{sl}_2$-module an SAYD module over $\mathfrak{sl}_2$.

Let $(\mathfrak{g}_1, \mathfrak{g}_2)$ be a matched pair of Lie algebras, with $\mathfrak{a} := \mathfrak{g}_1 \bowtie \mathfrak{g}_2$ as their double crossed sum Lie algebra. A vector space $M$ is a module over $\mathfrak{a}$ if and only if it is a module over $\mathfrak{g}_1$ and $\mathfrak{g}_2$, such that

\[
(m \cdot Y) \cdot X - (m \cdot X) \cdot Y = m \cdot (Y \triangleright X) + m \cdot (Y \triangleleft X)
\]

(2.25)

is satisfied. In the converse argument one considers the $\mathfrak{a}$ action on $M$ by

\[
m \cdot (X \oplus Y) = m \cdot X + m \cdot Y.
\]

(2.26)

For the comodule structures we have the following analogous result. If $M$ is a comodule over $\mathfrak{g}_1$ and $\mathfrak{g}_2$ via

\[
m \mapsto m_{[-1]} \otimes m_{[0]} \in \mathfrak{g}_1 \otimes M \quad \text{and} \quad m \mapsto m_{(-1)} \otimes m_{(0)} \in \mathfrak{g}_2 \otimes M,
\]

(2.27)

then we define the following linear map

\[
m \mapsto m_{[-1]} \otimes m_{[0]} + m_{(-1)} \otimes m_{(0)} \in \mathfrak{a} \otimes M.
\]

(2.28)

Conversely, if $M$ is a $\mathfrak{a}$-comodule via $\nabla_\mathfrak{a} : M \rightarrow \mathfrak{a} \otimes M$, then we define the linear maps with the help of projections.
Proposition 2.5. A vector space $M$ is an $\mathfrak{a}$-comodule if and only if it is a $\mathfrak{g}_1$-comodule and $\mathfrak{g}_2$-comodule such that

$$m_{[-1]} \otimes m_{[0]} = m_{(0)[1]} \otimes m_{(0)[0]}.$$  (2.30)

Proof. Assume first that $M$ is an $\mathfrak{a}$-comodule. By the $\mathfrak{a}$-coaction compatibility, we have

$$m_{[-2]} \wedge m_{[-1]} \otimes m_{[0]} + m_{[-1]} \wedge m_{[0]} \otimes m_{[0]} + m_{[-1]} \wedge m_{[0]} \otimes m_{[0]} = 0.$$  (2.31)

Applying the antisymmetrization map $\alpha : \mathfrak{a} \wedge \mathfrak{a} \rightarrow U(\mathfrak{a}) \otimes U(\mathfrak{a})$ we get

$$(m_{[-1]} \otimes 1) \otimes (1 \otimes m_{[0]} (-1)) \otimes m_{[0]} = (1 \otimes m_{[0]} (-1)) \otimes (m_{[-1]} \otimes 1) \otimes m_{[0]}$$

$$+ (1 \otimes m_{[-1]}) \otimes (m_{[0]} (-1) \otimes 1) \otimes m_{[0]} = (1 \otimes m_{[-1]}) \otimes (m_{[0]} (-1) \otimes 1) \otimes m_{[0]} = 0.$$  (2.32)

Finally, applying $\text{Id} \otimes \varepsilon_U(\mathfrak{g}_1) \otimes \varepsilon_U(\mathfrak{g}_2) \otimes \text{Id}$ on the both hand sides of the above equation to get the equation (2.30).

Let $m \mapsto m_{[-1]} \otimes m_{[0]} \in \mathfrak{a} \otimes M$ denote the $\mathfrak{a}$-coaction on $M$. Also let $p_1 : \mathfrak{a} \rightarrow \mathfrak{g}_1$ and $p_2 : \mathfrak{a} \rightarrow \mathfrak{g}_2$ be the projections onto the subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Then the $\mathfrak{a}$-coaction is

$$m_{[-1]} \otimes m_{[0]} = p_1(m_{[-1]}) \otimes m_{[0]} + p_2(m_{[-1]}) \otimes m_{[0]}.$$  (2.33)

Next, we shall prove that

$$m \mapsto p_1(m_{[-1]}) \otimes m_{[0]} \in \mathfrak{g}_1 \otimes M \quad \text{and} \quad m \mapsto p_2(m_{[-1]}) \otimes m_{[0]} \in \mathfrak{g}_2 \otimes M$$  (2.34)

are coactions. To this end, we observe that the

$$\alpha(p_1(m_{[-2]}) \wedge p_1(m_{[-1]})) \otimes m_{[0]} = (p_1 \otimes p_1)(\alpha(m_{[-2]} \wedge m_{[-1]})) \otimes m_{[0]} = 0,$$  (2.35)

for $M$ is an $\mathfrak{a}$-comodule.

Since the antisymmetrization map $\alpha : \mathfrak{g}_1 \wedge \mathfrak{g}_1 \rightarrow U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_1)$ is injective, we have

$$p_1(m_{[-2]}) \wedge p_1(m_{[-1]}) \otimes m_{[0]} = 0,$$  (2.36)

proving that $m \mapsto p_1(m_{[-1]}) \otimes m_{[0]}$ is a $\mathfrak{g}_1$-coaction. Similarly $m \mapsto p_2(m_{[-1]}) \otimes m_{[0]}$ is a $\mathfrak{g}_2$-coaction on $M$.

Conversely, assume that $M$ is a $\mathfrak{g}_1$-comodule and $\mathfrak{g}_2$-comodule such that the compatibility (2.30) is satisfied. Then obviously (2.31) is true, which is the $\mathfrak{a}$-comodule compatibility for the coaction (2.28).
We proceed by investigating the relations between AYD modules over the Lie algebras \( g_1 \) and \( g_2 \), and AYD modules over the double crossed sum Lie algebra \( a = g_1 \bowtie g_2 \).

**Proposition 2.6.** Let \((g_1, g_2)\) be a matched pair of Lie algebras, \( a = g_1 \bowtie g_2 \), and \( M \in \mathfrak{g}^{\text{conil}}M_a \). Then, \( M \) is an AYD module over \( a \) if and only if \( M \) is an AYD module over \( g_1 \) and \( g_2 \), and the following conditions are satisfied

\[
(m \cdot X)_{(-1)} \otimes (m \cdot X)_{(0)} = m_{(-1)} \triangleright X \otimes m_{(0)} + m_{(-1)} \otimes m_{(0)} \cdot X, \tag{2.37}
\]

\[
m_{(-1)} \triangleright X \otimes m_{(0)} = 0, \tag{2.38}
\]

\[
(m \cdot Y)_{[-1]} \otimes (m \cdot Y)_{[0]} = -Y \triangleright m_{[-1]} \otimes m_{[0]} + m_{[-1]} \otimes m_{[0]} \cdot Y, \tag{2.39}
\]

\[
Y \triangleleft m_{[-1]} \otimes m_{[0]} = 0, \tag{2.40}
\]

for any \( X \in g_1 \), \( Y \in g_2 \) and any \( m \in M \).

**Proof.** For \( M \in \mathfrak{g}^{\text{conil}}M_a \), assume that \( M \) is an AYD module over the double crossed sum Lie algebra \( a \) via the coaction

\[
m \mapsto m_{(-1)} \otimes m_{(0)} = m_{[-1]} \otimes m_{[0]} + m_{(-1)} \otimes m_{(0)}. \tag{2.41}
\]

As the \( a \)-coaction is locally conilpotent, by [16, Proposition 5.10] we have \( M \in U(a)^{\text{AYD}}U(a) \). Then since the projections

\[
\pi_1 : U(a) = U(g_1) \bowtie U(g_2) \rightarrow U(g_1), \quad \pi_2 : U(a) = U(g_1) \bowtie U(g_2) \rightarrow U(g_2) \tag{2.42}
\]

are coalgebra maps, we conclude that \( M \) is a comodule over \( U(g_1) \) and \( U(g_2) \). Finally, since \( U(g_1) \) and \( U(g_2) \) are Hopf subalgebras of \( U(a) \), AYD conditions on \( U(g_1) \) and \( U(g_2) \) are immediate, and thus \( M \) is an AYD module over \( g_1 \) and \( g_2 \).

We now prove the compatibility conditions (2.37), \ldots, (2.40). To this end, we will make use of the AYD condition for an arbitrary \( X \oplus Y \in a \) and \( m \in M \). On one hand side we have

\[
[m_{[-1]} \cdot X \oplus Y] \otimes m_{(0)} + m_{[-1]} \otimes m_{[0]} \cdot (X \oplus Y) =
\]

\[
\begin{align*}
[m_{[-1]} \oplus 0, X \oplus Y] \otimes m_{[0]} + [0 \oplus m_{(-1)}, X \oplus Y] \otimes m_{(0)} \\
+ m_{[-1]} \otimes m_{[0]} \cdot (X \oplus Y) + m_{(-1)} \otimes m_{(0)} \cdot (X \oplus Y) \\
= ([m_{[-1]} \cdot X] \triangleright m_{[-1]} \oplus (-Y \triangleleft m_{[-1]}) \otimes m_{[0]} \\
+ (m_{(-1)} \triangleright X \otimes m_{[-1]}, Y) + m_{(-1)} \triangleleft X \otimes m_{(0)} \\
+ (m_{[-1]} \oplus 0) \otimes m_{[0]} \cdot (X \oplus Y) + (0 \oplus m_{(-1)}) \otimes m_{(0)} \cdot (X \oplus Y) \\
= ((m \cdot X)_{[-1]} \oplus 0) \otimes (m \cdot X)_{[0]} + (0 \oplus m \cdot Y)_{(-1)} \otimes (m \cdot Y)_{[0]} \\
+ (-Y \triangleright m_{[-1]} \oplus (-m_{[-1]} \triangleleft X) \otimes m_{[0]} + (m_{(-1)} \triangleright X \oplus m_{(-1)} \triangleleft X) \otimes m_{[0]} \\
+ (m_{[-1]} \oplus 0) \otimes m_{[0]} \cdot Y + (0 \oplus m_{(-1)}) \otimes m_{(0)} \cdot X.
\end{align*}
\]
On the other hand,
\[
(m \cdot (X \oplus Y))_{[-1]} \otimes (m \cdot (X \oplus Y))_{(0)} = ((m \cdot X)_{[-1]} \oplus 0) \otimes (m \cdot X)_{(0)} + \\
((m \cdot Y)_{[-1]} \oplus 0) \otimes (m \cdot Y)_{(0)} + (0 \oplus (m \cdot X)_{(-1)}) \otimes (m \cdot X)_{(0)}
\] (2.44)
\[
+ (0 \oplus (m \cdot Y)_{(-1)}) \otimes (m \cdot Y)_{(0)}.
\]
Since \( M \) is an AYD module over \( g_1 \) and \( g_2 \), AYD compatibility (2.43) = (2.44) translates into
\[
((m \cdot Y)_{[-1]} \oplus 0) \otimes (m \cdot Y)_{(0)} + (0 \oplus (m \cdot X)_{(-1)}) \otimes (m \cdot X)_{(0)} = \\
(Y \triangleright m_{[-1]} \oplus -Y \langle m_{[-1]} \rangle \otimes m_{(0)} + (m_{(-1)} \triangleright X \oplus m_{(-1)} \langle X \rangle \otimes m_{(0)})
\] (2.45)
\[
+ (m_{[-1]} \oplus 0) \otimes m_{(0)} \cdot Y + (0 \oplus m_{(-1)}) \otimes m_{(0)} \cdot X.
\]
Finally, we set \( Y := 0 \) to get (2.37) and (2.38). The equations (2.39) and (2.40) are similarly implied by setting \( X := 0 \).

The converse argument is clear. \( \square \)

In general, if \( M \) is an AYD module over the double crossed sum Lie algebra \( a = g_1 \oplus g_2 \), then \( M \) is not necessarily an AYD module over the Lie algebras \( g_1 \) and \( g_2 \).

**Example 2.7.** Consider the Lie algebra \( sl_2 = \langle X, Y, Z \rangle \),
\[
[Y, X] = X, \quad [Z, X] = Y, \quad [Z, Y] = Z.
\] (2.46)

Then, \( sl_2 = g_1 \bowtie g_2 \) for \( g_1 = \langle X, Y \rangle \) and \( g_2 = \langle Z \rangle \).

In view of Example 2.3, the symmetric algebra \( M = S(sl_2^*) \) is a right-left AYD module over \( sl_2 \). The module structure is defined by the coadjoint action, that coincides with (2.21) since \( sl_2 \) is unimodular, and comodule structure is given by the Koszul coaction (2.22).

We now show that it is not an AYD module over \( g_1 \). Let \( \{\theta^X, \theta^Y, \theta^Z\} \) be a dual basis for \( sl_2 \). The linear map
\[
\nabla_{g_1} : M \rightarrow g_1 \otimes M, \quad m \mapsto X \otimes m\theta^X + Y \otimes m\theta^Y,
\] (2.47)
which is the projection onto the Lie algebra \( g_1 \), endows \( M \) with a left \( g_1 \)-comodule structure. However, the AYD compatibility on \( g_1 \) is not satisfied. Indeed, on one side we have
\[
\nabla_{g_1}(m \triangleleft X) = X \otimes (m \triangleleft X)\theta^X + Y \otimes (m \triangleleft X)\theta^Y,
\] (2.48)
and the other one we get
\[
[X, X] \otimes m\theta^X + [Y, X] \otimes m\theta^Y + X \otimes (m\theta^X) \triangleleft X + Y \otimes (m\theta^Y) \triangleleft X = \\
X \otimes (m \triangleleft X)\theta^X + Y \otimes (m \triangleleft X)\theta^Y - Y \otimes m\theta^Z.
\] (2.49)
Remark 2.8. Assume that the mutual actions of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are trivial. In this case, if $M$ is an AYD module over $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$, then it is an AYD module over $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

To see it, let us apply $p_1 \otimes \text{Id}_M$ on the both hand sides of the AYD condition (2.19) for $X \oplus 0 \in \mathfrak{a}$, where $p_1 : \mathfrak{a} \to \mathfrak{g}_1$ is the obvious projection. That is

$$p_1([m_{(-1)}, X \oplus 0]) \otimes m_{(0)} + p_1(m_{(-1)}) \otimes m_{(0)} \cdot (X \oplus 0) = p_1((m \cdot (X \oplus 0))_{(-1)}) \otimes (m \cdot (X \oplus 0))_{(0)}.$$  

(2.50)

Since in this case the projection $p_1 : \mathfrak{a} \to \mathfrak{g}_1$ is a map of Lie algebras, the equation (2.50) reads

$$[p_1(m_{(-1)}), X] \otimes m_{(0)} + p_1(m_{(-1)}) \otimes m_{(0)} \cdot X = p_1((m \cdot X)_{(-1)}) \otimes (m \cdot X)_{(0)},$$  

(2.51)

which is the AYD compatibility for the $\mathfrak{g}_1$-coaction. Similarly, one proves that $M$ is an AYD module over the Lie algebra $\mathfrak{g}_2$.

Let $\mathfrak{a} = \mathfrak{g}_1 \bowtie \mathfrak{g}_2$ be a double crossed sum Lie algebra and $M$ be an SAYD module over $\mathfrak{a}$. By the next example we show that $M$ is not necessarily stable over $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

Example 2.9. Consider the Lie algebra $\mathfrak{a} = gl_2 = \langle Y_1^1, Y_2^1, Y_1^2, Y_2^2 \rangle$ with a dual basis $\{\theta_1^1, \theta_1^2, \theta_2^1, \theta_2^2\}$.

We have a decomposition $gl_2 = \mathfrak{g}_1 \bowtie \mathfrak{g}_2$, where $\mathfrak{g}_1 = \langle Y_1^1, Y_2^1 \rangle$ and $\mathfrak{g}_2 = \langle Y_1^2, Y_2^2 \rangle$.

Let $M := S(gl_2^*)$ be the symmetric algebra as an SAYD module over $gl_2$ with the action (2.21) and the Koszul coaction (2.22) as in Example 2.3. Then the $\mathfrak{g}_1$-coaction on $M$ becomes

$$m \mapsto m_{(-1)} \otimes m_{(0)} = Y_1^1 \otimes m\theta_1^1 + Y_2^1 \otimes m\theta_2^1.$$  

(2.52)

Accordingly, since $\delta(Y_1^1) = 0 = \delta(Y_2^1)$ we have

$$\theta_2^1 \circ \theta_1^2 = -L_{Y_1^1} \theta_1^1 - L_{Y_2^2} \theta_1^2 = -\theta_1^2 \theta_1^1 \neq 0.$$  

(2.53)

We know that if a comodule over a Lie algebra $\mathfrak{g}$ is locally conilpotent then it can be lifted to a comodule over $U(\mathfrak{g})$. In the rest of this section, we are interested in translating Proposition 2.6 in terms of AYD modules over universal enveloping algebras.

Proposition 2.10. Let $\mathfrak{a} = \mathfrak{g}_1 \bowtie \mathfrak{g}_2$ be a double crossed sum Lie algebra and $M$ be a left comodule over $\mathfrak{a}$. Then $\mathfrak{a}$-coaction is locally conilpotent if and only if the corresponding $\mathfrak{g}_1$-coaction and $\mathfrak{g}_2$-coaction are locally conilpotent.

Proof. By (2.29) we know that $\nabla_a = \nabla_{\mathfrak{g}_1} + \nabla_{\mathfrak{g}_2}$. Therefore,

$$\nabla_a^2(m) = m_{(-2)} \otimes m_{(-1)} \otimes m_{(0)} + m_{(-2)} \otimes m_{(-1)} \otimes m_{(0)} + m_{(0)} \otimes m_{(-1)} \otimes m_{(0)}.$$  

(2.54)
By induction we assume that
\[
\mathbf{\nabla}^k_a(m) = m^{(-k+1)} \otimes m^{(-1)} + m^{(-k)} \otimes m_{[1]} + \sum_{p+q=k} m^{(-p)} \otimes m^{(-q)} \otimes m_{[0]},
\]
and we apply the coaction one more times to get
\[
\mathbf{\nabla}^{k+1}_a(m) = m^{(-k+1)} \otimes m^{(-1)} + m^{(-k)} \otimes m_{[1]} + \sum_{p+q=k} m^{(-p)} \otimes m^{(-q)} \otimes m_{[0]},
\]
and the result immediately implies the claim.

Let \( M \) be a locally conilpotent comodule over \( g_1 \) and \( g_2 \). We denote by
\[
M \to U(g_1) \otimes M, \quad m \mapsto m^{(-1)} \otimes m_{[0]},
\]
the lift of the \( g_1 \)-coaction and similarly by
\[
M \to U(g_2) \otimes M, \quad m \mapsto m^{(-2)} \otimes m_{[0]},
\]
the lift of the \( g_2 \)-coaction.

**Corollary 2.11.** Let \( a = g_1 \otimes g_2 \) be a double crossed sum Lie algebra and \( M \in \mathfrak{g} \text{-conil}\mathcal{M}_a \). Then the \( a \)-coaction lifts to the \( U(a) \)-coaction
\[
m \mapsto m^{(-1)} \otimes m^{(-2)} \otimes m_{[0]} \in U(g_1) \otimes U(g_2) \otimes M.
\]

**Proposition 2.12.** Let \( a = g_1 \otimes g_2 \) be a double crossed sum Lie algebra and \( M \in \mathfrak{g} \text{-conil}\mathcal{M}_a \). Then \( M \) is an AYD module over \( a \) if and only if \( M \) is a AYD module over \( g_1 \) and \( g_2 \), and the following conditions are satisfied for any \( m \in M \), any \( u \in U(g_1) \), and \( v \in U(g_2) \):
\[
(m \cdot u)^{-1} \otimes (m \cdot u)_{[0]} = m^{(-1)} \otimes u_{(1)} \otimes m_{[0]} \cdot u_{(2)},
\]
\[
m^{(-1)} \triangleright u \otimes m_{[0]} = u \otimes m,
\]
\[
(m \cdot v)^{-1} \otimes (m \cdot v)_{[0]} = S(v_{(2)}) \triangleright m^{(-1)} \otimes m_{[0]} \cdot v_{(1)},
\]
\[
v \triangleright m^{(-1)} \otimes m_{[0]} = v \otimes m.
\]
Proof. Let $M$ be AYD module over $a$. Since the coaction is conilpotent, it lifts to an AYD module over $U(a)$ by [16, Proposition 5.7]. We write the AYD condition of Hopf algebras (2.23) for $u \bowtie 1 \in U(a)$,

$$ (m \cdot u)_{\mathfrak{m}} \otimes (m \cdot u)_{\mathfrak{m}'} = (S(u_{(3)}) \otimes 1)(m_{\mathfrak{m}} \otimes m_{\mathfrak{m}'}) (u_{(1)} \otimes 1) \otimes m_{\mathfrak{m}'} \cdot (u_{(2)} \otimes 1) = S(u_{(4)})m_{\mathfrak{m}}(m_{\mathfrak{m}'} \cdot u_{(1)}) \otimes m_{\mathfrak{m}'} \cdot u_{(2)} \cdot u_{(3)}. $$

(2.64)

Applying $\varepsilon \otimes \text{Id} \otimes \text{Id}$ on the both hand sides of (2.64), we get (2.60). Similarly we get

$$ (m \cdot u)_{\mathfrak{m}} \otimes (m \cdot u)_{\mathfrak{m}'} = S(u_{(3)})m_{\mathfrak{m}}(m_{\mathfrak{m}'} \cdot u_{(1)}) \otimes m_{\mathfrak{m}'} \cdot u_{(2)}, $$

(2.65)

which yields the following equation after using AYD condition on the left hand side

$$ S(u_{(3)})m_{\mathfrak{m}}u_{(1)} \otimes m_{\mathfrak{m}'}u_{(2)} = S(u_{(3)})m_{\mathfrak{m}}(m_{\mathfrak{m}'} \cdot u_{(1)}) \otimes m_{\mathfrak{m}'} \cdot u_{(2)}. $$

(2.66)

This immediately implies (2.61). Switching to the Lie algebra $\mathfrak{g}_2$ and writing the AYD condition with a $1 \bowtie v \in U(a)$, we obtain (2.62) and (2.63).

Conversely, for $M \in \mathfrak{cone} \mathcal{M}_a$ which is also an AYD module over $\mathfrak{g}_1$ and $\mathfrak{g}_2$, assume that (2.60), . . . (2.63) are satisfied. Then $M$ is an AYD module over $U(\mathfrak{g}_1)$ and $U(\mathfrak{g}_2)$.

We show that (2.60) and (2.61) together imply the AYD condition for the elements of the form $u \bowtie 1 \in U(\mathfrak{g}_1) \bowtie U(\mathfrak{g}_2)$. Indeed,

$$ (m \cdot u)_{\mathfrak{m}} \otimes (m \cdot u)_{\mathfrak{m}'} = S(u_{(3)})m_{\mathfrak{m}}(m_{\mathfrak{m}'} \cdot u_{(1)}) \otimes m_{\mathfrak{m}'} \cdot u_{(2)}, $$

(2.67)

where the first equality follows from the AYD condition on $U(\mathfrak{g}_1)$, the second equality follows from the (2.60), and the last equality is obtained by using (2.61). Similarly, using (2.62) and (2.63) we prove the AYD condition for the elements of the form $1 \bowtie v \in U(\mathfrak{g}_1) \bowtie U(\mathfrak{g}_2)$. The proof is then complete, since the AYD condition is multiplicative.

The following generalization of Proposition 2.12 is now straightforward.

**Corollary 2.13.** Let $(U, V)$ be a mutual pair of Hopf algebras and $M$ a linear space. Then $M$ is an AYD module over $U \bowtie V$ if and only if $M$ is an AYD module over $U$ and $V$, and the following conditions are satisfied for any $m \in M$, any $u \in U$ and $v \in V$.

$$ (m \cdot u)_{\mathfrak{m}} \otimes (m \cdot u)_{\mathfrak{m}'} = m_{\mathfrak{m}} \cdot u_{(1)} \otimes m_{\mathfrak{m}'} \cdot u_{(2)}, $$

(2.68)

$$ m_{\mathfrak{m}} \cdot u \otimes m_{\mathfrak{m}'} = u \otimes m, $$

(2.69)

$$ (m \cdot v)_{\mathfrak{m}} \otimes (m \cdot v)_{\mathfrak{m}'} = S(v_{(2)}) \cdot m_{\mathfrak{m}'} \otimes m_{\mathfrak{m}'} \cdot v_{(1)}, $$

(2.70)

$$ v \cdot m_{\mathfrak{m}} \otimes m_{\mathfrak{m}'} = v \otimes m, $$

(2.71)
3 Lie-Hopf algebras and their SAYD modules

In this section we first recall the associated matched pair of Hopf algebras to a matched pair of Lie algebras from [15]. We then identify the AYD modules over the universal enveloping algebra of a double crossed sum Lie algebra with the YD modules over the corresponding bicrossed product Hopf algebra. Finally we prove that the only finite dimensional SAYD module over the Connes-Moscovici Hopf algebras is the one-dimensional one found in [2].

3.1 Lie-Hopf algebras

Let us first review the bicrossed product construction from [11]. Let \( U \) and \( F \) be two Hopf algebras. A linear map \( \triangleleft : U \to U \otimes F \), \( \triangleleft u = u^{(0)} \otimes u^{(1)} \), defines a right coaction and equips \( U \) with a right \( F^- \)comodule coalgebra structure, if the following conditions are satisfied for any \( u \in U \):

\[
\begin{align*}
(u^{(0)})_1 \otimes (u^{(0)})_2 \otimes u^{(1)} &= u^{(0)}_1 \otimes u^{(0)}_2 \otimes u^{(1)}_1 u^{(1)}_2, \\
\varepsilon(u^{(0)})u^{(1)} &= \varepsilon(u)1.
\end{align*}
\] (3.1)

We then form a cocrossed product coalgebra \( F \triangleleft U \). It has \( F \otimes U \) as underlying vector space and the coalgebra structure is given by

\[
\Delta(f \triangleleft u) = f^{(1)} \triangleleft u^{(0)}(1) \otimes f^{(1)} \triangleleft u^{(1)}(1) \triangleleft u^{(2)}, \quad \varepsilon(f \triangleleft u) = \varepsilon(f)\varepsilon(u). \] (3.2)

In a dual fashion, \( F \) is called a left \( U^- \)module algebra, if \( U \) acts from the left on \( F \) via a left action \( \triangleright : F \otimes U \to F \) which satisfies the following conditions for any \( u \in U \), and \( f, g \in F \):

\[
\begin{align*}
u \triangleright 1 &= \varepsilon(u)1, \\
u \triangleright (fg) &= (u^{(1)} \triangleright f)(u^{(2)} \triangleright g).
\end{align*}
\] (3.3)

This time we can endow the underlying vector space \( F \otimes U \) with an algebra structure, to be denoted by \( F \curlyeqwedge U \), with \( 1 \curlyeqwedge 1 \) as its unit and the product

\[
(f \curlyeqwedge u)(g \curlyeqwedge v) = f^{(1)} u^{(0)} \triangleright g \curlyeqwedge u^{(2)} v.
\] (3.4)

A pair of Hopf algebras \((F, U)\) is called a matched pair of Hopf algebras if they are equipped, as above, with an action and a coaction which satisfy the following compatibility conditions

\[
\begin{align*}
\Delta(u \triangleright f) &= u^{(0)}_1 \triangleright f^{(1)} \otimes u^{(1)}_1 \triangleright (u^{(2)}_1 \triangleright f^{(2)}), \\
\varepsilon(u \triangleright f) &= \varepsilon(u)\varepsilon(f) \quad (3.5) \\
\triangleleft(uv) &= u^{(0)}_1 v^{(0)} \otimes u^{(1)}_1 \triangleright v^{(1)}, \\
\triangleleft(1) &= 1 \otimes 1 \quad (3.6) \\
u^{(0)}_2 \otimes (u^{(1)} \triangleright f)u^{(1)}_2 &= u^{(0)}_1 \otimes u^{(1)}_1 \triangleright (u^{(2)}_2 \triangleright f). \quad (3.7)
\end{align*}
\]
for any \( u \in U \), and any \( f \in F \). We then form a new Hopf algebra \( F \triangleright \triangleleft U \), called the bicrossed product of the matched pair \((F,U)\). It has \( F \triangleright \triangleleft U \) as the underlying coalgebra and \( F \triangleright \triangleleft U \) as the underlying algebra. The antipode is given by

\[
S(f \triangleright \triangleleft u) = (1 \triangleright \triangleleft S(u^{(0)}))(S(f^{(1)}) \triangleright \triangleleft 1), \quad f \in F, \ u \in U.
\] (3.8)

Next, we recall Lie-Hopf algebras from [15]. A Lie-Hopf algebra produces a bicrossed product Hopf algebra such that one of the Hopf algebras involved is commutative and the other one is the universal enveloping algebra of a Lie algebra.

Let \( F \) be a commutative Hopf algebra on which a Lie algebra \( g \) acts by derivations. Then the vector space \( g \otimes F \) endowed with the bracket

\[
[X \otimes f, Y \otimes g] = [X,Y] \otimes fg + Y \otimes \varepsilon(f)X \triangleright g - X \otimes \varepsilon(g) Y \triangleright f
\] (3.9)

becomes a Lie algebra. Next, we assume that \( F \) coacts on \( g \) via \( \triangledown_g : g \to g \otimes F \). We say that the coaction \( \triangledown_g : g \to g \otimes F \) satisfies the structure identity of \( g \) if \( \triangledown_g : g \to g \otimes F \) is a Lie algebra map. Finally one uses the action of \( g \) on \( F \) and the coaction of \( F \) on \( g \) to define the following useful action of \( g \) on \( F \otimes F \):

\[
X \bullet (f^1 \otimes f^2) = \sum X^{(0)} \triangleright f^1 \otimes X^{(1)} f^2 + f^1 \otimes X \triangleright f^2.
\] (3.10)

We are now ready to define the notion of Lie-Hopf algebra.

**Definition 3.1.** [15]. Let a Lie algebra \( g \) act on a commutative Hopf algebra \( F \) by derivations. We say that \( F \) is a \( g \)-Hopf algebra if

1. \( F \) coacts on \( g \) and its coaction satisfies the structure identity of \( g \).
2. \( \Delta \) and \( \varepsilon \) are \( g \)-linear, that is \( \Delta(X \triangleright f) = X \bullet \Delta(f) \), \( \varepsilon(X \triangleright f) = 0 \), \( f \in F \) and \( X \in g \).

If \( F \) is a \( g \)-Hopf algebra, then \( U(g) \) acts on \( F \) naturally and makes it a \( U(g) \)-module algebra. On the other hand, we extend the coaction \( \triangledown_g \) of \( F \) on \( g \) to a coaction \( \triangledown_U \) of \( F \) on \( U(g) \) inductively via the rule (3.6).

As for the corresponding bicrossed product Hopf algebra, we have the following result.

**Theorem 3.2.** [15]. Let \( F \) be a commutative Hopf algebra and \( g \) be a Lie algebra. Then the pair \((F,U(g))\) is a matched pair of Hopf algebras if and only if \( F \) is a \( g \)-Hopf algebra.

A class of examples of Lie-Hopf algebras arises from matched pairs of Lie algebras. To be able to express such an example, let us recall first the definition of \( R(g) \), the Hopf algebra of representative functions on a Lie algebra \( g \).

\[
R(g) = \left\{ f \in U(g)^* \mid \exists I \subseteq \ker f \text{ such that } \dim(\ker f)/I < \infty \right\}.
\]
The finite codimensionality condition in the definition of $R(g)$ guarantees that for any $f \in R(g)$ there exist a finite number of functions $f'_i, f''_i \in R(g)$ such that for any $u^1, u^2 \in U(g)$,

$$f(u^1u^2) = \sum_i f'_i(u^1)f''_i(u^2).$$

(3.11)

The Hopf algebraic structure of $R(g)$ is summarized by:

$$\mu : R(g) \otimes R(g) \to R(g), \quad \mu(f \otimes g)(u) = f(u^{(1)})g(u^{(2)}),$$

(3.12)

$$\eta : \mathbb{C} \to R(g), \quad \eta(1) = \varepsilon,$$

(3.13)

$$\Delta : R(g) \to R(g) \otimes R(g),$$

(3.14)

$$\Delta(f) = \sum_i f'_i \otimes f''_i,$$

if $f(u^1u^2) = \sum_i f'_i(u^1)f''_i(u^2),$

$$S : R(g) \to R(g), \quad S(f)(u) = f(S(u)).$$

(3.15)

The following proposition produces a family of examples.

**Proposition 3.3.** [15]. For any matched pair of Lie algebras $(g_1, g_2)$, the Hopf algebra $R(g_2)$ is a $g_1$-Hopf algebra.

### 3.2 SAYD modules over Lie-Hopf algebras

Let us start with a very brief introduction to SAYD modules over Hopf algebras. Let $\mathcal{H}$ be a Hopf algebra. By definition, a character $\theta : \mathcal{H} \to \mathbb{C}$ is an algebra map. A group-like $\sigma \in \mathcal{H}$ is the dual object of the character, i.e., $\Delta(\sigma) = \sigma \otimes \sigma$. The pair $(\theta, \sigma)$ is called a modular pair in involution [4] if

$$\theta(\sigma) = 1, \quad \text{and} \quad S^0_\theta = Ad_\sigma,$$

(3.16)

where $Ad_\sigma(h) = \sigma h \sigma^{-1}$ and $S^0_\delta$ is defined by

$$S^0_\theta(h) = \theta(h^{(1)})S(h^{(2)}).$$

(3.17)

We recall from [7] the definition of a right-left stable-anti-Yetter-Drinfeld module over a Hopf algebra $\mathcal{H}$. Let $M$ be a right module and left comodule over a Hopf algebra $\mathcal{H}$. We say that it is stable-anti-Yetter-Drinfeld (SAYD) module over $\mathcal{H}$ if

$$\Box(m \cdot h) = S(h^{(3)})m^{(-1)}h^{(1)} \otimes m^{(0)} \cdot h^{(2)}, \quad m^{(0)} \cdot m^{(-1)} = v,$$

(3.18)

for any $m \in M$ and $h \in \mathcal{H}$. It is shown in [7] that any modular pair in involution defines a one dimensional SAYD module and all one dimensional SAYD modules come this way.

If $M$ is a module over a bicrossed product Hopf algebra $F \bowtie \mathcal{U}$, then by the fact that $F$ and $\mathcal{U}$ are subalgebras of $F \bowtie \mathcal{U}$ we can immediately conclude that $M$ is a module on $F$ and $\mathcal{U}$. More explicitly, we have the following elementary lemma, see [16, Lemma 3.4].
Lemma 3.4. Let \((F, U)\) be a matched pair of Hopf algebras and \(M\) a linear space. Then \(M\) is a right module over the bicrossed product Hopf algebra \(F \triangleright \lhd U\) if and only if \(M\) is a right module over \(F\) and a right module over \(U\), such that

\[
(m \cdot u) \cdot f = (m \cdot (u_{(1)} \triangleright f)) \cdot u_{(2)}. \tag{3.19}
\]

Let \((g_1, g_2)\) be a matched pair of Lie algebras and \(M\) be a module over the double crossed sum \(g_1 \bowtie g_2\) such that \(g_1 \bowtie g_2\)-coaction is locally conilpotent. Being a right \(g_1\)-module, \(M\) has a right \(U(g_1)\)-module structure. Similarly, since it is a locally conilpotent left \(g_2\)-comodule, \(M\) is a right \(R(g_2)\)-module. Then we define

\[
M \otimes R(g_2) \triangleright \lhd U(g_1) \rightarrow M,
\]

\[
m \otimes (f \triangleright \lhd u) \mapsto (m \cdot f) \cdot u = f(m_{(\overline{1})})m_{(\overline{0})} \cdot u. \tag{3.20}
\]

Corollary 3.5. Let \((g_1, g_2)\) be a matched pair of Lie algebras and \(M\) be an AYD module over the double crossed sum \(g_1 \bowtie g_2\) such that \(g_1 \bowtie g_2\)-coaction is locally conilpotent. Then \(M\) has a right \(R(g_2) \triangleright \lhd U(g_1)\)-module structure via (3.20).

Proof. For \(f \in R(g_2), u \in U(g_1)\) and \(m \in M\), we have

\[
(m \cdot u) \cdot f = f((m \cdot u)_{(\overline{1})})(m \cdot u)_{(\overline{0})} = f(m_{(\overline{1})} \triangleright u_{(1)})m_{(\overline{0})} \cdot u_{(2)}
\]

\[
= (u_{(1)} \triangleright f)(m_{(\overline{1})})m_{(\overline{0})} \cdot u_{(2)} = (m \cdot (u_{(1)} \triangleright f)) \cdot u_{(2)}. \tag{3.21}
\]

Here in the second equality we used Proposition 2.12. So by Lemma 3.4 the proof is complete. \(\square\)

Let us assume that \(M\) is a left comodule over the bicrossed product \(F \triangleright \lhd U\). Since the projections \(\pi_1 := \text{Id}_F \otimes \varepsilon_U : F \triangleright \lhd U \rightarrow F\) and \(\pi_2 := \varepsilon_F \otimes \text{Id}_U : F \triangleright \lhd U \rightarrow U\) are coalegbra maps, \(M\) becomes a left \(F\)-comodule as well as a left \(U\)-comodule via \(\pi_1\) and \(\pi_2\). Denoting these comodule structures by

\[
m \mapsto m^{(\overline{1})} \otimes m^{(\overline{0})} \in F \otimes M \quad \text{and} \quad m \mapsto m_{(\overline{1})} \otimes m_{(\overline{0})} \in U \otimes M, \tag{3.22}
\]

we mean the \(F \triangleright \lhd U\)-comodule structure is

\[
m \mapsto m^{(\overline{1})} \otimes m_{(\overline{1})} \otimes m_{(\overline{0})} \in F \triangleright \lhd U \otimes M. \tag{3.23}
\]

Lemma 3.6. Let \((F, U)\) be a matched pair of Hopf algebras and \(M\) a linear space. Then \(M\) is a left comodule over the bicrossed product Hopf algebra \(F \triangleright \lhd U\) if and only if it is a left comodule over \(F\) and a left comodule over \(U\), such that for any \(m \in M\)

\[
(m_{(\overline{0})})^{(0)} \otimes m_{(\overline{1})} \cdot (m_{(\overline{0})})^{(1)} \otimes m_{(\overline{0})} = m_{(\overline{1})} \otimes (m_{(\overline{0})})^{(\overline{1})} \otimes (m_{(\overline{0})})^{(\overline{0})}, \tag{3.24}
\]

where \(u \mapsto u^{(0)} \otimes u^{(1)} \in U \otimes F\) is the right \(F\)-coaction on \(U\).
Proof. Let assume that $M$ is a comodule over the bicrossed product Hopf algebra $F \triangleright \triangleleft U$. Then by the coassociativity of the coaction, we have

$$m^{(-2)} \triangleright (m^{(0)}_{[\bar{m}][\bar{m}]})^{(0)} \otimes (m^{(0)}_{[\bar{m}][\bar{m}]})^{(1)} \triangleright m^{(0)}_{[\bar{m}][\bar{m}]} \otimes m^{(0)}_{[\bar{m}][\bar{m}]},$$

(3.25)

By applying $\varepsilon_F \otimes \text{Id}_U \otimes \text{Id}_F \otimes \varepsilon_U \otimes \text{Id}_M$ on both hand sides of (3.25), we get

$$(m^{(0)}_{[\bar{m}][\bar{m}]})^{(0)} \otimes (m^{(0)}_{[\bar{m}][\bar{m}]})^{(1)} \otimes m^{(0)}_{[\bar{m}][\bar{m}]_{[\bar{m}][\bar{m}]} = m^{(0)}_{[\bar{m}][\bar{m}]} \otimes (m^{(0)}_{[\bar{m}][\bar{m}]})^{(0)} \otimes (m^{(0)}_{[\bar{m}][\bar{m}]})^{(0)}.$$ (3.26)

Conversely, assume that (3.24) holds for any $m \in M$. This results

$$m^{(-2)} \otimes (m^{(0)}_{[\bar{m}][\bar{m}]})^{(0)} \otimes m^{(0)}_{[\bar{m}][\bar{m}]} \triangleright (m^{(0)}_{[\bar{m}][\bar{m}]})^{(1)} \otimes m^{(0)}_{[\bar{m}][\bar{m}]} = m^{(-2)} \otimes m^{(0)}_{[\bar{m}][\bar{m}]} \otimes (m^{(0)}_{[\bar{m}][\bar{m}]})^{(0)} \otimes (m^{(0)}_{[\bar{m}][\bar{m}]})^{(0)}.$$ (3.27)

which implies (3.25) i.e., the coassociativity of the $F \triangleright \triangleleft U$-coaction. \qed

Corollary 3.7. Let $(g_1, g_2)$ be a matched pair of Lie algebras and $M$ be an AYD module over the double crossed sum $g_1 \triangleright \triangleleft g_2$ with locally finite action and locally conilpotent coaction. Then $M$ has a left $R(g_2) \triangleright \triangleleft U(g_1)$-comodule structure.

Proof. Since $M$ is a locally conilpotent left $g_1$-comodule, it has a left $U(g_1)$-comodule structure. On the other hand, being a locally finite right $g_2$-module, $M$ is a left $R(g_2)$-comodule $[8]$. By Proposition 2.12 we have

$$(m \cdot v)_{[\bar{m}][\bar{m}]} \otimes (m \cdot v)_{[\bar{m}][\bar{m}]} = S(v_{(2)}) \triangleright m^{(-2)} \otimes m_{[\bar{m}][\bar{m}]} \cdot v_{(1)},$$

(3.28)

or in other words

$$v_{(2)} \triangleright (m \cdot v_{(1)})_{[\bar{m}][\bar{m}]} \otimes (m \cdot v_{(1)})_{[\bar{m}][\bar{m}]_{[\bar{m}][\bar{m}]} = m^{(-2)} \otimes m_{[\bar{m}][\bar{m}]} \cdot v.$$ (3.29)

Using the $R(g_2)$-coaction on $M$ and $R(g_2)$-coaction on $U(g_1)$, we can translate this equality into

$$(m^{(-2)} \cdot (m^{(0)}_{[\bar{m}][\bar{m}]})^{(1)}(v)(m^{(0)}_{[\bar{m}][\bar{m}]})^{(0)} \otimes m^{(0)}_{[\bar{m}][\bar{m}]_{[\bar{m}][\bar{m}]} = m^{(-2)} \otimes (m_{[\bar{m}][\bar{m}]})^{(0)} ((m_{[\bar{m}][\bar{m}]})^{(0)}(v)).$$ (3.30)

Finally, by the non-degenerate pairing between $U(g_2)$ and $R(g_2)$ we get

$$(m^{(0)}_{[\bar{m}][\bar{m}]})^{(0)} \otimes m^{(0)}_{[\bar{m}][\bar{m}]} \cdot (m^{(0)}_{[\bar{m}][\bar{m}]})^{(1)} \otimes m^{(0)}_{[\bar{m}][\bar{m}]_{[\bar{m}][\bar{m}]} = m^{(-2)} \otimes (m_{[\bar{m}][\bar{m}]})^{(0)} \otimes (m_{[\bar{m}][\bar{m}]})^{(0)}),$$ (3.31)

i.e., the $R(g_2) \triangleright \triangleleft U(g_1)$-coaction compatibility. \qed

Our next challenge is to identify the Yetter-Drinfeld modules over Lie-Hopf algebras.

Let us recall that a right module left comodule $M$ over a Hopf algebra $\mathcal{H}$ is called a YD module if

$$h_{(2)}(m \cdot h_{(1)})_{(-1)} \otimes (m \cdot h_{(1)})_{(0)} = m_{(-1)} h_{(1)} \otimes m_{(0)} \cdot h_{(2)}$$

(3.32)

for any $h \in \mathcal{H}$ and any $m \in M$. 19
Proposition 3.8. Let \((\mathcal{F}, \mathcal{U})\) be a matched pair of Hopf algebras and \(M\) be a right module and left comodule over \(\mathcal{F} \triangleright \triangleright \mathcal{U}\) such that via the corresponding module and comodule structures it becomes a YD-module over \(\mathcal{U}\). Then \(M\) is a YD-module over \(\mathcal{F} \triangleright \triangleright \mathcal{U}\) if and only if \(M\) is a YD-module over \(\mathcal{F}\) via the corresponding module and comodule structures, and the following conditions are satisfied

\[
(m \cdot f)_{\triangleright} \otimes (m \cdot f)_{\triangleright} = m_{\triangleright} \otimes m_{\triangleright} \cdot f,
\]

(3.33)

\[
m_{\triangleright} f_{(1)} \otimes m_{\triangleright} f_{(2)} = m_{\triangleright} (m_{\triangleright} f_{(1)} \triangleright f_{(1)}) \otimes m_{\triangleright} f_{(2)},
\]

(3.34)

\[
m_{\triangleright} \otimes m_{\triangleright} \cdot u = (u_{(1)} \cdot (m \cdot (u_{(1)}^{(0)}))_{\triangleright} \otimes (m \cdot (u_{(1)}^{(0)}))_{\triangleright})
\]

(3.35)

\[
m_{\triangleright} u^{(0)} \otimes m_{\triangleright} \cdot u^{(1)} = m_{\triangleright} u \otimes m_{\triangleright}.
\]

(3.36)

Proof. First we assume that \(M\) is a YD module over \(\mathcal{F} \triangleright \triangleright \mathcal{U}\). Since \(\mathcal{F}\) is a Hopf subalgebra of \(\mathcal{F} \triangleright \triangleright \mathcal{U}\), \(M\) is a YD module over \(\mathcal{F}\).

Next, we prove the compatibilities (3.33), . . . , (3.36). Writing (3.32) for an arbitrary \(f \triangleright 1 \in \mathcal{F} \triangleright \triangleright \mathcal{U}\), we get

\[
(f_{(2)} \triangleright 1) \cdot ((m \cdot f_{(1)})_{\triangleright} \triangleright (m \cdot f_{(1)})_{\triangleright} \otimes (m \cdot f_{(1)})_{\triangleright}) = (m_{\triangleright} \triangleright m_{\triangleright}) \cdot (f_{(1)} \triangleright 1) \otimes m_{\triangleright} \cdot f_{(2)}.
\]

(3.37)

Using the YD condition on \(\mathcal{F}\) on the left hand side, we get

\[
f_{(2)} (m \cdot f_{(1)})_{\triangleright} \triangleright (m \cdot f_{(1)})_{\triangleright} \otimes (m \cdot f_{(1)})_{\triangleright} = m_{\triangleright} f_{(1)} \triangleright (m_{\triangleright} f_{(2)} \cdot m_{\triangleright} f_{(2)})_{\triangleright} \otimes (m_{\triangleright} f_{(2)} f_{(2)})_{\triangleright}
\]

(3.38)

\[
m_{\triangleright} (m_{\triangleright} f_{(1)} \triangleright f_{(1)}) \triangleright (m_{\triangleright} f_{(2)} \cdot m_{\triangleright} f_{(2)})_{\triangleright} \otimes (m_{\triangleright} f_{(2)} f_{(2)})_{\triangleright}.
\]

Now we apply \(\varepsilon_{\mathcal{F}} \otimes \text{Id}_{\mathcal{U}} \otimes \text{Id}_{\mathcal{M}}\) on the both hand sides of (3.38) to get (3.33). Similarly we apply \(\text{Id}_{\mathcal{F}} \otimes \varepsilon_{\mathcal{U}} \otimes \text{Id}_{\mathcal{M}}\) to get (3.34).

By the same argument, the YD compatibility of \(\mathcal{F} \triangleright \triangleright \mathcal{U}\) for an element of the form \(1 \triangleright 1 \in \mathcal{F} \triangleright \triangleright \mathcal{U}\), followed by the YD compatibility of \(\mathcal{U}\) yields (3.35) and (3.36).

Conversely, assume that \(M \in \mathcal{F} \mathcal{YD}_{\mathcal{F}}\) and (3.33), . . . , (3.36) are satisfied. We will prove that the YD condition over \(\mathcal{F} \triangleright \triangleright \mathcal{U}\) holds for the elements of the forms \(f \triangleright 1 \in \mathcal{F} \triangleright \triangleright \mathcal{U}\) and \(1 \triangleright u \in \mathcal{F} \triangleright \triangleright \mathcal{U}\). By (3.34), we have

\[
m_{\triangleright} f_{(1)} \triangleright (m_{\triangleright} f_{(2)} \cdot m_{\triangleright} f_{(2)})_{\triangleright} \otimes (m_{\triangleright} f_{(2)} f_{(2)})_{\triangleright}
\]

(3.39)

which, by using (3.33), implies the YD compatibility for the elements of the form \(f \triangleright 1 \in \mathcal{F} \triangleright \triangleright \mathcal{U}\).

Next, by (3.35) we have

\[
(u_{(1)} \cdot (m \cdot (u_{(1)}^{(0)}))_{\triangleright} \triangleright u_{(1)} \cdot (m \cdot (u_{(1)}^{(0)}))_{\triangleright} \otimes (m \cdot (u_{(1)}^{(0)}))_{\triangleright}) = m_{\triangleright} u_{(2)} (m_{\triangleright} \cdot u_{(1)})_{\triangleright} \otimes (m_{\triangleright} \cdot u_{(1)})_{\triangleright},
\]

(3.40)
which amounts to the YD compatibility for the elements of the form $\triangleright u \in \mathcal{F} \triangleright \mathcal{U}$ by using YD compatibility over $\mathcal{U}$ and (3.36).

Since YD condition is multiplicative, it is then satisfied for any $f \triangleright u \in \mathcal{F} \triangleright \mathcal{U}$, and hence we have proved that $M$ is YD module over $\mathcal{F} \triangleright \mathcal{U}$.

**Proposition 3.9.** Let $(\mathfrak{g}_1, \mathfrak{g}_2)$ be a matched pair of finite dimensional Lie algebras, $M$ an AYD module over the double crossed sum $\mathfrak{g}_1 \triangleright \mathfrak{g}_2$ with locally finite action and locally conilpotent coaction. Then, by the action (3.20) and the coaction (3.23), $M$ becomes a right-left YD module over $R(\mathfrak{g}_2) \triangleright \mathcal{U}(\mathfrak{g}_1)$.

**Proof.** We prove the proposition by verifying the conditions of Proposition 3.8. Since $M$ is an AYD module over $\mathfrak{g}_1 \triangleright \mathfrak{g}_2$ with a locally conilpotent coaction, it is an AYD module over $\mathcal{U}(\mathfrak{g}_1) \triangleright \mathcal{U}(\mathfrak{g}_2)$. In particular, it is a left comodule over $\mathcal{U}(\mathfrak{g}_1) \triangleright \mathcal{U}(\mathfrak{g}_2)$ with the following coaction as proved in Corollary 2.11

$$m \mapsto m_{(1)} \triangleright m_{(2)} \otimes m_{(3)} \in \mathcal{U}(\mathfrak{g}_1) \triangleright \mathcal{U}(\mathfrak{g}_2). \quad (3.41)$$

By the coassociativity of this coaction, we have

$$m_{(1)} \otimes m_{(2)} \otimes m_{(3)} = m_{(1)} \otimes m_{(1)} \otimes m_{(2)} \otimes m_{(3)}. \quad (3.42)$$

Thus, the application of $\text{Id}_{\mathcal{U}(\mathfrak{g}_1)} \otimes f \otimes \text{Id}_M$ on both hand sides results (3.33).

Using (2.63) and (3.42) we get

$$v_{(2)} (m \cdot v_{(1)}) \triangleright (m \cdot v_{(1)}) = (v_{(2)} \triangleleft (m \cdot v_{(1)}) \otimes (m \cdot v_{(1)})) \tekind{2} \otimes (m \cdot v_{(1)}) = (v_{(2)} \triangleleft (m \cdot v_{(1)})) \tekind{2} \otimes (m \cdot v_{(1)}). \quad (3.43)$$

Then applying $f \otimes \text{Id}_M$ to both sides and using the non-degenerate pairing between $R(\mathfrak{g}_2)$ and $\mathcal{U}(\mathfrak{g}_2)$, we conclude (3.34).

To verify (3.35), we use the $\mathcal{U}(\mathfrak{g}_1) \triangleright \mathcal{U}(\mathfrak{g}_2)$-module compatibility on $M$, i.e., for any $u \in \mathcal{U}(\mathfrak{g}_1)$, $v \in \mathcal{U}(\mathfrak{g}_2)$ and $m \in M$,

$$(m \cdot v) \cdot u = (m \cdot (v_{(1)} \triangleright u_{(1)})) \cdot (v_{(2)} \triangleleft u_{(2)}). \quad (3.44)$$

Using the non-degenerate pairing between $R(\mathfrak{g}_2)$ and $\mathcal{U}(\mathfrak{g}_1)$, we rewrite this equality as

$$m^{(2)} (v) m^{(3)} \cdot u = (m \cdot (v_{(1)} \triangleright u_{(1)})) \tekind{2} (v_{(2)} \triangleleft u_{(2)}) (m \cdot (v_{(1)} \triangleright u_{(1)})) \tekind{3}$$

$$= u_{(2)} \triangleright (m \cdot (v_{(1)} \triangleright u_{(1)})) \tekind{2} (v_{(2)} \triangleright (m \cdot (v_{(1)} \triangleright u_{(1)})) \tekind{2})$$

$$= (u_{(1)}) \tekind{1} (v_{(1)}) \tekind{2} (v_{(2)} \triangleright (m \cdot (v_{(1)}) \tekind{1} (m \cdot (u_{(1)})) \tekind{2} (v_{(2)}) (m \cdot (u_{(1)})) (m \cdot (u_{(1)})) \tekind{3} \tekind{4}$$

$$= [(u_{(1)}) \tekind{1} (v_{(2)} \triangleright (m \cdot (u_{(1)})) \tekind{1} (v)(m \cdot (u_{(1)})) \tekind{4}],$$

which means (3.35).
Using the $U(g_1) \Join U(g_2)$-coaction compatibility (3.42), together with (2.61), we have

$$m_{[-1]}(u^{(0)} \otimes m_{[\overline{0}]}) \cdot u^{(1)} = m_{[-1]}(u^{(0)} u^{(1)} (m_{[\overline{0}]}) \otimes m_{[\overline{0}]})$$

$$= m_{[-1]}(m_{[\overline{0}]}) \cdot u^{(1)} = m_{[\overline{0}]}) \otimes m_{[\overline{0}]})$$

$$= m_{[-1]}(u \otimes m_{[\overline{0}]})$$

which is (3.36).

We are now ready to express the main result of this section.

**Theorem 3.10.** Let $(F, U)$ be a matched pair of Hopf algebras such that $F$ is commutative and $U$ is cocommutative, and $\langle \cdot, \cdot \rangle : F \times V \to \mathbb{C}$ a non-degenerate Hopf pairing. Then $M$ is an AYD-module over $U \Join V$ if and only if $M$ is a YD-module over $F \Join U$ such that by the corresponding module and comodule structures it is a YD-module over $U$.

**Proof.** Let $M \in \mathcal{M}_{U \Join V}$. We first prove that $M \in \mathcal{M}_{U \Join V}$. By Proposition 3.8, we have (3.35). Evaluating both sides of this equality on an arbitrary $v \in V$ we get

$$m \cdot u = (m \cdot (v (1) \triangleright u (1))) \cdot (v (2) \triangleleft u (2)).$$

(3.47)

This proves that $M$ is a right module on the double crossed product $U \Join V$.

Next, we show that $M \in \mathcal{M}_{U \Join V}$. This time using (3.33) and the duality between right $F$-action and left $V$-coaction we get

$$f((m \cdot u)(\overline{v})) = f(m \cdot v)(\overline{u}) \cdot m(\overline{u}) \cdot u.$$}

(3.48)

Since the pairing is non-degenerate, we conclude that $M$ is a left comodule over $U \Join V$.

Finally, we prove that AYD condition over $U \Join V$ is satisfied by using Corollary 2.13, that is we show that (2.68), ..., (2.71) are satisfied.

Firstly, by considering the Hopf duality between the $F$ and $V$, the right $F \Join U$-module compatibility reads

$$f((m \cdot u)(\overline{v})) = f(m \cdot v)(\overline{u}) \cdot m(\overline{u}) \cdot u.$$}

(3.49)

Hence (2.68) holds.

Secondly, by (3.36) and the Hopf duality between $F$ and $V$, we get

$$m_{[-1]}(u^{(0)} u^{(1)} (m_{[\overline{0}]}) \otimes m_{[\overline{0}]}) = m_{[-1]}(m_{[\overline{0}]}) \cdot u \otimes m_{[\overline{0}]}) = m_{[-1]}(u \otimes m_{[\overline{0}]})$$

which immediately imply (2.69).
Finally, evaluating the left hand side of the evaluation \((3.34)\) on an arbitrary \(v \in V\), we get

\[
v_{(2)} \triangleright (m \cdot v_{(1)})_{\overline{-1}} \otimes (m \cdot v_{(1)})_{\overline{0}} = m_{\overline{-1}} \otimes m_{\overline{0}} \cdot v,
\]

which immediately implies \((2.70)\).

Finally, evaluating the left hand side of the evaluation \((3.34)\) on an arbitrary \(v \in V\), we get

\[
LHS = f_{(1)}(v_{(2)}) (m \cdot v_{(1)}) \cdot f_{(2)} = f_{(1)}(v_{(2)}) f_{(2)}((m \cdot v_{(1)})_{\overline{-1}})(m \cdot v_{(1)})_{\overline{0}} = f(m_{\overline{-1}} \cdot v_{(1)}) m_{\overline{0}} \cdot v_{(2)},
\]

and the right hand side turns into

\[
RHS = (m \cdot v_{(1)})_{\overline{-1}} \triangleright f_{(1)}(v_{(2)}) f_{(2)}((m \cdot v_{(1)})_{\overline{0}}(m \cdot v_{(1)})_{\overline{0}}(m \cdot v_{(1)})_{\overline{0}}) = f_{(1)}(v_{(4)} < (m_{\overline{0}} \cdot v_{(2)})_{\overline{-1}}) f_{(2)}(S(v_{(3)}) m_{\overline{-1}} v_{(1)}) (m_{\overline{0}} \cdot v_{(2)})_{\overline{0}},
\]

where on the third equality we use \((3.33)\). So we get

\[
m_{\overline{-1}} \cdot v_{(1)} \otimes m_{\overline{0}} \cdot v_{(2)} = [v_{(4)} < (m_{\overline{0}} \cdot v_{(2)})_{\overline{-1}}] S(v_{(3)}) m_{\overline{-1}} v_{(1)} \otimes (m_{\overline{0}} \cdot v_{(2)})_{\overline{0}}.
\]

Using the cocommutativity of \(V\), we conclude \((2.71)\).

Conversely, take \(M \in \mathcal{U}(\mathcal{A}YD_{\mathcal{U} \otimes V})\). Then \(M\) is a left comodule over \(\mathcal{F} \triangleright \mathcal{U}\) by \((2.70)\) and a right module over \(\mathcal{F} \triangleright \mathcal{U}\) by \((2.68)\). So by Proposition \(3.8\) it suffices to verify \((3.33), \ldots, (3.36)\).

Indeed, \((3.33)\) follows from the coaction compatibility over \(\mathcal{U} \bowtie V\). The condition \((3.34)\) is the consequence of \((2.71)\). The equation \((3.35)\) is obtained from the module compatibility over \(\mathcal{U} \bowtie V\). Finally, \((3.36)\) follows from \((2.69)\).

**Proposition 3.11.** Let \((\mathfrak{g}_1, \mathfrak{g}_2)\) be a matched pair of Lie algebras and \(M\) be an \(AYD\) module over the double crossed sum \(\mathfrak{g}_1 \bowtie \mathfrak{g}_2\) with locally finite action and locally conilpotent coaction. Assume also that \(M\) is stable over \(R(\mathfrak{g}_2)\) and \(U(\mathfrak{g}_1)\). Then \(M\) is stable over \(R(\mathfrak{g}_2) \triangleright \triangleright U(\mathfrak{g}_1)\).

**Proof.** For an \(m \in M\), using the \(U(\mathfrak{g}_1 \bowtie \mathfrak{g}_2)\)-comodule compatibility \((3.42)\), we get

\[
(m_{\overline{0}})_{\overline{0}} \cdot (m_{\overline{-1}} \cdot (m_{\overline{0}})_{\overline{0}}) = ((m_{\overline{0}})_{\overline{0}} \cdot m_{\overline{-1}}) \cdot (m_{\overline{0}})_{\overline{0}} = (m_{\overline{0}} \cdot m_{\overline{-1}})_{\overline{0}} \cdot (m_{\overline{0}} \cdot m_{\overline{-1}})_{\overline{0}} = m_{\overline{0}} \cdot m_{\overline{-1}} = m.
\]

\(\square\)

23
3.3 AYD modules over the Connes-Moscovici Hopf algebras

In this subsection we investigate the finite dimensional SAYD modules over the Connes-Moscovici Hopf algebras \( \mathcal{H}_n \). Let us first recall from [13] the bicrossed product decomposition of the Connes-Moscovici Hopf algebras.

Let \( \text{Diff}(\mathbb{R}^n) \) denote the group of diffeomorphisms on \( \mathbb{R}^n \). Via the splitting \( \text{Diff}(\mathbb{R}^n) = G \rtimes N \), where \( G \) is the group of affine transformation on \( \mathbb{R}^n \) and \( N = \{ \psi \in \text{Diff}(\mathbb{R}^n) \mid \psi(0) = 0, \psi'(0) = \text{Id} \} \), (3.57)

we have \( \mathcal{H}_n = \mathcal{F}(N) \triangleright \triangleleft U(\mathfrak{g}) \). Elements of the Hopf algebra \( \mathcal{F} := \mathcal{F}(N) \) are called regular functions. They are the coefficients of the Taylor expansions at \( 0 \in \mathbb{R}^n \) of the elements of the group \( N \). Here, \( \mathfrak{g} \) is the Lie algebra of the group \( G \) and \( U := U(\mathfrak{g}) \) is the universal enveloping algebra of \( \mathfrak{g} \).

On the other hand, by [5] the Lie algebra \( \mathfrak{a} \) of formal vector fields on \( \mathbb{R}^n \) admits the filtration

\[
\mathfrak{a} = L_{-1} \supseteq L_0 \supseteq L_1 \supseteq \ldots
\]

with the bracket

\[
[L_p, L_q] \subseteq L_{p+q}.
\]

(3.59)

Here, the subalgebra \( L_k \subseteq \mathfrak{a} \), \( k \geq -1 \), consists of the vector fields \( \sum f_i \partial / \partial x^i \) such that \( f_1, \ldots, f_n \) belongs to the \((k + 1)\)st power of the maximal ideal of the ring of formal power series. Then it is immediate to conclude

\[
gl_n = L_0/L_1, \quad L_{-1}/L_0 \cong \mathbb{R}^n, \quad \text{and} \quad \mathfrak{g}_1 = L_{-1}/L_0 \oplus L_0/L_1 \cong gl_n^{\text{aff}}.
\]

(3.60)

As a result, setting \( n := L_1 \), the Lie algebra \( \mathfrak{a} \) admits the decomposition \( \mathfrak{a} = \mathfrak{g} \oplus \mathfrak{n} \), and hence we have a matched pair of Lie algebras \((\mathfrak{g}, \mathfrak{n})\). The Hopf algebra \( \mathcal{F}(N) \) is isomorphic with \( R(\mathfrak{n}) \) via the following non-degenerate pairing

\[
\langle \alpha^i_{j_1, \ldots, j_p}, Z^k_{l_1, \ldots, l_q} \rangle = \delta_q^i \delta_{l_1}^{j_1} \ldots \delta_{l_q}^{j_q} \delta_{l_{p+1} \ldots l_p}^k.
\]

(3.61)

Here

\[
\alpha^i_{j_1, \ldots, j_p}(\psi) = \left. \frac{\partial^p}{\partial x^{j_1} \ldots \partial x^{j_p}} \right|_{x=0} \psi^i(x),
\]

(3.62)

and

\[
Z^k_{l_1, \ldots, l_q} = x^{k_1} \ldots x^{k_q} \frac{\partial}{\partial x^l}.
\]

(3.63)

We refer the reader to [2] for more details on this duality.

Let \( \delta \) be the trace of the adjoint representation of \( \mathfrak{g} \) on itself. Then it is known that \( \mathbb{C}_\delta \) is a SAYD module over the Hopf algebra \( \mathcal{H}_n \) [2].
Lemma 3.12. For any YD module over $\mathcal{H}_n$, the action of $\mathcal{U}$ and the coaction of $\mathcal{F}$ are trivial.

Proof. Let $M$ be a finite dimensional YD module over $\mathcal{H}_n = \mathcal{F} \triangleright \triangleleft \mathcal{U}$. One uses the same argument as in Proposition 3.8 to show that $M$ is a module over $\mathfrak{a}$. However we know that $\mathfrak{a}$ has no nontrivial finite dimensional representation by [5]. We conclude that the $\mathcal{U}$ action and the $\mathcal{F}$-coaction on $M$ are trivial.

Let us introduce the isotropy subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ by

$$\mathfrak{g}_0 = \left\{ X \in \mathfrak{g}_1 \mid Y \triangleright X = 0, \forall Y \in \mathfrak{g}_2 \right\} \subseteq \mathfrak{g}_1.$$

(3.64)

By the construction of $\mathfrak{a}$ it is obvious that $\mathfrak{g}_0$ is generated by $Z^i_j$. So $\mathfrak{g}_0 \cong \mathfrak{gl}_n$. By the definition of the coaction $\nabla^\mathcal{U}: \mathcal{U} \rightarrow \mathcal{U} \otimes \mathcal{F}$ we see that $U(\mathfrak{g}_0) = U^{\text{co}\mathcal{F}}$.

Lemma 3.13. For any finite dimensional YD module $M$ over $\mathcal{H}_n$ the coaction $\nabla : M \rightarrow \mathcal{H}_n \otimes M$ lands merely in $U(\mathfrak{g}_0) \otimes M$.

Proof. By Lemma 3.12 we know that $\mathcal{U}$-action and $\mathcal{F}$-coaction on $M$ are trivial. Hence, the left coaction $M \rightarrow \mathcal{F} \triangleright \triangleleft \mathcal{U} \otimes M$ becomes $m \mapsto 1 \triangleright \triangleleft m^{[-1]} \otimes m^{[0]}$. The coassociativity of the coaction

$$1 \triangleright \triangleleft m^{[-2]} \otimes 1 \triangleright \triangleleft m^{[-1]} \otimes m^{[0]} = 1 \triangleright \triangleleft (m^{[-2]})^{(0)} \otimes (m^{[-2]})^{(1)} \triangleright \triangleleft m^{[-1]} \otimes m^{[0]}$$

(3.65)

implies that

$$m \mapsto m^{[-1]} \otimes m^{[0]} \in \mathcal{U}^{\text{co}\mathcal{F}} \otimes M = U(\mathfrak{g}_0) \otimes M.$$

(3.66)

Lemma 3.14. Let $M$ be a finite dimensional YD module over the Hopf algebra $\mathcal{H}_n$ then the coaction of $\mathcal{H}_n$ on $M$ is trivial.

Proof. By Lemma 3.13 we know that the coaction of $\mathcal{H}_n$ on $M$ lands in $U(\mathfrak{g}_0) \otimes M$. Since $U(\mathfrak{g}_0)$ is a Hopf subalgebra of $\mathcal{H}_n$, it is obvious that $M$ is an AYD module over $U(\mathfrak{g}_0)$. Since $\mathfrak{g}_0$ is finite dimensional, $M$ becomes an AYD module over $\mathfrak{g}_0$.

Let us express the $\mathfrak{g}_0$-coaction for an arbitrary basis element $m^i \in M$ as

$$m^i \mapsto m^i^{[-1]} \otimes m^i^{[0]} = \alpha_{kq}^{ip} Z^q_p \otimes m^k \in \mathfrak{g}_0 \otimes M.$$ 

(3.67)

Then AYD condition over $\mathfrak{g}_0$ becomes

$$\alpha_{kq}^{ip} [Z^q_p, Z] \otimes m^k = 0.$$

(3.68)

Choosing an arbitrary $Z = Z_{\mathfrak{g}_0}^{pq} \in \mathfrak{gl}_n = \mathfrak{g}_0$, we get

$$\alpha_{kq}^{ip} Z^q_p \otimes m^k - \alpha_{kq}^{ip} Z_{\mathfrak{g}_0}^{pq} \otimes m^k = 0,$$

(3.69)
or equivalently
\[ \alpha_{kp}^{i}(Z_{q}^0 - Z_{q0}^0) + \sum_{q \neq 0} \alpha_{kq}^{i} Z_{q}^q - \sum_{p \neq p_{0}} \alpha_{kp}^{i} Z_{p}^p = 0. \] (3.70)

Thus for \( n \geq 2 \) we have proved that the \( g_{0} \)-coaction is trivial. Hence its lift as a U(\( g_{0} \))-coaction that we have started with is trivial. This proves that the \( \mathcal{U} \) coaction and hence the \( H_{n} \) coaction on \( M \) is trivial.

On the other hand, for \( n = 1 \), the YD condition for \( X \in H_{1} \) reads, in view of the triviality of the action of \( gl_{1}^{af} \),
\[ Xm_{(-1)} \otimes m_{(0)} + Z(m \cdot \delta_{1})_{(-1)} \otimes (m \cdot \delta_{1})_{(0)} = m_{(-1)} X \otimes m_{(0)}. \] (3.71)
By Lemma 3.13 we know that the coaction lands in \( U(gl_{1}^{af}) \). Together with the relation \([Z, X] = X\) this forces the \( H_{1} \)-coaction (and also the action) to be trivial.

**Lemma 3.15.** Let \( M \) be a finite dimensional YD module over the Hopf algebra \( H_{n} \). Then the action of \( H_{n} \) on \( M \) is trivial.

**Proof.** By Lemma 3.12 we know that the action of \( H_{n} \) on \( M \) is concentrated on the action of \( F \) on \( M \). So it suffices to prove that this action is trivial.

For an arbitrary \( m \in M \) and \( 1 \triangleright X_{k} \in H_{n} \), we write the YD compatibility. First we calculate
\[
\Delta_{2}(1 \triangleright X_{k}) = \\
(1 \triangleright 1) \otimes (1 \triangleright 1) \otimes (1 \triangleright X_{k}) + (1 \triangleright 1) \otimes (1 \triangleright X_{k}) \otimes (1 \triangleright 1) \\
+ (1 \triangleright X_{k}) \otimes (1 \triangleright 1) \otimes (1 \triangleright 1) + (\delta_{kp}^{q} \cdot 1 \triangleright 1) \otimes (1 \triangleright Y_{q}^{p}) \otimes (1 \triangleright 1) \\
+ (1 \triangleright 1) \otimes (\delta_{pq}^{k} \cdot 1) \otimes (1 \triangleright Y_{p}^{q}) + (\delta_{pq}^{k} \cdot 1) \otimes (1 \triangleright 1) \otimes (1 \triangleright Y_{p}^{q}).
\] (3.72)

Since, by Lemma 3.14, the coaction of \( H_{n} \) on \( M \) is trivial, the AYD condition can be written as
\[
(1 \triangleright 1) \otimes m \cdot X_{k} = S(1 \triangleright X_{k}) \otimes m + 1 \triangleright 1 \otimes m \cdot X_{k} + 1 \triangleright X_{k} \otimes m +
\delta_{pq}^{k} \cdot 1 \triangleright m \cdot Y_{p}^{q} - 1 \triangleright Y_{p}^{q} \otimes m \cdot \delta_{pq}^{k} - \delta_{pq}^{k} \cdot Y_{p}^{q} \otimes m =
\delta_{pq}^{k} \cdot 1 \otimes m \cdot Y_{p}^{q} + Y_{i}^{j} \cdot \delta_{jk}^{i} \cdot 1 \otimes m - 1 \triangleright Y_{p}^{q} \otimes m \cdot \delta_{pq}^{k}.
\] (3.73)

Therefore,
\[ m \cdot \delta_{pq}^{k} = 0. \] (3.74)

Finally, by the module compatibility on a bicrossed product \( F \triangleright U \), we get
\[ (m \cdot X_{l}) \cdot \delta_{pq}^{k} = m \cdot (X_{l} \triangleright \delta_{pq}^{k}) + (m \cdot \delta_{pq}^{k}) \cdot X_{l}, \] (3.75)
which in turn, by using one more time the triviality of the \( U(\mathfrak{g}_1) \)-action on \( M \), implies

\[
m \cdot \delta^p_{qkl} = 0. \tag{3.76}
\]

Similarly we have

\[
m \cdot \delta^p_{qkl_1 \ldots l_s} = 0, \quad \forall s \tag{3.77}
\]

This proves that the \( \mathcal{F} \)-action and a posteriori the \( \mathcal{H}_n \) action on \( M \) is trivial. \( \square \)

Now we prove the main result of this section.

**Theorem 3.16.** The only finite dimensional AYD module over the Connes-Moscovici Hopf algebra \( \mathcal{H}_n \) is \( \mathbb{C}_\delta \).

**Proof.** By Lemma 3.15 and Lemma 3.14 we know that the only finite dimensional YD module on \( \mathcal{H}_n \) is the trivial one. On the other hand, by the result of M. Staic in [17] we know that the category of AYD modules and the category of YD modules over a Hopf algebra \( H \) are equivalent provided \( H \) has a modular pair in involution \((\theta, \sigma)\). In fact the equivalence functor between these two categories are simply given by

\[
H \mathcal{YD}_H \ni M \mapsto \sigma M := M \otimes \sigma \mathbb{C}_\theta \in H \mathcal{AYD}_H. \tag{3.78}
\]

Since by the result of Connes-Moscovici in [2] the Hopf algebra \( \mathcal{H}_n \) admits a modular pair in involution \((\delta, 1)\), we conclude that the only finite dimensional AYD module on \( \mathcal{H}_n \) is \( \mathbb{C}_\delta \). \( \square \)

### 4 Hopf-cyclic cohomology with coefficients

Thanks to the results in the second section, we know all SAYD modules over a Lie-Hopf algebra \( (R(\mathfrak{g}_2), \mathfrak{g}_1) \) in terms of AYD modules over the ambient Lie algebra \( \mathfrak{g}_1 \bowtie \bowtie \mathfrak{g}_2 \). The next natural question is the Hopf cyclic cohomology of the bicrossed product Hopf algebra with coefficients in such a module \( \sigma M_\delta \), where \((\delta, \sigma)\) is the natural modular pair in involution associated to \((\mathfrak{g}_1, \mathfrak{g}_2)\) and \( M \) is a SAYD module over \( \mathfrak{g}_1 \bowtie \bowtie \mathfrak{g}_2 \). To answer this question we need to prove a van Est type theorem between Hopf cyclic complex of the Hopf algebra \( R(\mathfrak{g}_2) \bowtie U(\mathfrak{g}_1) \) with coefficients \( \sigma M_\delta \) and the relative perturbed Koszul complex of \( \mathfrak{g}_1 \bowtie \bowtie \mathfrak{g}_2 \) with coefficient \( M \) introduced in [16]. Actually we observe that our strategy in [15] can be improved to include all cases, not only the induced coefficients introduced in [15]. The main obstacle here is the \( R(\mathfrak{g}_2) \)-action and \( U(\mathfrak{g}_1) \)-coaction which prevent us from having two trivial (co)boundary maps. The first one is the Hochschild coboundary map of \( U(\mathfrak{g}_1) \) and the second one is the Connes boundary map of \( R(\mathfrak{g}_2) \). We observe that the filtration on \( \sigma M_\delta \) originally defined by Jara-Stefan in [10] is extremely helpful. In the first page of the spectral sequence associated to such filtration these two (co)boundary vanish and the situation descends to the case of [15].
4.1 Relative Lie algebra cohomology and cyclic cohomology of Hopf algebras

For a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and a right $\mathfrak{g}$-module $M$ we define the relative cochains by

$$C^q(\mathfrak{g}, \mathfrak{h}, M) = \left\{ \alpha \in C^q(\mathfrak{g}, M) = \bigwedge^q \mathfrak{g}^* \otimes M \mid \iota(X)\alpha = \mathcal{L}_X(\alpha) = 0, \ X \in \mathfrak{h} \right\},$$

(4.1)

where

$$\iota(X)(\alpha)(X_1, \ldots, X_q) = \alpha(X, X_1, \ldots, X_q),$$

(4.2)

$$\mathcal{L}_X(\alpha)(X_1, \ldots, X_q) = \sum (-1)^i \alpha([X, X_i], X_1, \ldots, \hat{X}_i, \ldots, X_q) + \theta(X_1, \ldots, X_q)X.$$

(4.3)

We can identify $C^q(\mathfrak{g}, \mathfrak{h}, M)$ with $\text{Hom}_{\mathfrak{h}}(\bigwedge^q (\mathfrak{g}/\mathfrak{h})^*, M)$ which is $(\bigwedge^q (\mathfrak{g}/\mathfrak{h})^* \otimes M)_{\mathfrak{h}}$, where the action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$ is induced by the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$.

It is checked in [1] that the Chevalley-Eilenberg coboundary $d_{CE} : C^q(\mathfrak{g}, M) \rightarrow C^{q+1}(\mathfrak{g}, M)$

$$d_{CE}(\alpha)(X_0, \ldots, X_q) = \sum (-1)^{i+j} \alpha([X_i, X_j], X_0 \ldots \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_q) + \sum (-1)^{i+j} \alpha(X_0, \ldots, \hat{X}_i, \ldots, X_q)X_i.$$

(4.4)

is well defined on $C^\bullet(\mathfrak{g}, \mathfrak{h}, M)$. We denote the homology of the complex $(C^\bullet(\mathfrak{g}, \mathfrak{h}, M), d_{CE})$ by $H^\bullet(\mathfrak{g}, \mathfrak{h}, M)$ and refer to it as the relative Lie algebra cohomology of $\mathfrak{h} \subseteq \mathfrak{g}$ with coefficients in $M$.

Next, we recall the perturbed Koszul complex $W(\mathfrak{g}, M)$ from [16]. Let $M$ be a right $\mathfrak{g}$-module and $S(\mathfrak{g}^*)$-module satisfying

$$(m \cdot X_j) \cdot \theta^t = m \cdot (X_j \triangleright \theta^t) + (m \cdot \theta^t) \cdot X_j.$$

(4.5)

Then $M$ is a module over the semi direct sum Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g}^* \triangleright\triangleleft \mathfrak{g}$.

Let $M$ be a module over the Lie algebra $\tilde{\mathfrak{g}}$. Then

- $M$ is called unimodular stable if $\sum_k (m \cdot X_k) \cdot \theta^k = 0$,
- $M$ is called stable if $\sum_k (m \cdot \theta^k) \cdot X_k = 0$.

(4.6)

By [16, Proposition 4.3], if $M$ is unimodular stable, then $M_\beta := M \otimes \mathbb{C}_\beta$ is stable over $\mathfrak{g}$. Here $\beta$ is the trace of the adjoint representation of the Lie algebra $\mathfrak{g}$ on itself.

For a unimodular stable right $\tilde{\mathfrak{g}}$-module $M$, the graded space $W^n(\mathfrak{g}, M) := \wedge^n \mathfrak{g}^* \otimes M$, $n \geq 0$, becomes a mixed complex with Chevalley-Eilenberg coboundary

$$d_{CE} : W^n(\mathfrak{g}, M) \rightarrow W^{n+1}(\mathfrak{g}, M)$$
and the Koszul differential
\[ d_K : W^n(\mathfrak{g}, M) \to W^{n-1}(\mathfrak{g}, M) \]
\[ \alpha \otimes m \mapsto \sum_i \iota_{X_i}(\alpha) \otimes m \otimes \theta^i. \] (4.7)

By [16, Proposition 5.13], a unimodular stable right ˜\mathfrak{g}-module is a right-left unimodular stable AYD module, unimodular SAYD in short, over the Lie algebra \( \mathfrak{g} \).

Finally we introduce the relative perturbed Koszul complex,
\[ W(\mathfrak{g}, \mathfrak{h}, M) = \left\{ f \in W(\mathfrak{g}, M) \mid \iota(Y)f = 0, \iota(Y)(d_{CE}f) = 0, \forall Y \in \mathfrak{h} \right\}. \] (4.8)

We have the following result.

**Lemma 4.1.** \( d_K(W(\mathfrak{g}, \mathfrak{h}, M)) \subseteq W(\mathfrak{g}, \mathfrak{h}, M) \).

**Proof.** For any \( \alpha \otimes m \in W^{n+1}(\mathfrak{g}, \mathfrak{h}, M) \) and any \( Y \in \mathfrak{h} \),
\[ \iota(Y)(d_K(\alpha \otimes m)) = \iota(Y)((-1)^n\iota(m_{-1})\alpha \otimes m_{[0]}) = (-1)^n\iota(Y)\iota(m_{-1})\alpha \otimes m_{[0]} \]
\[ = (-1)^{n-1}\iota(m_{-1})\iota(Y)\alpha \otimes m_{[0]} = d_K(\iota(Y)\alpha \otimes m) = 0 \] (4.9)

Similarly, using \( d_{CE} \circ d_K + d_K \circ d_{CE} = 0 \),
\[ \iota(Y)(d_{CE}(d_K(\alpha \otimes m))) = -\iota(Y)(d_K \circ d_{CE}(\alpha \otimes m)) = -d_K(\iota(Y)d_{CE}(\alpha \otimes m)) = 0. \] (4.10)

**Definition 4.2.** Let \( \mathfrak{g} \) be a Lie algebra, \( \mathfrak{h} \subseteq \mathfrak{g} \) be a Lie subalgebra and \( M \) be a unimodular SAYD module over \( \mathfrak{g} \). We call the homology of the mixed subcomplex \( (W(\mathfrak{g}, \mathfrak{h}, M), d_{CE} + d_K) \) the relative periodic cyclic cohomology of the Lie algebra \( \mathfrak{g} \) relative to the Lie subalgebra \( \mathfrak{h} \) with coefficients in unimodular stable right ˜\mathfrak{g}-module \( M \). We use the notation \( \tilde{H}^p(\mathfrak{g}, \mathfrak{h}, M) \).

In case of the trivial Lie algebra coaction, this cohomology becomes the relative Lie algebra cohomology.

We conclude this subsection by a brief account of cyclic cohomology of Hopf algebras. Let \( M \) be a right-left SAYD module over a Hopf algebra \( \mathcal{H} \). Let
\[ C^q(\mathcal{H}, M) := M \otimes \mathcal{H}^\otimes q, \quad q \geq 0. \] (4.11)

We recall the following operators on \( C^\bullet(\mathcal{H}, M) \)

- face operators \( \partial_i : C^q(\mathcal{H}, M) \to C^{q+1}(\mathcal{H}, M) \), \( 0 \leq i \leq q + 1 \)
- degeneracy operators \( \sigma_j : C^q(\mathcal{H}, M) \to C^{q-1}(\mathcal{H}, M) \), \( 0 \leq j \leq q - 1 \)
- cyclic operators \( \tau : C^q(\mathcal{H}, M) \to C^q(\mathcal{H}, M) \),
by
\[
\begin{align*}
\partial_0 (m \otimes h^1 \otimes \ldots \otimes h^q) &= m \otimes 1 \otimes h^1 \otimes \ldots \otimes h^q, \\
\partial_i (m \otimes h^1 \otimes \ldots \otimes h^q) &= m \otimes h^1 \otimes \ldots \otimes h^i \otimes h^{(i+1)} \otimes \ldots \otimes h^q, \\
\partial_{q+1} (m \otimes h^1 \otimes \ldots \otimes h^q) &= m_{(0)} \otimes h^1 \otimes \ldots \otimes h^q \otimes m_{(-1)}, \\
\sigma_j (m \otimes h^1 \otimes \ldots \otimes h^q) &= (m \otimes h^1 \otimes \ldots \otimes \varepsilon(h^{j+1}) \otimes \ldots \otimes h^q), \\
\tau (m \otimes h^1 \otimes \ldots \otimes h^q) &= m_{(0)} h^{(1)} \otimes S(h^{(2)}) \cdot (h^2 \otimes \ldots \otimes h^q \otimes m_{(-1)}),
\end{align*}
\]
where \( H \) acts on \( H^{\otimes q} \) diagonally.

The graded module \( C(H, M) \) endowed with the above operators is then a cocyclic module [6], which means that \( \partial_i, \sigma_j \) and \( \tau \) satisfy the following identities
\[
\begin{align*}
\partial_j \partial_i &= \partial_i \partial_j, & \text{if } i < j, \\
\sigma_j \sigma_i &= \sigma_i \sigma_{j+1}, & \text{if } i \leq j, \\
\sigma_j \partial_i &= \begin{cases} 
\partial_i \sigma_{j-1}, & \text{if } i < j \\
\Id & \text{if } i = j \text{ or } i = j + 1 \\
\partial_{i-1} \sigma_j & \text{if } i > j + 1,
\end{cases} \\
\tau \partial_i &= \partial_{i-1} \tau, & 1 \leq i \leq q \\
\tau \partial_0 &= \partial_{q+1}, \\
\tau \sigma_0 &= \sigma_{q+1} \tau = \Id.
\end{align*}
\]

One uses the face operators to define the Hochschild coboundary
\[
b : C^q(H, M) \rightarrow C^{q+1}(H, M), \quad \text{by} \quad b := \sum_{i=0}^{q+1} (-1)^i \partial_i. \tag{4.14}
\]

It is known that \( b^2 = 0 \). As a result, one obtains the Hochschild complex of the coalgebra \( H \) with coefficients in the bicomodule \( M \). Here, the right \( H \)-comodule structure on \( M \) defined trivially. The cohomology of \( (C^\bullet(H, M), b) \) is denoted by \( H_{\text{coalg}}^\bullet(H, M) \).

One uses the rest of the operators to define the Connes boundary operator,
\[
B : C^q(H, M) \rightarrow C^{q-1}(H, M), \quad \text{by} \quad B := \left( \sum_{i=0}^{q} (-1)^{q-i} \tau^i \right) \sigma_{q-1} \tau. \tag{4.15}
\]

It is shown in [3] that for any cocyclic module we have \( b^2 = B^2 = (b + B)^2 = 0 \). As a result, one defines the cyclic cohomology of \( H \) with coefficients in the SAYD module \( M \), which is denoted by \( HC^\bullet(H, M) \), as the total cohomology of the bicomplex
\[
C^{p,q}(H, M) = \begin{cases} 
M \otimes H^{\otimes q-p}, & \text{if } 0 \leq p \leq q, \\
0, & \text{otherwise}.
\end{cases} \tag{4.16}
\]
One also defines the periodic cyclic cohomology of $\mathcal{H}$ with coefficients in $M$, which is denoted by $HP^*(\mathcal{H}, M)$, as the total cohomology of direct sum total of the bicomplex

$$C^{p,q}(\mathcal{H}, M) = \begin{cases} M \otimes \mathcal{H}^{\otimes q-p}, & \text{if } p \leq q, \\ 0, & \text{otherwise}. \end{cases}$$

(4.17)

It can be seen that the periodic cyclic complex and hence the cohomology is $\mathbb{Z}_2$ graded.

### 4.2 Hopf-cyclic cohomology of Lie-Hopf algebras

Our first task in this subsection is to calculate the periodic cyclic cohomology of $R(\mathfrak{g})$ with coefficients in a general SAYD module. This will generalize our result in [15, Theorem 4.7], where the coefficients were the induced ones, i.e., those SAYD modules induced by a module over $\mathfrak{g}$.

In the second subsubsection we prove the main result of this paper. Roughly speaking, we identify the Hopf cyclic cohomology of a bicrossed product Hopf algebra, associated with a Lie algebra decomposition, with the Lie algebra cohomology of the ambient Lie algebra. The novelty here is the fact that we are able to prove such a van Est type theorem with the most general coefficients.

#### 4.2.1 Hopf algebra of representative functions

Let $M$ be a locally finite $\mathfrak{g}$-module and locally conilpotent $\mathfrak{g}$-comodule. We first define a left $R(\mathfrak{g})$-coaction on $M$. It is known, by [8] (see also [15]), that the linear map

$$M \to U(\mathfrak{g})^* \otimes M, \quad m \mapsto m^{(-1)} \otimes m^{(0)}$$

(4.18)

defined by the rule $m^{(-1)}(u)m^{(0)} = m \cdot u$, defines a left $R(\mathfrak{g})$-comodule structure

$$\nabla : M \to R(\mathfrak{g}) \otimes M$$

$$m \mapsto m^{(-1)} \otimes m^{(0)}.$$  

(4.19)

Then using the left $U(\mathfrak{g})$-comodule on $M$, we define the right $R(\mathfrak{g})$-module structure

$$M \otimes R(\mathfrak{g}) \to M$$

$$m \otimes f \mapsto m \cdot f := f(m^{(-1)})m^{(0)}.$$ 

(4.20)

**Proposition 4.3.** Let $M$ be locally finite as a $\mathfrak{g}$-module and locally conilpotent as a $\mathfrak{g}$-comodule. If $M$ is an AYD over $\mathfrak{g}$, then it is an AYD over $R(\mathfrak{g})$.

**Proof.** Since $M$ is AYD module over the Lie algebra $\mathfrak{g}$ with a locally conilpotent $\mathfrak{g}$-coaction, it is an AYD over $U(\mathfrak{g})$. 

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Let $m \in M$, $f \in R(\mathfrak{g})$ and $u \in U(\mathfrak{g})$. On one hand side of the AYD condition we have

\[ \nabla_{R(\mathfrak{g})}(m \cdot f)(u) = (m \cdot f) \cdot u = f(m_{[-1]})m_{[0]} \cdot u, \quad (4.21) \]

and on the other hand,

\[
\begin{align*}
(S(f_{(3)})m^{[-3]}f_{(1)})(u)m^{[0]} \cdot f_{(2)} &= \\
S(f_{(3)})(u_{(1)})m^{[-3]}(u_{(2)})f_{(1)}(u_{(3)})f_{(2)}((m^{[0]})_{[-1]})m^{[0]}_{[0]} &= \\
f_{(2)}(S(u_{(4)}))m^{[-3]}(u_{(2)})f_{(1)}(u_{(3)}(m^{[0]})_{[-1]})m^{[0]}_{[0]} &= \\
m^{[-3]}(u_{(2)})f(u_{(3)}(m^{[0]})_{[-1]})S(u_{(1)}))m^{[0]}_{[0]} &= \\
f(u_{(3)}(m^{[-1]}(u_{(2)}m^{[0]}))S(u_{(1)}))m^{[-1]}(u_{(2)})m^{[0]}_{[0]} &= \\
&= (4.22)
\end{align*}
\]

where we used the AYD condition on $U(\mathfrak{g})$ on the sixth equality. This proves that $M$ is an AYD module over $R(\mathfrak{g})$.

**Theorem 4.4.** Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{l}$ be a Levi decomposition. Let $M$ be a unimodular SAYD module over $\mathfrak{g}$ as a locally finite $\mathfrak{g}$-module and locally conilpotent $\mathfrak{g}$-comodule. Assume also that $M$ is stable over $R(\mathfrak{g})$. Then the periodic cyclic cohomology of $\mathfrak{g}$ relative to the subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ with coefficients in $M$ is the same as the periodic cyclic cohomology of $R(\mathfrak{g})$ with coefficients in $M$. In short,

\[ \widetilde{HP}(\mathfrak{g}, \mathfrak{h}, M) \cong HP(R(\mathfrak{g}), M) \quad (4.23) \]

**Proof.** Since $M$ is a unimodular stable AYD module over $\mathfrak{g}$, by Lemma 4.1 the relative perturbed Koszul complex $(W(\mathfrak{g}, \mathfrak{h}, M), d_{CE} + d_K)$ is well defined. On the other hand, since $M$ is locally finite as a $\mathfrak{g}$-module and locally conilpotent as a $\mathfrak{g}$-comodule, it is an AYD module over $R(\mathfrak{g})$ by Proposition 4.3. Together with the assumption that $M$ is stable over $R(\mathfrak{g})$, the Hopf-cyclic complex $(C(R(\mathfrak{g}), M), b + B)$ is well defined.

Since $M$ is a unimodular SAYD over $\mathfrak{g}$, $M_{\beta} := M \otimes \mathbb{C}_{\beta}$ is SAYD over $\mathfrak{g}$ by [16, Proposition 4.3]. Here $\beta$ is the trace of the adjoint representation of the Lie algebra $\mathfrak{g}$ on itself. Therefore, by [10, Lemma 6.2] we have the filtration $M = \cup_{p \in \mathbb{Z}} F_pM$ defined as $F_0M = M^{\text{co}U(\mathfrak{g})}$ and inductively

\[ F_{p+1}M/F_pM = (M/F_pM)^{\text{co}U(\mathfrak{g})}. \quad (4.24) \]

This filtration naturally induces an analogous filtration on the complexes as

\[ F_jW(\mathfrak{g}, \mathfrak{h}, M) = W(\mathfrak{g}, \mathfrak{h}, F_jM), \quad \text{and} \quad F_jC(R(\mathfrak{g}), M) = C(R(\mathfrak{g}), F_jM). \quad (4.25) \]
We now show that the (co)boundary maps $d_{CE}, d_K, b, B$ respect this filtration. To do so for $d_K$ and $d_{CE}$, it suffices to show that the $g$-action and $g$-coaction on $M$ respect the filtration; which is done by the same argument as in the proof of Theorem 6.2 in [16]. Similarly, to show that the Hochschild coboundary $b$ and the Connes boundary map $B$ respect the filtration we need to show that $R(g)$-action and $R(g)$-coaction respects the filtration.

Indeed, for an element $m \in F_p M$, writing the $U(g)$-coaction as 
\[ m \otimes m_{(-1)} \mapsto m \otimes m_{(-1)} \otimes m_{(0)}, \quad m_{(-1)} \otimes m_{(0)} \in U(g) \otimes F_{p-1} M, \] (4.26)
we get for any $f \in R(g)$
\[ m \cdot f = \varepsilon(f)m + f(m_{(-1)})m_{(0)} \in F_p M. \] (4.27)
This proves that $R(g)$-action respects the filtration. To prove that $R(g)$-coaction respects the filtration, we first write the coaction on $m \in F_p M$ as 
\[ m \mapsto \sum_i f^i \otimes m_i \in R(g) \otimes M. \] (4.28)
By [9] there are elements $u_j \in U(g)$ such that $f^i(u_j) = \delta^i_j$. Hence, for any $m_{i_0}$ we have
\[ m_{i_0} = \sum_i f^i(u_{i_0})m_i = m \cdot u_{i_0} \in F_p M. \] (4.29)
We have proved that $R(g)$-coaction respects the filtration.

Next, we write the $E_1$ terms of the associated spectral sequences. We have
\[
E_1^{i,j}(g, h, M) = H^{i+j}(F_j W(g, h, M)/F_{j-1} W(g, h, M)) = \bigoplus_{i+j=n \ mod \ 2} H^n(g, h, F_j M/F_{j-1} M),
\] (4.30)
where on the last equality we used the fact that $F_j M/F_{j-1} M$ has trivial $g$-coaction.

Similarly we have
\[
E_1^{i,j}(R(g), M) = H^{i+j}(F_j C(R(g), M)/F_{j-1} C(R(g), M)) = \bigoplus_{i+j=n \ mod \ 2} H^n_{coalg}(R(g), F_j M/F_{j-1} M),
\] (4.31)
where on the last equality we could use [15, Theorem 4.3] due to the fact that $F_j M/F_{j-1} M$ has trivial $g$-coaction, hence trivial $R(g)$-action.

Finally, under the hypothesis of the theorem, a quasi-isomorphism between $E_1$ terms is given by [15, Theorem 4.7].

Remark 4.5. If $M$ has a trivial $g$-comodule structure, then $d_K = 0$ and hence the above theorem descents to [15, Theorem 4.7].
4.2.2 Bicrossed product Hopf algebras

Let $M$ be an AYD module over a double crossed sum Lie algebra $g_1 \bowtie g_2$ with a locally finite $g_1 \bowtie g_2$-action and a locally conilpotent $g_1 \bowtie g_2$-coaction. Then by Proposition 3.9 $M$ is a YD module over the bicrossed product Hopf algebra $R(g_2) \bowtie U(g_1)$.

Let also $(\delta, \sigma)$ be an MPI for the Hopf algebra $R(g_2) \bowtie U(g_1)$, see [15, Theorem 3.2]. Then $\sigma M_\delta := M \otimes \sigma C_\delta$ is an AYD module over the bicrossed product Hopf algebra $R(g_2) \bowtie U(g_1)$.

Finally, let us assume that $M$ is stable over $R(g_2)$ and $U(g_1)$. Then $\sigma M_\delta$ is stable if and only if

$$m \otimes 1_C = (m^{(1)} \otimes 1_C) \cdot (m^{(2)} \triangleright m^{(3)})(\sigma \triangleright 1)$$

$$= (m^{(1)} \otimes 1_C) \cdot (m^{(2)} \triangleright 1)(1 \triangleright m^{(3)})(\sigma \triangleright 1)$$

$$= (m^{(1)} \cdot m^{(2)} \otimes 1_C) \cdot (1 \triangleright m^{(3)})(\sigma \triangleright 1)$$

$$= ((m^{(1)} \cdot m^{(2)} \triangleright m^{(3)})) \cdot m^{(4)} \delta(m^{(5)})(\otimes 1_C) \cdot (\sigma \triangleright 1)$$

$$= m \cdot \delta(m) \otimes 1_C. \quad (4.32)$$

Here, on the fourth equality we have used Proposition 3.11. In other words, $M$ satisfying the hypothesis of the Proposition 3.11, $\sigma M_\delta$ is stable if and only if

$$m \cdot \delta(m) \otimes 1_C = m \quad (4.33)$$

**Theorem 4.6.** Let $(g_1, g_2)$ be a matched pair of Lie algebras and $g_2 = h \times \mathfrak{l}$ be a Levi decomposition such that $h$ is $g_1$ invariant and it acts on $g_1$ by derivations. Let $M$ be a unimodular SAYD module over $g_1 \bowtie g_2$ with a locally finite $g_1 \bowtie g_2$-action and locally conilpotent $g_1 \bowtie g_2$-coaction. Finally assume that $\sigma M_\delta$ is stable. Then we have

$$HP(R(g_2) \bowtie U(g_1), \sigma M_\delta) \cong \widetilde{HP}(g_1 \bowtie g_2, h, M) \quad (4.34)$$

**Proof.** Let $C(U(g_1) \bowtie R(g_2), \sigma M_\delta)$ be the complex computing the Hopf-cyclic cohomology of the bicrossed product Hopf algebra $\mathcal{H} = R(g_2) \bowtie U(g_1)$ with coefficients in the SAYD module $\sigma M_\delta$.

By [13], Theorem 3.16, this complex is quasi-isomorphic with the total complex of the mixed complex $\mathcal{F}(\mathcal{H}, R(g_2); \sigma M_\delta)$ whose cylindrical structure is given by the following operators. The horizontal operators are

$$\tilde{\partial}_0(m \otimes \tilde{f} \otimes \tilde{u}) = m \otimes 1 \otimes f^1 \otimes \ldots \otimes f^p \otimes \tilde{u}$$

$$\tilde{\partial}_1(m \otimes \tilde{f} \otimes \tilde{u}) = m \otimes f^1 \otimes \ldots \otimes \Delta(f^1) \otimes \ldots \otimes f^p \otimes \tilde{u}$$

$$\tilde{\partial}_{p+1}(m \otimes \tilde{f} \otimes \tilde{u}) = m_{(0)} \otimes f^1 \otimes \ldots \otimes f^p \otimes \tilde{u}^{(-1)} m_{(-1)} \triangleright 1_{R(g_2)} \otimes \tilde{u}^{(0)}$$

$$\tilde{\sigma}_j(m \otimes \tilde{f} \otimes \tilde{u}) = m \otimes f^1 \otimes \ldots \otimes \varepsilon(f^{j+1}) \otimes \ldots \otimes f^p \otimes \tilde{u}$$

$$\tilde{\tau}(m \otimes \tilde{f} \otimes \tilde{u}) = m_{(0)} f^1 \otimes S(f^1_{(2)}) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \tilde{u}^{(-1)} m_{(-1)} \triangleright 1_{R(g_2)} \otimes \tilde{u}^{(0)}) \quad (4.35)$$
and the vertical operators are
\[ \uparrow \partial_0 (m \otimes \tilde{f} \otimes \tilde{u}) = m \otimes \tilde{f} \otimes 1 \otimes u^1 \otimes \ldots \otimes u^q \]
\[ \uparrow \partial_0 (m \otimes \tilde{f} \otimes \tilde{u}) = m \otimes \tilde{f} \otimes u^0 \otimes \ldots \otimes \Delta(u^1) \otimes \ldots \otimes u^q \]
\[ \uparrow \partial_{q+1} (m \otimes \tilde{f} \otimes \tilde{u}) = m_{(0)} \otimes \tilde{f} \otimes u^1 \otimes \ldots \otimes u^q \otimes \overline{m_{(-1)}} \]
\[ \uparrow \sigma (m \otimes \tilde{f} \otimes \tilde{u}) = m \otimes \tilde{f} \otimes u^1 \otimes \ldots \otimes \varepsilon(w^{i+1}) \otimes \ldots \otimes u^q \]
\[ \uparrow \tau (m \otimes \tilde{f} \otimes \tilde{u}) = m_{(0)} u^1_{(4)} S^{-1}(u^1_{(3)} \triangleright 1_{R(g_2)}) \otimes S(S^{-1}(u^2_{(2)} \triangleright 1_{R(g_2)}) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{m_{(-1)}})) \]
(4.36)

for any \( m \in \sigma M_\delta \).

Here,
\[ U(g_1)^q \otimes \to \mathcal{H} \otimes U(g_1)^q, \quad \tilde{u} \mapsto \tilde{u}^{(1)} \otimes \tilde{u}^{(0)} \]
(4.37)
arises from the left \( \mathcal{H} \)-coaction on \( U(g_1) \) that coincides with the original \( R(g_2) \)-coaction, [13, Proposition 3.20]. On the other hand, \( U(g_1) \cong \mathcal{H} \otimes R(g_2) \subseteq \mathcal{H} / \mathcal{H} R(g_2)^+ \) as coalgebras via the map \( (f \triangleright u) \otimes R(g_2) \ \lambda_C \mapsto \varepsilon(f)u \) and \( \overline{f} \triangleright u = \varepsilon(f)u \) denotes the corresponding class.

Since \( M \) is a unimodular SAYD module over \( g_1 \otimes g_2 \), it admits the filtration \( (F_p M)_p \in \mathbb{Z} \) defined similarly as before. We recall by Proposition 2.5 that \( g_1 \otimes g_2 \)-coaction is the summation of \( g_1 \)-coaction and \( g_2 \)-coaction. Therefore, since \( g_1 \otimes g_2 \)-coaction respects the filtration, we conclude that \( g_1 \)-coaction and \( g_2 \)-coaction respect the filtration. Similarly, since \( g_1 \otimes g_2 \)-action respects the filtration, we conclude that \( g_1 \)-action and \( g_2 \)-action respects the filtration. Finally, by a similar argument as in the proof of Theorem 4.4 we can say that \( R(g_2) \)-action and \( R(g_2) \)-coaction respect the filtration.

As a result, the (co)boundary maps \( d_{CE} \) and \( d_K \) of the complex \( W(g_1 \otimes g_2, \mathfrak{h}, M) \), and \( b \) and \( B \) from \( C(U(g_1) \triangleright R(g_2), \sigma M_\delta) \) respect the filtration.

Next, we proceed to the \( E_1 \) terms of the associated spectral sequences. We have
\[ E_1^{ij}(R(g_2) \triangleright U(g_1), \sigma M_\delta) = \]
\[ H^{i+j}(F_{\mathcal{J}} C(U(g_1) \triangleright R(g_2), \sigma M_\delta) / F_{\mathcal{J}-1} C(U(g_1) \triangleright R(g_2), \sigma M_\delta)), \]
(4.38)

where
\[ F_{\mathcal{J}} C(U(g_1) \triangleright R(g_2), \sigma M_\delta) / F_{\mathcal{J}-1} C(U(g_1) \triangleright R(g_2), \sigma M_\delta) = \]
\[ C(U(g_1) \triangleright R(g_2), F_{\mathcal{J}} \sigma M_\delta) / F_{\mathcal{J}-1} \sigma M_\delta). \]
(4.39)

Since
\[ F_{\mathcal{J}} \sigma M_\delta / F_{\mathcal{J}-1} \sigma M_\delta = \sigma (F_{\mathcal{J}} M / F_{\mathcal{J}-1} M)_\delta =: \sigma M_\delta \]
has a trivial \( g_1 \otimes g_2 \)-comodule structure, its \( U(g_1) \)-comodule structure and \( R(g_2) \)-module structure are also trivial. Therefore, it is an \( R(g_2) \)-SAYD in the sense of
[13]. In this case, by [13, Proposition 3.16], \( \mathcal{H}(R(g_2); \sigma M_\delta) \) is a bicyclic module and the cohomology in (4.38) is computed from the total of the following bicocyclic complex

\[
\begin{align*}
\sigma M_\delta \otimes U(g_1)^{\otimes 2} & \xrightarrow{\sigma M_\delta \otimes U(g_1)^{\otimes 2} \otimes R(g_2)} \sigma M_\delta \otimes U(g_1)^{\otimes 2} \otimes R(g_2)^{\otimes 2} \longrightarrow \cdots \\
\sigma M_\delta \otimes U(g_1) & \xrightarrow{\sigma M_\delta \otimes U(g_1) \otimes R(g_2)} \sigma M_\delta \otimes U(g_1) \otimes R(g_2)^{\otimes 2} \longrightarrow \cdots \\
\sigma M_\delta & \xrightarrow{\sigma M_\delta \otimes R(g_2)} \sigma M_\delta \otimes R(g_2)^{\otimes 2} \longrightarrow \cdots
\end{align*}
\]

Moreover, by [15, Proposition 5.1], the total of the bicomplex (4.40) is quasi-isomorphic to the total of the bicomplex

\[
\begin{align*}
\sigma M_\delta \otimes \Lambda^2_{g_1} & \xrightarrow{b_{R(g_2)}} \sigma M_\delta \otimes \Lambda^2_{g_1} \otimes R(g_2) \xrightarrow{b_{R(g_2)}} \sigma M_\delta \otimes \Lambda^2_{g_1} \otimes R(g_2)^{\otimes 2} \longrightarrow \cdots \\
\sigma M_\delta \otimes g_1^* & \xrightarrow{b_{R(g_2)}} \sigma M_\delta \otimes g_1^* \otimes R(g_2) \xrightarrow{b_{R(g_2)}} \sigma M_\delta \otimes g_1^* \otimes R(g_2)^{\otimes 2} \longrightarrow \cdots \\
\sigma M_\delta & \xrightarrow{b_{R(g_2)}} \sigma M_\delta \otimes R(g_2) \xrightarrow{b_{R(g_2)}} \sigma M_\delta \otimes R(g_2)^{\otimes 2} \longrightarrow \cdots
\end{align*}
\]

where \( b_{R(g_2)} \) is the coalgebra Hochschild coboundary with coefficients in the \( R(g_2) \)-comodule \( \sigma M_\delta \otimes \Lambda^3_{g_1} \).

Similarly,

\[
E_{i+j}^{1,i}(g_1 \bowtie g_2, h, M) = H^{i+j}(F_j W(g_1 \bowtie g_2, h, M) / F_{j-1} W(g_1 \bowtie g_2, h, M)) = H^{i+j}(W(g_1 \bowtie g_2, h, F_j M/ F_{j-1} M)) = \bigoplus_{i+j=n \mod 2} H^n(g_1 \bowtie g_2, h, F_j M/ F_{j-1} M)
\]

(4.42)

where the last equality follows from the fact that \( F_j M/ F_{j-1} M \) has a trivial \( g_1 \bowtie g_2 \)-comodule structure.

Finally, the quasi isomorphism between the \( E_1 \)-terms (4.38) and (4.42) is given by the Corollary 5.10 of [15].
Remark 4.7. In case of a trivial $g_1 \bowtie g_2$-coaction, this theorem becomes \cite[Corollary 5.10]{15}. In this case, $U(g_1)$-coaction and $R(g_2)$-action are trivial, therefore the condition (4.33) is obvious.

5 Illustration

In this section, first we exercise our method in Sections 3 to provide a highly nontrivial 4-dimensional SAYD module over $H_{1S}^{\text{cop}} \cong R(\mathbb{C}) \bowtie U(gl_1^{\text{aff}})$, the Schwarzian Hopf algebra, which is introduced in \cite{2}. The merit of this example is the nontriviality of the $R(\mathbb{C})$-action and the $U(gl_1^{\text{aff}})$-coaction which were assumed to be trivial for induced modules in \cite{15}. We then illustrate Theorem 4.6 by computing two hands of the theorem. At the end we explicitly compute the representative cocycles for these cohomology classes. From now on we denote $R(\mathbb{C})$ by $\mathcal{F}$, $U(gl_1^{\text{aff}})$ by $U$ and $H_{1S}^{\text{cop}}$ by $\mathcal{H}$.

5.1 A 4-dimensional SAYD module on the Schwarzian Hopf algebra

Let us first recall the Lie algebra $sl_2$ as a double crossed sum Lie algebra. We have $sl_2 = g_1 \bowtie g_2$, $g_1 = \mathbb{C}\langle X, Y \rangle$, $g_2 = \mathbb{C}\langle Z \rangle$, and the Lie bracket is

$$[Y, X] = X, \quad [Z, X] = Y, \quad [Z, Y] = Z. \quad (5.1)$$

Let us take $M = S(sl_2^*)^{[g]}$. By Example 2.4, $M$ is an SAYD over $sl_2$ via the coadjoint action and the Koszul coaction.

Writing $g_2^* = \mathbb{C}\langle \delta_1 \rangle$, we have $\mathcal{F} = R(g_2) = \mathbb{C}[\delta_1]$. Also, $U = U(g_1)$ and it is immediate to realize that $\mathcal{F} \bowtie U \cong \mathcal{H}_{1S}^{\text{cop}}$ \cite{12}.

Next, we construct the $\mathcal{F} \bowtie U$-(co)action explicitly and we verify that $(\sigma, \delta)$ is the canonical modular pair in involution associated to the bicrossed product $\mathcal{F} \bowtie U$ \cite{15}. By definition $\delta = Tr \circ ad_{g_1}$. Let us compute $\sigma \in \mathcal{F}$ from the right $\mathcal{F}$-coaction on $U$.

Considering the formula $[v, X] = v \triangleright X \oplus v \triangleleft X$, the action $g_2 \triangleleft g_1$ is given by

$$Z \triangleleft X = 0, \quad Z \triangleleft Y = Z. \quad (5.2)$$

Similarly, the action $g_2 \triangleright g_1$ is

$$Z \triangleright X = Y, \quad Z \triangleright Y = 0. \quad (5.3)$$

Dualizing the left action $g_2 \triangleright g_1$, we have the $\mathcal{F}$-coaction on $U$ as follows

$$\begin{align*}
\mathcal{U} &\rightarrow \mathcal{U} \otimes \mathcal{F}, \quad u \mapsto u^{(0)} \otimes u^{(1)} \\
X &\mapsto X \otimes 1 + Y \otimes \delta_1 \\
Y &\mapsto Y \otimes 1.
\end{align*} \quad (5.4)$$
Hence, by [15] Section 3.1
\[
\sigma = \det \begin{pmatrix} 1 & \delta_1 \\ 0 & 1 \end{pmatrix} = 1. \tag{5.5}
\]

On the other hand, by the Lie algebra structure of \( \mathfrak{g}_1 \cong gl_{1} \text{aff} \), we have
\[
\delta(X) = 0, \quad \delta(Y) = 1. \tag{5.6}
\]

Next, we express the \( \mathcal{F} \bowtie \mathcal{U} \)-coaction on \( M = S(sl_2^*) \) explicitly. A vector space basis of \( M \) is given by \( \{ 1_M, R^X, R^Y, R^Z \} \) and the \( \mathfrak{g}_1 \)-coaction (Kozsul) is
\[
M \rightarrow \mathfrak{g}_1 \otimes M, \quad 1_M \mapsto X \otimes R^X + Y \otimes R^Y, \quad R^i \mapsto 0, \quad i = X, Y, Z. \tag{5.7}
\]

Note that the application of this coaction twice is zero, thus it is locally conilpotent. Then the corresponding \( \mathcal{U} \) coaction is
\[
M \rightarrow \mathcal{U} \otimes M, \quad m \mapsto m^{\lceil \eta \rceil} \otimes m^{\lfloor \eta \rfloor} \tag{5.8}
\]
\[
1_M \mapsto 1 \otimes 1_M + X \otimes R^X + Y \otimes R^Y
\]
\[
R^i \mapsto 1 \otimes R^i, \quad i = X, Y, Z.
\]

To determine the left \( \mathcal{F} \)-coaction, we need to dualize the right \( \mathfrak{g}_2 \)-action. We have
\[
1_M \triangleleft Z = 0, \quad R^X \triangleleft Z = 0, \quad R^Y \triangleleft Z = R^X, \quad R^Z \triangleleft Z = R^Y, \tag{5.9}
\]

implying
\[
M \rightarrow \mathcal{F} \otimes M, \quad m \mapsto m^{\lceil \eta \rceil} \otimes m^{\lfloor \eta \rfloor} \tag{5.10}
\]
\[
1_M \mapsto 1 \otimes 1_M
\]
\[
R^X \mapsto 1 \otimes R^X
\]
\[
R^Y \mapsto 1 \otimes R^Y + \delta_1 \otimes R^X
\]
\[
R^Z \mapsto 1 \otimes R^Z + \delta_1 \otimes R^Y + \frac{1}{2} \delta_2 \otimes R^X.
\]

As a result, \( \mathcal{F} \bowtie \mathcal{U} \)-coaction appears as follows
\[
M \rightarrow \mathcal{F} \bowtie \mathcal{U} \otimes M, \quad m \mapsto m^{\lceil \eta \rceil} \bowtie m^{\lfloor \eta \rfloor} \otimes m^{\lfloor \eta \rfloor} \tag{5.11}
\]
\[
1_M \mapsto 1 \otimes 1_M + X \otimes R^X + Y \otimes R^Y
\]
\[
R^X \mapsto 1 \otimes R^X
\]
\[
R^Y \mapsto 1 \otimes R^Y + \delta_1 \otimes R^X
\]
\[
R^Z \mapsto 1 \otimes R^Z + \delta_1 \otimes R^Y + \frac{1}{2} \delta_2 \otimes R^X.
\]
Let us next determine the right $F \triangleright U$-action. It is enough to determine the $U$-action and $F$-action separately. The action of $U$ is directly given by

\[
\begin{align*}
1_M \triangleleft X &= 0, \quad 1_M \triangleleft Y = 0 \\
R^X \triangleleft X &= -R^Y, \quad R^X \triangleleft Y = R^X \\
R^Y \triangleleft X &= -R^Z, \quad R^Y \triangleleft Y = 0 \\
R^Z \triangleleft X &= 0, \quad R^Z \triangleleft Y = -R^Z.
\end{align*}
\]

(5.12)

To be able to see the $F$-action, we determine the $g_2$-coaction. This follows from the Koszul coaction on $M$, i.e.,

\[
\begin{align*}
M &\rightarrow U(g_2) \otimes M, \quad m \mapsto m_{(-1)} \otimes m_{(0)} \\
1_M &\mapsto 1 \otimes 1_M + Z \otimes R^Z \\
R^i &\mapsto 1 \otimes R^i, \quad i = X, Y, Z.
\end{align*}
\]

(5.13)

Hence, $F$-action is given by

\[
\begin{align*}
1_M \triangleleft \delta_1 &= R^Z, \quad R^i \triangleleft \delta_1 = 0, \quad i = X, Y, Z.
\end{align*}
\]

(5.14)

We will now check carefully that $M$ is a YD module over the bicrossed product Hopf algebra $F \triangleright U$. We leave to the reader to check that $M$ satisfies the conditions introduced in Lemma 3.4 and Lemma 3.6; that is $M$ is a module and comodule on $F \triangleright U$ respectively. We proceed to the verification of the YD condition on the bicrossed product Hopf algebra $F \triangleright U$.

By the multiplicative property of the YD condition, it is enough to check that the condition holds for the elements $X, Y, \delta_1 \in F \triangleright U$.

For simplicity of the notation, we write the $F \triangleright U$-coaction as $m \mapsto m_{(-1)} \otimes m_{(0)}$.

We begin with $1_M \in M$ and $X \in F \triangleright U$. On one hand we have

\[
\begin{align*}
X_{(2)} \cdot (1_M \triangleleft X_{(1)}{(-1)} \otimes (1_M \triangleleft X_{(1)})_{(0)}) &= (1_M \triangleleft X){(-1)} \otimes (1_M \triangleleft X)_{(0)} + X1_M{(-1)} \otimes 1_M{(0)} + \delta_1(1_M \triangleleft Y){(-1)} \otimes (1_M \triangleleft Y)_{(0)} \\
&= X \otimes 1_M + X^2 \otimes R^X + XY \otimes R^Y,
\end{align*}
\]

(5.15)

and on the other hand,

\[
\begin{align*}
1_M{(-1)}X_{(1)} \otimes 1_M{(0)} &\triangleleft X{(2)} = \\
1_M{(-1)}X \otimes 1_M{(0)} + 1_M{(-1)} \otimes 1_M{(0)} \triangleleft X + 1_M{(-1)}Y \otimes 1_M{(0)} \triangleleft \delta_1 &= \\
X \otimes 1_M + X^2 \otimes R^X + XY \otimes R^Y + X \otimes R^X \triangleleft X + Y \otimes R^Y \triangleleft X + Y \otimes 1_M \triangleleft \delta_1.
\end{align*}
\]

(5.16)

In view of $[Y, X] = X, \quad R^X \triangleleft X = -R^Y, \quad R^Y \triangleleft X = -R^Z$ and $1_M \triangleleft \delta_1 = R^Z$, we have the YD compatibility is satisfied for $1_M \in M$ and $X \in F \triangleright U$. 

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We proceed to check the condition for $1_M \in M$ and $Y \in F \triangleright U$. We have
\[
Y(2) \cdot (1_M \triangleleft Y(1))_{(-1)} \otimes (1_M \triangleleft Y(1))_{(0)} = \\
(1_M \triangleleft Y)_{(-1)} \otimes (1_M \triangleleft Y)_{(0)} + Y1_{M(-1)} \otimes 1_M = \tag{5.17}
Y \otimes 1_M + YX \otimes RX + Y^2 \otimes RY,
\]
and
\[
1_{M(-1)} Y(1) \otimes 1_{M(0)} \triangleleft Y(2) = \\
1_{M(-1)} Y \otimes 1_{M(0)} + 1_{M(-1)} \otimes 1_{M(0)} \triangleleft Y = 
\tag{5.18}
Y \otimes 1_M + XY \otimes RX + Y^2 \otimes RY + X \otimes RX \triangleleft Y.
\]
We use $[Y, X] = X$ and $RX \triangleleft Y = RX$, and hence the YD condition is satisfied for $1_M \in M$ and $Y \in F \triangleright U$.

For $1_M \in M$ and $\delta_1 \in F \triangleright U$ we have
\[
\delta_1(2)(1_M \triangleleft \delta_1(1))_{(-1)} \otimes (1_M \triangleleft \delta_1(1))_{(0)} = \\
(1_M \triangleleft \delta_1)_{(-1)} \otimes (1_M \triangleleft \delta_1)_{(0)} + \delta_1 1_{M(-1)} \otimes 1_{M(0)} = \tag{5.19}
1 \otimes R^2 + \delta_1 \otimes R^Y + \frac{1}{2} \delta_1^2 \otimes RX + \delta_1 \otimes 1_M + \delta_1 X \otimes RX + \delta_1 Y \otimes RY.
\]
On the other hand,
\[
1_{M(-1)} \delta_1(1) \otimes 1_{M(0)} \triangleleft \delta_1(2) = \\
1_{M(-1)} \delta_1 \otimes 1_{M(0)} + 1_{M(-1)} \otimes 1_{M(0)} \triangleleft \delta_1 = \tag{5.20}
\delta_1 \otimes 1_M + X \delta_1 \otimes RX + Y \delta_1 \otimes RY + 1 \otimes 1_M \triangleleft \delta_1.
\]
Thus, the YD condition for $1_M \in M$ and $\delta_1 \in F \triangleright U$ follows from $[X, \delta_1] = \frac{1}{2} \delta_1^2$,
$[Y, \delta_1] = \delta_1$ and $1_M \triangleleft \delta_1 = R^2$.

Next, we consider $RX \in M$ and $X \in F \triangleright U$. In this case we have,
\[
X(2)(RX \triangleleft X(1))_{(-1)} \otimes (RX \triangleleft X(1))_{(0)} = \\
(RX \triangleleft X)_{(-1)} \otimes (RX \triangleleft X)_{(0)} + XRX_{(-1)} \otimes RX_{(0)} + \delta_1 (RX \triangleleft Y)_{(-1)} \otimes (RX \triangleleft Y)_{(0)} = \\
-1 \otimes R^Y - \delta_1 \otimes RX + X \otimes RX + \delta_1 \otimes RX = \\
-1 \otimes R^Y + X \otimes RX. \tag{5.21}
\]
On the other hand,
\[
RX_{(-1)} X(1) \otimes RX_{(0)} \triangleleft X(2) = \\
RX_{(-1)} X \otimes RX_{(0)} + RX_{(-1)} \otimes RX_{(0)} \triangleleft X + RX_{(-1)} Y \otimes RX_{(0)} \triangleleft \delta_1 = \tag{5.22}
X \otimes RX + 1 \otimes RX \triangleleft X,
\]

40
and we have the equality in view of the fact that $R^X \triangleright X = -R^Y$.

For $R^X \in M$ and $Y \in \mathcal{F} \triangleright \mathcal{U}$, on one hand side we have

$$Y_{(2)} (R^X \triangleright Y_{(1)})_{(1)} \otimes (R^X \triangleright Y_{(1)})_{(0)} =$$

$$(R^X \triangleright Y)_{(1)} \otimes (R^X \triangleright Y)_{(0)} + YR^X_{(-1)} \otimes R^X_{(0)} =$$

$$1 \otimes R^X + Y \otimes R^X,$$

and on the other hand,

$$R^X_{(-1)} Y_{(1)} \otimes R^X_{(0)} \triangleright Y_{(2)} =$$

$$R^X_{(-1)} Y \otimes R^X_{(0)} + R^X \otimes R^X_{(0)} \triangleright Y =$$

$$Y \otimes R^X + 1 \otimes R^X \triangleright Y.$$

The equality is the consequence of $R^X \triangleright Y = R^X$.

For $R^X \in M$ and $\delta_1 \in \mathcal{F} \triangleright \mathcal{U}$ we have

$$\delta_{1(2)}(R^X \triangleright \delta_{1(1)})_{(-1)} \otimes (R^X \triangleright \delta_{1(1)})_{(0)} =$$

$$(R^X \triangleright \delta_{1(1)})_{(-1)} \otimes (R^X \triangleright \delta_{1(1)})_{(0)} + \delta_1 R^X_{(-1)} \otimes R^X_{(0)} =$$

$$\delta_1 \otimes R^X,$$

and

$$R^X_{(-1)} \delta_{1(1)} \otimes R^X_{(0)} \triangleright \delta_{1(2)} =$$

$$R^X_{(-1)} \delta_1 \otimes R^X_{(0)} + R^X_{(-1)} \otimes R^X_{(0)} \triangleright \delta_1 =$$

$$\delta_1 \otimes R^X + 1 \otimes R^X \triangleright \delta_1.$$

The result follows from $R^X \triangleright \delta_1 = 0$.

We proceed to verify the condition for $R^Y \in M$. For $R^Y \in M$ and $X \in \mathcal{F} \triangleright \mathcal{U}$, we have

$$X_{(2)} (R^Y \triangleright X_{(1)})_{(-1)} \otimes (R^Y \triangleright X_{(1)})_{(0)} =$$

$$(R^Y \triangleright X)_{(-1)} \otimes (R^Y \triangleright X)_{(0)} + XR^Y_{(-1)} \otimes R^Y_{(0)} + \delta_1 (R^Y \triangleright Y)_{(-1)} \otimes (R^Y \triangleright Y)_{(0)} =$$

$$-1 \otimes R^Z - \delta_1 \otimes R^Y - \frac{1}{2} \delta^2 \otimes R^X + X \otimes R^Y + X \delta_1 \otimes R^X,$$

as well as

$$R^Y_{(-1)} X_{(1)} \otimes R^Y_{(0)} \triangleright X_{(2)} =$$

$$R^Y_{(-1)} X \otimes R^Y_{(0)} + R^Y_{(-1)} \otimes R^Y_{(0)} \triangleright X + R^Y_{(-1)} Y \otimes R^Y_{(0)} \triangleright \delta_1 =$$

$$X \otimes R^Y + \delta_1 X \otimes R^X + 1 \otimes R^Y \triangleright X + \delta_1 \otimes R^X \triangleright X.$$

To see the equality, we use $[X, \delta_1] = \frac{1}{2} \delta^2$, $R^Y \triangleright X = -R^Z$ and $R^X \triangleright X = -R^Y$. 

Similarly for $R^Y \in M$ and $Y \in \mathcal{F} \triangleright \triangleright \mathcal{U}$, we have on one hand
\[
Y_{(2)} (R^Y \triangleleft Y_{(1)})_{(-1)} \otimes (R^Y \triangleleft Y_{(1)})_{(0)} = (R^Y \triangleleft Y)_{(-1)} \otimes (R^Y \triangleleft Y)_{(0)} + Y R^Y_{(-1)} \otimes R^Y_{(0)} = \tag{5.29}
Y \otimes R^Y + Y \delta_1 \otimes R^X,
\]
and on the other hand,
\[
R^Y_{(-1)} Y_{(1)} \otimes R^Y_{(0)} \triangleleft Y_{(2)} = R^Y_{(-1)} Y \otimes R^Y_{(0)} + R^Y_{(-1)} \otimes R^Y_{(0)} \triangleleft Y = \tag{5.30}
Y \otimes R^Y + \delta_1 Y \otimes R^X + \delta_1 \otimes R^X \triangleleft Y.
\]
Hence the equality by $[Y, \delta_1] = \delta_1$ and $R^X \triangleleft Y = R^X$.

Finally, for $R^Y \in M$ and $\delta_1 \in \mathcal{F} \triangleright \triangleright \mathcal{U}$ we have
\[
\delta_{1(2)} (R^Y \triangleleft \delta_{1(1)})_{(-1)} \otimes (R^Y \triangleleft \delta_{1(1)})_{(0)} = (R^Y \triangleleft \delta_1)_{(-1)} \otimes (R^Y \triangleleft \delta_1)_{(0)} + \delta_1 R^Y_{(-1)} \otimes R^Y_{(0)} = \tag{5.31}
\delta_1 \otimes R^Y + \delta_1^2 \otimes R^X,
\]
and
\[
R^Y_{(-1)} \delta_{1(1)} \otimes R^Y_{(0)} \triangleleft \delta_{1(2)} = R^Y_{(-1)} \delta_1 \otimes R^Y_{(0)} + R^Y_{(-1)} \otimes R^Y_{(0)} \triangleleft \delta_1 = \tag{5.32}
\delta_1 \otimes R^Y + \delta_1^2 \otimes R^X.
\]

Now we check the condition for $R^Z \in M$. For $R^Z \in M$ and $X \in \mathcal{F} \triangleright \triangleright \mathcal{U}$,
\[
X_{(2)} (R^Z \triangleleft X_{(1)})_{(-1)} \otimes (R^Z \triangleleft X_{(1)})_{(0)} = (R^Z \triangleleft X)_{(-1)} \otimes (R^Z \triangleleft X)_{(0)} + X R^Z_{(-1)} \otimes R^Z_{(0)} + \delta_1 (R^Z \triangleleft Y)_{(-1)} \otimes (R^Z \triangleleft Y)_{(0)} = X \otimes R^Z + X \delta_1 \otimes R^Y + \frac{1}{2} X \delta_1^2 \otimes R^X - \delta_1 \otimes R^Z - \delta_1^2 \otimes R^Y - \frac{1}{2} \delta_1^3 \otimes R^X. \tag{5.33}
\]

On the other hand,
\[
R^Z_{(-1)} X_{(1)} \otimes R^Z_{(0)} \triangleleft X_{(2)} = R^Z_{(-1)} X \otimes R^Z_{(0)} + R^Z_{(-1)} \otimes R^Z_{(0)} \triangleleft X + R^Z_{(-1)} Y \otimes R^Z_{(0)} \triangleleft \delta_1 = \tag{5.34}
X \otimes R^Z + \delta_1 X \otimes R^Y + \frac{1}{2} \delta_1^2 X \otimes R^X + \delta_1 \otimes R^Y \triangleleft X + \frac{1}{2} \delta_1^2 \otimes R^X \triangleleft X.
\]
Equality follows from $[X, \delta_1] = \frac{1}{2} \delta_1^2$, $R^Y \triangleleft X = -R^Z$ and $R^X \triangleleft X = -R^Y$.

Next, we consider $R^Z \in M$ and $Y \in \mathcal{F} \triangleright \triangleright \mathcal{U}$. In this case we have
\[
Y_{(2)} (R^Z \triangleleft Y_{(1)})_{(-1)} \otimes (R^Z \triangleleft Y_{(1)})_{(0)} = (R^Z \triangleleft Y)_{(-1)} \otimes (R^Z \triangleleft Y)_{(0)} + Y R^Z_{(-1)} \otimes R^Z_{(0)} = \tag{5.35}
-1 \otimes R^Z - \delta_1 \otimes R^Y - \frac{1}{2} \delta_1^2 \otimes R^X + Y \otimes R^Z + Y \delta_1 \otimes R^Y + \frac{1}{2} Y \delta_1^2 \otimes R^X,
\]
and on the other hand,

\[
\begin{align*}
R^Z_{(-1)} Y^{(1)} \otimes R^Z_{(0)} \triangleright Y^{(2)} &= \\
R^Z_{(-1)} Y \otimes R^Z_{(0)} + R^Z_{(-1)} \otimes R^Z_{(0)} \triangleright Y &= \\
Y \otimes R^Z + \delta_Y Y \otimes R^Y + \frac{1}{2} \delta_Y^2 Y \otimes R^X + 1 \otimes R^Z \triangleright Y + \frac{1}{2} \delta_Y^2 \otimes R^X \triangleright Y.
\end{align*}
\]

Equality follows from \([Y, \delta_1] = \delta_1, R^Z \triangleright Y = -R^Z\) and \(R^X \triangleright Y = R^X\).

Finally, we check the YD compatibility for \(R^Z \in M\) and \(\delta_1 \in F \triangleright \mathcal{U}\). We have

\[
\begin{align*}
\delta_1(2)(R^Z \triangleright \delta_1(1))_{(-1)} \otimes (R^Z \triangleright \delta_1(1))_{(0)} &= \\
(R^Z \triangleright \delta_1)_{(-1)} \otimes (R^Z \triangleright \delta_1)_{(0)} + \delta_1 R^Z_{(-1)} \otimes R^Z_{(0)} = & \\
\delta_1 \otimes R^Z + \delta_Y^2 \otimes R^Y + \frac{1}{2} \delta_Y^3 \otimes R^X,
\end{align*}
\]

and

\[
\begin{align*}
R^Z_{(-1)} \delta_1(1) \otimes R^Z_{(0)} \triangleright \delta_1(2) &= \\
R^Z_{(-1)} \delta_1 \otimes R^Z_{(0)} + R^Z_{(-1)} \otimes R^Z_{(0)} \triangleright \delta_1 &= \\
\delta_1 \otimes R^Z + \delta_Y^2 \otimes R^Y + \frac{1}{2} \delta_Y^3 \otimes R^X.
\end{align*}
\]

We have proved that \(M\) is a YD module over the bicrossed product Hopf algebra \(F \triangleright \mathcal{U} = \mathcal{H}_{1s}^{\text{cop}}\).

Let us now check the stability condition. Since in this case \(\sigma = 1, \sigma M_\delta\) has the same coaction as \(M\). Thus, \((m \otimes 1_\mathcal{C})_{(-1)} \otimes (m \otimes 1_\mathcal{C})_{(0)} \in F \triangleright \mathcal{U} \otimes \sigma M_\delta\) denoting the coaction, we have

\[
\begin{align*}
(1_M \otimes 1_\mathcal{C})_{(0)} \cdot (1_M \otimes 1_\mathcal{C})_{(-1)} &= (1_M \otimes 1_\mathcal{C}) \cdot 1 + (R^X \otimes 1_\mathcal{C}) \cdot X + (R^Y \otimes 1_\mathcal{C}) \cdot Y \\
&= 1_M \otimes 1_\mathcal{C} + R^X \cdot X(2) \delta(X(1)) \otimes 1_\mathcal{C} + R^Y \cdot Y(2) \delta(Y(1)) \otimes 1_\mathcal{C} \\
&= 1_M \otimes 1_\mathcal{C} + R^X \cdot X \otimes 1_\mathcal{C} + R^Y \cdot X \otimes 1_\mathcal{C} + R^Y \cdot Y \otimes 1_\mathcal{C} + R^Y \delta(Y) \otimes 1_\mathcal{C} \\
&= 1_M \otimes 1_\mathcal{C}.
\end{align*}
\]

\[
\begin{align*}
(R^X \otimes 1_\mathcal{C})_{(0)} \cdot (R^X \otimes 1_\mathcal{C})_{(-1)} &= (R^X \otimes 1_\mathcal{C}) \cdot 1 = R^X \otimes 1_\mathcal{C}.
\end{align*}
\]

\[
\begin{align*}
(R^Y \otimes 1_\mathcal{C})_{(0)} \cdot (R^Y \otimes 1_\mathcal{C})_{(-1)} &= (R^Y \otimes 1_\mathcal{C}) \cdot 1 + (R^X \otimes 1_\mathcal{C}) \cdot \delta_1 = R^Y \otimes 1_\mathcal{C}.
\end{align*}
\]

\[
\begin{align*}
(R^Z \otimes 1_\mathcal{C})_{(0)} \cdot (R^Z \otimes 1_\mathcal{C})_{(-1)} &= \\
&= (R^Z \otimes 1_\mathcal{C}) \cdot 1 + (R^Y \otimes 1_\mathcal{C}) \cdot \delta_1 + (R^X \otimes 1_\mathcal{C}) \cdot \frac{1}{2} \delta_Y^2 = R^Z \otimes 1_\mathcal{C}.
\end{align*}
\]

Hence the stability is satisfied.

We record our discussion in the following proposition.
Proposition 5.1. The four dimensional module-comodule

\[ M_\delta := M \otimes \mathbb{C}_\delta = \mathbb{C}\langle 1_M, R^X, R^Y, R^Z \rangle \otimes \mathbb{C}_\delta \]

is an SAYD module over the Schwarzian Hopf algebra \( \mathcal{H}_{15}^{cop} \), via the action and coaction

| \( \downarrow \) | \( X \) | \( Y \) | \( \delta_1 \) |
|---|---|---|---|
| 1 | 0 | 0 | \( \mathbb{R}^Z \) |
| \( R^X \) | \( -R^Y \) | 2\( R^X \) | 0 |
| \( R^Y \) | \( -R^Z \) | \( R^Y \) | 0 |
| \( R^Z \) | 0 | 0 | 0 |

\[ \triangledown : M_\delta \longrightarrow \mathcal{H}_{15}^{cop} \otimes M_\delta \]

Here, \( 1 := 1_M \otimes \mathbb{C}_\delta, \ R^X := R^X \otimes \mathbb{C}_\delta, \ R^Y := R^Y \otimes \mathbb{C}_\delta, \ R^Z := R^Z \otimes \mathbb{C}_\delta. \)

5.2 Computation of \( \widetilde{HP}(sl_2, S(sl_2^* )_{[2]} ) \)

This subsection is devoted to the computation of \( \widetilde{HP}(sl_2, M) \) by demonstrating explicit representatives of the cohomology classes. We know that the perturbed Koszul complex \( (W(sl_2, M), d_{CE} + d_K) \) computes this cohomology.

Being an SAYD over \( U(sl_2) \), \( M \) admits the filtration \( (F_p M) \) from \([10]\). Explicitly,

\[ F_0 M = \{ R^X, R^Y, R^Z \}, \quad F_p M = M, \quad p \geq 1. \]  (5.39)

The induced filtration on the complex is

\[ F_j(W(sl_2, M)) := W(sl_2, F_j M), \]  (5.40)

and the \( E_1 \) term of the associated spectral sequence is

\[ E_1^{i,j}(sl_2, M) = H^{i+j}(W(sl_2, F_j M)/W(sl_2, F_{j-1} M)) \cong H^{i+j}(W(sl_2, F_j M/F_{j-1} M)). \]  (5.41)

Since \( F_j M/F_{j-1} M \) has trivial \( sl_2 \)-coaction, the boundary \( d_K \) vanish on the quotient complex \( W(sl_2, F_j M/F_{j-1} M) \) and hence

\[ E_1^{i,j}(sl_2, M) = \bigoplus_{i+j \equiv \bullet \mod 2} H^*(sl_2, F_j M/F_{j-1} M). \]  (5.42)

In particular,

\[ E_1^{0,i}(sl_2, M) = H^i(W(sl_2, F_0 M)) \cong \bigoplus_{i \equiv \bullet \mod 2} H^*(sl_2, F_0 M) = 0. \]  (5.43)

The last equality follows from the Whitehead’s theorem (noticing that \( F_0 M \) is an irreducible \( sl_2 \)-module of dimension greater than 1). For \( j = 1 \) we have \( M/F_0 M \cong \mathbb{C} \) and hence

\[ E_1^{1,i}(sl_2, M) = \bigoplus_{i+1 \equiv \bullet \mod 2} H^*(sl_2), \]  (5.44)
which gives two cohomology classes as a result of Whitehead’s 1st and 2nd lemmas. Finally, by $F_p M = M$ for $p \geq 1$, we have $E^2_1(s_2, M) = 0$ for $j \geq 2$.

Let us now write the complex as

$$ W(sl_2, M) = W^{even}(sl_2, M) \oplus W^{odd}(sl_2, M), \quad (5.45) $$

where

$$ W^{even}(sl_2, M) = M \oplus (\wedge^2 s_2 * M), \quad W^{odd}(sl_2, M) = (s_2 * M) \oplus (\wedge^3 s_2 * M). \quad (5.46) $$

Next, we demonstrate the explicit cohomology cocycles of $\overline{HP}(sl_2, M)$. First, let us take $1_M \in W^{even}(sl_2, M)$. It is immediate to observe $d_{CE}(1_M) = 0$ as well as $d_K(1_M) = 0$. On the other hand, in the level of spectral sequence it descends to the nontrivial class 0 of the cohomology of $sl_2$. Hence, it is a representative of the even cohomology class.

Secondly, we consider

$$ (2 \theta^X \otimes R^Z - \theta^Y \otimes R^Y, \theta^X \wedge \theta^Y \wedge \theta^Z \otimes 1_M) \in W^{odd}(sl_2, M) $$

Here $\{\theta^X, \theta^Y, \theta^Z\}$ is the dual basis corresponding to the basis $\{X, Y, Z\}$ of $sl_2$. Let us show that it is a $d_{CE} + d_K$-cocycle. It is immediate that

$$ d_{CE}(\theta^X \wedge \theta^Y \wedge \theta^Z \otimes 1_M) = 0 \quad (5.47) $$

As for the Koszul differential,

$$ d_K(\theta^X \wedge \theta^Y \wedge \theta^Z \otimes 1_M) = \\
\iota_X(\theta^X \wedge \theta^Y \wedge \theta^Z) \otimes R^X + \iota_Y(\theta^X \wedge \theta^Y \wedge \theta^Z) \otimes R^Y + \iota_Z(\theta^X \wedge \theta^Y \wedge \theta^Z) \otimes R^Z = \\
\theta^Y \wedge \theta^Z \otimes R^X - \theta^X \wedge \theta^Z \otimes R^Y + \theta^X \wedge \theta^Y \otimes R^Z. \quad (5.48) $$

On the other hand, we have

$$ d_{CE}(\theta^X \otimes R^Z) = \theta^X \otimes \theta^Y \otimes R^Z - \theta^Y \otimes R^Z \cdot Y - \theta^Z \wedge \theta^X \otimes R^Z \cdot Z \\
= \theta^X \wedge \theta^Z \otimes R^Y, \quad (5.49) $$

and

$$ d_{CE}(\theta^Y \otimes R^Y) = \theta^X \wedge \theta^Z \otimes R^Y - \theta^X \wedge \theta^Y \otimes R^Y \cdot X - \theta^Z \wedge \theta^Y \otimes R^Z \cdot Z \\
= \theta^X \wedge \theta^Y \otimes R^Z + \theta^X \wedge \theta^Z \otimes R^Y + \theta^Y \wedge \theta^Z \otimes R^X. \quad (5.50) $$

Therefore,

$$ d_{CE}(2\theta^X \otimes R^Z - \theta^Y \otimes R^Y) = -\theta^X \wedge \theta^Y \otimes R^Z + \theta^X \wedge \theta^Z \otimes R^Y - \theta^Y \wedge \theta^Z \otimes R^X. \quad (5.51) $$

We also have

$$ d_K(2\theta^X \otimes R^Z - \theta^Y \otimes R^Y) = 2R^Z R^X - R^Y R^Y = 0 - 0 = 0. \quad (5.52) $$
Therefore, we can write
\[(d_{CE} + d_K)((2\theta^X \otimes R^Z - \theta^Y \otimes R^Y, \theta^X \land \theta^Y \land \theta^Z \otimes 1_M)) = 0.\] (5.53)

Finally we note that \((2\theta^X \otimes R^Z - \theta^Y \otimes R^Y, \theta^X \land \theta^Y \land \theta^Z \otimes 1_M)\) descends, in the \(E_1\)-level of the spectral sequence, to the cohomology class represented by the 3-cocycle
\[\theta^X \land \theta^Y \land \theta^Z.\]

Hence, it represents the odd cohomology class. Let us summarize our discussion so far

**Proposition 5.2.** The periodic cyclic cohomology of the Lie algebra \(sl_2\) with coefficients in SAYD module \(M := S(sl_2^*)^2\) is represented by
\[\widetilde{HP}^0(sl_2, M) = C\langle 1_M \rangle,\] (5.54)
\[\widetilde{HP}^1(sl_2, M) = C\langle (2\theta^X \otimes R^Z - \theta^Y \otimes R^Y, \theta^X \land \theta^Y \land \theta^Z \otimes 1_M) \rangle.\] (5.55)

### 5.3 Computation of \(HP(H_{1S}, M_\delta)\)

We now consider the complex \(C(\mathcal{U} \rhd \mathcal{F}, M_\delta)\), which computes the periodic Hopf cyclic cohomology
\[HP(H_{1S}^{\text{cop}}, M_\delta) = HP(\mathcal{F} \triangleright \mathcal{U}, M_\delta).\] (5.56)

We can immediately conclude that \(M_\delta\) is also a SAYD module over \(U(sl_2)\) with the same action and coaction due to the unimodularity of \(sl_2\). The corresponding filtration is then given by
\[F_0M_\delta = F_0M \otimes H_{\delta} = C\langle R^X, R^Y, R^Z \rangle \otimes H_{\delta}, \quad F_pM_\delta = M_\delta, p \geq 1.\] (5.57)

We will first derive a Cartan type homotopy formula for Hopf cyclic cohomology, as in [12]. One notes that in [12] the SAYD module was one dimensional. We have to adapt the homotopy formula to fit our situation. To this end, let
\[D_Y : \mathcal{H} \to \mathcal{H}, \quad D_Y(h) := hY.\] (5.58)

Obviously, \(D_Y\) is an \(\mathcal{H}\)-linear coderivation. Hence the operators
\[\mathcal{L}_{D_Y} : C^n(\mathcal{U} \rhd \mathcal{F}, M_\delta) \to C^n(\mathcal{U} \rhd \mathcal{F}, M_\delta)\]
\[\mathcal{L}_{D_Y}(m \otimes_{\mathcal{H}} c^0 \otimes \ldots \otimes c^n) = \sum_{i=0}^n m \otimes c^0_{\mathcal{H}} \otimes \ldots \otimes D_Y(c_i) \otimes \ldots \otimes c^n,\] (5.59)
\[\epsilon_{D_Y} : C^n(\mathcal{U} \rhd \mathcal{F}, M_\delta) \to C^{n+1}(\mathcal{U} \rhd \mathcal{F}, M_\delta)\]
\[\epsilon_{D_Y}(m \otimes_{\mathcal{H}} c^0 \otimes \ldots \otimes c^n) = (-1)^m m_{(0)} \otimes_{\mathcal{H}} c^0_{(2)} \otimes c^1 \otimes \ldots \otimes c^n \otimes m_{(-1)} D_Y(c^0_{(1)}),\] (5.60)
Let us first recall the isomorphism of cocyclic modules. By [13], we know that
\[
\Psi(\delta \otimes \mathcal{H} c_0 \otimes \ldots \otimes c^n) = (-1)^{n(i+1)} \epsilon(c^0)m_{(i)} \otimes \mathcal{H} c^{n-i+2} \otimes \ldots \otimes c^{n+1} \otimes m_{(-1)} c_1 \otimes \ldots \otimes m_{(-j+1)} e_{j-1} \otimes m_{(-j+i+1)} D_Y(c^{j+i+1}) \otimes m_{(-j+i+1)} e_{j+i+2} \otimes \ldots \otimes m_{(-n+i+1)} e_{n-i+1},
\]
(5.61)
satisfy, by [12, Proposition 3.7],
\[
[E_{DY} + e_{DY}, b + B] = \mathcal{L}_{DY}.
\]
(5.62)
We next obtain an analogous of [12, Lemma 3.8].

**Lemma 5.3.** We have
\[
\mathcal{L}_{DY} = I - \tilde{ad} Y,
\]
(5.63)
where
\[
\tilde{ad} Y(m_\delta \otimes \tilde{f} \otimes \tilde{u}) = m_\delta \otimes \tilde{ad} Y(\tilde{f} \otimes \tilde{u}) - (m \cdot Y)_\delta \otimes \tilde{f} \otimes \tilde{u},
\]
(5.64)
and \(m_\delta := m \otimes 1_C\).

**Proof.** Let us first recall the isomorphism
\[
\Theta := \Phi_2 \circ \Phi_1 \circ \Psi : C^{\bullet}_H(\mathcal{U} \rhd \mathcal{F}, M_\delta) \to \mathfrak{Z}^{\bullet,\bullet}
\]
(5.65)
of cocyclic modules. By [13], we know that
\[
\Psi(m_\delta \otimes \mathcal{H} u^0 \rhd f^0 \otimes \ldots \otimes u^n \rhd f^n) = \\
m_\delta \otimes \mathcal{H} u^{0(-n-1)} f^0 \otimes \ldots \otimes u^{0(-1)} \ldots \otimes u^{n(-1)} f^n \otimes u^{0(0)} \otimes \ldots \otimes u^{n(0)}
\]
\[
\Psi^{-1}(m_\delta \otimes \mathcal{H} f^0 \otimes \ldots \otimes f^n \otimes u^0 \otimes \ldots \otimes u^n) = \\
m_\delta \otimes \mathcal{H} u^{0(0)} \rhd S^{-1}(u^{0(-1)}) f^0 \otimes \ldots \otimes u^{n(0)} \rhd S^{-1}(u^{0(-1)} u^{1(-n)} \ldots u^{n(-1)}) f^n,
\]
(5.66)
\[
\Phi_1(m_\delta \otimes \mathcal{H} f^0 \otimes \ldots \otimes f^n \otimes u^0 \otimes \ldots \otimes u^n) = \\
m_\delta \cdot u^{0(2)} \otimes \mathcal{F} S^{-1}(u^{0(1)}) \triangleright (f^0 \otimes \ldots \otimes f^n) \otimes S(u^{0(3)}) \cdot (u^1 \otimes \ldots \otimes u^n)
\]
(5.67)
\[
\Phi_1^{-1}(m_\delta \otimes \mathcal{F} f^0 \otimes \ldots \otimes f^n \otimes u^1 \otimes \ldots \otimes u^n) = \\
m_\delta \otimes \mathcal{H} f^0 \otimes \ldots \otimes f^n \otimes 1_{U(g_1)} \otimes u^1 \otimes \ldots \otimes u^n,
\]
and
\[
\Phi_2(m_\delta \otimes \mathcal{F} f^0 \otimes \ldots \otimes f^n \otimes u^1 \otimes \ldots \otimes u^n) = \\
m_\delta \cdot f^{0(1)} \otimes S(f^{0(2)}) \cdot (f^1 \otimes \ldots \otimes f^n \otimes u^1 \otimes \ldots \otimes u^n)
\]
(5.68)
\[
\Phi_2^{-1}(m_\delta \otimes f^1 \otimes \ldots \otimes f^n \otimes u^1 \otimes \ldots \otimes u^n) = \\
m_\delta \otimes \mathcal{F} f^0 \otimes f^1 \otimes \ldots \otimes f^n \otimes u^1 \otimes \ldots \otimes u^n.
\]
Here, the left $\mathcal{H}$-coaction on $\mathcal{U}$ is the one corresponding to the right $\mathcal{F}$-coaction. Namely,  
\[ u^{(-1)} \otimes u^{(0)} = S(u^{(1)}) \otimes u^{(0)}. \]  
(5.69)
We also recall that  
\[
\Phi : \mathcal{H} \triangleright \triangleright \mathcal{U} \rightarrow \mathcal{U} \blacktriangleright \blacktriangleright \mathcal{F} \\
\Phi(f \blacktriangleleft u) = u^{(0)} \blacktriangleright f u^{(1)} \\
\Phi^{-1}(u \blacktriangleright f) = f S^{-1}(u^{(1)}) \blacktriangleleft u^{(0)}.
\]  
(5.70)
Therefore, we have
\[
\Theta \circ \mathcal{L}_{D_Y} \circ \Theta^{-1}(m_\delta \otimes f^1 \otimes \ldots \otimes f^n \otimes u^1 \otimes \ldots \otimes u^n) = \\
\Theta \circ \mathcal{L}_{D_Y} \circ \Psi^{-1} \circ \Phi_1^{-1} \circ \Phi_2^{-1}(m_\delta \otimes f^1 \otimes \ldots \otimes f^n \otimes u^1 \otimes \ldots \otimes u^n) = \\
\Theta \circ \mathcal{L}_{D_Y} \circ \Psi^{-1} \circ \Phi_1^{-1}(m_\delta \otimes \mathcal{F} f^1 \otimes \ldots \otimes f^n \otimes u^1 \otimes \ldots \otimes u^n) = \\
\Theta \circ \mathcal{L}_{D_Y} \circ \Psi^{-1}(m_\delta \otimes \mathcal{H}_1 \mathcal{F} f^1 \otimes \ldots \otimes f^n \otimes 1_U \otimes u^1 \otimes \ldots \otimes u^n) = \\
\Theta \circ \mathcal{L}_{D_Y}(m_\delta \otimes \mathcal{H}_1 \mathcal{F} f^1 \otimes \ldots \otimes f^n \otimes 1_U \otimes S^{-1}(u^{(-1)}) \triangleright f^1 \otimes \ldots \\
\ldots \otimes u^{n(0)} \triangleright S^{-1}(u^{(-n)} \ldots u^{n(-1)} \triangleright f^n) = \\
\Theta \circ \mathcal{L}_{D_Y}(m_\delta \otimes \mathcal{H}_1 \mathcal{F} u^1 \otimes \ldots \otimes u^{n(0)} \triangleright u^{n(1)} \ldots u^{n(n)} f^n),
\]  
(5.71)
where on the last equality we have used (5.69). In order to apply $\mathcal{L}_{D_Y}$, we make the observation
\[
\Phi \mathcal{D}_Y \Phi^{-1}(u \blacktriangleright f) = \Phi(f S^{-1}(u^{(1)}) \blacktriangleleft u^{(0)} Y) = \\
(u^{(0)} Y)^{(0)} \blacktriangleright f S^{-1}(u^{(1)})(u^{(0)} Y)^{(1)} = \\
u^{(0)(1)} Y^{(0)} \blacktriangleright f S^{-1}(u^{(1)})(u^{(0)(1)}) Y^{(1)} = \\
u^{(1)} Y^{(0)} \blacktriangleright f S^{-1}(u^{(1)(2)})(u^{(2)}(1)) Y^{(1)} = \\
y Y \blacktriangleright f,
\]
using the action-coaction compatibilities of a bicrossed product. Hence,
\[
\Theta \circ \mathcal{L}_{D_Y}(m_\delta \otimes \mathcal{H}_1 \mathcal{F} u^{(1)} f^1 \otimes \ldots \otimes u^{n(0)} \triangleright u^{n(1)} \ldots u^{n(n)} f^n) = \\
\Theta(m_\delta \otimes \mathcal{H} Y \blacktriangleright 1_{\mathcal{F}} \otimes u^{(0)} \triangleright u^{(1)} f^1 \otimes \ldots \otimes u^{n(0)} \triangleright u^{n(1)} \ldots u^{n(n)} f^n) + \\
\sum_{i=1}^{n} \Theta(m_\delta \otimes \mathcal{H} Y \blacktriangleright 1_{\mathcal{F}} \otimes u^{(0)} \triangleright u^{(1)} f^1 \otimes \ldots \\
\ldots \otimes u^{i(0)} \triangleright u^{i(1)} f^i \otimes \ldots \otimes u^{n(0)} \triangleright u^{n(1)} \ldots u^{n(n)} f^n).
\]  
(5.73)
We notice
\[
\Psi(m_\delta \otimes \mathcal{H} Y \blacktriangleright 1_{\mathcal{F}} \otimes u^{(0)} \triangleright u^{(1)} f^1 \otimes \ldots \otimes u^{n(0)} \triangleright u^{n(1)} \ldots u^{n(n)} f^n) = \\
m_\delta \otimes \mathcal{H} Y \blacktriangleright 1_{\mathcal{F}} \otimes Y \blacktriangleright (u^{(1)} f^1 \otimes \ldots \\
\ldots \otimes Y \blacktriangleright (u^{1(0)} \otimes u^{1(0)} \ldots u^{n(0)} f^n \otimes Y \otimes (u^{1(0)} \otimes \ldots \otimes u^{n(0)})) = \\
m_\delta \otimes \mathcal{H} 1_{\mathcal{F}} \otimes f^1 \otimes \ldots \otimes f^n \otimes Y \otimes u^1 \otimes \ldots \otimes u^n,
\]  
(5.74)
where on the last equality we have used (5.69). Similarly,

$$\Psi(m_\delta \otimes H 1_u \ll 1_F \otimes u^{(0)}_1 \ll u^{(1)}_1 f_1 \otimes \ldots$$

$$\ldots \otimes u^{(i)}_i Y \ll u^{(1)}_1 \ldots u^{(i)}_i f_i \otimes \ldots \otimes u^{(n)}_n Y \ll u^{(1)}_1 \ldots u^{(n)}_n f_n) =$$

(5.75)

$$m_\delta \otimes H 1_F \otimes f^1_1 \otimes \ldots f^n \otimes 1_u \otimes u^1 \otimes \ldots u^n.$$

Therefore,

$$\Theta(m_\delta \otimes H Y \ll 1_F \otimes u^{(0)}_1 \ll u^{(1)}_1 f_1 \otimes \ldots \otimes u^{(n)}_n Y \ll u^{(1)}_1 \ldots u^{(n)}_n f_n) +$$

$$\sum_{i=1}^n \Theta(m_\delta \otimes H 1_u \ll 1_F \otimes u^{(0)}_1 \ll u^{(1)}_1 f_1 \otimes \ldots$$

$$\ldots \otimes u^{(i)}_i Y \ll u^{(1)}_1 \ldots u^{(i)}_i f_i \otimes \ldots \otimes u^{(n)}_n Y \ll u^{(1)}_1 \ldots u^{(n)}_n f_n) =$$

$$\Phi_2 \circ \Phi_1(m_\delta \otimes H 1_F \otimes \bar{f} \otimes Y \otimes \bar{u} + m_\delta \otimes H 1_F \otimes \bar{f} \otimes 1_u \otimes \bar{u} \cdot Y) =$$

$$\Phi_2(m_\delta \cdot Y_2 \otimes F S^{-1}(Y_{(1)}) \triangleright (1_F \otimes \bar{f}) \otimes S(Y_{(3)}) \cdot \bar{u} + m_\delta \otimes F 1_F \otimes \bar{f} \otimes \bar{u} \cdot Y).$$

(5.76)

Considering the fact that $Y \in H$ is primitive, and hence $adY(f) = [Y, f] = Y \triangleright f,$ we conclude

$$\Phi_2(m_\delta \cdot Y_2 \otimes F S^{-1}(Y_{(1)}) \triangleright (1_F \otimes \bar{f}) \otimes S(Y_{(3)}) \cdot \bar{u} + m_\delta \otimes F 1_F \otimes \bar{f} \otimes \bar{u} \cdot Y) =$$

$$\Phi_2(-m_\delta \otimes F 1_F \otimes adY(\bar{f}) \otimes \bar{u} - m_\delta \otimes F 1_F \otimes \bar{f} \otimes Y \cdot \bar{u} +$$

$$m_\delta \cdot Y \otimes F 1_F \otimes \bar{f} \otimes \bar{u} + m_\delta \otimes F 1_F \otimes \bar{f} \otimes \bar{u} \cdot Y) =$$

$$m_\delta \cdot Y \otimes F \bar{f} \otimes \bar{u} - m_\delta \otimes F adY(\bar{f}) \otimes \bar{u} - m_\delta \otimes F \bar{f} \otimes adY(\bar{u}).$$

(5.77)

Finally, we recall $m_\delta \cdot Y = (m \cdot Y_{(1)})_\delta Y(Y_{(2)}) = (m \cdot Y)_\delta + m_\delta$ to finish the proof. □

**Lemma 5.4.** The operator $\tilde{ad}Y$ commutes with the horizontal operators (4.35) and the vertical operators (4.36).

**Proof.** We start with the horizontal operators. For the first horizontal coface, we have

$$\tilde{\partial}_0(\tilde{ad}Y(m_\delta \otimes \bar{f} \otimes \bar{u})) = \tilde{\partial}_0(m_\delta \otimes adY(\bar{f} \otimes \bar{u}) - (m \cdot Y)_\delta \otimes \bar{f} \otimes \bar{u}) =$$

$$m_\delta \otimes 1 \otimes adY(\bar{f} \otimes \bar{u}) - (m \cdot Y)_\delta \otimes 1 \otimes \bar{f} \otimes \bar{u} =$$

$$\tilde{ad}Y(\tilde{\partial}_0(m_\delta \otimes \bar{f} \otimes \bar{u})).$$

(5.78)

For $\tilde{\partial}_i$ with $1 \leq i \leq n,$ the commutativity is a consequence of $adY \circ \Delta = \Delta \circ adY$ on $F.$ To see this, we notice that

$$\Delta(adY(f)) = \Delta(Y \triangleright f) = Y^{(0)}_1 \triangleright f_1 \otimes (Y^{(1)}_1 \triangleright f_1 \otimes Y^{(2)}_1 \triangleright f_2)$$

$$= adY(f_1) \otimes f_2 + f_1 \otimes adY(f_2) = adY(\Delta(f)).$$

(5.79)
For the commutation with the last horizontal coface operator, we proceed as follows. First we observe
\[
\widetilde{ad}Y(\partial_{n+1}(m_\delta \otimes \tilde{f} \otimes \tilde{u})) = \widetilde{ad}Y(m_{(0)\delta} \otimes \tilde{f} \otimes \tilde{u}^{(-1)} m_{(-1)} \triangleright 1_F \otimes \tilde{u}^{(0)}) = \\
m_{(0)\delta} \otimes adY(\tilde{f}) \otimes [\tilde{u}^{(-1)} m_{(-1)}] \otimes \tilde{u}^{(0)} + m_{(0)\delta} \otimes \tilde{f} \otimes adY(\tilde{u}^{(-1)} m_{(-1)} \triangleright 1_F) \otimes \tilde{u}^{(0)} + \\
m_{(0)\delta} \otimes \tilde{f} \otimes [\tilde{u}^{(-1)} m_{(0)}] \otimes adY(\tilde{u}^{(0)}) - (m_{(0)} \cdot Y)_\delta \otimes \tilde{f} \otimes [\tilde{u}^{(-1)} m_{(-1)}] \otimes \tilde{u}^{(0)}.
\]
(5.80)

Next, for any \(h = g \triangleright u \in \mathcal{H}\) and \(f \in \mathcal{F}\), on one hand we have
\[
adY(h \triangleright f) = adY(g(u \triangleright f)) = adY(g)(u \triangleright f) + g(Yu \triangleright f),
\]
(5.81)
and on the other hand,
\[
adY(h \triangleright f) = (adY(g) \triangleright u + g \triangleright Y u - g \triangleright u Y) \triangleright f =
\]
\[
= adY(g)(u \triangleright f) + g(Yu \triangleright f) - g(uY \triangleright f).
\]

In other words,
\[
adY(h \triangleright f) = adY(h) \triangleright f + h \triangleright adY(f).
\]
(5.82)
Therefore we have
\[
m_{(0)\delta} \otimes \tilde{f} \otimes adY(\tilde{u}^{(-1)} m_{(-1)} \triangleright 1_F) \otimes \tilde{u}^{(0)} = \\
m_{(0)\delta} \otimes \tilde{f} \otimes adY(\tilde{u}^{(-1)} m_{(-1)} \triangleright 1_F) \otimes \tilde{u}^{(0)} + \\
m_{(0)\delta} \otimes \tilde{f} \otimes \tilde{u}^{(-1)} adY(m_{(-1)}) \triangleright 1_F \otimes \tilde{u}^{(0)}. \\
(5.83)
\]
Recalling (5.69) and the coaction - multiplication compatibility on a bicrossed product, we observe that
\[
adY(u^{(-1)} \otimes adY(u^{(0)})) = S(adY(u^{(1)})) \otimes adY(u^{(0)}) = \\
S(u^{(1)} Y^{(1)} (u^{(2)} \triangleright Y^{(1)})) \otimes S(u^{(1)} (u^{(2)} \triangleright Y^{(1)})) \otimes u^{(0)} Y^{(0)} = \\
S(u^{(1)}) \otimes Y u^{(0)} + S(Y \triangleright u^{(1)}) \otimes u^{(0)} - S(u^{(1)}) \otimes u^{(0)} Y,
\]
(5.84)
where we have used \(Y^{(0)} \otimes Y^{(1)} = Y \otimes 1\). This follows from \([Z,Y] = Z\) implying \(Z \triangleright Y = 0\).

By [14, Lemma 1.1] we also have \(S(Y \triangleright f) = Y \triangleright S(f)\) for any \(f \in \mathcal{F}\). Hence we can conclude
\[
adY(u^{(-1)} \otimes adY(u^{(0)})) = adY(u^{(-1)}) \otimes u^{(0)} + u^{(-1)} \otimes adY(u^{(0)}),
\]
(5.85)
which implies immediately that,
\[
adY(\tilde{u}^{(-1)} \otimes adY(\tilde{u}^{(0)})) = adY(\tilde{u}^{(-1)}) \otimes \tilde{u}^{(0)} + \tilde{u}^{(-1)} \otimes adY(\tilde{u}^{(0)}).
\]
(5.86)
Finally, by the right-left AYD compatibility of \( M \) over \( \mathcal{H} \) we have
\[
(m \cdot Y)_{(-1)} \otimes (m \cdot Y)_{(0)} = m_{(-1)} \otimes m_{(0)} \cdot Y - \text{adY} m_{(-1)} \otimes m_{(0)}. \tag{5.87}
\]

So, \( \tilde{\text{adY}} \) commutes with the last horizontal coface \( \partial_{n+1} \) as
\[
\tilde{\text{adY}}(\partial_{n+1}(m_{\delta} \otimes \tilde{f} \otimes \tilde{u})) = m_{(0)\delta} \otimes \text{adY}(\tilde{f}) \otimes \tilde{u}^{(-1)} m_{(-1)} \triangleright 1_{\mathcal{F}} \otimes \tilde{u}^{(0)} + \]
\[
m_{(0)\delta} \otimes \tilde{f} \otimes \text{adY}(\tilde{u})^{(-1)} m_{(-1)} \triangleright 1_{\mathcal{F}} \otimes \text{adY}(\tilde{u})^{(0)} - \]
\[
(m \cdot Y)_{(0)\delta} \otimes \tilde{f} \otimes \tilde{u}^{(-1)} (m \cdot Y)_{(0)} \triangleright 1_{\mathcal{F}} \otimes \tilde{u}^{(0)} = \]
\[
\partial_{n+1}(\text{adY}(m_{\delta} \otimes \tilde{f} \otimes \tilde{u})). \tag{5.88}
\]

It is immediate to observe the commutation \( \sigma_j \circ \tilde{\text{adY}} = \tilde{\text{adY}} \circ \sigma_j \) with the horizontal degeneracy operators.

We now consider the horizontal cyclic operator. Let us first note that
\[
m_{\delta} \cdot f = (m \cdot f)_{\delta}, \quad f \in \mathcal{F}. \tag{5.89}
\]

We then have
\[
\tilde{\text{adY}}(\tau(m_{\delta} \otimes \tilde{f} \otimes \tilde{u})) = \]
\[
\tilde{\text{adY}}((m_{(0)} \cdot f^{(1)}_{(1)})_{\delta} \otimes S(f^{(1)}_{(2)}) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \tilde{u}^{(-1)} m_{(-1)} \triangleright 1_{\mathcal{F}} \otimes \tilde{u}^{(0)})) = \]
\[
(m_{(0)} \cdot f^{(1)}_{(1)})_{\delta} \otimes \text{adY}(S(f^{(1)}_{(2)}) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \tilde{u}^{(-1)} m_{(-1)} \triangleright 1_{\mathcal{F}} \otimes \tilde{u}^{(0)})) + \]
\[
(m_{(0)} \cdot f^{(1)}_{(1)})_{\delta} \otimes S(f^{(1)}_{(2)}) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \text{adY}(\tilde{u})^{(-1)} m_{(-1)} \triangleright 1_{\mathcal{F}} \otimes \tilde{u}^{(0)})) + \]
\[
(m_{(0)} \cdot f^{(1)}_{(1)})_{\delta} \otimes S(f^{(1)}_{(2)}) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \text{adY}(\tilde{u})^{(-1)} m_{(-1)} \triangleright 1_{\mathcal{F}} \otimes \tilde{u}^{(0)})) - \]
\[
((m_{(0)} \cdot f^{(1)}_{(1)}) \cdot Y)_{\delta} \otimes S(f^{(1)}_{(2)}) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \tilde{u}^{(-1)} m_{(-1)} \triangleright 1_{\mathcal{F}} \otimes \tilde{u}^{(0)}). \tag{5.90}
\]

Next, by the commutativity of \( \text{adY} \) with the left \( \mathcal{H} \)-coaction on \( \mathcal{U} \) as well as with the antipode, we can immediately conclude
\[
\tilde{\text{adY}}(\tau(m_{\delta} \otimes \tilde{f} \otimes \tilde{u})) = \]
\[
(m_{(0)} \cdot f^{(1)}_{(1)})_{\delta} \otimes S(\text{adY}(f^{(1)}_{(2)})) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \tilde{u}^{(-1)} m_{(-1)} \triangleright 1_{\mathcal{F}} \otimes \tilde{u}^{(0)})) + \]
\[
(m_{(0)} \cdot f^{(1)}_{(1)})_{\delta} \otimes S(f^{(1)}_{(2)}) \cdot (\text{adY}(f^2 \otimes \ldots \otimes f^p) \otimes \tilde{u}^{(-1)} m_{(-1)} \triangleright 1_{\mathcal{F}} \otimes \tilde{u}^{(0)})) + \]
\[
(m_{(0)} \cdot f^{(1)}_{(1)})_{\delta} \otimes S(f^{(1)}_{(2)}) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \text{adY}(\tilde{u})^{(-1)} m_{(-1)} \triangleright 1_{\mathcal{F}} \otimes \text{adY}(\tilde{u})^{(0)})) + \]
\[
(m_{(0)} \cdot f^{(1)}_{(1)})_{\delta} \otimes S(f^{(1)}_{(2)}) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \text{adY}(\tilde{u})^{(-1)} m_{(-1)} \triangleright 1_{\mathcal{F}} \otimes \text{adY}(\tilde{u})^{(0)})) - \]
\[
((m_{(0)} \cdot f^{(1)}_{(1)}) \cdot Y)_{\delta} \otimes S(f^{(1)}_{(2)}) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \tilde{u}^{(-1)} m_{(-1)} \triangleright 1_{\mathcal{F}} \otimes \tilde{u}^{(0)}). \tag{5.91}
\]

Then by the module compatibility over the bicrossed product \( \mathcal{H} = \mathcal{F} \bowtie \mathcal{U} \), we have
\[
(m \cdot Y) \cdot f = (m \cdot f) \cdot Y + m \cdot \text{adY}(f). \tag{5.92}
\]
Therefore,
\[
\tilde{ad}Y(\tau (m_\delta \otimes \bar{f} \otimes \bar{u})) = \\
(m_{(0)} \delta \cdot adY(f_{(1)}^1) \otimes S(f_{(2)}^1) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \bar{u}^{(-1)} m_{(-1)} \triangleright \mathcal{F} \otimes \bar{u}^{(0)}) + \\
m_{(0)} \delta \cdot f_{(1)}^1 \otimes S(adY(f_{(2)}^1)) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \bar{u}^{(-1)} m_{(-1)} \triangleright \mathcal{F} \otimes \bar{u}^{(0)}) + \\
m_{(0)} \delta \cdot f_{(1)}^1 \otimes S(f_{(2)}^1) \cdot (adY(f^2 \otimes \ldots \otimes f^p) \otimes \bar{u}^{(-1)} m_{(-1)} \triangleright \mathcal{F} \otimes \bar{u}^{(0)}) + \\
m_{(0)} \delta \cdot f_{(1)}^1 \otimes S(f_{(2)}^1) \cdot (f^2 \otimes \ldots \otimes f^p \otimes adY(\bar{u})^{(-1)} m_{(-1)} \triangleright \mathcal{F} \otimes adY(\bar{u})^{(0)}) - \\
(m \cdot Y)_{(0)} \delta \cdot f_{(1)}^1 \otimes S(f_{(2)}^1) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \bar{u}^{(-1)} (m \cdot Y)_{(-1)} m_{(-1)} \triangleright \mathcal{F} \otimes \bar{u}^{(0)}).
\]
\[\text{(5.93)}\]

Finally, by the commutativity \(adY \circ \Delta = \Delta \circ adY\) on \(\mathcal{F}\) we finish as
\[
\tilde{ad}Y(\tau (m_\delta \otimes \bar{f} \otimes \bar{u})) = \\
m_{(0)} \delta \cdot adY(f_{(1)}^1) \otimes S(adY(f_{(2)}^1)) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \bar{u}^{(-1)} m_{(-1)} \triangleright \mathcal{F} \otimes \bar{u}^{(0)}) + \\
m_{(0)} \delta \cdot f_{(1)}^1 \otimes S(f_{(2)}^1) \cdot (adY(f^2 \otimes \ldots \otimes f^p) \otimes \bar{u}^{(-1)} m_{(-1)} \triangleright \mathcal{F} \otimes \bar{u}^{(0)}) + \\
m_{(0)} \delta \cdot f_{(1)}^1 \otimes S(f_{(2)}^1) \cdot (f^2 \otimes \ldots \otimes f^p \otimes adY(\bar{u})^{(-1)} m_{(-1)} \triangleright \mathcal{F} \otimes adY(\bar{u})^{(0)}) - \\
(m \cdot Y)_{(0)} \delta \cdot f_{(1)}^1 \otimes S(f_{(2)}^1) \cdot (f^2 \otimes \ldots \otimes f^p \otimes \bar{u}^{(-1)} (m \cdot Y)_{(-1)} \triangleright \mathcal{F} \otimes \bar{u}^{(0)}) \\
= \tau (adY(m_\delta \otimes \bar{f} \otimes \bar{u})).
\[\text{(5.94)}\]

We continue with the vertical operators. We see that
\[
\hat{\tau}_i \circ \tilde{ad}Y = \tilde{ad}Y \circ \hat{\tau}_i, \quad 0 \leq i \leq n
\[\text{(5.95)}\]
are similar to their horizontal counterparts. One notes that this time the commutativity \(adY \circ \Delta = \Delta \circ adY\) on \(\mathcal{U}\) is needed.

Commutativity with the last vertical coface operator follows, similarly as the horizontal case, from the AYD compatibility on \(M\) over \(\mathcal{H}\). Indeed,
\[
\tilde{ad}Y(\hat{\tau}_{n+1}(m_\delta \otimes \bar{f} \otimes \bar{u})) = \tilde{ad}Y(m_{(0)} \delta \otimes \bar{f} \otimes \bar{u} \otimes \overline{m_{(-1)}}) = \\
m_{(0)} \delta \otimes adY(\bar{f} \otimes \bar{u} \otimes \overline{m_{(-1)}}) + m_{(0)} \delta \otimes \bar{f} \otimes \bar{u} \otimes adY(m_{(-1)}) \\
- (m_{(0)} \cdot Y)_{\delta} \otimes \bar{f} \otimes \bar{u} \otimes \overline{m_{(-1)}} = \\
m_{(0)} \delta \otimes adY(\bar{f} \otimes \bar{u} \otimes \overline{m_{(-1)}}) - (m \cdot Y)_{(0)} \delta \otimes \bar{f} \otimes \bar{u} \otimes (m \cdot Y)_{(-1)} = \\
\hat{\tau}_{n+1}(\tilde{ad}Y(m_\delta \otimes \bar{f} \otimes \bar{u})).
\[\text{(5.96)}\]

Finally, we show the commutativity of \(\tilde{ad}Y\) with the vertical cyclic operator. First,
we notice that we can rewrite it as
\[
\begin{align*}
\uparrow \tau(m_\delta \otimes \tilde{f} \otimes \tilde{u}) &= (m_{(0)} \cdot u^1_{(4)}) \cdot S^{-1}(u^1_{(3)} \triangleright 1_\mathcal{F}) \otimes \\
S(S^{-1}(u^1_{(2)}) \triangleright 1_\mathcal{F}) \cdot \left( S^{-1}(u^1_{(1)}) \triangleright \tilde{f} \otimes S(u^1_{(0)}) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{m}_{(-1)}) \right) \\
&= m_{(0)} \cdot u^1_{(2)} \otimes S^{-1}(u^1_{(1)}) \triangleright \tilde{f} \otimes S(u^1_{(0)}) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{m}_{(-1)}) \\
&= (m_{(0)} \cdot u^1_{(3)}) \cdot \delta(u^1_{(2)}) \otimes S^{-1}(u^1_{(1)}) \triangleright \tilde{f} \otimes S(u^1_{(4)}) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{m}_{(-1)}).
\end{align*}
\]

Therefore we have
\[
\begin{align*}
\widetilde{\text{ad}}Y(\uparrow \tau(m_\delta \otimes \tilde{f} \otimes \tilde{u})) &= \\
\widetilde{\text{ad}}Y(m_{(0)} \cdot u^1_{(2)} \otimes S^{-1}(u^1_{(1)}) \triangleright \tilde{f} \otimes S(u^1_{(0)}) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{m}_{(-1)})) = \\
m_{(0)} \cdot u^1_{(2)} \otimes \text{ad}Y(S^{-1}(u^1_{(1)}) \triangleright \tilde{f} \otimes S(u^1_{(0)}) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{m}_{(-1)}) + \\
m_{(0)} \cdot u^1_{(2)} \otimes S^{-1}(u^1_{(1)}) \triangleright \tilde{f} \otimes \text{ad}Y(S(u^1_{(0)})) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{m}_{(-1)}) + \\
m_{(0)} \cdot u^1_{(2)} \otimes S^{-1}(u^1_{(1)}) \triangleright \tilde{f} \otimes S(u^1_{(0)}) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{\text{ad}Y(m_{(-1)})}) + \\
(m_{(0)} \cdot u^1_{(3)} Y) \cdot \delta(u^1_{(2)}) \otimes S^{-1}(u^1_{(1)}) \triangleright \tilde{f} \otimes S(u^1_{(4)}) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{m}_{(-1)}).
\end{align*}
\]

(5.97)

Recalling that
\[
\text{ad}Y(h \triangleright f) = \text{ad}Y(h) \triangleright f + h \triangleright \text{ad}Y(f),
\]
we then straightforwardly extend it to
\[
\text{ad}Y(h \triangleright \tilde{f}) = \text{ad}Y(h) \triangleright \tilde{f} + h \triangleright \text{ad}Y(\tilde{f}).
\]

(5.100)

As a result, we have
\[
\begin{align*}
m_{(0)} \cdot u^1_{(2)} \otimes \text{ad}Y(S^{-1}(u^1_{(1)}) \triangleright \tilde{f} \otimes S(u^1_{(0)}) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{m}_{(-1)})) = \\
m_{(0)} \cdot u^1_{(2)} \otimes S^{-1}(\text{ad}Y(u^1_{(1)})) \triangleright \tilde{f} \otimes S(u^1_{(0)}) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{m}_{(-1)}) + \\m_{(0)} \cdot u^1_{(2)} \otimes S^{-1}(u^1_{(1)}) \triangleright \text{ad}Y(\tilde{f}) \otimes S(u^1_{(0)}) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{m}_{(-1)}).
\end{align*}
\]

(5.101)

Next, we observe that
\[
\begin{align*}
-(m_{(0)} \cdot u^1_{(3)} Y) \cdot \delta(u^1_{(2)}) \otimes S^{-1}(u^1_{(1)}) \triangleright \tilde{f} \otimes S(u^1_{(4)}) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{m}_{(-1)}) = \\
(m_{(0)} \cdot \text{ad}Y(u^1_{(3)})) \cdot \delta(u^1_{(2)}) \otimes S^{-1}(u^1_{(1)}) \triangleright \tilde{f} \otimes S(u^1_{(4)}) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{m}_{(-1)}) - \\
(m_{(0)} \cdot Y) \cdot u^1_{(2)} \otimes S^{-1}(u^1_{(1)}) \triangleright \tilde{f} \otimes S(u^1_{(3)}) \cdot (u^2 \otimes \ldots \otimes u^q \otimes \overline{m}_{(-1)}),
\end{align*}
\]

(5.102)

where,
\[
(m \cdot \text{ad}Y(u_{(2)})) \delta(u_{(1)}) = m_\delta \cdot \text{ad}Y(u).
\]

(5.103)
Therefore we have
\[
\tilde{a}dY(\tau(m_\delta \otimes \tilde{f} \otimes \tilde{u})) = \\
m_{(0)}\delta \cdot u_{(2)}^1 \otimes S^{-1}(u_{(1)}^1) \triangleright \tilde{a}dY(\tilde{f}) \otimes S(u_{(3)}^1) \cdot (u^2 \otimes \ldots \otimes u^q \otimes m_{(-1)}^1) + \\
m_{(0)}\delta \cdot u_{(2)}^1 \otimes S^{-1}(\tilde{a}dY(u_{(1)}^1)) \triangleright \tilde{f} \otimes S(u_{(4)}^1) \cdot (u^2 \otimes \ldots \otimes u^q \otimes m_{(-1)}^1) + \\
m_{(0)}\delta \cdot \tilde{a}dY(u_{(2)}^1) \otimes S^{-1}(u_{(1)}^1) \triangleright \tilde{f} \otimes S(u_{(3)}^1) \cdot (u^2 \otimes \ldots \otimes u^q \otimes m_{(-1)}^1) + \\
m_{(0)}\delta \cdot u_{(2)}^1 \otimes S^{-1}(\tilde{a}dY(u_{(1)}^1)) \triangleright \tilde{f} \otimes S(a\tilde{y}(u_{(3)}^1)) \cdot (u^2 \otimes \ldots \otimes u^q \otimes m_{(-1)}^1) + \\
(m \cdot Y)_{(0)}\delta \cdot u_{(2)}^1 \otimes S^{-1}(u_{(1)}^1) \triangleright \tilde{f} \otimes S(u_{(3)}^1) \cdot (u^2 \otimes \ldots \otimes u^q \otimes (m \cdot Y)_{(-1)}^1).
\] (5.104)

Then the commutativity $\Delta \circ a\tilde{y} = a\tilde{y} \circ \Delta$ on $\mathcal{U}$ finishes the proof as
\[
\tilde{a}dY(\tau(m_{\delta} \otimes \tilde{f} \otimes \tilde{u})) = \\
m_{(0)}\delta \cdot u_{(2)}^1 \otimes S^{-1}(u_{(1)}^1) \triangleright \tilde{a}dY(\tilde{f}) \otimes S(u_{(3)}^1) \cdot (u^2 \otimes \ldots \otimes u^q \otimes m_{(-1)}^1) + \\
m_{(0)}\delta \cdot a\tilde{y}(u_{(2)}^1) \otimes S^{-1}(a\tilde{y}(u_{(1)}^1)) \triangleright \tilde{f} \otimes S(a\tilde{y}(u_{(3)}^1)) \cdot (u^2 \otimes \ldots \otimes u^q \otimes m_{(-1)}^1) + \\
(m \cdot Y)_{(0)}\delta \cdot u_{(2)}^1 \otimes S^{-1}(u_{(1)}^1) \triangleright \tilde{f} \otimes S(u_{(3)}^1) \cdot (u^2 \otimes \ldots \otimes u^q \otimes (m \cdot Y)_{(-1)}^1) = \tau(\tilde{a}dY(m_{\delta} \otimes \tilde{f} \otimes \tilde{u})).
\] (5.105)

For the generators $X, Y, \delta_1 \in \mathcal{H}$, it is already known that
\[
a\tilde{y}(Y) = 0, \quad a\tilde{y}(X) = X, \quad a\tilde{y}(\delta_1) = \delta_1.
\] (5.106)

We recall here the action of $Y \in sl_2$ as
\[
1_M \triangleleft Y = 0, \quad R^X \triangleleft Y = R^X, \quad R^Y \triangleleft Y = 0, \quad R^{\tilde{Z}} \triangleleft Y = -R^{\tilde{Z}}.
\] (5.107)

Hence we define the following weight on the cyclic complex by
\[
|Y| = 0, \quad |X| = 1, \quad |\delta_1| = 1, \quad |1_M| = 0, \quad |R^X| = -1, \quad |R^Y| = 0, \quad |R^{\tilde{Z}}| = 1,
\] (5.108)
we can express the following property of the operator $\tilde{a}dY$;
\[
\tilde{a}dY(m_{\delta} \otimes \tilde{f} \otimes \tilde{u}) = |m_{\delta} \otimes \tilde{f} \otimes \tilde{u}| m_{\delta} \otimes \tilde{f} \otimes \tilde{u},
\] (5.109)
where $|m_{\delta} \otimes \tilde{f} \otimes \tilde{u}| := |m| + |	ilde{f}| + |	ilde{u}|$. 

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Hence, the operator $\tilde{\text{ad}} Y$ acts as a grading (weight) operator. Extending the above grading to the cocyclic complex $3^{\bullet \bullet}$, we have
\[
3^{\bullet \bullet} = \bigoplus_{k \in \mathbb{Z}} 3^k,
\]
where
\[
3^k = \{ m_\delta \otimes \bar{f} \otimes \bar{u} \mid m_\delta \otimes \bar{f} \otimes \bar{u} = k \}.
\]
As a result of Lemma 5.4, we can say that $3^k$ is a subcomplex for any $k \in \mathbb{Z}$, and hence the cohomology inherits the grading. Namely,
\[
HP(H, M_\delta) = \bigoplus_{k \in \mathbb{Z}} H(3^k).
\]
Moreover, using Lemma 5.3 we conclude the following analogous of Corollary 3.10 in [12].

**Corollary 5.5.** The cohomology is captured by the weight 1 subcomplex, i.e.,
\[
H(3[1]) = HP(H, M_\delta), \quad H(3[k]) = 0, \quad k \neq 1.
\]

**Proposition 5.6.** The odd and even periodic Hopf cyclic cohomology of $H_{1S^{\text{cop}}}$ with coefficients in $M_\delta$ are both one dimensional. Their classes approximately are given by the following cocycles in the $E_1$ term of the natural spectral sequence associated to $M_\delta$.
\[
c^{\text{odd}} = 1 \otimes \delta_1 \in E_{1, \text{odd}}^1,
\]
\[
c^{\text{even}} = 1 \otimes X \otimes Y - 1 \otimes Y \otimes X + 1 \otimes Y \otimes \delta_1 Y \in E_{1, \text{even}}^1.
\]

Here, $1 := 1_M \otimes C_\delta$.

**Proof.** We have seen that all cohomology classes are concentrated in the weight 1 subcomplex. On the other hand, $E_1$ term of the spectral sequence associated to the above mentioned filtration on $M_\delta$ is
\[
E_{1,i}^j(H, M_\delta) = H_{i+j}(C(U \blacktriangleleft F, F_j M_\delta/F_{j-1} M_\delta)) \quad (5.116)
\]
where $F_0 M_\delta/F_{-1} M_\delta \cong F_0 M_\delta$, $F_1 M_\delta/F_0 M_\delta \cong C_\delta$ and $F_{j+1} M_\delta/F_j M_\delta = 0$ for $j \geq 1$.

Therefore,
\[
E_{1,0}^j(H, M_\delta) = 0, \quad E_{1,1}^j(H, M_\delta) = H^j(C(U \blacktriangleleft F, C_\delta)), \quad E_{1,j}^j(H, M_\delta) = 0, \quad j \geq 1.
\]

So the spectral sequence collapses at the $E_2$ term and we get
\[
E_{2,0}^0(H, M_\delta) \cong E_{\infty,0}^0(H, M_\delta) = 0, \quad (5.118)
\]
\[
E_{2,1}^0(H, M_\delta) \cong E_{\infty,1}^0(H) = F_1 H_{0}(C(U \blacktriangleleft F, M_\delta)) / F_0 H_{1}(C(U \blacktriangleleft F, M_\delta)). \quad (5.119)
\]
and
\[ E^{ij}_2(\mathcal{H}, M_\delta) \cong E^{ij}_\infty(\mathcal{H}, M_\delta) = 0, \quad j \geq 2. \] (5.120)

By definition of the induced filtration on the cohomology groups, we have
\[ F_1H^i(C(U \lhd F, M_\delta)) = H^i(C(U \lhd F, F_1M_\delta)) = H^i(C(U \lhd F, M_\delta)), \] (5.121)
and
\[ F_0H^i(C(U \lhd F, M_\delta)) = H^i(C(U \lhd F, F_0M_\delta)) \cong H^i(W(sl_2,F_0M)) = 0, \] (5.122)
where the last equality follows from the Whitehead’s theorem.

### 5.3.1 Construction of a representative cocycle for the odd class

In this subsection we first compute the odd cocycle in the total complex \( \text{Tot}^\bullet(F, U, M_\delta) \) of the bicomplex (4.40). Let us recall the total mixed complex
\[ \text{Tot}^\bullet(F, U, M_\delta) := \bigoplus_{p+q=\bullet} M_\delta \otimes F^\otimes p \otimes U^\otimes q, \] (5.123)
with the operators
\[
\begin{align*}
\rightarrow b_p &= \sum_{i=0}^{p+1} (-1)^i \partial_i, & \Uparrow b_q &= \sum_{i=0}^{q+1} (-1)^i \Uparrow \partial_i, & b_T &= \sum_{p+q=\bullet} \rightarrow b_p + (-1)^p \Uparrow b_q, \\
\rightarrow B_p &= \sum_{i=0}^{p-1} (-1)^i \sigma_{p-1} \rightarrow \tau^i, & \Uparrow B_q &= \sum_{i=0}^{q-1} (-1)^i \Uparrow \tau^i, & B_T &= \sum_{p+q=\bullet} \rightarrow B_p + (-1)^p \Uparrow B_q.
\end{align*}
\] (5.124)

**Proposition 5.7.** Let
\[ c' := 1 \otimes \delta_1 \in M_\delta \otimes F \] (5.126)
and
\[ c'' := R^X \otimes X + 2R^Z \otimes Y \in M_\delta \otimes U. \] (5.127)

Then \( c' + c'' \in \text{Tot}^1(F, U, M_\delta) \) is a Hochschild cocycle.

**Proof.** We start with the element \( c' := 1 \otimes \delta_1 \in M_\delta \otimes F \).

The equality \( \Uparrow b(c') = 0 \) is immediate to notice. Next, we observe that
\[
\begin{align*}
\rightarrow b(c') &= -R^X \otimes \delta_1 \otimes X - R^Y \otimes \delta_1 \otimes Y \\
&= -R^X \otimes \delta_1 \otimes X + R^Y \otimes \delta_1 \otimes Y + R^X \otimes \delta_1^2 \otimes Y - R^X \otimes \delta_1 \otimes Y - 2R^Y \otimes \delta_1 \otimes Y \\
&= \Uparrow b(R^Y \otimes X + 2R^Z \otimes Y).
\end{align*}
\] (5.128)
So, for the element $c'' := R^Y \otimes X + 2R^Z \otimes Y \in M \otimes U$, we have $\overrightarrow{b}(c') - \uparrow b(c'') = 0$.
Finally we notice $\overrightarrow{b}(c''') = 0$.

**Proposition 5.8.** The element $c' + c''' \in \text{Tot}^1(F, U, M)$ is a Connes cycle.

**Proof.** Using the action of $F$ and $U$ on $M$, we directly conclude that on one hand side we have $\uparrow B(c') = R^Z$, and on the other hand $\overrightarrow{B}(c''') = -R^Z$. 

Our next task is to send this cocycle to the cyclic complex $C^1(H, M)$. This is a two step process. We first use the Alexander-Whitney map

$$AW := \bigoplus_{p+q=n} AW_{p,q} : \text{Tot}^n(F, U, M) \to \mathfrak{z}^{n,n},$$

$$AW_{p,q} : F^\otimes p \otimes U^\otimes q \to F^\otimes p+q \otimes U^\otimes p+q$$

$$AW_{p,q} = (-1)^{p+q} \uparrow \partial_0 \uparrow \partial_0 \ldots \uparrow \partial_0 \partial_n \partial_{n-1} \ldots \partial_{p+1} \text{ (5.129)}$$

to pass to the diagonal complex $\mathfrak{z}^{*,*}(H, F, M)$. It is checked that

$$AW_{1,0}(c') = -1 \otimes \delta_1 \otimes 1 - R^X \otimes \delta_1 \otimes X - R^Y \otimes \delta_1 \otimes Y, \quad (5.130)$$
as well as

$$AW_{0,1}(c'') = -R^Y \otimes 1 \otimes X - 2R^Z \otimes 1 \otimes Y. \quad (5.131)$$

Summing them up we get

$$c_{\text{diag}}^{\text{odd}} := -1 \delta_1 \otimes 1 - R^X \delta_1 \otimes X - R^Y \delta_1 \otimes Y - R^Y \otimes 1 \otimes X - 2R^Z \otimes 1 \otimes Y. \quad (5.132)$$

Finally, via the quasi-isomorphism

$$\Psi : \mathfrak{z}^{*,*}(H, F, M) \to C^*(H, M)$$

$$\Psi(m \otimes f^1 \otimes \ldots \otimes f^n \otimes u^1 \otimes \ldots \otimes u^n)$$

$$= \sum m \otimes f^1 \triangleright u^1_{(0)} \otimes f^2 u^1_{(1)} \triangleright \ldots \otimes f^n u^1_{(n-1)} \otimes \ldots \otimes u^n_{(1)} \triangleright u^n \quad (5.133)$$

which is recalled from [15], we carry the element $c_{\text{diag}}^{\text{odd}} \in \mathfrak{z}^{2,2}(H, F, M)$ to

$$c^{\text{odd}} = -(1 \otimes \delta_1 + R^Y \otimes X + R^X \delta_1 X + R^Y \delta_1 Y + 2R^Z \otimes Y) \in C^1(H, M). \quad (5.134)$$

**Proposition 5.9.** The element $c^{\text{odd}}$ defined in (5.134) is a Hochschild cocycle.
Proof. We first calculate its images under the Hochschild coboundary \( b : C^1(\mathcal{H}, M_\delta) \rightarrow C^2(\mathcal{H}, M_\delta) \).

\[
b(1 \otimes \delta_1) = 1 \otimes 1_H \otimes \delta_1 - 1 \otimes \Delta(\delta_1) + 1 \otimes \delta_1 \otimes 1 + R^Y \otimes \delta_1 \otimes Y + R^X \otimes \delta_1 \otimes X
\]
\[
= R^Y \otimes \delta_1 \otimes Y + R^X \otimes \delta_1 \otimes X,
\]
\[
b(R^Y \otimes X) = R^Y \otimes 1_H \otimes X - R^Y \otimes \Delta(X) + R^Y \otimes X \otimes 1_H + R^X \otimes X \otimes \delta_1
\]
\[
= R^X \otimes X \otimes \delta_1 - R^Y \otimes Y \otimes \delta_1,
\]
\[
b(R^X \otimes \delta_1 X) = R^X \otimes 1_H \otimes \delta_1 X - R^X \otimes \Delta(\delta_1 X) + R^X \otimes \delta_1 X \otimes 1_H
\]
\[
= -R^X \otimes \delta_1 \otimes X - R^X \otimes \delta_1 Y \otimes \delta_1 - R^X \otimes X \otimes \delta_1 - R^X \otimes Y \otimes \delta_1^2,
\]
\[
b(R^Y \otimes \delta_1 Y) = R^Y \otimes 1_H \otimes \delta_1 Y - R^Y \otimes \Delta(\delta_1 Y) + R^Y \otimes \delta_1 Y \otimes 1_H + R^X \otimes \delta_1 Y \otimes \delta_1
\]
\[
= R^X \otimes \delta_1 Y \otimes \delta_1 - R^Y \otimes \delta_1 Y \otimes Y - R^Y \otimes Y \otimes \delta_1,
\]
\[
b(R^Z \otimes Y) = R^Z \otimes 1_H \otimes Y - R^Z \otimes \Delta(Y) + R^Z \otimes Y \otimes 1_H
\]
\[
+ R^Y \otimes Y \otimes \delta_1 + \frac{1}{2} R^X \otimes Y \otimes \delta_1^2
\]
\[
= R^Y \otimes Y \otimes \delta_1 + \frac{1}{2} R^X \otimes Y \otimes \delta_1^2.
\]

(5.135)

Now, summing up we get

\[
b(1 \otimes \delta_1 + R^Y \otimes X + R^X \otimes \delta_1 X + R^Y \otimes \delta_1 Y + 2 \cdot R^Z \otimes Y) = 0.
\]

(5.136)

\[\Box\]

Proposition 5.10. The Hochschild cocycle \( c^{\text{odd}} \) defined in (5.134) vanishes under the Connes boundary map.

Proof. The Connes boundary is defined on the normalized bi-complex by the formula

\[
B = \sum_{i=0}^n (-1)^n r^i \circ \sigma_{-1},
\]

(5.137)

where

\[
\sigma_{-1}(m_\delta \otimes h^1 \otimes \ldots \otimes h^{n+1}) = m_\delta \cdot h^1_{(1)} \otimes S(h^1_{(2)}) \cdot (h^2 \otimes \ldots \otimes h^{n+1})
\]

(5.138)

is the extra degeneracy. Accordingly,

\[
B(1 \otimes \delta_1 + R^Y \otimes X + R^X \otimes \delta_1 X + R^Y \otimes \delta_1 Y + 2 \cdot R^Z \otimes Y) =
\]
\[
1 \cdot \delta_1 + R^Y \cdot X + R^X \cdot \delta_1 X + R^Y \cdot \delta_1 Y + 2 \cdot R^Z \cdot Y =
\]
\[
R^Z - R^Z = 0.
\]

(5.139)

\[\Box\]
5.3.2 Construction of a representative cocycle for the even class

**Proposition 5.11.** Let
\[
c := 1 \otimes X \otimes Y - 1 \otimes Y \otimes X - R^X \otimes XY \otimes X - R^X \otimes Y \otimes X^2 + R^Y \otimes XY \otimes Y \\
+ R^Y \otimes X \otimes Y^2 - R^Y \otimes Y \otimes X \in M_\delta \otimes U^\otimes 2
\]  
(5.140)
and
\[
c'' := -R^X \otimes \delta_1 \otimes XY^2 + \frac{2}{3} R^X \otimes \delta_1^2 \otimes Y^3 + \frac{1}{3} R^Y \otimes \delta_1 \otimes Y^3 \\
- \frac{1}{4} R^X \otimes \delta_1^2 \otimes Y^2 - \frac{1}{2} R^Y \otimes \delta_1 \otimes Y^2 \in M_\delta \otimes F \otimes U.
\]  
(5.141)

Then \( c + c'' \in \text{Tot}^2(F, U, M_\delta) \) is a Hochschild cocycle.

**Proof.** We start with the element
\[
c := 1 \otimes X \otimes Y - 1 \otimes Y \otimes X - R^X \otimes XY \otimes X - R^X \otimes Y \otimes X^2 + R^Y \otimes XY \otimes Y \\
+ R^Y \otimes X \otimes Y^2 - R^Y \otimes Y \otimes X.
\]  
(5.142)

It is immediate that \( \overrightarrow{\partial}(c) = 0 \). To be able to compute \( \partial\overrightarrow{\partial}(c) \), we need to determine the following \( F \)-coaction.
\[
R^X \otimes (XY)^{(1,-1)} \otimes (XY)^{(0)} \otimes X^{(0)} \\
+ R^X \otimes (X^2)^{(1,-1)} \otimes Y \otimes (X^2)^{(0)} - R^Y \otimes (XY)^{(1,-1)} \otimes (XY)^{(0)} \otimes Y \\
- R^Y \otimes (XY)^{(1,-1)} \otimes X^{(0)} \otimes Y^2 + R^Y \otimes (XY)^{(1,-1)} \otimes Y \otimes X^{(0)}.
\]  
(5.143)

Hence, observing
\[
\nabla(X^2) = (X^2)^{(0)} \otimes (X^2)^{(1)} = X^{(1)} \otimes (X^{(0)} \otimes X^{(1)}(X_{2(1)} \triangleright X^{(1)})) \\
= X^2 \otimes 1 + 2XY \otimes \delta_1 + X \otimes \delta_1 + Y^2 \otimes \delta_1^2 + \frac{1}{2} Y \otimes \delta_1^2,
\]  
(5.144)

and
\[
\nabla(XY) = (XY)^{(0)} \otimes (XY)^{(1)} = X^{(0)} \otimes Y \otimes X^{(1)}, \quad XY \mapsto XY \otimes 1 + Y^2 \otimes \delta_1, \quad (5.145)
\]
we have
\[
\partial\overrightarrow{\partial}_0(c) = -R^X \otimes \delta_1 \otimes Y^2 \otimes X - R^X \otimes \delta_1 \otimes XY \otimes Y + R^X \otimes \delta_1^2 \otimes Y^2 \otimes Y \\
- 2R^X \otimes \delta_1 \otimes Y \otimes XY - R^X \otimes \delta_1 \otimes Y \otimes X + R^X \otimes \delta_1^2 \otimes X \otimes Y^2 \\
+ \frac{1}{2} R^X \otimes \delta_1^2 \otimes Y \otimes Y - R^X \otimes \delta_1 \otimes XY \otimes Y + R^Y \otimes \delta_1 \otimes Y^2 \otimes Y \\
+ R^X \otimes \delta_1^2 \otimes Y \otimes X \otimes Y^2 + R^Y \otimes \delta_1 \otimes Y \otimes Y^2 \\
+ R^X \otimes \delta_1^2 \otimes Y \otimes Y^2 + R^X \otimes \delta_1 \otimes Y \otimes X - R^Y \otimes \delta_1 \otimes Y \otimes Y - R^X \otimes \delta_1^2 \otimes Y \otimes Y.
\]  
(5.146)
It is now clear that
\[ b(c) = \begin{aligned} & 2 \left( R^X \otimes \delta_1 \otimes X \otimes Y^2 - \frac{2}{3} R^X \otimes \delta_1 \otimes Y^3 - \frac{1}{3} R^Y \otimes \delta_1 \otimes Y^3 \\
& + \frac{1}{4} R^X \otimes \delta_1 \otimes Y^2 + \frac{1}{2} R^Y \otimes \delta_1 \otimes Y^2 \right). \end{aligned} \] (5.147)

Therefore, for the element
\[ c' := -R^X \otimes \delta_1 \otimes X \otimes Y^2 + 2 \frac{2}{3} R^X \otimes \delta_1 \otimes Y^3 + \frac{1}{3} R^Y \otimes \delta_1 \otimes Y^3 \]
\[ - \frac{1}{4} R^X \otimes \delta_1 \otimes Y^2 - \frac{1}{2} R^Y \otimes \delta_1 \otimes Y^2, \] (5.148)

we have \( \bar{b}(c') + \bar{b}(c) = 0. \)

Finally we observe that,
\[ \bar{b}(c'') = R^X \otimes \delta_1 \otimes \delta_1 \otimes Y^3 - \frac{4}{3} R^X \otimes \delta_1 \otimes \delta_1 \otimes Y^3 + \frac{1}{3} R^X \otimes \delta_1 \otimes \delta_1 \otimes Y^3 \]
\[ + \frac{1}{2} R^X \otimes \delta_1 \otimes \delta_1 \otimes Y^2 - \frac{1}{2} R^X \otimes \delta_1 \otimes \delta_1 \otimes Y^2 = 0. \] (5.149)

\[ \square \]

**Proposition 5.12.** The element \( c + c'' \in \text{Tot}^2(F, U, M_\delta) \) vanishes under the Connes boundary map.

**Proof.** As above, we start with
\[ c := 1 \otimes X \otimes Y - 1 \otimes Y \otimes X - R^X \otimes XY \otimes X - R^X \otimes Y \otimes X^2 + R^Y \otimes XY \otimes Y \]
\[ + R^Y \otimes X \otimes Y^2 - R^Y \otimes Y \otimes X. \] (5.150)

To compute \( \bar{B} \), it suffices to consider the horizontal extra degeneracy operator \( \bar{\sigma} := \bar{\sigma}_1 \bar{\sigma}_1 \). We have,
\[ \bar{\sigma}_1(1 \otimes X \otimes Y) = 1 \cdot X_{(1)} \otimes S(X_{(2)})Y = -1 \otimes XY, \] (5.151)
and
\[ \bar{\sigma}_1(1 \otimes Y \otimes X) = 1 \cdot Y_{(1)} \otimes S(Y_{(2)})X = 1 \otimes X - 1 \otimes YX = -1 \otimes XY. \] (5.152)

Therefore we proved that \( \bar{\sigma}_1(1 \otimes X \otimes Y - 1 \otimes Y \otimes X) = 0. \) For the other terms in \( c \in M_\delta \otimes U \otimes U \) we proceed by
\[ \bar{\sigma}_1(R^X \otimes XY \otimes X) = R^X \cdot X_{(1)} Y_{(1)} \otimes S(Y_{(2)})S(X_{(2)})X \]
\[ = R^X \cdot XY \otimes X + R^X \otimes XY^2 - R^X \cdot X \otimes YX - R^X \cdot Y \otimes X^2 \]
\[ = R^Y \otimes XY + R^X \otimes YX^2 - 2R^X \otimes X^2, \] (5.153)
\[
\vec{\sigma}_{-1}(R^Y \otimes XY \otimes Y) = R^Y \cdot XY \otimes Y + R^Y \otimes YXY - R^Y \cdot X \otimes Y^2 - R^Y \cdot Y \otimes XY = R^Y \otimes XY^2 + R^Z \otimes Y^2; \\
(5.154)
\]

\[
\vec{\sigma}_{-1}(R^X \otimes Y \otimes X^2) = 2R^X \otimes X^2 - R^X \otimes YX^2, \\
(5.155)
\]

\[
\vec{\sigma}_{-1}(R^Y \otimes X \otimes Y^2) = -R^Z \otimes Y^2 - R^Y \otimes XY^2, \\
(5.156)
\]

and finally
\[
\vec{\sigma}_{-1}(R^Y \otimes Y \otimes X) = R^Y \otimes X - R^Y \otimes YX = -R^Y \otimes XY. \\
(5.157)
\]

This way we prove
\[
\vec{\sigma}_{-1}(-R^X \otimes XY \otimes X - R^X \otimes Y \otimes X^2 + R^Y \otimes XY \otimes Y + R^Y \otimes X \otimes Y^2 - R^Y \otimes Y \otimes X) = 0. \\
(5.158)
\]

Hence we conclude that \( \vec{\sigma}_1(c) = 0 \).

Since the action of \( \delta_1 \) on \( F_0M_\delta = \mathbb{C}\langle R^X, R^Y, R^Z \rangle \) is trivial, we have \( \vec{\delta}_1(c'') = 0 \).

The horizontal counterpart \( \vec{\delta}(c'') = 0 \) follows from the following observations. First we notice
\[
(R^X \otimes \delta_1^2) \cdot Y = R^X \cdot Y_{(1)} \otimes S(Y_{(2)}) \triangleright \delta_1^2 = 0, \\
(5.159)
\]

and secondly,
\[
(R^Y \otimes \delta_1) \cdot Y = R^Y \cdot Y_{(1)} \otimes S(Y_{(2)}) \triangleright \delta_1 = 0. \\
(5.160)
\]

Next, we send the element \( c + c'' \in \text{Tot}^2(F, U, M_\delta) \) to the cyclic complex \( C(H, M_\delta) \). As before, on the first step we use the Alexander-Whitney map to land in the diagonal complex \( \mathfrak{D}(H, F, M_\delta) \). To this end, we have
\[
AW_{0,2}(c) = \uparrow \partial_0 \uparrow \partial_0(c) \\
= 1 \otimes 1 \otimes 1 \otimes X \otimes Y - 1 \otimes 1 \otimes 1 \otimes X \otimes Y - R^X \otimes 1 \otimes 1 \otimes XY \otimes X \\
- R^X \otimes 1 \otimes 1 \otimes Y \otimes X^2 + R^Y \otimes 1 \otimes 1 \otimes XY \otimes Y + R^Y \otimes 1 \otimes 1 \otimes X \otimes Y^2 \\
- R^Y \otimes 1 \otimes 1 \otimes Y \otimes X, \\
(5.161)
\]

and
\[
AW_{1,1}(c) = \uparrow \partial_0 \vec{\partial}_1(c) \\
= -R^X \otimes 1 \otimes \delta_1 \otimes XY^2 \otimes 1 + \frac{2}{3}R^X \otimes 1 \otimes \delta_1^2 \otimes Y^3 \otimes 1 + \frac{1}{3}R^Y \otimes 1 \otimes \delta_1 \otimes Y^3 \otimes 1 \\
- \frac{1}{4}R^X \otimes 1 \otimes \delta_1^2 \otimes Y^2 \otimes 1 - \frac{1}{2}R^Y \otimes 1 \otimes \delta_1 \otimes Y^2 \otimes 1. \\
(5.162)
\]
As a result, we obtain the element
\[ c_{\text{diag}}^{\text{even}} = 1 \otimes 1 \otimes 1 \otimes X \otimes Y - 1 \otimes 1 \otimes 1 \otimes X \otimes X - R^X \otimes 1 \otimes 1 \otimes XY \otimes X \\
- R^X \otimes 1 \otimes 1 \otimes Y \otimes X^2 + R^Y \otimes 1 \otimes 1 \otimes XY \otimes Y + R^Y \otimes 1 \otimes 1 \otimes X \otimes Y^2 \\
- R^Y \otimes 1 \otimes 1 \otimes Y \otimes X - R^X \otimes 1 \otimes \delta_1 \otimes XY^2 \otimes 2 + \frac{2}{3}R^X \otimes 1 \otimes \delta_1^2 \otimes Y^3 \otimes 1 \\
+ \frac{1}{3}R^Y \otimes 1 \otimes \delta_1 \otimes Y^3 \otimes 1 - \frac{1}{4}R^X \otimes 1 \otimes \delta_1^2 \otimes Y^2 \otimes 1 - \frac{1}{2}R^Y \otimes 1 \otimes \delta_1 \otimes Y^2 \otimes 1. \] (5.163)

On the second step, we use the map (5.133) to obtain
\[ c_{\text{even}} = \Psi(c_{\text{diag}}^{\text{even}}) = 1 \otimes X \otimes Y - 1 \otimes Y \otimes X + 1 \otimes Y \otimes \delta_1 Y - R^X \otimes XY \otimes X \\
- R^X \otimes Y^2 \otimes \delta_1 X - R^X \otimes Y \otimes X^2 + R^Y \otimes XY \otimes Y + R^Y \otimes Y^2 \otimes \delta_1 Y \\
+ R^Y \otimes X \otimes Y^2 + R^Y \otimes \delta_1 Y \otimes Y^2 - R^Y \otimes Y \otimes X - R^X \otimes XY^2 \otimes \delta_1 \\
- \frac{1}{3}R^X \otimes Y^3 \otimes \delta_1^2 + \frac{1}{3}R^Y \otimes Y^3 \otimes \delta_1 - \frac{1}{4}R^X \otimes Y^2 \otimes \delta_1^2 - \frac{1}{2}R^Y \otimes Y^2 \otimes \delta_1, \] (5.164)
in \( C^2(\mathcal{H}, M_3) \).

**Proposition 5.13.** The element \( c_{\text{even}} \) defined in (5.164) is a Hochschild cocycle.

**Proof.** We first recall that
\[ b(1 \otimes X \otimes Y - 1 \otimes Y \otimes X + 1 \otimes Y \otimes \delta_1 Y) = \\
- R^X \otimes X \otimes Y \otimes X - R^Y \otimes X \otimes Y \otimes Y + R^X \otimes Y \otimes X \otimes X + R^Y \otimes Y \otimes X \otimes Y \\
- R^X \otimes Y \otimes \delta_1 \otimes Y \otimes X - R^Y \otimes Y \otimes \delta_1 Y \otimes Y. \] (5.165)

Next we compute
\[ b(R^X \otimes XY \otimes X) = -R^X \otimes X \otimes Y \otimes X - R^X \otimes Y \otimes X \otimes X - R^X \otimes Y^2 \otimes \delta_1 \otimes X \\
- R^X \otimes Y \otimes \delta_1 Y \otimes X + R^X \otimes XY \otimes Y \otimes \delta_1, \]
\[ b(R^X \otimes Y^2 \otimes \delta_1 X) = -2R^X \otimes Y \otimes Y \otimes \delta_1 X + R^X \otimes Y^2 \otimes \delta_1 \otimes X + R^X \otimes Y^2 \otimes X \otimes \delta_1 \\
+ R^X \otimes Y^2 \otimes \delta_1 Y \otimes \delta_1 + R^X \otimes Y^2 \otimes Y \otimes \delta_1^2, \]
\[ b(R^X \otimes Y \otimes X^2) = 2R^X \otimes Y \otimes X \otimes X + R^X \otimes Y \otimes XY \otimes \delta_1 \otimes \delta_1 + R^X \otimes Y \otimes Y \otimes X \delta_1 \otimes \delta_1 \\
+ R^X \otimes Y \otimes YX \otimes \delta_1 + R^X \otimes Y \otimes Y \otimes \delta_1 X + R^X \otimes Y \otimes Y^2 \otimes \delta_1^2, \]
\[ b(R^Y \otimes XY \otimes Y) = -R^Y \otimes X \otimes Y \otimes X - R^Y \otimes Y \otimes X \otimes Y - R^Y \otimes Y^2 \otimes \delta_1 \otimes Y \\
- R^Y \otimes Y \otimes \delta_1 Y \otimes X - R^X \otimes XY \otimes Y \otimes \delta_1, \]
\[ b(R^Y \otimes Y^2 \otimes \delta_1 Y) = -2R^Y \otimes Y \otimes Y \otimes \delta_1 Y + R^Y \otimes Y^2 \otimes \delta_1 \otimes Y + R^Y \otimes Y^2 \otimes Y \otimes \delta_1 \\
- R^X \otimes Y^2 \otimes \delta_1 Y \otimes \delta_1, \] (5.166)
as well as
\[
\begin{align*}
\text{b}(R^Y \otimes X \otimes Y^2) &= -R^Y \otimes Y \otimes \delta_1 \otimes Y^2 + 2R^Y \otimes X \otimes Y - R^X \otimes X \otimes Y^2 \otimes \delta_1, \\
\text{b}(R^Y \otimes Y \otimes \delta_1 Y^2) &= R^Y \otimes Y \otimes \delta_1 \otimes Y^2 + R^Y \otimes Y \otimes Y^2 \otimes \delta_1 + 2R^Y \otimes Y \otimes \delta_1 Y \otimes Y + 2R^Y \otimes Y \otimes Y \otimes \delta_1 Y - R^X \otimes Y \otimes \delta_1 Y^2 \otimes \delta_1, \\
\text{b}(R^Y \otimes Y \otimes X) &= R^Y \otimes Y \otimes Y \otimes \delta_1 - R^X \otimes Y \otimes X \otimes \delta_1, \\
\text{b}(R^X \otimes XY^2 \otimes \delta_1) &= -R^X \otimes X \otimes Y^2 \otimes \delta_1 - R^X \otimes Y^2 \otimes X \otimes \delta_1 - 2R^X \otimes XY \otimes Y \otimes \delta_1 \\
&- 2R^X \otimes Y \otimes XY \otimes \delta_1 - 2R^X \otimes Y^2 \otimes \delta_1 Y \otimes \delta_1 - R^X \otimes Y^3 \otimes \delta_1 \otimes \delta_1 \\
&- R^X \otimes Y \otimes \delta_1 Y^2 \otimes \delta_1, \\
\text{b}(R^X \otimes Y^3 \otimes \delta_1^2) &= -3R^X \otimes Y^2 \otimes Y \otimes \delta_1^2 - 3R^X \otimes Y \otimes Y^2 \otimes \delta_1^2 + 2R^X \otimes Y^3 \otimes \delta_1 \otimes \delta_1, \\
\text{b}(R^Y \otimes Y^3 \otimes \delta_1) &= -3R^Y \otimes Y^2 \otimes Y \otimes \delta_1 - 3R^Y \otimes Y \otimes Y^2 \otimes \delta_1 - R^X \otimes Y^3 \otimes \delta_1 \otimes \delta_1, \\
\text{b}(R^X \otimes Y^2 \otimes \delta_1^2) &= -2R^X \otimes Y \otimes Y \otimes \delta_1^2 + 2R^X \otimes Y^2 \otimes \delta_1 \otimes \delta_1, \\
\text{b}(R^Y \otimes Y^2 \otimes \delta_1) &= -2R^Y \otimes Y \otimes Y \otimes \delta_1 - R^X \otimes Y^2 \otimes \delta_1 \otimes \delta_1
\end{align*}
\]  

(5.167)

Summing up, we get the result. \(\square\)

\textbf{Proposition 5.14.} The Hochschild cocycle \(c^{\text{even}}\) defined in (5.164) vanishes under the Connes boundary map.

\textbf{Proof.} We will first prove that the extra degeneracy operator \(\sigma_{-1}\) vanishes on

\[
\begin{align*}
c := & 1 \otimes X \otimes Y - 1 \otimes Y \otimes X + 1 \otimes Y \otimes \delta_1 Y - R^X \otimes XY \otimes X \\
& - R^X \otimes Y^2 \otimes \delta_1 X - R^X \otimes Y \otimes X^2 + R^Y \otimes XY \otimes Y + R^Y \otimes Y^2 \otimes \delta_1 Y \\
&+ R^Y \otimes X \otimes Y^2 + R^Y \otimes Y \otimes \delta_1 Y^2 - R^Y \otimes Y \otimes X \in C^2(\mathcal{H}, M_{\delta}).
\end{align*}
\]
We observe that
\[
\sigma^{-1}(1 \otimes X \otimes Y - 1 \otimes Y \otimes X + 1 \otimes Y \otimes \delta_1 Y) = 0,
\]
\[
\sigma^{-1}(R^X \otimes XY \otimes X) = R^Y \otimes XY + R^X \otimes X^2 Y - R^X \otimes \delta_1 XY^2,
\]
\[
\sigma^{-1}(R^X \otimes Y^2 \otimes \delta_1 X) = R^X \otimes \delta_1 XY^2,
\]
\[
\sigma^{-1}(R^X \otimes Y \otimes X^2) = -R^X \otimes X^2 Y,
\]
\[
\sigma^{-1}(R^Y \otimes XY \otimes Y) = R^Z \otimes Y^2 + R^Y \otimes XY^2 - R^Y \otimes \delta_1 Y^3, \quad (5.169)
\]
\[
\sigma^{-1}(R^Y \otimes Y^2 \otimes \delta_1 Y) = R^Y \otimes \delta_1 Y^3,
\]
\[
\sigma^{-1}(R^Y \otimes X \otimes Y^2) = -R^Z \otimes Y^2 - R^Y \otimes XY^2 + R^Y \otimes \delta_1 Y^3,
\]
\[
\sigma^{-1}(R^Y \otimes Y \otimes \delta_1 Y^2) = -R^Y \otimes \delta_1 Y^3,
\]
\[
\sigma^{-1}(R^Y \otimes Y \otimes X) = -R^Y \otimes XY.
\]

As a result, we obtain \( \sigma^{-1}(c) = 0 \). On the second step, we prove that Connes boundary map \( B \) vanishes on
\[
c'' := -R^X \otimes XY^2 \otimes \delta_1 - \frac{1}{3} R^X \otimes Y^3 \otimes \delta_1^2 + \frac{1}{3} R^Y \otimes Y^3 \otimes \delta_1 \]
\[
- \frac{1}{4} R^X \otimes Y^2 \otimes \delta_1^3 + \frac{1}{2} R^Y \otimes Y^2 \otimes \delta_1 \in C^2(\mathcal{H}, M_6). \quad (5.170)
\]

Indeed, as in this case \( B = (\text{Id} - \tau) \circ \sigma^{-1} \), it suffices to observe that
\[
\sigma^{-1}(R^X \otimes XY^2 \otimes \delta_1) = -R^Y \otimes \delta_1 Y^2 - R^X \otimes \delta_1 XY^2 - \frac{1}{2} R^X \otimes \delta_1^2 Y^2 + R^X \otimes \delta_1^2 Y^3,
\]
\[
\sigma^{-1}(R^X \otimes Y^3 \otimes \delta_1^2) = -R^X \otimes \delta_1^2 Y^3,
\]
\[
\sigma^{-1}(R^Y \otimes Y^3 \otimes \delta_1) = -R^Y \otimes \delta_1 Y^3,
\]
\[
\sigma^{-1}(R^X \otimes Y^2 \otimes \delta_1^2) = R^X \otimes \delta_1^2 Y^2, \quad (5.171)
\]
\[
\sigma^{-1}(R^Y \otimes Y^2 \otimes \delta_1) = R^Y \otimes \delta_1 Y^2.
\]
together with
\[
\begin{align*}
\tau(R^Y \otimes \delta_1 Y) &= -R^Y \otimes \delta_1 Y^2 - R^X \otimes \delta_1^2 Y^2, \\
\tau(R^X \otimes \delta_1^2 Y) &= R^X \otimes \delta_1^2 Y^2, \\
\tau(R^Y \otimes \delta_1 Y^3) &= R^Y \otimes \delta_1 Y^3 + R^X \otimes \delta_1^2 Y^3, \\
\tau(R^X \otimes \delta_1 Y^3) &= -R^Y \otimes \delta_1 Y^2, \\
\tau(R^X \otimes \delta_1^2 Y^3) &= -R^Y \otimes \delta_1^2 Y^3, \\
\tau(R^X \otimes \delta_1 Y Y^2) &= R^Y \otimes \delta_1 Y^2 + R^X \otimes \delta_1 X Y^2 + \frac{1}{2} R^X \otimes \delta_1^2 Y^2 - R^X \otimes \delta_1^2 Y^3. \\
\end{align*}
\] (5.172)

We summarize our results in this section by the following theorem.

**Theorem 5.15.** The odd and even periodic Hopf cyclic cohomology of the Schwarzian Hopf algebra \(\mathcal{H}_{15}^{\text{cop}}\) with coefficients in the 4-dimensional SAYD module \(M_\delta = S(\mathfrak{sl}_2^*)\) are given by
\[
\begin{align*}
HP_{\text{odd}}(\mathcal{H}_{15}^{\text{cop}}, M_\delta) &= \mathbb{C} \Big\langle 1 \otimes \delta_1 + R^Y \otimes X + R^X \otimes \delta_1 X + R^Y \otimes \delta_1 Y + 2R^Z \otimes Y \Big\rangle, \\
\end{align*}
\] (5.173)
and
\[
\begin{align*}
HP_{\text{even}}(\mathcal{H}_{15}^{\text{cop}}, M_\delta) &= \mathbb{C} \Big\langle 1 \otimes X \otimes Y - 1 \otimes Y \otimes X + 1 \otimes Y \otimes \delta_1 Y - R^X \otimes X Y \otimes X \\
&\quad - R^X \otimes Y^2 \otimes \delta_1 X - R^X \otimes Y \otimes X^2 + R^Y \otimes X Y \otimes Y + R^Y \otimes Y^2 \otimes \delta_1 Y \\
&\quad + R^Y \otimes X \otimes Y^2 + R^Y \otimes Y \otimes \delta_1 Y^2 - R^Y \otimes Y \otimes X - R^X \otimes X Y^2 \otimes \delta_1 \\
&\quad - \frac{1}{3} R^X \otimes Y^3 \otimes \delta_1^2 + \frac{1}{3} R^Y \otimes Y^3 \otimes \delta_1 - \frac{1}{4} R^X \otimes Y^2 \otimes \delta_1^2 - \frac{1}{2} R^Y \otimes Y^2 \otimes \delta_1 \Big\rangle. \\
(5.174)
\end{align*}
\]

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