One dimensional constitutive model of isotropic magneto-sensitive rubber under shear deformation with amplitude, frequency and magnetic dependency

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Abstract. Modelling the amplitude, frequency and magnetic dependency of magneto-sensitive (MS) rubber in one dimensional case under shear deformation is proposed in this paper. MS rubber is a kind of smart materials which mainly consists of magnetizable particles and rubber matrix. The modulus of MS rubber changes rapidly after the applying of magnetic field which is referred as the magnetic dependency of MS rubber. Besides the magnetic dependency, there is an amplitude and frequency dependency of MS rubber. Specifically, the shear modulus of MS rubber under larger strain magnitude is smaller than the corresponding modulus under small strain and the shear modulus of MS rubber increases with increasing of frequency. A one-dimensional constitutive model consists of a fractional derivative element in parallel with a nonlinear kinematic elastoplastic model to depict the amplitude, frequency and magnetic dependency of MS rubber is developed in this paper. After comparison with the test result, it found that with eight parameters the amplitude, frequency and magnetic dependency can be depict precisely by the model developed. It is helpful for the predicting the dynamic behaviours of MS rubber-based vibration devices in the design phase and promotes MS rubber’s application in sound and vibration area.

1. Introduction
Magneto-sensitive (MS) rubber is a solid substitute for magnetorheological fluids with main components of magnetizable particles and non-magnetizable matrix. Due to the stiffness changing capacity, there are many researches of MS rubber’s possible application in vibration reduction area. For instance, Deng and Gong [1] developed a stiffness adaptable vibration absorber based on MS rubber. Test results showed that the vibration of the main structure can be reduced substantially by changing the stiffness of the MS rubber isolator to trace the frequency of the external harmonic loading. Wang and Kari [2] developed a MS rubber-based vibration isolator and change its stiffness by fuzzy logical controller. The simulation results revealed that compared with the traditional rubber-based vibration isolator without variable stiffness, the vibration isolation effect of MS rubber isolator is obviously enhanced. Similar results can be found in the articles by Jung [3], Sun [4] and Yang [5].

It is foreseeable that with the improvement of noise control requirements and progress in material science, the investigation of MS rubber will expand. A constitutive model which depicts the mechanical properties of MS rubber precisely is of great importance. There were many constitutive models of MS rubber, like the magnetic dipole-based models [6-7] and continuum mechanics-based models [8-9]. The drawback for the models mentioned is that only the magnetic-elastic and...
viscoelastic behaviours are simulated. However, the dynamic shear test by Lejon and Kari [10] shows except the magnetic-elastic and viscoelastic behaviours there is an amplitude dependency of mechanical behaviours of MS rubber. To be specific, the shear modulus of MS rubber decreases with the increasing of strain amplitude. However, the amplitude dependency is often neglected for the constitutive modelling of MS rubber which is unfavorable for the prediction of MS rubber’s mechanical properties during application. Therefore, a constitutive model with amplitude dependency of MS rubber is needed and necessary.

Normally the MS rubber vibration reduction devices [2-5] is constructed as follows: MS rubber is imbedded by two iron plates and works in a translational shear mode. A one-dimensional constitutive model to determine the shear stress strain relation is developed in this paper. The advantage of this model is that only eight parameters are needed to depict the amplitude, frequency and magnetic dependency of MS rubber which is helpful for the prediction of dynamic behaviours of MS rubber-based vibration reduction devices.

2. Constitutive model

The basic idea of the constitutive model is that the total shear stress is additively decomposed into viscoelastic $\tau^{ve}$ and elastoplastic stress $\tau^{ep}$ and the rheological configuration is visualized in figure 1. The meaning of material parameters in figure 1 will explained in the following section. Mathematically, the additive decomposition of the total shear stress is

$$\tau = \tau^{ve} + \tau^{ep}. \quad (1)$$

![Figure 1. Rheological configuration of the constitutive model.](image)

It is assumed that $\tau^{ve}$ is not affected by the magnetic field since physically the viscoelastic behaviour is mainly caused by the non-magnetizable rubber matrix. The measurement result that the loss factor of MS rubber is a relatively constant value with respect to the magnetic field supporting this assumption. The constitutive equation for the shear strain $\gamma$ and $\tau^{ve}$ is depicted by a fractional derivative viscoelastic model which is

$$\tau^{ve}(t) = \frac{b}{\Gamma(1-a)} \frac{d}{dt} \int_0^t \frac{\gamma(s)}{(t-s)^a} ds. \quad (2)$$

where $\Gamma$ is the gamma function and $a$ and $b$ are parameters for fractional derivative model. The restriction for $a$ is that $a \in [0,1]$. Compared with Generalized Maxwell model, only two parameters are needed for fractional derivative model to depict the viscoelastic behaviours with a higher accuracy for rubbers [11].

The elastoplastic stress is determined by a bounding surface nonlinear kinematic hardening model in series with an elastic spring. Symbols $G_e$, $S_{bounding}$ and $H_p$ are the parameters for the elastoplastic model. Measurement result reveals that the shear modulus and the magnetic induced shear modulus under a larger strain amplitude is smaller than the corresponding values under small strain amplitude. The conclusion that the amplitude dependency and magnetic dependency are correlated with each
other can be drawn. Therefore, magnetic dependency is introduced for the parameters $G_e$, $S_{\text{bounding}}$ and $H_p$. With considering the magnetic saturation of MS rubber, a hyperbolic tangent function is used

\[G_e = G_{\text{e0}} \left[ 1 + \kappa_1 \tanh \left( \frac{I_h}{I_s} \right) \right],\]

\[S_{\text{bounding}} = S_{\text{bounding0}} \left[ 1 + \kappa_2 \tanh \left( \frac{I_h}{I_s} \right) \right],\]

and

\[H_p = H_{p0} \left[ 1 + \kappa_3 \tanh \left( \frac{I_h}{I_s} \right) \right],\]

where $G_{\text{e0}}$, $S_{\text{bounding0}}$ and $H_{p0}$ are parameters under zero magnetic field. Symbols $\kappa_1$, $\kappa_2$ and $\kappa_3$ represent the magnetic field influencing factor. $I_h = H \cdot H$ with $H$ the magnetic field intensity is the magnetic field invariant. $I_s$ representing the magnetic saturation of MS rubber.

The bounding surface model within the elastoplastic model [12] is defined

\[ \Phi = \tau^{\text{ep, image}} - \beta - S_{\text{bounding}} = 0, \]

where \( \tau^{\text{ep, image}} \) is the image stress on the bounding surface which is determined by the kinematic rule and the extension direction of \( \tau^{\text{ep}} \). Let \([t_{n-1}, t_n]\) be the time interval of interest. Symbols with subscripts \( n \) and \( n-1 \) represent the quantities at the previous and current time step, respectively. Initially the plastic strain is locked at the current time step and the trial stress can be determined by

\[ \tau^{\text{ep, trial}} = G_e \left( \gamma_n - \gamma_{n-1}^p \right), \]

The extension direction for \( \tau^{\text{ep}} \) is obtained as

\[ m = \frac{\tau^{\text{ep, trial}} - \tau_{n-1}^{\text{ep, trial}}}{\tau_{n-1}^{\text{ep, trial}} - \tau_{n-1}^{\text{ep, trial}}}. \]

According to Mróz’s kinematic rule [13], \( \tau^{\text{ep, image}} \) is determined as the intersection between \( m \) and bounding surface in equation (6). Mathematically, it is

\[ \tau^{\text{ep, image}} = \tau^{\text{ep, trial}} + m \delta, \]

where \( \delta = \left[ m : \left( \tau^{\text{ep, trial}} - \beta_{n-1} \right) \right]^2 + S_{\text{bounding}}^2 \left( \tau^{\text{ep, trial}} - \beta_{n-1} \right)^2 \right]^{1/2} - m : \left( \tau^{\text{ep, trial}} - \beta_{n-1} \right) \). The normal direction from the center of the bounding surface \( \beta \) to \( \tau^{\text{ep, image}} \) is

\[ N_n = \frac{\tau^{\text{ep, image}} - \beta_{n-1}}{\left| \tau^{\text{ep, image}} - \beta_{n-1} \right|}. \]

After obtaining \( \tau^{\text{ep, image}} \), according to ref. [14] the plastic modulus $H_s$ which used to determine \( \tau^{\text{ep}} \) is obtained

\[ H_s = H_p \left( \frac{\delta}{\delta_m - \delta} \right), \]
where $\delta_m$ is the maximum distance between $\tau^{ep}$ and $\tau^{ep,\text{image}}$ during a smooth loading history. Initially, $\delta_m = S_{\text{bounding}}$. The value of $\delta_m$ updates when a drastic change of loading occurs. The updating criteria is if $m : N_{n-1} < 0$ or $\delta > \delta_m$, then $\delta_m = \delta$. The plastic hardening rule to determine the plastic multiplier $\lambda$ for bounding surface model is

$$H_s \lambda = \left( \tau_n^{ep} - \tau_{n-1}^{ep} \right) / N_n. \quad (12)$$

By equation (7) and the normality condition for associated plasticity that $\gamma_n^p = \lambda N_n$, the plastic multiplier $\lambda$ can be determined. Which is

$$\lambda = \frac{\left( \tau_n^{ep,\text{trial}} - \tau_{n-1}^{ep} \right)}{H_s + G_e}. \quad (13)$$

Subsequently, $\tau^{ep}$ and $\beta$ at the current time step are obtained by

$$\tau_n^{ep} = \tau_n^{ep,\text{trial}} - G_e \lambda N_n. \quad (14)$$

and

$$\beta_n = \beta_{n-1} + \frac{H_p}{H_s} \left( \tau_n^{ep} - \tau_{n-1}^{ep} \right). \quad (15)$$

### 3. Discussion and results

Measurement data from Lejon and Kari [10] was used for the parameter identification. 20 °C was chosen and the strain amplitudes are 0.0005, 0.0015 and 0.005. The frequency for the loading ranging from 200 to 900 Hz. More details for the measurement setup and method is referred to ref. [10]. Firstly, magnetic independent parameters include $G_0$, $S_{\text{bounding}0}$, $H_{p0}$, $a$ and $b$ were obtained by the nonlinear least square method with “lsqnonlin” function in matlab® (MATLAB Release 2015b, The MathWorks, Inc., Natick, Massachusetts, United States). The results are $G_0 = 4.747 \times 10^5$ Pa, $S_{\text{bounding}0} = 1.528 \times 10^5$ Pa, $H_{p0} = 2.765 \times 10^7$ Pa, $a = 0.283$ and $b = 0.330 \times 10^6$ Ns/m$^2$ with a relative error between measurement and simulation results 2.835%. After the first round of parameter identification, magnetic dependent parameters were identified under three kinds of magnetic field which are 0, 0.3 and 0.55T respectively. The results are $\kappa_s = 0.418$, $\kappa_z = 0$, $\kappa_z = 0.387$ and $I_s = (0.394T / \mu_0)^2$ with $\mu_0$ is the magnetic permeability in vacuum. The relative error between the measurement and simulation results is 3.501%. The comparison between the measurement and simulation result are shown from figure 2 to 4. It can be found that the match between the test and modelling result are quite good except the jump for the magnitude and loss factor in the frequency range from 600 to 700 Hz which is caused by the resonance of the test machine. Therefore, with the constitutive model developed in this paper, the amplitude, frequency and magnetic dependency of MS rubber can be predicted with a high accuracy.
Figure 2. Magnitude and loss factor of MS rubber under 0T magnetic field. Solids lines and symbols are experimental and simulation result, respectively. 0.0005, 0.0015 and 0.005 are strain amplitudes.

Figure 3. Magnitude and loss factor of MS rubber under 0.3T magnetic field. Solids lines and symbols are experimental and simulation result, respectively. 0.0005, 0.0015 and 0.005 are strain amplitudes.

Figure 4. Magnitude and loss factor of MS rubber under 0.55T magnetic field. Solids lines and symbols are experimental and simulation result, respectively. 0.0005, 0.0015 and 0.005 are strain amplitudes.

4. Conclusion
A new constitutive model of isotropic MS rubber under shear strain was proposed in this paper. Fractional derivative element and a nonlinear kinematic elastoplastic model was used to represent the frequency and amplitude dependency of MS rubber, respectively. The magnetic dependency is reflected by a hyperbolic tangent function with considering of the magnetic saturation. After parameter identification, the conclusion that the amplitude, frequency and magnetic dependency of MS rubber can be depict by this proposed constitutive model can be drawn. The work in this paper fills the research deficiency of amplitude dependency in constitutive modelling of MS rubber and sets a foundation for the prediction of MS rubber-based vibration reduction devices and the possible application of those devices in noise and vibration control area.
5. Reference

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