Light cone nucleon wave function in the quark-soliton model

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Abstract

The light-cone wave function of the nucleon is calculated in the limit $N_c \to \infty$ in the quark-soliton model inspired by the theory of the instanton vacuum of QCD. The technique of the finite time evolution operator is used in order to derive expressions for all components of the Fock vector describing the nucleon in the infinite momentum frame. It is shown that nucleon wave function for large $N_c$ can be expressed in terms of the wave function of the discrete level in the self-consistent meson field and light cone wave functions of 1, 2, etc mesons. The 3-quark components of the nucleon and $\Delta$-resonance are estimated. Wave function of the nucleon appears to be positive in the whole region of $x$ and it has rather small asymmetry. It differs strongly both from Chernyak-Zhitnitsky wave function and the asymptotic one. Large momentum transfer asymptotic of the electromagnetic and axial form factors is discussed.

1 Introduction

Light cone wave functions of hadrons were introduced in hadron physics many years ago \cite{1, 2, 3, 4, 5}. They contain virtually the full information which is necessary to describe hadron properties at high energies. The use of hadron wave functions in the context in QCD relies on the concept of factorization. Processes with hadrons are divided into 2 parts: i) the hard process-dependent parts are calculated according to perturbative QCD and ii) the soft process-independent part is usually encoded in soft functions, parton distributions, fragmentation functions, etc. Usually, a formal definition of the soft part can be formulated in terms of definite quark and gluon hadron matrix elements. Their logarithmic scale dependence is well understood in terms of corresponding evolution equations.
In principle, any soft part can be expressed in terms of the hadron wave function. For example, ordinary parton distributions is a sum of wave functions squared corresponding to a different number of quarks or gluons integrated over all momenta of partons except one. The analogous definition in terms of wave functions can be done for the fragmentation functions as well.

Another type of hadronic matrix elements is involved in the description of the elastic form factors or transition form factors. There one deals with the matrix elements of quark currents between hadron states which have very different initial and final momenta. The large difference of momenta in exclusive reactions effectively separates out the component of wave function with the minimal number of constituents, so gives an access to the simplest structures in hadron [1, 2, 3, 4, 5].

An intermediate situation between these two types of reactions occurs in processes like deeply virtual Compton scattering and hard meson production. In this case, the hadron operators involved are bilocal and their matrix elements are off-forward. The soft part of such processes is described by so-called generalized distributions. Recently it was shown that generalized distributions can also be easily presented in terms of light cone wave functions [6].

Unfortunately up to now light-cone wave functions of baryons (as well as mesons) in the low normalization point cannot be determined from the first principles of QCD. Perturbative theory is able to predict only so-called asymptotic wave functions which are normalized at arbitrary high normalization point.

Up to the present moment the light-cone nucleon wave functions were calculated only in the framework of QCD sum rules [7]. Of course, this can be considered only as a crude estimate. For this reason self-consistent models of the nucleon (more or less motivated by QCD) become highly desirable. Unfortunately, to the best of our knowledge, no calculation of the nucleon wave function was made in any self-consistent relativistic field-theoretical model.

In this paper we attempt to calculate light-cone wave functions at a low normalization point in the limit of large number of colours, $N_c \to \infty$. Even though in reality $N_c = 3$, the academic limit of large $N_c$ is known to be a useful guideline. It is a general QCD theorem that at large $N_c$ the nucleon is heavy and can be viewed as a classical soliton [8].

An example of the dynamical realization of this idea is given by the Skyrme model [9]. However, the Skyrme model is based on an unrealistic effective chiral Lagrangian. A far more realistic effective chiral Lagrangian is given by the functional integral over quarks in the background pion field [10]:

$$\exp(iS_{\text{eff}}[\pi(x)]) = \int D\psi D\bar{\psi} \exp \left(i \int d^4x \bar{\psi}(i\partial - MU \gamma_5)\psi \right).$$

Here $\psi$ is the quark field, $M$ is the effective quark mass which is due to the spontaneous breakdown of chiral symmetry (generally speaking, it is momentum-dependent) and $U$ is the $SU(2)$ chiral pion field. The NJL-type Lagrangian Eq. (1) has been derived from the instanton model of the QCD vacuum [11, 12], which provides a natural mechanism of chiral symmetry breaking and enables one to express the dynamical mass $M$ and the
ultraviolet cutoff $\Lambda$ intrinsic to eq. (1) through the $\Lambda_{\text{QCD}}$ parameter. Proposed on the basis of Lagrangian of eq. (1) the chiral quark-soliton model [12, 13, 14, 15] describes properties of baryons far better than the Skyrme model. For the recent status of the chiral quark-soliton model see reviews [16, 17].

Recently in the framework of quark-soliton the parton distributions [18, 19] and the generalized distributions [20] have been computed at a low normalization point. In this paper we complete the program of the investigation of high-energy properties of baryons in the quark-soliton model and present the framework for calculation of the light-cone wave of the nucleon.

We use the evolution operator technique in order to present wave functions of the nucleon state moving with the speed $V$. In the limit $V \to 1$ the corresponding wave function tends to the light-cone wave function in question. This method is convenient as the evolution operator can immediately be expressed as a functional integral with definite boundary conditions.

At large $N_c$ the functional integral representing the evolution operator can be evaluated using the saddle-point method. The saddle-point of the effective Lagrangian in eq. (1) in the sector with unity baryon charge corresponds to the nucleon-soliton [14]. The stationary solutions of saddle-point equations correspond to the nucleon in the rest frame. Owing to relativistic invariance of the equations of motion there is also infinite number of other solutions which describe the moving solitons. We need the solution which describes the nucleon in the infinite momentum frame, and we have to extract the corresponding quark-antiquark wave function. In the $N_c \to \infty$ limit it can be viewed as a product of the quark states in the time-dependent self-consistent pion field.

We show that the nucleon wave function at large $N_c$ is a product of $N_c$ one-quark wave functions of the valence quarks and the coherent exponential of quark-antiquark pairs corresponding to the sea quarks. As it should be in this limit, the nucleon wave function is completely factorized in colours. One-quark wave functions receive contribution both from the wave function of the discrete level in the mean pion field and the sea quarks. The quark-antiquark pair wave function can be expressed in terms of so-called Feynman Green function at finite time.

This structure of the light-cone nucleon wave function is rather general. Indeed, as was already said, nucleon is a soliton of an effective meson Lagrangian at large $N_c$. Hence its quark wave function is a product of states in the external field of all mesons. The model with Lagrangian of eq. (1) (which is, in turn, based on the theory of the instanton vacuum) is specific in two respects. First, as the size of the nucleon ($\sim 1/M$) is parametrically large as compared to the ordinary hadron scale ($\sim 1/\rho$, $\rho$ being the size of the instanton), only the lightest degrees of freedom (i.e. pions and constituent quarks) are important. Second, the number of gluons is suppressed in the instanton vacuum by the parameter $(M\rho)^2 \ll 1$ [18, 17], so gluons in this model do not participate in the formation of the nucleon wave function.

All what one needs to know in order to calculate the nucleon wave function in the model is the wave function of the discrete level (the solution of the Dirac equation in the external field) and the quark-antiquark pair wave function. Expanding the latter wave function in $\pi$-meson field, it is possible to present the wave function as convolution of the pion mean field with the quark-antiquark light-cone wave function of 1,2,\ldots pions. This
is to be expected, as the pions are the only agents inducing the interaction in the model.

In fact, light-cone wave functions of one and two pions were already considered in the
instanton vacuum (see Refs. [23] and [24], correspondingly) and appeared to describe the
data correctly. We discuss the connection with meson wave functions in the Section 3.

We formulate the scheme for calculation of the nucleon-soliton light-cone wave function
in Section 2 and in Section 3 we consider the wave functions in the infinite momentum
frame. The three-quark components of the nucleon wave functions (so-called distribution
amplitude) is calculated in Section 4. They appear to be almost symmetric (antisymmetric
part of the nucleon wave function is numerically small) smooth functions, which are far
from both the asymptotical wave function and the wave function of Chernyak-Zhitnicky
type [28, 29].

We also discuss physical observables (asymptotics of electromagnetic and axial form
factors) and evolution of the model wave function to the high $Q^2$. Asymptotics of the
form factors cannot be calculated explicitly as corresponding integrals depend strongly
on the region of small $x$ where the model is not valid. However we are able to calculate
ratios of the form factors which appear to be rather close to the experimental data.

2 Soliton wave function in field theory
at large $N_c$

In principle, calculation of the wave function of a given state in terms of quarks and
antiquarks is straightforward in the quantum field theory. However, usually this task is
too complicated. In fact, we know only one model field theory, namely, the Schwinger
model where the program of calculating of all wave functions was completed [21]. The
most direct way to obtain wave functions of any state is to calculate the evolution operator
$S(T)$ for the given theory and present it as a sum:

$$S(T) = \exp(-i\hat{H}T) = \sum_n e^{-iE_n T}|n\rangle\langle n|.$$  (2)

Here $|n\rangle$ is vector of certain state — the eigenfunction of Hamiltonian $H$. As it is well-
known, the evolution operator of eq. (2) can be expressed as a functional integral at finite
time with definite boundary conditions, namely ¹

¹We immediately write down the functional integral in the model of eq. (1) but, in fact, we have to start
from the full QCD. The corresponding functional integral is the integral over both the quark and gluon
field. In the model of the instanton vacuum this integral is saturated by instanton and antinstanton
gluon configuration. One can then integrate out gluons in this approximation. As it was shown in
Refs. [11, 12], one obtains the low-energy effective Lagrangian eq. (1). As the number of gluons in the
nucleon in the instanton model is parametrically suppressed, we should not put any boundary conditions
on gluon field. For this reason the derivation of the evolution operator in the low-energy limit repeats
literally the derivation of the low-energy effective lagrangian that was done in these papers. If one is
still interested in gluon components of the wave function, they should be traced directly from the general
expression for the full QCD evolution operator to which we have to apply the same approximations which
lead to the effective Lagrangian of eq. (1). This is straightforward in the instanton vacuum model. Gluon
components are of the order of $(M\rho)^2 \ll 1$ where $\rho$ is the instanton size. The same parameter was used
in the derivation of effective lagrangian eq. (1)
\begin{align}
S[T] &= \int_{\psi^{(-)=a}}^\psi^{(+)=b} D\psi(x) \int_{\tilde{\psi}^{(-)=a}}^\tilde{\psi}^{(+)=b} D\tilde{\psi}(x) \exp \left( i \int_0^T dt \, \mathcal{L}_{\text{eff}} \right),
\end{align}

where \( \mathcal{L}_{\text{eff}} \) is the effective Lagrangian of eq. (1). The operators \( a^\pm(p), b^\pm(p) \) are annihilation-creation operators for quarks-antiquarks which are defined through the expansion of the field \( \psi(x) \) into positive- and negative-frequency parts:

\begin{align}
\psi_a(x) &= \int \frac{d^3p}{(2\pi)^3} \sum_\lambda \sqrt{m} \frac{1}{P_0} \left( a^{(\lambda)}(\vec{p}) u^{(\lambda)}(\vec{p}) e^{i\vec{p} \cdot \vec{x}} + b^{(\lambda)^+}(\vec{p}) v^{(\lambda)}(\vec{p}) e^{-i\vec{p} \cdot \vec{x}} \right),
\end{align}

(here \( \lambda \) is quark polarization; \( u^{(\lambda)}(\vec{p}), v^{(\lambda)}(\vec{p}) \) are free plane wave spinors).

We do not impose any boundary conditions on the \( \pi \)-meson field. This field appears in the derivation of the effective Lagrangian as a result of bosonization [10], and it should not be considered as an elementary one. In fact, it is impossible to consider the light-cone wave function in terms of both quarks and \( \pi \)-mesons — it would be a kind of double-counting.

In the representation of eq. (3) the quark-antiquark creation operators \( a^+, b^+ \) anti-commute with annihilation operators \( a, b \), as they are related to different times 0 and \( T \). One has to calculate the functional integral of eq. (3) at the \textit{finite} time \( T \) as a functional of operators \( a^\pm, b^\pm \). Expanding in the exponentials \( \exp(-iE_nT) \) and factorizing creation operators from annihilation ones, we can find eigenfunctions of all states in terms of quarks-antiquarks (see, e.g. [21]).

In the large \( N_c \)-limit the functional integral over the \( \pi \)-meson field should be calculated in the saddle-point approximation. This procedure can be formulated as follows.

Let us first integrate over the quark field \( \psi(x) \). We divide fermion field into two parts, \( \tilde{\psi}(x) \) and \( \chi(x) \):

\begin{align}
\psi(x) = \tilde{\psi}(x) + \chi(x), \quad \tilde{\psi}(x) = \bar{\psi}(x) + \bar{\chi}(x).
\end{align}

Let us require that \( \tilde{\psi}(x) \) obeys the Dirac equation in the external \( \pi \)-meson field with corresponding boundary conditions:

\begin{align}
\left( i \gamma \partial - M e^{i\tilde{\gamma} \gamma_5} \right) \tilde{\psi} = 0,
\end{align}

\begin{align}
\tilde{\psi}^{(+)}(\vec{x}, t = 0) &= A(\vec{x}) \equiv \int \frac{d^3p}{(2\pi)^3} \bar{a}_\lambda(\vec{p}) u^\lambda(\vec{p}),
\end{align}

\begin{align}
\tilde{\psi}^{(-)}(\vec{x}, t = T) &= B^+(\vec{x}) \equiv \int \frac{d^3p}{(2\pi)^3} \bar{b}_\lambda^+(\vec{p}) v^\lambda(\vec{p}),
\end{align}

The field \( \chi(x) \) obeys zero boundary conditions.

A solution of the Dirac eq. (3) can be expressed in terms of the \textit{finite} time Feynman Green function \( G^{(T)}(\vec{x}, t|\vec{y}, 0) \)

\begin{align}
\tilde{\psi}(x, t) &= \int d^3y \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{P_0}{M}} \sum_\lambda \left[ G^{(T)}(\vec{x}, t|\vec{y}, 0) u^\lambda(\vec{p}) a_\lambda(\vec{p}) e^{i\vec{p} \cdot \vec{y}} + G^{(T)}(\vec{x}, t|\vec{y}, T) v^\lambda(\vec{p}) b_\lambda^+(\vec{p}) e^{-i\vec{p} \cdot \vec{y}} \right].
\end{align}
with an analogous expression for $\tilde{\psi}(\vec{x}, t)$.

The finite time Green function $G^{(T)}(\vec{x}, t|\vec{x}', t')$ is the solution of the Dirac equation in the external field which has vanishing negative-frequency part at $t = 0$ and vanishing positive-frequency part at $t = T$. At $T \to \infty$ finite time Green function reduces to the usual Feynman Green function. Feynman propagators for free particles appear to obey automatically the required boundary conditions at any $T$. Also it is easy to verify that the finite time Green function in the external field can be constructed as a sum of diagrams with free Feynman propagator interacting with external field. The only difference with the usual Feynman Green function is that all integrals in intermediate points should be carried only in the finite time interval $0 < t < T$ (for Feynman Green function one has to integrate over the whole interval $-\infty < t < \infty$).

Let us substitute the expansion of eq. (5) into the functional integral of eq. (3). One can see that variables $\tilde{\psi}$ and $\chi(x)$ are completely separated from each other. As a result we arrive at the following expression for the evolution operator (cf. [21]):

$$S[T, a^\pm, b^\pm] = \int D\pi(x)\text{Det}^{(T)}[\pi]S_{\text{ext}}[\pi, T]$$

(we used here eq. (7) for $\tilde{\psi}$). Here $S_{\text{ext}}[\pi, T]$ is the evolution operator in the given external field:

$$S_{\text{ext}}[\pi, T] = \exp\left\{ \int d^3x d^3y \left[ B(x)\gamma^0G^{(T)}(x0, yT)B^+(y) + B(x)\gamma^0G^{(T)}(x, y0)A(y) + A^+(x)\gamma^0G^{(T)}(xT, y0)A(y) + A^+(x)\gamma^0G^{(T)}(xT, T, y)B^+(y) \right] \right\},$$

where $\varepsilon \to +0$; the quantity $\text{Det}^{(T)}[\pi]$ is the finite-time determinant in the external field. It is the Gaussian functional integral over the fermion field $\chi(x)$ with zero boundary conditions. For this reason it does not depend on the operators $a^\pm, b^\pm$ being only the functional of pion field $\pi(x)$. It can also be presented in terms of the finite-time Green function:

$$\text{Det}^{(T)}[\pi] = \exp \left[ \int_0^M dM \int_0^T dt \int d^3x \text{Tr} \left( G^{(T)}(\vec{x}, T) iU^\gamma(\vec{x}, t) \right) \right].$$

(10)

Thus the evolution operator in the external field is the coherent exponential of the creation-annihilation operators. The nucleon is the lowest possible state in the sector with baryon charge $B = 1$. One can obtain its wave function by applying the evolution operator to any colourless state of $N_c$ quarks and taking the $T \to \infty$ limit. For example, the nucleon wave function $\Phi_N$ can be obtained from the state of free quarks:

$$c_N e^{-i\mathcal{M}_N T} \Phi_N(a^\pm, b^+) = \lim_{T \to \infty} S(T) \prod_{i=1}^{N_c} a^+_\alpha(p_i)|\Omega_0\rangle \sim$$

$$\int D\pi \text{Det}^{(T)}[\pi] \prod_{i=1}^{N_c} G_i^{(T)}(p_i, 0, k_i, T) a^+_\alpha(k) \exp[a^+(p)G^{(T)}(T, p, T, p')b^+(p')]$$

(11)

(we write the formula in a bit symbolical form, at the moment it is sufficient for our purposes). Here $\mathcal{M}_N$ is the nucleon mass and $c_N$ is the overlap between the initial
state of $N_c$ quarks and nucleon wave function. The functional integral of eq. (11) should be calculated in the saddle-point approximation. One has to find the pion field which extremizes the integrand. At large $T$ important factors are \(^2\)

$$\text{Det}^{(T)}[\pi] \sim \exp(-iE_{\text{field}}[\pi]T), \quad G_i^{(T)}(p_i, 0, k_i, T) \sim \exp(-iE_{\text{level}}[\pi]T),$$

(12)

where $E_{\text{field}}[\pi]$ is the energy of the Dirac continuum with the given pion field (proportional to $N_c$) and $E_{\text{level}}[\pi]$ is the energy of the (possible) discrete level for the quark in this field. In order to find the saddle-point one has to minimize the sum:

$$\mathcal{E}[\pi] = E_{\text{field}}[\pi] + N_cE_{\text{level}}[\pi]$$

(13)

in the presence of the pion field. It is exactly the condition which was used in constructing the nucleon-soliton in Refs [12, 13, 14, 15]. Both contributions to the total energy are of the order of $N_c$. The first is the full effective chiral Lagrangian (ECL) in the low-energy effective theory of eq. (1) calculated for a given pion field $\pi$.

The minimum of eq. (13) is achieved at some stationary pion field and corresponds to the nucleon at rest. The value of the energy in the minimum is the nucleon mass:

$$M_N = \min \mathcal{E}[\pi] \sim \mathcal{O}(N_c).$$

(14)

This minimum was found in Ref. [14, 15] and it corresponds to the hedgehog symmetry of the pion field:

$$\vec{\pi}^a(\vec{x}) = n^a P(r), \quad \vec{n} = \frac{r}{r},$$

(15)

where the profile function $P(r)$ is to be calculated numerically.

We can obtain the wave function of nucleon in the leading order of $N_c$, if we substitute the saddle-point field $\pi(x)$ of eq. (15) into eq. (11). In the higher orders, one has to express the general pion field $\pi(x, t) = \vec{\pi} + \pi^{\text{quant}}$ and then perform the Gaussian integration in $\pi^{\text{quant}}$ which appears to be $\pi^{\text{quant}} \sim \mathcal{O}(1/\sqrt{N_c})$. Thus it is possible to formulate the systematic perturbation theory in $1/N_c$. However, in this paper, we restrict ourselves to the leading order.

Let us stress that the procedure of the calculation of the nucleon wave function which we have formulated, is rather general. Indeed, it is a general QCD theorem that at large $N_c$, the nucleon is the soliton of some effective meson Lagrangian. Thus its wave function can always be presented in the form of eq. (11) where Green functions should be found in the self-consistent field of all mesons entering this effective Lagrangian. Of course, the exact low-energy meson Lagrangian is unknown. In the present work, we use the instanton vacuum model in order to fix this low-energy Lagrangian.

Let us also rewrite nucleon wave function in a different form. As explained above, nucleon can be described as $N_c$ valence quarks + Dirac continuum in the self-consistent external field. It is clear from eq. (11) that wave function of the Dirac continuum (i.e.

\(^2\)Operator exponential in eq. (11) does not contribute to the saddle-point eq. (13) as the Green function $G^{(T)}(T, p, T, p')$ does not contain exponential with the phase proportional to the time $T$. Let us note also that minimizing of eq. (13) gives the saddle-point for pion field only at times which are far from the end-points of the interval $(0, T)$. In fact this is enough for the calculation of wave function.
the state with all states with negative-energies occupied) is the coherent exponential of the quark-antiquark pairs:

$$|\Omega\rangle = \exp \left[ \sum_{\text{colour}} \int d^3x d^3y \ A^+(x) \ \gamma^0 G^{(T)}(xT - \varepsilon, yT) \ B^+(y) \right] |\Omega_0\rangle \equiv$$

$$\equiv \exp \left[ \sum_{\text{colour}} \frac{d^3p_1 d^3p_2}{(2\pi)^3} \ a_{\lambda_1}^+(\vec{p}_1) \ \Theta^{\lambda_1,\lambda_2}(\vec{p}_1, \vec{p}_2) \ b_{\lambda_2}^+(\vec{p}_1) \right] |\Omega_0\rangle,$$  

(16)

where $|\Omega_0\rangle$ is the vacuum of quarks-antiquarks. The function $\Theta^{\lambda_1,\lambda_2}(\vec{p}_1, \vec{p}_2)$ can be called the wave function of the quark-antiquark pair. We see that it is expressed in terms of the finite time Green function at equal times.

The nucleon itself is a state with one more level occupied by valence quarks, namely, a discrete level (with positive energy) which appears in the self-consistent external field. We can obtain the nucleon wave function by applying the operator which fills this discrete level to eq. (16):

$$\Phi_N = \prod_{\text{colour}} \int d^3x \psi^+(x) f_{\text{lev}}(x) |\Omega\rangle,$$  

(17)

where $f_{\text{lev}}(x)$ is the wave function of the discrete level (solution of Dirac equation in the external $\pi$-meson field). In order to express $\Phi_N$ in terms of quark-antiquark we have to expand $\psi$-operators according to eq. (4) and commute them with the exponential of eq. (16). As a result we get the following expression for the nucleon wave function ($A$ is the normalizing constant):

$$\Phi_N = A \prod_{\text{colour}} \int \frac{d^3p}{(2\pi)^3} \ F^\lambda(\vec{p}) \ a_{\lambda}^+(\vec{p}) \times$$

$$\times \exp \left[ \sum_{\text{colour}} \int \frac{d^3p_1 d^3p_2}{(2\pi)^3} \ a_{\lambda_1}^+(\vec{p}_1) \ \Theta^{\lambda_1,\lambda_2}(\vec{p}_1, \vec{p}_2) \ b_{\lambda_2}^+(\vec{p}_1) \right] |\Omega_0\rangle,$$  

(18)

Here $F^\lambda(\vec{p})$ is the one-quark wave function. It is a sum of two contributions:

$$F^\lambda(\vec{p}) = \int \frac{d^3p'}{(2\pi)^3} \sqrt{\frac{M}{\omega'}} \left[ u^{*\lambda}(\vec{p}) \ f_{\text{lev}}(\vec{p}) \ (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}') -$$

$$- \Theta^{\lambda,\lambda'}(\vec{p}, \vec{p}') \ v^{*\lambda'}(\vec{p}') \ f_{\text{lev}}(-\vec{p}') \right]$$  

(19)

The first contribution in eq. (19) is that of valence quarks, while the second term can be called the contribution of the sea quarks to one-quark wave function.

Let us point out the fact that the nucleon wave function is completely factorized in colours. In fact, it is a general theorem in the strict $N_c \rightarrow \infty$ limit.

### 3 Wave function of the nucleon in the IMF

It is well-known that the wave function in the rest frame has not too much physical sense \[5\]. The physical meaning can be ascribed only to the wave function of the fastly moving
nucleon (nucleon in the infinite momentum frame). Contrary to the wave function in the rest frame it can be accessed by measurements.

The common approach to the hadron wave functions is the formalism based on the light-cone quantization (see, e.g., review [22]). This formalism has many obvious advantages as compared to the approach based on Schrödinger wave functions. From the other side, the Schrödinger equal-time wave function can be defined in any frame, not necessarily on the light-cone. Also one can use well-trodden path which starts from the calculation of the evolution operator in the functional integral technique. This is why we use in this paper the last method. Light-cone wave function of the nucleon is, by definition, its wave function in the infinite momentum frame [5, 25].

The stationary saddle-point \( \bar{\pi}^a(\vec{x}) \) corresponds to the nucleon-soliton at rest. However, as the effective chiral Lagrangian is relativistically invariant, we are guaranteed that there are infinitely many solutions of saddle-point equations of motion which describe the nucleon moving in some direction with a speed \( \vec{V} \). The corresponding pion field is time-dependent and can be obtained from the stationary field by Lorentz transformation

\[
\pi^{(cd)}(\vec{x}, t) = \bar{\pi} \left( \frac{x - \vec{V}t}{\sqrt{1 - \vec{V}^2}} \right). \tag{20}
\]

In order to find all states of the Dirac continuum in the moving nucleon it is sufficient to solve the Dirac equation in the field of eq. (20). In particular, the wave function of the valence level can also be obtained as Lorentz-transformation:

\[
\Phi_{\text{lev}}(\vec{x}, t) = S[V] f_{\text{lev}}(x) \left( \frac{x - \vec{V}t}{\sqrt{1 - \vec{V}^2}} \right) \exp \left( -i \frac{\varepsilon t + \vec{V} \cdot \vec{x}}{\sqrt{1 - \vec{V}^2}} \right), \tag{21}
\]

where \( \varepsilon \) is the energy of the discrete level. Here \( S[V] \) is a matrix which transforms Lorentz indices

\[
S[V] = \exp(i \sigma_0 \omega), \quad \sigma_{\mu\nu} = \frac{1}{2} [\gamma_\mu, \gamma_\nu], \quad \tanh(\omega) = V. \tag{22}
\]

The wave function of the discrete level in stationary field \( \bar{\pi}(\vec{x}) \) of eq. (15) is a mixture of two wave functions with orbital angular momentum \( L = 0, h(r) \), and \( L = 1, j(r) \) [13, 14]:

\[
f^{\alpha f}_{\text{lev}}(\vec{x}) = \frac{1}{4\sqrt{\pi}} \begin{pmatrix} -h(r)\varepsilon^{\alpha f} + ij(r)\varepsilon \sigma_3 \varepsilon^{\beta f} \\ h(r)\varepsilon^{\alpha f} + ij(r)\varepsilon \sigma_3 \varepsilon^{\beta f} \end{pmatrix}. \tag{23}
\]

Here \( \alpha \) is a spinor index and \( f \) is flavour index. The spherically symmetric functions \( f(r) \) and \( g(r) \) were found by numerical integration of the Dirac equation.

Let us begin with the calculation of the contribution of the discrete level in the one-quark wave function (first term in eq. (19)) in the infinite momentum frame \( (V \to 1) \). We proceed in eq. (21) to momentum space and obtain from eq. (19):

\[
F^M_{\text{lev}}(p) = \sqrt{\frac{m}{\omega_p}} \int d^4k \: u^{*}_{\alpha}(p) S[V]_{\beta}^{\alpha} \bar{f}^{\alpha f}_{\text{lev}}(\vec{k}) \delta^{(2)}(p_\perp - k_\perp) \delta \left( \frac{k_3 - \varepsilon \sqrt{1 - V^2}}{p_3} \right),
\]

\[
\bar{f}^{\alpha f}_{\text{lev}}(\vec{k}) = \int d^3x \: e^{-i\vec{k} \cdot \vec{x}} f^{\alpha f}_{\text{lev}}(\vec{x}). \tag{24}
\]
Let us divide the quark momentum into the longitudinal and transverse parts with respect to the total nucleon momentum $P_N = M_N V / \sqrt{1 - V^2}$:

$$p = (z P_N, \vec{p}_\perp).$$

Also in the IMF it is convenient to use quark-antiquark operators normalized in a different way than the usual ones:

$$\{ \tilde{u}^\dagger (z_1, \vec{p}_1 \perp), \tilde{a} (z_2, \vec{p}_2 \perp) \} = \delta (z_1 - z_2) (2\pi)^2 \delta^{(2)} (\vec{p}_1 \perp - \vec{p}_2 \perp).$$

(We shall call a wave function the coefficient in front of quark operators normalized this way). We have from eq. (24)

$$\tilde{F}_{\text{lev}}^{\lambda f} (z, \vec{p}_\perp) = \frac{M_N}{P_N} \sqrt{\frac{M}{z}} \tilde{u} (\vec{p}) S [V] \tilde{f}_{\text{lev}} (\vec{p}) |_{p_3 = z M_N - \varepsilon}.$$  

We make use of the relation $\gamma^0 S [V] = S^{-1} [V] \gamma^0$ and apply Lorentz transformation to the free spinor $\tilde{u} (p)$:

$$\tilde{u}^{(\lambda)} (p) S^{-1} [V] = \tilde{u}^{\lambda} (\tilde{p}), \quad \tilde{p}_3 = \frac{p_3 + V \omega_p}{\sqrt{1 - V^2}}, \quad \tilde{\omega}_p = \frac{\omega_p + V p_3}{\sqrt{1 - V^2}},$$  

here $\omega_p = \sqrt{p^2 + M^2}$. In the limit $V \to 1$, the momentum $\tilde{p}_3$ is large, $\tilde{p}_3 \approx \tilde{\omega}_p \approx 2 z P_N^2 / M_N$. Wave functions of free fast-moving quarks become eigenfunctions of polarization operator $\gamma^0 \gamma^3$:

$$\tilde{u}^{(\lambda)} (\tilde{p}) = P \sqrt{\frac{z}{m_{M N}}} u^{(\lambda)}_0 (1 + \gamma^0 \gamma^3), \quad \tilde{u}^{(\lambda)} (\tilde{p}) = P \sqrt{\frac{z}{m_{M N}}} v^{(\lambda)}_0 (1 + \gamma^0 \gamma^3),$$  

where the quark-antiquark spinors with spin up and down are:

$$u^\dagger_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad u^\dagger_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad v^\dagger_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v^\dagger_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$  

In fact, the matrix $\frac{1}{2} (1 + \gamma^0 \gamma^3)$ is a projector on the state with definite chirality.

Finally we get for the contribution of valence quarks to the one-quark wave function

$$\tilde{F}_{\text{lev}}^{\lambda f} (z, \vec{p}_\perp) = \tilde{f}_{\text{lev}}^{\lambda f} (\vec{p}_\perp, z M_N - \varepsilon),$$  

where $\tilde{f}_{\text{lev}}^{\lambda f} (\vec{p})$ is Fourier transform of the level wave function. Using expression of eq. (23) we obtain:

$$F_{\text{lev}}^{\lambda f} (z, \vec{p}_\perp) = \frac{\sqrt{M_N \pi}}{p^2} \left[ (p_3 \tilde{\eta} (\vec{p}) - \tilde{p} \tilde{h} (\vec{p})) \sigma_3 + \tilde{\eta} (\vec{p}) \vec{p}_\perp \vec{\sigma}_3 \right]_{p_3 = z M_N - \varepsilon},$$  

where

$$\tilde{h} (p) = \int_0^\infty dr r^2 h (r) \sqrt{\frac{2 \sin pr}{pr}}; \quad \tilde{\eta} (p) = \int_0^\infty dr r^2 \tilde{j} (r) \sqrt{\frac{2 \sin pr - pr \cos pr}{pr^2}}.$$
We see that, as it should be, the dependence on the nucleon momentum $P$ is cancelled in the final expression for the level contribution of eq. (31) to the nucleon wave function. Also it is clear that this contribution remains stable in the large-$N_c$ limit in the main region of the quark momenta: $z_i \sim O(1/N_c)$, $p_\perp \sim O(1)$ (nucleon mass is proportional to the number of colours $N_c$, factor $\sqrt{M_N}$ is needed for the correct normalization of the one-quark wave function).

Let us proceed now with the sea quark contribution to the wave functions. According to eq. (19) it can be expressed in terms of the quark-antiquark pair wave function $\Theta^{\lambda,\lambda'}(\vec{p}, \vec{p'})$ which, in turn, is calculated through the finite-time Green function in the external pion field.

The equal-time Green function is a sum of diagrams in the pion field. The equal-time Green function is a sum of diagrams in the pion field. The se diagrams are similar to the ordinary Feynman graphs except for the fact that the integration in the intermediate times in the graph is going only in the interval $(0, T)$ (usually, one integrates over the whole interval $(-\infty, \infty)$). It is easy to show that these diagrams obey both the Dirac equation and necessary boundary conditions at $t = 0$ and $t = T$.

The diagrams represent the Green function as an expansion in the powers of the pion field. One can rearrange this series in such a way that the expansion is carried out in the powers of $(U\gamma^5 - 1)$ instead. It can be shown that the new expansion is in increasing powers of the gradients of the pion field.

Let us restrict ourselves by the first non-trivial term of this expansion. We call this approximation for the Green function an interpolation formula [13, 14]. It becomes exact in three limiting cases of the pion field: i) pion field is small $\pi(x) \ll 1$, ii) pion field is slowly varying function of coordinates with typical momenta $k_\pi \ll M$, iii) pion field momenta are large $k_\pi \gg M$. As a result, interpolation formula usually works rather well in the problems related to nucleon-soliton: its accuracy is typically around 10%.

We introduce the Feynman Green function in the mixed $p, t$ representation:

$$G(\vec{p}, t) = \frac{\omega_p \gamma^0 \text{sign}(t) - \vec{p} \vec{\gamma}}{2\omega_p}, \quad \omega_p = \sqrt{\vec{p}^2 + M^2}. \quad (32)$$

The first-order correction to the pair wave function in the pion field in the IMF is equal to

$$\Theta_{\lambda_1,\lambda_2}(p_1, p_2) = \int d^3k \sqrt{\frac{\omega_1 \omega_2}{m^2}} \delta \left( \frac{k}{\sqrt{1 - V^2}} - (p_1)_z - (p_2)_z \right) \delta (k_\perp - p_1 \perp - p_2 \perp) \times$$

$$\times u^{\lambda_1*}(p_1) \gamma_0 G^+(p_1) \frac{M\pi(k)\gamma_5}{\omega_1 + \omega_2 - \sqrt{1 - V^2}} G^-(p_2) v^{\lambda_2}(p_2), \quad (33)$$

signs $\pm$ label the sign of $t$ in the free Green functions. Let us denote by $z_1$ and $z_2$ the fractions of the nucleon momenta carried by the quark and antiquark correspondingly: $(p_{1,2})_z = z_{1,2} P_N$. The energy denominator in eq. (33) is small only if $1 > z_{1,2} > 0$.

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3We checked the accuracy of this formula, for example, in the calculation of the nucleon structure function [18] confronting it to the exact calculation accounting for all solutions of the Dirac continuum. This problem is, of course, rather close to the calculation of wave function. The interpolation formula was accurate enough.
Expanding it in the powers of nucleon momentum and summing over spinor indices with help of eq. (29), we obtain:

$$\Theta_{\lambda_1, \lambda_2}(z_1, z_2, p_{\perp 1}, p_{\perp 2}) = \left( M(z_1 + z_2)(\sigma_3)_{\lambda_1\lambda_2} + (z_2\vec{p}_1^\perp - z_1\vec{p}_2^\perp)(\vec{\sigma}^\perp)_{\lambda_1\lambda_2} \right) \times$$

$$\times \frac{\mathcal{M}_N M[\tilde{\pi}(\vec{p}_1 + \vec{p}_2)]_{p_{1z}=z_1, M_N, p_{2z}=z_2, M_N}}{(z_1 + z_2)z_1z_2\mathcal{M}_N^2 + (M^2 + p_{\perp 1}^2)z_2 + (M^2 + p_{\perp 2}^2)z_1},$$

where $$\sigma_i$$ are Pauli matrices. As it should be, the nucleon momentum $$P_N$$ is cancelled out in this expression. We see again that pair wave function has a finite limit at large $$N_\text{c}$$ — in the same sense as the one-quark wave function.

Collecting the terms in the perturbative expansion which correspond to the expansion of $$U^\gamma(x)$$ in powers of $$\pi$$ we can promote eq. (34) to the *interpolation formula*. It has the following form:

$$\Theta_{\lambda_1, \lambda_2}(z_1, z_2, p_{\perp 1}, p_{\perp 2}) = \frac{\mathcal{M}_N}{(z_1 + z_2)z_1z_2\mathcal{M}_N^2 + (M^2 + p_{\perp 1}^2)z_2 + (M^2 + p_{\perp 2}^2)z_1} \times$$

$$\times \left\{ (M(z_1 + z_2)(\sigma_3)_{\lambda_1\lambda_2} + (z_2\vec{p}_1^\perp - z_1\vec{p}_2^\perp)(\vec{\sigma}^\perp)_{\lambda_1\lambda_2} \tilde{\Pi}(\vec{p}_1 + \vec{p}_2) -$$

$$-i \left( M(z_2 - z_1)\delta_{\lambda_1\lambda_2} + i\varepsilon_{\alpha\beta}(z_1p_{2\alpha}^\perp - z_2p_{1\alpha}^\perp)(\sigma_3^\perp)_{\lambda_1\lambda_2} \tilde{\Sigma}(\vec{p}_1 + \vec{p}_2) \right\}_{p_{1z}=z_1, M_N}. (35)$$

Here $$\tilde{\Sigma}(\vec{k})$$ is the Fourier transform of the scalar component of $$U^\gamma(x)$$ and $$\tilde{\Pi}(\vec{k})$$ is its axial component. In the soliton mean field of eq. (15) they are equal to

$$\tilde{\Sigma}(\vec{k}) = \int d^3x e^{-i\vec{k}\vec{x}} (\cos P(r) - 1), \quad \tilde{\Pi}(\vec{k}) = \int d^3x e^{-i\vec{k}\vec{x}} (\vec{n}\vec{x}) \sin P(r),$$

(36)

The expression for the pair wave function of eq. (35) has a clear physical meaning. The factor in front of the parenthesis in this expression is, in fact, a light-cone wave function of the pion in the leading order in $$N_\text{c}$$ calculated in Ref. [23]. According to eq. (35) one has to convolute this wave function with a mean pion field in the nucleon in order to obtain quark-antiquark pair wave function. This *factorization* is a natural consequence of the large $$N_\text{c}$$ limit and strictly corresponds to the soliton nature of the nucleon in this

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4 We normalize eq. (34) in such a way that the pair wave function is equal to

$$\int \frac{dz_1 dz_2 dp_{1z}}{(2\pi)^2} \int \frac{dz_2 dp_{2z}}{(2\pi)^2} a^{+(\lambda_1)(z_1, p_{1z})} \Theta_{\lambda_1, \lambda_2}(z_1, z_2, p_{\perp 1}, p_{\perp 2}) b^{+(\lambda_2)}(z_2, p_{2z}),$$

where quark operators are normalized to

$$\{a^{+(\lambda_1)(z_1, p_{1z})}a^{(\lambda_2)(z_2, p_{2z})}\} = \delta_{\lambda_1\lambda_2} \delta(z_1 - z_2)(2\pi)^2 \delta^{(1)}(p_{1z} - p_{2z}),$$

$$\{b^{+(\lambda_1)(z_1, p_{1z})}b^{(\lambda_2)}(z_2, p_{2z})\} = \delta_{\lambda_1\lambda_2} \delta(z_1 - z_2)(2\pi)^2 \delta^{(2)}(p_{1z} - p_{2z}),$$

5 Pion light-cone wave function of Ref. [23] is obtained by substitution $$z_1 + z_2 = 1$$ (fractions should be measured relative to the momentum of the pion) and integrating over transverse momenta $$p_{1z}$$, $$p_{2z}$$, with the condition $$p_{1z} + p_{2z} = 0$$. It corresponds to one definite $$\gamma_5$$ component of the full wave function. Also, in Ref. [23] we took into account the dependence of the dynamical quark mass on the quark virtuality which is neglected here.
limit. Moreover, considering next terms in the expansion of the Green function one can prove this factorization also in all orders. For example, the second term is the product of the two-pion light-cone wave functions (calculated in our model in Ref. [24]) and the pion mean field squared. At last, according to eq. (19), the sea contribution to the one-quark wave function is a convolution of pair wave function with the antiquark contribution to the light-cone wave function of the discrete level. This formula is also guaranteed by the large \( N_c \) limit.

Projecting the conjugated wave function of the level onto antiquarks we obtain analogously to eq. (31)

\[
\bar{F}^{f,\text{val}}_\lambda(\vec{p}_\perp, z) = -\sqrt{M_N}\bar{v}^{(\lambda)}(1 + \gamma^0\gamma^3)f_{\text{lev}}(-\vec{p}_\perp, -zM_N - \varepsilon) = \\
-\frac{\sqrt{M_N\pi}}{p^2} \left[ (p_3j(p) - ph(p)(\sigma_1)\gamma^0 + j(p) (p_3(\sigma_3)\gamma^0 + ip_3\gamma^3) \right].
\]

In order to obtain the sea contribution to the 1-quark wave function we have to convolute the latter with the pair wave function of eq. (35):

\[
F^{f,\text{sea}}_\lambda(\vec{p}_\perp, z) = -\frac{\sqrt{M_N\pi}}{p^2} \int \frac{dz' d^2p'_\perp}{2(\pi)^2} \sqrt{\frac{z}{z'}} \Theta_\lambda\lambda'(z, \vec{p}_\perp, z', \vec{p}'_\perp) \bar{F}^{f}_\lambda(z', \vec{p}'_\perp).
\]

Again this contribution does not depend on \( N_c \) in the same manner as the level wave function of eq. (35). The total one-quark light-cone wave function is a sum of two contributions:

\[
F^f_\lambda(\vec{p}_\perp, z) = F^{f,\text{val}}_\lambda(\vec{p}_\perp, z) + F^{f,\text{sea}}_\lambda(\vec{p}_\perp, z)
\]

4 Distribution amplitudes of \( N \) and \( \Delta \)

Distribution amplitudes are, by definition, those components of the Fock vector of state for a given particle, which contain the lowest possible number of partons. For the nucleon the distribution amplitudes are its three-quark wave function on the light cone. As it is well-known, distribution amplitudes describe properties of the nucleon in hard exclusive processes.

The full wave function of the nucleon is given by eq. (18) and the wave function of the lowest component (with \( N_c \) quarks) is the product of \( N_c \) wave functions \( F^f_\lambda(\vec{p}_\perp, z) \) of eq. (39). However it is not the end of the story. The point is that the minimum in eq. (15) for the pion field, which corresponds to the nucleon, is degenerate. Any pion field which is the flavour (or space) rotation or the translation of the field of eq. (15) also gives a minimum of the action. In other words, integration over the pion field in the functional integral has zero modes which should be taken into account exactly.

It is well-known \([13, 18, 19]\) that the integration over translational zero modes leads to conservation of the momenta. In the context of our calculation of nucleon distribution amplitudes, this leads to the condition \( z_1 + \ldots + z_{N_c} = 1 \) and \( p_{1\perp} + \ldots + p_{N_c\perp} = 1 \). As to the integration over flavour rotations, it gives rise to the quantum numbers of the nucleon-soliton: the state with given quantum numbers (in \( SU(2) \) flavour group they are: spin
\(J\), isospin \(T\) and their projections \(J_3\) and \(T_3\) is obtained as a projection of the rotating soliton on the definite rotational wave function (which appears to be Wigner D-function).\(^6\)

As a result, accounting for rotational and translational zero modes, we obtain the following expression for the nucleon wave function:

\[
\Phi_{J_3;T_3}^T(z_1, \ldots, z_{N_c}, \vec{p}_{1\perp}, \ldots \vec{p}_{N_c\perp}) = A \int dR \sqrt{2J + 1} (-1)^{T + T_3} D_{T_3,\lambda_3}^{(J = T)}(R) \times 
\]

\[
\int \frac{dz_1 dp_{1\perp}}{(2\pi)^2} \ldots \int \frac{dz_{N_c} dp_{N_c\perp}}{(2\pi)^2} \delta \left(\sum z_i - 1\right) (2\pi)^2 \delta^{(2)} \left(\sum \vec{p}_i\perp\right) \times 
\]

\[
R_{\lambda_1}^{f_1} F_{\lambda_1}^{g_1}(\vec{p}_{1\perp}, z_1) \ldots R_{\lambda_Nc}^{f_{N_c}} F_{\lambda_Nc}^{g_{N_c}}(\vec{p}_{N_c\perp}, z_{N_c}) \quad (40)
\]

Here \(\Phi_{J_3;T_3}^T\) is the wave function for some state from the rotational band of the soliton, \(R\) is a rotational matrix from the flavour group (we consider flavour group \(SU(2)\) at the moment), \(A\) is the normalizing coefficient.

The integration over rotational and translational modes breaks down the factorization of the nucleon wave function into the product of one-quark wave functions. This is to be expected as rotation and translation are, strictly speaking, specific correction in \(N_c\). Let us note that the wave function of eq. (40) is symmetric under the exchange of quarks, as it should be, since the colour wave function of this state (omitted in this expression) is completely antisymmetric and, thus, corresponds to the colourless baryon.

In the realistic applications one has to put \(N_c = 3\) and discuss states with \(J = T = 1/2\) (nucleon) and \(J = T = 3/2\) (\(\Delta\)-resonance). The full wave function of these states is given by eq. (40). However usually we are interested in the so-called distribution amplitudes which are the wave functions integrated over all \(p\perp\):

\[
\phi(z_1, z_2, z_3) = \delta(z_1 + z_2 + z_3 - 1) \theta(z_1) \theta(z_2) \theta(z_3) \int \frac{d^2 p_{1\perp}}{(2\pi)^2} \ldots \Phi_{J_3;T_3}^T(z_1, \ldots) \quad (41)
\]

By definition, the normalization of the distribution amplitude is chosen in such a way that the total integral over all three \(z_i\) is equal to unity.

It is known \[26, 27\] that relativistic invariance and the symmetry considerations restrict the general form of the quark distribution for the nucleon in such a way that it depends on two scalar functions only: completely symmetric \(\phi_s(z_1, z_2, z_3)\) and antisymmetric \(\phi_a(z_1, z_2, z_3)\). For example, for the proton with spin up we have:

\[
\phi_p^T(z_1, z_2, z_3) = \frac{\phi_s(z_1, z_2, z_3)}{\sqrt{6}} (2 | u \uparrow d \downarrow u \uparrow > - | u \uparrow u \downarrow d \uparrow > - | d \uparrow u \downarrow u \uparrow > ) + 
\]

\[
+ \frac{\phi_a(z_1, z_2, z_3)}{\sqrt{2}} ( | u \uparrow u \downarrow d \uparrow > - | d \uparrow u \downarrow u \uparrow > ) \quad (42)
\]

Straitforward calculations based on eq. (40), of course, reproduce this general structure and give the concrete expressions for the symmetric and antisymmetric parts of the distribution amplitude. However, these formulae are somewhat cumbersome and we will not

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\(^6\)See, for example, Review \[16, 17\] for the discussion of the rotational quantization of baryon-soliton.
write them here. As an example, let us give only the expression for valence quark (level) contribution to the symmetric and antisymmetric parts of the quark distribution:

\[ \phi_s(z_1, z_2, z_3) = c_0 \int d^2x_\perp [2\tilde{g}(z_1, x_\perp)\tilde{g}(z_2, x_\perp)\tilde{g}(z_3, x_\perp) - x_\perp^2\tilde{j}(z_2, x_\perp)\left(\tilde{g}(z_1, x_\perp)\tilde{j}(z_3, x_\perp) + \tilde{g}(z_3, x_\perp)\tilde{j}(z_1, x_\perp)\right)], \]

(43)

\[ \phi_a(z_1, z_2, z_3) = \sqrt{3}c_0 \int d^2x_\perp x_\perp^2\tilde{j}(z_2, x_\perp)\left(\tilde{g}(z_1, x_\perp)\tilde{j}(z_3, x_\perp) - \tilde{g}(z_3, x_\perp)\tilde{j}(z_1, x_\perp)\right), \]

(44)

where \( \tilde{g} \) and \( \tilde{j} \) are the Fourier transforms of the level wave functions with \( l = 0 \) and \( l = 1 \) in the longitudinal direction:

\[ \tilde{g}(z, x_\perp) = \int dx_3 \exp (-ix_3(zM_N - \varepsilon)) \left[h(r) - \frac{ix_3}{r}j(r)\right], \]

\[ \tilde{j}(z, x_\perp) = \int dx_3 \exp (-ix_3(zM_N - \varepsilon)) \frac{ij(r)}{r} \]

(45)

and \( c_0 \) is a normalizing constant.

In the non-relativistic limit, \( j(r) \ll g(r) \) and only the symmetric wave function \( \phi_s(z) \) survives. The antisymmetric part is completely due to the relativistic effects and sea quarks.

The total wave function should be calculated with the account for sea quarks according to eq. (38). However, substituting the level and pair wave functions into eq. (38) we arrive at the integral which diverges at small \( z' \) due to the phase volume factor \( \sqrt{z/z'} \).

In fact, we have already met the singularity of this type in the calculations of pion wave function [23] and generalized distributions [20]. The reason for this singularity is simple: at small \( z \) the quark virtuality

\[ p_i^2 \approx \frac{M^2 + \vec{p}_\perp}{z_i} \]

(46)

becomes large and we cannot neglect any longer the dependence of the dynamical quark mass \( M(p) \) on this virtuality. The dependence of the constituent mass on the virtuality is known in the instanton vacuum model [11, 12, 17]; substituting \( M(p) \) in eq. (38) by this function we obtain the converging sea quarks contribution which tends to zero at \( z \to 0 \). As to the level contribution it remains non-zero at small \( z \).

The total wave function accounting for the sea quarks can also be represented in the form of eq. (44) where the functions \( \tilde{g} \), \( \tilde{j} \) receive contributions both from valence and sea quarks. Plots of the symmetric \( \phi_s(z_1, z_2, z_3) \) and antisymmetric \( \phi_a(z_1, z_2, z_3) \) parts of the nucleon wave functions are displayed in Fig.1 and Fig.2, correspondingly. It is seen from these curves that the antisymmetric part of the wave function is two orders of magnitude smaller than the symmetric one. This fact is of a numerical origin, parametrically both parts of the wave function are of the same order.

In fact, the quark distribution amplitudes calculated here are valid only in the region \( zN_c \sim 1 \). It is the main region for the nucleon wave function. However, approximations
Figure 1: Symmetric part of the nucleon quark distribution amplitude $\phi_s(z_1, z_2, z_3)$ as a function of $z_2$ at different $z_1$. 
Figure 2: Antisymmetric part of the nucleon quark distribution amplitude $\phi_a(z_1, z_2, z_3)$ as a function of $z_2$ at different $z_1$. 
used in the present paper break at the end-points, at $z_i = 0$ and $z_i = 1$ but for different reasons.

In the $z \to 1$ limit one quark is carrying almost all nucleon momentum and the momenta of all other quarks are necessary small. This limit contradicts the large $N_c$-approximation which leads to the factorized wave function corresponding to independent quarks. The $z \to 1$ asymptotics corresponds to rather rare configuration of the pion field and its contribution to the quark distribution is exponentially small in $N_c$.\(^7\)

Wave functions calculated here are neither valid in the limit $z \to 0$. The main reason is that at small $z$ the virtuality of nucleon constituents become large and we are driven out of the region of the applicability of chiral effective lagrangian of eq. 11. We have already mentioned that we have to take into account dependence of the dynamical mass of the quark $M(p)$ on its virtuality. However it can be shown that this effect does not lead to nullification of the quark distributions at $z = 0$. It seems that more important is an effect of gluons which are known to be produced intensively at small $z$, at least in perturbation theory. As a consequence, the relative contribution of pure 3-quark component decreases at small $z$. As it was already mentioned, all effects of this kind were neglected here due to the instanton vacuum parameter $(M\rho)^2 \ll 1$.

The small antisymmetric part of the nucleon wave function contradicts strongly to the commonly used parametrizations of quark distributions which are obtained on the basis of form factor data or QCD sum rules (see, e.g., [27]). Even more significantly our wave function differs from the Chernyak-Zhitnizky’s one [28, 29] which they suggested many years ago analyzing QCD sum rules. Their wave function has strong asymmetry $z_1 \leftrightarrow z_2$ and even changes sign as a function of $z_1$ and $z_2$ (This looks strange for the wave function of the nucleon which is the ground state in the sector with nonzero baryon number). Our nucleon wave function is much closer to the so-called asymptotic wave function (valid at arbitray high normalization point) which is

$$\phi_s(z_1, z_2, z_3) = 120 \, z_1 z_2 z_3, \quad \phi_u(z_1, z_2, z_3) = 0; \quad (47)$$

but still differs from it.

Data on the asymmetry of the nucleon wave function were reanalyzed recently in Ref. [39] in the framework of the QCD sum rules on the light-cone. According to the results of this analysis, the nucleon wave function is not very far from the asymptotics. Let us also note that the data on $\pi$-meson photoproduction on the threshold [40] also favor symmetric wave function.

Rotational symmetry of the nucleon-soliton allows one to calculate immediately quark distributions for $\Delta$-resonance as well. For $\Delta^+$-resonance with $J_3 = +1/2$ we obtain

$$\Phi^{\Delta^+}_{S_3=1/2}(z_1, z_2, z_3) = \frac{1}{\sqrt{3}} \phi_{\Delta}(z_1, z_2, z_3) \left( |u \uparrow u \downarrow d \uparrow> + |u \uparrow d \downarrow u \uparrow> - |d \uparrow u \downarrow u \uparrow> \right) \quad (48)$$

\(^7\)In principle, this situation can be also described by semiclassical methods but one has to find a new minimum in the functional integral over pion field corresponding to the limit $z \to 1$. It can be shown that corrections to the mean pion field become essential when parameter $1/N_c \ln(1 - z) \geq 1$. Let us note that this situation is rather general and one faces the same problem also in the calculation of structure functions at large $N_c$ [18, 19].
In other words ∆-resonance is characterized only by one symmetric wave function

\[ \phi_{\Delta}(z_1, z_2, z_3) = \frac{c_0}{\sqrt{2}} \int d^2x_\perp [\tilde{g}(z_1, x_\perp)\tilde{g}(z_2, x_\perp)\tilde{g}(z_3, x_\perp) + x_\perp^2 \tilde{j}(z_2, x_\perp) \left( \tilde{g}(z_1, x_\perp)\tilde{j}(z_3, x_\perp) + \tilde{g}(z_3, x_\perp)\tilde{j}(z_1, x_\perp) \right)] \] (49)

The main source of the experimental information on the nucleon wave functions is the asymptotics of form factors at large \( Q^2 \). Four form factors are measured: magnetic form factors of proton and neutron, axial nucleon form factor and the transitional \( \Delta \to N \) one. In the region of hard \( Q^2 \), the process is factorized into a product of the hard QCD part and corresponding wave function (see, e.g. \[25, 5\]). Asymptotics of the form factors are:

\[ g_p \equiv Q^4 G_{Mp} = 2f \int [dx][dy] \left[ T_1 \phi_s(x)\phi_s(y) + x \leftrightarrow y \right], \]
\[ g_n \equiv Q^4 G_{Mn} = -\frac{2f}{3} \int [dx][dy] \left[ (T_1 - T_2)\phi_s(x)\phi_s(y) + x \leftrightarrow y \right], \]
\[ g_A \equiv Q^4 g_A(Q^2) = \frac{2f}{3} \int [dx][dy] \left[ (4T_1 + T_2)\phi_s(x)\phi_s(y) + x \leftrightarrow y \right], \]
\[ g_{\Delta p} \equiv Q^4 G_{Mp\Delta} = \frac{2\sqrt{2}f}{3} \int [d^3x][d^3y] \left[ (T_1 - T_2)\phi_s(x)\phi_{\Delta}(y) + x \leftrightarrow y \right] \] (50)

(we give here the simplified version of hard kernels in the assumption that the antisymmetric part of the nucleon wave function is zero). Here \([dx]\) denotes the integration over the fractions of total momentum with the condition \( x_1 + x_2 + x_3 = 1 \), \( f = (16\pi\alpha_s/9)^2 \) and functions \( T_1 \) and \( T_2 \) are equal to

\[ T_1 = \frac{1}{x_3(1 - x_1)^2y_3(1 - y_1)^2} + \frac{1}{x_2(1 - x_1)^2y_2(1 - y_1)^2} - \frac{1}{x_2x_3(1 - x_3)y_2y_3(1 - y_1)}; \]
\[ T_2 = \frac{1}{x_1x_3(1 - x_1)y_1y_3(1 - y_3)}. \] (51)

Substituting our expressions for the wave functions into eq. (50) we see that integrals over \( z_i \) are divergent due to the end-point singularities both in the region \( z_i = 1 \) and \( z_i = 0 \), where our approximations done are not valid (see above). As to the divergence at \( z_i = 1 \), it does not look serious: we know anyway (from asymptotics of structure functions) that wave functions should behave at least as \( (1 - x)^{3/2} \). The singularities are weak and after any appropriate regularization the contribution of this region to the form factors is negligible.

A discussion of the singularities at \( x = 0 \) is much more involved. First, we do not know any strict QCD theorem which states that the wave function should be zero at \( z = 0 \). The divergence is linear at \( N_c = 3 \) but it becomes worse at larger number of colours. As a result, the main contribution to the integral comes from this region.

Thus, the considered form factors are rather bad objects to be treated in the framework of our model. We cannot calculate absolute value of the form factors but can say only
that they should be amplified by a large parameter (connected in turn with parameter \((M\rho)^{-1}\)). However we can calculate ratios of form factors using the fact that divergent part is universal for all form factors. In fact, this prediction is based only on the limit of \(N_c \rightarrow \infty\) and it does not use any concrete realization of the nucleon-soliton model.

Ratios of the form factors in the large \(N_c\) limit in comparison with results of the two popular parametrizations of the nucleon wave function are presented in Table 1. More extensive and complete analysis of the phenomenological implications of baryon distribution amplitudes can be found in [33, 34, 35]. In these works the corresponding analysis has been performed in the framework of so-called heterotic approach.

We see that, qualitatively, large \(N_c\) results agree with data\(^8\). Unfortunately the quality of the data is still rather bad. For recent comprehensive reviews of theory and phenomenology of hard exclusive reactions see Refs. [36, 37].

| Ratio         | Ch.-Zh. [28] | G.-St. [30] | Large \(N_c\) (this paper) | Experiment |
|---------------|--------------|-------------|-----------------------------|------------|
| \(g_n/g_p\)   | -0.48        | -0.1        | -1/3                        | -0.45 ± 0.1 [38] |
| \(g_A/g_p\)   | 1.53         | 1.13        | 4/3                         | 1.35 [31]  |
| \(g_{p\Delta}/g_p\) | 0.01         | 0.81        | \(\sqrt{2}/12\)            | < 0.3 [32] |

Table 1: Form factor ratios.

5 Conclusions

We have shown that the limit of large number of colours in QCD allows one to tell much about the light-cone wave function of the nucleon. In this limit the nucleon is a soliton and its quark wave functions are almost completely determined by the requirements of the large-\(N_c\) factorization. One needs to know only the wave function of the discrete level in the self-consistent meson field and one-quark, two-quark, etc. wave functions of the mesons which form this mean field. Then the factorization, which is valid at \(N_c \rightarrow \infty\), is enough to calculate the whole light-cone wave function of the nucleon and, in particular, its 3-quark component — the distribution amplitude.

Put in such a way, this statement looks like a strict theorem. In the present paper we realized this program for the concrete model of the nucleon structure of eq. (11) motivated by QCD instanton vacuum. The main drawback of the model is, of course, the fact that the gluon components of the wave function are parametrically suppressed (by the parameter \((M\rho)^2 \ll 1\)) while we are used to think that gluons play an essential role in the nucleon structure.

One should take into account that the distribution amplitudes calculated in this paper are normalized at a very low normalization point \((\mu \leq 600\text{ MeV})\). Evolving these functions even to 1 GeV will lead to significant amount of gluons in the wave function. Of course, calculations of the gluon components of the wave functions (as corrections in the density

\(^8\)Let us note that asymptotic wave function contradicts the data strongly and is not displayed here for this reason.
of instantons) are highly desirable. We plan to discuss the role of gluons in the nucleon-soliton wave function in the separate publication. For recent detailed works on evolution equations for baryon distribution amplitudes see Refs. [41, 42].

However, we believe that even in the leading order, the large $N_c$ nucleon wave function is interesting by itself. It was derived in the relativistically invariant field theory and therefore it obeys all general theorems and sum rules. It can be checked that the resulting wave function reproduces correctly the structure functions of the nucleon calculated in Ref. [18] and the generalized distributions of Ref. [20]. This is to be expected as they are calculated in the same model, under the same approximations. However, this demonstrates one more time that the model is self-consistent. On the other hand it is known that the structure functions obtained in Ref. [18] describe the data, at least qualitatively.

The main features of the nucleon wave functions considered in this paper are: i) relatively small contribution of the sea quarks to the component with the lowest number of partons ii) soft distribution in the transverse momenta of quarks with average value $\approx 400$ MeV iii) the form which is much closer to the asymptotic wave function than to the wave functions proposed on the basis of QCD sum rules [28] (no zeroes or change of a sign) but which is still rather far from the asymptotic form, iv) wave function is singular (does not vanish) at the end-points. These singularities can disappear as a result of evolution or if one takes into account corrections in $N_c$. Nevertheless it is natural to expect that factorization in the baryon channel will appear at higher $Q^2$ than in meson sector. It is quite probable that this is really the case in nature [39].

Let us note that the wave functions of mesons in the instanton vacuum appears to be rather close to asymptotic one in agreement with the data (see Ref. [23]). Baryon wave function in the same model is far from the asymptotics and this fact again corresponds to the data (asymptotics of the form factors). This is natural, as in the limit $N_c \rightarrow \infty$ of the nature of mesons and baryons is completely different. We conclude that this limit describes experimental wave functions at least on the qualitative level.

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