Entangling continuous variables with a qubit array

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We show that an array of qubits embedded in a waveguide can emit entangled pairs of microwave photon beams. The quadratures obtained from a homodyne detection of these outputs beams form a pair of correlated continuous variables similarly to the EPR experiment. The photon pairs are produced by the decay of plasmon-like collective excitations in the qubit array. The maximum intensity of the resulting beams is only bounded by the number of emitters. We calculate the excitation decay rate both into a continuum of photon state and into a one-mode cavity. We also determine the frequency of Rabi-like oscillations resulting from a detuning.

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Introduction: The steady improvement of superconducting electronics over the last two decades [1–3] cemented the place of Josephson effect-based devices among the leading platforms for quantum technologies (e.g., quantum computation) [4]. However, the control and observation of essential quantum correlations and entanglement necessary for the operation of these technologies remains a challenging task [5–8]. A convenient testbed for this research is provided by a superconducting qubit array embedded in a coplanar waveguide (a "1D quantum metamaterial" setup) [9–11]. Some of these metamaterials are predicted to display interesting nonlinear properties like the two-photon induced transparency [11–13], superradiance [14, 15] and lasing [16] but these results have been obtained using approximations short of a full QED treatment.

In this Letter we build a consistent theory for a linear array of qubits placed in a waveguide. Specifically, we consider a set of capacitively coupled transmons, but the general results are not going to be sensitive to the particular kind of a qubit. The Josephson junctions are arranged symmetrically in order to ensure a quadratic coupling to the electromagnetic field. The collective excitations are produced by abrupt changes of qubit electric charges, tantamount to a sudden modification of the photon dispersion relation in the waveguide - a quantum analogue of the emission of cosmological radiation in a curved space for spontaneous particle pair creations [17, 18] also refereed as dynamical Casimir effect [19]. The symmetry ensures that collective excitations of the array decay into the entangled microwave beams propagating along the waveguide in the opposite directions.

Compared to the prior art, the proposed mechanism does not involve the use of an external magnetic field [20] or a pump field within a waveguide [21]. It predicts quantum correlations at distance and therefore differs from other studies like the two photons correlations analyzed in [22–24], or sub- and superradiance in [25] or even phase transition [26].

Setup: The proposed scheme is presented in Fig.1 and Fig.2. An array of N transmon qubits is embedded in a ring waveguide at zero temperature. Starting at equilibrium, we adiabatically increase the potential of the qubit island to V, and then suddenly drop it to zero. The initial charge on the island is $q_0 = C_0 V$, where $C_0$ is the effective capacitance. Classically, the island charge oscillates at a frequency $\omega_0 = \sqrt{8E_Je^2/C_0}/\hbar$, where Josephson energy $E_J \gg e^2/C_0$ for a transmon. In the quantum case these oscillations will decay into the electromagnetic vacuum modes by producing two counterpropagating entangled beams. The subsequent action of a circulator passes these outputs on for a homodyne detection [27], that includes local oscillator mixing and frequency filtering, in order to determine their mutual EPR-type quantum correlations [28, 29]. The integral of the voltage pulse (flux [30]) and the total induced charge correspond to correlated (resp. anticorrelated) continuous variable quadratures.

Theoretical model: The lumped-elements scheme of the device is shown in Fig.2. The nth transmon’s quantum operators are its excess charge $\hat{q}_n = (2e)(\hat{n}_n - N_s)$ mea-
The Hamiltonian of the system is obtained by quantizing the different components’ contributions to the total energy [31], yielding:

$$\hat{H} = \sum_{n=1}^{N} V \hat{q}_{n} + \sum_{n'=1}^{N} \frac{C_{n,n'}^{-1}}{2} \hat{q}_{n'} \hat{q}_{n} - E_{J} \left[ \cos \left( \frac{2e}{\hbar} \left( \hat{\Phi}_{n} - \frac{\hat{\Phi}_{n}^{B}}{2} \right) \right) \right] + \cos \left( \frac{2e}{\hbar} \left( \hat{\Phi}_{n} + \frac{\hat{\Phi}_{n}^{B}}{2} \right) \right) + \frac{\Delta \hat{Q}_{n}^{2}}{2 \Delta C} + \frac{(\hat{\Phi}_{n+1}^{B} - \hat{\Phi}_{n}^{B})^{2}}{2 \Delta L}$$

(1)

where $E_{J}^{b}$ is the bare Josephson energy of a junction (see Fig. 2). The effective renormalized Josephson energy $E_{J} = E_{J}^{b}(0) \cos \left( 2e \hat{\Phi}_{n}/\hbar \right) \cos \left( e \hat{\Phi}_{n}^{B}/\hbar \right) |0\rangle$ is defined with respect to the vacuum energy state $|0\rangle$. The essentially nonlinear cosine interaction terms between the transmons and the electromagnetic modes result from the Josephson junctions and have been configured so as to be an even function of each field amplitude.

The transmons can be close enough to each other to be coupled through the mutual capacitances. Assuming the translational invariance of the ring, $C_{n,n'}$ depends only on the distance $n - n'$ modulo $N$ and it is convenient to define their Fourier components: $C_{k} = \sum_{n=1}^{N} e^{-i2\pi k(n-n')/N} C_{n,n'}/N$. Here the integer $k$ is defined modulo $N$, and the capacitance energy can be written as $E_{C,k} = (2e)^{2}/2C_{k}$. Under these conditions, we can rewrite the transmon operators in their "wavevec-

The creation-annihiliation operators $\hat{b}_{k}^{\dagger}$ and $\hat{b}_{k}$ describe plasmon-like collective excitations of charge motion with a wavenumber given by $K = 2\pi k/N$.

The charge and flux operators can be similarly defined through the electromagnetic field component as $\hat{\Phi}_{n}^{B} = \hat{\alpha}_{n}/\sqrt{\Delta C}$ and $\Delta \hat{Q}_{n} = \sqrt{\Delta C} \hat{\alpha}_{n}$. Here $\hat{\alpha}_{n}$ is the vector potential for an ideal waveguide consisting of two parallel infinite planes [13]. It can be expressed through...
the wavevector components:

\[ \hat{a}_n = \sum_{k=1}^{N} e^{i2\pi kn/N} \sqrt{\frac{\hbar}{2\omega_k N}} (\hat{a}_k + \hat{a}^\dagger_{-k}) \]  
\[ \dot{\hat{a}}_n = \sum_{k=1}^{N} e^{i2\pi kn/N} \sqrt{\frac{\hbar\omega_k}{2N}} (\hat{a}_k - \hat{a}^\dagger_{-k})/i \]

The first and second terms correspond respectively to the plasmon mode with energy spectrum \( \hbar \omega_k = \sqrt{4E_JE_{C,k}} \) and the photon mode with \( \omega_k = \sqrt{2(1 - \cos(K))} / (\Delta C \Delta L) + \omega_0' \), where \( \omega_0 = \sqrt{E_JE_{\Delta C}} \) and \( E_{\Delta C} = 2 \epsilon^2 / (\hbar^2 \Delta C) \). The plasmon spectrum is almost flat. The photon spectrum has a gap resulting from the Josephson energy contribution. Without it, the spectrum would be linear with a light speed \( c = D/\sqrt{\Delta C \Delta L} \) where \( D \) is the transmon interdistance. The third term is the quartic interaction responsible of for the coupling \( k + k' \leftrightarrow (k + l) + (k' - l) \) between the radiation and the transmon qubits and is negligible only if \( E_{\Delta C}/(\epsilon_k, \omega_k) \). Note that we neglect the quartic self-modulation terms responsible for anharmonicity \([2]\) for both fields since these are even weaker than the coupling.

**The proposed experiment:** We start by adiabatically applying to each qubit the potential \( V \), producing the initial charge with the non zero expectation \( \langle \hat{q}_n \rangle = C_0 V \) interpreted as displacement of the vacuum state. Then the potential is suddenly dropped to zero. The qubit islands begin discharging, emitting in the process entangled pairs of photons. The corresponding transmon mode is a coherent state with \( k = 0 \). Its amplitude at \( t = 0 \) is

\[ \varphi_0 \equiv \sqrt{\langle \hat{b}_0 \rangle} = i \sqrt{E_J/e_0} (eV/E_{C,0}). \]  

The qu-bit regime is recovered in the case of fainted coherent state. If parametrize the squeezed radiation mode with the squeezing amplitude \( r_k(t) \) and the phase \( \theta_k(t) \), the full Ansatz for the quantum state of radiation in the waveguide is \([32]\)

\[ |\Psi(t)\rangle = \prod_{k=1}^{N/2} e^{r_k (e^{-2i\theta_k} \hat{a}^\dagger_{-k} - e^{2i\theta_k} \hat{a}_k)} \hat{D}(t) |0\rangle, \]

with the displacement unitary transformation \( \hat{D}(t) = \exp(\sqrt{\mathcal{N}} \varphi_0 \hat{b}_0 - \sqrt{\mathcal{N}} \varphi \hat{b}_0) \). From the Lagrangian \( \mathcal{H}[|\Psi(t)\rangle(i\hbar)\partial_t - : \hat{H} : |\Psi(t)\rangle] \) we obtain the dynamical equations:

\[ i\dot{\varphi} = \epsilon_0 \varphi + \frac{E_{\Delta C} e_0}{8N} (\varphi + \varphi^*) \]
\[ \times \sum_{k=1}^{N/2} \cos(2r_k - 1 + \cos(2\theta_k) \sinh(2r_k)) \frac{1}{\hbar \omega_k} \]
\[ \dot{\theta}_k = \omega_k + \frac{E_{\Delta C} e_0}{16} (\varphi + \varphi^*)^{2 \frac{1}{2} + \cos(2\theta_k) \coth(2r_k)} \frac{1}{\hbar \omega_k}, \]
\[ \dot{r}_k = \frac{E_{\Delta C} e_0}{16} (\varphi + \varphi^*)^{2 \sin(2\theta_k)} \frac{1}{\hbar \omega_k}. \]

In the short time limit, assuming that all the capacitances are of the same order of magnitude and taking realistic values for the system parameters \( E_{C,0} \sim E_{\Delta C} \sim 10\text{GHz}, E_J \sim 1\text{THz} \), we can make the following direct estimates. The rate of squeezing is \( \dot{r}_k(0) \sim 10\mu\text{V} \). The relative charge leakage, \( \langle \Delta q_n \rangle/\langle q_n \rangle \sim E_J/E_{C,0}(eV/h)^2 \sim 10^2(\text{ns}^2) \), is quadratic in time. In the long time limit, the equations are solved in the rotating wave approximation. Then the decaying of two transmon excitations into two photons satisfies the number and energy conservation \( 2e_0 = 2\omega_k \). We consider two distinct cases of a transmon excitation decaying into either a continuum of photon modes or into a single mode.

**Decay into a continuum:** In the large-\( N \) limit and for weak squeezing \( r_k(t) \leq 1 \), we can approximate the phase by \( \theta_k(t) = \hbar \omega_k t - \pi/4 \). The solution for the transmon field in the continuum limit is then \( \varphi(t) \approx i e^{-i\epsilon t} \varphi_0/\sqrt{1 + 17} \) with a inverse power decay rate:

\[ \Gamma = \left( \frac{E_{\Delta C} eV}{16 E_{C,0}} \right)^2 \frac{E_J/h^3}{\sqrt{(\omega_0^2 - \epsilon_0^2)(\epsilon_0^2 - \omega_0^2)}} \approx \frac{C_0 V^2}{h}. \]

This rate corresponds to the capacitance energy perturbation introduced initially and has to be much less than the plasmon frequency \( \epsilon_0 \) (typically in the GHz-THz range) but much larger than any decoherence rate (in the MHz range) \([1]\). For the squeezing parameter, we
in the ring becomes large. We can then select only two
tangled modes ±k only in the Eqs. (9, 10, 11), which in-
interact with a resonant transmon mode. Other photon
modes are not perturbed.

Besides the interaction terms for the transition, an
additional modulation phase term affects the transition
frequency [21]. The maximum squeezing that
can be reached is $r_m \propto \ln(2N/|\varphi_0|^2)$ corresponding
to the total depletion $|\varphi(t)|^2 = 0$. For simplic-
ity, we shall assume the phase modulation term is con-
stant which implies the restriction to values $r_m \leq$
$r_m - \sqrt{N}/2e^{-\tau m}$. We define the dimensionless par-
parameters: $\epsilon = tE_{\Delta C}c/\hbar \omega_k$ and $\delta = \hbar \delta k 32\omega_k/\Delta C c_0$.

Two cases are considered:

1) No phase modulation: For short times, we note that
the fastest squeezing rate is achieved if the phase match-
ing condition $\delta = 4|\varphi_0|^2$ is satisfied. Using this condition,
the phase modulation can be neglected and the squeezing
parameter evolves towards $r_m$. The explicit expression is

$$r_m(t) = \frac{1}{2} \ln \left( \frac{e^{2r_m} + e^{-4\sinh(2r_m)t}/N}{1 + e^{2r_m} e^{-4\sinh(2r_m)t}/N} \right)$$

2) Weak depletion: When the the detuning is not
phase matched, the charge leakage from the island can
be neglected, i.e $|\phi(t)| \simeq |\varphi_0|$. For a small value of
detuning within the interval $-2|\varphi_0|^2 \leq \delta \leq 6|\varphi_0|^2$,
the photon number grows exponentially: $N_{ph}(t) =$
$\sinh^2(r_m(t)) = 8|\varphi_0|^2 \sinh^2(\Omega t)/\Omega^2$ with the character-
istic angular frequency $\Omega = \sqrt{(4|\varphi_0|^2)^2 - 4|\varphi_0|^4}$.

Outside this interval, the solution becomes $N_{ph}(t) =$
$8|\varphi_0|^2 \sin^2(\Omega t)/\Omega^2$, which corresponds to a Rabi-like oscil-
lation between the plasmon mode and the photon
modes. This Rabi-like superposition of a plasmon state
and an EPR photon state illustrates the rich possibilities
offered by this qubit line device for quantum design.

Conclusions: We propose the superconducting trans-
mon line embedded in a ring waveguide as a generator of
tangled beams of microwave radiation. Using the fully quantum description, we can describe the scatter-
ing process between photons and the collective transmon
excitation. We also show that high squeezing may be
obtained in the long wavelength regime, allowing for a
genuine EPR-like experiment in a microchip device. An
interesting extension of this design would be a paramet-
ric optical amplifier with the proposed setup for quantum imaging [21, 28].

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[32] See the Supplemental Materials for the mathematical details.