The $\mathcal{N} = 4$ Coset Model and the Higher Spin Algebra

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Abstract

By computing the operator product expansions between the first two $\mathcal{N} = 4$ higher spin multiplets in the unitary coset model, the (anti)commutators of higher spin currents are obtained under the large $(N, k)$ ’t Hooft-like limit. The free field realization with complex bosons and fermions is presented. The (anti)commutators for generic spins $s_1$ and $s_2$ with manifest $SO(4)$ symmetry at vanishing ’t Hooft-like coupling constant are completely determined. The structure constants can be written in terms of the ones in the $\mathcal{N} = 2$ $\mathcal{W}_\infty$ algebra found by Bergshoeff, Pope, Romans, Sezgin and Shen previously, in addition to the spin-dependent fractional coefficients and two $SO(4)$ invariant tensors. We also describe the $\mathcal{N} = 4$ higher spin generators, by using the above coset construction results, for general super spin $s$ in terms of oscillators in the matrix generalization of $AdS_3$ Vasiliev higher spin theory at nonzero ’t Hooft-like coupling constant. We obtain the $\mathcal{N} = 4$ higher spin algebra for low spins and present how to determine the structure constants, which depend on the higher spin algebra parameter, in general, for fixed spins $s_1$ and $s_2$. 
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L.1 The \( \mathcal{N} = 2 \) wedge subalgebra of \( \mathcal{W}_\infty^{\mathcal{N}=4}[\lambda] \) algebra
1 Introduction

The large $\mathcal{N} = 4$ holography in [1] has been proposed by constructing the matrix generalization of the Vasiliev higher spin theory on $AdS_3$ [2, 3] and comparing it with the two dimensional minimal model conformal field theories with large $\mathcal{N} = 4$ superconformal symmetry. The motivation for this proposal is based on the fact that the non-abelian version of Vasiliev higher spin theory might give some hints or evidences for the better description of type IIB string theory on the $AdS_3$ space where the internal seven space is a product of two three spheres and one sphere. The complete dual conformal field theory is not known as far as we know. The emergence of higher spin symmetry has not been fully clarified from the viewpoint of type IIB string theory, although there are some recent works in [4, 5, 6, 7, 8], where one observes that the infinite tower of modes becomes massless. One of the findings in [1] is that the spin contents with their multiplicity in the higher spin algebra of the Vasiliev higher spin theory, which contains an exceptional superalgebra $D(2, 1|1_{\lambda})$ as a subalgebra, are found [1]. That is, there exist seven spin one fields and eight fields of spin $s = \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \cdots$. However, the complete structure of the $\mathcal{N} = 4$ higher spin algebra (i.e., the (anti)commutators of $\mathcal{N} = 4$ higher spin generators) is not obtained so far. This fact allows us to further study this large $\mathcal{N} = 4$ holography more closely and is one of motivations of the current paper.

One way to observe the presence of the higher spin algebra (or its symmetry) is to study its two dimensional dual conformal field theory, where the affine Kac-Moody algebra can be obtained by using the adjoint spin-$1, \frac{1}{2}$ fields. In [9], by writing down the (higher spin) currents in terms of these fields living in the $\mathcal{N} = 4$ superconformal coset model, the operator product expansion (OPE) between the lowest (or first) $\mathcal{N} = 4$ higher spin multiplet and itself of $\mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda]$ algebra (in the notation of [10]) is determined. See also the relevant works in [11, 12, 13, 14, 15]. Note that the currents of the $\mathcal{N} = 4$ (linear) superconformal algebra become the ones of exceptional superalgebra $D(2, 1|1_{\lambda})$ (described by nine bosonic and eight fermionic generators), when the “wedge” condition is imposed. The next (or second) $\mathcal{N} = 4$ higher spin multiplet occurs in the right hand side of the above OPE.

We need to further compute the operator product expansions between these two $\mathcal{N} = 4$ higher spin multiplets in order to obtain the corresponding higher spin algebra, the

\[ \text{See also (7.2) for the precise relation between the \textquoteleft t Hooft coupling constant $\lambda$ and the higher spin algebra parameter $\mu$ (or $\nu$).} \]
(anti)commutators of $\mathcal{N} = 4$ higher spin generators, which can be determined by taking large $(N, k)$ 't Hooft-like limit together with the wedge condition, precisely.

We observe that the additional two (or third and fourth) $\mathcal{N} = 4$ higher spin multiplets arise in this construction.

In [10], the free field construction at $\lambda = 0$ for the large $\mathcal{N} = 4$ holography with $\mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda]$ algebra [1] is described by using $2N$-free complex bosons which transform as bifundamental $(N, \bar{2}) \oplus (\bar{N}, 2)$ of $U(N) \times U(K)$ with $K = 2$ and $2N$-free complex fermions which transform similarly under the $U(N) \times U(L)$ with $L = 2$. See also similar construction described in [16]. Note that in the $\mathcal{N} = 4$ superconformal coset model, the numerator of the coset contains $SU(N+2)$, while the denominator of the coset contains $SU(N)$. By taking the $U(N)$ invariant combinations [16, 10] of these fields, the higher spin currents can be determined as follows. 1) The four-(higher spin)currents of integer spins $s = 2, 3, 4, \cdots$ transforming as an adjoint representation of the $U(K = 2)$ Chan-Paton factor are obtained from the above bosons. 2) Other four-(higher spin)currents of integer spins $s = 1, 2, 3, \cdots$ transforming as an adjoint representation of the $U(L = 2)$ Chan-Paton factor can be determined from the fermions. Note that the bilinears of the bosons start at spin $s = 2$, contrary to the ones of the fermions, which can start at spin $s = 1$ [10]. By construction, there are only four spin-1 currents, compared to the field contents in the first paragraph. 3) The four-(higher spin)currents of half-integer spins $s = \frac{3}{2}, \frac{5}{2}, \cdots$, which transform as bifundamental representation $(2, 2)$ of the $U(K = 2) \times U(L = 2)$, are obtained from the above bosons and fermions. 4) Finally, other four-(higher spin)currents of half-integer spins $s = \frac{3}{2}, \frac{5}{2}, \cdots$ transform as bifundamental representation $(\bar{2}, 2)$. The (anti)commutators between the $\mathcal{N} = 4$ (higher spin) currents are obtained for low spins [10].

It is natural to ask whether one can determine these (anti)commutators for two higher spin currents of general spin $s_1$ and $s_2$ in the $\mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda = 0]$ algebra. Some time ago, Odake in [18] found the extended super algebra, where its generators with the particular level condition are written in terms of bilinears of the free complex bosonic and fermionic fields. The bosonic subsector is given by the sum of both $W^K_{\infty}$ algebra [19] and $W^L_{1+\infty}$ algebra [20] (in the notation of [18]). See also the relevant works in [21, 22, 23, 24]. As described in previous paragraph, the $K$-free complex bosons ($U(N)$ singlet) transform as the (anti)fundamental $\bar{K} \oplus K$ of $U(K)$ and the $L$-free complex fermions ($U(N)$ singlet) transform as the (anti)fundamental $\bar{L} \oplus L$ under the $U(L)$, in the notations of [10]. In the context of [10], we can interpret that the complex free bosons give rise to $W_{\infty}[1]$ algebra, whose wedge subalgebra is a bosonic higher spin algebra $hs[1]$ at $\lambda = 1$, while complex free fermions produce $W_{\infty}[0]$ algebra, whose wedge

\[ \text{By realizing the recent work of [17], it is an open problem to check whether the spin-1 current can be added or not in the free field construction.} \]
subalgebra is a bosonic higher spin algebra \( hs[0] \) at \( \lambda = 0 \), after decoupling the spin-1 current \([25]\). See also \([26]\). The \( K^2 \)-(higher spin)currents of integer spins \( s = 2, 3, 4, \cdots \) transform as an adjoint representation of the \( U(K) \) and are obtained from the free bosons. Similarly, the \( L^2 \)-(higher spin)currents from the free fermions of integer spins \( s = 1, 2, 3, \cdots \) transform as an adjoint representation of the \( U(L) \). Moreover, the \( 2KL \)-(higher spin)currents of half-integer spins \( s = \frac{3}{2}, \frac{5}{2}, \cdots \) transform as bifundamental representation \((K, \bar{L}) \oplus (\bar{K}, L)\) of the \( U(K) \times U(L) \), which can be obtained from the free bosons and fermions. The seven nontrivial (anti)commutators of \( \mathcal{N} = 4 \) (higher spin) multiplets are given in terms of a generalized hypergeometric function and some polynomials in the modes explicitly \([3]\).

In this paper, we observe that the free field construction in \([10]\) can be generalized to obtain the higher spin currents with \( K = L = 2 \), which lead to the manifest \( SO(4) \) symmetric (anti)commutators in the \( \mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda = 0] \) algebra (with arbitrary spins \( s_1 \) and \( s_2 \)) together with the structure constants found in \([24, 18]\). What happens for the case of nonzero \( \lambda \)? For the nonzero ‘t Hooft-like coupling constant \((\lambda \neq 0)\), we expect that the \( \mathcal{N} = 4 \) higher spin algebra can be constructed by oscillators studied in \([2, 3]\). In order to obtain this (unknown) higher spin algebra, we should resort to the (anti)commutators from the higher spin currents of \( \mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda] \) algebra in two dimensional conformal field theory by imposing the “wedge” condition together with the infinity limit of central charge (or infinity limit of \( N \)). Then all the nonlinear (and some linear) terms in \( \mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda] \) algebra vanish and we are left with linear terms in the (anti)commutators. For the case of \( \mathcal{W}_{\infty}^{\mathcal{N}=2}[\lambda] \) algebra (in the notation of \([10]\), the \( \mathcal{N} = 2 \) higher spin algebra for any spins \( s_1, s_2 \) is found in \([27]\). See also the relevant works in \([28, 29]\). According to \([30]\), the generators of wedge subalgebra of \( \mathcal{W}_{\infty}^{\mathcal{N}=2}[\lambda] \) algebra match with the ones of the \( \mathcal{N} = 2 \) higher spin algebra \([27]\). See also relevant works in \([31, 32, 33, 34, 35, 36]\). It is straightforward to construct the generators satisfying the \( \mathcal{N} = 2 \) higher spin algebra \( shs[\lambda] \) for generic \( \lambda \) from the viewpoint of oscillators by focusing on the (anti)commutators of the wedge subalgebra of \( \mathcal{W}_{\infty}^{\mathcal{N}=2}[\lambda] \) algebra \([3]\). Eventually this will lead to the findings in \([27]\). However, for the large \( \mathcal{N} = 4 \) holography associated with \( \mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda] \) algebra with nonzero \( \lambda \), there is no known higher spin algebra (as far as we know) from the beginning. We will observe that the subalgebra of this \( \mathcal{N} = 4 \) higher spin algebra \( shs_2[\lambda] \) (in the notation of \([10]\) with \( K = L = 2 \)) should contain the \( \mathcal{N} = 2 \) higher spin algebra \( shs[\lambda] \) in \([27]\) because the wedge subalgebra of \( \mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda] \)

\[ ^3 \text{In this paper, we mainly focus on the } K = L = 2 \text{ case associated with the large } \mathcal{N} = 4 \text{ holography.} \]

\[ ^4 \text{The bosonic subalgebra of } \mathcal{N} = 2 \text{ higher spin algebra } shs[\lambda] \text{ contains the two bosonic higher spin algebras, } hs[0] \text{ and } hs[1 - \lambda]. \text{ Then the wedge subalgebra of } \mathcal{W}_{\infty}^{\mathcal{N}=2}[0] \text{ algebra at } \lambda = 0 \text{ contains the bosonic higher spin algebras } hs[0] \text{ and } hs[1]. \]
algebra contains the one of $W_{\infty}^{N=2}[\lambda]$ algebra. Although there are oscillator construction in \cite{10} for the $N = 4$ higher spin algebra, the explicit expression for this algebra is not known.

We try to obtain the $N = 4$ higher spin generators with the help of oscillator formalism in the context of the matrix generalization of $AdS_3$ higher spin theory by using the corresponding (anti)commutators of $W_{\infty}^{N=4}[\lambda]$ algebra in the two dimensional conformal field theory, via the large $N = 4$ holography.

In this paper, we explicitly calculate the $s$-th $N = 4$ (higher spin) generators in terms of oscillators. We provide how to determine the (anti)commutators of the $N = 4$ higher spin algebra $shs_2[\lambda]$ for fixed spins $s_1$ and $s_2$.

We can write down the possible terms of the right hand side of the (anti)commutator. Then by using the general formula of the higher spin generators we explained above, we can express the (anti)commutator with unknown coefficients which depend on the $\lambda$ together with mode dependent factors (that can be obtained from the $\lambda = 0$ case). If the spins $s_1$ and $s_2$ are small, then we can do this computation by hand using the defining relations of the oscillators with the $2 \times 2$ matrix manipulations. However, the spins $s_1$ and $s_2$ become large, then this computation by hand is rather tedious and is not possible to express the (anti)commutator in closed form, although we can obtain the full expressions. We will provide how to generate $\lambda$-dependent structure constants appearing in the right hand side of (anti)commutator systematically, although the closed form for these structure constants is not known in this paper.

In section 2, we review on the OPE between the first $N = 4$ multiplet described in \cite{9}. In section 3, by using the explicit form for the first and second $N = 4$ multiplets found in \cite{9, 40}, we calculate the OPE between them and obtain the third $N = 4$ multiplet. In section 4, we compute the OPE between the second $N = 4$ multiplet. In section 5, we take the large $(N, k)$ ’t Hooft-like limit on the two OPEs obtained in previous sections and present the (anti)commutators. We also describe the subalgebra which has the $N = 2$ supersymmetry. In section 6, we describe the first and second $N = 4$ higher spin multiplets by using the free field construction studied in \cite{10}. After writing down the (anti)commutators satisfied by these fields, we present the (anti)commutators for general spins $s_1$ and $s_2$. In section 7, the oscillator realization for the first two $N = 4$ higher spin generators is given and furthermore, we present

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5. According to the result of \cite{1}, the higher spin algebra parameter $\mu$, which is a mass parameter of scalar field, is equal to $\lambda$. We use $shs_2[\lambda]$ rather than $shs_2[\mu]$. See also (7.2).

6. In the coset construction of \cite{10}, the free field construction can be obtained by taking large level limit. Recently, by studying the nontrivial adjoint currents of $U(K)$ together with the trivial higher spin currents, the structure of “rectangular” $W$ algebra is found in \cite{37, 38, 39}.
the $s$-th $\mathcal{N} = 4$ higher spin generators in terms of oscillators. In section 8, we summarize what we have obtained in this paper and the open problems are given. In Appendices A, B, and C, some detailed expressions appeared in the sections 2, 3, and 4 are given\[^7\].

We are using the Thielemans package \[^{\ref{41}}\] with mathematica \[^{\ref{42}}\]. An ancillary mathematica file, ancillary.nb, where the missing parts of Appendices are given explicitly, is included.

## 2 Review of $\mathcal{N} = 4$ unitary coset model

The Wolf space coset in the “supersymmetric” version with groups $G = SU(N + 2)$ and $H = SU(N) \times SU(2) \times U(1)$ is given by

$$\text{Wolf} = \frac{SU(N + 2)}{SU(N) \times SU(2) \times U(1)}, \tag{2.1}$$

where the group indices are described by

$$\begin{align*}
G \text{ indices} &: a, b, \cdots = 1, 2, \cdots, (N + 2)^2 - 1, \\
\overline{G} \text{ indices} &: \bar{a}, \bar{b}, \cdots = 1, 2, \cdots, 4N. \tag{2.2}
\end{align*}$$

In the bosonic version of Wolf space, there are $4N$-free fermions appearing in an extra $SO(4N)$ group in the numerator of the coset (2.1) at level $1^8$. The $\mathcal{N} = 1$ affine Kac-Moody algebra can be constructed from the adjoint spin-1 current and the spin-$\frac{1}{2}$ current of group $G = SU(N + 2)$. The operator product expansion between the (modified) spin-1 current $V^a(z)$ and the spin-$\frac{1}{2}$ current $Q^a(z)$ can be described as follows \[^{\ref{43}}\]:

$$\begin{align*}
V^a(z) V^b(w) &= \frac{1}{(z-w)^2} k g^{ab} - \frac{1}{(z-w)} f^{abc} V^c(w) + \cdots, \\
Q^a(z) Q^b(w) &= -\frac{1}{(z-w)} (k + N + 2) g^{ab} + \cdots. \tag{2.3}
\end{align*}$$

The metric can be obtained from $g_{ab} = \frac{1}{2c_G} f^{ac}_d f^{bd}_c$, where $c_G$ is the dual Coxeter number of the group $G$. The metric $g_{ab}$ is given by the generators of $G$ in the complex basis \[^{\ref{44}}\].

\[^7\] In Appendices D, E, and F, some expressions appeared in the sections 2, 3, and 4 for the OPEs are presented. In Appendix G, the (anti)commutators corresponding to Appendices D and E are described. In Appendix H, we provide the second $\mathcal{N} = 4$ higher spin multiplet in section 6. In Appendix I, the first $\mathcal{N} = 4$ higher spin multiplet in section 6 at $\lambda = 1$ is described. In Appendix J, the remaining (anti)commutators for the general spins $s_1$ and $s_2$ in section 6 are given. In Appendix K, the second $\mathcal{N} = 4$ higher spin generators in section 7 is presented. Finally in Appendix L, we explain how the $\mathcal{N} = 2$ higher spin algebra can be obtained from the results of section 7.

\[^8\] In \[^{\ref{10}}\], these are described by $2N$-complex fermions transforming as bifundamental $(N, 2) \oplus (\overline{N}, 2)$ under the $U(N) \times U(2)$.
as follows: $g_{ab} = \text{Tr}(T_a T_b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where $a, b = 1, 2, \cdots, (N + 2)^2 - 1$ from (2.2). The commutation relation for the $SU(N+2)$ generators in the fundamental representation is given by $[T_a, T_b] = f_{abc} T_c$.

### 2.1 The 11 currents of the large $\mathcal{N} = 4$ nonlinear superconformal algebra

The four supersymmetry currents of spin-$\frac{3}{2}$, $\hat{G}^0(z)$ and $\hat{G}^i(z)$, the six spin-1 currents of $SU(2) \times SU(2)$, $A^{\pm i}(z)$ and the spin-2 stress energy tensor $\hat{T}(z)$ can be described in terms of spin-1, $\frac{1}{2}$ currents [15, 16, 17] as follows:

\[
\begin{align*}
\hat{G}^0(z) &= \frac{i}{(k + N + 2)} g_{ab} Q^a V^b(z), \\
\hat{G}^i(z) &= \frac{i}{(k + N + 2)} h^i_{ab} Q^a V^b(z), \\
A^{+i}(z) &= -\frac{1}{4N} f^{abc} h^c_{ab} V^c(z), \\
A^{-i}(z) &= -\frac{1}{4(k + N + 2)} h^i_{ab} Q^a Q^b(z), \\
\hat{T}(z) &= \frac{1}{2(k + N + 2)^2} \left[ (k + N + 2) g_{ab} V^a V^b + k g_{ab} Q^a \partial Q^b + f^{abc} g_{ac} g_{bd} Q^c Q^d V^e(z) \right] - \frac{1}{(k + N + 2)} (A^{+i} + A^{-i})^2(z), \quad i = 1, 2, 3.
\end{align*}
\]

Note that the Wolf coset indices, $\bar{a}, \bar{b}, \cdots$, run over $\bar{a}, \bar{b}, \cdots = 1, 2, \cdots, 4N$ from (2.2). In the spin-2 stress energy tensor, the terms $A^{-i} A^{-i}$, which contain $(Q^a Q^b)(Q^a Q^d)(z)$, can be further simplified by using the defining relations (2.3). The three almost complex structures $h^i_{ab}$ are given by $4N \times 4N$ matrices [14] and they are antisymmetric and satisfy the quaternionic algebra [15]. Note that the Wolf space coset metric $g_{ab} \equiv h^0_{ab}$.

With the $SO(4)$ singlet $\hat{T}(z) \rightarrow \hat{L}(z)$, the spin-$\frac{3}{2}$ currents, transforming as the $SO(4)$ vector representation, are given by $\hat{G}^0(z) \rightarrow \hat{G}^2(z)$, $\hat{G}^i(z) \rightarrow \hat{G}^3(z)$, $\hat{G}^2(z) \rightarrow -\hat{G}^4(z)$, and $\hat{G}^3(z) \rightarrow \hat{G}^1(z)$. Furthermore, the six spin-1 currents, $\hat{T}^{\mu\nu}(z)$ transforming as the $SO(4)$ adjoint representation, can be obtained from the corresponding two spin-1 currents $A^{\pm i}(z)$ as follows [18, 19]: $A^{\pm 1}(z) \rightarrow \pm \frac{1}{2} \alpha^{\pm 1}_{\mu\nu} \hat{T}^{\mu\nu}(z)$, $A^{\pm 2}(z) \rightarrow \pm \frac{1}{2} \alpha^{\pm 2}_{\mu\nu} \hat{T}^{\mu\nu}(z)$, $A^{\pm 3}(z) \rightarrow \pm \frac{1}{2} \alpha^{\pm 3}_{\mu\nu} \hat{T}^{\mu\nu}(z)$, where the six $4 \times 4$ matrices $\alpha^{\pm i}_{\mu\nu}$ [19] relate the spin-1 currents $A^{\pm i}(z)$ to the spin-1 currents $\hat{T}^{\mu\nu}(z)$.

Then the large $\mathcal{N} = 4$ nonlinear superconformal algebra [50, 45, 51, 46] can be obtained explicitly using the above 11 currents as follows: $\hat{L}$, $\hat{G}^\mu$, and $\hat{T}^{\mu\nu}$. The nonlinear structure appears in the OPE between the spin-$\frac{3}{2}$ currents. Note that the two levels of $SU(2)$’s are given by $k$ and $N$ respectively.
2.2 The 16 currents of the large $\mathcal{N} = 4$ linear superconformal algebra

The explicit relation between the 16 currents of the large $\mathcal{N} = 4$ linear superconformal algebra and the 11 currents of the large $\mathcal{N} = 4$ nonlinear superconformal algebra is described by \[50\]

\[
T^{\mu\nu}(z) = \hat{T}^{\mu\nu}(z) + \frac{2i}{(2 + k + N)} \Gamma^{\mu} \Gamma^{\nu}(z),
\]
\[
G^\mu(z) = \hat{G}^\mu(z) + \frac{2i}{(2 + k + N)} U \Gamma^{\mu}(z)
+ \varepsilon^{\mu\nu\rho\sigma} \left[ \frac{4i}{3(2 + k + N)^2} \Gamma^{\nu} \Gamma^{\rho} \Gamma^{\sigma} - \frac{1}{(2 + k + N)} T^{\rho\sigma}\right](z),
\]
\[
L(z) = \hat{L}(z) - \frac{1}{(2 + k + N)} \left[ U U - \partial \Gamma^{\mu} \Gamma^{\mu}\right] (z).
\] (2.5)

In the last term of spin-$\frac{3}{2}$ currents in (2.5), the first relation of (2.5) should be inserted. Moreover, the four fermionic spin-$\frac{1}{2}$ currents are given by \[48, 44\]

\[
\Gamma^0(z) = -\frac{i}{4(N + 1)} h_{\dot{a} \dot{b}} \tilde{f}^{\dot{a}\dot{b}} c_{\dot{c}} Q^{\dot{c}}(z), \quad \Gamma^j(z) = -\frac{i}{4(N + 1)} h_{\dot{a} \dot{b}} \tilde{f}^{\dot{a}\dot{b}} Q^{\dot{c}}(z), j = 1, 2, 3,
\] (2.6)

where there is no sum over $j$ in the first equation of (2.6). We should change the index structures as follows: $\Gamma^0 \rightarrow -i \Gamma^2$, $\Gamma^1 \rightarrow -i \Gamma^3$, $\Gamma^2 \rightarrow i \Gamma^4$ and $\Gamma^3 \rightarrow -i \Gamma^1$ in order to use (2.5) from (2.6) with $SO(4)$ vector index. We introduce the coset $\text{Wolf} \times SU(2) \times U(1) = \frac{SU(N+2)}{SU(N)}$ by multiplying $SU(2) \times U(1)$ factor in (2.1) and the corresponding notation is as follows: $\tilde{a} = (\bar{a}, \hat{a})$, where the $\bar{a}$ index runs over $4N$ values as before and the index $\hat{a}$ associates with the $2 \times 2$ matrix corresponding to $SU(2) \times U(1)$ and runs over $4$ values. The bosonic spin-$1$ current is given by

\[
U(z) = -\frac{1}{4(N + 1)} h_{\bar{a} \bar{b}} \tilde{f}^{\bar{a}\bar{b}} c_{\bar{c}} Q^{\bar{c}}(z) - \frac{1}{2(k + N + 2)} f^{\bar{a} \bar{b}} c_{\bar{c}} Q^{\bar{c}}(z),
\] (2.7)

where there is no sum over the index $j$. Of course, the OPEs between the 11 currents in (2.4) and the spin-$\frac{1}{2}$ currents $\Gamma^\mu(z)$ are regular and similarly, the OPEs between the 11 currents and the spin-$1$ current $U(z)$ do not have any singular terms.

Therefore, the large $\mathcal{N} = 4$ linear superconformal algebra can be generated by (2.5), (2.6) and (2.7), together with the 11 currents obtained in previous subsection, in the $\mathcal{N} = 4$ coset $\frac{SU(N+2)}{SU(N)}$ model. Note that the two levels of $SU(2)$’s are given by $(k + 1)$ and $(N + 1)$ respectively contrary to the ones of nonlinear case. The central charge is given by

\[
c = \frac{6(k + 1)(N + 1)}{(k + N + 2)}.
\] (2.8)
Under the large \((N,k) \, 't \, Hooft\) limit, the central charge \((2.8)\) becomes 
\[ c = 6(1 - \lambda)N, \]
where the \('t \, Hooft\) coupling constant is defined as 
\[ \lambda \equiv \frac{(N+1)}{(k+N+2)}. \]
At \(\lambda = 0\), the central charge is 
\[ c = 6N. \]
For the infinity limit of \(k\), the central charge is given by 
\[ c = 6(N + 1) \] from \((2.8)\). \textsuperscript{52}

### 2.3 Operator product expansion between the first \(\mathcal{N} = 4\) higher spin multiplet

The single \(\mathcal{N} = 4\) super OPE studied in \textsuperscript{53} between the first \(\mathcal{N} = 4\) higher spin multiplet and itself can be summarized by \textsuperscript{9}

\[
\Phi^{(1)}(Z_1) \Phi^{(1)}(Z_2) = \frac{\theta_{12}^{-0}}{z_{12}^4} \frac{4kN(k-N)}{(2+k+N)^2} + \frac{\theta_{12}^{-i}}{z_{12}^3} Q^{(2),i}_i(Z_2) + \frac{\theta_{12}^{-0}}{z_{12}^3} Q^{(1),i}_i(Z_2)
\]

\[
+ \frac{1}{z_{12}^2} \frac{2kN}{(2+k+N)} + \frac{\theta_{12}^{-ij}}{z_{12}^2} Q^{(1),ij}_i(Z_2)
\]

\[
+ \frac{\theta_{12}^{-i}}{z_{12}^2} \left[ 2\partial Q^{(2),i}_i + Q^{(2),i}_i - \frac{(k-N)}{3(2+k+N)} Q^{(3),i}_i \right](Z_2)
\]

\[
+ \frac{\theta_{12}^{-0}}{z_{12}^2} \left[ \frac{3}{2} \partial Q^{(1),i}_i + Q^{(2),i}_i \right]
\]

\[
- \frac{8(k-N)}{(5+4k+4N+3kN)} \Phi^{(1)}(Z_2) + \frac{kN}{(2+k+N)} J^{4-0}
\]

\[
+ \frac{\theta_{12}^{-ij}}{z_{12}^2} \left[ \partial Q^{(1),ij}_i + Q^{(2),ij}_i \right](Z_2)
\]

\[
+ \frac{\theta_{12}^{-i}}{z_{12}^2} \left[ \frac{3}{2} \partial^2 Q^{(2),i}_i + \partial Q^{(2),i}_i + Q^{(3),i}_i \right](Z_2)
\]

\[
+ \frac{\theta_{12}^{-0}}{z_{12}^2} \left[ \partial^2 Q^{(1)} + \partial Q^{(2)} + Q^{(3)}_i \right]
\]

\[
- \frac{8(k-N)}{(5+4k+4N+3kN)} \partial(\Phi^{(1)}, \Phi^{(1)}) + \frac{kN}{(2+k+N)} J^{4-0}
\]

\[
- \frac{kN(k-N)}{(2+k+N)^2} \partial^2 J \right](Z_2) + \ldots . \tag{2.9}
\]

The nonlinear terms \(\Phi^{(1)}\Phi^{(1)}(Z_2)\) and its descendant term occur in this OPE \((2.9)\). In particular, the dependence of the first and second \(\mathcal{N} = 4\) higher spin multiplets can also arise
in the following quasi (super) primary fields:

\[
Q^{(2)}_{\frac{1}{2},i} = \frac{1}{2} D^i \Phi^{(2)} - \frac{4(k - N)}{(5 + 4k + 4N + 3kN)} \Phi^{(1)} D^i \Phi^{(1)} + J \text{ dependent terms},
\]

\[
Q^{(2)}_2 = 2 \Phi^{(2)} + J \text{ dependent terms}.
\]

The stress energy tensor \( J \) dependent terms in (2.10) are given in Appendix A. All the expressions for the quasi primary (super)fields are presented in (A.1) (with ancillary.nb). In obtaining (2.9), the fundamental OPEs in (A.2) are crucial. In Appendix D, the OPEs between the first \( \mathcal{N} = 4 \) multiplet are explicitly given after the large \((N, k)\) limit is taken. Furthermore, in (G.3), its (anti)commutators are presented explicitly. See also [9] for more details. We need to redefine the second \( \mathcal{N} = 2 \) higher spin multiplet (3.4) in next section.

3 Operator product expansion between the first and second \( \mathcal{N} = 4 \) higher spin multiplets

We construct the OPEs between the first and second \( \mathcal{N} = 4 \) higher spin multiplets in component approach and in \( \mathcal{N} = 4 \) superspace. The 16 = (1 + 4 + 6 + 4 + 1) higher spin currents of superspin \( s \) can be combined into one single \( \mathcal{N} = 4 \) super field as follows [9]:

\[
\Phi^{(s)} \equiv \left( \Phi^{(s)}_0, \Phi^{(s),i}_{\frac{1}{2}}, \Phi^{(s),ij}_1, \Phi^{(s),ij}_{\frac{3}{2}}, \Phi^{(s)}_2 \right), \quad i, j = 1, 2, 3, 4.
\]

(3.1)

The spins of each element are given by \( s, (s + \frac{1}{2}), (s + 1), (s + \frac{3}{2}), \) and \( (s + 2) \) respectively. The last two component fields in (3.1) are not quasi primary fields while the first three component fields are primary fields under the stress energy tensor \( L(z) \) of (2.5) [10]. After analyzing the component approach first and then we will end up with the single OPE in \( \mathcal{N} = 4 \) superspace.

3.1 The new higher spin current \( \tilde{\Phi}^{(3)}_0 \) of spin 3

Let us consider the OPE between the last component \( \Phi^{(1)}_2(z) \) with \( s = 1 \) and the first component \( \Phi^{(2)}_0(w) \) with \( s = 2 \) of (3.1). By adding the derivative term and other composite field

\[
\Phi^{(s)} \equiv \left( \Phi^{(s)}_0, \Phi^{(s),i}_{\frac{1}{2}}, \Phi^{(s),ij}_1, \Phi^{(s),ij}_{\frac{3}{2}}, \Phi^{(s)}_2 \right), \quad i, j = 1, 2, 3, 4.
\]

(3.1)

As in [54], the spin of the quasi primary field is given by the number of inside of the bracket. The subscript comes from the one in the first operator of the OPE in the component approach. See, for example, (A.2). We use the simplified notation \( J^{i=0} \equiv D^i D^2 D^3 D^4 J \).

We will use the simplified notations for the (higher spin)currents as follows:

\[
\tilde{T}^{ij} = \frac{1}{2!} \varepsilon_{ijkl} T^{kl}, \quad \tilde{G}^i = g^i - \frac{(k - N)}{(k + N + 2)} i \partial \Gamma^i, \quad \tilde{L} = L + \frac{(k - N)}{2(k + N + 2)} \partial U,
\]

\[
\tilde{\Phi}^{(s),ij}_1 = \frac{1}{2!} \varepsilon_{ijkl} \Phi^{(s),kl}_1.
\]

Also we have the \( \mathcal{N} = 4 \) stress energy tensor \( J = (-\Delta, i \Gamma^i, -i T^{ij}, -\tilde{G}^i, 2 \tilde{L}) \) with \( \partial \Delta \equiv -U \).
to the $\Phi^{(1)}_2$, we can make it to be a primary field as follows\footnote{We have the following relation}

$$
\Phi^{(s=1)}_2 \equiv \tilde{\Phi}^{(s=1)}_2 + p_1 \partial^2 \Phi^{(s=1)}_0 + p_2 L \Phi^{(s=1)}_0,
$$

(3.2)

where $\tilde{\Phi}^{(s=1)}_2$ is a primary field under the stress energy tensor $L(z)$. The coefficients $p_1$ and $p_2$ appearing in (3.2) were given in \cite{9} and they are

$$
p_1 \equiv -\frac{(k - N)(29 + 16k + 16N + 3kN)}{3(2 + k + N)(5 + 4k + 4N + 3kN)}, \quad p_2 \equiv \frac{8(k - N)}{(5 + 4k + 4N + 3kN)}.
$$

(3.3)

The higher spin-2 current in the second $\mathcal{N} = 4$ multiplet is redefined from the one (denoted by $\Phi^{(2)}_0$ in this paper) in \cite{9} (or [40]) as follows\footnote{The right hand side of (3.4) is proportional to the equation (6.37) of [40].}

$$
\Phi^{(2)}_0 = \hat{\Phi}^{(2)}_0 + r_0 \Phi^{(1)}_0 + r_1 L + r_2 T^{ij}T^{ij} + r_3 T^{ij}\tilde{T}^{ij} + r_4 T^{ij}\Gamma^i\Gamma^j + r_5 \tilde{T}^{ij}\Gamma^i\Gamma^j + r_6 UU + r_7 \partial\Gamma^i\Gamma^i + r_8 \varepsilon^{ijkl}\Gamma^i\Gamma^j\Gamma^k\Gamma^l,
$$

(3.4)

where the coefficients are given by

$$
r_0 \equiv -\frac{3}{2(2 + N)(2 + k + N)(5 + 4k + 4N + 3kN)}(100 + 187k + 118k^2 + 24k^3 + 303N + 505kN + 277k^2N + 46k^3N + 325N^2 + 455kN^2 + 195k^2N^2 + 20k^3N^2 + 148N^3 + 159kN^3 + 42k^2N^3 + 24N^4 + 16kN^4),
$$

$$
r_1 \equiv \frac{r_0}{3(4 + 3k + 3N + 2kN)}, \quad r_2 \equiv \frac{r_0}{6(4 + 3k + 3N + 2kN)}, \quad r_3 \equiv \frac{r_0}{6(4 + 3k + 3N + 2kN)}(k - N), \quad r_4 \equiv \frac{r_0}{3(2 + k + N)(4 + 3k + 3N + 2kN)}(-2i(4 + k + N)),
$$

$$
r_5 \equiv \frac{r_0}{3(2 + k + N)(4 + 3k + 3N + 2kN)}(-2i(k - N)), \quad r_6 \equiv \frac{r_0}{3(2 + k + N)(4 + 3k + 3N + 2kN)}(-4(2 + k)(2 + N)), \quad r_7 \equiv \frac{r_0}{3(2 + k + N)(4 + 3k + 3N + 2kN)}(2(20 + 7k + 7N + 2kN)), \quad r_8 \equiv \frac{r_0}{3(2 + k + N)^2(4 + 3k + 3N + 2kN)}(-(k - N)).
$$

Note that the coefficients $r_1, \ldots, r_8$ in (3.5) can be written in terms of $r_0$ by using the $\mathcal{N} = 4$ primary condition on the right hand side of (3.4). The second order pole of the OPE between $\Phi^{(1)}_2(z)$ and $\Phi^{(1)}_0(w)$ in [9] has the nonlinear term $\Phi^{(1)}_0 \Phi^{(1)}_0(w)$ as well as $\Phi^{(2)}_0(w)$. The coefficient $r_0$ can be fixed further by absorbing this nonlinear term. Note that the value $r_0$ appears in
the arXiv version of [9] (and is equal to the value $c_3$ in page 244). We also analyze the $\Phi_0^{(3)}$ case similarly.

Then we can calculate, from the result of the higher spin-3 current $\Phi_2^{(1)}(z)$ in [9] and the higher spin-2 current (3.4), the following OPE explicitly

$$
\Phi_2^{(1)}(z) \Phi_0^{(2)}(w) = \frac{1}{(z-w)^4} \left[ Q_2^{(1)} - (6 p_1 + 4 p_2) Q_0^{(1)} \right](w) \\
+ \frac{1}{(z-w)^3} \left[ \partial Q_2^{(1)} + Q_2^{(2)} - p_2 \partial Q_0^{(1)} \right](w) \\
+ \frac{1}{(z-w)^2} \left[ \frac{1}{2} \partial^2 Q_2^{(1)} + \frac{3}{4} \partial Q_2^{(2)} + Q_2^{(3)} - p_2 (2 \Phi_0^{(1)} \Phi_0^{(2)} + d_1^{0,2} L \Phi_0^{(1)}) \right](w) \\
+ \frac{1}{(z-w)} \left[ \frac{1}{6} \partial^3 Q_2^{(1)} + \frac{3}{10} \partial^2 Q_2^{(2)} + \frac{2}{3} \partial Q_2^{(3)} + Q_2^{(4)} - p_2 (2 \partial \Phi_0^{(1)} \Phi_0^{(2)}) \\
+ \Phi_0^{(1)} \partial \Phi_0^{(2)} + d_1^{0,2} \partial L \Phi_0^{(1)} \right](w) + \cdots, 
$$

(3.6)

where the coefficients $p_1$ and $p_2$ are given in (3.3). The coefficient appearing in the second order pole of (3.6) is a function of $N$ and $k$ as follows:

$$
\frac{d_1^{0,2}}{2 + k + N} = -\frac{16(k-N)(11 + 16k + 6k^2 + 16N + 17kN + 4k^2N + 6N^2 + 4kN^2)}{(2 + k + N)(4 + 3k + 3N + 2kN)(5 + 4k + 4N + 3kN)}. 
$$

(3.7)

We can calculate this OPE (3.6) for fixed $N = 3$ and after obtaining $\mathcal{N} = 4$ superspace description, where the fundamental 16 OPEs can be generalized to the full 256 OPEs, we will obtain all the structure constants which depend on $(N, k)$ explicitly from the Jacobi identity. The various quasi primary fields appearing in (3.6) are given in (B.1)\(^\text{13}\).

In particular, the second order pole of (3.6) has the following new primary higher spin-3 current, which cannot be written in terms of the known (higher spin) currents,

$$
Q_2^{(3)}(w) = w_{1,3} \Phi_0^{(3)}(w) + \cdots. 
$$

(3.8)

Note that the subscript 2 comes from the index 2 of $\Phi_2^{(1)}(z)$ in the left hand side of the OPE\(^\text{14}\). The remaining 25 terms in (3.8) are given in (B.1). We use the following quasi primary fields with their spins, $SO(4)$ indices $i, j$ and the subscript indicating the number of fermionic

\(^{13}\)Note that the quasi primary fields $Q^{(s)}_{ic}$ with $s_c = 0, \frac{1}{2}, 1, \frac{3}{2}$ and $2$, which do not have adjoint indices of group $G$, are nothing to do with the spin-$\frac{1}{2}$ currents appearing in (2.3) directly.

\(^{14}\)In the fundamental 16 OPEs we are describing, the higher spin-2 current $\Phi_0^{(2)}(w)$ is common. Then by specifying only the subscript of the component of first $\mathcal{N} = 4$ higher spin multiplet (in addition to the spin of quasi primary fields which tells us the pole of the singular term in the OPE), we can classify all the quasi primary fields uniquely (and independently).
coordinates in front of the quasi primary fields when we go to the $\mathcal{N} = 4$ superspace

\[
Q^{(1)}_0, \ldots, Q^{(2)}_n, i, \ldots; Q^{(1)}_1, Q^{(2)}_1, Q^{(3)}_1, i, \ldots;
\]

\[
Q^{(1)}_2, Q^{(2)}_2, Q^{(3)}_2, i, \ldots; Q^{(1)}_3, Q^{(2)}_3, Q^{(3)}_3, i, \ldots; Q^{(1)}_4, Q^{(2)}_4, Q^{(3)}_4, Q^{(4)}_i, Q^{(5)}_i, \ldots.
\]

(3.9)

Note that the spin of the field in (3.9) is given by the number inside the bracket of upper index. The corresponding $\mathcal{N} = 4$ super fields can be denoted by using the boldface symbols as in previous section (2.9) or (2.10).

Therefore, we observe that the lowest component of the third $\mathcal{N} = 4$ higher spin multiplet occurs in addition to the components of the first and second $\mathcal{N} = 4$ higher spin multiplets in (3.6).

### 3.2 The new higher spin current $\Phi^{(3)}_3$ of spin $\frac{7}{2}$

Let us consider the OPE between the higher spin-$\frac{5}{2}$ current $\Phi^{(1)}_x(z)$ of the first $\mathcal{N} = 4$ multiplet with $SO(4)$ vector index and the higher spin-2 current $\Phi^{(2)}_x(w)$ of the second $\mathcal{N} = 4$ multiplet with $SO(4)$ singlet, which is common to the fundamental 16 OPEs. It turns out that the OPE between them satisfies the following result

\[
\Phi^{(1)}_x(z) \Phi^{(2)}_x(w) = \frac{1}{(z-w)^3} Q^{(2)}_x(w) + \frac{1}{(z-w)^2} \left[ \frac{2}{3} \partial Q^{(2)}_x + Q^{(2)}_x - \frac{(k-N)}{3(2+k+N)} Q^{(2)}_x \right] (w) + \cdots.
\]

(3.10)

The third and second order poles can be written in terms of the known (higher spin) currents. This implies that there is no new primary field in these poles. On the other hand, the first order pole of (3.10) has the following quasi primary field

\[
Q^{(2)}_x = w_{1, x} \Phi^{(3)}_x + \cdots.
\]

(3.11)

A new higher spin-$\frac{7}{2}$ current of the third $\mathcal{N} = 4$ higher spin multiplet arises in (3.11) and the remaining 86 terms are given in (B.1) as before. We can easily check that the OPE between the spin-$\frac{3}{2}$ currents $G^i(z)$ and the higher spin-3 current $\Phi^{(3)}_0(w)$ of the third $\mathcal{N} = 4$ multiplet appearing in (3.8) provides the above higher spin-$\frac{7}{2}$ current $\Phi^{(3)}_x(w)$ at the first order pole (with minus sign). That is, the relevant OPE for this computation is given in Appendix C of [9]. See also Appendix G where the corresponding commutator is presented. Alternatively,
we can read off the corresponding OPEs. More precisely, we have the third equation of (G.2). This is one of the consistency checks for the validity of the OPE in (3.10).

In this case, the OPE described in (3.10) contains the components of the first three \( N = 4 \) higher spin multiplets as before. We expect that due to the \( SO(4) \) vector index \( i \) in (3.11), the \( N = 4 \) supersymmetric version of (3.11) can combine with the triple product of the fermionic coordinates in order to preserve the \( SO(4) \) singlet condition of the OPE. See (3.19).

3.3 Other remaining fundamental eleven OPEs

Let us calculate further OPEs. The OPE between the higher spin-2 current \( \Phi^{(1),ij}_1(z) \) of the first \( N = 4 \) multiplet with \( SO(4) \) adjoint index and the higher spin-2 current \( \Phi^{(2)}_0(w) \) of the second \( N = 4 \) multiplet with \( SO(4) \) singlet we described before can be described as

\[
\Phi^{(1),ij}_1(z) \Phi^{(2)}_0(w) = \frac{1}{(z-w)^2} Q^{(2),ij}_1(w) + \frac{1}{(z-w)} \left[ \frac{1}{2} \partial Q^{(2),ij}_1 + Q^{(3),ij}_1 \right](w) + \cdots \tag{3.12}
\]

where we have the following quasi primary field with the footnote 10

\[
Q^{(2),ij}_1(w) = w_{1,2} \Phi^{(1),ij}_1(w) + w_{6,2} \tilde{\Phi}^{(1),ij}_1(w) + \cdots \tag{3.13}
\]

The abbreviated six terms of (3.13) are given again in (B.1). There is no new primary field in this OPE (3.12). Of course, it is obvious that the other components \( \Phi^{(3),ij}_1(w) \) of the third \( N = 4 \) higher spin multiplet cannot appear in this OPE because the spins of the left hand side are not enough to allow us to have this higher spin-4 currents. We will see that the new higher spin-4 currents \( \Phi^{(3),ij}_1 \) can appear in the OPEs between the different higher spin currents. It is obvious to see that the composite field of spin-3 can arise in the first order pole of (3.12).

Let us consider the next OPE. The OPE between the higher spin-\( \frac{3}{2} \) current \( \Phi^{(1),i}_\frac{3}{2}(z) \) of the first \( N = 4 \) multiplet with \( SO(4) \) vector index and the higher spin-2 current \( \Phi^{(2)}_0(w) \) of the second \( N = 4 \) multiplet with \( SO(4) \) singlet can be summarized as follows:

\[
\Phi^{(1),i}_\frac{3}{2}(z) \Phi^{(2)}_0(w) = \frac{1}{(z-w)} Q^{(\frac{5}{2}),i}_\frac{3}{2}(w) + \cdots \tag{3.14}
\]

In this case, the first order pole can be expressed as the known (higher spin) currents (no new primary field occurs) and the quasi primary field in (3.14) contains the higher spin-\( \frac{5}{2} \) current of the first \( N = 4 \) higher spin multiplet as follows:

\[
Q^{(\frac{5}{2}),i}_{\frac{3}{2}}(w) = w_{1,\frac{5}{2}} \Phi^{(1),i}(w) + \cdots \tag{3.15}
\]

We can find other 13 terms of (3.15) in (B.1) as before.
Finally, the OPE between the lowest higher spin currents of the first and second $\mathcal{N} = 4$ higher spin multiplets can be written in terms of

$$\Phi_0^{(1)}(z) \Phi_0^{(2)}(w) = \frac{1}{(z-w)^2} Q_0^{(1)}(w) + \cdots,$$  

(3.16)

where the quasi primary field appears in (3.16) and is nothing but the higher spin-1 current as follows:

$$Q_0^{(1)}(w) = d_1^{0,2} \Phi_0^{(1)}(w),$$  

(3.17)

where the structure constant appearing in (3.17) is given by (3.7).

Then we observe that the OPEs in (3.12), (3.14) and (3.16) contain the components of the first $\mathcal{N} = 4$ higher spin multiplet. No new primary fields occur.

Therefore, the 16 fundamental OPEs, which are the OPEs between the first $\mathcal{N} = 4$ higher spin multiplet and the lowest component of the second $\mathcal{N} = 4$ higher spin multiplet, can be obtained from (3.6), (3.10), (3.12), (3.14) and (3.16). The total number of poles in these OPEs is given by 11. The $\mathcal{N} = 4$ supersymmetry allows us to read off the remaining $240 = (16^2 - 16)$ OPEs by going to $\mathcal{N} = 4$ superspace in next subsection.

### 3.4 The $\mathcal{N} = 4$ OPE between the first and the second $\mathcal{N} = 4$ higher spin multiplets

We would like to express the OPEs in component approach by using the $\mathcal{N} = 4$ superspace approach. Again the fundamental 16 OPEs (five kinds of OPEs between the first $\mathcal{N} = 4$ higher spin multiplet and the lowest component of the second $\mathcal{N} = 4$ higher spin multiplet) are given by (3.6), (3.10), (3.12), (3.14) and (3.16) and will determine the remaining 240 OPEs by using the $\mathcal{N} = 4$ supersymmetry described before. That is, we can generalize them in $\mathcal{N} = 4$ superspace by taking the following replacements [9] for the components of both $\mathcal{N} = 4$ stress energy tensor $J$ in the footnote [10] and the higher spin multiplet $\Phi^{(s)}$ in (3.11)

$$U(w) \rightarrow \partial J(Z_2), \quad \Gamma^i(w) \rightarrow -i D^i J(Z_2) \equiv -i J^i(Z_2),$$
$$T^{ij}(w) \rightarrow -\frac{i}{2!} \varepsilon^{ijkl} D^k D^l J(Z_2) \equiv -\frac{i}{2!} \varepsilon^{ijkl} J^{kl}(Z_2),$$
$$G^i(w) \rightarrow \frac{1}{3!} \varepsilon^{ijkl} D^k D^l D^i J(Z_2) \equiv \frac{1}{3!} \varepsilon^{ijkl} J^{ijkl}(Z_2),$$
$$L(w) \rightarrow \frac{1}{2 \cdot 4!} \varepsilon^{ijkl} D^i D^j D^k D^l J(Z_2) \equiv \frac{1}{2 \cdot 4!} \varepsilon^{ijkl} J^{ijkl}(Z_2),$$

\[ ^{15}\text{Compared to the total 136 OPEs in section 2.3, there are 256 = 16}^2 \text{ OPEs because we are considering two different $\mathcal{N} = 4$ higher spin multiplets.} \]
we can multiply the relevant fermionic coordinates together with singular terms. Due to the different number of fermionic coordinates, the total 11 singular terms in the above fundamental OPEs arise in \( \mathcal{N} = 4 \) superspace independently.

Then the single \( \mathcal{N} = 4 \) super OPE between the first and the second \( \mathcal{N} = 4 \) higher spin multiplets, like as in (2.9), can be summarized by

\[
\Phi^{(1)}(Z_1) \Phi^{(2)}(Z_2) = \frac{\theta_{12}^{0}}{z_{12}} \left[ Q_{2}^{(1)} - (6 p_1 + 4 p_2) Q_{0}^{(1)} \right] (Z_2) + \frac{\theta_{12}^{1}}{z_{12}} Q_{2}^{(1),i} (Z_2)
\]

and putting the relevant fermionic coordinates together with singular terms. Due to the different number of fermionic coordinates, the total 11 singular terms in the above fundamental OPEs arise in \( \mathcal{N} = 4 \) superspace independently.

As before, the coefficients \( p_1 \) and \( p_2 \) appearing in several places are given in (3.3) and the structure constant \( d_{ij}^{0,2} \) is given in (3.7). We present the quasi primary super fields in \( \mathcal{N} = 4 \) superspace in (3.19). The new primary third \( \mathcal{N} = 4 \) higher spin multiplet in (3.19) arise in the following quasi primary field

\[
Q_{2}^{(3)}(Z_2) = d_{1}^{0,2} \Phi^{(3)}(Z_2) + \cdots,
\]

As before, we simply put the fermionic coordinates to zero. For the second component with index \( i \), we multiply the super derivative \( D^i \) and then take the fermionic coordinates as zero. For the third component with \( i = 1, j = 2 \), we act the super derivatives \( D^1 D^1 D^4 \) and take \( \theta_{12}^{0} = \theta_{12}^{1} \) with minus sign. For the fourth component with \( i = 1 \), we can multiply \( D^2 D^3 D^4 \) and take \( \theta_{12}^{1} = 0 = \theta_{12}^{2} \) with minus sign. For the last component, we can multiply \( D^1 D^1 D^3 D^4 \) and take \( \theta_{12}^{1} = 0 = \theta_{12}^{2} \). For the other indices of the third and fourth components, similar analysis can be done.
where the abbreviated 25 terms are given in (B.2). This is $\mathcal{N} = 4$ supersymmetric version of previous component result in (3.8). Moreover, we have the descendant field of $\Phi^{(3)}(Z_2)$

$$Q^{(7),i}_{\frac{3}{2}}(Z_2) = d^{\frac{3}{2},1}_{I} D^{i} \Phi^{(3)}(Z_2) + \cdots,$$

(3.21)

which is the $\mathcal{N} = 4$ supersymmetric version of the component result in (3.11).

Therefore, the OPE in (3.19) contains the first three $\mathcal{N} = 4$ higher spin multiplets in addition to the stress energy tensor $J$: $\Phi^{(1)}$, $\Phi^{(2)}$ and $\Phi^{(3)}$. Note that the structure constants appearing in (3.19) are fixed by using the various Jacobi identities in the component approach. As explained before, all the 256 OPEs can be read off from the OPE (3.19) with the help of the footnote 16. We will describe some of them very briefly in next subsection.

3.5 The new higher spin currents $\Phi^{(3),lm}_{\frac{3}{2}}$, $\Phi^{(3),l}_{\frac{3}{2}}$ and $\Phi^{(3)}_2$, of spin $4, \frac{9}{2}, 5$

So far, we have discussed about the 16 fundamental OPEs. How do we observe (or obtain) the remaining components of the third $\mathcal{N} = 4$ higher spin multiplet? Because the third $\mathcal{N} = 4$ higher spin multiplet appears both in (3.20) and in (3.21), we can focus on these poles associated with them in the OPE (3.19).

For example, we can observe that the higher spin-4 currents $\Phi^{(3),mn}_{1}(w)$ arise the OPEs between the higher spin-2 currents $\Phi^{(1),ij}_{1}(z)$ and the higher spin-3 currents $\Phi^{(2),kl}_{1}(w)$. Let us fix the indices: $m = 1, n = 2$ and $i = k = 3, j = 1, l = 2$. The former higher spin currents can be obtained by multiplying the super derivatives $D^2_1 D^4_1$ (up to the overall numerical factor) into $\Phi^{(1)}(Z_1)$ and putting the fermionic coordinates to zero according to (3.18). On the other hand, the latter higher spin currents can be obtained by multiplying the super derivatives $D^2_1 D^4_1$ (up to the overall numerical factor) into $\Phi^{(2)}(Z_2)$ and putting the fermionic coordinates to zero.

Then it is easy to see, from the right hand side of the OPE in (3.19), that by splitting the two super derivatives $D^2_1 D^4_1$ into the piece of $\frac{g^{12}}{z_{12}}$ and the quasi primary field (3.21) respectively, we have $D^2_1 D^4_1$ and $D^2_2 D^4_2$ and we arrive at $\Phi^{(3),12}_{1}(w)$ after putting the vanishing fermionic coordinates, where we ignore all the numerical factors as well as signs. We can see the corresponding OPE in (E.1), where all the nonlinear terms (and some linear terms) disappear.

Similarly, the higher spin-$\frac{9}{2}$ currents $\Phi^{(3),l=1}_{\frac{3}{2}}(w)$ can be determined from the OPEs between the higher spin-2 currents $\Phi^{(1),ij}_{\frac{3}{2}}(z)$ and the higher spin-3 currents $\Phi^{(2),k}_{\frac{3}{2}}(w)$, where we fix the indices $i = k = 4$ and $j = 1$. As before, the former can be obtained by multiplying the super

\footnote{In this subsection, we do not care about the exact numerical factors and signs. We will demonstrate how we can observe the existence of the remaining components of the third $\mathcal{N} = 4$ higher spin multiplet we do not see so far. We refer to [9] for further detailed descriptions.}
derivatives $D_1^2 D_1^3$ into $\Phi^{(1)}(Z_1)$ and putting the fermionic coordinates to zero. The latter can be obtained by multiplying the super derivatives $D_2^1 D_2^2 D_2^3$ into $\Phi^{(2)}(Z_2)$ and putting the fermionic coordinates to zero. After we act the super derivatives $D_1^2 D_1^3 D_1^1$ on the pole $\frac{\delta_{12}^{t-n}}{z_{12}^{t-n}}$ and the super derivatives $D_2^2 D_2^3$ on the quasi primary field (3.21), we are left with $\Phi^{(3)}_{22}(w)$ (the index $n$ becomes 4) at the first order pole. The corresponding OPE can be found in (E.1) as before 18.

Of course, these new higher spin-4, $\frac{9}{2}, 5$ currents appear in other OPEs in (E.1). All the component results can be obtained from (3.19) by applying the super derivatives to both sides and putting the fermionic coordinates to zero. We present the 256 OPEs under the large ($N, k$) limit in (E.1). The ($N, k$) dependent structure constants will be attached in the ancillary.nb file. Its (anti)commutators appear in (G.4). When we compare the results [54] in an orthogonal coset model with the results of this paper, so far we do not observe any $SO(4)$ non-singlet $N = 4$ higher spin multiplet.

4 Operator product expansion between the second $N = 4$ multiplet and itself with $N = 5$

Since we do not complete this OPE for general $N$, we will present some parts of the OPE with fixed $N = 5$.

4.1 The new higher spin current of spin 4

Compared to $N = 3$ case where there is no new higher spin-4 primary current, the $N = 5$ case leads to the following new higher spin-4 current with the corresponding OPE

$$\Phi^{(2)}_2(z) \Phi^{(2)}_0(w) = \frac{1}{(z-w)^6} 8 \alpha e_0^{0,4} + \frac{1}{(z-w)^5} Q^{(1)}_2(w) + \frac{1}{(z-w)^4} \left[ \frac{3}{2} \partial Q^{(1)}_2 + Q^{(2)}_2 \right](w) + \frac{1}{(z-w)^3} \left[ \partial^2 Q^{(1)}_2 + \partial Q^{(2)}_2 + Q^{(3)}_2 \right](w) + \frac{1}{(z-w)^2} \left[ \frac{5}{12} \partial^3 Q^{(1)}_2 + \frac{1}{2} \partial^2 Q^{(2)}_2 + \frac{5}{6} \partial Q^{(3)}_2 + Q^{(4)}_2 \right](w).$$

18 Finally, the higher spin-5 current $\Phi^{(3)}_2(w)$ can be obtained from the OPEs between the higher spin-$\frac{5}{2}$ currents $\Phi^{(1),i}_{\frac{5}{2}}(z)$ and the higher spin-$\frac{7}{2}$ currents $\Phi^{(2),j}_{\frac{7}{2}}(w)$. Let us fix the index as $i = j = 1$. The former can be obtained by multiplying the super derivatives $D_1^2 D_1^3 D_1^1$ into $\Phi^{(1)}(Z_1)$ and putting the fermionic coordinates to zero. The latter can be obtained by multiplying the super derivatives $D_2^2 D_2^3 D_2^1$ into $\Phi^{(2)}(Z_2)$ and putting the fermionic coordinates to zero. After we act the super derivatives $D_1^2 D_1^3 D_1^1$ on $\frac{\delta_{12}^{t-n}}{z_{12}^{t-n}}$ and the super derivatives $D_2^2 D_2^3 D_2^1$ on the quasi primary field (3.21), where the index $n$ becomes 1, we are left with $\Phi^{(3)}_2(w)$ at the first order pole. We can find the corresponding OPE in (E.1).
\[ + \frac{1}{(z - w)} \left[ \frac{1}{8} \partial^4 Q_2^{(1)} + \frac{1}{6} \partial^3 Q_2^{(2)} + \frac{5}{14} \partial^2 Q_2^{(3)} + \frac{3}{4} \partial Q_2^{(4)} + Q_2^{(5)} \right](w) \]

\[ + p_1 \sum_{n=3}^4 \frac{1}{(z - w)^n} \left\{ \partial^2 \Phi_0^{(2)} \Phi_0^{(2)} \right\}_n(w) + p_2 \sum_{m=1}^4 \frac{1}{(z - w)^m} \left\{ (L \Phi_0^{(2)}) \Phi_0^{(2)} \right\}_m(w) \]

\[ + \cdots, \quad (4.1) \]

where \( p_1 \) and \( p_2 \) are in [9] and by substituting the value of \( s \) they are given by

\[ p_1 = \frac{(k - N)(55 + 29k + 29N + 3kN)}{(2 + k + N)(59 + 37k + 37N + 15kN)}, \quad p_2 = \frac{36(k - N)}{(59 + 37k + 37N + 15kN)}. \quad (4.2) \]

In particular, the new higher spin-4 current which is the lowest component of the fourth \( \mathcal{N} = 4 \) higher spin multiplet arises in the quasi primary field appearing in (4.1)

\[ Q_2^{(4)} = w_{0,4} \Phi_0^{(4)} + \cdots, \quad (4.3) \]

where the abbreviated parts of (4.3) are given in (C.5) with (C.6). The equivalent expression of (4.1) is found in (C.1). Other quasi primary fields can be found in Appendix C. Compared to the one in [54] where the orthogonal coset model is described, the OPE in (4.1) looks similar (in the context of the quasi primary fields and their coefficients in the OPE) but the first order pole in \( p_2 \) term is new in this paper.

### 4.2 Other OPEs

The remaining four kinds of fundamental OPEs, by following the method in previous section, are presented in Appendix C. In order to apply the Jacobi identities used in previous sections, we need to calculate the OPE between \( \Phi^{(1)}(Z_1) \) and \( \Phi^{(3)}(Z_2) \) which is beyond the scope of this paper. This is because the Jacobi identity between the three higher spin currents, \( \Phi^{(1)}, \Phi^{(2)} \) and \( \Phi^{(1)} \) leads to the above OPE because the OPE between \( \Phi^{(1)}(Z_1) \) and \( \Phi^{(2)}(Z_2) \) produces \( \Phi^{(3)}(Z_2) \). As long as the \( \mathcal{N} = 4 \) higher spin algebra is concerned, we will obtain the higher spin algebra associated with the OPE \( \Phi^{(2)}(Z_1) \) and \( \Phi^{(2)}(Z_2) \) and the OPE \( \Phi^{(1)}(Z_1) \) and \( \Phi^{(3)}(Z_2) \) in section 7.

### 5 Summary of coset construction

According to the observation of [1], the appropriate limit on the parameters \( N \) and \( k \) should be taken from the OPEs we have obtained so far in order to relate them to the classical asymptotic symmetry algebra in the \( AdS_3 \) higher spin theory with matrix generalization. If we further restrict the modes of the higher spin operators in the (nonlinear) classical asymptotic
symmetry algebra to the wedge modes, then we obtain the $\mathcal{N} = 4$ higher spin algebra which is called “wedge” subalgebra of $\mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda]$ algebra.

5.1 The large $(N, k)$ ’t Hooft-like limit and the nonlinear $\mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda]$ algebra

From the explicit OPEs found in previous sections 2, 3 (and 4), we can take the large $(N, k)$ ’t Hooft limit into the various structure constants by keeping the ’t Hooft-like coupling constant $\lambda \equiv \frac{(N+1)}{(k+N+2)}$ fixed. Before we are taking the infinity limit of $N$ after writing down the $k$ in terms of $\lambda$ and $N$ first, we are left with the nonlinear (and linear) terms together with the power of $\frac{1}{N}$ factor. Now we take the infinity limit of $N$, then all the nonlinear and some of the linear terms vanish. We present them in Appendices $D$ and $E$ and its (anti)commutators version will appear in Appendix $G$. From these results of Appendices, we restrict to use the wedge condition for the modes described before. Then all the expressions with typewriter font in Appendix $G$ disappear. Later, we will use these (anti)commutators to obtain the $\mathcal{N} = 4$ higher spin generators in terms of oscillators which will satisfy the above $\mathcal{N} = 4$ higher spin algebra.

There exists other limiting procedure which is the infinity limit of level $k$. In this case, the central charge is given by $c = 6(N + 1)$ around the discussion of (2.8). We obtain the free field construction [10] from the coset construction we have described so far. We will obtain the free field construction in section 6, where the central charge $c = 6N$ which is different from the above. Then we can expect that the extra piece in the coset construction should reflect the extra central charge $c = 6$ which is the difference between above two central charges.

5.2 The $\mathcal{N} = 2$ wedge subalgebra of $\mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda]$ algebra

Before we take the $U(2)$ matrix generalization of the AdS$_3$ Vasiliev higher spin theory, the original AdS$_3$ higher spin theory has $\mathcal{N} = 2$ supersymmetry. This implies that we should observe the corresponding $\mathcal{N} = 2$ higher spin algebra from the $\mathcal{N} = 4$ higher spin algebra, which obtained from the original higher spin theory by adding Chan-Paton factors (or $U(2)$ matrix generalization). In terms of oscillator formalism with explicit four $U(2)$ matrices, it will be obvious that we can choose the right candidate for the $\mathcal{N} = 2$ higher spin generators, among the $\mathcal{N} = 4$ higher spin generators, which will satisfy the above $\mathcal{N} = 2$ higher spin algebra.

We expect that by taking the particular combinations among the currents and the higher
spin currents,

\[ \Phi_0^{(1)} \rightarrow J, \quad \frac{1}{\sqrt{2}} \left( G^2 + \Phi_0^{(1,2)} \right) \rightarrow G^+, \]

\[ \frac{1}{\sqrt{2}} \left( G^2 - \Phi_0^{(1,2)} \right) \rightarrow G^-, \quad L \rightarrow L, \quad (5.1) \]

then the \( \mathcal{N} = 2 \) wedge subalgebra of \( \mathcal{W}_\infty^{N=4} [\lambda] \) algebra can be realized. The \( \text{SO}(4) \) index 2 in the above (5.1) is one of the choices among four values \( i = 1, 2, 3, 4 \). Once we take the spin-1 current \( J \) of \( \mathcal{N} = 2 \) superconformal algebra as the higher spin-1 current \( \Phi_0^{(1)} \), then the two spin-\( \frac{3}{2} \) currents \( G^\pm \) of \( \mathcal{N} = 2 \) superconformal algebra should be the linear combination like as (5.1) because the commutator between the higher spin-1 current \( \Phi_0^{(1)} \) and the spin-\( \frac{3}{2} \) currents \( G^i \) gives the higher spin-\( \frac{3}{2} \) currents \( \Phi_0^{(1,i)} \) and vice versa. See also (G.2). We can easily check that other (anti)commutators are satisfied by using the left hand side of (5.1). See also (L.1).

Moreover, the first \( \mathcal{N} = 2 \) higher spin multiplet in the context of \( \mathcal{N} = 2 \) wedge subalgebra of \( \mathcal{W}_\infty^{N=4} [\lambda] \) algebra can be obtained from the following combinations

\[ -\frac{\sqrt{3}}{8} \Phi_0^{(2)} \rightarrow W^{20}, \quad \frac{\sqrt{3}}{4\sqrt{2}} \left( \Phi_0^{(1,2)} - \frac{1}{2} \Phi_0^{(2,2)} \right) \rightarrow W^2+, \]

\[ \frac{\sqrt{3}}{4\sqrt{2}} \left( \Phi_0^{(1,2)} + \frac{1}{2} \Phi_0^{(2,2)} \right) \rightarrow W^2-, \quad -\frac{\sqrt{3}}{8} \Phi_0^{(1)} \rightarrow W^{21}. \quad (5.2) \]

Here we are using the notation of [55]. Note that the fermionic components of first and the second of \( \mathcal{N} = 4 \) higher spin multiplets are mixed. Similarly, the second \( \mathcal{N} = 2 \) higher spin multiplet can be obtained from the following combinations

\[ -\frac{1}{64} \Phi_0^{(3)} \rightarrow c_{22}^3 W^{30}, \quad \frac{1}{64\sqrt{2}} \left( 3 \Phi_0^{(2,2)} - \Phi_0^{(3,2)} \right) \rightarrow c_{22}^3 W^{3+}, \]

\[ \frac{1}{64\sqrt{2}} \left( 3 \Phi_0^{(2,2)} + \frac{1}{2} \Phi_0^{(3,2)} \right) \rightarrow c_{22}^3 W^{3-}, \quad -\frac{3}{64} \Phi_0^{(2)} \rightarrow c_{22}^3 W^{31}, \quad (5.3) \]

where \( c_{22}^3 \) is the structure constant appearing in the higher spin-3 current in the OPE between the higher spin-2 current and itself. We will come back to this issue together with Appendix L in section 7.

Then from these observations, we can further generalize to the next \( \mathcal{N} = 2 \) higher spin multiplet from the \( \mathcal{N} = 4 \) higher spin multiplets as follows:

\[ (s = 0, \quad \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, 1, 1, 3, 3, 2 : C^2, \frac{3}{2}, \frac{3}{2}, 2 : L), \]

\[ (s = 1 : \Phi_0^{(1)}, \quad \frac{3}{2}, \frac{3}{2}, \Phi_0^{(1,2)}, \frac{3}{2}, \frac{3}{2}, 2, 2, 2, 2, 2, 2, 2, \quad \frac{5}{2}, \frac{5}{2}, \Phi_0^{(1,2)}, \frac{5}{2}, \frac{5}{2}, 3 : \Phi_0^{(1)}), \]

\[ (s = 2 : \Phi_0^{(2)}, \quad \frac{5}{2}, \frac{5}{2}, \Phi_0^{(2,2)}, \frac{5}{2}, \frac{5}{2}, 3, 3, 3, 3, 3, \quad \frac{7}{2}, \frac{7}{2}, \Phi_0^{(2,2)}, \frac{7}{2}, \frac{7}{2}, 4 : \Phi_0^{(2)}), \]

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the vanishing 't Hooft coupling constant. After identifying some of the 16 currents of the multiplets are written in terms of free fields. Then the (anti)commutators for general spins as in the section 1, there are 2\(N\) free fields of spin \(s\) and \(\bar{s}\), instead of using \(\Phi^{(s)}(3)\), \(\Phi^{(s)}(4)\) symmetry are obtained explicitly from two different \(\mathcal{N} = 4\) higher spin multiplets of superspin \(s\) and \((s - 1)\) in (5.4). The first two, \(\Phi_0^{(a)}\) and \(\Phi_0^{(s).2}\) can be taken from the former, while the last two \(\Phi^{(s-1).2}_2\) and \(\Phi^{(a-1)}_2\) can be obtained from the latter. As observed before, the choice of \(SO(4)\) vector index in the fermionic field is arbitrary. In our convention, we take the index as 2. In section 7, we will see that the corresponding fermionic quantity has a \(2 \times 2\) identity matrix.

6 Free field realization with vanishing 't Hooft-like coupling constant

In this section, we describe the free field realization associated with the \(\mathcal{W}_{\mathcal{N}=4}^\infty[\lambda]\) algebra at the vanishing 't Hooft coupling constant. After identifying some of the 16 currents of the large \(\mathcal{N} = 4\) linear superconformal algebra in terms of free fields, the first and second \(\mathcal{N} = 4\) multiplets are written in terms of free fields. Then the (anti)commutators for general spins \(s_1, s_2\) with manifest \(SO(4)\) symmetry are obtained explicitly.

6.1 Some of the 16 currents in terms of free fields

As in the section 1, there are \(2N\)-free complex bosons transforming as bifundamental (\(\tilde{N}, 2\)) + (\(N, \bar{2}\)) of \(U(N) \times U(2)\) and \(2N\)-free complex fermions transforming similarly under the \(U(N) \times U(2)\). We follow the notations used in [10] and [18]. The complex bosons are denoted by \(\partial \phi^{i, \alpha}\) and \(\partial \bar{\phi}^{i, \bar{a}}\) where \(i, \bar{i} = 1, 2, \cdots, N\) and \(a, \bar{a} = 1, 2\). The complex fermions are denoted by \(\psi^{i, \alpha}\) and \(\bar{\psi}^{i, \bar{a}}\) where \(i, \bar{i} = 1, 2, \cdots, N\) and \(\alpha, \bar{\alpha} = 1, 2\). By constructing the bilinear terms between these free fields, we can obtain the following quasi (eight bosonic and eight fermionic) primary fields of spin \(h\) as follows:

\[
W_{F,h}^{\alpha \beta}(z) = n_{W_{F,h}} \sum_{k=0}^{h-1} \sum_{i=1}^{N} \delta_{i,i} (-1)^k \left( \frac{h-1}{k} \right)^2 (\partial^{h-k-1} \bar{\psi}^{i, \alpha} \partial^{k+1} \psi^{i, \beta})(z),
\]

\[
W_{B,h}^{\bar{a} \bar{b}}(z) = n_{W_{B,h}} \sum_{k=0}^{h-2} \sum_{i=1}^{N} \delta_{i,i} \left( \frac{h-1}{k+1} \right) \left( \frac{h-1}{k} \right) (\partial^{h-k-1} \bar{\phi}^{i, \bar{a}} \partial^{k+1} \phi^{i, \bar{b}})(z),
\]

\(^{19}\text{In this section we are using the notations for the spins } h_1, h_2 \text{ and } h, \text{ instead of using } s_1, s_2, \text{ and } s.\)
Each field has four components, 11, 12, 21, 22. As described in the section 1, there are only four spin-1 currents and there are eight currents of spins $s = \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$. The overall normalizations are taken from the ones in [24] 20.

$$n_{W,F,A} = \frac{2^{h-3}(h-1)!}{(2h-3)!!} q^{h-2}, n_{W,F,B} = \frac{2^{h-3}h!}{(2h-3)!!} q^{h-2}, n_{Q_h} = n_{Q_B} = \frac{2^{h-1}(h-1)!}{(2h-2)!!} q^{h-\frac{3}{2}}. \quad (6.2)$$

The binomial coefficients used in (6.1) are denoted by $\binom{n}{k} \equiv \frac{n!}{k!(n-k)!}$. The OPEs between the free fields are given by

$$\partial \bar{\phi}^{i\alpha}(z) \partial \phi^{j\beta}(w) = \frac{1}{(z-w)^2} \delta^{ij} \delta^{\alpha\beta} + \cdots, \quad \bar{\psi}^{i\bar{\alpha}}(z) \psi^{j\beta}(w) = \frac{1}{(z-w)} \delta^{ij} \delta^{\bar{\alpha}\beta} + \cdots. \quad (6.3)$$

The indices of (anti)fundamental representations of $U(N)$ in (6.1) are summed.

The stress energy tensor of spin 2 is given by the usual Sugawara construction from (6.1) and (6.2) as follows:

$$L = W_{B,2}^{11} + W_{B,2}^{22} + W_{F,2}^{11} + W_{F,2}^{22}. \quad (6.4)$$

The central charge is

$$c = N(2K + L) \bigg|_{K=L=2} = 6N. \quad (6.5)$$

This can be seen from the discussion of (2.8).

We can think of the four spin-$\frac{3}{2}$ currents of the large $\mathcal{N} = 4$ superconformal algebra as the linear combinations of $Q^a_{\frac{3}{2}}$ and $Q^a_{\frac{5}{2}}$. By using the fact that they are primary fields under the stress energy tensor (6.4) (corresponding to the second relation of (6.1)) and the defining equations for the OPEs between the spin-$\frac{3}{2}$ currents (corresponding to the fourth relation of (6.1)) which contain the stress energy tensor in the first order pole, we obtain the following four spin-$\frac{3}{2}$ currents of the large $\mathcal{N} = 4$ linear superconformal algebra

$$G^1 = -\frac{1}{2} (Q_{\frac{3}{2}}^{11} + i\sqrt{2} Q_{\frac{3}{2}}^{12} + 2i\sqrt{2} Q_{\frac{3}{2}}^{21} - 2 Q_{\frac{3}{2}}^{22} - 2 \bar{Q}_{\frac{3}{2}}^{11} - 2i\sqrt{2} Q_{\frac{3}{2}}^{12} - i\sqrt{2} Q_{\frac{3}{2}}^{21} + \bar{Q}_{\frac{3}{2}}^{22}),$$

---

20The $q$ can be arbitrary value.
\[ G^2 = \frac{i}{2} (Q_{\frac{3}{2}}^{11} + 2i\sqrt{2}Q_{\frac{3}{2}}^{21} - 2Q_{\frac{3}{2}}^{22} - 2Q_{\frac{3}{2}}^{11} - 2i\sqrt{2}Q_{\frac{3}{2}}^{12} + Q_{\frac{3}{2}}^{22}), \]
\[ G^3 = \frac{i}{2} (Q_{\frac{3}{2}}^{11} + i\sqrt{2}Q_{\frac{3}{2}}^{12} - 2Q_{\frac{3}{2}}^{22} - 2Q_{\frac{3}{2}}^{11} - i\sqrt{2}Q_{\frac{3}{2}}^{21} + Q_{\frac{3}{2}}^{22}), \]
\[ G^4 = \frac{1}{2} Q_{\frac{3}{2}}^{11} + Q_{\frac{3}{2}}^{22} + Q_{\frac{3}{2}}^{11} + \frac{1}{2} Q_{\frac{3}{2}}^{22}. \] (6.6)

Of course, there are 4! different choices for SO(4) vector indices. One of them is given by (6.6).

The six spin-1 currents of the large \( \mathcal{N} = 4 \) superconformal algebra appear as the linear combinations of \( W_{\bar{\alpha}\beta}^{\alpha} F_{\frac{1}{2}} \). We can obtain the following spin-1 currents from the previous OPEs between the spin-\( \frac{3}{2} \) currents by focusing on the second order pole

\[ \frac{1}{4q} (T^{14} + T^{23}) = i W_{F,1}^{11} - \sqrt{2} W_{F,1}^{12} - \sqrt{2} W_{F,1}^{21} - i W_{F,1}^{22}, \]
\[ \frac{1}{4q} (T^{13} + T^{42}) = -W_{F,1}^{11} - i \sqrt{2} W_{F,1}^{21} + W_{F,1}^{22}, \]
\[ \frac{1}{4q} (T^{12} + T^{34}) = W_{F,1}^{11} + i \sqrt{2} W_{F,1}^{12} - W_{F,1}^{22}. \] (6.7)

We can check that the remaining relations (the third, fifth and sixth in (G.1)) are satisfied. Note that there are only four independent spin-1 currents in the free field construction as mentioned before. Three of them are given in (6.7). We will observe the remaining one in next subsection soon.

### 6.2 The first \( \mathcal{N} = 4 \) higher spin multiplet in terms of free fields

We consider the first \( \mathcal{N} = 4 \) higher spin multiplet in terms of the free fields. We can construct the following higher spin-1 current by summing over the indices \[ \Phi_0^{(1)} = 4q \left( W_{F,1}^{11} + W_{F,1}^{22} \right). \] (6.8)

The overall factor can be determined by the commutator between this higher spin-1 current and the higher spin-\( \frac{3}{2} \) current, which leads to the spin-\( \frac{3}{2} \) current. See the second relation of (G.3). Then the four components of \( W_{\bar{\alpha}\beta}^{\alpha} F_{\frac{1}{2}} \) have the relations in (6.7) and (6.8).

Because we have seen the four spin-\( \frac{3}{2} \) currents from (6.6), the remaining four spin-\( \frac{3}{2} \) higher spin currents can be obtained by using the OPE between the spin-\( \frac{3}{2} \) currents (6.6) and the

\[ [\Phi_0^{(s)}] = q^{-s+2} \left( (s-1) W_{B,s}^{\alpha\alpha} - s W_{B,s}^{\alpha\alpha} \right) \] for general spin \( s \) where we added the \( q \) dependent rescaled overall factor and the first term which vanishes for \( s = 1 \).
higher spin-1 current in (6.8) (or the third relation of (G.2)) as follows \(^22\).

\[
\begin{align*}
\Phi^{(1),1}_{\frac{s}{2}} &= \frac{1}{2} \left( Q_{\frac{s}{2}}^1 + i\sqrt{2} Q_{\frac{s}{2}}^2 + 2i\sqrt{2} Q_{\frac{s}{2}}^3 - 2 Q_{\frac{s}{2}}^4 + 2i\sqrt{2} Q_{\frac{s}{2}}^5 + i\sqrt{2} Q_{\frac{s}{2}}^6 - Q_{\frac{s}{2}}^7 \right), \\
\Phi^{(1),2}_{\frac{s}{2}} &= -\frac{i}{2} \left( Q_{\frac{s}{2}}^1 + 2i\sqrt{2} Q_{\frac{s}{2}}^2 - 2 Q_{\frac{s}{2}}^3 + 2i\sqrt{2} Q_{\frac{s}{2}}^4 - Q_{\frac{s}{2}}^5 \right), \\
\Phi^{(1),3}_{\frac{s}{2}} &= -\frac{i}{2} \left( Q_{\frac{s}{2}}^1 + i\sqrt{2} Q_{\frac{s}{2}}^2 - 2 Q_{\frac{s}{2}}^3 + 2i\sqrt{2} Q_{\frac{s}{2}}^4 - Q_{\frac{s}{2}}^5 \right), \\
\Phi^{(1),4}_{\frac{s}{2}} &= -\frac{1}{2} Q_{\frac{s}{2}}^1 - Q_{\frac{s}{2}}^2 + \Phi^{(1),5}_{\frac{s}{2}} + \frac{1}{2} Q_{\frac{s}{2}}^7. 
\end{align*}
\]  
(6.9)

The defining relations (6.3) are used in this computation. Compared to the spin-\(\frac{s}{2}\) currents in (6.6), they are almost the same except some different signs in front of fields. In other words, the four components of \(Q_{\frac{s}{2}}^{\alpha\beta}\) and the four components of \(Q_{\frac{s}{2}}^{\alpha\beta}\) have relations with eight (higher spin) currents via (6.6) and (6.9).

We can proceed to obtain the next higher spin currents similarly. The six spin-2 higher spin currents are determined by using the OPE between the spin-\(\frac{s}{2}\) currents (6.6) and the higher spin-\(\frac{s}{2}\) current in (6.9) (or the fourth relation of (G.2)) and it turns out that \(^{23}\)

\[
\begin{align*}
\Phi^{(1),12}_{\frac{s}{2}} &= 2i W_{B,2}^{11} - \sqrt{2} W_{B,2}^{12} - 2i W_{B,2}^{22} - 2i W_{F,2}^{11} - 2\sqrt{2} W_{F,2}^{12} - 2i W_{F,2}^{22}, \\
\Phi^{(1),13}_{\frac{s}{2}} &= -2i W_{B,2}^{11} + 4\sqrt{2} W_{B,2}^{21} + 2i W_{B,2}^{22} + 2i W_{F,2}^{11} + 2\sqrt{2} W_{F,2}^{21} + 2i W_{F,2}^{22}, \\
\Phi^{(1),14}_{\frac{s}{2}} &= 2i W_{B,2}^{11} + i\sqrt{2} W_{B,2}^{12} + 4i\sqrt{2} W_{B,2}^{21} - 2i W_{B,2}^{22} - 2i W_{F,2}^{11} - 2i\sqrt{2} W_{F,2}^{12} \\
&- 2i\sqrt{2} W_{F,2}^{21} + 2i W_{F,2}^{22}, \\
\Phi^{(1),23}_{\frac{s}{2}} &= -2i W_{B,2}^{11} - i\sqrt{2} W_{B,2}^{12} - 4i\sqrt{2} W_{B,2}^{21} + 2i W_{B,2}^{22} + 2i W_{F,2}^{11} + 2i\sqrt{2} W_{F,2}^{12} \\
&- 2i\sqrt{2} W_{F,2}^{21} + 2i W_{F,2}^{22}, \\
\Phi^{(1),24}_{\frac{s}{2}} &= -2i W_{B,2}^{11} + 4\sqrt{2} W_{B,2}^{21} + 2i W_{B,2}^{22} - 2i W_{F,2}^{11} - 2\sqrt{2} W_{F,2}^{21} - 2i W_{F,2}^{22}, \\
\Phi^{(1),34}_{\frac{s}{2}} &= -2i W_{B,2}^{11} + \sqrt{2} W_{B,2}^{12} + 2i W_{B,2}^{22} + 2i W_{F,2}^{11} - 2\sqrt{2} W_{F,2}^{12} - 2i W_{F,2}^{22}. 
\end{align*}
\]  
(6.10)

We have seen the spin-2 current from (6.4) and we will see the remaining higher spin-2 current in next subsection. Therefore, we have explicit relations of four components of \(W_{F,2}^{\alpha\beta}\) and four components of \(W_{B,2}^{\alpha\beta}\) with eight (higher spin) currents.

We can use the OPE between the spin-\(\frac{s}{2}\) currents (6.6) and the higher spin-2 currents in

\(^22\) In this case, we have the general \(s\) dependent expression by multiplying the factor \(-q^{-s+1} \frac{(2s-1)}{4}\) in (6.9) and changing the upper index (1) to (s) and the subscript \(\frac{1}{2}\) to \((s \pm \frac{1}{2})\). For example, we have \(\Phi^{(s),1}_{\frac{1}{2}} = -q^{-s+1} \frac{(2s-1)}{8} (Q_{s+1}^{11} + \cdots)\) where the abbreviated part is the same as the seven terms with the subscript \(s + \frac{1}{2}\) in the first relation of (6.9) and so on.

\(^23\) By multiplying the overall factor \(-q^{-s+1} \frac{(2s-1)}{4}\) in (6.10) and changing the upper index (1) to (s) and the subscript 2 to \(s + 1\), the general expression for the spin \(s\) with subscript 1 can be obtained.
6.3 The second $\mathcal{N} = 4$ higher spin multiplet in terms of free fields

We can also obtain the second $\mathcal{N} = 4$ higher spin multiplet which can be realized by the free fields.

For the higher spin-2 current $\Phi^{(2)}$, we can take the four terms appearing in (6.4), where the first two terms have the same coefficient and the last two terms have the same coefficient. We should use the primary condition and the defining relations in (G.4) which contain this higher spin-2 current with the known expressions in the right hand side. For example, there are the first, sixth and eleventh terms in (G.4). Then we obtain the higher spin-2 current

$$
\tilde{\Phi}_{2}^{(1)} = -\frac{2}{q} (W_{\mathcal{B},3}^{11} + W_{\mathcal{B},3}^{22} + W_{\mathcal{F},3}^{11} + W_{\mathcal{F},3}^{22}).
$$

(6.12)

Note that the relative numerical coefficients are the same as the ones in (6.9). The remaining half of the higher spin-$\frac{5}{2}$ currents can be obtained later.

Finally, the single higher spin-3 current of the first $\mathcal{N} = 4$ higher spin multiplet is obtained from the OPE between the spin-$\frac{3}{2}$ currents (6.6) and the higher spin-$\frac{5}{2}$ currents in (6.11) (or the sixth relation of (G.2))

$$
\tilde{\Phi}_{3}^{(1)} = \Phi_{3}^{(1)} + i\sqrt{2}Q_{\frac{3}{2}}^{12} - 2Q_{\frac{3}{2}}^{22} - 2\bar{Q}_{\frac{3}{2}}^{11} - i\sqrt{2}\bar{Q}_{\frac{3}{2}}^{21} + \bar{Q}_{\frac{3}{2}}^{22}),
$$

(6.11)

This higher spin-3 current (6.12) looks like as the one in (6.4). The remaining seven higher spin-3 currents can be found in next subsection.

The 16 higher spin currents at $\lambda = 1$ are given in Appendix I.

24 We can generalize them by multiplying $q^{-s+2} \frac{(2s-1)}{4}$ in (6.11) and changing the upper index (1) to (s) and the subscript $\frac{5}{2}$ to $s + \frac{3}{2}$.

25 By multiplying $-q^{-s+2} \frac{(2s-1)}{4}$ in (6.12) and changing the upper index (1) to (s) and the subscript 3 to $s + 2$, we obtain the general expression $\tilde{\Phi}_{2}^{(s)} = q^{-s} \frac{(2s-1)}{4} (W_{\mathcal{B},3}^{11} + W_{\mathcal{F},3}^{22})$.

26 By considering the currents, $L, G^i (i = 1, 3, 4), G^2, A^{-1} = \frac{1}{4q}(T^{14} - T^{23})$, $A^{-2} = \frac{1}{4q}(T^{13} - T^{42})$, $A^{-3} = -\frac{1}{4q}(T^{12} - T^{34})$, and higher spin currents, $(-1)^s \Phi_{\frac{1}{2}}^{(1)}, (-1)^s \Phi_{\frac{1}{2}}^{(1),i} (i = 1, 3, 4), (-1)^s \Phi_{\frac{1}{2}}^{(1),2i} (i = 1, 3, 4), (-1)^s \Phi_{\frac{1}{2}}^{(1),i}, (-1)^s \Phi_{\frac{1}{2}}^{(1),2i}$, we obtain the same higher spin algebra for $\lambda = 0$. 

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explicitly and we present it in Appendix H. Or we can use the fourth relation of \((G.3)\) to obtain the higher spin-$\frac{5}{2}$ current first and then use the fourth relation of \((G.2)\) to obtain the higher spin-2 current.

For the other higher spin-$\frac{5}{2}$, 3, 7, 2, 4 currents, $\Phi^{(2),i}$, $\Phi^{(2),ij}$, $\tilde{\Phi}^{(2),i}$ and $\tilde{\Phi}^{(2),2}$, we apply the procedure done in previous subsection to those higher spin currents. We can determine the higher spin-$\frac{5}{2}$ currents from the third relation of \((G.2)\). Once we find these currents, we can substitute them into the fourth relation of \((G.2)\) and obtain six higher spin-3 currents. Now we can use the fourth relation to obtain the higher spin-$\frac{7}{2}$ currents. Finally, the sixth relation of \((G.2)\) leads to the higher spin-4 current. The complete expressions for these 16 higher spin currents are given in Appendix H.

In this way, we can determine any $\mathcal{N} = 4$ higher spin multiplet in terms of free fields.

We can check that the subalgebra from the complex bosons gives rise to $W_\infty[1]$ algebra \([56]\). We obtain the following spin-2, 3, 4 currents (where we fix $N = 3$ and it is straightforward to consider the general $N$ case)

$$
T = W^{11}_{B,2} + W^{22}_{B,2}, \quad W = \frac{1}{2\sqrt{6}q} \left( W^{11}_{B,3} + W^{22}_{B,3} \right),
$$

$$
U = \frac{\sqrt{4/12}}{32q^2} \left( W^{11}_{B,4} + W^{22}_{B,4} - \frac{96q^2}{41} \left( TT - \frac{3}{10} \partial^2 T \right) \right). \quad (6.13)
$$

Then it is easy to observe that the square of structure constant $C^4_{33}$, which is defined as the coefficient in front of spin-4 current in the OPE between the spin-3 current and itself, is given by $(C^4_{33})^2 = \frac{3584}{123}$. From the general expression appearing in \([56, 57, 58, 59]\),

$$(C^4_{33})^2 = \frac{64(c + 2)(\lambda - 3)(c(\lambda + 3) + 2(4\lambda + 3)(\lambda - 1))}{(5c + 22)(\lambda - 2)(c(\lambda + 2) + (3\lambda + 2)(\lambda - 1))}, \quad (6.14)
$$

we obtain $\lambda = 1$ at $c = 2 N K = 12$. Therefore, the nonlinear $W_\infty[1]$ algebra \((6.13)\) is realized by complex free bosons. See also the relevant paper \([60, 27]\).

\[^{27}\text{Similarly, the subalgebra from the complex fermions is equivalent to } W_\infty[0] \text{ algebra. With the following spin-1, 2, 3, 4 currents at the central charge } c = N L - 1 = 5 \text{ (by subtracting the contribution from the spin-1 current)},

$$
J = W^{11}_{F,1} + W^{22}_{F,1}, \quad T = W^{11}_{F,2} + W^{22}_{F,2} - \frac{4q^2}{3} JJ,
$$

$$
W = \frac{\sqrt{3}}{8q} \left( W^{11}_{F,3} + W^{22}_{F,3} - \frac{16q^2}{3} JT - \frac{64q^2}{27} JJJ \right),
$$

$$
U = \frac{\sqrt{4/15}}{32q^2} \left( W^{11}_{F,4} + W^{22}_{F,4} - \frac{64q^2}{15} J\partial^2 J + \frac{32q^4}{5} \partial J \partial J - 8q^2 JW - \frac{64q^2}{3} JJJ \right).
$$

28
6.4 The (anti)commutators in \( W_{\infty}^N=4[\lambda =0] \) algebra

Note that the quasi primary fields in (6.1) have \( N \) independent terms for fixed \( k \). Each term has its own nontrivial OPE according to (6.3). This implies that there are \( N \) independent (anti)commutators between the higher spin currents. We can apply the result of Odake in [18] to obtain \( N \) copies of the sum of the field dependent terms and the central term. Then again we can use the expressions in (6.1) to reexpress the above field dependent terms by collecting them in terms of the original quasi primary fields. The central terms can be added and it will contribute to the overall factor \( N \). Then we obtain the following (anti)commutators for general spin \( h_1, h_2 \)

\[
[(W_{F,h_1}^{\bar{a}\bar{b}})_m, (W_{F,h_2}^{\bar{c}\bar{d}})_n] = \sum_{h \geq 1} p_B^{h_1h_2h}(m,n) q^{h} \left( \delta^{\bar{c}\bar{b}} W_{F,h_1+h_2-2-h}^{\bar{a}\bar{b}} + (-1)^h \delta^{\bar{a}\bar{d}} W_{F,h_1+h_2-2-h}^{\bar{a}\bar{d}} \right)_{m+n} + c_{W,h_1} \delta^{\bar{a}\bar{d}} \delta^{\bar{c}\bar{b}} q^{2(h_1-2)} \delta_{m+n},
\]

\[
[(W_{B,h_1}^{\bar{a}\bar{b}})_m, (W_{B,h_2}^{\bar{c}\bar{d}})_n] = \sum_{h \geq 1} p_B^{h_1h_2h}(m,n) q^{h} \left( \delta^{\bar{c}\bar{b}} W_{B,h_1+h_2-2-h}^{\bar{a}\bar{b}} + (-1)^h \delta^{\bar{a}\bar{d}} W_{B,h_1+h_2-2-h}^{\bar{a}\bar{d}} \right)_{m+n} + c_{W,h_1} \delta^{\bar{a}\bar{d}} \delta^{\bar{c}\bar{b}} q^{2(h_1-2)} \delta_{m+n},
\]

\[
[(W_{F,h_1}^{\bar{a}})_m, (Q_{h_2}^{\gamma})_n] = \sum_{h \geq 1} q^{h_1h_2h}(m,n) q^{h} \delta^{\bar{a}\gamma} (Q_{h_1+h_2-2-h}^{\bar{a}})_{m+r},
\]

\[
[(W_{B,h_1}^{\bar{a}})_m, (Q_{h_2}^{\gamma})_n] = \sum_{h \geq 1} q^{h_1h_2h}(m,n) q^{h} \delta^{\bar{a}\gamma} (Q_{h_1+h_2-2-h}^{\bar{a}})_{m+r},
\]

\[
[(W_{F,h_1}^{\bar{a}})_m, (\bar{Q}_{h_2}^{\gamma})_n] = \sum_{h \geq 1} q^{h_1h_2h}(m,n) q^{h} \delta^{\bar{a}\gamma} (Q_{h_1+h_2-2-h}^{\bar{a}})_{m+r},
\]

\[
[(W_{B,h_1}^{\bar{a}})_m, (\bar{Q}_{h_2}^{\gamma})_n] = \sum_{h \geq 1} q^{h_1h_2h}(m,n) q^{h} \delta^{\bar{a}\gamma} (Q_{h_1+h_2-2-h}^{\bar{a}})_{m+r},
\]

\[
\{(Q_{h_1}^{\alpha})_r, (\bar{Q}_{h_2}^{\beta})_s\} = \sum_{h \geq 0} q^{h_1h_2h}(r,s) \delta^{\bar{a}\bar{b}} W_{F,h_1+h_2-1-h}^{\bar{a}\bar{b}} + \delta^{\bar{a}\bar{c}} W_{B,h_1+h_2-1-h}^{\bar{a}\bar{c}} q^{2(h_1-1)} \delta_{m+n}.
\]

Here the central terms in (6.16) are given by

\[
c_{W,h}(m) = N \frac{2^{(h-3)}((h-1)!)^2}{(2h-3)! (2h-1)!} \prod_{j=1-h}^{h-1} (m+j),
\]

\[
= \frac{128 q^6}{27} JJJJ - \frac{296 q^2}{47} (TT - \frac{3}{10} \partial^2 T),
\]

(6.15)

we can obtain \((C_{33}^4)^2 = \frac{756}{47}\) by using (6.14) which leads to \( \lambda = 0 \) at \( c = 5 \). Note that the OPEs between the spin-1 current and the spin-2, 3, 4 currents do not have any singular terms. Therefore, the nonlinear \( W_{\infty}[0] \) algebra in (6.15) generated by spins 2, 3, 4, \ldots is realized by complex free fermions. There is a free boson realization for \( W_{1+\infty} \) algebra [61].
In (6.17), there is an overall factor \( N \).

The mode dependent expressions appearing in (6.16) are described as follows:

\[
\begin{align*}
p_{F}^{h1h2}(m, n) &= \frac{1}{2(h+1)!} \phi_{h}^{h1, h2}(0, -\frac{1}{2}) N_{h}^{h1, h2}(m, n), \\
p_{B}^{h1h2}(m, n) &= \frac{1}{2(h+1)!} \phi_{h}^{h1, h2}(0, 0) N_{h}^{h1, h2}(m, n), \\
\phi_{F}^{h1h2}(m, r) &= \frac{(-1)^{h}}{4(h+2)!} \left( (h_{1} - 1) \phi_{h+1}^{h1, h2 + \frac{1}{2}}(0, 0) \right. \\
&\quad - \left. (h_{1} - h - 3) \phi_{h+1}^{h1, h2 + \frac{1}{2}}(0, -\frac{1}{2}) \right) N_{h}^{h1, h2}(m, n), \\
\phi_{B}^{h1h2}(m, r) &= \frac{-1}{4(h+2)!} \left( (h_{1} - h - 2) \phi_{h+1}^{h1, h2 + \frac{1}{2}}(0, 0) - (h_{1}) \phi_{h+1}^{h1, h2 + \frac{1}{2}}(0, -\frac{1}{2}) \right) N_{h}^{h1, h2}(m, n), \\
o_{F}^{h1h2}(r, s) &= \frac{4(-1)^{h}}{h!} \left( (h_{1} + h_{2} - 1 - h) \phi_{h}^{h1, h2 + \frac{1}{2}, h2 + \frac{1}{2}}(\frac{1}{2}, -\frac{1}{2}) \right. \\
&\quad - \left. (h_{1} + h_{2} - \frac{3}{2} - h) \phi_{h+1}^{h1, h2 + \frac{1}{2}, h2 + \frac{1}{2}}(\frac{1}{2}, -\frac{1}{2}) \right) N_{h-1}^{h1, h2}(m, n), \\
o_{B}^{h1h2}(r, s) &= \frac{-4}{h!} \left( (h_{1} + h_{2} - 2 - h) \phi_{h}^{h1, h2 + \frac{1}{2}, h2 + \frac{1}{2}}(\frac{1}{2}, -\frac{1}{2}) \right. \\
&\quad - \left. (h_{1} + h_{2} - \frac{3}{2} - h) \phi_{h+1}^{h1, h2 + \frac{1}{2}, h2 + \frac{1}{2}}(\frac{1}{2}, -\frac{1}{2}) \right) N_{h-1}^{h1, h2}(m, n).
\end{align*}
\]  

Moreover, we introduce following quantities

\[
N_{h}^{h1, h2}(m, n) = \sum_{l=0}^{h+1} (-1)^{l} \left( \begin{array}{c} h + 1 \\ l \end{array} \right) [h_{1} - 1 + m]_{h+1-l} [h_{1} - 1 - m]_{l} \times [h_{2} - 1 + n]_{l} [h_{2} - 1 - n]_{h+1-l},
\]

\[
\phi_{h}^{h1, h2}(x, y) = _{4}F_{3} \left[ \begin{array}{c} -\frac{1}{2} - x - 2y, \frac{3}{2} - x + 2y, -\frac{h+1}{2}, -\frac{h}{2} + x \\ -h_{1} + \frac{3}{2}, -h_{2} + \frac{3}{2}, h_{1} + h_{2} - h - \frac{3}{2} \end{array} ; 1 \right].
\]

The falling Pochhammer symbol \([a]_{n} \equiv a(a-1) \cdots (a-n+1)\) is used in (6.19). The previous notation for the binomial coefficients is used. The generalized hypergeometric function, with 4 upper arguments \(a_{i}\), 3 lower arguments \(b_{i}\) and variable \(z\), is defined as the series

\[
_4F_{3} \left[ \begin{array}{c} a_{1}, a_{2}, a_{3}, a_{4} \\ b_{1}, b_{2}, b_{3} \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{(a_{1})_{n}(a_{2})_{n}(a_{3})_{n}(a_{4})_{n} z^{n}}{(b_{1})_{n}(b_{2})_{n}(b_{3})_{n} n!},
\]

where the rising Pochhammer symbol \((a)_{n} \equiv a(a+1) \cdots (a+n-1)\) is used in (6.20). Due to the fact that there is a relation between the arguments and the variable, \(b_{1} + b_{2} + b_{3} = \)

30
\[a_1 + a_2 + a_3 + a_4 + z \text{ for the } \phi_h^{a_1, a_2}(x, y) \text{ in (6.19)}, \text{ the infinite series (6.20) for this particular case terminates [21].} \]

The quantity \( N_h^{a_1, a_2}(m, n) \) in (6.19) can be written in terms of Clebsch Gordan coefficients [21].

In summarizing, the (anti)commutators between the \( \mathcal{N} = 4 \) higher spin multiplets are basically described by (6.16). More precisely, they with manifest \( SO(4) \) symmetry are written in terms of the linear combinations of (6.16). Because we have the \( s \)-th \( \mathcal{N} = 4 \) higher spin multiplet in terms of free fields from the footnotes [21,25] (see also Appendix J where the inverse relations between them are given), we can calculate the (anti)commutators between them by using the relations in (6.16). Then we arrive at the following results [20], with simplified notations where the mode indices are ignored,

\[
\begin{align*}
[\Phi_0^{(h_1)}, \Phi_0^{(h_2)}] &= \frac{\delta^{(h_1+h_2-3)}}{(2h_1 + h_2 - 2h - 5)} \\
&\times \left( -h_1 h_2 p_{B, 2h}^{h_1 h_2} + (h_1 - 1)(h_2 - 1) p_{B, 2h}^{h_1 h_2} \right) \Phi_0^{(h_1 + h_2 - 2h - 2)} \Phi_0^{(h_1 + h_2 - 2h - 2)} \\
&+ \frac{2}{2(h_1 + h_2 - 2h)} (h_1 h_2 (h_1 + h_2 - 2h - 3) p_{B, 2h}^{h_1 h_2} \\
&+ (h_1 - 1)(h_2 - 1)(h_1 + h_2 - 2h - 2) p_{B, 2h}^{h_1 h_2} \Phi_0^{(h_1 + h_2 - 2h - 4)}) \\
&+ \delta^{h_1 h_2} \frac{N}{2^{2h_1 - 5} h_1!(2h_1 - (h_1 - 1)!) d^{h_1 - 1}} \sum_{j=-h_1+1}^{h_1-1} (m + j),
\end{align*}
\]

\[
\begin{align*}
[\Phi_0^{(h_1)}, \Phi_0^{(h_2), 1}] &= \frac{\delta^{(h_1+h_2-3)}}{(2h_1 + h_2 - 2h - 5)} \\
&\times \Phi_0^{(h_1 + h_2 - 2h - 2)} \\
&\times \delta^{h_1 h_2} \frac{N}{2^{2h_1 - 5} h_1!(2h_1 - (h_1 - 1)!) d^{h_1 - 1}} \sum_{j=-h_1+1}^{h_1-1} (m + j),
\end{align*}
\]

As observed in [18] where \( N \) is fixed by 1 in the context of coset model, we can check that by contracting the indices in the second relation of (6.16), the bosonic realization provides the \( W_\infty \) algebra [22], where the central charge is given by \( c = 2 \), with increased central charge by a factor \( NK \) according to (6.5). The fermionic realization with the first relation of (6.16) contains the subalgebra \( W_{1+\infty} \) algebra [24], where the central charge is given by \( c = 1 \), with increased central charge by a factor \( NL \). Note that the variable \( y \) of \( p_B^{h_1 h_2}(m, n) \) is equal to \(-\frac{1}{2}\) while the one of \( p_B^{h_1 h_2}(m, n) \) is equal to 0 in (6.18) and (6.19). The variable \( y \) corresponds to the variable \( s \) in [21,23].

In practice, we calculate the (anti)commutators for \( h_1, h_2 \leq 5 \) and read off the \( h_1, h_2 \) and \( h \)-dependences in the right hand side of the (anti)commutators.
we can obtain the following OPE corresponding to the first commutator in (6.21) as follows:

\[
\Phi_0^{(h_1)}(z)\Phi_0^{(h_2)}(w) = \frac{1}{(z-w)^{h_1+h_2}} \phi_{h_1,h_2}(0,0) M_{h_1 h_2}(m,n)\left( \frac{h_1}{2} \right) \left( \frac{h_2}{2} \right) \frac{1}{2(h_1 + h_2 - 2h - 2)}
\]

where the mode dependent piece in (6.21) is given in (6.18) and for simplicity, the third element appearing in the upper index of \(p_F, \cdots\) is located at the lower index. The remaining expressions for (anti)commutators are given in (1.1). The relations (G.3) and (G.4) can be obtained from these complete results by substituting \(h_1\) and \(h_2\). The \(m\) in the first relation is the mode of the first element \(\Phi_0^{(h_1)}\). We also use the notations in the footnote 10. We can also reexpress these (anti)commutators in terms of OPEs based on [21] [30]. As before, the
above relations (6.21) will be the fundamental ones because the other ones can be obtained from (6.21) in the OPE language. We can easily check that the coefficients of the higher spin currents having negative spins in (6.21) and (J.1) are vanishing.

In the first and the last commutators of (6.21), the right hand side contains the $SO(4)$ singlets with subscript 0 and 2. In the second and fourth commutators, the $SO(4)$ vectors with subscript $\frac{1}{2}$ and $\frac{3}{2}$ appear. In the third commutator, the $SO(4)$ adjoint with subscript 1 appears in the right hand side.

7 AdS$_3$ higher spin theory with matrix generalization

We consider the “deformed” oscillator construction corresponding to the coset construction in sections, 2, 3, and 4 associated with the $\mathcal{N} = 4$ higher spin multiplets under the large $(N,k)$ limit. The Lie algebra $shs[\lambda]$ is generated by $\hat{y}_a (\alpha = 1,2)$ with defining relations

$$[\hat{y}_\alpha, \hat{y}_\beta] = 2i \epsilon_{\alpha\beta}(1 + \nu k), \quad k \hat{y}_\alpha = -\hat{y}_\alpha k, \quad k^2 = 1. \quad (7.1)$$

The Chan-Paton factors are introduced and the generators of $\mathcal{N} = 4$ higher spin algebra denoted by $shs_2[\lambda]$ are given by the tensor product between the generators of the $\mathcal{N} = 2$ higher spin algebra $shs[\lambda]$ and $U(2)$ generators. There is a relation between the parameter in

$$+ \frac{2}{2(h_1 + h_2 - 2k)} - 9 \left( h_1 h_2 (h_1 + h_2 - 2h - 3) f_{1,2k}^{h_1 h_2} (\partial_z, \partial_w) + (h_1 - 1) (h_2 - 1) (h_1 + h_2 - 2h - 2) f_{1,2k}^{h_1 h_2} (\partial_z, \partial_w) \right) \frac{\hat{\Phi}_2^{(h_1 + h_2 - 2h - 4)} (w)}{(z - w)}. \quad (7.2)$$

Note that the central term is obtained from the one in the corresponding commutator by considering further the relation between the binomial coefficient and the product. The mode dependent parts in next singular terms $p_F$ and $p_B$ go to the differential operators $f_F$ and $f_B$ respectively. The fields are functions of $w$ together with the factor $\frac{1}{(z-w)}$. The other numerical factors in the commutator remain as the same with extra minus sign. We can obtain the OPEs for the other (anti)commutators similarly.

31 For example, $\phi_{h_1 + h_2 - 3}^{h_1, h_2 - 0}(0,0)$ is written as $4F_3 \left[ \frac{-h_1 + \frac{3}{2}, -h_2, 0, \frac{3}{2}}{-h_1 + \frac{3}{2}, -h_2 + \frac{3}{2}, \frac{3}{2}; 1} \right] = 3F_2 \left[ a \equiv -\frac{1}{2}, b \equiv -(h_1 + h_2 - 2), c \equiv -(h_1 + h_2 - 3), d \equiv -h_1 + \frac{3}{2}, e \equiv -h_2 + \frac{3}{2}; 1 \right]$ because the upper second argument is equal to the lower third argument and they are cancelled each other in (6.20). Therefore we are left with $3F_2$. Using the definition of $3F_2$, we have the factor $\frac{1}{\sqrt{\Gamma(-d-a) \Gamma(-c-a)}} = \sqrt{\Gamma(-h_1 + 2) \Gamma(-h_2 + 2)}$ which goes to zero for $h_1, h_2 > 1$.

Similarly, we have vanishing $\phi_{h_1 + h_2 - 3}^{h_1, h_2 + 2}(0,0)$ by shifting $h_2 \rightarrow h_2 + 2$.

32 In (G.3), the currents of the large $\mathcal{N} = 4$ superconformal algebra occur in the right hand side. By introducing the following notations $-\frac{1}{2} L \rightarrow \Phi_\frac{3}{2}^{(0)}$ and $\frac{1}{4} G^i \rightarrow \Phi_\frac{3}{2}^{(0),i}$, we can treat the currents and higher spin currents in (6.21) and (J.1) simultaneously.
the $\mathcal{N} = 4$ higher spin algebra and the one in the $\mathcal{N} = 4$ coset model as follows:

$$\nu = 2\lambda - 1, \quad \text{or} \quad \lambda = \frac{(\nu + 1)}{2} = \mu. \quad (7.2)$$

We construct the $\mathcal{N} = 4$ (higher spin) generators at generic $\lambda$ from the wedge subalgebra of the $\mathcal{W}_{\infty}^{N=4}[\lambda]$ algebra.33

### 7.1 The 16 generators and exceptional superalgebra $D(2, 1|\frac{\lambda}{1-\lambda})$

Spin-$\frac{3}{2}$ currents of the large $\mathcal{N} = 4$ superconformal algebra provide the eight fermionic operators for $\mathcal{N} = 4$ wedge algebra [1, 68] as follows:

$$G_{1/2}^1 = -\frac{1}{2} e^{i\pi\hat{y}_{3/2}} k \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad G_{1/2}^2 = \frac{i}{2} e^{i\pi\hat{y}_{3/2}} k \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$G_{1/2}^3 = \frac{i}{2} e^{i\pi\hat{y}_{3/2}} k \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad G_{1/2}^4 = \frac{1}{2} e^{i\pi\hat{y}_{3/2}} k \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (7.3)$$

Note that the second component of the spin-$\frac{3}{2}$ current has a $2 \times 2$ identity matrix in its expression. After calculating the anticommutators between these operators (7.3), we obtain the following result

$$\{G_r, G_\rho\} = \delta^{ij} 2 L_{r+\rho} - i(r - \rho) (T_{r+\rho}^{ij} - (2\lambda - 1) \tilde{T}_{r+\rho}^{ij}). \quad (7.4)$$

For the same index $i = j$, the spin-2 current of the large $\mathcal{N} = 4$ superconformal algebra gives the three bosonic operators of $\mathcal{N} = 4$ wedge algebra by using (7.1)

$$L_{+1} = \frac{1}{4i} \hat{y}_1 \hat{y}_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$L_0 = \frac{1}{8i} (\hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_1) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$L_{-1} = \frac{1}{4i} \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (7.5)$$

Moreover, for different index $i \neq j$, we can write down the six spin-1 operators as follows:

$$T_0^{12} = \frac{1}{2} k \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_0^{13} = \frac{1}{2} k \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$T_0^{14} = -\frac{i}{2} k \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T_0^{23} = \frac{i}{2} k \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$T_0^{24} = \frac{1}{2} k \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_0^{34} = -\frac{1}{2} k \otimes \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}. \quad (7.6)$$

33The product in the oscillators is a Moyal product [62]. On the other hand, the lone star product is introduced in [21]. Recently, in [29], these two products are equivalent to each other.
Note that for $\lambda = 0$, the last term of (7.4) can combine the second term. Then this leads to the similar relation as in (6.7).

We can check that the following commutators are satisfied

$$\begin{align*}
[L_m, L_n] &= (m - n) L_{m+n}, \\
[L_m, T^{ij}_n] &= 0, \\
[T^{ij}_m, T^{kl}_n] &= -i \delta^{ik} T^{jl} + i \delta^{il} T^{jk} + i \delta^{jk} T^{il} - i \delta^{jl} T^{ik}.
\end{align*}$$ (7.7)

The $\mathcal{N} = 4$ wedge algebra which generates nine bosonic and eight fermionic ones written in terms of (7.3), (7.5) and (7.6) is characterized by (7.4) and (7.7). We can easily see that the (anti)commutators in (G.1) will become the above $\mathcal{N} = 4$ wedge algebra by restricting the mode indices to the wedge indices.

7.2 The first $\mathcal{N} = 4$ generators of $\mathcal{N} = 4$ higher spin algebra $shs_2[\lambda]$

Let us start with the following higher spin-1 operator [1]

$$\Phi^{(1)}_{0,0} = (k + \nu) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (7.8)$$

Here the parameter $\nu$ has a relation with $\lambda$ via (7.2).

We can calculate the following commutators from (7.3) and (7.8) and read off the higher spin-$\frac{3}{2}$ operators appearing in the right hand side

$$[G^i_r, \Phi^{(1)}_{0,0}] = -\Phi^{(1),i}_{\frac{3}{2},r}. \quad (7.9)$$

This relation (7.9) is nothing but one of the $\mathcal{N} = 4$ primary conditions. See also (G.2). It turns out that the eight spin-$\frac{3}{2}$ operators are given by

$$\begin{align*}
\Phi^{(1),1}_{\frac{1}{2},\pm \frac{1}{2}} &= e^{i \frac{\pi}{4}} \hat{y}_{\pm \frac{1}{2} \pm \frac{1}{2}} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \Phi^{(1),2}_{\frac{1}{2},\pm \frac{1}{2}} &= -i e^{i \frac{\pi}{4}} \hat{y}_{\pm \frac{1}{2} \pm \frac{1}{2}} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\Phi^{(1),3}_{\frac{1}{2},\pm \frac{1}{2}} &= -i e^{i \frac{\pi}{4}} \hat{y}_{\pm \frac{1}{2} \pm \frac{1}{2}} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \Phi^{(1),4}_{\frac{1}{2},\pm \frac{1}{2}} &= -e^{i \frac{\pi}{4}} \hat{y}_{\pm \frac{1}{2} \pm \frac{1}{2}} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{align*} \quad (7.10)$$

The second element has a $2 \times 2$ identity matrix.

In order to obtain eighteen spin-2 operators, we should use the following anticommutators

$$\left\{ G^i_r, \Phi^{(1),ij}_{\frac{1}{2},\rho} \right\} = -\delta^{ij} (r - \rho) \Phi^{(1)}_{0,r+\rho} + \tilde{\Phi}^{(1),ij}_{1,r+\rho}. \quad (7.11)$$

Again, this is one of the $\mathcal{N} = 4$ primary conditions in component approach. For equal index $i = j$, there is no nontrivial relation. This will provide the following eighteen spin-2 operators
for different $i$ and $j$ indices as follows:

\[
\begin{align*}
\Phi^{(1),12}_{1,-1} &= -\hat{y}_2\hat{y}_2 k \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\Phi^{(1),12}_{1,+1} &= -\hat{y}_1\hat{y}_1 k \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\Phi^{(1),13}_{1,0} &= -\frac{1}{2}(\hat{y}_1\hat{y}_2 + \hat{y}_2\hat{y}_1) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\Phi^{(1),13}_{1,-1} &= -\hat{y}_2\hat{y}_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\Phi^{(1),14}_{1,0} &= i\hat{y}_2\hat{y}_1 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\Phi^{(1),14}_{1,+1} &= i\hat{y}_1\hat{y}_1 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\Phi^{(1),23}_{1,0} &= -\frac{i}{2}(\hat{y}_1\hat{y}_2 + \hat{y}_2\hat{y}_1) k \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\Phi^{(1),23}_{1,+1} &= -i\hat{y}_1\hat{y}_1 k \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\Phi^{(1),24}_{1,0} &= -\hat{y}_2\hat{y}_2 k \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\Phi^{(1),24}_{1,+1} &= -\hat{y}_1\hat{y}_1 k \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\Phi^{(1),34}_{1,0} &= -\frac{1}{2}(\hat{y}_1\hat{y}_2 + \hat{y}_2\hat{y}_1) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\Phi^{(1),34}_{1,+1} &= \hat{y}_1\hat{y}_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\]

There are no elements having a $2 \times 2$ identity matrix.

We can calculate the following commutators

\[
\left[ G^i_r, \Phi^{(1),jk}_{1,m} \right] = -\delta^{ij} (\Phi^{(1),k}_{\frac{1}{2}r+m} - \frac{1}{3} (2r - m)(2\lambda - 1) \Phi^{(1),k}_{\frac{1}{2}r+m}) \delta^{jk} (\Phi^{(1),j}_{\frac{1}{2}r+m} - \frac{1}{3} (2r - m)(2\lambda - 1) \Phi^{(1),j}_{\frac{1}{2}r+m}) + \varepsilon^{ijk}(2r - m) \Phi^{(1),l}_{\frac{1}{2}r+m}.
\]

which is one of the $\mathcal{N} = 4$ primary conditions mentioned before. This enables us to obtain the following sixteen spin-$\frac{5}{2}$ operators as follows:

\[
\begin{align*}
\tilde{\Phi}^{(1),1}_{\frac{1}{2},-\frac{1}{2}} &= -ie^{i\pi} \hat{y}_2\hat{y}_2 k \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\tilde{\Phi}^{(1),1}_{\frac{1}{2},-\frac{1}{2}} &= -\frac{i}{3} e^{i\pi} (\hat{y}_1\hat{y}_2\hat{y}_2 + \hat{y}_2\hat{y}_1\hat{y}_2 + \hat{y}_2\hat{y}_2\hat{y}_1) k \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\tilde{\Phi}^{(1),1}_{\frac{1}{2},+\frac{1}{2}} &= -\frac{i}{3} e^{i\pi} (\hat{y}_2\hat{y}_1\hat{y}_1 + \hat{y}_1\hat{y}_2\hat{y}_2 + \hat{y}_1\hat{y}_1\hat{y}_2) k \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\tilde{\Phi}^{(1),1}_{\frac{1}{2},+\frac{1}{2}} &= -ie^{i\pi} \hat{y}_1\hat{y}_1 k \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\tilde{\Phi}^{(1),2}_{\frac{1}{2},-\frac{1}{2}} &= -\frac{i}{3} e^{i\pi} (\hat{y}_1\hat{y}_2\hat{y}_2 + \hat{y}_2\hat{y}_1\hat{y}_2 + \hat{y}_2\hat{y}_2\hat{y}_1) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\tilde{\Phi}^{(1),2}_{\frac{1}{2},-\frac{1}{2}} &= -\frac{i}{3} e^{i\pi} (\hat{y}_2\hat{y}_2 + \hat{y}_2\hat{y}_1 + \hat{y}_2\hat{y}_2) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\]
\[
\Phi^{(1),2}_{\frac{3}{2}+\frac{1}{2}} = -\frac{1}{3} e^{i \vec{\pi}} (\dot{y}_1 \dot{y}_1 \dot{y}_1 + \dot{y}_1 \dot{y}_2 \dot{y}_2 + \dot{y}_1 \dot{y}_1 \dot{y}_2) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
\Phi^{(1),2}_{\frac{3}{2}+\frac{1}{2}} = -e^{i \vec{\pi}} \dot{y}_1 \dot{y}_1 \dot{y}_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 
\Phi^{(1),3}_{\frac{3}{2}+\frac{1}{2}} = -e^{i \vec{\pi}} \dot{y}_2 \dot{y}_2 \dot{y}_2 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
\[
\Phi^{(1),3}_{\frac{3}{2}+\frac{1}{2}} = -\frac{1}{3} e^{i \vec{\pi}} (\dot{y}_1 \dot{y}_1 \dot{y}_2 + \dot{y}_2 \dot{y}_1 \dot{y}_2 + \dot{y}_2 \dot{y}_2 \dot{y}_1) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
\[
\Phi^{(1),3}_{\frac{3}{2}+\frac{1}{2}} = -\frac{1}{3} e^{i \vec{\pi}} (\dot{y}_2 \dot{y}_1 \dot{y}_1 + \dot{y}_1 \dot{y}_2 \dot{y}_2 + \dot{y}_1 \dot{y}_1 \dot{y}_2) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
\[
\Phi^{(1),3}_{\frac{3}{2}+\frac{1}{2}} = -e^{i \vec{\pi}} \dot{y}_1 \dot{y}_1 \dot{y}_1 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\Phi^{(1),4}_{\frac{3}{2}+\frac{1}{2}} = i e^{i \vec{\pi}} \dot{y}_2 \dot{y}_2 \dot{y}_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
\Phi^{(1),4}_{\frac{3}{2}+\frac{1}{2}} = \frac{i}{3} e^{i \vec{\pi}} (\dot{y}_1 \dot{y}_2 \dot{y}_2 + \dot{y}_2 \dot{y}_1 \dot{y}_2 + \dot{y}_2 \dot{y}_2 \dot{y}_1) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
\Phi^{(1),4}_{\frac{3}{2}+\frac{1}{2}} = \frac{i}{3} e^{i \vec{\pi}} (\dot{y}_2 \dot{y}_1 \dot{y}_1 + \dot{y}_1 \dot{y}_2 \dot{y}_2 + \dot{y}_1 \dot{y}_1 \dot{y}_2) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
\Phi^{(1),4}_{\frac{3}{2}+\frac{1}{2}} = i e^{i \vec{\pi}} \dot{y}_1 \dot{y}_1 \dot{y}_1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

(7.14)

Note that the \( SO(4) \) index \( i = 2 \) case has a \( 2 \times 2 \) identity matrix.

From the following anticommutators coming from \( \mathcal{N} = 4 \) primary conditions with wedge conditions (see also (G.2))
\[
\{G_{r}^{i}, \Phi^{(1),j}_{\frac{3}{2}+\rho}\} = -\delta^{ij} \Phi^{(1)}_{2,r+\rho} - (3r - \rho)(\Phi^{(1),ij}_{1, r+\rho} - \frac{1}{3}(2\lambda - 1) \Phi^{(1),ij}_{1, r+\rho}),
\]
we can determine the five spin-3 operators as follows:
\[
\Phi^{(1)}_{2,-2} = -\dot{y}_2 \dot{y}_2 \dot{y}_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
\Phi^{(1)}_{2,-1} = -\frac{1}{4} (\dot{y}_1 \dot{y}_2 \dot{y}_2 + \dot{y}_2 \dot{y}_1 \dot{y}_2 + \dot{y}_2 \dot{y}_2 \dot{y}_1) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
\Phi^{(1)}_{2,0} = -\frac{1}{6} (\dot{y}_1 \dot{y}_2 \dot{y}_2 + \dot{y}_2 \dot{y}_1 \dot{y}_2 + \dot{y}_2 \dot{y}_2 \dot{y}_1 + \dot{y}_2 \dot{y}_1 \dot{y}_2 + \dot{y}_2 \dot{y}_2 \dot{y}_1 + \dot{y}_2 \dot{y}_1 \dot{y}_2) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
\Phi^{(1)}_{2,+1} = -\frac{1}{4} (\dot{y}_2 \dot{y}_1 \dot{y}_1 + \dot{y}_1 \dot{y}_2 \dot{y}_1 + \dot{y}_1 \dot{y}_1 \dot{y}_2 + \dot{y}_1 \dot{y}_1 \dot{y}_2) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
\Phi^{(1)}_{2,+2} = -\dot{y}_1 \dot{y}_1 \dot{y}_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(7.15)

These \( SO(4) \) singlets have a \( 2 \times 2 \) identity matrix in their expressions. We can check that the following commutator satisfies
\[
\left[G_{r}^{i}, \Phi^{(1)}_{2,m}\right] = -(4r - m) \Phi^{(1),j}_{\frac{3}{2}+r+m}.
\]

(7.16)
The relation (7.17) can be checked from (7.3), (7.16) and (7.14).

Therefore, the oscillator realization for the first $N = 4$ higher spin operators is summarized by (7.8), (7.10), (7.12), (7.14), and (7.16). Note that the oscillators $\hat{y}_\alpha$ appear symmetrically.

7.3 The second $N = 4$ generators of $\mathcal{N} = 4$ higher spin algebra $shs_2[\lambda]$

Let us consider the second $N = 4$ higher spin generators. For the higher spin-2 operator, we can find the higher spin-$\frac{5}{2}$ operator $\Phi^{(2),i}_{\frac{5}{2}}$ first. From the relation coming from the $\mathcal{N} = 4$ wedge algebra in (G.3)

$$\left[ \Phi^{(1)}_{0,m}, \Phi^{(1),i}_{\frac{5}{2},r} \right] = -\frac{1}{2} \Phi^{(2),i}_{\frac{5}{2},m+r},$$

we obtain sixteen higher spin-$\frac{5}{2}$ operators by calculating the left hand side of (7.18). After that, we can determine the three higher spin-2 operators by using (7.11) for upper index $s = 2$ (or the fourth relation of (G.2)) for equal index $i = j$. For the higher spin-3 operator, we can use (7.11) for $s = 2$ for different indices $i \neq j$. Then thirty higher spin-3 operators can be determined. The relation (7.13) for upper index $s = 2$ (or the fifth relation of (G.2)) is used for the twenty four higher spin-$\frac{7}{2}$ operators. Finally, the seven higher spin-4 operators can be obtained from the relation (7.15) for upper index $s = 2$.\[34]

7.4 The $s$-th $N = 4$ generators of $\mathcal{N} = 4$ higher spin algebra $shs_2[\lambda]$

From the results of the first and the second $N = 4$ higher spin generators (together with the third and fourth ones) in terms of oscillators, we obtain the following expressions for the $s$-th $N = 4$ higher spin generators

$$\Phi^{(s)}_{0,m} = \left[ \frac{(s-1-m)!(s-1+m)!}{(2s-2)!} \right] \hat{y}_1 \cdots \hat{y}_1 \hat{y}_2 \cdots \hat{y}_2 ((2s-1)\hat{k} + \nu) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Phi^{(s),1}_{\frac{s}{2},m} = \left[ \frac{(s-\frac{1}{2}-m)!(s-\frac{1}{2}+m)!}{(2s-2)!} \right] e^{i \frac{\pi}{4}} \hat{y}_1 \cdots \hat{y}_1 \hat{y}_2 \cdots \hat{y}_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\Phi^{(s),2}_{\frac{s}{2},m} = -i \left[ \frac{(s-\frac{1}{2}+m)!(s-\frac{1}{2}-m)!}{(2s-2)!} \right] e^{i \frac{\pi}{4}} \hat{y}_1 \cdots \hat{y}_1 \hat{y}_2 \cdots \hat{y}_2 k \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

\[34\] We present them explicitly in Appendix K where we can see a $2 \times 2$ identity matrix for the $SO(4)$ vector index $i = 2$. Compared to the first $\mathcal{N} = 4$ generators, the number of oscillators is increased with different overall factors. The form of $U(2)$ matrix elements remain the same and the oscillators $\hat{y}_\alpha$ appear symmetrically as before.
\[
\Phi_{\frac{1}{2}, m}^{(s), 3} = -i \left[ \frac{(s - \frac{1}{2} - m)!}{(2s - 2)!} \right] e^{i \pi \frac{\pi}{2}} y_1 \cdots y_1 \cdots y_2 \otimes \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right),
\]
\[
\Phi_{\frac{1}{2}, m}^{(s), 4} = - \left[ \frac{(s - \frac{1}{2} - m)!}{(2s - 2)!} \right] e^{i \pi \frac{\pi}{2}} y_1 \cdots y_1 \cdots y_2 \otimes \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right),
\]
\[
\Phi_{1, m}^{(s), 12} = - \left[ \frac{(s - m)!}{2s(2s - 2)!} \right] \hat{y}_1 \cdots \hat{y}_1 \cdots \hat{y}_2 \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right),
\]
\[
\Phi_{1, m}^{(s), 13} = - \left[ \frac{(s - m)!}{2s(2s - 2)!} \right] \hat{y}_1 \cdots \hat{y}_1 \cdots \hat{y}_2 \otimes \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right),
\]
\[
\Phi_{1, m}^{(s), 14} = i \left[ \frac{(s - m)!}{2s(2s - 2)!} \right] \hat{y}_1 \cdots \hat{y}_1 \cdots \hat{y}_2 \otimes \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right),
\]
\[
\Phi_{1, m}^{(s), 23} = -i \left[ \frac{(s - m)!}{2s(2s - 2)!} \right] \hat{y}_1 \cdots \hat{y}_1 \cdots \hat{y}_2 \otimes \left( \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right),
\]
\[
\Phi_{1, m}^{(s), 24} = - \left[ \frac{(s - m)!}{2s(2s - 2)!} \right] \hat{y}_1 \cdots \hat{y}_1 \cdots \hat{y}_2 \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),
\]
\[
\Phi_{1, m}^{(s), 34} = \left[ \frac{(s - m)!}{2s(2s - 2)!} \right] \hat{y}_1 \cdots \hat{y}_1 \cdots \hat{y}_2 \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right),
\]
\[
\Phi_{\frac{1}{2}, m}^{(s), 1} = -i \left[ \frac{(s + \frac{1}{2} - m)!}{2s(2s + 1)(2s - 2)!} \right] e^{i \pi \frac{\pi}{2}} \hat{y}_1 \cdots \hat{y}_1 \cdots \hat{y}_2 \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),
\]
\[
\Phi_{\frac{1}{2}, m}^{(s), 2} = - \left[ \frac{(s + \frac{1}{2} - m)!}{2s(2s + 1)(2s - 2)!} \right] e^{i \pi \frac{\pi}{2}} \hat{y}_1 \cdots \hat{y}_1 \cdots \hat{y}_2 \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right),
\]
\[
\Phi_{\frac{1}{2}, m}^{(s), 3} = - \left[ \frac{(s + \frac{1}{2} - m)!}{2s(2s + 1)(2s - 2)!} \right] e^{i \pi \frac{\pi}{2}} \hat{y}_1 \cdots \hat{y}_1 \cdots \hat{y}_2 \otimes \left( \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right),
\]
\[
\Phi_{\frac{1}{2}, m}^{(s), 4} = i \left[ \frac{(s + \frac{1}{2} - m)!}{2s(2s + 1)(2s - 2)!} \right] e^{i \pi \frac{\pi}{2}} \hat{y}_1 \cdots \hat{y}_1 \cdots \hat{y}_2 \otimes \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right),
\]
\[
\Phi_{2, m}^{(s)} = - \left[ \frac{(s + 1 - m)!}{2s(2s + 1)(2s + 2)(2s - 2)!} \right] \hat{y}_1 \cdots \hat{y}_1 \cdots \hat{y}_2 \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).
\]

The total number of the 16 higher spin operators is given by $16(1 + 2s)$. The difference between the number of $\hat{y}_1$ and the number of $\hat{y}_2$ is $2m$. The number of $\hat{y}_3$ for the higher spin generator of spin $\hat{s}$ is given by $2(\hat{s} - 1)$. We use the simplified notation for the symmetric
product of oscillators \( \hat{y}_\alpha \). For example, the expression \( \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 \) has ten terms as in \( \Phi^{(2),1}_{\frac{3}{2},-\frac{1}{2}} \) of Appendix K. The higher spin generators of \( SO(4) \) singlet and \( SO(4) \) vector index 2 have the \( 2 \times 2 \) identity matrix and they will consist of the \( \mathcal{N} = 2 \) higher spin algebra. The question is what is the \( \mathcal{N} = 4 \) higher spin algebra generated by (7.19) \(^{35}\).

### 7.5 The (anti)commutators in \( \mathcal{N} = 4 \) higher spin algebra \( shs_2[\lambda] \)

In Appendix G, we present the (anti)commutators between the \( \mathcal{N} = 4 \) higher spin multiplets we have constructed in previous sections. They originate from the results of OPEs in the coset construction. Because we do not know any OPEs for the general spins \( s_1, s_2 \), we cannot further obtain the corresponding (anti)commutators. The next step we can consider is to use the known expressions (7.19) of the \( \mathcal{N} = 4 \) higher spin generators in terms of oscillators. Is it possible to write down the \( \lambda \) dependent structure constants appearing in Appendix G in terms of any closed form using a special function like as the ones in section 6 for the \( \lambda = 0 \) case? According to the observation of \(^{29}\), the \( \mathcal{N} = 2 \) higher spin algebra \( shs[\lambda] \) can be written in terms of closed form by generalizing the mode dependent quantity in (6.19). Moreover, the nontrivial dependences of \( s_1, s_2, s \) and \( \lambda \) arise in the structure constants. Therefore, we expect that the \( \mathcal{N} = 4 \) higher spin algebra should be described in terms of closed form.

### 7.6 The \( \mathcal{N} = 2 \) higher spin algebra \( shs[\lambda] \)

As described in section 5, among 16 higher spin generators in (7.19), there are four higher spin generators, \( \Phi^{(s)}_{0,m}, \Phi^{(s),2}_{\frac{1}{2},m}, \Phi^{(s),2}_{\frac{3}{2},m} \) and \( \Phi^{(s)}_{2,m} \) which have \( U(2) \) identity matrix. We expect that they consist of their own closed subalgebra. In particular, from the identifications (5.2) and (5.3), we can calculate the corresponding (anti)commutators by using the quantities appearing in the left hand sides with the help of Appendix G. In Appendix L, we collect the relevant (anti)commutators from Appendix G. Moreover, the known (anti)commutators from \(^{10,55}\) are presented in terms of the notations of the right hand sides of (5.2) and (5.3). Under the wedge condition, we explicitly check that the \( \mathcal{N} = 2 \) wedge algebra coming from the first \( \mathcal{N} = 2 \) higher spin multiplet, in addition to the generators of \( \mathcal{N} = 2 \) superconformal algebra, can be realized by the ten higher spin operators coming from the first, second and third \( \mathcal{N} = 4 \) higher spin generators as well as the two operators from the large \( \mathcal{N} = 4 \) linear superconformal algebra \(^{36}\).

---

\(^{35}\)There is an overall factor \( \frac{1}{2} \) difference between the first \( \mathcal{N} = 4 \) higher spin multiplet in the coset model and the one in the oscillator formalism. Similarly, the relative factor \( -\frac{1}{3}(-\frac{6}{5}) \) difference appears in the second(third) \( \mathcal{N} = 4 \) higher spin multiplet of both descriptions.  

\(^{36}\)Moreover, according to the observation of \(^{10}\), the wedge subalgebra of \( \mathcal{W}_{\infty}^{N=2}[\lambda] \) algebra matches with the corresponding \( \mathcal{N} = 2 \) higher spin algebra studied in \(^{27}\). Therefore, we can conclude that the realization
Let us consider the simplest case. We can rewrite the commutator between the first of $\mathcal{N} = 2$ higher spin generators found by [30]. By using the $\mathcal{N} = 2$ higher spin algebra and expressing the right hand side in terms of the elements of $\mathcal{N} = 2$ higher spin generators (and the generators of $\mathcal{N} = 2$ superconformal algebra) as follows:

$$[W_{m}^{20}, W_{n}^{20}] = \frac{\sqrt{(1 - m)! (1 + m)! (1 - n)! (1 + n)!}}{6\sqrt{(1 - m - n)! (1 + m + n)!}} C_{m n, m n}^{111} \times \left( \frac{1}{3} \left( \lambda + 1 \right) (\lambda - 2) \left( (\lambda + 1) f_{TTT}^{111} - (\lambda - 2) f_{UUU}^{111} \right) L_{m + n} \right) + \frac{1}{\sqrt{3}} \left( (\lambda + 1)^2 f_{TTT}^{111} - (\lambda - 2)^2 f_{UUU}^{111} \right) W_{m + n}^{20},$$  

(7.20)

where the structure constants $f_{\lambda \lambda'}^{\lambda''}$, which depend on the $\lambda$, are given in [27] and $C_{\lambda \lambda'}^{\lambda''}$ are the $SL(2)$ Clebsch-Gordan coefficients. See also [30]. We also have other (anti)commutators in Appendix L. Therefore, we can observe that the $\mathcal{N} = 2$ higher spin generators of $\mathcal{N} = 4$ higher spin generators realized in (7.19) satisfy the corresponding $\mathcal{N} = 2$ higher spin algebra studied in [27]. We expect that the bosonic subalgebra should satisfy the higher spin algebra.

### 7.7 How to generate the $\mathcal{N} = 4$ higher spin algebra $shs_{2}[\lambda]$  

For the general spins $s_1$ and $s_2$, we can think of the (anti)commutators between two 16 higher spin generators and there are 256 (anti)commutators. Then how can we determine these nontrivial (anti)commutators? For the vanishing ’t Hooft-like coupling constant, there exist the previous relations in (6.21) and (7.11). The point is that we would like to determine the structure constants for the nonzero $\lambda$. As an example, we take the simplest one where $h_1 = 1$ and $h_2 = 3$. First of all, we do not know any OPEs from the coset construction. All we have is the explicit form for the $h$-th $\mathcal{N} = 4$ higher spin generators given by (7.19). In principle, we can calculate the various (anti)commutators from the first and the third $\mathcal{N} = 4$ higher spin generators, by using the relations (7.11). From the free field results in (6.21), we can take the same $SO(4)$ index structure and mode dependence and furthermore introduce the undetermined structure constants in the right hand side. Then we can compute the

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37 One representation of the Clebsch-Gordan coefficients through hypergeometric function is $C_{m m', m''}^{i j' i''} = \delta_{m + m', m''} \sqrt{(2j'' + 1)(\lambda + m')! (\lambda' + m'')! (j'' - j' - m')! (j'' - m'')! (j'' + j' - m')! (j'' + j' + m'')!} \sum_{j = 0}^{1} F_2 \left[ \begin{array}{c} -j + m, -j' - m', -j - j' + j'' \\ 1 + j'' - j - m', 1 + j'' - j' + m' \end{array} ; 1 \right]$, appearing in [21].
(anti)commutators by using the explicit forms in (7.19). We present the detailed computations in ancillary.nb file. 38

8 Conclusions and outlook

As in the abstract, the OPEs between the first and second $\mathcal{N} = 4$ higher spin multiplets are obtained in component and in $\mathcal{N} = 4$ superspace. By taking the large $(N, k)$ 't Hooft-like limit, we obtain the $\mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda]$ algebra for low spins. At $\lambda = 0$, the free field construction is determined and the corresponding $SO(4)$ symmetric $\mathcal{W}_{\infty}^{\mathcal{N}=4}[0]$ algebra is obtained for any spins $s_1$ and $s_2$. At $\lambda \neq 0$, the $\mathcal{N} = 4$ higher spin algebra $shs_2[\lambda]$ is determined by using the $\mathcal{N} = 4$ wedge subalgebra of $\mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda]$ algebra for low spins. We also present how to determine the structure constants of the $\mathcal{N} = 4$ higher spin algebra $shs_2[\lambda]$ for fixed spins $s_1$ and $s_2$ from the oscillator formalism.

We list the possible open problems along the line of this paper as follows:

- More OPEs

  It is an open problem to construct more OPEs (the OPE between the first and the third $\mathcal{N} = 4$ higher spin multiplets, the OPE between the second and the third $\mathcal{N} = 4$ higher spin multiplets, \cdots) and observe any new features beyond the corresponding wedge subalgebra. As the spin increases, the structure of the OPE becomes more complicated and the problem in this direction can be reduced if we can manage to express the singular terms in the given OPE in terms of known (higher spin) currents.

- Application of free field construction to type IIB string theory

  For the infinity limit of the level $k$, the free field construction was analyzed in the context of the higher spin theory and the string theory 52 where $(4N+4)$ free bosons and fermions are used. It would be interesting to analyze the free field construction obtained in this paper (under the large $(N, k)$ 't Hooft limit) and see how the higher spin symmetry can be embedded in the string theory.

- Any closed form for the (anti)commutators of higher spin algebra at generic $\lambda$

38 Several comments are in order. 1) In order to simplify the computation, we intentionally put the oscillator $\hat{y}_1$ to the left and the Klein operator $k$ to the right in the product of oscillators along the line of 41. 2) Also the (anti)commutators of the two tensor products are written in terms of sum of the tensor product between two product of oscillators and two $U(2)$ matrix product with appropriate minus or plus sign depending on the commutators or anticommutators. We can insert the general $s$-th $\mathcal{N} = 4$ higher spin multiplet in the mathematica program rather than fixed one.
Maybe we can generalize the work of \cite{28, 29} to the matrix computations and obtain the $\mathcal{N} = 4$ higher spin algebra at the general $\lambda$ in closed form for any spins $s_1, s_2$. Because we know the answer for $\lambda = 0$, we want to obtain the $\mathcal{N} = 4$ higher spin algebra $sh_{s_2}[\lambda]$ at nonzero $\lambda$ for general $s_1$ and $s_2$. Maybe we should examine the explicit cases for low spins and see how to express those structure constants with the possible spin dependence.

- **Orthogonal group**

  It is an open problem to describe the corresponding $\mathcal{N} = 4$ orthogonal coset model and to observe what the corresponding $\mathcal{N} = 4$ higher spin algebra, constructed from the oscillators, is. See also the relevant paper \cite{54}. Furthermore, according to \cite{10}, this case with fixed $K$ and $L$ which are different from 2 will be related to the explicit application of type IIB string theory for the specific ratio of two three spheres.

- **Nonsupersymmetric cases with general $(K, L)$ values**

  According to the observation of \cite{10}, the coset model can be generalized by take the arbitrary values for $K$ and $L$. It would be interesting to check whether there exists any nontrivial extended conformal algebra, although there is no supersymmetry, by calculating the OPEs for several $K$ and $L$ values in the coset model. One of the motivations in this direction is to consider the application of type IIB string theory with orthogonal coset model.

- **Construct the free fields with non $U(N)$ invariant**

  In this paper, we have restricted to focus on the $U(N)$ invariant quantity for the higher spin currents. However, we can construct the higher spin currents which have the explicit $U(N)$ indices. In other words, it is an open problem to understand the role of these higher spin currents with nonsinglet $U(N)$. See also the relevant papers on this direction \cite{19, 20}.

- **Any relations with the rectangular $W$ algebras**

  By allowing the group $G = SU(N + 2)$ to generalize to $G = SU(N + K)$ with $K \neq 2$, maybe we can see how the higher spin currents can be related to the ones of the rectangular $W$ algebras studied recently in \cite{37, 38, 39}. For the case of $K \geq 3$, there is a symmetric tensor of rank three in $SU(K)$. Maybe we can construct the higher spin currents (like as Sugawara construction) using these tensors.
• The classical asymptotic symmetry algebra $\mathcal{W}^{\mathcal{N}=4}_{\infty,cl}[^{\lambda}]$

According to the large $\mathcal{N} = 4$ holography in [1], there should be the relation between the nonlinear $\mathcal{W}^{\mathcal{N}=4}_{\infty}[^{\lambda}]$ algebra and the classical asymptotic nonlinear symmetry algebra (which can be obtained from $\text{shs}_2[^{\lambda}]$ by relaxing the wedge condition) of the $AdS_3$ bulk theory. It is an open problem to check whether the classical asymptotic symmetry algebra of the Vasiliev $AdS_3$ higher spin theory with matrix generalization can be reproduced from the $\mathcal{W}^{\mathcal{N}=4}_{\infty}[^{\lambda}]$ algebra in the coset model by taking the large central charge limit or not. The nonlinear terms in both sides should match with each other.

• Adding the bosonic spin-1 operators in the free field construction

Although we have not analyzed for this possibility fully in this paper, it would be interesting to observe whether we can add the bosonic spin-1 current in the free field construction described in section 6. Should we modify the stress energy tensor of spin-2? Can we also consider for the general $K$ and $L$ in order to have consistent closed algebra? Maybe we should also consider the $U(N)$ nonsinglet cases.

• Beyond the bilinear construction (cubic, quartic, · · ·) in the free field construction

In the construction of higher spin square [69, 70], it is possible to consider the free field construction beyond the bilinear terms. The corresponding algebra coming from higher spin currents will be, in general, nonlinear and it is interesting to obtain the nontrivial structure behind this rather complicated algebras. See also the relevant paper [17] in the context of horizontal algebra.

• The $\mathcal{N} = 3$ example

In [71], the $\mathcal{N} = 2$ supersymmetry is enhanced by taking the critical level which allows us to introduce the free fermionic fields. We expect that by introducing the matrix generalization in the $AdS_3$ higher spin theory, the corresponding higher spin algebra can be obtained from the oscillator formalism. It will be an open problem to study in detail. See also the relevant paper [64].

Acknowledgments

We would like to thank C. Peng for the free field construction and how to read off the $\lambda$ value from the free fields, S. Odake for his algebra and how to obtain it from the mode expansions of free fields and Y. Hikida for the large level limit and its relation to rectangular $W$ algebra. This research was supported by the Basic Science Research Program through
A Quasi (super)primary fields from section 2

We present the various quasi primary super fields appearing in (2.9) of the section 2.

\[ Q_{2}^{(2),ij} = c_{1}^{j}J^{4-i} + c_{2}^{j}J^{4-i} + c_{3}^{j}J^{4-i} + c_{4}^{j}J^{4-i} + c_{5}^{j}(J^{4-i}J^{1} + c_{6}^{j}J^{4-i}J^{1}) , \]

\[ Q_{3}^{(2),ij} = c_{1}^{j}J^{4-i} + c_{2}^{j}J^{4-i} + c_{3}^{j}J^{4-i} + c_{4}^{j}J^{4-i} + c_{5}^{j}(J^{4-i}J^{1} + c_{6}^{j}J^{4-i}J^{1}) + c_{7}^{j}(J^{4-i}J^{1} + c_{8}^{j}J^{4-i}J^{1}) , \]

\[ Q_{2}^{(2),ij} = c_{1}^{j}J^{4-i} + c_{2}^{j}J^{4-i} + c_{3}^{j}J^{4-i} + c_{4}^{j}J^{4-i} + c_{5}^{j}(J^{4-i}J^{1} + c_{6}^{j}J^{4-i}J^{1}) , \]

\[ Q_{2}^{(2),ij} = c_{1}^{j}J^{4-i} + c_{2}^{j}J^{4-i} + c_{3}^{j}J^{4-i} + c_{4}^{j}J^{4-i} + c_{5}^{j}(J^{4-i}J^{1} + c_{6}^{j}J^{4-i}J^{1}) . \]

We have complete expressions of (2.10) in (A.1).

\[ ^{39} \text{All the coefficients which are not in the Appendices are given explicitly in the attached ancillary.nb file.} \]
The fundamental five OPEs in component approach corresponding to (2.9) are described as

\[
\begin{align*}
\Phi_0^{(1)}(z) \Phi_0^{(1)}(w) &= \frac{1}{(z-w)^2} \epsilon^{0,2} + \cdots, \\
\Phi_{\frac{1}{2}}^{(1),i}(z) \Phi_0^{(1)}(w) &= \frac{1}{(z-w)} Q_{\frac{1}{2}}^{(\frac{1}{2}),i}(w) + \cdots, \\
\Phi_{1}^{(1),ij}(z) \Phi_0^{(1)}(w) &= \frac{1}{(z-w)^2} Q_{1}^{(1),ij}(w) + \frac{1}{(z-w)} \left[ \partial Q_{1}^{(1),ij} + Q_{1}^{(2),ij} \right](w) + \cdots, \\
\Phi_{\frac{3}{2}}^{(1),i}(z) \Phi_0^{(1)}(w) &= \frac{1}{(z-w)^3} Q_{\frac{3}{2}}^{(\frac{1}{2}),i}(w) \\
&\quad + \frac{1}{(z-w)^2} \left[ 2 \partial Q_{\frac{3}{2}}^{(\frac{1}{2}),i} + Q_{\frac{3}{2}}^{(\frac{3}{2}),i} - \frac{(k-N)}{3(2+k+N)} Q_{\frac{3}{2}}^{(\frac{3}{2}),i} \right](w) \\
&\quad + \frac{1}{(z-w)} \left[ \frac{3}{2} \partial^2 Q_{\frac{3}{2}}^{(1),i} + \partial Q_{\frac{3}{2}}^{(2),i} + Q_{\frac{3}{2}}^{(3),i} \right](w) + \cdots, \\
\Phi_{2}^{(1),i}(z) \Phi_0^{(1)}(w) &= \frac{1}{(z-w)^3} Q_{2}^{(1)}(w) \\
&\quad + \frac{1}{(z-w)^2} \left[ \frac{3}{2} \partial Q_{2}^{(1)} + Q_{2}^{(2)} \\
&\quad - \frac{8(k-N)}{(5+4k+4N+3kN)} (\Phi_0^{(1)} \Phi_0^{(1)} + \epsilon^{0,2} L) \right](w) \\
&\quad + \frac{1}{(z-w)} \left[ \partial^2 Q_{2}^{(1)} + \partial Q_{2}^{(2)} + Q_{2}^{(3)} \\
&\quad - \frac{8(k-N)}{(5+4k+4N+3kN)} \partial(\Phi_0^{(1)} \Phi_0^{(1)} + \epsilon^{0,2} L) \right](w) + \cdots. \quad (A.2)
\end{align*}
\]

The quasi primary fields appearing in (A.2) are summarized by

\[
\begin{align*}
Q_{\frac{1}{2}}^{(\frac{1}{2}),i} &= w_{1,\frac{1}{2}} \tilde{G}^i + w_{2,\frac{1}{2}} \partial \Gamma^i + w_{3,\frac{1}{2}} \tilde{T}^{i1} \Gamma^j + w_{4,\frac{1}{2}} U^i + w_{5,\frac{1}{2}} \varepsilon^{i1jk} \Gamma^j \Gamma^k \\
Q_{1}^{(1),ij} &= w_{1,1} \Gamma^i \Gamma^j + w_{2,1} \tilde{T}^{ij} + w_{3,1} \varepsilon^{i1jk} \Gamma^k \Gamma^1 + w_{4,1} T^{ij}, \\
Q_{1}^{(2),ij} &= w_{1,2} (\tilde{G}^i \Gamma^j - \tilde{G}^j \Gamma^i) + w_{2,2} \partial \tilde{T}^{ij} + w_{3,2} \partial (\Gamma^i \Gamma^j) + w_{4,2} \varepsilon^{i1jk} \partial(\Gamma^k \Gamma^1) \\
&\quad + w_{5,3} (\tilde{T}^{ik} \Gamma^j \Gamma^k - \tilde{T}^{jk} \Gamma^i \Gamma^k) + w_{6,4} U^i \Gamma^j, \\
Q_{\frac{3}{2}}^{(\frac{1}{2}),i} &= w_{1,\frac{3}{2}} \Gamma^i, \\
Q_{\frac{3}{2}}^{(1),ij} &= w_{1,\frac{3}{2}} \tilde{G}^i + w_{2,\frac{3}{2}} \partial \Gamma^i + w_{3,\frac{3}{2}} \tilde{T}^{i1} \Gamma^j + w_{4,\frac{3}{2}} U^i + w_{5,\frac{3}{2}} \varepsilon^{i1jk} \Gamma^j \Gamma^k + w_{6,\frac{3}{2}} \Gamma^j \Gamma^k + w_{7,\frac{3}{2}} \Gamma^j \Gamma^k, \\
Q_{\frac{3}{2}}^{(2),ij} &= w_{1,\frac{3}{2}} \Phi_0^{(2),i} + w_{2,\frac{3}{2}} \Phi_0^{(1),i} + w_{3,\frac{3}{2}} \tilde{T}^{i1} \Gamma^j + w_{4,\frac{3}{2}} U^i + w_{5,\frac{3}{2}} \partial U^i + w_{6,\frac{3}{2}} \partial \Gamma^i.
\end{align*}
\]

\[40\] The expressions with typewriter font in Appendices will vanish under the large \((N,k)\) limit. Some of the linear terms also vanish.

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\[
+ w_{7,\frac{i}{2}} ^{\text{T}ij} \partial \Gamma^j + w_{8,\frac{i}{2}} \Gamma^i \partial \Gamma^j + w_{9,\frac{i}{2}} ^{\text{T}ij} \Gamma^j + w_{10,\frac{i}{2}} \partial ^{\text{T}ij} \Gamma^j + w_{11,\frac{i}{2}} ^{\text{T}ij} \tilde{\Gamma}^j \\
+ w_{12,\frac{i}{2}} ^{\text{T}ij} \partial \Gamma^j + w_{13,\frac{i}{2}} ^{\text{T}ij} \Gamma^j + w_{14,\frac{i}{2}} \partial \tilde{\Gamma}^j + w_{15,\frac{i}{2}} \tilde{\Gamma}^j + \varepsilon^{ijkl} (w_{16,\frac{i}{2}} ^{\text{T}ij} \tilde{\Gamma}^k \tilde{\Gamma}^l + w_{17,\frac{i}{2}} \tilde{\Gamma}^j \partial \tilde{\Gamma}^k + w_{18,\frac{i}{2}} \tilde{\Gamma}^j \Gamma^k \partial \Gamma^l + w_{19,\frac{i}{2}} \tilde{\Gamma}^j \Gamma^k \partial \Gamma^l + w_{20,\frac{i}{2}} \tilde{\Gamma}^j \Gamma^k \partial \Gamma^l + w_{21,\frac{i}{2}} \tilde{\Gamma}^j \partial \Gamma^l) + w_{22,\frac{i}{2}} \tilde{\Gamma}^j \partial \Gamma^l \\
+ w_{23,\frac{i}{2}} \partial (\Gamma^j \Gamma^k \Gamma^l) + w_{24,\frac{i}{2}} \partial ^{\text{T}jk} \Gamma^l + w_{25,\frac{i}{2}} \tilde{\Gamma}^j \partial \Gamma^l \\
Q_2^{(1)} = w_{1,1} U \\
Q_2^{(2)} = w_{1,2} \Phi_0 (2) + w_{2,2} \tilde{L} + w_{3,2} \tilde{G}^i \Gamma^i + \varepsilon^{ijkl} (w_{4,2} \Gamma^i \Gamma^j \Gamma^k \Gamma^l + w_{5,2} \tilde{\Gamma}^i \Gamma^j \Gamma^k \Gamma^l) + w_{6,2} ^{\text{T}ij} \partial \Gamma^j \\
+ w_{7,2} ^{\text{T}ij} \Gamma^i \Gamma^j + w_{8,2} ^{\text{T}ij} \Gamma^i \Gamma^j + w_{9,2} \partial U + w_{10,2} \partial \Gamma^i + w_{11,2} \Gamma^i U \\
Q_2^{(3)} = w_{1,3} ^{\tilde{G}^i \partial \Gamma^i} + w_{2,3} \tilde{G}^i \partial \Gamma^i + w_{3,3} \partial \Gamma^i + w_{4,3} \partial U + w_{5,3} \partial \Gamma^i + w_{6,3} \partial (\tilde{T}^i \Gamma^i \Gamma^j) \\
+ w_{7,3} \Gamma^i \partial \tilde{T}^i + w_{8,3} \tilde{U} \partial \Gamma^i.
\]

We can obtain (A.3) from (A.1) by taking the appropriate procedures in (3.18). Note that the new higher spin-2 primary field $\Phi_0 (2)$ appears in the quasi primary field $Q_2^{(2)}$. See also (2.10) \[41\] There are also linear terms having the typewriter font. The fields with a tilde are defined in the footnote \[10\]

## B Quasi (super)primary fields from section 3

The various quasi primary fields in sections 3.1, 3.2, and 3.3 are presented as follows:

\[
Q_0^{(1)} = d_1^{0,2} \Phi_0 (1) (w), \\
Q_2^{(\frac{1}{2})} = w_{1,2} ^{\tilde{\Phi}(1)(\frac{1}{2})} + w_{2,2} \tilde{\Phi}(1)(\frac{1}{2}) + w_{3,2} \partial \tilde{\Phi}(1)(\frac{1}{2}) + w_{4,2} \partial \tilde{\Phi}(1)(\frac{1}{2}) + w_{5,2} \Gamma_i \partial \Phi(1)(\frac{1}{2}) + w_{6,2} \partial \tilde{\Phi}(1)(\frac{1}{2}) \\
+ w_{7,2} \partial \Gamma \Phi(1)(\frac{1}{2}) + w_{8,2} \partial \Gamma \Phi(1)(\frac{1}{2}) + w_{9,2} \partial \Gamma \tilde{\Phi}(1)(\frac{1}{2}) + w_{10,2} \partial \Gamma \Gamma \Phi(1)(\frac{1}{2}) + w_{11,2} \partial \Gamma \Gamma \Gamma \Phi(1)(\frac{1}{2}) + w_{12,2} \Gamma \partial \Gamma \Phi(1)(\frac{1}{2}) \\
+ w_{13,2} \Gamma \partial \Gamma \Gamma \Phi(1)(\frac{1}{2}) + w_{14,2} \Gamma \partial \Gamma \Gamma \Gamma \Phi(1)(\frac{1}{2}) \\
Q_2^{(2)} = w_{1,2} \tilde{\Phi}(1)(\frac{1}{2}) + w_{2,2} \Phi(1)(\frac{1}{2}) + w_{3,2} \tilde{\Phi}(1)(\frac{1}{2}) + w_{4,2} \partial \tilde{\Phi}(1)(\frac{1}{2}) + w_{5,2} \partial \Phi(1)(\frac{1}{2}) \\
+ \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2}) \\
+ \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2}) \\
Q_2^{(3)} = w_{1,3} \partial \Phi(1)(\frac{1}{2}) + w_{2,3} \partial \Phi(1)(\frac{1}{2}) + w_{3,3} \partial \Phi(1)(\frac{1}{2}) + w_{4,3} \partial \Phi(1)(\frac{1}{2}) + w_{5,3} \partial \Phi(1)(\frac{1}{2}) \\
+ \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2}) \\
+ \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2}) \\
+ \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2}) + \partial \Phi(1)(\frac{1}{2})}
\]

\[41\] All the other 120(= 136 – 16) OPEs can be read off from the relations \[9\] and \[20\] in the $\mathcal{N} = 4$ superspace. Then we obtain all the results in Appendix $H$ of \[9\].
+ $\varepsilon^{21k_1}(w_{16,3}(T^{1k_1}w_{1}^{1,1}) + r^{k_1}T^{1k_1}w_{1}^{1,1}) + w_{17,3}T^{k_1}w_{1}^{1,1} + w_{18,3}T^{k_1}w_{1}^{1,1}$
+ $w_{19,3}T^{1k_1}w_{1}^{1,1} + w_{20,3}(T^{1k_1}w_{1}^{1,1} + T^{k_1}w_{1}^{1,1}) + w_{21,3}T^{1k_1}w_{1}^{1,1} + w_{22,3}T^{k_1}w_{1}^{1,1})$
+ $w_{23,3}(T^{1k_1}w_{1}^{1,1} + T^{1k_1}w_{1}^{1,1}) + w_{24,3}(T^{k_1}T^{1k_1}w_{1}^{1,1} + T^{1k_1}T^{1k_1}w_{1}^{1,1}) + w_{25,3}T^{1k_1}w_{1}^{1,1})$
+ $w_{26,3}(T^{1k_1}T^{1k_1}w_{1}^{1,1} + T^{1k_1}T^{1k_1}w_{1}^{1,1}) + T^{1k_1}T^{1k_1}w_{1}^{1,1})$
+ $w_{28,3}(T^{1k_1}T^{1k_1}w_{1}^{1,1} + T^{1k_1}T^{1k_1}w_{1}^{1,1})$

$Q_{\frac{1}{2},i}^\frac{i}{j} = w_{1,3}T^{\Phi_1^1}w_{1}^{1,1} + w_{2,3}T^{\Phi_1^1}w_{1}^{1,1}$

$Q_{\frac{2}{3},i}^\frac{i}{j} = w_{1,5}T^{\Phi_1^1}w_{1}^{1,1} + w_{2,5}T^{\Phi_1^1}w_{1}^{1,1} + w_{3,5}T^{\Phi_1^1}w_{1}^{1,1} + w_{4,5}T^{\Phi_1^1}w_{1}^{1,1} + w_{5,5}T^{\Phi_1^1}w_{1}^{1,1}$
+ $w_{6,5}T^{\Phi_1^1}w_{1}^{1,1} + w_{7,5}T^{\Phi_1^1}w_{1}^{1,1} + w_{8,5}T^{\Phi_1^1}w_{1}^{1,1} + w_{9,5}T^{\Phi_1^1}w_{1}^{1,1}$
+ $w_{10,5}T^{\Phi_1^1}w_{1}^{1,1} + w_{11,5}T^{\Phi_1^1}w_{1}^{1,1}$

$Q_{\frac{3}{4},i}^\frac{i}{j} = w_{1,7}T^{\Phi_1^1}w_{1}^{1,1} + w_{2,7}T^{\Phi_1^1}w_{1}^{1,1} + w_{3,7}T^{\Phi_1^1}w_{1}^{1,1} + w_{4,7}T^{\Phi_1^1}w_{1}^{1,1} + w_{5,7}T^{\Phi_1^1}w_{1}^{1,1}$
+ $w_{6,7}T^{\Phi_1^1}w_{1}^{1,1} + w_{7,7}T^{\Phi_1^1}w_{1}^{1,1} + w_{8,7}T^{\Phi_1^1}w_{1}^{1,1} + w_{9,7}T^{\Phi_1^1}w_{1}^{1,1}$
+ $w_{10,7}T^{\Phi_1^1}w_{1}^{1,1} + w_{11,7}T^{\Phi_1^1}w_{1}^{1,1}$

$Q_{\frac{4}{5},i}^\frac{i}{j} = w_{1,9}T^{\Phi_1^1}w_{1}^{1,1} + w_{2,9}T^{\Phi_1^1}w_{1}^{1,1} + w_{3,9}T^{\Phi_1^1}w_{1}^{1,1} + w_{4,9}T^{\Phi_1^1}w_{1}^{1,1} + w_{5,9}T^{\Phi_1^1}w_{1}^{1,1}$
+ $w_{6,9}T^{\Phi_1^1}w_{1}^{1,1} + w_{7,9}T^{\Phi_1^1}w_{1}^{1,1} + w_{8,9}T^{\Phi_1^1}w_{1}^{1,1} + w_{9,9}T^{\Phi_1^1}w_{1}^{1,1}$
+ $w_{10,9}T^{\Phi_1^1}w_{1}^{1,1} + w_{11,9}T^{\Phi_1^1}w_{1}^{1,1}$

$Q_{\frac{5}{6},i}^\frac{i}{j} = w_{1,11}T^{\Phi_1^1}w_{1}^{1,1} + w_{2,11}T^{\Phi_1^1}w_{1}^{1,1} + w_{3,11}T^{\Phi_1^1}w_{1}^{1,1} + w_{4,11}T^{\Phi_1^1}w_{1}^{1,1} + w_{5,11}T^{\Phi_1^1}w_{1}^{1,1}$
+ $w_{6,11}T^{\Phi_1^1}w_{1}^{1,1} + w_{7,11}T^{\Phi_1^1}w_{1}^{1,1} + w_{8,11}T^{\Phi_1^1}w_{1}^{1,1} + w_{9,11}T^{\Phi_1^1}w_{1}^{1,1}$
+ $w_{10,11}T^{\Phi_1^1}w_{1}^{1,1} + w_{11,11}T^{\Phi_1^1}w_{1}^{1,1}$

$Q_{\frac{6}{7},i}^\frac{i}{j} = w_{1,13}T^{\Phi_1^1}w_{1}^{1,1} + w_{2,13}T^{\Phi_1^1}w_{1}^{1,1} + w_{3,13}T^{\Phi_1^1}w_{1}^{1,1} + w_{4,13}T^{\Phi_1^1}w_{1}^{1,1} + w_{5,13}T^{\Phi_1^1}w_{1}^{1,1}$
+ $w_{6,13}T^{\Phi_1^1}w_{1}^{1,1} + w_{7,13}T^{\Phi_1^1}w_{1}^{1,1} + w_{8,13}T^{\Phi_1^1}w_{1}^{1,1} + w_{9,13}T^{\Phi_1^1}w_{1}^{1,1}$
+ $w_{10,13}T^{\Phi_1^1}w_{1}^{1,1} + w_{11,13}T^{\Phi_1^1}w_{1}^{1,1}$

$Q_{\frac{7}{8},i}^\frac{i}{j} = w_{1,15}T^{\Phi_1^1}w_{1}^{1,1} + w_{2,15}T^{\Phi_1^1}w_{1}^{1,1} + w_{3,15}T^{\Phi_1^1}w_{1}^{1,1} + w_{4,15}T^{\Phi_1^1}w_{1}^{1,1} + w_{5,15}T^{\Phi_1^1}w_{1}^{1,1}$
+ $w_{6,15}T^{\Phi_1^1}w_{1}^{1,1} + w_{7,15}T^{\Phi_1^1}w_{1}^{1,1} + w_{8,15}T^{\Phi_1^1}w_{1}^{1,1} + w_{9,15}T^{\Phi_1^1}w_{1}^{1,1}$
+ $w_{10,15}T^{\Phi_1^1}w_{1}^{1,1} + w_{11,15}T^{\Phi_1^1}w_{1}^{1,1}$

$Q_{\frac{8}{9},i}^\frac{i}{j} = w_{1,17}T^{\Phi_1^1}w_{1}^{1,1} + w_{2,17}T^{\Phi_1^1}w_{1}^{1,1} + w_{3,17}T^{\Phi_1^1}w_{1}^{1,1} + w_{4,17}T^{\Phi_1^1}w_{1}^{1,1} + w_{5,17}T^{\Phi_1^1}w_{1}^{1,1}$
+ $w_{6,17}T^{\Phi_1^1}w_{1}^{1,1} + w_{7,17}T^{\Phi_1^1}w_{1}^{1,1} + w_{8,17}T^{\Phi_1^1}w_{1}^{1,1} + w_{9,17}T^{\Phi_1^1}w_{1}^{1,1}$
+ $w_{10,17}T^{\Phi_1^1}w_{1}^{1,1} + w_{11,17}T^{\Phi_1^1}w_{1}^{1,1}$
The full expressions of (3.8) and (3.11) are given in (B.1).

The quasi primary fields appearing in (3.19) of section 3.4 are presented as follows:

\[ Q^{(1)}_0 = d^{0,2}_1 \Phi^{(1)}, \]
\[
Q_{(3)}^{(1),i} = d_{1}^{\frac{1}{2}} D^{4-i} \Phi(1) + d_{2}^{\frac{1}{2}} \partial D^{4} \Phi(1) + d_{3}^{\frac{1}{2}} J^{4-i} \Phi(1) + d_{4}^{\frac{1}{2}} J^{1} \partial \Phi(1) + d_{5}^{\frac{1}{2}} J^{4} \Phi(1)
+ d_{6}^{\frac{1}{2}} J^{4-i} \Phi(1) + d_{7}^{\frac{1}{2}} \partial J J^{4} \Phi(1) + d_{8}^{\frac{1}{2}} J^{4-i} J D^{4} \Phi(1)
+ d_{9}^{\frac{1}{2}} J^{4} J D^{4} \Phi(1) + d_{10}^{\frac{1}{2}} J^{4} J^{4} \Phi(1) + d_{11}^{\frac{1}{2}} J^{4} J^{4} \Phi(1),
\]

\[
Q_{(1)}^{(1),ij} = d_{1}^{\frac{1}{2}} D^{4-i} \Phi(1) + d_{2}^{\frac{1}{2}} J^{4-i} \Phi(1) + d_{3}^{\frac{1}{2}} J^{1} \Phi(1) + d_{4}^{\frac{1}{2}} J^{1} \Phi(1)
+ d_{5}^{\frac{1}{2}} (J^{1} D^{4} \Phi(1) - J^{1} D^{1} \Phi(1)) + d_{6}^{\frac{1}{2}} D \partial \Phi(1) + d_{7}^{\frac{1}{2}} (J^{1} D^{4} \Phi(1) - J^{1} D^{1} \Phi(1))
+ d_{8}^{\frac{1}{2}} (J^{1} D^{4} D^{4} - J^{1} D^{4} D^{1} \Phi(1)) + d_{9}^{\frac{1}{2}} (J^{1} D^{4} D^{4} - J^{1} D^{4} D^{1} \Phi(1))
+ d_{10}^{\frac{1}{2}} (J^{1} D^{4} D^{4} - J^{1} D^{4} D^{1} \Phi(1)) + d_{11}^{\frac{1}{2}} (J^{1} D^{4} D^{4} - J^{1} D^{4} D^{1} \Phi(1))
+ d_{12}^{\frac{1}{2}} (J^{1} D^{4} D^{4} - J^{1} D^{4} D^{1} \Phi(1)) + d_{13}^{\frac{1}{2}} (J^{1} D^{4} D^{4} - J^{1} D^{4} D^{1} \Phi(1))
+ d_{14}^{\frac{1}{2}} (J^{1} D^{4} D^{4} - J^{1} D^{4} D^{1} \Phi(1)) + d_{15}^{\frac{1}{2}} D \partial \Phi(1) + \ldots,
\]

The complete expressions of (3.20) and (3.21) are given in (B.2). Note that all the terms in
these quasi primary fields in (B.2) have the dependence of $\mathcal{N} = 4$ higher spin multiplets $\Phi^{(n)}$ where $s = 1, 2, 3$ as well as the stress energy tensor $J$. We obtain (B.1) from (B.2) as before.

## C Quasi primary fields from section 4

The previous OPE in (4.1) can be described further as

$$
\phi_2^{(2)}(z) \phi_0^{(2)}(w) = \frac{1}{(z-w)^6} 8 \alpha e_0^{0,4} + \frac{1}{(z-w)^5} Q_2^{(1)}(w) 
$$

$$
+ \frac{1}{(z-w)^4} \left[ \frac{3}{2} \partial Q_2^{(1)} + Q_2^{(2)} - 6 p_1 Q_0^{(2)} - p_2 E_2^{(2)} \right](w) 
$$

$$
+ \frac{1}{(z-w)^3} \left[ \partial^2 Q_2^{(1)} + \partial Q_2^{(2)} + Q_2^{(3)} - p_1 \partial Q_0^{(2)} - p_2 E_2^{(3)} \right](w) 
$$

$$
+ \frac{1}{(z-w)^2} \left[ \frac{5}{12} \partial^3 Q_2^{(1)} + \frac{1}{2} \partial^2 Q_2^{(2)} + \frac{5}{6} \partial Q_2^{(3)} + Q_2^{(4)} - p_2 E_2^{(4)} \right](w) 
$$

$$
+ \frac{1}{(z-w)} \left[ \frac{1}{8} \partial^4 Q_2^{(1)} + \frac{1}{6} \partial^3 Q_2^{(2)} + \frac{5}{14} \partial^2 Q_2^{(3)} + \frac{3}{4} \partial Q_2^{(4)} + Q_2^{(5)} - p_2 E_2^{(5)} \right](w) 
$$

$$
+ \cdots. 
$$

(C.1)

where we introduce the last singular $p_2$ terms in (4.1) as

$$
E_2^{(4-n)} \equiv \left\{ (L \phi_0^{(2)}) \phi_0^{(2)} \right\}_{n+2}, \quad n = -1, 0, 1, 2, 
$$

(C.2)

and $p_1$ and $p_2$ are given in (4.2). Then we obtain (C.2) as follows:

$$
E_2^{(2)}(w) = (4 Q_0^{(2)} + e_0^{0,4} L)(w), 
$$

$$
E_2^{(3)}(w) = (\frac{5}{2} \partial Q_0^{(2)} + e_0^{0,4} \partial L)(w), 
$$

$$
E_2^{(4)}(w) = (\frac{1}{2} \partial^2 Q_0^{(2)} + L Q_0^{(2)} + \frac{1}{2} e_0^{0,4} \partial^2 L + 2 \phi_0^{(2)} \phi_0^{(2)})(w), 
$$

$$
E_2^{(5)}(w) = (-\frac{1}{24} \partial^3 Q_0^{(2)} + \frac{1}{2} L \partial Q_0^{(2)} + \partial L Q_0^{(2)} + \frac{1}{6} e_0^{0,4} \partial^3 L + \frac{3}{2} \partial (\phi_0^{(2)} \phi_0^{(2)}))(w). 
$$

(C.3)

Moreover, the remaining four kinds of fundamental OPEs can be described as

$$
\phi_0^{(2)}(z) \phi_0^{(2)}(w) = \frac{1}{(z-w)^6} e_0^{0,4} + \frac{1}{(z-w)^5} Q_0^{(2)}(w) + \frac{1}{(z-w)^4} \partial Q_0^{(2)}(w) + \cdots, 
$$

$$
\phi_0^{(2),i}(z) \phi_0^{(2)}(w) = \frac{1}{(z-w)^3} Q_0^{(2)i}(w) + \frac{1}{(z-w)^2} \left[ \frac{2}{3} \partial Q_0^{(2)i} + Q_0^{(2),i} \right](w) 
$$

$$
+ \frac{1}{(z-w)} \left[ \frac{1}{4} \partial^2 Q_0^{(2)i} + \frac{3}{5} \partial Q_0^{(2)i} + Q_0^{(2),i} \right](w) + \cdots, 
$$

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\[\Phi_{1,ij}^{(2)}(z) \Phi_{0}^{(2)}(w) = \frac{1}{(z-w)^4} Q_{1,ij}^{(1)}(w) + \frac{1}{(z-w)^3} \left[ \partial Q_{1,ij}^{(1)} + Q_{1,ij}^{(2)} \right](w) + \frac{1}{(z-w)^2} \left[ \frac{1}{2} \partial^2 Q_{1,ij}^{(1)} + \frac{3}{4} \partial Q_{1,ij}^{(2)} + Q_{1,ij}^{(3)} \right](w) + \frac{1}{(z-w)} \left[ \frac{1}{6} \partial^3 Q_{1,ij}^{(1)} + \frac{3}{10} \partial^2 Q_{1,ij}^{(2)} + \frac{2}{3} \partial Q_{1,ij}^{(3)} + Q_{1,ij}^{(4)} \right](w) - (i \leftrightarrow j) + \cdots,\]

\[\Phi_{\frac{3}{2},ij}^{(2)}(z) \Phi_{0}^{(2)}(w) = \frac{1}{(z-w)^5} Q_{\frac{3}{2},ij}^{(1)}(w) + \frac{1}{(z-w)^4} \left[ 2 \partial Q_{\frac{3}{2},ij}^{(1)} + Q_{\frac{3}{2},ij}^{(2)} \right](w) + \frac{1}{(z-w)^3} \left[ \frac{3}{2} \partial^2 Q_{\frac{3}{2},ij}^{(1)} + \partial Q_{\frac{3}{2},ij}^{(2)} + Q_{\frac{3}{2},ij}^{(3)} \right](w) + \frac{1}{(z-w)^2} \left[ \frac{2}{3} \partial^3 Q_{\frac{3}{2},ij}^{(1)} + \frac{1}{2} \partial^2 Q_{\frac{3}{2},ij}^{(2)} + \frac{4}{5} \partial Q_{\frac{3}{2},ij}^{(3)} + Q_{\frac{3}{2},ij}^{(4)} \right](w) + \frac{1}{(z-w)} \left[ \frac{5}{24} \partial^4 Q_{\frac{3}{2},ij}^{(1)} + \frac{1}{6} \partial^3 Q_{\frac{3}{2},ij}^{(2)} + \frac{1}{3} \partial^2 Q_{\frac{3}{2},ij}^{(3)} + \frac{5}{7} \partial Q_{\frac{3}{2},ij}^{(4)} + Q_{\frac{3}{2},ij}^{(5)} \right](w) + \frac{2 \alpha}{5} \sum_{n=2}^{4} \frac{1}{(z-w)^n} \left\{ \partial \Phi_{\frac{3}{2},ij}^{(2)} \Phi_{0}^{(2)} \right\}_n(w) + \cdots.\]  

(C.4)

We have the final quasi primary fields appearing in (C.1) with (C.3) and (C.4) as follows:

\[Q_{0}^{(2)} = w_{0,2} \Phi_{0}^{(2)}(w) + w_{1,2} \Phi_{0}^{(1)} \Phi_{0}^{(1)} + w_{2,2} \tilde{L} + w_{3,2} \partial U + w_{4,2} UU + w_{5,2} T^{ij} T^{ij} + w_{6,2} T^{ij} \tilde{T}^{ij} + w_{7,2} \Gamma^{i} \Gamma^{j} T^{ij} + w_{8,2} \Gamma^{i} \Gamma^{j} \tilde{T}^{ij} + w_{9,2} \Gamma^{i} \partial \Gamma^{i} + \varepsilon^{ijkl} w_{10,2} \Gamma^{i} \Gamma^{j} \Gamma^{k} \Gamma^{l},\]

\[Q_{\frac{1}{2}}^{(2)} = w_{1,\frac{1}{2}} \tilde{G}^{i} + w_{2,\frac{1}{2}} \partial \Gamma^{i} + w_{3,\frac{1}{2}} \Gamma^{i} U + \varepsilon^{ijkl} \left( w_{4,\frac{1}{2}} \Gamma^{j} T^{kl} + w_{5,\frac{1}{2}} \Gamma^{j} \Gamma^{k} \Gamma^{l} \right),\]

\[Q_{\frac{3}{2}}^{(2)} = w_{1,\frac{3}{2}} \Phi_{\frac{1}{2}}^{(2)} + w_{2,\frac{3}{2}} \Phi_{\frac{1}{2}}^{(1)} \Phi_{\frac{1}{2}}^{(1)} + w_{3,\frac{3}{2}} \partial \tilde{G}^{i} + w_{4,\frac{3}{2}} \partial \Gamma^{i} + w_{5,\frac{3}{2}} \partial U + w_{6,\frac{3}{2}} T^{ij} \partial \Gamma^{i} \partial U + w_{7,\frac{3}{2}} \partial T^{ij} U,\]

\[Q_{\frac{5}{2}}^{(2)} = w_{1,\frac{5}{2}} \Phi_{\frac{1}{2}}^{(2)} + w_{2,\frac{5}{2}} \partial \Phi_{\frac{1}{2}}^{(2)} + w_{3,\frac{5}{2}} \Phi_{\frac{1}{2}}^{(1)} \Phi_{\frac{1}{2}}^{(1)} + w_{4,\frac{5}{2}} \Phi_{\frac{1}{2}}^{(1)} \partial \Phi_{\frac{1}{2}}^{(1)} + \cdots + w_{20,\frac{5}{2}} U \Gamma^{i} \Phi_{0}^{(1)} \Phi_{0}^{(1)} + \cdots + w_{21,\frac{5}{2}} \partial \Phi_{0}^{(1)} \Phi_{0}^{(1)} + w_{22,\frac{5}{2}} \partial \tilde{L} \Gamma^{i} + w_{23,\frac{5}{2}} \tilde{L} \partial \Gamma^{i} + \cdots + w_{67,\frac{5}{2}} \tilde{T}^{ij} \tilde{T}^{ij} \partial \Gamma^{i} + \varepsilon^{ijkl} \left( w_{68,\frac{5}{2}} \Gamma^{j} \Phi_{0}^{(2),kl} + w_{69,\frac{5}{2}} \Gamma^{j} \Phi_{1}^{(2),l} + \cdots + w_{117,\frac{5}{2}} \Gamma^{j} T^{ij} T^{kl} \right)\]

\[Q_{1}^{(1,ij)} = w_{1,1} T^{ij} + w_{2,1} \Gamma^{i} \Gamma^{j} + \varepsilon^{ijkl} \left( w_{3,1} T^{kl} + w_{4,1} \Gamma^{k} \Gamma^{l} \right) - (i \leftrightarrow j),\]

\[Q_{1}^{(2,ij)} = w_{1,2} G^{ij} + w_{2,2} \partial (\Gamma^{i} \Gamma^{j}) + w_{3,2} U \Gamma^{j} + w_{4,2} \tilde{T}^{ijk} \Gamma^{i} \Gamma^{k} + w_{5,2} \tilde{T}^{ij} + \varepsilon^{ijkl} w_{6,2} \partial (\Gamma^{k} \Gamma^{l}),\]
\[ Q^{(3),ij}_1 = (i \leftrightarrow j), \]
\[ w_{1.3} \Phi_1^{(2),ij} + w_{2.3} \tilde{\Phi}_1^{(2),ij} + w_{3.3} \Phi_0^{(1),ij} \tilde{\Phi}_1^{(1),ij} + w_{4.3} \Phi_0^{(1),ij} \tilde{\Phi}_1^{(1),ij} + \ldots + w_{42.3} T^{kij} T^{kl} + \]
\[ + w_{43.3} T_t T^{kij} T^{kl} + \varepsilon^{ijkl} (w_{44.3} \Phi_1^{(1),k} \Phi_1^{(1),l} + w_{45.3} \Gamma^k \Phi_0^{(1),l} \Phi_1^{(1),l} + \ldots + w_{87.3} \Gamma^k \Gamma^l T^{kl} + w_{88.3} \varepsilon^{ijkl} \varepsilon^{r\rho\sigma} \Gamma^r \Gamma^\rho \Gamma^\sigma T^{kl}) + \]
\[ + (\varepsilon^{ijkl})^2 (w_{89.3} \Gamma^k T^{j} T^{kl} + \ldots + w_{92.3} \Gamma^k \Gamma^l T^{ij}) - (i \leftrightarrow j), \]
\[ Q^{(4),ij}_1 = w_{1.4} \Phi_1^{(2),ij} + w_{2.4} \Phi_0^{(1)} \Phi_1^{(1),ij} + w_{3.4} \Phi_0^{(1)} \Phi_1^{(1),ij} + \ldots + w_{126.4} \tilde{G}^l T^{ik} T^{kl} \Gamma^j \]
\[ + w_{127.4} \tilde{G}^l T^{ik} \Gamma^j \Gamma^l + w_{128.4} \tilde{G}^l T^{ik} T^{jl} \Gamma^k + \ldots + w_{153.4} \partial T^{ij} T^{kl} \Gamma^j \Gamma^k \]
\[ + \varepsilon^{ijkl} (w_{154.4} U T^{k} \Phi_1^{(2),l} + w_{155.4} \tilde{G}^k \Phi_1^{(2),l} + \ldots + w_{296.4} \partial^2 T^{ik} T^{jl} ) + \]
\[ + w_{297.4} \varepsilon^{ijkl} \varepsilon^{r\rho\sigma} T^{r} T^{\rho} T^{\sigma} U T^{k} T^l - (i \leftrightarrow j), \]
\[ Q^{(2),ij}_2 = w_{1.2} \Gamma^i, \]
\[ Q^{(3),ij}_2 = w_{1.2} \tilde{G}^i + w_{2.2} \partial T^i + w_{3.2} \Gamma^i U + w_{4.2} \Gamma^j T^j + \varepsilon^{ijkl} (w_{5.2} \Gamma^j T^{kl} + w_{6.2} \Gamma^j \Gamma^k \Gamma^l), \]
\[ Q^{(2),ij}_3 = w_{1.2} \Phi_1^{(2),ij} + w_{2.2} \Phi_1^{(2),ij} + w_{3.2} \Phi_0^{(1),ij} \Phi_1^{(1),ij} + \ldots + w_{29.2} \varepsilon^{ijkl} \Gamma^j \Phi_1^{(1),l} \Phi_1^{(1),l} \]
\[ + w_{35.2} \Gamma^k \Phi_0^{(1),l} \Phi_1^{(1),l} + w_{36.2} \Gamma^k \Gamma^l \Phi_0^{(1),l} \Phi_1^{(1),l} + \ldots + w_{93.2} \partial T^{ij} \Gamma^j \Gamma^l \]
\[ + \varepsilon^{ijkl} (w_{94.2} \Gamma^j T^{kl} \tilde{L} + w_{95.2} \Gamma^j T^{kl} \partial U + \ldots + w_{141.2} T^{ij} U T^{k} T^l ) + \]
\[ + w_{143.2} (\varepsilon^{ijkl})^2 T^{ij} T^{kl} \Gamma^k \Gamma^l , \]
\[ Q^{(1)}_2 = w_{1.1} U, \]
\[ Q^{(2)}_2 = w_{1.2} \Phi_0^{(2)} + w_{2.2} \Phi_1^{(0)} + w_{3.2} \tilde{L} + w_{4.2} \partial U + w_{5.2} U U + w_{6.2} T^{ij} T^j + w_{7.2} T^{ij} \tilde{T}^j \]
\[ + w_{8.2} \Gamma^i T^j + w_{9.2} \Gamma^j T^i + w_{10.2} \tilde{G}^i + w_{11.2} \Gamma^i \partial T^i + \varepsilon^{ijkl} w_{12.2} \Gamma^i \Gamma^j \Gamma^k \Gamma^l , \]
\[ Q^{(3)}_2 = w_{1.3} U \Phi_0^{(2)} + w_{2.3} \Phi_0^{(1)} \Phi_0^{(1)} + w_{3.3} \Gamma^i \Phi_0^{(2),i} + w_{4.3} \Gamma^i \Phi_0^{(1),i} + \partial U \]
\[ + w_{7.3} U U U + w_{8.3} \Gamma^i \partial \tilde{G}^i + w_{9.3} \partial \tilde{G}^i + w_{10.3} \Gamma^i \partial T^i + w_{11.3} \Gamma^i \partial T^i + w_{12.3} \Gamma^i T^{ij} \tilde{G}^j + w_{13.3} \Gamma^i T^{ij} \tilde{T}^j \]
\[ + w_{14.3} \Gamma^i T^{ij} U + w_{15.3} T^{ij} \tilde{T}^j U + w_{16.3} \Gamma^i \partial T^i + w_{17.3} (\partial \Gamma^i T^{ij} \tilde{T}^j + \partial \tilde{T}^{ij} + \varepsilon^{ijkl} (w_{18.3} \Gamma^{T} T^{jk} \tilde{G}^l + w_{19.3} \Gamma^i \Gamma^j \tilde{G}^l + w_{20.3} \Gamma^i \Gamma^j T^{kl} U + w_{21.3} T^{ij} T^{kl} U + w_{22.3} \Gamma^i \partial T^{kl} + w_{23.3} \partial T^{ij} T^{kl} + w_{24.3} \partial (\Gamma^i \Gamma^j \Gamma^k \Gamma^l) ) , \]
\[ Q^{(4)}_2 = w_{0.4} \Phi_0^{(4)} + w_{1.4} \Phi_0^{(2)} + w_{2.4} \Phi_0^{(2)} + w_{3.4} \tilde{L} \Phi_0^{(2)} + \ldots + w_{26.4} \Phi_1^{(1),i} \Phi_1^{(1),i} + \]
\[ + w_{27.4} \Phi_1^{(1),i} \Phi_1^{(1),i} \ldots + w_{31.4} \tilde{T}^{ij} T^{kl} \Gamma^k \Gamma^l + \Gamma^{ijkl} T^{ij} \tilde{T}^k \Gamma^l \]
\[ + \varepsilon^{ijkl} (w_{133.4} \tilde{L} \Gamma^i \Gamma^j \Gamma^k \Gamma^l + w_{134.4} \tilde{L} \Gamma^i \Gamma^{j} \Gamma^k \Gamma^l + \ldots + w_{168.4} \Gamma^i \partial \tilde{T}^{ij} \Gamma^l) , \]
where the central term in (C.1) contains

\[
\alpha = \frac{(k - N)}{2(k + N + 2)},
\]

\[
e_0^{0.4} = \frac{1}{(2 + k + N)^3(4 + 3k + 3N + 2kN)(5 + 4k + 4N + 3kN)^2} 64kN \times (-100 - 285k - 358k^2 - 246k^3 - 88k^4 - 12k^5 - 285N - 510kN - 275k^2N + 8k^3N + 56k^4N + 16k^5N - 358N^2 - 275kN^2 + 406k^2N^2 + 573k^3N^2 + 242k^4N^2 + 34k^5N^2 + 246N^3 + 8kN^3 + 573k^2N^3 + 510k^3N^3 + 155k^4N^3 + 12k^5N^3 - 88N^4 + 56kN^4 + 242k^2N^4 + 155k^3N^4 + 30k^4N^4 - 12N^5 + 16kN^5 + 34k^2N^5 + 12k^3N^5). \quad (C.6)
\]

Although the full expressions for the \( Q_{(9/2),0} \) of spin-\( 9/2 \) and the \( Q_{(5)} \) of spin-5 are not given in this Appendix, they are presented in ancillary.nb we attach and the complete expressions for the quasi primary fields can be found also. The \( \mathcal{N} = 4 \) version of quasi primary fields will appear in Appendix F.

**D** The OPE between the first \( \mathcal{N} = 4 \) higher spin multiplet in the component approach under the large \((N, k)\) limit

From the OPEs in [9] (or in section 2.3), we write down the following 15 kinds of OPEs under the large \((N, k)\) limit

\[
\Phi_0^{(1)}(z) \Phi_0^{(1)}(w) = -\frac{1}{(z-w)^2} 2N(\lambda - 1) + \cdots,
\]

\[
\Phi_0^{(1)}(z) \Phi_1^{(1),i}(w) = \frac{1}{(z-w)} G^i(w) + \cdots,
\]

\[
\Phi_0^{(1)}(z) \Phi_1^{(1),ij}(w) = -\frac{1}{(z-w)^2} 2i \left[ (2\lambda - 1) T^{ij} - \bar{T}^{ij} \right](w) + \cdots,
\]

\[
\Phi_0^{(1)}(z) \tilde{\Phi}_1^{(1),i}(w) = \frac{1}{(z-w)^3} 8i\lambda(\lambda - 1) \Gamma^i(w) \]
\[
- \frac{1}{(z-w)^2} 8 \left[ \frac{1}{3} (2\lambda - 1) G^i + i\lambda(\lambda - 1) \partial \Gamma^i \right](w) \]
\[
- \frac{1}{(z-w)} \frac{1}{2} \Phi_1^{(2),i}(w) + \cdots,
\]

\[\text{[42]}\text{Although there are nonlinear terms with the overall factors} \frac{1}{N}, \frac{1}{N^2}, \cdots \text{in the OPEs, the infinity limit of} N \text{leads to the fact that there will be no contributions from these nonlinear terms. We also take the infinity limit of} N \text{for the central terms. For the classical asymptotical symmetry algebra of AdS}_3 \text{higher spin theory, we should keep those nonlinear terms.} \]

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\[
\Phi_0^{(1)}(z) \Phi_2^{(1)}(w) = \frac{1}{(z - w)^3} 16 \lambda(\lambda - 1) U(w) + \ldots ,
\]
\[
\Phi_1^{(1),i}(z) \Phi_1^{(1),j}(w) = \frac{\delta^{ij}}{(z - w)^3} 4 N(\lambda - 1) + \frac{1}{(z - w)^2} \frac{2 i}{2} \left[ T^{ij} - (2\lambda - 1) \tilde{T}^{ij} \right](w) + \ldots ,
\]
\[
\Phi_1^{(1),i}(z) \Phi_1^{(1),jk}(w) = -\frac{1}{(z - w)^3} 8 i \lambda(\lambda - 1)(\delta^{ij} \Gamma^k - \delta^{ik} \Gamma^j)(w) + \ldots ,
\]
\[
\Phi_1^{(1),i}(z) \Phi_2^{(1)}(w) = -\frac{1}{(z - w)^3} 24 i \lambda(\lambda - 1) \Gamma^i(w) - \frac{1}{(z - w)^2} \frac{1}{2} \left[ \delta^{ij} \partial \Phi_0^{(2)} - \Phi_1^{(2),ij} \right](w) + \ldots ,
\]
\[
\Phi_1^{(1),ij}(z) \Phi_1^{(1),kl}(w) = \frac{1}{(z - w)^4} 12 N(\lambda - 1) (\delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk}) + 4 N(\lambda - 1)(2\lambda - 1) \varepsilon^{ijkl}
\]
\[
- \frac{1}{(z - w)^3} 4 i \left[ \left( \delta^{ik}((2\lambda - 1) \tilde{T}^{jl} + (2\lambda^2 - 2\lambda - 1) T^{jl})
\right.
\right.
\]
\[
- (k \leftrightarrow l) - (i \leftrightarrow j) + (i \leftrightarrow j, k \leftrightarrow l) \right] \right](w)
\]
\[
- \frac{1}{(z - w)^2} \left[ (\delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk}) 8 L - \varepsilon^{ijkl}(2 \Phi_0^{(2)} - \frac{8}{3} (2\lambda - 1) L)
\right]
\]
\[
+ \left( 2i \delta^{ik}((2\lambda - 1) \partial T^{jl} + (2\lambda^2 - 2\lambda - 1) \partial T^{jl}) \right)
\]
\[55\]
\[
\Phi_1^{(1),ij}(z) \bar{\Phi}_2^{(1),k}(w) = \frac{1}{(z-w)^4} \left[ 8i \lambda(\lambda-1)(2\lambda-1) \left( \delta^{ik} \Gamma^j - \delta^{jk} \Gamma^i \right) \right] (w) \\
+ \varepsilon^{ijkl} 24i \lambda(\lambda-1) \Gamma^l (w) \\
- \frac{1}{(z-w)^3} \left[ \frac{16}{3} (\lambda+1)(\lambda-2) \left( \delta^{ik} G^j - \delta^{jk} G^i \right) \right] (w) \\
- \frac{1}{(z-w)^2} \left[ \left( \delta^{ik} \left( \frac{1}{6} (2\lambda-1) \Phi_1^{(2),j} + \frac{16}{9} (\lambda+1)(\lambda-2) \partial G^j \right) \right) - (i \leftrightarrow j) \right] + \varepsilon^{ijkl} \frac{5}{2} \Phi_2^{(2),l} (w) \\
- \frac{1}{(z-w)} \left[ \left( \delta^{ik} \left( \frac{1}{2} \Phi_1^{(2),j} + \frac{1}{15} (2\lambda-1) \Phi_2^{(2),j} + \frac{4}{9} (\lambda+1)(\lambda-2) \partial^2 G^j \right) \right) - (i \leftrightarrow j) \right] + \varepsilon^{ijkl} \partial \Phi_2^{(2),l} (w) + \cdots ,
\]

\[
\Phi_1^{(1),ij}(z) \bar{\Phi}_1^{(1),k}(w) = \frac{1}{(z-w)^4} \left[ 16i \left( (2\lambda^2 - 2\lambda - 1) T^{ij} + (2\lambda - 1) \tilde{T}^{ij} \right) \right] (w) \\
+ \frac{1}{(z-w)^2} 3 \Phi_1^{(2),ij}(w) + \frac{1}{(z-w)} \partial \Phi_1^{(2),ij}(w) + \cdots ,
\]

\[
\bar{\Phi}_2^{(1),i}(z) \bar{\Phi}_2^{(1),j}(w) = \frac{1}{(z-w)^5} \delta^{ij} \frac{64}{3} N(\lambda-1)(\lambda+1)(\lambda-2) \\
+ \frac{1}{(z-w)^4} \delta^{ij} \frac{32i}{3} \left[ (\lambda+1)(\lambda-2) T^{ij} \right] (w) \\
- (\lambda+1)(\lambda-2)(2\lambda-1) \tilde{T}^{ij} \right] (w) \\
+ \frac{1}{(z-w)^3} \left[ \delta^{ij} \frac{8}{9} (3(2\lambda-1) \Phi_0^{(2)} - 20(\lambda+1)(\lambda-2) L) \right) \\
+ \frac{16i}{3} \left( (\lambda+1)(\lambda-2) \partial T^{ij} - (\lambda+1)(\lambda-2)(2\lambda-1) \partial \tilde{T}^{ij} \right] (w) \\
+ \frac{1}{(z-w)^2} \left[ \delta^{ij} \frac{4}{9} (3(2\lambda-1) \partial \Phi_0^{(2)} - 20(\lambda+1)(\lambda-2) \partial L) \right) \\
+ 3 \Phi_1^{(2),ij} - \frac{1}{3} (2\lambda-1) \bar{\Phi}_1^{(2),ij}
\]
\[ + \frac{16i}{9} \left( (\lambda + 1)(\lambda - 2) \partial^2 T^{ij} - (\lambda + 1)(\lambda - 2)(2\lambda - 1) \partial^2 \tilde{T}^{ij} \right) (w) \]

\[ + \frac{1}{(z - w)} \left[ \delta^{ij} \left( \frac{1}{2} \Phi_{\frac{1}{2}}^{(2)} + \frac{2}{5}(2\lambda - 1) \partial^2 \Phi_0^{(2)} - \frac{8}{3}(\lambda + 1)(\lambda - 2) \partial^2 L \right) \right] (w) \]

\[ + \frac{3}{2} \partial \Phi_{\frac{1}{2}}^{(2),ij} + \frac{1}{6} (2\lambda - 1) \partial \tilde{\Phi}_{\frac{1}{2}}^{(2),ij} \]

\[ + \frac{4i}{9} \left( (\lambda + 1)(\lambda - 2) \partial^3 T^{ij} - (\lambda + 1)(\lambda - 2)(2\lambda - 1) \partial^3 \tilde{T}^{ij} \right) (w) \]

\[ + \ldots, \]

\[ \tilde{\Phi}_{\frac{1}{2}}^{(1),i}(z) \tilde{\Phi}_{2}^{(1)}(w) = \frac{1}{(z - w)^4} \frac{80}{3} (\lambda + 1)(\lambda - 2) G^i \]

\[ - \frac{1}{(z - w)^3} \left[ \frac{4}{3} (2\lambda - 1) \Phi_{\frac{1}{2}}^{(2),i} - \frac{80}{9} (\lambda + 1)(\lambda - 2) \partial G^i \right] (w) \]

\[ + \frac{1}{(z - w)^2} \left[ \frac{7}{2} \Phi_{\frac{1}{2}}^{(2),i} - \frac{8}{15} (2\lambda - 1) \partial \Phi_{\frac{1}{2}}^{(2),i} + \frac{20}{9} (\lambda + 1)(\lambda - 2) \partial^2 G^i \right] \]

\[ + \frac{1}{(z - w)} \left[ \frac{3}{2} \partial \Phi_{\frac{1}{2}}^{(2),i} - \frac{2}{15} (2\lambda - 1) \partial^2 \Phi_{\frac{1}{2}}^{(2),i} + \frac{4}{9} (\lambda + 1)(\lambda - 2) \partial^3 G^i \right] \]

\[ + \ldots, \]

\[ \tilde{\Phi}_{2}^{(1)}(z) \tilde{\Phi}_{2}^{(1)}(w) = \frac{1}{(z - w)^6} \frac{320}{3} N(\lambda + 1)(\lambda - 1)(\lambda - 2) \]

\[ + \frac{1}{(z - w)^4} \left[ 8(2\lambda - 1) \Phi_{0}^{(2)} - \frac{320}{3} (\lambda + 1)(\lambda - 2) L \right] (w) \]

\[ + \frac{1}{(z - w)^3} \left[ 4(2\lambda - 1) \partial \Phi_{0}^{(2)} - \frac{160}{3} (\lambda + 1)(\lambda - 2) \partial L \right] (w) \]

\[ + \frac{1}{(z - w)^2} \left[ 4 \Phi_{2}^{(2)} + \frac{6}{5} (2\lambda - 1) \partial^2 \Phi_{0}^{(2)} - 16 (\lambda + 1)(\lambda - 2) \partial^2 L \right] (w) \]

\[ + \frac{1}{(z - w)} \left[ 2 \partial \Phi_{2}^{(2)} + \frac{4}{15} (2\lambda - 1) \partial^3 \Phi_{0}^{(2)} - \frac{32}{9} (\lambda + 1)(\lambda - 2) \partial^3 L \right] (w) \]

\[ + \ldots. \]  

\( (D.1) \)

We keep the leading terms in the central terms after the infinity limit of \( N \) in \( (D.1) \). The corresponding (anti)commutators will be given in \( (G.3) \) later. One of the reasons why we present these OPEs is that we need to know the structure constants appearing in the quasi primary fields in the right hand sides explicitly for the (anti)commutators. The relative coefficients appearing in all the descendant terms of the quasi primary fields are determined automatically. For example, from the fourth relation of \( (D.1) \) to the fourth relation of \( (G.3) \), we need to have the explicit structure constants of \( \Gamma^i, G^i, \) and \( \Phi_{\frac{1}{2}}^{(2),i} \) of the former: \( 8i \lambda(\lambda - 1), -\frac{8}{3}(2\lambda - 1), \) and \( -\frac{1}{2} \).
The OPE between the first and the second $\mathcal{N} = 4$ higher spin multiplets in the component approach under the large $(N, k)$ limit

From the description of section 3 (in particular, (3.19)), we summarize the complete 25 kinds of OPEs under the large $(N, k)$ limit as follows:

\[
\begin{align*}
\Phi^{(1)}_{0}(z) \Phi^{(2)}_{0}(w) & = + \cdots, \\
\Phi^{(1)}_{0}(z) \Phi^{(2),i}_{\frac{1}{2}}(w) & = - \frac{1}{(z-w)} \Phi^{(1),i}_{\frac{1}{2}}(w) + \cdots, \\
\Phi^{(1)}_{0}(z) \Phi^{(2),ij}_{1}(w) & = \frac{1}{(z-w)^2} \left[ \frac{8}{3}(2\lambda - 1) \Phi^{(1),ij}_{1} - 8 \Phi^{(1),ij}_{1} \right](w) + \cdots, \\
\Phi^{(1)}_{0}(z) \Phi^{(2),j}_{\frac{1}{2}}(w) & = \frac{1}{(z-w)^3} \left[ \frac{32}{3}(\lambda + 1)(\lambda - 2) \Phi^{(1),j}_{\frac{1}{2}}(w) \\
& + \frac{1}{(z-w)^2} \left[ \frac{64}{15}(2\lambda - 1) \Phi^{(1),j}_{\frac{1}{2}} - \frac{32}{9}(\lambda + 1)(\lambda - 2) \partial \Phi^{(1),j}_{\frac{1}{2}} \right](w) \\
& - \frac{1}{(z-w)^3} \Phi^{(3),j}_{\frac{1}{2}}(w) + \cdots, \\
\Phi^{(1)}_{0}(z) \Phi^{(2),i}_{2}(w) & = - \frac{1}{(z-w)^4} \Phi^{(3),i}_{2}(w) + \cdots, \\
\Phi^{(1),i}_{\frac{1}{2}}(z) \Phi^{(2),j}_{\frac{1}{2}}(w) & = - \frac{1}{(z-w)^2} \left[ 8 \Phi^{(1),ij}_{\frac{1}{2}} - \frac{8}{3}(2\lambda - 1) \Phi^{(1),ij}_{\frac{1}{2}} \right](w) \\
& - \frac{1}{(z-w)} \left[ 2\delta^{ij} \Phi^{(1)}_{\frac{1}{2}} + 2 \partial \Phi^{(1),ij}_{\frac{1}{2}} - \frac{2}{3}(2\lambda - 1) \partial \Phi^{(1),ij}_{\frac{1}{2}} \right](w) + \cdots, \\
\Phi^{(1),i}_{\frac{1}{2}}(z) \Phi^{(2),jk}_{\frac{1}{2}}(w) & = - \frac{1}{(z-w)^2} \left[ \frac{32}{3}(\lambda + 1)(\lambda - 2) \delta^{ij} \Phi^{(1),k}_{\frac{1}{2}} - \delta^{ik} \Phi^{(1),j}_{\frac{1}{2}} \right](w) \\
& - \frac{1}{(z-w)^2} \left[ 2(2\lambda - 1)\delta^{ij} \Phi^{(1),k}_{\frac{1}{2}} - \delta^{ik} \Phi^{(1),j}_{\frac{1}{2}} \right] + \varepsilon^{ijkl} 10 \Phi^{(1),l}_{\frac{1}{2}}(w) \\
& + \frac{1}{(z-w)} \left[ \delta^{ij} \left( \Phi^{(3),k}_{\frac{1}{2}} - \frac{12}{5}(2\lambda - 1)\partial \Phi^{(1),k}_{\frac{1}{2}} \right) - (j \leftrightarrow k) \right] \right](w) \\
& - \varepsilon^{ijkl} 12 \partial \Phi^{(1),l}_{\frac{1}{2}}(w) + \cdots, \\
\end{align*}
\]
\[
\Phi^{(1),i}(z) \tilde{\Phi}^{(2),j}(w) = \frac{1}{(z-w)^4} \delta^{ij} \frac{64}{3} (\lambda + 1)(\lambda - 2) \Phi^{(1)}_0(w) \\
- \frac{1}{(z-w)^3} \left[ \delta^{ij} \frac{32}{3} (\lambda + 1)(\lambda - 2) \partial \Phi^{(1)}_0 \\
+ \frac{128}{15} (2\lambda - 1) \Phi^{(1),ij} + \frac{32}{5} (\lambda + 2)(\lambda - 3) \tilde{\Phi}^{(1),ij}(w) \right] \\
- \frac{1}{(z-w)^2} \delta^{ij} \Phi^{(3)}_0 + \frac{1}{6} \frac{1}{(z-w)^2} \Phi^{(3),ij}_0(w) + \cdots,
\]

\[
\Phi^{(1),i}(z) \tilde{\Phi}^{(2)}(w) = \frac{1}{(z-w)^4} \frac{160}{3} (\lambda + 1)(\lambda - 2) \Phi^{(1),i}_\frac{1}{2}(w) \\
- \frac{1}{(z-w)^3} \left[ \frac{128}{15} (2\lambda - 1) \tilde{\Phi}^{(1),i}_\frac{1}{2} - \frac{160}{9} (\lambda + 1)(\lambda - 2) \partial \Phi^{(1),i}_\frac{1}{2}(w) \right] \\
+ \frac{1}{(z-w)^2} \frac{7}{6} \Phi^{(3),i}_\frac{1}{2}(w) + \frac{1}{6} \frac{1}{(z-w)^2} \partial \Phi^{(3),i}_\frac{1}{2}(w) + \cdots,
\]

\[
\Phi^{(1),ij}(z) \Phi^{(2)}(w) = \frac{1}{(z-w)^2} \left[ \frac{8}{3} (2\lambda - 1) \Phi^{(1),ij}_1 - 8 \Phi^{(1),ij}_1 + \Phi^{(3),ij}_1(w) \right] \\
+ \frac{4}{3} (2\lambda - 1) \partial \Phi^{(1),ij}_1 - 4 \partial \Phi^{(1),ij}_1 \right] (w) + \cdots,
\]

\[
\Phi^{(1),ij}(z) \Phi^{(2),k}(w) = \frac{1}{(z-w)^2} \left[ \frac{32}{3} (\lambda + 1)(\lambda - 2) \left( \delta^{ik} \Phi^{(1),j}_\frac{1}{2} - \delta^{jk} \Phi^{(1),i}_\frac{1}{2} \right) \right] (w) \\
- \frac{1}{(z-w)^2} \left[ \left( \delta^{ik} \left( \frac{2}{3} (2\lambda - 1) \tilde{\Phi}^{(1),j}_\frac{1}{2} - \frac{32}{9} (\lambda + 1)(\lambda - 2) \partial \Phi^{(1),j}_\frac{1}{2} \right) \\
- (i \leftrightarrow j) \right] + \varepsilon^{ijkl} 10 \tilde{\Phi}^{(1),l}_\frac{1}{2}(w) \\
- \frac{1}{(z-w)^2} \left[ \left( \delta^{ik} \left( \frac{1}{6} \Phi^{(3),j}_\frac{1}{2} + \frac{4}{15} (2\lambda - 1) \partial \Phi^{(1),j}_\frac{1}{2} \right) \\
- \frac{8}{9} (\lambda + 1)(\lambda - 2) \partial^2 \Phi^{(1),j}_\frac{1}{2} \right) - (i \leftrightarrow j) \right] + \varepsilon^{ijkl} 4 \partial \tilde{\Phi}^{(1),l}_\frac{1}{2}(w) \right] + \cdots,
\]

\[
\Phi^{(1),ij}(z) \Phi^{(2),kl}(w) = \frac{1}{(z-w)^4} \varepsilon^{ijkl} \frac{64}{3} (\lambda + 1)(\lambda - 2) \Phi^{(1)}_0(w) \\
+ \frac{1}{(z-w)^3} \left[ \left( \delta^{ik} \left( 2\lambda^2 - 2\lambda - 7 \right) \Phi^{(1),jl}_1 + (2\lambda - 1) \tilde{\Phi}^{(1),jl}_1 \right) \\
- (k \leftrightarrow l) - (i \leftrightarrow j) + (i \leftrightarrow j, k \leftrightarrow l) \right] (w) \\
- \frac{1}{(z-w)^2} \left[ \left( \delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk} \right) 12 \tilde{\Phi}^{(1)}_2 \\
- \left( \delta^{ik} \frac{4}{3} \left( 2\lambda^2 - 2\lambda - 7 \right) \partial \Phi^{(1),jl}_1 + (2\lambda - 1) \partial \tilde{\Phi}^{(1),jl}_1 \right) \\
- (k \leftrightarrow l) - (i \leftrightarrow j) + (i \leftrightarrow j, k \leftrightarrow l) \right) \right] (w) \\
- \left( \delta^{ik} \frac{4}{3} \left( 2\lambda^2 - 2\lambda - 7 \right) \partial \Phi^{(1),jl}_1 + (2\lambda - 1) \partial \tilde{\Phi}^{(1),jl}_1 \right) \\
- (k \leftrightarrow l) - (i \leftrightarrow j) + (i \leftrightarrow j, k \leftrightarrow l) \right) \right]
\]
\[ - \varepsilon^{ijkl} \left( \Phi_0^{(3)} - \frac{12}{5} (2\lambda - 1) \tilde{\Phi}_2^{(1)} \right) (w) \\
- \frac{1}{(z - w)} \left[ (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) 4 \partial \tilde{\Phi}_2^{(1)} + \left( \delta^i_k \left( \frac{1}{6} \Phi_4^{(3),j} - \frac{4}{15} (2\lambda^2 - 2\lambda - 7) \partial \Phi_4^{(3),j} - \frac{4}{15} (2\lambda - 1) \partial \tilde{\Phi}_4^{(1),j} \right) \right] (w) \\
+ \left( k \leftrightarrow l \right) - (i \leftrightarrow j) + (i \leftrightarrow j, k \leftrightarrow l) \right) \\
- \varepsilon^{ijkl} \left( \frac{1}{3} \partial \Phi_0^{(3)} - \frac{4}{5} (2\lambda - 1) \partial \tilde{\Phi}_2^{(1)} \right) (w) + \cdots, \\
\Phi_1^{(1),ij} (z) \tilde{\Phi}_2^{(2),k} (w) = \frac{1}{(z - w)^4} \left[ \frac{32}{5} (\lambda + 1)(\lambda - 2)(2\lambda - 1) (\delta^i_k \Phi_4^{(1),j} - \delta^j_k \Phi_4^{(1),i}) \right] (w) \\
+ \varepsilon^{ijk} \frac{160}{3} (\lambda + 1)(\lambda - 2) \Phi_4^{(1),j} (w) \\
+ \frac{1}{(z - w)^2} \left[ (\delta^i_k \left( \frac{1}{30} (2\lambda - 1) \Phi_4^{(3),j} - \frac{48}{25} (\lambda + 2)(\lambda - 3) \partial \tilde{\Phi}_4^{(1),j} \right) \\
- (i \leftrightarrow j) \right) + \varepsilon^{ijkl} \frac{7}{6} \Phi_4^{(3),i} \right] \\
- \frac{1}{(z - w)^4} \left[ \left( \delta^i_k \left( \frac{1}{6} \Phi_4^{(3),j} + \frac{1}{105} (2\lambda - 1) \partial \Phi_4^{(3),j} \right) \\
- \frac{8}{25} (\lambda + 2)(\lambda - 3) \partial^2 \Phi_4^{(1),j} \right) (i \leftrightarrow j) \right) + \varepsilon^{ijkl} \frac{1}{3} \partial \Phi_4^{3,i} (w) + \cdots, \\
\Phi_1^{(1),ij} (z) \tilde{\Phi}_2^{(2),k} (w) = -\frac{1}{(z - w)^4} \left[ \frac{64}{15} (17\lambda^2 - 17\lambda - 52) \Phi_4^{(1),ij} + \frac{128}{5} (2\lambda - 1) \tilde{\Phi}_4^{(1),ij} \right] (w) \\
+ \frac{1}{(z - w)^2} \left[ \frac{4}{3} \Phi_4^{(3),ij} (w) + \frac{1}{(z - w)^2} \frac{1}{3} \partial \Phi_4^{3,ij} (w) + \cdots, \\
\tilde{\Phi}_4^{(1),i} (z) \Phi_0^{(2),j} (w) = -\frac{1}{(z - w)^3} \frac{32}{3} (\lambda + 1)(\lambda - 2) \Phi_4^{(1),i} (w) \\
+ \frac{1}{(z - w)^2} \left[ \frac{8}{3} (2\lambda - 1) \tilde{\Phi}_4^{(1),i} - \frac{64}{9} (\lambda + 1)(\lambda - 2) \partial \Phi_4^{(1),i} \right] (w) \\
+ \frac{1}{(z - w)^3} \left[ \frac{1}{6} \Phi_4^{(3),i} + \frac{8}{5} (2\lambda - 1) \partial \tilde{\Phi}_4^{(1),i} \right. \\
- \frac{8}{3} (\lambda + 1)(\lambda - 2) \partial^2 \Phi_4^{(1),i} \left( w \right) + \cdots, \\
\tilde{\Phi}_4^{(1),i} (z) \tilde{\Phi}_4^{(2),j} (w) = -\frac{1}{(z - w)^4} \delta^{ij} \frac{64}{3} (\lambda + 1)(\lambda - 2) \Phi_0^{(1)} (w) \\
- \frac{1}{(z - w)^2} \frac{32}{3} \left[ \delta^{ij} (\lambda + 1)(\lambda - 2) \partial \Phi_0^{(1)} \right] \\
+ \frac{1}{(z - w)^3} \frac{8}{3} (\lambda + 1)(\lambda - 2) \partial \Phi_0^{(1)} \right) (w) + \cdots, \right]
\[
-(2\lambda + 1) \Phi_1^{(1),ij} - \frac{1}{3} (\lambda^2 - \lambda - 11) \Phi_1^{(1),ij} \bigg] (w)
+ \frac{1}{(z - w)^2} \left[ \delta^{ij} \left( \Phi_0^{(3)} - \frac{16}{15} (2\lambda - 1) \Phi_2^{(1)} - \frac{32}{9} (\lambda + 1)(\lambda - 2) \partial^2 \Phi_0^{(1)} \right) + \frac{16}{3} \left( (2\lambda - 1) \partial \Phi_1^{(1),ij} + \frac{1}{3} (\lambda^2 - \lambda - 11) \partial \Phi_1^{(1),ij} \right) \right] (w)
+ \frac{1}{(z - w)} \left[ \delta^{ij} \left( \frac{1}{2} \partial \Phi_0^{(3)} - \frac{8}{15} (2\lambda - 1) \partial \Phi_2^{(1)} \right) - \frac{8}{9} (\lambda + 1)(\lambda - 2) \partial^2 \Phi_0^{(1)} \right)
\]
- \frac{1}{6} \Phi_1^{(3),ij} + \frac{8}{5} \left( (2\lambda - 1) \partial^2 \Phi_1^{(1),ij} + \frac{1}{3} (\lambda^2 - \lambda - 11) \partial^2 \Phi_1^{(1),ij} \right) \bigg] (w)
+ \cdots ,
\]
\[
\Phi_2^{(1),i} (z) \Phi_2^{(2),jk} (w) = \frac{1}{(z - w)^4} \left[ \frac{32}{3} (\lambda + 1)(\lambda - 2)(2\lambda - 1) (\delta^{ij} \Phi_2^{(1),k} - \delta^{ik} \Phi_2^{(1),j}) \right.
+ \frac{160}{3} (\lambda + 1)(\lambda - 2) \varepsilon^{ijkl} \Phi_2^{(1),l} \bigg] (w)
\]
- \frac{1}{(z - w)^2} \left[ \left( \delta^{ij} \left( \frac{16}{3} (\lambda^2 - \lambda - 11) \Phi_2^{(3),k} \right) - \frac{32}{9} (\lambda + 1)(\lambda - 2) \partial \Phi_2^{(1),k} \right) \right.
\]
- \frac{8}{9} (\lambda + 1)(\lambda - 2)(2\lambda - 1) \partial^2 \Phi_2^{(1),k} \bigg] (w)
\]
+ \varepsilon^{ijkl} \left( \frac{7}{6} \Phi_2^{(3),l} - \frac{32}{15} (2\lambda - 1) \partial \Phi_2^{(3),l} - \frac{40}{9} (\lambda + 1)(\lambda - 2) \partial^2 \Phi_2^{(3),l} \right) \bigg] (w)
+ \frac{1}{(z - w)^2} \left[ \left( \delta^{ij} \left( \frac{1}{6} \Phi_2^{(3),k} - \frac{1}{42} (2\lambda - 1) \partial \Phi_2^{(3),k} \right) - \frac{8}{15} (\lambda^2 - \lambda - 11) \partial^2 \Phi_2^{(3),k} \right) \right.
\]
+ \frac{8}{45} (\lambda + 1)(\lambda - 2)(2\lambda - 1) \partial^3 \Phi_2^{(1),k} \bigg] (w) + \cdots ,
\]
\[
\Phi_2^{(1),i} (z) \Phi_2^{(2),jk} (w) = -\frac{1}{(z - w)^5} \delta^{ij} \frac{2048}{45} (\lambda + 1)(\lambda - 2)(2\lambda - 1) \Phi_0^{(1),ij} \bigg] (w)
\]
\[- \frac{1}{(z-w)^4} \left[ \frac{192}{5} (\lambda + 2)(\lambda - 3) \Phi_1^{(1),ij} \right] (w) \]
\[- \frac{64}{5} (\lambda + 2)(\lambda - 3)(2\lambda - 1) \Phi_1^{(1),ij} \right] (w) \]
\[+ \frac{1}{(z-w)^3} \left[ \delta^{ij} \left( \frac{16}{15} (2\lambda - 1) \Phi_0^{(3)} - \frac{336}{25} (\lambda + 2)(\lambda - 3) \Phi_2^{(1)} \right) \right] (w) \]
\[- \frac{48}{5} (\lambda + 2)(\lambda - 3) \Phi_1^{(1),ij} + \frac{16}{5} (\lambda + 2)(\lambda - 3)(2\lambda - 1) \Phi_1^{(1),ij} \right] (w) \]
\[+ \frac{1}{(z-w)^2} \left[ \delta^{ij} \left( \frac{16}{45} (2\lambda - 1) \Phi_1^{(3),ij} - \frac{336}{25} (\lambda + 2)(\lambda - 3) \Phi_1^{(1),ij} \right) \right] (w) \]
\[+ \frac{1}{(z-w)} \left[ \delta^{ij} \left( \frac{1}{6} \Phi_2^{(3)} + \frac{8}{105} (2\lambda - 1) \Phi_0^{(3)} \right) \right] (w) \]
\[- \frac{24}{25} (\lambda + 2)(\lambda - 3) \Phi_1^{(1),ij} \right] (w) \]
\[+ \frac{1}{2} \Phi_1^{(3),ij} - \frac{1}{30} (2\lambda - 1) \Phi_1^{(3),ij} - \frac{8}{25} (\lambda + 2)(\lambda - 3) \Phi_1^{(1),ij} \right] (w) \]
\[+ \frac{1}{(z-w)^3} \left[ \delta^{ij} \left( \frac{32}{45} (2\lambda - 1) \Phi_1^{(3),ij} + \frac{336}{25} (\lambda + 2)(\lambda - 3) \Phi_1^{(1),ij} \right) \right] (w) \]
\[+ \frac{1}{(z-w)^2} \left[ \frac{3}{2} \Phi_2^{(3),i} - \frac{64}{315} (2\lambda - 1) \Phi_2^{(3),i} \right] (w) \]
\[- \frac{56}{25} (\lambda + 2)(\lambda - 3) \Phi_1^{(1),ij} \right] (w) \]
\[+ \frac{1}{(z-w)} \left[ \frac{1}{2} \Phi_1^{(3),i} - \frac{4}{105} (2\lambda - 1) \Phi_1^{(3),i} \right] (w) \]
\[- \frac{8}{25} (\lambda + 2)(\lambda - 3) \Phi_1^{(1),ij} \right] (w) \]

$$\tilde{\Phi}_{1/2}^{(1),i}(z) \Phi_2^{(2)}(w) = \frac{1}{(z-w)^4} \left[ \frac{2048}{45} (\lambda + 1)(\lambda - 2)(2\lambda - 1) \Phi_1^{(1),i}(w) \right]$$

$$\Phi_2^{(1)}(z) \Phi_0^{(2)}(w) = -\frac{1}{(z-w)^4} \left[ \frac{64}{3} (\lambda + 1)(\lambda - 2) \Phi_0^{(1)}(w) \right]$$

$$\Phi_2^{(1)}(z) \Phi_0^{(2)}(w) = -\frac{1}{(z-w)^4} \left[ \frac{64}{3} (\lambda + 1)(\lambda - 2) \Phi_0^{(1)}(w) \right]$$

$$\Phi_2^{(1)}(z) \Phi_0^{(2)}(w) = -\frac{1}{(z-w)^4} \left[ \frac{64}{3} (\lambda + 1)(\lambda - 2) \Phi_0^{(1)}(w) \right]$$
\[
\begin{align*}
\tilde{\Phi}^{(1)}_2 (z) \Phi^{(2),i}_1 (w) &= - \frac{1}{(z-w)^4} \left[ 160 \Phi^{(3)}_0 + 16 \Phi^{(1)}_1 + 16 \Phi^{(1),i}_1 \right] (w) + \cdots , \\
\Phi^{(2),i}_2 (z) \Phi^{(1)}_1 (w) &= - \frac{1}{(z-w)^4} \left[ 16 \Phi^{(2)}_0 + 16 \Phi^{(1)}_1 + 16 \Phi^{(1),i}_1 \right] (w) + \cdots ,
\end{align*}
\]
\[
+ \frac{1}{(z-w)^4} \left[ \frac{16}{5} (2\lambda - 1) \Phi_0^{(3)} - \frac{2688}{25} (\lambda + 2)(\lambda - 3) \tilde{\Phi}_2^{(1)} \right](w) \\
+ \frac{1}{(z-w)^3} \left[ \frac{16}{15} (2\lambda - 1) \partial \Phi_0^{(3)} - \frac{896}{25} (\lambda + 2)(\lambda - 3) \partial \tilde{\Phi}_2^{(1)} \right](w) \\
+ \frac{1}{(z-w)^2} \left[ \frac{5}{3} \tilde{\Phi}_2^{(3)} + \frac{8}{35} (2\lambda - 1) \partial^2 \Phi_0^{(3)} \\
- \frac{192}{25} (\lambda + 2)(\lambda - 3) \partial^2 \tilde{\Phi}_2^{(1)} \right](w) \\
+ \frac{1}{(z-w)} \left[ \frac{2}{3} \partial \tilde{\Phi}_2^{(3)} + \frac{4}{105} (2\lambda - 1) \partial^3 \Phi_0^{(3)} \\
- \frac{32}{25} (\lambda + 2)(\lambda - 3) \partial^3 \tilde{\Phi}_2^{(1)} \right](w) + \cdots.
\]

(E.1)

Since we are considering the OPEs between the first and second $\mathcal{N} = 4$ higher spin multiplets in (E.1), we have more OPEs compared to the ones in Appendix D. As before, the corresponding (anti)commutators are presented in (G.4) later. There is no $\lambda$ factor in the structure constants in the right hand side of (E.1), contrary to the case of (D.1). Therefore, at $\lambda = 0$, all the terms in (E.1) survive.

**F The OPE between the second $\mathcal{N} = 4$ higher spin multiplet in the $\mathcal{N} = 4$ superspace with $N = 5$**

From the results of section 4 and Appendix C, we can summarize the complete OPE with $N = 5$ as follows:

\[
\Phi^{(2)}(Z_1) \Phi^{(2)}(Z_2) = \frac{1}{z_{12}^0} \epsilon_0^{0,4} + \frac{\theta_{12}^{4-0}}{z_{12}^0} 8 \alpha \epsilon_0^{0,4} + \frac{\theta_{12}^{4-i}}{z_{12}^0} Q_{1,0}^{(4),i}(Z_2) \\
+ \frac{\theta_{12}^{4-0}}{z_{12}^0} Q_{2}^{(1),ij}(Z_2) + \frac{\theta_{12}^{4-ij}}{z_{12}^0} Q_{1,ij}^{(1),ij}(Z_2) + \frac{\theta_{12}^{4-i}}{z_{12}^0} \left[ 2 \partial Q_{1}^{(1),ij} + Q_{1}^{(2),ij} + R_{1}^{(3),ij} \right](Z_2) \\
+ \frac{\theta_{12}^{4-i}}{z_{12}^0} \left[ \frac{3}{2} \partial^2 Q_{1}^{(1),ij} + \partial Q_{1}^{(2),i} + Q_{1}^{(3),ij} + R_{1}^{(3),ij} \right](Z_2) \\
+ \frac{\theta_{12}^{4-i}}{z_{12}^0} \left[ \frac{3}{2} \partial^2 Q_{1}^{(1),i} + \partial Q_{1}^{(2),i} + Q_{1}^{(3),ij} + R_{1}^{(3),ij} \right](Z_2) \\
+ \frac{\theta_{12}^{4-i}}{z_{12}^0} \left[ \frac{2}{3} \partial Q_{1}^{(3),ij} + Q_{1}^{(4),ij} \right](Z_2) + \frac{\theta_{12}^{4-i}}{z_{12}^0} \left[ \frac{1}{2} \partial^2 Q_{1}^{(1),ij} + \frac{3}{4} \partial Q_{1}^{(2),ij} + Q_{1}^{(3),ij} \right](Z_2) \\
+ \frac{\theta_{12}^{4-i}}{z_{12}^0} \left[ \frac{2}{3} \partial^2 Q_{1}^{(4),i} + \frac{1}{2} \partial^2 Q_{1}^{(5),i} + \frac{4}{5} \partial Q_{1}^{(6),i} + Q_{1}^{(7),i} + R_{1}^{(7),i} \right](Z_2)
\]
The various quasi super primary fields appearing in (F.1) are given by

\[
Q^{(2)}_0 = e_0^{0.2} \Phi^{(2)} + e_1^{0.2} \Phi^{(1)} \Phi^{(1)} + e_2^{0.2} J^{4-i} + e_3^{0.2} J^{4-i} + e_4^{0.2} \partial J \partial J + e_5^{0.2} J^{ij} J^{ij} \\
+ e_6^{0.2} J^{ij} J^{ij} \\
+ e_7^{0.2} J^{ij} J^{ij} + e_8^{0.2} J^{ij} J^{ij} + e_9^{0.2} J^{ij} J^{ij} + e_{10}^{0.2} e_i j k l J^{ij} J^{jk} J^{kl},
\]

\[
Q^{(2)}_2 = e_1^{0.2} \Phi^{(2)} + e_2^{0.2} \Phi^{(1)} \Phi^{(1)} + e_3^{0.2} \partial J^{4-i} + e_4^{0.2} \partial J^{4-i} + e_5^{0.2} J^{ij} J^{ij} \\
+ e_6^{0.2} J^{ij} J^{ij} \\
+ e_7^{0.2} J^{ij} J^{ij} + e_8^{0.2} J^{ij} J^{ij} + e_9^{0.2} J^{ij} J^{ij} + e_{10}^{0.2} e_i j k l J^{ij} J^{jk} J^{kl},
\]

\[
Q^{(2)}_2 = e_1^{0.2} \Phi^{(2)} + e_2^{0.2} \Phi^{(1)} \Phi^{(1)} + e_3^{0.2} \partial J^{4-i} + e_4^{0.2} \partial J^{4-i} + e_5^{0.2} J^{ij} J^{ij} \\
+ e_6^{0.2} J^{ij} J^{ij} \\
+ e_7^{0.2} J^{ij} J^{ij} + e_8^{0.2} J^{ij} J^{ij} + e_9^{0.2} J^{ij} J^{ij} + e_{10}^{0.2} e_i j k l J^{ij} J^{jk} J^{kl},
\]

\[
Q^{(2)}_2 = e_1^{0.2} \Phi^{(2)} + e_2^{0.2} \Phi^{(1)} \Phi^{(1)} + e_3^{0.2} \partial J^{4-i} + e_4^{0.2} \partial J^{4-i} + e_5^{0.2} J^{ij} J^{ij} \\
+ e_6^{0.2} J^{ij} J^{ij} \\
+ e_7^{0.2} J^{ij} J^{ij} + e_8^{0.2} J^{ij} J^{ij} + e_9^{0.2} J^{ij} J^{ij} + e_{10}^{0.2} e_i j k l J^{ij} J^{jk} J^{kl},
\]
\[Q_1^{(4),ij} = e_{1,1} D^{ij} \Phi (2) + e_{2,1} \Phi (1) \partial D^{ij} \Phi (1) + e_{3,1} \Phi (1) \partial D^{ij} \Phi (1) + e_{4,1} D^{ij} \Phi (1) + \ldots + e_{1,126} J^i J^j D^{ij} \Phi (2) + e_{1,127} J^i J^j D^{ij} \Phi (2) + \ldots + e_{1,153} J^i J^j D^{ij} \Phi (2) + \ldots + e_{1,296} \varepsilon^{ijkl} J^i J^j J^k J^l - (i \leftrightarrow j),\]

\[Q_2^{(4),i} = e_{1,1} J^i,\]

\[Q_2^{(4),i} = e_{1,2} J^i + e_{2,1} J^i + e_{3,1} J^i + e_{4,1} J^i + \ldots,\]

\[Q_2^{(4),i} = e_{1,3} D_i \Phi (2) + e_{2,3} D_i \Phi (2) + e_{3,3} D_i \Phi (1) + e_{4,3} J_i \Phi (1) + \ldots + e_{1,3} \varepsilon^{ijkl} J^i J^j J^k J^l,\]

\[Q_2^{(1)} = e_{1,5} \partial J,\]

\[Q_2^{(2)} = e_{1,4} \Phi (2) + e_{2,4} \Phi (1) \Phi (1) + e_{3,4} J^i - \partial J + e_{4,4} \partial J \partial J + e_{5,4} J_i J^j - i + e_{6,4} J_i J^j i + e_{7,4} J_i J^j i + e_{8,4} J_i J^j i + e_{9,4} J_i J^j i + e_{10,4} J_i J^j i + e_{11,4} J_i J^j i + e_{12,4} J_i J^j i + e_{13,4} J_i J^j i + e_{14,4} J_i J^j i + e_{15,4} J_i J^j i + e_{16,4} J_i J^j i + e_{17,4} J_i J^j i + e_{18,4} J_i J^j i + e_{19,4} J_i J^j i + e_{20,4} J_i J^j i,\]

\[Q_2^{(3)} = e_{1,2} J_i J^j i + e_{2,2} J_i J^j i + e_{3,2} J_i J^j i + e_{4,2} J_i J^j i + e_{5,2} J_i J^j i + e_{6,2} J_i J^j i + e_{7,2} J_i J^j i + e_{8,2} J_i J^j i + e_{9,2} J_i J^j i + e_{10,2} J_i J^j i + e_{11,2} J_i J^j i + e_{12,2} J_i J^j i + e_{13,2} J_i J^j i + e_{14,2} J_i J^j i + e_{15,2} J_i J^j i + e_{16,2} J_i J^j i + e_{17,2} J_i J^j i + e_{18,2} J_i J^j i + e_{19,2} J_i J^j i + e_{20,2} J_i J^j i,\]

\[Q_2^{(4)} = e_{0,4} \Phi (4) + e_{1,4} \Phi (2) (2) + e_{2,4} D^{0-0} \Phi (2) + e_{3,4} J^{0-0} \Phi (2) + \ldots + e_{26,4} D^{ij} \Phi (1) D^{ij} \Phi (1) + e_{27,4} D^{ij} \Phi (1) D^{ij} \Phi (1) + \ldots + e_{131,4} J_i J^j i J^j i J^j i J^j i + e_{132,4} J_i J^j i J^j i J^j i J^j i + \varepsilon^{ijkl} (e_{133,4} J^i J^j i J^j i J^j i J^j i + e_{134,4} J^i J^j i J^j i J^j i J^j i + \ldots + e_{168,4} J_i J^j i J^j i J^j i J^j i).\]
As in the component approach in Appendix C, the $Q^{(2),i}_{\frac{5}{2}}$ of super spin-$\frac{9}{2}$ and the $Q^{(5)}_{2}$ of super spin-5 are presented in ancillary.nb. Similarly, the higher spin currents $R_{\frac{5}{2}}^{(4),i}$, \ldots, $R_{2}^{(5)}$ can be found also

\[ R_{\frac{5}{2}}^{(4),i}(Z_{2}) \equiv -\frac{2\alpha}{5} \left( \frac{1}{4} \partial^{2} Q^{(4),i}_{\frac{5}{2}} + \frac{3}{5} \partial Q^{(4),i}_{\frac{5}{2}} + Q^{(4),i}_{\frac{5}{2}} \right)(Z_{2}), \]

\[ R_{\frac{3}{2}}^{(4),i}(Z_{2}) \equiv -\frac{4\alpha}{5} \left( \frac{2}{3} \partial Q^{(4),i}_{\frac{3}{2}} + Q^{(4),i}_{\frac{3}{2}} \right)(Z_{2}), \]

\[ R_{0}^{(4),i}(Z_{2}) \equiv -\frac{6\alpha}{5} Q^{(4),i}_{\frac{1}{2}}(Z_{2}), \]

\[ R_{2}^{(4-n)}(Z_{2}) \equiv -p_{2} E_{2}^{(4-n)}(Z_{2}) - p_{1} n(n + 1) Q_{0}^{(4-n)}(Z_{2}), \quad n = -1, 0, 1, 2. \quad (F.3) \]

We also have the following quantities corresponding to \( (C.3) \)

\[ E_{2}^{(2)}(Z_{2}) \equiv (4Q_{0}^{(2)} + \frac{e_{0,4}^{0,4}}{2} J^{4-0})(Z_{2}), \quad E_{2}^{(3)}(Z_{2}) \equiv (\frac{5}{2} \partial Q_{0}^{(2)} + \frac{e_{0,4}^{0,4}}{2} \partial J^{4-0})(Z_{2}), \]

\[ E_{2}^{(4)}(Z_{2}) \equiv (\frac{1}{2} \partial^{2} Q_{0}^{(2)} + \frac{1}{2} J^{4-0} Q_{0}^{(2)} + \frac{e_{0,4}^{0,4}}{4} \partial^{2} J^{4-0} + 2 \Phi^{(2)} \Phi^{(2)})(Z_{2}), \quad (F.4) \]

\[ E_{2}^{(5)}(Z_{2}) \equiv (-\frac{1}{24} \partial^{2} Q_{0}^{(2)} + \frac{1}{4} J^{4-0} \partial Q_{0}^{(2)} + \frac{1}{2} \partial J^{4-0} Q_{0}^{(2)} + \frac{e_{0,4}^{0,4}}{12} \partial^{3} J^{4-0} + \frac{3}{2} \partial(\Phi^{(2)} \Phi^{(2)}))(Z_{2}). \]

As emphasized before, all the coefficients appearing in (F.2), (F.3) and (F.4) depend on \( k \) with fixed \( N = 5 \). It is an open problem to obtain them for generic \( (N, k) \).

G \hspace{1cm} \textbf{The (anti)commutators from the coset construction of sections 2, 3, and 4}

In order to extract the \( \mathcal{N} = 4 \) higher spin algebra from the \( \mathcal{W}_{\infty}^{N=4}[\lambda] \) algebra in two dimensional conformal field theory, we should express the corresponding OPEs obtained in previous sections 2, 3 and 4 in terms of (anti)commutators by using the explicit formula in [72, 73].

By starting with the (anti)commutators of 16 currents and those between the 16 currents and 16 higher spin currents, we will present the (anti)commutators corresponding to Appendices \( A \) and \( B \).

G.1 \hspace{1cm} \textbf{The (anti)commutators between the 16 currents}

From the standard OPEs of the large \( \mathcal{N} = 4 \) superconformal algebra [9], we can write down them in terms of (anti)commutators under the large \( (N, k) \) ’t Hooft limit \[ 14 \]

\[ [L_{m}, L_{n}] = -m(m^{2} - 1) \frac{N(1-\lambda)}{2} \delta_{m+n} + (m - n) L_{m+n}, \]

[43] All the nonlinear terms appearing in this Appendix disappear under the infinity limit of \( N \) as before.

[44] The expressions with typewriter font will disappear after taking the wedge condition.
We obtain the \( \mathcal{N} = 4 \) wedge algebra, (7.4) and (7.7), by removing the parts having a typewriter font as in section 7.1.

### G.2 The (anti)commutators between the 16 currents and the s-th \( \mathcal{N} = 4 \) higher spin multiplet

From the standard \( \mathcal{N} = 4 \) primary condition for the \( \mathcal{N} = 4 \) higher spin multiplet in component approach \cite{9}, we can write down them in terms of (anti)commutators under the large \( (N, k) \)'t Hooft limit as follows:

\[
\begin{align*}
\{L_m, G^i_r\} &= (\frac{m}{2} - r) G^i_{m+r}, \\
\{L_m, T^{ij}_{0,m}\} &= -n T^{ij}_{m+n}, \\
\{G^i_r, G^j_r\} &= -\delta^{ij} \left( (4r^2 - 1) \frac{N(1 - \lambda)}{2} \delta_{r+\rho} - 2 L_{r+\rho} \right) - i (r - \rho) \left( T^{ij}_{r+\rho} - (2\lambda - 1) T^{ij}_{r+\rho} \right), \\
\{G^i_r, T^i_{m,j} \} &= \delta^{ij} \left( i G^k_{r+m} + m(2\lambda - 1) \Gamma^k_{r+m} \right) - \delta^{ik} \left( i G^j_{r+m} + m(2\lambda - 1) \Gamma^j_{r+m} \right) + \varepsilon^{ijkl} m \Gamma^l_{r+m}, \\
\{T^{ij}_{m,j}, T^{kl}_{n,k} \} &= m \delta_{m+n} \left( (\delta^{ik}\delta^{jl} - \delta^{jl}\delta^{ik}) \frac{N}{2\lambda} + \varepsilon^{ijkl} N(1 - \frac{1}{2\lambda}) \right) - i \delta^{ik} T^{jl} + i \delta^{jl} T^{ik} + i \delta^{jk} T^{il} - i \delta^{il} T^{jk}. \tag{G.1}
\end{align*}
\]

We obtain the \( \mathcal{N} = 4 \) wedge algebra, (7.4) and (7.7), by removing the parts having a typewriter font as in section 7.1.
we have only vanishing commutator for the higher spin-1 current as follows:

\[
[T_{ij}^{m}, \tilde{\Phi}^{(s)}_{2,n}] = 2(s + 1)i m \Phi^{(s),ij}_{1,m+n}.
\]

The \( \lambda \) dependence appears as \( (2\lambda - 1) \). We observe that there are terms having typewriter font in the right hand side and they will vanish under the wedge restriction. In particular, we have only vanishing commutator for the higher spin-1 current as follows:

\[
[L_m, \Phi^{(1)}_{0,n}] = -n \Phi^{(1)}_{0,m+n}.
\]

For the other higher spin currents, in general, we have the first and second relations of (G.2). The crucial point in (G.2) is that we can determine all the remaining 15 higher spin currents starting with the commutator between the spin-\( \frac{3}{2} \) current \( G^i \) and the lowest higher spin current \( \Phi^{(s)}_0 \).

\[ \text{G.3 The (anti)commutators between the first } \mathcal{N} = 4 \text{ higher spin multiplet} \]

The OPEs appearing in Appendix D can be written in terms of (anti)commutators by using the works of [72, 73] as follows:

\[
\begin{align*}
&\left[ \Phi^{(1)}_{0,m}, \Phi^{(1)}_{0,n} \right] = -2mN(\lambda - 1) \delta_{m+n}, \\
&\left[ \Phi^{(1)}_{0,m}, \Phi^{(1)}_{2,n} \right] = G^i_{m+n}, \\
&\left[ \Phi^{(1)}_{0,m}, \Phi^{(1)}_{3,n} \right] = 2i m \left( (1 - 2\lambda) T^{11}_{m+n} + \tilde{T}^{11}_{m+n} \right), \\
&\left[ \Phi^{(1)}_{0,m}, \tilde{\Phi}^{(1)}_{2,n} \right] = -\frac{1}{2} \Phi^{(2),i}_{3,m+n} - \frac{8}{3} m(2\lambda - 1) \delta^{i}_{m+n} + 4 i m(3m + 2r)\lambda(\lambda - 1) \Gamma^{i}_{m+n}, \\
&\left[ \Phi^{(1)}_{0,m}, \tilde{\Phi}^{(1)}_{0,n} \right] = 2m \tilde{\Phi}^{(2)}_{m+n} + 8m(2m + n)\lambda(\lambda - 1) U_{m+n}, \\
&\left\{ \Phi^{(1)}_{i,r}, \Phi^{(1)}_{j,s} \right\} = 2(r^2 - \frac{1}{4}) N(\lambda - 1) \delta_{r+s} - 2 \delta^{ij} L_{r+s} + i(r - s) \left( T^{ij}_{r+s} - (2\lambda - 1) \tilde{T}^{ij}_{r+s} \right), \\
&\left[ \Phi^{(1)}_{i,r}, \Phi^{(1)}_{j,k} \right] = \left( \delta^{ij} \left( \frac{1}{2} \Phi^{(2),k}_{r,m+r} + \frac{1}{3} (2r - m)(2\lambda - 1) G^{k}_{r+m} - i(4r^2 - 1)\lambda(\lambda - 1) \Gamma^{k}_{r+m} \right) \\
&- (j \leftrightarrow k) \right) + \varepsilon^{ijkl}(2r - m) G^l_{r+m}, \\
&\left\{ \Phi^{(1)}_{i,r}, \tilde{\Phi}^{(1)}_{j,s} \right\} = \delta^{ij} \left( \frac{1}{2} (-3r + s) \Phi^{(2)}_{0,r+s} - (4r^2 - 1)\lambda(\lambda - 1) U_{r+s} \right) \\
&+ \frac{1}{2} \tilde{\Phi}^{(2),ij}_{1,r+s} + \frac{1}{3} (4r^2 - 1) \left( 2(2\lambda - 1) T^{ij}_{r+s} + (\lambda + 1)(\lambda - 2) \tilde{T}^{ij}_{r+s} \right), \\
&\left[ \Phi^{(1)}_{i,r}, \tilde{\Phi}^{(1)}_{2,m} \right] = (2r - \frac{m}{2}) \Phi^{(2),i}_{2,r+m} \\
&+ i(4r^2 - 1) \left( \frac{2}{3} (2\lambda - 1) G^{i}_{r+m} - (4r + 3m)\lambda(\lambda - 1) \Gamma^{i}_{r+m} \right),
\end{align*}
\]
We observe the antisymmetric property of the $SO(4)$ indices. The central terms vanish under the wedge condition. Although the $\lambda$ (or $(1-\lambda)$) factor appears in the structure,
constants, those terms have typewriter font and they vanish by taking the wedge condition. The quadratic $\lambda$ terms appear under the wedge condition in the right hand side.

G.4 The (anti)commutators between the first and the second $\mathcal{N} = 4$ higher spin multiplets

The OPEs appearing in Appendix $E$ can be written in terms of (anti)commutators as follows:

\[
\begin{align*}
\left[ \Phi_{0,m}, \Phi_{0,n}^{(2)} \right] &= 0, \\
\left[ \Phi_{0,m}^{(1)}, \Phi_{0,n}^{(2)} \right] &= -2 \tilde{\Phi}_{0,m+n}^{(1)}, \\
\left[ \Phi_{0,m}^{(1)}, \Phi_{1,n}^{(2)} \right] &= 8m \left( \frac{1}{3} (2 \lambda - 1) \Phi_{1,m+n}^{(1),ij} - \tilde{\Phi}_{1,m+n}^{(1),ij} \right), \\
\left[ \Phi_{0,m}^{(1)}, \tilde{\Phi}_{2,n}^{(2)} \right] &= -\frac{1}{6} \Phi_{2,m+n}^{(3),i} + \frac{64}{15} m (2 \lambda - 1) \Phi_{2,m+n}^{(1),ij} + \frac{16}{9} m (5 m + 2 n) (\lambda + 1) (\lambda - 2) \tilde{\Phi}_{1,m+n}^{(1),ij}, \\
\left[ \Phi_{0,m}^{(1)}, \Phi_{2,n}^{(2)} \right] &= m \Phi_{0,m+n}^{(3),ij} + \frac{64}{15} m (2 \lambda - 1) \tilde{\Phi}_{2,m+n}^{(1),ij} - \frac{32}{9} m (5 m^2 + 5 m n + n^2 + 1) (\lambda + 1) (\lambda - 2) \Phi_{1,m+n}^{(1)}, \\
\left[ \Phi_{1,m}^{(1)}, \Phi_{0,n}^{(2)} \right] &= 2 \tilde{\Phi}_{1,m}^{(1),ij} \Phi_{0,n}^{(2)}, \\
\{ \Phi_{1,m}^{(1),i}, \Phi_{2,n}^{(2)} \} &= -2 \delta^{ij} \tilde{\Phi}_{2,r+s}^{(1)} - 2 (3 r - s) \left( \Phi_{1,1}^{(1),ij} \Phi_{2,r+s}^{(1),ij} - \frac{1}{3} (2 \lambda - 1) \Phi_{1,1}^{(1),ij} \right), \\
\left[ \Phi_{1,m}^{(1),ij}, \Phi_{2,n}^{(2)} \right] &= \left( \delta^{ij} \left( \frac{1}{6} \Phi_{2,r+m}^{(3),k} - \frac{2}{5} (4 r - m) (2 \lambda - 1) \Phi_{2,r+m}^{(1),k} \right) - (j \leftrightarrow k) \right) - \frac{4}{3} (4 r^2 - 1) (\lambda + 1) (\lambda - 2) \tilde{\Phi}_{1,r+s}^{(1),ij} \\
&+ 2 \varepsilon^{ijkl} (4 r - m) \tilde{\Phi}_{1,r+s}^{(1),ij}, \\
\{ \Phi_{1,m}^{(1),ij}, \Phi_{2,n}^{(2)} \} &= -\delta^{ij} \left( \left( \frac{1}{2} + r \right) \Phi_{1,1+r+s}^{(3),ij} - \frac{4}{9} (4 r^2 - 1) (5 r + 3 s) (\lambda + 1) (\lambda - 2) \Phi_{1,1+r+s}^{(1),ij} \right) + \frac{1}{6} \Phi_{1,1+r+s}^{(3),ij} - (4 r^2 - 1) \left( \frac{16}{15} (2 \lambda - 1) \Phi_{1,1+r+s}^{(1),ij} + \frac{4}{5} (\lambda + 2) (\lambda - 3) \tilde{\Phi}_{1,1+r+s}^{(1),ij} \right), \\
\left[ \Phi_{1,m}^{(1),ij}, \Phi_{2,n}^{(2)} \right] &= (r - m) \Phi_{1,1+r+s}^{(3),ij} - \frac{16}{15} (4 r^2 - 1) (2 \lambda - 1) \tilde{\Phi}_{1,1+r+s}^{(1),ij} - \frac{20}{9} (4 r^2 - 1) (2 r m + m) (\lambda + 1) (\lambda - 2) \Phi_{1,1+r+s}^{(1),ij}, \\
\left[ \Phi_{1,m}^{(1),ij}, \Phi_{0,n}^{(2)} \right] &= 4 (m - n) \left( \frac{1}{3} (2 \lambda - 1) \Phi_{1,m+n}^{(1),ij} - 3 \Phi_{1,m+n}^{(1),ij} \right), \\
\left[ \Phi_{1,m}^{(1),ij}, \Phi_{2,n}^{(2)} \right] &= \left( -\delta^{ik} \left( \frac{1}{6} \Phi_{2,m+r}^{(3),j} + \frac{2}{15} (3 m - 2 r) (2 \lambda - 1) \tilde{\Phi}_{2,m+r}^{(1),j} \right) - \frac{2}{9} (12 m^2 - 8 m r + 4 r^2 - 9) (\lambda + 1) (\lambda - 2) \Phi_{2,m+r}^{(1),j} \right) - (i \leftrightarrow j) \right)
\end{align*}
\]
\[
\begin{align*}
[\Phi_{1,m}, \Phi_{1,n}] &= -4(2m - n)(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) \Phi^{(1)}_{2,m+n} \\
&- \frac{1}{6} \delta^{ik} \Phi^{(3),j}_{1,m} - \delta^{il} \Phi^{(3),k}_{1,m} - \delta^{jk} \Phi^{(3),il}_{1,m+n} + \delta^{il} \Phi^{(3),ik}_{1,m+n} \\
&+ \left( \frac{\delta^{ik}}{15} (6m^2 - 3mn + n^2 - 4) ((2\lambda^2 - 2\lambda - 7) \Phi^{(1),j}_{1,m} + (2\lambda - 1) \Phi^{(1),l}_{1,m+n} \right) \\
&- (k \leftrightarrow l) - (i \leftrightarrow j) + (i \leftrightarrow j, k \leftrightarrow l) \\
&+ \varepsilon^{ijkl} \left( (2m - n) \left( \frac{1}{3} \Phi^{(3),j}_{0,m+n} - \frac{4}{5} (2\lambda - 1) \Phi^{(1),l}_{2,m+n} \right) \\
&- \frac{32}{9} m (m^2 - 1)(\lambda + 1)(\lambda - 2) \Phi^{(1)}_{0,m+n} \right), \\
[\Phi_{1,m}, \Phi^{(2),k}_{2,r}] &= \left( -\delta^{ik} \left( \frac{1}{6} \Phi^{(3),j}_{2,m} + \frac{1}{210} (5m - 2r)(2\lambda - 1) \Phi^{(3),j}_{2,m+r} \\
&- \frac{2}{25} (40m^2 - 16mr + 4r^2 - 25)(\lambda + 2)(\lambda - 3) \Phi^{(1),j}_{2,m+r} \\
&- \frac{16}{15} m (m^2 - 1)(\lambda + 1)(\lambda - 2)(2\lambda - 1) \Phi^{(1),j}_{2,m+r} \right) - (i \leftrightarrow j) \\
&- \varepsilon^{ijkl} \left( \frac{1}{6} (5m - 2r) \Phi^{(3),j}_{2,m} - \frac{80}{9} m (m^2 - 1)(\lambda + 1)(\lambda - 2) \Phi^{(1),l}_{2,m+r} \right), \\
[\Phi^{(1),ij}_{1,m}, \Phi^{(2),ij}_{2,n}] &= \frac{1}{3} (3m - n) \Phi^{(3),ij}_{1,m+n} \\
&- m (m^2 - 1) \left( \frac{32}{45} (17\lambda^2 - 17\lambda - 52) \Phi^{(1),ij}_{1,m} + \frac{64}{15} (2\lambda - 1) \Phi^{(1),ij}_{1,m+n} \right), \\
[\Phi^{(2),i}_{2,r}, \Phi^{(2),j}_{0,m}] &= \frac{1}{6} \Phi^{(3),j}_{2,r+m} + \frac{8}{15} (2r - 3m)(2\lambda - 1) \Phi^{(1),i}_{2,r+m} \\
&- \frac{2}{9} (4r^2 - 8rm + 12m^2 - 9)(\lambda + 1)(\lambda - 2) \Phi^{(1),i}_{2,r+m}, \\
\{\Phi^{(2),i}_{2,r}, \Phi^{(2),j}_{2,s}\} &= \delta^{ij} (m - n) \left( \frac{1}{2} \Phi^{(3)}_{0,r+s} - \frac{8}{15} (2\lambda - 1) \Phi^{(1)}_{2,r+s} \\
&- \frac{4}{9} (2r^2 + 2s^2 - 5)(\lambda + 1)(\lambda - 2) \Phi^{(1)}_{0,r+s} \right) - \frac{1}{6} \Phi^{(3),ij}_{1,r+s} \\
&+ 4(6r^2 - 8rs + 6s^2 - 9) \left( \frac{1}{15} (2\lambda - 1) \Phi^{(1),ij}_{1,r+s} + \frac{1}{45} (\lambda^2 - \lambda - 11) \Phi^{(1),ij}_{1,r+s} \right), \\
[\Phi^{(2),i}_{2,r}, \Phi^{(1),k}_{1,m}] &= \left( \delta^{ij} \left( \frac{1}{6} \Phi^{(3),j}_{2,r+m} + \frac{1}{126} (4r - 3m)(2\lambda - 1) \Phi^{(3),k}_{2,r+m} - \frac{4}{15} (4r^2 - 4rm + 2m^2 - 5)(\lambda^2 - \lambda - 11) \Phi^{(1),k}_{2,r+m} \\
&+ \frac{2}{45} (16r^3 - 12r^2m + 8rm^2 - 36r - 4m^3 + 19m) \right)
\end{align*}
\]
\[
\begin{align*}
&\times (\lambda + 1)(\lambda - 2)(2\lambda - 1) \Phi^{(1,k)}_{2,r+m} - (j \leftrightarrow k) \\
&+ \varepsilon^{ijkl}\left(- \frac{1}{6}(4r - 3m) \Phi^{(3,l)}_{2,r+m} + \frac{4}{15}(4r^2 - 4rm + 2m^2 - 5)(2\lambda - 1) \Phi^{(1,l)}_{2,r+m} \\
&+ \frac{2}{9}(16r^3 - 12r^2m + 8rm^2 - 36r - 4m^3 + 19m)(\lambda + 1)(\lambda - 2) \Phi^{(1,l)}_{2,r+m}\right) \\
&\{\tilde{\Phi}^{(1,i)}_{2,r'}, \tilde{\Phi}^{(2,j)}_{2,s}\} \\
&= \delta^{ij}\left(\frac{1}{6} \tilde{\Phi}^{(3)}_{2,r+s} + 2(40r - 32rs + 12s^2 - 45)\left(\frac{1}{315}(2\lambda - 1) \Phi^{(3)}_{0,r+s}ight) \\
&- \frac{1}{25}(\lambda + 2)(\lambda - 3) \tilde{\Phi}^{(1)}_{0,r+s} \\
&- \frac{256}{135}(x^2 - \frac{9}{4})(x^2 - \frac{1}{4}) + 4r - 3s)(\Phi^{(3,ij)}_{1,r+s} - \frac{1}{15}(2\lambda - 1) \tilde{\Phi}^{(3,ij)}_{1,r+s} \\
&- \frac{2}{25}(40r^3 - 24r^2s + 12rs^2 - 85r + 31s - 4s^3) \\
&\times \left((\lambda + 2)(\lambda - 3) (\Phi^{(1,ij)}_{1,r+s} - \frac{1}{3}(\lambda + 2)(\lambda - 3)(2\lambda - 1) \tilde{\Phi}^{(1,ij)}_{1,r+s}\right),
\end{align*}
\]
\[
\times \left( \frac{4}{105} (2\lambda - 1)\Phi_{0,m+n}^{(3)} - \frac{32}{25} (\lambda + 2)(\lambda - 3) \tilde{\Phi}_{2,m+n}^{(1)} \right) \\
- \frac{256}{135} m(m^2 - 1)(m^2 - 4)(\lambda + 1)(\lambda - 2)(2\lambda - 1) \Phi_{0,m+n}^{(1)}. \quad (G.4)
\]

There is no second $\mathcal{N} = 4$ higher spin multiplet in the right hand side of (G.4). Then, from the (anti)commutators having the components of first $\mathcal{N} = 4$ higher spin multiplet in the right hand side, we can obtain the information of the components of second $\mathcal{N} = 4$ higher spin multiplet in the left hand side. For example, see the second relation of (G.4). The cubic $\lambda$ terms appear under the wedge condition in the right hand side.

**H The second $\mathcal{N} = 4$ higher spin multiplet in terms free fields at $\lambda = 0$**

In section 6.2, we observed the first $\mathcal{N} = 4$ higher spin multiplet in terms of the free fields. For the second $\mathcal{N} = 4$ higher spin multiplet, we present them as follows:

\[
\Phi^{(2)}_0 = \frac{8}{3} (W_{B,2}^{11} + W_{B,2}^{22} - 2 W_{F,2}^{11} - 2 W_{F,2}^{22}),
\]

\[
\Phi^{(2),1}_\frac{q}{2} = -\frac{1}{q} (Q_{\frac{q}{2}}^{11} + i\sqrt{2} Q_{\frac{q}{2}}^{12} + 2i\sqrt{2} Q_{\frac{q}{2}}^{21} - 2Q_{\frac{q}{2}}^{22} + 2Q_{\frac{q}{2}}^{11} + 2i\sqrt{2} Q_{\frac{q}{2}}^{12} + i\sqrt{2} Q_{\frac{q}{2}}^{21} - Q_{\frac{q}{2}}^{22}),
\]

\[
\Phi^{(2),2}_\frac{q}{2} = \frac{1}{q} (i Q_{\frac{q}{2}}^{11} - 2\sqrt{2} Q_{\frac{q}{2}}^{12} - 2i Q_{\frac{q}{2}}^{22} + 2i Q_{\frac{q}{2}}^{11} - 2\sqrt{2} Q_{\frac{q}{2}}^{12} - i Q_{\frac{q}{2}}^{22}),
\]

\[
\Phi^{(2),3}_\frac{q}{2} = \frac{1}{q} (i Q_{\frac{q}{2}}^{11} - \sqrt{2} Q_{\frac{q}{2}}^{12} - 2i Q_{\frac{q}{2}}^{22} + 2i Q_{\frac{q}{2}}^{11} - \sqrt{2} Q_{\frac{q}{2}}^{21} - i Q_{\frac{q}{2}}^{22}),
\]

\[
\Phi^{(2),4}_\frac{q}{2} = \frac{1}{q} (Q_{\frac{q}{2}}^{11} + 2 Q_{\frac{q}{2}}^{22} - 2 Q_{\frac{q}{2}}^{11} - Q_{\frac{q}{2}}^{22}),
\]

\[
\Phi^{(2),12}_1 = \frac{2}{q} (-2i W_{B,3}^{11} + \sqrt{2} W_{B,3}^{12} + 2i W_{B,3}^{22} - 2i W_{F,3}^{11} + 2\sqrt{2} W_{F,3}^{12} + 2i W_{F,3}^{22}),
\]

\[
\Phi^{(2),13}_1 = \frac{2}{q} (2i W_{B,3}^{11} - 4\sqrt{2} W_{B,3}^{21} - 2i W_{B,3}^{22} + 2i W_{F,3}^{11} - 2\sqrt{2} W_{F,3}^{21} - 2i W_{F,3}^{22}),
\]

\[
\Phi^{(2),14}_1 = \frac{2}{q} (-2i W_{B,3}^{11} - i\sqrt{2} W_{B,3}^{12} - 4i\sqrt{2} W_{B,3}^{21} + 2 W_{B,3}^{22} + 2 W_{F,3}^{11} + 2i\sqrt{2} W_{F,3}^{12} + 2i\sqrt{2} W_{F,3}^{21}
- 2 W_{F,3}^{22}),
\]

\[
\Phi^{(2),23}_1 = \frac{2}{q} (2 W_{B,3}^{11} + i\sqrt{2} W_{B,3}^{12} + 4i\sqrt{2} W_{B,3}^{21} - 2 W_{B,3}^{22} + 2 W_{F,3}^{11} + 2i\sqrt{2} W_{F,3}^{12} + 2i\sqrt{2} W_{F,3}^{21}
- 2 W_{F,3}^{22}),
\]

\[
\Phi^{(2),24}_1 = \frac{2}{q} (2i W_{B,3}^{11} - 4\sqrt{2} W_{B,3}^{21} - 2i W_{B,3}^{22} - 2i W_{F,3}^{11} + 2\sqrt{2} W_{F,3}^{21} + 2i W_{F,3}^{22}),
\]

\[
\Phi^{(2),34}_1 = \frac{2}{q} (2i W_{B,3}^{11} - \sqrt{2} W_{B,3}^{12} - 2i W_{B,3}^{22} - 2i W_{F,3}^{11} + 2\sqrt{2} W_{F,3}^{12} + 2i W_{F,3}^{22}),
\]

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\[ \Phi_{\frac{3}{2}}^{(2),1} = \frac{1}{q^2} \left( Q_{\frac{1}{2}}^{11} + i \sqrt{2} Q_{\frac{1}{2}}^{12} + 2i \sqrt{2} Q_{\frac{1}{2}}^{21} - 2 Q_{\frac{1}{2}}^{22} - 2 \bar{Q}_{\frac{1}{2}}^{11} - 2i \sqrt{2} Q_{\frac{1}{2}}^{12} - i \sqrt{2} Q_{\frac{1}{2}}^{21} + \bar{Q}_{\frac{1}{2}}^{22} \right), \]
\[ \Phi_{\frac{3}{2}}^{(2),2} = -i \frac{1}{q^2} \left( Q_{\frac{1}{2}}^{11} + i \sqrt{2} Q_{\frac{1}{2}}^{12} - 2 Q_{\frac{1}{2}}^{22} - 2 \bar{Q}_{\frac{1}{2}}^{11} - 2i \sqrt{2} Q_{\frac{1}{2}}^{12} + \bar{Q}_{\frac{1}{2}}^{22} \right), \]
\[ \Phi_{\frac{3}{2}}^{(2),3} = -i \frac{1}{q} \left( Q_{\frac{1}{2}}^{11} + i \sqrt{2} Q_{\frac{1}{2}}^{12} - 2 Q_{\frac{1}{2}}^{22} - 2 \bar{Q}_{\frac{1}{2}}^{11} - 2i \sqrt{2} Q_{\frac{1}{2}}^{12} + \bar{Q}_{\frac{1}{2}}^{22} \right), \]
\[ \Phi_{\frac{3}{2}}^{(2),4} = -\frac{1}{q^2} \left( Q_{\frac{1}{2}}^{11} + 2 Q_{\frac{1}{2}}^{22} + 2 \bar{Q}_{\frac{1}{2}}^{11} + \bar{Q}_{\frac{1}{2}}^{22} \right), \]
\[ \Phi_{\frac{3}{2}}^{(2)} = \frac{4}{q^2} \left( W_{B,4}^{11} + W_{B,4}^{22} + W_{F,F,4}^{11} + W_{F,F,4}^{22} \right). \]

\textbf{H.1}

The structure of these higher spin currents looks like the one for the first \( \mathcal{N} = 4 \) higher spin multiplet in section 6.2. There are only sign changes in front of the fields for the \( SO(4) \) singlets. For the \( SO(4) \) nonsinglet case, the only overall factors differ. Once one of the components of the second \( \mathcal{N} = 4 \) higher spin multiplet is found by using some of the relations in \( \text{(G.4)} \), then the remaining 15 higher spin currents can be fixed with the help of spin-\( \frac{3}{2} \) currents as described before.

\section{I The first \( \mathcal{N} = 4 \) higher spin multiplet in terms free fields at \( \lambda = 1 \)}

For the \( \lambda = 1 \), we can analyze the description in section 6.1, 6.2, and 6.3 similarly. For some of the 16 currents, we have the following expressions

\[ L = W_{B,1}^{111} + W_{B,1}^{22} + W_{F,2}^{11} + W_{F,2}^{22}, \]
\[ \frac{1}{4q} \left( T^{14} - T^{23} \right) = i W_{F,1}^{11} - \sqrt{2} W_{F,1}^{12} - \sqrt{2} W_{F,1}^{21} - i W_{F,1}^{22}, \]
\[ \frac{1}{4q} \left( T^{13} - T^{42} \right) = -W_{F,1}^{11} - i \sqrt{2} W_{F,1}^{21} + W_{F,1}^{22}, \]
\[ \frac{1}{4q} \left( T^{12} - T^{34} \right) = -W_{F,1}^{11} - i \sqrt{2} W_{F,1}^{12} + W_{F,1}^{22}, \]
\[ G^1 = -\frac{1}{2} \left( Q_{\frac{3}{2}}^{12} + i \sqrt{2} Q_{\frac{3}{2}}^{12} + 2i \sqrt{2} Q_{\frac{3}{2}}^{21} - 2 Q_{\frac{3}{2}}^{22} - 2 \bar{Q}_{\frac{3}{2}}^{11} - 2i \sqrt{2} Q_{\frac{3}{2}}^{12} - i \sqrt{2} Q_{\frac{3}{2}}^{21} + \bar{Q}_{\frac{3}{2}}^{22} \right) \]
\[ - i \sqrt{2} Q_{\frac{3}{2}}^{21} + \bar{Q}_{\frac{3}{2}}^{22} \right), \]
\[ G^2 = -\frac{i}{2} \left( Q_{\frac{3}{2}}^{12} + 2i \sqrt{2} Q_{\frac{3}{2}}^{21} - 2 Q_{\frac{3}{2}}^{22} - 2 \bar{Q}_{\frac{3}{2}}^{11} - 2i \sqrt{2} Q_{\frac{3}{2}}^{12} + \bar{Q}_{\frac{3}{2}}^{22} \right), \]
\[ G^3 = \frac{i}{2} \left( Q_{\frac{3}{2}}^{12} + i \sqrt{2} Q_{\frac{3}{2}}^{21} - 2 Q_{\frac{3}{2}}^{22} - 2 \bar{Q}_{\frac{3}{2}}^{11} - i \sqrt{2} Q_{\frac{3}{2}}^{21} + \bar{Q}_{\frac{3}{2}}^{22} \right), \]
\[ G^4 = \frac{1}{2} Q_{\frac{3}{2}}^{11} + Q_{\frac{3}{2}}^{22} + \bar{Q}_{\frac{3}{2}}^{11} + \frac{1}{2} \bar{Q}_{\frac{3}{2}}^{22}. \]
We observe that there are some extra minus signs in the coefficients of \((I.1)\), compared to the \(\lambda = 0\) case in subsection 6.1.

For the first \(\mathcal{N} = 4\) higher spin multiplet, we obtain

\[
\begin{align*}
\Phi^{(1)}_0 &= 4q(W^{11}_{F,1} + W^{22}_{F,1}), \\
\Phi^{(1),1}_1 &= \frac{1}{2}(Q^{11}_{\frac{1}{2}} + i\sqrt{2}Q^{12}_{\frac{1}{2}} + 2i\sqrt{2}Q^{21}_{\frac{3}{2}} - 2Q^{22}_{\frac{1}{2}} + 2\bar{Q}^{11}_{\frac{1}{2}} + 2i\sqrt{2}\bar{Q}^{12}_{\frac{3}{2}} + i\sqrt{2}\bar{Q}^{21}_{\frac{3}{2}} - \bar{Q}^{22}_{\frac{1}{2}}), \\
\Phi^{(1),2}_1 &= \frac{i}{2}(Q^{11}_{\frac{1}{2}} + 2i\sqrt{2}Q^{21}_{\frac{3}{2}} - 2Q^{22}_{\frac{1}{2}} + 2\bar{Q}^{11}_{\frac{1}{2}} + 2i\sqrt{2}\bar{Q}^{12}_{\frac{3}{2}} - \bar{Q}^{22}_{\frac{1}{2}}), \\
\Phi^{(1),3}_1 &= -\frac{i}{2}(Q^{11}_{\frac{1}{2}} + i\sqrt{2}Q^{12}_{\frac{3}{2}} - 2Q^{22}_{\frac{1}{2}} + 2\bar{Q}^{11}_{\frac{1}{2}} + i\sqrt{2}\bar{Q}^{21}_{\frac{3}{2}} - \bar{Q}^{22}_{\frac{1}{2}}), \\
\Phi^{(1),4}_1 &= -\frac{1}{2}Q^{11}_{\frac{1}{2}} - Q^{22}_{\frac{1}{2}} + \bar{Q}^{11}_{\frac{1}{2}} + \frac{1}{2}\bar{Q}^{22}_{\frac{1}{2}}, \\
\Phi^{(1),12}_1 &= 2iW^{11}_{B,2} - \sqrt{2}W^{12}_{B,2} + 2iW^{22}_{B,2} + 2iW^{11}_{F,2} - 2\sqrt{2}W^{12}_{F,2} - 2iW^{22}_{F,2}, \\
\Phi^{(1),13}_1 &= 2iW^{11}_{B,2} - 4\sqrt{2}W^{21}_{B,2} - 2iW^{22}_{B,2} + 2iW^{11}_{F,2} - 2\sqrt{2}W^{21}_{F,2} - 2iW^{22}_{F,2}, \\
\Phi^{(1),14}_1 &= 2iW^{11}_{B,2} - i\sqrt{2}W^{12}_{B,2} + 4i\sqrt{2}W^{21}_{B,2} - 2W^{22}_{B,2} + 2W^{11}_{F,2} + 2W^{11}_{F,2} - 2i\sqrt{2}W^{12}_{F,2} + 2\sqrt{2}W^{12}_{F,2} - 2W^{22}_{F,2}, \\
\Phi^{(1),23}_1 &= -2iW^{11}_{B,2} - i\sqrt{2}W^{12}_{B,2} - 4i\sqrt{2}W^{21}_{B,2} - 2W^{22}_{B,2} - 2W^{11}_{F,2} - 2i\sqrt{2}W^{12}_{F,2} - 2\sqrt{2}W^{12}_{F,2} + 2W^{22}_{F,2}, \\
\Phi^{(1),24}_1 &= 2iW^{11}_{B,2} - 4\sqrt{2}W^{21}_{B,2} + 2iW^{22}_{B,2} + 2iW^{11}_{F,2} - 2\sqrt{2}W^{21}_{F,2} - 2iW^{22}_{F,2}, \\
\Phi^{(1),34}_1 &= 2iW^{11}_{B,2} - \sqrt{2}W^{12}_{B,2} - 2iW^{22}_{B,2} + 2iW^{11}_{F,2} + 2\sqrt{2}W^{12}_{F,2} + 2iW^{22}_{F,2}, \\
\Phi^{(1),1}_2 &= \frac{1}{2q}(Q^{11}_{\frac{1}{2}} + i\sqrt{2}Q^{12}_{\frac{3}{2}} + 2i\sqrt{2}Q^{21}_{\frac{3}{2}} - 2Q^{22}_{\frac{1}{2}} - 2i\sqrt{2}Q^{11}_{\frac{1}{2}} + i\sqrt{2}\bar{Q}^{21}_{\frac{3}{2}} + \bar{Q}^{22}_{\frac{1}{2}}), \\
\Phi^{(1),2}_2 &= \frac{i}{2q}(Q^{11}_{\frac{1}{2}} + 2i\sqrt{2}Q^{21}_{\frac{3}{2}} - 2Q^{22}_{\frac{1}{2}} - 2\bar{Q}^{11}_{\frac{1}{2}} - 2i\sqrt{2}\bar{Q}^{12}_{\frac{3}{2}} + \bar{Q}^{22}_{\frac{1}{2}}), \\
\Phi^{(1),3}_2 &= -\frac{i}{2q}(Q^{11}_{\frac{1}{2}} + i\sqrt{2}Q^{12}_{\frac{3}{2}} - 2Q^{22}_{\frac{1}{2}} - 2\bar{Q}^{11}_{\frac{1}{2}} - i\sqrt{2}\bar{Q}^{21}_{\frac{3}{2}} + \bar{Q}^{22}_{\frac{1}{2}}), \\
\Phi^{(1),4}_2 &= -\frac{1}{2q}(Q^{11}_{\frac{1}{2}} + 2Q^{22}_{\frac{1}{2}} + 2\bar{Q}^{11}_{\frac{1}{2}} + \bar{Q}^{22}_{\frac{1}{2}}), \\
\tilde{\Phi}^{(1)}_2 &= \frac{2}{q}(W^{11}_{B,3} + W^{22}_{B,3} + W^{11}_{F,3} + W^{22}_{F,3}). \quad (I.2)
\end{align*}
\]

The expression in \((I.2)\) looks similar to the ones in the subsection 6.2. For the next \(\mathcal{N} = 4\) higher spin multiplet, we can obtain similar results. We do not present them in this paper.
We present the remaining (anti)commutators described in section 6.4 as follows:

\[
\{\Phi_{\frac{h_1}{2}}^{(h_1)}\Phi_{\frac{h_2}{2}}^{(h_2)},\Phi_{\frac{h_1}{2}}^{(h_1-h_2-1)},\Phi_{\frac{h_2}{2}}^{(h_2-h_1-1)}\} = \delta^{ij} \sum_{h=0}^{\frac{h_1}{2}} \frac{(2h_1 - 1)(2h_2 - 1)}{32(h_1 + h_2 - 2h - 16)} \\
\times \left( \left( o_{\frac{h_1}{2}}^{(h_1)} - o_{\frac{h_2}{2}}^{(h_2)} \right) \Phi_{\frac{h_1}{2}}^{(h_1 + h_2)} \right) \\
- \frac{1}{2(h_1 + h_2 - 2h - 1)} \delta^{(h_1 + h_2 - 2h)}_0 \\
+ (h_1 + h_2 - 2h) \left( o_{\frac{h_1}{2}}^{(h_1 + h_2)} \right) \Phi_{\frac{h_1}{2}}^{(h_1 + h_2 - 2h)} \\
+ \frac{1}{2} \frac{(h_1 + h_2 - 2) (2h_1 - 1)(2h_2 - 1)}{32(h_1 + h_2 - 2h - 80)} \\
\times \left( \left( o_{\frac{h_1}{2}}^{(h_1) + \frac{1}{2}} - o_{\frac{h_2}{2}}^{(h_2) + \frac{1}{2}} \right) \Phi_{\frac{h_1}{2}}^{(h_1 + h_2 - 2h - 2),i} \right) \\
- \left( o_{\frac{h_1}{2}}^{(h_1) + \frac{1}{2}} + o_{\frac{h_2}{2}}^{(h_2) + \frac{1}{2}} \right) \Phi_{\frac{h_1}{2}}^{(h_1 + h_2 - 2h - 2),i} \\
- \delta^{h_1 h_2} \delta^{ij} N \frac{2^{2h_1 - 5} h_1! (h_1 - 1)!}{(2h_1 - 3)!!(2h_1 - 3)!! \prod_{j=-h_1}^{h_1 - 1} (r + j + \frac{1}{2})}, \right.
\]

\[\left[ \Phi_{\frac{h_1}{2}}^{(h_1)}\Phi_{\frac{h_2}{2}}^{(h_2)},\Phi_{\frac{h_1}{2}}^{(h_1-h_2-1)},\Phi_{\frac{h_2}{2}}^{(h_2-h_1-1)}\right] = \left[ -\delta^{ijk} \sum_{h=0}^{\frac{h_1}{2}} \frac{(2h_1 - 1)(2h_2 - 1)}{4(h_1 + h_2 - 2h - 10)} \\
\times \left( q_{\frac{h_1}{2}}^{(h_1)+\frac{1}{2}} - q_{\frac{h_2}{2}}^{(h_2)+\frac{1}{2}} \right) \Phi_{\frac{h_1}{2}}^{(h_1 + h_2 - 2),j} \\
- \left( q_{\frac{h_1}{2}}^{(h_1)+\frac{1}{2}} + q_{\frac{h_2}{2}}^{(h_2)+\frac{1}{2}} \right) \Phi_{\frac{h_1}{2}}^{(h_1 + h_2 - 2),j} \right] - (j \leftrightarrow k) \right]
\]

\[+ \epsilon^{jkl} \sum_{h=0}^{\frac{h_1}{2}} \frac{(2h_1 - 1)(2h_2 - 1)}{4(h_1 + h_2 - 2h - 10)} \\
\times \left( q_{\frac{h_1}{2}}^{(h_1)+\frac{1}{2}} + q_{\frac{h_2}{2}}^{(h_2)+\frac{1}{2}} \right) \Phi_{\frac{h_1}{2}}^{(h_1 + h_2 - 2),l} \\
- \left( q_{\frac{h_1}{2}}^{(h_1)+\frac{1}{2}} - q_{\frac{h_2}{2}}^{(h_2)+\frac{1}{2}} \right) \Phi_{\frac{h_1}{2}}^{(h_1 + h_2 - 2),l} \right), \]

\[\{\Phi_{\frac{h_1}{2}}^{(h_1)}\Phi_{\frac{h_2}{2}}^{(h_2)},\Phi_{\frac{h_1}{2}}^{(h_1-h_2-1)},\Phi_{\frac{h_2}{2}}^{(h_2-h_1-1)}\} = \delta^{ij} \sum_{h=0}^{\frac{h_1}{2}} \frac{(2h_1 - 1)(2h_2 - 1)}{32(h_1 + h_2 - 2h - 16)} \]
\[ \begin{align*}
&\times \left( \Phi^{(h_1+\frac{1}{2})(h_2+\frac{3}{2})} - \Phi^{(h_1+\frac{1}{2})(h_2+\frac{5}{2})} \right) \Phi_0^{(h_1+h_2-2h)} \\
&- \frac{2}{2(h_1 + h_2 - 2h) - 5} \left( (h_1 + h_2 - 2h - 1) \Phi_0^{(h_1+\frac{1}{2})(h_2+\frac{5}{2})} \right) \\
&+ (h_1 + h_2 - 2h) \Phi_0^{(h_1+\frac{1}{2})(h_2+\frac{3}{2})} \Phi_2^{(h_1+h_2-2h-2)} \\
&+ \frac{1}{2} (h_1 + h_2) \left( 2(h_1 - 1)(2h_2 - 1) \right) \\
&+ \sum_{n=0}^{h_1+h_2-2} \frac{(2h_1 - 1)(2h_2 - 1)}{32(h_1 + h_2 - 2h) - 16} \Phi_1^{(h_1+h_2-2h),ij} \\
&\times \left( \Phi_0^{(h_1+\frac{1}{2})(s_1+\frac{1}{2})} - \Phi_0^{(h_1+\frac{1}{2})(s_1+\frac{3}{2})} \right) \Phi_1^{(h_1+h_2-2h),ij} \\
&+ (h_1 + h_2 - 2h) \Phi_1^{(h_1+\frac{1}{2})(s_1+\frac{3}{2})} \Phi_2^{(h_1+h_2-2h-2)} \right) \\
&\left[ \Phi^{(h_1),i}, \Phi^{(h_2)} \right] = \frac{1}{2} (h_1 + h_2 - 2) \left( 2(h_1 - 1)(2h_2 - 1) \right) \\
&+ \sum_{n=0}^{h_1+h_2-2} \frac{(2h_1 - 1)(2h_2 - 1)}{4(h_1 + h_2 - 2h) - 10} \Phi_2^{(h_1+h_2-2h),i} \\
&\times \left( \Phi_0^{(h_2+\frac{1}{2})(h_1+\frac{1}{2})} + \Phi_0^{(h_2+\frac{1}{2})(h_1+\frac{3}{2})} \right) \Phi_2^{(h_1+h_2-2h-2),i} \\
&- \frac{1}{2} (h_1 + h_2 - 1) \left( 2(h_1 - 1)(2h_2 - 1) \right) \\
&+ \sum_{n=0}^{h_1+h_2-2} \frac{(2h_1 - 1)(2h_2 - 1)}{4(h_1 + h_2 - 2h) - 2} \Phi_2^{(h_1+h_2-2h),i} \\
&\times \left( \Phi_0^{(h_2+\frac{1}{2})(h_1+\frac{1}{2})} + \Phi_0^{(h_2+\frac{1}{2})(h_1+\frac{3}{2})} \right) \Phi_2^{(h_1+h_2-2h-2),i} \\
&\left[ \Phi^{(s_1),ij}, \Phi^{(s_2),kl} \right] = \left( \delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk} \right) \left( \Phi_0^{(h_1)+1}(h_2+1) \right) \\
&\times \left( \Phi_0^{(h_2)+1}(h_1+1) \right) \Phi_0^{(h_1+h_2-2h)} \\
&- \frac{2}{2(h_1 + h_2 - 2h) - 5} \left( (h_1 + h_2 - 2h - 1) \Phi_0^{(h_1+1)(h_2+1)} \right) \\
&+ (h_1 + h_2 - 2h) \Phi_0^{(h_1+1)(h_2+1)} \Phi_2^{(h_1+h_2-2h-2)} \\
&+ \varepsilon^{ijkl} \sum_{r=0}^{h_1+h_2-1} \frac{(2h_1 - 1)(2h_2 - 1)}{8(h_1 + h_2 - 2h) - 4} \Phi_2^{(h_1+h_2-2h-2)} \\
&\times \left( \Phi_0^{(h_2+1)(h_1+1)} + \Phi_0^{(h_2+1)(h_1+1)} \right) \Phi_0^{(h_1+h_2-2h)} \\
&- \frac{2}{2(h_1 + h_2 - 2h) - 5} \left( (h_1 + h_2 - 2h - 1) \Phi_0^{(h_1+1)(h_2+1)} \right) \\
&+ (h_1 + h_2 - 2h) \Phi_0^{(h_1+1)(h_2+1)} \Phi_2^{(h_1+h_2-2h-2)} \right) \\
&\right) \right) \\
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\]
\[
\begin{align*}
&+ \left[ -\delta^{ik} \sum_{r=-1}^{1} \left( \frac{1}{2}(h_1 + h_2 - 2)^2 \frac{(2h_1 - 1)(2h_2 - 1)}{8(h_1 + h_2 - 2h) - 20} \right. \\
&\times \left. \left( p_{F,2h+1}^{(h_1+1)(h_2+1)} + p_{B,2h+1}^{(h_1+1)(h_2+1)} \right) \Phi_1^{(h_1+h_2-2h),il} \\
&+ \left( p_{F,2h+1}^{(h_1+1)(h_2+1)} - p_{B,2h+1}^{(h_1+1)(h_2+1)} \right) \bar{\Phi}_1^{(h_1+h_2-2h),il} \right] \\
&- (k \leftrightarrow l) - (i \leftrightarrow j) + (i \leftrightarrow j, k \leftrightarrow l) \\
&- \delta^{h_1 h_2} \left( (\delta^{ik} \delta^{jl} - \delta^{ij} \delta^{kl}) \frac{N 2^{2h_1-5} h_1! (h_1 - 1)!}{(2h_1 - 3)!!} \right) \sum_{h=1}^{h} \left( \frac{1}{2(h_1 + h_2 - 2h) - 1} \right) \\
&- \varepsilon^{ijkl} \frac{N 2^{2h_1-5} h_1! (h_1 - 1)! (2h_1 - 1)}{(2h_1 + 1)(2h_1 - 3)!!} \prod_{j=-h_1}^{h_1} (m + j),
\end{align*}
\]

\[
\left[ \Phi_1^{(h_1),ij}, \bar{\Phi}_2^{(h_2),k} \right] = \left[ \delta^{ik} \frac{1}{2}(2h_1 - 1)(2h_2 - 1) \right. \\
\times \sum_{h=-1}^{1} \frac{1}{2(h_1 + h_2 - 2h) - 5} \left( p_{F,2h+1}^{(h_1+1)(h_2+\frac{3}{2})} - q_{B,2h+1}^{(h_1+1)(h_2+\frac{3}{2})} \right) \\
\times \left( p_{F,2h}^{(h_1+1)(h_2+\frac{3}{2})} - q_{B,2h}^{(h_1+1)(h_2+\frac{3}{2})} \right) \bar{\Phi}_2^{(h_1+h_2-2h),j} \\
+ \frac{1}{2} \left( p_{F,2h}^{(h_1+1)(h_2+\frac{3}{2})} + q_{B,2h}^{(h_1+1)(h_2+\frac{3}{2})} \right) \bar{\Phi}_2^{(h_1+h_2-2h),l} \\
\left. \sum_{h=-1}^{1} \frac{1}{2(h_1 + h_2 - 2h) - 1} \left( q_{F,2h}^{(h_1+1)(h_2+\frac{3}{2})} + q_{B,2h}^{(h_1+1)(h_2+\frac{3}{2})} \right) \right] \\
\times \left( \frac{1}{2} \delta^{(h_1+h_2-2h),l} \right),
\]

\[
\left[ \Phi_1^{(h_1),ij}, \bar{\Phi}_2^{(h_2)} \right] = - \sum_{h=0}^{\frac{1}{2}(h_1+h_2)} \frac{(2h_1 - 1)(2h_2 - 1)}{8(h_1 + h_2 - 2h) - 4} \\
\times \left( p_{F,2h}^{(h_2+2)(h_2+1)} + q_{B,2h}^{(h_2+2)(h_2+1)} \right) \Phi_1^{(h_1+h_2-2h),ij} \\
+ \left( p_{F,2h}^{(h_2+2)(h_2+1)} - q_{B,2h}^{(h_2+2)(h_2+1)} \right) \bar{\Phi}_1^{(h_1+h_2-2h),ij} \\
,\]

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\[ \left\{ \tilde{\Phi}_{\frac{1}{2}}^{(h_1),i}, \tilde{\Phi}_{\frac{1}{2}}^{(h_2),j} \right\} = -\delta^{ij} \sum_{h=0}^{\frac{1}{2}(h_1+h_2+1)} \frac{(2h_1-1)(2h_2-1)}{32(h_1+h_2-2h)+48} \times \left( o_{F,2h}^{(h_1+\frac{1}{2})(h_2+\frac{3}{2})} - o_{B,2h}^{(h_1+\frac{1}{2})(h_2+\frac{3}{2})} \right) \tilde{\Phi}_0^{(h_1+h_2-2h+2)} \]
\[ \times \left( \frac{2}{(h_1+h_2-2h+1)} \left( (h_1+h_2-2h+1) o_{F,2h}^{(h_1+\frac{1}{2})(h_2+\frac{3}{2})} \right) \right) + \left( h_1 + h_2 - 2h + 2 \right) o_{B,2h}^{(h_1+\frac{1}{2})(h_2+\frac{3}{2})} \right) \tilde{\Phi}_2^{(h_1+h_2-2h)} \right) \]
\[ \sum_{h=0}^{\frac{1}{2}(h_1+h_2)} \frac{(2h_1-1)(2h_2-1)}{32(h_1+h_2-2h)-16} \times \left( o_{F,2h+1}^{(h_1+\frac{1}{2})(h_2+\frac{3}{2})} - o_{B,2h+1}^{(h_1+\frac{1}{2})(h_2+\frac{3}{2})} \right) \tilde{\Phi}_1^{(h_1+h_2-2h),ij} \right) \]
\[ \times \left( \frac{2}{(h_1+h_2-2h+1)} \left( (h_1+h_2-2h+1) o_{F,2h+1}^{(h_1+\frac{1}{2})(h_2+\frac{3}{2})} \right) \right) + o_{B,2h+1}^{(h_1+\frac{1}{2})(h_2+\frac{3}{2})} \right) \tilde{\Phi}_2^{(h_1+h_2-2h),ij} \right) \]
\[ \delta^{h_1 h_2} \delta^{ij} \frac{N 2^{2h_1-3} h_1!(h_1+1)! (2h_1-1)^2}{(2h_1+1)!!(2h_1+1)!!} \prod_{j=-h_1-1}^{h_1} (r+j+\frac{1}{2}) , \]

\[
\left[ \tilde{\Phi}_{\frac{1}{2}}^{(h_1),i}, \tilde{\Phi}_{\frac{1}{2}}^{(h_2),j} \right] = -\sum_{h=1}^{\frac{1}{2}(h_1+h_2)} \frac{(2h_1-1)(2h_2-1)}{4(h_1+h_2-2h)-2} \times \left( q_{F,2h}^{(h_2+h_2)(h_1+\frac{3}{2})} + q_{B,2h}^{(h_2+h_2)(h_1+\frac{3}{2})} \right) \tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-2h),i} \]
\[ \left( q_{F,2h+1}^{(h_2+h_2+1)(h_1+\frac{3}{2})} + q_{B,2h+1}^{(h_2+h_2+1)(h_1+\frac{3}{2})} \right) \tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-2h),i} \right) , \]

\[
\left[ \tilde{\Phi}_{2}^{(h_1),i}, \tilde{\Phi}_{2}^{(h_2),j} \right] = -\sum_{h=0}^{\frac{1}{2}(h_1+h_2+2)} \frac{(2h_1-1)(2h_2-1)}{8(h_1+h_2-2h)+12} \times \left( p_{F,2h}^{(h_2+h_2+2)(h_1+\frac{3}{2})} - p_{B,2h}^{(h_2+h_2+2)(h_1+\frac{3}{2})} \right) \tilde{\Phi}_0^{(h_1+h_2-2h+2)} \]
\[ \times \left( \frac{2}{(h_1+h_2-2h+1)} \left( (h_1+h_2-2h+1) p_{F,2h}^{(h_1+2)(h_2+2)} \right) \right) + \left( h_1 + h_2 - 2h + 2 \right) p_{B,2h}^{(h_1+2)(h_2+2)} \right) \tilde{\Phi}_2^{(h_1+h_2-2h)} \right) \]
\[ + \delta^{h_1 h_2} \frac{N 2^{2h_1-3} h_1!(h_1+1)! (2h_1-1)^2}{(2h_1+1)!!(2h_1+1)!!} \prod_{j=-h_1-1}^{h_1} (r+j+\frac{1}{2}) . \tag{J.1} \]

Note that in the second, fourth, seventh and ninth of (J.1), the order of two modes appearing in \( q_F, q_B, p_F, p_B \) in the right hand side is reversed, compared to the ones in the left hand side. For example, the indices of \((r, m)\) appearing in the left hand side of the second of (J.1) occur
in the form of \( q_F(m, r) \) and \( q_B(m, r) \) in the right hand side. We can obtain \([\tilde{\Phi}^{(h_1),i}, \Phi^{(h_2)}_0]\) from the second expression of (6.21) by substituting \( h_1 \leftrightarrow h_2 \) with minus sign. Furthermore, the other nine (anti)commutators we do not present explicitly can be red off from the known ones similarly. As usual, the mode parameter appearing in the central terms refers to the one of the first higher spin current of the left hand side.\(^{45}\)

For convenience, we present the quasi primary fields in (6.1) in terms of the components of \( \mathcal{N} = 4 \) higher spin multiplets

\[
W_{F,h}^{11} = q^{h-2} \left( \frac{1}{2(1-2h)} \Phi_0^{(h)} + \frac{(h-1)}{(2h-5)(2h-1)} \bar{\Phi}_2^{(h-2)} \right) + q^{h-2} \frac{1}{2(2h-3)} (i \Phi_1^{(h-1),12}) - i \Phi_1^{(h-1),13} - i \Phi_1^{(h-1),14} - i \Phi_1^{(h-1),23} + i \Phi_1^{(h-1),24} + i \Phi_1^{(h-1),34},
\]

\[
W_{F,h}^{12} = q^{h-2} \frac{1}{\sqrt{2(2h-3)}} \left( \Phi_1^{(h-1),13} - i \Phi_1^{(h-1),14} - i \Phi_1^{(h-1),23} - i \Phi_1^{(h-1),24} \right),
\]

\[
W_{F,h}^{21} = -q^{h-2} \frac{1}{\sqrt{2(2h-3)}} \left( \Phi_1^{(h-1),12} + i \Phi_1^{(h-1),14} + i \Phi_1^{(h-1),23} + i \Phi_1^{(h-1),34} \right),
\]

\[
W_{F,h}^{22} = q^{h-2} \left( \frac{1}{2(1-2h)} \Phi_0^{(h)} + \frac{(h-1)}{(2h-5)(2h-1)} \bar{\Phi}_2^{(h-2)} \right) - q^{h-2} \frac{1}{2(2h-3)} (i \Phi_1^{(h-1),12}) - i \Phi_1^{(h-1),13} - i \Phi_1^{(h-1),14} - i \Phi_1^{(h-1),23} + i \Phi_1^{(h-1),24} + i \Phi_1^{(h-1),34},
\]

\[
W_{B,h}^{11} = -q^{h-2} \left( \frac{1}{2(1-2h)} \Phi_0^{(h)} - \frac{h}{(2h-5)(2h-1)} \bar{\Phi}_2^{(h-2)} \right) + q^{h-2} \frac{1}{2(2h-3)} (i \Phi_1^{(h-1),12}) - i \Phi_1^{(h-1),13} + i \Phi_1^{(h-1),14} - i \Phi_1^{(h-1),23} - i \Phi_1^{(h-1),24} - i \Phi_1^{(h-1),34},
\]

\[
W_{B,h}^{12} = q^{h-2} \frac{\sqrt{2}}{(2h-3)} \left( \Phi_1^{(h-1),13} + i \Phi_1^{(h-1),14} + i \Phi_1^{(h-1),23} + i \Phi_1^{(h-1),24} \right),
\]

\[
W_{B,h}^{21} = -q^{h-2} \frac{1}{\sqrt{2}(2h-3)} \left( \Phi_1^{(h-1),12} - i \Phi_1^{(h-1),14} + i \Phi_1^{(h-1),23} - i \Phi_1^{(h-1),34} \right),
\]

\[
W_{B,h}^{22} = -q^{h-2} \left( \frac{1}{2(1-2h)} \Phi_0^{(h)} - \frac{h}{(2h-5)(2h-1)} \bar{\Phi}_2^{(h-2)} \right) - q^{h-2} \frac{1}{2(2h-3)} (i \Phi_1^{(h-1),12}) - i \Phi_1^{(h-1),13} + i \Phi_1^{(h-1),14} - i \Phi_1^{(h-1),23} - i \Phi_1^{(h-1),24} - i \Phi_1^{(h-1),34},
\]

\(^{45}\) In the first, third and eighth anticommutators, the \( SO(4) \) singlets with subscript 0 and 2 and \( SO(4) \) adjoints appear in the right hand side of (6.1). The second and sixth commutators contain the \( SO(4) \) vectors with subscript \( \frac{1}{2} \) and \( \frac{3}{2} \). In the fourth and ninth commutators, the \( SO(4) \) vectors with subscript \( \frac{1}{2} \) and \( \frac{3}{2} \) appear. The fifth commutator contains the \( SO(4) \) singlets with subscript 0 and 2 and \( SO(4) \) adjoints. In the seventh commutator, the \( SO(4) \) adjoints with subscript 1 appear. Finally, the last commutator contains the \( SO(4) \) singlets with subscript 0 and 2. The coefficients of the higher spin currents having negative spins in (6.1) are vanishing. As in the footnote 51 we have \( \phi_{h_1+h_2+1}^{h_1+1,h_2+1}(0,0) = 0 = \phi_{h_1+2,h_2+2}^{h_1+1,h_2+1}(0,0) \). Moreover, \( \phi_{h_1+h_2+1}^{h_1+1,h_2} \left( \frac{1}{2}, -\frac{1}{4} \right) \) is written as \( 3 F_2 \left[ \begin{array}{c} \frac{1}{2}, -\frac{1}{2} \\ -h_1 + \frac{1}{2}, -h_2 + \frac{1}{2} \end{array} \right] ; 1 \right] \) which contains \( \frac{1}{\sqrt{\Gamma(-h_1+1) \Gamma(-h_2+1)}} \) and is zero. There are also vanishing \( \phi_{h_1+h_2}^{h_1+\frac{1}{2},h_2+\frac{1}{2}} \left( \frac{1}{2}, -\frac{1}{4} \right) \) and \( \phi_{h_1+h_2+1}^{h_1+\frac{1}{2},h_2+\frac{1}{2}} \left( \frac{1}{2}, -\frac{1}{4} \right) \).
\[ Q_{h+\frac{1}{2}}^{11} = q^{h-1} \frac{2}{(2h-1)} \left( \Phi^{(h),1} \frac{i}{2} - i \Phi^{(h),2} \frac{i}{2} - i \Phi^{(h),3} \frac{i}{2} + \Phi^{(h),4} \frac{i}{2} \right) \]
\[- q^{h-1} \frac{2}{(2h-3)} \left( \Phi^{(h-1),1} \frac{i}{2} - i \Phi^{(h-1),2} \frac{i}{2} - i \Phi^{(h-1),3} \frac{i}{2} + \Phi^{(h-1),4} \frac{i}{2} \right), \]
\[ Q_{h+\frac{1}{2}}^{12} = q^{h-1} \frac{2 \sqrt{2}}{(2h-1)} \left( i \Phi^{(h),1} \frac{i}{2} + \Phi^{(h),2} \frac{i}{2} \right) - q^{h-1} \frac{2 \sqrt{2}}{(2h-3)} \left( i \Phi^{(h-1),1} \frac{i}{2} + \Phi^{(h-1),2} \frac{i}{2} \right), \]
\[ Q_{h+\frac{1}{2}}^{21} = q^{h-1} \frac{\sqrt{2}}{(2h-1)} \left( i \Phi^{(h),1} \frac{i}{2} + \Phi^{(h),3} \frac{i}{2} \right) - q^{h-1} \frac{\sqrt{2}}{(2h-3)} \left( i \Phi^{(h-1),1} \frac{i}{2} + \Phi^{(h-1),3} \frac{i}{2} \right), \]
\[ Q_{h+\frac{1}{2}}^{22} = -q^{h-1} \frac{1}{(2h-1)} \left( \Phi^{(h),1} \frac{i}{2} - i \Phi^{(h),2} \frac{i}{2} - i \Phi^{(h),3} \frac{i}{2} - \Phi^{(h),4} \frac{i}{2} \right) \]
\[ + q^{h-1} \frac{1}{(2h-3)} \left( \Phi^{(h-1),1} \frac{i}{2} - i \Phi^{(h-1),2} \frac{i}{2} - i \Phi^{(h-1),3} \frac{i}{2} - \Phi^{(h-1),4} \frac{i}{2} \right), \]
\[ \bar{Q}_{h+\frac{1}{2}}^{11} = q^{h-1} \frac{1}{(2h-1)} \left( \Phi^{(h),1} \frac{i}{2} - i \Phi^{(h),2} \frac{i}{2} - i \Phi^{(h),3} \frac{i}{2} - \Phi^{(h),4} \frac{i}{2} \right) \]
\[ + q^{h-1} \frac{1}{(2h-3)} \left( \Phi^{(h-1),1} \frac{i}{2} - i \Phi^{(h-1),2} \frac{i}{2} - i \Phi^{(h-1),3} \frac{i}{2} - \Phi^{(h-1),4} \frac{i}{2} \right), \]
\[ \bar{Q}_{h+\frac{1}{2}}^{12} = q^{h-1} \frac{\sqrt{2}}{(2h-1)} \left( i \Phi^{(h),1} \frac{i}{2} + \Phi^{(h),3} \frac{i}{2} \right) + q^{h-1} \frac{\sqrt{2}}{(2h-3)} \left( i \Phi^{(h-1),1} \frac{i}{2} + \Phi^{(h-1),3} \frac{i}{2} \right), \]
\[ \bar{Q}_{h+\frac{1}{2}}^{21} = q^{h-1} \frac{2 \sqrt{2}}{(2h-1)} \left( i \Phi^{(h),1} \frac{i}{2} + \Phi^{(h),2} \frac{i}{2} \right) + q^{h-1} \frac{2 \sqrt{2}}{(2h-3)} \left( i \Phi^{(h-1),1} \frac{i}{2} + \Phi^{(h-1),2} \frac{i}{2} \right), \]
\[ \bar{Q}_{h+\frac{1}{2}}^{22} = -q^{h-1} \frac{2}{(2h-1)} \left( \Phi^{(h),1} \frac{i}{2} - i \Phi^{(h),2} \frac{i}{2} - i \Phi^{(h),3} \frac{i}{2} + \Phi^{(h),4} \frac{i}{2} \right) \]
\[ - q^{h-1} \frac{2}{(2h-3)} \left( \Phi^{(h-1),1} \frac{i}{2} - i \Phi^{(h-1),2} \frac{i}{2} - i \Phi^{(h-1),3} \frac{i}{2} + \Phi^{(h-1),4} \frac{i}{2} \right). \]

Note that the mixture of $h$-th, $(h-1)$-th and $(h-2)$-th of the $N = 4$ higher spin multiplets occurs.

**K  The second $N = 4$ higher spin generators in terms of oscillators**

We continue to calculate the second $N = 4$ higher spin multiplet in terms of oscillators by using the method in section 7 and we summarize them as follows:

\[ \Phi^{(2)}_{0,-1} = \hat{y}_2 \hat{y}_2 (3k + \nu) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi^{(2)}_{0,0} = \frac{1}{2} \left( \hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_1 \right) (3k + \nu) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]
\[ \Phi^{(2)}_{0,1} = \hat{y}_1 \hat{y}_1 (3k + \nu) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi^{(2),1}_{\frac{1}{2},-\frac{1}{2}} = 3 e^{i \frac{\pi}{2}} \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
\[ \Phi^{(2),1}_{\frac{3}{2},-\frac{1}{2}} = e^{i \frac{\pi}{2}} \left( \hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_1 \right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

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\[
\begin{align*}
\Phi^{(2),1}_{\frac{1}{2} + \frac{1}{4}} &= e^{i\frac{\pi}{4}} (\hat{y}_2 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_2 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_2) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\Phi^{(2),1}_{\frac{1}{2} - \frac{1}{4}} &= 3 e^{i\frac{\pi}{4}} \hat{y}_2 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Phi^{(2),2}_{\frac{1}{2} - \frac{1}{4}} = -3i e^{i\frac{\pi}{4}} \hat{y}_2 \hat{y}_2 \hat{y}_2 k \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\Phi^{(2),2}_{\frac{1}{2} + \frac{1}{4}} &= -i e^{i\frac{\pi}{4}} (\hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_1) k \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\Phi^{(2),2}_{\frac{1}{2} - \frac{1}{4}} &= -i e^{i\frac{\pi}{4}} (\hat{y}_2 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_2 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_2) k \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\Phi^{(2),2}_{\frac{1}{2} + \frac{1}{4}} &= -3i e^{i\frac{\pi}{4}} \hat{y}_2 \hat{y}_2 \hat{y}_2 k \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi^{(2),3}_{\frac{1}{2} - \frac{1}{4}} = -3i e^{i\frac{\pi}{4}} \hat{y}_2 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\Phi^{(2),3}_{\frac{1}{2} + \frac{1}{4}} &= -i e^{i\frac{\pi}{4}} (\hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_1) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\Phi^{(2),3}_{\frac{1}{2} + \frac{1}{4}} &= -i e^{i\frac{\pi}{4}} (\hat{y}_2 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_2 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_2) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\Phi^{(2),3}_{\frac{1}{2} + \frac{1}{4}} &= -3i e^{i\frac{\pi}{4}} \hat{y}_2 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Phi^{(2),4}_{\frac{1}{2} - \frac{1}{4}} = -3 e^{i\frac{\pi}{4}} \hat{y}_2 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\Phi^{(2),4}_{\frac{1}{2} + \frac{1}{4}} &= -e^{i\frac{\pi}{4}} (\hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_1) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\Phi^{(2),4}_{\frac{1}{2} + \frac{1}{4}} &= -3 e^{i\frac{\pi}{4}} \hat{y}_2 \hat{y}_2 \hat{y}_2 k \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Phi^{(2),12}_{1, -2} = -3 \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_2 k \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\Phi^{(2),12}_{1, -2} &= -\frac{3}{4} (\hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_1) k \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\Phi^{(2),12}_{1, 0} &= -\frac{1}{2} (\hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_1 \hat{y}_2 \hat{y}_1 \hat{y}_2 + \hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_1 + \hat{y}_2 \hat{y}_1 \hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_1 \hat{y}_2 \hat{y}_1) k \\
&\quad \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\Phi^{(2),12}_{1, 1} &= -\frac{3}{4} (\hat{y}_2 \hat{y}_1 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_2 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_2) k \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\Phi^{(2),12}_{1, 2} &= -3 \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_1 k \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Phi^{(2),13}_{1, -2} = -3 \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\Phi^{(2),13}_{1, -2} &= -\frac{3}{4} (\hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_1) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\Phi^{(2),13}_{1, 0} &= -\frac{1}{2} (\hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_1 \hat{y}_2 \hat{y}_1 \hat{y}_2 + \hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_1 + \hat{y}_2 \hat{y}_1 \hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_1 \hat{y}_2 \hat{y}_1) \\
&\quad \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\end{align*}
\]
\[
\Phi^{(2),13}_{1,1} = -\frac{3}{4} \left( \hat{y}_2 \hat{y}_1 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_2 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_2 \right) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
\Phi^{(2),13}_{1,1} = -3 \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Phi^{(2),14}_{1,-2} = 3i \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
\[
\Phi^{(2),14}_{1,-1} = \frac{3i}{4} \left( \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_1 \right) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
\[
\Phi^{(2),14}_{1,0} = \frac{i}{2} \left( \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_1 + \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_1 \right) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
\[
\Phi^{(2),14}_{1,1} = \frac{3i}{4} \left( \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_2 \hat{y}_1 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_1 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_1 \hat{y}_1 \right) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
\[
\Phi^{(2),23}_{1,-1} = \frac{3i}{4} \left( \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_1 \right) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
\[
\Phi^{(2),23}_{1,0} = \frac{i}{2} \left( \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_1 + \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_1 \right) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
\[
\Phi^{(2),23}_{1,-1} = \frac{3i}{4} \left( \hat{y}_2 \hat{y}_1 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_2 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_1 \hat{y}_1 \right) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
\[
\Phi^{(2),23}_{1,2} = -3i \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_1 k \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Phi^{(2),24}_{1,-2} = -3i \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_2 k \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
\Phi^{(2),24}_{1,-1} = -\frac{3}{4} \left( \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_1 \right) k \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
\Phi^{(2),24}_{1,0} = -\frac{1}{2} \left( \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_1 + \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_1 \right) k \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
\Phi^{(2),24}_{1,1} = -\frac{3}{4} \left( \hat{y}_2 \hat{y}_1 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_2 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_1 \hat{y}_1 \right) k \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
\Phi^{(2),24}_{1,-2} = -3 \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_1 k \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Phi^{(2),34}_{1,-2} = 3 \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
\[
\Phi^{(2),34}_{1,-1} = \frac{3}{4} \left( \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_1 \right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
\[
\Phi^{(2),34}_{1,0} = \frac{1}{2} \left( \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_2 + \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_1 \right),
\]

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\begin{align*}
\Phi^{(2),3}_{1,-\frac{1}{2}} &= \frac{3}{4} \begin{pmatrix}
\hat{y}_2 \hat{y}_1 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_2 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_1 + \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_2 \\
\hat{y}_2 \hat{y}_1 \hat{y}_1 \hat{y}_1 + \hat{y}_1 \hat{y}_2 \hat{y}_1 \hat{y}_2 + \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 + \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_2
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 \\
0 & -1 \\
0 & 1
\end{pmatrix},
\Phi^{(2),3}_{1,0} &= 3 \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_1 \otimes \begin{pmatrix}
1 & 0 \\
0 & -1 \\
0 & 1
\end{pmatrix},
\Phi^{(2),1}_{\frac{1}{2},-\frac{1}{2}} = -3i e^{i\pi} \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_2 k \otimes \begin{pmatrix}
1 & 0 \\
0 & -1 \\
0 & 1
\end{pmatrix},
\Phi^{(2),1}_{\frac{1}{2},0} = -\frac{3i}{5} e^{i\pi} \hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_2 k \otimes \begin{pmatrix}
1 & 0 \\
0 & -1 \\
0 & 1
\end{pmatrix},
\Phi^{(2),1}_{\frac{1}{2},\frac{1}{2}} = -\frac{3i}{10} e^{i\pi} \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 k \otimes \begin{pmatrix}
1 & 0 \\
0 & -1 \\
0 & 1
\end{pmatrix},
\Phi^{(2),1}_{\frac{1}{2},\frac{3}{2}} = -\frac{3i}{5} e^{i\pi} \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_1 k \otimes \begin{pmatrix}
1 & 0 \\
0 & -1 \\
0 & 1
\end{pmatrix},
\Phi^{(2),2}_{\frac{1}{2},-\frac{3}{2}} = -3 e^{i\pi} \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\Phi^{(2),2}_{\frac{1}{2},-\frac{1}{2}} = -\frac{3}{5} e^{i\pi} \hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\Phi^{(2),2}_{\frac{1}{2},\frac{1}{2}} = -\frac{3}{10} e^{i\pi} \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\Phi^{(2),2}_{\frac{1}{2},\frac{3}{2}} = -\frac{3}{5} e^{i\pi} \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_1 \otimes \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\Phi^{(2),3}_{\frac{1}{2},-\frac{1}{2}} = -3 e^{i\pi} \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix}
0 & 1 \\
1 & 0 \\
-1 & 0
\end{pmatrix},
\Phi^{(2),3}_{\frac{1}{2},0} = -\frac{3}{10} e^{i\pi} \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix}
0 & 1 \\
1 & 0 \\
-1 & 0
\end{pmatrix},
\Phi^{(2),3}_{\frac{1}{2},\frac{1}{2}} = -\frac{3}{5} e^{i\pi} \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_1 \otimes \begin{pmatrix}
0 & 1 \\
1 & 0 \\
-1 & 0
\end{pmatrix},
\Phi^{(2),3}_{\frac{1}{2},\frac{3}{2}} = -\frac{3}{5} e^{i\pi} \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_1 \otimes \begin{pmatrix}
0 & 1 \\
1 & 0 \\
-1 & 0
\end{pmatrix},
\Phi^{(2),4}_{\frac{1}{2},-\frac{1}{2}} = -3 e^{i\pi} \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_1 \otimes \begin{pmatrix}
0 & 1 \\
1 & 0 \\
-1 & 0
\end{pmatrix},
\Phi^{(2),4}_{\frac{1}{2},0} = \frac{3i}{5} e^{i\pi} \hat{y}_1 \hat{y}_2 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix}
0 & 1 \\
1 & 0 \\
-1 & 0
\end{pmatrix},
\Phi^{(2),4}_{\frac{1}{2},\frac{1}{2}} = \frac{3i}{10} e^{i\pi} \hat{y}_1 \hat{y}_1 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix}
0 & 1 \\
1 & 0 \\
-1 & 0
\end{pmatrix},
\Phi^{(2),4}_{\frac{1}{2},\frac{3}{2}} = \frac{3i}{5} e^{i\pi} \hat{y}_2 \hat{y}_2 \hat{y}_1 \hat{y}_1 \otimes \begin{pmatrix}
0 & 1 \\
1 & 0 \\
-1 & 0
\end{pmatrix},
\Phi^{(2),4}_{\frac{1}{2},-\frac{3}{2}} = 3i e^{i\pi} \hat{y}_1 \hat{y}_1 \hat{y}_1 \hat{y}_1 \otimes \begin{pmatrix}
0 & 1 \\
1 & 0 \\
-1 & 0
\end{pmatrix},
\Phi^{(2),-3} = -3 \hat{y}_2 \hat{y}_2 \hat{y}_2 \hat{y}_2 \otimes \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\end{align*}
In this Appendix, we select the relevant (anti)commutators appearing in Appendix algebra. The symmetrized product for the oscillators in these expressions (K.1) is used. For the third and more $\mathcal{N} = 4$ higher spin generators, we can use (7.19).

L The $\mathcal{N} = 2$ wedge subalgebra of $\mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda]$ algebra

After we consider the subalgebras of $\mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda]$, we will compare it with $\mathcal{N} = 2$ higher spin algebra.

L.1 The $\mathcal{N} = 2$ wedge subalgebra of $\mathcal{W}_{\infty}^{\mathcal{N}=4}[\lambda]$ algebra

In this Appendix, we select the relevant (anti)commutators appearing in Appendix G with $\mathcal{N} = 2$ supersymmetry. The $\mathcal{N} = 2$ superconformal algebra, by recalling the description of sections 5 and 7, is described by

\[
\begin{align*}
[L_m, L_n] &= (m-n) L_{m+n} - mn(m^2-1) \frac{N}{2} (\lambda - 1) \delta_{m+n}, \\
[L_m, \Phi_{0,n}^{(1)}] &= -n \Phi_{0,m+n}^{(1)}, \\
[L_m, G_{2r}^0] &= \frac{m}{2} - r) G_{2m+r}, \\
[L_m, G_{2r}^2] &= (m - r) G_{2m+r}, \\
[\Phi_{0,m}^{(1)}, G_{2r}^0] &= G_{2m+r}^0, \\
[\Phi_{0,m}^{(1)}, G_{2r}^2] &= G_{2m+r}^2, \\
{\{ \Phi_{2r}, G_{2s} \}} &= -2L_{r+s} - (4m^2-1) \frac{N}{2} (\lambda - 1) \delta_{r+s}, \\
{\{ G_{2r}, G_{2s} \}} &= 2L_{r+s} + (4m^2-1) \frac{N}{2} (\lambda - 1) \delta_{r+s}, \\
{\{ G_{2r}, G_{2s} \}} &= (r - s) \Phi_{0,r+s}^{(1)}. \tag{L.1}
\end{align*}
\]

The primary condition for the higher spin currents under the stress energy tensor can be summarized by

\[
\begin{align*}
[L_m, \Phi_{0,n}^{(2)}] &= (m-n) \Phi_{0,m+n}^{(2)}, \\
[L_m, \Phi_{2r}^{(2)}] &= (\frac{3m}{2} - r) \Phi_{2,m+r}^{(2)}, \\
[L_m, \tilde{\Phi}_{2r}^{(2)}] &= (\frac{3m}{2} - r) \tilde{\Phi}_{2,m+r}^{(2)}, \\
[L_m, \tilde{\Phi}_{0,n}^{(2)}] &= (2m-n) \tilde{\Phi}_{0,m+n}^{(2)}. \tag{L.2}
\end{align*}
\]
The remaining $\mathcal{N} = 2$ primary condition can be described as

\[
\begin{align*}
\left[ \Phi_{0,m}, \Phi_{0,n} \left( \frac{3}{2}, r \right) \right] &= 0, \\
\left[ \Phi_{0,m}, \Phi_{0,n} \left( \frac{3}{2}, s \right) \right] &= -2 \Phi_{0,m} \Phi_{0,n} \left( \frac{3}{2}, m+r \right), \\
\left[ \Phi_{0,m}, \Phi_{1,n} \left( \frac{5}{2}, r \right) \right] &= -\frac{1}{2} \Phi_{0,m} \Phi_{1,n} \left( \frac{5}{2}, m+r \right) + \frac{8}{3} \Phi_{0,m} \Phi_{1,n} \left( \frac{5}{2}, m+r \right), \\
\left[ \Phi_{0,m}, \Phi_{2,n} \left( \frac{7}{2}, r \right) \right] &= 2 \Phi_{0,m} \Phi_{2,n} \left( \frac{7}{2}, m+n+1 \right) + 8 \Phi_{0,m} \Phi_{2,n} \left( \frac{7}{2}, m+n+1 \right), \\
\left[ \Phi_{1,n}, \Phi_{2,n} \left( \frac{5}{2}, r \right) \right] &= -2 \Phi_{1,n} \Phi_{2,n} \left( \frac{5}{2}, m+r \right), \\
\left[ \Phi_{1,n}, \Phi_{2,n} \left( \frac{3}{2}, r \right) \right] &= -\frac{1}{2} \Phi_{1,n} \Phi_{2,n} \left( \frac{3}{2}, r \right), \\
\left[ \Phi_{1,n}, \Phi_{2,n} \left( \frac{3}{2}, s \right) \right] &= -\left( 3r - s \right) \Phi_{1,n} \Phi_{2,n} \left( \frac{3}{2}, r \right), \\
\left[ \Phi_{2,n}, \Phi_{2,n} \left( \frac{7}{2}, r \right) \right] &= -\Phi_{2,n} \Phi_{2,n} \left( \frac{7}{2}, r+m \right) - \Phi_{2,n} \Phi_{2,n} \left( \frac{7}{2}, r+m \right), \\
\left[ \Phi_{2,n}, \Phi_{2,n} \left( \frac{5}{2}, r \right) \right] &= -\Phi_{2,n} \Phi_{2,n} \left( \frac{5}{2}, r+m \right) + \frac{8}{3} \Phi_{2,n} \Phi_{2,n} \left( \frac{5}{2}, r+m \right)
\end{align*}
\]

(L.3)

The whole $\mathcal{N} = 2$ primary conditions are given by (L.2) and (L.3).

The (anti)commutators between the $\mathcal{N} = 2$ higher spin currents with the descriptions of sections 5 and 7 are summarized by

\[
\begin{align*}
\left[ \Phi_{0,m,n} \left( \frac{3}{2}, r \right), \Phi_{0,n} \left( \frac{3}{2}, s \right) \right] &= \frac{8}{3} \left( m-n \right) \left( 2\lambda - 1 \right) \Phi_{0,m,n} \left( \frac{3}{2}, m+r \right) - \frac{64}{9} \left( m-n \right) \left( \lambda + 1 \right) \left( \lambda - 2 \right) \delta_{m+n}, \\
\left[ \Phi_{0,m,n} \left( \frac{5}{2}, r \right), \Phi_{0,n} \left( \frac{5}{2}, s \right) \right] &= -\frac{1}{2} \Phi_{0,m,n} \Phi_{0,n} \left( \frac{5}{2}, m+r \right) + \frac{8}{15} \left( 3m-2r \right) \left( 2\lambda - 1 \right) \Phi_{0,m,n} \left( \frac{5}{2}, m+r \right), \\
\left[ \Phi_{0,m,n} \left( \frac{7}{2}, r \right), \Phi_{0,n} \left( \frac{7}{2}, s \right) \right] &= \frac{2}{6} \left( 12m^2-8mr+4r^2-9 \right) \left( \lambda + 1 \right) \left( \lambda - 2 \right) \Phi_{0,m,n} \left( \frac{7}{2}, m+r \right), \\
\left[ \Phi_{0,m,n} \left( \frac{7}{2}, r \right), \Phi_{2,n} \left( \frac{5}{2}, s \right) \right] &= \frac{2}{3} \left( 2m-n \right) \Phi_{0,m,n} \left( \frac{3}{2}, m+n+1 \right) + \frac{8}{15} \left( 2m-n \right) \left( 2\lambda - 1 \right) \Phi_{0,m,n} \left( \frac{3}{2}, m+n+1 \right), \\
\left[ \Phi_{1,n} \left( \frac{5}{2}, r \right), \Phi_{2,n} \left( \frac{3}{2}, s \right) \right] &= -\frac{1}{2} \left( 3r-s \right) \Phi_{1,n} \Phi_{2,n} \left( \frac{3}{2}, r \right), \\
\left[ \Phi_{1,n} \left( \frac{7}{2}, r \right), \Phi_{2,n} \left( \frac{5}{2}, s \right) \right] &= -\frac{8}{6} \left( 2\lambda - 1 \right) \Phi_{1,n} \Phi_{2,n} \left( \frac{5}{2}, r+m \right), \\
\left[ \Phi_{2,n} \left( \frac{7}{2}, r \right), \Phi_{2,n} \left( \frac{5}{2}, s \right) \right] &= \frac{1}{6} \left( 4r-3m \right) \Phi_{2,n} \Phi_{2,n} \left( \frac{3}{2}, s \right) + \frac{4}{15} \left( 4r^2-4rm+2m^2-5 \right) \left( 2\lambda - 1 \right) \Phi_{2,n} \Phi_{2,n} \left( \frac{3}{2}, s \right), \\
\left[ \Phi_{2,n} \left( \frac{5}{2}, r \right), \Phi_{2,n} \left( \frac{7}{2}, s \right) \right] &= \frac{2}{9} \left( 16r^3-12r^2+8rm^2-36r+19m-4m^3 \right) \left( \lambda + 1 \right) \left( \lambda - 2 \right) \Phi_{2,n} \Phi_{2,n} \left( \frac{5}{2}, r+m \right),
\end{align*}
\]

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Let us return to the work of [30]. We can collect the relevant (anti)commutators. By linear

\[(5.3)\]

we can check the corresponding ones written in terms of the notations in [30]

\[N\]

where we do not see the contributions from the outside of the wedge.

\[
\left\{ \tilde{\Phi}^{(1),2}_{\frac{3}{2},r}, \tilde{\Phi}^{(1),2}_{\frac{3}{2},s} \right\} = \frac{1}{2} \tilde{\Phi}^{(2)}_{2,r+s} + (6r^2 - 8rs + 6s^2 - 9) \left( \frac{1}{15} (2\lambda - 1) \Phi^{(2)}_{0,r+s} - \frac{4}{9} (\lambda + 1)(\lambda - 2) L_{r+s} \right) + (m^2 - \frac{1}{4}) (m^2 - \frac{9}{4} N) \frac{8}{9} (\lambda + 1)(\lambda - 2) \delta_{r+s},
\]

\[\left[ \tilde{\Phi}^{(1),2}_{\frac{3}{2},r}, \tilde{\Phi}^{(2),2}_{\frac{3}{2},m} \right] = \frac{1}{2} \left( 4r - 3m \right) \frac{16}{15} \left( 2\lambda - 1 \right) \Phi^{(2),2}_{0,m+n} - \frac{16}{9} (\lambda + 1)(\lambda - 2) L_{m+n} \right) + \frac{1}{9} (16r^3 - 12r^2m + 8rm^2 - 36r + 19m - 9m^2 + \lambda (\lambda + 1)(\lambda - 2) G_{r+m}^2.
\]

\[\left[ \tilde{\Phi}^{(1),2}_{2,m}, \tilde{\Phi}^{(1),2}_{2,n} \right] = 2 (m - n) \tilde{\Phi}^{(2)}_{2,m+n} \]

+ \left( m - n \right) \left( 2m^2 - mn + 2n^2 - 8 \right) - \frac{2}{15} (2\lambda - 1) \Phi^{(2)}_{0,m+n} - \frac{16}{9} (\lambda + 1)(\lambda - 2) L_{m+n} \right)

+ \frac{m(m^2 - 1)(m^2 - 4)}{9} \frac{8}{9} (\lambda + 1)(\lambda - 1)(\lambda - 2) \delta_{m+n}.

\[(L.4)\]

There are also two relations

\[\left[ \Phi^{(2)}_{0,m}, \Phi^{(2),2}_{\frac{3}{2},r} \right] = -2 \Phi^{(2),2}_{\frac{3}{2},m+r} + \frac{8}{15} (3m - 2r) (2\lambda - 1) \Phi^{(2),2}_{\frac{3}{2},m+r} - \frac{9}{4} (12m^2 - 8mr + 4r^2 - 9)(\lambda + 1)(\lambda - 2) G_{r+m}^2,
\]

\[\left\{ \Phi^{(2),2}_{\frac{3}{2},r}, \Phi^{(2),2}_{\frac{3}{2},\rho} \right\} = -2 \Phi^{(2),2}_{2,r+\rho} + (6r^2 - 8r\rho + 6\rho^2 - 9) \left( \frac{16}{9} (\lambda + 1)(\lambda - 2) L_{r+\rho} + \frac{4}{15} (2\lambda - 1) \Phi^{(2)}_{0,r+\rho} \right),
\]

\[(L.5)\]

where we do not see the contributions from the outside of the wedge.

**L.2 The N = 2 wedge subalgebra of WN∞=2[λ] algebra and the N = 2 higher spin algebra shs[λ]**

Let us return to the work of [30]. We can collect the relevant (anti)commutators. By linear

\[(L.1)\]

combinations of the above (anti)commutators in \[(L.1)-(L.5)\] with the help of \[(5.1), (5.2),\]

and \[(5.3)\], we can check the corresponding ones written in terms of the notations in [30]

\[[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0},
\]

\[[L_m, J_n] = -n J_{m+n} , \quad [L_m, G^\pm_{r}] = \left( \frac{m}{2} - r \right) G^\pm_{m+r} ,
\]

\[[J_m, J_n] = m \frac{c}{3} \delta_{m+n,0} , \quad [J_m, G^\pm_{r}] = \pm G^\pm_{m+r} ,
\]

\[\{G^+_r, G^-_s\} = 2L_{r+s} + (r - s) J_{r+s} + \frac{c}{3} \left( r^2 - \frac{1}{4} \right) \delta_{r+s,0} , \quad \{G^+_r, G^+_s\} = 0 ,
\]

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\[
\begin{align*}
[L_m, W_{n}^{s0}] & = [(s - 1)m - n] W_{m+n}^{s0}, \\
L_{m}, W_{n}^{s1} & = [sm - n] W_{m+n}^{s1}, \\
J_{m}, W_{n}^{s0} & = 0, \\
J_{m}, W_{n}^{s1} & = \pm W_{m+n}^{s1}, \\
G_{r}^{\pm}, W_{t}^{s \pm} & = \pm[(2s - 1)r - t] W_{r+t}^{s0} + 2 W_{r+t}^{s1}, \\
G_{r}^{\pm}, W_{t}^{s \pm} & = 0, \\
[\mathcal{G}_{r}^{\pm}, W_{n}^{s}] & = \frac{1}{2}[(2r + 1)s - 2 - m] W_{r+m}^{s \pm}, \\
W_{m}^{20}, W_{n}^{20} & = (m - n) \mathcal{A}^{[2]}_{m+n} + \frac{c}{12} m (m^2 - 1) \delta_{m+n,0}, \\
W_{m}^{21}, W_{n}^{21} & = \frac{c}{48} m (m^2 - 1) (m^2 - 4) \delta_{m+n,0} + (m - n) (2m^2 - mn + 2n^2 - 8) \mathcal{B}^{[2]}_{m+n} \\
+ (m - n) \mathcal{B}^{[3]}_{m+n} - 3 c^3_{W_{m+n}^{30}} \\
W_{m}^{20}, W_{n}^{21} & = \mathcal{C}^{[4]}_{m+n} + (2m - n) \left( \mathcal{C}^{[3]}_{m+n} - c^3_{W_{m+n}^{30}} \right) \\
W_{m}^{20}, W_{n}^{21} & = 6 m^2 - 3mn + n^2 - 4 \mathcal{C}^{[2]}_{m+n} + m (m^2 - 1) \mathcal{C}^{[1]}_{m+n}, \\
\{W_{r}^{2+}, W_{s}^{2-}\} & = \left( \mathcal{D}^{[2]}_{r+s} - 2 c^3_{W_{r+s}^{31}} \right) + (r - s) \left( \mathcal{D}^{[3]}_{r+s} - 3 c^3_{W_{r+s}^{30}} \right) \\
& + \left( 3 r^2 - 4 r s + 3 s^2 - \frac{9}{2} \right) \mathcal{D}^{[2]}_{r+s} + (r - s) \left( r^2 + s^2 - \frac{5}{2} \right) \mathcal{D}^{[1]}_{r+s} \\
& + \frac{c}{12} \left( r^2 - \frac{1}{4} \right) \left( r^2 - \frac{9}{4} \right) \delta_{r+s,0}, \\
\{W_{r}^{2+}, W_{s}^{2+}\} & = \mathcal{E}^{[4]}_{r+s}, \\
W_{m}^{20}, W_{r}^{2+} & = \left( \mathcal{F}^{[7/2]}_{r+m} - c^3_{W_{m+r}^{3+}} \right) + \left( \frac{3}{2} m - r \right) \Phi^{[5/2]}_{m+r} \\
& + \left( 3 m^2 - 2 mr + r^2 - \frac{9}{4} \right) \Phi^{[3/2]}_{m+r}, \\
W_{r}^{2+}, W_{m}^{21} & = \Psi^{[9/2]}_{r+m} + \left( 2 r - \frac{3}{2} m \right) \left( \mathcal{F}^{[7/2]}_{r+m} - c^3_{W_{r+m}^{3+}} \right) \\
& + \left( 2 r^2 - 2 rm + m^2 - \frac{5}{2} \right) \Psi^{[5/2]}_{r+m} \\
& + \left( 4 r^3 - 3 r^2 m + 2 rm^2 - m^3 - 9 r + \frac{19}{4} m \right) \Psi^{[3/2]}_{r+m}. \\
\end{align*}
\]  

The first four lines correspond to \(\text{(L.1)}\), the next five lines correspond to \(\text{(L.2)}\) and \(\text{(L.3)}\), and the last lines correspond to \(\text{(L.4)}\) and \(\text{(L.5)}\). Note that the quantities \(\mathcal{A}[s], \mathcal{B}[s], \mathcal{C}[s], \mathcal{D}[s], \mathcal{E}[s], \Phi[s]\) and \(\Psi[s]\) in \(\text{(L.6)}\) are found in \([53]\). Under the large \((N,k)\) 't Hooft-like limit, we have the following nonzero contributions with explicit \(\lambda\)-dependent terms

\[
\begin{align*}
\mathcal{A}^{[2]} & \rightarrow - \frac{1}{3} (\lambda + 1)(\lambda - 2) L - \frac{1}{\sqrt{3}} (2\lambda - 1) W^{20}, \\
\mathcal{B}^{[2]} & \rightarrow - \frac{1}{12} (\lambda + 1)(\lambda - 2) L - \frac{1}{20\sqrt{3}} (2\lambda - 1) W^{20}, \\
\mathcal{C}^{[3]} & \rightarrow - \frac{1}{5\sqrt{3}} (2\lambda - 1) W^{21}, \\
\mathcal{D}^{[3]} & \rightarrow \frac{1}{10\sqrt{3}} (2\lambda - 1) W^{21}, \\
\end{align*}
\]  

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\[\mathcal{O}^{[2]} \to -\frac{1}{6} (\lambda + 1)(\lambda - 2) L - \frac{1}{5\sqrt{3}} (2\lambda - 1) W^{2,0},\]
\[\mathcal{O}^{[1]} \to -\frac{1}{12} (\lambda + 1)(\lambda - 2) J, \quad \Phi^{[5/2]} \to -\frac{2}{5\sqrt{3}} (2\lambda - 1) W^{2,+},\]
\[\Phi^{[3/2]} \to -\frac{1}{12} (\lambda + 1)(\lambda - 2) G^+, \quad \Psi^{[5/2]} \to -\frac{1}{10\sqrt{3}} (2\lambda - 1) W^{2,+},\]
\[\Psi^{[3/2]} \to -\frac{1}{24} (\lambda + 1)(\lambda - 2) G^+.\] (L.7)

Therefore, by substituting \ref{eq:7.7} into \ref{eq:7.6}, our \(\mathcal{N} = 2\) wedge subalgebra of \(W^{4}\) algebra reproduces the \(\mathcal{N} = 2\) higher spin algebra \cite{27}, which is equal to \(\mathcal{N} = 2\) wedge subalgebra of \(W^{3}\) algebra from the observation of \cite{30}.

In \ref{eq:7.20}, we explicitly check that the structure constants in the right hand side can be written in terms of the ones in \cite{27}. There are also other (anti)commutators in addition to \ref{eq:7.20} as follows:

\[\left[ W^{2,0}_m, W^{2,+}_r \right] = -\frac{1}{6} \sqrt{(1-m)! (1+m)!} \left( \frac{3}{2} - r \right)! \left( \frac{3}{2} + r \right)! \times \left( \sum_{m'' = -\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{1}{2(\frac{1}{2} - m - r)! (\frac{3}{2} + m + r)!}} \left( \lambda + 1 \right) f_{T \Psi \Psi}^{\frac{1}{2} \frac{1}{2} \frac{1}{2}} + \left( \lambda - 2 \right) f_{U \Psi \Psi}^{\frac{1}{2} \frac{1}{2} \frac{1}{2}} \right) C_{m m''}^{\frac{1}{2} \frac{1}{2} \frac{1}{2}} G^{+}_{m''} \right),\]
\[\left[ W^{2,0}_m, W^{2,-}_r \right] = -\frac{1}{6} \sqrt{(1-m)! (1+m)!} \left( \frac{3}{2} - r \right)! \left( \frac{3}{2} + r \right)! \times \left( \sum_{m'' = -\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{1}{2(\frac{1}{2} - m - r)! (\frac{3}{2} + m + r)!}} \left( \lambda + 1 \right) f_{T \Psi \Psi}^{\frac{1}{2} \frac{1}{2} \frac{1}{2}} + \left( \lambda - 2 \right) f_{U \Psi \Psi}^{\frac{1}{2} \frac{1}{2} \frac{1}{2}} \right) C_{m m''}^{\frac{1}{2} \frac{1}{2} \frac{1}{2}} G^{-}_{m''} \right),\]
\[\left[ W^{2,0}_m, W^{3,1}_n \right] = \frac{1}{3\sqrt{30}} \sqrt{(1-m)! (1-m)! (2-n)! (2+n)!} \left( \frac{1}{2-m-n} (2-m+n)! \right) \times \left( \sum_{m'' = -\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{1}{2(\frac{1}{2} - m - r)! (\frac{3}{2} + m + r)!}} \left( \lambda + 1 \right) f_{T \Psi \Psi}^{\frac{1}{2} \frac{1}{2} \frac{1}{2}} + \left( \lambda - 2 \right) f_{U \Psi \Psi}^{\frac{1}{2} \frac{1}{2} \frac{1}{2}} \right) C_{m m''}^{\frac{1}{2} \frac{1}{2} \frac{1}{2}} W^{3,-}_{m''} \right),\]
\[
\times \left( 5\sqrt{3} c_{22,3} \left( (\lambda + 1) f_{TTT}^{122} - (\lambda - 2) f_{UUU}^{122} \right) C_{m_{m+n}}^{122} W_{m+n}^{30} \right. \\
- \left( 3(\lambda + 1)(\lambda - 3) f_{TTT}^{122} - 3(\lambda + 2)(\lambda - 2) f_{UUU}^{122} \right) C_{m_{m+n}}^{122} W_{m+n}^{21} \right) ,
\]

\[ [W_{r}^{2+}, W_{r}^{2+}] = 0 , \]

\[ [W_{r}^{2+}, W_{\rho}^{2+}] = \frac{1}{3} \sqrt{\frac{3}{2} - r} (\frac{3}{2} + r) \frac{1}{(\frac{3}{2} - \rho) (\frac{3}{2} + \rho)} ! \]

\[ \times \left( \frac{1}{(2\lambda - 1) \sqrt{(-r - \rho)! (r + \rho)!}} \right) \left( - \lambda f_{\bar{\Psi}T}^{\frac{3}{2} \frac{3}{2} 0} + (\lambda - 1) f_{\bar{\Psi}U}^{\frac{3}{2} \frac{3}{2} 0} \right) C_{r \rho m^\prime 0}^{\frac{3}{2} \frac{3}{2} 0} J_0 \]

\[ + \frac{1}{m_{m^\prime} = -1} \frac{2}{3(1 - r)!(1 + r)!} (\lambda - 2) f_{\bar{\Psi}T}^{\frac{3}{2} \frac{3}{2} 1} - (\lambda - 1) f_{\bar{\Psi}U}^{\frac{3}{2} \frac{3}{2} 1} C_{r \rho m^\prime 0}^{\frac{3}{2} \frac{3}{2} 1} L_{m^\prime} \]

\[ + \frac{1}{m_{m^\prime} = -1} \frac{2}{\sqrt{3(1 - r)! (1 + r)!}} (f_{\bar{\Psi}T}^{\frac{3}{2} \frac{3}{2} 1} - f_{\bar{\Psi}U}^{\frac{3}{2} \frac{3}{2} 1}) C_{r \rho m^\prime 0}^{\frac{3}{2} \frac{3}{2} 1} W_{m^\prime}^{20} \]

\[ + \frac{2}{m_{m^\prime} = -2} \frac{4 c_{22,3}}{\sqrt{(2 - r)! (2 + r)!}} (f_{\bar{\Psi}T}^{\frac{3}{2} \frac{3}{2} 2} - f_{\bar{\Psi}U}^{\frac{3}{2} \frac{3}{2} 2}) C_{r \rho m^\prime 0}^{\frac{3}{2} \frac{3}{2} 2} W_{m^\prime}^{30} \]

\[ - \frac{2}{m_{m^\prime} = -2} \frac{4\sqrt{3}}{5(2 - r)! (2 + r)!} (\lambda - 3) f_{\bar{\Psi}T}^{\frac{3}{2} \frac{3}{2} 2} - (\lambda + 2) f_{\bar{\Psi}U}^{\frac{3}{2} \frac{3}{2} 2} C_{r \rho m^\prime 0}^{\frac{3}{2} \frac{3}{2} 2} W_{m^\prime}^{21} \]

\[ + \frac{3}{m_{m^\prime} = -3} \frac{4 c_{22,3} c_{23,4}}{\sqrt{3(3 - r)! (3 + r)!}} (f_{\bar{\Psi}T}^{\frac{3}{2} \frac{3}{2} 3} - f_{\bar{\Psi}U}^{\frac{3}{2} \frac{3}{2} 3}) C_{r \rho m^\prime 0}^{\frac{3}{2} \frac{3}{2} 3} W_{m^\prime}^{40} \]

\[ - \frac{3}{m_{m^\prime} = -3} \frac{40 c_{22,3}}{7(3 - r)! (3 + r)!} (\lambda - 4) f_{\bar{\Psi}T}^{\frac{3}{2} \frac{3}{2} 3} - (\lambda + 3) f_{\bar{\Psi}U}^{\frac{3}{2} \frac{3}{2} 3} C_{r \rho m^\prime 0}^{\frac{3}{2} \frac{3}{2} 3} W_{m^\prime}^{31} \]

\[ [W_{r}^{2+}, W_{m}^{21}] = \frac{1}{12} \sqrt{(\frac{3}{2} - r)! (\frac{3}{2} + r)! (2 - m)! (2 + m)!} \]

\[ \times \left( \frac{1}{m_{m^\prime} = -1 - \frac{1}{2}} \frac{1}{\sqrt{2(\frac{1}{2} - r)! (\frac{1}{2} + r)!}} \left( f_{\bar{T}T}^{\frac{3}{2} \frac{3}{2} \frac{1}{2}} + f_{\bar{U}U}^{\frac{3}{2} \frac{3}{2} \frac{1}{2}} \right) C_{r \rho m^\prime 0}^{\frac{3}{2} \frac{3}{2} \frac{1}{2}} G_{m^\prime}^{+} \right. \\
- \frac{\frac{3}{2}}{m_{m^\prime} = -1 - \frac{1}{2}} \frac{\sqrt{3}}{\sqrt{(\frac{3}{2} - r)! (\frac{3}{2} + r)!}} \left( f_{\bar{T}T}^{\frac{3}{2} \frac{3}{2} \frac{3}{2}} + f_{\bar{U}U}^{\frac{3}{2} \frac{3}{2} \frac{3}{2}} \right) C_{r \rho m^\prime 0}^{\frac{3}{2} \frac{3}{2} \frac{3}{2}} W_{m^\prime}^{2+} \]

\[ - \frac{\frac{3}{2}}{m_{m^\prime} = -1 - \frac{1}{2}} \frac{10 \sqrt{2} c_{22,3}}{\sqrt{3(\frac{3}{2} - r)! (\frac{3}{2} + r)!}} \left( f_{\bar{T}T}^{\frac{3}{2} \frac{3}{2} \frac{5}{2}} + f_{\bar{U}U}^{\frac{3}{2} \frac{3}{2} \frac{5}{2}} \right) C_{r \rho m^\prime 0}^{\frac{3}{2} \frac{3}{2} \frac{5}{2}} W_{m^\prime}^{2+} \right) ,
\]

\[ [W_{r}^{2-}, W_{\rho}^{2-}] = 0 , \]

\[ [W_{r}^{2-}, W_{m}^{21}] = \frac{1}{12} \sqrt{(\frac{3}{2} - r)! (\frac{3}{2} + r)! (2 - m)! (2 + m)!} \]
\[
\begin{align*}
&\times \left( \sum_{m'' = -\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\sqrt{2(\frac{3}{2} - r - m)! (\frac{3}{2} + r + m)!}} \left( f_{\mathcal{T}^3\Phi^3}^2 + f_{\mathcal{U}^3\Phi^3}^2 \right) C_{r m m'}^2 G_{m''}^- \right) \\
&\quad - \sum_{m'' = -\frac{1}{2}}^{\frac{3}{2}} \frac{\sqrt{3}}{\sqrt{(\frac{3}{2} - r - m)! (\frac{3}{2} + r + m)!}} \left( f_{\mathcal{T}^3\Phi^3}^2 + f_{\mathcal{U}^3\Phi^3}^2 \right) C_{r m m'}^2 W_{m''}^- \right) \\
&\quad - \sum_{m'' = -\frac{1}{2}}^{\frac{3}{2}} \frac{10 \sqrt{2} c_{22,3}}{\sqrt{3(\frac{3}{2} - r - m)! (\frac{3}{2} + r + m)!}} \left( f_{\mathcal{T}^3\Phi^3}^2 + f_{\mathcal{U}^3\Phi^3}^2 \right) C_{r m m'}^2 W_{m''}^- \right), \\
&\left[ W_{m}^{21}, W_{n}^{21} \right] = \frac{1}{48} \sqrt{(2 - m)! (2 + m)! (2 - n)! (2 + n)} \\
&\times \left( \sum_{m'' = -1}^{2} \frac{2}{3(1 - m - n)! (1 + m + n)!} \left( (\lambda - 2) f_{\mathcal{T}^3\mathcal{U}^3\mathcal{T}^3}^{221} - (\lambda + 1) f_{\mathcal{U}^3\mathcal{U}^3\mathcal{U}^3}^{221} \right) C_{r m m'}^{221} L_{m''} \right) \\
&\quad + \sum_{m'' = -1}^{2} \frac{2}{3(1 - m - n)! (1 + m + n)!} \left( f_{\mathcal{T}^3\mathcal{T}^3\mathcal{T}^3}^{221} - f_{\mathcal{U}^3\mathcal{U}^3\mathcal{U}^3}^{221} \right) C_{r m m'}^{221} W_{m''}^{200} \right) \\
&\quad - \sum_{m'' = -3}^{3} \frac{40 c_{22,3}}{7(3 - m - n)! (3 + m + n)!} \left( (\lambda - 4) f_{\mathcal{T}^3\mathcal{T}^3\mathcal{T}^3}^{223} - (\lambda + 3) f_{\mathcal{U}^3\mathcal{U}^3\mathcal{U}^3}^{223} \right) C_{r m m'}^{223} W_{m''}^{31} \\
&\quad + \sum_{m'' = -3}^{3} \frac{4 c_{22,3} c_{23,4}}{\sqrt{3(3 - m - n)! (3 + m + n)!}} \left( f_{\mathcal{T}^3\mathcal{T}^3\mathcal{T}^3}^{223} - f_{\mathcal{U}^3\mathcal{U}^3\mathcal{U}^3}^{223} \right) C_{r m m'}^{223} W_{m''}^{10} \right), \quad (L.8)
\end{align*}
\]

where the \( \lambda \)-dependent structure constants in [27] are given by

\[
\begin{align*}
f_{\mathcal{T}^3\mathcal{T}^3\mathcal{T}^3}^{111} &= -2\sqrt{2}, \quad f_{\mathcal{U}^3\mathcal{U}^3\mathcal{U}^3}^{111} = -2\sqrt{2}, \\
f_{\mathcal{T}^3\Phi^3\Phi^3}^{1\frac{1}{2}\frac{1}{2}} &= \frac{1}{\sqrt{3}} (\lambda + 1)(\lambda - 2), \quad f_{\mathcal{T}^3\Phi^3\Phi^3}^{1\frac{1}{2}\frac{1}{2}} = \frac{1}{\sqrt{15}} (\lambda - 8), \\
f_{\mathcal{T}^3\Phi^3\Phi^3}^{1\frac{1}{2}\frac{1}{2}} &= -\frac{3}{\sqrt{5}}, \quad f_{\mathcal{U}^3\Phi^3\Phi^3}^{1\frac{1}{2}\frac{1}{2}} = -\frac{1}{\sqrt{3}} (\lambda + 1)(\lambda - 2), \\
f_{\mathcal{T}^3\Phi^3\Phi^3}^{1\frac{1}{2}\frac{1}{2}} &= -\frac{1}{\sqrt{15}} (\lambda + 7), \quad f_{\mathcal{U}^3\Phi^3\Phi^3}^{1\frac{1}{2}\frac{1}{2}} = \frac{3}{\sqrt{5}}, \\
f_{\mathcal{T}^3\Phi^3\Phi^3}^{1\frac{1}{2}\frac{1}{2}} &= -\frac{1}{\sqrt{3}} (\lambda + 1)(\lambda - 2), \quad f_{\mathcal{T}^3\Phi^3\Phi^3}^{1\frac{1}{2}\frac{1}{2}} = \frac{3}{\sqrt{5}}, \\
f_{\mathcal{T}^3\Phi^3\Phi^3}^{1\frac{1}{2}\frac{1}{2}} &= -\frac{1}{\sqrt{15}} (\lambda + 7), \quad f_{\mathcal{U}^3\Phi^3\Phi^3}^{1\frac{1}{2}\frac{1}{2}} = -\frac{3}{\sqrt{5}}, \\
f_{\mathcal{T}^3\mathcal{U}^3\mathcal{U}^3}^{1\frac{1}{2}\frac{1}{2}} &= -2\sqrt{2}, \quad f_{\mathcal{U}^3\mathcal{U}^3\mathcal{U}^3}^{1\frac{1}{2}\frac{1}{2}} = -2\sqrt{2}, \\
f_{\mathcal{T}^3\Phi^3\Phi^3}^{1\frac{1}{2}0} &= \frac{1}{2} \lambda (\lambda + 1)(\lambda - 2), \quad f_{\mathcal{T}^3\Phi^3\Phi^3}^{1\frac{1}{2}1} = \frac{1}{2\sqrt{5}} (\lambda + 1)(\lambda - 8), \\
f_{\mathcal{U}^3\Phi^3\Phi^3}^{1\frac{1}{2}0} &= \frac{1}{2} (\lambda + 4), \quad f_{\mathcal{U}^3\Phi^3\Phi^3}^{1\frac{1}{2}3} = -\frac{3}{2\sqrt{5}}.
\end{align*}
\]
\[
\begin{align*}
\mathcal{f}_{\Psi U}^{3,3,0} &= \frac{1}{2}(\lambda + 1)(\lambda - 1)(\lambda - 2), \\
\mathcal{f}_{\Psi U}^{3,3,2} &= -\frac{1}{2}(\lambda - 5), \\
\mathcal{f}_{\Psi U}^{3,3,3} &= -\frac{3}{2\sqrt{5}}, \\
\mathcal{f}_{\Psi \Psi}^{3,3,4} &= -\frac{1}{\sqrt{5}}(\lambda + 1)(\lambda - 2)(\lambda - 3), \\
\mathcal{f}_{\Psi \Psi}^{3,3,5} &= \frac{1}{\sqrt{5}}(\lambda + 4)(\lambda - 3), \\
\mathcal{f}_{\Psi \Psi}^{3,3,6} &= \frac{1}{\sqrt{5}}(\lambda + 1)(\lambda + 2)(\lambda - 2), \\
\mathcal{f}_{\Psi \Psi}^{3,3,7} &= -\sqrt{\frac{3}{35}}(\lambda + 10), \\
\mathcal{f}_{\Psi \Psi}^{3,3,8} &= -\sqrt{\frac{3}{35}}(\lambda + 4)(\lambda - 3), \\
\mathcal{f}_{\Psi \Psi}^{3,3,9} &= \frac{1}{\sqrt{5}}(\lambda + 1)(\lambda + 2)(\lambda - 2), \\
\mathcal{f}_{\Psi \Psi}^{3,3,10} &= -\sqrt{\frac{3}{35}}(\lambda + 10), \\
\mathcal{f}_{\Psi \Psi}^{3,3,11} &= 6\sqrt{\frac{2}{5}}(\lambda + 1)(\lambda - 3), \\
\mathcal{f}_{\Psi \Psi}^{3,3,12} &= -12\sqrt{\frac{2}{5}}, \\
\mathcal{f}_{\Psi \Psi}^{3,3,13} &= -12\sqrt{\frac{2}{5}}.
\end{align*}
\tag{L.9}
\]

The \(c_{22,3}\) and \(c_{23,4}\) are the structure constants. Note that the mode index \(m''\) in the right hand side of (L.8) is equal to the sum of two indices of the left hand side according to the definition in the footnote [37]. For the term of \(W^{40}\) in the anticommutator of higher spin-\(\frac{5}{2}\) generator and the higher spin-\(\frac{5}{2}\) generator, the coefficient is equal to zero because the structure constants \(f\) are equal to each other from (L.9). Then there is no contribution for this generator in this anticommutator. This feature also occurs at the last relation of (L.8) [46].

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