On A Possible Ground State for Quantum Gravity

G. Preparata and S.-S. Xue

Dipartimento di Fisica dell’Università di Milano
INFN - Sezione di Milano, Via Celoria 16, Milano, Italia

Abstract

A variational calculation is presented of the ADM-energy of the quantized gravitational field around a wormhole solution of the classical Einstein’s equations. One finds the energy of such state to be in general lower than the perturbative ground state, in which the quantized gravity field fluctuates around flat (Euclidean) space-time. As a result the strong indication emerges that a gas (or a lattice) of wormholes of Planck mass and average distance $l_p$, the Planck length, may be a good approximation of the Ground State of Quantum Gravity, some implications of which are reviewed.

November, 1997
PACS 04.60, ...(other pacs)

(*) Presented at the 8th Marcel Grossmann Meeting, Jerusalem, 21-28, June 1997
1 The background: the ground state of QCD

The problem of determining the Ground State (GS) – the vacuum – of a Quantum Field Theory (QFT) is of central importance for a correct assessment and understanding of the dynamics of its observable excited states. Unfortunately after almost 70 years of QFT the solution of this very difficult problem in the most general context still escapes us. On the other hand, it must be admitted, most theoretical physicists believe that a good approximation, at least for small coupling (such as in QED, and Asymptotically Free QCD), is provided by the Perturbative Ground State (PGS), where the quantized field oscillators perform their incoherent zero-point fluctuations. Indeed the remarkable successes of QED and the much more problematic ones of Perturbative QCD (PQCD) have convinced the vast majority of physicists that for practical purposes, at least within the Standard Model (SM), the problem of the vacuum is essentially solved.

Between 1984 and 1986 the present author has developed a research program\[1, 2\] aimed at ascertaining whether the PGS of QCD were really the good approximation to the vacuum of QCD that most believed, especially in view of the fact that the PGS leaves the fundamental question of colour confinement completely unanswered. Based on a gauge-invariant variational approach, this program led to the disquieting result that the PGS of QCD is “essentially unstable”, i.e., contrary to what is generally believed, there exists no space-time scale below which the PGS is a good approximation to real GS.

The basic idea behind such rather difficult calculation (which has been checked and confirmed by computer Monte Carlo simulations\[3\]) was to probe the stability of the PGS by analyzing the effective quantum potential – the energy density – of the QFT in a classical background field, solution of the classical Yang-Mills equations of QCD. Following a suggestion by G. Savvidy\[4\] a constant classical chromomagnetic field $B$ was chosen as background, and for $SU(2)$ the effective potential was found to differ from the patently faulty\[4\] result of the one-loop calculation by Savvidy\[4\](Λ is the ultraviolet cut-off),

$$E(B) = \frac{B^2}{2} - \frac{11}{4\pi^2}(gB)^2\ell n \left( \frac{\Lambda^2}{gB} \right),$$  

$$E(B) = \frac{11}{4\pi^2}(gB)^2\ell n \left( \frac{\Lambda^2}{gB} \right) + O[(gB)^2],$$

where the classical energy density $\frac{B^2}{2}$ is weakly screened by the zero-point oscillations of the “gluon” modes. Instead of eq.(1), the variational calculation was found to yield for $E(B)$:

\[1\] In fact, as pointed out by Nielsen and Olesen in 1978\[3\], Savvidy’s calculation had neglected, without justification, a set of field modes – the “unstable” modes” – for which the frequencies for small oscillations are imaginary.
where the classical term \( \frac{B^2}{2} \) turns out to be completely screened by the “unstable oscillators” of the quantized gauge-field, characterized by negative squared energies in the constant chromomagnetic background, neglected by Savvidy.

The change from eq. (1) to (2) has dramatic consequences for the structure of the GS, for the PGS is seen to lie above the new vacuum by the energy density (\( B^* \) is the value that minimizes \( E(B) \)),

\[
- \Delta E(B^*) = E(0) - E(B^*) = \frac{11A^4}{96\pi^2} \ln \left( \frac{\Lambda^2}{gB} \right)
\]

diverging like the fourth power of the ultraviolet cut-off \( \Lambda \). But one can argue that a better, and Lorentz-invariant approximation of the GS of QCD is a peculiar state, the Chromomagnetic Liquid (CML), consisting of Weiss domains of needle-like shape in rapid rotation, inside which there condenses a very strong \( (gB^* \approx \Lambda^2) \) chromomagnetic field.

It has been shown that the CML solves the problem of colour confinement, giving rise to a very good phenomenology of hadrons and their interactions.

\section{The variational problem in QG}

Looking at QG as a non-abelian gauge theory, whose gauge group is the Poincare group acting on the tangent spaces, it appears natural to try and probe with a similar strategy the stability of the troublesome PGS of QG. This alleged GS is, as we know, filled with the zero-point oscillations of the gravitational field on a flat, Minkowskian background, and is corrected in an uncontrollable way by the non-renormalizable interactions, stemming from Einstein’s action.

In order to define the problem in analogy with QCD we must first find a meaningful energy functional for the quantized gravitational field. This can be obtained from the ADM\cite{7} approach to QG in terms of the ADM mass

\[
E_{ADM} = \frac{1}{16\pi G} \int_{\partial \Sigma} dS^k \delta_{ij} (g_{ik,j} - g_{ij,k}),
\]

which represents the energy contained in a volume \( V \) measured by an asymptotic observer, residing in a pseudoeuclidean region. We now set

\[
g_{ik} = \eta_{ik} + h_{ik},
\]

where \( \eta_{ik} \) is a background metric and \( h_{ik} \) represents the field of quantum fluctuations of the metric around the background.

For \( \eta_{ik} \) we take the simplest non-trivial solution of sourceless Einstein’s gravity, the Schwarzschild metric \( \eta_{ik}^{(s)} \) of a “wormhole” (WH) of mass \( M \) and Schwarzschild radius \( 2m = 2GM \). However no fundamental new difficulty arises if one considers
the more complicated Reissner-Nordstrom and Ker metrics. The energy of space
around a WH in then given by

\[ E = M + \sum_{n \geq 2} \int d^3 x (N^{(0)} H^{(n)} + N_i^{(0)} H^{(n)}), \]  

(6)

where \( H^{(n)} \) and \( H_i^{(n)} \) are the \( n - th \) order expansion in \( h_{ij} \) of the Hamiltonian and of \(-2\pi i j | j \) (\( i \) denotes the covariant derivative with respect to the background field) respectively, while \( N^{(0)} \) is the lapse-function and \( N_i^{(0)} \) the shift-vector of the background metric. Our calculation will now consist in minimizing over the

variational Gaussian wavefunctions:

\[ \Psi_\Gamma[h_{ij}] = \exp - \frac{1}{4} \int \bar{\eta} h_{ij}(\bar{x}) \Gamma^{ijkl}(\bar{x}, \bar{y}) h_{kl}(\bar{y}), \]  

(7)

\( \Gamma^{ijkl}(\bar{x}, \bar{y}) \) being a variational function, the expectation value of the second order Hamiltonian:

\[ H^{(2)} = \int d^3 x N^{(0)} H^{(2)}(h_{ij}(x), -ih \frac{\delta}{\delta h_{ij}(x)}), \]  

(8)

i.e.

\[ \langle H^{(2)} \rangle_\Gamma \equiv \frac{\int [D \bar{h}] \Psi^* \Gamma[h_{ij}] H^{(2)} \left[ h_{ij}, -ih \frac{\delta}{\delta h_{ij}} \right] \Psi[h_{ij}]}{\int [D \bar{h}] \Psi^* \Gamma[h_{ij}] \left[ h_{ij}, -ih \frac{\delta}{\delta h_{ij}} \right] \Psi[h_{ij}]. \]  

(9)

Before briefly describing the calculation, let me observe that this will only have

a meaning if it shall be able to justify, \textit{a posteriori}, the neglect of the practically

intractable terms of the Hamiltonian for \( n \geq 3 \).

The class of gaussian functionals must be further restricted by the constraint

\[ -i \frac{\delta}{\delta \eta_{ij}} \Psi[h] | j = 0, \]  

(10)

required in ADM-approach by general covariance.

As in the Yang-Mills case the problem can be reduced to finding the eigenvalues and the eigenstates of the second order wave-operator (\( G \) is the Newton constant, \( G = l_p^2 \) and \( l_p = 10^{-33} \text{cm} \) is the Planck length)

\[ V^{(2)} = -\frac{1}{16\pi G} \eta^{1/2} (R^{(2)} + \frac{1}{2} h_k^l R^{(1)}) = h_{ij} \hat{Q}^{ijkl} h_{kl}, \]  

(11)

where \( \eta \) is the determinant of the background metric and \( R^{(1,2)} \) are the first- and second-order expansion in \( h_{ij} \) of the scalar of curvature. Setting,

\[ \hat{Q}^{ijkl} \Phi^{(\rho)}_{kl} = \lambda(\rho) \Phi^{(\rho)}_{ij}, \]  

(12)

where \( \Phi^{(\rho)}_{kl} \) is a complete orthonormal system of second rank tensors satisfying:

\[ \nabla_i \left( \frac{\Phi^i_j}{N} \right) = 0, \Phi^k = 0 \]  

(13)
we easily obtain for the minimum energy of the system WH plus quantized gravitational field:

\[ E = M + \frac{\hbar}{2} \sum_{\rho} \sqrt{\lambda(\rho)} + O(\hbar^2), \]  

(14)

provided, naturally, that \( \lambda(\rho) > 0 \).

3 The instability of flat space-time

In the course of a rather difficult and elaborate analysis\[8, 9\] aimed at diagonalizing the operator \( \hat{Q}^{ijkl} \) we have made the following discoveries:

1. There exist an \( S \)-wave mode with negative eigenvalue,

\[ \lambda = -\frac{1}{64m^2} = -\frac{1}{64(GM)^2} \]  

(15)

2. The stable modes at high energy (\( \lambda \) large and positive) in the WKB approximation in the Schwarzschild background are red-shifted and realize an energy gain with respect to the zero-point energy of the gravitational field over flat space-time,

\[ \Delta E(M) \approx -\frac{64\Lambda^4R^2}{\pi^3}GM\log\frac{R}{2GM}, \]  

(16)

where \( R \gg GM \) is the radius of the spherical volume around the WH, and \( \Lambda \) the ultraviolet cut-off.

It is already clear from the large energy gain (16) that flat-space-time is fundamentally unstable, the creation of a WH of mass \( M \ll R/G \) according to (16) leads to a large energy advantage that an open, quantized system cannot fail to exploit. But there is more. The existence of an unstable mode (see eq.(15)), like in the Yang-Mills case, implies the breakdown of the loop-expansion of the energy (see eq.(16)) in such a way that the negative \( O(\hbar) \)-oscillation is “stabilized” by the higher-order \( (O(\hbar^2) \) and higher) terms that have been neglected. This means that the unstable mode will in general produce a negative classical \( O(1) \) contribution to the energy. And, just like in the Yang-Mills case \[1, 2\], we expect that its contribution will just cancel the classical \( M \)-term of the theory. In fact we know that the classical ground state of General Relativity (GR) is just flat space-time and that its ADM energy is zero, thus the minimum value of the classical energy term of QG is obtained when \( M \) is precisely compensated by the unstable mode. In order to check how this could happen in practice we have compute the Riemann tensor of the metric

\[ g_{ij} = \eta_{ij}^{(s)} + \mu\Phi_{ij}^{(a)} \]  

(17)
where $\Phi^{(u)}_{ij}$ is the normalized wave-function of the unstable mode and $\mu$ is an adjustable amplitude. Averaging the components of the Riemann tensor over a spherical shell of width $m = GM$ outside the horizon of the WH for a well defined value of $\mu$ we have obtained values of its components much smaller than those for $\mu = 0$. This strongly indicates that the unstable mode around a WH adjusts its amplitude in such a way as to render the space outside the WH horizon as flat as possible and, as a consequence, the ADM energy as small as possible.

From this semi-quantitative analysis we may conclude that eq.(16) should be as adequate approximation of the difference $\Delta E(M, R)$ between the energy of the quantized gravitational field in a spherical region of radius $R$ fluctuating around a WH of mass $M \ll \frac{R}{2\alpha}$ and around flat space-time. Its negative sign shows unequivocally that flat space-time, the background of perturbative QG and of the Perturbative Ground State, cannot sustain a stable quantum dynamics of Einstein’s General Relativity (GR), thus excluding that the well known perturbative pathologies of GR may be of any relevance for QG, and of any support for the widely accepted idea that the quantization of GR is necessarily doomed to fail, requiring a geometry of space-time that goes beyond the simple one embodied in GR.

4 Gas of WH’s and the Planck lattice

We have just seen that the presence of a single WH of mass $M$, in a region of radius $R \gg 2MG$, realizes for the quantum gravitational field a definite large energy advantage [see eq.(16)], with respect to empty (classical) Euclidean space, which thus turns out to be quantum-mechanically unstable. However, the structure of eq.(16) makes it quite obvious that the state considered does by no means realize the GS of QG, i.e. the state of minimum energy, for the energy can be further lowered by considering a multiwormhole configuration for the classical solution of the sourceless Einstein equations. Even though, to my knowledge, we do not know the most general multiwormhole solution\footnote{For a particular configuration see ref.[10].}, the only important notion that we need in order to figure out a good candidate for the GS of QG is that a system of WH’s of ADM-mass $M$ (and horizon $2GM$) of average density $\left(\frac{1}{a}\right)^3$ ($a$ being the average inter WH distance) for $a \geq 4GM$ interacts through a two-body Newtonian potential $V \simeq -\frac{GMr^2}{a}$. Thus in order to pack a given volume of space $\Omega$ with as many elementary spherical regions of radius $\simeq \frac{a}{2}$, each realizing the energy gain appearing in eq.(17), it is necessary that the (semi)-classical system of WH’s does not collapse into a single giant WH. It is quite easy to find (qualitatively) the condition that the stability from collapse puts upon the mass $M$ of the WH’s. We need only to look at two WH’s as (extended) quantum particles of mass $M$ and solve the Schrödinger equation with reduced mass $\mu = \frac{M}{2}$ and Newtonian

tow-body potential $V = -\frac{GM^2}{r}$, whose well known solution yields for the “Bohr radius”

$$a_o = \frac{2}{GM^3},$$

which represents the average distance between the WH’s in the state of the lowest energy $E_o = 2M - \frac{GM^2}{4}$. Thus, provided

$$a_o \geq 4GM,$$

or

$$M \leq G^{-\frac{1}{2}} \left(\frac{1}{2}\right)^{-\frac{1}{4}},$$

the semi-classical system of WH’s will be stable against collapse and shall realize the energy density gain with respect to Euclidean space-time $[a \simeq 4GM = 2\frac{7}{4}l_p; M \simeq m_p \left(\frac{1}{2}\right)^{-\frac{1}{4}}, \kappa$ is a number of $O(1)]$

$$\frac{\Delta E}{V} \simeq -\kappa \frac{\Lambda^4}{\pi^3}.$$  

At this point it should be evident that the semi-classical state over which the quantized gravitational field fluctuates has just the features and the structure of the space-time foam, hypothesized many years ago by J.A. Wheeler[11]. It should likewise be evident that the observable gravitational field and for that matter all quantum fields, that live in the interstices of the fluctuating gas of WH’s, cannot be resolved in space regions of size smaller than the Planck length $l_p \simeq 10^{-33}$cm, and should be mathematically described by discrete fields defined on a lattice of lattice constant $a \simeq l_p$, the Planck lattice.

5 Conclusion

In spite of the considerable technical weaponry that has been necessary to address a fundamental problem such as determining the stability of the PGS of QG, and the structure of its possible GS, it is extremely pleasing that the final results of a long and difficult investigation[8, 9] are extremely simple, and loaded with far-reaching consequences and implications.

In a nutshell the main result of this analysis is that Euclidean space-time, the classical vacuum, once the metric field is allowed to preform quantum fluctuations (as required by the fundamental principles of quantum physics) becomes unstable against decaying into a gas of WH of mass $M \simeq G^{-\frac{1}{2}} = m_p$ and interwormhole distance $a \simeq G^{\frac{1}{2}} = l_p$. This remarkable metamorphosis, that the classical vacuum experiences under the “spell” of quantum fluctuations, has its origin in the simple fact that the red-shifts, that the gravitational waves are subject to in the surroundings of the WH’s, vastly decrease the energy density of the zero-point
fluctuations upon the classical, Euclidean vacuum. And this without paying the price, as in classical physics, of the ADM - wormhole mass $M$. Both stunning occurrences had been already encountered in a similar analysis of another non-abelian gauge theory, the $SU(2)$ Yang-Mills theory\cite{1,2}, that has finally led to a simple and physically transparent proof of colour confinement\cite{6}. And it is extremely reasonable that similar theoretical structures in the end provide us with (most likely) Ground States of similar highly non-perturbative character, and this, as far as I can judge, in a theoretically robust fashion.

What are the consequences and implications of what has been discovered (and summarily described in this talk)? The first and most important consequence is that at the Planck scale $l_p$ the familiar continuous space-time is found instead to be “foamy”, i.e. essentially discrete. Apart from realizing another prophecy of Berhard Riemann\cite{12}, this discovery gives us finally the long-sought momentum cut-off that is necessary to give a well-defined mathematical meaning to all QFT’s, that in continuous space-time have been plagued with the nasty ultraviolet divergences, that have puzzled great minds such as Dirac’s.

The second implication is that the cut-off at the Planck mass gives QG also a well defined, perturbative meaning, for the horrible looking, formerly non-renormalizable higher order (in the graviton fields) interaction terms of the Einstein’s action, yield now small corrections, calculable in principle. Finally the emergence of the “foamy” space-time, fully justifies a research program into the deep structure of the Standard Model\cite{3} that has already produced a number of interesting results.

References

[1] M. Consoli, G. Preparata, *Phys. Lett. B* 154, 411 (1985)

[2] G. Preparata, *Nuovo Cimento* A96 (1986) 366.

[3] L. Cosmai, G. Preparata, *Phys. Rev. Lett.* 57 (1986) 2613.

[4] G. K. Savvidy, *Phys. Lett.* B71 (1977) 133.

[5] H. Nielsen and P. Olesen, *Nucl. Phys.* B134 (1978) 376.

[6] G. Preparata, *Nuovo Cimento* 103A (1990) 1073.

[7] R. Arnowitt, S. Deser and C.W. Misner, *Phys. Rev.* 116 (1959) 1322.

[8] S. Cacciatori, G. Preparata, S. Rovelli, I. Spagnolatti and S.-S. Xue, hep-th/9701130, submitted to *Phys. Lett. B*

For review see ref.\cite{13}.
[9] G. Preparata, S. Rovelli, and S.-S. Xue submitted to Phys. Rev. D

[10] G.W. Gibbons, Malcolm J. Perry, Phys. Rev. D 22, 313 (1980)

[11] J.A. Wheeler, Geometrodynamics (Academic Press, New York, 1962).

[12] F.B. Riemann, Abandl. Ges Wiess. zu Göttingen (1868) 13.

[13] G. Preparata and S.-S. Xue, Nucl. Phys. B30 (1993) 674; an invited talk “Quantum Gravity, the Planck lattice and Standard Model”, Proceedings of VII Marcel Grossman meeting of General Relativity, Stanford, July 1994, and references therein.