Differential Equations for $\mathbb{F}_q$-Linear Functions

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Abstract

We study certain classes of equations for $F_q$-linear functions, which are the natural function field counterparts of linear ordinary differential equations. It is shown that, in contrast to both classical and $p$-adic cases, formal power series solutions have positive radii of convergence near a singular point of an equation. Algebraic properties of the ring of $F_q$-linear differential operators are also studied.

Key words: $F_q$-linear function; differential equation; radius of convergence; Noetherian ring; Ore domain
1 INTRODUCTION

A standard model of a non-discrete, locally compact field of a positive characteristic $p$ is the field $K$ of formal Laurent series with coefficients from the Galois field $F_q$, $q = p^v$, $v \in \mathbb{Z}_+$. Let $\overline{K}_c$ be a completion of an algebraic closure of $K$.

A function defined on a $F_q$-subspace $K_0$ of $K$, with values in $\overline{K}_c$, is called $F_q$-linear if $f(t_1 + t_2) = f(t_1) + f(t_2)$ and $f(\alpha t) = \alpha f(t)$ for any $t, t_1, t_2 \in K$, $\alpha \in F_q$.

$F_q$-linear functions often appear in analysis over $\overline{K}_c$; see, in particular, the works by Carlitz [4, 5], Wagner [19], Goss [8, 9], Thakur [17, 18], and the author [10, 11]. This class of functions includes, in particular, analogues of the exponential, logarithm, Bessel, and hypergeometric functions.

It has been noticed (see e.g. [18, 10]) that the above functions satisfy some equations, which can be seen as function field analogues of first and second order linear differential equations with polynomial coefficients. The role of a derivative is played by the operator

$$d = \sqrt{\tau} \circ \Delta$$

where $(\Delta u)(t) = u(xt) - xu(t)$, $x$ is a prime element in $K$. The operator $d$ is also basic in the calculus of $F_q$-linear functions developed in [11].

The meaning of a polynomial coefficient in the function field case is not a usual multiplication by a polynomial, but the action of a polynomial in the operator $\tau$, $\tau u = u^\theta$. This was most clearly demonstrated in [18], where an equation for the hypergeometric functions was derived.

The aim of this paper is to start a general theory of differential equations of the above kind. In fact we consider equations (or systems) with holomorphic coefficients. As in the classical theory, we have to make a distinction between the regular and singular cases.

In the analytic theory of linear differential equations over $\mathbb{C}$ a regular equation has a constant leading coefficient (which can be assumed equal to 1). A leading coefficient of a singular equation is a holomorphic function having zeroes at some points. One can divide the equation by its leading coefficient, but then poles would appear at other coefficients, and the solution can have singularities (not only poles but in general also essential singularities) at those points.

Similarly, in our case we understand a regular equation as the one with the coefficient 1 at the highest order derivative. As usual, a regular higher-order equation can be transformed into a regular first-order system. For the regular case we obtain a local existence and uniqueness theorem, which is similar to analogous results for equations over $\mathbb{C}$ or $\mathbb{Q}_p$ (for the latter see [12]). The only difference is a formulation of the initial condition, which is specific for the function field case.

The leading coefficient $A_m(\tau)$ of a singular $F_q$-linear equation of an order $m$ is a non-constant holomorphic function of the operator $\tau$. Now one cannot divide the equation

$$A_m(\tau)d^m u(t) + A_{m-1}(\tau)d^{m-1}u(t) + \ldots + A_0(\tau)u(t) = f(t)$$

for an $F_q$-linear function $u(t)$ (note that automatically $u(0) = 0$) by $A_m(\tau)$. If $A_m(\tau) = \sum_{i=0}^\infty a_{mi} \tau^i$, $a_{mi} \in \overline{K}_c$, then

$$A_m(\tau)d^m u = \sum_{i=0}^\infty a_{mi} (d^m u)^{q^i},$$
and even when $A_m$ is a polynomial, in order to resolve our equation with respect to $d^m u$ one has to solve an algebraic equation.

Thus for the singular case the situation looks even more complicated than in the classical theory. However we show that the behavior of the solutions cannot be too intricate. Namely, in a striking contrast to the classical theory, any formal series solution converges in some (sufficiently small) neighbourhood of the singular point $t = 0$. Note that in the $p$-adic case a similar phenomenon takes place for equations satisfying certain strong conditions upon zeros of indicial polynomials [3, 16]. In our case such a behavior is proved for any equation, which resembles the (much simpler) case [13] of differential equations over a field of characteristics zero, whose residue field also has characteristic zero.

We also study some algebraic properties of the ring of all polynomial differential operators, that is the ring generated by $K_c$, $\tau$, and $d$. It is interesting to compare our results with the ones for the case of characteristic zero [2, 7], and the ones for usual differential operators over a field of positive characteristic [14]. It appears that the main difference from the former case is caused by nonlinearity of $\tau$ and $d$, while the latter case is totally different. For example, the centre of our ring equals $F_q$, the centre of the ring of polynomial differential operators over a field $k$ with $\text{char } k = 0$ equals $k$. Meanwhile for the situation studied in [14] the centre is a “big” polynomial ring.

The author is grateful to David Goss for his constructive criticism, which helped much to improve the exposition.

2 ANALYTIC PROPERTIES

Let us introduce some notation. If $t \in K_c$,

$$t = \sum_{i=n}^{\infty} \theta_i x^i, \quad n \in \mathbb{Z}, \quad \theta_i \in F_q, \quad \theta_n \neq 0,$$

the absolute value $|t|$ is defined as

$$|t| = q^{-n}.$$

We preserve the notation $|\cdot|$ for the extension of the absolute value onto $K_c$. The norm $|P|$ of a matrix $P$ with elements from $K_c$ is defined as the maximum of absolute values of the elements.

We will use systematically the Carlitz factorial $D_i$, $i \geq 0$, defined as

$$D_i = [i][i-1]^q \ldots [1]^{q^{i-1}}, \quad i \geq 1; \quad D_0 = 1,$$

where $[i] = x^q - x$ (this notation should not be confused with the one for the commutator $[\cdot, \cdot]$).

It is easy to see [11] that

$$d \left( \frac{t^q}{D_i} \right) = \frac{t^{q^{i-1}}}{D_{i-1}}, \quad i \geq 1; \quad d(\text{const}) = 0;$$

(1)

$$\tau \left( \frac{t^{q^{i-1}}}{D_{i-1}} \right) = [i] \frac{t^q}{D_i}, \quad i \geq 1.$$

(2)

It is known [11, 13] that $[d, \tau] = [1]^{1/q}$.  

2.1 Equations without Singularities

Let us consider an equation

\[ dy(t) = P(\tau)y(t) + f(t) \]  

(3)

where for each \( z \in (\overline{K}_c)^m, t \in K, \)

\[ P(\tau)z = \sum_{k=0}^{\infty} \pi_k z^q, \quad f(t) = \sum_{j=0}^{\infty} \varphi_j \frac{t^q}{D_j}, \]  

(4)

\( \pi_k \) are \( m \times m \) matrices with elements from \( K_c \), \( \varphi_j \in (\overline{K}_c)^m \), and it is assumed that the series (4) have positive radii of convergence. The action of the operator \( \tau \) upon a vector or a matrix is defined component-wise, so that \( z^q = (z_1^q, \ldots, z_m^q) \) for \( z = (z_1, \ldots, z_m) \). Similarly, if \( \pi = (\pi_{ij}) \) is a matrix, we write \( \pi^q = (\pi_{ij}^q) \).

We will seek a \( F_q \)-linear solution of (3) in some neighbourhood of the origin, of the form

\[ y(t) = \sum_{i=0}^{\infty} y_i t^q \frac{q^i}{D_i}, \quad y_i \in (\overline{K}_c)^m, \]  

(5)

where \( y_0 \) is a given element, so that the “initial” condition for our situation is

\[ \lim_{t \to 0} t^{-1} y(t) = y_0. \]  

(6)

Note that a function (5), provided the series has a positive radius of convergence, tends to zero for \( t \to 0 \), so that the right-hand side of (3) makes sense for small \( |t| \).

**Theorem 1.** For any \( y_0 \in (\overline{K}_c)^m \) the equation (3) has a unique local solution of the form (5), which satisfies (6), with the series having a positive radius of convergence.

**Proof.** Making (if necessary) the substitutions \( t = c_1 t', \ y = c_2 y' \), with sufficiently small \( |c_1|, |c_2| \), we may assume that the coefficients in (4) are such that \( \varphi_j \to 0 \) for \( j \to \infty \),

\[ |\pi_k^q| \cdot q^{-\frac{q^{k+1}}{q-1}} \leq 1, \quad k = 0, 1, \ldots. \]  

(7)

Using (1) and (2) we substitute (5) into (3), which results in the recurrent formula for the coefficients \( y_i \):

\[ y_{l+1} = \sum_{n+k-l} \pi_k^q y_n^q [n+1]^q \ldots [n+k]^q \varphi_l^q, \quad l = 0, 1, 2, \ldots, \]  

(8)

where the expressions in square brackets are omitted if \( k = 0 \).

It is seen from (8) that a solution of (3), (6) (if it exists) is unique. Since \( |[n]| = q^{-1} \) for all \( n > 0 \), we find that

\[ [n+1]^q \ldots [n+k]^q = q^{-q^{k+\ldots+q}} = q^{-\frac{q^{k+1}}{q-1}}, \]
and it follows from (7),(8) that

\[ |y_{l+1}| \leq \max \left\{ |\varphi_l|^q, |y_0|^q, |y_1|^q, \ldots, |y_l|^q \right\}. \]

Since \( \varphi_n \to 0 \), there exists such a number \( l_0 \) that \( |\varphi_l| \leq 1 \) for \( l \geq l_0 \). Now either \( |y_l| \leq 1 \) for all \( l \geq l_0 \) (and then the series (5) is convergent in a neighbourhood of the origin), or \( |y_{l_1}| > 1 \) for some \( l_1 \geq l_0 \). In the latter case

\[ |y_{l+1}| \leq \max \left\{ |y_0|^q, |y_1|^q, \ldots, |y_l|^q \right\}, \quad l \geq l_1. \]

Let us choose \( A > 0 \) in such a way that

\[ |y_l| \leq A^q, \quad l = 1, 2, \ldots, l_1. \]

Then it follows easily by induction that \( |y_l| \leq A^q \) for all \( l \), which implies the convergence of (5) near the origin. ■

### 2.2 Singular Equations

We will consider scalar equations of arbitrary order

\[
\sum_{j=0}^{m} A_j(\tau) d^j u = f
\]  (9)

where

\[ f(t) = \sum_{n=0}^{\infty} \varphi_n t^q D_n, \]

\( A_j(\tau) \) are power series having (as well as the one for \( f \)) positive radii of convergence.

It will be convenient to start from the model equation

\[
\sum_{j=0}^{m} a_j \tau^j d^j u = f, \quad a_j \in \mathbb{R}, \quad a_m \neq 0.
\]  (10)

Suppose that \( u(t) \) is a formal solution of (10), of the form

\[
u(t) = \sum_{n=0}^{\infty} \frac{t^q D_n}{D_n}.
\]  (11)

Then

\[ a_0 \sum_{n=0}^{\infty} \frac{t^q D_n}{D_n} + \sum_{j=1}^{m} a_j \sum_{n=j}^{\infty} \frac{u_n [n-j+1] \ldots [n]}{D_n} \frac{t^q D_n}{D_n} = \sum_{n=0}^{\infty} \varphi_n \frac{t^q D_n}{D_n}.\]

Changing the order of summation we find that for \( n \geq m \)

\[ u_n \left( a_0 + \sum_{j=1}^{m} a_j [n-j+1] \ldots [n] \right) = \varphi_n. \]  (12)
Let us consider the expression

$$\Phi_n = a_0 + \sum_{j=1}^{m} a_j [n - j + 1] \ldots [n], \quad n \geq m.$$  

Using repeatedly the identity $[i]^q + [1] = [i + 1]$ we find that

$$\Phi^m_n = a^m_0 + \sum_{j=1}^{m} a^m_j \prod_{k=0}^{j-1} [n - k]^m = a^m_0 + \sum_{j=1}^{m} a^m_j \prod_{k=0}^{j-1} \left([n]^{q^m-k} - \sum_{l=1}^{k} [1]^{q^m-l}\right),$$

that is $\Phi^m_n = \Phi^{(m)}([n])$ where

$$\Phi^{(m)}(t) = a^m_0 + \sum_{j=1}^{m} a^m_j \prod_{k=0}^{j-1} \left(t^{q^m-k} - \sum_{l=1}^{k} [1]^{q^m-l}\right)$$

is a polynomial on $\overline{K_c}$ of a certain degree $N$ not depending on $n$. Let $\theta_1, \ldots, \theta_N$ be its roots. Then

$$\Phi^{(m)}([n]) = a^m_0 \prod_{\nu=1}^{N} ([n] - \theta_\nu).$$

As $n \to \infty$, $[n] \to -x$ in $\overline{K_c}$. We may assume that $\theta_\nu \neq [n]$ for all $\nu$, if $n$ is large enough. If $\theta_\nu \neq -x$ for all $\nu$, then for large $n$, say $n \geq n_0 \geq m$,

$$|\Phi^{(m)}([n])| \geq \mu > 0.$$

If $k \leq N$ roots $\theta_\nu$ coincide with $-x$, then

$$|\Phi^{(m)}([n])| \geq \mu q^{-k} q^n, \quad n \geq n_0.$$

Combining the inequalities and taking the root we get

$$|\Phi_n| \geq \mu_1 q^{-\mu_2} q^n, \quad n \geq n_0.$$

where $\mu_1, \mu_2 > 0$. Now it follows from (12) and (13) that the series (11) has (together with the series for $f$) a positive radius of convergence.

Turning to the general equation (9) we note first of all that one can apply an operator series

$$A(\tau) = \sum_{k=0}^{\infty} \alpha_k \tau^k$$

(even without assuming its convergence) to a formal series (11), setting

$$\tau^k u(t) = \sum_{n=0}^{\infty} u_n^k [n + 1]^{q^{k-1}} \ldots [n + k] \frac{t^q^{n+k}}{D_{n+k}}, \quad k \geq 1,$$

and

$$A(\tau) u(t) = \sum_{l=0}^{\infty} \frac{t^q^l}{D_l} \sum_{n+k=l} \alpha_k u_n^k [n + 1]^{q^{k-1}} \ldots [n + k]$$

where the factor $[n + 1]^{q^{k-1}} \ldots [n + k]$ is omitted for $k = 0$.

Therefore the notion of a formal solution (11) makes sense for the equation (9).

We will need the following elementary estimate.
Lemma. Let \( k \geq 2 \) be a natural number, with a given partition \( k = i_1 + \cdots + i_r \), where \( i_1, \ldots, i_r \) are positive integers, \( r \geq 1 \). Then

\[
q^{i_1 + \cdots + i_r} + q^{i_2 + \cdots + i_r} + \cdots + q^{i_r} \leq q^{k+1}.
\]

Proof. The assertion is obvious for \( k = 2 \). Suppose it has been proved for some \( k \) and consider a partition \( k + 1 = i_1 + \cdots + i_r \).

If \( i_1 > 1 \) then \( k = (i_1 - 1) + i_2 + \cdots + i_r \), so that

\[
q^{(i_1-1)+i_2+\cdots+i_r} + q^{i_2+\cdots+i_r} + \cdots + q^{i_r} \leq q^{k+1}
\]

whence

\[
q^{i_1+i_2+\cdots+i_r} + q^{i_2+\cdots+i_r} + \cdots + q^{i_r} \leq q^{k+2}.
\]

If \( i_1 = 1 \) then \( k = i_2 + \cdots + i_r \),

\[
q^{i_2+\cdots+i_r} + q^{i_3+\cdots+i_r} + \cdots + q^{i_r} \leq q^{k+1}
\]

and

\[
q^{i_1+\cdots+i_r} + q^{i_2+\cdots+i_r} + \cdots + q^{i_r} \leq 2q^{k+1} \leq q^{k+2}.
\]

Now we are ready to formulate our main result.

**Theorem 2.** Let \( u(t) \) be a formal solution (11) of the equation (9), where the series for \( A_j(\tau)z \), \( z \in K_c \), and \( f(t) \), have positive radii of convergence. Then the series (11) has a positive radius of convergence.

Proof. Applying (if necessary) the operator \( \tau \) a sufficient number of times to both sides of (9) we may assume that

\[
A_j(\tau) = \sum_{i=0}^{\infty} a_{ji} \tau^{i+j}, \quad a_{ji} \in K_c, \; j = 0, 1, \ldots, m,
\]

where \( a_{j0} \neq 0 \) at least for one value of \( j \). Let us assume, for example, that \( a_{m0} \neq 0 \) (otherwise the reasoning below would need an obvious adjustment). Denote by \( L \) the operator at the left-hand side of (9), and by \( L_0 \) its “principal part”,

\[
L_0 u = \sum_{j=0}^{m} a_{j0} \tau^j d^j u
\]

(the model operator considered above; we will maintain the notations introduced there). Note that \( L_0 \) is a linear operator.

As we have seen,

\[
L_0 \left( \frac{t^n}{D_n} \right) = \Phi_n \frac{t^n}{D_n}, \quad n \geq n_0,
\]
where $\Phi_n$ satisfies the inequality (13). This means that $L_0$ is an automorphism of the vector space $X$ of formal series

$$u = \sum_{n=n_0}^{\infty} u_n \frac{t^q^n}{D_n}, \quad u_n \in \mathbb{K}_c,$$

as well of its subspace $Y$ consisting of series with positive radii of convergence.

Let us write the formal solution $u$ of the equation (9) as $u = v + w$, where

$$v = \sum_{n=0}^{n_0-1} u_n \frac{t^q^n}{D_n}, \quad w = \sum_{n=n_0}^{\infty} u_n \frac{t^q^n}{D_n}.$$

Then (9) takes the form

$$Lw = g,$$

with $g = \sum g_n \frac{t^q^n}{D_n} \in Y$. In order to prove our theorem, it is sufficient to verify that $w \in Y$.

For any $y \in X$ we can write

$$Ly = (L_0 - L_1)y = L_0(I - L_0^{-1}L_1)y$$

where

$$L_1y = -\sum_{j=0}^{m} \sum_{i=1}^{\infty} a_{ji} t^{j+i} d^j y.$$  \hfill (15)

In particular, it is seen from (14) that

$$(I - L_0^{-1}L_1)w = L_0^{-1}g, \quad L_0^{-1}g \in Y.$$

Writing formally

$$(I - L_0^{-1}L_1)^{-1} = \sum_{k=0}^{\infty} (L_0^{-1}L_1)^k$$

and noticing that $L_0^{-1}L_1 : X \to \tau X$, we find that

$$w = \sum_{k=0}^{\infty} (L_0^{-1}L_1)^k h,$$  \hfill (16)

where $h = L_0^{-1}g = \sum_{n=n_0}^{\infty} h_n \frac{t^q^n}{D_n}$, $h_n = \Phi_n^{-1} g_n$, and the series in (16) converges in the natural non-Archimedean topology of the space $X$.

A direct calculation shows that for any $\lambda \in \mathbb{K}_c$

$$(L_0^{-1}L_1) \left( \lambda \frac{t^q^n}{D_n} \right) = -\sum_{i=1}^{\infty} \lambda \Phi_i^{-1} \Psi_i^{(n)} \frac{t^q^{n+i}}{D_{n+i}}$$
where
\[ \Psi_i^{(n)} = [n + 1]^{q_i} [n + 2]^{q_i-1} \cdots [n + i] \sum_{j=0}^{m} [n - j + 1]^{q_i} \cdots [n]^{q_i} a_{ji}, \]
and the coefficient at \( a_{j0} \) in the last sum is assumed to equal 1.

Proceeding by induction we get
\[
(L_0^{-1} L_1)^r \left( \frac{t^q}{D_n} \right) = (-1)^r \sum_{i_1, \ldots, i_r=1}^{n} \left( \Psi_{i_1}^{(n)} \right)^{q_{i_1}^{2+\cdots+i_r}} \left( \Psi_{i_2}^{(n+1)} \right)^{q_{i_2}^{3+\cdots+i_r}} \cdots \\
\times \left( \Psi_{i_r}^{(n+i_1+\cdots+i_{r-1})} \right)^{q_{i_r}^{1+\cdots+i_r}} \Phi_{n+i_1}^{-q_{i_1}^{2+\cdots+i_r}} \Phi_{n+i_1+i_2}^{-q_{i_2}^{3+\cdots+i_r}} \cdots \Phi_{n+i_1+\cdots+i_r}^{-1} \frac{t^{q_{n+i_1+\cdots+i_r}}}{D_{n+i_1+\cdots+i_r}}, \ r = 1, 2, \ldots .
\]

Substituting this into (16) and changing the order of summation we find an explicit formula for \( w(t) \):
\[
w(t) = \sum_{l=n_0}^{\infty} \frac{t^q}{D_l} \sum_{n_i+i_1+\cdots+i_r=l, n \geq n_0, i_r \geq 1} (-1)^r h_n^{q_i-n} \left( \Psi_{i_1}^{(n)} \right)^{q_{i_1}^{2+\cdots+i_r}} \left( \Psi_{i_2}^{(n+1)} \right)^{q_{i_2}^{3+\cdots+i_r}} \cdots \\
\times \left( \Psi_{i_r}^{(n+i_1+\cdots+i_{r-1})} \right)^{q_{i_r}^{1+\cdots+i_r}} \Phi_{n+i_1}^{-q_{i_1}^{2+\cdots+i_r}} \Phi_{n+i_1+i_2}^{-q_{i_2}^{3+\cdots+i_r}} \cdots \Phi_{n+i_1+\cdots+i_r}^{-1} (17)
\]

Observe that
\[
\left| \Psi_{i}^{(n)} \right| \leq (q^{-1})^{q_i-1+q_{i-2}+\cdots+1} \sup_j |a_{ji}|, \ \left| g_n \right| \leq M_1^n, \ \left| a_{ji} \right| \leq M_2^j,
\]
\( M_1, M_2 \geq 1 \) (due to positivity of the corresponding radii of convergence). We have
\[
\left| h_n^{q_i-n} \right| \leq \left| \Phi_n \right|^{-q_i^{1+\cdots+i_r}} M_1^j,
\]
and by the above Lemma
\[
\left| \Phi_n \right|^{-q_i^{1+\cdots+i_r}} \left| \Phi_{n+i_1} \right|^{-q_{i_1}^{2+\cdots+i_r}} \cdots \left| \Phi_{n+i_1+\cdots+i_r} \right|^{-1} \leq \mu_1 \left( q^{n+i_1+\cdots+i_r} + q^{n+i_2+\cdots+i_r} + \cdots + q^{n+i_1+\cdots+i_r} + q^r \right) \leq \mu_1 q^{n+1} \mu_2 q^n (q^{n+1}) \leq q^{\mu_3 q^{j+1}}, \ \mu_3 > 0.
\]

The Lemma also yields
\[
\left| \Psi_{i_1}^{(n)} \right|^{q_{i_1}^{2+\cdots+i_r}} \left| \Psi_{i_2}^{(n+1)} \right|^{q_{i_2}^{3+\cdots+i_r}} \cdots \left| \Psi_{i_r}^{(n+i_1+\cdots+i_{r-1})} \right| \leq M_2^{i_1+\cdots+i_r+q_{i_1}^{2+\cdots+i_r}+\cdots+q_{i_r}^{1+\cdots+i_r}} \leq M_2^{j+1}.
\]

Writing (17) as
\[
w(t) = \sum_{l=n_0}^{\infty} \frac{t^q}{D_l}
\]
we find that
\[
\limsup_{l \to \infty} |w_l|^{q-i} \leq \limsup_{l \to \infty} \left( q^{\mu_3 q^{j+1}} M_1^q M_2^{j+1} \right)^{q-l} < \infty,
\]
which implies positivity of the radius of convergence.

The function field analogue
\[ u(t) = \frac{2}{2F_1(a, b; c; t)}, \quad a, b, c \in \mathbb{Z}, \]
of the Gauss hypergeometric function, which was introduced by Thakur [17], satisfies the equation [18]
\[ A_2(\tau) \frac{d^2 u}{d\tau^2} + A_1(\tau) \frac{du}{d\tau} + A_0(\tau) u = 0, \]
where \( A_2(\tau) = (1 - \tau) \tau \), \( A_1(\tau) = \left[ -1 \right]^q + \left[ -b \right] + \left[ -c \right] \tau - \left[ -c \right], \) \( A_0(\tau) = -\left[ -a \right] \left[ -b \right] \) (note that the element \( [i] = x^q - x \in \overline{K}_c \) is defined for any \( i \in \mathbb{Z} \)). The radii of convergence for solutions of this equation are found explicitly in [18].

3 ALGEBRAIC PROPERTIES

In this section we consider the associative ring \( \mathcal{A} \) of “polynomial differential operators”, that is finite sums
\[ a = \sum_{i,j} \lambda_{ij} \tau^i d^j, \quad \lambda_{ij} \in \overline{K}_c. \tag{18} \]

Operations in \( \mathcal{A} \) are defined in the natural way, with the use of the commutation relations
\[ \tau \lambda = \lambda \tau, \quad d \lambda = \lambda^{1/q} d (\lambda \in \overline{K}_c), \quad d\tau - \tau d = [1]^{1/q}. \]

Note that a representation of an operator \( a \) in the form (18) is unique. Indeed, suppose that
\[ a = \sum_{i=0}^{m} \sum_{j=0}^{n} \lambda_{ij} \tau^i d^j = 0. \]

Let \( \psi_i(t) = \frac{t^q}{\tau^i} \). Using (1), (2), we find that
\[ 0 = a(\psi_0) = \sum_{i=0}^{m} \lambda_{i0} \tau^i \psi_0 = \lambda_{00} \psi_0 + \sum_{i=1}^{m} \lambda_{i0} [1]^{q^{-1}} [2]^{q^{-2}} \ldots [i] \psi_i \]
whence \( \lambda_{i0} = 0 \). Then we proceed by induction; if \( \lambda_{ij} = 0 \) for \( j \leq \nu < n \), then
\[ a(\psi_{\nu+1}) = \sum_{i=0}^{m} \sum_{j=\nu+1}^{n} \lambda_{ij} \tau^i d^j \psi_{\nu+1} = \sum_{i=0}^{m} \lambda_{i,\nu+1} \tau^i \psi_0, \]
so that \( \lambda_{i,\nu+1} = 0 \) \((i = 0, 1, \ldots, m)\) as before.

Some algebraic properties of the ring \( \mathcal{A} \) are collected in the following theorem.

**Theorem 3.** (i) The centre of the ring \( \mathcal{A} \) coincides with \( \mathbb{F}_q \).

(ii) The ring \( \mathcal{A} \) possesses no non-trivial two-sided ideals stable with respect to the mapping
\[ P \left( \sum_{i,j} \lambda_{ij} \tau^i d^j \right) = \sum_{i,j} \lambda_{ij}^q \tau^i d^j. \]
(iii) The ring \( A \) is Noetherian.
(iv) \( A \) is an Ore domain, that is \( A \) has no zero-divisors and \( Aa \cap Ab \neq \{0\}, aA \cap bA \neq \{0\} \) for all pairs of non-zero elements \( a, b \in A \).

**Proof.** (i) It is easily proved (by induction) that
\[
[d, \tau^i] = [i]^{1/q} \tau^{i-1}, \quad [d^j, \tau] = [j]^{q^{-j}} d^{j-1}
\]
for any natural numbers \( i, j \).

Suppose that \( a = \sum_{i=0}^{m} \sum_{j=0}^{n} \lambda_{ij} \tau^i d^j \) belongs to the centre of \( A \). Then \([\tau, a] = [d, a] = 0\).

Repeatedly using (19), we find that
\[
0 = [\tau, a] = \sum_{i=0}^{m} \sum_{j=0}^{n} (\lambda_{ij}^q - \lambda_{ij}) \tau^{i+1} d^j - \sum_{j=1}^{n} \lambda_{0j} [j]^{q^{-j}} d^{j-1} - \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \lambda_{i+1,j+1} [j+1]^{q^{-j}} \tau^{i+1} d^j
\]
whence
\[
\lambda_{0j} = 0, \quad j = 1, \ldots, n,
\]
\[
\lambda_{ij}^q - \lambda_{ij} - [j+1]^{q^{-j}} \lambda_{i+1,j+1} = 0, \quad i = 0, 1, \ldots, m-1; \quad j = 0, 1, \ldots, n-1.
\]

It follows from (20), (21) that \( \lambda_{ij} = 0 \) for \( i < j \). Next,
\[
0 = [d, a] = \sum_{i=1}^{m} \sum_{j=0}^{n} \left( \lambda_{ij}^{1/q} - \lambda_{ij} \right) \tau^i d^{j+1} - \sum_{i=0}^{m-1} \sum_{j=0}^{n} \lambda_{i+1,j+1}^{1/q} [j+1]^{1/q} \tau^i d^{j+1}
\]
\[
+ (\lambda_{00}^{1/q} - \lambda_{00}) d - \sum_{i=1}^{m} \lambda_{i0}^{1/q} [i]^{1/q} \tau^{i-1},
\]
so that
\[
\lambda_{i0} = 0, \quad i = 1, \ldots, n;
\]
\[
\lambda_{ij}^{1/q} - \lambda_{ij} - \lambda_{i+1,j+1}^{1/q} [i+1]^{1/q} = 0, \quad i = 1, \ldots, n-1; \quad j = 1, \ldots, i;
\]
\[
\lambda_{00}^{1/q} - \lambda_{00} - \lambda_{i1}^{1/q} [1]^{1/q} = 0.
\]

From (22), (23) we get \( \lambda_{ij} = 0 \) for \( i > j \). Raising (24) to the power \( q \) we can compare the resulting equality with (21) (with \( i = j = 0 \)). Then we find that \( \lambda_{11} = 0 \), and by virtue of (21) also \( \lambda_{ii} = 0, \quad i = 2, \ldots, m \). Finally, it follows from (24) that \( \lambda_{00}^{q} = \lambda_{00} \), so that \( a = \lambda_{00} \in F_q \).

(ii) Let \( D \) be a two-sided ideal in \( A \), \( PD \subset D \), containing a non-zero element
\[
a = \sum_{i=0}^{m} \sum_{j=0}^{n} \lambda_{ij} \tau^i d^j.
\]
Then $D$ contains the element $a_1 = P(a\tau) - \tau a$. It follows as above that

$$a_1 = \sum_{i=0}^{m} \sum_{j=1}^{n} \lambda^i_{ij} \tau^{i-j+1} d^{j-1}.$$ 

It is clear that either $\lambda_{ij} = 0$ for $j \geq 1$, or $a_1 \neq 0$, and the maximal degree of $d$ in $a_1$ is smaller by 1 than the one in $a$. Repeating the procedure (if necessary) we obtain a non-zero element of $D$ of the form

$$b = \sum_{i=0}^{m} \mu_i \tau^i.$$ 

If not all the coefficients $\mu_i$, $i \geq 1$, are equal to zero, we find a non-zero $b_1 \in D$, $b_1 = P(db) - bd$,

$$b_1 = \sum_{i=1}^{m} \mu_i[i] \tau^{i-1}.$$ 

After an appropriate repetition we obtain that $D$ contains a non-zero constant, so that $D = \mathcal{A}$. (iii) Let

$$A_\nu = \left\{ \sum_{i=0}^{m} \sum_{j=0}^{n} \lambda_{ij} \tau^i d^j \in \mathcal{A} : m + n \leq \nu \right\}.$$ 

The sequence $\{A_\nu\}$ of $K_c$-vector spaces is increasing, and we can define a graded ring

$$\text{gr} (\mathcal{A}) = A_0 \oplus A(1) \oplus A(2) \oplus \ldots$$ 

where $A(\nu) = A_\nu / A_{\nu-1}$, $\nu \geq 1$. The multiplication in $\text{gr} (\mathcal{A})$ is defined as follows. If $f \in A(\nu)$, $g \in A(k)$, $\varphi \in A_\nu$ and $\psi \in A_k$ are arbitrary representatives of $f$ and $g$ respectively, then $\varphi \psi \in A_{\nu+k}$, and we define $fg$ as the class of $\varphi \psi$ in $A(\nu+k)$. In can be checked easily that the multiplication is well-defined.

The ring $\text{gr} (\mathcal{A})$ is generated by the classes $\tau, d \in A(1)$ of the elements $\tau, d \in A_1$, and constants from $K_c$, with the commutation relations

$$\tau d - d \tau = 0, \quad \tau c = c^0 \tau, \quad d c = c^{1/4} d \quad (c \in K_c).$$ 

It follows from the generalization of the Hilbert basis theorem given in [15] that $\text{gr} (\mathcal{A})$ is a Noetherian ring.

Let $\mathcal{L}$ be a left ideal in $\mathcal{A}$. We have to prove that $\mathcal{L}$ is finitely generated as an $\mathcal{A}$-submodule. Let $\Gamma_\nu = A_\nu \cap \mathcal{L}$. Then $\{\Gamma_\nu\}$ is a filtration in $\mathcal{L}$. As above, we can construct a graded ring $\text{gr} (\mathcal{L})$, which is a left ideal in $\text{gr} (\mathcal{A})$. Since $\text{gr} (\mathcal{A})$ is Noetherian, $\text{gr} (\mathcal{L})$ has a finite system of generators $\sigma_1, \ldots, \sigma_n$, and we can write finite decompositions

$$\sigma_j = \sum_k \sigma_j(k), \quad \sigma_j(k) \in \Gamma_k / \Gamma_{k-1}.$$ 

Let $\mu_1, \ldots, \mu_N$ be the set of all non-zero elements $\sigma_j(k)$.
Denote by $\gamma$ the canonical imbedding $\Gamma_k \to \Gamma_k/\Gamma_{k-1} = \Gamma(k)$ extended to the mapping $L \to \text{gr}(L)$. Choose $m_i \in L$ in such a way that $\gamma(m_i) = \mu_i$. Then $m_i$ are generators of $L$, that is

$$\Gamma_k \subset \sum_{i=1}^{N} A m_i \quad \text{for each } k.$$

(25)

Indeed, (25) is obvious for $k = 0$. Suppose that

$$\Gamma_{k-1} \subset \sum_{i=1}^{N} A m_i \quad \text{for each } k.$$

(26)

Consider an element $l \in \Gamma_k \setminus \Gamma_{k-1}$. We have $\gamma(l) \in \Gamma(k)$,

$$\gamma(l) = \sum_{j=1}^{n} c_j \sigma_j, \quad c_j \in \text{gr}(A).$$

Writing each $c_j$ as a sum of homogeneous components

$$c_j = \sum_{\nu} c_j(\nu), \quad c_j(\nu) \in A(\nu),$$

and taking into account that $A(\nu) \Gamma(k) \subset \Gamma(k+\nu)$ for any $k, \nu$, we find that

$$\gamma(l) = \sum_{j+\nu=k} c_j(\nu) \sigma_j(k) = \sum_{j=0}^{k} c_j(k-j) \sigma_j(k).$$

Choosing $C_j \in A_{k-j}$ in such a way that $c_j(k-j)$ is a class of $C_j$ in $A(k-j)$, we obtain the inclusion

$$l - \sum_{j=0}^{k} C_j m_j' \in \Gamma_{k-1}$$

where $\{m_j'\}$ is a subset of $\{m_i\}_{i=1}^{N}$. Together with (26) this implies (25).

We have proved that $A$ is left Noetherian. The proof of the right Noetherianness is similar.

(iv) Let $ab = 0$ for

$$a = \sum_{i=0}^{m_1} \sum_{j=0}^{n_1} \lambda_{ij} \tau^i d^j, \quad b = \sum_{k=0}^{m_2} \sum_{l=0}^{n_2} \mu_{kl} \tau^k d^l,$$

and $a \neq 0, b \neq 0$, that is

$$\sum_{i=0}^{m_1} \lambda_{i} \tau^i \neq 0, \quad \sum_{k=0}^{m_2} \mu_{k} \tau^k \neq 0.$$

(27)

It follows from (19) that

$$d^i \tau^k = \tau^k d^i + O(d^{i-1}).$$
where \( O(d^{j-1}) \) means a polynomial in the variable \( d \), of a degree \( \leq j-1 \), with coefficients from the composition ring \( K_c\{\tau\} \) of polynomials in the operator \( \tau \).

Therefore the coefficient at \( d^{n_1+n_2} \) in the expression for the operator \( ab \) equals

\[
\sum_{i,k} \lambda_{in_1} \mu_{kn_2}^{q_{i-n_1}^{i+k}} \tau^{i+k} = P_1(P_2(\tau))
\]

where

\[
P_1(\tau) = \sum_{i=0}^{m_1} \lambda_{i} \tau^{i}, \quad P_2(\tau) = \sum_{k=0}^{m_2} \mu_{kn_2}^{q_{i-n_1}^{i+k}} \tau^{k},
\]

which contradicts (27), since the ring \( K_c\{\tau\} \) has no zero-divisors [9].

Now (iv) follows from (iii) (see Sect. 4.5 in [3]). However we will give also an elementary direct proof (which does not use the Hilbert basis theorem or its generalizations).

Let us prove the left Ore condition (the proof of the right condition is simpler and does not differ from the one in characteristic zero, see [2]). Thus let \( a,b \neq 0 \); we will prove that \( aA \cap bA \neq \{0\} \).

As above, we will use the filtration \( \{A_\nu\} \) in \( A \). Let \( \nu_1 \) be such a number that \( a,b \in A_{\nu_1} \). Then \( aA_\nu \subset A_{\nu+\nu_1}, bA_\nu \subset A_{\nu+\nu_1} \). Suppose that \( aA \cap bA = \{0\} \). Then in particular

\[
aA_\nu \cap bA_\nu = \{0\} \quad (28)
\]

Let us prove that the set \( \{a\tau^i d^j, \; i+j \leq \nu\} \) forms a basis of \( aA_\nu \), that is its elements are linearly independent. This assertion is evident for \( \nu = 0 \). Suppose that it holds for all \( \nu \leq \kappa - 1 \), and consider the case \( \nu = \kappa \). Let

\[
\sum_{i+j \leq \kappa} c_{ij} a\tau^i d^j = 0, \quad c_{ij} \in K_c.
\]

We may write

\[
a = \sum_{k+l \leq \kappa} \lambda_{kl} \tau^k d^l, \quad \lambda_{kl} \in K_c, \; k \leq \nu_1,
\]

where \( \lambda_{kl} \neq 0 \) at least for one couple \( (k,l) \) with \( k + l = \kappa \). Since

\[
d^i \tau^j \equiv \tau^i d^j \pmod{A_{i+l-1}},
\]

we find that

\[
\sum_{i+j = \kappa} c_{ij} \left( \sum_{k+l = \kappa} \lambda_{kl} \tau^k d^l \right) \tau^i d^j \equiv \sum_{i+j = \kappa} \sum_{k+l = \kappa} c_{ij} \lambda_{kl} \tau^{i+k} d^{i+l} \pmod{A_{i+k+j+l-1}}.
\]

By (29), this means that

\[
\sum_{i+j = \kappa} \sum_{k+l = \kappa} c_{ij} \lambda_{kl} \tau^{i+k} d^{i+l} = 0
\]

whence

\[
0 = \sum_{i=0}^{\kappa} \sum_{k=0}^{\kappa} \lambda_{k,k-k} \tau^{i+k} d^{N+k-(i+k)} = \sum_{m=0}^{N+k} \left( \sum_{i+k=m} c_{i,N-i} \lambda_{k,k-k} \right) \tau^m d^{N+k-m},
\]

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so that

\[ \sum_{i+k=m} c_{i,N-i} \lambda_{k,\kappa-k} = 0, \quad m = 0, 1, \ldots, N + \kappa. \]

The expression in the left-hand side coincides with the \( m \)-th coefficient of the product of two polynomials. Therefore \( c_{ij} = 0 \) for \( i + j = N \), the summation in (29) is actually performed for \( i + j \leq N - 1 \), and by the induction assumption \( c_{ij} = 0 \) for all \( i, j \).

Now \( \dim(aA_\nu) = \dim(bA_\nu) = \dim A_\nu \), and it follows from (28) that

\[ \dim(A_{\nu+\nu_1}) \geq \dim(aA_\nu \oplus bA_\nu) = 2 \dim(A_\nu). \] (30)

Note that

\[ \dim A_\nu = \text{card} \left\{ (i, j) : i + j \leq \nu \right\} = \frac{(\nu + 1)(\nu + 2)}{2}, \]

and we see that

\[ \frac{\dim(A_{\nu+\nu_1})}{\dim A_\nu} \rightarrow 1 \quad \text{as} \quad \nu \rightarrow \infty, \]

which contradicts (30).

\[ \blacksquare \]

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