Inverse problems for linear hyperbolic equations using mixed formulations

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Received 20 January 2015, revised 8 April 2015
Accepted for publication 22 April 2015
Published 19 May 2015

Abstract

We introduce a direct method which allows the solving of numerically inverse problems for linear hyperbolic equations. We first consider the reconstruction of the full solution of the equation posed in $\Omega \times (0, T)$—$\Omega$ being a bounded subset of $\mathbb{R}^N$—from a partial distributed observation. We employ a least-squares technique and minimize the $L^2$-norm of the distance from the observation to any solution. Taking the hyperbolic equation as the main constraint of the problem, the optimality conditions are reduced to a mixed formulation involving both the state to reconstruct and a Lagrange multiplier. Under usual geometric optic conditions, we show the well-posedness of this mixed formulation (in particular the inf–sup condition) and then introduce a numerical approximation based on space-time finite element discretization. We prove the strong convergence of the approximation and then discuss several examples for $N = 1$ and $N = 2$. The problem of the reconstruction of both the state and the source terms is also addressed.

Keywords: inverse problems, reconstruction of solution, hyperbolic systems, mixed formulations

(Some figures may appear in colour only in the online journal)

1. Introduction—inverse problems for the wave equation

Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ ($N \geq 1$) whose boundary $\partial \Omega$ is Lipschitz and let $T > 0$. We denote $Q_T = \Omega \times (0, T)$ and $\Sigma_T = \partial \Omega \times (0, T)$. We are concerned in this work with inverse type problems for the following linear hyperbolic equation,
\[
\begin{cases}
y_{tt} - \nabla \cdot (c(x) \nabla y) + d(x, t)y = f, & (x, t) \in Q_T \\
y = 0, & (x, t) \in \Sigma_T \\
(y(\cdot, 0), y_t(\cdot, 0)) = (y_{01}, y_{11}), & x \in \Omega.
\end{cases}
\]

(1)

We assume that \( c \in C^1(\overline{\Omega}, \mathbb{R}) \) with \( c(x) \geq c_0 > 0 \) in \( \overline{\Omega} \), \( d \in L^\infty(Q_T) \), \((y_{01}, y_{11}) \in H = L^2(\Omega) \times H^{-1}(\Omega) \) and \( f \in X = L^2(0, T; H^{-1}(\Omega)) \).

For any \((y_{01}, y_{11}) \in H \) and any \( f \in X \), there exists exactly one solution \( y \) to (1), with \( y \in C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega)) \) (see [21, theorem 8.2]).

In the following, for simplicity, we shall use the following notation:

\[ L y := y_{tt} - \nabla \cdot (c(x) \nabla y) + d(x, t)y \] (2)

and \( X := L^2(0, T; H^1_0(\Omega)) \).

Let now \( \omega \) be any non-empty open subset of \( \Omega \) and let \( q_T := \omega \times (0, T) \subset Q_T \). A typical inverse problem for (1) is the following: from an observation or measurement \( y_{\text{obs}} \) in \( L^2(q_T) \) on the sub-domain \( q_T \), we want to recover a solution \( y \) of the boundary value problem (1) which coincides with the observation on \( q_T \). Introducing the operator \( P : Z \rightarrow X \times L^2(q_T) \) defined by \( P y := (L y, y_{q_T}) \) where the space \( Z \) is defined by (3), the problem is reformulated as:

\[
\text{find } y \in Z \text{ solution of } Py = (f, y_{\text{obs}}). \quad (IP)
\]

From the unique continuation property for (1), if the set \( q_T \) satisfies some geometric conditions and if \( y_{\text{obs}} \) is a restriction to \( q_T \) of a solution of (1), then the problem is well-posed in the sense that the state \( y \) corresponding to the pair \((y_{\text{obs}}, f)\) is unique.

In view of the unavoidable uncertainties on the data \( y_{\text{obs}} \) (coming from measurements, numerical approximations, etc), the problem needs to be relaxed. In this respect, the most natural (and widely used in practice) approach consists in introducing the following extremal problem (of least-squares type)

\[
\left\{ \begin{array}{l}
\text{minimize over } H \\
J(y_{01}, y_{11}) := \frac{1}{2} \left\| y - y_{\text{obs}} \right\|_{L^2(q_T)}^2
\end{array} \right. \quad (LS)
\]

where \( y \) solves (1),

since \( y \) is uniquely and fully determined from \( f \) and the data \((y_{01}, y_{11})\). Here the constraint \( y - y_{\text{obs}} = 0 \) in \( L^2(q_T) \) is relaxed; however, if \( y_{\text{obs}} \) is a restriction to \( q_T \) of a solution of (1), then problems (LS) and (IP) coincide. A minimizing sequence for \( J \) in \( H \) is easily defined in terms of the solution of an auxiliary adjoint problem. Apart from a possible low decrease of the sequence near extrema, the main drawback when one wants to prove the convergence of a discrete approximation is that it is in general not possible to minimize over a discrete subspace of the set \{ \( y; L y - f = 0 \) \} subject to the equality (in \( X \)) \( L y - f = 0 \). Therefore, the minimization procedure first requires the discretization of the functional \( J \) and of the system (1); this raises the issue of the uniform coercivity property (typically here some uniform discrete observability inequality for the adjoint solution) of the discrete functional with respect to the approximation parameter. As far as we know, this delicate issue has received answers only for specific and somehow academic situations (uniform Cartesian approximation of \( \Omega \), constant coefficients in (1), etc). See [12, 18, 20, 24] and references therein.

More recently, a different method to solve inverse problems such as (IP) has emerged and it uses so called Luenberger observers: this consists in defining, from the observation on \( q_T \), an auxiliary boundary value problem whose solution possesses the same asymptotic behavior in time as the solution of (1); the use of the reversibility of the hyperbolic equation
then allows the reconstruction of the initial data \((y_0, y_1)\). See [8, 25] and references therein. But, for the same reasons, from a numerical point of view, these methods require one to prove uniform discrete observability properties.

In a series of works, Klibanov and co-workers use different approaches to solve inverse problems (see [19] and references therein): they advocate in particular the quasi-reversibility method which reads as follows: for any \(\varepsilon > 0\), find \(y \in \mathcal{A}\) the solution of

\[
\langle P_y, P_f \rangle_{X \times L^2(q_T)} + \varepsilon \langle y, f \rangle_{\mathcal{A}} = \langle (f, y_{obs}), P_f \rangle_{X \times L^2(q_T) \times X \times L^2(q_T)},
\]

for all \(f \in \mathcal{A}\), where \(\mathcal{A}\) denotes a Hilbert space subset of \(L^2(Q_T)\) so that \(Py \in X \times L^2(q_T)\) for all \(y \in \mathcal{A}\) and \(\varepsilon > 0\) is a Tikhonov-like parameter which ensures well-posedness. See, for instance, [14] where the lateral Cauchy problem for the wave equation with non-constant diffusion is addressed within this method. Note that \((QR)\) can be viewed as a least-squares problem since the solution \(y\) minimizes over \(\mathcal{A}\) the functional

\[
y \to | |Py - (f, y_{obs})| |^2_{X \times L^2(q_T)} + \varepsilon | |y| |^2_{\mathcal{A}}.
\]

Eventually, if \(y_{obs}\) is a restriction to \(q_T\) of a solution of \((1)\), the corresponding \(y\) converges in \(L^2(Q_T)\) toward to the solution of \((IP)\) as \(\varepsilon \to 0\). There, unlike in problem \((LS)\), the unknown is the state variable \(y\) itself (as it is natural for elliptic equations) so that any standard numerical methods based on a conformal approximation of the space \(\mathcal{A}\) together with appropriate observability inequalities allow one to obtain a convergent approximation of the solution. In particular, there is no need to prove discrete observability inequalities. See the book [2], and [5, 6] where a similar technique has been used recently to solve the inverse obstacle problem associated to the Laplace equation, which consists in finding an interior obstacle from boundary Cauchy data.

In the spirit of [6, 14, 19], we explore the direct resolution of the optimality conditions associated to the extremal problem \((LS)\), without the Tikhonov parameter while keeping \(y\) as the unknown of the problem. This strategy, which avoids any iterative process, has been successfully applied in the closely related context of the exact controllability of \((1)\) in [12] and [7, 10]. The idea is to take into account the state constraint \(Ly - f = 0\) with a Lagrange multiplier. This allows one to derive explicitly the optimality systems associated to \((LS)\) in terms of an elliptic mixed formulation and therefore reformulate the original problem. The well-posedness of a new such formulation is related to an observability inequality for the homogeneous solution of the hyperbolic equation.

Another important example of an inverse problem for \((1)\) (studied for instance in [26]) is the following: from the observation \(y_{obs}\) on the sub-domain \(q_T\), we want to recover not only the whole solution \(y\) but also the source term \(f\) such that \((1)\) holds and such that \(y\) and \(y_{obs}\) coincide in \(q_T\). In the framework of a least-squares approach, the relaxed problem reads as follows:

\[
\begin{cases}
\text{minimize over } H \times X & J(y_0, y_1, f) := \frac{1}{2} \| y - y_{obs} \|^2_{L^2(q_T)} \\
\text{where } (y, f) \text{ solves } (1).
\end{cases}
\]

\((LS_f)\)

Without an additional condition on \(f \in X\) this problem is not well-posed, since the pair \((y, f)\) corresponding to \(y_{obs}\) is not unique.

The outline of this paper is as follows. In section 2, we consider the least-squares problem \((P)\) and reconstruct the solution of the hyperbolic equation from a partial observation localized on a subset \(q_T\) of \(Q_T\). In section 2.1, we associate to \((P)\) the equivalent mixed formulation \((7)\) which relies on the optimality conditions of the problem. Assuming that \(q_T\) satisfies the classical geometric optic condition (hypothesis 1, see \((H)\)), we then show the
well-posedness of this mixed formulation, in particular, we check the Babuska–Brezzi inf–sup condition (see theorem 1). Interestingly, in section 2.2, we also derive an equivalent dual extremal problem, which reduces the determination of the state $y$ to the minimization of an elliptic functional with respect to the Lagrange multiplier. In section 3, we apply a similar procedure to address problem $(LS)$: specifically, in view of the non-uniqueness of the pair $(y, f)$, we consider an approximate reconstruction problem which aims to select among all the possible source terms the one of minimal $X$-norm. Section 4 is devoted to the numerical approximation through a conformal space-time finite element discretization. The strong convergence of the approximation $(y_h, f_h)$ is shown as the discretization parameter $h$ tends to zero. In particular, we discuss the discrete inf–sup property of the mixed formulation. We present numerical experiments in section 5 for $\Omega = (0, 1)$ and $\Omega \subset \mathbb{R}^2$, in agreement with the theoretical part. We consider, in particular, time dependent observation zones. Section 6 concludes with some perspectives on this study.

### 2. Recovering the solution from a partial observation: a mixed re-formulation of the problem

In this section, assuming that the initial $(y_0, y_1) \in H$ are unknown, we address the inverse problem $(IP)$. Without loss of generality, in view of the linearity of the system (1), we assume that the source term $f$ is zero: $f \equiv 0$ in $Q_T$. We consider the non-empty vector space $Z$ defined by

$$Z := \left\{ y \in C([0, T], L^2(\Omega)) \cap C^1([0, T], H^{-1}(\Omega)) \mid Ly \in X, y|_{\Sigma_0} = 0 \right\}$$

and then introduce the following hypothesis.

**Hypothesis 1.** There exists a constant $C_{\text{obs}} = C(\omega, T, ||c||_{C^1(\Omega)}, ||d||_{L^\infty(\Omega)})$ such that the following estimate holds:

$$\|y(\cdot, 0), y(t)(\cdot, 0)\|_H^2 \leq C_{\text{obs}} \left( \|y\|_{L^2(Q_T)}^2 + \|Ly\|_X^2 \right), \quad \forall y \in Z.$$

Condition $(H)$ is a generalized observability inequality for the solution of the hyperbolic equation: for constant coefficients, this estimate is known to hold if the triplet $(\omega, T, \Omega)$ satisfies a geometric optic condition. See [1]. In particular, $T$ should be large enough. Under the same condition, $(H)$ also holds in the non-cylindrical situation where the domain $\omega$ varies with respect to the time variable: see [7] for the one-dimensional case. For non-constant velocity $c$ and potential $d$, see [10] and references therein.

Then, within this hypothesis, for any $\eta > 0$, we define on $Z$ the bilinear form

$$\langle y, \varphi \rangle_Z := \int_{Q_T} y(t) \varphi(t) \, dx \, dt + \eta \int_0^T \langle Ly(t), L\varphi(t) \rangle_{H^{-1}(\Omega)} \, dt \quad \forall y, \varphi \in Z.$$  \hspace{1cm} (4)

Here, $\langle \cdot, \cdot \rangle_{H^{-1}(\Omega)}$ denotes the inner product in $H^{-1}(\Omega)$ defined by

$$\langle \varphi, \psi \rangle_{H^{-1}(\Omega)} = \int_{\Omega} \nabla (-\Delta)^{-1} \varphi(x) \cdot \nabla (-\Delta)^{-1} \psi(x) \, dx, \quad \forall \varphi, \psi \in H^{-1}(\Omega).$$

In view of $(H)$, this bilinear form defines a scalar product over $Z$. Moreover, with this scalar product, we easily obtain that $Z$ is a Hilbert space (see [7], corollary 2.4). We denote the corresponding norm by $||y||_Z := \sqrt{\langle y, y \rangle_Z}$.  

4
Then, we consider the following extremal problem:

\[
\begin{align*}
\inf J(y) &:= \frac{1}{2} \left\| y - y_{\text{obs}} \right\|^2_{L^2(Q_T)}, \\
\text{subject to} & \quad y \in W
\end{align*}
\]

where $W$ is the closed subspace of $Z$ defined by

\[
W := \{ y \in Z; Ly = 0 \text{ in } X \}
\]

and endowed with the norm of $Z$.

The extremal problem $(P)$ is well posed: the functional $J$ is continuous over $W$, it is strictly convex and is such that $J(y) \to +\infty$ as $\|y\|_W \to \infty$. Note also that the solution of $(P)$ in $W$ does not depend on $\eta$.

Recall that from the definition of $Z$, $Ly$ belongs to $X$. Moreover, the uniqueness of the solution does not hold any longer if the hypothesis $(H)$ is not fulfilled, for instance if $T$ is not large enough. Eventually, from $(H)$, the solution $y$ in $Z$ of $(P)$ satisfies $(y(\cdot, 0), \dot{y}(\cdot, 0)) \in H$, so that problem $(P)$ is equivalent to the minimization of $J$ with respect to $(\psi_0, \psi_1) \in H$ as in problem $(IP)$, section 1.

We also recall that, since $(1)$ is well posed, for any $z \in Z$ there exists a positive constant $C_{\Omega, T}$ such that

\[
\|z\|_{L^2(Q_T)} \leq C_{\Omega, T} \left( \left\| (z(\cdot, 0), \sigma(z(\cdot, 0))) \right\|_{L^2_{\text{obs}}} + \|Lz\|_{L^2_T} \right).
\]

This equality and $(H)$ imply that

\[
\|z\|_{L^2(Q_T)} \leq C_{\text{obs}} \left( \|z\|_{L^2(Q_T)} + (1 + C_{\text{obs}}) \|Lz\|_{L^2_T} \right), \quad \forall z \in Z.
\]

### 2.1. Direct approach

In order to solve $(P)$, we have to deal with the constraint equality which appears in the space $W$. Proceeding as in [12], we introduce a Lagrangian multiplier $\lambda \in X'$ and the following mixed formulation: find the $(y, \lambda) \in Z \times X'$ solution of

\[
\begin{align*}
\begin{cases}
\begin{align*}
& a(y, \varphi) + b(\varphi, \lambda) = l(\varphi), \quad \forall \varphi \in Z \\
& b(y, \varphi) = 0, \quad \forall \varphi \in X',
\end{align*}
\end{cases}
\end{align*}
\]

where

\[
\begin{align*}
a: Z \times Z \to \mathbb{R}, \quad & a(y, \varphi) := \int_{Q_T} y \varphi \, dxdt, \\
b: Z \times X' \to \mathbb{R}, \quad & b(y, \lambda) := \int_0^T \langle \lambda(t), Ly(t) \rangle_{H^1_0(\Omega); H^{-1}(\Omega)} dt, \\
l: Z \to \mathbb{R}, \quad & l(y) := \int_{\Omega_{\text{obs}}} y_{\text{obs}} y \, dxdt.
\end{align*}
\]

We recall that the duality product appearing in $(9)$ is defined by

\[
\langle \varphi, \psi \rangle_{H^1_0(\Omega); H^{-1}(\Omega)} = \int_{\Omega} V\varphi(x) \cdot V(-\Delta)^{-1}\psi(x) \, dx, \quad \forall \varphi \in H^1_0(\Omega), \psi \in H^{-1}(\Omega).
\]
System (7) is nothing other than the optimality system corresponding to the extremal problem (P). Specifically, the following result holds:

**Theorem 1.** Under the hypothesis (H),

1. The mixed formulation (7) is well-posed.
2. The unique solution \((y, \lambda) \in Z \times X'\) to (7) is the unique saddle-point of the Lagrangian
   \[ L: Z \times X' \to \mathbb{R} \text{ defined by} \]
   \[ L(y, \lambda) := \frac{1}{2} a(y, y) + b(y, \lambda) - l(y). \]
3. We have the estimate
   \[ \|y\|_{L^2(q)} \leq \|y_{obs}\|_{L^2(q)}, \quad \|\lambda\|_{X'} \leq 2\sqrt{C_{\Omega,T} + \eta} \|y_{obs}\|_{L^2(q)}. \]

**Proof.** We use classical results for saddle-point problems (see [4], chapter 4).

We easily obtain the continuity of the bilinear form \(a\) over \(Z \times Z\), the continuity of bilinear \(b\) over \(Z \times X'\) and the continuity of the linear form \(l\) over \(Z\). In particular, we obtain
\[
\|a\|_{Z \times Z} \leq \|y_{obs}\|_{L^2(q)}, \quad \|b\|_{Z \times X'} \leq 1, \quad \|l\|_{Z} \leq \eta^{-1/2}.
\]

Moreover, the kernel \(\mathcal{N}(b) = \{y \in Z; b(y, \lambda) = 0 \quad \forall \lambda \in X'\}\) coincides with \(W\): we easily obtain
\[
a(y, y) = \|y\|_{L^2(Z)}^2, \quad \forall y \in \mathcal{N}(b) = W.
\]
Therefore, in view of [4, theorem 4.2.2], it remains to check the inf–sup constant property: \(\exists \delta > 0\) such that
\[
\inf_{\lambda \in X'} \sup_{y \in Z} \frac{b(y, \lambda)}{\|y\|_{L^2(Z)} \|\lambda\|_{X'}} \geq \delta.
\]

We proceed as follows. For any fixed \(\lambda \in X'\), we define \(y_0 \in Z\) as the unique solution
\[
L_{\lambda_0} = -\Delta y + (y_0(\cdot, 0), y_{\partial T}(\cdot, 0)) = (0, 0) \text{ on } \Omega, \quad y_0 = 0 \text{ on } \Sigma_T.
\]
We obtain
\[
b(y_0, \lambda) = \|\lambda\|_{\mathcal{L}^2(q)}^2 \text{ and }
\]
\[
\|y_0\|_Z^2 = \|y_0\|_{L^2(q_T)}^2 + \eta \|\lambda\|_{X'}^2.
\]
From (5), we have the estimate \(\|y_0\|_{L^2(q)} \leq \sqrt{C_{\Omega,T} \|\lambda\|_{X'}}\) which implies, combined with the previous estimate, that
\[
\sup_{y \in Z} \frac{b(y, \lambda)}{\|y\|_{L^2(Z)} \|\lambda\|_{X'}} \geq \frac{b(y_0, \lambda)}{\|y_0\|_Z \|\lambda\|_{X'}} \geq \frac{1}{\sqrt{C_{\Omega,T} + \eta}} > 0
\]
and the result with \(\delta = (C_{\Omega,T} + \eta)^{-1/2}\).
The third point is the consequence of classical estimates (see [4], theorem 4.2.3.):

$$\|y\|_Z \leq \frac{1}{a_0} \|f\|_Z, \quad \|\lambda\|_X \leq \frac{1}{\delta} \left(1 + \frac{\|\ell\|}{a_0}\right) \|f\|_Z$$

where

$$a_0 := \inf_{y \in \mathcal{X}(b)} \frac{a(y, y)}{\|y\|_Z^2}. \quad (15)$$

Estimates (12) and the equality $a_0 = 1$ lead to the results. Eventually, from (12), we obtain that

$$\|\lambda\|_X \leq \frac{2}{\delta} \left\|y_{\text{obs}}\right\|_{L^2(q_T)}$$

and that $\delta \geq (C_{L,T} + \eta)^{-1/2}$ to obtain (11).

In practice, it is very convenient to 'augment' the Lagrangian (see [17]) and consider instead the Lagrangian $\mathcal{L}_r$ defined for any $r > 0$ by

$$\mathcal{L}_r(y, \lambda) := \frac{1}{2} a_r(y, y) + b(y, \lambda) - F(y),$$

$$a_r(y, y) := a(y, y) + r \|L y\|^2_X.$$  

Since $a_r(y, y) = a(y, y)$ on $W$, the Lagrangians $\mathcal{L}$ and $\mathcal{L}_r$ share the same saddle-point. The positive number $r$ is an augmentation parameter.

**Remark 1.** Assuming additional hypotheses on the regularity of the solution $\lambda$, specifically $L\lambda \in L^2(Q_T)$ and $(\lambda, \lambda_T)_{t=0,T} \in H^1_0(\Omega) \times L^2(\Omega)$, we easily prove, writing the optimality condition for $\mathcal{L}_r$, that the multiplier $\lambda$ satisfies the following relations:

$$\begin{cases} 
L\lambda &= -(y - y_{\text{obs}}) 1_{\omega} \quad \text{in } Q_T, \quad \lambda = 0 \quad \text{in } \Sigma_T, \\
\lambda &= \lambda_T = 0 \quad \text{on } \Omega \times \{0, T\}. 
\end{cases} \quad (16)$$

Therefore, $\lambda$ (defined in the weak sense) is an exact controlled solution of the wave equation through the control $-(y - y_{\text{obs}}) 1_{\omega} \in L^2(q_T)$.

- If $y_{\text{obs}}$ is the restriction to $q_T$ of a solution of (1), then the unique multiplier $\lambda$ must vanish almost everywhere. In that case, we have

$$\sup_{\lambda \in \mathcal{X}} \inf_{y \in \mathcal{Y}} \mathcal{L}_r(y, \lambda) = \inf_{y \in \mathcal{Y}} \mathcal{L}_r(y, 0) = \inf_{y \in \mathcal{Y}} J_r(y)$$

with

$$J_r(y) := \frac{1}{2} \left\|y - y_{\text{obs}}\right\|^2_{L^2(Q_T)} + \frac{r}{2} \|L y\|^2_X. \quad (17)$$

The corresponding variational formulation is then: find $y \in Z$ such that

$$a_r(y, \varphi) = \iint_{Q_T} y \varphi \, dx \, dt + r \int_0^T \langle Ly(t), L\varphi(t) \rangle_{H^{-1}(\Omega)} \, dt = l(\varphi), \quad \forall \varphi \in Z.$$

- In the general case, the mixed formulation can be rewritten as follows: find the $(z, \lambda) \in Z \times X'$ solution of
\[
\begin{aligned}
\left\{ \begin{array}{l}
\{ P_y, P_y^r \}_{X \times L^2(\Omega_T)} + \{ L_y, \lambda \}_{X', X} = \left\langle (0, y_{\text{obs}}), P_y^r \right\rangle_{X \times L^2(\Omega_T)}, \quad \forall \ y \in Z,
\{ L_y, \lambda \}_{X', X} = 0, \quad \forall \ \lambda \in X',
\end{array} \right.
\end{aligned}
\] (18)

with \( P_y := \left( \int L_y y, y \right)_{\Omega_T} \). Formulation (18) may be seen as generalization of the \((QR)\) problem (see \((QR)\)), where the variable \( \lambda \) is adjusted automatically (while the choice of the parameter \( \varepsilon \) in \((QR)\) is in general a delicate issue).

System (16) can be used to define a equivalent saddle-point formulation, which is very suitable at the numerical level. Precisely, we introduce—in view of (16)—the space
\[
\Lambda := \left\{ \lambda: \lambda \in C \left( [0, T]; H_0^1(\Omega) \right) \cap C^1 \left( [0, T]; L^2(\Omega) \right), L\lambda \in L^2(\Omega_T), \lambda(\cdot, 0) = \lambda_0(\cdot, 0) = 0 \right\}.
\]

Endowed with the scalar product
\[
\left\langle \lambda, \lambda' \right\rangle_{\Lambda} := \int_{\Omega_T} \left( \lambda \lambda' + L\lambda L\lambda' \right) \, dx \, dt,
\]
we check that, in view of assumption \((H)\), \( \Lambda \) is a Hilbert space. Then, for any parameter \( \alpha \in (0, 1) \), we consider the following mixed formulation: find \( (y, \lambda) \in Z \times \Lambda \) such that
\[
\begin{aligned}
a_{r,a}(y, \lambda) + b_a(y, \lambda) &:= l_{1,a}(\lambda), \quad \forall \ \lambda \in \Lambda, \\
b_a(y, \lambda) - c_a(\lambda, \lambda) &:= l_{2,a}(\lambda), \quad \forall \ \lambda \in \Lambda,
\end{aligned}
\] (19)

where
\[
\begin{aligned}
a_{r,a}: Z \times Z &\to \mathbb{R}, \quad a_{r,a}(y, \lambda) = (1 - \alpha) \int_{\Omega_T} y \lambda \, dx \, dt + \alpha \int_{\Omega_T} \left\langle L y(t), L y(t) \right\rangle_{H^1(\Omega)} \, dt, \\
b_a: Z \times \Lambda &\to \mathbb{R}, \quad b_a(y, \lambda) := \int_{\Omega_T} \left\langle \lambda(t), L y(t) \right\rangle_{H^1(\Omega)} \, dt - \alpha \int_{\Omega_T} L L y \lambda \, dx \, dt, \\
c_a: \Lambda \times \Lambda &\to \mathbb{R}, \quad c_a(\lambda, \lambda) := \int_{\Omega_T} \left\langle \lambda \lambda^* \right\rangle_{H^1(\Omega)} \lambda^* L \lambda \, dx \, dt, \\
l_{1,a}: Z &\to \mathbb{R}, \quad l_{1,a}(y) := (1 - \alpha) \int_{\Omega_T} y_{\text{obs}} y \, dx \, dt, \\
l_{2,a}: \Lambda &\to \mathbb{R}, \quad l_{2,a}(\lambda) := -\alpha \int_{\Omega_T} y_{\text{obs}} L \lambda \, dx \, dt.
\end{aligned}
\]

Proposition 1. Under the hypothesis \((H)\), for any \( \alpha \in (0, 1) \), the formulation (19) is well-posed. Moreover, the unique pair \((y, \lambda)\) in \( Z \times \Lambda \) satisfies
\[
\theta_1 \| y \|^2 + \theta_2 \| \lambda \|^2 \leq \frac{(1 - \alpha)^2}{\theta_1} + \frac{\alpha^2}{\theta_2} \| y_{\text{obs}} \|^2_{L^2(\Omega_T)},
\] (20)
with
\[ \theta_1 := \min \left( 1 - \alpha, \frac{r}{\eta} \right), \quad \theta_2 := \frac{\alpha}{1 + C_{\Omega,T}}. \]

Proof. We easily obtain the continuity of the bilinear forms \( a_{r,\alpha}, b_{\alpha} \) and \( c_{\alpha} \):
\[
|a_{r,\alpha}(y, \eta)| \leq \max \left( 1 - \alpha, \frac{r}{\eta} \right) \|y\|_L \|\eta\|_Z, \quad \forall \; y, \eta \in Z,
\]
\[
|b_{\alpha}(y, \lambda)| \leq \max \left( \alpha, \frac{1}{\sqrt{m}} \right) \|y\|_L \|\lambda\|_\Lambda, \quad \forall \; y \in Z, \forall \; \lambda \in \Lambda,
\]
\[
|c_{\alpha}(\lambda, \bar{\lambda})| \leq \alpha \|\lambda\|_\Lambda \|\bar{\lambda}\|_\Lambda, \quad \forall \; \lambda, \bar{\lambda} \in \Lambda
\]
and of the linear form \( l_1 \) and \( l_2 \):
\[
|l_1|_Z \leq (1 - \alpha) \|\chi_{\text{obs}}\|_{L^2(\Omega_T)} \text{ and } |l_2|_\Lambda \leq \alpha \|\chi_{\text{obs}}\|_{L^2(\Omega_T)}.
\]
Moreover, since \( \alpha \in (0, 1) \), we also obtain the coercivity of \( a_{r,\alpha} \) and of \( c_{\alpha} \): specifically, we check that \( a_{r,\alpha}(y, y) \geq \theta_1 \|y\|_L^2 \) for all \( y \in Z \) while, for any \( m \in (0, 1) \), by writing
\[
c_{\alpha}(\lambda, \bar{\lambda}) = \alpha \|L\lambda\|_{L^2(\Omega_T)}^2 = am \|L\lambda\|_{L^2(\Omega_T)}^2 + \alpha(1 - m) \|L\lambda\|_{L^2(\Omega_T)}^2 \\
\geq am \|L\lambda\|_{L^2(\Omega_T)}^2 + \frac{\alpha(1 - m)}{C_{\Omega,T}} \|\lambda\|_{L^2(\Omega_T)}^2 \\
\geq \alpha \min \left( m, \frac{1 - m}{C_{\Omega,T}} \right) \|\lambda\|_\Lambda^2.
\]
we obtain \( c_{\alpha}(\lambda, \bar{\lambda}) \geq \theta_2 \|\lambda\|_\Lambda^2 \) for all \( \lambda \in \Lambda \) with \( m = (1 + C_{\Omega,T})^{-1} \).

The result [4, proposition 4.3.1] implies the well-posedness of the mixed formulation (19) and the estimate (20).

The \( \alpha \)-term in \( L_{r,\alpha} \) is a stabilization term: it ensures a coercivity property of \( L_{r,\alpha} \) with respect to the variable \( \lambda \) and automatically ensures well-posedness. In particular, there is no need to prove any inf–sup property for the application \( b_{\alpha} \).

Proposition 2. If the solution \((y, \lambda) \in Z \times X'\) of 7 enjoys the property \( \lambda \in \Lambda \), then the solutions of (7) and (19) coincide.

Proof. The hypothesis of regularity and the relation (16) imply that the solution \((y, \lambda) \in Z \times X'\) of (7) is also a solution of (19). The result then follows from the uniqueness of the two formulations.

2.2. Dual formulation of the extremal problem (7)

As discussed at length in [12], we may also associate to the extremal problem (P) an equivalent problem involving only the variable \( \lambda \). Again, this is particularly interesting at the numerical level. This requires a strictly positive augmentation parameter \( r \).
For any $r > 0$, let us define the linear operator $P_r$ from $X'$ into $X'$ by

$$P_r \lambda := -\Delta^{-1}(Ly), \quad \forall \lambda \in X'$$

where $y \in Z$ is the unique solution to

$$a_r(y, \gamma) = b(\gamma, \lambda), \quad \forall \gamma \in Z. \quad (22)$$

The assumption $r > 0$ is necessary here in order to guarantee the well-posedness of (22). Specifically, for any $r > 0$, the form $a_r$ defines a norm equivalent to the norm on $Z$.

The following important lemma holds:

**Lemma 1.** For any $r > 0$, the operator $P_r$ is a strongly elliptic, symmetric isomorphism from $X'$ into $X'$.

**Proof.** From the definition of $a_r$, we easily obtain that $\|P_r \lambda\|_X \leq r^{-1} \|\lambda\|_{X'}$ and the continuity of $P_r$. Next, consider any $\lambda' \in X'$ and denote by $y'$ the corresponding unique solution of (22) so that $P_r \lambda' := -\Delta^{-1}(Ly')$. Relation (22) with $\gamma = y'$ then implies that

$$\int_0^T \langle P_r \lambda'(t), \lambda(t) \rangle_{H_0^1(\Omega)} \, dt = a_r(y, y') \quad (23)$$

and therefore the symmetry and positivity of $P_r$. The last relation with $\lambda' = \lambda$ and the observability estimate (H) imply that $P_r$ is also positive definite.

Finally, let us check the strong ellipticity of $P_r$, equivalently that the bilinear functional $(\lambda, \lambda') \mapsto \int_0^T \langle P_r \lambda(t), \lambda'(t) \rangle_{H_0^1(\Omega)} \, dt$ is $X'$-elliptic. Thus we want to show that

$$\int_0^T \langle P_r \lambda(t), \lambda(t) \rangle_{H_0^1(\Omega)} \, dt \geq C \|\lambda\|_{X'}^2, \quad \forall \lambda \in X' \quad (24)$$

for some positive constant $C$. Suppose that (24) does not hold; there exists then a sequence $\{\lambda_n\}_{n \geq 0}$ of $X'$ such that

$$\|\lambda_n\|_{X'} = 1, \quad \forall n \geq 0, \quad \lim_{n \to \infty} \int_0^T \langle P_r \lambda_n(t), \lambda_n(t) \rangle_{H_0^1(\Omega)} \, dt = 0.$$

Let us denote by $y_n$ the solution of (22) corresponding to $\lambda_n$. From (23), we then obtain that

$$\lim_{n \to \infty} r \left\|L y_n\right\|_X^2 + \left\|y_n\right\|_{L^2(\Omega)}^2 = 0. \quad (25)$$

From (22) with $\gamma = y_n$ and $\lambda = \lambda_n$, we have

$$\int_0^T \left\{ r(\Delta^{-1})L y_n(t) - \lambda_n(t), (\Delta^{-1})L \gamma(t) \right\}_{H_0^1(\Omega)} \, dt + \int_{\Omega} y_n \gamma \, dx \, dt = 0, \quad (26)$$

for every $\gamma \in Z$. We define the sequence $\{\Sigma_n\}_{n \geq 0}$ as follows:

$$\begin{cases}
L \Sigma_n = r L y_n + \Delta \lambda_n, & \text{in } Q_T, \\
\Sigma_n = 0, & \text{in } \Sigma_T, \\
\Sigma_n(\cdot, 0) = \Sigma_{n,t}(\cdot, 0) = 0, & \text{in } \Omega,
\end{cases}$$

so that, for all $n$, $\Sigma_n$ is the solution of the wave equation with zero initial data and source term $rL y_n + \Delta \lambda_n$ in $X$. The regularity of the solution of the hyperbolic equation implies that $\Sigma_n \in Z$ and, again from (5), we obtain $\|\Sigma_n\|_{L^2(Q_T)} \leq \sqrt{C_{\Omega,T}} \|rL y_n + \Delta \lambda_n\|_X$. Then, using (26) with
\[ \mathcal{Y} = \mathcal{S}_u \text{ we obtain} \]
\[ \left\| f \left( -\Delta^{-1} \right) L Y_0 - \lambda u \right\|_{\mathcal{C}} \leq \sqrt{C_{\mathcal{A}, T}} \left\| \nu \right\|_{L^2(T)} . \]

Then, from (25), we conclude that \( \lim_{\eta \to +\infty} \left\| \lambda_{\eta} u \right\|_{\mathcal{Y}} = 0 \) leading to a contradiction and to the strong ellipticity of the operator \( \mathcal{P} \).

The introduction of the operator \( \mathcal{P} \) is motivated by the following proposition:

**Proposition 3.** For any \( r > 0 \), let \( y_0 \in \mathcal{Z} \) be the unique solution of
\[ a_r \left( y_0, \mathcal{P} \right) = l(y), \quad \forall \mathcal{P} \in \mathcal{Z} \]
and let \( J^{**}_r : X' \to X' \) be the functional defined by
\[ J^{**}_r (\lambda) = \frac{1}{2} \int_0^T \left\{ P_r \lambda \left( t \right), \lambda \left( t \right) \right\}_{H^1_0(\Omega)} dt - b \left( y_0, \lambda \right) . \]

The following equality holds:
\[ \sup_{\lambda \in X'} \inf_{\mathcal{P} \in \mathcal{Z}} \mathcal{L}_s (y, \lambda) = - \inf_{\lambda \in X'} J^{**}_r (\lambda) + \mathcal{L}_s \left( y_0, 0 \right) . \]

The proof is standard, see, for instance, [12] in a similar context. This proposition reduces the search for \( y \), the solution of problem \( (\mathcal{P}) \), to the minimization of \( J^{**}_r \). The well-posedness is a consequence of the ellipticity of the operator \( \mathcal{P} \).

**Remark 2.** Assuming in addition that the domain \( \Omega \) is of class \( C^2 \), the results of this section apply if the distributed observation on \( q_r \) is replaced by a Neumann boundary observation on a sufficiently large subset \( \Sigma_r \) of \( \partial \Omega \times (0, T) \) (i.e. assuming \( \frac{\partial y}{\partial \nu} \bigg|_{\Sigma_r} = y_{\text{obs}} \in L^2(\Sigma_r) \) is known on \( \Sigma_r \)). This is due to the following generalized observability inequality: there exists a positive constant \( C_{\text{obs}} = C(\omega, T, ||c||_{L^1(T)}, ||d||_{L^\infty(\Omega)}) \) such that
\[ \left\| y \left( \cdot, 0 \right), y \left( \cdot, 0 \right) \right\|_{H^1_0(\Omega) \times L^2(\Omega)} \leq C_{\text{obs}} \left( \left\| \frac{\partial y}{\partial \nu} \right\|_{L^2(\Sigma_r)}^2 + ||L y||_{L^2(T)}^2 \right), \quad \forall y \in \mathcal{Z} \quad (27) \]
with \( \mathcal{Z} := \{ y : y \in C([0, T], H^1_0(\Omega)) \cap C^1([0, T], L^2(\Omega)), L y \in L^2(T) \} \).

This estimate holds if the triplet \( (Q_r, \Sigma_r, T) \) satisfies the geometric condition mentioned above (see [10] and references therein). In fact, it suffices to re-define the form \( a \) in (8) by \( a \left( y, \mathcal{P} \right) := \int_{\Sigma_r} \frac{\partial y}{\partial \nu} \frac{\partial \mathcal{P}}{\partial \nu} d\mathcal{S} \) and the form \( l \) by \( l(y) := \int_{\Sigma_r} \frac{\partial y}{\partial \nu} y_{\text{obs}} d\mathcal{S} \) for all \( y, \mathcal{P} \in \mathcal{Z} \).

**Remark 3.** We emphasize that the mixed formulation (7) has a structure very close to the one we obtain when we address—using the same approach—the null controllability of (1): more precisely, the control of the minimal \( L^2(q_r) \)-norm which drives to rest the initial data \( (y_0, \gamma) \in H^1_0(\Omega) \times L^2(\Omega) \) is given by \( v = \varphi 1_{q_r} \) where \( (\varphi, \lambda) \in \Phi \times L^2(0, T; H^1_0(\Omega)) \) solves the mixed formulation.
\[
\begin{aligned}
\begin{cases}
  a(\varphi, \varphi) + b(\varphi, \lambda) = l(\varphi), & \forall \varphi \in \Phi \\
  b(\varphi, \lambda) = 0, & \forall \lambda \in L^2(0, T; H^1_0(\Omega)),
\end{cases}
\end{aligned}
\]  

(28)

where

\[
\begin{align*}
a: \Phi \times \Phi & \to \mathbb{R}, \quad a(\varphi, \varphi) = \int_{q_T} \varphi(x, t) \varphi(x, t) \, dx \, dt \\
b: \Phi \times L^2(0, T; H^1_0(\Omega)) & \to \mathbb{R}, \quad b(\varphi, \lambda) = \int_0^T \langle L\varphi(t), \lambda(t) \rangle_{H^{-1}, H^1} \, dt \\
l: \Phi & \to \mathbb{R}, \quad l(\varphi) = -\langle \varphi_t(\cdot, 0), \varphi_0 \rangle_{H^{-1}(\Omega), H^1(\Omega)} + \int_0^1 \varphi(\cdot, 0) \gamma_1 \, dx.
\end{align*}
\]

with \( \Phi = \{ \varphi \in L^2(Q_T); L\varphi \in X; \varphi |_{\partial T} = 0 \} \). See [12].

**Remark 4.** Reversing the order of priority between the constraint \( y - y_{\text{obs}} = 0 \) in \( L^2(q_T) \) and \( L_T - f = 0 \) in \( X \), a possibility could be to minimize the functional \( y \to ||L_T - f||_{X} \) over \( y \in Z \) subject to the constraint \( y - y_{\text{obs}} = 0 \) in \( L^2(q_T) \) via the introduction of a Lagrange multiplier in \( L^2(q_T) \). The fact that the following inf–sup property, there exists \( \delta > 0 \) such that

\[
\inf_{\lambda \in L^2(q_T)} \sup_{y \in Z} \left( \int_{q_T} \lambda y \, dx \, dt \right) \geq \delta
\]

associated to the corresponding mixed formulation, holds true is, however, an open issue. On the other hand, if a \( \varepsilon \)-term is added as in (QR), this property is satisfied (see again the book [19]).

### 3. Recovering the source and the solution from a partial observation: a mixed re-formulation of the problem

Given a partial observation \( y_{\text{obs}} \) of the solution on the subset \( q_T \subset Q_T \), we now consider the reconstruction of the full solution as well as the source term \( f \) assumed in \( X \). We assume that the initial data \( (y_0, y_1) \in H^1 \) are unknown.

The situation is different with respect to the previous section, since without an additional assumption on \( f \), the couple \( (y, f) \) is not unique. Consider the case of a source \( f \) supported in a set which is near \( \partial \Omega \times (0, T) \) and disjoint from \( q_T \); from the finite propagation of the solution, the source \( f \) will not affect the solution \( y \) in \( q_T \). On the other hand, the determination of a couple \( (y, f) \) which solves (1) such that \( y \) coincides with \( y_{\text{obs}} \) is straightforward: it suffices to ‘extend’ \( y \) on \( Q_T \setminus q_T \) appropriately to preserve the boundary conditions, then compute \( L_T \) and recover a source term. However, we emphasize that, from a practical viewpoint, the extension of \( y_{\text{obs}} \) out of \( q_T \) is not obvious. Moreover, this strategy does not offer any control of the variable \( f \).

Additional assumptions are necessary to obtain the uniqueness of the pair \( (y, f) \). For instance, if the source term takes the form \( f(x, t) = \sigma(t) \mu(x) \), with \( \sigma \in C^1([0, T]) \), \( \sigma(0) \neq 0 \) and \( \mu \in L^2(\Omega) \), it is shown in [22, theorem 3.3] that the knowledge of the second derivative \( y_{xx} \) in \( q_T \) is enough to reconstruct uniquely the spatial term \( \mu \) of the source \( f \) and so the \( y \) solution of (1) (assuming in (1) that \( y_0 = y_1 = 0 \) and \( c := 1 \) and that \( \sigma \) is given). Within this hypothesis, we may introduce a least-squares type approach and derive a mixed formulation to reconstruct from \( y_{\text{obs}} I_{q_T} \) the unique pair \( (y, \mu) \). See [13] for the details in
the boundary situation. From a practical point of view, the knowledge of \(y_\tau\) on \(q_T\) is very probably too strong an assumption: consequently, here, we consider to be simpler a well-posed approximate reconstruction problem, without an additional assumption on \(f\).

We assume again that \((H)\) holds. We denote \(Y := Z \times X\) and define on \(Y\) the bilinear form, for any \(\epsilon, \eta > 0\)

\[
\left\langle (y, f), (\varphi, \tilde{f}) \right\rangle_y := \int_{q_T} y \varphi \, dx \, dt + \int_0^T \left( L_y(t) - f(t) \right) L_{\varphi}(t) - f(t) \right)_{H^{-1}(\Omega)} \, dt + \epsilon \int_0^T \left( f(t), \tilde{f}(t) \right)_{H^{-1}(\Omega)} \, dt, \quad \forall (y, f), (\varphi, \tilde{f}) \in Y.
\]

In view of \((H)\), this bilinear form defines a scalar product over \(Y\). Moreover, due to this scalar product, we easily obtain that \(Y\) is a Hilbert space (see [7]). We denote the corresponding norm by \(\| (y, f) \|_Y := \sqrt{\|y\|_{L^2(q_T)}^2 + \epsilon \|f\|_X^2} \), \(\forall (y, f) \in W\).

Then, for any \(\epsilon > 0\), we consider the following extremal problem:

\[
\begin{align*}
\inf_{y, f} J_{\epsilon}(y, f) &= \frac{1}{2} \|y - y_{\text{obs}}\|_{L^2(q_T)}^2 + \epsilon \|f\|_X^2, \\
\text{subject to} & \quad (y, f) \in W
\end{align*}
\]

where \(W\) is the closed subspace of \(Y\) defined by \(W := \{(y, f) \in Y; L_y - f = 0\}\) and endowed with the norm of \(Y\): specifically, it follows that \(\| (y, f) \|_W := \sqrt{\|\dot{y}\|_{L^2(q_T)}^2 + \epsilon \|f\|_X^2}, \quad \forall (y, f) \in W\).

The extremal problem \((P_\epsilon)\) is well posed: the functional \(J_{\epsilon}\) is continuous over \(W\), it is strictly convex and is such that \(J_{\epsilon}(y, f) \to +\infty\) as \(\| (y, f) \|_W \to \infty\). Note also that the solution of \((P_\epsilon)\) in \(W\) depends on \(\epsilon\) but not on \(\eta\).

Note also that if \(\epsilon = 0\), then \(J_{\epsilon}\) is a priori only convex leading possibly to distinct minima. This justifies the introduction of the \(\epsilon\)-term in the functional \(J_{\epsilon}\). We emphasize however that the \(\epsilon\)-term is not a regularization term as it does not improve the regularity of the state \(y\). In fact, by multiplying the cost \(J_{\epsilon}\) by \(\epsilon^{-1}\), \(\epsilon\) can be seen as a penalization parameter to obtain \(y - y_{\text{obs}}\) small in \(L^2(q_T)\) with a source term \(f\) of the minimal \(X\)-norm.

Eventually, from \((H)\), the solution \((y_{\epsilon}, f_{\epsilon})\) in \(W\) of \((P_\epsilon)\) satisfies \((y_{\epsilon}(\cdot, 0), y_{\epsilon}'(\cdot, 0)) \in H\), so that problem \((P_\epsilon)\) is again equivalent to the minimization of \(J_{\epsilon}\) with respect to \((y_{\epsilon}, y_{\epsilon}', f_{\epsilon}) \in H \times X\).

Proceeding as in section 2, we introduce a Lagrangian multiplier \(\lambda_{\epsilon} \in X'\) and the following mixed formulation: find the \((y_{\epsilon}, f_{\epsilon}, \lambda_{\epsilon}) \in Y \times X'\) solution of

\[
\begin{cases}
\begin{aligned}
a_{\epsilon} \left( (y_{\epsilon}, f_{\epsilon}), (\varphi, \tilde{f}) \right) + b \left( (y_{\epsilon}, f_{\epsilon}), \lambda_{\epsilon} \right) &= l(\varphi, \tilde{f}), \quad \forall (\varphi, \tilde{f}) \in Y \\
b \left( (y_{\epsilon}, f_{\epsilon}, \lambda_{\epsilon}) \right) &= 0, \quad \forall \lambda_{\epsilon} \in X'.
\end{aligned}
\end{cases}
\]

where

\[
\begin{align*}
a_{\epsilon}: Y \times Y &\to \mathbb{R}, \quad a_{\epsilon} \left( (y, f), (\varphi, \tilde{f}) \right) := \int_{q_T} y \varphi \, dx \, dt + \epsilon \left( f, \tilde{f} \right)_X, \\
b: Y \times X' &\to \mathbb{R}, \quad b((y, f), \lambda) := \int_0^T \left( L_y(t) - f(t) \right)_{H^{-1}(\Omega)} \, dt,
\end{align*}
\]
Theorem 2. Under the hypothesis (\(H\)), the following hold:

1. The mixed formulation (30) is well-posed.
2. The unique solution \((\gamma_\varepsilon, f_\varepsilon, \lambda_\varepsilon) \in Y \times X'\) is the saddle-point of the Lagrangian \(L_\varepsilon: Y \times X' \to \mathbb{R}\) defined by
   \[
   L_\varepsilon((y, f), \lambda) := \frac{1}{2} a_\varepsilon((y, f), (y, f)) + b((y, f), \lambda) - l(y, f).
   \]
   Moreover, the pair \((\gamma_\varepsilon, f_\varepsilon)\) solves the extremal problem \((P)\).
3. The following estimates hold:
   \[
   \left\| \begin{pmatrix} \gamma_\varepsilon \ f_\varepsilon \end{pmatrix} \right\|_Y = \left( \left\| \gamma_\varepsilon \right\|^2_{L^2(q_T)} + \varepsilon \left\| f_\varepsilon \right\|^2_X \right)^{1/2} \leq \left\| y_{\text{obs}} \right\|_{L^2(q_T)}
   \]
   and
   \[
   \left\| \lambda_\varepsilon \right\|_{L^2(q_T)} \leq 2 \sqrt{C_{\Omega,T} + \eta} \left\| y_{\text{obs}} \right\|_{L^2(q_T)}
   \]
   for some constant \(C_{\Omega,T} > 0\).

The proof is very close to the proof of theorem 1. In particular, the inf–sup property can be obtained by taking, for any \(\lambda \in X', f = 0\) and \(y\) as in (14) so that the inf–sup constant
\[
\delta_\varepsilon := \inf_{\lambda \in X', f \in Y} \sup_{(y, f) \in Y} \frac{b((y, f), \lambda)}{\| (y, f) \|_Y \left\| \lambda \right\|_X} \leq \frac{1}{\delta_\varepsilon}
\]
is bounded by above by \((C_{\Omega,T} + \eta)^{-1/2}\) uniformly with respect to \(\varepsilon\).

Note in particular that, if the optimality conditions are satisfied, the inequality (34) implies, since \(\varepsilon > 0\), that the equality \(\| y_\varepsilon - y_{\text{obs}} \|_{L^2(q_T)} = 0\) cannot hold if \(f_\varepsilon \neq 0\).

Remark 5. We may also prove the inf–sup property using the variable \(f\): for any \(\lambda \in X',\) we set \(y = 0\) and \(f = \Delta \lambda \in X\). We obtain
\[
\sup_{(y, f) \in Y} \frac{b((y, f), \lambda)}{\| (y, f) \|_Y \left\| \lambda \right\|_X} \geq \frac{b((0, \Delta \lambda), \lambda)}{\| (0, \Delta \lambda) \|_Y \left\| \lambda \right\|_X} = \frac{1}{\sqrt{\varepsilon + \eta}}
\]
so that \(\delta_\varepsilon \geq (\varepsilon + \eta)^{-1/2}\). Therefore, the estimate
\[
\left\| \lambda_\varepsilon \right\|_X \leq \frac{2}{\delta_\varepsilon} \left\| y_{\text{obs}} \right\|_{L^2(q_T)}
\]
implies that
\[
\left\| \lambda_\varepsilon \right\|_X \leq 2 \sqrt{\varepsilon + \eta} \left\| y_{\text{obs}} \right\|_{L^2(q_T)}
\]
(37).

Remark 6. The estimate (37) implies that the multiplier \(\lambda_\varepsilon\) vanishes in \(X'\) as \(\varepsilon + \eta \to 0^+\) (recall that \(\varepsilon\) and \(\eta\) can be chosen arbitrarily small in (4)).
Remark 7

(a) Assuming enough regularity on the solution $\lambda_\varepsilon$, specifically that $L\lambda_\varepsilon \in L^2(Q_T)$ and $(\lambda_\varepsilon, \lambda_{\varepsilon, t})_{t=0,T} \in H^1_0(\Omega) \times L^2(\Omega)$, we easily check that the multiplier $\lambda_\varepsilon$ satisfies the following relations:

$$
\begin{aligned}
L^0_\varepsilon &= -(y_\varepsilon - y_{\text{obs}})_w, \\
L^1_\varepsilon - f_\varepsilon &= 0, \\
\varepsilon f_\varepsilon + \Delta \lambda_\varepsilon &= 0 \quad \text{in } Q_T, \\
\lambda_\varepsilon &= 0 \quad \text{in } \Sigma_T, \\
\lambda_{\varepsilon, t} &= \lambda_{\varepsilon, t} = 0 \quad \text{on } \Omega \times [0, T].
\end{aligned}
$$

Therefore, $\lambda_\varepsilon$ is an exact controlled solution of the wave equation through the control $-(y_\varepsilon - y_{\text{obs}})_w$. Note that $f_\varepsilon$ may not be bounded in $X'$ uniformly w.r.t. $\varepsilon$ (contrary to the sequence $(\sqrt{\varepsilon} f_\varepsilon)_{\varepsilon > 0}$).

(b) The equality $L^1_\varepsilon = f_\varepsilon$ becomes $\varepsilon L^1_\varepsilon = -\Delta \lambda_\varepsilon$ and leads to $L(\varepsilon^{-1} L^1_\varepsilon) = -L\lambda_\varepsilon = (y_\varepsilon - y_{\text{obs}})_w$. Finally, $y_\varepsilon$ solves, at least in $D'$, the boundary value problem

$$
\begin{aligned}
L(\varepsilon^{-1} L^1_\varepsilon) + y_\varepsilon &= y_{\text{obs}}_w \quad \text{in } Q_T, \\
(\varepsilon L^1_\varepsilon) &= 0, \quad \text{in } \Omega \times [0, T] \\
y_\varepsilon &= 0, \quad \text{on } \Sigma_T
\end{aligned}
$$

or equivalently the variational formulation: find $y_\varepsilon \in Z$ (see (3)) solution of

$$
\varepsilon \int_0^T \left( L^1_\varepsilon(t), L^\sigma(t) \right)_{H^{-1}(\Omega)} dt + \int_{Q_T} y^\sigma \, d\sigma dt = \int_{Q_T} y_{\text{obs}} \, d\sigma \quad \forall \sigma \in Z
$$

which can actually be obtained directly from the cost $J_\varepsilon$, replacing from the beginning $f$ by the term $L^1_\varepsilon$. From the Lax–Milgram lemma, (38) is well-posed and the following estimates hold:

$$
\|y_\varepsilon\|_{L^2(Q_T)} \leq \|y_{\text{obs}}\|_{L^2(q_T)} + \sqrt{\varepsilon} \|L^1_\varepsilon\|_{L^2(q_T)} \leq \|y_{\text{obs}}\|_{L^2(q_T)}.
$$

This kind of variational formulation involving the fourth order term $L^1_\varepsilon L^1_\varepsilon$ has been derived and used in [10] in a controllability context.

For any $\varepsilon > 0$ and any $y_{\text{obs}} \in L^2(q_T)$, the method allows one to recover a couple $(y_\varepsilon, f_\varepsilon)$ such that $L^1_\varepsilon = f_\varepsilon$ in $Q_T$ and $y_\varepsilon$ is close to $y_{\text{obs}}$. In view of the loss of uniqueness, we have no information on the limit of the sequence as $\varepsilon \to 0$: the sequence may be unbounded at the limit in $L^2(Q_T) \times L^2(Q_T)$ even if $y_{\text{obs}}$ is the restriction to $q_T$ of a solution of (1).

Remark 8. Contrary to the inf–sup property, the coercivity of $a_\varepsilon$ over $\mathcal{X}(b)$ does not hold uniformly with respect to $\varepsilon$. Recall that the $\varepsilon$-term has been introduced to obtain a norm for $Y$. This forces us to add this term in the mixed formulation.

Remark 9. A fortiori, if the initial condition $(y_0, y_1) \in H$ is known, one may recover the pair $(y_\varepsilon, f_\varepsilon) \in Y$ from $y_{\text{obs}}$ and $(y_0, y_1)$. The procedure is similar; it suffices to define two additional Lagrange multipliers $\lambda_1 \in L^2(\Omega)$ and $\lambda_2 \in H^1_0(\Omega)$ to deal with the constraints $y(\cdot, 0) = y_0$ and $y_1(\cdot, 0) = y_1$, respectively. The extremal problem is now:
where $W$ is the closed subspace of $Y$ defined by
\[ W := \{ (u, f) \in Y; Ly - f = 0 \text{ in } X', (y(\cdot, 0), y'(\cdot, 0)) = (y_0, y_1) \text{ in } H \}. \]
The corresponding mixed formulation is: find the $((u_f, f_f), (\lambda_e, \lambda_{e,1}, \lambda_{e,2})) \in Y \times \Lambda$ solution of
\[
\begin{cases}
  a_e \left( (y_f, f_f), (\gamma, \overline{\tau}) \right) + b \left( (\gamma, \overline{\tau}), (\lambda_e, \lambda_{e,1}, \lambda_{e,2}) \right) = l_1(\gamma, \overline{\tau}), \\
  b \left( (y_f, f_f), (\lambda, \lambda_{1,}, \lambda_{2}) \right) = l_2(\lambda, \lambda_{1,}, \lambda_{2}),
\end{cases}
\]
where $a_e$ is given by (31) and
\[
b: Y \times \Lambda \to \mathbb{R}, \quad b \left( (y, f), (\lambda, \lambda_{1,}, \lambda_{2}) \right) = \int_0^T \langle \lambda(t), Ly(t) - f(t) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \, dt + \left\{ y(\cdot, 0), \lambda_1 \right\}_{L^2(\Omega)} + \left\{ y'(\cdot, 0), \lambda_2 \right\}_{H^{-1}(\Omega), H_0^1(\Omega)}.
\]
with $\Lambda := X' \times L^2(\Omega) \times H_0^1(\Omega)$. Using the estimate $(H)$, we easily show that this formulation is well-posed.

In view of remark 7 (a), we may also associate to the mixed formulation (30) a stabilized version, similarly to (19).

Again, it is very convenient to ‘augment’ the Lagrangian (see [17]) and consider instead the Lagrangian $\mathcal{L}_{e,r}$ defined for any $r > 0$ by
\[
\mathcal{L}_{e,r}((y, f), \lambda) := \frac{1}{2} a_{e,r}((y, f), (y, f)) + b(y, \lambda) - l(y, f),
\]
where $a_{e,r}((y, f), (y, f)) := \frac{1}{2} a_e((y, f), (y, f)) + r \| Ly - f \|^2_X.$

Since $a_e((y, f), (y, f))$ on $W$, the Lagrangian $\mathcal{L}_e$ and $\mathcal{L}_{e,r}$ share the same saddle-point. The positive number $r$ is an augmentation parameter. Similarly, proceeding as in section 2.2, we may also associate to the saddle-point problem $\inf_{(y, f) \in Y} \mathcal{L}_{e,r}((y, f), \lambda)$ a dual problem, which again reduces the search of the couple $(y_f, f_f)$, solution of problem $(P_r)$, to the minimization of an elliptic functional in $\lambda_e$.

**Proposition 4.** For any $r > 0$, let $(y_0, f_0) \in Y$ be the unique solution of
\[
a_{e,r} \left( (y_0, f_0), (\gamma, \overline{\tau}) \right) = l(\gamma, \overline{\tau}), \quad \forall (\gamma, \overline{\tau}) \in Y
\]
and let $\mathcal{P}_{e,r}^\lambda$ be the strongly elliptic and symmetric operator from $X'$ into $X'$ defined by
\[
\mathcal{P}_{e,r}^\lambda := -\Delta^{-1}(Ly - f) \quad \text{where } (y, f) \in Y \text{ is the unique solution to}
\]
\[ a_r \left( (y, f), (\varphi, \lambda) \right) = b \left( (\varphi, \lambda), \lambda \right), \quad \forall (\varphi, \lambda) \in Y. \tag{40} \]

Then, the following equality holds
\[ \sup_{\lambda \in \mathcal{X}(\gamma, f) \in Y} \inf_{\lambda \in \mathcal{X}} J^{**} (\lambda) = - \inf_{\lambda \in \mathcal{X}} J^{**} (\lambda), \]

where \( J^{**} : X' \to X' \) is the functional defined by
\[ J^{**} (\lambda) = \frac{1}{2} \int_0^T \left( \mathcal{P}_r (\lambda, \lambda) \right)_{H^1_0 (\Omega)} dt - b \left( (y_0, f_0), \lambda \right). \]

### 4. Numerical analysis of the mixed formulations

#### 4.1. Numerical approximation of the mixed formulation (7)

We consider the numerical analysis of the mixed formulation (7), assuming \( r > 0 \). We follow [12], see this source for the details.

Let \( Z_h \) and \( A_h \) be two finite dimensional spaces parametrized by the variable \( h \) such that \( Z_h \subset Z, A_h \subset X \) for every \( h > 0 \). Then, we can introduce the following approximated problems: find \( (y_h, \lambda_h) \in Z_h \times A_h \) solution of
\[
\begin{cases}
    a_r \left( y, \varphi \right) + b \left( \varphi, \lambda \right) = I \left( \lambda \right), & \forall \varphi \in Z_h \\
    b \left( y, \lambda \right) = 0, & \forall \lambda \in A_h.
\end{cases}
\tag{41}
\]

The well-posedness of this mixed formulation is again a consequence of two properties. The first is the coercivity of the bilinear form \( a_r \) on the subset
\[ \mathcal{N}_r (b) = \left\{ y_h \in Z_h; b \left( y_h, \lambda_h \right) = 0 \quad \forall \lambda \in A_h \right\}. \]

In fact, from the relation \( a_r (y, y) \geq (r/\eta) \| y \|_Z^2 \) for all \( y \in Z \), the form \( a_r \) is coercive on the full space \( Z \), and so a fortiori on \( \mathcal{N}_r (b) \subset Z_h \subset Z \).

The second property is a discrete inf–sup condition. Specifically, for any \( h > 0 \),
\[ \delta_h := \inf_{\lambda_h \in A_h} \sup_{y_h \in Z_h} \frac{b \left( y_h, \lambda_h \right)}{\| y_h \|_Z \| \lambda_h \|_{A_h}} > 0. \tag{42} \]

Let us assume that this condition holds, so that for any fixed \( h > 0 \) there exists a unique couple \( (y_h, \lambda_h) \) solution of (41). We then have the following estimate.

**Proposition 5.** Let \( h > 0 \). Let \( (y, \lambda) \) and \( (y_h, \lambda_h) \) be the solution of (7) and of (41), respectively. Let \( \delta_h \) be the discrete inf–sup constant defined by (42). Then,
\[
\| y - y_h \|_Z \leq 2 \left( 1 + \frac{1}{\sqrt{h} \delta_h} \right) d (y, Z_h) + \frac{1}{\sqrt{h}} d (\lambda, A_h), \tag{43}
\]
\[
\| \lambda - \lambda_h \|_A \leq 2 + \frac{1}{\sqrt{h} \delta_h} \frac{1}{\sqrt{h} \delta_h} d (y, Z_h) + \frac{3}{\sqrt{h} \delta_h} \frac{1}{\delta_h} d (\lambda, A_h). \tag{44}
\]
where \( d(\lambda, A_h) := \inf_{\lambda_h \in A_h} \| \lambda - \lambda_h \|_{X^*} \) and
\[
d(y, Z_h) := \inf_{\lambda_h \in Z_h} \left\| y - y_h \right\|_Z
= \inf_{\lambda_h \in Z_h} \left( \left\| y - y_h \right\|_{L^2(Q_h)}^2 + \eta \left\| L(y - y_h) \right\|_{L^2(Q)}^2 \right)^{1/2}.
\]

**Proof.** From the classical theory of approximation of saddle-point problems (see [4, theorem 5.2.2]) we have that
\[
\left\| y - y_h \right\|_Z \leq \left( \frac{2 \| a_r \|_{L^2(Q)}}{a_0} + \frac{2 \| a_r \|_{L^2(Q)} \| b \|_{L^2(Q)}}{a_0 \delta_h} \right) d(y, Z_h)
+ \frac{1}{a_0} d(\lambda, A_h) \tag{45}
\]
and
\[
\left\| \lambda - \lambda_h \right\|_{X^*} \leq \left( \frac{2 \| a_r \|_{L^2(Q)} \| b \|_{L^2(Q)}}{a_0 \delta_h} + \frac{\| a_r \|_{L^2(Q)} \| b \|_{L^2(Q)}}{a_0 \delta_h} \right) d(y, Z_h)
+ \frac{3 \| a_r \|_{L^2(Q)} \| b \|_{L^2(Q)}}{a_0 \delta_h} d(\lambda, A_h). \tag{46}
\]

Since, \( \| a_r \|_{L^2(Q)} \leq 1, a_0 = 1 \) and \( \| b \|_{L^2(Q)} \leq \eta^{-1/2} \), the result follows. \( \square \)

**Remark 10.** For \( r = 0 \), the discrete mixed formulation (41) is not well-posed over \( Z_h \times A_h \) because the form \( a_{r=0} \) is not coercive over the discrete kernel of \( b \): the equality \( b(y_h, \lambda_h) = 0 \) for all \( \lambda_h \in A_h \) does not imply in general that \( L y_h \) vanishes. Therefore, the term \( r \left\| L y_h \right\|_{L^2(Q)}^2 \), which appears in the Lagrangian \( \mathcal{L}_r \), may be understood as a stabilization term: for any \( h > 0 \), it ensures the uniform coercivity of the form \( a_r \) and vanishes at the limit in \( h \). We also emphasize that this term is not a regularization term as it does not add any regularity on the solution \( y_h \).

Let \( n_h = \dim Z_h, m_h = \dim A_h \) and let the real matrices \( A_{i,h} \in \mathbb{R}^{n_h \times n_h}, B_{i,h} \in \mathbb{R}^{m_h \times n_h}, J_h \in \mathbb{R}^{m_h \times m_h} \) and \( L_h \in \mathbb{R}^{n_h} \) be defined by

\[
\begin{align*}
&\left\{ a_r(y_h, \lambda_h) \right\}_{\mathbb{R}^{n_h \times n_h}} \forall y_h, \lambda_h \in Z_h, \\
&\left\{ b(y_h, \lambda_h) \right\}_{\mathbb{R}^{m_h \times n_h}} \forall y_h \in Z_h, \lambda_h \in A_h, \\
&\left\{ \int_{Q_h} \lambda_h \delta_h \, dx \, dt \right\}_{\mathbb{R}^{n_h \times n_h}} \forall \lambda_h, \delta_h \in \Lambda_h, \\
&\left( L(y_h) \right)_{\mathbb{R}^{n_h}} \forall y_h \in Z_h.
\end{align*}
\]

where \( \{ y_h \} \in \mathbb{R}^{n_h} \) denotes the vector associated to \( y_h \) and \( \langle \cdot, \cdot \rangle_{\mathbb{R}^{n_h \times n_h}} \) the usual scalar product over \( \mathbb{R}^{n_h} \). With these notations, the problem (41) reads as follows: find \( \{ y_h \} \in \mathbb{R}^{n_h} \)
and \( \{ \lambda_h \} \in \mathbb{R}^{m_h} \) such that

\[
\begin{pmatrix}
A_{r,h} & B_{r,h}^T \\
B_{r,h} & 0
\end{pmatrix}_{K^{n_h+1}} \begin{pmatrix}
\{ y_h \} \\
\{ \lambda_h \}
\end{pmatrix}_{K^{n_h+1}} = \begin{pmatrix}
L_h \\
0
\end{pmatrix}_{R^{n_h+q}} .
\] (48)

The matrix \( A_{r,h} \) as well as the mass matrix \( J_h \) are symmetric and positive definite for any \( h > 0 \) and any \( r > 0 \). On the other hand, the matrix of order \( m_h + n_h \) in (48) is symmetric but not necessarily definite.

We recall (see [4, theorem 3.2.1]) that the inf–sup property (42) is equivalent to the injective character of the matrix \( B_{r,h}^T \) of size \( n_h \times m_h \), that is \( \text{Ker}(B_{r,h}^T) = \{0\} \). If a necessary condition is given by \( m_h \leq n_h \), this property strongly depends on the choice and structure of the spaces \( Z_h \) and \( L_h \). In fact, the coercivity of \( a_r \) on \( L_h(b) \) implies the existence and uniqueness of a \( y_h \) solution of (41). This also implies the existence of \( \lambda_h \) but uniqueness only up to elements of \( \text{Ker}(B_{r,h}^T) \). We will discuss the property (42) numerically in remark 11 for a specific choice of approximation.

4.1.1. \( C^1 \)-finite elements and order of convergence for \( N = 1 \)

The finite dimensional and conformal space \( Z_h \) must be chosen such that \( L_y \) belongs to \( X = L^2(0,T;H^{-1}(\Omega)) \) for any \( y_h \in Z_h \). This is guaranteed, for instance, as soon as \( y_h \) possesses second-order derivatives in \( L^2(Q_T) \). As in [12], we consider a conformal approximation based on functions continuously differentiable with respect to both variables \( x \) and \( t \).

We introduce a triangulation \( T_h \) such that \( Q_T = \bigcup_{K \in T_h} K \) and we assume that \( \{ T_h \}_{h>0} \) is a regular family. We denote \( h := \max\{ \text{diam}(K), K \in T_h \} \), where \( \text{diam}(K) \) denotes the diameter of \( K \). Then, we introduce the space \( Z_h \) as follows:

\[
Z_h = \left\{ y_h \in C^1(\overline{Q_T}); y_h |_K \in \mathcal{P}(K) \quad \forall K \in T_h, \ y_h |_{\Sigma_T} = 0 \right\},
\] (49)

where \( \mathcal{P}(K) \) denotes an appropriate space of functions in \( x \) and \( t \). In this work, we consider two choices, in the one-dimensional setting for which \( Q \subset \mathbb{R}, Q_T \subset \mathbb{R}^2 \):

1. The Bogner–Fox–Schmit (BFS) \( C^1 \)-element defined for rectangles. It involves 16 degrees of freedom, namely the values of \( y_h, y_{h,t}, y_{h,x}, y_{h,xt} \) on the four vertices of each rectangle \( K \). Therefore \( \mathcal{P}(K) = \mathcal{P}_4 \times \mathcal{P}_4 \) is by definition the space of polynomial functions of order \( r \) in the variable \( \xi \). See [9, chapter 2, section 9, p 94].

2. The reduced Hsieh–Clough–Tocher (HCT) \( C^1 \)-element defined for triangles. This is a so-called composite finite element and involves 9 degrees of freedom, namely, the values of \( y_h, y_{h,st}, y_{h,t} \) on the three vertices of each triangle \( K \). See [9, chapter 7, section 46, p 285] and [3, 23] where the implementation is discussed.

We also define the finite dimensional space

\[
L_h = \left\{ \lambda_h \in C^0(\overline{Q_T}); \lambda_h |_K \in \mathcal{Q}(K) \quad \forall K \in T_h, \ \lambda_h |_{\Sigma_T} = 0 \right\}.
\]

where \( \mathcal{Q}(K) \) denotes the space of affine functions both in \( x \) and \( t \) on the element \( K \).

For any \( h > 0 \), we have \( Z_h \subset Z \) and \( L_h \subset L' \). We then have the following result:

**Proposition 6** BFS element for \( N = 1 \)—rate of convergence for the norm \( Z \). Let \( h > 0 \), let \( k \in \{ 1, 2 \} \) be a positive integer. Let \( (y, \lambda) \) and \( (y_h, \lambda_h) \) be the solutions of (7) and (41), respectively. If the solution \( (y, \lambda) \) belongs to \( H^{k+2}(Q_T) \times H^k(Q_T) \), then there exist two positive constants
\[ K_i = K_i \left( |\gamma|_{H^s(Q_T)}, \|\cdot\|_{C^s(\overline{Q_T})}, \|d\|_{L^n(Q_T)} \right), \quad i \in \{1, 2\}, \]

independent of \( h \), such that

\[ \|y - y_h\|_X \leq K_1 \frac{h^{k-1}}{\sqrt{n}} \left( \left( \frac{\sqrt{n}}{\delta_h} + \frac{1}{\delta_h} \right) \left( h^3 + \sqrt{n} h \right) + 1 \right), \quad (50) \]

\[ \|\lambda - \lambda_h\|_X \leq K_2 \frac{h^{k-1}}{\sqrt{n} \delta_h} \left( \left( \frac{\sqrt{n}}{\delta_h} + \frac{1}{\delta_h} \right) \left( h^3 + \sqrt{n} h \right) + 1 \right). \quad (51) \]

**Proof.** From [9, chapter 3, section 17], for any \( \lambda \in H^k(Q_T) \), \( k \leq 2 \), there exists \( C_1 = C_1(\|\lambda\|_{H^k(Q_T)}) \) such that

\[ \|\lambda - \Pi_{\lambda_h, \mathcal{T}_h}(\lambda)\|_X \leq C_1 h^{k-1}, \quad \forall h > 0 \]  

(52)

where \( \Pi_{\lambda_h, \mathcal{T}_h} \) designates the interpolant operator from \( X' \) to \( A_h \) associated to the regular mesh \( T_h \). Similarly, for any \( y \in H^{s+2}(Q_T) \), there exist \( C_2 = C_2(\|\gamma|_{H^s(Q_T)}) \) such that for every \( h > 0 \) we have

\[ \|y - \Pi_{\gamma_h, \mathcal{T}_h}(y)\|_{L^2(Q_T)} \leq C_2 h^{k+2}, \quad \|y - \Pi_{\gamma_h, \mathcal{T}_h}(y)\|_{H^s(Q_T)} \leq C_2 h^k. \]  

(53)

Then, observing that

\[ \left\| L_Y - L_{Y_h} \right\|_X \leq K \left( \|\cdot\|_{C^s(\overline{Q_T})}, \|d\|_{L^n(Q_T)} \right) \|y - y_h\|_{H^s(Q_T)}, \]  

(54)

for some positive constant \( K \), we obtain that

\[ d(y, Z_h) = \inf_{y_h \in Z_h} \left( \|y - y_h\|_{L^2(Q_T)} + \eta \|L_Y - L_{Y_h}\|_X \right)^{\frac{1}{2}} \leq C_2 \left( h^{k+2} + \eta K^2 (h^k)^2 \right)^{\frac{1}{2}} \leq C_2 \left( h^{k+2} + \sqrt{n} K h^3 \right) \]  

(55)

and then, from proposition 5, that

\[ \|y - y_h\|_X \leq 2 \left( 1 + \frac{1}{\sqrt{n} \delta_h} \right) C_2 \left( h^{k+2} + \sqrt{n} K h^3 \right) + \frac{1}{\sqrt{n}} C_1 h^{k-1}. \]

Similarly to (55), in view of (52), we have

\[ d(\lambda, \Lambda_h) = \inf_{\lambda_h \in \Lambda_h} \|\lambda - \lambda_h\|_X \leq C_1 h^{k-1}. \]

Applying again proposition 5, we obtain

\[ \|\lambda - \lambda_h\|_X \leq 2 \left( 1 + \frac{1}{\sqrt{n} \delta_h} \right) \frac{1}{\delta_h} C_2 \left( h^{k+2} + \sqrt{n} K h^3 \right) + \frac{3}{\sqrt{n}} C_1 h^{k-1}. \]

The last two estimates imply the proposition. \( \square \)
It remains now to deduce the convergence of the approximated solution $y_h$ for the $L^2(Q_T)$ norm: this is achieved by using the observability estimate (H). Specifically, we write that $(y-y_h)$ solves
\[
\begin{cases}
L(y-y_h) = -Ly_h & \text{in } Q_T \\
(y-y_h), (y-y_h)(0) \in H \\
y - y_h = 0 & \text{on } \Sigma_T.
\end{cases}
\]
Therefore using (6), there exists a constant $C(C_{\Omega,T}, C_{\text{obs}})$ such that
\[
\left\| y - y_h \right\|_{L^2(Q_T)} \leq C(C_{\Omega,T}, C_{\text{obs}}) \left( \left\| y - y_h \right\|_{L^2(Q_T)}^2 + \|Ly_h\|_{H^1}^2 \right)
\]
from which we deduce, in view of the definition of the norm $Z$, that
\[
\left\| y - y_h \right\|_{L^2(Q_T)} \leq C(C_{\Omega,T}, C_{\text{obs}}) \max \left( 1, \frac{2}{\sqrt{\eta}} \right) \left\| y - y_h \right\|_{Z}.
\]
Eventually, by coupling (56) and proposition 6, we obtain the following result:

**Theorem 3** BFS element for $N = 1$—rate of convergence for the norm $L^2(Q_T)$. Assume that the hypothesis (H) holds. Let $h > 0$, let $k \in \{1, 2\}$ be a positive integer. Let $(y, \lambda)$ and $(y_h, \lambda_h)$ be the solutions of (7) and (41), respectively. If the solution $(y, \lambda)$ belongs to $H^{k+2}(Q_T) \times H^k(Q_T)$, then there exist two positive constants $K = K(||y||_{H^{k+2}(Q_T)}$, $||\lambda||_{C^1(Q_T)}$, $||d||_{L^\infty(Q_T)}$, $C_{\Omega,T}$, $C_{\text{obs}}$), independent of $h$, such that
\[
\left\| y - y_h \right\|_{L^2(Q_T)} \leq K \max \left( 1, \frac{2}{\sqrt{\eta}} \right) \left( \left( \frac{h^2}{\sqrt{\eta}} \right) + 1 \right).
\]

**Remark 11.** Estimate (57) is not fully satisfactory as it depends on the constant $\delta_h$. In view of the complexity of both the constraint $Ly = 0$ and of the structure of the space $Z_h$, the theoretical estimation of the constant $\delta_h$ with respect to $h$ is a difficult problem. However, as discussed at length in [12, section 2.1], $\delta_h$ can be evaluated numerically for any $h$ as the solution of the following generalized eigenvalue problem (taking $\eta = \eta_r$, so that $a(y, y)$ is exactly $||y||_{Z_h}^2$):
\[
\delta_h = \inf \left\{ \sqrt{\delta} : B_hA_r^{-1}B_h^T \{ \lambda_h \} = \delta J_h \{ \lambda_h \}, \quad \forall \{ \lambda_h \} \in \mathbb{R}^{m_h} \setminus \{0\} \right\}
\]
where the matrices $A_r, B_h$ and $J_h$ are defined in (47).

Table 1 reports the values of $\delta_h$ for $r = 100, 1, 10^{-2}$, $h^{-2}$ and $r = h^{-1}$ for several values of $h, T = 2$, $\omega = (0.1, 0.3)$ and the BFS element. As in [12] where the boundary situation is considered, these values suggest that, asymptotically in $h$, the constant $\delta_h$ behaves as:
\[
\sqrt{r} \delta_h \approx C_r \quad \text{as } h \to 0^+
\]
with $C_r > 0$, a uniformly bounded constant w.r.t. $h$. Consequently, in view of (59), the right-hand side of the estimate (57) of $||y - y_h||_{L^2(Q_T)}$ behaves, taking $\eta = r$ and $r > 1$ so that $\max(1, \frac{1}{\sqrt{r}}) = 1$, as
and reaches its minimum for \( r = \frac{1}{h} \), leading to \( ||y - y_h||_{L^2(Q_T)} \leq Kh^{k-1/2} \).

Eventually, when the space \( Z_h \) is based on the HCT element, theorem \( 3 \) and remark \( 11 \) still hold for \( k = 1 \). From [9, chapter 7, section 48, p 295], we use that, for \( k \in \{0, 1\} \), there exists a constant \( C_2 > 0 \) such that
\[
||y - y_h||_{L^2(Q_T)} \leq C_2 h^{k+2}, \quad ||y - \Pi_{Z_h, r_h}(y)||_{L^2(Q_T)} \leq C_2 h^k.
\]

Then, we use that the error \( ||y - y_h||_{L^2(Q_T)} \) is again controlled by the error on the Lagrange multiplier \( \lambda \) through the term \( d(\lambda, \Lambda_h) \) in (43) to conclude.

### 4.2. Numerical approximation of the mixed formulation (19)

We address the numerical approximation of the stabilized mixed formulation (19) with \( a \in (0, 1) \) and \( r > 0 \). Let \( h \) be a real parameter. Let \( Z_h \) and \( \tilde{\Lambda}_h \) be two finite dimensional spaces such that
\[
Z_h \subset Z, \quad \tilde{\Lambda}_h \subset \Lambda, \quad \forall h > 0.
\]

The problem (19) becomes: find the \((y_h, \lambda_h) \in Z_h \times \tilde{\Lambda}_h\) solution of
\[
\begin{align*}
\left\{ \begin{array}{ll}
a_{1, a}(y_h, \varphi_h) + b_a(\lambda_h, \varphi_h) &= l_{1, a}(\varphi_h), & \forall \varphi_h \in Z_h \\
b_a(\tilde{\lambda}_h, y_h) - c_a(\lambda_h, \tilde{\lambda}_h) &= l_{2, a}(\tilde{\varphi}_h), & \forall \tilde{\varphi}_h \in \tilde{\Lambda}_h.
\end{array} \right.
\end{align*}
\]

Proceeding as in the proof of [4, theorem 5.5.2], we first easily show that the following estimate holds.

**Lemma 2.** Let \((y, \lambda) \in Y \times \Lambda\) be the solution of (19) and \((y_h, \lambda_h) \in Z_h \times \tilde{\Lambda}_h\) be the solution of (61). Then we have,
with \( \|a_{r,a}\|_{Z \times Z^T} \leq \max(1 - \alpha, \eta^{-1}r) \), \( \|b_{a}\|_{Z \times Z^T} \leq \max(\eta^{-1/2}, \alpha) \). Parameters \( \theta_1 \) and \( \theta_2 \) are defined in (20).

Concerning the space \( \tilde{\Lambda}_h \), since \( L^2_{\alpha} \) should belong to \( L^2(Q_T) \), a natural choice is

\[
\tilde{\Lambda}_h = \{ \lambda \in Z_h; \lambda(\cdot, 0) = \lambda_i(\cdot, 0) = 0 \}.
\]

where \( Z_h \subset Z \) is defined by (49). Then, using lemma 2 and the estimate (55), we obtain the following result.

**Proposition 7** BFS element for \( N = 1 \)—rates of convergence—stabilized mixed formulation. Let \( h > 0 \), let \( k \leq 2 \) be a positive integer and \( \alpha \in (0, 1) \). Let \((y, \lambda)\) and \((y_h, \lambda_h)\) be the solutions of (19) and (61), respectively. If \((y, \lambda)\) belongs to \( H^{k+2}(Q_T) \times H^{k+2}(Q_T) \), then there exists a positive constant \( K = K(\|y\|_{H^{k+2}(Q_T)}, \|\lambda\|_{H^{k+2}(Q_T)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(\overline{Q_T})}, \alpha, r, \eta) \) independent of \( h \), such that

\[
\|y - y_h\|^2 + \|\lambda - \lambda_h\|^2 \leq KH^k. \tag{64}
\]

In particular, arguing as in the previous section, we obtain

**Theorem 4** BFS element for the \( N = 1 \)—rates of convergence for the norm \( L^2(Q_T) \)—stabilized version. Assume that the hypothesis (H) holds. Let \( h > 0 \), let an integer \( k \leq 2 \). Let \((y, \lambda)\) and \((y_h, \lambda_h)\) be the solutions of (19) and (61), respectively. If the solution \((y, \lambda)\) belongs to \( H^{k+2}(Q_T) \times H^{k+2}(Q_T) \), then there exists a positive constant \( K = K(\|y\|_{H^{k+2}(Q_T)}, \|\lambda\|_{H^{k+2}(Q_T)}, \|c\|_{C^1(\overline{Q_T})}, \|d\|_{L^\infty(\overline{Q_T})}, \alpha, r, \eta) \) independent of \( h \) such that

\[
\|y - y_h\|_{L^2(Q_T)} \leq K \frac{h^k}{\sqrt{\eta}}. \tag{65}
\]

5. Numerical experiments

We now report and discuss some numerical experiments corresponding to mixed formulations (41) and (61) for \( N = 1 \) and \( N = 2 \).
5.1. One-dimensional case (N = 1)

We take $\Omega = (0, 1)$. In order to check the convergence of the method, we consider explicit solutions of (1). We define the smooth initial condition (see [8]):

\[
\begin{align*}
\psi(x) & = -\frac{1}{(x^2 + 1)^2}, \\
\phi(x) & = -\frac{1}{(2x - 1)^4}
\end{align*}
\]

and $f = 0$. The corresponding solution of (1) with $\epsilon = 0$ is given by

\[
y(x, t) = \sum_{k > 0} \left( a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sqrt{2} \sin(k\pi x)
\]

with

\[
a_k = \frac{32\sqrt{2}}{\pi^2 k^2} \left( \frac{\pi^2 k^2 - 12}{\pi^2 k^2} \right), \quad b_k = \frac{48\sqrt{2}}{\pi^4 k^4} \sin(\pi k/2), \quad k > 0.
\]

We also define the initial data in $H_0^1(\Omega) \times L^2(\Omega)$

\[
(\text{EX2}) \quad \begin{align*}
\psi_0(x) & = 1 - \left| 2x - 1 \right|, \\
\phi_0(x) & = 1_{(1/3,2/3)}(x), \quad x \in (0, 1)
\end{align*}
\]

for which the Fourier coefficients are

\[
a_k = \frac{4\sqrt{2}}{\pi^2 k^2} \sin(\pi k/2), \quad b_k = \frac{1}{\pi k} (\cos(\pi k/3) - \cos(2\pi k/3)), \quad k > 0.
\]

5.1.1. The cylindrical case: $q_T = \omega \times (0, T)$. We consider the case $\epsilon = 0$ described in section 2. We take $\omega = (0.1, 0.3)$ and $T = 2$ for which the inequality (H) holds true.

We recall that the direct method amounts to solving, for any $h$, the linear system (48). We use exact integration methods developed in [16] for the evaluation of the coefficients of the matrices. Moreover, the linear system is solved using the direct Gauss method.

We first consider the BFS finite element with uniform triangulation (each element $K$ of the triangulation $\mathcal{T}_h$ is a rectangle of lengths $\Delta x$ and $\Delta t$ so that $h = \sqrt{(\Delta x)^2 + (\Delta t)^2}$). Table 2

| $h$ | $\|y - y_h\|_{L^2(\Omega)}$ | $\|y\|_{L^2(\Omega)}$ | $\|y\|_{L^2(\Omega)}$ | $\|\lambda\|_{L^2(\Omega)}$ | $\kappa$ | dim ($\{\lambda_h\}$) | $\#$ CG iterates |
|-----|-----------------|-----------------|-----------------|-----------------|------|-----------------|-----------------|
| 7.01 $\times 10^{-2}$ | 5.95 $\times 10^{-2}$ | 4.58 $\times 10^{-2}$ | 2.24 $\times 10^{-2}$ | 1.10 $\times 10^{-2}$ | 5.52 $\times 10^{-3}$ | 861 | 27 |
| 3.53 $\times 10^{-2}$ | 1.76 $\times 10^{-2}$ | 8.83 $\times 10^{-3}$ | 4.42 $\times 10^{-3}$ | 5.62 $\times 10^{-3}$ | 3.21 $\times 10^{-3}$ | 3321 | 42 |
| 1.76 $\times 10^{-2}$ | 8.83 $\times 10^{-3}$ | 4.42 $\times 10^{-3}$ | 5.62 $\times 10^{-3}$ | 3.21 $\times 10^{-3}$ | 1.37 $\times 10^{-5}$ | 13041 | 70 |
| 8.83 $\times 10^{-3}$ | 4.42 $\times 10^{-3}$ | 5.62 $\times 10^{-3}$ | 3.21 $\times 10^{-3}$ | 1.37 $\times 10^{-5}$ | 6.89 $\times 10^{-6}$ | 51681 | 96 |
| 4.42 $\times 10^{-3}$ | 5.62 $\times 10^{-3}$ | 3.21 $\times 10^{-3}$ | 1.37 $\times 10^{-5}$ | 6.89 $\times 10^{-6}$ | 3.44 $\times 10^{-6}$ | 205761 | 90 |

Table 2. Example EX1—$r = 1, T = 2, \|y\|_{L^2(\Omega)} = 5.95 \times 10^{-2}$.
collects some norms with respect to $h$ for the initial data (EX1) for $r = 1$ and for $\Delta = \Delta t$. We observe a linear convergence for the variables $y_h$, $\lambda_h$ for the $L^2$-norm:

$$\frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} = \mathcal{O}(h^{1.03}), \quad \frac{\|y - y_h\|_{L^2(Q_T)}}{\|y\|_{L^2(Q_T)}} = \mathcal{O}(h^{0.98}), \quad \|\lambda - \lambda_h\|_{L^2(Q_T)} = \mathcal{O}(h^{0.98}).$$

In agreement with remark 1, since $y_{\text{obs}}$ is by construction the restriction to $Q_T$ of a solution of (1), the sequence $\lambda_h$ approximation of $\lambda$ vanishes as $h \to 0$. The $L^2$-norm of $L y_h$ also converges to 0 with $h$, with a lower rate:

$$\|L y_h\|_{L^2(Q_T)} = \mathcal{O}(h^{0.71}).$$

We also check that the minimization of the functional $J^*\star \star$ introduced in proposition 3 leads exactly to the same result: we recall that the minimization of the functional $J^*\star \star$ corresponds to the resolution of the associate mixed formulation by an iterative Uzawa type method. The minimization is performed using a conjugate gradient algorithm (see [12, section 2.2] for the algorithm). Each iteration amounts to solving a linear system involving the matrix $A_{r,h}$, which is sparse, symmetric and positive definite. The Cholesky method is used. The performance of the algorithm depends on the condition number of the operator $\nu\nu \nu_{\nu}$ defined by $\nu\nu \nu_{\nu} = ||\nu(P_\nu)||_{\nu\nu \nu_{\nu}(X')} ||\nu(P_\nu)^{-1}||_{\nu\nu \nu_{\nu}(X')}$: specifically, it is known that (see for instance [15]),

$$||\lambda^n - \lambda\nu\nu\nu_{\nu}||_{\nu\nu \nu_{\nu}} \leq 2 \sqrt{\nu(P_\nu)} \left( \frac{\nu(P_\nu) - 1}{\nu(P_\nu) + 1} \right)^n \nu\nu \nu_{\nu}||\nu\nu\nu_0 - \lambda\nu\nu\nu_{\nu}||_{\nu\nu \nu_{\nu}}, \quad \forall \ n \geq 1$$

where $\lambda$ minimizes $J^*\star \star$. As discussed in [12, section 4.4], the condition number of the operator $P_\nu$ restricted to $L_{\nu} \subset X'$ behaves asymptotically as $C_{\nu}^{-1} h^{-2}$. Table 2 reports the number of iterations of the conjugate gradient (CG) algorithm, initiated with $\lambda^0 = 0$ in $Q_T$. We take $\epsilon = 10^{-10}$ as a stopping threshold for the algorithm (the algorithm is stopped as soon as the norm of the residue $g^n$ given here by $L y^n$ satisfies $||g^n||_X \leq \epsilon ||g^n||_X$).

Table 2 reports the number of iterates to reach convergence, with respect to $h$. We observe that this number is sub-linear with respect to $h$, specifically, with respect to the dimension $m_h = \dim\{\lambda_h\}$ of the approximated problems. This renders this method very attractive from a numerical point of view.

From remark 6, we also check the convergence w.r.t. $h$ when we assume from the beginning that the multiplier $\lambda$ vanishes (see table 3). This amounts to minimizing the functional $J^\prime$, given by (17) or, equivalently, to performing exactly one iteration of the conjugate gradient algorithm we have just discussed. With $r = 1$, we observe a weaker

| $h$ | $7.01 \times 10^{-2}$ | $3.53 \times 10^{-2}$ | $1.76 \times 10^{-2}$ | $8.83 \times 10^{-3}$ | $4.42 \times 10^{-3}$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\|y - y_h\|_{L^2(Q_T)}$ | $9.74 \times 10^{-2}$ | $4.90 \times 10^{-2}$ | $2.84 \times 10^{-2}$ | $2.16 \times 10^{-2}$ | $2.01 \times 10^{-2}$ |
| $\|y - y_h\|_{L^2(Q_T)}$ | $8.35 \times 10^{-2}$ | $4.28 \times 10^{-2}$ | $2.18 \times 10^{-2}$ | $1.12 \times 10^{-2}$ | $6.21 \times 10^{-3}$ |
| $\|L y_h\|_{L^2(Q_T)}$ | $7.72 \times 10^{-3}$ | $1.11 \times 10^{-2}$ | $2.01 \times 10^{-2}$ | $3.40 \times 10^{-2}$ | $4.79 \times 10^{-2}$ |
to 0 in the norm
\[
\|y_h\|_{L^2(Q_T)} = 1.12 \times 10^{-1}
\]

\[
\|y - y_h\|_{L^2(Q_T)} = 1.21 \times 10^{-1} \quad 1.08 \times 10^{-1} \quad 1.34 \times 10^{-1} \quad 2.42 \times 10^{-1} \quad 5.19 \times 10^{-1}
\]

\[
\|y\|_{L^2(Q_T)} = 8.40 \times 10^{-2} \quad 4.34 \times 10^{-2} \quad 2.22 \times 10^{-2} \quad 1.12 \times 10^{-2} \quad 5.62 \times 10^{-3}
\]

\[
\|\lambda_h\|_{L^2(Q_T)} = 5.62 \times 10^{-2} \quad 2.77 \times 10^{-2} \quad 2.63 \times 10^{-2} \quad 2.25 \times 10^{-2} \quad 2.15 \times 10^{-2}
\]

\[
\|\lambda\|_{L^2(Q_T)} = 1.84 \times 10^{-5} \quad 9.48 \times 10^{-6} \quad 4.76 \times 10^{-6} \quad 2.38 \times 10^{-6} \quad 1.19 \times 10^{-6}
\]

\[
\kappa = 1.2 \times 10^{11} \quad 9.8 \times 10^{12} \quad 1.1 \times 10^{15} \quad 1.5 \times 10^{17} \quad 2.7 \times 10^{19}
\]

| h | 7.01 \times 10^{-2} | 3.53 \times 10^{-2} | 1.76 \times 10^{-2} | 8.83 \times 10^{-3} | 4.42 \times 10^{-3} |
| --- | --- | --- | --- | --- | --- |
| ||y - y_h||_{L^2(Q_T)} ||y||_{L^2(Q_T)} ||l||_{L^2(Q_T)} ||L\lambda_h||_{L^2(Q_T)} ||\lambda||_{L^2(Q_T)} # CG iterates | | | | | |
| 29 | 46 | 83 | 133 | 201 |

\[
\frac{||y - y_h||_{L^2(Q_T)}}{||y||_{L^2(Q_T)}} = \mathcal{O}(h^{0.574}), \quad \frac{||y - y_h||_{L^2(Q_T)}}{||y||_{L^2(Q_T)}} = \mathcal{O}(h^{0.94}).
\]

This example illustrates that the convergence of \(L\lambda_h\) to 0 in the norm \(L^2(0, T, H^{-1}(0, 1))\) is enough here to guarantee the convergence of the approximation \(y_h\): we obtain that 

\[
\|L\lambda_h\|_{L^2(0, T, H^{-1}(0, 1))} = \mathcal{O}(h^{1.5}) \quad \text{while} \quad \|L\lambda_h\|_{L^2(Q_T)} \quad \text{slightly increases}
\]

Obviously, in this specific situation, a larger \(r\) (acting as a penalty term) independent of \(h\) yields a lower \(\|L\lambda_h\|_{L^2(Q_T)}\) norm.

In contrast, we check that the convergence to 0 of \(\|y - y_h\|_{L^2(Q_T)}\) is lost when the inequality (H) is not satisfied: table 4 collects the norms w.r.t. \(h\) for the same data except the value \(T = 1\) (for which the uniqueness of the solution is lost). We observe that \(||y - y_h||_{L^2(Q_T)}\) increases as \(h \to 0\). As an illustration of the loss of uniqueness (as \(h\) tends to 0), these values also yield to a larger condition number \(\kappa\) of the matrix \(A_{r,h}\).

Similar conclusions hold with the less regular initial data (EX2). Numerical results are reported in table 5. We still observe a linear convergence w.r.t. \(h\) of \(||y - y_h||_{L^2(Q_T)}\), \(||y - y_h||_{L^2(Q_T)}\) and \(||\lambda_h||_{L^2(Q_T)}\). One notable difference is that the convergence rate is weaker for the norm \(||L\lambda_h||_{L^2(Q_T)}\).
Table 6. Example EX2—r = 1, T = 1, \( \| y \|_{L^2(\Omega_T)} = 1.104 \times 10^{-1} \).

| h | \( \| y - y_h \|_{L^2(\Omega_T)} \) | \( \| y \|_{L^2(\Omega_T)} \) | \( \| y \|_{L^2(\Omega_T)} \) | \( \| y - y_h \|_{L^2(\Omega_T)} \) |
|---|---|---|---|---|
|    | 7.01 \times 10^{-2} | 3.53 \times 10^{-2} | 1.76 \times 10^{-2} | 8.83 \times 10^{-3} |
| 2.93 \times 10^{-1} | 2.74 \times 10^{-1} | 4.15 \times 10^{-1} | 6.30 \times 10^{-1} | 1.21 |
| 7.62 \times 10^{-2} | 1.37 \times 10^{-2} | 5.76 \times 10^{-2} | 2.89 \times 10^{-2} | 2.41 \times 10^{-2} |
| 4.21 \times 10^{-2} | 5.97 \times 10^{-2} | 4.96 \times 10^{-2} | 4.96 \times 10^{-2} | 4.52 \times 10^{-2} |
| 2.87 \times 10^{-3} | 4.97 \times 10^{-3} | 2.32 \times 10^{-3} | 1.15 \times 10^{-5} | 5.76 \times 10^{-5} |

Table 7. Example EX2—r = 1, T = 2, \( \lambda \) fixed to zero.

| h | \( \| y - y_h \|_{L^2(\Omega_T)} \) | \( \| y \|_{L^2(\Omega_T)} \) | \( \| y \|_{L^2(\Omega_T)} \) | \( \| y - y_h \|_{L^2(\Omega_T)} \) |
|---|---|---|---|---|
|    | 7.01 \times 10^{-2} | 3.53 \times 10^{-2} | 1.76 \times 10^{-2} | 8.83 \times 10^{-3} |
| 2.93 \times 10^{-1} | 1.02 \times 10^{-1} | 5.27 \times 10^{-2} | 3.18 \times 10^{-2} | 2.48 \times 10^{-2} |
| 7.62 \times 10^{-2} | 1.34 \times 10^{-1} | 5.06 \times 10^{-2} | 2.37 \times 10^{-2} | 1.21 \times 10^{-2} |
| 6.65 \times 10^{-3} | 7.43 \times 10^{-2} | 7.43 \times 10^{-2} | 8.65 \times 10^{-2} | 1.10 \times 10^{-1} |
| 1.37 \times 10^{-2} | 7.43 \times 10^{-2} | 7.43 \times 10^{-2} | 8.65 \times 10^{-2} | 1.10 \times 10^{-1} |

Table 8. Example EX2—r = 1, T = 2, \( \alpha = 1/2 \), \( \| y \|_{L^2(\Omega_T)} = 5.95 \times 10^{-2} \), \( \| y \|_{L^2(\Omega_T)} = 1.59 \times 10^{-1} \).

| h | \( \| y - y_h \|_{L^2(\Omega_T)} \) | \( \| y \|_{L^2(\Omega_T)} \) | \( \| y \|_{L^2(\Omega_T)} \) | \( \| y - y_h \|_{L^2(\Omega_T)} \) |
|---|---|---|---|---|
|    | 7.01 \times 10^{-2} | 3.53 \times 10^{-2} | 1.76 \times 10^{-2} | 8.83 \times 10^{-3} |
| 2.93 \times 10^{-1} | 8.48 \times 10^{-2} | 4.01 \times 10^{-2} | 1.85 \times 10^{-2} | 8.66 \times 10^{-3} |
| 7.62 \times 10^{-2} | 2.80 \times 10^{-1} | 7.26 \times 10^{-2} | 2.61 \times 10^{-2} | 1.12 \times 10^{-2} |
| 5.05 \times 10^{-3} | 7.25 \times 10^{-2} | 6.59 \times 10^{-2} | 6.16 \times 10^{-2} | 5.58 \times 10^{-2} |
| 5.08 \times 10^{-2} | 4.11 \times 10^{-3} | 2.04 \times 10^{-3} | 1.49 \times 10^{-3} | 1.01 \times 10^{-3} |
| 7.37 \times 10^{-4} | \( \| L y_h \|_{L^2(\Omega_T)} = O(h^{0.125}) \). (67) |

Again, this is enough to guarantee the convergence of \( y_h \) toward a solution of the wave equation: recall that then \( \| L y_h \|_{L^2(\Omega_T)} = O(h^{0.125}) \). We also observe that the number of iterates in the CG algorithm remains largely sub-linear but is slightly larger: specifically, we have \( h \) CG iterates = \( O(h^{-0.71}) \). Table 6 illustrates the case \( T = 1 \) while table 7 illustrates the minimization of \( J_r \) (see 17), both for \( r = 1 \).

We end this section with some numerical results for the stabilized mixed formulation (61). The main difference is that the multiplier \( \lambda \) is approximated in a much richer space \( L^H \) (see 63) leading to larger linear system. Table 8 considers the case of the example EX2 with \( T = 2 \) and \( \alpha = 1/2 \). For comparison with the formulation (41), we take again \( r = 1 \). We
observe the convergence w.r.t $h$ and obtain slightly better rates and constants than in Table 5: in particular, we have $||y - y_h||_{L^2(Q_T)} \leq \mathcal{O}(h^{1.10})$. This is partially due to the fact that the space $\tilde{A}_h$ used for the variable $\lambda_h$ in (61) is richer than the space $A_h$ used in (41). Finally, we also check that—in contrast with the mixed formulation (41)—the positive parameter $r$ does not affect the numerical results.

We also emphasize that this variational method which requires a finite element discretization of the time-space $Q_T$ is particularly well-adapted to mesh optimization. Still for the example EX2, Figure 1 depicts a sequence of four distinct meshes of $Q_T = (0, 1) \times (0, T)$: the sequence is initiated with a coarse and regularly distributed mesh. The three other meshes are successively obtained by local refinement based on the norm of the gradient of $y_h$ on each triangle of $T_h$. As expected, the refinement is concentrated around the lines of singularity of $y_h$ traveling in $Q_T$, generated by the singularity of the initial position $y_0$. 

![Figure 1. Iterative refinement of the triangular mesh over $Q_T$ with respect to the variable $y_h$.](image)
The four meshes contain 792, 2108, 7902 and 14717 triangles, respectively (see table 9). The results obtained using the reduced HCT finite element method are reported in table 9.

| Mesh number | 1  | 2  | 3  | 4  |
|-------------|----|----|----|----|
| # elements  | 792| 2108| 7902| 14717 |
| # points    | 429| 1101| 4041| 7462 |
| $\|y - y_h\|_{L^2(\Omega_T)}$ | $1.34 \times 10^{-2}$ | $8.69 \times 10^{-3}$ | $6.01 \times 10^{-3}$ | $5.9 \times 10^{-3}$ |
| $\|\lambda_h\|_{L^2(\Omega_T)}$ | $1.14 \times 10^{-3}$ | $7.99 \times 10^{-6}$ | $5.02 \times 10^{-6}$ | $4.79 \times 10^{-6}$ |

Figure 2. Domain $q_T^1$ (a) and domain $q_T^2$ (b) triangulated using some coarse meshes.

The results obtained using the reduced HCT finite element method are reported in table 9.

5.1.2. The non-cylindrical case. We numerically illustrate the reconstruction of the state of the wave equation (1) from measurements $y_{obs}$ which are available in domains $q_T \subset Q_T$ non-constant in time (considered recently in [7] in a controllability context). Time dependent domains also appear for time under sampled observations (or measurements), see [11]. In the following we take $T = 2$ and $q_T$ to be one of the two following domains:
These two pairs $(T, q^i_T)$ $i = 1, 2$ satisfy the geometric optic condition, so that, using [7, proposition 2.1], inequality (73) holds true. Both domains $q^1_T$ and $q^2_T$ are displayed in figure 2 with the coarsest of the meshes used for the numerical experiments in this section.
We consider five levels of regular triangular meshes and use the reduced HCT finite element. We illustrate our method on the reconstruction of the solution of the wave equation corresponding to the initial data (EX2) considered in section 5.1.1.

Since domains $q^1_T$ and $q^2_T$ satisfy the geometric optic condition, we obtain similar results as in the case $q_T = \omega \times (0, T)$ studied in the previous section. More precisely, these results are reported in tables 10 and 11 for domains $q^1_T$ and $q^2_T$, respectively.

Note that the number of iterations needed for the conjugate gradient algorithm in order to achieve a residual smaller than $10^{-10}$ when we minimize the functional $J^*$ over $\Lambda_h$ is slightly larger than in the situations described in the previous section.

The exact solution $y$ corresponding to initial data (EX2) is displayed in figure 3(a) using the third mesh of the domain in figure 2(b). Figure 3(b) illustrates the solution $y_h$ of the mixed formulation (41), where the observation $y_{\text{obs}}$ is obtained as the restriction of $y$ to $q^2_T$.

| $h$ | $\|y - y_h\|_{L^2(q_T)}$ | $\|y\|_{L^2(q_T)}$ | $\|y - y_h\|_{L^2(q_T)}$ | $\|y\|_{L^2(q_T)}$ |
|-----|----------------------|----------------------|----------------------|----------------------|
| 1   | $1.38 \times 10^{-2}$ | $6.37 \times 10^{-3}$ | $2.64 \times 10^{-3}$ | $1.15 \times 10^{-3}$ |
| 2   | $1.27 \times 10^{-3}$ | $4.79 \times 10^{-3}$ | $2.02 \times 10^{-3}$ | $9.11 \times 10^{-4}$ |
| 3   | $3.86$                | $3.45$                | $3.36$                | $3.85$                |
| 4   | $6.37 \times 10^{-6}$ | $1.65 \times 10^{-6}$ | $3.88 \times 10^{-7}$ | $9.74 \times 10^{-8}$ |
| 5   | $2.02 \times 10^8$   | $2.62 \times 10^9$   | $2.05 \times 10^{10}$| $1.61 \times 10^{11}$|
| 6   | $554$                | $2135$                | $8381$                | $33209$               |
| 7   | $141$                | $331$                 | $720$                 | $1446$                |
| 8   | $3318$               | $3318$                | $3318$                | $3318$                |

Figure 4. (a) Example of sets $\Omega$ and $\omega$. (b) Example of mesh for $\Omega = (0, 1)^2$ and $T = 2$. 

We consider five levels of regular triangular meshes and use the reduced HCT finite element. We illustrate our method on the reconstruction of the solution of the wave equation corresponding to the initial data (EX2) considered in section 5.1.1.

Since domains $q^1_T$ and $q^2_T$ satisfy the geometric optic condition, we obtain similar results as in the case $q_T = \omega \times (0, T)$ studied in the previous section. More precisely, these results are reported in tables 10 and 11 for domains $q^1_T$ and $q^2_T$, respectively.

Note that the number of iterations needed for the conjugate gradient algorithm in order to achieve a residual smaller than $10^{-10}$ when we minimize the functional $J^*$ over $\Lambda_h$ is slightly larger than in the situations described in the previous section.

The exact solution $y$ corresponding to initial data (EX2) is displayed in figure 3(a) using the third mesh of the domain in figure 2(b). Figure 3(b) illustrates the solution $y_h$ of the mixed formulation (41), where the observation $y_{\text{obs}}$ is obtained as the restriction of $y$ to $q^2_T$. 

Table 11. Observation domain $q^2_T$. Example $\text{EX2} - r = 1, T = 2$, $\|y\|_{L^2(q_T)} = 2.75 \times 10^{-1}$, $\|y\|_{L^2(q_T)} = 5.87 \times 10^{-1}$. 

| $\|y - y_h\|_{L^2(q_T)}$ | $\|y\|_{L^2(q_T)}$ | $\|y - y_h\|_{L^2(q_T)}$ | $\|y\|_{L^2(q_T)}$ |
|----------------------|----------------------|----------------------|----------------------|
| $6.24 \times 10^{-2}$ | $3.12 \times 10^{-2}$ | $1.56 \times 10^{-2}$ | $7.8 \times 10^{-3}$ |
| $3.9 \times 10^{-3}$  | $3.9 \times 10^{-3}$  | $3.9 \times 10^{-3}$  | $3.9 \times 10^{-3}$  |

Figure 4. (a) Example of sets $\Omega$ and $\omega$. (b) Example of mesh for $\Omega = (0, 1)^2$ and $T = 2$. 

We consider five levels of regular triangular meshes and use the reduced HCT finite element. We illustrate our method on the reconstruction of the solution of the wave equation corresponding to the initial data (EX2) considered in section 5.1.1.

Since domains $q^1_T$ and $q^2_T$ satisfy the geometric optic condition, we obtain similar results as in the case $q_T = \omega \times (0, T)$ studied in the previous section. More precisely, these results are reported in tables 10 and 11 for domains $q^1_T$ and $q^2_T$, respectively.

Note that the number of iterations needed for the conjugate gradient algorithm in order to achieve a residual smaller than $10^{-10}$ when we minimize the functional $J^*$ over $\Lambda_h$ is slightly larger than in the situations described in the previous section.

The exact solution $y$ corresponding to initial data (EX2) is displayed in figure 3(a) using the third mesh of the domain in figure 2(b). Figure 3(b) illustrates the solution $y_h$ of the mixed formulation (41), where the observation $y_{\text{obs}}$ is obtained as the restriction of $y$ to $q^2_T$. 

Table 11. Observation domain $q^2_T$. Example $\text{EX2} - r = 1, T = 2$, $\|y\|_{L^2(q_T)} = 2.75 \times 10^{-1}$, $\|y\|_{L^2(q_T)} = 5.87 \times 10^{-1}$. 

| $h$ | $\|y - y_h\|_{L^2(q_T)}$ | $\|y\|_{L^2(q_T)}$ | $\|y - y_h\|_{L^2(q_T)}$ | $\|y\|_{L^2(q_T)}$ |
|-----|----------------------|----------------------|----------------------|----------------------|
| 1   | $1.38 \times 10^{-2}$ | $6.37 \times 10^{-3}$ | $2.64 \times 10^{-3}$ | $1.15 \times 10^{-3}$ |
| 2   | $1.27 \times 10^{-3}$ | $4.79 \times 10^{-3}$ | $2.02 \times 10^{-3}$ | $9.11 \times 10^{-4}$ |
| 3   | $3.86$                | $3.45$                | $3.36$                | $3.85$                |
| 4   | $6.37 \times 10^{-6}$ | $1.65 \times 10^{-6}$ | $3.88 \times 10^{-7}$ | $9.74 \times 10^{-8}$ |
| 5   | $2.02 \times 10^8$   | $2.62 \times 10^9$   | $2.05 \times 10^{10}$| $1.61 \times 10^{11}$|
| 6   | $554$                | $2135$                | $8381$                | $33209$               |
| 7   | $141$                | $331$                 | $720$                 | $1446$                |
| 8   | $3318$               | $3318$                | $3318$                | $3318$                |

Figure 4. (a) Example of sets $\Omega$ and $\omega$. (b) Example of mesh for $\Omega = (0, 1)^2$ and $T = 2$. 

We consider five levels of regular triangular meshes and use the reduced HCT finite element. We illustrate our method on the reconstruction of the solution of the wave equation corresponding to the initial data (EX2) considered in section 5.1.1.

Since domains $q^1_T$ and $q^2_T$ satisfy the geometric optic condition, we obtain similar results as in the case $q_T = \omega \times (0, T)$ studied in the previous section. More precisely, these results are reported in tables 10 and 11 for domains $q^1_T$ and $q^2_T$, respectively.

Note that the number of iterations needed for the conjugate gradient algorithm in order to achieve a residual smaller than $10^{-10}$ when we minimize the functional $J^*$ over $\Lambda_h$ is slightly larger than in the situations described in the previous section.

The exact solution $y$ corresponding to initial data (EX2) is displayed in figure 3(a) using the third mesh of the domain in figure 2(b). Figure 3(b) illustrates the solution $y_h$ of the mixed formulation (41), where the observation $y_{\text{obs}}$ is obtained as the restriction of $y$ to $q^2_T$.
5.2. Two-dimensional space case \((N = 2)\)

We now illustrate the method introduced in section 2 in the two-dimensional case. The procedure is similar but a bit more involved from a computational point of view, since \(Q_T\) is now a subset of \(\mathbb{R}^3\).

In order to approximate the mixed formulation (7), we consider a mesh \(T_h\) of the domain \(Q_T = \Omega \times (0, T)\) formed by triangular prisms. This mesh is obtained by extrapolating along the time axis a triangulation of the spatial domain \(\Omega\). See figure 4(b) for an example in the case \(\Omega = (0, 1)^2\) and figure 5(b) for an example in the case of non-rectangular domains \(\Omega\). For both examples, the extrapolation along the the time axis is uniform: the height of the prismatic elements \(\Delta t\) is constant.

Let \(Z_h\) be the finite dimensional space defined as follows

\[
Z_h = \left\{ \begin{array}{l}
\varphi_h = \psi(x_1, x_2) \theta(t) \in C^1(Q_T) \\
q_h = 0 \ 	ext{on} \ \Sigma_T \\
\psi \big|_{K_{x_1x_2}} \in \mathcal{P}(K_{x_1x_2}), \ \theta \big|_{K_t} \in \mathcal{P}(K_t)
\end{array} \right. \\
\text{for every} \ K = K_{x_1x_2} \times K_t \in T_h.
\]

\(\mathcal{P}(K_{x_1x_2})\) is the space of functions corresponding to the reduced HCT \(C^1\)-element (section 4.1.1); \(Q(K_t)\) is a space of degree three polynomials on the interval \(K_t\) of the form \([t_j, t_{j+1}]\) defined uniquely by their value and the value of their first derivative at the point \(t_j\) and \(t_{j+1}\). Therefore, \(Z_h\) is the finite element space obtained as a tensorial product between the reduced HCT finite element and cubic Hermite finite element. We check that on each element \(K = K_{x_1x_2} \times K_t\), the function \(q_h\) is determined uniquely in terms of the values of \(\Sigma_K := \{\varphi(a_i), \psi(a_i), q_{x_1}(a_i), q_{x_2}(a_i), q_t(a_i), q_{x_1x_2}(a_i), q_{x_2x_2}(a_i) \mid i = 1, \cdots, 6\}\) at the six nodes \(a_i\) of \(K\). Therefore, \(\dim \Sigma_K = 36\).
Similarly, let $\Lambda_h$ be the finite dimensional space defined by

$$\Lambda_h = \left\{ \begin{array}{l}
\varphi_h = \varphi(x_1, x_2) \theta(t) \in C^0(\Omega_T) \\
\varphi_h = 0 \text{ on } \Sigma_T \\
\varphi_h \mid_{K_{x_1 x_2}} \in P_1(K_{x_1 x_2}), \theta \mid_{K_t} \in Q_1(K_t)
\end{array} \right. \quad \text{for every } K = K_{x_1 x_2} \times K_t \in T_h,$$

(71)

where $P_1(K_{x_1 x_2})$ and $Q_1(K_t)$ are the spaces of degree one polynomials on the triangle $K_{x_1 x_2}$ and interval $K_t$, respectively.

For any $h$, we check that $Z_h \subset Z$ and that $\Lambda_h \subset \Lambda$.

5.2.1. Wave equation in a square. We first consider the case $\Omega$ defined by the unit square and again some explicit solutions used in [8]. Specifically, we define the following smooth initial condition:

$$\begin{align}
(\text{EX1-2D}) \quad & y_0(x_1, x_2) = 256x_1^2x_2^2(1 - x_1)^2(1 - x_2)^2 \\
& y_1(x_1, x_2) = (1 - [2x_1 - 1])(1 - [2x_2 - 1]) \\
& (x_1, x_2) \in \Omega. 
\end{align}$$

(72)

The corresponding solution of (1) with $c \equiv 1$, $d \equiv 0$ and $f \equiv 0$ is given by:

$$y(x_1, x_2, t) = \sum_{k,l \geq 0} \left( a_{kl} \cos(\mu_{kl} t) + \frac{b_{kl}}{\mu_{kl}} \sin(\mu_{kl} t) \right) \sin(k\pi x) \sin(l\pi y),$$

(73)

where $\mu_{kl} = \pi \sqrt{k^2 + l^2}$ for every $k, l \in \mathbb{Z}^+$ and

$$a_{kl} = 2^{10} \frac{\pi^2 k^2 - 12}{\pi^{10} k^5 l^5} \left( (-1)^k - 1 \right) \left( (-1)^l - 1 \right),$$

$$b_{kl} = 2^5 \frac{\sin \frac{\pi k}{2} \sin \frac{\pi l}{2}}{\pi^4 k^2 l^2}.$$
We also define the following initial data \((y_0, y_1) \in H^1_0(\Omega) \times L^2(\Omega)\):

\[
\text{(EX2-2D)} \quad \begin{cases}
y_0(x_1, x_2) = (1 - |2x_1 - 1|)(1 - |2x_2 - 1|) \\
y_1(x_1, x_2) = I_{\left(\frac{1}{3}, \frac{2}{3}\right)}(x_1, x_2)
\end{cases} \quad (x_1, x_2) \in \Omega.
\]

(74)

The Fourier coefficients of the corresponding solution are for all \(k, l \in \mathbb{Z}^n\):

\[
a_{kl} = \frac{2^5}{\pi^4k^2l^2} \sin \frac{\pi k}{2} \sin \frac{\pi l}{2}
\]

\[
b_{kl} = \frac{1}{\pi^2kl} \left( \cos \frac{\pi k}{3} - \cos \frac{2\pi k}{3} \right) \left( \cos \frac{\pi l}{3} - \cos \frac{2\pi l}{3} \right).
\]

In what follows, we consider \(\omega\) the subset of \(\Omega\) described in figure 4 (a) and given by:

\[
\omega = ((0, 0.2) \times (0, 1)) \cup ((0, 1) \times (0, 0.2))
\]

(75)

and take \(T = 2\). We check that the corresponding domain \(q_T = \omega \times (0, T)\) satisfies the geometric optic condition and, hence, inequality (H) holds. We consider three levels of meshes of \(Q_T\), labeled from 1 to 3 and containing the number of elements (prisms) and nodes listed in table 12.
For each of these meshes we solve the mixed formulation (41) with the term \( y_{\text{obs}} \) appearing in the right-hand side obtained as the restriction to \( q_T \) of the solution computed by (73) for initial data \( \text{EX1-2D} \) and \( \text{EX2-2D} \).

Table 13 concerns the example \( \text{EX1-2D} \). In this table we list the norm of the relative error between the exact solution \( y \) given by (73) and the solution \( y_h \) of the mixed formulation (7), the \( L^2 \)-norm of \( Ly_h \) and the \( L^2 \)-norm of the Lagrange multiplier \( \lambda_h \).

As theoretically stated in remark 1 and observed in numerical experiments in the case \( N = 1 \) (see, for instance, table 13), the Lagrange multiplier \( \lambda_h \) vanishes as \( h \to 0 \). In table 14 we display the results obtained by numerically solving the variational problem (7) obtained from the mixed formulation when \( \lambda_h = 0 \).

Tables 15 and 16 display the results obtained for the initial data specified by \( \text{EX2-2D} \), for the solutions \( (y_h, \lambda_h) \) of the mixed formulation and for the variational problem obtained when \( \lambda_h = 0 \), respectively.

The results are similar for both examples. In both cases we observe a linear convergence of \( y_h \) to \( y \) in the norm \( L^2 \) over \( Q_T \) when \( h \) goes to zero. Similarly, the norm \( \| \lambda_h \|_{L^2(Q_T)} \) linearly decreases as \( h \) goes to zero.

5.2.2. Wave equation in a non-rectangular domain of \( \mathbb{R}^2 \). Let \( \Omega \subset \mathbb{R}^2 \) be a domain with a regular boundary and \( \omega \) a non-empty subset with a regular boundary. An example of such a configuration is illustrated in figure 5(a). As in the previous section, we take \( T = 2 \) and we build a mesh formed by triangular prisms of the domain \( Q_T = \Omega \times (0, T) \). An example of such a mesh associated to the domain \( \Omega \) is displayed in figure 5(b). This mesh is composed on 17934 nodes distributed in 32140 prismatic elements (this mesh corresponds to mesh number 2 described in table 17).

![Figure 6. (a) Initial data \( y_0 \) given by (76). (b) Reconstructed initial data \( y_h(\cdot,0) \).](image-url)

### Table 17. Characteristics of the three meshes associated with \( Q_T \).

| Mesh number | 1    | 2    | 3    |
|-------------|------|------|------|
| Number of elements | 5730 | 32 1400 | 130 280 |
| Number of nodes    | 3432 | 17 934 | 69 864 |
| Height of elements \( (\Delta t) \) | 0.2  | 0.1  | 0.05 |
We consider three levels of meshes of the domain $QT$ formed by the number of prisms and containing the number of nodes reported in table 17. Comparing to the situation described in section 5.2.1, the eigenfunctions and eigenvectors of the Dirichlet Laplace operator defined on $\Omega$ are not explicitly available here. Consequently, from a given set of initial data, we first numerically solve the wave equation (1) using a standard time-marching method, from which we then extract an observation on $q_T$. Specifically, we use a $P_1$ finite element method in space coupled with a Newmark unconditionally stable scheme for the time discretization. Hence, we solve the wave equation on the same mesh which was extrapolated in time in order to obtain mesh number 2 of $QT$. This two-dimensional mesh contains 1704 nodes and 3257 triangles. The time discretization step is \( \Delta t = 10^{-2} \). We denote by $\hat{y}_h$ the solution obtained in this way for the initial data

$$\begin{cases}
-\Delta y_0 = 10, & \text{in } \Omega \\
y_0 = 0, & \text{on } \partial \Omega,
\end{cases} (76)$$

From $\hat{y}_h$, we generate the observation $y_{\text{obs}}$ as the restriction of $\hat{y}_h$ to $q_T$. Finally, from this observation we reconstruct $y_h$ as the solution of the mixed formulation (30) on each of the three meshes described in table 17. Table 18 display some norms of $y_h$ and $\lambda_h$ obtained for the three meshes and illustrates again the convergence of the reconstructed approximation.

| Mesh number | 1          | 2          | 3          |
|-------------|------------|------------|------------|
| $\|\hat{y}_h - y_h\|_{L^2(Q_T)}$ | $1.88 \times 10^{-1}$ | $8.04 \times 10^{-2}$ | $7.11 \times 10^{-2}$ |
| $\|y_h\|_{L^2(Q_T)}$ | 3.21 | 2.01 | 1.57 |
| $\|\lambda_h\|_{L^2(Q_T)}$ | $8.26 \times 10^{-5}$ | $3.62 \times 10^{-5}$ | $2.84 \times 10^{-5}$ |
| \# CG iterates | 52 | 167 | 400 |

We consider three levels of meshes of the domain $QT$ formed by the number of prisms and containing the number of nodes reported in table 17. Comparing to the situation described in section 5.2.1, the eigenfunctions and eigenvectors of the Dirichlet Laplace operator defined on $\Omega$ are not explicitly available here. Consequently, from a given set of initial data, we first numerically solve the wave equation (1) using a standard time-marching method, from which we then extract an observation on $q_T$. Specifically, we use a $P_1$ finite element method in space coupled with a Newmark unconditionally stable scheme for the time discretization. Hence, we solve the wave equation on the same mesh which was extrapolated in time in order to obtain mesh number 2 of $QT$. This two-dimensional mesh contains 1704 nodes and 3257 triangles. The time discretization step is \( \Delta t = 10^{-2} \). We denote by $\hat{y}_h$ the solution obtained in this way for the initial data

$$\begin{cases}
-\Delta y_0 = 10, & \text{in } \Omega \\
y_0 = 0, & \text{on } \partial \Omega,
\end{cases} (76)$$

From $\hat{y}_h$, we generate the observation $y_{\text{obs}}$ as the restriction of $\hat{y}_h$ to $q_T$. Finally, from this observation we reconstruct $y_h$ as the solution of the mixed formulation (30) on each of the three meshes described in table 17. Table 18 display some norms of $y_h$ and $\lambda_h$ obtained for the three meshes and illustrates again the convergence of the reconstructed approximation.

Figure 6(a) displays the solution $y_0$ of (76) and figure 6(b) displays the initial position $y_h(\cdot, 0)$ corresponding to the solution of our inverse problem. The error between these two functions is given by $\|y_0 - y_h(\cdot, 0)\|_{L^2(\Omega)} = 2.05 \times 10^{-2}$ which is consistent with the results reported in table 18.

6. Concluding remarks and perspectives

The mixed formulations we have introduced here in order to address inverse problems for the linear hyperbolic type equation seem original. These formulations are nothing other than the Euler systems associated to least-squares type functionals and depend on both the state to reconstruct and a Lagrange multiplier. This multiplier is introduced to take into account the state constraint $L_y - f = 0$ and turns out to be the controlled solution of a hyperbolic equation with the source term $(y - y_{\text{obs}}) \mathbf{1}_{q_T}$. This approach, recently used in a controllability context in [12], leads to a variational problem defined over time-space functional Hilbert spaces, without distinction between the time and space variables. The main ingredient that allows one to prove the well-posedness of the mixed formulations and
therefore the reconstruction of the solutions, is a generalized observability inequality, assuming here some geometric conditions on the observation zone.

At the practical level, the discrete mixed time-space formulation is solved in a systematic way in the framework of the finite element method. The approximation is conformal allowing one to obtain the strong convergence of the approximation as the discretization parameters tend to zero. In particular, we emphasize that there is no need, contrary to the classical approach, to prove some uniform discrete observability inequality: we simply use the observability equality on the finite dimensional discrete space. The resolution amounts to solving a sparse symmetric linear system: the corresponding matrix can be preconditioned if necessary and may be computed once for all as it does not depend on the observation $y_{obs}$.

Eventually, the space-time discretization of the domain allows an adaptation of the mesh so as to reduce the computational cost and capture the main features of the solutions. Similarly, this space-time formulation is very appropriate for the non-cylindrical situation.

In agreement with the theoretical convergence, the numerical experiments reported here display very good behavior and robustness of the approach: the reconstructed approximate solution converges strongly to the solution of the hyperbolic equation associated to the available observation. Note that from the continuous dependence of the solution with respect to the observation, the method is robust with respect to the possible noise on the data.

As mentioned in section 3, an additional assumption on the source term allows one to determine uniquely the pair $(y, f)$ from a partial measurement on $q_T$ or on a sufficiently large part $\Sigma_T$ of the boundary. For instance, from [26, theorem 2.1], assuming that the source term takes the form $f(x, t) = \sigma(t)\mu(x)$ with $\sigma \in C^1([0, T])$, $\sigma(0) \neq 0$ and $\mu \in H^{-1}(\Omega_T)$, then the following holds: there exists a positive constant $C$ such that

$$\|\mu\|_{H^{-1}(\Omega_T)}^2 \leq C \left( \left\| \frac{\partial y}{\partial \nu} \right\|_{L^2(\Sigma_T)}^2 + \|Ly - \sigma(t)\mu(x)\|_{L^2(Q_T)}^2 \right), \quad \forall (y, \mu) \in S$$

(77)

where $y$ solves (1) with $(y_0, y_1) \equiv 0$, $c = 1$ and $(\Sigma_T, T, Q_T)$ satisfies a geometric condition and $S$ denotes an appropriate functional space. Using this inequality (similar to $H$), we can study the mixed formulation associated to the Lagrangian from $S \times L^2(Q_T) \to \mathbb{R}$ defined by

$$\mathcal{L}(y, \mu, \lambda) = \frac{1}{2} \left\| \frac{\partial y}{\partial \nu} - y_{obs} \right\|_{L^2(\Sigma_T)}^2 + \int_{Q_T} \lambda(Ly - \sigma\mu) \, dx \, dt$$

to fully reconstruct $y$ and $\mu$ from $y_{obs}$ and $\sigma$. See [13] where this issue is investigated.

Eventually, since the mixed formulations rely essentially on a generalized observability inequality, it may be employed to any other observable system for which such a property is available: we mention in particular the parabolic case which is usually badly conditioned—in view of regularization properties—and for which direct and robust methods are certainly very advantageous.

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