STABLE ULRICH BUNDLES ON CUBIC FOURFOLDS

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Abstract. In this paper, we give necessary and sufficient conditions for the existence of Ulrich bundles on cubic fourfold $X$ of given rank $r$. As consequences, we show that for every integer $r \geq 2$ there exists a family of indecomposable rank $r$ Ulrich bundles on the certain cubic fourfolds, depending roughly on $r$ parameters, and in particular they are of wild representation type; special surfaces on the cubic fourfolds are explicitly constructed by Macaulay2; a new 19-dimensional family of projective ten-dimensional irreducible holomorphic symplectic manifolds associated to a certain cubic fourfold is constructed; and for certain cubic fourfold $X$, there exist arithmetically Cohen-Macaulay smooth surface $Y \subset X$ which are not an intersection $X \cap T$ for a codimension two subvariety $T \subset \mathbb{P}^5$.

1. Introduction

The study of moduli spaces of stable vector bundles of given rank and Chern classes on algebraic varieties is a very active topic in algebraic geometry. In recent years attention has focused on ACM bundles, that is vector bundles without intermediate cohomology. Some of the reasons, why the study of ACM bundles is important are:

- ACM bundles on the projective $n$-dimensional space are precisely the bundles which are direct sum of line bundles ([Hor64]).
- The $i$-th syzygy of a resolution of any vector bundle on hypersurfaces by split bundles, is an arithmetically Cohen–Macaulay bundle ([Eis80]).
- These sheaves correspond to maximal Cohen–Macaulay modules over the associated coordinate ring of hypersurfaces ([Bea00]).
- In [KRR09], ACM bundles on hypersurfaces have been used to provide counterexamples to a conjecture of Griffiths and Harris about whether subvarieties of codimension two of a hypersurface can be obtained by intersecting with a subvariety of codimension two of the ambient space.

Note that there have been numerous studies of rank 2 ACM bundles on surfaces and threefolds (see ([AM09, BF09, CF09, CM05, Mad00]) and the references in those papers), and a few studies of ACM bundles of higher rank [AG99, AM09, Mad05, Tru19, TY20b]. There have also been a few examples of indecomposable ACM bundles of arbitrarily high rank [PLT09]. But as far as we can tell, examples of stable ACM bundles of higher ranks are essentially unknown.

Very often the ACM bundles that we will construct and study will share another stronger property, namely they have the maximal possible number of global sections; they will be the so-called Ulrich bundles. Notice that Ulrich bundles exist on curves, linear determinantal varieties, hypersurfaces, and complete intersections. Moreover, Ulrich bundles are semistable in the sense of Gieseker, so once fix rank and Chern class, they may be parametrized by a quasi-projective scheme. In the case that $X$ is a hypersurface in $\mathbb{P}^n$ defined by a homogeneous form $f$, Ulrich bundles on $X$ correspond to linear determinantal descriptions of powers of $f$ ([Bea00]). This has been generalized to the case of arbitrary...
Theorem 1.1. [cf. Theorem 3.9] Let $F$ be a vector bundle of rank $r \geq 3$ on a cubic fourfold $X$ in $\mathbb{P}^5$. Then the following statements are equivalent.

1) $F$ is an Ulrich bundle of rank $r$.
2) $F$ is isomorphic to a vector bundle obtained from a certain surface $Y \subset X$ of degree $d = \frac{1}{2}(3r^2 - r)$ and sectional genus $r^3 - 2r^2 + 1$.
3) There exists a $3r \times 3r$ matrix $M$ of linear forms on $\mathbb{P}^5$ such that the sequence

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}^5}(r - 1) \overset{M}{\longrightarrow} \mathcal{O}_{\mathbb{P}^5}^r \longrightarrow F \longrightarrow 0.
$$

is exact.

Recall that varieties that admit only a finite number of indecomposable ACM bundles (up to twist and isomorphism) are called of finite representation type (see [DG01] and references herein). Varieties of finite representation type have been completely classified into a short list in [BGS87, Theorem C] and [EH88, p. 348]. They are three or less reduced points on $\mathbb{P}^2$, a projective space, a smooth quadric hypersurface $X \subseteq \mathbb{P}^n$, a cubic scroll in $\mathbb{P}^4$, the Veronese surface in $\mathbb{P}^5$ or a rational normal curve.

On the other extreme, we would find the varieties of wild representation type namely, varieties for which there exist $t$-dimensional families of non-isomorphic indecomposable ACM sheaves for arbitrary large $t$. In the case of dimension one, it is known that curves of wild representation type are exactly those of genus larger than or equal to two. In dimension two and three, Casanellas and Hartshorne showed in [CH11] that cubic surfaces and threefolds are of wild representation type. On the other hand, the first result for varieties of arbitrary dimension were obtained in [MRPL14], where the authors showed that Fano blow-ups of points in $\mathbb{P}^n$ are of wild representation type. In [MP13] it had already been proven that Bordiga or Castelnuovo surfaces were of wild representation type. However, a broader problem has been much less studied: which are the possible dimensions of families of ACM vector bundles of rank $r$, depending roughly on $r$ parameters, on certain projective surfaces. The main goal of this paper is to provide families of ACM vector bundles on a large range of a certain cubic fourfold, and in particular they are of wild representation type. Our source of examples will be the special cubic fourfold. Recall that a smooth cubic fourfold $X$ in $\mathbb{P}^5$ is special, if there is an embedding of a saturated rank-2 lattice

$$
L_\delta := \langle h^2, Y \rangle \hookrightarrow A(X),
$$

where $A(X)$ is the lattice of middle Hodge classes, $h \in \text{Pic}(X)$ is the hyperplane class, $Y$ is an algebraic surface not homologous to a complete intersection, and $\delta$ is the determinant of the intersection matrix of $\langle h^2, Y \rangle$. Note that the locus $C_\delta$ of special cubic fourfolds $X$ of a discriminant $\delta$ is an irreducible divisor which is nonempty if and only if $\delta > 6$ and $\delta \equiv 0, 2 \pmod{6}$ ([Has00]). With the above notation, we have the following theorem.

Theorem 1.2. There exists a cubic fourfold $X$ such that

1) $[X] \in C_{14} \cap C_{18}$.
2) For any $r \geq 2$, the moduli space of stable rank $r$ Ulrich bundles on a cubic fourfold $X$ is nonempty and smooth of dimension $r^2 + 1$.
3) $X$ is of wild representation type.

Recently, the study of some geometric properties of special ACM bundles on smooth cubic fourfolds has received considerable attention. In [LLMS18], the authors constructed an irreducible holomorphic symplectic eightfold $Z$ from the irreducible component of the Hilbert scheme of twisted cubic curves on each cubic fourfold $X$ not containing a plane. Moreover, $Z$ is isomorphic to an irreducible component
of a moduli space of Gieseker stable torsion sheaves or rank three torsion free sheaves. In [AL17] it was shown that $Z$ is deformation equivalent to a Hilbert scheme of four points on a K3 surface. After that, Lahoz et al. ([LLMS18]) was shown that $Z$ is birational to an irreducible component of a moduli space of Gieseker stable ACM bundles of rank six. In [LMS17], Lahoz, Macrì and Stellari studied ACM bundles on cubic fourfolds containing a plane exploiting the geometry of the associated quadric fibration and Kuznetsov’s treatment of their bounded derived categories of coherent sheaves. More precisely, they constructed the K3 surface naturally associated to the fourfold as a moduli space of Gieseker stable ACM bundles of rank four. In view of above results, we search for a moduli space of stable Ulrich bundles of rank 3, which is a ten-dimensional irreducible holomorphic symplectic manifold. More precisely, we are able to prove the next result.

**Theorem 1.3.** Let $X$ be a general special cubic fourfold in $\mathbb{C}^4$. Then the moduli space of stable rank 3 Ulrich bundles on $X$ with the Chern classes $c_1 = 3$ and $c_2 = 12$ is a smooth ten-dimensional holomorphically symplectic manifold.

Let us explain briefly how this paper is divided. This paper is divided into 4 sections. In Section 2 let us briefly note properties of ACM sheaves including the theory of matrix factorizations due to Eisenbud. We shall prove Theorem 1.1 and its corollaries in Section 3. In Section 4, as corollaries to the main theorem, we give to explicit construction of families of simple Ulrich vector bundles on a special cubic fourfold and remarks on a conjecture of Griffiths and Harris about whether subvarieties of codimension two of a hypersurface can be obtained by intersecting with a subvariety of codimension two of the ambient space. We construct a new 19-dimensional family of projective ten-dimensional irreducible holomorphic symplectic manifolds.

## 2. Preliminaries

Throughout this section, assume that $X$ is a closed subscheme in $\mathbb{P}^n$, $R$ is the polynomial ring $k[x_0, \ldots, x_n]$ over an algebraically closed field $k$ and $R_X$ is the coordinate ring of $X$. For any coherent sheaf $\mathcal{F}$, we denote by $H^i_1(X, \mathcal{F})$ the sum $\bigoplus_{l \in \mathbb{Z}} H^i(X, \mathcal{F}(l))$. Let us recall the definition of maximal Cohen-Macaulay $R_X$-modules and arithmetically Cohen-Macaulay coherent sheaves over $X$.

**Definition 2.1.** A graded $R_X$-module $E$ is a maximal Cohen-Macaulay module (MCM), if $\text{depth} E = \dim E = \dim_{R_X} E$. A closed subscheme $X \subseteq \mathbb{P}_k^n$ is arithmetically Cohen-Macaulay (ACM), if its homogeneous coordinate ring $R_X = R/I_X$ (where $I_X$ is the saturated ideal of $X$) is a Cohen-Macaulay ring. This is equivalent to saying $H^i_1(I_X, \mathcal{O}_{\mathbb{P}^n}) = 0$ and $H^i_1(O_X) = 0$, for any $0 < i < \dim X$. A coherent sheaf $\mathcal{E}$ on an ACM $X$ is an ACM sheaf, if it is locally Cohen-Macaulay on $X$ and $H^i_1(\mathcal{E}) = 0$, for any $0 < i < \dim X$.

Thanks to the graded version of the Auslander-Buchsbaum formula (for any finitely generated $R$-module $M$)

$$\text{pd}_RM = n + 1 - \text{depth} M,$$

where $\text{pd}_RM$ denotes the projective dimension of the $R$-module $M$. Moreover, we deduce that a subscheme $X \subseteq \mathbb{P}^n$ is ACM if and only if $\text{pd}(R_X) = \text{codim} X$. Hence, if $X \subseteq \mathbb{P}^n$ is a codimension $c$ ACM subscheme, a graded minimal free $R$-resolution of $I_X$ is of the form:

$$0 \longrightarrow F_c \overset{\varphi_c}{\longrightarrow} F_{c-1} \overset{\varphi_{c-1}}{\longrightarrow} \cdots \longrightarrow F_1 \overset{\varphi_1}{\longrightarrow} F_0 \longrightarrow R_X \longrightarrow 0,$$

where $F_0 = R$ and $F_i = \bigoplus_j R(-i - j)^{\beta_{ij}(X)}$, $1 \leq i \leq c$. The integers $\beta_{ij}(X)$ are called the graded Betti numbers of $X$ and they are defined by

$$\beta_{ij}(X) = \dim_k \text{Tor}^i(R_X, k)_{i+j}.$$

We construct the Betti diagram of $X$ writing in the $(i,j)$-th position the Betti number $\beta_{ij}(X)$. In this setting, minimal means that $\text{Im} \varphi_i \subseteq \mathfrak{m} F_{i-1}$. Therefore, the free resolution of $X$ is minimal if, after choosing a basis of $F_i$, the matrices representing $\varphi_i$ do not have any non-zero scalar.
Notice that there is a one-to-one correspondence between ACM sheaves on an ACM scheme $X$ and graded MCM $R_X$-modules that send $E$ to $H^0(X, E)$ (see [CH11, 2.1]). In the algebraic context, MCM modules have been extensively studied (see for example the book of Yoshino [Yos90]), as they reflect relevant properties of the corresponding ring. There has also been recent work on ACM bundles of small rank on particular varieties such as Fano threefolds, quartic threefolds and Grassmann varieties (see [AM09] and the references therein). When $X$ is a non-singular variety, which is going to be mainly our case, any coherent ACM sheaf on $X$ is locally free. For this reason, we are going to speak often of ACM bundles (since we identify locally free sheaves with their associated vector bundle).

It is well known that ACM sheaves provide a criterion to determine the complexity of the underlying variety. Indeed, this complexity can be studied in terms of the dimension and number of families of indecomposable ACM sheaves that it supports. Recently, inspired by an analogous classification for quivers and fork-algebras of finite type, the classification of any ACM variety as being of finite, tame or wild representation type (cf. [DG01] for the case of curves and [CH11] for the higher dimensional case) has been proposed. Let us recall the definitions:

**Definition 2.2.** An ACM scheme $X \subseteq \mathbb{P}^n$ is of finite representation type if it has, up to twist and isomorphism, only a finite number of indecomposable ACM sheaves. An ACM scheme $X \subseteq \mathbb{P}^n$ is of tame representation type if for each rank $r$, the indecomposable ACM sheaves of rank $r$ form a finite number of families of dimension at most one. On the other hand, $X$ will be of wild representation type if there exist $t$-dimensional families of non-isomorphic indecomposable ACM sheaves for arbitrary large $t$.

Varieties of finite representation type have been completely classified into a short list in [BGS87, Theorem C] and [EH88, p. 348]. They are three or less reduced points on $\mathbb{P}^2$, a projective space, a non-singular quadric hypersurface $X \subseteq \mathbb{P}^n$, a cubic scroll in $\mathbb{P}^4$, the Veronese surface in $\mathbb{P}^5$ or a rational normal curve. The only known example of a variety of tame representation type is the elliptic curve. On the other hand, so far only a few examples of varieties of wild representation type are known: curves of genus $g \geq 2$ (cf. [CH11]), del Pezzo surfaces and Fano blow-ups of points in $\mathbb{P}^n$ (the cases of the cubic surface and the cubic threefold have also been handled in [CH11]) and ACM rational surfaces on $\mathbb{P}^4$ ([MP13]).

Very often the ACM bundles that we will construct will share another stronger property, namely they have the maximal possible number of global sections; they will be the so-called Ulrich bundles. Let us end this section recalling the definition of Ulrich sheaves and summarizing the properties that they share and that will be needed in the sequel.

**Definition 2.3.** A coherent sheaf $\mathcal{F}$ on $X$ is said to be **initialized**, if

$$H^0(X, \mathcal{F}(-1)) = 0 \text{ and } H^0(X, \mathcal{F}) \neq 0.$$ 

If $\mathcal{F}$ is ACM, then there exists an integer $k$ such that $\mathcal{F}_{\text{init}} := \mathcal{F}(k)$ is initialized.

**Definition 2.4.** Given a projective scheme $X \subseteq \mathbb{P}^n$ and a coherent sheaf $\mathcal{F}$ on $X$. A vector bundle $\mathcal{F}$ is an Ulrich bundle, if $\mathcal{F}$ is ACM on $X$ and the initialized twist $\mathcal{F}_{\text{init}}$ of $\mathcal{F}$ satisfies $h^0(X, \mathcal{F}_{\text{init}}) = \text{rank}(\mathcal{F}) \deg(X)$.

When $X \subseteq \mathbb{P}^n$ is a hypersurface of degree $s$, the existence of Ulrich bundles is related to classical problems in algebraic geometry ([Bea00]). If $\text{rank}(\mathcal{F}) = 1$, then one has a determinantal presentation of $X := \{\det(M) = 0\}$, where $M = (t_{ij})$, $1 \leq i, j \leq s$ is a matrix of linear forms; a bundle $\mathcal{F}$ with $\text{rank}(\mathcal{F}) = 2$ corresponds to a Pfaffian equation of $X : \{\text{pf}(M) = 0\}$, where $M$ is a $(2s) \times (2s)$ skew-symmetric linear matrix (see [Bea00]). In this paper, we describe some criterions for determining when a cubic fourfold $X$ has an Ulrich bundle of given rank $r$. It is clear that each direct summand of an Ulrich bundle is also Ulrich. Thus, one can restrict the attention to indecomposable bundles, i.e. bundles that do not split as direct sum of bundles of smaller ranks. The study of indecomposable Ulrich bundles is a particularly in problem that could give some suggestions on the complexity of the embedding $X \subseteq \mathbb{P}^N$. 
Now let $X = V(f) \subseteq \mathbb{P}^n$ be a hypersurface cut out by a homogeneous form $f$ of degree $s$. It is well-known that a matrix factorization $(A, B)$ of $f$ induces a maximal Cohen-Macaulay module supported on $X$ by $\text{coker} A$. Conversely, if we have a maximal Cohen-Macaulay $R_X = R/(f)$-module, one has a matrix $A$ by reading off its length 1 resolution. Indeed, it forms a part of a matrix factorization of $f$. Therefore, there is a unique matrix $B$ such that $AB = BA = f \cdot \text{Id}$. As a conclusion, there is a bijection between the maximal Cohen-Macaulay modules and the equivalence classes of matrix factorizations of $f$.

Now, let us briefly recall Shamash’s construction (see [Sha69]). Suppose that $Y$ is a subscheme contained in hypersurface $X = V(f) \subseteq \mathbb{P}^n$. Let $R_Y$ be the coordinate ring of $Y$, and $F^*$ the minimal free $R$-resolution of $R_Y$. Since $Y \subseteq X$, we have a right exact sequence

$$\ldots \longrightarrow F_1 \otimes_R R_X \longrightarrow F_0 \otimes_R R_X \cong R_X \longrightarrow R_Y \longrightarrow 0.$$ 

Hence, there is an $R$-free resolution of $R_Y$ (possibly non-minimal)

$$\ldots \longrightarrow G_4 \oplus G_2(-s) \oplus G_0(-2s) \rightarrow G_3 \oplus G_1(-s) \rightarrow G_2 \oplus G_0(-s) \rightarrow G_1 \rightarrow G_0 \rightarrow R_Y \rightarrow 0,$$

where $G_i = F_i \otimes_R R_X$. It becomes eventually 2-periodic after the $(c-1)$-th step, where $c = \text{codim} R_Y$. Hence, it induces a matrix factorization of $f$. Such a matrix factorization provides a presentation of an ACM sheaf on $X$. Moreover, this construction allows us to control some extent the degrees of the entries of the corresponding minimal matrix factorization of $f$ induced by an $R_X$-module $R_Y$, if we know the Betti numbers of $R_Y$ as an $R$-module. Thus, the Betti numbers of the minimal periodic resolution is called the shape of the matrix factorization.

Let $X$ be a cubic fourfold containing a quintic Del Pezzo surface $T$. Then, $T$ have the minimal free resolution

$$(1) \quad 0 \longrightarrow \mathcal{O}_{ps}(-5) \longrightarrow \mathcal{O}_{ps}^5(-3) \longrightarrow \mathcal{O}_{ps}^5(-2) \longrightarrow \mathcal{I}_T \longrightarrow 0.$$ 

An easy way to produce matrix factorizations on a hypersurface $X = V(f)$ in $\mathbb{P}^5$ is to consider $\Gamma_\bullet (\mathcal{O}_T)$ as a module over $R = K[x_0, \ldots, x_5]$ annihilated by $f$. A matrix factorization of $f$ is given by the periodic part of a minimal free resolution of $\Gamma_\bullet (\mathcal{O}_T)$ as a module over $R_X = R/(f)$. In particular, the minimal resolution of $\Gamma_\bullet (\mathcal{O}_T)$ as a module over the homogeneous coordinate ring of a cubic fourfold $X \supset T$ is eventually 2-periodic with Betti numbers

$$\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & 5 & 6 & \cdot & \cdot & \cdot \\
2 & \cdot & \cdot & 6 & 6 & \cdot & \cdot \\
3 & \cdot & \cdot & \cdot & 6 & 6 & \cdot \\
\end{array}$$

Hence, $X$ has a matrix factorization of type $(\psi : \mathcal{O}_X^3(-3) \rightarrow \mathcal{O}_X^5(-1); \varphi : \mathcal{O}_X^5(-1) \rightarrow \mathcal{O}_X^5)$. Let $\mathcal{F} = \text{coker} \varphi$. Then $\mathcal{F}$ is an Ulrich bundle of rank 2.

To each $Y$ above as, we associate the vector bundle $\mathcal{F}$ by Serre’s construction:

$$0 \longrightarrow \mathcal{O}_X(-2) \stackrel{s}{\longrightarrow} \mathcal{F} \longrightarrow \mathcal{I}_{Y/X} \longrightarrow 0,$$

where $\mathcal{I}_{Y/X}$ is the ideal sheaf of $Y$ in $X$, and $s$ is a section of $\mathcal{F}$ such that $Y$ is its zero locus. Let $H$ be the class of a hyperplane section of $X$. Then, we have $c_1(\mathcal{F}) = 2H$ and $c_2(\mathcal{F}) = 5$. As follows from [Bea00, Bea02], the vector bundles $\mathcal{F}$ of this type have several other equivalent characterizations:

**Theorem 2.5.** Let $X$ be a cubic fourfold in $\mathbb{P}^5$ and $\mathcal{F}$ a rank 2 vector bundle on $X$. Then the following statements are equivalent.

1) $\mathcal{F}$ is an Ulrich bundle of rank 2.
2) $\mathcal{F}$ is isomorphic to a vector bundle obtained by Serre’s construction from a quintic Del Pezzo surface $T \subset X$. 


3) There exists a skew-symmetric 6 by 6 matrix $M$ of linear forms on $\mathbb{P}^5$ such that $F$ considered as a sheaf on $\mathbb{P}^5$ can be included in the following exact sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^5}^6(-1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^5}^6 \rightarrow F \rightarrow 0.$$  

3. Generalities on Ulrich bundles

The goal of this section is to extend Theorem 2.5 from [Bea00, Bea02] to Ulrich bundles of rank $r \geq 3$ on cubic fourfolds. First, we show how to associate an ACM surface to an Ulrich bundle on cubic fourfolds from $N$-type resolutions (or so-called Bourbaki sequences, c.f. [BHU87]). We recall the definition here.

**Definition 3.1.** Let $X \subset \mathbb{P}^n$ be an equidimensional scheme, and let $Y \subset X$ be a codimension 2 subscheme without embedded components. A coherent sheaf $\mathcal{L}$ on $X$ is *dissocié* if it is isomorphic to a direct sum $\bigoplus \mathcal{O}_X(a_i)$ for various $a_i \in \mathbb{Z}$. An $N$-type resolution of $Y$ is an exact sequence

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{N} \rightarrow \mathcal{J}_{Y,X} \rightarrow 0,$$

with $\mathcal{L}$ is dissocié, and $\mathcal{N}$ is a coherent sheaf satisfying $H^1_2(\mathcal{N}^\vee) = 0$, and $\text{Ext}^1(\mathcal{N}, \mathcal{O}_X) = 0$.

In the case that $X$ satisfies Serre’s condition $S_2$, and $H^1_2(\mathcal{O}_X) = 0$, then an $N$-type resolution of $Y$ exists (see [Har03, 2.12]). An $N$-type resolution is not unique but it is well known that any two $N$-type resolutions of the same subscheme are stably equivalent (see [Har03, 1.10]). In other words, if $\mathcal{N}$, and $\mathcal{N}'$ are two sheaves appearing in the middle of an $N$-type resolution of a subscheme $Y$, then there exist dissocié sheaves $\mathcal{L}_1, \mathcal{L}_2$, and an integer $a$ such that

$$\mathcal{N} \oplus \mathcal{L}_1 \cong \mathcal{N}'(a) \oplus \mathcal{L}_2.$$

The following theorem and its converse involve the relationship between Ulrich bundles and ACM surfaces.

**Theorem 3.2.** Let $F$ be a Ulrich bundle of rank $r \geq 2$ on a cubic fourfold $X$ and generated by its global sections. Then, for every $a \geq 1$, there is a surface $Y \subset X$ such that $Y$ has an $N$-type resolution

$$0 \rightarrow \mathcal{O}_X^{r-1}(-a) \rightarrow F \rightarrow \mathcal{J}_{Y/X}(b) \rightarrow 0$$

for some $b \in \mathbb{Z}$.

**Proof.** We denote by $M$ the module $H^0_*(F)$. Then, we have a short exact sequence

$$0 \rightarrow R^3r(-1) \rightarrow R^3r \rightarrow M \rightarrow 0.$$

Therefore, the Castelnuovo-Mumford regularity of $M$ is 0. Let $N$ be a graded submodule of $M$ generated by homogeneous elements of $M$ all of the same degree $a - 1$. It follows from the Castelnuovo-Mumford regularity of $M$ is 0 and $a \geq 1$ that $M/N$ is of finite length over $R$. We now observe that $N$ has a presentation

$$R^c(-a) \xrightarrow{\phi} R^b(-a + 1) \rightarrow N \rightarrow 0,$$

in which the presenting matrix $\phi$ has homogeneous entries with all the entries having degree 1. We then adjoin indeterminates $X_{ij}$ to $R$ for $1 \leq i \leq b$, and $1 \leq j \leq 5$ to obtain a new ring $A = R[Y_{ij}]$. We put $M_1 = A \otimes_R M$, and $N_1 = A \otimes_R N$. We may consider the matrix $\phi$ as an $A$-homomorphism $A^c(-a) \rightarrow A^b(-a + 1)$ presenting $N_1$, that is, we are identifying $A \otimes \phi$ with $\phi$. Let $\Phi$ be the matrix with elements $Y_{ij}$, we obtain an $A$-homomorphism $A^c(-a) \oplus A^{3r}(-a) \rightarrow A^b(-a + 1)$ via the matrix $\Psi = [\phi|\Phi]$, where the vertical line represents a matrix partitioning corresponding to the two direct summands of $A^c(-a) \oplus A^{3r}(-a)$. Let $E$ be the image of $A^{3r}(-a)$ in $N_1$ under the composition of the maps

$$A^c(-a) \oplus A^{3r}(-a) \xrightarrow{\Psi} A^b(-a + 1) \rightarrow N_1.$$
Then, $E$ is a free submodule of $N_1$ of rank $r$, and also $E$ is a graded submodule of $N_1$. Since $M$ is Cohen-Macaulay as $R_X$-module, $M_1$ is Cohen-Macaulay as $A$-module. It follows from $M_1/E$ being a graded $A$-module that its associated prime ideals are homogeneous. Since $\dim A \geq \dim R = 6$, and the $(x_0, \ldots, x_5)$-depth of $M$ over $R$ is equal to the $(x_0, \ldots, x_5)$-depth of $M_1$ over $A$, we get that depth$_{(x_0, \ldots, x_5)}(M_1/E) = 4$. Hence, because our field is infinite, there must be a form $g$ of degree one in $x_0, \ldots, x_5$ that is not a zero divisor on $M_1/F$.

Next we show that $A_g \otimes_A (N_1/E)$ is isomorphic to a normal prime ideal $I_g$ of $A_g$. Indeed, since degree of $g$ is one in $x_0, \ldots, x_5$, we have that $M_g$ at $N_g$ is a free $A_g$-module. Hence, $(N_1)_g = A_g \otimes_A N_g$ is also a free $A_g$-module. Thus, the sequence

$$A_g^r(-a) \xrightarrow{\Psi} A_g^b(-a + 1) \xrightarrow{(N_1)_g} 0$$

splits. After performing invertible row and column operations on $\Psi$, we obtain a $b \times c$ matrix of the form

$$\begin{pmatrix}
0 & I_t \\
0 & 0
\end{pmatrix}$$

where $I_t$ is the $t \times t$ identity matrix for some $t$. The effect of the row operations on the matrix $\Psi = [\phi|\Phi]$, is to obtain a matrix $\Psi' = [\phi'|\Phi']$, where $\Phi'$ consists of entries $y'_{ij}$ which are a new set of $3r \times b$ polynomial indeterminates over $R_g$. After applying further invertible column operations on $\Psi$, we obtain

$$\Psi' = \begin{pmatrix} 0 & I_t & 0 \\ 0 & 0 & y'_{ij} \end{pmatrix}$$

where the $y'_{ij}$ in the lower right-hand corner are a subset of the $y_{ij}$. Let $I_g$ be the $A_g$-ideal of projective dimension one, that is, the ideal generated by the maximal minors of the matrix $(y'_{ij})$ in the lower right-hand corner of $\Psi'$. Then, $A_g \otimes_A (N_1/E) \cong I_g$. Since $g$ is regular on $M_1/E$, we get that $M_1/E \cong I$. Since this matrix consists of indeterminates over $R_g$, it follows that $I_g$ is a normal prime ideal of $A_g$, and so $I$ is a normal prime ideal of $A$. Therefore, we have an exact sequence

$$0 \xrightarrow{} A(-a)^{3r} \xrightarrow{} M_1 \xrightarrow{} I \xrightarrow{} 0,$$

where $I$ is a homogeneous prime ideal of $A$ of height two, and $A/I$ is Cohen-Macaulay. Then, it follows from Flenner’s version of Bertini’s Theorem (see [Fle77, Section 5]) that there exists a linear regular sequence $h_1, \ldots, h_{3rb}$ on $A/I$. It follows that the sequence

$$0 \xrightarrow{} (A/(h_1, \ldots, h_{3rb})A)(-a)^{3r} \xrightarrow{} M_1/(h_1, \ldots, h_{3rb})M_1 \xrightarrow{} I/(h_1, \ldots, h_{3rb})I \xrightarrow{} 0$$

is exact. Since $M_1/(h_1, \ldots, h_{3rb})M_1 \cong M$ and $A/(h_1, \ldots, h_{3rb})A \cong R$, there is a codimension 2 subscheme $Y \subset X$ without embedded components such that $Y$ has an $N$-type resolution

$$0 \xrightarrow{} \mathcal{O}_X^{r-1}(-a) \xrightarrow{} \mathcal{F} \xrightarrow{} \mathcal{I}_{Y/X}(b) \xrightarrow{} 0$$

for some $b \in \mathbb{Z}$.

**Proposition 3.3.** Let $\mathcal{F}$ be an Ulrich bundle of rank $r \geq 3$ on the cubic fourfold $X$ and generated by its global sections. Then there is an ACM surface $Y \subset X$ of degree $d = \frac{1}{2}(3r^2 - r)$ and sectional genus $r^3 - 2r^2 + 1$ such that $Y$ has an $N$-type resolution

$$(2) \quad 0 \xrightarrow{} \mathcal{O}_X^{r-1} \xrightarrow{} \mathcal{F} \xrightarrow{} \mathcal{I}_{Y/X}(r) \xrightarrow{} 0.$$  

Moreover, $Y$ has the minimal free resolution

$$(3) \quad 0 \xrightarrow{} \mathcal{O}_X^{r-1}(-r - 3) \xrightarrow{} \mathcal{O}_X^{3r}(-r - 1) \xrightarrow{} \mathcal{O}_X^{2r+1}(-r) \oplus \mathcal{O}_X(-3) \xrightarrow{} \mathcal{I}_Y \xrightarrow{} 0.$$
Proof. Since \( F \) is an Ulrich bundle of rank \( r \), we can take \( r - 1 \) general sections of \( F \) so that the quotient of \( F \) by \( O_X^{-1} \) is torsion free. Therefore we have an exact sequence

\[
0 \to O_X^{-1} \to F \to \mathcal{I}_{Y/X}(D) \to 0
\]

where \( D = c_1(F) = rH \) is a certain divisor on \( X \) and \( Y \) is a surface of degree equal to \( c_2(F) \). As \( F \) is generated by its global sections, \( \mathcal{I}_{Y/X}(r) \) is also generated by global sections and \( h^0(\mathcal{I}_{Y/X}(r)) = 2r + 1 \).

On the other hand, \( F \) has the following minimal free resolution in \( \mathbb{P}^5 \)

\[
0 \to O_{\mathbb{P}^5}^3(-1) \xrightarrow{\varphi} O_{\mathbb{P}^5}^3 \to F \to 0
\]

by [CH11, Proposition 3.7]. Using the mapping cone procedure with the free resolutions of \( O_X \) and the exact sequence 2, \( Y \) has the minimal free resolution as required. Therefore \( Y \) is an ACM surface.

By [CH11, Proposition 3.7], we have \( \deg Y = \frac{1}{2}(3r^2 - r) \) and by Riemann–Roch, sectional genus of \( Y \) is \( r^3 - 2r^2 + 1 \). 

\[ \square \]

Remark 3.4. There have been many studies about ACM bundles of rank two in dimensions two and three in [AM09, BF09, CF09, CM05, Mad00]. The higher rank case has been investigated in [AG99, AM09, Mad05, Tru19, TY20b]. The papers [MRPL14] and [PLT09] give a few examples of indecomposable ACM bundles of arbitrarily high rank. The already mentioned papers [CH11, CH12] contain a systematic study of stable ACM bundles in higher rank on cubic surfaces and threefolds. A general existence result for Ulrich bundles on hypersurfaces is given in [BHU91]. But in general, the existence of Ulrich sheaves on a projective scheme \( X \) is not known.

Remark 3.5. Liaison has become an established technique in algebraic geometry. The greatest activity, however, has been in the last quarter-century beginning with the work of Peskine and Szpiro in 1974(see [PS74]). Liaison is a powerful tool for constructing examples.

Definition 3.6. We say \( Y_2 \) is obtained by an elementary biliaison of height \( m \) from \( Y_1 \), if there exists an ACM scheme \( Z \subset \mathbb{P}^n \), of dimension \( r + 1 \) containing \( Y_1 \) and \( Y_2 \), such that \( Y_2 \sim Y_1 + mH \) on \( Z \). (here \( \sim \) means linear equivalence of divisors on \( Z \) in the sense of [Har94].) The equivalence relation generated by elementary biliaisons is called simply biliaison. If \( X \) is a complete intersection scheme in \( \mathbb{P}^n \), we speak of CI-biliaison. If \( Y_1 \), \( Y_2 \), \( Z \) are contained in some projective scheme \( X \subset \mathbb{P}^n \), we speak of biliaison (CI-biliaison) on \( X \).

Casanellas and Hartshorne gave a criterion for when two codimension 2 subschemes \( Y_1 \), \( Y_2 \) of a normal ACM projective scheme \( X \) are in the same biliaison class (see [CH04, Theorem 3.1]). As application, we have the following corollary.

Corollary 3.7. Let \( F \) be an Ulrich bundle on a cubic fourfold \( X \). Then surfaces constructed from \( F \) as in Theorem 3.2 are ACM and belong to the same CI-biliaison equivalence class on \( X \).

Proof. It follows immediately from Theorem 3.2, Theorem 3.1 in [CH04]) and Proposition 3.3. 

Now we will prove a converse to the Proposition 3.3.

Theorem 3.8. Assume that \( Y \) is an ACM smooth surface with the minimal free resolution \( 3 \) for \( r \geq 3 \). Let \( X \) be a general cubic fourfold containing \( Y \). Then \( Y \) has an \( \mathcal{N} \)-type resolution with \( F \) being an Ulrich bundle of rank \( r \)

\[
0 \to O_X^{-1} \to F \to \mathcal{I}_{Y/X}(r) \to 0.
\]

Proof. Consider \( \Gamma_\bullet (\mathcal{O}_Y) \) as a \( R_X \)-module, the periodic part of its minimal free resolution yields, up to twist, matrix factorization of the form

\[
R^3r(-3) \xrightarrow{\varphi} R^3r(-1) \xrightarrow{\varphi} R^3r.
\]
Let $F^*$ and $\overline{F}$ be minimal free resolutions of the section ring $\Gamma_*(\mathcal{O}_Y)$ as an $R$-module, and $R_X$-module, respectively. Let $\varphi$ be the map $\overline{G}_4 \to \overline{G}_5$, and $F = \text{coker}\varphi(-3)$. Since $\deg \det \varphi = 1 \cdot (3r) = 3 \cdot r$, $F$ is an ACM sheaf of rank $r$, and so the Shamash resolution is

$$\cdots \to \mathcal{O}^3_Y(-r-4) \to \mathcal{O}^3_Y(-r-3) \to \mathcal{O}^3_Y(-r-1) \oplus \mathcal{O}^1_X(-3) \to \mathcal{O}^{2r+1}_X(r) \oplus \mathcal{O}^1_X(-3) \to \mathcal{I}_{Y/X} \to 0.$$ 

Thus, the Shamash resolution is always non-minimal, and in a minimal resolution a cancellation occurs, causing $r$ to drop rank in expected codimension 2. Applying again $\text{Hom}(\bullet, \omega_X)$, we get that $\alpha$ is injective. It follows from the Hilbert-Burch Theorem that we have an exact sequence

$$0 \to \mathcal{O}^{-1}_X(-r) \to \mathcal{F} \to \mathcal{O}^1_X(r) \to \mathcal{O}_Y(\ell) \to 0,$$

for some $\ell$. By applying again $\text{Hom}(\bullet, \omega_X)$ to this last exact sequence one gets that $\alpha^*(-r)$ is a presentation of $\omega_Y(-\ell)$. Hence, we have $\ell = 0$, as required. \hfill \square

**Theorem 3.9.** Let $F$ be a vector bundle of rank $r \geq 3$ on a cubic fourfold $X$ in $\mathbb{P}^5$. Then the following statements are equivalent.

1) $F$ is an Ulrich bundle of rank $r$.

2) $F$ is isomorphic to a vector bundle obtained from a surface $Y \subset X$ such that $\mathcal{I}_Y$ has the minimal free resolution 3.

3) There exists a $3r \times 3r$ matrix $M$ of linear forms on $\mathbb{P}^5$ such that the sequence

$$0 \to \mathcal{O}^{3r}_{\mathbb{P}^5}(-1) \to \mathcal{O}^{3r}_{\mathbb{P}^5} \to \mathcal{F} \to 0.$$

is exact.

*Proof.* 1) $\Rightarrow$ 2): This follows from Proposition 3.3.

2) $\Rightarrow$ 3): This follows from Theorem 3.8.

3) $\Rightarrow$ 1): This is trivial. \hfill \square

4. **Ulrich bundles of small rank**

First, let us describe a little more details on the existence of Ulrich bundles on smooth cubic fourfolds. In [KS20, Proposition 2.5], Kim and Schreyer showed that if $F$ is be an Ulrich bundle of rank $r$ on a very general cubic fourfold then $r$ is divisible by 3 and $r \geq 6$. Using invariant theory, Manivel [Man19, Proposition 2.2] found a family of very general cubic fourfolds having an Ulrich sheaf of rank 9. In [Bea00], Beauville showed that there exists a smooth cubic fourfold having an Ulrich bundle of rank 2. The set of such cubic fourfolds contains an open (but it is not open) subset of the divisor $C_{14}([\text{BRS19}])$. Hence, it is natural to ask the existence question for rank 3 Ulrich bundles on smooth cubic fourfolds. The goal of this section is to prove Theorem 4.7, which answers this question.

Now, let $\mathcal{H}_{0,4}$ be the component of the Hilbert scheme of curves of degree $d = 9$ and genus $g = 4$ in $\mathbb{P}^4$ which dominate the moduli spaces $\mathcal{M}_4$ of curves of genus 4.
Remark 4.2. In the computation above, we need exhibit explicit examples of modules, curves or 
\( \mathbb{Q} \)-surfaces defined over the rationals \( \mathbb{Q} \). We will explicitly construct such examples. Moreover, by inspection we find that the general \( \phi \)-curve is of maximal rank and homogeneous co-
ordinate ring \( R_C = R/I_C \) and the section ring \( \Gamma_0(\mathcal{O}_C) \) have minimal free resolutions with the following Betti tables:

\[
\begin{array}{cccc}
\theta & 0 & 1 & 2 & 3 & 4 \\
\theta & 1 & \cdot & \cdot & \cdot & \\
\theta & \cdot & \cdot & \cdot & \cdot & \\
\theta & \cdot & 18 & 9 & 1 & \\
\end{array}
\]

Proof. Assuming that the restriction map \( H^0(\mathcal{O}_{\mathbb{P}^4}(m)) \rightarrow H^0(\mathcal{O}_C(m)) \) has maximal rank. Then we have the following statements.

- The Hilbert series of the homogeneous coordinate ring of \( C \) is
  \[ H_C(t) = 1 + 5t + 15t^2 + 24t^3 + 33t^4 + 42t^5 + 11t^6 + 60t^7 + \ldots. \]
- The Hartshorne–Rao module
  \[ H^1(\mathcal{I}_C) \cong k(-1) \]
  is a vector space of dimension 1 concentrated in degree 1.
- The ideal sheaf \( \mathcal{I}_C \) is 3-regular.
- The homogeneous ideal \( I_C = H^0(\mathcal{I}_C) \) has a 11 generator in degree 3.
- The Hilbert numerateur has shape
  \[ (1-t)^4H_C(t) = 1 - 11t^3 + 18t^4 - 9t^6 + 1. \]

Hence, smooth maximal rank curves in \( \mathcal{H}_{9,4} \) have a Betti table as required. To establish that a general point in \( C \in \mathcal{H}_{9,4} \) is a maximal rank curve, it suffices to produce a single maximal rank example. We will explicitly construct such examples. Moreover, by inspection we find that the general \( C \) lies on a smooth cubic hypersurface \( X \).

Now let \( Z = \mathbb{P}^2(p_1, \ldots, p_{10}) \) be a blow-up of \( \mathbb{P}^2 \) in 10 points in general position by the complete linear system

\[ H_Z = (5; 2^2, 1^8) = 5L - E_1 - E_2 - \sum_{i=3}^{10} E_i. \]

Then \( Z \) is a smooth surface of degree 9 and sectional genus 4 in \( \mathbb{P}^6 \). Let \( q \in \mathbb{P}^6 \) be general point and \( \phi_q : \mathbb{P}^6 \dashrightarrow \mathbb{P}^5 \) the projection from \( q \) onto a hyperplane and let \( Y_1 = \phi_q(Z) \subseteq \mathbb{P}^5 \). Therefore, \( Y_1 \) is a surface of degree 9 and sectional genus 4 and \( K_{Y_1} = -1 \). Now we take a general hyperplane of \( Y_1 \), we get that curve as required.

The Betti table of the resolution of the section ring \( \Gamma_0(\mathcal{O}_C) \) as an \( R \)-module can be deduced with the same method, since \( H_{\Gamma_0}(\mathcal{O}_C)(t) = H_C(t) + t \).

\[ \square \]

Remark 4.2. In the computation above, we need exhibit explicit examples of modules, curves or surfaces defined over the rationals \( \mathbb{Q} \) or complex numbers \( \mathbb{C} \) satisfying some open conditions on their Betti numbers. Since such examples defined over an open part \( \text{Spec}\mathbb{Z} \), the same result holds for algebraically closed fields of positive characteristic except for possibly finitely many primes \( p \). Thus, in principle, one could try to verify the result one by one for the remaining primes, and then by semicontinuity, we get the same result holds for the rationals \( \mathbb{Q} \) or complex numbers \( \mathbb{C} \). In this paper, we provide explicit constructions/equations of examples with special geometric features. In an ancillary file (see [TY20]) we have implemented our constructions in the computer algebra software Macaulay2 [GS99].

Now let \( W_{12,10} \) be the irreducible component of the Hilbert scheme parametrizing projective Cohen-Macaulay smooth curves \( C \subseteq \mathbb{P}^4 \) of degree 12, and genus 10 which dominates the moduli space \( \mathcal{M}_{10} \).

Corollary 4.3. Let \( C \in W_{12,10} \) be a general point. Then \( C \) is of maximal rank and the homo-
geogeneous coordinate ring \( R_C = R/I_C \) and the section ring \( \Gamma_0(\mathcal{O}_C) \) have minimal free resolutions with the following Betti tables:

\[
\begin{array}{cccc}
\theta & 0 & 1 & 2 & 3 & 4 \\
\theta & 1 & \cdot & \cdot & \cdot & \\
\theta & \cdot & \cdot & \cdot & \cdot & \\
\theta & \cdot & 18 & 9 & 1 & \\
\end{array}
\]
Proof. As in Proposition 4.1, we can compute the expected Betti tables of the $R$-resolutions of $R/I_C$ and $\Gamma_\bullet(\mathcal{O}_C)$. \hfill \square

**Corollary 4.4.** A general cubic hypersurface in $\mathbb{P}^4$ contains a family of dimension 24 of curves of genus 10 and degree 12.

**Proof.** Let $C \in \mathcal{W}_{12,10}$ be a general point and let $X$ be a general cubic threefold containing it. Then $\chi(N_{C/X}) = 12(5 - 3) = 24$ by [ST18, Lemma 5.3]. We claim that $h^1(N_{C/X}) = 0$. It is sufficient to check this vanishing on one example, as can be done with the Macaulay2. Hence, $h^0(N_{C/X}) = 24$. Let $C$ be the space of cubic threefolds containing a general curve $C$ of genus 10 and degree 12, up to projective equivalences. Since $h^0(\mathbb{P}^4, I_C(3)) = 7$. We have $\dim C = \dim M_{10} + \rho(10, 4, 12) + 7 - 24 = 10$, where the Brill-Noether number $\rho(g, r, d) := g - (r + 1)(g + r - d)$. \hfill \square

**Proposition 4.5.** There exists a projective surface in $\mathbb{P}^5$ of degree 9 sectional genus 4 such that the homogeneous coordinate ring $R_Y = R/I_Y$ and the section ring $\Gamma_\bullet(\mathcal{O}_Y)$ have minimal free resolutions with the Betti tables as in Proposition 4.1. In particular, the minimal resolution of $\Gamma_\bullet(\mathcal{O}_Y)$ as a module over the homogeneous coordinate ring of a cubic fourfold $X \supset Y$ is eventually 2-periodic with Betti numbers

| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 0 | 1 | . | . | . | . | . |
| 1 | 1 | 5 | 1 | . | . | . |
| 2 | . | 1 | 9 | 9 | . | . |
| 3 | . | . | 9 | 9 | . | . |

**Proof.** Let $C$ be a smooth curve as in Proposition 4.1. Then it has minimal free resolution

$$
0 \longrightarrow R' \longrightarrow R'^9(-5) \xrightarrow{\psi} R'^{18}(-4) \xrightarrow{\varphi} R'^{11}(-3) \longrightarrow R' \longrightarrow R_C \longrightarrow 0,
$$

where $R' = k[x_0, \ldots, x_4]$. Recall that we say that a sequence of $R'$-modules

$$
\cdots \longrightarrow M_{i+1} \xrightarrow{\varphi_{i+1}} M_i \xrightarrow{\varphi_i} M_{i-1} \longrightarrow \cdots
$$

is exact at $M_i$ if and only if $\text{Im } \varphi_{i+1} = \text{Ker } \varphi_i$. The sequence is exact if it is exact at every $M_i$ in the sequence. Now we will construct surface $Y$ as follows.

- Let $A$ be an $11 \times 18$ matrix of indeterminates and $B$ be an $18 \times 9$ matrix of indeterminates. Let $T = k[A, B]$ be the polynomial ring generated by all the indeterminates $A_{ij}$ and $B_{ij}$ and $T' = R' \otimes T$. Consider a sequence $F_\bullet$ of $T'$-modules

$$
F_\bullet : T'^9(-5) \xrightarrow{\psi + B} T'^{18}(-4) \xrightarrow{\varphi + A} T'^{11}(-3).
$$

A sequence $F_\bullet$ of $T'$-modules is exact if and only if

$$
A \circ \psi + \varphi \circ B \equiv 0 \quad \text{and} \quad AB \equiv 0
$$
• Since $\psi$ and $\varphi$ are linear matrices over polynomial ring $R'$, the condition $A \circ \psi + \varphi \circ B \equiv 0$ gives a linear relationship on the indeterminates $A_{ij}$ and $B_{ij}$. Let $m + 1$ is numbers of independent variables on such linear relationship. Let $S = k[y_0, \ldots, y_m]$. Then the condition $AB \equiv 0$ defined a varieties $V$ such that a point of $V$ is correspondent to a smooth projective surface $Y(p)$ of degree 9 sectional genus 4. In the current case, they are $m = 14$ and $V$ is a cone in $\mathbb{P}^{14}$ of dimension 8 generated by 23 quadric elements.

Hence, we get that surface $Y$ has minimal free resolutions with the Betti tables as in Proposition 4.1. Moreover, the Hartshorne–Rao module:

$$H^1_\bullet(I_Y) \cong k(-1)$$

is a vector space of dimension 1 concentrated in degree 1. Since $H^1_\bullet(O_Y)(t) = H_Y(t) + t$, the Betti table of the resolution of the section ring as an $R$-module is as in Proposition 4.1. □

Recall that a smooth cubic fourfold $X$ in $\mathbb{P}^3$ is special, if there is an embedding of a saturated rank-2 lattice

$$L_\delta := \langle h^2, Y \rangle \hookrightarrow A(X),$$

where $A(X)$ is the lattice of middle Hodge classes, $h \in \text{Pic}(X)$ is the hyperplane class, $Y$ is an algebraic surface not homologous to a complete intersection, and $\delta$ is the determinant of the intersection matrix of $\langle h^2, Y \rangle$. In 2000, Hassett showed in [Has00] that the locus $C_3$ of special cubic fourfolds $X$ of a discriminant $\delta$ is an irreducible divisor which is nonempty if and only if $\delta > 6$ and $\delta \equiv 0, 2 \pmod{6}$. Now let $H_{12,10}$ be the irreducible component of the Hilbert scheme parametrizing projective Cohen-Macaulay smooth surfaces $Y \subseteq \mathbb{P}^5$ of degree 12, and sectional genus 10 which dominates the moduli space $W_{12,10}$. With the above notation, we have the following proposition.

**Proposition 4.6.** Let $Y \subseteq H_{12,10}$ be a general point. Then $Y$ is of maximal rank and the homogeneous coordinate ring $R_Y = R/I_Y$ and the section ring $\Gamma_\bullet(O_Y)$ have minimal free resolutions with the Betti tables 4. Moreover, if $X$ is a general cubic fourfold containing $Y$ then $X \in C_{18}$.

**Proof.** Assuming that the restriction map $H^0(O_{\hat{X}}(m)) \to H^0(O_Y(m))$ has maximal rank. Then we have the following statements.

• The Hilbert series of the homogeneous coordinate ring of $Y$ is

$$H_Y(t) = 1 + 6t + 21t^2 + 48t^3 + 87t^4 + 138t^5 + 201t^6 + 276t^7 + \ldots.$$  

• The Hartshorne–Rao module

$$H^1_\bullet(I_Y) \cong H^2_\bullet(I_Y) = 0.$$  

• The Hilbert numerator has shape

$$(1 - t)^5 H_Y(t) = 1 - 8t^3 + 9t^4 - 2t^6.$$  

Since $H_Y(t) = H^1_\bullet(O_Y)(t)$, the above Betti table of the resolution $F_\bullet$ of the section ring $\Gamma_\bullet(O_Y)$ as an $R$-module. Finally, we compute the Betti number of $\Gamma_\bullet(O_Y)$ as an $R_X$-module. The (possibly non-minimal) resolution has Betti table as above.

To show that the Betti tables are indeed the expected ones and that, a posteriori, a general surface $Y$ is of maximal rank, we only need to exhibit a concrete example, which we construct from Theorem 3.9 and Proposition 4.5. Now let $X$ be a general cubic fourfold containing $Y$. By computing the self-intersection of $Y \subseteq X$ using the formula from [Has00],

$$Y^2 = c_2(\mathcal{N}_Y/X) = 6H^2 + 3H \cdot K + K^2 - \chi_{top} = 54.$$  

Hence, we have $\delta = 3Y^2 - d^2 = 3 \cdot 54 - 12^2 = 18$. Hence $[X] \in C_{18}$. □
Theorem 4.7. Let $X$ be a general special cubic fourfold in $C_{18}$. Then the moduli space of stable rank 3 Ulrich bundles on a special cubic fourfold $X$ with first Chern class $c_1 = 3H$, where $H$ is the hyperplane class, and $c_2 = 12$ is nonempty and smooth of dimension 10.

Proof. By Proposition 4.6, moduli space of stable rank 3 Ulrich bundles on a special cubic fourfold $X$ is nonempty. Now we will show that a general $[X] \in C_{18}$ contains a surface $[Y] \in H_{12,10}$. In fact, $\dim H_{12,10} = 63$ and $H_{12,10}$ is generically smooth. Indeed, one can verify that $h^1(N_{Y/P^5}) = 0$ for $[Y] \in H_{12,10}$. Let $C \subseteq |H^0(P^5(3))| \cong P^{55}$ be the open set corresponding to smooth cubic hypersurfaces. Since $h^0(T_Y(3)) = 5$ so that the locus $D_{18} = \{([Y], [X]) : Y \subseteq X \} \subseteq H \times C$ has dimension $63 + 8 - 1 = 70$. The image of $\pi_2 : D_{18} \rightarrow C$ has dimension at most 54 because the general cubic does not contain any $Y$ belonging to $H$. For every $[X] \in \pi_2(D_{18})$ we have

$$\dim(\pi_2^{-1}([X])) \geq \dim(D_{18}) - \dim(\pi_2(D_{18})) \geq 70 - 54 = 16.$$ 

Since $h^0(N_{Y/X}) \geq \dim(Y \subset X)(\pi_2^{-1}([X]))$ for every $[Y \subset X] \in \pi_2^{-1}([X])$, to show that a general $X \in C_{18}$ contains a surface $Y$, it is sufficient to verify that $h^0(N_{Y/X}) = 16$ for a general $Y$ and for a smooth $X \in |H^0(T_Y(3))|$. We verified this via Maccaulay2 and we can conclude that $h^0(N_{Y/X}) = 16$.

To show that the moduli space is smooth of dimension 10, we just compute $h^1(\mathcal{F} \otimes \mathcal{F}^\vee)$ and $h^2(\mathcal{F} \otimes \mathcal{F}^\vee)$, which are 0. This is elementary (see the proof of Theorem 1.3).

\[\square\]

Remark 4.8. 1) As an application of the result above and [KM09, Theorem 4.3], one gets that the smooth locus of any moduli space of rank 3 Ulrich bundles on a special cubic fourfold $X \in C_{18}$ carries a closed symplectic form.

2) The Hilbert scheme parametrizing projective Cohen-Macaulay smooth surfaces of degree 12, and sectional genus 10 has least two components. In fact, let $X$ be a cubic fourfold containing an elliptic ruled surface $T$ as in Theorem 2 in [AHTVA19]. Consider $\Gamma_*(\mathcal{O}_T)$ as a $R_X$-module, the periodic part of its minimal free resolution yields, up to twist, matrix factorization of the form

$$R^0(-4) \oplus R(-3) \xrightarrow{\psi} R^0(-2) \oplus R(-3) \xrightarrow{\varphi} R^0(-1) \oplus R.$$ 

Thus there exists an ACM surface $Y$ such that homogeneous coordinate ring $R_Y = R/I_Y$ and the section ring $\Gamma_*(\mathcal{O}_Y)$ have minimal free resolutions with the Betti tables

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 |  |  |  |
| 1 |  |  |  |  |
| 2 |  | 8 | 9 | 1 |
| 3 |  | 1 | 2 |  |

Now, for the proof of Theorem 1.3 we need the following result.

Lemma 4.9. There exists a smooth cubic fourfold having both Ulrich bundles of rank 2 and 3.

Proof. Let $p_1, \ldots, p_{10}$ be points in general position in $P^2$ and $C_0$ a quadric curve passing four points $p_3, \ldots, p_6$. Let $Z = P^2(p_1, \ldots, p_{10})$ be a blow-up of $P^2$ in 10 points in general position by the complete linear system $H_Z = (5; 2^2, 1^8) = 5L - E_1 - E_2 - \sum_{i=3}^{10} E_i$. Then $Z$ is a smooth surface of degree 9 and sectional genus 4 in $P^6$. Let $C_1$ be the strict transform of $C_0$ on $Z$. Then $C_1$ is a smooth rational curve of degree 6.

Let $q$ be a point belong a trisecant line to the surface $Z$ and $\phi_q : P^6 \dashrightarrow P^5$ the projection from $q$ onto a hyperplane and let $Y = \phi_q(Z) \subseteq P^5$. Therefore, $Y$ is a surface of degree 9 and sectional genus 4 and $K_Y^2 = -1$. Moreover, $Y$ has the minimal free resolution with the Betti tables as in Proposition 4.1.

On the other hand, $C_2 = \phi_q(C_1) \subseteq P^5$ is a smooth curve on $Y$ with the minimal free resolution
Then $C_2$ is contained in a smooth rational normal scroll surface $S_1$ of degree 4. Moreover $S$ is contained in a smooth rational normal scroll cubic threefold $\Sigma$. Let $X$ be a smooth cubic fourfold which contains both surfaces $Y$ and $S_1$. Then there exists a quintic del Pezzo surface $S$ such that

$$X \cap \Sigma = S \cup S_1.$$ 

Then by Theorem 2.5 and Proposition 4.5, $X$ has Ulrich bundles of rank 2 and 3. Notice that, in an ancillary file (see [TY20]), we provide the explicit homogeneous ideal of surfaces $Y$, $S$ in the cubic fourfold $X$.

**Remark 4.10.** In the proof of Lemma 4.9, we give an explicit example of a triple $S, Y \subset X$ over a finite field with the help of Macaulay2 and then establishing the lemma in characteristic 0 with semi-continuity. By Theorem 2.5 and 4.7, $[X] \in C_{14} \cap C_{18}$. Since $S$ and $Y$ intersect transversally in 16 points, $X$ is a rational cubic fourfold in $C_{18}$ associated with a good sextic del Pezzo fibration (see [AHTVA19, Definition 11]) and with one nontrivial Brauer class. Notice that intersection $C_{14} \cap C_{18}$ has 9 irreducible components and H. Awada also gave another example of a rational cubic fourfold in $C_{18}$ associated with a good sextic del Pezzo fibration and with one nontrivial Brauer class (see [Awa19]).

**Proof of Theorem 1.3.** Let $X$ be as in Lemma 4.9. Then $[X] \in C_{14} \cap C_{18}$. Since $F$ is an Ulrich bundle of rank $r \geq 2$, it has the following minimal free resolution in $\mathbb{P}^3$:

\[
0 \longrightarrow O_{\mathbb{P}^3}(1) \xrightarrow{\varphi} O_{\mathbb{P}^3}^{3r} \longrightarrow F \longrightarrow 0
\]

by [CH11, Proposition 3.7]. Tensoring with $F^\vee$, we get a right exact sequence on $X$,

\[
0 \longrightarrow F^\vee(-1) \xrightarrow{\varphi} O_{\mathbb{P}^3}^{3r} \longrightarrow F \otimes F^\vee \longrightarrow 0.
\]

Now $F^\vee(2)$ is an Ulrich bundle, so $F^\vee$ and $F^\vee(-1)$ have no cohomology. It follows from cohomology sequences on $X$, that $H^2(F \otimes F^\vee) = H^3(F \otimes F^\vee) = 0$. Therefore, the moduli space is smooth. Furthermore, $\chi(F \otimes F^\vee) = \chi(F_H \otimes F_H^\vee) = -r^2$ by [CH11, Corollary 2.13], where $F_H$ is the general hyperplane section of $F$, since $c_1(F) = rH$. For $F$ stable or simple we have $h^0(F \otimes F^\vee) = 1$, and $h^2(F \otimes F^\vee) = h^3(F \otimes F^\vee) = 0$, so $h^1(F \otimes F^\vee) = r^2 + 1$ is the dimension of the moduli space.

It remains to show the existence. We proceed by induction on $r$, the cases $r = 2, 3$ by Lemma 4.9. So let $r \geq 4$, and choose $E$ stable of rank 2, and $F$ stable of rank $r - 2$, different from $E$. Then $h^i(E \otimes F^\vee) = 0$ for $i = 0, 2, 3$, so $h^1(E \otimes F^\vee) = -\chi(E \otimes F^\vee) = -\chi(E_H \otimes F_H^\vee) = 2(r - 2)$ by [CH11, Corollary 2.13]. In particular, this number is positive, so there exist non-split extensions

\[
0 \longrightarrow E \longrightarrow G \longrightarrow F \longrightarrow 0,
\]

and the new bundle $G$ will be a simple Ulrich bundle of rank $r$. We consider the modular family of these simple bundles, which will be smooth of dimension $r^2 + 1$ by the above observations. If the general simple bundle in this family is not stable, it must have the same splitting type as the ones just constructed. However, the dimension of the family of extensions above is

$$\dim E + \dim F + \dim(\text{Ext}^1(F, E)) - 1 = r^2 - 2r + 5.$$ 

Since $r \geq 4$, this number is strictly less than $r^2 + 1$. We conclude that the general simple bundle of rank $r$ is stable, so stable bundles exist. 

For a very general hypersurface $X \subseteq \mathbb{P}^3$ of degree at least 4, the Noether-Lefschetz theorem says that every curve $C \subset X$ is a complete intersection of $X$ with a surface in $\mathbb{P}^3$, i.e. $C = X \cap S$ where $S \subset \mathbb{P}^3$ is a surface. As a consequence of this theorem, any ACM line bundle on such an $X$ is the restriction of a line bundle on $\mathbb{P}^3$. One might ask whether this generalizes in some way to higher codimension...
and higher dimensional hypersurfaces. Motivated by this, Griffiths and Harris conjectured whether subvarieties \( Y \) of codimension two of a hypersurface \( X \subset \mathbb{P}^n \) can be obtained by intersecting with a subvariety of codimension two of the ambient space \( \mathbb{P}^n \). We shall call subvarieties \( Y \subset X \subset \mathbb{P}^n \) which are not intersections of \( X \) with any subvariety of codimension two of \( \mathbb{P}^n \) as \textit{distinguished}. C. Voisin very soon proved that a general threefold \( X \subset \mathbb{P}^4 \) always contains distinguished curves \( C \subset X \), thus proving that this conjecture is false. In [KRR09], it is shown that there exists a large class of distinguished ACM subvarieties in smooth hypersurfaces of dimension at least three and degree at least two. But such ACM subvarieties may not give smooth ones. The next goal in this section is to provide large families of distinguished ACM subvarieties on special cubic fourfolds.

**Theorem 4.11.** Let \( F \) be an Ulrich bundle on a cubic fourfold \( X \) and \( Y \) a surface constructed from \( F \) as in Theorem 3.2. Then \( Y \) is a distinguished ACM subvarieties on \( X \).

**Proof.** It follows immediately that Proposition 4 in [KRR09] and Theorem 3.9.

**Theorem 4.12.** Let \( X \) be a very general cubic fourfold in \( \mathbb{P}^5 \) containing a plane. Then the moduli space of stable rank 4 Ulrich bundles on a cubic fourfold \( X \) is nonempty and smooth of dimension 17.

**Proof.** The argument can be divided in a few parts.

**Step 1:** There exists stable rank 4 Ulrich bundles \( F \) on a cubic fourfold \( X \). Let \( Z = \mathbb{P}^2(p_1, \ldots, p_{10}) \) is a blow-up of \( \mathbb{P}^2 \) in 10 points in general position by the complete linear system

\[
H_{Z} = (6; 2^5, 1^3) = 6L - \sum_{i=1}^{5} 2E_i - \sum_{i=6}^{10} E_i.
\]

Then \( Z \) is smooth surface of degree 11 and sectional genus 5 in \( \mathbb{P}^7 \). Let \( \ell \subset \mathbb{P}^7 \) be general line, and \( \phi_\ell: \mathbb{P}^7 \rightarrow \mathbb{P}^5 \) be the projection from \( \ell \) onto a hyperplane. Put \( Y_1 = \phi_\ell(Z) \subset \mathbb{P}^5 \). Therefore, \( Y_1 \) is a surface of degree 11 and sectional genus 5 and \( R_{Y_1}^2 = -1 \). Moreover, homogeneous coordinate ring \( R_{Y_1} = R/I_{Y_1} \) have the following Betti table

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & 1 & \cdot & \cdot & \cdot \\
3 & \cdot & 25 & 65 & 63 & 28 & 5 \\
\end{array}
\quad
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
0 & 1 & \cdot & \cdot \\
1 & 2 & 7 & \cdot \\
2 & \cdot & 1 & 10 & 5 \\
\end{array}
\]

Let \( X \) be a unique smooth cubic fourfold containing the surface \( Y_1 \). Consider \( \Gamma_*(\mathcal{O}_{Y_1}) \) as a \( R_X \)-module, the periodic part of its minimal free resolution yields, up to twist, matrix factorization of the form

\[
R_X^{12} \xrightarrow{\psi} R_X^{12}(-1) \xrightarrow{\varphi} R_X^{12}(-3).
\]

Let \( F^* \) and \( \mathcal{G}^* \) be minimal free resolutions of the section ring \( \Gamma_*(\mathcal{O}_{Y_1}) \) as an \( R \)-module and an \( R_X \)-module, respectively. We denote by \( \varphi \) the syzygy map \( \mathcal{G}_3 \rightarrow \mathcal{G}_4 \) and \( F = \text{coker}\varphi(-3) \). Then \( F \) is rank 4 Ulrich bundles \( F \) on the cubic fourfold \( X \).

**Step 2:** If \( X \) is very general cubic fourfold in \( \mathbb{P}^5 \) containing a plane, then \( X \) contains a surface \( Y \) of degree 22 as in Theorem 3.9. Since \( F \) is rank 4 Ulrich bundles \( F \) on the cubic fourfold \( X \), by Theorem 3.9, there exists an ACM surface \( Y \) of degree 22 as in Theorem 3.9. Moreover, the irreducible component of the Hilbert scheme parametrizing the surfaces \( Y \subset \mathbb{P}^5 \) has dimension 91 and it is generically smooth. Indeed, one can verify that \( h^1(N_{Y/\mathbb{P}^5}) = 0 \). Let \( \mathbb{C} \subseteq |H^0(\mathbb{P}^5(3))| \cong \mathbb{P}^{55} \) be the open set corresponding to smooth cubic hypersurfaces. Since \( h^0(\mathcal{I}_Y(3)) = 1 \) so that the locus \( D_8 = \{(Y), [X]) : Y \subseteq X \} \subset \mathcal{H} \times C \) has dimension \( 91 + 1 - 1 = 91 \). The image of \( \pi_2: D_8 \rightarrow C \) has
dimension at most 54 because the general cubic does not contain any $Y$ belonging to $\mathcal{H}$. For every $[X] \in \pi_2(D_8)$ we have
\[
\dim(\pi_2^{-1}([X])) \geq \dim(D_8) - \dim(\pi_2(D_8)) \geq 91 - 54 = 37.
\]
Since $h^0(\mathcal{N}_{Y/X}) \geq \dim([Y \subset X] \pi_2^{-1}([X]))$ for every $[Y \subset X] \in \pi_2^{-1}([X])$, to show that a general $X \in \mathcal{C}_8$ contains a surface $Y$ it is sufficient to verify that $h^0(\mathcal{N}_{Y/X}) = 37$ for a general $Y$ and for a smooth $X \in |H^0(\mathcal{D}_2(3))|$, see also [Nue17] for a similar argument. We verified this via Macaulay2 and we can conclude that $h^0(\mathcal{N}_{Y/X}) = 37$. Since $[X] \in \mathcal{C}_8$ if and only if $X$ contains a plane, we get statement as required.

**Step 3:** *Dimension and stableness.* To complete the dimensioness statement of Theorem, we should prove that for stable rank 4 Ulrich bundles $\mathcal{F}$ on $X$, we have $h^1(\mathcal{F} \otimes \mathcal{F}^\vee) = 17$ and $h^2(\mathcal{F} \otimes \mathcal{F}^\vee) = h^3(\mathcal{F} \otimes \mathcal{F}^\vee) = 0$. For this we can use the same deformation argument as in the proof of Theorem 1.3.

By Theorem 4.7, there are no rank 2 or 3 Ulrich bundles on a very general cubic fourfold $X$ containing a plane $P$. Therefore, as observed in [CH12, Section 5], $\mathcal{F}$ is stable. 

**Corollary 4.13.** Let $X$ be a very general cubic fourfold in $\mathbb{P}^5$ containing a plane or a general special cubic fourfold in $\mathbb{C}^4_1$ or $\mathbb{C}^4_8$. Then $X$ contains distinguished smooth surfaces $Y$.

**Proof.** These facts follow immediately from Theorems 3.9, 4.7, 1.3, 4.11 and 4.12. 

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