Quantum-Mechanical Interpretation of Riemann Zeta Function Zeros

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ABSTRACT: We demonstrate that the Riemann zeta function zeros define the position and the widths of the resonances of the quantised Artin dynamical system. The Artin dynamical system is defined on the fundamental region of the modular group on the Lobachevsky plane. It has a finite volume and an infinite extension in the vertical direction that correspond to a cusp. In classical regime the geodesic flow in the fundamental region represents one of the most chaotic dynamical systems, has mixing of all orders, Lebesgue spectrum and non-zero Kolmogorov entropy. In quantum-mechanical regime the system can be associated with the narrow infinitely long waveguide stretched out to infinity along the vertical axis and a cavity resonator attached to it at the bottom. That suggests a physical interpretation of the Maass automorphic wave function in the form of an incoming plane wave of a given energy entering the resonator, bouncing inside the resonator and scattering to infinity. As the energy of the incoming wave comes close to the eigenmodes of the cavity a pronounced resonance behaviour shows up in the scattering amplitude.
Figure 1. The non-compact fundamental region $F$ of a finite area $\pi/3$ is represented by the hyperbolic triangle $ABD$. The vertex $D$ is at infinity of the $y$ axis and corresponds to a cusp. The edges of the triangle are the arc $AB$, the rays $AD$ and $BD$. The points on the edges $AD$ and $BD$ and the points of the arcs $AC$ with $CB$ are identified by the transformations $w = z + 1$ and $w = -1/z$ in order to form a closed non-compact surface $\tilde{F}$ of sphere topology by "gluing together" the opposite edges of the modular triangle [19].

1 Introduction

Hyperbolic systems have exponential instability of their trajectories and as such represent the most natural chaotic dynamical systems [7]. Of special interest are systems which are defined on closed surfaces of the Lobachevsky plane of constant negative curvature. An example of such system was introduced in 1924 by the mathematician Emil Artin [1]. The dynamical system is defined on the fundamental region of the Lobachevsky plane which is obtained by the identification of points congruent with respect to the modular group $\Gamma = SL(2, \mathbb{Z})$, a discrete subgroup of the Lobachevsky plane isometries [2–4]. The fundamental region $F$ in this case is an infinitely long non-compact hyperbolic triangle of finite area shown in Fig.1. The geodesic trajectories are bounded to propagate on the fundamental region and represent one of the most chaotic dynamical systems with exponential instability of its trajectories, mixing of all orders, Lebesgue spectrum and non-zero Kolmogorov entropy [5–18].

In a recent article [19] the authors investigated the behaviour of quantum-mechanical correlation functions of the quantised Artin system [20]. The solution of the time-independent Schrödinger equation $H\psi = E\psi$

$$-y^2(\partial_x^2 + \partial_y^2)\psi = E\psi$$

with periodic boundary conditions on the wave function with respect to the modular
group in the fundamental region $\mathcal{F}$ shown in Fig. 1 is:

$$\psi\left(\frac{az+b}{cz+d}\right) = \psi(z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$  \hspace{1cm} (1.2)

and it is defined by the Maass non-analytical automorphic function [22, 25–31, 34, 35].

One important novelty introduced in the article [19] was the representation of the Maass wave function [22] in terms of the natural physical variable $\tilde{y}$ which represents the distance in the vertical direction of the Lobachevsky plane $\int dy/y = \ln y = \tilde{y}$ and of the corresponding momentum $p$ [19]. This allows to represent the energy eigenfunctions obtained by Maass in the form which is appealing to the physical intuition [19]:

$$\psi_p(x, \tilde{y}) = e^{-ip\tilde{y}} + \frac{\theta(\frac{1}{2} + ip)}{\theta(\frac{1}{2} - ip)} e^{ip\tilde{y}} + \prod_{l=1}^{\infty} \tau_{ip}(l) K_{ip}(2\pi l e^{\tilde{y}}) \cos(2\pi lx),$$  \hspace{1cm} (1.3)

where

$$\theta(s) = \pi^{-s} \zeta(2s) \Gamma(s)$$  \hspace{1cm} (1.4)

is a product of Riemann zeta and gamma functions, $K$ is the modified Bessel's function

$$K_{ip}(y) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-y \cosh t} e^{ipt} dt,$$  \hspace{1cm} (1.5)

and

$$\tau_{ip}(n) = \sum_{a-b=n} (\frac{a}{b})^{ip}. $$  \hspace{1cm} (1.6)

The first two terms of the wave function (1.3) describe the incoming and outgoing plane waves. The plane wave $e^{-ip\tilde{y}}$ incoming from infinity of the $y$ axis on Fig. 2, the vertex $D$, elastically scatters on the boundary $ACB$ of the fundamental triangle $\mathcal{F}$. The reflection amplitude is a pure phase and is given by the expression in front of the outgoing plane wave $e^{ip\tilde{y}}$:

$$S = \frac{\theta(\frac{1}{2} + ip)}{\theta(\frac{1}{2} - ip)} = \exp[i2\delta(p)]. $$  \hspace{1cm} (1.7)

The rest of the wave function describes the standing waves $\cos(2\pi lx)$ in the $x$ direction between the boundaries $x = \pm1/2$ with the amplitudes $K_{ip}(2\pi l e^{\tilde{y}})$, which are exponentially decreasing with index $l$. The continuous energy spectrum is given by the formula [19]

$$E = p^2 + \frac{1}{4}. $$  \hspace{1cm} (1.8)
The incoming and outgoing plane waves. The plane wave $e^{-ip\tilde{y}}$ incoming from infinity of the $y$ axis on Fig. 1 (the vertex $D$) elastically scatters on the boundary $ACB$ of the fundamental triangle $\mathcal{F}$ on Fig. 1. The reflection amplitude is a pure phase and is given by the expression in front of the outgoing plane wave $e^{ip\tilde{y}}$. The rest of the wave function describes the standing waves in the $x$ direction between the boundaries $x = \pm 1/2$ with the amplitudes, which are exponentially decreasing.

In physical terms the system can be described as a narrow infinitely long waveguide stretched out to infinity along the vertical direction and a cavity resonator attached to it at the bottom $ACB$ (see Fig. 3). In order to support this interpretation we shall calculate the area of the fundamental region which is below the fixed coordinate $y_0 = e^{\tilde{y}_0}$:

$$\text{Area}(\mathcal{F}_0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{y_0}^{\tilde{y}_0} \frac{dy}{\sqrt{1-x^2}} = \frac{\pi}{3} - e^{-\tilde{y}_0}, \quad (1.9)$$

and confirm that the area above the ordinate $\tilde{y}_0$ is exponentially small: $e^{-\tilde{y}_0}$. The horizontal ($dy = 0$) size of the fundamental region also decreases exponentially in the vertical direction:

$$L_0 = \int ds = \int \frac{\sqrt{dx^2 + dy^2}}{y} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{y_0} = e^{-\tilde{y}_0}. \quad (1.10)$$

One can suggest therefore the following physical interpretation of the Maass wave function (1.3): The incoming plane wave $e^{-ip\tilde{y}}$ of energy $E = p^2 + \frac{1}{4}$ enters the "cavity resonator", bouncing back into the outgoing plane wave at infinity $e^{ip\tilde{y}}$. As the energy of the incoming wave $E = p^2 + \frac{1}{4}$ becomes close to the eigenmodes of the cavity resonator one should expect a pronounced resonance behaviour of the scattering amplitude.

To trace such behaviour let us consider the analytical continuation of the Maass wave function (1.3) to the complex energies $E$. The analytical continuation of the scattering amplitudes as a function of the energy $E$ considered as a complex variable allows to establish important spectral properties of the quantum-mechanical system. In particular, the method of analytic continuation allows to determine the real and
complex S-matrix poles. The real poles on the physical sheet correspond to the
discrete energy levels and the complex poles on the second sheet below the cut
correspond to the resonances in the quantum-mechanical system [36].

Indeed, the asymptotic form of the wave function can be represented in the
following form:

\[
\psi = A(E) e^{ip\tilde{y}} + B(E) e^{-ip\tilde{y}}, \quad p = \sqrt{E - 1/4},
\]

(1.11)

and to make the functions \(A(E)\) and \(B(E)\) single-valued one should cut the complex
plane along the real axis [36] starting from \(E = 1/4\). The complex plane with a cut
so defined is called a physical sheet. To left from the cut, at energies \(E_0 < 1/4\), the
wave function will take the following form:

\[
\psi = A(E) e^{-\sqrt{|E-1/4|}\tilde{y}} + B(E) e^{\sqrt{|E-1/4|}\tilde{y}},
\]

(1.12)

where the exponential factors are real and one of them decreases and the other
one increases at \(\tilde{y} \to \infty\). The bound states are characterised by the fact that the
corresponding wave function tends to zero at infinity \(\tilde{y} \to \infty\). This means that
the second term in (1.12) should be absent, and a discrete energy level \(E_0 < 1/4\)
corresponds to a zero of the \(B(E)\) function [36]:

\[
B(E_0) = 0.
\]

(1.13)

Because the energy eigenvalues are real, all zeros of \(B(E)\) on the physical sheet are
real.

\[\text{Figure 3.}\] The system can be described as a narrow (1.10) infinitely long waveguide
stretched to infinity along the vertical dierection and a cavity resonator attached to it at
the bottom \(ACB\).
The resonances $E_n - i\frac{\Gamma_n}{2}$ are located under the right hand side of the real axis.

Now consider a system which is unstable and therefore does not have a pure discrete spectrum, the motion of the system is unbounded and the energy spectrum is continuous [36]. The energy spectrum is quasi-discrete, consisting of smeared levels of a width $\Gamma$. In describing such states one should consider the wave functions which are diverging at infinity, describing a wave packet moving to infinity. Thus the boundary condition at infinity requires the presence of only outgoing waves. This boundary condition involves complex quantities and the energy eigenvalues in general are also complex [36]. With such boundary conditions the Hermitian energy operators can have complex eigenvalues of the form [36]

$$E = E_0 - i\frac{\Gamma}{2},$$

(1.14)

where $E_0$ and $\Gamma$ are both real and positive. The condition which defines the complex energy eigenvalues (1.14) lies in the requirement that at $E = E_0 - i\frac{\Gamma}{2}$ the incoming wave $e^{-i\tilde{p}y}$ in (1.11) is absent [36]:

$$B(E_0 - i\frac{\Gamma}{2}) = 0.$$  

(1.15)

The point $E_0 - i\frac{\Gamma}{2}$ is located under the right hand side of the real axis, see Fig.4. In order to reach that point without leaving the physical sheet one should move from the upper side of the cut anticlockwise. However in that case, the phase of the wave function changes its sign and the outgoing wave transforms into the incoming wave. In order to keep the outgoing character of the wave function one should cross the cut strait into the second sheet Fig.4. Expanding the function $B(E)$ near the quasi-discrete energy level (1.14) as $B(E) = (E - E_0 + i\frac{\Gamma}{2})b + \ldots$ one can get

$$\psi \approx b^* (E - E_0 - i\frac{\Gamma}{2}) e^{i\tilde{p}y} + b (E - E_0 + i\frac{\Gamma}{2}) e^{-i\tilde{p}y}$$

(1.16)

and the S-matrix will take the following form [36]

$$S = e^{2i\delta} = \frac{E - E_0 - i\Gamma/2}{E - E_0 + i\Gamma/2} e^{2i\delta_0}.$$  

(1.17)
where $e^{2i\delta_0} = b^*/b$. One can observe that moving throughout the resonance region the phase is changing by $\pi$.

Let us now consider the asymptotic behaviour of the wave function (1.3) at large $\tilde{y}$. The conditions (1.13) and (1.15) of the absence of incoming wave takes the form:

$$\theta\left(\frac{1}{2} - ip\right) = 0$$

(1.18)

and due to (1.4):

$$\theta\left(\frac{1}{2} - ip\right) = \frac{\zeta(1 - 2ip)\Gamma\left(\frac{1}{2} - ip\right)}{\pi^{\frac{1}{2} - ip}} = 0.$$  

(1.19)

The solution of this equation can be expressed in terms of zeros of the Riemann zeta function [21]:

$$\zeta\left(\frac{1}{2} - iu_n\right) = 0, \quad u_n > 0.$$  

(1.20)

Thus one should solve the equation

$$1 - 2ip_n = \frac{1}{2} - iu_n.$$  

(1.21)

The location of poles is therefore at the following values of the complex momenta

$$p_n = \frac{u_n}{2} - i\frac{1}{4}, \quad n = 1, 2, \ldots$$  

(1.22)

and at the corresponding complex energies (1.8):

$$E = p_n^2 + \frac{1}{4} = \left(\frac{u_n}{2} - i\frac{1}{4}\right)^2 + \frac{1}{4} = \frac{u_n^2}{4} + \frac{3}{16} - i\frac{u_n}{4}.$$  

(1.23)

Thus one can observe that there are resonances (1.14)

$$E = E_n - \frac{\Gamma_n}{2}$$

(1.24)

at the following energies and of the corresponding widths (1.23):

$$E_n = \frac{u_n^2}{4} + \frac{3}{16}, \quad \Gamma_n = \frac{u_n}{2}.$$  

(1.25)

The ratio of the width to the energy tends to zero [21]:

$$\frac{\Gamma_n}{E_n} = \frac{u_n}{2}/(\frac{u_n^2}{4} + \frac{3}{16}) \approx \frac{2}{u_n} \to 0$$  

(1.26)

and the resonances become infinitely narrow. The ratio of the width to the energy spacing between nearest levels is

$$\frac{\Gamma_n}{E_{n+1} - E_n} = \frac{2u_n}{(u_{n+1} + u_n)(u_{n+1} - u_n)} \approx \frac{1}{u_{n+1} - u_n}.$$  

(1.27)
As far as the zeros of the zeta function have the property to "repel", the difference $u_{n+1} - u_n$ can vanish with small probability [32, 33].

Thus one can conjecture the following representation of the S-matrix (1.7): 

$$S = e^{2i\delta} = \frac{\theta(\frac{1}{2} + ip)}{\theta(\frac{1}{2} - ip)} = \sum_{n=1}^{\infty} \frac{E - E_n - i\Gamma_n/2}{E - E_n + i\Gamma_n/2} e^{2i\delta_n}$$  \hspace{1cm} (1.28)

with yet unknown phases $\delta_n$. In order to justify the above representation of the S-matrix we shall try to find the location of the poles on the second Riemann sheet by using expansion of the S-matrix (1.7) at the "bumps" which occur along the real axis at energies 

$$E_n = \frac{u_n^2}{4} + \frac{3}{16}.$$  \hspace{1cm} (1.29)

The expansion will take the following form:

$$S|_{E \approx E_n} = \frac{\theta(\frac{1}{2} + i\sqrt{E - \frac{i}{4}})}{\theta(\frac{1}{2} - i\sqrt{E - \frac{i}{4}})}|_{E \approx E_n}$$

$$= \frac{\theta(\frac{1}{2} + i\sqrt{E_n - \frac{i}{4}}) + i\theta'(\frac{1}{2} + i\sqrt{E_n - \frac{i}{4}})}{\theta(\frac{1}{2} - i\sqrt{E_n - \frac{i}{4}}) + i\theta'(\frac{1}{2} - i\sqrt{E_n - \frac{i}{4}})} \frac{(E - E_n)}{(E - E_n + i\Gamma_n/2)}$$

$$= \frac{E - E_n}{E - E_n + i\Gamma_n/2} e^{2i\delta_n},$$  \hspace{1cm} (1.30)

where 

$$E'_n - i\Gamma_n'/2 = E_n - \frac{\theta(\frac{1}{2} - i\sqrt{E_n - \frac{i}{4}})}{\theta'(\frac{1}{2} - i\sqrt{E_n - \frac{i}{4}})}$$  \hspace{1cm} (1.31)

$$e^{2i\delta_n} = \frac{\theta'(\frac{1}{2} + i\sqrt{E_n - \frac{i}{4}})}{\theta'(\frac{1}{2} - i\sqrt{E_n - \frac{i}{4}})},$$

thus 

$$E'_n - E_n = -\Re \frac{\theta(\frac{1}{2} - i\sqrt{E_n - \frac{i}{4}})}{\theta'(\frac{1}{2} - i\sqrt{E_n - \frac{i}{4}})} - i\Gamma_n'/2 = -\Im \frac{\theta(\frac{1}{2} - i\sqrt{E_n - \frac{i}{4}})}{\theta'(\frac{1}{2} - i\sqrt{E_n - \frac{i}{4}})}$$  \hspace{1cm} (1.32)

and all quantities $E'_n$, $\Gamma'_n/2$ and $\delta_n$ are real.

Let us consider the first ten zeros of the zeta function which are known numerically [32, 33]. Using that information one can calculate the position of the resonances and their widths (1.25). In the Table 1 we present the values of energies and the widths of the resonances given by exact formula (1.25). In Table 2 the energies and widths are calculated using the approximation formulas (1.31). As one can see the approximation is consistent with the exact values within the two percent deviation.
Table 1. The table presents the numerical values \( u_n \) of the zeros of the Riemann zeta function. The corresponding energies and widths (1.25) are given on the second and the third columns.

| Position of zeros | Energies \( E_n \) | Width \( \Gamma_n \) |
|-------------------|-------------------|------------------|
| \( u_1 = 14.1347 \) | \( E_1 = 50.1351 \) | \( \Gamma_1 = 3.53368 \) |
| \( u_2 = 21.0220 \) | \( E_2 = 110.669 \) | \( \Gamma_2 = 5.25551 \) |
| \( u_3 = 25.0109 \) | \( E_3 = 156.573 \) | \( \Gamma_3 = 6.25271 \) |
| \( u_4 = 30.4249 \) | \( E_4 = 231.606 \) | \( \Gamma_4 = 7.60622 \) |
| \( u_5 = 32.9351 \) | \( E_5 = 271.367 \) | \( \Gamma_5 = 8.23377 \) |
| \( u_6 = 37.5862 \) | \( E_6 = 353.368 \) | \( \Gamma_6 = 9.39654 \) |
| \( u_7 = 40.9187 \) | \( E_7 = 418.773 \) | \( \Gamma_7 = 10.2927 \) |
| \( u_8 = 43.3271 \) | \( E_8 = 469.496 \) | \( \Gamma_8 = 10.8318 \) |
| \( u_9 = 48.0052 \) | \( E_9 = 576.311 \) | \( \Gamma_9 = 12.0013 \) |
| \( u_{10} = 49.7738 \) | \( E_{10} = 619.546 \) | \( \Gamma_{10} = 12.4435 \) |
| \( u_{10} = 49.7738 \) | \( E_{10} = 619.546 \) | \( \Gamma_{10} = 12.4435 \) |

Table 2. The table presents the numerical values \( u_n \) of the zeros of the Riemann zeta function. The energies and widths are calculated by using the approximation formulas (1.31).

| Position of zeros | Energies \( E'_n \) | Width \( \Gamma'_n \) |
|-------------------|-------------------|------------------|
| \( u_1 = 14.1347 \) | \( E'_1 = 51.2732 \) | \( \Gamma'_1 = 3.05908 \) |
| \( u_2 = 21.0220 \) | \( E'_2 = 112.487 \) | \( \Gamma'_2 = 4.32077 \) |
| \( u_3 = 25.0109 \) | \( E'_3 = 158.363 \) | \( \Gamma'_3 = 5.42025 \) |
| \( u_4 = 30.4249 \) | \( E'_4 = 234.382 \) | \( \Gamma'_4 = 5.79733 \) |
| \( u_5 = 32.9351 \) | \( E'_5 = 273.225 \) | \( \Gamma'_5 = 7.20321 \) |
| \( u_6 = 37.5862 \) | \( E'_6 = 356.546 \) | \( \Gamma'_6 = 7.5043 \) |
| \( u_7 = 40.9187 \) | \( E'_7 = 422.097 \) | \( \Gamma'_7 = 7.99925 \) |
| \( u_8 = 43.3271 \) | \( E'_8 = 471.764 \) | \( \Gamma'_8 = 9.44046 \) |
| \( u_9 = 48.0052 \) | \( E'_9 = 580.782 \) | \( \Gamma'_9 = 8.29622 \) |
| \( u_{10} = 49.7738 \) | \( E'_{10} = 621.9 \) | \( \Gamma'_{10} = 10.4703 \) |
| \( u_{10} = 49.7738 \) | \( E'_{10} = 621.9 \) | \( \Gamma'_{10} = 10.4703 \) |

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References

[1] Emil Artin, *Ein mechanisches system mit quasiergodischen bahnen*, E. Abh. Math. Semin. Univ. Hambg. (1924) 3: 170.

[2] Henri Poincaré *Théorie des Groupes Fuchsiennes*, Acta Mathematica, 1 (1882) 1-62.

[3] Henri Poincaré *Mémoire sur les Fonctions Fuchsiennes*, Acta Mathematica, 1 (1882) 193-294.

[4] Lazarus Fuchs, *Über eine Klasse von Funktionen mehrerer Variablen, welche durch Umkehrung der Integrale von Lösungen der linearen Differentialgleichungen mit rationalen Coefficienten entstehen*, J. Reine Angew. Math., 89 (1880) 151-169.

[5] J. Hadamard, *Les surfaces à courbures opposées et leur lignes géodesiques*, Liouville, Journal de Mathématique, 4 (1898) 27.

[6] G. Hedlund, *The dynamics of geodesic flow*, Bull. Am. Math. Soc. 45 (1939) 241-246.

[7] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds with negative curvature*, Trudy Mat. Inst. Steklov., Vol. 90 (1967) 3 - 210.

[8] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*. (Lecture Notes in Mathematics, no. 470: A. Dold and B. Eckmann, editors). Springer-Verlag (Heidelberg, 1975), 108 pp.

[9] A.N. Kolmogorov, *New metrical invariant of transitive dynamical systems and automorphisms of Lebesgue spaces*, Dokl. Acad. Nauk SSSR, 119 (1958) 861-865.

[10] A.N. Kolmogorov, *On the entropy per unit time as a metrical invariant of automorphism*, Dokl. Acad. Nauk SSSR, 124 (1959) 754-755.

[11] Ya.G. Sinai, *On the Notion of Entropy of a Dynamical System*, Doklady of Russian Academy of Sciences, 124 (1959) 768-771.

[12] D. Ruelle, *Thermodynamic Formalism*, Addison-Wesley, Reading, Mass., 1978.

[13] E. Hopf. *Statistik der Lösungen geodätischer Probleme vom unstabilen Typus. II.*, Math. Ann. 117 (1940) 590-608.

[14] E. Hopf, *Ergodic theory and the geodesic flow on surfaces of constant negative curvature*, Bull. Amer. Math. Soc., 77 (1971) 863-877.

[15] I.M. Gelfand and S.V. Fomin, *Geodesic flows on manifolds of constant negative curvature*, Uspekhi Mat. Nauk, 7 (1952) 118-137; Amer. Math. Soc. Translation 1 (1965) 49-65.

[16] H. Poghosyan, H. Babujian and G. Savvidy, *Artin Billiard Exponential Decay of Correlation Functions*, arXiv:1802.04543 [nlin.CD].

[17] G. Savvidy, *The Yang-Mills mechanics as a Kolmogorov K-system*, Phys. Lett. B 130 (1983) 303.
[18] G. Savvidy, *Classical and Quantum Mechanics of non-Abelian Gauge Fields*, Nucl. Phys. B 246 (1984) 302.

[19] H. Babujian, R. Poghossian and G. Savvidy, *Correlation Functions of Classical and Quantum Artin System defined on Lobachevsky Plane and Scrambling Time*, arXiv:1808.02132 [hep-th].

[20] L. D. Faddeev, *Feynman integral for singular Lagrangians*, Theor. Math. Phys. 1 (1969) 1 [Teor. Mat. Fiz. 1 (1969) 3]. doi:10.1007/BF01028566

[21] B. Riemann *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie, November 1859.

[22] H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen*, Math. Ann. 121, No 2 (1949), 141-183.

[23] A. Selberg, *On the zeros of Riemann’s zeta-function*, Skr. Norske Vid. Akad. Oslo I., 10 (1942) 59.

[24] A. Selberg, *Contributions to the theory of the Riemann zeta-function*, Arch. Math. Naturvid., 48 (1946) 89/155.

[25] W. Roelcke, *Über die Wellengleichung bei Grenzkreisgruppen erster Art*, Sitzungsber. Heidelberg. Acad. Wiss. 4 Abh. (1953/1956), 161-267.

[26] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetrical Riemannian spaces with applications to Dirichlet series*, Indian Journ. Math. Soc. 20 (1956) 47-87.

[27] A. Selberg, *Discontinuous groups and harmonic analysis*, Proceedings of Stockholm Mathematical Congress (1962).

[28] L.D. Faddeev, *Expansion in eigenfunctions of the Laplace operator on the fundamental domain of a discrete group on the Lobachevsky plane*, Trans. Moscow Math. Soc., 17 (1967) 357-386.

[29] L.D. Faddeev, A. B. Venkov and V. L. Kalinin *A non-arithmetic derivation of the Selberg trace formula*, J. Soviet Math., 8 2 (1977) 171-199.

[30] D.A. Hejhal, *The Selberg Trace Formula for PSL(2, R)*, Lecture Notes in Mathematics 548, Springer-Verlag Vol. 1 1976.

[31] A. Winkler, *Cusp forms and Hecke groups*, J. Reine Angew. Math. 386 (1988) 187

[32] A. M. Turing, *Some calculations of the Riemann zeta-function*, Proceedings of the London Mathematical Society, Third Series, 3 (1953) 99-117; doi:10.1112/plms/s3-3.1.99

[33] X. Gourdon, *The 10^{13} first zeros of the Riemann Zeta function, and zeros computation at very large height*, October 24-th 2004.

[34] D.A. Hejhal, *Eigenvalues of the Laplacian for PSL(2,Z) : some new results and computational techniques*, in International Symposium in Memory of Hua Loo-Keng (ed. by Gong, Lu, Wang, Yang), Science Press and Springer-Verlag 1 (1991) 59-102.
[35] D.A. Hejhal and B. Berg, *Some new results concerning eigenvalues of the non-Euclidean Laplacian for PSL(2, Z)*, Univ. of Minn. Math. Report No. **82-172** (1982) 7pp.

[36] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, 3rd Edition, eBook ISBN: 9781483149127, Imprint: Pergamon, Published Date: 23rd May 1977 (Chapter 17: ELASTIC COLLISIONS).