On the properties of the Ernst–Manko–Ruiz equatorially antisymmetric solutions

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Abstract

Two new equatorially antisymmetric solutions recently published by Ernst et al are studied. For both solutions the full set of metric functions is derived in an explicit analytic form and the behavior of the solutions on the symmetry axis is analyzed. It is shown in particular that two counter-rotating equal Kerr–Newman–NUT objects will be in equilibrium when the condition \(m^2 + \nu^2 = q^2 + b^2\) is verified, whereas two counter-rotating equal masses endowed with arbitrary magnetic and electric dipole moments cannot reach equilibrium under any choice of the parameters, so that a massless strut between them will always be present.

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1. Introduction

In the recent papers [1, 2], the notion of equatorial antisymmetry has been introduced for stationary axisymmetric electrovac spacetimes, and two new exact solutions of that type have been constructed within the framework of Sibgatullin’s method [3, 4]. Since Ernst et al presented their solutions only in terms of the Ernst complex potentials \(E\) and \(\Phi\) [5], it would be of interest to have the complete metrics related to those solutions because they could facilitate the analysis of physical properties of the new equatorially antisymmetric spacetimes. Another aspect of the Ernst–Manko–Ruiz (EMR) solutions requiring the knowledge of the respective metric fields is the following: these solutions describe some binary systems of two counter-rotating masses in the presence of the electromagnetic field, so with the aid of the analytical expressions for the metric functions it would be possible to consider the equilibrium problems of two counter-rotating constituents in the EMR spacetimes.
Bearing in mind the two main objectives above, in section 2 of the present paper the metrical fields of the EMR solutions will be derived using expansions of the determinantal formulæ of paper [6] in the $N = 2$ case. Furthermore, in section 3 the obtained analytical formulæ fully defining the EMR spacetimes will be utilized for the resolution of the equilibrium problem of two equal counter-rotating Kerr–Newman–NUT particles. Apart from the behavior of EMR solutions on the symmetry axis, the stationary limit surfaces, ring singularities and some limits of these solutions will be also considered.

2. The Ernst potentials and metric functions of EMR solutions

The equatorially antisymmetric EMR solutions belong to the $N = 2$ subclass of the analytically extended multisolution solution whose Ernst potentials $E$ and $\Phi$ in the general $N = 2$ case are given by the formulæ [6]

$$E = E_+/E_-, \quad \Phi = F/E_-,$$

$$E_\pm = \pm \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \pm \frac{r_f}{a - \beta_1} & \pm \frac{r_a}{a - \beta_1} & \cdots & \pm \frac{r_a}{a - \beta_1} \\ 0 & \frac{b_1(a_1)}{a - \beta_1} & \cdots & \frac{b_1(a_1)}{a - \beta_1} \\ 0 & \frac{b_2(a_1)}{a - \beta_1} & \cdots & \frac{b_2(a_1)}{a - \beta_1} \end{bmatrix}, \quad F = \begin{bmatrix} 0 & f(\alpha_1) & \cdots & f(\alpha_4) \\ -1 & \frac{r_f}{a - \beta_1} & \cdots & \frac{r_a}{a - \beta_1} \\ -1 & \frac{b_1(a_1)}{a - \beta_1} & \cdots & \frac{b_1(a_1)}{a - \beta_1} \\ 0 & \frac{b_2(a_1)}{a - \beta_1} & \cdots & \frac{b_2(a_1)}{a - \beta_1} \end{bmatrix},$$

$$r_n = \sqrt{\beta^2 + (z - \alpha_n)^2},$$

where $\beta_1$ are arbitrary complex parameters, $\alpha_n$ can take on arbitrary real values or occur in complex conjugate pairs, and a bar over a symbol means complex conjugation; $h_1(\alpha_n)$ and $f(\alpha_n)$ are constant objects defined as

$$h_1(\alpha_n) = \tilde{e}_1 + 2\tilde{f}_1 f(\alpha_n), \quad f(\alpha_n) = \frac{f_1}{\alpha_n - \beta_1} + \frac{f_2}{\alpha_n - \beta_2},$$

$e_1$ and $f_1$ being coefficients in the expressions of the potentials $E$ and $\Phi$ on the symmetry axis:

$$e(z) = 1 + \frac{e_1}{z - \beta_1} + \frac{e_2}{z - \beta_2}, \quad f(z) = \frac{f_1}{z - \beta_1} + \frac{f_2}{z - \beta_2}. $$

The first of the EMR solutions (henceforth referred to as solution I), representing two counter-rotating electrically and magnetically charged masses, is defined by the axis data

$$e(z) = \frac{z - k - m - i(a + v)}{z - k + m - i(a + v)} + \frac{z + k - m + i(a - v)}{z + k + m + i(a + v)},$$

$$f(z) = \frac{2(q + ib)z}{[z + k + m + i(a - v)][z - k + m + i(a + v)]},$$

where the parameters $m, a, q, b$ are the mass, angular momentum per unit mass, electric charge and magnetic charge, respectively; $v$ is the NUT parameter and $k$ is half the coordinate distance between the masses. Using (4) one easily obtains the quantities

$$e_1 = -\frac{2(m + i\nu)[k - m + i(a + v)]}{k + ia}, \quad e_2 = -\frac{2(m + i\nu)[k + m + i(a + v)]}{k + ia},$$

$$\beta_1 = k - m + i(a - v), \quad \beta_2 = -k - m - i(a + v),$$

$$f_1 = \frac{(q + ib)[k - m + i(a - v)]}{k + ia}, \quad f_2 = \frac{(q + ib)[k + m + i(a + v)]}{k + ia}. $$
by decomposing \( e(z) \) and \( f(z) \) into simple fractions, and also the corresponding parameters \( \alpha_n \), namely,

\[
\alpha_1 = \alpha_+, \quad \alpha_2 = \alpha_-, \quad \alpha_3 = -\alpha_-, \quad \alpha_4 = -\alpha_+,
\]

\[
\alpha_\pm = \sqrt{\delta \pm 2d}, \quad \delta = m^2 + k^2 + 3v^2 - a^2 - 2(q^2 + b^2),
\]

\[
d = [(m^2 + v^2 - q^2 - b^2)(k^2 + 2v^2 - a^2 - q^2 - b^2) - (mv - ka)^2]^{1/2},
\]

as roots of the algebraic equation

\[
e(z) + \bar{e}(z) + 2f(z)\bar{f}(z) = 0.
\]

Formulae (5) and (6) fully determine the respective quantities \( f(\alpha_n) \) and \( h_1(\alpha_n) \) that appear in the determinants (1). The potentials \( \mathcal{E} \) and \( \Phi \) calculated in [2] for the data (4) have the form

\[
\mathcal{E} = \frac{A - B}{A + B}, \quad \Phi = \frac{C}{A + B},
\]

\[
A = (m^2 + v^2 - q^2 - b^2)[(m^2 + k^2 + v^2 + a^2)^2 - 4(mk + av)^2]
\]

\[
\times (R_+ - R_-)(r_+ - r_-) - \delta \alpha_+\alpha_- (R_+ + R_-)(r_+ + r_-)
\]

\[
+ 2\alpha_\pm \alpha_\mp [(m^2 + v^2 - q^2 - b^2)(m^2 - k^2 + a^2 - v^2) + 2(mv - ka)^2]
\]

\[
\times (R_+ R_- + r_+ r_-) + 2i(dmv - ka)(R_+ R_- - r_+ r_-)),
\]

\[
B = 4d\alpha_+\alpha_- (m + iv)[(m^2 + v^2 - q^2 - b^2 + i(mv - ka)]
\]

\[
\times (R_+ + R_- - r_+ - r_-) - d(R_+ + R_- + r_+ + r_-),
\]

\[
C = 4d\alpha_+\alpha_- (q + ib)[(m^2 + v^2 - q^2 - b^2 + i(mv - ka)]
\]

\[
\times (R_+ + R_- - r_+ - r_-) - d(R_+ + R_- + r_+ + r_-),
\]

\[
R_\pm = \sqrt{\rho^2 + (z \pm \alpha_+)^2}, \quad r_\pm = \sqrt{\rho^2 + (z \pm \alpha_-)^2}.
\]

The other EMR equatorially antisymmetric solution (henceforth solution II), describing a pair of counter-rotating masses endowed with electric and magnetic dipole moments, arises from the axis data

\[
e(z) = \frac{z - k - m - i(a + v)}{z - k - m - i(a - v)} \frac{1}{z + k + m + i(a + v)},
\]

\[
f(z) = \frac{2(\chi + ic)}{[z + k + m + i(a - v)] (z + k + m + i(a + v))},
\]

so that the corresponding quantities \( e_1, e_2, \beta_1 \) and \( \beta_2 \) of these data are the same as in solution I, while \( f_1 \) and \( f_2 \) have the form

\[
f_1 = \frac{\chi + ic}{k + ia}, \quad f_2 = -\frac{\chi + ic}{k + ia}.
\]

For the parameters \( \alpha_n \) one has the expressions

\[
\alpha_1 = \alpha_+, \quad \alpha_2 = \alpha_-, \quad \alpha_3 = -\alpha_-, \quad \alpha_4 = -\alpha_+,
\]

\[
\alpha_\pm = \sqrt{\delta \pm 2d}, \quad \delta = m^2 + k^2 + 3v^2 - a^2,
\]

\[
d = \sqrt{(m^2 + v^2)(k^2 + 2v^2 - a^2) - (mv - ka)^2 - \chi^2 - c^2},
\]

with which the Ernst potentials defined by the axis data (9) were found to have the form [2]

\[
\mathcal{E} = \frac{A - B}{A + B}, \quad \Phi = \frac{C}{A + B},
\]

\[
A = [\delta((m^2 + v^2)^2 + (mv - ka)^2)] - d^2(3m^2 + k^2 + v^2 + a^2)
\]

\[
\times (R_+ - R_-)(r_+ - r_-) - \delta \alpha_+\alpha_- [\delta(m^2 + v^2) - \chi^2 - c^2].
\]
\[B = 4d\alpha_x\alpha_\nu(m + iv)[(m^2 + v^2) + i(mv - ka)]\]
\[(R_x + R_-)(r_+ + r_-) + 2\alpha_x\alpha_\nu[(m^2 + v^2)^2 + (mv - ka)^2 - d^2]\]
\[× (R_xR_- + r_+r_-) + 2id(mv - ka)(R_xR_- - r_+r_-)],\]
\[C = 4d(\chi + i\nu)[(m^2 + v^2) - i(mv - ka)[\alpha_x(R_x - r_+) - \alpha_\nu(r_- - r_+) + d(\alpha_xR_+ - \alpha_\nu(r_- - r_+))],\]
\[R_\pm = \sqrt{\rho^2 + (z \mp \alpha^2)^2}, \quad r_\pm = \sqrt{\rho^2 + (z \pm \alpha_x)^2}.\]  

The constants \(m, \alpha, \nu, k\) in (9)–(12) have the same meaning as in solution I, but \(c\) and \(\chi\) are the magnetic dipole and electric dipole parameters, respectively.

The metric functions \(f, \gamma\) and \(\omega\) which appear in the Papapetrou [7] axisymmetric stationary line element
\[dx^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2] - f(d\rho - \omega d\phi)^2\]  
can be calculated with the aid of the general formulae obtained in the paper [6]:
\[
\begin{align*}
f &= \frac{E_x\bar{E}_x + E_y\bar{E}_y + 2\bar{F}\bar{F}}{2E_x E_y}, \\
\omega &= \frac{2\text{Im}(E_x\bar{F} - \bar{E}_x G - F\bar{F})}{E_x E_y + E_y E_x + 2\bar{F}\bar{F}},
\end{align*}
\]
where the determinants \(G, H, I\) and \(K_0\) have the form
\[
G = \begin{vmatrix}
0 & r_1 + \alpha_1 - z & \cdots & r_{2N} + \alpha_2N - z \\
-1 & \alpha_1 - \beta_1 & \cdots & \alpha_{2N} - \beta_{2N} \\
0 & \alpha_1 - \beta_1 & \cdots & \alpha_{2N} - \beta_{2N} \\
0 & \alpha_1 - \beta_1 & \cdots & \alpha_{2N} - \beta_{2N}
\end{vmatrix}, \quad H = \begin{vmatrix}
z & 1 & \cdots & 1 \\
-\beta_1 & \alpha_1 - \beta_1 & \cdots & \alpha_{2N} - \beta_{2N} \\
-\beta_2 & \alpha_1 - \beta_1 & \cdots & \alpha_{2N} - \beta_{2N} \\
-\beta_2 & \alpha_1 - \beta_1 & \cdots & \alpha_{2N} - \beta_{2N}
\end{vmatrix},
\]
\[
I = \begin{vmatrix}
f_1 + f_2 & f(\alpha_1) & \cdots & f(\alpha_4) \\
z & 1 & \cdots & 1 \\
-\beta_1 & -1 & \cdots & 1 \\
-\beta_2 & -1 & \cdots & 1
\end{vmatrix}, \quad K_0 = \begin{vmatrix}
1 & \cdots & 1 \\
\alpha_1 - \beta_1 & \cdots & \alpha_{2N} - \beta_{2N} \\
\alpha_1 - \beta_1 & \cdots & \alpha_{2N} - \beta_{2N} \\
\alpha_1 - \beta_1 & \cdots & \alpha_{2N} - \beta_{2N}
\end{vmatrix}.
\]

It is clear that in all practical applications of the above formulae for the metric functions, determinants (15) should be expanded and then evaluated for some particular axis data with the aid of a computer program for analytical calculations. In the appendix, the reader can find the expansions of the determinants (15) which have proved to be most efficient for symbolic computer processing. The results of the calculations of metric functions with the help of formulae (A.1) and (A.2) are given below. The functions \(f, \gamma\) and \(\omega\) of both EMR solutions permit a unified representation, namely,
\[
\begin{align*}
f &= \frac{\Lambda\Lambda - B\bar{B} + C\bar{C}}{(A + B)(\Lambda + \bar{B})}, \\
\omega &= \frac{4\nu + \text{Im}[\bar{H}(A + B) - G(\Lambda + \bar{B}) - C\bar{I}]}{A\Lambda - B\bar{B} + C\bar{C}},
\end{align*}
\]

where
\[
\begin{align*}
\bar{A} &= A - 2\alpha_x\alpha_\nu(m + iv), \\
\bar{B} &= B - 2\alpha_x - 2i\alpha_\nu - 2i\alpha_x\alpha_\nu, \\
\bar{C} &= C - 2\alpha_x - 2i\alpha_\nu - 2i\alpha_x\alpha_\nu - 4\alpha_x\alpha_\nu m - 4\alpha_x\alpha_\nu v - 4i\alpha_x\alpha_\nu (mv - ka).
\end{align*}
\]
where the functions $A$, $B$, $C$, $r_\pm$, $r_\pm$ and quantities $\alpha_\pm$, $d$ are defined by formulæ (8) and (6) in the case of solution I, and by formulæ (12) and (11) in the case of solution II. The functions $G$, $H$ and $I$, entering the expression of $\omega$, have the following form:

$$G = -zB + 2(m + iv)A - 2d\alpha_+\alpha_-(m^2 + v^2 - q^2 - b^2)$$

$$\times [(\alpha_+ - \alpha_-)(R_+r_+ - R_-r_-) + (\alpha_+ + \alpha_-)(R_-r_+ - R_+r_-)]$$

$$+ 4d\alpha_+\alpha_-[d - m^2 - v^2 + q^2 + b^2 + i(ka - mv)]$$

$$\times [\alpha_+(m + iv)(R_- - R_+) - 2(m^2 + v^2 - q^2 - b^2)(R_+ + R_-)]$$

$$+ 4d\alpha_+\alpha_-[d + m^2 + v^2 - q^2 - b^2 - i(ka - mv)]$$

$$\times [\alpha_-(m + iv)(r_- - r_+) - 2(m^2 + v^2 - q^2 - b^2)(r_+ + r_-),$$

$$H = zA - 2(m + iv)B + \alpha_+\alpha_-(m - iv)$$

$$\times [(m^2 + v^2 - q^2 - b^2)[(\alpha_+ - \alpha_-)^2(R_+r_+ + R_-r_-)]$$

$$+ (\alpha_+ + \alpha_-)^2(R_+r_- + R_-r_+)] + 2d(m + iv)$$

$$\times [(\alpha_+ - \alpha_-)(R_+r_+ - R_-r_-) + (\alpha_+ + \alpha_-)(R_-r_+ - R_+r_-)]$$

$$+ 4(R_-R_+ + r_+r_-)[(m^2 + v^2 - q^2 - b^2)(k^2 - m^2 - v^2 - a^2)]$$

$$- 2(ka - mv)^2 + 8id(ka - mv)(R_+r_- - r_+r_-)]$$

$$\times [2(m - iv)(R_+ + R_-) + \alpha_+(R_+ - R_-)] - [m^2 + v^2 - q^2 - b^2 - i(ka - mv)]$$

$$\times [\alpha_-(r_+ - r_-) - \alpha_-(r_- - r_+)] - d[\alpha_+(R_+ - R_-) + \alpha_+(r_+ - r_-)],$$

$$I = 2(q + ib)(A + B) - [z + 2(m + iv)C + 2d\alpha_+\alpha_-\alpha_-(m - iv)(q + ib)$$

$$\times [(\alpha_+ - \alpha_-)(R_+r_+ - R_-r_-) + (\alpha_+ + \alpha_-)(R_-r_+ - R_+r_-)]$$

$$+ 4d\alpha_+\alpha_-\alpha_-(q + ib)[m^2 + v^2 - q^2 - b^2 - i(ka - mv)]$$

$$\times [2(m - iv)(R_+ + R_-) + \alpha_+(R_+ - R_-)] - [m^2 + v^2 - q^2 - b^2 - i(ka - mv)]$$

$$\times [\alpha_-(r_+ - r_-) + \alpha_-(r_- - r_+)] - d[\alpha_+(R_+ - R_-) + \alpha_+(r_+ - r_-)],$$

in the case of solution I, and

$$G = -zB + 2(m + iv)A + 2d[(\alpha_+ - \alpha_-)[(m^2 + v^2)\alpha_+\alpha_- + \chi^2 + c^2][R_+r_+ - R_-r_-]$$

$$+ (\alpha_+ + \alpha_-)[(m^2 + v^2)\alpha_+\alpha_- - \chi^2 - c^2][R_+r_- - R_-r_+]]$$

$$+ 4d\alpha_+\alpha_-(m + iv)[[(k + ia)^2 - (m + iv)^2][m^2 + v^2 - d + i(ka - mv)]$$

$$\times (r_+ - r_-) - 2\alpha_-(m - iv)[m^2 + v^2 - d - i(ka - mv)](r_+ + r_-)]$$

$$+ 4d\alpha_+\alpha_-(m + iv)[[(k + ia)^2 - (m + iv)^2][m^2 + v^2 + d + i(ka - mv)]$$

$$\times (r_+ - r_-) + 2\alpha_+(m - iv)(m^2 + v^2 - d - i(ka - mv))(R_+ + R_-),$$

$$H = zA - 2(m + iv)B + (m - iv)(\alpha_+ - \alpha_-)[(m^2 + v^2)\alpha_+\alpha_- + \chi^2 + c^2]$$

$$\times (R_+R_- + r_+r_-) + 2d\alpha_+\alpha_-(m + iv)(R_+R_- - r_+r_-)]$$

$$+ (m - iv)(\alpha_+ + \alpha_-) [(m^2 + v^2)\alpha_+\alpha_- - \chi^2 - c^2]$$

$$\times (R_+r_+ + R_-r_-) + 2d\alpha_+\alpha_-(m + iv)(R_+r_- - R_-r_+)]$$

$$- 4d\alpha_+\alpha_-(m - iv)[[(m^2 + v^2)^2 + (ka - mv)^2 - d^2][R_+R_- + r_+r_-]$$

$$- 2id(ka - mv)(R_+R_- - r_+r_-) + 4d\alpha_+\alpha_-(m + iv)$$

$$\times [m^2 + v^2 - i(ka - mv)][\alpha_-(r_+ - r_-) - \alpha_+(R_+ - R_-)]$$

$$+ d[\alpha_-(r_+ - r_-) + \alpha_+(R_+ - R_-)],$$

3 The formulæ of this paper have been obtained and checked using the Mathematica computer program [8].
in the case of solution II.

It is worth mentioning that an arbitrary additive constant in the expression of \( \omega \) in (16) was chosen in such a way that the constant \( \omega_0 \) in the definition of equatorially antisymmetric spacetimes [2] was equal to zero, i.e., \( \omega(\rho,z) = -\omega(\rho,-z) \) automatically.

3. The two-body equilibrium problem in EMR spacetimes

The expressions of the metric functions obtained in the previous section can be used for the analysis of the equilibrium problem of two counter-rotating constituents of the Kerr–Newman–NUT type. In order for these constituents to be in equilibrium, it is necessary that the conditions [9–12]

\[
\gamma = 0, \quad \omega = 0
\]  

(19)

are fulfilled on the part \( \rho = 0, |z| < \alpha_- \) of the symmetry axis that separates the particles (region II in figure 1). An important advantage of the equatorially antisymmetric systems is that the condition for \( \omega \) is fulfilled automatically in the region II, what can be checked directly with the aid of formulae (17) and (18), so that the condition for the metric function \( \gamma \) needs to be satisfied. The latter condition leads to the algebraic equation

\[
(m^2 + v^2 - q^2 - b^2)(k^2 + m^2 + a^2 + v^2)^2 - 4(km + av)^2 - \alpha_- \alpha_+ \delta = 0,
\]  

(20)
in the case of solution I, and to the algebraic equations

\[(\delta - \alpha \alpha_\gamma)(m^2 + \nu^2)^2 + (ka - m\nu)^2 + d^2(8 - 4(m^2 + \nu^2) - (1 \pm 2)\alpha \alpha_\gamma) = 0, \quad (21)\]

in the case of solution II. It should be mentioned that for both solutions \(\gamma = 0\) in the regions I and III, while \(\omega = 4\nu\) in the region I and \(\omega = -4\nu\) in the region III due to the presence of two semi-infinite NUT singularities described in the paper [19], the first singularity extending from \(\alpha_+\) to \(+\infty\) and the second from \(-\alpha_+\) to \(-\infty\). When \(\nu = 0\), both conditions (19) are verified identically in the regions I and III for solutions I and II because the latter solutions become asymptotically flat in that limit.

From (20) follows an important relation

\[m^2 + \nu^2 - q^2 - b^2 = 0, \quad (22)\]

at which two counter-rotating Kerr–Newman–NUT constituents are in equilibrium independently of the distance between them. Formula (22) generalizes the balance condition \(m^2 = q^2\) which is verified by two Majumdar–Papapetrou equal charged masses [13, 14] (condition \(m^2 = q^2\) also determines two spinning charged masses of the Perjés–Israel–Wilson type [15–17]).

Furthermore, it can be shown that in the case of solution I there are no equilibrium states other than those defined by the relation (22) because the equation

\[(k^2 + m^2 + a^2 + \nu^2)^2 - 4(km + a\nu)^2 - \alpha \alpha_\gamma \delta = 0 \quad (23)\]

has the roots that cause \(d = 0\) in the denominator of the function \(\gamma\) from (16) and, therefore, should be discarded. Indeed, first putting the term \(\alpha \alpha_\gamma \delta\) to the right-hand side of equation (23) and then taking square of the resulting equality, we arrive at the equation

\[[(k - m)^2 + (a - \nu)^2][(k + m)^2 + (a + \nu)^2] = 0, \quad (24)\]

in which the factor \(d^2\) has been canceled out. The two factors of the above equation become zero when

\[k = m, \quad a = \nu \quad \text{and} \quad k = -m, \quad a = -\nu, \quad (25)\]

and in both cases the solution I reduces to the ordinary NUT solution electrically and magnetically charged (see, e.g., [18]), that is, the two-body problem degenerates to the case of a single body.

In the case of solution II one can solve analytically the balance equations (21) in exactly the same way as equation (20) was solved for solution I. For each choice of the sign in (21) one then comes to an algebraic equation of higher order than the initial one which, however, factorizes into three factors, one of which is \(d^2\), and the other two permit, after equating them to zero, the solutions of the corresponding equations with respect to \(b^2\). The direct substitution of the expressions for \(b^2\) thus obtained into equations (21) shows that they do not satisfy the latter equations, thus being fictitious roots that must be discarded. The case \(d = 0\) must be discarded too on the same grounds as in the case of solution I. Therefore, in the equatorially antisymmetric systems of two magnetized masses there are no equilibrium states under any choice of parameters, and hence these masses are always supported by a massless strut [9] between them.

It should be emphasized that solutions I and II are asymptotically flat in the absence of the NUT parameter, in which case, by construction, they are regular on the upper and lower parts of the \(z\)-axis. In the presence of the NUT parameter, the conditions \(\gamma = 0, \omega \neq 0\) are verified on those parts of the axis for both solutions, giving rise to two semi-infinite
massive singularities of the NUT type [19]. We also mention that although it is tempting, following paper [20], to redefine the function $\gamma$, by adding a specific constant, in such a way that the condition $\gamma = 0$ is verified on the intermediate part of the symmetry axis, this would only change the single supporting intermediate strut $\gamma \neq 0$ of finite extension to a pair of semi-infinite struts $\gamma \neq 0$ of Israel’s type (massless in the absence of the NUT parameter), thus only worsening the situation and making the corresponding spacetimes frankly unphysical.

4. The multipole moments, basic limits, stationary limit surfaces and ring singularities

In paper [2] the multipole structure of solutions I and II was not studied, so it would be of interest to clarify this characteristic of the EMR solutions in some detail. We have calculated the first four mass, angular momentum, electric and magnetic multipole moments ($M_\nu$, $J_\nu$, $Q_\nu$, and $H_\nu$, respectively) as these were defined by Simon [21]. During the calculations we have used the Hoenselaers–Perjé procedure [22] rectified by Sotiriou and Apostolatos [23], yielding the following expressions for the multipole moments:

$$M_0 = 2m, \quad J_0 = 2v, \quad Q_0 = 2q, \quad H_0 = 2b, \quad M_1 = M_3 = 0, \quad J_1 = J_3 = 0, \quad Q_1 = Q_3 = 0, \quad H_1 = H_3 = 0$$

(26)

The above expressions support the physical meaning attributed to the parameters of EMR solutions in paper [2]. However, it may be observed that the rotational parameter $a$ in the EMR binary systems is not identical to the angular momentum per unit mass of the source, as it follows already in the pure vacuum case by considering the approximate double-Kerr solution (see formula (63) of [24]). From (26) and (27) follows that the main difference between the two solutions lies in the structure of the electromagnetic moments: in solution I the odd moments $Q_{2n+1}$ and $H_{2n+1}$ are equal to zero, whereas in solution II are equal to zero the even moments $Q_{2n}$ and $H_{2n}$.

In the absence of the electromagnetic field, both EMR solutions reduce to the same special vacuum spacetime for two counter-rotating Kerr–NUT masses, and this limit belongs to the double-Kerr family of solutions of Kramer and Neugebauer [25]. It is interesting that by further setting $k = m$, $a = v$, one arrives at the single NUT solution [26] with the total mass $2m$ and the NUT parameter $2v$ which, as was demonstrated by Manko and Ruiz [19], shares the property of being equatorially antisymmetric. Solution I reduces to the Bretón–Manko electrovac solution [27] for two counter-rotating Kerr–Newman masses when the NUT parameter $v$ and the magnetic charge $b$ are equal to zero. The physically most interesting subclass of solution II is defined by vanishing parameters $v$ and $\chi$, in which case this EMR solution represents two
counter-rotating magnetized masses, giving the first example of solutions of this kind known in the literature.

The stationary limit surfaces (SLS) which are defined by the equation \( f = 0 \), in the case of EMR spacetimes display several interesting properties worth mentioning here. In figures 2 and 3, we have plotted several particular SLS of solution I, while in figures 4 and 5 one can find particular SLS of solution II. Figures 2 and 4 clearly demonstrate a completely different evolution of SLS in solutions I and II as a function of the NUT parameter: in solution I (see figure 2) the SLS grows with increasing \( \nu \), while in solution II (figure 4) the increase of \( \nu \) causes the degeneration of SLS. In figures 3(A) and 5(A) the typical SLS of toroidal type show a clear similarity in the case of two hyperextreme constituents with positive masses for both solutions I and II. Nonetheless, in the case of constituents with negative masses the corresponding ring singularities develop differently. Indeed, it follows from figures 3(B) and (C) that ring singularities accompanying the negative mass in solution I are located outside the SLS, though very close to it. At the same time, looking at figures 5(B) and (C), one can see that the ring singularities of solution II, either in the subextreme or in the hyperextreme cases, lie on the SLS, exactly as in the pure vacuum case. We recall that ring singularities arise as solutions of the equation \( A + B = 0 \).
Figure 4. Particular SLS of solution II demonstrating the degeneration of SLS with growing NUT parameter \( \nu \). In all three cases \( m = 2, k = 3, a = 1, c = \chi = 1/2 \), while \( \nu \) varies in the following way: \( (A) \nu = 1/2, (B) \nu = 1, (C) \nu = 3/2 \).

Figure 5. Particular SLS of solution II: \( (A) \) the case of two hyperextreme constituents defined by \( m = 1, k = 3, a = 2, v = 1/4, c = \chi = 1/2 \); \( (B) \) two subextreme constituents with negative masses \( (m = -4, k = 3, a = 1, c = \chi = 1/2) \) and two ring singularities located at \( \rho \simeq 0.932, z \simeq \pm 5.908 \); \( (C) \) two hyperextreme constituents with negative masses \( (m = -2, k = a = 3, c = \chi = 1/2) \) and two ring singularities located at \( \rho \simeq 3.142, z \simeq \pm 3.836 \). The dots in \( (B) \) and \( (C) \) denote ring singularities.

5. Conclusions

The new symmetry discovered and described by Ernst et al [1] permits a systematic study of a large class of counter-rotating masses within the framework of general relativity. Thanks to the paper [1], and partly to the paper [19], we know for instance that the well-known NUT solution belongs to the family of equatorially antisymmetric spacetimes; it is also clear now that the word ‘antisymmetric’ applied more than a decade ago by Bretón and Manko to a system of two equal counter-rotating Kerr–Newman particles in [27] was quite appropriate. While the NUT solution represents a single body accompanied by two semi-infinite singularities, the new EMR 6-parameter solutions already describe the two-body systems endowed with singularities of the NUT type. In the present paper, we have constructed all metrical fields for both EMR solutions and have solved analytically the associated equilibrium problems, obtaining the genuine equilibrium states only for a particular subclass of solution I defined by the relation \( m^2 + \nu^2 = q^2 + b^2 \) (the two counter-rotating constituents then become hyperextreme). Some
physical properties of the EMR spacetimes have been also studied. As a final remark we would like to observe that in view of the physical differences existing in the equatorially symmetric case between the systems with even and odd numbers of particles (see, e.g., [28–30]) it may be interesting to consider the three-body equatorially antisymmetric solutions and compare them with the EMR two-body spacetimes.

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Appendix. Expansions of the determinants $E_\pm, F, G, H, I, K_0$

Since the determinants (1) and (15) contain the coordinate $\rho$ only through functions $r_n$, it is advantageous to expand these determinants over the two lines in which $r_n$ appear, using the Laplace rule. The resulting expressions for the determinants employed in the analytical computer codes for obtaining formulae (17) and (18) have the form

$$E_\pm = \Lambda \pm \Gamma = A \mp B,$$

$$\begin{align*}
\Lambda &= \sum_{1 \leq i < j \leq 4} (-1)^{i+j} r_i r_j (\alpha_i - \alpha_j) R_k \tilde{R}_{k} \tilde{R}_{j} [(\alpha_k - \tilde{\beta}_2) (\alpha_i - \tilde{\beta}_1) h_1(\alpha_k) h_2(\alpha_i) \\
&\quad - (\alpha_k - \tilde{\beta}_1) (\alpha_i - \tilde{\beta}_2) h_1(\alpha_i) h_2(\alpha_k)], \quad (k < l; \quad k, l \neq i, j)
\end{align*}$$

$$\begin{align*}
\Gamma &= \sum_{i=1}^{4} (-1)^{i+j} r_i R_j R_k \tilde{R}_k \tilde{R}_j [\tilde{R}_j f(\alpha_j) \\
&\quad \times [(\alpha_k - \tilde{\beta}_2) (\alpha_i - \tilde{\beta}_1) h_1(\alpha_k) h_2(\alpha_i) - (\alpha_k - \tilde{\beta}_1) (\alpha_i - \tilde{\beta}_2) h_1(\alpha_i) h_2(\alpha_k)] \\
&\quad - \tilde{R}_k f(\alpha_k) [(\alpha_j - \tilde{\beta}_2) (\alpha_i - \tilde{\beta}_1) h_1(\alpha_j) h_2(\alpha_i) - (\alpha_j - \tilde{\beta}_1) (\alpha_i - \tilde{\beta}_2) h_1(\alpha_i) h_2(\alpha_j)] \\
&\quad - (\alpha_j - \tilde{\beta}_1) (\alpha_i - \tilde{\beta}_2) h_1(\alpha_i) h_2(\alpha_j)], \quad (j < k < l; \quad j, k, l \neq i)
\end{align*}$$

$$\begin{align*}
G &= z\Gamma - (\beta_1 + \beta_2) \Lambda + \sum_{1 \leq i < j \leq 4} (-1)^{i+j} r_i r_j (\alpha_i^2 - \alpha_j^2) R_k \tilde{R}_k \tilde{R}_j \\
&\quad \times [(\alpha_k - \tilde{\beta}_2) (\alpha_i - \tilde{\beta}_1) h_1(\alpha_i) h_2(\alpha_i) - (\alpha_k - \tilde{\beta}_1) (\alpha_i - \tilde{\beta}_2) h_1(\alpha_i) h_2(\alpha_i)] \\
&\quad + \sum_{i=1}^{4} (-1)^{i+j} r_i R_j R_k \tilde{R}_k [\alpha_j \tilde{R}_j \\
&\quad \times [(\alpha_k - \tilde{\beta}_2) (\alpha_i - \tilde{\beta}_1) h_1(\alpha_k) h_2(\alpha_i) - (\alpha_k - \tilde{\beta}_1) (\alpha_i - \tilde{\beta}_2) h_1(\alpha_i) h_2(\alpha_k)]
\end{align*}$$
\[-a_k \bar{R}_j [(a_j - \tilde{\beta}_2)(a_k - \tilde{\beta}_1)h_1(a_j)h_2(a_k)]
\[-(a_j - \tilde{\beta}_1)(a_k - \tilde{\beta}_2)h_1(a_k)h_2(a_k)]]
\[+a_j \bar{R}_i [(a_j - \tilde{\beta}_2)(a_k - \tilde{\beta}_1)h_1(a_j)h_2(a_k)]
\[-(a_j - \tilde{\beta}_1)(a_k - \tilde{\beta}_2)h_1(a_k)h_2(a_k)]\],
\[H = z \Lambda - (\beta_1 + \beta_2)\Gamma + \sum_{i=1}^4 (-1)^i r_i \alpha_i R_j R_k \bar{R}_i \bar{R}_j \]
\[\times [\bar{R}_1[(a_j - \tilde{\beta}_2)(a_k - \tilde{\beta}_1)h_1(a_k)h_2(a_j)]
\[-(a_k - \tilde{\beta}_1)(a_j - \tilde{\beta}_2)h_1(a_j)h_2(a_k)]
\[+\bar{R}_2[(a_j - \tilde{\beta}_2)(a_k - \tilde{\beta}_1)h_1(a_k)h_2(a_j)]
\[-(a_j - \tilde{\beta}_1)(a_k - \tilde{\beta}_2)h_1(a_k)h_2(a_j)]\]
\[+ \sum_{1 \leq i < j \leq 4} (-1)^{i+j} r_i r_j \alpha_i R_j R_k \bar{R}_i \bar{R}_j \]
\[\times [\bar{R}_1[\tilde{e}_1(a_j - \tilde{\beta}_2)h_1(a_j) - \tilde{e}_2(a_j - \tilde{\beta}_2)h_1(a_j)]
\[-\tilde{e}_1[(a_k - \tilde{\beta}_1)h_2(a_k) - (a_k - \tilde{\beta}_2)h_2(a_k)]\]
\[+ \sum_{i=1}^4 (-1)^{i+1} r_i R_j R_k \bar{R}_i \bar{R}_j \]
\[\times [(\bar{R}_1[\tilde{e}_1(a_j - \tilde{\beta}_2)h_1(a_j) - \tilde{e}_2(a_j - \tilde{\beta}_2)h_1(a_j)]f(a_j)]
\[\bar{R}_1 R_i \bar{R}_i \]
\[-\tilde{e}_1[(a_k - \tilde{\beta}_1)h_2(a_k) - (a_k - \tilde{\beta}_2)h_2(a_k)]f(a_k)]\]
\[+ \sum_{i=1}^4 (-1)^i r_i \alpha_i R_j R_k \bar{R}_i \bar{R}_j \]
\[\times [(a_k - \tilde{\beta}_2)(a_k - \tilde{\beta}_1)h_1(a_k)h_2(a_k)]
\[-(a_k - \tilde{\beta}_1)(a_k - \tilde{\beta}_2)h_1(a_k)h_2(a_k)]\]
\[+ \sum_{1 \leq i < j \leq 4} (-1)^{i+j} \bar{R}_i \bar{R}_j f(a_i) - f(a_j)]\]
\[K_0 = \sum_{1 \leq i < j \leq 4} (-1)^{i+j} (\alpha_j - a_j) R_i R_j R_i \bar{R}_j [(a_k - \tilde{\beta}_2)(a_k - \tilde{\beta}_1)h_1(a_k)h_2(a_k)]
\[-(a_k - \tilde{\beta}_1)(a_k - \tilde{\beta}_2)h_1(a_k)h_2(a_k)], \quad (A.1)\]
where
\[R_k = (a_k - \tilde{\beta}_1) (a_k - \tilde{\beta}_2), \quad R_k = (a_k - \tilde{\beta}_1) (a_k - \tilde{\beta}_2) \quad (A.2)\]
are the constant objects first introduced by Manko and Ruiz for the vacuum soliton solution [31]. Note that the expressions (A.1) are given without the common factor \((\beta_1 - \beta_2)/\left(\prod_{n=1}^{n_4} R_n \tilde{R}_n\right)\).

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