Mysterious Properties of the Point at Infinity

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Abstract: From the viewpoints of the division by zero $1/0 = 0/0 = z/0 = 0$ and the division by zero calculus, we will examine the mysterious properties of the point at infinity in the sense of the Alexandroff one compactification of the complex plane which is realized by the stereographic projection.

Key Words: The point at infinity, stereographic projection, division by zero, division by zero calculus, $1/0 = 0/0 = z/0 = \tan(\pi/2) = 0$, Laurent expansion, conformal mapping center.

1 Introduction

The purposes of this paper are to introduce the very fundamental new concept of the point at infinity and to propose the related open problems.

The division by zero $1/0 = 0/0 = z/0$ itself will be quite clear and trivial with several natural extensions of the fractions against the mysteriously long history ([11]), as we can see from the concepts of the Moore-Penrose generalized inverse or the Tikhonov regularization method to the fundamental equation $az = b$ whose solutions lead to the definition $z = b/a$.

However, the result will show that for the elementary mapping

$$W = \frac{1}{z},$$

(1.1)

the image of $z = 0$ is $W = 0$ (should be defined from the form). This fact seems to be a curious one in connection with our well-established popular
image for the point at infinity on the Riemann sphere. As the representation
of the point at infinity of the Riemann sphere by the zero \( z = 0 \), we will see
some delicate relations between 0 and \( \infty \) which show a strong discontinuity
at the point of infinity on the Riemann sphere. We did not consider any value
of the elementary function \( W = 1/z \) at the origin \( z = 0 \), because we did not
consider the division by zero \( 1/0 \) in a good way. Many and many people
consider its value by the limiting like \( +\infty \) and \( -\infty \) or the point at infinity as
\( \infty \). However, their basic idea comes from continuity with the common and
natural sense or based on the basic idea of Aristotle. – For the related Greece
philosophy, see [17, 18, 19]. However, as the division by zero we will consider
the value of the function \( W = 1/z \) as zero at \( z = 0 \). We will see that this
new definition is valid widely in mathematics and mathematical sciences, see
([6, 7]) and the cited references for example. Therefore, the division by zero
will give great impacts to calculus, Euclidean geometry, differential equations,
analytic geometry, complex analysis and physics in the undergraduate level
and to our basic ideas for the space and universe. Here, we would like to
refer to some mysterious properties of the point at infinity.

2 Logical background

As stated in the introduction, our results are different from the very classical
results and ideas since Aristotle and Euclid. Therefore, we will check and
confirm simply our new mathematics.

At first, the number system containing the division by zero is established
as the Yamada field in [5] by a minor arrangement of the well-established
complex number field. The Yamada field is a natural and simple extension
of the complex field containing the division by zero.

The uniqueness theorem for the generalized fraction (division) is es-
stablished by Takahasi [2] under the very general assumption of the product
\( (a/b) \cdot (c/d) = (ac)/(bd) \). If this property is not satisfied, then we will not
be able to find any fundamental meanings of the fractions.

Our division by zero is also given as the Moore-Penrose generalized
inverse of the fundamental equation \( az = b \) whose solution is represented
by the generalized fraction \( z = b/a \). The Moore-Penrose generalized inverse
is a well-established general concept.

We gave many and many clear interpretations and applications of our
division by zero in the cited references.

2
We will be able to confirm our division by zero as the natural mathematics as the number system. However, we will need the concept of the division by zero calculus for applying the division by zero to functions. This will be given simply in the following way.

For any Laurent expansion around \( z = a \),

\[
f(z) = \sum_{n=-\infty}^{\frac{-1}{a}} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n,
\]

(2.1)
we obtain the identity, by the division by zero

\[
f(a) = C_0.
\]

(2.2)

For the correspondence (2.2) for the function \( f(z) \), we will call it the division by zero calculus. By considering the derivatives in (2.1), we can define any order derivatives of the function \( f \) at the singular point \( a \).

For functions, we can, in general, consider the applications of the division by zero in many ways, however, the above division by zero calculus is applicable in many cases, however, for the results obtained we have to check their senses. Indeed, we have the cases that we have good results and nonsense results.

For example, for the simple example for the line equation on the \( x, y \) plane

\[
ax + by + c = 0
\]
we have, formally

\[
x + \frac{by + c}{a} = 0,
\]
and so, by the division by zero, we have, for \( a = 0 \), the reasonable result

\[
x = 0.
\]

Indeed, for the equation \( y = mx \), from

\[
\frac{y}{m} = x,
\]
we have, by the division by zero, \( x = 0 \) for \( m = 0 \). This gives the case \( m = \infty \) of the gradient of the line. – This will mean that the equation
$y = mx$ represents the general line through the origin in this sense. This method was applied in many cases, for example see \[8, 9\].

However, from

$$\frac{ax + by}{c} + 1 = 0,$$

for $c = 0$, we have the contradiction, by the division by zero

$$1 = 0.$$  

Meanwhile, note that for the function $f(z) = z + \frac{1}{z}$, $f(0) = 0$, however, for the function

$$f(z)^2 = z^2 + 2 + \frac{1}{z^2},$$

we have $f^2(0) = 2$. Of course,

$$f(0) \cdot f(0) = \{f(0)\}^2 = 0.$$  

We can consider many ways applying the division by zero to functions. For many concrete examples, see \((6, 7)\) and the papers cited in the references. In particular, note that we can consider the division by zero for more general functions that are not restricted to analytic functions.

Before ending the background for this paper, we have to refer to the very important facts:

We have even the formal contradiction for the very classical result that the point at infinity is represented by $\infty$ and it is represented by zero. We were able to establish the fundamental relation between the point at infinity and 0, see \([9]\).

The inversion with respect to a circle of the center of the circle is given by the center of the circle, not the point at infinity. A line may be looked as a circle with radius zero and with center at the origin. See \((6, 7)\).

3 Many points at infinity?

When we consider a circle with center $P$, by the inversion with respect to the circle, the points of a neighborhood at the point $P$ are mapped to a neighborhood around the point at infinity except the point $P$. This property is independent of the radius of the circle. It looks that the point at infinity is depending on the center $P$. This will mean that there exist many points at infinity, in a sense.
4 Stereographic projection

The point at infinity may be realized by the stereographic projection as well known. However, the projection is dependent on the position of the sphere (the plane coordinates). Does this mean that there exist many points at infinity?

5 Laurent expansion

From the definition of the division by zero calculus, we see that if there exists a negative $n$ term in (2.1)

$$\lim_{z \to a} f(z) = \infty,$$

however, we have (2.2). The values at the point $a$ have many values, that are all complex numbers. At least, in this sense, we see that we have many points as the point of infinity.

In the sequel, we will show typical points at infinity.

6 Diocles’ curve of Carystus (BC 240? - BC 180?)

The beautiful curve $y^2 = \frac{x^3}{2a - x}$, $a > 0$

is considered by Diocles. By setting $X = \sqrt{2a - x}$ we have

$$y = \pm \frac{x^{(3/2)}}{\sqrt{2a - x}} = \pm \frac{(2a - X^2)^{(3/2)}}{X}.$$

Then, by the division by zero calculus at $X = 0$, we have a reasonable value 0.

Meanwhile, for the function $\frac{x^3}{2a - x}$, we have $-12a^2$, by the division by zero calculus at $x = 2a$. This leads to a wrong value.
7 Nicomedes’ curve (BC 280 - BC 210)

The very interesting curve
\[ r = a + \frac{b}{\cos \theta} \]
is considered by Nicomedes from the viewpoint of the 1/3 division of an angle. That has very interesting geometrical meanings. For the case \( \theta = \pm (\pi/2) \), we have \( r = a \), by the division by zero calculus.

Of course, the function is symmetric for \( \theta = 0 \), however, we have a mysterious value \( r = a \), for \( \theta = \pm (\pi/2) \). Look the beautiful graph of the function.

8 Newton’s curve (1642 - 1727)

Meanwhile, for the famous Newton curve
\[ y = ax^2 + bx + c + \frac{d}{x} \quad (a, d \neq 0), \]
of course, we have \( y(0) = c \).

Meanwhile, in the division by zero calculus, the value is determined by the information around any analytical point for an analytic function, as we see from the basic property of analytic functions.

At this moment, the properties of the values of analytic functions at isolated singular points are mysterious, in particular, in the geometrical sense.

9 Basic meanings of values at isolated singular points of analytic functions

Since the values of analytic functions at isolated singular points were given by the coefficients \( C_0 \) of the Laurent expansions (2.1) as the division by zero calculus. Therefore, their values may be considered as arbitrary ones by any sift of the image complex plane. Therefore, we can consider the values as zero in any Laurent expansions by shifts, as normalizations. However, if the Laurent expansions are determined by another normalizations, then the values will have their senses. We will examine such properties for the Riemann mapping function.
Let $D$ be a simply-connected domain containing the point at infinity having at least two boundary points. Then, by the celebrated theorem of Riemann, there exists a uniquely determined conformal mapping with a series expansion

$$W = f(z) = C_1 z + C_0 + \frac{C_{-1}}{z} + \frac{C_{-2}}{z^2} + \ldots, C_1 > 0,$$  \hspace{1cm} (9.1)

at the point at infinity which maps the domain $D$ onto the exterior $|w| > 1$ of the unit disc on the complex $W$ plane. We can normalize (9.1) as follows:

$$\frac{f(z)}{C_1} = z + \frac{C_0}{C_1} + \frac{C_{-1}}{C_1 z} + \frac{C_{-2}}{C_1 z^2} + \ldots.$$ \hspace{1cm} (9.2)

Then, this function $\frac{f(z)}{C_1}$ maps $D$ onto the exterior of a circle of radius $1/C_1$ and so, it is called the mapping radius of $D$ ([1], 80 page; [15], 83 page).

Meanwhile, from the normalization

$$f(z) - C_0 = C_1 z + \frac{C_{-1}}{z} + \frac{C_{-2}}{z^2} + \ldots,$$ \hspace{1cm} (9.3)

by the natural shift $C_0$ of the image plane, the unit circle is mapped to the unit circle with center $C_0$. Therefore, $C_0$ may be called as mapping center of $D$. The function $f(z)$ takes the value $C_0$ at the point at infinity in the sense of the division by zero calculus and now we have its natural sense as the mapping center of $D$. We have considered the value of the function $f(z)$ as infinity at the point at infinity, however, practically it was the value $C_0$. This will mean that in a sense the value $C_0$ is the farthest point from the point at infinity or from the image domain with the strong discontinuity. – Recall the mapping property of the fundamental function $W = 1/z$ of the unit disc $|z| < 1$.

The properties of mapping radius were investigated deeply in conformal mapping theory like estimations, extremal properties and meanings of the values, however, it seems that there is no information on the property of mapping center. See many books on conformal mapping theory or analytic function theory. See [15] for example.

10 Unbounded, however, bounded

We will consider the high

$$y = \tan \theta, \hspace{0.5cm} 0 \leq \theta \leq \frac{\pi}{2}$$
on the line \( x = 1 \). Then, the high \( y \) is unbounded, however, the high line (gradient) can not be extended beyond the \( y \) axis. The restriction is given by \( 0 = \tan(\pi/2) \).

Recall the stereographic projection of the complex plane. The points on the plane can be expanded in an unbounded way, however, all the points on the complex plane have to be corresponded to the points of the Riemann sphere. The restriction is the point at infinity which corresponds to the north pole of the Riemann sphere and the point at infinity is represented by 0.

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