For any ring $\Lambda$, we shall denote by $\Lambda$-Mod the category of all (left) $\Lambda$-modules, and by $\Lambda$-mod the full subcategory of finitely presented modules.

Let $\Lambda$ and $\Gamma$ be derived equivalent rings. That is, the unbounded derived categories $D(\Lambda$-Mod) and $D(\Gamma$-Mod) are equivalent as triangulated categories. Then many properties of $\Lambda$ and $\Gamma$ are shared. One example of this, first proved in [Ri2, Corollary 5.3], is that if $\Lambda$ is a symmetric finite-dimensional algebra over a field $k$, then so is $\Gamma$. In fact, the property of being symmetric can be easily characterized in terms of the derived category. Recall that an object of the derived category $D(\Lambda$-Mod) is called perfect if it is isomorphic to a bounded complex of finitely-generated projective $\Lambda$-modules, and that the perfect objects $P$ can be characterized as those that are compact, meaning that the functor $\text{Hom}_{D(\Lambda$-Mod)}(P, -)$ preserves arbitrary direct sums, and that if $\Lambda$ is a finite-dimensional $k$-algebra, then the objects $X$ of $D(\Lambda$-Mod) isomorphic to objects of $D(\Lambda$-mod) can be characterized as those objects such that, for every perfect object $P$, $\text{Hom}_{D(\Lambda$-Mod)}(P, X)$ is finite-dimensional. Thus such objects $P$ and $X$ are both intrinsically characterized in terms of the triangulated category $D(\Lambda$-Mod).

**Proposition 0.1.** A finite-dimensional $k$-algebra $\Lambda$ is symmetric if and only if, for every object $X$ of $D(\Lambda$-mod) and every perfect object $P$ of $D(\Lambda$-Mod), $\text{Hom}_{D(\Lambda$-Mod)}(P, M)$ and $\text{Hom}_{D(\Lambda$-Mod)}(M, P)$ are naturally dual as $k$-vector spaces.

**Proof.** It was shown in [Ri3, Corollary 3.2] that if $\Lambda$ is symmetric, then this duality holds. Conversely, if there is such a duality, then the case $P = X = \Lambda$ shows that there is an isomorphism between $\Lambda$ and its dual, which, by naturality, is an isomorphism of bimodules. Thus $\Lambda$ is symmetric. \hfill $\Box$

For the weaker property of being self-injective, there seems to be no such simple direct characterization in terms of the derived category. However, our main theorem in this paper is that, at least for finite-dimensional algebras over an algebraically closed field, the property of being self-injective is preserved by derived equivalence. The proof depends on a theorem of Huisgen-Zimmermann and Saorín [HS] on rigidity of tilting complexes. In Section 1 we show how this theorem also follows from the results of [Ri1].

1. **Rigidity for tilting complexes**

Let $k$ be an algebraically-closed field and let $\Lambda$ be a finite-dimensional $k$-algebra. In Corollary 9 of [HS], Huisgen-Zimmermann and Saorín prove a result that easily implies the following theorem.

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Theorem 1.1. Let \( \ldots, P^{-1}, P^0, P^1, \ldots \) be a sequence of finitely-generated projective \( \Lambda \)-modules, such that \( P^i = 0 \) for all but finitely many \( i \). Then up to isomorphism there are only finitely many tilting complexes
\[
P = \cdots \to P^{-1} \to P^0 \to P^1 \to \ldots
\]

Since, up to isomorphism, there is only a countable set of possibilities for the sequence \( \ldots, P^{-1}, P^0, P^1, \ldots \), and since every algebra derived equivalent to \( \Lambda \) is isomorphic to the endomorphism algebra of a tilting complex, the following corollary follows.

Corollary 1.2. Let \( \Lambda \) be a finite-dimensional algebra over an algebraically closed field \( k \). Then up to isomorphism there are only countably many tilting complexes for \( \Lambda \), and hence only countably many algebras derived equivalent to \( \Lambda \).

In particular, this rules out (at least over an uncountable field) the possibility of a family of non-isomorphic but derived equivalent algebras, parametrized by an algebraic curve or variety of higher dimension. A number of people have noticed that Theorem 1.1 also follows using the techniques of [Rl1], but we do not know of any reference to this in the literature, so let us sketch an alternative proof along these lines. The complexes of the form
\[
P = \cdots \to P^{-1} \to P^0 \to P^1 \to \ldots
\]
are parametrized by the \( k \)-rational points of an affine variety \( V \): namely, the closed sub-variety of
\[
\prod_{i \in \mathbb{Z}} \text{Hom}_\Lambda (P^i, P^{i+1})
\]
determined by the equations \( d^{i+1}d^i = 0 \).

Let \( k[V] \) be the coordinate ring of this variety. Then there is a natural complex
\[
\tilde{P} = \cdots \to P^{-1} \otimes_k k[V] \to P^0 \otimes_k k[V] \to P^1 \otimes_k k[V] \to \ldots
\]
of \( \Lambda \otimes_k k[V] \)-modules such that, if \( P_v \) is the complex corresponding to a \( k \)-rational point \( v \in V \) with associated maximal ideal \( m_v \), then
\[
P_v \cong \tilde{P} \otimes_{k[V]} (k[V]/m_v).
\]

For each \( k \)-rational point \( w \in V \), there is also a complex
\[
\tilde{P}_w = \tilde{P} \otimes_k k[V]
\]
with the same terms as \( \tilde{P} \), but with
\[
\tilde{P}_w \otimes_{k[V]} (k[V]/m_v) \cong P_w
\]
for every \( k \)-rational point \( v \in V \).

Now suppose that \( w \) is a \( k \)-rational point for which \( P_w \) is a tilting complex, and let \( \widehat{k[V]}_w \) be the completion of \( k[V] \) at the maximal ideal \( m_w \). Then by [Rl1] Theorem 3.3], the complexes \( \tilde{P} \otimes_{k[V]} \widehat{k[V]}_w \) and \( \tilde{P}_w \otimes_{k[V]} \widehat{k[V]}_w \) are isomorphic. Using the Artin Approximation Theorem as in the proof of [Rl1] Theorem 4.1], it follows that for some \( \acute{\text{e}} \text{tale} \) neighbourhood \( U \) of \( w \), there is an isomorphism between \( \tilde{P} \otimes_{k[V]} k[U] \) and \( \tilde{P}_w \otimes_{k[V]} k[U] \). In particular, for any \( k \)-rational point \( v \) in the open set that is the image of \( U \to V \),
\[
P_v = \tilde{P} \otimes_{k[V]} (k[V]/m_v) \cong \tilde{P}_w \otimes_{k[V]} (k[V]/m_v) \cong P_w.
\]
Thus, the complexes parametrized by the \( k \)-rational points of some open neighbourhood of \( w \) are all isomorphic to \( P_w \), and it follows that there can only be finitely many isomorphism classes of complexes \( P_v \) that are tilting complexes. In fact, as in Huisgen-Zimmermann and Saorín’s proof, it is only the property

\[
\text{Hom}_{\text{D}(\Lambda-\text{Mod})}(P, P[1]) = 0
\]

that is needed, so in fact there are only finitely many isomorphism classes of complexes \( P_v \) satisfying this weaker condition.

2. The main theorem

For a finite-dimensional \( k \)-algebra \( \Lambda \) we define the Nakayama functor

\[
\nu_\Lambda : \Lambda-\text{Mod} \to \Lambda-\text{Mod}
\]

by

\[
\nu_\Lambda(M) = \Lambda^\vee \otimes_\Lambda M,
\]

where \( \Lambda^\vee \) denotes the \( k \)-linear dual of \( \Lambda \), regarded as a \( \Lambda \)-bimodule. Thus if \( \Lambda \) is symmetric then the Nakayama functor is isomorphic to the identity functor. In general, \( \nu_\Lambda \) induces an equivalence between the categories of finitely generated projective and injective modules for \( \Lambda \).

It is a right exact functor, and has a total left derived functor

\[
L\nu_\Lambda : \text{D}(\Lambda-\text{Mod}) \to \text{D}(\Lambda-\text{Mod})
\]

where \( L\nu_\Lambda(X) \) is constructed in the usual way by applying the functor \( - \otimes_\Lambda X \) to a projective resolution of \( \Lambda^\vee \), or by applying the functor \( \Lambda^\vee \otimes_\Lambda - \) to a projective resolution of \( X \). If \( \Lambda \) is self-injective, however, \( \nu_\Lambda \) is exact, and so \( \nu_\Lambda = L\nu_\Lambda \). In this case \( \nu_\Lambda \) is a self-equivalence of the module category \( \Lambda-\text{Mod} \) and therefore induces a self-equivalence of the derived category \( \text{D}(\Lambda-\text{Mod}) \). In [Ri2 Proposition 5.2], it was shown that, if \( \Lambda \) and \( \Gamma \) are finite-dimensional \( k \)-algebras and

\[
F : \text{D}^-(\Lambda-\text{Mod}) \to \text{D}^-(\Gamma-\text{Mod})
\]

is an equivalence of triangulated categories, then, at least if we replace \( F \) by another “standard” equivalence that agrees with \( F \) on each object up to a (not-necessarily natural) isomorphism, the diagram

\[
\begin{array}{ccc}
\text{D}^-(\Lambda-\text{Mod}) & \overset{L\nu_\Lambda}{\longrightarrow} & \text{D}^-(\Lambda-\text{Mod}) \\
\downarrow & & \downarrow \\
\text{D}^-(\Gamma-\text{Mod}) & \overset{L\nu_\Gamma}{\longrightarrow} & \text{D}^-(\Gamma-\text{Mod})
\end{array}
\]

commutes up to isomorphism of functors, and so

\[
F(L\nu_\Lambda X) \cong L\nu_\Gamma(FX)
\]

for any object \( X \) of \( \text{D}^-(\Lambda-\text{Mod}) \).

**Theorem 2.1.** Let \( \Lambda \) and \( \Gamma \) be derived equivalent finite-dimensional algebras over an algebraically closed field \( k \), with \( \Lambda \) self-injective. Then \( \Gamma \) is self-injective.
Proof. Since \( \nu_T \) induces an equivalence between the categories of finitely generated projective and finitely generated injective \( \Gamma \)-modules, we just need to show that \( \nu_T(\Gamma) \) is projective.

Since \( \nu_{\Lambda} \) is a self-equivalence of the module category \( \Lambda\text{-Mod} \), some power \( \nu_r^\Lambda \) of the Nakayama functor has the property that \( \nu_r^\Lambda(P) \cong P \) for every projective \( \Lambda \)-module \( P \).

Let \( F : D^- (\Lambda\text{-Mod}) \to D^- (\Gamma\text{-Mod}) \) be a derived equivalence. Then \( F^{-1}(\Gamma) \) is isomorphic to some tilting complex \( T \) for \( \Lambda \), and \( \nu_r^\Lambda(T) \) is also a tilting complex which has components isomorphic to those of \( T \). Thus, for some multiple \( s \) of \( r \), \( \nu_s^\Lambda(T_i) \) is isomorphic to \( T_i \) for every indecomposable direct summand \( T_i \) of \( T \), since there are only finitely many such summands up to isomorphism.

By the discussion preceding the statement of Theorem 2.1, it follows that \( L\nu_T \) is a self-equivalence of \( D^- (\Gamma\text{-Mod}) \) and that \( L\nu_s^\Lambda(Q) \) is isomorphic to \( Q \) for every indecomposable projective \( \Gamma \)-module \( Q \). We shall prove that \( L\nu_s^\Lambda(Q) \) is isomorphic to a projective module concentrated in degree zero for all \( 0 \leq n \leq s \). For if not, then let \( n < s \) be maximal such that this is not true.

Since \( L\nu_T \) preserves the property of having zero homology in positive degrees, \( L\nu_s^\Lambda(Q) \) has zero homology in positive degrees. Let

\[
\cdots \to 0 \to Q^{-k} \to Q^{-k+1} \to \cdots \to Q^0 \to 0 \to \cdots
\]

be a minimal projective resolution of \( L\nu_s^\Lambda(Q) \) with \( k > 0 \) and \( Q^{-k} \neq 0 \). It is an indecomposable bounded complex of finitely generated projective modules, since \( L\nu_T \) is a self-equivalence of \( D^- (\Gamma\text{-Mod}) \) and therefore preserves the property of being an indecomposable perfect object.

Since \( \nu_T \) is an equivalence between the categories of finitely generated projective and injective \( \Gamma \)-modules, when we apply \( \nu_T \) we get an indecomposable complex

\[
\cdots \to 0 \to \nu_T Q^{-k} \to \nu_T Q^{-k+1} \to \cdots \to \nu_T Q^0 \to 0 \to \cdots
\]

of injective \( \Gamma \)-modules with \( \nu_T Q^{-k} \neq 0 \). But this complex is isomorphic in \( D(\Gamma\text{-Mod}) \) to \( L\nu_s^\Lambda(Q) \), which, by the assumption on \( n \), has homology concentrated in degree zero. Thus the differential

\[ \nu_T Q^{-k} \to \nu_T Q^{-k+1} \]

is injective, and therefore split, since \( \nu_T Q^{-k} \) is an injective module. But this contradicts the indecomposability of the complex. By contradiction it follows that \( \nu_T Q \) is a projective module for every indecomposable projective \( \Gamma \)-module \( Q \), and hence \( \Gamma \) is a self-injective algebra. \( \square \)

It would be interesting to find a simple direct characterization of self-injective algebras in terms of properties of their derived categories as we did for symmetric algebras in Proposition 0.1, but it is not clear that this can be done.

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