Hilbert Boundary Value Problems for Hyper Monogenic Functions on The Hyperplane

Pei Dang\textsuperscript{a}, Jinyuan Du\textsuperscript{b,c,†} Tao Qian\textsuperscript{d}

\textsuperscript{a}. Faculty of Information Technology, Macau University of Science and Technology, Macao
\textsuperscript{b}. Department of Mathematics, Wuhan University, Wuhan 430072, P. R. China
\textsuperscript{c}. School of Science, Linyi University, Linyi, Shandong 276000, P. R. China
\textsuperscript{d}. Faculty of Information Technology, Macau University of Science and Technology, Macao

Abstract. This paper systematically studies Hilbert boundary value problems for hyper monogenic functions on the hyperplane for the solutions being of any integer orders at the infinity, where the negative order cases are new even when restricted to the complex plane context. The explicit solution formulas are given and the solvability conditions are specified. The results are proved through using the Clifford symmetric extension method to reduce Hilbert boundary value problems to Riemann boundary value problems.

Keywords  Real part and imaginary part; Order at the infinite point; Hilbert boundary value problem; Symmetric extension.

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1 Introduction

As is well known, boundary value problems (BVPs) for analytic functions in the classical complex analysis form an important branch of mathematical analysis, and, due to its theoretical elegance and ample applications in physics and other subjects such as elastic theory, fracture mechanics, hydromechanics, etc., have been thoroughly studied over a long time. The theory of boundary value problems for analytic functions has been investigated systematically in the literature \cite{1-3}, and applications have been showed roundly in the monographs \cite{4,5}.

It has been proved that the function theory over Clifford algebra is an appropriate setting to generalize many aspects of the function theory of one complex variable to higher dimensions \cite{6-9}. It is natural that mathematicians hope to develop theories on boundary value problems for hyper monogenic functions, also called regular functions simply, in the hypercomplex analysis analogous to those for analytic functions in the classical complex analysis. In fact, some results on boundary value problems for analytic functions in the classical complex analysis have been generalized to regular functions in Clifford analysis (e.g., some articles listed in \cite{10}).
In [11] and [12], we discussed, respectively, Riemann boundary value problems on closed smooth surfaces and on the hyperplanes in detail. In this paper, we consider Hilbert boundary value problems (Hilbert BVPs) on the half-hyperplanes, such as the Poincaré upper half space. Hilbert BVPs are very important in pure mathematics and engineering practice. The discussion is rather difficult and complex. Even in the classical complex analysis, Hilbert boundary value problems on the real axis are also not completely discussed [12]. The researchers are restricted to the case of the bounded growth at the infinity, while for general growth orders at the infinity there have some obstacles. In [13], Z. Y. Xu and C. P. Zhou tried to solve Hilbert boundary value problems on the hyperplane. They generalized the classical results for Hilbert boundary value problems on the real axis to the hypercomplex analysis setting under the condition that the solutions are bounded at the infinity. In [13], Y. F. Gong and J. Y. Du continued to discuss Hilbert BVPs under the condition that the solutions have finite non-negative orders at the infinity by the symmetric extension method. The arguments of [13,14] contain some mistakes that will be pointed out below. To the authors’ knowledge, Hilbert BVPs with negative order at infinity have not been discussed in any paper, because the corresponding Riemann BVPs were not discussed before [12], so Hilbert BVPs with negative order at infinity are always an open problem.

In this paper, we will systematically discuss the Hilbert boundary value problems on the hyperplane for regular functions in Clifford analysis, including the cases of negative orders at the infinity. We will use the so-called symmetric extension method to solve the Hilbert BVPs. The paper is organized as follows. In §2 we review some of the necessary preparations in Clifford analysis. In §3, we formulate the Hilbert boundary value problems in the Poincaré upper half space. In §4, we introduce the symmetric extension of a regular function in the Poincaré upper half space and discuss its regularity. Then the Hilbert BVPs are sloved by converting them equivalently to the Riemann BVPs discussed in [12]. In §5, the solution formulas and the conditions of the Schwarz BVPs and the Hilbert BVPs for all orders at the infinity of the solutions are obtained. These results extend both the classical ones in complex analysis [1–3] and those in the Clifford analysis setting [13, 14].

2 Hypercomplex functions

We begin by recalling the necessary preliminary knowledge in Clifford algebra and Clifford analysis [6,7], which are used throughout this paper.

2.1 Clifford analysis

Let $C(V_n)$ be a $2^n$-dimensional real linear space. To expediently introduce the product on it, we write its basis by \( \{ e_A, A = (h_1, \ldots, h_r) \in \mathcal{P}N, 1 \leq h_1 < \cdots < h_r \leq n \} \), where $N$ stands for the set \( \{1, \ldots, n\} \) and \( \mathcal{P}N \) denotes the family of all the ordered subsets of $N$ in the above fixed way. Sometimes, $e_\emptyset$ is written as $e_0$ and $e_A$ as $e_{h_1 \cdots h_r}$ for $A = \{h_1, \ldots, h_r\} \in \mathcal{P}N$. The product on $C(V_n)$ is defined by

\[
\begin{align*}
    e_A e_B &= (-1)^{\#(A \cap B)} (-1)^{P(A,B)} e_{A \Delta B}, & \text{if } A, B \in \mathcal{P}N, \\
    \lambda \mu &= \sum_{A \in \mathcal{P}N} \sum_{B \in \mathcal{P}N} \lambda_A \mu_B e_A e_B, & \text{if } \lambda = \sum_{A \in \mathcal{P}N} \lambda_A e_A, \quad \mu = \sum_{A \in \mathcal{P}N} \mu_A e_A,
\end{align*}
\]

(2.1)
where the notation \( \#(A) \) denotes the number of the elements in \( A \) and \( P(A, B) = \sum_{j \in B} P(A, j) \) with \( P(A, j) = \#\{i, i \in A, i > j\} \), the symmetric difference set \( A \Delta B \) is also an ordered one in the above way, and \( \lambda_A \in \mathbb{R} \) is the coefficient of the \( e_A \)-component of the Clifford number \( \lambda \). It follows at once from the multiplication rule (2.1) that \( e_0 \) is the identity element written as \( 1 \) and in particular,

\[
\begin{align*}
\begin{cases}
  e_i^2 = -1, & \text{if } i = 1, \ldots, n, \\
  e_i e_j = -e_j e_i, & \text{if } 1 \leq i < j \leq n, \\
  e_{h_1} e_{h_2} \cdots e_{h_r} = e_{h_1 h_2 \cdots h_r}, & \text{if } 1 \leq h_1 < h_2 < \cdots < h_r \leq n.
\end{cases}
\end{align*}
\tag{2.2}
\]

It is clear that \( C(V_n) \) is a real linear and associative, non-commutative algebra by algebraically spanning the linear subspace \( V_n = \text{span}\{e_1, e_2, \ldots, e_n\} \). It is called the Clifford algebra over \( V_n \). The elements \( \lambda = \lambda_0 + \lambda_1 e_1 + \cdots + \lambda_n e_n \) for \( \lambda_0, \cdots, \lambda_n \in \mathbb{R} \) are called paravectors.

We frequently use the following defined involution:

\[
\begin{align*}
\overline{e_A} &= (-1)^{\#(A)(\#(A)+1)/2} e_A, \quad \text{if } A \in \mathcal{P} \mathcal{N}, \\
\overline{\lambda} &= \sum_{A \in \mathcal{P} \mathcal{N}} \lambda_A e_A, \quad \text{if } \lambda = \sum_{A \in \mathcal{P} \mathcal{N}} \lambda_A e_A.
\end{align*}
\tag{2.3}
\]

In the sequel, \( \lambda_A \) is also written as \( \llbracket \lambda \rrbracket_A \). In particular, the coefficient \( \lambda_0 \) is denoted by \( \lambda_0 \) or \( \llbracket \lambda \rrbracket_0 \), which is called the scalar part of the Clifford number \( \lambda \). An inner product \( (\cdot, \cdot) \) on \( C(V_n) \) is defined by putting for any \( \lambda \) and \( \mu \) in \( C(V_n) \)

\[
(\lambda, \mu) = \llbracket \lambda \overline{\mu} \rrbracket_0 = \sum_{A} \lambda_A \mu_A,
\tag{2.4}
\]

where \( \lambda = \sum_A \lambda_A e_A \), \( \mu = \sum_A \mu_A e_A \) and the symbol \( \sum_A \) is an abbreviation of \( \sum_{A \in \mathcal{P} \mathcal{N}} \). Thus, the corresponding norm on \( C(V_n) \) reads,

\[
|\lambda| = \sqrt{(\lambda, \lambda)} = \left[ \sum_A \lambda_A^2 \right]^{1/2}.
\tag{2.5}
\]

In such way, \( C(V_n) \) is a real Hilbert space and at the same time it is a Banach algebra with the equivalent norm

\[
|\lambda|_0 = 2^{n/2} |\lambda|,
\tag{2.6}
\]

that is

\[
|\lambda \mu|_0 \leq |\lambda|_0 |\mu|_0, \quad |\lambda \mu| \leq 2^{n+1} |\lambda| |\mu|.
\tag{2.7}
\]

In particular, if \( \lambda \) is a paravector and \( \mu \in C(V_n) \), then [9]

\[
|\lambda \mu| = |\mu \lambda| = |\lambda||\mu|.
\tag{2.8}
\]

Let \( \Omega \) be a non-empty subset of \( \mathbb{R}^{n+1} \). Hypercomplex functions \( f \) defined in \( \Omega \) and with values in \( C(V_n) \) will be considered, i.e., \( f : \Omega \rightarrow C(V_n) \). They are of the form

\[
f(w) = \sum_{A} f_A(w)e_A, \quad w = (w_0, w_1, \cdots, w_n) \in \Omega \subset \mathbb{R}^{n+1},
\tag{2.9}
\]
where the $f_A(w)$ is the $e_A$–component of $f(w)$. Obviously, the $f_A$’s are real–valued functions in $\Omega$, which are called the $e_A$–component functions of $f$. Whenever a property such as differentiability and continuity is ascribed to $f$, it is clear that in fact all the component functions $f_A$ possess the cited property. So the meaning $f \in C^{(r)}(\Omega, C(V_n))$ is very clear.

Obviously, $C(V_{n-1})$ is a subalgebra of $C(V_n)$, where $V_{n-1} = \text{span}\{e_1, e_2, \cdots, e_{n-1}\}$. Then, $\lambda \in C(V_n)$ has the unique decomposition [13,14]  
\begin{equation}
\lambda = x + e_n y^l \quad \text{where} \quad x, y^l \in C(V_{n-1}),
\end{equation}
i.e.,  
\begin{equation}
C(V_n) = C(V_{n-1}) \oplus e_n C(V_{n-1}).
\end{equation}

We define  
\begin{equation}
\text{Re}(\lambda) = x, \quad \text{Im}^l(\lambda) = y^l.
\end{equation}

**Remark 2.1** Similarly, $\lambda \in C(V_n)$ has also the unique decomposition  
\begin{equation}
\lambda = x + y^r e_n \quad \text{where} \quad x, y^r \in C(V_{n-1}).
\end{equation}

We also define  
\begin{equation}
\text{Re}(\lambda) = x, \quad \text{Im}^r(\lambda) = y^r.
\end{equation}

For clarity, we call, respectively, $y^l$ and $y^r$ in (2.10) and (2.13), the left and right imaginary part of $\lambda$. From (2.10) and (2.13), we have  
\begin{equation}
e_n y^l = y^r e_n, \quad \text{i.e.,} \quad e_n \text{Im}^l(\lambda) = \text{Im}^r(\lambda) e_n.
\end{equation}

It is clear that the decompositions (2.10) and (2.13) are generalizations of the representation of the classical complex numbers. In other words, (2.12) and (2.14) are the generalization of operators $\text{Re}$ and $\text{Im}$ acting on the complex numbers. From (2.11) and (2.13) we obviously have  
\begin{equation}
|\lambda|^2 = |x|^2 + |y^r|^2, \quad |\lambda|^2 = |x|^2 + |y^l|^2.
\end{equation}

For a hypercomplex function $f$ given by (2.9), we call, respectively,  
\begin{equation}
(\text{Re}f)(w) = \text{Re}(f(w)), \quad (\text{Im}^r f)(w) = \text{Im}^r(f(w)), \quad (\text{Im}^l f)(w) = \text{Im}^l(f(w)),
\end{equation}
the real part of $f$, the right and left imaginary part of $f$. In particular, if  
\begin{equation}
(\text{Im}^r f)^r = (\text{Im}^l f) = 0, \quad w = (w_0, w_1, \cdots, w_n) \in \Omega \subset \mathbb{R}^{n+1},
\end{equation}
i.e., $f: \Omega \rightarrow C(V_{n-1})$, then we say it be a $C(V_{n-1})$-valued function, briefly, a para real-valued function which mimics the case of the real-valued function in the classical complex analysis.

Sometimes, for clarity, we write  
\begin{equation}
\lambda = \sum_{A \in \mathcal{P}\{1, \cdots, n-1\}} \lambda_A e_A \quad \text{when} \quad \lambda \in C(V_{n-1}),
\end{equation}
where $\mathcal{P}\{1, \cdots, n-1\}$ is to denote the family of all ordered subsets of $\{1, \cdots, n-1\}$ in the similar way used in $\mathcal{P}N$. Thus, a hypercomplex function $f$ given in (2.9) may be re-written as
\[ f(w) = \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} u_A(w) e_A + \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} v_A^l(w) e_A e_n \]
\[ = \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} u_A(w) e_A + \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} v_A^r(w) e_n e_A, \tag{2.20} \]

where \( u_A, v_A^l \) and \( v_A^r \) are the real-valued functions, which are called respectively the component functions on \( C(V_{n-1}) \), \( e_n C(V_{n-1}) \) and \( C(V_{n-1}) e_n \). Obviously,
\[ (\text{Re} f)(w) \triangleq u(w) = \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} u_A(w) e_A, \quad x \in \Omega, \tag{2.21} \]
and
\[ (\text{Im} f)^l(w) \triangleq v^l(w) = \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} v_A^l(w) e_A, \quad w \in \Omega, \tag{2.22} \]
\[ (\text{Im} f)^r(w) \triangleq v^r(w) = \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} v_A^r(w) e_A, \quad w \in \Omega. \]

Obviously,
\[ v_A^r = (-1)^{\#(A)} v_A^l \quad \text{and} \quad e_n (\text{Im} f)^l(w) = (\text{Im} f)^r(w) e_n. \tag{2.23} \]

When \( w = (w_0, w_1, \ldots, w_n) \in \mathbb{R}^{n+1} \), we introduce the mapping
\[ \text{capital}: \ w \mapsto W = \sum_{i=0}^{n} w_i e_i, \tag{2.24} \]
which is a proper isomorphism between \( \mathbb{R}^{n+1} \) and the linear subspace \( \text{span}\{e_0, e_1, \ldots, e_n\} \) of \( C(V_n) \). In the sequel, we simply treat the capital \( W \) as \( w \). This is Vahlen’s choice \[15\,16\].

Thus, we have
\[ \text{Re} \ w = w_0 + w_1 e_1 + \cdots + w_{n-1} e_{n-1} \quad \text{while} \quad w = (w_0, w_1, \ldots, w_n) \in \mathbb{R}^{n+1}. \tag{2.25} \]
and
\[ \text{Im}^l(w) = \text{Im}^r(w) = w_n \triangleq \text{Im}(w) \quad \text{while} \quad w = (w_0, w_1, \ldots, w_n) \in \mathbb{R}^{n+1}. \tag{2.26} \]

In the following, if
\[ \text{Im}^l(\lambda) = \text{Im}^r(\lambda), \tag{2.27} \]
we will write both of them, without confusion, just as \( \text{Im}(\lambda) \).

We define
\[ \mathbb{R}^{n+1}_+ = \{ w, w \in \mathbb{R}^{n+1}, \text{Im}(w) > 0 \}, \quad \mathbb{R}^{n+1}_- = \{ w, w \in \mathbb{R}^{n+1}, \text{Im}(w) < 0 \}, \tag{2.28} \]
which are called, respectively, the Poincaré upper half space and the Poincaré lower half space, while the hyperplane
\[ \mathbb{R}^{n+1}_0 = \{ w, \text{Im}(w) = 0 \} \tag{2.29} \]
is called the parareal plane in \( \mathbb{R}^{n+1} \).

It is noted that the paravectors given in \( \mathbb{R}^{n+1}_0 \) play a treble role as elements of \( \mathbb{R}^n \) and \( \mathbb{R}^{n+1}_0 \) as well as \( C(V_{n-1}) \subset C(V_n) \).
2.2 Regular functions

Let \( \Omega \) be a domain of \( \mathbb{R}^{n+1} \). Introduce the following Dirac operator

\[
D = \sum_{k=0}^{n} e_k \frac{\partial}{\partial x_k} : C^{(r)}(\Omega, C(\mathbb{V}_n)) \rightarrow C^{(r-1)}(\Omega, C(\mathbb{V}_n)),
\]

whose actions on functions from the left and from the right are governed by the rules

\[
D[f] = \sum_{k=0}^{n} \sum_{A} e_k e_A \frac{\partial f_A}{\partial x_k}, \quad [f]D = \sum_{k=0}^{n} \sum_{A} e_A e_k \frac{\partial f_A}{\partial x_k}.
\]

**Definition 2.1** We say that a function \( f \in C^{(r)}(\Omega, C(\mathbb{V}_n)) \) (\( r \geq 1 \)) is left (right) regular or monogenic in \( \Omega \) if \( D[f] = 0 ([f]D = 0) \) in \( \Omega \). \( f \) is said to be biregular in \( \Omega \) if it is both left and right regular.

**Remark 2.2** Generally speaking, if

\[
f(w) = \sum_{j=1}^{m} f_j(w) \lambda_j \quad (f_j \in C(\Omega, \mathbb{R}) \text{ and } \lambda_j \in C(\mathbb{V}_n))
\]

then

\[
D[f] = \sum_{k=0}^{n} \sum_{j=1}^{m} e_k \lambda_j \frac{\partial f_j}{\partial w_k}, \quad [f]D = \sum_{k=0}^{n} \sum_{j=1}^{m} \lambda_j e_k \frac{\partial f_j}{\partial w_k}.
\]

**Example 2.1** Let

\[
E(w) = \frac{w}{|w|^{n+1}} = \frac{w^{-1}}{|w|^{n-1}}, \quad w \in \mathbb{R}^{n+1} \setminus \{0\}.
\]

Then \( E \) is biregular \[6,17\]. The function \( E \) is called the Cauchy kernel function.

**Example 2.2** The hypercomplex variables

\[
z_j = z_j(w) = w_j e_0 - w_0 e_j \quad (j = 1, \ldots, n)
\]

are biregular \[6,19-22\].

**Example 2.3** Let \( (\ell_1, \ldots, \ell_k) \in N^k \), i.e., \( \ell_j \)'s are \( k \) elements out of \( N \), where repetitions are allowed. We put

\[
V_{(\ell_1, \ldots, \ell_k)}(w) = \sum_{\pi(\ell_1, \ldots, \ell_k)} z_{\ell_1}(w) \cdots z_{\ell_k}(w), \quad w \in \mathbb{R}^{n+1},
\]

where the sum runs over all the permutations of \((\ell_1, \ldots, \ell_k)\), which is also the so-called the hypercomplex symmetric power. The hypercomplex symmetric power \( V_{(\ell_1, \ldots, \ell_k)} \) is sometimes also called the Fueter polynomial, being biregular \[6,18\].

**Example 2.4** All derivatives of \( E \)

\[
W_{(\ell_1, \ell_2, \ldots, \ell_k)}(w) = (-1)^k \frac{\partial^k E}{\partial w_{\ell_1} \partial w_{\ell_2} \cdots \partial w_{\ell_k}}(w), \quad w \in \mathbb{R}^{n+1} \setminus \{0\}
\]

are biregular \[6,19-22\], being sometimes called negative powers \[18\], and

\[
W_{(\ell_1, \ell_2, \ldots, \ell_k)}(w) = O\left(|w|^{-(n+k)}\right) \text{ near } \infty.
\]
3 Formulation of the Hilbert BVPs on $\mathbb{R}^{n+1}_+$

To suitably present and formulize Hilbert boundary value problems on the Poincaré upper half space $\mathbb{R}^{n+1}_+$, we must introduce a suitable statement for the growth condition at the infinity for the regular functions on the Poincaré upper half space $\mathbb{R}^{n+1}_+$.

3.1 Order at the infinity

Assume $F$ is a regular function on the Poincaré hyperplane $\mathbb{R}^{n+1}_+$ denoted as $\Phi \in \mathcal{M}(\mathbb{R}^{n+1}_+)$. We sometimes need the concept of the order of $F$ at the infinity.

Definition 3.1 Let $F \in \mathcal{M}(\mathbb{R}^{n+1}_+)$, $m$ be a integer. If

$$0 < \beta = \limsup_{w \in \mathbb{R}^{n+1}_+, w \to \infty} |w^{-m}F(w)| < +\infty,$$  \hspace{1cm} (3.1)

then $F$ is said to be of order $m$ at $w = \infty$, denoted it by $\text{Ord}(F, \infty) = m$.

Sometimes it is more convenient to write

$$|F(w)|^{\sup} \approx |G(w)| \text{ near } \infty \text{ when } a \leq \limsup_{w \to \infty} \frac{|F(w)|}{|G(w)|} \leq A,$$ \hspace{1cm} (3.2)

where $F$ and $G$ ($|G(w)| > 0$) are two hypercomplex functions defined near $\infty$, $A > a > 0$ are two constants.

The following lemma is an obvious fact.

Lemma 3.1 $|F(w)|^{\sup} \approx |w^k|$ near $w = \infty$ if and only if $\text{Ord}(F, \infty) = k$.

For boundary behavior of function $\Phi \in \mathcal{M}(\mathbb{R}^{n+1}_+)$ at the infinity, there are commonly three types of formulations:

(A) $\lim_{w \to \infty, w \in \mathbb{R}^{n+1}_+} w^{-(m+1)} \Phi(w) = 0$, namely $\Phi(w) = o(w^{m+1})$ near $w = \infty$, \hspace{1cm} (3.3)

(B) $\limsup_{w \in \mathbb{R}^{n+1}_+, w \to \infty} |w^{-m}| |\Phi(w)| = \beta$, namely $|\Phi(w)| = O(w^m)$ near $w = \infty$, \hspace{1cm} (3.4)

(C) $\text{Ord}(\Phi, \infty) = m$, namely $|\Phi(w)|^{\sup} \approx |w^k|$ near $w = \infty$ when $k \leq m$. \hspace{1cm} (3.5)

Remark 3.1 Obviously, (C) implies (B), while (B) implies (A). So, the condition (A) is the weakest. In [12], We used the condition (A) in Riemann boundary value problems with the hyperplane as jump surface, which was an innovation. In the present paper, we will still use it in our Hilbert boundary value problems on the Poincaré upper half space.

Some symbols will be used in the following [22]. Let

$$Z \langle w \rangle = (z_1(w), z_2(w), \cdots, z_n(w)) \text{ and } \alpha = [\alpha_1, \alpha_2, \cdots, \alpha_n],$$ \hspace{1cm} (3.6)

where $z_j$’s are the hypercomplex variables given in Example 2.2 and $\alpha_j$’s are nonnegative integers. Then the symmetry power $Z^\alpha$ in $\mathbb{R}^{n+1}$ is defined as the sum of all possible $z_i$ products of which each contains $z_i$ factor exactly $\alpha_i$ times. For example, for $n = 2$

$$(z_1, z_2)^{[0,0]} = 1, (z_1, z_2)^{[1,1]} = z_1 z_2 + z_2 z_1, (z_1, z_2)^{[2,0]} = z_1^2.$$ \hspace{1cm} (3.7)
3.2

Ord(only if see [11]) Lemma 3.2 (a hypercomplex symmetric polynomial of degree $m$ briefly as $f$ denoted by $\nu$.

Briefly, the $t$-Hölder index and the $t$-Hölder coefficient of $f$ are defined on $\Omega$ by:

$$|f(t)| \leq M|t-s|^\mu \quad (0 < \mu \leq 1)$$

(3.12)

for arbitrary points $t$, $s$ on $\Omega$, where $M$ and $\mu$ are constants, then $f$ is said to satisfy Hölder condition of order $\mu$, denoted by $f \in H^\mu(\Omega)$. The constants $\mu$ and $M$ are called, respectively, the Hölder index and the Hölder coefficient of $f$. If the order $\mu$ is not emphasized, it may be denoted briefly as $f \in H(\Omega)$.

Similarly,

$$Z^\alpha(w) = V(\ell_1,\ldots,\ell_k)(w) = \sum_{\pi(\ell_1,\ldots,\ell_k)} z_{\ell_1} \cdots z_{\ell_k}(w) \quad \text{with} \quad k = |\alpha|,$n

(3.10)

where $V(\ell_1,\ldots,\ell_k)$ is the Fueter polynomial given in Example 2.3. So $Z^\alpha$ is a biregular function [6].

**Definition 3.2** Let $m \geq 0$. We call

$$f(w) = \sum_{|\alpha| = 0}^m Z^\alpha(w) c_\alpha \quad \text{with} \quad \sum_{|\alpha| = m} c_\alpha \neq 0$$

(3.11)

a hypercomplex symmetric polynomial of degree $m$. In such case we denote $\text{Deg}(f) = m$.

**Lemma 3.2** (see [11]) Let $f$ be a hypercomplex symmetric polynomial. Then $\text{Deg}(f) = m$ if and only if $\text{Ord}(f, \infty) = m$.

### 3.2 $\hat{H}$ class of functions

In order to state the condition of the input function for Hilbert BVPs, we need to introduce some classes of hypercomplex functions used frequently in this paper.

**Definition 3.3** Assume $f$ is defined on $\Omega \subseteq \mathbb{R}^{n+1}$. If

$$|f(t) - f(s)| \leq M|t-s|^\mu \quad (0 < \mu \leq 1)$$

(3.12)

for arbitrary points $\xi, \zeta$ on $\Omega \setminus \{0\}$, where $M$ and $\mu$ are constants, then $f$ is said to satisfy $H_\dag$ condition of order $\mu$, denoted by $f \in H^\mu(\Omega)$. The constants $\mu$ and $M$ are called, respectively, the $\dag$-Hölder index and the $\dag$-Hölder coefficient of $f$. If the order $\mu$ is not emphasized, it may be denoted briefly as $f \in H^\mu_\dag(\Omega)$.

Sometimes we will use $\frac{1}{w}$ to represent $w^{-1}$ for $w \in \mathbb{R}^{n+1}$, which suggests the similarity with some results in the classical complex analysis.

**Definition 3.4** Assume $f$ is defined on $\Omega \subseteq \mathbb{R}^{n+1}$. If

$$|f(\xi) - f(\zeta)| \leq M\left|\frac{1}{\xi} - \frac{1}{\zeta}\right|^\mu \quad (0 < \mu \leq 1)$$

(3.13)

for arbitrary points $\xi, \zeta$ on $\Omega \setminus \{0\}$, where $M$ and $\mu$ are constants, then $f$ is said to satisfy $H^\mu_\dag$ condition of order $\mu$, denoted by $f \in H^\mu_\dag(\Omega)$. The constants $\mu$ and $M$ are called, respectively, the $\dag$-Hölder index and the $\dag$-Hölder coefficient of $f$. If the order $\mu$ is not emphasized, it may be denoted briefly as $f \in H^\mu_\dag(\Omega)$.
Definition 3.5 If \( f \in H^\mu(\Omega) \cap H^\mu_1(\Omega) \), then \( f \) is said to satisfy the \( \tilde{H} \) condition of order \( \mu \) on \( \Omega \), denoted by \( f \in H^\mu(\Omega) \) or briefly \( f \in \tilde{H}(\Omega) \).

The conditions (3.12) and (3.13) are, respectively, called the Hölder condition and \( \dagger \)-Hölder condition of the \( \tilde{H}(\Omega) \) class function \( f \). More discussions of the two classes of functions may be found in \[12\].

Definition 3.6 Let \( f \) be a function defined on \( \Omega \) with \( \infty \) as its cluster point. If
\[
\lim_{w \in \Omega, w \to \infty} f(w) = f(\infty),
\]
exists and
\[
|f(w) - f(\infty)| \leq \frac{M}{|w|^\mu} \left( 0 < \mu \leq 1 \right), \quad w \in \Omega \setminus \{0\},
\]
where \( M \) is a constant, then we say that \( f \) satisfies the pointwise \( \dagger \)-Hölder condition at the infinity in \( \Omega \), denoted by \( f \in H^\mu_1(\Omega, \infty) \), or also briefly \( f \in H^\mu_1(\infty) \) and \( f \in H_1(\infty) \).

In the sequel the following notations will be used. Let
\[
f_m(w) = w^m f(w).
\]

(1) If \( f_m \in H^\mu(\mathbb{R}^{n+1}_0) \), then we write \( f \in H^\mu_m(\mathbb{R}^{n+1}_0) \), or briefly \( f \in H_m(\mathbb{R}^{n+1}_0) \).

(2) If \( f_m \in H^\mu_1(\mathbb{R}^{n+1}_0) \), then we write \( f \in H^\mu_{m,1}(\mathbb{R}^{n+1}_0) \), or briefly \( f \in H_{m,1}(\mathbb{R}^{n+1}_0) \).

(3) If \( f_m \in H^\mu_1(\mathbb{R}^{n+1}_0, \infty) \), then we write \( f \in H^\mu_{m,1}(\mathbb{R}^{n+1}_0, \infty) \), or briefly \( f \in H_{m,1}(\infty) \).

The following classes of functions will also be used in Hilbert boundary value problems, of which the details can be found in \[12\]:
\[
\tilde{H}_m(\mathbb{R}^{n+1}_0) = H_m(\mathbb{R}^{n+1}_0) \cap H_{m,1}(\mathbb{R}^{n+1}_0),
\]
\[
\tilde{H}_{m,0}(\mathbb{R}^{n+1}_0) = \tilde{H}_m(\mathbb{R}^{n+1}_0) \cap \{ f, f_m(\infty) = 0 \}.
\]

3.3 Formulation of Hilbert boundary value problems

Let \( \Phi \) be a regular function defined on the Poincaré upper half space \( \mathbb{R}^{n+1}_+ \) which can extend continuously to the hyperplane \( \mathbb{R}^{n+1}_0 \). For clarity, we denote its boundary value by
\[
\Phi^+(t) = \lim_{w \to t, w \in \mathbb{R}^{n+1}_+} \Phi(w), \quad t \in \mathbb{R}^{n+1}_0.
\]

The Hilbert boundary value problem (or simply Hm problem). Find a left (right) regular function in \( \mathbb{R}^{n+1}_+ \) which can extend continuously to \( \mathbb{R}^{n+1}_0 \) such that
\[
\begin{cases}
(D[\Phi])(w) = 0 \quad &\text{for } w \in \mathbb{R}^{n+1}_+ \text{ (regularity)}, \\
\Re\left\{ \Phi^+(x) \lambda \right\} = c(x) \quad &\text{for } x \in \mathbb{R}^{n+1}_0 \text{ (boundary condition)}, \\
\Phi(w) = o(w^{m+1}) \quad &\text{near } \infty \text{ (growth condition)},
\end{cases}
\]
where \( \lambda \) is a given constant whose inverse exists, \( c \) is a given para real-valued function and
\[ c \in \begin{cases} \hat{H}(\mathbb{R}^{n+1}_0), & \text{when } m \geq 0 \ (\text{non-negative order}), \\ \hat{H}_{r,0}(\mathbb{R}^{n+1}_0), & \text{when } m < 0 \ \text{with } r = -(m + 1). \end{cases} \quad (3.21) \]

Obviously, the above BVPs are the direct generalizations of the classical Hilbert BVPs from complex plane to the space \( C(V_n) \). In the classical complex analysis, \( \lambda \) is allowed to be a complex function when \( \Phi \) is finite at the infinity, i.e., when \( m = 0 \).

A great number of literatures have discussed the solution methods of BVPs for some special case in Clifford analysis. One methodology is the so-called symmetric extension of which the main idea is to translate Hilbert BVPs (3.20) to the equivalent Riemann problems with some additional restriction condition. In the complex analysis, this approach has been quite successful \([1, 2]\), but in the higher dimensional spaces there exist many obstacles to implement this approach. Not all BVPs can be solved by the symmetric extension, especially for the case when \( \lambda \) in (3.20) is a hypercomplex function. For the case when \( \lambda \) is a hypercomplex function, it can still be translated to a Riemann problem, but an open problem. This method was tried by Xu and Zhou in \([13]\) with \( m = 0 \). Gong and Du solved the \( H_m \) problems with \( m \geq 0 \) in \([14]\) by using the symmetric extension method that generalizes the results in \([1, 2]\) directly to the Clifford setting.

The simplest \( H_m \) problem is the Schwarz problem, i.e., \( \lambda = 1 \). Let us start with some discussions on the Schwarz problem.

**The Schwartz boundary value problem \( S_m \).** Find a left (right) regular function \( \Phi \) in \( \mathbb{R}^{n+1}_+ \) that can extend continuously to \( \mathbb{R}^{n+1}_0 \) such that

\[
\begin{align*}
(D[\Phi])(w) &= 0 \quad \text{for } w \in \mathbb{R}^{n+1}_+ \ (\text{regularity}), \\
\Re\{\Phi^+(x)\} &= c(x) \quad \text{for } x \in \mathbb{R}^{n+1}_0 \ (\text{boundary condition}), \\
\Phi(w) &= o(w^{m+1}) \quad \text{near } \infty \ (\text{growth condition}),
\end{align*}
\]

where \( c \) is a given para real-valued function and \( c \) satisfies (3.21).

**Remark 3.2** Below, we generally assume that \( \Phi \) is left regular.

## 4 Symmetric extension and self-reflex action

In order to solve the \( H_m \) problem, we try to transfer it into an Jump problem \( R_m \) discussed in §6.3 of \([12]\). To do so, we introduce the symmetric extension and the self-reflex action.

### 4.1 Symmetric extension

Introduce the reflection operator, \( * : C(V_n) \rightarrow C(V_n) \), with respect to the hyperplane \( \mathbb{R}^{n+1}_0 \), by

\[ \lambda^* = \Re(\lambda) - e_n \Im^l(\lambda), \ \lambda \in C(V_n), \quad (4.1) \]

or

\[ \lambda^* = x - e_n y^l, \ x, y^l \in C(V_{n-1}), \quad (4.2) \]

where

\[ x = \Re(\lambda), \ y^l = \Im^l(\lambda). \quad (4.3) \]
Obviously, by \((2.15)\)
\[
\lambda^* = x - y^* e_n, \quad \lambda \in C(V_n),
\] \hfill (4.4)
where
\[
x = \text{Re}(\lambda), \quad y^* = \text{Im}^*(\lambda).
\] \hfill (4.5)
In particular,
\[
w^* = \sum_{j=0}^{n-1} w_j e_j - w_n e_n \quad \text{when} \quad w = \sum_{j=0}^{n} w_j e_j = \text{Re}(w) + \text{Im}(w) e_n \in \mathbb{R}^{n+1}.
\] \hfill (4.6)
We know that \(\lambda^*\) and \(\lambda\) are a pair of points symmetric to the hyperplane \(\mathbb{R}_0^{n+1}\) and
\[
(\lambda^*)^* = \lambda, \quad \lambda \in C(V_n).
\] \hfill (4.7)
Moreover, by \((2.3)\) and \((4.1)\),
\[
(w)^* = \overline{w^*}, \quad w \in \mathbb{R}^{n+1}.
\] \hfill (4.8)
For a function \(\Phi(w)\) in \(\mathbb{R}_+^{n+1}\), we define a function on \(\mathbb{R}_-^{n+1}\) by
\[
\Phi^*(w) = [\Phi(w^*])^*, \quad w \in \mathbb{R}_-^{n+1},
\] \hfill (4.9)
which is called its accompanying function. More specifically, if
\[
\Phi(w) = (\text{Re} \Phi)(w) + e_n (\text{Im}^t \Phi)(w), \quad w \in \mathbb{R}_+^{n+1},
\] \hfill (4.10)
with
\[
(\text{Re} \Phi)(w) = \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} u_A(w_0, \cdots, w_{n-1}, w_n) e_A,
\] \hfill (4.11)
and
\[
(\text{Im}^t \Phi)(w) = \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} v_A(w_0, \cdots, w_{n-1}, w_n) e_A,
\] \hfill (4.12)
where \(u_A\) and \(v_A\) are para real-valued functions, then
\[
\Phi^*(w) = (\text{Re} \Phi^*)(w) + e_n (\text{Im}^t \Phi^*)(w), \quad w \in \mathbb{R}_-^{n+1},
\] \hfill (4.13)
where
\[
(\text{Re} \Phi^*)(w) = \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} u_A(w^*) e_A = \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} u_A(w_0, \cdots, w_{n-1}, -w_n) e_A,
\] \hfill (4.14)
and
\[
(\text{Im}^t \Phi^*)(w) = -\sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} v_A(w^*) e_A = -\sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} v_A(w_0, \cdots, w_{n-1}, -w_n) e_A.
\] \hfill (4.15)
Similarly, if the original function \(\Phi(w)\) is defined in \(\mathbb{R}_-^{n+1}\), then its accompanying function \(\Phi^*(w)\) is determined by
\[
\Phi^*(w) = [\Phi(w^*)]^*, \quad w \in \mathbb{R}_+^{n+1}.
\] \hfill (4.16)
We easily see that
\[
(\Phi^*)^*(w) = \Phi(w), \quad w \in \mathbb{R}_+^{n+1} \left(\mathbb{R}_-^{n+1}\right),
\] \hfill (4.17)
that is to say, if $\Phi(w)$ is acted by the reflection operator with respect to $\mathbb{R}^{n+1}_0$ twice, then it returns to $\Phi(w)$ itself, or the reflection operator $*$ is idempotent.

Obviously, when (4.11) and (4.12) hold for $w \in \mathbb{R}^{n+1}_+$ then (4.14) and (4.15) also hold for $w \in \mathbb{R}^{n+1}$. 

**Remark 4.1** When the original function $\Phi$ is defined on $\mathbb{R}^{n+1}_+$ or $\mathbb{R}^{n+1}_-$, we have that

$$\Phi(w) = \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} u_A(w_0, \ldots, w_{n-1}, w_n) e_A + e_n \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} v_A(w_0, \ldots, w_{n-1}, w_n) e_A$$

(4.18)

is equivalent to

$$\Phi^*(w) = \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} u_A(w^*) e_A - e_n \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} v_A(w^*) e_A$$

(4.19)

$$= \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} u_A(w_0, \ldots, w_{n-1}, -w_n) e_A - \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} v_A(w_0, \ldots, w_{n-1}, w_n) e_A e_n.$$

**Lemma 4.1** ([12]) Let $f(w)$ and $g(w)$ be hypercomplex functions defined on $\mathbb{R}^{n+1}_+ \left( \mathbb{R}^{n+1}_- \right)$, then

$$|f^*(w^*)| = |f(w)|, \quad x \in \mathbb{R}^{n+1}_+ \left( \mathbb{R}^{n+1}_- \right),$$

(4.20)

as well as

$$\left[ f(w) g(w) \right]^* = f^*(w^*) g^*(w^*), \quad w \in \mathbb{R}^{n+1}_+ \left( \mathbb{R}^{n+1}_- \right).$$

(4.21)

If the inverse function $f^{-1}$ exists, then the inverse function of $f^*$ also exists, and

$$\left[ f^* \right]^{-1} = \left[ f^{-1} \right]^*,$$

(4.22)

more precisely,

$$\left[ f^* \right]^{-1}(s^*) = \left[ f^{-1} \right]^*(s^*), \quad s \in f\left( \mathbb{R}^{n+1}_+ \right) \left( f\left( \mathbb{R}^{n+1}_- \right) \right).$$

(4.23)

**Proof:** (4.21) comes directly from (4.11) and (4.19) by (2.16). Just to be clear, in (2.20) we rewrite $u_A$ and $v_A$ as, respectively, $u_{f,A}$ and $v_{f,A}^l$. Then,

$$\left[ f(w) g(w) \right]^* \quad \text{(say } w \in \mathbb{R}^{n+1}_+)$$

$$= \sum_{A,B \in \mathcal{P}\{1, \ldots, n-1\}} \left\{ u_{f,A}(w)e_A + e_n v_{f,A}^l(w)e_A \right\} \left\{ u_{g,B}e_B + e_n v_{g,B}^l(w)e_B \right\}^*$$

(4.24)

$$= \sum_{A,B \in \mathcal{P}\{1, \ldots, n-1\}} \left[ u_{f,A}(w)u_{g,B}(w) - (-1)^\#(A)v_{f,A}^l(w)v_{g,B}^l(w) \right] e_A e_B$$

$$- e_n \sum_{A,B \in \mathcal{P}\{1, \ldots, n-1\}} \left[ v_{f,A}^l(w)u_{g,B}(w) - (-1)^\#(A)u_{f,A}(w)v_{g,B}^l(w) \right] e_A e_B.$$
On the other hand, by (4.11), (4.12), (4.14), (4.15) and (4.2), we have
\[
\text{Re}\left[f^*g^*(w^*)\right] = \sum_{A,B \in P \{1, \ldots, n-1\}} \left[u_{f,A}(w)u_{g,B}(w) - (-1)^{#(A)}v_{f,A}(w)v_{g,B}(w)\right] e_A e_B, \tag{4.25}
\]
and
\[
\text{Im}\left[f^*g^*(w^*)\right] = -\sum_{A,B \in P \{1, \ldots, n-1\}} \left[(-1)^{#(A)}u_{f,A}(w)v_{g,B}(w) + v_{f,A}(w)u_{g,B}(w)\right] e_A e_B. \tag{4.26}
\]
(4.24), (4.25) and (4.26) imply (4.21).

Noting that
\[
w = (f^{-1} \circ f)(w) = f^{-1}(f(w)), \quad w \in \mathbb{R}^{n+1} \left(\mathbb{R}^{n+1}\right), \tag{4.27}
\]
we have, by (4.7) and (4.9),
\[
\left((f^{-1})^* \circ f^*\right)(w^*) = \left[f^{-1}\left(\left[f^*\left([w^*]^*\right)\right]\right)\right]^* = \left[f^{-1}(f(w))\right]^* = w^*, \quad w^* \in \mathbb{R}^{n+1} \left(\mathbb{R}^{n+1}_-\right), \tag{4.28}
\]
which is (4.22).

Introduce the operators
\[
\frac{\partial}{\partial x} = \sum_{j=0}^{n-1} e_j \frac{\partial}{\partial w_j} \quad \text{with} \quad x = \text{Re}(w) \tag{4.29}
\]
and
\[
\frac{\partial}{\partial y} = \frac{\partial}{\partial w_n} \quad \text{with} \quad y = \text{Im}(w). \tag{4.30}
\]
Their actions on functions from the left and from the right are governed by the rules
\[
\frac{\partial f}{\partial x} = \sum_{k=0}^{n-1} \sum_A e_k e_A \frac{\partial f_A}{\partial w_k}, \quad \frac{\partial f}{\partial x} = \sum_{k=0}^{n-1} \sum_A e_A e_k \frac{\partial f_A}{\partial w_k}. \tag{4.31}
\]

Lemma 4.2 (Cauchy-Riemann equations \[13, 14\])

(1) \( \Phi \) is left regular on \( \mathbb{R}^{n+1}_+ \left(\mathbb{R}^{n+1}_-\right) \) if and only if
\[
\frac{\partial[U]}{\partial x}(w) = \frac{\partial[V^l]}{\partial y}(w) \quad \text{and} \quad \frac{\partial[U]}{\partial y}(w) = -\frac{\partial[V^l]}{\partial x}(w), \tag{4.32}
\]
where
\[
U(w) = (\text{Re} \Phi)(w) \quad \text{and} \quad V^l(w) = (\text{Im}^l \Phi)(w) \tag{4.33}
\]
are, respectively, the real and the left imaginary parts of \( \Phi \).

(2) \( \Phi \) is right regular on \( \mathbb{R}^{n+1}_- \left(\mathbb{R}^{n+1}_+\right) \) if and only if
\[
\frac{\partial[U]}{\partial x}(w) = \frac{\partial[V^r]}{\partial y}(w) \quad \text{and} \quad \frac{\partial[U]}{\partial y}(w) = -\frac{\partial[V^r]}{\partial x}(w), \tag{4.34}
\]
where
\[
U(w) = (\text{Re} \Phi)(w) \quad \text{and} \quad V^r(w) = (\text{Im}^r \Phi)(w) \tag{4.35}
\]
are, respectively, the real and the right imaginary parts of \( \Phi \).
**Proof:** If $\Phi$ is left regular on $\mathbb{R}_+^{n+1}$, it is easy to see, by Remark 4.1, that

$$
\left( D[\Phi] \right) (w) \quad (w \in \mathbb{R}_+^{n+1})
$$

$$
= \sum_{k=0}^{n-1} e_k \, \left[ \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} e_k A \frac{\partial u_A}{\partial w_k} (w) + e_n \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} e_A \frac{\partial v^I_A}{\partial w_n} (w) \right] + e_n \left[ \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} e_A \frac{\partial u_A}{\partial w_n} (w) \right]
$$

$$
= \sum_{k=0}^{n-1} \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} e_k e_A \frac{\partial u_A}{\partial w_k} (w) - \sum_{k=0}^{n-1} \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} e_A \frac{\partial v^I_A}{\partial w_k} (w)
$$

$$
+ \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} e_n e_A \frac{\partial u_A}{\partial w_n} (w) + \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} e_n e_A \frac{\partial v^I_A}{\partial w_n} (w)
$$

$$
= \frac{\partial U}{\partial x} (w) - \frac{\partial [V^I]}{\partial y} (w) + e_n \left[ \frac{\partial U}{\partial y} (w) + \frac{\partial [V^I]}{\partial x} (w) \right] \quad \text{(by (4.29) and (4.30)),}
$$

which implies (1). The proof of (2) is similar. 

\[\Box\]

**Theorem 4.1 (Symmetry principle for regular functions)** If $\Phi$ is regular on $\mathbb{R}_+^{n+1}$, then $\Phi^*$ is also regular on $\mathbb{R}_+^{n+1}$.

**Proof:** By (4.18) and (4.19), we have

$$
U^*(w) = \text{Re} \, \Phi^*(w) = \text{Re} \, \Phi(w^*) = U(w^*), \quad w \in \mathbb{R}_-^{n+1},
$$

and

$$
(V^*)^I(w) = \text{Im} \, \Phi^*(w) = -\text{Im} \, \Phi(w^*) = -V^I(w^*), \quad w \in \mathbb{R}_-^{n+1}.
$$

So,

$$
\frac{\partial [U^*]}{\partial x} (w) = \sum_{k=0}^{n-1} e_k \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} e_A \frac{\partial u_A}{\partial w_k} (w_0, \ldots, w_n) = \frac{\partial [U]}{\partial x} (w_0, \ldots, w_n),
$$

$$
\frac{\partial [U^*]}{\partial y} (w) = \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} e_A \frac{\partial u_A}{\partial w_n} (w_0, \ldots, w_n) = \frac{\partial [U]}{\partial y} (w_0, \ldots, w_n),
$$

and

$$
\frac{\partial [(V^*)^I]}{\partial y} (w) = \sum_{k=0}^{n-1} e_k \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} e_A \frac{\partial v^I_A}{\partial w_k} (w_0, \ldots, -w_n) = \frac{\partial [V^I]}{\partial y} (w),
$$

$$
\frac{\partial [(V^*)^I]}{\partial x} (w) = \sum_{k=0}^{n-1} e_k \sum_{A \in \mathcal{P}\{1, \ldots, n-1\}} e_A \frac{\partial v^I_A}{\partial w_k} (w_0, \ldots, -w_n) = \frac{\partial [V^I]}{\partial x} (w).
$$

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By Theorem 4.1, (4.39), (4.40), (4.41), (4.42), we know that
\[ \Phi \text{ is (left) regular on } \mathbb{R}_+^{n+1} \left( \mathbb{R}_-^{n+1} \right) \iff \Phi^* \text{ is (left) regular on } \mathbb{R}_-^{n+1} \left( \mathbb{R}_+^{n+1} \right). \] (4.43)
Hence the symmetry principle for regular functions is proved.

By Remark 4.1, we may get the following results for boundary values.

**Lemma 4.3** If \( \Phi \) is defined in \( \mathbb{R}_+^{n+1} \) with the boundary value \( \Phi^+(t) \), then \( \Phi^* \) has the boundary value \( [\Phi^*]^-(t) \) and
\[ [\Phi^+]^*(t) = [\Phi^*]^-(t), \quad t \in \mathbb{R}_0^{n+1}. \] (4.44)
In other words,
\[ \Phi^+(t) + [\Phi^*]^-(t) = 2 \text{ Re}(\Phi^+(t)), \quad t \in \mathbb{R}_0^{n+1}. \] (4.45)

**Lemma 4.4** If \( \Phi \) is defined in \( \mathbb{R}_-^{n+1} \) with the boundary value \( \Phi^-(t) \), then \( \Phi^* \) also has the boundary value \( [\Phi^*]^+(t) \) and
\[ [\Phi^-]^*(t) = [\Phi^*]^+(t), \quad t \in \mathbb{R}_0^{n+1}. \] (4.46)
In other words,
\[ \Phi^-(t) + [\Phi^*]^+(t) = 2 \text{ Re}(\Phi^-(t)), \quad t \in \mathbb{R}_0^{n+1}. \] (4.47)

By Lemma 4.1 and Lemma 4.3, we immediately have the following theorem, which is the basis for the Hilbert boundary value problem being transferable into the Riemann boundary value problem.

If \( \Phi \) is defined in \( \mathbb{R}_+^{n+1} \left( \mathbb{R}_-^{n+1} \right) \), we call the function
\[ (\mathcal{E}[\Phi])(w) = \begin{cases} 
\Phi(w), & \text{when } w \in \mathbb{R}_+^{n+1} \left( \mathbb{R}_-^{n+1} \right), \\
\Phi^*(w), & \text{when } w \in \mathbb{R}_-^{n+1} \left( \mathbb{R}_+^{n+1} \right)
\end{cases} \] (4.48)
the symmetric extension of \( \Phi \) defined in \( \mathbb{R}_+^{n+1} \left( \mathbb{R}_-^{n+1} \right) \).

**Theorem 4.2** If \( \Phi \) is regular on \( \mathbb{R}_+^{n+1} \left( \mathbb{R}_-^{n+1} \right) \) and can be extended continuously to \( \mathbb{R}_0^{n+1} \), then its symmetric extension \( \mathcal{E}[\Phi] \) is the sectionally holomorphic function with \( \mathbb{R}_0^{n+1} \) as the jump surface (see [12]).

With the help of this theorem, the Hilbert BVP (3.20) will be converted equivalently to the Riemann BVP discussed in [12] with an additional restricting condition.

**Theorem 4.3** The Schwarz BVP (3.22) is equivalent to the following Riemann BVP (4.51) under the relationship
\[ \Psi(w) = (\mathcal{E}[\Phi])(w), \quad w \in \mathbb{R}_+^{n+1} \cup \mathbb{R}_-^{n+1} \] (4.49)
or
\[ \Phi(w) = \Psi \big|_{\mathbb{R}_+^{n+1}}(w) = \Psi^+(w), \quad w \in \mathbb{R}_+^{n+1}. \] (4.50)
**R\_m^* problem with the reflection condition.** Find a sectionally holomorphic function \( \Psi \), with \( \mathbb{R}^{n+1}_0 \) as its jump plane such that

\[
\begin{aligned}
\Psi^+(x) + \Psi^-(x) &= 2c(x), \quad x \in \mathbb{R}^{n+1}_0 \quad \text{(boundary value condition)}, \\
\Psi(w) &= o(w^{m+1}) \quad \text{near } \infty \quad \text{(growth condition at the infinity)}, \\
[\Psi^+]^*(x) &= [\Psi^+]^-(x), \quad x \in \mathbb{R}^{n+1}_0 \quad \text{(reflection condition)},
\end{aligned}
\] (4.51)

where \( m \) is some integer.

**The proof of Theorem 4.3.** Let \( \Phi \) be a solution of the Schwarz problem (3.22), by using Theorem 4.1, (4.20) and (4.45), we know that \( \Psi(w) = (\mathcal{E}[\Phi]) (w) \) is the solution of the \( R_m^* \) problem (4.51). And, in turn, if \( \Psi \) is the solution of the \( R_m^* \) problem (4.51), then

\[
\Phi(w) = \Phi^+(w) = \Psi^+(w), \quad w \in \mathbb{R}^{n+1}_+
\] (4.52)

is the solution of the Schwarz problem (3.22). In fact, by the reflection condition in (4.51) we have (4.17) which results in (3.22). \( \blacksquare \)

Now, the remaining question is how to solve the \( R_m^* \) problem (4.51). To do so, we introduce the self-reflex action of \( \Phi \) defined on \( \mathbb{R}^{n+1}_+ \cup \mathbb{R}^{n+1}_- \).

### 4.2 Self-reflex action

If \( \mathcal{U} \) is defined on \( \mathbb{R}^{n+1}_+ \cup \mathbb{R}^{n+1}_- \), say,

\[
\mathcal{U}(w) = \begin{cases} \\
\mathcal{U}^+(w), & w \in \mathbb{R}^{n+1}_+ \\
\mathcal{U}^-(w), & w \in \mathbb{R}^{n+1}_-
\end{cases}
\] (4.53)

then

\[
\mathcal{U}_\uparrow(w) = \begin{cases} \\
[\mathcal{U}^-]^*(w), & w \in \mathbb{R}^{n+1}_+ \\
[\mathcal{U}^+]^*(w), & w \in \mathbb{R}^{n+1}_-
\end{cases}
\] (4.54)

is called the reflective function of \( \mathcal{U} \). Obviously, by (4.17)

\[
\mathcal{U}(w) = (\mathcal{U}_\uparrow)^\downarrow(w), \quad w \in \mathbb{R}^{n+1}_+.
\] (4.55)

In particular, if

\[
\mathcal{U}_\downarrow(w) = \mathcal{U}(w), \quad w \in \mathbb{R}^{n+1}_+,
\] (4.56)

then we call \( \mathcal{U} \) a self-reflection function. Obviously, by (4.17),

\[
(\mathcal{S}[\mathcal{U}]) (w) = \frac{\mathcal{U}(w) + \mathcal{U}_\downarrow(w)}{2}
\] (4.57)

is a self-reflection function, which is called the self-reflection function of \( \mathcal{U} \). For the sake of convenience, we call the above steps from \( \Phi \), defined on \( \mathbb{R}^{n+1}_+ \cup \mathbb{R}^{n+1}_- \), to \( \mathcal{S}[\mathcal{E}[\Phi]] \) to be the so-called self-reflex action of \( \Phi \). From (4.56) and (4.57) we know that \( \Phi \) is a self-reflection function if and only if

\[
\Phi(w) = (\mathcal{S}[\mathcal{E}[\Phi]]) (w), \quad w \in \mathbb{R}^{n+1}_+ \cup \mathbb{R}^{n+1}_-.
\] (4.58)
**Example 4.1** From Example [2.1](#), we know that the restriction $E|_{\mathbb{R}^{n+1}_\pm}$ of the Cauchy kernel given in (2.34) is a self-reflection function.

**Example 4.2** Assume $f$ is a para real-valued function. Let

$$
\Psi(w) = \begin{cases}
(S[f])(w) = \frac{1}{\sqrt{n+1}} \int_{\mathbb{R}^{n+1}_0} E(x-w) \, d\sigma_f(x), & w \in \mathbb{R}^{n+1}_+,

-(S[f])(w) = -\frac{1}{\sqrt{n+1}} \int_{\mathbb{R}^{n+1}_0} E(x-w) \, d\sigma_f(x), & w \in \mathbb{R}^{n+1}_-,
\end{cases}
$$

(4.59)

where $S[f]$ is Cauchy type integral on the hyperplane $\mathbb{R}^{n+1}_0$, and

$$
\mathrm{d}S = \mathrm{d}x_0 \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} = e_n \, d\sigma
$$

(4.60)
is the elementary surface measure on the hyperplane $\mathbb{R}^{n+1}_0$ (see [12]). Then, by Example 4.1 and Lemma 4.1 its self-reflection function is

$$
\left(\mathcal{R}[\Psi]\right)(w) = (S[f])(w) = \frac{1}{\sqrt{n+1}} \int_{\mathbb{R}^{n+1}_0} E(x-w) \, d\sigma_f(x), \quad w \in \mathbb{R}^{n+1}_\pm.
$$

(4.61)

**Example 4.3** Let

$$
\Psi(w) = \begin{cases}
Z^\alpha(w) \, c_\alpha, & w \in \mathbb{R}^{n+1}_+,

-Z^\alpha(w) \, c_\alpha & w \in \mathbb{R}^{n+1}_-,
\end{cases}
$$

(4.62)

where $c_\alpha$ is a hypercomplex constant and $Z^\alpha$ is a Fueter polynomial given in (3.10). Then,

$$
\left(\mathcal{R}[\Psi]\right)(w) = Z^\alpha(w) \, \mathrm{Im} \, c_\alpha \, e_n, \quad w \in \mathbb{R}^{n+1}_\pm,
$$

(4.63)

which is called the para-imaginary coefficient polynomial.

We point out an obvious fact that, if $\Psi$ is the solution of $R^*_m$ problem (4.51) with a reflection condition, of course it is the solution of the following Riemann boundary value problem $R_m$ discussed in [12].

*Problem with no reflection condition.* Find a sectionally holomorphic function $\Psi$, with $\mathbb{R}^{n+1}_0$ as its jump plane, such that

$$
\begin{cases}
\Psi^+(x) + \Psi^-(x) = 2c(x), & x \in \mathbb{R}^{n+1}_0 \quad \text{(boundary value condition)},

\Psi(w) = o(w^{m+1}) \quad \text{near } \infty \quad \text{(growth condition at the infinity)}.
\end{cases}
$$

(4.64)

If $\Psi$ is both a solution of (4.64) and a reflexive function, then it is called a reflexive solution of (4.64). Under such case, by (4.55) the reflection condition in (4.51) automatically holds. So, we have the following result.

**Lemma 4.5** The reflexive solution of the $R_m$ problem with no reflection condition (4.64) surely is the solution of the $R^*_m$ problem with the reflection condition (4.51).

In turn, we also have the following result.
Lemma 4.6 If $\Psi$ is the solution of the $R_m$ problem without reflective condition (4.64), then its reflection function $\Psi\uparrow$ and the self-reflection function $R[\Psi]$ are the solutions of (4.51).

Thus, $R[\Psi]$ is the self-reflection solution of the $R_m^*$ problem with reflection condition in (4.51).

**Proof:** First, the reflection function $\Psi\uparrow$ is regular on $\mathbb{R}^{n+1}$ by Lemma 4.2. Secondly, using the reflection operator to both sides of the formulas in (4.64) we have, by (4.20) and (4.9),

$$\begin{cases}
(\Psi\uparrow)^+(x) + (\Psi\uparrow)^-(x) = 2c(x), & x \in \mathbb{R}^{n+1}_0 \text{ (boundary value condition)}, \\
\Psi\uparrow(w) = O(w^{m+1}) \text{ near } \infty \text{ (growth condition at the infinity)}. 
\end{cases} \quad (4.65)$$

This is to say that $\Psi\uparrow$ is also a solution of (4.64). So is the reflective function $R[\Psi]$ given in (4.57) with $\mathcal{U} = \Psi$, since $\Psi$ and $\Psi\uparrow$ are the solutions of (4.64).

**Theorem 4.4** The general solution of the Schwarz boundary value problem (3.22) should be

$$\Phi(w) = \left(R[\Psi]\right)(w) = \frac{\Psi(w) + \Psi\uparrow(w)}{2}, \quad w \in \mathbb{R}^{n+1}, \quad (4.66)$$

where $\Psi$ is the solution of the $R_m$ problem (4.64).

Thus, we have the principle of so-called self-reflex action by Theorem 4.2, Lemma 4.5 and Lemma 4.6. In short, the solutions of $R_m$ problem (4.64) with no reflection condition are derived from the solutions of the Schwarz boundary value problem (3.22) by the self-reflex action, or the solutions of the Schwarz boundary value problem (3.22) may be obtained from the solutions of $R_m$ problem without reflection condition (4.64) through two steps: first taking the self-reflex action and then taking the restriction on $\mathbb{R}^{n+1}_+$. 

5 Solutions of $H_m$ problem

Based on the reflexive principle, in order to solve $S_m$ problem (3.22) we only need to solve the $R_m$ problem (4.64), which is discussed in detail in [12].

5.1 Solutions of $R_m$ problem

For the convenience of reference, here we restate the results for $R_m$ problem as follows.

**Theorem 5.1** (see [12]) For the Riemann boundary value problem $R_m$ (4.64) the following four cases are a complete classification.

1. Let $m \geq 0$, $c \in \hat{H}^\mu(\mathbb{R}^{n+1}_0)$, then its general solution is

$$\Psi(w) = \begin{cases}
\Phi(w), & w \in \mathbb{R}^{n+1}_+, \\
-\Phi(w), & w \in \mathbb{R}^{n+1}_-, 
\end{cases} \quad (5.1)$$

where

$$\Phi(w) = (S[c])(w) + P_m(w) = \frac{1}{\sqrt{n+1}} \int_{\mathbb{R}^{n+1}} E(x - w)d\sigma c(x) + \sum_{|\alpha| = 0}^m \frac{1}{|\alpha|!} Z^\alpha(w) c_\alpha, \quad w \in \mathbb{R}^{n+1}_\pm, \quad (5.2)$$
where $P_m$ is arbitrary hypercomplex symmetric polynomial of degree not exceeding $m$ with $C^m_{n+m}$ free hypercomplex constants $c_\alpha$.

(2) Let $m = -1$ with $c \in \hat{H}(\mathbb{R}_0^{n+1})$, it has the unique solution

$$\Phi(w) = \begin{cases} 
(S[c])(w) = \frac{1}{\sqrt{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w) \, d\sigma(c(x), \quad w \in \mathbb{R}_+^{n+1}, \\
-(S[c])(w) = -\frac{1}{\sqrt{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w) \, d\sigma(c(x), \quad w \in \mathbb{R}_-^{n+1},
\end{cases} \quad (5.3)$$

if and only if

$$c(\infty) = \lim_{x \to \infty} c(x) = 0. \quad (5.4)$$

(3) Let $-n < m < -1$ and $r = -(m+1)$, with $c \in \hat{H}_{r,0}(\mathbb{R}_0^{n+1})$, it has the unique solution $[5.3]$.

(4) Let $m \leq -n$ and $r = -(m+1)$, with $c \in \hat{H}_{r,0}(\mathbb{R}_0^{n+1})$, it has the unique solution [5.3] if the $C^m_{n-m-1}$ conditions

$$\int_{\mathbb{R}_0^{n+1}} Z^\alpha(x) \, d\sigma(c(x) = 0, \quad |\alpha| = 0, 1, \cdots, -(n+1+m) \quad (5.5)$$

are fulfilled.

**Remark 5.1** The $R_m$ problem discussed in [12] is the jump problem for $\Phi$. Here the $R_m$ problem [4.64] for $\Psi$ is called the Szegö problem in some literature. They are slightly different and governed by the relation [5.1].

### 5.2 Solutions of $S_m$ problem

By Theorem 4.4, Theorem 5.1, Example 4.2 and Example 4.3 we get the following result.

**Theorem 5.2** The general solution of the Schwarz boundary value problem (3.22) should be as following four cases.

**Case 1.** When $m \geq 0$ and $c \in \hat{H}^m(\mathbb{R}_0^{n+1})$, then

$$\Phi(w) = (S[c])(w) = \frac{1}{\sqrt{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w) \, d\sigma(c(x) + \sum_{|\alpha|=0}^{m} \frac{1}{\alpha!} Z^\alpha(w) R_\alpha e_n \quad w \in \mathbb{R}_+^{n+1}, \quad (5.6)$$

where $R_\alpha$ are $C^m_{n+m}$ free para-real hypercomplex constants.

**Case 2.** When $m = -1$ with $c \in \hat{H}(\mathbb{R}_0^{n+1})$, it has the unique solution

$$\Phi(w) = (S[c])(w) = \frac{1}{\sqrt{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w) \, d\sigma(c(x), \quad w \in \mathbb{R}_+^{n+1},$$

(5.7)
if and only if
\[ c(\infty) = \lim_{x \to \infty} c(x) = 0. \]  \hspace{1cm} (5.8)

**Case 3.** When \(-n < m < -1\) and \(r = -(m+1)\), with \(c \in \widehat{H}_{r,0}(\mathbb{R}_0^{n+1})\), it has the unique solution \((5.7)\).

**Case 4.** When \(m \leq -n\) and \(r = -(m+1)\), with \(c \in \widehat{H}_{r,0}(\mathbb{R}_0^{n+1})\), it has the unique solution \((5.7)\) if the \(C^n_{-m-1}\) conditions

\[ \int_{\mathbb{R}_0^{n+1}} Z^\alpha(x) d\sigma c(x) = 0, \ |\alpha| = 0, 1, \cdots, -(n+1+m) \]  \hspace{1cm} (5.9)

or

\[ \int_{\mathbb{R}_0^{n+1}} Z^\alpha(x) c(x) dx_1 dx_2 \cdots dx_n = 0, \ |\alpha| = 0, 1, \cdots, -(n+1+m) \]  \hspace{1cm} (5.10)

are fulfilled.

### 5.3 Solutions of \(H_m\) problem

The Hilbert problem \((3.20)\) for the function \(\Phi\) may be directly translated into the Schwarz problem for the function \(\Psi\) by

\[
\begin{align*}
\Phi(w) &= \begin{cases} 
\Psi(w), & w \in \mathbb{R}_+^{n+1}, \\
\Psi(w)^\lambda, & w \in \mathbb{R}_-^{n+1}.
\end{cases}
\end{align*}
\]  \hspace{1cm} (5.11)

**Theorem 5.3** The general solution of the Hilbert boundary value problem \((3.20)\) should be as following four cases.

**Case 1.** When \(m \geq 0\) and \(c \in \widehat{H}^\mu(\mathbb{R}_0^{n+1})\), then

\[ \Phi(w) = \frac{1}{\sqrt{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w) d\sigma c(x)^\lambda \lambda^{-1} + \sum_{|\alpha|=0}^m \frac{1}{|\alpha|!} Z^\alpha(w) R_\alpha \lambda^{-1}, w \in \mathbb{R}_+^{n+1}, \]  \hspace{1cm} (5.12)

where \(R_\alpha\) are \(C^n_{n+m}\) free para-real hypercomplex constants.

**Case 2.** When \(m = -1\) with \(c \in \widehat{H}(\mathbb{R}_0^{n+1})\), it has the unique solution

\[ \Phi(w) = (S[c])^\lambda w^{-1} = \frac{1}{\sqrt{n+1}} \int_{\mathbb{R}_0^{n+1}} E(x-w) d\sigma c(x)^\lambda, w \in \mathbb{R}_+^{n+1}, \]  \hspace{1cm} (5.13)

if and only if

\[ c(\infty) = \lim_{x \to \infty} d(x) = 0. \]  \hspace{1cm} (5.14)
Case 3. When $-n < m < -1$ and $r = -(m+1)$, with $c \in \hat{H}_{r,0}(\mathbb{R}^{n+1}_0)$, it has the unique solution (5.13).

Case 4. Let $m \leq -n$ and $r = -(m+1)$. If $c \in \hat{H}_{r,0}(\mathbb{R}^{n+1}_0)$, then it has the unique solution (5.13) provided the $C^{n-m-1}$ conditions

$$\int_{\mathbb{R}^{n+1}_0} Z^{\alpha}(x) \, d\sigma \, c(x) = 0, \quad |\alpha| = 0, 1, \cdots, -(n + 1 + m)$$

or

$$\int_{\mathbb{R}^{n+1}_0} Z^{\alpha}(x) \, c(x) \, dx_1 \, dx_2 \cdots \, dx_n = 0, \quad |\alpha| = 0, 1, \cdots, -(n + 1 + m)$$

are fulfilled.

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