Abstract. As shown in a former article, General Relativity (GR) can be formulated and interpreted as the macroscopic limit of the multiple layer statistics concept derived from the horizon thermodynamics approach to gravity. A central feature of this formulation is the total entropy perceived by multiple observers. This so-called multiple-entropy (or m-entropy) $S$ has a one-to-one correspondence to the boundary action of GR. By itself, the expression for $S$ does not yet include matter. However, from the point of view of m-entropy, Einstein’s matter term appears as a constraining prescription with regard to the availability of quantum configurations. From this prescription, the characteristics of matter fields (first quantization) can be deduced. On the other hand, quantum fluctuations of $S$ around a given classical geometry are interpreted in terms of quantum matter. This interpretation also yields a value for the fundamental quantum of gravity which is related to the Planck constant via the Planck length.

1. Introduction
Since the thermodynamic properties of black holes have been found by Bekenstein [1] and Hawking [2], the thermodynamic approach of gravity has become an important branch of research [3]. On these lines, Jacobson [4] and Padmanabhan [5] have shown that Einstein’s field equations appear as a manifestation of the First Law applied to the null-horizon of the Rindler wedge. In this way, any null-surface acquires a local, observer-dependent temperature $T = \kappa/(2\pi)$ with surface gravity $\kappa$, and an entropy $S_{BH,JP} = A/(4L_p^2)$, where $A$ is the horizon surface area and $L_p$ is the Planck length.

This concept has recently been extended to describe multiple accelerated observers so as to obtain a temperature-independent, Boltzmann-like entropy which we call the m-entropy $S$ [7]. In fact, $S$ is equivalent to the boundary action of GR. In contrast to $S_{BH,JP}$, the temperature-independence of $S$ is of considerable advantage, as the underlying microscopic model is not restricted to local equilibrium. This allows for departures of the system from the classical law of GR and opens a door to quantum behaviour on a dimensionally reduced space. The reduced space has dimension 2+1, in correspondence to t’Hooft’s argument at the root of the holographic principle [8].

The present article develops on macroscopic departures from the vacuum (or maximum m-entropy) state. Due to the statistical properties of $S$, these departures are straight-forwardly
interpreted in terms of matter fields, the latter play the role of constraints. We derive some formal properties of matter (which is scalar in the context of GR) and investigate the arising fundamental quanta of gravity.

To the convenience of the reader, we also summarize in this article the concept of m-entropy which has been formerly derived using [3], the "boundary action" of GR (cf. Appendix A) and the multiple Rindler horizon (or multiple layer) description of gravity (cf. Appendix B). Its statistical interpretation is summarized in Appendix C. For the quantum properties of the m-entropy, we refer to [7], where we have sketched the derivation of quantum uncertainties in a system with a small number of quantum bits of gravity, as well as the derivation of some typical transition probabilities between quantum states.

2. Synopsis of the multiple layer formulation of GR

The starting point is the boundary action (omitting the cosmological constant and the matter term for simplicity)

$$S_{\partial V} = \frac{c^4}{16\pi G} \sum_A \varepsilon_A \int_{B_A} d^3 \partial V \sqrt{\gamma} K,$$

which is equivalent to the Einstein-Hilbert action describing GR (cf Appendix A / Section Appendix A). In [10], the integration is performed over the piece-wise smooth boundary $\partial V = \sum_A B_A$ of a compact 4-volume $V$, where every component $B_A$ is time- or space-like. Furthermore, $K = K_{ab}\gamma^{ab}$, and $\gamma_{ab}$ is the induced metric on $\partial V$ with determinant $\gamma$, and $K_{ab}$ is the exterior curvature on $\partial V$ with unit normal vector $n^a$, namely $K_{ab} = -\frac{1}{2} L_+ \gamma_{ab} = -g^{cd}_{\partial V} \nabla_c n_b$, $L_+ = L_n$, and $\varepsilon_A = n^a n_a$ on $B_A$. Equivalently, following the notation of [7], [10] can be written as

$$S_{\partial V} = \frac{c^4}{16\pi G} \int_{\partial V} N^c_{ab} f^{ab} d^3\Sigma_c, \quad f^{ab} = \sqrt{-g} g^{ab}, \quad N^c_{ab} d^3\Sigma_c = \varepsilon_A N^{ab} d^3 x,$$

where $N^c_{ab}$ is the momentum conjugate to $f^{ab}$ and $d^3\Sigma_c$ is the the 3-surface element covector. The boundary $\partial V$ can be divided into cuboids $\Delta \partial V$ which are small enough for geometric variations to be negligible across $\Delta \partial V$ and large enough for quantum gravity effects to be negligible. For the portion of boundary action contained by $\Delta \partial V$, it can be shown (cf Appendix B / Section Appendix B) [7, 9]:

$$S_{\partial V} \Delta \partial V = \frac{c^4 \varepsilon_A}{32\pi G} \int_{\Delta \partial V} d^3 \partial V \sqrt{\gamma} \gamma^{ab} L_+ \gamma_{ab}$$

$$= \frac{c^4 \varepsilon_A}{32\pi G} \sum_{k=1}^N \int_{A(x_k^w)} d^2 \tilde{x} \left[ \sqrt{\gamma(w)} \left| \frac{\partial u}{\partial x^w_k} \right| \Delta u \gamma^{ab}_{(w)} \nabla_{(w)} L_{(w)} \gamma_{(w)}^{ab} + \sqrt{\gamma(v)} \left| \frac{\partial v}{\partial x^w_k} \right| \Delta v \gamma^{ab}_{(v)} \nabla_{(v)} L_{(v)} \gamma_{(v)}^{ab} \right]$$

$$= \frac{c^4 \varepsilon_A}{32\pi G} \sum_{k=1}^N \sum_{w=\text{u, v}} \int_{\Delta \partial V} \int_{A(x^w_k)} d^2 \tilde{x} \sqrt{\gamma(w)} \left| \frac{\partial u}{\partial x^w_k} \right| \gamma^{ab}_{(w)} \nabla_{(w)} L_{(w)} \gamma_{(w)}^{ab}$$

$$= \sum_{k=1}^N \left[ \int_{\Sigma(w)(x^w_k)} \tilde{T}_{u} \ s \ du \ d^2 x + \int_{\Sigma(v)(x^w_k)} \tilde{T}_{v} \ s \ dv \ d^2 x \right],$$

where $\gamma^{ab}_{(w)}$ is the induced triad on $\partial V$, $\gamma^{ab}_{(w)}$ and $\gamma^{(w)}_{(v)}$ are the metric and inverse metric induced on the hypersurface $w = \text{constant}$, respectively, $w = u, v$ are null coordinates, $\tilde{x}^k = (u, v, x^2, x^3)$, $\tilde{T}_{w} = |\varepsilon_w^2| T_{w}$ and $T_{w}$ is the horizon temperature observed on the hypersurface $w = \text{constant}$. 
$S_{\partial V \Delta \partial V}$ can be interpreted as a temperature-independent Boltzmann-like entropy according to Equation (3) as soon as $\Delta w$ are at the limit of the classical spatial resolution. From the point of view of a set of Rindler observers, the total entropy in $\Delta \partial V$ (which we refer as m-entropy $S_{\partial V \Delta \partial V}$) is the sum of the entropies of all the null-strips contained in $\partial V$.

As a consequence, the boundary space splits into multiple 2d-layers (defined by a foliation across an arbitrary direction), and every layer consists of a collection of surface microstates $|q\rangle$, with no restriction about which fraction of the states is on which layer. It is only when requiring the number of microstate configurations per macroscopic geometry to be maximum that we obtain the stationarity condition on $S_{\partial V \Delta \partial V}$ which in turn yields the macroscopic limit (namely GR).

Multiple layer statistical gravity is not the first development for which some kind of analogy between the gravitational action and the notion of ”entropy” is suggested [10, 11, 12]. However, it is the first derivation of this analogy, by using multiple observer horizon thermodynamics. For quantum mechanics, such an analogy has been conjectured even earlier [13], this suggests some fundamental connection between gravity and matter.

3. Quasi-extremality condition as a matter generating mechanism

Although the stationary point of the boundary action is the vacuum geometry ($\gamma_{\text{vac}}^{ab}$), this geometry is very special. If we choose the numbers $N_K, n_K$ randomly, we will not expect to obtain the vacuum geometry but some other geometry with significantly lower m-entropy. We shall therefore explore the non-vacuum regime by constraining the range of available $\gamma_{ab} \neq \gamma_{\text{vac}}^{ab}$ and identify which geometries have stationary m-entropy. Since we do not have a preferred constraint function at our disposal, it will prove useful to apply a method different from the Lagrange multiplier method, we will call it the compensation method. Consider a family of distinct curves $\Gamma_\eta(\tau)$ in the neighbourhood of $\gamma_{\text{vac}}^{ab}$ in the space of induced metric $\gamma_{ab}$ with $0 \leq \tau \leq \epsilon$ positive and continuous parameter $\eta$, so that every curve starts at $\Gamma_\eta(0) = \gamma_{ab}^{\text{vac}}$ and reaches some point $\Gamma_\eta(\epsilon) \neq \gamma_{ab}^{\text{vac}}$. Our task is to impose the stationarity condition $\delta S|_{\gamma_{ab}=\Gamma_\eta(\epsilon)} = 0$ (up to pure gauge variations which leave $S$ constant). Or equivalently, we introduce a compensation term $S_m(\Gamma_\eta(\tau), \varphi(\tau))$ such that $\delta(S + S_m) = 0$. The idea is that $S_m$ should force $\gamma_{ab}$ to remain at a “fixed distance” from $\gamma_{ab}^{\text{vac}}$ via a distinct object $\varphi$ by compensating for the non-stationarity of $S$. The challenge is to find an appropriate general ansatz for $S_m$. There is a special case with simplified treatment,

$$ |\Gamma_\eta(\epsilon)) - \gamma_{ab}^{\text{vac}}| / |\gamma_{ab}^{\text{vac}}| \ll 1, \quad (5) $$

which we call the tenuous limit, where $|\gamma_{ab}|$ denotes the largest absolute component of $\gamma_{ab}$. In this limit, the compensation method is particularly suited, and it is also the observationally best accessible regime.

3.1. Tenuous limit

We expand the induced metric up to the first order in the real “distance” parameter $\tau$:

$$ \gamma_{ab}(\tau) \approx \eta_{ab} + \zeta_{ab}\tau, \quad \gamma^a_b(\tau) \approx \eta^a_b - \bar{\zeta}^a_b \tau, \quad (6) $$

where we define $\bar{\zeta}^{cd} = \eta^{ab}\zeta_{bc}\eta^{cd}$ so that $\gamma_{ab}\gamma_{bc} = \delta^{a}_c$ holds to first order in $\tau$. Next, we write the gravitational Lagrangian on the boundary to first order:

$$ L_g = \frac{c^4\varepsilon A}{32\pi G} \sqrt{|\gamma|} |\gamma_{ab}L_{\perp} \gamma_{ab}| \approx -\frac{c^4\varepsilon A}{32\pi G} \sqrt{|\gamma|} |\bar{\zeta}^{ab} L_{\perp} \zeta_{ab}| \tau^2. \quad (7) $$
Notice that the first order term in $\tau$ vanishes because $\gamma_{ab}(0)$ is a stationary point. Because $L_g$ is of second order in $\tau$, so must be the compensation term Lagrangian $L_m$. There are many possible ways to work out an ansatz for $L_m$. The easiest way is to reduce the number of variables of $L_g$ (to one) via a simplification and to replace the remaining variable by a metric-independent (i.e. entropy-constraining) function. We reduce (7) to

$$L_g \approx -\frac{c^4}{64\pi G} \sqrt{|\gamma|} L_\perp (\bar{\zeta}_{ab} \zeta_{ab}) \tau^2.$$  

In (8), we replace $-\bar{\zeta}_{ab} \zeta_{ab} \tau^2$ by a constraining scalar function $\sim \tau^2$ or, equivalently, by a squared scalar function $\phi^2$ (which is real-valued) to construct $L_m$, thus making $\phi$ proportional to $\tau$:

$$L_m \approx \vartheta \sqrt{|\gamma|} L_\perp \phi^2 = 2\vartheta \sqrt{|\gamma|} \varphi \nabla_\perp \varphi,$$  

where $\vartheta$ is an arbitrary constant (still to be determined). In general, $\varphi$ can be complex. Notice that the whole equation $\delta (S + S_m) = 0$ is linear in $\tau$, so that the objects $\bar{\zeta}_{ab}$ and $\zeta_{ab}$ at the stationary points of $S + S_m$ at fixed $\tau$ do not depend on the chosen value of $\tau$. It is sufficient to find $\zeta_{ab}$ at the stationary point for any $\tau$.

Because our starting point was GR on the boundary, we expect to find a spin-less matter field which can be compared to the Klein Gordon (KG) field. To make this comparison, we deduce the boundary KG term $S_{bKG}$ from the bulk KG Lagrangian $L_{KG}$ or action $S_{KG}$ (restricting ourselves for now to a real KG field $\varphi$), and use Gauss' Theoreme:

$$L_{KG} = \frac{1}{2} (\partial^\mu \varphi \partial_\mu \varphi - \frac{m^2 c^2}{\hbar^2} \varphi^2),$$

$$\delta S_{KG}/\delta \varphi = \int_V d^4x \sqrt{-g} (-\nabla^\mu \nabla_\mu - \frac{m^2 c^2}{\hbar^2}) \varphi + \delta S_{bKG}/\delta \varphi,$$  

$$S_{bKG} = \int_V d^4x \sqrt{-g} \nabla^\mu (\varphi \nabla_\mu \varphi) = \int_{\partial V} d^3x \sqrt{|\gamma|} \varphi \nabla_\perp \varphi.\quad (11)$$

We see that $S_m$ and $S_{bKG}$ are of the same mathematical form, which is good for our comparison. Considering $\varphi$ to be the same object in $S_m$ and $S_{bKG}$, we obtain

$$\vartheta = \frac{1}{2}.\quad (12)$$

### 3.2. Higher density

If (5) is not satisfied (higher density regime), it follows from the non-linearity of $S_g$ that a metric description similar to (6) would cause $\zeta_{ab}$ at the stationary points (while $\tau$ is fixed) to depend on $\tau$ in a more complex way:

$$\gamma_{ab}(\tau) \approx \eta_{ab} + \zeta_{ab}(\tau),$$

$$\gamma^a_b(\tau) \approx \eta^a_b - \bar{\zeta}^a_b(\tau),$$  

where $\bar{\zeta}^a_b (\eta_{bc} + \zeta_{bc}) = \eta^{ab} \zeta_{bc}$, namely, the tensorial structure of $\zeta_{ab}$ itself would depend on $\tau$ as well. Of course, we can proceed in analogy to the tenuous limit, as for (8,15), which leads to a non-uniform spectrum, if it can be defined at all.

Alternatively, we can avoid the complexity of a non-constant $\zeta_{ab}$ by expanding $\gamma_{ab}$ to higher order in $\tau$:
\[ \gamma_{ab}(\tau) \approx \eta_{ab} + \zeta_{(1)}^{ab} \tau + \zeta_{(2)}^{ab} \tau^2 + \ldots, \]

\[ \gamma^{ab}(\tau) \approx \eta^{ab} - \zeta_{(1)}^{ab} \tau + \zeta_{(2)}^{ab} \tau^2 + \ldots, \]

where each \( \zeta_{(i)}^{ab} \) can be computed (order by order) from the condition \( \gamma_{ab} \gamma_{bc} = \delta_a^c \). In this way, \( \mathcal{L}_g \) can be expanded to higher order in \( \tau \). As a next step, one could think of introducing multilinear forms for matter fields (not only bilinear as in the tenuous limit), e.g. the third order contribution would involve the matter field function \( \varphi^{(3)} \):

\[ \mathcal{L}_m^3 \sim \sqrt{\gamma} | \nabla \varphi^{(3)}_\perp |. \] (15)

The elegance of introducing constant objects \( \zeta_{(i)}^{ab} \) is, however, at the cost of an increase of the number of Euler-Lagrange equations, one per field function.

4. Quanta of matter, quanta of gravity, Planck area and canonical ensemble

Consider again a cuboid \( \Delta \partial V \) of the boundary and its foliation across \( x^K \) (\( K \) is a Minkowski index) into a maximum number \( N_K \) of pieces of 2d-surfaces (each appearing as a horizon to a Rindler observer) of area \( A_K \), i.e. \( n_K \) surface quanta, where \( N_K \) is defined by the achievable classical resolution along \( x^K \). If \( n_q \) defines the number of possible quantum states of every surface quantum bit on \( A_K \) and we consider pure microstates with respect to \( N_K, n_K \), the m-entropy of \( \Delta \partial V \) is given by (16) (see also Appendix C)

\[ s_K = N_K n_K \ln n_q. \] (16)

\( K \) is not used as a tensorial index in this expression. Equation (16) describes the quantum structure of GR via Rindler space. Although we can change \( N_K \) and \( n_K \) arbitrarily, it is only the change of the value of \( S \) which requires compensation via \( S_m \). Consider the curve \( \Gamma_s(\tau) \) (in \( \gamma_{ab} \)-space) made up of the points each satisfying the stationarity condition while \( \tau \) is held fixed. We assume in what follows that the tenuous limit holds. If we change \( s_K \) along \( \Gamma_s(\tau) \) by a number of quanta \( \Delta n \) in such a way that the change of \( \gamma_{ab} \) is no less than the smallest classically accessible difference, we have to modify \( \varphi \) accordingly (the matter content changes). Because \( \gamma_{ab} \) is quantized, so will be \( \varphi \), and \( \varphi \) changes by the same number \( \Delta n \) of quanta as does \( S \), every quantum change of gravity yields a quantum change in matter, and we can interpret the quantum fluctuations of gravity around \( \gamma_{ab} \) as being proportional (or equivalent) to quantum matter. Conversely, we can use the Planck constant \( \hbar \) (from matter quantization) to determine the quantization of \( \gamma_{ab} \) quantitatively, using (15, 11, 12) and

\[ \int_{\partial V} d^3x \sqrt{|\gamma|} \varphi \nabla_\perp \varphi = \hbar c, \] (17)

and

\[ \frac{1}{64\pi} \int_{\partial V} d^3x \sqrt{|\gamma|} \mathcal{L}_m \left( \bar{\zeta}_{ab} \zeta_{ab} \right) \tau^2 = \frac{\hbar G}{c^2} = L_p^2, \] (18)

which is the Planck area (up to a factor 64\( \pi \)), i.e. the expected fundamental constant of quantum gravity. Notice that, in the frame-work of our approach, the Planck area is not merely an estimated order of magnitude for quantum effects, but rather, it fixes the smallest unit of area itself. Because our approach does not require any microscopic model, but only the concept of a “constraining matter”, this result is quite general.

Equation (18) simplifies further if we consider the example of a scalar particle in the form of a monochromatic planar wave function with wave covector \( k_\mu = (k_0, k_\perp, 0, 0) \). The non-zero
elements of $\zeta_{ab}$ in the linear approximation must also yield a monochromatic plane wave function. If we choose $\partial V$ to be a hypercuboid with the two space-like hypersurfaces $\Sigma_{\pm}$ separated by $\pi/k_0$ (half a period) and the time-like hypersurfaces $\Pi_{\pm}$ perpendicular to $(0, k_\perp, 0)$ very near to each other (distance $\ll \pi/k_\perp$), so that $\Sigma_{\pm}$ are the only boundary components to contribute significantly to $S$, we obtain:

$$|\bar{\zeta}^a_{ab}\zeta^b_{ab}|_{\max}^2 = 32\pi \frac{L_p^2}{|\Sigma_+|k_0},$$

where $|\Sigma_+|$ is the 3-volume of $\Sigma_+$.

Another important consequence of our framework deals with the canonical ensemble treatment for a given cuboid $\partial V$ which is immersed in a bath of matter field ($\varphi$) playing the role of a “temperature” – we are not restricting ourselves to the tenuous limit here. Because $S$ represents an entropy, so does $S_m$. Therefore, the states of multiple layer gravity must obey the same statistical mechanics rules as do the quantum states of matter, i.e. the ensemble average entropy $s_N$ of arbitrary mixed multiple layer states of $\Delta \partial V$ with density matrix $\hat{\rho}$ and matter compensation term $s_m(\Delta \partial V, \varphi)$ is the von Neumann entropy of $\Delta \partial V$:

$$s_N = \text{tr} (\hat{\rho} \ln \hat{\rho}),$$

$$\rho_{ij} = \langle N_i, n_i \rangle \frac{\exp \left[ s_K(N_i, n_i) + s_m \right]}{\sum_k \exp \left[ s_K(N_i, n_i) + s_m \right]} \delta_{N_i, N_j} \delta_{n_i, n_j} \langle N_j, n_j \rangle,$$

where the notation $|N, n\rangle$ denotes m-entropy eigenstates with $N = N_K$ and $n = n_K$, respectively.

5. Conclusions

The above investigations have shown that an unambiguous concept of matter appears in the context of GR if its multiple layer statistical formulation is adopted while the m-entropy is constrained to be off-vacuum during the process of extremization. Multiple layer statistics originate from horizon thermodynamics from the perspective of multiple observers and therefore apply fairly generally. In this frame-work, quantum matter is intimately related to the quantum fluctuations of gravity. In the tenuous limit, the linearised constraining matter field can be identified with the scalar field in the low curvature limit of Quantum Field Theory, via the usual bilinear form. Beyond the tenuous limit, any attempt to linearise the matter field points towards a concept of multilinear forms. In our framework, the ratio between a quantum bit of gravity and a scalar particle has been found to be a constant multiple of the Planck area. If we take this observation literally, that would mean that the quantization of gravity has indirectly already been quantified since $\hbar$ has been determined. It should be stressed that no particular microscopic model of gravity has been used for this result. Finally, von Neumann’s quantum canonical ensemble statistics straightforwardly extends to the “matter bath” description of multiple layer gravity.

Appendix A. The boundary action of general relativity (GR)

The purely gravitational action (GR) can be expressed in several equivalent ways, e.g. as the Einstein-Hilbert action $S_H$ or the first order “covariant Einstein action” $S_1$ (we ignore here the cosmological constant) $\Box$:

$$S_H = \frac{c^4}{16\pi G} \int_V d^4x \sqrt{-g} \; R,$$

$$S_1 = \frac{c^4}{16\pi G} \left[ \int_V d^4x \sqrt{-g} \; R - \sum_A 2\varepsilon_A \int_{B_A} d^3x \sqrt{|\gamma|} \; K \right],$$

(A.1)  

(A.2)
with the symbols defined as in Section 2. The variation of (A.1) and (A.2) yields a bulk term containing the Einstein tensor and a boundary term

$$\delta S_{H|\partial V} = \frac{c^4}{16\pi G} \sum_A \varepsilon_A \int_{B_A} d^3x \, \gamma_{ab} \, \delta N^{ab}, \quad \delta S_{I|\partial V} = -\frac{c^4}{16\pi G} \sum_A \varepsilon_A \int_{B_A} d^3x \, N^{ab} \, \delta\gamma_{ab}, \quad (A.3)$$

with $N_{ab} = K_{ab} - K\gamma_{ab}$. We define the difference (and call it the boundary action)

$$S_{\partial V} = \frac{1}{2}(S_H - S_I) = \frac{c^4}{16\pi G} \sum_A \varepsilon_A \int_{B_A} d^3x \sqrt{\gamma} \, K$$

(A.4)

so that the stationary points of $S_{\partial V}$, $S_I$ and $S_H$ are the same in terms of $\gamma_{ab}$ and $N^{ab}$, and the physical systems determined by $S_{\partial V}$ and $S_H$ are equivalent.

**Appendix B. Multiple horizon thermodynamics**

Following 't Hooft’s quite general argument leading to the holographic principle and dimensional reduction \[\text{8}\], we shall be working with a 2+1-dimensional microscopic concept (boundary space). Starting from Rindler horizon thermodynamics \[\text{4,5}\], according to which a horizon acquires an observer-dependet temperature $T$ and entropy $S_{BH,JP}$, we can obtain a statistical model of gravity. Be $V$ a compact space-time region with non-null boundary $\partial V$. The boundary action $S_{\partial V}$ (A.3) can also be written as \[\text{6}\]

$$S_{\partial V} = \frac{c^4}{16\pi G} \int_{\partial V} N_{ab}^{c} f^{ab} d^3\Sigma_c, \quad f^{ab} = \sqrt{-g} \, g^{ab}, \quad N_{ab}^{c} d^3\Sigma_c = \varepsilon_A N^{ab} d^3x, \quad (B.1)$$

where $N_{ab}^{c}$ is the momentum conjugate to $f^{ab}$ and $d^3\Sigma_c$ is the 3-surface element covector.

We divide $\partial V$ into cuboids $\Delta\partial V \subset B_A \subset \partial V$ with edge lengths $L_e$ of each edge $e$ so that $L_p \ll L_e$ and $L_e$ is smaller than the typical variation scale of $\gamma_{ab}$ and $K_{ab}$. $S_{\partial V}$ then reads:

$$S_{\partial V} = \sum_{\Delta\partial V \subset \partial V} S_{\partial V} \Delta\partial V \quad (B.2)$$

with

$$S_{\partial V} \Delta\partial V = \frac{c^4}{16\pi G} \int_{\Delta\partial V} N_{ab}^{c} f^{ab} d^3\Sigma_c = \frac{c^4}{32\pi G} \sum_A \varepsilon_A \int_{\Delta\partial V} d^3x \sqrt{\gamma} \, \gamma^{ab} \mathcal{L}_{\perp\gamma_{ab}}. \quad (B.3)$$

In \[B.3\], $\gamma_{ab}$ can be diagonalised using an orthogonal transformation which is held fixed across $\Delta\partial V$, and we use the triads $e_i^{(a)}$ induced on $\partial V$ instead of $\gamma_{ab}$. Furthermore, we replace the coordinates $(x^a) = (x^+, x^{\|}, x^3, x^4)$ by null coordinates $(\bar{x}^a) = (\bar{x}^1 = u, \bar{x}^2 = v, x^3, x^4)$, where $x^{\|}$ is tangent to $\partial V$. Moreover, we split the range of integration in \[B.3\] into an interval $I = [x^{\|}, x^{\|}_+]$ and a piece of 2-surface $A(x^{\|})$ and the integral over $I$ can be written as a sum of $N$ strips of width $\Delta x^{\|}$ with $N = (x_+^{\|} - x_-^{\|})/\Delta x^{\|}$, provided that $\Delta x^{\|}$ is not larger than the classical spatial resolution.

In this way, it can be shown \[\text{7}\] that \[B.3\] can be written as

$$S_{\partial V} \Delta\partial V = \frac{c^4 \varepsilon_A}{16\pi G} \sum_{k=1}^N \int_{A(x^{\|}_k)} d^2x \, \sqrt{\gamma} \, \Delta x^{\|} \, \sigma, \quad (B.4)$$
where \( x_k^{||} = x^{||} + k \Delta x^{||} \), \( A(x_k^{||}) \) is the intersection of \( \Delta \partial V \) with the \((x^3, x^4)\)-surface at \( x^{||} = x_k^{||} \), and

\[
\Delta x^{||} \sigma = \Delta u e^{(a)u} \mathcal{L}_v e^{(w)v} + \Delta v e^{(a)v} \mathcal{L}_u e^{(w)u}, \tag{B.5}
\]

where \( e^{(a)w} \) are the triads induced on the hypersurface \( w = \) constant with \( w = u, v \). Diagonalising in turn \( e^{(a)w} \) and changing to metric notation yields

\[
S_{\partial V} = \frac{c^4 \varepsilon_A}{32 \pi G} \sum_{k=1}^{N} \int_{A(x_k^{||})} d^3 \tilde{x} \left[ \sqrt{\gamma(u)} \frac{e^2}{e_u} \Delta u \gamma^{ab} \mathcal{L}_v \gamma^{(w)}_a \mathcal{L}_u \gamma^{(w)}_b \right] + \int_{\Delta u} \int_{A(x_k^{||})} d^3 \tilde{x} \frac{e^2}{e_u} \gamma^{ab} \mathcal{L}_v \gamma^{(w)}_a \mathcal{L}_u \gamma^{(w)}_b \right], \tag{B.6}
\]

where \( \gamma^{(w)}_a \) and \( \gamma^{ab} \) are the metric and inverse metric induced on the hypersurface \( w = \) constant, respectively. Equation (B.6) expresses the boundary action with respect to multiple null surfaces (layers) which can be interpreted using horizon thermodynamics on null hypersurfaces \( \Sigma \), via the identity

\[
\frac{c^4}{16 \pi G} \int_{\Sigma} N_{ab} f^{ab} d^3 \Sigma_c = \int_T s d \lambda d^2 x, \tag{B.7}
\]

where \( s \) is the surface density of the entropy \( S_{BHJP} \) and \( T \) the temperature perceived by the Rindler observer on the horizon with parameter \( \lambda = \bar{u} \) or \( \bar{v} \). Therefore, (B.6) can be written as

\[
S_{\partial V} = \frac{1}{2} \sum_{k=1}^{N} \left[ \int_{\Sigma(u)(x_k^{||})} \tilde{T}_u s d u d^2 x + \int_{\Sigma(v)(x_k^{||})} \tilde{T}_v s d v d^2 x \right] \mathcal{L}_u \mathcal{L}_v \Delta \mathcal{L}_u + \mathcal{L}_v \mathcal{L}_u \Delta \mathcal{L}_v \right] d^2 x, \tag{B.8}
\]

where \( \tilde{T}_w = \frac{e^2}{e_u} T_w \) and \( w = u, v \). The multiple observer interpretation of (B.8) in terms of hidden information has been discussed in [7], and this justifies that \( S_{\partial V} \) be denoted as multiple observer entropy or m-entropy, in contrast to the single observer entropy \( S_{BHJP} \).

**Appendix C. Statistical gravity**

Because of the thermodynamic property of the boundary space, the quantum bits of the multiple layers can be interpreted as the building blocks of a statistical theory as follows. The cuboid \( \Delta \partial V \) is foliated across the direction defined by the Minkowski index vector \( x^K \) into \( N_K \) pieces of 2d-surfaces of area \( A_K \) restricted on \( \Delta \partial V \). The number \( N_K \) is given by the best resolution achievable classically.

The area \( A_K \) represents \( n_K \) quantum bits of \( n_q \) different states \(| q \rangle \) each \( (n_q \) a positive integer number), so that the number of microscopic states with \( (\text{unique}) \) numbers \( N_K, n_K \) in \( \Delta \partial V \) is

\[
\Omega_K(\Delta \partial V) = (n_q n_K)^{N_K} = \exp(s_K), \tag{C.1}
\]
and the m-entropy per volume $\Delta \partial V$ is

$$s_K = N_K n_K \ln n_q, \tag{C.2}$$

thus representing (the logarithm of) the number of pure microstates with respect to $N_K, n_K$. Further relations and a more detailed description can be found in [7].

As a generalisation, the multiple layer microstates are described by the states labeled $s^{(i)}_{(j)}$ for every quantum bit $i$ on every single layer $j$ with arbitrary direction of foliation defined by $x_K$:

$$| (s^{(1)}_{(1)}, s^{(2)}_{(1)}, \ldots, s^{(n_K)}_{(1)}); (s^{(1)}_{(2)}, \ldots, s^{(n_K)}_{(2)}); \ldots; (s^{(1)}_{(N_K)}, \ldots, s^{(n_K)}_{(N_K)}) \rangle. \tag{C.3}$$

A classical geometry (i.e. a solution of GR) corresponds to the macroscopic state with the highest probability. More precisely, if we divide the space of $\gamma_{ab}$ into small cells of equal size defined by a variation $\Delta \gamma_{ab}$, the classical geometry corresponds to the $\gamma_{ab}$ for which the cell contains the largest number of microstates. Therefore, to obtain a classical solution, we must set the variation of $S$ to zero with respect to $\gamma_{ab}$, i.e. we have to apply the variation principle to the boundary action or, equivalently, to the Einstein-Hilbert action.

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