Weyl Anomaly in NonRelativistic CFTs

Ali Davody

School of physics, Institute for Research in Fundamental Sciences (IPM)  
P.O. Box 19395-5531, Tehran, Iran

Department of Physics, Sharif University of Technology  
P.O. Box 11365-9161, Tehran, Iran  
davody@ipm.ir

Abstract

We study Weyl symmetry for non-relativistic conformal field theories on curved spatial spaces, and calculate its quantum anomaly. We show that there is no geometric anomaly, and the non-relativistic Weyl anomaly can appear only due to interaction. Also we study the anomaly by using the light-cone approach.

Dedicated to Amir Abbas Varshovi
1 Introduction

Recently non-relativistic version of AdS/CFT correspondence has received a lot of interest (a partial list of them is\(^\ast\)). Non-relativistic CFTs (NRCFTs), are invariant under Schrödinger group, which contains galilean group as a subgroup\(^2, 3\). Actually the Schrödinger group has two elements more than the galilean group: non-relativistic scale and special conformal transformation. Under nonrelativistic-scaling (NR-scale) time and space scale differently

\[
t \longrightarrow e^{\sigma} t \quad , \quad x \longrightarrow e^{\sigma} x
\]

and the special conformal transformation is given by

\[
t \longrightarrow \frac{t}{1 + \alpha t} \quad x \longrightarrow \frac{x}{1 + \alpha t}
\]

These transformations must combine with an appropriate change of fields for the action to be invariant. It is also known (see \(4\) and references therein) that the Schrödinger algebra can be derived from relativistic conformal algebra in one higher dimension. To see this, let us consider a complex massless scalar field in \(d + 1\) dimensions which is invariant under conformal group \(SO(d+1,2)\)

\[
S = \frac{1}{2} \int d^{d+1}x \partial^\mu \Phi \partial^\mu \Phi^\dagger.
\]

By going to the light-cone coordinate

\[
t = \frac{x^0 + x^d}{\sqrt{2}} \quad , \quad \xi = \frac{x^0 - x^d}{\sqrt{2}},
\]

and taking \(\Phi\) to have a definite momentum in \(\xi\) direction, \(\Phi = e^{iM\xi} \Psi\), we arrive at

\[
S = \int dt d^{d-1}x \left(-iM \Psi^\dagger \partial_t \Psi + iM \Psi \partial_t \Psi^\dagger + \partial_i \Psi \partial^i \Psi^\dagger \right)
\]

which is the free Schrödinger action in \((d - 1) + 1\) dimensions and is invariant under Schrödinger group. Therefore the Schrödinger group may be thought of as a subgroup of conformal group in one higher dimension that does not mix modes with different momentum \(M\) along the null direction \(\xi\)\(^4\).

An important quantity in field theories is energy-momentum tensor, which must be traceless in Weyl invariant theories on a curved space. Actually this symmetry is anomalous and one interesting feature of AdS/CFT correspondence is that the Weyl anomaly of boundary field theory can be computed from gravity dual\(^6\). In analogy with relativistic case, the energy-momentum tensor of NRCFTs must obey the trace identity, and one may expect that by using the holographic renormalization for non-relativistic backgrounds (see\(^7\)) an anomaly appears in the Ward identity of energy-momentum tensor.

\(^\ast\) It was also shown that this non-relativistic correspondence can be generalize to the fermionic systems\(^5\).
So it is interesting to consider NRCFTs on curved special space and study the issue of Weyl anomaly for these theories.

Let us review some aspects of the CFTs on curved spaces. Indeed it is well known that \( \text{Weyl} \otimes \text{diff} \) invariant theories are invariant under the conformal group of the background. To see this, consider the Weyl transformation

\[
G_{\mu\nu} \rightarrow e^\sigma G_{\mu\nu},
\]

and note also that under the conformal transformations the metric changes by a factor which can be absorbed by a Weyl transformation. Motivated by this definition of Weyl transformation, we will define the NR-Weyl symmetry which guarantees that the theory becomes invariant under NR-scale transformation on flat space.

As in the relativistic case where scale invariance implies that the improved energy-momentum tensor is traceless,

\[
\Theta^\mu_\mu = 0,
\]

the NR-scale symmetry implies that the non-relativistic trace of improved energy-momentum tensor is zero too

\[
2T^{00} - T^i_i = 0.
\]

On the other hand it is known that relativistic Weyl symmetry breaks at the quantum level,

\[
\Theta^\mu_\mu = A(G_{\mu\nu}),
\]

where the anomaly \( A \) depends on the geometry of space. A natural question is whether the equation holds at the quantum level? Actually we find that NRCFTs do not admit Geometric Anomaly, i.e. NR-Weyl anomaly is zero for free NRCFTs on curved spatial space, though NR-Weyl anomaly can appear in the interacting theories. This is different from the relativistic case where the free CFTs admit Weyl anomaly as well.

The organization of this paper is as follows. In the next section we give a definition of NR-Weyl symmetry. In section 3 we calculate the NR-Weyl anomaly. In section 4 we analyze anomaly from light-cone point of view. We end in section 5 with conclusions and some comments.

## 2 Weyl Symmetry in NRCFTs

When a relativistic field theory on curved space has Weyl symmetry, it becomes a CFT on flat space. In a similar way we can define NR-Weyl symmetry for NR-field theory on curved spacial space in such way that a NR-Weyl invariant filed theory becomes NRCFT on the flat space. In other words NR-scale symmetry must be in the residual symme-
tries of NR-Weyl invariant field theory when the metric becomes flat. Now consider the following transformation

\[ h'_{ij} = e^\sigma h_{ij}, \quad t' = e^\sigma t \]  

(2.1)

where \( h_{ij} \) is the metric of special space and \( \sigma \) is a constant. Suppose a field theory is invariant under the above transformation, it is easy to see that it becomes a NR-CFT in flat space. Actually under NR-scaling (1.1), the time and flat metric transform as \( t \to e^\sigma t \), \( \delta_{ij} \to e^{\sigma} \delta_{ij} \) and this factors can be absorbed by a NR-Weyl transformation such that the action remain invariant. So symmetry under (2.1) guaranties the field theory becomes invariant under non-relativistic scale transformation (1.1) in flat space. Hence we take the (2.1) for the definition of NR-Weyl transformation.

To proceed let us consider the simplest case; free Schrödinger field in 2+1 dimensions on a curved spatial space

\[ S_{\text{free}} = \int dt d^2x \sqrt{\eta L} = \int dt d^2x \sqrt{h} (iM\psi^\dagger \partial_t \psi - iM\partial_t \psi^\dagger \psi - h^{ij} \partial_j \psi^\dagger \partial_i \psi), \]  

(2.2)

where \( M \) is mass of the particle. This action is invariant under NR-Weyl transformation (2.1) if \( \psi \) transforms in the following way:

\[ \psi'(x, t') = e^{-\frac{\sigma}{2}} \psi(x, t) \]  

(2.3)

NR-Weyl symmetry implies that

\[ \delta h_{ij} \frac{\partial (\sqrt{\eta L})}{\partial h_{ij}} + \partial \mu \left[ \frac{\partial (\sqrt{\eta L})}{\partial (\partial_\mu \psi^\dagger)} \delta \psi + \frac{\partial (\sqrt{\eta L})}{\partial (\partial_\mu \psi)} \delta \psi^\dagger + \sqrt{\eta L} \delta x^\mu \right] = 0. \]  

(2.4)

The first term is spatial part of canonical energy-momentum tensor

\[ T^{ij}_f = \frac{2}{\sqrt{\eta}} \delta S}{\sqrt{\eta} \delta h_{ij}} = h^{ij} (iM\psi^\dagger \partial_i \psi - iM\partial_i \psi^\dagger \psi - \partial^i \psi^\dagger \partial_k \psi) + \partial^i \psi^\dagger \partial^j \psi + \partial^j \psi^\dagger \partial^i \psi. \]  

(2.5)

By making use of this expression, the equation (2.4) can be recast to the following form

\[ \frac{1}{2} h_{ij} T^{ij} - iM\psi^\dagger \partial_t \psi + iM\partial_t \psi^\dagger \psi = 0. \]  

(2.6)

One could also add interacting terms to the action which preserve the NR-Weyl symmetry. For example the following interactions are NR-Weyl invariant:

\[ \int d^2x \sqrt{h} \frac{g}{4} \psi^\dagger \psi^\dagger \psi \psi, \quad \int d^2x \sqrt{h} d^2y \sqrt{h} \psi^\dagger(x) \psi^\dagger(x) \frac{1}{|x-y|^2} \psi(y) \psi(y) \]

the first one describes Non-relativistic bosons interacting via a \( \delta \) function potential with strength \( g \), while the second one describes non-relativistic particles interacting through a \( \frac{1}{r^2} \) potential. We now examine the validity of classical identity (2.6) at the quantum level.

\[ \text{here we use relativistic notation but only raising and lowering of spatial index is meaningful.} \]
3 NR-Weyl Anomaly

Quantum anomaly comes from the renormalization of the theory. Actually the matrix elements of energy-momentum tensor are divergent and must be regularized before taking trace in (2.6), and it is not clear whether the regularized energy-momentum tensor obeys NR-trace identity (2.6). Let us first consider the free field (2.2), which can be expanded in terms of energy eigenstates, $\psi'_n$:

$$\Psi(x,t) = c \sum_n a_n \psi_n(x) e^{-i\omega_n t} \quad (3.1)$$

According to canonical quantization we have (by appropriate choice of $c$ in (3.1))

$$[a_n, a^\dagger_m] = \delta_{n,m}. \quad (3.2)$$

The vacuum state is defined by $a_n |0\rangle = 0$, and the exited states can be built by acting on $|0\rangle$ with the creation operators. Since there is no negative energy in non-relativistic case the mode expansion (3.1) does not contain creation operator, $a^\dagger$, the expectation value of free energy-momentum tensor (2.5) between any states is finite and so NR-trace identity (2.6) holds at the quantum level for free NRCFTs. In other words unlike relativistic case, there is no Geometric anomaly in NRCFT's.

On the other hand for interacting theories, classical symmetry can be broken due to quantum corrections. Indeed NR-Weyl transformation (2.1) includes time scaling and thus the scale of energy is changed by this transformation. If $\beta$ function is nonzero, this implies that the shift of the coupling constant, $g$, under NR-Weyl transformation is

$$g \rightarrow g - \sigma \beta(g), \quad (3.3)$$

and the Lagrangian is changed in the following way

$$\mathcal{L} \rightarrow \mathcal{L} - \sigma \beta(g) \frac{\partial \mathcal{L}}{\partial g}, \quad (3.4)$$

So that the classical identity (2.6) will be corrected by the quantum correction as follows (12):

$$2T^{tt} - T^i_i = A = -\frac{\beta}{\sqrt{\hbar}} \frac{\partial \mathcal{L}}{\partial g}. \quad (3.5)$$

Therefore the NR-Weyl anomaly is given by $\beta$ function which depends on the details of interaction and the geometry of space. To clarify how the geometry of space enters the calculating of $\beta$ function, we will compute the $\beta$ function for an interacting bosonic system on a sphere in the next section.

3.1 Anyons on sphere

Consider a system of bosonic particles in $2 + 1$ dimensions interacting via a $\delta$-function potential on a sphere. The action of this system can be derived by taking NR limit from

\footnote{Note that we have written the action and energy-momentum tensor in the normal ordering form.}
Figure 1: Two-point function

The relativistic $\lambda \phi^4$ theory, and is given by (we set $M = 1$)

$$S = \int dt d^2x \sqrt{h} (i \bar{\Psi} \partial_t \Psi - i \bar{\Psi} \partial_x \Psi - h^{ij} \partial_i \Psi \partial_j \bar{\Psi} + \frac{g}{4} \bar{\Psi} \bar{\Psi} \Psi \Psi)$$  \hspace{1cm} (3.6)

The free field solution on sphere can be expanded in terms of spherical harmonic functions $Y_{l,m}(\theta, \phi)$:

$$\Psi(x, t) = \frac{1}{R} \sum_{l,m} a_{l,m} Y_{l,m}(\theta, \phi) e^{-iE_l t}$$  \hspace{1cm} (3.7)

where $E_l = \frac{l(l+1)}{2R^2}$ and $R$ is radius of the sphere. The Feynman propagator in position space is

$$D_F(x, t, x', t') = \langle 0 | T\bar{\Psi}(x, t)\Psi(x', t') | 0 \rangle = \frac{1}{R^2} \theta(t - t') \sum_{l,m} Y_{l,m}(\theta, \phi) Y_{l,m}^*(\theta', \phi') e^{-iE_l (t - t')}$$

while in the momentum space it is given by

$$D_F(l, m, \omega, l', m', \omega') = \int d^2x \sqrt{h} d^2x' \sqrt{h} dt dt' e^{i(\omega t - \omega' t')} Y_{l,m}(x) Y_{l',m'}(x') \langle 0 | T\bar{\Psi}(x, t)\Psi(x', t') | 0 \rangle = R^2 (2\pi) \delta(\omega - \omega') \delta_{l,l'} \delta_{m,m'} \frac{1}{\omega - E_l}$$  \hspace{1cm} (3.9)

Now consider the two-point function (see Fig 1)

$$\langle \Omega | T\bar{\Psi} \bar{\Psi} | \Omega \rangle = \langle 0 | T\bar{\Psi} \bar{\Psi} \exp(-i \int dt d^2x \sqrt{h} H_I) | 0 \rangle_{\text{connected}}$$  \hspace{1cm} (3.10)

due to $\theta$ function in (3.8) and normal ordering in action (3.6) all loop contributions are zero. So the mass and field are not renormalized.

To evaluate the $\beta$ function, we must compute the four-point function at one-loop level by inserting a renormalization condition at scale $\mu$, which we assume to be

$$\tilde{G}(l_1, l_2, l_3, l_4, m_1, m_2, m_3, m_4) \bigg| l_2 = l_4 = 0, l_1 = l_3 = \mu = -ig$$  \hspace{1cm} (3.11)

$$m_1 = m_2 = m_3 = m_4 = 0$$
Figure 2: four-point function

where \( \tilde{G} \) is the four-point function in the momentum space. Up to one-loop order we have (see Fig2)

\[
\tilde{G}(\mu) = -ig - \frac{ig}{2R^2} \sum_{l_5, l_6} \int \frac{d\omega_5}{2\pi i} \frac{1}{(\omega_5 - E_{l_5})(\mu - \omega_5 - E_{l_6})} \int d^2z \sqrt{h} Y_{l_1,0}^* Y_{l_0,0}^* (z) Y_{l_5,m_5} Y_{l_6,m_6} (z) \\
\int d^2w \sqrt{h} Y_{l_1,0}^* (w) Y_{l_0,0}^* Y_{l_5,m_5} (w) Y_{l_6,m_6}^* (w)
\]

\[
= -ig - \frac{ig^2}{8\pi R^2} \sum_{l_5, l_6} \frac{1}{E_{l_5} + E_{l_6} - \mu} \frac{(2l_5 + 1)(2l_6 + 1)}{(2l + 1)} (l_5, l_6, 0, 0|l, 0)^2
\]

where \( \Lambda \) is an ultraviolet cut off. Renormalization condition (3.11) implies that the counterterm, \( \delta g \), must be

\[
\delta g = \frac{g^2}{8\pi MR^2} \sum_{l_5, l_6} \frac{1}{E_{l_5} + E_{l_6} - \mu} \frac{(2l_5 + 1)(2l_6 + 1)}{(2l + 1)} (l_5, l_6, 0, 0|l, 0)^2. \tag{3.12}
\]

leading to a non zero \( \beta \) function defining by \( \beta(g) = \mu \frac{\partial g}{\partial \mu} \), and the NR-Weyl anomaly is given by:

\[
2T^{tt} - T^{ii} = -\frac{\beta(g)}{4} \Psi^\dagger \Psi \Psi
\]

(3.13)

It is interesting to study \( R \to \infty \) limit of NR-Weyl anomaly. To get a meaningful result in \( R \to \infty \) limit, we must take angular momentum \( l \) to be large. Also we approximate the sum in (3.12) by integration and calculate the integral for large \( l_5 \). By using the asymptotic expression for Clebsh-Gordon coefficients (13)

\[
(l_5, l_6, 0, 0|l, 0)^2 \approx \frac{2}{\pi l_5 \sin \theta}
\]

(3.14)

with \( \theta \) is the angel between \( l \) and \( l_5 \), the equation (3.12) becomes:

\[
\delta g = \frac{g^2}{8\pi} \int \frac{d\theta dl_5}{l_5 - l \cos \theta} \frac{dl_5}{l_5} \approx \frac{g^2}{8\pi} \ln \frac{4\Lambda^2}{\mu^2}. \tag{3.15}
\]

---

We drop the energy conserving \( \delta \) function and normalization due to integration over \( Y_{l,m}\)'s , and also external propagators.
The anomaly can be read from (3.13) as follows
\[ A = \frac{g^2}{16\pi} \Psi \Psi \Psi \Psi \] (3.16)
in agreement with the result of (9).

4 Light-Cone Weyl Anomaly

As noted in the introduction, NRCFTs can be derived from CFTs in one higher dimension by light-cone procedure. In this section we generalize this approach to the curved space. Since CFTs on curved space are Weyl invariant, let us start with the Weyl⊗diff-invariant complex massless scalar filed in 3+1 dimensions
\[ S_{1+3} = \frac{1}{2} \int d^4x \sqrt{G} (G_{\mu\nu} \partial_{\mu} \Phi \partial_{\nu} \Phi^\dagger - \frac{1}{6} R \Phi \Phi^\dagger) \] (4.1)
This action is invariant under relativistic Weyl transformation
\[ G^{\mu,\nu} \rightarrow e^{\sigma(x)} G^{\mu,\nu}, \quad \Phi(x) \rightarrow e^{-\sigma(x)/2} \Phi(x) \] (4.2)
which implies that the energy-momentum tensor is traceless
\[ \Theta_{\mu} = 0, \] (4.3)
This identity receives quantum anomaly
\[ \Theta_{\mu} = A(G_{\mu\nu}) \] (4.4)
which depends on the geometry of space-time.

To study NR-field theory on curved spatial space we choose the following form for the metric
\[ ds^2 = -dx_0^2 + dx_3^2 + h_{ij} dx^i dx^j. \] (4.5)
Going to the light cone coordinates
\[ ds^2 = -2dt d\xi + h_{ij} dx^i dx^j \] (4.6)
and assuming that the \( \Phi \) has a definite light-cone momentum \( \Phi = e^{iM\xi} \Psi \), the equation (4.1) becomes
\[ S_{1+2} = \int dt d^2x \sqrt{h} (iM \Psi \Psi \dagger - iM \Psi \partial_i \Psi^\dagger + h^{ij} \partial_i \Psi \partial_j \Psi^\dagger - \frac{1}{6} R_{\Psi \Psi} \Psi^\dagger) \] (4.7)
From the equation (4.3) or by using Noether current, one can show that the energy-momentum tensor of above action, \( T_{\mu\nu} \), satisfies the following equation
\[ 2T^{tt} - T_i^i = 0. \] (4.8)
If the light-cone procedure works in quantum level, the equation (4.4) should reduce to

$$2T^{tt} - T^i_i = \mathcal{A}(h_{ij}).$$  \hspace{1cm} (4.9)

On the other hand we saw in section 2 that NR-Weyl anomaly occurs only in interacting theories. Actually by the same approach of section 2 one can show that NR-Weyl anomaly is zero for (4.7) and indeed the equation (4.9) is not correct. Thus the light-cone procedure dose not lead to the correct result at the quantum level.

Therefore a natural question would be what is the symmetry of the action (4.7)? Since we have started from a Weyl⊗diff-invariant action and then the metric has been fixed in the form of (4.5) and also we have kept a sector with a definite light-cone momentum, the residual symmetry of the equation (4.7) consists of those transformations, O, that satisfy the following conditions

$$O \in CKV(\tilde{G}) \quad \text{and} \quad [O, P_\xi] = 0$$  \hspace{1cm} (4.10)

where CKV(\tilde{G}) is the set of conformal Killing vectors of the metric (4.6).

In order for the action (4.7) to have nontrivial symmetry \*, the metric $h_{ij}$ must have CKV. If we write (4.7) on a compact 2 dimensional manifold, we have only two choices, $S^2$ with 6 CKV and $T^2$ whith 2 CKV. The conformal Killing vectors of (4.6) which satisfy (4.10) for $S^2$ are

$$H = \partial_t, \quad P_\xi = \partial_\xi$$
$$L_1 = \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi, \quad L_2 = \cos \phi \partial_\theta + \cot \theta \sin \phi \partial_\phi, \quad L_3 = \partial_\phi$$  \hspace{1cm} (4.11)

so the symmetry algebra of (4.7) on sphere is $SU(2) \times U(1) \times U(1)$. In other words, the action is only invariant under the isometry of sphere, not by those C.K.Vs which give a factor to the metric of sphere. Also we have $U(1) \times U(1) \times U(1)$ symmetry for torus.

## 5 Discussions

With definition of Weyl transformation in NR field theories, we have shown that NR-Weyl anomaly is not a geometric effect and only appears in interacting theories. An important difference between the Weyl symmetry which we have defined in (2.1) and those in the relativistic case is that the relativistic Weyl transformation is local, while the NR-Weyl transformation is not local in the sense that $\sigma$ in (2.1) is not a function of position. Actually the free action is not invariant under local NRWeyl transformation. It would be interesting to write down an action with local NR-Weyl symmetry and study the theory on flat space with conformal symmetry in special space(14)( see (15) for affine extension a class of nonrelativistic algebras including non centrally-extended Schrodinger algebra and Galilean Conformal Algebra (GCA) in 2+1 dimensions).

We have also discussed anomaly from light-cone point of view, where we have observed that the light cone procedure dose not work at the quantum level and it would be interesting to explore the exact relation between NRCFTs and CFTs in one higher dimension at the quantum level (14).

\*\*time translation is the symmetry of (4.7) for every $h_{ij}$
It would be also very interesting to extend the results of this paper to Lifshitz-like theories which do not admit galilean symmetry but have anisotropic scaling symmetry

\[ t \longrightarrow \lambda^z t \quad x^i \longrightarrow x^i \]  

(5.1)

For example for the action

\[ S = \int dt d^d x (\dot{\phi}^2 - (\nabla \phi)^2) \]  

(5.2)

due to the presence of the negative energy, one would expect that such theories admit geometric anomaly. (see [10] for calculation of Weyl anomaly for a four-dimensional \( z=3 \) Lifshitz scalar coupled to Horava’s theory of anisotropic gravity)

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