Abstract
Given a polynomial function $f : \mathbb{R}^n \to \mathbb{R}$ and an unbounded closed semi-algebraic set $S \subset \mathbb{R}^n$, we show that the conditions listed below are characterized exactly in terms of the so-called tangency variety of the restriction of $f$ on $S$:

- $f$ is bounded from below on $S$;
- $f$ attains its infimum on $S$;
- The sublevel sets $\{x \in S \mid f(x) \leq \lambda\}$ for $\lambda \in \mathbb{R}$ are compact;
- $f$ is coercive on $S$.

Besides, we also provide some stability criteria for boundedness and coercivity of $f$ on $S$.

Keywords Boundedness · Coercivity · Compactness · Critical points · Existence of minimizers · Polynomial · Semi-Algebraic · Stability · Sub-levels · Tangencies

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1 Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function and $S$ an unbounded closed semi-algebraic subset of $\mathbb{R}^n$. Consider the optimization problem

$$\text{minimize } f(x) \quad \text{subject to } \quad x \in S. \quad (P)$$

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 Dedicated to Professor Gue Myung Lee on the occasion of his 70th birthday.
In this paper we are interested in the following questions:

1. When is $f$ bounded from below on $S$?
2. Suppose that $f$ is bounded from below on $S$. When does the problem $(P)$ have a solution?
3. Given a real number $\lambda \in \mathbb{R}$, when is the sublevel set $\{x \in S \mid f(x) \leq \lambda\}$ compact?
4. When is $f$ coercive on $S$?
5. Suppose that $f$ is bounded from below on $S$. Let $g : S \to \mathbb{R}$ be a continuous function. When is $f + g$ bounded from below on $S$?
6. Suppose that $f$ is coercive on $S$. Let $g : S \to \mathbb{R}$ be a continuous function. When is $f + g$ coercive on $S$?

These questions are not easy to answer. In fact, concerning the first question, Shor [30] writes

“Checking that a given polynomial function is bounded from below is far from trivial.”

Nie, Demmel, and Sturmfels in the paper [28] (see also [7]) propose a method for finding the global infimum of a polynomial function via sum of squares relaxations under the assumption that the optimal value is attained; in the conclusion section of the paper, the authors write:

“This paper proposes a method for minimizing a multivariate polynomial $f(x)$ over its gradient variety. We assume that the infimum $f^*$ is attained. This assumption is nontrivial, and we do not address the (important and difficult) question of how to verify that a given polynomial $f(x)$ has this property.”

Indeed, very recently, Ahmadi and Zhang showed in [1] that the testing attainment of the optimal value of a polynomial optimization problem is strongly NP-hard. Furthermore, they prove that the following conditions, which are sufficient for the existence of optimal solutions to the problem $(P)$, are strongly NP-hard to test:

- There is some $\lambda \in \mathbb{R}$ such that the sublevel set $\{x \in S \mid f(x) \leq \lambda\}$ is nonempty compact.
- The polynomial $f$ is coercive on $S$.

In different lines of development, we also would like to mention that the coercivity of polynomials defined on basic closed semi-algebraic sets and its relation to the Fedoryuk and Malgrange conditions are analyzed by Hà and Pham [12] (see also [18]), while a sufficient condition for the coercivity of polynomials on $\mathbb{R}^n$ is provided by Jeyakumar, Lasserre, and Li [15]. A connection between the coercivity of polynomials on $\mathbb{R}^n$ and their Newton polytopes is given by Bajbar and Stein [3]. For coercive polynomials, the order of growth at infinity and how this relates to the stability of coercivity with respect to perturbations of the coefficients are considered by Bajbar and Stein [4] and by Bajbar and Behrends [2].

In this paper, we show that the questions stated in the beginning of this section can be answered completely based on the information contained in the so-called tangency variety of the restriction of $f$ on $S$. It is worth noting that tangencies play an important
role in semi-algebraic optimization, see the papers [9, 11, 18, 29] and the monograph [13] for more details.

We would like to note that the results presented in the paper may be generalized for definable sets and definable functions in polynomially bounded o-minimal structures (see [32] for more on the subject). However, to lighten the exposition, we do not pursue this extension here.

The rest of this paper is organized as follows. Section 2 contains some basic results about semi-algebraic functions/sets. Some definitions and preliminaries concerning optimality conditions and tangencies are presented in Sect. 3; in particular, some properties of tangencies are new and are of interest by themselves. The main results are given in Sect. 4. Finally, several examples are provided in Sect. 5.

2 Semi-algebraic geometry

We start this section with some words about our notation. We suppose $1 \leq n \in \mathbb{N}$ and abbreviate $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ by $x$. The space $\mathbb{R}^n$ is equipped with the usual scalar product $\langle \cdot, \cdot \rangle$ and the corresponding Euclidean norm $\| \cdot \|$. Let $B_R := \{ x \in \mathbb{R}^n | \| x \| \leq R \}$ and $S_R := \{ x \in \mathbb{R}^n | \| x \| = R \}$. By convention, the infimum of the empty set is $+\infty$.

Now, we recall some notions and results of semi-algebraic geometry, which can be found in [5] and [13, Chapter 1].

**Definition 2.1** A subset $S$ of $\mathbb{R}^n$ is called *semi-algebraic* if it is a finite union of sets of the form

$$\{ x \in \mathbb{R}^n | f_i(x) = 0, \ i = 1, \ldots, k; f_i(x) > 0, \ i = k + 1, \ldots, p \},$$

where all $f_i$ are polynomials. In other words, $S$ is a union of finitely many sets, each defined by finitely many polynomial equalities and inequalities.

A map $f : S \to \mathbb{R}^m$ is said to be *semi-algebraic* if its graph

$$\{(x, y) \in S \times \mathbb{R}^m | y = f(x)\}$$

is a semi-algebraic set.

A major fact concerning the class of semi-algebraic sets is its stability under linear projections.

**Theorem 2.2** (Tarski–Seidenberg theorem) The image of any semi-algebraic set $S \subset \mathbb{R}^n$ under a projection to any linear subspace of $\mathbb{R}^n$ is a semi-algebraic set.

**Remark 2.3** As an immediate consequence of the Tarski–Seidenberg Theorem, we get semi-algebraicity of any set $\{ x \in A : \exists y \in B, (x, y) \in C \}$, provided that $A$, $B$, and $C$ are semi-algebraic sets in the corresponding spaces. Also, $\{ x \in A : \forall y \in B, (x, y) \in C \}$ is a semi-algebraic set as its complement is the union of the complement of $A$ and the set $\{ x \in A : \exists y \in B, (x, y) \notin C \}$. Thus, if we have a finite collection of
semi-algebraic sets, then any set obtained from them with the help of a finite chain of quantifiers is also semi-algebraic.

The following well-known lemmas will be of great importance for us.

Lemma 2.4 (monotonicity lemma) Let \( f : (a, b) \to \mathbb{R} \) be a semi-algebraic function. Then there are finitely many points \( a = t_0 < t_1 < \cdots < t_k = b \) such that the restriction of \( f \) to each interval \((t_i, t_{i+1})\) is analytic, and either constant, or strictly increasing or strictly decreasing.

Lemma 2.5 (growth dichotomy lemma) Let \( f : (0, \infty) \to \mathbb{R} \) be a semi-algebraic function with \( f(t) \neq 0 \) for all \( t \in (0, \infty) \). Then there exist constants \( a \neq 0 \) and \( \alpha \in \mathbb{Q} \) such that \( f(t) = at^\alpha + o(t^\alpha) \) as \( t \to 0^+ \).

Lemma 2.6 (semi-algebraic choice) Let \( S \) be a semi-algebraic subset of \( \mathbb{R}^n \times \mathbb{R}^m \). Denote by \( \pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) the projection on the first \( m \) coordinates. Then there is a semi-algebraic map \( f : \pi(S) \to \mathbb{R}^n \) such that \((t, f(t)) \in S\) for all \( t \in \pi(S) \).

Lemma 2.7 (curve selection lemma) Consider a semi-algebraic set \( S \subset \mathbb{R}^n \) and a point \( x \in \mathbb{R}^n \) that is a cluster point of \( S \). Then there exists an analytic semi-algebraic curve \( \phi : (0, \infty) \to \mathbb{R}^n \) with \( \lim_{t \to 0^+} \phi(t) = x \) and with \( \phi(t) \in S \) for \( t \in (0, \infty) \).

Lemma 2.8 (curve selection lemma at infinity) Let \( S \subset \mathbb{R}^n \) be a semi-algebraic set, and let \( f := (f_1, \ldots, f_p) : \mathbb{R}^n \to \mathbb{R}^p \) be a semi-algebraic map. Assume that there exists a sequence \( \{x^\ell\}_{\ell \geq 1} \subset S \) such that \( \lim_{\ell \to \infty} \|x^\ell\| = \infty \) and \( \lim_{\ell \to \infty} f(x^\ell) = y \in (\mathbb{R}^p)^c \), where \( \mathbb{R}^c := \mathbb{R} \cup \{\pm \infty\} \). Then there exists an analytic semi-algebraic curve \( \phi : (R, +\infty) \to \mathbb{R}^n \) such that \( \phi(t) \in S \) for all \( t > R \), \( \lim_{t \to +\infty} \|\phi(t)\| = \infty \) and \( \lim_{t \to +\infty} f(\phi(t)) = y \).

Lemma 2.9 (path connectedness) The following statements hold.

(i) Every semi-algebraic set has a finite number of connected components and each such component is semi-algebraic.

(ii) Every connected semi-algebraic set \( S \) is semi-algebraically path connected: for every points \( x, y \in S \), there exists a piecewise smooth and continuous semi-algebraic curve \( \phi : [0, 1] \to \mathbb{R}^n \) lying in \( S \) such that \( \phi(0) = x \) and \( \phi(1) = y \).

In the sequel we will make use of Hardt’s semi-algebraic triviality. We present a particular case—adapted to our needs—of a more general result: see [5, 14, 32] for the statement in its full generality.

Theorem 2.10 (Hardt’s semi-algebraic triviality) Let \( S \) be a semi-algebraic set in \( \mathbb{R}^n \) and \( f : S \to \mathbb{R} \) a continuous semi-algebraic map. Then there are finitely many points \( -\infty = t_0 < t_1 < \cdots < t_k = +\infty \) such that \( f \) is semi-algebraically trivial over each the interval \((t_i, t_{i+1})\), that is, there exists a semi-algebraic set \( F_i \subset \mathbb{R}^n \) and a semi-algebraic homeomorphism \( h_i : f^{-1}(t_i, t_{i+1}) \to (t_i, t_{i+1}) \times F_i \) such that the composition \( h_i \) with the projection \((t_i, t_{i+1}) \times F_i \to (t_i, t_{i+1}), \ (t, x) \mapsto t \), is equal to the restriction of \( f \) to \( f^{-1}(t_i, t_{i+1}) \).

We close this section with the following simple observation.
Corollary 2.11 Let $S \subset \mathbb{R}^n$ be a semi-algebraic set. Then $S$ is unbounded if and only if there exists a real number $R > 0$ such that the set $S \cap S_t$ is nonempty for all $t > R$.

**Proof** It suffices to show the necessity. Assume that the set $S$ is unbounded, i.e., there exists a sequence $\{x^\ell\}_{\ell \geq 1} \subset S$ such that $\lim_{\ell \to \infty} \|x^\ell\| = \infty$. Applying Lemma 2.8 for the semi-algebraic set $S$ and the semi-algebraic map $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \|x\|$, we get a continuous (semi-algebraic) curve $\phi : [R', +\infty) \to \mathbb{R}^n$ such that $\phi(t) \in S$ for all $t \geq R'$ and $\lim_{t \to +\infty} \|\phi(t)\| = \infty$. Now it is easy to see that the real number $R := \|\phi(R')\| + 1$ has the desired properties. \quad \Box

3 Preliminary results

In order to formulate and prove the main results of the paper, several auxiliary results are established in this section.

From now on let $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, l$, $j = 1, \ldots, m$, be polynomial functions and assume that the set

$$S := \{x \in \mathbb{R}^n \mid g_1(x) = 0, \ldots, g_l(x) = 0, h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}$$

is nonempty and unbounded.

3.1 Critical points

Consider the optimization problem stated in the introduction:

minimize $f(x)$ subject to $x \in S$. \quad (P)

It is well-known that the standard first order necessary conditions for optimality in the problem (P) are the following; see [16].

**Theorem 3.1** (Fritz-John optimality conditions) If $x \in S$ is an optimal solution of the problem (P), then there exist real numbers $\kappa, \lambda_i, i = 1, \ldots, l$, and $\nu_j, j = 1, \ldots, m$, not all zero, such that

$$\kappa \nabla f(x) - \sum_{i=1}^l \lambda_i \nabla g_i(x) - \sum_{j=1}^m \nu_j \nabla h_j(x) = 0,$$

$$\nu_j h_j(x) = 0, \quad \nu_j \geq 0, \quad \text{for } j = 1, \ldots, m.$$

Notice that if $\kappa = 0$, the above conditions are not very informative about a minimizer and so, we usually make an assumption called a constraint qualification to ensure that $\kappa \neq 0$. A constraint qualification—probably the one most often used in the design of algorithms—is defined as follows; see [25].
Definition 3.2 We say that the Mangasarian–Fromovitz constraint qualification (MFCQ) holds on $S$ if for every $x \in S$, the gradient vectors $\nabla g_i(x)$, $i = 1, \ldots, l$, are linearly independent and there exists a vector $v \in \mathbb{R}^n$ such that

$$\langle \nabla g_i(x), v \rangle = 0 \quad \text{for} \quad i = 1, \ldots, l$$

and

$$\langle \nabla h_j(x), v \rangle > 0 \quad \text{for} \quad j \in J(x),$$

where $J(x) := \{ j \in \{1, \ldots, m\} \mid h_j(x) = 0 \}$ - the set of active constraint indices.

Remark 3.3 The condition (MFCQ) holds generically. Indeed, by the Sard theorem, it is not hard to show that there exists an open and dense semi-algebraic set $\mathcal{U} \subset \mathbb{R}^l \times \mathbb{R}^m$ such that for any $(a, b) \in \mathcal{U}$, the corresponding set

$S(a, b) := \{ x \in \mathbb{R}^n \mid g_1(x) = a_1, \ldots, g_l(x) = a_l, h_1(x) \geq b_1, \ldots, h_m(x) \geq b_m \}$

is regular, i.e., for each $x \in S(a, b)$, the gradient vectors $\nabla g_i(x)$, $i = 1, \ldots, l$, and $\nabla h_j(x)$, $j \in \{ j \mid h_j(x) = b_j \}$ are linearly independent; in particular, the condition (MFCQ) holds on $S(a, b)$. For more details, we refer the reader to [6, 13, 31].

Under the assumption that (MFCQ) holds on $S$, we may obtain the more informative optimality conditions due to Karush, Kuhn and Tucker (and called the KKT optimality conditions) where the real number $\kappa$ in Theorem 3.1 can be taken to be 1; see [17, 19].

Theorem 3.4 (KKT optimality conditions) Let (MFCQ) hold on $S$. If $x \in S$ is an optimal solution of the problem (P), then there exist real numbers $\lambda_i, i = 1, \ldots, l$, and $\nu_j, j = 1, \ldots, m$, such that

$$\nabla f(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) = 0,$$

$$\nu_j h_j(x) = 0, \quad \nu_j \geq 0, \quad \text{for} \quad j = 1, \ldots, m.$$

The KKT optimality conditions lead to the following definition.

Definition 3.5 We define the set of critical points of $f$ on $S$ to be the set:

$$\Sigma(f, S) := \{ x \in S \mid \text{there exist } \lambda_i, \nu_j \in \mathbb{R} \text{ such that}$$

$$\nabla f(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) = 0,$$

$$\nu_j h_j(x) = 0, \quad \nu_j \geq 0, \quad \text{for} \quad j = 1, \ldots, m \}.$$

Remark 3.6 (i) In the case $S = \mathbb{R}^n$, we have

$$\Sigma(f, \mathbb{R}^n) = \{ x \in \mathbb{R}^n \mid \nabla f(x) = 0 \},$$
which is the usual set of critical points of $f$.

(ii) By Theorem 2.2, $\Sigma(f, S)$ is a semi-algebraic set. Furthermore, in light of Theorem 3.4, every optimal solution of the problem (P) belongs to $\Sigma(f, S)$ provided that (MFCQ) holds on $S$. Moreover, we have:

**Lemma 3.7** The following statements hold:

(i) If the condition (MFCQ) holds on $S$, then the set $\Sigma(f, S)$ is closed.

(ii) The restriction of $f$ on each connected component of $\Sigma(f, S)$ is constant. In particular, $f(\Sigma(f, S))$ is a finite set.

**Proof** (i) Let $\{x^\ell\}_{\ell \geq 1} \subset \Sigma(f, S)$ be a sequence converging to some point $x \in \mathbb{R}^n$. We will show that $x \in \Sigma(f, S)$.

Indeed, we have $x \in S$ because $S$ is closed. Furthermore, by definition, there exists a sequence $\{(\lambda^\ell, \nu^\ell)\}_{\ell \geq 1} \subset \mathbb{R}^l \times \mathbb{R}^m$ such that

$$\nabla f(x^\ell) - \sum_{i=1}^l \lambda^\ell_i \nabla g_i(x^\ell) - \sum_{j=1}^m \nu^\ell_j \nabla h_j(x^\ell) = 0,$$

$$v_j^\ell h_j(x^\ell) = 0, \quad v_j^\ell \geq 0, \quad j = 1, \ldots, m.$$

Hence, by a standard argument, it suffices to show that the sequence $\{(\lambda^\ell, \nu^\ell)\}_{\ell \geq 1}$ is bounded.

Suppose to the contrary that the sequence $\{(\lambda^\ell, \nu^\ell)\}_{\ell \geq 1}$ is unbounded, or equivalently, $\| (\lambda^\ell, \nu^\ell) \| \to +\infty$ as $\ell \to +\infty$. Passing to a subsequence, we may assume that the following limit exists:

$$(\lambda, \nu) := \lim_{\ell \to +\infty} \frac{(\lambda^\ell, \nu^\ell)}{\| (\lambda^\ell, \nu^\ell) \|}.$$

Then it is not hard to see that the following conditions hold:

$$\sum_{i=1}^l \lambda_i \nabla g_i(x) + \sum_{j=1}^m \nu_j \nabla h_j(x) = 0, \quad (1)$$

$$v_j h_j(x) = 0, \quad v_j \geq 0, \quad j = 1, \ldots, m, \quad \text{and} \quad \| (\lambda, \nu) \| = 1. \quad (2)$$

On the other hand, since the condition (MFCQ) holds on $S$, there exists a vector $v \in \mathbb{R}^n$ such that

$$\langle \nabla g_i(x), v \rangle = 0 \quad \text{for} \quad i = 1, \ldots, l \quad \text{and} \quad \langle \nabla h_j(x), v \rangle > 0 \quad \text{for} \quad j \in J(x).$$

Combined with (1), we obtain

$$0 = \sum_{i=1}^l \lambda_i \langle \nabla g_i(x), v \rangle + \sum_{j=1}^m \nu_j \langle \nabla h_j(x), v \rangle$$
\[
= \sum_{j=1}^{m} v_j \langle \nabla h_j(x), v \rangle.
\]

Consequently, by (2), \( v_j = 0 \) for all \( j = 1, \ldots, m \), and so the gradient vectors \( \nabla g_i(x), i = 1, \ldots, l \), are linearly dependent, which is a contradiction.

(ii) (Cf. [13, Theorem 2.3].) Since \( \Sigma(f, S) \) is semi-algebraic, it has a finite number of connected components. Hence it suffices to show that the restriction of \( f \) on each connected component of \( \Sigma(f, S) \) is constant. To this end, let \( \phi: [0, 1] \to \mathbb{R}^n \) be a smooth semi-algebraic curve lying in \( \Sigma(f, S) \). Applying Lemma 2.6 for the (nonempty) semi-algebraic set

\[
\tilde{\Sigma} := \left\{ (t, x, \lambda, \nu) \in [0, 1] \times S \times \mathbb{R}^l \times \mathbb{R}^m \mid x = \phi(t), \quad \nabla f(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} v_j \nabla h_j(x) = 0, \quad v_j h_j(x) = 0, \quad v_j \geq 0, \quad j = 1, \ldots, m \right\}
\]

and the projection \( \tilde{\Sigma} \to [0, 1], (t, x, \lambda, \nu) \mapsto t \), we get a semi-algebraic curve \( (\lambda, \nu): [0, 1] \to \mathbb{R}^l \times \mathbb{R}^m \) satisfying the following conditions:

\[
\begin{align*}
\nabla f(\phi(t)) - \sum_{i=1}^{l} \lambda_i(\phi(t)) \nabla g_i(\phi(t)) - \sum_{j=1}^{m} v_j(\phi(t)) \nabla h_j(\phi(t)) & \equiv 0, \quad (3) \\
g_i(\phi(t)) & \equiv 0, \quad i = 1, \ldots, l, \quad (4) \\
v_j(\phi(t)) h_j(\phi(t)) & \equiv 0, \quad v_j(\phi(t)) \geq 0, \quad j = 1, \ldots, m. \quad (5)
\end{align*}
\]

Since the functions \( v_j \) and \( h_j \circ \phi \) are semi-algebraic, it follows from the monotonicity lemma (Lemma 2.4) that there is a partition \( 0 =: t_1 < \cdots < t_N := 1 \) of \( [0, 1] \) such that on each interval \( (t_\ell, t_{\ell+1}) \) these functions are analytic and either constant or strictly monotone, for \( \ell \in \{1, \ldots, N-1\} \). By (5), then either \( v_j(t) \equiv 0 \) or \( (h_j \circ \phi)(t) \equiv 0 \) on \( (t_\ell, t_{\ell+1}) \). In particular, we have

\[
v_j(t) \frac{d}{dt}(h_j \circ \phi)(t) \equiv 0, \quad j = 1, \ldots, m.
\]

It follows from (3), (4), and (5) that

\[
\frac{d}{dt}(f \circ \phi)(t) = \begin{pmatrix}
\nabla f(\phi(t)) & \frac{d\phi(t)}{dt}
\end{pmatrix}
\]

\[
= \sum_{i=1}^{l} \lambda_i(\phi(t)) \begin{pmatrix}
\nabla g_i(\phi(t)) & \frac{d\phi(t)}{dt}
\end{pmatrix} + \sum_{j=1}^{m} v_j(\phi(t)) \begin{pmatrix}
\nabla h_j(\phi(t)) & \frac{d\phi(t)}{dt}
\end{pmatrix}
\]
\[ = \sum_{i=1}^{l} \lambda_i(t) \frac{d}{dt} (g_i \circ \phi)(t) + \sum_{j=1}^{m} v_j(t) \frac{d}{dt} (h_j \circ \phi)(t) = 0. \]

So \( f \) is constant on the curve \( \phi \).

Finally, since \( \Sigma(f, S) \) is semi-algebraic, each its connected component is semi-algebraically path connected (see Lemma 2.9): any two points can be joined by a piecewise smooth and continuous semi-algebraic curve. Therefore, the restriction of \( f \) on each connected component of \( \Sigma(f, S) \) is constant. \( \square \)

### 3.2 Tangencies

Consider the problem \( (P) \). By definition, we have

\[
\inf_{x \in S} f(x) \leq \inf_{x \in \Sigma(f, S)} f(x),
\]

and the inequality can be strict as shown in the following example.

**Example 3.8** Let \( S := \mathbb{R}^2 \) and \( f(x, y) := (xy - 1)^2 + y^2 \). We have \( f > 0 \) on \( \mathbb{R}^2 \) and

\[
f\left(\ell, \frac{1}{\ell}\right) = \frac{1}{\ell^2} \rightarrow 0 \text{ as } \ell \rightarrow \infty.
\]

Hence

\[
\inf_{(x, y) \in \mathbb{R}^2} f(x, y) = 0.
\]

Note that

\[
\nabla f(x, y) = (0, 0) \iff (x, y) = (0, 0).
\]

Therefore,

\[
\inf_{(x, y) \in \mathbb{R}^2} f(x, y) = 0 < 1 = f(0, 0).
\]

Assume that the problem \( (P) \) has no optimal solution. Then there exists a sequence \( \{x^\ell\} \subset S \) such that

\[
\lim_{\ell \to +\infty} \|x^\ell\| = +\infty \quad \text{and} \quad \lim_{\ell \to +\infty} f(x^\ell) = \inf_{x \in S} f(x).
\]

Since the set \( \{x \in S \mid \|x\|^2 = \|x^\ell\|^2\} \) is nonempty compact, the optimization problem

\[
\text{minimize } f(x) \quad \text{subject to } x \in S \text{ and } \|x\|^2 = \|x^\ell\|^2
\]

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has at least one optimal solution, say, \( y^\ell \). In light of Theorem 3.1, for each \( \ell \) there exist \( \kappa, \lambda_i, \nu_j, \mu \in \mathbb{R} \), not all zero, such that

\[
\kappa \nabla f(y^\ell) - \sum_{i=1}^{l} \lambda_i \nabla g_i(y^\ell) - \sum_{j=1}^{m} \nu_j \nabla h_j(y^\ell) - \mu y^\ell = 0, \text{ and}
\]

\[
v_j h_j(y^\ell) = 0, \quad \nu_j \geq 0, \quad j = 1, \ldots, m.
\]

This observation leads to the following definition.

**Definition 3.9** By the tangency variety of \( f \) on \( S \) we mean the set

\[
\Gamma_1(f, S) := \{ x \in S | \text{there exist } \kappa, \lambda_i, \nu_j, \mu \in \mathbb{R}, \text{ not all zero, such that}
\]

\[
\kappa \nabla f(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) - \mu x = 0, \text{ and}
\]

\[
v_j h_j(x) = 0, \quad \nu_j \geq 0, \quad j = 1, \ldots, m \},
\]

Geometrically, the tangency variety \( \Gamma(f, S) \) contains all points \( x \in S \) where the level sets of the restriction of \( f \) on \( S \) are tangent to the sphere in \( \mathbb{R}^n \) centered in the origin with radius \( \|x\| \). (See Figs. 1 and 2 below.)

**Remark 3.10** (i) We can replace the tangency variety \( \Gamma(f, S) \) by the set

\[
\Gamma_a(f, S) := \{ x \in S | \text{there exist } \kappa, \lambda_i, \nu_j, \mu, \text{ not all zero, such that}
\]

\[
\kappa \nabla f(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) - \mu (x - a) = 0, \text{ and}
\]

\[
v_j h_j(x) = 0, \quad \nu_j \geq 0, \quad j = 1, \ldots, m \},
\]

where \( a \in \mathbb{R}^n \). Then all subsequent results still hold with obvious modifications. The advantage is that, under certain assumptions on \( S \), there exists an open and dense semi-algebraic set \( \mathcal{U} \subset \mathbb{R}^n \) such that for any \( a \in \mathcal{U} \), \( \Gamma_a(f, S) \setminus \Sigma(f, S) \) is a one-dimensional submanifold of \( \mathbb{R}^n \); confer [10, Lemma 2.1].

(ii) In the case \( S = \mathbb{R}^n \), we have

\[
\Gamma(f, \mathbb{R}^n) = \left\{ x \in \mathbb{R}^n | \text{rank} \left( \frac{\nabla f(x)}{x} \right) \leq 1 \right\}.
\]
By definition, $\Sigma(f, S) \subset \Gamma(f, S)$. Furthermore, we have
\[
\inf_{x \in S} f(x) = \inf_{x \in \Gamma(f, S)} f(x) \leq \inf_{x \in \Sigma(f, S)} f(x).
\]

Let us illustrate with some examples.

**Example 3.11** Let $f(x, y) := x^2 - 2y$ and $S := \mathbb{R}^2$. We have $\Sigma(f, \mathbb{R}^2) = \emptyset$ and the tangency variety $\Gamma(f, \mathbb{R}^2)$ is given by the equation:
\[
x(y + 1) = 0.
\]

For any $a \geq 0$, we have $(\pm a, -1), (0, r) \in \Gamma(f, \mathbb{R}^2)$ where $r := \sqrt{a^2 + 1}$. The level sets $f^{-1}(a^2 + 2)$ and $f^{-1}(-2r)$ are tangent to the circle $x^2 + y^2 = r^2$ at the points $(\pm a, -1)$ and $(0, r)$, respectively. (See Fig. 1.)

**Example 3.12** Let $f(x, y) := x^4 + y^4 - x^2y^2$ and $S := \mathbb{R}^2$. We have $\Sigma(f, \mathbb{R}^2) = \{(0, 0)\}$ and the tangency variety $\Gamma(f, \mathbb{R}^2)$ is given by the equation:
\[
xy(x^2 - y^2) = 0.
\]

For any $a > 0$, we have $(\pm a, 0), (0, \pm a), (\pm a, \pm a), (\pm a, \mp a) \in \Gamma(f, \mathbb{R}^2)$, and the level set $f^{-1}(a^4)$ is tangent to the circle $x^2 + y^2 = a^2$ at the points $(\pm a, 0), (0, \pm a)$ and is tangent to the circle $x^2 + y^2 = 2a^2$ at the points $(\pm a, \pm a)$ and $(\pm a, \mp a)$. (See Fig. 2.)

The next lemma was shown in [13, Lemma 2.4] under the so-called the linear independence constraint qualification which is clearly stronger than (MFCQ).
The following statements hold:

(i) The tangency variety $\Gamma(f, S)$ is a nonempty and unbounded semi-algebraic set.

(ii) Assume that (MFCQ) holds on $S$. Then there exists a real number $R > 0$ such that for any $x \in \Gamma(f, S) \setminus B_R$ there exist real numbers $\lambda_i, \nu_j, \mu$ satisfying the following conditions:

\[
\nabla f(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) - \mu x = 0, \quad \nu_j h_j(x) = 0, \quad \nu_j \geq 0, \quad j = 1, \ldots, m.
\]

Proof

(i) By the Tarski–Seidenberg theorem (Theorem 2.2), $\Gamma(f, S)$ is a semi-algebraic set. The semi-algebraic set $S$ is unbounded and closed. Hence, in view of Corollary 2.11, there exists a real number $R > 0$ such that for any $t > R$, the set $\{x \in S \mid \|x\|^2 = t^2\}$ is nonempty compact and so the optimization problem

\[
\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in S \quad \text{and} \quad \|x\|^2 = t^2
\]

has at least one optimal solution, say, $\phi(t)$. Thanks to Theorem 3.1, we have $\phi(t) \in \Gamma(f, S)$. This, together with the fact that $\lim_{t \to +\infty} \|\phi(t)\| = \lim_{t \to +\infty} t = +\infty$, implies that $\Gamma(f, S)$ is nonempty and unbounded.

(ii) Let $x \in \Gamma(f, S)$. By definition, there exist real numbers $\kappa, \lambda_i, \nu_j, \mu$ such that

\[
\kappa \nabla f(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} \nu_j \nabla h_j(x) - \mu x = 0, \\
\nu_j h_j(x) = 0, \quad \nu_j \geq 0, \quad j = 1, \ldots, m.
\]
Thus, it suffices to show that $\kappa \neq 0$, provided that $x \in \Gamma(f, S), \|x\| \gg 1$.

Suppose to the contrary that there exists a sequence $\{(x^\ell, \lambda^\ell, \nu^\ell, \mu^\ell)\} \subset \tilde{\Gamma}$ such that $\lim_{\ell \to \infty} \|x^\ell\| = +\infty$, where we put

$$
\tilde{\Gamma} := \left\{ (x, \lambda, \nu, \mu) \in S \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R} \mid \sum_{i=1}^l |\lambda_i| + \sum_{j=1}^m v_j + |\mu| = 1 \right\}.
$$

Applying the curve selection lemma at infinity (Lemma 2.8) for the semi-algebraic function

$$
\tilde{\Gamma} \to \mathbb{R}, \quad (x, \lambda, \nu, \mu) \mapsto \|x\|,
$$

we get an analytic semi-algebraic curve $(\phi, \lambda, \nu, \mu): (R, +\infty) \to \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}$ satisfying the following conditions:

(a) $\phi(t) \in S$ for $t \in (R, +\infty)$;
(b) $\|\phi(t)\| \to +\infty$ as $t \to +\infty$;
(c) $\sum_{i=1}^l \lambda_i(t) \nabla g_i(\phi(t)) + \sum_{j=1}^m v_j(t) \nabla h_j(\phi(t)) + \mu(t) \phi(t) \equiv 0$;
(d) $v_j(t) h_j(\phi(t)) \equiv 0, \quad v_j(t) \geq 0, \quad j = 1, \ldots, m$; and
(e) $\sum_{i=1}^l |\lambda_i(t)| + \sum_{j=1}^m v_j(t) + |\mu(t)| \equiv 1$.

Since the functions $v_j$ and $h_j \circ \phi$ are semi-algebraic, it follows from the monotonicity lemma (Lemma 2.4) that for $R > 0$ large enough, these functions are either constant or strictly monotone. Then, by (d), we can see that either $v_j(t) \equiv 0$ or $(h_j \circ \phi)(t) \equiv 0$; in particular,

$$
v_j(t) \frac{d}{dt} (h_j \circ \phi)(t) \equiv 0, \quad j = 1, \ldots, m.
$$

Hence, it follows from (c) that

$$
0 = \sum_{i=1}^l \lambda_i(t) \left( \nabla g_i(\phi(t)), \frac{d\phi}{dt} \right) + \sum_{j=1}^m v_j(t) \left( \nabla h_j(\phi(t)), \frac{d\phi}{dt} \right) + \mu(t) \left( \phi(t), \frac{d\phi}{dt} \right)
$$

$$
= \sum_{i=1}^l \lambda_i(t) \frac{d}{dt} (g_i \circ \phi)(t) + \sum_{j=1}^m v_j(t) \frac{d}{dt} (h_j \circ \phi)(t) + \frac{\mu(t)}{2} \frac{d\|\phi(t)\|^2}{dt}
$$

$$
= \frac{\mu(t)}{2} \frac{d\|\phi(t)\|^2}{dt}.
$$
So \( \mu(t) = 0 \). On the other hand, since the condition \( \text{(MFCQ)} \) holds on \( S \), there exists \( v(t) \in \mathbb{R}^n \) such that

\[
\langle \nabla g_i(\phi(t)), v(t) \rangle = 0 \quad \text{for } i = 1, \ldots, l \quad \text{and} \quad \langle \nabla h_j(\phi(t)), v(t) \rangle > 0 \quad \text{for } j \in J(\phi(t)).
\]

Combined with (c), we obtain

\[
0 = \sum_{i=1}^{l} \lambda_i(t) \langle \nabla g_i(\phi(t)), v(t) \rangle + \sum_{j=1}^{m} v_j(t) \langle \nabla h_j(\phi(t)), v(t) \rangle
\]

\[
= \sum_{j=1}^{m} v_j(t) \langle \nabla h_j(\phi(t)), v(t) \rangle.
\]

Consequently, by (d), \( v_j(t) = 0 \) for all \( j = 1, \ldots, m \), and so the gradient vectors \( \nabla g_i(\phi(t)), i = 1, \ldots, l \), are linearly dependent, which is a contradiction. \( \square \)

Applying Hardt’s triviality theorem (Theorem 2.10) for the continuous semi-algebraic function

\[ \rho : \Gamma(f, S) \to \mathbb{R}, \quad x \mapsto \|x\|, \]

we find a real number \( R > 0 \) such that the restriction map

\[ \rho|_{\Gamma(f, S) \setminus B_R} : \Gamma(f, S) \setminus B_R \to (R, +\infty) \]

is semi-algebraic trivial fibration, i.e., there exists a semi-algebraic set \( F \subset \mathbb{R}^n \) and a semi-algebraic homeomorphism

\[ h : \Gamma(f, S) \setminus B_R \to (R, +\infty) \times F \]

which makes the following diagram commute:

\[
\begin{array}{ccc}
\Gamma(f, S) \setminus B_R & \xrightarrow{h} & (R, +\infty) \times F \\
\downarrow \rho & & \downarrow \pi \\
(R, +\infty) & \xrightarrow{id} & (R, +\infty)
\end{array}
\]

where \( \pi \) is the projection on the first component of the product and \( id \) is the identity map.

Since \( F \) is semi-algebraic, the number of its connected components, say, \( p \), is finite. Then \( \Gamma(f, S) \setminus B_R \) has exactly \( p \) connected components, say, \( \Gamma_1, \ldots, \Gamma_p \), and each such component is an unbounded semi-algebraic set. Moreover, for all \( k = 1, \ldots, p \) and all \( t > R \), the sets \( \Gamma_k \cap S_t \) are connected. Corresponding to each \( \Gamma_k \), let

\[ f_k : (R, +\infty) \to \mathbb{R}, \quad t \mapsto f_k(t), \]

\( \square \) Springer
be the function defined by $f_k(t) := f(x)$, where $x \in \Gamma_k \cap \mathbb{S}_t$. The definition is well-posed as shown in the following lemma.

**Lemma 3.14** Assume that (MFCQ) holds on $S$. For all $R$ large enough, the following statements hold:

(i) All the functions $f_k$ are well-defined and semi-algebraic.
(ii) Each function $f_k$ is either constant or strictly monotone.
(iii) The function $f_k$ is constant if and only if $\Gamma_k \subset \Sigma(f, S)$.

**Proof** We choose $R$ large enough so that the conclusion of Lemma 3.13 holds.

(i) Fix $k \in \{1, \ldots, p\}$, and take any $t > R$. We will show that the restriction of $f$ on $\Gamma_k \cap \mathbb{S}_t$ is constant. To see this, let $\phi: [0, 1] \to \mathbb{R}^n$ be a smooth semi-algebraic curve such that $\phi(\tau) \in \Gamma_k \cap \mathbb{S}_t$ for all $\tau \in [0, 1]$. By definition, we have

$$\|\phi(\tau)\| \equiv t \quad \text{and} \quad g_i(\phi(\tau)) \equiv 0, \quad i = 1, \ldots, l.$$  

(6)

Consider the semi-algebraic set

$$\tilde{\Gamma} := \{ (\tau, x, \lambda, v, \mu) \in [0, 1] \times S \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R} \mid x = \phi(\tau), \quad \nabla f(x) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x) - \sum_{j=1}^{m} v_j \nabla h_j(x) - \mu x = 0, \quad v_j h_j(x) = 0, \quad v_j \geq 0, \quad j = 1, \ldots, m \}.$$  

By Lemma 3.13, $\tilde{\Gamma}$ is nonempty. Applying Lemma 2.6 for the set $\tilde{\Gamma}$ and the projection $\tilde{\Gamma} \to [0, 1], (\tau, x, \lambda, v, \mu) \mapsto \tau$, we get a semi-algebraic curve $(\lambda, v, \mu): [0, 1] \to \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}$ satisfying the following conditions:

$$\nabla f(\phi(\tau)) - \sum_{i=1}^{l} \lambda_i(\tau) \nabla g_i(\phi(\tau)) - \sum_{j=1}^{m} v_j(\tau) \nabla h_j(\phi(\tau)) - \mu(\tau) \phi(\tau) \equiv 0, \quad v_j(\tau) h_j(\phi(\tau)) \equiv 0, \quad v_j(\tau) \geq 0, \quad j = 1, \ldots, m.$$  

(7)

(8)

Since the functions $v_j$ and $h_j \circ \phi$ are semi-algebraic, it follows from the monotonicity lemma (Lemma 2.4) that there is a partition $0 =: \tau_1 < \cdots < \tau_N := 1$ of $[0, 1]$ such that on each interval $(\tau_\ell, \tau_{\ell+1})$ these functions are analytic and either constant or strictly monotone, for $\ell \in \{1, \ldots, N - 1\}$. By (8), then either $v_j(\tau) \equiv 0$ or $(h_j \circ \phi)(\tau) \equiv 0$ on $(\tau_\ell, \tau_{\ell+1})$. In particular, we have

$$v_j(\tau) \frac{d}{dt} (h_j \circ \phi)(\tau) \equiv 0, \quad j = 1, \ldots, m.$$  

It follows from (6), (7), and (8) that
\[
\frac{d}{d\tau} (f \circ \phi) (\tau) = \left\{ \nabla f (\phi (\tau)), \frac{d\phi (\tau)}{d\tau} \right\} \\
= \sum_{i=1}^{l} \lambda_i (\tau) \left\{ \nabla g_i (\phi (\tau)), \frac{d\phi (\tau)}{d\tau} \right\} + \sum_{j=1}^{m} \nu_j (\tau) \left\{ \nabla h_j (\phi (\tau)), \frac{d\phi (\tau)}{d\tau} \right\} + \mu (\tau) \left\{ \phi (\tau), \frac{d\phi (\tau)}{d\tau} \right\} \\
= \sum_{i=1}^{l} \lambda_i (\tau) \frac{d}{d\tau} (g_i \circ \phi) (\tau) + \sum_{j=1}^{m} \nu_j (\tau) \frac{d}{d\tau} (h_j \circ \phi) (\tau) + \mu (\tau) \frac{1}{2} \frac{d \| \phi (\tau) \|^2}{d\tau} \\
= 0.
\]

So \( f \) is constant on the curve \( \phi \).

On the other hand, since the set \( \Gamma_k \cap S_t \) is connected semi-algebraic, it is path connected, and so any two points in \( \Gamma_k \cap S_t \) can be joined by a piecewise smooth and continuous semi-algebraic curve (see Lemma 2.9). It follows that the restriction of \( f \) on \( \Gamma_k \cap S_t \) is constant. Finally, by the Tarski–Seidenberg theorem (Theorem 2.2), the function \( f_k \) is semi-algebraic.

(ii) Increasing \( R \) (if necessary) and applying the monotonicity lemma (Lemma 2.4), the claim follows.

(iii) Necessity. We argue by contradiction: assume that the function \( f_k \) is constant but there exists a point \( x^* \in \Gamma_k \setminus \Sigma (f, S) \). Since the set \( \Sigma (f, S) \) is closed (by Lemma 3.7(i)) and since the restriction \( \rho |_{\Gamma (f, S) \setminus B_R} : \Gamma (f, S) \setminus B_R \to (R, +\infty), \ x \mapsto \| x \|, \) is semi-algebraic trivial fibration, we can find a sequence \( \{ x^\ell \}_{\ell \geq 1} \subset \Gamma_k \setminus \Sigma (f, S) \) satisfying the following conditions:

(a) \( x^\ell \) tends to \( x^* \) as \( \ell \) tends to \( +\infty \); and

(b) \( \| x^\ell \| > \| x^* \| \) for all \( \ell \geq 1 \).

By the curve selection lemma (Lemma 2.7), there exists an analytic semi-algebraic curve \( (\phi, \lambda, \nu, \mu) : [a, b] \to \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}, \phi (a) = x^*, \) such that for all \( t \in [a, b], \) the following conditions hold:

\[
\phi (t) \in \Gamma_k \setminus \Sigma (f, S), \ \| \phi (t) \| \equiv t > R, \\
\nabla f (\phi (t)) - \sum_{i=1}^{l} \lambda_i (t) \nabla g_i (\phi (t)) - \sum_{j=1}^{m} \nu_j (t) \nabla h_j (\phi (t)) - \mu (t) \phi (t) \equiv 0, \\
\nu_j (t) h_j (\phi (t)) \equiv 0, \ \nu_j (\tau) \geq 0, \ j = 1, \ldots, m.
\]

Note that the function \( f \circ \phi \) is just \( f_k \), and so, it is constant (by the assumption). Then a simple calculation shows that

\[
0 = \frac{d (f \circ \phi) (t)}{dt} = \left\{ \nabla f (\phi (t)), \frac{d\phi (t)}{dt} \right\} = \mu (t) \frac{1}{2} \frac{d \| \phi (t) \|^2}{dt} = \mu (t) t.
\]

\( \text{Springer} \)
Hence, $\mu(t) \equiv 0$, and so the curve $\phi$ lies in $\Sigma(f, S)$, which is a contradiction. Therefore, $\Gamma_k \subset \Sigma(f, S)$.

**Sufficiency.** We know from Lemma 3.7(ii) that the restriction of $f$ on each connected component of the set $\Sigma(f, S)$ is constant. Hence, the function $f_k = f|_{\Gamma_k}$ is constant because $\Gamma_k$ is a connected subset of $\Sigma(f, S)$.

In light of Lemma 3.14, we have associated to the function $f$ a finite number of functions $f_k$ of a single variable, each function $f_k$ is either constant or strictly monotone. In particular, the following limits exist:

$$\lambda_k := \lim_{t \to +\infty} f_k(t) \in \mathbb{R} \cup \{\pm \infty\} \quad \text{for} \quad k = 1, \ldots, p.$$ 

As an application, we get the next corollary (see also [9, Lemma 2.2], [11, Proposition 3.2], and [24, Theorem 1.5]). Let

$$T_\infty(f, S) := \{\lambda \in \mathbb{R} | \text{there exists a sequence } \{x^\ell\}_{\ell \geq 1} \subset \Gamma(f, S) \text{ such that}$$

$$\|x^\ell\| \to +\infty \quad \text{and} \quad f(x^\ell) \to \lambda\},$$

and we call it the set of tangency values at infinity of $f$ on $S$.

**Corollary 3.15** Assume that (MFCQ) holds on $S$. We have

$$T_\infty(f, S) = \left\{ \lim_{t \to +\infty} f_k(t) \mid k = 1, \ldots, p \right\} \cap \mathbb{R}.$$ 

In particular, the set $T_\infty(f, S)$ is finite.

**Proof** By the curve selection lemma at infinity (Lemma 2.8), we can see that a real number $\lambda$ belongs to $T_\infty(f, S)$ if and only if there exists an analytic semi-algebraic curve $\phi: (R', +\infty) \to \mathbb{R}^n$ lying in $\Gamma(f, S)$ with $R' \geq R$ such that

$$\lim_{\tau \to +\infty} \|\phi(\tau)\| = +\infty \quad \text{and} \quad f(\phi(\tau)) = \lambda.$$ 

Increasing $R'$ if necessary, we may assume that the curve $\phi$ lies in $\Gamma_k$ for some $k \in \{1, \ldots, p\}$. Consequently, we get $f(\phi(\tau)) = f_k(\|\phi(\tau)\|)$ for all $\tau > R'$. Then the desired conclusion follows. $\square$

**Remark 3.16** It is worth emphasizing that the finiteness of the set of tangency values at infinity plays an important role in solving numerically polynomial optimization problems, see [9, 11]. For more details on the subject, we refer the reader to the survey [23] and the monographs [13, 21, 22, 26] with the references therein.

**Corollary 3.17** Assume that (MFCQ) holds on $S$. If $f$ is bounded from below on $S$, then

$$\inf_{x \in S} f(x) = \min\{\lambda \mid \lambda \in f(\Sigma(f, S)) \cup T_\infty(f, S)\}.$$
Proof If \( f \) attains its infimum on \( S \), then \( f_* := \inf_{x \in S} f(x) \in f(\Sigma(f, S)) \) because of Theorem 3.4. Otherwise, the argument given before Definition 3.9 shows that \( f_* \in T_{\infty}(f, S) \). In both cases, we have

\[
f_* \geq \min \{ \lambda \mid \lambda \in f(\Sigma(f, S)) \cup T_{\infty}(f, S) \},
\]

from which follows the desired conclusion.

For each \( t > R \), we have \( S \cap S_t \) is a nonempty compact semi-algebraic set. Hence, the function

\[
\psi : (R, +\infty) \to \mathbb{R}, \quad t \mapsto \psi(t) := \min_{x \in S \cap S_t} f(x),
\]

is well-defined, and moreover, it is semi-algebraic because of the Tarski–Seidenberg theorem (Theorem 2.2).

The following lemma is simple but useful.

Lemma 3.18 For \( R \) large enough, the following statements hold:

(i) Any two of the functions \( \psi, f_1, \ldots, f_p \) either coincide or are distinct.

(ii) \( \psi(t) = \min_{k=1, \ldots, p} f_k(t) \) for all \( t > R \).

(iii) There is an index \( k \in \{1, \ldots, p\} \) such that \( \psi(t) = f_k(t) \) for all \( t > R \).

Proof (i) This is an immediate consequence of the monotonicity lemma (Lemma 2.4) and the semi-algebraicity of the functions in question.

(ii) By construction, for all \( t > R \) we have

\[
\Gamma(f, S) \cap S_t = \bigcup_{k=1}^{p} \Gamma_k \cap S_t.
\]

Therefore,

\[
\psi(t) = \min_{x \in S \cap S_t} f(x) = \min_{x \in \Gamma(f, S) \cap S_t} f(x) = \min_{k=1, \ldots, p} \min_{x \in \Gamma_k \cap S_t} f(x) = \min_{k=1, \ldots, p} f_k(t),
\]

where the second equality follows from Theorem 3.1.

(iii) This follows from items (i) and (ii).

Recall that

\[
\lambda_k := \lim_{t \to +\infty} f_k(t) \in \mathbb{R} \cup \{ \pm \infty \} \quad \text{for} \quad k = 1, \ldots, p.
\]

In view of Lemma 3.14(iii), if \( f_k \equiv \lambda_k \), then \( \lambda_k \in f(\Sigma(f, S)) \). Furthermore, by Lemma 3.18, the limit \( \lim_{t \to +\infty} \psi(t) \) exists and equals to \( \lambda_k \) for some \( k \).

Lemma 3.19 We have

\[
\lim_{t \to +\infty} \psi(t) = \min_{k=1, \ldots, p} \lambda_k.
\]
Proof Indeed, by Lemma 3.18, \( \psi(t) \leq f_k(t) \) for all \( t > R \) and all \( k = 1, \ldots, p \). Letting \( t \to +\infty \), we get

\[
\lim_{t \to +\infty} \psi(t) \leq \min_{k=1,\ldots,p} \lambda_k.
\] (9)

On the other hand, by Lemma 3.18 again, there exists an index \( k \in \{1, \ldots, p\} \) such that \( \psi \equiv f_k \), and so

\[
\lim_{t \to +\infty} \psi(t) = \lambda_k.
\]

Combining this with the inequality (9), we get the desired conclusion. \( \square \)

We finish this section with the following observation.

Lemma 3.20 We have

\[
\lim_{t \to +\infty} \psi(t) \geq \inf_{x \in S} f(x)
\] with the equality if \( f \) does not attain its infimum on \( S \).

Proof Indeed, we have for all \( t > R \),

\[
\psi(t) = \min_{x \in S \cap S_t} f(x) \geq \inf_{x \in S} f(x).
\]

Letting \( t \to +\infty \), we get \( \lim_{t \to +\infty} \psi(t) \geq \inf_{x \in S} f(x) \).

Now suppose that \( f \) does not attain its infimum on \( S \); then there exists a sequence \( \{x^\ell\}_{\ell \geq 1} \subset S \) such that

\[
\lim_{\ell \to +\infty} \|x^\ell\| = +\infty \quad \text{and} \quad \lim_{\ell \to +\infty} f(x^\ell) = \inf_{x \in S} f(x).
\]

On the other hand, by definition, it is clear that \( \psi(\|x^\ell\|) \leq f(x^\ell) \) for all \( \ell \) large enough. Therefore, \( \lim_{t \to +\infty} \psi(t) \leq \inf_{x \in S} f(x) \), and so the desired conclusion follows. \( \square \)

Note that in the above lemma we do not assume that \( f \) is bounded from below on \( S \).

4 Main results

In this section, we give some answers to the questions stated in the introduction section. Recall that \( f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, l, j = 1, \ldots, m \), are polynomial functions and that the set

\[
S := \{ x \in \mathbb{R}^n \mid g_1(x) = 0, \ldots, g_l(x) = 0, h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \}
\]

is nonempty and unbounded. From now on we will assume that (MFCQ) holds on \( S \).
Keeping the notations as in the previous section, we know that \( \Gamma(f, S) \setminus \mathbb{B}_R \) has exactly \( p \) connected components \( \Gamma_1, \ldots, \Gamma_p \), and each such component is an unbounded semi-algebraic set. Corresponding to each \( \Gamma_k \), the functions

\[
f_k : (\mathbb{R}, +\infty) \to \mathbb{R}, \quad t \mapsto f_k(t),
\]

are well-defined, and so are the real numbers

\[
\lambda_k := \lim_{t \to +\infty} f_k(t) \in \mathbb{R} \cup \{\pm \infty\}.
\]

Also, recall that the function \( \psi : (\mathbb{R}, +\infty) \to \mathbb{R} \) is defined by

\[
\psi(t) := \min_{x \in S \cap \Sigma} f(x).
\]

Here and in the following, \( R \) is chosen large enough so that the conclusions of Lemmas 3.13, 3.14, and 3.18 hold.

### 4.1 Boundedness

In this subsection we present necessary and sufficient conditions for the boundedness from below of the objective function \( f \) on the feasible set \( S \).

**Theorem 4.1** The function \( f \) is bounded from below on \( S \) if and only if it holds that

\[
\min_{k=1,\ldots,p} \lambda_k > -\infty.
\]

**Proof** Indeed, by Lemmas 3.19 and 3.20, we have

\[
\min_{k=1,\ldots,p} \lambda_k \geq \inf_{x \in S} f(x)
\]

with the equality if \( f \) is not bounded from below on \( S \). \( \square \)

In what follows we let

\[
K := \{ k \mid f_k \text{ is not constant} \}.
\]

**Remark 4.2** Observe that \( K \neq \{1, \ldots, p\} \) if and only if the set \( \Sigma(f, S) \) of critical points of \( f \) on \( S \) is unbounded. Indeed, we first assume that \( K \neq \{1, \ldots, p\} \). By definition, the function \( f_k \) is constant for some \( k \notin K \). In view of Lemma 3.14(iii), \( \Gamma_k \subset \Sigma(f, S) \), and so \( \Sigma(f, S) \) is unbounded. Conversely, assume that \( \Sigma(f, S) \) is unbounded. Then \( \Sigma(f, S) \setminus \mathbb{B}_R \) contains an unbounded connected component, say, \( X \). By Lemma 3.7(ii), the restriction of \( f \) on \( X \) is constant. On the other hand, since...
\( \Sigma(f, S) \subset \Gamma(f, S) \), we have \( X \subset \Gamma_k \) for some \( k \). By Corollary 2.11, for \( t \) sufficiently large, \( X \cap S_t \) is nonempty and so

\[
   f_k(t) = f|_{\Gamma_k \cap S_t} = f|_{X \cap S_t},
\]

which yields \( f_k \) is a constant function. Hence \( K \neq \{1, \ldots, p\} \).

By the growth dichotomy lemma (Lemma 2.5), we can assume that each function \( f_k, k \in K \), is developed into a fractional power series of the form

\[
   f_k(t) = a_k t^{\alpha_k} + \text{lower order terms in } t \quad \text{as } t \to +\infty,
\]

where \( a_k \in \mathbb{R} \setminus \{0\} \) and \( \alpha_k \in \mathbb{Q} \).

**Theorem 4.3** The function \( f \) is bounded from below on \( S \) if and only if for any \( k \in K \),

\[
   \alpha_k > 0 \implies a_k > 0.
\]

**Proof** In light of Lemma 3.7, \( f(\Sigma(f, S)) \) is a finite subset of \( \mathbb{R} \). By Lemma 3.14, if \( k \notin K \), then \( f|_{\Gamma_k} = \lambda_k \) and \( \Gamma_k \subset \Sigma(f, S) \), which yield \( \lambda_k \in f(\Sigma(f, S)) \), and so \( \lambda_k \) is finite. Therefore, in view of Theorem 4.1, \( f \) is bounded from below on \( S \) if and only if it holds that

\[
   \lambda_k = \lim_{t \to +\infty} f_k(t) > -\infty \quad \text{for all } k \in K.
\]

Then the desired conclusion follows immediately from the definition of \( \alpha_k \) and \( a_k \). \( \Box \)

**Remark 4.4** Following [8] and [20] we can say that the exponents \( \alpha_k \) are tangency exponents of \( f|_S \) at infinity at \( \lambda_k \).

### 4.2 Existence of optimal solutions

In this subsection we provide necessary and sufficient conditions for the existence of optimal solutions to the problem (P). We start with the following result.

**Theorem 4.5** The function \( f \) attains its infimum on \( S \) if and only if it holds that

\[
   \Sigma(f, S) \neq \emptyset \quad \text{and} \quad \min_{x \in \Sigma(f, S)} f(x) \leq \min_{k \in K} \lambda_k.
\]

**Proof** Note that \( f(\Sigma(f, S)) \) is a finite subset of \( \mathbb{R} \) (see Lemma 3.7).

**Necessity.** Let \( f \) attain its infimum on \( S \), i.e., there exists a point \( x^* \in S \) such that

\[
   f(x^*) = \inf_{x \in S} f(x).
\]

In light of Theorem 3.4, \( x^* \in \Sigma(f, S) \) and so \( \Sigma(f, S) \) is nonempty.
On the other hand, for all $t > R$ we have

$$\inf_{x \in S} f(x) \leq \min_{x \in S \cap \mathbb{C}} f(x) = \psi(t) = \min_{k=1, \ldots, p} f_k(t),$$

where the last equality follows from Lemma 3.18(ii). Therefore,

$$f(x^*) \leq f_k(t) \quad \text{for} \quad k = 1, \ldots, p.$$  

Letting $t \to +\infty$, we get

$$f(x^*) \leq \min_{k=1, \ldots, p} \lambda_k \leq \min_{k \in K} \lambda_k.$$  

**Sufficiency.** By the assumption, we have

$$-\infty < \min_{x \in \Sigma(f, S)} f(x) \leq \min_{k \in K} \lambda_k.$$  

On the other hand, it is clear from Lemma 3.14(iii) that $\lambda_k \in f(\Sigma(f, S))$ for all $k \not\in K$ and so

$$\min_{\lambda \in f(\Sigma(f, S))} \lambda \leq \min_{k \not\in K} \lambda_k.$$  

Therefore,

$$-\infty < \min_{x \in \Sigma(f, S)} f(x) = \min_{\lambda \in f(\Sigma(f, S))} \lambda \leq \min_{k=1, \ldots, p} \lambda_k,$$  

which together with Theorem 4.1 yields that $f$ is bounded from below on $S$. Now, assume that $f$ does not attain its infimum on $S$. Then

$$\min_{x \in \Sigma(f, S)} f(x) > \inf_{x \in S} f(x).$$  

Moreover, by Lemmas 3.19 and 3.20, we have

$$\inf_{x \in S} f(x) = \lim_{t \to +\infty} \psi(t) = \min_{k=1, \ldots, p} \lambda_k.$$  

Consequently,

$$\min_{x \in \Sigma(f, S)} f(x) > \min_{k=1, \ldots, p} \lambda_k.$$  

Thanks to Lemma 3.14(iii), we know that $\lambda_k \in f(\Sigma(f, S))$ for all $k \not\in K$. Therefore

$$\min_{x \in \Sigma(f, S)} f(x) > \min_{k \in K} \lambda_k,$$  

which contradicts the assumption that $\min_{x \in \Sigma(f, S)} f(x) \leq \min_{k \in K} \lambda_k$.  

\[ \square \]
Corollary 4.6  The set of all optimal solutions of the problem (P) is nonempty compact if and only if it holds that
\[ \Sigma(f, S) \neq \emptyset, \quad \min_{x \in \Sigma(f, S)} f(x) \leq \min_{k \in K} \lambda_k, \quad \text{and} \quad \min_{x \in \Sigma(f, S)} f(x) < \min_{k \neq K} \lambda_k. \]

Proof  Recall that, by Theorem 3.4, if the problem (P) has optimal solution, then
\[ \min_{x \in \Sigma(f, S)} f(x) = \inf_{x \in S} f(x). \]

Necessity. Take any \( k \notin K \). Then \( f|_{\Gamma_k} \equiv \lambda_k \). Since \( \Gamma_k \) is unbounded, our assumption implies that
\[ \min_{x \in \Sigma(f, S)} f(x) = \inf_{x \in S} f(x) < \lambda_k, \]
which, together with Theorem 4.5, yields the desired conclusion.

Sufficiency. In view of Theorem 4.5, it suffices to show that the set of all optimal solutions of the problem (P) is bounded. Suppose to the contrary that the semi-algebraic set
\[ \left\{ x \in S \setminus \mathbb{B}_R \mid f(x) = \inf_{x \in S} f(x) \right\} \]
is unbounded. By Lemma 2.9(i), this set must contain an unbounded (semi-algebraic) connected component, say, \( X \). Observe that
\[ X \subset \Sigma(f, S) \subset \Gamma(f, S). \]
Therefore, \( X \subset \Gamma_k \) for some \( k \in \{1, \ldots, p\} \). Thanks to Corollary 2.11, for all \( t \) large enough, the set \( X \cap S_t \) is nonempty and so
\[ f_k(t) = f|_{\Gamma_k \cap S_t} = f|_{X \cap S_t}. \]
Consequently, \( f_k \) is constant inf \( x \in S \) \( f(x) \), which yields \( k \notin K \) and \( \lambda_k = \inf_{x \in S} f(x) \), in contradiction to the assumption that
\[ \inf_{x \in S} f(x) = \min_{x \in \Sigma(f, S)} f(x) < \lambda_k. \]
The corollary is proved.

Recall that the set \( T_\infty(f, S) \) of tangency values at infinity of \( f \) on \( S \) is a (possibly empty) finite set in \( \mathbb{R} \) (see Corollary 3.15). Furthermore, we have

Theorem 4.7  Suppose that \( f \) is bounded from below on \( S \). Then \( f \) attains its infimum on \( S \) if and only if it holds that
\[ \Sigma(f, S) \neq \emptyset \quad \text{and} \quad \min_{x \in \Sigma(f, S)} f(x) \leq \min_{\lambda \in T_\infty(f, S)} \lambda. \]
We only consider the case \( T \cap \neq \emptyset \); the case \( T \cap = \emptyset \) is proved similarly (see also Theorem 4.9 below).

By Lemma 3.14(iii), \( \lambda_k \in f(\Sigma(f, S)) \) for all \( k \neq K \). Consequently,

\[
\min_{x \in \Sigma(f, S)} f(x) \leq \min_{k \neq K} \lambda_k.
\]

Therefore,

\[
\min_{x \in \Sigma(f, S)} f(x) \leq \min_{k \in K} \lambda_k \iff \min_{x \in \Sigma(f, S)} f(x) \leq \min_{k = 1, \ldots, p} \lambda_k.
\]

On the other hand, since \( f \) is bounded from below on \( S \), it follows from Corollary 3.15 that

\[
\min_{\lambda \in T \cap} \lambda = \min_{k = 1, \ldots, p} \lambda_k.
\]

Now, applying Theorem 4.5, we get the desired conclusion. \( \square \)

### 4.3 Compactness of sublevel sets

Recall that the function \( \psi : (R, +\infty) \to R, t \mapsto \psi(t) \), is defined by

\[
\psi(t) := \min_{x \in S \cap S_t} f(x).
\]

In view of Lemmas 3.14 and 3.18, the function \( \psi \) is either constant or strictly monotone. Consequently, the following limit exists:

\[
\lambda_* := \lim_{t \to +\infty} \psi(t) \in R \cup \{+\infty\}.
\]

We also note from Lemma 3.19 that

\[
\lambda_* = \min_{k = 1, \ldots, p} \lambda_k.
\]

For each \( \lambda \in R \), we write

\[
\mathcal{L}(\lambda) := \{x \in S \mid f(x) \leq \lambda\}.
\]

It is easy to see that if \( \mathcal{L}(\lambda) \) is nonempty compact for some \( \lambda \), then the infimum \( \inf_{x \in S} f(x) \) of \( f \) on \( S \) is finite and attained.

**Theorem 4.8** Suppose that \( f \) is bounded from below on \( S \). The following statements hold:

(i) If \( \lambda > \lambda_* \), then \( \mathcal{L}(\lambda) \) is unbounded.

(ii) If \( \lambda < \lambda_* \), then \( \mathcal{L}(\lambda) \) is compact.
(iii) Assume that $\lambda = \lambda_*$. Then $\mathcal{L}(\lambda)$ is compact if and only if the function $\psi$ is strictly decreasing.

**Proof** (i) Assume that $\lambda > \lambda_*$. Since $\lim_{t \to +\infty} \psi(t) = \lambda_*$, we have $\psi(t) < \lambda$ for all $t$ sufficiently large. On the other hand, in view of Lemma 3.18, there exists an index $k \in \{1, \ldots, p\}$ such that $\psi \equiv f_k$. Moreover, by definition, for each $t > R$, there exists $\phi(t) \in \Gamma_k \cap \mathcal{S}_t \subset S$ such that $f_k(\phi(t)) = f(\phi(t))$. Therefore, for all $t$ sufficiently large,

$$f(\phi(t)) = f_k(t) = \psi(t) < \lambda.$$ 

For such $t$, we have $\phi(t) \in \mathcal{L}(\lambda)$. Since $\lim_{t \to +\infty} \|\phi(t)\| = \lim_{t \to +\infty} t = +\infty$, it follows that the set $\mathcal{L}(\lambda)$ is unbounded.

(ii) Assume that $\lambda < \lambda_*$. By contradiction, suppose that $\mathcal{L}(\lambda)$ is unbounded. In view of Corollary 2.11, the set $\mathcal{L}(\lambda) \cap \mathcal{S}_t$ is nonempty for all $t$ sufficiently large. For such $t$, we have

$$\psi(t) = \min_{x \in \mathcal{S}_t} f(x) = \min_{x \in \mathcal{L}(\lambda) \cap \mathcal{S}_t} f(x) \leq \lambda.$$ 

Therefore

$$\lambda_* = \lim_{t \to +\infty} \psi(t) \leq \lambda,$$

which contradicts our assumption.

(iii) Assume that $\lambda = \lambda_*$. We first assume that $\mathcal{L}(\lambda)$ is compact. Then, for $t > R$ large enough, the set $\mathcal{L}(\lambda) \cap \mathcal{S}_t$ is empty and so $f > \lambda$ on the (nonempty) compact set $\mathcal{S}_t \cap S$. For such $t$, we have

$$\psi(t) = \min_{x \in \mathcal{S}_t} f(x) > \lambda = \lambda_*,$$

Since $\lim_{t \to +\infty} \psi(t) = \lambda_*$, it follows that the function $\psi$ is strictly decreasing.

Finally, assume that $\mathcal{L}(\lambda)$ is unbounded. Then, by Corollary 2.11 again, for $t$ large enough, the set $\mathcal{L}(\lambda) \cap \mathcal{S}_t$ is nonempty and so

$$\psi(t) = \min_{x \in \mathcal{S}_t} f(x) = \min_{x \in \mathcal{L}(\lambda) \cap \mathcal{S}_t} f(x) \leq \lambda = \lambda_*.$$

Hence, the function $\psi$ is either constant $\lambda_*$ or strictly increasing. \qed

### 4.4 Coercivity

In this subsection we give necessary and sufficient conditions for the coercivity of $f$ on $S$. Here and in the following, we say that $f$ is coercive on $S$ if for every sequence $\{x^\ell\} \subset S$ such that $\|x^\ell\| \to +\infty$, we have $f(x^\ell) \to +\infty$. It is well known that if $f$ is coercive on $S$, then all sublevel sets of $f$ on $S$ are compact, and so $f$ achieves its infimum on $S$. 

[Springer]
Theorem 4.9 The following statements are equivalent:

(i) The function $f$ is coercive on $S$.
(ii) $K = \{1, \ldots, p\}$ and $\alpha_k > 0$ and $a_k > 0$ for all $k = 1, \ldots, p$.
(iii) $\lambda_* = +\infty$.
(iv) The function $f$ is bounded from below on $S$ and $T_\infty(f, S) = \emptyset$.

Proof (i) $\Rightarrow$ (ii): In fact, if $K \neq \{1, \ldots, p\}$ then the function $f_k$ is constant $\lambda_k$ for some $k \notin K$, and so there exists a curve $\phi: (R, +\infty) \to \Gamma_k$ such that
\[ \|\phi(t)\| = t \quad \text{and} \quad f(\phi_k(t)) = f_k(t) = \lambda_k \quad \text{for all } t > R. \]
Clearly, this contradicts the assumption that $\lim_{x \in S, \|x\| \to \infty} f(x) = +\infty$. Hence, $K = \{1, \ldots, p\}$.

Now, it is easy to see that $\alpha_k > 0$ and $a_k > 0$ for all $k = 1, \ldots, p$.

(ii) $\Rightarrow$ (iii): The assumption implies that for all $k = 1, \ldots, p$, $\lambda^* = \lim_{t \to +\infty} f_k(t) = +\infty$.

Hence $\lambda_* = \min_{k=1,\ldots,p} \lambda_k = +\infty$.

(iii) $\Rightarrow$ (iv): Since $\lim_{t \to +\infty} \psi(t) = \lambda_* = +\infty$, it follows from Lemma 3.19 and Theorem 4.1 that $f$ is bounded from below on $S$. Moreover, we have $T_\infty(f, S) = \emptyset$, which follows from Corollary 3.15.

(iv) $\Rightarrow$ (i): By contradiction, assume that $f$ is not coercive on $S$. Then the limit
\[ \lambda_* := \lim_{t \to +\infty} \psi(t) \]
is finite because $f$ is bounded from below on $S$. On the other hand, in view of Lemma 3.19, $\lambda_* = \lambda_k = \lim_{t \to +\infty} f_k(t)$ for some $k \in \{1, \ldots, p\}$. By Corollary 3.15, then $\lambda_* \in T_\infty(f, S)$, which contradicts to the assumption that $T_\infty(f, S) = \emptyset$. □

Remark 4.10 Recall that a continuous map $F: X \to \mathbb{R}^m$ (with $X$ being a closed subset of $\mathbb{R}^n$) is proper if the pre-image of a compact set is compact, which is equivalent to requiring that the function $X \to \mathbb{R}$, $x \mapsto \|F(x)\|$, is coercive. Hence, Theorem 4.9 also provides criteria for the properness of semi-algebraic maps.

4.5 Stability

In this subsection, we show some stability properties for semi-algebraic functions.

Given two numbers $\epsilon > 0$ and $\alpha \in \mathbb{R}$, let $F_{\epsilon,\alpha}(S)$ denote the set of all semi-algebraic functions $g: S \to \mathbb{R}$, for which there exists $R' > 0$ such that
\[ |g(x)| \leq \epsilon \|x\|^\alpha \quad \text{for any} \quad x \in S \quad \text{and} \quad \|x\| \geq R'. \]

Remark 4.11 Note that for any $\epsilon > 0$ and any $\alpha \in \mathbb{R}$, the set $F_{\epsilon,\alpha}(S)$ is nonempty. For example, it is easy to see that the function $g: S \to \mathbb{R}$, $x \mapsto (1 + \|x\|)^\beta$, belongs to $F_{\epsilon,\alpha}(S)$, where $\beta$ is an integer number less than $\alpha$. □
Recall that for each $k \in K$, we have asymptotically as $t \to +\infty$,

$$f_k(t) = a_k t^{\alpha_k} + \text{lower order terms in } t,$$

where $a_k \in \mathbb{R} \setminus \{0\}$ and $\alpha_k \in \mathbb{Q}$. Let

$$\alpha_* := \min_{k=1, \ldots, p} \alpha_k,$$

where $\alpha_k := 0$ for $k \notin K$. The following result gives a stability property for the boundedness from below of semi-algebraic functions.

**Theorem 4.12** Assume that $f$ is bounded from below on $S$. The following two assertions hold:

(i) There exists $\epsilon > 0$ such that for all $\alpha \leq \alpha_*$ and all $g \in \mathcal{F}_{\epsilon, \alpha}(S)$, the function $f + g$ is bounded from below on $S$.

(ii) For all $\epsilon > 0$ and all $\alpha > \max\{0, \alpha_*\}$, there exists a continuous function $g \in \mathcal{F}_{\epsilon, \alpha}(S)$ such that the function $f + g$ is not bounded from below on $S$.

**Proof** (i) The claim is clear in the case $\alpha_* \leq 0$. So assume that $\alpha_* > 0$. Then none of the functions $f_k$ is constant, i.e., $K = \{1, \ldots, p\}$. By Theorem 4.3, $\alpha_k > 0$ and $a_k > 0$ for all $k = 1, \ldots, p$. Hence, there exist constants $c > 0$ and $R' > R$ such that

$$f_k(t) \geq c t^{\alpha_*} \quad \text{for all} \quad t \geq R'.$$

Consequently, we have for all $x \in S$ with $\|x\| \geq R'$,

$$f(x) \geq \min_{y \in S, \|y\|=\|x\|} f(y) = \psi(\|x\|) = \min_{k=1, \ldots, p} f_k(\|x\|) \geq c \|x\|^{\alpha_*}.$$

Let $\epsilon := \frac{\epsilon}{2}$. Take any $\alpha \leq \alpha_*$ and let $g : S \to \mathbb{R}$ be a continuous semi-algebraic function such that

$$|g(x)| \leq \epsilon \|x\|^\alpha \quad \text{for any} \quad x \in S \quad \text{and} \quad \|x\| \geq R'.$$

We have for all $x \in S$ with $\|x\| \geq R'$,

$$f(x) + g(x) \geq c \|x\|^\alpha_* - \epsilon \|x\|^\alpha \geq c \|x\|^\alpha_* - \epsilon \|x\|^\alpha_* = \epsilon \|x\|^\alpha_*.$$

Clearly, this implies that the function $f + g$ is bounded from below on $S$.

(ii) Let $\epsilon > 0$ and $\alpha > \max\{0, \alpha_*\}$. Take any rational number $\beta$ with $\alpha \geq \beta > \max\{0, \alpha_*\}$. Define the function $g : S \to \mathbb{R}$ by $g(x) := -\epsilon \|x\|^\beta$. Then $g$ is continuous semi-algebraic and belongs to $\mathcal{F}_{\epsilon, \alpha}(S)$.

On the other hand, by definition, $\alpha_* = \alpha_k$ for some $k \in \{1, \ldots, p\}$. Furthermore, for each $t > R$, there exists $\phi(t) \in \Gamma_k \cap S_t$ such that $f_k(t) = f(\phi(t))$. Note that if
\(k \notin K\), then \(\alpha_k = 0\) and \(f_k = \lambda_k\); otherwise we have asymptotically as \(t \to +\infty\),

\[
f_k(t) = a_k t^{\alpha_k} + \text{lower order terms in } t,
\]

where \(a_k \in \mathbb{R} \setminus \{0\}\). Letting

\[
c := \begin{cases} 
\lambda_k & \text{if } k \notin K, \\
\alpha_k & \text{otherwise},
\end{cases}
\]

we can write

\[
f(t) = c t^{\alpha_*} + \text{lower order terms in } t.
\]

It follows that

\[
f(t) + g(t) = -\epsilon t^\beta + c t^{\alpha_*} + \text{lower order terms in } t,
\]

which tends to \(-\infty\) as \(t\) tends to \(+\infty\). Hence, the function \(f + g\) is not bounded from below on \(S\).

The next result gives a stability criterion for the coercivity of semi-algebraic functions.

**Theorem 4.13** Assume that \(f\) is coercive on \(S\). The following two assertions hold:

(i) There exists \(\epsilon > 0\) such that for all \(\alpha \leq \alpha_*\) and all \(g \in \mathcal{F}_{\epsilon,\alpha}(S)\), the function \(f + g\) is coercive on \(S\).

(ii) For all \(\epsilon > 0\) and all \(\alpha > \alpha_*\), there exists a continuous function \(g \in \mathcal{F}_{\epsilon,\alpha}(S)\) such that the function \(f + g\) is not coercive on \(S\).

**Proof** (i) By Theorem 4.9, we know that \(K = \{1, \ldots, p\}\) and \(\alpha_* > 0\). Then the rest of the proof is analogous to that of Theorem 4.12.

(ii) Let \(\epsilon > 0\) and \(\alpha > \alpha_*\). Take any rational number \(\beta\) with \(\alpha \geq \beta > \alpha_*\). Define the function \(g: S \to \mathbb{R}\) by \(g(x) := -\epsilon \|x\|^\beta\). Clearly, \(g\) is semi-algebraic continuous and belongs to \(\mathcal{F}_{\epsilon,\alpha}(S)\). Moreover, as in the proof of Theorem 4.12, we can see that the function \(f + g\) is not bounded from below on \(S\), and so, it is not coercive on \(S\).

We finish this section by noting that it is not true that if \(f\) attains its infimum on \(S\), then there exists \(\epsilon > 0\) such that for all \(\alpha \leq \alpha_*\) and all \(g \in \mathcal{F}_{\epsilon,\alpha}(S)\), the function \(f + g\) attains its infimum on \(S\).

**Example 4.14** Let \(f(x, y) := x^2\) and \(S := \mathbb{R}^2\). Clearly, \(f\) is bounded from below and attains its infimum on \(S\). A direct calculation shows that \(\alpha_* = 0\). Furthermore, for all \(\epsilon > 0\) and all integers \(\alpha \leq 0\), we have \(g(x, y) := \epsilon (1 + \|(x, y)\|^{\alpha-1}) \in \mathcal{F}_{\epsilon,\alpha}(S)\) and the function \(f + g\) is bounded from below but does not attain its infimum on \(S\).
5 Examples

In this section we provide examples to illustrate our main results. For simplicity in presentation, we consider the case where $S := \mathbb{R}^2$ and $f$ is a polynomial function in two variables $(x, y) \in \mathbb{R}^2$. By definition, then

$$
\Sigma(f, \mathbb{R}^2) := \{(x, y) \in \mathbb{R}^2 \mid \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0\},
$$

$$
\Gamma(f, \mathbb{R}^2) := \{(x, y) \in \mathbb{R}^2 \mid y \frac{\partial f}{\partial x} - x \frac{\partial f}{\partial y} = 0\}.
$$

Example 5.1 Let $f(x, y) := x^3 - 3y^2$. We have $\Sigma(f, \mathbb{R}^2) = \{(0, 0)\}$, and the tangency variety $\Gamma(f, \mathbb{R}^2)$ is given by the equation:

$$
3x^2y + 6xy = 0.
$$

Hence, for $R > 2$, the set $\Gamma(f, \mathbb{R}^2) \setminus \mathbb{B}_R$ has six connected components:

$$
\Gamma_{\pm 1} := \{(0, \pm t) \mid t \geq R\},
\Gamma_{\pm 2} := \{(-2, \pm t) \mid t \geq R\},
\Gamma_{\pm 3} := \{(\pm t, 0) \mid t \geq R\}.
$$

Consequently,

$$
f|_{\Gamma_{\pm 1}} = -3t^2,
\quad f|_{\Gamma_{\pm 2}} = -8 - 3t^2,
\quad f|_{\Gamma_{\pm 3}} = \pm t^3.
$$

It follows that $K = \{\pm 1, \pm 2, \pm 3\}$ and

$$
\lambda_{\pm 1} = \lambda_{\pm 2} = -\infty \quad \text{and} \quad \lambda_{\pm 3} = \pm \infty.
$$

Therefore, by Theorem 4.1, $f$ is bounded neither from below nor from above. (See Fig. 3.)

Example 5.2 Let us consider the Motzkin polynomial (see [27])

$$
f(x, y) := x^2y^4 + x^4y^2 - 3x^2y^2 + 1,
$$

which is nonnegative on $\mathbb{R}^2$. A simple calculation shows that

$$
\Sigma(f, \mathbb{R}^2) = \{x = 0\} \cup \{y = 0\} \cup \{(1, 1), (1, -1), (-1, 1), (-1, -1)\},
$$

$$
\Gamma(f, \mathbb{R}^2) := \{(x, y) \in \mathbb{R}^2 \mid x^2y + xy^2 = 0\}.
$$
Fig. 3 (a) The tangency variety of the polynomial $f(x, y) := x^3 - 3y^2$. (b) The level sets $f^{-1}(\lambda)$ for $\lambda < 0, \lambda = 0$ and $\lambda > 1$ are red, green and blue, respectively (color figure online).

and the tangency variety $\Gamma(f, \mathbb{R}^2)$ is given by the equation:

$$0 = \left(4x^3y^2 + 2xy^2 - 6xy - 2x^4y + 4x^2y^3 - 6x^2y\right) x$$

$$= xy \left(x^2 - y^2\right) \left(6 - 2(x^2 + y^2)\right).$$

Hence, for $R > \sqrt{3}$, the set $\Gamma(f, \mathbb{R}^2) \setminus \mathbb{B}_R$ has eight connected components:

$$\Gamma_{\pm 1} := \{(\pm t, 0) \mid t \geq R\},$$
$$\Gamma_{\pm 2} := \{(0, \pm t) \mid t \geq R\},$$
$$\Gamma_{\pm 3} := \{(\pm t, \pm t) \mid t \geq R\},$$
$$\Gamma_{\pm 4} := \{(\pm t, \mp t) \mid t \geq R\}.$$

Consequently,

$$f|_{\Gamma_{\pm 1}} = f|_{\Gamma_{\pm 2}} = 1,$$
$$f|_{\Gamma_{\pm 3}} = f|_{\Gamma_{\pm 4}} = 2t^6 - 3t^4 + 1.$$

It follows that $T_\infty(f, \mathbb{R}^2) = \{1\}$, $K = \{\pm 3, \pm 4\}$, and

$$\lambda_{\pm 1} = \lambda_{\pm 2} = 1,$$
$$\lambda_{\pm 3} = \lambda_{\pm 4} = +\infty,$$
$$\alpha_{\pm 3} = \alpha_{\pm 4} = 6.$$

Therefore, in light of Theorems 4.1 and 4.5, $f$ is bounded from below and attains its infimum. By Corollary 4.6, the set of optimal solutions of the problem...
The tangency variety of the Motzkin polynomial $f(x, y) = x^2y^4 + x^4y^2 - 3x^2y^2 + 1.$

(b) The level sets $f^{-1}(\lambda)$ for $0 < \lambda < 1, \lambda = 1$ and $\lambda > 1$ are red, green and blue, respectively (color figure online).

\[
\inf_{(x, y) \in \mathbb{R}^2} f(x, y) \text{ is nonempty compact. In fact, we can see that this set is }
\]
\[
f^{-1}(0) = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}.
\]

Moreover, from Theorem 4.8 we have

* If $\lambda < 0$, then $\mathcal{L}(\lambda)$ is empty;
* If $0 \leq \lambda < 1$, then $\mathcal{L}(\lambda)$ is nonempty compact;
* If $\lambda > 1$, then $\mathcal{L}(\lambda)$ is non-compact;
* If $\lambda = 1$, then the set $\mathcal{L}(\lambda)$ is non-compact because the function $\psi$ is constant 1. In fact, $\mathcal{L}(1)$ contains the following unbounded set:

\[
f^{-1}(1) = \{x = 0\} \cup \{y = 0\} \cup \{x^2 + y^2 = 3\}.
\]

Finally, by Theorem 4.9, the polynomial $f$ is not coercive. A direct computation shows that the level set $f^{-1}(1)$ is connected while the level sets $f^{-1}(\lambda)$ for $\lambda \geq 0$ and $\lambda \neq 1$ have four connected components. Furthermore, the level set $f^{-1}(\lambda)$ shrinks to four points $(\pm 1, \pm 1)$ as $\lambda$ tends to $0^+$. (See Fig. 4.)

**Example 5.3** Let $f(x, y) := (xy - 1)^2 + y^2$ be the polynomial considered in Example 3.8. Then the tangency variety $\Gamma(f, \mathbb{R}^2)$ is given by the equation:

\[
2(-x^3y + xy^3 + x^2 - xy - y^2) = 0.
\]

We can see that for $R$ large enough, the set $\Gamma(f, \mathbb{R}^2) \setminus B_R$ has eight connected components:

---

1. The computations are performed with the software Maple, using the command “puiseux” of the package “algcurves” for the rational Puiseux expansions.
Fig. 5 (a) The tangency variety of the polynomial $f(x, y) = (xy - 1)^2 + y^2$. (b) The level sets $f^{-1}(\lambda)$ for $0 < \lambda < 1$, $\lambda = 1$ and $\lambda > 1$ are red, green and blue, respectively (color figure online).

\[ \Gamma_{\pm 1} : \begin{align*}
x &= s^{-1} - \frac{1}{6}s - \frac{11}{72}s^3, \\
y &= s^{-1} + \frac{1}{3}s + \frac{11}{36}s^3,
\end{align*} \]
\[ \Gamma_{\pm 2} : \begin{align*}
x &= \frac{1}{3}s + \frac{1}{9}s^3, \\
y &= 3s^{-1} - \frac{2}{7}s - \frac{2}{9}s^3,
\end{align*} \]
\[ \Gamma_{\pm 3} : \begin{align*}
x &= 3s^{-1} + \frac{1}{6}s - \frac{7}{216}s^3, \\
y &= -3s^{-1} - \frac{1}{5}s + \frac{7}{108}s^3,
\end{align*} \]
\[ \Gamma_{\pm 4} : \begin{align*}
x &= \frac{3}{2}s^{-1} - \frac{1}{3}s + \frac{2}{27}s^3, \\
y &= \frac{2}{3}s - \frac{4}{27}s^3,
\end{align*} \]

where $s \to \pm 0$. Then substituting these expansions in $f$ we get

\[ f|_{\Gamma_{\pm 1}} = s^{-4} - \frac{2}{3}s^{-2} + \frac{14}{9} + \frac{77}{216}s^2 + \cdots, \]
\[ f|_{\Gamma_{\pm 2}} = 9s^{-2} - 4 - \frac{8}{9}s^2 + \cdots, \]
\[ f|_{\Gamma_{\pm 3}} = 81s^{-4} + 54s^{-2} + 4 - \frac{133}{72}s^2 + \cdots, \]
\[ f|_{\Gamma_{\pm 4}} = \frac{4}{9}s^2 - \frac{16}{243}s^6 + \cdots. \]

It follows that $T_\infty(f, \mathbb{R}^2) = \{0\}$, $K = \{\pm 1, \pm 2, \pm 3, \pm 4\}$, and
\[ \lambda_{\pm 1} = \lambda_{\pm 2} = \lambda_{\pm 4} = +\infty, \quad \lambda_{\pm 3} = 0. \]

In light of Theorem 4.1, $f$ is bounded from below. Note that $\Sigma(f, \mathbb{R}^2) = \{(0, 0)\}$ and
\[ f(0, 0) = 1 > 0 = \min_{k=1,2,3,4} \lambda_{\pm k}. \]

Hence, by Theorem 4.5, $f$ does not attain its infimum. Furthermore, in view of Corollary 3.17, we have
\[ \inf_{(x, y) \in \mathbb{R}^2} f(x, y) = 0. \]
A direct computation shows that the level set \( f^{-1}(1) \) is connected while the level sets \( f^{-1}(\lambda) \) for \( \lambda > 0 \) and \( \lambda \neq 1 \) have two connected components. Furthermore, the level set \( f^{-1}(\lambda) \) vanishes at infinity as \( \lambda \) tends to \( 0^+ \). (See Fig. 5.)

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