Time regularity for local weak solutions of the heat equation on local Dirichlet spaces

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Abstract

We study the time regularity of local weak solutions of the heat equation in the context of local regular symmetric Dirichlet spaces. Under two basic and rather minimal assumptions, namely, the existence of certain cut-off functions and a very weak $L^2$ Gaussian type upper-bound for the heat semigroup, we prove that the time derivatives of a local weak solution of the heat equation are themselves local weak solutions. This applies, for instance, to the local weak solutions of a uniformly elliptic symmetric divergence form second order operator with measurable coefficients. We describe some applications to the structure of ancient local weak solutions of such equations which generalize recent results of [8] and [31].

1 Introduction

When $-P$ is the infinitesimal generator of a self-adjoint strongly continuous semigroup of operators $H_t = e^{-tP}$ acting on a Hilbert space $H$, spectral theory implies the time regularity of any (global) solution $u(t) = H_t u_0$ of the equation $(\partial_t + P)u = 0$ with initial data $u_0 \in H$. When $H = L^2(X, m)$ and $-P$ is associated with a bilinear form $\mathcal{E}$ so that $\mathcal{E}(f, g) = \int f P g dm$ for enough functions $f, g$, it is often very useful to consider the concept of local weak solution of the equation $(\partial_t + P)u = 0$ in $I \times \Omega \subset \mathbb{R} \times X$, in some appropriate sense. Such definition goes roughly as follows. A local weak solution $u$ is a function defined on $I \times \Omega$ which MUST belong (locally) to a certain function space $\mathcal{F}$ (in the most classical case, $\mathcal{F}$ is related to the Sobolev space) and satisfies

$$- \int_{I \times \Omega} u \partial_t \phi \, dt \, dm + \int_I \mathcal{E}(u, \phi) \, dt = 0 \quad (1.1)$$

for all “test functions” $\phi$ compactly supported in $I \times \Omega$. The precise nature of the space $\mathcal{F}$ and of the space of test functions to be used here are important part

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of such definition. When dealing with such a definition, the time regularity of a
local weak solution is not automatic. Formally, one expects the time derivative
of a local weak solution to be a local weak solution of (1.1) but the problem lies
with the a priori requirement that \( v = \partial_t u \) belongs locally to the space \( \mathcal{F} \).

Consider the classical case when \( P \) is a symmetric locally uniformly elliptic
second order operator with measurable coefficients \( (a_{ij}(x))_{i,j=1}^n \) so that

\[
\mathcal{E}(f,g) = \int \sum_{i,j} a_{ij}(x) \partial_i f(x) \partial_j g(x) \, dx.
\]

The basic assumption, local uniform ellipticity, means that for any compact
subset \( K \) there are \( \epsilon_K > 0 \) and \( C_K < \infty \) such that

\[
\max_{i,j} \sup_K |a_{ij}| \leq C_K \text{ and } \sum_{i,j} a_{ij} \xi_i \xi_j \geq \epsilon_K \| \xi \|^2_2, \quad \forall \xi = (\xi_i)_{i=1}^n.
\]

A local weak solution of \((\partial_t + P)u = 0\) in \((a,b) \times \Omega\) is an element \( u \in L^2_{loc}((a,b) \to W^{1,2}_{loc}(\Omega))\) such that

\[
- \int_a^b \int_{\Omega} u(t,x) \partial_t \phi(t,x) \, dx \, dt + \int_a^b \int_{\Omega} \sum_{i,j} a_{ij} \partial_i u(t,x) \partial_j \phi(t,x) \, dx \, dt = 0
\]

for all functions \( \phi \in C^\infty((a,b) \times \Omega) \) with compact support in \((a,b) \times \Omega\).

One consequence of the general results proved in this paper is that the it-
erated time derivatives \( v_k(t,x) = \partial_t^k u(t,x) \) of any local weak solution \( u \) of the
equation above are themselves local weak solutions of the same equation in \((a,b) \times \Omega\). This follows for instance from the
following more general theorem. In this statement we assume that \((X,m)\) is a
locally compact separable Hausdorff space and \( m \) is a positive radon measure
with full support.

**Theorem 1.1.** Assume \( (\mathcal{E}, \mathcal{F}) \) is a symmetric strictly local regular Dirichlet
form on \( L^2(X,m) \) whose intrinsic pseudo-metric is a continuous metric which
induces the topology of \( X \). For any local weak solution \( u \) of the associated heat
equation in \((a,b) \times \Omega\), the iterated time derivatives \( v_k = \partial_t^k u \) are themselves local
weak solutions of the same heat equation in \((a,b) \times \Omega\).

Although this theorem excludes fractal sets such as the Sierpinski Gasket
and the Sierpinski Carpet (on such examples, the intrinsic pseudo-distance is
identically equal to 0) as well as some infinite dimensional examples (e.g., on the
infinite dimensional torus in cases when the intrinsic pseudo-distance is infinite
almost surely), these cases are in fact also covered by our more general results.
Indeed, only two related types of assumptions play a key part in our results:

- The existence of good cut-off functions (in a sense that is somewhat weaker
  than most conditions of this type that exist in the literature);
A very weak $L^2$-Gaussian bound, namely, the fact that for any integers $m, k = 0, 1, 2, \ldots$, and any disjoint compact sets $V_1, V_2$

$$t^{-m} \sup_{\phi_1, \phi_2} \int_X \phi_2 \partial^k_t H_t \phi_1 dm \to 0 \quad (as \ t \to 0)$$

where the sup is taken over all functions $\phi_1, \phi_2$ supported respectively in $V_1, V_2$ and with $L^2$-norm at most 1.

As an application of our results, we extend two recent structure theorems regarding ancient weak solutions, [8, 22, 31]. The first result of this type describes very general conditions under which any ancient (local) weak solutions with “polynomial growth” must be of the form $u(t, x) = \sum_{k=1}^d t^k u_k(x)$ where all $u_k$ are of polynomial growth, $u_d$ is a harmonic function, and other $u_k$’s satisfy $\Delta u_k = (k+1)u_{k+1}$ in a weak sense. The integer $d$ is related to the given growth degree of $u$. The second result describes very general conditions under which any ancient weak solution of “exponential growth” is real analytic in time.

The general approach we take is to utilize the heat semigroup to study the time regularity properties of local weak solutions of the heat equation. The basic idea to derive hypoellipticity type results from properties of the heat semigroup goes back to Kusuoka and Stroock’s paper [19] which is written in the context of the heat equation associated with Hörmander sums of squares of vector fields on Euclidean spaces. It was also implemented in [5] to study distributional solutions of the Laplace equation on the infinite dimensional torus and other infinite dimensional compact groups.

This approach differs from the classical hypoellipticity viewpoint in the primary role it gives to the properties of the fundamental solution of the heat equation (here, in the very minimal form of the heat semigroup itself) while traditional studies of hypoellipticity treat all solutions equally and are then used to deduce the basic regularity of the fundamental solution. In this paper we generalize this heat semigroup approach to hypoellipticity to the general setting of Dirichlet spaces on metric measure spaces. One natural goal is to cover rougher structures that make smoothness more elusive. Here, we treat a purely $L^2$-theory. In a sequel of this paper, we will further utilize this method to study the local boundedness and continuity properties of local weak solutions of the heat equation (the $L^\infty$ type properties) under additional assumptions.

This work is organized as follows. In Section 2, we introduce our general Dirichlet space setup and define the relevant notion of local weak solutions. In Section 3, we introduce and discuss our two main hypotheses, the existence of certain cut-off functions and the notion of very weak $L^2$-Gaussian bound. We state in Section 4 the main theorems proved in this paper and give a sketch of the proof of the main result that conveys the main ideas while avoiding many long necessary computations and technical details. In Section 5 we give a complete proof of the main theorems stated in Section 4. Section 6 is devoted to the results concerning the structure of ancient (local weak) solutions. Section 7 discusses briefly several typical examples that illustrate the results of this paper in a variety of different contexts. Lastly Section 8 provides tools to verify that
the very weak $L^2$-Gaussian bound is satisfied under rather weak assumptions involving the existence of cut-off functions, as well as the proofs for some lemmas regarding cut-off functions.

We remark that, in this paper, the Dirichlet forms we treat are symmetric, and are not time dependent. The independence on time is a crucial assumption for us, as we take advantage of the smoothness of the heat semigroup in time. The symmetry assumption can probably be replaced by some form of the sector condition but we leave this to a further study. For related but different results (under stronger assumptions) for nonsymmetric or time dependent Dirichlet spaces, we refer to [28, 27] and [21].

2 Dirichlet spaces and local weak solutions

2.1 Dirichlet spaces

We briefly review some concepts and properties related to Dirichlet forms. A classical reference for (symmetric) Dirichlet forms is [14]. Let $(X, d, m)$ be a metric measure space where $X$ is locally compact, separable, and Hausdorff, $m$ is a Radon measure on $X$ with full support, and $d$ is some metric on $X$ that we will omit writing in the rest of the paper since we do not use it explicitly. Let $(\mathcal{E}, \mathcal{F})$ be a symmetric, regular, local Dirichlet form on $L^2(X, m)$, where $\mathcal{F}$ denotes the domain of $\mathcal{E}$. By definition, a (symmetric) Dirichlet form is a closed symmetric form that further satisfies the Markov property. Here the term symmetric form refers to any symmetric, nonnegative definite, densely defined bilinear form. This form is closed, that is, its domain $\mathcal{F}$ equipped with the $\mathcal{E}_1$ norm

$$\|f\|_{\mathcal{E}_1} := \left( \mathcal{E}(f, f) + \int_X f^2 dm \right)^{1/2}.$$

By assumption, the domain $\mathcal{F}$ of $\mathcal{E}$ equipped with the $\mathcal{E}_1$ norm is a Hilbert space. A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called regular if $C_c(X) \cap \mathcal{F}$ is dense in $C(X)$ in the sup norm and dense in $\mathcal{F}$ in the $\mathcal{E}_1$ norm. Any subset $\mathcal{C} \subset C_c(X) \cap \mathcal{F}$ that is dense in these two senses is called a core of $\mathcal{E}$. A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called local if $\mathcal{E}(u, v) = 0$ for $u, v \in \mathcal{F}$ whenever $\text{supp}\{u\}$ and $\text{supp}\{v\}$ are disjoint and compact.

Regular Dirichlet forms satisfy the Beurling-Deny decomposition formula, and as a corollary, a regular local Dirichlet form $(\mathcal{E}, \mathcal{F})$ admits the decomposition formula

$$\mathcal{E}(u, v) = \int_X d\Gamma(u, v) + \int_X uv \, dk.$$

Here $dk$ is a positive Radon measure, called the killing measure. The energy measure $\Gamma$ is a (Radon) measure-valued bilinear form which is first defined for any $u$ in $\mathcal{F} \cap L^\infty(X)$ by

$$\int_X \phi \, d\Gamma(u, u) := \mathcal{E}(\phi u, u) - \frac{1}{2} \mathcal{E}(u^2, \phi).$$
for any \( \phi \in \mathcal{F} \cap C_c(X) \), and extended to any pair \( u, v \in \mathcal{F} \cap L^\infty(X) \) by polarization. For \( u \in \mathcal{F}, \) the energy measure of \( u \) is the limit of the energy measures associated with the truncation functions \( ((u \wedge n) \vee -n) \) as \( n \to \infty \).

As a generalization of the classical energy integral \( \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx \) in \( \mathbb{R}^n \), that is, intuitively as a measure given by gradients, the energy measure satisfies the following properties.

- (Leibniz rule) For any \( u, v, w \in \mathcal{F} \) with \( uv \in \mathcal{F} \) (e.g. \( u,v \in \mathcal{F} \cap L^\infty \)),
  \[
  d\Gamma (uv,w) = u \, d\Gamma (v,w) + v \, d\Gamma (u,w) .
  \]

- (Chain rule) For any \( u,v \in \mathcal{F}, \) any \( \Phi \in C^1(\mathbb{R}) \) with bounded derivative and satisfies \( \Phi(0) = 0 \),
  \[
  d\Gamma (\Phi(u),v) = \Phi'(v) \, d\Gamma (u,v) .
  \]

- (Cauchy-Schwartz inequality) For any \( f,g,u,v \in \mathcal{F} \cap L^\infty \) (more generally, when \( u,v \in \mathcal{F} \cap L^\infty \) and \( f \in L^2(X,\Gamma(u,u)), \, g \in L^2(X,\Gamma(v,v)) \))
  \[
  \int fg \, d\Gamma (u,v) \leq \left( \int f^2 \, d\Gamma (u,u) \right)^{1/2} \left( \int g^2 \, d\Gamma (v,v) \right)^{1/2}
  \leq \frac{C}{2} \int f^2 \, d\Gamma (u,u) + \frac{1}{2C} \int g^2 \, d\Gamma (v,v) .
  \]

This last inequality holds for any \( C > 0 \). The corresponding measure version holds too, namely,
  \[
  |fg| \, d\Gamma (u,v) \leq \frac{C}{2} f^2 \, d\Gamma (u,u) + \frac{1}{2C} g^2 \, d\Gamma (v,v) .
  \]

- (Strong locality) For any \( u,v \in \mathcal{F}, \) if on some open set \( U \subset X, \, v \equiv C \) for some constant \( C \), then
  \[
  1_U \, d\Gamma (u,v) = 0 .
  \]

Associated with any Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) there are a corresponding Markov semigroup \( (H_t)_{t>0}, \) its (infinitesimal) generator \( -P \) with dense domain \( \mathcal{D}(P), \) and a Markov resolvent \( (G_\alpha)_{\alpha>0} \) (in the sense of [14, page 15]). The semigroup \( H_t \) and resolvent \( G_\alpha \) have domain \( L^2(X,m) \) and the domain \( \mathcal{D}(P) \) of \( P \) is dense in \( \mathcal{F} \) w.r.t. the \( \mathcal{E}_1 \) norm. These are self-adjoint operators, and by spectral theory, \( -P \) has a spectral resolution \( (E_\lambda)_{\lambda \geq 0} \) such that, for any \( t > 0, \)
  \[
  -PH_t = \int_0^\infty \lambda e^{-\lambda t} \, dE_\lambda .
  \]

As a consequence, for any \( k \in \mathbb{N}, \)
  \[
  \left\| \partial_t^k H_t \right\|_{L^2(X) \to L^2(X)} = \left\| P^k H_t \right\|_{L^2(X) \to L^2(X)} \leq (k/et)^k .
  \]
For the resolvents \((G_\alpha)_{\alpha > 0}\), recall that
\[
\|\alpha G_\alpha\|_{L^2(X) \to L^2(X)} \leq 1,
\]
and that for any \(u \in L^2(X, m)\), \(\alpha G_\alpha u \to u\) as \(\alpha \to +\infty\) in the \(L^2\) sense. For any function \(u_0 \in L^2(X, m)\), \(u(t, x) := H_t u_0(x)\) is smooth in \(t > 0\), and solves
\[
\partial_t u = -Pu
\]
in the strong sense (i.e. \(\lim_{h \to 0} \frac{u(t+h, \cdot) - u(t, \cdot)}{h} = -Pu(t, \cdot)\) in \(L^2(X, m)\)).

Given the notations above, our main goal in this section is to define local weak solutions of the heat equation (with appropriate right-hand side \(f\))
\[
(\partial_t + P) u = f.
\]

### 2.2 Function spaces associated with \((E, F)\)

To properly discuss candidate functions for local weak solutions, and later their properties, we first introduce some function spaces associated with \((E, F)\). In choosing notations for these function spaces, we mostly follow [28] with a few exceptions that we will remark on later. Among these function spaces there are two prevalent types, one type consists of functions that have compact support (all with subscript "\(c\)"); and the other type of functions that locally satisfy the required properties (all with subscript "\(\text{loc}\)").

Recall that the inclusion \(F \subset L^2(X)\) is dense. After equating \(L^2(X)\) with its dual w.r.t. the \(L^2\) inner product, we get the Hilbert triple
\[
F \subset L^2(X) \subset F' \quad \text{(2.1)}
\]
in which the inclusions are dense and continuous. Intuitively, the “\(~c\)" spaces are on the “\(F\)” end, and the “\(~\text{loc}\)" spaces are on the “\(F'\)” (dual space) end. We will consider the dual spaces of “\(~c\)" spaces too.

We now give precise definitions of these spaces, organized in pairs, starting with the following:

\[
F_c(X) = \{ f \in F \mid f \text{ has compact (essential) support} \}
\]
and

\[
F_{\text{loc}}(X) = \{ f \in L^2_{\text{loc}}(X) \mid \forall \text{ compact } K \subset X \ \exists f^t \in F \text{ s.t. } f^t = f \text{ a.e. on } K \}.
\]

Given any open subset \(U \subset X\), we define

\[
F_c(U) = \{ f \in F \mid f \text{ has compact (essential) support in } U \};
\]

\[
F_{\text{loc}}(U) = \{ f \in L^2_{\text{loc}}(U) \mid \forall \text{ compact } K \subset U \ \exists f^t \in F \text{ s.t. } f^t = f \text{ a.e. on } K \}.
\]

**Remark 2.1.** When \(U \neq X\), by definition, there is an injection \(i : F_c(U) \hookrightarrow F_c(X)\), and clearly \(F_{\text{loc}}(X) \hookrightarrow F_{\text{loc}}(U)\) by restriction to \(U\). Note, however, that \(F_{\text{loc}}(U)\) is not a subspace of \(F_{\text{loc}}(X)\).
Fix some open set $U \subset X$ and some open interval $I = (a, b) \in \mathbb{R}$. $a < b$ are two arbitrary real numbers. In the sequel, when there is no ambiguity, we use notation $u(t, \cdot)$ as an abbreviation for $u(t, \cdot)$. More precisely, this means for any fixed $t$, we consider $u(t)$ as a function of $y$, denoted by $u^t$. Note that this is not any power of $u$, or the time derivative of $u$ (which is denoted in the sequel by $\partial_t u$). Consider the following function spaces associated to $\mathcal{E}$ and involving time and space. In defining these spaces, we switch freely between two viewpoints. In the first one, elements in these spaces are viewed as functions of time and space. In the second one, they are viewed as maps from the time interval $I$ to some (spatial) function space. The rigorous setup for the latter viewpoint is the theory of Bochner integrals, for which we refer to [30].

First, we fix the notation for the “base space”

$$\mathcal{F} (I \times X) := L^2 (I \to \mathcal{F}).$$

**Remark 2.2.** $L^2 (I \to \mathcal{F})$ is the completion of the space of bounded continuous functions $C_b (I \to \mathcal{F})$ under the $||\cdot||_{L^2 (I \to \mathcal{F})}$ norm

$$||u||_{L^2 (I \to \mathcal{F})} = \left( \int_I ||u^t||_{L^2(E)}^2 \, dt \right)^{1/2}.$$ 

We use the notation $\mathcal{F} (I \times X)$ to clarify the definition of the spaces $\mathcal{F}_c (I \times U)$, $\mathcal{F}_{loc} (I \times U)$ below. See also Remark 2.3.

Based on the “base space” $\mathcal{F} (I \times X)$, we define

$$\mathcal{F}_c (I \times U) := \{ u \in \mathcal{F} (I \times X) \mid u \text{ is compactly supported in } I \times U \}$$

and

$$\mathcal{F}_{loc} (I \times U) := \{ u \in L^2_{loc} (I \times U) \mid 
\forall I' \subset I, \forall U' \subset U, \exists u^t \in \mathcal{F} (I \times X) \text{ s.t. } u^t = u \text{ on } I' \times U' \text{ a.e.} \}.$$ 

The first two spaces $\mathcal{F} (I \times X), \mathcal{F}_c (I \times U)$ are subspaces of $L^2 (I \times X)$ and $L^2 (I \times U)$, respectively. We identify the $L^2$ spaces with their own duals (under the $L^2$ inner product), and denote the dual spaces of $\mathcal{F} (I \times X), \mathcal{F}_c (I \times U)$ under the $L^2$-inner-product by $(\mathcal{F} (I \times X))^\prime$, $(\mathcal{F}_c (I \times U))^\prime$.

**Remark 2.3.** $(\mathcal{F} (I \times X))^\prime = (L^2 (I \to \mathcal{F}))^\prime = L^2 (I \to \mathcal{F}^\prime)$.

**Remark 2.4.** Here our notations are slightly different from the ones used in other papers (e.g. [28][15]). In the definition of $\mathcal{F} (I \times X)$, we do not require the functions to further be in $W^{1,2} (I \to \mathcal{F}^\prime)$ (functions with time derivatives in the distribution sense that belong to $L^2 (I \to \mathcal{F}^\prime)$). The reason we consider the function spaces defined above instead of the ones obtained by taking the intersection with $W^{1,2} (I \to \mathcal{F}^\prime)$, is to put minimum assumptions on the definition of local weak solution. We will show that, under our definition and hypotheses, such local weak solutions automatically satisfies better properties. In particular,
we explain at the end of this section that under a very natural assumption on existence of cut-off functions, and when we consider the right-hand side \( f \) to be locally in \( L^2(I \to \mathcal{F}') \), our choice of definition of local weak solutions agrees with the definition used in other papers. This is proved by adapting the proof of Lemma 1 in [13].

To include more time derivatives we introduce the following notations for function spaces

\[
\mathcal{F}^k(I \times X) := W^{k,2}(I \to \mathcal{F});
\]

\[
\mathcal{F}^k_c(I \times U) := \{ u \in \mathcal{F}^k(I \times X) \mid u \text{ is compactly supported in } I \times U \};
\]

\[
\mathcal{F}^k_{loc}(I \times U) := \{ u \in L^2_{loc}(I \times U) \mid \forall I' \in I, \forall U' \in U, \exists u^\# \in \mathcal{F}^k(I \times X) \text{ s.t. } u^\# = u \text{ on } I' \times U' \text{ a.e.} \}.
\]

**Remark 2.5.** In general, we say a function \( u \) is locally in some function space if for any compact set, there exists a function \( w \) in the said function space such that \( w = u \text{ m} - \text{a.e. on the compact set.} \)

### 2.3 Notion of local weak solutions

For any symmetric, local, regular Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \( L^2(X, m) \), we define the following notion of local weak solutions of the associated heat equation (below \( -P, (H_t)_{t \geq 0} \) are the corresponding generator and semigroup as before).

**Definition 2.6 (Local weak solution).** Given some open subset \( U \subset X \), and given a function \( f \) locally in \( L^2(I \to \mathcal{F}') \), we say \( u \) is a local weak solution of the heat equation \( (\partial_t + P)u = f \) on \( I \times U \), if \( u \in \mathcal{F}^k_{loc}(I \times U) \), and for any \( \varphi \in \mathcal{F}^c_c(I \times U) \cap C^\infty_c(I \rightarrow \mathcal{F}), \)

\[
-\int_I \int_X u \cdot \partial_t \varphi \, dm \, dt + \int_I \mathcal{E}(u, \varphi) \, dt = \int_I <f, \varphi>_{\mathcal{F}', \mathcal{F}} \, dt.
\]

Here \( u \) in the integral is understood as \( u^\# \) as in the definition for \( \mathcal{F}^k_{loc}(I \times U) \) (relative to the support of \( \varphi \)). We take this convention throughout this paper. Note that \( \mathcal{E}(u, \varphi) \) is well-defined (independent of the choice of \( u^\# \)) by the local property of \( \mathcal{E} \).

We remark that we can define local weak solutions for more general right-hand side \( f \), e.g. \( f \in (\mathcal{F}^c_c(I \times U))' \). But in the propositions and theorems in this paper we always put more restrictions on \( f \) than \( f \) locally in \( L^2(I \to \mathcal{F}') \), and moreover the results are interesting even for the case \( f \equiv 0 \), so here in the definition we do not aim to consider the most general right-hand side. With this choice, Definition 2.6 will be shown to be equivalent to the following variant, under a natural assumption on the existence of certain cut-off functions. As mentioned above and in Remark 2.4, the following definition is often adopted in the literature.

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Definition 2.7 (Local weak solution, variant). Given some open subset $U \subset X$, and given $f$ locally in $L^2(I \to \mathcal{F}^t)$, $u$ is a local weak solution of the heat equation, if $u$ is locally in $L^2(I \to \mathcal{F}) \cap W^{1,2}(I \to \mathcal{F}^t)$ with compact support in $I \times U$, for any $J \subset I$,

$$\int_J \int_X \partial_t u \varphi F' dmdt + \int_J \mathcal{E}(u, \varphi) dt = \int_J <f, \varphi> F'. \quad (2.3)$$

Under a natural assumption on existence of some type of cut-off functions (Assumption 2.8 below), the two notions of local weak solutions defined above actually agree.

Note that in general

$$\mathcal{F}_c(I \times U) \cdot \mathcal{F}_{loc}(I \times U) \not\subset \mathcal{F}_c(I \times U),$$

roughly because $\mathcal{F}$ is not an algebra. What we want to assume is that there is a subset of $\mathcal{F}_c(I \times U) \cap \mathcal{C}(I \times U)$ that contains enough functions, each of which brings functions in $\mathcal{F}_{loc}(I \times U)$ to $\mathcal{F}_c(I \times U)$ by multiplication (these can be thought of as cut-off functions with some nice properties). We denote this subset of cut-off functions by $\mathcal{C}(I \times U)$. Observe that we just need the existence of an analogous subset $\mathcal{C}(U) \subset \mathcal{F}_c(U)$, and then to construct $\mathcal{C}(I \times U)$, we take products of functions in $\mathcal{C}(U)$ with standard cut-off functions in $\mathcal{C}_\infty(\mathbb{R})$.

The following assumption makes precise what we want to require from the set $\mathcal{C}(U) \subset \mathcal{F}_c(U)$.

Assumption 2.8. There exists a subset $\mathcal{C}(U) \subset \mathcal{F}_c(U) \cap \mathcal{C}(U)$ such that

(i) for any pair of open sets $V \subset U \subset X$, there exists a function $\varphi \in \mathcal{C}(U)$ such that $\varphi = 1$ on $V$, and $\text{supp}\{\varphi\} \subset U$;

(ii) for any $\varphi \in \mathcal{C}(U)$, any $u \in \mathcal{F}_{loc}(U)$, the product $\varphi u \in \mathcal{F}_c(U)$.

Remark 2.9. The requirement (i) in Assumption 2.8 is standard, and easily fulfilled when the Dirichlet form is regular. The requirement (ii) is nontrivial, and in general, only the products of functions in $\mathcal{F} \cap L^\infty(X)$ are guaranteed to belong to $\mathcal{F}$.

We now state the equivalence of the two definitions for local weak solutions.

Lemma 2.10 (Equivalence of definitions of local weak solutions). Under Assumption 2.8, when $f$ is locally in $L^2(I \to \mathcal{F}^t)$, Definition 2.6 is equivalent to Definition 2.7.

Proof. The proof follows essentially that of [13, Lemma 1].

3 Main hypotheses

3.1 Assumption on existence of cut-off functions

For a pair of open sets $V \subset U \subset X$, by a cut-off function for the pair $V \subset U$ we mean a function $\eta \in \mathcal{F} \cap \mathcal{C}(X)$ such that $\eta = 1$ on $V$, and $\text{supp}\{\eta\} \subset U$. 
Such cut-off functions always exist for any pair of open sets $V \subset U$ in a regular Dirichlet space, see [14]. For results in this paper what we need is the existence of cut-off functions that further have controlled energy, and we explain what this means in the following assumption.

**Assumption 3.1 (existence of nice cut-off functions).** There exists some topological basis $\mathcal{TB}$ of $X$ such that for any pair of open sets $V \subset U$, $U, V \in \mathcal{TB}$, for any $0 < C_1 < 1$, there exists some constant $C_2(C_1, U, V) > 0$, and some cut-off function $\eta$ for the pair $V \subset U$, such that for any $v \in F$,

$$
\int_X v^2 d\Gamma (\eta, \eta) \leq C_1 \int_X \eta^2 d\Gamma (v, v) + C_2(C_1, U, V) \int_{\text{supp}(\eta)} v^2 dm. \quad (3.1)
$$

We call such $\eta$ functions **nice cut-off functions**.

When we do not want to emphasize the dependence of $C_2(C_1, U, V)$ on $C_1$, $U, V$, we write (3.1) as

$$
\int_X v^2 d\Gamma (\eta, \eta) \leq C_1 \int_X \eta^2 d\Gamma (v, v) + C_2 \int_{\text{supp}(\eta)} v^2 dm. \quad (3.2)
$$

**Remark 3.2.** We will later show that in Assumption 3.1 the condition $U, V \in \mathcal{TB}$ for some topological basis $\mathcal{TB}$ is “redundant”, in the sense that Assumption 3.1 implies automatically that nice cut-off functions in the sense of (3.1) exist for any pair of open sets $V \subset U$. We also remark the assumption has a straightforward equivalent form that for any pair of precompact open sets $U, V$ with disjoint closures, i.e. $U \cap V = \emptyset$, for any $C_1$ between 0 and 1, there exists a cut-off function $\eta$ such that $\eta = 1$ on $U$, $\eta = 0$ on $V$, and there exists some constant $C_2(C_1, U, V) > 0$, such that for any $v \in F$,

$$
\int_X v^2 d\Gamma (\eta, \eta) \leq C_1 \int_X \eta^2 d\Gamma (v, v) + C_2(C_1, U, V) \int_{\text{supp}(\eta)} v^2 dm. \quad (3.3)
$$

Let $\eta(x)$ be a nice cut-off function, and let $l(t)$ be a smooth function on $\mathbb{R}$ with compact support, then the product $\eta(x)l(t)$ is a function in $\mathcal{F}_c(I \times X)$. We call such product functions **nice product cut-off functions**, and we denote such functions by $\tilde{\eta}(t, x) := \eta(x)l(t)$.

**Remark 3.3.** When there exists a cut-off function $\eta$ for the pair $V \subset U$, whose corresponding energy measure is absolutely continuous w.r.t. $m$, and $d\Gamma(\eta, \eta)/dm$ is bounded, i.e.

$$
d\Gamma(\eta, \eta) \leq C(U, V)dm \quad (3.4)
$$

for some $C(U, V) < \infty$, then $\eta$ satisfies (3.2) with $C_1 = 0$, $C_2 = C(U, V)$. This trivially implies that $\eta$ satisfies (3.1) with any $0 < C_1 < 1$, and $C_2(C_1, U, V) = C(U, V)$ (independent of $C_1$). We say in this special case that the cut-off function $\eta$ has bounded gradient.
Conversely, if (3.1) can be extended to hold true for $C_1 = 0$ and $C_2(0,U,V) < \infty$, for some cut-off function $\eta$, then $\eta$ has bounded gradient.

In particular, when the intrinsic pseudo-distance of the Dirichlet space,

$$\rho_X(x,y) = \sup \{ \varphi(x) - \varphi(y) \mid \varphi \in \mathcal{F}_{\text{loc}}(X) \cap C(X), \ d\Gamma(\varphi, \varphi) \leq dm \},$$

(3.5)
is a continuous metric that induces the same topology of $X$, the Dirichlet space satisfies Assumption 3.1 with cut-off functions with bounded gradient, and the cut-off functions can be explicitly constructed using the intrinsic distance. cf. [28].

**Remark 3.4.** Typical examples of Dirichlet spaces that satisfy Assumption 3.1 but do not possess cut-off functions with bounded gradient are some fractal spaces, including for example the Sierpinski gasket and the Sierpinski carpet.

For fractal spaces, usually the existence of nice cut-off functions is guaranteed as consequences of other properties like sub-Gaussian upper bounds satisfied by the Dirichlet space (heat kernel). In general, in such cases, there are no simple explicit constructions of the cut-off functions satisfying (3.2). For references we mention [1] and [2].

We first show the cut-off functions in Assumption 3.1 indeed satisfy the conditions in Assumption 2.8.

**Lemma 3.5.** Any nice cut-off function $\varphi$ in the sense of (3.2) satisfies (ii) in Assumption 2.8, namely, let $U \subseteq X$ be some open set such that $\text{supp} \{ \varphi \} \subseteq U$, then for any $u \in \mathcal{F}_{\text{loc}}(U)$, the product $\varphi \cdot u \in \mathcal{F}_{c}(U)$.

**Proof.** The support of the product function $\varphi \cdot u$ is clearly contained in $U$. To show $\varphi \cdot u \in \mathcal{F}$, recall that $u \in \mathcal{F}_{\text{loc}}(U)$ means $u$ is in $L^2_{\text{loc}}(U)$, and satisfies for any $V \Subset U$, there exists some $u^t$ in $\mathcal{F}$ such that $u^t = u$ m-a.e. on $V$. Pick some open set $V$ such that $\text{supp} \{ \varphi \} \subset V \Subset U$, and fix some $u^t \in \mathcal{F}$ that agrees with $u$ m-a.e. on $V$. Then

$$||\varphi u^t||_{E_1}^2 = \int_X (\varphi u^t)^2 \ dm + \int_X d\Gamma(\varphi u^t, \varphi u^t) + \int_X (\varphi u^t)^2 \ dk \leq \int_X (\varphi u^t)^2 \ dm + \int_X (\varphi u^t)^2 \ dk + 2 \left[ \int_X \varphi^2 \ d\Gamma(u^t, u^t) + \int_X (u^t)^2 \ d\Gamma(\varphi, \varphi) \right].$$

The first two terms are clearly finite, the third term is bounded above by $E_1(u^t, u^t)$ up to some constant, and the last term is finite due to (3.2). Hence $||\varphi u^t||_{E_1} < +\infty$, and $\varphi u = \varphi u^t \in \mathcal{F}_{c}(U)$. \qed

So far the examples we have described satisfy Assumption 3.1 for all pairs of open sets $V \Subset U$. And the reason in Assumption 3.1 we only require nice cut-off functions to exist for pairs of open sets in some topological basis $\mathcal{T \mathcal{B}}$ is to make the assumption easy to check for some infinite dimensional examples, like the infinite dimensional torus or infinite product of Sierpinski gaskets.

In the next lemma we state the automatic extension of existence of nice cut-off functions for general pairs of open sets, given Assumption 3.1. And we postpone the proof to the Appendix.
Lemma 3.6. Suppose Assumption 3.4 holds. Then for any two open sets $U, V$ with $V \subseteq U$, any constant $0 < C_1 < 1$, there exists a constant $C_2(C_1, U, V)$, and a nice cut-off function in the sense of (3.2). In particular, $U, V$ are not necessarily in $\mathcal{T}B$.

Given any nice cut-off function and any function in the domain $F$, by Lemma 3.5, their product belongs to $F$. The energy of the product function satisfies the following estimate, which we later refer to as the gradient inequality.

Lemma 3.7 (gradient inequality). Let $\eta$ be a nice cut-off function, and let $v \in F$. Then

$$
\int_X d\Gamma (\eta v, \eta v) \leq \frac{1}{1 - 2C_1} \int_X d\Gamma (v, v) + \frac{C_2}{1 - 4C_1} \int_{\text{supp}(\eta)} v^2 \, dm, \quad (3.6)
$$

where $C_1, C_2$ are associated with $\eta$ as in (3.1).

Note that the right-hand side of the inequality can be written as $L^2$ integrals when $v \in \mathcal{D}(P)$. Indeed, the first integral is equal to $\int_X \eta^2 v \, P v \, dm$. The point of the Lemma is to bound the energy of the product function $\eta v$ on the left-hand side by such $L^2$ integrals on the right-hand side (when $v \in \mathcal{D}(P)$).

It is easy to check the validity of this lemma in the special case when the cut-off function has bounded gradient. In this case, by expanding $\int_X d\Gamma (\eta v, \eta v)$ by the product rule and utilizing the upper bound $d\Gamma (\eta, \eta) / dm \leq M$, we get

$$
\int_X d\Gamma (\eta v, \eta v) \leq \int_X d\Gamma (v, v) + M \int_{\text{supp}(\eta)} v^2 \, dm, \quad (3.7)
$$

which is exactly (3.6) with $C_1 = 0, C_2 = M$.

In the general case, when the cut-off function does not have bounded gradient ($C_1$ in (3.1) must be taken as positive), (3.6) is less obvious, and we give the proof below.

Proof for Lemma 3.7

$$
\int_X d\Gamma (\eta v, \eta v) = \int_X \eta^2 d\Gamma (v, v) + \int_X v^2 d\Gamma (\eta, \eta) + 2 \int_X \eta v d\Gamma (\eta, v)
$$

$$
\geq \int_X \eta^2 d\Gamma (v, v) + \int_X v^2 d\Gamma (\eta, \eta) - \frac{1}{2} \int_X \eta^2 d\Gamma (v, v) - 2 \int_X v^2 d\Gamma (\eta, \eta)
$$

$$
= \frac{1}{2} \int_X \eta^2 d\Gamma (v, v) - \int_X v^2 d\Gamma (\eta, \eta)
$$

$$
\geq \left( \frac{1}{2} - C_1 \right) \int_X \eta^2 d\Gamma (v, v) - C_2 \int_{\text{supp}(\eta)} v^2 \, dm.
$$

Hence when $C_1 < \frac{1}{2}$,

$$
\int_X \eta^2 d\Gamma (v, v) \leq \frac{1}{2 - C_1} \int_X d\Gamma (\eta v, \eta v) + \frac{C_2}{2 - C_1} \int_{\text{supp}(\eta)} v^2 \, dm. \quad (3.8)
$$
On the other hand,
\[
\int_X d\Gamma (\eta v, \eta v) = \int_X d\Gamma (\eta^2 v, v) + \int_X v^2 d\Gamma (\eta, \eta)
\leq \int_X d\Gamma (\eta^2 v, v) + C_1 \int_X \eta^2 d\Gamma (v, v) + C_2 \int_{\text{supp}(\eta)} v^2 dm.
\]

Substituting the upper bound in (3.8) for \(\int_X \eta^2 d\Gamma (v, v)\) here, we get
\[
\int_X d\Gamma (\eta v, \eta v) \leq \int_X d\Gamma (\eta^2 v, v) + C_2 \int_{\text{supp}(\eta)} v^2 dm.
\]

When \(C_1 < \frac{1}{4}\), this implies
\[
\int_X d\Gamma (\eta v, \eta v) \leq \frac{1 - 2C_1}{1 - 4C_1} \int_X d\Gamma (\eta^2 v, v) + \frac{C_2}{1 - 4C_1} \int_{\text{supp}(\eta)} v^2 dm.
\]

\square

In applications we do not care about the exact constants, so in the following we consider \(C_1 < \frac{1}{8}\) and (3.6) implies
\[
\int_X d\Gamma (\eta v, \eta v) \leq 2 \int_X d\Gamma (\eta^2 v, v) + 2C_2 \int_{\text{supp}(\eta)} v^2 dm,
\]
and since \(d\Gamma\) is nonnegative, we also have
\[
\mathcal{E} (\eta v, \eta v) \leq 2\mathcal{E} (\eta^2 v, v) + 2C_2 \int_{\text{supp}(\eta)} v^2 dm.
\]

3.2 \(L^2\) Gaussian upper bound

In our treatment of the \(L^2\) time regularity of local weak solutions, we rely much on the heat semigroup, which is smooth in time. Roughly speaking, we use the heat semigroup to construct an approximate sequence to a local weak solution \(u\), and we show that this approximate sequence converges to \(u\) in some weak sense, and forms a Cauchy sequence in some \(\mathcal{F}^n(I \times X)\) space. These two statements together then imply \(u\) is (locally) in the space \(\mathcal{F}^n(I \times X)\). To show the approximate sequence is Cauchy, we need to use the following \(L^2\) version of Gaussian type upper bound for the heat semigroup.

**Assumption 3.8.** For any two open sets \(V_1, V_2 \subset X\) with \(\overline{V_1} \cap \overline{V_2} = \emptyset\), let \(\mathcal{A}(V_1, V_2) := \{(g_1, g_2) \mid \text{supp}\{g_i\} \subset V_i, \|g_i\|_{L^2(V_i, m)} \leq 1, i = 1, 2\}\). For any \(a \geq 0\), any \(n \in \mathbb{N}\),
\[
\lim_{t \to 0^+} \left( \sup_{(g_1, g_2) \in \mathcal{A}(V_1, V_2)} \left\{ \frac{1}{t^a} |< \partial_t^n H_t g_1, g_2 >| \right\} \right) = 0.
\]
To simplify notation we denote

\[ G_{V_1, V_2}(a, n, t) := \sup \left\{ \frac{1}{t^{a}} \left| \partial_{n}^{a} H_{t}g_1, g_2 \right| \left| (g_1, g_2) \in \mathcal{A}(V_1, V_2) \right. \right\} . \]

We remark that the \( L^2 \) Gaussian type upper bound is often automatically satisfied by the heat semigroup. For example, when there are enough cut-off functions with bounded gradient \( (3.4) \), or when the general assumption (Assumption \( 3.1 \)) holds with \( C_2(C_1, U, V) = C(U, V)C_1^{-\alpha} \) for some \( \alpha > 0 \), then the \( L^2 \) Gaussian bound for the semigroup holds. We also note that the \( L^2 \) Gaussian type bound above is a very weak Gaussian upper bound. For example, from this bound itself we cannot tell if the heat semigroup even admits a density, and even if we assume there is a density, neither can we say anything about the pointwise estimate of the density function. On the other hand, when there is the pointwise Gaussian or sub-Gaussian upper bound, then the \( L^2 \) Gaussian bound is a very weak consequence, hence we still name it “\( L^2 \) Gaussian type upper bound”, after the name of the classical pointwise Gaussian or sub-Gaussian upper bound.

More precisely, under Assumption \( 3.1 \) with cut-off functions with bounded gradient \( (3.4) \), one can define the distance between sets (cf. \( [16] \)): for any two measurable, precompact sets \( U, V \),

\[
| < H_t g_1, g_2 | \leq \exp \left\{ - \frac{d(U, V)^2}{4t} \right\}.
\]

The proofs for showing various kinds of Gaussian upper bounds follow from the so-called Davies’ method, cf. eg. \( [10] \). Then to generalize the upper bound for terms like \( | < \partial^n H_t g_1, g_2 | \), one can use for example the complex analysis method from \( [9] \), or the method in \( [11] \).

However, when the existence of nice cut-off functions with bounded gradient is not guaranteed, there could be disjoint, closed measurable sets \( U, V \) with distance \( d(U, V) = 0 \) (because roughly speaking the only functions with bounded gradient are constant functions), and then this distance notion will not be helpful in getting a Gaussian type upper bound.

Under Assumption \( 3.1 \) with cut-off functions satisfying the general inequality \( (3.1) \), or in the equivalent form of Assumption \( 3.1 \) satisfying \( (3.3) \) (see Remark \( 3.2 \)), when furthermore \( C_2 \) depends on \( C_1 \) in the specific form \( C_2(C_1, U, V) = \)
we can show the following $L^2$ Gaussian type bound

$$|<H_t g_1, g_2>| \leq \exp\left\{ -\left( \frac{1}{4^{\alpha+1}C(V_1,V_2)t} \right)^{1+2\alpha} \right\}. \tag{3.12}$$

Here again $V_1, V_2$ are two precompact measurable subsets of $X$ with $\overline{V_1} \cap \overline{V_2} = \emptyset$, and $(g_1, g_2) \in A(V_1, V_2)$.

We call both (3.11) and (3.12) Gaussian type upper bounds. Note that (formally) if we take $\alpha = 0$ and $C(V_1, V_2) = d(V_1, V_2)$ in (3.12), then we recover (3.11). In Appendix we give a proof for (3.12), as well as how this implies a similar bound for $|<\partial_t^n H_t g_1, g_2>|$.

4 Statement of the main results and overview of proof

4.1 Statement of the main results

In this section we state our results on the time regularity property of local weak solutions of the heat equation $$(\partial_t + P) u = f.$$ Our main result is that the regularity in time of $u$ is as good as that of the right-hand side $f$. Note that as a local weak solution on some $I \times U \subset I \times X$, $u$ satisfies the prerequisite $u \in F_{\text{loc}}^n(I \times U)$, so any of its “$F(I \times X)$ representative” $u^\sharp$ automatically has distributional time derivatives of any order. The challenge hence lies in showing that these time derivatives belong to $\mathcal{F}(I \times X) = L^2(I \to F)$. As suggested by the desired conclusion (i.e., time derivatives being in $L^2(I \to F)$), this entire article is based on the structural properties of Dirichlet forms (the Beurling-Deny decomposition formula) and the spectral theory for self-adjoint operators. Our main theorem is the following.

**Theorem 4.1.** Let $(X, m)$ be a metric measure space and $(\mathcal{E}, \mathcal{F})$ be a symmetric, regular, local Dirichlet form satisfying Assumption 3.1 (existence of nice cut-off functions). Assume the associated heat semigroup $(H_t)_{t > 0}$ satisfies Assumption 3.8 (the $L^2$ Gaussian type upper bound). Given $U \subset X$, $I = (a, b) \subset \mathbb{R}$ and a function $f$ that is locally in $W^{n, 2}(I \to L^2(U))$, let $u$ be a local weak solution of $(\partial_t + P) u = f$ on $I \times U$. Then $u$ is in $F_{\text{loc}}^n(I \times U)$.

In short, Theorem 4.1 claims that if the right-hand side $f$ of the heat equation locally has time derivatives up to order $n$, then so does the local weak solution $u$, and its time derivatives up to order $n$ locally belong to $L^2(I \to \mathcal{F})$. An important implication of Theorem 4.1 is that the time derivatives of $u$ (up to the order $n$) are local weak solutions of the heat equation $$(\partial_t + P) \partial_t^k u = \partial_t^k f.$$
Corollary 4.2. Under the hypotheses in Theorem 4.1, if \( f \) is locally in the space \( W^{n,2}(I \to L^2(U)) \), then for any \( 1 \leq k \leq n \), \( \partial_t^k u \) is a local weak solution of

\[
(\partial_t + P) \partial_t^k u = \partial_t^k f.
\]

In particular, if \( u \) is a local weak solution of \((\partial_t + P) u = 0\) on \( I \times U \), then all time derivatives \( \partial_t^k u \) of \( u \), \( 1 \leq k < \infty \), are local weak solutions of the same heat equation on \( I \times U \).

4.2 Sketch of proof for a special case of Theorem 4.1

In the next two sections we prove Theorem 4.1 and Corollary 4.2. In this section, we give a simplified outline in a special case - when \( X \) is compact, and we consider any local weak solution \( u \) of the heat equation on \( I \times X \). The rigorous proof in the next section is built on this outline but takes into consideration the complications brought in by noncompactness of the space \( X \) and the restriction on some open subset \( U \subset X \). In this more general context, the existence of nice cut-off functions and the \( L^2 \) Gaussian type upper bound become essential. In the special case where \( X \) is compact and \( u \) is a local weak solution on the full time-space cylinder \( I \times X \), since \( F_c(X) = F = F_{\text{loc}}(X) \), we know that \( u \) itself is in the domain of the Dirichlet form, and in particular, in \( L^2(X) \). The spaces \( F_c(I \times X) \), \( F(I \times X) \), \( F_{\text{loc}}(I \times X) \) are different due to the inclusion of the open time interval \( I = (a,b) \). The statement of the theorem is much shortened as we do not need to assume the existence of nice cut-off functions, nor that the heat semigroup satisfies the \( L^2 \) Gaussian type upper bound. In the proof we do need to multiply \( u \) with some smooth cut-off function in time, but in the outline below we ignore that technicality and pretend the functions are globally good in time.

Recall that we take the following convention. For any function \( g(s,x) \), we write \( g^\tau(x) := g(s,x) \).

Proposition 4.3 (special case). Let \((X,m)\) be a compact metric measure space and \((\mathcal{E},\mathcal{F})\) be a symmetric, regular, local Dirichlet form. Given \( I = (a,b) \in \mathbb{R} \) and a function \( f \) that is locally in \( W^{1,2}(I \to L^2(X)) \), let \( u \) be a local weak solution of \((\partial_t + P) u = f \) on \( I \times X \). Then \( u \) is locally in \( F^1(I \times X) \).

Outline. We consider the function

\[
u_\tau(s,x) := \int_I \rho_\tau(s-t)H_{s-t}u^t(x)\, dt.
\]

The integral makes sense as a Bochner integral. Here \( \tau > 0 \), and \( \rho_\tau(r) = \frac{1}{\tau}\rho(\frac{r}{\tau}) \), where \( \rho \) is some smooth nonnegative cut-off function on \( \mathbb{R} \) supported in \((1,2)\), and with total integral equal to 1. Note that when there is the notion of convolution and when \( H_t \) admits a density function (heat kernel), the approximate sequence above is exactly the convolution in time and space of \( u \) and the heat kernel (with a cut-off function in time).
Since $H_t$ is smooth in time, it is easy to show that $u_{\tau}$ is smooth in time. More precisely, for any $\tau > 0$, $u_{\tau} \in C^\infty(I \to \mathcal{F})$. And it is routine to show that $u_{\tau}$ converges to $u$ in $L^2(I \times X)$ as $\tau$ tends to 0. So to prove the proposition, it suffices to show $\{u_{\tau}\}_{\tau>0}$ is Cauchy in $W^{1,2}(I \to \mathcal{F}) = \mathcal{F}(I \times X)$.

To this end, it is enough to show that $\|\partial_{\tau} u_{\tau}\|_{W^{1,2}(I \to \mathcal{F})}$ is integrable in $\tau$ near 0, because then

$$
\int_0^\gamma \|\partial_{\tau} u_{\tau}\|_{W^{1,2}(I \to \mathcal{F})} \, d\tau \to 0 \text{ as } \gamma \to 0,
$$

and thus $\{u_{\tau}\}$ is a Cauchy sequence in $W^{1,2}(I \to \mathcal{F})$. We first estimate $\|\partial_{\tau} \partial_s u_{\tau}\|_{L^2(I \times X)}$. Note that $\partial_{\tau} \rho_{\tau}(r) = -\partial_{s} \bar{\rho}_{\tau}(r)$, where $\bar{\rho}_{\tau}(r) = \frac{\tau}{r} \rho(\frac{r}{\tau})$. By duality,

$$
\|\partial_{\tau} \partial_s u_{\tau}\|_{L^2(I \times X)} = \sup_{\|\varphi\|_{L^2(I \times X)} \leq 1, \varphi \in C^\infty_c(I \to L^2(X))} \left\{ \int_I \left[ \partial_s [\partial_t \bar{\rho}_{\tau}(s-t)H_{s-t}] u(t) \right] dt, \varphi > 0 \right\}
$$

$$
= \sup_{\|\varphi\|_{L^2(I \times X)} \leq 1, \varphi \in C^\infty_c(I \to L^2(X))} \left\{ \int_I \int_I \int_X u(t) \partial_s \partial_t [\bar{\rho}_{\tau}(s-t)H_{s-t}] \varphi(s,x) \, dmdsdt \right\}.
$$

From the second line to the third line we used Fubini theorem and the self-adjointness of $H_t$ to move $\partial_s [\partial_t \bar{\rho}_{\tau}(s-t)H_{s-t}]$ from the “$u$” side to “$\varphi$” side, and then use product rule to redistribute $\partial_t$. Then, since $\partial_t H_{s-t} = PH_{s-t}$, and $u$ is a local weak solution of $(\partial_t + P)u = f$ on $I \times X$, we get the above two terms in the brackets together, modulo a cut-off function in time that we omitted in this proof, equals $-\int_I \int_I \int_X f(s,x) \cdot \partial_s [\bar{\rho}_{\tau}(s-t)H_{s-t}] \varphi(s,x) \, dmdsdt$, and by rewriting $\partial_s [\bar{\rho}_{\tau}(s-t)H_{s-t}]$ as $-\partial_t [\bar{\rho}_{\tau}(s-t)H_{s-t}]$ and use integration by parts, we get

$$
\|\partial_{\tau} \partial_s u_{\tau}\|_{L^2(I \times X)}
$$

$$
= \sup_{\|\varphi\|_{L^2(I \times X)} \leq 1, \varphi \in C^\infty_c(I \to L^2(X))} \left| \int_I \int_I \int_X \partial_t f(t,x) \cdot \bar{\rho}_{\tau}(s-t)H_{s-t} \varphi(s,x) \, dmdsdt \right|.
$$

Here we did not consider the boundary term, but that is not a problem once we add in the cut-off function in time in the rigorous proof in the next section. By using the $W^{1,2}(I \to L^2(X))$ norm of $f$, and note that $\sup_{s \in I} \int_I \bar{\rho}_{\tau}(s-t) \, dt = 1$, we conclude that $\|\partial_{\tau} \partial_s u_{\tau}\|_{L^2(I \times X)}$ is bounded above independent of $\tau$. 

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To estimate \( ||\partial_x \partial_y u_\tau||_{L^2(I \rightarrow F)} \), note that for any \( \tau > 0 \), \( s \in I \), \( \partial_x \partial_y u_\tau \) belongs to \( D(P) \). Thus
\[
\left( \int_I E(\partial_x \partial_y u_\tau, \partial_x \partial_y u_\tau) \, ds \right)^{1/2} = \left( \int_I \int_X \partial_x \partial_y u_\tau \cdot P(\partial_x \partial_y u_\tau) \, dmds \right)^{1/2} \leq ||\partial_x \partial_y u_\tau||_{L^2(I \times X)}^{1/2} \leq ||P(\partial_x \partial_y u_\tau)||_{L^2(I \times X)}^{1/2}.
\]
And since \( \sup_{0<\tau<1} ||\partial_x \partial_y u_\tau||_{L^2(I \times X)} < \infty \) from above, it suffices to show
\[
||P(\partial_x \partial_y u_\tau)||_{L^2(I \times X)} \lesssim \frac{1}{\tau}.
\]
Running the estimate for \( ||\partial_x \partial_y u_\tau||_{L^2(I \times X)} \) again with \( H_{s-t} \) replaced by \( PH_{s-t} \), we can get the desired estimate. \( \square \)

5 Proof of the main results

5.1 Proof of Theorem 4.1 - general strategy

In this section we prove Theorem 4.1. We decompose the proof in a few steps. First, we define the approximate sequence (now, with proper nice cut-off functions inserted) to the local weak solution \( u \), and show the approximate sequence is Cauchy in some \( W^{k,2}(I \rightarrow F) \) space. Next, we show that the sequence converges to \( u \) in the \( L^2 \) sense (this step does not make use of the fact that \( u \) is a local weak solution).

More precisely, to show that \( u \in \mathcal{F}^n_{loc} (I \times U) \), by definition, for any \( J \times V \subseteq I \times U \), we show there exists some \( v \in \mathcal{F}^n (I \times X) \) such that \( v = u \) a.e. on \( J \times V \). Equivalently, let \( \tilde{v} (s,x) := \psi(x) u(s) \) be some nice product cut-off function such that \( \tilde{v} \equiv 1 \) on some \( J_{\tilde{v}} \times V_{\tilde{v}} \) where \( J \times V \subseteq J_{\tilde{v}} \times V_{\tilde{v}} \), and \( \{ \tilde{v} \} \subseteq I_{\tilde{v}} \times U_{\tilde{v}} \subseteq \tilde{I} \times U \). Our notational choice is that \( J, V \) are proper subsets of \( I, U \), and subscripts mark which function these sets are “affiliated with”. We show there exists some function in \( \mathcal{F}^n (I \times X) \) that equals to \( \tilde{v} u \) over \( J \times V \). Recall that \( \mathcal{F}^n (I \times X) \) is defined as \( W^{n,2} (I \rightarrow F) \). To find such a function in \( W^{n,2} (I \rightarrow F) \), we construct a family of functions that is Cauchy in \( W^{n,2} (I \rightarrow F) \) and consider their limit. Let
\[
\bar{u}_\tau (s,x) := \int_I \rho(\tau-t) H_{s-t} (\eta' u^\dagger) (x) \, dt.
\] (5.1)

Here \( \rho \) is defined as in the last section, that is, \( \rho(t) \in C_c^\infty (1,2) \) is some positive bounded function satisfying \( \int_1^2 \rho(t) \, dt = 1 \), and \( \rho_\tau(t) \) is defined as \( \rho_\tau(t) = \frac{\tau}{\tau} \rho \left( \frac{t}{\tau} \right) \) (\( \tau > 0 \)). Then \( \{ \rho_\tau \} \subseteq (\tau,2\tau) \). By inspection, \( \partial_\tau \rho_\tau(t) = -\partial_\tau \eta_\tau(t) \), where \( \eta_\tau(t) = \frac{t}{\tau} \rho \left( \frac{t}{\tau} \right) \). Let \( \eta(y,t) = \eta(y) t \) be another nice product cut-off function which is 1 over some neighborhood of the support of \( \tilde{v} \). More precisely, \( \eta \equiv 1 \) on some \( J_\eta \times V_\eta \) where \( J \times V \subseteq J_\eta \times V_\eta \subseteq J_{\tilde{v}} \times V_{\tilde{v}} \), and \( \{ \eta \} \subseteq I_\eta \times U_\eta \).
for some $I_\tau \times U_\tau \in I \times U$. We claim that the family $\{ \tilde{\psi}u_\tau \}$ is Cauchy in $W^{n,2}(I \to F)$, and hence has a limit in the same function space. Later we show $\tilde{\psi}u_\tau \to \tilde{\psi}u = \tilde{\psi}u$ in $L^2(I \times X)$, so the two limit functions must equal $m$-a.e. In other words, the “$L^2$ limit” $\tilde{\psi}u$ in fact belongs to $W^{n,2}(I \to F)$. Note also that $\tilde{\psi}u = u$ $m$-a.e. on $J \times V$, thus the statement in Theorem 4.1 follows.

To show $\{ \tilde{\psi}u_\tau \}$ is Cauchy in $W^{n,2}(I \to F)$, we first show that for each $\tau > 0$, $\tilde{\psi}u_\tau \in C^\infty(I \to F)$. It then suffices to prove the following two propositions.

Proposition 5.1. Under the hypotheses in Theorem 4.1, for any nice product function $\tilde{\psi}$ supported in $I \times U$, any $0 \leq k \leq n$,

$$\max_{0 \leq k \leq n} \sup_{0 < \tau < 1} ||\partial_\tau \partial_s^k (\tilde{\psi}u_\tau)||_{L^2(I \times X)} < +\infty.$$ 

Proposition 5.2. Under the hypotheses in Theorem 4.1, for any nice product function $\tilde{\psi}$ supported in $I \times U$, any $0 \leq k \leq n$,

$$\left( \int_I E(\partial_\tau \partial_s^k (\tilde{\psi}u_\tau), \partial_\tau \partial_s^k (\tilde{\psi}u_\tau)) \, ds \right)^{1/2} \lesssim \frac{1}{\sqrt{\tau}}.$$ 

These two propositions together show that

$$\int_0^\gamma ||\partial_\tau (\tilde{\psi}u_\tau)||_{W^{n,2}(I \to F)} \, d\tau \lesssim \int_0^\gamma \frac{1}{\sqrt{\tau}} \, d\tau \to 0 \text{ as } \gamma \to 0,$$

and hence the family $\{ \tilde{\psi}u_\tau \}$ is Cauchy in $W^{n,2}(I \to F)$.

To show $\tilde{\psi}u_\tau \in C^\infty(I \to F)$, we note that for any fixed $\tau > 0$ and $m \in \mathbb{N}$,

$$\int_I \rho_\tau(s-t) ||P^m H_{s-t}(\tilde{\eta}u_\tau)||_{L^2(X)} \, dt \lesssim \frac{1}{\tau^m} ||\rho_\tau||_{L^\infty} ||\tilde{\eta}u||_{L^2(I \times X)} < \infty. \quad (5.2)$$

It follows that all $\partial_s^m(\tilde{u}_\tau)$ are well-defined as Bochner integrals, and are in $L^\infty(I \to F)$. Hence $\tilde{u}_\tau \in C^\infty(I \to F)$. The conclusion that $\tilde{\psi}u_\tau \in C^\infty(I \to F)$ then follows from the gradient inequality (Lemma 3.7).

We next prove Proposition 5.1. We present the proof in two steps. In the first step we express and split $||\partial_\tau \partial_s^k (\tilde{\psi}u_\tau)||_{L^2(I \times X)}$ into three parts, and in the second step we estimate each part and show that they are all bounded independent of $0 < \tau < 1$ and $0 \leq k \leq n$.

### 5.2 Proof of Proposition 5.1 - Step 1

Consider the nice product cut-off function $\tilde{\eta}$ defined as in the general strategy subsection. Recall that $u$ is understood as some fixed $u^\sharp \in F(I \times X)$ with $u^\sharp = u$ on $I_\tau \times U_\tau$, some neighborhood of the support of $\tilde{\eta}$. We first compute $\partial_\tau \tilde{u}_\tau(s,x)$.

$$\partial_\tau \tilde{u}_\tau(s,x) = \int_I \partial_\tau \rho_\tau(s-t) H_{s-t}(\tilde{\eta}^t u^\sharp)(x) \, dt$$

$$= \int_I \partial_t \rho_\tau(s-t) \cdot H_{s-t}(\tilde{\eta}^t u^\sharp)(x) \, dt.$$
Recall (from last section) that here \( \overline{\rho}_r(s - t) = \frac{\nu(t)}{\nu(t)} \rho_r(s - t) = \frac{\nu(t)}{\nu(t)} \rho(s - t) \). Let \( \mathcal{T} := \{ \varphi \mid \| \varphi \|_{L^2(\mathbb{R}^N)} \leq 1, \varphi \in C^\infty_c (I \to L^2(X)) \} \), and recall that \( \overline{w}(s, x) = w(s)\psi(x) \). We have

\[
\left\| \partial_s \partial^k_s \left( \overline{w} \overline{u}_r \right) \right\|_{L^2(I \times X)} = \sup_{\varphi \in \mathcal{T}} \left\langle \psi \partial_s \partial^k_s \left( w(s)\overline{u}_r \right), \varphi \right\rangle \quad \text{for} \quad \varphi \in L^2(I \times X)
\]

\[
= \sup_{\varphi \in \mathcal{T}} \int_I \int_I \int_I \int_I \int_I \left\{ \int_I \partial^k_s \left[ w(s) \left( \partial_r \overline{\rho}_r(s - t) \right) H_{s - t} \right] \left( \overline{\eta}^I u^I \right) (x) \ dt \right\} \psi(x) \varphi(s, x) \ dm(x) \ ds
\]

\[
= \sup_{\varphi \in \mathcal{T}} \int_I \int_I \int_I \int_I \int_I \left\{ \int_I \left( \overline{\eta}^I u^I \right) (x) \cdot \partial^k_s \left[ w(s) \left( \partial_r \overline{\rho}_r(s - t) \right) H_{s - t} \right] (\psi \varphi^*) (x) \ dm(x) \ dt \right\} \ ds \ dt \ dt \ ds
\]

\[
:= \text{(intermediate)}.
\]

The last line is by the Fubini Theorem (changing integration order from \( \int_I \int_I \int_I dtdmds \) to \( \int_I \int_I \int_I X \ dmdtds \)) and by the self-adjointness of \( H_{s - t} \). Next we use the product rule for \( \partial_t \) to rewrite \( w(s) \left( \partial_r \overline{\rho}_r(s - t) \right) H_{s - t} \) in the square bracket as \( \partial_t \left( w(s)\overline{\rho}_r(s - t)H_{s - t} \right) - w(s)\overline{\rho}_r(s - t)\partial_t H_{s - t} \), the above then further equals to

\[
\left\| \partial_s \partial^k_s \left( \overline{w} \overline{u}_r \right) \right\|_{L^2(I \times X)} = \text{(intermediate)}
\]

\[
= \sup_{\varphi \in \mathcal{T}} \left\{ \int_I \int_I \int_I \int_I \int_I \int_I \left\{ \int_I \left( \overline{\eta}^I u^I \right) (x) \cdot \partial_t \left[ w(s)\overline{\rho}_r(s - t) H_{s - t} \right] (\psi \varphi^*) (x) \ dm(x) \ dt \right\} \ ds \ dt \ dt \ ds
\]

In the last line, since \( \partial_t H_{s - t} = PH_{s - t} \), the second term equals

\[
= \int_I \int_I \int_I \int_I \int_I \left\{ \int_I \left( \overline{\eta}^I u^I \right) (x) \cdot \partial_t \left[ w(s)\overline{\rho}_r(s - t) H_{s - t} \right] (\psi \varphi^*) (x) \ dm(x) \ dt \right\} \ ds \ dt \ ds
\]

Substituting back to the above computation, we have \( \left\| \partial_s \partial^k_s \left( \overline{w} \overline{u}_r \right) \right\|_{L^2(I \times X)} \) equals

\[
\sup_{\varphi \in \mathcal{T}} \left\{ \int_I \int_I \int_I \int_I \int_I \int_I \left\{ \int_I \left( \overline{\eta}^I u^I \right) (x) \cdot \partial_t \left[ w(s)\overline{\rho}_r(s - t) H_{s - t} \right] (\psi \varphi^*) (x) \ dm(x) \ dt \right\} \ ds \ dt \ dt \ ds
\]

To simplify notation we let

\[
v_{k, r} (s, t, x) := \partial^k_s \left( w(s)\overline{\rho}_r(s - t) H_{s - t} \right) (\psi \varphi^*) (x).
\]

\[
(5.3)
\]
It is clear that for any fixed $\tau > 0$, $v_{k,\tau} \in L^2(I \times I \times X)$, and for any $s, t \in I$, $v_{s,t}^{k,\tau} \in D(P)$. The result of the whole computation above can be written as

$$\left\lVert \partial_{\tau} \partial_k (\psi u_{\tau}) \right\rVert_{L^2(I \times X)} = \sup_{\varphi \in T} \left\{ \int_I \int_I \int_X \eta(t,x) u(t,x) \cdot \partial_t [v_{k,\tau}(s,t,x)] \, dm(x) dt ds - \int_I \int_I \int_X E(\eta(t,\cdot) u(t,\cdot), v_{k,\tau}(s,t,\cdot)) \, dm(x) dt ds \right\}. \quad (5.4)$$

Recall that $u$ is a local weak solution on $I \times U$. If in (5.4) $\eta$ is not grouped with $u$ but appears on the same side with $v_{k,\tau}$, then (5.4) is exactly $\int_I <f, \eta v_{s,t}^{k,\tau}> ds$ (the pairing is $L^2(I \times X)$ pairing). This observation inspires us to write (5.4) as this term plus the difference, and then estimate them each separately. More precisely, we have

$$\left\lVert \partial_{\tau} \partial_k (\psi u_{\tau}) \right\rVert_{L^2(I \times X)} = (5.4) \leq \sup_{\varphi \in T} \lvert A_k(\tau, \varphi) \rvert + \sup_{\varphi \in T} \lvert B_k(\tau, \varphi) \rvert + \sup_{\varphi \in T} \lvert C_k(\tau, \varphi) \rvert,$$

where

$$A_k(\tau, \varphi) = \int_I \int_I \int_X (\eta^t u^t) \cdot \partial_t [v_{s,t}^{k,\tau}] \, dm dt ds - \int_I \int_I \int_X u^t \cdot \partial_t [\eta^t v_{s,t}^{k,\tau}] \, dm dt ds,$$

$$B_k(\tau, \varphi) = \int_I \int_I \int_X E(\eta^t u^t, v_{s,t}^{k,\tau}) \, dt ds + \int_I \int_I \int_X E(u^t, \eta^t v_{s,t}^{k,\tau}) \, dt ds,$$

$$C_k(\tau, \varphi) = \int_I \int_I \int_X \eta(t,x) u(t,x) \cdot v_{k,\tau}(s,t,x) \, dm(x) dt ds.$$

5.3 Proof of Proposition 5.1 - Step 2

Next we estimate $\lvert A_k(\tau, \varphi) \rvert$, $\lvert B_k(\tau, \varphi) \rvert$, $\lvert C_k(\tau, \varphi) \rvert$ individually. We will see that the upper bounds we find for $A_k$, $B_k$, $C_k$ usually involve some $L^2$ or $E_1$ norms of the local weak solution $u$ on some precompact subsets of $I \times X$ (hence the norms are well-defined). To conveniently express these norms of $u$, we introduce a nice (product) cut-off function that lives in (i.e. has compact...
support in) $I \times U$ and is flat 1 on some open set that covers the supports of all other cut-off functions in the whole proof. We denote this cut-off function by $\Psi(t,x) = n(t)\overline{\Psi}(x)$. It can be determined after all other nice (product) cut-off functions in the proof for Theorem 4.1 are being introduced.

For $A_k(\tau, \varphi)$, note that $\partial_t[\tilde{\eta}(t,x)]$ is only nonzero for $t \in (J_{\tilde{\eta}})^c$ (away form where $\tilde{\eta} \equiv 1$), and $s \in I_{\tilde{\eta}} \cap J_{\tilde{\eta}}$ because of $w(s)$. Therefore for small $\tau$ (more precisely, $\tau < d(I_{\tilde{\eta}}, (J_{\tilde{\eta}})^c)/2 = : c_0$), we have

$$\partial_t[\tilde{\eta}(t,x)] v_{k,\tau}(s,t,x) \equiv 0,$$

so $A_k(\tau, \varphi) = 0$ for $\tau < c_0$. For $\tau \geq c_0$, substituting in $v_{k,\tau}$, we get

$$|A_k(\tau, \varphi)| = \left| \int \int_X u^t \cdot \partial_t[\tilde{\eta}^t] \cdot \partial_s^k (w(s)\overline{\varphi}(s-t)) \psi \right| dmdtds \leq 2^k ||\psi||_{C^1(I)} \times \int_{I_{\tilde{\eta}}} \int_{I_{\tilde{\eta}}} ||u^t||_{L^2(U_{\eta})} \max_{0 \leq a, b \leq k} \left\{ \left| \partial_s^a (w(s)\overline{\varphi}(s-t)) \right| \left| \partial_s^b \overline{\varphi}(s-t) \right| \right\} dt ds,$$

Using $||w||_{C^0/\tau^{k+2}}$ to bound sup$_{s,t} \max_{a,b} |\partial_s^a (w(s)\overline{\varphi}(s-t))|$, and note that the remaining part in the integral is bounded by

$$\int_{I_{\tilde{\eta}}} \int_{I_{\tilde{\eta}}} ||u^t||_{L^2(U_{\eta})} \max_{0 \leq a, b \leq k} \left\{ \left| \partial_s^a \overline{\varphi}(s-t) \right| \right\} dt ds \leq \int \int_{I_{\tilde{\eta}}} ||u^t||_{L^2(U_{\eta})} \max_{0 \leq a, b \leq k} \left\{ ||\partial_s^a \overline{\varphi}(s-t) ||_{L^2(I \times X)} \right\} dt ds,$$

after combining the bounds $||PH_{s-t}||_{2\to 2} \lesssim \frac{1}{s-t}, \tau < s-t < 2\tau$, and $\tau \geq c_0$, we conclude that

$$|A_k(\tau, \varphi)| \leq \frac{2^k C_{\overline{\varphi}, \overline{\psi}, \rho}(\tau, \varphi)}{\tau^{2k+2}} \int_{I_{\tilde{\eta}}} ||\varphi||_{L^2(I \times X)} ds \int_{I_{\tilde{\eta}}} ||u^t||_{L^2(U_{\eta})} dt \leq \bar{C}(k, c_0, \tilde{\eta}, \overline{\psi}, \rho) ||\varphi||_{L^2(I \times X)} ||\overline{\Psi}||_{L^2(I \times X)}.$$

Here the constant $\bar{C}(k, c_0, \tilde{\eta}, \overline{\psi}, \rho)$ depends only on the two cut-off functions $\eta$, $\overline{\psi}$, the function $\rho$ (note that $c_0 = d(I_{\tilde{\eta}}, (J_{\tilde{\eta}})^c)/2$ depends on the two functions), and the sum of the binomial coefficients that is bounded by $2^k$, so

$$\max_{0 \leq k \leq n} \bar{C}(k, c_0, \tilde{\eta}, \overline{\psi}, \rho) < \infty.$$

Denote some fixed upper bound by $C_A$, and recall that we take supremum over the functions $\varphi$ with $||\varphi||_{L^2(I \times X)} \leq 1$. Hence

$$\max_{0 \leq k \leq n} \sup_{0 < \tau < \frac{1}{2}} \sup_{0 < \tau < 1} |A_k(\tau, \varphi)| \leq C_A(n, \tilde{\eta}, \overline{\psi}, \rho) \cdot ||\overline{\Psi}||_{L^2(I \times X)} \cdot$$

$$\leq \frac{2^k C_{\overline{\varphi}, \overline{\psi}, \rho}(\tau, \varphi)}{\tau^{2k+2}} \int_{I_{\tilde{\eta}}} ||\varphi||_{L^2(I \times X)} ds \int_{I_{\tilde{\eta}}} ||u^t||_{L^2(U_{\eta})} dt \leq \bar{C}(k, c_0, \tilde{\eta}, \overline{\psi}, \rho) ||\varphi||_{L^2(I \times X)} ||\overline{\Psi}||_{L^2(I \times X)}.$$

(5.5)
For $B_k (τ, φ)$, observe that $η(t, y) = l(t)η(y)$ and $η ≡ 1$ on $V_τ$, so by the strong locality of the energy measure $dΓ$, the two terms in $B_k (τ, φ)$,

$$1_{V_τ} dΓ \left( η^i u^t, v_{k, τ}^{s, t} \right) = 1_{V_τ} dΓ \left( u^t, η^i v_{k, τ}^{s, t} \right).$$

In other words, we have

$$dΓ \left( η^i u^t, v_{k, τ}^{s, t} \right) - dΓ \left( u^t, η^i v_{k, τ}^{s, t} \right) = dΓ \left( η^i u^t, Φ v_{k, τ}^{s, t} \right) - dΓ \left( u^t, Φ v_{k, τ}^{s, t} \right) \quad (5.6)$$

for any “bowl-shaped” $Φ$ that equals 0 inside $V_τ$ and becomes 1 before it reaches the boundary of $V_τ$, provided the products of the functions are still in the domain $F$. To later utilize the $L^2$ Gaussian type upper bound to estimate, we take $Φ$ to be a nice cut-off function “disjointly supported” from $ψ$. More precisely, recall that $V_τ \equiv U_Ω \equiv V_τ \equiv U_Ω$. Let $V', U'$ be two open sets that sit in the middle of this chain, and let $V'', U''$ be two open sets at the right end of the chain, i.e.

$$V_Ω \equiv U_Ω \equiv V' \equiv U' \equiv V_τ \equiv U_Ω \equiv V'' \equiv U'' \equiv U.$$

Let $V_φ := V'' \setminus U'$, and $U_φ := U'' \setminus V'$. Then $V_φ \equiv U_φ$, and there exists a nice cut-off function that is 1 on $V_φ$ and 0 on $U_φ$. We fix such a function and denote it by $Φ$. The existence of $Φ$ is guaranteed by Lemma 3.6, or we can take the difference of two nice cut-off functions and show that the difference still satisfies $\begin{cases} 3.2 \end{cases}$. The nice cut-off function $Φ$ then satisfies equation $\begin{cases} 5.6 \end{cases}$, and has disjoint support from $ψ$. We thus have

$$|B_k (τ, φ)| = \left| - \int_I \int_X dΓ \left( η^i u^t, Φ v_{k, τ}^{s, t} \right) dt ds + \int_I \int_I \int_X dΓ \left( u^t, Φ v_{k, τ}^{s, t} \right) dt ds \right| ,$$

where by the Cauchy-Schwartz inequality and Hölder inequality, we have

$$\int_I \int_X dΓ \left( η^i u^t, Φ v_{k, τ}^{s, t} \right) dt ds \leq \left( \int_I \int_X dΓ \left( η^i u^t, η^i u^t \right) \right)^{1/2} \left( \int_X \left( Φ v_{k, τ}^{s, t}, Φ v_{k, τ}^{s, t} \right) \right)^{1/2} dt ds$$

$$\leq \left( \int_I \int_X \left( \left( Φ v_{k, τ}^{s, t}, Φ v_{k, τ}^{s, t} \right) dt ds \right)^{1/2} \left( \int_I \int_X \left( \left( Φ v_{k, τ}^{s, t}, Φ v_{k, τ}^{s, t} \right) dt ds \right)^{1/2} \right)^{1/2},$$

and similarly (recall that $Ψ$ equals to 1 on the supports of all other nice cut-off functions)

$$\int_I \int_I \int_X dΓ \left( u^t, Φ v_{k, τ}^{s, t} \right) dt ds = \int_I \int_I \int_X dΓ \left( Ψ u^t, Φ v_{k, τ}^{s, t} \right) dt ds \leq \left( \int_I \int_I \left( \left( Φ v_{k, τ}^{s, t}, Φ v_{k, τ}^{s, t} \right) dt ds \right)^{1/2} \left( \int_I \int_I \left( Φ v_{k, τ}^{s, t}, Φ v_{k, τ}^{s, t} \right) dt ds \right)^{1/2} \right).$$
Recall that \( v \) which is essentially most identical, so we only do it for that.

Here \( \Phi \) is associated with \( \eta \). Hence (3.9) applied to the nice cut-off function \( \Phi \), and it remains to estimate \( \left( \int_{I^2} \mathcal{E} \left( \Phi \eta^{s,t}_k, \Phi v^{s,t}_k \right) \right)^{1/2} \). The estimate for the two integrals are almost identical, so we only do it for \( \left( \int_{I^2} \mathcal{E} \left( \Phi \eta^{s,t}_k, \Phi v^{s,t}_k \right) \right)^{1/2} \) here.

Recall that \( v^{s,t}_k \in \mathcal{D}(P) \), we first want to move \( \Phi \eta \) to one side in order to rewrite the \( \mathcal{E} \) integral as an \( L^2 \) integral with \( P v^{s,t}_k \). To this end we apply the gradient inequality (Lemma [3.7]). Using (3.9) applied to the nice cut-off function \( \Phi \eta \), we can bound \( \int_{I^2} \mathcal{E} \left( \Phi \eta^{s,t}_k, \Phi v^{s,t}_k \right) \) by

\[
\int_{I^2} \mathcal{E} \left( \Phi \eta^{s,t}_k, \Phi v^{s,t}_k \right) \, dtds \leq 2 \times \left( \int_{I^2} \left| \int_{X} (\Phi \eta)^2 v^{s,t}_k \cdot P v^{s,t}_k \, dm \right| dtds + C_2 \int_{I^2} \int_{\text{supp}(\Phi \eta)} (\eta^{s,t}_k)^2 \, dm \, dtds \right)
\]

where \( C_2 \) is associated with \( \Phi \eta \). Recall that by (5.3),

\[ v^{s,t}_k(s,t,x) = \partial_s^k (w(s) \bar{p}_r(s-t) H_{s-t})(\psi \varphi^a)(x), \]

which is essentially \( P^a H_{s-t}(\psi \varphi^a) \) for \( 0 \leq a \leq k \) (up to the derivatives of \( w(s) \bar{p}_r(s-t) \) which are bounded by some multiple of \( 1/\tau^{k+1} \)). Moreover, by construction \( \Phi \) and \( \psi \) have disjoint supports, hence the two pairs of functions \( (\Phi \eta)^2 v^{s,t}_k \psi \varphi^a \) and \( 1 \Phi \eta v^{s,t}_k \) with \( \psi \varphi^a \) have disjoint supports, respectively. Thus we can apply the \( L^2 \) Gaussian type upper bound and get

and then to estimate \( \left\| 1 \Phi \eta \eta^{s,t}_k \right\|_{L^2(I \times I \times X)} \), note that

\[
\left\| 1 \Phi \eta \eta^{s,t}_k \right\|_{L^2(I \times I \times X)}^2 \leq \int_{I} \int_{I} 1 \Phi \eta \eta^{s,t}_k(s,t,x) \cdot \partial_s^k (w(s) \bar{p}_r(s-t) H_{s-t})(\psi \varphi^a)(x) \, dm(x) dtds \leq 2 \times \left\| w \rho \right\|_{C^k} (k+1) \left\| 1 \Phi \eta \eta^{s,t}_k \right\|_{L^2(I \times I \times X)} \left\| \psi \varphi \right\|_{L^2(I \times I \times X)} |I|^{1/2},
\]

Hence

\[
|B_k (\tau, \varphi)| \leq C \left( \left\| \eta \right\|_{L^2(I \times \mathcal{F})} + \left\| \nabla \eta \right\|_{L^2(I \times \mathcal{F})} \times \left( \int_{I^2} \mathcal{E} \left( \Phi \eta^{s,t}_k, \Phi v^{s,t}_k \right) \right)^{1/2} + \left( \int_{I^2} \mathcal{E} \left( \Phi \eta^{s,t}_k, \Phi v^{s,t}_k \right) \right)^{1/2} \right],
\]
where the left-hand side and the right-hand side have a common factor \(||1_{\varphi} v_{k,\tau}||\).

So

\[
\int_1^t \int_I \mathcal{E} \left( \Phi \psi^t \phi^{s,t}, \Phi \psi^t \phi^{s,t} \right) \, dt ds \\
\leq C \left( k, \eta, \psi, \rho, \Phi \right) G(k + 1, k, \tau)^2 \left( ||\psi||_{L^2(I \times X)}^2 \right).
\]

Since \( \sup_{0<\tau<1} G(k + 1, k, \tau) \) is clearly finite, we obtain the estimate for \( B_k (\tau, \varphi) \)

\[
\max \sup_{0 \leq k \leq n} \sup_{0 < \tau < 1} \sup_{\varphi \in \mathcal{T}} |B_k (\tau, \varphi)| \\
\leq C_B \left( n, \eta, \psi, \rho, \Phi \right) \left( \left( ||\eta_u||_{L^2(I \rightarrow F)} + \left| \left| \xi u \right| \right|_{L^2(I \rightarrow F)} \right) \right) (5.7)
\]

where \( C_B \left( n, \eta, \psi, \rho, \Phi \right) \) is some constant.

Last, we estimate the term \( C_k (\tau, \varphi) \). The idea is to use the product rule for differentiation in time (\( \partial_t \)) to expand and rewrite (in the last line we switch some \( \partial_t \) derivatives to \( \partial_s \) derivatives)

\[
v_{k,\tau}^{s,t} = \partial_s^k \{ w(s) \phi_{\tau} (s - t) H_{s-t} \} (\psi \varphi^s) \\
= \sum_{a=0}^{k} \binom{k}{a} \partial_s^{k-a} w(s) \cdot \partial_s^a (\phi_{\tau} (s - t) H_{s-t}) (\psi \varphi^s) \\
= -\sum_{a=0}^{k} \binom{k}{a} \partial_s^{k-a} w(s) \cdot \partial_s^a (\phi_{\tau} (s - t) H_{s-t}) (\psi \varphi^s),
\]

and then move all the \( \partial_t \) on \( \phi_{\tau} (s - t) H_{s-t} \), \( 0 \leq b \leq k \), to \( f \), using integration by parts. Thus we have

\[
|C_k (\tau, \varphi)| = \left| \int_1^t \int_X f(t, x) \cdot \eta(t, x) \partial_t^k \{ w(s) \phi_{\tau} (s - t) H_{s-t} \} (\psi \varphi^s) (x) \, dmdt ds \right|
\]

\[
= \left| \sum_{a=0}^{k} \binom{k}{a} \int_1^t \int_X \partial_t^{k-a} w(s) \cdot <\partial_t^a (\eta^t f^t), \phi_{\tau} (s - t) H_{s-t} (\psi \varphi^s) >_{L^2(I \times X)} ds \right|
\]

\[
\leq 2^k \max_{0 \leq a \leq k} \left| \int_1^t \int_X \partial_t^a (\eta^t f^t) \cdot \phi_{\tau} (s - t) H_{s-t} (\psi \varphi^s) \, dmdt \right|.
\]

In the second equality we used integration by parts in \( t \). For any \( 0 \leq a \leq k \), note that

\[
\left| \int_1^t \int_X \partial_t^a (\eta^t f^t) \cdot \phi_{\tau} (s - t) H_{s-t} (\psi \varphi^s) \, dmdt \right|
\]

\[
\leq \int_1^t ||\partial_t^a (\eta^t f^t)||_{L^2(X)} \cdot \left| \int_1^t \phi_{\tau} (s - t) H_{s-t} (\psi \varphi^s) \, ds \right| \left| \int_1^t \phi_{\tau} (s - t) H_{s-t} (\psi \varphi^s) \, ds \right| \, dt
\]

\[
\leq \int_1^t ||\partial_t^a (\eta^t f^t)||_{L^2(X)} \cdot \left| \int_1^t \phi_{\tau} (s - t) ||H_{s-t} (\psi \varphi^s)||_{L^2(X)} \, ds \right| \, dt.
\]
Then after using the Cauchy-Schwartz inequality, and note that one factor, 
\[
\left( \int \left| \partial_t^k (\psi f^t) \right|^2_{L^2(X)} dt \right)^{1/2},
\]
is bounded by \( \|\eta f\|_{W^{k,2}(I \rightarrow L^2(X))} \), and the other factor, 
\[
\left[ \int \left( \int \bar{\rho}_\tau (s-t) \|H_{s-t} (\psi f^s)\|_{L^2(X)} \right)^2 \|H_{s-t} (\psi f^s)\|_{L^2(X)} \ dx \right]^{1/2},
\]
can be bounded using Jensen’s inequality (moving the power 2 inside on \( \|H_{s-t} (\psi f^s)\|_{L^2(X)} \)), we have

\[
\int \int_X \partial_t^k (\psi f^t) \cdot \bar{\rho}_\tau (s-t) H_{s-t} (\psi f^s) \ dx \ dm dt \leq \frac{1}{\tau} \eta f \cdot \psi f \ .
\]

Here \( \int \bar{\rho}_\tau (s-t) \ dx \leq 2 \) is clear once we substitute in \( \bar{\rho}_\tau (s-t) = \frac{\tau - 1}{\tau} \rho \left( \frac{s-t}{\tau} \right) \) and recall that \( \int \rho = 1 \), and recall that for \( \rho \left( \frac{s-t}{\tau} \right) \) to be nonzero, \( 1 < \frac{s-t}{\tau} < 2 \). Hence

\[
\max_{0 \leq k \leq n} \sup_{0 < \tau < 1} \sup_{\varphi \in T} |C_k (\tau, \varphi)| \leq C \left( n, \eta, \psi \right) \cdot \|\eta f\|_{W^{k,2}(I \rightarrow L^2(X))},
\]

for some constant \( C \left( n, \eta, \psi \right) \).

In the above estimates for \( A_k, B_k, C_k \), we kept terms like \( \|\eta f\|_{W^{k,2}(I \rightarrow L^2(X))}, \|u\|_{L^2(I \rightarrow \mathcal{F})} \), since \( u, f \) are only assumed to be locally in those function spaces.

If we take any representative \( u^2, f^2 \) we can bound those norms by the corresponding norms of \( u^2 \) and \( f^2 \).

Combining the estimates \( \text{(5.5)} \), \( \text{(5.7)} \), and \( \text{(5.8)} \) for \( A_k (\tau, \varphi), B_k (\tau, \varphi), \) and \( C_k (\tau, \varphi) \) completes the proof for Proposition 5.1. To finish with the proof for \( \{\bar{\psi} u_{\tau}\} \) being Cauchy in \( W^{n,2}(I \rightarrow \mathcal{F}) \), we still need to prove Proposition 5.2.

**5.4 Proof of Proposition 5.2**

We want to show for \( 0 \leq k \leq n \),

\[
\int \mathcal{E} \left( \partial_s \partial_s^k \left( \bar{\psi} u_{\tau} \right), \partial_s \partial_s^k \left( \bar{\psi} u_{\tau} \right) \right) \ dx \lesssim \frac{1}{\tau}.
\]

Note that \( \bar{\psi} u_{\tau} \) is in the domain of \( P \), we apply the gradient inequality (Lemma 3.7) to bound this \( \mathcal{E} \) integral of product functions by \( L^2 \) integrals as below. We
we replace \( \tilde{\psi} < \tau < \) bounded independent of 0. The first estimate for \( \nu \) have

\[
\int_I \mathcal{E} \left( \partial_t \partial_s^k (\tilde{w}u + \tilde{v}) \right) \, ds
= \int_I \mathcal{E} \left( \psi \partial_t \partial_s^k (w(s)\tilde{u}) + \psi \partial_t \partial_s^k (w(s)\tilde{v}) \right) \, ds
\leq 2 \int_I \mathcal{E} \left( \psi^2 \partial_t \partial_s^k (w(s)\tilde{u}) + \psi \partial_t \partial_s^k (w(s)\tilde{v}) \right) \, ds
+ 2C_2 \int_I \int_{\text{supp}(u)} (\partial_t \partial_s^k (w(s)\tilde{u}))^2 \, dmds.
\]

Here \( C_2 \) is associated with \( \psi \), and the proof for Proposition 5.1 implies that the second term is bounded, namely,

\[
C_2 \int_I \int_{\text{supp}(u)} (\partial_t \partial_s^k (w(s)\tilde{u}))^2 \, dmds = C_2 \left\| \partial_t \partial_s^k (w(s)\tilde{u}) \right\|_{L^2(I \times U)} \leq M_1
\]

for some constant \( M_1 \) independent of 0 < \( \tau < 1 \).

To estimate the first term, \( \int_I \mathcal{E} \left( \psi^2 \partial_t \partial_s^k (w(s)\tilde{u}) + \partial_t \partial_s^k (w(s)\tilde{v}) \right) \, ds \), note that \( \tilde{u} \in D(P) \), so

\[
\int_I \mathcal{E} \left( \psi^2 \partial_t \partial_s^k (w(s)\tilde{u}) + \partial_t \partial_s^k (w(s)\tilde{v}) \right) \, ds
= \int_I \int_X \psi^2 \partial_t \partial_s^k (w(s)\tilde{u}) \cdot \partial_t \partial_s^k (w(s)\tilde{v}) \, ds
\leq \left\| \partial_t \partial_s^k (\tilde{w}u + \tilde{v}) \right\|_{L^2(I \times X)} \left\| \partial_t \partial_s^k (\tilde{w}P\tilde{u}) \right\|_{L^2(I \times X)}.
\]

The first \( L^2 \) norm is exactly the quantity treated in Proposition 5.1. It is bounded independent of 0 < \( \tau < 1 \). To estimate the second \( L^2 \) norm,

\[
\left\| \partial_t \partial_s^k (\tilde{w}P\tilde{u}) \right\|_{L^2(I \times X)}
\]

we replace \( \tilde{u} \) by \( P\tilde{u} \) in the proof of Proposition 5.1, and by the same arguments, \( \left\| \partial_t \partial_s^k (\tilde{w}P\tilde{u}) \right\|_{L^2(I \times X)} \) breaks into three parts \( A_k(t, \varphi), B_k(t, \varphi), C_k(t, \varphi) \), and the estimates for \( A_k \) and \( B_k \) look almost identical to those for \( A_k \) and \( B_k \). We write about the estimate for \( C_k(t, \varphi) \) here. The only difference is that instead of using \( \|H_t\|_{2 \to 2} \leq 1 \) as in the estimate for \( C_k \), we use \( \|PH_t\|_{2 \to 2} \leq 1/e\tau \) here. Substituting back the expression for \( v_{k,\tau} \), we have

\[
C_k(t, \varphi) = \int_I \int_X f(t, x) \cdot \bar{\eta}(t, x) P\nu_{k,\tau} (s, t, x) \, dm(x) \, dt ds
= \int_I \int_X f(t, x) \cdot \bar{\eta}(t, x) \partial_s^k (w(s)\bar{\nu}_s(s - t)PH_{s-t} (\psi \varphi^s)) (x) \, dm(x) \, dt ds.
\]

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As in the estimate for \( C_k \), the estimate for \( C_k' \) comes down to estimate

\[
\max_{0 \leq a \leq k} \left| \int I_X \partial_t^a (\pi^t f) \cdot \int_I \pi_t(s-t) PH_{s-t} (\psi \varphi^s) \, ds \, dt \right|
\]

\[
\leq ||\pi f||_{W^{k,2}(I \rightarrow L^2(X))} \cdot \left[ \int_I \int_I \pi_t(s-t) ||PH_{s-t} (\psi \varphi^s)||^2 \, ds \, dt \right]^{1/2}
\]

\[
\leq \frac{2}{c_T} ||\pi f||_{W^{k,2}(I \rightarrow L^2(X))} \cdot \||\psi \varphi||_{L^2(I \times X)}.
\]

Hence

\[
||\partial_\tau \partial_t^k (\psi P\tilde{u}_\tau)||_{L^2(I \times X)} = \sup_{||\varphi||_{L^2(I \times X)} \leq 1} <\psi \partial_\tau \partial_t^k (w(s) P\tilde{u}_\tau), \varphi >_{L^2}
\]

\[
\leq \sup_{\varphi \in \mathcal{T}} |A_k(\tau, \varphi)| + \sup_{\varphi \in \mathcal{T}} |B_k(\tau, \varphi)| + \sup_{\varphi \in \mathcal{T}} |C_k(\tau, \varphi)| \lesssim \frac{1}{\tau}.
\]

And hence (5.9) follows.

### 5.5 Convergence of the approximate sequence in \( L^2 \) sense

Proposition 5.1 and 5.2 together imply that the approximate sequence \( \{\tilde{u}_\tau\} \) is Cauchy in \( W^{n,2}(I \rightarrow \mathcal{F}) \). As we explained at the beginning of this section, to finish with the proof for Theorem 4.1 it suffices to show that the approximate sequence converges to \( \psi u \) in some weak sense. We prove the following slightly more general result.

For any function \( w \) in \( L^2(I \times X) \), any \( s \in I \), for any \( \tau > 0 \), define

\[
(A_\tau w)(s, x) := \int_I \rho_\tau(s-t) H_{s-t}(w^t)(x) \, dt.
\]  

(5.10)

When \( \tau \) is not small enough, \( A_\tau w \) is the zero function. Similar to showing \( \tilde{u}_\tau \in C^\infty(I \rightarrow \mathcal{F}) \) for any \( \tau > 0 \), we can show for any \( \tau > 0 \), \( A_\tau w \in C^\infty(I \rightarrow \mathcal{F}) \).

More precisely, it belongs to \( C^\infty(I \rightarrow \mathcal{F}) \).

**Proposition 5.3.** Let \( (H_t)_{t \geq 0} \) be any strongly continuous semigroup. Then \( A_\tau w \) defined as in (5.10) converges to \( w \) in \( L^2(I \times X) \), for any \( w \) in \( L^2(I \times X) \).

**Proof.** In this proposition we treat the larger class of semigroups \( H_t \) that are only assumed to be strongly continuous (not necessarily satisfying the Markov property and corresponding to a Dirichlet form), as roughly the same proof works under the weaker assumption. These \( H_t \) satisfy that there exists some \( M > 0, \omega > 0 \), so that

\[
||H_t||_{L^2(X) \rightarrow L^2(X)} \leq M e^{\omega t}.
\]

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We first show that for any $w$ in $C_c \left( I \to L^2(X) \right)$, $A_r w$ converges to $w$ in $L^2(I \times X)$. Then as $C_c \left( I \to L^2(X) \right)$ is dense in $L^2(I \times X)$, and
\[
\sup_{0 < r < 1} \| A_r \|_{L^2(I \times X) \to L^2(I \times X)} < +\infty,
\]
the statement holds for all $w$ in $L^2(I \times X)$. For $w \in C_c \left( I \to L^2(X) \right)$, we have
\[
\| A_r w - w \|_{L^2(I \times X)} = \left\| \int_I \rho_r (\cdot - t) \left[ H_{s-t} (w^t) - w^s \right] dt \right\|_{L^2(I \times X)} + \left\| \left( 1 - \int_I \rho_r (\cdot - t) dt \right) w \right\|_{L^2(I \times X)}.
\]
In the second term, since for $s, t \in I = (a, b)$, $s - b < s - t < s - a$, we know that $\int_I \rho_r (s - t) dt = 1$ only when $s - a \geq 2 \tau$. So $1 - \int_I \rho_r (s - t) dt$ is nonzero only when $a < s < a + 2 \tau$, which is an interval of length $2 \tau$. And since $w$ is in $C_c \left( I \to L^2(X) \right)$, we conclude that
\[
\left\| \left( 1 - \int_I \rho_r (\cdot - t) dt \right) w \right\|_{L^2(I \times X)} \to 0 \quad \text{as} \quad \tau \to 0.
\]
For the first term, we first write
\[
\int_I \rho_r (s - t) \left[ H_{s-t} (w^t) - w^s \right] dt = \int_I \rho_r (s - t) H_{s-t} (w^t - w^s) dt + \int_I \rho_r (s - t) [H_{s-t}(w^s) - w^s] dt.
\]
Since $\| \cdot \|_{L^2(I \times X)} = \left\| \| \cdot \|_{L^2(X)} \right\|_{L^2(I)}$, the $L^2$ norm of the first part,
\[
\| \int_I \rho_r (s - t) H_{s-t} (w^t - w^s) dt \|_{L^2(I \times X)},
\]
is bounded by
\[
\int_\tau^{2\tau} \rho_r (r) \left\| H_r (w^{s-r} - w^s) \right\|_{L^2(I \times X)} dr = \int_\tau^{2\tau} \rho_r \left\| \| H_r (w^{s-r} - w^s) \|_{L^2(X)} \right\|_{L^2(I)} dr \leq \int_\tau^{2\tau} \rho_r (r) \left\| H_r (w^{s-r} - w^s) \right\|_{L^2(I \times X)} dr \leq C \sup_{s \in I, \tau < r < 2\tau} \| w^{s-r} - w^s \|_{L^2(X)} \to 0 \quad \text{as} \quad \tau \to 0.
\]
The $L^2$ norm of the second part has upper bound
\[
\left\| \int_I \rho_r (s - t) [H_{s-t}(w^s) - w^s] dt \right\|_{L^2(I \times X)} \leq C \sup_{s \in I, \tau < r < 2\tau} \left\| H_r (w^s) - w^s \right\|_{L^2(X)},
\]
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where the right-hand side, because of the supremum taken over \( s \in I \) (in other words, \( w^s \) is not a fixed function in \( L^2(X) \)), needs further estimate. We first note that for any fixed \( r > 0 \), any \( s, t \in I \),

\[
\|H_r (w^s) - w^s\|_{L^2(X)} \\
\leq \|H_r (w^s - w^t)\|_{L^2(X)} + \|H_r (w^t) - w^t\|_{L^2(X)} + \|w^t - w^s\|_{L^2(X)} \\
\leq 2Me^{\omega r} \|w^t - w^s\|_{L^2(X)} + \|H_r (w^t) - w^t\|_{L^2(X)}.
\]

For any \( \epsilon > 0 \), any \( s \in I \), there is some \( \tau_0 (s) > 0 \) such that

1. for any \( r < \tau_0 (s) \), \( \|H_r (w^s) - w^s\|_{L^2(X)} < \epsilon \) (since \( w^s \in L^2(X) \)), and
2. \( \|w^t - w^s\|_{L^2(X)} < \epsilon \), for any \( |s - t| < \tau_0 (s) \) (since \( w \in C_c (I \to L^2(X)) \)).

Since \( I \) is compact and \( \mathcal{T} \subset \bigcup_{s \in \mathcal{T}} B (s, \tau_0 (s)) \) (here \( B (s, \tau_0 (s)) := (s - \tau_0 (s), s + \tau_0 (s)) \)), we can find some \( \{ B (s_k, \tau_0 (s_k)) \}_{k=1}^N \) as a finite cover for \( I \). Hence we can find some fixed \( \tau_0 (\tau_0 = \min_{1 \leq k \leq N} \{ \tau_0 (s_k) \}) \) such that

1. for any \( r < \tau_0 \), any \( s_k, 1 \leq k \leq N \), \( \|H_r (w^{s_k}) - w^{s_k}\|_{L^2(X)} < \epsilon \), and
2. for any \( s \in I \), there exists some \( s_k \) such that \( s \in B (s_k, \tau_0 (s_k)) \), and so

\( \|w^s - w^{s_k}\|_{L^2(X)} < \epsilon \).

Therefore,

\[
\sup_{s \in I, \tau < r < 2\tau} \|H_r (w^s) - w^s\|_{L^2(X)} \to 0 \quad \text{as} \quad \tau \to 0.
\]

This completes the proof for Proposition 5.3. \( \square \)

Note that for the local weak solution \( u \) in Theorem 4.1, the function \( \tilde{u}_r \) is exactly \( A_r (\eta u) \). So Proposition 5.3 applies to \( \tilde{u}_r \), and it follows that \( \psi \tilde{u}_r \to \psi u \) in \( L^2 (I \times X) \) as \( \tau \to 0 \). This completes the proof for Theorem 4.1.

### 5.6 Proof of Corollary 4.2

In this subsection we prove Corollary 4.2, which says essentially that time derivatives of local weak solutions of the heat equation are still local weak solutions.

**Proof for Corollary 4.2** By Theorem 4.1, \( u \) belongs to \( \mathcal{F}^u_{\text{loc}} (I \times U) \). And by definition of local weak solution on \( I \times U \), for any test function \( \varphi \) (and hence \( \partial^k_t \varphi \) for any \( 1 \leq k \leq n \)) in \( \mathcal{F} (I \times U) \cap C^\infty_c (I \to \mathcal{F}) \),

\[
- \int_I \int_X u \partial_t^{k+1} \varphi \, dmdt + \int_I \int_X \mathcal{E} \left( u, \partial_t^k \varphi \right) \, dt = \int_I \int_X f \partial_t^k \varphi \, dmdt. \quad (5.11)
\]

To show \( \partial^k_t u \) is a local weak solution (4.1), intuitively it suffices to do integration by parts \( k \) times to move \( \partial^k_t \) to the \( u \) and \( f \) sides of the integrals. We now justify this procedure.
Integration by parts for the first and third integrals in (5.11) are straightforward. We only describe the first step and the remaining is clear by induction. By Fubini-Tonelli Theorem, suppose supp(φ) ⊂ J × V ⊂ I × U, since
\[ \int_I \int_X |u \partial_t^{k+1} \varphi| \, dt \, dm \leq ||u||_{L^2(J \times U)} \cdot ||\varphi||_{W^{k+2,2}(I \to L^2(U))} < \infty, \]
we can switch the order of integration and get
\[- \int_I \int_X u \partial_t^{k+1} \varphi \, dt \, dm = - \int_I \int_X u \partial_t^{k+1} \varphi \, dt \, dm = \int_I \int_X \partial_t u \partial_t^{k+1} \varphi \, dt \, dm, \]
where the second equality is by integration by parts and that φ is compactly supported in time. The same works for the integral
\[ \int_I \int_X f \partial_t^k \varphi \, dt \, dm = - \int_I \int_X \partial_t f \partial_t^{k-1} \varphi \, dt \, dm. \]
For the second term in (5.11), to do integration by parts we want to first convert the \( E \)-integral into an \( L^2 \) type integral in order to switch order of integration. To this end, for each fixed \( t \in I \), we consider the approximate sequence \( \{ \beta G_\beta (\partial_t^k \varphi^i) \}_{\beta > 0} \), where \( G_\beta \) is the resolvent associated with the semigroup and Dirichlet form. Recall that \( \beta G_\beta \) is a contraction on \( L^2(X) \), and maps \( L^2(X) \) to \( \mathcal{D}(P) \). So for any fixed \( t \in I \), all \( \beta G_\beta (\partial_t^k \varphi^i) \in \mathcal{D}(P) \), and \( \beta G_\beta (\partial_t^k \varphi^i) \to \partial_t^k \varphi^i \) in \( \mathcal{E}_1 \)-norm as \( \beta \to \infty \). We now show this convergence is uniform in \( t \), i.e., \( \beta G_\beta (\partial_t^k \varphi^i) \to \partial_t^k \varphi^i \) in \( L^\infty(I \to \mathcal{F}) \) as \( \beta \to \infty \). Since \( \varphi \in C^\infty_c(I \to \mathcal{F}) \),
\[ ||\beta G_\beta (\partial_t^k \varphi^0) - (\partial_t^k \varphi^0)||_{\mathcal{E}_1} \leq ||\beta G_\beta (\partial_t^k \varphi^{t_0} - \partial_t^k \varphi^{t_1})||_{\mathcal{E}_1} + ||\beta G_\beta (\partial_t^k \varphi^{t_1}) - (\partial_t^k \varphi^{t_1})||_{\mathcal{E}_1} + ||(\partial_t^k \varphi^{t_1}) - (\partial_t^k \varphi^{t_1})||_{\mathcal{E}_1}, \]
We look at each term separately. The first term equals
\[ \left\{ \left( ||\beta G_\beta (\partial_t^k \varphi^{t_0} - \partial_t^k \varphi^{t_1})||_{L^2}^2 + ||\beta G_\beta P^{1/2} (\partial_t^k \varphi^{t_0} - \partial_t^k \varphi^{t_1})||_{L^2}^2 \right) \right\}^{1/2}, \]
and is thus bounded above by \( ||\partial_t^k \varphi^{t_0} - \partial_t^k \varphi^{t_1}||_{\mathcal{E}_1} \) (\( \beta G_\beta \) is an \( L^2 \)-contraction). So this term is small when \( t_0 \) and \( t_1 \) are close, regardless of the value of \( \beta \). The third term is small when \( t_0 \) and \( t_1 \) are close, and the second term tends to 0 when \( \beta \) tends to infinity. So by partitioning \( J \) (recall that supp(φ) ⊂ J × V) into finitely many thin enough sub-intervals, pick one point \( t_i \) in each piece, and consider the maximum of \( \beta \) such that \( ||\beta G_\beta (\partial_t^k \varphi^{t_i}) - (\partial_t^k \varphi^{t_i})||_{\mathcal{E}_1} \) are all small, then for any \( t \in J \), \( ||\beta G_\beta (\partial_t^k \varphi^i) - (\partial_t^k \varphi^i)||_{\mathcal{E}_1} \) is small. In other words, as \( \beta \to \infty \),
\[ \beta G_\beta \partial_t^k \varphi \to \partial_t^k \varphi \text{ in } L^\infty(I \to \mathcal{F}) \]
(and in \( C(I \to \mathcal{F}) \)). Hence
\[ \int_I E(u, \partial_t^k \varphi) \, dt = \lim_{\beta \to \infty} \int_I E(u, \beta G_\beta (\partial_t^k \varphi^i)) \, dt = \lim_{\beta \to \infty} \int_I \int_X u P(\beta G_\beta (\partial_t^k \varphi^i)) \, dt \, dm. \]
Since $P = G^{-1} - \beta$, $P\beta G = \beta - \beta^2 G$ satisfies $\|P\beta G\|_{L^2 \to L^2} \leq 2\beta < \infty$, it follows that $\beta G$ maps $C^m (I \to \mathcal{F})$ to $C^m (I \to \mathcal{D}(P))$ for any $m \in \mathbb{N}$, and

$$
\partial_t (P\beta G \varphi^t) = \lim_{\Delta t \to 0} \frac{P\beta G (\varphi^{t+\Delta t} - \varphi^t)}{\Delta t} = P\beta G \left( \frac{\varphi^{t+\Delta t} - \varphi^t}{\Delta t} \right) = P\beta G \partial_t \varphi^t.
$$

The limits in the above line are $L^2$ limits. The Fubini-Tonelli Theorem still applies to $\int_I \int_X u P (\beta G \beta (\partial_t \varphi^t)) \, dmdt$, and since $P\beta G \partial_t \varphi^t = \partial_t (P\beta G \varphi^t)$,

$$
\int_I \int_X u P (\beta G \beta (\partial_t \varphi^t)) \, dmdt = (-1)^k \int_X \partial_t^k u P (\beta G \beta \varphi^t) \, dt dm = (-1)^k \int_I \mathcal{E} (\partial_t^k u, \beta G \varphi^t) \, dt,
$$

by integration by parts ($k$ times). Therefore

$$
\int_I \int_X \mathcal{E} (u, \partial_t^k \varphi) \, dt = \lim_{\beta \to \infty} \int_I \int_X u P (\beta G \beta (\partial_t^k \varphi^t)) \, dmdt = \lim_{\beta \to \infty} (-1)^k \int_I \mathcal{E} (\partial_t^k u, \beta G \varphi) \, dt = (-1)^k \int_I \mathcal{E} (\partial_t^k u, \varphi) \, dt.
$$

In summary, after $k$ times of integration by parts, (5.11) becomes

$$
(-1)^{k+1} \int_I \int_X \partial_t^k u \partial_t \varphi \, dmdt + (-1)^k \int_I \mathcal{E} (\partial_t^k u, \varphi) \, dt = (-1)^k \int_I \int_X \partial_t^k f \varphi \, dmdt,
$$

and thus $\partial_t^k u$ is a local weak solution of (4.1) on $I \times U$. The statement in Corollary 4.2 for $f = 0$ then follows. \qed

6 Ancient solutions

6.1 Statement of results

In this section we generalize the results in [31] on the structure of ancient solutions of the heat equation to the setting of Dirichlet spaces. As usual we assume that $(\mathcal{E}, \mathcal{F})$ is symmetric and regular, and instead of local, we assume it is strongly local. We call a local weak solution $u$ of $(\partial_t + P)u = 0$ on $(-\infty, b) \times X$ for some $b > 0$ an ancient (local weak) solution. We assume $(X, \mathcal{E}, \mathcal{F})$ satisfies the assumption on existence of nice cut-off functions (Assumption 3.1), and the following further assumption.

**Assumption 6.1.** For any precompact open set $V \subseteq X$, any $C_1 > 0$, any $n \in \mathbb{N}_+$, there exists

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(1) an exhaustion of $X$, $\{W^n_{V,i}\}_{i \in \mathbb{N}_+}$, with each set covering $V$. That is, $\{W^n_{V,i}\}_{i \in \mathbb{N}_+}$ is a sequence of increasing open sets, satisfying

$$V \subset W^n_{V,1}, \quad W^n_{V,i} \Subset W^n_{V,i+1}, \quad \bigcup_{i=1}^{\infty} W^n_{V,i} = X.$$  

(2) a sequence of cut-off functions $\{\varphi^n_{V,i}\}_{i \in \mathbb{N}_+}$, where each $\varphi^n_{V,i} = \varphi_i$ is a cut-off function for the pair $W^n_{V,i} \subset W^n_{V,i+1}$, i.e. $\varphi_i = 1$ on $W^n_{V,i}$, $\text{supp}(\varphi_i) \subset W^n_{V,i+1}$. And $\varphi_i$ further satisfies that for any $v \in F$,

$$\int_X v^2 \, d\Gamma(\varphi_i, \varphi_i) \leq C_1 \int_X \varphi_i^2 \, d\Gamma(v, v) + \frac{1}{n} \int_{\text{supp}(\varphi_i)} v^2 \, dm. \quad (6.1)$$

In our theorems below, we only require the existence of an exhaustion and cut-off functions corresponding to some particular small value of $C_1$.

When $X$ is compact, Assumption 6.1 trivially holds because we can take all $W^n_{V,i}$ to be the whole space $X$, and take all $\varphi_i$ to be the constant function 1. For noncompact spaces, in the most classical setting $\mathbb{R}^d$ with the standard Dirichlet form, if $V \subset B(0; R)$ where $B(0; R)$ stands for the ball of radius $R$ centered at the origin, we can take $W^n_{V,i} = B(0; R + c i^{1/2})$ for some $c \geq 1$. Then the nice cut-off functions $\varphi_i$ for each pair $W^n_{V,i} \subset W^n_{V,i+1}$ satisfies

$$d\Gamma(\varphi_i, \varphi_i) \leq \frac{1}{e^n} \, dm,$$

which implies (6.1) with $C_1 = 0$.

In the following theorems we consider two types of ancient solutions, one with polynomial $L^2$ growth bound, and the other with exponential $L^2$ growth bound. We first remark that for any ancient local weak solution $u$, by Theorem 4.1 $u$ is locally in $W^{\infty, 2}(\mathbb{R}^d) \to F)$, and in particular, $u$ is locally in $C^\infty((\mathbb{R}^d) \to L^2(X))$. As generalizations of results in [8] [31], we have the following theorems on the structure of ancient solutions in the Dirichlet space setting.

**Theorem 6.2.** Let $(X, m)$ be a metric measure space, and let $(E, F)$ be a symmetric, regular, strongly local Dirichlet form on $X$. When $X$ is not compact, assume the Dirichlet space $(X, E, F)$ satisfies Assumption 3.1 and Assumption 6.1. Let $(H_t)_{t \geq 0}$ and $-P$ be the corresponding semigroup and generator. Let $b > 0$ be an arbitrary number. Let $u$ be a local weak solution of $(\partial_t + P)u = 0$ on $(-\infty, b) \times X$. Suppose $u$ satisfies the $L^2$ polynomial growth condition, namely, for any open subset $V \subset X$, for any $i \in \mathbb{N}_+$, for $C_1 = \frac{1}{136}$, there exist positive constants $b_u, d_u, C_{u,V,i} > 0$ ($b_u, d_u$ are independent of $V, i$), such that for any $T > 1$, $n \in \mathbb{N}_+$,

$$\left( \int_{[-T,0] \times W^n_{V,i}} |u(t, x)|^2 \, dm \, dt \right)^{1/2} \leq C_{u,V,i} \max \left\{ T^{-d_u + \frac{1}{2}}, n^{b_u} \right\}. \quad (6.2)$$
Then there exists some $N > 0$ such that for any $k > N$,
\[ \partial^k_t u = 0. \]

More precisely, $u$ is a polynomial in time, with
\[
 u(t, x) = u(0, x) + \partial_t u(0, x) t + \partial^2_t u(0, x) \frac{t^2}{2!} + \cdots + \partial^N_t u(0, x) \frac{1}{N!} t^N. 
\]

Here $N = \lfloor d_u \rfloor$, the largest integer not exceeding $d_u$.

For ancient solutions of the exponential growth type, we only need one sequence of exhaustion to get sufficient estimates, so we fix $n = 1$ and some precompact open set $V_0$, and consider the sequence $W_1 V_0, i = W_i$ only. Here as in the previous theorem, the $C_{1,136}$ is taken as $C_1 = \frac{1}{136}$, a small enough constant, the exact value of which is not essential (as can be seen from the proofs of the two theorems given in the next two sections).

**Theorem 6.3.** Let $(X, m)$ be a metric measure space, and let $(E, F)$ be a symmetric, regular, strongly local Dirichlet form on $X$. When $X$ is not compact, assume the Dirichlet space $(X, E, F)$ satisfies Assumption 3.1 and Assumption 6.1. Let $(H_t)_{t > 0}$ and $-P$ be the corresponding semigroup and generator. Let $b > 0$ be an arbitrary number. Let $u$ be a local weak solution of $(\partial_t + P)u = 0$ on $(-\infty, b) \times X$. Suppose $u$ satisfies the $L^2$ exponential growth condition, namely, there exists some $c_u > 0$, such that for any $T > 1$, any $i \in \mathbb{N}$,
\[
 \int_{[-T, 0] \times W_i} |u(t, x)|^2 \, dm \, dt \leq e^{c_u(T+i)}. \tag{6.3}
\]

Then $u$ is analytic in $t \in (-\infty, 0]$, in the sense that for any precompact open set $V \subset X$,
\[
 \left\| u(t, \cdot) - \sum_{i=1}^k \frac{\partial^i_t u(0, \cdot)}{i!} \frac{t^i}{i!} \right\|_{L^2(V)} \to 0, \quad k \to \infty, \tag{6.4}
\]

and the convergence is uniform in $t \in [a, 0]$ for any $a < 0$.

We first make some remarks about the two theorems.

**Remark 6.4.** For Theorem 6.2 if we denote $\frac{1}{n!} \partial^k_t u(0, x) = u_k(x)$, and let $N = [d_u]$, then $\{u_k\}_{k=0}^N$ satisfies
\[
 -Pu_k(x) = u_{k+1}(x), \quad \text{for } 0 \leq k \leq N - 1, \\
 -Pu_N(x) = 0, 
\]

both in the sense that (consider $u_{N+1} = 0$) for any $\varphi \in F_c(X)$,
\[
 \mathcal{E}(u_k, \varphi) = \int_X u_{k+1} \varphi \, dm, 
\]
for any \(1 \leq k \leq N\). We call \(u_k\) a local weak solutions of \(-Pu_k = u_{k+1}\) on \(X\). Moreover, all \(u_k\) satisfy the \(L^2\) growth bound that for any precompact open set \(V \Subset X\), for any \(i, n \in \mathbb{N}_+\), there exist constants \(C_{u,V,i} > 0\) and \(b_u > 0\) (independent of \(V, i\)), such that

\[
\left( \int_{W^V_i} |u_k(x)| \right)^{1/2} \leq C_{u,V,i} n^{b_u}.
\]

**Remark 6.5.** In Theorem 6.3, if we write \(u(t, x) = \sum_{k=0}^{\infty} a_k(x) t^k\) where the two sides equal in the above \(L^2\) sense, then the \(a_k(x)\) functions are \(a_k(x) = \partial_t^k u(0, x)\).

Using a Caccioppoli type estimate for local weak solutions, namely, for any local weak solution \(v\) of the heat equation \((\partial_t + P)v = 0\) on \((-\infty, c) \times X\) for some \(c > 0\),

\[
\sup_{t \in [-T,0]} \int_{W_i} |u(t, x)|^2 \, dm \leq \int_{[-T,0] \times W_i} |u(t, x)|^2 \, dmdt,
\]

where \(W_i\) is defined as in Theorem 6.3. By taking \(v\) to be \(\partial_t^k u(t, x)\) which by Corollary 4.2 are local weak solutions, and by using the inequality in Proposition 6.8 given in the next section, we can get that \(a_k(x)\) satisfies the \(L^2\) bound

\[
\int_{W_i} |a_k(x)|^2 \, dm \leq C_k e^{c_u(T+i+5k)}
\]

for any \(T > 0\), any \(i \in \mathbb{N}_+\). Here \(C_k\) is some constant that only depends on \(k\), and it can be taken as \(C_k = 400^k\).

**Remark 6.6.** Most of the conclusions in Theorems 6.2 and 6.3 are in the \(L^2\) sense. If the (essential) supremum of a local weak solution over each time-space cylinder can be controlled by the \(L^2\) integral of the local weak solution over the same cylinder, then we can make all conclusions in Theorem ?? (m-a.e.) pointwise conclusions. For example, some ultracontractivity property of the heat semigroup is sufficient for this purpose.

As a corollary for Theorem 6.2, we recover in the current setting the dimension result in [8] under an additional condition on the polynomial growth of the \(W^V_{V,i}\) sets. We first define the function spaces. For each \(d, b \in \mathbb{N}_+\), let \(P_{d,b}(X)\) denote the vector space of all ancient (local weak) solutions \(u\) of \((\partial_t + P)u = 0\) on \((-\infty, c) \times X\) for some \(c > 0\), that satisfy for some precompact open set \(V \Subset X\), for any \(n, i \in \mathbb{N}_+\), there exists some constant \(C_{u,V,i} > 0\), such that

\[
\sup_{[-T,0] \times W^V_i} |u(t, x)| \leq C_{u,V,i} \max \{T^d, n^b \}.
\]

(6.5)

Let \(\mathcal{H}_{d,b}(X)\) denote the vector space of all local weak solutions \(v\) of \(Pv = 0\) on \(X\) with polynomial growth bound

\[
\sup_{x \in W^V_i} |v(x)| \, dm \leq D_{v,V,i}n^b.
\]

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Here $V$ is the same precompact open set as above, and $D_{v,\iota} > 0$ is some constant for any $i \in \mathbb{N}_+$.

**Corollary 6.7.** Under the hypotheses of Theorem 6.2, and further assume that for some precompact open set $V \Subset X$, for any $n, i \in \mathbb{N}_+$, the sets $W_{V,i}^n$ satisfy some polynomial volume growth bound

$$m(W_{V,i}^n) \leq E_{V,i} n^a$$

where $E_{V,i}, a > 0$ are constants. Then

$$\dim \mathcal{P}_{d,b}(X) \leq (d+1) \dim \mathcal{H}_b(X).$$

**Proof.** Take any $u \in \mathcal{P}_{d,b}(X)$. Note that (6.5) and (6.6) together imply the $L^2$ growth condition (6.2) with $d_u = d$, $b_u = b + \frac{a}{2}$. Hence by Theorem 4.1, $u$ is a polynomial in time with $\partial^k_t u = 0$ for $k > d_u = d$. As in Remark 6.4, denote $u_k(x) = \sum_{j=0}^d b_j^k u(t_j, x)$. By the discussion in [8], for any fixed $t_0, t_1, \cdots, t_d \in [-\frac{1}{2}, 0]$ that are distinct, there exist numbers $b_j^k \geq 0$ such that for any $0 \leq k \leq d$,

$$u_k(x) = \sum_{j=0}^d b_j^k u(t_j, x).$$

Since all $|t_j| < 1$, and $u \in \mathcal{P}_{d,b}(X)$, for any precompact open set $V \Subset X$, any $i, n \in \mathbb{N}_+$,

$$\sup_{x \in W_{V,i}^n} |u_k(x)| \leq \max_{0 \leq j \leq d} |b_j^k| \cdot C_{u,V,i} n^b.$$ 

This implies that $u_k \in H_b(X)$. And by the arguments in [8], it follows that

$$\dim \mathcal{P}_{d,b}(X) \leq (d+1) \dim \mathcal{H}_b(X).$$

We make some final remarks about the two assumptions on existence of cut-off functions, Assumption 6.1 and Assumption 3.1.

First, Assumption 3.1 focuses on for any fixed pair of open sets $V \Subset U$, in particular they could be very close to each other, for any small $C_1$, the existence of a cut-off function for the pair $V \subset U$ that satisfies (3.1). There $C_2$ depends on $C_1, U, V$ and is usually a large number when $C_1$ is small and $U, V$ are close. And the cut-off function can be intuitively thought of as a steep function. In contrast, in Assumption 6.1 the focus is for any fixed beginning set $V \Subset X$ and fixed $C_1$, for small $C_2$ ($C_2 = \frac{1}{8}$ for large $n$), the existence of an exhaustion and cut-off functions for each pair of adjacent open sets. Intuitively, for large $n$, the sets in the exhaustion are far apart, and the cut-off functions have flat shapes.

Regarding the validity of Assumption 6.1, we remark that in general Dirichlet spaces that have some notion of distance that interacts well with the energy measure this assumption is satisfied. Roughly speaking, for large $n$, to find $W_{V,i}^n$’s and $\varphi_i$’s, we just require $W_{V,i}^n$ and the complement of $W_{V,i+1}^n$ to be separated by a large enough distance. For example, consider a Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ that admits “nice metric cut-off functions”, namely, there exists

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some distance \( d \) that defines the same topology of \( X \), such that for any pair of open sets \( V \subset U \), for any \( 0 < C_1 < 1 \), there exists some nice cut-off function \( \varphi \) satisfying for any \( v \in \mathcal{F} \),

\[
\int_X v^2 d\Gamma(\varphi, \varphi) \leq C_1 \int_X \varphi^2 d\Gamma(v, v) + C(C_1) \cdot d(V, U_c)^{-\beta} \int_{\text{supp}(\varphi)} v^2 dm,
\]

where \( C(C_1) \) is some positive function of \( C_1 \). Assume \( V \subset B(0; R) \). Then we can take \( W_{i, j} = B(x_0; R + ain^{1/\beta}) \), for any \( a \) satisfying \( a^\beta \geq C(\frac{1}{136}) \). Concretely,

1. when the Dirichlet space admits a nice intrinsic distance, it is a special case of the case above and \( \beta = 2 \);

2. when the Dirichlet space is the standard Dirichlet form on the Sierpinski gasket, and \( d \) is the Euclidean metric, the discussion above applies with \( \beta = \frac{\log 5}{\log 2} \), which is the walk dimension \( d_w \) of the Sierpinski gasket.

### 6.2 Proof of Theorem 6.2 and Theorem 6.3

#### 6.2.1 Overview and a key estimate

There are two difficulties in generalizing the structure results on ancient heat equation solutions to the setting of Dirichlet spaces. The first difficulty is introduce proper assumptions on the existence of cut-off functions in order to adapt estimates such that

\[
| \int fg \nabla v \cdot \nabla w \, dx | \leq \left( \int (fg)^2 \, dx \right)^{1/2} \left( \int |\nabla v|^2 |\nabla w|^2 \right)^{1/2}
\]

to the setting of energy measures, especially when the energy measure is singular with respect to the measure \( m \) in the metric measure space \((X, m)\). The more essential difficulty is about whether the time derivatives of an ancient local weak solution is still an ancient weak solution, and this is addressed by the main part of this paper with an affirmative answer (Corollary 4.2).

In this section we state the key estimate and use it to prove the two theorems stated in the previous section. In the following proposition we do not need the Dirichlet space to satisfy Assumption 3.1. The estimate is about bounding the \( L^2 \) integral of time derivatives of an ancient solution \( u \) over some time-space cylinder by the \( L^2 \) integral of \( u \) over some larger time-space cylinder, where the spatial sets are ones in an exhaustion of \( X \). Here we treat the exhaustion with less precision in the sense that we do not specify an initial precompact open set \( V \), and instead of specifying the constant \( \frac{1}{n} \) in (6.1) of Assumption 6.1 we consider an exhaustion with cut-off functions to correspond to constants \( C_1 \) and \( C \), in other words, for some exhaustion denoted by \( \{ W_i \} \) together with cut-off functions \( \{ \varphi_i \} \), each \( \varphi_i \) is a cut-off function for the pair \( \overline{W_i} \subset W_{i+1} \), and satisfies for any \( v \in \mathcal{F} \),

\[
\int_X v^2 d\Gamma(\varphi_i, \varphi_i) \leq C_1 \int_X \varphi_i^2 d\Gamma(v, v) + C \int_{\text{supp}(\varphi_i)} v^2 dm.
\]
We call such \( \{ W_i \} \), \( \{ \varphi_i \} \) an exhaustion of \( X \) corresponding to \( C_1, C \).

**Proposition 6.8.** Let \( (X,m) \) be a metric measure space, and let \( (E,F) \) be a symmetric, regular, strongly local Dirichlet form on \( X \). When \( X \) is not compact, further assume the Dirichlet space \( (X,E,F) \) satisfies Assumption 6.1. Let \( (H_t)_{t>0} \) and \( -P \) be the corresponding semigroup and generator. Let \( b > 0 \) be an arbitrary number. Let \( u \) be a local weak solution of \( (\partial_t + P)u = 0 \) on \( (-\infty, b) \times X \), i.e. \( u \) is an ancient solution. Let \( J = [c,0] \subset (-\infty,b) \) be any finite subset with a fixed right end. Take \( C_1 = \frac{1}{136} \) and fix an arbitrary \( C > 0 \). Let \( \{ W_i \}_{i \in \mathbb{N}_+}, \{ \varphi_i \}_{i \in \mathbb{N}_+} \) be an exhaustion corresponding to \( C_1, C \). Then for any \( k \in \mathbb{N}_+ \), \( i \in \mathbb{N}_+ \),

\[
\int_J \int_{W_i} (\partial_t^k u)^2 \, dmdt \leq \left( 100 \left( C + \frac{1}{r} \right)^2 \right)^k \int_{J-2k} \int_{W_{1+3k}} u^2 \, dmdt.
\]

Here \( J_- := [c-s,0] \) for any \( s > 0 \).

We now use this proposition to prove Theorem 6.2 and Theorem 6.3.

### 6.2.2 Proof of Theorem 6.2

To show \( \partial_t^k u = 0 \) for \( k \) large enough, we show that the \( L^2 \) integral of such \( \partial_t^k u \) over any large open set is zero. For any large open set \( [-T,0] \times V \) where \( V \subset X \), recall that for such a precompact open set \( V \) and for any \( n \in \mathbb{N}_+ \), Assumption 6.1 guarantees the existence of an exhaustion \( \{ W_{V,i}^n \}_{i \in \mathbb{N}_+} \) of \( X \) with cut-off functions \( \varphi_i \) for each pair \( W_{V,i}^n \subset W_{V,i+1}^n \) corresponding to \( C_1 = \frac{1}{136} \) in (6.1). Applying Proposition 6.8 to \( \{ W_{V,i}^n \} \) and \( \{ \varphi_i \} \), note that \( C = \frac{1}{n} \). For \( i = 1 \), for \( J = [-T,0] \subset (-\infty,0] \), by taking \( r = n \), we have

\[
\int_J \int_{W_{V,1}^n} (\partial_t^k u)^2 \, dmdt \leq \left( 400 \frac{1}{n^2} \right)^k \int_{J-2k} \int_{W_{V,1+3k}^n} u^2 \, dmdt.
\]

Our strategy is to let \( n \) tend to infinity in the above inequality, note that \( i \) is fixed as \( i = 1 \). Here letting \( n \) tend to infinity has the effect of taking adjacent balls (i.e. \( W_{V,i}^n \) v.s. \( W_{V,i+1}^n \)) with bigger and bigger distance from each other, so that the right-hand side, which is bounded by some rational function in \( n \), tend to zero as \( n \) tends to infinity.

More precisely, we have

\[
\int_{[-T,0] \times W_{V,1}^n} (\partial_t^k u)^2 \, dmdt \leq \left( \frac{400^k C_{u,V,1+3k} \max \left\{ \left( \lceil T \rceil + 2kn \right)^{d_u+\frac{1}{2}}, n^{b_u} \right\} }{n^{2k}} \right)^2.
\]

Since for any \( 2k > d_u + \frac{1}{2} + b_u \), the right-hand side tends to 0 as \( n \) tends to infinity (\( i, k \) are fixed), by discussion above, we conclude that for \( 2k > d_u + \frac{1}{2} + b_u \),

\[
\partial_t^k u = 0.
\]
This concludes the proof that $u$ is a polynomial in $t$. Then applying the growth bound (6.2) to $u$ in the explicit polynomial form, we conclude that $\partial^k_t u = 0$ for $k > d_u$.

6.2.3 Proof for Theorem 6.3

By Taylor expansion formula (expansion in $t$), for any fixed $x$, any $t < 0$,

$$u(t, x) = \sum_{i=0}^{k} \frac{\partial^i u(0, x)}{i!} t^i + \int_{0}^{t} \partial^{k+1}_s u(s, x) \frac{(t-s)^k}{k!} ds.$$ 

So to prove the statement in Theorem 6.3, we want to prove for any precompact open set $V$,

$$\int_{V} \left( \int_{0}^{t} \partial^{k+1}_s u(s, x) \frac{(t-s)^k}{k!} ds \right)^2 dm(x) \rightarrow 0 \quad (6.7)$$

as $k \to \infty$, uniformly in $t \in [a, 0]$ for any fixed $a < 0$. We can first bound the integral by

$$\left| \int_{V} \left( \int_{0}^{t} \partial^{k+1}_s u(s, x) \frac{(t-s)^k}{k!} ds \right)^2 dm(x) \right| \leq \frac{1}{|t| (k!)} \left| \int_{V} \int_{0}^{t} \left( \partial^{k+1}_s u(s, x) \cdot (t-s)^k \right)^2 ds dm(x) \right| \leq \frac{|t|^k}{|t| (k!)} \int_{V} \int_{0}^{t} \left( \partial^{k+1}_s u(s, x) \right)^2 ds dm.$$ 

(6.8)

Recall the notation introduced in the statement of Theorem 6.3, i.e. $W_j := W_{V_{0,j}}$ for some fixed $V_0 \in X$. Intuitively by fixing $n = 1$ (or any fixed integer), we are looking at open sets whose sizes grow linearly. Since $V \in X$, and $\{W_j\}$ is an exhaustion of $X$, there exists some $j_0$ such that for all $j \geq j_0$, $V \subset W_j$.

By Proposition 6.8 for any $r > 0$,

$$\int_{t}^{0} \int_{W_j} \left( \partial^{k+1}_s u(s, x) \right)^2 dm dt \leq \left( 100 \left( 1 + \frac{1}{r} \right)^2 \right)^{k+1} \int_{t}^{0} \int_{s(k+1)}^{W_{j+3(k+1)}} u(s, x)^2 dm dt.$$ 

And by the exponential growth assumption (6.3) on $u$, we conclude that (take for example $r = 1$) for any $t \in [a, 0]$, where $a$ is any fixed negative constant,

$$\int_{t}^{0} \int_{W_j} \left( \partial^{k+1}_s u(s, x) \right)^2 dm dt \leq (400)^{k+1} e^{c_u(|a|+j+3(k+1))}.$$ 

Substituting this bound back to (6.8), note that $V \subset W_j$,

$$\left| \int_{V} \left( \int_{0}^{t} \partial^{k+1}_s u(s, x) \frac{(t-s)^k}{k!} ds \right)^2 dm(x) \right| \leq \frac{|a|^k}{k!} \cdot (400)^{k+1} e^{c_u(|a|+j+3(k+1))} \rightarrow 0 \quad (k \to \infty).$$
This completes the proof of (6.4), and shows the convergence is uniform for 
\( t \in [a, 0] \) for any \( a < 0 \).

### 6.3 Proof of the key estimate

In this subsection we give the proof for Proposition 6.8. Proposition 6.8 follows from the following proposition by iteration.

**Proposition 6.9.** Let \((X, m)\) be a metric measure space, and let \((\mathcal{E}, \mathcal{F})\) be a symmetric, regular, strongly local Dirichlet form on \( X \). When \( X \) is not compact, further assume the Dirichlet space \((X, \mathcal{E}, \mathcal{F})\) satisfies Assumption 6.1. Let \((H_t)_{t > 0}\) and \( -P \) be the corresponding semigroup and generator. Let \( b > 0 \) be an arbitrary number. Let \( u \) be a local weak solution of \((\partial_t + P)u = 0\) on \(( -\infty, b) \times X \). Let \( J = [c, 0] \subset (-\infty, b) \) be any finite subset with a fixed right end. Take \( C_1 = \frac{1}{16} \) and fix an arbitrary \( C > 0 \). Denote an exhaustion corresponding to \( C_1, C \) by \( \{W_i\}_{i \in N_+}, \{\varphi_i\}_{i \in N_+} \). Then for any \( r > 0 \), there exist constants \( K_1, K_2 \) (dependent on \( C \) and \( r \)) such that for any \( i \in N_+ \),

\[
\int_{J-s} \int_{W_i} (\partial_t u)^2 \, dmdt \leq K_1(C, r) \int_{J-r} \int_{W_{i+2}} d\Gamma(u, u) \, dt \\
\leq K_2(C, r) \int_{J-2r} \int_{W_{i+3}} u^2 \, dmdt.
\]

Here \( J-s := [c - s, 0] \) for any \( s > 0 \), and

\[
K_1(C, r) = 8 \left( C + \frac{1}{r} \right), \quad K_2(C, r) = 100 \left( C + \frac{1}{r} \right)^2.
\]

To prove this proposition, we first give a technical lemma.

**6.3.1 A technical lemma**

Last we state and prove a technical lemma we used in the proof for Proposition 6.9.

**Lemma 6.10.** Let \( \varphi \) be any nice cut-off function that satisfies for any \( v \in \mathcal{F}_{loc}(X) \),

\[
\int_X v^2 \, d\Gamma(\varphi, \varphi) \leq C_1 \int_X \varphi^2 \, d\Gamma(v, v) + C_2 \int_{\text{supp}(\varphi)} v^2 \, dm,
\]

where \( C_1, C_2 \) are some constants associated with \( \varphi \). Assume \( C_1 < \frac{1}{68} \) (this number is not important). Then

\[
\int_X \varphi^2 v^2 \, d\Gamma(\varphi, \varphi) \leq \frac{C_2}{1 - 68C_1} \int \varphi^2 v^2 \, dm.
\]
The last inequality says when there is the same cut-off function with bounded energy in both the integrand and in the energy measure, the net effect is the same as having a cut-off function with bounded gradient in the energy measure. This is easy to check for the special case $\int_X \phi^2 \, d\Gamma(\varphi, \varphi)$. And here we generalize this observation for $\int_X \phi^2 v^2 \, d\Gamma(\varphi, \varphi)$.

Proof. First, since $\phi$ is a nice cut-off function with associated constants $C_1, C_2$,

$$
\int_X \phi^2 v^2 \, d\Gamma(\varphi, \varphi) \leq C_1 \int_X \phi^2 \, d\Gamma(\varphi v, \varphi v) + C_2 \int \phi^2 v^2 \, dm. \quad (6.9)
$$

The first term $\int_X \phi^2 \, d\Gamma(\varphi v, \varphi v)$ equals

$$
\int_X \phi^2 \, d\Gamma(\varphi v, \varphi v) = \int_X \phi^2 \, d\Gamma(\varphi^2 v, v) + \int_X \phi^2 v^2 \, d\Gamma(\varphi, \varphi)
= \int_X d\Gamma(\varphi^2 v, \varphi) - \int_X 2\varphi v \, d\Gamma(\varphi^2 v, \varphi) + \int_X \phi^2 v^2 \, d\Gamma(\varphi, \varphi), \quad (6.10)
$$

where the middle term is bounded by

$$
\left| -\int_X 2\varphi v \, d\Gamma(\varphi^2 v, \varphi) \right| \leq b \int_X \phi^2 v^2 \, d\Gamma(\varphi, \varphi) + \frac{1}{b} \int_X d\Gamma(\varphi^2 v, \varphi^2 v)
$$

for any $b > 0$. Substituting this bound back in (6.10), we get (for any $a > 0$)

$$
\int_X \phi^2 \, d\Gamma(\varphi v, \varphi v) \leq (1 + \frac{1}{b}) \int_X d\Gamma(\varphi^2 v, \varphi^2 v) + (1 + b) \int_X \phi^2 v^2 \, d\Gamma(\varphi, \varphi)
\leq (1 + \frac{1}{b}) \cdot \left[ 2a \int_X \phi^2 \, d\Gamma(\varphi v, \varphi v) + \frac{2}{a} \int_X \phi^2 v^2 \, d\Gamma(\varphi, \varphi) \right] + (1 + b) \int_X \phi^2 v^2 \, d\Gamma(\varphi, \varphi).
$$

If we take $b = 1$, and $a = \frac{1}{8}$, then

$$
\int_X \phi^2 v^2 \, d\Gamma(\varphi, \varphi) \leq 68 \int_X \phi^2 v^2 \, d\Gamma(\varphi, \varphi).
$$

Then by (6.9),

$$
\int_X \phi^2 v^2 \, d\Gamma(\varphi, \varphi) \leq 68C_1 \int_X \phi^2 v^2 \, d\Gamma(\varphi, \varphi) + C_2 \int \phi^2 v^2 \, dm,
$$

and by taking $C_1$ small enough, we have

$$
\int_X \phi^2 v^2 \, d\Gamma(\varphi, \varphi) \leq \frac{C_2}{1 - 68C_1} \int \phi^2 v^2 \, dm.
$$

□
6.3.2 Proof of Proposition 6.9

Let \( l \) be a bump function in time (on \((-\infty,0]\)) that equals to 1 on \([c,0]\) and has compact support in \((c-r,0]\), with \(||l'||_\infty \leq \frac{2}{3}\). It can be easily extended into a function in \(C_c^\infty(\mathbb{R})\), in the following we only use its part on \((-\infty,0]\). By Assumption 6.1, for each \( i \) there exists a nice cut-off function \( \phi_i \) for the pair \( W_i \subset W_{i+1} \) such that for any \( v \in \mathcal{F} \),

\[
\int_X v^2 d\Gamma(\phi_i, \phi_i) \leq C_1 \int_X \phi^2_i d\Gamma(v, v) + C \int_{\text{supp}(\phi_i)} v^2 dm. \tag{6.11}
\]

(When \( X \) is compact, we can take all sets \( W_i \) to be \( X \) and \( \phi_i \) to be the constant function 1.) Here \( C_1 \) is \( \frac{1}{136} \), and the actual value is not important. We keep \( C_1 \) in our computations and plug in \( C_1 = \frac{1}{136} \) later. Note that the product \( \phi_i(x)l(t) \) is a nice cut-off function for the pair \((c,0] \times W_i \in (c-r,0] \times W_{i+1}\).

We first note that

\[
\int_{J_{-r}} \int_{W_{i+1}} 2u \partial_t u \phi^2_i l^2 \ dm dt + \int_{J_{-r}} \int_{W_{i+1}} u^2 \phi^2_i (l^2)' \ dm dt \\
= \int_{J_{-r}} \partial_t \left( \int_{W_{i+1}} u^2 \phi^2_i l^2 \right) dt = \left( \int_{W_{i+1}} u^2 \phi^2_i l^2 \right) \bigg|_{t=0} \geq 0 \tag{6.12}
\]

On the other hand, since \( u \) is an ancient local weak solution of the heat equation \((\partial_t + P)u = 0\),

\[
\int_{J_{-r}} \int_{W_{i+1}} 2u \partial_t u \phi^2_i l^2 \ dm dt = -2 \int_{J_{-r}} l^2 E(u, u \phi^2_i) dt \\
= -2 \int_{J_{-r}} \int_{W_{i+1}} \phi^2_i l^2 \ d\Gamma(u, u) dt - 4 \int_{J_{-r}} \int_{W_{i+1}} \phi_i u \ d\Gamma(\phi_i, u) dt \\
\leq - \int_{J_{-r}} \int_{W_{i+1}} \phi^2_i l^2 \ d\Gamma(u, u) dt + 4 \int_{J_{-r}} \int_{W_{i+1}} u^2 l^2 \ d\Gamma(\phi_i, \phi_i) dt,
\]

where the last line is by Cauchy-Schwartz inequality. Combining (6.11), (6.12), and (6.11), we get

\[
(1 - 4C_1) \int_{J_{-r}} \int_{W_{i+1}} l^2 \phi^2_i d\Gamma(u, u) dt \leq (4C + \frac{4}{r}) \int_{J_{-r}} \int_{W_{i+1}} u^2 dm dt.
\]

This implies

\[
\int_{J_{-r}} \int_{W_{i+1}} d\Gamma(u, u) dt \leq (1 - 4C_1)^{-1} \left( 4C + \frac{4}{r} \right) \int_{J_{-r}} \int_{W_{i+1}} u^2 dm dt. \tag{6.13}
\]

Next we estimate the \( L^2 \) norm of \( \partial_t u \), which by Corollary 4.2 is also a local weak solution on \((-\infty, b) \times X\). We have

\[
\int_{J_{-r}} \int_{W_{i+1}} (\partial_t u \phi_i l)^2 dm dt = - \int_{J_{-r}} l^2 E(u, \partial_t u \phi^2_i) dt,
\]
and by the product rule and chain rule, this equals

$$- \int_{J_{\rightarrow}} l^2 \int_{W_{i+1}} \varphi_i^2 d\Gamma(u, \partial_t u) dt - \int_{J_{\rightarrow}} l^2 \int_{W_{i+1}} 2\varphi_i \partial_t u d\Gamma(u, \varphi_i) dt.$$  \hspace{1cm} (6.14)

We will show that

$$\int_{W_{i+1}} \varphi_i^2 d\Gamma(\partial_t u, u) = \frac{1}{2} \partial_t \left( \int_{W_{i+1}} \varphi_i^2 d\Gamma(u, u) \right)$$

to replace the first term in (6.14). This follows from the estimate

$$\left| \left( \int_X f d\Gamma(v, v) \right)^{1/2} - \left( \int_X f d\Gamma(w, w) \right)^{1/2} \right| \leq \left( \int_X f d\Gamma(v - w, v - w) \right)^{1/2}$$

where $f$ is any nonnegative, bounded, Borel function, and $v, w \in F$ (cf. Chapter 3 in [14]). This implies that if $v_n \to v$ in $E_1$ norm (or just in the $E$-energy), then

$$\lim_{n \to \infty} \int_X f d\Gamma(v_n, w) = \int_X f d\Gamma(v, w).$$

Here $w \in F$, and $f$ is as above. By taking $v_n(x) := \frac{u(t+1/n,x) - u(t,x)}{1/n}, \ w(x) := u(t,x)$, and $f := \varphi_i^2$, we conclude that for any $t \in (-\infty, b)$,

$$\frac{1}{2} \partial_t \left( \int_{W_{i+1}} \varphi_i^2 d\Gamma(u, u) \right) = \int_{W_{i+1}} \varphi_i^2 d\Gamma(\partial_t u, u).$$

Thus the first term in (6.14) equals

$$- \int_{J_{\rightarrow}} l^2 \int_{W_{i+1}} \varphi_i^2 d\Gamma(u, \partial_t u) dt = \frac{1}{2} \int_{J_{\rightarrow}} l^2 \partial_t \left( \int_{W_{i+1}} \varphi_i^2 d\Gamma(u, u) \right) dt$$

$$= \frac{1}{2} \int_{J_{\rightarrow}} \partial_t \left( \int_{W_{i+1}} \varphi_i^2 d\Gamma(u, u) \right) dt + \frac{1}{2} \int_{J_{\rightarrow}} \left( l^2 \right)^{1/2} \int_{W_{i+1}} \varphi_i^2 d\Gamma(u, u) dt$$

$$\leq \frac{1}{2} \int_{J_{\rightarrow}} \left( l^2 \right)^{1/2} \int_{W_{i+1}} \varphi_i^2 d\Gamma(u, u) dt.$$  \hspace{1cm} (6.15)

To estimate the second term in (6.14), note that there is a nice cut-off function $\phi_{i+1}$ for the pair $W_{i+1} \subset W_{i+2}$ such that for any $v \in F$,

$$\int_X v^2 d\Gamma(\phi_{i+1}, \phi_{i+1}) \leq C_1 \int_X \varphi_{i+1}^2 d\Gamma(v, v) + C \int_{\text{supp}(\phi_{i+1})} v^2 dm.$$  \hspace{1cm} (6.16)

In particular, $\phi_{i+1} \equiv 1$ on $W_{i+1}$. By inserting in $\phi_{i+1}$, we have

$$- \int_{J_{\rightarrow}} l^2 \int_{W_{i+1}} 2\varphi_i \partial_t u d\Gamma(u, \varphi_i) dt \leq - \int_{J_{\rightarrow}} l^2 \int_{W_{i+2}} 2\varphi_i \phi_{i+1} \partial_t u d\Gamma(u, \varphi_i) dt$$

$$\leq \epsilon \int_{J_{\rightarrow}} l^2 \int_{W_{i+2}} \phi_i^2 (\partial_t u)^2 d\Gamma(\phi_{i+1}, \phi_{i+1}) dt + \frac{1}{\epsilon} \int_{J_{\rightarrow}} l^2 \int_{W_{i+2}} \phi_{i+1}^2 d\Gamma(u, u) dt$$
for any $\epsilon > 0$. By a Lemma below (Lemma 6.10), we have

$$\int_{W_{i+2}} \varphi_i^2 (\partial_t u)^2 \ d\Gamma(\varphi_i, \varphi_i) \leq \frac{C}{1 - 68C_1} \int_{W_{i+2}} \varphi_i^2 (\partial_t u)^2 \ dm,$$

thus

$$\left| - \int_{J_{-r}} l^2 \int_{W_{i+1}} 2\varphi_i \partial_t u \ d\Gamma(u, \varphi_i) \ dt \right| \leq \epsilon \frac{C}{1 - 68C_1} \int_{J_{-r}} l^2 \int_{W_{i+2}} \varphi_i^2 (\partial_t u)^2 \ dmdt$$

$$+ \frac{1}{\epsilon} \int_{J_{-r}} l^2 \int_{W_{i+2}} \varphi_{i+1}^2 \ d\Gamma(u, u) \ dt. \quad (6.17)$$

Now we plug in $C_1 = \frac{1}{132}$, and take $\epsilon = \frac{1}{4C}$, then by (6.14), (6.15), and (6.17),

$$\int_{J_{-r}} \int_{W_{i+1}} (\partial_t u \varphi_i l)^2 \ dmdt$$

$$= - \int_{J_{-r}} l^2 \int_{W_{i+1}} \varphi_i^2 \ d\Gamma(u, \partial_t u) - \int_{J_{-r}} l^2 \int_{W_{i+1}} 2\varphi_i \partial_t u \ d\Gamma(u, \varphi_i) \ dt$$

$$\leq \frac{1}{2} \int_{J_{-r}} (l^2)' \int_{W_{i+1}} \varphi_i^2 \ d\Gamma(u, u) \ dt + \frac{1}{2} \int_{J_{-r}} l^2 \int_{W_{i+2}} \varphi_i^2 (\partial_t u)^2 \ dmdt$$

$$+ 4C \int_{J_{-r}} l^2 \int_{W_{i+2}} \varphi_{i+1}^2 \ d\Gamma(u, u) \ dt.$$

Recall that $\text{supp} \{\varphi_i\} \subset W_{i+1}$, we hence have

$$\int_{J_{-r}} \int_{W_{i+1}} (\partial_t u \varphi_i l)^2 \ dmdt$$

$$\leq \int_{J_{-r}} (l^2)' \int_{W_{i+1}} \varphi_i^2 \ d\Gamma(u, u) \ dt + 8C \int_{J_{-r}} l^2 \int_{W_{i+2}} \varphi_{i+1}^2 \ d\Gamma(u, u) \ dt.$$

And this implies

$$\int_{J} \int_{W_i} (\partial_t u)^2 \ dmdt \leq \left( 8C + \frac{4}{r} \right) \int_{J_{-r}} \int_{W_{i+2}} \ d\Gamma(u, u) \ dt. \quad (6.18)$$

Applying (6.13) for $J_{-r}$ and $W_{i+2}$ with $C_1 = \frac{1}{136}$, and combining (6.13) and (6.18), we obtain

$$\int_{J} \int_{W_i} (\partial_t u)^2 \ dmdt \leq \left( 8C + \frac{4}{r} \right) \int_{J_{-2r}} \int_{W_{i+3}} \ d\Gamma(u, u) \ dt$$

$$\leq \left( 8C + \frac{4}{r} \right) \frac{34}{33} \left( 4C + \frac{4}{r} \right) \int_{J_{-2r}} \int_{W_{i+3}} u^2 \ dmdt.$$
Denote $K_1(C, r) := 8 \left( C + \frac{1}{r} \right) > 8C + \frac{4}{r}$ and $K_2(C, r) := 100 \left( C + \frac{1}{r} \right)^2 > K_1(C, r) \frac{32}{49} \left( 4C + \frac{4}{r} \right)$. This completes the proof for Proposition 6.9. Note that by taking $C$ small and $r$ large enough, we can make the coefficients $K_1(C, r)$ and $K_2(C, r)$ as small as needed.

Straightforward iterations lead to Proposition 6.8.

7 Examples

In this section we list examples to which our theorems apply. We group them according to the types of nice cut-off functions they admit. Note that the properties we require on the nice cut-off functions only involve the energy measures associated with the Dirichlet form (the strongly local part of the Dirichlet form), so in the following we describe examples of strongly local Dirichlet forms, but our theorems apply to any local Dirichlet form with their strongly local part belonging to the following examples.

7.1 Dirichlet spaces with good intrinsic distance

In [28], Sturm showed that in a symmetric, strongly local, regular Dirichlet space, when the topology induced by the intrinsic distance (3.5), that is,

$$
\rho_X(x, y) = \sup \{ \varphi(x) - \varphi(y) \mid \varphi \in \mathcal{F}_{\text{loc}}(X) \cap C(X), \ d\Gamma(\varphi, \varphi) \leq dm \},
$$

is equivalent to the original topology on $X$, one can use the intrinsic distance to construct nice cut-off functions with bounded gradient. More precisely, for $V \subset U \subset X$, define

$$
\eta(x) := \frac{\left( \frac{1}{\sqrt{2}} \rho_X(V, U^c) - \rho_X(x, V) \right)}{\frac{1}{\sqrt{2}} \rho_X(V, U^c)}.
$$

Clearly $\eta = 1$ on $V$ and supp{$\eta$} $\subset$ $U$. Further, $\eta$ is in $\mathcal{F}_{\text{loc}}(X) \cap C(X)$, and

$$
d\Gamma(\eta, \eta) \leq \frac{2}{d(V, U^c)^2} dm. \quad (7.1)
$$

These results are from Lemma 1.9 in [28]. It clearly follows that such Dirichlet spaces satisfy Assumptions 3.1 and 6.1 (pick the exhaustion $\{W_{r_i}^n\}_{i=1}^\infty$ to be given by balls with radii $r_i$ that increase fast enough). By Lemma 8.1 these Dirichlet spaces clearly satisfy the $L^2$ Gaussian type upper bound. Thus all results in this paper apply to this type of examples which include:

1. Weighted Riemannian manifold with Dirichlet form associated with any uniformly elliptic operator with bounded measurable coefficients. See, e.g. [26]. This includes the example we described in the Introduction, and we remark that all results in this paper hold when the operator is only locally uniformly elliptic.
(2) Riemannian polyhedra under minimal local assumptions (cf. [12, 25] and [7]).

(3) Alexandrov spaces and their Dirichlet space structures as considered for instance in [24, 20].

7.2 Fractal type Dirichlet spaces

For fractal spaces, Assumption 3.1 is a nontrivial hypothesis to check. It is well known that in many fractal spaces the only functions in $\mathcal{F}_{loc}(X) \cap C(X)$ are constant functions (cf. e.g. [19]), so fractal spaces in general do not possess cut-off functions with bounded gradient. More generally, in a recent paper [18], it was shown that for a very general class of Dirichlet spaces, two-sided off-diagonal heat kernel estimates with walk-dimension strictly larger than two implies the singularity of the energy measures with respect to the symmetric measure.

On the other hand, many fractal spaces admit cut-off functions satisfying the inequality (3.2) in Assumption 3.1. For example, the Sierpinski gasket and its non-compact extension as in the following pictures both satisfy Assumption 3.1. (The picture on left ($SG$) is from Wikipedia, and the picture on right ($ISG$) is obtained by shifting copies of $SG$.)

We remark that the existence of cut-off functions satisfying (3.2) on such examples is highly nontrivial, and although their existence is known (cf. [1]), there is in general no direct geometric construction of such cut-off functions. For example, in [1] the authors showed that fractal spaces that satisfy some version of parabolic Harnack inequality must admit cut-off functions that satisfy some more specific version of the inequality (3.1). A typical more specific dependence of $C_2$ on $C_1, U, V$ is that $C_2 \sim C_1^{-\alpha}d_X(V, U^c)^{-\beta}$ for some distance $d_X(V, U^c)$ between $V, U^c$, and for some constants $\alpha, \beta > 0$. In [2], the authors proved that this dependence is in fact a fairly general case, that for a class of functions called regular scale functions, denoted by $\Psi$, there exists some distance $d_\Psi$ that defines the same topology (being so-called quasisymmetric with the original distance of the fractal space), such that the fractal space admits cut-off function satisfying the so-called cut-off energy inequality $CS(\Psi)$ (a special form of (3.1)).
where $C_2$ is expressed in terms of the $\Psi$ function, more general than powers), if and only if the fractal space admits cut-off functions satisfying (3.1) with $C_2 \sim C_1^{-\alpha} d_\Psi(V,U)^{-\beta}$, which are power functions with respect to the distance $d_\Psi$. For all these fractal spaces, Lemma 8.1 guarantees they satisfy the $L^2$ Gaussian type upper bound. And because of the distance $d_X$, similar to the first type of examples, we can check that Assumption 6.1 is satisfied. Hence all results in this paper apply.

7.3 Infinite product of Dirichlet spaces of the first two types

The first examples we have in mind for this type of examples are the infinite dimensional torus $\mathbb{T}^\infty$ and the infinite product of Sierpinski gaskets $\mathcal{SG}^\infty$, the first one being a special case of the class of locally compact connected metrizable (infinite dimensional) groups, cf. [6], and the second one the simplest of the infinite product of compact fractal spaces. A general treatment of the elliptic diffusion on (compact) infinite product spaces like $\mathbb{T}^\infty$ is [3], and their results apply more generally to anomalous diffusion on infinite products of (compact) fractal spaces too. To have some noncompact examples we can consider the Iwasawa’s example (cf. [17] [6]), or replace one piece of Sierpinski gasket in the product $\mathcal{SG}^\infty$ by the infinite Sierpinski gasket $\mathcal{ISG}$.

On a locally compact connected metrizable group $G$ that is unimodular, one usually starts with a heat (convolution) semigroup, or a (left-invariant) Laplacian of the form $L = -\sum a_{ij}X_iX_j$, where $(a_{ij})_{i,j=1}^\infty$ is symmetric and positive definite, and $\{X_i\}_{i=1}^\infty$ is a projective basis of the left-invariant vector fields on $G$ (in the projective Lie algebra of $G$), and then consider the associated (left-invariant) Dirichlet form. Depending on the coefficients, the Dirichlet form may or may not have nondegenerate intrinsic distance.

For general product spaces that have rougher differential structures, like $\mathcal{SG}^\infty$, it is easier and more convenient to consider only the “diagonal Dirichlet form”, namely, for any diagonal matrix $(a_{ii})_{i=1}^\infty$ with all $a_{ii} > 0$, consider

$$
\mathcal{E}(f,g) = \sum_{i=1}^\infty a_{ii} \int \mathcal{E}_i(f,g) d \left( \otimes m_j \right). 
$$

(7.2)

Here $\mathcal{E}_i$ stands for the standard Dirichlet form on the $i$th factor of $\mathcal{G}$, $m_j$ stands for the normalized Hausdorff measure on the $j$th factor of $\mathcal{SG}$, and $f,g$ are proper functions.

This third type of examples does not satisfy a property often satisfied in the previous two types of examples, namely, for these infinite dimensional spaces, the volume doubling property (local or global) cannot hold. In some cases these infinite product examples do possess nondegenerate intrinsic distance that defines the same topology (e.g. on $\mathbb{T}^\infty$ where the coefficient matrix for the Laplacian is diagonal and satisfies $\sum a_{ii}^{-2} < \infty$) in which case Assumption 3.1 and Assumption 6.1 follows. But, more generally, one can show that the cut-off function assumptions (Assumption 3.1 and Assumption 6.1) are satisfied using
the fact that each factor in the infinite product possess nice cut-off functions in
the sense required.

More precisely, since the product topology is generated by cylindric sets (sets
that are direct product of open sets of the first few factors, and the whole space
for all remaining factors), for pairs of cylindric sets it is easy to construct a nice
cut-off function taking product of nice cut-off functions for pairs of open sets
on the first few factors, namely,
\[ \varphi(x) := \prod_{i=1}^{N_V} \varphi_i(x_i). \] (7.3)

We verify this for the simpler case when the Dirichlet form is defined as in (7.2)
(for the group case this is when the coefficient matrix is diagonal, and there is
no drift part in the Laplacian).

Suppose \( \varphi_i(x_i) \) is a nice cut-off function on the \( i \)-th factor \( X_i \) of the infinite
product space \( X = \prod_i X_i \), satisfying for any \( v \in D(E_i) \),
\[ \int v^2 d\Gamma_i(\varphi_i, \varphi_i) \leq C_1 \int \varphi_i^2 d\Gamma(v, v) + C_2 \int_{\text{supp}(\varphi_i)} v^2 dm_i. \] (7.4)

Here \( \Gamma_i \) represents the energy measure on \( X_i \), and \( C_1, C_2 \) are the same for all
factors \( X_i \). Then for any \( f \in D(E) \), for the function \( \varphi \) defined as in (7.3),
\[ \int_X f^2 d\Gamma(\varphi, \varphi) \]
\[ = \sum_{i=1}^{\infty} a_{ii} \int_{X_i} \left( \int_{X_i} f^2 d\Gamma_i(\varphi_i, \varphi_i) \right) \left( \prod_{j \neq i, j=1}^{N_V} \varphi_j(x_j) \right)^2 d(\otimes_{j \neq i} m_j) \]
\[ \leq \sum_{i=1}^{\infty} a_{ii} \left[ C_1 \int_{X_i} \left( \int_{X_i} (\varphi_i)^2 d\Gamma_i(f, f) \right) \left( \prod_{j \neq i, j=1}^{N_V} \varphi_j(x_j) \right)^2 d(\otimes_{j \neq i} m_j) \right] + C_2 \int_{\text{supp}(\varphi)} f^2 dm \]

In the last line we bounded the product of \( \varphi_i \)'s by 1. Then note that since
\[ \int_X \varphi^2 d\Gamma(f, f) = \sum_{i=1}^{\infty} a_{ii} \int_{X_i} \int_{X_i} \varphi^2 d\Gamma(f, f) d(\otimes_{j \neq i} m_j), \]
we conclude
\[ \int_X f^2 d\Gamma(\varphi, \varphi) \leq C_1 \int_X \varphi^2 d\Gamma(f, f) + C_2 \int_{\text{supp}(\varphi)} f^2 dm. \]

Thus these infinite product spaces satisfy Assumption 3.1. Using the topological
basis of cylindric open sets, we can also easily check that these infinite product
spaces satisfy Assumption 6.1. And again by Lemma 8.1 these spaces satisfy the \( L^2 \) Gaussian type upper bound. We remark that here we do not have additional requirements on the coefficient matrix \((a_{ii})\) except that all \( a_{ii} > 0\), or for the infinite dimensional group case, that the coefficient matrix is positive definite.

**Remark 7.1.** On infinite dimensional compact groups, when the Laplacian \( L \) is bi-invariant, one can define more function spaces associated with \( L \) that capture the smoothness of functions, and define corresponding distributional solutions of the heat equation \( (\partial_t + L)u = 0 \). These are broader classes of solutions than the local weak solutions we consider in this paper. In the new settings one can consider the time regularity and other spatial regularity properties of the distributional solutions of the heat equation, under more assumptions on the associated heat (convolution) semigroup, cf. [5][4]. In a sequel paper we will show that for these bi-invariant Laplacians \( L \) and some other differential operators that have comparable Dirichlet forms, the distributional solutions are smooth, with repeated time and spatial derivatives that still belong to the function spaces associated with \( L \). These results provide generalizations of the results in [5] and describe hypoellipticity type properties of \( \partial_t + L \).

8 The weak Gaussian bound and other lemmas

8.1 The weak Gaussian bound

In this subsection we record a modification of the classical proof for \( L^2 \) Gaussian bound (when there are cut-off functions with bounded gradient) that proves an \( L^2 \) Gaussian type upper bound only assuming the existence of cut-off functions satisfying (3.1) with \( C_2(C_1, U, V) = C_1^{-\alpha}C(U, V) \) for some \( \alpha > 0 \). For references that discuss about stronger (sub)-Gaussian estimates under stronger assumptions, we mention [10][23]. The last part in this subsection about transitioning to estimates on derivatives of the heat semigroup is a straightforward modification of the methods in [9].

The following is the main lemma for \( L^2 \) Gaussian type upper bound. And the proof for it is very close to for example the beginning part of the proof in [23].

**Lemma 8.1.** Suppose the Dirichlet space \((X, m, \mathcal{E}, \mathcal{F})\) satisfies Assumption 3.1 (see Remark 3.2 for its equivalent form), with dependence \( C_2 = C_1^{-\alpha}C(U, V) \) for some \( \alpha > 0 \), \( C(U, V) > 0 \), for any precompact open sets \( U, V \subset X \) with disjoint closures. Then for any \( f, g \in L^2(X) \) with \( \text{supp} \{f\} \subset U, \text{supp} \{g\} \subset V, \)

\[
| < H_t f, g > | \leq \exp \left\{ - \left( \frac{1}{4C(U, V)t} \right)^{\frac{1}{1+2\alpha}} \right\} \|f\|_{L^2} \|g\|_{L^2}.
\]

(8.1)

Here \(<,>\) represents the \( L^2 \) inner product on \( X \).

When there exists enough nice cut-off functions with bounded gradient (which can be thought of as corresponding to \( \alpha = 0 \)), Lemma 8.1 is a classical result obtained from the so-called Davies’ Method. We adapt it to include
the case when there only exists nice cut-off functions with bounded energy (as specified in the statement above). In the proof we refer to the cut-off functions with the specified dependence $C_2(C_1, U, V) = C_1^{-\alpha} C(U, V)$ in short as nice cut-off functions.

**Proof.** For any fixed $\lambda > 0$, any nice cut-off function $\phi$, consider the following perturbed semigroup

$$H_t^{\lambda \phi} f := e^{-\lambda \phi} H_t (e^{\lambda \phi} f).$$

For any $f, g \in L^2(X)$ with $\text{supp} \{f\} \subset U$, $\text{supp} \{g\} \subset V$ for some precompact open sets $U, V \subset X$, and $U \cap V = \emptyset$, let $\phi$ be some nice cut-off function in the sense of (3.3) such that $\phi = 1$ on $U$ and $\phi = 0$ on $V$, and with associated constants $C_1$ and $C_2 = C_1^{-\alpha} C(U, V)$. We pick $\phi$ so that $C_1 < \frac{1}{2}$. First observe that

$$| \langle H_t^{\lambda \phi} f, g \rangle | = e^\lambda | \langle H_t f, g \rangle |.$$  \hfill (8.2)

On the other hand,

$$| \langle H_t^{\lambda \phi} f, g \rangle | \leq \left\| H_t^{\lambda \phi} f \right\|_{L^2} \cdot \|g\|_{L^2}.$$ 

We estimate $\left\| H_t^{\lambda \phi} f \right\|_{L^2}$ by looking at its (square’s) time derivative first.

$$\frac{d}{dt} \left( \left\| H_t^{\lambda \phi} f \right\|_{L^2}^2 \right) = \int_X 2 \left( H_t^{\lambda \phi} f \right) \frac{d}{dt} H_t^{\lambda \phi} f \, dm$$

$$= \int_X 2 \left( H_t^{\lambda \phi} f \right) e^{-\lambda \phi} \frac{d}{dt} H_t (e^{\lambda \phi} f) \, dm = -2 \mathcal{E} \left( e^{-\lambda \phi} H_t^{\lambda \phi} f, e^{\lambda \phi} H_t^{\lambda \phi} f \right)$$

$$= -2 \mathcal{E} \left( H_t^{\lambda \phi} f, H_t^{\lambda \phi} f \right) + 2\lambda^2 \int_X \left( H_t^{\lambda \phi} f \right)^2 \, d\Gamma(\phi, \phi). \hfill (8.3)$$

Since $\phi$ is a nice cut-off function associated with $C_1, C_2$, we have

$$\int_X \left( H_t^{\lambda \phi} f \right)^2 \, d\Gamma(\phi, \phi)$$

$$\leq C_1 \int_X \phi^2 \, d\Gamma \left( H_t^{\lambda \phi} f, H_t^{\lambda \phi} f \right) + C_2 \int_{\text{supp}(\phi)} \left( H_t^{\lambda \phi} f \right)^2 \, dm$$

$$\leq C_1 \mathcal{E} \left( H_t^{\lambda \phi} f, H_t^{\lambda \phi} f \right) + C_2 \int_{\text{supp}(\phi)} \left( H_t^{\lambda \phi} f \right)^2 \, dm$$

Substituting this bound back to (8.3), we get

$$\frac{d}{dt} \left( \left\| H_t^{\lambda \phi} f \right\|_{L^2}^2 \right) = -2 \mathcal{E} \left( H_t^{\lambda \phi} f, H_t^{\lambda \phi} f \right) + 2\lambda^2 \int_X \left( H_t^{\lambda \phi} f \right)^2 \, d\Gamma(\phi, \phi)$$

$$\leq (-2 + 2\lambda^2 C_1) \mathcal{E} \left( H_t^{\lambda \phi} f, H_t^{\lambda \phi} f \right) + 2\lambda^2 C_2 \int_{\text{supp}(\phi)} \left( H_t^{\lambda \phi} f \right)^2 \, dm.$$
When \(-2 + 2\lambda^2C_1 \leq 0\) \((C_1 \leq \frac{1}{4\gamma})\), we can drop the first term and get
\[
\frac{d}{dt} \left( \left\| H_t^{\lambda\phi} f \right\|_{L^2}^2 \right) \leq 2\lambda^2C_2 \left\| H_t^{\lambda\phi} f \right\|_{L^2(X)}^2.
\]
Observe that at \(t = 0\), \(\left\| H_t^{\lambda\phi} f \right\|_{L^2}^2 \bigg|_{t=0} = \|f\|_{L^2}^2\), so Gronwall’s inequality gives
\[
\left\| H_t^{\lambda\phi} f \right\|_{L^2}^2 \leq \|f\|_{L^2}^2 \exp\left(2\lambda^2C_2 t\right).
\]
Combining this with (8.2), we have
\[
| <H_t f, g> | \leq e^{-\lambda} \left\| H_t^{\lambda\phi} f \right\|_{L^2(X)} \|g\|_{L^2(X)} \leq \|f\|_{L^2} \|g\|_{L^2(X)} \exp\left(-\lambda + 2\lambda^2C_2 t\right).
\]
Take \(C_1 = \frac{1}{4\gamma}\), and let
\[
\lambda = \left( \frac{1}{4C(U,V)t} \right)^{\frac{1}{1+2\alpha}},
\]
then (since \(C_2 = C_1^{-\alpha}C(U,V)\))
\[
\lambda = 4\lambda^2C_2 t > 2\lambda^2C_2 t,
\]
and
\[
| <H_t f, g> | \leq \|f\|_{L^2} \|g\|_{L^2(X)} \exp\left\{- \left( \frac{1}{4C(U,V)t} \right)^{\frac{1}{1+2\alpha}} \right\}.
\]

\(\square\)

**Remark 8.2.** When \(C_2\) has the more explicit dependence \(C_2(C_1, U, V) = C_1^{-\alpha}d_X(U, V)^{-\beta}\) for some \(\alpha, \beta > 0\), and some distance \(d_X\) on \(X\) that defines the same topology, substituting \(C(U, V) = d_X(U, V)^{-\beta}\) in the above \(L^2\) Gaussian bound, we get the \(L^2\) version of the sub-Gaussian upper bound. For example, for fractals with walk dimension \(d_w\), \(C_2 \sim C_1^{-\frac{d_w}{2}}d_X(U, V)^{-d_w}\) (cf. [23]), then in our expression, \(\alpha = \frac{d_w}{2} - 1\), \(\beta = d_w\), and the exponential term in the upper bound for \(| <H_t f, g> |\) is exactly \(\exp\left\{- \left( \frac{d_X(U,V)^{d_w}}{4t} \right)^{\frac{1}{1+2\alpha}} \right\}\).

Next we want to estimate \(| \partial_t^k H_t f, g > |\), and the estimate essentially follows from a straightforward adaptation of Proposition 2.2 in [9]. For another approach on obtaining estimates on time derivatives of \(<H_t f, g>\), cf. [11].

**Lemma 8.3.** Suppose that \(F\) is an analytic function on \(C_+\). Assume that, for given numbers \(A, B, \gamma > 0\), \(a \geq 0\),
\[
|F(z)| \leq B, \quad \forall z \in \mathbb{C},
\]
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and for some $0 < a \leq 1$,
\[ |F(t)| \leq Ae^{at}e^{-\left(\frac{\gamma}{t}\right)^a}, \quad \forall t \in \mathbb{R}_+. \]

Then
\[ |F(z)| \leq B \exp \left( -\text{Re} \left( \left( \frac{\gamma}{z} \right)^a \right) \right), \quad \forall z \in \mathbb{C}_+. \quad (8.4) \]

When $a = 1$, this is exactly Proposition 2.2 in [9], and the proof for Lemma 8.3 is close to that of the proposition in [9]. Here we follow their use of the notation $\mathbb{C}_+$ for the right half plane.

**Lemma 8.4** ($L^2$ Gaussian upper bound). Under the hypotheses in Lemma 8.1, for any $f, g \in L^2(X)$ with $\text{supp}\{f\} \subset U$, $\text{supp}\{g\} \subset V$, where $U, V$ are precompact open sets with disjoint closures,
\[ |\langle \partial^n \mathcal{H}_t f, g \rangle| \leq n! \frac{2^n}{n!} ||f||_{L^2} ||g||_{L^2} \exp \left\{ - \left( \frac{1}{2C(U,V)t} \right)^{\frac{1}{1+2\alpha}} \right\}. \quad (8.5) \]

**Proof.** Let $F(t) := \langle \mathcal{H}_t f, g \rangle$. By spectral calculus, for any $z \in \mathbb{C}$ with $\text{Re}(z) > 0$,
\[ \mathcal{H}_z v = \int_{0}^{+\infty} e^{-z\lambda} d\lambda \]

is well-defined for all $v \in L^2$, and hence $F(z)$ can be analytically extended to $z \in \mathbb{C}_+$. Moreover,
\[ ||\mathcal{H}_z f||_{L^2}^2 = \int_{0}^{\infty} e^{-2\text{Re}(z)\lambda} d\lambda \langle E_{\lambda} f, f \rangle \leq ||f||_{L^2}^2, \]

so $F(z)$ satisfies $|F(z)| \leq ||f||_{L^2} ||g||_{L^2}$. Lemma 8.1 says
\[ |F(t)| \leq \exp \left\{ - \left( \frac{1}{4C(U,V)t} \right)^{\frac{1}{1+2\alpha}} \right\} ||f||_{L^2} ||g||_{L^2}. \]

So by Lemma 8.3
\[ |F(z)| \leq ||f||_{L^2} ||g||_{L^2} \exp \left( -\text{Re} \left( \left( \frac{\gamma}{z} \right)^a \right) \right), \quad (8.6) \]

where $\gamma = \frac{1}{4C(U,V)}$.

Recall that in complex analysis we have the expression for the $n$th derivative of $F(z)$ using the integral over some circle around $z$,
\[ F^{(n)}(z) = \frac{n!}{2\pi i} \int_{C} \frac{F(\xi)}{(\xi - z)^{n+1}} d\xi = \frac{n!}{2\pi} \int_{0}^{2\pi} \frac{F(z + re^{i\theta})}{re^{in\theta}} dr. \quad (8.7) \]
Consider $z = t \in \mathbb{R}_+$. Take for example $r = \frac{t}{2}$. Then (8.6) gives the bound

$$|F\left(t + \frac{t}{2}e^{i\theta}\right)| \leq \|f\|_{L^2} \|g\|_{L^2} \exp\left(-\text{Re}\left[\frac{\gamma}{t + \frac{t}{2}e^{i\theta}}\right]\right)$$

$$\leq \|f\|_{L^2} \|g\|_{L^2} \exp\left(-\left(\frac{2\gamma}{t}\right)^{\frac{1}{1+2\alpha}}\right).$$

Substituting this bound in (8.7), we get

$$|F^{(n)}(t)| = |<\partial^n_t H, f, g>| \leq n! \frac{2^n}{t^n} \|f\|_{L^2} \|g\|_{L^2} \exp\left(-\left(\frac{2\gamma}{t}\right)^{\frac{1}{1+2\alpha}}\right).$$

(8.8)

### Proof.

The energy measure $d\Gamma(\eta_1 + \eta_2, \eta_1 + \eta_2)$ equals

$$d\Gamma(\eta_1 + \eta_2, \eta_1 + \eta_2) = d\Gamma(\eta_1, \eta_1) + 2d\Gamma(\eta_1, \eta_2) + d\Gamma(\eta_2, \eta_2).$$

#### In the application of the Gaussian upper bound in the proofs in previous sections, the exact form of the upper bounds are not essential, we only need the property that the upper bound, divided by any power of $t$, tends to 0 as $t$ tends to 0. Hence we take Assumption 3.8 in previous sections.

### 8.2 Other lemmas

In this subsection we prove Lemma 3.6 on existence of nice cut-off functions for general pairs of open sets. Starting with the existence of nice cut-off functions on a topological basis $\mathcal{T}B$ in the sense of Assumption 3.1, we now construct nice cut-off functions on any pair of open sets $V \subseteq U$ (Lemma 3.6). In the next two lemmas we first discuss the properties of the sum and product of two nice cut-off functions. By taking maximum if necessary, we assume all cut-off functions correspond to the same $C_1, C_2$.

#### Lemma 8.5 (Sum of nice cut-off functions). For any two nice cut-off functions $\eta_1, \eta_2$ for some pairs of open sets $V_1 \subseteq U_2, V_2 \subseteq U_2$, respectively, where $V_1, U_1, V_2, U_2$ are all subsets of $X$, their sum $\eta := \eta_1 + \eta_2$ is still a nice cut-off function satisfying

$$\int_X v^2 d\Gamma(\eta_1 + \eta_2, \eta_1 + \eta_2) \leq 2C_1 \int_X (\eta_1 + \eta_2)^2 d\Gamma(v, v) + 4C_2 \int_{\text{supp}(\eta_1 + \eta_2)} v^2 dm. \quad (8.9)$$

#### Proof.

The energy measure $d\Gamma(\eta_1 + \eta_2, \eta_1 + \eta_2)$ equals

$$d\Gamma(\eta_1 + \eta_2, \eta_1 + \eta_2) = d\Gamma(\eta_1, \eta_1) + 2d\Gamma(\eta_1, \eta_2) + d\Gamma(\eta_2, \eta_2).$$
For any \( v \in \mathcal{F} \),

\[
\int_X v^2 \, d\Gamma (\eta_1 + \eta_2, \eta_1 + \eta_2) = \int_X v^2 \, d\Gamma (\eta_1, \eta_1) + 2 \int_X v^2 \, d\Gamma (\eta_1, \eta_2) + \int_X v^2 \, d\Gamma (\eta_2, \eta_2) \leq 2 \int_X v^2 \, d\Gamma (\eta_1, \eta_1) + 2 \int_X v^2 \, d\Gamma (\eta_2, \eta_2) \leq 2 \left[ C_1 \int_X \eta_1^2 \, d\Gamma (v, v) + C_2 \int_{\text{supp}(\eta_2)} v^2 \, dm + C_1 \int_X \eta_2^2 \, d\Gamma (v, v) + C_2 \int_{\text{supp}(\eta_1)} v^2 \, dm \right] \leq 2C_1 \int_X (\eta_1 + \eta_2)^2 \, d\Gamma (v, v) + 4C_2 \int_{\text{supp}(\eta_1 + \eta_2)} v^2 \, dm.
\]

The last line comes from \( \eta_1, \eta_2 \geq 0 \), and \( \text{supp}(\eta_1), \text{supp}(\eta_2) \subset \text{supp} \{ \eta_1 + \eta_2 \} \).

**Lemma 8.6 (Product of nice cut-off functions).** If \( 0 \leq C_1 < \frac{1}{4} \), for any two nice cut-off functions \( \eta_1, \eta_2 \) for some pairs of open sets \( V_1 \subset U_2, V_2 \subset U_2 \), respectively, where \( V_1, U_1, V_2, U_2 \) are all subsets of \( X \), the product function \( \eta := \eta_1 \eta_2 \) is still a nice cut-off function satisfying

\[
\int_X v^2 \, d\Gamma (\eta \eta_2, \eta_1 \eta_2) \leq 16C_1 \int_X \eta_1^2 \eta_2^2 \, d\Gamma (v, v) + 4C_2 \int_{\text{supp}(\eta_1 \eta_2)} v^2 \, dm. \tag{8.10}
\]

**Proof.** Using the product rule for the energy measure, \( d\Gamma (\eta_1 \eta_2, \eta_1 \eta_2) \) equals

\[
d\Gamma (\eta_1 \eta_2, \eta_1 \eta_2) = \eta_1^2 \, d\Gamma (\eta_2, \eta_2) + 2 \eta_1 \eta_2 \, d\Gamma (\eta_1, \eta_2) + \eta_2^2 \, d\Gamma (\eta_1, \eta_1).
\]

Then by Cauchy-Schwartz inequality, for any \( v \in \mathcal{F} \),

\[
\int_X v^2 \, d\Gamma (\eta_1 \eta_2, \eta_1 \eta_2) \leq 2 \int_X v^2 \eta_1^2 \, d\Gamma (\eta_2, \eta_2) + 2 \int_X v^2 \eta_2^2 \, d\Gamma (\eta_1, \eta_1), \tag{8.11}
\]

and for any \( \beta > 0 \),

\[
\int_X v^2 \eta_1^2 \, d\Gamma (\eta_2, \eta_2) + \int_X v^2 \eta_2^2 \, d\Gamma (\eta_1, \eta_1) \leq C_1 \left[ \int_X \eta_1^2 \, d\Gamma (\eta_1 v, \eta_1 v) + \int_X \eta_1^2 \, d\Gamma (\eta_2 v, \eta_2 v) \right] + C_2 \int_{\text{supp}(\eta_1 \eta_2)} v^2 \, dm \leq C_1 \left[ 2 (1 + \beta) \int_X \eta_1^2 \eta_2^2 \, d\Gamma (v, v) + \left( 1 + \frac{1}{\beta} \right) \int_X \eta_1^2 v^2 \, d\Gamma (\eta_2, \eta_2) \right] + \left( 1 + \frac{1}{\beta} \right) \int_X \eta_2^2 v^2 \, d\Gamma (\eta_1, \eta_1) + C_2 \int_{\text{supp}(\eta_1 \eta_2)} v^2 \, dm.
\]

So

\[
\left( 1 - C_1 \left( 1 + \frac{1}{\beta} \right) \right) \left[ \int_X v^2 \eta_1^2 \, d\Gamma (\eta_2, \eta_2) + \int_X v^2 \eta_2^2 \, d\Gamma (\eta_1, \eta_1) \right] \leq 2C_1 (1 + \beta) \int_X \eta_1^2 \eta_2^2 \, d\Gamma (v, v) + C_2 \int_{\text{supp}(\eta_1 \eta_2)} v^2 \, dm.
\]

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For $C_1 < \frac{1}{4}$, we can take $\beta = 1$, then $\frac{2C_2(1+\beta)}{1-C_1(1+\frac{3}{2})} = \frac{4C_1}{1-2C_1} < 8C_1$, and

$$
\int_X v^2 \eta_1^2 \, d\Gamma (\eta_2, \eta_2) + \int_X v^2 \eta_2^2 \, d\Gamma (\eta_1, \eta_1) \leq \frac{8C_1}{1-2C_1} \int_X v^2 \eta_1 \, d\Gamma (v, v) + 2C_2 \int_{\text{supp}(\eta_1\eta_2)} v^2 \, dm.
$$

(8.12)

Combining (8.11) and (8.12), we get (8.10).

To show that Assumption 3.1 can be extended to pairs of general open sets, we use a construction similar to the standard construction of partitions of unity to obtain cut-off functions for general pairs of open sets and then check the so-obtained functions satisfy (3.1). We first state the following lemma on using open sets in the basis $TB$ to cover any compact set.

**Lemma 8.7.** For any compact set $K \subset X$ and any open neighborhood $U$ of $K$ ($K \subset U \subset X$), there exist two finite open covers $C_1 = \{U_1, U_2, \cdots, U_n\}$ and $C_2 = \{V_1, V_2, \cdots, V_m\}$, such that all $U_i, V_i$ are elements in $TB$, $K \subset \bigcup_{i=1}^n U_i \subset \bigcup_{j=1}^m V_j \subset U$, and $C_2$ is subordinate to $C_1$, i.e. for any $V_i \in C_2$, there exists some $U_j \in C_1$ such that $V_i \subset U_j$.

**Proof.** For any point $p \in K$, there exists an open neighborhood $U_p \in TB$ such that $p \in U_p \subset U$ since $TB$ is a topology basis and $X$ is regular (to ensure there is some $U_p$ that is precompact in $U$). Then $\{U_p \mid p \in K\}$ is an open cover of $K$, which has a finite sub-cover $C_1 = \{U_{p_1}, U_{p_2}, \cdots, U_{p_n}\}$. We rename $U_p$ as $U_j$.

Now we construct $C_2$ from $C_1$. For any point $p \in K$, there exists some $U_j$ ($j = 1, 2, \cdots, n$) such that $p \in U_j$. Then there exists some smaller open neighborhood $V_p \in TB$ such that $p \in V_p \subset U_j$. $\{V_p \mid p \in K\}$ is an open cover of $K$, and let $\{V_{p_1}, V_{p_2}, \cdots, V_{p_m}\}$ be a finite sub-cover, then this gives the $C_2$ open cover we wanted, after renaming $V_p$ as $V_i$.

Next we proceed to prove the lemma on the automatic extension of the applicability of Assumption 3.1 from pairs of open sets in a topological basis to all open sets.

**Proof of Lemma 3.6.** For any pair of open sets $V \subset U$, for any $0 < C_0 < 1$, we want to construct a nice cut-off function $\psi$ for the pair $V \subset U$ with the given number $C_0$ as the associated constant $C_1$ in (3.2). Pick another open set $V'$ such that $V \subset V' \subset U \subset X$. Applying Lemma 8.7 to the compact set $K = \overline{V}$ with open neighborhood $U$, we get two finite open covers $C_1 = \{O_1, \cdots, O_n\}$, and $C_2 = \{\Omega_1, \cdots, \Omega_m\}$ such that $C_2$ is subordinate to $C_1$, and that both cover $\overline{V}$ and are contained in $U$. Applying Lemma 8.7 to the compact set $U \setminus V'$ with open neighborhood $X \setminus \overline{V}$, we get two more finite open covers $C'_1 = \{O'_1, \cdots, O'_n\}$, and $C'_2 = \{\Omega'_1, \cdots, \Omega'_m\}$, such that $C'_2$ is subordinate to $C'_1$, that both cover $U \setminus V'$, and are contained in $X \setminus \overline{V}$.

Note that for each pair $O_i \subset \Omega_j$ or $O'_i \subset \Omega'_j$, for any $0 \leq C < 1$, by Assumption 3.1, there exists some nice cut-off function $\eta$ for the pair $O_i \subset \Omega_j$.
with $C_1 = C$ in (3.2), and some nice cut-off function $\varphi$ for the pair $O'_1 \subset \Omega'_j$ with $C_1 = C$. Since all $C_1, C_2, C'_1, C'_2$ are finite covers, there are finitely many $\eta$'s and $\varphi$'s. We re-index these nice cut-off functions as $\eta_1, \ldots, \eta_r$ and $\varphi_1, \ldots, \varphi_k$. Let

$$\eta := \sum_{i=1}^k \eta_i, \quad \varphi := \sum_{j=1}^r \varphi_j.$$ 

Then $1 \leq \varphi \leq k + r$ on $U$, and $\varphi = \eta$ on $V$, since all $\varphi_i$ vanish on $V$. Hence $\eta/\varphi$ is well-defined on $U$, and becomes 0 before it reaches the boundary of $U$ since $\eta$ is supported in $U$. By extending the quotient by 0 outside $U$, we obtain the function $\psi$ satisfying

$$\psi(x) = \begin{cases} \frac{\eta}{\varphi}, & x \in U, \\ 1, & x \in V, \\ 0, & x \in U^c \end{cases}$$

Hence it remains to show $\psi$ satisfies (3.2). By the lemmas on the sum and product of nice cut-off functions, we only need to show $1/\varphi$ satisfies (3.2) for $u \in \mathcal{F}$ with support in $U$ (since $\psi$ is supported in $U$). For any $u \in \mathcal{F}$ with support in $U$,

$$\int u^2 \, d\Gamma \left( \frac{1}{\varphi}, \frac{1}{\varphi} \right) = \int u^2 \cdot \left( \frac{1}{\varphi^2} \right)^2 \, d\Gamma (\varphi, \varphi) \leq \int u^2 \, d\Gamma (\varphi, \varphi) \leq C_1 \int \varphi^2 \, d\Gamma (u, u) + C_2 \int_{\text{supp}(\varphi)} u^2 \, dm,$$

where $C_1 = 2(k + r)C$ is obtained from the lemma on sum of nice product functions and our definition of $\varphi$, and $C_2$ can be computed correspondingly. Moreover, since $1 \leq \varphi \leq k + r$, $1 \leq \varphi^2 \leq (k + r)^2$, we get $\varphi \leq (k + r)^2 / \varphi$ on $U$, and hence

$$\int u^2 \, d\Gamma \left( \frac{1}{\varphi}, \frac{1}{\varphi} \right) \leq C_1 \int \varphi^2 \, d\Gamma (u, u) + C_2 \int_{\text{supp}(\varphi)} u^2 \, dm \leq C_1 \int \frac{(k + r)^4}{\varphi^2} \, d\Gamma (u, u) + C_2 \int_{\text{supp}(\varphi)} u^2 \, dm,$$

which is indeed of the form (3.2). In order to get the given number $C_0$ as the $C_1$ for $\psi$ in the inequality (3.2), we can adjust $\eta_j$ and $\varphi_i$ by multiplying with proper constant if necessary, or rely on the self-improving property of the existence of cut-off functions satisfying (3.2) with smaller $C_1$'s to conclude the existence of such a nice cut-off function (cf. [1]). \qed
References

[1] Sebastian Andres and Martin T. Barlow. Energy inequalities for cutoff functions and some applications. *J. Reine Angew. Math.*, 699:183–215, 2015.

[2] Martin T. Barlow and Mathav Murugan. Stability of the elliptic Harnack inequality. *Ann. of Math. (2)*, 187(3):777–823, 2018.

[3] A. Bendikov and L. Saloff-Coste. Elliptic diffusions on infinite products. *J. Reine Angew. Math.*, 493:171–220, 1997.

[4] A. Bendikov and L. Saloff-Coste. Spaces of smooth functions and distributions on infinite-dimensional compact groups. *J. Funct. Anal.*, 218(1):168–218, 2005.

[5] A. Bendikov and L. Saloff-Coste. Hypoelliptic bi-invariant Laplacians on infinite dimensional compact groups. *Canad. J. Math.*, 58(4):691–725, 2006.

[6] Alexander Bendikov and Laurent Saloff-Coste. Invariant local Dirichlet forms on locally compact groups. *Ann. Fac. Sci. Toulouse Math. (6)*, 11(3):303–349, 2002.

[7] Alexander Bendikov, Laurent Saloff-Coste, Maura Salvatori, and Wolfgang Woess. The heat semigroup and Brownian motion on strip complexes. *Adv. Math.*, 226(1):992–1055, 2011.

[8] Tobias Holck Colding and William P. Minicozzi II. Optimal bounds for ancient caloric functions, 2019.

[9] Thierry Coulhon and Adam Sikora. Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem. *Proc. Lond. Math. Soc. (3)*, 96(2):507–544, 2008.

[10] E. B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

[11] E. B. Davies. Non-Gaussian aspects of heat kernel behaviour. *J. London Math. Soc. (2)*, 55(1):105–125, 1997.

[12] J. Eells and B. Fuglede. *Harmonic maps between Riemannian polyhedra*, volume 142 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001. With a preface by M. Gromov.

[13] Nathaniel Eldredge and Laurent Saloff-Coste. Widder’s representation theorem for symmetric local Dirichlet spaces. *J. Theoret. Probab.*, 27(4):1178–1212, 2014.

[14] Masatoshi Fukushima, Yoichi Oshima, and Masayoshi Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended edition, 2011.
[15] Pavel Gyrya and Laurent Saloff-Coste. Neumann and Dirichlet heat kernels in inner uniform domains. Astérisque, (336):viii+144, 2011.
[16] Masanori Hino and José A. Ramírez. Small-time Gaussian behavior of symmetric diffusion semigroups. Ann. Probab., 31(3):1254–1295, 2003.
[17] Kenkichi Iwasawa. On some types of topological groups. Ann. of Math. (2), 50:507–558, 1949.
[18] Naotaka Kajino and Mathav Murugan. On (non-)singularity of energy measures under full off-diagonal heat kernel estimates, 2019.
[19] S. Kusuoka and D. Stroock. Applications of the Malliavin calculus. II. J. Fac. Sci. Univ. Tokyo Sect. IA Math., 32(1):1–76, 1985.
[20] Kazuhiro Kuwae, Yoshiroh Machigashira, and Takashi Shioya. Sobolev spaces, Laplacian, and heat kernel on Alexandrov spaces. Math. Z., 238(2):269–316, 2001.
[21] Janna Lierl. Local behavior of solutions of quasilinear parabolic equations on metric spaces. arXiv e-prints, page arXiv:1708.06329, Aug 2017.
[22] Fanghua Lin and Q. S. Zhang. On ancient solutions of the heat equation. Comm. Pure Appl. Math., 72(9):2006–2028, 2019.
[23] Mathav Murugan and Laurent Saloff-Coste. Davies’ method for anomalous diffusions. Proc. Amer. Math. Soc., 145(4):1793–1804, 2017.
[24] Yukio Otsu and Takashi Shioya. The Riemannian structure of Alexandrov spaces. J. Differential Geom., 39(3):629–658, 1994.
[25] Melanie Pivarski and Laurent Saloff-Coste. Small time heat kernel behavior on Riemannian complexes. New York J. Math., 14:459–494, 2008.
[26] Laurent Saloff-Coste. Uniformly elliptic operators on Riemannian manifolds. J. Differential Geom., 36(2):417–450, 1992.
[27] K. T. Sturm. Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality. J. Math. Pures Appl. (9), 75(3):273–297, 1996.
[28] Karl-Theodor Sturm. Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations. Osaka J. Math., 32(2):275–312, 1995.
[29] Masayoshi Takeda. On a martingale method for symmetric diffusion processes and its applications. Osaka J. Math., 26(3):605–623, 1989.
[30] J. Wloka. Partial differential equations. Cambridge University Press, Cambridge, 1987. Translated from the German by C. B. Thomas and M. J. Thomas.
[31] Qi S. Zhang. A note on time analyticity for ancient solutions of the heat equation, 2019.