Homogenization of the Stokes equation with mixed boundary condition in a porous medium

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Abstract: We homogenize stationary incompressible Stokes flow in a periodic porous medium. The fluid is assumed to satisfy a no-slip condition on the boundary of solid inclusions and a normal stress (traction) condition on the global boundary. Under these assumptions, the homogenized equation becomes the classical Darcy law with a Dirichlet condition for the pressure.

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Keywords: homogenization; Darcy's law; normal stress boundary condition; traction condition; porous media

1. Introduction

Various physical phenomena can be described in terms of fluid flow in porous media. It occurs e.g. in the study of filtration in sandy soils or blood circulation in capillaries, see Bear (1975) and Hornung (1997) for more examples and motivation. In the study of such processes, one would like to find some averaged characteristics of the flow, e.g. permeability, macro-velocity, and macro-pressure. To obtain such quantities, there exist several mathematical approaches collectively referred to as homogenization theory (see e.g. Allaire, 1989; Hornung, 1997; Lions, 1981; Sanchez-Palencia, 1980; Tartar, 1980 and the references therein) as well as heuristic methods based on phase averaging (Whitaker, 1986).

In the present study, we consider the flow of an incompressible fluid in a perforated domain which depends on a small parameter $\varepsilon$. More precisely, let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ which is perforated by $\varepsilon$-periodically distributed obstacles. The fluid part of $\Omega$ is denoted as $\Omega'$. The boundary $\partial \Omega' = \partial \Omega \cup S'$ of the domain $\Omega'$ consists of two disjoint parts: $S'$—the boundary of micro-inclusions

ABOUT THE AUTHORS

John Fabricius, Elena Miroshnikova and Peter Wall are mathematicians who work in the fields of partial differential equations and homogenization theory. The present work is part of an ongoing project devoted to the mathematical modeling of fluid flow in thin or porous domains. The common feature of boundary value problems that arise in such contexts is the presence of a microstructure that is characterized by a small parameter. Since such models are complicated to analyze, one can study the asymptotic behavior of solutions using various notions of convergence. This often leads to simpler models that in an averaged sense give accurate descriptions of the flow.

PUBLIC INTEREST STATEMENT

Darcy’s law is a simple mathematical equation that describes the flow of a viscous fluid through a porous medium, e.g. a bed of sand or rocks. On the microscopic level such flows can be very complicated, but on average the fluid will flow from high-pressure regions to low-pressure regions. The amount of fluid that can pass through a porous medium is determined by its permeability which depends both on the fractional volume of the pores as well as their shape. While Darcy’s law was originally formulated based on experiments, it can also be derived as an approximation of more general fluid models via homogenization. This provides a sound theoretical basis for the equations that govern flow in porous media.
(interior boundary)—and $\partial \Omega$—the exterior boundary. The flow of an incompressible fluid in $\Omega^\varepsilon$ is assumed to be governed by the Stokes equations. Moreover, we impose a standard no-slip condition on $S^\varepsilon$, whereas a normal stress is prescribed on the global boundary $\partial \Omega$. The non-standard (but physically appropriate) stress condition is the main novelty of the present analysis. All previous studies related to flow in porous media impose the no-slip condition on the global boundary as well.

The mixed boundary value problem for the Stokes system in non-perforated domains has been studied in Maz'ya and Rossmann (2007) and Fabricius (2016). An important consequence of the mixed boundary condition is that both pressure and velocity are unique. More precisely, uniqueness of pressure follows from the normal stress condition on $\partial \Omega$ whereas uniqueness of velocity follows from the no-slip condition on $S^\varepsilon$. It is well known, that if a Dirichlet condition is imposed on the whole boundary, then the pressure can only be determined up to a constant. Correspondingly, if a Neumann condition is imposed on the whole boundary, the pressure is unique but the velocity can only be determined up to a rigid body velocity.

Our objective is to homogenize the Stokes system under the mixed boundary condition, i.e. we want to find the effective (or macroscopic) set of equations that govern pressure and velocity in the limit as the parameter $\varepsilon$ tends to zero. This includes

1. to define a suitable notion of convergence for solutions defined on a sequence of perforated domains $\Omega^\varepsilon$ as $\varepsilon \to 0$;
2. to deduce the limit (homogenized) boundary value problem in the unperforated domain $\Omega$; and
3. to rigorously derive the corresponding Darcy law in terms of homogenized problem solution.

To accomplish this, we use the two-scale convergence technique introduced by Nguetseng (1989), see also Allaire (1992) and Lukkassen, Nguetseng, and Wall (2002). Tartar (1980) proved the homogenization result corresponding to the Dirichlet problem for the Stokes system using the method of oscillating test functions. A proof of that result based on two-scale convergence has been given by Allaire in Hornung (1997, Ch. 3). Tartar’s result has been developed in various directions, e.g. by starting from a more general model for fluid flow (Mikelic, 1991); by relaxing the geometrical requirements on the domain $\Omega^\varepsilon$ (Allaire, 1989; Feireisl, Namlyeyeva, & Nečasová, 2016); by considering a compressible fluid Masmoudi (2002) or a non-Newtonian fluid (Bourgeat & Mikelic, 1996; Kalousek, 2016).

To our knowledge, all previous mathematical studies concern the Dirichlet problem for the velocity field. This leads to Darcy’s equation with a Neumann condition, which implies that the homogenized pressure distribution can only be determined up to a constant. However, in applications where the flow is driven by a pressure gradient (Srinivasan, Bonito, & Rajagopal, 2013), or more generally a prescribed boundary stress (traction) (Larese, Rossi, & Oñate, 2015), it is natural to use a Dirichlet condition for the pressure in Darcy’s law. For a thorough discussion regarding boundary conditions for flow in porous media, we refer to Chapter 7 of Bear (1975). The present analysis shows that a normal stress condition for the Stokes system implies a Dirichlet condition for the pressure in Darcy’s law. The latter condition seems more accurate from a physical point of view, because it gives a unique homogenized pressure. Since this is also the condition that is used in many applications, our result provides a rigorous link between a microscopic model and its macroscopic counterpart that has not been explained before.

2. Preliminaries
To formulate the problem and state the main results, we first describe the class of domains where the flow is considered, introduce some function spaces and state some versions of well-known results that are used in the subsequent analysis.
2.1. Perforated domain $\Omega^\varepsilon$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary $\partial \Omega$. For any $\varepsilon > 0$, set

$$\Omega(\varepsilon) = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > 2\varepsilon \}, \quad I(\varepsilon) = \Omega(\varepsilon) \cap \varepsilon \mathbb{Z}^n,$$

Furthermore, let $\omega$ be an unbounded Lipschitz domain of $\mathbb{R}^n$ with $Q$-periodic structure, where $Q = (-1/2, 1/2)^n$ denotes the unit cube in $\mathbb{R}^n$. Set

$$Q' = \omega \cap Q \quad (\text{fluid part of } Q)$$

$$Q^s = Q \setminus Q' \quad (\text{solid part of } Q).$$

We assume a positive distance between $Q^s$ and the boundary $\partial Q$. Thus $\partial Q f = \partial Q \cup S$, where $S = \partial \omega \cap Q$ denotes the boundary of $Q^s$. The domain $\Omega^\varepsilon$ (see Figure 1) is defined by

$$\Omega^\varepsilon = \Omega \setminus \left( \bigcup_{i \in I(\varepsilon)} Q^s_i \right),$$

where $Q^s_i$ is the set $Q^s$ scaled by a factor $\varepsilon$ and translated by the vector $i$. Thus there exists so called “safety region” in the neighborhood of $\partial \Omega$ without inclusions. From the figure, one sees that $\partial \Omega^\varepsilon$ consists of two disjoint parts: $\partial \Omega$ (the exterior boundary) and $S^\varepsilon = \partial Q^\varepsilon \cap \Omega$ (the interior boundary of solid inclusions).

2.2. Function spaces

Given $u = (u_1, \ldots, u_n) \in H^1(\Omega; \mathbb{R}^n)$ we denote as $\nabla u$, $e(u)$ the $n \times n$ matrix valued functions defined by

$$\nabla u = \left( \frac{\partial u_i}{\partial x_j} \right), \quad e(u) = \frac{1}{2} (\nabla u + (\nabla u)^\top) = \left( \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \right).$$

In other words, $e(u)$ is the symmetric part of $\nabla u$.

For a given set $\Sigma \subset \partial \Omega$, we define $H^1(\Omega, \Sigma; \mathbb{R}^n)$, as the subspace of $H^1(\Omega; \mathbb{R}^n)$ consisting of all functions $u$ that have zero trace on $\Sigma$.

Consequently, $H^1_0(\Omega; \mathbb{R}^n) = H^1(\Omega, \partial \Omega; \mathbb{R}^n)$. The letter $V$ is reserved for divergence free function spaces. Thus, we define

$$V(\Omega) = \{ u \in H^1(\Omega; \mathbb{R}^n) : \nabla \cdot u = 0 \text{ in } \Omega \}$$

$$V(\Omega, \Sigma) = \{ u \in V(\Omega) : u = 0 \text{ on } \Sigma \}$$

$$V_0(\Omega) = V(\Omega, \partial \Omega).$$

$L^2(\Omega)$ is defined as the set of all $p$ in $L^2(\Omega)$ such that $\int_\Omega p \, dx = 0$. For the unit cube $Q$, $H^1_{\text{per}}(Q; \mathbb{R}^n)$ denotes the closure of the space of smooth $Q$-periodic functions in $H^1$-norm. To distinguish between macro- and micro-variables, the following differential operators will be used:

![Figure 1. Structure of perforated domain $\Omega^\varepsilon$.](image)
\[ \nabla_x = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right), \quad \nabla_y = \left( \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right), \quad (x, y) \in \Omega \times Q. \]

The space \( H_{\text{div}}(\Omega; \mathbb{R}^n) \) consists of functions \( F \in L^2(\Omega; \mathbb{R}^{n \times n}) \) such that \( \nabla \cdot F \in L^2(\Omega; \mathbb{R}^n) \). Since \( \partial \Omega \) is assumed to be of class \( C^{0,1} \) the outward unit normal \( \hat{n} \) is defined almost everywhere on \( \partial \Omega \) and there exists a continuous normal trace operator

\[ \gamma_n : H_{\text{div}}(\Omega; \mathbb{R}^n) \to H^{-1/2}(\partial \Omega; \mathbb{R}^n) \]

such that \( \gamma_n(F) = F\hat{n} \) for any \( F \in C^1(\overline{\Omega}; \mathbb{R}^{n \times n}) \) and the divergence theorem holds

\[ \int_{\Omega} F : \nabla u + (\nabla \cdot F) \cdot u \, dx = \langle \gamma_n(F), u \rangle_{H^{1/2}(\partial \Omega; \mathbb{R}^n), H^{-1/2}(\partial \Omega; \mathbb{R}^n)} \]

for any \( F \in H_{\text{div}}(\Omega; \mathbb{R}^n) \), \( u \in H^1(\Omega; \mathbb{R}^n) \). The operator \( \gamma_n \) is in fact onto (see Boyer & Fabrie, 2013, pp. 248–249; Temam, 1979, pp. 9–20). For simplicity, we shall write \( F\hat{n} \) in place of \( \gamma_n(F) \) and \( \langle \cdot, \cdot \rangle_{\Omega} \) in place of \( \langle \cdot, \cdot \rangle_{H^{1/2}(\partial \Omega; \mathbb{R}^n), H^{-1/2}(\partial \Omega; \mathbb{R}^n)} \).

### 2.3. Auxiliary results in function analysis

The stress boundary condition implies some essential difficulties compared to the classical no-slip condition. To deal with the stress condition, we need some generalizations of classical results. For Lipschitz domains, this is achieved by following the approach based on Nečas’ inequality described in Boyer and Fabrie (2013), Nečas (2012) and Tartar (2006). In particular, we need a version of Korn’s inequality for perforated domains and a “strong” version of de Rham’s theorem.

#### 2.3.1. Korn’s inequality

We need to control the constant in the classical Korn inequality (see e.g. Ciarlet, 2010) for vector functions defined in \( \Omega \) satisfying a homogeneous Dirichlet condition on the boundary of solid inclusions.

**Theorem 1**  \textbf{[Korn’s inequality]} There exists a constant \( C > 0 \) independent of \( \epsilon \) such that for all sufficiently small \( \epsilon > 0 \) any function \( u \in H^1(\Omega; \mathbb{R}^n) \) satisfies

\[
\|u\|_{H^1(\Omega; \mathbb{R}^n)} \leq C \|e(u)\|_{L^2(\Omega; \mathbb{R}^n)}; \tag{2}
\]

\[
\|u\|_{H^1(\Omega; \mathbb{R}^n)} \leq C \|\xi\|_{L^2(\Omega; \mathbb{R}^n)}; \tag{3}
\]

The proof is based on covering the domain \( \Omega \) by a finite collection of cubes \( Q_{i, M} \), \( M > 1, i \in I(\epsilon) \) and applying the Korn inequality (see e.g. Theorem 3.3 in Duvaut, 1976, Ch. 3) locally on each cube (see Miroshnikova, 2016 for details).

#### 2.3.2. Bogovskiĭ operator

To prove a strong version of de Rham’s theorem, we use the notion of a so called Bogovskiĭ operator (see Bogovskiĭ, 1979, or Galdi 2011, Ch. III.3).

**Theorem 2**  \textbf{For any} \( f \in L^2(\Omega) \), \textbf{there exists} \( v \in H^1(\Omega; \mathbb{R}^n) \) such that

\[ f = \nabla \cdot v \quad \text{in} \ \Omega. \]  \tag{4}

Moreover, \( v \) is unique in \( H^1(\Omega; \mathbb{R}^n) \) modulo \( V(\Omega) \) and

\[ \|v\|_{H^1(\Omega; \mathbb{R}^n)} \leq C\omega \|f\|_{L^2(\Omega; \mathbb{R}^n)}. \]
Remark 1  Theorem 2 is usually formulated with \[ \int_{\Omega} f \, dx = 0, \quad v \in H^1_0(\Omega; \mathbb{R}^n) \] (see Bogovskii, 1979; Galdi, 2011, Ch. III.3). The present result follows from the classical result and the simple observation that there exists \( \hat{\varphi} \in H^1_0(\Omega; \mathbb{R}^n) \) such that

\[ \nabla \cdot \hat{\varphi} = 1. \]

For example, \( \hat{\varphi}(x) = \frac{n}{|\Omega|} - \frac{1}{x} \) has this property.

Remark 2  Theorem 2 asserts the existence of a bounded linear operator \( B: L^2(\Omega) \to H^1(\Omega; \mathbb{R}^n) / V(\Omega) \) such that \( \nabla \cdot (Bf) = f \).

2.3.3. The strong de Rham theorem

The key result to prove existence of a pressure corresponding to an incompressible flow is de Rham's theorem (1984) (see e.g. Boyer & Fabrie, 2013, pp. 241–245). It can be formulated as follows:

**Theorem 3**  \([\text{de Rham's theorem}]\) Suppose \( G \in L^2(\Omega; \mathbb{R}^{n \times n}) \) and \( g \in L^2(\Omega; \mathbb{R}^n) \) satisfy

\[ \int_{\Omega} G : \nabla_v v + g \cdot v \, dx = 0 \quad \forall v \in V(\Omega). \]

Then there exists a unique \( p \in L^2(\Omega) \) such that

\[ \int_{\Omega} (-p \mathbf{I} + G) : \nabla_v v + g \cdot v \, dx = 0 \quad \forall v \in H^1_0(\Omega; \mathbb{R}^n). \]

Moreover, \( p \) satisfies the estimate

\[ \|p\|_{L^2(\Omega; \mathbb{R}^n)} \leq C \left( \|G\|_{L^2(\Omega; \mathbb{R}^{n \times n})}^2 + \|g\|_{L^2(\Omega; \mathbb{R}^n)}^2 \right)^{1/2}. \]

Remark 3  Note that \( p \) is unique only because of the artificial requirement that \( \int_{\Omega} p \, dx = 0 \). If this condition is omitted, then \( p \) is only unique up to a constant.

The following result is a variant of Theorem 3 for periodic functions that can be deduced from Temam (1979, p. 14).

**Theorem 4**  Suppose \( G \in L^2(\Omega \times Q; \mathbb{R}^{n \times n}) \) and \( g \in L^2(\Omega \times Q; \mathbb{R}^n) \) satisfy

\[ \int_{\Omega \times Q} G : \nabla_v v + g \cdot v \, dx = 0 \quad \forall v \in L^2(\Omega; V_{\text{per}}(Q', S)). \]

Then there exists \( p \in L^2(\Omega \times Q; \mathbb{R}^n) \) such that

\[ \int_{\Omega \times Q} (-p \mathbf{I} + G) : \nabla_v v + g \cdot v \, dx = 0 \quad \forall v \in L^2(\Omega; H^1_{\text{per}}(Q', S; \mathbb{R}^n)). \]

However, as observed in Boyer and Fabrie (2013, Ch. IV), Theorem 3 is not directly applicable when the boundary condition is of Neumann type. The abstract problem that must be solved is to characterize \( V(\Omega)^\prime \), i.e. the bounded linear functionals on \( H^1(\Omega; \mathbb{R}^n) \) that vanish on \( V(\Omega) \). This can be done by considering the transpose \( B^\prime: H^1(\Omega; \mathbb{R}^n) / V(\Omega)^\prime \to L^2(\Omega) \) of the Bogovskii operator (see Remark 1). In concrete terms, we have the following result:
Theorem 5  [The strong de Rham theorem] Suppose that $F \in L^2(\Omega; \mathbb{R}^{n,n})$ and $f \in L^2(\Omega; \mathbb{R}^n)$ satisfy
\[
\int_{\Omega} F : \nabla v + f \cdot v \, dx = 0 \quad \forall v \in V(\Omega). \tag{5}
\]
Then there exists a unique $p \in L^2(\Omega)$ such that
\[
\langle (-pI + F) \hat{h}, v \rangle_{\Omega} = \int_{\Omega} (-pI + F) \nabla v + f \cdot v \, dx = 0
\]
for all $v \in H^1(\Omega; \mathbb{R}^n)$. Moreover
\[
\|p\|_{L^2(\Omega; \mathbb{R}^n)} \leq C \left( \|F\|_{L^2(\Omega; \mathbb{R}^{n,n})}^2 + \|f\|_{L^2(\Omega; \mathbb{R}^n)}^2 \right)^{1/2}, \tag{6}
\]
where the constant $C$ depends only on $\Omega$.

3. Model and results

The steady flow of a viscous fluid at low Reynolds number can be modeled by the Stokes equation. Consider the following boundary value problem:

\[
\begin{aligned}
\nabla \cdot (-pI + 2\mu e(u')) + f &= 0 & & \text{in } \Omega', \\
\nabla \cdot u' &= 0 & & \text{in } \Omega', \\
(-pI + 2\mu e(u')) \hat{h} &= -p^b \hat{n} & & \text{on } \partial \Omega, \\
u' &= 0 & & \text{on } \Sigma',
\end{aligned}
\tag{7}
\]

where $\mu > 0$ is a kinematic viscosity coefficient, $f \in L^2(\Omega; \mathbb{R}^n)$ is an external force acting on the unit mass of fluid, $p^b \in H^{1/2}(\partial\Omega)$ is an external stress (known functions), $p' \in \mathbb{R}$ is a scalar function of the fluid pressure, and $u'$ is vector valued function of the fluid velocity (unknown functions). As mentioned in the introduction, the originality in the problem formulation resides in the non-standard boundary condition imposed on $\partial\Omega$ which is of Neumann type. Homogenizing (7) means that we study the asymptotic behavior of $u'$ and $p'$ as $\varepsilon$ tends to zero.

Remark 4  Since $p^b$ belongs to $H^{1/2}(\partial\Omega)$, there exists $\rho_b \in H^1(\Omega)$ such that $\rho(\rho_b) = p^b$ on $\partial\Omega$. Further we keep notation $p^b$ for $\rho_b$.

We say that a pair $(u', p')$ of functions $u' \in V(\Omega', S')$ and $p' \in L^2(\Omega')$ is a weak solution of the problem (7) if for any test function $v \in H^1(\Omega', S'; \mathbb{R}^n)$, the following integral equality holds:
\[
\int_{\Omega'} (-p'I + 2\mu e(u')) : \nabla v - f \cdot v \, dx = -\langle p^b \hat{n}, v \rangle_{\partial\Omega}, \tag{8}
\]
which is equivalent to
\[
\int_{\Omega'} (-p'I + 2\mu e(u')) : \nabla v - \tilde{f} \cdot v \, dx = 0 \quad \forall v \in H^1(\Omega', S'; \mathbb{R}^n), \tag{9}
\]
with $\tilde{p'} = p' - p^b$, $\tilde{f} = f - \nabla p^b$.

The main results of this paper are the following two theorems.

Theorem 6  [Existence, uniqueness, and extension] For each $\varepsilon > 0$ the boundary value problem (7) has a unique weak solution $u' \in V(\Omega', S')$, $p' \in L^2(\Omega')$. Moreover, there exist extensions
such that $U^e \in V(\Omega)$ and $P^e \in L^2(\Omega)$ satisfy

$$
\epsilon^{-2} \left\| U^e \right\|_{L^2(\Omega; \mathbb{R}^n)} + \epsilon^{-1} \left\| e(U^e) \right\|_{L^2(\Omega; \mathbb{R}^{n \times n})} \leq C, \\
\left\| P^e \right\|_{L^2(\Omega)} \leq C, 
$$

(10)

(11)

where the constant $C$ is independent of $\epsilon$.

Let the homogenized permeability tensor $K = (K_{ij})$ be defined as

$$
K_{ij} = \frac{1}{|Q'|} \int_{Q'} \nabla w^i \cdot \nabla w^j \, dy, \quad i, j = 1, \ldots, n, 
$$

(12)

where $(w^i, q^i), i = 1, \ldots, n$, are the unique solutions in $H^1_{\text{per}}(Q^\prime; \mathbb{R}^n) \times L^2(Q^\prime)$ of the cell problems

$$
\begin{align*}
\nabla q^i - \Delta w^i &= e^i & \text{in } Q^\prime, \\
\nabla \cdot w^i &= 0 & \text{in } Q^\prime, \\
w^i &= 0 & \text{in } Q^s, 
\end{align*}
$$

(13)

and $e^i, i = 1, \ldots, n$, denote the standard basis vectors in $\mathbb{R}^n$.

**Theorem 7** [Darcy’s law] The extended solution $(U^e, P^e)$ of (7) satisfies: velocity $\epsilon^{-2} U^e$ converges weakly in $L^2(\Omega; \mathbb{R}^n)$ to $v \in L^2(\Omega; \mathbb{R}^n)$ and the pressure $P^e$ converges strongly in $L^2(\Omega)$ to $p \in H^2(\Omega)$, where $v$ and $p$ are related through Darcy’s law

$$
\begin{align*}
\nabla \cdot v &= 0 & \text{in } \Omega, \\
nabla \cdot (K \nabla p) &= f - \nabla \cdot \bar{u} & \text{in } \Omega, \\
\nabla \cdot v &= 0 & \text{in } \Omega, \\
p &= p_b & \text{on } \partial \Omega, 
\end{align*}
$$

(14)

and $K$ is given by (12).

**Remark 5** The above Darcy law agrees with that of Hornung (1997) and Tartar (1980) except for the boundary condition. In Hornung (1997) and Tartar (1980) the Dirichlet condition $u^e = 0$ on $\partial \Omega$ implies the Neumann condition $\nabla \cdot \bar{u} = 0$ on $\partial \Omega$ in Darcy’s law. In the present study, we start with the Neumann condition $\left(-\mu' I + 2 \mu e(u^e)\right)\bar{u} = -p^b \bar{n}$ on $\partial \Omega$ (prescribed normal stress) and end up with the Dirichlet condition $p = p^b$ on $\partial \Omega$ in Darcy’s law. Thus there is a dual correspondence between Dirichlet and Neumann conditions in the original problem and the homogenized problem.

**4. Proof of Theorem 6**

The proof is divided into two parts. Since the existence, uniqueness and extension of the velocity are obtained as in the Dirichlet case (see Allaire, 1992; Hornung, 1997) we do not give the details. The most interesting part here concerns the uniqueness of the pressure, where the strong de Rham theorem is essentially applied.
4.1. Velocity

Equation (8) provides the bilinear form $B$ acting on the space $V(\Omega', S')$:

$$B[u, v] = 2\mu \int_{\Omega} e(u) : e(v) \, dx.$$  

The inequality (3) allows us to apply the Lax–Milgram lemma (see Evans, 1998, p. 297) to obtain a unique solution $u^r \in V(\Omega', S')$ of (7).

It is natural to extend $u^r$ by zero in $\Omega \setminus \Omega'$ since this is compatible with the no-slip boundary condition on $S'$:

$$U^r = \begin{cases} u^r & \text{in } \Omega', \\ 0 & \text{in } \Omega \setminus \Omega' \end{cases}, \quad U^r \in H^1(\Omega; \mathbb{R}^n).$$

The estimates (10) are achieved by applying (1), (2) to (8) when $v = u^r$.

4.2. Pressure

The existence of pressure is based on the construction of a restriction operator $H^1(\Omega; \mathbb{R}^n) \to H^1(\Omega', S'; \mathbb{R}^n)$ (see Allaire, 1992; Mikelić, 1991; Tartar, 1980) and the strong de Rham theorem. Note that the restriction operator that appears in these papers is defined on $H^1_0(\Omega; \mathbb{R}^n)$, but since the construction is local it works also in our case.

**Lemma 1** There exists a linear continuous operator $R: H^1(\Omega; \mathbb{R}^n) \to H^1(\Omega', S'; \mathbb{R}^n)$ such that

$$R(v) = v \text{ in } \Omega' \quad \text{if } v = 0 \text{ on } S',$$

$$\nabla \cdot R(v) = 0 \text{ in } \Omega' \quad \text{if } \nabla \cdot v = 0 \text{ in } \Omega.$$

(see Hornung, 1997, pp. 52–53 and Miroshnikova, 2016 for details)

Using the same approach as for Theorem 2, one can show that

$$\{ \nabla \cdot v : v \in H^1(\Omega', S'; \mathbb{R}^n) \} = L^2(\Omega'). \quad (15)$$

To prove the existence of pressure in $\Omega$, we define

$$L^r(v) = \int_{\Omega} 2\mu \nabla u^r : \nabla v \, dx - \tilde{f} \cdot R(v) \, dx, \quad v \in H^1(\Omega; \mathbb{R}^n),$$

which is a bounded linear functional on $H^1(\Omega; \mathbb{R}^n)$. In view of the strong de Rham theorem (Theorem 5), there exists a unique $\bar{P}^r \in L^2(\Omega)$ such that

$$L^r(v) = \int_{\Omega} \bar{P}^r \cdot \nabla v \, dx \quad \text{and} \quad \|\bar{P}^r\|_{L^2(\Omega)} < C. \quad (16)$$

By choosing $v \in H^1(\Omega; \mathbb{R}^n)$ such that $v = 0$ on $S'$ it is clear that $\bar{P}^r = \bar{P}^r_{|\Omega'}$, the restriction of $\bar{P}^r$ to $\Omega'$ is a solution of (9). Uniqueness of $\bar{P}^r$ follows from (15) and the equality

$$\int_{\Omega} \bar{P}^r \cdot \nabla v \, dx = \int_{\Omega} \bar{P}^r \cdot R(v) \, dx.$$
By density arguments (see Theorem 2) from (16), we obtain the uniqueness of $\hat{P}$. To show that $P^* = \hat{P} + p^b$ is a constant in $\Omega \setminus Q'$ one can choose $\nu \in H^1(\Omega; \mathbb{R}^n)$ such that $\text{supp}(\nu) \subset Q_{\iota}, i \in I(\varepsilon)$, and use properties of $\nabla \cdot R_1(\nu)$ (see Hornung, 1997, p. 53).

5. Homogenization

5.1. Two-scale convergence

Definition 1 A bounded sequence $\{u'\} \subset L^2(\Omega; \mathbb{R}^n)$ is said to two-scale converge to a limit $u \in L^2(\Omega \times Q; \mathbb{R}^n)$ if for any function $\varphi \in C^\infty_c(\Omega(\mathbb{C}_\text{per}(Q; \mathbb{R}^n)))$
\[
\lim_{\varepsilon \to 0} \int_\Omega u'(x) \cdot \varphi(x, \frac{x}{\varepsilon}) \, dx = \int_\Omega u(x, y) \cdot \varphi(x, y) \, dx \, dy.
\]
Moreover, a bounded sequence $\{f^*_\varepsilon\}$ in $H^{-1}(\Omega; \mathbb{R}^n)$ is said to two-scale converge to a limit $f \in H^{-1}(\Omega \times Q; \mathbb{R}^n)$ if for any function $\varphi \in C^\infty_c(\Omega(\mathbb{C}_\text{per}(Q; \mathbb{R}^n)))$
\[
\lim_{\varepsilon \to 0} (f^*_\varepsilon, \varphi(\frac{x}{\varepsilon}))_{H^{-1}(\Omega; \mathbb{R}^n)} = (f, \varphi)_{H^{-1}(\Omega \times Q; \mathbb{R}^n)}.
\]

Theorem 8 Let $\{u'\}$ be some bounded sequence in $L^2(\Omega; \mathbb{R}^n)$. Then there exists a subsequence $\{u'_{\varepsilon}\}$ which two-scale converges to $u \in L^2(\Omega \times Q; \mathbb{R}^n)$.

The proof can be found in Lukkassen et al. (2002).

The following result is a consequence of Allaire (1992, Prop. 1.14).

Theorem 9 Let $\{u'\}$ be a sequence in $H^1(\Omega; \mathbb{R}^n)$ such that for any $u'$
\[
\|u'\|_{L^2(\Omega; \mathbb{R}^n)} + \varepsilon \|e(u')\|_{L^2(\Omega; \mathbb{R}^n)} \leq \varepsilon^2 C,
\]
where the constant $C$ is independent of $\varepsilon$. Then there exists a function $u \in L^2(\Omega; H^1_\text{per}(Q; \mathbb{R}^n))$ such that, up to a subsequence, $\{\varepsilon^{-2} u'\}$ and $\{\varepsilon^{-1} e(u')\}$ two-scale converge to $u$ and $e_j(u) = 1/2 \left( \nabla_j u + (\nabla_j u)^T \right)$, respectively.

5.2. Homogenized problem

The following convergence result holds:

Theorem 10 Let $U^*$ and $P^*$ be as in Theorem 6. Then
\begin{enumerate}
\item $\{\varepsilon^{-2} U'\}$ two-scale converges to $u \in L^2(\Omega \times Q; \mathbb{R}^n)$;
\item $\{\varepsilon^{-1} e(U')\}$ two-scale converges to $e_j(u) \in L^2(\Omega \times Q; \mathbb{R}^n)$;
\item $\{P^*\}$ converges strongly to $p$ in $L^1(\Omega)$;
\item $\varepsilon \nabla P^*$ two-scale converges to $\nabla_j p_1 \in H^{-1}(\Omega \times Q; \mathbb{R}^n)$,
\end{enumerate}
where $u, p, p_1$ constitute the unique solution
\[
(u, p, p_1) \in L^2(\Omega; H^1_\text{per}(Q; \mathbb{R}^n)) \times H^1(\Omega \times \Omega; \mathbb{R}^n) \times L^2(\Omega; L^2(Q'))
\]
of the following two-pressure Stokes problem:

\[
\begin{aligned}
\nabla_x p + \nabla_y p_1 + \mu \Delta_y u &= f & \text{in } & \Omega \times Q', \\
\nabla_y \cdot u &= 0 & \text{in } & \Omega \times Q', \\
\n\nabla_x \cdot \int_0^1 u dy &= 0 & \text{in } & \Omega, \\
\n\nabla_y \cdot u &= 0 & \text{in } & \Omega \times Q^a, \\
\np &= p^b & \text{on } & \partial \Omega, \\
\nu, p_1 & \text{ are } Q - \text{periodic with respect to } y.
\end{aligned}
\]  

(17)

Proof. From Theorem 9 and estimate (10), it follows that all two-scale limits exist. Moreover, since \(U^\epsilon\) is zero in \(\Omega'\) and divergence free it satisfies

\[
\int_\Omega U^\epsilon \cdot \nabla \psi \left( x, \frac{x}{\epsilon} \right) dx = 0 \quad \forall \psi \in C^\infty_c(\Omega') \cap C^\infty(\Omega). 
\]  

(18)

By standard procedure (Allaire, 1991; Hornung, 1997) choosing different types of test function \(\psi \in C^\infty_c(\Omega') \cap C^\infty(\Omega)\) in (18) and passing to the limit \(\epsilon \to 0\) we obtain

\[
\begin{aligned}
\nabla_y u(x,y) &= 0 \quad (x,y) \in \Omega \times Q, \\
\nabla_y \cdot \int_0^1 u(x,y) dy &= 0 \quad x \in \Omega, \\
\nu(x,y) &= 0 \quad (x,y) \in \Omega \times Q^a.
\end{aligned}
\]

Since \(\{\hat{p}^i\}\) is bounded (see (11)), it has a two-scale limit \(\tilde{p} \in L^2(\Omega \times Q)\).

Consider the weak formulation (9), i.e.

\[
\int_\Omega \left( -\hat{p} I + 2\mu \epsilon(\hat{u}^\epsilon) : \nabla \psi \left( x, \frac{x}{\epsilon} \right) - f \cdot \psi \left( x, \frac{x}{\epsilon} \right) \right) dx = 0,
\]

(19)

with \(\psi(x,y) \in C^\infty(\Omega \times \mathbb{R}^n)\) such that \(\psi = 0\) in \(\Omega \times Q^a\).

Multiplying (19) with \(\epsilon\) and passing to the limit \(\epsilon \to 0\) in (19), we obtain

\[
\int_{\Omega \times Q^a} \tilde{p}(x,y) \nabla_y \cdot \psi(x,y) dy dx = 0.
\]

Since \(Q^a\) is connected we conclude that \(\tilde{p}\) depends only on \(x\), i.e. \(p \in L^2(\Omega)\).

Next step is to use test functions that are divergence-free with respect to \(y\). By passing to the limit in (19), we obtain

\[
\int_{\Omega \times Q^a} -\tilde{p} \nabla_x \cdot \psi + 2\mu \epsilon_j(u) : \nabla \psi - f \cdot \psi dy dx = 0.
\]

(20)

Choosing \(\psi(x,y) = \phi(x) \Phi(y)\), \(\phi \in C^\infty(\Omega)\), \(\Phi \in C^\infty(\Omega' \cap \mathbb{R}^n)\), so that \(\Phi = 0\) in \(Q^a\), \(\nabla \cdot \Phi = 0\) and \(\int_{Q^a} \Phi dx = e^i, i = 1, \ldots, n\) (the existence of such functions \(\Phi\) is shown in Miroshnikova (2016)) gives

\[
\int_{\Omega} -\tilde{p} \nabla_x \phi \cdot \left( \Phi dy \right) + \phi \left( \int_{Q^a} 2\mu \epsilon_j(u) : \nabla_y \Phi - f \cdot \Phi dx \right) dx = 0,
\]
which is equivalent to
\[
\int_{\Omega} -\bar{\rho} \frac{\partial \phi}{\partial x_i} + G' \phi \, dx = 0 \quad \forall \phi \in C^\infty(\Omega),
\]
where
\[
G' = \int_{\Omega'} 2\mu \varepsilon_y(u) : \nabla \psi' - \bar{f} \cdot \psi' \, dy.
\]
Since \( G' \in L^2(\Omega) \) it follows from Brezis (2005, Proposition IX.18) that
\[
\bar{\rho} \in H^1_0(\Omega) \quad \text{with} \quad \frac{\partial \bar{\rho}}{\partial x_i} = -G'.
\] (21)
Thus (20) can be written as
\[
\int_{\Omega \times \Omega'} 2\mu \varepsilon_y(u) : \nabla \psi + (\nabla \bar{\rho} - \bar{f}) \cdot \psi \, dy \, dx = 0.
\]
By Theorem 4 there exists \( p_1 \in L^2(\Omega L^2_0(\Omega')) \), such that
\[
\int_{\Omega \times \Omega'} (-p_1 I + 2\mu \varepsilon_y(u)) : \nabla \psi + (\nabla \bar{p} - \bar{f}) \cdot \psi \, dy \, dx = 0
\] (22)
for all \( \psi \in C^\infty(\Omega; C^\infty_{per}(\Omega'; \mathbb{R}^n)) \) such that \( \psi = 0 \) in \( \Omega \times Q' \).
Since \( \bar{\rho} = p - p^b \) where \( p \) is the two-scale limit of \( \{p'\} \), we can write (22) as
\[
\int_{\Omega \times \Omega'} (-p_1 I + 2\mu \varepsilon_y(u)) : \nabla \psi + (\nabla p - f) \cdot \psi \, dy \, dx = 0.
\]
By similar arguments, the two-scale convergence of \( \varepsilon \nabla p' \) in \( H^1 \) follows directly and the strong convergence of \( P' \) in \( L^2(\Omega) \) was shown by Hornung (1997), Allaire (1992). This completes the proof. \( \square \)

The only statement that has to be proved is Darcy’s law, formulated in Theorem 7. Now it is a direct corollary of Theorem 10.

5.3. Proof of Theorem 7
The existence of extensions \( U' \) and \( P' \) and their convergence have been demonstrated in Theorems 6 and 10 correspondingly. There remains to derive the Darcy law 14.

From the first equation in (17), by linearity we have
\[
u(x, y) = \frac{1}{\mu} \sum_{i=1}^n \left( f_i(x) - \frac{\partial p(x)}{\partial x_i} \right) w_i(y), \quad (x, y) \in \Omega \times Q',
\]
\[
p_i(x, y) = \sum_{i=1}^n \left( \frac{\partial p(x)}{\partial x_i} - f_i(x) \right) q_i(y), \quad (x, y) \in \Omega \times Q',
\]
where \( w_i', q_i' \), \( i = 1, \ldots, n \), are solutions of corresponding cell-problems (13).

Averaging \( u \) over \( Q \) yields
\[
\bar{u} = \frac{1}{\mu} K (f - \nabla p), \quad \text{where} \ K \ \text{is as in (12)}.
\]
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