Nonlinear Interactions in Spherically Polarized Alfvénic Turbulence

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Turbulent magnetic field fluctuations observed in the solar wind often maintain a constant magnitude condition accompanied by spherically polarized velocity fluctuations; these signatures are characteristic of large-amplitude Alfvén waves. Nonlinear energy transfer in Alfvénic turbulence is typically considered in the small-amplitude limit where the constant magnitude condition may be neglected; in contrast, nonlinear energy transfer in the large-amplitude limit remains relatively unstudied. We develop a method to analyze finite-amplitude turbulence through studying fluctuations as constant magnitude rotations in the stationary wave (de Hoffmann-Teller) frame, which reveals that signatures of finite-amplitude effects exist deep into the MHD range. While the dominant fluctuations are consistent with spherically-polarized large-amplitude Alfvén waves, the subdominant mode is relatively compressible. Signatures of nonlinear interaction between the finite-amplitude spherically polarized mode with the subdominant population reveal highly aligned transverse components. In theoretical models of Alfvénic turbulence, alignment is thought to reduce nonlinearity; our observations require that alignment is sufficient to either reduce shear nonlinearity such that non-Alfvénic interactions may be responsible for energy transfer in spherically polarized states, or that counter-propagating fluctuations maintain anomalous coherence, which is a predicted signature of reflection-driven turbulence.

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Introduction

Spacecraft observations of the solar wind provide much of the context for our understanding of nonlinear interactions occurring in plasma environments [1]. Although the solar wind is collisionless, many of its dynamics can be understood using fluid approximations, such as magnetohydrodynamics (MHD) [2]. To understand nonlinear interactions in MHD, the velocity \( \mathbf{v} \) and magnetic field \( \mathbf{B} \) are cast into the Elsasser variables \( z^\pm = \mathbf{v} \pm \mathbf{b} \) with \( \mathbf{b} = B/\sqrt{\mu_0 \rho} \) and mass density \( \rho \) [3]. Assuming incompressibility, which is motivated by our observations,

\[
\partial_t z^\pm = -z^\mp \cdot \nabla z^\pm - \frac{1}{\rho} \nabla (\rho v_n + \frac{\rho}{2} b^2).
\] (1)

The Elsasser fluctuations can be identified as travelling parallel \( z^- \) and anti-parallel \( z^+ \) to the mean magnetic field \( \mathbf{b}_0 \) [3, 4]. Fluctuations in the solar wind, defined as \( \delta \mathbf{x} = \mathbf{x} - \mathbf{x}_0 \), are dominated by fluctuations perpendicular to \( \mathbf{b}_0 \) with \( \delta \mathbf{v}_\perp \approx \pm \delta \mathbf{b}_\perp \) [5]. In the small-amplitude limit, Alfvén waves are approximated by transverse linear plane-waves polarized perpendicular to both wavevector \( \mathbf{k} \) and the background field \( \mathbf{b}_0 \). This limit is well described by Reduced MHD (RMHD), where \( z^\pm \) perturbations are perpendicular, \( z^\pm = z_\perp^\pm \), polarized perpendicular to their wave vector \( \mathbf{k} \), and can be written using the Elsasser potentials \( \zeta^\pm \) as \( z^\pm = b_0 \times \nabla_\perp \zeta^\pm \) [12, 21]. Observations in the solar wind are often dominated by one of \( \delta z_\perp^\pm \) [5]—a condition that is commonly known as imbalance [6]. In the solar wind, imbalance favors the anti-sunward propagating Alfvén wave population [7, 8]. In turn, the subdominant Elsasser variable corresponds to a sunward propagating mode that interacts nonlinearly with the dominant mode via the \( \delta z^\mp \cdot \nabla \delta z^\pm \) term in Equation (1). This mutual shearing of counter-propagating fluctuations in MHD is thought to generate a cascade, similar to neutral-fluid turbulence, that results in inertial range energy transfer [9, 20].

Theoretical consideration of magnetized turbulence is often performed in a small-amplitude limit [9, 20]; however, the solar wind is subject to large-amplitude \(|\delta \mathbf{b}|/|\mathbf{b}_0| \sim 1 \) fluctuations that maintain

\[ |\mathbf{b}| = |\mathbf{b}_0 + \delta \mathbf{b}| = \text{const}, \]

and thus appear spherically-polarized [25, 26]. This condition is characteristic of large-amplitude Alfvén waves, which are not entirely perpendicular to \( \mathbf{b}_0 \), but can acquire a component parallel to \( \mathbf{b}_0 \) in order to maintain constant magnitude of the total magnetic field [26, 29]. The small-amplitude limit negates the higher-order corrections that maintain constant total magnitude.

Observations from Parker Solar Probe (PSP) reveal that both the constant-magnitude condition and high-Alfvénicity are pronounced in the inner-heliosphere [30, 38], and are consistent with the large-amplitude Alfvén mode; however, these studies mostly omit di-
cussion of observed finite-amplitude signatures in turbulence. Both magnetic field and velocity fluctuations show finite-amplitude signatures of spherical polarization \cite{30,40} consistent with the transverse Alfvén mode \cite{29}. While spherical polarization of the velocity fluctuations is independent of (Galilean) reference frame, constant-magnitude of the total velocity vector is only maintained in the frame at the center of spherical polarization \cite{29,30,41}, associated with the de Hoffmann-Teller frame (dHTf) \cite{41}. Analysis in the dHTf enables fluctuations to be characterized in terms of constant magnitude rotations corresponding to large-amplitude Alfvén waves \cite{29,40,43}. Formally, the dHTf minimizes the convected electric field $E = -v \times B$, which is ideally zero in a stationary frame of transverse electromagnetic waves. The existence of an empirically measurable dHTf indicates significant alignment between magnetic and velocity fluctuations, which constrains turbulent energy transfer.

In this Letter, we discuss the nonlinear interactions observed in spherically polarized turbulence, which we measure to be highly aligned in nature. The relation between alignment and nonlinearity is clear when RMHD is written using the Elsasser potentials

$$\partial_t \nabla_\perp^2 \zeta^\pm \propto \{\zeta^+, \nabla_\perp^2 \zeta^-\} + \{\zeta^-, \nabla_\perp^2 \zeta^+\} = \nabla_\perp \{\zeta^+, \zeta^-\},$$

with $\{A, B\} = (\nabla \perp A \times \nabla \perp B \cdot \hat{b}_0$. Because $\{A, B\}$ vanishes if $\nabla \perp A$ and $\nabla \perp B$ are parallel, the final term vanishes for aligned $z^\pm$. The other nonlinear terms are significantly reduced if contours of $\nabla_\perp^2 \zeta^\pm$ are approximately aligned with $\zeta^\mp$, a condition that is satisfied for perturbations that look locally like sheets or tubes. In the theory of dynamic alignment, cascading turbulent structures are sheet-like and $\delta z^+$ and $\delta z^-$ shear themselves into alignment (i.e., becoming more parallel at smaller scales), such that $\delta z^+ \cdot \nabla \delta z^+$ is reduced by a factor $\sim \sin \phi_k$, where $\phi_k$ is the alignment angle between $\delta z^+$ and $\delta z^-$ at scale $k$. This depletes the nonlinearity towards smaller scales, flattening $\sim k^{-5/3}$, energy spectrum to $\sim k^{-3/2}$, which is often observed in the solar wind \cite{29,33,43}.

The dynamic alignment argument neglects the fact that perfectly aligned fluctuations may have significant nonlinearity. Because $\delta z^+ \cdot \nabla \delta z^+ = \delta z^+ \cdot \nabla \delta z^- - \nabla \times (\delta z^+ \times \delta z^-)$ it is possible to have both $z^\pm \propto z_k^\pm$ and significant nonlinearity as long as $\delta z^\pm \cdot \nabla \delta z^\pm \approx \delta z^\pm \cdot \nabla \delta z^\pm$. In essence, it is possible that $z_k^+ \propto z_k^\pm (\zeta^+ \propto \zeta^-)$, while not satisfying a specific topology, in which case aligned fluctuations have significant nonlinearity. Here, we show that measurements of highly aligned $z^\pm$ suggests either (i) that large-amplitude fluctuations are sheet or tube-like structures and nonlinearity is significantly reduced to an extent that non-Alfvén modes contribute significantly to nonlinear energy transfer, or (ii) $z^-$ and $z^+$ are “anomalously coherent” (nearly proportional), but that alignment does not affect nonlinear energy transport.

\textbf{Data} \quad \text{PSP provides measurements of the inner heliosphere using the electromagnetic FIELDS \cite{53} and Solar Wind Electron Alpha and Proton (SWEAP, \cite{49}) instrument suites. We study a stream from PSP peri-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{(a) Velocity measurements in dHTf (green), magnetic field fluctuations in Alfvén units (black). (b) Elsasser variables $z^\pm$ constructed in the dHTf (black +, blue -).}
\end{figure}

\textbf{Turbulent Signatures in the dHTf} \quad \text{Fig. 1(a) shows measurements of $b$ and $v' = v - v_{dHT}$, with projections}
on each 2D plane. \( \mathbf{v}_{dHT} \) is found through minimizing

\[
E^2 = \sum_i ((\mathbf{v}_{dHT} - \mathbf{v}_i) \times \mathbf{B}_i)^2
\]

with respect to \( \mathbf{v}_{dHT} \). Fig. 1(b) shows the dHTf Elsasser variables defined as \( z^+ = \mathbf{v} + \frac{1}{2} \mathbf{B} \), with \( \mathbf{v}_{dHT} = [-71, 23, -342] \) km/s (\( \mathbf{v} = \mathbf{v}_{sw} = [-99, 26, -265] \) km/s. The Alfvén speed is \( \mathbf{v}_A = \mathbf{b} = [19, 10, 97] \) km/s. The mean solar wind speed is \( |\mathbf{v}_{sw}| = 295 \) km/s, with central half of values \( 265 < |\mathbf{v}_{sw}| < 310 \) km/s. Fig. 1(b) shows that \( z^+ \) is well approximated by a spherical surface, with constant magnitude \( |z^+| = 221 \) km/s. In contrast, \( z^- \) has significant compressibility. The magnetic field is taken as frame invariant and no substantial effects were found when introducing kinetic Alfvén speed normalizations [43, 54].

A local, scale-dependent, dHTf, \( \bar{\mathbf{v}}_{l,t+t} \), is constructed using two point increments and averages

\[
\Delta \mathbf{x} = \mathbf{x}(t + \tau) - \mathbf{x}(t) \quad \bar{x} = \frac{\mathbf{x}(t) + \mathbf{x}(t + \tau)}{2}
\]

in \( \mathbf{v} \) and \( \mathbf{b} \) for every increment pair [55],

\[
E^2_{x,t+t} = \left( [\bar{\mathbf{v}} - \Delta \mathbf{v}] \times (\mathbf{b} + \Delta \mathbf{b}) \right)^2
\]

\[
\frac{\partial E^2}{\partial \mathbf{v}} = 0.
\]

**Rotational Increments**  In the small-amplitude limit, perpendicular and parallel increments discern between Alfvénic and compressible fluctuations [12, 18, 23, 56, 57]

\[
\Delta \mathbf{x}_\parallel(t, \tau) = (\Delta \mathbf{x}(t, \tau) \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}}
\]

\[
\Delta \mathbf{x}_\perp(t, \tau) = \Delta \mathbf{x}(t, \tau) - \Delta \mathbf{x}_\parallel(t, \tau),
\]

\[
\bar{x} = \frac{x}{|x|}.
\]

In contrast, the finite-amplitude Alfvén mode is associated with a constant magnitude rotation, we thus introduce a **rotational increment**:

\[
\Delta \mathbf{x}(t, \tau) = \mathbf{x}(t) - \mathbf{R}_{t,t+\tau} \mathbf{x}(t),
\]

\[
\mathbf{R}_{t,t+\tau} = (\bar{x}(t + \tau) \bar{x}(t))^T
\]

which is the part of \( \Delta \mathbf{x} \) involving only a rotation of \( \mathbf{x} \). The rotational increment is only sensible to define for \( \mathbf{v} \) and \( z^+ \) fluctuations in the dHTF, where the velocity fluctuations have (approximately) constant magnitude. Outside of the dHTF, a constant magnitude rotation in velocity includes contribution from the offset-center of the spherical surface into the estimate of \( \Delta \mathbf{v}_{R} \). Each rotational increment for \( \mathbf{v}' \) and \( z'^+ \) is defined in its local, scale dependent, \( \bar{\mathbf{v}}_{l,t+t} \) frame. The non-rotational portion of the increment \( \Delta \mathbf{x}_C \) is

\[
\Delta \mathbf{x}_C = \Delta \mathbf{x} - \Delta \mathbf{x}_R.
\]

**FIG. 2** (a-b) Structure functions \( S_2 \) for finite-amplitude increments (\( \Delta \mathbf{x}, \Delta \mathbf{x}_R, \Delta \mathbf{x}_C \) for \( \mathbf{b} \) (red), \( \mathbf{v}' \) (green) and \( z'^+ \) (black) normalized to \( |z'| \); total \( \Delta z'^+ \) is in blue. Power-law fits to structure functions of \( \Delta z'^+ \) are in black lines. (c) Small amplitude \( \Delta z'^+ \) (black), and \( z'^- \) (blue) increments. Perpendicular \( \perp \) and parallel \( \parallel \) increments are shown respectively as + and ×. As described in the text, a bound on instrumental noise determined from unaligned increments in \( \mathbf{v} \) is shown in orange. Panels (d) and (e) show the ratio of \( S_2^2/S_2 \) and \( S_2^2/S_2 \) for \( \Delta z'^+ \), \( \Delta \mathbf{v}' \) and \( \Delta \mathbf{b} \).

Fig. 2(a) shows the square-root of the trace of the second order structure functions \( S_2 = \langle \Delta \mathbf{x}' \Delta \mathbf{x}' \rangle \) for finite-amplitude increments in \( \mathbf{b} \) and \( \mathbf{v}' \). The amplitudes are normalized to the magnitude \( |z'| = 221 \) km/s. The non-rotational portion of the velocity fluctuations \( \Delta \mathbf{v}_C \) is significantly larger than \( \Delta \mathbf{b}_C \). Fig. 2(b) shows total, rotational, and non-rotational increments in \( z'^+ \) normalized to \( |z'| \). Alfvénic fluctuations in \( z'^- \) are not ex-
Nonlinearities

Nonlinear turbulent interactions of finite-amplitude Alfvén waves are poorly understood, yet key to understanding how these fluctuations evolve and dissipate. We study the strength of nonlinear interactions using cross-terms $\langle |\Delta x||\Delta y| \rangle$ [13, 22]; this assumes that nonlinear interactions are local in scale. The strength of the nonlinearity associated with transverse shear Alfvén wave interactions is given as $\langle |\Delta z^+||\Delta z^-| \rangle$. Similarly, we estimate shear interactions not associated with Alfvén modes using $\langle |\Delta z^-||\Delta z^+| \rangle$. Fig. 3 shows the mean magnitude quantities of nonlinear terms. The transverse nonlinearity $\langle |\Delta z^+||\Delta z^-| \rangle$ dominates across MHD scales.

Nonlinear term $[\text{km/s}]^2$

Nonlinear interactions of Alfvénic turbulence in a large-amplitude limit, in which the dominant mode corresponds to constant magnitude rotations [20, 29]. The Alfvénic constant $|B|$ state is a nonlinear equilibrium solution satisfied in physical space that affects all scales contributing significantly to $\delta B$. Our development of rotational increments shows that the constant magnitude condition, and large amplitude fluctuations $\delta B/B > 0.3$, observed across scales, such that if the shear nonlinearity is depleted according to $\delta z^+ \cdot \nabla \delta z^+ \sim \nabla \times (\delta z^+ \times \delta z^+)$, then the non-transverse terms become as strong as the Alfvénic nonlinearity. Reduction in the Alfvénic nonlinearity indicates that the rotational $\Delta z^+$ is highly aligned with $\Delta z^-$.

The sensitivity of our results to instrumental noise is tested by computing unaligned fluctuations in $\Delta v_L$, as $\epsilon_v = \Delta v_L \times \Delta b_L$, as a bound on measurement uncertainty (i.e., assuming any unaligned $\Delta v_L$ is entirely noise). The rms value of $\epsilon_v$ at smallest scales ($\tau_{\text{min}}$ = 7s) is 5.4 km/s, which is an upper-bound on the error and corresponds to $\sqrt{\epsilon_v^2} \sim 0.02(\text{km/s})$. The smallest scale noise is $\sqrt{3\epsilon_v^2} \sim 9$ km/s or $\sqrt{5\epsilon_v^2} \sim 0.05(\text{km/s})$. Smallest $\Delta z^-$ fluctuations are at $\sim (0.1 \text{km/s})$. Fig. 2(c) shows that the alignment strongly affects the shear Alfvén-wave nonlinearity. This effect is

$$\Delta b^2 = b^2(t + \tau) - b^2(t),$$

and thermal pressure gradients assuming $p_{th} = c_s^2 \rho$

$$c_s^2/b_0^2 = \beta = 2\frac{\mu_0 n_p}{B^2}(2/3T_{\perp} + 1/3T_{\parallel}),$$

$$\Delta p_{th} = \beta b_0^2 \frac{\Delta \rho}{\rho},$$

where $T_{\perp, \parallel}$ is defined using proton core and beam populations [31]. The median $\beta = 0.66$ with the central half of values between 0.48 < $\beta$ < 0.75. Though the isothermal equation of state is likely not well satisfied, and full compressible treatment requires additional terms in Equation 1, Fig. 3 shows the dominance of shear nonlinearities, justifying an incompressible approximation.

We study the effect of alignment on nonlinearity using the terms $\langle |\Delta z^+||\Delta z^-| \rangle$ and $\langle |\Delta z^+||\Delta z^-| \rangle$ [13, 22, 23, 67]. Fig. 2 shows that the alignment strongly affects the shear Alfvén-wave nonlinearity. This effect is
can be maintained deep into scales traditionally associated with an “inertial range” (Fig. 2a–c) [9, 11]. Our observations suggest that fluctuations across all MHD scales may maintain correlations to keep |B| constant [59].

While incompressible shearing between counterpropagating Alfvén waves is often invoked in describing magnetized Alfvénic turbulence, theories are typically developed in limits that omit correlations that maintain the constant magnitude condition [9, 11, 16, 29]. The presence of correlations between that maintain constant magnitude throughout MHD-range fluctuations, suggest that finite-amplitude dynamics [68, 70] are likely important nonlinear processes. Finite-amplitude effects may contribute to the growth and evolution of MHD-scale compressible fluctuations, as well as kinetic processes [77, 78].

Our observations show that large-amplitude fluctuations in the solar wind are highly aligned and that measured nonlinearities are not in agreement with common understandings of alignment on Alfvénic turbulence [17]. Across all scales, the Alfvénic (rotational) increments of $z^+$ are aligned with Alfvénic (perpendicular) increments of $z^-$, such that $\langle |\Delta z^+_R \times \Delta z^-_R| \rangle \ll \langle |\Delta z^+_R| \Delta z^-_R \rangle$. Strong alignment only affects the Alfvénic part of $z^-$: the parallel $z^-$ increment is not strongly aligned ($\langle |\Delta z^+_R \times \Delta z^-_R| \rangle \approx \langle |\Delta z^+_{Rz} \Delta z^-_{Rz}| \rangle$), such that $\langle |\Delta z^+_R \times \Delta z^-_R| \rangle \approx \langle |\Delta z^+_{Rz} \times \Delta z^-_{Rz}| \rangle$ even though $\langle |\Delta z^-_R|^2 \rangle < \langle |\Delta z^-_{Rz}|^2 \rangle$. The nonlinearity associated with the magnetic pressure is weak relative to the shear interactions, suggesting that the waves relax to the constant magnitude state efficiently [61]. Recent-work shows that small perturbations in magnetic pressure may be an effect of expanding large-amplitude Alfvén waves with oblique wave-vectors [60].

Interpreting these results hinges on the relevance of alignment to strength of nonlinear interactions. If the Alfvénic nonlinearity is depleted by $\sim \sin \phi_z$ [17], then our results imply that the non-Alfvénic nonlinearity (involving $z^0$) is approximately equal to the Alfvénic shear-nonlinearity, challenging the basis for RMHD turbulence phenomenologies. If alignment is not directly related to the nonlinearity, which occurs if $\delta z^+ \cdot \nabla \delta z^\perp \approx \delta z^\perp \cdot \nabla \delta z^+$, then our results imply that $z^\perp$ and $z^+$ are anomalously coherent, despite propagating in opposite directions. Either way, our results do not support a dynamically aligned cascade phenomenology often invoked to explain observed spectra [16, 18]. Our results further suggest that alignment has little bearing on the development of inertial range turbulence from outer scales [56, 57]. The possibility that nonlinearity is unrelated to alignment, and that the Elsasser variables are anomalously coherent, resembles theories of reflection-driven turbulence [15, 47], where $z^-$ is strongly aligned with $z^+$ because it is driven directly by wave reflection from large-scale gradients. In any case, our results question the relevance of RMHD turbulence theories to the inner-heliosphere and provide constraints on dynamics of finite-amplitude turbulence.

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