Gravitationally collapsing stars in $f(R)$ gravity

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The gravitational dynamics of a collapsing matter configuration which is simultaneously radiating heat flux is studied in $f(R)$ gravity. Two particular functional forms in $f(R)$ gravity are considered to show that it is possible to envisage boundary conditions such that the end state of the collapse has a weak singularity and that the matter configuration radiates away all of its mass before collapsing to reach the central singularity.

Keywords: Gravitational collapse. $f(R)$ gravity.

1 Introduction

The general theory of relativity (GR) is an impeccably robust relativistic theory of gravity. It forms the basis for our understanding of gravitational phenomenon at small as well as large scales [1, 2]. However, it is well known that GR cannot be the ultimate theory of gravitation since it has a well defined regime of validity; for example, understanding past and future spacetime singularities are beyond the reach of GR. Suitable modification(s) of GR is(are) essential to understand singularities, or to resolve them. It is believed that a quantum theory of gravity may lead to solution to the problems affecting GR [2, 3, 4, 5, 6]. Naturally, in absence of any consensus on the theory of quantum gravity, modified theories of gravity with quantum corrections are also of interest. The Einstein- Hilbert action may be thought of as only a low energy contribution and higher curvature terms consistent with the diffeomorphism invariance may become relevant as one goes to higher energies. Higher curvature corrections should leave imprints at low energy scales which become important for low energy physics too [5, 6, 7]. Out of these alternate theories, we shall study the $f(R)$ model since it has been found interesting in the cosmological studies as well. These $f(R)$ gravity models are thought of as alternate to the dark energy models [8, 9, 10]. The standard way to construct these $f(R)$ theories is to replace the Einstein- Hilbert Lagrangian by a well defined

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function of Ricci scalar, \( f(R) \) (for general relativity \( f(R) = R \)) \[11\]. For a detailed review of the motivation, validity of various functional forms \( f(R) \), applications as well as shortcomings of these gravity theories have been extensively analyzed \[12, 13, 14, 15, 16, 17, 18\].

The purpose of the present work is to construct, in \( f(R) \) gravity, spacetimes formed due to gravitational collapse which admit gravitationally weak spacetime singularities \[19\]. Usually, gravitationally collapsing (spherical) matter configurations have density and curvatures diverging at the center. However, it is possible to arrange the matter and geometric variables (without violating energy conditions) in a way that during the collapse, the configuration radiates away mass (in the form of heat flux) at such a rate that the matter boundary never reaches its Schwarzschild horizon. Hence, no horizon is formed at the star boundary and the matter center remains visible to the observers at infinity. A further consequence of these conditions is that the matter center admits a weak singularity and all the mass is radiated away before the star collapses to the central singularity. This collapsing configuration is such that the energy density, radial and tangential pressure, pressure anisotropy, heat flux, remain regular and positive throughout the collapse. The luminosity and adiabatic index are also regular and positive, and admits maximum value when the star approaches the singularity. Thus, for an observer at infinity observing the collapse, the star shall become extremely bright, reaching its maximum luminosity before turning off, indicating that it has radiated off all its mass. Solutions of such kind are not unknown and is possible in the Newtonian gravity as well. Consider a star in the Newtonian gravity which is extremely heavy to be supported by the Pauli exclusion principle alone. So, when the gravitation contraction takes place, thermal pressure may balance to some extent. But since there is no event horizon, the star shall continue to radiate all the gravitational energy to infinity. Hence all the matter contained in the star shall be converted to thermal radiation and radiated off. We shall show here that configurations of similar nature are also possible in \( f(R) \) gravity. The matter configuration we consider in this paper satisfy the equations of motion of \( f(R) \) gravity.

From a theoretical point of view, gravitational collapse in the \( f(R) \) gravity is important, and recently there has been an increasing interest to understand whether the nature of collapse is altered in the modified gravity theories. For example, gravitational collapse in GR show that the collapse outcome depends upon, among other quantities, the choices of mass profiles and velocity profiles of the collapsing matter. In the context of inhomogeneous LTB models in GR, these issues have been considered in great detail for various matter models including dust and viscous fluids \[20\]. In \[21\] the authors studied the stellar collapse of the homogeneous dust cloud in \( f(R) \) gravity. Earlier studies have looked into various aspects of radiating stellar collapse under the \( f(R) \) regime, for different forms of \( f(R) \) function, different matter and density distributions \[22\] - \[31\]. The aspects of the numerical simulation of the gravitational collapse under \( f(R) \) regime has been studied in \[32\] - \[33\]. Different type of \( f(R) \) models has been considered in the last few decades. However, only those \( f(R) \) models should be considered physically viable which are in agreement with the standard cosmological observations. Here, we consider two models of \( f(R) \) gravity as \((a) \ f(R) = R + \lambda R^2 \) \[34\] and \((b) \ f(R) \sim R^{n+1} \) \[27\]. The interior collapsing spacetime is smoothly matched with the exterior Vaidya spacetime over a timelike surface \( \Sigma \) \[35\] - \[36\]. A well-known way to study spherical gravitational matter configurations (and stars) is by using the Karmarkar condition.
These conditions determine gravitational potentials for static and non-static system both in general relativity as well as in modified gravity regimes \cite{31,40,42}. We must mention here that similar studies on \( f(R) \) gravity have been carried out in \cite{31}. However, the solutions obtained there are restrictive in the sense that one off the metric function have been kept constant to derive the values of other metric function. On the other hand we shall show that such restrictions are not necessary. Karmarkar condition expresses relationship between metric functions. Naturally, these forms of metric functions are arbitrary, and dependent on one’s choice. Recently in \cite{43}, it has been argued that this arbitrariness may be removed if metric functions are related to matter variables. For example, it has been shown that a specific form of pressure anisotropy (difference of the radial and tangential pressures, denoted by \( \Delta = p_t - p_r \)), gives rise to unambiguous set of gravitational potentials. In the present study, we shall use this particular approach to evaluate the metric functions.

The stability of this model is also analyzed and it is found that a faraway observer will see a source whose luminosity is exponentially increasing until a time when it shuts off quickly. This is due to the fact that the total mass of the star radiates linearly and, as the star reaches its maximum luminosity there is no mass left to radiate. The evolution of the temperature profiles during stellar collapse is also studied since they play an important role in the study of transport processes in radiative gravitational collapse \cite{44,45,46,47,48,49,50,51,52}.

The paper is organised as follows. In section 2, we give the field equations of \( f(R) \) gravity and the junction conditions for smooth matching of the interior and the exterior spacetimes across the timelike hypersurface \( \Sigma \). This section also includes the solutions of the \( f(R) \) field equations along with the explicit expressions for physical quantities. We inspect the physical relevance of our exact solutions by verifying the energy conditions. The stability criteria and discussion about the luminosity and adiabatic index, radial and transverse velocity are carried out in section 2.1. In section 2.2 we study the temperature profiles of the radiative stellar collapse. In section 3 includes the discussion of the results accompanied with concluding remarks.

2 Field Equations and Matching Conditions

The action for the \( f(R) \) gravity is obtained by replacing the standard Einstein- Hilbert Lagrangian by a well defined function of Ricci scalar \cite{13}

\[
\mathcal{S} = \frac{1}{2} \int \sqrt{-g} \left[ f(R) + 2\mathcal{L}_M(g_{\mu\nu}, \Psi_m) \right] d^4x,
\]

where \( \Psi_m \) refers collectively to all matter fields, \( \mathcal{L}_M \) is the Lagrangian density of the matter fields \( \Psi_m \), \( g \) is the determinant of the metric tensor \( g_{\mu\nu} \), \( R \) is the Ricci scalar curvature and \( f(R) \) is the generic function of Ricci scalar defining the theory under consideration and (using units with \( c = 1 = 8\pi G \)). Varying the action (1) with respect to the metric tensor \( g_{\mu\nu} \) yields the following
field equations:

\[ F(R) R_{\mu \nu} - \frac{1}{2} f(R) g_{\mu \nu} - (\nabla_\mu \nabla_\nu - g_{\mu \nu} \Box) F(R) = T^M_{\mu \nu}, \quad (2) \]

where \( F(R) = d f(R)/dR \), and \( \Box \equiv \nabla_\mu \nabla^\mu \). This equation may also be rewritten as

\[ R_{\mu \nu} - (1/2) g_{\mu \nu} R = F(R)^{-1} (T^M_{\mu \nu} + T^D_{\mu \nu}) , \quad (3) \]

where the left side of the equation (3) is the usual Einstein tensor, \( T^M_{\mu \nu} \) and \( T^D_{\mu \nu} \) are the energy momentum tensor and effective energy momentum tensor having the form as:

\[ T^M_{\mu \nu} = (p_\rho + p_r) u_\mu u_\nu + p(g_{\mu \nu} + (p_r - p_\rho) X_\mu X_\nu + q_\mu u_\nu + q_\nu u_\mu) , \quad (4) \]
\[ T^D_{\mu \nu} = (1/2) [f(R) - RF(R)] g_{\mu \nu} + (\nabla_\mu \nabla_\nu - g_{\mu \nu} \Box) F(R). \quad (5) \]

Here, \( \rho, p_r \) and \( p_\rho \) are the energy density, radial pressure and the tangential pressure respectively. Also, \( q^\mu u^\mu, X^\mu \) represents the radial heat flow vector, 4-velocity vector and spacelike 4-vector respectively, which satisfy \( u_\mu u^\mu = -X_\mu X^\mu = -1 \) and \( u_\mu X^\mu = u_\mu q^\mu = 0 \).

We now consider a general non-static shear free spherically symmetric spacetime metric given by the following form

\[ ds^2 = -a(r)^2 dt^2 + b(r)^2 s(t)^2 (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) . \quad (6) \]

The forms of \( u^\mu, X^\mu \) and \( q^\mu \) in terms of the metric (6) are

\[ u^\mu = a^{-1} \delta^\mu_0, \quad X^\mu = (b s)^{-1} \delta^\mu_1; \quad q^\mu = (b s)^{-1} X^\mu, \quad (7) \]

The magnitude of the expansion scalar \( \Theta \) and Ricci scalar for the metric (6) have the form

\[ \Theta = \nabla_\mu u^\nu = \frac{3 \dot{s}}{a s}, \quad (8) \]
\[ R = 6 \frac{\ddot{s} + \dot{s}^2}{a^2 s^2} - \frac{2}{b^2 s^2} \left[ \frac{a''}{a} - \frac{b'^2}{b^2} + \frac{a' b'}{a b} + 2 \frac{b'}{b} + 2 \left(\frac{a'}{a} + \frac{2 b'}{b}\right) \right] . \quad (9) \]

The field equations in \( f(R) \) gravity for the metric (6), energy momentum tensor (4), (5) and (7) are

\[ \rho = \frac{F(R)}{s^2} \left[ \frac{3 \ddot{s}^2}{a^2} - 1 \frac{b^4}{b^2} \left( \frac{2 b'^2}{b^2} + \frac{4 b'}{r b} \right) + \left( \frac{f - RF}{2} \right) + 3 \frac{\dot{F}}{s a^2} \right] - 1 \frac{F'}{b^2 s^2} \left[ \frac{F''}{a^2} \left( \frac{b'}{b} + 2 \frac{a'}{r} \right) - \left( \frac{f - RF}{2} \right) \right] , \quad (10) \]
\[ p_r = \frac{F}{s^2} \left[ -\frac{1}{a^2} \left( 2 s \ddot{s} + \dot{s}^2 \right) + \frac{1}{b^2} \left( \frac{2 a' b'}{a b} + \frac{2}{r} \left( \frac{a'}{a} + \frac{b'}{b} \right) \right) - \left( \frac{f - RF}{2} \right) \right] \]
\[ - \frac{\dot{F}}{a^2} \left( \frac{\ddot{F}}{F} + \frac{2 \dot{s}}{s} \right) + \frac{F'}{b^2 s^2} \left( \frac{a'}{a} + \frac{2}{r} + \frac{2 b'}{b} \right) , \quad (11) \]
\[ p_t = \frac{F}{s^2} \left[ -\frac{1}{a^2} (2s \ddot{s} + \dot{s}^2) + \frac{1}{b^2} \left( \frac{a''}{a} + \frac{b''}{b} - \frac{b' \dot{b}}{b^2} + \frac{1}{r} \left( \frac{a'}{a} + \frac{b'}{b} \right) \right) \right] - \left( \frac{f - RF}{2} \right) \]

\[-\frac{\dot{F}}{a^2} \left( \frac{\dot{F}}{F} + \frac{2s}{s} \right) + \frac{1}{b^2 s^2} \left( F'' + F' \left( \frac{a'}{a} + \frac{1}{r} \right) \right), \quad (12)\]

\[ q = -\frac{2a' \dot{s}}{a^2 b^2 s^3} + \frac{1}{a^2 b^2 s^2} \left( \dot{F}' - \frac{\dot{F} a'}{a} - \frac{\dot{s} F'}{s} \right), \quad (13)\]

where prime and dot are the derivatives with respect to \( r \) and \( t \) respectively.

Let us consider the junction conditions for the smooth matching of the interior manifold \( M^- \) (equation (6) considered above) with the exterior manifold \( M^+ \) across timelike hypersurface \( \Sigma \), at \( r = r_b \). As described in [53, 54], the junction conditions for the \( f(R) \) gravity requires the matching of several geometric quantities other than the induced metric \( (h_{ij}) \) and the extrinsic curvature \( (K_{ij}) \). In fact, it has been established that in \( f(R) \) gravity, the following variables must be matched at the boundary:

\[ [h_{ij}]^+ = 0, \quad (14)\]

\[ F(R) [K_{ij} - (1/3)K h_{ij}]^+ = 0, \quad (15)\]

\[ [K]^+ = 0, \quad (16)\]

\[ (\partial F(R)/\partial R) [\partial_r R]^+ = 0, \quad (17)\]

\[ [R]^+ = 0, \quad (18)\]

where \( \tau \) represents the proper time of the timelike hypersurface, and \( K \) is the trace of the extrinsic curvature. Out of these five conditions, for the set of \( f(R) \) theories under consideration, which are \( f(R) = R + \lambda R^2 \), and \( f(R) = \alpha' R^2 \), it is sufficient to match the metric, the extrinsic curvature, the Ricci scalar, and the derivative of the Ricci as given above, determined from either sides.

The Vaidya spacetime in the outgoing coordinate is taken to be our exterior spacetime \( M^+ \) [35]

\[ ds_+^2 = - \left[ 1 - \frac{2M(v)}{r} \right] dv^2 - 2 dv dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (19)\]

For our later convenience, let us define the proper time, \( d\tau = a(r) \Sigma dt \). The junction condition as given (14), implies the following conditions

\[ r_{\Sigma}(v) = (r b s)_{\Sigma}, \quad (20)\]

\[ \left( \frac{dv}{d\tau} \right)^{-2}_{\Sigma} = \left( 1 - \frac{2M}{r} + \frac{2}{r} \frac{dr}{dv} \right)_{\Sigma}, \quad (21)\]

where \( \tau \) represents the proper time defined on the hypersurface \( \Sigma \). The normal vector fields to \( \Sigma \) are given by

\[ n_i^- = [0, (b s)_{\Sigma}, 0, 0], \quad n_i^+ = \left[ 1 - \frac{2M}{r} + \frac{2}{r} \frac{dr}{dv} \right]^{-\frac{1}{2}}_{\Sigma} \left[ -\frac{dr}{dv} \delta_i^0 + \delta_i^1 \right]_{\Sigma}. \quad (22)\]
The extrinsic curvatures for metrics (6) and (19) are given by

\[ K_{-\tau\tau} = -\left[ \frac{a'}{a b s} \right]_{\Sigma}, K_{\theta\theta} = \left[ r b s \left( 1 + \frac{r b'}{b} \right) \right]_{\Sigma}, \]  

(23)

\[ K_{+\tau\tau} = \left[ \frac{d^2 v}{d\tau^2} \left( \frac{dv}{d\tau} \right)^{-1} - \left( \frac{dv}{d\tau} \right) M \right]_{\Sigma}, K_{\theta\theta}^+ = \left[ \left( \frac{dv}{d\tau} \right) \left( 1 - \frac{2M}{r} \right) r - r \frac{dr}{d\tau} \right]_{\Sigma}, \]  

(24)

\[ K_{\phi\phi}^- = \sin^2 \theta K_{\theta\theta}^- , K_{\phi\phi}^+ = \sin^2 \theta K_{\theta\theta}^+. \]  

(25)

Now, from the junction condition on \( K_{ij} \) (because of the conditions that \( K \) must satisfy on the hypersurface, matching \( K_{ij} \) is enough), we get the following. From the equality for the \( \theta \theta \) components at hypersurface \( \Sigma \), and the equations (20) and (21), we obtain

\[ \left[ r b s \left( 1 + \frac{r b'}{b} \right) \right]_{\Sigma} = \left[ \left( \frac{dv}{d\tau} \right) \left( 1 - \frac{2M}{r} \right) r - r \frac{dr}{d\tau} \right]_{\Sigma}, \]  

(26)

and the total energy inside the boundary hypersurface \( \Sigma \), given by the Misner-Sharp mass, denoted by \( 2m \) (such that \( m = M \) on the matching hypersurface) \(^{55, 56}\), where

\[ m_{\Sigma} = \left[ \frac{r^3 s^2 b^3 s}{2 a^2} - \frac{r^3 s b' \frac{2}{2 b}}{2 - r^2 s b'} \right]_{\Sigma}. \]  

(27)

Now, again from the matching of the \( K_{+\tau\tau} = K_{-\tau\tau} \) component we have the following equation:

\[ -\left[ \frac{a'}{a b s} \right]_{\Sigma} = \left[ \frac{d^2 v}{d\tau^2} \left( \frac{dv}{d\tau} \right)^{-1} - \left( \frac{dv}{d\tau} \right) M \right]_{\Sigma}, \]  

(28)

and, substituting the relation between proper and coordinate time along with the eqns. (20) and (27) into the eqn. (26) we have

\[ \left( \frac{dv}{d\tau} \right)_{\Sigma} = \left( 1 + \frac{r b'}{b} + r \frac{b s}{a} \right)^{-1}. \]  

(29)

Now, differentiating (29) with respect to the \( \tau \) and using eqns (27) and (29), we can rewrite (28).

Further, comparing with equations (11) and (13) we have the following useful form

\[ (p_r + T_{rr}^D + b s T_{tr}^D)_{\Sigma} = (q b s)_{\Sigma}. \]  

(30)

where,

\[ T_{rr}^D = \left( \frac{f - RF}{2} \right) + \frac{\dot{F}}{a^2} \left( \frac{\dot{F} + 2\dot{s}}{s} \right) - \frac{F'}{b^2 s^2} \left( \frac{a' - 2}{a r} + \frac{2b'}{b} \right), \]  

(31)

\[ T_{tr}^D = \frac{1}{a^2 b^2 s^2} \left( \frac{\dot{F} + \ddot{F}a' - \dot{s} F'}{s} \right), \]  

(32)
are the dark source terms. From equation (30), it is found that just like for general relativity, the radial pressure does not vanish at the boundary but, instead is proportional to the dissipative as well as radiative dark source terms. The extra terms $T_{rr}^D$ and $T_{tr}^D$ on the LHS of equation (30) are the dark source term and may appear due to the higher order curvature geometry of the collapsing sphere [31].

Let us now move to match $K$ as given in (14). The expressions for the trace of extrinsic curvatures on the either sides lead to the following matching condition on the hypersurface:

$$[p_r + T_{rr}^D + b s T_{tr}^D - q b s] = 2 [M - m]_\Sigma,$$

and naturally, this condition is identically satisfied due to the abovementioned equations. The matching of the Ricci and its proper time derivative gives the following conditions which are to be satisfied for the metric of the internal manifold (at the hypersurface $\Sigma$):

$$s \ddot{s} + \dot{s}^2 = \frac{a^2}{3b^2} \left[ \frac{a''}{a} + 2 \frac{b'}{b} + a'b' + \frac{2}{r} \left( \frac{a'}{a} + 2 \frac{b'}{b} \right) - \frac{b'^2}{b^2} \right],$$

$$3 \dot{s} \ddot{s} + s^2 = 0.$$  \hspace{1cm} (34)

The metric of the internal manifold must be chosen so as to satisfy the two conditions in (34). To determine metric functions according to all these junction conditions, it is necessary to use some auxiliary conditions. We shall see below that these equations are consistent with a collapsing time dependent internal metric. In fact, one may argue that junction conditions indeed force such a possibility. Additionally, we must also ascertain the physical viability of the spacetime metric. From equations (11), (12) and (6), the pressure anisotropy factor $\Delta = p_t - p_r$ has the form

$$\Delta = \frac{F''}{b^2 s^2} - \frac{F'}{b^2 s^2} \left( \frac{2b'}{b} + \frac{1}{r} \right) + \frac{\dot{F}'}{b^2 s^2} \left( \frac{2a'b'}{ab} - \frac{a''}{a} + \frac{a'}{ra} \right).$$  \hspace{1cm} (35)

The general expression for the shear free spacetime as given in (35) is has the complicated form.

To find the solution of the metric functions and mathematical simplicity, we take an adhoc form of the pressure anisotropy $\Delta$ to be:

$$\Delta = \frac{F''}{b^2 s^2} - \frac{F'}{b^2 s^2} \left( \frac{2b'}{b} + \frac{1}{r} \right) - \frac{F}{b^2 s^2} \left[ \frac{2a'b'}{ab} - \frac{a''}{a} + \frac{a'}{ra} \right].$$  \hspace{1cm} (36)

Although, we have chosen this form of the anisotropy in pressure $\Delta$ for the mathematical simplicity, later we will see that they represents the physically viable solutions of the potentials. Also, this choice of $\Delta$ is physically significant, such that $\Delta$ is regular throughout the collapse. It must be noted that this choice of the anisotropy (36) reduces the total pressure anisotropic equation (35) as differential equation of only one function, given by

$$0 = \frac{1}{s^2 b^2} \left( \frac{b''}{b} - \frac{2b'^2}{b^2} - \frac{b'}{rb} \right),$$  \hspace{1cm} (37)
The form of the function $b(r)$ is

$$b(r) = -2[C_3 r^2 + 2 C_4]^{-1}, \quad (38)$$

where $C_3$ and $C_4$ are constant of integration.

Let us now use the fact that under certain conditions, a $(n + 1)$-dimensional space can be embedded into a pseudo Euclidean space of dimension $(n + 2)$ \[37\]. Thus the necessary and sufficient condition for any Riemannian space to be an embedding class I is the Karmarkar condition \[38, 39\],

$$R_{rtrt} R_{\theta \phi \theta \phi} - R_{r \theta r \theta} R_{t \phi t \phi} = 0. \quad (39)$$

The non vanishing components of the Riemann tensor for the metric (6) are

$$R_{rtrt} = a^2 \left( a'' - \frac{b^2 s^2}{a^2} s'' - \frac{a' b'}{a b} \right), \quad (40)$$

$$R_{\theta \phi \theta \phi} = r^2 b^2 s^2 \left( \frac{b'^2}{a^2} s^2 - \frac{2 b'}{r b} - \frac{b'^2}{b^2} \right) \sin^2 \theta, \quad (41)$$

$$R_{r \theta r \theta} = r^2 b^2 s^2 \left( \frac{b'^2}{a^2} s^2 - \frac{b'}{r b} - \frac{b''}{b^2} + \frac{b'^2}{b^2} \right), \quad (42)$$

$$R_{t \phi t \phi} = r^2 a^2 b \left( \frac{a'}{r a} - \frac{b^2 s}{a^2} s + \frac{a' b'}{a b} \right) \sin^2 \theta, \quad (43)$$

$$R_{\theta r t \theta} = \frac{r^2 b^2 s}{a} a', \quad (44)$$

$$R_{\phi r t \phi} = \sin^2 \theta R_{\theta r t \theta}. \quad (45)$$

Using the expressions for Riemannian tensors from eqns \[40\]- \[45\] into the eqn \[39\] we have

$$0 = b^2 s^2 b^3 \left( \frac{a''}{a} - 2 \frac{a' b'}{a b} + \frac{a'^2}{a^2} - \frac{a'}{r a} \right) - r^2 b^3 s^2 \left( \frac{b''}{b} - \frac{2 b'^2}{b^2} - \frac{b'}{r b} \right)$$

$$+ r^2 a b' \left( \frac{b'}{b} + \frac{1}{r} \right) - r^2 a b' \left( \frac{b'^2}{b^2} + 2 \frac{b'}{r b} \right) + r a b' \left( \frac{b'}{r b} + 2 \frac{b'^2}{b^2} \right). \quad (46)$$

For a given form of metric function $b(r)$ \[38\], the class I condition in equation \[46\] is nonlinear. A physical relevant collapsing model must satisfy \[30\] and \[46\] simultaneously. It must be noted that simplest choice of solutions of \[30\] is a linear solution \[57\]

$$s(t) = -C_Z t, \quad (47)$$

$C_Z > 0$. The form of the other metric function $a(r)$ is obtained by using equation \[38\] and \[47\] into the class 1 condition \[46\]

$$a(r) = \frac{1}{2 \sqrt{2 C_3 C_4}} \left[ C_2^2 (C_1 b(r) + 4 C_2 C_3)^2 - 4 C_2^2 \right]^{1/2} \quad (48)$$
where $C_1$ and $C_2$ are integration constants. Surprisingly, the quantity in the numerator inside the square root, arises naturally from the matching of the Ricci scalar (and it’s derivative), given in (34). These forms of the solutions of the gravitational potentials are same as obtained in [43] for shear free spacetime. In [43], it has been shown that for the static case, Karmarkar condition together with the pressure isotropy yields the Schwarzschild [58] like form of the metric functions. Also, it has been shown that these set of gravitational potentials are the special class of those found in [59]. Thus, although we have assume this particular form of $\Delta$ (36) for the mathematical simplicity, represents the physically viable solutions.

It is now instructive to rewrite the physical quantities of the matter cloud in terms of the metric variables for a better understanding of the dynamics of spacetime during the collapse process. These expressions have been written in detail in the Appendix. The boundary condition (30) in the view of these equations in the Appendix, (66)-(68) becomes

\[
2s \ddot{s} + s^2 - 2x s \dot{s} + b s T_{rt}^D = y - T_{rr}^D,
\]

where $T_{rr}^D$ and $T_{rt}^D$ are given by equations (31) and (32) respectively and the quantities $x$ and $y$ are

\[
x = \left( \frac{a'}{b} \right) \Sigma, \quad y = \left( \frac{a^2}{b^2} \left[ \frac{b'}{b^2} + \frac{2}{r} \frac{b'}{b} + \frac{2a'b'}{ab} + \frac{2a'}{r} \right] \right) \Sigma.
\]

(50)

The metric functions $a(r)$ and $b(r)$ should not vanish during the collapsing phenomena, since otherwise the metric shall become degenerate. This also implies that their signatures remain unchanged. For second metric potential to be positive i.e. $a(r) > 0$ we must have, from (48), that

\[
C^2_Z < C^4_4 \left[ \frac{C_1}{C_3} r^2 + 2C_4 - 2C_2 C_3 \right]^2.
\]

This equation also implies that at the center of the cloud, $r = 0$, we must have $C_Z < C_1 - 2C_2 C_3 C_4$.

The graphical representations of the physical quantities (65)-(68) shows that they are well defined throughout the stellar collapse for both the $f(R)$ models. Figures 1a and 1b, 2a and 2b, 3a and 3b, 4a and 4b shows that the density, radial, tangential pressures and pressure anisotropy are positive and regular throughout the collapse for both $f(R) = R + \lambda R^2$ and $f(R) = R^{n+1}/(n+1)$ with $\lambda = 0.001$, $n = 1$. As seen from the figures 5a and 5b, for both the $f(R)$ models, the heat flux increase as the collapse starts and remains positive throughout the collapse.
Figure 1: (a) and (b) shows the plots of the density $\rho$ w.r.t. time $t$ and radial $r$ coordinates for $f(R) = R + \lambda R^2$ with $\lambda = 0.001$ and $f(R) = R^{n+1}/(n+1)$ with $n = 1$ respectively. For both $f(R)$ models, it remains regular as well as positive throughout the collapse.

Figure 2: (a) and (b) shows the plots of the radial pressure $p_r$ w.r.t. time $t$ and radial $r$ coordinates for $f(R) = R + \lambda R^2$ with $\lambda = 0.001$ and $f(R) = R^{n+1}/(n+1)$ with $n = 1$ respectively. For both $f(R)$ models, it remains regular as well as positive throughout the collapse.
Figure 3: (a) and (b) shows the plots of the tangential pressure $p_t$ w.r.t. time $t$ and radial $r$ coordinates for $f(R) = R + \lambda R^2$ with $\lambda = 0.001$ and $f(R) = R^{n+1}/(n+1)$ with $n = 1$ respectively. For both $f(R)$ models, it remains regular as well as positive throughout the collapse.

Figure 4: (a) and (b) shows the plots of the pressure anisotropy $\Delta$ w.r.t. time $t$ and radial $r$ coordinates for $f(R) = R + \lambda R^2$ with $\lambda = 0.001$ and $f(R) = R^{n+1}/(n+1)$ with $n = 1$ respectively. For both $f(R)$ models, it remains regular and positive throughout the collapse.
Figure 5: (a) and (b) shows the plots of the radial heat flux $q$ w.r.t. time $t$ and radial $r$ coordinates for $f(R) = R + \lambda R^2$ with $\lambda = 0.001$ and $f(R) = R^{n+1}/(n+1)$ with $n = 1$ respectively. For both the $f(R)$ models, it remains positive throughout the collapse.

Figure 6: (a) Plot of the expansion scalar $\Theta$ w.r.t. time $t$ and radial $r$ coordinates. At the beginning of collapse $\Theta$ has zero value and it starts decreasing and remains negative throughout the collapse. (b) Plot of the mass of the collapsing star w.r.t. time $t$, and it shows that mass radiates linearly.
A related quantity of importance in this study is the total luminosity visible to an observer at infinity, which may be defined in terms of the mass loss from the boundary surface:

\[ L_\infty = - \left( \frac{dm}{dv} \right)_\Sigma = \left[ \frac{r^2 s^2 b^2 p_r}{2} \left( 1 + \frac{r b'}{b} + \frac{r b s}{a} \right)^2 \right]_\Sigma , \]  

(52)

where we have used the equations (11), (27) and (28). Now, as soon as the black hole is formed, by definition, the luminosity of the surface is zero. From the above equation, this implies that sufficient condition for the formation of a black hole is

\[ \left[ 1 + \frac{r b'}{b} + \frac{r b s}{a} \right]_\Sigma = 0. \]  

(53)

Naturally, for any static observer at asymptotic infinity, the redshift diverges at the time of formation of the black hole.

![Figure 7](image.png)

Figure 7: (a) and (b) shows the plots of luminosity \[52\] at \( r = r_\Sigma = 1 \), w.r.t. time \( t \) for \( f(R) = R + \lambda R^2 \) with \( \lambda = 0.001 \) and \( f(R) = R^{n+1}/(n + 1) \) with \( n = 1 \) respectively.

To show that these spacetime solutions are physically viable, we show that they satisfy the energy conditions as well. Indeed, all the energy conditions namely weak (W), null (N), dominant (D) and strong (SEC) hold good for the collapsing star. In the following we list these conditions [58, 60]

**E1:** \( (\rho + p_r)^2 - 4q^2 \geq 0 \) \hspace{1cm} (D/S/W)

**E2:** \( \rho - p_r \geq 0 \) \hspace{1cm} (D)

**E3:** \( \rho - p_r - 2p_t + \sqrt{(\rho + p_r)^2 - 4q^2} \geq 0 \) \hspace{1cm} (D)

**E4:** \( \rho - p_r + \sqrt{(\rho + p_r)^2 - 4q^2} \geq 0 \) \hspace{1cm} (W/D)
Figure 8: (a), (b) and (c) shows the plots of energy conditions $E_1$, $E_2$ and $E_3$ w.r.t. time $t$ and radial $r$ coordinates for $f(R) = R + \lambda R^2$ with $\lambda = 0.001$ respectively.

\[ E_5 : \quad \rho - p_r + 2p_t + \sqrt{(\rho + p_r)^2 - 4q^2} \geq 0 \quad \text{(D/W/S)} \]
\[ E_6 : \quad 2p_t + \sqrt{(\rho + p_r)^2 - 4q^2} \geq 0 \quad \text{(S)} \]

The star should also satisfy
\[ E_7 : \quad \rho > 0, \quad p_r > 0, \quad p_t > 0, \quad \rho' < 0, \quad p_r' < 0, \quad p_t' < 0. \]

It is clear from the above conditions that $E_1$, $E_2$, $E_3$ and $E_7$ are enough to validate the physical conditions existing inside the star.

For the radiating collapsing stellar models in $f(R)$ gravity, figures 8a, 8b and 8c show that the energy conditions are positive and regular throughout the interior of the star for the $f(R) = R + \lambda R^2$ model with $\lambda = 0.001$. Also, from figures 9a, 9b and 9c, the energy conditions are also satisfied for $f(R) = R^{n+1}/(n + 1)$ model with $n = 1$.

Figure 9: (a), (b) and (c) shows the plots of energy conditions $E_1$, $E_2$ and $E_3$ w.r.t. time $t$ and radial $r$ coordinates for $f(R) = R^{n+1}/(n + 1)$ with $n = 1$ respectively.
2.1 Stability Criteria

The study of dynamical instability (stability) of spherical stellar system shows that for adiabatic index $\Gamma < 4/3$ ($\Gamma > 4/3$) the stellar system becomes unstable (stable) as the weight of the stellar system increase much faster (remains less than) than that of its pressure \[61\]. Also, the causality condition imposes certain constraints on the dynamics of the stellar system such that inside the star, the radial $V_r$ and the transverse $V_t$ components of the speed of sound should be less than the speed of the light ($c = 1$), so that $0 \leq V_r \leq 1$ and $0 \leq V_t \leq 1$ \[62\]. Thus, to check the stability/instability of the collapsing stellar system, we need to study the behavior of the important physical quantities, adiabatic index, sound of the speed which are defined as \[36, 62, 63\]

\[\Gamma_{\text{eff}} = \left[ \frac{\partial (\ln p_r)}{\partial (\ln \rho)} \right]_{\Sigma}, \quad V_r = \frac{dp_r}{d\rho}, \quad V_t = \frac{dp_t}{d\rho}\] \hspace{1cm} (54)

Although stability may be understood from the behaviour of the pressure and density variables, the quantities in (54) and (55) are considered to be better to establish stability. For $f(R) = R + \lambda R^2$ model with $\lambda = 0.001$, figures 7a and 10a shows that the total luminosity and the adiabatic index are positive and increasing. Note that the adiabatic index attains a maximum value where the luminosity is maximum. This behavior of the luminosity and adiabatic index can be interpreted as follows. Any static observer at asymptotic infinity will see an exponentially radiating radial source until a time when luminosity reaches its maximum value after which it instantaneously

Figure 10: (a) and (b) shows the plots of the effective adiabatic index (54) at $r = r_\Sigma$, w.r.t. time $t$ for $f(R) = R + \lambda R^2$ with $\lambda = 0.001$ and $f(R) = R^{n+1}/(n + 1)$ with $n = 1$ respectively.
turn off. This is due to the fact that the total mass of the star radiates linearly as seen from the figure 6b and when the star reaches its maximum luminosity, there is no mass left to radiates and hence the observer at rest at infinity will see sudden turn off of the light source. The similar kind of behavior were obtained in [63]. Figure 10a shows that the effective adiabatic index is positive and less than 4/3 which implies that the considered stellar system is unstable and representing the collapsing scenario [61]. For \( f(R) = R^{n+1}/(n+1) \) model with \( n = 1 \), similar behavior of luminosity is obtained as that of for the first model. Figure 10b shows that the effective adiabatic index is constant function of time, and is positive and less than 4/3, which implies it represents the collapsing scenario. As we have shown graphically that the star radiates all its mass before reaching at the singularity. So, there are no trapped surfaces formed during the collapse. Which implies that neither the black hole nor naked singularity are the end state of the collapse.

2.2 Thermal Properties

Earlier studies have shown that relaxation effects are important to understand dissipative gravitational collapse [45, 46, 47, 50, 64]. To study the temperature profiles, we consider the transport equation for the metric \( (6) \) given by [44, 45, 65]

\[
\tau h^\nu_\mu \dot{q}_\nu + q_\mu = -k \left( h^\nu_\mu \nabla_\nu T + T \dot{u}_\mu \right) \tag{56}
\]

\[
\tau (qbs)_t + q a b s = -\frac{k (aT)_\sigma}{bs}, \tag{57}
\]

where, \( \alpha > 0, \beta > 0, \gamma > 0 \) and \( \sigma > 0 \), & \( h^\mu_\nu = g^\mu_\nu + u^\mu u^\nu \). Also,

\[
\tau_c = (\alpha/\gamma) (T)^{-\sigma}, \quad k = \gamma \tau_c T^3, \quad \tau = \tau_c (\beta \gamma)/\alpha \tag{58}
\]

where \( \tau_c \) is the mean collision time, \( k \) is thermal conductivity and \( \tau \) represents the relaxation time respectively [45, 52]. The quantity \( \tau \) measures the strength of relaxational effects and is called the causality index. The values \( \tau = 0 \) or \( \beta = 0 \) represents the noncausal temperature profile. Using conditions in equation \( (58) \), the the causal heat transport equation \( (57) \) becomes

\[
\beta T^{-\sigma} (qbs)_t + q a b s = -\frac{\alpha (aT)_\sigma}{bs} T^{3-\sigma}. \tag{59}
\]

The noncausal solution of the heat transport equation \( (59) \), with \( \beta = 0 \) i.e. \( \tau = 0 \) are [52]

\[
(aT)^4 = -\frac{4}{\alpha} \int a^4 q b^2 s^2 dr + G(t), \quad \sigma = 0 \tag{60}
\]

\[
\ln (aT) = -\frac{1}{\alpha} \int q b^2 s^2 dr + G(t), \quad \sigma = 4 \tag{61}
\]
Figure 11: (a) & (b) shows the plots of temperature profiles of the collapsing stellar system w.r.t. radial coordinate $r$ for $\sigma = 0$ and $f(R) = R + \lambda R^2$ with $\lambda = 0.001$ and $f(R) = R^{n+1}/(n + 1)$ with $n = 1$ respectively.

The causal solution of the above heat transport equation (59) are

$$
(a T)^4 = -\frac{4}{\alpha} \left[ \beta \int a^3 b s(q b s)_{,t} dr + \int a^4 q b^2 s^2 dr \right] + G(t), \quad \sigma = 0 \tag{62}
$$

$$
(a T)^4 = -\frac{4\beta}{\alpha} \exp \left( - \int \frac{4 q b^2 s^2}{\alpha} dr \right) \int a^3 b s(q b s),t dr \exp \left( \int \frac{4 q b^2 s^2}{\alpha} dr \right) + G(t) \exp \left( - \int \frac{4 q b^2 s^2}{\alpha} dr \right), \quad \sigma = 4 \tag{63}
$$

where $G(t)$ appears as a function of integration and is determined by following boundary condition

$$
(T^4)_{\Sigma} = \left( \frac{L_\infty}{4\pi \delta r^2 b^2 s^2} \right)_{\Sigma}. \tag{64}
$$

where $L_\infty$ is the total luminosity for an observer at infinity given by (52) and $\delta > 0$ is constant. Both figures 11a and 11b shows that, for both the $f(R)$ models, both the temperatures are same at the boundary. At the later stages of gravitational collapse, due to the relaxation effects, the stellar system deviates from thermodynamical equilibrium, and so, the causal and noncausal temperature profiles differ inside the interior of the star. This behavior can be seen from the figures 11a and 11b that with $\beta > 0$. 

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3 Discussion of the results

In this paper, we investigated the dynamics of a collapsing stellar system in $f(R)$ gravity. The interior spacetime has been smoothly matched with the Vaidya metric across a timelike hypersurface. Incidentally, as has been noted earlier too, the matching conditions for the $f(R)$ gravity is highly restrictive, since the geometric variables which are to matched here not only includes induced metric and the extrinsic curvatures, but also the trace of the extrinsic curvatures, and the Ricci scalar along with it’s time derivative. However, we have shown that all these matching conditions can be carried out consistently, leading to a spacetime solution which admits a collapsing scenario in which the matter cloud radiates heat flux, in such a manner that the entire matter is radiated out without forming a black hole. Although similar solutions have been reported earlier for GR, our solution incorporates these features into the collapsing models in the $f(R)$ gravity, and more specifically for two particular theories: $f(R) = R + \lambda R^2$, and $f(R) = R^{n+1}/(n + 1)$.

For both these $f(R)$ models, we have analyzed the physical quantities, energy density (65), radial pressure (66) and tangential pressure (67), pressure anisotropy (36) and it can be seen from the Figs. 1a and 1b, 2a and 2b, 3a and 3b, 4a and 4b that they are regular and positive throughout the collapse. From Fig. 5a and 5b it is clear that the radial heat flux (68) is finite and positive throughout collapse. In particular, for $f(R) = R + \lambda R^2$ model with $\lambda = 0.001$, figures 7a and 10a show that both total luminosity and the effective adiabatic index are positive and increasing and have maximum value where luminosity is maximum. This behavior of the luminosity and adiabatic index can be interpreted as follows: an observer at rest at infinity will see a exponential radiating radial source until it reaches time when luminosity reaches its maximum value and then instantaneous turn off of the radial source. This happens since the total mass of the star radiates linearly as seen from the figure 6b and when the star reaches its maximum luminosity, there is no mass left to radiate and hence the observer at rest at infinity will see sudden turn off of the light source. The similar kind of behavior were obtained in GR also [63]. The figure 10a also shows that the effective adiabatic index is positive and less than 4/3 which implies that the considered stellar system is unstable and represents a collapsing phenomena. Also note that the figures 8a, 8b, and 8c show that the energy conditions are positive and regular throughout the interior of the star. For $f(R) = R^{n+1}/(n + 1)$ model with $n = 1$, similar behavior of the luminosity is obtained as well. Figure 10a shows that the effective adiabatic index is constant function of time, and is positive and less than 4/3, which implies it represents the collapsing scenario. As we have shown graphically that the star radiates all its mass before reaching at the singularity. So, there are no trapped surfaces formed during the collapse. Which implies that neither the black hole nor naked singularity are the end state of the collapse. Also, from figures 9a, 9b, and 9c the energy conditions are also satisfied for $f(R) = R^{n+1}/(n + 1)$ model with $n = 1$. These graphs show that these model under consideration are physically viable. Also, the results obtained here reduces to those obtained for the general relativity regime for $f(R) = R$ [43].

Let us now comment on the nature of the central singularity. First, we note that the Ricci scalar (9) together with (17) imply that it diverges at $t = 0$, when all the matter has been radiated away. So, naturally the question arises if this central curvature singularity is gravitationally strong.
If that is so, this would lead to an example of a naked singularity. The sufficient condition for the singularity to be weak is that the curvature scalars go as $t^{-2}$, which is precisely the case here. So, this solution represents a physically viable model where the spherically symmetric collapsing matter cloud undergoes gravitational collapse, which alongside also radiates away mass in the form of heat flux. The flux is radiated at such a rate that no horizon is ever formed and the central singularity is naked but gravitationally weak in nature.

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Data availability
This manuscript has no associated data. This is a theoretical study and does not contain any experimental data.

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**Appendix**

In this appendix, we give the detail expressions of the physical quantities of the collapsing matter cloud in terms of the metric functions. More precisely, we give the values for (10)-(13), and other quantities like the expansion scalar (8) and the Misner-Sharp mass function (27).

\[
\rho = \frac{6FC_3C_4^3}{S_1C_2^2Z^2} \left[ C_1 - 2C_2C_3 (2C_4 + C_3r^2) \right]^2 + \frac{3\dot{s}F}{sa^2} - \frac{F'}{b^2s^2} + \frac{f - RF}{2} - \frac{F''}{b^2s^2} \left[ \frac{b'}{b} + \frac{2}{r} \right], \tag{65}
\]

\[
p_r = \frac{FC_3C_4^3}{S_1C_2^2Z^2} \left[ 2C_1C_2C_3 (12C_4^2 + 4C_3C_4r^2 - C_3r^2) - C_1^2 (4C_4 - C_3r^2) - 8C_2^2C_3^2C_4 (2C_4 + C_3r^2)^2 \right] - \left( \frac{f - RF}{2} \right) - \frac{\dot{F}}{a^2} \left( \frac{\ddot{F}}{F} + \frac{2\dot{s}}{s} \right) + \frac{F'}{b^2s^2} \left( \frac{a'}{a} + \frac{2}{r} + \frac{2b'}{b} \right), \tag{66}
\]

\[
p_t = \frac{FC_3C_4^3}{S_1C_2^2Z^2} \left[ C_1^4C_4^4 (C_3r^2 - 4C_4) + 2C_1^3C_2C_3C_4^2 (-3C_2^2r^4 + 8C_3C_4r^2 + 28C_4^2) \right] + \frac{F'}{r b^2 s^2}
+ \frac{2FC_1^2C_2C_3^3C_4^4}{S_1C_2^2Z^2} \left[ (C_3r^2 + 2C_4)^2 [6C_2^2C_3^2C_4^2 (C_3r^2 - 6C_4) + C_2^2 (2C_4 - C_3r^2)] \right] + \frac{F'a'}{a b^2 s^2}
+ \frac{2FC_1C_2C_3^3C_4^4}{S_1C_2^2Z^2} \left[ (C_3r^2 + 2C_4)^3 (C_2^2 (C_3r^2 - 6C_4) - 4C_2^2C_3^2C_4^2 (C_3r^2 - 10C_4)) \right] + \frac{F''}{b^2 s^2}
- \frac{8FC_1^2C_2^3C_3C_4^3}{S_1C_2^2Z^2} \left[ (C_3r^2 + 2C_4) (4C_2^2C_3^2C_4^2 - C_2^3) \right] - \left( \frac{f - RF}{2} \right) - \frac{\dot{F}}{a^2} \left( \frac{\ddot{F}}{F} + \frac{2\dot{s}}{s} \right), \tag{67}
\]

\[
q = - \left[ \frac{C_3^{5/2}C_4^{3/2}}{C_2^3Z^3} \cdot \frac{4\sqrt{2r}FC_1}{C_2^2Z^3} \left[ \frac{4C_2C_3 - \frac{2C_2}{C_3r^2 + 2C_4}}{C_2^2 (4C_2C_3 - \frac{2C_3}{C_3r^2 + 2C_4})^2} - 4C_2^2 \right]^{3/2} \right] + \frac{1}{a^2 b^2 s^2} \left[ \dddot{F} - \frac{\dot{F}a'}{a} - \frac{\ddot{F}s}{s} \right], \tag{68}
\]
\[ \Theta = \frac{6\sqrt{C_3 C_4}}{t \sqrt{C_4^2 \left( 2C_2 C_3 - \frac{C_1}{C_3 r^2 + 2C_4} \right) - 2C_Z^2}}, \tag{69} \]

\[ m = \frac{8t r^3 C_3 C_4 C_Z}{(C_3 r^2 + 2C_4)^3} \left[ \frac{2C_2 C_3 \left( C_3 r^2 + 2C_4 \right) - C_1}{2 \left( C_3 r^2 + 2C_4 \right) \left( C_2 C_3 C_4^2 - C_Z^2 \right) - C_1 C_4^2} \right], \tag{70} \]

\[ S_1 = C_4^2 \left( 2C_2 C_3 - C_1 / (2C_4 + C_3 r^2) \right)^2 - C_Z^2. \]