Corrections to de Sitter entropy through holography

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The holographic entanglement entropy is computed for an entangling surface that coincides with the horizon of a boundary de Sitter metric. This is achieved through an appropriate slicing of anti-de Sitter space and the implementation of a UV cutoff. The entropy is equal to the Wald entropy for an effective action that includes the higher-curvature terms associated with the conformal anomaly. The UV cutoff can be expressed in terms of the effective Planck mass and the number of degrees of freedom of the dual theory. The entanglement entropy takes the expected form of the de Sitter entropy, including logarithmic corrections.

Keywords: Entropy; Holography.

The relation between entanglement and gravitational entropy in spaces that contain horizons can shed light to the nature of the latter. The divergent part of the entanglement entropy scales with the area of the entangling surface \(\mathcal{A}\). This feature hints at a connection with the gravitational entropy when this surface is identified with a horizon. In the context of the AdS/CFT correspondence [2], the conjecture of [3, 4] states that the entanglement entropy takes the expected form of the de Sitter entropy, including logarithmic corrections.

We consider the slicing of AdS space that results in a metric with a static dS boundary:

\[
d s_{d+2}^2 = \frac{R^2}{z^2} \left[ dz^2 + \left( 1 - \frac{1}{4} H^2 z^2 \right)^2 \left( -(1 - H^2 \rho^2) dt^2 + \frac{d\rho^2}{1 - H^2 \rho^2} + \rho^2 d\Omega_{d-1}^2 \right) \right],
\]

where \(0 \leq \rho \leq 1/H\) covers a static patch for \(d > 1\). There are two such patches in the global geometry, with \(\rho = 0\) corresponding to the “North” and “South poles”. For \(d = 1\), \(\rho\) can be negative and each static patch is covered by \(-1/H \leq \rho \leq 1/H\). All the coordinates in the above expressions, as well as \(H\), are taken to be dimensionless, with \(R\) the only dimensionful parameter. The dimensionality of the various quantities can be reinstated by multiplication with the appropriate powers of \(R\). In particular, the physical Hubble scale is \(H/R\). Eq. (1) is a particular example of a Fefferman-Graham parametrization of AdS space [5].

The minimal surface \(\gamma_A\) in the bulk can be determined through the minimization of the area

\[
\text{Area}(\gamma_A) = R^d S^{d-1} \int d\rho \rho^{d-1} \left( 1 - \frac{1}{4} H^2 z^2 \right)^{d-1} \sqrt{\frac{(1 - \frac{1}{4} H^2 z^2)^2}{1 - H^2 \rho^2} + \left( \frac{dz(\rho)}{d\rho} \right)^2},
\]

with \(S^{d-1}\) the volume of the \((d-1)\)-dimensional unit sphere. Through the definitions \(\sigma = \sin^{-1}(H\rho)\), \(w = 2 \tanh^{-1}(Hz/2)\), the above expression becomes

\[
\text{Area}(\gamma_A) = R^d S^{d-1} \int d\sigma \frac{\sin^{d-1}(\sigma)}{\sinh^2(w)} \sqrt{1 + \left( \frac{dw(\sigma)}{d\sigma} \right)^2}.
\]

Minimization of the area results in the differential equation

\[
\tan(\sigma) \tanh(w) w'' + (d - 1) \tanh(w) \left( (w')^3 + w' \right) + d \tan(\sigma) \left( w'^2 + 1 \right) = 0,
\]

whose solution is

\[
w(\sigma) = \cosh^{-1} \left( \frac{\cos(\sigma)}{\cos(\sigma_0)} \right).
\]

For \(\sigma_0 \to 0\) the solution becomes \(w(\sigma) = \sqrt{\sigma_0^2 - \sigma^2}\), reproducing the known expression for \(H = 0\) [3, 4]. For \(\sigma_0 \to \pi/2\) the solution approaches the boundary at the location of the horizon with \(dw/d\sigma \to -\infty\). We also have \(w(0) \to \infty\) in this limit. In fig. [1] we depict the solution for increasing values of \(\sigma_0\).
The total area is dominated by the region near the boundary. Cutting off the range of \( z \) at \( z = \epsilon \), gives for the divergent part

\[
\text{Area}(\gamma_A) = R^d S^{d-1} I(\epsilon) = R^d S^{d-1} \int_{H \epsilon} \frac{dw}{\sinh^d(w)}.
\]

The leading contribution to \( I(\epsilon) \) is \( I(\epsilon) = 1/((d-1)H^{d-1} \epsilon^{d-1}) \) for \( d \neq 1 \), and \( \log(1/(H \epsilon)) \) for \( d = 1 \). In [5] it was argued that a connection with the entropy associated with a gravitational background can be established if we define the effective Newton’s constant for the boundary theory following [7]:

\[
G_{d+1} = (d-1)\epsilon^{d-1} \frac{G_{d+2}}{R},
\]

with \((d-1)\epsilon^{d-1}\) replaced by \(1/\log(1/\epsilon)\) for \(d = 1\). This definition is natural within an effective theory that implements consistently a cutoff procedure by eliminating the part of AdS space corresponding to \( z < \epsilon \). Such a framework is provided, for example, by the Randall-Sundrum (RS) model [8]. As pointed out in [7], the cutoff dependence of the four-dimensional Newton’s constant is not apparent in [8] because the metric is rescaled by \(\epsilon^2\).

In order to see how eq. (7) may arise, we consider the AdS parametrization used in [8]:

\[
ds_{d+2}^2 = e^{-2kr_c \phi} \eta_{\mu\nu} dx^{\mu} dx^{\nu} + r_c^2 d\phi^2.
\]

It corresponds to a flat slicing of AdS with a Fefferman-Graham coordinate \( z = \exp(k r_c \phi) \) and AdS radius \( R = 1/k \). The parameter \( r_c \) determines the distance between the positive-tension brane located at \( \phi = 0 \) and the negative-tension brane at \( \phi = \pi \). The orbifold construction assumes a reflection symmetry around \( \phi = 0 \), or \( z = 1 \), with an identification of the points \( \phi = \pm \pi \). Taking the limit \( r_c \to \infty \) removes the negative-tension brane. Gravitational excitations are considered by taking \( \eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu} \) and treating \( h_{\mu\nu} \) as a perturbation. The curvature scalar of the background has a \( \delta \)-function singularity at \( \phi = 0 \), which leads to the localization of a massless excitation mode around this point. The effective Newton’s constant for the low-energy theory is

\[
G_{d+1}^{-1} = 2G_{d+2}^{-1} \int_0^{\pi} d\phi e^{-(d-1)k\phi} = \frac{2}{(d-1)k} G_{d+2}^{-1},
\]

for \( r_c \to \infty \), with the factor of 2 coming from the two copies of AdS in the construction. In order to reproduce eq. [7] we need to place the positive tension brane at \( \phi = -\pi \), the negative tension brane at \( \phi = 0 \), and identify the points \( \phi = -2\pi \) and 0. The limit \( r_c \to \infty \) now takes the positive tension brane arbitrarily close to the AdS boundary and results in an effective Newton’s constant given by twice the value of eq. [7] with \( \epsilon = \exp(-kr_c \pi) \). The factor of 2 in eq. [9] does not appear in [7] because the latter accounts for only one copy of AdS space, as opposed to two in the RS scenario. A similar factor of 2 would appear in the total entropy, with the two factors cancelling out in the final expression. We shall carry out our analysis based on eq. [7] in order to be consistent with holographic renormalization.
The use of eq. (7) gives a leading contribution to the entropy

\[
S_{\text{dS}} = \frac{\text{Area}(\gamma_A)}{4G_{d+2}} = \frac{R^d S^{d-1}}{4G_{d+2}(d-1)H^{d-1}e^{d-1}} = \frac{S^{d-1}}{4G_{d+1}} \left( \frac{R}{H} \right)^{d-1} = \frac{A_H}{4G_{d+1}},
\]

with \( A_H \) the area of the horizon. This expression reproduces the gravitational entropy of [9] and agrees with the findings of [10]. Notice that, according to our conventions, the physical Hubble scale is \( H/R \). The result is valid for \( d = 1 \) as well, with \( 1/((d-1)e^{d-1}) \) replaced by \( \log(1/\epsilon) \) and \( S^0 = 2 \), because the horizons of the global dS2 geometry consist of 2 points [7].

We turn next to the subleading divergences that may appear in the integral \( I(\epsilon) \) for \( \epsilon \to 0 \). Even though we focus here on the dominant terms, it must be pointed out that the solution [3] is exact, so that, in principle, the entropy can be computed exactly through eq. (3), including the finite terms. For \( d = 3 \) we have

\[
I(\epsilon) = \frac{1}{2H^2\epsilon^2} + \frac{1}{2} \log(H\epsilon) + O(\epsilon^0),
\]

for \( d = 2 \) there are no singular subleading terms, while for \( d = 1 \) the logarithmic term, already accounted for in eq. [10], is the leading divergence with no subleading ones. For \( d > 3 \) we obtain subleading power-law divergences for odd \( d + 1 \), augmented by a logarithmic one for even \( d + 1 \). The implication of eq. (11) is that the dS entropy in four dimensions is proportional to the area of the horizon, with a coefficient that receives a logarithmic correction:

\[
S_{\text{dS}} = \frac{A_H}{4G_4} \left( 1 + H^2\epsilon^2 \log H\epsilon \right),
\]

where we have made use of eq. (7).

The correction must be attributed to higher-curvature terms in the effective gravitational action. For a four-dimensional boundary theory (\( d = 3 \)), on which we focus, the logarithmic dependence on the cutoff implies a connection with the conformal anomaly of the dual theory. The most straightforward way to obtain the effective action for our setup is through known results in the context of holographic renormalization [11][13]. The bulk metric of a five-dimensional asymptotically AdS space is written in a Fefferman-Graham expansion [6] as

\[
ds^2 = R^2 \left( \frac{d^2}{z^2} + g_{ij}(x,z)dx^i dx^j \right)
\]

\[
g(x,z) = g(0) + z^2 g(2) + z^4 g(4) + z^4 \log z^2 h(4) + O(z^5).
\]

A solution is then obtained order by order. The on-shell gravitational action is regulated by restricting the bulk integral to the region \( z > \epsilon \). The divergent terms are subtracted through the introduction of appropriate counterterms. In this way a renormalized effective action is obtained, expressed in terms of the induced metric \( \gamma_{ij} \) on the surface at \( z = \epsilon \).

The holographic entanglement entropy displays divergences similar to those of the on-shell action for \( \epsilon \to 0 \). In our approach the entropy is not renormalized. We assume that the cutoff \( \epsilon \) is physical and we incorporate it in the effective couplings. This amounts to employing the regulated form of the effective action, without the subtraction of divergences. The leading terms, which would diverge for \( \epsilon \to 0 \), can be found in the counterterm action of holographic renormalization. They are expressed in terms of the induced metric \( \gamma_{ij} \). It is apparent from eqs. (13) that \( \gamma_{ij} \) on a surface at \( z = \epsilon \) includes a factor \( \epsilon^{-2} \). In the RS model this factor is displayed explicitly and determines the relative size of the couplings of the effective theory at various locations of the brane. Using the results of [12] and extracting the \( \epsilon^{-2} \) factor from \( \gamma_{ij} \), we can express the leading terms of the regulated action as

\[
S = \frac{R^3}{16\pi G_5} \int d^4x \sqrt{-g} \left[ \frac{6}{\epsilon^2} + \frac{1}{2\epsilon^2} R - \frac{1}{4} \log \epsilon \left( R_{ij}R^{ij} - \frac{1}{3} R^2 \right) \right],
\]

where we have multiplied with an appropriate power of \( R \) so that all quantities within the integral are dimensionless, in agreement with our earlier conventions. We have also adapted the results of [12] to a metric of Lorentzian signature. The first term corresponds to a cosmological constant, which must be (partially) cancelled by vacuum energy localized on the surface at \( z = \epsilon \), such as the brane tension in the RS model [8]. The second term is the standard Einstein term if the effective Newton’s constant \( G_4 \) is defined according to eq. (7) with \( d=3 \). The third term is responsible for the holographic conformal anomaly. The structure of the effective action is the same for the standard RS model in bulk space with Einstein gravity, apart from rescalings that set the location of the brane at a finite distance equal to \( R \) from the boundary [14] [15].
Assuming that the cosmological constant can be adjusted to the desired value, the action (14) supports a dS solution. This can be checked explicitly through the Einstein equations, in which the contribution from the anomaly term vanishes for a dS background. The gravitational entropy must take into account the presence of the third term in eq. (14). This can be achieved by computing the Wald entropy (16), which gives the horizon entropy in theories with higher curvature interactions. For the action (14) the Wald entropy reduces to

$$S_{\text{Wald}} = \frac{A_H}{4G_4} - \frac{R^3}{32G_5} \log \epsilon \int d^2y \sqrt{h} \left( 2R^{ij} \gamma_{ij} - \frac{4}{3} R \right),$$

(15)

where the integration is over the horizon, with induced metric $h$, and $\gamma_{ij}$ denotes the metric in the transverse space. We have made use of eq. (7) and of the fact that the physical Hubble scale is $H/R$. This result is in qualitative agreement with [15], even though the logarithmic term is replaced by a constant there, as the RS brane is located at a finite distance from the boundary. For a dS background the term in the parenthesis in the r.h.s. is constant and the integration reproduces the area of the horizon. We obtain

$$S_{\text{Wald}} = \frac{A_H}{4G_4} (1 + H^2 \epsilon^2 \log \epsilon).$$

(16)

The correction to the dS entropy in eq. (16) is in agreement with the singular part of the correction provided by the holographic calculation (12).

The logarithmic term in eq. (12) can be compared with the known logarithmic correction to the black-hole entropy [18], which is of the form $S_{BH} = S_{BH} + C \log a$, where $S_{BH}$ is the Bekenstein-Hawking entropy [19] and $a$ the black-hole size parameter, such that the horizon area scales $\sim a^3$ in four dimensions. The constant $C$ is related to the conformal anomaly and depends on the number of massless scalar, Dirac and vectors fields of the theory. The field content of the $N = 4$ supersymmetric $SU(N)$ gauge theory in the large-$N$ limit includes $n_S = 6N^2$ scalars, $n_F = 2N^2$ Dirac fermions and $n_V = N^2$ vectors. The Weyl-squared and Euler-density terms in the divergent part of the effective action have opposite coefficients and combine into a term

$$S = -\frac{\beta}{16\pi^2} \Gamma \left( 2 - \frac{d+1}{2} \right) \int d^4x \sqrt{-\gamma} \left( \mathcal{R}_{ij} \mathcal{R}^{ij} - \frac{1}{3} \mathcal{R}^2 \right),$$

(17)

with $\beta = -(n_S + 11n_F + 62n_V)/360 = -N^2/4$. The divergence of $\Gamma(2-(d+1)/2)$ in dimensional regularization in the limit $d+1 \to 4$ corresponds to a $\log(1/\epsilon^2)$ divergence in the cutoff regularization we are using, with $\epsilon$ a length scale. A comparison of the third term in eq. (14) and the above expression reproduces the standard relation $G_5 = \pi R^3/(2N^2)$ between the bulk Newton’s constant and the central charge of the dual CFT. With this relation, the dimensionful momentum cutoff for $d = 3$ can be expressed as $(\epsilon_N R)^{-2} = 2G_5/(R^3 G_4) = 8\pi^2 m_{Pl}^2/N^2$, with $m_{Pl}^2 = 1/(8\pi G_4)$. Now eq. (12) for $d = 3$ can be cast in the form

$$S_{dS} = \frac{A_H}{4G_4} + N^2 \log(H \epsilon_N) = \frac{A_H}{4G_4} + N^2 \log \left( \frac{N}{\sqrt{8\pi}} \frac{H/R}{m_{Pl}} \right),$$

(18)

where $H/R$ is the physical Hubble scale. This expression is completely analogous to the black-hole result [18], with the horizon size parameter measured in units of the UV cutoff. It is also in agreement with the calculation of the logarithmic part of the holographic entanglement entropy in [21].

Our derivation of the dS entropy is consistent with the expectation that the entropy associated with gravitational horizons can be understood as entanglement entropy if Newton’s constant is induced by quantum fluctuations of matter fields [22]. In the context of the AdS/CFT correspondence the bulk degrees of freedom correspond to the matter fields of the dual theory. Within a construction that implements a UV cutoff, such as the RS model [8], the Einstein action arises through the integration of these bulk degrees of freedom. This is demonstrated explicitly by the expression (9) for the effective Newton’s constant. The leading contribution to the entropy has a universal form that depends only on the horizon area, because the same degrees of freedom contribute to the entropy and Newton’s constant. Also, the detailed nature of the UV cutoff does not affect the leading contribution. The particular features of the underlying theory, such as the number of degrees of freedom become apparent at the level of the subleading corrections to the entropy. However, a level of universality still persists: the coefficient of the logarithmic correction is determined by the central charge of the theory, and is independent of the regularization.

It has been argued that the presence of a UV cutoff is automatic even in theories in which gravity is not induced [23]. In particular, for a theory with a large number of degrees of freedom $N_{dof}$, the UV momentum cutoff is expected to be proportional to $m_{Pl}/\sqrt{N_{dof}}$. Our analysis reproduces this relation for a UV cutoff $1/(\epsilon_N R)$ and $N_{dof} \sim N^2$. It seems likely then that the connection between entanglement and gravitational entropy is not limited to the case of induced gravity.
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