Holographic superconductivity of a critical Fermi surface

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We derive holographic superconductivity from a Hamiltonian that describes pairing of two-dimensional electrons near a ferromagnetic quantum-critical point. At low energies the theory maps onto a four-dimensional gravity description with Lifshitz spacetime and dynamic scaling exponent \( z = 3/2 \). The curved spacetime is due to powerlaw correlations of the critical normal state. The Lifshitz anisotropy is caused by phase-space constraints near the Fermi surface. The pairing instabilities obtained in Lifshitz space and from the Eliashberg formalism are found to be identical. We also formulate the holographic map for values of the dynamic scaling exponent \( 1 < z < \infty \). Our result provides an explicit realization of the holographic correspondence in two dimensions.

The holographic correspondence of \( d + 1 \)-dimensional quantum-field theory and gravity theory in \( d + 2 \) dimensions\(^1\)\(^–\)\(^3\) is a powerful tool to understand many-body systems in the strong-coupling limit\(^4\)\(^–\)\(^8\). It relates theories of strongly interacting particles and of gravity, where the extra holographic dimension is not manifest in the original field theory. Even when no quasiparticle descriptions apply, the correspondence allows for an understanding of transport processes\(^9\)\(^–\)\(^12\) and offers insights into broken symmetry states such as charge-density waves\(^13\), \(^14\) or superconductivity\(^15\)\(^–\)\(^17\).

Despite this formidable power of the holographic duality, it remains enigmatic to a considerable portion of the condensed-matter physics community. Legitimate questions are: Why is there a gravity formulation of a condensed matter problem? What is the precise meaning of the holographic dimension in terms of variables of a given many-body Hamiltonian? Which gravitational theory should one use to describe a physical situation? Deriving holography from a concrete many-body problem is one avenue to address these questions. Progress has been made for the zero-dimensional Sachdev-Ye-Kitaev (SYK) model\(^18\)\(^–\)\(^21\). At low energies, the SYK model is governed by the same theory as two-dimensional gravity\(^22\)\(^–\)\(^24\). Another development is Ref.\(^25\), where holographic superconductivity in anti-de Sitter space AdS\(_2\) was derived from the superconducting Yukawa-SYK model of Ref.\(^26\). To have a derivation for finite spatial dimensions would reveal how quantum-critical time and length scales become intertwined.

The situation is particularly challenging for a critical system with a Fermi surface, i.e., a compressible state of matter. Here, the highly symmetrical AdS\(_{d+2}\) does not seem to be appropriate. Therefore one adds a fixed charge to the gravity formulation, leading to a finite compressibility. This yields a Reissner-Nordström black hole, a geometry that factorizes at low energies into AdS\(_2\) \( \otimes \mathbb{R}_d \)\(^4\)\(^–\)\(^8\). Now correlations become completely local in space and non-local only in time\(^27\), \(^28\), which is somewhat unrealistic for many condensed matter settings. Intermediate between these two limits is a state called Lifshitz gravity\(^29\)\(^–\)\(^37\), characterized by a dynamic scaling exponent \( z \geq 1 \), relating typical energies and momenta \( \omega_{\text{typ}} \sim k_{\text{typ}}^{\frac{3}{2}} \)\(^38\). It holds \( z = 1 \) for AdS\(_{d+2}\) and \( z \rightarrow \infty \) for AdS\(_2\) \( \otimes \mathbb{R}_d \)\(^8\).

In this paper, we derive the theory of holographic superconductivity\(^15\)\(^–\)\(^17\) from a two-dimensional model of electrons that interact with an Ising-ferromagnetic degree of freedom. It generalizes Ref.\(^25\) to finite space dimensions. We find a gravitational formulation of the problem in terms of a Lifshitz gravity with dynamic scaling exponent \( z = 3/2 \). The curved spacetime is endowed by the powerlaw behavior of the quantum-critical normal state. Superconductivity of this critical state is then described by a Ginzburg-Landau theory

\[
S_{\text{sc}} = \int d^4x \sqrt{\hat{g}} (\partial_\mu \psi^* \partial^\mu \psi + m \psi^* \psi)
\]

that lives in the emergent spacetime with metric tensor \( \hat{g}_{\mu \nu} \) and \( \hat{g} = \det \hat{g}_{\mu \nu} \), as expected from the holographic correspondence\(^15\)\(^–\)\(^17\). The extra dimension describes the internal dynamics of Cooper pairs formed from the critical normal state. The instability due to Breitenlohner and Freedman\(^39\) towards condensation of \( \psi \) in negatively curved space is shown to be identical to the onset of pairing that follows from the coupled Eliashberg equations\(^40\). First, we will formulate and solve our many-body problem in an appropriate large-\( N \) limit. Here we will reproduce known results for the critical state and for superconductivity, yet using a more controlled approach, formulated in terms bilocal collective fields\(^41\). In a second step we will use these fields and formulate an explicit holographic map. We also generalize the theory to generic \( 1 < z < \infty \).

The model: The many-body Hamiltonian of our analysis combines the large-\( N \) formulation of the Yukawa-SYK model\(^26\), \(^42\) with the theory of fermions, coupled to quantum-critical Ising-ferromagnetic fluctuations in \( d \) = 2\(^43\)\(^–\)\(^49\):

\[
H = \sum_{p i \sigma} \varepsilon_p c_{pi \sigma}^\dagger c_{pi \sigma} + \frac{1}{2} \sum_{ql'} (\pi_{q l'} \pi_{-q l'} + \omega_{ql'}^2 \phi_{q l'} \phi_{-q l'}) + \frac{1}{N} \sum_{pq,ij|\sigma\sigma'} g_{ij} c_{p+q+i \sigma}^\dagger c_{j \sigma'} \phi_{-q l'} + h.c.
\]

\( c_{pi \sigma}^\dagger \) creates a fermion of momentum \( p \) and energy \( \varepsilon_p = \)
$\frac{\omega^2}{2m} - \varepsilon_F$ and spin $\sigma$. Other $\varepsilon_p$ with finite Fermi-surface curvature are equally possible. For each $(p, \sigma)$ there is another flavor index $i = 1 \ldots N$, introduced to formulate a controlled theory. $\phi_{ql}$ is a neutral boson, odd under time reversal, with momentum $q$, and $\omega_q^2 \approx \omega_{q0}^2 + c^2 q^2$, and flavor index $l = 1, \ldots M$. $\pi_{ql}$ is the momentum conjugate to $\phi_{ql}$. Models similar to Eq.(2) were put forward for purely electronic[50], for Ising nematic[51], and for Dirac systems[52]. We consider the Ising ferromagnet as it avoids the superconducting first-order transition of the Heisenberg limit[47] and suppressed fluctuations due to acoustic phonons of the Ising-nematic state[53, 54].

The Yukawa coupling $g_{ijkl}$ is a random all-to-all interaction in flavor space. It is the same for all lattice sites and times, i.e. space and time translation invariance are not broken[50–52]. The random coupling constants are complex and Gaussian distributed, where $\left(1 - \frac{\xi_0}{2}\right) g^2/2$ and $\alpha g^2/4$ are the widths of the real and imaginary parts of $g_{ijkl}$, respectively[55]. We will see that sufficiently small $\alpha$ yields a superconducting solution while superconductivity cannot occur at large-$N$ for $\alpha = 1$. Hence, $\alpha$ plays the role of a pair-breaking parameter and determines the coupling constant in the pairing channel $g^2_\alpha = g^2(1 - \alpha)$. We take the limit of large $N$ and $M$, where $\mu = M/N$ is finite. The bare magnetic correlation length $\xi_0 = c/\omega_0$ gets renormalized by coupling to fermions and diverges at a ferromagnetic quantum-critical point (QCP). Without randomness of the $g_{ijkl}$, Eq.(2) was discussed in Refs.[43–49]. As pointed out in Refs.[56, 57], a proper large-$N$ formulation of these models is rather subtle. These complications are avoided for random $g_{ijkl}$.

**Bilocal fields and large-$N$:** The large-$N$ analysis of the model follows closely Refs.[50–52]. Instead of fermions and bosons, one introduces bilocal collective fields

$$G_{\sigma\sigma'}(x, x') = \frac{1}{N} \sum_{i=1}^{N} c_{\sigma}(x) c_{\sigma'}(x'),$$

$$D(x, x') = \frac{1}{M} \sum_{i=1}^{M} \phi_{i}(x) \phi_{i}(x').$$

$x = (x, \tau)$ comprises space and imaginary time. For $\alpha \neq 1$ one must also include bilocal pairing fields

$$F_{\sigma\sigma'}(x, x') = \frac{1}{N} \sum_{i=1}^{N} c_{\sigma}(x) c_{\sigma'}(x')$$

as well as $F^\dagger$ with $c$ replaced by $c^\dagger$. For details of this derivation, consult the appendix[58]. The usage of these bilocal fields will be very helpful for our formulation of a holographic theory[41]. Eqns.(3) and (4) are enforced via Lagrange-multiplier fields $\Sigma$, $\Pi$, and $\Phi$ that depend on the same coordinates. This allows writing the disorder-averaged interaction term as

$$S_{\text{int}}/M = \frac{g^2}{2} \int_{x,x'} \text{tr} \left( \hat{\sigma}^z \hat{G}(x, x') \hat{\sigma}^z \hat{G}(x', x) \right) D(x, x')$$

$$- \frac{g^2}{2} \int_{x,x'} \text{tr} \left( \hat{\sigma}^z \hat{F}(x, x') \hat{\sigma}^z \hat{F}^\dagger(x', x) \right) D(x, x').$$

$\Sigma$ and $\Phi$ denote the trace over spin indices, hats refer to matrices in spin space, and $\text{tr} = \int d^2x dx'$. Now the original fermions and bosons can be integrated out, yielding a theory exclusively in terms of bilocal fields:

$$S/N = -\frac{1}{2} \text{Tr} \log \left( \hat{g}_0^{-1} - \Sigma \right) + \frac{\mu}{2} \text{Tr} \log \left( d_0^{-1} - \Pi \right)$$

$$- \frac{1}{2} \text{Tr} \hat{G} \otimes \Sigma + \frac{\mu}{2} \text{Tr} \hat{D} \otimes \Pi + S_{\text{int}}/N.$$  (5)

We use $\text{Tr}\hat{A} \otimes \hat{B} = \int_{xx'} \text{tr} \left( \hat{A}(x, x') \hat{B}(x', x) \right)$, $\hat{G}$ and $\Sigma$ are matrices in Nambu space such that $\text{Tr}$ stands for an additional trace over the Nambu components[58]. While similar to a Luttinger-Ward functional[59–61], the bilocal fields in Eq.(5) are genuine dynamic variables. At large-$N$ and fixed $\mu = M/N$ the saddle-point equations $\delta S/\delta G = 0$ etc. become exact. At the saddle point, the fields only depend on $x - x'$, i.e. $G_{ab}(x, x') = G_{ab}(x - x')$ and behave like propagators and self energies. Fourier transformation to momentum and frequency variables $p = (p, \omega)$ yields the Eliashberg equations[58]:

$$\hat{\Sigma}(p) = -\mu g^2 \int_{p'} \hat{\sigma}^z \hat{G}(p') \hat{\sigma}^z D(p - p'),$$

$$\hat{\Phi}(p) = \mu g^2 \int_{p'} \hat{\sigma}^z \hat{F}(p') \hat{\sigma}^z D(p - p'),$$

$$\Pi(q) = -g^2 \int_{p} \text{tr} \left[ \hat{\sigma}^z \hat{G}(p) \hat{\sigma}^z \hat{G}(p + q) \right]$$

$$+ \frac{g^2}{2} \int_{p} \text{tr} \left[ \hat{\sigma}^z \hat{F}(p) \hat{\sigma}^z \hat{F}^\dagger(p + q) \right].$$  (6)

These equations fully agree with and justify one-loop diagrammatic treatments of the corresponding non-random system[43–49]. In what follows we will use the low-energy, normal state solutions of these equations at the QCP and consider small fluctuations on top of it, i.e. $\Sigma = \Sigma_{\text{sp}} + \delta \Sigma$ and similar for all fields, where $\Sigma_{\text{sp}}$ is a solution of Eq.(6). We focus on Gaussian fluctuations in $F$ and $\Phi$ which decouple from other fluctuations due to the $U(1)$ invariance of the normal state.

**Normal state saddle at the QCP:** If the saddle point values of $F$ and $\Phi$ vanish, the system is in its normal state. To avoid cluttering equations we measure momenta in units of the Fermi momentum $p_F$ and energies in units of $\varepsilon_F = v_F p_F$[58]. At the QCP, the bosonic and fermionic self energies take the form

$$\Pi_q(\epsilon) = \omega_q^2 - 2g^2 \rho_F |\epsilon|/q,$$  (7)

$$\Sigma_p(\epsilon) = -i \lambda \text{sign}(\epsilon) |\epsilon|^{2/3}.$$  (8)

Here, $\rho_F$ is the density of states while $\lambda = \frac{\omega_q}{2v_F^2} \left(\frac{\sqrt{2\pi \rho_F q^2}}{\omega_q}\right)^{1/3}$ is a dimensionless coupling constants.

At low energy, the propagators are determined by $1/G_P(\epsilon) \approx -\varepsilon_p - \Sigma_p(\epsilon)$ and $1/D_q(\epsilon) \approx \omega_q^2 - \Pi_q(\epsilon)$. The derivation of these results from Eq.(6) is well established[43–47]. Eqns.(7) and (8) also agree with
findings from sign-problem free Quantum Monte Carlo calculations for the corresponding non-random Ising ferromagnet [48, 49]. Notice, fermions and bosons described by (7) and (8) have different dynamic scaling exponents. Balancing $|e|/q$ against $q^2$ yields for bosons $z_{bos} = 3$ [62, 63], while comparing $\Sigma(\epsilon) \sim |\epsilon|^{2/3}$ of (8) with the kinetic energy $v_F p_\perp$ gives $z_{\text{ter}} = 3/2$. $p_\perp$ is the momentum perpendicular to the Fermi surface. It is not obvious which of the two, if any, is the dynamical critical exponent of the problem.

**Onset of superconductivity:** The onset of superconductivity at the QCP can be analyzed from the linearized gap equation for $\hat{\Phi}(\rho) = \sum_{\alpha=0}^{3} \hat{\Phi}^{(\alpha)}(\epsilon) i\sigma^\alpha$, one finds an attractive interaction of equal spin states, i.e. triplet pairing. For $|p| - |p_F|$ the anomalous self energy only depends on the angle $\theta_p$ of $\mathbf{p}$ and can be expanded in harmonics, where $l \in \mathbb{Z}$ is the angular momentum. The gap equation is diagonal in $l$. Different angular momenta are almost degenerate, where the degeneracy is lifted only due to effects that are sub-leading at small energies. Including these sub-leading effects, the leading channel is $p$-wave triplet pairing with $l = \pm 1$. In this channel the frequency dependence of $\Phi(\epsilon)$ follows from the linearized Eliashberg equation

$$\Phi(\epsilon) = \frac{1 - \alpha}{3} \int d\epsilon' \frac{\Phi(\epsilon')}{|\epsilon'|^{2/3}|\epsilon - \epsilon'|^{1/3}}$$

only determined by the pair-breaking parameter $\alpha$. This equation was analyzed in Refs. [40, 55]. It is solved via powerlaw ansatz and the superconducting ground state survives until $\alpha$ reaches a critical strength $\alpha^*$, determined by $1 = \frac{1}{3} \int_{-\infty}^{\infty} d\epsilon (1 - 3\alpha/2)^{1/3}$ which yields $\alpha_* \approx 0.879618$ [58].

**Pairing fluctuations:** Next we analyze the leading Gaussian fluctuations $S_{\text{sc}}$ of the action $S$ with regards to the bilocal pairing fields $F$ and $\Phi$. This describes pairing fluctuations of the critical normal state and determines the instability towards superconductivity. It is useful to Fourier transform $F_k, k' = (\tau, \epsilon')$ and $\Phi_k, k' = (\tau, \epsilon')$ with regards to the relative time $\tau - \tau'$, which leads to the frequency $\epsilon$ and the absolute time $(\tau + \tau')/2$, with corresponding frequency $\omega$, respectively. Similarly, for momenta we use $k = p + p'$ and $q = (p - p')/2$. The dominant coupling is via transferred momenta tangential to the Fermi surface. Hence, near the Fermi surface, only the direction $\theta_q$ of $q$ is important and we expand, in analogy to the gap equation, in harmonics

$$\tilde{F}_{k,q}(\omega, \epsilon) = \sum_{l\alpha} F^{(l,\alpha)}_{k}(\omega, \epsilon) i\sigma^y \sigma^\alpha \sqrt{2\pi} e^{i\theta_q}$$

and same for $\tilde{\Phi}_{k,q}(\omega, \epsilon)$. Integrating out $\tilde{\Phi}^{(l,\alpha)}_{k}$ we obtain a collective field theory of the leading channel with amplitude $F_k(\omega, \epsilon) = F^{(\pm,1,\epsilon,\lambda)}_{k}(\omega, \epsilon)$:

$$S_{\text{sc}}/N = \int_{k,\omega} F^*_{k} (\omega, \epsilon) \chi^{-1}_{k}(\omega, \epsilon) F_k(\omega, \epsilon)$$

with $\lambda_p = \lambda(1-\gamma)(1-\eta)$ and particle-particle propagator:

$$\chi_k(\omega, \epsilon) = \frac{\chi}{2 \epsilon^2} \frac{\epsilon - |\omega|/2}{\sqrt{(|\epsilon + \lambda_p^2|^{1-\gamma} + |\epsilon - \omega|^{1-\gamma})^2 + (\epsilon^2 \lambda)^2}}$$

For our problem of Eq. (2) the exponent $\gamma$ takes the value $\gamma = 1/3$. To put our findings in more general context, we have written Eq. (10) and (11) for exponents $0 < \gamma < 1$ that follow for a $d$-dimensional system with normal state self energy $\Sigma(\epsilon) \sim |\epsilon|^{1-\gamma}$ and boson propagator $\int_{q} D_q(\epsilon) \sim |\epsilon|^{-1}$, averaged over momenta tangential to the Fermi surface [40, 64]. Examples are $\gamma = 1/2$ for $d = 2$ spin-density wave instabilities [65, 66] and $\gamma = 0^+$, i.e. $\Sigma(\epsilon) \sim \epsilon \log \epsilon$, for $d = 3$ color superconductivity due to gluon exchange [67] or for three-dimensional spin fluctuations [46, 68]. $\gamma \rightarrow 1^-$, i.e. $\Sigma(\epsilon) \sim \log \epsilon$, follows for $d = 3$ massless bosons at very strong coupling [68]. $\gamma = 1/3$ also describes composite fermions at half-filled Landau levels [69], emergent gauge fields [70-73] or nematic transitions [74, 75] in two dimensions. For each of these problems one can formulate a Hamiltonian similar to Eq. (2).

**Holographic map:** The above Gaussian action of the dominant mode $F_k(\omega, \epsilon)$ depends on $d + 2$ coordinates, just like the scalar field of the holographic theory. Indeed, the field

$$\psi(k, \omega, \zeta) = \frac{c_0(\omega/c_1)}{1 + (z\omega/m)^2} F_k(\omega, c_1/\zeta),$$

with constants $m^2, c_0$, and $c_1$ listed below and with exponent

$$z = (1 - \gamma)^{-1},$$

can be given a geometric interpretation. Inserting Eq. (12) into Eq. (10) it follows at small $\omega$ and small gradients w.r.t. $\zeta$ that the Gaussian pairing fluctuations can be written as [58]:

$$S_{\text{sc}}/N = \frac{1}{z^d} \int d^d k d\omega d\zeta (2\pi)^d+1 \zeta^{d-1} z \left( \frac{\partial \psi^*}{\partial \zeta} \frac{\partial \psi}{\partial \zeta} \right) + \left( \omega^2 + A k^2 \zeta^2 z^2 + \frac{m^2}{z^2} \right) |\psi|^2$$

We introduced $A = 8c_0^2/\gamma \lambda^2$ as well as the two contributions $m_1^2 = \frac{\lambda^2}{c_0^2} z^2$ and $m_2^2 = -\frac{\alpha}{\zeta^2} - \frac{1}{4} (d + z)^2$ to the total mass $m^2 = m_1^2 + m_2^2$ that originate from the
first and second term in Eq.(10), respectively. To simplify some of the coefficients in $S_{sc}$, we used $c_2^2 = \frac{\lambda^2}{2m_1^2}$ and $c_0^2 = \frac{\lambda^2}{16\pi z_0^2} c_1^{-2\gamma}$. The positive coefficients $a_\gamma$ and $b_\gamma$ follow from the expansion $r_w = \int_{-\infty}^{\infty} dx \frac{|x|^{\gamma}}{|1-|x||^{1-\gamma/2}} \approx a_\gamma - b_\gamma \frac{v_{eff}^2}{|z|}$.

Eq.(14) is the key result of our analysis. Transforming back from momenta and energies $(k, \omega)$ to coordinates and time $(x, \tau)$, we can bring Eq.(14) in the form Eq.(1) of a scalar field in gravitational space with metric

$$ds^2 = \tilde{g}_{\mu \nu} dx^\mu dx^\nu = z^2 \left( \frac{d\xi^2}{\zeta^2} + \frac{dx^2}{\zeta^2/z^2} \right).$$

This metric describes a Lifshitz geometry\cite{29-36}, with dynamic scaling exponent $z$. To see this, rescale $\zeta \to s\zeta$, $\tau \to s\tau$ and $x_i \to s^{1/2} x_i$ with parameter $s$, which ensures invariance of the line element.

The origin of the gravity description of $H$ in Eq.(2) is the power-law dependence of the particle-particle propagator and the boson propagator that enter Eq.(10). The holographic variable $\zeta$ is identified as the inverse of the frequency related to the relative time of the bilocal pair field. Somewhat similar to a renormalization group scale, it probes the internal dynamics of the Cooper pair. On the other hand, $\tau$ corresponds to the total time in Eq.(15) and describes excitations and the non-equilibrium dynamics of the pair as a whole. The total momentum leads to the coordinates $x$ that describes inhomogeneous pairing states. However, the relative momentum $q$, that also determines the symmetry of Cooper pairs, does not appear in the holographic theory; there is only one extra dimension. Near $p_F$ the dependence on $q$ is frozen by the Fermi-surface kinematics.

For the Ising-ferromagnetic QCP with $\gamma = 1/3$ we obtain $z = 3/2$. Hence, the natural dynamic scaling exponent of the holographic theory is not $z_{hos} = 3$ but $z_{bos} = 3/2$. Indeed, balancing $\Sigma(\epsilon) \sim |\epsilon|^{-1}$ against $v_F p_{\perp}$ leads to $\epsilon \sim p_{\perp}^2$ with $z$ of Eq.(13) for all $\gamma$. If $\gamma \to 0^+$ (i.e. $z = 1 + \gamma^+$), we obtain AdS$_d + 2$, while for $\gamma \to 1^-$ (i.e. $z \to \infty$) we have AdS$_2 \otimes \mathbb{R}_d$. $z$ of Eq.(13) can already be deduced from the pairing response $\chi_{k}(\omega, \epsilon)$ of Eq.(11).

At $T = 0$, superconductivity disappears when $m^2 = m^2_{BP}$, where $m^2_{BP} = -\frac{1}{4}(d + z)^2$ is indeed the Breitenlohner-Freedman bound\cite{39} for the Lifshitz geometry\cite{37}. This condition determines a critical pair-breaking strength $\alpha_s(z) = 1 - 2 e/4z_{+1/2}$, with coefficient $\alpha_s$ introduced above. This is identical to the condition that follows from the Eliashberg Eq.(9). For example, for $z = 3/2$ we find the same value for $\alpha_s$ as given above. For the AdS$_d \otimes \mathbb{R}_d$ phase with $z \to 1$, superconductivity is rather robust, with $\alpha_s \approx 1 - \frac{e}{\pi^2}$. On the other hand for an almost local problem with AdS$_2 \otimes \mathbb{R}_d$ where $z \to \infty$, pairing is very fragile with $\alpha_s \approx \frac{5}{4} \log 4 z^{-1}$. Notice, $m^2_{BP}$, which is due to the pairing interaction, is always more negative than $m^2_{BP}$. It is the contribution $m^2_{BP} > 0$ due to particle-particle excitations of a non-Fermi liquid that drives the ground state into the normal phase. Notice, the expansion of $r_w$ w.r.t. $w$ corresponds to an expansion w.r.t. $(\zeta - i m_{BF}) \psi$\cite{58} as appropriate for anti-de Sitter spaces.

In Eq.(14) we have no gauge field in the gravitational bulk, i.e. $\partial_\xi \zeta$ instead of $\partial_\xi i 2e A_\xi$ with corresponding electric field $E$. This is consistent with the holographic map of the Yukawa-SYK model where one finds $E = 0$ at particle-hole symmetry\cite{25}, a symmetry that is emergent for a system with Fermi surface. In addition, there is no hyper-scaling violation in Eq.(15), which would amount to a global factor $s^{\theta/(ds)}$ upon rescaling\cite{7}. Thus, we obtain $\theta = 0$.

The Lifshitz geometry is not a solution of a pure gravity problem, i.e. it must be caused by the back-reaction of some matter fields. Numerous, rather different proposals that all lead to Lifshitz spacetimes have been made\cite{29-37}. The back-reaction must be due to gravitational duals of operators made from fermions and bosons of our Hamiltonian Eq.(2). This could offer a hint as to which mechanism for Lifshitz gravities is more appropriate. The absence of electromagnetic bulk fields due to particle-hole symmetry, argues against an electromagnetic screening mechanism like the feedback of the charged scalar itself\cite{31} or the formation of an electron star\cite{33}; see also Ref.[34]. Since our Ising ferromagnetic mode is neutral, a more natural scenario might be the theory of Ref.[30, 37].

Finally, we comment on the denominator $1 + z^2 \zeta^2 \omega^2/m^2$ in Eq.(12). It is necessary because of the negative sign in front of the $(\omega/\epsilon)^2$ term of an expansion of $\chi_{k}^{-1}(\omega, \epsilon)$ of Eq.(11). This does not signal an instability towards high-frequency solutions as the physical susceptibility $\int d\epsilon \chi_p(\omega, \epsilon)$ is still largest for $\omega \to 0$. It merely signals that $F_{\epsilon}(\omega, \epsilon)$ is not an ideal variable to perform manipulations, while $\psi$ is much better behaved. For the Yukawa-SYK problem, the same issue occurred and was regularized using a Radon transformation\cite{24, 25}. This suggests that Eq.(12) is a correspondence valid only at low energies and that a more general relationship must be non-local.

In summary, we analyzed a controlled large-$N$ theory of fermions at a two-dimensional Ising-ferromagnetic quantum critical point, where critical fluctuations give rise to $p$-wave triplet superconductivity. We then showed that superconducting fluctuations of this model behave like those of a holographic superconductor in four dimensional Lifshitz gravity with dynamic scaling exponent $z = 3/2$. We also considered more general systems where the non-Fermi liquid is governed by a fermionic dynamic exponent $z > 1$ and showed that the holographic map continues to be a Lifshitz geometry. Quantum-critical superconductivity, as it follows from an analysis of the Eliashberg equations, is shown to be equivalent to holographic superconductivity. This bridge between two rather different perspectives of the pairing problem for systems without quasiparticles might answer some of the questions posed in the introduction and comes with the potential to use insights and techniques of one community and use it to solve open problems in the other.
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Appendix A: The derivation of the effective action and the Saddle-Point equations

In this appendix we derive the effective action and saddle point equations of the Hamiltonian given in Eq. (2). We consider a slightly more general problem with fermion spinor $c_i$ coupled to bosonic fields $\phi_{ls}(x)$ via the bilinear $c_i^\dagger(x) \hat{\gamma}_s c_j(x)$ with $S$ Hermitian matrices $\hat{\gamma}_s$ in spinor space, where $s = 1, \cdots, S$. $x = (x, \tau)$ comprises space and imaginary time coordinates. For the problem in the main text we have $S = 1$ and $\hat{\gamma} = \hat{\sigma}^2$ such that $c_i(x)$ is a two-component spinor in spin space. Hence, we consider a problem with Yukawa coupling

$$S_{\text{int}} = \frac{1}{N} \sum_{i,j,l,s} g_{ijl} \int_x c_i^\dagger(x) \hat{\gamma}_s c_j(x) \phi_{ls}(x) + \text{h.c.}.$$  \hspace{1cm} (A1)

There are $N$ additional flavor indices $i = 1 \cdots N$ for the fermions and $M$ for bosons, $l = 1 \cdots M$. For the real and imaginary parts of the random coupling constant $g_{ijl} = g'_{ijl} + ig''_{ijl}$ we use the correlation functions

$$g_{ijl}g'_{ij'l'} = g^2 \left(1 - \frac{\alpha}{2}\right) \delta_{ll'} \delta_{jj'} \delta_{ij'} + (\delta_{ij} \delta_{jj'}) \delta_{ij'} + (\delta_{ij} \delta_{jj'}),$$

$$g_{ijl}g''_{ij'l'} = g'^2 \frac{\alpha}{2} \delta_{ll'} \delta_{jj'} - (\delta_{jj'} \delta_{ij'}),$$

$$g_{ijl}g''_{ij'l'} = 0.$$ \hspace{1cm} (A2)

Performing the averaging over the $g_{ijl}$ using the replica trick leads to

$$S_{\text{int}} = -\frac{g^2}{2N^2} \int_{xx'} \sum_{ijl,s,s',AB} \sum_{c_i^A}(x) \hat{\gamma}_s c_j A(x) \phi_{lsA}(x) c_j^B(x') \hat{\gamma} s' c_{lsB}(x')$$

$$- \frac{g''}{2N^2} \int_{xx'} \sum_{ijl,s,s',AB} \sum_{c_i^A}(x) \hat{\gamma}_s c_j A(x) \phi_{lsA}(x) c_j^B(x') \hat{\gamma} s' c_{lsB}(x') \phi_{ls'B}(x').$$ \hspace{1cm} (A3)

Here, the coupling constant in the pairing channel $g'' = g^2 (1 - \alpha)$ includes the pair-breaking parameter $\alpha$. A and $B$ are replica indices. As usual, we assume a trivial structure in replica space and drop replica indices in what follows. Next we define bilinear fermion fields

$$G_{ab}(x, x') = \frac{1}{N} \sum_{i=1}^N c_{ia}(x) c_{ib}^\dagger(x'),$$

$$F_{ab}(x, x') = \frac{1}{N} \sum_{i=1}^N c_{ia}(x) c_{ib}(x'),$$

$$F_{ab}(x, x') = \frac{1}{N} \sum_{i=1}^N c_{ia}(x) c_{ib}^\dagger(x'),$$

$$D_{ss'}(x, x') = \frac{1}{M} \sum_{i=1}^M \phi_{ls}(x) \phi_{ls'}(x').$$ \hspace{1cm} (A4)

The interaction term then becomes

$$S_{\text{int}} = \frac{M g^2}{2} \sum_{ss'} \int_{xx'} \text{Tr} \left( \hat{\gamma}_s \hat{G}(x, x') \hat{\gamma}_s \hat{G}^\dagger(x', x) \right) D_{ss'}(x, x')$$

$$- \frac{M g''}{2} \sum_{ss'} \int_{xx'} \text{Tr} \left( \hat{\gamma}_s \hat{F}(x, x') \hat{\gamma}_s \hat{F}^\dagger(x', x) \right) D_{ss'}(x, x').$$ \hspace{1cm} (A5)

$\text{tr}$ stands for the trace of the spinor indices $a, b$ of the fermions. We will also use it w.r.t. the boson components $s$ and $s'$.

We can now integrate out fermions and bosons to obtain the effective action in terms of the bilocal fields

$$S/N = \frac{\mu}{2} \text{Tr} \log \left(D^{-1}_0 - \Pi_0 \right) - \frac{1}{2} \text{Tr} \log \left(\hat{G}_0^{-1} - \hat{\Sigma} \right)$$

$$+ \frac{\mu}{2} \text{Tr} \hat{D} \otimes \hat{\Pi} - \frac{1}{2} \text{Tr} \hat{G} \otimes \hat{\Sigma} + S_{\text{int}}/N,$$ \hspace{1cm} (A6)
where $\mu = M/N$. We use the notation:

$$\text{Tr} \hat{A} \otimes \hat{B} = \int_{xx'} \text{tr} \left( \hat{A}(x, x') \hat{B}(x', x) \right),$$

(A7)

and Tr for an additional trace in Nambu space. Here we use

$$\hat{G}(x, x') = \begin{pmatrix} \hat{G}(x, x') & \hat{F}(x, x') \\ \hat{F}^\dagger(x, x') & \hat{G}(x, x') \end{pmatrix},$$

(A8)

as well as

$$\hat{\Sigma}(x, x') = \begin{pmatrix} \Sigma(x, x') & \Phi(x, x') \\ \Phi^\dagger(x, x') & \Sigma(x, x') \end{pmatrix}.$$

(A9)

with $\hat{A}(x, x') = -\hat{A}^T(x', x)$. In the limit of large $N$ we obtain the saddle-point equations

$$\hat{\Sigma}(x, x') = g^2 \mu \sum_{ss'} \hat{\gamma}_s \hat{G}(x, x') \hat{\gamma}_{s'} D_{ss'} (x, x'),$$

$$\hat{\Phi}(x, x') = -g^2 \mu \sum_{ss'} \hat{\gamma}_s \hat{F}(x, x') \hat{\gamma}_{s'}^T D_{ss'} (x, x'),$$

$$\Pi_{ss'} (x, x') = -g^2 \text{tr} \left[ \hat{\gamma}_s \hat{G}(x, x') \hat{\gamma}_{s'} \hat{G}(x', x) \right] + g^2 \mu \text{tr} \left[ \hat{\gamma}_s \hat{F}(x, x') \hat{\gamma}_{s'}^T \hat{F}^\dagger (x', x) \right].$$

(A10)

At the stationary point we assume translation invariance in space and time, i.e. $G_{ab}(x, x') = G_{ab}(x-x')$ etc. and perform a Fourier transformation to momentum and frequency variables $p = (p, \epsilon)$. We then obtain the Eliashberg equations

$$\hat{\Sigma}(p) = g^2 \mu \sum_{ss'} \int d\epsilon \hat{\gamma}_s \hat{G}(p) \hat{\gamma}_{s'} D_{ss'} (p-p'),$$

$$\hat{\Phi}(p) = -g^2 \mu \sum_{ss'} \int d\epsilon \hat{\gamma}_s \hat{F}(p) \hat{\gamma}_{s'}^T D_{ss'} (p-p'),$$

$$\Pi_{ss'} (q) = -\int d\epsilon \left\{ g^2 \text{tr} \left[ \hat{\gamma}_s \hat{G}(p) \hat{\gamma}_{s'} \hat{G}(p+q) \right] - g^2 \mu \text{tr} \left[ \hat{\gamma}_s \hat{F}(p) \hat{\gamma}_{s'}^T \hat{F}^\dagger (p+q) \right] \right\}.$$

(A11)

In addition we have the Dyson equations with $q = (q, \epsilon)$:

$$D_{ss'}^{-1} = (\epsilon^2 + \omega_q^2) \delta_{ss'} - \Pi_{ss'} (q)$$

(A12)

for the bosonic and

$$\hat{G}(p)^{-1} = \hat{G}_0(p)^{-1} - \hat{\Sigma}(k)$$

(A13)

for the fermionic propagators.

Appendix B: Analysis of the saddle point equations

1. Analysis in the normal state

Let us first analyze these equations in the normal state. We perform the momentum integration as

$$\int_p \cdots = \frac{1}{N} \sum_p \cdots = \rho_F \int_{\varepsilon_F} d\varepsilon_p \int_{2\pi} \frac{d\theta_p}{2\pi} \cdots,$$

(B1)

where $N$ is the number of lattice sites, $\varepsilon_p$ is the energy, and $\theta_p$ the direction of the momentum $p$. $\rho_F = \frac{1}{2\pi} m_e a_0^2 \varepsilon_F$ is the density of states multiplied by the Fermi energy $\varepsilon_F$, where $a_0$ is the lattice constant of the crystal and $m_e$ the electron mass. $\rho_F$ is a dimensionless quantity.
The repeating theme in the analysis is to assume and show in the end that typical bosonic momenta are slower than fermionic ones which implies that the fermionic self energy weakly depends on the momentum perpendicular to the Fermi surface. Hence, we assume that \( \Sigma_p (\epsilon) \approx \Sigma_\theta_p (\epsilon) \), independent on the deviation of \( |p| \) from \( p_F \), i.e. independent of \( \varepsilon_p \). Then it follows for the boson self energy:

\[
\Pi_q (\epsilon) = \Pi_{q \to 0} (0) - \frac{2g^2 \rho_F}{\varepsilon_F} \int_0^{2\pi} d\theta_p \frac{2\pi}{\varepsilon_F} \left( \frac{\varepsilon_p}{\varepsilon_{p_F}} \right)^2 + \cos^2 (\theta_p - \theta_q),
\]

\[
\approx \Pi_{q \to 0} (0) - \frac{2g^2 \rho_F}{\varepsilon_F} |\epsilon| \frac{|\epsilon|}{v_F q}.
\]  

(B2)

If we introduce the renormalized correlation length

\[
\xi = \frac{c}{\sqrt{\omega_0^2 - \Pi_{q \to 0} (0)}},
\]

(B3)

we obtain for the bosonic propagator

\[
D_q (\epsilon) = \frac{1}{\varepsilon^2 (\xi^{-2} + q^2) + \frac{2g^2 \rho_F |\epsilon|}{\varepsilon_F} v_F q}.
\]  

(B4)

At the quantum-critical point we have \( \Pi_{q \to 0} (0) = \omega_0^2 \) such that \( \xi \to \infty \) and the boson becomes massless.

Now one can evaluate the fermionic self energy of the normal state:

\[
\Sigma_p (\epsilon) = -\mu g^2 \int \frac{d^2 p'}{(2\pi)^2} \int \frac{d\epsilon'}{2\pi i} \frac{D_{p-p'} (\epsilon - \epsilon')}{\epsilon' - \varepsilon_p - \Sigma_{p'} (\epsilon')}.
\]

(B5)

We focus on the critical point \( \xi^{-1} = 0 \). Under the conditions for the fermionic self energy outlined above, we can assume that \( D_{p-p'} (\epsilon) \) is dominated by the momentum dependence that is tangential to the Fermi surface, expressed in terms of \( \theta_p \) and \( \theta_{p'} \). Then we can perform the integral over \( \varepsilon_{p'} \) using

\[
\int \frac{d\varepsilon_{p'}}{i\epsilon' - \Sigma_{\theta_{p'}} (\epsilon') - \varepsilon_{p'}} = -i\pi \text{sign} (\epsilon'),
\]

(B6)

independent on the actual form of the self energy. This analysis also illustrates that the precise form of the dispersion is not very important, as long as it is sufficiently generic, i.e. without nested pieces of the Fermi surface and the band width is large compared to the typical excitation energy. The resulting angular integration is dominated by small \( \varphi = \theta_p - \theta_{p'} \) and yields for the self energy

\[
\Sigma_{\theta_p} (\epsilon) = -i\lambda \frac{\varepsilon_{\frac{1}{3}}}{3} \frac{\text{sign} (\epsilon')}{|\epsilon - \epsilon'|^{1/3}}
\]

\[
= -i\lambda \text{sign} (\epsilon) \frac{\varepsilon_{\frac{1}{3}}}{3} |\epsilon|^{2/3},
\]

(B7)

with the dimensionless constant

\[
\lambda = \frac{\mu}{2\sqrt{3}} \left( \frac{2g^2 \rho_F v_F^3}{\varepsilon_F c^2} \right)^{2/3}.
\]

(B8)

This result is indeed independent on momentum.

Eqn. (B2) at \( \xi^{-1} = 0 \) and (B7) are the expressions for the bosonic and fermionic self energies given in the main text.

2. Analysis in the superconducting state

Next we analyze the linearized gap equation for the anomalous self energy

\[
\tilde{\Phi} (p) = -\mu g_p^2 \int_{p'} \tilde{\sigma} \tilde{G} (p') \tilde{\Phi} (p') \tilde{G} (p') \tilde{\sigma} D (p - p').
\]

(B9)
The Pauli principle implies $\Phi_{\sigma'} (p) = -\Phi_{\sigma'} (-p)$. We expand the anomalous self energy in spin space, using $p = (p, \epsilon)$ for the momentum and energy parts of $p$:

$$
\Phi_{\sigma\sigma'} (p) = \sum_{\mu=0}^{3} \Phi_{\mu\mu} (\epsilon) i (\sigma'\sigma^\mu)_{\sigma\sigma'}.
$$

(B10)

For a system with inversion symmetry, the pairing function has well defined parity. Hence, it is either odd in momentum $\Phi_{\mu\mu} (\epsilon) = -\Phi_{\mu\mu} (\epsilon)$ and even in energy $\Phi_{\mu\mu} (\epsilon) = \Phi_{\mu\mu} (\epsilon)$ or the other way around. In what follows we consider the even-frequency option and find

$$
\Phi_{\mu\mu} (\epsilon) = s_{\mu} \int_{p'} d\epsilon_{p'} G_{p'} (\epsilon_{p'}) \Phi_{\mu\mu} (\epsilon_{p'}) G_{p'} (\epsilon_{p' - \epsilon}) (\epsilon_{p' - \epsilon}),
$$

(B11)

where we used $\frac{1}{2} \text{tr} (\sigma^\mu \sigma^\nu \sigma^\tau \sigma \sigma^\iota) = s_{\mu} \delta_{\nu\iota}$ with $s_0 = s_2 = -1$ and $s_1 = s_3 = 1$. Hence for even frequency pairing and small momentum transfer we find triplet pairing $\Phi_{p} (\epsilon)$ and $\Phi_{p} (\epsilon)$. In a real crystalline lattice the degeneracy between these two states may or may not be lifted. For our purposes, where we only analyze the Gaussian part of the action, this effect is not particularly important and we will assume that the degeneracy is lifted and consider only one triplet pairing state which we call $\Phi_{p} (\epsilon)$, determined by the gap equation

$$
\Phi_{p} (\epsilon) = \mu \int_{p'} d\epsilon_{p'} G_{p'} (\epsilon_{p'}) G_{p'} (\epsilon_{p' - \epsilon}) (\epsilon_{p' - \epsilon}).
$$

(B12)

We again use that $\Phi_{p} (\epsilon)$ depends only on the angle $\theta_{p}$ and perform the integration over $\epsilon_{p'}$:

$$
\int d\epsilon_{p'} G_{p'} (\epsilon_{p'}) G_{p'} (\epsilon_{p'}) = \int \frac{1}{\pi} \frac{\epsilon_{p}^2 + \lambda^2 \epsilon_{p}^{4/3} |\epsilon'|^{4/3}}{\epsilon_{p}^{1/3} |\epsilon'|^{2/3}}.
$$

(B13)

It follows with $(p - p')^2 \approx 2\pi^2 (1 - \cos (\theta - \theta'))$ that

$$
\int \frac{d\theta'}{2\pi} \frac{\Phi_{p} (\epsilon_{p'}) \sqrt{2} (1 - \cos (\theta - \theta'))}{(2 - 2 \cos (\theta - \theta'))^{3/2}} = \frac{g^{2} \rho_{F} |\epsilon|^{3/2}}{\epsilon_{p}^{1/2} |\epsilon'|^{3/2}}.
$$

(B14)

To proceed we expand in harmonics

$$
\Phi_{\mu} (\epsilon) = \sum_{l=-\infty}^{\infty} \Phi_{l} (\epsilon) e^{l \theta},
$$

(B15)

keeping in mind that only odd $l$ are allowed as we are analyzing odd-parity triplet pairing. This yields

$$
\Phi_{l} (\epsilon) = \frac{\mu_{F}^{2} \rho_{F}}{2 \epsilon_{p}^{1/3} |\epsilon|^{2/3}} \int \frac{d\epsilon_{p} (\epsilon_{p} - \epsilon_{p}')}{|\epsilon_{p}^{1/2} |\epsilon'|^{2/3}} \Phi_{l} (\epsilon_{p'}),
$$

(B16)

where

$$
d_{l} (\epsilon) = \frac{\sqrt{2} (1 - \cos \varphi) e^{l \varphi}}{2 |\epsilon|^{3/2}} + \frac{2 g^{2} \rho_{F} |\epsilon|^{3/2}}{\epsilon_{p}^{1/2} |\epsilon'|^{3/2}}.
$$

(B17)

At small energies follows

$$
d_{l} (\epsilon) = \frac{2}{3 \sqrt{3}} \left( \frac{\epsilon_{p}^{1/3} \epsilon_{p}^{1/3}}{2g^{2} \rho_{F} \epsilon_{p}^{1/3}} \right)^{1/3} \frac{|\epsilon|^{1/3}}{\epsilon_{p}^{1/3}} - \frac{|l|}{2} + O \left( \frac{\kappa |\epsilon|^{1/3}}{\epsilon_{p}} \right).
$$

(B18)
The second term implies that the most attractive channel corresponds to \( l = \pm 1 \), i.e. \( p \)-wave pairing. We pick this channel and determine the frequency dependence of \( \Phi (\epsilon) \) from the linearized gap equation

\[
\Phi (\epsilon) = \frac{1 - \alpha}{3} \int d\epsilon' \frac{\Phi (\epsilon')}{|\epsilon'|^{2/3} |\epsilon' - \epsilon|^{1/3}}.
\] (B19)

Following Ref.[55] one makes a powerlaw ansatz \( \Phi (\epsilon) \sim |\epsilon|^\frac{1}{2} e^{i\theta} \) and finds that superconductivity survives until \( \alpha \) reaches a critical pair-breaking strength \( \alpha_* \approx 0.879618 \) that follows from

\[
1 = \frac{1}{3} \int_{-\infty}^{\infty} dx \frac{1 - \alpha}{|1 - x|^{1/3} |x|^{5/6}}.
\] (B20)

Then the transition temperature vanishes as \( T_c \sim \exp \left(-\frac{A}{\sqrt{\alpha_* - \alpha}}\right) \). Right at \( \alpha = \alpha^* \) holds \( \Phi (\epsilon) \propto |\epsilon|^{-\frac{1}{2}} \left( 1 + \frac{1}{6} \log \frac{\epsilon_0}{|\epsilon|} \right) \).

**Appendix C: Gaussian pairing fluctuations**

In this appendix we formulate the theory of Gaussian fluctuations relative to the normal state. This lays the grounds for the derivation of the holographic action. We keep the discussion rather general for as long as it makes sense and specify the concrete interaction and pairing state of our problem further below.

We expand the action Eq.(5) up to second order in pairing terms, i.e.

\[
S_{sc}/N = -\text{Tr} \left( \hat{F}^\dagger \otimes \hat{\Phi} + \hat{\Phi} \otimes \hat{F}^\dagger \right)
- \mu g_p^2 \sum_{ss'} \int_{x,x'} \text{tr} \left( \tilde{\gamma}_s \hat{F} (x, x') \tilde{\gamma}_s^T \hat{F}^\dagger (x', x) \right) D_{ss'} (x, x')
+ \text{Tr} \left( \hat{G} \otimes \hat{\Phi}^\dagger \otimes \hat{\Phi} \right). \tag{C1}
\]

After Fourier transformation to momentum and frequency coordinates, where \( k \) refers to the variable conjugate to \( (x + x')/2 \) and \( q \) conjugate to \( x - x' \) we obtain:

\[
S_{sc}/N = -\frac{1}{2} \int_{kq} \text{tr} \left( \hat{F}^\dagger (k, q) \hat{\Phi} (k, q) + h.c. \right)
- \frac{\mu g_p^2}{2} \sum_{ss'} s_kq \text{tr} \left( \tilde{\gamma}_s \hat{F}^\dagger (k, q) \tilde{\gamma}_s^T \hat{F} (k, q') \right) D_{ss'} (q - q')
- \frac{1}{2} \int_{kq} \text{tr} \left( \hat{G} \left( -\frac{k}{2} + q \right) \hat{\Phi}^\dagger (k, q) \hat{\Phi} (k, q) \hat{G} \left( -\frac{k}{2} - q \right) \right). \tag{C2}
\]

From the analysis of the gap equation we already know that the dependence of the pairing function on the momentum \( q \) and on the energy \( \epsilon \) is rather different. Hence, at this point the usage of combined space-time variables \( q = (q, \epsilon) \) or \( k = (k, \omega) \) seizes to be efficient. In what follows we expand the pairing functions with respect to some complete set of functions \( \hat{\eta}_q^{(L)} \) that depend on \( q \) and the spin structure

\[
\hat{\Phi} (k, q) = \hat{\Phi}_{k,q} (\omega, \epsilon) = \sum_{L} \Phi_{kL} (\omega, \epsilon) \hat{\eta}_q^{(L)},
\]

\[
\hat{F} (k, q) = \hat{F}_{k,q} (\omega, \epsilon) = \sum_{L} F_{kL} (\omega, \epsilon) \hat{\eta}_q^{(L)}. \tag{C3}
\]

We insert this expansion and obtain

\[
S_{sc} = \frac{1}{2} \int \left( \Phi_{kL}^\dagger (\omega, \epsilon) F_{kL} (\omega, \epsilon) + \Phi_{kL} (\omega, \epsilon) F_{kL}^\dagger (\omega, \epsilon) \right)
- \mu g_p^2 \int F_{kL}^\dagger (\omega, \epsilon) D_{LL'} (\epsilon - \epsilon') F_{kL'} (\omega, \epsilon')
- \frac{1}{2} \int \Phi_{kL}^\dagger (\omega, \epsilon) \chi_{kLL'} (\omega, \epsilon) \Phi_{kL'} (\omega, \epsilon), \tag{C4}
\]
where we introduced
\[ D_{LL'}(\epsilon) = \frac{1}{2} \int_{q,q'} \text{tr} \left( \hat{\gamma}_q \hat{\gamma}_{q'}^{(L)} \right) D_{q-q'}(\epsilon), \] (C5)
as well as
\[
\chi_{k,LL'}(\omega,\epsilon) = \int_q \text{tr} \left( \hat{\gamma}_q^{(L)} \hat{\gamma}_{q'}^{(L')} \right) G_{\frac{\omega}{2}-q}^{\frac{\omega}{2}} G_{\frac{\omega}{2}+q}^{\frac{\omega}{2}} \left( \omega - \epsilon \right) G_{\frac{\omega}{2}+q}^{\frac{\omega}{2}} \left( \omega + \epsilon \right). \] (C6)

Now we are in a position to integrate out the \( \Phi_{kL}(\omega,\epsilon) \) and obtain
\[
S^{(sc)} + \int F_{kL}^{\dagger}(\omega,\epsilon) \chi_k^{-1}(\omega,\epsilon)_{LL'} F_{kL'}(\omega,\epsilon)
- \mu g^2 \int F_{kL}^{\dagger}(\omega,\epsilon) D_{LL'}(\epsilon - \epsilon') F_{kL'}(\omega,\epsilon'). \] (C7)

This formulation of the problem is really only useful if we can find a set of functions \( \hat{\gamma}_q^{(L)} \) that simultaneously diagonalize the \( \chi_{k=0,LL'}(\omega=0,\epsilon) \) and \( D_{LL'}(\epsilon) \) for all \( \epsilon \). We can then expand in this basis for small \( \omega \) and \( \k \). To proceed start from
\[
\hat{\gamma}_p^{(l,\alpha,n)} = i \hat{y} \hat{\gamma} \hat{n}^{\epsilon} e^{i \hat{q} \hat{n}} T_n \left( \frac{|p| - p_F}{p_F} \right). \] (C8)

Hence the index \( L = (l, \alpha, n) \) consists of the angular momentum \( l \in \mathbb{Z} \) and the spin index of the Cooper pair \( \alpha \), with \( \alpha = 0 \) for a singlet and \( \alpha \in \{x, y, z\} \) for triplets. In addition, the \( T_n(y) \) is some complete set of polynomials where \( n \) is the order of the polynomial. We use the insight that we obtained from the solution of the linearized gap equation. Here we found that only the leading polynomial \( T_n(1) = 1 \) in \( y = (|p| - p_F) / p_F \) contributes for momenta near the Fermi surface. From now on we drop the index \( n = 0 \). Now we are in a position to evaluate the matrix elements. It follows
\[
D_{\alpha l,\alpha' l'}(\epsilon) = s_\alpha \delta_{ll'} \delta_{\alpha \alpha'} d_{l}(\epsilon), \] (C9)
with \( d_l(\epsilon) \) of Eq.(B18).

We can generalize our approach and consider a self energy of the normal state that behaves as
\[
\Sigma_p(\epsilon) = -i \lambda \text{sign}(\epsilon) \varepsilon_F^{\gamma} |\epsilon|^{1-\gamma} \] (C10)
while the matrix of the pairing interaction is at leading order
\[
D_{\alpha l,\alpha' l'}(\epsilon) \sim \delta_{ll'} \delta_{\alpha \alpha'} |\epsilon|^{-\gamma}. \] (C11)

Then we have
\[
\chi_{k=0,\alpha l,\alpha' l'}(\omega,\epsilon) = \rho_F \int \frac{d\theta_p}{2\pi} \int \frac{d\varepsilon_p}{\varepsilon_p + \varepsilon_{p+k} + \Sigma(\frac{| \omega |}{2} + \varepsilon)} \delta_{\alpha \alpha'} e^{-i \theta_p (l-l')} \frac{1}{\varepsilon_p + \Sigma(\frac{| \omega |}{2} - \varepsilon)} \] (C12)
For \( l = l' \) and \( \alpha = \alpha' \) follows
\[
\chi_k(\omega,\epsilon) = \frac{4\pi}{\lambda \varepsilon_F^{1+\gamma}} \frac{\theta \left( |\epsilon| - |\frac{\omega}{2} | \right)}{\sqrt{\left( |\epsilon + \frac{\omega}{2} |^{1-\gamma} + |\epsilon - \frac{\omega}{2} |^{1-\gamma} \right)^2 + \left( \frac{\omega}{\lambda \varepsilon_F} \right)^2}} \] (C13)
Off diagonal elements in \( l \) and \( l' \) only occur for finite momenta and one can perform a small \( k \) expansion to include those. Expanding for small \( k \) and \( \omega \) then yields the result given in Eq.(11).
Appendix D: The holographic map

In this appendix we give the details of the holographic map. To keep the equations shorter we use, just like in the main text, a system of units where energies are measured in units of \( \varepsilon_F \) and momenta in units of \( p_F \).

The Gaussian theory of the pairing fluctuations theory takes the following form

\[
S_{sc} = S_{sc,1} + S_{sc,2}
\]  

(D1)

where

\[
S_{sc,1} = \int_{k\omega} F_k^* (\omega, \epsilon) \chi_k^{-1} (\omega, \epsilon) F_k (\omega, \epsilon)
\]  

(D2)

with

\[
\chi_k^{-1} (\omega, \epsilon) = \frac{\lambda}{2\pi} |\epsilon|^{1-\gamma} \left( 1 - \frac{\gamma}{8} \frac{(\omega/\epsilon)^2}{\lambda^2 |\epsilon|^{2-2\gamma}} \right),
\]  

(D3)

as well as

\[
S_{sc,2} = -\lambda_p \int_{k\omega,\epsilon'} \frac{F_k^* (\omega, \epsilon) F_k (\omega, \epsilon')}{|\epsilon - \epsilon'|^\gamma},
\]  

(D4)

with \( \lambda_p = \frac{1}{2} \lambda (1 - \gamma) (1 - \alpha) \).

At the saddle point we obtain

\[
\Phi_k (\omega, \epsilon) = \lambda_p \int d\epsilon' \frac{\chi_k (\omega, \epsilon') \Phi_k (\omega, \epsilon')}{|\epsilon - \epsilon'|^\gamma}.
\]  

(D5)

in terms of the anomalous self energy

\[
\Phi_k (\omega, \epsilon) = \chi_k^{-1} (\omega, \epsilon) F_k (\omega, \epsilon).
\]  

(D6)

Notice, the equation for \( \Phi_k (\omega, \epsilon) \) is nonlocal only with respect to the relative energy \( \epsilon \). For \( \omega = 0 \) and \( \mathbf{k} = 0 \) this is the usual linearized Eliashberg equation

\[
\Phi (\epsilon) = \frac{\lambda_p}{\lambda} \int d\epsilon' \frac{\Phi (\epsilon)}{|\epsilon - \epsilon'|^\gamma} |\epsilon'|^{1-\gamma}.
\]  

(D7)

For \( \gamma = 1/3 \) this agrees with Eq.(B19). At a superconducting quantum-critical point, which is reached for a threshold strength of the pairing interaction, one can make the power-law ansatz \( \Phi (\epsilon) \sim |\epsilon|^{-\gamma/2} \) which yields the condition

\[
\frac{\lambda}{\lambda_p} = a_\gamma \equiv \int_{-\infty}^{\infty} dx \frac{1}{|1-x|^\gamma |x|^{1-\gamma/2}}.
\]  

(D8)

Below we will see that the same criterion also follows from the holographic analysis at the Breitenlohner-Freedman bound.

1. **On the sign of the \((\omega/\epsilon)^2\)-term in \(\chi_k^{-1}(\omega,\epsilon)\)**

Before we perform the holographic map, we comment on the curious minus sign in front of the \( \frac{\gamma}{8} \frac{(\omega/\epsilon)^2}{\lambda^2 |\epsilon|^{2-2\gamma}} \) in \( \chi_k^{-1} (\omega, \epsilon) \). At first glance this looks like an instability towards large \( \omega \) configurations \( F_k (\omega, \epsilon) \). However, in a Gaussian theory this depends on the appropriate source field or boundary conditions. An external Josephson coupling

\[
H_J = \int_{x,\tau} J_{\sigma\sigma'} (x, \tau) \sum_i c_{i\sigma}^\dagger (\tau) c_{i\sigma'}^\dagger (\tau)
\]  

(D9)

couples to \( \int dt F_k (\omega, \epsilon) \). The bare susceptibility of relevance is

\[
\tilde{\chi}_q (\omega) = \int \frac{d\epsilon}{\pi} \chi_q (\omega, \epsilon).
\]  

(D10)
One easily sees that $\dot{\chi}_q (\omega) = \dot{\chi}_0 (0) - a_\omega |\omega|^\gamma - a_k k^{\frac{2}{\gamma}}$ is largest at $\omega = 0$, where $a_\omega$ and $a_k$ are positive. Here $\dot{\chi}_0 (0) = \frac{\lambda c_1}{2 c_0}$ is finite.

The situation is somewhat similar to a Gaussian theory with action

$$S_0 = \int_{\omega < \Lambda} d\omega (m^2 + \omega^2) \phi_\omega \phi_\omega,$$  \hspace{1cm} (D11)

which is stable for $m^2 > 0$. Let us now introduce the new field $f_\omega = \phi_\omega (1 + \omega^2/\Lambda^2)$. At small momenta this theory becomes

$$S_0 = \int_{\omega < \Lambda} d\omega \left( m^2 - \left( \frac{2m^2}{\Lambda^2} - 1 \right) \omega^2 \right) f_\omega f_\omega,$$  \hspace{1cm} (D12)

which looks unstable for $m^2 > \frac{1}{2} \Lambda^2$. However, large $\omega$ fluctuations in $f_\omega$ will be suppressed for the original field $\phi_\omega$, which is simply the more reasonable variable. The situation is very similar to our case where the scale $\Lambda$ is played by $\epsilon$, see the cut off coefficient $\theta \left( |\epsilon| - \frac{|\omega|}{2} \right)$ in Eq. (C13).

2. Holographic map for $S_\text{sc1}$

We will first perform the map for the field

$$\psi (k, \omega, \zeta) = c_0 \zeta^{(d-1)(1-\gamma)} F_k (\omega, c_1/\zeta).$$  \hspace{1cm} (D13)

The additional denominator in Eq.(12) will be included at the end. Inserting Eq.(D13) into the action $S_{\text{sc1}}$ yields

$$S_{\text{sc1}} = \frac{c_1}{c_0^2} \int_{kw \zeta} \zeta^{-(d-1)(1-\gamma)} \chi_1^{-1} (\omega, c_1/\zeta) |\psi (k, \omega, \zeta)|^2.$$  \hspace{1cm} (D14)

If we now use the explicit expression for $\chi^{-1}$ we get

$$S_{\text{sc1}} = \frac{\lambda c_1^{2-\gamma}}{2\pi c_0^2} \int_{kw \zeta} \zeta^{-d(1-\gamma)} \left( \frac{1}{\zeta^2} - \frac{\gamma}{8} \left( \frac{\omega}{c_1} \right)^2 + \left( \frac{k}{\lambda c_1^{1-\gamma} \zeta^\gamma} \right)^2 \right) |\psi (k, \omega, \zeta)|^2.$$  \hspace{1cm} (D15)

We fix the constant $c_1$ such that the coefficient in front of $\omega^2$ is $-(1-\gamma)^d$ which yield

$$c_1^\gamma = \frac{\lambda}{2\pi c_0^2} \frac{\gamma (1-\gamma)^d}{8}.$$  \hspace{1cm} (D16)

Then we get

$$S_{\text{sc1}} = (1-\gamma)^d \int_{kw \zeta} \zeta^{-d(1-\gamma)} \left( \frac{m_1^2 (1-\gamma)^2}{\zeta^2} - \omega^2 + A k^2 \right) |\psi (k, \omega, \zeta)|^2,$$  \hspace{1cm} (D17)

with $m_1^2 = \frac{8c_1^2}{\gamma(1-\gamma)}$ and $A = \frac{8c_1^{2\gamma}}{\gamma\lambda}$.  

3. Holographic map for $S_\text{sc2}$

Next we analyze $S_{\text{sc2}}$. Let us drop all the irrelevant coordinates and consider the action

$$S_F = - \int \frac{d\epsilon d\epsilon'}{(2\pi)^2} \frac{F (\epsilon) F (\epsilon')}{|\epsilon - \epsilon'|}.$$  \hspace{1cm} (D18)

We perform a Mellin transform

$$F (\epsilon) = \int_{-\infty}^{\infty} dw w f_w |\epsilon|^{-i\epsilon - 1 + \gamma/2}.$$  \hspace{1cm} (D19)
with inverse
\[ f_w = \int_{-\infty}^{\infty} \frac{de}{4\pi} |e|^{iw-\gamma/2} F(e). \] (D20)

Then we obtain
\[ S_F = -\int \frac{dwdw'}{(2\pi)^2} f_w^* f_{w'} R_{w,w'} \] (D21)

with
\[ R_{w,w'} = \int_{-\infty}^{\infty} d\epsilon d\epsilon' \frac{|\epsilon|^{-iw-1+\gamma/2} |\epsilon'|^{-iw'-1+\gamma/2}}{|\epsilon - \epsilon'|^{\gamma}}. \] (D22)

We first perform the integration over \( \epsilon' \) which yields
\[ R_{w,w'} = r_w \int_{-\infty}^{\infty} \frac{d\epsilon}{|\epsilon|} |\epsilon|^{-i(w-w')}, \] (D23)

with
\[ r_w = \int_{-\infty}^{\infty} dx \frac{|x|^{iw}}{|1 - x|^{\gamma} |x|^{1-\gamma/2}} = \frac{\pi^2}{2 \cos \left( \frac{\pi \gamma}{2} \right) \Gamma \left( \gamma \right) \left| \Gamma \left( 1 + iw - \frac{\gamma}{2} \right) \sinh \left( \frac{(2z - i)\pi}{4} \right) \right|^2}. \] (D24)

Performing the integration over \( \epsilon \) is straightforward and yields
\[ R_{w,w'} = 4\pi r_w \delta(w-w'). \] (D25)

Hence, the Mellin transform diagonalizes the action \( S \). At small \( w \) holds
\[ r_w = a_\gamma - b_\gamma w^2, \] (D26)

where the two positive coefficients are
\[ a_\gamma = \frac{\pi^2}{2 \Gamma \left( \gamma \right) \cos \left( \frac{\pi \gamma}{2} \right) \left( \sin \left( \frac{\pi \gamma}{2} \right) \Gamma \left( 1 - \frac{\gamma}{2} \right) \right)^2}, \]
\[ b_\gamma = \frac{\pi^2 \left( \pi^2 - 4\psi^{(1)}(1 - \frac{\gamma}{2}) \sin^2 \left( \frac{\pi \gamma}{2} \right) \right)}{8 \Gamma \left( \frac{\gamma}{2} \right) \Gamma^2 \left( 1 - \frac{\gamma}{2} \right) \sin \left( \frac{\pi \gamma}{2} \right) \cos \left( \frac{\pi \gamma}{2} \right)}. \] (D27)

Thus, we obtain
\[ S_F = \int \frac{dw}{\pi} (-a + bw^2) |f_w|^2. \] (D28)

Alternatively, we will analyze a theory of the form
\[ S_\psi = \int d\zeta \zeta^{-d(1-\gamma)} |\partial_\zeta \psi|^2, \] (D29)

which anticipates in the holographic action. To diagonalize the term we perform a corresponding Mellin transform
\[ \psi(\zeta) = \int_{-\infty}^{\infty} ds \varphi_s \zeta^{1+d(1-\gamma)} - is, \] (D30)

with inverse
\[ \varphi_s = \int_{0}^{\infty} \frac{d\zeta}{2\pi} \psi(\zeta) \zeta^{is - \frac{1+d(1-\gamma)}{2}}. \] (D31)
Then holds
\[ \partial_\zeta \psi = \int_{-\infty}^{\infty} ds \varphi_s \left( \frac{1 + d (1 - \gamma)}{2} - is \right) \zeta^{\frac{d(1-\gamma)-1}{2} - is}, \]
and we get
\[ S_\psi = \int ds ds' \varphi_s \varphi_s \left( \frac{1 + d (1 - \gamma)}{2} + is \right) \left( \frac{1 + d (1 - \gamma)}{2} - is' \right) \int_0^\infty \frac{d\zeta}{\zeta} \zeta^{i(s-s')} \].

If we now use that
\[ \int_0^\infty \frac{d\zeta}{\zeta} \zeta^{i(s-s')} = \int_{-\infty}^{\infty} d\log \zeta e^{i(s-s')} \log \zeta = 2\pi \delta (s - s') , \]
we obtain
\[ S_\psi = 2\pi \int ds \frac{(1 + d (1 - \gamma))}{4} + s^2 |\varphi_s|^2 \].

Hence, the two problems described by \( S_F \) and \( S_\psi \) are both diagonalized by a Mellin transform and then take a very similar form. Notice, the expansion of \( r_w \) with respect to \( w \) is not a gradient expansion in \( \partial_\zeta \psi \). Instead, it is an expansion for small \( (\partial_\zeta - \frac{1 + d (1 - \gamma)}{2}) \psi \) which is the appropriate expansion for curved space. Here the coefficient \( 1/2 \) determines the Breitenlohner-Freedman bound for the mass.

The similarity in the action suggests to make the identification
\[ \varphi_s = c_0 f_{-s} . \]

The opposite sign is due to the fact that large \( \zeta \) and small \( \epsilon \) refer to low energies. We can perform the inverse Mellin transform of this equation, which allows us to identity
\[ \psi (\zeta) = c_0 \zeta^{\frac{(d-1)(1-\gamma)}{2}} F \left( \frac{1}{\zeta} \right) . \]

Hence, it follows
\[ S_F = \frac{1}{c_0^2} \int \frac{ds}{\pi} (-a_\gamma + b_\gamma s^2 |\varphi_s|^2 \]
\[ = \frac{b_\gamma}{2\pi c_0^2} S_\psi - \frac{b_\gamma}{\pi c_0^2} \left( \frac{a_\gamma}{b_\gamma} + \frac{(1 + d (1 - \gamma))}{4} \right) \int ds |\varphi_s|^2 \].

To identify the last term, we use the inverse Mellin transform
\[ \int ds |\varphi_s|^2 = \frac{1}{2\pi} \int_0^\infty d\zeta \zeta^{-2-d(1-\gamma)} |\psi|^2 \].

Thus, we showed that
\[ S_F = - \int \frac{d\epsilon d\epsilon'}{(2\pi)^2} \frac{F (\epsilon) F (\epsilon')}{|\epsilon - \epsilon'|^2} \]
\[ = \frac{b_\gamma}{2\pi c_0^2} \int d\zeta \zeta^{-d(1-\gamma)} |\partial_\zeta \psi|^2 \]
\[ - \frac{b_\gamma}{2\pi c_0^2} \left( \frac{a_\gamma}{b_\gamma} + \frac{(1 + d (1 - \gamma))}{4} \right) \int_0^\infty d\zeta \zeta^{-2-d(1-\gamma)} |\psi|^2 . \]

This identification was done using the map Eq.(D37), i.e. \( \epsilon = 1/\zeta \). It is convenient to introduce instead \( \epsilon = c_1/\zeta \), i.e. to use
\[ \psi (\zeta) = c_0 \zeta^{\frac{(d-1)(1-\gamma)}{2}} F \left( \frac{c_2}{\zeta} \right) . \]
The argument $c_1/\zeta$ instead of $1/\zeta$ merely changes the relation between the two models to

$$S_F = \frac{b_\gamma c^{2-\gamma}_1}{2\pi c^2_0} \int d\zeta \zeta^{d(1-\gamma)} |\partial_\zeta \psi|^2$$

$$- \frac{b_\gamma c^{2-\gamma}_1}{2\pi c^2_0} \left( \frac{a_\gamma}{b_\gamma} + \frac{(1 + d (1 - \gamma))^2}{4} \right) \int_0^\infty d\zeta \zeta^{-2+d(1-\gamma)} |\psi|^2. \quad (D42)$$

Notice Eq.(D41) is indeed identical to the one we used for $S_{sc_1}$; see Eq.(D13). With this result we find

$$S_{sc_2} = \lambda_p \frac{b_\gamma c^{2-\gamma}_1}{2\pi c^2_0} \int k_\omega \zeta \zeta^{-d(1-\gamma)} |\partial_\zeta \psi|^2$$

$$- \lambda_p \frac{b_\gamma c^{2-\gamma}_1}{2\pi c^2_0} \left( \frac{a_\gamma}{b_\gamma} + \frac{(1 + d (1 - \gamma))^2}{4} \right) \int k_\omega \zeta^{-2+d(1-\gamma)} |\psi|^2. \quad (D43)$$

We can now fix $c_0$ to yield a coefficient $(1 - \gamma)^d$ for the gradient term:

$$c_0^2 = \lambda_p \frac{b_\gamma c^{2-\gamma}_1}{2\pi} (1 - \gamma)^{-d}, \quad (D44)$$

Combined with the earlier result for $c_1^0$ this implies $c_1^0 = \frac{\lambda}{\lambda_p \bar{b}_\gamma} \gamma$. With these conventions follows.

$$S_{sc_2} = \int k_\omega \zeta \zeta^{-d(1-\gamma)} \left( |\partial_\zeta \psi|^2 + \frac{m^2_2 (1 - \gamma)^2}{\zeta^2} |\psi|^2 \right), \quad (D45)$$

where the contribution to the mass from the bosonic pairing interaction is

$$m^2_2 = - \frac{a_\gamma}{b_\gamma (1 - \gamma)^2} - \frac{(1 + d (1 - \gamma))^2}{4 (1 - \gamma)^2}. \quad (D46)$$

With the expression for $c_1^2$ we can also simplify our result for the contribution to the mass from particle-particle propagator:

$$m_1^2 = \frac{\lambda}{\lambda_p \bar{b}_\gamma} \frac{1}{(1 - \gamma)^2} \quad (D47)$$

and for the coefficient in front of the $k^2$ term

$$A = \frac{(8/\lambda)^{1-\gamma}}{\lambda^2} \left( \frac{\lambda}{\lambda_p \bar{b}_\gamma} \right)^\gamma$$

4. Final result for the holographic map and identification with Lifshitz spacetime

If we add both contributions $S_{sc_1}$ and $S_{sc_2}$ we finally obtain with $m^2 = m_1^2 + m_2^2$ that

$$S_{sc} = \int k_\omega \zeta \zeta^{-d(1-\gamma)} \left( |\partial_\zeta \psi|^2 + \left( -\omega^2 + \frac{A k^2}{\zeta^2} + \frac{m^2 (1 - \gamma)^2}{\zeta^2} \right) |\psi|^2 \right). \quad (D48)$$

This is almost the result given in the main text. The difference is the sign in front of the $\omega^2$ term. As discussed above, this can be fixed by using the map given in Eq.(12) with additional denominator $\left(1 + \frac{\zeta^2 \omega^2}{m^2 (1 - \gamma)^2}\right)$ and expanding at small $\omega$ and $k$ and gradients in $\zeta$ as we had been doing anyway. Then we obtain the desired result:

$$S_{sc} = \int k_\omega \zeta \zeta^{-d(1-\gamma)} \left( |\partial_\zeta \psi|^2 + \left( \omega^2 + \frac{A k^2}{\zeta^2} + \frac{m^2 (1 - \gamma)^2}{\zeta^2} \right) |\psi|^2 \right). \quad (D49)$$
To make the connection to a gravitational theory explicit, we consider

$$S = \int d^{d+2}x \sqrt{g} \left( g^{\mu\nu} \partial_{\mu} \psi^* \partial_{\nu} \psi + m^2 \psi^* \psi \right).$$

(D50)

For the coordinates of a Lifshitz gravity one often uses coordinates\cite{29, 37}

$$ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = \frac{dr + dx' \cdot dx'}{r^2} + \frac{d\tau'^2}{r^{2z}}$$

(D51)

with holographic direction $r$. In our formulation we use instead

$$\zeta = r^z \quad \tau = z \tau' \quad x = zx'$$

such that

$$ds^2 = z^{-2} \left( \zeta^{-2} (d\zeta^2 + d\tau^2) + \zeta^{-2/z} dx \cdot dx \right).$$

(D53)

Hence, we have $g_{\zeta\zeta} = g_{\tau\tau} = z^{-2}\zeta^{-2}$, and $g_{i\bar{i}} = z^{-2} \zeta^{-2/z}$ where $i = 1, \cdots, d$ refers to the $d$ spatial coordinates. For the determinant follows $g = \det (g_{\mu\nu}) = z^{-4d} \zeta^{-4d/z}$. It now follows

$$S = \int \frac{d\zeta d\tau d^d x}{(2\pi)^{d+1} z^d \zeta^{d/z}} \left( \zeta^2 |\partial_{\zeta} \psi|^2 + \zeta^2 |\partial_{\tau} \psi|^2 + \zeta^{2/z} \sum_{i=1}^{d} |\partial_i \psi|^2 + \frac{m^2}{z^2} |\psi|^2 \right).$$

(D54)

We perform a Fourier transform w.r.t. $\tau$ and $x$ to $\omega$ and $k$ and obtain

$$S = \int \frac{d\omega dk d^d x}{(2\pi)^{d+1} z^d \zeta^{d/z}} \left( \omega^2 + \zeta^2 |\omega|^2 + \zeta^{2(1-z)} k^2 + \frac{m^2}{z^2} |\psi|^2 \right).$$

(D55)

With the identification $z = 1/(1-\gamma)$ this is exactly the expression Eq.(D49) or Eq.(14) of the main text.

A $T = 0$ transition from a superconducting to a normal state takes place when the mass $m^2$ equals the Breitenlohner-Freedman bound $m^2_{BF} = -(d+z)^2/4$. The condition is

$$\lambda = a_\gamma,$$

(D56)

where

$$a_\gamma = \int_{-\infty}^{\infty} dx \frac{1}{|1 - x|^\gamma |x|^{1-\gamma/2}},$$

(D57)

identical to the result that we obtained from the linearized Eliashberg equation. Hence, we have a critical pair-breaking strength

$$\alpha_\star(\gamma) = 1 - \frac{2}{(1-\gamma/a_\gamma)}.$$ 

(D58)

For $\gamma = 1/3$, relevant to the quantum-critical Ising ferromagnet, it holds

$$\alpha_\star = 1 - \frac{(6\sqrt{3} - 9) \Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{8}{3}\right)^2}{4\pi^2} \approx 0.879618.$$ 

(D59)

For $\gamma \to 0$, i.e. $z \to 1$, superconductivity is rather robust, with $\alpha_\star \approx 1 - \frac{1}{2}$. On the other hand for an almost local problem with $\gamma \to 1$, i.e. for $z \to \infty$, pairing is very fragile with $\alpha_\star \approx \frac{(\pi/2 + \log 4)}{1-\gamma} (1-\gamma)$.

Finally we give the numerical values of some of the coefficients for $d = 2$, $\gamma = 1/3$ and $\lambda = 1$ at $\alpha = \alpha^*$: It holds $A \approx 2.55156$ for the prefactor of the $k^2$ term and $c_1 \approx 0.03466$ as well as $c_0 \approx 0.0950746$. 

