SPHERICALLY SYMMETRIC ACCRETION FLOWS: MINIMAL MODEL WITH MAGNETOHYDRODYNAMIC TURBULENCE

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ABSTRACT

The first spherical accretion model was developed 55 years ago, but the theory is still far from being complete. The real accretion flow was found to be time dependent and turbulent. This paper presents the minimal magnetohydrodynamic (MHD) spherical accretion model that separately deals with turbulence. Treatment of turbulence is based on simulations of several regimes of collisional MHD. The effects of freezing-in amplification, dissipation, dynamo action, isotropization, and constant magnetic helicity are self-consistently included. The assumptions of equipartition and magnetic field-isotropy are released. The correct dynamics of magnetized flow is calculated. Diffusion, convection, and radiation are not accounted for. Two different types of radiatively inefficient accretion flows are found: (1) a transonic nonrotating flow and (2) a flow with effective transport of angular momentum outward. The nonrotating flow has an accretion rate several times smaller than Bondi rate, because turbulence inhibits accretion. The flow with angular momentum transport has an accretion rate about 10–100 times smaller than the Bondi rate. The effects of highly helical turbulence, states of outer magnetization, and different equations of state are discussed. The flows were found to be convectively stable on average, despite the fact that gas entropy increases inward. The proposed model has a small number of free parameters and the following attractive property. Inner density in the nonrotating magnetized flow was found to be several times lower than density in a nonmagnetized accretion. However, a density that is several times lower is still required to explain the observed low infrared luminosity and low Faraday rotation measure of accretion onto Sgr A*.

Subject headings: accretion, accretion disks — Galaxy: center — MHD — turbulence

Online material: color figures

1. INTRODUCTION

The dynamics of magnetized accretion flows is a major topic of astrophysical research. The problem can be solved with two different approaches: numerical and analytical. Each of them has specific difficulties, so these methods can be applied together for a better result.

Realistic numerical simulations require a lot of computational time to model even the isotropic case (Lazarian 2006). Convergence of properties of the isotropic turbulence is reached only when the computational domain has more than 1024 cells in each dimension (Ladeinde & Gaitonde 2004; Biskamp 2003). Nonisotropic simulations of this size were not performed. It is also very difficult to model the system with a large range of scales. The system then possesses vastly different timescales. Existing simulations of accretion flows are either axisymmetric ( McKinney 2006) or consider a rather small domain close to the object (Hawley & Balbus 2002; Igumenshchev 2006). In addition, simulations should be run for a sufficiently long time, or several runs should be made to obtain average quantities, e.g., accretion rate, power of emitted radiation.

Analytical models do not suffer from a need to average if they are based on averaged quantities. However, to build a reasonable model is itself difficult. No unified method exists for combining insights in physics and mathematics into a perfect analytical model. That is why the zoo of approximations of astrophysical flows is so huge.

In particular, many analytical treatments were devised for accretion: the spherically symmetric treatment (Bondi 1952; Meszaros 1975; Coker & Melia 2000; Beskin & Karpov 2005), the standard disk (Shakura & Syunyaev 1973), the advection-dominated accretion flow (ADAF; Narayan & Yi 1995) with its variation the hot luminous accretion flow (Yuan 2001), adiabatic inflow-outflow solutions (ADIOS; Blandford & Begelman 1999), the convection-dominated accretion flow (CDAF; Narayan et al. 2000; Quataert & Gruzinov 2000a), and Jet-ADAF (Yuan et al. 2002). Their aim is to describe essentially the same process, axisymmetric plasma inflow onto a compact source. Some models include the effects the others miss. Examples include the energy transport in CDAF and outflows in ADIOS. Some effects are not treated properly in any approximation.

The magnetic field is a main source of uncertainty and the cause of mistakes in theory of accretion flows. Two assumptions are usually posed to incorporate it into the model. First, the magnetic field is considered to be isotropic (Coker & Melia 2000; Narayan & Yi 1995). Then magnetic pressure and magnetic energy density may be put (Narayan & Yi 1995) into the dynamical equations. Second, the ratio of magnetic field energy density to gas thermal energy density is set to be constant. This is called thermal equipartition assumption. These two ideas are at the very least unproven and may not even work. The magnetic field is predominantly radial in the spherical inflow (Shvartsman 1971) because of the freezing-in condition and is predominantly toroidal in the disk (Hawley & Balbus 2002) because of magnetorotational instability.

In a good model, the direction and strength of the magnetic field should be determined self-consistently. Nonisotropy of the magnetic field requires special dynamics. Dynamical equations were partially derived more than 20 years ago ( Scharlemann 1983), but did not receive much attention or were considered erroneous ( Beskin & Karpov 2005).

Such a model may offer a natural explanation of certain accretion patterns. Accretion onto Sgr A* gives an excellent opportunity for testing. Our Galaxy is proven to host a supermassive
black hole (SMBH) named Sgr A* in its center (Ghez et al. 2003; Shen 2006). This black hole accretes matter and emits radiation with characteristic low-luminosity spectrum (Narayan et al. 1998). This spectrum was satisfactorily explained with the combination of two models: jet or nonthermal (Yuan et al. 2003) radio emission and X-rays with infrared (IR) radiation coming from a conventional ADAF flow. However, the large number of free parameters allows one to fit any spectrum well. A model with no remaining free parameters is an ultimate goal of the ongoing study.

Partial progress in building a self-consistent accretion model is made in this paper, which is organized as follows. An averaged spherical magnetohydrodynamic (MHD) model with turbulence is devised in § 2. This approximate model employs the characteristic length scale about the size of the region of interest. Coefficients are taken from several hydrodynamic (HD) and MHD simulations. External sources sustain turbulence at large radii, whereas turbulence is self-sustained in the converging flow at small radii. Necessary boundary conditions are discussed in § 3 for a general flow and for Sgr A*. Results in § 4 are followed by the discussion of the model in § 5. Observational implications in § 6 are supplemented with prospects for future work and the conclusion in § 7. This paper has several appendices.

2. SPHERICAL MODEL

I base all calculations on a MHD system of equations (Landau et al. 1984). The viscous terms are retained where they do not vanish in the limit of vanishing viscosity. The quantities in the following equations are fully dependent on time and coordinates. The general mass flux equation reads

$$\frac{\partial \rho}{\partial t} + \nabla (\rho \mathbf{V}) = 0,$$

where $\mathbf{V}$ is fluid velocity. Force balance is described by the Navier-Stokes equation

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \nabla) \mathbf{V} = - \frac{\nabla p}{\rho} - \nabla \phi_g - \frac{[\mathbf{B} \times (\nabla \times \mathbf{B})]}{4\pi \rho} + \nu \nabla^2 \mathbf{V},$$

where $\phi_g$ is gravitational potential and $\nu$ is kinematic viscosity. The last term is responsible for finite energy dissipation through a Kolmogorov cascade (Landau & Lifshitz 1987). The momentum equation is a combination of equations (1) and (2),

$$\frac{\partial (\rho \mathbf{V}_i)}{\partial t} = - \frac{\partial}{\partial x_k} \left[ \rho \delta_{ik} + \rho \mathbf{V}_i V_k + \frac{1}{4\pi} \left( \frac{1}{2} \mathbf{B}^2 \delta_{ik} - B_i B_k \right) \right]$$

$$- \frac{\partial \phi_g}{\partial x_k} + \nu (\nabla^2 \mathbf{V}),$$

(3)

The energy equation,

$$\frac{\partial (\rho \mathbf{V}_i \mathbf{V}_i/2 + \rho c^2 + \mathbf{B}^2/8\pi)}{\partial t} = - \nabla \left\{ \rho \mathbf{V} \left( \frac{\mathbf{V}^2}{2} + \phi_g + w \right) + \frac{1}{4\pi} \left[ \mathbf{B} \times (\mathbf{V} \times \mathbf{B}) \right] \right\} + \text{viscous},$$

(4)

includes information about the equation of state. Here $\varepsilon$ is the gas internal energy density, $w = \varepsilon + \int \mathbf{p} d\mathbf{V}/\rho$ is the gas specific enthalpy. The viscous term is responsible for diffusion. Magnetic field evolution is described by the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \nu_M \Delta \mathbf{B},$$

(5)

with magnetic diffusivity $\nu_M$. The magnetic field is solenoidal as well as an incompressible random velocity field,

$$\mathbf{\nabla} \cdot \mathbf{B} = 0, \quad \mathbf{\nabla} \cdot \mathbf{u} = 0.$$

(6)

2.1. Dynamics

Spherical accretion is the simplest pattern of all symmetric setups. We need to solve the basic model first to move then to a more realistic pattern. Construction of the minimal maximally symmetric model is the subject of the following study.

I employ the natural spherical coordinates $(r, \theta, \phi)$ for the problem and average over angular variables $(\theta, \phi)$. The results depend only on the radial variable $r$ and not on time $t$ in the assumption that angular averaging is the same as time averaging. I need now to determine the essential quantities and derive their closed system of equations.

The essential quantities of a nonmagnetized solution in Bondi (1952) are the inflow speed $v(r)$, density $\rho(r)$, and temperature $T(r)$. The turbulent magnetized case requires several more quantities. As I release the assumption of isotropy, there are two special directions: along the radial vector $e_r$ and perpendicular to the radial vector. To describe realistic MHD turbulence, I need at least six quantities: squares of radial and perpendicular magnetic fields $B_r^2$ and $B_\theta^2$, squares of radial and perpendicular random fluid speeds $u_r^2$ and $u_\theta^2$, the characteristic length scale $L$, and a dimensionless magnetic helicity $\xi$. The last quantity is described in detail in the corresponding subsection (§ 2.4). For simplicity I consider random velocity to be isotropic and denote it as $u(r)$.

Total velocity of a fluid parcel,

$$\mathbf{V}(r, \theta, \phi, t) = v(r) \mathbf{e}_r + \mathbf{u}(r, \theta, \phi, t),$$

(7)

is a sum of averaged inflow speed $v(r)$ and instantaneous random velocity $\mathbf{u}(r, \theta, \phi, t)$, where by definition angular average of turbulent velocity vanishes,

$$\int \mathbf{u}(r, \theta, \phi, t) d\Omega = 0.$$

(8)

The general continuity equation (eq. [1]) can be averaged with the aid of equations (6) and (8) to

$$4\pi \rho(r) v(r)r^2 = \dot{M},$$

(9)

where $\dot{M}$ is the mass accretion rate.

I derive the averaged force equation from the general momentum equation (eq. [3]). Tensor $\rho \mathbf{V}_i \mathbf{V}_k$ averages out into the diagonal form $\rho v^2 g_r + \rho u^2 g_r/3$. Because there are no sources of a magnetic field (eq. [6]) and spherical geometry is assumed, no regular magnetic field exists. Following Scharlemann (1983), I add $B_r \nabla B / (4\pi \rho)$ to the radial magnetic force $F_r = (\mathbf{B} \times (\nabla \times \mathbf{B}))/ (4\pi \rho)$, average over the solid angle, and then set $B_\theta = B_{\perp}$ and $B_\phi = B_{\parallel}$. Cross terms with $(B_r B_r, B_r B_\theta)$, and $(B_\phi B_\phi)$ cancel on average over the solid angle. Finally, I obtain

$$F_r = \frac{(r^4 B_r^4)'}{8\pi \rho r^2} - \frac{(r^2 B_\parallel^2)'}{4\pi \rho r^2}$$

(10)

for the magnetic force. I denote radial derivatives by $(\ldots)'$. I omit the bulk viscosity term that results from $\nabla \Delta V$. The Paczynski-Wiita gravitational potential (Paczynski & Wiita 1980)

$$\phi_g = - \frac{r_g v^2}{2(r - r_g)}$$

(11)
is used to imitate the effects of general relativity, where
\[ r_g = \frac{2GM}{c^2} \]  
(12)

is a Schwarzschild radius of an object with mass \( M \). I take gas pressure to be that of an ideal gas \( p = \rho RT/\mu \), where \( \mu \) is a mean molecular weight. Combining all the terms, I come to the averaged force equation
\[ \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} - \frac{v_r v_\theta}{r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial r} \right) + \frac{\rho u^2}{2} \left( \frac{r v_r^2}{2} + \frac{1}{4\pi \rho r^2} - \frac{(r^2 B_r^2)}{8\pi \rho r^2} \right) = 0. \]  
(13)

The averaged energy advection equation can be derived directly from the general energy equation (eq. [4]). The enthalpy term should include a contribution from random fluid motions as well as from gas. Isotropic random motions of fluid exert isotropic pressure \( p_{\text{rand}} = \rho u^2/3 \) and have the internal energy density \( e_{\text{rand}} = u^2/2 \). The total enthalpy \( w \) is
\[ w = w_{\text{gas}} + w_{\text{rand}}, \]
where \( w_{\text{gas}} = RT \left( \frac{f_e a_e(T) + f_i a_i(T) + 1}{\mu} \right) \),
and \( w_{\text{rand}} = \frac{5}{6} u^2 \).  
(14)

Fractions of electrons \( f_e \approx 0.54 \) and ions \( f_i \approx 0.46 \) are calculated for a gas with twice the solar abundance of elements. Such a high concentration of helium and metals was assumed by Baganoff et al. (2003) for the spectrum fitting of Sgr A*. The corresponding mean molecular weight is \( \mu \approx 0.7 \) g cm\(^{-3} \). The integral heat capacity per particle \( a_e(T) \) and \( a_i(T) \) are different for electrons and ions. Ions are nonrelativistic down to \( r_g \) (Narayan & Yi 1995); therefore, \( a_i(T) = 3/2 \). The general expression (Chandrasekhar 1957) should be used for thermal relativistic particles \( a_e(T) = \Theta^{-1} \left[ 3K_3(\Theta^{-1}) + K_1(\Theta^{-1})^{-1} \right]/[4K_2(\Theta^{-1})^{-1} - 1] \), where \( \Theta = kT/m_e c^2 \) is the dimensionless temperature, and \( K_n \) is a modified Bessel function of the second kind. The expression for nonrelativistic enthalpy is
\[ w_{\text{NR}} = \frac{5RT}{2\mu} + \frac{5}{6} u^2. \]  
(15)

This is valid in the limit \( \Theta \ll 1 \). Time derivatives in the energy equation (eq. [4]) vanish under averaging. This equation takes the form \( \nabla q = 0 \), where \( q \) is the energy flux. Part of flux proportional to random velocity \( u \) averages out, because turbulence is incompressible and \( u \) is zero on average (eq. [8]). Applying the continuity relation (eq. [9]), I finally obtain
\[ \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} - \frac{v_r v_\theta}{r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial r} \right) + \rho u^2 \left( \frac{r v_r^2}{2} + \frac{1}{4\pi \rho r^2} - \frac{(r^2 B_r^2)}{8\pi \rho r^2} \right) = 0, \]  
(16)

where again \( B_r^2 = B_r^2 = B_z^2 \). I assumed the term \( \int [B \times (u \times B)] d\Omega \) to also be zero along with all viscous energy transfer terms. I limit this study to advection-dominated flows by deliberately cutting off diffusion and convection (see Appendix C).

By subtracting the force equation (eq. [13]) from the energy advection equation (eq. [16]), I get the heat balance equation that reads in the nonrelativistic limit
\[ \frac{R}{\mu} \left( \frac{3}{2} T - \frac{\rho_0}{\rho} T \right) + \left[ \frac{u^2}{2} \right]_r - \frac{\rho_0 u^2}{3} \]
\[ + \frac{\rho u^2}{4\pi r^2} \left( \frac{B_r^2}{\rho} \right)_r + \frac{1}{8\pi \rho r^2} 
\]
\[ \times \left( r^4 B_r^2 \right)_r = 0, \]  
(17)

similar to entropy conservation in HD. Work done by gas is represented by \(-\frac{\rho_0 u^2}{3}\). The first term has exactly the form of the second if I make the substitution of the mean square particles velocity
\[ u_p^2 = \frac{3RT}{\mu}. \]  
(18)

Work done by the magnetic field enters the expression as derivatives of \( \rho \) and \( r \) in the magnetic part.

2.2. Evolution of Turbulence

Dynamics is the only part of ideal Bondi problem (Bondi 1952). In reality, flow always has some small-scale turbulence that exerts back reaction on the mean flow. The magnitude of back-reaction terms should be determined from additional equations that describe the evolution of random magnetic field and fluid motions. Since no complete theory of turbulence exists, I make a lot of approximations. The model is adjusted to agree with the results of several numerical simulations. I also apply analytical tests similar to that in Ogilvie (2003) to assure the model reproduces the basic properties of observed turbulence.

I need a nonideal induction equation (eq. [5]) and Navier-Stokes equation (eq. [2]) to derive how turbulence evolves. My goal is to compile reasonable equations on average squares of the radial magnetic field \( B_r^2 \), perpendicular magnetic field \( B_z^2 \), and isotropic velocity \( u^2 \). I also need equations on the characteristic length scale of turbulence \( L \) and dimensionless magnetic helicity \( \xi \).

The radial part of the induction equation (eq. [5]) easily gives the equation on \( B_r^2 \), when the former is multiplied by \( 2B_r \) and averaged over the solid angle,
\[ 2B_r \frac{\partial B_r}{\partial r} = 2B_r \left[ \nabla \times (u \times B) \right]_r + 2B_r \left[ \nabla \times (u \times B) \right]_r + 2\nu_{\text{eff}} B_r \left( \Delta B \right)_r, \]  
(19)

where indices \((.,.)\) without primes denote the radial parts. The left-hand side vanishes as do all the time derivatives. The first term on the right-hand side represents the uniform increase of the magnetic field due to flux freezing. I combine it with the continuity equation (eq. [9]) to eliminate \( \nu \)-derivatives. The second term is the dynamo action. It cannot be easily averaged. The characteristic turbulence length scale \( L \) may be used to approximate derivatives
\[ \frac{\partial B_i}{\partial x_k} \sim \frac{B_i}{L} e_k \]  
and \[ \frac{\partial u_i}{\partial x_k} \sim \frac{u_i}{L} e_k, \]  
(20)

where \( e_k \) is the unit vector. Then we arrive at the dynamo action with the characteristic timescale \( \tau_{\text{dyn}} = c_{\text{Bo}} \tau_{\text{edd}} \) about eddy turnover time \( \tau_{\text{edd}} = u/L \). The averaged expression is quadratic in
the magnetic field. I take the coefficient to be \(c_{Bu1}\) at any \(B_i\), 
and \(c_{Bu2}\) at any \(B_i B_j\) with \(i \neq k\). The final form of the dynamo 
term reads 
\[2B_i \left[ \nabla \times (\mathbf{u} \times \mathbf{B}) \right]_r = \left[ c_{Bu1} B_i^2 + c_{Bu2} (B_i + B_j) B_j \right] u / L,\]
where 
\[B_r = \sqrt{B_i^2} \quad \text{and} \quad B_{\theta} = B_{\phi} = B_\perp = \sqrt{B_i^2} \quad \text{(21)}\]
are the rms magnetic fields. The last term on the right-hand side 
of equation (19) represents magnetic field dissipation. The 
dissipation term \(v_A M_\Delta B\) of the induction equation (eq. [5]) is macroscopic 
in turbulence, even for vanishing magnetic diffusivity \(v_A\) 
(Biskamp 2003). I approximate radial dissipation to have a timescale \(\tau_{diss} = c_{BB} T_A\), 
about the Alfven timescale \(T_A = v_A / L\). The averaged expression is also quadratic in the magnetic field. 
I take the coefficient to be \(c_{BB1}\) at any \(B_i\), 
and \(c_{BB2}\) at any \(B_i B_j\) with \(i \neq k\). Finally, 
\[v_A M_\Delta B = v_A \left[ c_{BB1} B_i^2 + c_{BB2} (B_i + B_j) B_j \right] / L.\]
Collecting all the terms, I obtain 
\[\frac{v}{r^2} \frac{\partial (B_i^2 r^1)}{\partial r} = -\frac{(c_{Bu1} B_i^2 + 2 c_{Bu2} B_j B_j) u}{L} + \frac{(c_{BB1} B_i^2 + 2 c_{BB2} B_j B_j) v_{A\perp}}{L} \quad \text{(22)}\]
for the radial magnetic field in the absence of external energy 
Sources.
The perpendicular part of the induction equation (eq. [5]), for example, \(\theta\) part, gives the equation on \(B_\theta\) when equation (5) is multiplied by \(B_\theta\) and averaged over the solid angle. The flux-freezing condition for the perpendicular field is different from that for the radial field; \(B_\theta B_r = \text{const.} \) represents perpendicular 
flux freezing. I repeat the calculations made for the radial field \(B_r\) 
to find dynamo and dissipation terms. The dynamo term takes the form 
\[\left[ c_{Bu1} B_\theta^2 + c_{Bu2} (B_\theta + B_j) B_j \right] u / L.\] 
The dissipation term is 
\[v_A \left[ c_{BB1} B_\theta^2 + c_{BB2} (B_\theta + B_j) B_j \right] / L \] 
with a perpendicular Alfven timescale for dissipation. I have \(B_\perp = B_\theta = B_\phi = B_\theta \) \(r\).
Finally, I obtain 
\[\frac{v}{r^2} r \frac{\partial \left( B_i^2 r^1 \right)}{\partial r} = -\frac{(c_{Bu1} B_i^2 + 2 c_{Bu2} B_j B_j) u}{L} + \frac{(c_{BB1} B_i^2 + 2 c_{BB2} B_j B_j) v_{A\perp}}{L}, \quad \text{(23)}\]
where the continuity equation (eq. [9]) is used. Radial \(v_{A\perp}\) and 
perpendicular \(v_{A\perp}\) Alfven speeds and random velocity \(u\) 
are respectively 
\[v_{A\perp} = \frac{\sqrt{B_i^2}}{4 \pi \rho}, \quad v_{A\perp} = \frac{\sqrt{B_i^2}}{4 \pi \rho}, \quad \text{and} \quad u = \sqrt{\frac{u^2}{4 \pi \rho}}. \quad \text{(24)}\]
Coefficients \(c_{Bu1}\), \(c_{Bu2}\), \(c_{BB1}\), and \(c_{BB2}\) are yet to be determined.
The evolution equation for squared random fluid velocity \(u^2\) 
can be found from momentum equation (eq. [3]), when it is multiplied 
by \(2u\) and averaged over the solid angle. Potential energy 
and pressure terms average out and only three terms are left 
\[2u \left[ (\nabla \nabla) V + \frac{\nabla (\rho V)}{\rho} \right] = 2u \left[ B_j (B_i \times \nabla B_j) \right] u + 2 \frac{u^2}{2} \Delta u. \quad \text{(25)}\]
I apply the same averaging procedure as was used for the magnetic 
field evolution equations (22) and (23). The final result is 
\[\frac{1}{2} \frac{v}{r^2} r \frac{\partial \left( u^2 r^1 \right)}{\partial r} = \frac{2 c_{Bu1} u^3}{L} \cdot \frac{\Delta u}{L} = \frac{2}{2} \left( \frac{u^2}{2} \right) + 2 \frac{u^2}{2} \Delta u. \quad \text{(26)}\]
with additional three coefficients \(c_{Bu1}\), \(c_{Bu2}\), and \(c_{BB2}\). Some of 
these and other \(c_{\perp}\)-like coefficients can be taken from numerical 
simulations of isotropic turbulence, and some of them can be inferred 
from analytical tests. They may not simply be set to convenient 
values like Ogilvie (2003) did.

2.3. Correspondence to Numerical Simulations

Isotropic turbulence is studied quite thoroughly in numerical 
simulations. Some results are reproduced by a number of researchers 
(see Biskamp 2003 for the review). That is why I believe 
in these results and base a model on them. Three simulations 
of different turbulence regimes can provide four conditions that 
let me uniquely determine four combinations of coefficients \(c_{\perp}\). 
These regimes are decaying HD turbulence, decaying MHD turbulence, 
and dynamo growth of a small seed magnetic field. I assume 
that \(c_{\perp}\) are constants independent of regime and extend 
the derived model to any anisotropic case.

Let me consider my model in an isotropic incompressible case 
of box turbulence. In these settings \(B_\theta^2 = B_\phi^2 = B_\perp^2\). The squared 
magnetic field \(B^2\) equals \(3 B_\perp^2\). Transition to the comoving 
frame of averaged inflow in turbulence evolution equations (22), 
(23), and (26) is done by stating \(d/dt = -i \omega / r\). Now I should 
write time derivatives instead of radius derivatives and set \(r = \text{const} \) 
since matter is not moving anywhere from the box. I obtain 
equations of evolution of isotropic turbulent Alfven speed \(v_{A}\) 
and isotropic turbulent velocity \(u\), 
\[(u^2)' = \frac{\dot{c}_{Bu}}{L} v_{A} u - \frac{\dot{c}_{Bu}}{L} v_{A} u^3, \quad \text{(27)}\]
\[(v_{A}' = \frac{c_{BB} v_{A}^2}{L} - \frac{\dot{c}_{BB}}{L} v_{A} u^3. \quad \text{(27)}\]
Here \(v_{A} = \sqrt{B_i^2 / 4 \pi \rho}\) and \(p = \text{const} \). Coefficients with hats are 
\(\hat{c}_{Bu} = c_{Bu1} + 2 c_{Bu2}, \quad \hat{c}_{BB} = \frac{c_{BB1} + 2 c_{BB2}}{\sqrt{3}}, \quad \text{(28)}\]
in terms of the previously defined \(c_{\perp}\).

I have the freedom to set \(L\), because it enters the equations 
only in combination, \(c_{\perp} / L\), but the \(c_{\perp}\) are not yet determined. 
For the simplicity of further derivation, I take \(L(r)\) to be 
the effective size of energy containing eddies for isotropic 
incompressible turbulence, 
\[u^2 = \int_{2 \pi / L}^{\infty} |u_k|^2 dk \quad \text{and} \quad \frac{v_{A}}{A^2} = \int_{2 \pi / L}^{\infty} |v_{Ak}|^2 dk. \quad \text{(29)}\]
Isotropic decay of HD turbulence is the simplest simulation. 
The convenient constant of decay is Kolmogorov constant \(C_{HD}\). 
It is defined as 
\[C_{HD} = E_k k^{5/3} \epsilon^{-2/3}, \quad \text{where} \quad \epsilon = \frac{d}{dt} \left( \frac{u^2}{2} \right) \quad \text{and} \quad E_k = \frac{|u_k|^2}{2}. \quad \text{(30)}\]
where $E_k$ is the energy spectrum, $\epsilon$ is a decay rate. The Kolmogorov constant was found to be $C_{HD} \approx 1.65$ in the large set of simulations (Sreenivasan 1995). I substitute this number into equation (30) and evaluate the first integral in equation (29) to find
\[
\hat{c}_{uu} = \frac{4\pi}{(3C_{HD})^{3/2}} \approx 1.14
\]
for the isotropic formulae in equation (27).
Isotropic decay of MHD turbulence gives two conditions. The MHD Kolmogorov constant is defined similarly to the HD case in equation (27) as
\[
C_{MHD} = E_k k^{5/3} e^{-2/3}, \quad \text{where} \quad \epsilon = -\frac{d}{dt} \left( \frac{u^2 + v_A^2}{2} \right)
\]
and $E_k = \frac{|u_k|^2 + |v_A|^2}{2}$. (32)

MHD turbulence is more difficult to model numerically, but the value of $C_{MHD} \approx 2.2$ is rather rigorous (Biskamp 2003). In addition, the kinetic energy was found to decay in exactly the same rate as the magnetic energy. Evaluation of the sum of two integrals (eq. [29]) with definitions from equation (32) and known $C_{MHD}$ yields
\[
\hat{c}_{BB} - \hat{c}_{Bu} = \hat{c}_{uu} - \hat{c}_{uB} \approx 2\pi \left( \frac{2}{3C_{MHD}} \right)^{3/2} \approx 1.05.
\]

Dynamo simulations explore the regime $v_A^2 \ll u^2$. Exponential growth of a small magnetic field corresponds to some value of coefficient $\hat{c}_{Bu}$ in equation (27) as
\[
B^2 \propto \exp \left( \hat{c}_{Bu} \frac{uB}{L} \right).
\]

External driving is purely mechanical for $v_A^2 \ll u^2$, so an external source of a magnetic field does not alter the picture of field amplification by a dynamo. The characteristic length scale in dynamo simulations is usually the size of energy-containing $L$ consistent with equation (29), so renormalization of the length scale is not required. Older simulations (Kida et al. 1991) have found $b = 0.39$, which corresponds to $\hat{c}_{Bu} \approx 0.61$. Later results (Schekochihin et al. 2004) indicate values a bit higher, $\hat{c}_{Bu} \approx 0.7$, that I use for my model. Finally,
\[
\hat{c}_{Bu} = 0.70, \quad \hat{c}_{BB} = 1.75, \quad \hat{c}_{uu} = 1.14, \quad \hat{c}_{uB} = 0.09.
\]

The values of four $\hat{c}_{ct}$ (eq. [35]) are not enough to obtain all seven coefficients $c_{ct}$ in equations (22), (23), and (26) with definitions from equation (28). However, the application of common sense analytical conditions to a nonisotropic system of equations puts some additional constrains on $c_{ct}$ that allows me to complete the model with as little guessing as possible.

Analytic tests are described in Appendix A. This completes the derivation and verification of turbulence evolution equations (22), (23), and (26) with coefficients
\[
c_{BB1} = 3.03, \quad c_{BB2} = 0.00, \quad c_{Bu1} = 0.41, \quad c_{Bu2} = 0.29, \quad c_{uu} = 1.14, \quad c_{uB1} = 0.09, \quad c_{uB2} = 0.00
\]
that I obtain summarizing equations (28), (35), (A11), and (A12). However, not all major effects have been included so far.

\[\text{10.4. Magnetic Helicity}\]

Certain correlation called magnetic helicity may strongly influence magnetic field dissipation. This quantity is defined as
\[
H = \int_V (A\mathbf{B})dV, \quad (37)
\]
where $A$ is a vector potential with a defined gauge condition (Biskamp 2000). The time derivative of magnetic helicity is very small compared to the time derivative of magnetic energy in high Reynolds number astrophysical plasma (Biskamp 2003),
\[
\frac{dH}{dE_M} \ll 1.
\]

Constancy of magnetic helicity defines the rules of selective decay. Magnetic energy $E_M$ decays in free turbulence down to nonzero value, allowed by constant magnetic helicity $H = \text{const}$. The final force-free configuration has zero random kinetic energy $E_k$ and has aligned current density and magnetic field $j\mathbf{B}$ (Biskamp 2003).

However, the derived system of turbulence evolution equations (eq. [A1]) and, therefore, equations (22), (23), and (26) cannot handle selective decay. The decay of magnetic energy must be modified in order to have the transition to zero dissipation rate at a certain $v_A$, and $v_A$, as a function of magnetic helicity $H$. First, I should employ the proper magnetic helicity constancy. Then I should quantify the relation between critical $v_A$, $v_A$, and $H$.

Let me consider the region $S$ that evolves together with the mean flow of fluid. This region has the constant angle boundaries $\theta = \text{const} \phi = \text{const}$. Its radial elongation $L_r$ scales as the inflow velocity: $L_r \propto v$. The region $S$ contains a constant mass $m = \text{const}$ of matter, because matter flux through its boundaries is zero by definition. If I neglect diffusion by random velocity, frozen magnetic field lines do not move through the boundaries of the region. Because of this, magnetic helicity in $S$ is constant $H = \text{const}$ (Biskamp 2003).

The simplest order-of-magnitude relation between magnetic energy $E_M$ and $H$ is
\[
E_M L_H = H = \text{const}
\]
in the region $S$, where $L_H$ is magnetic helicity characteristic length scale (Biskamp 2003). As the magnetic field decays in turbulence, $L_H$ grows according to equation (39).

I can parameterize $L_H$ to be a fraction of $L$
\[
L_H = \xi L, \quad (40)
\]
Volume of the region of interest $S$
\[
V = \frac{m}{\rho} \quad (41)
\]
with $m = \text{const}$. Total magnetic energy $E_M$ is
\[
E_M = \frac{V}{8\pi} (B_r^2 + 2B_\perp^2). \quad (42)
\]
I substitute equations (40), (41), and (42) into equation (39) and use the definitions from equation (24) of Alfvén velocities to come to
\[
L(v_A^2 + 2v_\perp^2)\xi = \text{const}. \quad (43)
\]
Now I need to include $\xi$ into the turbulence evolution equations (22), (23), and (26) so that they can handle selective decay. The natural limit of $L_H$ growth is the characteristic size of energy containing eddies $L$. So the $\xi \ll 1$ regime corresponds to nonhelical turbulence, and the $\xi \sim 1$ regime corresponds to turbulence, where magnetic helicity significantly inhibits dissipation. The $\xi \gg 1$ regime does not occur. The basic way to modify the equations is to decrease the magnetic field decay rate by a smooth multiplier $f(\xi) < 1$. For qualitative agreement with experiment (Biskamp 2003), I can employ

$$f(\xi) = \exp(-\xi),$$

(44)

which means that magnetic energy dissipation timescale becomes $\exp(\xi)$ times larger. Terms with both $u$ and $v_{Ar}$ and $v_{Al}$ in magnetic field evolution equations (22) (23), respectively, do not need to be modified, since random velocity energy decays to zero and these terms do not matter. However, I multiply the term with both random velocity and Alfvén speed in the turbulent velocity evolution equation (eq. [26]) by $\exp(-\xi)$ to make random velocity $u$ decay to zero.

2.5. System of Equations with Source Terms

With only minor corrections, the final system of equations can be written down. In general, turbulence has external sources of energy that sustain finite magnetic and kinetic energies, even in cases of box turbulence. I can add source terms to an incompressible system (eq. [A1]) and consequently to the system of compressible equations (22), (23), (26).

Equation (A1) with coefficients from equations (35) and (36), a modifier from equation (44), and source terms reads

$$\frac{d(v_{Ar}^2)}{dt} = \frac{[0.70v_{Ar}^2 + 0.58(v_{Al} - v_{Ar})v_{Ar}]}{L}u - \frac{3.03v_{Ar}^3}{L} \exp(-\xi) + c_0 \frac{v_p^3}{L},$$

(45a)

$$\frac{d(v_{Al}^2)}{dt} = \frac{[0.70v_{Al}^2 + 0.29(v_{Ar} - v_{Al})v_{Al}]}{L}u - \frac{3.03v_{Al}^3}{L} \exp(-\xi) + c_1 \frac{v_p^3}{L},$$

(45b)

$$\frac{d(u^2)}{dt} = 0.09(v_{Ar}^2 + 2v_{Al}^2)u \exp(-\xi) - 1.14u^3 + c_2 \frac{v_p^3}{L},$$

(45c)

where $v_p$ is the mean square particle speed (eq. [18]) and $c_0, c_1$, and $c_2$ are dimensionless coefficients. These coefficients determine the rates of external energy input into turbulent fields.

I denote by $\sigma$ the ratio of total turbulent energy to thermal energy,

$$\sigma = \frac{E_K + E_M}{E_{th}}, \quad \text{so } \frac{3RT}{2\mu} = \frac{\sigma v_p^2}{2} = \frac{u^2}{2} + \frac{v_{Ar}^2}{2} + \frac{v_{Al}^2}{2}. \quad (46)$$

Unlike conventional plasma magnetization, magnetization $\sigma$ with equation (46) includes the energy of random fluid motions.

In the dynamic equilibrium of constant $v_{Ar}, v_{Al}, u$, and known $\xi$, the system in equations (45a)–(45c) gives three algebraic equations for ratios $v_{Ar}/v_p, v_{Al}/v_p$, and $u/v_p$ as functions of $c_0, c_1,$ and $c_2$. To estimate $c_0, c_1,$ and $c_2$, I take stationary driven isotropic turbulence with kinetic energy $E_K$ equal to magnetic energy $E_M$. Isotropic turbulence of interest has $v_{Ar} = v_{Al} = u/\sqrt{3}$.

Such turbulence occurs far from the central object, where outer magnetization is a constant $\sigma$. Solving the system of equations (eqs. [45a]–[45c]), I obtain using equation (46)

$$c_0 = c_1 \approx 0.124\sigma^{3/2}, \quad c_2 = 3c_0 \approx 0.371\sigma^{3/2} \quad (47)$$

in case $\xi = 0$. I apply these values to turbulence with $\xi > 0$. The total external energy input $Q_\perp$ into $E_K$ and $E_M$ is

$$Q_\perp \approx 0.742\sigma^{3/2} \frac{v_p^3}{L}. \quad (48)$$

This energy adds up to thermal gas energy after being processed through turbulence. However, I do not adjust my dynamical equations (13) and (16) for $Q_\perp$. I self-consistently omit external heating and radiative or diffusive cooling. This omission is physically justified sufficiently far from the central object, where cooling $Q_H$ balances external heating $Q_\perp$. It is also justified in the inner region, where both $Q_H$ and $Q_\perp$ are negligible compared to the internal driving and energy advection. Internal driving represents build-up of self-sustained turbulence in a converging flow due to conservation of magnetic flux (Coker & Melia 2000).

Only the size $L$ of energy containing eddies should be specified to complete the derivation of a closed system of equations. In the case when energy input $Q_\perp$ does not matter, the problem has only one relevant scale that is the size of the system $r$. Therefore, I can set $L$ to be the fraction of the radius

$$L = \gamma r \quad (49)$$

with the proportionality constant $\gamma$ about unity. However, energy input from external sources $Q_\perp$ is relatively large far from the central source. This causes a medium with constant $Q_\perp, v_p,$ and $\sigma$ to have a constant size of the largest eddies

$$L = L_\infty = \text{const} \quad (50)$$

because of equation (48). This equality holds for radii larger than some $r_0 \approx L_\infty/\gamma$. I introduce a function with a smooth transition from equation (49) for $r \ll r_0$ to equation (50) for $r \gg r_0$.

$$L(r) = L_\infty \left(1 - \exp \left(-\frac{\gamma r}{L_\infty} \right) \right). \quad (51)$$

This completes the derivation and verification of eight equations (9), (13), (16), (22), (23), (26), (43), and (51) with coefficients from equations (35), (36), and (47) on eight quantities $L(r), \xi(r), \nu(r), u(r), v_{Ar}(r), v_{Al}(r), T(r),$ and $\rho(r)$ that are the characteristic turbulent length scale, normalized magnetic helicity, matter inflow velocity, turbulent velocity, radial Alfvén speed, perpendicular Alfvén speed, temperature, and density, respectively. I rewrite the equations once again in terms of the named quantities:

$$4\pi \nu \frac{v^2}{2r} = \dot{M}, \quad (52a)$$

$$\nu_{\nu} + \frac{\nu_{\tau} c^2}{2(r - r_s)} + \frac{\nu_{\nu}}{\mu} + \frac{\nu_{\tau}c^2}{3\rho} \quad \nu_{\nu_{\nu}} + \frac{(r^2 \rho v_{Ar}^2)}{pr^2} + \frac{(r^2 \rho v_{Al}^2)}{2pr^4} = 0, \quad (52b)$$

$$\nu_{\nu} + \frac{\nu_{\tau} c^2}{2(r - r_s)} + w' + \frac{5}{3} w' + 2 \left(\nu_{v_{Al}} \right)' = 0. \quad (52c)$$
The values of boundary conditions and parameters for the equations to date show no existence of external energy input $Q_{\pm}$ and outer magnetization $\sigma_{\pm}$. However, $Q_{+}$ is not known. I assume for simplicity $L_{\infty} = \gamma r_{B}$ so that $L$ changes its behavior near $r_{B}$ together with temperature and density.

The Bondi radius is about $r_{B} \approx 3 \times 10^{-5} r_{g}$ for our Galactic center (Ghez et al. 2003). The properties of gas at $3 r_{B}$ are somewhat constrained from observations. I take the values for a uniformly-emitting gas model with temperature $T_{\infty} \approx 1.5 \times 10^{4}$ K and electron and total number densities $n_{e,\infty} = 26$ cm$^{-3}$ and $n_{\infty} = 48$ cm$^{-3}$ (Baganoff et al. 2003) at $r_{B} = 3 r_{B}$, which corresponds to $5^{\circ}$ in the sky. The presence of a dense cold component can make the average temperature much lower and the average density much higher (Cuadra et al. 2006), but I am leaving these uncertainties for future research.

Expanding and colliding hyper-Alfvénic stellar winds provide magnetic field into the region. Its strength near Bondi radius is not known. Only the very general estimate can be made. Matter magnetization is likely to be lower than the saturation value of $\sigma_{\infty} = 1$. I take the values in the range $\sigma_{\infty} = 0.001$–1 to cover all reasonable magnetization states of matter at $3 r_{B}$. If the magnetic field is rather a product of decay than dynamo amplification, then the local dimensionless helicity $\xi$ may be close to unity. I cover the range $\xi_{\infty} = 0.001$–0.5 in simulations to determine the possible dynamical significance of nonzero magnetic helicity.

3. Boundary Conditions and Parameters

The system of equations (52a)–(52h) consists of five differential and three algebraic equations and should be integrated inward from some outer boundary at $r_{c}$. This requires knowledge of at least eight constants. Seven of them are the values “at infinity” $L_{\infty}, T_{\infty}, \rho_{\infty}, u_{\infty}, v_{A\infty}, v_{A\perp \infty}$, and $v_{A\perp \infty}$. The eighth is the accretion rate $M$. It is usually determined by some extra condition and is not adjustable. I assume isotropic turbulence with $E_{K} = E_{M}$ at the outer boundary. Therefore,

$$v_{A\infty} = v_{A\perp \infty} = \left( \frac{RT_{\infty} \sigma_{\infty}}{\mu} \right)^{1/2} \text{ and } u_{\infty} = \left( \frac{3RT_{\infty} \sigma_{\infty}}{\mu} \right)^{1/2} ,$$

and I have one model parameter $\sigma_{\infty}$ instead of three velocities $v_{A\infty}, v_{A\perp \infty}$, and $u_{\infty}$. Another adjustable parameter of the model is $\gamma$, which determines the size of energy-containing eddies $L$ near the object (eq. [52h]).

Parameter $\gamma$ is not free in principle, but its value cannot be determined within the proposed theory. Neither does there exist anisotropic MHD simulations that could provide $\gamma$. All simulations to date show $\gamma$ to be within 0.2–2 (Tennekes & Lumley 1972; Landau & Lifshitz 1987; Biskamp 2003) in both the HD and MHD cases. I assume the same range of $\gamma$ in my calculations.

3.1. Outer Medium Transition

The Bondi radius

$$r_{B} = r_{g} \frac{c^{2}}{c_{\infty}^{2}} \text{ with } c_{\infty} = \left( \frac{5RT_{\infty}}{3\mu} \right)^{1/2}$$

is the natural length scale of the spherical accretion flow (Bondi 1952). The density $\rho$ and temperature $T$ of plasma are constant for radii $r \gg r_{B}$, because gravitational energy and gas regular kinetic energy are negligible there compared to gas internal energy (Bondi 1952). The averaged magnetic field and averaged random velocity are also constant for $r \gg r_{B}$, because constant external energy input balances the dissipation in this region. As a consequence, $\xi = \xi_{\infty}$ and $L = L_{\infty}$ for $r \gg r_{B}$.

I set the outer boundary at $r_{c} = 3 r_{B}$, where matter is almost uniform. The length scale $L_{\infty}$ should be determined from a known external energy input $Q_{+}$ and outer magnetization $\sigma_{\pm}$. However, $Q_{+}$ is not known. I assume for simplicity $L_{\infty} = \gamma r_{B}$, so that $L$ changes its behavior near $r_{B}$ together with temperature and density.

The Bondi radius is about $r_{B} \approx 3 \times 10^{-5} r_{g}$ for our Galactic center (Ghez et al. 2003). The properties of gas at $3 r_{B}$ are somewhat constrained from observations. I take the values for a uniformly-emitting gas model with temperature $T_{\infty} \approx 1.5 \times 10^{4}$ K and electron and total number densities $n_{e,\infty} = 26$ cm$^{-3}$ and $n_{\infty} = 48$ cm$^{-3}$ (Baganoff et al. 2003) at $r_{B} = 3 r_{B}$, which corresponds to $5^{\circ}$ in the sky. The presence of a dense cold component can make the average temperature much lower and the average density much higher (Cuadra et al. 2006), but I am leaving these uncertainties for future research.

Expanding and colliding hyper-Alfvénic stellar winds provide magnetic field into the region. Its strength near Bondi radius is not known. Only the very general estimate can be made. Matter magnetization is likely to be lower than the saturation value of $\sigma_{\infty} = 1$. I take the values in the range $\sigma_{\infty} = 0.001$–1 to cover all reasonable magnetization states of matter at $3 r_{B}$. If the magnetic field is rather a product of decay than dynamo amplification, then the local dimensionless helicity $\xi$ may be close to unity. I cover the range $\xi_{\infty} = 0.001$–0.5 in simulations to determine the possible dynamical significance of nonzero magnetic helicity.

3.2. Transition to Rotationally Supported Flow

The system of equations (52a)–(52h) has the same property as the spherically symmetric system of HD equations (Bondi 1952); the subsonic solution exists for all accretion rates $\dot{M}$ up to the maximum $\dot{M}^{*}$, the transonic solution is valid for only the value $\dot{M}^{*}$, and no solution exists for $\dot{M} > \dot{M}^{*}$. The solution with

$$\dot{M} = \dot{M}^{*} \text{ (for the transonic solution)}$$

is preferable, because it has the highest rate of energy transfer toward the equilibrium state of the system matter SMBH. The same argument is valid for a general HD nozzle (Landau & Lifshitz 1987). It is reasonable to expect that maximum mass flux solution for system with magnetic field from equations (52a)–(52h) also obeys the condition from equation (55). However, even a small amount of angular momentum can change the picture.

Every real astrophysical accretion flow has nonzero specific angular momentum at the outer boundary

$$l = l_{g} c, \text{ or equivalently, } l = v_{K} \Omega_{g} r_{g}$$

(56)
where \( r_{\text{cir}} \) is a radius where matter becomes rotationally supported and \( v_{\text{K, cir}} \) is the Keplerian velocity at \( r_{\text{cir}} \). The general Newtonian expression for Keplerian velocity at radius \( r \) is
\[
 v_{\text{K}} = \frac{c}{2r} \sqrt{\frac{r_g}{2r}}.
\]  
(57)

At larger radii, \( r > r_{\text{cir}} \), the angular momentum exerts a relatively small force \( F_l \propto l^2/r^3 \) on plasma, since \( F_l \) decreases with radius faster than the gravitational force \( F_g \propto r_g / r^2 \). Numerical simulations (Cuadra et al. 2006) suggest \( r_{\text{cir}} \sim 3 \times 10^9 r_g \) for our Galactic center. 

When angular momentum (eq. [56]) is large, \( \lambda \gg 1 \), it should be able to travel outward through the outer quasi-spherical solution by means of the \( r \phi \) component of the stress tensor \( t_{\alpha \beta} \). The angular averaged form of this component is
\[
 t_{\phi \phi} = \frac{\langle B_r B_\perp \rangle}{4\pi} \Omega,
\]  
(58a)

where I neglect the kinetic part for the estimate. It can be transformed with the aid of the Schwartz formula \( \langle x^2 \rangle \leq \sqrt{\langle x^2 \rangle^2 \sqrt{\langle y^2 \rangle^2}} \) into the inequality
\[
 t_{\phi \phi} \leq \frac{B_r B_\perp}{4\pi} \]  
(58b)

with definitions (eq. [21]) of rms \( B_r \) and \( B_\perp \). Let us take a disk (Shakura & Syunyaev 1973) with height \( H \) and write the angular momentum transfer equation as
\[
 \frac{d(r^2 H t_{\phi \phi})}{dr} = 0.
\]  
(59a)

The result of the integration is (Gammie & Popham 1998)
\[
 M_l = 4\pi H r^2 t_{\phi \phi},
\]  
(59b)

in the large dimensionless angular momentum, \( \lambda \gg 1 \), case (Gammie & Popham 1998). I take specific angular momentum \( l \) from equation (56) and the accretion rate to be
\[
 M = 2\pi r H v_c.
\]  
(60)

I substitute the angular momentum \( l \) from equation (56), the accretion rate \( M \) from equation (60), the Alfvén speeds from equation (24), the Keplerian velocity from equation (57), and the inequality (58b) on \( t_{\phi \phi} \) into an angular momentum transfer equation (eq. [59b]) to obtain
\[
 \frac{r v_{\text{K, cir}}}{r v_{\text{L, cir}}} \sqrt{\frac{r_{\text{cir}}}{r}} = 2\chi, \quad \chi \leq 1,
\]  
(61a)

that should be valid at any radius \( r \). Sometimes, this inequality is valid for \( r > r_{\text{cir}} \) if it is valid at \( r_{\text{cir}} \), so inequality (61a) can in some cases be simplified to
\[
 \frac{r v_{\text{K, cir}}}{r v_{\text{L, cir}}} \leq 2 \text{ at } r_{\text{cir}}.
\]  
(61b)

The height of the disk \( H \) cancels out of the final expression; thus, inequalities (61a) and (61b) are approximately valid, even for flows with \( H \approx r \). Such flows are likely to describe the realistic transition region from outer quasi-spherical inflow to inner rotational solution. There are no extra degrees of freedom to put conditions on the surface of the compact object, so I consider the object to be effectively a black hole.

The condition of angular momentum transport from equations (61a) and (61b) may be stronger than the maximum accretion rate condition from equation (55). This depends on the value of the specific angular momentum \( l \) and the viscous \( \alpha \)-parameter (Shakura & Syunyaev 1973). The viscous \( \alpha \) is approximately \( \alpha \sim \sigma^2 \) according to my definitions in equations (46) and (61a). If \( \sigma \gg 0.5 \), then accretion proceeds without the direct dynamical effect of rotation (Narayan et al. 1997). Thus, two types of solutions are possible:

1. maximum accretion rate solutions that describe radial flows with a small angular momentum \( \lambda \sim r_{\text{cir}} \) or large viscosity \( \chi \sigma > 0.5 \) (§ 4.1),
2. or flows with the rotational support that work for a large angular momentum \( \lambda \gg r_{\text{cir}} \) and small viscosity \( \chi \sigma < 0.5 \) (§ 4.2).

Inequalities (61a) and (61b) give a crude estimate of the inflow velocity and accretion rate \( M \), since it assumes a specific angular momentum to be constant down to \( r_{\text{cir}} \). As matter travels to \( r_{\text{cir}} \), the amount of specific angular momentum left becomes smaller. Nevertheless, I calculate the solutions with effective angular momentum transport using inequalities (61a) and (61b) to illustrate the dependence of the accretion rate on model parameters for the rotating flow.

4. RESULTS

4.1. Maximum Rate Solution

Let me first disregard the angular momentum transport condition from equations (61a) and (61b) and calculate the flow with small angular momentum \( \lambda \ll r_{\text{cir}} \), when mean rotation is not dynamically important.

The system of equations I solve (eqs. [52a]–[52h]) can be rewritten as
\[
 \frac{F_i}{F_l} = \frac{N_i(F, r)}{D}, \quad \text{for } i = 1, 2, \ldots, 8,
\]  
(62)

where \( F_i(r) \) are eight functions I solve for, \( N_i(F, r) \) are function-and radius-dependent numerators, and
\[
 D = 1 - \frac{v_s^2}{V_s^2}
\]  
(63)

is a common denominator. The critical velocity \( V_s \) is
\[
 V_s^2 = c_g^2 + 2c_{\text{A, cir}}^2, \quad \text{where } c_g^2 = c_s^2 + \frac{5u^2}{3}.
\]  
(64)

The effective sound speed \( c_g \) is equal to that of plasma with effective particle velocity \( v_{\text{eff}} = v_p^2 + u^2 \).

According to the maximum rate condition from equation (55), I search for a smooth solution that has a sonic point at some radius \( r_s \). The condition at \( r_s \) is \( D(r_s) = 0 \). A zero denominator requires all the numerators \( N_i(F, r) \) to be zero at \( r_s \). It can be shown from the system from equations (52a)–(52h) that all eight conditions \( N_i(F(r_s), r_s) = 0 \) collapse into just one, which indicates that the maximum accretion rate solution is smooth. Two equalities,
\[
 D(r_s) = 0 \quad \text{and} \quad N_1(F(r_s), r_s) = 0,
\]  
(65)

give the missing eighth condition on \( M \) for the system from equations (52a)–(52h) and the sonic radius \( r_s \). Thus, I have seven
conditions at the boundary at $3r_B$ and one condition somewhere in the region. I employ the shooting method to search for the $M$ and $r_s$ that satisfy the relation from equation (65). I obtain the Bondi HD model (Bondi 1952) if I set all Alfvén velocities and turbulent velocity to zero and use a nonrelativistic prescription for enthalpy $w_{NR}$ (eq. [52c]). Therefore, the accretion rate $\dot{M}$ equals the Bondi accretion rate of monatomic gas $\dot{M}_B = \frac{\pi}{4} r_s^2 c^4 \rho_{\infty} \left( \frac{3\mu}{5RT_{\infty}} \right)^{3/2} \approx 4 \times 10^{-6} M_\odot \text{ yr}^{-1}$ (66) in the limiting case of no turbulence. The number is calculated for the black hole in our Galactic center with $r_s = 1.1 \times 10^{12}$ cm (Ghez et al. 2003), $T = 1.5 \times 10^7$ K, and $n \approx 48$ cm$^{-3}$ (Baganoff et al. 2003). The accretion rate $\dot{M}$ appears to be lower than $\dot{M}_B$ when the turbulent energy is nonzero (Fig. 1).

Inhibition of accretion by turbulence has the following explanation. First, the energy of the magnetic field increases inward, therefore it exerts back-reaction force, stopping the matter (Shvartsman 1971). Second, the magnetic field serves as a very effective mechanism of energy conversion from gravitational to thermal via dissipation of turbulence (Igumenshchev & Narayan 2002). A larger amount of thermal energy corresponds to a larger gas pressure that also stops matter. Within the deduced model I can estimate the actual decrease of the accretion rate $\dot{M}$ from the Bondi value $\dot{M}_B$.

I take my reference model to have the values $\gamma = 1$, $\sigma_{\infty} = 1$, and $\xi_{\infty} = 0.025$ of, correspondingly, the dimensionless scale of turbulence, outer magnetization, and outer magnetic helicity. The found accretion rates are $0.14\dot{M}_B$ for a nonrelativistic equation of state and $0.24\dot{M}_B$ for a relativistic equation of state. I can now consider the whole ranges of all three parameters and explain the observed correlations between them and accretion rate $\dot{M}$.

Larger flow magnetization $\sigma_{\infty}$ results in a lower accretion rate $\dot{M}$. A larger magnetic field and turbulent velocity field exerts a larger back-reaction force on matter. In addition, the transformation of gravitational energy into thermal happens more readily if magnetization is larger. A larger thermal energy means a larger gas pressure and larger back-reaction force on matter striving to fall onto the central object.

Several factors lead to higher magnetization. A larger outer magnetization $\sigma_{\infty}$ makes magnetization in the entire flow larger. Then a larger dissipation length scale $\gamma$ allows for a smaller dissipation of the magnetic field. A larger magnetic helicity $\xi$ also lowers the magnetic energy dissipation and leads to a larger magnetization $\sigma$. These correlations can be observed in Figure 1. The increase of the relative length scale of energy containing eddies $\gamma$ from 0.2 to 2 results (Fig. 1a) in about a twofold drop in accretion rate $\dot{M}$.

**Fig. 1.—Maximum accretion rate solution.** Shown is the dependence of the accretion rate in units of Bondi rate on dimensionless parameters: (a) characteristic length scale $\gamma$, (b) outer magnetic helicity $\xi_{\infty}$, and (c) outer matter magnetization $\sigma_{\infty}$. Also shown is the dependence (d) of sonic radius on outer magnetization $\sigma_{\infty}$. I take the reference model to have the following values of parameters: $\gamma = 1$, $\sigma_{\infty} = 1$, and $\xi_{\infty} = 0.025$. One parameter is varied to make one plot. A nonrelativistic one-temperature EOS (dashed line) vs. relativistic one-temperature EOS (solid line) are shown. [See the electronic edition of the Supplement for a color version of this figure.]
rate $\dot{M}$. The accretion rate stays constant (Fig. 1b) at small values of outer magnetic helicity $\xi_{\infty}$. However, $\dot{M}$ drops an order of magnitude as turbulence approaches a highly helical state at the outer boundary $3r_B$ with $\xi_{\infty}$ close to 0.5. The dependence of $\dot{M}$ on the outer magnetization $\sigma_{\infty}$ is not quite steep; the accretion rate gradually decreases by about 4 times as the outer magnetization increases 3 orders of magnitude from 0.001 to 1. Surprisingly, the accretion rate does not rise to $\dot{M}_{\text{B}}$ (Fig. 1c), even for a very small outer magnetization $\sigma_{\infty} \approx 0.001$ for the nonrelativistic equation of state. Even a small outer magnetic field increases inward and influences flow dynamics.

The accretion rate is systematically about 40% higher (Fig. 1) for the relativistic equation of state (solid line) compared to the nonrelativistic equation of state (dashed line), because a magnetized system has some properties of a nonmagnetized one. The formula for Bondi mass accretion rate from equation (66) is valid only for nonrelativistic monatomic gas that has an adiabatic index $\Gamma = 5/3$. The accretion rate is higher for lower $\Gamma$ and is about 3 times larger (Shapiro & Teukolsky 1983) in case of ultrarelativistic particles with adiabatic index $\Gamma = 4/3$. The accretion rate $\dot{M}$ is determined by equation (65) at a sonic radius $r_s$ that is smaller than $10^4r_g$ (Fig. 1d). Electrons become relativistic at the somewhat larger radius of about $10^3 r_g$ in the solutions of the system of equations (52a)–(52h). This leads to a gas adiabatic index $\Gamma$ (the magnetic field is disregarded) lower than 5/3 at the sonic point $r = r_s$. Thus, the accretion rate is considerably larger in the relativistic equation of state case.

It is also instructive to trace the dependence of sonic radius $r_s$ on parameters. The sonic radius for HD accretion of nonrelativistic monatomic gas is equal to several Schwarzschild radii, $r_s = 2$–$10 r_g$ (Beskin & Pidoprygora 1995). The sonic radius is a considerable fraction of $r_B$ for a gas with an adiabatic index $\Gamma$ substantially smaller than 5/3 for nonmagnetized accretion (Bondi 1952). Magnetized accretion has the same properties. A nonrelativistic equation of state (EOS; Fig. 1) results in a very small sonic radius $r_s = 7$–$11 r_g$ (Fig. 1d). The sonic radius for a relativistic EOS (dashed line) is $r_s = 300$–$1200 r_g$, about the radius where electrons become relativistic $r \sim 10^4 r_g$. The value of the sonic radius drops several times as plasma’s outer magnetization $\sigma_{\infty}$ increases from 0.001 to 1. As the outer magnetization $\sigma_{\infty}$ increases, the accretion rate drops (Fig. 1c), because density $\rho$ and gas inflow speed $v$ decrease. Then effective sound speed $V_s$ equals the inflow speed $v$ at a point closer to the black hole.

The inflow velocity $v$ as well as other characteristic velocities of the flow are depicted in Figure 2 as functions of radius $r$ for the reference model with $\sigma_{\infty} = 1$, $\gamma = 1$, and $\xi_{\infty} = 0.025$. All velocities are normalized to the freefall speed

$$v_g = \frac{c}{\sqrt{\frac{r_g}{r - r_g}}}. \quad (67)$$

I also normalize the perpendicular Alfvén velocity $v_{A,\perp}$ and turbulent speed $u$ to one dimension. The horizontal line in Figure 2 corresponds to radial dependence $r^{-1/2}$.

The inflow velocity $v$ monotonically increases inward, whereas sound speed $c_g$ monotonically decreases with almost the same rate at the sonic point. Radial Alfvén velocity $v_{A,\text{r}}$, perpendicular Alfvén velocity $v_{A,\perp}$, and turbulent velocity $u$ (Fig. 2) start out as constants from the outer boundary at $3r_B$, where turbulence is sustained by external pumping. Then these velocities increase and deviate from one another. Radial Alfvén velocity $v_{A,\text{r}}$ appears to be much larger than $v_{A,\perp}$ and $u$ in the inner accretion region. This fulfills the expectations of earlier models (Shakura & Syunyaev 1973; Scharlemann 1983; Beskin & Karpov 2005).

At small radius, turbulence is driven by freezing-in amplification of the magnetic field and random velocity. Left-hand sides of turbulence evolution equations (52d), (52e), and (52f) dominate over corresponding terms with external driving for radius $r \lesssim 10^4 r_g$. Internal driving of $v_{A,\text{r}}$ is much more effective than the driving of $v_{A,\perp}$, and $u$. Therefore, radial Alfvén velocity $v_{A,\text{r}}$ is larger than other two speeds. This refutes any model with an isotropic magnetic field.

Several pairs of lines intersect on the velocity plot (Fig. 2). I consider three main intersection points for the reference model with $\sigma_{\infty} = 1$, $\gamma = 1$, $\xi_{\infty} = 0.025$, and a relativistic EOS (Fig. 2a). The crossing of the inflow velocity $v$ and sound speed $c_g$ occurs almost at the sonic point at $r_s$, determined by equation (65) with critical velocity $V_s$ (eq. [64]). No plasma waves can escape from within the region with high inflow velocity $v > V_s$ at the sonic point $r_s \approx 6 \times 10^{-4} r_B$, $c_g \approx V_s$, because of low magnetization $\sigma \approx 20\%$ in that region (Fig. 3a). The Alfvén point is determined by the equality $v = v_{A,\text{r}}$ at radius $r_A$. Alfvén waves cannot escape from within the region where the inflow speed is greater than the radial Alfvén speed $v_{A,\text{r}}$. The equality holds at the relatively large radius $r_A \approx 0.03 r_B$. The third combination of the same three speeds also gives a characteristic intersection point. Radial Alfvén speed $v_{A,\text{r}}$ increases faster inward and becomes equal to the sound speed $c_g$ at about $r \approx 4 r_g$. Further relative increase of $v_{A,\text{r}}$ leads to a magnetic-energy–dominated flow, which can be traced on magnetization plot (Fig. 3a).
The inflow speed $v$ from the freefall scaling $r^{-1/2}$ makes a density profile in the magnetized flow different from that in a standard advection-dominated accretion flow (ADAF). I consider the flow where energy is only advected inward. Nevertheless, I obtain

$$\rho \propto r^{-\zeta}, \quad \text{where } \zeta \approx 1.25,$$

almost independently on the parameters or the EOS, somewhat shallower than $\rho \propto r^{-1.5}$ in ADAF.

The only question left is how well this flow with maximum accretion rate can describe the real situation with large angular momentum $l$. Given the solution of the system from equations (52a)–(52h), I can check whether the condition for effective angular momentum transport inequalities (61a) and (61b) holds. Inequalities (61a) and (61b) break when evaluated for the maximum rate solution with parameters $\xi_{\infty}$, $\sigma_{\infty}$, and $\gamma$ within the chosen ranges and circularization radius $r_{\text{circ}} > r_g$. This means that a flow with a maximum accretion rate is unable to effectively transport the angular momentum outward. The same conclusion can be made simpler. The transport of angular momentum is a magnetic process. So, $l$ can be transported only by Alfvén waves. However, Alfvén waves cannot escape from the region within $r_A \approx 0.03r_B$ from the compact object, which makes angular momentum transport impossible, even from quite large radius.

4.2. Solution with Effective Angular Momentum Transport

A solution with a large outer angular momentum $l \gg r_c c$ and small viscosity may have properties substantially different from those of the maximum rate solution. The actual details of the solution and allowed accretion rate depend on how this angular momentum is transported. For the simple estimate I suppose that the accretion rate is determined by the equality in the angular momentum transport inequalities (61a) and (61b). The maximum accretion rate $M$ for inequalities (61a) and (61b) appears to be about 2 orders of magnitude lower than Bondi rate $M_B$ (eq. [66]). I add one parameter in modeling: an unknown circularization radius $r_{\text{circ}}$ for specific angular momentum $l$ (eq. [56]). I take it to be $r_{\text{circ}} = 10^3r_g$ for the reference model. Plots of the accretion rate verses model parameters are shown in Figure 4. Dependencies for the rotating solution (Fig. 4) have the opposite slopes to those for the maximum rate solution in Figure 1. Accretion rate $M$ increases with increasing outer magnetization $\sigma_{\infty}$ (Fig. 4b) and increasing outer magnetic helicity $\xi_{\infty}$ (Fig. 4c). Both effects lead to higher plasma magnetization $\sigma$. I showed in § 4.1 that the magnetic field plays an inhibiting role on matter inflow and that the larger the magnetic field is, the smaller the accretion rate $M$. However, the correlation between the magnetic field and accretion rate is the opposite in the rotating flow case. The accretion rate quantitatively agrees with the relation for ADAF flows $M \sim \alpha M_B \sim \sigma_{\infty} M_B$ (Narayan et al. 1997) with $\sigma \approx 0.01$ at $r_{\text{circ}}$ (Fig. 6a).

The inflow speed $v$ allowed by inequalities (61a) and (61b) is proportional to the product of the radial Alfvén speed $v_A$ and perpendicular Alfvén speed $v_{AL}$. A larger magnetic field results in a larger transport of angular momentum outward, so a larger inflow velocity $v$ and a larger accretion rate are possible. Larger outer magnetization $\sigma_{\infty}$ and a larger outer magnetic helicity $\xi_{\infty}$ both lead to higher magnetization $\sigma$ and a higher magnetic field. The inhibiting effect of a magnetic field is smaller in cases...
having lower accretion rates $\dot{M}$ and lower inflow velocities $v$. A lower $v$ results in a lower relative driving of turbulence, which makes the magnetic field weaker. A weaker magnetic field has a weaker influence on dynamics. In summary, a larger magnetic field $B$ results in a larger accretion rate $\dot{M}$ when it needs to transfer angular momentum.

The dependence of $\dot{M}$ on length scale $L$ is obscured by the dependence of external driving on $L$. The accretion rate $\dot{M}$ is smaller for a smaller magnetic field, but the state of low magnetization can be achieved in two different ways. First, the magnetic field decays faster when $L$ decreases. However, the plasma at circularization radius $r_{\text{cir}} = 10^3 r_g$ is still partially influenced by the outer boundary conditions. Internal driving does not depend on $L$, whereas external driving is stronger and the magnetization $\sigma$ is higher when $L$ is small. The two described effects balance each other and make the accretion rate $\dot{M}$ almost independent of the dimensionless length scale $\gamma$ (Fig. 4a).

The accretion rate $\dot{M}$ decreases with the decrease of the circularization radius $r_{\text{cir}}$ (Fig. 4d) for a nonrelativistic EOS. To explain this, I trace on Figure 5b all the quantities that enter the angular momentum transport inequalities (61a) and (61b) for the reference model. Velocities normalized by the freefall speed (eq. [67]) are shown in Figure 5b. Inflow speed $v$ and the radial Alfvén velocity $v_A$, reach freefall scaling at about $0.02 r_B$. Only perpendicular Alfvén velocity $v_{A\perp}$ has a different dependence on distance from the central object for $r < 0.02 r_B$. Because $v_{A\perp}$ decreases with radius, the allowed $v$ and $\dot{M}$ are smaller for a smaller circularization radius.

However, the accretion rate increases for small circularization radii for one-temperature EOS (Fig. 4d, solid line). This is the consequence of the decreasing gas adiabatic index, when electrons reach relativistic temperatures. Solutions with lower adiabatic index are known to have larger accretion rates (Bondi 1952) that are equivalent to the lower inflow speeds $v$ in the solutions for the fixed-matter inflow rate. Velocity $v$ (Fig. 5a) starts deviating down from the self-similar $r^{-1/2}$ solution at approximately $10^3 r_g$, making the solutions with higher $M$ possible. In fact, inequalities (61a) and (61b) for the solutions with small $r_{\text{cir}}$ become critical at some fixed point $r_d > r_{\text{cir}}$ instead of reaching equality at $r_{\text{cir}}$ (inequality [61b]). Therefore, according to inequality (61a), the maximum value of the inflow speed grows with the decrease of circularization radius as $v \propto r_{\text{cir}}^{-1/2}$, explaining the rise of the accretion rate for small $r_{\text{cir}}$ (Fig. 4d, solid line) for a one-temperature EOS.

The solution for a nonrelativistic EOS, in turn, possesses its own feature. Self-similar flow (see Appendix B) settles in at $10^3 r_g$, making the accretion rate almost independent on the circularization radius (Fig. 4d). Magnetic helicity $\xi$ in such a flow is a number of about unity that is consistent with the self-similar solution obtained in Appendix B. The self-similar flow cannot be
established for a one-temperature EOS, because relativistic effects become important before it establishes and break self-similarity.

In fact, magnetization $\sigma$ and magnetic helicity $\xi$ (Fig. 6) are not constant at small radii for the correct one-temperature EOS, because these relativistic corrections work. At about $0.01r_B$ magnetization reaches an almost constant level $\sigma \approx 0.02$ (Fig. 6a) and then starts to slightly decrease, because equilibrium $\sigma$ for matter with a lower gas adiabatic index $\Gamma < 5/3$ is lower. Magnetic helicity $\xi$ behaves (Fig. 6b) in a manner opposite to magnetization $\sigma$: magnetic helicity reaches $\xi \approx 1.5$ at $0.01r_B$ and starts to slightly increase as the radius decreases.

5. DISCUSSION OF THE MODEL

I present the sophisticated analytical model to determine the properties of spherical magnetized accretion. The common assumptions of magnetic field isotropy and thermal equipartition are released, but many assumptions are still left. As is usual in fluid dynamics, a lot of simplifications are made during the course of elaboration. The validity of almost everything can be questioned. The system of equations (52a)–(52h) may not describe the real flow (§ 5.1) or may have some inaccuracies (§ 5.2). Gas cooling may not be neglected (§ 5.3). Convection and diffusion may change the flow structure (§ 5.4). The EOS was also found to influence the dynamics (§ 5.5). Let me discuss all these topics and determine the practical significance of the model.

5.1. Real Flow

The presented model is partially applicable to the real systems. It may describe some gas flows onto supermassive black holes in low luminosity galactic centers, in particular in the center of our Galaxy. These flows are geometrically thick (Narayan & Yi 1995) and may have low angular momentum (Moscibrodzka et al. 2006). However, the real flows may have properties that my model cannot handle in its current state. First of all, the sources of matter and external driving should be explicitly accounted for. Second, the self-consistent angular momentum transport theory is needed.

The material is mainly supplied to the central parsec of the Milky Way by stellar winds (Quataert 2004). The wind-producing stars have a broken power-law distribution as a function of radius $r/r_B$ (Baganoff et al. 2003). Some stars are as close to the central black hole as $0.1r_B$ (Ghez et al. 2003). The material is mainly supplied to the central parsec of the Milky Way by stellar winds (Quataert 2004). This Bondi radius coincides with the radius where inflow starts to dominate outflow in numerical simulations with the accretion rate $M \sim 10^{-6} M_\odot$ yr$^{-1}$ (Cuadra et al. 2006). The maximum accretion rate in the solution with zero angular momentum is $0.2M_B \approx 10^{-6} M_\odot$ yr$^{-1}$ and $0.01M_B$ for the rotating flow. Therefore, the transition from the outflow to the inflow happens at $r \gtrsim 10^3 r_g$.

I can show that outflow from $r \gtrsim 10^3 r_g$ does not change the accretion rate from the calculated value. Outflows substantially alter the value and the sign of inflow velocity $v$ in the system of
equations (52a)–(52h). However, the differences in inflow velocity do not influence any other quantity as long as three conditions are satisfied:

1. \( v \) is much smaller than the gas particle velocity \( v_p \) and bulk kinetic energy of the gas is negligible in the outflow region,
2. the external driving \( Q_\perp \) dominates over the internal driving in the outflow region,
3. and the condition on \( M \) is set in the inflow region.

The first two conditions are satisfied down to \( r \sim 10^3 r_g \) (Figs. 2 and 5). The third condition holds for the maximum rate solution, because the condition on \( M \) is set at the sonic point about \( 10^3 r_g \) from the central object. It also holds for the solution with angular momentum transport, because the condition on \( M \) is usually set at the inner boundary \( 10^3 r_g \). All three above conditions hold, hence outflows of stellar winds do not substantially change the accretion rate or any quantity in the system.

5.2. Treatment of Magnetic Field

The long history of accretion theory has many accepted models based on ideas, extended beyond the area of applicability of these ideas. For example, general relativity was substituted with the Paczynski-Wiita gravitational potential (Paczynski & Wiita 1980; Shakura & Syunyaev 1973). The magnetic field was long treated similar to the normal matter (Narayan & Yi 1995; Coker & Melia 2000). The displacement current was neglected in the magnetic field dynamics that allowed the treatment of the magnetic field without an electric field (Scharlemann 1983). System of viscous equations describe viscosity with a single parameter (Shakura & Syunyaev 1973; Landau & Lifshitz 1987; Landau et al. 1984; Biskamp 2003). Gyrokinetics is used to solve the problems with non-Maxwellian distribution functions (Sharma et al. 2007), and power-law nonthermal electrons are usually present in plasma (Yuan et al. 2002).

The model described above is extended in several ways, mainly with regard to the magnetic field. The isotropic MHD system of turbulent equations (eqs. (27)) describes the real box collisional turbulence quite well, because it corresponds to a convergent set of simulations. Collisionality assumes that the medium behaves like the normal matter (Narayan & Yi 1995; Coker & Melia 2000). The equation of magnetic helicity evolution from equation (52g) holds only if I assume an even redistribution of the field along the angular direction is about \( 10^3 r_g \) (Beskin & Karpov 2005). However, changes in these coefficients do not lead to a dramatically different accretion rate or flow structure. Setting \( c_{BB2} = c_{BB1} \) instead of \( c_{BB2} = 0 \) leads to only 10% of a \( M \) change for the reference model. All seven introduced coefficients \( c_{ex} \) may themselves depend on the anisotropy of the magnetic field. The details of anisotropic MHD are still debatable.

The first of all, I need to introduce arbitrary coefficients \( c_{BB1}, c_{BB2}, \) and \( c_{BB3} \) to describe the isotropization of an anisotropic magnetic field and anisotropic energy transfer between magnetic field and fluid motions. Reasonable values of these coefficients were taken to satisfy rather loose analytical tests (Appendix A). However, changes in these coefficients do not lead to a dramatically different accretion rate or flow structure. Setting \( c_{BB2} = c_{BB1} \) instead of \( c_{BB2} = 0 \) leads to only 10% of a \( M \) change for the reference model. All seven introduced coefficients \( c_{ex} \) may themselves depend on the anisotropy of the magnetic field. The details of anisotropic MHD are still debatable.

Second, the presented theory is not general relativistic. The accretion rate \( M \) appears to be insensitive to the choice of the gravitational potential. The condition on \( M \) is set at about \( 10^3 r_g \) in a case with a relativistic EOS and zero angular momentum \( \ell \). The sonic point is situated close to the black hole at \( r_s \sim 5–10 r_g \) for a nonrelativistic EOS, but a 1% increase of \( M \) leads to the sonic point at \( r_s > 100 r_g \), independent of the method of mimicking general relativity. However, the region near the black hole is important, because part of synchrotron IR radiation as well as a part of radio emission comes from several Schwarzschild radii (Narayan et al. 1998; Falcke & Markoff 2000; Marrone et al. 2007). Thus, to fully constrain the theory by observations, general relativistic MHD is a must.

Third, magnetic helicity \( H \) involves numerous complications. Magnetic helicity evolves in the region that is frozen into the matter. The distance \( L_{\parallel} \) between the radial boundaries of this region is proportional to inflow velocity \( v \); thus, \( L_{\parallel} \) increases with increasing \( r \) and at some point \( L_{\parallel} > r \), whereas the size in the angular direction is about \( L = \gamma r \). A part of the region is getting sucked into the black hole, whereas a part is still situated at a fairly large radius \( r \). The equation of magnetic helicity evolution from equation (52g) holds only if I assume an even redistribution of magnetic helicity over the mass of the plasma. This holds for a frozen magnetic field, but in reality diffusion and convection are present. Diffusion may change the results for \( H \) (eq. (52g)) as well as for the entire flow pattern. I also leave these uncertainties for future research.

Fourth, it was recently suggested by Beskin & Karpov (2005) that ions and electrons should be viewed in accretion as being confined by magnetic field lines. This is the opposite of the standard picture where magnetic field lines are frozen into matter (Scharlemann 1983). The former case has a higher heating rate of matter under contraction (Beskin & Karpov 2005), because of conservation of the first adiabatic invariant \( I = 3p_i^2 / (2eE) = \text{const} \) (Landau & Lifshitz 1975). Here \( p_i \) is a particle’s momentum in the direction perpendicular to \( B \). However, only highly
magnetized flows with magnetization $\sigma > 1$ conserve $I$. Non-linear collective interactions of particles in low-$\sigma$ plasma are likely to isotropize their distribution. When particles are heated isotropically under contraction, general MHD (eqs. [1]–[6]) works (Landau et al. 1984) and the heating rate stays unchanged. Magnetization in computed models is below unity (Figs. 3a and 6a). Thus, application of the first adiabatic invariant conservation to magnetized accretion flow seems irrelevant.

Finally, the mean rotation of the flow also creates anisotropy. Because the inner gas rotates faster than the outer, magnetorotational instability (MRI) works. It produces the additional driving of a magnetic field that may be concurrent to other sources. MRI (Hawley & Balbus 2002) has a timescale

$$\tau_{\text{MRI}} = - \left[ \frac{d(I/r^2)}{dr} \right]^{-1}. \quad (71)$$

When the MRI timescale becomes larger then the dynamic timescale $\tau_{\text{dyn}} = r/v$, a field amplification occurs mainly because of a regular shear tangential motion, instead of regular radial motion. MRI may be crucial, even in the region without rotational support. The full consideration of effects of angular momentum on the flow is the subject of the next study.

5.3. Radiative Cooling

The system of equations (52a)–(52h) describes the accretion flow, where all the energy is stored in the same piece of matter where it initially was. There is no energy loss by diffusive or radiative cooling. However, I examine whether such a model is realistic.

Let me estimate the radiative cooling first. Line cooling is more effective than bremsstrahlung cooling for temperatures of $T_\infty \approx 1.5 \times 10^7 K$. The line cooling function is $\Lambda \approx 6 \times 10^{-21} n^2 (T/10^7 K)^{-0.7}$ ergs cm$^{-3}$ s$^{-1}$ (Sutherland & Dopita 1993). Thus, the characteristic cooling time $\tau_{\text{cool}}$ is

$$\tau_{\text{cool}} = \frac{3RT\rho}{2\Lambda \mu} \approx 1 \times 10^{12} \text{ s} \quad (72)$$

for our Galactic center accretion. The dynamic timescale $\tau_{\text{dyn}} = r/v$ for accretion with the rate $\dot{M} = 0.1\dot{M}_B$ (eq. [66]) is

$$\tau_{\text{dyn}} = \frac{pr^3}{\dot{M}} \approx 5 \times 10^{10} \text{ s} \quad (73)$$

with the continuity equation (9) at a radius $r = r_B$ (eq. [54]). The cooling time is about 20 times larger than the inflow time in the region where outflows dominate. Nevertheless, anisotropy of stellar winds may lead to the significant cooling of some clumps of matter (Cuadra et al. 2005). This may trigger the formation of a disk (Cuadra et al. 2006). Careful calculation with line cooling is yet to be done.

5.4. Convection and Diffusion

The system of equations (52a)–(52h) does not include diffusive or convective transport of quantities. Thus, the system represents advection-dominated flow, where the magnetic field and gas can exchange energy between each other. The exact model would include the transport of momentum, energy, magnetic field, and magnetic helicity that may or may not influence the dynamics.

First of all, any type of convective or diffusive motion would happen at a speed $v_c$ not exceeding the maximum of turbulent speeds, the radial Alfvén speed $v_c < v_{A,R}$. This leads to the transition from a convection-dominated to an advection-dominated flow at several dozens $r_g$ in the case containing rotation (Abramowicz et al. 2002). Correspondingly, inflow speed $v$ becomes large $v \sim v$ (Gammie & Popham 1998). The transport becomes ineffective at $r \leq r_A$, where $r_A$ is the radius of the Alfvén point. According to Figure 2a, the Alfvén point in my spherical solutions lies at $r_A \approx 0.3r_B$. Thus, diffusion and convection are strongly suppressed in the inner flow. By the same reasoning, magnetothermal instability (MTI; Parrish & Stone 2005) is not supposed to play any role for spherical inflow, but may play a role in a case containing rotation. For the nonconductive convective stability criterion, see Appendix C.

However, the speed of electrons $v_e$ may overcome the speed of sound $c_s$, so electron conduction may principle transport energy from within $r_A$ (Johnson & Quataert 2007). It is yet unclear whether electron conduction is suppressed at high inflow velocity $v > v_{A,R}$, because electrons may be bound to the field lines of a tangled magnetic field. The efficiency of conduction is a free parameter. If the efficiency is close to maximum and conduction is not inhibited, then the accretion rate may be 1–2 orders of magnitude lower than the Bondi rate $\dot{M}_B$ (Johnson & Quataert 2007); thus, the accretion rate would be limited by conduction and not by back reaction of the magnetic field. Other processes of energy transport (Parrish & Stone 2005) may kick in for lower accretion rates. The correct calculation with a magnetic field and better prescription for conductivity is yet to be done.

5.5. Equation of State

The difference in the accretion rate $\dot{M}$ between one-temperature relativistic and one-temperature nonrelativistic EOSs is up to 40% for a maximum rate solution (§ 4.1) and up to several times for a solution with effective angular momentum transport (§ 4.2). The solution with a smaller gas adiabatic index $\Gamma$ has larger accretion rate $\dot{M}$ (Shapiro & Teukolsky 1983). The gas adiabatic index gradually falls from $\Gamma = 5/3$ to 1.43 in the relativistic EOS case as matter approaches the black hole.

However, the electron temperature $T_e$ is unlikely to be equal to ion temperature $T_i$. Electron temperature $T_e$ is usually modeled to be lower than $T_i$ (Narayan & Yi 1995). This two-temperature model has lower gas pressure support and larger gas adiabatic index $\Gamma$ than a one-temperature model with $T = T_i$. Lower gas pressure leads to a higher accretion rate, larger $\Gamma$ leads to a lower accretion rate. The combination of these two effects is expected to change the accretion rate by about the same 40% as between relativistic and nonrelativistic one-temperature EOSs. The exact details depend on the two-temperature model chosen.

6. OBSERVATIONS

The proposed quasi-spherical magnetized accretion model is aimed at explaining plasma flow onto supermassive black hole Sgr A* in our Galactic center. Many observations of this source are made. These observations reasonably agree with the results of my model.

A common misconception about Chandra X-ray observations of Sgr A* exists in the literature. X-rays mainly originate in the region that lies further than the Bondi radius $r_B$ from the central object. Thus, characteristic density $\rho_s$, and temperature $T_\infty$ far from the black hole can be found (Baganoff et al. 2003). If one knows the mass $M$, this automatically gives the Bondi accretion rate $\dot{M}_B$ (eq. [66]). However, the accretion rate is not necessarily determined by equation (66), unlike what some papers suggest (Bower et al. 2005). In my model, the accretion rate $\dot{M}$ is independent of radius and is smaller than $\dot{M}_B$.

IR (Eckart et al. 2006) and radio (Shen 2006) observations are difficult to interpret, because fluxes in these diapasons depend
strongly on the accretion model. The density of matter $\rho$ is better constrained by observations than the accretion rate $\dot{M}$. The general agreement (Yuan et al. 2002) is that density $\rho$ should be lower than it is in the Bondi solution $\rho_B$ in the region close to the black hole. Solutions with outflows (Yuan et al. 2003) and convectively dominated flows (Quataert & Gruzinov 2000a) were invented to explain this lower density. A magnetized solution without angular momentum does the same job well. Let me consider the reference magnetized model with $\sigma_\infty = 1$, $\gamma = 1$, $\xi_\infty = 0.025$, $l = 0$, and a one-temperature relativistic EOS. The ratio of density in a reference magnetized model to density in a nonmagnetized solution is

$$\frac{\rho_{\text{magn}}}{\rho_{\text{nonmagn}}} \approx 0.27 \text{ at } 10r_g. \quad (74)$$

Density in a magnetized model is much lower than that in a nonmagnetized one. However, all types of models can be made to fit the data by adjusting the temperature (Quataert & Gruzinov 2000b), whether advection or convection or outflow dominated.

Faraday rotation of submillimeter radiation offers a good differentiation mechanism between ADAF flows and flows with outflows or convection. The rotation measure is proportional to both magnetic field and electron density and has a relativistic temperature factor (Marrone et al. 2007). Model $B$ predicts magnetization $\sigma = 0.7$ and number density $n = 2 \times 10^{19} \text{ cm}^{-3}$ at $3r_g$ that is consistent with (Hawley & Balbus 2002). The observed Faraday rotation measure is $\text{RM} = -6 \times 10^{5} \text{ rad m}^{-1}$ (Marrone et al. 2007). Fitting the relativistic rotation measure for temperature gives $T_e = 4 \times 10^{10} \text{ K}$ in excellent agreement with Sharma et al. (2007). The accretion rate in the reference model is about $9 \times 10^{-7} \text{ M}_\odot \text{ yr}^{-1}$, which is 30 times lower than that in Sharma et al. (2007). However, the electron density in my model is close to that in the rotating model (Sharma et al. 2007), because inflow velocity in the rotating model is $\alpha$ times lower. For densities to agree, I need $\alpha \sim 0.03$, which is somewhat smaller than that found in numerical simulations $\alpha \geq 0.2$ (Hawley & Balbus 2002). This means my solution overestimates density $n$ by about a factor of 5, which results in an IR flux larger than that observed (Eckart et al. 2006). The effects of angular momentum transport, outflows (Yuan et al. 2002), or conduction (Johnson & Quataert 2007) must come into play to allow for a successful fitting for both IR flux and Faraday rotation measure.

7. CONCLUSIONS

Although many ways of dealing with inefficient accretion were invented, my approach is substantially different from all previous efforts. I elaborated a model that

1. has very few free parameters,
2. self-consistently includes averaged turbulence, combining geometrical effects of freezing-in amplification with dissipation,
3. ties evolution of random magnetic field and random velocity field to numerical simulations,
4. connects outer externally supported turbulence to inner self-sustained turbulence,
5. predicts the accretion rates $\dot{M}$ and flow patterns for the flows with negligible angular momentum,
6. and gives the order-of-magnitude estimate of $\dot{M}$ for large angular momentum flows.

The model predicts

1. an accretion rate $\dot{M}$ of a magnetized fluid $0.2 - 0.7$ of the Bondi rate $\dot{M}_B$, even for a small outer magnetization $\sigma_\infty$,
2. a subequipartitioned magnetic field in the outer part of the flow and a superequipartition in the inner part,
3. a density several times lower than that in the Bondi model near the central object, which with the addition of other effects would explain the observations of Sgr A*,
4. a half an order of magnitude effect of different equations of state on the accretion rate,
5. the unimportance of magnetic helicity conservation,
6. and the ineffectiveness of convection. Convection and diffusion should be accounted for together.

The next version of the model will include

1. more anisotropic effects, in particular, magnetorotational instability,
2. two-temperature equations of state,
3. a full treatment of angular momentum transport,
4. and the diffusion of momentum, heat, and the magnetic field.

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APPENDIX A

ANALYTICAL TESTS

Let me consider my model in the anisotropic incompressible case of box turbulence. I substitute $-v \partial/\partial r = d/dt$ in equations (22), (23), and (26) and set $r = \text{const}$. The box has infinite volume. I express some number of unknown $c_{xx}$ in terms of known $\tilde{c}_{xx}$ from equation (28). The system now reads

$$\frac{d(v_{A_r}^2)}{dt} = \left[\tilde{c}_{Ba}(v_{A_r} + 2\tilde{c}_{Ba}(v_{A_r} - v_{A\perp})v_{A\perp})u - \sqrt{3}\tilde{c}_{BB}v_{A\perp} + c_{BB}(v_{A_r} - v_{A\perp})\right]v_{A\perp}^2/L, \quad (A1a)$$

$$\frac{d(v_{A\perp}^2)}{dt} = \left[\tilde{c}_{Ba}(v_{A\perp} + 2\tilde{c}_{Ba}(v_{A_r} - v_{A\perp})v_{A\perp})u - \sqrt{3}\tilde{c}_{BB}v_{A\perp} + c_{BB}(v_{A_r} - v_{A\perp})\right]v_{A\perp}^2/L, \quad (A1b)$$

$$\frac{d(u^2)}{dt} = \left[\tilde{c}_{uu}(v_{A_r}^2 + 2v_{A\perp}^2 - v_{A\perp}^2)u - c_{uu}u^3\right]/L. \quad (A1c)$$

I need to determine three coefficients $c_{BB}$, $c_{BB}$, and $c_{uu}$ and prove the entire system (eqs. [A1a]–[A1c]) makes sense.

There are three kinds of analytical tests divided by the degree of their certainty. The tests from the first group have solid physical grounds. The tests from the second group represent how turbulence is believed to work, these are the general relations with clear physical insight. The third group of tests consists of the order-of-magnitude relations and the disputable ideas.
The tests of the first group are proven to work. Only one test of this kind can be applied to our system. This is the energy decay test. The total energy of free incompressible turbulence decreases with time, because energy decrease corresponds to the increase of entropy of the system gas/magnetic field (Landau et al. 1984),

$$\frac{d}{dt} \left( \frac{v_{\perp}^2 + 2v_{\perp\perp}^2 + u^2}{2} \right) < 0 \text{ for at least one of } v_{\perp\perp}, v_{\perp\perp}, u \text{ nonzero.} \quad (A2)$$

I take the sum with proper coefficients of the right-hand sides of the system (A1a)–(A1c). Then I maximize it with respect to $v_{\perp\perp}/v_{\perp\perp}$ and $v_{\perp}/u$. I find that when

$$2c_{Bu2} + c_{uB2} \geq -2.2, \quad (A3)$$

the total energy decreases with time for any nonzero $v_{\perp\perp}, v_{\perp\perp}$, and $u$. Let me remind the reader that all these velocities are nonnegative according to definitions in equation (24). Inequality (A3) is weak. Some tests from the second and the third categories constrain $c_{uB2}$ and $c_{Bu2}$ better, thus making inequality (A3) valid.

The typical test of the second category deals with dynamo amplification of the anisotropic field. Dynamo action not only amplifies the magnetic field, but also isotropizes it. I take the isotropization condition to be

$$\frac{d(v_{\perp\perp} - v_{\perp\perp})}{dt}(v_{\perp\perp} - v_{\perp\perp}) \leq 0. \quad (A4)$$

Taking expressions for derivatives from the system (A1a)–(A1c), I arrive at

$$(\dot{c}_{Bu} - 3c_{Bu2})u - \sqrt{3}c_{BB2}(v_{\perp\perp} + v_{\perp\perp}) + c_{BB2}(2v_{\perp\perp} + v_{\perp\perp}) \leq 0. \quad (A5a)$$

This condition should hold when any speed in inequality (A5a) is much larger than the two others. Therefore, inequality (A5a) is equivalent to

$$\dot{c}_{Bu} < 3c_{Bu2}, \quad c_{BB2} < \frac{\sqrt{3}}{2}c_{BB}. \quad (A5b)$$

Another second-category dynamo test states that the magnetic field should always increase if the dynamo operates without dissipation or any energy transfer. This occurs when Alfvén speeds are much smaller than the turbulent velocity field $u$. A positive amplification condition then reads

$$\frac{dv_{\perp\perp}}{dt} > 0, \quad \frac{dv_{\perp\perp}}{dt} > 0. \quad (A6)$$

Taking the expressions for derivatives from the system (A1a)–(A1c) and applying the limit $v_{\perp\perp} \ll u$ and $v_{\perp\perp} \ll u$, I find that the inequality (A6) is valid for any balance between $v_{\perp\perp}$ and $v_{\perp\perp}$ when

$$\dot{c}_{Bu} > 2c_{Bu2}. \quad (A7)$$

Equations (A5b) and (A7) give tight constrains on $c_{Bu2}$.

A similar test exists for the random velocity. The magnetic field is supposed to increase the turbulent velocity in the limit $v_{\perp\perp} \gg u$. The corresponding condition

$$\frac{d}{dt} \left( \frac{u^2}{2} \right) > 0 \text{ for } v_{\perp\perp} \sim v_{\perp\perp} \gg u \quad (A8)$$

reduces for the system of equations (A1a)–(A1c) to the condition of constant positive acceleration that an initially steady magnetic field applies to the matter. Finally,

$$c_{uB2} < \dot{c}_{uB}. \quad (A9)$$

The decay of isotropic MHD turbulence offers the following test of the second kind. Numerical simulations show equality of the magnetic field dissipation rate and the random velocity dissipation rate from equation (33) when the initial magnetic energy equals the initial kinetic energy. However, this equality should be stable, otherwise kinetic and magnetic energy would diverge from each other after any perturbation and equality of $u$ and $v_{\perp\perp}$ would not have been observed. The stability condition is

$$\frac{d(v_{\perp\perp}^2 + 2v_{\perp\perp}^2 - u^2)}{dt}(v_{\perp\perp}^2 + 2v_{\perp\perp}^2 - u^2) < 0 \quad (A10)$$

for $v_{\perp\perp} = v_{\perp\perp} = u$. 

The are no more proven or justified assumptions I can make. I need to make use of inequalities (A3), (A5b), (A7), (A9), and (A10) and apply unjustified tests. I take the value of \( \epsilon_{Ba2} \) to be in the middle of the allowed interval

\[
\epsilon_{Ba2} = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{3} \right) \epsilon_{Ba} \approx 0.29.
\] (A11)

The value of \( \epsilon_{AB} \) is small compared to the values of other coefficients. If \( \epsilon_{AB2} \ll (-\epsilon_{AB}) \), then the kinetic energy \( u^2 \) grows rapidly for the anisotropic magnetic field, which is unphysical. Turbulent velocity may be expected to increase, regardless of the direction of the magnetic field in equation (A1c). This idea leads to \( |\epsilon_{AB2}| < \epsilon_{AB} \). I take

\[
\epsilon_{AB2} = 0
\] (A12)

for the simple estimate. A similar estimate allows me to set

\[
\epsilon_{BB2} = 0.
\] (A13)

In this case the isotropization of the magnetic field has a timescale about equal to the dissipation timescale.

**APPENDIX B**

**SELF-SIMILAR SOLUTION**

Let me describe the self-similar solution, when the differential system of equations (52a)–(52h) can be reduced to the algebraic system. I set the proper scalings of quantities with radius and make weak additional assumptions. I introduce the standard dimensionless variables \( T(x) \), \( \rho(x) \), \( L(x) \), \( aa(x) \), \( bb(x) \), \( pp(x) \), and \( vel(x) \) to replace, respectively, \( T(r) \), \( \rho(r) \), \( L(r) \), \( u(r) \), \( v_{A}(r) \), \( v_{A\perp}(r) \), and \( v(r) \) as follows:

\[
T(r) = T_{\infty} T(x), \quad \rho(r) = vel(x) \left[ \frac{2RT(x)}{\mu} \right]^{1/2}, \quad L(r) = (r/x)L(x),
\]

\[
u(r) = aa(x) \left[ \frac{2RT(x)}{\mu} \right]^{1/2}, \quad v_{A}(r) = bb(x) \left[ \frac{2RT(x)}{\mu} \right]^{1/2}, \quad v_{A\perp}(r) = pp(x) \left[ \frac{2RT(x)}{\mu} \right]^{1/2}.
\] (B1)

The radius is normalized to the Bondi radius (eq. [54]) as \( r = r_{B} x \). The natural power-law radial dependencies of these quantities (eq. [B1])

\[
T(x) = T_{SS} x^{-1}, \quad vel(x) = v_{SS} x^{-1/2}, \quad L(x) = \gamma, \quad aa(x) = u_{SS} x^{-1/2}, \quad bb(x) = v_{A\perp SS} x^{-1/2}, \quad pp(x) = v_{A\perp SS} x^{-1/2}
\] (B2)

make my system of equations (52a)–(52h) independent of \( x \) under the following restrictions:

1. gravity is Newtonian,
2. external turbulence driving is negligible,
3. and the EOS is nonrelativistic.

These assumptions are valid in the intermediate region \( 10^3 r_{g} \leq x \leq 0.1 r_{B} \). Gravity is Newtonian for \( r \gg r_{g} \). Turbulence driving is mainly internal for \( r \leq 0.1 r_{B} \) (see §4.1 and 4.2 and Figures 1b and 5b). Electrons become relativistic at around \( 10^6 r_{g} \). The found range of \( r \) where all of the above assumptions hold is small. I can instead consider a nonrelativistic EOS with \( w = w_{NR} \) (eq. [52c]) everywhere. This makes a standard self-similar solution possible from 0.1 \( r_{B} \) down to several Schwarzschild radii \( r_{g} \).

The dimensionless magnetic helicity \( \xi \) appears to be constant in the self-similar regime. Equations (52h), (52g), and (46) lead to

\[
\xi = \frac{3\sigma_{\infty}}{4T_{SS}(v_{A\perp SS}^2 + 2v_{A\perp SS}^2)} \xi_{\infty}.
\] (B3)

The continuity equation (eq. [52a]) can be used to obtain the scaling of the density \( \rho \sim x^{-3/2} \). The heat balance equation (eq. [17]) reduces to the equality of radial and total perpendicular magnetic fields

\[
v_{SS}^2 = 2v_{A\perp SS}^2
\] (B4)

The Euler equation (eq. [52b]) gives the formula for the self-similar temperature

\[
T_{SS} = 5/(15 + 10u_{SS}^2 + 9v_{A\perp SS}^2 + 6v_{A\perp SS}^2 + 6v_{SS}^2).
\] (B5)

The turbulence evolution equations (22), (23), and (26) are now treated without source terms. They respectively give three relations

\[
2u_{SS}v_{A\perp SS}c_{Bu11} - 2v_{A\perp SS}c_{BB11} \exp(-\xi) + 4u_{SS}c_{Bu22}v_{A\perp SS} + 3v_{A\perp SS}c_{SS1}v_{A\perp SS} = 0,
\]

\[
2u_{SS}(v_{A\perp SS}c_{Bu22} + (c_{Bu11} + c_{Bu22})v_{A\perp SS}) - v_{A\perp SS}(2c_{BB11} \exp(-\xi)v_{A\perp SS} + 3c_{SS1}v_{A\perp SS} = 0,
\]

\[
-u_{SS}c_{uu} + c_{BB11} \exp(-\xi)(v_{A\perp SS}^2 + 2v_{A\perp SS}^2) = 0
\] (B6)
where the definitions of Alfvén and turbulent velocities from equation (24) are used.

Let me first set the magnetic helicity to zero, $\xi = 0$, and consider the four equations in equations (B4) and (B6) on four velocities $v_{SS}, u_{SS}, \gamma_{SS}, \gamma_{SS}, v_{A,SS}, \gamma_{A,SS}, v_{A,SS}, \gamma_{A,SS}$. The only solution of this system has all the velocities identical zeroes. No self-similar solution is possible for zero magnetic helicity $\xi$.

However, the nonlinear algebraic system of equations on $\xi$ and velocities in equations (B3), (B4), and (B6) possesses a nontrivial self-similar solution. For the full system from equations (52a)–(52h), I need the additional condition to determine the accretion rate and solve for radial dependencies of quantities. This condition is either a condition for the maximum accretion rate from equation (64) or a condition for the effective angular momentum transport from inequalities (61a) and (61b). I can transform both into self-similar form. The maximum $M$ condition (eq. [64]) reads

$$5 + 10u_{SS}^2 + 12v_{A,SS} = 6v_{SS},$$  \hspace{1cm} (B7a)

The effective angular momentum transport condition (eqs. [61a] and [61b]) gives

$$\frac{\sqrt{5/3}v_{SS}}{4v_{A,SS}v_{A,SS}v_{SS}} \leq 1,$$  \hspace{1cm} (B7b)

regardless of the circularization radius $r_{cirk}$. Let me first find the self-similar solution in the large angular momentum case. I solve the equality in equation (B7b) and five equations (B3), (B4), (B5), and (B6) for seven quantities: $\xi, T_{SS}, u_{SS}, v_{A,SS}, v_{A,SS}, \gamma_{SS}, \gamma_{SS}$, and the product $\sigma \xi$. I normalize the results to freefall velocity (eq. [67]) to be able to directly compare these with the numbers in Figure 5b:

$$c(r) = 0.58, \frac{u(r)}{\sqrt{3}v_{SS}} = 0.0094, \frac{v_{A,r}(r)}{v_{r}(r)} = 0.041, \frac{v_{A,\perp}(r)}{v_{\perp}(r)} = 0.029, \frac{v(r)}{v_{\perp}(r)} = 0.0033, \sigma_{\infty} \xi = 0.00718$$  \hspace{1cm} (B8)

for $r \gg r_g$. Figure 5b shows profiles of velocities for the reference model with $\sigma_{\infty} = 1, \xi_{\infty} = 0.025, and \gamma = 1$. The actual velocities on the inner boundary at $r = 3 \times 10^{-3} r_B = 90 r_g$ are

$$c(r) = 0.58, \frac{u(r)}{\sqrt{3}v_{SS}} = 0.0033, \frac{v_{A,r}(r)}{v_{r}(r)} = 0.076, \frac{v_{A,\perp}(r)}{v_{\perp}(r)} = 0.024, \frac{v(r)}{v_{\perp}(r)} = 0.0051.$$  \hspace{1cm} (B9)

The reference model has $\sigma_{\infty} \xi = 0.025$, which is about 3 times larger than in the self-similar solution (eq. [B8]); the magnetic field in the reference model is stronger. Therefore, higher values of all characteristic velocities are expected in the actual solution (eq. [B9]). I obtain the inflow velocity $v$ and radial Alfvén speed $v_{A,r}$, correspondingly, 1.5 and 1.8 times higher for the solution in equation (B9). Sonic speeds are the same in the self-similar (eq. [B8]) and actual (eq. [B9]) solutions, because almost all gravitational energy goes into thermal energy in both cases. However, the perpendicular Alfvén velocity $v_{A,\perp}$ and turbulent velocity $u$ do not qualitatively agree with the self-similar solution. They are correspondingly 1.2 and 2.8 times lower in the actual solution (eq. [B9]). The naive estimate for the accretion rate is

$$4\pi \rho_{\infty}(r_B)^2 r_B^2 \approx 0.05 \dot{M}_B.$$  \hspace{1cm} (B10)

This appears to be 8 times larger than the actual accretion rate $0.0061 \dot{M}_B$. The velocity near the Bondi radius (eq. [54]) is much smaller than the self-similar value, which leads to an overestimate of $\dot{M}$. Thus, the self-similar solution can give order-of-magnitude estimates for all characteristic velocities of the flow and even for the accretion rate $\dot{M}$. However, the self-similar solution has only two free parameters instead of three, because $\sigma_{\infty} \xi$ is treated as one constant. Therefore, the solution of the full system from equations (52a)–(52h) is required to probe the entire parameter space and to achieve more precise results.

The self-similar solution does not exist for a nonrotating flow when the condition from equation (B7a) holds. The formal solution of equations (B7a), (B3), (B4), and (B6) leads to a negative product $\sigma_{\infty} \xi$. The absence of a self-similar solution in this case is reasonable, since the actual solution does not exhibit self-similar scalings (Fig. 1b).

APPENDIX C

CONVECTION

Let me elaborate on the stability criterion against convection in my model. As I noted in the main text (§ 5.4), small-scale perturbations of quantities are smeared out by diffusion. Thus, the high-frequency analysis by Scharlemann (1983) is not appropriate to determine the convective stability. The timescale of diffusion $\tau_{\text{diff}}$ is

$$\tau_{\text{diff}} \sim \frac{h}{u},$$  \hspace{1cm} (C1)

where $l$ is the scale of perturbation. As $h$ decreases, the diffusion time also decreases and becomes smaller than the perturbation growth timescale $\tau_{\text{grow}}$. If $\tau_{\text{diff}} < \tau_{\text{grow}}$, convection is ineffective, which is likely to happen at small scales $h$. Thus, I need to consider the motion of the large blobs of the size $h \sim L$.

I consider a blob of plasma displaced at some small $\Delta r$ from its equilibrium position (Fig. 7). The density of the blob itself changes by $\Delta \rho_{\text{blob}}$ when it is moved. The density of the outer medium changes by $\Delta \rho_{\text{fluid}}$ between the two positions of the blob. The goal is to
calculate the difference in density differences $\Delta \rho_{\text{fluid}} - \Delta \rho_{\text{blob}}$ between the outer medium and the blob. The positive difference $\Delta \rho_{\text{fluid}} - \Delta \rho_{\text{blob}} > 0$ for a positive $\Delta r > 0$ implies convective instability. The rising blob of gas is rarefied compared to the fluid and is buoyant. The results for $\Delta \rho$ may be affected by an external driving that is somewhat artificial in my model. Thus, I need to calculate $\Delta \rho$ in the inner accretion region where external driving is not important. The motion of the blob is adiabatic and governed by the same adiabatic dynamical equations (52b) and (52c) as the rest of the fluid. I neglect energy, which is associated with gas regular velocity $v$.

The term $v^2$ cannot be neglected in the region where $v$ approaches sound speed $c_s$. However, convection ceases if $v \sim c_s$ (Narayan et al. 2002). I denote by index A the physical quantities in the blob and by index F quantities in the rest of the fluid.

The Euler equation (eq. [52b]) results in the following equations on differences in the blob,

$$\frac{R}{\mu} \Delta_A(\rho T) + \frac{1}{3} \Delta_A(\rho u^2) + \frac{1}{r^2} \Delta_A(r^2 \rho v_\bot^2) - \frac{1}{2r^4} \Delta_A(r^4 \rho v_\bot^2) = 0,$$

and in the fluid,

$$\frac{R}{\mu} \Delta_F(\rho T) + \frac{1}{3} \Delta_F(\rho u^2) + \frac{1}{r^2} \Delta_F(r^2 \rho v_\bot^2) - \frac{1}{2r^4} \Delta_F(r^4 \rho v_\bot^2) = 0.$$

In both equations I take variations between quantities at $r + \Delta r$ and $r$. I introduce the difference operator

$$\Delta(\ldots) = \Delta r(\ldots) - \Delta_A(\ldots)$$

and calculate the variations of all quantities between the fluid and the blob. Subtracting equation (C2b) from equation (C2a), I find the radial pressure balance in the first order in $\Delta r$,

$$\frac{R}{\mu} \Delta(\rho T) + \frac{1}{3} \Delta(\rho u^2) + \Delta(\rho v_\bot^2) - \frac{1}{2} \Delta(\rho v_\bot^2) = 0.$$

The blob of plasma should also be in equilibrium in the perpendicular direction, not only in the radial direction. I use the same technique as I used to derive the radial force equation (eq. [13]) from the general momentum equation (eq. [3]) to deduce it. The component $\theta$ of the magnetic force in equations (2) and (3) reads $F_\theta = \frac{B \times (\nabla \times B)}{\mu_0(4\pi \rho)}$. I subtract $B_\theta(\nabla \times B)/(4\pi \rho)$ from it and average over the $\phi$-direction. I obtain

$$F_\theta = \frac{(B_\theta^2)_{\theta}}{8\pi \rho r}$$
for $B_0^2 = B_a^2$ and $B_\perp B_\parallel = 0$ on average over $\phi$. The final form of the force balance in the $\theta$-direction is

$$
\frac{\partial}{\partial \theta} \left( R \mu \frac{T}{\rho} + \frac{1}{3} \rho u^2 + \frac{1}{2} \rho v^2 \right) = 0.
$$

(C6)

The perpendicular force balance (eq. [C6]) has the same form in any direction perpendicular to the radial vector owing to the symmetry of the problem. I apply operator $\Delta$ (eq. [C3]) to the integral form of the perpendicular pressure balance and get

$$
\frac{R}{\mu} \Delta \left( \frac{1}{3} \Delta (\rho T) + \frac{1}{2} \Delta (\rho v^2) \right) = 0.
$$

(C7)

The heat balance equation (eq. [17]) gives the third relation

$$
\frac{R}{\mu} \left( \frac{3}{2} \Delta T - \frac{\Delta \rho}{\rho} \rho \right) + 2 \left( u \Delta u - \frac{u^2}{3} \Delta \rho \right) + \rho \Delta \left( \frac{v^2_{\perp}}{\rho} \right) + \frac{1}{2} \Delta \left( \rho v^2_{\parallel} \right) = 0.
$$

(C8)

The expansion or contraction of a blob is nonuniform. Perpendicular $b$ and parallel $a$ sizes (Fig. 7) deform in different ways. The continuity equation for the fluid (eq. [9]) can be written as

$$
\frac{\Delta \rho}{\rho} + \frac{\Delta v}{v} + 2 \frac{\Delta r}{r} = 0.
$$

(C9a)

I consider the parcel with constant mass, $m = \rho V$. Therefore,

$$
\frac{\Delta \rho}{\rho} + \frac{\Delta a}{a} + 2 \frac{\Delta b}{b} = 0
$$

(C9b)

is the continuity relation for the parcel. Finally, I subtract equation (C9a) from equation (C9b) and obtain

$$
\frac{\Delta \rho}{\rho} + \frac{\Delta r}{r} + 2 \frac{\Delta r}{r} = 0
$$

(C10)

for the change in density according to equation (C3). The inflow velocity $v$ is clearly associated with the fluid, but I omit subscript $F$ at $v$. I also omit subscript $A$ at the dimensions of the blob.

Now I need to quantify the variation of the turbulent magnetic field and the random velocity. I assume that the blob moves at a speed $V(r)$ much higher than the inflow velocity $V'(r) \gg v(r)$; therefore, the magnetic field does not dissipate in the parcel. Differences of the turbulence evolution equations (52d), (52e), and (52f) are

$$
2u \Delta u - 2 \frac{u^2}{3} \frac{\Delta \rho}{\rho} = \frac{\Delta r}{vL} \left[ c_{uw} u^3 - c_B B_{11} \left( v_{\parallel}^2 + 2 v_{\perp}^2 \right) u \exp (-\xi) \right],
$$

(C11a)

$$
\Delta \left( \rho v_{\perp}^2 \right) + 4 \rho v_{\perp}^2 \left( \frac{\Delta r}{r} - \frac{\Delta b}{b} \right) = \frac{\Delta r}{vL} \left[ c_{BB_11} v_{\perp}^3 \exp (-\xi) - (c_{B11} v_{\perp}^2 + 2 c_{B22} v_{\perp}^2) u \right],
$$

(C11b)

$$
\Delta \left( \rho v_{\perp}^2 \right) + 2 \rho v_{\perp}^2 \left( \frac{\Delta r}{r} - \frac{\Delta a}{a} - \frac{\Delta b}{b} \right) = \frac{\rho \Delta r}{vL} \left[ c_{BB_11} v_{\perp}^3 \exp (-\xi) - (c_{B11} + c_{B22}) v_{\perp}^2 - c_{B22} v_{\perp}^2 - c_{B22} v_{\perp}^2 \right] u.
$$

(C11c)

The magnetic helicity variation does not directly influence the dynamics of the blob. Solving the system of seven equations (C4), (C7), (C8), (C10), (C11a), (C11b), and (C11c) on seven quantities $\Delta T, \Delta \rho, \Delta v_A, \Delta v_{\perp}, \Delta \mu, \Delta a, \Delta b$, I obtain

$$
\frac{\Delta \rho_{\text{correct}}}{\rho \Delta r} \approx 2.02 \exp (-\xi) v_{\parallel} v_{\perp} (v_{\parallel} + 2 v_{\perp}) - u \left[ 0.39 (v_{\parallel}^2 + v_{\perp}^2) + v_{\perp} (1.21 v_{\perp}^2 - 0.63 u^2) \right] \frac{c_{L}^2}{v_{\perp}^2 + v_{\perp}^2}.
$$

(C12)

The actual expression is much longer. I take only the largest terms in the numerator and the denominator.

Let me compare this result (eq. [C12]) with the naive estimate, when magnetic field dissipation increases the gas internal energy only (Bisnovatyi-Kogan & Ruzmaikin 1974), and the gas pressure balance is used instead of parallel and perpendicular pressure balances (C4), (C7). The gas pressure balance is

$$
\Delta (\rho T) = 0.
$$

(C13)

The naive heat balance (eq. [16]) for the unit mass is

$$
\frac{R}{\mu} \left( \frac{3}{2} \rho \Delta T - T \Delta \rho \right) \approx \frac{\Delta r}{vL} \left[ 0.41 v_{\parallel}^2 u + 1.16 v_{\parallel} v_{\perp} + 1.4 v_{\perp}^2 - 3.03 (v_{\parallel} + 2 v_{\perp}) \exp (-\xi) - 1.14 u^3 \right]
$$

(C14)
Eliminating $\Delta T$ from equations (C13) and (C14), I find

$$\frac{1}{\rho} \frac{\Delta \rho_{\text{naive}}}{\Delta r} \approx \frac{0.61 (r_\alpha^3 + 2r_\perp^3) \exp(-\xi) + 0.23 u^3 - 0.82 r_\perp^2 u - 0.23 v_{\theta \perp} u_{\perp} - 0.28 v_{\perp}^2}{c_s^2 L_\perp}. \quad (C15)$$

I evaluate the convective derivatives of density (eqs. [C12] and [C15]) in the inner region of the reference solution with angular momentum transport (§ 4.2). Parameters of the reference model are $\xi_\perp = 0.025$, $\sigma_\perp = 1$, $\gamma = 1$, and a nonrelativistic EOS. Corresponding velocities are shown in Figure 5(b). I take the values (from eq. [B9]) of velocities and the magnetic helicity on the inner boundary of integration at $r = 3 \times 10^{-3} r_B \approx 90 r_g$. The change in density appears to be negative $\Delta \rho < 0$ for $\Delta r > 0$ in the result of the full calculation (eq. [C12]). The naive calculation shows positive $\Delta \rho > 0$ for $\Delta r > 0$.

$$\frac{\Delta \rho_{\text{correct}}}{\Delta \rho_{\text{naive}}} \approx -0.2. \quad (C16)$$

The naive calculation suggests that the flow is convectively unstable, whereas the full calculation under reasonable assumptions indicates a convectively stable flow.

The calculated result (eq. (C16)) is applicable only to the inner regions of solution with angular momentum transport (§ 4.2). Excluded external driving is important in the outer regions. In turn, the solution with the maximum accretion rate has a large inflow velocity $v$ that approaches the gas sound speed $c_s$, and convection is suppressed (§ 5.4). As a bottom line, either flow appears to be convectively stable on average or convection is suppressed in all calculated solutions without electron conductivity.

However, numerical simulations by Igumenshchev (2006) of nonrotating flows find evidence of convection. This convection may be physical. My model averages heat from all dissipation events over the fluid. Local reconnection events can lead to burst-type local heating that leads to the buoyancy of blobs. In addition, magnetic buoyancy and diffusion play an important role in the transfer processes (Igumenshchev 2006). The correct inclusion of convection, magnetic buoyancy, and diffusion is the subject of future studies.

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