HERMITIAN AZUMAYA MODULES AND ARITHMETIC CHERN CLASSES

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Abstract. We compute arithmetic Chern classes of sheaves on an arithmetic surface \(X\) associated to a Hermitian Azumaya algebra.

Introduction

Let \(\pi : X \to Y\) be an arithmetic surface and let \(A\) be an Azumaya algebra on \(X\). In [Ree15] we introduced the notion of Hermitian Azumaya algebras and Hermitian Azumaya modules by equipping certain sheaves with Hermitian metrics. Using this, we defined a Deligne pairing \(\langle M, N \rangle_A\) for Hermitian \(A\)-line bundles \(M\) and \(N\), which generalized the classical Deligne pairing for line bundles on \(X\).

In this note we introduce arithmetic Chern classes for Hermitian Azumaya modules and compute the first arithmetic Chern class of the Deligne pairing for \(A\)-line bundles.

Our main result shows, that we can compute the first arithmetic Chern class of the \(A\)-Deligne pairing in terms of the arithmetic \(A\)-Chern classes. Explicitly is says:

\[ \hat{c}_1(\langle M, N \rangle_A) = -\pi_*(\hat{c}_1^A(M) \hat{c}_1^A(N)). \]

The structure of this paper is as follows: In section 1 we recall some facts about Hermitian vector spaces, the construction of Hermitian inner products on associated vector spaces and isometric vector spaces. In section 2 we recall the definition of Hermitian Azumaya algebras and compute the first arithmetic Chern class of a Hermitian Azumaya algebra. In the final section 3 we introduce arithmetic Chern classes for Hermitian Azumaya modules and proof the main result.

1. Hermitian vector spaces

Definition 1.1. A Hermitian vector space \(V\) is a pair \((V, h)\), where \(V\) is a finite dimensional \(\mathbb{C}\)-vector space and \(h : V \times V \to \mathbb{C}\) is a Hermitian inner product.

The inner product \(h\) induces the so called Riesz isomorphism between \(V\) and the dual space \(V^\vee = Hom_{\mathbb{C}}(V, \mathbb{C})\) defined by

\[ \theta_V : V \to V^\vee, v \mapsto h(-, v). \]

Note that this a conjugate linear isomorphism. Using this isomorphism we define the associated dual hermitian vector space \(V^\vee\) to be the pair \((V^\vee, h^\vee)\), where \(h^\vee\) is the induced dual inner product defined by:

\[ h^\vee(f, f') := \overline{h(\theta_V^{-1}(f), \theta_V^{-1}(f'))} \quad \text{for} \quad f, f' \in V^\vee. \]

This inner product gives the Riesz isomorphism \(\theta_{V^\vee} : V^\vee \to V^{\vee\vee}\).

Iterating this construction we get the induced bidual Hermitian vector space \(V^{\vee\vee} = (V^{\vee\vee}, h^{\vee\vee})\).

Given two Hermitian vector space \(V = (V, h)\) and \(W = (W, k)\) we say \(\Psi : V \to W\) induces an isometry of Hermitian vector spaces if \(\Psi\) is an isomorphism of \(\mathbb{C}\)-vector spaces.

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and further more $Ψ^*k = h$, that is we have, for any two $v, v' ∈ V$ the equality $h(v, v') = k(Ψ(v), Ψ(v'))$.

**Lemma 1.2.** The natural isomorphism $ι : V → V^{**}$ induces an isometry of Hermitian vector spaces

$$ι : V → V^{**}.$$  

**Proof.** We note that we have $ι = θ_V^{**} ◦ θ_V$. This can be see as follows: for arbitrary $v ∈ V$ and $f ∈ V^*$ we have

$$((θ_V^{**} ◦ θ_V)(v))(f) = h^*(f, θ_V(v))$$

$$= h(θ_V^{-1}(f), θ_V^{-1}(θ_V(v)))$$

$$= h(v, θ_V^{-1}(f))$$

$$= f(v) = ι(v)(f).$$

Using this fact we can compute that we have $ι^*h^{**} = h$, hence $ι$ is an isometry between $V$ and $V^{**}$.  

Equip the tensor product $V ⊗ W$ with the induced tensor product metric $h ⊗ k$ defined on elementary tensors by:

$$h ⊗ k(v ⊗ w, v' ⊗ w') := h(v, v')k(w, w').$$

This construction defines the Hermitian vector space $V ⊗ W$.

**Remark 1.3.** The Riesz isomorphism $θ_{V ⊗ W}$ induced by $h ⊗ k$ has the following decomposition:

$$θ_{V ⊗ W} = α ◦ (θ_V ⊗ θ_W).$$

Here $α : V^* ⊗ W^* → (V ⊗ W)^*$ is the natural isomorphism defined on elementary tensors by

$$f ⊗ g ↦ (v ⊗ w ↦ f(v)g(w)).$$

**Lemma 1.4.** The natural isomorphism $α : V^* ⊗ W^* → (V ⊗ W)^*$ induces an isometry of Hermitian vector spaces

$$α : V^* ⊗ W^* → (V ⊗ W)^*.$$  

**Proof.** We have to show that $α^*(h ⊗ k)^* = h^* ⊗ k^*$. Using the aforementioned decomposition of $θ_{V ⊗ W}$ we have:

$$α^*(h ⊗ k)^*(f ⊗ g, f' ⊗ g') = h ⊗ k(θ_{V ⊗ W}^{-1}(α(f ⊗ g)), θ_{V ⊗ W}^{-1}(α(f' ⊗ g')))$$

$$= h ⊗ k(θ_V^{-1} ◦ θ_W^{-1}(f ⊗ g), θ_V^{-1} ◦ θ_W^{-1}(f' ⊗ g'))$$

$$= h(θ_V^{-1}(f), θ_V^{-1}(f'))k(θ_W^{-1}(g), θ_W^{-1}(g'))$$

$$= h^*(f, f')k^*(g, g').$$

**Remark 1.5.** We also note that the natural isomorphism

$$V ⊗ W → W ⊗ V$$

induces an isometry of Hermitian vector spaces.

Composing all these natural isomorphisms we get a natural isomorphism

$$V^* ⊗ W ≅ V^* ⊗ W^{**} ≅ W^{**} ⊗ V^* ≅ (W^* ⊗ V)^*$$

which, as we showed, induces in fact an isometry of Hermitian vector spaces:

$$V^* ⊗ W ≅ (W^* ⊗ V)^*.$$
Using the natural isomorphisms
\[ V^\vee \otimes W \cong \text{Hom}(V, W) \quad \text{and} \quad W^\vee \otimes V \cong \text{Hom}(W, V), \]
the isomorphism \([\ ]\) gives rise to a natural isomorphism
\[ \text{Hom}(V, W) \xrightarrow{\cong} \text{Hom}(W, V)^\vee. \]
It is well known that this isomorphism is nothing but the trace map, i.e. it comes from the perfect pairing:
\[ \text{Hom}(V, W) \times \text{Hom}(W, V) \xrightarrow{(\ -) \circ (-)} \text{Hom}(V, V) \xrightarrow{\text{tr}} \mathbb{C} \]
and it maps \( \phi : V \to W \in \text{Hom}(V, W) \) to \( \text{tr}((-) \circ \phi) \in \text{Hom}(W, V)^\vee \).

The natural isomorphism \( V^\vee \otimes W \xrightarrow{\cong} \text{Hom}(V, W) \) also defines a Hermitian inner product on \( \text{Hom}(V, W) \) which in turn defines the Hermitian vector space \( \text{Hom}(V, W) \).

Putting all these steps together we have proven:

**Theorem 1.6.** The trace map \( \text{tr} : \text{Hom}(V, W) \to \text{Hom}(W, V)^\vee \) induces an isometry of Hermitian vector spaces
\[ \text{tr} : \text{Hom}(V, W) \xrightarrow{\cong} \overline{\text{Hom}(W, V)}^\vee. \]

**Remark 1.7.** All results in this section can be immediately generalized to Hermitian vector bundles, that is pairs \((E, h)\), where \( E \) is a complex vector bundle on a smooth manifold and \( h \) is a Hermitian metric on \( E \), that is a Hermitian inner product on each fiber.

### 2. Hermitian Azumaya algebras

**Definition 2.1.** We call a two-dimensional integral regular projective flat \( \mathbb{Z} \)-scheme \( X \) an arithmetic surface and denote the structure morphism by \( \pi : X \to Y \), here we define \( Y := \text{Spec}(\mathbb{Z}) \).

We denote the generic fiber of \( \pi \) by \( X_{\mathbb{Q}} \) and the base change to \( \mathbb{C} \) by \( X_{\mathbb{C}} \). Both \( X_{\mathbb{Q}} \) and \( X_{\mathbb{C}} \) are smooth curves over \( \mathbb{Q} \) resp. \( \mathbb{C} \).

Given a sheaf of \( \mathcal{O}_X \)-modules \( \mathcal{F} \), then we denote the induced sheaves on \( X_{\mathbb{Q}} \) and \( X_{\mathbb{C}} \) by \( \mathcal{F}_{\mathbb{Q}} \) and \( \mathcal{F}_{\mathbb{C}} \).

The associated Riemann surface \( S \) of \( X \) is given by \( S = X_{\mathbb{C}}(\mathbb{C}) \). \( S \) comes endowed with a Hermitian metric \( \omega \), which is Kähler. If \( \mathcal{F} \) is a locally free \( \mathcal{O}_X \)-module, then we denote the induced vector bundle on \( S \) by \( F \).

**Definition 2.2.** A Hermitian vector bundle \( \overline{E} \) on \( X \) is a pair \((E, h)\), where \( E \) is a locally free sheaf on \( X \) and \( h \) is a Hermitian metric on the induced vector bundle \( E \) on \( S \), which is invariant under the complex conjugation on \( S \).

**Remark 2.3.** If we use a Hermitian metric in the following, we will always assume that the metric is invariant with respect to the complex conjugation on \( S \), see [Sou92, Definition IV.4.1.4.].

Let \( \mathcal{A} \) be an Azumaya algebra on the arithmetic surface \( X \), that is a matrix algebra in the étale topology, then we have
\[ \mathcal{A}_{\mathbb{C}} \cong \text{End}_{\mathcal{O}_{X_{\mathbb{C}}}}(\mathcal{E}) \]
for some locally free sheaf \( \mathcal{E} \) on \( X_{\mathbb{C}} \). This fact is due to Tsen’s theorem. The sheaf \( \mathcal{E} \) induces a vector bundle \( E \) on \( S \), especially we have \( A \cong \text{End}(E) \cong E^\vee \otimes E \) on \( S \).

Choosing a Hermitian metric \( h \) on \( E \) induces the metric \( h^\vee \otimes h \) on \( E^\vee \otimes E \), and hence also a metric \( h^A \) on \( A \). This leads to the following definition:

**Definition 2.4.** Let \( X \) be an arithmetic surface. A Hermitian Azumaya algebra \( \overline{\mathcal{A}} \) over \( X \) is a pair \((A, h^A)\), here \( A \) is Azumaya algebra on \( X \) and \( h^A \) a Hermitian metric on the associated vector bundle constructed as described above.
Theorem 2.5. Let $X$ be an arithmetic surface and let $\mathcal{A}$ be a Hermitian Azumaya algebra on $X$, then we have:

$$2\hat{c}_1(\mathcal{A}) = 0 \in CH^1(X).$$

Proof. The trace pairing for the algebra $\mathcal{A}$ on $X$, given by

$$\text{tr} : \mathcal{A} \to \mathcal{A}^\vee, \ a \mapsto (b \mapsto \text{tr}(ba)),$$

is an isomorphism. This can be checked étale locally, where it reduces to the fact that $\mathcal{A}$ becomes a matrix algebra and this algebra is self-dual with respect to the trace map.

Now the Hermitian Azumaya algebra $\mathcal{A}$ induces a Hermitian vector bundle $\mathcal{A}^\vee$ via the dual metric. Theorem 1.6 says that the trace induces an isometric isomorphism

$$\text{tr} : \mathcal{A} \cong \text{End}(E) \xrightarrow{\sim} \text{End}(E)^\vee \cong \mathcal{A}^\vee,$$

by our choice of the metric on the Hermitian Azumaya algebra.

We especially have

$$\text{tr}^*((h^A)^\vee) = h^A,$$

i.e. $h^A$ is the metric induced by the trace map and $(h^A)^\vee$.

On the one hand we now compute

$$\hat{c}_1(\mathcal{A}^\vee) = \hat{c}_1(\mathcal{A}).$$

This follows from the above and [GS90, Proposition 1.2.5, Theorem 4.8.(ii)] since $h^A$ is the metric induced by the metric on $\mathcal{A}^\vee$ via the trace map.

On the other hand we have, using [GS90, 4.9.],

$$\hat{c}_1(\mathcal{A}^\vee) = -\hat{c}_1(\mathcal{A}).$$

Combining these results gives:

$$\hat{c}_1(\mathcal{A}) = \hat{c}_1(\mathcal{A}^\vee) = -\hat{c}_1(\mathcal{A}),$$

or equivalently

$$2\hat{c}_1(\mathcal{A}) = 0.$$

\[\square\]

Corollary 2.6. Let $X$ be an arithmetic surface and let $\mathcal{A}$ be a Hermitian Azumaya algebra on $X$, then $\hat{c}_1(\mathcal{A}) = 0$ if $CH^1(X)$ is torsion-free.

Corollary 2.7. Let $X$ be an arithmetic surface and let $\mathcal{A}$ be a Hermitian Azumaya algebra on $X$, then $\hat{c}_1(\mathcal{A}) = 0 \in CH^1(X)_\mathbb{Q}$, especially $\hat{c}_1(\mathcal{A}) = rk(\mathcal{A}) - \hat{c}_2(\mathcal{A}) \in CH^1(X)_\mathbb{Q}$.

3. The first arithmetic Chern class of the Deligne pairing

Let $X$ be an arithmetic surface. If $\mathcal{A}$ is a Hermitian Azumaya algebra on $X$ and $\mathcal{M}$ is a locally projective left $\mathcal{A}$-module, then using $\mathcal{A}$ and Morita equivalence, we see that

$$\mathcal{M}_C = \mathcal{E} \otimes_{X_C} \mathcal{M}'$$

for some locally free sheaf $\mathcal{M}'$ on $X_C$ which induces a vector bundle $\mathcal{M}'$ on $S$, so the induced vector bundle $\mathcal{M}$ on $S$ is given by $\mathcal{M} = E \otimes \mathcal{M}'$.

The vector bundle $E$ still comes with the Hermitian metric $h$ and we furthermore pick a Hermitian metric $h'$ on $\mathcal{M}'$. The tensor product metric of $h$ and $h'$ yields a Hermitian metric $h^\mathcal{M} := h \otimes h'$ on $\mathcal{M}$. This defines a Hermitian locally free sheaf $\mathcal{M} = (\mathcal{M}, h^\mathcal{M})$, which also has the structure of a locally projective left $\mathcal{A}$-module. This suggests the following definition:

Definition 3.1. Let $X$ be an arithmetic surface and let $\mathcal{A}$ be a Hermitian Azumaya algebra over $X$. A Hermitian Azumaya module $\mathcal{M}$ is a couple $(\mathcal{M}, h^\mathcal{M})$ where $\mathcal{M}$ is a locally projective $\mathcal{A}$-module and $h^\mathcal{M}$ is a Hermitian metric on the associated vector bundle on $S$ chosen as described above.
Assume $\mathcal{M}$ and $\mathcal{N}$ are two Hermitian Azumaya modules, then $\text{Hom}_A(\mathcal{M}, \mathcal{N})$ is locally free as an $\mathcal{O}_X$-module as both modules are locally projective over $A$. We remember that we have $\mathcal{A}_C \cong \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$ on $X_C$ and $\mathcal{M}_C \cong \mathcal{E} \otimes \mathcal{M}'$ as well as $\mathcal{N}_C \cong \mathcal{E} \otimes \mathcal{N}'$ by Morita equivalence. This equivalence also gives a natural isomorphism

$$\text{Hom}_{\mathcal{A}_C}(\mathcal{M}_C, \mathcal{N}_C) \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{M}', \mathcal{N}').$$

The vector bundle associated to the last sheaf is isomorphic to $(\mathcal{M}')^\vee \otimes \mathcal{N}'$.

This bundle comes naturally equipped with the Hermitian metric $(h')^\vee \otimes h''$, the one given by the Hermitian metrics $h'$ and $h''$ on the vector bundles $\mathcal{M}'$ and $\mathcal{N}'$. So we also have a Hermitian metric $h(\mathcal{M}, \mathcal{N})$ on the vector bundle associated to $\text{Hom}_A(\mathcal{M}, \mathcal{N})$. This construction defines the Hermitian vector bundle

$$\text{Hom}_A(\mathcal{M}, \mathcal{N}) := (\text{Hom}_A(\mathcal{M}, \mathcal{N}), h(\mathcal{M}, \mathcal{N})).$$

The trace map and the tensor-hom adjunction give the following natural isomorphism of locally free $\mathcal{O}_X$-modules:

$$A \otimes \text{Hom}_A(\mathcal{M}, \mathcal{N}) \cong A^\vee \otimes \text{Hom}_A(\mathcal{M}, \mathcal{N})$$

$$\cong \text{Hom}_A(A \otimes \mathcal{M}, \mathcal{N})$$

$$\cong \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \text{Hom}_A(\mathcal{A}, \mathcal{N})) \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}).$$

**Lemma 3.2.** The natural isomorphism $A \otimes \text{Hom}_A(\mathcal{M}, \mathcal{N}) \rightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ induces an isometry of Hermitian vector bundles

$$A \otimes \text{Hom}_A(\mathcal{M}, \mathcal{N}) \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}).$$

**Proof.** Looking at all the isomorphisms and the naturally induced metrics on the vector bundles on $S$, this boils down to the following isometry

$$(E^\vee \otimes E \otimes M'^\vee \otimes N', h'^\vee \otimes h \otimes (h')^\vee \otimes h'') \cong ((E \otimes M')^\vee \otimes E \otimes N', (h \otimes h')^\vee \otimes h \otimes h'')$$

which follows from 1.4 and 1.3. □

**Corollary 3.3.** Let $X$ be an arithmetic surface and let $\overline{\mathcal{A}}$ be a Hermitian Azumaya algebra on $X$, if $\mathcal{M}$ and $\mathcal{N}$ are Hermitian Azumaya modules, the we have the following equality:

$$\overline{\text{ch}}(\overline{\mathcal{A}}) \cdot \text{ch}(\text{Hom}_A(\mathcal{M}, \mathcal{N})) = \overline{\text{ch}}(\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}))$$

**Proof.** By 3.2 the Hermitian vector bundles $A \otimes \text{Hom}_A(\mathcal{M}, \mathcal{N})$ and $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ are isometric, so the properties of the arithmetic Chern classes give the desired result. □

**Definition 3.4.** Let $X$ be an arithmetic surface and let $\overline{\mathcal{A}}$ be a Hermitian Azumaya algebra on $X$, then we define the arithmetic $\mathcal{A}$-Chern character of a Hermitian Azumaya module $\mathcal{M}$ by:

$$\overline{\text{ch}}^A(\mathcal{M}) := \text{ch}(\mathcal{M}) \cdot \overline{\text{ch}}(\mathcal{A})^{-\frac{1}{2}}$$

and the first arithmetic $\mathcal{A}$-Chern class by

$$\hat{c}_1^A(\mathcal{M}) := (\overline{\text{ch}}^A(\mathcal{M}))^{(1)}.$$  

**Remark 3.5.** By the definition of the arithmetic $\mathcal{A}$-Chern character and by the choice of the induced metrics on all sheaves involved, we see:

$$\overline{\text{ch}}(\text{Hom}_A(\mathcal{M}, \mathcal{N})) = \overline{\text{ch}}^A(\mathcal{M}) \cdot \overline{\text{ch}}^A(\mathcal{N}).$$

Explicitly we have

$$\overline{\text{ch}}(\mathcal{A})^{-\frac{1}{2}} = \frac{1}{\sqrt{rk(\mathcal{A})}} - \frac{1}{2\sqrt{rk(\mathcal{A})^3}} \hat{c}_2(\mathcal{A}).$$

This computation shows:

$$\hat{c}_1^A(\mathcal{M}) = \frac{1}{\sqrt{rk(\mathcal{A})}} \hat{c}_1(\mathcal{M}).$$
A similar explicit computation is possible for \( \hat{c}^A(\mathcal{M}) \).

Since \( \mathcal{A} \) is an Azumaya algebra, \( rk(\mathcal{A}) \) is a square, so all these classes are well defined in \( CH(X)_Q \).

Now we want to compute some arithmetic Chern classes, using the arithmetic Riemann-Roch theorem, due to Gillet and Soulé, see [GS92]. As \( X \) is an arithmetic surface, we have \( dim(X) - dim(Y) = 1 \), so we can use the following version of the arithmetic Riemann-Roch theorem for a Hermitian vector bundle \( \mathcal{E} \) on \( X \), see [GS91] Conjecture 1.5):

\[
\hat{c}_1(\hat{\lambda}(\mathcal{E})) = \pi_*(\hat{c}_h(\mathcal{E}) \hat{T}d(\overline{T_{X/Y}}))^{(1)} - a(rk(\mathcal{E})(1 - g)(4\zeta'(-1) - \frac{1}{6}))
\]

Here \( \hat{\lambda}(\mathcal{E}) \) is the 'determinant of the cohomology', a Hermitian line bundle on \( Y \) defined by

\[
\lambda(\mathcal{E}) := \bigotimes_{i \geq 0} det(R^i\pi_*(\mathcal{E})^{(-1)}),
\]

and the line bundle \( \lambda(\mathcal{E}) \) is equipped with the Quillen metric \( h_Q \) induced by the Hermitian metric on \( \mathcal{E} \). Furthermore \( g \) is the genus of \( X_Q \) and \( \zeta'(-1) \) is the value of the derivative of the Riemann zeta function at \(-1\).

**Definition 3.6.** Let \( X \) be an arithmetic surface and \( \overline{\mathcal{A}} \) be a Hermitian Azumaya algebra on \( X \). If \( (\mathcal{M}, \mathcal{N}) \) is a pair of Hermitian Azumaya modules, then we define the \( \mathcal{A} \)-Deligne pairing of the pair \( (\mathcal{M}, \mathcal{N}) \) as the Hermitian line bundle on \( Y \) given by:

\[
\langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}} := \lambda_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \otimes \lambda_{\mathcal{A}}(\mathcal{M}, \mathcal{A})^{(-1)} \otimes \lambda_{\mathcal{A}}(\mathcal{A}, \mathcal{N})^{(-1)} \otimes \lambda_{\mathcal{A}}(\mathcal{A}, \mathcal{A}),
\]

where

\[
\lambda_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) = \lambda(Hom_{\mathcal{A}}(\mathcal{M}, \mathcal{N})).
\]

**Theorem 3.7.** Let \( X \) be an arithmetic surface and let \( \overline{\mathcal{A}} \) be a Hermitian Azumaya algebra on \( X \). If \( \overline{\mathcal{M}} \) and \( \overline{\mathcal{N}} \) are Hermitian \( \mathcal{A} \)-line bundles, that is \( rk(\mathcal{M}) = rk(\mathcal{A}) = rk(\mathcal{N}) \), then there is the following equality in \( CH^1(X)_Q \):

\[
\hat{c}_1(\langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}}) = -\pi_*(\hat{c}_1^A(\mathcal{M}) \hat{c}_1^A(\mathcal{N})).
\]

**Proof.** By the properties of \( \hat{c}_1 \) we have:

\[
\hat{c}_1(\langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}}) = \hat{c}_1(\lambda_{\mathcal{A}}(\mathcal{M}, \mathcal{N})) - \hat{c}_1(\lambda_{\mathcal{A}}(\mathcal{M}, \mathcal{A})) - \hat{c}_1(\lambda_{\mathcal{A}}(\mathcal{A}, \mathcal{N})) + \hat{c}_1(\lambda_{\mathcal{A}}(\mathcal{A}, \mathcal{A})).
\]

Using the arithmetic Riemann-Roch theorem [3.5.3 and 3.5.5] one gets:

\[
\hat{c}_1(\langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}}) = \pi_*(c_h(\overline{\mathcal{M}}) - c_h(\overline{\mathcal{N}}) - c_h(\overline{\mathcal{A}})) - c_h(\overline{\mathcal{A}}) - \hat{T}d(\overline{T_{X/Y}})^{(1)}
\]

since the analytic terms of the form \( a(-) \) cancel each other.

Using the definitions of \( \hat{c}_1 \) one computes:

\[
\hat{c}_1(\langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}}) = \pi_*(c_h(\overline{\mathcal{M}}) - c_h(\overline{\mathcal{N}}) - c_h(\overline{\mathcal{A}})) - \hat{T}d(\overline{T_{X/Y}})^{(1)}
\]

since \( rk(\mathcal{A}) = rk(\mathcal{M}) = rk(\mathcal{N}) \).

So using the definition of \( \hat{c}_1^A \) and \( \hat{T}d \) we finally get:

\[
\hat{c}_1(\langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}}) = -\pi_*(\hat{c}_1^A(\mathcal{M}) \hat{c}_1^A(\mathcal{N})).
\]

**Remark 3.8.** The sign in the formula is not surprising, since the Deligne pairing for Hermitian Azumaya modules is the dual of the classical Deligne pairing in the case \( \overline{\mathcal{A}} = \overline{\mathcal{O}}_X \), see [Ree15] Theorem 2.7.]
REFERENCES

[GS90] Henri Gillet and Christophe Soulé. Characteristic classes for algebraic vector bundles with Hermitian metric I. The Annals of Mathematics, 131(1):163–203, 1990.

[GS91] Henri Gillet and Christophe Soulé. Analytic torsion and the arithmetic Todd genus, with an appendix by D. Zagier. Topology, 30(1):21–54, 1991.

[GS92] Henri Gillet and Christophe Soulé. An arithmetic Riemann-Roch theorem. Inventiones mathematicae, 110:473–543, 1992.

[Ree15] Fabian Reede. A Deligne pairing for Hermitian Azumaya modules, 2015.

[Sou92] Christophe Soulé. Lectures on Arakelov geometry. Cambridge University Press, 1992. Written with D. Abramovich, J.-F. Burnol & J. Kramer.

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