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Phys. Rev. A 95, 022333 — Published 23 February 2017
DOI: 10.1103/PhysRevA.95.022333
The overarching framework between Gaussian quantum discord and Gaussian quantum illumination

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(Dated: January 30, 2017)

We cast the problem of illuminating an object in a noisy environment into a communication protocol. A probe is sent into the environment, and the presence or absence of the object constitute a signal encoded on the probe. The probe is then measured to decode the signal. We calculate the Holevo information and bounds to the accessible information between the encoded and received signal with two different Gaussian probes—an EPR state and a coherent state. We also evaluate the Gaussian discord consumed during the encoding process with the EPR probe. We find that the Holevo quantum advantage, defined as the difference between the Holevo information obtained from the EPR and coherent state probes is approximately equal to the discord consumed. These quantities become exact in the typical illumination regime of low object reflectivity and low probe energy. Hence we show that discord is the resource responsible for the quantum advantage in Gaussian quantum illumination.

I. INTRODUCTION

Quantum illumination is a simple target detection scheme, first proposed by Lloyd for photonic qubits [1]. It harnesses entanglement in a quantum state of light to better infer the presence or absence of a weakly reflecting object flooded by white noise. The protocol distinguished itself in displaying quantum advantage, even in regimes so noisy that no entanglement survives. It presented a remarkable deviation from the conventional view that quantum technologies are fragile, displaying advantage only in carefully engineered environments which ensure little or no loss of entanglement. Since its original inception, quantum illumination has gained significant scientific interest. Many variants have been proposed, including some that make use of Gaussian states in the continuous variable regime [2–4] and inspiring a number of different experimental realizations [5–8]. The phenomenon has also seen applications outside metrology, where quantum illumination has been harnessed to provide security against passive eavesdropping in the setting of secure communication [9].

Quantum illumination challenges the conventional view that entanglement alone can explain all quantum advantage. It joins a particularly surprising class of protocols that appear to thrive in noisy, possibly entanglement-breaking environments [10, 11]. What other quantum resources then, could help us better understand its noisy resilience? Quantum discord [12–14] which quantifies correlations beyond entanglement is considered a likely candidate. Unlike entanglement, discord is far more robust, and can also survive in highly noisy conditions [15]. In fact, Weedbrook et al. have shown such a relation for discrete variables [16]. Specifically, they showed that the performance advantage of quantum illumination – in terms of extra accessible information about whether an object is present – can be directly related to the amount of discord in the illumination protocol that survives after being subjected to entanglement-breaking noise. Does a similar relationship hold for continuous variables?

The aim of this work is to answer this question. We extend the framework relating discord and illumination to the continuous variable regime. This involves understanding how these relations generalize when a number of conditions specific to the discrete scenario no longer hold. The paper is organized as follows. In section II we describe the illumination protocol and the quantifiers of performance. In section III we describe discord and how it relates to quantum illumination. In section IV we present and discuss our results, demonstrating that there is a general relationship between discord and the quantum advantage of illumination in the continuous variable regime.

II. THE ILLUMINATION FRAMEWORK

A. Setup

The illumination framework is described as follows: Bob wishes to determine whether an object is located in a noisy environment. He sends a quantum state, referred to as the probe, to the location. If an object is present, part of the probe will be reflected back to Bob, along with some background noise. If the object is not present, Bob receives only the background noise. Bob may have another state called the idler, which was ini-
We are interested in quantum illumination in the continuous variable setting, where the probe and idler are Gaussian states. For single-mode illumination, Bob uses a coherent state $\rho_\alpha$, where $\alpha$ is its amplitude. For quantum illumination, Bob uses an EPR state described by

$$\rho_{EPR} = |\psi_{EPR}\rangle \langle \psi_{EPR}|,$$  

where

$$|\psi_{EPR}\rangle = \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} (-\lambda)^n |n\rangle_A |n\rangle_B.$$  

(1)

where $\lambda = \tanh(r)$, and $r$ is the squeezing parameter.

Illumination can also be recast as a communication protocol. Let us suppose that Alice is in control of the object, and she would like to communicate with Bob. She can do so by encoding a binary alphabet via the control of the object, such as in the Morse code. The message she sends to Bob can be described by realizations of a random variable $X$, where if $X = 0$ Alice places the object in the noisy environment, and if $X = 1$ Alice removes the object. Let $p_x$ be the prior probability that $X = x$, and let $p_0 = p_1$, i.e. let both hypotheses be equally likely to occur. Let $\rho(x)$ denote the state received by Bob when $X = x$. Noise is injected into the probe state before Alice encodes the value of $X$. This is shown diagrammatically in Fig. 1(c) and (d). We model the object as a beam splitter with reflectivity $\epsilon$. The environment noise state $\rho_{env}$ is a thermal state with mean photon number $\bar{n}_{env}$, where $\rho_{env}(\bar{n}) = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(n!^2)} |n\rangle \langle n|$. When the object is present, the environment noise is multiplied by a factor of $1/(1-\epsilon)$ such that the mean number of noise photons arriving at the detector is the same as when the object is absent. This approach has been adopted by [17] to avoid a ‘shadowing effect’ – so that the object is not detected by a reduction in the number of noise photons arriving at the detector. The typical illumination scenario that has greatest quantum advantage is for the regime of low object reflectivity and high noise, i.e. $\epsilon \ll 1$ and $\bar{n} \ll \bar{n}_{env}$ where $\bar{n}$ is the mean photon number of the probe. We term this as the intense white noise limit.

Consider Fig. 1(c) and (d). After the noise injection, the entanglement is reduced or lost all together, before any information is encoded within the probe. In fact, for all the settings studied in section [14] the entanglement after noise injection is strictly zero. Nevertheless we see a quantum advantage. Thus, quantum entanglement itself does not give a complete picture on why illumination thrives in such noise. Our goal here is to see if discord will give us additional insight.

In the next subsection, we will use the communication formalism to study the amount of information that Alice can communicate to Bob under different settings. This provides a measure for assessing the performance of illumination under these settings.

B. Quantifiers of performance

We consider two quantifiers of performance of illumination: the accessible information and Holevo information.
Let $\mathcal{M} = \{E_k\}$ be a set of positive operator-valued measures (POVMs) that represent mathematically the outcome of a measurement. These are non-negative, self-adjoint operators satisfying $\sum_k E_k = 1$, where the subscript $k$ labels the outcome of the measurement. The probability of the measurement outcome $k$ on a state $\rho^{(x)}$ is then given by $q_k = \text{Tr}(\rho^{(x)} E_k)$. Let this be governed by random variable $K_M$. In the communication setting described in the last subsection, the amount of information obtained by Bob after measurement of the state $\rho^{(x)}$ is given by the mutual information,

$$I_{\text{mut}}(X,K_M) = \sum_k \sum_{x=0}^1 p_x q_k^{(x)} \log \left( \frac{q_k^{(x)}}{q_k} \right),$$

(2)

where $q_k = \sum_{x=0}^1 p_x q_k^{(x)}$. The accessible information is the maximization of the mutual information over all POVMs:

$$A\left(\rho^{(0)},\rho^{(1)}\right) = \max_{\mathcal{M}} I_{\text{mut}}(X,K_M).$$

(3)

The accessible information quantifies Bob’s knowledge when each $\rho^{(x)}$ from $N$ trials is measured separately using an optimal POVM. In the context of communication, illumination can be regarded as classical information exchange over a noisy channel. By the Shannon’s noisy-illumination can be regarded as classical information exchanged over a noisy channel. By the Shannon’s noisy-channel coding theorem [18], Alice and Bob communicate by random variable $K_M$. In the context of communication, POVMs:

is the maximization of the mutual information over all upper and lower bounds found by Fuchs and Caves [19]. The lower bound, hereby referred to as the Fuchs’s lower bound is

$$I_{\text{lower}} = \text{Tr}\left\{ p_0 \rho^{(0)} \ln \left[ \mathcal{L}_\rho(\rho^{(0)}) \right] + p_1 \rho^{(1)} \ln \left[ \mathcal{L}_\rho(\rho^{(1)}) \right] \right\},$$

(4)

where $\mathcal{L}$ is the lowering superoperator given by

$$\mathcal{L}_\rho(\Delta) = \sum_{\{j,k|\lambda_j+\lambda_k\neq0\}} \left[ \frac{2}{\lambda_j(p_1)+\lambda_k(p_1)} \times \langle \psi_j(p_1)|\Delta|\psi_j(p_1)\rangle \langle \psi_k(p_1)|\psi_k(p_1)\rangle \right],$$

(5)

and where $\Delta = \rho^{(1)} - \rho^{(0)}$. $\lambda_j(p_1)$ and $|\psi_j(p_1)\rangle$ are the eigenvalues and eigenvectors of $\rho = (1-p_1)\rho^{(0)} + p_1\rho^{(1)}$. The Fuchs upper bound $I_{\text{upper}}$, is found by numerically solving the differential equation

$$\frac{d^2 I_{\text{upper}}(p_1)}{dp_1^2} = \sum_{\{j,k|\lambda_j+\lambda_k\neq0\}} \left[ \frac{2}{\lambda_j(p_1)+\lambda_k(p_1)} \times |\langle \psi_j(p_1)|\Delta|\psi_k(p_1)\rangle|^2 \right]$$

(6)

subject to:

$$I_{\text{upper}}(0) = I_{\text{upper}}(1) = 0.$$  

(7)

The other figure of merit we consider is the Holevo information [20]. It is given by

$$\chi(\rho^{(0)},\rho^{(1)}) = S \left( \sum_{x=0}^1 p_x \rho^{(x)} \right) - \sum_{x=0}^1 p_x S(\rho^{(x)})$$

(8)

where $S(\rho)$ is the Von Neumann entropy of the quantum state $\rho$. The Holevo information is the maximum communication rate Bob can obtain, provided he stores all of the $N$ states and then performs a joint measurement upon all of the states. From the Holevo-Schumacher-Westmoreland theorem [21, 22] this information rate is obtainable when $N \rightarrow \infty$.

C. Three cases of illumination and quantum advantage

Three cases, together with three pairs of accessible information and Holevo information are relevant for our assessment of the illumination scheme (Figure 1(a)) in the communication framework. They are as follows:

Case 1. Quantum illumination with joint measurement: $A_q$ and $\chi_q$ are the accessible information and Holevo information, respectively for Bob when two mode EPR states are used as probes and idlers for illumination. Any arbitrary joint measurement over the two modes is allowed.

Case 2. Quantum illumination with local measurements: $A_c$ and $\chi_c$ are the average accessible information and Holevo information for Bob with EPR state as the probe and idler, under the restriction that Bob must perform the optimal Gaussian local measurement on mode B, followed by an arbitrary local measurement on mode A. The measurement on mode B is optimal in the sense that it maximizes the amount of accessible information/Holevo information Bob receives. In this case, Bob only takes advantage of the classical correlations of the EPR state. This enables a direct comparison to case 1, when both quantum and classical correlations are utilized.

Case 3. Single-mode illumination: $A_s$ and $\chi_s$ are the accessible information and Holevo information, respectively when Bob uses a single mode coherent state with a fixed amplitude $\alpha$ as the illumination probe.

The quantum advantage is defined as the difference between the performance of quantum illumination and single-mode illumination protocol. The protocols are compared for scenarios where the probe states have coinciding energy. This constraint allows for fair comparison,
as it is always possible to detect the presence of an object with any fixed accuracy by using a sufficiently energetic probe. The quantum advantage in terms of accessible information is $A_N - A$, and the Holevo information quantum advantage is $\chi_q - \chi_s$, where each information quantity is evaluated over the probe with mean photon number $\bar{n}$. As we shall show in this paper, these quantum advantages can be linked to the discord consumed in the illumination protocol.

III. DISCORD AND QUANTUM ILLUMINATION

Quantum discord is a measure of the nonclassical correlations between two quantum states. It arises from the difference between quantum analogs of two distinct definitions of the classical mutual information \[12\] \[13\]:

\[
I(A : B) = S(A) + S(B) - S(AB) \tag{9}
\]

\[
J(A | B) = S(A) - \min_{\{\Pi_k\}} \sum p_b S(A | b) \tag{10}
\]

where $\Pi_k$ is the positive-operator valued measure (POVM) of the outcome $b$, $p_b$ is the probability of that outcome, and $S(A | b)$ is the entropy of the state conditioned on the outcome $b$. The discord is then

\[
\delta(A | B) = I(A : B) - J(A | B) = S(B) - S(AB) + \min_{\{\Pi_k\}} \sum p_b S(A | b) , \tag{11}
\]

where the minimization is done over all possible POVMs on mode B. In the special case that the domain of this minimization is restricted to Gaussian measurements, then the discord is known as the Gaussian discord \[23\] \[24\]. It was recently shown that for a large class of Gaussian states, Gaussian quantum discord is equal to quantum discord \[25\]. Henceforth we denote the Gaussian discord: $\delta^G(A | B)$ with a superscript G.

We now consider the evolution of the discord when quantum illumination is described by Fig. 1(c) and (d). After the noise injection step, Alice is left with state $\rho$, which can encode information to send to Bob. We note that this state may have no entanglement due to the noise injection \[17\]. Alice encodes the value of $X$ on the state by performing the operation $O_x$ on $\rho$, resulting in a state $\rho^{(x)} = O_x(\rho)$ with discord $\delta^{(x)}(A | B)$.

Let us decompose the discord of $\rho$, $\delta(A | B)$ into three components:

\[
\delta(A | B) = \delta_{\text{loss}} + \delta(A | B) + \delta_{\text{con}}(A | B) \tag{12}
\]

The first component $\delta_{\text{loss}}$ is the amount of discord lost to the environment during the encoding process. This can be evaluated by first defining

\[
\delta^{(x)}_{\text{loss}} = \delta(A | B) - \delta^{(x)}(A | B) \tag{13}
\]

as the loss of discord for each possible value of $x$ that Alice can encode, and then taking the weighted average over the probability of encoding that $x$. This results in

\[
\delta_{\text{loss}} = \sum_x p_x \delta^{(x)}_{\text{loss}} \tag{14}
\]

The second component $\delta(A | B)$ is the discord of $\tilde{\rho} = \rho_0 \rho^{(0)} + \rho_1 \rho^{(1)}$, the state after encoding. This is the state seen by Bob who is oblivious to the value of $X$.

We term the remaining component the consumed discord $\delta_{\text{con}}(A | B)$, and represents the discord in $\rho$ that remains unaccounted for. In prior literature, it was proposed to capture the amount of discord consumed to encode the value of $X$ on the state $\rho$ \[10\]. For the special case where encodings were unitary, such that $\delta^{(x)} = 0$, $\delta_{\text{con}}(A | B)$ was related to the advantage of using coherent interactions \[26\]. It is also interesting to note that $\delta_{\text{con}}(A | B)$ also coincides with the the extra discord Bob sees between $A$ and $B$, should he learn the value of $X$.

In quantum illumination, when $X = 0$, Alice performs an identity operation, thus $\delta^{(0)}(A | B) = \delta(A | B)$ and $\delta_{\text{loss}} = 0$. When $X = 1$, Alice performs a swap operation between mode $A$ of $\rho$ with the environment noise, destroying all correlations between the two modes. All discord is lost and $\delta^{(1)}_{\text{loss}} = \delta(A | B)$. Putting this together, the average discord loss is thus $\delta_{\text{loss}} = p_0 \delta(A | B)$. Hence the consumed discord for quantum illumination is

\[
\delta_{\text{con}}(A | B) = p_0 \delta^{(0)}(A | B) - \delta(A | B). \tag{15}
\]

IV. METHOD AND RESULTS

In section IV A we first derive a general result that if certain conditions are fulfilled, the discord consumed is equal to the Holevo information quantum advantage. In section IV B we numerically calculate the illumination information quantities. In section IV C we numerically evaluate the consumed discord and compare it to the quantum advantages. Our main result is that for continuous variable quantum illumination, the consumed discord is approximately equal to the Holevo information quantum advantage.

A. Analytic Result

We prove the following theorem:

**Theorem 1.** Let $\rho^{(0)}_{AB}$ and $\rho^{(1)}_{AB}$ be two arbitrary two mode states. If the following conditions are met:

1. Mode $B$ is the same for both states: $\rho^{(0)}_{B} = \rho^{(1)}_{B}$
   where $\rho^{(x)}_{B} = \text{Tr}_A(\rho^{(x)}_{AB})$ where $\text{Tr}_A$ denotes the partial trace over subsystem $A$.

2. $\rho^{(1)}_{AB}$ is a product state: $\rho^{(1)}_{AB} = \rho^{(1)}_{A} \otimes \rho^{(1)}_{B}$
3. The Holevo information of local measurement $\chi_c$, the discord of $\hat{\rho}_{AB} = p_0\rho_{AB}^{(0)} + p_1\rho_{AB}^{(1)}$, and the discord of $\rho_{AB}^{(0)}$ are achieved by the same measurement, then $\delta_{\text{con}}(A|B) = \chi_q - \chi_c$, where

$$\chi_q = \chi(\rho_{AB}^{(0)}, \rho_{AB}^{(1)}) \quad \chi_c = \max_{\{P_b\}} \sum_b p_b \chi(\rho_{A|b}^{(0)}, \rho_{A|b}^{(1)}),$$

where $p_b$ is the probability of measuring outcome $P_b$ on subsystem $B$, and $\rho_{A|b}^{(c)}$ are the states of subsystem $A$ conditioned on that outcome.

Proof. Let $\{P_b\}$ be the measurement in condition 3 that simultaneously optimizes $\chi_c$, as well as the discord of states $\hat{\rho}_{AB}$ and $\rho_{AB}^{(0)}$. The measurement outcome probability is

$$p_b = \text{Tr}(\{(P_b \otimes I)\rho_{AB}^{(0)}\}) = \text{Tr}(\{(P_b \otimes I)\rho_{AB}^{(1)}\}),$$

where we have used condition 1. The resulting conditional states are

$$\rho_{A|b}^{(c)} = \frac{\text{Tr}_B(P_b \rho_{AB}^{(c)} \rho_{AB}^{(c)})}{p_b}.$$

Our goal is to prove $\delta_{\text{con}}(A|B) = \chi_q - \chi_c$. Because of condition 2, $\delta^{(1)}(A|B) = 0$, as so the consumed discord is

$$\delta_{\text{con}}(A|B) = p_0\delta^{(0)}(A|B) - \delta(A|B)$$

$$= p_0(S(\rho_B^{(0)}) - S(\rho_{AB}^{(0)}) + \sum_b p_b S(\rho_{A|b}^{(0)}))$$

$$- S(\hat{\rho}_B) + S(\hat{\rho}_{AB}) - \sum_b p_b S(\hat{\rho}_{A|b}).$$

We also have that:

$$\chi_q - \chi_c = S(\hat{\rho}_{AB}) - p_0 S(\rho_{AB}^{(0)}) - p_1 S(\rho_{AB}^{(1)})$$

$$+ \sum_b p_b (-S(\hat{\rho}_{A|b}) + p_0 S(\rho_{A|b}^{(0)}) + p_1 S(\rho_{A|b}^{(1)})).$$

This leads to

$$\delta_{\text{con}}(A|B) - (\chi_q - \chi_c)$$

$$= p_0 S(\rho_B^{(0)}) - S(\hat{\rho}_B) + p_1 S(\rho_{AB}^{(1)}) - \sum_b p_b p_1 S(\rho_{A|b}^{(1)}).$$

From condition 1 we have that $\rho_B^{(0)} = \rho_B^{(1)} = \hat{\rho}_B$. From condition 2, $\rho_{AB}^{(1)}$ is a product state, so $S(\rho_{AB}^{(1)}) = S(\rho_A^{(1)}) + S(\rho_B^{(1)})$ and $\rho_{A|b}^{(1)} = \rho_A^{(1)}$. So this becomes

$$\delta_{\text{con}}(A|B) - (\chi_q - \chi_c)$$

$$= S(\rho_B^{(0)})(p_0 - 1 + p_1) + S(\rho_A^{(1)})(p_1 - p_1)$$

$$= 0.$$
FIG. 2. Information versus object reflectivity \(\epsilon\) when probe has mean photon number (a) 0.5 and (b) 0.01. The environment noise has mean photon number 4. Each plot has two insets showing zoomed portions. Insets (ii) show the upper and lower bounds for \(A_q\), the true value of lying somewhere in the shaded region. Insets (iii) show that \(\chi_s, \chi_c, A_s, A_c\) differ slightly, despite appearing as a single line in the main plot.

In the communication context, Alice can communicate with Bob with a higher bit-rate if Bob uses a probe entangled with an idler instead of a coherent state probe.

From Fig. 2 we see that the performance of a coherent state probe is approximately equal to performance of an EPR probe when a local Gaussian measurement is performed on the mode B. However, \(A_s\) is slightly higher than \(A_c\) (and \(\chi_s\) slightly higher than \(\chi_c\)), because \(A_s\) is a concave function of energy (see appendix B). By considering the ratio of \(A_s\) and \(A_c\), we find that their relative difference approaches zero in both the limits \(\epsilon \to 0\) and \(\bar{n} \to 0\). This indicates that there is no advantage to using a distribution of single-mode Gaussian states for the probe, which, under the masking of strong environmental noise, gives an approximately equal knowledge about a weakly reflecting object as using a single mode coherent state probe.

C. Relating Quantum Advantage to Discord Consumed

To calculate the consumed discord \(\delta_{con}(A|B)\), we need to compute the discord of states \(\rho^{(0)}\) and \(\tilde{\rho}\) when the entangled state \(\rho_{EPR}\) is used as probe and idler. \(\rho^{(0)}\), the resulting state when Alice does nothing, is a Gaussian state whose discord is equal to the Gaussian discord, and additionally this discord is obtained when the measurement is a heterodyne measurement [27]. The state after
encoding $\rho$, however, is not Gaussian, thus the same rule does not apply. Unfortunately, calculating the discord of a general state is an NP-hard problem [28], so there is no method to calculate it efficiently. Here, we simplify the problem by restricting ourselves to Gaussian discord and calculate the consumed Gaussian discord $\delta_{\text{con}}^G (A|B)$ instead. This is just Eq. (15) with the discords replaced with Gaussian discords.

The Gaussian discord of state $\tilde{\rho}$ was obtained by numerically optimizing Eq. (11) over Gaussian measurements. It was found that the optimal point occurs when the measurement is a heterodyne measurement. The two discord values $\delta^{G(0)} (A|B)$ and $\delta^G (A|B)$ are then substituted into Eq. (15) to obtain the consumed Gaussian discord.

Due to the optimality of the Gaussian discord of state $\rho^{(0)}$, and the fact that Gaussian discord is an upper bound for the discord for state $\tilde{\rho}$, the consumed Gaussian discord is a lower bound of the consumed discord, i.e. $\delta_{\text{con}}^G (A|B) \leq \delta_{\text{con}} (A|B)$. A plot of the $\delta_{\text{con}}^G (A|B)$ compared to the information differences is shown in Fig. 4.

As discussed in Sec. IV A, since a heterodyne measurement on mode B optimizes $\delta^{(0)} (A|B)$, and numerical results show that this is the case for $\delta (A|B)$ and $\chi_c$, from theorem 1, $\delta^{(0)} (A|B) = \chi_q - \chi_c$. Numerical calculation of $\delta^{(0)} (A|B)$ and $\chi_q - \chi_c$ agree within the precision of the calculation, further verifying the theorem.

From Fig. 4, we see that the difference in Holevo information between quantum illumination ($\chi_q - \chi_c$) and single-mode illumination ($\chi_q - \chi_s$) differ by 1.3% for $\bar{n} = 0.5$ and 0.005% for $\bar{n} = 0.01$ when $\epsilon = 0.3$. The percentage difference approaches zero when $\epsilon \to 0$. Since $\delta_{\text{con}}^G (A|B) = \chi_q - \chi_c$, this leads us to the conclusion that in limit of low reflectivity and low probe energy, $\chi_q - \chi_s$ converges to the Gaussian discord consumed. Hence, discord encoded can suitably explain the quantum advantage of quantum illumination, if quantum illumination is viewed as a communication problem with access to devices such as quantum memory.

On the other hand, $A_q - A_s$, which quantifies the performance advantage for quantum illumination in the single copy measurement case is more relevant from a practical point of view since this does not require the storage of quantum states [3]. From Fig. 4 we see that $\delta_{\text{con}}^G (A|B)$ is greater than $A_q - A_s$ and $A_q - A_c$. This discrepancy is mainly due to the difference between the Holevo information $\chi_q$ and the accessible information $A_q$ for the states involved in quantum illumination. Hence, measuring each illumination event separately does not fully harness the benefits offered by the discord. However, it is sufficient to provide some quantum advantage over single-mode illumination.

D. Quantum advantage versus probe energy

There is nothing special about our choice of probe energies of 0.01 and 0.5 used in the previous sections. To demonstrate this, Fig. 5 shows the illumination performance, quantum advantage and consumed Gaussian discord for probe mean photon numbers in the range 0 to 0.1, while the object reflectivity is kept constant at 0.1. There is always a quantum advantage, and the consumed Gaussian discord is approximately equal to the quantum advantage in terms of Holevo information.

E. Comparison to discrete variables

It is worth comparing continuous variable (CV) illumination to discrete variable (DV) illumination [10]. In discrete variables, the environmental noise is often described as white noise. This scenario is not realistic in continuous variables, as it corresponds to a thermal state at infinite temperature, and thus is of unbounded energy. Using a maximally mixed environment noise for DV illumination has the consequence that all pure state probes yield the same information for single-mode illumination. This is clearly not the case for any physically relevant cases of CV illumination, where a coherent state with a high energy generally performs better than a coherent state with low energy.

The probe used for quantum illumination for DV illumination is a maximally entangled state. Again, this state in CV illumination would have unbounded energy. A maximally entangled probe and idler, and a maximally mixed environment mean that $\rho^{(0)}$ and $\rho^{(1)}$ commute in DV illumination. Hence, the Holevo information and accessible information are equal. This is not the case for CV illumination. From Fig. 6 we see the differences between $A_q$ and $\chi_q$ can be significant, though deviations between $A_c$ and $\chi_c$ remain small. Quantum advantage,
In DV illumination performing a local measurement on the idler first, followed by a local measurement on the probe yields identical information as single-mode illumination. For CV illumination this is only approximately true; these two quantities approach equality in the limit of low reflectivity and low probe energy.

Finally, in DV illumination, the consumed discord is exactly equal to the Holevo information quantum advantage and the accessible information quantum advantage. We found for CV illumination, this approximately holds for the Holevo information, but not for the accessible information. The differences between DV and CV illumination are summarized in table I.

V. CONCLUSION

In [10], it has been shown that quantum discord coincide exactly with quantum advantage in a DV quantum illumination. Here, we complete the picture by extending the framework to CV quantum illumination [2]. To this end, we numerically calculated the performance enhancement quantum illumination has over single-mode illumination and compared it to the Gaussian discord of the system. We derived an analytic result showing that $\delta_{\text{con}}^G(A|B) = \chi_q - \chi_c$ provided condition 3 of theorem 1 is met. Our main result is that the quantum advantage in terms of Holevo information matches the consumed discord in the limit of low probe energy and low object reflectivity ($\bar{n} \rightarrow 0$ and $\epsilon \rightarrow 0$). This is in agreement with the DV counterpart, which analogously assumes an maximally entropic illumination environment.
Several remarks on relation with other works are in order. In deriving our results, we have demonstrated that a joint measurement over the returning probe and idler is necessary to exploit the surviving quantum correlation to determine the non-unitary encoding. Similar to [26], a coherent interaction is required to unlock the information encoded via unitary discord consumption. The discrepancy between the quantum advantage offered by Holevo information and accessible information is in concordance with recent findings, where the improvement of error probability of quantum illumination over single-mode illumination is limited to 3 dB (out of maximum gain of 6 dB) for single copies separate measurement in the intense white noise limit [3, 4].

We note other efforts in quantifying the source of enhancement in quantum illumination-like protocols. In [29], mutual information is used to quantify the advantage offered by entangled source over correlated thermal source. Gaussian discriminating strength is proposed to distinguish the absence or presence of a set of unitary operation in [30, 31]. The role of correlation in the improvement of channel loss detection is also established by linking discord to the performance numerically [32]. Meanwhile, several other cryptographic and metrological variants of illumination has been proposed and demonstrated recently [6, 9], which we envisage our framework would shed light in understanding the discord’s role in the their quantum enhancement.

**Acknowledgements:** We are grateful for funding from the National Research Foundation of Singapore and in particular NRF Award No. NRF–NRFF2016–02, from the John Templeton Foundation Grant 53914 “Occam’s Quantum Mechanical Razor: Can Quantum theory admit the Simplest Understanding of Reality?”, the Foundational Questions Institute and the Australian Research Council Centre of Excellence for Quantum Computation and Communication Technology (Project number CE110001027). This material is based on research supported in part by the Singapore National Research Foundation under NRF Award No. NRF–NRFF2013-01. ST acknowledges support from the Air Force Office of Scientific Research under grant FA2386-15-1-4082.
FIG. 5. The accessible information and Holevo information quantities (top) and consumed Gaussian discord and quantum advantage (bottom) versus the mean energy of the probe. The environment noise mean photon number is 4 and object reflectivity $\epsilon = 0.1$. 
Appendix A: Suboptimality of coherent state probe

A coherent state is not the optimal state to use for single-mode illumination. Small perturbations were made on a coherent state, such that the mean photon number was maintained. Fig. [?] shows a histogram of the Holevo information when the perturbed states were used in illumination. Some of the perturbed states resulted in a Holevo information greater than that achieved with a coherent state. Hence, a coherent state is not the

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FIG. 6. One million small perturbations were made on a coherent state. This is a histogram of the Holevo information when the top 500,000 states are used in single-mode illumination. The vertical line shows the Holevo information of the coherent state. Parameters are $n_{\text{env}} = 4$, $\epsilon = 0.5$, $\bar{n} = 0.5$. Optimal probe to use in single-mode illumination. However, we hypothesize it is close to optimal. The problem of finding the optimal probe is too difficult to calculate, so this hypothesis is difficult to prove.

If the probe is restricted to a Gaussian state, as in Gaussian single-mode illumination, the coherent state is still not optimal. Using a squeezed coherent state with a tiny squeezing can result in an increased accessible information (as can be seen in Fig. 7) but the improvement is negligible. Hence, a coherent state is approximately optimal for Gaussian single-mode illumination.

**Appendix B: Calculating $\chi_c$ and $A_c$ from integration of $\chi_s$ and $A_s$**

From Fig. 8 we see that $A_s$ is a concave function of energy. If $A_s$ were a perfect linear function of energy, $A_c$ and $A_s$ would be equal. As can be seen from Fig. 8, $A_s$ as a function of energy becomes more linear as the $\epsilon$ approaches zero. Hence, this suggests that $A_c$ and $A_s$ become equal as $\epsilon$ approaches zero. The same applies to...
\( \chi_c \) and \( \chi_s \).