EMERGENCE OF QUANTUM DYNAMICS FROM CHAOS:
THE CASE OF PREQUANTUM CAT MAPS

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ABSTRACT. Faure and Tsuji have recently proposed a new quantization theory for symplectic Anosov diffeomorphisms. The method combines prequantization with the study of the Pollicott-Ruelle resonances of a suitably defined transfer operator. Its main appeal lies in its naturalness: the quantum behavior appears dynamically in the correlation functions of the prequantum transfer operator. In this paper, we apply the full theory to the hyperbolic symplectic automorphisms of the \(2^n\)-dimensional torus, the so-called cat maps. The main theorem gives an explicit relation between the resonances of the prequantum transfer operator and the usual Weyl spectrum of the quantum evolution operators. The case \(n = 1\) was introduced in [Fau07]. We also provide the first concrete prequantization of all the cat maps.

1. INTRODUCTION

Early in the inception of quantum mechanics, it became clear that the differences with classical mechanics were too drastic, both in the predictions and the mathematical formalism, for them not to warrant a deeper explanation. We now understand, in what has come to be known as Bohr’s correspondence principle, that the classical world emerges as a limiting case of its quantum counterpart. Celebrated results in this direction include Egorov’s theorem and the WKB approximation.

In recent years, work by Faure and Tsuji has hinted at a potential relationship in the other direction: they have shown that quantum dynamics can emerge from the long-term behavior of both discrete [Fau07, FT15] and continuous [FT21] chaotic classical systems. Such a correspondence, going from classical to quantum, is interesting because quantum mechanics is considered to be the more fundamental physical theory. Could it be that the quantum effects that we experimentally observe are in fact manifestations of an underlying chaotic yet deterministic (nonlocal) system?

Their observations are closely related to the problem of quantization. Heuristically, this is a procedure for going from a classical system to its quantum analog. It is quite natural to stare at a classical system and dwell on the question: what underlying quantum system is giving rise to my observations? A first remark is that this question is often ill-defined because two quantum systems can give rise to the same classical dynamics as limiting cases. This leads to a wealth of questions, making the theory of
quantization very rich from a mathematical point of view. Is there a procedure more canonical than others? Can we reduce, if not remove, the arbitrariness in our choices? As the physical systems evolve, for how long is the classical to quantum correspondence preserved? Could we have guessed quantum mechanics from classical dynamics?

Until recently, there have been two general approaches to delve into questions of this sort: the algebraic theory of deformation quantization and the geometric analog known as geometric quantization. These two formalizations are dual to each other, much like the Heisenberg and Schrödinger pictures in quantum mechanics.

It turns out that the first step of geometric quantization, known as prequantization, can be compelling in its own right. Roughly speaking, prequantum dynamics is the same as classical dynamics but with the introduction of complex phases. It should feel natural to introduce complex phases because these govern interference effects, a key characteristic of wave and quantum dynamics. The evolution of these phases over time is dictated by the classical trajectories. More precisely, prequantization is equivalent to picking a contact $U(1)$-extension of the classical dynamics.

The Hilbert space of wave functions resulting from prequantization is too large to be considered the final product of a proper quantization procedure. Heuristically, this is because prequantization does not include an uncertainty principle. The problem is remedied in geometric quantization by a choice of a polarization. Nevertheless, such a step involves an arbitrary choice and limits the class of functions that can be quantized.

Faure and Tsujii’s recent work shows that, for chaotic classical systems, there might be a better way to achieve a full quantization theory. It would seem that the missing ingredient was the long-term dynamics of the prequantum transfer operator. The method is worked out for symplectic Anosov diffeomorphisms in [FT15] and contact Anosov flows in [FT21]. In this paper, we apply the full theory to the hyperbolic symplectic automorphisms of the $2n$-dimensional torus, the so-called cat maps. In Theorem 1, we show that the Pollicott-Ruelle resonances of the prequantum transfer operator are related to the usual Weyl spectrum of the quantum evolution operators. We then show in Theorem 2 how quantum behavior appears dynamically in the correlation functions.

This is akin to the results in [Fau07] with $n = 1$. There are, however, several differences. First, [Fau07] only considers cat maps obtained as the time-1 flow of a quadratic Hamiltonian on $\mathbb{R}^2$. This assumption is not too restrictive in the case $n = 1$ because the group $\text{Sp}(2, \mathbb{R})$ coincides with $\text{SL}(2, \mathbb{R})$, and each of its elements can hence be written as $\pm e^X$ for some $X \in \mathfrak{sl}(2, \mathbb{R})$. In higher dimensions, the assumption becomes much stronger. Second, we aim to clarify many of the statements around the uniqueness of the objects, the necessity for parity conditions on the matrix coefficients, and so on, which were previously ignored or underexplained. Finally, instead of using the complex line bundle terminology, we adopt the language of principal $U(1)$-bundles to harmonize the concepts and notation with the more recent work [FT15].
Quantum cat maps are toy models that have been extensively used to explore various topics in quantum chaos [Zel87, DE93, BDB96, FNDB03]. A question that arises is whether the prequantum transfer operator approach can shed light on the semiclassical behaviour of the eigenstates of the quantum dynamics, which has been the central problem in the study of quantum cat maps. As a result, another objective of this paper is to give a concrete realization of all the prequantum cat maps. Part of it is done in the previous work [Zel97] on Toeplitz quantization, but only a special subgroup is explicitly prequantized. By being as explicit as possible in our constructions, we hope to set the ground for further investigation.

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2. Statement of the results

The phase space of interest for us is the $2n$-dimensional torus $T^{2n} := \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. We will use coordinates $x = (q, p)$ to represent a point on either phase space $T^{2n}$ or its universal cover $\mathbb{R}^{2n}$. We endow $T^{2n}$ with the symplectic 2-form $\omega = \sum_{j=1}^{n} dq_j \wedge dp_j$.

A matrix $M \in \text{Sp}(2n, \mathbb{Z})$ descends to a symplectic automorphism of $T^{2n}$, which we again denote by $M$. When the matrix $M$ is hyperbolic, i.e., it has no eigenvalues on the unit circle, the resulting automorphism is Anosov. Motivated by the physics literature, we call it a classical cat map. These maps are interesting because they exhibit strong chaotic properties such as ergodicity and mixing. They serve as discrete proxies for the geodesic flow on a negatively curved Riemannian manifold.

We are interested in studying their prequantization.

**Definition 2.1.** A prequantum bundle $(P, \pi, \alpha)$ over $(T^{2n}, \omega)$ is a smooth principal $U(1)$-bundle $\pi : P \to T^{2n}$ equipped with a principal $U(1)$-connection $\alpha \in \Omega^1(P, i\mathbb{R})$ such that the curvature 2-form $\Omega = d\alpha$ satisfies

$$\Omega = 2\pi i (\pi^* \omega).$$

(2.1)

Given a hyperbolic matrix $M \in \text{Sp}(2n, \mathbb{Z})$, a $U(1)$-equivariant lift $\widetilde{M} : P \to P$ preserving $\alpha$ is called a prequantum cat map.

We will verify that such objects exist and discuss to what degree they are unique.

**Remark.** We earlier characterized prequantization as a contact $U(1)$-extension of the Hamiltonian dynamics. Let us justify this. If we define $\alpha' := \frac{1}{2\pi} \alpha \in \Omega^1(P)$, then

$$dV_P := \frac{1}{n!} \alpha' \wedge (d\alpha')^\wedge n = \alpha' \wedge \pi^* dV_{T^{2n}}$$

is a non-degenerate volume form on $P$, making $\alpha'$ a contact form preserved by $\widetilde{M}$. 
Prequantum dynamics are best described by a transfer operator.

**Definition 2.2.** The prequantum transfer operator \( \hat{M} : C^\infty(P) \to C^\infty(P) \) associated to a prequantum cat map \( \tilde{M} : P \to P \) is defined as

\[
\hat{M}u := u \circ \tilde{M}^{-1}, \quad u \in C^\infty(P).
\]

This type of operator is pervasive in the study of dynamical systems because, once extended by duality to \( D'(P) \), it corresponds to the push-forward evolution of probability distributions by the dynamics. To avoid dealing with chaotic individual trajectories, it is common to consider the behavior of correlation functions

\[
C_{u,v}(t) := \int_P (u \circ \tilde{M}^{-t}) \cdot \nabla dV_P, \quad u, v \in C^\infty(P),
\]

as \( t \to +\infty \). These functions measure the loss of memory or asymptotic independence.

One nice consequence of the \( U(1) \)-equivariance of \( \tilde{M} : P \to P \) is that the transfer operator \( \hat{M} \) commutes with the action of \( U(1) \) on \( C^\infty(P) \). As a result, it respects a natural decomposition into Fourier modes with respect to the \( U(1) \) action.

**Definition 2.3.** For a given \( N \in \mathbb{Z} \), the space of functions in the \( N \)th Fourier mode is

\[
C_N^\infty(P) := \{ u \in C^\infty(P) \mid u(e^{2\pi is}p) = e^{2\pi iNs}u(p) \text{ for all } p \in P, s \in \mathbb{R} \}.
\]

We denote the restriction of the prequantum transfer operator to \( C_N^\infty(P) \) by \( \hat{M}_N \).

To understand the long-time behavior of the correlation functions, it hence suffices to study the correlation functions of each operator \( \hat{M}_N \). Indeed, we have

\[
C_{u,v}(t) = \langle \hat{M}^t u, v \rangle_{L^2(P)} = \sum_{N \in \mathbb{Z}} \langle \hat{M}_N^t u_N, v_N \rangle_{L^2(P)},
\]

where \( u_N, v_N \in C_N^\infty(P) \) are the \( N \)th Fourier components of the functions \( u, v \in C^\infty(P) \).

**Remarks.**

(i) Complex conjugation commutes with \( \hat{M} \) and maps \( C_N^\infty(P) \) to \( C_{-N}^\infty(P) \). It is thus enough to study the case \( N \geq 0 \).

(ii) When \( N = 0 \), we are effectively dealing with smooth functions on the torus (and classical cat maps). Using their Fourier series, we can show (see [BS01, p. 946]) that we have super-exponential decay of correlations, i.e., for any \( 0 < \rho < 1 \),

\[
C_{u,v}(t) = \int_P u dV_P \int_P \nabla dV_P + O_{u,v}(\rho^t).
\]

We will therefore restrict our attention to \( N \geq 1 \).

(iii) The space \( C_N^\infty(P) \) can be identified with the space of smooth sections of an associated Hermitian complex line bundle \( L^{\otimes N} \) over \( \mathbb{T}^{2n} \) (i.e., the \( N \)th tensor power of a line bundle \( L \to \mathbb{T}^{2n} \)) equipped with a covariant derivative. This is the point of view adopted in [Fau07] and most references on geometric quantization. We preferred to avoid this identification to simplify the presentation.
Following [Rue86], we take the power spectrum, which for us is the Fourier transform \( \hat{C}_{u,v} \) of \( C_{u,v} \) restricted to \( t > 0 \), with \( u, v \in \mathcal{C}_N(P) \). By [FT15], we know that \( \hat{C}_{u,v} \) is analytic in the region \( \{ \lambda \in \mathbb{C} \mid \text{Im } \lambda < 0 \} \) and has a meromorphic extension to \( \mathbb{C} \) with poles of finite rank. The location and rank of its poles do not depend on the observables \( u \) and \( v \). The Pollicott-Ruelle resonances are defined as the complex numbers \( e^{i\lambda} \) such that \( \lambda \in \mathbb{C} \) is a pole of this meromorphic extension. The long-time behavior of \( C_{u,v}(t) \) is governed by these resonances.

The microlocal method of Faure-Tsuji in [FT15] alternatively defines Pollicott-Ruelle resonances for each operator \( \hat{M}_N \) (which we extend to \( (\mathcal{C}_N(P))^* \) by duality) as the points \( \lambda \in \mathbb{C} \) for which the bounded operator
\[
\lambda - \hat{M}_N : \mathcal{H}_N^r \to \mathcal{H}_N^r, \quad c' < |\lambda|,
\]
is not invertible. Here \( \mathcal{C}_N(P) \subset \mathcal{H}_N \subset (\mathcal{C}_N(P))^* \) is a certain anisotropic Sobolev space, and \( 0 < c < 1 \) is a fixed constant independent of \( r > 0 \) (see [FT15, Theorem 1.3.1] for details). Resonances and their associated eigenspaces are shown to be independent of the parameter \( r \). The relation with our definition is given by the formula
\[
\hat{C}_{u,v}(\lambda) = \sum_{t=1}^{\infty} e^{-i\lambda t} C_{u,v}(t) = \sum_{t=1}^{\infty} e^{-i\lambda t} \langle \hat{M}_N^t u, v \rangle_{L^2(P)} = \langle (e^{i\lambda} - \hat{M}_N)^{-1} u, v \rangle_{L^2(P)},
\]
valid for \( \text{Im } \lambda < 0 \) and \( u, v \in \mathcal{C}_N(P) \).

The first result in this paper is a description of these resonances. It is interesting to find systems for which resonances are explicit as they cannot be computed in general. What is more, Theorem 1 exposes a link between the resonances of the transfer operators \( \hat{M}_N \) and the eigenvalues \( \sigma(M_{N,\theta}) \) of the usual quantum cat maps \( M_{N,\theta} \) obtained through Weyl quantization (parametrized by \( N \in \mathbb{N}^* \) and \( \theta \in \mathbb{T}^{2n} \)) and acting on the \( N^n \)-dimensional Hilbert space \( \mathcal{H}_{N,\theta} \) (see Section 5.1 for their definition).

To cleanly state the relationship, we impose the condition \( \varphi_M = 0 \), where \( \varphi_M \) is an element of \( \{0,1\}^{2n} \) uniquely defined for each cat map \( M \) via Lemma 3.2.

**Theorem 1.** Let \( M \in \text{Sp}(2n,\mathbb{Z}) \) be hyperbolic. We can pick \( E \in \text{GL}(n,\mathbb{R}) \) satisfying \( \|E^{-1}\| < 1 \) and \( |\det E| > 1 \) such that \( \hat{M} \) is equivalent to
\[
\begin{pmatrix}
E & 0 \\
0 & (E^T)^{-1}
\end{pmatrix}
\]
up to symplectic conjugation. If \( \varphi_M = 0 \), then for any prequantum transfer operator \( \hat{M} \) and parameter \( N \in \mathbb{N}^* \), the Pollicott-Ruelle resonances of the restriction \( \hat{M}_N \) are
\[
|\det E|^{-\frac{1}{2}} \cdot \{\lambda_j\}_{j \in \mathbb{N}} \cdot \sigma(M_{N,\theta}) \subseteq \mathbb{D},
\]
where \( \{\lambda_j\}_{j \in \mathbb{N}} \) are the eigenvalues of the operator
\[
L_E u := u \circ E^{-1}, \quad u \in \mathcal{C}_\infty(\mathbb{R}^n).
\]
The latter are contained in annuli indexed by $k \in \mathbb{N}$ corresponding to the restriction of $L_E$ to the space of homogeneous polynomials on $\mathbb{R}^n$ of order $k$ and given by
\[
\{ z \in \mathbb{C} \mid \|E\|^{-k} \leq |z| \leq \|E^{-1}\|^k \}.
\] (2.2)

**Remark.** (i) Note that the annuli in (2.2) can intersect each other, except for the outermost one which is isolated because $0 < |\lambda_j| < \lambda_0 = 1$ for all $j \in \mathbb{N}^*$. (ii) We have $\sigma(M_{N,0}) \subseteq S^1$ and $|\sigma(M_{N,0})| = N^n$ counting multiplicities. Out of the operators $M_{N,\theta}$, only the one with $\theta = 0$ appears because of the condition $\varphi_M = 0$. (iii) The case $n = 1$ is treated in [Fau07]. As discussed above, however, the paper only deals with a restricted class of cat maps. Moreover, instead of imposing the condition $\varphi_M = 0$, it chooses to restrict the result to even $N$. (iv) We only provide a semi-explicit description of the sequence $\{\lambda_j\}_{j \in \mathbb{N}}$, but our approach leaves the door open to their computation (based on the Jordan block decomposition of $E$) for the interested reader. When $n = 1$, for instance, we recover the fact that $\lambda_j = \lambda^{-j}$, where $\lambda > 1$ is the largest eigenvalue of $M$.

The main incentive to study resonances is that the correlation functions can be expressed as asymptotic expansions over them, up to exponentially small error (see [FT15, Theorem 1.6.3]). Theorem 2 makes precise what we mean by the emergence of quantum dynamics: it tells us that $C_{u,v}$ eventually starts behaving like quantum correlation functions, i.e., quantum elements of the quantum propagators.

**Theorem 2.** Assume $\varphi_M = 0$, pick $E \in \text{GL}(n, \mathbb{R})$ as in Theorem 1, and let $N \in \mathbb{N}^*$. For any $u, v \in C^\infty_N(P)$, there exist $\hat{u}, \hat{v} \in \mathcal{H}_N,\theta$ such that
\[
C_{u,v}(t) = \left| \det E \right|^{-\frac{1}{2}} \langle M_{N,0}^t \hat{u}, \hat{v} \rangle_{\mathcal{H}_{N,0}} + o_{u,v}(\left| \det E \right|^{-\frac{1}{2}}) \quad \text{as } t \to +\infty.
\]

The argument will rely on the fact that, as $t \to +\infty$, the external prequantum resonances on the circle of radius $|\det E|^{-\frac{1}{2}}$ dominate the behavior. This is why the quantization scheme of Faure-Tsuji [FT15, Definition 1.3.6] consists of spectrally projecting the transfer operator onto the outermost annulus of the Pollicott-Ruelle spectrum.

Putting together all the frequencies $N \in \mathbb{Z}$, we thus recover the mixing property of the system while unveiling quantum oscillations around equilibrium.

**Corollary 2.4.** Assume $\varphi_M = 0$. For any $u, v \in C^\infty(P)$, there are quantum states $\hat{u}_N, \hat{v}_N \in \mathcal{H}_{|N|,\theta}, N \in \mathbb{Z}^*$, such that, as $t \to +\infty$, we have
\[
C_{u,v}(t) = \int_P u \, dV_P \int_P \pi \, dV_P + \left| \det E \right|^{-\frac{1}{2}} \sum_{N \in \mathbb{Z}^*} \langle M_{|N|,0}^t \hat{u}_N, \hat{v}_N \rangle_{\mathcal{H}_{|N|,0}} + o_{u,v}(\left| \det E \right|^{-\frac{1}{2}}).
\]
3. Prequantization of classical cat maps

In this section, we construct the objects introduced in Definition 2.1. We also discuss their varying degrees of uniqueness.

3.1. Construction of the prequantum bundle. The reduced \((2n+1)\)-dimensional Heisenberg group \(\mathbb{H}_n^{\text{red}}\) plays an important role. As a reference, the Heisenberg group and its representations are thoroughly introduced in [Fol89, Chapter 1]. We recall that the group \(\mathbb{H}_n^{\text{red}}\) simply consists of the set \(\mathbb{R}^{2n} \times \mathbb{S}^1\) endowed with the group law

\[
(x, e^{2\pi i}) \cdot (x', e^{2\pi i'}) := \left( x + x', e^{2\pi i(s + s' + \frac{1}{2}\sigma(x, x'))} \right),
\]

where \(\sigma : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}\) is the usual symplectic bilinear form on \(\mathbb{R}^{2n}\) defined by

\[
\sigma(x, x') := \langle p, q' \rangle - \langle p', q \rangle.
\]

This group first makes an appearance because it serves as the canvas on which to build a non-trivial principal \(U(1)\)-bundle over \(T^{2n}\). More concretely, if we define the lattice \(\Gamma \subset \mathbb{H}_n^{\text{red}}\) as the image of the group homomorphism \(\mathbb{Z}^{2n} \to \mathbb{H}_n^{\text{red}}\) given by

\[
w \mapsto (w, e^{\pi i Q(w)}),
\]

where the quadratic form \(Q\) on \(\mathbb{R}^{2n}\) is defined by

\[
Q(x) := \langle q, p \rangle, \quad (3.1)
\]

then we obtain a principal \(U(1)\)-bundle by considering \(\pi : \Gamma \setminus \mathbb{H}_n^{\text{red}} \to T^{2n}\). The map \(\pi\) is nothing but the projection onto the first factor. The center of \(\mathbb{H}_n^{\text{red}}\) is the \(\mathbb{S}^1\) factor, and its \(U(1)\)-action generates the fibers of the principal \(U(1)\)-bundle.

In the following statement, we take two prequantum bundles \((P, \pi, \alpha)\) and \((P', \pi', \alpha')\) over \((T^{2n}, \omega)\) to be equivalent if there exists a \(U(1)\)-equivariant bundle isomorphism \(\varphi : P \to P'\) such that \(\varphi^* \alpha' = \alpha\).

**Proposition 3.1.** Any prequantum bundle over \((T^{2n}, \omega)\) is equivalent to exactly one of \((\Gamma \setminus \mathbb{H}_n^{\text{red}}, \pi, \alpha_\kappa), \kappa \in [0, 1)^{2n}\), where \(\alpha_\kappa\) is the principal \(U(1)\)-connection given by

\[
\alpha_\kappa := 2\pi i \left( ds + \frac{1}{2} \sum_{j=1}^{n} (q_j dp_j - p_j dq_j) + \sum_{j=1}^{n} (\kappa_j dp_j - \kappa_{j+n} dq_j) \right).
\]

**Proof.** Each \(\alpha_\kappa\) is well-defined on the quotient \(\Gamma \setminus \mathbb{H}_n^{\text{red}}\) because it is left-invariant under the action of \(\Gamma\). This becomes even easier to check if we more succinctly write

\[
\alpha_\kappa = 2\pi i \left( ds + \frac{1}{2} \sigma(dx, x) + \sigma(dx, \kappa) \right).
\]
Note that the normalization $\alpha_\kappa(\frac{\partial}{\partial x}) = 2\pi i$ is respected, and the $U(1)$ action leaves $\alpha_\kappa$ invariant, so we indeed have $\alpha_\kappa \in \Omega^1(\Gamma \setminus \mathbb{H}_n^{\text{red}}, i\mathbb{R})$. The condition

$$d\alpha_\kappa = 2\pi i(\pi^*\omega)$$

is also satisfied, as desired. This confirms that we are dealing with an explicit family of prequantum bundles over $(\mathbb{T}^{2n}, \omega)$ parametrized by $\kappa \in [0,1)^{2n}$.

All other choices of prequantum bundles are equivalent to $(\Gamma \setminus \mathbb{H}_n^{\text{red}}, \pi)$ up to bundle isomorphism because the first Chern class is always $[\omega] \in H^2(\mathbb{T}^{2n}, \mathbb{Z})$. We can hence fix the principal $U(1)$-bundle to be $(\Gamma \setminus \mathbb{H}_n^{\text{red}}, \pi)$ and consider principal $U(1)$-connections up to gauge transformation.

The gauge transformations of $(\Gamma \setminus \mathbb{H}_n^{\text{red}}, \pi)$ are in one-to-one correspondence with $C^\infty(\mathbb{T}^{2n}, U(1))$. Given $f$ in the latter space, the corresponding gauge transformation is $g_f(x, e^{2\pi i s}) := (x, f(x)e^{2\pi i s})$. Note that $g_f$ is well-defined on the quotient because we can first define it on $\mathbb{H}_n^{\text{red}}$ and then notice that it commutes with the action of $\Gamma$. We will use the fact that, for a principal $U(1)$-connection $\alpha$, we have $g_f^*\alpha = \alpha + \pi^*(f^{-1}df)$.

Let $\alpha \in \Omega^1(\Gamma \setminus \mathbb{H}_n^{\text{red}}, i\mathbb{R})$. The difference $\alpha - \alpha_0$ is invariant by the action of $U(1)$, so we may write $\alpha - \alpha_0 = 2\pi i(\pi^*\beta)$ for some $\beta \in \Omega^1(\mathbb{T}^{2n})$. If $\alpha$ satisfies the curvature condition (2.1), then $\beta$ must be closed. As a result, using the Hodge decomposition, we can write $\beta = dr + h$, where $r \in C^\infty(\mathbb{T}^{2n}, \mathbb{R})$ and $h \in \Omega^1(\mathbb{T}^{2n})$ is harmonic and unique. Now using the gauge corresponding to $e^{-2\pi i r}$, we can get rid of the $dr$ part and assume that $\beta = h$. Harmonic forms on $\mathbb{T}^{2n}$ have the form $h = \langle y, dx \rangle$ with $y \in \mathbb{R}^{2n}$. There is a unique $w \in \mathbb{Z}^{2n}$ such that $y - w \in [0,1)^{2n}$, so we can use the gauge corresponding to $x \mapsto e^{-2\pi i(x,w)}$ to bring $h$ to another harmonic 1-form with coefficients in $[0,1)^{2n}$. By combining two gauges, we can thus find a gauge $g$ such that $g^*\alpha = \alpha_\kappa$ for some $\kappa \in [0,1)^{2n}$. This completes the proof since tracing through the uniqueness statements shows that $\alpha_\kappa$ is equivalent to $\alpha_{\kappa'}$ up to gauge transformation if and only if $\kappa = \kappa'$.

\[\square\]

3.2. Construction of the prequantum cat maps. For a given matrix $M \in \text{Sp}(2n, \mathbb{Z})$, it will be convenient to express it in block form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$ 

We recall that $M \in \text{Sp}(2n, \mathbb{Z})$ if and only if $A^T D - C^T B = I$, $A^T C = C^T A$, and $B^T D = D^T B$. We will need the following lemma.

Lemma 3.2. For each $M \in \text{Sp}(2n, \mathbb{Z})$, there exists a unique $\varphi_M \in \{0,1\}^{2n}$ such that

$$Q(M^{-1}w) - Q(w) = \sigma(\varphi_M, w) \mod 2\mathbb{Z} \quad \text{for all } w \in \mathbb{Z}^{2n},$$

with the quadratic form $Q$ defined in (3.1).
Proof. Let us define a map $\mathbb{Z}^{2n} \to \mathbb{Z}/2\mathbb{Z}$ by

$$w \mapsto Q(M^{-1}w) - Q(w) \mod 2\mathbb{Z}.$$  

It is easy to check that this is a group homomorphism by using the relation

$$Q(w + w') = Q(w) + Q(w') + \sigma(w, w') \mod 2\mathbb{Z} \quad \text{for all } w, w' \in \mathbb{Z}^{2n},$$

as well as the fact that $M \in \text{Sp}(2n, \mathbb{Z})$. The existence and uniqueness of $\varphi_M$ follow. □

Remark. i) We note that the map $M \to \varphi_M$ satisfies

$$\varphi_M^{-1} = M^{-1} \varphi_M \mod (2\mathbb{Z})^{2n} \quad \text{and} \quad \varphi_{MM'} = \varphi_M + M \varphi_{M'} \mod (2\mathbb{Z})^{2n}.$$  

ii) We also notice that

$$Q(M^{-1}w) - Q(w) = \langle CD^T m, m \rangle + \langle AB^T n, n \rangle \mod 2\mathbb{Z}$$

for all $w = (m, n) \in \mathbb{Z}^{2n}$.

Therefore, if the integer matrices $CD^T$ and $AB^T$ have even entries, we have $\varphi_M = 0$.

iii) In the case $n = 1$, we may explicitly write

$$\varphi_M = (CD, AB) \mod (2\mathbb{Z})^2.$$  

Motivated by this, we say that cat maps with $\varphi_M = 0$ are in checkerboard form.

Armed with this notation, we proceed to give a concrete realization of the prequantum cat maps.

Proposition 3.3. Let $M \in \text{Sp}(2n, \mathbb{Z})$ be hyperbolic, and define $\kappa \in \mathbb{R}^{2n}$ by

$$\kappa := \frac{1}{2}(I - M)^{-1}\varphi_M.$$  

(3.3)

The map $\widetilde{M} : \Gamma \setminus \mathbb{H}_{n}^{\text{red}} \to \Gamma \setminus \mathbb{H}_{n}^{\text{red}}$ given by

$$\widetilde{M}(x, e^{2\pi i s}) := \left(Mx, e^{2\pi i \left(s + \frac{1}{2}\sigma(dx, x) + \varphi_M(dx, x)\right)}\right)$$

is the unique (up to a global phase) $\text{U}(1)$-equivariant lift of $M : \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ preserving the principal $\text{U}(1)$-connection

$$\alpha_{\kappa} := 2\pi i \left(ds + \frac{1}{2}\sigma(dx, x) + \varphi_M(dx, x)\right).$$

Moreover, there is no $\text{U}(1)$-equivariant lift of $M$ preserving $\alpha_{\kappa'}$ if $\kappa' \neq \kappa$.

Proof. The map $\widetilde{M}$ is well-defined on the quotient $\Gamma \setminus \mathbb{H}_{n}^{\text{red}}$ because it preserves the subgroup $\Gamma$ when seen as an automorphism of $\mathbb{H}_{n}^{\text{red}}$. Indeed, for any $w \in \mathbb{Z}^{2n}$, we have

$$\left(Mw, e^{\pi i (Q(w) + \sigma(\varphi_M, Mw))}\right) = \left(Mw, e^{\pi i Q(Mw)}\right)$$

thanks to relation (3.2). The fact that $\widetilde{M}$ is a $\text{U}(1)$-equivariant lift of $M$ is then clear.
Since $M$ is hyperbolic, the definition of $\tilde{\kappa}$ makes sense. We claim that $\tilde{M}$ preserves the principal $U(1)$-connection $\alpha_{\tilde{\kappa}}$. To see why this is true, note that writing $(x', e^{2\pi is'}) = \tilde{M}(x, e^{2\pi is})$ is equivalent to having $x' = Mx$ and $s' = s + \frac{1}{2}\sigma(\varphi_M, Mx)$. As a result, by leveraging the relation (3.3), we obtain

$$
\tilde{M}^*\alpha_{\tilde{\kappa}} = 2\pi i \left( ds' + \frac{1}{2}\sigma(dx', x') + \sigma(dx', \tilde{\kappa}) \right) 
= 2\pi i \left( ds + \frac{1}{2}\sigma(M^{-1}\varphi_M, dx) + \frac{1}{2}\sigma(dx, x) + \sigma(dx, M^{-1}\tilde{\kappa}) \right) 
= 2\pi i \left( ds + \frac{1}{2}\sigma(dx, x) + \sigma(dx, \tilde{\kappa}) \right).
$$

Any other $U(1)$-equivariant lift of $M$ must be of the form $(x, e^{2\pi is}) \mapsto (Mx, e^{2\pi i(s+h(x))})$ for some $h \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$ with the right properties. If such a lift preserves a connection of the form $\alpha_{\tilde{\kappa}'}$ for some $\tilde{\kappa}' \in \mathbb{R}^{2n}$, then a calculation analogous to (3.4) shows that $\nabla h$ is constant, so $h$ must be an affine function. Since we are ignoring global phases, we can take $h$ to be linear. But then the fact that the lift is well-defined on $\Gamma \setminus \mathbb{H}^{red}_n$ imposes $h(x) = \frac{1}{2}\sigma(\varphi_M, Mx)$. This gives us the uniqueness result. By doing another calculation like (3.4), it also gives us the last statement of the proposition. \hfill \Box

Let us put together the existence and uniqueness statements from this section. Prequantum bundles over $(\mathbb{T}^{2n}, \omega)$ exist, and they are equivalent to exactly one of $(\Gamma \setminus \mathbb{H}^{red}_n, \pi, \alpha_{\kappa})$ with $\kappa \in [0, 1)^{2n}$. For any hyperbolic $M \in \text{Sp}(2n, \mathbb{Z})$, there is a unique $\tilde{\kappa} \in \mathbb{R}^{2n}$ and a lift $\tilde{M}$ (unique up to global phase) such that $\alpha_{\tilde{\kappa}}$ is preserved by $\tilde{M}$. As in the proof of Proposition 3.1, an explicit gauge $g_f$ sends $\alpha_{\tilde{\kappa}}$ to $\alpha_{\kappa}$, where $\kappa \in [0, 1)^{2n}$ is the fractional part of $\tilde{\kappa} \in \mathbb{R}^{2n}$. The conjugated lift $g_f\tilde{M}g_f^{-1}$ then preserves $\alpha_{\kappa}$.

Any other prequantum cat map on $(\Gamma \setminus \mathbb{H}^{red}_n, \pi)$ must preserve a principal $U(1)$-connection equivalent to one of $\alpha_{\kappa}$ with $\kappa \in [0, 1)^{2n}$. But then we can conjugate it via a gauge transformation to the unique (up to global phase) prequantum cat map preserving $\alpha_{\tilde{\kappa}}$ with $\tilde{\kappa} \in \mathbb{R}^{2n}$ satisfying (3.3). It follows in particular that $\kappa$ has to be the fractional part of $\tilde{\kappa}$.

Conjugation by gauge transformations does not affect the prequantum transfer operator since the latter commutes with the action of $U(1)$, so it is enough for us to consider the prequantum cat maps explicitly described in Proposition 3.3.

4. THE RELATIONSHIP BETWEEN PREQUANTUM AND QUANTUM HILBERT SPACES

We turn our attention to the functions on the prequantum bundle $\Gamma \setminus \mathbb{H}^{red}_n$. We want to characterize them because they play a fundamental role in the description of the dynamics. This effort will turn out to surface an intimate link between the prequantum and quantum Hilbert spaces.
We think of a function in \( \mathcal{C}^\infty(\Gamma \setminus \mathbb{H}_n^\text{red}) \) as a smooth \( \Gamma \)-invariant function on \( \mathbb{H}_n^\text{red} \). Unraveling the definitions, this is nothing but a function \( u \in \mathcal{C}^\infty(\mathbb{R}^{2n} \times S^1) \) satisfying the condition
\[
u \mapsto u(\nu, e^{2\pi i s}) = e^{2\pi i s} \quad \text{for all } w \in \mathbb{Z}^{2n}. \tag{4.1}
\]
From this point of view, belonging to the subspace \( \mathcal{C}_\nu^\infty(\Gamma \setminus \mathbb{H}_n^\text{red}) \) simply adds the restriction of observing
\[
u \mapsto u(\nu, e^{2\pi i s}) = e^{2\pi i N s} u(x, 1).
\]
We notice that the Lebesgue measure is left and right translation invariant on \( \mathbb{H}_n^\text{red} \). It follows that it is the Haar measure on \( \mathbb{H}_n^\text{red} \), and the group is unimodular. Likewise, the Lebesgue measure is the unique Haar measure such that \( \Gamma \setminus \mathbb{H}_n^\text{red} \) has total measure one. We hence suppress it from the notation in what follows. We can endow \( \mathcal{C}^\infty(\Gamma \setminus \mathbb{H}_n^\text{red}) \) with the inner product obtained by integrating over a fundamental domain for \( \Gamma \) such as the unit cube \( [0,1)^{2n+1} \). This yields the Hilbert space \( L^2(\Gamma \setminus \mathbb{H}_n^\text{red}) \).

**Definition 4.1.** For each \( N \in \mathbb{Z} \), the *prequantum Hilbert space* \( \tilde{\mathcal{H}}_N \) is defined to be the \( L^2 \)-completion of \( \mathcal{C}_\nu^\infty(\Gamma \setminus \mathbb{H}_n^\text{red}) \).

We have already implicitly noted the orthogonal decomposition
\[
L^2(\Gamma \setminus \mathbb{H}_n^\text{red}) = \bigoplus_{N \in \mathbb{Z}} \tilde{\mathcal{H}}_N.
\]

4.1. **The Schrödinger representation.** How may one concretely construct a function on the prequantum bundle \( \Gamma \setminus \mathbb{H}_n^\text{red} \)? We take our hint from the condition (4.1).

If we were given a representation \( \rho \) of \( \mathbb{H}_n^\text{red} \) on \( \mathcal{S}'(\mathbb{R}^n) \) and a tempered distribution \( \nu \in \mathcal{S}'(\mathbb{R}^n) \) that was \( \Gamma \)-invariant in the sense that
\[
\rho\left( w, e^{2\pi i Q(w)} \right) \nu = \nu \quad \text{for all } w \in \mathbb{Z}^{2n}, \tag{4.2}
\]
then we could obtain a whole family of functions in \( \mathcal{C}^\infty(\Gamma \setminus \mathbb{H}_n^\text{red}) \) by considering
\[
(x, e^{2\pi i s}) \mapsto \langle \nu, \rho(x, e^{2\pi i s}) f \rangle_{\mathcal{D}'(\mathbb{R}^n)}
\]
for any \( f \in \mathcal{S}(\mathbb{R}^n) \). In order to land in the subspace \( \mathcal{C}_\nu^\infty(\Gamma \setminus \mathbb{H}_n^\text{red}) \), we want the representation \( \rho \) to satisfy \( \rho(0, e^{2\pi i s}) = e^{2\pi i N s} I \).

We are thus naturally led to consider the Schrödinger representation \( \rho_N \) of \( \mathbb{H}_n^\text{red} \) on \( L^2(\mathbb{R}^n) \). We recall (see [Fol89, Section 1.3]) that this representation, defined by
\[
\rho_N(x, e^{2\pi i s}) f(y) := e^{2\pi i N (s + (p, y) - \frac{1}{2} Q(x))} f(y - q), \quad f \in \mathcal{S}(\mathbb{R}^n), \tag{4.3}
\]
is unitary, faithful, and irreducible for \( N \geq 1 \). We also have the characteristic commutator formula
\[
\rho_N(x, e^{2\pi i s}) \rho_N(x', e^{2\pi i s'}) = e^{2\pi i N \sigma(x, x')} \rho_N(x', e^{2\pi i s'}) \rho_N(x, e^{2\pi i s}). \tag{4.4}
\]
As a small parenthesis, which we shall revisit at a later time, we further point out that the Schrödinger representation is intimately related to quantum translations. Indeed, after extending the Schrödinger representation to $S'(\mathbb{R}^n)$ by duality, each operator $\rho_N(x) := \rho_N(x, 1)$
is called a *quantum translation*. Note in particular that $\rho_N(x)$ is a unitary operator on $L^2(\mathbb{R}^n)$. Even though the map $x \mapsto \rho_N(x)$ is not a group homomorphism because
$$\rho_N(x)\rho_N(x') = e^{\pi i \sigma(x, x')} \rho_N(x + x'),$$
there are two main reasons we still speak of quantum translations. Both justifications rely on the key observation that, if we set the semiclassical parameter
$$h := \frac{1}{2\pi N},$$
then by [Zwo12, Theorem 4.7] we have
$$\rho_N(x) = \text{Op}_h \left( \exp \left( \frac{i}{h} \sigma(x, z) \right) \right) = \exp \left( -\frac{i}{h} \text{Op}_h(\sigma(z, x)) \right).$$
Since the classical translation by $x \in \mathbb{R}^{2n}$ is obtained as the time-$1$ flow generated by the Hamiltonian $z \mapsto \sigma(z, x)$, it is reasonable to call $\rho_N(x)$ a quantum translation: in light of the previous relation, it is nothing but the time-$1$ propagator generated by the corresponding quantum Hamiltonian.

A second reason to call each operator $\rho_N(x)$ a quantum translation is that it satisfies the following exact Egorov property:
$$\rho_N^{-1}(x) \text{Op}_h(a) \rho_N(x) = \text{Op}_h(a(\cdot + x)) \quad \text{for all } a \in S(1). \quad (4.5)$$

We refer the reader to [Zwo12] for any necessary refreshers on Weyl quantization on $\mathbb{R}^n$, symbol classes such as $S(1)$, and standard results in semiclassical analysis.

### 4.2. Appearance of the quantum Hilbert space.

Building on the observations of the previous section, we notice that we can slightly alleviate the condition (4.2) and still obtain functions on the prequantum bundle. Indeed, it would be enough if there existed $\theta \in \mathbb{R}^{2n}$ such that
$$\rho_N \left( w, e^{\pi i Q(w)} \right) \nu = e^{\pi i \sigma(\theta, w)} \nu \quad \text{for all } w \in \mathbb{Z}^{2n}$$
because then, by (4.4), we could again obtain functions in $C^\infty_N(\Gamma \setminus \mathbb{H}_n^{\text{red}})$ by considering
$$(x, e^{2\pi is}) \mapsto \left\langle \rho_N \left( \frac{x}{(2\pi i)^s} \right) \nu, \rho_N(x, e^{2\pi is}) f \right\rangle_{D' (\mathbb{R}^n)}. \quad (4.6)$$
It hence behooves us to understand the following subspace.

**Definition 4.2.** Given $N \in \mathbb{N}^*$ and $\theta \in \mathbb{T}^{2n}$, we define the *quasi-periodic distributions*
$$\mathcal{H}_{N, \theta} := \left\{ \nu \in S'(\mathbb{R}^n) \mid \rho_N \left( w, e^{\pi i Q(w)} \right) \nu = e^{2\pi i \sigma(\theta, w)} \nu \quad \text{for all } w \in \mathbb{Z}^{2n} \right\}.$$
Intuitively, these are \( \mathbb{Z}^n \)-periodic (up to a phase) tempered distributions on \( \mathbb{R}^n \) whose \( \hbar \)-Fourier transform is also \( \mathbb{Z}^n \)-periodic (again, up to phase). Here the relation between the semiclassical parameters \( \hbar > 0 \) and \( N \in \mathbb{N}^* \) is still \( \hbar = (2\pi N)^{-1} \).

As it turns out, once we endow \( \mathcal{H}_{N,\theta} \) with an inner product, we obtain nothing short of the usual quantum Hilbert space, sometimes also called the space of quantum states. To see how we may define an inner product, we start by noticing that it is finite dimensional. In what follows, we shall write

\[
\mathbb{Z}_N := \{0, \ldots, N - 1\}.
\]

**Lemma 4.3.** The space \( \mathcal{H}_{N,\theta} \) is \( N^n \)-dimensional with a basis given by \( e_j^\theta, \ j \in \mathbb{Z}_N^n \), where for \( \theta = (\theta_1, \theta_2) \in \mathbb{R}^{2n} \) we define

\[
e_j^\theta(y) := N^{-\frac{n}{2}} \sum_{k \in \mathbb{Z}^n} e^{-2\pi i \langle \theta_2, k \rangle} \delta \left(y - k - \frac{j - \theta_1}{N}\right).
\]

(4.7)

**Remark.** The distributions \( e_j^\theta \) satisfy the identities

\[
e_j^\theta + w = e_{j-m}^\theta \quad \text{for all } w = (m, n) \in \mathbb{Z}^{2n},
e_{j+l}^\theta = e^{2\pi i \langle \theta_2, l \rangle} e_j^\theta \quad \text{for all } l \in \mathbb{Z}^n.
\]

(4.8)

We refer to the proof of [DJ23, Lemma 2.5] for a justification. Note that, while the space \( \mathcal{H}_{N,\theta} \) is canonically defined for \( \theta \in \mathbb{T}^{2n} \), the basis \( \{e_j^\theta\} \) depends on a choice of representative \( \theta_1 \in \mathbb{R}^n \). We may turn each \( \mathcal{H}_{N,\theta} \) into a Hilbert space by defining an inner product characterized by \( \{e_j^\theta\} \) becoming an orthonormal basis. As follows from (4.8), this choice of inner product only depends on \( \theta \in \mathbb{T}^{2n} \).

Thanks to the commutator formula (4.4) and our choice of inner product on the Hilbert spaces \( \mathcal{H}_{N,\theta} \), each quantum translation \( \rho_N(x) \) is a unitary transformation from \( \mathcal{H}_{N,\theta} \) onto \( \mathcal{H}_{N,\theta-Nx} \). Since we will use this observation a few times, we give it a name.

**Definition 4.4.** For \( N \in \mathbb{N}^* \) and \( \theta \in \mathbb{T}^{2n} \), we denote by \( T_{N,\theta} : \mathcal{H}_{N,\theta} \rightarrow \mathcal{H}_{N,\theta} \) the unitary transformation given by

\[
T_{N,\theta} := \rho_N \left(\frac{\{\theta\}}{N}\right),
\]

(4.9)

where \( \{\theta\} \in [0, 1)^{2n} \) denotes the fractional part of any representative of \( \theta \) in \( \mathbb{R}^{2n} \).

4.3. **Decomposition of the prequantum Hilbert spaces.** Having unveiled the nature of the \( \Gamma \)-invariant tempered distributions using the Schrödinger representation, we elevate the observation (4.6) to a definition.

**Definition 4.5.** For \( N \in \mathbb{N}^* \) and \( \theta \in \mathbb{T}^{2n} \), we denote by

\[
U_{N,\theta} : \mathcal{H}_{N,\theta} \otimes \mathcal{S}(\mathbb{R}^n) \rightarrow C^\infty_N(\Gamma \setminus \mathbb{H}_{n}^{\text{red}})
\]
the pairing characterized by
\[ \nu \otimes f \mapsto N^\frac{n}{2} \langle T_{N,0} \nu, \rho_N(\cdot) f \rangle_{D'(\mathbb{R}^n)}. \]

Using (4.3) and (4.7), we note in particular that, for each \( f \in \mathcal{S}(\mathbb{R}^n) \), we have
\[ U_{N,0}(e_j^0 \otimes f)(x, e^{2\pi i s}) = \sum_{k \in \mathbb{Z}^n} \rho_N(x, e^{2\pi i s}) f \left( k + \frac{j}{N} \right) = e^{2\pi i N(s + \varphi_{\rho_N}(x) - \frac{1}{2} Q(x))} \sum_{k \in \mathbb{Z}^n} e^{2\pi i N(p,k)} f \left( k + \frac{j}{N} - q \right). \] (4.10)

Therefore, the map \( U_{N,0} \) is closely related to the Fourier-Wigner and Weil-Brazin transforms, well-known objects used in the harmonic analysis of Heisenberg nilmanifolds (see for instance [Fol89, Section 1.4] or [Fol04, Section 3] for a discussion).

The next result captures the idea that the prequantum Hilbert spaces \( \tilde{\mathcal{H}}_N \), which are infinite-dimensional, "contain" the finite-dimensional quantum Hilbert spaces \( \mathcal{H}_{N,\theta} \).

**Proposition 4.6.** Each operator \( U_{N,\theta} \) initially defined on \( \mathcal{H}_{N,\theta} \otimes \mathcal{S}(\mathbb{R}^n) \) extends to \( \mathcal{H}_{N,\theta} \otimes L^2(\mathbb{R}^n) \) as a unitary transformation onto \( \tilde{\mathcal{H}}_N \).

**Proof.** We have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{H}_{N,\theta} \otimes L^2(\mathbb{R}^n) & \xrightarrow{T_{N,\theta} \otimes I} & \mathcal{H}_{N,0} \otimes L^2(\mathbb{R}^n) \\
\downarrow U_{N,\theta} & & \downarrow U_{N,0} \\
\tilde{\mathcal{H}}_N & & \tilde{\mathcal{H}}_N \\
\end{array}
\]

As a result, it suffices to consider the case \( \theta = 0 \).

We first show the existence of a unique bounded extension. For \( f, g \in \mathcal{S}(\mathbb{R}^n) \), equation (4.10) and Fubini’s theorem give us
\[
\langle U_{N,0}(e_j^0 \otimes f), U_{N,0}(e_l^0 \otimes g) \rangle_{\tilde{\mathcal{H}}_N} = N^n \langle \langle e_j^0, \rho_N(\cdot) f \rangle_{D'(\mathbb{R}^n)}, \langle e_l^0, \rho_N(\cdot) g \rangle_{D'(\mathbb{R}^n)} \rangle_{L^2([0,1)^{2n})}
= \sum_{k, k' \in \mathbb{Z}^n} \int_{[0,1)^n} e^{2\pi i (p,N(k-k') + j-l)} \, dp 
\int_{[0,1)^n} f \left( k + \frac{j}{N} - q \right) \overline{g} \left( k' + \frac{l}{N} - q \right) \, dq
= \delta_{j=l} \sum_{k \in \mathbb{Z}^n} \int_{[0,1)^n - k} f \left( \frac{j}{N} - q \right) \overline{g} \left( \frac{j}{N} - q \right) \, dq
= \delta_{j=l} \int_{\mathbb{R}^n} f \left( \frac{j}{N} - q \right) \overline{g} \left( \frac{j}{N} - q \right) \, dq
= \langle e_j^0, e_l^0 \rangle_{\mathcal{H}_{N,0}} \langle f, g \rangle_{L^2(\mathbb{R}^n)}. \]
This also shows that $U_{N,0}$ is unitary.

To see why it is surjective onto $\tilde{\mathcal{H}}_N$, we want to show that each $u \in \tilde{\mathcal{H}}_N$ can be written as $u = \sum_{j \in \mathbb{Z}_N^n} U_{N,0}(e_0^j \otimes f_j)$ with $f_j \in L^2(\mathbb{R}^n)$. In what follows, we adapt the proof of [Tha09, Theorem 3.6].

Let $v(x) := u(p, -q, 1)$. For each $j \in \mathbb{Z}_N^n$ and $k \in \mathbb{Z}^n$, we further introduce
\[
v_{j,k}(x) := e^{-\pi i (k,p)} e^{-\frac{2\pi i}{N} (j,k)} v \left( q + \frac{k}{N}, p \right) \quad \text{and} \quad v_j := N^{-n} \sum_{k \in \mathbb{Z}_N^n} v_{j,k}.
\]
This gives us $v = \sum_{j \in \mathbb{Z}_N^n} v_j$ because
\[
\sum_{j,k \in \mathbb{Z}_N^n} v_{j,k}(x) = \sum_{k \in \mathbb{Z}_N^n} e^{-\pi i (k,p)} v \left( q + \frac{k}{N}, p \right) \left( \sum_{j \in \mathbb{Z}_N^n} e^{-\frac{2\pi i}{N} (j,k)} \right) = N^n v(x).
\]
By the $\Gamma$-invariance of $u$ described in (4.1), we note that
\[
v(x + w) = e^{\pi i N (Q(w) + \sigma(x,w))} v(x) \quad \text{for all} \ w \in \mathbb{Z}^n. \tag{4.11}
\]
A quick calculation shows that the same property is satisfied by each $v_j$. Next, for all $l \in \mathbb{Z}^n$, we claim that
\[
v_{j}(q + l, p) = e^{\pi i (l,p)} e^{\frac{2\pi i}{N} (j,l)} v_j(x). \tag{4.12}
\]
To see why this is true, note that applying property (4.11) yields
\[
v_{j,k + Nl}(x) = e^{-\pi i (k + Nl, p)} e^{-\frac{2\pi i}{N} (j,k)} v \left( q + \frac{k}{N} + l, p \right).
\]
and hence
\[
\sum_{k \in \mathbb{Z}_N^n} v_{j,k + l} = \sum_{k \in \mathbb{Z}_N^n} v_{j,k}.
\]
This allows us to write
\[
v_{j} \left( q + \frac{l}{N}, p \right) = N^{-n} \sum_{k \in \mathbb{Z}_N^n} e^{-\pi i (k,p)} e^{-\frac{2\pi i}{N} (j,k)} v \left( q + \frac{k + l}{N}, p \right) = e^{\pi i (l,p)} e^{\frac{2\pi i}{N} (j,l)} N^{-n} \sum_{k \in \mathbb{Z}_N^n} v_{j,k + l}(x) = e^{\pi i (l,p)} e^{\frac{2\pi i}{N} (j,l)} v_j(x),
\]
as desired. Property (4.12) implies that the function
\[
g_{j}(x) := e^{-2\pi i (j,q)} e^{-\pi i Q(x)} v_j(x)
\]
is $\frac{1}{N}$-periodic in the $q$ variables. As a result, it admits a decomposition of the form

$$g_j(x) = \sum_{k \in \mathbb{Z}^n} c_k(p)e^{2\pi iN(k,q)},$$

where $c_k$ are the Fourier coefficients given by

$$c_k(p) := \int_{[0,\frac{1}{N})^n} e^{-2\pi iN(k,q)}g_j(x) \, dq.$$ 

Since property (4.11) implies that $g_j(q, p-k) = g_j(x)e^{2\pi iN(k,q)}$, we get $c_k(p-k) = c_0(p)$. This leads to

$$v_j(x) = e^{2\pi i(j,q)}e^{\pi iNQ(x)} \sum_{k \in \mathbb{Z}^n} c_0(p + k)e^{2\pi iN(k,q)}.$$ 

In other words, comparing with (4.10), we have $v_j(x) = U_{N,0}(e_0^j \otimes f_j)(p, -q, 1)$, where

$$f_j(y) := c_0\left(y + \frac{j}{N}\right).$$

We conclude by noticing that $f_j \in L^2(\mathbb{R}^n)$ and, as we wanted,

$$u(x, e^{2\pi is}) = e^{2\pi iNs}v(-p, q) = e^{2\pi iNs} \sum_{j \in \mathbb{Z}^n_j} v_j(-p, q) = \sum_{j \in \mathbb{Z}^n_j} U_{N,0}(e_0^j \otimes f_j)(x, e^{2\pi is}).$$

The following corollary tells us that each $U_{N,\theta}$ "preserves the smooth observables."

**Corollary 4.7.** Each operator $U_{N,\theta}$ restricts to an isomorphism

$$\mathcal{H}_{N,\theta} \otimes \mathcal{S}(\mathbb{R}^n) \rightarrow C^\infty_N(\Gamma \setminus \mathbb{H}_n^{\text{red}}).$$

**Proof.** Again, it suffices to check for $U_{N,0}$. Stepping through the proof of Proposition 4.6, we note that each $f_j$ belongs to $\mathcal{S}(\mathbb{R}^n)$ if $u \in \mathcal{H}_N$ is taken to be smooth.

\[ \square \]

5. Study of the prequantum transfer operator

We have just shown that the prequantum Hilbert space $\mathcal{H}_N$ is unitarily equivalent to each tensor product $\mathcal{H}_{N,\theta} \otimes L^2(\mathbb{R}^n)$. We now turn our attention to the transfer operator associated to a prequantum cat map. In principle, there is no reason this operator, once conjugated to act on $\mathcal{H}_{N,\theta} \otimes L^2(\mathbb{R}^n)$, should behave nicely with respect to the tensor product decomposition. Nevertheless, we will show that, under the appropriate quantization condition between the classical cat map $M$ and the parameters $N$ and $\theta$, ...
not only does the conjugated transfer operator also decompose as a tensor product, but the component acting on $\mathcal{H}_{N,\theta}$ is precisely the usual quantum cat map $M_{N,\theta}$.

5.1. Quantum cat maps. In this section, we very briefly review quantum cat maps, referring the reader to [DJ23, Section 2.2] for more details. As we have done before, we fix $N \in \mathbb{N}^*$ and set $h := (2\pi N)^{-1}$ so that we may use the two semiclassical parameters interchangeably depending on what notation is more convenient or standard.

The first building block is the quantization of observables on the torus $T^{2n}$. Let us consider $a \in C^\infty(T^{2n})$. This is nothing but a smooth $\mathbb{Z}^{2n}$-periodic function on $\mathbb{R}^{2n}$. As a result, the observable belongs to the symbol class $S(1)$, so its Weyl quantization $\text{Op}_h(a)$ is a bounded operator on $L^2(\mathbb{R}^n)$. The key observation is that, since $a$ is $\mathbb{Z}^{2n}$-periodic, the exact Egorov property (4.5) yields the commutation relations

$$\text{Op}_h(a)\rho_N(w) = \rho_N(w)\text{Op}_h(a) \quad \text{for all } w \in \mathbb{Z}^{2n}.$$  

Thanks to this feature, it then follows that the operator $\text{Op}_h(a)$, which we extend to $S'(\mathbb{R}^n)$ by duality as usual, preserves each quantum Hilbert space $\mathcal{H}_{N,\theta}$. We can thus define the quantizations

$$\text{Op}_{N,\theta}(a) := \text{Op}_h(a)|_{\mathcal{H}_{N,\theta}} : \mathcal{H}_{N,\theta} \to \mathcal{H}_{N,\theta},$$

which depend smoothly on $\theta \in T^{2n}$.

We next consider the quantization of a symplectic integral matrix $M \in \text{Sp}(2n, \mathbb{Z})$. If $M = e^X$ for some $X \in \text{sp}(2n, \mathbb{R})$, then $M$ describes the time-1 dynamics of a Hamiltonian flow on $\mathbb{R}^{2n}$. Indeed, the flow generated by the quadratic Hamiltonian $H(z) := \sigma(Xz, z)$ is $z(t) = e^{tX}z_0$. The quantum propagator $M_N := \exp\left(-\frac{i}{\hbar}\text{Op}_h(H)\right)$ is then a unitary operator on $L^2(\mathbb{R}^n)$ satisfying

$$M_N^{-1}\text{Op}_h(a)M_N = \text{Op}_h(a \circ M) \quad \text{for all } a \in S(1).$$

In particular, we have the following intertwining relation with quantum translations:

$$M_N^{-1}\rho_N(x)M_N = \rho_N(M^{-1}x) \quad \text{for all } x \in \mathbb{R}^{2n}.\quad (5.2)$$

One consequence of this property is that

$$M_N(\mathcal{H}_{N,\theta}) \subseteq \mathcal{H}_{N,M\theta + \frac{N}{2}\varphi_M}.$$ 

If we choose $\theta \in T^{2n}$ such that

$$(I - M)\theta = \frac{N}{2}\varphi_M \quad \text{mod } \mathbb{Z}^{2n},$$

we hence obtain a unitary operator

$$M_{N,\theta} := M_N|_{\mathcal{H}_{N,\theta}} : \mathcal{H}_{N,\theta} \to \mathcal{H}_{N,\theta}.$$
When $M$ is hyperbolic, a parameter $\theta \in T^{2n}$ satisfying the quantization condition (5.3) is guaranteed to exist, and we call $M_{N,\theta}$ a quantum cat map. It then becomes clear from the property (5.1) that we also have an exact Egorov property on the torus:

$$M_{N,\theta}^{-1}\text{Op}_{N,\theta}(a)M_{N,\theta} = \text{Op}_{N,\theta}(a \circ M) \quad \text{for all } a \in C^\infty(T^{2n}).$$

What happens if $M \in \text{Sp}(2n,\mathbb{Z})$ is not in the image of the Lie algebra $\mathfrak{sp}(2n,\mathbb{R})$ under the exponential map? We point out that all we really needed in order to define the quantum cat maps was the property (5.1).

For each $M \in \text{Sp}(2n,\mathbb{R})$, denote by $M_{N,M}$ the set of all unitary transformations $M_{N} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ satisfying the property (5.1). These transformations exist, they are unique up to a complex phase, and they preserve $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ (see [Zwo12, Theorem 11.9]). The set

$$\mathcal{M}_N := \bigcup_{M \in \text{Sp}(2n,\mathbb{R})} \mathcal{M}_{N,M}$$

is a subgroup of the unitary transformations of $L^2(\mathbb{R}^n)$ called the metaplectic group. The map $M_N \to M$ is a group homomorphism $\mathcal{M}_N \to \text{Sp}(2n,\mathbb{R})$. See [Fol89, Chapter 4] for more details.

In the general case, we thus define our quantization procedure by using the same procedure as before but by choosing a metaplectic operator $M_N$. Since we are not concerned with the complex phase (we care about spectral properties), the specific choice does not matter. By uniqueness up to a phase, we know that the definition of the quantum cat maps using Weyl quantization agrees with the more general one.

5.2. Factorization of the prequantum transfer operator. We tackle the study of the prequantum transfer operator $\widehat{M}_N : \mathcal{H}_N \to \mathcal{H}_N$ associated to a classical cat map $M \in \text{Sp}(2n,\mathbb{Z})$. More precisely, seeking to better understand the dynamics generated by $\widehat{M}_N$, we examine the unitarily equivalent operator $U_{N,\theta}^{-1}\widehat{M}_NU_{N,\theta}$:

$$\begin{array}{ccc}
\mathcal{H}_{N,\theta} \otimes L^2(\mathbb{R}^n) & \longrightarrow & \mathcal{H}_{N,\theta} \otimes L^2(\mathbb{R}^n) \\
U_{N,\theta} & & U_{N,\theta}^{-1} \\
\mathcal{H}_N & \xrightarrow{\widehat{M}_N} & \mathcal{H}_N
\end{array}$$

(5.4)

The explicit formulas quickly get involved and are not very elucidating. We instead take a small turnabout. Let us introduce the operator $P_{N,\theta} : S(\mathbb{R}^n) \to \mathcal{H}_{N,\theta}$ given by

$$P_{N,\theta} := \sum_{w \in \mathbb{Z}^{2n}} e^{-2\pi i \sigma(\theta,w)} \rho_N(w, e^{\pi i Q(w)}).$$

(5.5)

It is well-defined because the series converges in $S'(\mathbb{R}^n)$ and

$$\rho_N(w, e^{\pi i Q(w)})P_{N,\theta} = e^{2\pi i \sigma(\theta,w)}P_{N,\theta} \quad \text{for all } w \in \mathbb{Z}^{2n},$$
so we indeed land in the quantum Hilbert space $\mathcal{H}_{N,\theta}$.

Through an explicit computation using (4.3), (4.7), and the Poisson summation formula, one can check that the operator $P_{N,\theta}$ defined by (5.5) can also be written as

$$P_{N,\theta}f = \sum_{j \in \mathbb{Z}_N} \langle e_j(\theta_1, \ldots, \theta_2), f \rangle_{\mathcal{D}'(\mathbb{R}^n)} e_j^\theta$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. (5.6)

**Lemma 5.1.** The operator $P_{N,\theta} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{H}_{N,\theta}$ is surjective.

**Proof.** The commutator formula (4.4) combined with definitions (4.9) and (5.5) yield

$$T_{N,\theta}^{-1} P_{N,0} T_{N,\theta} = P_{N,\theta}.$$

Thus it suffices to consider the case $\theta = 0$. We argue similarly to [DJ23, Lemma 2.6].

Let $\nu \in \mathcal{H}_{N,0}$ and define

$$R\nu(y) := N^{\frac{n}{2}} \int_{T^n} \langle T_{N,(Ny,-\theta_2)} \nu, e_0^{(-Ny,\theta_2)} \rangle_{\mathcal{H}_{N,(0,\theta_2)}} d\theta_2, \quad y \in \mathbb{R}^n.$$

One can check that $R\nu \in \mathcal{S}(\mathbb{R}^n)$ using a non-stationary phase argument and the following consequence of (4.8):

$$R\nu(y - k) = R(e^{2\pi i \langle \theta_2, k \rangle} \nu)(y) \quad \text{for all } k \in \mathbb{Z}^n.$$

We then compute

$$\langle e_j^0, R\nu \rangle_{\mathcal{D}'(\mathbb{R}^n)} = N^{-\frac{n}{2}} \sum_{k \in \mathbb{Z}^n} R\nu\left( k + \frac{j}{N} \right)$$

$$= \sum_{k \in \mathbb{Z}^n} \int_{T^n} \langle T_{N,(0,-\theta_2)} \nu, e_0^{(-N(k-j),\theta_2)} \rangle_{\mathcal{H}_{N,(0,\theta_2)}} d\theta_2$$

$$= \sum_{k \in \mathbb{Z}^n} \int_{T^n} e^{-2\pi i \langle \theta_2, k \rangle} \langle T_{N,(0,-\theta_2)} \nu, e_j^{(0,\theta_2)} \rangle_{\mathcal{H}_{N,(0,\theta_2)}} d\theta_2$$

$$= \langle \nu, e_j \rangle_{\mathcal{H}_{N,0}},$$

where we’ve used (4.8) again as well as the convergence of the Fourier series of the function $\theta_2 \mapsto \langle T_{N,(0,-\theta_2)} \nu, e_j^{(0,\theta_2)} \rangle_{\mathcal{H}_{N,(0,\theta_2)}}$. By (5.6), this yields $P_{N,0} R\nu = \nu$, as desired.

The following property will also be useful.

**Lemma 5.2.** Let $M \in \text{Sp}(2n,\mathbb{Z})$ be hyperbolic, fix $N \in \mathbb{N}^*$, and choose $\theta \in \mathbb{T}^{2n}$ satisfying (5.3). Then, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{S}(\mathbb{R}^n) & \xrightarrow{P_{N,\theta}} & \mathcal{S}(\mathbb{R}^n) \\
\downarrow P_{N,\theta} & & \downarrow P_{N,\theta} \\
\mathcal{H}_{N,\theta} & \xrightarrow{M_{N,\theta}} & \mathcal{H}_{N,\theta}
\end{array}$$
Proof. As noted before, there always exists \( \theta \in \mathbb{T}^{2n} \) satisfying condition (5.3) because the matrix \( M \) is hyperbolic by assumption. Using properties (3.2) and (5.3), we obtain

\[
P_{N,\theta} = \sum_{w \in \mathbb{Z}^{2n}} e^{-2\pi i \sigma(M\theta, Mw)} \rho_N(w, e^{\pi i (Q(Mw) + \sigma(\varphi_M, Mw)))}
\]

\[
= \sum_{w \in \mathbb{Z}^{2n}} e^{-2\pi i (\sigma(\theta, Mw) - \frac{1}{2} \sigma(\varphi_M, Mw)))} \rho_N(w, e^{\pi i (Q(Mw) + \sigma(\varphi_M, Mw)))})
\]

\[
= \sum_{w \in \mathbb{Z}^{2n}} e^{-2\pi i \sigma(\theta, Mw)} \rho_N(w, e^{\pi i Q(Mw)}).
\]

Therefore, using the intertwining relation (5.2) yields

\[
M_{N,\theta} P_{N,\theta} = \sum_{w \in \mathbb{Z}^{2n}} e^{-2\pi i \sigma(\theta, Mw)} M_N \rho_N(w, e^{\pi i Q(Mw)})
\]

\[
= \sum_{w \in \mathbb{Z}^{2n}} e^{-2\pi i \sigma(\theta, Mw)} \rho_N(Mw, e^{\pi i Q(Mw)}) M_N
\]

\[
= P_{N,\theta} M_N.
\]

Armed with this, we then want to find an operator on \( S(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n) \) making the following diagram commute:

\[
\begin{array}{ccc}
S(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n) & \longrightarrow & S(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n) \\
P_{N,\theta} \otimes I & \downarrow & P_{N,\theta} \otimes I \\
\mathcal{H}_{N,\theta} \otimes L^2(\mathbb{R}^n) & \longrightarrow & \mathcal{H}_{N,\theta} \otimes L^2(\mathbb{R}^n) \\
U_{N,\theta} & \downarrow & U_{N,\theta} \\
\tilde{\mathcal{H}}_N & \longrightarrow & \tilde{\mathcal{H}}_N \\
\end{array}
\]

(5.7)

We do this when \( \varphi_M = 0 \). The trick lies in using metaplectic operators.

**Proposition 5.3.** If \( \varphi_M = 0 \), then \( M_N \otimes M_N \) makes the diagram (5.7) commute.

**Proof.** Since \( M_{N,0} \) is a unitary transformation of the finite-dimensional space \( \mathcal{H}_{N,0} \), we can use an orthonormal basis of eigenvectors to check that

\[
KM_{N,0}K = M_{N,0},
\]

where \( K \) is the complex conjugation operator. Indeed, the complex conjugate of an eigenvector to a given eigenvalue is an eigenvector for the conjugate eigenvalue.

We recall that, from Proposition 3.3, we have

\[
(\tilde{M}_N u)(x, e^{2\pi i s}) = u(M^{-1} x, e^{2\pi i s}).
\]
Therefore, for all \( f, g \in S(\mathbb{R}^n) \), we can apply the intertwining relation (5.2), the fact that \( M_N \) is unitary on \( L^2(\mathbb{R}^n) \), property (5.8), and Lemma 5.2, to write

\[
\hat{M}_N(U_{N,0}(P_N,0 f \otimes g))(x, e^{2\pi i s}) = N^{\frac{n}{2}} \langle P_N,0 f, \rho_N (M_N^{-1} x, e^{2\pi i s}) g \rangle_{D'(\mathbb{R}^n)}
\]

\[
= N^{\frac{n}{2}} \langle P_{N,0} f, M_N^{-1} \rho_N (x, e^{2\pi i s}) M_N g \rangle_{D'(\mathbb{R}^n)}
\]

\[
= N^{\frac{n}{2}} \langle K M_N K P_{N,0} f, \rho_N (x, e^{2\pi i s}) M_N g \rangle_{D'(\mathbb{R}^n)}
\]

\[
= N^{\frac{n}{2}} \langle P_{N,0} M_N f, \rho_N (x, e^{2\pi i s}) M_N g \rangle_{D'(\mathbb{R}^n)}
\]

This shows that

\[
\hat{M}_N(U_{N,0}(P_N,0 f \otimes g)) = U_{N,0}(P_{N,0} M_N f \otimes M_N g).
\]

Applying Lemmas 5.1 and 5.2, we thus obtain the following corollary.

**Corollary 5.4.** If \( \varphi_M = 0 \), we have

\[
U_{N,0}^{-1} \hat{M}_N U_{N,0} = M_{N,0} \otimes M_N
\]

on \( H_{N,0} \otimes L^2(\mathbb{R}^n) \). This completes the commutative diagram (5.4).

### 5.3. Pollicott-Ruelle spectrum of the prequantum transfer operator.

In this section, we prove Theorems 1 and 2. We will use the tensor product decomposition from Corollary 5.4 to show that, when \( \varphi_M = 0 \), the Pollicott-Ruelle resonances of \( \hat{M}_N \) are given by \( e^{i\varphi_j} e^{i\lambda} \), where \( e^{i\varphi_j}, \varphi_j \in \mathbb{R} \), are the eigenvalues of \( M_{N,0} \) on \( H_{N,0} \) and \( e^{i\lambda}, \lambda \in \mathbb{C} \), are the Pollicott-Ruelle resonances of the operator \( M_N \).

The resonances of \( M_N \) are defined via correlation functions \( C_{u,v} \) as in Section 2 but with observables \( u \) and \( v \) in \( S(\mathbb{R}^n) \). We have not yet shown that these are well-defined.

The strategy is to reduce the problem to a study of the transfer operator

\[
L_E u := u \circ E^{-1}
\]

associated to an expanding linear map \( E : \mathbb{R}^n \to \mathbb{R}^n \). As a reminder, we say that an invertible matrix \( E \in \text{GL}(n, \mathbb{R}) \) is expanding if \( \| E^{-1} \| < 1 \).

Then, [FT15, Proposition 3.4.6] tells us that the transfer operator \( L_E \) has a well-defined discrete Pollicott-Ruelle spectrum contained in annuli indexed by \( k \in \mathbb{N} \) and given by (2.2). Each annulus corresponds to the restriction of \( L_E \) to the space of homogeneous polynomials on \( \mathbb{R}^n \) of order \( k \).

**Proposition 5.5.** Let \( M \in \text{Sp}(2n, \mathbb{R}) \) be hyperbolic. There exists an expanding matrix \( E \in \text{GL}(n, \mathbb{R}) \) with \( | \text{det} E | > 1 \) such that each metaplectic operator \( M_N \) on \( L^2(\mathbb{R}^n) \) is unitarily equivalent to the unitary operator \( | \text{det} E |^{-\frac{1}{2}} L_E \) on \( L^2(\mathbb{R}^n) \). The equivalence preserves \( S(\mathbb{R}^n) \).
Proof. The first step of our argument is to show that there exists $D \in \text{Sp}(2n, \mathbb{R})$ and an expanding $E \in \text{GL}(n, \mathbb{R})$ such that

$$
D^{-1}MD = \begin{pmatrix} E & 0 \\ 0 & (E^T)^{-1} \end{pmatrix}.
$$

To see why this is true, we consider the stable and unstable subspaces $E^s, E^u \subset \mathbb{R}^{2n}$ associated to the hyperbolic matrix $M$. For any $x, y \in E^s$ (resp. $E^u$) and $n \in \mathbb{Z}$, we have $\omega(x, y) = \omega(M^nx, M^ny)$. Therefore, upon taking the limit $n \to \infty$ (resp. $n \to -\infty$), we note that $\omega(x, y) = 0$. This implies that $E^s$ and $E^u$ are isotropic. Since $\mathbb{R}^{2n} = E^s \oplus E^u$, it follows by a count of dimensions that both subspaces are Lagrangian.

Since $\text{Sp}(2n, \mathbb{R})$ acts transitively on the set of pairs of transverse Lagrangians, there exists $D \in \text{Sp}(2n, \mathbb{R})$ simultaneously mapping $E^s$ to $\mathbb{R}^n \oplus \{0\}$ and $E^u$ to $\{0\} \oplus \mathbb{R}^n$. The stable and unstable subspaces are preserved by $M$, which acts as an expanding map on $E^u$ and a contraction on $E^s$. This implies that we must have the relation (5.9) for some expanding $E \in \text{GL}(n, \mathbb{R})$. It is then clear that $|\det E| > 1$.

Since the map $M_N \to M$ from the metaplectic group to $\text{Sp}(2n, \mathbb{R})$ is a group homomorphism, we have $(D^{-1}MD)_N = D^{-1}_NMD_N$ up to a complex phase. This tells us in particular that the operator $M_N : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is unitarily equivalent to $(D^{-1}MD)_N : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Note that $D_N$ preserves the subspace $\mathcal{S}(\mathbb{R}^n)$.

We claim that, up to a complex phase, we have

$$
(D^{-1}MD)_N = |\det E|^{-\frac{1}{2}}L_E.
$$

By [Zwo12, Theorem 11.9], this is equivalent to asking that

$$
L_E^{-1}\text{Op}_h(a)L_E = \text{Op}_h(a \circ (D^{-1}MD)) \quad \text{for all } a \in S(1).
$$

This can be readily checked by using the definition

$$
[\text{Op}_h(a)u](x) := (2\pi h)^{-n} \int e^{\frac{i}{2h}(\xi, x-y)}a\left(\frac{x-y}{2h}, \xi\right) u(y) \, d\xi \, dy, \quad u \in \mathcal{S}(\mathbb{R}^n),
$$

the change of variables formula, and an argument by density.

We can now collect the results of the previous sections.

Proof of Theorem 1. Suppose that $\varphi_M = 0$, and fix the parameter $N \in \mathbb{N}^*$. Let $\{\nu_j \mid j \in \mathbb{Z}_N^*\}$ be an orthonormal basis of $\mathcal{H}_{N,0}$ consisting of eigenvectors of the unitary operator $M_{N,0}$. We will suppose that $M_{N,0}\nu_j = e^{i\varphi_j}\nu_j$, with $\varphi_j \in \mathbb{R}$.

Let $u, v \in \mathcal{C}_N^\infty(\Gamma \setminus \mathbb{H}_n^{red})$. By Corollary 4.7, there exist $a_j, b_j \in \mathbb{C}$ and functions $f_j, g_j \in \mathcal{S}(\mathbb{R}^n)$ such that $U_{N,0}^{-1}u = \sum_{j \in \mathbb{Z}_N^*} a_j\nu_j \otimes f_j$ and $U_{N,0}^{-1}v = \sum_{j \in \mathbb{Z}_N^*} b_j\nu_j \otimes g_j$. 
Using Corollary 5.4 and the fact that $U_{N,0}$ is unitary, we may hence calculate

$$C_{u,v}(t) = \langle \hat{M}_N^t u, v \rangle_{\mathcal{H}_N}$$

$$= \langle U_{N,0}(M_{N,0}^t \otimes M_{N}^t)U_{N,0}^{-1}u, v \rangle_{L^2(P)}$$

$$= \langle (M_{N,0}^t \otimes M_{N}^t)U_{N,0}^{-1}u, U_{N,0}v \rangle_{\mathcal{H}_{N,0} \otimes L^2(\mathbb{R}^n)}$$

$$= \sum_{j,j' \in \mathbb{Z}_N^n} a_j \overline{b}_{j'} \langle M_{N,0}^j \nu_j, \nu_{j'} \rangle_{\mathcal{H}_{N,0}} \langle M_{N}^j f_j, g_{j'} \rangle_{L^2(\mathbb{R}^n)}$$

$$= \sum_{j \in \mathbb{Z}_N^n} a_j \overline{b}_j e^{i\phi_j t} \langle M_{N}^j f_j, g_j \rangle_{L^2(\mathbb{R}^n)}$$

(5.10)

It follows that

$$\hat{C}_{u,v}(\lambda) = \sum_{t=1} e^{-i\lambda t} C_{u,v}(t) = \sum_{j \in \mathbb{Z}_N^n} a_j \overline{b}_j \langle (e^{-i\phi_j} e^{i\lambda} - M_N)^{-1} f_j, g_j \rangle_{L^2(\mathbb{R}^n)}$$

In light of Corollary 4.7, this shows that $\hat{C}_{u,v}$ has a meromorphic extension to the plane $\mathbb{C}$ and $\lambda \in \mathbb{C}$ is a pole if and only if $e^{i\lambda} = e^{i\phi_j} e^{i\lambda'}$ for some $j \in \mathbb{Z}_N^n$ and $e^{i\lambda'}$ a Pollicott-Ruelle resonance of $M_N$. By Proposition 5.5, the resonances of $M_N$ are the same as those of $|\det E|^{-\frac{1}{2}} L_E$.

□

Proof of Theorem 2. Using the fact that $M_N$ and $L_E$ have the same Pollicott-Ruelle resonances (rescaled by a factor of $|\det E|^{-\frac{1}{2}}$), we note that we can spectrally project on the isolated resonance 1 corresponding to constant functions to write

$$\langle M_N^t f, g \rangle_{L^2(\mathbb{R}^n)} = |\det E|^{-\frac{1}{2}} + o(|\det E|^{-\frac{1}{2}}), \quad f, g \in \mathcal{S}(\mathbb{R}^n).$$

By (5.10), we can hence write

$$C_{u,v}(t) = |\det E|^{-\frac{1}{2}} \langle M_{N,0}^t \hat{u}, \hat{v} \rangle_{\mathcal{H}_{N,0}} + o_{u,v}(|\det E|^{-\frac{1}{2}})$$

for some $\hat{u}, \hat{v} \in \mathcal{H}_{N,0}$, as desired. □

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