A PRIORI ESTIMATES FOR A GENERALISED
MONGE-AMPÈRE PDE ON SOME COMPACT KÄHLER
MANIFOLDS

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Abstract. We study a fully nonlinear PDE involving a linear combination of symmetric polynomials of the Kähler form on a Kähler manifold. A $C^0$ a priori estimate is proven in general and a gradient estimate is proven in certain cases. Independently, we also provide a method-of-continuity proof via a path of Kähler metrics to recover the existence of solutions in some of the known cases. Known results are then applied to an analytic problem arising from Chern-Weil theory and to a special Lagrangian-type equation arising from mirror symmetry.

1. Introduction

Consider the following fairly general equation on a compact Kähler manifold $(X, \omega)$.

\[ \omega^n = \sum_{k=1}^{n} \alpha_k \wedge \omega^{n-k}, \]

where $\alpha_k \geq 0$ are closed smooth positive $(k,k)$-forms such that $\alpha_{k_0} > 0$ for at least one $k_0$ (in particular, $k_0$ can be equal to $n$), $\phi$ is a smooth function such that $\omega \phi = \omega + \sqrt{-1} \partial \bar{\partial} \phi > 0$, and $\omega$ satisfies the so-called “cone condition” $n\omega^{n-1} - \sum_{k=1}^{n} (n-k)\alpha_k \wedge \omega^{n-k-1} > 0$ and the consistency condition $\int \omega^n = \sum_{k=1}^{n} \int \alpha_k \wedge \omega^{n-k}$. Our notion of positivity of $(p,p)$-forms is explained in section 2.

Notice that if $\alpha_k = \delta_{k,n} \eta$ then equation 1 boils down to the Calabi-Yau theorem [23]. In its full generality, equation 1 and its cousins arise in the representation problem of Chern-Weil theory [17], canonical metrics in Kähler geometry [21, 20, 11, 12, 10, 8, 6], symplectic geometry [18] and mirror symmetry in string theory [13, 15]. So far, in most places where it has been studied a flow technique (like the J-flow) was used to study it. In Wei Sun’s paper [20] the method of continuity was used but it used non-Kähler metrics to prove openness. In this paper we aim to prove a priori estimates and solve 1 in some cases.

Our first result is an a priori $C^0$ estimate on $\phi$ under general assumptions (proposition 2.6). We remark that a paper of Székelyhidi [21] deals with an equation that overlaps with the generalised Monge-Ampère equation in some cases (like the J-flow case). In [21] the ABP estimate is used to prove a $C^0$-estimate but we use Yau’s Moser iteration argument. In the course of setting up the method of continuity we indicate a proof of a theorem (theorem 1.4 in [20] when $\psi$ is a constant which is also
a theorem of Fang-Lai-Ma \cite{11} in remark 2.3 using a method of continuity that
uses only Kähler metrics. This may potentially be of independent interest. Then
we proceed to prove a gradient estimate for 1 in a special case.

**Theorem 1.1.** Let \((X, \omega)\) be a compact Kähler manifold. Assume that \(\chi\) is another
Kähler form such that \(\chi\) has non-negative bisectional curvature. Assume that \(\alpha = f \chi^{n-1} + (\sqrt{-1})^{n-1} (-1)^{n(n-1)/2} \sum_{a=1}^{N} f_a \Phi_a \wedge \overline{\Phi}_a\) is a closed smooth \((n-1, n-1)\)-
form \((N\) is an arbitrary natural number) which satisfies \(-C\alpha \leq \nabla_X \alpha \leq C\alpha\) for
all real \(\chi\)-unit vectors \(X\), where \(f \geq 0, f_a \geq 0\) are smooth functions, \(\Phi_a\) are smooth
\((n-1, 0)\)-forms, \(C > 0\) is a constant, and \(\nabla\) is the canonical connection induced
by \(\chi\). Let \(\eta > 0\) be an \((n, n)\)-form. Assume that \(\int \omega^n = \int \alpha \wedge \omega + \int \eta\) and that
\(n \omega^{n-1} - \alpha > 0\). For a smooth function \(\phi\), denote \(\omega_\phi = \omega + \sqrt{-1} \partial \bar{\partial} \phi\). Consider the
equation.

\[\omega_\phi^n = \alpha \wedge \omega_\phi + \eta.\]

The following hold.

(1) Gradient estimate A smooth solution \(\phi\) of equation 3 satisfies \(\|\phi\|_{C^1} \leq C\)
where \(C\) depends only on the coefficients.

(2) Partial Laplacian estimate and existence If \(\alpha\) is parallel with respect to \(\chi\)
then \(\frac{\alpha \wedge \omega_\phi}{\chi} \leq C\). In addition, if \(\alpha > 0\), i.e., if \(f > 0\), then a unique smooth
solution exists satisfying \(\omega_\phi > 0\) and \(n \omega_\phi^{n-1} - \alpha > 0\).

**Remark 1.2.** The existence part of theorem 1.1 actually follows from a far more
general theorem of Wei Sun \cite{20}. However, the \textit{a priori} estimates are new in
the case of \(\alpha\) being degenerate. This is perhaps the main point of the theorem.
The assumptions of theorem 1.1 might look restrictive but actually even as stated
the technique used to prove them might potentially be useful in \cite{18} where similar
assumptions are in force (essentially \cite{18} deals with domains in \(\mathbb{R}^n\) but the difficulty
is that the equation there is degenerate elliptic).

In situations involving equations like 1 dealing with the J-flow or in the special
case of 1 mentioned in remark 2.4 one can actually avoid the gradient estimate by
proving the laplacian estimate directly. In fact, thanks to the work of Tosatti and
Weinkove \cite{22}, just proving the estimate \(\Delta \phi \leq C e^{A(\phi - \inf \phi)}\) is enough to guarantee
a \(C^2\) bound on \(\phi\). Usually the technique behind proving such estimates is to use the
maximum principle on an appropriately chosen function. For instance, one choice
\cite{20} is \(\psi = e^w (\Delta_X \phi + \text{tr}_X \omega)\) where \(w = -A \phi + f(\phi)\) is chosen judiciously. This
method was pioneered by Aubin \cite{1} and Yau \cite{23}. The major difficulty here is that
in general, equation 1 is not a symmetric polynomial in the hessian. This problem
is exacerbated if we allow \(\alpha_k\) to be degenerate. Therefore it is not clear that some
inequalities in the spirit of \cite{20, 6, 11} work in this setting.

Independently, we apply the main result in \cite{20} to prove two theorems. The first
one deals with Chern-Weil theory.

**Theorem 1.3.** Let \((V, h_0)\) be a hermitian rank-k holomorphic vector bundle over
a compact Kähler manifold \((X, \omega)\). Denote the curvature of the Chern connection
of \(h_0\) by \(F_0\) and define \(\Theta_0 = \frac{\sqrt{-1} F_0}{2\pi}\). Let \(\eta\) be an \((n, n)\)-form on \(X\) representing
the top Chern character class, i.e., $[\eta] = [\text{tr} (\Theta_0)^n]$. Define the forms $\alpha_i$ inductively according to

$$\alpha_1 = n \left( \omega - \frac{1}{k} \text{tr} (\Theta_0) \right)$$

$$\alpha_p = - \left( \frac{n}{p} \right) \frac{1}{k} \text{tr} (\Theta_0)^p + \left( \frac{n}{p} \right) \omega^p - \sum_{i=1}^{p-1} \left( \frac{n-i}{p-i} \right) \alpha_i \omega^{p-i} \forall \ 2 \leq p \leq n-1$$

$$\alpha_n = \frac{n}{k} \text{tr} (\Theta_0)^n + \omega^n - \sum_{i=1}^{n-1} \alpha_i \omega^{n-i}.$$  

(3)

Assume that there exists a hermitian metric $\chi$, constants $c_i \geq 0$, and a smooth function $\psi$ such that $\alpha_i = c_i \chi^i \psi$ and $\sum c_i > 0$. Also assume that $\omega$ satisfies

$$n\omega^{n-1} - \sum_{i=1}^{n-1} (n-i)\alpha_i \omega^{n-i-1} > 0.$$  

Then there exists a smooth metric $h = h_0 e^{-2\pi\phi}$, unique up to constant multiples such that the top Chern character form of $h$ is $\eta$, i.e., $\eta = \text{tr} (\Theta_h)^n$.

Some examples of the applicability of a very restricted version of theorem 1.3 are given in [17].

The second one deals with a special-Lagrangian type equation motivated from mirror symmetry.

**Theorem 1.4.** Let $L$ be a holomorphic line bundle over a compact Kähler manifold $(X, \omega)$. Let $\hat{\theta}$ be defined by the equation $\text{Im} (\omega + 2\pi \sqrt{-1} c_1(L))^3 = \tan(\hat{\theta}) \text{Re} (\omega + 2\pi \sqrt{-1} c_1(L))^3$. Assume that $\tan(\hat{\theta}) > 0$. Also assume that there exists a metric $h_0$ on $L$ whose curvature $F_0$ is such that the $(1,1)$-form $\Omega = \sqrt{-1} F_0 - \omega \tan(\hat{\theta})$ satisfies

1. $\Omega > 0$, and
2. $\Omega^2 - \omega^2 \sec^2 \hat{\theta} > 0$.

Then there exists a smooth metric $h = h_0 e^{-\phi}$, unique up to constant multiples satisfying

$$\text{Im} (\omega - F_0)^3 = \tan(\hat{\theta}) \text{Re} (\omega - F_0^3).$$  

(4)

We remark that since the theorem 1.4 does not require non-negative bisectional curvature, it is in some cases more general than the result in [13]. In particular, it may be applied to the Calabi-Yau 3-folds that are of interest to physicists. We give an example in section 5.

Here is a more detailed outline of the paper. In section 2 we set up the method of continuity, prove uniqueness and a uniform estimate, indicate a proof of the theorem in [20], and also prove that upper bounds on $\omega_\phi$ lead to uniform ellipticity. Owing to the non-symmetric nature of the equation, this is actually somewhat non-trivial. In sections 3 and 4 we prove further $a\ priori$ estimates in the special case of the equation in theorem 1.1. In section 5 we prove theorems 1.3 and 1.4.

2. Setup of the method of continuity and the uniform estimate

Before proceeding further, we define a notion of positivity of $(p,p)$-forms.
**Definition 2.1.** Let \((X, \chi)\) be a hermitian manifold. A smooth \((p,p)\)-form \(\alpha_p\) is (strictly) positive if 
\[
\alpha_p = f \chi^p + (\sqrt{-1})^p(-1)^{p(p-1)/2} \sum_{k=1}^{N} f_k \Phi_k \wedge \Phi_k
\]
where \(f\) is a (strictly) positive smooth function, \(f_k \geq 0\) are positive smooth functions, and \(\Phi_k\) are smooth \((p,0)\)-forms. Moreover, we write \(\gamma \geq 0\) if \(\gamma\) is positive and \(\alpha > 0\) if it is strictly so.

**Remark 2.2.** Perhaps a more natural definition would be to require that \(\alpha_p\) define a hermitian non-negative bilinear form on \(\Lambda^p T^{(1,0)} X\). However, one can easily see that this is equivalent to our definition. In particular, the wedge product of strictly positive forms is strictly positive.

In order to solve \(\Pi\) we employ the method of continuity. In whatever follows we assume that on \((X, \omega)\) for at least one value of \(k_0\), \(\alpha_{k_0} > \delta \omega^{k_0}\) for a positive constant \(\delta\). Consider the following family of equations parametrised by \(t \in [0, 1]\).

\[
(5) \quad \omega^n = t \sum_{k=1}^{n-1} \alpha_k \wedge \omega^{n-k} + \alpha_n b_t c^{1-t},
\]

where \(\omega_{\phi_t} = \omega + \sqrt{-1} \partial \bar{\partial} \phi_t\) is a Kähler form, \(\alpha_k\) are \(d\)-closed positive \((k,k)\)-forms and \(\alpha_n\) is a strictly positive \((n,n)\)-form, \(c = \int \frac{\omega^n}{\alpha_n}\), and \(b_t\) is a normalising constant chosen so that the integrals are equal on both sides, i.e., \(b_t = c^{t-1} \left( \int \frac{\omega^n}{\alpha_n} \right) = c^{t-1} \left( \int (1-t) \omega^n + t \alpha_n \right)\). Thus \(b_t c^{1-t} \geq 1\).

Let \(\mathcal{T}\) be the set of \(t \in [0, 1]\) where equation \(5\) has a unique smooth solution \(\phi_t\) such that \(\int \phi_t \omega^n = 0\), \(\omega_{\phi_t} > 0\), and \(n \omega^n_{\phi_t} - t \sum (n-k) \alpha_k \wedge \omega^{n-k}_{\phi_t} > 0\). \(\mathcal{T}\) is non-empty because at \(t = 0\) the equation is the usual Monge-Ampère equation which has a solution thanks to \([23]\). As usual, we need to prove that \(\mathcal{T}\) is both, open and closed.

**Openness :** Let \(\mathcal{C}\) be the set of \(C^{0,\beta}\) zero-average functions \(\phi\) such that \(\omega_{\phi} > 0\) where the background metric used to define the Banach spaces and the average is \(\omega\). We proceed to define a smooth map \(T\) from \(\mathcal{B}\) (where \(\mathcal{B}\) is an open subset of \(\mathcal{C} \times [0, 1]\)) such that \(n \omega^n_{\phi} - t \sum (n-k) \alpha_k \wedge \omega^{n-k}_{\phi} > 0\) to \(C^{0,\beta}\) top forms \(\gamma > 0\) such that \(\int \gamma = 1\). It is given by

\[
T(\phi, t) = \frac{\omega^n - t \sum_{k=1}^{n-1} \alpha_k \omega^{n-k}_{\phi}}{\int \omega^n - t \sum_{k=1}^{n-1} \alpha_k \omega^{n-k}}. \quad \text{The derivative } DT \text{ at the point } (\phi_0, \alpha) \text{ evaluated on the vector } (u, 0) \text{ is computed to be } DT_{\phi_0, \alpha}(u, 0) = \frac{(n \omega^n_{\phi_0} - t \sum (n-k) \alpha_k \wedge \omega^{n-k}_{\phi_0}) \wedge dd^c u}{\int \omega^n - t \sum \alpha_k \omega^{n-k}}. \quad \text{It is easily seen to be a self-adjoint elliptic operator. By the Fredholm alternative, we can solve the PDE if the}
\]

\footnote{Unless specified otherwise, we use positive in the French sense to mean non-negative.}
right hand side is orthogonal to its kernel. Its kernel (by the maximum principle) consists of constants. Thus by the implicit function theorem on Banach manifolds, on the level set \( T^{-1}(\frac{\alpha}{\theta_n}) \) we can locally solve for \( \phi \) as a smooth function of \( t \).

**Closedness:** If \( t_j \to t \), we need to prove that a subsequence \( \phi_j \to \phi \) in \( C^{2,\gamma} \), \( \omega_{\phi} > 0 \), and \( n\omega_{\phi_j}^{n-1} - t_j(n-k)\alpha_k \wedge \omega_{\phi_j}^{n-k-1} > 0 \). By the usual bootstrap argument this implies that \( \phi \) is smooth. The Arzela-Ascoli theorem shows that it is enough to prove \textit{a priori} \( C^{2,\gamma} \) estimates in order to show convergence of \( \phi_j \to \phi \). The following argument shows that the limiting \( \phi \) satisfies the other conditions.

**Lemma 2.3.** If \( \|\phi_j\|_{C^2} \leq C \), then \( \omega_{\phi_j} \geq R \omega > 0 \) and \( n\omega_{\phi_j}^{n-1} - t_j(n-k)\alpha_k \wedge \omega_{\phi_j}^{n-k-1} \geq R\omega^{n-1} > 0 \) where the positive constant \( R \) depends on \( C \).

**Proof.** Recall that by assumption \( \alpha_{k_0} \geq \delta \omega_{k_0} \) for some \( k_0 \) and some constant \( \delta > 0 \). Equation 5 implies that

\[
1 \geq \delta \frac{\omega_{k_0} \wedge \omega_{\phi_j}^{n-k_0}}{\omega_{\phi_j}^n} \geq \delta k_0!(n-k_0)! \sum \frac{1}{\lambda_{i_1} \ldots \lambda_{i_{k_0}}},
\]

where \( \lambda_i \) are the eigenvalues of \( \omega_{\phi_j} \) with respect to \( \omega \). Hence we see that \( \omega_{\phi_j} \geq R \omega \) for some \( R > 0 \) depending on the upper bound on \( \omega_{\phi_j} \). Indeed, if the smallest eigenvalue \( \lambda_1 \) (with respect to \( \omega \)) of \( \omega_{\phi_j} \) becomes arbitrarily small then the right hand side of the equation above becomes arbitrarily large because \( \omega_{\phi_j} \) is bounded above by assumption.

Likewise, at an arbitrary point \( p \), let \( v \) be a unit \((1,0)\)-form with respect to \( \omega \). Choose coordinates so that \( \omega_{\phi_j} \) is diagonal with eigenvalues 1 and \( v = c\frac{\partial}{\partial \bar{z}^k} \) at \( p \). Note that \( c \) is bounded below and above because \( \omega_{\phi_j} \) is. Then

\[
\left( n\omega_{\phi_j}^{n-1} - t_j \sum_{k}(n-k)\alpha_k \wedge \omega_{\phi_j}^{n-k-1} \right) \wedge v \wedge \bar{v} = c^2 \left( n\omega_{\phi_j}^{n-1} - t_j \sum_{k}(n-k)\alpha_k \wedge \omega_{\phi_j}^{n-k-1} \right)_{2233...n\bar{n}}
\]

where if the multi-index \( I = (i_1, \ldots, i_{n-k-1}) \) then \( I^0 \) is the multi-index consisting of \( k \) other numbers in \( 1, 2, \ldots, n \). Equation 5 implies that

\[
n! = t_j \sum_{\|W\|=n-k} \sum_{|I|=n-k-1} (n-k)! (\alpha_k)_{W^0} = t_j \sum_{|W|=n-k-1} \sum_{|I|=n-k} (n-k)! (\alpha_k)_{W^0}
\]

\[
\Rightarrow n! - t_j \sum_{k} \sum_{|I|=n-k-1} (n-k)! (\alpha_k)_{I^0} = t_j \sum_{|W|=n-k-1} \sum_{|I|=n-k} (n-k)! (\alpha_k)_{W^0}
\]

\[
\geq t_j \delta(n-k_0)! \sum_{|W|=n-k_0-1} (\omega_{k_0})_{W^0} \geq R.
\]

The last equation implies that \( \left( n\omega_{\phi_j}^{n-1} - t_j \sum_{k}(n-k)\alpha_k \wedge \omega_{\phi_j}^{n-k-1} \right) \wedge v \wedge \bar{v} \geq R \) for some \( R > 0 \) depending on the upper bound on \( \omega_{\phi_j} \).

**Remark 2.4.** At this juncture, if in equation 5 we substitute \( \alpha_k = \psi_k c_k \chi^k \forall 1 \leq k \leq n-1, \alpha_n = (c_n + \epsilon) \chi^n \) (where \( \chi \) is a Kähler metric), such that \( c_k \geq 0 \forall 1 \leq k \leq n-1 \).
\[ k \leq n, \epsilon > 0 \text{ and } \psi = \int \omega - \int (\epsilon + c_n)\chi^n \] are constants such that \( \sum_{k=1}^{n} c_k > 0 \) and

\[ \int \omega^n = \sum_{k=1}^{n} c_k \chi^k \wedge \omega^{n-k}, \] then the \( C^{2,\gamma} \) \textit{a priori} estimates in [20, 6] guarantee that the resulting equation has a smooth solution \( \phi_\epsilon \). We note that the \textit{a priori} estimates up to the second order in [20, 6] do not depend on \( \epsilon \). Actually, using the Evans-Krylov theorem and the fact that (by lemma 2.3) the equation is uniformly elliptic we have \( C^{2,\gamma} \) \textit{a priori} estimates independent of \( \epsilon \). Therefore, up to a subsequence \( \phi_\epsilon \rightarrow \phi \) in \( C^{2,\beta} \) as \( \epsilon \rightarrow 0 \). Hence we recover the main theorem in [20] in the Kähler case via a continuity path that passes only through Kähler metrics and more importantly, openness is easy to prove (as opposed to [20]).

As mentioned earlier, lemma 2.3 shows that all we have to do in order to solve equation (1) is to prove \textit{a priori} \( C^{2,\gamma} \) estimates on \( \phi \). We prove a general \( C^0 \) estimate on \( \phi \) here.

**The uniform estimate** : Before proceeding further, we prove a lemma about concavity of certain potentially non-symmetric functions of the Kähler form.

**Lemma 2.5.** The function \( \omega \rightarrow \alpha_k \wedge \omega^{n-k} \) is a convex function of Kähler forms if \( \alpha_k \geq 0 \).

**Proof.** Recall that \( \alpha_k = f \chi^k + \sqrt{-1}^{-1} f (t \omega_1 + (1-t) \omega_2) \). Choosing normal coordinates for \( \chi \) we see that \( \omega \rightarrow f \chi^k \wedge \omega^{n-k} \) is convex by standard theory. Furthermore, let \( \omega_1 \) and \( \omega_2 \) be two Kähler forms. At the point under consideration choose coordinates so that \( \omega_1 \) is Euclidean and \( \omega_2 \) is diagonal with eigenvalues \( \lambda_j \). Therefore for some positive constant \( C \) we have

\[
\frac{\Phi_i \wedge \Phi_j \wedge (t \omega_1 + (1-t) \omega_2)^{n-k}}{(t \omega_1 + (1-t) \omega_2)^n} = C \sum_{|I|=n-k} \left( \Phi_i \wedge \Phi_j \right) f \left( t + (1-t) \lambda_I \right) / \left( t + (1-t) \lambda_n \right).
\]

It is now easy to deduce the desired result from expression (6) and the fact that \( \det(A) \) is convex as a function of positive-definite matrices \( A \).

**Proposition 2.6.** A smooth solution of equation (1) satisfies \( \| \phi \|_{C^0(X)} \leq C \) where \( C \) depends only on \( \omega \), bounds on the coefficients of the equation, and the positive lower bound on \( \alpha_{k_0} \).

**Proof.** We follow Yau’s by-now-classical [23] technique adapted from [16]. In whatever follows, unless otherwise specified, all controlled constants are denoted by \( C \). Without loss of generality we may change the normalisation of \( \phi \) so that \( \sup \phi = -1 \). Let \( \phi = -\phi \). We will find an upper bound on \( \phi \) using Moser iteration (as usual). Let \( \Theta = \omega^n - \sum_{k=1}^{n} \alpha_k \wedge \omega^{n-k} \). Subtracting \( \Theta \) on both sides
from \( \omega^n - \sum_{k=1}^n \alpha_k \wedge \omega^{n-k}_\phi = 0 \), multiplying by \( \phi^p \) and integrating we see that

\[
\int_X \phi^p \int_0^1 \frac{d}{dt} \left( \omega^n_{t\phi} - \sum_{k=1}^n \alpha_k \wedge \omega^{n-k}_{t\phi} \right) dt \leq C \| \phi_- \|^p_{L^p}.
\]

\[
\Rightarrow - \int_0^1 \int_X \phi^p \left( n \omega^{n-1}_{t\phi} - \sum_{k=1}^n (n-k) \alpha_k \wedge \omega^{n-k-1}_{t\phi} \right) \sqrt{-1} \partial \bar{\partial} \phi_- dt \leq C \| \phi_- \|^p_{L^p}.
\]

\[
\Rightarrow \int_0^1 \int_X \sqrt{-1} \partial \bar{\partial} \phi_{-}^{n+1} \wedge \bar{\partial} \phi_{-}^{n+1} \wedge \left( n \omega^{n-1}_{t\phi} - \sum_{k=1}^n (n-k) \alpha_k \wedge \omega^{n-k-1}_{t\phi} \right) dt
\]

\[
\leq \frac{C(p+1)^2}{p} \| \phi_- \|^p_{L^p} \leq C(p+1) \| \phi_- \|^p_{L^p}.
\]

At a point \( q \) we choose coordinates normal coordinates \( z^i \) for \( \omega \) so that \( \partial \bar{\partial} \phi_- \) is proportional to \( \partial \bar{z}^p \). This means that at \( q \)

\[
\partial \phi_{-}^{n+1} \wedge \bar{\partial} \phi_{-}^{n+1} \wedge \left( n \omega^{n-1}_{t\phi} - \sum_{k=1}^n (n-k) \alpha_k \wedge \omega^{n-k-1}_{t\phi} \right)
\]

\[
= \partial \phi_{-}^{n+1} \bar{\partial} \phi_{-}^{n+1} \left( n \omega^{n-1}_{t\phi} - \sum_{k=1}^n (n-k) \alpha_k \wedge \omega^{n-k-1}_{t\phi} \right)_{2\bar{z}, n\bar{\bar{z}}}
\]

\[
= \partial \phi_{-}^{n+1} \bar{\partial} \phi_{-}^{n+1} \left( \sum_{k=1}^n (n-k) \alpha_k \wedge \omega^{n-k-1}_{t\phi} \right)_{2\bar{z}, n\bar{\bar{z}}}
\]

\[
1 - \frac{\sum_{k=1}^n (n-k) \alpha_k \wedge \omega^{n-k-1}_{t\phi}}{n \omega^{n-1}_\phi}_{2\bar{z}, n\bar{\bar{z}}}
\]

Now we restrict ourselves to the subspace spanned by \( \partial_2, \ldots, \partial_n \). To emphasize this we denote the restriction of any form \( \beta \) by \( \tilde{\beta} \). Now we proceed as in [20]. Note that \( \omega_{t\phi} = t \omega_{\phi} + (1-t) \omega \) and that \( \det(A)^{1/n} \) is concave as a function of positive-definite \( n \times n \) matrices \( A \). Therefore \( \tilde{\omega}_{t\phi}^{n-1} \geq n \omega^{n-1} \phi + (1-t) n \omega^{n-1} \phi \). Likewise, lemma [20] shows that

\[
1 - \frac{\sum_{k=1}^n (n-k) \alpha_k \wedge \omega^{n-k-1}_{t\phi}}{n \omega^{n-1}_{t\phi}} \geq t \left( 1 - \frac{\sum_{k=1}^n (n-k) \alpha_k \wedge \omega^{n-k-1}_{\phi}}{n \omega^{n-1}_{\phi}} \right)
\]

\[
+ (1-t) \left( 1 - \frac{\sum_{k=1}^n (n-k) \alpha_k \wedge \omega^{n-k-1}_{\phi}}{n \omega^{n-1}_{\phi}} \right)
\]

Therefore,

\[
\frac{1}{n+1} \int_X \sqrt{-1} \partial \phi_{-}^{n+1} \wedge \bar{\partial} \phi_{-}^{n+1} \wedge \left( n \omega^{n-1}_{t\phi} - \sum_{k=1}^n (n-k) \alpha_k \wedge \omega^{n-k-1}_{t\phi} \right) \leq C(p+1) \| \phi_- \|^p_{L^p}
\]

\[
\Rightarrow \| \nabla (\phi^{(p+1)/2}_-)^2 \|_{L^2} \leq C(p+1) \| \phi_- \|^p_{L^p}.
\]
From this point onwards, the proof is standard. (See [16] for instance.) □

**Uniqueness**: If \( \phi_1 \) and \( \phi_2 \) are two smooth solutions of equation (1) such that
\[
\int \phi_1 \omega^n = \int \phi_2 \omega^n, \quad \omega_{\phi_1} > 0 \quad \text{and} \quad n \omega_{\phi_1}^{n-1} - \sum_{k=1}^{n-1} (n-k) \alpha_k \wedge \omega_{\phi_1}^{n-k} > 0,
\]
then upon subtraction we get
\[
\int_0^1 \frac{d}{dt} \left( \omega_{\phi_1 + (1-t)\phi_2} - \sum_{k=1}^n \alpha_k \wedge \omega_{\phi_1 + (1-t)\phi_2}^{n-k} \right) dt = 0
\]
(7)
\[
\Rightarrow \left( \int_0^1 \left( n \omega_{\phi_1 + (1-t)\phi_2}^{n-1} - \sum_{k=1}^{n-1} (n-k) \alpha_k \wedge \omega_{\phi_1 + (1-t)\phi_2}^{n-k-1} \right) dt \right) \wedge \sqrt{-1} \partial \bar{\partial} (\phi_1 - \phi_2) = 0
\]
The proof of proposition 2.6 shows that equation 7 is elliptic. Thus the maximum principle implies that \( \phi_1 = \phi_2 \).

3. THE GRADIENT ESTIMATE

From now onwards we restrict ourselves to solving a special case of equation (1) on the Kähler manifold \((X, \omega)\) where \( \omega \) satisfies the cone condition. Firstly, let \( \frac{1}{b} \omega \leq \chi \leq b\omega \) be an arbitrary Kähler metric on \( X \) having nonnegative bisectional curvature. As mentioned in the introduction, we aim at solving
\[
\omega^n_{\phi} = \alpha \wedge \omega_{\phi} + \eta,
\]
(8)
where \( \alpha = f \chi^{n-1} + \sum_{a=1}^N f_a \Phi_a \wedge \bar{\Phi}_a \) and \( \eta > 0 \). In addition we assume that \(-C\alpha \leq \nabla_X \alpha \leq C\alpha\) where \( X \) is a real \( \chi \)-unit vector. Also, from now onwards we write \( \eta = h \chi^n \) where \( h > 0 \) is a smooth function, \( \alpha \wedge \beta \) locally as \( \chi^n A^{ki} \beta_{kl} \) for a non-negative matrix \( A \), and \( \omega \) locally as \( \omega_{ij} dz^i d\bar{z}^j \) where \( \omega \) is used (by abuse of notation) to denote both, the Kähler potential as well as the metric itself.

In order to prove a gradient bound on \( \phi \) we use Blocki’s technique [2]. Denote by \( \nabla \) the Levi-Civita connection associated to \( \chi \). Let \( \psi = \ln(|\nabla \phi|^2) - \gamma(\phi) \) where \( \gamma(t) = \frac{1}{2} \ln(2x+1) \) is chosen so that \( \gamma' > E > 0 \) and \(-\gamma'' + (\gamma')^2 > Q > 0 \) for two positive constants \( E \) and \( Q \). At the maximum point \( p \) of \( \psi \), \( \nabla \psi = 0 \) and \( \psi_{ki} \) is negative semi-definite. Without loss of generality we may assume that \( |\nabla \phi(p)| \geq N \) for any \( N \). Choosing normal coordinates for \( \chi \) at \( p \) so that \( \omega_{\phi} \) is diagonal with eigenvalues \( \lambda_i \) we obtain
\[
0 = \psi_k(p) = \sum_{i} \frac{\phi_{ik} \phi_i + \phi_i \phi_{ik}}{|\nabla \phi|^2} - \gamma' \phi_k
\]
(9)
\[
\psi_{kl}(p) = -(\gamma'' + (\gamma')^2) \phi_k \phi_l - \gamma' \phi_{kl} + \sum_{i} \frac{\phi_{ik} \phi_i + \phi_i \phi_{ik} + \phi_{ik} \phi_{il} + \phi_i \phi_{ikl}}{|\nabla \phi|^2} + \frac{\chi_{kl} \phi_i \phi_j}{|\nabla \phi|^2}.
\]
Rewriting equation (8) as $1 = \frac{n\omega_\phi}{\omega_\phi^n} + \frac{n}{\omega_\phi}$, differentiating once, and multiplying by $\phi_i$, at the point $p$ we obtain the following (after using the assumption on $\nabla \alpha$).

$0 \geq -L^k\phi (\omega_{ki} + \phi_{ki}) - C|\nabla \phi|$

where $L^k = -\frac{A^k}{\chi^i\ldots\ldots\chi^k} + \frac{\delta_{ki}}{\chi_k}$. Multiplying equation (9) by $L^k$ and using equation (11) we obtain,

$0 \geq QL^k\phi_k\phi_i - \gamma' L^k(\omega_\phi)_{ki} + \gamma' L^k\omega_{ki} + L^k\chi^{ij}\phi_i\phi_j - \sum L^k (\omega_{ki} + \omega_{ki}) - \frac{L^k|\nabla \omega_{ki}|}{|\nabla \phi|^2} - \frac{C}{|\nabla \phi|}.$

Noting that $L^k(\omega_\phi)_{ki} = -\frac{\omega_\phi}{\omega_\phi^n} + n$ (which is larger than $n - 1$ and less than $n$), and the assumption on the bisectional curvature $\chi^{\alpha\beta} \geq 0$ for all $\alpha, \beta$ we get

$0 \geq QL^k\phi_k\phi_i + \gamma' L^k\omega_{ki} - \gamma' \left( n - \frac{\alpha\omega_\phi}{\omega_\phi^n}) - 2 \frac{L^k|\nabla \omega_{ki}|}{|\nabla \phi|^2} - \frac{C}{|\nabla \phi|} \right).$

Now we multiply on both sides by $\omega_\phi^n$ and define $\tilde{L}^k = \frac{\omega_\phi^n}{n!}L^k = A^k - \delta^k\chi^i\ldots\ldots\chi^k$. We get

$0 \geq Q\tilde{L}^k\phi_k\phi_i + \left( \gamma' - \frac{C}{|\nabla \phi|} \right) \tilde{L}^k\omega_{ki} - \gamma' \left( n(h + A^k(\omega_\phi)_{ki}) - A^k(\omega_\phi)_{ki} - \frac{C\omega_\phi^n}{|\nabla \phi|} \right)

\Rightarrow C \geq Q\tilde{L}^k\phi_k\phi_i + \left( \gamma' - \frac{C}{|\nabla \phi|} \right) \left( -A^k\omega_{ki} + \frac{n\omega_\phi^{n-1}\omega_\phi}{\omega_\phi^n} (h + A^k(\omega_\phi)_{ki}) \right)$

$- (n - 1)\gamma' A^k(\omega_\phi)_{ki} - \frac{C(h + A^k(\omega_\phi)_{ki})}{|\nabla \phi|}.$

Note that inequality (12) implies that at $p$, the expression $\frac{n\omega_\phi^{n-1}\omega_\phi}{\omega_\phi^n}$ is bounded above.

At $p$ if we can prove that $\tilde{L}^k \geq T\chi^{k\bar{k}} > 0$ then we will have a gradient estimate on $\phi$. Actually, if we manage to prove that $\Delta_\chi \phi$ is bounded above, then by lemma 2.3 we are done. If we just prove that $\frac{n\omega_\phi^{n-1}\omega_\phi}{\omega_\phi^n} > n - 1 + \varepsilon$ for some uniform positive constant $\varepsilon$ then inequality (12) implies that $\omega_\phi^n$ is bounded above and hence by the lower bound on $\omega_\phi$ so is $\Delta_\chi \phi$ above. Indeed, the following lemma coupled with this observation completes the proof of the gradient estimate.

Lemma 3.1. If at a point $q$, $\omega_\phi \geq R\omega > 0$, $\Delta_\chi \phi \to \infty$ then $\frac{n\omega_\phi^{n-1}\omega_\phi}{\omega_\phi^n} > n - 1 + \varepsilon$ for some uniform positive constant $\varepsilon$.

Proof. Without loss of generality we assume that $\lambda_1 \geq \lambda_2 \ldots \lambda_n$. Equation (8) implies that at least $\lambda_n$ is bounded above at $p$. Notice that the cone condition implies

$(n\omega_\phi^{n-1})_{22\ldots n\bar{n}} > n!A^{k\bar{k}}.$

By the classical Hadamard inequality for matrices (see [7] for instance) $\omega_{22\ldots n\bar{n}} \geq \frac{(\omega_\phi^{n-1})_{22\ldots n\bar{n}}}{(n-1)!}$. Thus

$\omega_{22\ldots n\bar{n}} > A^{k\bar{k}}.$
Solving for $\lambda_1$ from equation 8 we see that $\lambda_1 = f + \sum_{k \neq 1} A^{kk}$. This coupled with the lower bound on the $\lambda_i$ and the assumption that $\lambda_1 \to \infty$ implies that $\lambda_2 \ldots \lambda_n \to A^{11}$. Therefore,

$$\frac{n\omega_{\phi}^{n-1}}{\omega_{\phi}^{n}} = \sum_{k=1}^{n} \frac{\omega_{kk}}{\lambda_k} \to \sum_{k=2}^{n} \frac{\omega_{kk}}{\lambda_k} \geq \frac{(n-1) \left( \prod_{k=2}^{n} \frac{\omega_{kk}}{\lambda_k} \right)^{1/(n-1)}}{(n-1) \left( \prod_{k=2}^{n} \frac{\omega_{kk}}{A^{11}} \right)^{1/(n-1)}},$$

where we used the AM-GM inequality. Using inequality 14 we are done. \(\square\)

4. Higher order estimates

In this section we prove the partial Laplacian estimate. In addition to the assumptions in section 3 we assume that $\alpha$ is parallel with respect to $\chi$.

Remark 4.1. It is but natural to wonder whether there are any forms $\alpha$ that satisfy the desired requirements other than multiples of $\chi^{n-1}$. If $X$ is a complex torus and $\chi$ the flat metric, then $\alpha = dz^1 \wedge d\bar{z}^1 \ldots dz^{n-1} \wedge d\bar{z}^{n-1}$ furnishes a non-trivial degenerate example. In general, one can take a locally hermitian symmetric space or a product of any 2 manifolds with $\chi$ being the product metric to produce lots of examples using $(1,1)$-forms. (Note that in our case we also need $\chi$ to have nonnegative bisectional curvature.) In fact, it is known that on manifolds other than local products or locally hermitian symmetric spaces the only such forms are indeed multiples of $\chi^{n-1}$. According to Bryant [4], this result follows from the classification of Riemannian holonomy groups.

Partial Laplacian bound: We now prove an upper bound on $\frac{\alpha \omega_{\phi}}{\chi^n}$. As in [17] we use the function $\Psi = \frac{\alpha \omega_{\phi}}{\chi^n} - \mu \phi$ where $\mu$ is a constant that will be chosen later. If we prove that $\Psi$ is bounded above then we are done. As before, at the maximum point $p$ of $\Psi$, $\Psi_k = 0$ and $\Psi_{k\bar{l}}$ is negative semi-definite. We choose normal coordinates for $\chi$ at $p$ and make sure that $\omega_{\phi}$ is diagonal at $p$ with eigenvalues $\lambda_i$. Differentiation of $\Psi$ yields the following.

$$0 = \nabla_k \Psi = \Psi_k = \frac{\alpha \omega_{\phi,k}}{\chi^n} - \mu \phi_k$$

$$\Psi_{k\bar{l}} = \nabla_k \nabla_{\bar{l}} \Psi = \frac{\alpha \nabla_k \nabla_{\bar{l}} \omega_{\phi}}{\chi^n} - \mu \nabla_k \nabla_{\bar{l}} \phi \geq \frac{\alpha \omega_{\phi,k\bar{l}}}{\chi^n} - \mu (\omega_{\phi})_{k\bar{l}} + \mu \omega_{k\bar{l}}$$

(15)
Differentiating $1 = \frac{\omega_n}{\omega_\phi} + \frac{\eta}{\omega_\phi}$ twice, multiplying by $A^{k\bar{l}}$ and summing over $k = l$ we obtain

$$0 = \frac{\alpha \omega_{\phi,k}}{\omega_\phi^n} - \frac{n \omega_n^{n-1} \omega_{\phi,k}}{\omega_\phi^n} + \eta \frac{k}{\omega_\phi^n}$$

$$(16)$$

$$0 \geq \frac{A^{k\bar{l}} \alpha \nabla_\phi^2 \nabla_\phi \omega_\phi}{\omega_\phi^n} - \frac{A^{k\bar{l}} \omega_n^{n-1} \nabla_\phi^2 \nabla_\phi \omega_\phi}{\omega_\phi^n} - A^{k\bar{l}} \eta \frac{k}{\omega_\phi^n} + \frac{A^{k\bar{l}} \eta \omega_n^{n-1} \omega_{\phi,k}}{\omega_\phi^n} + \frac{\nabla_\phi^2 \nabla_\phi \eta}{\omega_\phi^n}$$

$$(17)$$

where we used lemma 2.5. At this juncture we use equations 15 and 16 to get

$$C \geq \frac{A^{k\bar{l}} \alpha \omega_{\phi,k\bar{l}}}{\omega_\phi^n} - \frac{A^{k\bar{l}} \omega_n^{n-1} \omega_{\phi,k\bar{l}}}{\omega_\phi^n} - C \left( \frac{\alpha \omega_\phi}{\omega_\phi^n} + n \right) - A^{k\bar{l}} \eta \frac{k}{\omega_\phi^n} + \frac{A^{k\bar{l}} \eta \omega_n^{n-1} \omega_{\phi,k\bar{l}}}{\omega_\phi^n} - C \eta \frac{n \omega_n^{n-1} \omega_{\phi,k\bar{l}}}{\omega_\phi^n}$$

$$(18)$$

We multiply equation 15 by $L^{k\bar{l}}$ and sum to obtain (after substituting in 18)

$$C \geq -\mu \left( \frac{\omega_n^{n-1} \omega_{\phi,k\bar{l}}}{\omega_\phi^n} \right) - L^{k\bar{l}} \mu (\omega_\phi) \omega_{\phi,k\bar{l}}$$

$$= -\mu \left( \frac{\omega_n^{n-1} \omega_{\phi,k\bar{l}}}{\omega_\phi^n} \right) - \left( n - \frac{\alpha \omega_\phi}{\omega_\phi^n} \right) + \left( \frac{n \omega_n^{n-1} \omega_{\phi,k\bar{l}}}{\omega_\phi^n} - \frac{\alpha \omega_\phi}{\omega_\phi^n} \right)$$

$$(19)$$

Since $\eta > 0$ we know that $\omega_n^{n-1} \omega_{\phi,k\bar{l}}$ is bounded from below. Moreover, $0 \leq \frac{\alpha \omega_\phi}{\omega_\phi^n} \leq 1$. Therefore $\frac{\omega_n^{n-1} \omega_{\phi,k\bar{l}}}{\omega_\phi^n} \leq C$. This implies a lower bound, $\omega_\phi > R \omega > 0$. Since we are assuming that $\Delta_\phi \phi \to \infty$, $\frac{C}{\mu} \geq \epsilon$. Therefore $\frac{C}{\mu} \geq \epsilon$. Choosing $\mu$ to be large enough we arrive at a contradiction. This proves the partial Laplacian estimate.

In the case when $\alpha > 0$ the partial Laplacian estimate implies an estimate on $\Delta_\phi \phi$.

$C^{2,\gamma}$ estimates: The previously established partial Laplacian bound when $\alpha > 0$ implies by lemma 2.3 that the equation is uniformly elliptic. In fact, it also implies that $1 = \frac{\alpha \omega_\phi}{\omega_\phi^n} + \frac{n}{\omega_\phi^n}$ is uniformly elliptic. Lemma 2.5 implies that the equation is also convex. Thus the (complex version of) the Evans-Krylov theory 19 is applicable and furnishes a $C^{2,\gamma}$ estimate. This completes the proof of theorem 1.1.
5. Applications

5.1. Representation of the top Chern character.

Given a \((k,k)\) form \(\eta\) representing the \(k\)th Chern character class \(\text{tr}((\Theta)^k)\) of a vector bundle on a compact complex manifold (where \(\sqrt{-1} \frac{\partial}{\partial x} \frac{\partial}{\partial y} = \Theta\) and \(F\) is the curvature of a connection), it is natural to ask whether there is a metric \(h\) on the vector bundle whose induced Chern connection realises

\[
\text{tr} (\Theta^k) = \eta.
\]

As phrased this question seems almost intractable. It is not even obvious as to whether there is any connection satisfying this requirement, leave aside a Chern connection. Work along these lines was done by Datta in [9] using the h-principle. Therefore, it is more reasonable to ask whether equality can be realised for the top Chern character form. To restrict ourselves further we ask whether any given metric \(h_0\) may be conformally deformed to \(h = h_0 e^{-\phi}\) satisfying the desired requirement. In the case of a line bundle \(L\) (where the only choice we have is conformal deformations) equation (20) boils down to the PDE

\[
\text{tr} \left( \Theta_0 + \sqrt{-1} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \phi \right)^n = \eta.
\]

When \(\Theta_0 > 0\) and \(\eta > 0\) this is the usual Monge-Ampère equation solved by Yau [23]. In general, one gets a complicated fully nonlinear PDE which reduces to equation (1) in some cases. It is clear from the case of a line bundle that for the general case of a vector bundle, unfortunately quite a few potentially unnatural positivity requirements will have to be made on the curvature \(\Theta_0\) and the form \(\eta\). Note that the local problem was dealt with in [14]. In [17] an existence result was proven on complex 3-tori. Using the result in [20] we prove theorem 1.3.

**Proof of theorem 1.3** According to theorem 1.1 in [20] the equation

\[
\omega_\phi^n = e^b \psi \sum_{i=1}^n c_i \chi^i \omega_\phi^{n-i}
\]

on a compact Kähler manifold \((X, \omega)\) has a unique smooth solution \(\phi, b\) satisfying \(\omega_\phi > 0\) and \(n \omega_\phi^{n-1} - \psi \sum_{i=1}^{n-1} \omega_\phi^{n-i-1} > 0\) if there exists a smooth function \(v\) such that

\[
\omega_v^n \leq \psi \sum_{i=1}^n c_i \chi^i \omega_v^{n-i}.
\]

Actually, if \(\int_X \omega^n = \int_X \psi \sum_{i=1}^n c_i \chi^i \omega^{n-i}\) and \(\psi \chi^i\) is closed, then one can choose \(v = 0\) and \(\tilde{\psi} = e^{-b} \psi\) where \(b\) is small enough for condition (22) to hold. This shows that under such circumstances, equation (21) has a unique solution with \(b = 0\). Indeed,
expanding equations\textsuperscript{[3]} and \textsuperscript{[21]} (with $b = 0$) we get

$$\sum_{r=0}^{n} \binom{n}{r} \frac{1}{k} \text{tr} (\Theta_0^r) \left( \sqrt{-1} \partial \bar{\partial} \phi \right)^{n-r} - \frac{n}{k} = 0$$

$$\sum_{r=0}^{n} \omega^r \left( \sqrt{-1} \partial \bar{\partial} \phi \right)^{n-r} - \psi \sum_{k=1}^{n} \sum_{r=0}^{n-k} c_k \chi^k \binom{n-k}{r} \omega^r \left( \sqrt{-1} \partial \bar{\partial} \phi \right)^{n-k-r} = 0$$

(23)

Comparing the two equations we get the desired result. \hfill \Box

5.2. A special Lagrangian type equation.

According to superstring theory the spacetime of the universe is constrained to be a product of a compact Calabi-Yau three-fold and a four dimensional Lorentzian manifold. A “duality” relates the geometry of this Calabi-Yau manifold with another “mirror” Calabi-Yau manifold. From a differential geometry standpoint this maybe thought of (roughly) as a relationship between the existence of “nice” metrics on a line bundle on one Calabi-Yau manifold and special Lagrangian submanifolds of the other Calabi-Yau manifold. Using the Fourier-Mukai transform, Leung-Yau-Zaslow showed \textsuperscript{[15]} that this mirror symmetry implies that equation \textsuperscript{[4]} ought to be satisfied in some cases. In \textsuperscript{[13]}, Jacob and Yau showed that given an ample line bundle $L$ over a compact Kähler manifold with non-negative orthogonal bisectional curvature, $L^k$ admits a solution to equation \textsuperscript{[4]} However, the assumption of non-negative orthogonal bisectional curvature is not desirable if one wants to apply such a result to general Calabi-Yau manifolds. Here we attempt to partially address this issue by restricting our attention to 3-folds.

Proof of theorem \textsuperscript{1.4}: Equation \textsuperscript{4} can be written using $\Theta = \sqrt{-1} F$ as

$$-\Theta^3 + 3 \omega^2 \Theta = \tan(\hat{\theta}) \left( \omega^3 - 3 \Theta^2 \omega \right).$$

Grouping terms together we see that it is equivalent to

$$\Omega^3 + 3 \omega^2 \Omega \text{sec}^2(\hat{\theta}) - 2 \omega^3 \tan(\hat{\theta}) \text{sec}^2(\hat{\theta}) = 0,$$

(24)

where $\Omega_\phi = \Omega + \sqrt{-1} \partial \bar{\partial} \phi = \Theta_0 - \omega \tan(\hat{\theta}) + \sqrt{-1} \partial \bar{\partial} \phi$. Comparing this equation to the theorem in \textsuperscript{[20]} we see that if $\Omega > 0$, $\tan(\hat{\theta}) > 0$, and the cone condition $\Omega^2 - \omega^2 \sec^2(\hat{\theta}) > 0$ is satisfied, then the equation has a unique smooth solution up to a constant multiple. \hfill \Box

The conditions imposed on $\Omega$ in theorem \textsuperscript{1.4} are reminiscent of the “stability” condition in \textsuperscript{[13]}. Here is a concrete example of a Calabi-Yau manifold where the theorem is applicable:

Let $X$ be $\mathbb{C} \Lambda \times K$ where $K$ is a projective K3 surface with Picard group generated by an ample line bundle $L$ (for example $K$ can be a non-singular degree 4 surface in $\mathbb{P}^3$). Let $\omega$ be the product of the flat metric on the torus and the unique Calabi-Yau metric in the Kähler class $[L] + \epsilon [\gamma]$ where $[\gamma]$ is any cohomology class and $\epsilon$ is chosen to be small enough (as to how small can be determined easily) for the
example to work. Endow $L$ with a metric $h_0$ with positive curvature $F_0$. Choose $k$ to be large enough so that for $(L^k, h^k_0)$, $\tan(\hat{\theta}_k) > 0$. Indeed,

$$\tan(\hat{\theta}_k) = \frac{\int (k^3 \Theta^3_0 - 3k \Theta_0 (\Theta_0 + \epsilon \gamma)^2)}{\int (3k^2 \Theta^2_0 (\Theta_0 + \epsilon \gamma) - (\Theta_0 + \epsilon \gamma)^3)}.$$

So if $k \geq 2$ and $\epsilon$ small enough, then $\tan(\hat{\theta}) > 0$. Note that as $k \to \infty$, $\tan(\hat{\theta}_k)$ grows linearly in $k$.

Notice that if $\Theta_0 > 2\omega \tan(\hat{\theta})$ then the cone condition is definitely satisfied for large enough $k$. Indeed,

$$\Omega^2 - \omega^2 \sec^2(\hat{\theta}) = \Theta^2_0 + \omega^2 \tan^2(\hat{\theta}) - 2\Theta_0 \omega \tan(\hat{\theta}) - \omega^2 \sec^2(\hat{\theta})$$

$$= \Theta^2_0 - 2\Theta_0 \omega \tan(\hat{\theta}) - \omega^2 = \Theta_0 (\Theta_0 - 2\omega \tan(\hat{\theta})) - \omega^2$$

Hence, our requirement boils down to making sure that $k\Theta_0 - 2\omega \tan(\hat{\theta}_k) > 0$ for large $k$.

(25) $k\Theta_0 - 2\omega \tan(\hat{\theta}_k) = k\Theta_0 - 2(\Theta_0 + \epsilon \gamma)\frac{\int (k^3 \Theta^3_0 - 3k \Theta_0 (\Theta_0 + \epsilon \gamma)^2)}{\int (3k^2 \Theta^2_0 (\Theta_0 + \epsilon \gamma) - (\Theta_0 + \epsilon \gamma)^3)}$

For large enough $k$ we just have to look at

(26) $k\Theta_0 - 2\frac{2}{3}k(\Theta_0 + \epsilon \gamma)\frac{\int \Theta^3_0}{\Theta^3_0 (\Theta_0 + \epsilon \gamma)}$

which is obviously positive if $\epsilon$ is small enough.

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