HILBERT SPACES BUILT ON A SIMILARITY AND ON DYNAMICAL RENORMALIZATION

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Abstract. We develop a Hilbert-space framework for a number of general multi-scale problems from dynamics. The aim is to identify a spectral theory for a class of systems based on iterations of a non-invertible endomorphism. We are motivated by the more familiar approach to wavelet theory which starts with the two-to-one endomorphism \( r: z \mapsto z^2 \) in the one-torus \( \mathbb{T} \), a wavelet filter, and an associated transfer operator. This leads to a scaling function and a corresponding closed subspace \( V_0 \) in the Hilbert space \( L^2(\mathbb{R}) \). Using the dyadic scaling on the line \( \mathbb{R} \), one has a nested family of closed subspaces \( V_n, n \in \mathbb{Z} \), with trivial intersection, and with dense union in \( L^2(\mathbb{R}) \). More generally, we achieve the same outcome, but in different Hilbert spaces, for a class of non-linear problems. In fact, we see that the geometry of scales of subspaces in Hilbert space is ubiquitous in the analysis of multiscale problems, e.g., martingales, complex iteration dynamical systems, graph-iterated function systems of affine type, and subshifts in symbolic dynamics. We develop a general framework for these examples which starts with a fixed endomorphism \( r \) (i.e., generalizing \( r(z) = z^2 \) ) in a compact metric space \( X \). It is assumed that \( r: X \to X \) is onto, and finite-to-one.

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1. INTRODUCTION

We study a class of endomorphisms \( r: X \to X \), where \( X \) is a metric space. The endomorphism is assumed onto, and finite-to-one. We build a spectral theory on a Hilbert space associated naturally with \( (X, r) \). Our focus is on the case when \( X \) is assumed to carry a certain strongly invariant measure \( \rho \), see (2.3).

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Continuing our earlier work [13] we consider basis constructions in a general context of dynamical systems; the case of endomorphisms, i.e., non-reversible dynamics. Our framework will include wavelet bases, as well as algorithmic basis constructions in Hilbert spaces built on fractals or on Julia sets of rational functions in one complex variable. In fact, these examples motivated our results.

First recall that in the real variable case of standard wavelets (in one or several variables, i.e., the $d$-dimensional Lebesgue measure), there is a separate generalizations of standard dyadic wavelets, again based on translation and scaling: See for example [3] for such an approach to the construction of generalized wavelet bases in the Hilbert space $L^2(\mathbb{R}^d)$, i.e., of orthogonal bases in $L^2(\mathbb{R}^d)$, or just frame wavelet bases, but still in $L^2(\mathbb{R}^d)$.

It is the purpose of this paper to develop a geometric context of this viewpoint which applies to any kind of dynamics which is based on an iterated scale of self-similarity. Hence our paper will offer a Hilbert-space framework which goes beyond the setting of scale similarity, and our results will offer a new viewpoint even in the case of the more familiar selfsimilarity which is based on a cascade of affine scales.

The best know instance of this is $d = 1$, and dyadic wavelets [10]. In that case, the two operations on the real line $\mathbb{R}$ are translation by the group $\mathbb{Z}$ of the integers, and scaling by powers of 2, i.e., $x \mapsto 2^j x$, as $j$ runs over $\mathbb{Z}$. This is the approach to wavelet theory which is based on multiresolutions and filters from signal processing. In higher dimensions $d$, the scaling is by a fixed matrix, and the translations by the rank-$d$ lattice $\mathbb{Z}^d$. Again we will need scaling by all integral powers. We view points $x$ in $\mathbb{R}^d$ as column vectors, and we then consider the group of scaling transformations, $x \mapsto A^j x$ as $j$ ranges over $\mathbb{Z}$.

Suitable spectral conditions will be imposed on $A$. In particular we note that if $A$ is integral, i.e., the entries in $A$ are in $\mathbb{Z}$, then $x \mapsto Ax$ passes to the quotient $\mathbb{R}^d/\mathbb{Z}^d$. Since $\mathbb{R}^d/\mathbb{Z}^d$ is a copy of the compact $d$-torus $\mathbb{T}^d$ via a familiar identification, we see that $A$ induces an endomorphism $r_A$ in $\mathbb{T}^d$. If further $A$ is invertible, then $r_A$ is finite-to-one, and maps $\mathbb{T}^d$ onto itself. In fact, for every $x$ in $\mathbb{T}^d$, the inverse image $r_A^{-1}(x)$ has cardinality $= |\det A| =: N$.

\begin{equation}
(1.1) \quad \frac{1}{\sqrt{|\det A|}} \varphi(A^{-1} x) = \sum_{k \in \mathbb{Z}^d} a_k \varphi(x - k), \quad (x \in \mathbb{R}^d).
\end{equation}

So our starting point is a given finite-to-one endomorphism $r : X \to X$ in a compact space $X$. Our aim is three-fold: (1) to build an associated Hilbert space which admits wavelet decompositions; (2) to show that the corresponding computations can be done with a geometric algorithm; and finally (3) we offer concrete examples from dynamics where our approach leads to new insight. So in addition to the endomorphism $(X, r)$, our initial setup will include a scalar function $m_0$; an analogue of the function from wavelet theory which determines low-pass filters.

Details: Set $W(x) := |m_0|^2/\# r^{-1}(x)$. We say that $m_0$ satisfies a low-pass condition if $W(0) = 1$. (In the special case of (1.1) above, the relationship between the function $m_0$ and the coefficients $\{a_k\}$ is that the $a_k$ numbers will be the $d$-Fourier coefficients of $m_0$ when $m_0$ is viewed as a function on the compact quotient $X = \mathbb{R}^d/\mathbb{Z}^d$. This explains the summation over $\mathbb{Z}^d$ in (1.1).)

Suppose for some $p$, and $x \in X$, that $r^p(x) = x$. Then we say that the finite set of points $C = \{x, r(x), \ldots, r^{p-1}(x)\}$ is a cycle. A cycle $C$ is called a $W$-cycle if $W(y) = 1$ for all $y \in C$. 
We will extend to the context of endomorphisms the following general principle from wavelets in the Hilbert space $L^2(\mathbb{R}^d)$: A generalized wavelet basis (also called a Parseval-frame, see e.g., [3]) will have the stronger orthonormal basis (ONB) property when the only one $W$-cycle is $C = \{0\}$. On the other hand, the presence of non-trivial $W$-cycles is consistent with wavelet systems that form frame-bases. The reader is referred to [6] for details regarding these more general wavelet bases. It was proved in [6] that the presence of $W$-cycles is consistent with a class of certain super-wavelets. This wavelet basis involves an additional cyclic structure which we will develop in the paper.

This setup arose earlier for the familiar linear multiresolution analysis (MRA) approach to wavelets: Recall [10] that dyadic wavelets represent a special basis for the Hilbert space $L^2(\mathbb{R})$, but they are generated by a subspace $V_0$ in $L^2(\mathbb{R})$ which is the closed linear span of a single function $\varphi$ and its translates by the integers $\mathbb{Z}$. The function $\varphi$ satisfies a certain scaling identity

\begin{equation}
\frac{1}{\sqrt{2}} \varphi(x/2) = \sum_{k \in \mathbb{Z}} a_k \varphi(x - k), \quad (x \in \mathbb{R}).
\end{equation}

which implies that the scaling operator $Uf(x) := 1/\sqrt{2}f(x/2)$ maps $V_0$ into itself. A solution $\varphi$ is called a scaling function. Using a terminology from optics, we say that functions on $\mathbb{R}$ represent signals or images, and that the subspace $V_0$ initializes a fixed resolution.

A special case: $X = T = \{z \in \mathbb{C} \mid |z| = 1\} = \mathbb{R}/\mathbb{Z}$, $r(z) = z^2$, and $m_0$ is the function on $T$ with Fourier coefficients equal to the masking coefficients $a_k$ from (1.2), i.e., $m_0(z) = \sum_{k \in \mathbb{Z}} a_k z^{k}$. The function $m_0$ is called a filter function because of an analogy to a setting in signal processing. One of the axioms for $m_0$ (the quadrature-mirror-filter axiom) from wavelet theory amounts to the fact that the associated linear operator, $S_0 h(z) := m_0(z)h(z^2)$ is isometric in $L^2(T, \text{Haar measure})$; see Figure 1.

In this paper, we state a version of the scaling identity (1.2), for the case of an endomorphism $r : X \to X$, and we show that it admits a solution in certain Hilbert spaces built on $(X, r)$. It turns out that the variant of (1.2) which arises by the Fourier transform, i.e.,

\begin{equation}
\sqrt{2} \hat{\varphi}(t) = m_0(e^{it/2})\hat{\varphi}(t/2), \quad (t \in \mathbb{R}),
\end{equation}

is more suggestive of the generalization we have in mind; see Theorem 2.14 for details.

While the standard MRA approach to wavelets (see [17]) restricts the functions $m_0$ in (1.2) by assuming that $m_0$ is in some regularity class, e.g., is Lipschitz, we
shall not do this here. Moreover, there is a rich class of wavelet systems where $m_0$ is typically only known \textit{a priori} to be $L^\infty$. This is the case, for example, for the frequency localized wavelets studied in [3] and [21] (In this last case, $m_0$ is in fact matrix-valued.)

The scaling identity (1.2) implies that there is a natural intertwining of the isometry $S_0$ on $L^2(\mathbb{T})$ with the restriction of $U$ to the subspace $V_0$ in $L^2(\mathbb{R})$. A second axiom for $m_0$ from wavelet theory (called low-pass) implies that $S_0$ is a \textit{pure} shift isometry, i.e., that the intersection $S_0^n(L^2(\mathbb{T}))$, for $n$ in $\mathbb{N}$ is $\{0\}$, see Figure 1. Because of the intertwining relation, this fact guarantees that the standard functions that make up a wavelet basis really do form a basis for the whole Hilbert space $L^2(\mathbb{R})$. See Figure 1. The purity of $S_0$ is what yields a certain martingale system, i.e., a nested family of spaces, or of sigma-algebras.

It is the purpose of this paper to generalize this setting to that of endomorphisms, and to realize a natural scaling function, as a generating vector in a Hilbert space which corresponds to $L^2(\mathbb{R})$ for the special case of wavelets. For this purpose we introduce a solenoid $X_\infty$ built on $X$, and a family of repelling cycles for the system $(X,r,m_0)$. Our Hilbert space is built as an $L^2$-space on certain infinite paths starting at $X$. In Theorem 2.14 we solve the corresponding scaling identity, and write the scaling function as an infinite product. As one should expect by analogy to wavelets, a central theme in our present analysis is a characterization of those filter functions $m_0$ on $X$ for which the scaling identity has non-trivial solutions in a Hilbert space of functions of $X_\infty$.

A concrete example of this wavelet technique used on a particular graph dynamical system (The Golden mean shift) is presented in Proposition 2.18 below. Our aim is to present this as a systematic tool for dynamics outside the traditional context of wavelets in $L^2(\mathbb{R})$.

In recent papers [11, 13], the co-authors have adapted this MRA technique to a related but different problem, the problem of creating a spectral theory for a class of non-linear iterated function systems (IFS), but in those cases, there is not a direct analogue to the scaling identity. Our construction here parallels the one we outlined briefly for the standard dyadic MRA wavelet constructions [17]. (We have sketched the standard wavelet construction only in the dyadic case, and only in one dimension, i.e., for $\mathbb{R}$, but it is known that this construction carries over \textit{mutatis mutandis} to $\mathbb{R}^d$ with $d > 1$, and when $x \mapsto 2x$, is replaced with matrix scaling $x \mapsto Ax$ in $\mathbb{R}^d$ where $A$ is a $d$ by $d$ matrix over $\mathbb{Z}$ with eigenvalues $\lambda$ such that $|\lambda| > 1$. Moreover our results apply to the kind of multiwavelets studied recently in [3].)

Our present paper is not about $\mathbb{R}^d$-wavelets but instead about a class of non-linear dynamics $r: X \to X$. Specifically, now we start with $r: X \to X$, and the function $m_0$ is defined on $X$. We will also call $m_0$ a \textit{filter function} because of known analogy to subband filtering in signal processing. When $m_0$ is given then $S_0$ given by $S_0 h(x) := m_0(x) h(r(x))$ is isometric in $L^2(X)$, subject to a technical condition on $m_0$. So by Wold’s theorem [31], it is then the orthogonal sum of a shift operator $S$ and a unitary operator $U_0$; i.e., the Hilbert space $L^2(X)$ on which $S_0$ acts is the direct sum of two Hilbert spaces $H(S)$ and $H(U_0)$ such that (i) each space invariant for $S_0$, (ii) the restriction to $H(S)$ is a shift $S$, and the restriction to $H(U_0)$ a unitary operator. We say that $S_0$ is \textit{pure} if $H(U_0) = \{0\}$. (See Figure...
This will be equivalent to the fact that the intersection of the multiresolution subspaces is trivial.

This means that $S_0$ is itself a shift operator on $L^2(X)$. Our theorem 3.9 gives a simple condition for the isometry $S_0$ to be pure.

The first step in our construction is an extension from the initial endomorphism $r: X \to X$, to a new invertible system $\hat{r}: X_\infty \to X_\infty$, i.e., with $\hat{r}$ invertible on $X_\infty$. When $r$ is assumed finite-to-one, this can be done such that there is a quotient $X_\infty/X$ which becomes a Cantor space. The extension space $X_\infty$ is called a solenoid.

For the case when $(X, r)$ is a one-sided subshift [24], we work out (in Section 2.4) an explicit model for this solenoid.

In fact the notion of a solenoid (for the study of dynamics of an endomorphism and extension to an automorphism) was used already in a pioneering paper by Lawton [19] in 1973. Lawton considered groups with expansive automorphisms; see also [20]. Motivated by applications, we note that our present analysis is not restricted to groups.

The use of solenoids in the study of particular systems with scale similarity was initiated in the paper [8], and was continued in [7]. The context of [8] is a class of algebraic irrational numbers and an associated $C^*$-algebraic crossed product. In a general context of non-linear dynamics, this work was continued in [11, 13].

2. Covariant representations

Let $X$ be a compact metric space with a non-invertible endomorphism $r: X \to X$ such that $r$ is measurable, onto and finite to one, i.e., $0 < \#r^{-1}(x) < \infty$ for all $x \in X$.

We have shown in [12] and [13] that, for certain filter functions $m_0$ on $X$, one can construct multiresolutions and scaling functions in Hilbert spaces of functions on $X_\infty$ (see (2.1)).

In [13] we proved that to get useful multiresolutions, the function $m_0$ must have certain extreme cycles (see Definition 2.10). In this case the measure on $X_\infty$ is actually supported on a smaller set $N_C$ (see (2.4) below).

2.1. The ground space. An (infinite) path starting at $x$ is a sequence $(z_1, z_2, \ldots)$ of points in $X$ such that $r(z_1) = x$, $r(z_{n+1}) = z_n$ for $n \geq 1$. We denote by $\Omega_x$ the set of paths starting at $x$. We denote by $X_\infty$ the set of all paths, 

$$X_\infty = \bigcup_{x \in X} \Omega_x. \quad (2.1)$$

Note that a path $(z_1, z_2, \ldots)$ in $\Omega_x$ can be identified with the doubly infinite sequence $(z_n)_{n \in \mathbb{Z}}$, where $z_0 := x$ and $z_{-n} = r^n(x)$ for $n \geq 0$.

$X_\infty \subset X^\mathbb{Z}$ inherits the usual Tychonoff topology from $X^\mathbb{Z}$.

The maps $\theta_n: X_\infty \to X$ are defined for all $n \in \mathbb{Z}$, by

$$\theta_n((z_k)_{k \in \mathbb{Z}}) = z_n.$$ 

The endomorphism $r$ can be extended to the automorphism $\hat{r}$ defined on $X_\infty$ by

$$\hat{r}(z_n)_{n \in \mathbb{Z}} = (z_{n-1})_{n \in \mathbb{Z}}.$$

These maps satisfy the following relations:

$$\theta_n \circ \hat{r} = \theta_{n-1}, \quad \theta_0 \circ \hat{r} = r \circ \theta_0.$$
For a function \( g \) on \( X \) we define
\[
(2.2) \quad g^{(n)}(x) := g(x)g(r(x)) \cdots g(r^{n-1}(x)), \quad (n \geq 1).
\]

For a function \( \xi \) on \( X_\infty \), we define \( \xi^{(0)} = 1 \),
\[
\xi^{(n)} := \xi \circ \hat{r} \cdots \xi \circ \hat{r}^{n-1},
\]
and if \( \xi \) is not vanishing on \( X_\infty \), then
\[
\xi^{(-n)} = \frac{1}{\xi \circ \hat{r}^{-1} \xi \circ \hat{r}^{-2} \cdots \xi \circ \hat{r}^{-n}}, \quad (n \geq 1).
\]

We can identify functions \( g \) on \( X \) with functions on \( X_\infty \) by \( g \mapsto g \circ \theta_0 \). (Note that the two definitions for \( g^{(n)} \) will coincide.)

Consider \( r : X \to X \) and suppose \( \rho \) is a strongly invariant probability measure on \( X \), i.e.,
\[
(2.3) \quad \int_X f(x) \, d\rho(x) = \int_X \frac{1}{\#r^{-1}(x)} \sum_{y \in r^{-1}(x)} f(y) \, d\rho(x), \quad (f \in L^\infty(\rho)).
\]

Let \( C = \{x_0, x_1, \ldots, x_{p-1}\} \subset X \) be a cycle of length \( p \), i.e., the points \( x_i \) are distinct and \( r(x_{i+1}) = x_i, r(x_0) = x_{p-1} \).

We define the set
\[
(2.4) \quad N_C(x) := \{\omega = (z_1, z_2, \ldots) \in \Omega_x \mid \lim_{n \to \infty} z_{pn} \in C\}.
\]

For each \( x \in X \) and \( \omega = (z_1, z_2, \ldots) \in N_C(x) \), define \( i(\omega) \in \{0, \ldots, p-1\} \) by \( i(\omega) := i \) if \( \lim_{k \to \infty} z_{kp} = x_i \).

An inspection reveals that each \( N_C(x) \) is countable.

Define the measure \( \lambda_C \) on \( X_\infty \) by
\[
(2.5) \quad \int_{X_\infty} f \, d\lambda_C = \int_X \sum_{\omega \in N_C(x)} f(\omega) \, d\rho(x).
\]

To simplify the notation we write \( c(x) = \#r^{-1}(r(x)) \).

**Proposition 2.1.** For all \( \xi \in L^1(X_\infty, \lambda_C) \) and \( n \in \mathbb{Z} \), we have
\[
\int_{X_\infty} \xi^{(n)} \circ \hat{r}^n \, d\lambda_C = \int_{X_\infty} \xi \, d\lambda_C.
\]

**Proof.** It is enough to prove this for \( n = 1 \), the rest follows by induction.
\[
\int_{X_\infty} \xi \circ \hat{r} \, d\lambda_C = \int_X \#r^{-1}(r(x)) \sum_{\omega \in N_C(x)} \xi(\hat{r}(x, \omega)) \, d\rho(x) =
\[
\int_X \frac{1}{\#r^{-1}(x)} \sum_{y \in r^{-1}(x)} \#r^{-1}(r(y)) \sum_{\omega \in N_C(y)} \xi(\hat{r}(y, \omega)) \, d\rho(x) =
\[
\int_X \sum_{\omega \in N_C(x)} \xi(x, \omega) \, d\rho(x) = \int_{X_\infty} \xi \, d\lambda_C.
\]

\( \square \)
2.2. The operators. In this subsection we show that when \((X, r)\) is given as above, there is a natural covariant representation \((U, \pi)\) acting on the Hilbert space \(L^2(X_\infty, \lambda_\mathbb{C})\), i.e., with \(r\) inducing a unitary operator \(U\), and \(\pi\) a representation of \(L^\infty(X)\) by multiplication operators, such that the relation (2.8) is satisfied on \(L^2(X_\infty, \lambda_\mathbb{C})\). (The operators on \(L^2(X_\infty, \lambda_\mathbb{C})\) are equipped with the strong operator topology (SOT).)

Let \(\alpha_0, \ldots, \alpha_{p-1}\) be a set of complex numbers of absolute value 1.

Let \(U\) be the operator on \(L^2(X_\infty, \lambda_\mathbb{C})\) defined by
\[
U\xi(x, \omega) = \alpha_{i(\omega)} \sqrt{\#r^{-1}(r(x))}\xi \circ \hat{r}(x, \omega), \quad (\xi \in L^2(X_\infty, \lambda_\mathbb{C}), x \in X, \omega \in \Omega_x).
\]

For \(f \in L^\infty(X, \rho)\) define the operator \(\pi(f)\) on \(L^2(X_\infty, \lambda_\mathbb{C})\) by
\[
\pi(f)\xi(x, \omega) = f(x)\xi(x, \omega), \quad (\xi \in L^2(X_\infty, \lambda_\mathbb{C}), x \in X, \omega \in \Omega_x).
\]

**Proposition 2.2.** The operator \(U\) is unitary, \(\pi\) is a representation of the algebra \(L^\infty(X, \rho)\) and the following covariance relation is satisfied:
\[
U\pi(f)U^{-1} = \pi(f \circ r), \quad (f \in L^\infty(X, \rho)).
\]

For any function \(f \in L^\infty(X, \rho)\) and \(n \geq 1\), the operator \(U^{-n}\pi(f)U^n\) is the operator of multiplication by \(f \circ \theta_n\). The union of the algebras \(\{U^{-n}\pi(f)U^n \mid f \in L^\infty(X, \rho)\}\) is SOT-dense in the algebra \(L^\infty(X_\infty, \lambda_\mathbb{C})\) (seen as multiplication operators on \(L^2(X_\infty, \lambda_\mathbb{C})\)). An operator \(S\) on \(L^2(X_\infty, \lambda_\mathbb{C})\) commutes with \(U\) and \(\pi(f)\) for all \(f \in L^\infty(X, \rho)\) if and only if there exists a function \(F \in L^\infty(X_\infty, \lambda_\mathbb{C})\) such that \(F = F \circ \hat{r}\) and \(S\) is the operator of multiplication by \(F\).

**Proof.** The fact that \(U\) is an isometry follows form Proposition 2.1.

The inverse of \(U\) is
\[
U^{-1}\xi(x, \omega) = \alpha_{i(\hat{r}(\omega))} \sqrt{\#r^{-1}(x)}\xi \circ r^{-1}(x, \omega).
\]

Some simple computations prove the other relations.

Note that the algebra \(\{U^{-n}\pi(f)U^n \mid f \in L^\infty(X, \rho)\}\) is the algebra of operators of multiplication by functions which depend only on the first \(n\) coordinates. Since any function in \(L^\infty(X, \rho)\) can be pointwise and uniformly boundedly approximated by such functions, it follows that the union of these algebras is dense in \(L^\infty(X_\infty, \lambda_\mathbb{C})\).

Since \(L^\infty(X_\infty, \lambda_\mathbb{C})\) is maximal abelian, if \(S\) commutes with \(U\) and \(\pi\) then \(S\) commutes with the multiplication operators, so it is a multiplication operator itself, \(S = MF\). Since \(S\) commutes with \(U\), it follows that \(F = F \circ \hat{r}\). 

Our formula for the measure \(\lambda_\mathbb{C}\) in (2.5) shows that the Hilbert space \(L^2(X_\infty, \lambda_\mathbb{C})\) fibers over functions on \(X\) as follows: for a dense space of functions \(\xi, \eta \in L^2(X_\infty, \lambda_\mathbb{C})\), the sum
\[
\langle \xi \mid \eta \rangle(x) := \sum_{\omega \in N_C(x)} \xi(\omega) \overline{\eta(\omega)}
\]
defines a \(C(X)\)-valued inner product as in (2.8), (16) and
\[
\int_X \langle \xi \mid \eta \rangle(x) \, d\rho(x) = \langle \xi \mid \eta \rangle_{L^2(X_\infty, \lambda_\mathbb{C})}.
\]
2.3. A direct integral decomposition. We now resume our discussion of cycles $C$ for the endomorphism. The cycle $C = \{x_0, \ldots, x_{p-1}\}$ generates $p$ points in $X_\infty$:

$$\omega_C := (x_0, x_1, \ldots, x_{p-1}, x_0, \ldots)$$

and $r^k(\omega_C), k \in \{1, \ldots, p - 1\}$, i.e., $\omega_C$ is the path that goes through the cycle infinitely many times. We may write $\omega_C := C \cdot C \cdots = C^\infty$.

**Definition 2.3.** A fixed point $x_0$ for $r$ is called repelling if there is $0 < c < 1$ and $\delta > 0$ such that for all $x \in X$ with $d(x, x_0) < \delta$, one has $d(r(x), x_0) > c^{-1}d(x, x_0)$.

A cycle $C = \{x_0, \ldots, x_{p-1}\}$ is called repelling if each point $x_i$ is repelling for $r^p$.

**Definition 2.4.** A subset $A$ of $X_\infty$ is called a cross section if for every path $\omega \in \cup_{x \in \mathbb{N}_C(x)} \{r^k(\omega_C) | k \in \{0, \ldots, p - 1\}\}$, the intersection $A \cap \{r^k(\omega) | k \in \mathbb{Z}\}$ contains exactly one point.

**Proposition 2.5.** If $C$ is repelling, and $r$ is continuous at the points in $C$, then there exists a cross section.

**Proof.** Using the continuity of $r$ and the fact that the cycle is repelling, we can find a small $\delta > 0$ and $0 < c < 1$ such that $r^i(B(x_0, \delta)) \cap B(x_0, \delta) = \emptyset$, for $i \in \{1, \ldots, p - 1\}$, $r^{-p}(x_0) \cap B(x_0, \delta) = \{x_0\}$, and such that $d(r^p(x), x_0) \geq c^{-1}d(x, x_0)$ for $x \in B(x, \delta)$.

Define

$$A := \{(z_k)_{k \in \mathbb{Z}} \in X_\infty \mid z_0 \in r^p(B(x_0, \delta)) \setminus B(x_0, \delta), \text{ and } z_{kp} \in B(x_0, \delta) \text{ for } k \geq 1\}.$$ 

It is enough to prove that, for every path $(z_k)_{k \in \mathbb{Z}}$ in $\mathbb{N}_C(x)$, except the special ones $\omega_C$ and the others, there is a unique $k_0 \in \mathbb{Z}$ such that

$$\text{(2.9)} \quad z_{k_0} \in r^p(B(x_0, \delta)) \setminus B(x_0, \delta), \text{ and } z_{kp} \in B(x_0, \delta) \text{ for } k \geq 1.$$ 

Since $\omega$ is in $\mathbb{N}_C(x)$, the sequence $\{z_{kp}\}$ converges to one of the points $x_i$. Then, using the continuity of $r$, $\{z_{kp+p-1}\}$ converges to $x_0$.

Take the first $k_0 \in \mathbb{Z}$ such that $z_{kp+k_0} \in B(x_0, \delta)$, for all $k \geq 1$. We still have to justify why there is a first one.

If not, then $z_{-kp+k_0} \in B(x_0, \delta)$ for all $k \geq 0$. Then, using the fact that $x_0$ is repelling for $r^p$, there is a $c$ such that $0 < c < 1$, and for all $k \geq 0$

$$\delta > d(z_{-kp+k_0}, x_0) = d(r^{kp}(z_{k_0}), x_0) \geq c^{-k}d(z_{k_0}, x_0).$$

But then let $k \to \infty$, and obtain that $z_{k_0} = x_0$. So, $z_{k_0-l} = r^l(x_0) = x_l \mod p$ for all $l \geq 0$. Also, since $z_{k_0+p} \in B(x_0, \delta) \cap r^{-p}(x_0)$, we get $z_{k_0+p} = x_0$. By induction we obtain then that $\omega$ is one of the special points in the orbit of $\omega_C$, which yields the contradiction.

Since $z_{k_0+p} \in B(x_0, \delta)$, $z_{k_0} = r^p(z_{k_0+p})$ is in $r^p(B(x_0, \delta))$ but not in $B(x_0, \delta)$. Since $z_{k_0+kp} \in B(x_0, \delta)$ for $k \geq 1$, this proves that $k_0 + kp$ does not have the property (2.9) for $k \geq 1$.

Since $z_{k_0} \not\in B(x_0, \delta)$, this proves that $k_0 - kp$ does not have the property (2.9) for $k \geq 1$.

Since for $k \geq 0$, $z_{k_0+kp+p} \in B(x_0, \delta)$, it follows that for $i \in \{1, \ldots, p - 1\}$, one has

$$z_{k_0+kp+i} \in r^{p-i}(B(x_0, \delta))$$

so it is not in $B(x_0, \delta)$, and therefore $k$ does not satisfy (2.9) when $k \not\equiv k_0 \mod p$.

This proves that $A$ is a cross section. \qed
Our present notion of cross section, and our next theorem are motivated in part by an earlier theorem of Lim, Packer, and Taylor [21] on direct integral decompositions of a class of wavelet representations: This is the class of wavelets for which the Fourier transform $\hat{\psi}$ of the wavelet mother-function $\psi$ is the indicator function of a measurable subset in $\mathbb{R}^d$. Both our present direct integral decomposition theorem, and that in [21] are motivated by Mackey’s theory of unitary representations of non-abelian groups. In fact, our representation of the covariant system $(U, \pi)$ may be viewed as a single representation of a certain non-abelian crossed product $\mathbb{R}^\infty := C(X_\infty) \rtimes F\mathbb{Z}$ (see [3]), and our simultaneous direct integral decomposition of $U$ and $\pi$ in Theorem 2.6 below, is also a direct integral decomposition of a single representation of the crossed product group.

Assume now that $A$ is a cross section. For each $\omega \in A$, define the operators $U_\omega$ and $\pi_\omega(f)$, $f \in L^\infty(X, \rho)$ on $l^2(\mathbb{Z})$ by

$$U_\omega \xi(k) = \alpha_i(\omega^k) \xi(k + 1), \quad (\xi \in l^2(\mathbb{Z}), k \in \mathbb{Z}),$$

$$\pi_\omega(f) \xi(k) = f(\omega^k) \xi(k), \quad (\xi \in l^2(\mathbb{Z}), k \in \mathbb{Z}).$$

The representation $\pi_\omega$ extends to a representation of $C(X_\infty)$ defined by

$$\pi_\omega(f) \xi(k) = f(\hat{\omega}(\xi))(k), \quad (f \in C(X_\infty), \xi \in l^2(\mathbb{Z}), k \in \mathbb{Z}).$$

The covariance relation is satisfied:

$$U_\omega \pi_\omega(f) U_\omega^{-1} = \pi_\omega(f \circ \hat{\omega}), \quad (f \in C(X_\infty)).$$

**Theorem 2.6.** Let $A$ be a cross section, and assume that $\rho(C) = 0$. The map $\Phi: L^2(X_\infty, \lambda_C) \to L^2(A, \lambda_C) \otimes l^2(\mathbb{Z})$ defined by

$$(\Phi(f))(\omega, k) = \sqrt{\epsilon(k)} \delta_0(f(\hat{\omega}))(k), \quad (f \in L^2(X_\infty, \lambda_C), \omega = (z_k)_{k \in \mathbb{Z}} \in A, k \in \mathbb{Z}),$$

is an isometric isomorphism which intertwines the operators $U$ and $\int_A U_\omega d\lambda_C(\omega)$, and also the representations $\pi$ and $\int_A \pi_\omega d\lambda_C(\omega)$. The representations $(U_\omega, \pi_\omega)$ on $l^2(\mathbb{Z})$ are irreducible for all $\omega \in A$.

**Proof.** The fact that $\Phi$ is isometric follows from Proposition 2.4. The inverse of $\Phi$ is defined as follows: for each $\omega \in \cup_x N_C(x)$, (except the special ones which have measure 0, so do not matter), there exists a unique $k(\omega) \in \mathbb{Z}$ and $\eta(\omega) \in A$ such that $\omega = \hat{\omega}^k(\eta(\omega))$. Then

$$\Phi^{-1}(f)(\omega) = \frac{1}{\sqrt{\epsilon(k(\omega))(\eta(\omega)_0)}} f(\eta(\omega), k(\omega)).$$

Everything follows by direct computation.

We prove now that the representation $(U_\omega, \pi_\omega)$ is irreducible for all $\omega = (z_k)_{k \in \mathbb{Z}} \in A$.

Note first that $\pi_\omega(f)$ is a diagonal operator $F$ with entries $F_{kk} = f(\omega^{-k})$, $k \in \mathbb{Z}$.

We claim that for $k \neq l$ big enough, we have $z_k \neq z_l$. Since $\omega$ is in $N_C(z_0)$, it follows that $z_{kp}$ converges to one of the points of the cycle. Also, for $k$ big enough, the points $z_k$ cannot be in $C$, because this path $\omega_C$ was removed from $A$. Suppose now that for any $m$ we can find $k, l \geq m$, such that $k > l$ and $z_k = z_l$. Then this implies that $z_k$ is periodic, therefore also $z_{k-1} = r(z_k), z_{k-2}, \ldots, z_l$ are periodic, and since $m$ is arbitrary, it follows that all the points $z_m$ are periodic. The orbit of the two periodic points $z_0$ and $z_1$ intersect, because $\hat{\omega}^k(z_k) = z_0$, therefore the two orbits must be the same. Thus the path $(z_k)_{k \in \mathbb{Z}}$ is an infinite repetition of the
periodic orbit of \( z_0 : (z_0, z_1, \ldots, z_{p_0-1}, z_0, z_1, \ldots) \). However, this cannot converge to the cycle \( C \).

Take now \( k \neq l \) small enough. Then \( z_{-k} \neq z_{-l} \) so we can pick a function continuous function \( f \) such that \( F_{kk} = f(z_{-k}) \neq f(z_{-l}) = F_{ll} \). If \( T = (T_{ij})_{i,j \in \mathbb{Z}} \) is an operator on \( l^2(\mathbb{Z}) \) that commutes with \( U_{\omega} \) and \( \pi_{\omega} \), then \( T_{kl}F_{ll} = F_{kk}T_{kl} \). So \( T_{kl} = 0 \) for \( k \neq l \) small enough.

Note that \( (U_{\omega}^{-1}\pi_{\omega}(f)U_{\omega}\xi)(k) = f(z_{-k+1})\xi(k) \), so the conjugation with \( U_{\omega} \) shifts the diagonal entries of \( \pi_{\omega}(f) \). Therefore, with the previous argument, we obtain that \( T_{kl} = 0 \) for all \( k \neq l \). So \( T \) is a diagonal operator. Since, \( T \) commutes with \( U_{\omega} \), we obtain that \( T_{kk} = T_{k+1,k+1} \). So \( T \) is a constant multiple of the identity. This proves that the representation \( (U_{\omega}, \pi_{\omega}) \) is irreducible.

We show in Theorem 2.7 below that the harmonic analysis of covariant systems \((U, \pi)\) as in (2.8) is completely equivalent to that of single representations \( \hat{\pi} \) of a certain \( C^* \)-algebraic crossed product \( \mathfrak{A}_x \). With this identification \( (U, \pi) \leftrightarrow \hat{\pi} \), we note in particular that the operators in the commutant of the pair \((U, \pi)\) coincide with the commutant of the representation \( \hat{\pi} \). Our main conclusion in Theorem 2.7 is that the representation \( \hat{\pi} \) of \( \mathfrak{A}_x \) is faithful, i.e., that the kernel of \( \hat{\pi} \) is trivial.

**Theorem 2.7.** Assume that for every \( x \in X \), there exists a path \( (z_i)_{i \geq 1} \) that starts at \( x \) and with \( \lim_{i \to \infty} z_{pi} \in C \), i.e., \( N_C(x) \) is non-empty. Then the operators \( U \) and \( M_f \), \( f \in C(X_{\infty}) \) on \( L^2(X_{\infty}, \lambda_C) \) form a faithful representation of the crossed-product \( \mathfrak{A}_x := C(X_{\infty}) \rtimes_f \mathbb{Z} \).

**Proof.** The \( C^* \)-algebraic crossed product \( \mathfrak{A}_x \) is the \( C^* \)-algebraic completion of formal symbols \( \{ (f, k) \mid f \in C(X_{\infty}), k \in \mathbb{Z} \} \) with product

\[
(f, k) \cdot (g, l) = (f \circ \hat{\pi}^k, k+l), \quad (f, g) \in C(X_{\infty}), k,l \in \mathbb{Z}.
\]

The representation is defined by

\[
\hat{\pi} : (f, k) \mapsto \pi(f)U^k.
\]

We saw in Proposition 2.2 and its proof that the covariance relation is satisfied, so we have to check only that this representation is faithful. If not, using a result from [12], we see that there is a non-zero element in \( \mathfrak{A}_x \) of the form \( (\sum_{k \in \mathbb{Z}} c_k(f, k)) \) with \( \sum_{k \in \mathbb{Z}} |c_k| < \infty, f \in C(X_{\infty}) \) such that this element is mapped to 0 under \( \hat{\pi} \).

This means that \( \pi(f) \sum_{k \in \mathbb{Z}} c_k U^k = 0 \). With Theorem 2.6 it follows that for almost all \( \omega \in A \) and all \( \xi \in l^2(\mathbb{Z}) \), \( l \in \mathbb{Z} \), one has

\[
f(\hat{\pi}^l(\omega)) \sum_{k \in \mathbb{Z}} c_k \xi(k+l) = 0.
\]

Take \( \xi = \delta_{l} \) and get \( f(\hat{\pi}^l(\omega))c_{l-1} = 0 \) which implies that either \( f(\hat{\pi}^l(\omega)) = 0 \) for all \( l \), or \( c_l = 0 \) for all \( l \). But, if \( c_l = 0 \) for all \( l \) then this contradicts that the element in the crossed-product in non-zero.

Thus, for almost all \( \omega = (z_i)_{i \in \mathbb{Z}} \in A \), we have that \( f(\hat{\pi}^l(\omega)) = 0 \) for all \( l \).

This implies that \( f = 0 \) on almost all \( \cup_x N_C(x) \). We know that non-empty open sets in \( X \) have positive \( \rho \)-measure (see [13]). Hence, since the measure on each \( N_C(x) \) is atomic, every non-empty open set in \( \cup_x N_C(x) \) has positive measure. This implies that \( f \) has to be 0 on all \( \cup_x N_C(x) \). We claim that this set is dense in \( X_{\infty} \).

Take \( \omega := (z_1, z_2, \ldots) \in X_{\infty} \) and \( n \geq 1 \) fixed. Since \( N_C(z_n) \) is not empty, there exists a path \( (y_1, y_2, \ldots) \) that starts at \( z_n \) and is convergent to the cycle. Then,
$(z_1, z_2, \ldots, z_n, y_1, y_2, \ldots)$ is in $N_C(x)$ and coincides with the initial path on the first $n$ components. Thus, the path $\omega$ can be approximated with paths in $N_C(x)$.

Hence $\cup_0 N_C(x)$ is dense in $X_\infty$, and this implies that $f = 0$. The contradiction yields the result.

□

Remark 2.8. [Iteration of rational functions]

Let $r: S^2 \to S^2$ be a rational function viewed as an endomorphism of the Riemann sphere $S^2$, or $C^\infty = C \cup \{\infty\}$; and suppose the degree of $r$ is bigger than 2. Let $X = X(r)$ be the Julia set of $r$, i.e., $X$ is the complement of the largest open subset $U$ such that $\{r^n|_U | n \geq 1\}$ is a normal family. It is known that $X$ is non-empty, compact, and that $(X, r)$ carries a unique strongly invariant probability measure; see [5] and [9].

Our present general result for cycles are motivated by the following specific theorems for rational mappings:

Let $r$ be a rational mapping of degree at least 2.

- Let $C$ be a $p$-cycle for $r$. Then $C$ is repelling if and only if $|(r^p)'(z)| > 1$ for all $z \in C$. Moreover, $(r^p)'(z) = \prod_{w \in C} r'(w)$, $z \in C$, so $(r^p)'$ has the same value for all points $z$ on the cycle $C$.
- Every repelling cycle $C$ lies in the Julia set $X$.
- The Julia set $X$ is the closure of the repelling periodic points, see [25, Theorem 3.1].

2.4. The scaling function. We now turn to a theorem which is analogous to the existence theorem for the scaling function in the classical theory of wavelets. As outlined in [10], the wavelet scaling function $\varphi$ in $L^2(\mathbb{R})$ depends on a filter function $m_0$ defined on $T = \mathbb{R}/\mathbb{Z}$. In fact, in the wavelet theory, it is the Fourier transform $\hat{\varphi}$ which is an infinite product of scaled versions of $m_0$. As is well known, this representation requires that the function $m_0$ satisfies two conditions: one is called the quadrature condition, and the second is called the low-pass condition. Both of these conditions are motivated directly from the probabilistic interpretation that $|m_0|^2$ enjoys in signal processing.

In our theorem below we identify the analogue of these two conditions for the function $m_0: X \to \mathbb{C}$ which is associated to an endomorphism $r: X \to X$. The quadrature condition takes the form (2.10), and the low-pass condition takes the form (2.11). The reason for the name quadrature is that $r(z) = z^2$ in the wavelet case, and the reason for the name low-pass, is that points on $T = \mathbb{R}/\mathbb{Z}$ correspond to frequencies, and $x = 0$ is the lowest frequency.

In the general setting of the endomorphism $r$, the analogue of low frequencies are points in cycles $C$ for $r$, and in this setting low-pass means that $|m_0|^2$ attains its maximum on $C$. This is exactly what condition (2.11) is saying.

In the case of endomorphism, we will therefore expect to represent a scaling function as an infinite product built out of $m_0$ and iterated shifts applied to $m_0$. The fact that this can be done is the main conclusion in the theorem.

Definition 2.9. A complex valued function $f$ on a metric space $X$ is called $\beta$-Lipschitz at a point $x_0$ if there is a non-decreasing function $\beta: [0, \infty) \to [0, \infty)$
such that for all $A > 0$ and $c < 1$,
\[
\sum_{k=1}^{\infty} \beta(Ac^k) < \infty,
\]
and
\[
|f(x) - f(x_0)| \leq \beta(d(x, x_0)),
\]
for all $x$ in some neighborhood of $x$.

**Definition 2.10.** Let $W : X \to [0, 1]$ be a given function, and set
\[
R_W f(x) := \sum_{r(y) = x} W(y)f(y), \quad (x \in X).
\]
We say that $R_W$ is a **transfer operator**, or a **Ruelle operator**. A function $h$ on $X$ is said to be **harmonic** with respect to $R_W$ if $R_W h = h$. A cycle $C$ for $r$ is said to be a $W$-cycle if $W(x) = 1$ for all $x \in C$.

The operator in Definition 2.10 plays a role in many areas of mathematics and physics. Some of its recent uses are highlighted in Ref. [30], where it is key to Ruelle’s thermodynamical formalism.

**Lemma 2.11.** There is a unique family of measures $P_x$ supported on $\Omega_x$, $x \in X$, satisfying the following relation: for all measurable sets $E \subset \Omega_x$ and all $x \in X$
\[
\sum_{r(y) = x} W(y)P_y(\Omega_y \cap \hat{r}^{-1}(E)) = P_x(\Omega_x \cap E).
\]

**Proof.** It is enough to define $P_x$ on cylinder sets: for a fixed $(a_1, a_2, \ldots, a_n, \ldots) \in \Omega_x$ and $n \geq 2$,
\[
E := \{(z_1, z_2, \ldots) \in \Omega_x \mid z_1 = a_1, \ldots, z_n = a_n\}
\]
Then $P_x(E \cap \Omega_x) = \prod_{k=1}^{n} W(a_k)$.

The extension of $P_x$ to the sigma-algebra generated by the cylinder sets now follows from Kolmogorov’s theorem. See [13] for more details.

Also, for $y \in r^{-1}(x)$, one has that $\hat{r}^{-1}(E) \cap \Omega_y$ is empty unless $y = a_1$, in which case $P_y(\hat{r}^{-1}(E) \cap \Omega_y) = \prod_{k=2}^{n} W(a_k)$. The lemma follows. □

**Lemma 2.12.** The function $h_C(x) := P_x(N_C(x))$ is harmonic with respect to $R_W$.

**Proof.** We have the following disjoint union $\cup_{r(y) = x} N_C(y) = \hat{r}^{-1}(N_C(x))$. The lemma follows then from Lemma 2.11. Indeed,
\[
(R_W h_C)(x) = \sum_{r(y) = x} W(y)h_C(y) = \sum_{r(y) = x} W(y)P_y(\Omega_y \cap N_C(y))
\]
\[
= \sum_{r(y) = x} W(y)P_y(\Omega_y \cap \hat{r}^{-1}N_C(x)) = P_x(\Omega_x \cap N_C(x)) = h_C(x).
\]
□

**Definition 2.13.** We call $h_C(x) := P_x(N_C(x))$ the **harmonic function** associated to the cycle $C$. See also [13] for more details.

In our next theorem, we prove that each repelling cycle $C$ generates a covariant operator system on the corresponding Hilbert space $L^2(\lambda_C)$. Moreover, under two conditions on a given filter function $m_0$, we show that the corresponding scaling equation has a natural solution $\hat{\varphi}_C$ in $L^2(\Omega, \lambda_C)$. 
Theorem 2.14. Assume now that the cycle $C$ is repelling and the functions $r$ and $x \mapsto \#r^{-1}(x)$ are continuous at the points in $C$. Let $m_0 \in L^\infty(X, \rho)$ be a function which is $\beta$-Lipschitz at the points in $C$ and which satisfies the conditions

\begin{equation}
\frac{1}{\#r^{-1}(x)} \sum_{y \in r^{-1}(x)} |m_0(y)|^2 = 1, \quad (x \in X),
\end{equation}

and

\begin{equation}
m_0(x_i) = \alpha_i \sqrt{\#r^{-1}(r(x_i))}, \quad (i \in \{0, \ldots, p-1\}).
\end{equation}

Define the function $\hat{\varphi}$ by

\begin{equation}
\hat{\varphi}(x, (z_k)_{k \geq 1}) := \prod_{k=1}^\infty \frac{\alpha^{-1}_{(\omega_k)+1} m_0(z_k)}{\sqrt{\#r^{-1}(r(z_k))}}, \quad (x \in X, (z_k)_{k \geq 1} \in N_C(x)).
\end{equation}

Then $\hat{\varphi}$ is in $L^2(X_\infty, \lambda_C)$ and it satisfies the following relation:

\begin{equation}
U \hat{\varphi} = \pi(m_0) \hat{\varphi}.
\end{equation}

If $W(x) := |m_0(x)|^2 / \#r^{-1}(r(x))$, then $C$ is a W-cycle, and if $h_C$ is the harmonic function associated to this W-cycle, then

\begin{equation}
\langle \pi(f) \hat{\varphi} \mid \hat{\varphi} \rangle = \int_X f h_C \, d\rho.
\end{equation}

Set $V_0 := \{ f \in L^\infty(X, \rho) \mid |f| \leq 1 \}$, and $V_n := U^{-n} V_0$, for $n \in \mathbb{Z}$. Then $V_n \subset V_{n+1}$,

\[ \bigcup_{n \in \mathbb{Z}} V_n = L^2(X_\infty, \lambda_C), \quad \bigcap_{n \in \mathbb{Z}} V_n = \{0\}. \]

Proof. First we check that the infinite product (2.12) is convergent. Take $x \in X$, $\omega = (z_1, z_2, \ldots) \in N_C(x)$. Then the sequence $\{z_k\}$ converges to one of the points of the cycle $C$, namely $x_{i(\omega)}$. Applying $r$, which is continuous at these points, we obtain that, for all $l$, the sequence $\{z_{k+p+1}\}$ is convergent to $x_{i(\omega)+l}$.

Now we use the fact that the cycle is repelling. For $k$ large enough, the path $\omega$ enters the neighborhood where the cycle is repelling (see Definition 2.3). Therefore, there are constants $0 < c_l < 1$, $0 \leq m_l < \infty$ such that for $k$ large enough

\[ d(z_{k+p+1}, x_{i(\omega)+l}) \leq c_l^k m_l, \quad \text{for all } l \in \{0, \ldots, p-1\}. \]

Take $c = \max \{ c_l \} \in (0, 1)$ and $M := c^{-p} \max \{ m_l \}$. Then for $k$ large enough

\[ d(z_k, x_{i(\omega)+k}) \leq c^k M. \]

Since the function $\#r^{-1}(r(\cdot))$ is continuous at the points of the cycle, we get that for $k$ large, $\#r^{-1}(r(z_k)) = \#r^{-1}(r(x_{i(\omega)+k})) =: A_k \geq 1$.

Let $\beta$ be the function given by the $\beta$-Lipschitz condition for $m_0$ at all the points of the cycle (Take the minimum of these functions over all the points of the cycle). Using the condition (2.11), we have:

\[ \frac{\alpha^{-1}_{(\omega)+1} m_0(z_k)}{\sqrt{\#r^{-1}(r(z_k))}} - 1 \leq \beta(d(z_k, x_{i(\omega)+k})) \leq \beta(c^k M). \]

This implies that the sum over the terms on the left-hand side of this inequality is convergent, which in turn implies that the infinite product is absolutely convergent.
Next we check (2.14). It is clear that $C$ is a $W$-cycle. Also note that

$$|\hat{\varphi}(x, (z_k)_{k \geq 1})|^2 = \prod_{k=1}^{\infty} W(z_k) = P_x(\{ (z_k)_{k \geq 1} \}).$$

(See [13]). Then

$$(2.15) \quad h_C(x) = P_x(N_C(x)) = \sum_{\omega \in N_C(x)} |\hat{\varphi}(x, \omega)|^2,$$

and equation (2.14) follows. Since $h_C \leq 1$, this also implies that $\hat{\varphi}$ is in $L^2(X_\infty, \lambda_C)$.

We check the equation (2.13). For $\omega = (z_1, z_2, \ldots) \in N_C(x)$,

$$U\hat{\varphi}(x, \omega) = \sqrt{\# r^{-1}(r(x))} \alpha_i(\omega) \prod_{k=1}^{\infty} \frac{\alpha_i(r(\omega))^{-1} m_0(z_{k-1})}{\sqrt{\# r^{-1}(r(z_{k-1}))}} = \sqrt{\# r^{-1}(r(x))} \alpha_i(\omega) m_0(x) \prod_{k=2}^{\infty} \frac{\alpha_i(r(\omega))^{-1} m_0(z_{k-1})}{\sqrt{\# r^{-1}(r(z_{k-1}))}} = m_0(x) \hat{\varphi}(x, \omega).$$

The scaling equation (2.13) and the covariance equation (2.8) imply that $V_{-1} \subset V_0$. This implies that the sequence of subspaces $\{V_n\}$ is increasing.

To check that their union is dense, we note that the closure of this union is invariant for $U$ and for $\pi(f), f \in L^\infty(X, \rho)$. Therefore the projection $P$ onto this space is in the commutant $\{U, \pi\}'$. But, then, with Proposition 2.2 we obtain a function $F$ in $L^\infty(X_\infty, \lambda_C)$ such that $F = F \circ \hat{r}$ and $P = M_F$. In particular, $F\hat{\varphi} = \hat{\varphi}$ and $F$ is the characteristic function of some set $F$ which is invariant for $\hat{r}$.

However, the previous argument shows that, if $\omega = (z_1, z_2, \ldots) \in N_C(x)$ has $z_i$ close enough to the cycle $C$, for all $i \geq 1$, then $\hat{\varphi}(x, \omega)$ is close to 1. Now, take $\omega = (z_1, z_2, \ldots) \in N_C(x) \setminus F$. Then $\hat{r}^{-n}(\omega)$ is outside $F$. Because $\omega \in N_C(x)$, for $n$ large enough, all the points $z_{n+1}, z_{n+2}, \ldots$ are close to the cycle, so $\hat{\varphi}(\hat{r}^{-n}(\omega))$ is close to 1. But $\hat{\varphi}(\hat{r}^{-n}(\omega)) = \hat{\varphi}(\hat{r}^{-n}(\omega))1_x(\hat{r}^{-n}(\omega)) = 0$, a contradiction. It follows that $F$ has complement of measure 0 so $P = M_F$ is the identity, and therefore the union of the multiresolution subspaces is dense.

It remains to check that the intersection $\cap V_n$ is trivial. We use the following lemma:

**Lemma 2.15.** Define $\mathcal{J}: L^2(X, h_C \, d\rho) \to V_0$ by

$$\mathcal{J}(f) = \pi(f)\hat{\varphi}, \quad (f \in L^\infty(X, \rho)).$$

Define the operator $S_0$ on $L^2(X, h_C \, d\rho)$ by $S_0 f = m_0 f \circ r$. Then $\mathcal{J}$ is an isometric isomorphism such that $U \mathcal{J} = \mathcal{J} S_0$.

The proof of the lemma requires just some simple computations. The fact that $S_0$ is an isometry is proved in Theorem 3.8.

With this lemma, the assertion follows from Theorem 3.8. □

Let $(X, \mathcal{B}, \rho)$ be a measure space with $\rho$ some fixed probability measure defined on the sigma-algebra $\mathcal{B}$ on $X$. Let $\pi$ be a representation of $L^\infty(X, \mathcal{B})$ on a Hilbert space $\mathcal{H}$, and suppose that the measure $f \mapsto \langle \pi(f)\psi | \psi \rangle$ is absolutely continuous with respect to $\rho$ for all $\psi \in \mathcal{H}$, i.e., there exists $h_\psi \in L^1(X, \mathcal{B})$ such that $\langle \pi(f)\psi | \psi \rangle = \int_X f h_\psi \, d\rho$, $f \in L^\infty(X, \mathcal{B})$.

By the spectral multiplicity theorem ([3], [15], [31]), there is a measurable function $d: X \to \{1, 2, \ldots, \infty\}$ such that if $X_k := \{x \in X \mid d(x) \geq k\}$, then the
spectral representation of \( \pi \) takes the form of an isometric isomorphism \( \mathcal{J}: H \to \sum_{k \geq 1} L^2(X_k, \mathfrak{B}, \rho) \), such that \( \mathcal{J}_k \pi(f) \psi = f \mathcal{J}_k \psi = M_f \mathcal{J}_k \psi \) for all \( f \in L^\infty(X, \mathfrak{B}) \) and all \( \psi \in \mathcal{H} \).

We say that \( \mathcal{D} \) is the multiplicity function of the representation \( \pi \).

**Corollary 2.16.** Let \( V_0 \subset L^2(X_\infty, \lambda_C) \), be the subspace from Lemma 2.15 and Theorem 2.14, and let \( \pi_n \), \( n \in \mathbb{N} \), be the restriction of the representation \( \pi \) of \( L^\infty(X, \mathfrak{B}) \) to \( U^{-n}V_0 \). Then

\[
d_{U^{-n}V_0}(x) = \sum_{r^n(y) = x} d_{V_0}(y) = \sum_{r^n(y) = x} \chi_{E_C}(y) = \#(r^{-n}(x) \cap E_C).
\]

**Proof.** Since \((\pi(f)\hat{\varphi}|\hat{\varphi}) = \int_X f h_C d\rho\), it follows that \( d_{V_0}(x) = \chi_{\{z \in X | h_C(z) \neq 0 \}} =: \chi_{E_C} \).

From 2.12, we know that

\[
d_{U^{-n}V_0}(x) = \sum_{r^n(y) = x} d_{V_0}(y) = \sum_{r^n(y) = x} \chi_{E_C}(y) = \#(r^{-n}(x) \cap E_C).
\]

**Example 2.17.** Let \( A \) be a square matrix with \( 0-1 \) entries. Suppose every column of \( A \) contains at least one entry 1. Then we show that the two systems \((X_\infty, \hat{r})\) and \((X, r)\) may be realized as two-sided, respectively one-sided subshifts.

Let \( I \) be the index set for the rows and columns of \( A \). Let

\[
X_\infty(A) := \{(x_i)_{i \in \mathbb{Z}} \in I^\mathbb{Z} | A(x_i, x_{i+1}) = 1 \}.
\]

Let

\[
\hat{r}((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}.
\]

Define \( \theta_0((x_i)_{i \in \mathbb{Z}}) = (x_i)_{i \geq 0} \), and set \( X(A) = \theta_0(X_\infty(A)) \).

Then there is an endomorphism \( r = r_A: X(A) \to X(A) \) such that \( r \circ \theta_0 = \theta_0 \circ \hat{r} \).

Specifically,

\[
x = (x_i)_{i \in \mathbb{Z}} = \ldots x_{-2}x_{-1}x_0x_1x_2 \ldots
\]

with \( x_i \in I \);

\[
\theta_0((x_i)_{i \in \mathbb{Z}}) = (x_i)_{i \geq 0} = x_0x_1x_2 \ldots;
\]

and \( r(x_0x_1x_2 \ldots) = (x_1x_2x_3 \ldots) \) for \( x \in X(A) \).

For \( x, y \in X(A) \), let \( x \wedge y \) be the longest initial block in \( I \times I \times \ldots \) common to both \( x \) and \( y \), and let \( |x \wedge y| \) be the length of this block. Let \( 0 < c < 1 \), and set \( d_c(x, y) = c^{|x \wedge y|} \). Then \( d_c \) is a metric, and \((X(A), d_c)\) is a compact metric space whose open sets are generated by the cylinder sets in \( X(A) \). Moreover, \( d_c(r(x), r(y)) \leq c^{-1}d_c(x, y) \) holds for all \( x, y \in X(A) \). If \( x \in X(A) \), then \( r^{-1}(x) = \{(ix) | A(i, x_0) = 1 \} \), and for the transfer operator \( \mathcal{L}_A: C(X(A)) \to C(X(A)) \),

\[
(\mathcal{L}_A f)(x) = \frac{1}{\#r^{-1}(x)} \sum_{r(y) = x} f(y),
\]

we have

\[
(\mathcal{L}_A f)(x) = \frac{1}{\# \{i | A(i, x_0) = 1 \}} \sum_{A(i, x_0) = 1} f(ix).
\]

By [2], there is a unique probability measure \( \rho = \rho_A \) on \( X(A) \) such that \( \rho(\mathcal{L}_A f) = \rho(f) \) for all \( f \in C(X(A)) \); i.e., \( \rho = \rho_A \) is the unique strongly invariant probability measure on \( X(A) \).
It follows that all the results in this setting apply; in particular, if $C \subset X(A)$ is a cycle, then $L^2(X_\infty(A), \lambda_C)$ is defined by

$$\int_{X_\infty(A)} |f|^2 \, d\lambda_C = \int_{X(A)} \sum_{\omega \in N_C(x)} |f(\omega)|^2 \, d\rho(x) < \infty.$$ 

Note also that $N_C(x)$ consists of doubly infinite words in $X_\infty(A)$ that start with an infinite repetition of the cycle $C$. Specifically, for $x = (x_0x_1x_2 \ldots) \in X(A)$, $N_C(x) = \{(\omega_i)_{i \in \mathbb{Z}} \in X_\infty(A) \mid \exists k \in \mathbb{N} \text{ such that } (\omega_i)_{i \leq k} = C^\infty, (\omega_i)_{-k \leq i < -1} \text{ is some finite word, and } \omega_i = x_i \text{ for } i \geq 0\}$.

We now turn to a concrete example. Let the index set $I$ be $\{1, 2\}$ and let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. This is called the golden mean shift (see [22, page 37]).

**Proposition 2.18.** Let $m_0$ be the function on $X(A)$ determined by

(2.16) \[ m_0(11 \ldots) = \sqrt{2}, \quad m_0(21 \ldots) = 0, \quad m_0(12 \ldots) = 1, \]

and $C$ be the cycle $\{111 \ldots\}$. Then $m_0$ satisfies (2.10) and (2.11) and defines a scaling vector $\hat{\phi} \in L^2(X_\infty, \lambda_C)$ with $h_C = 1$.

**Proof.** It is easy to verify the two conditions (2.10) and (2.11). The scaling function $\hat{\phi}$ is defined by the infinite product (2.12). (We set $\alpha_i = 1$.) If $\omega$ is in $N_C(x_0x_1 \ldots)$, then it has the form $111x_{-n}x_{-n+1} \ldots x_{-1}x_0 \omega x_1 \ldots$. Note that if one of the letters $x_{-k}$ is $2$ ($k \geq 1$) then the next one has to be $1$. Therefore, shifting the word to the right will bring the $21$ to the central position, and $m_0$ is $0$ on words that start with $21$. Therefore the infinite product is non-zero only when $x_{-k} = 1$ for all $k \geq 1$. Then, an analysis of the possibilities for $x_0$ shows that $\hat{\phi} = 1$ in this case. Therefore $\hat{\phi}(\omega) = 1$, if $x_{-k} = 1$ for all $k \geq 1$, and $\hat{\phi}(\omega) = 0$ otherwise.

Then by (2.15)

$$h_C(x_0x_1 \ldots) = \sum_{\omega \in N_C(x_0x_1 \ldots)} |\hat{\phi}(\omega)|^2 = 1.$$ 

An interesting consequence of (2.10) and (2.11) for this example is that an admissible $m_0$ cannot be of the form $m_0 = \sqrt{2} \chi_E$ for a subset $E$ of $X(A)$ (because $|m_0(21 \ldots)|^2 = 1$). This contrasts a known wavelet, the Shannon wavelet, see [17 and [3].

### 3. Ergodic properties and the Wold decomposition

In our analysis of the intersection of the multiresolution spaces $V_n$, we are forced to study some convergence properties for the measure $\rho$ and the filter $m_0$. The main tool in this study will be Doob’s convergence theorems for reversed martingales, see e.g. [27].

Even though we are mainly interested in the strongly invariant measure $\rho$, our analysis works in the following more general case.

**Definition 3.1.** Let $V \geq 0$ be a measurable function on $X$ such that

$$\sum_{r(y) = x} V(y) = 1, \quad (x \in X).$$
A probability measure $\nu$ on $X$ such that

\[ (3.1) \quad \int_X f \, d\nu = \int_X \sum_{r(y)=x} V(y)f(y) \, d\nu(x), \quad (f \in L^1(X,\nu)). \]

is called a Perron-Frobenius measure for the corresponding Ruelle operator

\[ (R_V f)(x) := \sum_{r(y)=x} V(y)f(y), \quad (x \in X). \]

For example, when $V(x) = 1/#r^{-1}(r(x))$, then (3.1) is equivalent to the strong invariance of $\nu$.

Note that a Perron-Frobenius measure $\nu$ is also invariant for $r$, because

\[ \int_X f \circ r \, d\nu = \int_X \sum_{r(y)=x} V(y)f(r(y)) \, d\nu(x) = \int_X f(x) \sum_{r(y)=x} V(y) \, d\nu(x) = \int_X f \, d\nu. \]

Let $\mathcal{B}$ be the sigma-algebra of measurable subsets of $X$.

**Definition 3.2.** Let $\mathcal{B}$ be a sigma-algebra on $X$ and let $\nu$ be a probability measure on $(X,\mathcal{B})$. Let $\mathcal{C} \subset \mathcal{B}$ be a sub-sigma-algebra. Then the $\mathcal{C}$-conditional expectation $E_{\mathcal{C}}$ is defined by

\[ \int_X E_{\mathcal{C}}fg \, d\nu = \int_X fg \, d\nu, \]

for $f \in L^1(\mathcal{B},\nu)$, $g \in L^\infty(\mathcal{C})$; and $E_{\mathcal{C}}L^1(\mathcal{B},\nu) = L^1(\mathcal{C},\nu)$.

**Proposition 3.3.** The operator $E_n^V$ defined on $L^1(X,\nu)$ by

\[ E_n^V(f)(x) = \sum_{r^n(y)=r^n(x)} V^{(n)}(y)f(y), \quad (x \in X), \]

defines the conditional expectation of $\mathcal{B}$ with respect to $r^{-n}(\mathcal{B})$.

**Proof.** First note that if a function $g$ on $X$ is $r^{-n}(\mathcal{B})$-measurable, then $g(x) = g(y)$ whenever $r^n(x) = r^n(y)$. Take now $g \in L^2(r^{-n}(\mathcal{B}))$ and $f \in L^1(\mathcal{B})$. Using the invariance of $\nu$ and (3.1), we have

\[ \int_X E_n^V(f)g \, d\nu = \int_{r^n(x)=r^n(y)} \sum_{r^n(y)=x} V^{(n)}(y)f(y)g(x) \, d\nu(x) \]

\[ = \int_X \sum_{r^n(y)=x} V^{(n)}(y)f(y)g(y) \, d\nu(x) = \int_X fg \, d\nu. \]

This shows that $E_n^V$ is the conditional expectation.

We note the relation between the Ruelle operator $R_V$ and the conditional expectation $E_n^V$:

\[ (3.2) \quad E_n^V(f) = (R_V^n) \circ r^n, \quad (n \geq 1, f \in L^1(X,\nu)). \]

The sigma-algebras $r^{-n}(\mathcal{B})$ form a decreasing sequence, and we denote their intersection by $\mathcal{B}_\infty$. Denote by $E^V_\infty$ the conditional expectation of $\mathcal{B}$ with respect to $\mathcal{B}_\infty$. Doob’s theorems for reverse martingales can be applied now directly and we obtain the following theorem:

**Theorem 3.4.** If $f \in L^p(X,\nu)$, $(1 \leq p < \infty)$, then $E_n^V(f)$ converges pointwise $\nu$-a.e. and in $L^p(X,\rho)$ to $E_\infty^V(f)$. 

Definition 3.5. We say that \( r \) is averaging (with respect to the measure \( \nu \)), if \( L^1(\mathcal{B}_\infty) \) contains only functions which are constant \( \nu \)-a.e., (or, equivalently, the sigma-algebra \( \mathcal{B}_\infty \) contains only sets of \( \nu \)-measure 0 or 1).

Proposition 3.6. If \( r \) is averaging with respect to \( \nu \) then it is also ergodic with respect to \( \nu \).

Proof. If \( A \) is a completely invariant set for \( r \) then, for any two points \( x, y \) such that \( r^n(x) = r^n(y) \) for some \( n \geq 0 \), \( \chi_A(x) = \chi_A(r^n(x)) = \chi_A(r^n(y)) = \chi_A(y) \), so \( \chi_A \in L^1(\mathcal{B}_\infty) \), therefore \( \nu(A) \) is 0 or 1.

Corollary 3.7. If \( r \) is averaging with respect to \( \nu \), then for all \( f \in L^p(X, \nu) \), \( 1 \leq p < \infty \), the sequence \( E_n^r(f) \) converges pointwise \( \nu \)-a.e. and in \( L^p(X, \nu) \) to \( \int_X f \, d\nu \).

Next, we will derive an ergodic property for a function \( m_0 \) satisfying (2.10).

Theorem 3.8. Assume that the strongly invariant measure \( \rho \) is ergodic with respect to \( r \). Let \( m_0 \in L^\infty(X, \rho) \) be a function that satisfies (2.10) and such that \( |m_0| \neq 1 \) on a set of positive measure. Then

\[
A := \int_X \ln |m_0(x)| \, d\rho(x) \in [-\infty, 0).
\]

Then

\[
\lim_{n \to \infty} |m_0(x) \cdots m_0(r^{n-1}(x))|^{1/n} = e^A, \quad \text{for } \rho\text{-a.e. } x \in X.
\]

Proof. We have, using the strong invariance of \( \rho \):

\[
\int_X \ln |m_0(x)| \, d\rho(x) = \frac{1}{2} \int_X \ln |m_0(x)|^2 \, d\rho(x) = \int_X \frac{1}{\#r^{-1}(x)} \sum_{r(y) = x} \ln |m_0(y)|^2 \, d\rho(x)
\]

\[
= \int_X \ln \left( \prod_{r(y) = x} |m_0(y)|^2 \right) \, d\rho(x)
\]

\[
\leq \int_X \ln \left( \frac{1}{\#r^{-1}(x)} \sum_{r(y) = x} |m_0(y)|^2 \right) \, d\rho(x) = \int_X \ln(1) = 0.
\]

If we have equality in this chain, then we get that for \( \rho \)-a.e., \( x \in X \), \( |m_0(y)| = |m_0(y')| \) for all \( y, y' \in r^{-1}(x) \), which implies that

\[
1 = \frac{1}{\#r^{-1}(r(x))} \sum_{r(y) = r(x)} |m_0(y)|^2 = |m_0(x)|, \quad \text{for a.e. } x.
\]

This contradicts the hypothesis. Thus \( A \in (-\infty, 0) \).

Assume now, that \( A > -\infty \). Then, using Birkhoff’s ergodic theorem, we obtain that

\[
\lim_{n \to \infty} \ln \left( \frac{|m_0(x) \cdots m_0(r^{n-1}(x))|^{1/n}}{e^A} \right) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |m_0(r^k(x))| - A
\]

\[
= \int_X \ln |m_0(x)| \, d\rho(x) - A = 0.
\]

This yields the conclusion in the case \( A > -\infty \).
When \( A = -\infty \), take \( 0 > B > -\infty \) arbitrary and choose a bounded measurable function \( f \), with \( |f| \geq |m_0| \) and such that \( -\infty < \int_X \ln |f(x)| \, d\rho(x) = B \). Then apply the previous argument to conclude that \( |f(x) f(r(x)) \cdots f(r^{n-1}(x))|^{1/n} \) converges a.e. to \( e^B \). Then

\[
\limsup_n |m_0(x) \cdots m_0(r^{n-1}(x))|^{1/n} \leq e^B
\]

and, as \( B \) is arbitrary this implies that the limit is \( e^{-\infty} = 0 \). \( \square \)

With these results, we are now able to derive the result about the Wold decomposition \( \mathcal{W} \) of the isometry \( S_0 \) associated to \( m_0 \).

**Theorem 3.9.** Let \( \rho \) be a strongly invariant measure for \( r \). Let \( m_0 \in L^\infty(X, \rho) \) be a function that satisfies (2.10). Let \( h \in L^\infty(X, \rho) \) be a function such that \( h \geq 0 \) and

\[
\frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} |m_0(y)|^2 h(y) = h(x), \quad (x \in X).
\]

Then the operator \( S_0 \) on \( L^2(X, h \, d\rho) \) defined by

\[
S_0 f = m_0 f \circ r
\]

is an isometry.

Assume in addition that \( r \) is averaging with respect to \( \rho \), and that \( |m_0| \neq 1 \) on a set of positive measure \( \rho \). Then

\[
\bigcap_{k \geq 1} S_0^k (L^2(X, h \, d\rho)) = \{0\}.
\]

**Proof.** The fact that \( S_0 \) is an isometry follows from the fact that \( \rho \) is strongly invariant and from the relation \( c_0 \):

\[
\int_X |m_0(x)|^2 |f(r(x))|^2 h(x) \, d\rho(x) = \int_X \frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} |m_0(y)|^2 |f(r(y))|^2 h(y) \, d\rho(x)
\]

\[
= \int_X |f(y)|^2 \, d\rho(x).
\]

Denote, by

\[
c(x) := \frac{1}{\#r^{-1}(r(x))}, \quad (x \in X).
\]

Note that

\[
R_{m_0}^k f(x) = \sum_{r^k(y)=x} c^{(k)}(y) |m_0^{(k)}(y)|^2 f(y),
\]

where \( R_{m_0} \) is the Ruelle operator associated to \( W(x) := |m_0(x)|^2 / \#r^{-1}(r(x)) \).

In particular

\[
\sum_{r^k(y)=x} c^{(k)}(y) |m_0^{(k)}(y)|^2 = 1.
\]
Take now $\xi \in \cap_k S_k^2(L^2(X, h \, dp))$. Then for all $k \geq 1$, there exists $f_k \in L^2(X, h \, dp)$ such that $\xi = m_0^{(k)} f_k \circ r^k$. This implies that for all $x \in X$:

$$|\xi(x)|^2 = |m_0^{(k)}(x)|^2 |f_k(r^k(x))|^2 \sum_{r^k(y) = r^k(x)} c^{(k)}(y) |m_0^{(k)}(y)||f_k(r^k(y))|^2$$

$$= |m_0^{(k)}(x)|^2 \sum_{r^k(y) = r^k(x)} c^{(k)}(y) |m_0^{(k)}(y)||f_k(r^k(y))|^2$$

$$= |m_0^{(k)}(x)|^2 \sum_{r^k(y) = r^k(x)} c^{(k)}(y) |\xi(y)|^2$$

$$= |m_0^{(k)}(x)|^2 E_\xi(|\xi|^2).$$

With Theorem 3.8 and Corollary 3.7 if we let $k \to \infty$, we can conclude that $\xi = 0$, $\rho$-a.e. This proves the theorem.

\[ \square \]

**Remark 3.10.** It is conceivable that the last conclusion in Theorem 3.9 above may hold slightly more generally; possibly when only ergodicity is assumed for $(X, r, \rho)$. But for the applications we have in mind, our present assumption of strong invariance is appropriate, i.e., the averaging assumption we place on the system $(X, r, \rho)$.

### 3.1 Some conditions for $r$ to be averaging.

We will give some necessary conditions for $r$ to averaging. For this we will relate the expectation $E_\nu^V$ to the Ruelle operator $R_V$.

Just as before, assume $V \geq 0$ is a measurable function such that

$$\sum_{r(y) = x} V(y) = 1, \quad (x \in X),$$

and let $\nu$ be a measure such that

$$\int_X R_V f \, d\nu = \int_X f \, d\nu.$$

**Proposition 3.11.** Suppose there exists a family of functions $\mathcal{F}$ which is dense in $L^1(X, \nu)$ such that for all $f \in \mathcal{F}$,

$$\lim_{n \to \infty} \|R_V^n(f) - \int_X f \, d\nu\|_1 = 0.$$

Then, for all $f \in L^1(X, \nu)$,

$$\lim_{n \to \infty} R_V^n(f) = \int_X f \, d\nu = E_\nu^V(f).$$

In particular $r$ is averaging with respect to $\nu$.

**Proof.** Take $f \in L^1(X, \nu)$, and $\epsilon > 0$. There exists $g \in \mathcal{F}$, such that $\|f - g\|_1 < \epsilon$. Then, using the fact that $\nu$ is invariant for $r$, and also for $R_V$, we have, with the aid of (3.2):

$$\|E_\nu^V(f) - \int_X f \, d\nu\|_1 = \|R_V^0 f - \int_X f \, d\nu\|_1$$

$$\leq \|R_V^0(f - g)\|_1 + \|R_V^n g - \int_X g \, d\nu\|_1 + \|\int_X (g - f) \, d\nu\|_1$$

$$\leq 2\|f - g\|_1 + \|R_V^n g - \int_X g \, d\nu\|_1 < 3\epsilon,$$
for $n$ large enough. This proves the first assertion. Since $E^\infty(f)$ is constant for all $f \in L^1(X,\nu)$, it follows that $L^1(\mathcal{B}_\infty)$ contains only constant functions so $r$ is averaging. □

Remark 3.12. The conditions of Proposition 3.11 are satisfied in many cases of interest. This is a consequence of Ruelle’s theorem (see [1], [14]). For example, if $r$ is locally expanding (i.e., there exists $b > 0$ and $\lambda > 1$ such that $d(r(x), r(y)) \geq \lambda d(x, y)$ when $d(x, y) < b$), and mixing (i.e., for every open set $U$ in $X$, there exists $n$ such that $r^n(U) = X$), and if $V > 0$ and is Lipschitz, then $\mathcal{F}$ can be taken to be the set of continuous functions, and $R^V f$ converges uniformly to $\int_X f \, d\nu$, where $\nu$ is the unique probability measure invariant for $R^V$.

In particular, this is satisfied, for subshifts of finite type.

Also, consider the case when $r$ is a rational map on $\mathbb{C}$ and $X$ is its Julia set. Take $V = 1/N$ where $N$ is the degree of the map $r$. Then $\nu = \rho$ is the unique strongly invariant measure and we may take again $\mathcal{F}$ to be the set of continuous functions (see [23]).

Given our assumptions above, the existence and the uniqueness of the measure $\nu$ follows from the conclusion in Ruelle’s theorem, applied to $R^V$.

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