A Modified Spectral Phase Conjugation Algorithm for Quadratic Programming with Linear Constraint Problems-application to Municipal Solid Waste Management

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Abstract. In this study, a modified spectral phase conjugation algorithm (MSPCA) for quadratic programming with linear constraint (QPLC) is developed for waste-management systems. MSPCA is applied to a municipal solid waste (MSW) management problem. The results indicate that reasonable solutions have been generated for supporting MSW management, which supply more information for managers to make decisions.

Introduction

Many factors lead to nonlinearity in municipal solid waste (MSW) management systems, which may intensify the complexities in decision making process [1-3]. Thus, effective algorithm which can deal with nonlinearity in the optimization analysis is desired [4, 5]. Quadratic programming (QP) is a useful optimization method for dealing with nonlinearities in systems analysis. For example, Tøndel et al. obtained explicit solutions to constrained linear model predictive control problems by solving multi-parametric quadratic programs [6]. An inexact piecewise quadratic programming model was proposed by Sun to handle both nonlinearity and uncertainty [7]. Kong et al. developed a duality theorem-based algorithm for inexact quadratic programming in municipal solid waste management systems [8]. Wichapa and Khokhajaikiat proposed a new multi-objective facility location problem model in order to select new suitable locations for infectious waste disposal [9].

This study is to develop a modified spectral phase conjugation algorithm (MSPCA) for quadratic programming with linear constraint (QPLC). MSPCA will be applied to a problem of municipal solid waste management. The solutions are suitable to real MSW problems and will provide more information for making better decisions.

Methodology

Formulation of a Linear Constraint Programming Problem

Consider a quadratic programming with linear constraint (QPLC) problem [10]:

$$\min \ f(x) \quad \text{s.t.} \quad Ax = b$$

where $f : R^n \rightarrow R$ is continuously partially differentiable, $A \in R^{m \times n}$ is row full rank, $b \in R^m$. $D$ and $S$ denote the feasible region and the set of feasible direction of the QPLC problem, respectively.

$$D = \{ x \in R^n | Ax = b \}$$

$$S = \{ d \in R^n | Ad = 0 \}$$

The coefficient matrix $A$ can be decomposed as:
\( A = (B, N) \) \tag{4}

where \( B \in \mathbb{R}^{m \times m} \) is nonsingular. Let \( x = (x^B, x^N)^T \) for \( x \in D \). So we have \( x^B = B^{-1}b - B^{-1}Nx^N \).

Therefore, the QPLC problem (1) can be transformed to an unconstrained optimization problem shown below:

\[
\min F(x^N) = f(x^B, x^N) = f(B^{-1}b - B^{-1}Nx^N, x^N) \tag{5}
\]

**Modified Spectral Phase Conjugation Algorithm (MSPCA) for Quadratic Programming with Linear Constraint**

**Establishment of MSPCA.** The previous section shows that it is needed to solve model (5) to find the optimal solution of model (1). In this paper, a modified spectral conjugation algorithm (MSPCA) is proposed to solve model (5). Let \( x_k \) as current iteration point. Solve the following QPLC problem.

\[
\min \nabla f(x_k)^T d \\
\text{s.t.} \quad Ad = 0, \|d^N\| \leq 1 \tag{6}
\]

The solution of model (6) is \( \bar{d} = (\bar{d}^B, \bar{d}^N)^T \). Therefore, for any \( d = (d^B, d^N)^T \in \mathbb{R}^n \) that satisfies \( Ad = 0 \), we have

\[
(B, N) \begin{pmatrix} d^B \\ d^N \end{pmatrix} = 0 \tag{7}
\]

\[
d^B = -B^{-1}Nd^N. \tag{8}
\]

And the objective function of model (6) can be transformed as follows:

\[
\nabla f(x_k)^T d = \left( \nabla_{x_k^B} f \left( x_k^B \left( x_k^N \right), x_k^N \right)^T, \nabla_{x_k^N} f \left( x_k^B \left( x_k^N \right), x_k^N \right)^T \right) \begin{pmatrix} d^B \\ d^N \end{pmatrix} \\
= \nabla_{x_k^B} f \left( x_k^B \left( x_k^N \right), x_k^N \right)^T d^B + \nabla_{x_k^N} f \left( x_k^B \left( x_k^N \right), x_k^N \right)^T d^N \\
= \nabla_{x_k^B} f \left( x_k^B \left( x_k^N \right), x_k^N \right)^T ( -B^{-1}Nd^N ) + \nabla_{x_k^N} f \left( x_k^B \left( x_k^N \right), x_k^N \right)^T d^N \\
= \left( \nabla_{x_k^B} f \left( x_k^B \left( x_k^N \right), x_k^N \right) - \left( -B^{-1}N \right)^T \nabla_{x_k^N} f \left( x_k^B \left( x_k^N \right), x_k^N \right) \right)^T d^N \\
= \nabla F(x_k^N)^T d^N. \tag{9}
\]

Based on the Gauchy-Schwarz inequality, as to \( \forall d = \begin{pmatrix} d^B \\ d^N \end{pmatrix} \in \mathbb{R}^n, \|d^N\| \leq 1 \), we have

\[
\nabla F(x_k^N)^T d^N \geq -\|\nabla F(x_k^N)\| \|d^N\| \geq -\|\nabla F(x_k^N)\|. \tag{10}
\]

Therefore, \( \bar{d}_k = (\bar{d}_k^B, \bar{d}_k^N)^T \) can be got.
\[
\begin{align*}
\vec{d}_k^N &= -\frac{\nabla F(x_k^N)}{\|\nabla F(x_k^N)\|} \\
\vec{d}_k^\theta &= -B^{-1}N\vec{d}_k^N
\end{align*}
\]

Let \( z_k = \nabla f(x_k) \vec{d}_k = \nabla F(x_k^N)^T \vec{d}_k^N = -\|\nabla F(x_k^N)\| \), \( g_k = z_k \vec{d}_k \), the search direction \( d_k \) can be established as follows:

\[
d_k = \begin{cases} 
-g_k, & k = 0 \\
-\theta_k g_k + \beta_k d_{k-1}, & k \geq 1.
\end{cases}
\]

The determined search direction \( d_k \) is a feasible direction at \( x_k \). The detail procedure of MSPCA can be summarized as follows:

Step 1. Set constant \( \varepsilon > 0 \), point of feasible areas \( x_0 \in D \) and \( k = 0 \).

Step 2. Solve Model (6) and get the corresponding solution \( \vec{d}_k \). Let \( z_k = \nabla f(x_k) \vec{d}_k \) and \( g_k = z_k \vec{d}_k \). If \( z_k \leq \varepsilon \) is satisfied, stop. The solution of Model (1) is \( x_k \). Otherwise, go to Step 3.

Step 3. Determine the search direction \( d_k \) by Equation (13).

Step 4. Determine \( \alpha_k \) through linear searching method to satisfy \[ f(x_k + \alpha_k d_k) = \min f(x_k + \alpha d_k). \]

Step 5. Let \( x_{k+1} = x_k + \alpha_k d_k \) and \( k = k + 1 \). Then back to Step 2.

**Global Convergence Analysis.** Hypothesis \( A \) : (1) \( \Omega = \{x \in R^n \mid f(x) \leq f(x_0)\} \) is bounded; (2) \( f(x) \) is continuously differentiable and the corresponding gradient is lipschitz continuous in the area \( N \) of \( \Omega \), i.e. \( \exists L > 0, \text{s.t.} \|g(x) - g(y)\| \leq L \|x - y\|(x, y \in N) \).

Theorem 1. If the objective function \( f(x) \) satisfy the hypothesis A, the range of points generated by MSPCA under exact linear searching method satisfy that:

\[
\liminf_{k \to \infty} \|z_k\| = 0
\]

Proof: Based on Equation (13), we have:

\[
\begin{pmatrix} d_k^\theta \\ d_k^N \end{pmatrix} = \theta_k \begin{pmatrix} g_k^\theta \\ g_k^N \end{pmatrix} + \beta_k \begin{pmatrix} d_{k-1}^\theta \\ d_{k-1}^N \end{pmatrix}
\]

Specially, we have \( d_k^N = -\theta_k g_k^N + \beta_k d_{k-1}^N \). Two scenes are discussed according to the values of
\( \beta_k \):

1. When \( \beta_k = 0 \), the value of \( \theta_k \) is 1 (i.e. \( \theta_k = 1 \)). Therefore, \( \lim_{k \to \infty} \|z_k\| = 0 \) is satisfied.

2. When \( \beta_k \neq 0 \), suppose that Equation (17) is not true, i.e. \( \exists \epsilon > 0, s.t. \)

\[ \|z_k\| \geq \epsilon, \quad \forall k \geq 0 \]

Thus, we have

\[
\begin{align*}
\|d_k\|^2 &= \theta_k^2 \left\| g_k^N \right\|^2 - 2 \theta_k \beta_k \left( g_k^N \right)^T d_{k-1} + \beta_k^2 \left\| d_{k-1} \right\|^2 \\
&= \theta_k^2 \left\| g_k^N \right\|^2 + \beta_k^2 \left\| d_{k-1} \right\|^2
\end{align*}
\]

Equation (20) can be transformed to:

\[
\frac{\|d_k\|^2}{\left\| \nabla F(x_k^N) \right\|^2} = \frac{\theta_k^2 \left\| g_k^N \right\|^2 + \beta_k^2 \left\| d_{k-1} \right\|^2}{\left\| \nabla F(x_k^N) \right\|^2} \tag{21}
\]

(a) when \( \frac{z_k^T z_{k-1}}{z_k^T d_{k-1}} \geq 0 \), we have \( \beta_k = \frac{z_k^T}{z_k^T d_{k-1}} \) and \( \theta_k = 1 \). Therefore, Equation (21) can be transformed to:

\[
\frac{\|d_k\|^2}{\left\| \nabla F(x_k^N) \right\|^2} = \frac{1}{\left\| \nabla F(x_k^N) \right\|^2} + \frac{1}{\left\| \nabla F(x_k^N) \right\|^2} = \sum_{i=1}^{k} \frac{1}{\left\| \nabla F(x_k^N) \right\|^2} \tag{22}
\]

Then

\[
\frac{\|d_k\|^2}{\left\| \nabla F(x_k^N) \right\|^2} \leq \frac{k}{\epsilon^2} \tag{23}
\]

(b) when \( \frac{z_k^T z_{k-1}}{z_k^T d_{k-1}} \leq 0 \), we have \( \beta_k = \frac{z_k^T (z_k - z_{k-1})}{\|z_{k-1}\|^2} \) and \( \theta_k = 1 \). Therefore, Equation (21) can be transformed to:

\[
\frac{\|d_k\|^2}{\left\| \nabla F(x_k^N) \right\|^2} \leq \frac{k}{\epsilon^2} \tag{24}
\]

Synthesize (23) and (24), we have

\[
\|d_k\|^2 \leq \frac{k}{\epsilon^2} \left\| \nabla F(x_k^N) \right\|^4 \tag{25}
\]

\( d_k \) is feasible direction and \( d_k^\beta = -B^{-1} N d_k^N \). Therefore, \( \exists \epsilon > 0, s.t. \)

\[
\|d_k^\beta\|^2 \leq \epsilon \|d_k\|^2 \leq \frac{c k}{\epsilon^2} \left\| \nabla F(x_k^N) \right\|^4 \tag{26}
\]

Then
\[ \|d_k\|^2 \leq \|d_k^0\|^2 + \|d_k^\varepsilon\|^2 \leq \frac{c+1}{\varepsilon^2} k \|\nabla F(x_k^\varepsilon)\|^2 \]

Equation (27) is equivalent to:

\[ \frac{\|\nabla F(x_k^\varepsilon)\|^2}{\|d_k\|^2} \geq \frac{\varepsilon^2}{(c+1)k} \]

Then

\[ \sum_{k=0}^{\infty} \frac{\|\nabla F(x_k^\varepsilon)\|^2}{\|d_k\|^2} \geq \sum_{k=0}^{\infty} \frac{\varepsilon^2}{(c+1)k} = \infty \]

This is conflict to Zoutendijk. The theorem proving is finished.

### Numerical Example

A simple QPLC problem is used as an example to verify the proposed algorithm. Consider a QPLC problem as follows:

\[
\begin{align*}
\min & \quad f(x) = (x_1 - 1)^2 + (x_2 - 2)^2 \\
\text{s.t.} & \quad x_1 + x_2 + x_3 = 2 \\
& \quad -2x_1 + 3x_2 + x_4 = 3
\end{align*}
\]

The point of feasible areas is \( x_0 = (0, 0, 2, 3)^T \). Solutions of the above model can be derived, \( f = 1.8524e^{-009} \), which occurs at \( x_1^* = 1.0000 \), \( x_2^* = 2.0000 \), \( x_3^* = -0.9999 \) and \( x_4^* = -1.0000 \).

### Application

A hypothetical case is developed to illustrate the applicability of linear constraint programming method with MSPCA in MSW management system. A landfill and an incinerator and two cities are included in this hypothetical case. The capacity of the landfill is 0.90 million tons. The capacity of the incinerator is 400 tons/day. The incinerator generates residues of approximately 35% of the incoming waste streams, and its revenue from energy sale is $16 per ton of combusted waste. Operating costs of landfill and incinerator are 55 and 78 ($/ton), respectively. The waste generations of City1 and City 2 are 350 and 300 (tons/day), respectively. The planning horizon is 1 year. The waste transportation costs from cities to landfill and incinerator are functions of waste flow (shown in Figure 1 and Figure 2), leading to a QPLC problem.
The objective is to allocate waste flows to suitable waste management facilities to minimize the net system cost. The constraints contain the relationships among decision variables and capacities of waste management facilities. Consequently, we have:

\[
\begin{align*}
\min f &= \sum_{i=1}^{2} \sum_{j=1}^{2} L(\alpha_{ij}x_{ij} + \beta_{ij})x_{ij} + \sum_{i=1}^{2} \sum_{j=1}^{2} L \cdot OP_{ij}x_{ij} \\
&\quad + \sum_{j=1}^{2} L \cdot FE(\left(\lambda_{j}x_{j} + FE + \mu_{j}\right) + OP_{j})x_{j} - \sum_{j=1}^{2} L \cdot RE \cdot x_{j} \\
\text{s.t.} \quad \sum_{j=1}^{2} L(x_{ij} + FE \cdot x_{j}) &\leq TL \\
\sum_{j=1}^{2} x_{ij} &\leq TE \\
x_{ij} &\geq 0, \quad \forall i, j
\end{align*}
\]

where \( f \) is total cost; \( i \) is type of waste management facility (\( i = 1 \) for landfill, and \( i = 2 \) for incinerator); \( x_{ij} \) is waste flow from city to facility \( i \) (ton/day); \( L \) is length of period (day); \( \alpha_{ij}x_{ij} + \beta_{ij} \) is transportation cost which is a function of waste flow ($/ton); \( OP_{ij} \) is operating cost of facility \( i \) ($/ton); \( FE \) is residue flow rate; the corresponding transportation cost form incinerator to landfill is \( \lambda_{j}x_{j} + \mu_{j} \) which is a function of waste flow ($/ton); \( RE \) is revenue from incinerator ($/ton); \( TL \) and \( TE \) is capacity of landfill and incinerator, respectively. \( WG \) is waste generation in the city (ton/day). MSPCA can be used to solve Model (31). The optimal objective function and the corresponding solutions of decision variables are presented in Table 1.

| \( ij \) | Facility | City | Allowable waste flow (tons/day) |
|--------|----------|------|---------------------------------|
| 11     | Landfill | 1    | 108.69                          |
| 12     | Landfill | 2    | 300                             |
| 21     | Incinerator | 1 | 241.31                          |
| 22     | Incinerator | 2 | 0                               |

The minimized system cost ($) \( f \) = 0.22 \times 10^8

Figure 3 shows the waste flows allocated from cities to landfill and incinerator. In detail, City 1 would use two waste management facilities to deal with its wastes. The waste flow from City 1 to landfill is 108.69 tons/day in the planning horizon. The waste flow to incinerator is 241.31 tons/day. City 2 would only use landfill to deal with the waste flows (i.e. 300 tons/day). The minimized system cost is $0.22 \times 10^8$. The optimized decision variables are presented in Table 1.
cost is $f = 0.22 \times 10^8$. The results can provide more information for decision makers. Global optimum solutions of QPLC problems can be obtained using the proposed algorithm.

![Figure 3. Waste flows allocated from cities to facilities.](image)

**Conclusions**

In this study, a modified spectral phase conjugation algorithm (MSPCA) has been proposed to deal with quadratic programming with linear constraint (QPLC) problems. The application to a case study of MSW management indicates that MSPCA can efficiently deal with nonlinearity in the objective function and linearity in the constraint. The results will supply more information for decision makers in association with cost minimization.

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