Quantum Double Models coupled to matter fields: some remarks and an algebraic dualization procedure

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Abstract

In this paper, we construct a new generalization of a class of discrete two-dimensional models, the so-called Quantum Double Models, by adding matter fields (i.e., new qunits) to the lattices on which these models are defined. As these matter fields were added to the lattice faces and coupled to the gauge fields by using a co-action homomorphism, this new generalization can be interpreted as the algebraic dual of the first one, where these same matter fields were added to the lattice vertices and coupled to the same gauge fields by using a group action. By evaluating the algebraic and topological orders of this new generalization, we prove that, similar to what happens in the first one, a new phenomenon of quasiparticle confinement (which is analogous to that of quark confinement) may appear again: this always happens when the co-action homomorphism between the matter and gauge groups is non-trivial. Consequently, the group homomorphism (that defines this co-action) not only classifies all the different models of this new generalization, but also suggests that they can be interpreted as two-dimensional restrictions of the 2-lattice gauge theories.

1. Introduction

One of the current issues of interdisciplinary research involves models and technologies that try to support some kind of quantum computing \cite{1, 2, 3, 4, 5}. And since the original purpose of the Quantum Computation is to construct a generalization of the Classical Computation \cite{6, 7} by using and manipulating \textit{quantum bits} (qubits) \cite{8}, some of these models are theoretically proposed associating these qubits with the edges of some lattice $L_2$. In general, this lattice is usually chosen as the one that discretizes some two-dimensional compact orientable manifold $\mathcal{M}_2$ to avoid any problems in reading data encoded by these qubits. However, a crucial advantage of using lattices that discretize these two-dimensional compact orientable manifolds is the possibility of dealing with models that, because they

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have a topological order \[9\], can perform some fault-tolerant quantum computation \[10, 11\]. This is precisely the case of models such as the \textit{Toric Code} and its natural generalization, so-called \textit{Quantum Double Models} (\(D(G)\)), which uses quantum nits (qunits) instead of qubits \[11, 12, 13\].

Once the \(D(G)\) models do not associate any qunits with the vertices or/and faces that define its lattice, one paper was published a few years ago \[14\] where these models were coupled to new qunits associated with the lattice vertices (\(D_M(G)\)). And as the \(D(G)\) models can be understood in terms of a gauge theory \[12\], these new qunits were purposely denoted as \textit{matter fields}, similarly to what was done in Ref. \[15\], where lattice gauge theories were coupled to fixed-length scalar (Higgs) fields allocated on the lattice vertices. Yet, although this Ref. \[14\] has explored important features of these \(D_M(G)\) models, not a word was said about their topological order: the only thing that was said was that, as the \(D_M(G)\) magnetic quasiparticles increase the energy of the system when they are transported through the lattice, the ground state of these \(D_M(G)\) models does not necessarily depend on the first homotopy group \(\pi_1(M_2)\).

By virtue of this increase of energy, another paper was published shortly thereafter, presenting a new class of toy models (\(H_M/C(Z_N)\)) where this increase no longer happens \[16\]. This class of (Abelian) \(H_M/C(Z_N)\) models can be interpreted as a subclass of the \(D_M(G)\) models where (i) the gauge group \(G\) is the Abelian group \(Z_N\) and (ii) the operators, which measure the holonomies around the lattice faces in the \(D_M(G)\) models, were excluded from their Hamiltonian\(^{1}\). And although it seems a little strange to build this subclass, which have the same gauge group dependence as the \(D(G)\) models, without the operators that measure these holonomies, these \(H_M/C(Z_N)\) models have, at least, a very interesting property: some (matter) excitations, which can be created by manipulating the matter fields, exhibit non-Abelian fusion rules \[16, 18\]. Therefore, since the only difference between the \(H_M/C(Z_N)\) and \(D_M(Z_N)\) models (i.e., the \(D_M(G)\) models where \(G = Z_N\)) concerns the operators that measure the holonomies above-mentioned, it is not wrong to affirm that these (matter) excitations are also present in the \(D_M(Z_N)\) models. In this fashion, since the \(H_M/C(Z_N)\) and (therefore) \(D_M(G)\) vacuum states can be defined by filling all the lattice vertices with these (matter) excitations, it is not difficult to demonstrate that the topological order of these \(H_M/C(Z_N)\) and \(D_M(G)\) models is associated with the second group of homology \(H_2(M_2)\) \[19, 20\].

Nevertheless, in contrast to the \(D(G)\) models \[21\], all these generalizations cannot be interpreted as self-dual models and, therefore, a natural question that arises is: how to use these \(D_M(G)\) models as a basis for defining a self-dual generalization of the \(D(G)\) models where, for instance, qunits are associated with all the faces and vertices of \(L_2\)? By the way, can a generalization of the \(D(G)\) models, intentionally defined by using the dual framework of these \(D_M(G)\) models, show us if it is possible to construct this self-dual

\(^{1}\)At this point, we need to make an addendum: after all, although Ref. \[14\] refers to its models by using the same notation as Ref. \[16\], here we prefer to use the \("D_M(G)\"\) notation not only to differentiate the models of these two works, but also to highlight the fact that the models of Ref. \[14\] satisfy the same Drinfeld’s quantum double algebra \[17\] of the \(D(G)\) models \[11\].
generalization? And in order to answer these questions, this paper is rightly devoted to the analysis of a new class of models \( (D^K(G)) \) that is intentionally defined by using the dual framework of these \( D_M(G) \) models. That is, this new generalization of the \( D(G) \) models, whose construction/definition will be detailed in Section 3, have the same gauge structure as them, but has the matter qunits attached only with the centre of all the faces of \( L_2 \), since all these centres can be interpreted as the vertices of a dual lattice \( L_2^* \) [22]. However, as we need to do this construction/definition (and, consequently, analyse it) based on what we already know about the \( D_M(G) \) models, we will intentionally use the next Section to do a thorough review of these models. Just for the sake of simplicity, we will consider that \( L_2 \) and, consequently, \( L_2^* \) are square lattices, even though we know that all the considerations that will be presented in this paper can be applied to general lattices.

2. A brief overview on the \( D_M(G) \) models

According to what we already said in the Introduction, the \( D_M(G) \) models are straightforward generalizations of the \( D(G) \) models and there are, at least, two reasons that justify this statement. The first one is the fact that, besides the \( D_M(G) \) models being defined by

- taking the same oriented lattice \( L_2 \) of the \( D(G) \) models, which has \( N_\ell \) edges and \( N_v \) vertices, and discretizes a two-dimensional compact orientable manifold \( M_2 \), and
- associating an \( N \)-dimensional Hilbert space \( H_N \) with the \( \ell \)-th edge of \( L_2 \) in the same way as in the \( D(G) \) models,

these models also take advantage of the \( M \)-dimensional Hilbert spaces \( H_M \), which are assigned to all the lattice vertices in order to define their (total) Hilbert space as

\[
H_N \otimes \cdots \otimes H_N \otimes H_M \otimes \cdots \otimes H_M
\]

In the case of the Hilbert space \( H_N \), it is possible to affirm that its basis is \( B_\ell = \{ |g\rangle : g \in G \} \), where \( G \) is a group. In the case of the Hilbert space \( H_M \), whose basis is \( B_\alpha = \{ |\alpha\rangle : \alpha \in S \} \) with \( S = 0, 1, \ldots, M - 1 \), one thing we can say is that it can be interpreted as a (left) \( C(G) \)-module [23] because there is a multiplication (group action) \( \mu : G \times S \to S \) that defines how the \( G \) group acts on the \( B_\alpha \) elements.

The second reason, which justifies why the \( D_M(G) \) models are straightforward generalizations of the \( D(G) \) models, is the fact that the \( D_M(G) \) Hamiltonian operator

\[
H_{D_M(G)} = -\sum_v A_v - \sum_f B_f - \sum_\ell C_\ell
\]

is given by the linear superposition of the operators

\[
A_v = \frac{1}{|G|} \sum_{g \in G} A_v^{(g)}, \quad B_f = B_f^{(0)} \quad \text{and} \quad C_\ell = C_\ell^{(0)}
\]
Figure 1: Piece of an oriented square lattice $\mathcal{L}_2$ that supports the $D_M(G)$ models where we see: (i) the rose-coloured sector ($S_v$) centred in the $v$-th vertex; (ii) the baby blue coloured sector ($S_f$) highlighting the $f$-th face; and (iii) the light orange coloured sector ($S_\ell$) centred in the $\ell$-th edge of this lattice to which the two vertices, which limit this edge, belong. Here, the highlighted edges (in black) correspond to Hilbert subspaces $\mathcal{H}_N$ in which, for instance, the vertex (the rose-coloured sector), the face (the baby blue coloured sector) and the edge (the light orange coloured sector) operators mentioned in (3) act effectively. An analogous comment applies to the vertices highlighted with Greek letters: i.e., they correspond to Hilbert subspaces $\mathcal{H}_M$ in which, for instance, only these vertex and edge operators act effectively.

whose components are given by Figure 2. Here, the operators $A_v$, $B_f$ and $C_\ell$, which are denoted by vertex, face and edge operators respectively, act non-identically only in the subspaces highlighted in Figure 1. Thus, by noting that the group action is such that

$$\mu(g, 0) = 0,$$

it is not difficult to conclude that the $D(G)$ models can be interpreted as a subclass of the $D_M(G)$ models: after all, as the $D(G)$ Hamiltonian is given by [11]

$$H_{D(G)} = -\sum_v A_v - \sum_f B_f,$$

it is not hard to see that, when $M = 1$,

- the $D_M(G)$ vertex and face operators are reduced to those of the $D_M(G)$ models, whose components are given by Figure 3, and

- all the eigenvalues of the edge operator are equal to 1 and, consequently,

$$H_{D(G)} - H_{D_1(G)} = N_\ell.$$
\[ A_v^{(g)} \left| \frac{a}{d} \rightarrow \frac{b}{c} \alpha \right\rangle = \sum_{\gamma} \delta (\mu (g, \alpha), \gamma) \left| \frac{g \alpha}{c g^{-1}} \rightarrow \frac{g b}{c g^{-1}} \right\rangle \]

\[ B_f^{(h)} \left| \frac{a}{b} \frac{c}{d} \right\rangle = \delta (h, a^{-1} b^{-1} c d) \left| \frac{a}{b} \frac{c}{d} \right\rangle \]

\[ C^{(\alpha)}_{\ell} \left| \frac{a}{b} \frac{\alpha}{\beta} \right\rangle = \delta (\mu (a, \alpha), \beta) \left| \frac{a}{b} \frac{\alpha}{\beta} \right\rangle \]

Figure 2: Definition of the components \( A_v^{(g)} \), \( B_f^{(h)} \) and \( C^{(\alpha)}_{\ell} \) in terms of their effective action on \( L_2 \), where the group element \( a \) is indexing an \( \left| a \right\rangle \) basis element of the Hilbert space \( H_N \) and the symbol \( \alpha \) indexes an \( \left| \alpha \right\rangle \) basis element of the Hilbert space \( H_M \). Here, \( \delta (x, y) \) should be interpreted as a Kronecker delta that was written differently for the sake of intelligibility (i.e., \( \delta (x, y) = \delta_{xy} \)).

\[ A_v^{(g)} \left| \begin{array}{c} a \\ c \\ d \\ a \\ b \\ c \\ d \end{array} \right\rangle = \begin{array}{c} \frac{a}{d} \\ \frac{c}{d} \end{array} \]

\[ B_f^{(h)} \left| \begin{array}{c} a \\ c \\ d \\ b \\ a \\ c \\ d \end{array} \right\rangle = \delta (h, a^{-1} b^{-1} c d) \left| \begin{array}{c} a \\ c \\ d \\ b \\ a \\ c \\ d \end{array} \right\rangle \]

Figure 3: Definition of the components \( A_v^{(g)} \) and \( B_f^{(h)} \), which define the \( D(G) \) vertex and face operators, in terms of their effective action on \( L_2 \). Note that, since these \( D(G) \) vertex and face operators are also expressed as \( A_v^{(g)} = \frac{1}{|G|} \sum_{g \in G} A_v^{(g)} \) and \( B_f^{(h)} = B_f^{(0)} \) respectively, the definition of these components makes it very clear that the \( D(G) \) and \( D_M(G) \) face operators are exactly the same.

2.1. Some considerations about the vertex, face and edge operators

As a matter of fact, it is due to this “correspondence principle” (which can be recognized between the \( D(G) \) and \( D_M(G) \) models by taking \( M = 1 \)) that we can affirm that the vertex and face operators, which define the Hamiltonian (2), have the same properties of those that define the Hamiltonian (5). After all, just as with the \( D(G) \) vertex operator, the \( D_M(G) \) vertex operator can also be interpreted as an operator that performs a new kind of gauge transformations because its action does not modify the holonomies around the lattice faces. In accordance with the Figure 2, these holonomies are measured by the face components \( B_f^{(h)} \) and, in the case of the face operator that defines the Hamiltonian operators (2) and (5), it measures flat connections in the \( D(G) \) and \( D_M(G) \) models [12]; i.e., \( B_f \equiv B_f^{(0)} \) measures trivial holonomies characterized by \( h = 0 \) along the faces.

With regard to the edge components \( C^{(\alpha)}_{\ell} \) that appear in Figure 2, it is possible to affirm that they (and, consequently, the edge operator that defines the Hamiltonian (2)) work...
literally as comparators; i.e., $C_{\ell}^{(\alpha)}$ compares two neighbouring matter fields by checking whether they are aligned from the $\mu$ point of view. In this fashion, as the general idea behind the creation of these $D_M(G)$ models is to serve as a prototype for some physical system(s) that support(s) the creation and annihilation of quasiparticles, this ability to compare that $C_{\ell}^{(\alpha)}$ possesses is very welcome: after all, by noting that, when all the gauge qunits are “turned off” by the adoption of a trivial group $G = \{0\}$, these $D_M(G)$ models are reduced to Potts models [24], it is not wrong to say that these $D_M(G)$ models can be interpreted as the $D(G)$ models coupled to the Potts models.

Another thing that is also not wrong to say about these $D_M(G)$ models is that the logic behind their definition is purely computational. After all, as qunits are the quantum mechanical generalizations of (classical) bits in classical information processing, attach them to a lattice is a way of organizing the scanning procedure (performed by the vertex, face and edge operators) in search of energy excitations (in the sectors that give these operators their names). As a matter of fact, as these operators $A_v$, $B_f$ and $C_\ell$ are projectors (i.e., as they are all operators that have eigenvalues equal to 0 and 1), they can be interpreted as “counting operators” and this is exactly what justifies the expression of the Hamiltonian operator as the linear superposition (2). And as this expression (2) makes it clear that the lowest energy of the $D_M(G)$ models is

$$E_0 = - (N_v + N_f + N_\ell),$$

it is not hard to conclude that the $D_M(G)$ vacuum states are such that

$$A_v |\xi_0\rangle = |\xi_0\rangle, \quad B_f |\xi_0\rangle = |\xi_0\rangle \quad \text{and} \quad C_\ell |\xi_0\rangle = |\xi_0\rangle$$

hold for all the $N_v$ vertices, $N_f$ faces and $N_\ell$ edges that structure $L_2$. Note that, by taking into account the definition of the vertex operator in (3), it is not difficult to see that one of these vacuum states is

$$|\xi_0^{(1)}\rangle = \prod_{\ell} C_\ell \prod_f B_f \prod_v A_v \left[ |0\rangle \otimes \ldots \otimes |0\rangle \right]_{N_\ell \text{ times}} \otimes \left[ |0\rangle \otimes \ldots \otimes |0\rangle \right]_{N_v \text{ times}}.$$

After all, since one of its summands is

$$\left[ |0\rangle \otimes \ldots \otimes |0\rangle \right]_{N_v \text{ times}} \otimes \left[ |0\rangle \otimes \ldots \otimes |0\rangle \right]_{N_\ell \text{ times}},$$

the others are all the different gauge transformations that can be performed on (8) preserving not only all the trivial holonomies around the lattice faces (i.e., those such that $A_v |\xi_0^{(1)}\rangle = |\xi_0^{(1)}\rangle$ and $B_f |\xi_0^{(1)}\rangle = |\xi_0^{(1)}\rangle$), but also the alignment of all the matter fields (i.e., those such that $C_\ell |\xi_0^{(1)}\rangle = |\xi_0^{(1)}\rangle$).

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\[2\] Here it is important to note that, as we just said that the logic behind the definition of the $D_M(G)$ models is purely computational, we are deliberately taking a notation where the elements of the group $G$ are symbolized by using natural numbers. Therefore, since 0 must be interpreted as the neutral element of $G$, it is worth stressing that the product $0 \cdot g$ (and $g \cdot 0$) is only equal to 0 when $g = 0$. 

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2.1.1. How can non-vacuum states be defined?

Since we have just mentioned that these operators (3) are projectors, it is important to pay attention to the fact that they are not the only projectors that define these $D_M (G)$ models. Scilicet, if we resymbolize the vertex operator in (3) by $A_{v,0}$, it is possible to demonstrate that there are other vertex operators

$$A_{v,1}, A_{v,2}, \ldots, A_{v,N-2} \text{ and } A_{v,N-1}$$

that allow us to recognize

$$\mathfrak{A}_v = \{A_{v,0}, A_{v,1}, \ldots, A_{v,N-2}, A_{v,N-1}\} ,$$

$$\mathfrak{B}_f = \{B_{f,0}, B_{f,1}, \ldots, B_{f,N-2}, B_{f,N-1}\} \text{ and }$$

$$\mathfrak{C}_\ell = \{C_{\ell,0}, C_{\ell,1}, \ldots, C_{\ell,M-2}, C_{\ell,M-1}\}$$

as three complete sets of orthogonal projectors onto Hilbert space (1). Here, $B_{f,h} \equiv B_f^{(h)}$ and $C_{\ell,\alpha} \equiv C_{\ell}^{(\alpha)}$. That is, all the operators that are contained in $\mathfrak{A}_v$, $\mathfrak{B}_f$ and $\mathfrak{C}_\ell$

(a) have eigenvalues equal to 0 and 1,

(b) satisfy the relations

$$A_{v',J'} \circ A_{v'',J''} = A_{v',J'} \circ A_{v'',J''} = \delta_{v',v''} \cdot \delta_{J',J''} \cdot A_{v',J''} ,$$

$$B_{f',L'} \circ B_{f'',L''} = B_{f',L'} \circ B_{f'',L''} = \delta_{f',f''} \cdot \delta_{L',L''} \cdot B_{f',L''} ,$$

$$C_{v',N'} \circ C_{v'',N''} = C_{v',N'} \circ C_{v'',N''} = \delta_{v',v''} \cdot \delta_{N',N''} \cdot C_{v'',N'} ,$$

$$A_{v',J'} \circ B_{f',L'} = B_{f',L'} \circ A_{v',J'} = 0 ,$$

$$A_{v',J'} \circ C_{v',N'} = C_{v',N'} \circ A_{v',J'} = 0 \text{ and }$$

$$B_{f',L'} \circ C_{v',N'} = C_{v',N'} \circ B_{f',L'} = 0$$

not only for all the values of $J^{(l)}, L^{(l)} = 0, 1, \ldots, N - 1$ and $\Lambda^{(l)} = 0, 1, \ldots, M - 1$, but also for all the vertices, faces and edges of $L_2$, and

(c) are such that

$$\sum_{J=0}^{N-1} A_{v,J} = 1_v , \sum_{L=0}^{N-1} B_{f,L} = 1_f \text{ and } \sum_{\Lambda=0}^{M-1} C_{\ell,\Lambda} = 1_{\ell} .$$

Note that all these properties not only ensure that the $D_M (G)$ models are exactly solvable [14], but are also in full agreement with the Quantum Mechanics requirements [25] because, as any $D_M (G)$ vacuum state $| \xi_0 \rangle$ satisfies (6) for all the $N_v$ vertices, $N_f$ faces and $N_\ell$ edges that structure $L_2$, this allows us to decompose the Hilbert space (1) into the direct sum

$$\mathfrak{H} = \mathfrak{H}_M^{(0)} \oplus \mathfrak{H}_M^{\perp} .$$

(9)
Here, $\mathcal{H}^{(0)}$ and $\mathcal{H}^\perp$ are the orthogonal subspaces that contains all the $D_M(G)$ vacuum and non-vacuum states respectively [18]. And in the case of these non-vacuum states, which do not satisfy at least one of the conditions in (6), the energy excitations that characterize them are created through the action of operators $W_{\ell}^{(J,L,N)}$ and $W_{v}^{(J,N)}$ that are respectively such that

\begin{align}
W_{\ell}^{(J,L,N)} \circ A_{v,0} &= A_{v,J} \circ W_{\ell}^{(J,L,N)}, \\
W_{\ell}^{(J,L,N)} \circ B_{f,0} &= B_{f,L} \circ W_{\ell}^{(J,L,N)}, \\
W_{\ell}^{(J,L,N)} \circ C_{\ell,0} &= C_{\ell,\Lambda} \circ W_{\ell}^{(J,L,N)},
\end{align}

\begin{align}
W_{v}^{(J,N)} \circ A_{v,0} &= A_{v,J} \circ W_{v}^{(J,N)} \quad \text{and} \\
W_{v}^{(J,N)} \circ C_{\ell,0} &= C_{\ell,\Lambda} \circ W_{v}^{(J,N)}.
\end{align}

Thus, by remembering that the general idea behind the creation of these $D_M(G)$ models is to make them serve as a prototype for some physical system(s), it is essential that the respective energy excitations $q^{(J,L,N)}$ and $Q^{(J,N)}$ that these operators $W_{\ell}^{(J,L,N)}$ and $W_{v}^{(J,N)}$ create can be interpreted as quasiparticles. And in order for this to happen, it is essential that these energy excitations be at least such that

\begin{align}
q^{(J',L',N')} \times q^{(J'',L'',N'')} &= q^{(J'',L'',N'')} \times q^{(J',L',N')}, \\
q^{(J',L',N')} \times Q^{(J',N')} &= Q^{(J',N')} \times q^{(J',L',N')} \quad \text{and} \\
Q^{(J',N')} \times Q^{(J'',N'')} &= Q^{(J'',N'')} \times Q^{(J',N')}.
\end{align}

2.2. General properties of the $D_M(G)$ models

Due to the “correspondence principle” $D_M(G)|_{|M=1} = D(G)$, it is not difficult to conclude that the $D_M(G)$ models support the same quasiparticles as the $D(G)$ models\(^3\). However, although the fusion rules (12a) of all the $D(G)$ quasiparticles are preserved in the $D_M(G)$ models, the quasiparticles that are detectable by $B_v$ may acquire a new property when $\mu$ is a non-trivial group action. And what property is this? It is the property that we will interpret as confinement.

2.2.1. The Toric Code coupled to matter fields as an example

The best way to understand this confinement property is by taking, only as an example, the Toric Code coupled to matter fields with $M = 2$: i.e., the Abelian $D_2(\mathbb{Z}_2)$ model, where

\[^3\text{In other words, although we are using a different notation to symbolize the quasiparticles that are produced by the operators } W_{\ell}^{(J,L,N)} \text{ that satisfy (10), these quasiparticles are exactly the same ones that can be identified in the } D(G) \text{ models.} \]
the gauge group $G$ is the cyclic group $\mathbb{Z}_2$. In fact, this model is an excellent example for several reasons and, certainly, the main one is related to the fact that the matrix representation of its vertex, face and edge operators is given by [26]

\begin{align}
A_{v,J} &= \frac{1}{2} \sum_{g \in \mathbb{Z}_2} (-1)^{-Jg} \cdot M_v(g) \prod_{\ell' \in S_v} (\sigma_{\ell'}^x)^g, \\
B_{f,L} &= \frac{1}{2} \sum_{g \in \mathbb{Z}_2} (-1)^{Lg} \prod_{\ell' \in S_f} (\sigma_{\ell'}^z)^g \quad \text{and} \\
C_{\ell,\Lambda} &= \frac{1}{2} \sum_{g \in \mathbb{Z}_2} (-1)^{\Lambda g} \cdot M_\ell(g) \prod_{v \in S_\ell} (\sigma_{v}^z)^g,
\end{align}

where [14, 16]

\begin{align}
M_v(g) &= (\sigma_v^x)^g \quad \text{and} \\
M_\ell(g) &= (\sigma_\ell^x)^g.
\end{align}

After all, since the set

$$\{M(g) : g \in \mathbb{Z}_2\},$$

which is composed by matrices that represent the group action $\mu$, must also be interpreted as a matrix representation of $\mathbb{Z}_2$ [16], something that is no longer difficult to observe is, for instance, that all these operators (13) satisfy all the conditions (a), (b) and (c).

By the way, another thing that is no longer difficult to observe is that, due to the fact that the operators (13a) and (13b) are represented with the help of the Pauli matrices $\sigma^x$ and $\sigma^z$ [27], the operators $W_{\ell}^{(J,L,\Lambda)}$ that create (pairs of) quasiparticles in this $D_2(\mathbb{Z}_2)$ model can be represented by

\begin{equation}
W_{\ell}^{(J,L,\Lambda)} = (\sigma_\ell^z)^J \circ (\sigma_\ell^x)^L \quad \text{or} \quad W_{\ell}^{(J,L,\Lambda)} = (\sigma_\ell^x)^L \circ (\sigma_\ell^z)^J.
\end{equation}

That is, they are the same operators that produce the quasiparticles of the $D(\mathbb{Z}_2)$ model. However, what is most important to note here is that, unlike the $D(\mathbb{Z}_2)$ model, all the

\begin{equation*}
W_{\ell}^{(1,0,0)} = \sigma_\ell^z, \quad W_{\ell}^{(0,1,1)} = \sigma_\ell^x \quad \text{and} \quad W_{\ell}^{(1,1,1)} = \sigma_\ell^x \circ \sigma_\ell^z = \sigma_\ell^z \circ \sigma_\ell^x.
\end{equation*}

we wrote in Ref. [20], because it places more emphasis on the fact that these operators need to be expressed in terms of those that compose the Hamiltonian (5). After all, it is always good to remember that, as well as in QFT (where Hamiltonians can be expressed in the Fock representation by using the creation $a^\dagger$ and annihilation $a$ operators [28]), the entire $D(\mathbb{Z}_2)$ energy spectrum can also be well understood from [20]

- the knowledge of the ground state of these models, and
- the excitations created by the action of the operators that compose its Hamiltonian on this ground state.

Note that write (15) is also very welcome because, when $J = L = \Lambda = 0$, the operator $W_{\ell}^{(J,L,\Lambda)}$ can be identified as those that produces (a pair of) vacuum quasiparticles.
quasiparticles \( q^{(J,1,1)} \) (i.e., all the quasiparticles that are produced by \( W^{(J,1,1)}_\ell \) and, therefore, are detected by the vertex operator \( C_{\ell,0} \)) can be considered as confined in this \( D_2(\mathbb{Z}_2) \) model. After all, although it is perfectly possible to transport these quasiparticles in the \( D_2(\mathbb{Z}_2) \) model, this transport increases the energy of the system and, as this increase is not welcome for several physical reasons, we need to ignore the fact that this transport is mathematically possible and consider that all these quasiparticles \( q^{(J,1,1)} \) are confined (i.e., that all they “cannot” be transported).

As a matter of fact, it is interesting to note that there is no mistake in asserting that the main reason that led us to call this phenomenon as “confinement” is its analogy with the phenomenon of quark confinement. After all, although it is not impossible to move one quark away from another/others with which it defines a hadron, it is well known that, as this quark moves away from another/others, the potential energy of this hadronic system increases until the moment that Nature creates an additional meson (i.e., a hadron usually composed of a quark-antiquark pair) in order to conserve the energy of the system\(^5\) [30]. Thus, by noting that

- the potential energy between a quark and an antiquark in a meson increases linearly with the distance between them [31] and

- the action of any operator \( W^{(J,L,\Lambda)}_\ell \) produces one pair, which is composed of one quasiparticle \( q^{(J,L,\Lambda)}_+ \) and its anti-quasiparticle \( q^{(J,L,\Lambda)}_- \), where before it was a vacuum,

this endorses the analogy mentioned above because, as shown in Figure 4, the energy associated with a pair of quasiparticles \( q^{(0,1,1)}_+ \) and \( q^{(0,1,1)}_- \) also increases linearly with the distance between them. Given these facts, and by remembering that the general idea behind the creation of these \( D_M(G) \) models is to serve as a prototype for some physical system(s), there is no way not to recognize, from the perspective of elementary particle physics, that the action of any operator \( W^{(J,1,1)}_\ell \) produces a kind of prototype of a meson.

2.2.2. Is there any \( D_M(\mathbb{Z}_2) \) model whose ground state degeneracy depends on \( \pi_1(\mathcal{M}_2) \)?

One of the consequences of this quasiparticle confinement is that, unlike what happens with the \( D(G) \) models, the \( D_M(G) \) ground state degeneracy does not necessarily depend on the order of the fundamental group \( \pi_1 \) associated with \( \mathcal{M}_2 \). And, once again, the best example, which makes it very clear that this confinement may lead to a ground state that is independent of the order of \( \pi_1(\mathcal{M}_2) \), is the \( D_2(\mathbb{Z}_2) \) model because an operator

\[
O_1(\gamma^*) = \prod_{\ell' \in \gamma^*} W^{(0,1,1)}_{\ell'},
\]

which acts on a set \( \gamma^* \) of edges that intersect any closed dual path, always leads to a non-vacuum state.

\(^5\)In plain English, Nature prefers to convert this energy increase into mass-energy of a new quark-antiquark pair and this is exactly what, for example, justifies the appearance of the jets (i.e., spray of new hadrons) in the various experiments involving the collision of high-energy hadrons [29].
Figure 4: Piece of a same lattice region at two different times in the $D_2 (Z_2)$ model. In the first instant $t_1$ (above) we have a pair of quasiparticles $q^{(0,1,1)}_+$ and $q^{(0,1,1)}_-$ (blue outlined and purposely indexed with the “+” and “−” symbols respectively), which were created by the action of a single operator $W^{(0,1,1)}_\ell$. Here, the single orange dot corresponds to the unique vacuum violation detected by $C_{\ell,0}$. In the second instant $t_2 > t_1$ (below) we have these same quasiparticles, but after one of them has been transported away from the other due to the action of operators $W^{(0,1,1)}_\ell'$ on all the edges that intersect the dual path (highlighted in dashed black). And in this latter case, we have new (six) orange dots: one for each edge involved in this transport, making clear the linearity related to the growth of the system energy in this transport. Note that, although the quasiparticles $q^{(J,L;\Lambda)}$ of the $D_2 (Z_2)$ model can be identified as their own anti-quasiparticles, we prefer to construct this figure in this way, by using these “+” and “−” symbols, since this scenario is (also) the reality of several $D_M (Z_N)$ models where this identification does not necessarily happen.

However, when we analyse the Toric Code coupled to matter fields with $M > 2$, we may find a situation that is somewhat different. This is just what happens when we analyse the
The $D_3(\mathbb{Z}_2)$ model, whose vertex operators can always be represented as

\[ A_{v,J} = \frac{1}{2} \sum_{g \in \mathbb{Z}_2} (-1)^{-Jg} \cdot M_v(g) \prod_{\ell' \in S_v} (\sigma^x_{\ell'})^g, \]  

(17)

where\(^6\) [14, 16]

\[ M_v(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_v(1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]  

(18)

After all, by noting that the set\(^7\)

\[ \text{Fix}_\mu = \{ |\alpha\rangle_v \in H_M : M_v(g)|\alpha\rangle_v = |\alpha\rangle_v \quad \text{for all} \quad g \in \mathbb{Z}_3 \} \]  

(19)

of points of $H_M$ fixed by the group action $\mu$ is non-empty, it is not difficult to conclude that the $D_3(\mathbb{Z}_2)$ ground state degeneracy is dependent of the order of $\pi_1(M_2)$. And in order to understand this conclusion, the first thing we need to do is observe that, as

- all the $D_M(\mathbb{Z}_2)$ face operators are represented by (13b) and
- all the $D_M(G)$ edge operators cannot perform any permutation between the gauge or matter fields,

the (18) allows us to recognize that this $D_3(\mathbb{Z}_2)$ model has two vacuum states\(^8\)

\[ |\xi_0^{(0)}\rangle = \prod_{\ell'} C_{\ell'} \prod_{f'} B_{f'} \prod_{v'} A_{v'} \left( \bigotimes_{\ell \in \mathcal{L}_2} |0\rangle \right) \otimes \left( \bigotimes_{v \in \mathcal{L}_2} |0\rangle \right) \quad \text{and} \]  

(20a)

\[ |\xi_0^{(2)}\rangle = \prod_{\ell'} C_{\ell'} \prod_{f'} B_{f'} \prod_{v'} A_{v'} \left( \bigotimes_{\ell \in \mathcal{L}_2} |0\rangle \right) \otimes \left( \bigotimes_{v \in \mathcal{L}_2} |2\rangle \right) \]  

(20b)

because there is no transformation, which can be expressed as a product of the operators $A_v$, $B_f$ and $C_\ell$, that can connect these two vacuum states (20).

Once all these $D_M(\mathbb{Z}_N)$ models need to be defined respecting the same “correspondence principle” mentioned on page 5, the second thing we need to do is observe that the same operators (15) create (pairs of) quasiparticles in this $D_3(\mathbb{Z}_2)$ model\(^9\). Similarily, it is also not difficult to observe that the operators

\[ W_v^{(g,0)} = M_v(g) \]  

(21)

---

\(^6\)In this paper, we are considering the same single-qubit computational basis states of Refs. [16] and [18], where the vector (ket) $|n\rangle$, with $n$ being a natural number, can be represented by a column matrix whose $n$-th row contains the number 1 while the others are filled with the number 0.

\(^7\)Here, we are using the index $v$ only to emphasize that $|\alpha\rangle$ is an element associated with a lattice vertex.

\(^8\)Note that (20a) is just a more streamlined way of writing the same vacuum state (7).

\(^9\)In fact, these operators (15) create (pairs of) quasiparticles in all the $D_M(\mathbb{Z}_2)$ models because all these models have the same gauge structure as the Toric Code.
also need to be listed among those are able to create (matter) excitations in this model because, among other things, they define the $D_3(Z_2)$ Hamiltonian\textsuperscript{10}. However, in spite of the action of (21) on the vacuum state (20a) is identical to that of the operators (14a) on the unique $D_2(Z_2)$ vacuum state (7), there is a “problem” here: after all, as (18) permutes $|0\rangle_v \leftrightarrow |1\rangle_v$ but fixes $|2\rangle_v$, these operators (21) are completely unable to create any (matter) excitation on the vacuum state (20b). That is, the only operators that compose the $D_3(Z_2)$ Hamiltonian and can excite this vacuum state (20b) are the operators (15). And although this “problem” exists, it is precisely it that allows us to understand the relationship between the $D_2(Z_2)$ ground state degeneracy and the cardinality of (19) and, consequently, to prove why this degeneracy is dependent of the order of $\pi_1(M_2)$. After all, as $|\tilde{\xi}(2,\lambda_0)\rangle = 1$ because (18) fixes $|2\rangle_v$, and this fixing makes the $D_2(Z_2)$ vertex operator $C_{L,0}$ unable to detect any energy excitation created by $W^{(J,L,0)}_L$ on the vacuum state (20b), it becomes clear that all the quasiparticles created by these operators on this vacuum state are not confined. Thus, as the action of an operator (16) on the vacuum state (20b) does not lead to an excited state, it is not difficult to conclude that, when the set of non-contractile curves that generate the fundamental group of $M_2$ (i.e., the set of non-contractile curves that generate the first homotopy group $\pi_1(M_2)$) is non-empty, all the vacuum states

$$|\xi_0^{(2,\tilde{\lambda})}\rangle = \prod_{p=1}^{s} [O_1(\tilde{\gamma}_p^*)]^{\lambda_p} |\xi_0^{(2)}\rangle,$$

where $\tilde{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{s-1}, \lambda_s) \neq (0, 0, \ldots, 0, 0)$, are topologically independent of each other and, by definition, with respect to the vacuum states (20)\textsuperscript{11}. Here, $\lambda_p = 0, 1$ and $\tilde{\gamma}_p^*$ is a closed dual path (analogous to that highlighted in Figure 4) that should be interpreted as the discretization of $C_{p^*}$.  

2.2.3. How does the $D_{M}(G)$ ground state degeneracy depends on the topology of $M_2$?

Note that, although we have taken the $D_3(Z_2)$ model only as an example in order to show that there is a Toric Code, coupled to matter fields, whose ground state degeneracy depends on $\pi_1(M_2)$, the fact that the same proof presented above can be adapted for the others $D_{M}(G)$ models. In what way? With the help of the operators

$$O_L(\tilde{\tau}^*) = \prod_{e' \in \Gamma^{+}_{\tilde{\tau}}} W^{(0,L,0)}_{e'} \prod_{e' \in \Gamma^{-}_{\tilde{\tau}}} \left(W^{(0,L,0)}_{e'}\right)^\dagger,$$

where $\tilde{\tau}^*$ is a non-contractile closed dual path that crosses all the edges that define the set $\Gamma^{+}_{\tilde{\tau}} \cup \Gamma^{-}_{\tilde{\tau}}$, which is composed of the union of the (disjoint) subsets $\Gamma^{+}_{\tilde{\tau}}$ and $\Gamma^{-}_{\tilde{\tau}}$ whose edges

\textsuperscript{10}This comment is in full agreement with the one we already made in the footnote on page 9.

\textsuperscript{11}This condition $\tilde{\lambda} \neq \tilde{0} = (0, 0, \ldots, 0, 0)$ is of paramount importance because, when $\tilde{\lambda} = \tilde{0}$, the vacuum state $|\xi_0^{(2,\tilde{\lambda})}\rangle$ is reduced to (20b).
have counterclockwise and clockwise orientations respectively. After all, when we notice, for instance, that all the $D_M(G)$ models, where

$$M_v(\gamma) |\alpha\rangle_v = |\alpha\rangle_v,$$

have $L$ different operators $W_{\ell}^{(0,L,0)}$ that define (24) as an operator that does not modify the energy of their systems, we conclude that all these models have a set of vacuum states

$$|\xi_0^{(\alpha,\tilde{\lambda},\tilde{L})}\rangle = \prod_{p=1}^{s} [O_{L_p}(\gamma_p)]^{\lambda_p} \prod_{\ell'} C_{\ell'} \prod_{f'} B_{f'} \prod_{v'} A_{v'} \left(\bigotimes_{\ell \in L_2} |0\rangle\right) \otimes \left(\bigotimes_{v \in L_2} |\alpha\rangle\right)$$

(25)

that is degenerate as a function of $\pi_1(M_2)$. Here,

$$\tilde{L} = (L_1, L_2, \ldots, L_{s-1}, L_s) \quad \text{and} \quad \tilde{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{s-1}, \lambda_s)$$

with $\lambda_p = 0, 1$. And this is a very interesting conclusion because, from the point of view of the “correspondence principle” mentioned on page 5, it makes it clear that the $D(G)$ models are not particular cases of the $D_M(G)$ models only when $M = 1$, but also when $\mu$ acts trivially on some $\mathcal{H}_M$.

Nevertheless, there is no way not to recognize that the allocation of matter fields on the lattice vertices makes it possible to define, in all these $D_M(G)$ models, vacuum states by putting all these vertices with the same value $|\alpha\rangle$. This happens with all the $D_M(G)$ models, especially with those that have a trivial group action $\mu$: the only difference, between the $D_M(G)$ models that have a trivial action from those that do not, is that, in the former case (i.e., when $\mu$ is a trivial action), all the vacuum states

$$|\xi_0^{(\alpha)}\rangle = \prod_{\ell'} C_{\ell'} \prod_{f'} B_{f'} \prod_{v'} A_{v'} \left(\bigotimes_{\ell \in L_2} |0\rangle\right) \otimes \left(\bigotimes_{v \in L_2} |\alpha\rangle\right)$$

(26)

are independent of each other while, in the latter case (i.e., when $\mu$ is a non-trivial action), some of these states (26) (or perhaps all of them) can be connected by using some transformation that can be expressed as a product of the operators $A_v$, $B_f$ and $C_{\ell}$. Note that:

- in the former case, the $D_M(G)$ ground state is $n$-fold degenerated, where

$$n = |\text{Fix}_\mu| \cdot \mathcal{D}_{D(G)}$$

(27)

\[12\] Note that, in the case of the $D_M(Z_2)$ models, these operators (24) reduce to

$$O_{\ell} (\gamma^*) = \prod_{\ell' \in \gamma} W_{\ell}^{(0,1,0)}$$

because $(W_{\ell}^{(0,1,0)})^\dagger = W_{\ell}^{(0,1,0)} = \sigma_0$. That is, if we ignore the fact that $\gamma^*$ is a non-contractile closed dual path, it is quite remarkable that this result “coincides” with (16).
is the product between the cardinality of
\[ \mathfrak{Fix}_\mu = \{ |\alpha\rangle_v \in \mathcal{H}_M : M_v (g) |\alpha\rangle_v = |\alpha\rangle_v \text{ for all } g \in G \} \]
and \( d_{D(G)} \), which is the number of vacuum states that define the \( D(G) \) ground state;

- in the latter case, the \( n \)-fold degeneracy of the \( D_M(G) \) ground state is characterized by
\[ n = n_{\text{orb}} + |\mathfrak{Fix}_\mu| \cdot d_{D(G)} \]
because the non-trivial actions of the \( D_M(G) \) models can also define \( n_{\text{orb}} \) orbits containing more than one element (i.e., \( k \)-cycles where \( k > 1 \)).

However, as this allocation of matter fields on the lattice vertices allows us to define vacuum states by putting all these vertices with the same value \( |\alpha\rangle \), it is also impossible not to recognize that the \( D_M(G) \) ground state degeneracy also depends on the second group of homology \( \mathcal{H}_2(M_2) \) \[19, 20\]. By the way, as \( \mathcal{H}_2(M_2) \) measures the amount of 2-cycles of \( M_2 \) that cannot be considered as 2-boundaries of \( M_2 \) (i.e., \( \mathcal{H}_2(M_2) \) measures the amount of non-contractile two-dimensional compact orientable manifolds that are embedded in \( M_2 \)) \[20\], it is not difficult to show that the topological order of these \( D_M(G) \) models only becomes experimentally evident in some cases, such as where, for instance, these models are defined on a union of \( n \) two-dimensional compact connected orientable manifolds. And in order to prove this, it is important to pay attention to the statement of Theorems 1 and 2 below, whose proofs are in Ref. \[19\].

**Theorem 1.** Let \( M_2 \) be a compact connected surface (i.e., a two-dimensional manifold) without boundary. If \( M_2 \) is orientable, then \( \mathcal{H}_2(M_2) \simeq \mathbb{Z} \). If \( M_2 \) is not orientable, then \( \mathcal{H}_2(M_2) \simeq 0 \).

**Theorem 2.** Let \( K_1 \) and \( K_2 \) be connected cell complexes with \( K_1 \cap K_2 \neq \emptyset \). Then, if \( K = K_1 \cup K_2 \),
\[ \mathcal{H}_k(K) = \mathcal{H}_k(K_1) \oplus \mathcal{H}_k(K_2) \]
for all the values \( k = 0, 1, 2, \ldots \).

After all, by considering two lattices that have no common element, which discretize two disjoint two-dimensional compact orientable manifolds \( M_2^{(1)} \) and \( M_2^{(2)} \), these two theorems allow us to conclude that
\[ \mathcal{H}_2(M_2^{(1)} \cup M_2^{(2)}) = \mathbb{Z} \oplus \mathbb{Z} \]
and, therefore, that the ground state degeneracy of any \( D_M(G) \) model, which is defined on these two lattices, is \( n^2 \), where \( n \) is the number of vacuum states that this same model has in a single lattice. Thus, by extending this result to the case where these \( D_M(G) \) models...
Here, we have two two-dimensional lattices $L_2^{(1)}$ and $L_2^{(2)}$ that can be used in order to define a $D_M(G)$ model whose ground state is $n^2$-fold degenerated. And in the case of the two different colours that are being used here, they were chosen only to highlight that the two independent subsystems (that can be identified one in each lattice) are in independent vacuum configurations. Note that, although these lattices discretize two two-dimensional spheres, we can also take lattices that discretize two two-dimensional tori. After all, as all two-dimensional compact connected orientable manifold is homeomorphic to a two-dimensional sphere or to a connected sum of two-dimensional tori [19], this does not contradict the conclusions we can draw from Theorems 1 and 2.

are defined on a disjoint union of $n$ of these lattices, we can conclude that their ground state degeneracy is

$$\underbrace{n \cdot \ldots \cdot n}_{n \text{ times}} = n^n.$$  

That is, the $D_M(G)$ ground state degeneracy is a function of the quantity of sets $Z$ that appear in

$$\mathcal{H}_2(M_2^{(1)} \cup \ldots \cup M_2^{(n)}) = \underbrace{Z \oplus \ldots \oplus Z}_{n \text{ times}}$$

Note that an illustration of how this works is shown in Figure 5, where we see a $D_M(G)$ model, with $n > 1$, that is defined by using two two-dimensional lattices $L_2^{(1)}$ and $L_2^{(2)}$. After all, by supposing that these lattices have no common element, we can put the two independent subsystems (that can be identified one in each lattice) in independent vacuum configurations, which leads to a ground state $n^2$-fold degenerated.

2.2.4. Algebraic order and the presence of non-Abelian fusion rules

Given the fact that it is the group action that determines whether the vacuum states (26) are all independent of each other or not, one thing that is clear is that the $D_M(G)$ models not only have a topological order, but that they also have an algebraic order. This algebraic order is not exactly a novelty since, for instance, the vacuum state (7) is clearly expressed in terms of the vertex and edge operators in (3) (i.e., in terms of operators that explicitly depend on the group action). And one of the interesting aspects of this algebraic order is related to the fact that it is always possible to represent $\mu$ as

$$M(g) = \begin{pmatrix} A_1(g) & 0 \\ 0 & A_2(g) \end{pmatrix},$$  

(28)
where $\mathcal{A}_1$ and $\mathcal{A}_2$ are block diagonal representations of the gauge group $G$. After all, as (28) defines, at least, two orbits that have no elements in common, it makes it clear that, when $M > N \geq 2$, the $D^M (G)$ ground state must be algebraically degenerated [18].

By the way, a good example where all these features are evident is the $D_3 (\mathbb{Z}_2)$ model, which has even been partially analysed in Subsubsection 2.2.2: after all, in addition to being possible to rewrite the operators (18) as

$$M_v (g) = \begin{pmatrix} (\sigma^x)^g & 0 \\ 0^T & I \end{pmatrix},$$

(29)

where

- $\sigma^x$ (i.e., $X = \sum_{h \in \mathbb{Z}_2} |(h + 1) \mod 2 \rangle \langle h|$) generates a faithful representation of the gauge group $\mathbb{Z}_2$ and
- $I$ is the identity matrix of order 1 that generates a trivial representation of this same group,

the number $n'$ of vacuum states (20) is equal to the number of orbits (one 2-cycle and one 1-cycle) [26] that (29) defines. Yet, another thing that characterizes this algebraic order (and that, therefore, deserves our attention here) brings us to the same “problem” that was mentioned on page 13. And what is this “problem”? It is the “problem” related to the fact that, in spite of the vacuum state (20b) supports the presence of the $D (\mathbb{Z}_2)$ quasiparticles, the equations (11) show that none of the operators (21) are capable of creating any (matter) excitation on it. Nevertheless, in accordance with these same equations (11), the only way to make (20b) not useless from the matter (fields) point of view, and consequently to go from (20b) to (20a) and vice versa, is through the operator

$$W_v^{(2,0)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & a \end{pmatrix},$$

(30)

where $a$ is a complex number, because

$$W_v^{(2,0)} |0\rangle_v = W_v^{(2,0)} |1\rangle_v = |2\rangle_v \quad \text{and} \quad W_v^{(2,0)} |2\rangle_v = |0\rangle_v + |1\rangle_v + a \cdot |2\rangle_v.$$

(31)

Note that, from the physical point of view, the presence of this operator, among those who create (matter) excitations in this $D_3 (\mathbb{Z}_2)$ model, is very welcome. After all, if we consider that these two vacuum states (20) correspond to two phases that can coexist in the same energy regime, it is possible to go from one phase to another, and vice versa, via a condensation mechanism [18]. And within this context, it is also important to note that, although Ref. [14] has not discussed the need to make transitions among the states that define the $D^M (G)$ ground state, it makes an interesting observation: it observes that there

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Note that, when the set (22) is empty, $n' = n$ because the orbit of a 1-cycle is a fixed point of $\mu$. 17
are some linear combinations of (20) and (26) that are equivalent to vacuum states where, for instance, all the lattice vertices have the same matter field $|0⟩ + |1⟩ + a \cdot |2⟩$ with $a = 1$. In this fashion, by noting that $W_v^{(2,0)}$ is the only operator that can lead directly to these vacuum states (where all the lattice vertices have the same matter field $|0⟩ + |1⟩ + a \cdot |2⟩$ with $a = 1$), this observation only reinforces the need for this operator to be present in the $D_3(Z_2)$ model\(^\text{14}\).

For the sake of completeness, it is also worth noting that, as $W_v^{(2,0)}$ is such that

$$W_v^{(2,0)} \circ W_v^{(2,0)} = \begin{pmatrix} 1 & 1 & a \\ 1 & 1 & a \\ a & a & 2 + a^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + a \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & a \end{pmatrix}$$

(32)

and this composition must necessarily be associated with the fusion rule between the (matter) excitations produced by $W_v^{(2,0)}$, it becomes clear that this $D_3(Z_2)$ model supports non-Abelian fusion rules [12]. And particularly this is an interesting finding because, unlike the $D(G)$ models, these $D_M(G)$ models show that it is not necessary to deal with a non-Abelian group $G$ in order to obtain non-Abelian fusion rules. Note that, although we have explored only the $D_M(Z_2)$ models as examples in this Section, it is possible to demonstrate that non-Abelian fusion rules are always needed, for instance, when the gauge group action is represented by (28), so that there can be transitions between/among the $D_M(G)$ vacuum states [18].

3. A dualization procedure on the $D_M(G)$ models

A notable advantage we have because we have built these $D_M(G)$ models is that they can be used as a basis for building a new class of models that, for instance, can also be interpreted as a generalization of the $D(G)$ models. And one of these new classes, which we will denote by using $D^K(G)$, is the one where the $D(G)$ models are coupled to new matter fields that now appear in the faces and no longer in the vertices of $L_2$.

In order to understand why we are interested in building these $D^K(G)$ models, it is of paramount importance to remember, for instance, that the $D(G)$ models have a property that the $D_M(G)$ models do not have: they are self dual models [21]. From the physical point of view, this means, for instance, that, for each quasiparticle that is detected by a $D(G)$ vertex operator $A_v$, there is always another one, with the same properties, that is detected by a $D(G)$ face operator $B_f$ and vice versa. And one of the reasons why this happens is

\(^{14}\)At this point, it may be interesting to note that, since the operators $B_f$ and $C_ℓ$ do not perform any transformations on the gauge or/and matter fields, it does not matter to write the states that define the $D_M(G)$ ground states as we are doing or as Ref. [14] does when, for instance, it writes the five states that define the $D_3(Z_2)$ ground state by using only the operator $A_v$. However, as we want to emphasize the fact that the $D_M(G)$ ground states belong to $S_0^{(0)}$, we find it more instructive to interpret them as a result of the action of $\prod_ℓ C_ℓ \prod_f B_f \prod_v A_v$ on the “seed” (8).

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related to the fact that, when $\mathcal{L}_2$ is a lattice that discretizes some two-dimensional compact orientable manifold, there is a one-to-one correspondence between all its vertices (faces) and all the faces (vertices) of its dual lattice $\mathcal{L}_2^\ast$, and vice-versa.

Based on this finding, it seems interesting to realize a dualization procedure on the $D_M(G)$ models in order to evaluate the main properties of these new $D^K(G)$ models, which we can get by using this one-to-one correspondence [22]. Of course, this procedure will not define a class of self dual models as the $D_M(G)$ models are not either. However, if this procedure leads us to new models that are dual to these $D_M(G)$ models, all of them together may show us how these self dual models, where qunits are also associated with all the faces and vertices of $\mathcal{L}_2$, can be defined. And the first conclusion we can draw from this one-to-one correspondence, which is exemplified in Figure 6, is that the Hamiltonian operator of these $D^K(G)$ models must necessarily be defined as

$$H_{D^K(G)} = -\sum_v A'_v - \sum_f B'_f - \sum_\ell C'_\ell . \quad (33)$$

After all, as this Hamiltonian needs to be dual to (2), it is crucial that it also be defined in terms of a linear superposition of three different operators ($A'_v$, $B'_f$ and $C'_\ell$) that can
perform scans (on the vertices, faces and edges sectors of $\mathcal{L}_2$ respectively) in search of energy excitations.

3.1. Some considerations about the vertex, face and edge operators

By the way, as these new $D^K (G)$ models also need to be interpreted as generalizations of the $D (G)$ models, the first thing we can talk about $A'_v$, $B'_f$ and $C'_\ell$ is that all of them need to act on the same Hilbert (sub)space

$$\mathcal{H}_N \otimes \ldots \otimes \mathcal{H}_N$$

that was already associated with $\mathcal{L}_2$ in the $D (G)$ and $D_M (G)$ models. Note that it is a natural consequence of the fact that these $D (G)$ and $D^K (G)$ models need to respect a “correspondence principle”, which allows us to obtain the former as special cases of the latter.

Given this “correspondence principle”, the second thing we can talk about these operators concerns specifically the operator $A'_v$, which needs to act on the vertex sectors of $\mathcal{L}_2$. After all, as these vertex sectors do not include any matter fields, this “correspondence principle” requires that $A'_v$ must necessarily be identified as the same vertex operator $A_v$ of the $D (G)$ models. Note that this is a requirement that, for instance, cannot be extended to the face ($B'_f$) and new edge ($C'_\ell$) operators because they act on the sectors that include the new matter fields.

3.1.1. How can we define the face operator $B'_f$?

Something we already know about the face operators of the $D (G)$ and $D_M (G)$ models is that they measure the holonomies around the lattice faces. And in the specific case of the face operator $B_f$ that define the Hamiltonian of these two models, it measures the flat connections. That is, when $B_f$ acts on $\mathcal{L}_2$ and detects an energy excitation, what it is doing (from the geometrical point of view) is detecting a local deformation in this lattice.

Faced with this fact, and especially given the need to respect the requirements of the “correspondence principle” mentioned above, it is impossible not to recognize that $B'_f$ also needs to measure the “trivial holonomies” around the lattice faces. However, as Figure 6 already makes it clear that there are matter fields on all these lattice faces, this “trivial holonomy” it is able to measure may not be exactly the same trivial holonomy that $B_f$ is able to measure.

In order to understand what is this “trivial holonomy” that $B'_f$ needs to measure, it is worth noting that this Figure 6 also helps us to recognize that there is another fact that needs to be taken into account to define this operator. What fact is this? It is the fact that the operators $A_v$, $B_f$ and $C_\ell$ of the $D_M (G)$ models need to be dual to the operators $B'_f$, $A'_v$ and $C'_\ell$ of the $D^K (G)$ models respectively, and vice-versa. But while this Figure 6 only points to the need for this duality from a geometrical point of view, we also need to take into account that this dualization procedure also needs to be algebraic: after all, remember that the $D_M (G)$ models were created, by coupling the $D (G)$ models to matter fields allocated on the lattice vertices, with the help of a group action $\mu : G \times S \to S$. 

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Therefore, since $B'_{f}$ must necessarily be interpreted as the algebraic dual of the vertex operator in (3), it becomes clear that its definition can be done by using a co-action

$$\tilde{\alpha} \mapsto \mathcal{F}(\tilde{\alpha}) = f(\tilde{\alpha}) \otimes \tilde{\alpha},$$

(35)

where $\tilde{\alpha}$ and $f(\tilde{\alpha})$ must be elements of $\tilde{S}$ and $G$ respectively. Note that, just to distinguish these new matter fields from those that were allocated on the vertices of $L_2$, we will assume that these new matter fields belong to a Hilbert space $\tilde{\mathcal{H}}_K$ whose basis is $B_{f} = \{ |\tilde{\alpha} \rangle : \tilde{\alpha} \in \tilde{S} \}$, where $\tilde{S} = 0, 1, \ldots, K - 1$.

3.1.2. What can be said about the “trivial holonomies” that $B'_{f}$ measures?

As a matter of fact, as the result $\mu(g, \alpha) = \gamma$ is incorporated into the gauge transformations performed by the $D_M(G)$ vertex operators, it is essential that the result of (35) is also incorporated into the “trivial holonomies” that $B'_{f}$ is able to measure. And one of the things we need to do, in order to understand how this incorporation should be done, is to note that all the different holonomies, which can be measured around the lattice faces by the $D(G)$ and $D_M(G)$ face operators, are defined as

$$h = a^{-1}b^{-1}cd.$$ 

(36)

That is, as $h$ is calculated by using the same binary operation that defines $G$ as a group, all the $N - 1$ deformations, which can be identified on each face of $L_2$ due to the presence of some energy excitation, can be characterized by an element $h$ of $G$.

Given this algebraic fact, it is impossible not to recognize that the “trivial holonomies”, which $B'_{f}$ needs to be able to measure, also need to be characterized by an element of $G$: scilicet, by $h = 0$. Nonetheless, since this definition (36) of holonomy needs to be slightly modified in these $D^K(G)$ models (as otherwise their face operators will not be able to identify the presence of the dual (matter) excitations on the faces of $L_2$), it is also impossible not to recognize that this can be done with the help of the function $f$: after all, as $f(\tilde{\alpha})$ is an element of $G$, it is not hard to see, for instance, that

$$h' = f(\tilde{\alpha}) \cdot h = f(\tilde{\alpha}) \cdot a^{-1}b^{-1}cd$$

(37)

is also an element of $G$. In this fashion, as this “fake holonomy” (37) can be reduced to the true holonomy (36) in some special cases, it seems convenient to take $B'_{f,h'} \equiv B'^{(h')}_{f}$ (whose components $B'^{(h')}_{f}$ are defined in Figure 7) as the $D^K(G)$ face operators.

3.1.3. How can we define the edge operator $C'_{\ell}$?

Another important fact, which points to the convenience of taking $B'_{f,h'} \equiv B'^{(h')}_{f}$ as the $D^K(G)$ face operators, is that all of them commute with all the vertex operators inherited from the $D(G)$ models. And undoubtedly this fact is extremely relevant because, in order to make these $D^K(G)$ models exactly solvable, it is essential that all these operators are interpreted as projectors. However, as (33) shows us that the $D^K(G)$ Hamiltonian is also
\[ A'_v^{(g)} \left| \begin{array}{c} a \\ d \\ b \\ c \end{array} \right\} = \left| \begin{array}{c} gb \\ g \alpha \\ d \end{array} \right\] \]

\[ B'_f^{(h')} \left| \begin{array}{c} a \\ b \\ c \\ d \end{array} \right\} = \delta(h', \tilde{f}(\tilde{\alpha}) \cdot a^{-1} b^{-1} c d) \left| \begin{array}{c} a \\ b \\ c \\ d \end{array} \right\] \]

Figure 7: Definition of the components \( A'_v^{(g)} \) and \( B'_f^{(h')} \), which define the operators \( A'_v = \frac{1}{|G|} \sum_{g \in G} A'_v^{(g)} \) and \( B'_f = B'_f^{(h')} \) of these new \( D^K(G) \) models, in terms of their effective action on \( \mathcal{L}_2 \).

\[ A'_v \circ B'_f^{(h')} \left| \begin{array}{c} a \\ d \\ b \\ c \\ r \end{array} \right\} = \delta(h', \tilde{f}(\tilde{\alpha}) \cdot b^{-1} c^{-1} m r) \cdot A'_v \left| \begin{array}{c} a \\ d \\ b \\ c \\ r \end{array} \right\] \]

\[ = \frac{1}{|G|} \sum_{g \in G} \delta(h', \tilde{f}(\tilde{\alpha}) \cdot b^{-1} c^{-1} m r) \left| \begin{array}{c} gb \\ g \alpha \\ d \end{array} \right\] \]

\[ B'_f^{(h')} \circ A'_v \left| \begin{array}{c} a \\ d \\ b \\ c \\ r \end{array} \right\} = \frac{1}{|G|} \sum_{g \in G} B'_f^{(h')} \left| \begin{array}{c} gb \\ a \\ c \\ r \end{array} \right\] \]

\[ = \frac{1}{|G|} \sum_{g \in G} \delta(h', \tilde{f}(\tilde{\alpha}) \cdot (gb)^{-1} (cg^{-1})^{-1} m r) \left| \begin{array}{c} gb \\ g \alpha \\ d \end{array} \right\] \]

Figure 8: Proof that the operators \( A'_v \) and \( B'_{f,h} \equiv B'_f^{(h')} \) commute because the elements of \( G \) are such that \((gb)^{-1} (cg^{-1})^{-1} = b^{-1} (g^{-1} g) c^{-1} = b^{-1} c^{-1}\). Here, \( A'_v \) and \( B'_{f,h} \) act only on the vertex and face sectors whose intersection is not empty because, when this intersection is empty, these operators commute by definition.

defined by an operator \( C'_\ell' \), which acts only on the edge sectors of \( \mathcal{L}_2 \), it becomes clear that this \( C'_\ell' \) also needs to commute with these operators for the same reason.

Although we have not yet presented an expression for this operator \( C'_\ell' \), one thing is certain about it: \( C'_\ell' \) must be dual to the operator \( C_\ell \). But although Figure 6 is already showing us this need from the geometrical point of view, what does it mean to say that \( C'_\ell' \) must be dual to \( C_\ell \) from the algebraic point of view? Based on the dual relationship
\[ C'_{\ell}(\tilde{\lambda}) \left| \bar{\alpha} \cdots \bar{\beta} \right\rangle = \left| \tilde{\alpha}' \cdots \tilde{\beta}' \right\rangle \]

Figure 9: Definition of the components $C'_{\ell}(\tilde{\lambda})$ that define the edge operator (38). Note that, since $C'_{\ell}(\tilde{\lambda})$ is defined by taking $a' = f(\tilde{\lambda}) \cdot a$, $\tilde{\alpha}' = \tilde{\lambda}^{-1} * \tilde{\alpha}$ and $\tilde{\beta}' = \tilde{\beta} * \tilde{\lambda}$, this shows that $C'_{\ell}$ actually performs transformations in the gauge and matter fields on which it acts.

between the $D_M(G)$ ($D^K(G)$) vertex and $D^K(G)$ ($D_M(G)$) face operators, when we say that $C'_{\ell}$ must be dual to $C_{\ell}$, we are saying that, while $C_{\ell}$ just compares two matter fields (without performing any transformation on them), $C'_{\ell}$ must necessarily perform some kind of transformation in the gauge and dual matter fields on which it acts. And given this scenario, the expression that seems to best fit our needs is

\[ C'_{\ell} = \frac{1}{|\tilde{S}|} \sum_{\lambda \in \tilde{S}} C'_{\ell}(\tilde{\lambda}) , \quad (38) \]

whose components are defined in Figure 9.

3.1.4. What are the requirements for $A'_{\nu}$, $B'_{\nu}$ and $C'_{\ell}$ to be projectors?

Of course, there are several reasons that justify this expression (38) and we will discuss some of them in the following pages. And, within this context, it is interesting to observe that one of the conditions, which make this operator $C'_{\ell}$ a projector, is that $\text{Im}(f)$ must necessarily be the centre of $G$ [26]. After all, in accordance with Figure 10, the only way to cancel $[A'_{\nu}, C'_{\ell}]$ is by taking

\[ f(\tilde{\gamma}) \cdot g = g \cdot f(\tilde{\gamma}) . \]

This need, for $\text{Im}(f)$ to be identified as the centre of $G$, is also reinforced by Figure 11 since it shows us that

\[ f(\tilde{\beta} * \tilde{\lambda}) \cdot a^{-1} \cdot [f(\tilde{\lambda}) \cdot b]^{-1} = f(\tilde{\beta} * \tilde{\lambda}) \cdot ab^{-1} \cdot [f(\tilde{\lambda})]^{-1} = f(\tilde{\beta}) \cdot a^{-1}b^{-1} \quad (39) \]

needs to be satisfied for $[B'_{\nu}, C'_{\ell}]$ to vanish. Here, $[f(\tilde{\beta})]^{-1}$ is the inverse of the (group) element $f(\tilde{\beta})$. And once the assumption that $\text{Im}(f)$ is the centre of $G$ allows us to conclude that (39) is equivalent to

\[ f(\tilde{\beta} * \tilde{\lambda}) \cdot [f(\tilde{\lambda})]^{-1} = f(\tilde{\beta}) , \quad (40) \]

this also allows us to reach another conclusion: for $[B'_{\nu}, C'_{\ell}]$ to vanish, $f$ needs to be a group homomorphism. Note that, if $f$ is a group homomorphism, and consequently $\tilde{S}$ is a group, its properties

\[ f(0) = 0 , \quad [f(\tilde{\alpha})]^{-1} = f(\tilde{\alpha}^{-1}) = f^{-1}(\tilde{\alpha}) \quad \text{and} \quad f(\tilde{\alpha}_1) \cdot f(\tilde{\alpha}_2) = f(\tilde{\alpha}_1 * \tilde{\alpha}_2) \quad (41) \]
\[ A'_{v} \circ C'_{\ell} \left\| d \xrightarrow{a} \frac{\hat{\beta}}{c} \frac{\hat{\alpha}}{b} \right\| = \frac{1}{|S|} \sum_{\lambda \in \hat{S}} A'_{v} \left\| d \xrightarrow{a} \frac{\hat{\beta}}{c} \frac{f(\hat{\lambda})}{b} \right\| = \frac{1}{|G|} \frac{1}{|S|} \sum_{\hat{\lambda} \in \hat{S}} \sum_{g \in G} \left( g \cdot a \right) \left( g \cdot \hat{\lambda} \right) \left( g \cdot f(\hat{\lambda}) \right) \left( g \cdot b \right) \]

\[ C'_{\ell} \circ A'_{v} \left\| d \xrightarrow{a} \frac{\hat{\beta}}{c} \frac{\hat{\alpha}}{b} \right\| = \frac{1}{|G|} \sum_{g \in G} C'_{\ell} \left\| \sum_{\hat{\lambda} \in \hat{S}} \sum_{g \in G} \left( g \cdot a \right) \left( g \cdot \hat{\lambda} \right) \left( g \cdot f(\hat{\lambda}) \right) \left( g \cdot b \right) \right\| \]

Figure 10: Result of action of the operators \( A'_{v} \circ C'_{\ell} \) and \( C'_{\ell} \circ A'_{v} \) on the lattice \( L_2 \), from which it is clear that \( [A'_{v}, C'_{\ell}] \) will only be equal to zero if \( f(\hat{\lambda}) \) belongs to the centre of \( G \). Analogous to what has already been observed in Figure 8, \( A'_{v} \) and \( C'_{\ell} \) act only on the vertex and edge sectors whose intersection is not empty because, when this intersection is empty, these operators commute by definition. Note that the order in which the summations are performed is irrelevant.

ensure not only that \( [A'_{v}, C'_{\ell}] = [B'_{f}, C'_{\ell}] = 0 \), but also confirm the interpretation of \( C'_{\ell} \) as a projector. After all, if \( f \) is indeed a group homomorphism, this ensures that the requirements

\[
\hat{\alpha}'' = (\hat{\lambda})^{-1} * \hat{\alpha}' = (\hat{\lambda})^{-1} * \hat{\lambda}^{-1} * \hat{\alpha} = (\hat{\lambda} * \hat{\lambda})^{-1} * \hat{\alpha} , \tag{42a}
\]

\[
\hat{\beta}'' = \hat{\beta}' * \hat{\lambda} = \hat{\beta}' * (\hat{\lambda} * \hat{\lambda}') \quad \text{and} \tag{42b}
\]

\[
a'' = f(\hat{\lambda}') * a = f(\hat{\lambda}') * f(\hat{\lambda}) * a = f(\hat{\lambda} * \hat{\lambda}) * a , \tag{42c}
\]

which need to be satisfied in the double action of \( C'_{\ell}(\hat{\lambda}) \) being shown in Figure 12, are respected. In this way, by

- remembering that the \( D^K (G) \) vertex operators were inherited from the \( D (G) \) models and

- noting that the double action of \( B'_{f,h} \) (on the same face sector of \( L_2 \)) shows that it is, in fact, a projector because

\[
\delta(h', f(\hat{\alpha}) * a^{-1}b^{-1}cd) \cdot \delta(h', f(\hat{\alpha}) * a^{-1}b^{-1}cd) = \delta(h', f(\hat{\alpha}) * a^{-1}b^{-1}cd) ,
\]

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we can conclude that the operators, which define the Hamiltonian (33), can be chosen as

$$A'_v = \frac{1}{|G|} \sum_{g \in G} A_v^{(g)} \quad B'_f = \tilde{B}'_f^{(0)} \quad \text{and} \quad C'_\ell = \frac{1}{|S|} \sum_{\lambda \in S} C'_\ell^{(\lambda)} \quad (43)$$

since, with them, the $D^K(G)$ models become exactly solvable.

3.1.5. The dual behaviour of the edge operator $C'_\ell$ as a comparator

Given everything we have done so far, perhaps you, the reader, have an important question: are these operators (43) really the most suitable to define these $D^K(G)$ models? Moreover, how can we justify the existence of the co-action (35)? And what are the transformations that $C'_\ell$ performs?

In order to begin to answer these questions, it is worth noting that the second one can be answered easily. How? By taking into account, for instance, that the $D(G)$ models are Hamiltonian realizations of lattice gauge theories based (i) on an involutive Hopf algebra $C(G)$ [32] and (ii) on finite quantum groupoids (i.e. on a weak Hopf algebra) [33]. More specifically, it is possible to affirm that the $D(G)$ Hamiltonian realizes a representation of the Drinfeld’s quantum double [17] of these involutive Hopf algebras [34]. Thus, by noting not only that the underlying algebra with involution is a star-algebra [35], but also that it is possible to describe the $D(G)$ models based on star-quantum groupoids [33, 36], the construction of the $D^K(G)$ models, with the use of the co-action homomorphism (35), is
Figure 12: Scheme related to the double action of the edge operator $C'_\ell$ on the same edge sector of $\mathcal{L}_2$. Here, $\tilde{\alpha}'', \tilde{\beta}''$ and $a''$ are given by the expressions (42), which reinforce the need for $f$ to be a group homomorphism.

justified. After all, in addition to being possible to prove that, whenever a group $G$ acts on a ring $\mathcal{A}$ that can be interpreted as an star-algebra, there is a co-action $F : \mathcal{A} \to C(G) \otimes \mathcal{A}$, Ref. [37] shows some examples that make it very clear that such co-action homomorphism can be defined.

As naive as it sounds, another important thing we need to mention here is that, since the centre of any group is an Abelian subgroup [26], the fact that the image of a group homomorphism $f : \tilde{S} \to G$ is the centre of $G$ implies that $\tilde{S}$ is an Abelian group. And why is it important to mention this here? Because this is one of the things that helps us to understanding what are the transformations that $C'_\ell$ performs. How? By noting that, when $f$ is a group homomorphism such that $\tilde{S}$ and $\text{Im}(f)$ are two finite Abelian groups, the basis

\[
\left\{ |\tilde{\alpha}^{-1} \ast \tilde{\alpha}, f(\tilde{\lambda}) \cdot g, \tilde{\beta} \ast \tilde{\lambda} \rangle : g \in G\text{ and } \tilde{\alpha}, \tilde{\beta}, \tilde{\lambda} \in \tilde{S} \right\}
\]

can be obtained through the unitary transformations

\[
|g'\rangle = \frac{1}{|G|} \sum_{g \in G} \omega_{g'}(g) |g\rangle \text{ and } |\tilde{\alpha}'\rangle = \frac{1}{|\tilde{S}|} \sum_{\tilde{\alpha} \in \tilde{S}} \chi_{\tilde{\alpha}'}(\tilde{\alpha}) |\tilde{\alpha}\rangle,
\]

(44)

where $\omega_{g'}(g)$ and $\chi_{\tilde{\alpha}'}(\tilde{\alpha})$ are the characters of $G$ and $\tilde{S}$ respectively. That is, although the action (38) is clearly such that

\[
C'_\ell |\tilde{\alpha}, g, \tilde{\beta}\rangle = \frac{1}{|\tilde{S}|} \sum_{\tilde{\lambda} \in \tilde{S}} |\tilde{\lambda}^{-1} \ast \tilde{\alpha}, f(\tilde{\lambda}) \cdot g, \tilde{\beta} \ast \tilde{\lambda}\rangle,
\]

(45)

the substitution of the relations (44) into (45) allow us to recognize that this action is also such that

\[
C'_\ell |\tilde{\alpha}', g', \tilde{\beta}'\rangle = \frac{1}{|\tilde{S}|} \sum_{\tilde{\lambda} \in \tilde{S}} \tilde{\chi}_{\tilde{\alpha}'}(\tilde{\lambda}) \omega_{g'}(f(\tilde{\lambda})) \chi_{\tilde{\beta}'}(\tilde{\lambda}) |\tilde{\alpha}, g, \tilde{\beta}\rangle.
\]

(46)

Given this result, it is important to note that, since $\tilde{S}$ and $\text{Im}(f) \subseteq G$ are two finite Abelian groups, the Fourier transform $\hat{f} \in L(\tilde{S}^*)$ is such that

\[
\hat{f}(\chi) = \sum_{\tilde{\lambda} \in \tilde{S}} f(\tilde{\lambda}) \chi(\tilde{\lambda}) \text{ and } f(\tilde{\lambda}) = \frac{1}{|\tilde{S}|} \sum_{\chi \in \tilde{S}^*} \hat{f}(\chi) \chi(\tilde{\lambda}),
\]

(47)

26
where the dual group ˜\(S^*\) is isomorphic to ˜\(S\) [26, 38, 39, 40]. After all, by noting that an expression of the sort ˜\(\chi_{\{\bar{\alpha}',\bar{\beta}'\}}(\bar{\lambda})\) is also a character, the substitution of (47) into (46) yields

\[
C'_{\{\bar{\alpha}',g',\bar{\beta}'\}} = \frac{1}{|\hat{S}|} \sum_{\chi_{\bar{\lambda}} \in \hat{S}} [w_{g'} \circ f](\chi_{\bar{\lambda}}) \left\{ \frac{1}{|\hat{S}|} \sum_{\bar{\lambda} \in \hat{S}} \bar{\chi}_{\{\bar{\alpha}',\bar{\beta}'\}}(\bar{\lambda}) \chi_{\lambda}(\bar{\lambda}) \right\} |\bar{\alpha}',g,\bar{\beta}'\rangle
\]

In other words, although the exact form of the index \{\bar{\alpha}',\bar{\beta}'\} depends on the nature of ˜\(S\), it is undeniable that \(C'_{\{\bar{\alpha}',g',\bar{\beta}'\}}\) can also be interpreted as an operator that compares matter fields differently, which only becomes clear when this operator acts on a diagonal basis. And this different way of comparing the two matter fields, which belong to the same edge sector, rests on the Pontryagin duality, which ensures that there is a one-to-one correspondence between the characters \(\chi_{\lambda}\) and the elements of \(\hat{S}\) [41].

4. General properties of these \(D^K(G)\) models

As discussed in the last Subsection, there is no way not to conclude that the operators (43) show the dual behaviour necessary to define the \(D^K(G)\) models as dual to the \(D_M(G)\) models. And given this scenario, there are some things we still need to do/say in order to define these new models consistently. What are these things?

4.1. Some considerations on the vacuum and non-vacuum states

The first thing we can do here is, for instance, recognize that, as these operators are projectors, the \(D^K(G)\) ground state energy is also

\[
E_0 = -(N_v + N_f + N_\ell)
\]

because the \(D^K(G)\) vacuum states are such that

\[
A'_{\bar{\nu}}|\tilde{\xi}_0\rangle = |\tilde{\xi}_0\rangle, \quad B'_{\bar{f}}|\tilde{\xi}_0\rangle = |\tilde{\xi}_0\rangle \quad \text{and} \quad C'_{\bar{\ell}}|\tilde{\xi}_0\rangle = |\tilde{\xi}_0\rangle
\]

hold for all the \(N_v\) vertices, \(N_f\) faces and \(N_\ell\) edges of \(L_2\). One of these vacuum states is

\[
|\tilde{\xi}_0^{(0)}\rangle = \prod_{\bar{\ell}} C'_{\bar{\ell}} \prod_{\bar{f}} B'_{\bar{f}} \prod_{\bar{v}} A'_{\bar{\nu}} |0 \otimes \ldots \otimes |0\rangle \otimes |0 \otimes \ldots \otimes |0\rangle \text{ \(N_\ell\) times \(N_f\) times}
\]

Note that, since we have just mentioned that these operators (43) are projectors, it is important to pay attention to the fact that, like the \(D_M(G)\) models, these operators are not
the only projectors that define these $D^K(G)$ models. And by resymbolizing the vertex and edge operators in (43) by $A'_{v,0}$ and $C'_{\ell,0}$ respectively, we can easily demonstrate that they are elements of the complete sets

$$\mathfrak{A}_v = \{ A'_{v,0}, A'_{v,1}, \ldots, A'_{v,N-2}, A'_{v,N-1} \} ,$$

$$\mathfrak{B}_f = \{ B'_{f,0}, B'_{f,1}, \ldots, B'_{f,N-2}, B'_{f,N-1} \}$$

and

$$\mathfrak{C}_\ell = \{ C'_{\ell,0}, C'_{\ell,1}, \ldots, C'_{\ell,K-2}, C'_{\ell,K-1} \}$$

of orthogonal projectors onto Hilbert space

$$\mathfrak{H}_N \otimes \ldots \otimes \mathfrak{H}_N \otimes \mathfrak{H}_K \otimes \ldots \otimes \mathfrak{H}_K .$$

That is, by paraphrasing what we have already said in the Subsubsection 2.1.1, all the operators that are contained in these complete sets

(a) have eigenvalues equal to 0 and 1,

(b) satisfy the relations

$$A'_{v',J'} \circ A'_{v'',J''} = A'_{v'',J''} \circ A'_{v',J'} = \delta_{v',v''} \cdot \delta_{J',J''} \cdot A'_{v,J} ,$$

$$B'_{f,L'} \circ B'_{f',L''} = B'_{f',L''} \circ B'_{f,L'} = \delta_{f',f''} \cdot \delta_{L',L''} \cdot B'_{f,K} ,$$

$$C'_{\ell,N} \circ C'_{\ell',N'} = C'_{\ell',N'} \circ C'_{\ell,N} = \delta_{\ell',\ell''} \cdot \delta_{N,N'} \cdot C'_{\ell,N'} ,$$

$$A'_{v',J'} \circ B'_{f',L'} = B'_{f',L'} \circ A'_{v',J'} = 0 ,$$

$$A'_{v',J'} \circ C'_{\ell,N} = C'_{\ell,N} \circ A'_{v',J'} = 0$$

and

$$B'_{f',L'} \circ C'_{\ell,N} = C'_{\ell,N} \circ B'_{f',L'} = 0$$

not only for all the values of $J^{(\ell)}, L^{(\ell)} = 0, 1, \ldots, N-1$ and $\Lambda^{(\ell)} = 0, 1, \ldots, K-1$, but also for all the vertices, faces and edges of $\mathcal{L}_2$, and

(c) are such that

$$\sum_{J=0}^{N-1} A'_{v,J} = 1_v , \quad \sum_{L=0}^{N-1} B'_{f,L} = 1_f$$

and

$$\sum_{\Lambda=0}^{K-1} C'_{\ell,\Lambda} = 1_\ell .$$

And why is it important to make these observations on these set elements? Because all the energy excitations, which characterize the $D^K(G)$ non-vacuum states, need to be created by the action of operators $\tilde{W}_{\ell}^{(J,L,\Lambda)}$ and $\tilde{W}_{\ell}^{(J,\Lambda)}$ that must be respectively such that

$$\tilde{W}_{\ell}^{(J,L,\Lambda)} \circ A'_{v,0} = A'_{v,J} \circ \tilde{W}_{\ell}^{(J,L,\Lambda)} ,$$

$$\tilde{W}_{\ell}^{(J,L,\Lambda)} \circ B'_{f,0} = B'_{f,L} \circ \tilde{W}_{\ell}^{(J,L,\Lambda)} ,$$

$$\tilde{W}_{\ell}^{(J,L,\Lambda)} \circ C'_{\ell,0} = C'_{\ell,\Lambda} \circ \tilde{W}_{\ell}^{(J,L,\Lambda)} ,$$

$$28$$
Thus, by noting that the general idea behind the creation of these $D^K(G)$ models is also to make them serve as a prototype for some physical system(s), it is crucial that the energy excitations $\tilde{q}^{(J,L,A)}$ and $\tilde{Q}^{(J,A)}$, which are created by the operators $\tilde{W}^{(J,L,A)}_{f}$ and $\tilde{W}^{(J,A)}_{f}$ respectively, can be interpreted as quasiparticles. That is, it is crucial that these energy excitations be at least such that\(^{15}\)

\[
\begin{align*}
\tilde{q}^{(J',L',A')} & \times \tilde{q}^{(J'',L'',A'')} = \tilde{q}^{(J',L',A')} \times \tilde{q}^{(J'',L'',A'')}, \\
\tilde{q}^{(J',L',A')} & \times \tilde{Q}^{(J',A')} = \tilde{Q}^{(J',A')} \times \tilde{q}^{(J',L',A')} \quad \text{and} \\
\tilde{Q}^{(J',A')} & \times \tilde{Q}^{(J'',A'')} = \tilde{Q}^{(J',A')} \times \tilde{Q}^{(J'',A')}.
\end{align*}
\]

4.2. The matrix representation of the $D^K(G)$ vertex, face and edge operators

Given not only that the groups $\tilde{S}$ and $G$ we are dealing with are finite, but also that $\tilde{S}$ and $\text{Im}(f) \subseteq G$ are Abelian groups, it is interesting to pay attention to the $D^K(G)$ models where $G$ is a cyclic group. After all, in addition to allowing us to compare these models with the $D(Z_N)$ and $D_M(Z_N)$ models, their vertex, face and edge operators have a well-defined representation: this representation is given by

\[
\begin{align*}
A_{v,J}' = & \frac{1}{|G|} \sum_{g \in Z_N} \chi_{1+J}(g) \cdot \left( \prod_{v' \in S_v^c} X_{v'}^g \right) \left( \prod_{v' \in S_v^c} X_{v'}^{-g} \right), \\
B_{f,L}' = & \frac{1}{|G|} \sum_{g \in Z_N} \chi_{1+L}(g) \cdot F_f(g) \left( \prod_{v' \in S_v^c} Z_{v'}^g \right) \left( \prod_{v' \in S_v^c} Z_{v'}^{-g} \right) \quad \text{and} \\
C_{e,A}' = & \frac{1}{|S|} \sum_{\gamma \in \tilde{S}} \tilde{\chi}_{1+\Lambda}(\gamma) \cdot (\tilde{X}_{f_1}^{1+\gamma} \otimes F_\ell(\gamma) \otimes (\tilde{X}_{f_2})^{1+\gamma},
\end{align*}
\]

where

(i) $S_v^c$ and $S_v^c$ are disjoint edge subsets of $S_v$, whose edge orientations pointing in and out of the $v$-th vertex respectively,

(ii) $S_f^c$ and $S_f^c$ are disjoint edge subsets of $S_f$, whose edges have counterclockwise and clockwise orientations respectively, and

\(^{15}\)As we have already noted in the footnote on page 8, it is worth reinforcing that, although we are symbolizing by $\tilde{q}^{(J,L,A)}$ all the quasiparticles that are produced by the operators $\tilde{W}^{(J,L,A)}_{f}$, they are exactly the same quasiparticles that can be identified in the $D(G)$ models.
And in the case of this matrix representation (53), it leads us to
\[ X = \sum_{h \in \mathbb{Z}_N} |(h + 1) \text{ mod } N \rangle \langle h|, \quad Z = \sum_{h \in \mathbb{Z}_N} \omega^h |h \rangle \langle h|, \]
\[ \tilde{X} = \sum_{\tilde{\alpha} \in \mathbb{Z}_K} |(\tilde{\alpha} + 1) \text{ mod } K \rangle \langle \tilde{\alpha}| \quad \text{and} \quad \tilde{Z} = \sum_{\tilde{\alpha} \in \mathbb{Z}_K} \tilde{\omega}^{\tilde{\alpha}} |\tilde{\alpha} \rangle \langle \tilde{\alpha}|, \]

since the "correspondence principle" dictates that these \( D^K(\mathbb{Z}_N) \) models must be reduced to the \( D(\mathbb{Z}_N) \) models in special cases. Here, \( \omega = e^{i(2\pi/N)} \) and \( \tilde{\omega} = e^{i(2\pi/K)} \) are the generators of the gauge (\( G = \mathbb{Z}_N \)) and matter (\( \tilde{S} = \mathbb{Z}_K \)) groups\(^{16}\).

Now, with respect to the matrices \( F_f(g) \) and \( F_{\ell}(\tilde{\gamma}) \) that appear in (53), they represent how the group homomorphism \( f \) couples the \( D(\mathbb{Z}_N) \) models to the matter fields. And in order for us to understand the main properties of these models, it is important to pay attention to the statements of Theorems 3 and 4 below, whose proofs are in Refs. [42] and [43] respectively.

### Theorem 3
The number of group homomorphisms from \( \mathbb{Z}_K \) into \( \mathbb{Z}_N \) is \( \gcd(K,N) \).

### Theorem 4
Every group homomorphism \( f: \mathbb{Z}_K \to \mathbb{Z}_N \) can be completely determined by
\[ f([\tilde{\alpha}]) = [n\tilde{\alpha}] , \]
where \( n \) is a natural number that assumes values other than zero if, and only if, \( nK \) is a natural number divisible by \( N \).

That is, in accordance with these statements, all the \( D^K(\mathbb{Z}_N) \) models have, at least, a description where \( F_f(g) \) and \( F_{\ell}(\tilde{\gamma}) \) are identity matrices: after all, for all the values of \( N \) and \( K \), there will always be a group homomorphism
\[ f([\tilde{\alpha}]) = [0] \]
that maps all the elements of \( \mathbb{Z}_K \) to the identity element of \( \mathbb{Z}_N \). Also, when these \( D^K(\mathbb{Z}_N) \) models are such that \( N \) and \( K \) are coprime numbers, the only way to define these models is by using this trivial group homomorphism (56).

---

\(^{16}\)Although we have not evaluated the commutation of these operators (53) when \( J \neq 0 \) and \( \Lambda \neq 0 \), note that the only difference that exists, between them and those with \( J = \Lambda = 0 \), concerns the characters that multiply each of the components \( A^{(g)}_i \) and \( C^{(i)}_\ell \). And as these characters are constants that commute with each other, there is no way not to conclude that
\[ [A'_{i,j}, B_{f,L}] = [A'_{i,j}, C_{\ell,A}] = [B_{f,L}, C_{\ell,A}] = 0 \]
holds not only for all the values of \( J^{(t)}(g), L^{(s)}(g) = 0,1,\ldots,N-1 \) and \( \Lambda^{(t)}(g) = 0,1,\ldots,M-1 \), but also for all the vertices, faces and edges of \( L_2 \).
4.2.1. A first comment on the $D^K(\mathbb{Z}_N)$ ground state degeneracy

When we come across this description, where all these models are defined by using (56), one of the things that we can say about them is that their “fake holonomies” (37) reduce to the true holonomies (36). And since this reduction allows us to identify the operators $B'_{f,L}$ as the same face operators of the $D(\mathbb{Z}_N)$ models, there is no way not to conclude, for instance, that all these $D^K(\mathbb{Z}_N)$ models support the same quasiparticles as the $D(\mathbb{Z}_N)$ models.

Note that this conclusion is not surprising because, by definition, the “correspondence principle” already requires that the $D^K(\mathbb{Z}_N)$ models support these quasiparticles in some way. However, what is most striking in this correspondence that must necessarily exist between these $D^K(\mathbb{Z}_N)$ models, which are defined by using (56), and the $D(\mathbb{Z}_N)$ models is directly related to the fact that the operators

\[ B'_{f,L} = \frac{1}{|G|} \sum_{g \in \mathbb{Z}_N} \chi_{1+L}(g) \cdot 1_f \left( \prod_{v' \in S_f^g} Z_{v'}^g \right) \left( \prod_{v'' \in S_f^g} Z_{v''}^{-g} \right) \]  

and

\[ C'_{\ell,\Lambda} = \frac{1}{|S|} \sum_{\tilde{\gamma} \in \tilde{S}} \tilde{\chi}_{1+\Lambda}(\tilde{\gamma}) \cdot (\tilde{\chi}_{f_1}^\dagger)^{\tilde{\gamma}} \otimes 1_\ell \otimes (\tilde{\chi}_{f_2})^{\tilde{\gamma}}, \]  

(where $I_\ell$ and $1_f$ are identity matrices of order $N$ and $K$ respectively) cannot detect any matter and gauge excitations respectively. After all, since

- none of these operators is able to detect any change $|\tilde{\alpha}'\rangle_f \leftrightarrow |\tilde{\alpha}''\rangle_f$ \(^{17}\) and

- none of the operators $(\tilde{\chi}_{f})^{\tilde{\gamma}}$ (which make all these changes $|\tilde{\alpha}'\rangle_f \leftrightarrow |\tilde{\alpha}''\rangle_f$) can be expressed as a product involving the vertex, face and edge operators,

we can conclude that all states

\[ |\tilde{\gamma}_{0}\rangle = \prod_{v'} C_{v'} \prod_{f'} B'_{f'} \prod_{v''} A_{v''} \left( \bigotimes_{\ell \in L_2} |0\rangle \right) \otimes \left( \bigotimes_{f \in L_2} |0\rangle \right) \otimes |\tilde{\alpha}\rangle_{f''}, \]  

which are defined by taking $\tilde{\alpha} = 0, 1, \ldots, K - 1$, are independent of each other.

In this fashion, since the inability of (57b) to detect gauge excitations implies, for instance, that all the quasiparticles $\tilde{q}^{(J,L,\Lambda)}$ can be transported without increasing/decreasing the energy of the system, it is not difficult to conclude that the action of an operator

\[ \tilde{O}_L(\tilde{\gamma}^\dagger) = \prod_{v' \in \Gamma_2} \tilde{W}_{v'}^{(0,L,0)} \prod_{v'' \in \Gamma_2} \left( \tilde{W}_{v''}^{(0,L,0)} \right)^{\dagger} \]

\(^{17}\)By paraphrasing the footnote on page 12: here, we are using the index $f$ only to emphasize that $|\tilde{\alpha}\rangle$ is an element associated with a face of $L_2$.
does not lead to an excited state when it acts on any of the vacuum states (58). And how this allows us to recognize that all the vacuum states 

\[ |\xi_0^{(\tilde{\alpha}, \tilde{\lambda}, \tilde{L})}\rangle = s \prod_{p=1}^{s} \left[ \tilde{O}_{L_p}(\tilde{\tau}_p^*) \right]^{\lambda_p} |\tilde{\xi}_0^{(\tilde{\alpha})}\rangle \]

are topologically independent of each other due to the non-contractility of \( \gamma^* \) [20], it is also not difficult to conclude that all these \( D^K(Z_N) \) models, which are defined by using (56), can be interpreted as the same \( D(Z_N) \) models, but with a ground state that is

\[ \tilde{n} = |\ker(f)| \cdot d_{D(Z_N)} \]  

- fold degenerated\(^{18}\).

Note that, by the point of view of the duality that we want to explore between the \( D_M(Z_N) \) and \( D^K(Z_N) \) models, all these conclusions are welcome because, as the co-action (35) that this trivial group homomorphism (56) defines can be always induced by a trivial (sub)group action

\[ \tilde{\mu}_f(f(\tilde{\alpha}), \tilde{\gamma}) = \tilde{\gamma} , \]  

it is precisely this that reinforces this duality. After all, as this trivial (sub)group action is represented by the same matrix

\[ F_f(g) = 1_f \]

that composes (57a) and, therefore, defines the set

\[ \mathfrak{S}\mathfrak{i}\mathfrak{r}_{\tilde{\mu}} = \{ |\tilde{\alpha}\rangle_f \in \mathfrak{S}_{\tilde{\mu}} : F_f(g) |\tilde{\alpha}\rangle_f = |\tilde{\alpha}\rangle_f \text{ for all } g \in Z_N \} \]  

of points of \( \mathfrak{S}_{\tilde{\mu}} \) that are fixed by (60), it is very clear that all the models, which are dual to these trivial \( D^K(Z_N) \) models (i.e., which are dual to the \( D_M(Z_N) \) models that are defined by using a trivial group action), can be interpreted as the same \( D(Z_N) \) models, but with ground states that are

\[ n = |\mathfrak{S}\mathfrak{i}\mathfrak{r}_{\tilde{\mu}}| \cdot d_{D(Z_N)} \]  

-fold degenerated. Note that (62) corresponds to the same expression (27) that defines the degree of degeneracy of the ground states of all the trivial \( D_M(Z_N) \) models.

4.3. What happens when \( f \) is not a trivial group homomorphism?

Given that we already know a lot about the \( D^K(Z_N) \) models, it is time to analyse the main properties of these models when \( f \) is a non-trivial group homomorphism. And in order to start this analysis, it might be interesting to start by taking an example: the \( D^2(Z_2) \) model. After all, in view of what was stated by Theorem 3, there are two ways to define this model:

\(^{18}\)Here, we are using the same notation used on page 15, now to refer to the number \( d_{D(Z_N)} \) of vacuum states that define the \( D(Z_N) \) ground state.
one, which we just presented in Subsubsection 4.2.1 by using a trivial group homomorphism, that can be interpreted as the same $D(Z_2)$ model, but with an algebraically degenerated ground state; and

another that, because it needs to be defined by using a non-trivial group homomorphism, has vertex, face and edge operators represented by

$$A'_{v,J} = \frac{1}{|G|} \sum_{g \in \mathbb{Z}_N} (-1)^{-J_g} \cdot \prod_{\ell \in S_v} (\sigma_\ell^x)^g, \quad (63a)$$

$$B'_{f,L} = \frac{1}{|G|} \sum_{g \in \mathbb{Z}_N} (-1)^{L_g} \cdot F_f(g) \prod_{\ell \in S_f} (\sigma_\ell^z)^g \quad \text{and} \quad (63b)$$

$$C'_{\ell,\Lambda} = \frac{1}{|S|} \sum_{\tilde{\gamma} \in \tilde{S}} (-1)^{A_\ell} \cdot F_\ell(\tilde{\gamma}) \prod_{f \in S_\ell} (\sigma_f^x)^{\tilde{\gamma}} \quad (63c)$$

respectively, where $F_f(g)$ and $F_\ell(\tilde{\gamma})$ cannot be identity matrices.

In this fashion, by noting that Theorem 4 guarantees that the only non-trivial group homomorphism $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ that exists is

$$f(0) = 0 \quad \text{and} \quad f(1) = 1, \quad (64)$$

the fact that $\text{Im}(f) = G$ allows us to conclude that, in this [2nd] way, we have

$$F_f(g) = (\sigma_f^z)^g \quad \text{and} \quad F_\ell(\tilde{\gamma}) = (\sigma_\ell^x)^{\tilde{\gamma}}. \quad (65)$$

And by according to this picture, it is not difficult to recognize that, in this [2nd] way, the state

$$\left| \tilde{\xi}_0^{(\tilde{\alpha})} \right> = \prod_{\ell'} C_{\ell'} \prod_{f'} B_{f'} \prod_{\nu'} A_{\nu'} \left( \bigotimes_{f' \in L_2} |0\rangle \right) \otimes \left( \bigotimes_{f \in L_2} |0\rangle \right) \otimes |\tilde{\alpha}\rangle_{f''}$$

with $\tilde{\alpha} \neq 0$ (i.e., with $\tilde{\alpha} = 1$) cannot be interpreted as a vacuum state. After all, as all the operators

$$(\sigma_f^{x,z})^g \quad \text{and} \quad (\sigma_\ell^{x,z})^{\tilde{\gamma}},$$

which compose the vertex, face and edge operators (63) (and, consequently, the Hamiltonian (33)), create quasiparticles in this model, it is not difficult to recognize that

$$\tilde{W}^{(1,0)}_f = \sigma_f^x$$

(which satisfies (51) with $L = 1$ and $\Lambda = 0$) creates a quasiparticle $\tilde{Q}^{(1,0)}$ throughout a permutation $|0\rangle_f \leftrightarrow |1\rangle_f$.  

\textsuperscript{19}See the comments made in the footnote on page 9.

33
4.3.1. Are there “confined” quasiparticles in the \( D^K (\mathbb{Z}_N) \) models?

Another point that deserves to be mentioned here is that, in addition to this group isomorphism (64) has defined a \( D^2 (\mathbb{Z}_2) \) model that does not have an algebraically degenerated ground state, (64) also made \( C'_\ell \) able to detect the pairs of quasiparticles \( \tilde{q}(1,L,1) \) that are produced by

\[
\tilde{W}'(1,L,1) = \sigma_x^\ell \circ (\sigma_x^\ell)^L \quad \text{or} \quad \tilde{W}'(1,L,1) = (\sigma_x^\ell)^L \circ \sigma_x^\ell.
\] (66)

That is, the operator \( C'_\ell \) is capable of detecting the same pair of quasiparticles that are detected individually by the operator \( A'_v \). And why does this deserve to be mentioned here? Because this situation is entirely analogous to that of the \( D^2 (\mathbb{Z}_2) \) model analysed in Subsubsection 2.2.1. After all, contrary to what happens, for instance, in the \( D (\mathbb{Z}_2) \) model, where it is possible to transport these quasiparticles \( \tilde{q}(1,L,1) \) without changing the energy of the system, this is not possible in this \( D^2 (\mathbb{Z}_2) \) model: whenever the transport of these quasiparticles occurs, the energy of the system increases when \( f \) is defined by (64). Thus, as this increase is not welcome for several physical reasons, we need to do the same thing we did before: that is, we need to ignore the fact that the transport of these quasiparticles \( \tilde{q}(1,L,1) \) is mathematically possible and consider that all they are confined (i.e., that all they “cannot” be transported).

In view of this “confinement”, it is not wrong to say that this \( D^2 (\mathbb{Z}_2) \) model, where \( f \) is a group isomorphism, has properties that are dual to those of the \( D^2 (\mathbb{Z}_2) \) model that was discussed in Subsubsection 2.2.1. After all, it is quite clear, for instance, that

- while, in the \( D^2 (\mathbb{Z}_2) \) model, the “confined” quasiparticles are detected by the face operator \( B_f \),
- here, in the \( D^2 (\mathbb{Z}_2) \) model, the “confined” quasiparticles are detected by the vertex operator \( A'_v \), which is dual to \( B_f \).

However, there is, at least, one aspect of this \( D^2 (\mathbb{Z}_2) \) model that seems to spoil this duality. What is this aspect? Is it the aspect that is related precisely to the fact that these “confined” quasiparticles \( \tilde{q}(1,0,1) \) are not detected by \( B'_f \). And why does this seem to spoil the duality that exists between the \( D^2 (\mathbb{Z}_2) \) and \( D^2 (\mathbb{Z}_2) \) models? Because, as these quasiparticles are not detected by any of the operators that measure the (“fake”) holonomies around the lattice faces, this means that their production cannot be associated with any type of local deformation of \( L_2 \). In this way, by noting that the action of an operator

\[
O_1 (\tau) = \prod_{\ell' \in \tau} \tilde{W}'(1,0,1) ,
\] (67)

on a set of edges that form a non-contractile closed path \( \tau \), does not have the slightest importance for the determination of vacuum states that are topologically independent of \( |\tilde{\xi}_0\rangle = \prod_{\ell} C_\ell \prod_{f'} B'_f \prod_{v'} A'_v \left( \bigotimes_{\ell \in L_2} |0\rangle \otimes \bigotimes_{f \in L_2} |0\rangle \right) ,
\] (68)

\footnote{Observe that (68) is just a more streamlined way of writing the same vacuum state (49).}
Figure 13: Piece of a same lattice region at two different times in the $D^2 (\mathbb{Z}_2)$ model, whose similarity with Figure 4 makes clear one of the dual aspects that exists between this model and the $D^2 (\mathbb{Z}_2)$ model. After all, note that, in the first instant $t_1$ (above), we have a pair of quasiparticles $\tilde{q}^{(1,0,1)}$ and $\tilde{q}^{(1,0,1)}$ (red outlined and purposely indexed with the “+” and “−” symbols respectively), which were created by the action of a single operator $\tilde{W}^{(1,0,1)}(1,0,1)$, where the single green dot corresponds to the unique vacuum violation detected by $C'_{t,0}$. Now, in the second instant $t_2 > t_1$ (below) we have these same quasiparticles after one of them has been transported away from the other due to the action of operators $\tilde{W}^{(1,0,1)}(1,0,1)$ on all the edges that intersect the dual path (highlighted in dashed black). Note that, in this latter case, we have new fifteen green dots: one for each edge involved in this transport, making clear the linearity related to the growth of the system energy in this transport.

the fact that the quasiparticles $\tilde{q}^{(1,0,1)}$ are “confined” does not prevent the $D^2 (\mathbb{Z}_2)$ ground state from depending on the first homotopy group $\pi_1 (\mathcal{M}_2)$. In other words, the ground state of this $D^2 (\mathbb{Z}_2)$ model, where $f$ is given by (64), is made up of all vacuum states

$$|\xi_0^{(0,\lambda)}\rangle = \prod_{p=1}^{s} \left[ \hat{O}_1 (\bar{\tau}_p) \right]^{\lambda_p} |\tilde{\xi}_0^{(0)}\rangle$$
since all these vacuum states are topologically independent of each other due to the non-contractility of $\gamma^*$. 

4.3.2. The $D^N (Z_N)$ models as another example

Given all that we have just understood about the $D^2 (Z_2)$ model, it is not difficult to conclude, for instance, that all the other $D^N (Z_N)$ models, where $\phi$ is a group homomorphism (i.e., where $\phi$ is a group homomorphism (55) with $N = K$ and $n = 1$), have the same properties that were listed in the last two Subsections. After all, since this group homomorphism requires that

$$F_\phi (g) = (\tilde{Z}_\phi)^g \quad \text{and} \quad F_\phi (\tilde{\gamma}) = (X_\phi)^\tilde{\gamma},$$

we can conclude that:

I. All the operators $(X_\ell)^g$, $(Z_\ell)^g$, $(\tilde{X}_\ell)^g$ and $(\tilde{Z}_\ell)^g$,

which compose the vertex, face and edge operators (13) (and, consequently, the Hamiltonian (33)), create quasiparticles in this model.

II. Once the operators $\tilde{W}_\phi^{(g,0)} = (X_\ell)^g$

(which satisfy (51) with $L = g$ and $\Lambda = 0$) can make all the changes $|\tilde{\alpha}'\rangle_f \leftrightarrow |\tilde{\alpha}''\rangle_f$

that are allowed between the elements of $B_f$, a state

$$|\tilde{\xi}_0 (\tilde{\alpha})\rangle = \prod_{\ell'} C_{\ell'} \prod_{f'} B_{f'} \prod_{v'} A_{v'} \left( \bigotimes_{\ell \in \mathcal{L}_2} |0\rangle \right) \otimes \left( \bigotimes_{f \in \mathcal{L}_2} |0\rangle \right) \otimes |\tilde{\alpha}\rangle_f^g,$$

with $\tilde{\alpha} \neq 0$, cannot be interpreted as a vacuum state.

III. All the quasiparticles $\tilde{q}^{(0,L,0)}$, which are created by an operator

$$\tilde{W}_\ell^{(0,L,0)} = (X_\ell)^L,$$

can be transported without increasing/decreasing the energy of the system, while the others $\tilde{q}^{(J,L,\Lambda)}$, which are produced by any operator

$$\tilde{W}_\ell^{(J,L,\Lambda)} = (Z_\ell)^J \circ (X_\ell)^L \quad \text{or} \quad \tilde{W}_\ell^{(J,L,\Lambda)} = (X_\ell)^J \circ (Z_\ell)^J$$

with $J \neq 0$, should be regarded as “confined”.

IV. As a consequence of items II. and III., the ground state of these $D^N (Z_N)$ models are made up of

$$|\xi_0^{(0,L)}\rangle = \prod_{p=1}^s \left( \tilde{O}_{L_p} (\gamma^*_p) \right)^{\lambda_p} \prod_{\ell'} C_{\ell'} \prod_{f'} B_{f'} \prod_{v'} A_{v'} \left( \bigotimes_{\ell \in \mathcal{L}_2} |0\rangle \right) \otimes \left( \bigotimes_{f \in \mathcal{L}_2} |0\rangle \right)$$

since all these vacuum states are topologically independent of each other due.
However, something that becomes quite clear from Theorems 3 and 4 is that, except that $N$ is a prime number, all these $D^N(Z_N)$ models can also be defined by using a $\mathfrak{f}$ that is neither a trivial group homomorphism nor a group isomorphism. This is, for example, the case of the $D^4(Z_4)$ model that, in addition to being able to be defined by using these two group homomorphisms, can also be defined by using $\mathfrak{f}(0) = \mathfrak{f}(2) = 0$ and $\mathfrak{f}(1) = \mathfrak{f}(3) = 2$.

$$\mathfrak{f}(0) = \mathfrak{f}(2) = 0 \text{ and } \mathfrak{f}(1) = \mathfrak{f}(3) = 2.$$ (72)

4.3.3. And what may happen when $\mathfrak{f}$ is not a group isomorphism?

Although Theorem 4 shows us that another non-trivial group homomorphism, which can also be used to define this $D^4(Z_4)$ model, is given by $\mathfrak{f}(0) = 0$, $\mathfrak{f}(1) = 3$, $\mathfrak{f}(2) = 2$ and $\mathfrak{f}(3) = 1$,

the group homomorphism (72) is a little more interesting because

$$|\ker(\mathfrak{f})| > 1 \text{ and } |\text{Im}(\mathfrak{f})| = 2.$$ After all, besides (73) is not very different from the group isomorphism $\mathfrak{f} : Z_4 \to Z_4^{21}$, one of the things that this (72), where $\ker(\mathfrak{f}) = \{0, 2\}$, allows us to see is that the ground state of this model is defined by

$$\begin{align*}
|\xi_0(\bar{\alpha},\bar{\lambda},\bar{L})\rangle &= \prod_{p=1}^{s} \left[ \tilde{O}_{L_p}(\bar{\gamma}_p) \right] \left[ \prod_{p} C_p \prod_{p'} B_{p'} \prod_{\nu} A_{\nu} \left( \bigotimes_{\ell \in L_2} |0\rangle \right) \otimes \left( \bigotimes_{f \in L_2} |0\rangle \right) \right] \otimes |\bar{\alpha}\rangle_{f''}.
\end{align*}$$ (75)

where $\alpha \in \ker(\mathfrak{f})$. In other words, we are faced with a $D^4(Z_4)$ model that has an algebraically degenerated ground state, but where this algebraic degeneracy is neither a maximum nor a minimum.

Another interesting aspect of this $D^4(Z_4)$ model, which is defined by using (72), is related to the fact that

$$F_{\ell}(g) = (\bar{Z}_f^2)^g \text{ and } F_{\ell}(\bar{\gamma}) = (X_2^\ell)^{\bar{\gamma}}.$$ (76)

Why? Because, when we substitute these matrices into (53), it is not difficult to see that not all quasiparticles, which are detected individually by the operator $A'_\nu$, can be considered as “confined”. And how can we see it? By noting that

- the quasiparticles, which are detected by the operator $A'_\nu$, are produced by

$$\bar{W}_{\ell}^{(g,0,\lambda)} = (Z_2)^g, \text{ and }$$

Note that, as this group isomorphism is defined by

$$\begin{align*}
\mathfrak{f}(0) = 0, \mathfrak{f}(1) = 1, \mathfrak{f}(2) = 2 \text{ and } \mathfrak{f}(3) = 3,
\end{align*}$$ (74)

the only difference between it and (73) can be justified in terms of a permutation.
the edge operators can be represented by
\[ C_{\ell,\Lambda}' = \frac{1}{4} \sum_{\tilde{\gamma} \in \tilde{S}} (i)^{\Lambda_{\tilde{f}_1}} \cdot (\tilde{X}_{\tilde{f}_1}^\dagger)^{\tilde{\gamma}} \otimes (X_2^2)^{\tilde{\gamma}} \otimes (\tilde{X}_{\tilde{f}_2})^{\tilde{\gamma}}, \]

where \( i = e^{i(2\pi/4)} \) is the generator of the matter group.

After all, as the generator of the gauge group is also equal to \( i \) in this case where \( N = 4 \) and, therefore, the operators (54a) are such that
\[ Z^g X^h = i^{([g+h] \mod(4))} X^h Z^g, \]
it is not difficult to conclude that all the quasiparticles \( \tilde{q}^{(2,0,0)} \), which are produced by an operator \( \tilde{W}_\ell^{(2,0,0)} \), are “unconfined” (i.e., these quasiparticles can be transported without increasing/decreasing the energy of the system) because
\[ Z^2 X^2 = i^{[4 \mod(4)]} X^2 Z^2 = X^2 Z^2. \]

Consequently, as there are \( N - 1 \) quasiparticles \( \tilde{q}^{(2,L,0)} \) that are produced by the operators
\[ \tilde{W}_\ell^{(g,L,0)} = (Z_\ell)^g \circ (X_\ell)^L \quad \text{or} \quad \tilde{W}_\ell^{(g,L,0)} = (X_\ell)^L \circ (Z_\ell)^g \]
through a fusion
\[ \tilde{q}^{(2,2,0)} = \tilde{q}^{(2,0,0)} \times \tilde{q}^{(0,2,0)} = \tilde{q}^{(0,2,0)} \times \tilde{q}^{(2,0,0)} \]
between the quasiparticles \( \tilde{q}^{(2,0,0)} \) and \( \tilde{q}^{(0,L,0)} \), all these quasiparticles \( \tilde{q}^{(2,L,0)} \) are also interpreted as “unconfined” since all \( \tilde{q}^{(0,L,0)} \) are also “unconfined”.

4.4. The ground state degeneracy and the classifiability of the \( D^K(\mathbb{Z}_N) \) models

In view of what we have just seen in the last Subsubsection, the question that you, the reader, may be asking is: is there some rule to determine when the \( D^K(\mathbb{Z}_N) \) models have quasiparticles \( \tilde{q}^{(j,L,0)} \) that are “unconfined”? And in order for us to answer this question, it is interesting to pay attention, for instance, to the \( D^K(\mathbb{Z}_N) \) models that are defined by using a trivial group homomorphism (56). Why? Because trivial group homomorphisms \( \tilde{f} : \mathbb{Z}_K \to \mathbb{Z}_N \) always map every element of \( \mathbb{Z}_K \) to the identity element of \( \mathbb{Z}_N \). Thus, as it is precisely the result of this mapping (i.e., the identity element of \( \mathbb{Z}_N \)) that needs to change the gauge fields on which the edge operators \( C_{\ell,\Lambda}' \) act, there is no change and, therefore, these operators become unable to detect any gauge excitations. In other words, all the quasiparticles \( \tilde{q}^{(j,L,0)} \) are “unconfined” only when the \( D^K(\mathbb{Z}_N) \) models are defined by using trivial group homomorphisms.

Note that, although the \( D^4(\mathbb{Z}_4) \) model discussed above was not defined by using a trivial group homomorphism, its group homomorphism (72) places it in a situation that, in some way, is comparable to this one. After all, unlike the (73) and (74), the group homomorphism (72) defines two distinct equivalence classes: videlicet,
\[ [0] = \{ a \in \mathbb{Z}_N : a \equiv 0 \mod(4) \} \quad \text{and} \quad [2] = \{ a \in \mathbb{Z}_N : a \equiv 2 \mod(4) \} \]
since (72) is nothing more than the same group homomorphism (55) where \( n = 2 \). And why is it important to pay attention to the fact that (72) defines two distinct equivalence classes? Because (72) is just one example of a group homomorphism that can do this: other functions (55), which can also do this, can be identified whenever \( K \) and \( N \) are two even numbers. How? By considering that \( N = 2n \): after all, as \( K \) is also an even number and, therefore, \( nK \) will always be divisible by \( N \), the Theorem 4 guarantees the existence of the group homomorphism

\[
f([\tilde{\alpha}]) = [n\tilde{\alpha}],
\]

which can be used to define the two distinct equivalence classes

\[
[0] = \{ a \in \mathbb{Z}_{2n} : a \equiv 0 \mod (2n) \} \quad \text{and} \quad [n] = \{ a \in \mathbb{Z}_{2n} : a \equiv n \mod (2n) \}.
\]

And the main consequence of this is that, whenever we define a \( D^K(\mathbb{Z}_{2n}) \) model where \( K \) is an even number, this group homomorphism will lead us to

\[
F_f(g) = (\tilde{Z}^n_f)^g \quad \text{and} \quad F_\ell(\tilde{\gamma}) = (X^n_\ell)^{\tilde{\gamma}},
\]

and, therefore, all the quasiparticles produced by

\[
\tilde{W}_f^{(n,L,0)} = (Z_\ell)^n \circ (X_\ell)^L \quad \text{or} \quad \tilde{W}_\ell^{(n,L,0)} = (X_\ell)^L \circ (Z_\ell)^n
\]

will never be detected by the \( D^K(\mathbb{Z}_{2n}) \) edge operators as long as

\[
Z^nX^n = i^{[(n+n)\mod(2n)]}X^nZ^n = X^nZ^n.
\]

And as \(|\text{Im}(f)| \) is equal to the number of equivalence classes that \( f \) defines, this explains why we take, as an example, this \( D^4(\mathbb{Z}_4) \) model where \(|\text{Im}(f)| = 2 \). That is, as much as we have highlighted the fact that \(|\ker(f)| > 1 \), the necessary condition for the existence of “unconfined” quasiparticles in the \( D^K(\mathbb{Z}_N) \) models is that \(|\text{Im}(f)| \leq 2 \).

4.4.1. What can we say about the quasiparticles that are created by manipulating matter fields?

Notwithstanding, the information that \(|\ker(f)| > 1 \) is still relevant because it is precisely this \(|\ker(f)| \) that computes the number of (matter) excitations, which are created by manipulating matter fields, that are not able to locally deform \( L_2 \). And although we still have not said a word about all the (matter) excitations \( \tilde{Q}^{(J,\Lambda)} \) that can be created in these \( D^K(\mathbb{Z}_N) \) models, they are not as surprising as the (matter) excitations of the \( D_M(\mathbb{Z}_N) \) models: after all, as all the operators

\[
\tilde{W}_f^{(L,\Lambda)} = (\tilde{X}_f)^L \circ [F_f(g)]^\Lambda \quad \text{and} \quad \tilde{W}_f^{(L,\Lambda)} = [F_f(g)]^\Lambda \circ (\tilde{X}_f)^L
\]

that produce them can be identified in the expressions of the \( D^K(\mathbb{Z}_N) \) face and edge operators, it is not difficult to conclude that
• $\tilde{Q}^{(J,\Lambda)}$ have Abelian fusion rules, because we always have that
$$F_f(g) = (\tilde{Z}_n^g)^g$$
where $n$ takes the values that satisfy Theorem 4, and
• the action of these operators (79), with $\Lambda = 0$, is sufficient to perform transitions between/among the $D^K(Z_N)$ vacuum states.

Note that, just by looking at the vacuum states
$$|\xi_{\tilde{\alpha},\tilde{\lambda},\tilde{L}}\rangle = \prod_{p=1}^n [\tilde{O}_{L_p}(\pi_p^*)]^{\lambda_p} \prod_{\ell} C_{\ell'}^{\ell} \prod_{f'} B_{f'} \prod_{v'} A_{v'} \left( \bigotimes_{\ell \in L_2} |0\rangle \right) \otimes \left( \bigotimes_{f \in L_2} |0\rangle \right) \otimes |\tilde{\alpha}\rangle_{f''}$$
that define the $D^K(Z_N)$ ground state, this sufficient condition makes it very clear that all the (matter) excitations, which are not able to locally deform $L_2$, are produced by the operators $\tilde{W}_f(0,0)$ that reduce the “fake holonomy” (37) to the true holonomy (36).

Based on these findings, it is not difficult to conclude that all these $D^K(Z_N)$ models can be completely classified in terms of an ordered 3-tuple $(N, K, n)$ as follows:

(a) $(N, K, 0)$

All the $D^K(Z_N)$ models, which are characterized by an ordered 3-tuple where $n = 0$, have ground states with an algebraic degeneracy $|\ker(f)|$ that is maximal (i.e., it is equal to $K$). As a consequence of this maximality, all the manipulations, which can be done on the matter fields, do not (locally) deform $L_2$ and, therefore, do not change the energy of the system. In this way, it is valid to affirm that all these models, with $(N, K, 0)$, are nothing more than the same $D(Z_N)$ models, but with an algebraically degenerated ground state$^{22}$.

(b) $(N, N, N)$

When $n = N$, all the $D^K(Z_N)$ model have an algebraic degeneracy $|\ker(f)|$ that is minimal (i.e., it is equal to 1). And as one of the consequence of this minimality is that $|\text{Im}(f)| = N$, we can affirm that, although these models house all the $D(Z_N)$ quasiparticles among their energy excitations, all the quasiparticles that are detected by the $D^K(Z_N)$ vertex operators are “confined”. Note that, since this minimality also implies that all the (matter) excitations $\tilde{Q}_f^{(L,\Lambda)}$, where $(L, \Lambda) \neq (0, 0)$, are detectable by the $D^K(Z_N)$ face and edge operators, the ground state of all these models can only be indexed by $\tilde{\alpha} = 0$.

$^{22}$Note that, as there is no way to manipulate the matter fields when $K = 1$, the Hamiltonian of any $D^1(Z_N)$ model (i.e., of any $D^K(Z_N)$ model that is classified as $(N, 1, 0)$) is given by
$$H_{D^1(Z_N)} = -\sum_v A_v - \sum_f B_f - \sum_\ell \mathbb{1}_{f_1} \otimes \mathbb{1}_\ell \otimes \mathbb{1}_{f_2} \Rightarrow H_{D(Z_N)} - H_{D^1(Z_N)} = N_\ell,$$
which only reinforces the existence of a “correspondence principle” between the $D^K(Z_N)$ and $D(Z_N)$ models.
In this case, where this ordered 3-tuple is different from \((N, K, 0)\) or \((N, N, N)\), it is possible to affirm that the \(D^K (Z_N)\) models may have intermediate properties between those of \((a)\) and \((b)\). After all, although these models may be perfectly defined by using group homomorphisms that confine all the quasiparticles \(\tilde{q}^{(J,L,\Lambda)}\) with \(J \neq 0\), whenever \(K\) is an even number and \(N = 2n\) we can also define such models by using (77): i.e., whenever \(K\) is an even number and \(N = 2n\), we can define the \(D^K (Z_{2n})\) models where all the quasiparticles \(\tilde{q}^{(J,L,0)}\), with \(J \in [0]\), are unconfined. As a consequence of this partial deconfinement, the algebraic degeneracy of the \(D^K (Z_{2n})\) ground state is neither a maximum nor a minimum because, for instance, they are indexed by \(1 < |\ker (f)| < K\) values of \(\tilde{\alpha}\).

4.4.2. Does the degree of degeneracy of the \(D^K (Z_N)\) ground state depend on the (sub)set \(\text{Im} (f)\)?

Although we have said that the cardinality of \(\ker f\) is relevant for determining the degree \(\tilde{n}\) of degeneracy of the \(D^K (Z_N)\) ground states, we have not yet presented the formula for this \(\tilde{n}\) when \(f\) is not a trivial group homomorphism. So, the natural question we can ask here is: does this formula exist?

In order to understand the answer to this question, it is interesting that we remember, for instance, that we have already managed to determine this formula when we analysed the trivial \(D^K (Z_N)\) models. And an interesting aspect of the formula (59) that we found is that it clearly shows is that there is a dual correspondence between the trivial \(D_M (Z_N)\) and \(D^K (Z_N)\) models. After all, in accordance with what has been said on page 32, all the trivial group homomorphisms (56), which are used to define the trivial \(D^K (Z_N)\) models, always are induced by a trivial (sub)group action (60) that maximizes \(|\text{Fix} \mu|\). But what happens when, for instance, the \(D^K (Z_N)\) models can be defined by using non-trivial group homomorphisms?

One of the things that happens is that, since all these non-trivial group homomorphisms are induced by non-trivial (sub)group actions \(\tilde{\mu}_f : \text{Im} (f) \times \tilde{S} \to \tilde{S}\) that define only \(k\)-cycles where \(k > 1\), this fact allows us to conclude that the dual correspondence, between the non-trivial \(D_M (Z_N)\) and \(D^K (Z_N)\) models, is not so perfect. Why? Because, for instance, non-trivial \(D_M (Z_N)\) models can be defined by using non-trivial group actions that can define 1-cycles. In plain English, no \(D_M (Z_N)\) model, which is defined by using a non-trivial action \(\mu\) where \(|\text{Fix} \mu|\) is not 0, can be interpreted as the dual of any \(D^K (Z_N)\) model.

Nevertheless, it is also worth noting that, in accordance with the definition of the \(D^K (Z_N)\) face and edge operators, the elements of \(\text{Im} (f)\) must also act on the elements of \(G\). After all, since \(\text{Im} (f)\) is a normal subgroup of \(G\), it is not difficult to conclude that this other (sub)group action \(\tilde{\mu}_f : \text{Im} (f) \times G \to G\) allows us to interpret its \(k\)-cycles as elements of the quotient group \(G/\text{Im} (f)\). And what does it mean? This means that all the quasiparticles inherited from the \(D (Z_N)\) models are divided into equivalence classes. Thus, by noting that

- the \(D (G)\) models have ground states that are \(|G|^{2g}\)-fold degenerated, where \(g\) is the
genus of $\mathcal{M}_2$ [12], and

• all the quasiparticles $\tilde{q}^{(0, L, 0)}$, which are only detected by the face operators (53b), are divided into $|\mathbb{Z}_N/\text{Im}(f)|$ equivalence classes,

it becomes clear that the degree of degeneracy of the $D^K(\mathbb{Z}_N)$ ground states is given by

$$\tilde{n} = |\ker(f)| \cdot |\mathbb{Z}_N/\text{Im}(f)|^{2g}.$$ (81)

Note that this result is in full agreement with the formula (59) because, when $f$ is a trivial group homomorphism, $G/\text{Im}(f) = G$.

5. Final remarks

In accordance with everything we have just seen, it is perfectly possible to realize a dualization procedure on the $D_M(G)$ models and, thus, obtain another class $D^K(G)$ of exactly solvable models that can also be interpreted as a generalization of the $D(G)$ models. However, if we overlay these $D_M(G)$ and $D^K(G)$ models in order to define a more general class, whose Hamiltonian operator is given by

$$H_{\text{total}} = H_{D_M(G)} + H_{D^K(G)} ,$$

this more general class does not necessarily define self-dual models. After all, unlike the $D_M(\mathbb{Z}_N)$ models that, for instance, can support quasiparticles with non-Abelian fusion rules, all the quasiparticles of the $D^K(G)$ models exhibit only Abelian fusion rules. And one of the reasons why these quasiparticles, with such non-Abelian fusion rules, appear in the $D_M(\mathbb{Z}_N)$ models and not in the $D^K(G)$ models is that, in the former, these non-Abelian fusion rules are necessary to perform transitions between/among the different phases that coexist in the ground state. That is, something that always is needed when, for instance, $\mu$ is a non-trivial group action with $|\text{Fix}_\mu| \neq 0$.

Another striking difference between these $D_M(\mathbb{Z}_N)$ and $D^K(G)$ models is that, while the former may differ from the $D(\mathbb{Z}_N)$ models when $M$ and $N$ are coprime numbers, the latter cannot do the same when $K$ and $N$ are coprime numbers. After all,

• as $K$ and $N$ index the cyclic groups $\mathbb{Z}_K$ and $\mathbb{Z}_N$ respectively, and

• the $D^K(\mathbb{Z}_N)$ models are only different from the $D(\mathbb{Z}_N)$ models when (55) is a non-trivial group homomorphism,

Theorem 4 guarantees the identification of the $D^K(\mathbb{Z}_N)$ models as the $D(\mathbb{Z}_N)$ models, but with an algebraically degenerated ground state, because the only group homomorphism that exists, when $K$ and $N$ are coprime numbers, is the trivial one.

Nevertheless, it is interesting to point out that the way we constructed these $D^K(\mathbb{Z}_N)$ models, by taking the dual algebraic framework of the $D^M(\mathbb{Z}_N)$ models, allows us to recognize several dual properties between these $D^M(\mathbb{Z}_N)$ and $D^K(\mathbb{Z}_N)$ models. And the main one has to do with the presence of confined quasiparticles in both models: whenever we, for instance, define
• the \( D_N (\mathbb{Z}_N) \) models by taking \( \{ M (g) : g \in \mathbb{Z}_N \} \) as the faithful representation of \( \mathbb{Z}_N \), all the quasiparticles, which are interpreted as gauge excitations and are detected by their face (vertex) operators, are confined (unconfined), and

• the \( D^N (\mathbb{Z}_N) \) models by using a group isomorphism \( f \), all the quasiparticles, which are interpreted as gauge excitations and are detected by their vertex (face) operators (i.e., which are detected by the operators that are dual to the \( D_N (\mathbb{Z}_N) \) face (vertex) operators), are confined (unconfined).

It is clear that such confined quasiparticles cannot be identified only in these circumstances where \( M = K = N \): such an identification can be done whenever the group actions, which define the \( D^M (\mathbb{Z}_N) \) models and that which induces a non-trivial group homomorphism (55), define \( k \)-cycles where \( k > 1 \). And as can be seen in Section 2, all these \( D^M (\mathbb{Z}_N) \) models, where \( \{ M (g) : g \in \mathbb{Z}_N \} \) is neither a trivial nor a faithful representation of \( \mathbb{Z}_N \), may be defined by using group actions that make a part of these quasiparticles, which are detected by their face operators, to be unconfined. Note that this comment also applies to the \( D^K (\mathbb{Z}_N) \) models: in accordance with the complete classification presented for these models at the end of the Subsubsection 4.4.1, there are non-trivial group homomorphisms \( f \) that make a part of these quasiparticles, which are detected by the \( D^K (\mathbb{Z}_N) \) vertex operators, to be unconfined.

Given all that can be listed among the properties of these \( D_M (G) \) and \( D^K (G) \) models, it is impossible not to recognize that both have algebraic and topological orders. And by the way, another thing that is also impossible not to recognize is that the dual behaviour, which we were able to identify in these models, opened a window for us to explore them from the physical point of view: after all, since the \( D (G) \) models were intentionally created by exploiting an analogy that allows us to identify the quasiparticles, which are detected by the vertex and face operators of (5), as electronic and magnetic respectively [11, 12, 44, 45], all these aforementioned confinements seem to be pointing to the possibility of exploiting these \( D_M (G) \) and \( D^K (G) \) models to describe, for instance, superconductors (or, at least, perfect diamagnets) and topological insulators. And, certainly, we will explore this in a future work.

But, although these \( D^K (G) \) models have been successfully constructed, something that also deserves our attention is that there is no impediment, a priori, to define new generalizations of these \( D (G) \) and \( D_M (G) \) models without the artifice of an algebraic dualization procedure. Therefore, a relevant question that we can ask ourselves is whether any of these new generalizations are able to lead us to the same results as the \( D^K (G) \) models. And a possibility, which we can explore in order to answer this question, is the one where \( f \) defines a crossed module [46]: i.e., the one where \( f \) is a group homomorphism that, together with a group action \( \tilde{\mu} : G \times \tilde{S} \rightarrow \tilde{S} \), respects two conditions

\[
f (\tilde{\mu} (g, \tilde{\alpha})) = g f (\tilde{\alpha}) g^{-1} \quad \text{and} \quad \tilde{\mu} (f (\tilde{\alpha}), \tilde{\beta}) = \tilde{\alpha} \tilde{\beta} \tilde{\alpha}^{-1},
\]

where the second is known as the Peiffer condition [47, 48]. Note that the group homomorphisms, which define the \( D^K (G) \) models, satisfy these two conditions when the group action
that induces it is trivial because $G$ and $\tilde{S}$ are Abelian groups. And the possible advantage of taking $f$ as the group homomorphism that defines a crossed module lies in the fact that it seems possible to recover the $D^K(G)$ models as a special case of the higher lattice gauge theories [49], which are based on the higher-dimensional category theory [50, 51, 52]. A good example of this can be found in Ref. [53], where a 2-lattice gauge theory is defined by using a three-dimensional lattice in which we can measure 1- and 2-holonomies: after all, while the 1-holonomy is identified as the same “fake holonomy” (37), which is preserved by the gauge transformations that the operator $A'_v$ performs, the 2-holonomy [54] is preserved by the action of the operator

$$\prod_{\ell \in S_\ell} C'_\ell,$$

which corroborates with the perception that the edge operator (38) is nothing more than a component of another, which is capable of performing another kind of gauge transformations. Note that, if $\tilde{f}$ is the group homomorphism that defines a crossed module $\mathcal{G} = (G, \tilde{S}; f, \tilde{\mu})$, the first and second homotopy groups of this crossed module can be defined as $\pi_1(\mathcal{G}) = G / \text{Im } f = \text{coker } (f)$ and $\pi_2(\mathcal{G}) = \ker (f) = \pi_2(G)$ respectively [55], whose orders define the formula (81). We will also return to this topic in another future work.

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### References

[1] P. Benioff: *J. Stat. Phys.* **22**, 563 (1980).
[2] D. Deutsch: *Proc. R. Soc. Lond. A* **400**, 97 (1985).
[3] D. Loss, D. P. DiVincenzo: *Phys. Rev. A* **57**, 120 (1998).
[4] J. I. Cirac, P. Zoller: *Nature* **404**, 579 (2000).
[5] S. Akama: *Elements of Quantum Computing: History, Theories and Engineering Applications* (Springer International Publishing Switzerland, Cham 2015).
[6] M. Nielsen, I. Chuang: *Quantum Computation and Quantum Information*(Cambridge University Press, Cambridge 2000).
[7] M. L. Bellac: *A Short Introduction to Quantum Computation and Quantum Computation* (Cambridge University Press, Cambridge 2006).
[8] B. Schumacher: *Phys. Rev. A* **51** (4), 2738 (1995).
[9] X.-G. Wen: *Int. J. Mod. Phys. B* **44**, 239 (1999).
[10] E. Dennis, A. Kitaev, A. Landahl, J. Preskill: *J. Math. Phys.* **43**, 4452 (2002).
[11] A. Yu. Kitaev: *Annals Phys.* **303**, 2 (2003).
[12] J. K. Pachos: *Introduction to Topological Quantum Computation* (Cambridge University Press, New York 2012).
[50] I. Bucur, A. Deleanu: *Introduction to the Theory of Categories and Functors* (John Wiley & Sons Ltd, London 1970).

[51] E. Cheng, A. Lauda: *Higher-Dimensional Categories: An Illustrated Guidebook*. Available at http://www.dpmms.cam.ac.uk/~elgc2/guidebook/

[52] S. Awodey: *Category Theory* (Oxford University Press, New York 2006).

[53] A. Bullivant, M. Calçada, Z. Kádár, P. Martin, J. Faria Martins: *Phys. Rev.* B95, 155118 (2017).

[54] H. Abbaspour, F. Wagemann: *On 2-holonomy* (Notes de cours de l’Université de Nantes, Nantes 2012).

[55] R. Costa de Almeida: *Ordem topológica em sistemas tridimensionais e simetria de 2-gauge* (USP Master’s Thesis, São Paulo 2017).