Calibrating the Learning Rate for Adaptive Gradient Methods to Improve Generalization Performance

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Abstract

Although adaptive gradient methods (AGMs) have fast speed in training deep neural networks, it is known to generalize worse than the stochastic gradient descent (SGD) or SGD with momentum (S-Momentum). Many works have attempted to modify AGMs so to close the gap in generalization performance between AGMs and S-Momentum, but they do not answer why there is such a gap. We identify that the anisotropic scale of the adaptive learning rate (A-LR) used by AGMs contributes to the generalization performance gap, and all existing modified AGMs actually represent efforts in revising the A-LR. Because the A-LR varies significantly across the dimensions of the problem over the optimization epochs (i.e., anisotropic scale), we propose a new AGM by calibrating the A-LR with a softplus function, resulting in the SADAM and SAMSGRAD methods. These methods have better chance to not trap at sharp local minimizers, which helps them resume the dips in the generalization error curve observed with SGD and S-Momentum. We further provide a new way to analyze the convergence of AGMs (e.g., ADAM, SADAM, and SAMSGRAD) under the nonconvex, non-strongly convex, and Polyak-Łojasiewicz conditions. We prove that the convergence rate of ADAM also depends on its hyper-parameter epsilon, which has been overlooked in prior convergence analysis. Empirical studies support our observation of the anisotropic A-LR and show that the proposed methods outperform existing AGMs and generalize even better than S-Momentum in multiple deep learning tasks.

1 Introduction

Many machine learning problems can be formulated as the minimization of an objective function $f$ of the form: $\min_{x \in \mathbb{R}^d} f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x)$, where both $f$ and $f_i$ are typically nonconvex in deep learning. Stochastic gradient descent (SGD), its variants such as SGD with momentum (S-Momentum) \cite{7, 21, 20, 22}, and adaptive gradient methods (AGMs) \cite{6, 12, 25} play important roles in solving the minimization problem due to their simplicity and wide applicability. In particular, AGMs often exhibit fast initial progress in training and are easy to implement in solving large scale optimization problems. The updating rule of AGMs can be generally written as follows:

$$x_{t+1} = x_t - \eta_t \odot m_t,$$

(1)

where $\odot$ calculates element-wise product of the first-order momentum $m_t$ and the learning rate (LR) $\eta_t$. Note that the learning rate here is no longer a scalar as in SGD, but a vector of the same length of $x$. There is fairly an agreement on how to compute $m_t$, which is a convex combination of previous $m_{t-1}$ and current stochastic gradient $g_t$, i.e., $m_t = \beta_1 m_{t-1} + (1 - \beta_1) g_t$, $\beta_1 \in [0, 1]$.

\footnote{Code is available at https://github.com/neilliang90/Sadam.git.}

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The LR consists of two parts: the base learning rate (B-LR) \( \eta_b \) is a scalar which can be constant or decay over iterations. In this work we consider the B-LR as constant \( \eta \). The adaptive learning rate (A-LR), \( \frac{1}{\sqrt{\eta^2}} \), varies adaptively across dimensions of the problem, where \( \eta \in \mathbb{R}^d \) is the second-order momentum calculated as a combination of previous and current squared stochastic gradients. If \( g_t = [g_{t,1}, g_{t,2}, \cdots, g_{t,d}]^T \), the squared gradient \( g_t^2 = [g_{t,1}^2, g_{t,2}^2, \cdots, g_{t,d}^2]^T \). Unlike the first-order momentum, the formula to estimate the second-order momentum varies in different AGMs. As the core technique in AGMs, A-LR opens a new regime of controlling LR, and allows the algorithm to move with different step sizes along the search direction at different coordinates.

The first known AGM is ADAGRAD [6] where the second-order momentum is estimated as \( v_t = \sum_{i=1}^t g_i^2 \). It works well in sparse settings, but the A-LR often decays rapidly for dense gradients. To tackle this issue, ADADELTA [25], RMSPROP [19], ADAM [12] have been proposed to use exponential moving averages of past squared gradients, i.e., \( v_t = \beta_2 v_{t-1} + (1 - \beta_2) g_t^2, \beta_2 \in [0, 1] \) (equivalently, \( v_t = \sum_{i=1}^t (1 - \beta_2) \beta_2^{t-i} g_i^2 \) and calculate the A-LR by \( \frac{1}{\sqrt{v_t + \epsilon}} \) where \( \epsilon > 0 \) is used in case that \( v_t \) vanishes to zero. In particular, ADAM has become the most popular optimizer in the deep learning area due to its effectiveness in early training stage. Nevertheless, it has been empirically shown that ADAM generalizes worse than S-Momentum to unseen data [8, 23, 9], and even fails to converge in some cases [17, 15]. AGMs decrease the objective value rapidly in early iterations, and then stay at a plateau whereas SGD and S-Momentum continue to show dips in the training error curves, and thus continue to improve test accuracy over iterations. It is essential to understand what happens to A-LR in the later learning process, so we can modify AGMs to enhance their generalization performance. There have been two lines of research for this purpose: (1) to theoretically understand and prove the convergence of an AGM in nonconvex settings; and (2) to empirically design practical AGMs that can perform better on test data.

Recently, a few modified AGMs have been developed, such as, AMSGRAD [17], YOGI [24], and ADABOUND [15]. The AMSGRAD is the first method to theoretically address the non-convergence issue of ADAM by taking the largest second-order momentum estimated in the past iterations, i.e., \( v_t = \max \{ v_{t-1}, \tilde{v}_t \} \) where \( \tilde{v}_t = \beta_2 \tilde{v}_{t-1} + (1 - \beta_2) g_t^2 \) and proves its convergence in the convex case. The analysis is later extended to other AGMs (such as RMSPROP and AMSGRAD) in nonconvex settings [26, 4, 1, 13]. YOGI claims that the past \( g_t^2 \)'s are forgotten in a fairly fast manner in ADAM and proposes \( v_t = v_{t-1} - (1 - \beta_2) \text{sign}(v_{t-1} - g_t^2) g_t^2 \) to adjust the decay rate of A-LR to improve performance. However, the parameter \( \epsilon \) in the A-LR is adjusted to \( 10^{-3} \) in YOGI, instead of \( 10^{-8} \) in the default setting of ADAM, so \( \epsilon \) dominates the A-LR in later iterations when \( v_t \) becomes small and can be responsible for performance improvement. The hyper-parameter \( \epsilon \) has rarely been discussed previously and our analysis shows that the convergence rate is closely related to \( \epsilon \), which is further verified in our experiments. PADAM [3, 24] claims that the A-LR in ADAM and AMSGRAD are “overadapted”, which leads to the generalization gap, and proposes to replace the A-LR updating formula by \( 1/((v_t)^p + \epsilon) \) where \( p \in (0, 1/2) \) and uses this A-LR in the AMSGRAD. ADABOUND confines the LR to a predefined range by applying \( \text{Clip}(\frac{\epsilon}{\sqrt{v_t}}, \eta_l, \eta_r) \), where LR values outside the interval \([\eta_l, \eta_r]\) are clipped to the interval edges.

\textbf{Figure 1:} The range of the A-LR used by ADAM over iterations in four different settings: (a) CNN on MNIST, (b) ResNet20 on CIFAR-10, (d) DenseNets on CIFAR-10. We show the min, max, median, and the 25 and 75 percentiles of the A-LR across the problem dimensions (the elements in \( \sqrt{\frac{\eta}{\epsilon}} \)).

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\textsuperscript{2}The PADAM in [3] actually used AMSGRAD, and for clear comparison, we named it PAMSGRAD. In our experiments, we also compared with the ADAM that used the A-LR formula in [3], which we named PADAM.
We investigate how the A-LR in ADAM varies over time and across problem dimensions, and plot four examples in Figure 1 (see more figures in Appendix) where we run ADAM to optimize a convolutional neural network (CNN) on the MNIST dataset, and ResNets or DenseNets on the CIFAR dataset. At each epoch, we compute the min, max, median, and the 25 and 75 percentiles of the step sizes in √v_i/∥x_i∥. The curves in Figure 1 have very irregular shapes, and the median value is hardly placed in the middle of the range. As a general trend, the A-LR becomes larger when v_t approaches 0 over iterations. At some coordinates, the A-LR quickly reaches the maximum 10^{8} (because we use \( \epsilon = 10^{-8} \) in the figure). This anisotropic scale of A-LR across the dimensions makes it difficult to determine the B-LR \( \eta \) and creates numerical instability. As all of the gradient-based methods rely on local function behavior at \( x_t \) to determine the search, a very large step size makes the algorithm depart from the local area. By analyzing all previous modified AGMs that aim to close the generalization gap, we find that all these works can be summarized into one technique: constraining the A-LR, \( 1/(\sqrt{v_t} + \epsilon) \), to a reasonable range. Specifically, Yogi uses \( \epsilon = 10^{-3} \), PADAM uses smaller \( \rho \), and ADABOUND thresholds the elements of the A-LR into \([\eta_l, \eta_r]\), to adjust the A-LR. However, a more effective way is to calibrate the A-LR according to an activation function (such as those used in deep learning) rather than hard-thresholding the A-LR at all coordinates (Yogi and ADABOUND). PADAM essentially regulates the A-LR very much in the same way as our method (see figures in Appendix).

In this paper, our contributions are summarized as follows:

- We study AGMs from a new perspective: the range of the A-LR. Through experimental studies, we find that the A-LR is always anisotropic. This anisotropy may lead the algorithm to focus on a few dimensions (those with large A-LR), which may consequently trap the process at a sharp local minimizer, and exacerbate generalization performance. We analyze the existing modified AGMs to help explain how they close the generalization gap.

- We propose to calibrate the A-LR using the softplus function that has a hyper-parameter \( \beta \), which can be combined with any AGM. In this paper, we combine it with ADAM and AMSGRAD to form the SADAM and SAMSGRAD methods. Empirical evaluations show that they obviously increase test accuracy, and outperform many AGMs and even S-Momentum in multiple deep learning models.

- Theoretically, we are the first to include hyper-parameter \( \epsilon \) into the convergence analysis and clearly show that the convergence rate is upper bounded by a term that has \( \frac{1}{\mathcal{L}} \), verifying prior observations that \( \epsilon \) affects ADAM’s performance empirically. We provide a new approach to convergence analysis of AGMs under the nonconvex, non-strongly convex, or Polyak-Lojasiewicz (P-L) condition, which recovers the same convergence rate as the SGD and S-Momentum in terms of maximum iteration \( T \) as \( O(1/\sqrt{T}) \) rather than known \( O(\log T/\sqrt{T}) \) in [4].

**Notation.** For any vectors \( a, b \in \mathbb{R}^d \), we use \( a \odot b \) for element-wise product, \( a^2 \) for element-wise square, \( \sqrt{a} \) for element-wise square root, \( a/b \) for element-wise division; we use \( a^k \) to denote element-wise power of \( k \), and \( ||a|| \) to denote its \( l_2 \)-norm. We use \( \langle a, b \rangle \) to denote their inner product, \( \max\{a, b\} \) to compute element-wise maximum. \( e \) is the Euler number, \( \log(\cdot) \) denotes logarithm function with base \( e \), and \( O(\cdot) \) to hide constants which do not rely on the problem parameters.

2 Revisit ADAM

In this section, we revisit ADAM method and analyze why AGMs generalize worse than S-Momentum and how to resolve the issue. First, we restudy \( \epsilon \) which is always underestimated, however plays an essential role in ADAM. Then, we notice the inconsistency updating between momentums: \( m_t \) and \( v_t \) and observe the anisotropic property of A-LR. The entanglement of these two factors causes "small learning rate dilemma" [3], which corresponds to bad generalization performance of ADAM. Second, we propose to use softplus function on top of ADAM to tackle this issue. The behavior of softplus function and its effectiveness will be explored.

Sensitive to \( \epsilon \). As a hyper-parameter in AGMs, \( \epsilon \) is originally introduced to avoid the zero denominator issue when the second-order momentum \( v_t \) goes to 0, and has never been studied in the convergence analysis of AGMs. However, it has been empirically observed that AGMs can be sensitive to the choice of \( \epsilon \) [1][24]. As shown in Figure 1, a smaller \( \epsilon = 10^{-8} \) leads to a wide span of the A-LR across the different dimensions, whereas a bigger \( \epsilon = 10^{-13} \) as used in Yogi, reduces the
After calibrating A-LR. This calibration brings the following benefits: (1) constraining extreme large-valued A-LR well regulates the distribution of the A-LR. Anisotropic A-LR. As shown in Figure 1, the range of A-LR across the problem dimensions is diminished to zero may be different from the speed of \( \sqrt{v_t} \) diminishing to zero due to the different exponential moving average of past gradients. Besides, noise generated in stochastic algorithms has nonnegligible influence to the learning process. When \( m_t \) and \( v_t \) are close to zero, the stochastic noise can not be ignored anymore and may cause some coordinates in A-LR to be large whereas the corresponding coordinates in \( m_t \) are not small.

Anisotropic A-LR. As shown in Figure 1 the range of A-LR across the problem dimensions is anisotropic for AGMs. The elements in the A-LR vary significantly across dimensions and there are always some coordinates in the A-LR of AGMs that reach the maximum determined by \( \epsilon \). We argue that the anisotropy of A-LR together with the different speed of \( m_t \) and \( v_t \) approaching 0 will lead to "small learning rate dilemma". Based on the two observations, the B-LR, \( \eta \), should be set small enough so that the LR \( \frac{\eta}{\sqrt{v_t + \epsilon}} \) is appropriate; or otherwise some coordinates will have very large updates because the corresponding A-LR’s are big, likely resulting in performance oscillation [13]. However, very small \( \eta \) may harm the later stage of the learning process since the small magnitude of \( m_t \) multiplying with a small step size (at some coordinates) will be too small to escape sharp local minima, which has been shown to lead to poor generalization [10] [2] [14]. Further, in many deep learning tasks, stage-wise policies are often taken to decay the LR after several epochs, thus making the LR even smaller and leading to the "small learning rate dilemma". To address the dilemma, it is essential to control the A-LR, especially when stochastic gradients get close to 0. Currently, although methods like YOGI, ADABOUND, and PAdam may reduce the A-LR range, they may not well regulate the distribution of the A-LR.

The softplus function helps. As a well-studied function, \( \text{softplus}(x) = \frac{1}{\beta} \log(1 + e^{\beta x}) \) is known to keep large values unchanged (behaved like function \( y = x \)) while smoothing out small values (see Figure 2(a)). The target magnitude to be smoothed out can be adjusted by a hyper-parameter \( \beta \). In our algorithms, we introduce \( \text{softplus}(\sqrt{v_t}) = \frac{1}{\beta} \log(1 + e^{\beta \sqrt{v_t}}) \) to smoothly calibrate the A-LR. This calibration brings the following benefits: (1) constraining extreme large-valued A-LR in some coordinates (corresponding to the small-values in \( v_t \)) while keeping others untouched with appropriate \( \beta \). For the undesirable large values in the A-LR, the softplus function condenses them smoothly instead of hard thresholding. For other coordinates, the A-LR largely remains unchanged; (2) remove the sensitive parameter \( \epsilon \) because the softplus function can be lower-bounded by a nonzero number when used on non-negative variables, \( \text{softplus}(\epsilon) \geq \frac{1}{\beta} \log 2 \).

After calibrating \( \sqrt{v_t} \) with a softplus function, the anisotropic A-LR becomes much more regulated (see figures in Section 3 and Appendix), and we clearly observe improved training and test performance (Figure 2(b) and (c) and more figures in Appendix). We name this method "SADAM" to represent the softplus ADAM. More empirical evaluations in Section 5 show that the proposed methods significantly improve the generalization performance of ADAM and AMSGRAD.
3 Algorithms

In this section, we develop two new variants of AGMs: SADAM and SAMSGRAD (Algorithms 1 and 2), which are developed based on ADAM and AMSGrad, respectively. The key difference lies in the way to design the adaptive functions, instead of using the generalized square root function only, we apply softplus\((\cdot)\) on top of the square root of the second-order momentum, which serves to regulate A-LR’s anisotropic behavior and replace the tolerance parameter \(\epsilon\) for which there may not be much guidance to tune in practice in the original AGMs by the hyper-parameter \(\beta\) used in the softplus function. Our approach opens up a new direction to examine other activation functions (not limited to the softplus function) to calibrate the A-LR, which we leave for future investigation.

**Algorithm 1 SADAM**

**Input:** \(x_1 \in \mathbb{R}^d\), learning rate \(\{\eta_t\}_{t=1}^T\), parameters \(0 \leq \beta_1, \beta_2 < 1\).

**Initialize** \(m_0 = 0, v_0 = 0\)

**for** \(t = 1\) to \(T\) **do**

- Compute stochastic gradient \(g_t\)
  \(m_t = \beta_1 m_{t-1} + (1 - \beta_1) g_t\)
  \(v_t = \beta_2 v_{t-1} + (1 - \beta_2) g_t^2\)

- \(x_{t+1} = x_t - \eta_t \text{softplus}(\sqrt{v_t}) \odot m_t\)

**end for**

**Algorithm 2 SAMSGRAD**

**Input:** \(x_1 \in \mathbb{R}^d\), learning rate \(\{\eta_t\}_{t=1}^T\), parameters \(0 \leq \beta_1, \beta_2 < 1, \beta\).

**Initialize** \(m_0 = 0, \tilde{v}_0 = 0\)

**for** \(t = 1\) to \(T\) **do**

- Compute stochastic gradient \(g_t\)
  \(m_t = \beta_1 m_{t-1} + (1 - \beta_1) g_t\)
  \(\tilde{v}_t = \beta_2 \tilde{v}_{t-1} + (1 - \beta_2) g_t^2\)
  \(v_t = \max\{v_{t-1}, \tilde{v}_t\}\)

- \(x_{t+1} = x_t - \eta_t \text{softplus}(\sqrt{v_t}) \odot m_t\)

**end for**

Figure 3: Behavior of the A-LR in the SADAM method with different choices of \(\beta\) (CNN on the MNIST data).

In the new algorithms, the first-order momentum \(m_t\) is still used to compute the search direction at each iteration, and the second-order momentum \(v_t\) is used to adaptively control the step size for each coordinate along the search direction. The default values of the hyper-parameters are recommended as \(\beta_1 = 0.9, \beta_2 = 0.999\), which are similar to what has been recommended for ADAM and AMSGrad. For clarity, we omit the bias correction step proposed in the original ADAM. However, our arguments and convergence analysis in Section 4 are applicable to the bias correction version as
We now analyze the convergence of $A^v$ we take a commonly used assumption $\beta$ well [12, 5, 24]. Using the softplus function, we introduce a new hyper-parameter $\beta$, which performs as a controller to smooth out anisotropic A-LR values, and connect the ADAM and S-Momentum methods automatically. When $\beta$ is set to be small, SADAM and SAMSGRAD perform similarly to S-Momentum; when $\beta$ is set to be big, softplus($\sqrt{v_t}$) = $\frac{1}{\beta} \log(1 + e^{\beta \sqrt{v_t}}) \approx \frac{1}{\beta} \log(e^{\beta \sqrt{v_t}}) = \sqrt{v_t}$, and the updating formula becomes $x_{t+1} = x_t - \frac{\eta}{\sqrt{v_t}} m_t$, which is degenerated into the original AGMs. The hyper-parameter $\beta$ can be well tuned to achieve the best performance for different datasets and tasks. Based on our empirical observations, we recommend to use $\beta = 50$. For fair comparison in our experiments, we fixed $\beta = 50$ even though we observed better performance with other choices for some specific tasks (see Figures 2 and 3 and more figures in Appendix).

The proposed SADAM and SAMSGRAD can be treated as members of a class of AGMs that use the softplus (or another suitable activation) function to better adapt the step size. It can be readily combined with any other AGM, e.g., RMSROP, YOGI, and PADAM. These methods may easily go back to the original ones by choosing a big $\beta$. The proposed calibration scheme regulates the massive range of A-LR in AGMs back down to a moderate scale (e.g., (0, 80) if $\beta = 50$). The median of A-LR in different dimensions is now well positioned to the middle of the 25-75 percentile zone.

4 Convergence analysis

We now analyze the convergence of ADAM and SADAM. Besides other assumptions described next, we take a commonly used assumption $v_t \geq v_{t-1}$, which holds in AMSGRAD and SAMSGRAD. The main theorem characterizes the convergence under the nonconvex condition, and analysis for the non-strongly convex, and P-L condition (which includes the strongly convex condition) can be derived following the same line of arguments. The comprehensive proof is presented in Appendix.

**Optimization terminology.** In the convex setting, the optimality gap, $f(x_t) - f^*$, is examined where $x_t$ is the iterate at iteration $t$, and $f^*$ is the optimal value attained at $x^*$ assuming that $f$ does have a minimum. When $f(x_t) - f^* \leq \delta$, it is said that the method reaches an optimal solution with $\delta$-accuracy. However, in the study of AGMs, the average regret $\frac{1}{T} \sum_{t=1}^{T} (f(x_t) - f^*)$ (where $T$ is pre-specified) is used to approximate the optimality gap to define $\delta$-accuracy. Note that $T$ corresponds to $\delta$ in the sense that it is the number of iterations required to reach $\delta$-accuracy. Our analysis moves one step further to examine if $f \left( \frac{1}{T} \sum_{t=1}^{T} x_t \right) - f^* \leq \delta$ by applying Jensen’s inequality to the regret. In the nonconvex setting, finding the global minimum or even local minimum is NP-hard, so optimality gap is not examined. Rather, it is common to evaluate if a first-order stationary point has been achieved [16,17,24]. More precisely, we evaluate if $E\|\nabla f(x_t)\|^2 \leq \delta$ (e.g., in the analysis of SGD [21]). The convergence rate of SGD is $O(1/\sqrt{T})$ in both non-strongly convex and nonconvex settings. Requiring $O(1/\sqrt{T}) \leq \delta$ yields the maximum number of iterations $T = O(1/\delta^2)$. Thus, SGD can obtain a $\delta$-accurate solution in $O(1/\delta^2)$ steps in the non-strongly convex and nonconvex settings. Our theoretical results recover the rate of SGD and S-Momentum in terms of $T$.

**Assumption 1.** The loss $f_i$ occurred on each training example $i$ and the objective $f$ satisfy:

1. **L-smoothness.** $\forall x, y \in \mathbb{R}^d, \forall i \in \{1, ..., n\}, \|\nabla f_i(x) - \nabla f_i(y)\| \leq L \|x - y\|.$
2. **Gradient bounded.** $\forall x \in \mathbb{R}^d, \|\nabla f(x)\| \leq G, G \geq 0$.
3. **Variance bounded.** $\forall x \in \mathbb{R}^d, t \geq 1, E[g_t] = \nabla f(x_t), E[\|g_t - \nabla f(x_t)\|^2] \leq \sigma^2$.

**Definition 1.** Suppose $f$ has the global minimum, denoted as $f^* = f(x^*)$. Then for any $x, y \in \mathbb{R}^d$,

1. **Non-strongly convex.** $f(y) \geq f(x) + \nabla f(x)^T (y - x)$.
2. **Polyak-Łojasiewicz (P-L) condition.** $\exists \lambda > 0$ such that $\|\nabla f(x)\|^2 \geq 2\lambda(f(x) - f^*)$.
3. **Strongly convex.** $\exists \mu > 0$ such that $f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2}\|y - x\|^2$.

We have an important lemma here to highlight that every coordinate in the A-LR is both upper and lower bounded at all iterations, which is consistent with empirical observations (Figure 1), and forms the foundation of our proof. Most of the existing convergence analysis follows the line in [17] to first project the sequence of the iterates into a minimization problem as $x_{t+1} = x_t - \frac{\eta}{\sqrt{v_t}} m_t = \min_x \|v_t^{1/4} (x - (x_t - \frac{\eta}{\sqrt{v_t}} m_t))\|$, and then examine if $\|v_t^{1/4} (x_{t+1} - x^*)\|$ decreases
over iterations. Hence, $\epsilon$ is not discussed in this line of proof because it is not included in the step size. Using the bounds of the A-LR, we can find the convergence rate for reaching an $x$ that satisfies $E[\|\nabla f(x_t)\|^2] \leq \delta$ in the nonconvex setting. Then, by the definitions of convexity and the P-L condition, we can derive the optimality gap from the stationary point in the convex and P-L settings.

**Lemma 4.1. (Bounded A-LR.)** With Assumption [1] for any $t \geq 1$, $j \in [1, d]$, $\beta_2 \in [0, 1]$, and $\epsilon$ in ADAM, $\beta$ in SADAM, anisotropic A-LR is bounded in AGMs,

$$\text{ADAM has } (\mu_1, \mu_2)\text{-bounded A-LR: } \mu_1 \leq \frac{1}{\sqrt{\epsilon}} \mu_2, \quad \text{SADAM has } (\mu_3, \mu_4)\text{-bounded A-LR: } \mu_3 \leq \frac{1}{\text{softplus}(\sqrt{\epsilon})} \mu_4,$$

where $0 < \mu_1 \leq \mu_2$, and $0 < \mu_3 \leq \mu_4$.

**Remark 4.2.** Besides the square root function and softplus function, the A-LR calibrated by any positive monotonically increasing function can be bounded, and the bounds are related to $\epsilon$ or $\beta$. The fact that A-LR is bounded in any dimension and at any iteration is essential to prompt us to design a new way for convergence analysis, and can be extended to other AGMs as long as LR is bounded.

We now describe our main results on the convergence rates of ADAM and SADAM in the nonconvex case, which depend on the maximum number of iterations $T$, $\epsilon$ (for ADAM), and $\beta$ (for SADAM). We clearly show that the convergence rate is proportional to $\frac{1}{\sqrt{T}}$ or $\beta^2$.

**Theorem 4.3. [Nonconvex ADAM]** Suppose $f(x)$ is a nonconvex function that satisfies Assumption [1] and $v_t \geq v_{t-1}$, $\forall t$. Let $\eta_t = \eta = O\left(\frac{1}{\sqrt{T}}\right)$, the ADAM method has

$$\min_{t=1, \ldots, T} E[\|\nabla f(x_t)\|^2] \leq O\left(\frac{1}{c^2 \sqrt{T}} + \frac{d}{cT} + \frac{d}{c^2 T \sqrt{T}}\right).$$

**Theorem 4.4. [Nonconvex SADAM]** Suppose $f(x)$ is a nonconvex function that satisfies Assumption [1] and $v_t \geq v_{t-1}$, $\forall t$. Let $\eta_t = \eta = O\left(\frac{1}{\sqrt{T}}\right)$, the SADAM method has

$$\min_{t=1, \ldots, T} E[\|\nabla f(x_t)\|^2] \leq O\left(\frac{\beta^2}{\sqrt{T}} + \frac{d \beta^2}{T \sqrt{T}}\right).$$

**Remark 4.5.** The convergence rate of SADAM relies on $\beta$, which can be a much smaller number ($\beta = 50$ as recommended) than $\frac{1}{\sqrt{T}}$ (commonly $\epsilon = 10^{-8}$ in AGMs), showing that our methods have a better convergence rate than ADAM. When $\beta$ is huge, SADAM’s rate is comparable to the classic ADAM. When $\beta$ is small, the convergence rate will be $O\left(\frac{1}{\sqrt{T}}\right)$ which recovers that of SGD [2].

**Corollary 4.5.1.** If $\epsilon$ or $\beta$ is treated as a constant, then the ADAM, SADAM (and SAMSGRAD) methods with fixed $L$, $\sigma$, $G$, $\beta_1$, and $\eta = O\left(\frac{1}{\sqrt{T}}\right)$, have complexity of $O\left(\frac{1}{\sqrt{T}}\right)$, and thus call for $O\left(\frac{1}{\sqrt{T}}\right)$ iterations to achieve $\delta$-accurate solutions.

**Theorem 4.6. [Non-strongly convex]** Suppose $f(x)$ is a convex function that satisfies Assumption [1] and $v_t \geq v_{t-1}$, $\forall t$. Assume that $E[\|x_m - x_n\|^2] \leq D$, $\forall m \neq n$, let $\eta_t = \eta = O\left(\frac{1}{\sqrt{T}}\right)$. ADAM and SADAM have $f(x_t) - f^* \leq O\left(\frac{1}{\sqrt{T}}\right)$, where $\tilde{x}_t = \frac{1}{T} \sum_{s=1}^{T} x_t$.

**Remark 4.7.** The accurate convergence rate will be $O\left(\frac{d \beta}{\sqrt{T}^{\frac{3}{2}}}\right)$ for ADAM and $O\left(\frac{d \beta^2}{\sqrt{T}}\right)$ for SADAM with fixed $L$, $\sigma$, $G$, $\beta_1$, $D$, $D_\infty$. Some works may specify additional sparsity assumptions on stochastic gradients, and in other words, require $\sum_{t=1}^{T} \sum_{j=1}^{d} \|g_{t,j}\| \leq \sqrt{dT}$ [6] [17] [26] [13] to reduce the order from $d$ to $\sqrt{d}$. Some works may use the element-wise bounds $\sigma_j$ or $G_j$, and apply $\sum_{j=1}^{d} \sigma_j = \sigma$ and $\sum_{j=1}^{d} G_j = G$ to hide $d$. In our work, we do not assume sparsity, so we use $\sigma$ and $G$ throughout the proof. Otherwise, those techniques can also be used to hide $d$ from our convergence rate.

**Corollary 4.7.1.** If $\epsilon$ or $\beta$ is treated as a constant, the ADAM, SADAM (and SAMSGRAD) methods with fixed $L$, $\sigma$, $G$, $\beta_1$, and $\eta = O\left(\frac{1}{\sqrt{T}}\right)$ in the convex case will call for $O\left(\frac{1}{\sqrt{T}}\right)$ iterations to achieve $\delta$-accurate solutions.

**Theorem 4.8. [P-L condition]** Suppose $f(x)$ satisfies the P-L condition (with parameter $\lambda$) and Assumption [1] in the convex case, and $v_t \geq v_{t-1}$, $\forall t$. Let $\eta_t = \eta = O\left(\frac{1}{\sqrt{T}}\right)$.

- **ADAM has the convergence rate:** $E[\|f(x_{T+1}) - f^*\|] \leq \left(1 - \frac{2 \mu_1}{\lambda}\right)^T E[\|f(x_1) - f^*\|] + O\left(\frac{1}{\sqrt{T}}\right)$;
- **SADAM has the convergence rate:** $E[\|f(x_{T+1}) - f^*\|] \leq \left(1 - \frac{2 \mu_2}{\lambda}\right)^T E[\|f(x_1) - f^*\|] + O\left(\frac{1}{\sqrt{T}}\right)$.

The P-L condition is weaker than strongly convex, and for the strongly-convex case, we have:
with S-Momentum, leaving a clear generalization gap exactly same as what is previously reported. The CIFAR-10 dataset is tested on Residual Neural Network with 20 layers (ResNets 20) and 56 layers (ResNets 56) [8], and DenseNets with 40 layers [9] respectively. We train CNN on the MNIST dataset is tested on a CNN with 5 hidden layers (architecture details are provided in Appendix). Experimental setup. We use two datasets for image classifications: MNIST and CIFAR-10. The MNIST dataset is tested on a CNN with 5 hidden layers (architecture details are provided in Appendix). All algorithms perform grid search for hyper-parameters to choose from \{1, 0.1, 0.01, 0.001, 0.0001\} for B-LR, \{0.9, 0.99\} for \(\beta_1\) and \{0.99, 0.999\} for \(\beta_2\). For algorithm-specific hyper-parameters, the recommended values are used, such as \(p = \frac{3}{8}\) in PADAM and PAMSGRAD, dynamic bounds \(\eta_t = 0.1 - \frac{0.1}{(1 - \beta_1) t + 1}\), \(\eta_r(t) = 0.1 + \frac{0.1}{(1 - \beta_1) t + 1}\) for ADABOUND and AMSBOUND. Notice that, comparison under such a setting is fair because: i) our framework can be used on top of any existing AGMs, and fully recover the original AGM by adjusting \(\beta\) to be large (such as \(\beta = 1000\) or larger) and achieve higher test accuracy by adjusting \(\beta\); ii) \(\beta\) is fixed to 50 in SADAM and SAMSGrad for fair comparison, though we do observe fine-tuning \(\beta\) can achieve better test accuracy most of the time. All experiments are repeated for 6 times to obtain the mean and standard deviation for each algorithm.

MNIST. Figure 4 shows the learning curve for all baseline algorithms and our algorithms on both training and test datasets. As expected, all methods can reach the zero loss quickly, while for test accuracy, our SAMSGrad shows increase in test accuracy and outperforms competitors.

CIFAR-10. Using the PyTorch framework, we first run the ResNets 20 model and results are shown in Table I. The original AGMs (ADAM and AMSGrad) have lower test accuracy in comparison with S-Momentum, leaving a clear generalization gap exactly same as what is previously reported. By adjusting \(c\), [24] shows the generalization gap can be reduced. PADAM and PAMSGrad close the generalization gap by tuning the parameter \(p\). For our methods, SADAM and SAMSGrad

5 Experiments

In this section, we empirically compare SADAM and SAMSGrad against several state-of-the-art optimizers including S-Momentum, ADAM, AMSGrad, Yogi, PADAM, PAMSGrad, ADABOUND, and AMSBOUND. Our methods show great efficacy on several standard benchmarks in both training and testing results, and outperform most optimizers in terms of generalization performance.

Experimental setup. We use two datasets for image classifications: MNIST and CIFAR-10. The MNIST dataset is tested on a CNN with 5 hidden layers (architecture details are provided in Appendix). The CIFAR-10 dataset is tested on Residual Neural Network with 20 layers (ResNets 20) and 56 layers (ResNets 56) [8], and DenseNets with 40 layers [9] respectively. We train CNN on the MNIST data for 100 epochs and ResNets/DenseNets on CIFAR-10 for 300 epochs, with a weight decay factor of \(5 \times 10^{-4}\) and a batch size of 128. For the CIFAR-10 tasks, we use a fixed multi-stage LR decaying scheme: the B-LR decays by 0.1 at the 150-th epoch and 225-th epoch, which is a popular decaying scheme and used in many works [11][18]. All algorithms perform grid search for hyper-parameters to choose from \{1, 0.1, 0.01, 0.001, 0.0001\} for B-LR, \{0.9, 0.99\} for \(\beta_1\) and \{0.99, 0.999\} for \(\beta_2\). For algorithm-specific hyper-parameters, the recommended values are used, such as \(p = \frac{3}{8}\) in PADAM and PAMSGrad, dynamic bounds \(\eta_t = 0.1 - \frac{0.1}{(1 - \beta_1) t + 1}\), \(\eta_r(t) = 0.1 + \frac{0.1}{(1 - \beta_1) t + 1}\) for ADABOUND and AMSBOUND. Notice that, comparison under such a setting is fair because: i) our framework can be used on top of any existing AGMs, and fully recover the original AGM by adjusting \(\beta\) to be large (such as \(\beta = 1000\) or larger) and achieve higher test accuracy by adjusting \(\beta\); ii) \(\beta\) is fixed to 50 in SADAM and SAMSGrad for fair comparison, though we do observe fine-tuning \(\beta\) can achieve better test accuracy most of the time. All experiments are repeated for 6 times to obtain the mean and standard deviation for each algorithm.

MNIST. Figure 4 shows the learning curve for all baseline algorithms and our algorithms on both training and test datasets. As expected, all methods can reach the zero loss quickly, while for test accuracy, our SAMSGrad shows increase in test accuracy and outperforms competitors.

CIFAR-10. Using the PyTorch framework, we first run the ResNets 20 model and results are shown in Table I. The original AGMs (ADAM and AMSGrad) have lower test accuracy in comparison with S-Momentum, leaving a clear generalization gap exactly same as what is previously reported. By adjusting \(c\), [24] shows the generalization gap can be reduced. PADAM and PAMSGrad close the generalization gap by tuning the parameter \(p\). For our methods, SADAM and SAMSGrad

\[\text{Corollary 4.8.1. [strongly convex]} \] Suppose \(f(x)\) is \(\mu\)-strongly convex function that satisfies Assumption 7 and \(v_t \geq v_{t-1} \forall t\). Let \(\eta_t = \eta = O\left(\frac{1}{T}\right)\).

- ADAM has the convergence rate: \(E[f(x_{T+1}) - f^*] \leq (1 - 2\mu T) E[f(x_1) - f^*] + O\left(\frac{1}{T}\right)\);
- SADAM has the convergence rate: \(E[f(x_{T+1}) - f^*] \leq (1 - 2\mu T) E[f(x_1) - f^*] + O\left(\frac{1}{T}\right)\).

In summary, our methods share the same convergence rate as ADAM, and enjoy even better convergence speed if comparing the common values chosen for the parameters \(\epsilon\) and \(\beta\). Our convergence rate recovers that of SGD and S-Momentum in terms of \(T\) for a small \(\beta\).
clearly close the gap, and SADAM achieves the best test accuracy among competitors. We further test all methods on ResNets 56 with greater network depth, and the overall performance of each algorithm has been improved. The proposed methods successfully address the bad generalization performance of AGMs and achieve the best test accuracy. For the experiments with DenseNets, we use a DenseNet with 40 layers and a growth rate $k = 12$ without bottleneck, channel reduction, or dropout. The results are reported in the last column of Table 1. SAMSGRAD still achieves the best test performance, and the proposed two methods largely improve the performance of ADAM and AMSGRAD and close the gap with S-Momentum.

## 6 Conclusion

We have examined adaptive gradient methods from a new perspective that is driven by the observation that the adaptive learning rates are anisotropic at each iteration. We propose to calibrate the adaptive learning rates using a softplus function. We combine this calibration scheme with the widely-used ADAM and AMSGRAD methods and empirical evaluations show obvious improvement on their generalization performance in multiple deep learning tasks. Using this calibration scheme, we replace the hyper-parameter $\epsilon$ in the original methods by a new parameter $\beta$ in the softplus function. A new mathematical model has been proposed to analyze the convergence of adaptive gradient methods. Our analysis shows that the convergence rate is related to $\epsilon$ or $\beta$, which has not been previously revealed, and the dependence on $\epsilon$ or $\beta$ helps us justify the advantage of the proposed methods. In the future, the calibration scheme can be designed based on other suitable activation functions, and used in conjunction with any other adaptive gradient method to improve generalization performance.
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Appendix

7 More empirical results

7.1 A-LR range of ADAM-type methods

Besides the A-LR range of ADAM method, which has shown in main paper, we further want to study more other ADAM-type methods, and do experiments focus on AMSGRAD, PADAM, and PAMSGRAD on different tasks (Figure 5, 6, and 7). AMSGRAD also has extreme large-valued coordinates, and will encounter the "small learning rate dilemma" as well as ADAM. With partial parameter $p$, the value range of A-LR can be largely narrow down, and the maximum range will be reduced around $10^2$ with PADAM, and less than $10^2$ with PAMSGRAD. This reduced range, avoiding the "small learning rate dilemma", may help us understand what "trick" works on ADAM’s A-LR can indeed improve the generalization performance. Besides, the range of A-LR in YOGI, ADABOUND and AMSBOUND will be reduced or controlled by specific $\epsilon$ or clip function, we don’t show more information here.

Figure 5: A-LR range of AMSGRAD (a), PADAM (b), and PAMSGRAD (c) on MNIST.

Figure 6: A-LR range of AMSGRAD (a), PADAM (b), and PAMSGRAD (c) on ResNets 20.

Figure 7: A-LR range of AMSGRAD (a), PADAM (b), and PAMSGRAD (c) on DenseNets.
7.2 Parameter $\beta$ reduces the range of A-LR

The main paper has discussed about \textit{softplus} function, and mentions that it does help to constrain large-valued coordinates in A-LR while keep others untouched, here we give more empirical support. No matter how does $\beta$ set, the modified A-LR will have a reduced range. By setting various $\beta$’s, we can find an appropriate $\beta$ that performs the best for specific tasks on datasets. Besides the results of A-LR range of SADAM on MNIST with different choices of $\beta$, we also study SADAM and SAMSGRAD on ResNets 20 and DenseNets.

![Figure 8: The range of A-LR: $1/\text{softplus}(\sqrt{v_t})$ over iterations for SADAM on MNIST with different choices of $\beta$. The maximum ranges in all figures are compressed to a reasonable smaller value compared with $10^8$.](image)

![Figure 9: The range of A-LR: $1/\text{softplus}(\sqrt{v_t})$, $v_t = max\{v_{t-1}, \tilde{v}_t\}$ over iterations for SAMSGRAD on MNIST with different choice of $\beta$. The maximum ranges in all figures are compressed to a reasonable smaller value compared with those of AMSGRAD on MNIST.](image)
Figure 10: The range of A-LR: $\frac{1}{\text{softplus}(\sqrt{v_t})}$ over iterations for SADAM on ResNets 20 with different choices of $\beta$.

Figure 11: The range of A-LR: $\frac{1}{\text{softplus}(\sqrt{v_t})}$, $v_t = \max\{v_{t-1}, \tilde{v}_t\}$ over iterations for SAMSGRAD on ResNets 20 with different choices of $\beta$. 
Here we do grid search to choose appropriate $\beta$ from $\{10, 50, 100, 200, 500, 1000\}$. In summary, with $\text{softplus}$ function, SADAM and SAMSGRAD will narrow down the range of A-LR, make the A-LR vector more regular, avoiding "small learning rate dilemma" and finally achieve better performance.
7.3 Parameter $\beta$ matters in both training and testing

After studying existing ADAM-type methods, and effect of different $\beta$ in adjusting A-LR, we focus on the training and testing accuracy of our softplus framework, especially SADAM and SAMSGRAD, with different choices of $\beta$.

![Figure 14: Performance of SADAM on CIFAR-10 with different choice of $\beta$.](image1)

![Figure 15: Performance of SAMSGRAD on CIFAR-10 with different choice of $\beta$.](image2)
8 Architecture used in our experiments

Here we mainly introduce the MNIST architecture with Pytorch used in our empirical study, ResNets and DenseNets are well-known architectures used in many works and we do not include details here.

| layer setting | layer |
|---------------|-------|
| self.conv1 = nn.Conv2d(1, 6, 5) | F.relu(self.conv1(x)) |
| self.conv2 = nn.Conv2d(6, 16, 5) | F.relu(self.conv2(x)) |
| x.view(-1, 16*4) | F.relu(self.fc1(x)) |
| self.fc1 = nn.Linear(16*4*4, 120) | F.relu(self.fc2(x)) |
| self.fc2 = nn.Linear(120, 84) | x = self.fc3(x) |
| self.fc3 = nn.Linear(84, 10) | F.log_softmax(x, dim=1) |

9 Theoretical analysis details

We analyze the convergence rate of ADAM and SADAM under different cases, and derive competitive results of our methods. The following table gives an overview of stochastic gradient methods convergence rate under various conditions, in our work we provide a different way of proof compared with previous works and also associate the analysis with hyperparameters of ADAM methods.

| Optimizer | SC (P-L condition) | Convex | Nonconvex |
|-----------|-------------------|--------|-----------|
| SGD [7, 21] | $O(\frac{1}{T})$ | $O(\frac{1}{\sqrt{T}})$ | $O(\frac{1}{\sqrt{T}})$ |
| S-Momentum [20, 22] | $O(\frac{1}{T})$ | $O(\frac{1}{\sqrt{T}})$ | $O(\frac{1}{\sqrt{T}})$ |
| RMSProp [1, 24] | $O(\frac{1}{\sqrt{T}})$ | $O(\frac{1}{\sqrt{T}})$ | $O(\frac{1}{\sqrt{T}})$ |
| ADAM [12, 4] | $O(\frac{1}{T})$ | $O(\frac{1}{\sqrt{T}})$ | $O(\frac{1}{\sqrt{T}})$ |
| ADAM (Our work) | $O(\frac{1}{T})$ | $O(\frac{1}{\sqrt{T}})$ | $O(\frac{1}{\sqrt{T}})$ |
| SADAM (Our work) | $O(\frac{1}{T})$ | $O(\frac{1}{\sqrt{T}})$ | $O(\frac{1}{\sqrt{T}})$ |

9.1 Prepared lemmas

**Lemma 9.1.** For any vectors $a, b, c \in \mathbb{R}^d$, $< a, b \odot c > = < a \odot \sqrt{b}, c \odot \sqrt{b} >$, here $\odot$ is element-wise product, $\sqrt{b}$ is element-wise square root.

**Proof.**

\[
< a, b \odot c > = \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} \begin{pmatrix} b_1 c_1 \\ \vdots \\ b_d c_d \end{pmatrix} = a_1 b_1 c_1 + \cdots + a_d b_d c_d
\]

\[
< a \odot \sqrt{b}, c \odot \sqrt{b} > = \begin{pmatrix} a_1 \sqrt{b_1} \\ \vdots \\ a_d \sqrt{b_d} \end{pmatrix} \begin{pmatrix} \sqrt{b_1} c_1 \\ \vdots \\ \sqrt{b_d} c_d \end{pmatrix} = a_1 b_1 c_1 + \cdots + a_d b_d c_d
\]
Lemma 9.2. For any vector $a$, we have
\[ \|a^2\|_\infty \leq \|a\|^2. \]  

(2)

Lemma 9.3. For unbiased stochastic gradient, we have
\[ E[\|g_t\|^2] \leq \sigma^2 + G^2. \]  

(3)

Proof. From gradient bounded assumption and variance bounded assumption,
\[ E[\|g_t\|^2] = E[\|g_t - \nabla f(x_t) + \nabla f(x_t)\|^2] = E[\|g_t - \nabla f(x_t)\|^2 + \|\nabla f(x_t)\|^2] \leq \sigma^2 + G^2. \]

\[ \Box \]

Lemma 9.4. All momentum-based optimizers using first momentum $m_t = \beta_1 m_{t-1} + (1 - \beta_1)g_t$ will satisfy
\[ E[\|m_t\|^2] \leq \sigma^2 + G^2. \]  

(4)

Proof. From the updating rule of first momentum estimator, we can derive
\[ m_t = \Sigma^t_{i=1} (1 - \beta_1)\beta_1^{t-i} g_i. \]  

(5)

Let $\Gamma_t = \Sigma^t_{i=1} \beta_1^{t-i} = \frac{1 - \beta_1^t}{1 - \beta_1}$, by Jensen inequality and Lemma 9.3,
\[ E[\|m_t\|^2] = E[\|\Sigma^t_{i=1} (1 - \beta_1)\beta_1^{t-i} g_i\|^2] = \Gamma_t^2 E[\|\Sigma^t_{i=1} (1 - \beta_1)\beta_1^{t-i} g_i\|^2] \leq \Gamma_t^2 \Sigma^t_{i=1} (1 - \beta_1)^2 \beta_1^{t-i} \Gamma_i \leq \Gamma_t (1 - \beta_1)^2 \Sigma^t_{i=1} \beta_1^{t-i} (\sigma^2 + G^2) \leq \sigma^2 + G^2. \]

\[ \Box \]

Lemma 9.5. Each coordinate of vector $v_t = \beta_2 v_{t-1} + (1 - \beta_2)g_t^2$ will satisfy
\[ E[v_{t,j}] \leq \sigma^2 + G^2, \]  

where $j \in [1, d]$ is the coordinate index.

Proof. From the updating rule of second momentum estimator, we can derive
\[ v_{t,j} = \Sigma^t_{i=1} (1 - \beta_2)\beta_2^{t-i} g_{i,j}^2 \geq 0. \]  

(6)

Since the decay parameter $\beta_2 \in [0, 1)$, $\Sigma^t_{i=1} (1 - \beta_2)\beta_2^{t-i} = 1 - \beta_2^t \leq 1$. From Lemma 9.3,
\[ E[v_{t,j}] = E[\Sigma^t_{i=1} (1 - \beta_2)\beta_2^{t-i} g_{i,j}^2] \leq \Sigma^t_{i=1} (1 - \beta_2)\beta_2^{t-i} (\sigma^2 + G^2) \leq \sigma^2 + G^2. \]

\[ \Box \]

This relaxation can be applied to all AGMs, also including ADAGRAD. In fact, this upper bound actually are not the same in ADAM with exponentionally weighted average (i.e., $\Sigma^t_{i=1} (1 - \beta_2)\beta_2^{t-i} g_{i,j}^2$) and ADAGRAD with geometrically sum (i.e., $\Sigma^t_{i=1} g_{i,j}^2$), because ADAM will have tighter $E[v_{t,j}]$ due to momentum parameter (i.e., $\beta_2 \in [0, 1]$) compared with ADAGRAD’s, here we use the same relaxation to get a unified result. Then in the following lemma, ADAM actually have larger $\mu_1$ compared with ADAGRAD. Take the following nonconvex analysis for example, ADAM will have tighter bound compared with ADAGRAD dues to larger $\mu_1$ and unchanged $\mu_2$, verifying that exponentially weighted moving average do help in the learning rate scheme.

And we can derive the following important lemma:
Lemma 9.6. (Bounded A-LR) For any $t \geq 1, j \in [1, d], \beta_2 \in [0, 1]$, and fixed $\epsilon$ in ADAM and $\beta$ defined in softplus function in SADAM, the following bounds always hold:

ADAM has $(\mu_1, \mu_2)$—bounded A-LR:

$$\mu_1 \leq \frac{1}{\sqrt{v_{t,j}^l + \epsilon}} \leq \mu_2;$$  (7)

SADAM has $(\mu_3, \mu_4)$—bounded A-LR:

$$\mu_3 \leq \frac{1}{\text{softplus}(\sqrt{v_{t,j}^l})} \leq \mu_4;$$  (8)

where $0 < \mu_1 \leq \mu_2, 0 < \mu_3 \leq \mu_4$. For brevity, we use $\mu_1, \mu_u$ denoting the lower bound and upper bound respectively, and both ADAM and SADAM will be analysis with the help of $(\mu_1, \mu_u)$.

Proof. For ADAM, let $\mu_1 = \frac{1}{\sqrt{\sigma_t + \epsilon^2}}, \mu_2 = \frac{1}{\sigma_t}$, then we can get the result in [1].

For SADAM, notice that $\text{softplus}()$ is a monotone increasing function, and $\sqrt{v_{t,j}^l}$ is both upper-bounded and lower-bounded, then we have [8], where $\mu_3 = \frac{1}{\frac{1}{\beta} \log(1 + e^{\beta - 1})}, \mu_4 = \frac{1}{\frac{1}{\beta} \log(1 + e^{\beta - 1})}$. □

Lemma 9.7. Define $z_t = x_t + \frac{\beta_t}{1 - \beta_1}(x_t - x_{t-1}), \forall t \geq 1, 1 \leq 1, 1 \leq 0, 1 \leq 1$. Let $\eta_t = \eta_t$ then the following updating formulas hold:

Gradient-based optimizer

$$z_t = x_t, \quad z_{t+1} = z_t - \eta g_t;$$  (9)

ADAM optimizer

$$z_{t+1} = z_t + \frac{\eta \beta_t}{1 - \beta_1} \left( \frac{1}{\sqrt{v_{t-1}^l} + \epsilon} - \frac{1}{\sqrt{v_t^l} + \epsilon} \right) \odot m_{t-1} - \frac{\eta}{\sqrt{v_t^l} + \epsilon} \odot g_t;$$  (10)

SADAM optimizer

$$z_{t+1} = z_t + \frac{\eta \beta_t}{1 - \beta_1} \left( \frac{1}{\text{softplus}(\sqrt{v_{t-1}^l})} - \frac{1}{\text{softplus}(\sqrt{v_t^l})} \right) \odot m_{t-1} - \frac{\eta}{\text{softplus}(\sqrt{v_t^l})} \odot g_t. \quad \text{(11)}$$

Proof. We consider the ADAM optimizer and let $\beta_1 = 0$, we can easily derive the gradient-based case.

$$z_{t+1} = z_t + \frac{\beta_t}{1 - \beta_1} (x_{t+1} - x_t)$$

$$z_{t+1} = z_t + \frac{\eta}{1 - \beta_1} \left( \frac{1}{\sqrt{v_{t-1}^l} + \epsilon} \odot m_{t-1} - \frac{\eta}{1 - \beta_1} \sqrt{v_{t-1}^l} + \epsilon \odot g_t \right)$$

$$= z_t + \frac{\eta \beta_t}{1 - \beta_1} \left( \frac{1}{\text{softplus}(\sqrt{v_{t-1}^l})} \odot m_{t-1} - \frac{1}{\text{softplus}(\sqrt{v_t^l})} \odot g_t \right).$$

Similarly, consider the SADAM optimizer:

$$z_{t+1} = z_t + \frac{\beta_t}{1 - \beta_1} (x_{t+1} - x_t) - \frac{\beta_t}{1 - \beta_1} (x_t - x_{t-1})$$

$$= z_t + \frac{\eta}{1 - \beta_1} \left( \frac{1}{\text{softplus}(\sqrt{v_t^l})} \odot m_t - \frac{\beta_t}{1 - \beta_1} \text{softplus}(\sqrt{v_{t-1}^l}) \odot m_{t-1} \right)$$

$$= z_t + \frac{\eta \beta_t}{1 - \beta_1} \left( \frac{1}{\text{softplus}(\sqrt{v_{t-1}^l})} - \frac{1}{\text{softplus}(\sqrt{v_t^l})} \right) \odot m_{t-1} - \frac{\eta}{\text{softplus}(\sqrt{v_t^l})} \odot g_t.$$ □
Lemma 9.8. As defined in Lemma 9.7, with the condition that \( v_t \geq v_{t-1} \), i.e., AMSGrad and SAMSGGrad, we can derive the bound of distance of \( \|z_{t+1} - z_t\|^2 \) as follows:

**ADAM optimizer**

\[
E[\|z_{t+1} - z_t\|^2] \leq \frac{2\eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E[\sum_{j=1}^{d} \left( \frac{1}{\sqrt{v_{t-1,j}} + \epsilon} \right)^2 - \left( \frac{1}{\sqrt{v_{t,j}} + \epsilon} \right)^2] + 2\eta^2 \mu_2^2 (\sigma^2 + G^2) \tag{12}
\]

**SAMSG optimizer**

\[
E[\|z_{t+1} - z_t\|^2] \leq \frac{2\eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E[\sum_{j=1}^{d} \left( \frac{1}{\text{softplus}(\sqrt{v_{t-1,j}})} \right)^2 - \left( \frac{1}{\text{softplus}(\sqrt{v_{t,j}})} \right)^2] + 2\eta^2 \mu_2^2 (\sigma^2 + G^2) \tag{13}
\]

**Proof.** ADAM case:

\[
E[\|z_{t+1} - z_t\|^2] = E[\left\| \frac{\eta \beta_1}{1 - \beta_1} \left( \frac{1}{\sqrt{v_{t-1}} + \epsilon} - \frac{1}{\sqrt{v_t} + \epsilon} \right) \odot m_{t-1} - \frac{\eta}{\sqrt{v_t} + \epsilon} \odot g_t \right\|^2] \\
\leq 2E[\left\| \frac{\eta \beta_1}{1 - \beta_1} \left( \frac{1}{\sqrt{v_{t-1}} + \epsilon} - \frac{1}{\sqrt{v_t} + \epsilon} \right) \odot m_{t-1} \right\|^2 + 2E[\left\| \frac{\eta}{\sqrt{v_t} + \epsilon} \odot g_t \right\|^2] \\
\leq \frac{2\eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E[\sum_{j=1}^{d} \left( \frac{1}{\sqrt{v_{t-1,j}} + \epsilon} - \frac{1}{\sqrt{v_{t,j}} + \epsilon} \right)^2] + 2\eta^2 \mu_2^2 (\sigma^2 + G^2) \\
\leq \frac{2\eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E[\sum_{j=1}^{d} \left( \frac{1}{\text{softplus}(\sqrt{v_{t-1,j}})} \right)^2 - \left( \frac{1}{\text{softplus}(\sqrt{v_{t,j}})} \right)^2] + 2\eta^2 \mu_2^2 (\sigma^2 + G^2)
\]

The first inequality holds because \( \|a - b\|^2 \leq 2\|a\|^2 + 2\|b\|^2 \), the second inequality holds because Lemma 9.3 and 9.4 and Lemma 9.6, the third inequality holds because \( (a - b)^2 \leq a^2 - b^2 \) when \( a \geq b \), and in our assumption, we have \( v_t \geq v_{t-1} \) holds.

**SAMSG case:**

\[
E[\|z_{t+1} - z_t\|^2] = E\left[\left\| \frac{\eta \beta_1}{1 - \beta_1} \left( \frac{1}{\text{softplus}(\sqrt{v_{t-1}})} - \frac{1}{\text{softplus}(\sqrt{v_t})} \right) \odot m_{t-1} - \frac{\eta}{\text{softplus}(\sqrt{v_t})} \odot g_t \right\|^2\right] \\
\leq 2E\left[\left\| \frac{\eta \beta_1}{1 - \beta_1} \left( \frac{1}{\text{softplus}(\sqrt{v_{t-1}})} - \frac{1}{\text{softplus}(\sqrt{v_t})} \right) \odot m_{t-1} \right\|^2 \right] \\
+ 2E[\left\| \frac{\eta}{\text{softplus}(\sqrt{v_t})} \odot g_t \right\|^2] \\
\leq \frac{2\eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E\left[\sum_{j=1}^{d} \left( \frac{1}{\text{softplus}(\sqrt{v_{t-1,j}})} - \frac{1}{\text{softplus}(\sqrt{v_{t,j}})} \right)^2\right] + 2\eta^2 \mu_2^2 (\sigma^2 + G^2) \\
\leq \frac{2\eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E\left[\sum_{j=1}^{d} \left( \frac{1}{\text{softplus}(\sqrt{v_{t-1,j}})} \right)^2 - \left( \frac{1}{\text{softplus}(\sqrt{v_{t,j}})} \right)^2\right] + 2\eta^2 \mu_2^2 (\sigma^2 + G^2)
\]

Because the **softplus** function is monotone increasing function, therefore, the third inequality holds as well. □
**Lemma 9.9.** As defined in Lemma [9.7] with the condition that \( v_t \geq v_{t-1} \), we can derive the bound of the inner product as follows:

**ADAM optimizer**

\[
- \mathbb{E}[\langle \nabla f(z_t) - \nabla f(x_t), \eta \frac{g_t}{\sqrt{v_t} + \epsilon} \rangle] \leq \frac{1}{2} L^2 \eta^2 \mu_2^2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 (\sigma^2 + G^2) + \frac{1}{2} L^2 \eta^2 \mu_2 (\sigma^2 + G^2);
\]

(14)

**SADAM optimizer**

\[
- \mathbb{E}[\langle \nabla f(z_t) - \nabla f(x_t), \eta \text{softplus}(\sqrt{v_t}) \circ g_t \rangle] \leq \frac{1}{2} L^2 \eta^2 \mu_2^2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 (\sigma^2 + G^2) + \frac{1}{2} L^2 \eta^2 \mu_2 (\sigma^2 + G^2).
\]

(15)

**Proof.** Since the stochastic gradient is unbiased, then we have \( \mathbb{E}[g_t] = \nabla f(x_t) \).

**ADAM case:**

\[
- \mathbb{E}[\langle \nabla f(z_t) - \nabla f(x_t), \eta \frac{g_t}{\sqrt{v_t} + \epsilon} \rangle] \\
\quad \leq \frac{1}{2} \mathbb{E}[\| \nabla f(z_t) - \nabla f(x_t) \|^2] + \frac{1}{2} \mathbb{E}[\| \eta \frac{g_t}{\sqrt{v_t} + \epsilon} \|^2] \\
\quad \leq \frac{L^2}{2} \mathbb{E}[\| z_t - x_t \|^2] + \frac{1}{2} \mathbb{E}[\| \eta \frac{g_t}{\sqrt{v_t} + \epsilon} \|^2] \\
\quad = \frac{L^2}{2} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 \mathbb{E}[\| x_t - x_{t-1} \|^2] + \frac{1}{2} \mathbb{E}[\| \eta \frac{g_t}{\sqrt{v_t} + \epsilon} \|^2] \\
\quad = \frac{L^2}{2} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 \mathbb{E}[\| \eta \frac{g_t}{\sqrt{v_t} + \epsilon} \|^2] \| m_{t-1} \|^2] + \frac{1}{2} \mathbb{E}[\| \eta \frac{g_t}{\sqrt{v_t} + \epsilon} \|^2] \\
\quad \leq \frac{1}{2} L^2 \eta^2 \mu_2^2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 (\sigma^2 + G^2) + \frac{1}{2} L^2 \eta^2 \mu_2 (\sigma^2 + G^2)
\]

The first inequality holds because \( \frac{1}{2} a^2 + \frac{1}{4} b^2 \geq - \langle a, b \rangle \), the second inequality holds for L-smoothness, the last inequalities hold due to Lemma [9.4] and [9.6].

Similarly, for **SADAM**, we also have the following result:

\[
- \mathbb{E}[\langle \nabla f(z_t) - \nabla f(x_t), \eta \text{softplus}(\sqrt{v_t}) \circ g_t \rangle] \\
\quad \leq \frac{1}{2} \mathbb{E}[\| \nabla f(z_t) - \nabla f(x_t) \|^2] + \frac{1}{2} \mathbb{E}[\| \eta \text{softplus}(\sqrt{v_t}) \circ g_t \|^2] \\
\quad \leq \frac{L^2}{2} \mathbb{E}[\| z_t - x_t \|^2] + \frac{1}{2} \mathbb{E}[\| \eta \text{softplus}(\sqrt{v_t}) \circ g_t \|^2] \\
\quad = \frac{L^2}{2} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 \mathbb{E}[\| x_t - x_{t-1} \|^2] + \frac{1}{2} \mathbb{E}[\| \eta \text{softplus}(\sqrt{v_t}) \circ g_t \|^2] \\
\quad = \frac{L^2}{2} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 \mathbb{E}[\| \eta \text{softplus}(\sqrt{v_t}) \circ g_t \|^2] \| m_{t-1} \|^2] + \frac{1}{2} \mathbb{E}[\| \eta \text{softplus}(\sqrt{v_t}) \circ g_t \|^2] \\
\quad \leq \frac{1}{2} L^2 \eta^2 \mu_2^2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 (\sigma^2 + G^2) + \frac{1}{2} L^2 \eta^2 \mu_2 (\sigma^2 + G^2).
\]

\[\square\]

### 9.2 ADAM convergence in nonconvex setting

**Proof.** All the analyses hold true under the condition: \( v_t \geq v_{t-1} \). From L-smoothness and Lemma [9.7] we have
Applying the bound of $m_t$ and $\nabla f(z_t)$,

$$f(z_{t+1}) \leq f(z_t) + \langle \nabla f(z_t), z_{t+1} - z_t \rangle + \frac{L}{2} \|z_{t+1} - z_t\|^2$$

$$= f(z_t) + \frac{\eta \beta_1}{1 - \beta_1} (\nabla f(z_t), (\frac{1}{\sqrt{v_{t-1}} + \epsilon} - \frac{1}{\sqrt{v_t} + \epsilon}) \odot m_{t-1})$$

$$- \langle \nabla f(z_t), \frac{\eta}{\sqrt{v_t} + \epsilon} \odot g_t \rangle + \frac{L}{2} \|z_{t+1} - z_t\|^2$$

Take expectation on both sides,

$$E[f(z_{t+1}) - f(z_t)] \leq \frac{\eta \beta_1}{1 - \beta_1} E[\langle \nabla f(z_t), (\frac{1}{\sqrt{v_{t-1}} + \epsilon} - \frac{1}{\sqrt{v_t} + \epsilon}) \odot m_{t-1} \rangle]$$

$$- E[\langle \nabla f(z_t), \frac{\eta}{\sqrt{v_t} + \epsilon} \odot g_t \rangle] + \frac{L}{2} E[\|z_{t+1} - z_t\|^2]$$

$$= \frac{\eta \beta_1}{1 - \beta_1} E[\langle \nabla f(z_t), (\frac{1}{\sqrt{v_{t-1}} + \epsilon} - \frac{1}{\sqrt{v_t} + \epsilon}) \odot m_{t-1} \rangle]$$

$$- E[\langle \nabla f(z_t) - \nabla f(x_t), \frac{\eta}{\sqrt{v_t} + \epsilon} \odot g_t \rangle] - E[\langle \nabla f(x_t), \frac{\eta}{\sqrt{v_t} + \epsilon} \odot g_t \rangle]$$

$$+ \frac{L}{2} E[\|z_{t+1} - z_t\|^2]$$

Plug in the results from prepared lemmas, then we have,

$$E[f(z_{t+1}) - f(z_t)] \leq \frac{\eta \beta_1}{1 - \beta_1} E[\langle \nabla f(z_t), (\frac{1}{\sqrt{v_{t-1}} + \epsilon} - \frac{1}{\sqrt{v_t} + \epsilon}) \odot m_{t-1} \rangle]$$

$$+ \frac{L}{2} \eta^2 \mu_2^2 (\frac{\beta_1}{1 - \beta_1})^2 (\sigma^2 + G^2) + \frac{1}{2} \eta^2 \mu_2^2 (\sigma^2 + G^2) - E[\langle \nabla f(x_t), \frac{\eta}{\sqrt{v_t} + \epsilon} \odot g_t \rangle]$$

$$+ \frac{L \eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E[\sum_{j=1}^{d} (\frac{1}{\sqrt{v_{t-1,j}} + \epsilon})^2 - (\frac{1}{\sqrt{v_{t,j}} + \epsilon})^2] + L \eta^2 \mu_2^2 (\sigma^2 + G^2)$$

Applying the bound of $m_t$ and $\nabla f(z_t)$,

$$E[f(z_{t+1}) - f(z_t)] \leq \frac{\eta \beta_1}{1 - \beta_1} G \sqrt{\sigma^2 + G^2} E[\sum_{j=1}^{d} (\frac{1}{\sqrt{v_{t-1,j}} + \epsilon})^2 - (\frac{1}{\sqrt{v_{t,j}} + \epsilon})^2]$$

$$+ \frac{L}{2} \eta^2 \mu_2^2 (\frac{\beta_1}{1 - \beta_1})^2 (\sigma^2 + G^2) + \frac{1}{2} \eta^2 \mu_2^2 (\sigma^2 + G^2) - E[\langle \nabla f(x_t), \frac{\eta}{\sqrt{v_t} + \epsilon} \odot g_t \rangle]$$

$$+ \frac{L \eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E[\sum_{j=1}^{d} (\frac{1}{\sqrt{v_{t-1,j}} + \epsilon})^2 - (\frac{1}{\sqrt{v_{t,j}} + \epsilon})^2] + L \eta^2 \mu_2^2 (\sigma^2 + G^2)$$

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By rearranging,

\[
E[\langle \nabla f(x_t), \frac{\eta}{\sqrt{\epsilon_t}} \otimes g_t \rangle] \leq E[f(z_t) - f(z_{t+1})] + \frac{\eta \beta_1}{1 - \beta_1} G \sqrt{\sigma^2 + G^2} E[\sum_{j=1}^{d} \frac{1}{\sqrt{\epsilon_{t-1,j}} + \epsilon} - \frac{1}{\sqrt{\epsilon_{t,j}} + \epsilon}]
\]

\[
+ \frac{1}{2} L^2 \eta^2 \mu_2^2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 (\sigma^2 + G^2) + \frac{1}{2} \eta^2 \mu_2^2 (\sigma^2 + G^2)
\]

\[
+ \frac{L \eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E[\sum_{j=1}^{d} \left( \frac{1}{\sqrt{\epsilon_{t-1,j}} + \epsilon} \right)^2 - \left( \frac{1}{\sqrt{\epsilon_{t,j}} + \epsilon} \right)^2] + L \eta^2 \mu_2^2 (\sigma^2 + G^2)
\]

For the LHS above:

\[
E[\langle \nabla f(x_t), \frac{1}{\sqrt{\epsilon_t}} \otimes g_t \rangle] \geq E[\sum_{\{j: \nabla f(x_{t,j})g_{t,j} \geq 0\}} \mu_1 \nabla f(x_{t,j})g_{t,j} + \sum_{\{j: \nabla f(x_{t,j})g_{t,j} < 0\}} \mu_2 \nabla f(x_{t,j})g_{t,j}]
\]

\[
\geq E[\sum_{\{j: \nabla f(x_{t,j})g_{t,j} \geq 0\}} \mu_1 \nabla f(x_{t,j})^2 + \sum_{\{j: \nabla f(x_{t,j})g_{t,j} < 0\}} \mu_2 \nabla f(x_{t,j})^2]
\]

\[
\geq \mu_1 \|\nabla f(x_t)\|^2
\]

Then we obtain:

\[
\eta \mu_1 \|\nabla f(x_t)\|^2 \leq E[f(z_t) - f(z_{t+1})] + \frac{\eta \beta_1}{1 - \beta_1} G \sqrt{\sigma^2 + G^2} E[\sum_{j=1}^{d} \frac{1}{\sqrt{\epsilon_{t-1,j}} + \epsilon} - \frac{1}{\sqrt{\epsilon_{t,j}} + \epsilon}]
\]

\[
+ \frac{1}{2} L^2 \eta^2 \mu_2^2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 (\sigma^2 + G^2) + \frac{1}{2} \eta^2 \mu_2^2 (\sigma^2 + G^2)
\]

\[
+ \frac{L \eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E[\sum_{j=1}^{d} \left( \frac{1}{\sqrt{\epsilon_{t-1,j}} + \epsilon} \right)^2 - \left( \frac{1}{\sqrt{\epsilon_{t,j}} + \epsilon} \right)^2] + L \eta^2 \mu_2^2 (\sigma^2 + G^2)
\]

Divide \( \eta \mu_1 \) on both sides:

\[
\|\nabla f(x_t)\|^2 \leq \frac{1}{\eta \mu_1} E[f(z_t) - f(z_{t+1})] + \frac{\beta_1}{1 - \beta_1} \mu_1 G \sqrt{\sigma^2 + G^2} E[\sum_{j=1}^{d} \frac{1}{\sqrt{\epsilon_{t-1,j}} + \epsilon} - \frac{1}{\sqrt{\epsilon_{t,j}} + \epsilon}]
\]

\[
+ \frac{1}{2 \mu_1} L^2 \eta^2 \mu_2^2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 (\sigma^2 + G^2) + \frac{1}{2 \mu_1} \eta^2 \mu_2^2 (\sigma^2 + G^2)
\]

\[
+ \frac{L \eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2 \mu_1} E[\sum_{j=1}^{d} \left( \frac{1}{\sqrt{\epsilon_{t-1,j}} + \epsilon} \right)^2 - \left( \frac{1}{\sqrt{\epsilon_{t,j}} + \epsilon} \right)^2] + \frac{L \eta^2 \mu_2^2}{\mu_1} (\sigma^2 + G^2)
\]

Summing from \( t = 1 \) to \( T \), where \( T \) is the maximum number of iteration,

\[
\sum_{t=1}^{T} \|\nabla f(x_t)\|^2 \leq \frac{1}{\eta \mu_1} E[f(z_1) - f^*] + \frac{\beta_1}{1 - \beta_1} \mu_1 G \sqrt{\sigma^2 + G^2} E[\sum_{j=1}^{d} \frac{1}{\sqrt{\epsilon_{T-1,j}} + \epsilon} - \frac{1}{\sqrt{\epsilon_{T,j}} + \epsilon}]
\]

\[
+ \frac{T}{2 \mu_1} L^2 \eta^2 \mu_2^2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 (\sigma^2 + G^2) + \frac{T}{2 \mu_1} \eta^2 \mu_2^2 (\sigma^2 + G^2)
\]

\[
+ \frac{L \eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2 \mu_1} E[\sum_{j=1}^{d} \left( \frac{1}{\sqrt{\epsilon_{T-1,j}} + \epsilon} \right)^2 - \left( \frac{1}{\sqrt{\epsilon_{T,j}} + \epsilon} \right)^2] + \frac{T \eta^2 \mu_2^2}{\mu_1} (\sigma^2 + G^2)
\]
Since $v_0 = 0$, $\mu_2 = \frac{1}{\epsilon}$, we have
\[
\sum_{t=1}^{T} \|\nabla f(x_t)\|^2 \leq \frac{1}{\eta t} E[f(z_t) - f^*] + \frac{\beta_1 d}{(1 - \beta_1)\mu_1} G\sqrt{\sigma^2 + G^2(\mu_2 - \mu_1)} \\
+ \frac{T}{2\mu_1} L^2 \eta \mu_2^2 (\frac{\beta_1}{1 - \beta_1})^2 (\sigma^2 + G^2) + \frac{T}{2\mu_1} \eta \mu_2^2 (\sigma^2 + G^2) \\
+ \frac{L\eta \beta_1^2 d(\sigma^2 + G^2)}{(1 - \beta_1)^2 \mu_1^2} (\mu_2^2 - \mu_1^2) + \frac{T L\eta \mu_2^2}{\mu_1} (\sigma^2 + G^2)
\]

Divided by $\frac{1}{T}$,
\[
\frac{1}{T} \sum_{t=1}^{T} \|\nabla f(x_t)\|^2 \leq \frac{1}{\eta t} E[f(z_t) - f^*] + \frac{\beta_1 d}{(1 - \beta_1)\mu_1 T} G\sqrt{\sigma^2 + G^2(\mu_2 - \mu_1)} \\
+ \frac{1}{2\mu_1} L^2 \eta \mu_2^2 (\frac{\beta_1}{1 - \beta_1})^2 (\sigma^2 + G^2) + \frac{1}{2\mu_1} \eta \mu_2^2 (\sigma^2 + G^2) \\
+ \frac{L\eta \beta_1^2 d(\sigma^2 + G^2)}{(1 - \beta_1)^2 \mu_1 T} (\mu_2^2 - \mu_1^2) + \frac{L\eta \mu_2^2}{\mu_1} (\sigma^2 + G^2) \\
\leq \frac{1}{\eta \mu_1 T} E[f(z_t) - f^*] + \frac{\beta_1 d}{(1 - \beta_1)\mu_1 T} (\mu_2 - \mu_1) \\
+ \frac{1}{2\mu_1} L^2 \eta \mu_2^2 (\frac{\beta_1}{1 - \beta_1})^2 + \eta \mu_2^2 \frac{L\beta_1^2 d(\mu_2^2 - \mu_1^2)}{(1 - \beta_1)^2 \mu_1 T} + \frac{L\mu_2^2}{\mu_1} (\sigma^2 + G^2)
\]

The second inequality holds because $G\sqrt{\sigma^2 + G^2} \leq \sigma^2 + G^2$.

Setting $\eta = \frac{1}{\sqrt{T}}$, let $x_0 = x_1$, then $z_1 = x_1$, $f(z_1) = f(x_1)$ we derive the final result:
\[
\min_{t=1,\ldots,T} E[\|\nabla f(x_t)\|^2] \leq \frac{1}{\mu_1 \sqrt{T}} E[f(x_1) - f^*] + \frac{\beta_1 d}{(1 - \beta_1)\mu_1 T (\mu_2 - \mu_1)} \\
+ \frac{L^2 \mu_2^2 (\frac{\beta_1}{1 - \beta_1})^2}{2\mu_1 \sqrt{T}} + \frac{\mu_2^2}{2\mu_1 \sqrt{T}} + \frac{L\beta_1^2 d(\mu_2^2 - \mu_1^2)}{(1 - \beta_1)^2 \mu_1 T \sqrt{T}} + \frac{L\mu_2^2}{\mu_1 \sqrt{T}} (\sigma^2 + G^2) \\
= C_1 \frac{1}{\sqrt{T}} + C_2 \frac{1}{T} + C_3 \frac{1}{T \sqrt{T}}
\]

where
\[
C_1 = \frac{1}{\mu_1} [f(x_1) - f^*] + \frac{L^2 \mu_2^2 (\frac{\beta_1}{1 - \beta_1})^2 + \mu_2^2}{2\mu_1} + \frac{L\mu_2^2}{\mu_1} (\sigma^2 + G^2)
C_2 = \frac{\beta_1 (\mu_2 - \mu_1) d}{(1 - \beta_1)\mu_1},
C_3 = \frac{L\beta_1^2 d(\mu_2^2 - \mu_1^2)}{(1 - \beta_1)^2 \mu_1}.
\]

With fixed $L$, $\sigma$, $G$, $\beta_1$, we have $C_1 = O(\frac{1}{\sqrt{T}})$, $C_2 = O(\frac{d}{T})$, $C_3 = O(\frac{d}{T \sqrt{T}})$. Therefore,
\[
\min_{t=1,\ldots,T} E[\|\nabla f(x_t)\|^2] \leq O\left(\frac{1}{e^2 \sqrt{T}} + \frac{d}{e T} + \frac{d}{e^2 T \sqrt{T}}\right)
\]

Thus, we get the sublinear convergence rate of ADAM in nonconvex setting, which recovers the well-known result of SGD [7] in nonconvex optimization in terms of $T$.  

\[\square\]
Remark 9.10. The leading item from the above convergence is $C_1/\sqrt{T}$, $\epsilon$ plays an essential role in the complexity, and we derive a more accurate order $O(\frac{1}{\sqrt{v_{t+1}}})$. At present, $\epsilon$ is always underestimated and considered to be not associated with accuracy of the solution [24]. However, it is closely related with complexity, and with bigger $\epsilon$, the computational complexity should be better. This also supports the analysis of A-LR: $\frac{1}{\sqrt{v_{t+1}}}$ of ADAM in our main paper.

In some other works, people use $\sigma_i$ or $G_i$ to show all the element-wise bound, and then by applying $\sum_{j=1}^{d} \sigma_i := \sigma$, $\sum_{j=1}^{d} G_i := G$ to hide $d$ in the complexity. Here in our work, we didn’t specify write out $\sigma_i$ or $G_i$, instead we use $\sigma, G$ through all the procedure. Also some works require sparsity to derive a tight bound $\sum_{j=1}^{d} \|g_{t,j}\|^2 \leq \sqrt{d}\|g_t\|^2$, however, this sparsity for ADAM is a little strong, and we don’t use this constraint in our analysis.

9.3 SADAM convergence in nonconvex setting

As SADAM also has constrained bound pair $(\mu_1, \mu_4)$, we can learn from the proof of ADAM method, which provides us a general framework of such kind of adaptive methods.

Similar to the ADAM proof, from L-smoothness and Lemma 9.7, we have

Proof. All the analyses hold true under the condition: $v_t \geq v_{t-1}$. From L-smoothness and Lemma 9.7 we have

$$f(z_{t+1}) \leq f(z_t) + \langle \nabla f(z_t), z_{t+1} - z_t \rangle + \frac{L}{2} \| z_{t+1} - z_t \|^2$$

$$= f(z_t) + \frac{\eta \beta_1}{1 - \beta_1} \langle \nabla f(z_t), \left(1 - \frac{1}{\text{softplus}(\sqrt{v_{t-1}})} \right) \odot m_{t-1} \rangle - \langle \nabla f(z_t), \frac{\eta}{\text{softplus}(\sqrt{v_t})} \odot g_t \rangle + \frac{L}{2} \| z_{t+1} - z_t \|^2$$

Taking expectation on both sides, and plug in the results from prepared lemmas, then we have,

$$E[f(z_{t+1}) - f(z_t)]$$

$$\leq \frac{\eta \beta_1}{1 - \beta_1} E[\langle \nabla f(z_t), \left(1 - \frac{1}{\text{softplus}(\sqrt{v_{t-1}})} \right) \odot m_{t-1} \rangle]$$

$$- E[\langle \nabla f(z_t), \frac{\eta}{\text{softplus}(\sqrt{v_t})} \odot g_t \rangle] + \frac{L}{2} E[\| z_{t+1} - z_t \|^2]$$

$$\leq \frac{\eta \beta_1}{1 - \beta_1} E[\langle \nabla f(z_t), \left(1 - \frac{1}{\text{softplus}(\sqrt{v_{t-1}})} \right) \odot m_{t-1} \rangle]$$

$$- E[\langle \nabla f(z_t), \frac{\eta}{\text{softplus}(\sqrt{v_t})} \odot g_t \rangle]$$

$$+ \frac{\ln^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E \left( \sum_{j=1}^{d} \frac{1}{\text{softplus}(\sqrt{v_{t-1,j}})} \right)^2 - \left( \frac{1}{\text{softplus}(\sqrt{v_{t,j}})} \right)^2] + \ln^2 \mu_4^2 (\sigma^2 + G^2)$$

$$= \frac{\eta \beta_1}{1 - \beta_1} G(\sigma^2 + G^2) E \left( \sum_{j=1}^{d} \frac{1}{\text{softplus}(\sqrt{v_{t-1,j}})} \right) \odot \left( \frac{1}{\text{softplus}(\sqrt{v_{t,j}})} \right)$$

$$- E[\langle \nabla f(z_t) - \nabla f(x_t), \frac{\eta}{\text{softplus}(\sqrt{v_t})} \odot g_t \rangle] - E[\langle \nabla f(x_t), \frac{\eta}{\text{softplus}(\sqrt{v_t})} \odot g_t \rangle]$$

$$+ \frac{\ln^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E \left( \sum_{j=1}^{d} \frac{1}{\text{softplus}(\sqrt{v_{t-1,j}})} \right)^2 - \left( \frac{1}{\text{softplus}(\sqrt{v_{t,j}})} \right)^2] + \ln^2 \mu_4^2 (\sigma^2 + G^2)$$
\[ \begin{align*}
\eta \mu & \quad \text{iteration,} \\
\eta \mu & \quad v \\
\text{Since} \\
\text{For the LHS above:} \\
\text{By rearranging,} \\
E[(\nabla f(x_t), \eta \text{softplus}(\sqrt{v_{t,j}}) \odot g_t)] & \leq E[f(z_t) - f(z_{t+1})] + \frac{\eta \beta_1}{1 - \beta_1} G \sqrt{\sigma^2 + G^2} E[\sum_{j=1}^{d} \frac{1}{\text{softplus}(\sqrt{v_{t-1,j}})} - \frac{1}{\text{softplus}(\sqrt{v_{t,j}})}] \\
& + \frac{L \eta^2 \mu_4}{2} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 (\sigma^2 + G^2) + \frac{\eta^2 \mu_4}{2} (\sigma^2 + G^2) \\
& + \frac{L \eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E[\sum_{j=1}^{d} \frac{1}{\text{softplus}(\sqrt{v_{t-1,j}})}^2 - \frac{1}{\text{softplus}(\sqrt{v_{t,j}})}^2] + L \eta^2 \mu_4^2 (\sigma^2 + G^2)
\end{align*} \]

Then we obtain:
\[ \eta \mu_3 \| \nabla f(x_t) \|^2 \leq E[f(z_t) - f(z_{t+1})] + \frac{\eta \beta_1}{1 - \beta_1} G \sqrt{\sigma^2 + G^2} E[\sum_{j=1}^{d} \frac{1}{\text{softplus}(\sqrt{v_{t-1,j}})} - \frac{1}{\text{softplus}(\sqrt{v_{t,j}})}] \\
& + \frac{L \eta^2 \mu_4}{2} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 (\sigma^2 + G^2) + \frac{\eta^2 \mu_4}{2} (\sigma^2 + G^2) \\
& + \frac{L \eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E[\sum_{j=1}^{d} \frac{1}{\text{softplus}(\sqrt{v_{t-1,j}})}^2 - \frac{1}{\text{softplus}(\sqrt{v_{t,j}})}^2] + L \eta^2 \mu_4^2 (\sigma^2 + G^2)
\]

Divide \( \eta \mu_4 \) on both sides and then sum from \( t = 1 \) to \( T \), where \( T \) is the maximum number of iteration,
\[ \sum_{t=1}^{T} \| \nabla f(x_t) \|^2 \leq \frac{1}{\eta \mu_3} E[f(z_1) - f^*] + \frac{\beta_1}{1 - \beta_1} G \sqrt{\sigma^2 + G^2} E[\sum_{j=1}^{d} \frac{1}{\text{softplus}(\sqrt{v_{0,j}})} - \frac{1}{\text{softplus}(\sqrt{v_{T,j}})}] \\
& + \frac{L \eta T \mu_4}{2 \mu_3} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 (\sigma^2 + G^2) + \frac{\eta T \mu_4}{2 \mu_3} (\sigma^2 + G^2) \\
& + \frac{L \eta T \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2 \mu_3} E[\sum_{j=1}^{d} \frac{1}{\text{softplus}(\sqrt{v_{0,j}})}^2 - \frac{1}{\text{softplus}(\sqrt{v_{T,j}})}^2] + \frac{L \eta^2 T \mu_4^2 (\sigma^2 + G^2)}{\mu_3}
\]

Since \( v_0 = 0, \frac{1}{\text{softplus}(0)} = \mu_4 \), we have
With fixed $L, \sigma, G, \beta_1$,

Thus, we get the sublinear convergence rate of $S_\text{ADAM}$ in nonconvex setting, which is the same order of ADAM and recovers the well-known result of SGD [7] in nonconvex optimization in terms of $T$. 

$$\sum_{t=1}^{T} \|\nabla f(x_t)\|^2 \leq \frac{1}{\eta_3 T} \sum_{t=1}^{T} \|f(x_t) - f^*\|^2 + \beta_1 d \frac{1}{(1 - \beta_1) \mu_3} G \sqrt{\sigma^2 + G^2}$$

Divided by $\frac{1}{T}$,

$$\frac{1}{T} \sum_{t=1}^{T} \|\nabla f(x_t)\|^2 \leq \frac{1}{\eta_3 T} \sum_{t=1}^{T} \|f(x_t) - f^*\|^2 + \beta_1 d \frac{1}{(1 - \beta_1) \mu_3} G \sqrt{\sigma^2 + G^2}$$

Setting $\eta = \frac{1}{\sqrt{T}}$, let $x_0 = x_1$, then $z_1 = x_1$, $f(z_1) = f(x_1)$ we derive the final result for SADAM method:

$$\min_{t=1, \ldots, T} E[\|\nabla f(x_t)\|^2] \leq \frac{1}{\eta_3 T} E[\|f(x_1) - f^*\|^2] + \frac{\beta_1 d}{(1 - \beta_1) \mu_3} (\mu_4 - \mu_3)$$

Setting $\eta = \frac{1}{\sqrt{T}}$, let $x_0 = x_1$, then $z_1 = x_1$, $f(z_1) = f(x_1)$ we derive the final result for SADAM method:

$$\min_{t=1, \ldots, T} E[\|\nabla f(x_t)\|^2] \leq \frac{1}{\mu_3 \sqrt{T}} E[\|f(x_1) - f^*\|^2] + \frac{\beta_1 d}{(1 - \beta_1) \mu_3} (\mu_4 - \mu_3)$$

$$\min_{t=1, \ldots, T} E[\|\nabla f(x_t)\|^2] \leq \frac{1}{\mu_3 \sqrt{T}} E[\|f(x_1) - f^*\|^2] + \frac{\beta_1 d}{(1 - \beta_1) \mu_3} (\mu_4 - \mu_3)$$

$$\min_{t=1, \ldots, T} E[\|\nabla f(x_t)\|^2] \leq \frac{1}{\mu_3 \sqrt{T}} E[\|f(x_1) - f^*\|^2] + \frac{\beta_1 d}{(1 - \beta_1) \mu_3} (\mu_4 - \mu_3)$$

$$\min_{t=1, \ldots, T} E[\|\nabla f(x_t)\|^2] \leq \frac{1}{\mu_3 \sqrt{T}} E[\|f(x_1) - f^*\|^2] + \frac{\beta_1 d}{(1 - \beta_1) \mu_3} (\mu_4 - \mu_3)$$

$$\min_{t=1, \ldots, T} E[\|\nabla f(x_t)\|^2] \leq \frac{1}{\mu_3 \sqrt{T}} E[\|f(x_1) - f^*\|^2] + \frac{\beta_1 d}{(1 - \beta_1) \mu_3} (\mu_4 - \mu_3)$$

With fixed $L, \sigma, G, \beta_1$, we have $C_1 = O(\beta^2)$, $C_2 = O(d \beta)$, $C_3 = O(d \beta^2)$. Therefore,

$$\min_{t=1, \ldots, T} E[\|\nabla f(x_t)\|^2] \leq O\left(\frac{\beta^2}{\sqrt{T}} + \frac{d \beta}{T} + \frac{d \beta^2}{T \sqrt{T}}\right)$$
Remark 9.11. The leading item from the above convergence is $C_1 / \sqrt{T}$; $\beta$ plays an essential role in the complexity, and a more accurate convergence should be $O(\frac{\beta \log (1 + \varepsilon^3)}{\sqrt{T}})$. When $\beta$ is chosen big, this will become $O(\frac{\beta^2}{\sqrt{T}})$, somehow behave like ADAM’s case as $O(\frac{1}{\sqrt{T}})$, which also guides us to have a range of $\beta$; when $\beta$ is chosen small, this will become $O(\frac{1}{\sqrt{T}})$, the computational complexity will get close to SGD case, and $\beta$ is a much smaller number compared with $1/\epsilon$, proving that SADAM converges faster. This also supports the analysis of range of A-LR: $1/\text{softplus}(\sqrt{\epsilon})$ in our main paper.

9.4 Non-strongly convex

In previous works, convex case has been well-studied in adaptive gradient methods. AMSGRAD and later methods PAMSGRAD both use a projection on minimizing objective function, here we want to show a different way of proof in non-strongly convex case. For consistency, we still follow the construction of sequence $\{z_t\}$.

Starting from convexity:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x).$$

Then, for any $x \in \mathbb{R}^d$, $\forall t \in [1, T]$, we have

$$\langle \nabla f(x), x_t - x^* \rangle \geq f(x_t) - f^*,$$

where $f^* = f(x^*)$, $x^*$ is the optimal solution.

Proof. ADAM case:

In the updating rule of ADAM optimizer, $x_{t+1} = x_t - \frac{\eta_t}{\sqrt{v_t} + \epsilon} \odot m_t$, setting stepsize to be fixed, $\eta_t = \eta$, and assume $v_t \geq v_{t-1}$ holds. Using previous results,

$$E[\|z_{t+1} - x^*\|^2]$$

$$= E[\|z_t + \frac{\eta \beta_1}{1 - \beta_1} \left( \frac{1}{\sqrt{v_{t-1}} + \epsilon} - \frac{1}{\sqrt{v_{t-1}} + \epsilon} \right) \odot m_{t-1} - \frac{\eta}{\sqrt{v_t} + \epsilon} \odot g_t - x^*\|^2]$$

$$= E[\|z_t - x^*\|^2] + E[\| \frac{\eta \beta_1}{1 - \beta_1} \left( \frac{1}{\sqrt{v_{t-1}} + \epsilon} - \frac{1}{\sqrt{v_{t-1}} + \epsilon} \right) \odot m_{t-1} - \frac{\eta}{\sqrt{v_t} + \epsilon} \odot g_t\|^2]$$

$$+ 2E[\langle \frac{\eta \beta_1}{1 - \beta_1} \left( \frac{1}{\sqrt{v_{t-1}} + \epsilon} - \frac{1}{\sqrt{v_{t-1}} + \epsilon} \right) \odot m_{t-1}, z_t - x^* \rangle] - 2E[\langle \frac{\eta}{\sqrt{v_t} + \epsilon} \odot g_t, z_t - x^* \rangle]$$

$$\leq E[\|z_t - x^*\|^2] + 2 \frac{\eta^2 \beta_1^2}{(1 - \beta_1)^2} E[\| \frac{1}{\sqrt{v_{t-1}} + \epsilon} \odot m_{t-1}\|^2] + 2\eta^2 E[\| \frac{1}{\sqrt{v_t} + \epsilon} \odot g_t\|^2]$$

$$+ 2 \frac{\eta \beta_1}{1 - \beta_1} (\frac{1}{\sqrt{v_{t-1}} + \epsilon} - \frac{1}{\sqrt{v_{t-1}} + \epsilon}) \odot m_{t-1} - x^* \rangle - 2\eta E[\langle \frac{1}{\sqrt{v_t} + \epsilon} \odot g_t, z_t - x^* \rangle]$$

$$\leq E[\|z_t - x^*\|^2] + 2 \frac{\eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E\left[\left( \frac{1}{\sqrt{v_{t-1}} + \epsilon} \right)^2 - \left( \frac{1}{\sqrt{v_{t-1}} + \epsilon} \right)^2 \right] + 2\eta^2 \mu_2^2 (\sigma^2 + G^2)$$

$$+ 2 \frac{\eta \beta_1}{1 - \beta_1} E[\langle \frac{1}{\sqrt{v_{t-1}} + \epsilon} \odot m_{t-1}, z_t - x^* \rangle - 2\eta E[\langle \frac{1}{\sqrt{v_t} + \epsilon} \odot g_t, z_t - x^* \rangle]$$

The first inequality holds due to $\|a - b\|^2 \leq 2\|a\|^2 + 2\|b\|^2$, the second inequality holds due to Lemma 9.3, 9.4, 9.6.
Since $a, b \geq \frac{1}{2\eta} a^2 + \frac{1}{2} b^2$, 
\[
2E[(\frac{1}{\sqrt{v_{t-1}} + \epsilon} - \frac{1}{\sqrt{v_t} + \epsilon}) \odot m_{t-1}, z_t - x^*)] \\
\leq \frac{1}{\eta} E[\| (\frac{1}{\sqrt{v_{t-1}} + \epsilon} - \frac{1}{\sqrt{v_t} + \epsilon}) \odot m_{t-1} \|^2] + \eta E[\| z_t - x^* \|^2] \\
\leq \frac{1}{\eta} (\sigma^2 + G^2) E[\sum_{j=1}^{d} (\frac{1}{\sqrt{v_{t-1,j}} + \epsilon})^2 - (\frac{1}{\sqrt{v_{t,j} + \epsilon}})^2] + \eta E[\| z_t - x^* \|^2]
\]

From the definition of $z_t$ and convexity, 
\[
(\nabla f(x_t), x_t - x^*) \geq f(x_t) - f^* \geq 0
\]

\[
-2\eta E[(\frac{1}{\sqrt{v_t} + \epsilon}) \odot g_t, z_t - x^*)] \\
= -2\eta E[(\frac{1}{\sqrt{v_t} + \epsilon}) \odot g_t, x_t - x^* + \frac{\beta_1}{1 - \beta_1}(x_t - x_{t-1})] \\
= -2\eta E[(\frac{1}{\sqrt{v_t} + \epsilon}) \odot g_t, x_t - x^*] - \frac{2\eta \beta_1}{1 - \beta_1} E[(\frac{1}{\sqrt{v_t} + \epsilon}) \odot g_t, x_t - x_{t-1}] \\
= -2\eta E[(\frac{1}{\sqrt{v_t} + \epsilon}) \odot g_t, x_t - x^*] - \frac{2\eta^2 \beta_1^2}{1 - \beta_1} E[(\frac{1}{\sqrt{v_t} + \epsilon}) \odot g_t, \frac{1}{\sqrt{v_{t-1}} + \epsilon} \odot m_{t-1}] \\
\leq -2\eta \mu_1 (\nabla f(x_t), x_t - x^*) + \frac{2\eta^2 \beta_1 \mu_2^2}{(1 - \beta_1)} (\sigma^2 + G^2) \\
\leq -2\eta \mu_1 (f(x_t) - f^*) + \frac{2\eta^2 \beta_1 \mu_2^2}{(1 - \beta_1)} (\sigma^2 + G^2)
\]

Plugging in previous two inequalities:
\[
E[\| z_{t+1} - x^* \|^2] \\
\leq E[\| z_t - x^* \|^2] + 2\eta^2 \beta_1^2 (\sigma^2 + G^2) E[\sum_{j=1}^{d} (\frac{1}{\sqrt{v_{t-1}} + \epsilon})^2 - (\frac{1}{\sqrt{v_t} + \epsilon})^2] + 2\eta^2 \mu_2^2 (\sigma^2 + G^2) \\
+ \frac{\beta_1 (\sigma^2 + G^2)}{1 - \beta_1} E[\sum_{j=1}^{d} (\frac{1}{\sqrt{v_{t-1,j}} + \epsilon})^2 - (\frac{1}{\sqrt{v_{t,j} + \epsilon}})^2] + \frac{\eta^2 \beta_1}{1 - \beta_1} E[\| z_t - x^* \|^2] \\
- 2\eta \mu_1 (f(x_t) - f^*) + \frac{2\eta^2 \beta_1 \mu_2^2}{(1 - \beta_1)} (\sigma^2 + G^2)
\]

By rearranging:
\[
2\eta \mu_1 (f(x_t) - f^*) \\
\leq E[\| z_t - x^* \|^2] - E[\| z_{t+1} - x^* \|^2] + 2\eta^2 \beta_1^2 (\sigma^2 + G^2) E[\sum_{j=1}^{d} (\frac{1}{\sqrt{v_{t-1}} + \epsilon})^2 - (\frac{1}{\sqrt{v_t} + \epsilon})^2] \\
+ 2\eta^2 \mu_2^2 (\sigma^2 + G^2) + \frac{\beta_1 (\sigma^2 + G^2)}{1 - \beta_1} E[\sum_{j=1}^{d} (\frac{1}{\sqrt{v_{t-1,j}} + \epsilon})^2 - (\frac{1}{\sqrt{v_{t,j} + \epsilon}})^2] + \frac{\eta^2 \beta_1}{1 - \beta_1} E[\| z_t - x^* \|^2] \\
+ \frac{2\eta^2 \beta_1 \mu_2^2}{(1 - \beta_1)} (\sigma^2 + G^2)
\]

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Divide $2\eta_{\mu_1}$ on both sides,

$$f(x_t) - f^* \leq \frac{1}{2\eta_{\mu_1}}(E[\|z_t - x^*\|^2] - E[\|z_{t+1} - x^*\|^2]) + \frac{\eta_{\beta_1}^2(\sigma^2 + G^2)}{(1 - \beta_1)^2\mu_1} E[\sum_{j=1}^{d}(\frac{1}{\sqrt{v_{t-1} + \epsilon}})^2 - (\frac{1}{\sqrt{v_t + \epsilon}})^2]
+ \frac{\eta_{\beta_1}^2}{\mu_1} (\sigma^2 + G^2) + \frac{\beta_1(\sigma^2 + G^2)}{2\eta_{\mu_1}(1 - \beta_1)} E[\sum_{j=1}^{d}(\frac{1}{\sqrt{v_{t-1,j} + \epsilon}})^2 - (\frac{1}{\sqrt{v_{t,j} + \epsilon}})^2]
+ \frac{\eta_{\beta_1}}{2\mu_1(1 - \beta_1)} E[\|z_t - x^*\|^2] + \frac{\eta_{\beta_1}\mu_1^2}{(1 - \beta_1)^2\mu_1} (\sigma^2 + G^2)
$$

Assume that $\forall t$, $E[\|x_t - x^*\| \leq D$, for any $m \neq n$, $E[\|x_m - x_n\| \leq D_{\infty}$ hold, then $E[\|z_t - x^*\|^2]$ can be bounded.

$$E[\|z_1 - x^*\|^2] = E[\|x_1 - x^*\|^2] \leq D^2\tag{17}
E[\|z_t - x^*\|^2] = E[\|x_t - x^* + \frac{\beta_1}{1 - \beta_1}(x_t - x_{t-1})\|^2]
\leq 2E[\|x_t - x^*\|^2] + \frac{2\beta_1^2}{(1 - \beta_1)^2} E[\|x_t - x_{t-1}\|^2]
\leq 2D^2 + \frac{2\beta_1^2}{(1 - \beta_1)^2} D_{\infty}^2\tag{18}$$

Thus:

$$f(x_t) - f^* \leq \frac{1}{2\eta_{\mu_1}}(E[\|z_t - x^*\|^2] - E[\|z_{t+1} - x^*\|^2]) + \frac{\eta_{\beta_1}^2(\sigma^2 + G^2)}{(1 - \beta_1)^2\mu_1} E[\sum_{j=1}^{d}(\frac{1}{\sqrt{v_{t-1} + \epsilon}})^2 - (\frac{1}{\sqrt{v_t + \epsilon}})^2]
+ \frac{\eta_{\beta_1}^2}{\mu_1} (\sigma^2 + G^2) + \frac{\beta_1(\sigma^2 + G^2)}{2\eta_{\mu_1}(1 - \beta_1)} E[\sum_{j=1}^{d}(\frac{1}{\sqrt{v_{t-1,j} + \epsilon}})^2 - (\frac{1}{\sqrt{v_{t,j} + \epsilon}})^2]
+ \frac{\eta_{\beta_1}}{2\mu_1(1 - \beta_1)} E[\|z_t - x^*\|^2] + \frac{\eta_{\beta_1}\mu_1^2}{(1 - \beta_1)^2\mu_1} (\sigma^2 + G^2)
$$

Summing from $t = 1$ to $T$,

$$\sum_{t=1}^{T}(f(x_t) - f^*) \leq \frac{1}{2\eta_{\mu_1}}(E[\|z_1 - x^*\|^2] - E[\|z_T - x^*\|^2]) + \frac{\eta_{\beta_1}^2(\sigma^2 + G^2)}{(1 - \beta_1)^2\mu_1} E[\sum_{j=1}^{d}(\frac{1}{\sqrt{v_{0} + \epsilon}})^2 - (\frac{1}{\sqrt{v_T + \epsilon}})^2]
+ \frac{\eta_{\beta_1}^2}{\mu_1} (\sigma^2 + G^2) + \frac{\beta_1(\sigma^2 + G^2)}{2\eta_{\mu_1}(1 - \beta_1)} E[\sum_{j=1}^{d}(\frac{1}{\sqrt{v_{0,j} + \epsilon}})^2 - (\frac{1}{\sqrt{v_{T,j} + \epsilon}})^2]
+ \frac{\eta_{\beta_1}^2}{\mu_1(1 - \beta_1)} + \frac{\eta_{\beta_1}^2 D_{\infty}^2 T}{\mu_1(1 - \beta_1)^3} + \frac{\eta_{\beta_1}^2 T}{\mu_1} (\sigma^2 + G^2)
\leq \frac{1}{2\eta_{\mu_1}}D^2 + \frac{\eta_{\beta_1}^2}{(1 - \beta_1)^2\mu_1} (\mu_1^2 - \mu_1^2) + \frac{\eta_{\beta_1}^2}{\mu_1} (\sigma^2 + G^2) + \frac{\beta_1 d(\sigma^2 + G^2)}{2\eta_{\mu_1}(1 - \beta_1)} (\mu_2^2 - \mu_1^2)
+ \frac{\eta_{\beta_1}^2}{\mu_1(1 - \beta_1)} + \frac{\eta_{\beta_1}^2 D_{\infty}^2 T}{\mu_1(1 - \beta_1)^3} + \frac{\eta_{\beta_1}^2 T}{\mu_1} (\sigma^2 + G^2)
$$

The second inequality is based on the fact that, when iteration $t$ reaches the maximum number $T$, $x_t$ is the optimal solution, $z_T = x^*$. 

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By Jensen’s inequality,
\[ \frac{1}{T} \sum_{t=1}^{T} (f(x_t) - f^*) \geq f(\bar{x}_T) - f^*, \]
where \( \bar{x}_T = \frac{1}{T} \sum_{t=1}^{T} x_t. \)

Then,
\[
f(\bar{x}_T) - f^* \leq \frac{D^2}{2\eta\mu_1 T} + \frac{\eta \beta_2^2 d(\sigma^2 + G^2)}{(1 - \beta_1)^2 \mu_1 T} (\mu_2 - \mu_1^2) + \frac{\eta \mu_2^2}{\mu_1} (\sigma^2 + G^2) + \frac{\beta_1 d(\sigma^2 + G^2)}{2\eta \mu_1 (1 - \beta_1) T} (\mu_2 - \mu_1^2) \\
+ \frac{\eta \beta_1 D^2}{\mu_1 (1 - \beta_1)} + \frac{\eta \beta_1^2 D_{\infty}^2}{\mu_1 (1 - \beta_1)^3} + \frac{\eta \beta_1 \mu_2^2}{(1 - \beta_1) \mu_1} (\sigma^2 + G^2)
\]

By plugging the stepsize \( \eta = O\left(\frac{1}{\sqrt{T}}\right) \), we complete the proof of ADAM in non-strongly convex case.

\[
f(\bar{x}_T) - f^* \leq \frac{D^2}{2\eta \mu_3 T} + \frac{\beta_2^2 d(\sigma^2 + G^2)}{(1 - \beta_1)^2 \mu_3 T} (\mu_4 - \mu_3^2) + \frac{\mu_3^2}{\mu_3 \sqrt{T}} (\sigma^2 + G^2) + \frac{\beta_1 d(\sigma^2 + G^2)}{2\eta \mu_3 (1 - \beta_1) T} (\mu_4 - \mu_3^2) \\
+ \frac{\eta \beta_1 D^2}{\mu_3 (1 - \beta_1)} + \frac{\eta \beta_1^2 D_{\infty}^2}{\mu_3 (1 - \beta_1)^3} + \frac{\eta \beta_1 \mu_2^2}{(1 - \beta_1) \mu_3} (\sigma^2 + G^2)
\]

**Remark 9.12.** The leading item of convergence order of ADAM should be \( O\left(\frac{\tilde{C}}{\sqrt{T}}\right) \), where \( \tilde{C} = \frac{D^2}{2\mu_1} + \frac{\mu_2^2}{\mu_1^2} (\sigma^2 + G^2) + \frac{\beta_2^2 d(\sigma^2 + G^2)}{2\mu_1 (1 - \beta_1)^2} (\mu_2^2 - \mu_1^2) + \frac{\beta_1 d(\sigma^2 + G^2)}{2\mu_1 (1 - \beta_1)^2} (\mu_2^2 - \mu_1^2) \). With fixed \( L, \sigma, G, \beta_1, D, D_{\infty}, \tilde{C} = O\left(\frac{1}{\sqrt{T}}\right) \), which also contains \( \epsilon \) as well as dimension \( d \), here with bigger \( \epsilon \), the order should be better; this also supports the discussion in our main paper.

The analysis of SADAM is similar to ADAM, by replacing the bounded pairs \((\mu_1, \mu_2)\) with \((\mu_3, \mu_4)\), we briefly give convergence result below.

**Proof.** SADAM case:

\[
f(\bar{x}_T) - f^* \leq \frac{D^2}{2\eta \mu_3 T} + \frac{\beta_2^2 d(\sigma^2 + G^2)}{(1 - \beta_1)^2 \mu_3 T} (\mu_4 - \mu_3^2) + \frac{\mu_3^2}{\mu_3 \sqrt{T}} (\sigma^2 + G^2) + \frac{\beta_1 d(\sigma^2 + G^2)}{2\mu_3 (1 - \beta_1) T} (\mu_4 - \mu_3^2) \\
+ \frac{\eta \beta_1 D^2}{\mu_3 (1 - \beta_1)} + \frac{\eta \beta_1^2 D_{\infty}^2}{\mu_3 (1 - \beta_1)^3} + \frac{\eta \beta_1 \mu_2^2}{(1 - \beta_1) \mu_3} (\sigma^2 + G^2)
\]

By plugging the stepsize \( \eta = O\left(\frac{1}{\sqrt{T}}\right) \), we get the convergence rate of SADAM in non-strongly convex case.

\[
f(\bar{x}_T) - f^* \leq \frac{D^2}{2\eta \mu_3 T} + \frac{\beta_2^2 d(\sigma^2 + G^2)}{(1 - \beta_1)^2 \mu_3 T} (\mu_4 - \mu_3^2) + \frac{\mu_3^2}{\mu_3 \sqrt{T}} (\sigma^2 + G^2) + \frac{\beta_1 d(\sigma^2 + G^2)}{2\mu_3 (1 - \beta_1) T} (\mu_4 - \mu_3^2) \\
+ \frac{\beta_1 D^2}{\mu_3 (1 - \beta_1) \sqrt{T}} + \frac{\beta_1^2 D_{\infty}^2}{\mu_3 (1 - \beta_1)^3 \sqrt{T}} + \frac{\beta_1 \mu_2^2}{(1 - \beta_1) \mu_3} (\sigma^2 + G^2) \\
= O\left(\frac{1}{\sqrt{T}}\right) + O\left(\frac{1}{T \sqrt{T}}\right) = O\left(\frac{1}{\sqrt{T}}\right).
\]

For brevity,

\[ f(\bar{x}_T) - f^* = O\left(\frac{1}{\sqrt{T}}\right). \]
Remark 9.13. The leading item of convergence order of SADAM should be $O(\frac{\sqrt{T}}{T})$, where $\tilde{C} = \frac{D^2}{2\mu_3} + \frac{\mu_3^2}{\mu_3^2}(\sigma^2 + G^2) + \frac{\beta_1 D^2(\sigma^2 + G^2)}{\mu_3(1 - \beta_1)} + \frac{\beta_1^2 D^2}{\mu_3(1 - \beta_1)} + \frac{\beta_1^2 D^2}{\mu_3(1 - \beta_1)}(\sigma^2 + G^2)$. With fixed $L, \sigma, G, \beta_1, D, D_\infty$, $\tilde{C} = O(d\beta \log(1 + e^\beta)) = O(d\beta^2)$, with small $\beta$, the SADAM will be similar to SGD convergence rate, and $\beta$ is a much smaller number compared with $1/\epsilon$, proving that SADAM method performs better than ADAM in terms of convergence rate.

9.5 P-L condition

Suppose that strongly convex assumption holds, we can easily deduce the P-L condition (see Lemma 9.14), which shows that P-L condition is much weaker than strongly convex condition. And we further prove the convergence of ADAM-type optimizer (ADAM and SADAM) under the P-L condition in non-strongly convex case, which can be extended to the strongly convex case as well.

Lemma 9.14. Suppose that $f$ is continuously differentiable and strongly convex with parameter $\gamma$. Then $f$ has the unique minimizer, denoted as $f^* = f(x^*)$. Then for any $x \in \mathbb{R}^d$, we have

$$\|\nabla f(x)\|^2 \geq 2\gamma(f(x) - f^*).$$

Proof. From strongly convex assumption,

$$f^* \geq f(x) + \nabla f(x)^T(x^* - x) + \frac{\gamma}{2}\|x^* - x\|^2$$

$$\geq f(x) + \min_\xi(\nabla f(x)^T \xi + \frac{\gamma}{2}\|\xi\|^2)$$

$$= f(x) - \frac{1}{2\gamma}\|\nabla f(x)\|^2$$

Letting $\xi = x^* - x$, when $\xi = -\frac{\nabla f(x)}{\gamma}$, the quadratic function can achieve its minimum. \qed

Theorem 5.3 Suppose $f(x)$ satisfies Assumption 1 and PL condition (with parameter $\lambda$) in non-strongly convex case and $v_t \geq v_{t-1}$. Let $\eta_t = \eta = O(\frac{1}{T})$, ADAM and SADAM have convergence rate

$$E[f(x_t) - f^*] \leq O(\frac{1}{T}).$$

Proof. ADAM case:

Starting from L-smoothness, and borrowing the previous results we already have

$$E[f(z_{t+1}) - f(z_t)] \leq \frac{\eta \beta_1}{1 - \beta_1}G\sqrt{\sigma^2 + G^2}E(\sum_{j=1}^{d}\frac{1}{v_{t-1,j} + \epsilon} - \frac{1}{\sqrt{v_{t,j} + \epsilon}}$$

$$+ \frac{L^2 \eta^2 \mu_3^2}{2}(\frac{1}{1 - \beta_1})^2(\sigma^2 + G^2) + \frac{\eta^2 \mu_3^2}{2}(\sigma^2 + G^2) - E\langle \nabla f(x_t), \frac{\eta}{\sqrt{v_t + \epsilon}} \circ g_t \rangle$$

$$+ \frac{L \eta^2 \beta_1^2(\sigma^2 + G^2)}{(1 - \beta_1)^2}E(\sum_{j=1}^{d}\frac{1}{v_{t-1,j} + \epsilon} - \frac{1}{\sqrt{v_{t,j} + \epsilon}})^2 + \frac{L \eta^2 \beta_1^2(\sigma^2 + G^2)}{(1 - \beta_1)^2}$$

$$E\langle \nabla f(x_t), \frac{1}{\sqrt{v_t + \epsilon}} \circ g_t \rangle \geq \mu_1 \|\nabla f(x_t)\|^2$$

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Therefore, we get:

\[
E[f(z_{t+1}) - f(z_t)] \leq \frac{\eta \beta_1}{1 - \beta_1} G\sqrt{\sigma^2 + G^2} E\left[\sum_{j=1}^{d} \frac{1}{\sqrt{v_{t-1,j} + \epsilon}} - \frac{1}{\sqrt{v_{t,j} + \epsilon}}\right]
\]

\[
+ \frac{L^2 \eta^2 \mu^2_2}{2} (\frac{\beta_1}{1 - \beta_1})^2 (\sigma^2 + G^2) + \frac{\eta^2 \mu^2_2}{2} (\sigma^2 + G^2) - \eta \mu_1 \|\nabla f(x_t)\|^2
\]

\[
+ \frac{L \eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E\left[\sum_{j=1}^{d} \left(\frac{1}{\sqrt{v_{t-1,j} + \epsilon}} - \frac{1}{\sqrt{v_{t,j} + \epsilon}}\right)^2\right] + L \eta^2 \mu_2^2 (\sigma^2 + G^2)
\]

From P-L condition assumption,

\[
E[f(z_{t+1})] \leq E[f(z_t)] + \frac{\eta \beta_1}{1 - \beta_1} G\sqrt{\sigma^2 + G^2} E\left[\sum_{j=1}^{d} \frac{1}{\sqrt{v_{t-1,j} + \epsilon}} - \frac{1}{\sqrt{v_{t,j} + \epsilon}}\right]
\]

\[
+ \frac{L^2 \eta^2 \mu^2_2}{2} (\frac{\beta_1}{1 - \beta_1})^2 (\sigma^2 + G^2) + \frac{\eta^2 \mu^2_2}{2} (\sigma^2 + G^2) - 2\lambda \eta \mu_1 E[f(x_t) - f^*]
\]

\[
+ \frac{L \eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E\left[\sum_{j=1}^{d} \left(\frac{1}{\sqrt{v_{t-1,j} + \epsilon}} - \frac{1}{\sqrt{v_{t,j} + \epsilon}}\right)^2\right] + L \eta^2 \mu_2^2 (\sigma^2 + G^2)
\]

From convexity,

\[
f(z_{t+1}) \geq f(x_{t+1}) + \frac{\beta_1}{1 - \beta_1} \langle \nabla f(x_{t+1}), x_{t+1} - x_t \rangle
\]

\[
= f(x_{t+1}) + \frac{\beta_1}{1 - \beta_1} \langle \nabla f(x_{t+1}), \frac{\eta}{\sqrt{v_t + \epsilon}} \odot m_t \rangle
\]

From L-smoothness,

\[
f(z_t) \leq f(x_t) + \frac{\beta_1}{1 - \beta_1} \langle \nabla f(x_t), x_t - x_{t-1} \rangle + \frac{L}{2} (\frac{\beta_1}{1 - \beta_1})^2 \|x_t - x_{t-1}\|^2.
\]

Then we can obtain

\[
E[f(x_{t+1})] + \frac{\beta_1}{1 - \beta_1} E[\langle \nabla f(x_{t+1}), \frac{\eta}{\sqrt{v_t + \epsilon}} \odot m_t \rangle]
\]

\[
\leq E[f(x_t)] + \frac{\beta_1}{1 - \beta_1} E[\langle \nabla f(x_t), x_t - x_{t-1} \rangle] + \frac{L}{2} (\frac{\beta_1}{1 - \beta_1})^2 E[\|x_t - x_{t-1}\|^2]
\]

\[
+ \frac{\eta \beta_1}{1 - \beta_1} G\sqrt{\sigma^2 + G^2} E\left[\sum_{j=1}^{d} \frac{1}{\sqrt{v_{t-1,j} + \epsilon}} - \frac{1}{\sqrt{v_{t,j} + \epsilon}}\right]
\]

\[
+ \frac{L^2 \eta^2 \mu^2_2}{2} (\frac{\beta_1}{1 - \beta_1})^2 (\sigma^2 + G^2) + \frac{\eta^2 \mu^2_2}{2} (\sigma^2 + G^2) - 2\lambda \eta \mu_1 E[f(x_t) - f^*]
\]

\[
+ \frac{L \eta^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E\left[\sum_{j=1}^{d} \left(\frac{1}{\sqrt{v_{t-1,j} + \epsilon}} - \frac{1}{\sqrt{v_{t,j} + \epsilon}}\right)^2\right] + L \eta^2 \mu_2^2 (\sigma^2 + G^2)
\]
Then,

\begin{align*}
E[f(x_t)] + \frac{\beta_1}{1 - \beta_1} E[\nabla f(x_t), \frac{\eta}{\sqrt{v_{t-1}} + \epsilon} \otimes m_{t-1}] + \frac{L\eta^2}{2} (\frac{\beta_1}{1 - \beta_1})^2 E\| \frac{1}{\sqrt{v_t} + \epsilon} \otimes m_t \|^2 \\
+ \frac{\eta \beta_1}{1 - \beta_1} G \sqrt{\sigma^2 + G^2} E[\sum_{j=1}^{d} \frac{1}{\sqrt{v_{t-1,j}} + \epsilon} - \frac{1}{\sqrt{v_{t,j}} + \epsilon}] \\
+ \frac{L^2 \eta^2 \mu^2_2}{2} (\frac{\beta_1}{1 - \beta_1})^2 (\sigma^2 + G^2) + \frac{\eta^2 \mu^2_2}{2} (\sigma^2 + G^2) - 2\lambda \eta \mu_1 E[f(x_t) - f^*] \\
+ \frac{L^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E[\sum_{j=1}^{d} (\frac{1}{\sqrt{v_{t-1,j}} + \epsilon})^2 - (\frac{1}{\sqrt{v_{t,j}} + \epsilon})^2] + \frac{L\eta^2 \mu^2_2 (\sigma^2 + G^2)}{2}
\end{align*}

By rearranging,

\begin{align*}
E[f(x_{t+1})] &\leq E[f(x_t)] + \frac{\beta_1 \eta}{1 - \beta_1} (E[\nabla f(x_t), \frac{1}{\sqrt{v_{t-1}} + \epsilon} \otimes m_{t-1}] - E[\nabla f(x_{t+1}), \frac{1}{\sqrt{v_t} + \epsilon} \otimes m_t]) \\
&\quad + \frac{L\eta^2}{2} (\frac{\beta_1}{1 - \beta_1})^2 E\| \frac{1}{\sqrt{v_t} + \epsilon} \otimes m_t \|^2 + \frac{\eta \beta_1}{1 - \beta_1} G \sqrt{\sigma^2 + G^2} E[\sum_{j=1}^{d} \frac{1}{\sqrt{v_{t-1,j}} + \epsilon} - \frac{1}{\sqrt{v_{t,j}} + \epsilon}] \\
&\quad + \frac{L^2 \eta^2 \mu^2_2}{2} (\frac{\beta_1}{1 - \beta_1})^2 (\sigma^2 + G^2) + \frac{\eta^2 \mu^2_2}{2} (\sigma^2 + G^2) - 2\lambda \eta \mu_1 E[f(x_t) - f^*] \\
&\quad + \frac{L^2 \beta_1^2 (\sigma^2 + G^2)}{(1 - \beta_1)^2} E[\sum_{j=1}^{d} (\frac{1}{\sqrt{v_{t-1,j}} + \epsilon})^2 - (\frac{1}{\sqrt{v_{t,j}} + \epsilon})^2] + \frac{L\eta^2 \mu^2_2 (\sigma^2 + G^2)}{2}
\end{align*}

From the fact $\pm < a, b > \leq \frac{1}{2} a^2 + \frac{1}{2} b^2$, and Lemma 9.1 [9.4].

\begin{align*}
E[\nabla f(x_t), \frac{1}{\sqrt{v_{t-1}} + \epsilon} \otimes m_{t-1}] &= E[\nabla f(x_{t+1}) \odot \sqrt{\frac{1}{\sqrt{v_{t-1}} + \epsilon}}, m_t \odot \sqrt{\frac{1}{\sqrt{v_{t-1}} + \epsilon}}] \\
&\leq \frac{G^2 \mu_2}{2} + \frac{(\sigma^2 + G^2) \mu_2}{2} \leq (\sigma^2 + G^2) \mu_2
\end{align*}

Similar,

\begin{align*}
-E[\nabla f(x_{t+1}), \frac{1}{\sqrt{v_t} + \epsilon} \otimes m_t] &= -E[\nabla f(x_{t+1}) \odot \sqrt{\frac{1}{\sqrt{v_{t-1}} + \epsilon}}, m_t \odot \sqrt{\frac{1}{\sqrt{v_{t-1}} + \epsilon}}] \\
&\leq \frac{G^2 \mu_2}{2} + \frac{(\sigma^2 + G^2) \mu_2}{2} \leq (\sigma^2 + G^2) \mu_2
\end{align*}

Then,
\[ E[f(x_{t+1})] \leq E[f(x_t)] + \frac{2\beta_1 \eta \mu_2}{1 - \beta_1} (\sigma^2 + G^2) + \frac{L \eta^2 \mu_2^2}{2} (\frac{\beta_1}{1 - \beta_1})^2 (\sigma^2 + G^2) \]
\[ + \frac{\eta \beta_1}{1 - \beta_1} G \sqrt{\sigma^2 + G^2} E\left[ \sum_{j=1}^{d} \frac{1}{\sqrt{v_{t-1,j} + \epsilon}} - \frac{1}{\sqrt{v_{t,j} + \epsilon}} \right] \]
\[ + \frac{L^2 \eta^2 \mu_2^2}{2} (\frac{\beta_1}{1 - \beta_1})^2 (\sigma^2 + G^2) + \frac{\eta^2 \mu_2^2}{2} (\sigma^2 + G^2) - 2\lambda \eta \mu_1 E[f(x_t) - f^*] \]
\[ + \frac{L \eta^2 \beta_1^2}{(1 - \beta_1)^2} (\sigma^2 + G^2) E\left[ \sum_{j=1}^{d} \frac{1}{\sqrt{v_{t-1,j} + \epsilon}} - \frac{1}{\sqrt{v_{t,j} + \epsilon}} \right] + L \eta^2 \mu_2^2 (\sigma^2 + G^2) \]

\[ E[f(x_{t+1}) - f^*] \leq (1 - 2\lambda \eta \mu_1) E[f(x_t) - f^*] + \frac{2\beta_1 \eta \mu_2}{1 - \beta_1} (\sigma^2 + G^2) + \frac{L \eta^2 \mu_2^2}{2} (\frac{\beta_1}{1 - \beta_1})^2 (\sigma^2 + G^2) \]
\[ + \frac{\eta \beta_1}{1 - \beta_1} G \sqrt{\sigma^2 + G^2} E\left[ \sum_{j=1}^{d} \frac{1}{\sqrt{v_{t-1,j} + \epsilon}} - \frac{1}{\sqrt{v_{t,j} + \epsilon}} \right] \]
\[ + \frac{L^2 \eta^2 \mu_2^2}{2} (\frac{\beta_1}{1 - \beta_1})^2 (\sigma^2 + G^2) + \frac{\eta^2 \mu_2^2}{2} (\sigma^2 + G^2) \]
\[ + \frac{L \eta^2 \beta_1^2}{(1 - \beta_1)^2} (\sigma^2 + G^2) E\left[ \sum_{j=1}^{d} \frac{1}{\sqrt{v_{t-1,j} + \epsilon}} - \frac{1}{\sqrt{v_{t,j} + \epsilon}} \right] + L \eta^2 \mu_2^2 (\sigma^2 + G^2) \]
\[ \leq (1 - 2\lambda \eta \mu_1) E[f(x_t) - f^*] + \frac{2\beta_1 \eta \mu_2}{1 - \beta_1} + \frac{L \eta^2 \mu_2^2}{2} (\frac{\beta_1}{1 - \beta_1})^2 \]
\[ + \frac{\eta \beta_1}{1 - \beta_1} E\left[ \sum_{j=1}^{d} \frac{1}{\sqrt{v_{t-1,j} + \epsilon}} - \frac{1}{\sqrt{v_{t,j} + \epsilon}} \right] + \frac{L^2 \eta^2 \mu_2^2}{2} (\frac{\beta_1}{1 - \beta_1})^2 + \frac{\eta^2 \mu_2^2}{2} \]
\[ + \frac{L \eta^2 \beta_1^2}{(1 - \beta_1)^2} E\left[ \sum_{j=1}^{d} \frac{1}{\sqrt{v_{t-1,j} + \epsilon}} - \frac{1}{\sqrt{v_{t,j} + \epsilon}} \right] + L \eta^2 \mu_2^2 (\sigma^2 + G^2) \]

The last inequality holds because \( G \sqrt{\sigma^2 + G^2} \leq \sigma^2 + G^2 \).

Let
\[ \theta = 1 - 2\lambda \eta \mu_1 \]
\[ \Theta_t = \left( \frac{2\beta_1 \eta \mu_2}{1 - \beta_1} + \frac{L \eta^2 \mu_2^2}{2} (\frac{\beta_1}{1 - \beta_1})^2 + \frac{\eta \beta_1}{1 - \beta_1} E\left[ \sum_{j=1}^{d} \frac{1}{\sqrt{v_{t-1,j} + \epsilon}} - \frac{1}{\sqrt{v_{t,j} + \epsilon}} \right] + \frac{L^2 \eta^2 \mu_2^2}{2} (\frac{\beta_1}{1 - \beta_1})^2 \right) \]
\[ + \frac{\eta^2 \mu_2^2}{2} + \frac{L \eta^2 \beta_1^2}{(1 - \beta_1)^2} E\left[ \sum_{j=1}^{d} \frac{1}{\sqrt{v_{t-1,j} + \epsilon}} - \frac{1}{\sqrt{v_{t,j} + \epsilon}} \right] + L \eta^2 \mu_2^2 (\sigma^2 + G^2) \]

then we have
\[ E[f(x_{t+1}) - f^*] \leq \theta E[f(x_t) - f^*] + \Theta_t. \]

Let \( \Phi_t = E[f(x_t) - f^*] \), then \( \Phi_1 = E[f(x_1) - f^*] \),

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\[ \Phi_{t+1} \leq \theta \Phi_t + \Theta_t \leq \theta^2 \Phi_{t-1} + \theta \Theta_{t-1} + \Theta_t \]
\[ \vdots \]
\[ \leq \theta^t \Phi_1 + \theta^{t-1} \Theta_1 + \ldots + \theta \Theta_{t-1} + \Theta_t \]
\[ \theta < 1 \]
\[ \leq \theta^t \Phi_1 + \Theta_1 + \ldots + \Theta_{t-1} + \Theta_t. \]

Let \( t = T \),
\[ \Phi_{T+1} \leq \theta^T \Phi_1 + \Theta_1 + \ldots + \Theta_{T-1} + \Theta_T \]
\[ \leq \theta^T \Phi_1 + \left( \frac{2\beta_1 \eta \mu_4 T}{1 - \beta_1} + \frac{L \eta^2 \mu_2^2 T}{2} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 + \frac{\eta \beta_1}{1 - \beta_1} E \sum_{j=1}^{d} \frac{1}{\sqrt{v_{0,j} + \epsilon}} - \frac{1}{\sqrt{v_{T,j} + \epsilon}} \right) \]
\[ + \frac{L \eta^2 \beta_1^2}{2} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 + \frac{\eta^2 \mu_2^2 T}{2} \]
\[ + \frac{L \eta^2 \beta_1^2}{(1 - \beta_1)^2} E \sum_{j=1}^{d} \left( \frac{1}{\sqrt{v_{0,j} + \epsilon}} \right)^2 - \left( \frac{1}{\sqrt{v_{T,j} + \epsilon}} \right)^2 \right] + L \eta^2 \mu_2^2 T)(\sigma^2 + G^2) \]
\[ \leq \theta^T \Phi_1 + \left( \frac{2\beta_1 \eta \mu_3 T}{1 - \beta_1} + \frac{L \eta^2 \mu_4^2 T}{2} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 + \frac{\eta \beta_1 d}{1 - \beta_1} (\mu_4 - \mu_3) + \frac{L \eta^2 \mu_2^2 T}{2} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 \]
\[ + \frac{\eta^2 \mu_2^2 T}{2} + \frac{L \eta^2 \beta_1^2 d}{(1 - \beta_1)^2} (\mu_4^2 - \mu_3^2) + L \eta^2 \mu_2^2 T)(\sigma^2 + G^2) \]
\[ = \theta^T \Phi_1 + O(\eta T) + O(\eta^2 T) + O(\eta) + O(\eta^2) \]

From the above inequality, \( \eta \) should be set less than \( O\left(\frac{1}{T}\right) \) to ensure all items in the RHS small enough.

Set \( \eta = \frac{1}{T^2} \), then \( \theta = 1 - 2\lambda \eta \mu_1 = 1 - \frac{2\lambda \mu_1}{T^2} \)
\[ \Phi_{T+1} = \theta^T \Phi_1 + O\left(\frac{1}{T}\right) + O\left(\frac{1}{T^3}\right) + O\left(\frac{1}{T^2}\right) + O\left(\frac{1}{T^4}\right) \]
\[ = \theta^T \Phi_1 + O\left(\frac{1}{T}\right) \rightarrow 0 \]

With appropriate \( \eta \), we can derive the convergence rate under P-L condition (strongly convex) case.

The proof of SADAM is exactly same as ADAM, by replacing the bounded pairs \((\mu_1, \mu_2)\) with \((\mu_3, \mu_4)\), and we can also get:
\[ \Phi_{T+1} \leq \theta^T \Phi_1 + \Theta_1 + \ldots + \Theta_{T-1} + \Theta_T \]
\[ \leq \theta^T \Phi_1 + \left( \frac{2\beta_1 \eta \mu_4 T}{1 - \beta_1} + \frac{L \eta^2 \mu_2^2 T}{2} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 + \frac{\eta \beta_1}{1 - \beta_1} E \sum_{j=1}^{d} \frac{1}{\sqrt{\text{softplus}(v_{0,j})}} - \frac{1}{\sqrt{\text{softplus}(v_{T,j})}} \right) \]
\[ + \frac{L \eta^2 \beta_1^2}{2} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 + \frac{\eta^2 \mu_2^2 T}{2} \]
\[ + \frac{L \eta^2 \beta_1^2}{(1 - \beta_1)^2} E \sum_{j=1}^{d} \left( \frac{1}{\sqrt{\text{softplus}(v_{0,j})}} \right)^2 - \left( \frac{1}{\sqrt{\text{softplus}(v_{T,j})}} \right)^2 \right] + L \eta^2 \mu_2^2 T)(\sigma^2 + G^2) \]
\[ \leq \theta^T \Phi_1 + \left( \frac{2\beta_1 \eta \mu_3 T}{1 - \beta_1} + \frac{L \eta^2 \mu_4^2 T}{2} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 + \frac{\eta \beta_1 d}{1 - \beta_1} (\mu_4 - \mu_3) + \frac{L \eta^2 \mu_2^2 T}{2} \left( \frac{\beta_1}{1 - \beta_1} \right)^2 \]
\[ + \frac{\eta^2 \mu_2^2 T}{2} + \frac{L \eta^2 \beta_1^2 d}{(1 - \beta_1)^2} (\mu_4^2 - \mu_3^2) + L \eta^2 \mu_2^2 T)(\sigma^2 + G^2) \]
\[ = \theta^T \Phi_1 + O(\eta T) + O(\eta^2 T) + O(\eta) + O(\eta^2) \]
By setting appropriate $\eta$, we can also prove the SADAM converges under PL condition (and strongly convex).

Set $\eta = O(\frac{1}{T})$,

$$E[f(x_{T+1}) - f^*] \leq (1 - \frac{2\lambda \mu_3}{T^2})^T E[f(x_1) - f^*] + O(\frac{1}{T}).$$

Overall, we have proved ADAM algorithm and SADAM in all commonly used conditions, our designed algorithms always enjoy the same convergence rate compared with ADAM, and even get better results with appropriate choice of $\beta$ defined in softplus function. The proof procedure can be easily extended to other adaptive gradient algorithms, and theoretical results support the discussion and experiments in our main paper.