Positive-entropy Hamiltonian systems on Nilmanifolds via scattering

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Received 12 February 2013, revised 28 July 2014
Accepted for publication 19 August 2014
Published 19 September 2014

Abstract

Let $\Sigma$ be a compact quotient of $T_4$, the Lie group of $4 \times 4$ upper triangular matrices with unity along the diagonal. The Lie algebra $\mathfrak{t}_4$ of $T_4$ has the standard basis $\{X_{ij}\}$ of matrices with 0 everywhere but in the $(i, j)$ entry, which is unity. Let $g$ be the Carnot metric, a sub-Riemannian metric, on $T_4$ for which $X_{i,i+1}$, $(i = 1, 2, 3)$, is an orthonormal basis. Montgomery, Shapiro and Stolin showed that the geodesic flow of $g$ is algebraically non-integrable. This paper proves that the geodesic flow of that Carnot metric on $T/\Sigma$ has positive topological entropy and its Euler field is real-analytically non-integrable. It extends earlier work by Butler and Gelfreich.

Keywords: sub-Riemannian geometry, nilmanifold, topological entropy, geodesic flows

(Some figures may appear in colour only in the online journal)
Mathematics Subject Classification: 37J30, 53C17, 53C30, 53D25

1. Introduction

Let $G$ be a connected nilpotent Lie group with discrete subgroup $D$ and let $\Sigma = G/D$ be the corresponding homogeneous space. Each homogeneous (sub-)Riemannian metric on $G$ induces a locally homogeneous metric on $\Sigma$. These left-invariant geometries are interesting both geometrically and dynamically. A basic question is

**Question 1.1.** Which left-invariant geodesic flows on a compact nilmanifold have zero topological entropy?

Let $T_n$ be the nilpotent group of upper triangular $n \times n$ real matrices with unity on the diagonal. Montgomery, Shapiro and Stolin [7] investigate the geodesic flow of a Carnot metric on $T_4$; they show that it reduces to the Yang–Mills Hamiltonian flow which is known
to be algebraically non-integrable [10, 11]. In [3], metrics on compact quotients of the 3-step nilpotent Lie group $T_d \oplus T_1$ are constructed whose geodesic flows have positive topological entropy. In [4], Butler & Gelfreich showed that there are Riemannian and sub-Riemannian metrics on $T_1$ which have positive topological entropy. Numerical analysis in that paper suggested that the Carnot metric of Montgomery, Shapiro and Stolin has a horseshoe, hence positive topological entropy, and is analytically non-integrable. This note proves those numerical results are, in fact, correct. In that paper, a Melnikov integral is expressed in terms of scattering data for a second-order scalar differential equation; in the present paper, this scattering data is explicitly computed in terms of $\Gamma$-functions.

The Lie algebra of $T_d$, $t_d$, has the standard basis consisting of those $4 \times 4$ matrices $X_{ij}$ with a unit in the $i$-th row and $j$-th column, $i < j$, and zeros everywhere else. We will restrict attention to those structures $\langle \cdot, \cdot \rangle$ where $\langle X_{ij}, X_{kl} \rangle = b_{ij}$ when $i = k$, $j = l$ and zero otherwise. The standard Riemannian metric has $b_{ij} = 1$ for all $i$, $j$; the standard Carnot sub-Riemannian metric studied in [7] has $b_{12} = b_{23} = b_{34} = 1$ and all other coefficients zero.

The topological entropy of a uniformly continuous self-map $f$ of a metric space $(X, d)$ on a compact subset $K \subset X$ is defined by Bowen as follows: for a natural number $n$, define the Bowen distance between $x, y \in X$, $d_n(x, y)$, to be the maximum of $d(f^i(x), f^i(y))$ for $0 \leq i \leq n$. A set $E \subset K$ is $(n, \epsilon)$-separated if each pair of distinct points $x, y \in E$ has $d_n(x, y) \geq \epsilon$; let $N(n, \epsilon)$ be the maximum cardinality of an $(n, \epsilon)$-separated subset of $K$. The topological entropy is then

$$h_{\text{top}}(f; K) := \lim_{\epsilon \to 0^+} \lim_{n \to \infty} \frac{1}{n} \log N(n, \epsilon).$$

The topological entropy of $f$ is defined as the supremum over all such ‘local entropies’. It is easy to see that when $g : (X, d) \to (X', d')$ is a submetry that is a semi-conjugacy of $f$ with $f'$, then $h_{\text{top}}(f) \geq h_{\text{top}}(f')$, i.e. the topological entropy of the factor $f'$ is a lower bound for that of $f$ [2]. By definition, the topological entropy of a flow is the topological entropy of its time-1 map.

**Theorem 1.1.** Let $\Sigma$ be a homogeneous space of $T_d$. The topological entropy of the geodesic flows of the standard Riemannian and Carnot metrics is positive.

This theorem is proven by reducing the flows to Hamiltonian flows on $t_d^*$, the dual of the Lie algebra $t_d$ of $T_d$. The Poisson sub-algebra of left-invariant Hamiltonians on $T^*T_d$ is naturally identified with the Hamiltonians on $t_d^*$ with the natural Poisson structure. The Lie group’s co-adjoint action is by Poisson automorphisms and a co-adjoint orbit $O \subset t_d^*$ is a symplectic submanifold to which all such Hamiltonian vector fields are tangent. A quadratic Hamiltonian $h : t_d^* \to \mathbb{R}$ is diagonal if it is expressed as $h = \sum a_{ij} X_{ij}^2$ for some constants $a_{ij}$.

**Theorem 1.2.** Let $h : t_d^* \to \mathbb{R}$ be a diagonal Hamiltonian with $a_{13} a_{34} = a_{12} a_{24}$ and let $E_h$ be the Hamiltonian, or Euler, vector field of $h$ on $t_d^*$. There is an open set of regular co-adjoint orbits $O \subset t_d^*$ such that $E_h|O$ has a horseshoe. In particular, the Euler vector field of the standard Carnot metric has positive topological entropy and is real analytically non-integrable.

Theorem 1.2 is proven by expressing a Melnikov integral as a quadratic form in 2-variables with coefficients that are obtained by solving a scattering problem; these coefficients are naturally expressed in terms of $\Gamma$-functions involving a parameter, called $\alpha$ below, that depends on the metric coefficients $a_{ij}$ and the co-adjoint orbit. Note that [4] asserts that the horseshoe exists on all but countably many regular co-adjoint orbits; this is inaccurate. That paper shows the horseshoe exists for all but countably many real values of the invariant $\alpha$; however, $\alpha$ may
be imaginary on an open set. This is explained in figure 1. As noted in [4], when $a_{13} = 0$, the invariant $\alpha$ is constant and one cannot therefore conclude that there is a horseshoe on any of the co-adjoint orbits. The standard Carnot metric of [7] falls into this case ($\alpha = 1$). The present paper uses an alternative approach that shows the existence of a horseshoe for all non-zero real values of $\alpha$. This is strong enough to prove the existence of a horseshoe on an open set of co-adjoint orbits, even when $a_{13}$ vanishes. It remains an open question if the Euler vector field has a horseshoe on a co-adjoint orbit where $\alpha$ is imaginary.

Theorem 1.2 implies, from the structural stability of the horseshoe, that there is an open set $W$ of quadratic Hamiltonians on $T_q^4$ each of which has a horseshoe; further, the $\text{Aut}(T_q^4)$ orbit of $W$ has this property, too. This motivates the following:

**Question 1.2.** Does there exist a quadratic Hamiltonian on $T_q^4$ which induces a non-degenerate (sub-)Riemannian structure on $T_q^4$ and which is completely integrable or has zero topological entropy?

If one drops the non-degeneracy condition, then the answer is trivially yes to both questions, as witnessed by $h = X_{14}^2$, which is a Casimir.

**1.1. Outline**

This note is organized as follows: section 2 reviews the derivation of the Melnikov form from [4]; section 3 computes the integrals that arise in the Melnikov form in terms of the scattering matrices at $\pm \infty$ in a general scattering problem; section 4 demonstrates the non-degeneracy of the Melnikov form for the particular form arising from section 2 and completes the proof of theorems 1.1 and 1.2.
2. Background

This section recalls a number of facts about left-invariant Hamiltonian systems on the cotangent bundle of a Lie group; see also [6, 4].

2.1. Poisson geometry of left-invariant Hamiltonians

A Poisson manifold is a smooth manifold \( M \) such that \( C^\infty(M) \) is equipped with a skew-symmetric bracket \( \{\cdot,\cdot\} \) that makes \( (C^\infty(M), \{\cdot,\cdot\}) \) into a Lie algebra of derivations of \( C^\infty(M) \). The centre of \( (C^\infty(M), \{\cdot,\cdot\}) \) is the set of Casimirs. A Casimir is a first integral of all Hamiltonian vector fields.

The dual space of a Lie algebra gives an example of a Poisson manifold that is not (in general) a symplectic manifold. Let \( g \) be a finite-dimensional real Lie algebra and let \( g^* \) be the dual vector space of \( g \). \( T^*_p g^* \) is identified with \( g \) for all \( p \in g^* \). The Poisson bracket on \( g^* \) is defined for all \( f, h \in C^\infty(g^*) \) and \( p \in g^* \) by

\[
\{ f, h \}(p) := -\langle p, [df_p, dh_p] \rangle,
\]

where \( \langle \cdot, \cdot \rangle : g^* \times g \to \mathbb{R} \) is the natural pairing. Recall that for \( \xi \in g \), \( \text{ad}^*_g \xi : g^* \to g^* \) is the linear map defined by \( \langle \text{ad}^*_g \xi p, \eta \rangle = -\langle p, [\xi, \eta] \rangle \). \( \text{ad}^* : g \to gl(g^*) \) is the co-adjoint representation. For any \( h \in C^\infty(g^*) \), the Hamiltonian vector field \( E_h = \{\cdot, h\} \) equals \( -\text{ad}^*_{dh(p)} p \).

Let \( G \) be a connected Lie group whose Lie algebra is \( g \). The adjoint representation of \( G \) on \( g \), \( \text{Ad}_g \xi = \frac{d}{dt} \bigg\vert_{t=0} g \exp(t\xi)g^{-1} \), induces the co-adjoint representation \( \langle \text{Ad}^*_g p, \xi \rangle = \langle p, \text{Ad}_g^{-1} \xi \rangle \) for all \( p \in g^*, g \in G \) and \( \xi \in g \). As each vector field \( p \to \text{ad}^*_g p \) is Hamiltonian on \( g^* \), with linear Hamiltonian \( h_\xi(p) = -\langle p, \xi \rangle \), the co-adjoint action of \( G \) on \( g^* \) preserves the Poisson bracket. The orbits of the co-adjoint action are called the co-adjoint orbits; a co-adjoint orbit is regular if it is a regular common level set for a maximally independent set of Casimirs. Each co-adjoint orbit is a homogeneous \( G \)-space, and every Hamiltonian vector field on \( g^* \) is tangent to each co-adjoint orbit. For this reason, the Poisson bracket \( \{\cdot, \cdot\}_g^* \) restricts to each co-adjoint orbit, and is non-degenerate on each co-adjoint orbit. Thus, the co-adjoint orbits are naturally symplectic manifolds. A Casimir is necessarily constant on each co-adjoint orbit, and in many cases (as in this paper) each regular co-adjoint orbit is the common level set of all Casimirs.

The Hamiltonian flow of a left-invariant Hamiltonian \( H \) on \( T^*G \) has the equations of motion:

\[
X_H(g, p) = \begin{cases} 
\dot{g} = T_g L_g dh(p), \\
\dot{p} = -\text{ad}^*_{dh(p)} p,
\end{cases}
\]

(3)

The reduction of the vector field \( X_H \) to \( g^* \) is the Euler vector field \( E_h \).

2.2. Poisson geometry of \( T^*T_4 \)

The Lie algebra of \( T_4 \) is

\[
\mathfrak{t}_4 = \begin{bmatrix}
0 & x & z & w \\
0 & 0 & y & u \\
0 & 0 & 0 & v \\
0 & 0 & 0 & 0
\end{bmatrix} : u, v, w, x, y, z \in \mathbb{R}
\].
Let $p_*$ be the coordinate functions on $t^*_0$ dual to the above coordinates on $t_0$. The Poisson bracket on $t^*_0$ is:

\[
\{p_*, p_y\} = -p_x, \quad \{p_*, p_u\} = -p_w, \quad \{p_*, p_v\} = -p_w.
\]

(4)

There are two independent Casimirs of $t^*_0$ are $K_1(p) = p_w, K_2(p) = p_w p_3 - p_3 p_w$. Let $K : t^*_0 \rightarrow \mathbb{R}^3$ be defined by $K = (K_1, K_2)$. The level sets of $K$ are the co-adjoint orbits of $T_3$’s action on $t^*_0$ and will be denoted by $O_k$, where $k = (k_1, k_2)$. We will say that $O_k$ is a regular co-adjoint orbit if $k_1 \neq 0$.

**Lemma 2.1.** Each regular co-adjoint orbit $O_k$ is symplectomorphic to $T^* \mathbb{R}^2$ equipped with its canonical symplectic structure.

**Proof.** Indeed, the right-hand column of the commutation relations (2.2) shows that when $k_1 = p_w \neq 0$, the coordinates $(p_*, p_u, p_v)$ are conformally symplectic and the first column is a consequence of $K_2 = k_2$ constant on $O_k$. See [4].

2.3. The Hamiltonians

Let $a_{ij} \geq 0$ be constants such that $a_{12}a_{23}a_{34} \neq 0$ and $a_{13}a_{34} = a_{12}a_{24}$. Define

\[
4h(p) = a_{12}p_1^2 + a_{23}p_2^2 + a_{13}p_3^2 + a_{24}p_4^2 + a_{34}p_3^2 + a_{14}p_4^2.
\]

(5)

As shown in [4], there is a change of coordinates that transforms $h|O_k$ to

\[
2h_k = (x^2 - \xi X^2 + v X^2) + (y^2 + \omega Y^2 + v Y^2 - 2vX^2Y^2),
\]

(6)

where $(x, X, y, Y)$ are canonically conjugate coordinates, $\xi = -(a_{13}a_{34}k_1^2 + a_{23}k_2\sqrt{a_{12}a_{34}})$, $\omega = \xi + 2a_{13}a_{34}k_2^2 = a_{13}a_{34}k_1^2 - a_{23}k_2\sqrt{a_{12}a_{34}}$.

The Casimirs may be rescaled to $c_1 = \sqrt{a_{13}a_{34}}k_1$ and $c_2 = a_{23}k_2\sqrt{a_{12}a_{34}}$. In this case, $\xi = -c_1^2 - c_2$ and $\omega = c_1^2 - c_2$. The ratio $\sqrt{\frac{\xi}{\omega}}$ is negative when $\xi < 0 < \omega$, that is, when $c_1^2 > c_2 > -c_1^2$. Otherwise, either $\xi < \omega < 0$ or $0 < \xi < \omega$. If $\epsilon > 0$, the energy level $|h_k = \epsilon|$ intersects the set $S$ of Casimir values where the ratio $\sqrt{\frac{\xi}{\omega}} > 0$, i.e. where the origin is a saddle-centre for $h_k$ (figure 1). In the degenerate case where $a_{13} = 0$, $c_1 \equiv 0$ and the ratio $\omega/\xi \equiv 1$, except when $c_2 = 0$, where it is undefined.

2.3.1. The geodesic flow and Euler equations. When $\frac{\omega}{\xi} > 0$, there is a second change of variables that transforms $h_k$ into a constant multiple of

\[
2h = x^2 + (X^2 - \frac{1}{\sqrt{\frac{\xi}{\omega}}})^2 + y^2 + \epsilon^2 Y^2 - 2X^2Y^2.
\]

(7)

For all $\epsilon > 0$, the rescaling $(y, Y) \mapsto (y, Y)/\sqrt{\epsilon}$ transforms the Hamiltonian vector-field of $h$ (equation (7)) to the non-Hamiltonian vector-field

\[
\lambda^\epsilon = \begin{cases} 
\dot{X} = x, \\
\dot{y} = X - 2X^3 + 2\epsilon XY^2, \\
\dot{Y} = -a^2 + 2X^2 \end{cases} \quad Y - 2\epsilon Y^3.
\]

(8)

2.3.2. The normally hyperbolic manifold $S$. The plane

\[
S = \{ (x, X, Y) : x = X = 0 \}
\]

is $\lambda^\epsilon$-invariant for all $\epsilon$. As shown in [4], $S$ is normally hyperbolic for all $\epsilon$. 

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2.3.3. The stable and unstable manifolds of \( S \). The function \( h : T^* \mathbb{R}^2 \rightarrow \mathbb{R} \), \( h = x^2 + (X^2 - \frac{1}{2})^2 \) is a first integral of \( \lambda_0 \). The set \( h^{-1}(\frac{1}{2}) \) is the stable and unstable manifold of \( S \), which we denote by \( W_0^\pm(S) \). On \( W_0^\pm(S) \), the flow of \( \lambda_0 \) satisfies

\[
X = \pm \text{sech}(t + t_0), \quad x = \mp \tanh(t + t_0) \text{sech}(t + t_0),
\]

\[
Y = c_0 Y_0(t + t_0) + c_1 Y_1(t + t_0), \quad y = \dot{Y},
\]

where \( X(0) = \pm \text{sech}(t_0), x(0) = \mp \tanh(t_0) \text{sech}(t_0)^2 \) and \( \{Y_j\} \) is a basis of solutions to the initial-value problem

\[
\dot{Y} + \left[ \alpha^2 - 2 \text{sech}(t)^2 \right] Y = 0,
\]

such that \( Y(0) = Y_{0,j}, \dot{Y}(0) = \dot{Y}_{0,j} \)

and \( W = Y_{0,0} \dot{Y}_{0,1} - Y_{0,1} \dot{Y}_{0,0} \neq 0 \)

while \( Y(0) = c_0 Y_0(t_0) + c_1 Y_1(t_0), y(0) = c_0 \dot{Y}_0(t_0) + c_1 \dot{Y}_1(t_0) \). The particular choice of basis is discussed in section 4.

Given a basis of solutions, this determines a coordinate system \((t_0, c_0, c_1)\) on the stable and unstable manifolds \( W_0^\pm(S) \) of \( S \) for \( \lambda_0 \).

2.3.4. The Melnikov function. By well-known results [5], the perturbed stable and unstable manifolds of \( S \), \( W_0^\pm(S) \) for \( \lambda_0 \) are, on compact sets, graphs over \( W_0^\pm(W) \). The Melnikov function \( m \) measures the \( O(\epsilon) \) separation of these graphs. In this case, it is a quadratic form in the coordinates \( c_0, c_1 \) [4]:

\[
m(t_0, c_0, c_1) = m_0 c_0^2 + 2 m_01 c_0 c_1 + m_1 c_1^2
\]

where

\[
m_{ij} = \int_{\tau \in \mathbb{R}} \dot{q}(\tau) Y_i(\tau) Y_j(\tau) \, d\tau, \quad \text{and} \quad q(\tau) = 2 \text{sech}(\tau)^2.
\]

3. Improper integrals via scattering

In [4], the coefficients \( m_{ij} \) are computed in terms of the asymptotic phase angle between an even and odd solution to (10). This section examines an alternative route to computing the coefficients \( m_{ij} \) for a general class of scattering problems and integrals like those in (12).

Definition 3.1. Two functions \( f, g \in C^1(\mathbb{R}) \) are said to be asymptotically equal at \( +\infty \), written \( f \simeq g \), if, for each \( \epsilon > 0 \), there is a \( T > 0 \) such that

\[
t \geq T \implies |f(t) - g(t)| + |f'(t) - g'(t)| < \epsilon.
\]

The definition of asymptotic equality at \(-\infty\) is similar and denoted by \( \simeq \).

Let \( q \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}) \) satisfy \( q \simeq 0 \) and \( \alpha > 0 \). Since \( q \simeq 0 \), there are solutions \( w_j^\pm \), \( j \in \{0, 1\} \), to the differential equation

\[
\dot{w} + [\alpha^2 - q(t)] w = 0
\]

such that

\[
w_j^\pm(t) \simeq \exp((-1)^j i \alpha t).
\]

Given two solutions \( w_0, w_1 \) to (14)--which are not necessarily the solutions (15)--, let

\[
J_{\pm} = \lim_{t \rightarrow \infty} \frac{1}{\alpha^2} \dot{w}_0(t) \dot{w}_1(t) + \alpha^2 w_0(t) w_1(t).
\]

Since, for each \( \sigma \in \{\pm\} \), \( \{w_{0\sigma}, w_{1\sigma}\} \) is a basis of the solution space to (14), there are constants \( a_{ij}^{\sigma} \) such that

\[
\begin{bmatrix}
w_0 \\
w_1
\end{bmatrix} = \begin{bmatrix}
a_{00}^{\sigma} & a_{01}^{\sigma} \\
a_{10}^{\sigma} & a_{11}^{\sigma}
\end{bmatrix} \begin{bmatrix}
w_0^\sigma \\
w_1^\sigma
\end{bmatrix}.
\]

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Lemma 3.1. The limits (16) exist and are equal to
\[ J_\sigma = 2 \alpha^2 \left[ a_{01}^\sigma w_{10}^\sigma + a_{00}^\sigma w_{11}^\sigma \right], \quad \text{for } \sigma \in \{\pm\}. \] (18)

Proof. Let \( \sigma = + \) or \(-\). One sees that
\[
\dot{w}_0 w_1 + \alpha^2 \dot{w}_0 w_1
= (a_{00}^\sigma \dot{w}_0^\sigma + a_{01}^\sigma \dot{w}_1^\sigma) (a_{10}^\sigma \dot{w}_0^\sigma + a_{11}^\sigma \dot{w}_1^\sigma) + \alpha^2 (a_{00}^\sigma \dot{w}_0^\sigma + a_{01}^\sigma \dot{w}_1^\sigma) (a_{10}^\sigma \dot{w}_0^\sigma + a_{11}^\sigma \dot{w}_1^\sigma)
\]
where
\[
\tilde{\sigma} - \alpha^2 (a_{00}^\sigma \exp(i\alpha t) - a_{01}^\sigma \exp(-i\alpha t)) (a_{10}^\sigma \exp(i\alpha t) - a_{11}^\sigma \exp(-i\alpha t))
+ \alpha^2 (a_{00}^\sigma \exp(i\alpha t) + a_{01}^\sigma \exp(-i\alpha t)) (a_{10}^\sigma \exp(i\alpha t) + a_{11}^\sigma \exp(-i\alpha t))
\]
which implies (18).

\[ \square \]

Theorem 3.1. Let \( w_0, w_1 \) be solutions to (14). The integral
\[ I = \int_{-\infty}^\infty \dot{q}(t) w_0(t) w_1(t) \, dt \] (19)
exists and equals \( L_+ - L_- \), that is,
\[ I = 2 \alpha^2 \left[ a_{01}^\sigma w_{10}^\sigma + a_{0\sigma}^\sigma w_{11}^\sigma - a_{01}^\sigma a_{10}^\sigma - a_{00}^\sigma a_{11}^\sigma \right]. \] (20)

Proof. The proof is similar to that in [4]. Let \( I_\sigma = \sigma \int_{-\infty}^{\infty} \dot{q}(t) w_0(t) w_1(t) \, dt \) for \( \sigma \in \{\pm\} \), so that \( I = I_+ + I_- \). Integration by parts shows that \( I_+ = -I_+ + C \) and \( I_- = -I_- - C \) where \( C = \int \alpha^2 q(0) w_0(0) w_1(0) + w_0(0) w_1(0) \) is a boundary datum.

The following is useful in computing the Melnikov coefficients \( m_{ij} \) (12).

Corollary 3.1. Let \( I_{ij} \) denote the integral (19) with \( w_0 = w_i^- \) and \( w_1 = w_j^- \) for \( i, j \in \{0, 1\} \). Then
\[ I_{ii} = -4 \alpha^2 a_{0i}^\sigma a_{i1}^\sigma, \quad I_{01} = I_{10} = 2 \alpha^2 (1 - a_{00}^\sigma a_{11}^\sigma - a_{01}^\sigma a_{10}^\sigma) \] (21)
where \( [a_{ij}^\sigma] \) is the connection matrix (17). Moreover
\[ \det[I_{ij}] = -4 \alpha^4 \left( 1 - a_{00}^\sigma a_{11}^\sigma - a_{01}^\sigma a_{10}^\sigma \right)^2 - 4 a_{00}^\sigma a_{11}^\sigma a_{01}^\sigma a_{10}^\sigma. \] (22)

4. The scattering coefficients and splitting of the invariant manifolds

To compute the \( m_{ij} \) in (12), it is useful to transform the differential equation (10) into a form that reveals its solubility in terms of hypergeometric functions. Substitution of \( z = \tanh(t) \) transforms the differential equation (10) into the Legendre differential equation [9, p 324]
\[ (1 - z^2) Y'' - 2z Y' + \left( \nu(v + 1) - \frac{\mu^2}{1 - z^2} \right) Y = 0, \] (23)
where \( \mu = i \alpha, \nu = -\frac{1}{2} + \frac{\sqrt{\alpha^2 - 1}}{2} \) and \( v = \frac{\dot{\theta}}{\theta} \).

Let \( F(a; b; c; z) \) be the \((2,1)\) hypergeometric function with moduli \( a, b, c \in \mathbb{C} \) and argument \( z \in \mathbb{C} [9, p 281], [8, section 15.2.1] \). There are four privileged solutions to (23) that are expressed in terms of these hypergeometric functions, namely,
\[
Y_0^+ = \left[ \frac{1 + z}{1 - z} \right]^\frac{\nu}{2} F \left( a, b; c; \frac{1 - z}{2} \right), \quad Y_1^+ = \left[ \frac{1 + z}{1 - z} \right]^{-\frac{\nu}{2}} F \left( a, b; c; \frac{1 - z}{2} \right),
\]
\[
Y_0^- = \left[ \frac{1 - z}{1 + z} \right]^\frac{\nu}{2} F \left( a, b; c; \frac{1 + z}{2} \right), \quad Y_1^- = \left[ \frac{1 - z}{1 + z} \right]^{-\frac{\nu}{2}} F \left( a, b; c; \frac{1 + z}{2} \right).
\] (24)
where \( a = \frac{1}{4} + \frac{\sqrt{3}}{2} \), \( b = \bar{a} \) and \( c = 1 - i\alpha \) [9, p 286]. (The notation is explained thus: the group \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) acts by automorphisms of (23) by changing the sign of \( a \) and \( z \) independently. From the fact that as \( \pm t \to \infty, \pm z \to 1 \) and \( F(a, b, c; 0) = 1 \), it is apparent that
\[
Y_0^+ \sim \exp(iat) \quad Y_0^- \sim \exp(-iat)
\]
\[
Y_0^+ \sim \exp(iat) \quad Y_0^- \sim \exp(-iat),
\]
(viewed as functions of \( t = \tanh^{-1}(z) \).

The linear transformation rules for hypergeometric functions [1, 15.3.3, 15.3.6] imply the relations
\[
Y_0^- = AY_0^+ + BY_1^+ \quad Y_1^- = \bar{B}Y_0^+ + \bar{A}Y_1^+ \quad \text{where}
\]
\[
A = \frac{\Gamma(c) \Gamma(1-c)}{\Gamma(a) \Gamma(b)} \quad B = \frac{\Gamma(c) \Gamma(c-1)}{\Gamma(c-a) \Gamma(c-b)}
\]
so the connection matrices are
\[
[a_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad [a_{ij}'] = \begin{bmatrix} A & B \\ \bar{B} & \bar{A} \end{bmatrix}.
\]

**Lemma 4.1.** Let \( s = \sqrt{7}/2 \). Then
\[
|B|^2 = \frac{\cosh(2\pi \alpha) + \cosh(2\pi s)}{\cosh(2\pi \alpha) - 1}, \quad |A|^2 = \left( \frac{\cosh(\pi s)}{\sinh(\pi \alpha)} \right)^2.
\]
So, \(|B|\) exceeds unity for all \( \alpha \in \mathbb{R}, \alpha \neq 0 \) and \(|A|/|B|\) is maximized at \( \alpha = 0 \) and decreases monotonically to 0 as \( \alpha \to \infty \).

**Proof.** Assume that \( x \neq 0 \). The reflection formula for the \( \Gamma \)-function implies that
\[
|\Gamma(\frac{1}{2} + ix)|^2 = \frac{\pi}{\cosh(\pi x)}, \quad |\Gamma(ix)|^2 = \frac{\pi}{|x\sinh(\pi x)|} \quad \text{and} \quad |\Gamma(1 + ix)|^2 = \frac{\pi |x|}{|x\sinh(\pi x)|} \quad [1, \S 6.1.17, 6.1.29–31].
\]
Then, since \( c = 1 - i\alpha \) and \( c - a = \frac{1}{2} + i(-\alpha - s) \),
\[
|B|^2 = \left| \frac{\Gamma(1-i\alpha) \Gamma(-i\alpha)}{\Gamma(\frac{1}{2} + i(-\alpha - s)) \Gamma(\frac{1}{2} + i(-\alpha + s))} \right|^2 = \frac{\cosh(\pi (-\alpha-s)) \cosh(\pi (-\alpha+s))}{\sinh(\pi \alpha)^2}
\]
which yields the first part of (28) and implies \(|B| \geq 1\) and \( > 1 \) if \( \alpha \neq 0 \). A similar computation shows the second part. This implies that
\[
|A|^2/|B|^2 = \cosh(\pi s)^2 / (\cosh(\pi s)^2 + \cosh(\pi \alpha)^2 - 1) \leq 1
\]
and \(< 1\) when \( \alpha \neq 0 \) and decreases monotonically as \( \alpha \to \infty \).

**Theorem 4.1.** The Melnikov form with coefficients \( m_{ij} \) (12) for the basis \( Y_j = Y_j^- \) is non-degenerate and indefinite for all \( \alpha \in \mathbb{R}, \alpha \neq 0 \).

**Proof.** When corollary 3.1 is applied, with the connection coefficients in (26), one computes that \( m_{ij} = I_{ij} \) so
\[
\det[m_{ij}] = -4 \alpha^4 ((1 - |A|^2 - |B|^2)^2 - 4|A|^2|B|^2).
\]
By hypothesis, \( \alpha \neq 0 \), so \( \det[m_{ij}] \) vanishes iff \(|A| \pm |B| = \pm 1\). By lemma 4.1, the only possible equation to be satisfied is \(|A| = |B| - 1\). If this latter equation is satisfied,
then

\[ \cosh(\pi s) + \sinh(\pi \alpha) = \cosh(\pi s) + \cosh(\pi \alpha)^2 - 1, \]

which implies that \( \alpha = 0 \). Therefore, \( \det[m_{ij}] \neq 0 \) for all real, non-zero \( \alpha \).

Indefiniteness of the Melnikov form follows from the even-ness of the potential \( q(t) = -2 \sech(t)^2 \): non-trivial even and odd solutions to (10) exist and the Melnikov form vanishes on these solutions by theorem 3.1.

\[ \square \]

Remark 4.1. Let us compare the results for the Melnikov form (12) to that obtained in [4]. The Melnikov form \( [m_{ij}] \) relative to the basis \( \{Y_0^+, Y_1^+\} \) has been computed to be

\[ M_y = -2\alpha^2 \left[ \begin{array}{ccc} 2AB & |B|^2 + |A|^2 - 1 \\ |B|^2 + |A|^2 - 1 & 2\bar{A}\bar{B} \end{array} \right]. \]  

(30)

Let \( W_0, W_1 \) be a pair of solutions whose Wronskian matrix is the identity at \( z = 0 \); in particular, \( W_0 \) (resp. \( W_1 \)) is an even (resp. odd) solution. The Melnikov form in this basis is equal to

\[ M_w = 2\alpha \cot(\beta) \times \left[ \begin{array}{ccc} 0 & 1 \\ 1 & 0 \end{array} \right]. \]  

(31)

The change of variables formula for quadratic forms implies that \( \det M_y = (-2i\alpha)^2 \times \det M_w \), where the first term on the right arises from the Wronskian of \( \{Y_0^+, Y_1^+\} \). This implies that

\[ 4 \cot(\beta)^2 = 4|A|^2|B|^2 - (1 - |A|^2 - |B|^2)^2. \]  

(32)

From this, and lemma 4.1, one can numerically compute the phase angle \( \beta \) and the integral \( I \) as functions of \( \alpha \). These are depicted in figure 2(b). In [4, figure 1], these quantities were determined by numerically solving the initial-value problem 10. The absolute and relative errors between the closed form solutions from equation (32) and the numerical approximations in [4] are depicted in figure 3. This figure shows the approximations are extremely good, with a mean absolute error of approximately \( 1.6 \times 10^{-9} \). One can also show that \( I(0^+) = -2 \cosh(\sqrt{7}\pi/2)/\pi \cong -20.317 \), which is in close agreement with figure 2(b).\(^1\)

\(^1\) The angle \( B \) reported in [4, figure 1] equals \( \pi + \beta \) in the present paper.
Figure 3. Error in numerical integration of $I(\alpha)$ versus closed form.

Acknowledgment

The author thanks Adri Olde Daalhuis for his helpful comments on an early draft of this paper.

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