A singular elliptic problem involving fractional $p$-Laplacian and a discontinuous critical nonlinearity

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Abstract
In this article, we prove the existence of solutions to a nonlinear nonlocal elliptic problem with a singularity and a discontinuous critical nonlinearity which is given as follows.

$$(-\Delta)^s_p u = \mu g(x, u) + \frac{\lambda}{u^\gamma} + H(u - \alpha)u^{p^*_s - 1}, \text{ in } \Omega$$

$$u > 0, \text{ in } \Omega,$$

$$u = 0, \text{ in } \mathbb{R}^N \setminus \Omega,$$

(0.1)

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s \in (0, 1)$, $2 < p < \frac{N}{s}$, $\gamma \in (0, 1)$, $\lambda, \mu > 0$, $\alpha \geq 0$ is real, $H$ is the Heaviside function, i.e. $H(a) = 0$ if $a \leq 0$, $H(a) = 1$ if $a > 0$ and $p^*_s = \frac{Np}{N-сп}$ is the fractional critical Sobolev exponent.

Under suitable assumptions on the function $g$, we prove the existence of solution to the problem. Furthermore, we show that as $\alpha \to 0^+$, the sequence of solutions of (0.1) for each such $\alpha$ converges to a solution of the problem for which $\alpha = 0$.

Keywords: Fractional $p$-Laplacian, Heaviside function, Mountain pass theorem, Critical exponent, Singularity.

AMS Classification: 35R11, 35J75, 35J60, 46E35.

1. Introduction
We will study the existence of solution to the following nonlinear, nonlocal problem involving a singularity and a discontinuous critical nonlinearity.

$$(-\Delta)^s_p u = \mu g(x, u) + \frac{\lambda}{u^\gamma} + H(u - \alpha)u^{p^*_s - 1}, \text{ in } \Omega$$

$$u > 0, \text{ in } \Omega,$$

$$u = 0, \text{ in } \mathbb{R}^N \setminus \Omega,$$

(P$_\alpha$)

We impose the hypotheses on $g$ which are as follows.

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Preprint submitted to Elsevier March 16, 2021
(g1) The function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a $N$-measurable function and $g(x, u) = 0$ if $u \leq 0$.

(g2) There exist $K > 0$ and $r \in (p, p^*_s)$ such that $|g(x, u)| \leq K(1 + |u|^{r-1})$ for every $u \geq 0$.

(g3) There exists $b > p$ and $v > 0$ such that for all $u \geq v$

$$0 < bG(x, u) = b \int_0^u g(x, \tau)d\tau \leq ug(x, u).$$

(g4) There exists $\beta > 0$ (that will be fixed later) such that $H(u - \beta) \leq g(x, u)$ uniformly in $\Omega \times (0, \infty)$.

(g5) Let $\lambda_1$ be the first eigen value of $(-\Delta)^p$ defined in (2.2). Then $\lim_{u \to 0} \frac{g(x, u)}{u^{p-1}} \leq \lambda_1$ uniformly in $\Omega$.

A prototype of $g$ satisfying the assumptions $(g1) - (g5)$ is $H(t - \beta)t^{r-1}/\beta^r$.

The problems of type $(P_{\alpha})$ having discontinuous nonlinearities have many applications in free boundary problems of mathematical physics. For instance, obstacle problem, Elenbaas equations, the seepage surface problem etc. Refer [3, 11, 12, 13] for further details.

Elliptic problems involving critical and discontinuous nonlinearities can be treated by different techniques. Amongst these methods, variational methods for nondifferentiable functionals, dual variational principle, Palais principle of symmetric criticality for locally Lipschitz functional, lower-upper solution method, theory of multivalued mappings and global branching are a few well known techniques. Badiale & Tarantello in [6] studied the following class of problem using variational methods with lower-upper solution methods.

$$(-\Delta)u = \delta H(u - \alpha) + u^{2^*_s-1}, \text{ in } \Omega$$

$$u = 0, \text{ on } \partial \Omega.$$  \hfill (1.1)

Here $2^*_s = 2N/(N-2)$, $\delta, \alpha > 0$ and $H$ is the Heaviside function. Later, the authors in [2] generalized the work of [6] in $\mathbb{R}^N$. Badiale in [5] proved the existence result for the critical elliptic problem given by

$$(-\Delta)u = g(u) + u^{2^*_s-1}, \text{ in } \Omega$$

$$u = 0, \text{ on } \partial \Omega,$$  \hfill (1.2)

where $g$ can have discontinuities. The authors in [16] and [18] extended the result of [5] for a Kirchhoff type problem involving critical Caffarelli-Kohn-Nirenberg growth and for a Schrödinger-Kirchhoff equation, respectively. Recently, Santos & Tavares in [17] considered the problem

$$\mathcal{L}_Ku = g(x, u) + H(u - \alpha)u^{2^*_s-2}u, \text{ in } \Omega$$

$$u = 0, \text{ in } \mathbb{R}^N \setminus \Omega,$$  \hfill (1.3)

$$u \geq 0, \text{ in } \Omega,$$

where $2^*_s = 2N/(N-2s)$, $\alpha > 0$, $g$ is a discontinuous function and $\mathcal{L}_K$ is a nonlocal operator

$$\mathcal{L}_Ku(x) = \iint_{\mathbb{R}^{2N}} (u(x+y) + u(x-y) - 2u(x))K(y)dy.$$
They used the nonsmooth version of Mountain pass theorem to investigate the existence and the behavior of solution for problem (1.3). We also cite [1, 3, 4, 7, 11, 13, 14, 32, 34] and the references therein for readers to have a glimpse of the problems of the type as in (1.1) – (1.3).

Inspired by the above works, specifically [5, 6, 11, 12, 17] we analyze our problem (P_α). The problem (P_α) with singularity, critical and discontinuous nonlinearities is a new and first work in the literature, at least to our knowledge. But we find enormous works dealing with the following class of problems involving singularity and critical nonlinearity given by

\[-\Delta u = \lambda g_1(x) \frac{u^q}{u^\gamma} + \delta u^{q-1}, \text{ in } \Omega \]

\[u = 0, \text{ in } \mathbb{R}^N \setminus \Omega,\]

\[u > 0, \text{ in } \Omega,\]  

where \(\lambda, \delta > 0, q \in (1, p^*_s], g_1 > 0\) is bounded. Several techniques like variational method, concentration compactness method, Nehari manifold method etc. have been applied to study the problems of type (1.4) for both local and nonlocal cases. Refer [15, 21, 22, 23, 24, 25, 27, 28, 30, 31] and the bibliography therein.

The main result of this article is the following.

**Theorem 1.1.** Let \((g_1) - (g_5)\) hold. Then

1. there exist \(\bar{\alpha}, \bar{\lambda}, \bar{\mu} > 0\) such that for every \(a \in (0, \bar{\alpha})\), every \(\lambda \in (0, \bar{\lambda})\) and every \(\mu \in (0, \bar{\mu})\) the problem \((P_\alpha)\) admits at least one nontrivial weak solution \(u_\alpha\). Furthermore, the lebesgue measure of the set \(\{x \in \Omega : u_\alpha > \alpha\}\) is positive

2. for any sequence \(\alpha_n \in (0, \bar{\alpha})\) with \(\alpha_n \to 0^+\), we have, up to a subsequential level, \(u_{\alpha_n} \to u_0\) in \(W^{s,p}_0(\Omega)\), where \(u_0\) is a weak solution of the problem \((P_0)\), i.e.

\[-\Delta u = \mu g(x, u) + \frac{\lambda}{u^\gamma} + u^{p^*_s-1}, \text{ in } \Omega\]

\[u = 0, \text{ in } \mathbb{R}^N \setminus \Omega,\]

\[u > 0, \text{ in } \Omega.\]  

The proof of the main result, i.e Theorem 1.1, has been splitted into two sections. Section 3 is devoted to the first part of Theorem 1.1, i.e. the existence of a weak solution \(u_\alpha\) to \((P_\alpha)\). In Section 4, we examine the nature of the sequence \((u_\alpha)\) as \(\alpha \to 0^+\) and prove the second part of Theorem 1.1.

2. **Nonsmooth critical point theory**

Let us fix \(0 < s < 1, 2 < p < \frac{N}{s}\), \(\Omega\) be an open and bounded domain of \(\mathbb{R}^N\). We denote \(Q = (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\Omega^c \times \Omega^c)\) where \(\Omega^c = \mathbb{R}^N \setminus \Omega\). We define the fractional Sobolev space by

\[W^{s,p}(\Omega) = \{u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable : } u|_\Omega \in L^p(\Omega), \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} dy dx < \infty\} \]
equipped with the norm
\[ \|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left( \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\,dydx \right)^{1/p}. \]

We further define the space
\[ W_0^{s,p}(\Omega) = \{ u \in W^{s,p}(\Omega) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \} \]
and \((W_0^{s,p}(\Omega), \| \cdot \|_{W_0^{s,p}(\Omega)})\) is a reflexive Banach space where the fractional Sobolev norm is given by
\[ \|u\|_{W_0^{s,p}(\Omega)}^p = \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\,dydx. \]

Given below are a few well known embedding results for the space \(W_0^{s,p}(\Omega)\).

**Theorem 2.1** ([33]). The following results hold for the fractional Sobolev space \(W_0^{s,p}(\Omega)\).

1. If \(\Omega\) has a continuous boundary, then the embedding \(W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)\) is compact for every \(q \in [1, p^*_s)\).

2. The embedding \(W_0^{s,p}(\Omega) \hookrightarrow L^{p^*_s}(\Omega)\) is continuous.

We now define the best constant \(S_{s,p} > 0\) given by
\[ S_{s,p} = \inf_{u \in W_0^{s,p}(\Omega) \setminus \{0\}} \frac{\int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\,dydx}{(\int_\Omega |u|^{p^*_s}dx)^{\frac{1}{p^*_s}}} \tag{2.1} \]
and \(S_{s,p}\) is well-defined due to Theorem 2.1.

**Theorem 2.2** ([8], Theorem 4.1 of [19]). Let \(s \in (0, 1)\) and \(p > 1\). Then the eigenvalue problem
\[ (-\Delta)_s u = \lambda |u|^{p-2}u, \text{ in } \Omega \]
\[ u = 0, \text{ in } \mathbb{R}^N \setminus \Omega \]
possesses a smallest eigenvalue \(\lambda_1 > 0\) given by
\[ \lambda_1 = \min_{u \in W_0^{s,p}(\Omega) : \|u\|_{L^p(\Omega)} = 1} \|u\|_{W_0^{s,p}(\Omega)}^p. \tag{2.2} \]

Let \(V\) be a Banach space. Then a functional \(J : V \to \mathbb{R}\) is said to be locally Lipschitz continuous if for any \(u \in V\) there exists an open neighborhood \(N := N_u \subset V\) and some constant \(C := C_N > 0\) such that
\[ |J(u_1) - J(u_2)| \leq C \|u_1 - u_2\|_V, \quad u_1, u_2 \in V. \]

We define the directional derivative of \(J\) at \(u\) in the direction of \(z \in V\) by
\[ \tilde{J}(u; z) = \lim_{h \to 0} \lim_{\xi \to 0} \frac{J(u + h + \xi z) - J(u + h)}{\xi}. \]
Thus, \( \tilde{J}(u; \cdot) \) is convex, continuous and its subdifferential at \( w \in V \) is the set
\[
\partial \tilde{J}(u; w) = \{ \nu \in V^* : \tilde{J}(u; z) \geq \tilde{J}(u; w) + \langle \nu, z - w \rangle, \; z \in V \}.
\]
Here \( \langle \cdot, \cdot \rangle \) is the duality pair between \( V \) and \( V^* \). The generalized gradient of \( J \) at \( u \) is defined by
\[
\partial J(u) = \{ \nu \in V^* : \langle \nu, z \rangle \leq \tilde{J}(u; z), \; z \in V \}
\]
and is convex and weak*-compact. Clearly, \( \partial J(u) \) is nonempty and is the subdifferential of \( \partial \tilde{J}(u; 0) \) as \( \tilde{J}(u; 0) = 0 \).

We say \( \bar{u} \) to be a critical point of \( J \) if \( 0 \in \partial J(\bar{u}) \) and \( c \in \mathbb{R} \) to be a critical value of \( J \) if \( J(\bar{u}) = c \) for some critical point \( \bar{u} \in V \). Moreover, if \( J \) is a \( C^1 \) functional then \( \partial J(u) = \{ J'(u) \} \). We now denote
\[
\Lambda_J(u) = \min \{ \| \nu \|_{V^*} : \nu \in \partial J(u) \}.
\]
The following result is the Mountain Pass Theorem for locally Lipschitz non-differentiable functional.

**Theorem 2.3** ([26, 32]). Let \( V \) be a Banach space and \( J \) be a locally Lipschitz functional with \( J(0) = 0 \). Assume that there exists \( \rho_1, \rho_2 > 0 \) and \( \sigma \in V \) such that

1. \( J(u) \geq \rho_2, \) for every \( u \in V; \; \|u\|_V = \rho_1 \),
2. \( J(\sigma) < 0 \) and \( \| \sigma \|_V > \rho_1 \).

Let
\[
c = \inf \max_{\zeta \in \Gamma} J(\zeta(t)), \; \Gamma = \{ \zeta \in C([0, 1]; V) : \zeta(0) = 0, \zeta(1) = \sigma \}.
\]
Then \( c \geq \rho_2 \) and there exists a Palais-Smale \((PS)_c\) sequence \((u_n) \subset V\) satisfying
\[
J(u_n) \to c \text{ and } \Lambda_J(u_n) \to 0.
\]

Moreover, if \( J \) satisfies the nonsmooth Palais-Smale \((PS)_c\) condition, i.e. every \((PS)_c\) sequence has a convergent subsequence, then \( c \) is a critical value of \( J \).

**Proposition 2.4** ([26]). Let \((u_n) \subset V\) and \((\theta_n) \subset V^*\) with \( \theta_n \in \partial J(u_n) \). If \( u_n \to u \) in \( V \) and \( \theta_n \rightharpoonup \theta \) in \( V^* \), then \( \theta \in \partial J(u) \).

A function \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is said to be a \( N \)-measurable function if for every \( u \in W^{s,p}_0(\Omega) \) the function \( g(\cdot, u(\cdot)) : V \to \mathbb{R} \) is measurable (refer [11]). Let \( g(x, \cdot) \in L^1_{loc}(\Omega) \), then we denote
\[
\underline{g}(x, u) = \lim_{\delta \downarrow 0} \inf_{|v-u|<\delta} g(x, v), \; \overline{g}(x, u) = \lim_{\delta \downarrow 0} \sup_{|v-u|<\delta} g(x, v).
\]
We now provide the notion of weak solution to \((P_\alpha)\) (influenced by [17]). We say a function \( u \in W^{s,p}_0(\Omega) \) to be a weak solution of \((P_\alpha)\) if \( u > 0 \) a.e. in \( \Omega \), \( u^{-\gamma} \in L^1_{loc}(\Omega) \) and there exist \( \eta_\alpha \in L^{\frac{r}{r-\gamma}}(\Omega) \) and \( \theta_\alpha \in L^{\frac{p}{p-\gamma}}(\Omega) \) such that for every \( \varphi \in W^{s,p}_0(\Omega) \)
\[
\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}}(\varphi(x) - \varphi(y))dxdy - \mu \int_{\Omega} \eta_\alpha \varphi dx - \int_{\Omega} \theta_\alpha \varphi dx - \lambda \int_{\Omega} \varphi \gamma d\varphi = 0. \tag{2.3}
\]
Here \( \eta_\alpha(x) \in [\underline{g}(x, u(x)), \overline{g}(x, u(x))] \) and \( \theta_\alpha(x) \in [\underline{f}_\alpha(u(x)), \overline{f}_\alpha(u(x))] \) a.e. in \( \Omega \) with \( f_\alpha(t) = H(t - \alpha)t^{p-1} \).
3. Existence result for \( P_\alpha \) - Proof of Theorem 1.1 (1)

The energy functional corresponding to the problem \( P_\alpha \) is \( J_\alpha^0 : W_0^{s,p}(\Omega) \rightarrow \mathbb{R} \) defined by

\[
J_\alpha^0(u) = \frac{1}{p} \int_{\mathbb{R}^2N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx dy - \frac{\lambda}{1-\gamma} \int_\Omega (u^+)^{1-\gamma} - \mu \int_\Omega G(x, u) - \int_\Omega F_\alpha(u),
\]

where \( G(x, u) = \int_0^u g(x, \tau) \, d\tau \), \( F_\alpha(u) = \int_0^u f_\alpha(\tau) \, d\tau \) with \( f_\alpha(u) = H(u-\alpha)u^{p_\ast-1} \). Let us denote

\[
I_\lambda^0(u) = \frac{1}{p} \int_{\mathbb{R}^2N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx dy - \frac{\lambda}{1-\gamma} \int_\Omega (u^+)^{1-\gamma}, \quad I_\mu(u) = \int_\Omega G(x, u), \quad I_\alpha(u) = \int_\Omega F_\alpha(u).
\]

We provide some properties of the functionals \( I_\mu \) and \( I_\alpha \) in the following lemma and these results can be proved by following the arguments of Lemma 3.1 of [16].

**Lemma 3.1.** The functionals \( I_\mu : L^r(\Omega) \rightarrow \mathbb{R}, \ I_\alpha : L^{p_\ast}(\Omega) \rightarrow \mathbb{R} \) are locally Lipschitz functionals and they satisfy the followings.

1. \( \partial I_\mu(u) \subset [g(x, u(x))/\varphi(x, u(x))] \) a.e. in \( \Omega \).
2. \( \partial I_\alpha(u) \subset [f_\alpha(u(x)), \varphi_\alpha(u(x))] \) a.e. in \( \Omega \).
3. \[
[f_\alpha(u), \varphi_\alpha(u)] = \begin{cases} 
\{0\}, & \text{if } u < a \\
[0, u^{p_\ast-1}], & \text{if } u = a \\
\{u^{p_\ast-1}\}, & \text{if } u > a.
\end{cases}
\]

**Remark 3.2.** The inclusion (1) of Lemma 3.1 imply that if \( \eta \in \partial I_\mu(u) \) then \( \eta \in L^{r-1}(\Omega) \) and \( \eta(x) \subset [g(x, u(x))/\varphi(x, u(x))] \) a.e. in \( \Omega \). The same argument follows for (2).

The functional \( J_\alpha^0 \) is not of class \( C^1 \) due to the presence of a singularity. To tackle this issue, we follow the truncation technique. Let us consider the problem

\[
(-\Delta)_w^\gamma u = \frac{\lambda}{u^{\gamma}}, \quad \text{in } \Omega \\
u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \\
\ D^\gamma u > 0, \quad \text{in } \Omega.
\]

According to Canino et al. in [10], we have the existence result for \( (3.3) \) as given below.

**Lemma 3.3.** Let \( \gamma \in (0, 1) \) and \( \lambda > 0 \). Then \( (3.3) \) admits a unique nontrivial weak solution \( u_\lambda \) in \( W_0^{s,p}(\Omega) \) such that for every \( \omega \subset \subset \Omega \) we have \( \text{ess inf}_\omega u_\lambda > 0 \).

We now define

\[
\psi(x, t) = \begin{cases} 
t^{-\gamma}, & \text{if } t > u_\lambda \\
u_\lambda^{-\gamma}, & \text{if } t \leq u_\lambda.
\end{cases}
\]
where $u$ is the unique solution to (3.3). Further, define a function $I_\lambda : W_0^{s,p}(\Omega) \to \mathbb{R}$ by

$$I_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy - \lambda \int_\Omega \Psi(x,u) dx$$

(3.5)

where $\Psi(x,t) = \int_0^t \psi(x,\tau)d\tau$.

We now consider the following cutoff problem.

$$(-\Delta)^s_p w = \mu g(x,w) + \lambda \psi(x,w) + H(w - \alpha)w^{p_s-1}, \text{ in } \Omega$$

$$w = 0, \text{ in } \mathbb{R}^N \setminus \Omega,$$

$$w > 0, \text{ in } \Omega.$$  

(3.6)

A function $w \in W_0^{s,p}(\Omega)$ is said to be a weak solution of (3.6) if $w > 0$ a.e. in $\Omega$ and there exist $\eta_\alpha \in L^{s,p}(\Omega)$ and $\theta_\alpha \in L^{s,p}(\Omega)$ such that for every $\varphi \in W_0^{s,p}(\Omega)$

$$\int_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))}{|x - y|^{N+sp}}(\varphi(x) - \varphi(y))dx dy - \mu \int_\Omega \eta_\alpha \varphi dx - \lambda \int_\Omega \psi(x,w)\varphi dx = 0,$$

where $\eta_\alpha(x) \in [g(x,w(x)),\overline{g}(x,w(x))]$ and $\theta_\alpha(x) \in [f_\alpha(w(x)),\overline{f}_\alpha(w(x))]$ a.e. in $\Omega$.

The associated functional of (3.6) is $J_\alpha : W_0^{s,p}(\Omega) \to \mathbb{R}$ defined by

$$J_\alpha(w) = I_\lambda(w) - I_\mu(w) - I_\alpha(w).$$

(3.8)

The functionals $I_\lambda, I_\mu$ and $I_\alpha$ are given in (3.1) and (3.3). By Lemma 3.1, the functional $J_\alpha$ is a locally Lipschitz functional on $W_0^{s,p}(\Omega)$ and thus using Proposition 1.3.12 and 1.3.13 of [20] we obtain

$$\partial J_\alpha(w) \subset \{I_\lambda'(w)\} - \partial I_\mu(w) - \partial I_\alpha(w), \text{ for all } w \in W_0^{s,p}(\Omega).$$

(3.9)

It is easy to check that if $w$ is a weak solution to (3.6) with $w \geq u_\lambda$ a.e. in $\Omega$, then $w$ is also a weak solution to (3.6). Hence, with the help of a comparison principle and non-smooth variational approach we prove our main result.

**Lemma 3.4.** Any sequence $(w_n) \subset W_0^{s,p}(\Omega)$ satisfying $J_\alpha(w_n) \to c$ and $\Lambda_{J_\alpha}(w_n) \to 0$ is bounded in $W_0^{s,p}(\Omega)$.

**Proof.** Suppose $(v_n) \subset (W_0^{s,p}(\Omega))'$ is a sequence with $\|v_n\|_{(W_0^{s,p}(\Omega))'} = \Lambda_{J_\alpha}(w_n)$, where $(W_0^{s,p}(\Omega))'$ is the dual of $W_0^{s,p}(\Omega)$. This implies $v_n \in \partial J_\alpha(w_n)$. From (3.9), there exist $\eta_\alpha \in \partial I_\mu(w_n)$ and $\theta_\alpha \in \partial I_\alpha(w_n)$ that satisfy

$$\langle v_n, \varphi \rangle = \int_{\mathbb{R}^{2N}} \frac{|w_n(x) - w_n(y)|^{p-2}(w_n(x) - w_n(y))}{|x - y|^{N+sp}}(\varphi(x) - \varphi(y))dx dy - \lambda \int_\Omega \psi(x,w_n)\varphi dx$$

$$- \mu \langle \eta_\alpha, \varphi \rangle - \langle \theta_\alpha, \varphi \rangle, \forall \varphi \in W_0^{s,p}(\Omega).$$

(3.10)

By (3.8), (3.10) and Lemma 3.1, we get

$$J_\alpha(w_n) - \frac{1}{b} \langle v_n, w_n \rangle = \left(\frac{1}{p} - \frac{1}{b}\right) \|w_n\|^p_{W_0^{s,p}(\Omega)} + \frac{\lambda}{b} \int_\Omega \psi(x,w_n)w_n - \lambda \int_\Omega \Psi(x,w_n)$$

$$+ \mu \int_\Omega \left(\frac{1}{b} \eta_\alpha w_n - G(x,w_n)\right) + \int_\Omega \left(\frac{1}{b} \theta_\alpha w_n - F_\alpha(w_n)\right).$$

(3.11)
According to (g3),
\[
\int_{\Omega} \left( \frac{1}{b} \eta_n w_n - G(x, w_n) \right) \geq \int_{\Omega} \left( \frac{1}{b} \eta_n w_n - G(x, w_n) \right)
\]  \hspace{1cm} (3.12)
and using (g1), (g2) we obtain the following uniform bound.
\[
\left| \int_{\{w_n \leq v\}} \left( \frac{1}{b} \eta_n w_n - G(x, w_n) \right) \right| \leq K \left[ \left( \frac{1}{b} + 1 \right) v + \left( \frac{1}{b} + \frac{1}{r} \right) v^r \right] |\Omega| = C_1, \forall n \in \mathbb{N}. \hspace{1cm} (3.13)
\]
From Lemma 3.1 (3), we rewrite the last term of (3.11) as follows.
\[
\int \frac{1}{b} \theta_n w_n - F_n(w_n) = \left( \frac{1}{b} - \frac{1}{p_s^*} \right) \int_{\{w_n > \alpha\}} |w_n|^{p_s^*} + \int_{\{w_n = \alpha\}} \frac{1}{b} \theta_n w_n + \frac{1}{p_s^*} \int_{\{w_n > \alpha\}} \alpha^{p_s^*}. \hspace{1cm} (3.14)
\]
We have used the fact that \( F_n(t) = \chi_{\{t \geq \alpha\}} \frac{1}{p_s^*} (|t|^{p_s^*} - \alpha^{p_s^*}) \) for all \( t \in \mathbb{R} \) to obtain (3.14).
On combining (3.11) - (3.14) we get
\[
J_\alpha(w_n) - \frac{1}{b} \langle v_n, w_n \rangle \geq \left( \frac{1}{p} - \frac{1}{b} \right) \|v_n\|_{W_0^{s,p}(\Omega)}^{p_s^*} - \frac{\lambda}{1 - \gamma} \|w_n\|_{L^{1-\gamma}(\Omega)}^{1-\gamma} - \mu C_1 + \left( \frac{1}{b} - \frac{1}{p_s^*} \right) \int_{\{w_n > \alpha\}} |w_n|^{p_s^*}. \hspace{1cm} (3.15)
\]
Using Theorem 2.1 and (2.1) in (3.15) we establish the following.
\[
J_\alpha(w_n) - \frac{1}{b} \langle v_n, w_n \rangle \geq \left( \frac{1}{p} - \frac{1}{b} \right) \|v_n\|_{W_0^{s,p}(\Omega)}^{p_s^*} - \frac{\lambda}{1 - \gamma} \|w_n\|_{L^{1-\gamma}(\Omega)}^{1-\gamma} - \mu C_1 + \left( \frac{1}{b} - \frac{1}{p_s^*} \right) \int_{\{w_n > \alpha\}} |w_n|^{p_s^*}. \hspace{1cm} (3.16)
\]
We already have \( J_\alpha(w_n) = c + o_n(1) \) and \( \|v_n\|_{W_0^{s,p}(\Omega)^p} = o_n(1) \). Thus,
\[
J_\alpha(w_n) - \frac{1}{b} \langle v_n, w_n \rangle \leq |J_\alpha(w_n)| + \frac{1}{b} \|v_n\|_{W_0^{s,p}(\Omega)^p} \|w_n\|_{W_0^{s,p}(\Omega)} \leq c + 1 + \|w_n\|_{W_0^{s,p}(\Omega)} + o_n(1). \hspace{1cm} (3.17)
\]
Since \( p < b < p_s^* \) and \( \alpha > 0 \), from (3.16) and (3.17), we obtain
\[
c + 1 + \|w_n\|_{W_0^{s,p}(\Omega)} + o_n(1) \geq \left( \frac{1}{p} - \frac{1}{b} \right) \|v_n\|_{W_0^{s,p}(\Omega)}^{p_s^*} - \frac{\lambda}{1 - \gamma} \|w_n\|_{L^{1-\gamma}(\Omega)}^{1-\gamma} - \mu C_1. \hspace{1cm} (3.18)
\]
With the consideration of the above inequality (3.18), we conclude that \( (w_n) \) is a bounded sequence in \( W_0^{s,p}(\Omega) \).

**Proposition 3.5.** Let \( (w_n) \subset W_0^{s,p}(\Omega) \) be a non-smooth (PS)_c sequence such that \( J_\alpha(w_n) \to c \) and \( \Lambda_{\Lambda_n}(w_n) \to 0 \) with
\[
c < \left( \frac{1}{p} - \frac{1}{b} \right) S_{s,p}^{p_s^*} - \left( \frac{1}{p} - \frac{1}{b} \right) \frac{\lambda}{1 - \gamma} \|w_n\|_{W_0^{s,p}(\Omega)}^{p_s^*} \left( \frac{1 + \gamma}{p_s^*} \right)^{\frac{p_s^*}{1 + \gamma} - \frac{p}{1 + \gamma}} - \mu C_1 = c^*, \hspace{1cm} (3.19)
\]
where \( S_{s,p} \) is defined in (2.1) and \( C_1 \) is given in (3.13). Then \( J_\alpha \) satisfies the non-smooth (PS)_c condition, i.e. \( (w_n) \) admits a strongly convergent subsequence. Further, \( c^* > 0 \) for a sufficiently small \( \mu \).
Proof. According to Lemma 3.4, the (PS) sequence \((w_n)\) is bounded in \(W_0^{s,p}(\Omega)\). Let \(v_n, \eta_n\) and \(\theta_n\) are same as used in the proof of Lemma 3.4. Then by \((g_2)\) and Lemma 3.1, we establish that \((\eta_n)\) and \((\theta_n)\) are bounded in \(L^{s,p}(\Omega)\) and \(L^{p^*_s}(\Omega)\), respectively. Hence, up to a subsequence,

\[
\|w_n - w\|_{W_0^{s,p}(\Omega)}^{p^*_s} \rightarrow 0.
\]

If \(M = 0\), then it implies \(w_n \rightarrow w\) in \(W_0^{s,p}(\Omega)\) as \(n \rightarrow \infty\) and hence the proof. Thus, we assume \(M > 0\). Noting that \((3.20)\) indicates

\[
\int_{\Omega} \frac{|w_n|^{p^*_s - 1} |\chi_{\{w_n > 0\}}| \varphi dx}{|x-y|^{N+2s}} \rightarrow \int_{\Omega} \frac{|w|^{p^*_s - 1} |\chi_{\{w > 0\}}| \varphi dx}{|x-y|^{N+2s}}, \forall \varphi \in L^{p^*_s}(\Omega).
\]

Then, by Brézis-Lieb Lemma (see \cite{9}, Theorem 1), we have

\[
\|w_n\|_{W_0^{s,p}(\Omega)}^{p^*_s} = \|w_n - w\|_{W_0^{s,p}(\Omega)}^{p^*_s} + \|w\|_{W_0^{s,p}(\Omega)}^{p^*_s} + o_n(1),
\]

\[
\int_{\Omega} |w_n|^{p^*_s - 1} |\chi_{\{w_n > 0\}}| \varphi dx \rightarrow \int_{\Omega} |w|^{p^*_s - 1} |\chi_{\{w > 0\}}| \varphi dx, \forall \varphi \in L^{p^*_s}(\Omega).
\]

Since \(w_n \rightarrow w\) strongly in \(L^q(\Omega)\) for any \(1 < q < p^*_s\) and \(\int_{\Omega} \psi(x, w_n) < \int_{\Omega} \frac{1}{p^*_s}(\Omega)\) (see the definition of \(\psi\) given in \((3.4)\)), we obtain

\[
\|w_n - w\|_{W_0^{s,p}(\Omega)}^{p^*_s} \leq o_n(1).
\]

Using Lemma 3.1 \((g_2)\) and \((3.22) - (3.24)\) we obtain

\[
o_n(1) = \langle v_n, w_n - w \rangle
\]

\[
= \|w_n\|_{W_0^{s,p}(\Omega)}^{p^*_s} - \int_{\mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^{p^*_s - 2} (w_n(x) - w_n(y)) (w(x) - w(y)) dx dy}{|x-y|^{N+2s}}
\]

\[
- \int_{\Omega} \psi(x, w_n)(w_n - w) - \int_{\Omega} \theta_n (w_n - w) + o_n(1)
\]

\[
= \|w_n - w\|_{W_0^{s,p}(\Omega)}^{p^*_s} - \int_{\Omega} |w_n|^{p^*_s - 1} (w_n - w) \chi_{\{w_n > 0\}} dx + o_n(1)
\]

\[
= \|w_n - w\|_{W_0^{s,p}(\Omega)}^{p^*_s} - \int_{\Omega} |w_n|^{p^*_s - 1} \chi_{\{w_n > 0\}} dx + \int_{\Omega} |w|^{p^*_s - 1} \chi_{\{w > 0\}} dx + o_n(1)
\]

\[
= \|w_n - w\|_{W_0^{s,p}(\Omega)}^{p^*_s} - \int_{\Omega} |w_n \chi_{\{w_n > 0\}}|^{p^*_s} dx + o_n(1).
\]

This implies

\[
\|w_n - w\|_{W_0^{s,p}(\Omega)}^{p^*_s} \leq \int_{\Omega} |w_n \chi_{\{w_n > 0\}}|^{p^*_s} dx + o_n(1).
\]

From \((2.1), (3.21), (3.22)\) and \((3.25)\) we have

\[
S_{s,p} \leq \frac{\|w_n - w\|_{W_0^{s,p}(\Omega)}^{p^*_s}}{\left(\int_{\Omega} |w_n \chi_{\{w_n > 0\}}|^{p^*_s} dx\right)^{p/p^*_s}} \leq M^\frac{p_s}{p^*_s} + o_n(1)
\]
and hence $M > S_{s,p}^{\frac{\lambda}{\bar{\mu}}} + o_n(1)$. According to (3.15) and the fact that $p < b < p^*_s$, we get

$$J_{\alpha}(w_n) - \frac{1}{b} \langle v_n, w_n \rangle \geq \left( \frac{1}{p} - \frac{1}{b} \right) \|w_n\|_{W_0^{s,p}(\Omega)}^p - \frac{\lambda}{1 - \gamma} \|w_n\|_{L^{1-\gamma}(\Omega)}^{1-\gamma} - \mu C_1,$$  

(3.27)

where $b$ is given in (g3) and $C_1$ is obtained in (3.13) which is independent of $\alpha$. Since $J_{\alpha}(w_n) = c + o_n(1)$ and $\|v_n\|_{W_0^{s,p}(\Omega)} = o_n(1)$, using (3.20) – (3.22), (3.26), (3.27), Theorem 2.1 (2.1) and Young’s inequality we obtain

$$c \geq \left( \frac{1}{p} - \frac{1}{b} \right) (M + \|w\|_{W_0^{s,p}(\Omega)}^p) - \frac{\lambda}{1 - \gamma} \|w\|_{W_0^{s,p}(\Omega)}^{1-\gamma} \mu C_1 + o_n(1)$$

$$\geq \left( \frac{1}{p} - \frac{1}{b} \right) (M + \|w\|_{W_0^{s,p}(\Omega)}^p) - \frac{\lambda}{1 - \gamma} \|w\|_{W_0^{s,p}(\Omega)}^{1-\gamma} \mu C_1 + o_n(1)$$

$$\geq \left( \frac{1}{p} - \frac{1}{b} \right) \left( \frac{\lambda}{1 - \gamma} \|w\|_{W_0^{s,p}(\Omega)}^{1-\gamma} \mu C_1 + o_n(1) \right)$$

$$= c^* + o_n(1).$$

(3.28)

The above inequality (3.28) is a contradiction to (3.19). Thus, $M = 0$ and $w_n \to w$ strongly in $W_0^{s,p}(\Omega)$. Moreover, $c^* > 0$ for a sufficiently small $\mu > 0$. 

In the next lemma, we prove the functional $J_{\alpha}$ satisfies all the hypotheses of Theorem 2.3 with a suitable choice of $\beta$ given in (g4).

**Lemma 3.6.** Let (g1) – (g5) hold. Then there exist $\rho_1 > 0$, $\bar{\mu} \in (0, 1)$, $\bar{\lambda} = \bar{\lambda}(\rho_1) > 0$, $\rho_2 > 0$, $m^* > 0$ and $\sigma \in W_0^{s,p}(\Omega)$ such that for every $\alpha > 0$, every $\lambda \in (0, \bar{\lambda}]$ and every $\mu \in (0, \bar{\mu}]$ we have

1. $\sup_{m \in [0,m^*]} J_{\alpha}(m \sigma) < \left( \frac{1}{p} - \frac{1}{b} \right) S_{s,p}^{\frac{\lambda}{\bar{\mu}}} - \left( \frac{1}{p} - \frac{1}{b} \right) \left( \frac{\lambda}{1 - \gamma} \|\sigma\|_{W_0^{s,p}(\Omega)}^{1-\gamma} \right) \mu C_1$ where $S_{s,p}$ is defined in (2.1) and $C_1$ is given in (3.13).

2. $J_{\alpha}(w) \geq \rho_2$ for every $w \in W_0^{s,p}(\Omega)$, $\|w\|_{W_0^{s,p}(\Omega)} = \rho_1$, where $\rho_1$ and $\rho_2$ are independent of $\alpha$.

3. $J_{\alpha}(m^* \sigma) < 0$ for $\|m^* \sigma\|_{W_0^{s,p}(\Omega)} > \rho_1$.

**Proof.** Let us fix $\sigma \in W_0^{s,p}(\Omega)$ with $\sigma > 0$ in $\Omega$ and $\|\sigma\|_{W_0^{s,p}(\Omega)} = 1$. By the hypothesis (g4) we can write

$$J_{\alpha}(m \sigma) = \frac{1}{p} m^p - \int_{\{m \sigma > \beta\}} \sigma \, dx + \beta \{\{m \sigma > \beta\} \}, \forall m \geq 0.$$  

(3.29)

It is easy to check that there exist $\bar{\mu} \in (0, 1)$ and $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ and $\mu \in (0, \bar{\mu}]$ we have

$$\left( \frac{1}{p} - \frac{1}{b} \right) S_{s,p}^{\frac{\lambda}{\bar{\mu}}} \geq \left( \frac{1}{p} - \frac{1}{b} \right) \left( \frac{\lambda}{1 - \gamma} \|\sigma\|_{W_0^{s,p}(\Omega)}^{1-\gamma} \right) \mu C_1 > 0.$$
Hence, we choose $m^* > 0$ such that
\[
\frac{1}{p}(m^*)^p - m^* \int_{\{m > \beta\}} \sigma dx < 0 \tag{3.30}
\]
and
\[
\frac{(m^*)^p}{p} < \left(\frac{1}{p} - \frac{1}{b}\right) S_{s,p}^N - \left(\frac{1}{p} - \frac{1}{b}\right) \frac{1}{p-1+\gamma} \lambda \left(\frac{1}{1-\gamma} |\Omega| \rho_1^{s-1+\gamma} S_{s,p}^{-1+\gamma}\right) - \mu C_1. \tag{3.31}
\]
Since $m^*$ does not depend on $\beta$, we get
\[
\int_{\Omega} \sigma dx = \int_{\{m^* > \beta\}} \sigma dx + o_\beta(1). \tag{3.32}
\]
Combining (3.29) – (3.32) we choose $\beta > 0$ very small such that $J_\alpha(m^*\sigma) < 0$ for every $\sigma > 0$. This proves (3) and also (1). According to $(g_1)$, $(g_3)$ and the fact that $J_\alpha(0) = 0$, there exist $\overline{K}$ and $c_K > 0$ (independent of $\alpha$) such that $\overline{K} < \lambda_1$ and
\[
|g(x,t)| \leq \frac{\overline{K}}{p}\|t\|^{p-1} + c_K \|t\|^{r-1}, \forall t \in \mathbb{R}.
\]
Thus, from (2.2) for every $w \in W_0^{s,p}(\Omega)$
\[
\int_{\Omega} G(x,w)dx \leq \frac{\overline{K}}{p} \int_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^p}{|x-y|^{N+sp}}dx dy + \frac{c_K}{r} \int_{\Omega} |w|^r dx. \tag{3.33}
\]
Hence,
\[
J_\alpha(w) \geq \frac{1}{p} \left(1 - \frac{\mu \overline{K}}{\lambda_1}\right) \|w\|_{W_0^{s,p}(\Omega)}^p - C_1 c_K \|w\|_{W_0^{s,p}(\Omega)} - C_2 \|w\|_{W_0^{s,p}(\Omega)}^r - \frac{\lambda}{1-\gamma} |\Omega| \frac{\rho_1^{s-1+\gamma}}{r} S_{s,p}^{-1+\gamma} \|w\|_{W_0^{s,p}(\Omega)}^{1-\gamma}, \tag{3.34}
\]
for some constant $C_1, C_2 > 0$ independent of $\alpha$. Since $1 - \gamma < 1 < p < r < p_\ast^\alpha$, the function
\[
h(t) = \frac{1}{p} \left(1 - \frac{\mu \overline{K}}{\lambda_1}\right) t^{p-1+\gamma} - C_1 c_K t^{r-1+\gamma} - C_2 t^{p_\ast^\alpha-1+\gamma}, \quad t \in [0,1]
\]
admits a maximum at some $\rho_1 \in (0,1]$ small enough, i.e. $\max_{t \in [0,1]} h(t) = h(\rho_1) > 0$. Therefore, let
\[
\lambda^{**} = \frac{1-\gamma}{2 |\Omega| \frac{\rho_1^{s-1+\gamma}}{r} S_{s,p}^{-1+\gamma}} h(\rho_1),
\]
then for every $w \in W_0^{s,p}(\Omega)$ with $\|w\|_{W_0^{s,p}(\Omega)} = \rho_1 \leq 1$ and for every $\lambda \in (0, \lambda^{**})$, we have $J_\alpha(w) \geq \rho_1^{1-\gamma} h(\rho_1)/2 = \rho_2$. Let $\tilde{\lambda} = \min\{\lambda^*, \lambda^{**}\}$. Then with $\lambda \in (0, \tilde{\lambda})$ and $\mu \in (0, \tilde{\mu})$, we conclude the proof. \qed

**Proposition 3.7.** Let $(g_1) - (g_5)$ are satisfied. Then there exist $\bar{\alpha}, \bar{\lambda}, \bar{\mu} > 0$ such that for every $a \in (0, \bar{\alpha})$, every $\lambda \in (0, \tilde{\lambda})$ and every $\mu \in (0, \tilde{\mu})$ the problem $[P_\alpha]$ admits at least one nontrivial weak solution $u_\alpha$. Furthermore, the lebesgue measure of the set $\{x \in \Omega : u_\alpha > \alpha\}$ is positive.
Proof. Let

\[ c_\alpha = \inf_{\zeta \in \Gamma} \max_{t \in [0,1]} J_\alpha(\zeta(t)) \]

and \( \Gamma = \{ \zeta \in C([0,1]; W^{s,p}_0(\Omega)) : \zeta(0) = 0, \zeta(1) = m^* \sigma \} \),

where \( m^*, \sigma, \rho_1, \rho_2, \tilde{\lambda}, \tilde{\mu} \) are obtained in Lemma 3.6. Since \( J_\alpha \) satisfy the hypothesis of Theorem 2.3 (refer Lemma 3.6), we guarantee the existence of a sequence \((u_n) \subset W^{s,p}_0(\Omega)\) that satisfy \( J_\alpha(u_n) = c_\alpha + o_n(1) \) and \( \Lambda J_\alpha(u_n) = o_n(1) \). By (1) and (2) of Lemma 3.6 we also have

\[ \rho_2 \leq c_\alpha < \left( \frac{1}{p} - \frac{1}{b} \right) S^{\frac{\alpha}{p}}_{s,p} - \left( \frac{1}{p} - \frac{1}{b} \right) \left( \frac{\lambda}{1 - \gamma} |\Omega| \right)^{\frac{1 - \gamma}{p}} \frac{1}{S^{\frac{\alpha}{p}}_{s,p}} - \mu C_1, \quad \forall \alpha > 0. \]  

(3.35)

From Proposition 3.5 there exists \( w_\alpha \in W^{s,p}_0(\Omega) \) such that, up to a subsequence, \( w_n \to w_\alpha \) in \( W^{s,p}_0(\Omega) \) as \( n \to \infty \). This implies \( J_\alpha(w_\alpha) = c_\alpha \) and \( 0 \in \partial J_\alpha(w_\alpha) \). Thus, by (3.9) and Lemma 3.11 there exist \( \eta_\alpha \in L^\infty(\Omega) \) and \( \theta_\alpha \in L^{s,p}\gamma(\Omega) \) such that

\[ \int_{\mathbb{R}^2N} \frac{|w_\alpha(x) - w_\alpha(y)|^{p - 2}(w_\alpha(x) - w_\alpha(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + sp}} \, dx \, dy = \lambda \int_{\Omega} \psi(x, w_\alpha) \varphi \, dx + \mu \langle \eta_\alpha, \varphi \rangle + \langle \theta_\alpha, \varphi \rangle, \]

(3.36)

for every \( \varphi \in W^{s,p}_0(\Omega) \), where \( \eta_\alpha(x) \in [g(x, w_\alpha(x)), \overline{g}(x, w_\alpha(x))] \geq 0 \) and \( \theta_\alpha(x) \in [f(w_\alpha(x)), \overline{f}(w_\alpha(x))] \geq 0 \) a.e. in \( \Omega \). According to the strong maximum principle (Lemma 2.3, [29]) we have \( w_\alpha > 0 \) a.e. in \( \Omega \).

This proves that \( w_\alpha \) is a weak solution to (3.6). By the weak comparison principle for fractional \( p \)-Laplacian (Lemma 3.1, [22]), we conclude that \( u_\lambda \leq w_\alpha \) a.e. in \( \Omega \). This implies \( \psi(x, w_\alpha) = w_\alpha^{-\gamma} \) a.e. in \( \Omega \) and \( w_\alpha = u_\alpha \) is a weak solution to (P\(_\alpha\)).

The next claim is to prove that the set \( \{ x \in \Omega : w_\alpha(x) > \alpha \} \) has positive Lebesgue measure in \( \mathbb{R}^N \). We prove this claim by method of contradiction. For this let us assume that the set \( \{ x \in \Omega : w_\alpha(x) > \alpha \} \) is of zero Lebesgue measure in \( \mathbb{R}^N \). Thus, \( w_\alpha(x) \leq \alpha \) a.e. in \( \Omega \).

From Lemma 3.1, (3.33) and (3.36) we obtain

\[ \|w_\alpha\|_{W^{s,p}_0(\Omega)}^p = \mu \int_{\Omega} \eta_\alpha w_\alpha + \int_{\{w_\alpha = \alpha\}} \theta_\alpha w_\alpha + \lambda \int_{\Omega} \psi(x, w_\alpha) w_\alpha \]

\[ \leq \mu \int_{\Omega} \left( |K\alpha^p + c_K\alpha^r| \right) dx + \int_{\{w_\alpha = \alpha\}} \alpha^{p^*_s} + \lambda \int_{\Omega} \alpha^{1 - \gamma}. \]

Since \( J_\alpha(w_\alpha) = c_\alpha \) by Lemma 3.6 and (3.33), for \( \alpha > 0 \) small enough, we establish

\[ \rho_2 \leq \mu (\|K + c_K\| + 1 + \lambda) |\Omega| \alpha^{1 - \gamma}. \]

This contradicts the fact that \( \rho_2 \) is independent of \( \alpha \). Thus, there exists \( \bar{\alpha} > 0 \) small such that for any \( \alpha \in (0, \bar{\alpha}) \) the set \( \{ x \in \Omega : w_\alpha(x) > \alpha \} \) has positive Lebesgue measure.

\[ \square \]

4. Proof of Theorem 1.1 (2)

Let \( u_\alpha \) be a nontrivial weak solution to (P\(_\alpha\)) given in Proposition 3.7. In this section, we prove the second part of Theorem 1.1 i.e. we examine the nature of \( (u_\alpha) \) as \( \alpha \to 0^+ \).

Consider the functional \( J_0^0 : W^{s,p}_0(\Omega) \to \mathbb{R} \) associated to (P\(_0\)) defined by

\[ J_0^0(u) = \frac{1}{p} \int_{\mathbb{R}^2N} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy - \lambda \int_{\Omega} u^{1 - \gamma} - \mu \int_{\Omega} G(x, u) - \frac{1}{p^*_s} \int_{\Omega} (u^+)^{p^*_s}, \quad \forall u \in W^{s,p}_0(\Omega). \]
Let us define
\[ c_0 = \inf_{\zeta \in \Gamma} \max_{t \in [0, 1]} J_0^0(\zeta(t)) \] and \( \Gamma = \{ \zeta \in C([0, 1]; W_0^{s,p}(\Omega)) : \zeta(0) = 0, \zeta(1) = m^*\sigma \} \),
(4.1)
where \( m^*, \sigma \) as obtained in Lemma 3.6

**Lemma 4.1.** \( \lim_{\alpha \to 0^+} c_\alpha = c_0 \geq \rho_2 \), where \( c_\alpha, c_0 \) and \( \rho_2 \) are given in Proposition 3.7, (4.1) and Lemma 3.6, respectively.

**Proof.** Since \( F_\alpha(t) = \chi_{t \geq \alpha} \frac{1}{p_\alpha} (|t|^{p_\alpha} - \alpha^{p_\alpha}) \), we obtain
\[
\left| \frac{1}{p_\alpha} \int_\Omega (u^+)^{p_\alpha} - \int_\Omega F_\alpha(u) \right| = \left| \frac{1}{p_\alpha} \int_\Omega (u^+)^{p_\alpha} \chi_{\{u \leq \alpha\}} + \frac{1}{p_\alpha} \int_\Omega \alpha^{p_\alpha} \chi_{\{u > \alpha\}} \right|
\leq \frac{2\alpha^{\rho_\alpha} |\Omega|}{p_\alpha}.
\]
(4.2)
Clearly, \( J_\alpha^0(u) \leq J_\alpha^0(u) \), for all \( u \in W_0^{s,p}(\Omega) \). Thus, \( c_0 \leq c_\alpha \), for all \( \alpha > 0 \). According to (4.2) we establish
\[
J_\alpha^0(u) = J_\alpha^0(u) + o_\alpha(1), \ \forall \ u \in W_0^{s,p}(\Omega),
\]
(4.3)
where \( o_\alpha(1) \to 0 \) as \( \alpha \to 0^+ \) independently of \( u \). This gives,
\[
J_\alpha^0(\zeta(t)) = J_\alpha^0(\zeta(t)) + o_\alpha(1), \ \forall \ \zeta \in \Gamma, \ t \in [0, 1]
\]
(4.4)
and hence \( c_\alpha = c_0 + o_\alpha(1) \).
With the consideration of (4.4) and Lemma 3.6 (3), we conclude that \( \lim_{\alpha \to 0^+} c_\alpha = c_0 \geq \rho_2 \). \( \square \)

**Proposition 4.2.** For any sequence \( \alpha_n \in (0, \bar{\alpha}) \) with \( \alpha_n \to 0^+ \), we have, up to a subsequence, \( u_{\alpha_n} \to u_0 \) in \( W_0^{s,p}(\Omega) \), where \( u_0 \) is a nontrivial weak solution to the problem (P_0).

**Proof.** Let \( u_\alpha \) be the weak solution to (P_0) given in Proposition 3.7. Thus, \( J_\alpha^0(u_\alpha) = J_\alpha(u_\alpha) = c_\alpha \) and
\[
\int_{\mathbb{R}^{2N}} \frac{|u_\alpha(x) - u_\alpha(y)|^{p-2}(u_\alpha(x) - u_\alpha(y))(\varphi(x) - \varphi(y))dx dy - \mu \int_\Omega \eta_\alpha \varphi dx - \int_\Omega \theta_\alpha \varphi dx - \lambda \int_\Omega \chi_{\Omega} \varphi dx = 0,
\]
(4.5)
where \( \eta_\alpha(x) \in [g(x, u_\alpha(x)), \overline{g}(x, u_\alpha(x))] \) and \( \theta_\alpha(x) \in [f_\alpha(u_\alpha(x)), \overline{f}_\alpha(u_\alpha(x))] \) a.e. in \( \Omega \) with \( f_\alpha(t) = H(t - \alpha) t^{p_\alpha - 1} \).
Consider the sequence \( (w_n) \subset W_0^{s,p}(\Omega) \) obtained in Proposition 3.7 with \( w_n \to u_\alpha \) in \( W_0^{s,p}(\Omega) \). From (3.18) we have
\[
c_\alpha + 1 + \|w_n\|_{W_0^{s,p}(\Omega)} \geq \left( \frac{1}{p} - \frac{1}{b} \right) \|w_n\|_{W_0^{s,p}(\Omega)}^p - \frac{\lambda}{1 - \gamma} |\Omega|^{\frac{p - 1 - \gamma}{p}} S_{s,p}^{-\gamma} \|w_n\|_{W_0^{s,p}(\Omega)}^{1 - \gamma} - \mu C_1, \ \forall \ \alpha > 0,
\]
where \( C_1 \) is independent of \( \alpha \) (refer (3.13)). Thus,
\[
c_\alpha + 1 + \|w_\alpha\|_{W_0^{s,p}(\Omega)} \geq \left( \frac{1}{p} - \frac{1}{b} \right) \|w_\alpha\|_{W_0^{s,p}(\Omega)}^p - \frac{\lambda}{1 - \gamma} |\Omega|^{\frac{p - 1 - \gamma}{p}} S_{s,p}^{-\gamma} \|w_\alpha\|_{W_0^{s,p}(\Omega)}^{1 - \gamma} - \mu C_1, \ \forall \ \alpha > 0
\]
and the sequence \((u_\alpha)\) is uniformly bounded in \(W^{s,p}_0(\Omega)\). By \((g_2)\) and Lemma 3.1 we establish that \((\eta_\alpha)\) and \((\theta_\alpha)\) are bounded in \(L^{r^*}(\Omega)\) and \(L^{p^*_\alpha-1}(\Omega)\), respectively. Consider a sequence \((\alpha_n) \subset (0, \bar{\alpha})\) with \(\alpha_n \to 0^+\). Hence, up to a subsequence,

\[
 u_{\alpha_n} \rightharpoonup u_0 \text{ in } W^{s,p}_0(\Omega), \quad u_{\alpha_n}(x) \to u_0(x) \text{ a.e. in } \Omega, \quad u_{\alpha_n} \to u_\alpha \text{ in } L^q(\Omega) \text{ for any } 1 \leq q < p^*_\alpha, \\
 \eta_{\alpha_n} \rightharpoonup \eta_0 \text{ in } L^{r^*}(\Omega) \text{ and } \theta_{\alpha_n} \rightharpoonup \theta_0^{p^*_\alpha-1} \text{ in } L^{p^*_\alpha-1}(\Omega).
\]

(4.6)

We already have \(u_{\alpha_n} \geq u_\Lambda\) for all \(n \in \mathbb{N}\) (refer Proposition 3.7), where \(u_\Lambda\) is a weak solution to (3.6) given in Lemma 3.3. Thus, by combining (4.5) and (4.6) we pass the limit \(\alpha_n \to 0^+\) to obtain

\[
 \int_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^{p-2}(u_0(x)-u_0(y))\varphi(x)-\varphi(y))dxdy-\mu \int_{\Omega} \eta_0 \varphi dx - \int_{\Omega} u_0^\gamma \varphi dx - \lambda \int_{\Omega} \frac{\varphi}{u_0^\gamma} dx = 0,
\]

for every \(\varphi \in W^{s,p}_0(\Omega)\). From Proposition 2.4, \(\eta_0 \in \partial I_\mu(u_0)\).

According to (3.35) we have

\[
 \rho_2 \leq J_{\alpha_n}^0(u_{\alpha_n}) = c_\alpha < \left(\frac{1}{p-\frac{1}{b}}\right)^{\frac{\gamma}{s^*p}} \left(\frac{1}{p-\frac{1}{b}}\right)^{-\frac{1+\gamma}{p-1+\gamma}} \left(\frac{\lambda}{1-\gamma}\right)^{\frac{\gamma}{p^*\gamma-1}} S_{s^*p}^{-\frac{1}{p}} - \mu C_1, \quad \forall \alpha_n > 0
\]

(4.7)

and from (4.3),

\[
 c_\alpha = J_{\alpha_n}^0(u_{\alpha_n}) = J_0^0(u_{\alpha_n}) + o_{\alpha_n}(1).
\]

(4.8)

Considering (4.7), (4.8) and then following the proof of Proposition 3.5, we get

\[
 u_{\alpha_n} \to u_0 \text{ in } W^{s,p}_0(\Omega) \text{ as } \alpha_n \to 0^+.
\]

(4.9)

Combining (4.9), Lemma 4.1 and (4.3), we have \(J_0^0(u_0) = c_0 \geq \rho_2 > 0\). This proves that \(u_0\) is a nontrivial weak solution to \((P_0)\) and we conclude the proof.

\[\square\]

**Acknowledgement**

The author Debajyoti Choudhuri thanks the grant received from Council of Scientific and Industrial Research (CSIR), India for the research grant (09/983(0013)/2017-EMR-I). The author Akasmika Panda thanks the financial assistantship received from the Ministry of Human Resource Development (M.H.R.D.), Govt. of India.

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