The Sum Degree Distance and the Product Degree Distance of Generalized Transformation Graphs $G^{ab}$

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Abstract. In this contribution, we consider line splitting graph $L_s(G)$ of a graph $G$ as transformation graph $G^{++}$ of $G^{ab}$. We investigate the sum degree distance $DD_s(G)$ and product degree distance $DD_p(G)$ of transformation graph $G^{ab}$, which are weighted version of Wiener index. The Transformation graphs of $G^{ab}$ are $G^{++}$, $G^{--}$, $G^{-+}$ and $G^{+-}$. 

1. Introduction

Throughout this paper, we consider finite, un-directed, simple, connected, r-regular graphs with vertex set $V(G) = \{v_1, v_2, v_3, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, e_3, ..., e_m\}$. For the undefined terminologies we refer[8].

The degree of vertex in a graph $G$ is denoted by $\deg_G(v)$ or $d_G(v)$ and the distance between two vertices $v_i$ and $v_j$, denoted by $dist_G(v_i, v_j)$ or $d_G(v_i, v_j)$, is the length of a shortest path between the vertices $v_i$ and $v_j$ in $G$. The shortest $v_i\sim v_j$ path is often called a geodesic. The diameter of a connected graph $G$ is the length of any longest geodesic. The graphs considered in this construction are with $diam \leq 2$. The degree of an edge $e_i$ in $G$ is the number of edges adjacent to $e_i$ and is denoted by $deg_G(e_i)$. The degree of edge in a graph $G$ is

$$deg_G(e_i) = deg_G(uv) = deg_G(u) + deg_G(v) - 2.$$ 

Topological indices and graph invariants based on distances between vertices of a graph are widely used in mathematical chemistry[2], which are due to their correlations with physical, chemical and thermodynamic parameters of chemical compounds.

One of the oldest and well studied distance based graph invariant associated with a connected graph $G$ is the Wiener number $W(G)$, also termed as Wiener index in chemical or mathematical chemistry literature, which is defined in [13] as the sum of distances over all unordered vertex pairs in $G$,

$$W(G) = \sum_{i<j} d(v_i, v_j) \quad (1)$$

Which was first time introduced by Wiener. Initially, the Wiener index $W(G)$ was considered as a molecular structure descriptor used in chemical applications, but soon it attracted the interest of pure mathematicians[1,3,5,14,15].

Eventually a number of modifications of the Wiener index were proposed, which are as follows.

$$DD_s(G) = \sum_{u, v \in V(G)} [deg_G(u) + deg_G(v)]d_G(u, v) \quad (2)$$

$$DD_p(G) = \sum_{u, v \in V(G)} [deg_G(u) \cdot deg_G(v)]d_G(u, v) \quad (3)$$
The graph invariants defined in (2) and (3) have all been much studied in the past. The invariant $DD_+$ was first time introduced by Dobrynin and Kochetova[4] and named as sum-degree distance. Later the same quantity was examined under the name "Schulz index" [7]. For mathematical research on degree distance see[9,12] and the references cited therein. A remarkable property of $DD_+$ is that in the case of trees of order $n$, the identity $DD_+=4W-n(n-1)$ holds [10].

Gutman [7] proved that the multiplicative variant of the degree distance, namely $DD_*, i.e., from (2), obeys an analogous relation: $DD_*=4W-(2n-1) (n-1)$. This latter quantity is sometimes referred to as the "Gutman index"[6], but here we call it product-degree distance.

The open neighborhood $N(e_i)$ of an edge $e_i$ in $E(G)$ is the set of edges adjacent to $e_i$

$$i.e., \quad N(e_i)=\{e_i/e_i, e_j \text{ are adjacent in } G \}.$$

For each edge $e_i$ of $G$, a new vertex $e'_i$ is taken and the resulting set of vertices is denoted by $E'(G)$.

The line Splitting graph $L_s(G)$ of a graph $G$ is defined as the graph having vertex set $E(G) \cup E'(G)$ with two vertices adjacent if they correspond to adjacent edges of $G$ or one corresponds to an element $e'_i$ of $E'(G)$ and the other to an element $e_j$ of $E(G)$, and $e_j$ is in $N(e_i)$. This concept was introduced by Kulli and Biradar in[11].

2 Generalized Transformation Graphs $G^{ab}$

Let $G=(V, E)$ be a graph. Let $\alpha, \beta$ and $\alpha', \beta'$ be the element of $E(G)$ and $E'(G)$ respectively. We say that the associativity of $\alpha$ and $\beta$ is +, if they are adjacent in $G$ otherwise is - and the associativity of $\alpha$ and $\beta'$ or $\alpha'$ and $\beta$ is +, if $\alpha$ is the neighborhood point of $\beta$ or $\beta$ is neighborhood point of $\alpha$ in $G$, otherwise is -.

Let $ab$ be a 2-permutation of the set $\{+,-\}$. We say that $\alpha$ and $\beta$ corresponds to the first term $a$ of $ab$, and $\alpha, \beta \in E(G)$. Whereas $\alpha$ and $\beta'$ or $\beta$ and $\alpha'$ corresponds to the both first and second term of $ab$ and $\alpha', \beta' \in E'(G)$.

The transformation graph $G^{ab}$ of a graph $G$ is the graph with vertex set $E(G) \cup E'(G)$. $\alpha$ and $\beta$ or $\alpha$ and $\beta'$ or $\beta$ and $\alpha'$ are adjacent if and only if the following conditions holds;

* $\alpha, \beta \in E(G)$, $\alpha$ and $\beta$ are adjacent in $G$ if $a=+$ otherwise $a=-$.

** $\alpha, \beta \in E(G)$ and $\alpha', \beta' \in E'(G)$, if $a$ neighborhood points of $\beta$ or $\beta$ is neighborhood point of $\alpha$ in $G$ then $b=+$ otherwise $b=-$.

Since there are four distinct 2-permutations of $\{+,-\}$, we obtain 4-graphical transformations of $G$. Here we consider $G^{++}$, which is nothing but line splitting graph of $G$ and the other generalized transformation graphs are $G^{++}, G^{+-}$ and $G^{-+}$.

Note that, in this paper we consider graphs with $n \geq 5$ for $G^{++}$ and $G^{-+}$ and in particular for $G^{++}$ and $G^{-+}$ we consider graphs with $n > 5$ and having atleast three edges $e_i, e_j$ and $e_w \in E(G)$; $i,j,w=1,2,3,..,m$ and $i\neq j \neq w$ such that $e_i$ and $e_j$ are non adjacent edges and $e_w$ is non adjacent to $e_i$ and $e_j$.

The aim of present work is to obtain the expression for the sum degree distance and product degree distance of the generalized transformation graphs $G^{ab}$.

3. Results

In this section we obtain the sum degree distance and product degree distance of the transformation graphs $G^{ab}$, which is line splitting graph i.e., $G^{++}$, and its generalized transformation graphs $G^{++}, G^{+-}, G^{-+}$.
We start by stating the following propositions and observations, needed for proving our main results.

**Proposition 3.1** Let G be an \((n,m)\) graph. Then by the definition order of \(G^{ab}\) is 2m and

1. The size of \(G^{+} \) is \(-m + \frac{1}{2} nr^2 + 2m(r - 1)\).

2. The size of \(G^{-} \) is \(-m^2 + \frac{1}{2} (nr^2) - 2mr\).

3. The size of \(G^{++} \) is \(\frac{m}{2} [m + 2r - 3]\).

4. The size of \(G^{--} \) is \(\frac{3}{2} [m - 2r + 1]\).

**Proof.** Let G be a \((n,m)\)-graph with regular degree \(r\), then

1. \(E(G^{+}) = E(L(G)) + \sum_{i=1}^{m} \deg_{G}(e_i)\)
   
   \[= -m + \frac{1}{2} \sum_{i=1}^{m} d_i^2 + \sum_{i=1}^{m} \deg_{G}(e_i)\]
   
   \[= -m + \frac{1}{2} nr^2 + m(2r - 2) \quad [\because G \text{ is } r-\text{regular graph}]\]
   
   \[E(G^{+}) = -m + \frac{1}{2} nr^2 + 2m(r - 1).\]

2. \(E(G^{-}) = \sum_{(uv) \in E(G)} \text{the edges which are not incident to } u \text{ and } v \text{ in } G\)
   
   \[= -m + \frac{1}{2} nr^2 + m(m - 2r + 1) \quad [\because G \text{ is } r-\text{regular graph}]\]
   
   \[= -m + \frac{1}{2} nr^2 + m^2 - 2mr + m\]
   
   \[E(G^{-}) = m^2 + \frac{1}{2} nr^2 - 2mr.\]

3. \(E(G^{++}) = \frac{1}{2} \sum_{(uv) \in E(G)} \text{the edges which are not incident to } u \text{ and } v \text{ in } G + \sum_{i=1}^{m} \deg_{G}(e_i)\)
   
   \[= \frac{1}{2} m(m - 2r + 1) + m(\deg_{G}(u) + \deg_{G}(v) - 2)\]
   
   \[= \frac{1}{2} m^2 - mr + \frac{1}{2} m + m(2r - 2) \quad [\because G \text{ is } r-\text{regular graph}]\]
   
   \[E(G^{++}) = \frac{m}{2} [m + 2mr - 3].\]

4. \(E(G^{--}) = \frac{1}{2} \sum_{(uv) \in E(G)} \text{the edges which are not incident to } u \text{ and } v \text{ in } G + \sum_{(uv) \in E(G)} \text{the edges which are not incident to } u \text{ and } v \text{ in } G\)
   
   \[= \frac{1}{2} m(m - 2r + 1) + m(m - 2r + 1) \quad [\because G \text{ is } r-\text{regular graph}]\]
   
   \[E(G^{--}) = \frac{3}{2} m[m - 2r + 1].\]
Proposition 3.2 Let G be an (n,m) graph. Then the degree of vertices $e_i$ and $e_i'$ of $G^{ab}$ are,

(i) $d_{G^{++}}(e_i) = 4(r - 1)$ and $d_{G^{++}}(e_i') = 2(r - 1)$.

(ii) $d_{G^{--}}(e_i) = (m - 1)$ and $d_{G^{--}}(e_i') = m - 2r + 1$.

(iii) $d_{G^{+-}}(e_i) = (m - 1)$ and $d_{G^{+-}}(e_i') = 2(r - 1)$.

(iv) $d_{G^{-+}}(e_i) = 2(m - 2r + 1)$ and $d_{G^{-+}}(e_i') = (m - 2r + 1)$.

Proof. Let G be a (n,m)-graph with regular degree r, then

(i) $d_{G^{++}}(e_i) = 2\deg_G(e_i) = 2(2r - 2) = 4(r - 1)$ and $d_{G^{++}}(e_i') = \deg_G(e_i) = 2r - 2 = 2(r - 1)$.

(ii) $d_{G^{--}}(e_i) = \deg_G(e_i) + (m - 2r + 1) = 2r - 2 + m - 2r + 1 = (m - 1)$ and $d_{G^{--}}(e_i') = \deg_G(e_i) = (2r - 2) = 2(r - 1)$.

(iii) $d_{G^{+-}}(e_i) = (m - 2r + 1) + \deg_G(e_i) = m - 2r + 2r - 2 = (m - 1)$ and $d_{G^{+-}}(e_i') = \deg_G(e_i) = (2r - 2) = 2(r - 1)$.

(iv) $d_{G^{-+}}(e_i) = 2($The total number of edges which are not incident to $u$ and $v$ in $G$ and $uv = e_i$) $= 2(m - 2r + 1)$ and $d_{G^{-+}}(e_i') = $ The total number of edges which are not incident to $u$ and $v$ in $G$ and $uv = e_i$ $= (m - 2r + 1)$.

We use Proposition 3.2 for the following observations.

Observation A.

1. G be any (n, m) graph.
   If $d_{G^{++}}(e_i, e_i') = 1$, then
   $$\sum_{(e_i, e_i') \in F(G^{++})}[\deg_{G^{++}}(e_i) + \deg_{G^{++}}(e_i')]d_{G^{++}}(e_i, e_i') \text{ in } G^{++} = 8m(r - 1)^2.$$ 

2. Let G be any (n, m) graph.
   If $d_{G^{++}}(e_i, e_i') = 2$, then
   $$\sum_{(e_i, e_i') \in F(G^{++})}[\deg_{G^{++}}(e_i) + \deg_{G^{++}}(e_i')]d_{G^{++}}(e_i, e_i') \text{ in } G^{++} = 16(r - 1)(m^2 + m - 2mr).$$ 

3. Let G be any (n, m) graph.
   If $d_{G^{++}}(e_i', e_i') = 1$, then
   $$\sum_{(e_i', e_i') \in F(G^{++})}[\deg_{G^{++}}(e_i') + \deg_{G^{++}}(e_i')]d_{G^{++}}(e_i', e_i') \text{ in } G^{++} = 12m(r - 1)^2.$$ 

   If $d_{G^{++}}(e_i', e_i') = 2$, then
   $$\sum_{(e_i', e_i') \in F(G^{++})}[\deg_{G^{++}}(e_i') + \deg_{G^{++}}(e_i')]d_{G^{++}}(e_i', e_i') \text{ in } G^{++} = 12m(r - 1)(m - 2r + 2).$$ 

   If $d_{G^{++}}(e_i', e_i') = 3$ when $r = 2$, then
   $$\sum_{(e_i', e_i') \in F(G^{++})}[\deg_{G^{++}}(e_i') + \deg_{G^{++}}(e_i')]d(e_i', e_i') \text{ in } G^{++} = 12m(r - 1).$$
Theorem 3.3. For any \((n, m)\) graph \(G\) with \(r \geq 2\),
if \(r = 2\) then
\[
DD_+(G^{++}) = 4(r - 1)[m(7m + 5 - 11r) + 3m + nr^2 + 2\sum_{k=1}^{m}(k - 1)]
\]
(*)

And if \(r > 2\) then
\[
DD_+(G^{++}) = 4(r - 1)[m(7m + 5 - 11r) + nr^2 + 2\sum_{k=1}^{m}(k - 1)]
\]
(**)

Proof. Let \(G\) be any \((n,m)\)-graph. From Proposition 3.1, \(G^{++}\) contains \(2m\) vertices and
\((-m + \frac{1}{2}nr^2 + 2m(r - 1))\) edges.

From (2), we have
\[
DD_+(G) = \sum_{u,v \in V^+(G)} [\deg_G(u) + \deg_G(v)]d_G(u,v)
\]

Therefore,
\[
DD_+(G^{++}) = \sum_{(e_i,e_j) \in E^+(G^{++})} [\deg_{G^{++}}(e_i) + \deg_{G^{++}}(e_j)]d_{G^{++}}(e_i,e_j) +
\sum_{(e_i,e'_j) \in E^+(G^{++})} [\deg_{G^{++}}(e_i) + \deg_{G^{++}}(e'_j)]d_{G^{++}}(e_i,e'_j) +
\sum_{(e'_i,e'_j) \in E^+(G^{++})} [\deg_{G^{++}}(e'_i) + \deg_{G^{++}}(e'_j)]d_{G^{++}}(e'_i,e'_j).
\]

Applying observation A to the above equation,
when \(r = 2\),
\[
DD_+(G^{++}) = 8m(r - 1)^2(-m + \frac{1}{2}nr^2) + 16m(r - 1)(m - 2r + 1) + 12m(r - 1)^2 + 12m(r - 1)
\]
\((m-2r+2)+12m(r-1)+8(r-1)\sum_{k=1}^{m}(k - 1).
\]

and \(r > 2\),
\[
DD_+(G^{++}) = 8m(r - 1)^2(-m + \frac{1}{2}nr^2) + 16m(r - 1)(m - 2r + 1) + 12m(r - 1)^2 + 12m(r - 1)
\]
\((m-2r+2)+8(r-1)\sum_{k=1}^{m}(k - 1).
\]

On simplification, we get (*) and (**)
i.e.,
\[
DD_+(G^{++}) = 4(r - 1)[m(7m + 5 - 11r) + 3m + nr^2 + 2\sum_{k=1}^{m}(k - 1)]
\]
and
\[
DD_+(G^{++}) = 4(r - 1)[m(7m + 5 - 11r) + nr^2 + 2\sum_{k=2}^{m}(k - 1)].
\]

Observation B.
1. Let \(G\) be any \((n,m)\) graph.
If \(d_{G^{++}}(e_i,e_j) = 1\), then
\[
\sum_{(e_i,e_j) \in E^+(G^{++})} [\deg_{G^{++}}(e_i) \cdot \deg_{G^{++}}(e_j)]d_{G^{++}}(e_i,e_j) \text{ in } G^{++} = 16m (r - 1)^2(-m + \frac{1}{2}nr^2).
\]

If \(d_{G^{++}}(e_i,e_j) = 2\), then
\[
\sum_{(e_i,e_j) \in E^+(G^{++})} [\deg_{G^{++}}(e_i) \cdot \deg_{G^{++}}(e_j)]d_{G^{++}}(e_i,e_j) \text{ in } G^{++} = 32 (r - 1)^2(m^2 + m - 2mr).
\]

2. Let \(G\) be any \((n,m)\) graph.
If \(d_{G^{++}}(e_i,e'_j) = 1\), then
\[
\sum_{(e_i,e'_j) \in E^+(G^{++})} [\deg_{G^{++}}(e_i) \cdot \deg_{G^{++}}(e'_j)]d_{G^{++}}(e_i,e'_j) \text{ in } G^{++} = 16m (r - 1)^3 m.
\]
If $d_G^{++}(e_i, e'_j) = 2$, then
\[
\sum_{(e_i, e'_j) \in \mathcal{P}(G^{++})} [\deg_{G^{++}}(e_i) \cdot \deg_{G^{++}}(e'_j)]d_G^{++}(e_i, e'_j) \text{ in } G^{++} = 16m (r-1)^2(m-2r+2).
\]

3. Let $G$ be any $(n,m)$ graph.
If $d_G^{++}(e_i, e'_j) = 2$, then
\[
\sum_{(e_i, e'_j) \in \mathcal{P}(G^{++})} [\deg_{G^{++}}(e_i) \cdot \deg_{G^{++}}(e'_j)]d_G^{++}(e_i, e'_j) \text{ in } G^{++} = 8(r-1)^2 \sum_{k=2}^m (k-1).
\]

If $d_G^{++}(e_i, e'_j) = 3$, when $r = 2$, then
\[
\sum_{(e_i, e'_j) \in \mathcal{P}(G^{++})} [\deg_{G^{++}}(e'_j) \cdot \deg_{G^{++}}(e'_j)]d_G^{++}(e_i, e'_j) \text{ in } G^{++} = 12m(r-1)^2.
\]

**Theorem 3.4.** For any $(n,m)$ graph $G$ with $r \geq 2$,

- when $r = 2$
  \[
  DD_*(G^{++}) = 8(r-1)^2[2m(3m-5r+2) + nr^2 + 3m + \sum_{k=2}^m (k-1)] \quad \text{(*)}
  \]
  and when $r > 2$
  \[
  DD_*(G^{++}) = 8(r-1)^2[2m(3m-5r+2) + nr^2 + 3m + \sum_{k=2}^m (k-1)] \quad \text{(***)}
  \]

**Proof.** Let $G$ be any $(n,m)$-graph. From Proposition 3.1, $G^{++}$ contains $2m$ vertices and $(-m + \frac{1}{2}nr^2 + 2m(r-1))$ edges.

From (3), we have
\[
DD_*(G) = \sum_{u,v \in G} [\deg_G(u) \cdot \deg_G(v)]d_G(u,v)
\]

Therefore,
\[
DD_*(G^{++}) = \sum_{(e_i, e'_j) \in \mathcal{P}(G^{++})} [\deg_{G^{++}}(e_i) \cdot \deg_{G^{++}}(e'_j)]d_{G^{++}}(e_i, e'_j) + \sum_{(e_i, e'_j) \in \mathcal{P}(G^{++})} [\deg_{G^{++}}(e_i) \cdot \deg_{G^{++}}(e'_j)]d_{G^{++}}(e_i, e'_j) + \sum_{(e_i, e'_j) \in \mathcal{P}(G^{++})} [\deg_{G^{++}}(e'_j) \cdot \deg_{G^{++}}(e'_j)]d_{G^{++}}(e_i, e'_j).
\]

Applying observation B to the above equation, when $r = 2$,
\[
DD_*(G^{++}) = 16(r-1)^2(-m + \frac{1}{2}) + 32(r-1)^2(m^2 + m - 2mr) + 16m(r-1)^3 + 16m(r-1)^2 \]
\[
(m-2r+2) + 12m(r-1)^2 + 8(r-1)^2 \sum_{k=2}^m (k-1).
\]

When $r > 2$
\[
DD_*(G^{++}) = 16(r-1)^2(-m + \frac{1}{2}) + 32(r-1)^2(m^2 + m - 2mr) + 16m(r-1)^3 + 16m(r-1)^2 \]
\[
(m-2r+2) + 8(r-1)^2 \sum_{k=2}^m (k-1).
\]

On simplification, we get (*) and (***)
\[
DD_*(G^{++}) = 8(r-1)^2[2m(3m-5r+2) + nr^2 + 3m + \sum_{k=2}^m (k-1)]
\]

and
\[
DD_*(G^{++}) = 8(r-1)^2[2m(3m-5r+2) + nr^2 + \sum_{k=2}^m (k-1)].
\]
Observation C.
1. Let $G$ be any $(n,m)$ graph.
   If $d_{G^+}(e_i,e_j)=1$, then
   \[ \sum_{(e_i,e_j) \in F(G^+)} [\deg_{G^+}(e_i) + \deg_{G^+}(e_j)]d_{G^+}(e_i,e_j) \text{ in } G^+ = 2(m-1)(-m + \frac{1}{2}nr^2). \]
   If $d_{G^+}(e_i,e_j)=2$, then
   \[ \sum_{(e_i,e_j) \in F(G^+)} [\deg_{G^+}(e_i) + \deg_{G^+}(e_j)]d_{G^+}(e_i,e_j) \text{ in } G^+ = 4(m-1)(m^2 + m - 2mr). \]

2. Let $G$ be any $(n,m)$ graph.
   If $d_{G^+}(e_i,e_j)=1$, then
   \[ \sum_{(e_i,e_j') \in F(G^+)} [\deg_{G^+}(e_i) + \deg_{G^+}(e_j')]d_{G^+}(e_i,e_j') \text{ in } G^+ = 2(m-r)(m-2r+1). \]
   If $d_{G^+}(e_i,e_j)=2$, then
   \[ \sum_{(e_i,e_j') \in F(G^+)} [\deg_{G^+}(e_i) + \deg_{G^+}(e_j')]d_{G^+}(e_i,e_j') \text{ in } G^+ = 8(m-r)(r-1). \]
   If $d_{G^+}(e_i,e_j)=3$, then
   \[ \sum_{(e_i,e_j') \in F(G^+)} [\deg_{G^+}(e_i) + \deg_{G^+}(e_j')]d_{G^+}(e_i,e_j') \text{ in } G^+ = 6(m-r). \]

3. Let $G$ be any $(n,m)$ graph.
   If $d_{G^+}(e_i',e_j')=2$, then
   \[ \sum_{(e_i',e_j') \in F(G^+)} [\deg_{G^+}(e_i') + \deg_{G^+}(e_j')]d_{G^+}(e_i',e_j') \text{ in } G^+ = 4(m-2r+1)\sum_{k=2}^m (k-1). \]

Theorem 3.5. For any $(n,m)$ graph $G$,
\[ DD_+(G^+)=2m(m^3-5)-1+mr[4m-6m-4r+8]nr^2+4(m-2r+1)\sum_{k=1}^m (k-1). \]

Proof. Let $G$ be any $(n,m)$-graph. From Proposition 3.1, $G^+$ contains $2m$ vertices and
\[ (-m^2 + \frac{1}{2}nr^2 - 2mr) \] edges.

From (2), we have
\[ DD_+(G) = \sum_{u,v \in F(G)} [\deg_G(u) + \deg_G(v)]d_G(u,v) \]
Therefore,
\[ DD_+(G^+) = \sum_{(e_i,e_j) \in F(G^+)} [\deg_{G^+}(e_i) + \deg_{G^+}(e_j)]d_{G^+}(e_i,e_j) + \sum_{(e_i,e_j') \in F(G^+)} [\deg_{G^+}(e_i) + \deg_{G^+}(e_j')]d_{G^+}(e_i,e_j') + \sum_{(e_i',e_j') \in F(G^+)} [\deg_{G^+}(e_i') + \deg_{G^+}(e_j')]d_{G^+}(e_i',e_j') \]

Applying observation C to the above equation, we get
\[ DD_+(G^+) = 2m(m-1)(r-1) + 4(m-1)(m^2 + m - 2mr) + 2(m-r)(m-2r+1) + 8m(m-r)(r-1) + 6m(m-r) + 2(m-2r+1)\sum_{k=1}^m (k-2). \]

On simplification,
\[ DD_+(G^+) = 2m[m(m-3)-1] + mr[4m-6m-4r+8]nr^2+4(m-2r+1)\sum_{k=1}^m (k-1). \]
Observation D.

1. Let $G$ be any $(n,m)$ graph.
   If $d_{G^+}(e_i,e_j) = 1$, then
   \[
   \sum_{(e_i,e_j) \in V(G^+)} [\deg_{G^+}(e_i) \cdot \deg_{G^+}(e_j)] d_{G^+}(e_i,e_j) \text{ in } G^+ = (m-1)^2(-m + \frac{1}{2}nr^2).
   \]
   If $d_{G^+}(e_i,e_j) = 2$, then
   \[
   \sum_{(e_i,e_j) \in V(G^+)} [\deg_{G^+}(e_i) \cdot \deg_{G^+}(e_j)] d_{G^+}(e_i,e_j) \text{ in } G^+ = 2(m-1)^2(m^2 + m - 2mr).
   \]

2. Let $G$ be any $(n,m)$ graph.
   If $d_{G^+}(e_i,e_j') = 1$, then
   \[
   \sum_{(e_i,e_j') \in V(G^+)} [\deg_{G^+}(e_i) \cdot \deg_{G^+}(e_j')] d_{G^+}(e_i,e_j') \text{ in } G^+ = (m^2 - 2mr + 2r - 1)\text{.}
   \]
   If $d_{G^+}(e_i,e_j') = 2$, then
   \[
   \sum_{(e_i,e_j') \in V(G^+)} [\deg_{G^+}(e_i) \cdot \deg_{G^+}(e_j')] d_{G^+}(e_i,e_j') \text{ in } G^+ = 4(m^2 - 2mr + 2r - 1)(r-1)m.
   \]
   If $d_{G^+}(e_i,e_j') = 3$, then
   \[
   \sum_{(e_i,e_j') \in V(G^+)} [\deg_{G^+}(e_i) \cdot \deg_{G^+}(e_j')] d_{G^+}(e_i,e_j') \text{ in } G^+ = 3m(m^2 - 2m + 2r - 1)\text{.}
   \]

3. Let $G$ be any $(n,m)$ graph.
   If $d_{G^+}(e_i',e_j') = 2$, then
   \[
   \sum_{(e_i',e_j') \in V(G^+)} [\deg_{G^+}(e_i') \cdot \deg_{G^+}(e_j')] d_{G^+}(e_i',e_j') \text{ in } G^+ = 2(m-2r+2)^2 \sum_{k=2}^{m} (k-1)^2\text{.}
   \]

Theorem 3.6. For any $(n,m)$ graph $G$,

\[
DD_1(G^+) = m[m^2(3m-3) - 7(m+1)] + \frac{1}{2} nr^2(m^2 - m + 1) - 2mr[2m(3-r) + 2r - 3] + 2(m-2r+2)^2 \sum_{k=2}^{m} (k-1)^2.
\]

Proof. Let $G$ be any $(n,m)$-graph. From Proposition 3.1, $G^+$ contains 2m vertices and

\[-m^2 + \frac{1}{2}(nr^2) - 2mr\text{ edges.}\]

From (3), we have

\[
DD_1(G) = \sum_{u,v \in V(G)} [\deg_G(u) \cdot \deg_G(v)] d_G(u,v)
\]

Therefore,

\[
DD_1(G^+) = \sum_{(e_i,e_j) \in V(G^+)} [\deg_{G^+}(e_i) \cdot \deg_{G^+(e_j)}] d_{G^+}(e_i,e_j) + \sum_{(e_i,e_j') \in V(G^+)} [\deg_{G^+}(e_i) \cdot \deg_{G^+(e_j')}] d_{G^+}(e_i,e_j') + \sum_{(e_i',e_j') \in V(G^+)} [\deg_{G^+}(e_i') \cdot \deg_{G^+(e_j')}] d_{G^+}(e_i',e_j').
\]

Applying observation D to the above equation, we get

\[
DD_1(G^+) = (m-1)^2(-m + \frac{1}{2}nr^2) + 2(m-1)^2(m^2 + m - 2mr) + (m^2 - 2mr + 2r - 1)(m - 2r + 1) + 4(m^2 - 2mr + 2r - 1)(r-1)m + 3m(m^2 - 2m + 2r - 1) + 2(m-2r+2)^2 \sum_{k=2}^{m} (k-1)^2.
\]
On simplification,
\[
DD_+(G^+) = m[m^2(3m - 3) - 7(m + 1)] + \frac{1}{2} nr^2(m^2 - m + 1) - 2mr[2m(3 - r) + 2r - 3] + 2(m - 2r + 2)^2 \sum_{k=2}^{m} (k - 1)^2.
\]

**Observation E.**
1. Let \( G \) be any \((n,m)\) graph.
   If \( d_{G^+}(e_i,e_j) = 1 \), then
   \[
   \sum_{(e_i,e_j) \in E(G^+)} [\text{deg}_{G^+}(e_i) + \text{deg}_{G^+}(e_j)]d_{G^+}(e_i,e_j) \text{ in } G^+ = (m-2r+1)(m-1)m.
   \]
   If \( d_{G^+}(e_i,e_j) = 2 \), then
   \[
   \sum_{(e_i,e_j) \in E(G^+)} [\text{deg}_{G^+}(e_i) + \text{deg}_{G^+}(e_j)]d_{G^+}(e_i,e_j) \text{ in } G^+ = (m-1)(r-1)m.
   \]
2. Let \( G \) be any \((n,m)\) graph.
   If \( d_{G^+}(e_i,e_j') = 1 \), then
   \[
   \sum_{(e_i,e_j') \in E(G^+)} [\text{deg}_{G^+}(e_i) + \text{deg}_{G^+}(e_j')]d_{G^+}(e_i,e_j') = 2m(m+2r-3)(r-1).
   \]
   If \( d_{G^+}(e_i,e_j') = 2 \), then
   \[
   \sum_{(e_i,e_j') \in E(G^+)} [\text{deg}_{G^+}(e_i) + \text{deg}_{G^+}(e_j')]d_{G^+}(e_i,e_j') = 2m(m+2r-3)(m-2r+2).
   \]
   If \( d_{G^+}(e_i,e_j') = 3 \), then
   \[
   \sum_{(e_i,e_j') \in E(G^+)} [\text{deg}_{G^+}(e_i) + \text{deg}_{G^+}(e_j')]d_{G^+}(e_i,e_j') = 3m(m+2r-3).
   \]
3. Let \( G \) be any \((n,m)\) graph.
   If \( d_{G^+}(e_i',e_j) = 2 \), then
   \[
   \sum_{(e_i',e_j) \in E(G^+)} [\text{deg}_{G^+}(e_i') + \text{deg}_{G^+}(e_j)]d(e_i',e_j) \text{ in } G^+ = 8(r-1)\sum_{k=2}^{m} (k - 1).
   \]

**Theorem 3.7.** For any \((n,m)\) graph \( G \),
\[
DD_+(G^+) = m[m(3m-2)-15] - nr[m-4r-17] + 8(r-1)\sum_{k=2}^{m} (k - 1).
\]

**Proof.** Let \( G \) be any \((n,m)\)-graph, from the Proposition 3.1, \( G^+ \) contains 2m vertices and \( \frac{m}{2} [m+2r-3] \) edges.

From (2), we have
\[
DD_+(G) = \sum_{u,v \in V(G)} [\text{deg}_G(u) + \text{deg}_G(v)]d_G(u,v)
\]

Therefore,
\[
DD_+(G^+) = \sum_{(e_i,e_j) \in E(G^+)} [\text{deg}_{G^+}(e_i) + \text{deg}_{G^+}(e_j)]d_{G^+}(e_i,e_j) + \sum_{(e_i,e_j') \in E(G^+)} [\text{deg}_{G^+}(e_i) + \text{deg}_{G^+}(e_j')]d_{G^+}(e_i,e_j') + \sum_{(e_i',e_j') \in E(G^+)} [\text{deg}_{G^+}(e_i') + \text{deg}_{G^+}(e_j')]d_{G^+}(e_i',e_j').
\]

Applying observation E to the above equation, we get
\[
DD_+(G^+) = (m-2r+1)(m-1)m + (m-1)(r-1)m + 2m(m+2r-3)(r-1) + 2m(m+2r-3)(m-2r+2) + 3m(m+2r-3) + 8(r-1)\sum_{k=2}^{m} (k - 1).
\]
On simplification,

\[ DD_+(G^{-}) = m[m(3m - 2) - 15] - mr[m - 4r - 17] + 8(r - 1) \sum_{k=2}^{m} (k - 1). \]

**Observation F.**

1. Let \( G \) be any \((n,m)\) graph.
   - If \( d_{G^{-}}(e_i, e_j) = 1 \), then
     \[
     \sum_{(e_i, e_j) \in V(G^{-})}[\deg_{G^{-}}(e_i) \cdot \deg_{G^{-}}(e_j)]d_{G^{-}}(e_i, e_j) \quad \text{in} \quad G^{-} = \frac{1}{2} (m - 1)^2 (m^2 - 2mr + m). \]
   - If \( d_{G^{-}}(e_i, e_j) = 2 \), then
     \[
     \sum_{(e_i, e_j) \in V(G^{-})}[\deg_{G^{-}}(e_i) \cdot \deg_{G^{-}}(e_j)]d_{G^{-}}(e_i, e_j) \quad \text{in} \quad G^{-} = 2(m - 1)^2(r - 1)m. \]

2. Let \( G \) be any \((n,m)\) graph.
   - If \( d_{G^{-}}(e_i, e_j') = 1 \), then
     \[
     \sum_{(e_i, e_j') \in V(G^{-})}[\deg_{G^{-}}(e_i) \cdot \deg_{G^{-}}(e_j')]d_{G^{-}}(e_i, e_j') \quad \text{in} \quad G^{-} = 4m(mr - m - r + 1)(r - 1). \]
   - If \( d_{G^{-}}(e_i, e_j') = 2 \), then
     \[
     \sum_{(e_i, e_j') \in V(G^{-})}[\deg_{G^{-}}(e_i) \cdot \deg_{G^{-}}(e_j')]d_{G^{-}}(e_i, e_j') \quad \text{in} \quad G^{-} = 4m(mr - m - r - 1)(m - 2r + 2). \]
   - If \( d_{G^{-}}(e_i, e_j') = 3 \), then
     \[
     \sum_{(e_i, e_j') \in V(G^{-})}[\deg_{G^{-}}(e_i) \cdot \deg_{G^{-}}(e_j')]d_{G^{-}}(e_i, e_j') \quad \text{in} \quad G^{-} = 6m(mr - m - r + 1). \]

3. Let \( G \) be any \((n,m)\) graph.
   - If \( d_{G^{-}}(e_i', e_j') = 2 \), then
     \[
     \sum_{(e_i', e_j') \in V(G^{-})}[\deg_{G^{-}}(e_i') \cdot \deg_{G^{-}}(e_j')]d_{G^{-}}(e_i', e_j') \quad \text{in} \quad G^{-} = 8(r - 1)^2 \sum_{k=2}^{m} (k - 1). \]

**Theorem 3.8.** For any \((n,m)\) graph \( G \),

\[ DD_+(G^{-}) = \frac{1}{2} m[m(m - 13) - 5m + 8] + mr(m(5m - 4r + 8) + 4r - 13) + 
8(r - 1)^2 \sum_{k=2}^{m} (k - 1). \]

**Proof.** Let \( G \) be any \((n,m)\)-graph. From Proposition 3.1, \( G^{-} \) contains \( 2m \) vertices and \( \frac{m}{2} \) \([m + 2r - 3]\) edges.

From (3), we have

\[ DD_+(G) = \sum_{u,v \in V(G)} [\deg_G(u) \cdot \deg_G(v)]d_G(u,v) \]

Therefore,

\[ DD_+(G^{-}) = \sum_{(e_i, e_j) \in V(G^{-})}[\deg_{G^{-}}(e_i) \cdot \deg_{G^{-}}(e_j)]d_{G^{-}}(e_i, e_j) + 
\sum_{(e_i, e_j') \in V(G^{-})}[\deg_{G^{-}}(e_i) \cdot \deg_{G^{-}}(e_j')]d_{G^{-}}(e_i, e_j') + 
\sum_{(e_i', e_j') \in V(G^{-})}[\deg_{G^{-}}(e_i') \cdot \deg_{G^{-}}(e_j')]d_{G^{-}}(e_i', e_j'). \]

Applying observation F to the above equation, we get

\[ DD_+(G^{-}) = \frac{1}{2} (m - 1)^2 (m^2 - 2mr + m) + 2(m - 1)^2(r - 1)m + 4m(mr - m - r + 1)(r - 1) \]
+ 4m(mr - m - r - 1)(m - 2r + 2) + 6m(mr - m - r + 1) + 8(r - 1)^2 \sum_{k=2}^{m}(k - 1).

On simplification,

\[ DD_+(G^+) = \frac{1}{2} m[m^2(13) - 5m + 8] + m[5m(m - 4r + 8) + 4r - 13] + 8(r - 1)^2 \sum_{k=2}^{m}(k - 1). \]

**Observation G.**

1. Let \( G \) be any \((n,m)\) graph.
   If \( d^-_G(e_i, e_j) = 1 \), then
   \[ \sum_{(e_i, e_j) \in E(G^-)}[\text{deg}^-_G(e_i) + \text{deg}^-_G(e_j)]d^-_G(e_i, e_j) \]
   in \( G^- = 2m(m - 2r + 1)^2 \).
   If \( d^-_G(e_i, e_j) = 2 \), then
   \[ \sum_{(e_i, e_j) \in E(G^-)}[\text{deg}^-_G(e_i) + \text{deg}^-_G(e_j)]d^-_G(e_i, e_j) \]
   in \( G^- = 8(m - 2r + 1)^2(r - 1) \).

2. Let \( G \) be any \((n,m)\) graph.
   If \( d^-_G(e_i, e'_j) = 1 \), then
   \[ \sum_{(e_i, e'_j) \in E(G^-)}[\text{deg}^-_G(e_i) + \text{deg}^-_G(e'_j)]d^-_G(e_i, e'_j) \]
   in \( G^- = 3m(m - 2r + 1)^2 \).
   If \( d^-_G(e_i, e'_j) = 2 \), then
   \[ \sum_{(e_i, e'_j) \in E(G^-)}[\text{deg}^-_G(e_i) + \text{deg}^-_G(e'_j)]d^-_G(e_i, e'_j) \]
   in \( G^- = 6m(2r - 1)(m - 2r + 1)^2 \).

3. Let \( G \) be any \((n,m)\) graph.
   If \( d^-_G(e'_i, e'_j) = 2 \), then
   \[ \sum_{(e'_i, e'_j) \in E(G^-)}[\text{deg}^-_G(e'_i) + \text{deg}^-_G(e'_j)]d^-_G(e'_i, e'_j) \]
   in \( G^- = 4(m - 2r + 1) \sum_{k=2}^{m}(k - 1) \).

**Theorem 3.9.** For any \((n,m)\) graph \( G \),

\[ DD_+(G^-) = m[5m^2 - 4m - 9] - 4mr[5r - 7] + 4(m - 2r + 1) \sum_{k=2}^{m}(k - 1). \]

**Proof.** Let \( G \) be any \((n,m)\)-graph. From proposition 3.1, \( G^- \) contains 2m vertices and \( \frac{3}{2} m[2r + 1] \) edges.

From (2), we have

\[ DD_+(G) = \sum_{u,v \in V(G)}[\text{deg}_G(u) + \text{deg}_G(v)]d_G(u,v) \]

Therefore,

\[ DD_+(G^-) = \sum_{(e_i, e_j) \in E(G^-)}[\text{deg}^-_G(e_i) + \text{deg}^-_G(e_j)]d^-_G(e_i, e_j) \]
\[ + \sum_{(e'_i, e'_j) \in E(G^-)}[\text{deg}^-_G(e'_i) + \text{deg}^-_G(e'_j)]d^-_G(e'_i, e'_j) \]
\[ + \sum_{e_i, e'_j \in E(G^-)}[\text{deg}^-_G(e'_i) + \text{deg}^-_G(e'_j)]d^-_G(e'_i, e'_j). \]

Applying observation G to the above equation, we get

\[ DD_+(G^-) = 2m(m - 2r + 1)^2 + 8(m - 2r + 1)(r - 1) + 3m(m - 2r + 1)^2 + 6m(2r - 1)(m - 2r + 1) \]
\[ + 4(m - 2r + 1) \sum_{k=2}^{m}(k - 1) \]

\[ DD_+(G^-) = m[5m^2 - 4m - 9] - 4mr[5r - 7] + 4(m - 2r + 1) \sum_{k=2}^{m}(k - 1). \]
On simplification, 
\[ DD_\ast (G^{-}) = m[5m^2 - 4m - 9] - 4mr[5r - 7] + 4(m - 2r + 1) \sum_{k=2}^{m} (k - 1). \]

**Observation H.**
1. Let \( G \) be any \((n,m)\) graph.
   If \( d_{g^-}(e_1,e_2) = 1 \), then
   \[ \sum_{(e_j,e_j) \in V(G^-)} [deg_{g^-}(e_j) \cdot deg_{g^-}(e_j)]d_{g^-}(e_j,e_j) \] in 
   \( G^- = 2m(m - 2r + 1)^3 \).
   If \( d_{g^-}(e_1,e_2) = 2 \), then
   \[ \sum_{(e_j,e_j) \in V(G^-)} [deg_{g^-}(e_j) \cdot deg_{g^-}(e_j)]d_{g^-}(e_j,e_j) \] in 
   \( G^- = 8m(m - 2r + 1)^2(r - 1). \)
2. Let \( G \) be any \((n,m)\) graph.
   If \( d_{g^-}(e_1,e_2) = 1 \), then
   \[ \sum_{(e_j,e_j) \in V(G^-)} [deg_{g^-}(e_j) \cdot deg_{g^-}(e_j)]d_{g^-}(e_j,e_j) \] in 
   \( G^- = 2m(m - 2r + 1)^3 \).
   If \( d_{g^-}(e_1,e_2) = 2 \), then
   \[ \sum_{(e_j,e_j) \in V(G^-)} [deg_{g^-}(e_j) \cdot deg_{g^-}(e_j)]d_{g^-}(e_j,e_j) \] in 
   \( G^- = 4m(m - 2r + 2)^2(2r - 1). \)
3. Let \( G \) be any \((n,m)\) graph.
   If \( d_{g^-}(e_1,e_2) = 2 \), then
   \[ \sum_{(e_j,e_j) \in V(G^-)} [deg_{g^-}(e_j) \cdot deg_{g^-}(e_j)]d_{g^-}(e_j,e_j) \] in 
   \( G^- = 2m(m - 2r + 1)^3 \sum_{k=2}^{m} (k - 1). \)

**Theorem 3.10.** For any \((n,m)\) graph \( G \),
\[ DD_\ast (G^-) = 2m[m^2(2m - 3) - 5m + 6] - 4mr[3m^2 - 4r^2 + 10mr - 14m + 12r - 13] + 
2(m - 2r + 1)^3 \sum_{k=2}^{m} (k - 1). \]

**Proof.** Let \( G \) be any \((n,m)\)-graph. From Proposition 3.1, \( G^- \) contains \( 2m \) vertices and
\[ \frac{3}{2} m - 2r + 1 \] edges.

From (3), we have
\[ DD_\ast (G) = \sum_{u,v \in V(G)} [deg_{G}(u) \cdot deg_{G}(v)]d_{G}(u,v) \]

Therefore,
\[ DD_\ast (G^-) = \sum_{(e_j,e_j) \in V(G^-)} [deg_{g^-}(e_j) \cdot deg_{g^-}(e_j)]d_{g^-}(e_j,e_j) + 
\sum_{(e_j,e_j) \in V(G^-)} [deg_{g^-}(e_j) \cdot deg_{g^-}(e_j)]d_{g^-}(e_j,e_j) + 
\sum_{(e_j,e_j) \in V(G^-)} [deg_{g^-}(e_j) \cdot deg_{g^-}(e_j)]d_{g^-}(e_j,e_j). \]

Applying observation H the above equation, we get
\[ DD_\ast (G^-) = 2m(m - 2r + 1)^3 + 8m(m - 2r + 1)^2(r - 1) + 2m(m - 2r + 1)^3 + 
4m(m - 2r + 2)^2(2r - 1) + 2(m - 2r + 1)^3 \sum_{k=2}^{m} (k - 1). \]

On simplification,
\[ DD_\ast (G^-) = 2m[m^2(2m - 3) - 5m + 6] - 4mr[3m^2 - 4r^2 + 10mr - 14m + 12r - 13] + 
2(m - 2r + 1)^3 \sum_{k=2}^{m} (k - 1). \]
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