THE RESULTS CONCERNING JORDAN DERIVATIONS

Byung Do Kim*

Abstract. Let $R$ be a 3!-torsion free semiprime ring, and let $D : R \to R$ be a Jordan derivation on a semiprime ring $R$. In this case, we show that $[D(x), x]D(x) = 0$ if and only if $D(x)[D(x), x] = 0$ for every $x \in R$. In particular, let $A$ be a Banach algebra with $\text{rad}(A)$. If $D$ is a continuous linear Jordan derivation on $A$, then we see that $[D(x), x]D(x) \in \text{rad}(A)$ if and only if $[D(x), x]D(x) \in \text{rad}(A)$ for all $x \in A$.

1. Introduction

Throughout, $R$ represents an associative ring and $A$ will be a real or complex Banach algebra. We write $[x, y]$ for the commutator $xy - yx$ for $x, y$ in a ring. Let $\text{rad}(R)$ denote the (Jacobson) radical of a ring $R$. And a ring $R$ is said to be (Jacobson) semisimple if its Jacobson radical $\text{rad}(R)$ is zero. See [1] for the more details.

A ring $R$ is called $n$-torsion free if $nx = 0$ implies $x = 0$. Recall that $R$ is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$.

An additive mapping $D$ from $R$ to $R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And an additive mapping $D$ from $R$ to $R$ is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$.

Johnson and Sinclair [4] proved that any linear derivation on a semisimple Banach algebra is continuous. A result of Singer and Wermer [5] states that every continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. From these two results, we can conclude that there are no nonzero linear derivations on a commutative semisimple Banach algebra.

Received March 10, 2016; Accepted October 17, 2016.

2010 Mathematics Subject Classification: Primary 16N60, 16W25, 17B40.

Key words and phrases: Banach algebra, Jordan derivation, prime and semiprime ring, (Jacobson) radical.
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Vukman [7] proved the following: let \( R \) be a 2-torsion free prime ring. If \( D : R \to R \) is a derivation such that \([D(x), x]D(x) = 0\) for all \( x \in R \), then \( D = 0 \).

Moreover, using the above result, he proved that the following holds: let \( A \) be a noncommutative semisimple Banach algebra. Suppose that \([D(x), x]D(x) = 0\) holds for all \( x \in A \). In this case, \( D = 0 \).

In this paper, our first aim is to prove the following result in the ring theory in order to apply it to the Banach algebra theory:

let \( R \) be a 3!-torsion free semiprime ring, and let \( D : R \to R \) be a Jordan derivation on a semiprime ring \( R \). In this case, we show that \([D(x), x]D(x) = 0\) if and only if \([D(x), x]D(x) = 0\) for every \( x \in R \). In particular, let \( A \) be a Banach algebra with \( \text{rad}(A) \) and if \( D \) is a continuous linear Jordan derivation on \( A \), then we see that \([D(x), x]D(x) \in \text{rad}(A)\) if and only if \([D(x), x]D(x) \in \text{rad}(A)\) for all \( x \in A \).

2. Preliminaries

The following lemma is due to Chung and Luh [3].

**Lemma 2.1.** Let \( R \) be a \( n! \)-torsion free ring. Suppose there exist elements \( y_1, y_2, \cdots, y_{n-1}, y_n \) in \( R \) such that \( \sum_{k=1}^{n} k^t y_k = 0 \) for all \( t = 1, 2, \cdots, n \). Then we have \( y_k = 0 \) for every positive integer \( k \) with \( 1 \leq k \leq n \).

The following theorem is due to Brešar [2].

**Theorem 2.2.** Let \( R \) be a 2-torsion free semiprime ring and let \( D : R \to R \) be a Jordan derivation. In this case, \( D \) is a derivation.

And the following theorem is proved by Vukman in [7] under the condition of the prime ring \( R \).

**Theorem 2.3.** Let \( R \) be a 3!-torsion free prime ring. Let \( D : R \to R \) be a Jordan derivation on \( R \). In this case, we see that if \([D(x), x]D(x) = 0\) for every \( x \in R \), then \( D(x) = 0 \) for all \( x \in R \).

3. Main results

We need the following notations. After this, by \( S_m \) we denote the set \( \{ k \in \mathbb{N} \mid 1 \leq k \leq m \} \) where \( m \) is a positive integer. When \( R \) is a
The results concerning Jordan derivations ring, we shall denote the maps $B : R \times R \to R$, $f, g : R \to R$ by $B(x, y) \equiv [D(x), y] + [D(y), x]$, $f(x) \equiv [D(x), x]$, $g(x) \equiv [f(x), x]$ for all $x, y \in R$ respectively. And we have the basic properties:

\[
B(x, y) = B(y, x),
\]

\[
B(x, yz) = B(x, y)z + yB(x, z) + D(y)[z, x] + [y, x]D(z)
\]

for all $x, y \in R$ and $z \in R$.

**Theorem 3.1.** Let $R$ be a 3!-torsionfree semiprime ring. Let $D : R \to R$ be a Jordan derivation on $R$. In this case, it follows that

\[
[D(x), x]D(x) = 0 \iff D(x)[D(x), x] = 0
\]

for every $x \in R$.

**Proof.** (Necessity)

It is sufficient to prove the noncommutative case of $R$.

Assume that

\[
[D(x), x]D(x) = f(x)D(x) = 0, \ x \in R.
\]

Replacing $x + ty$ for $x$ in (3.1), we have

\[
[D(x + ty), x + ty]D(x + ty) = f(x)D(x) + t\{B(x, y)D(x) + f(x)D(y)\} + t^2H(x, y)
\]

\[
+t^3f(y)D(y) = 0, \ x, y \in R, \ t \in S_2
\]

where $H$ denotes the term satisfying the identity (3.2).

From (3.1) and (3.2), we obtain

\[
t\{B(x, y)D(x) + f(x)D(y)\} + t^2H(x, y) = 0, \ x, y \in R, \ t \in S_2.
\]

Since $R$ is 2!-torsionfree, by Lemma 2.1 the relation (3.3) yields

\[
B(x, y)D(x) + f(x)D(y) = 0, \ x, y \in R.
\]

Writing $yx$ for $y$ in (3.4), we get

\[
B(x, yx)D(x) + f(x)D(yx) = B(x, y)xD(x) + 2yf(x)D(x) + [y, x]D(x)^2 + f(x)D(y)x
\]

\[
+f(x)yD(x) = 0, \ x, y \in R.
\]

From (3.1) and (3.5), we obtain

\[
B(x, y)xD(x) + [y, x]D(x)^2 + f(x)D(y)x + f(x)yD(x)
\]

\[
= 0, \ x, y \in R.
\]
Left multiplication of (3.1) by $D(x)$ gives
$$-2D(x)f(x)f(y) - D(x)[y, x]f(x) + D(x)g(x)yD(x) = 0, \quad x, y \in R.$$ (3.14)
From (3.1) and (3.14), we arrive at
$$D(x)f(x)yf(x) + (g(x)D(x) - D(x)g(x))yD(x) = 0, \quad x, y \in R.$$ (3.15)
Right multiplication of (3.15) by $D(x)$ leads to
$$D(x)f(x)yf(x)D(x) + (g(x)D(x) - D(x)g(x))yD(x)^2 = 0, \quad x, y \in R.$$ (3.16)
From (3.1) and (3.16), we obtain
$$g(x)D(x) - D(x)g(x)yD(x)^2 = 0, \quad x, y \in R.$$ (3.17)
Writing \( yD(x) \) for \( y \) in (3.15), we have
\[
D(x)f(x)yD(x)f(x) + (g(x)D(x) - D(x)g(x))yD(x)^2 = 0, \ x, y \in R.
\]
(3.18)

From (3.17) and (3.18), we get
\[
D(x)f(x)yD(x)f(x) = 0, \ x, y \in R.
\]
(3.19)

Since \( R \) is semiprime, (3.19) gives
\[
D(x)f(x) = 0, \ x \in R.
\]

Therefore the necessity is proved.

The inverse statement is symmetrically proved in the expressions proved.

(Sufficiency)
Suppose
\[
D(x)[D(x), x] = D(x)f(x) = 0, \ x \in R.
\]
(3.20)

Replacing \( x + ty \) for \( x \) in (3.20), we have
\[
D(x + ty)[D(x + ty), x + ty] = D(x)f(x) + t[D(y)f(x) + D(x)B(x, y)] + t^2I(x, y)
\]
(3.21)

where \( I \) denotes the term satisfying the identity (3.21).

From (3.20) and (3.21), we obtain
\[
t[D(y)f(x) + D(x)B(x, y)] + t^2I(x, y) = 0, \ x, y \in R, \ t \in S_2.
\]
(3.22)

Since \( R \) is 2!-torsionfree, by Lemma 2.1 the relation (3.22) yields
\[
D(y)f(x) + D(x)B(x, y) = 0, \ x, y \in R.
\]
(3.23)

Writing \( xy \) for \( y \) in (3.23), we get
\[
xD(y)f(x) + D(x)yf(x) + D(x)xB(x, y) + 2D(x)f(x)y + D(x)^2[y, x] = 0, \ x, y \in R.
\]
(3.24)

From (3.20) and (3.24), it follows from that
\[
xD(y)f(x) + D(x)yf(x) + D(x)xB(x, y) + D(x)^2[y, x] = 0, \ x, y \in R.
\]
(3.25)

Left multiplication of (3.23) by \( x \) leads to
\[
xD(y)f(x) + xD(x)B(x, y) = 0, \ x, y \in R.
\]
(3.26)
From (3.25) and (3.26), we get
\[(3.27)\]
\[D(x)yx f(x) + f(x)B(x, y) + D(x)^2[y, x] = 0, \ x, y \in R.\]

Right multiplication of (3.27) by \(x\) yields
\[(3.28)\]
\[D(x)yx f(x) + f(x)B(x, y)x + D(x)^2[y, x]x = 0, \ x, y \in R.\]

Replacing \(yx\) for \(y\) in (3.27), we have
\[(3.29)\]
\[D(x)yx f(x) + f(x)B(x, y)x + 2f(x)yx f(x) + f(x)[y, x]D(x)\]
\[+ D(x)^2[y, x]x = 0, \ x, y \in R.\]

From (3.28) and (3.29), we obtain
\[(3.30)\]
\[D(x)yg(x) - 2f(x)yx f(x) - f(x)[y, x]D(x) = 0, \ x, y \in R.\]

Right multiplication of (3.30) by \(D(x)\) gives
\[(3.31)\]
\[D(x)yg(x)D(x) - 2f(x)yx f(x)D(x) - f(x)[y, x]D(x)^2\]
\[= 0, \ x, y \in R.\]

Putting \(yD(x)\) instead of \(y\) in (3.30), it is obvious that
\[(3.32)\]
\[D(x)yg(x) - 2f(x)yg(x) - f(x)[y, x]D(x)^2\]
\[- f(x)yx f(x)D(x) = 0, \ x, y \in R.\]

From (3.20) and (3.32), we have
\[(3.33)\]
\[D(x)yg(x)D(x) - f(x)[y, x]D(x)^2 - f(x)yx f(x)D(x)\]
\[= 0, \ x, y \in R.\]

From (3.31) and (3.33), we get
\[(3.34)\]
\[D(x)y\{g(x)D(x) - D(x)g(x)\} - f(x)yx f(x)D(x)\]
\[= 0, \ x, y \in R.\]

Left multiplication of (3.34) by \(D(x)\) leads to
\[(3.35)\]
\[D(x)^2yg\{g(x)D(x) - D(x)g(x)\} - D(x)f(x)yx f(x)D(x)\]
\[= 0, \ x, y \in R.\]

From (3.20) and (3.35), we obtain
\[(3.36)\]
\[D(x)^2yg\{g(x)D(x) - D(x)g(x)\} = 0, \ x, y \in R.\]

Substituting \(D(x)y\) for \(y\) in (3.34), we have
\[(3.37)\]
\[D(x)^2yg\{g(x)D(x) - D(x)g(x)\} - f(x)D(x)yx f(x)D(x)\]
\[= 0, \ x, y \in R.\]
From (3.36) and (3.37), we obtain

\begin{equation}
    f(x)D(x)yf(x)D(x) = 0, \; x, y \in R.
\end{equation}

Since \( R \) is semiprime, (3.38) gives

\begin{equation}
    f(x)D(x) = 0, \; x \in R.
\end{equation}

Therefore the sufficiency is proved. \( \square \)

We obtain the equivalent property of continuous Jordan derivations on Banach algebras as the application to the Banach algebra theory.

**Theorem 3.2.** Let \( A \) be a Banach algebra with \( \text{rad}(A) \). Let \( D : A \rightarrow A \) be a continuous linear Jordan derivation. Then we obtain

\[ [D(x), x]D(x) \in \text{rad}(A) \iff D(x)[D(x), x] \in \text{rad}(A) \]

for every \( x \in A \).

**Proof.** It suffices to prove the case that \( A \) is noncommutative. By the result of B.E. Johnson and A.M. Sinclair [4] any linear derivation on a semisimple Banach algebra is continuous. Sinclair [5] has proved that every continuous linear Jordan derivation on a Banach algebra leaves the primitive ideals of \( A \) invariant. Hence for any primitive ideal \( P \subseteq A \) one can introduce a derivation \( D_P : A/P \rightarrow A/P \), where \( A/P \) is a prime and factor Banach algebra, by \( D_P(\hat{x}) = D(x) + P, \; \hat{x} = x + P \).

By the assumption that \( [D(x), x]D(x) \in \text{rad}(A), \; x \in A \), we obtain

\[ [D_P(\hat{x}), \hat{x}]D_P(\hat{x}) = 0 \iff D_P(\hat{x})[D_P(\hat{x}), \hat{x}] = 0, \; \hat{x} \in A/P, \] since all the assumptions of Theorem 3.1 are fulfilled. Thus we see that

\[ [D(x), x]D(x) \in P \iff D(x)[D(x), x] \in P \]

for every \( x \in A \) and all primitive ideals of \( A \). Therefore we conclude that

\[ [D(x), x]D(x) \in \text{rad}(A) \iff D(x)[D(x), x] \in \text{rad}(A) \]

for every \( x \in A \). \( \square \)

As a special case of Theorem 3.2 we get the following result which characterizes commutative semisimple Banach algebras.

**Corollary 3.3.** Let \( A \) be a semisimple Banach algebra. Then we have

\[ [[y, x], x][y, x] = 0 \iff [y, x][[y, x], x] = 0 \]

for every \( x, y \in A \).
Acknowledgment

The author wishes to thank the referees for their valuable comments.

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