Rigorous path integrals for supersymmetric quantum mechanics: completing the path integral proof of the index theorem

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Abstract

Many introductory courses in quantum mechanics include Feynman’s time-slicing definition of the path integral, with a complete derivation of the propagator in the simplest of cases. However, attempts to generalize this, for instance to non-quadratic potentials, encounter formidable analytic issues in showing the successive approximations in fact converge to a definite expression for the path integral. The present work describes how to carry out the analysis for a class of Lagrangians broad enough to include the evolution, in imaginary time, of spinors constrained to live on a Riemannian manifold. For these Lagrangians, the successive time-slicing approximations converge. The limit provides a definition of the path integral which agrees with the imaginary-time Feynman propagator. With this as the definition, the steepest-descent approximation to the path integral for twisted \(N = 1/2\) supersymmetric quantum mechanics is provably correct. These results complete a new proof of the Atiyah-Singer index theorem for the twisted Dirac operator.

Keywords: Path integral, Supersymmetry, Quantum mechanics, Index theorem

Introduction

Elaborating on an argument due to Witten\(^1\), Alvarez-Gaumé\(^2\) evaluates path integrals for supersymmetric quantum mechanics (SUSYQM) using what has become a familiar argument, the short version of which is that for cohomological quantum theories stationary phase is exact. Friedan and Windey\(^3\) extend this to a “twisted” version.

In slightly more detail, the path integral in question is

\[
\int e^{-S(\sigma, \Psi, \Psi^\dagger, t)} \mathcal{D}\Psi \mathcal{D}\Psi^\dagger \mathcal{D}\sigma d\Psi^\dagger d\Psi dx
\]

where the action \(S\) is the time integral of the SUSYQM Lagrangian, and the integral is over the space of paths which start at a pair \((y, \psi_y)\), consisting of a point \(y\) in a Riemannian manifold \(M\) and an associated spinor \(\psi_y\), and after a time interval of length \(t\) end at the pair \((x, \psi_x)\). The paths consist of a standard path \(\sigma\) in \(M\) and, at each point of \(\sigma\), a spinor \(\Psi\) and a dual spinor \(\Psi^\dagger\). A close reading of Alvarez-Gaumé’s treatment of the Euclidean theory for SUSYQM on \(M\) reveals the argument rests on two key properties of the path integral:

1. The path integral represents the heat kernel of a Laplacian operator on a known bundle.
2. Steepest-descent (i.e., the standard path integral technique of stationary phase but in the Euclidean realm) provides an approximation to the path integral valid for small \(t\).

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Taking an appropriate supertrace of the heat kernel, which corresponds to taking the integral over loops rather than paths with fixed endpoints, calculates a known topological invariant. For instance, in the $N = 1$ theory this is the Euler characteristic, while in the $N = 1/2$ theory it is the index of the Dirac operator. Necessarily, such invariants do not depend on the time parameter $t$ appearing in the path integral. Thus, assuming the above properties hold, taking $t$ to 0 in the steepest-descent approximation to the path integral over loops gives the exact value of the path integral, and with it the supertrace of the heat kernel. Calculating this integral with the supertrace gives a path integral proof of the corresponding index theorem. In the above cases these are the Gauss-Bonnet-Chern theorem and the Atiyah-Singer index theorem, respectively.

From a mathematical viewpoint, these elegant arguments must be taken to be merely heuristic, because the relevant path integrals themselves have not been carefully defined. Most crucially, the stated properties have been proven for rigorous definitions of path integrals only in settings much simpler than those of SUSYQM on manifolds.

The following, which may be thought of as an explanation of some analytic details missing from the derivations in [1], [2] and [3], describes the rigorous mathematical construction, based on Feynman’s original time-slicing definition [4], of path integrals for a large class of Lagrangians; namely, those corresponding to generalized Laplacians on bundles on manifolds. The class is large enough to include the Lagrangian for the twisted version of $N = 1/2$ SUSYQM of Friedan and Windey[3] from which they derive the Atiyah-Singer index theorem for the twisted Dirac operator, which includes the others a special cases. The present construction of the path integral agrees with the heat kernel of the generalized Laplacian, so Property 1 holds true. In the special case of twisted $N = 1/2$ SUSYQM on loops, bounds on the error between the time-slicing approximation based on a given partition of $t$ and the path integral suffice to interchange the fine partition and the small-$t$ limit. This interchange of limits leads to a proof of the validity of the steepest descent approximation, which is Property 2, to sufficiently high order in $t$ to complete a new proof of the Atiyah-Singer index theorem for the twisted Dirac operator.

Starting from the Lagrangian, several choices go into defining the time-slicing approximation to the path integral. These include choices of Riemann sum approximations to the action, and terms which might explicitly depend on $\hbar$. While the various choices should be equivalent in the sense of the existence of a fine-partition limit, and its value, the argument in Sect. 2.3 that the approximate path integrals converge to the heat kernel depends on a particular estimate which constrains the choice of time-slicing approximation. Remark 2.2 discusses this in some detail, in particular identifying many choices leading to the same limiting path integral. Presumably, choices satisfying the additional constraint make the rate of convergence manifestly faster.

**Other approaches to rigorous path integrals and index theorems**

The elegant heuristic arguments have inspired a variety of mathematically rigorous approaches. Bismut[5, 6] uses stochastic techniques with the heat equation to give a proof of the index theorem in the spirit of the physics argument. Getzler[7, 8], who does not directly construct path integrals, gives an index theorem proof using the theory of pseudo-differential operators to provide the estimates suggested by these arguments. Rogers[9] uses stochastic techniques to construct an explicit supersymmetric path integral for the heat kernel on manifolds whose Riemannian metric is Euclidean outside of a bounded region. This suffices to reproduce the path integral proof of the GBC theorem for arbitrary compact manifolds, since the argument
only depends on the short-time behavior of the restriction of the heat kernel to the diagonal. In later work\[10, 11\], she extends these techniques to prove the twisted Hirzebruch index theorem, from which follows the full index theorem. Andersson and Driver\[12\] use stochastic techniques to construct a version of the bosonic path integral on curved space.

The present paper describes an argument, which Fine and Sawin\[13\] presents in more technical terms, whose innovation is to make rigorous Feynman’s time-slicing procedure in constructing the supersymmetric path integral, thereby representing the heat kernel for the any generalized Laplacian on an arbitrary compact Riemannian manifold as a path integral, and to obtain, from this representation, its short-time approximation on the diagonal in the special case of $N = 1/2$ SUSYQM.

1 The action for a generalized Laplacian on the Grassman algebra of a vector bundle and a corresponding time-slicing approximate kernel

1.1 Generalized Laplacians as Hamiltonians and the corresponding Lagrangians

Let $M$ be a Riemannian manifold, with metric $g$, and let $V$ be a vector bundle over $M$. Let $\nabla$ be a generalized Laplacian; that is, a second-order a second-order elliptic operator on sections of $V$ which in local coordinates has the form

$$\Delta = g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + A^i \frac{\partial}{\partial x_i} + B,$$

with $A^i$ and $B$ valued in Matrix_{n,n}. The goal is to construct path integrals for quantum mechanics on $M$ with (imaginary-time) Hamiltonian given by $\Delta$. Generalized Laplacians include Hamiltonians with non-trivial potentials, while generalizing to vector bundles allows for additional structure such as spinors.

If the data $(M, g, V, \Delta)$ are not all smooth, or $M$ is not compact, require that the data be tame in the following sense:\[3\]

**Definition 1** An atlas of charts for $V$ over $M$ is tame if

- All derivatives of $g$ and $g^{-1}$ of order $0 \leq k \leq 6$, expressed in the coordinates of each chart, are uniformly bounded in the supremum norm on all charts.

- There is a $D_0 > 0$ such that the ball of radius $D_0$ around any point is contained in a single chart.

The tuple $(M, g, V)$ is tame if it admits a tame atlas. If $\Delta$ is a generalized Laplacian, and if there is a tame atlas so that the derivatives of order $0 \leq k \leq 2$ of $A^i$ and $B$ in all charts are uniformly bounded in the supremum norm, then say that $(M, g, V, \Delta)$ is tame.

\[3\]Noncompact manifolds arise in the argument for the convergence of the approximate path integrals, because the key properties prove to be local. The noncompact manifold $\mathbb{R}^m$ provides the easiest setting in which to formulate these properties. Noncompact manifolds, again $\mathbb{R}^m$, appear in Sect. 3.2.1 which extends local data to ultimately prove the steepest descent expression gives the correct small-$t$ behavior of the path integral. This is also the case Rogers treats using stochastic quantization \[10, 11\].
If $M$ is compact and all the data is smooth, then $(M, g, V, \Delta)$ is necessarily tame. Tameness is a technical restriction ensuring in more general settings that the manifold, metric, bundle and generalized Laplacian are sufficiently smooth to apply the convergence arguments of Sect. 2.

In passing to the imaginary-time formalism, the time evolution operator associated to the Hamiltonian $\Delta$ becomes the heat operator $e^{-t\Delta/2}$ taking an initial configuration $f_0(x)$ to the corresponding solution $f(x, t)$ of the heat equation

$$\frac{1}{2} \Delta f = \frac{\partial f}{\partial t}.$$ 

Indeed, the heat equation is the imaginary-time Schrödinger equation governing the evolution of the wave-function $f(x, t)$ from its initial value $f_0(x)$. Here $x \in M$ is a point of $M$ while $f(x, t)$ is a point in the fiber of $V$ over $x$. In local coordinates, these are specified by the coordinates $x^i$ of a point in $\mathbb{R}^n$ and the components $f^a(x, t)$ of a vector, relative to a particular basis $e_a(x)$, in the vector space defining $V$. In the cases of most interest, the bundle will actually have the form $X = \Lambda V$ for some other bundle $V$ and the sections can be written as $f(x, \psi, t)$ for $\psi$ a Grassman-valued, or anti-commuting, section of $V$; see Sect. 1.3 below.

Berline, Getzler and Vergne [14] observe that every generalized Laplacian can be written locally as

$$\Delta^V = g^{ij} \left[ \nabla_{\partial_i} V^j - \Gamma^j_{ik} \nabla_{\partial_k} V^j \right] V,$$  \hspace{1cm} (1.1)

where $\nabla^V$ is the covariant derivative defined by a connection on $V$, $\nabla^V_{\partial_i} (\partial_j) = \Gamma^k_{ij} \partial_k$ defines the Christoffel symbols for the Levi-Civita connection on the tangent bundle, and $V$ is a section of $\text{End}(V)$. (In local coordinates, relative to the basis $e_a(x)$, $V$ is a matrix-valued function $V^a_b(x)$.)

Let $\sigma$ denote a path in $M$ with parameter $s$, $\Psi$ a lift of $\sigma$ to $V^*$, the dual vector bundle, and $\Psi^\dagger$ a lift to $V$. The Lagrangian corresponding to $\Delta$ is then

$$L(\sigma, \Psi, \Psi^\dagger, s) = \frac{1}{2} \langle \dot{\sigma}, \dot{\sigma} \rangle + i \langle \Psi^\dagger, \nabla^V_{\partial_s} \Psi \rangle - \frac{i}{2} \langle V \Psi^\dagger, \Psi \rangle.$$ 

In local coordinates, with $\nabla^V_{\partial_i} e_b = A_i^b e_a$, this is

$$L(\sigma, \Psi, \Psi^\dagger, s) = \frac{1}{2} g_{ij} \dot{\sigma}^i \dot{\sigma}^j + i \langle \Psi^\dagger, A^b_{ab} \dot{\sigma}^a \Psi \rangle - \frac{i}{2} V_b (\Psi^\dagger)_a \Psi^b.$$ 

Here the dot refers to the derivative with respect to $s$, $\sigma^i$ is evaluated at $s$, while $g_{ij}$, $\Psi^\dagger$, $\Psi$, $A^a_{ab}$, and $V_b^a$ are evaluated at $\sigma(s)$. The action is just the time integral of the Lagrangian,

$$S(\sigma, \Psi, \Psi^\dagger, t) = \int_0^t L(\sigma, \Psi, \Psi^\dagger, s) \, ds.$$ 

1.2 Feynman’s time-slicing approximation to the path integral as a kernel

Consider the case of the usual Laplacian on functions (corresponding to $V = M \times \mathbb{R}$ and $V = 0$); that is, the Euclidean version of the Hamiltonian for a bosonic particle moving in $M$. Its heat kernel is a function $K_{\text{heat}}(x, y; t)$ of a pair of points of $M$ and the time parameter $t$ determined by

$$\left( e^{t\Delta/2} f_0 \right)(x, t) = \int K_{\text{heat}}(x, y; t) f_0(y) \, dy,$$  \hspace{1cm} (1.2)
where the integral is with respect the Riemannian volume form on $M$. In other words, $K_{\text{heat}}$ implements the heat operator as an integral kernel. The path integral
\[ \int e^{-\int_0^t L \, ds} d\sigma, \]
where $\sigma : [0, t] \to M$ is a path from $y$ to $x$, and $L(\sigma, \dot{\sigma}, s)$ is the Lagrangian, should be equal to $K_{\text{heat}}(x, y; t)$. (Here $\dot{\sigma} = \frac{d\sigma}{dt}$.) The argument for this is Feynman’s time-slicing interpretation of the path integral: Partition $[0, t]$ into subintervals of length $t_i$ for $i = 1, 2 \ldots n$. Write the path integral as a product of $n$ such integrals, where $y_{i-1}$ and $y_i$ are the starting and ending values, respectively, of the path in the $i$th integral, and the product is integrated over all the repeated points of $M$. In each of these path integrals, replace the integral of $L$ over the subinterval of length $t_i$ with an approximation $\hat{L}(y_i, y_{i-1}; t_i) t_i$. Require that $\sum \hat{L}(y_i, y_{i-1}; t_i) t_i$ be a Riemann sum converging under refinement to $\int_0^t L \, ds$. The choice of $\hat{L}$ determines an approximate heat kernel $K(x, y; t) = (2\pi t)^{-m/2} e^{-\hat{L}(x,y;t)t}$ and an approximate path integral; namely, the kernel product
\[ \int e^{-\int_0^t L \, ds} d\sigma \approx \int K(x, y_{n-1}; t_n)K(y_{n-1}, y_{n-2}; t_{n-1}) \cdots K(y_1, y_1) dy_{n-1} \cdots dy_1 \quad (1.3) \]
of $n$ copies of $K$. If $K$ happens to have the semigroup property, then the approximation is independent of the choice of partition; the convergence of the approximate path integral is immediate in this case. The Riemann sum requirement suggests that if $t$ itself is small enough, the trivial partition with $n = 1$ should give a good approximation to the fine-partition limit; hence, $K$ should be close to the actual heat kernel when $t$ is small. In the special case $M = \mathbb{R}^m$ and $V = 0$, which corresponds to a free particle in flat spacetime, defining the approximate kernel $K$ by $\hat{L}(x, y; t) t = \int_0^t L(\sigma_{cl}, \dot{\sigma}_{cl}; s) \, ds$ where $\sigma_{cl}$ is the path obeying the classical equations of motion subject to $\sigma_{cl}(0) = y$ and $\sigma_{cl}(t) = x$, happens to make $K$ exactly the heat kernel $K_{\text{heat}}$. This is a semigroup, and it is immediate that the approximations converge to a limit path integral which is $K_{\text{heat}}(x, y; t)$. If $V$ is quadratic, and $M$ is still $\mathbb{R}^m$, explicitly calculating the successive approximations is straight-forward, and the resulting expressions converge to the heat kernel. However, in the general setting, the time-slicing approximations may fail to converge as the partitions become finer. Even if they are known to converge, there is a separate question of whether the limiting kernel is the heat kernel. For example, on a more general compact manifold $M$, even for the Laplacian on functions, choosing $\hat{L}$ analogously leads to an approximate kernel $K$ for which, although the approximate path integrals converge, the limiting kernel is not the heat kernel for this Laplacian. To get the desired Laplacian requires modifying $\hat{K}$ by correction terms, which, as in physical units they enter at higher powers of $\hbar$, may be thought of as resolving operator-ordering ambiguities.

1.3 GRASSMAN-VALUED VARIABLES

The Lagrangian for SUSYQM refers to spinors, which are sections of the Grassman algebra of a certain vector bundle, which can be expressed as functions of Grassman-valued variables as follows: If $f(v_1, \ldots, v_n)$ is a multilinear function of $\mathbb{V}^*$ for some vector space $\mathbb{V}$, then the antisymmetrization of $f$ represents an element of $\Lambda^n \mathbb{V}$ To say $\psi$ is a Grassman variable valued in $\mathbb{V}^*$, means that the expression $f(\psi, \ldots, \psi)$, represents that element. More generally, write $f(\psi)$ for a linear combination of forms of various degrees, i.e. a multiform. If $\mathbb{V}$ has an inner
product the Berezin Integral $\int f(\psi)d\psi$ is the coefficient of the canonical top-degree element of $\Lambda V$ in $f(\psi)$. See [15] for a standard reference on Grassman variables; [10] and [16] give examples relevant to SUSYQM.

### 1.4 The action

Suppose $(M, g, V, \nabla, V)$ are, respectively, a Riemannian manifold, its metric, a vector bundle over $M$, a connection on $V$ and a section of $\text{End}(V)$. Recall these are the ingredients required to define a generalized Laplacian $\Delta^V$ acting on sections of $V$. Let $\mathcal{X} = \Lambda V$, promote $\nabla$ and $V$ to a connection and operator on $\mathcal{X}$ using $\nabla(a\wedge b) = \nabla(a)\wedge b + a\wedge \nabla(b)$ and $V(a\wedge b) = V(a)\wedge b + a\wedge V(b)$, and let $\Delta^X$ be the generalized Laplacian associated to $\mathcal{X}$. For each point $x \in M$ let $\psi_x$ be a Grassman variable valued in $\mathcal{V}^*_x$, so as to write kernels on $\mathcal{X}$ as superkernels $K(x, y, \psi_x, \psi_y)$. Here $K$ acts on a section of $\mathcal{X}$, which is represented by a superfunction $f(x, \psi_x)$, as

$$(K \ast f)(x, \psi_x) = \int \int K(x, y, \psi_x, \psi_y)f(y, \psi_y)d\psi_y dy.$$ 

Let $\sigma(s)$ be a path in $M$, let $\Psi, \Psi^\dagger$ be Grassman variables valued in lifts of $\sigma$ to $\mathcal{V}^*$ and $\mathcal{V}$ respectively, and consider the action

$$\int \frac{1}{2}(\dot{\sigma}, \dot{\sigma}) + i \left\langle \Psi^\dagger, \nabla_s^V \Psi \right\rangle - i \frac{1}{2} \left\langle V \Psi^\dagger, \Psi \right\rangle ds. \quad (1.4)$$

To construct a time-slicing approximate kernel, consider a small interval of parameter length $t$, and approximate the path connecting $x$ and $y$ by a geodesic. This gives $\int \frac{1}{2}(\dot{\sigma}, \dot{\sigma}) dt \sim (x_0/t, x_0/t) t/2 \sim \frac{|x_0|^2}{2} / (2t)$, where $x_0 \in T_xM$ satisfies $\exp_0(x_0) = x$. Assuming $\Psi^\dagger$ and $\nabla_x \Psi$ are covariantly slowly varying, $\int i \left\langle \Psi^\dagger, \nabla_s^V \Psi \right\rangle ds \sim i \left\langle \Psi^\dagger(t_y), \nabla^V_y \Psi(t_x) - \Psi(t_y) \right\rangle = i \left\langle \psi_y^\dagger, \nabla^V_y \psi_x - \psi_y \right\rangle$ and $\int i \left\langle V \Psi^\dagger, \Psi \right\rangle ds \sim i \left\langle \Psi^\dagger(t_y), tV^*(y) \Psi(t_y) \right\rangle \sim i \left\langle \psi_y^\dagger, t\nabla^V_y V^*(x) \psi_x \right\rangle$. This suggests an approximate heat kernel

$$K_{\Delta^X}(x, y, \psi_x, \psi_y; t) = \oint H_D(x, y; t)e^{-\frac{\text{Ricci}(x_0, x_0)}{12} - \frac{t}{12} 1 + i \left\langle \psi_y^\dagger, \nabla^V_x 1 - tV^*(x)/2 \right\rangle \psi_x - \psi_y} d\psi_y^\dagger. \quad (1.5)$$

Here

$$H_D(x, y; t) = \chi_{<D}(x, y)(2\pi t)^{-m/2} e^{-|y-x|^2/(2t)}, \quad (1.6)$$

where $\chi_{<D}$ provides a cut-off away from the diagonal

$$\chi_{<D}(x, y) = \begin{cases} 1 & \text{if } d(x, y) < D \\ 0 & \text{else,} \end{cases}$$

and $D > 0$ is small enough that there is in fact a unique geodesic between $x$ and $y$. The kernel $H_D$, which for the Euclidean metric agrees with the flat-space heat kernel for $d(x, y) < D$, will serve as the basic kernel to which to compare all others.

The Ricci and scalar curvature terms do not follow directly from the approximation to the action. Rather, referring to Rem. 2.2, they correspond to the resolution of the operator-ordering ambiguity that gives $\Delta^X$ as the operator whose kernel is the path integral with this Lagrangian, and, among such choices, they are of the particular form to make $K_{\Delta^X}$ an approximate heat kernel for $\Delta^X$ in the technical sense required for the convergence arguments of Sect. 2.3 below. These ensure, under the tameness assumptions of Def. 1, the time-slicing approximations to the path integral converge pointwise to the heat kernel for $\Delta^X$.
1.5 The Dirac operator & twisted $N = 1/2$ SUSYQM

1.5.1 The Dirac operator

Heuristically, the path integral for twisted $N = 1/2$ SUSYQM in imaginary time is related to the kernel of the heat operator for a Laplacian which is the square of the twisted Dirac operator [3]. To define the twisted Dirac operator for a manifold, recall some Clifford algebra facts and terminology as detailed for instance in Ch. 3 of [14]. If $M$ is a Riemannian manifold, define $\mathcal{C} = C(T^* M)$ to be the bundle which at each point $x \in M$ is the complexified $\mathbb{Z}/2\mathbb{Z}$-graded (and $\mathbb{Z}$-filtrated) algebra generated by $T^*_x M$, subject to the relation

$$v^* \cdot w^* + w^* \cdot v^* = -2(v^*, w^*).$$  \hspace{1cm} (1.7)

A Clifford module is a graded vector bundle $\mathcal{V}$ over $M$ with a graded homomorphism $c_{\mathcal{V}} : \mathcal{C} \rightarrow \text{End}(\mathcal{V})$. $\Lambda(T^* M)$ is a Clifford module with the action $c_{\Lambda}(v^*)\alpha = v^* \wedge \alpha - i_v(\alpha)$ where $v$ is dual to $v^*$ in the inner product.

If $M$ is even-dimensional and spin, the spinor bundle $\mathcal{S} = \Lambda P$, where $P$ is a polarization of the complexified cotangent bundle of $M$ is a Clifford module. Indeed, with this action, $\mathcal{C} \cong \text{End}(\mathcal{S})$, and any Clifford module can be written as $\mathcal{V} = \mathcal{S} \otimes \mathcal{T}$, where $\mathcal{T}$ is a vector bundle on which $\mathcal{C}$ acts trivially.

For $\mathcal{V}$ a Clifford module, a connection $\nabla^{\mathcal{V}}$ is a Clifford connection if, for any vector field $X$ and section $Y$ of $T^* M$,

$$[\nabla_X^{\mathcal{V}}, c_{\mathcal{V}}(Y)] = c_{\mathcal{V}}(\nabla_X^{\mathcal{C}} Y).$$  \hspace{1cm} (1.8)

(The bracket on the left-hand side is graded.) In the case where $M$ is even-dimensional and spin, any Clifford connection $\nabla^{\mathcal{V}}$ can be written as

$$\nabla = \nabla^S \otimes 1 + 1 \otimes \nabla^T$$  \hspace{1cm} (1.9)

for some connection $\nabla^T$ on $\mathcal{T}$ and the Levi-Civita connection $\nabla^S$ on $\mathcal{S}$. If $M$ is even-dimensional but not spin, the Clifford action is still faithful and the curvature of a Clifford connection still decomposes as $R + F^T$, where $R$ is Riemannian curvature and $F^T$ is the component of the curvature in $\text{End}_{\gamma_0 M}(\mathcal{V})$. [14](Props. 3.35,3.40 & 3.43).

If $\mathcal{V}$ is a Clifford module and $\nabla^{\mathcal{V}}$ a Clifford connection, the twisted Dirac operator is

$$D^{\mathcal{V}} = c_{\mathcal{V}}(dx^i) \nabla_{\partial_i}^{\mathcal{V}}.$$  \hspace{1cm} (1.10)

In the case of $\mathbb{R}^n$ and trivial $\mathcal{V}$, writing $c_{\mathcal{V}}(dx^i) = \gamma^i$ which acts on spinors of a given type, this is the standard Dirac operator $D = \gamma^i \partial_i$. The square of $D^{\mathcal{V}}$ is a generalized Laplacian $\Delta^{\mathcal{V}}$ with section $V = c_{\mathcal{V}}(F^T) - \tau/4$, where $c_{\mathcal{V}}$ acts on two-forms by $c_{\mathcal{V}}(v^* \wedge w^*) = \frac{1}{2}[c_{\mathcal{V}}(v^*) c_{\mathcal{V}}(w^*) - c_{\mathcal{V}}(w^*) c_{\mathcal{V}}(v^*)]$. That is, with this choice of $V$,

$$\Delta^{\mathcal{V}} = (D^{\mathcal{V}})^2.$$  \hspace{1cm} (1.11)

In the special case $\mathcal{V} = \mathcal{S}$, the operator $D^{\mathcal{V}}$ is the ordinary Dirac operator.
1.5.2 The Lagrangian for twisted \( N = 1/2 \) SUSYQM and a time-slicing approximation to the corresponding path integral

If \( M \) is even-dimensional and spin and \( T \) is a bundle over \( M \) with a connection whose curvature is \( F \), define twisted \( N = 1/2 \) SUSYQM via the action

\[
S_{\text{twisted}} = \int_0^1 \left( \frac{1}{2} \langle \dot{\sigma}, \dot{\sigma} \rangle + i \langle \Psi^\dagger, \nabla_S^S \Psi \rangle + i \langle \Pi^\dagger, \nabla_s^T \Pi \rangle - \frac{i}{2} \langle F(\Psi, \Psi)\Pi^\dagger, \Pi \rangle \right) ds, \tag{1.12}
\]

for \( \Psi \) and \( \Psi^\dagger \) Grassman-valued lifts of \( \sigma \) to \( \mathcal{P}^* \) and \( \mathcal{P} \) respectively, and \( \Pi \) and \( \Pi^\dagger \) Grassman-valued lifts to \( \mathcal{T}^* \) and \( \mathcal{T} \) respectively. This action was first written down by Friedan and Windey [3] (with slightly different normalization conventions). If \( T \) is the trivial bundle it reduces to the action for \( N = 1/2 \) SUSYQM of [2].

Discretize as above to get a kernel on \( \hat{V} = S \otimes \Lambda T \)

\[
K_{\text{twisted}} = \oint H_D(x, y; t) e^{-\text{Ric}(x_y, x_y)/12 - \text{tr} \gamma/12} \times e^{i\langle \psi_y^\dagger, \nabla^{S^*} \psi_x - \psi_y \rangle + i\langle \eta_y^\dagger, \nabla^{S^*} \eta_x - \eta_y \rangle + i\langle \eta_y^\dagger, \nabla^{T^*} \eta_x^\dagger, \eta_x + \tau/4 \rangle \eta_x + \tau/4 \rangle} / 2 \, d\eta_y \, d\psi_y^\dagger, \tag{1.13}
\]

where the parallel transports are with respect to the connections \( \nabla^S \) and \( \nabla^T \). As in the general case, the terms with parallel transport represent, under Berezin integration, the kernel of \( e^{-H_D(x, y; t)} \Psi_y^\dagger \), with this parallel transport being with respect to the connection on \( \hat{V} \). Thus the discretization is exactly the approximate heat kernel \( K_{\Delta V} \) of Eq. (1.5) with the choice \( V = c(F) - \tau/4 \).

2 Convergence results for path integrals for generalized Laplacians

2.1 Kernels, local geometry, and the \( t \)-norm

As noted above, if the time-slicing approximate kernel happens to be the heat kernel (as is the case for the free theory in Euclidean space) then the semigroup property makes the convergence automatic. In a more general case, the idea is to first show that the successive approximations of Eq. (1.3) approach some limit as the partitions become finer, and then to show that limit is the heat kernel. To see whether a limit exists, think of the effect of subdividing one interval in a given partition. This replaces a term of the form \( K(x, y; t) \) with \( \int K(x, z; t_1)K(z, y; t_2) \, dz \) where \( t_1 + t_2 = t \). Writing \( \int K(x, z; t_1)K(z, y; t_2) \, dz = K(x, y; t_1 + \epsilon(x, y, t_1, t_2)) + \epsilon \), where \( \epsilon \) denotes an error term reflecting the failure of \( K \) to be a semigroup, the question of convergence boils down to keeping track of how the error terms propagate. That is, how big are terms like \( \int \epsilon(x, z; t_1, t_2)K(z, y; t_3) \, dz \) and \( \int \epsilon(x, z; t_1, t_2)\epsilon(y, z; t_3, t_4) \, dz \)? Here “big” should mean as compared to the kernels \( K \) themselves.

There are two issues: The obvious one is that as the partitions become finer, the number of error terms, and of their products with other kernels and each other, increases. The error terms must decrease, in some measure of their size, quickly enough as their time arguments decrease to ensure the sum of the errors does not build up to be infinitely large as the partitions become finer. The more subtle issue is the kernels \( K \) relative to which the error terms should be small
are actually families of operator kernels, with parameter $t$, and that even the “best” example, the heat kernel, is singular on the diagonal ($x = y$ in $K(x, y; t)$) as $t \to 0$. Thus, in asking whether a given error term is “big”, the comparison will be to something that may be singular in places. Likewise, away from the diagonal the heat kernel vanishes rapidly as $t \to 0$ or as the distance between $x$ and $y$ increases, so “small” should be in comparison to something with this rapid decay. Sect. 2.1.3 defines a family $\mathcal{E}_{B, D}(t)$ of kernels with this behavior, and a “$t$-norm” which takes these features into account.

Placing an appropriate bound in the $t$-norm on the failure of a time-slicing approximate kernel to be a semigroup suffices to prove the convergence of the approximate path integrals under refinement. An additional bound on the failure of the approximate kernel to satisfy the heat equation will ensure that the limiting kernel is indeed the heat kernel.

2.1.1 Notation and some facts about local geometry in $R^n$ with a non-Euclidean metric

The positive aspect of comparing error terms with kernels having the extreme behavior noted above is that the convergence arguments turn out to be entirely local thanks to the rapid decay away from the diagonal. With this in mind, consider first an open set $O \subset R^n$ with smooth Riemannian metric $g$ (not necessarily Euclidean). For technical reasons related to the convergence argument to follow, require that all derivatives of order $k$ of $g$ and of $g^{-1}$ are bounded in supremum norm for $0 \leq k \leq 5$.

Let $d(x, y)$ be the distance between $x, y \in O$ in this metric. For $v \in R^m$, $x \in O$ and $t \in R$ the geodesic through $x$ with tangent $v$ at $x$ and parameter $t$ proportional to arc length defines the exponential map $\exp_x tv$. If $y \in O$ is close enough to $x$ that there is a unique minimal geodesic connecting them, define $y_x = \exp_x^{-1} y$; that is, $y_x$ is the vector at $x$ whose exponential gives $y$.

Let $(\cdot, \cdot)_x$ denote the inner product with respect to $g$ at $x \in O$, and let $|\cdot|_x$ denote the corresponding norm. If the vectors inside the norm or inner product are of the form $y_x$, or the point at which this is computed is otherwise understood from context, drop the subscript. Write $d_g y = \det_g^{1/2}(g) dy$, where $dy$ is standard Lebesgue measure on $R^m$ restricted to $O$, and write $dy_x$ for Lebesgue measure on $O$ with respect to the inner product given by $g$ at $x$; that is, the metric measure at $x$ pulled back to $y$ by $\exp_x^{-1}$.

Henceforth to say that a quantity, such as $D$ in the following lemma, “depends on the metric bounds” will mean that quantity is a function of the assumed bounds on the supremum norm of $g$, $g^{-1}$ and their first five derivatives (as well as on the dimension $m$). The concern is that, in later arguments which require rescaling the metric, preserving these bounds should be sufficient to preserve the estimates which follow here.

Even with the general metric, nearby points in $O$ behave a lot like points in Euclidean space, as regards length and integration. Specifically, direct arguments based on Rauch’s comparison theorem [17] show there is is a $D > 0$ depending on the metric bounds such that, for $x, y, z \in O$

---

4Without the heat equation bound, the cumulative effect of errors in the semigroup property may not spoil convergence, but will in general allow the limiting kernel to differ from the original time-slicing approximate kernel; the bound ensures the limiting kernel is in fact the heat kernel.

5While $d(x, y)$ is of course symmetric in the two points, the notation here and in Eq. (1.6) suggests thinking of $x$ as fixed and $y$ as variable, which is natural in the context of a kernel acting via integration as in Eq. (1.2). Section 2.3, which applies the Laplacian to specific kernels, reverses this to allow the operator to act on the first variable, as is natural in this context. The switch is purely a matter of convention.
with \(d(x,y), d(y,z), d(x,z) < D\), there is a unique minimal geodesic connecting \(x\) and \(y\), \(y_x\) depends smoothly on \(x\) and \(y\), and \(y-x\) depends smoothly on \(x\) and on \(y_x\). Moreover,

\[
y - x = y_x + O\left(|y_x|^2\right) \tag{2.1}
\]

\[
|z_x|^2 = |z_y|^2 + |x_y|^2 - 2(x_y, z_y) + O\left(|x_y|^2 |z_y|^2\right) \tag{2.2}
\]

\[
\frac{d_y y}{dy_x} = 1 + O\left(|y_x|^2\right) \tag{2.3}
\]

where for example \(O\left(|x_y|^2 |z_y|^2\right)\) indicates the difference between the left-hand side and the truncated Taylor series is bounded by a constant (depending on the metric bounds) times \(|x_y|^2 |z_y|^2\) (as each of these tends towards zero).

### 2.1.2 The operator norm, and the “kernel” norm

Given \(n \in \mathbb{N}\), let \(f: O \to \mathbb{R}^n\), \(f^*: O \to (\mathbb{R}^n)^*\) and \(K: O \times O \to \text{Matrix}_{n,n}\). Notice \(f\) and \(f^*\), as functions from \(O\) to \(\mathbb{R}^n\) or \((\mathbb{R}^n)^*\), are local expressions of sections of vector bundles, and \(K\) represents kernels of the left or right operators (on the space of such functions) whose actions are given by

\[
K * f(x) = \int_O K(x,y) \cdot f(y) dy_x \\
f^* * K(y) = \int_O f^*(x) \cdot K(x,y) dy_x \tag{2.4}
\]

where \(\cdot\) represents the matrix product. The kernel of the operator product of the operators represented by \(K\) and \(J\) is the \(*\)-product

\[
J * K(x,z) = \int_O J(x,y) \cdot K(y,z) dy_x. \tag{2.5}
\]

The matrix norm sends \(K\) to a nonnegative function \(|K|\) on \(O \times O\). Use this to define

\[
|K|_{\text{op}} = \max\left(\sup_x \int_O |K(x,y)| dy_y, \sup_y \int_O |K(x,y)| dy_x\right),
\]

which is the maximum of the operator norms of \(K\) acting on the left and the right. Define the kernel norm by

\[
|K|_{\text{ker}} = \max(|K|_{\text{op}}, |K|_{\infty}).
\]

Notice \(|J * K|_{\text{ker}} \leq |J|_{\text{ker}} |K|_{\text{ker}}\) and \(|J * K|_{\text{ker}} \leq |J|_{\text{op}} |K|_{\text{ker}}\).

### 2.1.3 Two families of kernels and the \(t\)-norm

Now begin to explore classes of kernels whose relation to \(H_D\) are increasingly tenuous, to delineate the extent to which they retain key properties of the heat kernel under kernel products. This exploration culminates in the definition of a class of kernels \(E_{B,D}(t)\) against which to compare error terms like those above, as well as others arising from the failure of the time-slicing
approximation to satisfy the heat equation. Appropriate bounds on these errors, expressed in terms of the “t-norm”, which measures the ratio of the error to elements of $E_{B,D}(t)$, will ensure the approximate path integrals converge with sufficient rapidity to the heat kernel of the given Laplacian. For $B$ large enough, $D$ small enough, and $t$ small enough for the right-hand side to make sense (each depending on the bounds of the metric and the preceding quantities), define

$$K_{B,D}(x, y; t) = e^{B|x-y|^2/(5m)}H_D(x, y; t),$$

(2.6)

This allows $K_{B,D}$ to grow much faster than $H_D$ away from the diagonal (for fixed $t$). Nevertheless, for $0 < t_1, t_2$, and $t = t_1 + t_2$,

$$\chi_{<D}[K_{B,D}(t_1) * K_{B,D}(t_2)] \leq e^{Bt_1t_2/4}K_{B,D}(t),$$

$$\|\chi_{>D}[K_{B,D}(t_1) * K_{B,D}(t_2)]\|_{\text{ker}} \leq t^2e^{-D^2/9t},$$

(2.7)

which gives control over kernel products both near and far from the diagonal. Moreover, as an operator,

$$|K_{B,D}(t)|_{\text{op}} \leq e^{Bt}.$$  

(2.8)

The derivation of these inequalities uses the facts about local geometry from Eqs. 2.2 and 2.3 to bound the Gaussian integrals implicit in Eqs. 2.7 and 2.8.

Now smear out $K_{B,D}$ a little in time and allow some additional growth in $t$ to define a class of kernels $E_{B,D}(t)$ which will be almost closed under the $*$ product and whose products, away from the diagonal, decay rapidly with decreasing $t$, in the kernel norm, as do those of the heat kernel.

**Definition 2** For $B, D, t > 0$ define $E_{B,D}(t)$ to be the set of all kernels $K$ for which there exists a probability measure $d\mu$ on the interval $[1, 2]$ such that

$$|K(x, y)| \leq e^{B\sqrt{t}}\int K_{B,D}(x, y; \alpha t)d\mu_{\alpha},$$

(2.9)

where $K_{B,D}$ is the particular one-parameter family of kernels defined in Eq. (2.6).

Note that $K_{B,D}(t)$ itself is in $E_{B,D}(t)$. Direct estimates using the above properties of $K_{B,D}$ lead to the following precise statements about this class: If $B$ is large enough, $D$ is small enough, and $T$ is small enough (each depending on the bounds of the metric and the previous quantities) and if $K_1$ and $K_2$ are one-parameter families of kernels with $K_1(t), K_2(t) \in E_{B,D}(t)$ for $t < T$, then, for $0 < t_1, t_2$ and $t = t_1 + t_2 < T$

$$|K_1(t)|_{\text{op}} \leq e^{1.1B\sqrt{T}}$$

(2.10)

and

$$\chi_{<D}K_1(t_1) * K_2(t_2) \in e^{B\sqrt{t_1t_2}/4}E_{B,D}(t)$$

$$\|\chi_{>D}K_1(t_1) * K_2(t_2)\|_{\text{ker}} \leq t^2e^{-D^2/(20t)}.$$  

(2.11)

Continue to enlarge the class of kernels which behave well under kernel products to allow an additional “tail” behavior for larger $t$ to get the final class of kernels against which to measure various error terms:
Definition 3 For $B,D,t > 0$ define $E^t_{B,D}(t)$ to be the set of all kernels which can be written as $K + J$ where $K \in E_{B,D}(t)$ and $|J|_{ker} \leq te^{-D^2/(20t)}$.

The advantage of incorporating the tail behavior into the class of kernels is products of kernels in $E^t_{B,D}(t)$ follow easily from the definitions: If $B$ is large enough, $D$ is small enough and $T$ is small enough (each depending only on the bounds of the metric and the previous quantities) and if $K_1$ and $K_2$ are one-parameter families of kernels with $K_1(t), K_2(t) \in E^t_{B,D}(t)$ for all $t < T$, then, for $0 < t_1, t_2$ and $t = t_1 + t_2 < T$

$$\|K_1(t)\|_{op} \leq e^{2B\sqrt{T}},$$  

(2.12)

$$|K_i(x,y;t)| \leq 2(2\pi t)^{-m/2}e^{-d(x,y)^2/(4t)} + te^{-D^2/(20t)},$$  

(2.13)

and

$$K_1(t_1) * K_2(t_2) \in e^{B\sqrt{T}}E^t_{B,D}(t).$$  

(2.14)

The class $E^t_{B,D}(t)$ and its properties explicitly depend on choices of constants $B, D$ and $T$ (the last as an upper bound for $t$). The relation of these constants to the bounds on the metric and the relation between these constants are as follows: There is a minimum $B$ and a maximum $D$ and $T$ to make the above properties hold, and these numbers depend only on the supremum of the first few derivatives of the metric and its inverse (and on the dimension $m$), a fact that will be crucial in Sect. 3.2.2. Choosing a larger $B$ would make the maximum $D$ and $T$ smaller, but these would still exist. If one chose an even smaller $D$, the maximum $T$ would be smaller still. In the definition of approximate semigroup and approximate heat kernel below, the choice of constants will further depend on the family of kernels being considered.

The properties of $E^t_{B,D}(t)$ provide the basis upon which to define a norm, which indeed is the motivation for defining this class:

Definition 4 For given $B,D,t > 0$ define the $t$-norm $\|K\|_{(t)}$ to be the smallest positive real number such that $K/\|K\|_{(t)} \in E^t_{B,D}(t)$ if it exists. (Otherwise set $\|K\|_{(t)} = \infty$.)

The advantage of using this norm is its behavior under the kernel product: If $B$ is large enough, $D$ is small enough and $t$ is small enough (each depending only on the bounds of the metric and the previous constants), then for the associated $t$-norm and for families of kernels $K_1$, and $K_2$,

$$\|K_1(t_1) * K_2(t_2)\|_{(t)} \leq e^{B\sqrt{T}}\|K_1\|_{(t_1)}\|K_2\|_{(t_2)},$$  

(2.15)

At the same time, the new norm is related to the more obvious norms via

$$\|K_i\|_{op} \leq e^{2B\sqrt{T}}\|K_i\|_{(t)},$$  

(2.16)

and

$$|K_i(x,y;t)| \leq \|K_i\|_{(t)}\left[2(2\pi t)^{-m/2}e^{-d(x,y)^2/(4t)} + te^{-D^2/(20t)}\right];$$  

(2.17)

In particular, there is an $A_2 > 0$ such that

$$\|K_i(t)\|_{\infty} \leq A_2 t^{-m/2}\|K_i\|_{(t)}.$$  

(2.18)

The above are all restatements or immediate consequences of the properties of kernels in the class $E^t_{B,D}(t)$. 

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2.2 Approximate semigroups \& approximate heat kernels

2.2.1 Approximate semigroups on $O \subset \mathbb{R}^n$

With the $t$-norm in hand, define an approximate semigroup with constants $(B,C,D,T)$ as a family of kernels $K(t)$ for which

$$\|K(t)\|_{(t)} \leq 1$$

and, given $0 < t_1, t_2$ and $t = t_1 + t_2 < T$,

$$\|K(t_1) * K(t_2) - K(t)\|_{(t)} \leq Ct^{3/2}.$$  \hfill (2.20)

Notice Eq. (2.19) means an approximate semigroup must be in the class $\mathcal{E}_{B,D}(t)$ for all $t < T$. Some easy estimates show the condition keeping $K * K$ close to $K$ reduces to a condition only on the piece $\tilde{K}(t) \in \mathcal{E}(t)$ in the decomposition $K(t) = \tilde{K}(t) + J(t)$.

2.2.2 Approximate heat kernel on $O \subset \mathbb{R}^n$

Now consider kernels in $\mathcal{E}'(t)$ with additional conditions tying them to a generalized Laplacian. Specifically, given a generalized Laplacian $\Delta$ define an approximate heat kernel for $\Delta$ with constants $(B,C,D,T)$, all positive, as a family of kernels $K(t)$ whose members are differentiable to first order in $t \in (0, T)$ and to second order in the spatial variables, and satisfy, for $t < T$ and the $t$-norm with constants $(B,D)$,

$$\|K(t)\|_{(t)} \leq 1,$$  \hfill (2.21)

for all $f: O \to \mathbb{R}^n$

$$\lim_{t \to 0} K(t) * f = f,$$  \hfill (2.22)

$$\lim_{t \to 0} \frac{K(t) * f - f}{t} = \frac{\Delta f}{2},$$  \hfill (2.23)

(both pointwise),

$$\left\| \frac{\partial}{\partial x} K(x,y;t) \right\|_{(t)} \leq B/t,$$  \hfill (2.24)

and

$$\left\| \frac{1}{2} \Delta_x - \frac{\partial}{\partial t} \right\| K(x,y;t) \left\|_{(t)} \leq Ct^{1/2},$$  \hfill (2.25)

where $\Delta_x$ acts from the left on $\text{End}(\mathbb{R}^n)$ and $\Delta^*_y$ acts from the right via $\int_O \Delta^*_y[h^*(y)] \cdot f(y) \, dy = \int_O h^*(y) \cdot \Delta_y[f(y)] \, dy$. The first condition again ensure $K(t)$ is in $\mathcal{E}'(t)$. The next ensures the operator $K$ defines will implement the initial condition expected of the heat operator. Eq. (2.23) says this operator agrees with the heat operator as $t \to 0$; whereas, Eq (2.25) bounds the failure of $K$ to satisfy the heat equation (which the heat kernel would) away from $t = 0$. The bound of Eq. (2.24), along with those of Eq. (2.25) and the observation regarding checking Eq. (2.20), combine with straight-forward estimates to show an approximate heat kernel in the sense of this definition satisfies Eq. (2.20) and is thus also an approximate semigroup. (The constants $C$, $D$ and $T$ may need to be refined in passing from the approximate heat kernel to the approximate semigroup.)
2.2.3 Approximate semigroups and approximate heat kernels on tame manifolds

Recall that so far everything has taken place on an open set $O \subset \mathbb{R}^m$. To define the $t$-norm for kernels on the tame manifolds of Sec. 1.1, simply observe that on any tame atlas, for sufficiently large $B$ and sufficiently small $D$, there is a sufficiently small $t$ such that the $t$-norm with constants $(B,D)$ can be defined on each chart. Define $|K|_t$ to be the supremum of the $t$-norms of its image in each chart. If $(M,g,V)$ is tame the $t$-norm defined in terms of any tame atlas will satisfy Eqs. (2.15) and (2.16) for sufficiently large $B$ and sufficiently small $D$. Then extend the definition of an approximate semigroup to be a family of kernels $K(t)$ on $V$ for which $(M,g,V)$ admits a tame atlas on each chart of which $K$ is represented as an approximate semigroup. Require of course that the constant $D$ implicit in the definition of an approximate semigroup on the chart be less than the constant $D_0$ appearing in the definition of a tame atlas. Extend the definition of approximate heat kernel analogously. The bounds of Eqs. (2.15)-(2.20) extend to any approximate semigroup on $V$. Again because all the previous results were local, an approximate heat kernel for some $\Delta$ on $V$ is an approximate semigroup. The constants $(B,C,D,T)$ of this approximate semigroup can be taken to depend only on the corresponding constants for the approximate heat kernel and the bounds on the defining atlas.

**Remark 2.1** While it suffices for the rest of the work, the dependence of the structures defined on the choice of tame atlas might distress the mathematically inclined reader. However, there is a natural notion of the comparability of tame structures, which simply involves requiring that the diffeomorphisms between charts induced by the identity on $V$ have all derivatives up to the appropriate order uniformly bounded. It is then straightforward if laborious to check that the $t$-norms associated to compatible tame atlases are comparable (each bounded by a multiple of the other), that families of kernels that are approximate semigroups or heat kernels with respect to one atlas are the same with respect to the other, and therefore that the limit results of the following section depend only on the “tame equivalence class” of the vector bundle, Riemannian manifold and operator.

2.2.4 The twisted $N = 1/2$ SUSYQM time-slicing approximation as an approximate heat kernel

Return at last to the time-slicing approximate heat kernel $K_{\Delta \mathcal{X}}$ of Eq. (1.5) to check it is in fact an approximate heat kernel in the specific sense of the preceding definitions. First use standard properties of integration with Grassman variables to see the quantity

$$\int e^{i\langle \psi^+_y, [1 - tV^*(x)/2] \psi_x - \psi_y \rangle} d\psi^+_y$$

is, up to terms in $O(t^2)$, the superkernel for the operator

$$e^{-tV(x)/2\mathcal{Q}_x}: \mathcal{X}_y \to \mathcal{X}_x,$$

which is the extension of $e^{-tV(x)/2\mathcal{Q}_x}: V_y \to V_x$. As the addition of terms of order $O(t^2)$ $K$ will affect neither whether a kernel $K$ is an approximate heat kernel, nor convergence of kernel products nor the fine-partition limit, consider the kernel\(^6\)

$$K_{\Delta}(x,y;t) = H_D(x,y;t)e^{-\text{Ricci}(x,y)/12 - t/12 - tV(x)/2\mathcal{Q}_x}.$$

\(^6\)Up to the above-mentioned irrelevant terms, $K_{\Delta}$ is just $K_{\Delta \mathcal{X}}$ of Eq. (1.5), but written in a more general form that would describe a kernel on a bundle $\mathcal{X}$ that need not be of the form $\Lambda V$. 

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Checking that \( K_\Delta \) is an approximate heat kernel means checking it satisfies equations Eqs. (2.21) through (2.25). These are all local conditions, so pick \( y \in M \) and work in Riemann normal coordinates centered at \( y \). That is, choose an orthonormal basis for \( T_yM \), and notice each point \( x \in M \) near \( y \) is the value of the exponential map at a unique vector \( x \in T_yM \) near \( 0 \). (The \( x \) was \( x_y \) in Sect. 2.1.1; the subscript is implicit here where there is no danger of confusion.)

The components of \( x \) with respect to the chosen basis define the Riemann normal coordinates of the point \( x \). Some thought about the the geodesic equation as a system of ODE’s, which is also the basis of the facts laid out in Sect. 2.1.1, says tameness implies \( g_{ij} \) in Riemann normal coordinates has bounded \( k \)th derivatives for \( 0 \leq k \leq 4 \).

If \( X \) and \( Y \) are tangent vectors at \( x \in M \) let \( R_x[X, Y] \) be the Riemannian curvature (endomorphism on \( T_xM \), Ricci\(_x\)(\( X, Y \)) be the Ricci curvature, and \( \tau_x \) be the scalar curvature. The coordinate derivatives \( \partial_i \) for \( i = 1, \ldots, m \) at each \( x \in M \) near \( y \in M \) form a basis of \( T_xM \) and define vector fields in a neighborhood of \( y \) (commuting but not in general orthonormal). At \( y \) these agree with the original choice of orthonormal basis. Define a second basis \( e_i \in T_xM \) (orthonormal but not commuting as vector fields) by parallel transporting the same orthonormal basis of \( T_yM \) along a minimal geodesic from \( y \) to (nearby) \( x \). The two bases are related by \([14] (\text{Prop. 1.28})\)

\[
e_i = \delta_i^j + \frac{1}{6} R_{iklj} x^k x^l \partial_j + \mathcal{O}(\|x\|^3) \tag{2.27}
\]

where \( R_{iklj} \partial_j = R_y[\partial_i, \partial_k] \partial_l \) defines the coordinates of the curvature at \( y \). If \( g_{ij}(x) = (\partial_i, \partial_j)_x \), with inverse \( g^{ij}(x) \), and \( \Gamma_{ij}^k(x) \), \( \partial_k = \nabla_{\partial_i} \partial_j \), Eq. (2.27) implies

\[
g_{ij}(x) = \delta_{ij} + \frac{1}{3} R_{iklj} x^k x^l + \mathcal{O}(\|x\|^3) \tag{2.28}
\]
\[
g^{ij}(x) = \delta^{ij} - \frac{1}{3} R_{klij} x^k x^l + \mathcal{O}(\|x\|^3) \tag{2.29}
\]
\[
\Gamma_{ij}^k(x) = -\frac{1}{3} [R_{ij}^k + R_{jik}] x^l + \mathcal{O}(\|x\|^2) \tag{2.30}
\]
\[
det^{1/2} g(x) = 1 + \frac{1}{6} R_{iklj} x^k x^l + \mathcal{O}(\|x\|^3) \tag{2.31}
\]

freely raising and lowering indices using \( g_{ij}(0) = \delta_{ij} \). At \( y \), abbreviate Ricci\(_y\)(\( \partial_i, \partial_j \)) as \( R_{iklj} \). At \( x \) use the estimations implied by \( \mathcal{O}(\|x\|^p) \) above depend only on the bounds on \( g_{ij} \) and its derivatives up to order three. Trivialize the bundle \( \mathcal{X} \) in a ball of radius \( D \) around \( y \) by identifying \( \mathcal{X}_x \) with \( \mathcal{X}_y \) via parallel transport along the unique minimal geodesic connecting \( y \) and \( x \). Then as in Prop. 1.8 of \([14]\)

\[
\nabla^X_i = \partial_i + \frac{1}{2} x^j F^{\mathcal{X}}_{ij} + \mathcal{O}(\|x\|^2) \tag{2.32}
\]

where \( F_{ij} \) is the curvature of \( \nabla^X \) evaluated at \( y \) in the \( \partial_i \wedge \partial_j \) direction, and the bound depends on the bound on the coefficients of \( \nabla \) to order \( 2 \).

From its definition in Eq. (2.26), \( K_\Delta(x, y; t) = [1 + \mathcal{O}(\|x\|) + \mathcal{O}(t)] H_D(x, y; t) \) Use the estimate

\[
d(x, y)^k H_D(x, y; t) \leq 2^{(m+k)/2}(k/e)^{k/2} k^{k/2} H_D(x, y; 2t), \tag{2.33}
\]
which follows readily for $k \in \mathbb{N}$ from $x^k e^{-x^2/2} \leq (k/e)^{k/2}$, to convert the $x$ dependence to $t^{1/2}$ at the expense of doubling the time, giving

$$
K_\Delta(x, y; t)H_D(x, y; t) = H_D(x, y; t) + O(t^{1/2}) H_D(x, y; 2t).
$$

Then, for sufficiently large $B$, small enough $t$ and an appropriate $\mu$,

$$
|K_\Delta(t)| \leq H_D(t) + \left(e^{2Bt^{1/2}} - 1\right) H_D(2t) \\
\leq e^{Bt^{1/2}} \left[e^{-Bt^{1/2}} H_D(t) + \left(1 - e^{-Bt^{1/2}}\right) H_D(2t)\right] = e^{Bt^{1/2}} \int_0^2 H_D(\alpha t) d\mu_\alpha \in \mathcal{E}_{B,D}(t)
$$

by Def. 2. Thus, $|K_\Delta(t)| \leq 1$, verifying Eq. (2.21) of the definition of an approximate heat kernel.

Using Eqs. (2.28)-(2.32) and the antisymmetry of $F^X$,

$$
\Delta = g^{ij} \left[ \nabla^X_i \nabla^X_j - \Gamma^k_{ij} \nabla^k \right] - V
$$

$$
= g^{ij} \left[ \partial_i \partial_j + \frac{1}{2} F^{X^k}_{ij} + \frac{1}{2} x^k (F^{X^k} \partial_j + F^{X^k} \partial_i) - \Gamma^k_{ij} \partial_k + O(|x|) + O\left(|x|^2\right) \partial_t \right] - V
$$

$$
= \partial_t - \frac{1}{3} R_{kij} x^k \partial_j - x^k F^{X^k} \partial_i - V + \frac{2}{3} \text{Ricci}_j x^i \partial_j + O(|x|) + O\left(|x|^2\right) \partial_t + O\left(|x|^3\right) \partial_i \partial_j.
$$

Compute

$$
\frac{\partial}{\partial t} K_\Delta(x, y; t) = \left[-\frac{m}{2t} + \frac{|x|^2}{2t^2} - \frac{r}{12} - \frac{V}{2}\right] K_\Delta(x, y; t),
$$

$$
\partial_{i,x} K_\Delta(x, y; t) = \left[-\frac{x_i}{t} - \frac{\text{Ricci}_{ij} x^j}{6} + O(t)\right] K_\Delta(x, y; t),
$$

$$
\partial_{i,x} \partial_{j,x} K_\Delta(x, y; t) = \left[-\frac{\delta_{ij}}{t} - \frac{\text{Ricci}_{ij} x^j}{6} + \frac{x_i x_j}{t^2} + \frac{x^i \text{Ricci}_{ik} x^k + x^j \text{Ricci}_{ik} x^k}{6t} + O(t + |x|^2)\right] K_\Delta(x, y; t),
$$

so

$$
\left[ \frac{\partial}{\partial t} - \frac{1}{2} \Delta \right] K_\Delta = \left[-\frac{m}{2t} + \frac{|x|^2}{2t^2} - \frac{r}{12} - \frac{V}{2} + \frac{m}{2t} + \frac{r}{12} - \frac{|x|^2}{2t^2} - \frac{x^i \text{Ricci}_{ij} x^j}{6t}\right] K_\Delta(x, y; t)
$$

$$
- \frac{x^i \text{Ricci}_{ij} x^j}{6t} + \frac{1}{6t^2} R_{kij} x^k x^i x^j + \frac{1}{2t^2} x^k F^{X^k} x^i + \frac{V}{2} + \frac{x^i \text{Ricci}_{ij} x^j}{3t}
$$

$$
+ O\left(|x| + |x|^3/t + |x|^5/t^2 + t\right) K_\Delta(x, y; t)
$$

$$
= O\left(|x| + |x|^3/t + |x|^5/t^2 + t\right) K_\Delta(x, y; t)
$$

after taking into account the antisymmetry of $F^X$ and the fourfold symmetry of $R$. Again using the estimate of Eq. (2.33), the right-hand side has $t$-norm bounded by a multiple of $t^{1/2}$, verifying the first line of Eq. (2.25). Since the Laplace-Beltrami operator is self-adjoint, $\Delta^*$ is the operator associated to $g$, $\nabla^\dagger$ and $V^\dagger$, where $\dagger$ represents the canonical map sending $\text{End}(\mathbb{R}^n)$ to $\text{End}(\mathbb{R}^n)^*$. So for the second line of Eq. (2.25) it suffices to observe that $K_{\Delta^*}(x, y; t) =$
$K^1_{\Delta}(y,x;t) + O(|x|_y^3 + |x|_y^t)$. This estimate follows from the tameness assumption which more directly implies Ricci$(y_x, y_x) - Ricci(y_y, x_y) = O(|x|_y^3)$, $x_x - x_y = O(|x|_y|)$, and $V(y) - (\mathcal{P}^y_x)^{-1}V(x)\mathcal{P}^y_x = O(|x|_y)$, with the bounds depending on the bounds on the metric. Eq. (2.25) now follows.

For Eq. (2.22), let $f$ be a smooth function on $O$ valued in $\mathbb{R}^n$. Then, working in Riemann normal coordinates around $x$ with the the bundle trivialized by parallel transport in radial directions,

$$\lim_{t \to 0} \int K_\Delta(x, y; t) \cdot f(y) dy = \int H_D(x, y; t)f(y)\left[1 + O\left(|x|_y^2\right) + O(t)\right] dy_x$$

$$= f(x) + O(t).$$

Similarly, for Eq. (2.23) it suffices by the Mean Value Theorem to show $\lim_{t \to 0} \frac{\partial}{\partial t} K_\Delta \ast f = \frac{1}{2}\Delta f$. In Riemann normal coordinates

$$\lim_{t \to 0} \frac{\partial}{\partial t} K_\Delta \ast f(x) = \lim_{t \to 0} \int \left[-\frac{m}{2t} + \frac{|x|_y^2}{2t^2} - \frac{r}{t} - \frac{V}{2}\right] K_\Delta(x, y; t)f(y) dy$$

$$= \lim_{t \to 0} \frac{1}{2} [\partial_t, \partial_t - V] f(x) + O\left(t^{1/2}\right) = \frac{1}{2}\Delta f$$

by straightforward Gaussian integrals. Finally Eq. (2.24) follows for appropriate $B$ from the above calculation for $\partial_t K_\Delta$.

**Remark 2.2** The calculations verifying Eq. (2.25) shed some light on the role of the Ricci and scalar curvature terms in the definition of $K_\Delta$. Absent the Ricci term and scalar terms, Eq. (2.37) would gain a net $\frac{\partial}{\partial t}$ Ricci$(x_y, x_y)$, coming from the expansion of the metric and Christoffel symbols, which would persist as an $O(1)$ term in the $t$-norm, so Eq. (2.25) would fail to hold. Adding in just the Ricci term would cancel this; however, it would introduce an extra $\frac{1}{2t}$ to Eq. (2.37), again an $O(1)$ term in the $t$-norm, and again spoiling Eq. (2.25). This scalar curvature term appears due to a general phenomenon, familiar from purely quadratic path integrals in $\mathbb{R}^n$, where, for any self-adjoint linear operator $A$, $\int e^{-\frac{1}{2t} \int_0^t |\dot{\sigma}|^2 d\sigma} \left[\int_0^t (A, \dot{\sigma}) d\sigma\right] D\sigma = \text{tr} A$. In more generality, a tedious calculation checks that, for an approximate heat kernel $K$, the modified kernel $K_A = K(1 + [x, Ax] - tr A)$ has the property that $K_A^{*n} = K^{*n}(1 + c \text{tr} A)$, where $c$ depends on the precise nature of the partition, but, under mild restrictions, vanishes as the partition is refined.

The combination appearing in $K_\Delta$ thus cancels out the Ricci curvature terms, without adding new scalar terms. More generally, adding a term of the form $ax + b(Ricci(x_y, x_y) - v)$ to the exponent in $K_\Delta$ adds error terms $ax^2 + 2\|b(Ricci(x_y, x_y) - v)\|_{(t)} \in O(1)$ to the right-hand side. If $a = 0$, these terms eventually cancel out under refinement and do not affect the fine-partition limit. If $a \neq 0$, redefining $\Delta$ by the addition of $ax$ would cancel out the first term. In units where $h$ is not 1, this addition is actually $av^2$ and thus is a quantum correction to the Hamiltonian. This correction presumably would correspond to a different resolution of the operator-ordering ambiguity inherent in promoting $g_{ij} \rho^i \rho^j$ to an operator.

### 2.3 Convergence and the Limit of Approximate Path Integrals

Having confirmed that Feynman’s time-slicing prescription, adjusted to satisfy the heat equation for the correct Laplacian up to errors of order $t^{1/2}$ in the $t$-norm, leads to an approximate
kernel $K_D$, it remains to check that the corresponding approximate path integrals converge to a definite limiting kernel, and to check this limit is the heat kernel. Let $P = (t_1, t_2, \ldots, t_k)$ be a partition, of size $|P|$, of $t > 0$; that is, $t_i > 0$, $\sum_i t_i = t$ and $|P| = \max_i t_i$. Then the time-sliced approximate path integral based on $K_D$ at partition $P$ is, suppressing the spatial variables,

$$K_D^P(t) = K_D(t_1) * K_D(t_2) * \cdots * K_D(t_k).$$

(2.38)

This approximation should get better as the partition $P$ becomes finer in the following precise sense: If $P$ is a partition of $t$ and $P'$ is a partition of $t'$, then the concatenation $PP'$ is a partition of $t + t'$; if $P_i$ is a partition of $t_i$ for $1 \leq i \leq k$, then the partition $P_1 P_2 \cdots P_k$ is a refinement of $P = (t_1, \ldots, t_k)$. Note that, by definition, if $Q$ is a refinement of $P$ then $|Q| < |P|$.

### 2.3.1 Convergence of approximations to the path integral based on an approximate semigroup

The convergence of approximate path integrals will follow from a Cauchy property, which says that once $P$ and $P'$ are small enough that for given $P, P'$ and $P''$, the concatenation $PP'$ is a partition of $t + t'$, if $P_i$ is a partition of $t_i$ for $1 \leq i \leq k$, then the partition $P_1 P_2 \cdots P_k$ is a refinement of $P = (t_1, \ldots, t_k)$. Note that, by definition, if $Q$ is a refinement of $P$ then $|Q| < |P|$.

The required Cauchy property would say there is an $A > 0$ depending on $B, C$ and $D$ such that, if $T$ is chosen small enough,

$$\|K^{*Q}(t) - K^{*P}(t)\|_{(t)} < A t^{5/4} |P|^{1/4}$$

(2.39)

for all refinements $Q$ of all partitions $P$ of $t < T$. To derive this Cauchy property, first notice Eq. 2.20, the defining property of an approximate semigroup, readily extends to a version with three terms

$$\|K(t_1) * K(t_2) * K(t_3) - K(t)\|_{(t)} \leq ct^{3/2}.$$

Write $t = t_1 + t_2 + t_3$, with $0 \leq t_i \leq t/2$ for $i = 1, 3$ and choose partitions $Q_i$ of $t_i$ again for $i = 1, 3$ so that $Q = Q_1(t_2)Q_3$. Then

$$\|K^{*Q}(t) - K(t)\|_{(t)} =$$

$$\|\left[ K^{*Q_1}(t_1) - K(t_1) \right] * K(t_2) * K(t_3) + K(t_1) * K(t_2) * \left[ K^{*Q_3}(t_3) - K(t_3) \right]$$

$$+ \left[ K^{*Q_1}(t_1) - K(t_1) \right] * K(t_2) * \left[ K^{*Q_3}(t_3) - K(t_3) \right] + K(t_1) * K(t_2) * K(t_3) - K(t) \|_{(t)}.$$ 

Induction on the number of entries in $Q$, combined with Eqs. (2.15) and (2.19) and a choice of $T$ small enough that for given $b_1$ and $c_1$ (coming from these inequalities) $c_1 e^{b_1 t^{1/2}} t^{3/2}$ is less than $(1 - 2^{-1/4})$, leads to

$$\|K^{*Q}(t) - K(t)\|_{(t)} \leq c_1 e^{b_1 t^{1/2}} t^{3/2},$$

(2.40)

for some positive constants $c_1$ and $b_1$. Now write

$$\|K^{*Q}(t) - K^{*P}(t)\|_{(t)} = \|K^{*Q_1}(t_1) * K^{*Q_2}(t_2) * K^{*Q_3}(t_3) - K^{*P_1}(t_1) * K(t_2) * K^{*P_3}(t_3)\|_{(t)},$$

for $P = P_1(t_2)P_3$ analogous to the refinement of $Q$ above and $Q = Q_1Q_2Q_3$ where $Q_2$ is any partition of $t_2$. Another induction argument, similar to and using the previous result, verifies Eq. (2.39).
Eq. (2.39) says that for any sequence of partitions $P_1 = (t), P_2, \ldots$ for sufficiently small $t$, with each partition a refinement of the previous and with $|P_i| \to 0$ as $i \to \infty$, $K^{*P_i}(x, y, t)$ is a Cauchy sequence in the $t$-norm. Eq. (2.18) relating the norms guarantees this sequence is Cauchy in the supremum norm and so by completeness converges to some $K^\infty(x, y; t)$. Then, for any partition $P$ of $t$,

$$\|K^{*P}(t) - K^\infty(t)\|_{(t)} = \|K^{*P}(t) - K^{*P_i}(t) + K^{*P_i}(t) - K^{*P_i}(t) + K^{*P_i}(t) - K^\infty(t)\|_{(t)},$$

for any sequence $P_i$ of common refinements of $P$ and $P_i$. The triangle inequality for the $t$-norm, two applications of Eq. (2.39), and the convergence of the $K^{*P_i}(t)$ give

$$\|K^\infty(t) - K^{*P}(t)\|_{(t)} \leq A t^{5/4} |P|^{1/4}$$

(2.41)

Using the relation between norms, and some judicious choices of partitions related to $P$ through refinement, this leads to

$$\|K^\infty(t) - K^{*P}(t)\|_{\infty} \leq A_1 t e^{B_1 t} |P|^{D_1}$$

(2.42)

for some set of constants $A_1, B_1, D_1$ and $T_1$ (depending on the previous constants and the dimension $m$) and for all $P$ with $|P| < T_1$ and all $t$. The $t |P|^{D_1}$ dependence arises from expressing the change from $P$ to a refinement $Q$ (Eq. (2.39)) as a sequence of smaller changes chosen to exchange the $t$-dependence in the bound in Eq. (2.18) (describing the relation between the $t$ and supremum norms) for a combined $t$ and $|P|$-dependence. The $e^{B_1 t}$ ultimately derives from the bound on the operator norm in terms of the $t$-norm appearing in Eq. (2.16), and a bound, for appropriate partitions $Q$, of the form

$$\|K^{*Q}(t)\|_{(t)} \leq e^{b t^{1/2}}$$

(2.43)

coming from $K^{*Q} = K^{*Q} - K + K$ and Eq. (2.40).

Taking $|P|$ to 0 shows

$$K^\infty(t) = \lim_{|P| \to 0} K^{*P}(t)$$

(2.44)

That is, the limit under successive refinements of the approximate path integrals based on the approximate semigroup $K$ does exist. Therefore $K^\infty(t)$ provides a rigorous definition for the path integral based on a first approximation $K$ which may be chosen, as above, to be compatible with Feynman’s time-slicing prescription for a given generalized Laplacian. Note the argument depends only on choosing $K$ to be an approximate semigroup; however, for generic choices of $K$, the limit $K^\infty$ would not have an interpretation as a path integral, as the successive approximations would not correspond to time-slicing in any sense.

### 2.3.2 When the approximate semigroup is an approximate heat kernel, the limiting kernel is the heat kernel

If $K$ is indeed an approximate heat kernel, such as the specific choice $K_\Delta$ above coming from time-slicing, it is necessarily an approximate semigroup, so the approximations $K^{*P}$ to the path integral will converge to a path integral $K^\infty$. The question remains how $K^\infty$ relates to the heat kernel for the generalized Laplacian associated to $K$. 
Most of the answer follows from allowing the kernel \( K^\infty \), for \( K \) any approximate heat kernel on a tame vector bundle \( \mathcal{X} \), to act as a distribution on a sections \( f \) of \( \mathcal{X} \). This is \( f(t) = K^\infty(t) \ast f \). Consider first the small-\( t \) limit of \( f(t) \). If \( K^\infty(t) \) is to agree with the heat kernel, as a distribution, then this limit should just be \( f \). Assuming \( f \) is smooth and bounded on each coordinate patch, using the trivial partition in Eq. (2.41) and then Eq. (2.16) to relate the \( t \)-norm to the operator norm gives, for \( t < T \),

\[
\| f(t) - K(t) \ast f \|_\infty \leq At^{3/2}e^{2B\sqrt{T}} \| f \|_\infty
\]

from which it follows that \( f(t) \) satisfies the correct initial condition.

To see why the heat equation holds, first note the terminology “approximate semigroup” is accurate in that the limiting kernel \( K^\infty \), or in other words the path integral, is a semigroup: \( K^\infty(t) = K^\infty(t_1) \ast K^\infty(t_2) \), for \( t = t_1 + t_2 \) and \( t_1, t_2 > 0 \). This follows immediately from considering the limit of \( K(t_1) \ast K(t_2) \) under refinements of the partition \((t_1, t_2)\) of \( t \). That means in particular \( f(t + \tau) = K^\infty(\tau) \ast f(t) \) for small \( \tau \). Using this in the definition of the \( t \)-derivative,

\[
\left| \frac{\partial f(t)}{\partial t} - \frac{1}{2} \Delta f(t) \right| = \left| \lim_{\tau \to 0} \frac{K^\infty(\tau) \ast f(t) - f(t)}{\tau} - \frac{1}{2} \Delta f(t) \right|
\]

\[
\leq \left| \lim_{\tau \to 0} \frac{K(\tau) \ast f(t) - f(t)}{\tau} - \frac{1}{2} \Delta f(t) \right| + \lim_{\tau \to 0} \frac{A\tau^{3/2}e^{2B\sqrt{T}} \| f(t) \|_\infty}{\tau}.
\]

That the first term is 0 is the requirement of Eq. (2.23) in the definition of an approximate heat kernel, so \( f(t) \) is the unique solution of heat equation with the initial condition \( f \). Thus, the path integral \( K^\infty(t) \) agrees with the heat kernel as a distribution. Technically, this is only true for \( t < T \). However, larger \( t \) can be partitioned as some \( Q = (t_1, \ldots, t_k) \) with each \( t_i < T \), and then the semigroup property ensures \( K^\infty(t) = (K^\infty)^Q(t) \). Thus, \( K^\infty \) is a distributional heat kernel for all \( t > 0 \). Finally, since \( \Delta \) is elliptic, standard results on elliptic regularity [18] imply \( K^\infty(x, y; t) \) is smooth in \( x, y \), and \( t \) and thus is the heat kernel of \( \Delta \).

That is, the limit of the kernel products of any approximate heat kernel is well-defined and agrees pointwise with the heat kernel. In particular, the choice \( K_\Delta \) of Eq. (2.26) shows Feynman’s time-slicing prescription applied to the action of Eq. (1.4) leads to a well-defined path integral over paths with fixed endpoints, and this path integral is equal to the heat kernel.

3  The Atiyah-Singer index theorem for the twisted Dirac operator from the twisted \( N = 1/2 \) SUSYQM path integral

3.1  The heuristic argument

Let \( H_\Delta^\psi(x, y, \psi_x, \psi_y, \eta_x, \eta_y; t) \) denote the heat kernel for the twisted Dirac operator of Eq. (1.10) on sections of the bundle \( \hat{\mathcal{V}} = \mathcal{S} \otimes \Lambda T \). The relevant supertrace of the heat kernel is

\[
\text{str} H_\Delta^\psi = \int H_\Delta^\psi(x, x, \psi, \psi, \eta, \eta; t) d\psi dx.
\]

Now-standard arguments due to Witten[1] in the language of supersymmetry and McKean & Singer[19] in the mathematics literature say this supertrace, which is a sum over the eigenvalues
but with those on odd-degree subspaces counting negative, computes the index of the Dirac operator.\footnote{This would be the Dirac operator on $\mathcal{V}$. For that on $S \times T$, restrict the heat kernel to the degree-one piece in $T$ and take the trace in $\text{End}(T)$.} The reason is that the operator $(D_+ + D_-)$ provides an isomorphism between even-degree and odd-degree eigenspaces for non-zero eigenvalues. Thus, in the supertrace these contributions cancel, leaving only $\dim \ker \Delta_+ - \dim \ker \Delta_-$, which is the index. Notice the index depends only the heat kernel along the diagonal, and, indeed, only on its degree-$m$ component as a form on $M$.

The heuristic Property 1 of the introduction would imply the path integral with the action $S_{\text{twisted}}$ of Eq. 1.12 and spinor paths going from $(y, \psi_y, \eta_y)$ to $(x, \psi_x, \eta_x)$ in time $t$ agrees with $H_{\Delta t}(x, y, \psi_y, \psi_x, \eta_x, \eta_y; t)$. (The results of Sect. 2.3 say this is in fact true for the rigorous path integral $K_{\Delta t}^\infty$.) With this, $\text{str}H_{\Delta t}$ is the path integral taken over loops.

Now Property 2 would say the steepest descent approximation gives the small-$t$ behavior of the path integral over loops. Since the index does not depend on $t$, this small-$t$ approximation computes the index as the integral over $(x, \psi)$ of the steepest descent approximation to path integral on loops based at this point. The resulting equation expresses the topological index as an integral over $M$, the integral over $\psi$ serving to pick out the top-form piece. This is the index theorem.

To compute the steepest descent approximation, expand the action $S_{\text{twisted}}$ about its minimum, after rescaling the paths according to their expected contribution for small-$t$, and discard terms of order higher than 1 in $t$. The result is an approximate action $S_q$ which is purely quadratic in the paths. The path integral taken over based loops with action $S_q$ then reduces heuristically to ratios of the determinants of the differential operators appearing in $S_q$.

### 3.2 A rigorous version of rescaling

#### 3.2.1 Reduction in a neighborhood of the diagonal to a trivial bundle in $\mathbb{R}^m$

Work locally in the bundle $\mathcal{V} = S \otimes \Lambda T$ over $M$ (henceforth dropping the hats). Let $x_0 \in M$. Endow a ball of radius $D_1 > 0$ around $x_0$ with Riemann normal coordinates, and identify the restriction of $\mathcal{V}$ to this ball with $\mathcal{V}_{x_0}$ via parallel transport along minimal geodesics. This defines a metric $g_1$, a trivial bundle $\mathcal{V}_1$, and a connection $\nabla^1$ over a neighborhood of the origin in $\mathbb{R}^m$, all with bounded derivatives up to order four. Extend all of these to all of $\mathbb{R}^m$ so that the derivatives remain bounded and so that both $\nabla^1$ and the Levi-Civita connection $\nabla^{g_1}$ continue to be 0 on radial directions. Let $\mathcal{C}$ denote the Clifford algebra $\mathcal{C}(T_{x_0}^* M)$ at $x_0 = 0$, whose action on $\mathcal{V}_{x_0}$ splits it into $S \otimes T$, where $S$ is the spinor representation of $\mathcal{C}$ and $\mathcal{C}$ acts trivially on $T$. $\mathcal{V}_1$ can be identified with the trivial bundle $S \otimes T$ over $\mathbb{R}^m$. Identifying the Clifford algebra at any point in $\mathbb{R}^m$ with $\mathcal{C}$ by radial translation gives it an action on $S \otimes T$ that makes $\nabla^1$ a Clifford connection agreeing with $\nabla^V$ in the ball of radius $D_1$. In fact then $\nabla^1 = \nabla^{g_1} \otimes 1 + 1 \otimes \nabla^T$, where $\nabla^{g_1}$ is the Levi-Civita connection on $S$ and $\nabla^T$ is some connection on $T$ with curvature $F^T$. The choice $V_1 = \mathcal{C}(F^T) - r_1/4$ defines a Dirac operator $D_1$ on $(g_1, S \otimes T \times \mathbb{R}^m, \nabla^1)$, and a generalized Laplacian $\Delta_1 = (D_1)^2$, whose associated approximate heat kernel $K_1 = K_{\Delta_1}$ can be identified with $K_{\text{twisted}}$ of Eq. (1.13) in that ball by the obvious isomorphism.

On the other hand, given any pair of approximate semigroups, each on a tame bundle over a tame Riemannian manifold, if there is an bundle isomorphism respecting the tameness structure under which the two semigroups are identified (via pullback) in some neighborhood of a given point, then the corresponding path integrals will agree on the diagonal at that point up to terms.
that are exponentially damped as $t \to 0$. That is, letting $\Phi$ denote the isomorphism, there are real positive constants $c$, $d$ and $T$, depending on the constants $B_i$, $C_i$, and $D_i$ for $i = 1, 2$ required to define the semigroups, such that

$$|K_1^\infty(x, x; t) - K_2^\infty(\Phi(x), \Phi(x); t)| \leq ce^{-d/t}. \quad (3.1)$$

The argument for this starts by breaking $P$ up according to its intervals as $P = P_j(t_j)P_{j'}$, and writing $K_1^{tP} - K_2^{tP} = \sum_j K_1^{tP_j} * K_1(t_j) * K_2^{tP_{j'}} - K_2(t_j) * K_2^{tP_{j'}}$. If the left-hand side is being evaluated at a pair of points on which the two semigroups agree, the equation is unaffected by adding the assumption that in each term the lone $K_1$, and hence the preceding $K_1^{tP_j}$, is evaluated at $(y_{j-1}, y_j; t_j)$ for $y_{j-1}$ outside the neighborhood of agreement and $y_j$ inside, while in $K_2(\Phi(y_{j-1}), \Phi(y_j); t_j)$ the point $y_{j-1}$ is inside and $y_j$ outside this neighborhood. With this added assumption, a bound analogous to that of Eq. (2.43) on the growth of kernel products in the $t$-norm for a semigroup and Eq. (2.17) relating the $t$-norm to the supremum norm lead to $|K_1^{tP}(x, x; t) - K_2^{tP}(\Phi(x), \Phi(x); t)| \leq c_1e^{-d/t}$. The convergence result for semigroups, specifically Eq. (2.41), readily gives the agreement in the path integrals on the diagonal.

The upshot is that to understand the behavior, on the diagonal for short times, of the path integral based on the approximate kernel $K_{twisted}$ it suffices to understand that of the path integral based on $K_1$ in the simpler setting of the trivial bundle $V_1$ on $\mathbb{R}^m$.

### 3.2.2 Rescaling the local kernel

Because $V_1$ is trivial, $K_1$ can be taken to be a function on $\mathbb{R}^m \times \mathbb{R}^m$ with values in $\text{End}(\mathcal{S}) \otimes \text{End}(\mathcal{T}) \sim C \otimes \text{End}(\mathcal{T})$. The Clifford algebra action $c_A$ on $\Lambda^2s\tau_M$ maps $K_1$ to a function with values in $\text{End}(\Lambda^2s\tau_M) \otimes \text{End}(\mathcal{T})$. Mildly abuse notation to let $K_1$ also refer to this function.

Rescale by a parameter $0 < r \leq 1$ as follows: Define $\phi_r : \mathbb{R}^m \to \mathbb{R}^m$ by $\phi_r(x) = rx$, and define $\psi_r : \Lambda^2s\tau_M \to \Lambda^2s\tau_M$ for elements $\alpha$ of a given degree by $\psi_r(\alpha) = r^{de(g)}\alpha$. For the metric, define $g_r = r^{-2}\phi_r^*(g_1)$. This extends continuously to $g_0 = g_1.0$, where, by construction, $g_1.0(v, w) = (v, w)$, the standard inner product on $\mathbb{R}^m$. Finally, for $K(x, y; t)$ a kernel on the bundle $\Lambda^2s\tau_M \times \mathcal{T}$ over $\mathbb{R}^m$, define

$$\Phi_r[K](x, y; t) = r^m\psi_r^{-1}K(rx, ry; r^2t)\psi_r. \quad (3.2)$$

Write

$$K_r = \Phi_r(K_1)$$

for the rescaled version of $K_1$.

The family of metrics has the properties (extending each formula by continuity to $r = 0$):

$$g_{r,x}(v, w) = g_{1,rx}(v, w)$$

$$d_{g_r}(x, y) = r^{-1}d_{g_1}(rx, ry)$$

$$(y_x)_{g_r} = r^{-1}((y)_rx)_{g_1}$$

$${\text{Ricci}}_r(y_x, y_x) = {\text{Ricci}}_1((y)_rx, (y)_rx)$$

$$t_r = r^2t_1$$

$$d_{g_r} = r^{-m}d_{g_1}(ry).$$
Direct calculation shows the rescaling commutes with the kernel product. Further, as the constants $B_1$ and $D_1$ in the definition of an approximate semigroup and the $t$-norm depend only on the supremum of the metric $g_1$, and the rescaling from $g_1$ to $g_r$ cannot change the supremum, these constants will work for any of the $g_r$, in the sense that there is a $t$-norm which is independent of $r$. In fact, by making $D$ a fixed fraction of $D_1$, a straightforward argument shows there are constants such that for all $0 < r \leq 1$, $\|\Phi_r(K)\|_{(t)} \leq \|K\|_{(t)}$. From this, directly checking the definition shows $K_r$ is an approximate semigroup, so the path integral $K_r^\infty$ based on $K_r$ will be well-defined. Further, since rescaling commutes with the kernel products defining the approximate path integrals,

$$K_r^\infty = \Phi_r(K_1^\infty).$$

That is, rescaling the path integral based on the approximate kernel $K_1$, which up to exponentially-damped terms agrees on the diagonal with the heat kernel for $\Delta^V$, gives the same result as basing the path integral on the approximate kernel $K_r$.

If this extends to $r = 0$, it will say any aspect of the heat kernel (on the diagonal) which can be calculated from the $r = 0$ limit of the rescaling applied to the path integral can in fact be calculated by a presumably simpler path integral based on the $r = 0$ limit of $K_r$. There will still be two issues:

1. Does the rescaling limit of the path integral retain enough information to compute the supertrace?
2. Does the rescaling limit of $K_r$ lead, via the refinement limit of its products, to a computable path integral?

Address the second question first, by considering what happens to

$$K_r = \lim_{r \to 0} r^m (2\pi t)^{-m/2} e^{-[d_{g_r}(x,y)]^2/(2t)}$$

$$\times e^{-\text{Ricci}((x,y)-(x,y))/12+\frac{t}{r^2}} F_{ij}^T(rx)\psi_r^{-1} c(dx_i)c(dx_j)\psi_r^{-1} \psi_r^T x^i x^j \psi_r$$

as $r \to 0$. Since $g_1 = g_0 + \mathcal{O}(|x|^2)$ both curvature terms vanish in the limit, and $d_{g_r}(x,y) \to |x - y|$. Direct calculation shows $\lim_{r \to 0} \psi_r^{-1} r c(dx)\psi_r = dx$, so

$$\lim_{r \to 0} F_{ij}^T(rx)\psi_r^{-1} r^2 c(dx_i)c(dx_j)\psi_r / 2 = F,$$

where

$$F = \frac{1}{2} F_{ij}^T(x_0)dx^i \wedge dx^j$$

defines $F$ as an element of $\Lambda T^*_{x_0} M \otimes \text{End}(T)$ (that is, a 2-form at $x_0$ taking values in linear transformations on the vector space $T$). Finally, in $\psi_r^{-1} \psi_o^T x^i x^j \psi_r$, with the bundle being trivialized radially at the origin, the parallel transport from $rx$ to $ry$ is the holonomy of the geodesic triangle from $0$ to $rx$ to $ry$ to $0$. In $T$, this holonomy differs from 1 by a quantity proportional to the area enclosed, which is $\mathcal{O}(r^2)$ and will thus vanish in the limit. For the $\Lambda(T^*_{x_0} M)$ piece, the holonomy is an element of the spin group and therefore an exponential of a degree-two element of $C$. This Clifford element in turn is the image under $c$ of the two-form generating the holonomy
about the same geodesic triangle with respect to the Levi-Civita connection. Standards results [20] say this is \((R \cdot \mathbf{r}, \mathbf{r} - \mathbf{r}) / 4 + \mathcal{O}(r^3)\), where analogously to \(F\),

\[
R^i_k = \frac{1}{2} R^l_{ijk}(x_0) dx^i \wedge dx^j
\]
defines \(R \in \Lambda T^* M \otimes \text{End}(T_0 M)\). Thus, this piece is the exponential of the image under \(c\) of \((R \cdot \mathbf{r}, \mathbf{r} - \mathbf{r}) / 4 + \mathcal{O}(r^3)\). Conjugation by \(\psi_r\) will reduce the power of \(r\) by two, giving

\[
\lim_{r \to 0} \psi_r^{-1} \frac{\partial}{\partial \mathbf{x}} \psi_r = e^{(R \mathbf{x}, \mathbf{y} - \mathbf{x}) / 4}.
\]

Putting this all together,

\[
\lim_{r \to 0} K_r = K_0,
\]

where

\[
K_0(x, y; t) = (2\pi t)^{-m/2} e^{-|y - x|^2 / (2t)} e^{R(x, y - x) / 4 - t F / 2}.
\]

The kernel \(K_0\) is in fact a time-slicing approximation for the path integral with action \(S_q\), as anticipated by the heuristic application of steepest descent. Using Lebesgue Dominated Convergence it is straight-forward to show that for a fixed partition \(P\) of any \(t > 0\)

\[
\lim_{r \to 0} K^{SP}_r(0, 0; t) = K_0^{SP}(0, 0; t).
\]

That is, approximate path integrals based on \(K_r\) go to those based on \(K_0\) in the rescaling limit. The starting point is to observe \(K_0\) and \(K_r\) are bounded by \(C_1 H(x, y; C_2 t)\) for some \(C_1, C_2\) where

\[
H(x, y; t) = (2\pi t)^{-m/2} e^{-d_0^2(x, y)^2 / (2t^2)},
\]

which in turn follows from the same bound on \(K_1\).

Unfortunately, the appearance of \(R \cdot \mathbf{x}\) in the exponential, and the fact that \(\mathbf{x}\) is free to range over all of \(\mathbb{R}^m\) though \(|\mathbf{x} - \mathbf{y}| < D\), means that \(K_0(t)\) will not satisfy the definition of an approximate semigroup, so the preceding convergence results do not immediately apply to ensure the refinement limit \(K_0^\infty\), and hence the path integral even exist. On the other hand, it should be a standard result on path integrals with quadratic actions that the path integral based on \(K_0\) is well-defined and agrees with the heat kernel for a Laplacian compatible with the Lagrangian whose action is \(S_q\). For a proof in the language of kernel products see [13]. The precise statement is

\[
\lim_{|P| \to 0} K^{SP}_0 = K_0^\infty
\]

converges pointwise, and is the heat kernel for the operator

\[
\Delta = \frac{\partial^2}{\partial x_i \partial x_i} + \frac{1}{2} R^i_{\ j\ k\ l}(x_0) \frac{\partial}{\partial x_i} - F + |R \cdot \mathbf{x}|^2 / 16.
\]

Standard results on heat kernels for this Laplacian, which physically is just the Hamiltonian for a particle in a constant magnetic field, give the explicit value on the diagonal:

\[
K_0^\infty(0, 0; t) = (2\pi t)^{-m/2} \det^{1/2} \left( \frac{t R/4}{\sinh(t R/4)} \right) e^{-t F / 2}.
\]

The ratio of determinants on the right-hand side is that predicted by the heuristic path integral with action \(S_q\).
3.2.3 Along the diagonal, the small-\(t\) behavior of the full path integral agrees with that of the rescaling limit

Return now to the question of whether the rescaling limit captures enough of the small-\(t\) behavior to compute the supertrace of the full heat kernel (Question 1 above). Working directly from the heat equation, it is easy to see that \(K_t^\infty(0,0;t) = \sum_{i=0}^{(m+2)/2} \alpha_i t^{-m/2} + \mathcal{O}(t)\) for \(\alpha_i \in \mathcal{C} \otimes \text{End}(\mathcal{T})\), where each \(\alpha_i\) is of degree \(2i\) in the Clifford filtration. Thus, \(c_\Lambda(\alpha_i)\) is of degree at most \(2i\) as an element of \(\text{End}(\Lambda_T^* M) \otimes \text{End}(\mathcal{T})\). The supertrace, which includes an integration over \(\mathcal{M}\), will pick out the degree-\(m\) piece of \(K_t^\infty(0,0;t)\). This piece comes from \(\alpha_m + \alpha_{(m+2)/2} t + \mathcal{O}(t)\).

As \(t\) goes to 0, the supertrace thus sees only \(c_\Lambda(\alpha_m)\), and that, only the piece of degree \(m\).

To see this is also exactly the piece that survives the rescaling limit, first apply the rescaling, which takes the term \(c_\Lambda(\alpha_i) t^{i-m/2}\) to \(\psi^{-1} c_\Lambda(\alpha_i) \psi r^{2i-m/2}\). Since \(c_\Lambda(\alpha_i)\) is a sum of terms of degrees up to \(2i\), and conjugation by \(\psi\) multiplies a term of degree \(k\) by \(r^{-k}\), the result is an overall factor of \(r^{2i-k}\). As \(r \to 0\), only the “top” piece, of degree \(2i\), will survive. Moreover, the last term in the series, where \(i = m/2 + 1\) and so the top piece of \(\alpha_i\) has degree \(m+2\), will go to 0 after rescaling and taking \(r \to 0\). Likewise, the \(\mathcal{O}(t)\) corrections vanish in this rescaling limit.

In short,

\[
\lim_{r \to 0} \Phi_r[K_t^\infty](0,0;t) = \sum_{i=0}^{m/2} \rho(\alpha_i) t^{i-m/2},
\]

where \(\rho\) takes an element \(\alpha_i\) of Clifford degree \(2i\) to the form of degree \(2i\) corresponding to the top-form piece of \(c_\Lambda(\alpha_i)\). In particular, the degree-\(m\) piece of the rescaling limit of the path integral agrees with that of the small-\(t\) limit of the heat kernel (path integral) \(K_t^\infty(0,0;t)\), so the rescaling limit indeed captures enough of the full heat kernel to calculate the small-\(t\) limit of the supertrace.

3.3 The index theorem

It remains to check the rescaling limit of the path integral is the same as the path integral based on the rescaling limit of the approximate heat kernel; that is, to check

\[
\lim_{r \to 0} \Phi_r[K_t^\infty](0,0;t) = K_t^\infty(0,0;t).
\]

For fixed \(t\), there is a choice of \(P\) making both \(K_t^r P(0,0;t)\) arbitrarily close to \(K_t^\infty(0,0;t)\) and \(K_t^r P(0,0;t)\) arbitrarily close to \(\Phi_r[K_t^\infty](0,0;t)\), for all \(r \in (0,1]\). With \(P\) fixed, there is a choice of \(r\) making \(K_t^r P(0,0;t)\) close to \(K_t^\infty(0,0;t)\). That means choosing this \(P\) and \(r\) combination will make \(\Phi_r[K_t^\infty](0,0;t)\) arbitrarily close to \(K_t^\infty(0,0;t)\), which is the statement of convergence.

Putting this all together,

\[
\text{str}K_t^\infty(t) = \lim_{r \to 0} \text{str}K_t^r(t) = \int_M (2\pi t)^{-m/2} \det^{1/2} \left( \frac{tR/4}{\sinh(tR/4)} \right) e^{-tR/2}.
\]

This is the Atiyah-Singer index theorem for the twisted Dirac operator. In fact, the argument says something about the lower-degree terms in the expansion for the heat kernel; namely, writing

\[
P(t) = \sum_{k=0}^\infty A_k t^{k-m/2}.
\]

\[\text{This relies on the fact that the constants making } K, \text{ an approximate semigroup do not depend on } r.\]
for the Laurent series in $t$ asymptotic to the diagonal of the heat kernel $K_N^\infty(x_0, x_0; t)$ of $\Delta = D^2$,

$$
\rho(P(t)) = \sum_{k=0}^{m/2} \rho(A_k) t^{k-m/2} = (2\pi)^{-m/2} \frac{1}{\det^{1/2}(1/2(tR/4))} e^{-tF/2}.
$$

This statement is the local form of the index theorem.

4 Some conclusions

This completes the work of filling in the details to apply Feynman’s time-slicing prescription to define the path integral for twisted $N = 1/2$ SUSYQM on a Riemannian manifold and to check it has the properties Witten, Alvarez-Gaumé, Friedan and Windey assume in their path integral proofs of index theorems. The definition, and the representation of the heat kernel as a path integral, extends to generalized Laplacians on any vector bundle. The method of proof appears to single out a particular resolution of the operator-ordering ambiguity inherent in passing from the Lagrangian to a Hamiltonian as one providing faster convergence of the approximations to the path integral.

This approach to proving the convergence of time-slicing approximate path integrals may apply to other settings, particularly those where exact evaluations of the path integral are feasible. These include two-dimensional Yang-Mills, which in fact is known to reduce to quantum mechanics [21], Chern-Simons theory [22], and cohomological field theories [23, 24]. The path integral arguments in the latter are closely analogous to those for SUSYQM.

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