On finite groups with elements of prime power orders*

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Abstract

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All groups considered are finite.

G. Higman in [1] studied the groups in which every element has prime power order. For convenience, we call them EPPO-groups. Later, Feit, Hall and Thompson et al investigated the structure of CN-groups, which is more extensive than EPPO-groups and firstly concluded that the order of a non-solvable CN-group is even (see [2]). Suzuki in [3] studied the class of CIT-groups, which is wider than the class of EPPO-groups, and in which the order of the centralizer of an element with order 2 is a 2-group. Suzuki’s paper [3] is very profound, which almost classified the non-solvable CIT-groups and proved the equivalence of non-solvable CN-groups and non-solvable CIT-groups. Since the order of a CIT-group is even, the EEPO-groups, in particular the solvable EPPO-groups of odd orders, are not contained the class of CIT-groups.

In this paper, we continue discussing the structure of groups in which every element has prime power order and independently obtain some more detailed results than Higman’s results in [1]. In addition, by Suzuki’s results in [3], we determine the types of non-solvable EPPO-groups, which imply an interesting result, that is, the smallest non-abelian simple group $A_5$ could be characterized only by its two ‘orders’, that is, the characteristic property of $A_5$ is: (1) The order of group contains at least three prime factors, and (2) the order of every non-identity element is prime.

Throughout this paper, we denote by $\pi(G)$ the set of all primes dividing the order of a group $G$ and denote by $G_p$ a Sylow $p$-subgroup of $G$ for some $p \in \pi(G)$.

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1 General Properties

The following observation is clear.

**Theorem 1.1.** The subgroups and quotients of an EPPO-group are also EPPO-groups.

**Theorem 1.2.** Let \( G \) be a group. Then the following statements are equivalent:

1. \( G \) is an EPPO-group.
2. For any non-trivial \( x \) and \( y \) in \( G \) with \( (|x|, |y|) = 1, xy \neq yx \).
3. For any non-trivial \( x \) and \( y \) in \( G \) with \( (|x|, |y|) = 1, C_G(x) \cap C_G(y) = 1 \).
4. For any non-trivial \( p \)-subgroup \( A \leq G, C_G(A) \leq P \), where \( p \) is a prime dividing the order of \( G \) and \( P \) is a Sylow \( p \)-subgroup of \( G \).

**Proof.** (1)⇒(2): Let \( a, b \in G \) such that \(|a| = p^α \) and \(|b| = q^β \), where \( p \neq q \). If \( ab = ba \), then \(|ab| = p^α q^β \), a contradiction because \( G \) is an EPPO-group.

(2)⇒(3): Let \( a, b \in G \) such that \((|a|, |b|) = 1\). Suppose that \( C_G(a) \cap C_G(b) \neq 1 \). Then for some \( 1 \neq g \in C_G(a) \cap C_G(b) \), we have that either \((|g|, |a|) = 1 \) or \((|g|, |b|) = 1 \) by the hypothesis. But since \( g \) centralizes both \( a \) and \( b \), this yields a contradiction.

(3)⇒(4): Note that
\[
C_G(A) = \bigcap_{y \in A} C_G(y).
\]

If \( A \) is a \( p \)-group, then for any nontrivial element \( y \in A \), \( C_G(y) \) is also a \( p \)-group by the assumption. Thus, \( C_G(A) \) is a \( p \)-group by the observation above and consequently there exists a Sylow \( p \)-subgroup \( P \) of \( G \) such that \( C_G(A) \leq P \).

Next, we suppose that there exists at least two different primes \( p \) and \( q \) dividing the order of \( A \). Let \( a \) be an element of order \( p \) in \( A \) and \( b \) be an element with order \( q \) in \( A \). Then \( C_G(a) \cap C_G(b) = 1 \) by the hypothesis and so \( C_G(A) = \bigcap_{y \in A} C_G(y) = 1 \), as desired.

(4)⇒(1): If \( G \) is not an EPPO-group, then we may assume that \( G \) contains an element \( g \) of order \( p^α q^β \) with \( α \neq 0 \) and \( β \neq 0 \). It is clear that \( g \leq C_G(⟨g⟩) \neq 1 \), a contradiction by the assumption of (4). \( \square \)

**Theorem 1.3.** Let \( G \) be a non-cyclic metacyclic EPPO-group such that \(|\pi(G)| \geq 2\). Then \( G = \langle a, b \rangle \) such that \( a^{p^α} = 1, b^{q^β} = 1, b^{-1}ab = a^r \), where \( p \) and \( q \) are different primes such that the exponent of \( r \mod p^α \) is \( q^β \).

**Proof.** We first prove the necessity.

Let \( G \) be a non-cyclic metacyclic EPPO-group. Then \( G' \) and \( G/G' \) are cyclic and moreover are EPPO-groups by the hypothesis. Therefore \(|G'| = p^α \) and \(|G/G'| = q^β \) for some \( p, q \in \pi(G) \). Since
\(|\pi(G)| \geq 2\), we see that \(p \neq q\) and so \(G\) is of order \(p^a q^\beta\) such that every Sylow subgroup of \(G\) is cyclic. By [4, Theorem 9.4.3], we have that

\[
G = \langle a, b | a^{p^\alpha} = 1, b^{q^\beta} = 1, p \neq q, b^{-1}ab = a^r, (p, r - 1) = 1, r^{q^\beta} \equiv 1 \pmod{p^\alpha} \rangle.  \tag{*}
\]

We claim that \(p^\alpha | (r^{q^\beta} - 1)\). If not, suppose that for some \(0 \leq s < \beta\), we have \(p^\alpha | (r^{q^\beta} - 1)\). Then \(b^{-q^s} ab^q = a^r = a\), which implies that \(ab^q = b^q a\), contradicting that \(G\) is an EPPO-group. Hence our claim holds. It follows that \(p \nmid r - 1\). In fact, if \(p | (r - 1)\), then we can write \(r = kp + 1\). Then \(r^{q^\alpha - 1} = (kp + 1)^{q^\alpha - 1}\), form which we conclude that \(r^{p^\alpha - 1} \equiv 1 \pmod{p^\alpha}\). By the argument above, we get that \(q^\beta | p^\alpha - 1\), a contradiction because \(p \neq q\). Thus the necessity of this theorem is proved.

Now we prove the sufficiency. By [4, Theorem 9.4.3], the Sylow subgroups of a group \(G\) satisfying relation \((\ast)\) are all cyclic and so \(G\) is a metacyclic group such that \(G = \langle a \rangle \times \langle b \rangle\). Let \(g = b^x a^y\) be any element of \(G\), where \(x\) and \(y\) are natural numbers. If \(x = 0\), then \(g\) is an element of prime power order. Suppose that \(x \neq 0\). Then

\[
(b^x a^y)^{q^\beta} = b^{xq^\beta} a^{(r^{q^\beta - 1} + r^{q^\beta - 2} + \cdots + 1)y}.
\]

Since

\[
p^\alpha | r^{q^\beta} - 1 = (r - 1)(r^{q^\beta - 1} + r^{q^\beta - 2} + \cdots + 1),
\]

and \(p \nmid r - 1\), we have

\[
p^\alpha | (r^{q^\beta - 1} + r^{q^\beta - 2} + \cdots + 1),
\]

which implies that \((b^x a^y)^{q^\beta} = 1\). Thus, \(G\) is an EPPO-group and therefore \(G\) is noncyclic. \(\square\)

**Theorem 1.4.** Let \(G\) be an EPPO-group and \(H\) be a subgroup of \(G\) such that \(|H|, d) = 1\) for a natural number \(d > 1\). Then \(|H|\) divides the number of elements of order \(d\) in \(G\).

**Proof.** If \(d\) does not divide the order of \(G\), then the result is trivial. Hence we may assume that \(d||G|\). let \(\Omega\) denote the set of elements of order \(d\) in \(G\). Take an element \(a \in \Omega\) and let \(C_1\) denote the orbit of \(a\) by the conjugate action of \(H\) on \(\Omega\). By Theorem 1.2, we have that \(N_H(a) = C_H(a) = 1\) and so \(|C_1| = |H : C_H(a)| = |H|\) by [4, Theorem 1.6.1]. Let \(b \in \Omega \setminus C_1\). Then we have other \(H\)-orbit \(C_2\) of size \(\beta |H|\). Continuing like this, we have that \(\Omega = \bigcup_{i=1}^{t} C_i\), where each \(C_i\) is an \(H\)-orbit of size \(H\). It follows that \(|H||\Omega|\), implying the result. \(\square\)

**Remark 1.** After 30 years, we find that Theorem 1.4 is also the characteristic property of an EPPO-group. See [A.A. Buturlakin, Rulin Shen and Wujie Shi, Siberian Math. J., 58, no.3(2017), 405-407.].
Corollary 1.1. Let $G$ be an EPPO-group and $N$ be a normal subgroup of $G$. Then

$$
\prod p^\alpha \mid (|N| - 1),
$$

where $p^\alpha$ is the $p$-part of $|G|$ and $p \notin \pi(N)$.

2 Solvable EPPO-groups

Lemma 2.1. Let $G$ be a non-abelian solvable EPPO-group. Then $G$ has a normal abelian subgroup which is not contained the center of $G$.

Theorem 2.1. A solvable EPPO-group $G$ is an $M$-group. That is, the every irreducible representation of $G$ is a monomial representation

Proof. This proof is similar to [5, Theorem 16].

Lemma 2.2. Let $N$ be an elementary abelian group of order $q^m$ and $H$ be an EPPO-group such that $(|H|, |N|) = 1$. Suppose that $H$ acts on $N$ and denote by $G = N \rtimes H$ the semidirect product of $N$ by $H$. Then $G$ is an EPPO-group if and only if $H$ acts faithfully on $N$, and for any non-trivial element $h \in H$, 1 is not an eigenvalue of $h$ as a linear transformation on $N$.

Proof. We first prove the necessity. Suppose that $G$ is an EPPO-group. Let $h$ and $a$ be non-trivial elements in $H$ and $N$ respectively. Then $a^h = h^{-1}ah$ by the definition of semidirect. If $h$ has a characteristic root 1, then there exists an element $1 \neq a_1 \in N$ such that $a_1^h = 1 \cdot a_1 = a_1$ and therefore $a_1h = ha_1$, a contradiction. In addition, it is easy to see that $H$ acts faithfully on $N$.

Now we prove the sufficiency.

By the hypothesis, $H$ is an EPPO-group and so $|h| = p^\alpha$ for some $p \in \pi(H)$. Observing that $h$ is a linear transformation on $N$, $h$ is a root of the polynomial $\lambda^{p^\alpha} - 1$. Assume that $h \neq 1$. Since 1 is not an eigenvalue of $h$ as a linear transformation on $N$, the minimal polynomial of $h$ is $\lambda^{p^\alpha-1} + \lambda^{p^\alpha-2} + \cdots + 1$. Let $ha$ be any element in $G$ with $h \in H$ and $a \in N$. If $h = 1$, then $(ha)^q = 1$. If $h \neq 1$ and $h^{p^\alpha} = 1$, then

$$(ha)^{p^\alpha} = h^{p^\alpha} \cdot a^{[h^{p^\alpha-1} + h^{p^\alpha-2} + \cdots + 1]} = 1 \cdot a^{0} = 1,$$

which shows that $G$ is an EPPO-group.

Lemma 2.3. Let $M$ be an $n \times n$ monomial matrix and $M^p = I_n$ for a prime $p$. If $M$ is not a diagonal matrix, then 1 is an eigenvalue of $M$.

Proof. Let $M$ be an $n \times n$ monomial matrix. Since $M^p = I_n$, we have that $|M| \neq 0$ and so in each row and column of $M$ there is exactly one non-zero element. Denote by $a_{i\sigma(i)}$ the non-zero element
of $M$ in the $i$th row, where $\sigma$ is a permutation on the set \{1, 2, \ldots, n\} and the order of $\sigma$ is $p$ by the hypothesis. Therefore $\sigma = \sigma_1 \sigma_2 \cdots \sigma_s$, where every $\sigma_i$ is a cyclic permutation of order $p$. Since $M$ is not a diagonal matrix, there exists at least one non-trivial $\sigma_i$ and without loss of generality, one can assume that $\sigma_1 = (12 \cdots p)$ and so

$$M = \begin{pmatrix} C & 0 \\ 0 & * \end{pmatrix},$$

where

$$C = \begin{pmatrix} 0 & a_{12} & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_{23} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & 0 & a_{p-1,p} \\ a_{p1} \end{pmatrix}.$$ 

Since $M^p = I_n$, we have that $C^p = I_n$, which induces that $a_{12}a_{23} \cdots a_{p-1,p}a_{p1} = 1$. Thus

$$|I - C| = \begin{vmatrix} 1 & -a_{12} \\ 1 & -a_{23} \\ \vdots & \vdots \\ -a_{p1} \end{vmatrix} = 1 - a_{12}a_{23} \cdots a_{p-1,p}a_{p1} = 0.$$ 

It follows that 1 is an eigenvalue of $C$ and is also an eigenvalue of $M$. 

Combining Lemma 2.2 and Lemma 2.3, we get the following result.

**Theorem 2.2.** Let $G$ be an EPPO-group with a non-trivial normal $q$-subgroup $Q$ for a prime $q$. Let $H$ be a solvable subgroup of $G$ such that $(|H|, q) = 1$. Then $H$ is either a cyclic $p$-group or a generalized quaternion group. In particular, if $p \neq q$, the Sylow $p$-subgroups of $G$ are cyclic or generalized quaternion and furthermore, if $G$ is solvable, the order of $G$ is $p^\alpha q^\beta$ for some positive integers $\alpha$ and $\beta$.

**Proof.** Let $N$ be a minimal $H$-invariant subgroup contained in $Q$. Then $N$ is an elementary $q$-group. Since $G$ is an EPPO-group, we have that $H$ is also an EPPO-group by Theorem 1.1. It follows that the representation of $H$ on $N$ is a monomial representation. Let $h \in H$ and let $h$ be of prime order. We claim that the representation matrix of $h$ is diagonal. If not, by Lemmas 2.2 and 2.3, we obtain that $G$ is not an EPPO-group, contradicting our assumption on $G$. Thus, if $h$ and $k$ are two elements in $H$ of prime orders, then $hk = kh$. Hence $H$ must be a $p$-group some prime $p$ since $H$ is an EPPO-group. Furthermore, by [6, Theorem 7.24], we conclude that $H$ is a cyclic group or a generalized quaternion group.
Suppose that $G$ is solvable. Then, for prime $q$, $G$ has a Hall $q'$-subgroup $H$ and consequently by the foregoing arguments, $H$ must be a $p$-group for some $p \in \pi(G)$. Therefore $|G| = p^\alpha q^\beta$.

**Theorem 2.3.** Let $G$ be an EPPO-group. Then the following statements hold.

1. If $G$ has a generalized quaternion Sylow $2$-subgroup, then $G$ is of order $2^\alpha q^\beta$ with $\beta \geq 0$ and the Sylow $q$-subgroup of $G$ is normal in $G$.

2. If $G$ has a non-trivial normal $q$-subgroup $Q$ for a prime $q$, then $G$ is solvable provided one of the following holds.

   (i) $q$ is an odd prime.

   (ii) $Q$ is the Sylow $q$-subgroup of $G$.

   (iii) $G$ has an abelian Sylow $2$-subgroup.

**Proof.** (1) By the hypothesis and [7], we have that there exists a normal subgroup of $N$ of odd order such that $G/N$ has a central element of order 2. Since $G/N$ is an EPPO-group, $G$ must be a 2-group. By the oddness of $N$, we know that $G$ is solvable. By Theorem 2.2, the order of $G$ is $2^\alpha q^\beta$ and $N$ is a normal Sylow $q$-subgroup in $G$.

(2) Let $|G| = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s}q^\beta$. By (1), we may assume that the Sylow 2-subgroups of $G$ are not generalized quaternion groups.

   (i) Suppose that $q$ is an odd prime. If $p_i \neq 2$ for $1 \leq i \leq s$, then $G$ is of odd order and so $G$ is solvable. If $p_i = 2$ for some $i$, then by Theorem 2.2, the Sylow 2-subgroups of $G$ are cyclic. It follows from Burnside’s Theorem (see [4, Theorem 14.3.1]) that $G$ has a normal 2-complement and so $G$ is solvable.

   (ii) If $Q$ is a Sylow $q$-subgroup of $G$, then $Q$ is a normal Sylow $q$-subgroup of $G$. By Theorem 2.2, we have that every Sylow $p$-subgroup of $G$ is cyclic for $p \neq q$. Thus, $G/Q$ is a metacyclic group and therefore $G$ is solvable.

   (iii) Let $P$ be a Sylow 2-subgroup of $G$ and suppose that $P$ is abelian. If $q \neq 2$, then the result follows from case (i). If $q = 2$, then $C_G(Q)$ is a Sylow 2-subgroup of $G$ by the hypothesis on $G$. Since $C_G(Q)$ is normal in $G$, we obtain that $G$ is solvable.

Thus, the proof is complete.

**Lemma 2.4.** Let $G$ be a group of order $p^\alpha q^\beta$, $P$ be a Sylow $p$-subgroup of $G$ and $Q$ be a Sylow $q$-subgroup of $G$. Suppose that $P$ is cyclic and $Q$ is minimal normal in $G$. If $C_G(Q) = Q$, then $\beta$ is the exponent of $q^{(\text{mod } p^\alpha)}$. Especially, if $G$ is a EPPO-group, then $C_G(Q) = Q$ is satisfied.

**Proof.** Since $Q$ is an elementary abelian group of order $q^\beta$ and $G = PQ$, we have that $Q$ is also a minimal $P$-invariant subgroup by the minimality of $Q$. Then $P$ acts irreducibly and faithfully on $Q$. Set $P = \langle a \rangle$. Suppose that under some basis of $Q$, the representation matrix of $a$ is $A$. By [3] Ch.3,
Theorem 2], we know that the characteristic polynomial of $A$ is equal to its minimum polynomial. Since the order of $a$ is $p^\alpha$, we have that $A^{p^\gamma} = I_\beta$, which implies that the characteristic polynomial $f(x)$ of $A$ divides $x^{p^\gamma} - 1$, where $f(x) \in \mathbb{F}_q[x]$ and $\mathbb{F}_q$ denotes the finite field with $q$ elements. Let $k(q^\beta)$ denote the splitting field of $f(x)$ over $\mathbb{F}_q$. Then all characteristic roots of $f(x)$ are $p^\alpha$th roots of unity in $k(q^\beta)$ and $f(x)$ has no multiple root in $k(q^\beta)$. We claim that $f(x)$ is irreducible over $\mathbb{F}_q$. If not, suppose that $f(x) = f_1(x)f_2(x)$, then it is easy to see that $A$ is similar to the following block matrix

$$ \begin{pmatrix} A_1 & \vphantom{A_2} \\ \vphantom{A_1} A_2 \end{pmatrix}, $$

where $f_i(x)$ is the characteristic polynomial for $A_i$, contradicting the irreducibility of $A$. It follows that $f(x)$ is irreducible over $\mathbb{F}_q$.

Let $\omega$ be a root of $f(x)$ in $k(q^\beta)$. Then the number of conjugate elements with $\omega$ is $\beta$. Let $\zeta$ generate $k(q^\beta)$. Then the map $\phi : \zeta \to \zeta^q$ generates the automorphism group of $k(q^\beta)$. It is clear that $\phi(\omega) = \omega^q$. Since the order of $A$ is $p^\alpha$, we get that the order of $\omega$ is also $p^\alpha$. This implies that the conjugacy class containing $\omega$ consists of $\omega^0, \omega^1, \ldots, \omega^{q^r-1}$ where $q^r \equiv 1 \pmod{p^\alpha}$. Thus $\beta = r$, which completes the proof.

\[\square\]

**Theorem 2.4.** Let $G$ be a solvable EPPO-group such that $|\pi(G)| > 1$ and $Q$ be a maximal normal $q$-subgroup of $G$ for some $q \in \pi(G)$. Then

1. Suppose that $G_2$ is not a generalized quaternion group, where $G_2$ is a Sylow $2$-subgroup of $G$. Then $G/Q$ is a meta-cyclic group. Let $|G| = p^\alpha q^\beta$. Then $G/Q$ is of order $p^\alpha q^\gamma$ with $q^\gamma | (p - 1)$ and the chief series of $G$ is as follows:

$$ p, \ldots, p; q\gamma, \ldots, q\gamma; b\gamma, i = 1, \ldots, k, \quad \gamma < b, $$

where $p^\alpha | (q^b - 1)$. If $Q$ is the Sylow $q$-subgroup of $G$, then the chief series of $G$ is as follows:

$$ p, \ldots, p; q\alpha, \ldots, q\alpha; k\beta, \quad \beta = kb, $$

and the length of nilpotency class of $Q$ is bounded by $k$.

2. If the Sylow $2$-subgroups of $G$ are generalized quaternion groups, then $G$ has the following chief series

$$ 2, \ldots, 2; q\alpha, \ldots, q\alpha; b, $$

where $b_i > 1$, $b|b_i$ for $i = 1, \ldots, k$, where $b$ is the exponent of $q \pmod{2^\alpha-1}$.
Proof. (1) Since \( G \) is an EPPO-group, we have that \( |G| = p^a q^\beta \), where \( p, q \in \pi(G) \) and \( p \neq q \). If \( Q \) is a Sylow \( q \)-subgroup of \( G \), then \( G/Q \) is of order \( p^a \) and by Theorem 2.2, \( G/Q \) is a cyclic group. If \( |Q| < q^\beta \), then \( G/Q \) has a cyclic Sylow \( p \)-subgroup since the Sylow \( p \)-subgroups of \( G \) are cyclic. By our hypothesis, \( G/Q \) has no normal Sylow \( q \)-subgroup since the order of each chief factor is as \( q \) of \( k \).

By induction, we obtain that the chief series of \( G \) is cyclic, we observe that the chief series of \( G \) is a Sylow \( q \)-subgroup of \( G \). By Corollary 2.4, we have that \( b \) is the exponent of \( q(\text{mod } p^a) \). Considering the factor group \( G/C_{s-1} \), we observe that the chief series of \( G/C_{s-1} \) is
\[
G/C_{s-1} > \cdots > C_{a-1}/C_{s-1} > Q/C_{s-1} > C_{a+1}/C_{s-1} > \cdots > C_{s-2}/C_{s-1} > 1.
\]
By induction, we obtain that the chief series of \( G \) has the following type:
\[
p^{\alpha}_1, \ldots, p^{\alpha}_i, q^{\beta}_1, \ldots, q^{\beta}_k, \quad \beta = kb,
\]
where \( b \) is the exponent of \( q(\text{mod } p^a) \). Since \( C_{s-1} \leq Z(Q) \), the analogous argument induces that \( C_{s-2}/C_{s-1} \leq Z(G/C_{s-1}) \). By induction, we conclude that \( C_i/C_{i+1} \) is contained in \( Z(Q/C_{i+1}) \) with \( i = \alpha + 1, \ldots, s - 1 \). It follows that \( Q > C_{a+1} > \cdots > C_{s-1} > 1 \) is a central series of \( Q \) with length of \( k \). Therefore the nilpotency class is bounded by \( k \).

Now, we may assume that \( Q \) is not a Sylow \( q \)-subgroup of \( G \). Then \( G/Q \) is a metacyclic group of order \( p^a q^{\gamma} \). Let \( H/Q \) be a Sylow \( p \)-subgroup of \( G/Q \). Then \( H/Q \) is normal in \( G/Q \). By Corollary 1.1, we have that \( q^{\gamma}(p - 1) \) and \( G \) has a normal series
\[
G > H > Q > 1,
\]
where the corresponding indices are \( q^{\gamma}, p^a, q^{3-\gamma} \), respectively. We can refine above normal series as
\[
G > \cdots > H > \cdots > Q > C_1 > \cdots > C_k > 1,
\]
where the orders of each chief factor is as
\[
q, \ldots, q; p, \ldots, p; q^{b_1}, \ldots, q^{b_k}.
\]
At last, we consider the following normal series of $H$:
\[ H > \cdots > Q > C_1 > \cdots C_k > 1. \]
Observing that $Q$ is a normal Sylow $q$-subgroup of $H$, we see that the orders of chief factor of $H$ are
\[ p, \ldots, p; q^b, \ldots, q^b. \]
Refining the chief series of $H$ above, we get that $b|b_i$ for $1 \leq i \leq k$.

Since $q^\gamma|(p - 1)$ and $p^\alpha|(q^b - 1)$, we have that
\[ q^\gamma \leq p - 1 \leq p^\alpha - 1 < p^\alpha \leq q^b - 1 < q^b, \]
and so $\gamma < b$. (??)

(2) Let $P$ be a Sylow 2-subgroup of $G$ and suppose that $P$ is a generalized quaternion group. It follows from Theorem 2.3 that $G$ has a normal Sylow $q$-subgroup $Q$. Since $P$ has a cyclic subgroup $K$ of order $2^{\alpha - 1}$, we have that $H = KQ$ is normal in $G$, and
\[ G > H > Q > 1 \]
is a normal series of $G$. Using a similar argument as in (1), we conclude that $G$ has a chief series with every factor having order as
\[ 2, \ldots, 2; q^{b_1}, \ldots, q^{b_k}, \]
where $b|b_i$ for $i = 1, \ldots, k$ and $b$ is the exponent of $q(mod 2^{\alpha - 1})$. Now we prove that each $b_i > 1$ and it suffices to prove $b_k > 1$ by induction. Let $C_K$ be a minimal normal subgroup of $G$ and assume that $|C_K| = q$. Write $L = PC_K$. Since $L$ is an EPPO-group, we have that $C_K$ is centralized by itself in $L$ and thus $L/C_K \simeq P$ is isomorphic to a subgroup of Aut($C_K$), contradicting that Aut($C_K$) is cyclic. Hence $b_k > 1$ and the result follows. \(\Box\)

Notice that Theorem 2.4 is a refinement of Theorem 1 in [1].

**Corollary 2.1.** Let $G$ be an EPPO-group. Then $G$ is supersolvable if and only if $G$ has a normal subgroup of order $q$ with $q \in \pi(G)$.

**Proof.** It suffices to prove the sufficiency part.

Assume first that $q$ is an odd prime. By (2) in Theorem 2.3, $G$ is a solvable group of order $p^\alpha q^\beta$. By Corollary 1.1, $p^\alpha$ divides $q - 1$, then the exponent of $q(mod p^\alpha)$ is 1, and so the Sylow $p$-subgroups of $G$ are cyclic. Let $|G/Q| = p^\alpha q^{\gamma}$. Then, by Theorem 2.4, we have $\gamma = 0$, which indicates that $Q$ is a normal Sylow $q$-subgroup of $G$. Since $p^\alpha|(q - 1)$, by Theorem 2.4 again, $G$ has a chief series such that every factor has order as the following:
\[ p, \ldots, p; q, \ldots, q. \]
implying that $G$ is supersolvable, as wanted.

If $q = 2$, then $G$ has a central element of order 2 and it follows that $G$ is a 2-group by the hypothesis. Thus, the result is clear. □

3 Non-solvable EPPO-groups

From the result of M. Suzuki ([3] Part 3, Theorem 5), we need only discuss the $ZT$-groups, $G = PSL_2(q)$ with $q$ a Fermat primes or a Mersenne primes, $q = 4, 9, PSL_3(4)$ and $M_9$.

Lemma 3.1. A $ZT$-group $G$ is an EPPO-group if and only if $G$ is isomorphic to $PSL_2(2^2)$, $PSL_2(2^3)$, $Sz(2^3)$ or $Sz(2^5)$.

Lemma 3.2. Let $G = PSL_2(q)$ with $q$ a Fermat primes or a Mersenne primes. Then $G$ is an EPPO-group if and only if $q = 5, 7, 17$.

Lemma 3.3. $PSL_2(9)$ and $M_9$ are EPPO-groups.

Lemma 3.4. $PSL_3(4)$ is an EPPO-group.

Remark 2. Through concrete calculations we get Lemma 3.1 to Lemma 3.4. Notice that the orders of cyclic subgroups of the above groups all are prime powers.

Theorem 3.1. Suppose that $G$ is a non-solvable EPPO-group. Then one of the following holds.

1. $G$ is isomorphic to $PSL_2(q)$ for $q = 7, 9$ or $PSL_3(4)$ or $M_9$.

2. There exists a normal 2-subgroup $T$ such that $G/T$ is isomorphic to

   i. $PSL_2(q)$ for $q = 5, 8, 17$. In this case, the class length of $T$ is not greater than 2 and $|T|-1$ is divisible by $3 \cdot 5, 3^2 \cdot 7, 3^2 \cdot 17$ respectively.

   ii. $Sz(2^3)$ with $5 \cdot 7 \cdot 13||(|T| - 1)$.

   iii. $Sz(2^5)$ with $5^2 \cdot 31 \cdot 41||(|T| - 1)$.

Theorem 3.2. Let $G$ be a non-abelian simple EPPO-group. Then $G$ is one of the following groups:

\[ A_5, PSL_2(7), PSL_2(8), PSL_2(9), PSL_2(17), PSL_3(4), Sz(2^3), Sz(2^5). \]

Theorem 3.3. Let $G$ be a group. Then $G \simeq A_5$ if and only if there are at least 3 different primes in $\pi(G)$ and the order of each non-trivial element in $G$ is a prime.

Remark 3. Let $G$ be a finite group. Then $G \simeq A_5$ if and only if $\pi_e(G) = \{1, 2, 3, 5\}$, where $\pi_e(G)$ denote the set of element orders of $G$.

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