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Physical optics solution for the scattering of a partially-coherent wave from a statistically rough material surface

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Abstract: The scattering of a partially-coherent wave from a statistically rough material surface is investigated via derivation of the scattered field cross-spectral density function. Two forms of the cross-spectral density are derived using the physical optics approximation. The first is applicable to smooth-to-moderately rough surfaces and is a complicated expression of source and surface parameters. Physical insight is gleaned from its analytical form and presented in this work. The second form of the cross-spectral density function is applicable to very rough surfaces and is remarkably physical. Its form is discussed at length and closed-form expressions are derived for the angular spectral degree of coherence and spectral density radii. Furthermore, it is found that, under certain circumstances, the cross-spectral density function maintains a Gaussian Schell-model form. This is consistent with published results applicable only in the paraxial regime. Lastly, the closed-form cross-spectral density functions derived here are rigorously validated with scatterometer measurements and full-wave electromagnetic and physical optics simulations. Good agreement is noted between the analytical predictions and the measured and simulated results.

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1. Introduction

Rough surface scattering has been an active area of research for a half century. The published research on the subject can generally be divided into two main groups. The first deals with rough surface scattering research performed by the RF/microwave community for synthetic aperture radar and remote sensing applications. Predominately concerned with fully-coherent, monochromatic plane-wave scattering from rough surfaces (with some exceptions), the common approaches employed by the RF/microwave community to the rough surface scattering problem are physical optics (PO) [1–7], perturbation [4, 8], and computational/full-wave [9–14] methods. The interested reader is referred to Elfouhaily and Guérin [15], Warnick and Chew [16], Maradudin [17], Ogilvy [18], and Nieto-Vesperinas [19] for excellent summaries of these techniques.
The second of the two main groups deals with rough surface scattering research performed by the optics community. Initially, this work mainly dealt with how incoherent light interacted with surfaces, modeled via the bidirectional reflectance distribution function (BRDF), for applications in passive visible/near-IR remote sensing and computer graphics [20–24]. More recently, with the proliferation of laser-based systems (LIDAR/LADAR and directed energy), the interaction of coherent laser light with rough surfaces, in particular the statistical behaviors of the resulting speckle patterns, gained considerable interest in such fields as metrology and remote sensing. Some of the early notable research in this area was performed by Dainty [25], Goodman [26], Parry [27], Fujii and Asakura [28,29], Pedersen [30], and Yoshimura et al. [31].

Since the presence of speckle is typically detrimental in applications involving coherent light, techniques for suppressing speckle naturally followed. While there are many such techniques [32], due predominately to the work of Wolf [33–35], the use of partially-coherent light, instead of laser light, in active illumination systems is becoming tremendously popular for speckle suppression. In particular, much literature is dedicated to the properties of partially-coherent light whose cross-spectral density (CSD) function possesses a Gaussian Schell-model (GSM) form [33,35–39]. In regards to the scattering of partially-coherent light, most of the current literature deals with scattering from low-contrast scatterers, i.e., for scatterers in which the Born approximation is valid [33,35,36,40–45]. Far less work has been performed analyzing the scattering of partially-coherent light from rough metallic surfaces. Of the work that has been published, the following approaches are common: the phase-screen model [32,46–48], ABCD matrices [47–51], and the coherent-mode representation [52].

The purpose of this rough surface scattering work is to extend the traditional, fully-coherent approaches to cases involving partially-coherent illumination. Previous work by the authors derived analytical forms for the scattering of partially-coherent beam from a statistically rough perfectly-reflecting surface using the PO approximation [53]. In this work, the previously derived scattering solutions are generalized to material surfaces and rigorously verified via experiment and simulation.

Two forms of the scattered field CSD function are derived and discussed. The first, applicable to surfaces of smooth-to-moderate roughness, is expressed in terms of an infinite series. While its behavior depends in a complex way on source and surface parameters, its analytical form is physically intuitive. The second form of the scattered field CSD function is applicable to very rough surfaces. This form of the CSD function is incredibly physical and, under certain circumstances, maintains a GSM form, which is in agreement with the literature cited above valid in the paraxial regime. As such, closed-form expressions are derived for the angular spectral degree of coherence (SDoC) and spectral density (SD) radii. These expressions are, in general, complicated functions of both the source and surface parameters. It is demonstrated that for many scenarios of interest, the SDoC radius can be safely approximated as a function of just the source parameters and the SD radius can be simplified to a function of just the surface parameters.

To verify the theoretical analysis, experimental and Monte Carlo simulation results using a full-wave electromagnetic technique (all the physics of the wave/surface interaction are included) and one based on the PO approximation are presented and compared to the predictions of the analytical models. This paper is concluded with a summary of the work and contributions presented.

2. Methodology

The scattering geometry utilized in this analysis is shown in Fig. 1. The surface height is described by the function \( h(x) \) with mean, standard deviation, and correlation length equal to 0, \( \sigma_h \), and \( \ell_h \), respectively. The surface (of length \( 2L \)) is illuminated by a partially-coherent beam.
Fig. 1. Scattering geometry of a one-dimensional (the surface and source excitation are invariant in the \( z \) direction) rough surface of length \( 2L \). The medium below the rough interface is electrically defined by the permittivity \( \varepsilon \) and permeability \( \mu \); the medium above the rough interface is vacuum \((\varepsilon_0, \mu_0)\). The rough surface height is described by the function \( h(x) \); the mean, standard deviation, and correlation length of the surface are \( 0, \sigma_h, \) and \( \ell_h \), respectively. The point \((-x_s, y_s)\) denotes the location of the source plane origin. The observation vector \( \rho_{\rho \rho} = \hat{x} x + \hat{y} y \) points from the rough surface origin to the observation point. The vector \( \rho_s = \hat{x} x_s - \hat{y} y_s \) points from the source plane origin to the rough surface origin.

(parameters \( w_s \) and \( \ell_s \) defined below) emanating from the source plane specified by the coordinates \((u, v, z)\). Its origin, relative to the rough surface coordinate system, is located at the point \((-x_s, y_s)\). The observation vector \( \rho = \hat{x} x + \hat{y} y \) points from the rough surface origin to the observation point. The medium below the rough interface is electrically described by the permittivity \( \varepsilon \) and permeability \( \mu \); the medium above the rough interface is vacuum \((\varepsilon_0, \mu_0)\).

A GSM form for the incident field CSD is used to model the partially-coherent illumination, viz.,

\[
W^i(u_1, u_2) = \langle E^i(u_1) E'^i(u_2) \rangle = E_0^2 \exp \left( -\frac{u_1^2 + u_2^2}{4w_s^2} \right) \exp \left[ -\frac{(u_1 - u_2)^2}{2\ell_s^2} \right],
\]  

where \( w_s \) and \( \ell_s \) are the source radius and source coherence radius, respectively and the functional dependence of \( W^i, E_0, w_s, \) and \( \ell_s \) on the radian frequency \( \omega \) is omitted for brevity \([33,35]\). Both s-pol and p-pol scattering solutions are derived below. Note that the rough surface and the source excitation are invariant in the \( z \) direction resulting in a two-dimensional scattering problem. As mentioned by Johnson \([17]\), light scattered from one-dimensional surfaces demonstrates the same physical behaviors as light scattered from two-dimensional surfaces. The mechanisms which are not captured in two-dimensional scattering problems are cross-polarized scattering and out-of-plane scattering \([17]\). In this way, the expressions derived in this paper are equivalent to scalar-wave scattering solutions.

2.1. PO expression for the scattered field

For the sake of brevity, the analysis to follow focuses on perpendicular polarization. The solution for parallel polarization is provided at the end of the section. For perpendicular polarization,
Utilizing the plane-wave spectrum representation of electromagnetic fields [54], the incident electric and magnetic fields become

\[ E' = \frac{1}{2\pi} \int_{-\infty}^{\infty} T_e'(k_u') \exp \left( -jk' \cdot \rho \right) \exp \left( -jk' \cdot \rho \right) dk_u' \quad v > 0 \]

\[ H' = \frac{1}{2\pi} \int_{-\infty}^{\infty} k' \times T_e'(k_u') \exp \left( -jk' \cdot \rho \right) \exp \left( -jk' \cdot \rho \right) dk_u' \quad v > 0 \]

where \( \rho \) is a vector that points from the source plane origin to the rough surface origin, \( k' = \hat{u} k_u' + \hat{v} k_v' \), and

\[ T_e'(k_u') = \frac{1}{\lambda} \int_{-\infty}^{\infty} E_z(\rho) \exp (jk_u' \rho \cdot \rho) \, d\rho. \]

Note that \(|k'| = k_0 = 2\pi/\lambda\), thus \( k_u' = \sqrt{k_0^2 - (k_u')^2} \).

The scattered field in the far zone can be found by utilizing the transverse components of the far-field vector potentials \( N \) and \( L \) [55] and the equivalent electric \( J \) and magnetic \( M \) currents induced on the rough surface by the incident field, namely,

\[ E' \approx \sqrt{\frac{\sqrt{H_0}}{8\pi \sqrt{\varepsilon}}} \left[ -\frac{H_0}{\varepsilon_0} N_t + L_t \right] \]

\[ N_t = (\hat{\phi} \cdot \hat{z}) \int_C J(\rho') \exp (jk_0 \rho' \cdot \rho') \, d\rho' \]

\[ L_t = (\hat{\phi} \cdot \hat{z}) \int_C M(\rho') \exp (jk_0 \rho' \cdot \rho') \, d\rho' \]

where \( \rho' = \hat{x}' \cdot \hat{y} + \hat{y}(x') \). Note that the integral is over the parameterized surface contour \( C \) with \( dc \) equal to the arc length, i.e., \( dc = dx' \sqrt{1 + [h'(x')]^2} \).

Using the PO approximation, the equivalent currents induced by the incident field on the material surface are

\[ J \approx (1 - r_\perp) \hat{n} \times H' \mid_{x=h(x')} \]

\[ M \approx (1 + r_\perp) E' \times \hat{n} \mid_{x=h(x')} \]

Here, \( \hat{n} \) is the unit outward normal to the surface given by

\[ \hat{n} = \frac{\hat{y} - \hat{y}h'(x)}{\sqrt{1 + [h'(x')]^2}} = \frac{\hat{y} - \hat{y} h_x}{\sqrt{1 + h_x^2}} \]

and \( r_\perp \) is the s-pol Fresnel complex amplitude reflection coefficient. Substitution of Eqs. (6) and (7) into Eq. (5) and subsequent simplification yields the following expression for \( E_z' \):

\[ E_z' = \frac{\exp \left( -jk_0 \rho + j\frac{\pi}{2} \right)}{2\pi \sqrt{8\pi k_0 \rho}} \int_{-\infty}^{\infty} T_e'(k_u') e^{-jk'_u \rho} \int_{-L}^{L} H_\perp(x', k_u') e^{ik'_u \rho'} \rho' \, d\rho' \, dk_u' \]

\[ \]
where $\mathbf{v} = k_0 \hat{\mathbf{p}} - \mathbf{k}'$ and
\[
\mathcal{H}_\perp (x', k'_s) = (1 - r'_s) \left( k'_s \hat{\mathbf{u}} - k'_s \hat{\mathbf{v}} \right) \cdot (\hat{\mathbf{x}} + \hat{\mathbf{y}} h_{r'}) + k_0 \left( 1 + r'_s \right) \hat{\mathbf{p}} \cdot (\hat{\mathbf{y}} - \hat{\mathbf{x}} h_{r'}). \tag{9}
\]
Note that $r'_s$ is primed because it is a function of surface coordinates $(x', h(x'))$.

No analytical solution has yet been found for the integral over the rough surface coordinate $x'$ in Eq. (8) because of $\mathcal{H}_\perp$, which is a complicated function of $h(x')$ and its derivative $h_{r'}$ [1]. This integral is typically evaluated using the stationary-phase approximation in which $h_{r'} \approx -v_s/v_r [1, 19]$. This approximation physically dictates that reflection from the rough surface is locally specular, i.e., local diffraction effects are excluded [1, 17–19, 23]. Applying this approximation to Eq. (8) and rearranging the integrals results in
\[
E^s_i = \frac{\exp \left( -j k_0 \rho + j \frac{k_0^2}{2} \right)}{2\pi \sqrt{8\pi k_0 \rho}} \int \int \mathcal{H}_\perp (k'_s) T_{r's} (k'_s) e^{-jk'_s \rho} e^{j\mathbf{v} \cdot \mathbf{p}' d\mathbf{k}' d\mathbf{k}'_s}. \tag{10}
\]
This expression represents the $s$-pol scattered electric field given one random incident field and one random surface realization. The only assumptions that have been made thus far are that observation is in the far field, the rough surface is such that the PO approximation for the currents holds, and shadowing/masking and multiple scattering can be safely neglected [18, 19].

2.2. Scattered field cross-spectral density function

Applying the definition of the autocorrelation of a random process to Eq. (10) yields
\[
W^{xx} (\rho_1, \rho_2) = \langle E^s_i (\mathbf{p}_1) E^{*s}_i (\mathbf{p}_2) \rangle
= \frac{e^{j k_0 (\rho_2 - \rho_1)}}{32\pi^2 k_0 \rho_1 \rho_2} \int \int \int \mathcal{H}_\perp \mathcal{H}_\perp^{*} e^{j(k'_s - k'_s') \rho} \langle T_{r's}^{r's} \rangle \langle e^{j\mathbf{v} \cdot \mathbf{p}'} e^{-j\mathbf{v} \cdot \mathbf{p}''} \rangle d\mathbf{k}' d\mathbf{k}'_s d\mathbf{x}' d\mathbf{x}'_2, \tag{11}
\]
where the subscripts 1 and 2 imply that the associated expression is a function of $x'_1, \rho'_1, k'_s$ or $x'_2, \rho'_2, k'_s$ (whichever is applicable), respectively, and the expectations are computed over the ensembles of rough surface and incident field realizations. In arriving at the above expression, it has been assumed that the incident field plane-wave spectrum is statistically independent of the rough surface; this assumption is physically intuitive. Note that the autocorrelation of the incident field plane-wave spectrum is equivalent to the Fourier transform of the CSD function given in Eq. (1). The second autocorrelation expression in Eq. (11), after some simple manipulation, is equivalent to the joint characteristic function of the random variables $h(x'_1)$ and $h(x'_2)$ [23]. The incident field term and the rough surface term are represented as $\Phi (k'_u_1, k'_u_2)$ and $\chi (k'_s_1, k'_s_2; x'_1, x'_2)$ in the analysis to follow.

The integrals over the incident field plane-wave spectrum can be approximated using the method of stationary phase [33]. Using the method of stationary phase to evaluate the plane-wave spectrum integrals has two implications. The first is that $k'_s \gg k'_u$ and is therefore approximated as $k'_s \approx k_0 - (k'_u)^2 / (2k_0)$ for all “phase” terms (i.e., $\exp[j \cdot \cdot \cdot]$) and as $k'_r \approx k_0$ for all amplitude terms. This physically implies that the incident field is highly directionally being predominantly directed along the $\mathbf{v}$ direction in Fig. 1. The second implication is that the distance from the rough surface to the source plane $\rho_s$ must be much greater than $L$—this is generally the case. To provide some idea of how much greater $\rho_s$ must be than $L$, $\rho_s = 10L$, $\rho_s = 20L$, and $\rho_s = 50L$ are approximately 95%, 97.5%, and 99% accurate, respectively. Note that these two implications mean that the $k'_s \hat{\mathbf{v}}$ term in $\mathcal{H}_\perp$, see Eq. (9), diminishes at the rate of $1/\rho_s$.
and therefore can be neglected. Recall that this term corresponded to the \( \hat{v} \) component of the incident magnetic field implying that the field incident on the rough surface is transverse electromagnetic (in particular, TEM\(^+\)) when the method of stationary phase is utilized. Applying the method of stationary phase and substituting in \( \Phi'(k_{\rho 1}^i, k_{\rho 2}^i) \) produces

\[
W^i(\rho_1, \rho_2) = \frac{E_0^2 k_0^2}{16\pi \rho_s \sqrt{\rho_1 \rho_2}} \frac{\exp\left(\frac{j k_0^2 \left(\hat{x} \cdot \hat{u}\right)^2}{\rho_s^2}\right)}{\rho_1^2 + \rho_2^2} \int_{-L}^{L} \int_{-L}^{L} \exp\left[-\frac{j k_0^2 \left(\hat{x} \cdot \hat{u}\right)^2}{2 \rho_s^2}ight] \left(x_1^2 + x_2^2\right) \exp\left[2b \frac{k_0^2 \left(\hat{x} \cdot \hat{u}\right)^2}{\rho_s^2} x_1 x_2\right] \exp\left[k_0 \left(\hat{x} \cdot \hat{u}\right) / \rho_s \left(x_1^2 - x_1' x_2'ight)\right] e^{j k_0 \left(\hat{x} \cdot \hat{u}\right) / \rho_s \left(x_1' x_2\right)} \right. \\
+ \left. \chi \left(k_0^i \left(\hat{x} \cdot \hat{u}\right) x_1', k_0^i \left(\hat{x} \cdot \hat{u}\right) x_2'; \xi_1, \xi_2\right) e^{j k_0 \left(\hat{x} \cdot \hat{u}\right) / \rho_s \left(x_1' x_2\right)} \right] \, dx_1 \, dx_2
\]  

(12)

where the symbols used above are defined in the appendix.

To simplify Eq. (12) further, a form for the rough surface characteristic function \( \chi \) must be chosen. A very common choice for the statistical distribution of the rough surface is to assume that the surface heights are Gaussian distributed and Gaussian correlated, i.e.,

\[
p_{H_1, H_2}(h_1, h_2) = \frac{1}{2\pi \sigma_h^2 \sqrt{1 - \Gamma^2}} \exp\left[-\frac{h_1^2 - 2\Gamma h_1 h_2 + h_2^2}{2\sigma_h^2 (1 - \Gamma^2)}\right] \Gamma(x_1 - x_2) = \exp\left[-\frac{(x_1 - x_2)^2}{\ell_h^2}\right].
\]  

(13)

Historically, Gaussian-Gaussian (G-G) models for rough surfaces were chosen for analytical convenience [32]. Recent profilometer measurements of sandblasted metallic surfaces showed that the stretched exponential-stretched exponential (SE-SE) model [56] more accurately represented surfaces roughened by random industrial processes [57]. Unfortunately, no general analytical form for the SE joint characteristic function exists. Although the results in [57] indicated that SE-SE models were superior, G-G models were fairly good approximations for the surfaces that were measured. Note that, in addition to G-G and SE-SE, numerous other surface models can be found throughout the literature. Fourier transforming the above joint probability density function yields the desired characteristic function:

\[
\chi(k_{\rho 1}, k_{\rho 2}; x_1, x_2) = \exp\left[-\frac{\sigma_h^2}{2} (v_{x_1} + v_{x_2})\right] \exp\left[\frac{\sigma_h^2 v_{y_1} v_{y_2}}{\ell_h^2} \frac{(x_1 - x_2)^2}{\ell_h^2}\right].
\]  

(14)

Substitution of \( \chi \) into Eq. (12), performing the common variable transformation \( x_d = x_1 - x_2' \) and \( x_a = x_1' + x_2' \), and subsequent simplification yields

\[
W^i(\rho_1, \rho_2) = \frac{E_0^2 k_0^2}{32\pi \rho_s \sqrt{\rho_1 \rho_2}} \frac{\exp\left(\frac{j k_0^2 \left(\hat{x} \cdot \hat{u}\right)^2}{\rho_s^2}\right)}{\rho_1^2 + \rho_2^2} \int_{-2L}^{2L} \exp\left[-\frac{(a - b) k_0^2 \left(\hat{x} \cdot \hat{u}\right)^2}{2 \rho_s^2} x_a\right] \exp\left[\frac{k_0}{2} (\vartheta_{x_1} - \vartheta_{x_2}) x_a\right] \\

\int_{-2L}^{2L} \exp\left[-\frac{(a + b) k_0^2 \left(\hat{x} \cdot \hat{u}\right)^2}{2 \rho_s^2} x_d\right] \exp\left[\frac{k_0 \sigma_h^2}{2} \vartheta_{x_1} \vartheta_{x_2} \exp\left(-\frac{x_a^2}{\ell_h^2}\right)\right] \\
\int_{|x_a| = 2L} \exp\left[-\frac{j k_0 \left(\hat{x} \cdot \hat{u}\right) x_a}{2 \rho_s}\right] \exp\left[-\frac{j k_0 \left(\hat{x} \cdot \hat{u}\right) x_a}{2 \rho_s} x_a\right] \, dx_a \, dx_a.
\]  

(15)
At this point it is necessary to handle the exponential term containing the surface autocorrelation function. Classically, this has been done in one of two ways. The first is to expand the exponential term in a Taylor series and proceed with the evaluation of the integrals [23]. Mathematically, this approach is applicable to all surfaces; however, computationally (because the series is slowly convergent), this approach is limited to smooth-to-moderately rough surfaces and is investigated in the next section. The other approach involves expanding the surface autocorrelation function in a Taylor series and only retaining the first two terms, i.e., $\exp \left( -x^2_\delta / \ell_h^2 \right) \approx 1 - x^2_\delta / \ell_h^2$ [3, 23]. This treatment is applicable to very rough surfaces and is discussed in Section 2.2.2.

2.2.1. Smooth-to-moderately rough surfaces

As briefly discussed in the previous section, further progress can be made on Eq. (15) by expanding the exponential term containing the surface autocorrelation function in a Taylor series, specifically,

$$\exp \left[ k_0^2 \sigma_h^2 \partial_{x_1} \partial_{x_2} \exp \left( -x^2_\delta / \ell_h^2 \right) \right] = \sum_{m=0}^{\infty} \frac{(k_0^2 \sigma_h^2 \partial_{x_1} \partial_{x_2})^m}{m!} \exp \left( -m x^2_\delta / \ell_h^2 \right).$$  

Substitution of this series into Eq. (15) and subsequent evaluation of the integral over $x_\delta$ results in

$$W^s (\rho_1, \rho_2) = \frac{E^2 \mathcal{Q}}{16 \sqrt{2 \pi} \rho_1 \rho_2 (a^2 - b^2)} \exp \left[ -\frac{k_0^2 \sigma_h^2}{2} \left( \partial_{x_1}^2 + \partial_{x_2}^2 \right) \right] \sum_{m=0}^{\infty} \frac{(k_0^2 \sigma_h^2 \partial_{x_1} \partial_{x_2})^m}{m!}$$

$$\int_{-2L}^{2L} \exp \left\{ -\frac{k_0^2 (\hat{x} \cdot \hat{u})^2 x_\delta^2}{2 \rho_s^2} \frac{(\hat{a} - \hat{b})^2}{\rho_s^2} + \frac{\ell_h^2 (\hat{x} \cdot \hat{u})^2}{4 \mathcal{D}_m} \right\}$$

$$\exp \left\{ \frac{k_0 x_a}{2} \left[ \frac{k_0 \rho_s \ell_h^2 (\hat{x} \cdot \hat{u})^2}{2 \mathcal{D}_m} + j \left( \partial_{x_1} + \partial_{x_2} \right) \right] \right\}$$

$$\text{Re} \left\{ \frac{\erf \left( \frac{(2L - |x_a|)}{\sqrt{2 \rho_s^2 \ell_h^2}} + j k_0 \rho_s \ell_h}{\sqrt{2 \rho_s^2 \ell_h} \mathcal{D}_m} \right[ \frac{(\hat{x} \cdot \hat{u})^2}{2 \rho_s^2} x_a - \left( \partial_{x_1} + \partial_{x_2} \right) \right]\right\} dx_\delta,$$

where $\mathcal{D}_m = k_0^2 (\hat{x} \cdot \hat{u})^2 \ell_h^2 (\hat{a} + \hat{b}) + 2m \rho_s^2$.

In order to evaluate the remaining integral and arrive at a closed-form solution, the complex error function (erf) must be properly handled. While it is possible to determine the order of the error function argument by using physical insight and examining the associated exponential functions in Eq. (17), it is much simpler to examine the integrand in Eq. (15) and determine the condition at which the $x_\delta$ integration limits can be extended from $[|x_a| - 2L, 2L - |x_a|]$ to $(-\infty, \infty)$. A necessary condition for this approximation occurs when

$$\exp \left\{ -x^2_\delta \left[ \frac{(\hat{a} + \hat{b}) k_0^2 (\hat{x} \cdot \hat{u})^2}{2 \rho_s^2} + \frac{m}{\ell_h^2} \right] \right\} > \delta,$$

where $\delta$ is a user-defined parameter and denotes the point at which the exponential function no longer has significant value. Taking note that the minimum of the exponential function's
argument occurs when \( m = 0 \) and that the maximum value \( x_d \) takes on in the integration is \( 2L \), the following condition can be derived:

\[
L > \frac{\rho_s}{k_0 w_1 \sqrt{2|\hat{x} \cdot \hat{u}|}} \sqrt{-\ln \delta}.
\]  

(19)

The above condition physically means that the projected fully-coherent incident beam size must “fit” on the rough surface for the \( \text{erf} \approx 1 \), or equivalently, the \( x_d \) integration limits in Eq. (15) \([|x_d| \leq 2L, 2L - |x_d|] \approx (-\infty, \infty)\). How well the incident beam “fits” is determined by \( \delta \)—the smaller the \( \delta \), the more accurate the approximation. If the projected fully-coherent incident beam size does not “fit” on the rough surface, the \( \text{erf} \neq 1 \) and the remaining expression must be evaluated numerically.

Assuming that the above condition holds, the remaining integral in Eq. (17) can be evaluated, namely,

\[
W^s (\rho_1, \rho_2) = \frac{E_0^2 k_0 \ell h \rho_1 e^{k_0 (\rho_2 - \rho_1)}}{16|\hat{x} \cdot \hat{u}| \sqrt{\rho_1 \rho_2 (a^2 - b^2)}} \exp \left[ -\frac{k_0^2 \sigma_h^2}{2} (\vartheta_{1x}^2 + \vartheta_{2x}^2) \right]
\]

\[
\sum_{m=0}^{\infty} \frac{\rho_s^2 \kappa_{m}^2 |\hat{x} \cdot \hat{u}|^2 b + m \rho_s^2}{\alpha_m^2 (\hat{x} \cdot \hat{u})^2} \exp \left\{ \frac{\rho_s^2 \kappa_{m}^2 |\hat{x} \cdot \hat{u}|^2}{\alpha_m^2 (\hat{x} \cdot \hat{u})^2} (\vartheta_{1x} - \vartheta_{2x})^2 \right\} \exp \left[ \frac{j \rho_s^2 \kappa_{m}^2}{2 \alpha_m} (\vartheta_{1x} - \vartheta_{2x}) \right],
\]  

(20)

where \( \alpha_m = 4 \alpha_m (\hat{a} - \hat{b}) + \rho_s^2 \kappa_{m}^2 |\hat{x} \cdot \hat{u}|^2 \). A similar analysis to that performed above can be performed for the error functions in this expression. The condition for the sum of the \( \text{erf} \)'s \( \approx 2 \) is

\[
L > \frac{\rho_s}{k_0 w_1 \sqrt{2|\hat{x} \cdot \hat{u}|}} \sqrt{-\ln \delta},
\]  

(21)

where \( \alpha = \ell \omega/\lambda \). This condition physically means that the projected partially-coherent beam size must “fit” on the rough surface in order to safely assume that the sum of the \( \text{erf} \)'s \( \approx 2 \). This condition is more stringent than Eq. (19).

The physical interpretation of Eq. (20) is obscured somewhat by the summation; however, the exponential term on the second line of the expression generally drives the angular extent of the scattered SD. The first exponential term on the third line generally determines the angular extent over which the scattered field is correlated, i.e., the SD/CoC radius. The SD and SD/CoC definitions can be found in Refs. [33, 35, 36]. When one considers a fully-coherent incident Gaussian beam \( (\ell \omega \rightarrow \infty) \) and quasi-monochromatic light [58], the coherent scattered irradiance, i.e., \( I^s = \left| \langle E^s \rangle \right|^2 \), can be determined quite easily from Eq. (20) by considering just the \( m = 0 \) term of the series and setting \( \rho_1 = \rho_2 = \rho \). With some minor algebraic manipulation, one arrives at the two-dimensional form of the coherent scattered irradiance expression derived by Wang et al. [2]. One can also derive from Eq. (20) the incoherent scattered irradiance \( I^s \) for a fully-coherent incident Gaussian beam and quasi-monochromatic light. Note that \( I^s \) is
equivalent to the variance of the scattered field, i.e., \( I_s^i = I^r - I_c^r \), where \( I^r = \langle E_s^r E_s^r \rangle \) is the average scattered irradiance. This expression also simplifies to the form derived by Wang et al. [2].

2.2.2. Very rough surfaces

Consider the form of the joint characteristic function:

\[
\chi = \exp \left\{ -\frac{k_0^2 \sigma_h^2}{2} (\theta y_1 + \theta y_2) \exp \left( -\frac{x_d^2}{\ell_h^2} \right) \right\}
\]

\[
= \exp \left\{ -\frac{k_0^2 \sigma_h^2}{2} \theta y_1 \left[ \frac{\theta y_1}{\theta y_2} + \frac{\theta y_2}{\theta y_1} - 2\exp \left( -\frac{x_d^2}{\ell_h^2} \right) \right] \right\}.
\]

In the case of very rough surfaces, i.e., \( k_0^2 \sigma_h^2 \theta y_1 \theta y_2 / 2 \gg 1 \), \( \chi \) only has significant value when \( \theta y_1 / \theta y_2 + \theta y_2 / \theta y_1 - 2\exp \left( -x_d^2 / \ell_h^2 \right) \approx 0 \). Considering that \( \theta y_1 / \theta y_2 \approx \theta y_2 / \theta y_1 \), \( \theta y_1 / \theta y_2 + \theta y_2 / \theta y_1 - 2\exp \left( -x_d^2 / \ell_h^2 \right) \approx 0 \) is only possible for small \( x_d \). Thus, it makes sense to expand \( -x_d^2 / \ell_h^2 \) and retain the first two terms, i.e., \( \exp \left( -x_d^2 / \ell_h^2 \right) \approx 1 - x_d^2 / \ell_h^2 \). Substituting this into Eq. (15) and carrying out the integrations, one arrives at the desired result for the scattered field CSD function:

\[
W^s(\rho_1, \rho_2) = \frac{E_0 k_0 \ell_h^0 \rho_0 e^{i k_0 (p_2 - p_1) \cdot \hat{r} \perp}}{16 |\hat{\mathbf{x}} \cdot \hat{\mathbf{u}}| \sqrt{\rho_1 \rho_2 (a^2 - b^2)}} \exp \left\{ -\frac{k_0^2 \sigma_h^2}{2} (\theta y_1 - \theta y_2)^2 \right\} \frac{1}{\sqrt{\mathcal{S}^{2}}} 
\]

\[
\exp \left\{ \frac{k_0^2 \rho_0^2}{4 \mathcal{S}^{2}} \left( 1 - \rho_0^2 \ell_h^0 (\hat{\mathbf{x}} \cdot \hat{\mathbf{u}}) \right)^2 \left( \theta y_1^2 + \theta y_2^2 \right) \right\} \exp \left\{ \frac{k_0^2 \rho_0^2}{2 \mathcal{S}^{2}} \left( \theta y_1 - \theta y_2 \right)^2 \right\}
\]

\[
\left\{ \begin{array}{c}
\text{erf} \left\{ \frac{L_0 |\hat{\mathbf{x}} \cdot \hat{\mathbf{u}}|}{\rho_s} \sqrt{2 \mathcal{S}^{2} + k_0 |\hat{\mathbf{x}} \cdot \hat{\mathbf{u}}| \rho_0^2 \ell_h^0 (\theta y_1 + \theta y_2) + j \rho_s |\hat{\mathbf{x}} \cdot \hat{\mathbf{u}}| \sqrt{\mathcal{S}^{2}} (\theta y_1 - \theta y_2) \right\} \\
+ \text{erf} \left\{ \frac{L_0 |\hat{\mathbf{x}} \cdot \hat{\mathbf{u}}|}{\rho_s} \sqrt{2 \mathcal{S}^{2} - k_0 |\hat{\mathbf{x}} \cdot \hat{\mathbf{u}}| \rho_0^2 \ell_h^0 (\theta y_1 + \theta y_2) - j \rho_s |\hat{\mathbf{x}} \cdot \hat{\mathbf{u}}| \sqrt{\mathcal{S}^{2}} (\theta y_1 - \theta y_2) \right\}
\end{array} \right.
\]

where \( \mathcal{S} = k_0 \rho_0^2 \ell_h^0 (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}) \) and \( \mathcal{S} = 4 \mathcal{S} (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}) + \rho_0^2 \ell_h^0 (\hat{\mathbf{x}} \cdot \hat{\mathbf{u}})^2 \). If Eq. (21) is satisfied, then the sum of the erf’s \( \approx 2 \).

The above expression for the scattered field CSD function is remarkably physical. The first exponential term on the second line of Eq. (23) is predominately responsible for the angular extent of the scattered SD. This exponential term is a function of the sum of the squares of the projected observation angles \( \theta y_1 \) and \( \theta y_2 \). The exponential term on the third line of Eq. (23) determines the SDoC radius. Note that this term is a function of the difference of the projected observation angles, i.e., \( [\theta y_1 - \theta y_2] = [\sin \theta_1' - \sin \theta_2'] \). Thus, it could be stated that the scattered field CSD function maintains its GSM form with respect to \( \theta y_1 \) and \( \theta y_2 \) if Eq. (21) is satisfied.

Because of the magnitude of the argument of the “correlation” exponential (specifically, the \( k_0^2 \rho_0^2 \sigma_h^2 \theta y_1 \theta y_2 \) term), \( \sin \theta y_1' \approx \sin \theta y_2' \), implying that \( \theta y_1' \approx \theta y_2' \), for the exponential to have a significant value. This implies that the correlation exponential is approximately a function of \( \Delta \theta' = \theta y_1' - \theta y_2' \). Since this term is predominately responsible for the behavior of the SDoC, the SDoC is also a function (approximately) of \( \Delta \theta' = \theta y_1' - \theta y_2' \). This is in agreement with the findings of previous studies which were restricted to the paraxial regime.
2.3. Angular spectral degree of coherence radius

In this section, a theoretical form for the angular SDoC radius is derived from Eq. (23). Note that because of the summation in Eq. (20), it is not possible to derive an expression for the angular SDoC radius using that form of the scattered field CSD function.

Assuming Eq. (21) is satisfied and setting the exponential term on line three of Eq. (23) equal to 1/e results in the following expression:

$$|\hat{\vartheta}_1 - \hat{\vartheta}_2|_{1/e} = |\sin \theta'_1 - \sin \theta'_2|_{1/e} \approx \left| \mathbf{\hat{s}} \cdot \mathbf{\hat{u}} \right| \sqrt{8 \frac{w_s^2 \ell_s^2}{\ell_s^2 + 4w_s^2} + 2 \frac{\rho_s^2 \ell_s^2 (\mathbf{\hat{s}} \cdot \mathbf{\hat{u}})^2}{k_0^2} \left( \ell_s^2 w_s^2 (\mathbf{\hat{s}} \cdot \mathbf{\hat{u}})^2 + 2 \rho_s^2 \sigma_h^2 \hat{\vartheta}_1 \hat{\vartheta}_2 \right)}.$$  (24)

As discussed above, the magnitude of the argument of the correlation exponential term is very large; thus, \(\theta'_1 \approx \theta'_2\) for the correlation term to have a significant value. Note that this physically implies that the scattered field is correlated for observation points separated by very small angles. The fact that \(\theta'_1 \approx \theta'_2\) motivates setting \(\theta'_1 = \theta'_2 + \Delta \theta'\) and expanding \(\sin \theta'_1\) in a Taylor series about \(\theta'_2 = \theta'\). After some simple algebra, the expression for the angular SDoC radius becomes

$$|\Delta \theta'|_{1/e} \approx \frac{\cos \theta'}{\rho_s \cos \theta'} \sqrt{8 \frac{w_s^2 \ell_s^2}{\ell_s^2 + 4w_s^2} + 2 \frac{1}{k_0^2 \sigma_h'^2 (1 + \cos \theta' / \cos \theta'_0)^2} \left( \ell_s^2 w_s^2 (\mathbf{\hat{s}} \cdot \mathbf{\hat{u}})^2 + 2 \rho_s^2 \sigma_h^2 \hat{\vartheta}_1 \hat{\vartheta}_2 \right)}.$$  (25)

where \(\alpha = \ell_s/w_s\) (previously defined), \(\Omega = w_s/\rho_s\) is the half-angle subtended by the source viewed from the rough surface, and \(\sigma_h' = \sqrt{2} \sigma_h/\ell_s\) is the surface slope standard deviation [22, 23]. Note that the same procedure applied to \(\hat{\vartheta}_1\) and \(\hat{\vartheta}_2\) (discussed above) has been applied to \(\hat{\vartheta}_1\) and \(\hat{\vartheta}_2\) in deriving the above expression.

The term involving \(\Omega\) and \(\sigma_h'\) can be neglected when one considers that typical values for these parameters range from approximately \(10^{-4}\) rad and 0.05–0.5 rad, respectively. Note that the first term under the radical contains only source parameters, while the second term contains mostly rough surface parameters. In many scenarios of interest the source term is much greater than the rough surface term; thus, factoring out the source term and expanding the resulting radical in a Taylor series yields

$$|\Delta \theta'|_{1/e} \approx 2\Omega \sqrt{\frac{2}{1 + (2/\alpha)^2} \cos \theta'} + \Omega \sqrt{\frac{2}{k_0^2 w_s^2 4 \sigma_h'^2 (1 + \cos \theta' / \cos \theta'_0)^2} \cos \theta'}.$$  (26)

The first term is entirely composed of source parameters and therefore represents the source contribution to the angular SDoC radius. Because of the presence of \(\sigma_h'\), the second term provides a small correction to the angular SDoC radius due to the roughness of the surface. In most cases, the angular SDoC radius can be safely approximated by utilizing only the first term making the expression dependent only on the properties of the source illumination. This is consistent with the classic narrow-band, fully-coherent result derived by Goodman [26]. Note that when \(\theta' \approx \theta'\), i.e., when observation is in the specular direction, the dependence of the expression on observation and incident angles disappears.
2.4. Angular spectral density radius

As is the case with the angular SDoC radius, an expression for the angular SD radius can only be derived using the very rough surface form of the scattered field CSD function in Eq. (23). Furthermore, the expression can only be derived for near-normal incidence.

Assuming Eq. (21) is satisfied, letting \( \theta' = 0 \) and \( \theta'_\parallel = \theta'_\perp = \theta'_{1/e} \), and setting the first exponential term on line two of Eq. (23) equal to 1/e yields the following expression after some simplification:

\[
\sin^2 \theta'_{1/e} = 2\Omega^2 + 2\sigma^2_e \left( 1 + \cos \theta'_{1/e} \right) + \frac{1 + (2/\alpha)^2}{2k_0^2 w_s^2},
\]

where all symbols have been previously defined. With liberal use of trigonometric identities and some algebra, one arrives at

\[
\cos^4 \left( \frac{\theta'_{1/e}}{2} \right) - \frac{1}{1 + 2\sigma^2_e} \cos^2 \left( \frac{\theta'_{1/e}}{2} \right) + \frac{2\Omega^2 + \frac{1 + (2/\alpha)^2}{2k_0^2 w_s^2}}{4 + 8\sigma^2_e} = 0.
\]

One quickly recognizes that the above expression is a quadratic equation in terms of \( \cos^2 \left( \frac{\theta'_{1/e}}{2} \right) \). Letting \( \psi = \cos^2 \left( \frac{\theta'_{1/e}}{2} \right) \), implying that \( \theta'_{1/e} = 2\cos^{-1} \sqrt{\psi} \) (only the positive \( \psi \) root makes physical sense), and utilizing the quadratic formula, yields the following roots for \( \psi \):

\[
\psi = \frac{1}{2 + 4\sigma^2_e} \left[ 1 \pm \sqrt{1 - \left( 1 + 2\sigma^2_e \right) \left( 2\Omega^2 + \frac{1 + (2/\alpha)^2}{2k_0^2 w_s^2} \right)} \right].
\]

The second term in the radicand is much less than one; therefore, only the “plus” root has significant value. Choosing the “plus” root, expanding the radical in a Taylor series, and applying \( \theta'_{1/e} = 2\cos^{-1} \sqrt{\psi} \) produces

\[
\theta'_{1/e} \approx 2\cos^{-1} \left[ \frac{1}{1 + 2\sigma^2_e} \sqrt{1 - \frac{1}{4} \left( 1 + 2\sigma^2_e \right) \left( 2\Omega^2 + \frac{1 + (2/\alpha)^2}{2k_0^2 w_s^2} \right)} \right].
\]

In Eq. (30), the first term in the argument of the inverse cosine depends solely on the characteristics of the rough surface and it physically represents the rough surface contribution to the angular extent of the SD. The second argument contains terms dealing with the source parameters and provides a small correction to the angular SD radius due to the size and coherence of the source. In most cases, the angular SD radius can be approximated using only the first term making the expression solely dependent on the roughness of the surface.

2.5. Parallel-polarization cross-spectral density function

Considering that the rough surface’s features are large compared to wavelength, intuition dictates that the p-pol scattered field should have the same general behavior as the s-pol scattered field. This is the case. The only difference between the s-pol and p-pol CSD functions is the parameter \( T \). Setting \( T^\perp = T^\parallel \) in Eqs. (20) and (23) yields the p-pol scattered field CSD functions. Expressions for \( T^\parallel \) and \( T^\perp \) can be found in the appendix.
3. Validation of analytical solutions

In this section, the analytical solutions for the scattered field CSD functions derived above are validated via experiments, full-wave electromagnetic simulations, and PO simulations. For the experimental validation, the Complete Angle Scatter Instrument (CASI) [60] at the Air Force Institute of Technology was used. Measurements of the scattered SD $S^\theta$ versus $\theta^\prime$ from a LabSphere Infragold [61] 5.08 cm $\times$ 5.08 cm coupon were made at $\lambda = 3.39 \mu m$ (HeNe MWIR laser) and $\theta^\prime = 20^\circ$, $40^\circ$, and $60^\circ$. Before the scattering measurements were made, the surface statistics of the Infragold coupon were determined using a KLA Tencor Alpha-Step IQ surface profiler [62]. Four 1 cm scans (step size 0.2 $\mu m$), performed along different directions, were taken and analyzed. The measured surface height standard deviation $\sigma$, surface slope standard deviation $\sigma^\prime$, and correlation length $\ell_\theta$ were 11.09 $\mu m$, 0.2441 rad, and 116.9 $\mu m$, respectively.

For the full-wave electromagnetic simulations, the Method of Moments (MoM) [63] was used to solve the following impedance-boundary-condition-based s-pol electric field integral equation for the unknown electric current $J$:

$$
E^s_z (\mathbf{r}) = \frac{\mu}{\varepsilon} J_z (\mathbf{r}) + j k_0 \sqrt{\mu_0 \varepsilon_0} A^s_z (\mathbf{r}) + \left( \frac{\partial F_y (\mathbf{r})}{\partial x} - \frac{\partial F_x (\mathbf{r})}{\partial y} \right) \mathbf{r} \in C,
$$

where $A$ and $F$ are the magnetic and electric vector potentials, respectively [64]. Once $J$ had been found, the scattered field was determined via convolution with the far-field form of the free-space Green’s function. A narrow-band, fully-coherent Gaussian beam was used for the incident field $E^i_z$, $S^\theta$ versus $\theta^\prime$ and the modulus of the SDoC $|\mu^s|$ versus $\Delta \theta^\prime$ were calculated using the scattered field predicted from 500 rough surface realizations at $\lambda = 3.39 \mu m$, the Infragold surface statistics stated above, and a complex index of refraction $n - jk$ for gold of 1.995 $-$ j20.95 [65]. The 500 rough surface realizations were generated using the method outlined by Yura and Hanson [66].

Both the experimental and full-wave simulations used narrow-band, fully-coherent incident fields. To verify the predictions of the analytical solutions for partially-coherent incident fields, PO Monte Carlo simulations were also performed. To model the random nature of the incident field, 500 source plane field instances were generated using the phase screen method described by Xiao and Voelz [67]. For the random surface, 500 rough surface instances were generated using the technique detailed in Ref. [66]. The 500 source plane field instances were propagated to and evaluated at each rough surface instance. The scattered fields were then computed numerically using the relations in Eq. (5). The end result of this process was a set of 500$^2$ scattered fields, from which $|\mu^s|$ and $S^\theta$ were computed. These simulations were performed at $\lambda = 3.39 \mu m$ using hypothetical rough aluminum surfaces $n - jk = 5.366 - j33.33$ [65] with varying degrees of surface roughness. In addition, the properties of the incident field (size and coherence) were also varied to examine their effects.

3.1. Results

Figure 2 shows the normalized $S^\theta$ CASI measurement, MoM simulation, PO analytical solution, and PO simulation results versus $\theta^\prime$. The simulations were setup to best match the CASI experimental scenario, i.e., $\rho_s = 185$ cm and $\omega_s = 0.5$ mm. For MoM computational reasons, $L = 2.5$ mm, which resulted in an incident beam which over-illuminated the rough surface by less than 2%. While all the traces possess a Gaussian-like shape, there is a significant difference between the measured and MoM trace widths versus the PO results. As briefly discussed in Section 2.2, previous work had found that the SE-SE model more accurately represented surfaces roughened by random industrial processes [57]. If the Infragold sample measured and analyzed here adhered to the G-G model of rough surfaces, $\sigma^\prime = \sqrt{2} \sigma / \ell_\theta \approx 0.1342$ rad. The
measured $\sigma_h'$ was roughly twice that value. Thus, in this case, the G-G model is not an ideal choice. Note that in these MoM simulation results, SE-SE surfaces were used, hence the good agreement with the CASI results. SE-SE surfaces could have been used in the PO simulations; however, as discussed previously, it would not have been possible to present analytical results. These results are presented for two main reasons. The first is to verify the analytical solutions which, based on the excellent agreement between the PO simulation and PO analytical results, is clearly accomplished. The other is to demonstrate the validity of the ubiquitous G-G model of rough surfaces. As can be seen, the model tends to underpredict the width of the scattered SD. Nevertheless, considering that analytical scattering solutions are possible using the G-G model, this is likely an acceptable drawback.

To truly validate the PO analytical solutions, MoM simulations were also performed using hypothetical Infragold G-G surfaces. These results, along with PO simulation and PO analytical solution results, are shown in Fig. 3. The source distance $p_s$, source radius $w_s$, and surface 1/2 length $L$ were the same as above. The figure reports the normalized $S'$ versus $\theta'$ in Fig. 3(a) and

![Figure 2](image-url)
Fig. 3. MoM simulation, analytical PO solution, and PO simulation results of normalized $S'_s$ versus $\theta'$ [(a)] and $|\mu'|$ versus $\Delta \theta'$ [(b) and (c)] for hypothetical Infragold G-G surfaces. In (a) and (b), the hypothetical Infragold surface possessed the measured $\sigma_h$ and $\ell_h$ values of 11.09 $\mu$m and 116.9 $\mu$m, respectively. Since this hypothetical surface qualifies as a very rough surface, $\theta_{1/e}'$ and $|\Delta \theta'|_{1/e}$, given in Eqs. (30) and (26), are also plotted as vertical dashed line in (a) and (b), respectively. In (c), $\sigma_h$ is varied while $\ell_h$ is held constant at $8\lambda$. |

In (a) and (b), the theoretical angular SD and SDoC radii, given in Eqs. (30) and (26), respectively, are also plotted as vertical dashed line in Figs. 3(a) and 3(b). Note the excellent agreement between the MoM simulation, PO simulation, and analytical PO solution results. In Fig. 3(c), $|\mu'|$ is shown for hypothetical Infragold G-G surfaces which possessed $\sigma_h = 0.05\lambda$, 0.1$\lambda$, and 0.25$\lambda$ with $\ell_h$ held constant at $8\lambda$. Surfaces with these parameters do not qualify as very rough surfaces and therefore, the angular SDoC radii are not shown. Again, note the very good agreement among the results.

Having validated the PO analytical solutions (as well as the PO simulation procedure) via experiment and full-wave electromagnetic simulations which utilized narrow-band, fully-coherent incident fields, attention can now be turned to validating the PO solutions via PO simulations...
Fig. 4. $S_s$ versus $\theta_r$ at normal incidence, i.e., $\theta_i = 0^\circ$—(a) $\alpha = 2$, (b) $\alpha = 1$, (c) $\alpha = 0.5$, and (d) $\alpha = 0.25$. The solid traces are the PO analytical predictions, i.e., Eq. (20) or Eq. (23) (whichever is applicable); the circles are the PO simulation results. The vertical dashed lines in the figures mark the locations of $\theta_{1/e}$, namely, Eq. (30).

using partially-coherent incident fields. These results are reported in Figs. 4 and 5. Figure 4 shows the theoretical (solid curves) and simulated (circles) $S_s$ for $\sigma_m = 0.01 \text{ rad}$, 0.05 rad, 0.1 rad, and 0.25 rad (with $\ell_h = 20\lambda$, $\sigma_h = \ell_h \sigma_m / \sqrt{2}$, and $L = 0.35 \text{ m}$) versus $\theta'$ at normal incidence. Figures 4(a)–4(d) depict these curves for $\alpha = 2$, 1, 0.5, and 0.25 (with $w_s = 5 \text{ mm}$), respectively, i.e., a coherent source to a relatively incoherent source. The vertical dashed lines in the figures mark the locations of $\theta_{1/e}$ given by Eq. (30). Note the excellent agreement between the simulated and theoretical predictions. Also note that although the coherence properties of the source are very different for the curves plotted in the figures, the $S_s$ for the very rough surface curves $\sigma_m = 0.05 \text{ rad}$, 0.1 rad, and 0.25 rad are nearly identical. This physically implies that $S_s$ (or the angular spread of the spectral density) for very rough surfaces is driven by surface properties not source parameters. Recall that this was theoretically predicted.

Figure 5 shows the theoretical (solid traces) and simulated (circles) $|\mu'|$ for $\alpha = 2$, 1, 0.5, and 0.25 (with the same $w_s$ as above) versus $\Delta \theta'$. Figures 5(a)–5(d) show these curves for $\sigma_m = 0.01 \text{ rad}$, 0.05 rad, 0.1 rad, and 0.25 rad (with the same $\ell_h$, $\sigma_h$, and $L$ as above), respectively,
Fig. 5. $|\mu|\text{ versus } \Delta \theta = (a) \sigma_{\mu} = 0.01 \text{ rad}, (b) \sigma_{\mu} = 0.05 \text{ rad}, (c) \sigma_{\mu} = 0.1 \text{ rad}, \text{ and (d) } \sigma_{\mu} = 0.25 \text{ rad. The solid traces are the PO analytical predictions, i.e., Eq. (20) or Eq. (23) (whichever is applicable); the circles are the PO simulation results. The vertical dashed lines in the figures mark the locations of } |\Delta \theta|_{1/e}, \text{ namely, Eq. (26).}

i.e., a smooth surface to a very rough surface. The vertical dashed lines in the figures mark the locations of $|\Delta \theta|_{1/e}$ given by Eq. (26). Recall that Eq. (26) is applicable to very rough surfaces, a condition not met by the surfaces whose results are shown in Fig. 5(a). Note the excellent agreement between the Monte Carlo simulation results and the theoretical predictions. Also note that $|\Delta \theta|_{1/e} \text{ depend almost exclusively on the coherence properties of the source } \alpha, \text{ as predicted.}

4. Conclusion

The scattering of a partially-coherent field from a statistically rough surface was investigated. This work significantly extended previous efforts by incorporating the effects of material parameters ($\varepsilon$ and $\mu$) in the analysis and by rigorously validating the derived solutions via experiment and extensive simulation. Two forms of the scattered-field CSD function were derived in this work using the PO approximation. The first, applicable to smooth-to-moderately rough surfaces, was represented as an infinite series. While being a rather complicated expression...
dependent on both source and surface parameters, physical insight was gleaned from its analytical form. The second form of the scattered field CSD function was applicable to very rough surfaces. This expression was examined at length to include derivations of the angular SDoC and SD radii. It was noted that under certain circumstances, this form of the CSD function maintained a GSM form in agreement with published results valid only in the paraxial regime. Lastly, the closed-form expressions for the scattered field CSD functions were rigorously validated with scatterometer measurements, as well as, MoM and PO Monte Carlo simulations. The analytical predictions were found to be in good agreement with the measurement and simulation results.

Appendix

The symbols first used in Eq. (12) are

\[ \mathcal{I}^+ = \mathcal{H}_{1,2}^{+} \]

\[ \mathcal{H}_{1,2} = \left( \hat{x} \cdot \hat{u} - \frac{\partial_{x_{1,2}}}{\partial_{y_{1,2}}} \hat{y} \cdot \hat{u} + \frac{\partial_{x_{1,2}}}{\partial_{y_{1,2}}} \hat{y} \cdot \hat{x} , \hat{x} \cdot \hat{p}_{1,2} + \frac{\partial_{x_{1,2}}}{\partial_{y_{1,2}}} \hat{x} \cdot \hat{p}_{1,2} \right) \],

(32)

where

\[ \partial_{x_{1,2}} = \hat{x} \cdot \hat{p}_{1,2} - \hat{x} \cdot \hat{v} \]

\[ \partial_{y_{1,2}} = \hat{y} \cdot \hat{p}_{1,2} - \hat{y} \cdot \hat{v} \]

\[ a = \frac{1}{4w_{1}^2} + \frac{1}{2l_{1}^2} \]

\[ b = \frac{1}{2l_{1}^2} \]

(33)

For the p-pol case,

\[ \mathcal{I}^{\perp} = \mathcal{H}_{1,2}^{x_{1,2}} \]

\[ \mathcal{H}_{1,2}^{\perp} = \left( \hat{y} \cdot \hat{p}_{1,2} + \frac{\partial_{x_{1,2}}}{\partial_{y_{1,2}}} \hat{x} \cdot \hat{p}_{1,2} - \hat{x} \cdot \hat{u} + \frac{\partial_{x_{1,2}}}{\partial_{y_{1,2}}} \hat{x} \cdot \hat{u} \right) \],

(34)

\[ -r_{\perp,1,2} \left( \hat{y} \cdot \hat{p}_{1,2} + \frac{\partial_{x_{1,2}}}{\partial_{y_{1,2}}} \hat{x} \cdot \hat{p}_{1,2} + \hat{x} \cdot \hat{u} - \frac{\partial_{x_{1,2}}}{\partial_{y_{1,2}}} \hat{x} \cdot \hat{u} \right) \]

where \( r_{\perp} \) is the p-pol Fresnel complex amplitude reflection coefficient. Note that \( r_{\perp,\parallel} = -1 \) and 1 for a perfect electric conductor (i.e., a perfect reflector) and a perfect magnetic conductor, respectively.

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