MULTIPLE-INTEGRAL REPRESENTATIONS
OF VERY-WELL-POISED HYPERGEOMETRIC SERIES

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The aim of this note is to connect two objects: very-well-poised hypergeometric series

\[ F_k(h) = F_k(h_0; h_1, \ldots, h_k) := \frac{\Gamma(1 + h_0) \prod_{j=1}^{k} \Gamma(h_j)}{\prod_{j=1}^{k} \Gamma(1 + h_0 - h_j)} \times k+2F_{k+1}\left(h_0, 1 + \frac{1}{2}h_0, h_1, \ldots, h_k\right) \left| (-1)^{k+1}\right) \]

\[ = \sum_{\mu = 0}^{\infty} (h_0 + 2\mu) \frac{\prod_{j=0}^{k} \Gamma(h_j + \mu)}{\prod_{j=0}^{k} \Gamma(1 + h_0 - h_j + \mu)} (-1)^{(k+1)\mu}, \]  

(1)

and multiple integrals

\[ J_k(a, b) = J_k\left(a_0, a_1, \ldots, a_k \right) \]

\[ := \int \cdots \int_{[0,1]^k} \frac{\prod_{j=1}^{k} x_j^{a_j-1}(1 - x_j)^{b_j - a_j - 1}}{Q_k(x_1, x_2, \ldots, x_k)^{a_0}} \, dx_1 \, dx_2 \cdots dx_k \]  

(2)

with \( Q_0 := 1 \) and

\[ Q_k = Q_k(x_1, x_2, \ldots, x_k) := 1 - (1 - (\cdots (1 - (1 - x_k)x_{k-1}) \cdots) x_2)x_1 \]

\[ = 1 - x_1Q_{k-1}(x_2, \ldots, x_k) = Q_{k-1}(x_1, \ldots, x_{k-1}) + (-1)^k x_1x_2 \cdots x_k \]  

(3)

for \( k \geq 1. \)

\[ ^1 \text{An extract from my contribution “Well-poised hypergeometric service for diophantine problems of zeta values” to Actes des 12èmes rencontres arithmétiques de Caen.} \]
Theorem. For each $k \geq 1$, there holds the identity

$$
\prod_{j=1}^{k+1} \frac{\Gamma(1 + h_0 - h_j - h_{j+1})}{\Gamma(h_1) \Gamma(h_{k+2})} \cdot F_{k+2}(h_0; h_1, \ldots, h_{k+2}) = J_k \left( h_1, h_2, h_3, \ldots, h_{k+1} \right),
$$

provided that

$$1 + \text{Re } h_0 > \frac{2}{k + 1} \cdot \sum_{j=1}^{k+2} \text{Re } h_j,$$

$$\text{Re}(1 + h_0 - h_{j+1}) > \text{Re } h_j > 0 \quad \text{for } j = 2, \ldots, k + 1,$$

$$h_1, h_{k+2} \neq 0, -1, -2, \ldots.$$

Remark. Condition (5) is required for the absolute convergence of the series (1) in the unit circle (and, in particular, at the point $(-1)^{k+1}$), while condition (6) ensures the convergence of the corresponding multiple integral (2). The restriction (7) can be removed by the theory of analytic continuation if we write $\Gamma(h_j + \mu)/\Gamma(h_j)$ for $j = 1, k + 2$ as Pochhammer’s symbol $(h_j)_\mu$ when summing in (1).

In the case of integral parameters $h$, the quantities (1) are known to be $\mathbb{Q}$-linear forms in even/odd zeta values depending on evenness/oddness of $k \geq 4$. Therefore, if positive integral parameters $a$ and $b$ satisfy the additional condition

$$b_1 + a_2 = b_2 + a_3 = \cdots = b_{k-1} + a_k,$$

then the quantities (2) are $\mathbb{Q}$-linear forms in even/odd zeta values. Specialization $a_j = n + 1$ and $b_j = 2n + 2$ gives one the coincidence of multiple integrals and well-poised hypergeometric series conjectured by us in [Zu2, Section 9]. Denoting the corresponding integrals by $J_{k,n}$ and using our arithmetic results [Zu1, Lemmas 4.2–4.4] we then conclude that

$$D_{n+1}^{k+1} \Phi_n^{-1} J_{k,n} \in \mathbb{Z} \zeta(k) + \mathbb{Z} \zeta(k-2) + \cdots + \mathbb{Z} \zeta(3) + \mathbb{Z} \quad \text{for } k \text{ odd},$$

where

$$\Phi_n := \prod_{p < n} p, \quad \lim_{n \to \infty} \frac{\log \Phi_n}{n} = \psi(1) - \psi\left(\frac{2}{3}\right) - \frac{1}{2} = 0.24101875 \ldots$$
(\{\cdot\}\) denotes the fractional part of a number), that is closed enough to Vasilyev’s conjectural inclusions [V]. The choice \(a_j = rn + 1\) and \(b_j = (r + 1)n + 2\) in (2) (or, equivalently, \(h_0 = (2r + 1)n + 2\) and \(h_j = rn + 1\) for \(j = 1, \ldots, k + 2\) in (1)) with the integer \(r \geq 1\) depending on a given odd integer \(k\) presents almost the same linear forms in odd zeta values as considered by T. Rivoal in [R] for proving his remarkable result on infiniteness of irrational numbers in the set \(\zeta(3), \zeta(5), \zeta(7), \ldots\).

In addition, we have to mention, under hypothesis (8), the obvious stability of the quantity

\[
\frac{F_{k+2}(h_0; h_1, \ldots, h_{k+2})}{\prod_{j=1}^{k+2} \Gamma(h_j)} = \frac{J_k(a, b)}{\prod_{j=1}^{k+1} \Gamma(h_j) \cdot \prod_{j=1}^{k+1} \Gamma(1 + h_0 - h_j - h_{j+1})} \cdot \frac{J(a, b)}{\prod_{j=1}^{k} \Gamma(a_j) \cdot \Gamma(b_1 + a_2 - a_0 - a_1) \cdot \prod_{j=1}^{k} \Gamma(b_j - a_j)}
\]

under the action of the \((h\text{-trivial})\) group \(\mathfrak{G}\) of order \((k + 2)!\) containing all permutations of the parameters \(h_1, \ldots, h_{k+2}\). This fact can be applied for number-theoretical applications. In the cases \(k = 2\) and \(k = 3\) the change of variables \((x_{k-1}, x_k) \mapsto (1 - x_k, 1 - x_{k-1})\) in (2) produces an additional transformation \(c\) of both (2) and (1); for \(k \geq 4\) this transformation is not yet available since condition (8) is broken. The groups \(\langle \mathfrak{G}, c \rangle\) of orders 120 and 1920 for \(k = 2\) and \(k = 3\) respectively are known [RV1], [RV2]; G. Rhin and C. Viola make a use of these groups to discover nice estimates for the irrationality measures of \(\zeta(2)\) and \(\zeta(3)\). Finally, we want to note that the group \(\mathfrak{G}\) can be easily interpreted as the permutation group of the parameters

\[
e_{0l} = h_l - 1, \quad 1 \leq l \leq k + 2, \quad e_{jl} = h_0 - h_j - h_l, \quad 1 \leq j < l \leq k + 2
\]

(see [Zu2, Section 9] for details).

**Lemma 1.** Theorem is true if \(k = 1\).

**Proof.** Thanks to a limiting case of Dougall’s theorem,

\[
F_3(h_0; h_1, h_2, h_3) = \frac{\Gamma(h_1) \Gamma(h_2) \Gamma(h_3) \Gamma(1 + h_0 - h_1 - h_2 - h_3)}{\Gamma(1 + h_0 - h_1 - h_2) \Gamma(1 + h_0 - h_1 - h_3) \Gamma(1 + h_0 - h_2 - h_3)}
\]

(see, e.g., [B, Section 4.4, formula (1)]), provided that \(1 + \Re h_0 > \Re(h_1 + h_2 + h_3)\) and \(h_j\) is not a non-positive integer for \(j = 1, 2, 3\). On the other hand, the integral on the right of (4) has Euler type, that is

\[
J_1\left(\begin{array}{c} h_1, \ h_2 \\ 1 + h_0 - h_3 \end{array}\right) = \int_0^1 x^{h_2 - 1}(1 - x)^{h_0 - h_2 - h_3} \frac{1 - x^{h_1}}{(1 - x)^{h_1}} \, dx
\]

\[
= \frac{\Gamma(h_2) \Gamma(1 + h_0 - h_1 - h_2 - h_3)}{\Gamma(1 + h_0 - h_1 - h_3)}
\]
provided that $1 + \text{Re} \, h_0 > \text{Re} (h_1 + h_2 + h_3)$ and $\text{Re} \, h_2 > 0$. Therefore, multiplying equality (9) by the required product of gamma-functions we deduce identity (4) if $k = 1$.

Remark. If we arrange about $J_0(a_0)$ to be 1, the claim of Theorem remains valid if $k = 0$ thanks to another consequence of Dougall’s theorem [B, Section 4.4, formula (3)].

Lemma 2 [N, Section 3.2]. Let $a_0, a, b \in \mathbb{C}$ and $t_0 \in \mathbb{R}$ be numbers satisfying the conditions

$$\text{Re} \, a_0 > t_0 > 0, \quad \text{Re} \, a > t_0 > 0, \quad \text{and} \quad \text{Re} \, b > \text{Re} \, a_0 + \text{Re} \, a.$$ 

Then for any non-zero $z \in \mathbb{C} \setminus (1, +\infty)$ the following identity holds:

$$\int_0^1 x^{a-1} (1-x)^{b-a-1} \frac{\Gamma(b-a)}{(1-zx)^{a_0}} \, dx = \frac{\Gamma(b-a)}{\Gamma(a_0)} \cdot \frac{1}{2\pi i} \int_{-t_0-i\infty}^{-t_0+i\infty} \frac{\Gamma(a_0+t) \Gamma(a+t) \Gamma(-t)}{\Gamma(b+t)} (-z)^t \, dt, \quad (10)$$

where $(-z)^t = |z|^t e^{it \text{arg}(-z)}$, $-\pi < \text{arg}(-z) < \pi$ for $z \in \mathbb{C} \setminus [0, +\infty)$ and $\text{arg}(-z) = \pm \pi$ for $z \in (0, 1]$. The integral on the right-hand side of (10) converges. In addition, if $|z| \leq 1$, both integrals in (10) can be identified with the absolutely convergent Gauss hypergeometric series

$$\frac{\Gamma(a) \Gamma(b-a)}{\Gamma(b)} \cdot {}_2F_1 \left( \begin{matrix} a_0, a \\ b \end{matrix} \middle| \frac{z}{b} \right) = \frac{\Gamma(b-a)}{\Gamma(a_0)} \sum_{\nu=0}^{\infty} \frac{\Gamma(a_0+\nu) \Gamma(a+\nu)}{\nu! \Gamma(b+\nu)} z^\nu.$$

Set $\varepsilon_k = 0$ for $k$ even and $\varepsilon_k = 1$ or $-1$ for $k$ odd.

Lemma 3. For each integer $k \geq 2$, there holds the relation

$$J_k \left( \begin{matrix} a_0, a_1, \ldots, a_{k-1}, a_k \\ b_1, \ldots, b_{k-1}, b_k \end{matrix} \right) = \frac{\Gamma(b_k-a_k)}{\Gamma(a_0)} \cdot \frac{1}{2\pi i} \int_{-t_0-i\infty}^{-t_0+i\infty} \frac{\Gamma(a_0+t) \Gamma(a_k+t) \Gamma(-t)}{\Gamma(b_k+t)} \cdot e^{\varepsilon_k \pi i t} \cdot J_{k-1} \left( \begin{matrix} a_0 + t, a_1 + t, \ldots, a_{k-1} + t \\ b_1 + t, \ldots, b_{k-1} + t \end{matrix} \middle| \frac{z}{b_k} \right) dt,$$

provided that $\text{Re} \, a_0 > t_0 > 0$, $\text{Re} \, a_k > t_0 > 0$, $\text{Re} \, b_k > \text{Re} \, a_0 + \text{Re} \, a_k$, and the integral on the left converges.

Proof. We start with mentioning that the first recursion in (3) and inductive arguments yield the inequality

$$0 < Q_k(x_1, x_2, \ldots, x_k) < 1 \quad \text{for} \quad (x_1, x_2, \ldots, x_k) \in (0, 1)^k \quad \text{and} \quad k \geq 1. \quad (11)$$
By the second recursion in (3), 
\[ Q_k = Q_{k-1} \cdot (1 - z_k) \]
for \( k \geq 2 \), where
\[ z = \frac{(-1)^{k+1}x_1 \cdots x_{k-1}}{Q_{k-1}(x_1, \ldots, x_{k-1})}. \]

For each \((x_1, \ldots, x_{k-1}) \in (0, 1)^{k-1}\), the number \( z \) is real with the property \( z < 0 \) for \( k \) even and \( 0 < z < 1 \) for \( k \) odd, since in the last case we have
\[ z = \frac{x_1 \cdots x_{k-1}}{Q_{k-1}(x_1, \ldots, x_{k-2}, x_{k-1})} = \frac{x_1 \cdots x_{k-1}}{Q_{k-2}(x_1, \ldots, x_{k-2}) + x_1 \cdots x_{k-1}} < 1 \]
by (11). Therefore, splitting the integral (2) over \([0, 1]^k = [0, 1]^{k-1} \times [0, 1]\) and applying Lemma 2 to the integral
\[ \int_0^1 \frac{x_k^{a_k-1}(1 - x_k)^{b_k-a_k-1}}{(1 - z x_k)^{a_0}} \, dx_k \]
we arrive at the desired relation.

**Proof of Theorem.** The case \( k = 1 \) is considered in Lemma 1. Therefore we will assume that \( k \geq 2 \), identity (4) holds for \( k - 1 \), and, in addition, that
\[ 1 + \text{Re} h_0 > \frac{2}{k} \sum_{j=1}^{k+1} \text{Re} h_j, \quad \text{Re} h_{k+2} < 1. \tag{12} \]
The restrictions (12) can be easily removed from the final result by the theory of analytic continuation.

By the inductive hypothesis, for \( t \in \mathbb{C} \) with \( \text{Re} t < 0 \), we deduce that
\[ J_{k-1} \left( h_1 + t, \quad h_2 + t, \quad h_3 + t, \quad \ldots, \quad h_k + t \right) \]
\[ = \prod_{j=1}^{k} \frac{\Gamma(1 + h_0 - h_j - h_{j+1})}{\Gamma(1 + h_0 - h_j - h_{j+1})} \cdot F_{k+1}(h_0 + 2t; h_1 + t, \ldots, h_{k+1} + t) \]
\[ = \prod_{j=1}^{k} \frac{\Gamma(1 + h_0 - h_j - h_{j+1})}{\Gamma(1 + h_0 - h_j - h_{j+1})} \cdot \frac{1}{2\pi i} \int_{-s_0 - i\infty}^{-s_0 + i\infty} (h_0 + 2t + 2s) \]
\[ \times \frac{\Gamma(h_0 + 2t + s) \prod_{j=1}^{k+1} \Gamma(h_j + t + s) \Gamma(-s)}{\prod_{j=1}^{k+1} \Gamma(1 + h_0 - h_j + t + s)} e^{\varepsilon_{k-1} \pi i s} \, ds, \tag{13} \]
where the real number \( s_0 > 0 \) satisfies the conditions
\[ \text{Re}(h_0 + 2t) > s_0, \quad \text{Re}(1 + \frac{1}{2}h_0 + t) > s_0, \quad \text{Re}(h_j + t) > s_0 \quad \text{for} \ j = 1, \ldots, k+1, \]
and the absolute convergence of the last Barnes-type integral follows from [N, Lemma 3]. Shifting the variable \( t + s \mapsto s \) in (13) (with a help of the equality \( e^{\varepsilon_{k+1} \pi i t} \).
\[ e^{\varepsilon_{k-1} \pi i s} = e^{\varepsilon_{k-1} \pi i (t+s) \cdot e^{\varepsilon_{1} \pi it}}, \]

applying Lemma 3, and interchanging double integration (thanks to the absolute convergence of the integrals) we conclude that

\[
J_k\left(h_1, h_2, h_3, \ldots, h_k, h_{k+1} \right)
= \frac{1}{\prod_{j=1}^{k+1} \Gamma(1 + h_0 - h_j - h_{j+1})} \\
\times \frac{1}{2\pi i} \int_{-s_1+i\infty}^{-s_1-i\infty} (h_0 + 2s) \prod_{j=1}^{k+1} \Gamma(h_j + s) e^{\varepsilon_{k-1} \pi is} \\
\times \frac{1}{2\pi i} \int_{-t_0+i\infty}^{-t_0-i\infty} \Gamma(-s + t) \Gamma(h_0 + s + t) \Gamma(-t) e^{\varepsilon_{1} \pi it} ds ds,
\]

where \( s_1 = s_0 + t_0 \). Since \( \text{Re} h_{k+2} < 1 \) and \( h_{k+2} \neq 0, -1, -2, \ldots \), the last Barnes-type integral has the following closed form by Lemma 2:

\[
\frac{1}{2\pi i} \int_{-t_0-i\infty}^{-t_0+i\infty} \Gamma(-s + t) \Gamma(h_0 + s + t) \Gamma(-t) e^{\varepsilon_{1} \pi it} ds
\]

Substituting this final expression in (14) we obtain

\[
J_k\left(h_1, h_2, h_3, \ldots, h_k, h_{k+1} \right)
= \frac{1}{\prod_{j=1}^{k+1} \Gamma(1 + h_0 - h_j - h_{j+1})} \\
\times \left( \frac{1 - i \cot \pi h_{k+2}}{2} \right) e^{-\pi is} + \left( \frac{1 + i \cot \pi h_{k+2}}{2} \right) e^{\pi is}.
\]

If \( k \) is even, we take \( \varepsilon_{k-1} = -1 \) in the first integral and \( \varepsilon_{k-1} = 1 \) in the second one. Therefore the both integrals are equal to

\[
\int_{-s_1-i\infty}^{-s_1+i\infty} (h_0 + 2s) \prod_{j=0}^{k+2} \Gamma(h_j + s) \Gamma(-s) e^{\varepsilon_{k-1} \pi is} ds = 2\pi i \cdot F_{k+2}(h_0; h_1, \ldots, h_{k+2})
\]
that gives the desired identity (4). The proof of Theorem is complete.

Another family of multiple integrals

\[ S(z) := \int \cdots \int_{[0,1]^k} \prod_{j=1}^k \frac{x_j^{a_j-1}(1-x_j)^{b_j-a_j-1}}{\prod_{i=1}^m (1-zx_1x_2\cdots x_i)^{c_i}} \, dx_1 \, dx_2 \cdots dx_k, \tag{15} \]

\[ 1 \leq r_1 < r_2 < \cdots < r_m = k, \]

is known due to works of V. Sorokin [S1], [S2]. Recently, S. Zlobin [Zl1], [Zl2] has proved (in more general settings) that the integrals (2) can be reduced to the form (15) with \( z = 1 \). Therefore, Theorem gives one a way to reduce the integrals \( S(1) \) to the very-well-poised hypergeometric series (1) under certain conditions on the parameters \( a_j, b_j, c_i, \) and \( r_i \) in (15). In addition, Zlobin [Zl1] shows that, for integral parameters in (15) satisfying natural restrictions of convergence, the integral \( S(z) \) is a \( \mathbb{Q}[z^{-1}] \)-linear combination of modified multiple polylogarithms

\[ \sum_{n_1 \geq n_2 \geq \cdots \geq n_l \geq 1} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_l^{s_l}} \quad \text{with} \quad s_j \geq 1, \; s_j \in \mathbb{Z}, \; j = 1, \ldots, l, \]

where \( 0 \leq s_1 + s_2 + \cdots + s_l \leq k \) and \( 0 \leq l \leq m \).

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