Abstract: In this study, we first establish a \((p, q)\)-integral identity involving the second \((p, q)\)-derivative, and then, we use this result to prove some new midpoint-type inequalities for twice-\((p, q)\)-differentiable convex functions. It is also shown that the newly established results are the refinements of the comparable results in the literature.

Keywords: Hermite–Hadamard inequality; \((p, q)\)-calculus; convex functions

1. Introduction

In convex functions theory, the Hermite–Hadamard (HH) inequality is very important, which was discovered by C. Hermite and J. Hadamard independently (see [1] (p. 137)).

\[
\Pi \left( \frac{\pi_1 + \pi_2}{2} \right) \leq \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(x) dx \leq \frac{\Pi(\pi_1) + \Pi(\pi_2)}{2}
\]

where \(\Pi\) is a convex function. In the case of concave mappings, the above inequality is satisfied in reverse order. For more recent refinements of Inequality (1), one can consult [2,3].

On the other hand, several works in the field of \(q\)-analysis are being carried out, beginning with Euler, in order to achieve mastery of the mathematics that drives quantum computing. The link between physics and mathematics is referred to as \(q\)-calculus. It has a wide range of applications in mathematics, including number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, and other disciplines, as well as mechanics, relativity theory, and quantum theory [4,5]. Quantum calculus also has many applications in quantum information theory, which is an interdisciplinary area that encompasses computer science, information theory, philosophy, and cryptography, among other areas [6,7]. Euler is thought to be the inventor of this significant branch of mathematics. In Newton’s work on infinite series, he used the \(q\)-parameter. Jackson [8,9] was the first to present the \(q\)-calculus, who knew calculus without limits in a methodical manner. In 1966, Al-Salam [10] introduced a \(q\)-analogue of the \(q\)-fractional integral and \(q\)-Riemann–Liouville fractional. Since then, the related research has gradually increased. In particular, in 2013, Tariboon introduced the \(\pi_1 D_q\)-difference operator and \(q\pi_1\)-integral...
in [11]. In 2020, Bermudo et al. introduced the notion of the \( \pi_2 D_q \)-derivative and \( q^{\pi_2} \)-integral in [12]. Sadjang generalized to quantum calculus and introduced the notions of post-quantum calculus or simply \((p, q)\)-calculus in [13]. Soontharanon et al. [14] studied the concepts of fractional \((p, q)\)-calculus later on. In [15], Tung and Göv gave the post-quantum variant of the \( \pi_1 D_q \)-difference operator and \( q^{\pi_1} \)-integral. Recently, in 2021, Chu et al. introduced the notions of the \( \pi_2 D_{p,q} \)-derivative and \((p, q)\)^{\pi_2}-integral in [16].

Many integral inequalities have been studied using quantum and post-quantum integrals for various types of functions. For example, in [17–22], the authors used \( \pi_1 D_q \)-derivatives and \( q^{\pi_1} \)-intervals to prove HH integral inequalities and their left-right estimates for convex and coordinated convex functions. In [23], Noor et al. presented a generalized version of quantum integral inequalities. For generalized quasi-convex functions, Nwaeze et al. proved certain parameterized quantum integral inequalities in [24]. Khan et al. proved the quantum HH inequality using the Green function in [25]. Budak et al. [26], Ali et al. [27], and Vivas-Cortez et al. [28] developed new quantum Simpson and quantum Newton-type inequalities for convex and coordinated convex functions. For quantum Ostrowski inequalities for convex and coordinated convex functions, one can consult [29,30]. Kunt et al. [31] generalized the results of [19] and proved HH-type inequalities and their left estimates using the \( \pi_1 D_{p,q} \)-difference operator and \((p, q)\)^{\pi_2}-integral. Recently, Latif et al. [32] found the right estimates of the HH-type inequalities proven by Kunt et al. [31]. To prove Ostrowski’s inequalities, Chu et al. [16] used the concepts of the \( \pi_2 D_{p,q} \)-difference operator and \((p, q)\)^{\pi_2}-integral. Recently, Vivas-Cortez et al. [33] generalized the results of [12] and proved the HH-type inequalities and their left estimates using the \( \pi_2 D_{p,q} \)-difference operator and \((p, q)\)^{\pi_2}-integral.

Inspired by the ongoing studies, we use the \((p, q)\)-integral to develop some new post-quantum midpoint-type inequalities for \((p, q)\)-differentiable convex functions. We also show that the newly developed inequalities are extensions of some previously known inequalities.

The following is the structure of this paper: Section 2 provides a brief overview of the fundamentals of \( q \)-calculus, as well as other related studies in this field. In Section 3, we go over the concepts of \((p, q)\)-calculus, as well as some recent research in this field. The midpoint-type inequalities for twice \((p, q)\)-differentiable functions via \((p, q)\)-integrals are described in Section 4. The relationship between the findings reported here and similar findings in the literature is also taken into account. Section 5 concludes with some recommendations for future research.

## 2. Quantum Derivatives and Integrals

In this portion, we recall a few known definitions in \( q \)-calculus. Throughout the paper, we use real numbers \( p \) and \( q \) such that \( 0 < q < p \leq 1 \). Set the following notation (see [5]):

\[
[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \ldots + q^{n-1},
\]

\[
(1 - s)_q^n = (s, q)_n = \prod_{i=0}^{n-1} (1 - q^i s).
\]

The \( q \)-Jackson integral of a mapping \( \Pi \) from zero to \( \pi_2 \) was given by Jackson [9], which is defined as:

\[
\int_0^{\pi_2} \Pi(\varpi) \, d_q \varpi = (1 - q)\pi_2 \sum_{n=0}^{\infty} q^n \Pi(\pi_2 q^n)
\]
provided that the sum converges absolutely. Moreover, over the interval \([\pi_1, \pi_2]\), he gave the following integral of a mapping \(\Pi\):

\[
\int_{\pi_1}^{\pi_2} \Pi(\tau) \, d_q \tau = \int_{0}^{\pi_2} \Pi(\tau) \, d_q \tau - \int_{0}^{\pi_1} \Pi(\tau) \, d_q \tau .
\]

**Definition 1** ([11]). The \(q_{\pi_1}\)-derivative of mapping \(\Pi : [\pi_1, \pi_2] \to \mathbb{R}\) is defined as:

\[
\pi_1 D_q \Pi(\tau) = \frac{\Pi(\tau) - \Pi(q\tau + (1-q)\pi_1)}{(1-q)(\tau - \pi_1)}, \quad \tau \neq \pi_1 .
\]  \(3\)

For \(\tau = \pi_1\), we state \(\pi_1 D_q \Pi(\pi_1) = \lim_{\tau \to \pi_1} \pi_1 D_q \Pi(\tau)\) if it exists and it is finite.

**Definition 2** ([12]). The \(q_{\pi_2}\)-derivative of mapping \(\Pi : [\pi_1, \pi_2] \to \mathbb{R}\) is given as:

\[
\pi_2 D_q \Pi(\tau) = \frac{\Pi(q\tau + (1-q)\pi_2) - \Pi(\tau)}{(1-q)(\pi_2 - \tau)}, \quad \tau \neq \pi_2 .
\]  \(4\)

For \(\tau = \pi_2\), we state \(\pi_2 D_q \Pi(\pi_2) = \lim_{\tau \to \pi_2} \pi_2 D_q \Pi(\tau)\) if it exists and it is finite.

**Definition 3** ([11]). The \(q_{\pi_1}\)-definite integral of mapping \(\Pi : [\pi_1, \pi_2] \to \mathbb{R}\) on \([\pi_1, \pi_2]\) is defined as:

\[
\int_{\pi_1}^{\pi_2} \Pi(\tau) \, d_q \tau = (1-q)(\tau - \pi_1) \sum_{n=0}^{\infty} q^n \Pi(q^n \tau + (1-q^n)\pi_1), \quad \tau \in [\pi_1, \pi_2].
\]  \(5\)

On the other side, the following concept of the \(q\)-definite integral was stated by Bermudo et al. [12]:

**Definition 4** ([12]). The \(q_{\pi_2}\)-definite integral of mapping \(\Pi : [\pi_1, \pi_2] \to \mathbb{R}\) on \([\pi_1, \pi_2]\) is given as:

\[
\int_{\pi_1}^{\pi_2} \Pi(\tau) \, d_q \tau = (1-q)(\pi_2 - \tau) \sum_{n=0}^{\infty} q^n \Pi(q^n \tau + (1-q^n)\pi_2), \quad \tau \in [\pi_1, \pi_2].
\]  \(6\)

3. Post-Quantum Derivatives and Integrals

In this section, we review some fundamental notions and notations of \((p, q)\)-calculus.

We set the notation (see [13]):

\[
[n]_{p,q} = \frac{p^n - q^n}{p - q} .
\]

The \([n]_{p,q}!\), \(\left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q}!\) and \((1 - s)^n_{p,q}\) are called the \((p,q)\)-factorial, \((p,q)\)-binomial and \((p,q)\)-power, respectively, and are expressed as (see [13]):

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q}! = \frac{[n]_{p,q}!}{[n-k]_{p,q}!} [k]_{p,q}! ,
\]

\[
\left[ \begin{array}{c} \frac{n}{k} \end{array} \right]_{p,q}! = \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q}! \frac{[n-k]_{p,q}!}{[k]_{p,q}!} .
\]
and:

\[(1 - s)^n_{p,q} = \prod_{i=0}^{n-1} \left(p^i - q^i s\right).\]

**Definition 5 ([13]).** The \((p,q)\)-derivative of mapping \(\Pi : [\pi_1, \pi_2] \to \mathbb{R}\) is given as:

\[
D_{p,q}\Pi(\varkappa) = \frac{\Pi(p\varkappa) - \Pi(q\varkappa)}{(p - q)\varkappa}, \quad \varkappa \neq 0.
\] (7)

**Definition 6 ([15]).** The \((p,q)_{\pi_1}\)-derivative of mapping \(\Pi : [\pi_1, \pi_2] \to \mathbb{R}\) is given as:

\[
\pi_1 D_{p,q}\Pi(\varkappa) = \frac{\Pi(p\varkappa + (1 - p)\pi_1) - \Pi(q\varkappa + (1 - q)\pi_1)}{(p - q)(\varkappa - \pi_1)}, \quad \varkappa \neq \pi_1.
\] (8)

For \(\varkappa = \pi_1\), we state \(\pi_1 D_{p,q}\Pi(\pi_1) = \lim_{\varkappa \to \pi_1} \pi_1 D_{p,q}\Pi(\varkappa)\) if it exists and it is finite.

**Definition 7 ([16]).** The \((p,q)_{\pi_2}\)-derivative of mapping \(\Pi : [\pi_1, \pi_2] \to \mathbb{R}\) is given as:

\[
\pi_2 D_{p,q}\Pi(\varkappa) = \frac{\Pi(p\varkappa + (1 - q)\pi_2) - \Pi(q\varkappa + (1 - p)\pi_2)}{(p - q)(\pi_2 - \varkappa)}, \quad \varkappa \neq \pi_2.
\] (9)

For \(\varkappa = \pi_2\), we state \(\pi_2 D_{p,q}\Pi(\pi_2) = \lim_{\varkappa \to \pi_2} \pi_2 D_{p,q}\Pi(\varkappa)\) if it exists and it is finite.

**Remark 1.** It is clear that if we use \(p = 1\) in (8) and (9), then the equalities (8) and (9) reduce to (3) and (4), respectively.

**Definition 8 ([13]).** The definite \((p,q)\)-integral of mapping \(\Pi : [0, \pi_2]\) is stated as:

\[
\int_0^{\pi_2} \Pi(\tau) \, d_{p,q}\tau = (p - q)\pi_2 \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Pi\left(\frac{q^n}{p^{n+1}}\pi_2\right).\] (10)

Moreover, the definite \((p,q)\)-integral of mapping \(\Pi : [\pi_1, \pi_2]\) is stated as:

\[
\int_{\pi_1}^{\pi_2} \Pi(\tau) \, d_{p,q}\tau = \int_0^{\pi_2} \Pi(\tau) \, d_{p,q}\tau - \int_0^{\pi_1} \Pi(\tau) \, d_{p,q}\tau.
\]

**Definition 9 ([15]).** The definite \((p,q)_{\pi_1}\)-integral of mapping \(\Pi : [\pi_1, \pi_2]\) is stated as:

\[
\int_{\pi_1}^{\pi_2} \Pi(\tau) \, d_{p,q}\tau = (p - q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Pi\left(\frac{q^n}{p^{n+1}}\pi_2 + \left(1 - \frac{q^n}{p^{n+1}}\right)\pi_1\right).\] (11)

**Definition 10 ([16]).** The definite \((p,q)_{\pi_2}\)-integral of mapping \(\Pi : [\pi_1, \pi_2]\) is stated as:

\[
\int_{\pi_2}^{\pi_1} \Pi(\tau) \, d_{p,q}\tau = (p - q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Pi\left(\frac{q^n}{p^{n+1}}\pi_2 + \left(1 - \frac{q^n}{p^{n+1}}\right)\pi_1\right).\] (12)

**Remark 2.** It is evident that if we pick \(p = 1\) in (11) and (12), then the equalities (11) and (12) change into (5) and (6), respectively.

**Remark 3.** If we take \(\pi_1 = 0\) and \(\pi_2 = 1\) in (11), then we have:

\[
\int_0^1 \Pi(\tau) \, d_{p,q}\tau = (p - q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Pi\left(\frac{q^n}{p^{n+1}}\right).
\]
Similarly, by taking \( \kappa = \pi_1 = 0 \) and \( \pi_2 = 1 \) in (12), then we obtain that:

\[
\int_0^1 \Pi(\tau)^1 d_{p,q}\tau = (p - q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1} \Pi \left(1 - \frac{q^n}{p^{n+1}}\right)}.
\]

**Lemma 1** ([33]). We have the following equalities:

\[
\int_{\pi_1}^{\pi_2} (\pi_2 - \pi_1)^{\alpha} \; d_{p,q}\pi = \frac{\alpha \pi_2^{\alpha+1}}{\alpha + 1}
\]

\[
\int_{\pi_1}^{\pi_2} (\pi - \pi_1)^{\alpha} \; d_{p,q}\pi = \frac{\alpha \pi_2^{\alpha+1}}{\alpha + 1}
\]

where \( \alpha \in \mathbb{R} - \{-1\} \).

**4. Post-Quantum Midpoint-Type Inequalities**

In this section, we prove some new midpoint-type inequalities for twice-(\(p, q\))-differentiable mappings via the \((p, q)\)-integral.

Let us begin with the following lemma.

**Lemma 2.** Let \( \Pi : I = [\pi_1, \pi_2] \rightarrow R \) be a \((p, q)\)-differentiable function on the interior of \(I\). If \( \pi_2 D_{p,q}^2 \Pi \) is continuous and integrable on \(I\), then the following equality holds:

\[
\frac{(\pi_2 - \pi_1)}{2} \left[ \int_0^{\frac{1}{\pi_2 - \pi_1}} (p^2 q + (1 - p q t)^2) + p(1 - q t)^2 \right] \pi_2 D_{p,q}^2 \Pi(t \pi_1 + (1 - t) \pi_2) d_{p,q} t \\
+ \int_{\frac{1}{\pi_2 - \pi_1}}^{\frac{1}{\pi_2 - \pi_1}} \frac{1}{\pi_2 - \pi_1} \pi_2 D_{p,q}^2 \Pi(t \pi_1 + (1 - t) \pi_2) d_{p,q} t \\
= \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(\kappa) \pi_2 d_{p,q} \kappa - \Pi \left(\frac{p \pi_1 + q \pi_2}{2} \right).
\]

**Proof.** Consider from Definition 7:

\[
\pi_2 D_{p,q}^2 \Pi(t \pi_1 + (1 - t) \pi_2) = \pi_2 D_{p,q} \left[ \frac{\Pi(q t \pi_1 + (1 - q t) \pi_2) - \Pi(p t \pi_1 + (1 - p t) \pi_2)}{(p - q)(\pi_2 - \pi_1)} \right]
\]

\[
= \frac{p q (q^2 t^2 + (1 - q^2 t^2) \pi_2) - (2 \pi_1) \Pi(p q t \pi_1 + (1 - p q t) \pi_2) + q \Pi(p^2 t \pi_1 + (1 - p^2 t) \pi_2)}{p q (p - q)^2 (\pi_2 - \pi_1)^2 t^2}
\]

From Definition 8, we have:

\[
\int_{\frac{1}{\pi_2 - \pi_1}}^{\frac{1}{\pi_2 - \pi_1}} \frac{1}{\pi_2 - \pi_1} \pi_2 D_{p,q}^2 \Pi(t \pi_1 + (1 - t) \pi_2) d_{p,q} t \\
+ \int_{0}^{\frac{1}{\pi_2 - \pi_1}} (p^2 q + (1 - p q t)^2) + p(1 - q t)^2 \right] \pi_2 D_{p,q}^2 \Pi(t \pi_1 + (1 - t) \pi_2) d_{p,q} t
\]
\[
\begin{align*}
= & \int_0^1 p(1-qt)^2_{p,q} D_{p,q}^2 \Pi(t \tau_1 + (1-t) \tau_2) \, d_{p,q} t \\
& + \int_0^1 \left( p^2 q^3 t^2 - (1-pqt)^2_{p,q} \right) D_{p,q}^2 \Pi(t \tau_1 + (1-t) \tau_2) \, d_{p,q} t \\
= & \frac{1}{q(p-q)^2(\tau_2 - \tau_1)^2} \\
& \times \int_0^1 \left( \frac{1}{t^2} \right) \left( p\Pi(q^2 t \tau_1 + (1-q^2 t) \tau_2) - [2]_{p,q} \Pi(pqt \tau_1 + (1-pqt) \tau_2) \right) \, d_{p,q} t \\
& \times \left( p\Pi(q^2 t \tau_1 + (1-q^2 t) \tau_2) - [2]_{p,q} \Pi(pqt \tau_1 + (1-pqt) \tau_2) \right) \, d_{p,q} t \\
= & \frac{1}{q(p-q)^2(\tau_2 - \tau_1)^2} \left( l_1 + \frac{1}{p} l_2 \right).
\end{align*}
\]

We calculate the integrals \( I_1 \) and \( I_2 \) by using the Definition 10. Let us consider:

\[
\begin{align*}
I_1 &= \int_0^1 \left( \frac{1}{t^2} \right) \left( p\Pi(q^2 t \tau_1 + (1-q^2 t) \tau_2) - [2]_{p,q} \Pi(pqt \tau_1 + (1-pqt) \tau_2) \right) \, d_{p,q} t \\
&= \frac{1}{p-q} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left( q^2 \Pi \left( q^{n+1} p^{n+1} \tau_1 + (1 - q^{n+1} \frac{q^n}{p^{n+1}}) \tau_2 \right) \\
&- [2]_{p,q} \Pi \left( q^n p^{n+1} \tau_1 + (1 - q^n \frac{q^{n+1}}{p^{n+1}}) \tau_2 \right) \right) \\
&= \frac{1}{p-q} \sum_{n=0}^{\infty} \left( 1 - q^{n+1} \frac{q^n}{p^{n+1}} \right) \left( q^{n+2} \Pi \left( q^{n+2} p^{n+1} \tau_1 + (1 - q^{n+2} \frac{q^n}{p^{n+1}}) \tau_2 \right) \\
&- [2]_{p,q} \Pi \left( q^{n+1} p^{n+1} \tau_1 + (1 - q^{n+1} \frac{q^n}{p^{n+1}}) \tau_2 \right) \right) \\
&= \left( p-q \right) \sum_{n=0}^{\infty} \left( 1 - q^{n+1} \frac{q^n}{p^{n+1}} \right) \left( q^{n+2} \Pi \left( q^{n+2} p^{n+1} \tau_1 + (1 - q^{n+2} \frac{q^n}{p^{n+1}}) \tau_2 \right) \\
&- [2]_{p,q} \Pi \left( q^{n+1} p^{n+1} \tau_1 + (1 - q^{n+1} \frac{q^n}{p^{n+1}}) \tau_2 \right) \right) \\
&= \left( p-q \right) \left[ p \sum_{n=0}^{\infty} \left( 1 - q^{n+1} \frac{q^n}{p^{n+1}} \right) \Pi \left( q^{n+2} p^{n+1} \tau_1 + (1 - q^{n+2} \frac{q^n}{p^{n+1}}) \tau_2 \right) \\
&- [2]_{p,q} \sum_{n=0}^{\infty} \left( 1 - q^{n+1} \frac{q^n}{p^{n+1}} \right) \Pi \left( q^{n+1} p^{n+1} \tau_1 + (1 - q^{n+1} \frac{q^n}{p^{n+1}}) \tau_2 \right) \right] \\
&+ q \sum_{n=0}^{\infty} \left( 1 - q^{n+1} \frac{q^n}{p^{n+1}} \right) \Pi \left( q^n p^{n-1} \tau_1 + (1 - q^n \frac{q^{n+1}}{p^{n-1}}) \tau_2 \right).
\end{align*}
\]
\[
= (p - q) \left[ p \sum_{n=2}^{\infty} \frac{(1 - q^{n-2})^2 \Pi_q \left( \frac{q^n}{p^{n-2}} \tau_1 + (1 - \frac{q^n}{p^{n-1}}) \tau_2 \right)}{q^{n-2} p^{n-1}} \right]
\]

\[
- [2]_{p,q} \sum_{n=1}^{\infty} \frac{(1 - q^{n-1})^2 \Pi_q \left( \frac{q^n}{p^{n-1}} \tau_1 + (1 - \frac{q^n}{p^{n-1}}) \tau_2 \right)}{q^{n-1} p^{n-1}}
\]

\[
+ q \sum_{n=0}^{\infty} \frac{(1 - q^{n})^2 \Pi_q \left( \frac{q^n}{p^{n-1}} \tau_1 + (1 - \frac{q^n}{p^{n-1}}) \tau_2 \right)}{q^n p^{n+1}}
\]

\[
= (p - q) \left[ p \sum_{n=0}^{\infty} \frac{(1 - q^{n+2})^2 \Pi_q \left( \frac{q^n}{p^{n-2}} \tau_1 + (1 - \frac{q^n}{p^{n-1}}) \tau_2 \right)}{q^{n+2} p^{n-2}} \right]
\]

\[
- [2]_{p,q} \sum_{n=0}^{\infty} \frac{(1 - q^{n+1})^2 \Pi_q \left( \frac{q^n}{p^{n-1}} \tau_1 + (1 - \frac{q^n}{p^{n-1}}) \tau_2 \right)}{q^{n+1} p^{n-1}}
\]

\[
+ q \sum_{n=0}^{\infty} \frac{(1 - q^{n+1})^2 \Pi_q \left( \frac{q^n}{p^{n-1}} \tau_1 + (1 - \frac{q^n}{p^{n-1}}) \tau_2 \right)}{q^{n+1} p^{n+1}}
\]

\[
= (p - q) \sum_{n=0}^{\infty} \Pi_q \left( \frac{q^n}{p^{n-1}} \tau_1 + (1 - \frac{q^n}{p^{n-1}}) \tau_2 \right)
\]

\[
\times \left( p \frac{(1 - q^{n+2})^2 \Pi_q \left( \frac{q^n}{p^{n-2}} \tau_1 + (1 - \frac{q^n}{p^{n-1}}) \tau_2 \right)}{q^{n+2} p^{n-2}} - [2]_{p,q} \frac{(1 - q^{n+1})^2 \Pi_q \left( \frac{q^n}{p^{n-1}} \tau_1 + (1 - \frac{q^n}{p^{n-1}}) \tau_2 \right)}{q^{n+1} p^{n-1}} + q \frac{(1 - q^{n+1})^2 \Pi_q \left( \frac{q^n}{p^{n-1}} \tau_1 + (1 - \frac{q^n}{p^{n-1}}) \tau_2 \right)}{q^{n+1} p^{n+1}} \right)
\]

\[
= q(p - q)(p - q)^2 [2]_{p,q} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Pi_q \left( \frac{q^n}{p^{n-1}} \tau_1 + (1 - \frac{q^n}{p^{n-1}}) \tau_2 \right)
\]

\[
= \frac{q(p - q)^2 [2]_{p,q}}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \Pi_q(\xi) d\xi d_{\xi, q} \xi
\]

Now, consider:
\[ I_2 = \frac{1}{[2]_{p,q}} \int_0^{t_2} \frac{(pq^3 t^2 - (1 - pq t)^2)_{p,q}}{t^2} \left( -\Pi(pq t \pi_1 + (1 - q^2 t) \pi_2) + q \Pi(pq t \pi_1 + (1 - p^2 t) \pi_2) \right) \, dp \, dq \, t \]

\[ = \frac{p - q}{[2]_{p,q}} \sum_{n=0}^{\infty} q^n \left( \frac{p^2 q^3 \left( \frac{q^n}{[2]_{p,q} p^{n+\pi}} \right)^2 - (1 - pq \frac{q^n}{[2]_{p,q} p^{n+\pi}})^2}{p^n} \right) \]

\[ \times \left( p \Pi \left( \frac{q^{n+2}}{[2]_{p,q} p^{n+\pi}} \pi_1 + (1 - \frac{q^{n+2}}{[2]_{p,q} p^{n+\pi}}) \pi_2 \right) - [2]_{p,q} \Pi \left( \frac{p q^{n+1}}{[2]_{p,q} p^{n+\pi}} \pi_1 + (1 - \frac{p q^{n+1}}{[2]_{p,q} p^{n+\pi}}) \pi_2 \right) + q \Pi \left( \frac{q^n}{[2]_{p,q} p^{n-\pi}} \pi_1 + (1 - \frac{q^n}{[2]_{p,q} p^{n-\pi}}) \pi_2 \right) \right) \]

\[ = \frac{p - q}{[2]_{p,q}} \sum_{n=0}^{\infty} p \left( \frac{p^2 q^3 \left( \frac{q^n}{[2]_{p,q} p^{n+\pi}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^n}{[2]_{p,q} p^{n+\pi}} \right)^2}{p^n} \right) \]

\[ \times \Pi \left( \frac{q^{n+2}}{[2]_{p,q} p^{n+\pi}} \pi_1 + (1 - \frac{q^{n+2}}{[2]_{p,q} p^{n+\pi}}) \pi_2 \right) - \sum_{n=0}^{\infty} \frac{[2]_{p,q} \left( p^2 q^3 \left( \frac{q^n}{[2]_{p,q} p^{n+\pi}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^n}{[2]_{p,q} p^{n+\pi}} \right)^2 \right)}{p^n} \]

\[ \times \Pi \left( \frac{q^{n+1}}{[2]_{p,q} p^{n+\pi}} \pi_1 + (1 - \frac{q^{n+1}}{[2]_{p,q} p^{n+\pi}}) \pi_2 \right) + \sum_{n=0}^{\infty} q \left( p^2 q^3 \left( \frac{q^n}{[2]_{p,q} p^{n+\pi}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^n}{[2]_{p,q} p^{n+\pi}} \right)^2 \right) \]

\[ \times \Pi \left( \frac{q^n}{[2]_{p,q} p^{n-\pi}} \pi_1 + (1 - \frac{q^n}{[2]_{p,q} p^{n-\pi}}) \pi_2 \right) \]

\[ = \frac{p - q}{[2]_{p,q}} \sum_{n=0}^{\infty} \frac{p \left( p^2 q^3 \left( \frac{q^{n-2}}{[2]_{p,q} p^{n-\pi}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{n-2}}{[2]_{p,q} p^{n-\pi}} \right)^2 \right)}{p^n} \]

\[ \times \Pi \left( \frac{q^n}{[2]_{p,q} p^{n-\pi}} \pi_1 + (1 - \frac{q^n}{[2]_{p,q} p^{n-\pi}}) \pi_2 \right) \]
\[-p \left( p^2 q^3 \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \Pi \left( \frac{p}{[2]_{p,q}} \pi_1 + (1 - \frac{p}{[2]_{p,q}}) \pi_2 \right) \]
\[-p^2 q^3 (q^{-1})^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-1}}{[2]_{p,q}} \right) \]
\[\times \Pi \left( \frac{q}{[2]_{p,q}} \pi_1 + (1 - \frac{q}{[2]_{p,q}}) \pi_2 \right) \]
\[-\sum_{n=0}^{\infty} [2]_{p,q} \left( p^2 q^3 \left( \frac{q^n}{p^{n+1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^n}{[2]_{p,q} p^{n+1}} \right) \right) \]
\[\times \Pi \left( \frac{q^n}{[2]_{p,q} p^{n+1}} \pi_1 + (1 - \frac{q^n}{[2]_{p,q} p^{n+1}}) \pi_2 \right) \]
\[\Pi \left( \frac{p}{[2]_{p,q}} \pi_1 + (1 - \frac{p}{[2]_{p,q}}) \pi_2 \right) \]
\[\sum_{n=0}^{\infty} \left( p^2 q^3 \left( \frac{q^n}{p^{n+1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^n}{[2]_{p,q} p^{n+1}} \right) \right) \]
\[\times \left( \frac{p}{[2]_{p,q}} \pi_1 + (1 - \frac{p}{[2]_{p,q}}) \pi_2 \right) \]
\[= \frac{(p - q)}{[2]_{p,q}} \sum_{n=0}^{\infty} \Pi \left( \frac{q^n}{[2]_{p,q} p^{n+1}} \pi_1 + (1 - \frac{q^n}{[2]_{p,q} p^{n+1}}) \pi_2 \right) \]
\[\times \left( p^2 q^3 \left( \frac{q^n-2}{p^{n+1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^n-2}{[2]_{p,q} p^{n+1}} \right) \right) \]
\[-\sum_{n=0}^{\infty} \left( p^2 q^3 \left( \frac{q^n}{p^{n+1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^n}{[2]_{p,q} p^{n+1}} \right) \right) \]
\[\times \left( p^2 q^3 \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{p}{[2]_{p,q}} \pi_1 + (1 - \frac{p}{[2]_{p,q}}) \pi_2 \right) \]
\[\times \left( -p^2 q^3 \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{q}{[2]_{p,q}} \pi_1 + (1 - \frac{q}{[2]_{p,q}}) \pi_2 \right) \]
\[\times \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
\[\times \left( \frac{q^{-2}}{p^{-1}} \right)^2 - [2]_{p,q} \left( 1 - pq \frac{q^{-2}}{[2]_{p,q} p^{-1}} \right) \right) \]
Theorem 1. Suppose that the assumptions of Lemma 2 hold. If $|\tau_2 D_{p,q}\Pi|$ is convex on $[\tau_1, \tau_2]$, then we have the inequality:

$$
\left| \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \Pi(\tau_1) \tau_2 d_{q,t} \right| 
\leq \frac{(\tau_2 - \tau_1)^2}{2} \left[ \Theta_1(p,q) |\tau_2 D_{p,q}\Pi(\tau_1)| + \Theta_2(p,q) |\tau_2 D_{p,q}\Pi(\tau_2)| \right],
$$

where:

$$
\Theta_1(p,q) = \frac{p^3 |2|_{p,q} (p^3 |2|_{p,q} - p^2 - q^2)}{|2|_{p,q} |3|_{p,q} |4|_{p,q}}.
$$

Remark 4. In Lemma 2, if we set $p = 1$, then we obtain the following identity:

$$
\frac{(\tau_2 - \tau_1)^2}{2q^2} \int_{0}^{\pi_q} q^2 \tau_2 D_{q}^2 \Pi(t \tau_1 + (1 - t) \tau_2) d_q t
\]

Thus, the proof is complete. ☐

This was proven by Ali et al. in [18] (Lemma 5).
Axioms 2022, 11, 46

\[ \Theta_2(p, q) = \frac{[3]_{p,q}[4]_{p,q}((1 - 2p)(2)_p^2 - 1) + q[2]_{p,q}}{[2]_{p,q}[3]_{p,q}[4]_{p,q}} + \frac{(p^3q^22^2_{p,q} + p^3[2]_{p,q}^2 - p[2]_{p,q}[4]_{p,q} - q[4]_{p,q})}{[2]_{p,q}^2[3]_{p,q}[4]_{p,q}}. \]

**Proof.** By taking the modulus of (13), we have:

\[ \left| \frac{1}{\pi_2 - \pi_1} \int_0^{\pi_2} \Pi(\pi)^{\pi_2}d_{p,q}\pi - \Pi\left(\frac{p\pi_1 + q\pi_2}{2}_{p,q}\right) \right| \]

\[ \leq \frac{(\pi_2 - \pi_1)^2}{[2]_{p,q}} \int_0^{\pi_2} \left( (p^3q^2)\pi^2 - (1 - pqt)^2_{p,q} + p(1 - qt)^2_{p,q} \right) |\pi_2^2D_{p,q}^2\Pi(\pi_1)| |d_{p,q}t \]

\[ + \int_0^{1} \frac{1}{p_{p,q}} (p(1 - qt)^2_{p,q} |\pi_2^2D_{p,q}^2\Pi(\pi_1) + p(1 - qt)^2_{p,q}(1 - t)| |\pi_2^2D_{p,q}^2\Pi(\pi_2)|) |d_{p,q}t \].

Now, by using the convexity of $|\pi_2^2D_{p,q}^2\Pi|$, we have:

\[ \left| \frac{1}{\pi_2 - \pi_1} \int_0^{\pi_2} \Pi(\pi)^{\pi_2}d_{p,q}\pi - \Pi\left(\frac{p\pi_1 + q\pi_2}{2}_{p,q}\right) \right| \]

\[ \leq \frac{(\pi_2 - \pi_1)^2}{[2]_{p,q}} \int_0^{\pi_2} \left( \left( (p^3q^2\pi^2 - (1 - pqt)^2_{p,q} + p(1 - qt)^2_{p,q}) t |\pi_2^2D_{p,q}^2\Pi(\pi_1)| \right) + (p^3q^2\pi^2 - (1 - pqt)^2_{p,q} + p(1 - qt)^2_{p,q})(1 - t) |\pi_2^2D_{p,q}^2\Pi(\pi_2)| \right) |d_{p,q}t \]

\[ + \int_0^{1} \frac{1}{p_{p,q}} (p(1 - qt)^2_{p,q} t |\pi_2^2D_{p,q}^2\Pi(\pi_1) + p(1 - qt)^2_{p,q}(1 - t)| |\pi_2^2D_{p,q}^2\Pi(\pi_2)|) |d_{p,q}t \].

Now, we compute the integrals appearing on the right side of (15) using Lemma 1. We have:

\[ \int_0^1 (p^3q^2\pi^2 - (1 - pqt)^2_{p,q} + p(1 - qt)^2_{p,q}) \pi d_{p,q}t \]

\[ = \int_0^1 (p^3 - p) t + pq^3 \pi^3 \pi d_{p,q}t = \frac{[2]_{p,q}[4]_{p,q}(p^2 - p) + pq^3}{[2]_{p,q}[4]_{p,q}} \]

\[ \int_0^1 (p^3q^2\pi^2 - (1 - pqt)^2_{p,q} + p(1 - qt)^2_{p,q})(1 - t) \pi d_{p,q}t \]

\[ = \int_0^1 (-pq^3 + pq^3\pi^2 - (p^2 - p) t + (p^2 - p)) \pi d_{p,q}t \]

\[ = \int_0^1 (-pq^3 + pq^3\pi^2 - (p^2 - p) t + (p^2 - p)) \pi d_{p,q}t \]
Remark 5. In Theorem 1, if we set $p = 1$, then we obtain the following inequality:

\[
\leq \left( \frac{\tau_2 - \tau_1}{2} \right)^2 \left[ 2q + 4q^2 + 2q^3 \right] \mid \pi_2 D_{\pi_1}^2 \Pi(\tau_1) \mid + \left( -q - q^2 + 2q^3 + 4q^4 + 3q^5 + q^6 \right) \mid \pi_2 D_{\pi_1}^2 \Pi(\tau_2) \mid.
\]

This was proven by Ali et al. [18] (Theorem 3).

Theorem 2. Suppose that the assumptions of Lemma 2 hold. If $\mid \pi_2 D_{\pi_1} \Pi \mid^r$, $r \geq 1$ is convex on $[\tau_1, \tau_2]$, then we have the inequality:

\[
\leq \left( \frac{\tau_2 - \tau_1}{2} \right)^2 \left[ \Theta_1^{-\frac{1}{2}} (p, q) \left( \Theta_3(p, q) \mid \pi_2 D_{\pi_1}^2 \Pi(\tau_1) \mid + \Theta_4(p, q) \mid \pi_2 D_{\pi_1}^2 \Pi(\tau_2) \mid \right)^{\frac{1}{2}} + \Theta_5^{-\frac{1}{2}} (p, q) \left( \Theta_3(p, q) \mid \pi_2 D_{\pi_1}^2 \Pi(\tau_1) \mid + \Theta_6(p, q) \mid \pi_2 D_{\pi_1}^2 \Pi(\tau_2) \mid \right)^{\frac{1}{2}} \right],
\]
Axioms 2022, 11, 46

\[ \Theta_3(p, q) = \frac{[2]_{p,q}[4]_{p,q}(p^2 - p) + pq^3}{[2]_{p,q}[4]_{p,q}}, \]
\[ \Theta_4(p, q) = \frac{-pq^3[3]_{p,q} + pq^3[2]_{p,q}[4]_{p,q} - (p^2 - p)([2]_{p,q}^2 - 1)[2]_{p,q}[3]_{p,q}[4]_{p,q}}{[2]_{p,q}[3]_{p,q}[4]_{p,q}}, \]
\[ \Theta_5(p, q) = \frac{pq^3[3]_{p,q}([2]_{p,q}^2 - 1) - pq[2]_{p,q}^2[4]_{p,q}([2]_{p,q}^3 - 1) + p^2[2]_{p,q}[3]_{p,q}[4]_{p,q}([2]_{p,q}^2 - 1)}{[2]_{p,q}[3]_{p,q}[4]_{p,q}}, \]
\[ \Theta_6(p, q) = p \left[ \frac{-q^3([2]_{p,q}^2 - 1)[3]_{p,q} + (pq + q^2)[2]_{p,q}^2[4]_{p,q}([2]_{p,q}^3 - 1) - (pq + p^2)[2]_{p,q}^3[4]_{p,q}([2]_{p,q}^2 - 1) + p[2]_{p,q}^2[3]_{p,q}[4]_{p,q}([2]_{p,q} - 1)}{[2]_{p,q}[3]_{p,q}[4]_{p,q}} \right]. \]
\[ \Theta_7(p, q) = \frac{[2]_{p,q}^3[3]_{p,q}(p^2 - p) + pq^3}{[2]_{p,q}^3[3]_{p,q}}, \]
\[ \Theta_8(p, q) = \frac{pq^3([2]_{p,q}^3 - 1)}{[3]_{p,q}2^3_{p,q}} + pq \left( 1 - \frac{2^2_{p,q}}{[2]_{p,q}} \right) + \frac{p^2([2]_{p,q}^2 - 1)}{[2]_{p,q}}. \]

**Proof.** By taking the modulus of (13) and applying the power mean inequality, we have:

\[
\left| \frac{1}{\pi_2 - \pi_1} \int_{p^2\pi_1 + (1-p^2)\pi_2}^{\pi_2} \Pi(x)^{\pi_2} d_{p,q} x \right| - \Pi \left( \frac{p \pi_1 + q \pi_2}{2} \right) \leq \frac{(\pi_2 - \pi_1)^2}{[2]_{p,q}} \left( \int_{0}^{\pi_2} (p^2 q \pi_2^2 - (1 - pqt)_{p,q}^2 + p(1 - qt)_{p,q}^2) \left| \frac{\pi_2}{[2]_{p,q}} D_{p,q}^2 \Pi(t \pi_1 + (1 - t) \pi_2) \right| d_{p,q} t \right) \]

\[
+ \frac{1}{\pi_2} \left( \int_{0}^{\pi_2} (p(1 - q t)_{p,q}^2 \left| \frac{\pi_2}{[2]_{p,q}} D_{p,q}^2 \Pi(t \pi_1 + (1 - t) \pi_2) \right| d_{p,q} t \right) \]

\[
\leq \frac{(\pi_2 - \pi_1)^2}{[2]_{p,q}} \left( \int_{0}^{\pi_2} (p^2 q \pi_2^2 - (1 - pqt)_{p,q}^2 + p(1 - qt)_{p,q}^2) d_{p,q} t \right) \]

\[
\times \left( \int_{0}^{\pi_2} (p^2 q \pi_2^2 - (1 - pqt)_{p,q}^2 + p(1 - qt)_{p,q}^2) \left| \frac{\pi_2}{[2]_{p,q}} D_{p,q}^2 \Pi(t \pi_1 + (1 - t) \pi_2) \right| d_{p,q} t \right) \]

\[
+ \left( \frac{1}{\pi_2} \int_{0}^{\pi_2} (p(1 - q t)_{p,q}^2 \left| \frac{\pi_2}{[2]_{p,q}} D_{p,q}^2 \Pi(t \pi_1 + (1 - t) \pi_2) \right| d_{p,q} t \right) \]

Applying the convexity of $\frac{\pi_2}{[2]_{p,q}} D_{p,q}^2 \Pi$, $r \geq 1$, we have:
\[
\left| \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \frac{\tau_2}{\tau_1} d_{p,q} \right| - \left| \frac{p\tau_1 + q\tau_2}{[2]_{p,q}} \right| \\
\leq \left( \frac{\tau_2 - \tau_1}{[2]_{p,q}} \right)^2 \left\{ \int_0^1 \left( \frac{1}{[2]_{p,q}} + \left( p^2 q^3 t^2 - (1 - pqt)^2 + p(1 - qt)^2 \right) d_{p,q} t \right) \right\}^{1 - \frac{1}{r}} \\
\times \left\{ \frac{1}{[2]_{p,q}} + \left( p(1 - qt)^2 \right) \right\}^{1 - \frac{1}{r}} \\
+ \left( \frac{1}{[2]_{p,q}} \right) \left\{ \int_0^1 \left( p(1 - qt)^2 \left[ t \right][\tau_2 D_{p,q} \Pi(\tau_1)]^r + (1 - t)[\tau_2 D_{p,q} \Pi(\tau_2)]^r \right) d_{p,q} t \right\}^{1 - \frac{1}{r}} \\
= \left( \frac{\tau_2 - \tau_1}{[2]_{p,q}} \right)^2 \left\{ \Theta_5(p,q) \left( \Theta_3(p,q) \left[ \tau_2 D_{p,q}^2 \Pi(\tau_1) \right]^r + \Theta_4(p,q) \left[ \tau_2 D_{p,q}^2 \Pi(\tau_2) \right]^r \right) \right\}^{1 - \frac{1}{r}} \\
+ \Theta_6(p,q) \left( \Theta_5(p,q) \left[ \tau_2 D_{p,q}^2 \Pi(\tau_1) \right]^r + \Theta_6(p,q) \left[ \tau_2 D_{p,q}^2 \Pi(\tau_2) \right]^r \right) \right\}^{1 - \frac{1}{r}} \\
\right]
\]

Thus, the proof is complete. \( \square \)

**Remark 6.** In Theorem 2, if we set \( p = 1 \), then Theorem 2 becomes Theorem 5 in [18].

**Theorem 3.** Suppose that the assumptions of Lemma 2 hold. If \( \tau_2 D_{p,q} \Pi \), \( r > 1 \) is convex on \( [\tau_1, \tau_2] \), then we have the inequality:

\[
\left| \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \frac{\tau_2}{\tau_1} d_{p,q} \right| - \left| \frac{p\tau_1 + q\tau_2}{[2]_{p,q}} \right| \\
\leq \left( \frac{\tau_2 - \tau_1}{[2]_{p,q}} \right)^2 \left\{ \Theta_0(p,q) \left( \Theta_3(p,q) \left[ \tau_2 D_{p,q}^2 \Pi(\tau_1) \right]^r + \Theta_4(p,q) \left[ \tau_2 D_{p,q}^2 \Pi(\tau_2) \right]^r \right) \right\}^{1 - \frac{1}{r}} \\
\times \left\{ \Theta_5(p,q) \left[ \tau_2 D_{p,q}^2 \Pi(\tau_1) \right]^r + \Theta_6(p,q) \left[ \tau_2 D_{p,q}^2 \Pi(\tau_2) \right]^r \right\} \right\}^{1 - \frac{1}{r}} \\

\text{where} \quad \frac{1}{r} + \frac{1}{r} = 1,
\]

\[\Theta_0(p,q) = \int_0^1 \left( p^2 q^3 t^2 - (1 - pqt)^2 + p(1 - qt)^2 \right) d_{p,q} t\]
and:

\[ \Theta_{10}(p, q) = \int_{\pi^2}^{1} p^s (1 - qt)^{2s} \pi d_{p,q}t. \]

**Proof.** By taking the modulus of (13) and applying Hölder’s inequality, we have:

\[ \left| \int_{\pi^2}^{1} \frac{\pi}{\pi^2 - \pi^2} p^{\pi^2} (1 - pt)^{\pi^2} d_{p,q}t \right| \]

\[ \leq \frac{(\pi^2 - \pi^2)^2}{[2]_{p,q}} \left[ \left( \int_{0}^{1} (p^2 q^2 t^2 - (1 - pqt)^2) (p^2)_{p,q} + p(1 - qt)^2_{p,q} \right)^{\pi^2} \right]^{\frac{1}{2}} \]

\[ \times \left( \int_{0}^{1} \left| \nabla^2_{p,q} \Pi(t, p, q) \right| d_{p,q}t \right)^{\frac{1}{2}} \]

\[ + \left( \int_{0}^{1} \nabla^2_{p,q} \Pi(t, p, q) \right)^{\frac{1}{2}} \left( \int_{0}^{1} \left| \nabla^2_{p,q} \Pi(t, p, q) \right| d_{p,q}t \right)^{\frac{1}{2}}. \]

Applying the convexity of \( |\nabla^2_{p,q} \Pi|', r > 1, \) we have:

\[ \left| \int_{\pi^2}^{1} \frac{\pi}{\pi^2 - \pi^2} p^{\pi^2} (1 - pt)^{\pi^2} d_{p,q}t \right| \]

\[ \leq \frac{(\pi^2 - \pi^2)^2}{[2]_{p,q}} \left[ \left( \int_{0}^{1} (p^2 q^2 t^2 - (1 - pqt)^2) (p^2)_{p,q} + p(1 - qt)^2_{p,q} \right)^{\pi^2} \right]^{\frac{1}{2}} \]

\[ \times \left( \int_{0}^{1} \left| \nabla^2_{p,q} \Pi(t, p, q) \right| d_{p,q}t \right)^{\frac{1}{2}} \]

\[ + \left( \int_{0}^{1} \nabla^2_{p,q} \Pi(t, p, q) \right)^{\frac{1}{2}} \left( \int_{0}^{1} \left| \nabla^2_{p,q} \Pi(t, p, q) \right| d_{p,q}t \right)^{\frac{1}{2}}. \]
\[
\frac{(\pi_2 - \pi_1)^2}{[2]_{p,q}} \left[ \Theta_5^q(p,q) \left( \frac{|\pi_2 D_{p,q}^2 \Pi(\pi_1)|^r}{[2]_{p,q}} + \frac{|\pi_2 D_{p,q}^2 \Pi(\pi_2)|^r}{[2]_{p,q}} \right)^{\frac{1}{r}} \right] + \Theta_{10}^q(p,q) \left( \frac{|\pi_2 D_{p,q}^2 \Pi(\pi_1)|^r}{[2]_{p,q}} + \frac{|\pi_2 D_{p,q}^2 \Pi(\pi_2)|^r}{[2]_{p,q}} \right)^{\frac{1}{r}} \right].
\]

Thus, the proof is complete. \(\square\)

**Remark 7.** In Theorem 3, if we set \(p = 1\), then Theorem 3 becomes Theorem 4 in [18].

5. Concluding Remarks

In this work, we established some new midpoint-type \((p,q)\)-integral inequalities for twice-(\(p,q\))-differentiable convex functions. In the last twenty-five years, quantum calculus (or \(q\)-calculus) has served as a link between mechanics and physics. Physicists make up the majority of scientists who utilize \(q\)-calculus today. Many scientists, in fact, employ \(q\)-calculus as a mathematical model in their studies. In the theory of quantum gravity, \(q\) can be thought of as a parameter connected to the exponential of the cosmological constant. Quantum gravity can be divided into two types:

(i) There is no gravity when \(q = 1\), which is the situation of classical quantum mechanics;

(ii) When \(q \neq 0\), we have quantum mechanics, in which the energy density of the vacuum is non-zero.

Other areas of study in physics and related fields where \(q\)-calculus is being employed include: (1) the \(q\)-Coulomb problem and the \(q\)-hydrogen atom; (2) quantum hydrodynamics; (3) the Wess–Zumino model; (4) string theory; (5) Electroweak interaction; (6) Knot theory; (7) quantum cardiodynamics; (8) special relativity; (9) Newtonian quantum gravity; (10) quantum field theory; (11) elementary particle physics and chemical physics; (12) molecular and nuclear spectroscopy; (13) general relativity.

For several other applications of \(q\)-calculus in physics, one can read [4]. A good book on “Quantum Gravity” [34] is recommended for those interested in these topics. The fact that fluid dynamics resemble quantum physics is no coincidence. In truth, quantum physics and classical physics can both reflect various parts of the same physical process; for example, see [35].

**Author Contributions:** Conceptualization, T.S., G.M., M.A.A., C.P., I.B.S. and P.A.; methodology, T.S., G.M., M.A.A., C.P., I.B.S. and P.A.; validation, T.S., G.M., M.A.A., C.P., I.B.S. and P.A.; investigation, T.S., G.M., M.A.A., C.P., I.B.S. and P.A.; writing—original draft preparation, T.S., G.M., M.A.A., C.P., I.B.S. and P.A.; writing—review and editing, T.S., G.M., M.A.A., C.P., I.B.S. and P.A.; supervision, T.S. and M.A.A.; funding acquisition, T.S. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was funded by King Mongkut’s University of Technology North Bangkok, Contract No. KMUTNB-63-KNOW-22.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** We thank the Referees and Editor for their valuable comments. The third author is thankful to Ch. Sohail Tariq (Project Director GCUF Sahiwal Campus) and Ch. Faraz Hanif (Project Director GCUF Sahiwal Campus) for providing good research environment at GCUF Sahiwal Campus.

**Conflicts of Interest:** The authors declare no conflict of interest.
Axioms 2022, 11, 46

References

1. Pečarić, J.E.; Proschan, F.; Tong, Y.L. Convex Functions, Partial Orderings and Statistical Applications; Academic Press: Boston, MA, USA, 1992.

2. Khan, M.B.; Zaini, H.G.; Treanta, S.; Soliman, M.S.; Nanlaopon, K. Riemann-Liouville Fractional Integral Inequalities for Generalized Pre=Invex Functions of Interval-Valued Settings Based upon Pseudo Order Relation. Mathematics 2022, 10, 204. [CrossRef]

3. Khan, M.B.; Treanta, S.; Soliman, M.S.; Zaini, H.G.; Nanlaopon, K. Some Hadamard-Fejer Type Inequalities for LR-Convex Interval-Valued Functions. Fractal Fract. 2022, 6, 6. [CrossRef]

4. Ernst, T. A Comprehensive Treatment of q-Calculus; Springer: Basel, Switzerland, 2012.

5. Jackson, F.H. On a q-definite integrals. Q. J. Pure Appl. Math. 1910, 41, 193–203.

6. Al-Salam, W. Some fractional q-integrals and q-derivatives. Proc. Edinb. Math. Soc. 1966, 15, 135–140. [CrossRef]

7. Kac, V.; Cheung, P. Quantum Calculus; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2001.

8. Ernst, T. The History of Q-Calculus And New Method; Department of Mathematics, Uppsala University: Uppsala, Sweden, 2000.

9. Jackson, F.H. On a q-definite integrals. Q. J. Pure Appl. Math. 1910, 41, 193–203.

10. Al-Salam, W. Some fractional q-integrals and q-derivatives. Proc. Edinb. Math. Soc. 1966, 15, 135–140. [CrossRef]

11. Tariboon, J.; Ntouyas, S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. Adv. Differ. Equ. 2013, 2013, 1–19. [CrossRef]

12. Bermudo, S.; Körüs, P.; Valdés, J.N. On q-Hermite–Hadamard inequalities for general convex functions. Acta Math. Hung. 2020, 162, 364–374. [CrossRef]

13. Sadjang, P.N. On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas. Results Math. 2018, 73, 1–21.

14. Soontharanon, J.; Sithiwirathatham, T. On Fractional (p,q)-Calculus. Adv. Differ. Equ. 2020, 2020, 1–18. [CrossRef]

15. Tunç, M.; Göv, E. Some integral inequalities via (p,q)-calculus on finite intervals. RGMIA Res. Rep. Coll. 2016, 19, 1–12. [CrossRef]

16. Chu, Y.-M.; Awan, M.U.; Talib, S.; Noor, M.A.; Noor, K.I. New post quantum analogues of Ostrowski-type inequalities using new definitions of left-right (p,q)-derivatives and definite integrals. Adv. Differ. Equ. 2020, 2020, 1–15. [CrossRef]

17. Ali, M.A.; Budak, H.; Abbas, M.; Chu, Y.-M. Quantum Hermite–Hadamard-type inequalities for functions with convex absolute values of second q-derivatives. Adv. Differ. Equ. 2021, 2021, 1–12. [CrossRef]

18. Ali, M.A.; Alp, N.; Budak, H.; Chu, Y.-M.; Zhang, Z. On some new quantum midpoint type inequalities for twice quantum differentiable convex functions. Open Math. 2021, 19, 427–439. [CrossRef]

19. Alp, N.; Sarikaya, M.Z.; Kurt, M.; İşcan, İ. q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. J. King Saud Univ.-Sci. 2018, 30, 193–203. [CrossRef]

20. Budak, H. Some trapezoid and midpoint type inequalities for newly defined quantum integrals. Proyecciones 2021, 40, 199–215. [CrossRef]

21. Budak, H.; Ali, M.A.; Tarhanaci, M. Some new quantum Hermite–Hadamard-like inequalities for coordinated convex functions. J. Optim. Theory Appl. 2020, 186, 899–910. [CrossRef]

22. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum estimates for Hermite–Hadamard inequalities. Appl. Math. Comput. 2015, 251, 675–679. [CrossRef]

23. Noor, K.I.; Noor, M.A.; Awan, M.U. Some integral inequalities via preinvex functions. Appl. Math. Comput. 2015, 269, 242–251. [CrossRef]

24. Nwaeze, E.R.; Tameru, A.M. New parameterized quantum integral inequalities via η-quasiconvexity. Adv. Differ. Equ. 2019, 2019, 1–12. [CrossRef]

25. Khan, M.A.; Noor, M.; Nwaeeze, E.R.; Chu, Y.-M. Quantum Hermite–Hadamard inequality by means of a Green function. Adv. Differ. Equ. 2020, 2020, 1–20. [CrossRef]

26. Budak, H.; Erden, S.; Ali, M.A. Simpson and Newton type inequalities for convex functions via newly defined quantum integrals. Math. Meth. Appl. Sci. 2020, 44, 378–390. [CrossRef]

27. Ali, M.A.; Budak, H.; Zhang, Z.; Yıldırım, H. Some new Simpson’s type inequalities for co-ordinated convex functions in quantum calculus. Math. Meth. Appl. Sci. 2021, 44, 4515–4540. [CrossRef]

28. Vivas-Cortez, M.; Ali, M.A.; Kashuri, A.; Sial, I.B.; Zhang, Z. Some New Newton’s Type Integral Inequalities for Co-Ordinated Convex Functions in Quantum Calculus. Symmetry 2020, 12, 1476. [CrossRef]

29. Ali, M.A.; Chu, Y.-M.; Budak, H.; Akkurt, A.; Yıldırım, H. Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables. Adv. Differ. Equ. 2021, 2021, 1–26. [CrossRef]

30. Ali, M.A.; Budak, H.; Akkurt, A.; Chu, Y.-M. Quantum Ostrowski type inequalities for twice quantum differentiable functions in quantum calculus. Open Math. 2021, 19, 440–449. [CrossRef]

31. Kunt, M.; İşcan, İ.; Alp, N.; Sarikaya, M.Z. (p,q)-Hermite–Hadamard inequalities and (p,q)-estimates for midpoint inequalities via convex quasi-convex functions. Rev. Real Acad. Cienc. Exactas Fisicas Nat. Ser. A Matemáticas 2018, 112, 969–992. [CrossRef]

32. Latif, M.A.; Kunt, M.; Dragomir, S.S.; İşcan, İ. Post-quantum trapezoid type inequalities. AIMS Math. 2020, 5, 4011–4026. [CrossRef]
33. Vivas-Cortez, M.; Ali, M.A.; Budak, H.; Kalsoom, H.; Agarwal, P. Some New Hermite–Hadamard and Related Inequalities for Convex Functions via $(p,q)$-Integral. *Entropy* 2021, 23, 828. [CrossRef] [PubMed]

34. Rovelli, C. *Quantum Gravity*; Cambridge Monograph on Mathematical Physics; Cambridge University Press: Cambridge, UK, 2004.

35. Sengar, R.S.; Sharma, M.; Trivedi, A. Fractional calculus applied in solving instability phenomenon in fluid dynamics. *Int. J. Civ. Eng. Technol.* 2015, 6, 34–44.