ALL STRICTLY EXCEPTIONAL COLLECTIONS IN $\mathcal{D}_{\text{coh}}(\mathbb{P}^n)$
CONSIDER CONSIST OF VECTOR BUNDLES

LEONID POSITSELSKI

1. Introduction

Let $k$ be a field and $\mathcal{D}$ be a $k$-linear triangulated category; we will denote, as usually, $\text{Hom}^i(X, Y) = \text{Hom}(X, Y[i])$ and $\text{Hom}^*(X, Y) = \bigoplus_i \text{Hom}^i(X, Y)$. An object $E \in \text{Ob} \mathcal{D}$ is called exceptional if one has $\text{Hom}^s(E, E) = 0$ for $s \neq 0$ and $\text{Hom}^0(E, E) = k$. A finite sequence $E$ of exceptional objects $E_1, \ldots, E_n$ is called an exceptional collection if $\text{Hom}^*(E_i, E_j) = 0$ for $i > j$. A collection $E$ is called full if it generates $\mathcal{D}$ in the sense that any object of $\mathcal{D}$ can be obtained from $E_i$ by the operations of shift and cone. The Grothendieck group $K_0(\mathcal{D})$ of a triangulated category $\mathcal{D}$ generated by an exceptional collection $E$ is the free $\mathbb{Z}$-module generated by the classes of $E_1, \ldots, E_n$, so any full exceptional collection consists of $n = \text{rk} K_0(\mathcal{D})$ objects. Moreover, it is explained in the paper [4] that (under some technical restriction which is usually satisfied) a triangulated category $\mathcal{D}$ generated by an exceptional collection $E$ is equivalent to the derived category of modules over the differential graded algebra corresponding to $E$.

Let $(E_1, E_2)$ be an exceptional pair; the left and right mutated objects $L_{E_1}E_2$ and $R_{E_2}E_1$ are defined as the third vertices of exceptional triangles

\[ E_2[-1] \longrightarrow L_{E_1}E_2 \longrightarrow \text{Hom}^*(E_1, E_2) \otimes E_1 \longrightarrow E_2 \]
\[ E_1 \longrightarrow \text{Hom}^*(E_1, E_2)^* \otimes E_2 \longrightarrow R_{E_2}E_1 \longrightarrow E_1[1]. \]

This definition was given in the papers [2, 3]; it was shown that the mutated collections

\[ E_1, \ldots, E_{i-2}, L_{E_{i-1}}E_i, E_{i-1}, E_{i+1}, \ldots, E_n \]
\[ E_1, \ldots, E_{i-2}, E_i, R_{E_i}E_{i-1}, E_{i+1}, \ldots, E_n \]

remain exceptional (and full) and that the left and right mutations are inverse to each other. Mutations defined this way form an action of the Artin’s braid group $B_n$ with $n$ strings on the set of all isomorphism classes of exceptional collections of $n$ objects.

There is a central element $\phi \in B_n$ that corresponds to the rotation action of the circle on the space of $n$-point configurations in $\mathbb{C}$. Its action on exceptional collections can be described as follows. Let $E_{n+1} = R^{n-1}E_1$ be the object obtained by successive left mutations of $E_1$ through $E_2, \ldots, E_n$. Then it follows that the collection
$E_2, \ldots, E_{n+1}$ is also exceptional. Proceeding in this way, we obtain the collection $E_3, \ldots, E_{n+2}$, and so on, constructing an infinite sequence of exceptional objects $E_1, E_2, E_3, \ldots$ with the property that any $n$ sequential objects $E_i, \ldots, E_{i+n-1}$ form an exceptional collection. Using left mutations, we can continue it to the negative indices: $E_0 = L^{n-1}E_n$, $E_{-1} = L^{n-1}E_{n-1}$, and so on. This sequence is called a helix.

The action of $\phi$ on exceptional collections shifts it $n$ times to the left:

$$\phi(E_1, \ldots, E_n) = (E_{-n+1}, \ldots, E_0).$$

The point is that this shift can be extended to an exact auto-equivalence of the category $\mathcal{D}$. Namely, the Serre functor for a triangulated category $\mathcal{D}$ is a covariant functor $F: \mathcal{D} \rightarrow \mathcal{D}$ for which there is a natural isomorphism

$$\text{Hom}^\bullet(U, V) = \text{Hom}^\bullet(V, FU)^\ast.$$

It is shown in [3] that one has $E_{i-n} = F(E_i)[-n + 1]$ for a full helix $\mathcal{E}$ in $\mathcal{D}$.

Now let us turn to exceptional collections in the derived category $\mathcal{D}_{\text{coh}}^b(X)$ of coherent sheaves on a smooth projective algebraic variety $X$. In this case the Serre functor has the form $F(U) = U \otimes \omega_X[\dim X]$, where $\omega_X$ is the canonical line bundle. In the initial works of A. Gorodentsev and A. Rudakov [1], they considered exceptional collections consisting of pure sheaves, not complexes. Therefore, such mutations were not defined for any exceptional collections, but only under the conditions that some maps are injective or surjective. For example, we see that the helix generated by a full exceptional collection of sheaves will not consist of sheaves unless its period $n$ is equal to $\dim X + 1$.

Conversely, it was shown by A. Bondal [3] that all mutations of a full exceptional collection of $\dim X + 1$ sheaves in $\mathcal{D}_{\text{coh}}^b(X)$ (that is, for a variety with $\text{rk} K_0(X) = \dim X + 1$) consist of pure sheaves again. Indeed, the statement that $R_{E_2}E_1$ is a sheave follows immediately from the isomorphism $R_{E_2}E_1 = L_{E_3} \cdots L_{E_n}E_{n+1}$, where all the objects $E_1, \ldots, E_{n+1}$ are pure sheaves. It is also easy to see that in this case mutations preserve the property of a full exceptional collection to consist of locally free sheaves. Note that for any projective variety one has $\text{rk} K_0(X) \geq \dim X + 1$, since the cycles of self-intersection of $\mathcal{O}(1)$ are linearly independent over $\mathbb{Q}$; the equality holds for $\mathbb{P}^m$, odd-dimensional quadrics, and some others.

The principal problem of the theory of mutations of exceptional bundles on $\mathbb{P}^m$ is to prove that their action on full exceptional collections of vector bundles is transitive. More generally, it was conjectured in [5] that the action of the semidirect product group $B_n \ltimes \mathbb{Z}^n$ generated by mutations and shifts on full exceptional collections in any triangulated category $\mathcal{D}$ is transitive. The second half of this latter conjecture for smooth projective varieties with $\text{rk} K_0(X) = \dim X + 1$ states that any full exceptional collection in $\mathcal{D}_{\text{coh}}^b(X)$ consists of shifts of vector bundles. In this paper we prove this last statement under the following additional restriction. An exceptional collection is said to be strictly exceptional if one has $\text{Hom}^s(E_i, E_j) = 0$ for $s \neq 0$. 
Theorem. Let $X$ be a smooth projective variety for which $n = \text{rk} K_0(X) = \dim X + 1$. Then for any strictly exceptional collection $E_1, \ldots, E_n$ generating $D^b_{\text{coh}}(X)$ the objects $E_i$ are locally free sheaves shifted on the same number $a \in \mathbb{Z}$ in $D$.

Conversely, it was shown in [3] that any full exceptional collection of $\dim X + 1$ sheaves on a smooth projective variety is strictly exceptional.

In particular, if a full exceptional collection on a variety with $\text{rk} K_0(X) = \dim X + 1$ consists of pure sheaves, then these sheaves are locally free. On the other hand, it follows that the property of a full exceptional collection in a triangulated category of this kind to be strictly exceptional is preserved by mutations; moreover, all strictly exceptional collections in these categories are geometric in the sense of [5].

At last, our methods provide an approach to the results on recovery of algebraic varieties from the derived categories of coherent sheaves, alternative to the one given by Bondal–Orlov [6].

Corollary. Suppose the canonical sheaf of a smooth projective variety $X$ is either ample or anti-ample. Then the standard $t$-structure on the derived category $D^b_{\text{coh}}(X)$ can be recovered (uniquely up to a shift) from the triangulated category structure.

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2. Reduction to a Local Problem

The next result is due to A. Bondal and A. Polishchuk [5].

Proposition. Suppose a helix $\{E_i, i \in \mathbb{Z}\}$ in a triangulated category $\mathcal{D}$ is generated by a strictly exceptional collection $E_1, \ldots, E_n$. Then one has $\text{Hom}^s(E_i, E_j) = 0$ for $s > 0$ and $i \leq j \in \mathbb{Z}$, as well as for $s < n - 1$ and $i \geq j \in \mathbb{Z}$.

Proof: First note that the Serre duality isomorphisms

$$\text{Hom}^s(E_i, E_j) = \text{Hom}^{n-1-s}(E_{j+n}, E_i)^*$$

mean that two statements are equivalent to each other; let us prove the first one. The simplest way is to identify $\mathcal{D}$ with the derived category of modules over the homomorphism algebra $A = \bigoplus_{k,l=1}^n A_{kl}$, $A_{kl} = \text{Hom}(E_k, E_l)$ of our strictly exceptional collection, so that the objects $E_i$ correspond to the projective $A$-modules $P_i = \bigoplus_k A_{kl}$ for $1 \leq l \leq n$. Since the Serre functor provides $n$-periodicity isomorphisms $\text{Hom}^s(E_i, E_j) = \text{Hom}^s(E_{i+n}, E_{j+n})$, we can assume that $1 \leq i \leq n$. Let $j = k + Nn$ for some $1 \leq k \leq n$; then we have $E_j = F^{-N} E_i [N(n-1)]$. The Serre functor on $D^b(\text{mod-}A)$ has the form $F(M) = \text{Hom}_A(\text{RHom}_A(M, A), \mathbb{k})$ and $F^{-1}(M) = \text{RHom}_A(\text{Hom}_A(M, \mathbb{k}), A)$; since the homological dimension of $A$ is not greater than $n - 1$, we obtain $E_j \in D^{\leq 0}(\text{mod-}A)$ for $j \geq 1$. Since $E_i$ are projective
for $1 \leq i \leq n$, the assertion follows. A direct, but more complicated calculation from [5] allows to avoid the additional condition on $D$. \hfill \square

Proof of Theorem: First let us show that $X$ is a Fano variety. We give a simple strengthening of the argument from [5]. Since $\text{rk} \text{Pic}(X) = 1$, there are only three types of invertible sheaves: ample ones, antiample ones, and sheaves of finite order. We have to prove that $\omega^{-1}$ is ample; it is enough to show that $H^0(\omega^N) = 0$ for all $N > 0$. Let us denote by $\mathcal{H}^s(U)$ the cohomology sheaves of a complex $U$. Since $E_1, \ldots, E_n$ generate $D$, it is clear that there exists $i$ and $s$ such that $\text{supp} \mathcal{H}^s(E_i) = X$. Let we have a nonzero section $f \in H^0(\omega^N)$; it induces a morphism $E_i \rightarrow E_i \otimes \omega^N$ which is nonzero since its restriction to $\mathcal{H}^s$ is. But we have $E_i \otimes \omega^N = E_{i-Nn}$ which provides a contradiction with Proposition.

Remark 1: More generally, one can see that the canonical sheave $\omega$ cannot be of finite order for a variety $X$ admitting a full exceptional collection. Indeed, the action of invertible sheaves on $K_0(X)$ is unipotent with respect to the filtration by the dimensions of supports, thus in the case in question the action of $\omega$ on $K_0(X)$ must be trivial. But this action (skew-)symmetrizes the canonical bilinear form $\chi([U],[V]) = \sum (-1)^s \dim \text{Hom}^s(U,V)$ on $K_0(X)$. In the basis of $K_0$ corresponding to an exceptional collection, the matrix of this form is upper-triangular with units on the diagonal, so it cannot be skew-symmetric and if it is symmetric then it is positive. The latter is impossible since one has $\chi([\mathcal{O}_x],[\mathcal{O}_x]) = 0$ for the structure sheave $\mathcal{O}_x$ of a point $x \in X$.

We will essentially use the tensor structure on $\mathcal{D}_{\text{coh}}^b(X)$. Namely, let

$$\mathcal{R}\text{Hom} : \mathcal{D}^{\text{opp}} \times \mathcal{D} \rightarrow \mathcal{D}$$

be the derived functor of local homomorphisms of coherent sheaves; it can be calculated using finite locally free resolvents. We have $\text{Hom}^s(U,V) = H^s(\mathcal{R}\text{Hom}(U,V))$, where $H$ denotes the global sheave’s cohomology. Let $i, j \in \mathbb{Z}$ be fixed and $N$ be large enough; one has

$$\text{Hom}^s(E_i, E_{j+Nn}) = H^s(\mathcal{R}\text{Hom}(E_i, E_j \otimes \omega^{-N})) = H^s(C_{ij} \otimes \omega^{-N}),$$

where we denote $C_{ij} = \mathcal{R}\text{Hom}(E_i, E_j)$. Since $\omega^{-1}$ is ample, for large $N$ we have $H^{>0}H^s(C_{ij} \otimes \omega^{-N}) = 0$, hence by the spectral sequence

$$H^s(C_{ij} \otimes \omega^{-N}) = H^0H^s(C_{ij} \otimes \omega^{-N}).$$

Using the property of ample sheaves again, we see that $\text{Hom}^s(E_i, E_{j+Nn})$ is nonzero iff $H^s(C_{ij})$ is. Let $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{>0}$ denote the subcategories of $\mathcal{D}_{\text{coh}}^b(X)$ defined in the standard way. Comparing the last result with Proposition, we finally obtain $C_{ij} \in \mathcal{D}^{\leq 0}$.

Remark 2: Now we can show easily that our exceptional collection is geometric [5]. Indeed, using the duality $\mathcal{R}\text{Hom}(V,U) = \mathcal{R}\text{Hom}(\mathcal{R}\text{Hom}(U,V), \mathcal{O})$, one obtains
\[ C_{ij} = \mathcal{R}\text{Hom}(C_{ji}, \mathcal{O}) \] and since \( \mathcal{R}\text{Hom}(\mathcal{D}^{\leq 0}, \mathcal{O}) \subset \mathcal{D}^{\geq 0} \), it follows that \( C_{ij} \) are pure sheaves. Therefore \( \text{Hom}^{\leq 0}(E_i, E_j) = H^{\leq 0}(C_{ij}) = 0 \) for any \( i \) and \( j \).

The following local statement allows to finish the proof.

**Main Lemma.** Let \( E \in \mathcal{D}^{b}_{\text{coh}}(X) \) be a coherent complex on a smooth algebraic variety \( X \) such that \( \mathcal{R}\text{Hom}(E, E) \in \mathcal{D}^{< 0} \). Then \( E \) is a (possibly shifted) locally free sheaf.

It only remains to show that all of \( E_i \) are placed in the same degree in \( \mathcal{D} \), which is true since they are locally free and \( \mathcal{R}\text{Hom}(E_i, E_j) \) is placed in degree 0.

**Proof of Corollary:** The functor of twisting on \( \omega \) on the derived category can be recovered in terms of the Serre functor. Let \( \omega \) be anti-ample. According to Main Lemma, an object \( E \in \mathcal{D}^{b}_{\text{coh}}(X) \) is a (possibly shifted) locally free sheaf when \( E \) is surjective. Thus, the inclusion \( P \rightarrow E \) is surjective. Since \( \omega \) is anti-ample, \( \mathcal{R}\text{Hom}(E, E \otimes \omega^{-N}) = 0 \) for \( s \neq 0 \) and \( N \) large enough. For a nonzero vector bundle \( E \) and \( U \in \mathcal{D} \) one has \( E \in \mathcal{D}^{> 0} \) iff \( \text{Hom}^{< 0}(E_i, U \otimes \omega^{-N}) = 0 \) for large \( N \), and the same for \( \mathcal{D}^{< 0} \).

### 3. The Proof of Main Lemma

**Lemma 1.** If \( E \in \text{Coh}(X) \) and \( \mathcal{R}\text{Hom}(E, \mathcal{O}) \) are pure sheaves placed in degree 0, then \( E \) is locally free.

**Proof:** Let \( 0 \rightarrow P_k \rightarrow P_{k-1} \rightarrow \ldots \rightarrow P_0 \rightarrow 0 \) be a locally free resolvent of \( E \). Since \( \mathcal{H}\text{om}^{b}(E, \mathcal{O}) = 0 \), we see that the morphism \( \mathcal{H}\text{om}(P_{k-1}, \mathcal{O}) \rightarrow \mathcal{H}\text{om}(P_k, \mathcal{O}) \) is surjective. Thus, the inclusion \( P_k \rightarrow P_{k-1} \) is locally split and the quotient sheaf \( P_{k-1}/P_k \) is locally free, which allows to change our resolvent to a shorter one.

Let \( U \otimes \mathcal{L} \) denote the derived functor of tensor product over \( \mathcal{O}_X \) on \( \mathcal{D}^{b}_{\text{coh}}(X) \); then one has \( \mathcal{R}\text{Hom}(U, V) = \mathcal{R}\text{Hom}(U, \mathcal{O}) \otimes \mathcal{L} \).

**Lemma 2.** Let \( E, F \in \mathcal{D}^{b}_{\text{coh}}(X) \); suppose \( E \otimes \mathcal{L} F \in \mathcal{D}^{< 0} \). Then for any \( i + j \geq 0 \) one has \( \text{supp} \mathcal{H}^i(E) \cap \text{supp} \mathcal{H}^j(F) = \emptyset \).

**Proof:** Proceed by decreasing induction on \( i + j \). Consider the Künneth spectral sequence

\[
E_2^{pq} = \bigoplus_{i+j=q} \text{Tor}_{-p}(\mathcal{H}^i(E), \mathcal{H}^j(F)) \Longrightarrow \mathcal{H}^{p+q}(E \otimes \mathcal{L} F).
\]

If the intersection of supports is nonzero, then it is easy to see that \( \mathcal{H}^r E \otimes \mathcal{H}^r F \neq 0 \), thus \( E_2^{0,0} \neq 0 \). This term can be only killed by some \( E^{-r,q+r,-1} \), where \( r \geq 2 \); but it follows from the induction hypothesis that \( E_2^{0,0} = 0 \).

**Proof of Main Lemma:** Let \( F = \mathcal{R}\text{Hom}(E, \mathcal{O}) \); then one has \( \mathcal{R}\text{Hom}(E, E) = E \otimes \mathcal{L} F \). Using a shift, we can assume that \( E \in \mathcal{D}^{\leq 0} \) and \( \mathcal{H}^0(E) \neq 0 \); then \( F \in \mathcal{D}^{\geq 0} \) and \( \mathcal{H}^0 F = \mathcal{H}\text{om}(\mathcal{H}^0 E, \mathcal{O}) \). By Lemma 2, we have \( \text{supp} \mathcal{H}^0 E \cap \text{supp} \mathcal{H}^0 F = \emptyset \). Clearly, one can assume that \( X \) is irreducible. First let us show that \( \text{supp} \mathcal{H}^0(E) = X \). Indeed, in the other case it is clear that \( \mathcal{H}^0 F = 0 \) and the restriction of \( F \) on
$X \setminus \text{supp} \mathcal{H}^> \, F$ is acyclic while the restriction of $E$ is not, which contradicts the local nature of $\mathcal{RHom}$. Thus we have $\text{supp} \mathcal{H}^0(E) = X$, which implies $\mathcal{H}^> \, F = 0$ and $F \in \text{Coh}(X)$. It follows that $E = \mathcal{RHom}(F, \mathcal{O}) \in D^> \, 0$ and $E \in \text{Coh}(X)$. By Lemma 1, $E$ is locally free. \hfill \Box

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Independent University of Moscow

E-mail address: posic@mccme.ru