Self-Consistent Theory of Normal-to-Superconducting Transition

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ABSTRACT

I study the normal-to-superconducting (NS) transition within the Ginzburg-Landau (GL) model, taking into account the fluctuations in the $m$-component complex order parameter $\psi_\alpha$ and the vector potential $\vec{A}$ in the arbitrary dimension $d$, for any $m$. I find that the transition is of second-order and that the previous conclusion of the fluctuation-driven first-order transition is an artifact of the breakdown of the $\epsilon$-expansion and the inaccuracy of the $1/m$-expansion for physical values $\epsilon = 1$, $m = 1$. I compute the anomalous $\eta(d, m)$ exponent at the NS transition, and find $\eta(3, 1) \approx -0.38$. In the $m \to \infty$ limit, $\eta(d, m)$ becomes exact and agrees with the $1/m$-expansion. Near $d = 4$ the theory is also in good agreement with the perturbative $\epsilon$-expansion results for $m > 183$ and provides a sensible interpolation formula for arbitrary $d$ and $m$.

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Normal-to-\textit{neutral}-superfluid transition is one of the best understood second-order phase transitions with an unparalleled agreement between theory, simulations and experiments. In contrast the problem of the normal-to-\textit{charged}-superfluid i.e. the normal-to-superconducting (NS) transition is significantly more challenging. This problem was first studied twenty years ago by Halperin, Lubensky and Ma (HLM) \cite{HLM} with the GL model generalized to \( m \) complex component \( \psi_\alpha \) superconducting order parameter. Using renormalization group (rg) together with the first order expansion in \( \epsilon = 4 - d \) and \( 1/m \) to treat the gauge field and the order parameter critical fluctuations, these authors found that the charge is a relevant operator that grows in the long wavelength limit. Although for an unphysically large number of order parameter components, \( m > 365.9/2 \approx 183 \), the Heisenberg fixed point (which controls the neutral superfluid transition) was found to be unstable to a new critical point at a finite value of the charge, for physical superconductors \((m = 1)\) no new perturbative critical point was found to terminate this charge instability. The authors interpreted these runaway rg flows as a signal of a fluctuation-driven first-order phase transition, providing a first example in which the fluctuations modify the order of the transition. Similar conclusions were also reached for the scalar electrodynamics, exactly in four dimensions in the context of quantum field theory \cite{2}.

Although this interpretation is believed to be correct near \( d = 4 \), the conclusion of the first-order transition for the extreme type-II superconductors in \( d = 3 \), is most certainly suspect. Following the original work of Ref.\cite{HLM} Dasgupta and Halperin \cite{3} studied the problem on the lattice. Using duality arguments together with Monte Carlo techniques, they found that a 3d superconductor exhibits a second-order transition in the universality class of the (inverted) XY-model. Later Monte Carlo simulations in 3d further demonstrated that the nature of the transition changes from first- to second-order as one goes from the type-I to the extreme type-II superconductor. \cite{4}

In high \( T_c \) superconductors the thermal fluctuations are enhanced and lead to an increase of the critical region by several orders of magnitude as compared to the conventional superconductors. Unfortunately, even in these materials, the size of the critical region is still much too small to experimentally resolve the question of existence of fluctuation-driven first-order NS transition, and therefore the nature of the transition appears to be an academic question. However, it is believed that the nematic-to-smectic-A (NA) transition in liquid crystals is described by a model very similar to the GL gauge theory of the NS transition, \cite{5} and therefore the same conclusions apply to this system. \cite{6} In contrast to superconductors, however, the NA transition is estimated to have a critical region
and the size of the fluctuation-driven first-order transition to be within the experimentally accessible range. Since the NA transition appears experimentally [7], and predicted theoretically [8] to be continuous, I take this as a further indication of the breakdown of the perturbative $\epsilon$-expansion in 3d and question the conclusions of Ref.[1].

In this Letter, I reexamine the problem of the NS transition with analytical methods that do not rely on the perturbative expansion in $\epsilon$ or $1/m$. Using a non-perturbative method which amounts to solving approximate Dyson equations for arbitrary $d$ and $m$, [9] I find a nontrivial critical fixed point ($\epsilon \neq 0$) that controls the NS transition. When the order parameter and the gauge field fluctuations are taken into account the Heisenberg critical point ($\epsilon = 0$) controlling the neutral superfluid transition is found to be unstable to this new critical point. I therefore show that in contrast to the previous conclusions based on the $\epsilon$-expansions, the 3d type-II superconductors undergo a second-order NS transition, consistent with the consensus described above. Besides being an independent prediction for the nature of the NS transition in 3d, corroborating the findings of Ref.[3], my approach has the advantage of working in arbitrary dimension and therefore sheds light on the question of how the 3d behavior is connected to the findings near $d = 4$.

Within the GL description, the generalized superconductor is defined by the free-energy functional $F[\psi_\alpha, \vec{A}]$ of the $m$-complex-component superconducting order parameter $\psi_\alpha$ and the electromagnetic vector potential $\vec{A}$

$$
\frac{F[\psi_\alpha, \vec{A}]}{k_B T} = \int d^d x \left[ |(\vec{\nabla} - iq_o \vec{A})\psi_\alpha|^2 + r_o |\psi_\alpha|^2 + \frac{1}{2} u_o (|\psi_\alpha|^2)^2 + \frac{1}{8\pi \mu_o} (\vec{\nabla} \times \vec{A})^2 \right],
$$

(1)

where $r_o \sim (T - T_c)/T_c$, $q_o = 2e/\hbar c$, and $\mu_o$ is magnetic permeability of the normal metal. The choice of the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$, leads to few simplifications.

I study the critical behavior of the NS transition within the self-consistent screening approximation (SCSA) that has previously been quite successfully applied to a variety of other problems. [9] The approximation builds on the $1/m$-expansion for general dimensionality $d$. [10] One writes downs the large $m$ limit expressions for the renormalized interactions and propagators in terms of the bare ones and then replaces all the bare quantities by the renormalized ones thereby obtaining the large $m$ limit of Dyson equations for the renormalized interactions and propagators. The advantage of this method is that in the limit $m \to \infty$ it reduces to the exact $1/m$ result. Furthermore, while the straight $1/m$-expansion diverges for $m \to 0$ and therefore cannot be taken seriously quantitatively
for the physical value of \( m = 1 \), the SCSA is perfectly well behaved in this limit and is therefore quantitatively more trustworthy for real superconductors.

To simplify the analysis it is convenient to integrate out the gauge field, which can formally be done exactly since \( \vec{A} \) appears at most quadratically Eq.(1) , giving \( F_{\text{tot}}/k_B T = \int_x \left[ \nabla \psi_\alpha |^2 + r_o |\psi_\alpha|^2 + \frac{1}{2} u_o (|\psi_\alpha|^2)^2 - \frac{1}{2} J_i(x) D_{ij}(x) J_j(x) + \frac{1}{2} T r \log(D_{ij}^{-1}) \right] \). The last two terms are the long-range effective current-current interaction and the functional determinant generated from integration over \( \vec{A} \), respectively. \( J_i(x) = i q_o (\psi_\alpha^* \nabla_i \psi_\alpha - \psi_\alpha^* \nabla_i \psi_\alpha^*) \) is the paramagnetic current, \( D_{ij}^{-1} = \left[ -\nabla^2/(4 \pi \mu_o) + 2 q_o^2 \psi_\alpha^* \psi_\alpha \right] P_{ij}^T \) is the inverse of the gauge field propagator and can be read off from Eq.(1) . The treatment of type-I superconductors is relatively simple because the relevant temperature range lies well outside the critical region, and therefore the order parameter fluctuations can be ignored.\(^{[6]} \) By minimizing the effective free energy in the standard way \(^{[10]} \) I obtain a gauge field corrected mean-field theory describing the first-order NS transition previously found in Ref.\(^{[1]} \).

A more interesting and challenging regime is that of the type-II superconductors where the fluctuations in the order parameter field are strong and must be carefully taken into account. To treat this case I expand the free energy functional in powers of \( \psi_\alpha \) to quartic order, and because of the smallness of the order parameter near the NS transition I ignore the higher order corrections. I thereby obtain an effective field theory in terms of \( \psi_\alpha \) alone, with long-range self-interactions, described by an effective free energy \( F_{eff}[\psi_\alpha] \)

\[
\frac{F_{eff}[\psi_\alpha]}{k_B T} = \int_k \psi_\alpha^*(k)(k^2 + r_o)\psi_\alpha(k) + \frac{1}{2} \int_{k_1, k_2, p} U_o(k_1, k_2, p) \psi_\alpha^*(k_1 - p) \psi_\alpha(k_1) \psi_\beta^*(k_2 + p) \psi_\beta(k_2) ,
\]

expressed in Fourier space with \( \psi_\alpha(k) = \int d^d x \psi_\alpha(x) e^{-i k x} \) and \( \int_k = \int d^d k/(2 \pi)^d \). The effective long-range vertex of the quartic interaction is \( U_o(k_1, k_2, p) = u_o - f_o k_1 k_2 P_{ij}^T(p)/p^2 \), where \( f_o = 16 \pi \mu_o q_o^2 \) is the bare effective charge and \( P_{ij}^T(p) = \delta_{ij} - p_i p_j/p^2 \).

Using this effective free energy I write down the coupled Dyson equations for the renormalized \( \psi_\alpha \) propagator \( G(k) \) and the renormalized quartic interactions \( u(p) \) and \( f(p) \)

\[
G^{-1}(k) = G_o^{-1}(k) + \int_p U(k, k - p, p) G(k - p) ,
\]

\[
u(p) = \frac{u_o}{1 + u_o \Pi_u(p)} , \quad f(p) = \frac{f_o}{1 + f_o \Pi_f(p)} ,
\]

where \( \Pi_u(p) = m \int_{p'} G(p') G(p - p') \) and \( \Pi_f(p) = -m P_{ij}^T(p)/(d - 1)/p^2 \int_{p'} p'_i p'_j G(p') G(p - p') \) are the polarization bubbles. The diagrammatic version of these equations is displayed
in fig. [3]. I look for the long-wavelength-limit solutions of the above integral equations for \( G(k) \), \( u(p) \) and \( f(p) \). In general this can be done numerically with the simplification that near a critical point there are only two relevant length scales, \( k^{-1} \) and the correlation length \( \xi \), and therefore for example \( G^{-1}(k) = k_\eta^\eta k^{2-\eta} g(k\xi) \), with \( g(x) \) being the scaling function and \( k_c \) is a constant that depends on the microscopics of the model. However, exactly at criticality, \( r = 0 \), the correlation length \( \xi \) diverges and the scaling function \( g(x \to \infty) = 1 \). In this case \( k^{-1} \) is the only relevant length scale with correlation functions assuming even a simpler scaling form. In particular \( G^{-1}(k) = k_\eta^\eta k^{2-\eta} \), integral equations above can be solved exactly, and \( \eta \) determined analytically, as I demonstrate below.

Substituting the simplified scaling form for \( G^{-1}(k) \) into \( \Pi_u(p) \) and \( \Pi_f(p) \) I find,

\[
\Pi_u(p) = mI_0(1-\eta/2,1-\eta/2)k_c^{-2\eta}p^{d-4+2\eta}, \quad \Pi_f(p) = mI_02(1-\eta/2,1-\eta/2)k_c^{-2\eta}p^{d-4+2\eta},
\]

where I defined integrals, \( I_0(a,b) = \int \rho'(\hat{\rho} - p')^{-a}/p'^{-b} = \Gamma(a + b - d/2)\Gamma(d/2 - a)\Gamma(d/2 - b)/(4\pi)^{d/2}\Gamma(a)/\Gamma(b)\Gamma(d - a - b) \), \( I_{ij}(a,b) = \int \rho'_i\rho'_j(\hat{\rho} - p')^{-a}\rho'^{-b} = \delta_{ij}I_0(a,b) + \hat{\rho}_i\hat{\rho}_jI_{22}(a,b), \quad I_{02}(a,b) = -\Gamma(a + b - d/2 - 1)\Gamma(d/2 - a + 1)\Gamma(d/2 - b + 1)/2/(4\pi)^{d/2}\Gamma(a)/\Gamma(b)\Gamma(d - a - b + 2), \quad I_{22}(a,b) = \Gamma(a + b - d/2)\Gamma(d/2 - a)\Gamma(d/2 - b + 2)/(4\pi)^{d/2}\Gamma(a)/\Gamma(b)\Gamma(d - a - b + 2), \) and \( \hat{\rho} \) is a unit vector.

I first look at the Heisenberg critical point by setting \( f_o = 0 \), which automatically leads to \( f(p) = 0 \) from Eq.(34). Assuming that \( 4 - d > 2\eta \), (this assumption will be satisfied by the solution for \( \eta \) for \( d < 4 \); for \( d > 4 \) the Gaussian fixed point result is recovered) in the long wavelength limit, \( p \to 0 \), \( \Pi_u(p) \) dominates over the 1 in the denominator of Eq.(34) and the renormalized \( u(p) \) interaction reduces to a universal function,

\[
u(p) = \Pi_u^{-1}(p) = \frac{1}{mI_0(1-\eta/2,1-\eta/2)k_c^{-2\eta}p^{-d+4-2\eta}}. \tag{5}
\]

Using the renormalized version of the quartic interaction together with above equations and the scaling form for \( G(k) \) in Eq.(34), I find that \( \eta \) is determined by \( m = I_0(1-\eta/2,\eta + d/2 - 2)/I_0(1-\eta/2,1-\eta/2) \). For the physical superfluids with \( m = 1 \) and \( d = 3 \) this result leads to \( \eta \approx 0.125 \). The implicit equation for \( \eta(d,m) \) can also be expanded in \( \epsilon = 4 - d \) or \( 1/m \) in which case I obtain \( \eta_\epsilon = \epsilon^2/(4m) \) (arbitrary \( m \)) and \( \eta_m = 4/(3\pi^2 m) \) (\( d = 3 \)), respectively. The large \( m \) limit of this result, \( \eta_m \), by construction agrees exactly with the direct \( 1/m \)-expansion result. However, the \( d \to 4 \) limit of SCSA, \( \eta_\epsilon \), does not get the \( m \) dependence quite correctly when compared to the leading order in the \( \epsilon \)-expansion, where the result is \( \eta = \epsilon^2(1 + m)/(8 + 2m)^2 \). I note that this disagreement with \( \epsilon \)-expansion for
small \( m \) is expected from the fact that SCSA does not correctly account for triple vertex renormalization which are taken into account in the direct \( \epsilon \)-expansion for arbitrary \( m \).\[10\]

I now apply the above calculations to the full problem of the NS transition. Allowing now for a nonzero charge, \( f_0 \neq 0 \), I arrive at the charged analogs of Eqs.(3). The expression for \( u(p) \) is the same, and \( f(p) \) in the asymptotic limit reduces to

\[
f(p) = \Pi_f^{-1}(p) = \frac{1}{m I_0^2(1 - \eta/2, 1 - \eta/2)} k_c^{2\eta} p^{-d+4-2\eta}.
\] (6)

Substituting these screened interactions into \( U \), defined by the renormalized version of \( U_0 \) and then using Eq.(3a) I obtain,

\[
k_c^\eta k^{2-\eta} = \frac{k_c^\eta}{m} \int_p \left( \frac{I_0^{-1}(1 - \eta/2, 1 - \eta/2)}{(k-p)^2 - \eta p^{d-4+2\eta}} - \frac{I_2^{-1}(1 - \eta/2, 1 - \eta/2)k_i k_j P_{ij}(p)}{(k-p)^2 - \eta p^{d-2+2\eta}} \right).
\] (7)

Performing above integrals leads to the equation which determines \( \eta(d, m) \) at the new superconducting critical point,

\[
m = \frac{I_0(1 - \eta/2, \eta + d/2 - 2)}{I_0(1 - \eta/2, 1 - \eta/2)} - \frac{I_0(1 - \eta/2, \eta + d/2 - 1) + I_0(1 - \eta/2, \eta + d/2) - I_2^1(1 - \eta/2, \eta + d/2)}{I_0(1 - \eta/2, 1 - \eta/2)}.
\] (8)

The above implicit result for \( \eta(d, m) \) reduces to \( \eta_c = -9\epsilon/(m - 18) \) in the limit of \( d \to 4 \), for arbitrary \( m \). In the regime where HLM find the NS critical fixed point \( (m > 183) \), this \( \epsilon \)-expansion result is less than within 10% of their exact (to \( O(\epsilon) \) ) value of \( \eta_c^{HLM} = -9\epsilon/m \).\[1\] It is important to note that although the complete SCSA result for \( \eta \) (Eq.(8)) is well behaved as a function of \( m \) (see fig.[2]), it breaks down at a critical value of \( m_c = 18 \), when expanded in \( \epsilon \). This suggests that the dissappearance of the critical point and the runaway rg flows for \( m < m_c \approx 183 \) in the direct \( \epsilon \)-expansion of Ref.[1] should be interpreted as the breakdown of the \( \epsilon \)-expansion rather than the fluctuation driven first-order transition.

Expanding the result, Eq.(8) , for large \( m \) in powers of \( 1/m \) I recover the results of HLM in this limit, obtaining \( \eta_m = -20/\pi^2 m \approx -2.026/m \), for \( d = 3 \). It is important to note, however, that the direct \( 1/m \)-expansion leads to the value of \( \eta_m = -2.026 \) (for \( d = 3, m = 1 \) ) that lies outside the physical range \( \eta > 2 - d = -1 \). In contrast the SCSA approximation gives a sensible result of \( \eta = -0.38 \) for real superconductors, that is well within this physical range. For \( d > 4 \), I recover the Gaussian fixed point, as expected since
the upper-critical dimension for the NS transition is $d_{uc} = 4$. The SCSA then serves as a physical interpolation between large $m$ and small $\epsilon$ behavior.

In conclusion, I find that the self-consistently modified large $m$- expansion for the generalized theory of the NS transition leads to a nontrivial critical point for 3d superconductors. In contrast to the original HLM interpretation I predict a second-order transition for type-II superconductors in $d = 3$, in qualitative agreement with the work of Dasgupta and Halperin and with the related theoretical and experimental findings for the NA transition. The results suggest a break down of the $\epsilon$-expansion below a critical value of $m$, while the actual NS transition remains continuous.

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**Figure Captions**

Fig. 1. Graphical representation of SCSA for renormalized propagator and interaction.

Fig. 2. $\eta^{SCSA}$ (for $d = 3$, full curve) plotted as a function of $m$, showing improvement at small $m$ compared to the direct $1/m$-expansion result, $\eta_m = -20/(\pi^2 m)$ (dashed curve). The inset shows $\eta^{SCSA}$ as a function of $m$ for various values of $d$. 