On the maximum value of conflict-free vertex-connection number of graphs*

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Abstract A path in a vertex-colored graph is called conflict-free if there is a color used on exactly one of its vertices. A vertex-colored graph is said to be conflict-free vertex-connected if any two vertices of the graph are connected by a conflict-free path. The conflict-free vertex-connection number, denoted by \( vcfc(G) \), is defined as the smallest number of colors required to make \( G \) conflict-free vertex-connected. Li et al. [10] conjectured that for a connected graph \( G \) of order \( n \), \( vcfc(G) \leq vcfc(P_n) \). We confirm that the conjecture is true and pose a relevant conjecture concerning the conflict-free connection number introduced by Czap et al. in [6].

Keywords: Conflict-free connection; Conflict-free vertex-connection; Tree

1 Introduction

We consider simple, finite and undirected graphs only, and refer to the book [1] for undefined notation and terminology. Let \( G = (V, E) \) be a finite graph with vertex set \( V \) and edge set \( E \). The size of \( G \), denoted by \( e(G) \), is \( |E| \). The degree of a vertex \( v \), denoted by \( d_G(v) \), is the number of edges which are incident with \( v \) in \( G \). As usual, \( \delta(G) \) and \( \Delta(G) \) denote the minimum degree and the maximum degree of \( G \), respectively. A subgraph \( H \) of \( G \) is a spanning subgraph of \( G \) if \( V(H) = V(G) \).

We use \( K_n \), \( P_n \), and \( K_{1,n-1} \) to denote the complete graph, the path, and the star of order \( n \), respectively.

Very recently, Czap et al. introduced the concept of conflict-free connection in [6]. A path in an edge-colored graph is called conflict-free if there is a color used on exactly one of its edges. An edge-colored graph is said to be conflict-free connected if any two vertices of the graph are connected by a conflict-free path. The conflict-free connection number of a connected graph, denoted by \( cfc(G) \), is defined as the smallest number of colors required to make \( G \) conflict-free connected.

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Motivated by the above mentioned concepts, as a natural counterpart of a conflict-free connection number, Li et al. [10] introduced the concept of conflict-free vertex-connection number. A path in a vertex-colored graph is called conflict-free if there is a color used on exactly one of its vertices. A vertex-colored graph is said to be conflict-free vertex-connected if any two vertices of the graph are connected by a conflict-free path. The conflict-free vertex connection number, denoted by $vcfc(G)$, is defined as the smallest number of colors required to make $G$ conflict-free vertex-connected. Note that for a nontrivial connected graph $G$ of order $n$,

$$2 \leq vcfc(G) \leq n$$ (1)

Li et al. determined the conflict-free vertex connection number of almost all graphs by showing the following result.

**Theorem 1.1.** (Li et al. [10]) Let $G$ be a connected graph $G$ of order at least three. Then $vcfc(G) = 2$ if and only if $G$ is 2-connected or it has only one cut vertex.

So, a basic question arise: what is the maximum value of the conflict-free vertex connection numbers of all graphs of order $n$?

It can be observed in [10] that for a nontrivial connected graph $G$, if $H$ is a spanning subgraph of $G$, then $vcfc(H) \geq vcfc(G)$. In particular, for any spanning tree $T$ of $G$, $vcfc(T) \geq vcfc(G)$. Thus the maximum value of the conflict-free vertex connection numbers must be achieved by some tree of order $n$. It can be checked that $vcfc(K_{1,n-1}) = 2$ for any $n \geq 2$. In particular, Li et al. showed that

**Theorem 1.2.** (Li et al. [10]) For an integer $n \geq 2$, $vcfc(P_n) = \lceil \log_2(n + 1) \rceil$.

A $k$-ranking of a connected graph $G$ is a labeling of its vertices with labels $1, \ldots, k$ such that every path between any two vertices with the same label $i$ in $G$ contains at least one vertex with label $j > i$. A graph $G$ is said to be $k$-rankable if it has a $k$-ranking. The minimum $k$ for which $G$ is $k$-rankable is denoted by $r(G)$. Iyer et al. [9] showed that for a tree of order $n \geq 3$, $r(T) \leq \log_2\frac{n}{2}$. Li et al. [10] showed that

**Theorem 1.3.** (Li et al. [10]) For a tree of order $n \geq 3$, $vcfc(T) \leq \log_2\frac{n}{2}$.

Further, Li et al. [10] conjectured that

**Conjecture 1.4.** For a connected graph $G$ of order $n$, $vcfc(G) \leq vcfc(P_n)$.

The aim of this note is to prove the conjecture. We refer to [2, 3, 4, 5, 8, 12] for some relevant works on conflict-free coloring of graphs.
2 The proof

We begin with the following key lemma. For convenience, we denote by moc($T - v$) the maximum value of the orders of all components of $T - v$.

**Lemma 2.1.** Let $T$ be a tree of order $n \geq 3$. If $n$ is odd, then there exists a vertex $v$ with moc($T - v$) $\leq \frac{n-1}{2}$.

**Proof.** Choose a vertex $v_0 \in V(T)$ such that moc($T - v_0$) = min\{moc($T - v$) | $v$ run over all non-leaf vertices of $T$\}. We claim that $v_0$ is the vertex $v$, as we required. If it is not, then there exists a component of $T - v_0$, say $T_1$, has order $n_1 > \frac{n-1}{2}$. Note that $n_1 = moc(T - v_0)$. Let $v_1$ be the neighbor of $v_0$ in $T_1$. Let us consider the orders of the components of $T - v_1$. The component of $T - v_0v_1$ containing $v_0$ is a component of $T - v_1$ having order with $n - n_1 < n - \frac{n-1}{2} = \frac{n+1}{2}$ (implying that $n - n_1 \leq \frac{n-1}{2}$). Moreover, since all other components of $T - v_1$ is a proper subgraph of $T_1$, their orders are less than the order of $T_1$. It follows that moc($T - v_1$) $< n_1 = moc(T - v_0)$, contradicting the choice of $v_0$. This shows that the claim is true, and thus the result follows. 

Now we are ready to prove Conjecture 1.4.

**Theorem 2.2.** For a tree $T$ of order $n$, vcfc($T$) $\leq$ vcfc($P_n$).

**Proof.** We show it by induction on $n$. The result is trivially true when $n = 2$, and now assume that $n \geq 3$. Then there exists an integer $k \geq 2$ such that $2^{k-1} \leq n \leq 2^k - 1$. Let $T$ be a spanning tree of $G$. As we have seen before, vcfc($G$) $\leq$ vcfc($T$). By Theorem 1.2, vcfc($P_n$) = $k$. So it suffices to show that vcfc($T$) $\leq$ $k$.

By Lemma 2.1, there exists a vertex $v \in V(G)$ with moc($T - v$) $\leq 2^{k-1} - 1$ (If necessary, by adding some pendent vertices to $T$, one can make the resulting tree $T'$ have $2^{k-1} - 1$ vertices, and apply Lemma 2.1 to $T'$). Let $T_1, \ldots, T_l$ be all components of $T - v$, and $n_i = |V(T_i)|$ for each $i$. By the induction hypothesis, vcfc($T_i$) $\leq k - 1$ for each $i$. Taking a conflict-free coloring of $T_i$ using colors in $\{1, \ldots, k - 1\}$ for each $i$, and color the vertex $v$ by $k$, we obtain a conflict-free coloring of $T$ using colors in $\{1, \ldots, k\}$. This proves vcfc($T$) $\leq k$, and thus vcfc($G$) $\leq$ vcfc($P_n$).

3 Further research

In this note, we focuss on the conflict-free vertex-connection number, and combining some known results, we have shown that for a connected graph of order $n$,

$$2 \leq vcfc(G) \leq \lceil \log_2(n + 1) \rceil,$$
where the lower bound can be achieved by 2-connected graphs of order \( n \) and the upper bound can be achieved by \( P_n \).

In [6], Czap et al. determined the conflict-free connection number of all graphs by showing that for a noncomplete 2-connected graph \( G \), \( cfc(G) = 2 \). It was further extended in [2] by Chang et al. showing that for a noncomplete 2-edge-connected graph \( G \), \( cfc(G) = 2 \).

Clearly, for an integer \( n \geq 2 \), \( K_n \) is the unique connected graph \( G \) of order \( n \) with \( cfc(G) = 1 \). On the other hand, \( cfc(K_{1,n-1}) = n - 1 \). Observe that for a nontrivial connected graph \( G \), if \( H \) is a spanning subgraph of \( G \), then \( cfc(H) \geq cfc(G) \). In particular, for any spanning tree \( T \) of \( G \), \( cfc(T) \geq cfc(G) \). Thus the maximum value of the conflict-free connection numbers must be achieved by some tree of order \( n \). Actually, (see [10]) for a nontrivial connected graph \( G \) of order \( n \),

\[
1 \leq cfc(G) \leq n - 1,
\]

with the left hand side of equality if and only if \( G \cong K_n \), and with the right hand side of equality if and only if \( G \cong K_{1,n-1} \).

It is an interesting problem to decide that among all trees of order \( n \), which one has the least conflict-free connection number? Czap et al. [6] showed that for an integer \( n \geq 2 \), \( cfc(P_n) = \lceil \log_2 n \rceil \). We pose the following conjecture.

**Conjecture 3.1.** For a tree \( T \) of order \( n \), \( cfc(T) \geq \lceil \log_2 n \rceil \).

Another interesting problem, posed by Li [11], is the complexity for determining the conflict-free connection number or the conflict-free vertex-connection number of a graph. Yang [13] designed a polynomial-time algorithm to determine the conflict-free connection number of a tree.

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