Randomized Exploration for Reinforcement Learning with General Value Function Approximation

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Abstract

We propose a model-free reinforcement learning algorithm inspired by the popular randomized least squares value iteration (RLSVI) algorithm as well as the optimism principle. Unlike existing upper-confidence-bound (UCB) based approaches, which are often computationally intractable, our algorithm drives exploration by simply perturbing the training data with judiciously chosen i.i.d. scalar noises. To attain optimistic value function estimation without resorting to a UCB-style bonus, we introduce an optimistic reward sampling procedure. When the value functions can be represented by a function class $\mathcal{F}$, our algorithm achieves a worst-case regret bound of $O(\text{poly}(d_E H) \sqrt{T})$ where $T$ is the time elapsed, $H$ is the planning horizon and $d_E$ is the eluder dimension of $\mathcal{F}$. In the linear setting, our algorithm reduces to LSVI-PHE, a variant of RLSVI, that enjoys an $\tilde{O}(\sqrt{d^3 H^3 T})$ regret. We complement the theory with an empirical evaluation across known difficult exploration tasks.

1. Introduction

The exploration-exploitation trade-off is a core problem in reinforcement learning (RL): an agent may need to sacrifice short-term rewards to achieve better long-term returns. A good RL algorithm should explore efficiently and find a near-optimal policy as quickly and robustly as possible. A big open problem is the design of provably efficient exploration when general function approximation is used to estimate the value function, i.e., the expectation of long-term return. In this work, we propose an exploration strategy inspired by the popular Randomized Least Squares Value Iteration (RLSVI) algorithm (Osband et al., 2016b; Russo, 2019; Zanette et al., 2020a) as well as by the optimism principle (Brafman & Tennenholtz, 2001; Jaksch et al., 2010; Jin et al., 2018; 2020; Wang et al., 2020), which is efficient in both statistical and computational sense, and can be easily plugged into common RL algorithms, including UCB-VI (Azar et al., 2017), UCB-Q (Jin et al., 2018) and OPPO (Cai et al., 2019).

The main exploration idea is the well-known “optimism in the face of uncertainty (OFU)” principle, which leads to numerous upper confidence bound (UCB)-type algorithms. These algorithms compute statistical confidence regions for the model or the value function, given the observed history, and perform the greedy policy with respect to these regions, or upper confidence bounds. However, it is costly or even intractable to compute the upper confidence bound explicitly, especially for structured MDPs or general function approximations. For instance, in Wang et al. (2020), computing the confidence bonus requires sophisticated sensitivity sampling and a width function oracle. The computational cost hinders the practical application of these UCB-type algorithms.

Another recently rediscovered exploration idea is Thompson sampling (TS) (Thompson, 1933; Osband et al., 2013). It is motivated by the Bayesian perspective on RL, in which we have a prior distribution over the model or the value function; then we draw a sample from this distribution and compute a policy based on this sample. Theoretical guarantees exist for both Bayesian regret (Russo & Van Roy, 2013) and worst-case regret (Agrawal & Jia, 2017) for this approach. Although TS is conceptually simple, in many cases the posterior is intractable to compute and the prior may not exist at all. Recently, approximate TS, also known as randomized least squares value iteration (RLSVI) or following the perturbed leader (Kveton et al., 2019), has received significant attention due to its good empirical performance. It has been proven that RLSVI enjoys sublinear worst-case or frequentist regret in tabular RL, by simply adding Gaus-
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sian noise on the reward (Russo, 2019; Agrawal et al., 2020). However, in the improved bound for tabular MDP (Agrawal et al., 2020) and linear MDP (Zanette et al., 2020a), the uncertainty of the estimates still needs to be computed in order to perform optimistic exploration; it is unknown whether this can be removed. Moreover, this computation is difficult to do in the general function approximation setting.

In this work, we propose a novel exploration idea called optimistic reward sampling, which combines OFU and TS organically. The algorithm is surprisingly simple: we perturb the reward several times and act greedily with respect to the maximum of the estimated state-action values. The intuition is that after the perturbation, the estimate has a constant probability of being optimistic, and sampling multiple times guarantees that the maximum of these sampled estimates is optimistic with high probability. Thus, our algorithm utilizes approximate TS to achieve optimism.

Similar algorithms have been shown to work empirically, including SUNRISE (Lee et al., 2020), NoisyNet (Fortunato et al., 2017) and bootstrapped DQN (Osband et al., 2016). However, the theoretical analysis of perturbation-based exploration is still missing. We prove that it enjoys near optimal regret $O(\sqrt{Hd^3T})$ for linear MDP and the sampling time is only $M = O(d)$. We also prove similar bounds for the general function approximation case, by using the notion of eluder dimension (Russo & Van Roy, 2013; Wang et al., 2020). In addition, this algorithm is computationally efficient, as we no longer need to compute the upper confidence bound. In the experiments, we find that a small sampling time $M$ is sufficient to achieve good performance, which suggests that the theoretical choice of $M = O(d)$ is too conservative in practice.

Optimistic reward sampling can be directly plugged into most RL algorithms, improving the sample complexity without harming the computational cost. The algorithm only needs to perform perturbed regression. To our best knowledge, this is the first online RL algorithm that is both computationally and statistically efficient with linear function approximation and general function approximation. We hope optimistic reward sampling can be a large step towards bridging the gap between algorithms with strong theoretical guarantees and those with good computational performance.

2. Preliminaries

We begin by introducing some necessary notations. For any positive integer $n$, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$. For any set $A$, $\langle \cdot, \cdot \rangle_A$ denotes the inner product over set $A$. For a positive definite matrix $A \in \mathbb{R}^{d \times d}$ and a vector $x \in \mathbb{R}^d$, we denote the norm of $x$ with respect to matrix $A$ by $\|x\|_A = \sqrt{x^T A x}$. We denote the cumulative distribution function of the standard Gaussian by $\Phi(\cdot)$. For function growth, we use $\tilde{O}(\cdot)$, ignoring poly-logarithmic factors.

We consider episodic MDPs of the form $(S, A, H, P, r)$, where $S$ is the (possibly uncountable) state space, $A$ is the (possibly uncountable) action space, $H$ is the number of steps in each episode, $P = \{P_h\}_{h=1}^H$ are the state transition probability distributions, and $r = \{r_h\}_{h=1}^H$ are the reward functions. For each $h \in [H]$, $P_h(\cdot | s, a)$ is the transition kernel over the next states if action $a$ is taken at state $s$ during the $h$-th time step of the episode. Also, $r_h : S \times A \to [0, 1]$ is the deterministic reward function at step $h$.

A policy $\pi$ is a collection of $H$ functions $\{\pi_h : S \to \Delta(A)\}_{h \in [H]}$ where $\Delta(A)$ denotes probability simplex over action space $A$. We denote by $\pi(\cdot | s)$ the action distribution of policy $\pi$ for state $s$, and by $\pi^*$ the optimal policy, which maximizes the value function defined below.

The value function $V_h^\pi : S \to \mathbb{R}$ at step $h \in [H]$ is the expected sum of remaining rewards until the end of the episode, received under $\pi$ when starting from $s_h = s$,

$$V_h^\pi(s) = \mathbb{E}_\pi \left[ \sum_{h'=h}^H r_{h'}(s_{h'}, a_{h'}) \mid s_h = s \right].$$

The action-value function $Q_h^\pi : S \times A \to \mathbb{R}$ is defined as the expected sum of rewards given the current state and action when the agent follows policy $\pi$ afterwards,

$$Q_h^\pi(s, a) = \mathbb{E}_\pi \left[ \sum_{h'=h}^H r_{h'}(s_{h'}, a_{h'}) \mid s_h = s, a_h = a \right].$$

We denote $V_h^\pi(s) = V_h^{\pi^*}(s)$ and $Q_h^\pi(s, a) = Q_h^{\pi^*}(s, a)$. Moreover, to simplify notation, we denote $[P_h V_{h+1}](s, a, s') = \mathbb{E}_{s'' \sim P_h(\cdot | s, a)} V_{h+1}(s'')$.

Recall that value functions obey the Bellman equations:

$$Q_h^\pi(s, a) = (r_h + \mathbb{E}_{s' \sim P_h(\cdot | s, a)} V_{h+1}(s')),$$

$$V_h^\pi(s) = (Q_h^\pi(s, \cdot) \pi_h(\cdot | s))_{A},$$

$$V_{H+1}^\pi(s) = 0.$$

The aim of the agent is to learn the optimal policy by acting in the environment for $K$ episodes. Before starting each episode $k \in [K]$, the agent chooses a policy $\pi^k$ and an adversary chooses the initial state $s^k_1$. Then, at each time step $h \in [H]$, the agent observes $s^k_h \in S$, picks an action $a^k_h \in A$, receives a reward $r_h(s^k_h, a^k_h)$ and the environment transitions to the next state $s^k_{h+1} \sim P_h(\cdot | s^k_h, a^k_h)$.

The episode ends after the agent collects the $H$-th reward and reaches the state $s^k_{H+1}$. The suboptimality of an agent can be measured by its regret, the cumulative difference of optimal

\footnote{We assume the reward function is deterministic for notational convenience. Our results can be straightforwardly generalized to the case when rewards are stochastic.}
Algorithm 1 $\mathcal{F}$-LSVI-PHE

1: Set $M$ to be a fixed integer.
2: For episode $k = 1, 2, \ldots, K$
3: Receive the initial state $s^h_1$.
4: Set $V_h^k(s) = 0$ for all $s \in \mathcal{S}$.
5: For step $h = H, H - 1, \ldots, 1$
6: For $m = 1, 2, \ldots, M$
7: Sample i.i.d. Gaussian noise $\xi_h^{r,m} \sim N(0, \sigma^2_h)$. 
8: Perturbed dataset: $\mathcal{D}_h^{k,m} \leftarrow \{(s_h^i, a_h^i, r_h^i + \xi_h^{r,m}) \}_{i \in [k-1]}$.
9: Set $\tilde{f}_h^{k,m} \leftarrow \arg\min_{f \in \mathcal{F}} L(f | \mathcal{D}_h^{k,m})^2 + \lambda \tilde{R}(f)$.
10: Set $Q_h^{k,m}(\cdot, \cdot) \leftarrow \min \{ \max_{a \in \mathcal{A}} Q_h^{k,m}(\cdot, a), H - h + 1 \}$.
11: Take action $a_h^k \leftarrow \max_{a \in \mathcal{A}} Q_h^{k,m}(s_h^k, a)$.
12: Observe reward $r_h^k(s_h^k, a_h^k)$, get next state $s_h^{k+1}$.

and achieved return, which after $K$ episodes is

$$\text{Regret}(K) = \sum_{k=1}^{K} \left[ V^*_1(s_1^k) - V^{k}_1(s_1^k) \right].$$ (2)

**Additional notations.** The performance of function $f$ on dataset $\mathcal{D} = \{(x_i, y_i)\}_{i \in [|\mathcal{D}|]}$ is defined by $L(f | \mathcal{D}) = \left( \sum_{i=1}^{|\mathcal{D}|} (f(x_i) - y_i)^2 \right)^{1/2}$. The empirical $\ell_2$ norm of function $f$ on input set $\mathcal{Z} = \{x_i\}_{i \in [|\mathcal{Z}|]}$ is defined by $\|f\|_\mathcal{Z} = \left( \sum_{i=1}^{|\mathcal{Z}|} f(x_i)^2 \right)^{1/2}$. Given a function class $\mathcal{F} \subseteq \{f : X \rightarrow \mathbb{R}\}$, we define the width function given some input $x$ as $w(\mathcal{F}, x) = \max_{f, f' \in \mathcal{F}} f(x) - f'(x)$.

3. Algorithm: LSVI-PHE

In this section, we lay out our algorithm LSVI-PHE\(^2\) (Algorithm 1), an optimistic modification of RLSVI, where the optimism is realized by, what we will call, optimistic reward sampling. To describe our algorithm and facilitate its analysis in Section 4, we first define the perturbed least squares regression. We add noises on the regression target and the regularizer to achieve enough randomness in all directions of the regressor.

**Definition 3.1** (Perturbed Least Squares). Consider a function class $\mathcal{F} : X \rightarrow \mathbb{R}$. For an arbitrary dataset $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^{|\mathcal{D}|}$, a regularizer $R(f) = \sum_{j=1}^{|\mathcal{D}|} p_j(f)$ where $p_j(\cdot)$ are functionals, and positive constant $\sigma$, the perturbed dataset and perturbed regularizer are defined as

$$\tilde{\mathcal{D}}_\sigma = \{(x_i, y_i + \xi_i)\}_{i=1}^{|\mathcal{D}|}, \quad \tilde{R}_\sigma(f) = \sum_{j=1}^{|\mathcal{D}|} [p_j(f) + \xi_j]^2,$$

where $\xi_i$ and $\xi_j$ are i.i.d. zero-mean Gaussian noises with variance $\sigma^2$. For a loss function $L$, the corresponding perturbed least squares regression solution is

$$\tilde{f}_\sigma = \arg\min_{f \in \mathcal{F}} L(f | \tilde{\mathcal{D}}_\sigma) + \lambda \tilde{R}_\sigma(f).$$

Within each episode $k \in [K]$, at each time-step $h$, we perturb the dataset by adding zero mean random Gaussian noise to the reward in the replay buffer $\{(s_h^i, a_h^i, r_h^i)\}_{i \in [k-1]}$ and the regularizer before we solve the perturbed regularized least-squares regression. At each time step $h$, we repeat the process for $M$ (to be specified in Section 4) times and use the maximum of the regressor as the optimistic estimate of the state-action value function. Concretely, we set $V_h^{k+1} = 0$ and calculate $Q_h^{k+1} \hat{Q}_h^{k+1}, \ldots, Q_h^{k+1}$ iteratively as follows.

For each $h \in [H]$ and $m \in [M]$, we solve the following perturbed regression problem,

$$\tilde{f}_h^{k,m} \leftarrow \arg\min_{f \in \mathcal{F}} L(f | \tilde{\mathcal{D}}_h^{k,m})^2 + \lambda \tilde{R}(f).$$ (3)

We set $Q_h^{k,m}(\cdot, \cdot) = \tilde{f}_h^{k,m}(\cdot, \cdot)$ and define

$$Q_h^{k,m}(\cdot, \cdot) = \min \{ \max_{m \in [M]} \{Q_h^{k,m}(\cdot, \cdot)\}, H - h + 1 \}. \quad \text{(4)}$$

We then choose the greedy policy with respect to $Q_h^k$ and collect a trajectory data for the $k$-th episode. We repeat the procedure until all the $K$ episodes are completed.

3.1. LSVI-PHE with Linear Function Class

We now present LSVI-PHE when we consider linear function class (see Algorithm 2). In this case, the following proposition shows that, adding scalar Gaussian noise to the reward is equivalent to perturbing the least-squares estimate using $d$-dimensional multivariate Gaussian noise.

**Proposition 3.2.** In line 9 of Algorithm 2, conditioned on all the randomness except $\{\xi_h^{r,j}\}_{j \in [k-1]} \times [M]$ and $\{\xi_h^{k,j}\}_{j \in [M]}$, the estimated parameter $\hat{\theta}_h^{k,j}$ satisfies

$$\hat{\theta}_h^{k,j} - \hat{\theta}_h^{k,j} \sim N(0, \sigma^2(L_h^{k,j})), $$

where $\hat{\theta}_h^{k,j} = (L_h^{k,j})^{-1}(\sum_{i=1}^{k-1}[r_h^i + V_h^{k+1}(s_h^{i+1})]\phi(s_h^i, a_h^i))$ is the unperturbed regressor.

Intuitively, adding a zero-mean multivariate Gaussian noise on the parameter $\hat{\theta}_h^k$ can guarantee that $\hat{Q}_h^k$ is optimistic with constant probability. By repeating this procedure multiple times, this constant probability can be amplified to arbitrary high probability.
4. Theoretical Analysis

For the analysis we will need the concept of the eluder dimension due to (Russo & Van Roy, 2013). Let $\mathcal{F}$ be a set of real-valued functions with domain $\mathcal{X}$. For $f \in \mathcal{F}, x_1, \ldots, x_t \in \mathcal{X}$, introduce the notation $f|_{\{x_1, \ldots, x_t\}} = (f(x_1), \ldots, f(x_t))$. We say that $x \in \mathcal{X}$ is $\epsilon$-independent of $x_1, \ldots, x_t \in \mathcal{X}$ given $\mathcal{F}$ if there exists $f, f' \in \mathcal{F}$ such that $||(f - f')|_{\{x_1, \ldots, x_t\}}|\leq \epsilon$ while $f(x) - f'(x) > \epsilon$.

Definition 4.1 (Eluder dimension, (Russo & Van Roy, 2013)). The eluder dimension $\dim_{\mathcal{E}}(\mathcal{F}, \epsilon)$ of $\mathcal{F}$ at scale $\epsilon$ is the length of the longest sequence $(x_1, \ldots, x_n)$ in $\mathcal{X}$ such that for some $\epsilon' \geq \epsilon$, for any $2 \leq t \leq n$, $x_t$ is $\epsilon'$-independent of $(x_1, \ldots, x_{t-1})$ given $\mathcal{F}$.

For a more detailed introduction of eluder dimension, readers can refer to (Russo & Van Roy, 2013; Osband & Van Roy, 2014; Wang et al., 2020; Ayoub et al., 2020).

4.1. Assumptions for General Function Approximation

For our general function approximation analysis, we make a few assumptions first. To emphasize the generality of our assumptions, in Section 4.1.1, we show that our assumptions are satisfied by linear function class.

Our algorithm (Algorithm 1) receives a function class $\mathcal{F} \subseteq \{f : \mathcal{S} \times \mathcal{A} \to [0, H]\}$ as input and furthermore, similar to (Wang et al., 2020; Ayoub et al., 2020), we assume that for any $V : \mathcal{S} \to [0, H]$, upon applying the Bellman backup operator, the output function lies in the function class $\mathcal{F}$. Concretely, we have the following assumption.

**Assumption A.** For any $V : \mathcal{S} \to [0, H]$ and for any $h \in [H]$, $v_h + P_h V \in \mathcal{F}$, i.e. there exists a function $f_V \in \mathcal{F}$ such that for all $(s, a) \in \mathcal{S} \times \mathcal{A}$ it satisfies

$$f_V(s, a) = v_h(s, a) + P_h V(s, a).$$

We emphasize that many standard assumptions in the RL theory literature such as tabular MDPs (Jaksch et al., 2010; Jin et al., 2018) and Linear MDPs (Yang & Wang, 2019; Jin et al., 2020) are special cases of Assumption A. In the appendix, we consider a misspecified setting and show that even when (5) holds approximately, Algorithm 1 achieves provable regret bounds.

We further assume that our function class has bounded covering number.

**Assumption B.** For any $\epsilon > 0$, there exists an $\epsilon$-cover $C(\mathcal{F}, \epsilon)$ with bounded covering number $\mathcal{N}(\mathcal{F}, \epsilon)$.

Next we define anti-concentration width, which is a function of the function class $\mathcal{F}$, dataset $D$ and noise variance $\sigma^2$.

**Definition 4.2 (Anti-concentration Width Function).** For a loss function $L(\cdot, \cdot)$ and dataset $D$, let $\hat{f} = \arg \min_{f \in \mathcal{X}} L(f | D)^2 + \lambda R(f)$ be the regularized least squares solution and $f_\sigma = \arg \min_{f \in \mathcal{X}} L(f | \mathcal{D}_\sigma)^2 + \lambda R_\sigma(f)$ be the perturbed regularized least-squares solution. For a fixed $\nu \in (0, 1)$, let $g_\nu : \mathcal{X} \to \mathbb{R}$ be a function such that for any input $x$:

$$g_\nu(x) = \sup \left\{ g \in \mathbb{R} : \mathbb{P} \left( \hat{f}_\sigma(x) \geq f(x) + g \right) \geq \nu \right\}.$$

We call $g_\nu(\cdot)$ the anti-concentration width function.

In words, $g_\nu(\cdot)$ is the largest value some $g \in \mathbb{R}$ can take such that the probability that $\hat{f}_\sigma$ is greater than $f + g$ is at least $\nu$.

We assume that for a concentrated function class, there exists a $\sigma$ such that the anti-concentration width is larger than the function class width.

**Assumption C (Anti-concentration).** Given the input $X = \{x_i\}_{i=1}^n$ of dataset $D$ and some arbitrary positive constant $\beta$, we define a function class $\mathcal{F}_{\beta} = \{f : \|f - \hat{f}\|_X^2 + \lambda R(f - \hat{f}) \leq \beta \}$. We assume that there exists a $\sigma$ such that $g_\sigma(x) \geq w(\mathcal{F}_{\beta}, x)$, for all inputs $x$ and $\sigma' \geq \sigma$.

This assumption guarantees that the randomized perturbation over the regression target has large enough probability of being optimistic. This assumption is satisfied by the linear function class. For more details, see Section 4.1.1.

Our next assumption is on the regularizer function $R(\cdot)$.

**Assumption D (Regularization).** We assume that our regularizer $R(\cdot)$ has several basic properties.
• \( R(f) + R(f') \geq cR(f + f') \) for some positive constant \( c > 0 \), for all \( f, f' \in \mathcal{F} \).

• \( R(f) = R(-f) \geq 0 \), for all \( f \in \mathcal{F} \).

• For any \( V : S \to [0, H] \), \( R(r + PV) \leq B \) for some constant \( B \in \mathbb{R} \).

For Assumption E, we set \( \mu = \|Pw\|_\infty \) (Linear MDP, (Yang & Wang, 2019; Jin et al., 2020)). We set \( \hat{f} = \arg \min_{f \in \mathcal{F}} L(f | D)^2 + \lambda R(f) \) to be \( f_\theta^* \). Then we have

\[
\hat{\theta} = \arg \min_{\theta} \sum_{t=1}^{k-1} (\phi(s^*_t, a^*_t)\theta - r^*_t)^2 + \lambda \|\theta\|^2
\]

where \( \Lambda^k_h = \sum_{t=1}^{k-1} \phi(s^*_t, a^*_t)\phi(s^*_t, a^*_t)^\top + \lambda I \). Similarly, we set \( f_\sigma = \arg \min_{f \in \mathcal{F}} L(f | D)^2 + \lambda R(f) \). Then we have

\[
\tilde{\theta} = (\Lambda^k_h)^{-1} \sum_{t=1}^{k-1} (r^*_t + \xi_t)\phi(s^*_t, a^*_t) + (\Lambda^k_h)^{-1} \sum_{j=1}^{d} \xi'_j e_j
\]

\[
\sim N(\hat{\theta}, \sigma^2 (\Lambda^k_h)^{-1}).
\]

For Definition 4.2, we set \( v = \Phi(-1) \). Using the anti-concentration property of Gaussian distribution, it is straightforward to show that for any \( (s, a) \in S \times A \):

\[
\mathbb{P} \left( f^*_\sigma(s, a) \geq \tilde{f}_\sigma(s, a) + \sigma \|\phi(s, a)\| (\Lambda^k_h)^{-1} \right) = v.
\]

So we have \( g_\sigma(s, a) \geq \sigma \|\phi(s, a)\| (\Lambda^k_h)^{-1} \) from Definition 4.2.

For Assumption C, the function class \( \mathcal{F}_{D,\beta} = \{ f : L(f - \tilde{f} | D)^2 + \lambda R(f - \tilde{f}) \leq \beta \} \) is equivalent to \( \Theta_{D,\beta} = \{ \theta : (\theta - \hat{\theta})^\top \Lambda^k_h (\theta - \hat{\theta}) \leq \beta \} \). So the width on the state-action pair \( (s, a) \) is \( 2\sqrt{\beta \|\phi(s, a)\| (\Lambda^k_h)^{-1}} \). If we set \( \sigma = 2\sqrt{\beta} \), we have

\[
g_\sigma(s, a) \geq w(\mathcal{F}_{D,\beta}, s, a).
\]

For Assumption D, as \( R(f_\theta) = \|\theta\|^2 \) is a \( \ell_2 \) norm function, the first two properties are direct to show with constant \( c = 1/2 \). For the third property, we have that

\[
g(s, a) = r(s, a) + P(s, a)V = \phi(s, a)(\mu + \sum_{s'} V(s')\mu(s')).
\]

So we have \( g = g_\theta \) where \( \theta = w + \sum_{s'} V(s')\mu(s') \) and

\[
\|\theta\|^2 \leq 2Hd.
\]

For Assumption E, we set \( \theta_f : f = f_\theta_f, \theta_v : v = v_\theta \) and \( \Theta_{F,f} = \{ \theta_f : f \in \mathcal{F} \} \) to be the parameterization. From Assumption D, we have \( \|\theta_v\|^2 \leq 2Hd \). In addition, we have \( \lambda R(f - v) = \lambda \|\theta - \theta_v\|^2 \leq \beta \). Then we have

\[
\Theta_{F,f} \subseteq \{ \theta_f : \|\theta_f - \theta_v\|^2 \leq \beta/\lambda, \|\theta_v\|^2 \leq 2Hd \}
\]

\[
\{ \theta_f : \|\theta_f\|^2 \leq 2\beta/\lambda + 4Hd \}.
\]
As shown in (Russo & Van Roy, 2013), this \( F \) has eluder dimension \( \tilde{O}(d) \).

### 4.2. Regret bound for General Function Approximation

First, we specify our choice of the noise variance \( \sigma^2 \) in the algorithm. We prove certain concentration properties of the regularized regressor \( \hat{f}_h \) so that the condition in Assumption C holds. Thus we can choose an appropriate \( \sigma \) such that the Assumption C is satisfied. A more detailed description is provided in the appendix.

Our first lemma is about the concentration of the regressor. A similar argument appears in (Wang et al., 2020) but their result does not include regularization, which is essential in our randomized algorithm to ensure exploration in all directions.

**Lemma 4.4** (Informal Lemma on Concentration). **Under Assumptions A, B, C, D, and E**, let \( F^k,m = \{ f \in F ||f - \hat{f}_h^k,m||_n^2 + \lambda R(f - \hat{f}_h^k,m) \leq \beta(F, \delta) \} \), where \( Z_h^k = \{ (s_h^k, a_h^k) \}_{r \in [k-1]} \), and

\[
\beta(F, \delta) = \tilde{O}\left((H + \sigma)^2 \log N(F, 1/T)\right).
\]

With high probability, for all \( (k, h, m) \in [K] \times [H] \times [M] \), we have

\[
\tau_h(\cdot^r, \cdot) + P_h V_h^{k+1}(\cdot, \cdot) \in F^k,m.
\]

This lemma shows that the perturbed regularized regression still enjoys concentration.

Our next lemma shows that LSVI-PHE is optimistic with high probability.

**Lemma 4.5** (Informal Lemma on Optimism). **Let**

\[
M = \ln \left( \frac{T|S||A|}{\delta} \right) / \ln \left( \frac{1}{1 - \epsilon} \right).
\]

With probability at least \( 1 - \delta \), for all \( (s, a, h, k) \in S \times A \times [H] \times [K] \), we have

\[
Q^*_h(s, a) \leq Q_h^k(s, a).
\]

With optimism, the regret is known to be bounded by the sum of confidence width (Wang et al., 2020). As Assumption E assumes that all the confidence region is in a bounded function class in the measure of eluder dimension, we can adapt proof techniques from (Wang et al., 2020) and prove our final result.

**Theorem 4.6** (Informal Theorem). **Under Assumptions A, B, C, D, and E**, with high probability, Algorithm 1 achieves a regret bound of

\[
\text{Regret}(K) \leq \tilde{O}\left( \sqrt{\text{dim}_\epsilon(F, 1/T)\beta(F, \delta)HT} \right),
\]

where

\[
\beta(F, \delta) = \tilde{O}\left((H + \sigma)^2 \log N(F, 1/T)\right).
\]

The theorem shows that our algorithm enjoys sublinear regret and have polynomial dependence on the horizon \( H \), noise variance \( \sigma^2 \) and eluder dimension \( \text{dim}_\epsilon(F, 1/T) \), and have logarithmic dependence on the covering number of the function class \( N(F, 1/T) \).

### 4.3. Regret bound for linear function class

Now we present the regret bound for Algorithm 2 under the assumption of linear MDP setting. In the appendix, we provide a simple yet elegant proof of the regret bound.

**Theorem 4.7.** Let \( M = d \log(\delta/9)/\log \Phi(1) \), \( \sigma = \tilde{O}(H\sqrt{d}) \), and \( \delta \in (0, 1] \). Under linear MDP assumption from Definition 4.3, the regret of Algorithm 2 satisfies

\[
\text{Regret}(T) \leq \tilde{O}(d^{3/2}H^{3/2}\sqrt{T}),
\]

with probability at least \( 1 - \delta \).

**Remark 4.8.** Under linear MDP assumption, this regret bound is at the same order as the LSVI-UCB algorithm from (Jin et al., 2020) and \( \sqrt{dH} \) better than the state-of-the-art TS-type algorithm (Zanette et al., 2020a). The only work that enjoys a \( \sqrt{d} \) better regret is (Zanette et al., 2020b), which requires solving an intractable optimization problem.

**Remark 4.9.** Along with being a competitive algorithm in statistical efficiency, we want to emphasize that our algorithm has good computational efficiency. LSVI-PHE with linear function class only involves linear programming to find the greedy policy while LSVI-UCB (Jin et al., 2020) requires solving a quadratic programming. The optimization problem in OPT-RLSVI (Zanette et al., 2020a) is hard too because the Q-function there is a piecewise continuous function and in one piece, it includes the product of the square root of a quadratic term and a linear term.

### 5. Numerical Experiments

We run our experiments on RiverSwim (Strehl & Littman, 2008), DeepSea (Osband et al., 2016b) and sparse MountainCar (Brockman et al., 2016) environments as these are considered to be hard exploration problems where \( \epsilon \)-greedy is known to have poor performance. For both RiverSwim and DeepSea experiments, we make use of linear features. The objective here is to compare an exploration method that randomizes the targets in the history (LSVI-PHE) with an exploration method that computes upper confidence bounds given the history (LSVI-UCB) (Jin et al., 2020; Cai et al., 2019). For the continous control MountainCar environment, we use neural-network as function approximator to implement LSVI-PHE. The objective here is to compare deep
RL variant of LSVI-PHE against other popular deep RL algorithms specifically designed to tackle exploration task.

5.1. Measurements

We plot the per episode return of each algorithm to benchmark their performance. As the agent begins to act optimally the per episode return begins to converge to the optimal, or baseline, return. The per episodes returns are the sum of all the rewards obtained in an episode. We also report the performance of LSVI-PHE when $\sigma^2$ is fixed and $M$ varies.

5.2. Results for RiverSwim

A diagram of the RiverSwim environment is shown in the Appendix. RiverSwim consists of $S$ states lined up in a chain. The agent begins in the leftmost state $s_1$ and has the choice of swimming to the left or to the right at each state. The agent’s goal is to maximize its return by trying to reach the rightmost state which has the highest reward. Swimming to the left, with the current, transitions the agent to the left deterministically. Swimming to the right, against the current, stochastically transitions the agent and has relatively high probability of moving right toward the goal state. However, because the current is strong there is a high chance the agent will stay in the current state and a low chance the agent will get swept up in the current and transition to the left. Thus, smart exploration is required to learn the optimal policy in this environment. We experiment with the variant of RiverSwim where $S = 12$ and $H = 40$. For this experiment, we swept over the exploration parameters in both LSVI-UCB (Jin et al., 2020) and LSVI-PHE and report the best performing run on a 12 state RiverSwim.

LSVI-UCB computes confidence widths of the following form $\beta \left| \phi(s, a) \right|_\Sigma^{-1}$, where $\phi(s, a) \in \mathbb{R}^d$ are the features for a given state-action pair and $\Sigma \in \mathbb{R}^{d \times d}$ is the empirical covariance matrix. We sweep over $\beta$ for LSVI-UCB and $\sigma^2$ for LSVI-PHE, where $M$ is chosen according to our theory (Theorem 4.7). We sweep over these parameters to speed up learning as choosing the theoretically optimal choices for $\beta$ and $\sigma^2$ often leads to a more conservative exploration policy which is slow to learn. As shown in Figure 1, the best performing LSVI-PHE achieves similar performance to the best performing LSVI-UCB on the 12 state RiverSwim environment.

5.3. Results for DeepSea

DeepSea (Osband et al., 2016b) consists of $S = N \times N$ states arranged in a grid, where $N$ is the depth of the sea. The agent begins at the top leftmost state in the grid $s_1$ and has the choice of moving down and left or down and right at each state. Once the agent reaches the bottom of the sea it transitions back to state $s_1$. The agent’s goal is to maximize its return by reaching the bottom right most state.

The agent gets a small negative reward for transitioning to the right while no reward is given if the agent transitions to the left. Thus, smart exploration is required; otherwise the agent will rarely go right the necessary amount of time to reach the goal state. We run our experiments on a $10 \times 10$ DeepSea environment. As shown in Figure 2, the best performing LSVI-PHE achieves similar performance to the best performing LSVI-UCB on DeepSea. We also vary $M$ given a fixed $\sigma^2 = 5 \times 10^{-3}$ for LSVI-UCB and $\sigma^2 = 5 \times 10^{-5}$ for LSVI-PHE.

These experiments on hard exploration problems highlight that we are able to simulate optimistic exploration, as in UCB, by perturbing the targets multiple times and taking the max over the perturbations to boost the probability of an optimistic estimate. If we are willing to sweep over $M$,
the number of times we perturb the history, and $\sigma^2$, we can then get a faster algorithm that still performs well in practice. If we let $M = 1$ and $\sigma^2 = 1$ then LSVI-PHE reduces to RLSVI and we would get the same performance as in (Osband et al., 2016b).

### 5.4. Results for MountainCar

We further evaluated LSVI-PHE on a continuous control task which requires exploration: sparse reward variant of continuous control MountainCar from OpenAI Gym (Brockman et al., 2016). This environment consists of a 2-dimensional continuous state space and a 1-dimensional continuous action space $[-1, 1]$. The agent only receives a reward of $+1$ if it reaches the top of the hill and everywhere else it receives a reward of $0$. We set the length of the horizon to be $1000$ and discount factor $\gamma = 0.99$.

For this setting, we compare four algorithms: LSVI-PHE, DQN with epsilon-greedy exploration, Noisy-Net DQN (Fortunato et al., 2017) and Bootstrapped DQN (Osband et al., 2016a). Our experiments are based on the baseline implementations of (Lan, 2019). As neural network, we used a multi-layer perceptron with hidden layers fixed to $[32, 32]$. The size of the replay buffer was $10,000$. The weights of neural networks were optimized by Adam (Kingma & Ba, 2014) with gradient clip 5. We used a batch size of 32. The target network was updated every 100 steps. The best learning rate was chosen from $[10^{-3}, 5 \times 10^{-4}, 10^{-4}]$. For LSVI-PHE, we set $M = 8$ and we chose the best value of $\sigma$ from $[10^{-4}, 10^{-3}, 10^{-2}]$. Results are shown in Figure 4.

### 6. Related Works

**RL with Function Approximation.** Many recent works have studied RL with function approximation, especially in the linear case (Jin et al., 2020; Cai et al., 2019; Zanette et al., 2020a;b; Wang et al., 2020; Ayoub et al., 2020; Foster et al., 2020; Jiang et al., 2017; Sun et al., 2019). Under the assumption that the agent has access to a well-designed feature extractor, these works design provably efficient algorithms for linear MDPs and linear kernel MDPs. LSVI-UCB (Jin et al., 2020), the first work with both polynomial runtime and polynomial sample complexity with linear function approximation, has a regret of $\tilde{O}(\sqrt{d^3 H^3 T})$. The state-of-the-art regret bound is $\tilde{O}(H d \sqrt{T})$, achieved by ELEANOR (Zanette et al., 2020b). However, ELEANOR needs to solve an optimization problem in each episode, which is computationally intractable. Wang et al. (2019) introduces a new expressivity condition named optimistic closure for generalized linear function approximation under which they propose a variant of optimistic LSVI with regret bound $\tilde{O}(d H^{3/2} T)$. Wang et al. (2020); Ayoub et al. (2020) focus on online RL with general function approximation and their analysis is based on the eluder dimension (Russo & Van Roy, 2013). Other complexity measures of general function classes include disagreement coefficient (Foster et al., 2020), Bellman rank (Jiang et al., 2017) and Witness rank (Sun et al., 2019).

**Thompson Sampling.** Thompson Sampling (Thompson, 1933) was proposed almost a century ago and rediscovered several times. Strens (2000) was the first work to apply TS to RL. Osband et al. (2013) provides a Bayesian regret bound and Agrawal et al. (2016); Ouyang et al. (2017) provide
worst case regret bounds for TS.

Randomized least-squares value iteration (RLSVI), proposed in Osband et al. (2019), uses random perturbations to approximate the posterior. Recently, several works focused on the theoretical analysis of RLSVI (Russo, 2019; Zanette et al., 2020b; Agrawal et al., 2020). Russo (2019) provides the first worst-case regret $\tilde{O}(H^{5/2}S^{3/2}/\sqrt{AT})$ for tabular MDP and Agrawal et al. (2020) improves it to $\tilde{O}(H^2S\sqrt{AT})$. Zanette et al. (2020a) proves $\tilde{O}(H^2d^2\sqrt{T})$ regret bound for linear MDP. However, Agrawal et al. (2020); Zanette et al. (2020a) both need to compute the confidence width as a warm-up stage, which is complicated and computationally costly.

7. Conclusion

In this work, we propose an algorithm LSVI-PHE for online RL with function approximation based on optimistic sampling. We prove the theoretical guarantees of LSVI-PHE and through experiments also demonstrate that it performs competitively against previous algorithms. We believe optimistic sampling provides a new provably efficient exploration paradigm in RL and it is practical in complicated real-world applications. We hope our work can be one step towards filling the gap between theory and application.

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A. LSVI-PHE with General Function Approximations

A.1. Noise

In the section, we specify how to choose \( \sigma \) in Algorithm 1. Note that we use \( \xi_{h,k}^\tau,m \) for the noise added in episode \( k \), timestep \( h \), data from episode \( \tau < k \) and sampling time \( m \). Similarly, \( \xi_{h,k}^\tau,m \) is for episode \( k \), timestep \( h \), regularizer \( p_i(\cdot) \) and sampling time \( m \). We set \( \lambda = 1 \) in our algorithm. By Lemma A.6, there exists \( \beta'(\mathcal{F}, \delta) \) such that with probability at least \( 1 - \delta \), for all \( (k, h) \in [K] \times [H] \), we have

\[
\mathbb{E}_{f_h} \left( f_h^k(r, \cdot) \right) := r(\cdot; \cdot) + P_h V_h^{k+1}(\cdot, \cdot) \in \mathcal{F}_h^k,
\]

where \( \mathcal{F}_h^k = \{ f \in \mathcal{F} | \| f - f_h^k \|_{\mathcal{F}_h^k} + R(f - f_h^k) \leq \beta'(\mathcal{F}, \delta) \} \). By Assumption C, for each \( \mathcal{F}_h^k \), there exists a \( \sigma_{h,k} \) such that

\[
g_{\sigma_{h,k}}(s, a) \geq w(\mathcal{F}_h^k, s, a).
\]

We define \( \sigma = \max_{k \in [K], h \in [H]} \sigma_{h,k} \) to be the maximum standard deviation of the added noise.

A.2. Concentration

We first define few filtrations and good events that we will use in the proof of lemmas in this section.

**Definition A.1** (Filtrations). We denote the \( \sigma \)-algebra generated by the set \( \mathcal{G} \) using \( \sigma(\mathcal{G}) \). We define the following filtrations

\[
\mathcal{G}^k \overset{\text{def}}{=} \sigma \left( \{(s^i_t, a^i_t, r^i_t) \}_{i \in [k-1] \times [H]} \cup \left( \{ \xi_{i,t}^{i,j} \}_{i \in [k], t \in [H] \times [M] \times [k-1]} \cup \{ \xi_{i,t}^{i,j} \}_{i \in [k], t \in [H] \times [M] \times [k-1]} \right) \right),
\]

\[
\mathcal{G}_{h,1}^k \overset{\text{def}}{=} \sigma \left( \{ (s^i_t, a^i_t, r^i_t) \}_{i \in [k]} \cup \left( \{ \xi_{i,t}^{i,j} \}_{i \in [k], t \in [H], j \in [M]} \cup \{ \xi_{i,t}^{i,j} \}_{i \in [k], t \in [H], j \in [M]} \right) \right),
\]

\[
\mathcal{G}_{h,2}^k \overset{\text{def}}{=} \sigma \left( \{ (s^i_t, a^i_t, r^i_t) \}_{i \in [k]} \right).
\]

**Definition A.2** (Good events). For any \( \delta > 0 \), we define the following random events

\[
\mathcal{G}^k(\xi, \delta) \overset{\text{def}}{=} \left\{ \max_{i \in [k], j \in [M]} \left| \xi_{i,h}^{i,j} \right| \leq \sqrt{\gamma_k(\delta)} \right\},
\]

\[
\mathcal{G}(K, H, \delta) \overset{\text{def}}{=} \bigcap_{k \leq K, h \leq H} \mathcal{G}^k(\xi, \delta),
\]

where \( \gamma_k(\delta) \) is some constant to be specified in Lemma A.3.

**Notation:** To simplify our presentation, in the remaining part of this section, we always denote \( \sqrt{\mathcal{G}} := \sqrt{\gamma_k(\delta)} \).

The next lemma shows that the good event defined in Definition A.2 happens with high probability.

**Lemma A.3.** For good event \( \mathcal{G}(K, H, \delta) \) defined in Definition A.2, if we set \( \sqrt{\mathcal{G}} = O(\delta) \), then it happens with probability at least \( 1 - \delta \).

**Proof.** Recall that \( \xi_{i,t}^{i,j} \) is a zero-mean Gaussian noise with variance \( \sigma_{i,t}^2 \). By the concentration of Gaussian distribution (Lemma D.1), with probability \( 1 - \delta' \), we have

\[
|\xi_{i,t}^{i,j}| \leq \sigma_{i,t} \sqrt{2 \log(1/\delta')} \leq \sigma \sqrt{2 \log(1/\delta')},
\]

The same result holds for \( \xi_{i,t}^{i,j} \). We complete the proof by setting \( \delta' = \delta/(K + D)MHK \) and using union bound. \( \square \)

In Definition 3.1, for a regularizer \( R(f) = \sum_{j=1}^D p_j(f)^2 \), where \( p_j(\cdot) \) are functionals, we defined the perturbed regularizer as \( \bar{R}_\sigma(f) = \sum_{j=1}^D [p_j(f) + \xi_{i,t}^j]^2 \) with \( \xi_{i,t}^j \) being i.i.d. zero-mean Gaussian noise with variance \( \sigma^2 \). Note that in the algorithm, the variance of the noise for the regularizer is the same as the dataset, which is \( \sigma_{h,k}^2 \). Recall from Assumption D that for any \( V : \mathcal{S} \to [0, H] \), our regularizer \( R \) satisfies \( R(r + PV) \leq B \) for some constant \( B \in \mathbb{R} \).

Our next lemma establishes a bound on the perturbed estimate of a single backup.
Lemma A.4. Consider a fixed $k \in [K]$ and a fixed $h \in [H]$. Let $Z_h^k = \{(s_h^\tau, a_h^\tau)\}_{\tau \in [k-1]}$ and $\tilde{D}_h^{k,V} = \{(s_h^\tau, a_h^\tau, r_h^\tau + \xi_h^\tau + V(s_{h+1}^\tau))\}_{\tau \in [k-1]}$. Define $\tilde{F}_{h,V} = \arg \min_{f \in \mathcal{F}} \|f\|_{D_h^{k,V}}^2 + R(f)$. Conditioned on the good event $\mathcal{G}(K, H, \delta)$, with probability at least $1 - \delta$, for a fixed $V : S \rightarrow [0, H]$ and any $V' : S \rightarrow [0, H]$ with $\|V' - V\|_\infty \leq 1/T$, we have

$$
\left\|\tilde{f}_{h,V'}(\cdot, \cdot) - r_h(\cdot, \cdot) - P_h V'(\cdot, \cdot)\right\|_{Z_h^k}^2 + R \left(\tilde{f}_{h,V'}(\cdot, \cdot) - r_h(\cdot, \cdot) - P_h V'(\cdot, \cdot)\right)
\leq c' \left[H + 1 + \sqrt{\gamma_k} \sqrt{\log(2/\delta)} + \log N(\mathcal{F}, 1/T) + \sqrt{B + \gamma_k BD}\right] \cdot
$$

for some constant $c'$. Here $B$ is the bound on the regularizer (Assumption D) and $D$ is the number of regularizers (Definition 3.1). Define this event as $\mathcal{E}_{h,V}(\delta)$.

Proof. Recall that for notational simplicity, we denote $[\mathbb{P}_h V_{h+1}(s, a) = \mathbb{E}_{s' \sim \mathbb{P}_h(\cdot | s, a)} V_{h+1}(s')]$. Now consider a fixed $V : S \rightarrow [0, H]$, and define

$$f_V(\cdot, \cdot) := r_h(\cdot, \cdot) + P_h V(\cdot, \cdot).$$

(6)

For any $f \in \mathcal{F}$, we consider $\sum_{\tau \in [k-1]} \chi_h^\tau(f)$ where

$$\chi_h^\tau(f) := 2(f(s_h^\tau, a_h^\tau) - f_V(s_h^\tau, a_h^\tau)) (f_V(s_h^\tau, a_h^\tau) - r_h^\tau(s_h^\tau, a_h^\tau) - \xi_h^\tau - V(s_{h+1}^\tau)).$$

Recalling the definition of the filtration $\mathcal{G}_{h+1}^\tau$ from Definition A.1, we note

$$\mathbb{E}[\chi_h^\tau(f)|\mathcal{G}_{h+1}^\tau] = \mathbb{E}[2(f(s_h^\tau, a_h^\tau) - f_V(s_h^\tau, a_h^\tau))(f_V(s_h^\tau, a_h^\tau) - r_h^\tau(s_h^\tau, a_h^\tau) - \xi_h^\tau - V(s_{h+1}^\tau))|\mathcal{G}_{h+1}^\tau]
= 2(f(s_h^\tau, a_h^\tau) - f_V(s_h^\tau, a_h^\tau)) \mathbb{E}[(f_V(s_h^\tau, a_h^\tau) - r_h^\tau(s_h^\tau, a_h^\tau) - \xi_h^\tau - V(s_{h+1}^\tau))|\mathcal{G}_{h+1}^\tau]
= 2(f(s_h^\tau, a_h^\tau) - f_V(s_h^\tau, a_h^\tau))(f_V(s_h^\tau, a_h^\tau) - r_h^\tau(s_h^\tau, a_h^\tau) - P_h V(s_h^\tau, a_h^\tau))
= 0.$$

In addition, conditioning on the good event $\mathcal{G}(K, H, \delta)$, we have

$$\left|\chi_h^\tau(f)\right| \leq 2(H + 1 + \sqrt{\gamma_k}) |f(s_h^\tau, a_h^\tau) - f_V(s_h^\tau, a_h^\tau)|.$$

As $\chi_h^\tau(f)$ is a martingale difference sequence conditioned on the filtration $\mathcal{G}_{h+1}^\tau$, by Azuma-Hoeffding inequality, we have

$$\mathbb{P}\left[\left|\sum_{\tau \in [k-1]} \chi_h^\tau(f)\right| \geq \epsilon\right] \leq 2\exp\left(-\frac{\epsilon^2}{8(H + 1 + \sqrt{\gamma_k})^2 \|f - f_V\|_{Z_h^k}^2}\right).$$

Now we set

$$\epsilon = \sqrt{8(H + 1 + \sqrt{\gamma_k})^2 \log \left(\frac{2N(\mathcal{F}, 1/T)}{\delta}\right)} \|f - f_V\|_{Z_h^k}^2
\leq 4(H + 1 + \sqrt{\gamma_k}) \|f - f_V\|_{Z_h^k} \sqrt{\log(2/\delta) + \log N(\mathcal{F}, 1/T)}.$$

With union bound, for all $g \in \mathcal{C}(\mathcal{F}, 1/T)$, with probability at least $1 - \delta$ we have

$$\left|\sum_{\tau \in [k-1]} \xi_h^\tau(g)\right| \leq 4(H + 1 + \sqrt{\gamma_k}) \|f - f_V\|_{Z_h^k} \sqrt{\log(2/\delta) + \log N(\mathcal{F}, 1/T)}.$$
Thus, for all $f \in \mathcal{F}$, there exists $g \in C(\mathcal{F}, 1/T)$ such that $\|f - g\|_\infty \leq 1/T$ and

$$\left| \sum_{(r) \in [k-1]} \chi_h^r(f) \right| \leq \sum_{(r) \in [k-1]} \chi_h^r(g) + 2(H + 1 + \sqrt{\gamma_k})$$

$$\leq 4(H + 1 + \sqrt{\gamma_k})\|g - f_V\|_{Z_k^h} \sqrt{\log (2/\delta) + \log N(\mathcal{F}, 1/T)} + 2(H + 1 + \sqrt{\gamma_k})$$

$$\leq 4(H + 1 + \sqrt{\gamma_k})(\|f - f_V\|_{Z_k^h} + 1) \sqrt{\log (2/\delta) + \log N(\mathcal{F}, 1/T)} + 2(H + 1 + \sqrt{\gamma_k}).$$

For $V' : \mathcal{S} \to [0, H]$ such that $\|V - V'\|_\infty \leq 1/T$, we have $\|f_{V'} - f_V\|_\infty \leq \|V' - V\|_\infty \leq 1/T$.

For any $f \in \mathcal{F}$, we have

$$\|f\|_{D_{h,V'}}^2 - \|f_{V'}\|_{D_{h,V'}}^2 = |f - f_{V'}|_{Z_k^h}^2 + 2 \sum_{(s_h^r, a_h^r) \in Z_k^h} (f(s_h^r, a_h^r) - f_{V'}(s_h^r, a_h^r))(f_{V'}(s_h^r, a_h^r) - r_h^r(s_h^r, a_h^r) - \xi_h - V'(s_{h+1}^r))$$

$$\geq |f - f_{V'}|_{Z_k^h}^2 + 2 \sum_{(s_h^r, a_h^r) \in Z_k^h} (f(s_h^r, a_h^r) - f_V(s_h^r, a_h^r))(f_V(s_h^r, a_h^r) - r_h^r(s_h^r, a_h^r) - \xi_h - V(s_{h+1}^r))$$

$$- 4(H + 1 + \sqrt{\gamma_k})\|V' - V\|_\infty |Z_k^h|$$

$$\geq |f - f_{V'}|_{Z_k^h}^2 - 4(H + 1 + \sqrt{\gamma_k})(|f - f_V|_{Z_k^h} + 1) \sqrt{\log (2/\delta) + \log N(\mathcal{F}, 1/T)} - 6(H + 1 + \sqrt{\gamma_k})$$

$$\geq |f - f_{V'}|_{Z_k^h}^2 - 4(H + 1 + \sqrt{\gamma_k})(|f - f_{V'}|_{Z_k^h} + 2) \sqrt{\log (2/\delta) + \log N(\mathcal{F}, 1/T)} - 6(H + 1 + \sqrt{\gamma_k}).$$

In addition, using Assumption D, we have the approximate triangle inequality for the perturbed regularizer:

$$\bar{R}(f) - \bar{R}(f_{V'}) = \sum_{i} [\xi_i(f)| \xi_i|_2^2 - \sum_{i} [\xi_i(f_{V'})|^2$$

$$= R(f) - R(f_{V'}) + 2 \sum_{i} \xi_i^2(p_i(f) - p_i(f_{V'}))$$

$$\geq cR(f - f_{V'}) - 2R(f_{V'}) - 2 \sum_{i} \sqrt{\gamma_k}p_i(f_{V'})$$

$$\geq cR(f - f_{V'}) - 2B - 2\sqrt{\gamma_k}\sqrt{BD}.$$
To prove the above argument, we use the inequality that if we have \(x^2 + y \leq ax + b\) for positive \(a, b, y\), then \(x \leq a + \sqrt{b}\) and \(x^2 + y \leq (a + \sqrt{b})^2\). In addition, we can remove \(c\) by replacing \(c'\) with \(c'/\min\{1, c\}\) and then we get our final bound. □

**Lemma A.5** (Confidence Region). Let \(\mathcal{F}^{k,m}_h = \{f \in \mathcal{F} \| f - \tilde{f}^{k,m}_h \|_2^2 + R(f - \tilde{f}^{k,m}_h) \leq \beta(\mathcal{F}, \delta)\}\), where

\[
\beta(\mathcal{F}, \delta) = c' \left[ (H + 1 + \sqrt{k}) \sqrt{\log \left( \frac{2}{\delta} \right)} + \log \mathcal{N}(\mathcal{F}, 1/T) + \sqrt{B + \sqrt{\gamma_k BD}} \right]^2.
\]  

Conditioned on the event \(\mathcal{G}(K, H, \delta)\), with probability at least \(1 - \delta\), for all \((k, h, m) \in [K] \times [H] \times [M]\), we have

\[
r_h(\cdot, \cdot) + P_h V^k_{h+1}(\cdot, \cdot) \in \mathcal{F}^{k,m}_h.
\]

**Proof.** First note that for a fixed \((k, h, m) \in [K] \times [H] \times [M]\),

\[
\mathcal{Q} = \{\min\{f(\cdot, \cdot), H\} \mid f \in \mathcal{C}(\mathcal{F}, 1/T)\} \cup \{0\}
\]

is a \((1/T)\)-cover of \(Q^{k,m}_{h+1}(\cdot, \cdot)\). This implies \(\mathcal{Q}\) is also a \((1/T)\)-cover of \(Q^k_{h+1}(\cdot, \cdot)\). This further implies

\[
\mathcal{V} = \{\max_{\alpha \in A} q(\cdot, \alpha) \mid q \in \mathcal{Q}\}
\]

is a \(1/T\) cover of \(V^k_{h+1}(\cdot)\) where we have \(\log(|\mathcal{V}|) = \log \mathcal{N}(\mathcal{F}, 1/T)\).

For the remaining part of the proof, we condition on \(\bigcap_{V \in \mathcal{V}} \mathcal{E}_{h,V}(\delta/|\mathcal{V}| TM)\), where \(\mathcal{E}_{h,V}(\delta)\) is the event defined in Lemma A.4. By Lemma A.4 and union bound, we have \(\Pr \left[ \bigcap_{V \in \mathcal{V}} \mathcal{E}_{h,V}(\delta/(8|\mathcal{V}| TM)) \right] \geq 1 - \delta/(8TM)\).

Let \(V \in \mathcal{V}\) such that \(\|V - V^k_{h+1}\|_\infty \leq 1/T\). By Lemma A.4 we have

\[
\left\| \tilde{f}^{k,m}_h(\cdot, \cdot) - r_h(\cdot, \cdot) - P_h V^k_{h+1}(\cdot, \cdot) \right\|_2^2 + R(\tilde{f}^{k,m}_h(\cdot, \cdot) - r_h(\cdot, \cdot) - P_h V^k_{h+1}(\cdot, \cdot))
\]

\[
\leq c' \left[ (H + 1 + \sqrt{k}) \sqrt{\log \left( \frac{2}{\delta} \right)} + \log \mathcal{N}(\mathcal{F}, 1/T) \right]^2,
\]

where \(c'\) is some absolute constant. By union bound, for all \((k, h, m) \in [K] \times [H] \times [M]\) we have \(r_h(\cdot, \cdot) + P_h V^k_{h+1}(\cdot, \cdot) \in \mathcal{F}^{k,m}_h\) with probability \(1 - \delta\). □

The last lemma guarantees that \(r_h(\cdot, \cdot) + P_h V^k_{h+1}(\cdot, \cdot)\) lies in the confidence region \(\mathcal{F}^{k,m}_h\) with high probability. Note that the confidence region \(\mathcal{F}^{k,m}_h\) is centered at \(\tilde{f}^{k,m}_h\), which is the solution to the perturbed regression problem defined in (3). For the unperturbed regression problem and its solution as center of the confidence region, we get the following lemma as a direct consequence of Lemma A.5.

**Lemma A.6**. Let \(\mathcal{F}^k_h = \{f \in \mathcal{F} \| f - \tilde{f}^k_h \|_2^2 + R(f - \tilde{f}^k_h) \leq \beta'(\mathcal{F}, \delta)\}\), where

\[
\beta'(\mathcal{F}, \delta) = c' \left[ (H + 1) \sqrt{\log \left( \frac{2}{\delta} \right)} + \log \mathcal{N}(\mathcal{F}, 1/T) + \sqrt{B} \right]^2.
\]

With probability at least \(1 - \delta\), for all \((k, h, m) \in [K] \times [H] \times [M]\), we have

\[
r_h(\cdot, \cdot) + P_h V^k_{h+1}(\cdot, \cdot) \in \mathcal{F}^k_h.
\]

**Proof.** This is a direct implication of Lemma A.5 with zero perturbation. □
A.3. Optimism

In this section, we will show that \( \{Q^k_h\}_{(h,k) \in [H] \times [K]} \) is optimistic with high probability. Formally, we have the following lemma.

**Lemma A.7.** Set \( M = \ln(T) / \ln(1 - \epsilon) \) in Algorithm 1. Conditioned on the event \( \mathcal{G}(K, H, \delta) \), with probability at least \( 1 - \delta \), for all \( s \in \mathcal{S}, a \in \mathcal{A}, h \in [H], k \in [K] \), we have

\[
Q^*_h(s, a) \leq Q^k_h(s, a).
\]

**Proof.** For timestep \( H + 1 \), we have \( Q^k_{H+1} = Q^*_{H+1} = 0 \). By Lemma A.6, there exists \( \beta'(\mathcal{F}, \delta) \) such that with probability at least \( 1 - \delta \), for all \( (k, h) \in [K] \times [H] \), we have

\[
f^k_h(\cdot, \cdot) := r_h(\cdot, \cdot) + P_h V^k_{h+1}(\cdot, \cdot) \in \mathcal{F}^k_h,
\]

where \( \mathcal{F}^k_h = \{ f \in \mathcal{F} \mid \| f - \hat{f}^k_h \|_{\mathbb{Z}}^2 + R(f - \hat{f}^k_h) \leq \beta'(\mathcal{F}, \delta) \} \).

Using notations introduced in Definition 4.2, let \( g^k_{h,a} \) be a function such that \( \hat{f}^{k,m}_h(s, a) \geq \hat{f}(s, a) + g^k_{h,a}(s, a) \) holds with probability at least \( v \). We set \( M = \ln(T) / \ln(1 - \epsilon) \) and then \( \hat{f}^{k,m}_h(s, a) \geq \hat{f}(s, a) + g^k_{h,a}(s, a) \) with probability at least

\[
1 - (1 - v)^M = 1 - \frac{\delta}{T|\mathcal{S}||\mathcal{A}|},
\]

for any \( (k, h) \in [K] \times [H] \) and \( (s, a) \in \mathcal{S} \times \mathcal{A} \). By union bound, we have \( \hat{f}^{k,m}_h(s, a) \geq \hat{f}(s, a) + g^k_{h,a}(s, a) \) for all \( (k, h) \in [K] \times [H] \) and \( (s, a) \in \mathcal{S} \times \mathcal{A} \) with probability at least \( 1 - \delta \) and we have

\[
\hat{f}^k_h(s, a) = \max_{m \in [M]} \hat{f}^{k,m}_h(s, a)
\]

\[
\geq \hat{f}^k_h(s, a) + g^k_{h,a}(s, a)
\]

\[
\geq \hat{f}^k_h(s, a) + w(\mathcal{F}^k_h)
\]

\[
\geq f^k_h(s, a),
\]

where the second inequality is from Assumption C and the choice of \( \sigma \) as discussed in Appendix A.1. The last inequality follows from the definition of the width function and the previous observation that \( f^k_h(\cdot, \cdot) \in \mathcal{F}^k_h \) with probability at least \( 1 - \delta \). Now we induct on \( h \) from \( h = H \) to 1.

\[
Q^*_h(s, a) = \min\{r_h(s, a) + P_h V^*_{h+1}(s, a), H\}
\]

\[
= \min\{f^k_h(s, a) + P_h (V^*_{h+1} - V^k_{h+1})(s, a), H\}
\]

\[
\leq \min\{f^k_h(s, a) + P_h (V^*_{h+1} - V^k_{h+1})(s, a), H\}
\]

\[
= Q^k_h(s, a).
\]

Thus,

\[
V^*_h(s) = \max_a Q^*_h(s, a) \leq \max_a Q^k_h(s, a) = V^k_h(s).
\]

where the second inequality is from \( V^*_{h+1} \leq V^k_{h+1} \), which is implied by induction.

A.4. Regret Bound

We are now ready to provide the regret bound for Algorithm 1. The next lemma upper bounds the regret of the algorithm by the sum of the width functions.

**Lemma A.8 (Regret decomposition).** Denote \( b^k_h(s, a) = w(\mathcal{F}^k_h, s, a) \). Conditioned on the event \( \mathcal{G}(K, H, \delta) \), with probability at least \( 1 - \delta \), we have
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\[ \text{Regret}(K) \leq \sum_{k=1}^{K} \sum_{h=1}^{H} b_h^k(s_h^k, a_h^k) + \sum_{k=1}^{K} \sum_{h=1}^{H} c_h^k, \]

where \( \zeta_h^k = P(s_h^k, a_h^k)(V_{h+1} - V_{h+1}^{\pi^k}) - (V_h(s_{h+1}) - V_{h+1}^{\pi} (s_{h+1})) \) is a martingale difference sequence with respect to the filtration \( G_{h:2} \).

**Proof.** We condition on the good events in Lemma A.5. For all \((k, h, m) \in [K] \times [H] \times [M], \) we have

\[ \|r_h(\cdot, \cdot) + P_h V_{h+1}^k (\cdot, \cdot) - \tilde{f}_h^{k,m}\|_{Z_k^h}^2 + R(r_h(\cdot, \cdot) + P_h V_{h+1}^k (\cdot, \cdot) - \tilde{f}_h^{k,m}) \leq \beta(\mathcal{F}, \delta). \]

Recall that \( \mathcal{F}_h^k = \{ f \mid \|r_h(\cdot, \cdot) + P_h V_{h+1}^k (\cdot, \cdot) - f\|_{Z_k^h}^2 + R(r_h(\cdot, \cdot) + P_h V_{h+1}^k (\cdot, \cdot) - \tilde{f}_h^{k,m}) \leq \beta(\mathcal{F}, \delta) \} \) is the confidence region. Then for \((k, h, m) \in [K] \times [H] \times [M], \) \( \tilde{f}_h^{k,m} \in \mathcal{F}_h^k. \) Defining \( b_h^k(s, a) = w(\mathcal{F}_h^k, s, a), \) for all \((k, h, m) \in [K] \times [H] \times [M], \) we have,

\[ b_h^k(s, a) \geq |r(s, a) + P(s, a)V_{h+1}^k - \tilde{f}_h^{k,m}(s, a)|. \]

As \( Q_h^k(s, a) = \min_\{\max_{m \in M}\}\{ \tilde{f}_h^{k,m}(\cdot, \cdot), H - h + 1\}, \) we have

\[ b_h^k(s, a) \geq |r(s, a) + P(s, a)V_{h+1}^k - Q_h^k(s, a)|. \]

By Lemma A.7 and standard telescoping argument, we have

\[ \text{Regret}(K) \leq \sum_{k=1}^{K} V_1^k(s_1^k) - V_1^{\pi^k}(s_1^k) \]
\[ = \sum_{k=1}^{K} Q_1^k(s_1^k, a_1^k) - Q_1^{\pi^k}(s_1^k, a_1^k) \]
\[ = \sum_{k=1}^{K} Q_1^k(s_1^k, a_1^k) - (r(s_1^k, a_1^k) + P(s_1^k, a_1^k)V_2^k) + (r(s_1^k, a_1^k) + P(s_1^k, a_1^k)V_2^k) - Q_1^{\pi^k}(s_1^k, a_1^k) \]
\[ \leq \sum_{k=1}^{K} b_1^k(s_1^k, a_1^k) + P(s_1^k, a_1^k)(V_2^k - V_2^{\pi^k}) \]
\[ = \sum_{k=1}^{K} b_1^k(s_1^k, a_1^k) + (V_2^k(s_2^k) - V_2^{\pi^k}(s_2^k)) + \zeta_1^k \]
\[ \leq \sum_{k=1}^{K} \sum_{h=1}^{H} b_h^k(s_h^k, a_h^k) + \sum_{k=1}^{K} \sum_{h=1}^{H} \zeta_h^k. \]

**Lemma A.9** (Time inhomogeneous version of Lemma 10 in (Wang et al., 2020)). Let \( \mathcal{F}' \) be a subset of function class \( \mathcal{F}, \) consisting of all \( f \in \mathcal{F} \) such that

\[ \|f - v\|_{Z_2}^2 + R(f - v) \leq \beta(\mathcal{F}, \delta), \]

where \( v = r + PV \) as in Assumption E and \( \beta(\mathcal{F}, \delta) \) as defined in Lemma A.5. With probability at least \( 1 - \delta, \) we have

\[ \sum_{k=1}^{K} \sum_{h=1}^{H} b_h^k(s_h^k, a_h^k) \leq H + 4H^3 \dim_{\mathcal{F}'}(1/T) + H \sqrt{c \dim_{\mathcal{F}'}(1/T)} K \beta(\mathcal{F}, \delta), \]

for some absolute constant \( c > 0. \)
As a result, our bound implies a \( \tilde{F} \) (Assumption 3 in (Wang et al., 2020)) for function class \( \mathcal{F} \).

Under all the assumptions, with probability at least 1 - \( \delta \), we have

\[
\text{Regret}(K) \leq 4H^3 \text{dim}_\varepsilon(\mathcal{F}, 1/T) + \sqrt{\text{dim}_\varepsilon(\mathcal{F}, 1/T) \beta(\mathcal{F}, \delta) KT},
\]

where

\[
\beta(\mathcal{F}, \delta) = c' \left( (H + 1 + \sigma) \sqrt{\log (2/\delta) + \log N(\mathcal{F}, 1/T)} + \sqrt{B + \sigma BD} \right)^2,
\]

for some constant \( c' \).

\[\text{Remark A.11. For linear MDP, as shown in Section 4.1.1, we have}\]

\[
\sigma = 2\sqrt{\beta(\mathcal{F}, \delta)} = c' \left( (H + 1) \sqrt{\log (2/\delta) + \log N(\mathcal{F}, 1/T)} + \sqrt{B} \right)^2,
\]

\[B = 2HD\text{ and } D = d.\]

As a result, our bound implies a \( \tilde{O}(\sqrt{H^3d^3T}) \) regret bound for linear MDP.

\[\text{B. GFA With Model Misspecification}\]

**Assumption F.** (Assumption 3 in (Wang et al., 2020)) For function class \( \mathcal{F} \), there exists a real number \( \zeta \), such that for any \( V : \mathcal{S} \to [0, H] \), there exists \( g_V \in \mathcal{F} \) which satisfies

\[
\max_{(s, a) \in \mathcal{S} \times \mathcal{A}} \left| g_V(s, a) - r(s, a) - \sum_{s' \in \mathcal{S}} P(s' | s, a) V(s') \right| \leq \zeta.
\]

In addition, we assume \( g_V \) satisfies Assumption D, i.e. \( R(g_V) \leq B \).
Lemma B.1. Consider a fixed \(k \in [K]\) and a fixed \(h \in [H]\). Let \(Z_h^k = \{(s_h^r, a_h^r)\}_{r \in [k-1]}\) and \(\bar{D}_{h,V}^k = \{(s_h^r, a_h^r, r_h^r + \xi_h^r + V(s_{h+1}^r))\}_{r \in [k-1]}\). Define \(\bar{f}_{h,V}^k = \arg\min_{f \in \mathcal{F}} \|f\|^2_{\bar{Z}_h^k} + R(f)\). Conditioned on the good event \(G(K, H, \delta)\), with probability at least \(1 - \delta\), for a fixed \(V : S \rightarrow [0, H]\) and any \(V' : S \rightarrow [0, H]\) with \(\|V' - V\|_\infty \leq 1/T\), we have

\[
\left\|\bar{f}_{h,V'}(\cdot, \cdot) - r_h(\cdot, \cdot) - P_h V'(\cdot, \cdot)\right\|^2_{\bar{Z}_h^k} + R(\bar{f}_{h,V'}(\cdot, \cdot) - r_h(\cdot, \cdot) - P_h V'(\cdot, \cdot)) \leq \epsilon\left[(H + 1 + \sqrt{\gamma_k})\sqrt{\log(2/\delta) + \log N(\mathcal{F}, 1/T)} + \sqrt{B + \sqrt{\gamma_k BD + \zeta K(H + \sqrt{\gamma_k})}}\right]^2,
\]

for some constant \(\epsilon\).

Proof. Recall that for notational simplicity, we denote \([P_{h,V,h+1}] V(s, a)\) by \(E_{s' \sim P_{h,V,h+1}} V_{h+1}(s')\). Now consider a fixed \(V : S \rightarrow [0, H]\), and define

\[
f_V(\cdot, \cdot) = r_h(\cdot, \cdot) + P_h V(\cdot, \cdot).
\]

By Assumption F, there exists \(g_V \in \mathcal{F}\) such that

\[
\max_{(s,a) \in S \times A} \left| g_V(s,a) - f_V(s,a) \right| \leq \zeta.
\]

For any \(f \in \mathcal{F}\), consider

\[
\chi_h^k = 2f(s_h^r, a_h^r) - f_V(s_h^r, a_h^r)(f_V(s_h^r, a_h^r) - r_h(s_h^r, a_h^r) - \xi_h^r - V(s_{h+1}^r)).
\]

First we show that \(\chi_h^k(f)\) is a martingale difference sequence with respect to the filtration \(G^k_{h,1}\).

\[
E[\chi_h^k(f)|G^k_{h,1}] = E[2f(s_h^r, a_h^r) - f_V(s_h^r, a_h^r)(f_V(s_h^r, a_h^r) - r_h(s_h^r, a_h^r) - \xi_h^r - V(s_{h+1}^r))|G^k_{h,1}] = 2f(s_h^r, a_h^r) - f_V(s_h^r, a_h^r)E[(f_V(s_h^r, a_h^r) - r_h(s_h^r, a_h^r) - \xi_h^r - V(s_{h+1}^r))|G^k_{h,1}] = 2f(s_h^r, a_h^r) - f_V(s_h^r, a_h^r)(f_V(s_h^r, a_h^r) - r_h(s_h^r, a_h^r) - P_h V(s_h^r, a_h^r)) = 0.
\]

In addition, conditioning on good events \(G(K, H, \delta)\), we have

\[
|\chi_h^k(f)| \leq 2(H + 1 + \sqrt{\gamma_k})|f(s_h^r, a_h^r) - f_V(s_h^r, a_h^r)|.
\]

As \(\chi_h^k(f)\) is a martingale difference sequence conditioned on the filtration \(G^k_{h,1}\), by Azuma-Hoeffding inequality, we have

\[
P\left[ \sum_{r \in [k-1]} \chi_h^k(f) \geq \epsilon \right] \leq 2\exp\left(-\frac{\epsilon^2}{8(H + 1 + \sqrt{\gamma_k})^2 \|f - f_V\|^2_{\bar{Z}_h^k}}\right).
\]

Now we set

\[
\epsilon = \sqrt{\frac{8(H + 1 + \sqrt{\gamma_k})^2 \log\left(\frac{2N(\mathcal{F}, 1/T)}{\delta}\right)}{\delta}} \|f - f_V\|^2_{\bar{Z}_h^k} \leq 4(H + 1 + \sqrt{\gamma_k})\|f - f_V\|^2_{\bar{Z}_h^k} \sqrt{\log(2/\delta) + \log N(\mathcal{F}, 1/T)}.
\]

With union bound, for all \(g \in C(\mathcal{F}, 1/T)\), with probability at least \(1 - \delta\) we have

\[
\left| \sum_{(r) \in [k-1]} \xi_h^k(g) \right| \leq 4(H + 1 + \sqrt{\gamma_k})\|f - f_V\|^2_{\bar{Z}_h^k} \sqrt{\log(2/\delta) + \log N(\mathcal{F}, 1/T)}.
\]
Thus, for all $f \in \mathcal{F}$, there exists $g \in C(\mathcal{F}, 1/T)$ such that $\|f - g\|_{\infty} \leq 1/T$ and ,

\[
\left| \sum_{(\tau) \in [k-1]} \chi_{k}(f) \right| \leq \left| \sum_{(\tau) \in [k-1]} \chi_{k}(g) \right| + 2(H + 1 + \sqrt{\gamma_k})
\]

\[
\leq 4(H + 1 + \sqrt{\gamma_k})\|f - f_{V}\|_{2}^{1/\gamma_k} \leq \log (2/\delta) + \log \mathcal{N}(\mathcal{F}, 1/T) + 2(H + 1 + \sqrt{\gamma_k})
\]

\[
\leq 4(H + 1 + \sqrt{\gamma_k})(\|f - f_{V}\|_{2} + 1) \log (2/\delta) + \log \mathcal{N}(\mathcal{F}, 1/T) + 2(H + 1 + \sqrt{\gamma_k})
\]

For $V' : S \rightarrow [0, H]$ such that $\|V - V'\|_{\infty} \leq 1/T$, we have $\|f_{V'} - f_{V}\|_{\infty} \leq \|V' - V\|_{\infty} \leq 1/T$.

For any $f \in \mathcal{F}$, we have

\[
\|f\|_{2}^{2}_{P_{k}, V'} - \|f_{V'}\|_{2}^{2}_{P_{k}, V'} = \|f - f_{V'}\|_{2}^{2}_{P_{k}, V'} + \sum_{(s, a, k) \in P_{k}^{\tau},(\tau, h) \in [k-1] \times [H]} \chi_{h}(f) - \chi_{h}(f_{V'})
\]

\[
\geq \|f - f_{V'}\|_{2}^{2}_{P_{k}, V'} + \sum_{(s, a, k) \in P_{k}^{\tau},(\tau, h) \in [k-1] \times [H]} \chi_{h}(f) - \chi_{h}(f_{V'}) - 4(H + 1 + \sqrt{\gamma_k})\|V' - V\|_{\infty} \leq 1/T
\]

\[
\leq \|f - f_{V'}\|_{2}^{2}_{P_{k}, V'} + \sum_{(s, a, k) \in P_{k}^{\tau},(\tau, h) \in [k-1] \times [H]} \chi_{h}(f_{V'}) - \chi_{h}(f_{V'}) - 4(H + 1 + \sqrt{\gamma_k})\|V' - V\|_{\infty} \leq 1/T
\]

\[
\geq \|f - f_{V'}\|_{2}^{2}_{P_{k}, V'} + \sum_{(s, a, k) \in P_{k}^{\tau},(\tau, h) \in [k-1] \times [H]} \chi_{h}(f) - \chi_{h}(f_{V'}) - 4(H + 1 + \sqrt{\gamma_k})\|V' - V\|_{\infty} \leq 1/T
\]

In addition, by Assumption D, we have

\[
\bar{R}(f) - \bar{R}(f_{V'}) = \sum_{i} [p_{i}(f) - \xi_{i}] - \sum_{i} [p_{i}(f_{V'}) - \xi_{i}] = R(f) - R(f_{V'}) - 2 \sum_{i} \xi_{i}(p_{i}(f) - p_{i}(f_{V'}))
\]

\[
\geq cR(f - f_{V'}) - 2R(f_{V'}) - 2 \sum_{i} \sqrt{\gamma_k} p_{i}(f_{V'})
\]

\[
\geq cR(f - f_{V'}) - 2B - 2\sqrt{\gamma_k} \sqrt{BD}.
\]

Summing the above two inequalities we have

\[
\|f\|_{2}^{2}_{P_{k}, V'} + \bar{R}(f) - \|f_{V'}\|_{2}^{2}_{P_{k}, V'} - \bar{R}(f_{V'}) \geq \|f - f_{V'}\|_{2}^{2}_{P_{k}, V'} + cR(f - f_{V'}) - C,
\]

where $C = 4(H + 1 + \sqrt{\gamma_k})(\|f - f_{V'}\|_{2}^{2} + 1) \log (2/\delta) + \log \mathcal{N}(\mathcal{F}, 1/T) + 6(H + 1 + \sqrt{\gamma_k}) + 2B + 2\sqrt{\gamma_k} \sqrt{BD}$. 
Now we try to replace the $f_{\nu'}$ in the RHS with $g_{\nu'}$.

$$
\|f_{\nu'}\|_{Z_{k,h,v}}^2 - \|g_{\nu'}\|_{Z_{k,h,v}}^2
= \sum_{r \in [k-1]} (f_{\nu'}(s^r_h, a^r_h) - (r^r_h + \xi^r_h + V(s_{h+1}^r)))^2 - \sum_{r \in [k-1]} (g_{\nu'}(s^r_h, a^r_h) - (r^r_h + \xi^r_h + V(s_{h+1}^r)))^2
= \sum_{r \in [k-1]} (f_{\nu'}(s^r_h, a^r_h) - g_{\nu'}(s^r_h, a^r_h))(f_{\nu'}(s^r_h, a^r_h) + g_{\nu'}(s^r_h, a^r_h) - 2(r^r_h + \xi^r_h + V(s_{h+1}^r)))
\geq -\zeta K(4H + 2\sqrt{\gamma_k}).
$$

By the boundedness of the regularizer (Assumption D), we have

$$
\|f_{\nu'}\|_{Z_{k,h,v}}^2 + R(f_{\nu'}) - \|g_{\nu'}\|_{Z_{k,h,v}}^2 - \tilde{R}(g_{\nu'}) \geq -\zeta K(4H + 2\sqrt{\gamma_k}) - B.
$$

Thus we have

$$
\|f\|_{Z_{k,h,v}}^2 + \tilde{R}(f) - \|g_{\nu'}\|_{Z_{k,h,v}}^2 - \tilde{R}(g_{\nu'}) \geq \|f_{\nu'}\|_{Z_{k,h,v}}^2 + \tilde{R}(f) - \|f_{\nu'}\|_{Z_{k,h,v}}^2 - \tilde{R}(f_{\nu'}) - \zeta K(4H + 2\sqrt{\gamma_k}) - B
\geq \|f - f_{\nu'}\|_{Z_{k,h,v}}^2 + cR(f - f_{\nu'}) - C - \zeta K(4H + 2\sqrt{\gamma_k}) - B.
$$

As $\tilde{f}_{h,v'}$ is the minimizer of $\|f\|_{Z_{k,h,v}}^2 + \tilde{R}(f)$ for $f \in \mathcal{F}$ and note that $g_{\nu'} \in \mathcal{F}$, we have

$$
\|\tilde{f}_{h,v'} - f_{\nu'}\|_{Z_{k,h,v}}^2 + cR(\tilde{f}_{h,v'} - f_{\nu'})
\leq c' \left((H + 1 + \sqrt{\gamma_k})\sqrt{\log(2/\delta) + \log N(\mathcal{F},1/T)} + \sqrt{B + \sqrt{\gamma_k BD} + \zeta K(H + \sqrt{\gamma_k})}\right)^2.
$$

To prove the above argument, we use the inequality that if we have $x^2 + y \leq ax + b$ for positive $a, b, y$, then $x \leq a + \sqrt{b}$ and $x^2 + y \leq (a + \sqrt{b})y$. In addition, we can remove $c$ by replacing $c'$ with $c' \min\{1, c\}$ and then we get the final bound.

\[ \square \]

**Lemma B.2.** (Misspecified Confidence Region) Let $\mathcal{F}_k^{k,m} = \{f \in \mathcal{F} | \|f - \tilde{f}_h^{k,m}\|_{Z_{h,k}}^2 + R(f - \tilde{f}_h^{k,m}) \leq \beta(\mathcal{F}, \delta)\}$, where

$$
\beta(\mathcal{F}, \delta) = c' \left((H + 1 + \sqrt{\gamma_k})\sqrt{\log(2/\delta) + \log N(\mathcal{F},1/T)} + \sqrt{B + \sqrt{\gamma_k BD} + \zeta K(H + \sqrt{\gamma_k})}\right)^2.
$$

Conditioned on the event $\mathcal{G}(K, H, \delta)$, with probability at least $1 - \delta$, for all $(k, h, m) \in [K] \times [H] \times [M]$, we have

$$
r_h(\cdot, \cdot) + P_{hV_{h+1}^k}(\cdot, \cdot) \in \mathcal{F}_h^{k,m}.
$$

\[ \square \]

**Theorem B.3.** Under all the assumptions, with probability at least $1 - \delta$, Algorithm 1 achieves a regret bound of

$$
\text{Regret}(K) \leq 4H^3\dim_e(\mathcal{F}, 1/T) + \sqrt{\dim_e(\mathcal{F}, 1/T)\beta(\mathcal{F}, \delta)HT},
$$

where

$$
\beta(\mathcal{F}, \delta) = c' \left((H + 1 + \sigma)\sqrt{\log(2/\delta) + \log N(\mathcal{F},1/T)} + \sqrt{B + \sigma \sqrt{BD} + \zeta K(H + \sigma)}\right)^2,
$$

for some constant $c'$.

\[ \square \]

**Proof.** With Lemma B.1, the proof is same as Lemma A.5.

**Theorem B.3.** Under all the assumptions, with probability at least $1 - \delta$, Algorithm 1 achieves a regret bound of

$$
\text{Regret}(K) \leq 4H^3\dim_e(\mathcal{F}, 1/T) + \sqrt{\dim_e(\mathcal{F}, 1/T)\beta(\mathcal{F}, \delta)HT},
$$

where

$$
\beta(\mathcal{F}, \delta) = c' \left((H + 1 + \sigma)\sqrt{\log(2/\delta) + \log N(\mathcal{F},1/T)} + \sqrt{B + \sigma \sqrt{BD} + \zeta K(H + \sigma)}\right)^2,
$$

for some constant $c'$.

\[ \square \]

**Proof.** With Lemma B.2, the proof is same as Theorem A.10.
C. LSVI-PHE with linear function approximation

In this section, we prove Theorem 4.7. Our analysis specialized to linear MDP setting is simpler and may provide additional insights. In addition, compared to GFA setting, we improve the bound for $M$ and it no longer depends on $|S|$ or $|A|$. We first introduce the notation and few definitions that are used throughout this section. Upon presenting lemmas and their proofs, finally we combine the lemmas to prove Theorem 4.7.

**Definition C.1** (Model prediction error). For all $(k, h) \in [K] \times [H]$, we define the model prediction error associated with the reward $r^k_h$,

$$I^k_h(s, a) = r^k_h(s, a) + \mathbb{E}_h V^k_{h+1}(s, a) - Q^k_h(s, a).$$

This depicts the prediction error using $V^k_{h+1}$ instead of $V^k_{h+1}$ in the Bellman equations (1).

**Definition C.2** (Unperturbed estimated parameter). For all $(k, h) \in [K] \times [H]$, we define the unperturbed estimated parameter as

$$\widehat{\theta}^k_h = (\Lambda^k_h)^{-1} \left( \sum_{\tau=1}^{k-1} [r^\tau_h + V^k_{h+1}(s^\tau_h, a^\tau_h)] \phi(s^\tau_h, a^\tau_h) \right).$$

Moreover, we denote the difference between the perturbed estimated parameter $\widehat{\theta}^k_{h,j}$ and the unperturbed estimated parameter $\widehat{\theta}^k_h$ as

$$\zeta^k_{h} = \widehat{\theta}^k_{h,j} - \widehat{\theta}^k_h.$$

C.1. Concentration

Our first lemma characterizes the difference between the perturbed estimated parameter $\widehat{\theta}^k_{h,j}$ and the unperturbed estimated parameter $\widehat{\theta}^k_h$.

**Proposition C.3** (restatement of Proposition 3.2). In step 9 of Algorithm 2, conditioned on all the randomness except $\{\epsilon^k_{h,i,j}\}_{i,j \in [k-1] \times |M|}$ and $\{\xi^h_j\}_{j \in [M]}$, the estimated parameter $\widehat{\theta}^k_{h,j}$ satisfies

$$\zeta^k_{h} = \widehat{\theta}^k_{h,j} - \widehat{\theta}^k_h \sim N(0, \sigma^2 (\Lambda^k_h)^{-1}),$$

where $\widehat{\theta}^k_h = (\Lambda^k_h)^{-1} \left( \sum_{\tau=1}^{k-1} [r^\tau_h + V^k_{h+1}(s^\tau_h, a^\tau_h)] \phi(s^\tau_h, a^\tau_h) \right)$ is the unperturbed estimated parameter from Definition C.2.

**Proof.** From Algorithm 2, note that

$$\widehat{\theta}^k_{h,j} = (\Lambda^k_h)^{-1} (\eta^k_h + \epsilon^k_{h,j})$$

$$= (\Lambda^k_h)^{-1} \left( \sum_{\tau=1}^{k-1} [r^\tau_h + V^k_{h+1}(s^\tau_h, a^\tau_h)] \phi(s^\tau_h, a^\tau_h) \right) + (\Lambda^k_h)^{-1} \left( \sum_{\tau=1}^{k-1} \epsilon^k_{h,j} \phi(s^\tau_h, a^\tau_h) \right)$$

$$= \widehat{\theta}^k_h + (\Lambda^k_h)^{-1} \left( \sum_{\tau=1}^{k-1} \epsilon^k_{h,j} \phi(s^\tau_h, a^\tau_h) \right).$$

Since $\epsilon^k_{h,j} \sim N(0, \sigma^2)$, note that for $\tau \in [k-1]$,

$$\epsilon^k_{h,j} \phi(s^\tau_h, a^\tau_h) \sim N(0, \sigma^2 \phi(s^\tau_h, a^\tau_h) \phi(s^\tau_h, a^\tau_h)^\top).$$

Now, since $\xi^k_{h} \sim N(0, \sigma^2 \lambda I_d)$,

$$(\Lambda^k_h)^{-1} \left( \sum_{\tau=1}^{k-1} \epsilon^k_{h,j} \phi(s^\tau_h, a^\tau_h) \right) \sim (\Lambda^k_h)^{-1} \cdot N \left( 0, \sigma^2 \left( \sum_{\tau=1}^{k-1} \phi(s^\tau_h, a^\tau_h) \phi(s^\tau_h, a^\tau_h)^\top + \lambda I_d \right) \right)$$

$$\sim (\Lambda^k_h)^{-1} \cdot N \left( 0, \sigma^2 \Lambda^k_h \right)$$

$$\sim N(0, \sigma^2 (\Lambda^k_h)^{-1}).$$
Thus, we have
\[
\zeta_h^{k,j} = \hat{\theta}_h^k - \tilde{\theta}_h^k \sim N(0, \sigma^2(\Lambda_h^k)^{-1}).
\]

\[\Box\]

**Lemma C.5** (Lemma B.1 in Jin et al., 2020). Under Definition 4.3 of linear MDP, for any fixed policy \(\pi\), let \(\{\theta_h^k\}_{h \in [H]}\) be the corresponding weights such that \(Q_h^k(s, a) = (\phi(s, a), \theta_h^k)\) for all \((s, a, h) \in \mathcal{S} \times \mathcal{A} \times [H]\). Then for all \(h \in [H]\), we have
\[
\|\theta_h^k\| \leq 2H\sqrt{d}.
\]

Our next lemma states that the unperturbed estimated weight \(\hat{\theta}_h^k\) is bounded.

**Lemma C.5.** For any \((k, h) \in [K] \times [H]\), the unperturbed estimated weight \(\hat{\theta}_h^k\) in Definition C.2 satisfies
\[
\|\hat{\theta}_h^k\| \leq 2H\sqrt{kd/\lambda}.
\]

**Proof.** We have
\[
\|\hat{\theta}_h^k\| = \left\| (A_h^k)^{-1} \sum_{\tau = 1}^{k-1} \left[ r_h^\tau(s_h^\tau, a_h^\tau) + V_h^{k+1}(s_h^{k+1}) \right] \cdot \phi(s_h^\tau, a_h^\tau) \right\|
\]
\[
= \left\| (A_h^k)^{-1} \sum_{\tau = 1}^{k-1} \left[ r_h^\tau(s_h^\tau, a_h^\tau) + \max_{a \in \mathcal{A}} Q_h^{k+1}(s_h^{k+1}, a) \right] \cdot \phi(s_h^\tau, a_h^\tau) \right\|
\]
\[
\leq \frac{1}{\sqrt{\lambda}} \sqrt{k - 1} \left( \sum_{\tau = 1}^{k-1} \left[ \|r_h^\tau(s_h^\tau, a_h^\tau) + \max_{a \in \mathcal{A}} Q_h^{k+1}(s_h^{k+1}, a) \| \cdot \|\phi(s_h^\tau, a_h^\tau)\|_{(\Lambda_h^k)^{-1}} \right]^2 \right)^{1/2}
\]
\[
\leq \frac{2H}{\sqrt{\lambda}} \sqrt{k - 1} \left( \sum_{\tau = 1}^{k-1} \|\phi(s_h^\tau, a_h^\tau)\|_{(\Lambda_h^k)^{-1}}^2 \right)^{1/2}
\]
\[
\leq 2H\sqrt{kd/\lambda}.
\]

Here, the first inequality follows from Lemma D.5. The second inequality follows from the truncation of \(Q_h^k\) to the range \([0, H - h + 1]\) in Line 11 of Algorithm 2. The last inequality is due to Lemma D.3.

For the ease of exposition, we now define the values \(\beta_k(\delta), \nu_k(\delta)\) and \(\gamma_k(\delta)\) which we use to define our high confidence bounds.

**Definition C.6** (Noise bounds). For any \(\delta > 0\) and some large enough constants \(c_1, c_2\) and \(c_3\), let
\[
\sqrt{\beta_k(\delta)} \overset{\text{def}}{=} c_1 H \sqrt{d\log(Hdk/\delta)},
\]
\[
\sqrt{\nu_k(\delta)} \overset{\text{def}}{=} c_2 H \sqrt{d\log(Hdk/\delta)},
\]
\[
\sqrt{\gamma_k(\delta)} \overset{\text{def}}{=} c_3 \sqrt{d\nu_k(\delta) \log(d/\delta)}.
\]

**Definition C.7** (Noise distribution). In Algorithm 2, we set the following values for \(\sigma\)
\[
\sigma_k = 2\sqrt{\nu_k(\delta)}.
\]

Thus for all \(j \in [M]\), we have,
\[
\{\epsilon_h^{k,j}\} \sim N(0, 4\nu_k(\delta)(\Lambda_h^k)^{-1}).
\]

Now, we define some events based on the characterization of the random variable \(\epsilon_h^{k,j}\) as defined in Definition C.2.

**Definition C.8** (Good events). For any \(\delta > 0\), we define the following random events
\[
G_h^k(\zeta, \delta) \overset{\text{def}}{=} \left\{ \max_{j \in [M]} \|\epsilon_h^{k,j}\|_{\Lambda_h^k} \leq \sqrt{\gamma_k(\delta)} \right\},
\]
\[
G(K, H, \delta) \overset{\text{def}}{=} \bigcap_{k \leq K} \bigcap_{h \leq H} G_h^k(\zeta, \delta).
\]
Next, we present our main concentration lemma in this section.

**Lemma C.9.** Let $\lambda = 1$ in Algorithm 2. For any fixed $\delta > 0$, conditioned on the event $G(K, H, \delta)$, we have for all $(k, h) \in [K] \times [H]$, 

$$
\left\| \sum_{\tau=1}^{k-1} \phi(s_{h, \tau}^k, a_{h, \tau}^k) \left[ (V_{h+1}^k - \mathbb{P}_h V_{h+1}^k)(s_{h, \tau}^k, a_{h, \tau}^k) \right] \right\|_{(\Lambda_h^k)^{-1}} \leq c_1 H \sqrt{d \log \left( H d k / \delta \right)},
$$

with probability at least $1 - \delta$ for some constant $c_1 > 0$.

**Proof.** From Lemma C.5, we know, for all $(k, h) \in [K] \times [H]$, we have $\| \hat{\theta}_{h+1}^k \| \leq 2H \sqrt{k d / \lambda}$. In addition, by construction of $\Lambda_{h+1}^k$, the minimum eigenvalue of $\Lambda_{h+1}^k$ is lower bounded by $\lambda$. Thus we have $\sqrt{\lambda} \| \hat{\theta}_{h+1}^k \| \leq \| \epsilon_{h+1}^k \| \Lambda_{h+1}^k \leq \sqrt{\gamma_k(\delta)}$. Finally, triangle inequality implies, $\| \hat{\theta}_{h+1}^k \| = \| \hat{\theta}_{h+1}^k + c_{h+1}^k \| \leq 2H \sqrt{k d / \lambda} + \sqrt{\gamma_k(\delta) / \lambda}$ for all $j \in [M]$. Combining Lemma D.6 and Lemma D.8, we have that, for any $\varepsilon > 0$ and $\delta > 0$, with probability at least $1 - \delta$,

$$
\left\| \sum_{\tau=1}^{k-1} \phi(s_{h, \tau}^k, a_{h, \tau}^k) \left[ (V_{h+1}^k - \mathbb{P}_h V_{h+1}^k)(s_{h, \tau}^k, a_{h, \tau}^k) \right] \right\|_{(\Lambda_h^k)^{-1}} \\
\leq \left( 4H^2 \frac{d}{2} \log \left( \frac{k + 1}{\lambda} \right) + d \log \left( 1 + 4H \sqrt{k d / \lambda + 2\sqrt{\gamma_k(\delta) / \lambda}} \right) + \frac{1}{\delta} + \frac{8k^2 \varepsilon^2 d}{\lambda} \right)^{1/2} \\
\leq \left( 4H^2 \frac{d}{2} \log \left( \frac{k + 1}{\lambda} \right) + d \log \left( \frac{3(2H \sqrt{k d / \lambda + \sqrt{\gamma_k(\delta) / \lambda}})}{\varepsilon} \right) + \frac{1}{\delta} + \frac{8k^2 \varepsilon^2 d}{\lambda} \right)^{1/2} \\
\leq 2H \frac{d}{2} \log \left( \frac{k + 1}{\lambda} \right) + d \log \left( \frac{3(2H \sqrt{k d / \lambda + \sqrt{\gamma_k(\delta) / \lambda}})}{\varepsilon} \right) + \frac{1}{\delta} + 2\sqrt{2k \varepsilon} + \frac{2\sqrt{2k \varepsilon}}{\sqrt{\lambda}}.
$$

(12)

Setting $\lambda = 1$, $\varepsilon = H \sqrt{d / k}$ and substituting $\sqrt{\gamma_k(\delta)} = c_3 \sqrt{d \log(d / \delta) / (d / \delta)} \leq c_4 H d \log \left( H d k / \delta \right)$ for some constant $c_4 > 0$, we get

$$
\left\| \sum_{r=1}^{k-1} \phi(s_{h, r}^k, a_{h, r}^k) \left[ (V_{h+1}^k - \mathbb{P}_h V_{h+1}^k)(s_{h, r}^k, a_{h, r}^k) \right] \right\|_{(\Lambda_h^k)^{-1}} \\
\leq 2H \sqrt{d} \left[ \frac{1}{2} \log(k + 1) + \log(1 / \delta) + \log \left( \frac{3(2H \sqrt{d k / \lambda} + c_4 H d \log \left( H d k / \delta \right))}{H \sqrt{d}} \right) \right]^{1/2} + 2\sqrt{2H \sqrt{d}} \\
\leq c_1 H \sqrt{d \log \left( H d k / \delta \right)}.
$$

(13)

for some constant $c_1 > 0$.

□

**Lemma C.10.** Let $\lambda = 1$ in Algorithm 2. For any $\delta > 0$, conditioned on the event $G(K, H, \delta)$, for any $(h, k) \in [K] \times [H]$ and $(s, a) \in S \times A$, we have

$$
\left| \phi(s, a) \left[ \hat{\theta}_{h}^k - r_{h}^k(s, a) - \mathbb{P}_h V_{h+1}^k(s, a) \right] \right| \leq c_2 H \sqrt{d \log \left( H d k / \delta \right)} \left\| \phi(s, a) \right\|_{(\Lambda_h^k)^{-1}},
$$

with probability $1 - \delta$, where $c_2 > 0$ is a constant.

**Proof.** Let us denote the inner product over $S$ by $\langle \cdot, \cdot \rangle_S$. Using linear MDP assumption for transition kernel from
Using (14) we obtain,

\[
\phi(s, a)^\top \tilde{\theta}_h^k - r_h^k(s, a) - (P_h V_{h+1}^k)(s, a) = (P_h V_{h+1}^k)(s, a) - \phi(s, a)^\top (\Lambda_h^k)^{-1} \sum_{r=1}^{k-1} \left[ r_h^k(s_r^h, a_r^h) + V_{h+1}^k(s_{h+1}^r) \right] \cdot \phi(s_r^h, a_r^h) - r_h^k(s, a)
\]

\[
- \phi(s, a)^\top (\Lambda_h^k)^{-1} \sum_{r=1}^{k-1} \phi(s_r^h, a_r^h)(P_h V_{h+1}^k)(s_r^h, a_r^h) + \lambda I(\mu_h, V_{h+1}^k)
\]

\[
= \phi(s, a)^\top (\Lambda_h^k)^{-1} \left[ \sum_{r=1}^{k-1} \phi(s_r^h, a_r^h)(V_{h+1}^k - P_h V_{h+1}^k)(s_r^h, a_r^h) \right] + \lambda \phi(s, a)^\top (\Lambda_h^k)^{-1} \mu_h(\mu_h, V_{h+1}^k)
\]

\[
= \phi(s, a)^\top (\Lambda_h^k)^{-1} \left[ \sum_{r=1}^{k-1} \phi(s_r^h, a_r^h)(V_{h+1}^k - P_h V_{h+1}^k)(s_r^h, a_r^h) \right] + \lambda \phi(s, a)^\top (\Lambda_h^k)^{-1} \mu_h(\mu_h, V_{h+1}^k)
\]

where in the last line we rely on the definition of \( P_h \).

Using (14) we obtain,

\[
\phi(s, a)^\top \tilde{\theta}_h^k - r_h^k(s, a) - (P_h V_{h+1}^k)(s, a) = \phi(s, a)^\top (\Lambda_h^k)^{-1} \sum_{r=1}^{k-1} \left[ r_h^k(s_r^h, a_r^h) + V_{h+1}^k(s_{h+1}^r) \right] \cdot \phi(s_r^h, a_r^h) - r_h^k(s, a)
\]

\[
- \phi(s, a)^\top (\Lambda_h^k)^{-1} \sum_{r=1}^{k-1} \phi(s_r^h, a_r^h)(P_h V_{h+1}^k)(s_r^h, a_r^h) + \lambda I(\mu_h, V_{h+1}^k)
\]

\[
= \phi(s, a)^\top (\Lambda_h^k)^{-1} \left[ \sum_{r=1}^{k-1} \phi(s_r^h, a_r^h)(V_{h+1}^k - P_h V_{h+1}^k)(s_r^h, a_r^h) \right] + \lambda \phi(s, a)^\top (\Lambda_h^k)^{-1} \mu_h(\mu_h, V_{h+1}^k)
\]

(15)

In the following we will analyze the each of the three terms in (15) separately and derive high probability bound for each of them.

**Term (i).** Since \((\Lambda_h^k)^{-1} \succ 0\), by Cauchy-Schwarz inequality and Lemma C.9, with probability at least \(1 - \delta\), we have

\[
\phi(s, a)^\top (\Lambda_h^k)^{-1} \left[ \sum_{r=1}^{k-1} \phi(s_r^h, a_r^h)(V_{h+1}^k - P_h V_{h+1}^k)(s_r^h, a_r^h) \right] 
\]

\[
\leq \left\| \sum_{r=1}^{k-1} \phi(s_r^h, a_r^h)(V_{h+1}^k - P_h V_{h+1}^k)(s_r^h, a_r^h) \right\| (\Lambda_h^k)^{-1} \left\| \phi(s, a) \right\| (\Lambda_h^k)^{-1} 
\]

\[
\leq \sqrt{\beta_k(\delta)} \left\| \phi(s, a) \right\| (\Lambda_h^k)^{-1}.
\]
Term (ii). Note that
\[
\phi(s, a) \hat{\nabla}^T (\Lambda^k_h)^{-1} \left( \sum_{r=1}^{k-1} r_h^r(s_h^r, a_h^r) \phi(s_h^r, a_h^r) \right) - r_h^k(s, a)
\]
\[
= \phi(s, a) \hat{\nabla}^T (\Lambda^k_h)^{-1} \left( \sum_{r=1}^{k-1} r_h^r(s_h^r, a_h^r) \phi(s_h^r, a_h^r) \right) - \phi(s, a) \hat{\nabla} w_h
\]
\[
= \phi(s, a) \hat{\nabla}^T (\Lambda^k_h)^{-1} \left( \sum_{r=1}^{k-1} r_h^r(s_h^r, a_h^r) \phi(s_h^r, a_h^r) - \Lambda^k_h w_h \right)
\]
\[
= \phi(s, a) \hat{\nabla}^T (\Lambda^k_h)^{-1} \left( \sum_{r=1}^{k-1} r_h^r(s_h^r, a_h^r) \phi(s_h^r, a_h^r) - \sum_{r=1}^{k-1} \phi(s_h^r, a_h^r) \phi(s_h^r, a_h^r)^T w_h - \lambda I w_h \right)
\]
\[
= \phi(s, a) \hat{\nabla}^T (\Lambda^k_h)^{-1} \left( \sum_{r=1}^{k-1} r_h^r(s_h^r, a_h^r) \phi(s_h^r, a_h^r) - \sum_{r=1}^{k-1} \phi(s_h^r, a_h^r) r_h^r(s_h^r, a_h^r) - \lambda I w_h \right)
\]
\[
= -\lambda \phi(s, a) \hat{\nabla}^T (\Lambda^k_h)^{-1} w_h,
\]
where in the penultimate step, we used the fact \(r_h(s, a) = \langle \phi(s, a), w_h \rangle\) from Definition 4.3. Applying Cauchy-Schwarz inequality we obtain,
\[
-\lambda \phi(s, a) \hat{\nabla}^T (\Lambda^k_h)^{-1} w_h \leq \lambda \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}} \| w_h \|_{(\Lambda^k_h)^{-1}}
\]
\[
\leq \sqrt{\lambda} \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}} \| w_h \|_2
\]
\[
\leq \sqrt{\lambda d} \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}}.
\]
(17)

Here the second inequality follows by observing that the smallest eigenvalue of \(\Lambda^k_h\) is at least \(\lambda\) and thus the largest eigenvalue of \((\Lambda^k_h)^{-1}\) is at most \(1/\lambda\). The last inequality follows from Definition 4.3. Combining (17) and (18) we get
\[
\phi(s, a) \hat{\nabla}^T (\Lambda^k_h)^{-1} \left( \sum_{r=1}^{k-1} r_h^r(s_h^r, a_h^r) \phi(s_h^r, a_h^r) \right) - r_h^k(s, a) \leq \sqrt{\lambda d} \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}}.
\]
(19)

Term (iii). Similar to (18), applying Cauchy-Schwarz inequality, we get
\[
-\lambda \phi(s, a) \hat{\nabla}^T (\Lambda^k_h)^{-1} \langle \mu_h^k, V_{k+1}^h \rangle_s \leq \lambda \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}} \| \mu_h^k, V_{k+1}^h \|_s \| \Lambda^k_h \|^{-1}
\]
\[
\leq \sqrt{\lambda} \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}} \| \mu_h^k, V_{k+1}^h \|_2
\]
\[
\leq \sqrt{\lambda} \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}} \left( \sum_{s=1}^{d} \| \mu_h^s \|^2 \right)^{1/2} \| V_{k+1}^h \|_\infty
\]
\[
\leq H \sqrt{\lambda d} \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}}.
\]
(20)

Here the second inequality follows using the same observation we did for term (ii). The last inequality follows from \(\sum_{s=1}^{d} \| \mu_h^s \|^2 \leq d\) in Definition 4.3 and the clipping operation performed in Line 2 of Algorithm 2. Now combining (16), (19) and (20), and letting \(\lambda = 1\), we get,
\[
\| \phi(s, a) \hat{\nabla}^k_h - r_h^k(s, a) - P_h V_{k+1}^h(s, a) \| \leq (\sqrt{d} + H \sqrt{d} + \sqrt{d}) \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}}
\]
\[
= (c_1 H \sqrt{d} \log(H d/k) + H \sqrt{d} + \sqrt{d}) \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}}
\]
\[
\leq c_2 H \sqrt{d} \log(H d/k) \| \phi(s, a) \|_{(\Lambda^k_h)^{-1}},
\]
(21)
with probability $1 - \delta$ for some constant $c_2 > 0$.

In addition, if we set $\theta^k_h : \phi(\cdot, \cdot)\top \theta^k_h = r^k_h(\cdot, \cdot) + \mathbb{P}_h V^k_{h+1}(\cdot, \cdot)$ to be the true parameter and $\Delta \theta^k_h = \theta^k_h - \hat{\theta}^k_h$ to be the regression error, then from the analysis above we can derive that $||\Delta \theta^k_h||_{A^k_h} \leq \sqrt{\nu_k(\delta)} = c_2 H \sqrt{d \log(H/dk/\delta)}$. □

**Lemma C.11** (stochastic upper confidence bound). Let $\lambda = 1$ in Algorithm 2. For any $\delta > 0$, conditioned on the event $G(K, H, \delta)$, for any $(h, k) \in [H] \times [K]$ and $(s, a) \in \mathcal{S} \times \mathcal{A}$, with probability at least $1 - (\delta + c^M_0)$, we have

$$l^k_h(s, a) \leq 0,$$

and

$$-l^k_h(s, a) \leq \left( \sqrt{\nu_k(\delta)} + \sqrt{\gamma_k(\delta)} \right) \left\| \phi(s, a) \right\|_{(A^k_h)^{-1}},$$

where $c_0 = \Phi(1)$.

**Proof.** Applying Lemma C.10, for any $(h, k) \in [H] \times [K]$ and $(s, a) \in \mathcal{S} \times \mathcal{A}$, we have,

$$|r^k_h(s, a) + \mathbb{P}_h V^k_{h+1}(s, a) - \phi(s, a)\top \hat{\theta}^k_h| \leq c_2 H \sqrt{d \log(H/dk/\delta)} = \sqrt{\nu_k(\delta)} \left\| \phi(s, a) \right\|_{(A^k_h)^{-1}},$$

with probability at least $1 - \delta$.

As we are conditioning on the event $G(K, H, \delta)$, for any $(h, k) \in [H] \times [K]$ and $(s, a) \in \mathcal{S} \times \mathcal{A}$, we have

$$\max_{j \in [M]} |\phi(s, a)\top \zeta^k_hj| \leq \sqrt{\gamma_k(\delta)} \left\| \phi(s, a) \right\|_{(A^k_h)^{-1}}.$$  (26)

Now from the definition of model prediction error, using (24) and (26), we get, with probability $1 - \delta$,

$$-l^k_h(s, a) = Q^k_h(s, a) - r^k_h(s, a) - \mathbb{P}_h V^k_{h+1}(s, a)$$

$$= \min_{j \in [M]} \phi(s, a)\top (\hat{\theta}^k_h + \zeta^k_hj, H) - r^k_h(s, a) - \mathbb{P}_h V^k_{h+1}(s, a)$$

$$\leq \max_{j \in [M]} \phi(s, a)\top (\hat{\theta}^k_h + \zeta^k_hj) - r^k_h(s, a) - \mathbb{P}_h V^k_{h+1}(s, a)$$

$$= \max_{j \in [M]} \phi(s, a)\top \zeta^k_hj - \max_{j \in [M]} \phi(s, a)\top \hat{\theta}^k_h$$

$$\leq \left( \sqrt{\nu_k(\delta)} + \sqrt{\gamma_k(\delta)} \right) \left\| \phi(s, a) \right\|_{(A^k_h)^{-1}},$$  (27)

Set $\theta^k_h : \phi(\cdot, \cdot)\top \theta^k_h = r^k_h(\cdot, \cdot) + \mathbb{P}_h V^k_{h+1}(\cdot, \cdot)$ to be the true parameter and $\Delta \theta^k_h = \theta^k_h - \hat{\theta}^k_h$ to be the regression error. By the concentration part, conditioning on good events, we have $||\Delta \theta^k_h||_{A^k_h} \leq \sqrt{\nu_k(\delta)}$ and $||\zeta^k_hj||_{A^k_h} \leq \sqrt{\gamma_k(\delta)}$ for all $j \in [M]$.

For all $(h, k) \in [H] \times [K]$ and any $(s, a) \in \mathcal{S} \times \mathcal{A}$, we have

$$l^k_h(s, a) = Q^k_h(s, a) + \mathbb{P}_h V^k_{h+1}(s, a) - Q^k_h(s, a)$$

$$= r^k_h(s, a) + \mathbb{P}_h V^k_{h+1}(s, a) - \min_{j \in [M]} \{ H, \max_{j \in [M]} \phi(s, a)\top (\hat{\theta}^k_h + \zeta^k_hj) \} + \max_{j \in [M]} \phi(s, a)\top \zeta^k_hj, 0 \}$$

$$\leq \max_{j \in [M]} \phi(s, a)\top \Delta \theta^k_h + \max_{j \in [M]} \phi(s, a)\top \zeta^k_hj, 0 \}$$

Now we prove that with high probability, $\max_{j \in [M]} \phi(s, a)\top \zeta^k_hj - \phi(s, a)\top \Delta \theta^k_h \geq 0$ for all $(s, a) \in \mathcal{S} \times \mathcal{A}$. Note that the inequality still holds if we scale $\phi(s, a)$. Now we assume all $\phi(s, a)$ satisfy $\left\| \phi(s, a) \right\|_{(A^k_h)^{-1}} = 1$. Define $C(\varepsilon)$ to be a
\(\epsilon\)-cover of the ellipsoid \(\{\phi||\phi||_{(\Lambda^k)^{-1}} = 1\}\) with respect to norm \(\cdot ||(\Lambda^k)^{-1}\) and \(\log |C(\epsilon)| = O(d \log(\frac{1}{\epsilon}))\). For all \(j \in [M]\), we have,

\[
\{\xi^k_{h,j}\} \sim N(0, 4\nu_k(\delta)(\Lambda^k)^{-1})
\]

Thus, for all \(j \in [M]\) and for all \(\phi \in C(\epsilon)\), we have

\[
\{\phi^T \xi^k_{h,j}\} \sim N(0, 4\nu_k(\delta)||\phi||^2_{(\Lambda^k)^{-1}}).
\]

Now, for all \(j \in [M]\) and for all \(\phi \in C(\epsilon)\), we have

\[
P\left(\phi^T \xi^k_{h,j} - 2\sqrt{\nu_k(\delta)||\phi||_{(\Lambda^k)^{-1}}} \geq 0\right) = \Phi(-1).
\]

Now

\[
P\left(\max_{j \in [M]} \phi^T \xi^k_{h,j} - 2\sqrt{\nu_k(\delta)||\phi||_{(\Lambda^k)^{-1}}} \geq 0\right) \geq 1 - (1 - \Phi(-1))^M
\]

\[
= 1 - \Phi(1)^M
\]

\[
= 1 - \epsilon^M_0,
\]

By union bound, with probability \(1 - |C(\epsilon)|\epsilon^M_0\), the above bound holds for all elements in \(C\) simultaneously.

Now condition on the previous event, for \(\phi = \phi(s,a)\), we can find a \(\phi' \in C(\epsilon)\) such that \(||\phi - \phi'||_{(\Lambda^k)^{-1}} \leq \epsilon\). Define \(\Delta \phi = \phi - \phi'\).

\[
\phi^T \xi^k_{h,j} - \phi'^T \xi^k_{h,j} - \phi^T \Delta \theta^k_{h,j} + \Delta \phi^T \xi^k_{h,j} + \Delta \phi^T \Delta \theta^k_{h,j}
\]

\[
\geq \phi'^T \xi^k_{h,j} - 2\sqrt{\nu_k(\delta)||\phi'||_{(\Lambda^k)^{-1}}} + \sqrt{\nu_k(\delta)||\phi'||_{(\Lambda^k)^{-1}}} - \epsilon||\xi^k_{h,j}||_{\Lambda^k} - \epsilon||\Delta \theta^k_{h,j}||_{\Lambda^k}
\]

\[
\geq \phi'^T \xi^k_{h,j} - 2\sqrt{\nu_k(\delta)||\phi'||_{(\Lambda^k)^{-1}}} + \sqrt{\nu_k(\delta)||\phi'||_{(\Lambda^k)^{-1}}} - \epsilon\sqrt{\nu_k(\delta)} - \epsilon\sqrt{\nu_k(\delta)}
\]

Set \(\epsilon = \frac{\sqrt{\nu_k(\delta)}}{\sqrt{\nu_k(\delta)} + \sqrt{\nu_k(\delta)}} = \tilde{O}(\frac{1}{\sqrt{d}})\) and we have, with probability \(1 - |C(\epsilon)|\epsilon_0^M\),

\[
\max_{j \in [M]} \phi^T \xi^k_{h,j} - \phi'^T \theta^k_{h,j} \geq \max_{j \in [M]} \phi'^T \xi^k_{h,j} - 2\sqrt{\nu_k(\delta)||\phi'||_{(\Lambda^k)^{-1}}}
\]

\[
\geq 0.
\]

Finally we have conditioning on good event \(\mathcal{G}(K, H, \delta)\), with probability at least \(1 - |C(\epsilon)|\epsilon_0^M\), for all \((s, a) \in S \times A\), \(l_h^k(s, a) \leq 0\). As \(\log |C(\epsilon)| = \tilde{O}(d \log(\frac{1}{\epsilon}))\), we can set \(M = \tilde{O}(\frac{d \log(1/\epsilon)}{\log(1/\epsilon_0)})\) to have probability \(1 - \delta\).

\(\square\)

**C.2. Regret Bound**

**Definition C.12 (Filtrations).** We denote the \(\sigma\)-algebra generated by the set \(\mathcal{G}\) using \(\sigma(\mathcal{G})\). We define the following filtrations:

\[
\mathcal{F}_h^{k} \overset{\text{def}}{=} \sigma\left(\{(s_t^i, a_t^i, r_t^i)\}_{i, t \in [k-1] \times [H]} \cup \{\xi^k_{h,j}\}_{i, t, j \in [k-1] \times [H] \times [M]}\right),
\]

\[
\mathcal{F}_{h,1}^{k} \overset{\text{def}}{=} \sigma\left(\mathcal{F}_h^{k} \cup \{(s_t^i, a_t^i, r_t^i)\}_{i \in [k]} \cup \{\xi^k_{h,j} : t \leq h, \ 1 \leq j \leq M\}\right),
\]

\[
\mathcal{F}_{h,2}^{k} \overset{\text{def}}{=} \sigma\left(\mathcal{F}_{h,1}^{k} \cup \{x_{h+1}^k\}\right).
\]
Lemma C.13 (Lemma 4.2 in (Cai et al., 2019)). It holds that

\[
\text{Regret}(T) = \sum_{k=1}^{K} \left( V^*_1(s^k_1) - V^{\pi^k}_1(s^k_1) \right)
\]

\[
= \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \langle Q^k_h(s_h, \cdot), \pi^*_h(\cdot | s_h) \rangle - \langle Q^k_h(s_h, \cdot), \pi^h(\cdot | s_h) \rangle \right] | s_1 = s^k_1 |
\]

\[
+ \sum_{k=1}^{K} \sum_{h=1}^{H} D^k_h + \sum_{k=1}^{K} \sum_{h=1}^{H} M^k_h
\]

\[
+ \sum_{k=1}^{K} \sum_{h=1}^{H} \left( \mathbb{E}_{\pi^*} \left[ \langle t^k_h(s_h, a_h), s^k_1 \rangle - t^k_h(s_h, a_h) \rangle \right], \right) \tag{29}
\]

where

\[
D^k_h := \langle (Q^k_h - Q^k_{\pi^*})(s^k_h, \cdot), \pi^*_h(\cdot | s_h) \rangle - \langle Q^k_h - Q^k_{\pi^*})(s^k_h, a^k_h), (30)
\]

\[
M^k_h := \mathbb{E}_{h}((V^k_{h+1} - V^{\pi^k}_{h+1}))(s^k_h, a^k_h) - (V^k_{h+1} - V^{\pi^k}_{h+1}))(s^k_h). \tag{31}
\]

Lemma C.14. For the policy \( \pi^k_h \) at time-step \( k \) of episode \( h \), it holds that

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}_{\pi^*} \left[ \langle Q^k_h(s_h, \cdot), \pi^*_h(\cdot | s_h) \rangle - \langle Q^k_h(s_h, \cdot), \pi^h(\cdot | s_h) \rangle \right] | s_1 = s^k_1 | \leq 0, \tag{32}
\]

where \( T = HK \).

Proof. Obvious from the observation that \( \pi^k_h \) acts greedily with respect to \( Q^k_h \). Note that if \( \pi^k_h = \pi^* \) then the difference is 0. Else the difference is negative since \( \pi^*_h \) is deterministic with respect to its action-values meaning it takes a value of 1 where \( \pi^*_h \) would take a value of 0 and \( Q^k_h \) would have the greatest value at the state-action pair that \( \pi^k_h \) equals one.

Lemma C.15 (Bound on Martingale Difference Sequence). For any \( \delta > 0 \), it holds with probability \( 1 - 2\delta/3 \) that

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} D^k_h + \sum_{k=1}^{K} \sum_{h=1}^{H} M^k_h \leq 2\sqrt{2HT \log(3/\delta)}. \tag{33}
\]

Proof. Recall that

\[
D^k_h := \langle (Q^k_h - Q^k_{\pi^*})(s^k_h, \cdot), \pi^*_h(\cdot | s_h) \rangle - \langle Q^k_h - Q^k_{\pi^*})(s^k_h, a^k_h), (30)
\]

\[
M^k_h := \mathbb{E}_{h}((V^k_{h+1} - V^{\pi^k}_{h+1}))(s^k_h, a^k_h) - (V^k_{h+1} - V^{\pi^k}_{h+1}))(s^k_h). \tag{31}
\]

Note that in line 2 of Algorithm 2, we truncate \( Q^k_h \) to the range \( [0, H - h] \). Thus for any \( (k, t) \in [K] \times [H] \), we have, \( |D^k_h| \leq 2H \). Moreover, since \( \mathbb{E}[D^k_h | F^k_{h+1}] = 0 \), \( D^k_h \) is a martingale difference sequence. So, applying Azuma-Hoeffding inequality we have with probability at least \( 1 - \delta/3 \),

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} D^k_h \leq \sqrt{2HT \log(3/\delta)}, \tag{34}
\]

where \( T = KH \).
Applying Azuma-Hoeffding inequality we have with probability at least $1 - \delta/3$,

$$\sum_{k=1}^K \sum_{t=1}^H \mathcal{M}_h^k \leq \sqrt{2H^2T \log(3/\delta)}.$$  \hspace{1cm} (35)

Applying union bound on (34) and (35) gives (33) and completes the proof.

Lemma C.16. Let $\lambda = 1$ in Algorithm 2. For any $\delta > 0$, conditioned on the event $\mathcal{G}(K, H, \delta)$, we have,

$$\sum_{k=1}^K \sum_{h=1}^H \mathbb{E}^{\pi_\cdot} \left[ l_h^k (s_h, a_h)|s_1 = s_1^k \right] - l_h^k (s_h, a_h) \leq \left( \sqrt{\nu_K(\delta)} + \sqrt{\gamma_K(\delta)} \right) \sqrt{2dHT \log(1 + K)},$$

with probability $1 - (\delta + c_0^M)$.

Proof. By Lemma C.11, with probability $1 - (\delta + c_0^M)$ it holds that

$$\sum_{k=1}^K \sum_{h=1}^H \mathbb{E}^{\pi_\cdot} \left[ l_h^k (s_h, a_h)|s_1 = s_1^k \right] \leq 0,$$  \hspace{1cm} (37)

and

$$\sum_{k=1}^K \sum_{h=1}^H -l_h^k (s_h, a_h) \leq \sum_{k=1}^K \sum_{h=1}^H \left( \sqrt{\nu_K(\delta)} + \sqrt{\gamma_K(\delta)} \right) \| \phi(s_h, a_h) \|_{(A_k^h)^{-1}}$$

$$\leq \left( \sqrt{\nu_K(\delta)} + \sqrt{\gamma_K(\delta)} \right) \sum_{k=1}^K \sum_{h=1}^H \| \phi(s_h, a_h) \|_{(A_k^h)^{-1}}$$

$$\leq \left( \sqrt{\nu_K(\delta)} + \sqrt{\gamma_K(\delta)} \right) H \sqrt{2dK \log(1 + K)}$$

$$= \left( \sqrt{\nu_K(\delta)} + \sqrt{\gamma_K(\delta)} \right) \sqrt{2dHT \log(1 + K)}. \hspace{1cm} (38)$$

Here the second inequality follows from the fact that both $\nu_k(\delta)$ and $\gamma_k(\delta)$ are increasing in $k$. The third and the fourth inequalities follow from Cauchy-Schwarz inequality and Lemma D.4. Combining (37) and (38) completes the proof.

Lemma C.17 (Good event probability). For any $K \in \mathbb{N}$ and any $\delta > 0$, we would have the event $\mathcal{G}(K, H, \delta')$ with probability at least $1 - \delta$, where $\delta' = \delta/M^T$.

Proof. By Lemma D.2, we have, for any fixed $t$ and $k$, the event $\mathcal{G}^k_h (\xi, \delta')$ occurs with probability at least $1 - M\delta'$. Recall from Definition C.8 that,

$$\mathcal{G}(K, H, \delta') = \bigcap_{k \leq K} \bigcap_{h \leq H} \mathcal{G}^k_h (\xi, \delta').$$

Now taking union bound over all $(t, k) \in [H] \times [K]$, we have

$$\mathbb{P}\left( \bigcap_{k \leq K} \bigcap_{h \leq H} \mathcal{G}^k_h (\xi, \delta') \right) \geq 1 - M^T \delta' = 1 - \delta,$$

which completes the proof.
Theorem C.18. Let $\lambda = 1$, $\sigma = O(H \sqrt{d})$ and $M = d \log(\delta/9)/\log c_0$, where $c_0 = \Phi(1)$ and $\delta \in (0, 1]$. Under Definition 4.3, the regret of Algorithm 2 satisfies

$$\text{Regret}(T) \leq O(d^{3/2} H^{3/2} \sqrt{T}),$$

with probability at least $1 - \delta$.

Proof of Theorem C.18. Let $\delta' = \delta/9$. From Lemma C.17, the event $\mathcal{G}(K, H, \delta')$ happens with probability $1 - \delta'$. Combining Lemma C.16 and Lemma C.17 we have that the event $\mathcal{G}(K, H, \delta')$ occurs and it holds that

$$\sum_{k=1}^K \sum_{h=1}^H \mathbb{E}_{\pi^*} \left[ I_k^h(s_h, a_h) | s_1 = s_1^k \right] - I_k^h(s_h, a_h^* h) \leq \left( \sqrt{\nu_K(\delta')} + \sqrt{\gamma_K(\delta')} \right) \sqrt{2dHT \log(1 + K)}, \quad (39)$$

with probability at least $(1 - \delta')(1 - (\delta' + c_0^M))$. Note that $c_0^M = \delta'$ and $(1 - \delta')(1 - (\delta' + c_0^M)) > 1 - 3\delta' = 1 - \delta/3$.

The martingale inequalities from Lemma C.15 happens with probability $1 - 2\delta/3$.

Applying union bound on (32), (33) and (39) gives the final regret bound of $O(d^{3/2} H^{3/2} \sqrt{T})$ completes the proof.

D. Auxiliary lemmas

This section presents several auxiliary lemmas and their proofs.

D.1. Gaussian Concentration

Lemma D.1 (Gaussian Concentration (Vershynin, 2018)). Consider a $d$-dimensional multivariate normal distribution $\eta \sim N(0, A\Lambda^{-1})$ where $A$ is a scalar. For any $\delta > 0$, with probability $1 - \delta$,

$$\|\eta\|_A \leq c \sqrt{dA \log(d/\delta)},$$

where $c$ is some absolute constant. For $d = 1$, we have $c = \sqrt{2}$.

Lemma D.2. Consider a $d$-dimensional multivariate normal distribution $N(0, A\Lambda^{-1})$ where $A$ is a scalar. Let $\eta_1, \eta_2, \ldots, \eta_M$ be $M$ independent samples from the distribution. Then for any $\delta > 0$

$$\mathbb{P} \left( \max_{j \in [M]} \|\eta_j\|_A \leq c \sqrt{dA \log(d/\delta)} \right) \geq 1 - M\delta,$$

where $c$ is some absolute constant.

Proof. From Lemma D.1, for a fixed $j \in [M]$, with probability at least $1 - \delta$ we would have

$$\|\eta\|_A \leq c \sqrt{dA \log(d/\delta)}.$$

Applying union bound over all $M$ samples completes the proof.

D.2. Inequalities for summations

Lemma D.3 (Lemma D.1 in (Jin et al., 2020)). Let $\Lambda_k = \lambda I + \sum_{i=1}^t \phi_i \phi_i^\top$, where $\phi_i \in \mathbb{R}^d$ and $\lambda > 0$. Then it holds that

$$\sum_{i=1}^t \phi_i^\top (\Lambda_k)^{-1} \phi_i \leq d.$$

Lemma D.4 (Lemma 11 in (Abbasi-Yadkori et al., 2011)). Using the same notation as defined in this paper

$$\sum_{k=1}^K \|\phi(s^k_h, a^k_h)\|^2_{(\Lambda_k)^{-1}} \leq 2d \log \left( \frac{\lambda + K}{\lambda} \right).$$
Lemma D.5. Let $A \in \mathbb{R}^{d \times d}$ be a positive definite matrix where its largest eigenvalue $\lambda_{\text{max}}(A) \leq \lambda$. Let $x_1, \ldots, x_k$ be $k$ vectors in $\mathbb{R}^d$. Then it holds that
\[
\left\| A \sum_{i=1}^{k} x_i \right\| \leq \sqrt{k \lambda \left( \sum_{i=1}^{k} \| x_i \|_A^2 \right)^{1/2}}.
\]

**Proof.** For any vector $v \in \mathbb{R}^d$,
\[
\|Av\| = \|A^{1/2}A^{1/2}v\| \\
\leq \|A^{1/2}\| \|A^{1/2}v\| \\
= \|A^{1/2}\| \|v\|_A.
\]
Here the inequality follows from the definition of the operator norm $\|A^{1/2}\|$. Moreover, $\|A^{1/2}\| \leq \sqrt{k}$ since $\lambda_{\text{max}}(A) \leq \lambda$. Thus,
\[
\left\| A \sum_{i=1}^{k} x_i \right\| \leq \sqrt{k \lambda \left( \sum_{i=1}^{k} \| x_i \|_A^2 \right)^{1/2}}. 
\]
Now by Cauchy-Schwarz inequality,
\[
\left\| \sum_{i=1}^{k} x_i \right\|_A^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} x_i^T A x_j \\
\leq \sum_{i=1}^{k} \sum_{j=1}^{k} \| x_i \|_A \| x_j \|_A \\
= \left( \sum_{i=1}^{k} \| x_i \|_A \right)^2 \\
\leq k \sum_{i=1}^{k} \| x_i \|_A^2. 
\]
Combining (40) and (41), proves the lemma. □

### D.3. Covering numbers and self-normalized processes

**Lemma D.6 (Lemma D.6 in (Jin et al., 2020)).** Let $\{s_t\}_{t=1}^{\infty}$ be a stochastic process on state space $S$ with corresponding filtration $\{\mathcal{F}_t\}_{t=1}^{\infty}$. Let $\{\phi_t\}_{t=1}^{\infty}$ be an $\mathbb{R}^d$-valued stochastic process where $\phi_t \in \mathcal{F}_{t-1}$, and $\|\phi_t\| \leq 1$. Let $\Lambda_k = \lambda I + \sum_{t=1}^{k} \phi_t \phi_t^T$. Then for any $\delta > 0$, with probability at least $1 - \delta$, for all $k \geq 0$, and any $V \in \mathcal{V}$ with $\sup_{s \in S} |V(s)| \leq H$, we have
\[
\left\| \sum_{i=1}^{k} \phi_t \{V(s_t) - \mathbb{E}[V(s_t) | \mathcal{F}_{t-1}]\} \right\|_{\Lambda_k^{-1}}^2 \leq 4H^2 \left[ \frac{d}{2} \log \left( \frac{k + \lambda}{\lambda} \right) + \log \frac{N_\varepsilon}{\delta} \right] + \frac{8k^2 \varepsilon^2}{\lambda},
\]
where $N_\varepsilon$ is the $\varepsilon$-covering number of $\mathcal{V}$ with respect to the distance $\text{dist}(V, V') = \sup_{s \in S} |V(s) - V'(s)|$.

**Lemma D.7** (Covering number of Euclidean ball, (Vershynin, 2018)). For any $\varepsilon > 0$, the $\varepsilon$-covering number, $N_\varepsilon$, of the Euclidean ball of radius $B > 0$ in $\mathbb{R}^d$ satisfies
\[
N_\varepsilon \leq \left( 1 + \frac{2B}{\varepsilon} \right)^d \leq \left( \frac{3B}{\varepsilon} \right)^d.
\]

**Lemma D.8.** Consider a class of functions $\mathcal{V} : S \to \mathbb{R}$ which has the following parametric form
\[
V(\cdot) = \left\langle \min\{\phi(\cdot)^T \theta, H\}^+, \pi(\cdot | \cdot) \right\rangle_A.
\]
where the parameter \( \theta \) satisfies \( \| \theta \| \leq B \) and for all \((s, a) \in S \times A\), we have \( \| \phi(s, a) \| \leq 1 \). If \( N_{V, \varepsilon} \) denotes the \( \varepsilon \)-covering number of \( V \) with respect to the distance \( \text{dist}(V, V') = \sup_{s \in S} |V(s) - V'(s)| \), then

\[
\log N_{V, \varepsilon} \leq d \log (1 + 2B/\varepsilon) \leq d \log (3B/\varepsilon).
\]

Proof. Consider any two functions \( V_1, V_2 \in \mathcal{V} \) with parameters \( \theta_1 \) and \( \theta_2 \), respectively. Note that \( \min \{ \cdot, H \} \) is a contraction mapping. Thus we have

\[
\begin{align*}
\text{dist}(V_1, V_2) & \leq \sup_{s} |\phi(s, \cdot)^\top \theta_1 - \phi(s, \cdot)^\top \theta_2| \\
& \leq \sup_{\phi : \|\phi\| \leq 1} |\phi^\top (\theta_1 - \theta_2)| \\
& \leq \sup_{\phi : \|\phi\| \leq 1} \|\theta_1 - \theta_2\|_2 \|\phi\|_2 \\
& = \|\theta_1 - \theta_2\|,
\end{align*}
\]

where the second inequality follows from the triangle inequality and the third inequality follows from the Cauchy-Schwarz inequality.

If \( N_{\theta, \varepsilon} \) denotes the \( \varepsilon \)-covering number of \( \{ \theta \in \mathbb{R}^d | \| \theta \| \leq B \} \), Lemma D.7 implies

\[
N_{\theta, \varepsilon} \leq \left( 1 + \frac{2B}{\varepsilon} \right)^d \leq \left( \frac{3B}{\varepsilon} \right)^d.
\]

Let \( C_{\theta, \varepsilon} \) be an \( \varepsilon \)-cover of \( \{ \theta \in \mathbb{R}^d | \| \theta \| \leq B \} \) with cardinality \( N_{\theta, \varepsilon} \). Consider any \( V_1 \in \mathcal{V} \). By (42), there exists \( \theta_2 \in C_{\theta, \varepsilon} \) such that \( V_2 \) parameterized by \( \theta_2 \) satisfies \( \text{dist}(V_1, V_2) \leq \varepsilon \). Thus we have

\[
\log N_{V, \varepsilon} \leq \log N_{\theta, \varepsilon} \leq d \log (1 + 2B/\varepsilon) \leq d \log (3B/\varepsilon),
\]

which concludes the proof. \( \square \)

E. Experiment Details

In this section we include the figure for the RiverSwim environment from (Osband et al., 2013).

Figure 5: The 6 state RiverSwim environment (Osband et al., 2013). State \( s_1 \) has a small reward while state \( s_5 \) has a large reward. The action whose transition is denoted with a dashed arrow deterministically moves the agent left. The other action is stochastic, and with relative high probability moves the agent towards the goal state \( s_6 \). This action represents swimming against the current, hence the name RiverSwim.