GLOBAL EXISTENCE FOR A NONLINEAR SYSTEM WITH FRACTIONAL LAPLACIAN IN BANACH SPACES

MIGUEL LOAYZA AND PAULO F. S. SILVA

Abstract. We consider the Cauchy problem for the fractional power dissipative equation

\[ u_t + \left(-\Delta\right)^{\beta/2}u = F(u), \quad \text{where} \quad \beta > 0 \quad \text{and} \quad F(u) = B(u, ..., u) \quad \text{and} \quad B \text{ is a multilinear form on a Banach space } E. \]

We show a global existence result assuming some properties of scaling degree of the multilinear form and the norm of the space \( E \). We extend the ideas used for the treating of the equation to determine the global existence for the system

\[ u_t + \left(-\Delta\right)^{\beta/2} = F(v), \quad v_t + \left(-\Delta\right)^{\beta/2} = G(u) \]

where \( F(u) = B_1(u, ..., u) \), \( G(v) = B_2(v, ..., v) \)

Keywords: fractional power equation, Banach spaces, global solution.

1. Introduction

Let \( n \in \mathbb{N}, \beta > 0 \). We consider the Cauchy problem for the semilinear fractional power equation

\[
\begin{aligned}
\partial_t u + \left(-\Delta\right)^{\beta/2} u &= F(u), \quad \text{in } \mathbb{R}^n \times (0, \infty) \\
u(0) &= u_0 \quad \text{in } \mathbb{R}^n,
\end{aligned}
\tag{1.1}
\]

where \( F(u) = B(u, ..., u) \) and \( B : E^p \to \mathbb{R} \) is a \( p \)-linear form, where \( E \) is a Banach space, \( p > 1 \) and \( E^p = E \times ... \times E(p \text{ times}) \). We also assume that the \( p \)-linear form \( B \) acts on \( u \) only with respect to the spacial variable.

The problem (1.1) models several classical problems, for example

(1) The semilinear fractional power dissipative equation

\[ u_t + \left(-\Delta\right)^{\beta/2} u = \nu u^p. \]

(2) The generalized Hamilton-Jacobi equation

\[ u_t + \left(-\Delta\right)^{\beta/2} = \nabla u \cdot \nabla u. \]

When \( \beta = 2 \), we have the Hamilton-Jacobi equation.

(3) The generalized Navier-Stokes equation

\[ u_t + \left(-\Delta u\right)^{\beta/2} - (u \cdot \nabla)u + \nabla P = 0, \quad \nabla \cdot u = 0. \]

(4) The generalized convection-diffusion equation

\[ u_t + \left(-\Delta u\right)^{\beta/2} = a \cdot \nabla (u^p), \quad a \in \mathbb{R}^n, a \neq 0 \]

The case \( \beta = 2 \) for the semilinear dissipative equation correspond to the semilinear heat equation and has been studied extensively, see for instance [10], [4], [6], [20], [21], [22]. For \( \beta \neq 2 \) see for example [16], [23], [15]. For the nonlinear Hamilton-Jacobi equation the local well posedness in Lebesgue spaces has been discussed in [2]. Concerning the generalized

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Navier-Stokes equation see [5]. Global well posedness including self-similar solutions and the large time behavior have been proved for convection-diffusion in [11] and [9].

Local and global existence and large time behavior for solutions of problem (1.1), with $\beta = 2$ and $B$ a bilinear form on $E \times E$, were studied in a general context in [12]. Specifically, it is assumed that

(i) The norm $\| \cdot \|_E$ has scaling degree equal to $a$, that is

$$\| u_\lambda \|_E = \lambda^a \| u \|_E$$

(1.2)

for each $u \in E, \lambda > 0$ such that $u_\lambda \in E$, where $u_\lambda(x) = u(\lambda x)$ for $x \in \mathbb{R}^n$.

(ii) The bilinear form $B$ has the following scaling property

$$B((u_1)_\lambda, (u_2)_\lambda) = \lambda^b [B(u_1, u_2)]_\lambda$$

for some $b \in \mathbb{R}, \lambda > 0$, and for every $u_i \in E$ so that $(u_i)_\lambda \in E$ $(i = 1, 2)$.

(iii) The Banach spaces $E$ is adequate to problem (1.1), that is,

(a) $S \subset E \subset S'$ with continuous injections.

(b) The norm is translations invariant, that is, $\| u(\cdot + x) \|_E = \| u \|_E$ for all $u \in E$ and $x \in \mathbb{R}^n$.

(c) For all $u, v \in E$, $B(u, v) \in S'$ and there exist $T_0 > 0$ and a function $w \in L^1(0, T_0)$, $w > 0$ such that

$$\| S(t)B(u, v) \| \leq w(t) \| u \| \| v \|,$$

for any $u, v \in E$. Here, $(S(t))_{t \geq 0}$ is the heat semigroup.

Henceforth, $S$ denotes the space of Schwartz rapidly decreasing functions and $S'$ its dual space, that is, the space of tempered distributions. Some examples of adequate spaces are the Lebesgue space $L^p(\mathbb{R}^n)$, the Marcinkiewicz weak $L^p(\mathbb{R}^n)$ space, the Lorentz space $L^{p,q}(\mathbb{R}^n)$ and the Morrey space $M^p(\mathbb{R}^n)$.

With these concepts it was established the following local existence result.

**Theorem 1.1** ([12]). Let $\beta = 2$, $E$ a Banach space and let $B$ be a bilinear form on $E \times E$ with scaling degree $\sigma < 2$. Let $r > n(2 - \sigma)$ and $0 \leq \alpha < \min\{1, 2 - \sigma - n/r, \sigma + n/p\}$. Suppose that $E$ has the following properties:

(i) $E$ is adequate to problem (1.1);

(ii) The norm $\| \cdot \|_E$ has a scaling degree $-\frac{n}{r}$;

(iii) $S(t) : E \to L^q(\mathbb{R}^n)$ is a bounded operator for every $t > 0$ and some $q \in [1, \infty]$

Let $u_0 \in BE^\alpha$. There exists $T = T(u_0)$ and a unique local in time solution of problem (1.1) on $[0, T)$, which is unique in the space

$$\mathcal{F}([0, T), BE^\alpha) \cap \{ u : (0, T) \to E; \sup_{0 < t < T} t^{\alpha/2} \| u(t) \|_E < \infty \}.$$  

(1.3)

The space $\mathcal{F}([0, T], BE^\alpha) = \{ u \in L^\infty((0, T), BE^\alpha); u(t) \to u(0) \text{ as } t \to 0 \text{ in } \mathcal{S}' \}$. The spaces $BE^\alpha$ is given by

$$BE^\alpha = \{ f \in S'; \| f \|_{BE^\alpha} = \sup_{t > 0} t^{\alpha/2} \| S(t)f \|_E < \infty \}.$$  

(1.4)

For the global existence we have.
We begin with the local existence for problem (1.1).

Our result about global existence is the following.

**Theorem 1.4.** Let $B$ a $p-$linear form with scaling degree $\sigma$ and $E \in X$ an adequate Banach space to problem (1.1) with scaling degree $a$, where

$$\frac{\beta - \sigma}{p - 1} - \frac{\beta}{p} < -a < \frac{\beta - \sigma}{p - 1} \quad \text{and} \quad \alpha = \frac{\beta - \sigma}{p - 1} + a. \quad (1.8)$$
Let $M, R > 0$ such that $R + pKM^{p-1} < M$ with $K > 0$ given explicitly by (3.11). Then for every $u_0 \in BE^\alpha$ with $\|u_0\|_{BE^\alpha} \leq R$, there exists a unique global solution $u$ of problem (1.1) satisfying
\[
\|u\| := \sup_{t > 0} \|u(t)\|_{BE^\alpha} + \sup_{t > 0} t^{\frac{\alpha}{p}} \|u(t)\|_E \leq M. \tag{1.9}
\]
Moreover, if $v$ is other global solution for problem (1.7) satisfying (1.9) and with initial data $v_0$ with $\|v_0\|_{BE^\alpha} \leq R$, then
\[
\|u - v\| \leq [1 - p(2M)^{p-1}K]^{-1}\|u_0 - v_0\|_{BE^\alpha}.
\]
In addition, if $pK_1M^{p-1} < 1$, where $K_1$ is given by (3.8), we have
\[
\lim_{t \to \infty} t^{\frac{\alpha}{p}/2}\|u(t) - v(t)\|_E = 0
\]
if and only if $\lim_{t \to \infty} t^{\frac{\alpha}{p}/2}\|S_\beta(u_0 - v_0)\|_E = 0$

**Remark 1.5.** Here are some comments concerning Theorem 1.4.

(i) It is clear that if $\beta = p = 2$, Theorem 1.4 reduces to Karch’s result.

We use our arguments to analyze the semilinear fractional power system
\[
\begin{aligned}
&\partial_t u + (-\Delta)^{\beta/2} u = B_1(v, \ldots, v), \text{ in } \mathbb{R}^n \times (0, \infty) \\
&\partial_t v + (-\Delta)^{\beta/2} v = B_2(u, \ldots, u), \text{ in } \mathbb{R}^n \times (0, \infty) \\
&u(0) = u_0, v(0) = v_0 \text{ in } \mathbb{R}^n,
\end{aligned} \tag{1.10}
\]
where the $B_1, B_2$ are multi-linear forms defined on Banach spaces. Problem (1.10) for $\beta = 2$ has been considered by various authors, see for example, [8, 19].

As in the problem (1.1), if $u_0 \in BE^{\alpha_1}, v_0 \in BE^{\alpha_2}$ and $BE^{\alpha_1}, BF^{\alpha_2}$ are Banach spaces, we say that $(u, v) \in L^\infty((0, T); BE^{\alpha_1}) \times L^\infty((0, T); BF^{\alpha_2})$ is a solution of the problem (1.10) if verifies, in some sense, the following system
\[
\begin{aligned}
&u(t) = S_\beta(t)u_0 + \int_0^t S_\beta(t - \tau)B_1(v, \ldots, v)d\tau, \\
v(t) = S_\beta(t)v_0 + \int_0^t S_\beta(t - \tau)B_2(u, \ldots, u)d\tau,
\end{aligned} \tag{1.11}
\]
for every $t \in (0, T)$.

On the global existence for problem (1.10) we have the following result.

**Theorem 1.6.** Let $E, F$ be Banach spaces, $B_1 : E^q \to \mathbb{R}$ and $B_2 : F^p \to \mathbb{R}$ two forms with scaling degree $\sigma_1$ and $\sigma_2$ respectively. Assume that $E, F \in X$ have scaling degree $a$ and $b$ respectively and that $E \times F$ is adequate to system (1.10). Let $pq > 1$,
\[
\alpha_1 = \frac{\beta(1 + q)}{pq - 1} + a - \frac{\sigma_1 + q\sigma_2}{pq - 1}, \quad \alpha_2 = \frac{\beta(1 + p)}{pq - 1} + b - \frac{\sigma_2 + p\sigma_1}{pq - 1}. \tag{1.12}
\]
Suppose that

(i) $\alpha_1, \alpha_2 > 0$.
(ii) $\alpha_1 + qb < a + \sigma_1$ and $\alpha_2 + pa < b + \sigma_2$.
(iii) $\alpha_1 < q\alpha_2, \alpha_2 < p\alpha_1$. 

Let $M, R > 0$ so that $R + qM^qK_1 + pM^pK_2 < M$. Then, for any $\Phi = (u_0,v_0) \in BE^{\alpha_1} \times BF^{\alpha_2}$ verifying

$$\mathcal{N}(\Phi) := \|u_0\|_{BE^{\alpha_1}} + \|v_0\|_{BF^{\alpha_2}} \leq R,$$

there exists an unique solution $U = (u,v)$ for system (1.11) such that

$$\|U\| := \sup_{t>0} t^{\frac{p}{qM^q-1}} \|u(t)\|_E + \sup_{t>0} t^{\frac{p}{pM^p-1}} \|v(t)\|_F + \|v(t)\|_{BF^{\alpha_2}} \leq M.$$

Moreover, if $\overline{U} = (\overline{u}, \overline{v})$ is a solution for problem (1.11) verifying $\|\overline{U}\| \leq M$ with initial data $\Phi = (\overline{u}_0, \overline{v}_0)$ which verify $\mathcal{N}(\Phi) \leq R$, then

$$\|U - \overline{U}\| \leq [1 - \left( qM^q-1K_1 + pM^p-1K_2 \right)^{-1} \mathcal{N}(\Phi - \Phi)].$$

**Remark 1.7.** Here are some comment on Theorem 1.6

(i) If $\sigma_1 = \sigma_2 = \sigma$ and $b = \left[(p+1)/(q+1)\right]a$, then $\alpha_1 = \left[(\beta - \sigma)(q+1)/(pq - 1)\right]$, $\alpha_2 = \left[(p+1)/(q+1)\right] \alpha_1$ and conditions (i)-(iv) are reduced to

$$(\beta - \sigma) \frac{q+1}{pq - 1} - \frac{\beta}{p} \gamma < -a < (\beta - \sigma) \frac{q+1}{pq - 1},$$

where $\gamma = \min\{1, [p(q+1)]/[q(p+1)]\}$. In particular, for $p = q$ we have condition (1.8).

(ii) Local existence for problem (1.11) can be obtained modifying slightly the proof of Theorem 1.6. To do this, we assume that

(a) $\alpha_1, \alpha_2 > 0$, $\alpha_1 - q\alpha_2 + \beta + qb > a + \sigma_1$, $-p\alpha_1 + \alpha_2 + \beta + pa > b + \sigma_2$.

(b) $\beta + qb > a + \sigma_1$, $\beta + pa > b + \sigma_2$.

(c) $\beta > q\alpha_2$, $\beta > p\alpha_1$.

(d) $a + \sigma_1 > a + q\beta$ and $b + \sigma_2 > \sigma_2 + pa$.

2. Preliminary results

In this section we extend for $\beta \neq 2$ the definitions considered by Karch in [12, 13] for $\beta = 2$. The arguments used in the proof of Propositions 2.3, 2.4 and 2.11 are similar to Karch’s arguments, but since we are considering situations in that $\beta$ can be different to two we give them for completeness.

2.1. Scaling properties. Let $(E, \| \cdot \|_E)$ be a Banach space which can be imbedded continuously in $S'$. We say that the norm $\| \cdot \|_E$ has scaling degree equal to $a$ if equality (1.2) holds. What follows are some examples of Banach spaces with its respective scaling degrees:

(i) Lebesgue spaces, $L^p(\mathbb{R}^n)$: $\|u_\lambda\|_{L^p} = \lambda^{-\frac{np}{p}} \|u\|_{L^p}$, $1 \leq p \leq +\infty$,

(ii) Lorentz spaces, $L^{p,q}(\mathbb{R}^n)$: $\|u_\lambda\|_{L^{p,q}} = \lambda^{-\frac{np}{p}} \|u\|_{L^{p,q}}$, $1 \leq p, q \leq +\infty$,

(iii) Morrey Homogeneous spaces, $L^p_q(\mathbb{R}^n)$: $\|u_\lambda\|_{L^p_q} = \lambda^{-\frac{np}{p}} \|u\|_{L^p_q}$, $1 \leq q \leq p \leq +\infty$,

(iv) Besov Homogeneous spaces, $B^s_{p,q}(\mathbb{R}^n)$: $\|u_\lambda\|_{B^s_{p,q}} = \lambda^{s-\frac{np}{p}} \|u\|_{B^s_{p,q}}$, $1 \leq p, q \leq +\infty$ and $s < 0$
For a study of these spaces see [3] and [11].

We say that a $p-$linear form $B$ defined on $E^p = E \times E \times \ldots \times E (p \text{ times})$ has scaling degree equal to $\sigma$ if

$$B((u_1)_\lambda, \ldots, (u_p)_\lambda) = \lambda^\sigma [B(u_1, \ldots, u_p)]_\lambda,$$

for every $u_1, u_2, \ldots, u_p \in E$, $\lambda > 0$ and $(u_i)_\lambda = u_i(\lambda \cdot)$, $i = 1, 2, \ldots, p$.

Some examples of $p-$linear forms, with its respective scaling degree $\sigma$, are given below:

(i) $B(u_1, \ldots, u_p) = u_1 \cdot \ldots \cdot u_p$, has scaling degree $\sigma = 0$.
(ii) $B(u_1, \ldots, u_p) = (u_1 \cdot \ldots \cdot u_{p-1}) \cdot \nabla u_p$ with $a \in \mathbb{R}^n$ has scaling degree $\sigma = 1$.
(iii) $B(u_1, u_2) = \nabla u_1 \cdot \nabla u_2$ has scaling degree $\sigma = 2$.

In the following result we establish some scaling relations.

**Proposition 2.1.** Let $\lambda > 0$ and $u \in S'$. Then, $(-\Delta)^{\beta/2}u_\lambda = \lambda^\beta [(-\Delta)^{\beta/2}u]_\lambda$, $S_\beta(t)u_\lambda = [S_\beta(\lambda^\beta t)u]_\lambda$ and $t^{\frac{n}{2}}K_\beta(t, t^{\frac{1}{2}}x) = K_\beta(1, x)$ for $t > 0$, $x \in \mathbb{R}^n$.

**Proof.** By the change of variable $\xi' = t^{1/\beta} \xi$ we have

$$K_\beta(t, t^{\frac{1}{2}}x) = \int_{\mathbb{R}^n} e^{2\pi i (t^{\frac{1}{2}} x, \xi')} e^{-t|\xi'|^\beta} d\xi'$$

$$= t^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{2\pi i (x, \xi')} e^{-|\xi'|^\beta} d\xi'.$$

By (1.5) and the change of variables $y' = \lambda y$ and $\xi' = \lambda^{-1} \xi$ we conclude

$$[(-\Delta)^{\beta/2}u_\lambda](x) = \int_{\mathbb{R}^n} e^{2\pi i (x, \xi')} |\xi'|^\beta \mathcal{F}u_\lambda(\xi) d\xi$$

$$= \int_{\mathbb{R}^n} e^{2\pi i (x, \xi')} |\xi'|^\beta \left[ \int_{\mathbb{R}^n} e^{-2\pi i (\xi, y)} u(\lambda y) dy \right] d\xi$$

$$= \lambda^\beta \int_{\mathbb{R}^n} e^{2\pi i (\lambda x, \xi')} |\xi'|^\beta \left[ \int_{\mathbb{R}^n} e^{-2\pi i (\xi, y')} u(y') \lambda^{-n} dy' \right] \lambda^n d\xi'$$

$$= \lambda^\beta [(-\Delta)^{\beta/2}u](\lambda x)$$

$$= \lambda^\beta [(-\Delta)^{\beta/2}u_\lambda](x)$$

From (1.6)

$$[S_\beta(t)u_\lambda](x) = \int_{\mathbb{R}^n} \mathcal{F}^{-1}(e^{-t|\xi'|^\beta})(x-y) u(\lambda y) dy$$

$$= \int_{\mathbb{R}^n} \mathcal{F}^{-1}(e^{-t|\xi'|^\beta})(x-\lambda^{-1} y') u(y') \lambda^{-n} dy'$$

$$= \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} e^{2\pi i (x-\lambda^{-1} y', \xi')} e^{-t|\xi'|^\beta} u(y') \lambda^{-n} dy' \right] \varphi(y') \lambda^{-n} dy'$$

$$= \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} e^{2\pi i (x-y', \xi')} e^{-t\lambda y'} e^{-t|\xi'|^\beta} \varphi(y') \lambda^{-n} dy' \right] d\xi'$$

$$= \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} e^{2\pi i (x-y', \xi')} e^{-t\lambda |\xi'|^\beta} \varphi(y') \lambda^{-n} dy' \right] u(y') dy'$$

$$= \left[ \mathcal{F}^{-1}(e^{-t\lambda |\xi'|^\beta} \varphi)(y') \right] u(y')$$

$$= [S_\beta(\lambda^\beta t)u_\lambda](x)$$

When $u$ is an homogeneous function we have the following result.
Corollary 2.2. If $E$ is a Banach space with scaling degree equal to $a$ and $u$ is an homogeneous function with degree equal to $\theta$, then $S_\beta(t)u = \lambda^{-\theta} [S_\beta(\lambda^t u)]_\lambda$ and $\|S_\beta(1)u\|_E = t^{(a-\theta)/\beta} \|S_\beta(t)u\|_E$.

Proposition 2.3. Let $E_a$ and $E_b$ be Banach spaces with scaling degree $a$ and $b$ respectively. Assume that there exists $t_0 > 0$ such that $S_\beta(t_0) : E_a \to E_b$ is bounded, where $S_\beta(t)$ is defined by (1.6). Then there exists $C = C(a,b,\beta,t_0) > 0$ so that

$$\|S_\beta(t)u\|_{E_b} \leq Ct^{\frac{\beta}{a-b}} \|u\|_{E_a}.$$ 

for all $t > 0$. Moreover, if $E_b \subset S'$ is imbedded continuously and $b < a$, then $E_a = \{0\}$.

Proof. Since $S_\beta(t_0)$ is bounded, there exists $C' > 0$ such that $\|S_\beta(t_0)u\|_{E_b} \leq C' \|u\|_{E_a}$, for every $u \in E_a$. By Proposition 2.1, we have

$$\|S_\beta(\lambda t_0)u\|_{E_b} \leq C' \|\lambda^{\frac{\beta}{a-b}} u\|_{E_a}$$

Thus, $\|S_\beta(\lambda^t t_0)u\|_{E_b} \leq C' \lambda^{\frac{\beta}{a-b}} \|u\|_{E_a}$ for every $\lambda > 0$. Setting $\lambda = (t/t_0)^{\frac{1}{\beta}}$ we conclude that $\|S_\beta(t)u\|_{E_b} \leq Ct^{\frac{\beta}{a-b}} \|u\|_{E_a}$ for every $t > 0$ with $C = C't_0^{\frac{\beta(b-a)}{\beta}}$.

Note that if $b < a$ and $u \in E_a$, then $S_\beta(t)u \to 0$ as $t \to 0$ in $E_b$. Since $E_b$ is imbedded continuously in $S'$ follows that $S_\beta(t)u \to 0$ as $t \to 0$ in $S'$. Since $S_\beta(t)u \to u$ in $S'$ as $t \to 0$, by uniqueness, we conclude that $u = 0$.

Proposition 2.4. Let $E$ be a Banach space with scaling degree $a$. Suppose that for some $t_0 > 0$ and $r_0 \in [1, +\infty]$ the operator $S_\beta(t_0) : E \to L^{r_0}(\mathbb{R}^n)$ is bounded. Then $S_\beta(t) : E \to L^r(\mathbb{R}^n)$ for all $t > 0$ and $r \geq r_0$. Moreover, there exists a constant $C > 0$ which does not depend of $t > 0$ and

$$\|S_\beta(t)u\|_{L^r} \leq Ct^\left(\frac{1}{r_0} - \frac{\theta}{a}\right) \|u\|_{E},$$

for all $t > 0$ and $u \in E$.

Proof. Since $S_\beta(t_0) : E \to L^{r_0}(\mathbb{R}^n)$ is bounded we have that $\|S_\beta(t_0)u\|_{L^{r_0}} \leq C \|u\|_E$ for every $u \in E$ and some constant $C > 0$. As $L^{r_0}(\mathbb{R}^n)$ has scaling degree $-n/r_0$, by Proposition 2.3, we have $\|S_\beta(t)u\|_{L^{r_0}} \leq Ct^\left(\frac{1}{r_0} - \frac{\theta}{a}\right) \|u\|_{r_0}, \ 1 \leq r_0 \leq r \leq +\infty$ and $t > 0$. Thus

$$\|S_\beta(t)u\|_{L^r} \leq C(t/2)^{-\frac{n}{2} \left(\frac{1}{r_0} - \frac{\theta}{a}\right)} \|S_\beta(t/2)u\|_{L^{r_0}} \leq C(t/2)^{-\frac{n}{2} \left(\frac{1}{r_0} - \frac{\theta}{a}\right)} (t/2)^{\frac{n}{2} \left(\frac{1}{r_0} - \frac{\theta}{a}\right)} \|u\|_{E} = C't^{\frac{n}{2} \left(\frac{1}{r_0} - \frac{\theta}{a}\right)} \|u\|_{E},$$

for all $t > 0$ and $u \in E$.

Remark 2.5. From here on, we denote by $X$ the class of Banach spaces $E'$ which can be imbedded continuously in $S'$ such that there exist $t_0$ and $r_0 \in [1, +\infty]$ and $S_\beta(t_0) : E \to L^{r_0}(\mathbb{R}^n)$ is bounded.
2.2. The space $BE^a$. Let $\alpha \geq 0$ and let $E$ be a nontrivial Banach space which can be imbedded continuously in $S'$. We define $BE^a$ as

$$BE^a = \{ u \in S'; \| u \|_{BE^a} = \sup_{t>0} t^{\alpha/\beta} \| S_\beta(t)u \|_E < \infty \}. \quad (2.3)$$

It is clear that $BE^a$ is a linear space and $\| \cdot \|_{BE^a}$ defines a norm on $BE^a$. Indeed, $\| u \|_{BE^a} = 0$ implies $\| S_\beta(t)u \|_E = 0$, for every $t > 0$. Since $E$ is imbedded continuously in $S'$ is continuous we conclude that $S_\beta(t)u \to 0$ in $S'$. Since $S_\beta(t)u \to u$ in $S'$ as $t \to 0$, follows that $u = 0$. The other axioms are easily verified.

**Remark 2.6.** (i) If $E$ is a Banach space with scaling degree $a$, then $\| \cdot \|_{BE^a}$ has scaling degree $a - \alpha$. Indeed, for $\lambda > 0$ and Proposition 2.7,

$$\| u_\lambda \|_{BE^a} = \sup_{t>0} t^{\alpha/\beta} \| S_\beta(t)u_\lambda \|_E = \sup_{t>0} t^{\alpha/\beta} \| S_\beta(\lambda^\beta t)u_\lambda \|_E$$

$$= \lambda^\alpha \sup_{t>0} t^{\alpha/\beta} \| S_\beta(\lambda^\beta t)u \|_E$$

$$= \lambda^{(a-\alpha)} \sup_{t>0} (\lambda^\beta t)^{\alpha/\beta} \| S_\beta(\lambda^\beta t)u \|_E = \lambda^{(a-\alpha)} \| u \|_{BE^a}.$$

(ii) If the norm $\| \cdot \|$ is translation invariant, then $\| \cdot \|_{BE^a}$ is also one. Indeed,

$$\| T_y u \|_{BE^a} = \sup_{t>0} t^{\alpha/\beta} \| S_\beta(t)T_y u \|_E = \sup_{t>0} t^{\alpha/\beta} \| T_{-y} S_\beta(t)u \|_E$$

$$= \sup_{t>0} t^{\alpha/\beta} \| S_\beta(t)u \|_E = \| u \|_{BE^a}.$$

Our objective now is to show that $BE^a$ is a Banach space. To do this, we consider some properties of the homogeneous Besov space $\dot{B}_{p,q}^\gamma(\mathbb{R}^n)$ with $\gamma < 0$.

The following result was proved in [16](Proposition 2.1).

**Proposition 2.7.** Let $1 \leq r, s \leq +\infty, \gamma < 0$ and $0 < \beta < +\infty$. Then $u \in \dot{B}_{r,s}^\gamma(\mathbb{R}^n)$ if and only if

$$\| u \|_{\dot{B}_{r,s}^\gamma} = \left\{ \begin{array}{l}
\left[ \int_0^{+\infty} \left( t^{-\gamma/\beta} \| S_\beta(t)u \|_{L^r} \right)^s \frac{dt}{t} \right]^{1/s}, \quad \text{if } 1 \leq s < \infty,
\sup_{t>0} t^{-\gamma/\beta} \| S_\beta(t)u \|_{L^r}, \quad \text{if } s = +\infty.
\end{array} \right.$$

is finite.

Note that if $E = L^r(\mathbb{R}^n)$ the set $BE^a, \alpha > 0$, is exactly the space $\dot{B}_{r,\infty}^{-\alpha}(\mathbb{R}^n)$.

In the next result we establish a continuous imbedding of $BE^a$ in a homogeneous Besov space.

**Proposition 2.8 (Imbedding in Besov spaces).** Let $E$ be a nontrivial Banach space continuously imbedded in $S'$. Let $\alpha \geq 0$, $E \in \mathcal{X}$ with scaling degree $a$. There exist $r > 1$ such that the imbedding $BE^a \subset \dot{B}_{r,\infty}^\gamma(\mathbb{R}^n)$ is continuous with $\gamma = n/r + a - \alpha < 0$.

**Proof.** Since $E \in \mathcal{X}$, let $t_0 > 0$ and $r_0 \in [1, +\infty]$ so that $S_\beta(t_0) : E \to L^{r_0}(\mathbb{R}^n)$ is bounded. From Proposition 2.3 we conclude that $a + n/r_0 \leq 0$. Let $r > r_0$ so that $a + n/r < 0$. From Proposition 2.4 there exists $C > 0$ such that

$$\| S_\beta(t)u \|_{L^r} = \| S_\beta(t/2)S_\beta(t/2)u \|_{L^r} \leq C(t/2)^{\frac{1}{r}(a + \frac{n}{r})} \| S_\beta(t/2)u \|_E.$$
Multiplying this inequality by \( t^{-\gamma/\beta} \) with \( \gamma = n/r + a - \alpha \) we obtain

\[
t^{-\frac{\gamma}{\beta}}\|S_\beta(t)u\|_{L^r} \leq C2^{-\frac{\gamma}{\beta}}(t/2)^{\frac{\gamma}{\beta}}\|S_\beta(t/2)u\|_E = C'(t/2)^{\frac{\gamma}{\beta}}\|S_\beta(t/2)u\|_E.
\]

Hence, we get

\[
sup_{t > 0} t^{-\gamma/\beta}\|S_\beta(t)u\|_{L^r} \leq C'\|u\|_{BE^\alpha}.
\]

Now, the result follows from Proposition 2.7.

Finally, we show that \( BE^\alpha \) is a Banach space.

**Proposition 2.9 (Completeness of \( BE^\alpha \)).** Let \( E \) be a nontrivial Banach space continuously imbedded in \( S' \). Let \( \alpha \geq 0, E \in \mathcal{X} \) with scaling degree \( a \). Then the space \( BE^\alpha \) is a Banach space. Furthermore, there exists \( r > 1 \) so that the imbedding \( BE^\alpha \subset B^{\gamma}_{r,\infty}(\mathbb{R}^n) \) is continuous, with \( \gamma = n/r + a - \alpha \).

**Proof.** Since \( E \subset S' \) is a normed linear space it is sufficient to show that \( BE^\alpha \) is complete. Let \( (u_n)_{n \in \mathbb{N}} \) a Cauchy sequence in \( BE^\alpha \). For every \( t > 0 \) the sequence \( (S_\beta(t)u_n)_{n \in \mathbb{N}} \) is a Cauchy sequence in \( E \) because

\[
\|S_\beta(t)u_n - S_\beta(t)u_m\|_E \leq t^{-\frac{\gamma}{\beta}}\|u_n - u_m\|_{BE^\alpha}.
\]

Hence, since \( E \) is a Banach space we have \( u(t) := \lim_{n \to \infty} S_\beta(t)u_n \) in \( E \). Using the fact that the embedding \( E \subset S' \) is continuous, we conclude that \( S_\beta(t)u_n \to u(t) \) in \( S' \).

On the other hand, since \( (u_n)_{n \in \mathbb{N}} \) is a Cauchy sequence in \( BE^\alpha \) we obtain a constant \( C > 0 \) such that \( \|u_n(t)\|_E \leq Ct^{-\frac{\gamma}{\beta}} \) for every \( t > 0 \). Thus, \( \|u(t)\|_E \leq Ct^{-\frac{\gamma}{\beta}} \) for every \( t > 0 \).

We show now that there exists \( v \in S' \) so that \( u(t) = S_\beta(t)v \). From Proposition 2.9 the embedding \( BE^\alpha \subset B^{\gamma}_{r,\infty} \) is continuous for some \( r > 1, \gamma = n/r + a - \alpha \). Therefore, the Cauchy sequence \( (u_n)_{n \in \mathbb{N}} \) converges in \( B^{\gamma}_{r,\infty} \) for some function \( v \). Since \( B^{\gamma}_{r,\infty} \subset S' \) we have that \( u_n \to u \) and \( S_\beta(t)u_n \to S_\beta(t)v \) in \( S' \). By uniqueness we conclude that \( u(t) = S_\beta(t)v \).

### 2.3. Adequate spaces.

Let \( E \) be a Banach space and \( B \) a \( p \)-linear form defined on \( E^p \). We say that \( E \) is adequate to problem (1.1) if

(i) \( S \subset E \subset S' \) both with continuous injections.

(ii) The norm \( \| \cdot \|_E \) is invariant by translations, that is, \( \|Tyu\|_E = \|u\|_E \) for every \( u \in E \), \( y \in \mathbb{R}^n \) and \( Tyu = u(-y) \).

(iii) For every \( u_1, \ldots, u_p \in E \), we have \( B(u_1, \ldots, u_p) \in S' \). Moreover,

\[
\|S_\beta(t)B(u_1, \ldots, u_p)\|_E \leq \omega(t)\prod_{i=1}^p \|u_i\|_E
\]

where \( \omega : (0, +\infty) \to (0, +\infty) \) and \( \omega \in L^1(0, T) \) for \( 0 < T < +\infty \).

**Remark 2.10.** If \( (E, \| \cdot \|_E) \) is a Banach space satisfying conditions (i) and (ii), then

\[
\|S_\beta(t)u\|_E \leq \|K_{\beta}(1, \cdot)\|_{L^i} \|u\|_E,
\]

since \( K_{\beta}(1, \cdot) \in L^1(\mathbb{R}^n) \), see Lemma 2.1 of [10]. In particular, \( E \) is adequate to problem (1.1) for \( B(u, \ldots, u) = u \).

In the next result we establish estimates for \( S_\beta(t)B(u_1, \ldots, u_p) \) in the spaces \( E \) and \( BE^\alpha \).

**Proposition 2.11.** Let \( B \) be a \( p \)-linear form with scaling degree \( \sigma \) and let \( E \) be a Banach space adequate to problem (1.1) with scaling degree \( a \).
(i) There exists a constant $C_1 = C_1(a, \sigma, \beta, p) > 0$ such that

$$\|S_\beta(t)B(u_1, ..., u_p)\|_E \leq C_1 t^{(p-1)a-\sigma}/\beta \prod_{i=1}^{p} \|u_i\|_E. \quad (2.5)$$

(ii) Assume that $0 \leq \alpha \leq -(p-1)a + \sigma$. Then there exists $C_2 = C_2(a, \sigma, \beta, p) > 0$ such that

$$\|S_\beta(t)B(u_1, ..., u_p)\|_{BE^\alpha} \leq C_2 t^{(p-1)a-\sigma}/\beta \prod_{i=1}^{p} \|u_i\|_E. \quad (2.6)$$

Proof. (i) Let $u_i \in E_a$, $i = 1, ..., p$ and $t_0 > 0$. Since $B$ has scaling degree $\sigma$, we obtain from Proposition 2.1, $S_\beta(t_0)B((u_1)_\lambda, ..., (u_p)_\lambda) = \lambda^\sigma[S_\beta(\lambda^\beta t_0)B((u_1), ..., (u_p))]$. Hence, using Proposition 2.1, the facts that $B$ is adequate and $E$ has scaling degree $a$, we have

$$\lambda^{\sigma + a} \|S_\beta(\lambda^\beta t_0)B(u_1, ..., u_p)\|_E = \|\lambda^\sigma[S_\beta(\lambda^\beta t_0)B(u_1, ..., u_p)]\|_E$$

$$= \|S_\beta(t_0)B((u_1)_\lambda, ..., (u_p)_\lambda)\|_E$$

$$\leq \omega(t_0) \prod_{i=1}^{p} \|(u_i)_\lambda\|_E$$

$$= \omega(t_0) \lambda^{\alpha a} \prod_{i=1}^{p} \|u_i\|_E.$$

Thus, $\|S_\beta(\lambda^\beta t_0)B(u_1, ..., u_p)\|_E \leq \omega(t_0)\lambda^{[p-1]a-\sigma} \prod_{i=1}^{p} \|u_i\|_E$, for all $\lambda > 0$. Setting $\lambda = (t/t_0)^{\frac{1}{\beta}}$ with $t > 0$, we obtain (2.5) with $C_1 = \omega(t_0)t_0^{-\frac{a}{\beta}}$.

(ii) We first claim that there exists a constant $C' = C'(t) > 0$ such that for all $\tau > 0$,

$$\tau^\frac{\alpha}{\beta} \|S_\beta(\tau)S_\beta(t)B(u_1, ..., u_p)\|_E \leq C'(t) \prod_{i=1}^{p} \|u_i\|_E.$$

Indeed, since $\alpha \geq 0$, (2.4) holds and $B$ is adequate, we have for $0 < \tau \leq 1$,

$$\tau^\frac{\alpha}{\beta} \|S_\beta(\tau)S_\beta(t)B(u_1, ..., u_p)\|_{E_a} \leq \|S_\beta(\tau)S_\beta(t)B(u_1, ..., u_p)\|_{E_a}$$

$$\leq \tilde{C}\|S_\beta(t)B(u_1, ..., u_p)\|_{E_a}$$

$$\leq \tilde{C}\omega(t) \prod_{i=1}^{p} \|u_i\|_{E_a}.$$

On the other hand, from bound (2.4), estimate (2.5) and $\alpha + (p-1)a - \sigma \leq 0$ we obtain for $\tau > 1$,

$$\tau^\frac{\alpha}{\beta} \|S_\beta(\tau)S_\beta(t)B(u_1, ..., u_p)\|_E = \tau^\frac{\alpha}{\beta} \|S_\beta(t)S_\beta(\tau)B(u_1, ..., u_p)\|_E$$

$$\leq \tilde{C}\tau^\frac{\alpha}{\beta} \|S_\beta(\tau)B(u_1, ..., u_p)\|_E$$

$$\leq \tilde{C}\tau^{\alpha + (p-1)a - \sigma} \prod_{i=1}^{p} \|u_i\|_E$$

$$\leq \tilde{C} \prod_{i=1}^{p} \|u_i\|_E.$$
Thus, the claim holds for \( C'(t) = \max \{ Cw(t), \tilde{C} \} \).

Fix now \( t_0 > 0 \). Since

\[
C'(t_0) \lambda^p \prod_{i=1}^{p} \| u_i \|_E = C'(t_0) \prod_{i=1}^{p} \| (u_i)_{\lambda} \|_E
\]

\[
\geq \tau^\frac{\alpha}{\beta} \| S_\beta(\tau)S_\beta(t_0)B((u_\lambda)_1, ..., (u_\lambda)_p) \|_E
\]

\[
= \lambda^{\sigma + \alpha} \tau^\frac{\alpha}{\beta} \| S_\beta(\lambda^\beta \tau)S_\beta(\lambda^\beta t_0)B(u_1, ..., u_p) \|_E
\]

\[
= \lambda^{\sigma + \alpha - \alpha} \| S_\beta(\lambda^\beta \tau)S_\beta(\lambda^\beta t_0)B(u_1, ..., u_p) \|_E
\]

we have \((\lambda^\beta \tau)^{\alpha/\beta} \| S_\beta(\lambda^\beta \tau)S_\beta(\lambda^\beta t_0)B(u_1, ..., u_p) \|_E \leq C'(t_0)\lambda^{\alpha + (p-1)\sigma} \prod_{i=1}^{p} \| u_i \|_E\). Taking the supreme on \( \tau \) and setting \( \lambda = (t/t_0)^{1/\beta} \) we get (2.6) with \( C_2 = C'(t_0) t_0^{1-\beta} \).

3. Local and global existence for problem (1.1)

The existence, local and global, of solutions for problem (1.1) is based in following abstract result.

**Lemma 3.1.** Assume that \( X \) is a Banach space and \( A: X \times ... \times X \to X \) is a \( p \)-linear form \((p > 1)\) verifying

\[
\| A(u_1, ..., u_p) \| \leq K \prod_{i=1}^{p} \| u_i \|, \tag{3.1}
\]

for all \( u_i \in X, \ i = 1, ..., p \) and for some constant \( K > 0 \). Let \( M, R > 0 \) such that

\[
R + pKM^p < M. \tag{3.2}
\]

Then, for every \( y \in X \) with \( \| y \| \leq R \) the equation

\[
u = y + A(u, ..., u) \tag{3.3}
\]

has a unique solution \( u \in X \) and \( \| u \| \leq M \). Moreover, the solution \( u \) depends continuously in the sense that, if \( v \) is a solution of (3.3), with \( y_1 \) in place of \( y \), and \( \| y \| \leq R, \| v \| \leq M \), then

\[
\| u - v \| \leq (1 - pKM^{p-1})^{-1} \| y - y_1 \|. \tag{3.4}
\]

**Proof.** Set \( B_M = \{ u \in X; \| u \| \leq M \} \). Consider the mapping \( \mathcal{G}_y: B_M \to X \) defined by \( \mathcal{G}_y(u) = y + A(u, ..., u) \). Since \( A \) is \( p \)-linear and verify inequality (3.1) we deduce

\[
\| A(u, ..., u) - A(v, ..., v) \| \leq K \left( \sum_{k=0}^{p-1} \| u \|^{p-1-k} \| v \|^{k} \right) \| u - v \|.
\]

Hence,

\[
\| \mathcal{G}_y(u) - \mathcal{G}_y(v) \| \leq \| y - y_1 \| + K \left( \sum_{k=0}^{p-1} \| u \|^{p-1-k} \| v \|^{k} \right) \| u - v \|. \tag{3.5}
\]

Setting \( y_1 = v = 0 \) in (3.5), we conclude by (3.2) that \( \| \mathcal{G}_y u \| \leq \| y \| + K \| u \|^p \leq R + KM^p < M \). From (3.5), for \( y_1 = y \) we have that \( \| \mathcal{G}_y(u) - \mathcal{G}_y(v) \| \leq pKM^{p-1} \| u - v \| \), where \( KM^{p-1} < 1 \), by (3.2). Therefore, \( \mathcal{G}_y \) is a strict contraction on \( B_M \). Thus, \( \mathcal{G}_y \) has a fixed point. The continuous dependence follows directly from (3.5).
Lemma 3.2. Let \( p > 1, \beta > 0, a, \sigma \in \mathbb{R} \) and \( \alpha = (\beta - \sigma)/(p - 1) + a \). Assume that \( 0 < \alpha + (p - 1)a < \sigma \). Then,

(i) \( \beta + (p - 1)a = \sigma + (p - 1)\alpha \).

(ii) \( \beta + (p - 1)a > \sigma, \beta > p\alpha \) and \( \beta + (p - 1)a > \sigma - \alpha \).

Proof. It follows directly since

\[
\beta + (p - 1)a - \sigma = (p - 1)\alpha > 0
\]

\[
\beta - p\alpha = -(p - 1)a + \sigma - \alpha > 0
\]

\[
\beta + (p - 1)a - \sigma - \alpha = p\alpha > 0.
\]

Proof of the Theorem 1.4. Let \( u_0 \in BE^\alpha \) and

\[
X = L^\infty((0, \infty); BE^\alpha) \cap \left\{ u : (0, \infty) \rightarrow E; \sup_{t > 0} t^{\alpha/\beta} \| u(t) \|_E < \infty \right\}
\]

with the norm \( \| u \|_X = \sup_{t > 0} \| u(t) \|_{BE^\alpha} + \sup_{t > 0} t^{\alpha/\beta} \| u(t) \|_E \). For \( u_i \in E, i = 1, 2, \ldots, p \) set

\[
y = S_\beta(t)u_0 \text{ and } A(u_1, \ldots, u_p)(t) = \int_0^t S_\beta(t - \tau)B(u(\tau), \ldots, u(\tau))d\tau.
\]

(3.6)

Since \( E \in X \), by Proposition 2.9, \( BE^\alpha \) is a Banach space. Therefore, \( X \) is also a Banach space. From Proposition 2.11 and Lemma 3.2

\[
t^\alpha \| A(u_1, \ldots, u_p)(t) \|_E \leq t^\alpha \int_0^t C_1(t - \tau) \left( \prod_{i=1}^p \| u_i(\tau) \|_{BE^\alpha} \right) d\tau
\]

\[
\leq t^\alpha \left( \int_0^t C_1(t - \tau) \left( \prod_{i=1}^p \| u_i(\tau) \|_E \right) d\tau \right) \leq t^{\alpha/(\alpha - \beta)} \left( \prod_{i=1}^p \| u_i \|_X \right)
\]

\[
= K_1 \prod_{i=1}^p \| u_i \|_X,
\]

(3.7)

where

\[
K_1 = \int_0^1 C_1(1 - \tau) \left( \prod_{i=1}^p \| u_i \|_X \right) d\tau.
\]

(3.8)

Similarly, by Proposition 2.11 and the definition of \( \alpha \) we obtain

\[
\| A(u_1, \ldots, u_p) \|_{BE^\alpha} \leq C_2 \int_0^t (t - \tau) \left( \prod_{i=1}^p \| u_i(\tau) \|_{BE^\alpha} \right) d\tau
\]

\[
\leq C_2 t^{\alpha/(\alpha - \beta)} \left( \prod_{i=1}^p \| u_i \|_X \right) \int_0^1 (1 - \tau) \left( \prod_{i=1}^p \| u_i \|_X \right) d\tau
\]

(3.9)

\[
= K_2 \prod_{i=1}^p \| u_i \|_X
\]
where
\[ K_2 = \int_0^1 C_2(1 - \tau)^{-\frac{(p-1)a-\sigma}{\beta}} \tau^{-\frac{p\alpha}{\beta}} d\tau. \] (3.10)
Lemma 3.2 provides that \( K_1, K_2 < \infty \). Hence, taking
\[ K = K_1 + K_2 \] (3.11)
we conclude that \( \|A(u_1, ..., u_p)\|_X \leq K \prod_{i=1}^p \|u_i\|_X \). From Lemma 3.1 the global existence and continuous dependence follows.

To show, the asymptotic behavior we argue as [12]. Arguing as (3.7) and using (3.5) it is possible to conclude
\[
t^\alpha/\beta \|u(t) - v(t)\|_E \leq t^\alpha/\beta \|S_\beta(t)(u_0 - v_0)\|_E + C_1 \int_0^t (t - \tau)^{(p-1)a-\sigma-\sigma/\beta} \left( \sum_{k=0}^{p-1} \|u\|_{BE}^{p-1-k} \|v\|_E^{k} \right) \|u(\tau) - v(\tau)\|_E d\tau
\]
\[
\leq C_1 p M^{p-1} \int_0^1 (1 - \tau)^{(p-1)a-\sigma-\sigma/\beta} \tau^{-\frac{p\alpha}{\beta}} \left[ (\tau t)^{\alpha/\beta} \|u(\tau t) - v(\tau t)\|_E \right] d\tau.
\]
Similarly,
\[
t^\alpha/\beta \|u(t) - v(t) - S_\beta(t)(u_0 - v_0)\|_E \leq C_1 p M^{p-1} \int_0^1 (1 - \tau)^{(p-1)a-\sigma-\sigma/\beta} \tau^{-\frac{p\alpha}{\beta}} \left[ (\tau t)^{\alpha/\beta} \|u(\tau t) - v(\tau t)\|_E \right] d\tau.
\]
From these estimates and Lemma 6.1 of [12] the conclusion follows.

Proof of Theorem 1.3 Let \( u_0 \in BE^\alpha \) and
\[ X_T = L^\infty((0, T); BE^\alpha) \cap \left\{ u : (0, T) \to E; \sup_{0 < t < T} t^\alpha/\beta \|u(t)\|_E < \infty \right\} \]
with the norm \( \|u\| = \sup_{0 < t < T} \|u(t)\|_{BE^\alpha} + \sup_{0 < t < T} t^\alpha/\beta \|u(t)\|_E \). For \( u_i \in E, i = 1, 2, ..., p \) set \( y \) and \( A \) given by (3.6). Arguing as in the derivation of (3.7) we obtain
\[
t^\alpha/\beta \|A(u_1, ..., u_p)\|_E \leq K_1 T^{1+\frac{(p-1)(a-\alpha)-\sigma}{\beta}} \prod_{i=1}^p \|u_i\|_X,
\]
where \( K_1 \) is given by (3.8). Similarly, arguing as in the derivation of (3.9) we conclude
\[
\|A(u_1, ..., u_p)\|_{BE^\alpha} \leq K_2 T^{1+\frac{(p-1)(a-\alpha)-\sigma}{\beta}} \prod_{i=1}^p \|u_i\|_X,
\]
where \( K_2 \) is given by (3.10). Hypotheses guarantee that constants \( K_1, K_2 \) are finite and that \( 1 + [(p-1)(a-\alpha) - \sigma]/\beta > 0 \). Thus, we have \( \|A(u_1, ..., u_p)\|_a \leq K \prod_{i=1}^p \|u_i\|_X \), with \( K = (K_1 + K_2) T^{1+\frac{(p-1)(a-\alpha)-\sigma}{\beta}} \). From Lemma 3.1 we have the desired result.
4. Global existence for system (1.11)

We extend the concept of adequate space for problem (1.1) given in subsection 2.3. Let $E$ and $F$ be Banach spaces and let $B_1$ and $B_2$ be $q$–linear form and $p$–linear form respectively. We say that $E \times F$ is adequate to system (1.10), if

(i) The inclusions $\mathcal{S} \subset E, F \subset \mathcal{S}'$ are continuous.
(ii) The norms $\| \cdot \|_E$ and $\| \cdot \|_F$ are invariants for translations.
(iii) $B_1(v_1, ..., v_q), B_2(u_1, ..., u_p) \in \mathcal{S}'$, for every $u_i \in E$, $i = 1, ..., p$ and $v_j \in F$, $j = 1, ..., q$, and

$$
\|S_\beta(t)B_1(v_1, ..., v_q)\|_E \leq \omega_1(t) \prod_{i=1}^{q} \|v_i\|_F,
$$

$$
\|S_\beta(t)B_2(u_1, ..., u_p)\|_F \leq \omega_2(t) \prod_{i=1}^{p} \|u_i\|_E
$$

where $\omega_1, \omega_2 : (0, +\infty) \rightarrow (0, +\infty)$ so that $\omega_1, \omega_2 \in L^1(0, T)$, for every $0 < T < +\infty$.

If we consider $B_1(v_1, ..., v_q) = u_1u_2 \cdots u_p$ and $B_2(u_1, ..., u_p) = u_1u_2 \cdots u_p$, then the spaces $L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n)$ and $L^{(r,s_1)}(\mathbb{R}^n) \times L^{(s,s_1)}(\mathbb{R}^n)$, with $s = s(q + 1)/(p + 1) > n(pq - 1)/(\beta + 1)$, are adequate to system (1.10). This fact, follows from the following estimates

$$
\|S_\beta(t)B_1(v_1, v_2, ..., v_q)\|_r \leq Ct^{-\frac{\sigma_1(q - 1)}{\beta}} \|v_1v_2 \cdots v_q\|_q
$$

$$
\leq Ct^{-\frac{\sigma_1(q - 1)}{\beta}} \prod_{i=1}^{q} \|v_i\|_s,
$$

$$
\|S_\beta(t)B_1(v_1, v_2, ..., v_q)\|_{(r,s_1)} \leq Ct^{-\frac{\sigma_1(q - 1)}{\beta}} \|v_1v_2 \cdots v_q\|_{(s_1,s_1)}
$$

$$
\leq Ct^{-\frac{\sigma_1(q - 1)}{\beta}} \prod_{i=1}^{q} \|v_i\|_{(s,s_1)}.
$$

Lemma 4.1. Let $B_1$ and $B_2$ be $q$ and $p$–linear forms with scaling degree $\sigma_1$ and $\sigma_2$ respectively. Assume that $E$ and $F$ are Banach spaces with scaling degree $\alpha$ and $\beta$ respectively and that $E \times F$ is adequate to system (1.10). Then,

(i) There exist positive constants $C_1$ and $C_2$ such that

$$
\|S_\beta(t)B_1(v_1, ..., v_q)\|_E \leq C_1t^{(q\beta-a-\sigma_1)/\beta} \prod_{i=1}^{q} \|v_i\|_F
$$

$$
\|S_\beta(t)B_2(u_1, ..., u_p)\|_E \leq C_2t^{(p\beta-b-\sigma_2)/\beta} \prod_{i=1}^{p} \|u_i\|_E
$$

(ii) If $\alpha_1, \alpha_2 > 0$ and

$$
\alpha_1 + q\beta \leq a + \sigma_1, \alpha_2 + p\beta \leq b + \sigma_2.
$$
Then there exist positive constants $C_1$ and $C_2$ such that
\[
\|S_\beta(t)B_1(v_1, \ldots, v_q)\|_{BE^{\alpha_1}} \leq C_1 t^{(\alpha_1-a+q b-\sigma_1)/\beta} \prod_{i=1}^q \|v_i\|_F
\]
\[
\|S_\beta(t)B_2(u_1, \ldots, u_p)\|_{BE^{\alpha_2}} \leq C_2 t^{(\alpha_2-b+p a-\sigma_2)/\beta} \prod_{i=1}^p \|u_i\|_E
\]

Proof. The proof follows the same arguments used in the proof of Proposition 2.11.

We need also of the following technical result.

Lemma 4.2. Let $E$ and $F$ be Banach spaces. Consider the system
\[
\begin{align*}
  x &= x_0 + B_1(y, \ldots, y) \\
  y &= y_0 + B_2(x, \ldots, x)
\end{align*}
\]
where $B_1 : F \times \cdots \times F \to E$ and $B_2 : E \times \cdots \times E \to F$ are a $p$–linear and a $q$–linear forms respectively. Assume that there exist $K_1, K_2 > 0$ such that
\[
\|A_1(y_1, \ldots, y_q)\|_E \leq K_1 \prod_{i=1}^q \|y_i\|_F
\]
\[
\|A_2(x_1, \ldots, x_p)\|_F \leq K_2 \prod_{i=1}^p \|x_i\|_E
\]

Let $M, R > 0$ verifying
\[
R + qM^q K_1 + pM^p K_2 < M.
\]

Then, for every $(x_0, y_0) \in E \times F$ so that $\|(x_0, y_0)\|_{E \times F} = \|x_0\|_E + \|y_0\|_F \leq R$, there exists a unique solution $(x, y) \in E \times F$ for the system \((4.1)\) such that $\|(x, y)\|_{E \times F} \leq M$. Moreover, if $\|(\bar{x}, \bar{y})\|_{E \times F} \leq R$ and $(\bar{x}, \bar{y})$ is the corresponding solution of \((4.1)\) with $\|(\bar{x}, \bar{y})\|_{E \times F} \leq M$, then
\[
\|(x, y) - (\bar{x}, \bar{y})\|_{E \times F} \leq [1 - (qM^{q-1} K_1 + pM^{p-1} K_2)]^{-1} \|(x_0, y_0) - (\bar{x}_0, \bar{y}_0)\|_{E \times F}.
\]

Proof. Set $B_M = \{(x, y) \in E \times F; \|(x, y)\|_{E \times F} = \|x\|_E + \|y\|_F \leq M\}$. Define $G_{(x_0, y_0)} : B_M \to E \times F$ by $G_{(x_0, y_0)}(x, y) = (F_{x_0}(x, y), G_{y_0}(x, y))$, with
\[
\begin{align*}
  F_{x_0}(x, y) &= x_0 + A_1(y, \ldots, y), \\
  G_{y_0}(x, y) &= y_0 + A_2(x, \ldots, x).
\end{align*}
\]

Since
\[
\|A_1(y, \ldots, y) - A_1(\bar{y}, \ldots, \bar{y})\|_E \leq K_1 \left( \sum_{k=0}^{q-1} \|y\|_F^{q-1-k} \|\bar{y}\|_F^k \right) \|y - \bar{y}\|_F,
\]
\[
\|A_2(x, \ldots, x) - A_2(\bar{x}, \ldots, \bar{x})\|_F \leq K_2 \left( \sum_{k=0}^{p-1} \|x\|_E^{p-1-k} \|\bar{x}\|_E^k \right) \|x - \bar{x}\|_E,
\]
we obtain
\[
\|F_{x_0}(x, y) - F_{\bar{x}_0}(\bar{x}, \bar{y})\|_E \leq \|x_0 - \bar{x}_0\|_E + qM^{q-1} K_1 \|y - \bar{y}\|_F,
\]
\[
\|G_{y_0}(x, y) - G_{\bar{y}_0}(\bar{x}, \bar{y})\|_F \leq \|y_0 - \bar{y}_0\|_F + pM^{p-1} K_2 \|x - \bar{x}\|_E.
\]
Consider \(X, Y\), \(x \in X\) and \(y \in Y\). The similar argument can be used to show the other inequalities.

**Proof.**

Follows directly since

\[
\beta > q\alpha
\]

Thus, for \((x_0, y_0) = (\bar{x}, \bar{y})\) we have

\[
\|G(x_0, y_0)(x, y) - G(x_0, y_0)(\bar{x}, \bar{y})\|_{E \times F} \leq \|G(x_0, y_0)(x, y) - (\bar{x}, \bar{y})\|_{E \times F}.
\]

Therefore, from inequality (4.4), \(G(x_0, y_0)\) is a strict contraction.

On the other hand, for \((\bar{x}, \bar{y}) = (0, 0)\)

\[
\|G(x_0, y_0)(x, y)\|_{E \times F} \leq \|G(x_0, y_0)(x, y)\|_{E \times F} + \{qM^q - 1K_1 + pM^p - 1K_2\} \|((x, y) - (\bar{x}, \bar{y}))\|_{E \times F} \leq R + qM^qK_1 + pM^pK_2 \leq M.
\]

So, \(G(x_0, y_0)(B_M) \subseteq B_M\) and the existence follows by fixed point theorem.

Continuous dependence follows from (4.4).

**Lemma 4.3.** Let \(p, q \geq 1\) such that \(pq > 1\), \(\beta > 0, a, b, \sigma_1, \sigma_2 \in \mathbb{R}\). Set

\[
\alpha_1 = \frac{\beta(q + 1)}{pq - 1} + a - \frac{\sigma_1 + q\sigma_2}{pq - 1}, \quad \alpha_2 = \frac{\beta(p + 1)}{pq - 1} + b - \frac{\sigma_2 + p\sigma_1}{pq - 1}.
\]

Suppose that

(i) \(\alpha_1, \alpha_2 > 0\).

(ii) \(\alpha_1 + qb < a + \sigma_1, \alpha_2 + pa < b + \sigma_2\).

(iii) \(\alpha_1 < q\alpha_2, \alpha_2 < p\alpha_1\).

Then,

(i) \(\beta - a + \alpha_1 - \alpha_2q = -qb + \sigma_1, \beta - b - \alpha_1p + \alpha_2 = -pa + \sigma_2\).

(ii) \(\beta - a > -qb + \sigma_1, \beta - b > -pa + \sigma_2\).

(iii) \(\beta > q\alpha_2, \beta > p\alpha_1\).

(iv) \(\beta + \alpha_1 - a > -qb + \sigma_1, \beta + \alpha_2 - b > -pa + \sigma_2\).

**Proof.**

Follows directly since

\[
\beta - a + qb - \sigma_1 = q\alpha_2 - \alpha_1
\]

\[
\beta - q\alpha_2 = -\alpha_1 + a - qb + \sigma_1
\]

\[
\beta + \alpha_1 - a + qb - \sigma_1 = q\alpha_2.
\]

Similar argument can be used to show the other inequalities.

**Proof of Theorem 1.6** Let \(\alpha_1, \alpha_2\) defined by (1.12), and let \((u_0, v_0) \in BE^{\alpha_1} \times BF^{\alpha_2}\). Set \(x_0 = S_\beta(t)u_0, y_0 = S_\beta(t)v_0\) and

\[
A_1(v_1, \ldots, v_q)(t) = \int_0^t S_\beta(t - \tau)B_1(v_1(\tau), \ldots, v_q(\tau))d\tau,
\]

\[
A_2(u_1, \ldots, u_p)(t) = \int_0^t S_\beta(t - \tau)B_2(u_1(\tau), \ldots, u_p(\tau))d\tau.
\]

Consider \(X, Y\) given by

\[
X = \left\{ u : (0, +\infty) \to E; \|u\|_X = \sup_{t > 0} t^{\alpha_1/\beta} \|u(t)\|_E + \sup_{t > 0} \|u(t)\|_{BE^{\alpha_1}} \leq +\infty \right\},
\]

\[
Y = \left\{ v : (0, +\infty) \to F; \|v\|_2 = \sup_{t > 0} t^{\alpha_2/\beta} \|v(t)\|_F + \sup_{t > 0} \|v(t)\|_{BF^{\alpha_2}} \leq +\infty \right\}.
\]
Since $E, F \in X$, from Proposition \ref{prop:Banach_spaces}, we conclude that $BE^{\alpha_1}$ and $BF^{\alpha_2}$ are Banach spaces. Therefore, $(X, \| \cdot \|_1)$ and $(Y, \| \cdot \|_2)$ are Banach spaces. Thus, $X \times Y$ is also a Banach space with the norm

$$
\|(u, v)\|_{X \times Y} := \|u\|_X + \|v\|_Y.
$$

Since $\alpha_1 + qa < a + \sigma_1$ and $\alpha_2 + pa \leq b + \sigma_2$, Lemma \ref{lemma:4.1} can be used. Note that

$$
\beta + \alpha_1 + qa - a - \sigma_1 - \alpha_2 q = 0, \quad \beta + \alpha_2 + pa - b - \sigma_2 - p \alpha_1 = 0.
$$

By Lemma \ref{lemma:4.1}(i) we have

$$
t^{\alpha_1/\beta} \|A_1(v_1, ..., v_q)(t)\|_E \leq K_1 \left( \prod_{i=1}^{q} \|v_i\|_2 \right),
$$

(4.5)

$$
t^{\alpha_2/\beta} \|A_2(u_1, ..., u_p)(t)\|_F \leq K_2 \left( \prod_{i=1}^{p} \|u_i\|_1 \right),
$$

(4.6)

where

$$
K_1 = C_1 \int_0^1 (1-s)^{\frac{q a - a - \sigma_1}{\beta}} s^{-\frac{\alpha_2}{\beta}} ds < +\infty,
$$

$$
K_2 = C_2 \int_0^1 (1-s)^{\frac{p a - b - \sigma_2}{\beta}} s^{-\frac{p \alpha_1}{\beta}} ds < +\infty.
$$

By Lemma \ref{lemma:4.1}(ii) we have

$$
\|A_1(v_1, ..., v_q)(t)\|_{BE^{\alpha_1}} \leq \bar{K}_1 \left( \prod_{i=1}^{q} \|v_i\|_2 \right),
$$

(4.7)

$$
\|A_2(u_1, ..., u_p)(t)\|_{BF^{\alpha_2}} \leq \bar{K}_2 \left( \prod_{i=1}^{p} \|u_i\|_1 \right),
$$

(4.8)

where

$$
\bar{K}_1 = C_1 \int_0^1 (1-s)^{\frac{\alpha_1 + qa - a - \sigma_1}{\beta}} s^{-\frac{\alpha_2}{\beta}} ds,
$$

$$
\bar{K}_2 = C_2 \int_0^1 (1-s)^{\frac{\alpha_2 + pa - b - \sigma_2}{\beta}} s^{-\frac{p \alpha_1}{\beta}} ds,
$$

Therefore,

$$
\|A_1(v_1, ..., v_q)\|_1 \leq K'_1 \left( \prod_{i=1}^{p} \|v_i\|_2 \right),
$$

$$
\|A_2(u_1, ..., u_p)\|_2 \leq K'_2 \left( \prod_{i=1}^{p} \|u_i\|_1 \right),
$$

where $K'_i = K_i + \bar{K}_i$, $i = 1, 2$. A finiteness of $K_1, K_2, \bar{K}_1, \bar{K}_2$ are consequences of Lemma \ref{lemma:4.3}. Moreover, the right side of estimates (4.5)-(4.8) do not depend of $t$. Thus, the result follows from Lemma \ref{lemma:4.2}. 
REFERENCES

[1] J. Aguirre, M. Escobedo and E. Zuazua, Self-similar solutions of a convection diffusion equations and related elliptic problems, Comm. Partial Differential Equations 15, 139-157 (1990).

[2] M. Ben-Artzi, P. Souplet and F. B. Weissler. The local theory for viscous Hamilton Jacobi equations in Lebesgue space, J. Math. Pures Appl. (9) 81, 343-378, (2002).

[3] J. Bergh, J. Lköstom, Interpolation Spaces, an introduction, Springer-Verlag, New York, 1976.

[4] H. Brezis and T. Cazenave, A nonlinear heat equation with singular initial data, J. Anal. Math. 68 (1996), 277-304.

[5] M. Cannone and G. Karch, About the regularized Navier-Stokes equations, J. Math. Fluid Mech. 7 (2005) 1-28.

[6] T. Cazenave and F. B. Weissler, Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations, Math. Z. 228, 83-120, (1998).

[7] A. S. Chandrasekhar, Stochastic problems in physics and astronomy, Rev. Mod. Phys. vol.15 (1943), 1-89.

[8] M. Escobedo, M.A. Herrero, Boundedness and blow up for a semilinear reaction diffusion system, J. Differential Equations, 89 (1991), 176-202.

[9] M. Escobedo and E. Zuazua, Large time behavior for convection-diffusion equations in $\mathbb{R}^N$, J. Differential Equations 100, 119-161 (1991).

[10] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. A Math. 16 (1966) 105-113.

[11] L. Grafakos, Modern Fourier Analysis, Ed. 2a, Springer-Verlag, New York, 2009.

[12] G. Karch, Scaling in nonlinear parabolic equations, J. Math. Anal. Appl. 234, 534-558 (1999).

[13] G. Karch, Scalling in nonlinear parabolic equations: locality versus globality

[14] A. A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Diferential Equations, Vol. 204 (North-Holland Mathematics Studies), 2006.

[15] C. Miao, B. Yuan, Solutions to some nonlinear parabolic equations in pseudomeasure spaces, Math. Nachr. 280 (2007) 171-186.

[16] C. Miao, B. Yuan, B. Zhang, Well-posedness of the Cauchy problem for the fractional power dissipative equations, Nonlinear Anal. 68, 461-484 (2008)

[17] C. A. Roberts, W. E. Olmstead, Blow-up in a subdiffusive medium of infinite extent. Fract. Calc. Appl. Anal. 12 (2009), no. 2, 179-194.

[18] S. Snoussi, S. Tayachi, Fred B. Weissler, Asymptotically self-similar global solutions of a semilinear parabolic equation with a nonlinear gradient term, Proc. Royal Soc. Edinburgh Sect. A., 129 (1999), 419-440.

[19] S. Snoussi, S. Tayachi, Global existence, asymptotic behavior and self-similar solutions for a class of semilinear parabolic systems, Nonlinear Analysis, 48 (2002), 13-35.

[20] S. Snoussi, S. Tayachi, Fred B. Weissler, Asymptotically self-similar global solutions of a general semilinear heat equation, Math. Ann. 321,(2001) 131-155.

[21] F. B. Weissler, semilinear evolution equations in Banach spaces, J. Funct. Anal. 32 (1979), 277-296.

[22] F. B. Weissler, Local existence and nonexistence for semilinear parabolic equations in $L^p$, Indiana Univ. Math. J. 29 (1980), 79-102.

[23] G. Wu, Jia Yuan, Well-posedness of the Cauchy problem for the fractional power dissipative equation in critical Besov spaces, J. Math. Anal. Appl. 340, 1326-1335 (2008).

Departamento de Matemática, Universidade Federal de Pernambuco - UFPE, 50740-540, Recife, PE, Brazil

E-mail address: miguel@mat.ufpe.br

Departamento de Matemática, Universidade Federal do Rio Grande do Norte - UFRN, 50078-970, Natal, PE, Brazil

E-mail address: paulorfss@ccet.ufrn.br