Rainbow Turán number of even cycles, repeated patterns and blow-ups of cycles

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Abstract

The rainbow Turán number $\text{ex}^*(n, H)$ of a graph $H$ is the maximum possible number of edges in a properly edge-coloured $n$-vertex graph with no rainbow subgraph isomorphic to $H$. We prove that for any integer $k \geq 2$, $\text{ex}^*(n, C_{2k}) = O(n^{1+1/k})$. This is tight and establishes a conjecture of Keevash, Mubayi, Sudakov and Verstraëte. We use the same method to prove several other conjectures in various topics. First, we prove that there exists a constant $c$ such that any properly edge-coloured $n$-vertex graph with more than $cn(\log n)^4$ edges contains a rainbow cycle. It is known that there exist properly edge-coloured $n$-vertex graphs with $\Omega(n \log n)$ edges which do not contain any rainbow cycle. Secondly, we prove that in any proper edge-colouring of $K_n$ with $o(n^{r-1} \cdot (k-1)/k)$ colours, there exist $r$ colour-isomorphic, pairwise vertex-disjoint copies of $C_{2k}$. This proves in a strong form a conjecture of Conlon and Tyomkyn, and a strengthened version proposed by Xu, Zhang, Jing and Ge. Finally, we answer a question of Jiang and Newman by showing that there exists a constant $c = c(r)$ such that any $n$-vertex graph with more than $cn^{2-1/r}(\log n)^{7/r}$ edges contains the $r$-blowup of an even cycle.

1 Introduction

In this paper we develop a method that allows us to find cycles with suitable extra properties in graphs with sufficiently many edges. We give applications in three different areas, which are introduced in the next three subsections.

1.1 Rainbow Turán numbers

For a family of graphs $\mathcal{H}$, the Turán number (or extremal number) $\text{ex}(n, \mathcal{H})$ is the maximum number of edges in an $n$-vertex graph which does not contain any $H \in \mathcal{H}$ as a subgraph. When $\mathcal{H} = \{H\}$, we write $\text{ex}(n, H)$ for the same function. This function is determined asymptotically by the Erdős–Stone–Simonovits [7, 6] theorem when $H$ has chromatic number at least 3. However, for bipartite graphs $H$, even the order of magnitude of $\text{ex}(n, H)$ is unknown in general. For example, a result of Bondy and Simonovits [2] states that $\text{ex}(n, C_{2k}) = O(n^{1+1/k})$, but a matching lower bound is only known when $k \in \{2, 3, 5\}$.

A variant of this function was introduced by Keevash, Mubayi, Sudakov and Verstraëte in [17]. In an edge-coloured graph, we say that a subgraph is rainbow if all its edges are of different colour. The rainbow Turán number of the graph $H$ is then defined to be the maximum number of edges in a properly edge-coloured $n$-vertex graph that does not contain a rainbow $H$ as a subgraph. This number is denoted by $\text{ex}^*(n, H)$. Clearly, $\text{ex}^*(n, H) \geq \text{ex}(n, H)$ for every $n$ and $H$. Keevash, Mubayi, Sudakov and Verstraëte proved, among other things, that for any non-bipartite graph $H$, we have $\text{ex}^*(n, H) = (1 + o(1))\text{ex}(n, H)$. Hence, the most challenging case again seems to be when $H$ is bipartite. Keevash, Mubayi, Sudakov

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and Verstraëte showed that \( \text{ex}^*(n, K_{s,t}) = O(n^{2-1/s}) \), which is tight when \( t > (s-1)! \) [18, 1]. The function has also been studied for trees (see [15, 10, 16]). About even cycles, Keevash, Mubayi, Sudakov and Verstraëte proved the following lower bound.

**Theorem 1.1** (Keevash–Mubayi–Sudakov–Verstraëte [17]). For any integer \( k \geq 2 \),

\[
\text{ex}^*(n, C_{2k}) = \Omega(n^{1+1/k}).
\]

They conjectured that this is tight.

**Conjecture 1.2** (Keevash–Mubayi–Sudakov–Verstraëte [17]). For any integer \( k \geq 2 \),

\[
\text{ex}^*(n, C_{2k}) = \Theta(n^{1+1/k}).
\]

They have verified their conjecture for \( k \in \{2, 3\} \). For general \( k \), Das, Lee and Sudakov proved the following upper bound.

**Theorem 1.3** (Das–Lee–Sudakov [5]). For every fixed integer \( k \geq 2 \),

\[
\text{ex}^*(n, C_{2k}) = O\left(n^{1+\left(1+\varepsilon_k\right)\ln k} / k\right),
\]

where \( \varepsilon_k \to 0 \) as \( k \to \infty \).

In this paper we prove Conjecture 1.2 by establishing the following result.

**Theorem 1.4.** For any integer \( k \geq 2 \), we have

\[
\text{ex}^*(n, C_{2k}) = O(n^{1+1/k}).
\]

The theta graph \( \theta_{k,t} \) is the union of \( t \) paths of length \( k \) which share the same endpoints but are pairwise internally vertex-disjoint. We remark that our proof can be easily modified to show that \( \text{ex}^*(n, \theta_{k,t}) = O(n^{1+1/k}) \) for any fixed \( k \) and \( t \).

Keevash, Mubayi, Sudakov and Verstraëte also asked how many edges a properly edge-coloured \( n \)-vertex graph can have if it does not contain any rainbow cycle. They constructed such graphs with \( \Omega(n \log n) \) edges. Note that this is quite different from the uncoloured case, since any \( n \)-vertex acyclic graph has at most \( n - 1 \) edges. Das, Lee and Sudakov proved that if \( \eta > 0 \) and \( n \) is sufficiently large, then any properly edge-coloured \( n \)-vertex graph with at least \( n \exp \left((\log n)^{1+\eta/2}\right) \) edges contains a rainbow cycle. We prove the following improvement.

**Theorem 1.5.** There exists an absolute constant \( C \) such that if \( n \) is sufficiently large and \( G \) is a properly edge-coloured graph on \( n \) vertices with at least \( C n (\log n)^4 \) edges, then \( G \) contains a rainbow cycle of even length.

### 1.2 Colour-isomorphic even cycles in proper colourings

Conlon and Tyomkyn [4] have initiated the study of the following problem. We say that two subgraphs of an edge-coloured graph are colour-isomorphic if there is an isomorphism between them preserving the colours. For an integer \( r \geq 2 \) and a graph \( H \), they write \( f_r(n, H) \) for the smallest number \( C \) so that there is a proper edge-colouring of \( K_n \) with \( C \) colours containing no \( r \) vertex-disjoint colour-isomorphic copies of \( H \). They proved various general results about this function, such as the following upper bound.

**Theorem 1.6** (Conlon–Tyomkyn [4]). For any graph \( H \) with \( v \) vertices and \( e \) edges,

\[
f_r(n, H) = O\left(\max\left(n, n^{\frac{r-2}{(r-1)p}}\right)\right).
\]
Regarding even cycles, they established the following result.

**Theorem 1.7** (Conlon–Tyomkyn [4]). \( f_2(C_6) = \Omega(n^{4/3}) \).

One of the several open problems they posed is the following question.

**Question 1.8** (Conlon–Tyomkyn [4]). Is it true that for every \( \varepsilon > 0 \), there exists \( k_0 = k_0(\varepsilon) \) such that, for all \( k \geq k_0 \), \( f_2(n, C_{2k}) = \Omega(n^{2-\varepsilon}) \)?

Later, Xu, Zhang, Jing and Ge made a more precise conjecture.

**Conjecture 1.9** (Xu–Zhang–Jing–Ge [19]). For any \( k \geq 3 \), \( f_2(n, C_{2k}) = \Omega(n^{2-\frac{2}{k}}) \).

We prove this conjecture in a more general form.

**Theorem 1.10.** Let \( k, r \geq 2 \) be fixed integers. Then \( f_r(n, C_{2k}) = \Omega(n^{r-1} \log(n) \frac{4k}{k+1}) \).

### 1.3 Turán number of blow-ups of cycles

For a graph \( F \), the \( r \)-blowup of \( F \) is the graph obtained by replacing each vertex of \( F \) with an independent set of size \( r \) and each edge of \( F \) by a \( K_{r,r} \). We write \( F[r] \) for this graph.

The systematic study of the Turán number of blow-ups was initiated by Grzesik, Janzer and Nagy [12]. They proved that for any tree \( T \) we have ex\( (n, T[r]) = O(n^{2-1/r}) \). They have also made the following general conjecture.

**Conjecture 1.11** (Grzesik–Janzer–Nagy [12]). Let \( r \) be a positive integer and let \( F \) be a graph such that ex\( (n, F) = O(n^{2-\alpha}) \) for some \( 0 \leq \alpha \leq 1 \) constant. Then

\[
ex(n, F[r]) = O(n^{2-\frac{\alpha}{r}}).
\]

Their result mentioned above proves this conjecture when \( F \) is a tree. It is easy to see that the conjecture holds also when \( F = K_{s,t} \) and \( \alpha = 1/s \).

In the case of forbidding all \( r \)-blowups of cycles, an earlier question was formulated by Jiang and Newman [13]. To state this question, we write \( C[r] = \{C_{2k}[r] : k \geq 2\} \).

**Question 1.12** (Jiang–Newman [13]). Is it true that for any positive integer \( r \) and any \( \varepsilon > 0 \), ex\( (n, C[r]) = O(n^{2-\frac{1}{r} + \varepsilon}) \)?

We answer this question affirmatively in a stronger form.

**Theorem 1.13.** For any positive integer \( r \),

\[
ex(n, C[r]) = O(n^{2-1/r} (\log n)^{7/r}).
\]

Erdős–Rényi random graphs show that ex\( (n, C[r]) = \Omega(n^{2-1/r}) \). It would be interesting to decide whether the logarithmic factor in Theorem 1.13 can be removed.

Finally, we establish an upper bound for the Turán number when only one blowup cycle is forbidden.

**Theorem 1.14.** For any integers \( r \geq 1 \) and \( k \geq 2 \), we have

\[
ex(n, C_{2k}[r]) = O\left(n^{2-\frac{1}{r} + 1\frac{1}{r} + \frac{1}{r+1} - 1} (\log n)^{\frac{1}{r+1} - \frac{1}{2}}\right).
\]

This is still quite a long way from the conjectured \( \text{ex}(n, C_{2k}[r]) = O(n^{2-\frac{1}{r} + \frac{1}{r}}) \).

The rest of this paper is organised as follows. In Section 2, we prove Theorem 1.4. In Section 3, we prove Theorem 1.5. In Section 4, we prove Theorem 1.10. The proofs of
Theorem 1.13 and Theorem 1.14 are given in Section 5. We give some concluding remarks and open problems in Section 6.

While we see no implication relations between our results, the proofs in the three topics (rainbow Turán numbers, colour-isomorphic cycles and blow-ups of cycles) are very similar. In order to avoid repeating the same argument many times, we give the full proofs in the case of rainbow Turán problems, but we often only sketch the proofs in the sections on colour-isomorphic cycles and blow-ups of cycles. Nevertheless, we always indicate the necessary twists and in one case we give a proof in the appendix.

2 Rainbow cycles of length $2k$

Notation. In what follows, for graphs $H$ and $G$ we write $\text{hom}(H, G)$ for the number of graph homomorphisms $V(H) \to V(G)$. $P_k$ will denote the path with $k$ edges and we use the convention $C_2 = P_1$. For vertices $x, y \in V(G)$, $\text{hom}_{x,y}(P_\ell, G)$ denotes the number of walks of length $\ell$ in $G$ between $x$ and $y$. Moreover, $\text{hom}_x(P_\ell, G)$ denotes the number of walks of length $\ell$ in $G$ starting at $x$. We will write $d_G(x)$ (or $d(x)$ if $G$ is clear) for the degree of the vertex $x$ in $G$ and we write $N_G(x)$ or $N(x)$ for the neighbourhood of $x$. Finally, we write $\delta(G)$ and $\Delta(G)$ for the minimum and maximum degree of $G$, respectively. Logarithms are base 2 unless stated otherwise.

2.1 Finding suitable short cycles

Our goal in this section is to develop a method for finding ‘suitable’ cycles of given length. This is done in two steps. In this subsection we prove that there exist ‘suitable’ cycles of length at most $2k$, while in the next subsection we push the method further to make sure that we get cycles of length exactly $2k$. We have been deliberately vague about what we mean by a ‘suitable’ cycle. In this section it will mean rainbow cycle, but in later sections it will have different meanings. For example, in both Section 4 and Section 5 we will work in auxiliary graphs whose vertices are sets, and a ‘suitable’ cycle in these auxiliary graphs will be one whose vertices are disjoint sets.

Our first key lemma is an upper bound on the number of those (homomorphic) $2\ell$-cycles which are not suitable. With a slight abuse of terminology, we call a homomorphism $H \to G$ a homomorphic copy of $H$ in $G$. That is, a homomorphic copy of $C_{2\ell}$ is a tuple $(x_1, \ldots, x_{2\ell}) \in V(G)^{2\ell}$ such that $x_1x_2, x_2x_3, \ldots, x_{2\ell}x_1 \in E(G)$. A rainbow homomorphic copy of $H$ is one in which the images of distinct edges of $H$ have different colour.

Lemma 2.1. Let $\ell \geq 2$ be a positive integer and let $G$ be a properly edge-coloured graph. Then the number of homomorphic copies of $C_{2\ell}$ which are not rainbow is at most

$$16\ell (\ell \Delta(G) \text{hom}(C_{2\ell-2}, G) \text{hom}(C_{2\ell}, G))^{1/2}.$$  

Proof. Let $c(e)$ be the colour of the edge $e \in E(G)$. We want to prove that the number of $(x_1, x_2, \ldots, x_{2\ell}) \in V(G)^{2\ell}$ with $x_1x_2, \ldots, x_{2\ell}x_1 \in E(G)$ such that $c(x_1x_2), \ldots, c(x_{2\ell}x_1)$ are not all distinct is at most $16\ell (\ell \Delta(G) \text{hom}(C_{2\ell-2}, G) \text{hom}(C_{2\ell}, G))^{1/2}$. By symmetry, it suffices to prove that the number of $(x_1, x_2, \ldots, x_{2\ell}) \in V(G)^{2\ell}$ with $x_1x_2, \ldots, x_{2\ell}x_1 \in E(G)$ such that $c(x_1x_2) = c(x_i x_{i+1})$ for some $2 \leq i \leq \ell+1$ is at most $8 (\ell \Delta(G) \text{hom}(C_{2\ell-2}, G) \text{hom}(C_{2\ell}, G))^{1/2}$.

For a positive integer $s$, let $\alpha_s$ be the number of walks of length $\ell-1$ in $G$ whose endpoints $y$ and $z$ have $2^{s-1} \leq \text{hom}_{y,z}(P_{\ell-1}, G) < 2^s$ and let $\beta_s$ be the number of walks of length $\ell$ in $G$ whose endpoints $y$ and $z$ have $2^{s-1} \leq \text{hom}_{y,z}(P_{\ell}, G) < 2^s$. Clearly,

$$\sum_{s \geq 1} \alpha_s 2^{s-1} \leq \text{hom}(C_{2\ell-2}, G)$$  

(1)
and
\[ \sum_{s \geq 1} \beta_s 2^{s-1} \leq \text{hom}(C_{2\ell}, G). \] (2)

For positive integers \( s \) and \( t \), write \( \gamma_{s,t} \) for the number of homomorphic copies \( x_1 x_2 \ldots x_{2\ell} x_1 \) of \( C_{2\ell} \) such that \( c(x_1 x_2) = c(x_1 x_{i+1}) \) for some \( 2 \leq i \leq \ell + 1 \), \( 2^{s-1} \leq \text{hom}_{x_1, x_{i+1}}(P_{\ell-1}, G) < 2^s \) and \( 2^{t-1} \leq \text{hom}_{x_2, x_{i+2}}(P_{\ell}, G) < 2^t \). Observe that \( \gamma_{s,t} \leq \alpha_s \Delta(G) \cdot 2^t \). Indeed, if \( x_1 x_2 \ldots x_{2\ell} x_1 \) is a homomorphic \( C_{2\ell} \) with \( 2^{s-1} \leq \text{hom}_{x_1, x_{i+1}}(P_{\ell-1}, G) < 2^s \) and \( 2^{t-1} \leq \text{hom}_{x_2, x_{i+2}}(P_{\ell}, G) < 2^t \), then there are at most \( \alpha_s \) ways to choose \( (x_{i+2}, x_{i+3}, \ldots, x_{2\ell}, x_1) \), given such a choice there are at most \( \Delta(G) \) choices for \( x_2 \), and given these there are at most \( 2^t \) choices for \( (x_3, \ldots, x_{\ell+1}) \). On the other hand, \( \gamma_{s,t} \leq \beta_t \cdot \ell \cdot 2^s \). Indeed, there are at most \( \beta_t \) ways to choose \( (x_2, \ldots, x_{\ell+2}) \). Given such a choice, there are at most \( \ell \) possibilities for \( x_1 \), since \( c(x_1 x_2) = c(x_1 x_{i+1}) \) for some \( 2 \leq i \leq \ell + 1 \), the edges \( x_2 x_3, \ldots, x_{\ell+1} x_{\ell+2} \) are already fixed and \( c \) is a proper colouring. Finally, there are at most \( 2^s \) ways to complete this to a suitable homomorphic copy of \( C_{2\ell} \).

Clearly, the total number of homomorphic copies \( x_1 x_2 \ldots x_{2\ell} x_1 \) of \( C_{2\ell} \) with \( c(x_1 x_2) = c(x_1 x_{i+1}) \) for some \( 2 \leq i \leq \ell + 1 \) is \( \sum_{s,t \geq 1} \gamma_{s,t} \). We give an upper bound for this sum as follows. Let \( q \) be the integer for which \( \left( \frac{\ell}{\text{hom}(C_{2\ell}, G)} \right)^{1/2} \leq 2^q < 2 \left( \frac{\ell \text{hom}(C_{2\ell}, G)}{\text{hom}(C_{2\ell-2}, G)} \right)^{1/2} \).

Now, using \( \gamma_{s,t} \leq \beta_t \cdot \ell \cdot 2^s \) and equation (2),
\[
\sum_{s,t : s \leq t - q} \gamma_{s,t} \leq \ell \sum_{s,t : s \leq t - q} 2^s \beta_t \leq \ell \cdot \sum_{t \geq 1} 2^{t-1} \beta_t \leq \ell \cdot 2^{q+1} \text{hom}(C_{2\ell}, G)
\]
\[ \leq 4(\ell \Delta(G) \text{hom}(C_{2\ell-2}, G) \text{hom}(C_{2\ell}, G))^{1/2}. \]

Also, using \( \gamma_{s,t} \leq \alpha_s \cdot \Delta(G) \cdot 2^t \) and equation (1),
\[
\sum_{s,t : s > t - q} \gamma_{s,t} \Delta(G) \sum_{s,t : s > t - q} 2^t \alpha_s \leq \Delta(G) \sum_{s \geq 1} 2^{t+q} \alpha_s \leq \Delta(G) 2^{t+1} \text{hom}(C_{2\ell-2}, G)
\]
\[ \leq 4(\ell \Delta(G) \text{hom}(C_{2\ell-2}, G) \text{hom}(C_{2\ell}, G))^{1/2}. \]

Thus,
\[
\sum_{s,t \geq 1} \gamma_{s,t} \leq 8(\ell \Delta(G) \text{hom}(C_{2\ell-2}, G) \text{hom}(C_{2\ell}, G))^{1/2}.
\]
This completes the proof. \( \square \)

**Corollary 2.2.** Let \( k \geq 2 \) be an integer and let \( G \) be a properly edge-coloured non-empty graph on \( n \) vertices with \( \text{hom}(C_{2k}, G) \geq 2^{8k-3} k^3 n \Delta(G)^k \). Then \( G \) contains a rainbow cycle of length at most \( 2k \).

**Proof.** Let \( \ell \) be the smallest positive integer satisfying
\[
\text{hom}(C_{2\ell}, G) \geq 2^{8k-3} k^3 n \Delta(G)^\ell.
\]
This is well-defined and \( \ell \leq k \) by the condition of the lemma. Since \( \text{hom}(C_2, G) = 2e(G) \leq n \Delta(G) \), we have \( \ell \geq 2 \).

Note that
\[
\text{hom}(C_{2\ell-2}, G) < 2^{8(\ell-1) k^3 (\ell-1)} n \Delta(G)^{\ell-1} \leq \frac{\text{hom}(C_{2\ell}, G)}{2^{8k^3} \Delta(G)} \leq \frac{\text{hom}(C_{2\ell}, G)}{2^{8k^3} \Delta(G)},
\]
so by Lemma 2.1, the number of homomorphic copies of \( C_{2\ell} \) which are not rainbow is less than \( \text{hom}(C_{2\ell}, G) \).

Hence, there is at least one homomorphic copy of \( C_{2\ell} \) in \( G \) which is rainbow. This implies the existence of a rainbow cycle. Indeed, the homomorphic \( C_{2\ell} \) uses every edge of \( G \) at most once (since it is rainbow), so it is a circuit. Thus, it has a subgraph which is a cycle. Clearly, this is a rainbow cycle. \( \square \)
The next lemma is another instance of an upper bound for the number of certain kind of non-suitable homomorphic copies of $C_{2\ell}$, namely non-injective ones. In what follows, an injectively homomorphic copy of $C_{2\ell}$ is a homomorphic copy $(x_1, x_2, \ldots, x_{2\ell})$ of $C_{2\ell}$ where the vertices $x_1, \ldots, x_{2\ell}$ are distinct. That is, it is a labelled genuine $C_{2\ell}$.

**Lemma 2.3.** Let $\ell \geq 2$ be a positive integer and let $G$ be a graph. Then the number of homomorphic, but not injective copies of $C_{2\ell}$ in $G$ is at most

$$16\ell (\ell \Delta(G) \hom(C_{2\ell-2}, G) \hom(C_{2\ell}, G))^{1/2}.$$ \[Proof.\] The proof is almost identical to the proof of Lemma 2.1. The only difference is that instead of bounding those homomorphic copies $(x_1, x_2, \ldots, x_{2\ell})$ with $c(x_1x_2) = c(x_ix_{i+1})$ for some $2 \leq i \leq \ell + 1$, we bound those with $x_1 = x_i$ for some $2 \leq i \leq \ell + 1$. All details go through exactly the same way. \[\square\]

### 2.2 Finding a cycle of given length

In this subsection we develop the necessary tools to find a suitable cycle of length exactly $2k$ (rather than length at most $2k$ as in Corollary 2.2).

We will need the following lemma.

**Lemma 2.4.** Let $H$ be a bipartite graph and suppose that it does not contain a non-empty subgraph with minimum degree at least $d$. Then the largest eigenvalue of $H$ is at most $2d \sqrt{\Delta(H)}$.

We defer its simple proof until the next subsection and proceed with the main part of the argument. The next lemma is an easy corollary of Lemma 2.4. It will be used to compare $\hom_x(C_{2\ell-2}, G)$ with $\hom_x(C_{2\ell}, G)$, where $\hom_x(C_{2j}, G)$ denotes the number of homomorphic copies $(x_1, x_2, \ldots, x_{2j})$ of $C_{2j}$ with $x_1 = x$.

**Lemma 2.5.** Let $H$ be a bipartite graph with parts $Y$ and $Z$. Let $f : Y \to \mathbb{R}$ be a function and let $g(z) = \sum_{y \in N_H(z)} f(y)$ for every $z \in Z$. Suppose that $H$ does not contain a non-empty subgraph with minimum degree at least $d$. Then

$$\sum_{y \in Y} f(y)^2 \geq \frac{1}{4d \Delta(H)} \sum_{z \in Z} g(z)^2.$$ \[The next lemma is one of our key results.\]

**Lemma 2.6.** Let $k$ be a fixed positive integer and let $G$ be a properly edge-coloured non-empty graph on $n$ vertices. Suppose that for some $2 \leq \ell \leq k$ we have

$$\hom(C_{2\ell}, G) \geq c_k \Delta(G) \hom(C_{2\ell-2}, G),$$

where $c_k = 2^{18} k^7$. Then $G$ contains a rainbow $C_{2k}$.

**Proof.** Call a pair $(x_1, x_{\ell+1})$ of vertices nice if the number of rainbow injectively homomorphic copies of $C_{2\ell}$ of the form $x_1x_2 \ldots x_{2\ell}x_1$ is greater than $(1 - \frac{1}{(2\ell)^2}) \left( \hom_{x_1,x_{\ell+1}}(P, G) \right)^2$. Observe that the total number of homomorphic copies of $C_{2\ell}$ of the form $x_1x_2 \ldots x_{2\ell}x_1$ is $\hom_{x_1,x_{\ell+1}}(P, G)^2$, so this means that the proportion of those which are not injective or not rainbow is less than $\frac{1}{(2\ell)^2}$. Hence, if we choose two walks of length $\ell$ between $x_1$ and $x_{\ell+1}$ randomly with replacement, then the probability that their concatenation is a non-injective or non-rainbow homomorphic copy of $C_{2\ell}$ is less than $\frac{1}{(2\ell)^2}$. In particular, if we choose $4k$ random walks of length $\ell$ between $x_1$ and $x_{\ell+1}$ with replacement, then with positive probability any two of these walks form a rainbow, injectively homomorphic copy of $C_{2\ell}$. Hence,
there exist at least $4k$ pairwise internally vertex-disjoint paths between $x_1$ and $x_{\ell+1}$ such that no colour appears more than once on these paths.

By Lemmas 2.1 and 2.3, the number of non-rainbow or non-injective homomorphic copies of $C_{2\ell}$ in $G$ is at most

$$32\ell^{3/2} (\Delta(G) \hom(C_{2\ell-2}, G) \hom(C_{2\ell}, G))^{1/2} \leq \frac{32\ell^{3/2}}{c_k^{1/2}} \hom(C_{2\ell}, G).$$

Hence,

$$\sum_{(x_1, x_{\ell+1}) \text{ not nice}} \frac{1}{4k^2} \hom_{x_1, x_{\ell+1}}(P_{\ell}, G)^2 \leq \frac{32\ell^{3/2}}{c_k^{1/2}} \hom(C_{2\ell}, G),$$

so, using $\sum_{x_1, x_{\ell+1} \in V(G)} \hom_{x_1, x_{\ell+1}}(P_{\ell}, G)^2 = \hom(C_{2\ell}, G)$, we have

$$\sum_{(x_1, x_{\ell+1}) \text{ nice}} \hom_{x_1, x_{\ell+1}}(P_{\ell}, G)^2 \geq \left(1 - \frac{4k}{2} \frac{32\ell^{3/2}}{c_k^{1/2}} \right) \hom(C_{2\ell}, G) > \frac{1}{2} \hom(C_{2\ell}, G) \geq \frac{c_k}{2} \Delta(G) \hom(C_{2\ell-2}, G).$$

Thus, there exists some $x \in V(G)$ such that

$$\sum_{z \in V(G); (x, z) \text{ is nice}} \hom_{x, z}(P_{\ell}, G)^2 > \frac{c_k}{2} \Delta(G) \hom_x(C_{2\ell-2}, G), \quad (3)$$

where $\hom_x(C_{2\ell-2}, G)$ denotes the number of homomorphic copies $(x_1, \ldots, x_{2\ell-2})$ of $C_{2\ell-2}$ with $x_1 = x$. Let $Z = \{z \in V(G) : (x, z) \text{ is nice}\}$ and let $Y = V(G)$. Consider the bipartite graph $H$ with parts $Y$ and $Z$, defined by $G$. (We view $Y$ and $Z$ as disjoint sets even though they overlap as subsets of $V(G)$.)

Suppose that $H$ does not contain a subgraph with minimum degree at least $4k$. Let $f(y) = \hom_{x,y}(P_{\ell-1}, G)$ for every $y \in Y = V(G)$ and define $g$ as in Lemma 2.5. By that lemma with $d = 4k$,

$$\sum_{y \in Y} f(y)^2 \geq \frac{1}{16k\Delta(H)} \sum_{z \in Z} g(z)^2 \geq \frac{1}{16k\Delta(G)} \sum_{z \in Z} g(z)^2.$$  

However, $g(z) = \sum_{y \in N_G(z)} \hom_{x,y}(P_{\ell-1}, G) = \hom_{x,z}(P_{\ell}, G)$, so, using equation (3),

$$\sum_{y \in Y} f(y)^2 \geq \frac{1}{16k\Delta(G)} \sum_{z \in Z} \hom_{x,z}(P_{\ell}, G)^2 > \frac{c_k}{32k} \hom_x(C_{2\ell-2}, G).$$

However, $\sum_{y \in Y} f(y)^2 = \hom_x(C_{2\ell-2}, G)$, which is a contradiction.

Thus, $H$ contains a subgraph with minimum degree at least $4k$. Then we can greedily find a rainbow path of length $2k - 2\ell$ in $G$ which avoids $x$ and which have both endpoints in $Z$. Let this path be $Q$ with endpoints $z_1$ and $z_2$. Since $(x, z_1)$ is a nice pair, there exist at least $4k$ pairwise internally vertex-disjoint paths of length $\ell$ between $x$ and $z_1$ such that any colour appears at most once on these paths. Thus, by avoiding the vertices and colours on $Q$, we can choose a path $Q_1$ of length $\ell$ between $x$ and $z_1$ in a way that the concatenation of $Q_1$ and $Q$ is a rainbow path of length $2k - \ell$. Moreover, since $(x, z_2)$ is a nice pair, we can extend this path to a rainbow cycle of length $2k$.

**Corollary 2.7.** Let $k$ be a fixed positive integer and let $G$ be a properly edge-coloured non-empty graph on $n$ vertices. Suppose that for some $2 \leq j \leq k$ we have

$$\hom(C_{2j}, G) = \omega \left(n \Delta(G)^j\right).$$

Then, for $n$ sufficiently large, $G$ contains a rainbow $C_{2k}$.
Proof. Choose $L = \omega(1)$ such that \( \text{hom}(C_2, G) \geq L^n \Delta(G)^j \). Let $\ell$ be the smallest positive integer satisfying \( \text{hom}(C_{2\ell}, G) \geq L^n \Delta(G)^j \). Clearly $\ell \leq j \leq k$, and since $\text{hom}(C_2, G) \leq n \Delta(G)$, we have $\ell \geq 2$. Now $\text{hom}(C_{2\ell}, G) \geq L \Delta(G) \text{hom}(C_{2\ell-2}, G)$, so by Lemma 2.6, $G$ contains a rainbow $C_{2k}$. \qed

Corollary 2.7 shows in particular that if we have many homomorphic cycles of length $k$ and the maximum degree is not too large, then there exists a rainbow $C_{2k}$. Since large average degree implies the existence of many homomorphic cycles, it is useful for us to pass to a subgraph which is nearly regular. We say a graph $G$ is $K$-almost regular if $\Delta(G) \leq K \delta(G)$. We will use the following lemma of Jiang and Seiver, which is a slight modification of a much earlier result by Erdős and Simonovits [8].

**Lemma 2.8** (Jiang–Seiver [14]). Let $\varepsilon, c$ be positive reals, where $\varepsilon < 1$ and $c \geq 1$. Let $n$ be a positive integer that is sufficiently large as a function of $\varepsilon$. Let $G$ be a graph on $n$ vertices with $e(G) \geq cn^{1+\varepsilon}$. Then $G$ contains a $K$-almost regular subgraph $G'$ on $m \geq n^{\frac{c+2}{1+2\varepsilon}}$ vertices such that $e(G') \geq \frac{2}{5} m^{1+\varepsilon}$ and $K = 20 \cdot 2^{\frac{1}{1+2\varepsilon}} + 1$.

We are now ready to prove $\text{ex}^*(n, C_{2k}) = O(n^{1+1/k})$.

**Proof of Theorem 1.4.** By Lemma 2.8, it suffices to prove that for any fixed $K$, if $G'$ is a properly edge-coloured $K$-almost regular graph on $m$ vertices with minimum degree $\delta = \omega(m^{1/k})$, then, for $m$ sufficiently large, $G'$ contains a rainbow $C_{2k}$.

It is well known that $C_{2k}$ satisfies Sidorenko’s conjecture, so

$$
\text{hom}(C_{2k}, G') \geq \frac{\text{hom}(K_2, G')^{2k}}{m^{2k}} \geq \delta^{2k} \geq \frac{\delta^k}{mK^k} m \Delta(G')^k.
$$

Then $\text{hom}(C_{2k}, G') = \omega(m \Delta(G')^k)$, so by Corollary 2.7, $G'$ contains a rainbow $C_{2k}$. \qed

### 2.3 The proof of Lemma 2.4

It remains to prove Lemma 2.4.

**Lemma 2.9.** Let $H$ be a bipartite graph with parts $Y$ and $Z$. Suppose that $H$ does not contain a non-empty subgraph with minimum degree at least $d$. Then there exist bipartite graphs $H_1, H_2$ both with parts $Y$ and $Z$ such that $E(H)$ is the disjoint union of $E(H_1)$ and $E(H_2)$, every vertex in $Y$ has degree less than $d$ in $H_1$ and every vertex in $Z$ has degree less than $d$ in $H_2$.

**Proof.** Since $H$ has minimum degree less than $d$, there is a vertex $u$ in $H$ which has degree less than $d$. If $u \in Y$, let every edge in $H$ of the form $uv$ belong to $H_1$, otherwise let every edge of the form $uv$ belong to $H_2$. Set $H' = H - u$.

Since $H'$ has minimum degree less than $d$, there is a vertex $u'$ in $H'$ which has degree less than $d$. If $u' \in Y$, let every edge in $H$ of the form $u'v$ belong to $H_1$, otherwise let every edge of the form $u'v$ belong to $H_2$. Set $H'' = H' - u'$. Continue this procedure until all edges are placed in $H_1$ or $H_2$. It is easy to see that these graphs are suitable. \qed

The next two lemmas are well known.

**Lemma 2.10.** Let $H$ be a bipartite graph with parts $Y$ and $Z$ so that every vertex in $Y$ has degree at most $D_1$ and every vertex in $Z$ has degree at most $D_2$. Then the largest eigenvalue of $H$ is at most $\sqrt{D_1 D_2}$.

**Lemma 2.11.** Let $A$ and $B$ be symmetric real matrices with largest eigenvalues $\lambda$ and $\mu$. Then the largest eigenvalue of $A + B$ is at most $\lambda + \mu$. 

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Proof of Lemma 2.4. Define graphs $H_1$ and $H_2$ as in Lemma 2.9. By Lemma 2.10, both $H_1$ and $H_2$ have largest eigenvalue at most $\sqrt{d\Delta(H)}$. Hence, by Lemma 2.11, the largest eigenvalue of $H$ is at most $2\sqrt{d\Delta(H)}$. □

3 Rainbow cycles of arbitrary length

In this section we prove Theorem 1.5. We will use Corollary 2.2, but we first have to find a ‘regular enough’ subgraph. Using Corollary 2.2, one can show that there exists a constant $C$ such that any $C$-almost regular graph on $n$ vertices with at least $Cn(\log n)^3$ edges contains a rainbow cycle. Unfortunately, we think that it is not possible to find a $O(1)$-almost regular subgraph on $m = \omega(1)$ vertices with $\omega(m(\log m)^3)$ edges in an arbitrary $n$-vertex graph with $\omega(n(\log n)^3)$ edges. The next two lemmas give us a suitable subgraph for which Corollary 2.2 is applied, but we lose a log $n$ factor on the way, that is why we need $Cn(\log n)^4$ edges in Theorem 1.5.

Lemma 3.1. Let $d$ be sufficiently large and let $G$ be a graph on $n$ vertices with average degree $d$. Then there exists a non-empty bipartite subgraph $G'$ of $G$ with parts $X$ and $Y$ such that $e(G') \geq |X| \cdot \frac{\Delta(G')}{80}$ and $e(G') \geq |Y| \cdot \frac{d}{10 \log n}$.

Proof. By passing to a suitable subgraph, we may, without loss of generality, assume that every subgraph of $G$ has average degree at most $d$.

Let $A$ be the set consisting of the $\lceil n/2 \rceil$ largest degree vertices in $G$ and let $B = V(G) \setminus A$.

Suppose first that $e(G[B]) \geq \frac{e(G)}{10}$. Then we may partition $B$ into sets $X$ and $Y$ such that $e(G[X,Y]) \geq \frac{e(G)}{20} = \frac{nd}{10}$. Let $G' = G[V(G) \setminus \{A\}]$. Any vertex in $B$ has degree at most $\frac{2e(G)}{n} = \frac{nd}{n} \leq 2d/20$ in $G$, so $\Delta(G') \leq 2d$. Since $|X|, |Y| \leq n/2$, $G'$ satisfies the conditions in the lemma.

Hence, we may assume that $e(G[B]) < \frac{e(G)}{10}$. Suppose that $e(G[A]) \geq \frac{6e(G)}{10}$. Then $G[A]$ has larger average degree than $G$, which is a contradiction. Thus, $e(G[A]) < \frac{6e(G)}{10}$ and so $e(G[A,B]) \geq \frac{3e(G)}{10}$.

Let $A_{low} = \{x \in X : |N_G(x) \cap B| \leq \frac{d}{20}\}$ and let $A' = A \setminus A_{low}$. Clearly, $e(G[A_{low}, B]) \leq \frac{e(G)}{10}$, so $e(G[A', B]) \geq \frac{e(G)}{20}$. For $0 \leq i \leq \lfloor \log n \rfloor$, let $A_i = \{x \in A' : 2^i \leq |N_G(x) \cap B| < 2^{i+1}\}$. The sets $A_i$ partition $A'$, so there exists some $i$ such that $e(G[A_i, B]) \geq \frac{e(G[A', B])}{\log n + 1} \geq \frac{e(G)}{10 \log n} \geq |B| \cdot \frac{d}{10 \log n}$.

Let $X = A_i$, $Y = B$ and $G' = G[X,Y]$. The last inequality from the previous paragraph gives that $e(G') \geq |X| \cdot \frac{d}{10 \log n}$. Since every $x \in A_i$ has $d/20 < d_{G'}(x) \leq 2^{i+1}$, we have $d_{G'}(x) > 2^i$. But every $y \in B$ has $d_{G'}(y) \leq d_G(y) \leq 2d$, so $\Delta(G') \leq 40 \cdot 2^{i+1}$. However, for every $x \in A_i$, we have $d_{G'}(x) \geq 2^i$, so $e(G') \geq |X| \cdot 2^i \geq |X| \cdot \frac{\Delta(G')}{80}$. □

Lemma 3.2. Let $d$ be sufficiently large and let $G$ be a graph on $n$ vertices with average degree $d$. Then there exists a non-empty bipartite subgraph $G''$ of $G$ with parts $X$ and $Y$ such that for every $x \in X$, we have $d_{G''}(x) \geq \frac{\Delta(G'')}{160}$ and for every $y \in Y$, we have $d_{G''}(y) \geq \frac{d}{20 \log n}$.

Proof. By Lemma 3.1, we may choose a non-empty bipartite subgraph $G'$ with parts $X'$ and $Y'$ such that $e(G') \geq |X'| \cdot \frac{\Delta(G')}{80}$ and $e(G') \geq |Y'| \cdot \frac{d}{10 \log n}$. Now perform the following simple algorithm: as long as there is a vertex in $X'$ which has degree less than $\frac{\Delta(G')}{160}$ in the current graph, or there is a vertex in $Y'$ which has degree less than $\frac{d}{20 \log n}$ in the current graph, then discard one such vertex. Let the final graph be $G''$ and let its parts be $X$ and $Y$. Clearly we have $d_{G''}(x) \geq \frac{\Delta(G'')}{160} \geq \frac{\Delta(G''')}{160}$ for every $x \in X$ and $d_{G''}(y) \geq \frac{d}{20 \log n}$ for every $y \in Y$. Finally, $G''$ is non-empty since the number of edges discarded by the algorithm is less than $|X| \cdot \frac{\Delta(G'')}{160} + |Y| \cdot \frac{d}{20 \log n} \leq e(G')$. □
Now we prove that the subgraph we find by Lemma 3.2 has many homomorphic $C_{2k}$’s.

**Lemma 3.3.** Let $G$ be a bipartite graph with parts $X$ and $Y$ such that $d(x) \geq s$ for every $x \in X$ and $d(y) \geq t$ for every $y \in Y$. Then, for every positive integer $k$, 
\[
\text{hom}(C_{2k}, G) \geq s^k t^k.
\]

**Proof.** If $k$ is even, then $\text{hom}(P_k, G) \geq |X|^{k/2} t^{k/2}$. Hence,
\[
\text{hom}(C_{2k}, G) \geq \sum_{x, x' \in X} \text{hom}_{x, x'}(P_k, G)^2 \geq \frac{1}{|X|^2} \left( \sum_{x, x' \in X} \text{hom}_{x, x'}(P_k, G) \right)^2 \geq \left( \frac{\text{hom}(P_k, G)}{|X|} \right)^2 \geq s^k t^k.
\]

Now suppose that $k$ is odd. Without loss of generality, we may assume that $|X|s \geq |Y|t$. Note that $\text{hom}(P_k, G) \geq |X|s^{k+1} t^{-k}$. Hence,
\[
\text{hom}(C_{2k}, G) \geq \sum_{x \in X, y \in Y} \text{hom}_{x, y}(P_k, G)^2 \geq \frac{1}{|X||Y|} \left( \sum_{x \in X, y \in Y} \text{hom}_{x, y}(P_k, G) \right)^2 \geq \frac{\text{hom}(P_k, G)^2}{|X||Y|} \geq s^{k+1} t^{-1} \geq s^k t^k.
\]

\[\square\]

**Lemma 3.4.** Let $d$ be sufficiently large and let $G$ be a graph on $n$ vertices with average degree $d$. Then there exists a non-empty bipartite subgraph $G''$ of $G$ such that for every positive integer $k$,
\[
\text{hom}(C_{2k}, G'') \geq \left( \frac{d}{20 \log n} \right)^k \left( \frac{\Delta(G'')}{160} \right)^k.
\]

**Proof.** This follows immediately from Lemma 3.2 and Lemma 3.3. \[\square\]

**Proof of Theorem 1.5.** Let $n$ be sufficiently large and let $G$ be a properly edge-coloured graph on $n$ vertices with at least $Cn(\log n)^4$ edges, where $C = 2^{100}$. Let $k = \lceil \log n \rceil$.

By Lemma 3.4, $G$ has a non-empty bipartite subgraph $G''$ such that
\[
\text{hom}(C_{2k}, G'') \geq \left( \frac{C}{10 (\log n)^2} \right)^k \left( \frac{\Delta(G'')}{160} \right)^k \geq 2^{50k} k^{3k} \Delta(G'')^k \geq 2^{8k} k^{3k} n \Delta(G'')^k.
\]

Then, by Corollary 2.2, $G''$ contains a rainbow cycle. It has even length because $G''$ is bipartite. \[\square\]

## 4 Colour-isomorphic cycles

In this section we prove Theorem 1.10. Throughout the section, let $k$ and $r$ be fixed.

**Definition 4.1.** Given an edge-colouring of $K_n$, define an auxiliary graph $G_0$ as follows. Let the vertex set of $G_0$ be the set of $r$-vertex subsets of $V(K_n)$, i.e. let $V(G_0) = V(K_n)^{(r)}$. Now let $U$ and $V$ be joined by an edge if $U \cap V = \emptyset$ and there is a monochromatic matching between $U$ and $V$.

We will prove that if $K_n$ is coloured with $o(n^{\frac{r-1}{r-1} + \frac{k}{r-1}})$ colours, then there exists a copy of $\theta_{k, r+1}$ in $G_0$ in which the vertices are pairwise disjoint as subsets of $V(K_n)$. This implies that there exist $r$ colour-isomorphic, pairwise vertex-disjoint copies of $C_{2k}$. Indeed, let $X, Y_{i,j}$ for $1 \leq i \leq k-1, 1 \leq j \leq r! + 1$ and $Z$ be pairwise disjoint $r$-subsets of $V(K_n)$ with $X$
joined to $Y_{1,j}$ in $G_0$ for $1 \leq j \leq r! + 1$, $Y_{i,j}$ joined to $Y_{i+1,j}$ for every $1 \leq i \leq k - 2$ and every $1 \leq j \leq r! + 1$ and $Y_{k-1,j}$ joined to $Z$ for every $1 \leq j \leq r!+1$. For each $1 \leq j \leq r!+1$, pair each vertex in $X$ with the vertex in $Z$ that we get to if we follow the edges in the monochromatic matchings between $X, Y_{1,j}, Y_{2,j}, \ldots, Y_{k-1,j}, Z$. This gives, for each $1 \leq j \leq r!+1$, a bijection between $X$ and $Z$. Since there are $r!$ bijections between two sets of size $r$, two of these bijections must be identical, say the one corresponding to $j_1$ and the one corresponding to $j_2$. Then $X, Y_{1,j_1}, \ldots, Y_{k-1,j_1}, Z, Y_{k-1,j_2}, \ldots, Y_{1,j_2}$ and the monochromatic matchings between them provide $r$ colour-isomorphic, pairwise vertex-disjoint copies of $C_{2k}$.

**Lemma 4.2.** If $K_n$ is properly edge-coloured with $o(n^{r-1} \frac{k-1}{r})$ colours, then $c(G_0) = \omega(n^{r+r/k})$.

**Proof.** By the convexity of the function $f(r)$, the number of monochromatic $r$-matchings in $K_n$ is $\omega \left( n^{r-1} \frac{k-1}{r} \cdot (n^{2-\frac{k-1}{r}})^r \right) = \omega(n^{r+r/k})$. Any monochromatic $r$-matching gives rise to an edge in $G_0$ and any edge in $G_0$ is counted at most $r$ times, so the statement of the lemma follows.

For the rest of the proof, we fix a proper edge-colouring of $K_n$ with $o(n^{r-1} \frac{k-1}{r})$ colours and define $G_0$ as above. Since $G_0$ has $N := \binom{n}{r}$ vertices and $\operatorname{ex}(N, \theta_{k,r!+1}) = O(N^{1+1/k})$ (see [11]), it is already clear by Lemma 4.2 that $G_0$ contains a copy of $\theta_{k,r!+1}$. What we will prove is that this $\theta_{k,r!+1}$ can be chosen in a way that the vertices are pairwise disjoint sets.

The following simple lemma will be useful for making sure that the vertices are disjoint sets.

**Lemma 4.3.** Let $x, y \in V(G_0)$. Then the number of $z \in V(G_0)$ such that $xz \in E(G_0)$ and $z \cap y \neq \emptyset$ is at most $r^2$.

**Proof.** Since $y$ is a set of size $r$, there are $r$ ways to specify which element $v \in y$ will be contained in $z$. Given this choice, there are $r$ ways to choose the colour of the monochromatic matching between $x$ and $z$ since it must be the colour of $uv$ for some $u \in x$. Given these two choices, $z$ is uniquely determined (if exists) since the colouring of $K_n$ is proper.

The next lemma is analogous to Lemma 2.1.

**Lemma 4.4.** Let $\ell \geq 2$ be a positive integer and let $G$ be a subgraph of $G_0$. Then the number of homomorphic copies of $C_{2\ell}$ in $G$ in which the vertices are not pairwise disjoint (as subsets of $V(K_n)$) is at most

$$16\ell \left( r^2 \ell \Delta(G) \operatorname{hom}(C_{2\ell-2}, G) \operatorname{hom}(C_{2\ell}, G) \right)^{1/2}.
$$

The proof is nearly identical to that of Lemma 2.1, so it is only briefly sketched here. As in Lemma 2.1, we count the number of $(x_1, \ldots, x_{2\ell}) \in V(G)^{2\ell}$ with $x_1 x_2, \ldots, x_{2\ell} x_1 \in E(G)$ such that $x_i \cap x_i \neq \emptyset$ for some $2 \leq i \leq \ell + 1$. The only minor difference is that given $x_2, \ldots, x_{\ell+2}$, there are at most $r^2 \ell$, rather than $\ell$ ways to choose $x_1$. Indeed, there are $\ell$ ways to choose $i$ such that $x_i \cap x_i \neq \emptyset$, and, given any such choice, by Lemma 4.3, there are at most $r^2$ ways to choose $x_1$.

The next lemma is analogous to Lemma 2.6.

**Lemma 4.5.** Let $G$ be a non-empty subgraph of $G_0$ and suppose that for some $2 \leq \ell \leq k$ we have

$$\operatorname{hom}(C_{2\ell}, G) = \omega \left( \Delta(G) \cdot \operatorname{hom}(C_{2\ell-2}, G) \right).
$$

Then, for $n$ sufficiently large, $G$ contains a $\theta_{k,r!+1}$ in which the vertices are pairwise disjoint sets.
The proof of this lemma is very similar to that of Lemma 2.6 and is given in the appendix, but let us list here the three minor differences.

First, whenever in the proof of Lemma 2.6 we said ‘rainbow, injectively homomorphic copy of $C_{2\ell}$’, we now say ‘homomorphic copy of $C_{2\ell}$ in which the vertices are pairwise disjoint sets’.

We very slightly modify the definition of a ‘nice pair’ such that between any nice pair of vertices in $G$ we find $r|V(\theta_{k,r^{t+1}})|$ paths of length $\ell$, such that the vertices of $G$ involved in these paths are pairwise disjoint sets in $V(K_n)$.

The last difference is that we now find a subgraph of $H$ with sufficiently large minimum degree so that (using Lemma 4.3) we can greedily embed a spider with $r!+1$ legs of length $k-\ell$ in $H$ whose vertices are pairwise disjoint sets, and such that all the legs have endpoints which form nice pairs with $x$. (A spider with $t$ legs of length $s$ is the union of $t$ paths of length $s$ which share one endpoint but are pairwise vertex-disjoint apart from that.) Then we can extend this spider to a copy of $\theta_{k,r^{t+1}}$ in $G$ in which the vertices are pairwise disjoint sets.

**Corollary 4.6.** Let $G$ be a subgraph of $G_0$ on $m$ vertices and suppose that for some $2 \leq j \leq k$ we have

$$\text{hom}(C_{2j}, G) = \omega\left(m\Delta(G)^j\right).$$

Then, for $n$ sufficiently large, $G$ contains a $\theta_{k,r^{t+1}}$ in which the vertices are pairwise disjoint sets.

The proof of this is identical to that of Corollary 2.7.

We are now in a position to prove Theorem 1.10. Suppose that $K_n$ is properly edge-coloured with $o(n^{\frac{r-1}{2}})$ colours. By Lemma 4.2, we have $e(G_0) = \omega(N^{1+1/k})$, where $N = |V(G_0)| = \binom{m}{2}$. By Lemma 2.8, $G_0$ has a $K$-almost regular subgraph $G$ on $m = \omega(1)$ vertices with minimum degree $\delta = \omega(m^{1/k})$ such that $K = O(1)$. Now $\text{hom}(C_{2k}, G) \geq \delta^{2k} = \omega(m\Delta(G)^k)$, so by Corollary 4.6, $G_0$ contains a $\theta_{k,r^{t+1}}$ in which the vertices are pairwise disjoint sets. As we have discussed after Definition 4.1, this guarantees the existence of $r$ colour-isomorphic, pairwise vertex-disjoint copies of $C_{2k}$.

## 5 Blow-ups of cycles

In this section we prove Theorem 1.13 and Theorem 1.14.

**Definition 5.1.** Given a graph $G$, define an auxiliary graph $G_0$ as follows. Let the vertex set of $G_0$ be the set of $r$-vertex subsets of $V(G)$, i.e. let $V(G_0) = V(G)^r$. Now let $U$ and $V$ be joined by an edge if $U \cap V = \emptyset$ and $uv \in E(G)$ for every $u \in U$ and $v \in V$.

For the rest of the proof, we fix a positive integer $r$ and a graph $G$, and define $G_0$ as above. In order to find a copy of $C_{2k}[r]$ in $G$, we need to find a copy of $C_{2k}$ in $G_0$ in which the vertices are disjoint as subsets of $V(G)$. The next lemma will be useful for making sure that the vertices in our cycles are disjoint sets, and it plays the role of Lemma 4.3 from the previous section.

**Lemma 5.2.** Let $x, y \in V(G_0)$. Then the number of $z \in V(G_0)$ such that $xz \in E(G_0)$ and $z \cap y \neq \emptyset$ is at most $r r^{t+1} d_{G_0}(x)^{1-1/r}$.

**Proof.** There are $r$ ways to choose the element of $y$ that should belong to $z$, so it suffices to prove that for any $v \in V(G)$, the number of neighbours of $x$ in $G_0$ that contain $v$ is at most $r d_{G_0}(x)^{1-1/r}$. Let $d$ be the size of the common neighbourhood (in $G$) of the vertices in $x$. There are $\binom{d-1}{r-1}$ ways to choose the $r-1$ vertices in $z$ that are different from $v$. Since

$$\binom{d-1}{r-1} \leq r^{r-1} \binom{d}{r}^{1-1/r} = r d_{G_0}(x)^{1-1/r},$$

the proof is complete. \qed
The next lemma is analogous to Lemma 2.1 and Lemma 4.4.

**Lemma 5.3.** Let $\ell \geq 2$ be a positive integer and let $\mathcal{G}$ be a bipartite subgraph of $\mathcal{G}_0$ with parts $X_1$ and $X_2$ such that every $x \in X_1$ has $d_{\mathcal{G}_0}(x) \leq D_1$ and every $x \in X_2$ has $d_{\mathcal{G}_0}(x) \leq D_2$, where $D_1 \leq D_2$. Then the number of homomorphic copies of $C_{2\ell}$ in $\mathcal{G}$ in which the vertices are not pairwise disjoint (as subsets of $V(\mathcal{G})$) is at most

$$32\ell \left(r^{\ell+1}D_1^{1-1/r}D_2 \hom(C_{2\ell-2}, \mathcal{G}) \hom(C_{2\ell}, \mathcal{G})\right)^{1/2}.$$ 

The proof of this lemma is similar to that of Lemma 2.1, but not quite identical, so we give a sketch of the proof.

**Sketch of proof.** We want to prove that the number of $(x_1, x_2, \ldots, x_{2\ell}) \in V(\mathcal{G})^{2\ell}$ with $x_1, x_2, \ldots, x_{2\ell} \in E(\mathcal{G})$ such that $x_1, x_2, \ldots, x_{2\ell}$ are not all disjoint is at most

$$32\ell \left(r^{\ell+1}D_1^{1-1/r}D_2 \hom(C_{2\ell-2}, \mathcal{G}) \hom(C_{2\ell}, \mathcal{G})\right)^{1/2}.$$ 

By symmetry, it suffices to prove that the number of $(x_1, x_2, \ldots, x_{2\ell}) \in V(\mathcal{G})^{2\ell}$ with $x_1, x_2, \ldots, x_{2\ell} \in E(\mathcal{G})$ such that $x_1 \cap x_i \neq \emptyset$ for some $2 \leq i \leq \ell + 1$ is at most $16 \left(r^{\ell+1}D_1^{1-1/r}D_2 \hom(C_{2\ell-2}, \mathcal{G}) \hom(C_{2\ell}, \mathcal{G})\right)^{1/2}$. 

For a positive integer $s$, let $\alpha_s$ be the number of walks of length $\ell - 1$ in $\mathcal{G}$ whose endpoints $y$ and $z$ have $2s-1 \leq \hom_y(z)(P_{\ell-1}, \mathcal{G}) < 2s$ and let $\beta_s$ be the number of walks of length $\ell$ in $\mathcal{G}$ whose endpoints $y$ and $z$ have $2s-1 \leq \hom_y(z)(P_{\ell}, \mathcal{G}) < 2s$. 

For positive integers $s$ and $t$, write $\gamma_{s, t}$ for the number of homomorphic copies $x_1x_2\ldots x_{2\ell}x_1$ of $C_{2\ell}$ such that $x_1 \in X_1$, $x_1 \cap x_i \neq \emptyset$ for some $2 \leq i \leq \ell + 1$, $2s-1 \leq \hom_{x_1, x_1+t-1}(P_{\ell}, \mathcal{G}) < 2s$, and $2t-1 \leq \hom_{x_2, x_2+t-2}(P_{\ell}, \mathcal{G}) < 2t$, and write $\gamma'_{s, t}$ for the number of homomorphic copies $x_1x_2\ldots x_{2\ell}x_1$ of $C_{2\ell}$ such that $x_1 \in X_2$, $x_1 \cap x_i \neq \emptyset$ for some $2 \leq i \leq \ell + 1$, $2s-1 \leq \hom_{x_1, x_1+t-1}(P_{\ell}, \mathcal{G}) < 2s$, and $2t-1 \leq \hom_{x_2, x_2+t-2}(P_{\ell}, \mathcal{G}) < 2t$.

Here comes the main difference compared to the proof of Lemma 2.1. Observe that $\gamma_{s, t} \leq \alpha_s \cdot D_1 \cdot 2^{t}$. Indeed, if $x_1x_2\ldots x_{2\ell}x_1$ is a homomorphic copy of $C_{2\ell}$ with $x_1 \in X_1$, $2s-1 \leq \hom_{x_1, x_1+t-1}(P_{\ell}, \mathcal{G}) < 2s$, and $2t-1 \leq \hom_{x_2, x_2+t-2}(P_{\ell}, \mathcal{G}) < 2t$, then there are at most $\alpha_s$ ways to choose $(x_{t+2}, x_{t+3}, \ldots, x_{2\ell}, x_1)$, given such a choice, as $x_1 \in X_1$, there are at most $D_1$ choices for $x_2$, and given these there are at most $2^t$ choices for $(x_3, \ldots, x_{t-1})$. On the other hand, $\gamma'_{s, t} \leq \beta_t \cdot r^{\ell+1}D_2^{1-1/r} \cdot 2^t$. Indeed, there are at most $\beta_t$ ways to choose $(x_2, \ldots, x_{t+2})$. By Lemma 5.2, given such a choice, there are at most $\ell r^{\ell+1}d_{\mathcal{G}_0}(x_2)^{1-1/r} \leq \ell r^{\ell+1}D_2^{1-1/r}$ possibilities for $x_1$ (since $x_1 \cap x_i \neq \emptyset$ for some $2 \leq i \leq \ell + 1$). Finally, there are at most $2^t$ ways to complete this to a suitable homomorphic copy of $C_{2\ell}$. Similarly, $\gamma'_{s, t} \leq \alpha_s \cdot D_2 \cdot 2^{t}$ and $\gamma'_{s, t} \leq \beta_t \cdot \ell r^{\ell+1}D_1^{1-1/r} \cdot 2^t$.

Now similarly to the proof of Lemma 2.1, we can prove that

$$\sum_{s, t \geq 1} \gamma_{s, t} \leq 8 \left(r^{\ell+1}D_1D_2^{1-1/r} \hom(C_{2\ell-2}, \mathcal{G}) \hom(C_{2\ell}, \mathcal{G})\right)^{1/2}$$

and

$$\sum_{s, t \geq 1} \gamma'_{s, t} \leq 8 \left(r^{\ell+1}D_1^{1-1/r}D_2 \hom(C_{2\ell-2}, \mathcal{G}) \hom(C_{2\ell}, \mathcal{G})\right)^{1/2}.$$ 

Hence, the total number of homomorphic copies of $C_{2\ell}$ in $\mathcal{G}$ in which the vertices are not pairwise disjoint is

$$\sum_{s, t \geq 1} \gamma_{s, t} + \gamma'_{s, t} \leq 16 \left(r^{\ell+1}D_1^{1-1/r}D_2 \hom(C_{2\ell-2}, \mathcal{G}) \hom(C_{2\ell}, \mathcal{G})\right)^{1/2}. \square$$

Now we want to find a bipartite subgraph $\mathcal{G}$ in $\mathcal{G}_0$ which has many homomorphic cycles but whose vertices have not too large degree in $\mathcal{G}_0$. 

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Lemma 5.4. Let $G_0$ have average degree $d > 0$. Then there exist $D_1, D_2 \geq \frac{d}{4}$ and a non-empty bipartite subgraph $G$ in $G_0$ with parts $X_1$ and $X_2$ such that for every $x \in X_1$, we have $d_G(x) \geq \frac{D_1}{256r^2(\log n)^2}$ and $d_G(x) \leq D_1$, and for every $x \in X_2$, we have $d_G(x) \geq \frac{D_2}{256r^2(\log n)^2}$ and $d_G(x) \leq D_2$.

Proof. Let $N$ and $e$ denote the number of vertices and edges in $G_0$, respectively. Observe that the number of edges in $G_0$ incident to vertices of degree at most $d/4$ is at most $N d/4 = e/2$. Hence, a random partitioning of all vertices with degree at least $d/4$ shows that there exist disjoint sets $A$ and $B$ in $V(G_0)$ such that for every $v \in A \cup B$ we have $d_{G_0}(v) \geq d/4$ and the number of edges in $G_0[A, B]$ is at least $e/4$. For each $1 \leq i \leq \lfloor r \log n \rfloor$, let $A_i = \{v \in A : 2^{i-1} \leq d_{G_0}(v) < 2^i\}$ and let $B_i = \{v \in B : 2^{i-1} \leq d_{G_0}(v) < 2^i\}$. Note that the $A_i$’s partition $A$. Indeed, $\Delta(G_0) \leq \binom{n}{r} \leq n^r$. Similarly, the $B_i$’s partition $B$. Hence, there exist $i, j$ such that $e(G_0[A, B]) \geq \frac{e}{4|\lfloor r \log n \rfloor|^2} \geq \frac{e}{16r^2(\log n)^2}$.

Note that $|A_i| 2^{i-1} \leq 2e(G_0) = 2e$, so $|A_i| \leq \frac{2e}{2^{i-1}}$. Thus, the average degree of the vertices in $A_i$ in the graph $G_0[A_i, B_j]$ is at least $\frac{2e}{2^{i-1}}$. Similarly, the average degree of the vertices in $B_j$ in the same graph is at least $\frac{2e}{2^{j-1}}$. Thus, by a standard vertex removal argument, there exist non-empty $X_1 \subset A_i$ and $X_2 \subset B_j$ such that for $G = G_0[X_1, X_2]$, we have $d_G(x) \geq \frac{2^{j-1}}{128r^2(\log n)^2}$ for every $x \in X_1$ and $d_G(x) \geq \frac{2^{i-1}}{128r^2(\log n)^2}$ for every $x \in X_2$. Take $D_1 = 2^i$ and $D_2 = 2^j$. Since $d/4 \leq d_{G_0}(v) < 2^i$ holds for every $v \in X_1 \cup A$, we have $D_1 \geq d/4$. Similarly, $D_2 \geq d/4$.

The following supersaturation result guarantees that $G_0$ has enough edges, and is our final ingredient to the proof of Theorem 1.13.

Lemma 5.5 (Erdős–Simonovits [9]). There exist positive constants $c = c(r), \gamma = \gamma(r)$ such that any graph on $n$ vertices with $e > c \cdot n^{2-1/r}$ edges contains at least $\gamma n^{2 - 2/r}$ copies of $K_{r,r}$.

Proof of Theorem 1.13. Let $G$ be an $n$-vertex graph with $\omega(n^{2-1/r}(\log n)^{7/r})$ edges. We will prove that if $n$ is sufficiently large, then $G$ contains an $r$-blownup cycle. By Lemma 5.5, $G_0$ has $\omega(n^r(\log n)^{7r})$ edges, so it has average degree $\omega((\log n)^{7r})$. By Lemma 5.4, there exist $D_1, D_2 = \omega((\log n)^{7r})$ and a non-empty bipartite subgraph $G$ in $G_0$ with parts $X_1$ and $X_2$ such that for every $x \in X_1$, we have $d_G(x) \geq \frac{D_1}{256r^2(\log n)^2}$ and $d_G(x) \leq D_1$, and for every $x \in X_2$, we have $d_G(x) \geq \frac{D_2}{256r^2(\log n)^2}$ and $d_G(x) \leq D_2$. Without loss of generality, we may assume that $D_1 \leq D_2$.

By Lemma 3.3, for every positive integer $k$ we have

$$\text{hom}(C_{2k}, G) \geq \left(\frac{D_1}{256r^2(\log n)^2}\right)^k \left(\frac{D_2}{256r^2(\log n)^2}\right)^k = \left(\frac{D_1^{1/r}}{2^{16r^4(\log n)^2}}\right)^k \left(D_1^{1-1/r}D_2\right)^k.$$

Let $k = \lfloor \log n \rfloor$. Since $D_1 = \omega((\log n)^{7r})$, we have

$$\left(\frac{D_1^{1/r}}{2^{16r^4(\log n)^2}}\right)^k \geq \left(\frac{n}{r}\right)^k (L(\log n)^3)^k$$

for some $L = \omega(1)$. Then

$$\text{hom}(C_{2k}, G) \geq \left(\frac{n}{r}\right)^k (L(\log n)^3)^k D_1^{1-1/r}D_2^k.$$

Let $\ell$ be the smallest positive integer such that

$$\text{hom}(C_{2\ell}, G) \geq \left(\frac{n}{r}\right)^k (L(\log n)^3)^k D_1^{1-1/r}D_2^\ell.$$
Clearly, $\ell \leq k$. Moreover, since $G$ has at most $\binom{\nu}{r}$ vertices and maximum degree at most $D_2$, we have $\ell \geq 2$. Now note that

$$\text{hom}(C_{2\ell-2}, G) \leq \frac{\text{hom}(C_{2\ell}, G)}{L(\log n)^3 D_1^{1-\ell/r} D_2}.$$  

Hence, by Lemma 5.3, the number of homomorphic copies of $C_{2\ell}$ in $G$ in which the vertices are not pairwise disjoint is less than

$$\frac{32n^{-\frac{\ell+1}{2}} \ell^{3/2}}{L^{1/2}(\log n)^{3/2}} \text{hom}(C_{2\ell}, G).$$

Since $\ell \leq k \leq \log n$ and $L = \omega(1)$, this is less than $\text{hom}(C_{2\ell}, G)$ provided that $n$ is sufficiently large. Thus, there exists a homomorphic copy of $C_{2\ell}$ in $G$ in which the vertices are pairwise disjoint subsets of $V(G)$. This gives a $C_{2\ell[r]}$ in $G$. \qed

We will now prove Theorem 1.14. The key step is the following lemma, which is similar to Lemma 2.6 and Lemma 4.5 from the previous sections, but very slightly more involved.

**Lemma 5.6.** Let $\ell \geq 2$ and $k \geq \ell$ be fixed integers and let $G$ be a bipartite subgraph of $G_0$ with parts $X_1$ and $X_2$ such that every $x \in X_1$ has $d_{G_0}(x) \leq D_1$ and every $x \in X_2$ has $d_{G_0}(x) \leq D_2$, where $D_1 \leq D_2$. Assume that

$$\text{hom}(C_{2\ell}, G) = \omega\left(D_1^{1-\ell/r} D_2 \text{hom}(C_{2\ell-2}, G)\right).$$

Then, for $n$ sufficiently large, $G$ contains a copy of $C_{2k}$ in which the vertices are pairwise disjoint subsets of $V(G)$. In particular, $G$ contains a copy of $C_{2k[r]}$.

To prove this lemma, we need the following strengthening of Lemma 2.5.

**Lemma 5.7.** Let $H$ be a bipartite graph with parts $Y$ and $Z$. Let $f : Y \to \mathbb{R}$ be a function and let $g(z) = \sum_{y \in N_H(z)} f(y)$ for every $z \in Z$. Assume that $d_H(y) \leq D_1$ for every $y \in Y$ and that $d_H(z) \leq D_2$ for every $z \in Z$. Also suppose that $H$ does not contain a subgraph $H'$ with parts $Y' \subset Y$ and $Z' \subset Z$ such that for every $y \in Y'$, we have $d_{H'}(y) \geq d_1$ and for every $z \in Z$, we have $d_{H'}(z) \geq d_2$. Then

$$\sum_{y \in Y} f(y)^2 \geq \min\left(\frac{1}{4d_1 D_2}, \frac{1}{4D_1 d_2}\right) \sum_{z \in Z} g(z)^2.$$  

The proof of Lemma 5.7 is similar to the proof of Lemma 2.5 and is omitted.

Let us briefly sketch the proof of Lemma 5.6. It is nearly identical to the proof of Lemma 2.6 up to the definition of $H$, the only difference is that we replace each ‘rainbow, injectively homomorphic copy of $C_{2\ell}$’ by ‘$C_{2\ell}$ in which the vertices are disjoint sets’. Let us define the parts of $H$ very slightly differently: let $H$ have parts $Y$ and $Z$ where $Z = \{z \in V(G) : \{x, z\} \text{ is nice}\}$ and let $Y$ be the set of vertices in $G$ which have a neighbour in the set $Z$. Since there is a walk of length $\ell$ from $x$ to any element of $Z$, and $G$ is bipartite, we have either $Y \subset X_1$ and $Z \subset X_2$ or $Y \subset X_2$ and $Z \subset X_1$. In the former case we use Lemma 5.7 to find a subgraph of $H$ with parts $Y' \subset Y$ and $Z' \subset Z$ such that every $y \in Y'$ has $d_{H'}(y) = \omega(D_1^{1-\ell/r})$ and every $z \in Z'$ has $d_{H'}(z) = \omega(D_2^{1-\ell/r})$. In the latter case we use Lemma 5.7 to find a subgraph of $H$ with parts $Y' \subset Y$ and $Z' \subset Z$ such that every $y \in Y'$ has $d_{H'}(y) = \omega(D_2^{1-\ell/r})$ and every $z \in Z'$ has $d_{H'}(z) = \omega(D_1^{1-\ell/r})$. Then, using Lemma 5.2, we can greedily find a path of length $2k - 2\ell$ in which the vertices are disjoint from each other and from $x$ and which has endpoints in $Z$. Then we can extend this to a cycle of length $2k$ through $x$ in which the vertices are disjoint sets.
Proof of Theorem 1.14. Let $G$ be a graph with $\omega \left( n^{2 - \frac{1}{r} + \frac{1}{r}} \right)$ edges. By Lemma 5.5, $G_0$ has average degree $\omega \left( n^{\frac{1}{r}} \right)$. By Lemma 5.4, $G_0$ has a bipartite subgraph $G$ with parts $X_1$ and $X_2$ such that for every $x \in X_i$ we have $d_G(x) \geq \frac{D_i}{256n^2(\log n)^2}$ and $d_{G_0}(x) \leq D_i$, where $D_i = \omega \left( n^{\frac{1}{r}} \right)$. Using Lemma 3.3, we have $\text{hom}(C_{2k}, G) \geq \Omega \left( \frac{D_i^k}{(\log n)^{4k}} \right) \geq \omega \left( (D_1^{1-1/r} D_2)^{k-1/n} D_2 \right)$. So there exists some $2 \leq \ell \leq k$ with $\text{hom}(C_{2\ell}, G) = \omega \left( (D_1^{1-1/r} D_2) \text{hom}(C_{2\ell-2}, G) \right)$. By Lemma 5.6, $G$ contains $C_{2k}[r]$ as a subgraph. \hfill \Box

6 Concluding remarks

Rainbow cycles. We have shown that for a sufficiently large constant $C$, any properly edge-coloured $n$-vertex graph with at least $Cn(\log n)^4$ edges contains a rainbow cycle. However, the best known construction of a graph without a rainbow cycle has only $\Theta(n^{0.68})$ edges. One such example, found by Keevash, Mubayi, Sudakov and Verstraëte [17], is the $m$-dimensional cube whose vertices are the subsets of $\{1, 2, \ldots, n\}$ where $A$ is joined to $A \setminus \{i\}$ for every $i \in A$. The colour of the edge between $A$ and $A \setminus \{i\}$ is $i$. This graph has $2^n$ vertices and $\frac{1}{4}m2^m$ edges and it has no rainbow cycle. Examples with more than $0.58n\log n$ edges were also found by Keevash, Mubayi, Sudakov and Verstraëte.

Colour-isomorphic cycles. Recall that $f_r(n, H)$ is the smallest number $C$ so that there is a proper edge-colouring of $K_n$ with $C$ colours containing no $r$ vertex-disjoint colour-isomorphic copies of $H$. We have shown that $f_r(n, C_{2k}) = \Omega \left( n^{\frac{r-1}{r} - \frac{1}{r}} \right)$. Note that our result becomes trivial when $r \geq k$ since $f_r(n, H) \geq n - 1$ holds for any $r$ and $H$ (as any proper colouring of $K_n$ must use at least $n - 1$ colours).

The best general upper bound comes from the probabilistic construction that is used in Theorem 1.6 and says that $f_r(n, C_{2k}) = O \left( n^{\frac{r-1}{r} - \frac{1}{r}} \right)$. Another result of Conlon and Tyomkyn [4, Theorem 1.4], proved by a variant of Bukh’s random algebraic method [3], states that if $H$ contains a cycle, then there exists $r$ such that $f_r(n, H) = O(n)$. It would be interesting to decide what the smallest such $r$ is when $H = C_{2k}$. Our result shows that we must have $r \geq k$. This question was studied in the case $H = C_4$ by Xu, Zhang, Jing and Ge [19], who showed that $f_r(n, C_4) = \Theta(n)$ for any $r \geq 3$.

Blow-ups of cycles. We have shown that $\text{ex}(n, C[r]) = O(n^{2-1/r}(\log n)^{7/r})$. On the other hand, a random graph with edge probabilities $p = \frac{n^{-1/r}}{2}$ contains no $r$-blowup cycles with probability at least $1/2$, so $\text{ex}(n, C[r]) = \Omega(n^{2-1/r})$. We pose the following question.

Question 6.1. Let $r \geq 2$. Is it true that $\text{ex}(n, C[r]) = \Theta(n^{2-1/r})$?

Finally, regarding a single forbidden blowup cycle, we reiterate the conjecture of Grzesik, Janzer and Nagy [12] stating that $\text{ex}(n, C_{2k}[r]) = O(n^{2-1/r + \frac{1}{r}})$.

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A Appendix

Proof of Lemma 4.5. Let $s = r|V(\theta_{k,r+1})|$. For a graph $F$, call a homomorphic copy of $F$ in $G$ good if the images of the vertices of $F$ are disjoint sets (as subsets of $V(K_n)$). In particular, any such copy is an injectively homomorphic copy of $F$. Call a pair $(x_1, x_{\ell+1})$ of vertices in $G$ nice if the number of good copies of $C_{2\ell}$ of the form $x_1, x_2, \ldots, x_{2\ell}, x_1$ is greater than $(1 - \frac{1}{(\ell)^2}) (\text{hom}_{x_1, x_{\ell+1}}(P_{\ell}, G))^2$. Observe that the total number of homomorphic copies of $C_{2\ell}$ of the form $x_1, x_2, \ldots, x_{2\ell}, x_1$ in $G$ is $\text{hom}_{x_1, x_{\ell+1}}(P_{\ell}, G)^2$, so this means that the proportion of those which are not good is less than $\frac{1}{(\ell)^2}$. In particular, if we choose $s$ random walks of length $\ell$ between $x_1$ and $x_{\ell+1}$ with replacement, then with positive probability any two of these walks form a good copy of $C_{2\ell}$. Hence, there exist at least $s$ pairwise internally disjoint paths between $x_1$ and $x_{\ell+1}$ such that the vertices involved in these paths are pairwise disjoint sets in $V(K_n)$.

By Lemma 4.4, the number of non-good copies of $C_{2\ell}$ in $G$ is

$$O_{r, \ell} \left( (\Delta(G) \text{hom}(C_{2\ell - 2}, G) \text{hom}(C_{2\ell}, G))^{1/2} \right) \leq o(\text{hom}(C_{2\ell}, G)).$$

Hence,

$$\sum_{(x_1, x_{\ell+1}) \text{ not nice}} \frac{1}{(\ell)^2} \text{hom}_{x_1, x_{\ell+1}}(P_{\ell}, G)^2 = o(\text{hom}(C_{2\ell}, G)),$$

so, using $\sum_{x_1, x_{\ell+1} \in V(G)} \text{hom}_{x_1, x_{\ell+1}}(P_{\ell}, G)^2 = \text{hom}(C_{2\ell}, G)$, we have

$$\sum_{(x_1, x_{\ell+1}) \text{ nice}} \text{hom}_{x_1, x_{\ell+1}}(P_{\ell}, G)^2 \geq (1 - o(1)) \text{hom}(C_{2\ell}, G) > (1 - o(1))L \Delta(G) \text{hom}(C_{2\ell - 2}, G)$$

for some $L = \omega(1)$.

Thus, there exists some $x \in V(G)$ such that

$$\sum_{z \in V(G) : (x, z) \text{ is nice}} \text{hom}_{x, z}(P_{\ell}, G)^2 > (1 - o(1))L \Delta(G) \text{hom}_{x}(C_{2\ell - 2}, G). \quad (4)$$

Let $Z = \{ z \in V(G) : (x, z) \text{ is nice} \}$ and let $Y = V(G)$. Consider the bipartite graph $H$ with parts $Y$ and $Z$, defined by $G$. (We view $Y$ and $Z$ as disjoint sets even though they overlap as subsets of $V(G)$.)

Suppose that $H$ does not contain a subgraph with minimum degree at least $r^2 k(r! + 1)$. Let $f(y) = \text{hom}_{x, y}(P_{\ell - 1}, G)$ for every $y \in Y = V(G)$ and define $g$ as in Lemma 2.5. By that lemma with $d = r^2 k(r! + 1)$,

$$\sum_{y \in Y} f(y)^2 \geq \frac{1}{4d \Delta(H)} \sum_{z \in Z} g(z)^2 \geq \frac{1}{4d \Delta(G)} \sum_{z \in Z} g(z)^2.$$

However, $g(z) = \sum_{y \in N_G(z)} \text{hom}_{x, y}(P_{\ell - 1}, G) = \text{hom}_{x, z}(P_{\ell}, G)$, so, using equation (4),

$$\sum_{y \in Y} f(y)^2 \geq \frac{1}{4d \Delta(G)} \sum_{z \in Z} \text{hom}_{x, z}(P_{\ell}, G)^2 > \frac{(1 - o(1))L}{4d} \text{hom}_{x}(C_{2\ell - 2}, G).$$

However, $\sum_{y \in Y} f(y)^2 = \text{hom}_{x}(C_{2\ell - 2}, G)$, which is a contradiction, as $L = \omega(1)$ and $n$ is sufficiently large.

Thus, $H$ contains a subgraph with minimum degree at least $r^2 k(r! + 1)$. Then, by Lemma 4.3 we can greedily find in $H$ a spider whose vertices are disjoint (as subsets of $V(K_n)$) from $x$ and from each other and which has $r! + 1$ legs of length $k - \ell$ such that the endpoints of
these legs are in $Z$. Let this spider be $S$ with endpoints $z_1, z_2, \ldots, z_{r!+1}$. Since for every $i$, $(x, z_i)$ is a nice pair, there exist at least $s = r|V(\theta_{k,r!+1})|$ paths of length $\ell$ between $x$ and $z_i$ such that all the internal vertices in these paths are distinct and pairwise disjoint from each other. Hence, we can choose paths of length $\ell$ between $x$ and $z_i$ for every $1 \leq i \leq r! + 1$ such that all the vertices involved are disjoint from the vertices of $S$ and from each other. Then the union of these paths with $S$ gives a suitable $\theta_{k,r!+1}$. \square