A THEORY OF COMPLEX OSCILLATORY INTEGRALS:
A CASE STUDY

JAMES WRIGHT

Abstract. In this paper we develop a theory for oscillatory integrals with complex phases. When \( f : \mathbb{C}^n \to \mathbb{C} \), we evaluate this phase function on the basic character \( e(z) := e^{2\pi i x} e^{2\pi i y} \) of \( \mathbb{C} \simeq \mathbb{R}^2 \) (here \( z = x + iy \in \mathbb{C} \) or \( z = (x, y) \in \mathbb{R}^2 \)) and consider oscillatory integrals of the form

\[
I = \int_{\mathbb{C}^n} e(f(\xi)) \phi(\xi) d\xi
\]

where \( \phi \in C^\infty_c(\mathbb{C}^n) \).

Unfortunately basic scale-invariant bounds for the oscillatory integrals \( I \) do not hold in the generality that they do in the real setting. Our main effort is to develop a perspective and arguments to locate scale-invariant bounds in (necessarily) less generality than we are accustomed to in the real setting.

1. Introduction

Oscillatory integral estimates known as van der Corput estimates are very useful in a wide range of areas. It states that for every integer \( k \geq 1 \), there is a constant \( C_k \) such that whenever \( f : [a, b] \to \mathbb{R} \) is a smooth function satisfying \( |f^{(k)}| \geq 1 \) on \([a, b]\), then

\[
\left| \int_a^b e^{2\pi i \lambda f(x)} \, dx \right| \leq C_k |\lambda|^{-1/k}.
\]  

(1)

Here \( \lambda \in \mathbb{R} \) is a real parameter. When \( k = 1 \) a monotonicity condition is needed on \( f' \). The example \( f(x) = x^k \) shows the sharpness of the exponent \( 1/k \).

The usefulness of these estimates lies in the uniformity of the constant \( C_k \) which depends only on \( k \). As a consequence the quantitative hypothesis \( |f^{(k)}(x)| \geq 1 \) for \( x \in [a, b] \) can be relaxed to \( |f^{(k)}(x)| \geq \mu \) for \( x \in [a, b] \) and the estimate scales accordingly. Applying (1) to \( g = \mu^{-1} f \) (so that \( |g^{(k)}| \geq 1 \) holds on \([a, b]\)), we have

\[
\left| \int_a^b e^{2\pi i \lambda f(x)} \, dx \right| \leq C_k |\mu|^{-1/k},
\]  

(2)

a conclusion we would not be able to deduce if the constant \( C_k \) in (1) depended on the phase \( f \). For lack of better terminology, we will call the bound in (1) or (2) a scale-invariant bound. See [13] or [14] for further details.
The situation changes when we move from the real field $\mathbb{R}$ to the complex field $\mathbb{C}$ and consider complex differentiable phases $f : D \to \mathbb{C}$ where $D$ is some domain in $\mathbb{C}^n$. Here we consider oscillatory integrals of the form

$$I_\phi(f) = \int_{\mathbb{C}^n} e(f(z)) \phi(z) \, dz$$

(3)

where $\phi \in C^\infty_c(\mathbb{C}^n)$ is a smooth cut-off function and $e(z) = e^{2\pi ix} e^{2\pi iy}$ with $z = x + iy \in \mathbb{C}$ is the basic character on the locally compact abelian group $\mathbb{C} \simeq \mathbb{R}^2$.

The study of these oscillatory integrals is not to be confused with the beautiful theory of complex oscillatory integrals defined by evaluating the above complex phase $f$ on the basic character $t \to e^{2\pi it}$ from $\mathbb{R}$ which has a unique analytic extension from $\mathbb{R}$ to $\mathbb{C}$ and is very useful in developing a geometric-invariant theory for real oscillatory integrals. See [4].

The oscillatory integral $I_\phi(f)$ is a real oscillatory integral with the real phase $\text{Re}(f) + \text{Im}(f)$. It would be nice if we had complex versions of van der Corput estimates where scale-invariant bounds for $I_\phi(f)$ are derived from a condition that some complex derivative of $f$ is bounded below. We would then be able to apply such estimates to the fourier transform $\hat{\sigma}$ of measures $\sigma$ in $\mathbb{C}^n$ since we can write

$$\hat{\sigma}(Tw) = \int_{\mathbb{C}^n} e(\langle w, z \rangle) \, d\sigma(z)$$

for some sympletic (and hence measure-preserving) transformation $T$ on $\mathbb{C}^n$. Here $\langle w, z \rangle = w_1z_1 + \cdots + w_nz_n$. Therefore questions regarding $L^p$ norms of $\hat{\sigma}$ (which arise in the fourier restriction problem for example) can then be investigated using complex van der Corput estimates. See Section 15 for details.

Unfortunately the complex analogue of (1) does not hold. When $n = 1$, suppose that $f$ is a complex differentiable function satisfying $|f^{(k)}| \geq 1$ on the unit disc $D = \{ z \in \mathbb{C} : |z| \leq 1 \}$, say. Then a scale-invariant bound

$$\left| \int_{\mathbb{C}} e(\lambda f(z)) \phi(z) \, dz \right| \leq C_{k, \phi} |\lambda|^{-2/k}$$

with a constant $C_{k, \phi}$ only depending on $k$ and $\phi \in C^\infty_c(D)$ does not hold. Considering the example $f(z) = z^k$ shows the exponent $2/k$ is optimal.

The examples illustrating this lack of scale-invariance are non-polynomial. For polynomials phases, we are able to adapt and extend arguments from elementary number theory to establish scale-invariant bounds for oscillatory integrals $I_\phi(f)$ in (3).

Fix $\phi \in C^\infty_c(\mathbb{C}^n)$ and define

$$H_{f, \phi} = \inf_{z \in \text{supp}(\phi)} H_f(z)$$

where $H_f(z) = \max_{|\alpha| \geq 1} \left( |\partial^\alpha f(z)|/|\alpha|^{1/|\alpha|} \right)$.

for polynomials $f \in \mathbb{C}[X_1, \ldots, X_n]$. Notation involving complex polynomials and partial derivatives will be defined in Section 8.

**Theorem 1.1.** If $f \in \mathbb{C}[X_1, \ldots, X_n]$ has degree $d$, then

$$|I_\phi(f)| \leq C_{d, n, \phi} H_{f, \phi}^{-2}.$$
Here $C_{d,n,\phi}$ depends only on $d, n$ and $\phi$.

The real version for polynomials $f \in \mathbb{R}[X_1, \ldots, X_n]$ can be found in [1]. It is possible to use the real version (after all our phase is $\text{Re}(f) + \text{Im}(f) \in \mathbb{R}[X_1, \ldots, X_{2n}]$) but we would only obtain the bound $I_\phi(f) \lesssim_{d,n,\phi} H_f^{-1}$ which has limited use in applications.

As an immediate consequence of Theorem 1.1, we have the following scale-invariant version of (1) for complex polynomials.

**Corollary 1.2.** Suppose $f \in \mathbb{C}[X_1, \ldots, X_n]$ has degree $d$ and satisfies $|\partial^\alpha f| \geq \mu$ on the support of $\phi \in C_\infty^\infty(C^n)$ for some partial derivative $\partial^\alpha$. Then

$$|I_\phi(\lambda f)| \leq C_{d,n,\phi} |\mu \lambda|^{-2/|\alpha|}.$$ 

The bound in Theorem 1.1 has a number of applications. For example a complex version of a robust oscillatory integral estimate with polynomial phases $f \in \mathbb{R}[X]$ due to Phong and Stein [11] can be established. Let $f \in \mathbb{C}[X]$ be a complex polynomial of degree $d$ and consider the derivative $f'(z) = a \prod_{1 \leq j \leq L} (z - w_j)^{m_j}$ where $\{w_j\}$ are the distinct roots of $f'$.

**Proposition 1.3.** We have

$$|I_\phi(f)| \leq C_{d,\phi} \max_{1 \leq j \leq L} \min_{C \supseteq w_j} \left[ \frac{1}{a \prod_{w_k \not\in C} (w_j - w_k)^{m_k}} \right]^{2/(S(C) + 1)}$$

where the minimum is taken over all root clusters $C \subseteq \{w_k\}$ containing $w_j$ and $S(C) = \sum_{w_k \in C} m_k$.

This opens the door to establish complex versions of results based on the Phong-Stein bound.

Another application involves the fourier extension operator for the complex moment curve $z \rightarrow (z, z^2, \ldots, z^d)$ which we can view as a 2-surface in $\mathbb{R}^{2d}$. The oscillatory integral operator

$$\mathcal{E} b \left( \overline{w} \right) = \int_C e(w_1 z + \cdots + w_d z^d) b(z) \phi(z) \, dz$$

differs from the fourier extension operator of the 2-surface defined by the complex moment curve by a sympletic transformation.

**Proposition 1.4.** Set $q_d = 0.5(d^2 + d) + 1$. Then $\mathcal{E}1 \in L^q(\mathbb{C}^d)$ if and only if $q > q_d$.

If $\mathcal{E} b \left( \overline{w} \right)$ is defined with respect to the sparse polynomial $w_1 z^{k_1} + \cdots + w_d z^{k_d}$ where $K := k_1 + \cdots + k_d < (0.5)(k_d(k_d + 1))$, then $\mathcal{E}1 \in L^q(\mathbb{C}^d)$ if and only if $q > K$.

The real analogue of Proposition 1.4 is due to Arkhipov, Chubarikov and Karatsuba; see [2] and [3].
Structure of the paper. In the following section we illustrate the lack of scale-invariant bounds for general complex functions. In Sections 3 and 4, we motivate and develop the theory of sublevel sets for complex differentiable functions and in Section 5 we introduce the $H$ functional in the complex setting and illustrate its usefulness in the theory of sublevel sets. In Sections 6 and 7, we develop the theory for oscillatory integrals with complex polynomial phases. In Sections 8-11, we give the proofs of the main results for sublevel sets and oscillatory integrals (Theorem 1.1), reducing matters to a structural sublevel set statement which we establish in Sections 12 and 13. In Section 14, we give the proof of Proposition 1.3 and in Sections 15-19, we give the proof of Proposition 1.4.

Notation. We use the notation $A \lesssim B$ between two positive quantities $A$ and $B$ to denote $A \leq CB$ for some constant $C$. We sometimes use the notation $A \lesssim k B$ to emphasise that the implicit constant depends on the parameter $k$. We sometimes use $A = O(B)$ to denote the inequality $A \lesssim B$. Furthermore, we use $A \ll B$ to denote $A \leq \delta B$ for a sufficiently small constant $\delta > 0$ whose smallness will depend on the context.

Acknowledgement. We thank Rob Fraser, John Green and Jonathan Hickman for enlightening conversations on the topics of this paper.

2. An illustration

Let us begin with a simple illustration. Suppose we have a function $f$ with a large derivative, say $|f'| \geq 1$ everywhere in some region $I$. Therefore $f$ is not stationary and we expect the sublevel set $S_\epsilon = \{z \in I : |f(z)| \leq \epsilon\}$ to be small when $\epsilon$ is small. Consider two points $z_0, z_1 \in S$ so that $f$ is small at these two points but we know that $f'$ is large at all points $z_t := z_0 + t(z_1 - z_0)$, $t \in [0, 1]$ on the line segment from $z_0$ to $z_1$. Of course these two bits of information can be connected by the fundamental theorem of calculus:

$$\int_0^1 f'(z_t) \, dt \, (z_1 - z_0) = f(z_1) - f(z_0).$$

Taking absolute values, we have

$$\left| \int_0^1 f'(z_t) \, dt \right| |z_1 - z_0| = |f(z_1) - f(z_0)| \leq 2\epsilon$$

by a simple use of the triangle inequality and since $z_0, z_1 \in S_\epsilon$. But we know $|f'(z_t)| \geq 1$ for all $0 \leq t \leq 1$ and so it seems we are one step away from deducing that the diameter of $S_\epsilon$ is at most $2\epsilon$.

But we have not specified what world we are living in. The above discussion makes sense for real functions $f$ whose derivative is always larger than 1 but it also makes sense for complex functions whose complex derivative $f'$ is everywhere large in absolute value. With a little imagination, the above discussion makes sense for a wide range of functions defined over disparate fields with an absolute value $|\cdot|$.
The point of this illustration is that the real world is a very nice world to live in because the underlying field \( \mathbb{R} \) is not only a complete field, but it is also an ordered field which gives rise to the \textit{intermediate value theorem} from elementary calculus. This simple result can pack a powerful punch at times.

So for the time being, let us suppose that we are living in the real world looking at a real function \( f \) on the real line with a large derivative. Then by the intermediate value theorem, we conclude that either \( f' \geq 0 \) is always nonnegative or \( f' \leq 0 \) is always nonpositive and hence

\[
\int_0^1 |f'(z_t)| \, dt = |\int_0^1 f'(z_t) \, dt|
\]

so that we can move the absolute value sign inside the integral for free! And so indeed we are just one step away from concluding that \( \text{diam}(S_\epsilon) \leq 2\epsilon \) for all \( t \in [0, 1] \). The inequality \( \text{diam}(S_\epsilon) \leq 2\epsilon \) is not only a nice structural statement for the sublevel set \( S_\epsilon \) (it implies in particular the measure bound \( |S_\epsilon| \leq 2\epsilon \)), it is \textit{scale-invariant} as described in the Introduction; namely, we can relax the condition that \( |f'| \geq 1 \). For general \( f \), set \( \mu = \inf_{I} |f'| \) and scale \( g = \mu^{-1} f \) (if \( \mu > 0 \)), noting \( |g'| \geq 1 \) on \( I \) and applying the above diameter bound to \( g \), we see that \( \text{diam}(S_\epsilon) \leq 2\epsilon/\mu \). This inequality remains true if \( \mu = 0 \). Such a general statement is not possible if we only knew that \( \text{diam}(S_\epsilon) \leq C\epsilon \) where \( C \) depends on \( f \).

Scale-invariant inequalities are very powerful. For example the scale-invariant measure bound \( |S_\epsilon| \leq 2\epsilon \) whenever \( |f'| \geq 1 \) on \( I \) almost implies by itself that the scale-invariant bound \( |S_\epsilon| \leq C_k \epsilon^{1/k} \) holds whenever \( |f^{(k)}| \geq 1 \) on \( I \). The standard induction on \( k \) argument needs one additional \textit{a priori} structural statement for \( S_\epsilon = \{ z \in I : |f(z)| \leq \epsilon \} \) when \( f^{(k)} \) does not vanish on \( I \); namely, that \( S_\epsilon \) is the union of at most \( k \) intervals. This is yet another consequence of the intermediate value theorem or the order structure of \( \mathbb{R} \). The same story holds for oscillatory integrals with a real phase \( f \). One proves a scale-invariant bound when \( |f'| \geq 1 \) everywhere and then uses this bound to prove a bound whenever \( |f^{(k)}| \geq 1 \). These are the van der Corput estimates we mentioned in the Introduction. All this from the order structure of the real field \( \mathbb{R} \). See [14] for more details.

Now let us move from the real world to the complex world. The inequality [4] still holds for complex functions whose complex derivative satisfies \( |f'| \geq 1 \) on \( I \). How far are we from concluding that \( \text{diam}(S_\epsilon) \leq 2\epsilon \) without the use of the intermediate value theorem coming from the order structure of the reals?

The problem here is that the real and imaginary parts of \( f \) can conspire to produce many zeros of \( f \) in our region \( I \) while \( f \) still retains the property \( |f'| \geq 1 \) everywhere on \( I \). Consider the function

\[
f(z) = \frac{e^N(e^{Nz} - 1)}{N}
\]

for some large \( N \geq 1 \). We have

\[
|f'(z)| = e^{N(\text{Re}(z)+1)} \geq 1 \quad \text{for all } z \in I := \{ z \in \mathbb{C} : \text{Re}(z) \geq -1 \}.
\]

On the other hand, \( f(z) = 0 \) for infinitely many \( z \) on the line \( \text{Re}(z) = 0 \); precisely for \( z = x + iy \) with \( x = 0 \) and \( y = 2\pi k/N \) for every \( k \in \mathbb{Z} \). Since zeros of \( f \) are
clearly contained in any sublevel set $S_\epsilon$, we see that $\text{diam}(S_\epsilon) = \infty$ for every $\epsilon > 0$. This is not an artifact that $I$ is an unbounded region since we could restrict our attention to the unit square $\mathcal{U} = \{z = x + iy : -1 \leq x, y \leq 1\}$ and conclude that $\text{diam}(S_\epsilon \cap \mathcal{U}) \sim 2$. In particular $\text{diam}(S_\epsilon \cap \mathcal{U}) \not \to 0$ as $\epsilon \to 0$ which is in sharp contrast to the situation over the real field $\mathbb{R}$.

A diameter bound is stronger than a measure bound which is often what is required in applications. Let us modify the example above, moving the zeros from the line $\text{Re}(z) = 0$ where $|f'|$ is exponential in $N$ to the line $\text{Re}(z) = -1$ where $|f'(z)| \equiv 1$. We shift the example to
\[
f(z) = \frac{e^{N(z+1)} - 1}{N}\]
so that the zeros of $f$ now lie on the line $\text{Re}(z) = -1$ but $|f'(z)| \geq 1$ still holds on the half-plane $I = \{z \in \mathbb{C} : \text{Re}(z) \geq -1\}$. A simple calculation shows that
\[
|\{z \in \mathcal{U} : |f(z)| \leq \epsilon\}| \sim N\epsilon^2, \text{ if } \epsilon < N^{-1}
\]
for the sublevel set $S_\epsilon \cap \mathcal{U}$ on the unit square $\mathcal{U}$. Hence there is no scale-invariant bound for sublevel sets of the form $\{|f| \leq \epsilon\} \leq C\epsilon^2$ for general complex differentiable functions with $|f'| \geq 1$ and where $C$ is a universal constant. A bound in terms of $\epsilon^2$ is optimal and is the natural bound; when $f' \neq 0$, the function $f$ is an open map and locally 1-1 and so we expect, as the above example shows, the sublevel set to be a union of $\epsilon$ discs centred at the zeros of $f$, at least when $\epsilon > 0$ is small enough.

Considering $g(z) = |f(z)|^2$ where $f$ is the example above shows that there are no scale-invariant bounds of the form $\{|g| \leq \epsilon\} \leq C\epsilon$ which hold for some universal constant $C$ and every complex differentiable function $g$ satisfying $|g''(z)| \geq 1$ on $\mathcal{U}$.

### 3. Complex sublevel set bounds

The parameter $N$ in the counterexample $f = f_N$ above for a scale-invariant sublevel set bound $\{|f| \leq \epsilon\} \leq C\epsilon^2$ can be viewed as the number of zeros $f$ on the unit square $\mathcal{U}$. But it can also be viewed as the logarithm of the $L^\infty$ norm of $f$, $\log \|f\|_{L^\infty}$, on any larger disc, say $D_2 = \{z : |z| \leq 2\}$. These two points of view are connected by Jensen’s formula from basic complex analysis which has the following consequence; we have
\[
\# \{z \in D_r : f(z) = 0\} \lesssim_r \log \|f\|_{L^\infty(D_2)}
\]
for any $r < 2$ and for general complex differentiable functions $f$ on $D_2$ with $|f(0)| \geq 1$. Note that equality occurs for the exponential example above. In some sense, the bound $\#$ for our exponential example is sharp.

**Proposition 3.1.** Suppose $f \in \mathcal{H}(D_2)$ with $|f(z)| \leq M$ for $z \in D_2$. Let $N$ denote the number of zeros of $f$ in the disc $D_{5/4}$. If $64\epsilon \leq M^{-1}$, then
\[
\{|z \in D : |f(z)| \leq \epsilon\} \leq 10N\epsilon^2.
\]
If we restrict our attention to complex polynomials \( f \in \mathbb{C}[X] \) of degree \( d \), then the bound (7) becomes \( C_d \epsilon^2 \) and we might think this gives us a scale-invariant bound for the class of complex polynomials \( f \) of degree at most \( d \) (scaling \( \mu f \) by a constant \( \mu \) does not change the degree of the polynomial \( f \)). But this is not the case due to the smallness condition \( 64 \epsilon \leq M^{-1} \) which is necessary in general by the above example.

Proposition 3.1 can be extended to higher derivatives.

**Proposition 3.2.** Let \( f \in \mathcal{H}(\mathbb{D}) \) with \( |f(z)| \leq M \) for all \( z \in \mathbb{D} \). Suppose that \( |f^{(k)}(z)| \geq 1 \) for all \( z \in \mathbb{D} \). Then if \( \epsilon \ll_k M^{-(2k-1)} \),
\[
\{ z \in \mathbb{D} : |f(z)| \leq \epsilon \} \lesssim_n M^{2(k-1)/k} N' \epsilon^{2/k}
\]
where \( N' \) now denotes the number of zeros of \( f, f', \ldots \) and \( f^{(k-1)} \) in the disc \( \mathbb{D}_{5/4} \).

Propositions 3.1 and 3.2 are proved using variants of Hensel’s lemma from elementary number theory and adapting them to the archimedean setting of the real \( \mathbb{R} \) or complex \( \mathbb{C} \) field. In nonarchimedean settings, the triangle inequality we used above to show \( |f(z_1) - f(z_0)| \leq 2\epsilon \) for points \( z_0, z_1 \in \{ |f| \leq \epsilon \} \) improves to \( |f(z_1) - f(z_0)| \leq \epsilon \) and this is key to run a Hensel-type argument which is an iterative scheme to find an actual zero of \( f \), starting with an approximate zero \( z \in \{ |f| \leq \epsilon \} \). Losing a factor of 2 at each step in the iteration results in an unacceptable exponential loss \( 2^n \) at the \( n \)th stage. However the example \( (e^{N(z+1)} - 1)/N \) tells us that we can only expect a good bound for small \( \epsilon \ll N \). The argument can then be adjusted to hide/move the factors of 2 into the smallness of \( \epsilon \).

Importantly, these Hensel-type arguments have the advantage of being local in nature; the nondegeneracy condition \( f' \neq 0 \) (or \( f^{(k)} \neq 0 \)) is only needed for points in the sublevel set. The iteration scheme remains essentially in the sublevel set. Hence the bound (8) in Proposition 3.2 can be improved to a measure bound of a local sublevel set:
\[
\{ z \in \mathbb{D} : |f^{(k)}(z)| \geq 1, \ |f(z)| \leq \epsilon \} \lesssim_{k,M} \epsilon^{2/k} \quad \text{when} \quad \epsilon \ll_{k,M} 1.
\]
In fact we can go further and deduce a structural statement about these local sublevel sets; they are contained in a union of \( \epsilon^{1/k} \) discs centred at the zeros of \( f \) and its derivatives.

This is the key to develop a theory and guide us to look for scale-invariant bounds. Not only are the arguments local but we can and should be bounding local quantities, whether they be sublevel sets or oscillatory integrals. Propositions 3.1 and 3.2 are not central to the development of our theory and so we provide proofs of these results in an appendix to this paper.

4. Moving from analytic functions to polynomials

The argument at the outset for bounding the diameter of the sublevel set \( \{ |f| \leq \epsilon \} \) under a global condition \( |f'| \geq 1 \) no longer applies to the the local sublevel set.
\{ z : |f^{(k)}(z)| \geq 1, |f(z)| \leq \epsilon \}, even in the real setting. If we consider the real function
\[ f(x) = \frac{2\sin(Nx)}{N} \text{ on } [0, 1], \tag{10} \]
we see that
\[ |\{ x \in [0, 1] : |f'(x)| \geq 1, |f(x)| \leq \epsilon \}| \sim \begin{cases} N\epsilon, & \text{if } \epsilon < N^{-1} \\ 1 & \text{if } N^{-1} \leq \epsilon \leq 1. \end{cases} \tag{11} \]
This is the real version of the example (9). Here the problem is that the function \( f \) and its derivative \( f' \) are conspiring to produce many zeros of \( f \) at places where the
\[ \text{derivative } f' \text{ is large. Hence, even in the real setting, there is no scale-invariant bound for local sublevel sets for general differentiable functions.} \]

In both the real and the complex case, the examples (5) and (10) are non-polynomial. If \( f \in \mathbb{R}[X] \) has degree at most \( d \), then the order structure of \( \mathbb{R} \) can once again be used to show that the local sublevel set
\[ S_{\text{loc}} = \{ x \in [0, 1] : |f^{(k)}(x)| \geq 1, |f(x)| \leq \epsilon \} \]
is a union \( \bigcup I \) of at most \( O_d(1) \) intervals and on each \( I \), we can apply the argument relying on the intermediate value theorem to obtain the scale-invariant bound
\[ |S_{\text{loc}}| \leq C_d \epsilon^{1/k}. \]

There is an additional, less obvious, feature of the Hensel-like argument establishing bounds such as (9). The smallness condition \( \epsilon \ll k, M \) 1 can be removed if our function \( f \) has the property that some derivative of bounded order has the global bound \( |f^{(n)}(z)| \gtrsim M^{n/(n+1)} \) from below. Polynomials have this property and in fact satisfy a stronger property – see Lemma 12.1 below.

As a consequence, we can establish the following bound.

**Proposition 4.1.** Let \( f \in \mathbb{C}[X] \) be a complex polynomial of degree \( d \). Then there is a constant \( C_d \), depending only on \( d \), such that
\[ |\{ z \in \mathbb{D} : |f^{(k)}(z)| \geq 1, |f(z)| \leq \epsilon \}| \leq C_d \epsilon^{2/k} \tag{12} \]
holds for any \( k \geq 1 \).

The bound (12) in Proposition 4.1 is scale-invariant. As an immediate consequence, we see that
\[ |\{ z \in \mathbb{D}_R : |f^{(k)}(z)| \geq \mu, |f(z)| \leq \epsilon \}| \leq C_d (\epsilon/\mu)^{2/k} \tag{13} \]
holds for any \( f \in \mathbb{C}[X] \) of degree at most \( d \), for any \( k \geq 1 \) and for any \( R, \mu, \epsilon > 0 \).

Here we use the notation \( \mathbb{D}_R(z_0) = \{ z \in \mathbb{C} : |z-z_0| \leq R \} \) to denote the disc of radius \( R \) with centre \( z_0 \). Also we denote \( \mathbb{D}_R(0) \) by \( \mathbb{D}_R \) and \( \mathbb{D} \) denotes the unit disc. When we move to higher dimensions, we will denote \( \mathbb{B}_R(z_0) = \{ z \in \mathbb{C}^n : |z-z_0| \leq R \} \) as the ball in \( \mathbb{C}^n \) with centre \( z_0 \) and radius \( R \) with similar conventions for balls centred at the origin and \( \mathbb{B}^n \) denotes the unit ball in \( \mathbb{C}^n \).

The local nature of the arguments allow us to extend the above result to polynomials of several variables. Furthermore we are able to treat local sublevel sets defined
by general linear partial differential operators $Lf(\z) = \sum_{1 \leq |\alpha| \leq k} c_\alpha(\z) \partial^\alpha f(\z)$ with general bounded measurable coefficients $\{c_\alpha(\z)\}$.

**Proposition 4.2.** Let $f \in \mathbb{C}[X_1, \ldots, X_n]$ be a complex polynomial in $n$ variables of degree $d$ and let $L$ be as above. Then there is a constant $C_{d,n,L}$, depending only on $d,n$ and the $L^\infty$ norms of the coefficients $\{c_\alpha(\z)\}$, such that

$$\left|\{z \in \mathbb{B}^n : |L f(z)| \geq \mu, |f(z)| \leq \epsilon\right| \leq C_{d,n,L} \epsilon^{k/2}$$

holds for any $\mu, \epsilon > 0$. If $L = \sum_{|\alpha| = k} c_\alpha(\z) \partial^\alpha$ is homogeneous of degree $k$, then

$$\left|\{z \in \mathbb{B}_R^n : |L f(z)| \geq \mu, |f(z)| \leq \epsilon\right| \leq C_{d,n,L} R^{2(n-1)} \epsilon^{k/2}$$

holds for any $R, \mu, \epsilon > 0$.

**Remarks:** The bounds (14) and (15) are scale-invariant. In particular to prove these bounds, by scaling $f$ by $\mu^{-1} f$, we can reduce to the case $\mu = 1$. Furthermore, by a change of variables $\z = \frac{R}{1}$, the proof of (15) can be reduced to the case $R = 1$ which is then subsumed by (14).

The polynomial $f(\z) = g(z_1)$ could depend only on one variable and so we see that the exponent $2/k$ is optimal (consider $g(z_1) = z_1^2$ and $L = \partial_1^2$). And for the same reason, we see why the factor $R^{2(n-1)}$ is present.

When we consider general partial differential operators, even in the real $\mathbb{R}$ setting, there are no scale-invariant bounds for general analytic functions. By Runge’s theorem, one can show that for every $\epsilon > 0$, there is an analytic real-valued function $f$ with $\Delta f \geq 1$ on $\mathbb{D}$ such that $\{|z \in \mathbb{D} : |f(z)| \leq \epsilon\} \geq 1$. Here $\Delta = \partial_1^2 + \partial_2^2$ is the Laplace operator.

## 5. The $H$ functional

In the study of oscillatory integrals for real-valued phases, there is a very useful functional which we now consider for complex functions $f$:

$$H_{f,R} := \inf_{z \in \mathbb{B}_R^n} H_f(z) \quad \text{where} \quad H_f(z) = \max_{|\alpha| \geq 1} \left( |\partial^\alpha f(z) / |\alpha|^{1/|\alpha|} \right).$$

The estimate in Proposition 4.2 can be extended in the following way.

**Proposition 5.1.** Let $n, d \in \mathbb{N}$ be given. There exists a constant $C_{d,n}$, depending only on $d$ and $n$, such that for any $f \in \mathbb{C}[X_1, \ldots, X_n]$ of degree $d$ and for any $R > 0$, we have

$$\sup_{a \in \mathbb{C}} \left|\{z \in \mathbb{B}_R^n : |f(z) - a| \leq 1\} \right| \leq C_{d,n} R^{2(n-1)} H_{f,R}^{-2}.$$  \hspace{1cm} (16)

There is a corresponding local version:

$$\sup_{a \in \mathbb{C}} \left|\{z \in \mathbb{B}_R^n : H_f(z) \geq H, |f(z) - a| \leq 1\} \right| \leq C_{d,n} R^{2(n-1)} H^{-2}$$

holds for any $H > 0$. 


The bounds (16) and (17) are not scale-invariant due to the nonlinear nature of the functional $H_f$. However the bound (17) implies the scale-invariant bounds (14) and (15). In fact, as remarked above, the bound (15) reduces to (14) and furthermore, we may assume $μ = 1$. Finally to prove (14) with $μ = 1$, we may also assume $ε ≤ 1$; otherwise, the trivial bound gives the desired result.

Setting $P = ε^{-1}f ∈ C[X_1, \ldots, X_n]$, we have

$$\{z ∈ B^N : |Lf(z)| ≥ 1, |f(z)| ≤ ε\} ⊂ \bigcup_{1 ≤ |α| ≤ k} \{z ∈ B^N : |α^n P(z)| ≥ (Aε)^{-1}, |P(z)| ≤ 1\}$$

where $A = C_{d,n} \max_n ||c_α||_∞$. For each $1 ≤ |α| ≤ k$, let $S_α$ denote the corresponding set on the right-hand side above. Note that

$$S_α ⊂ \{z ∈ B^N : H_P(z) ≥ (Aε)^{-1/|α|}, |P(z)| ≤ 1\}$$

and so (17) with $a = 0$ implies $|S_α| ≲_{d,n} \max(1, A^2) \epsilon^{2/|α|}$. But $\epsilon^{2/|α|} ≤ \epsilon^{2/k}$ since $ε ≤ 1$ and therefore

$$\left| \left\{ \left\{z ∈ B^N : |Lf(z)| ≥ 1, |f(z)| ≤ ε\right\}\right| \leq C \epsilon^{2/k}$$

where $C$ depends only on $d, n$ and the $L^∞$ norms of the coefficients $\{c_α\}$. Hence (14) holds.

The case $n = 1$. Consider the case of polynomials of a single variable $f ∈ C[X]$, say of degree $d$. Let $z_⋆ ∈ D_R$ be a point where $H_{f,R} = H_f(z_⋆)$. Then for $a = f(z_⋆)$, we have $|f(z) - a| ≤ 1$ whenever $|z - z_⋆| ≤ (4H_{f,R})^{-1}$. If fact $|f(z) - a| =

$$|f(z) - f(z_⋆)| = \left| \sum_{k=1}^{d} f^{(k)}(z_⋆) / k! (z - z_⋆)^k \right| ≤ \sum_{k=1}^{d} (2H_f)^k (4H_f)^{-k} ≤ \sum_{k=1}^{∞} 2^{-k} = 1.$$\

Hence if $Λ_R(f) := \min(R, H_{f,R}^{-1})$, then

$$Λ_R(f)^2 ≤ |D_R| \sup_{a ∈ C} |\{z ∈ D_R : |f(z) - a| ≤ 1\}| ≲_d Λ_R(f)^2,$$

illustrating the usefulness of the functional $H_f$ for sublevel set bounds. In particular, we have

$$\min(R, H_{f,R}^{-1})^2 \sim_d \sup_{a ∈ C} |\{z ∈ D_R : |f(z) - a| ≤ 1\}| \quad (18)$$

and we can apply Proposition 3.3 from [9] (see also [12]) in the complex field setting which states the following: if $f(z) = a \prod_{j=1}^{m} (z - w_j)^{r_j} ∈ C[X]$, then

$$\bigcup_{j=1}^{m} \left[ B_{2^{r_j}}(w_j) \cap D_R \right] ≤ \left\{ z ∈ D_R : |f(z)| ≤ 1 \right\} \subseteq \bigcup_{j=1}^{m} \left[ B_{2^{r_j}}(w_j) \cap D_R \right]$$

where

$$r_j = \min_{C \ni w_j} \left[ \left| a \prod_{k \notin C} (w_j - w_k)^{r_k} \right|^{1/S(C)} \right].$$

Here the minimum is taken over all root clusters $C ⊂ \{w_1, \ldots, w_m\}$ containing $w_j$ and $S(C) = \sum_{k \notin C} e_k$. Hence

$$\max_{j ∈ J} \left( R_j \inf_{w_j ∈ C} \left[ \left| a \prod_{k \notin C} (w_j - w_k)^{r_k} \right|^{1/S(C)} \right] \right)^{1/S(C)} ≲_d \min(R, H_{f,R}^{-1})^2$$

and

$$\sum_{j=1}^{m} \max_{w_j ∈ C} \left( R_j \inf_{w_j ∈ C} \left[ \left| a \prod_{k \notin C} (w_j - w_k)^{r_k} \right|^{1/S(C)} \right] \right)^{1/S(C)} \leq \min(R, H_{f,R}^{-1})^2.$$
holds where \( J = \{1 \leq j \leq m : B_{2^{-d_j}}(w_j) \cap \mathbb{D}_R \neq \emptyset \} \).

Momentarily we will see a similar but stronger relationship between \( H_f \) and the roots of the derivative \( f' \) of our polynomial.

6. Oscillatory integrals with complex-valued phases

In the Introduction we introduced oscillatory integrals

\[
I_\phi(f) = \int_{\mathbb{C}^n} e^{i(f(\zeta))} \phi(\zeta) d\zeta
\]

with complex phases \( f : \mathbb{C}^n \to \mathbb{C} \). Here \( \phi \in C_\infty^\infty(\mathbb{C}^n) \) is a smooth cut-off function.

The integrals \( I_\phi(f) \) are connected to complex sublevel sets in the same way that real oscillatory integrals are connected to real sublevel sets. Consider the sublevel set

\[
S_{a, \epsilon}^R = \{ \zeta \in \mathbb{B}_R^n : |f(\zeta) - a| \leq \epsilon \}
\]

and nonnegative functions \( \phi_R \in C_\infty^\infty(\mathbb{C}^n) \) such that \( \phi_R \equiv 1 \) on \( \mathbb{B}_R^n \) and \( \psi \in C_\infty^\infty(\mathbb{C}) \) such that \( \psi \equiv 1 \) on \( \mathbb{D} \). Then

\[
|S_{a, \epsilon}^R| \leq \int_{\mathbb{C}^n} \psi((f(\zeta) - a)/\epsilon) \phi_R(\zeta) d\zeta = \int_{\mathbb{C}} \hat{\psi}(w)e(-aw/\epsilon) \left[ \int_{\mathbb{C}^n} e(wf(\zeta)/\epsilon) \phi_R(\zeta) d\zeta \right] dw
\]

where \( \hat{\psi}(w_1 + iw_2) = 2\hat{\psi}(u, v) \) and \( w_1 = (u + v)/2 \) and \( w_2 = (u - v)/2 \). In fact if we set \( u = |\text{Re}f|/\epsilon \) and \( v = |\text{Im}f|/\epsilon \), then by the Fourier inversion formula,

\[
\psi(u, v) = \int_{\mathbb{R}^2} \hat{\psi}(u, v)e^{2\pi i (uw + vw)} du dv = 2 \int_{\mathbb{C}} \hat{\psi}(w_1 + iw_2)e((w_1 + iw_2)f(\zeta)) dw.
\]

Therefore we have

\[
|S_{a, \epsilon}^R| \leq \int_{\mathbb{C}} |\hat{\psi}(w)| |I_{\phi_R}(wf/\epsilon)| dw \quad (19)
\]

where \( \hat{\psi} \) is a Schwartz function on \( \mathbb{C} \). Hence bounds for \( I_{\phi_R}(wf) \) give bounds for \( S_{a, \epsilon}^R \).

When \( n = 1 \), suppose we have a general complex differentiable function \( f \) on the disc \( \mathbb{D}_2 \) which satisfies \( |f''(z)| \geq 1 \) on \( \mathbb{D}_2 \). Then a scale-invariant bound of the form

\[
|I_\phi(\lambda f)| \leq C_\phi |\lambda|^{-1}
\]

where \( \phi \in C_\infty^\infty(\mathbb{D}_2) \) cannot hold with a constant \( C_\phi \) only depending on \( \phi \). If such a scale-invariant bound were true, then (19) would imply the scale-invariant bound

\[
|S_{a, \epsilon}^1| \leq C_\phi \int_{\mathbb{C}} \min(1, \epsilon/|w|) |\hat{\psi}(w)| dw \lesssim_\phi \epsilon
\]

for sublevel sets which we have observed is impossible.

We can relate derivatives of the real phase \( \Phi(z) = \text{Re}f(z) + \text{Im}f(z) \) defining \( I_\phi(f) \) to complex derivatives of \( f \) when \( n = 1 \). By the Cauchy-Riemann equations, we have \( ||\nabla \Phi(z)|| = \sqrt{2} |f'(z)| \) where \( || \cdot || \) is the Euclidean norm on \( \mathbb{R}^2 \). Furthermore we have \( |\det(\text{Hess} \Phi(z))| = 4 |f''(z)|^2 \).
Corollary 1.2. Let $\phi$ be a normalised bump function. As a consequence of Theorem 1.1 we have the following scale-invariant bound. For any $f \in \mathbb{C}[X_1, \ldots, X_n]$ of degree $d$ and for any $\phi \in C_0^\infty(\mathbb{C}^n)$, we have

$$|I_\phi(f)| \leq C H_{f,\phi}^{-2}$$

where $C = C_{d,n,\phi}$ only depends on $d, n$ and $\phi$.

Remark: If our smooth cut-off function is of the form $\phi_R(\underline{z}) = \varphi(R^{-1} \underline{z})$ for some normalised bump function $\varphi$ (say $\text{supp}(\varphi) \subseteq \mathbb{B}^n$ or $\varphi \equiv 1$ on $\mathbb{B}^n$ and $\text{supp}(\varphi) \subseteq \mathbb{B}^n_2$), then a change of variables shows

$$I_{\phi_R}(f) = R^{2n} I_{\varphi}(Q_R)$$

where $Q(w) = f(Rw)$ and since $H_{Q,1} = RH_{f,R}$, we see that (20) implies

$$|I_{\phi_R}(f)| \leq C_{d,n,\varphi} R^{2(n-1)} \text{min}(R, H_{f,R}^{-1})^2.$$  

(21)

As a consequence of Theorem 1.1 we have the following scale-invariant bound.

**Corollary 1.2.** Let $P \in \mathbb{C}[X_1, \ldots, X_n]$ have degree $d$ and suppose $|\partial^\alpha P(\underline{z})/\alpha!| \geq 1$ for $\underline{z} \in \text{supp}(\phi)$. Then for $\lambda \in \mathbb{C}$,

$$|I_\phi(\lambda P)| \leq C_{d,n,\phi} |\lambda|^{-2/|\alpha|}.$$  

(22)

**Proof:** We apply Theorem 1.1 to the polynomial $f(\underline{z}) = \lambda P(\underline{z})$. Note that for all $\underline{z} \in \text{supp}(\phi)$, we have $|\partial^\alpha f(\underline{z})/\alpha!| = |\lambda| |\partial^\alpha P(\underline{z})/\alpha!| \geq |\lambda|$ and so

$$H_{f(\underline{z})/\alpha!} \geq |\partial^\alpha f(\underline{z})/\alpha!| \geq |\lambda|,$$

implying $H_{f,\phi} \geq |\lambda|^{1/|\alpha|}$. Hence the bound (20) implies (22). \qed

Now consider a polynomial $f \in \mathbb{C}[X]$ in one variable and fix $R > 0$. Putting (21) together with (18) and (19), we have the following observations: let $\phi_R(z) = \varphi(R^{-1}z)$ where $\varphi \in C_0^\infty(\mathbb{D})$ such that $\varphi \equiv 1$ on $\mathbb{D}_1/2$. Then

$$|I_{\phi_R}(f)| \lesssim_{d,\varphi} \text{min}(R, H_{f,R}^{-1})^2 \lesssim \sup_{a \in \mathbb{C}} \{ |\{ z \in \mathbb{D} : |f(z) - a| \leq 1 \} | \}$$

and so

$$|I_{\phi_R}(f)| \lesssim_{d,\varphi} \int_{\mathbb{C}} |\psi(w)||I_{\varphi_R}(wf)| \, dw$$

where $\psi$ is some fixed Schwartz function on $\mathbb{C}$ (whose fourier transform is nonnegative and larger than 1 on $\mathbb{D}$).

So here we have succeeded in controlling oscillatory integrals with a general polynomial phase (in one variable) by the measure of sublevel sets. This reverses the
usual relationship and one can deduce oscillatory integral bounds from sublevel set bounds. In particular we can bound individual oscillatory integrals by an average of oscillatory integrals.

7. Sharpness of (20) in Theorem 1.1 when \( n = 1 \)

The real analogue of the bound (20) can be found in the book [1] where it is shown that

\[
\left| \int_{[0,1]^n} e^{2\pi i f(x)} \, dx \right| \leq C_d,n H_f^{-1} \tag{23}
\]

holds for any \( f \in \mathbb{R}[X_1, \ldots, X_n] \) of degree at most \( d \). Here \( H_f = H_{f,1} \). The bound (24) is proved by applying the classical van der Corput estimates, together with many applications of the intermediate value theorem.

When \( n = 1 \), it is shown in [1] that given any \( f \in \mathbb{R}[X] \), there is a \( c = c(f) \in [0,1] \) such that

\[
\left| \int_a^b e^{2\pi i f(x)} \, dx \right| \sim_d H_f^{-1} \tag{24}
\]

whenever \( H_f \geq 1 \). The proof of (24) relies heavily on the order structure of the reals \( \mathbb{R} \) and we do not know how to prove an analogous statement with truncations for oscillatory integrals \( I_\phi(f) \) with complex-valued phases. However in this section, an alternative to (24) will be proposed which unfortunately will have limited use.

The asymptotic bound (24) for some \( c \in [0,1] \) shows the sharpness of the bound (23) and it can be used to compare the bound (23) to other known bounds which are robust under truncations of oscillatory integrals.

For instance, if the derivative \( f'(x) = a \prod_{j=1}^m (x - z_j)^{e_j} \) has distinct roots \( \{z_j\} \) with \( \sum_j e_j = d - 1 \) where \( d = \deg(f) \), then a sharp bound due to Phong and Stein [11] is the following: for any \( a < b \),

\[
\left| \int_a^b e^{2\pi i f(x)} \, dx \right| \leq C_d \max_{1 \leq k \leq m} \min_{C \subseteq \{z_k\}} \left[ \frac{1}{a \prod_{j \in C} (z_k - z_j)^{c_j}} \right]^{1/(S(C)+1)} \tag{25}
\]

where the minimum is taken over all root clusters \( C \subseteq \{z_j\} \) containing \( z_k \) and \( S(C) := \sum_{z_j \in C} e_j \). The constant \( C_d \) only depends on the degree of \( f \) and can be taken to be independent of \( a \) and \( b \). Therefore by (24), the bound (25) implies

\[
\min_k \max_{C \subseteq \{z_k\}} \left[ |a \prod_{z_j \in C} (z_k - z_j)^{c_j} \right]^{1/(S(C)+1)} \leq C_d H_f. \tag{26}
\]

In Section 14 we will give a direct proof of (26) which will also hold for complex polynomials \( f \in \mathbb{C}[X] \). As a consequence, Theorem 1.1 will imply a complex version of the Phong-Stein bound as stated in Proposition 1.3.

When \( n = 1 \), we saw that \( \sup_{a \in \mathbb{C}} |S_{a,R}^{R_1}| \sim_d \min(R, H^{-1}_{f,R}) \) and so the functional \( H_{f,R} \) is precisely the right quantity which controls the measure of sublevel sets.
where \( S_{n,1} \). Here we will see the usefulness of a variant of \( H_{f,\phi} \):

\[
J_{f,\phi} = \inf_{\mathbf{z} \in \text{supp}(\phi)} J_{f}(\mathbf{z}) \quad \text{where} \quad J_{f}(\mathbf{z}) = \max_{|\alpha| \geq 2} (|\partial^\alpha f(\mathbf{z})/\alpha!|^{1/|\alpha|}).
\]

For sharp cut-offs, we write \( J_{f,R} = \inf_{\mathbf{z} \in B_R} J_{f}(\mathbf{z}) \). We have \( J_{f,\phi} \leq H_{f,\phi} \) and so Theorem 1.1 implies

\[
|I_\phi(f)| \leq C_{d,n,\phi} J_{f,\phi}^{-2}
\]

which is a bound that does not depend on the linear coefficients of \( f \) and so gives bounds which are robust under linear perturbations of the phase. This is useful in many problems.

Furthermore if our cut-off is of the form \( \phi_R(z) = \varphi(R^{-1}z) \) for some normalised bump function \( \varphi \), then as in (21), we have

\[
|I_{\phi_R}(f)| \leq C_{d,n,\varphi} R^{2(n-1)} \min(R, J_{f,R}^{-1})^2.
\]  

For \( f \in C[X_1, \ldots, X_n] \), we write \( f_{a,b}(\mathbf{z}) = f(\mathbf{z}) - a - b \cdot z \) where \( a \in \mathbb{C} \) and \( b \in \mathbb{C}^n \). We note that \( J_{f_{a,b}} = J_{f,R} \) and since \( J_{f,R} \leq H_{f,R} \), we see that (10) in Proposition 5.1 implies

\[
\sup_{a,b} \left| \{z \in B_R : |f(\mathbf{z}) - a - b \cdot z| \leq 1 \} \right| \leq C_{d,n} R^{2(n-1)} \min(R, J_{f,R}^{-1})^2.
\]  

Consider the case \( n = 1 \) and \( f \in C[X] \). Let \( z_* \in D_R \) be a point where \( J_{f,R} = J_{f,R}(z_*) \). Then for \( a = f(z_*) - f'(z_*) z_* \) and \( b = f'(z_*) \), we have \( |f(z) - a - bz| \leq 1 \) whenever \( |z - z_*| \leq (4J_{f,R})^{-1} \). If fact

\[
|f(z) - f(z_*) - f'(z_*) (z - z_*)| = \left| \sum_{k=2}^{d} \frac{f^{(k)}(z_*) k! (z - z_*)^k}{k!} \right| \leq \sum_{k=2}^{d} (2J_{f,R})^k (4J_{f,R})^{-k}
\]

which is at most \( 1/2 \). Hence by (29), if \( \Omega_R(f) := \min(R, J_{f,R}^{-1}) \),

\[
\Omega_R(f)^2 \lesssim |D(4J_{f,R})^{-1}(z_*) \cap D_R| \leq \sup_{a,b \in \mathbb{C}} |\{z \in D_R : |f(z) - a - bz| \leq 1\}| \lesssim_d \Omega_R(f)^2
\]

and so

\[
\Omega_R(f)^2 \sim_d \sup_{a,b \in \mathbb{C}} |\{z \in D_R : |f(z) - a - bz| \leq 1\}|.
\]  

This illustrates the usefulness of the \( J \) functional for sublevel set bounds.

Fix \( \varphi \in C_c^\infty(\mathbb{D}) \) such that \( \varphi \equiv 1 \) on \( D_{1/2} \) and set \( \phi_R(z) = \varphi(R^{-1}z) \). Arguing as in (19), we have

\[
\sup_{a,b} \left| \{z \in D_R : |f(z) - a - bz| \leq 1\} \right| \leq \sup_{b \in \mathbb{C}} \int_{\mathbb{C}} \sim_{w} |\tilde{\psi}(w)| |I_{\phi_{2R}}(w f_b)| \, dw
\]

where \( f_b(z) = f(z) - bz \).

Now set

\[
\alpha_{f,R,\varphi} = \sup_{w,b} \left[ \Omega_{2R}(w f)^{-2} |I_{\phi_{2R}}(w f_b)| \right].
\]
From [25], we have $\alpha_{f,R,\varphi} \leq C_{d,\varphi}$ and we seek a lower bound. Importantly for us, the $J$ functional scales like $J_w^{-2} \leq |w|^{-1} J_w^{-2}$ for $w \in \mathbb{D}$ and $|w|^{-1}$ is integrable on $\mathbb{D}$. We note that $J_{w,f}\geq J_{w,f,R} \geq \min(|w|^{1/2},|w|^{1/d})$ $J_{f,2R}$ and so

$$\sup_{b \in \mathbb{C}} \int_\mathbb{C} |\tilde{\psi}(w)||I_{\varphi_{2R}}(w,f_b)| dw \leq \alpha_{f,R,\varphi} \Omega_{2R}(f)^2 \int_\mathbb{C} |\tilde{\psi}(w)| \frac{1}{\min(|w|,|w|^{2/d})} dw,$$

which implies, by (30) and (31),

$$\Omega_R(f)^2 \lesssim_d \int_\mathbb{C} |\tilde{\psi}(w)||I_{\varphi_{2R}}(w,f_b)| dw \lesssim_d \alpha_{f,R,\varphi} \Omega_{2R}(f)^2$$

and so

$$\left[ \min(1,J_{Q,1}^{-1}) \right]^2 \lesssim_\varphi \left[ \min(1,J_{Q,2}^{-1}) \right]^2 \lesssim_d \alpha_{f,R,\varphi}$$

where $Q(w) = f(Rw)$. Here we used the observation $J_{Q,1} = RJ_{f,R}$.

As a consequence, we see that $c_d \leq \alpha_{f,R,\varphi} \leq C_{d,\varphi}$ where

$$c_d := \inf_{Q \in \mathcal{P}_d} \left[ \min(1,J_{Q,1}^{-1}) \right]^2$$

and $\mathcal{P}_d$ is the space of all polynomials $Q \in \mathbb{C}[X]$ of degree at most $d$. Hence for any $f \in \mathbb{C}[X]$ of degree at most $d$, there exist $w,b \in \mathbb{C}$ such that

$$c_d \min(R,J_{w,f,R}^{-1})^2 \leq \int_\mathbb{C} e(|f(z) - bz|) \phi(R^{-1}z) dz \leq C_{d,\varphi} \min(R,J_{w,f,R}^{-1})^2.$$

This is our alternative to (24).

Unfortunately this nice general lower bound for oscillatory integrals is not very useful since the constant $c_d$ turns out to be zero whenever $d \geq 3!$ For $f(z) = a + bz + cz^2$, we have $J_{f,R} = |c|^{1/2}$ for any $R$ and so $c_2 = 1$. To see $c_d = 0$ when $d \geq 3$, consider $f(z) = a(z - 3/2)^d$ and note that $J_{Q,2} \leq |a|^{1/d}$ but $\max_{k \geq 2} |a|^{1/k} \lesssim J_{Q,1}$. Hence $\min(1,J_{Q,1})/\min(1,J_{Q,2}) \lesssim |a|^{-(d-2)/2d}$ as $|a| \to \infty$.

However there are certain subspaces of polynomials of degree at most $d$ where a uniform lower bound is possible. For example when $d = 4$, the subspace $\mathcal{P} := \{a + bz + cz^3 + dz^4\}$ of quartics with no quadratic term has the property that

$$J_{Q,1} \sim J_{Q,2} \sim \max(|c|^{1/3},|d|^{1/4})$$

for all $Q \in \mathcal{P}$. Hence

$$\inf_{Q \in \mathcal{P}} \frac{\min(1,J_{Q,1}^{-1})}{\min(1,J_{Q,2}^{-1})} \sim 1$$

and so the above boxed statement can be applied to those $f$ in the subspace $\mathcal{P}$.

**Proofs of Proposition 5.1 and Theorem 1.1**

We now give the details of the proofs of the main sublevel set bound, Proposition 5.1, and the main oscillatory integral bound, Theorem 1.1.
Let $C[X_1, \ldots, X_n]$ denote the space of complex polynomials in $n$ variables. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we set

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

so that a polynomial $P \in \mathbb{C}[X_1, \ldots, X_n]$ of degree at most $d$ can be written as

$$P(z) = \sum_{|\alpha| \leq d} c_\alpha z^\alpha.$$

Here $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

For a multi-index $\alpha \in \mathbb{N}^n$ and $P \in \mathbb{C}[X_1, \ldots, X_n]$, we set

$$\partial^\alpha P := \frac{\partial^{|\alpha|} P}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}$$

and $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$.

Let $V_{n,k}$ denote the complex vector space of homogeneous polynomials in $\mathbb{C}[X_1, \ldots, X_n]$ of degree $k$. So $Q \in V_{n,k}$ means $Q(z) = \sum_{|\alpha| = k} b_\alpha z^\alpha$. The hermitian inner product

$$\langle P, Q \rangle := \sum_{|\alpha| = k} \alpha! a_\alpha \overline{b_\alpha}$$

gives $V_{n,k}$ a Hilbert space structure. If $Q(\partial) = \sum_{|\alpha| = k} b_\alpha \partial^\alpha$ denotes the corresponding differential operator, then note that

$$Q(\partial)P(z) \equiv \langle P, Q \rangle$$

where $Q(z) = \sum_{|\alpha| = k} b_\alpha z^\alpha$.

We need the following well-known fact (see for example, [14]).

**Lemma 8.1.** Let $d(n,k)$ denote the dimension of the Hilbert space $V_{n,k}$. There exists a sequence of unit vectors $u_j, j = 1, \ldots, d(n,k)$ in $\mathbb{C}^n$ such that

$$Q_j(z) := (u_j \cdot z)^k, \quad j = 1, \ldots, d(n,k)$$

does a basis for $V_{n,k}$.

**Proof.** It suffices to show that $\text{Span}\{ (u \cdot z)^k : u \in \mathbb{C}^n \} = V_{n,k}$. Suppose not. Then since

$$V_{n,k} = M \oplus M^\perp$$

where $M = \text{Span}\{ (u \cdot z)^k : u \in \mathbb{C}^n \}$, we can find a non-zero $P \in M^\perp$. In particular, we have $\langle P, Q_u \rangle = 0$ for all $u \in \mathbb{C}^n$ where $Q_u(z) = (u \cdot z)^k$. Hence

$$Q_u(\partial)P(z) = (u \cdot \nabla)^k P(z) \equiv 0 \quad \text{for all} \quad u \in \mathbb{C}^n.$$

Note that

$$f^{(k)}(0) = (u \cdot \nabla)^k P(0) = 0 \quad \text{where} \quad f(z) = P(zu) = \left[ \sum_{|\alpha| = k} a_\alpha u^\alpha \right] z^k,$$

implying $P(u) = 0$ for all $u \in \mathbb{C}^n$. Hence we arrive at the contradiction $P = 0$. □
As a consequence of Lemma 8.1, we see that for every \( \alpha \in \mathbb{N}^n \) with \( |\alpha| = k \),
\[
 z^\alpha = d(n,k) \sum_{j=1}^{d(n,k)} c_j (u_j \cdot z)^k
\]
for some choice of coefficients \( c_j = c_j(\alpha) \in \mathbb{C} \). Hence we can write
\[
 \partial^\alpha = d(n,k) \sum_{j=1}^{d(n,k)} c_j (u_j \cdot \nabla)^k
\]
for every \( \alpha \) with \( |\alpha| = k \).

9. A STRUCTURAL SUBLEVEL SET STATEMENT

Here we state a key sublevel set bound central to the proofs of Proposition 5.1 and Theorem 1.1.

**Proposition 9.1.** Let \( Q \in \mathbb{C}[X] \) have degree \( d \) and \( z_0 \in \mathbb{D} \). Suppose \( |Q^{(k)}(z_0)| \geq 1 \) for some \( k \geq 1 \) and \( |Q(z_0)| \leq \epsilon \) for some \( 0 < \epsilon \leq \epsilon_d \) where \( \epsilon_d \) is a sufficiently small positive constant, depending only on the degree of \( Q \). Then there exists a zero \( z_* \) of \( Q^{(j)} \) for some \( 0 \leq j \leq d \) such that \( |z_0 - z_*| \lesssim d\epsilon^{1/k} \).

Proposition 9.1 has the following consequence.

**Corollary 9.2.** Let \( P \in \mathbb{C}[X] \) be a polynomials of degree \( d \) and set
\[
 Z := \{ z \in \mathbb{C} : P^{(j)}(z) = 0 \text{ for some } 0 \leq j \leq d \}.
\]
Then there is a \( C_d > 0 \), depending only on \( d \), such that
\[
 \{ z \in \mathbb{D}_R : |P(z)| \leq \epsilon, |P^{(k)}(z)| \geq \mu \} \subseteq \bigcup_{z_* \in Z} \mathbb{D} C_d (\epsilon/\mu)^{1/k} (z_*) \tag{33}
\]
holds for any \( R, \epsilon, \mu > 0 \).

Hence there is a constant \( C_d \) such that
\[
 \left| \{ z \in \mathbb{D}_R : |P(z)| \leq \epsilon, |P^{(k)}(z)| \geq \mu \} \right| \leq C_d (\epsilon/\mu)^{2/k} \tag{34}
\]
holds for all \( R, \epsilon, \mu > 0 \). This is the bound (13) stated after Proposition 4.1.

**Proof.** We may suppose the local sublevel set
\[
 S_R := \{ z \in \mathbb{D}_R : |P(z)| \leq \epsilon, |P^{(k)}(z)| \geq \mu \}
\]
is nonempty. We will make a reduction to the case \( \epsilon/(\mu R^k) \leq \epsilon_d \) where \( \epsilon_d \) is the sufficiently small constant appearing in Proposition 9.1. This will allow us to invoke Proposition 9.1. We note that to prove (34), this reduction is immediate since the trivial bound \( |S_R| \leq \pi R^2 \) implies (33) when \( \epsilon/(\mu R^k) \geq \epsilon_d \).
First consider the case $\epsilon/(\mu R^k) \geq \epsilon_d$. In this case, we claim that there is a zero $z_\ast \in \mathcal{Z}$ such that $|z_\ast| \leq B_d(\epsilon/\mu)^{1/k}$ for some sufficiently large $B_d$. If this is true, then for any $z \in S_R$,

$$|z - z_\ast| \leq R + B_d(\epsilon/\mu)^{1/k} \leq |1/\epsilon_d + B_d|(\epsilon/\mu)^{1/k}$$

and so $z \in \mathbb{D}_{C_d(\epsilon/\mu)^{1/k}}(z_\ast)$ for $C_d = B_d + \epsilon_d^{-1}$, establishing (33).

To prove the claim, we argue by contradiction and suppose that $|z_\ast| \geq B_d(\epsilon/\mu)^{1/k}$ for all $z_\ast \in \mathcal{Z}$. Factor

$$P(z) = a_dz^d + \cdots + a_0 = a_d(z - z_1) \cdots (z - z_d)$$

into linear factors and order the roots $|z_d| \leq |z_{d-1}| \leq \cdots \leq |z_1|$. If $B_d$ is large enough, we have $|z - z_\ast| \leq |z_\ast|$ for all $z \in \mathbb{D}_R$ and $z_\ast \in \mathcal{Z}$. Hence for any $z \in \mathbb{D}_R$, $|P(z)| \sim_d |a_d z_1 z_2 \cdots z_d|$ and in particular

$$|a_d z_1 \cdots z_d| \lesssim_d \epsilon$$

(35)

since we are assuming $S_R$ is nonempty. Next note that

$$P^{(k)}(z)/k! = a_k + (k + 1)a_{k+1}z + \cdots + \binom{d}{k} a_d z^{d-k} = c_k a_d (z - \eta_1) \cdots (z - \eta_{d-k})$$

and so for all $z \in \mathbb{D}_R$, $|P^{(k)}(z)| \sim_d |a_d \eta_1 \cdots \eta_{d-k}| = |a_k|$. But $a_k = \pm a_d s_{d-k}(z_1, \ldots, z_d)$ where $s_j$ is the $j$th elementary symmetric polynomial in $d$ variables. Hence

$$|s_{d-k}(z_1, \ldots, z_d)| \lesssim_d |z_{d-k} \cdots z_1|$$

by our ordering of the roots. Hence

$$\mu \leq |P^{(k)}(z)| \lesssim_d |a_d z_{d-k} \cdots z_1|$$

(36)

since we are assuming $S_R$ is nonempty. Putting (33) and (36) together, we have

$$\mu |a_d z_1 \cdots z_d| \lesssim_d \mu \epsilon \leq \epsilon |a_d z_{d-k} \cdots z_1|$$

and so $|z_{d-k} \cdots z_{d+k+1}| \lesssim_d \epsilon/\mu$. But since $|z_j| \geq B_d(\epsilon/\mu)^{1/k}$ for every $j$, we see $B_d(\epsilon/\mu) \lesssim_d \epsilon/\mu$ which is impossible if $B_d$ is chosen large enough.

It remains to treat the case $\epsilon/(\mu R^k) \leq \epsilon_d$. Write any $z \in \mathbb{D}_R$ as $z = Rw$ where $w \in \mathbb{D}$ and consider the polynomial $Q(w) = (\mu R^k)^{-1}P(Rw)$. Fix any $z \in S_R$ so that $|Q(w)| \leq \epsilon/(\mu R^k)$ and $|Q^{(k)}(w)| \geq 1$. Since $\epsilon' = \epsilon/(\mu R^k) \leq \epsilon_d$, we can apply Proposition 9.2 to conclude there is a zero $w_\ast$ of $Q^{(j)}$ for some $0 \leq j \leq d$ (so that $z_\ast = Rw_\ast \in \mathcal{Z}$) such that $|w - w_\ast| \lesssim_d \epsilon^{1/k} = R^{-1}(\epsilon/\mu)^{1/k}$. Hence $|z - z_\ast| = R|w - w_\ast| \lesssim_d (\epsilon/\mu)^{1/k}$, establishing (33). □

Proposition 9.2 is a consequence of the following higher order Hensel lemma which in turn is an extension of Proposition 2.1 in [17] from the nonarchimedean setting.

**Lemma 9.3.** Let $\phi \in \mathcal{H}(\mathbb{D}_2)$ with $M = M_\phi := \max_{z \in \mathbb{D}_2} |\phi(z)|$. Fix $L \geq 1$. Suppose $z_0 \in \mathbb{D}$ is a point where $\phi^{(k)}(z_0) \neq 0$ for each $1 \leq k \leq L$. Set $\delta := |\phi(z_0)\phi'(z_0)^{-1}\phi^{(L)}(z_0)^{-1}|$ and for $1 \leq k \leq L - 1$, set $\delta_k := |\phi^{(k+1)}(z_0)\phi^{(k)}(z_0)^{-1}\phi(z_0)\phi'(z_0)^{-1}|$. 


Suppose $\delta_k \lesssim_L 1$, $2 \leq k \leq L - 1$ and suppose $|\phi(z_0)\phi'(z_0)^{-1}| \leq 1/8$. Then if $\delta M, \delta_1 \ll_L 1$, there is a $z \in \mathbb{D}_{5/4}$ such that

(a) $\phi(z) = 0$ and (b) $|z - z_0| \leq 2|\phi(z_0)\phi'(z_0)^{-1}|$.

Remark: When $L = 1$, there are no $\delta_k$ for $1 \leq k \leq L - 1$ and the conditions reduce to the single condition $M \delta = M|\phi(z_0)\phi'(z_0)^{-2}| \leq 1/64$.

We postpone the proofs of Proposition 9.1 and Lemma 9.3 until Section 12 and Section 13 respectively.

10. Proof of Proposition 5.1

Here we show how Proposition 5.1 follows from Corollary 9.2.

Let $P \in \mathbb{C}[X_1, \ldots, X_n]$ be a polynomial of degree at most $d$. Our aim is to establish the bound (17) from Proposition 5.1. A simple scaling argument (just make the change of variables $\tilde{z} = Rz$) shows that we may assume $R = 1$. We reproduce the statement of (17) when $R = 1$ for the convenience of the reader: for any $a \in \mathbb{C}$,

$$|\{z \in \mathbb{B}^n : |H_P(z)| \geq H, |P(z) - a| \leq 1\}| \leq C_{d,n} H^{-2}.$$ 

The bound (17) will follow from the bound (34) in Corollary 9.2.

Note that

$$S(H) := \{z \in \mathbb{B}^n : H_P(z) \geq H, |P(z) - a| \leq 1\} \subseteq \bigcup_{\alpha} S_{\alpha}(H)$$

where

$$S_{\alpha}(H) = \{z \in S(H) : H_P(z) = |\partial^\alpha P(z)/\alpha!|^{1/|\alpha|}\}.$$ 

For $\alpha$ with $|\alpha| = k$, Lemma 5.1 implies there exists $(a_1, \ldots, a_{d(n,k)}) \in \mathbb{C}^{d(n,k)}$ such that

$$z^\alpha = a_1(z_1 \cdot \tilde{z})^k + \cdots + a_{d(n,k)}(\tilde{z}^{d(n,k)} \cdot \tilde{z})^k$$

or

$$\partial^\alpha = a_1(\tilde{u}_1 \cdot \nabla)^k + \cdots + a_{d(n,k)}(\tilde{u}_{d(n,k)} \cdot \nabla)^k.$$ 

Hence there exists $c_{d,n} > 0$ such that whenever $z \in \mathbb{C}^n$ satisfies $|\partial^\alpha P(z)| \geq H^k$, there exists a $j = j(z)$ such that $|(\tilde{u}_j \cdot \nabla)^k P(z)| \geq c_{d,n} H^k$. Therefore for each $\alpha$ with $|\alpha| = k$,

$$S_{\alpha}(H) \subseteq \bigcup_{j=1}^{d(n,k)} \{z \in S(H) : |(\tilde{u}_j \cdot \nabla)^k P(z)| \geq c_{d,n} H^k\} =: \bigcup_{j=1}^{d(n,k)} S_{\alpha,j}(H).$$

It suffices to bound each $S_{\alpha,j}(H)$. Let $R_j : \mathbb{C}^n \to \mathbb{C}^n$ be an orthogonal transformation such that $R_j \tilde{u}_j = (1, 0, \ldots, 0)$. We make the change of variables $w = R_j z$ so that

$$|S_{\alpha,j}(H)| \leq |\{w \in \mathbb{B}^n : |\partial^k Q_j(w)| \geq c_{d,n} H^k, |Q_j(w)| \leq 1\}|$$
where \( Q_j = P_j \circ R_j^{-1} - a \) is still a polynomial of degree at most \( d \). Now fix \( w' \in B^{n-1} \) and define \( Q(w) = Q_j(w, w') \), a complex polynomial in one variable of degree at most \( d \). Consider the slice

\[
S^w_{\alpha,j}(H) := \left\{ w \in \mathbb{D} : |Q^{(k)}(w)| \geq c_{d,n}H^k, |Q(w)| \leq 1 \right\}
\]

so that

\[
|S_{\alpha,j}(H)| \leq \int_{B^{n-1}} |S^w_{\alpha,j}(H)| dw' \leq \int_{B^{n-1}} \left| \left\{ w \in \mathbb{D} : |f^{(k)}(w)| \geq 1, |f(w)| \leq \delta \right\} \right| dw'
\]

where \( f(w) := Q(w)/(c_{d,n}H^k) \in \mathcal{P}_d \) and \( \delta = (c_{d,n}H^k)^{-1} \). For each fixed \( w' \in B^{n-1} \), we apply (34) to conclude

\[
|S_{\alpha,j}(H)| \leq C_d \int_{B^{n-1}} \delta^{2/k} dw' \leq C_{d,n}H^{-2},
\]

finishing the proof of Proposition 5.1.

11. The Proof of Theorem 1.1

For \( \phi \in C^\infty_c(\mathbb{C}^n) \) and \( P \in \mathbb{C}[X_1, \ldots, X_n] \), we set

\[
I_\phi(P) = \int_{\mathbb{C}^n} e(P(z)) \phi(z) \, dz
\]

and redefine (slightly)

\[
H_{P,\phi} := \inf_{z \in \text{supp}(\phi)} \max_{|\alpha| \geq 1} |\partial^\alpha P(z)|^{1/|\alpha|},
\]

dropping the factorials for notational convenience. Since the cut-off \( \phi \) is fixed, we will also drop the subscript \( \phi \) and write \( H_P \) instead of \( H_{P,\phi} \), again for notational convenience.

Our aim is to establish the bound

\[
|I_\phi(P)| \leq C_{d,n,\phi} H_P^{-2}. \tag{37}
\]

**Remark:** In the application establishing Proposition 1.4, it will be important to track the dependence of the constant \( C_{d,n,\phi} \) in (37) on \( \phi \) more precisely. The proof will show (see (43)) that

\[
C_{d,n,\phi} \lesssim_N \left[ 1 + J_P^{-N} \|\phi\|_{C^N} \right] \tag{38}
\]

for any \( N \geq 4 \). Here

\[
J_P = \inf_{z \in \text{supp}(\phi)} J(z) := \inf_{z \in \text{supp}(\phi)} \max_{2 \leq |\alpha| \leq d} |\partial^\alpha P(z)|^{1/|\alpha|}
\]

so that \( H_P = \inf_{z \in \text{supp}(\phi)} (|\nabla P(z)|, J(z)) \).

Set \( r(z) := 1/J(z) \).
Lemma 11.1.

1. There exists $c_{d,n} > 0$ such that if $\epsilon \leq c_{d,n}$ and $z = w + \epsilon r(w)u$ with $u \in \mathbb{B}^n$, then
   \[
   (1/2) J(w) \leq J(z) \leq 2 J(w).
   \] (39)

2. For all $A > 0$, there exists a constant $C = C_{A,d,n}$ such that if $z = w + Ar(w)u$ with $u \in \mathbb{B}^n$, then
   \[
   J(z) \leq C J(w).
   \] (40)

Proof. Let $a = \epsilon$ or $A$ and suppose $J(w) = |\partial^\alpha P(w)|^{1/|\alpha|}$ for some $2 \leq |\alpha| =: k_0 \leq d$. For any $\alpha$ with $|\alpha| = k$ and $2 \leq k \leq d$, set $Q = \partial^\alpha P$. Hence
   \[
   Q(z) = Q(w) + \sum_{1 \leq |\beta| \leq d-k} \frac{1}{|\beta|!} \partial^\beta Q(w) \left( (ar(w))^{|\beta|} w^\beta. \right.
   \]

For $1 \leq |\beta| \leq d - k$,
   \[
   |\partial^\beta Q(w) (ar(w))^{|\beta|} = |\partial^{\alpha + \beta} P(w)| |\alpha|^{|\beta|} \frac{1}{|\partial^\alpha P(w)||^{k/k_0}} \leq a^{|\beta|} |\partial^\alpha P(w)|^{k/k_0}
   \]

and therefore
   \[
   \left| \sum_{1 \leq |\beta| \leq d-k} \frac{1}{|\beta|!} \partial^\beta Q(w) (ar(w))^{|\beta|} \right| \leq |\partial^\alpha P(w)|^{k/k_0} \sum_{\ell=1}^{d-k} a\ell \sum_{|\beta| = \ell} \frac{1}{|\beta|!}
   \]

   \[
   = |\partial^\alpha P(w)|^{k/k_0} \left( \sum_{\ell=1}^{d-k} \frac{(an)^\ell}{\ell!} \right) \leq (e^{an} - 1) |\partial^\alpha P(w)|^{k/k_0}.
   \]

Now suppose $a = A$. Then
   \[
   |\partial^\alpha P(z)| \leq |\partial^\alpha P(w)|^{k/k_0} + (e^{an} - 1) |\partial^\alpha P(w)|^{k/k_0} \leq C |\partial^\alpha P(w)|^{k/k_0}
   \]

and so for all $\alpha$ with $2 \leq |\alpha| \leq d$,
   \[
   |\partial^\alpha P(z)|^{1/|\alpha|} \leq C J(w), \text{ implying } J(z) \leq C J(w).
   \]

For $a = \epsilon$, we have $e^{an} - 1 \leq 1/2$ for $c_{d,n}$ small enough and so for all $\alpha$ with $2 \leq |\alpha| \leq d$,
   \[
   |\partial^\alpha P(z)|^{1/|\alpha|} \leq 2 J(w), \text{ implying } J(z) \leq 2 J(w).
   \]

Furthermore when $\alpha = \alpha_0$,
   \[
   |\partial^\alpha P(z)| \geq |\partial^\alpha P(w)| - (e^{an} - 1)|\partial^\alpha P(w)| \geq (1/2)|\partial^\alpha P(w)|
   \]

and so
   \[
   (1/2) J(w) \leq |\partial^\alpha P(z)|^{1/|\alpha_0|} \leq J(z).
   \]

For any $\epsilon < c_{d,n}$, we have by the standard Vitali covering lemma,
   \[
   \text{supp}(\phi) \subseteq \bigcup_{j=1}^{M} \mathbb{B}_{3\epsilon r(z_j)}(z_j) \text{ where each } z_j \in \text{supp}(\phi) \text{ and } \{\mathbb{B}_{\epsilon r(z_j)}(z_j)\}_{j=1}^{M} \text{ are pairwise disjoint.}
   \]
Lemma 11.2. For any $C > 0$, there exists $N = N_{C,d,n,\epsilon} \in \mathbb{N}$ such that every

$$
\psi \in \bigcup_{j=1}^{M} B^n_{C^r(z_j)}(z_j)
$$

lies in at most $N$ of these balls.

Proof. First we note that for $z \in \mathbb{B}^n_{C^r(z_j)}(z_j)$, we have

$$
J(z) \leq C'J(z_j) \quad \text{or} \quad r(z_j) \leq C'r(z)
$$

by Lemma 11.1 part 2. The constant $C'$ depends on $C,d$ and $n$. We may assume that $C > \epsilon$. Now fix $z_\ast \in B_{C^r(z_j)}(z_j)$ and choose $w_\ell = z_j + t_\ell(z_\ast - z_j) \in B^n_{C^r(z_j)}(z_j)$ with $1 \leq \ell \leq L$ and $t_\ell \in [0,1]$ such that $|z_j - w_1| = \epsilon r(z_j)$,

$$
|w_1 - w_2| = \epsilon r(w_1), \ldots, |w_{L-1} - w_L| = \epsilon r(w_{L-1}) \quad \text{and} \quad |w_L - z_\ast| \leq \epsilon r(w_L).
$$

Set $w_0 = z_j$ and $w_{L+1} = z_\ast$. Note that $(\ast)$ implies $r(z_j) \leq C'r(w_\ell)$ for every $0 \leq \ell \leq L+1$ and so

$$
er(z_j)L \leq C'\epsilon \sum_{\ell=0}^{L-1} r(w_\ell) = C'\sum_{\ell=0}^{L-1} |w_\ell - w_{\ell+1}| \leq C'\epsilon |z_j - z_\ast| \leq CC'r(z_j),
$$

implying $L \lesssim_{C,C',\epsilon} 1$.

Next by Lemma 11.1 part 1., we have

$$(1/2)J(w_{\ell-1}) \leq J(w_\ell) \quad \text{or} \quad r(w_\ell) \leq 2r(w_{\ell-1})$$

for $1 \leq \ell \leq L$. Hence

$$r(z_\ast) = r(w_{L+1}) \leq 2r(w_L) \leq 2^2r(w_{L-1}) \leq \cdots \leq 2^{L+1} r(w_0) = 2^{L+1} r(z_j)$$

and so

$$r(z_\ast) \lesssim_{C,C',\epsilon} r(z_j). \tag{\dagger}$$

Set

$$A := \{1 \leq j \leq M : z_\ast \in \mathbb{B}^n_{C^r(z_j)}(z_j)\}$$

so that by $(\ast)$,

$$\bigcup_{j \in A} \mathbb{B}^n_{\epsilon r(z_j)}(z_j) \subseteq \mathbb{B}^n_{2C'r(z_\ast)}(z_\ast).$$

Hence by the disjointness of $\{\mathbb{B}^n_{\epsilon r(z_j)}(z_j)\}_{j=1}^{M}$ and $(\dagger)$,

$$\#A r(z_\ast)^{2n} \lesssim_{C,C',\epsilon} \sum_{j \in A} |\mathbb{B}^n_{\epsilon r(z_j)}(z_j)| \leq [2C'r(z_\ast)]^{2n},$$

implying $\#A \lesssim_{C,d,n,\epsilon} 1$. \qed

Now fix $\psi \in C^\infty_c(\mathbb{B}^n_2)$ such that $\psi \equiv 1$ on $\mathbb{B}^n$ and set

$$\Psi(z) := \sum_{j=1}^{M} \psi((3\epsilon)^{-1}r(z_j)^{-1}(z - z_j)), $$

the sum having bounded overlap by Lemma 11.2 and so \( \Psi \) is bounded. Note that for every \( z \in \text{supp}(\phi) \), we have \( z \in \mathbb{B}^n_{\epsilon r(z)}(z_j) \) for some \( 1 \leq j \leq M \), implying \( \psi((3\epsilon)^{-1}r(z_j)^{-1}(z - z_j)) = 1 \) and hence \( \Psi \geq 1 \) on \( \text{supp}(\phi) \). Also

\[
\text{supp}(\Psi) \subseteq \bigcup_{j=1}^M \mathbb{B}^n_{\epsilon r(z)}(z_j).
\]

We decompose the oscillatory integral

\[
I = \sum_{j=1}^M \int_{\mathbb{C}^n} e(P(z)) \varphi_j(z) \, dz =: \sum_{j=1}^M I_j
\]

where

\[
\varphi_j(z) := \psi_j((3\epsilon)^{-1}r(z_j)^{-1}(z - z_j)) \phi(z) \Psi^{-1}(z).
\]

Let

\[
A := \{ 1 \leq j \leq M : |\nabla P(z_j)| \leq J(z_j) \}.
\]

Fix \( j \in A \) and let \( z \in \text{supp}(\varphi_j) \). Then \( z = z_j + 6\epsilon r(z_j)u \) for some \( u \in \mathbb{B}^n \). Suppose \( J(z_j) = |\partial^\alpha P(z_j)|^{1/|\alpha|} \) for some \( 2 \leq |\alpha| \leq d \). Then since \( \epsilon < \epsilon_{d,n} \) and \( \epsilon_{d,n} \) is small, we see by the proof of Lemma 11.1 part 1,

\[
(1/2)|\partial^\alpha P(z)|^{1/|\alpha|} \leq |\partial^\alpha P(z_j)|^{1/|\alpha|} \leq 2|\partial^\alpha P(z)|^{1/|\alpha|}
\]

which implies \( J(z_j) \leq 2J(z) \). Furthermore the proof of Lemma 11.1 also shows that

\[
|\nabla P(z)| \leq 2|\partial^\alpha P(z)|^{1/|\alpha|} \leq 4|\partial^\alpha P(z)|^{1/|\alpha|}
\]

and therefore

\[
|\nabla P(z)| \leq 4|\partial^\alpha P(z)|^{1/|\alpha|} \leq 4J(z).
\]

Hence

\[
\left| \sum_{j \in A} I_j \right| \leq \int_{\mathcal{S}} \left| \sum_{j \in A} \varphi_j(z) \right| \, dz
\]

where \( \mathcal{S} := \{ z \in \text{supp}(\phi) : |\nabla P(z)| \leq 4J(z) \} \). The sum \( \sum_{j \in A} \varphi_j(z) \) is uniformly bounded and so

\[
\left| \sum_{j \in A} I_j \right| \lesssim |\mathcal{S}|. \tag{41}
\]

For \( z \in \mathcal{S} \), we note that

\[
J(z) \sim H(z) := \max_{1 \leq k \leq d} \max_{|\alpha| = k} |\partial^\alpha P(z)|^{1/|\alpha|}.
\]

Since \( |\nabla P(z)| \sim \max_{1 \leq j \leq n} |\partial_j P(z)| \), we see that

\[
\mathcal{S} \subseteq \bigcup_{j=1}^n \{ z \in \mathcal{S} : |\nabla P(z)| \sim |\partial_j P(z)| \} =: \bigcup_{j=1}^n \mathcal{S}_j.
\]

Note that \( \mathcal{S}_j = \{ z \in \mathcal{S} : |\partial_j P(z)| \leq 4J(z) \sim H(z) \} \). Since

\[
J(z) = \max_{2 \leq |\alpha| \leq d} |\partial^\alpha P(z)|^{1/|\alpha|},
\]

we can decompose each \( \mathcal{S}_j \) further; we have

\[
\mathcal{S}_j \subseteq \bigcup_{2 \leq |\alpha| \leq d} \{ z \in \mathcal{S}_j : J(z) = |\partial^\alpha P(z)|^{1/|\alpha|} \} =: \bigcup_{2 \leq |\alpha| \leq d} \mathcal{S}_{j,\alpha}.
\]
For each $1 \leq j \leq n$ and $2 \leq |\alpha| \leq d$, we decompose

$$\mathcal{G}_{j,\alpha} \subseteq \bigcup_{r,\ell \geq 0} \{ z \in \mathcal{G}_j : 2^\ell |\partial_j P(z)| \sim |\partial^\alpha P(z)|^{1/|\alpha|} \sim 2^\ell H_P \}$$

Hence $|\mathcal{G}_{j,\alpha}| \leq \sum_{r,\ell \geq 0} |\mathcal{G}_{j,\alpha}^{r,\ell}|$ where

$$\mathcal{G}_{j,\alpha}^{r,\ell} := \{ z \in \mathcal{G}_j : 2^\ell |\partial_j P(z)| \sim |\partial^\alpha P(z)|^{1/|\alpha|} \sim 2^\ell H_P \}.$$  

We apply Proposition 4.2 to $Q(z) = \partial_j P(z)$ to conclude

$$|\mathcal{G}_{j,\alpha}^{r,\ell}| \leq C_{d,n} \left[ \frac{2^{r-\ell} H_P}{(2^\ell H_P)^{|\alpha|}} \right]^{2/|\alpha| - 1} = C_{d,n} 2^{-2\ell(|\alpha| - 1) - 2 - 2r} H_P^{-2}$$

which sums in $r \geq 0$ and $\ell \geq 0$ to produce the bound $|\mathcal{G}_{j,\alpha}| \leq C_{d,n} H_P^{-2}$ for every $1 \leq j \leq n$ and $2 \leq |\alpha| \leq d$. Hence $|\mathcal{G}| \lesssim_{d,n} H_P^{-2}$ and so (41) implies

$$|\sum_{j \in A} I_j| \lesssim_{d,n} H_P^{-2}.$$  \hspace{1cm} (42)

We now turn to the sum $\sum_{j \in B} I_j$ where

$$B := \{ 1 \leq j \leq M : J(z_j) \leq |\nabla P(z_j)| \}.$$  

For each $j \in B$, we make a change of variables in the integral $I_j$ to write it as

$$I_j = r(z_j)^{2n} \int_{\mathbb{S}^n} e(P(z_j + r(z_j)w)\psi((3\epsilon)^{-1} w)\tilde{\phi}(w) dw$$

where $\tilde{\phi}(w) = [\phi/\psi](z_j + r(z_j)w)$. We expand

$$P(z_j + r(z_j)w) = P(z_j) + r(z_j)\nabla P(z_j) \cdot w + \sum_{2 \leq |\alpha| \leq d} \frac{1}{\alpha!} \partial^\alpha P(z_j) r(z_j)^{|\alpha|} w^\alpha$$

so that

$$I_j = r(z_j)^{2n} e(P(z_j)) \int_{\mathbb{S}^n} e(\Lambda_j Q(w))\psi((3\epsilon)^{-1} w)\tilde{\phi}(w) dw$$

where $\Lambda_j = r(z_j)|\nabla P(z_j)|$ and

$$Q(w) := A_j \cdot w + \sum_{2 \leq |\alpha| \leq d} B_\alpha w^\alpha.$$  

Here $A_j = \nabla P(z_j)/|\nabla P(z_j)|$ and

$$B_\alpha := \Lambda_j^{-1} \frac{1}{\alpha!} \partial^\alpha P(z_j) r(z_j)^{|\alpha|}$$

Since $j \in B$, we see that $\Lambda_j \geq 1$ and each $|B_\alpha| \lesssim 1$. Hence, since $|w| \leq 6\epsilon$ and $\epsilon > 0$ is small, we have $|\nabla Q(w)| \gtrsim 1$ and $\|Q\|_{C^d} \lesssim 1$ on the support of $\psi((3\epsilon)^{-1})$, we have

$$|I_j| \lesssim_N r(z_j)^{2n} \Lambda_j^{-N}$$

for all $N \geq 1$.

**Remark:** In fact integration by parts shows

$$|I_j| \lesssim_N r(z_j)^2[1 + r(z_j)^{N} \|\phi\|_{C^N} \Lambda_j^{N} \lesssim_N r(z_j)^2[1 + J_P^{-N} \|\phi\|_{C^N} \Lambda_j^{N}$$

where $J_P = \inf_{\phi \in \text{supp}(\phi)} J(z)$. This gives the dependence on the cut-off $\phi$ made in the Remark (38) at the beginning of this section.
From Lemma 11.1, we know that \( r(z) \) is roughly constant on the support of \( \psi((3e)^{-1}) \). Furthermore, using \( J(z_j) \leq |\nabla P(z_j)| \) for \( j \in B \), we see that \( |\nabla P(z)| \) is roughly constant on \( \psi((3e)^{-1}) \) (the proof is precisely the same as in Lemma 11.1). Therefore

\[
|I_j| \lesssim_{N,\epsilon} \int_{\mathbb{C}^{n}} \left[ \frac{J(z)}{|\nabla P(z)|} \right]^{N} \psi((3e)^{-1}r(z_j)^{-1}(z - z_j)) \, dz
\]

and so

\[
\left| \sum_{j \in B} I_j \right| \lesssim_{N,\epsilon} \int_{B} \left[ \frac{J(z)}{|\nabla P(z)|} \right]^{N} \, dz
\]

where \( B = \{ z \in \mathbb{B}^{n} : J(z) \lesssim |\nabla P(z)| \} \). Exactly as in the sum over \( A \), we may decompose

\[
B \subseteq \bigcup_{j, \alpha, r, \ell \geq 0} B_{j,\alpha}^{r,\ell}
\]

where \( B_{j,\alpha}^{r,\ell} := \{ z \in B : 2^{r} |\partial^{\alpha} P(z)|^{1/(\alpha)} \sim |\partial_{j} P(z)| \sim 2^{r} H_{P} \} \), implying

\[
\left| \sum_{j \in B} I_j \right| \lesssim_{N,\epsilon,\alpha} \sum_{\ell = 2}^{n} \sum_{|\alpha| \leq \ell} \sum_{r, \ell \geq 0} 2^{-N\ell} |B_{j,\alpha}^{r,\ell}|.
\]

(44)

For fixed \( 1 \leq j \leq n \) and \( 2 \leq |\alpha| \leq d \), we apply Proposition 11.2 to \( Q(z) = \partial_{j} P(z) \) to conclude

\[
|B_{j,\alpha}^{r,\ell}| \lesssim_{d,\alpha,\epsilon} 2^{2r} H_{P} \left( \frac{2^{r} H_{P}}{(2r - 2) H_{P}^{k}} \right)^{2/|\alpha|} = C_{d,\alpha,\epsilon} 2^{2r} H_{P}^{1/(\alpha) - 1} 2^{-2r} H_{P}^{-2}.
\]

Inserting this bound into (44) with \( N > 2d/(d - 1) \) implies that

\[
\left| \sum_{j \in B} I_j \right| \lesssim_{d,\alpha,\epsilon} H_{P}^{-2}
\]

which, together with (12), completes the proof of (37) and hence Theorem 1.1.

**Proofs of Proposition 9.1 and Lemma 9.3**

Here we give the proofs of Proposition 9.1 and Lemma 9.3.

12. **Proof of Proposition 9.1**

Let us recall the statement of Proposition 9.1. Let \( Q \in \mathbb{C}[X] \) have degree \( d \). Let \( z_{0} \in \mathbb{D} \) and \( 0 < \epsilon \leq \epsilon_{d} \) for a sufficiently small \( \epsilon_{d} > 0 \). Suppose \( |Q^{(j)}(z_{0})| \geq 1 \) and \( |Q(z_{0})| \leq \epsilon \). Our aim is to find a zero \( z_{*} \) of \( Q^{(j)} \) for some \( 0 \leq j \leq d \) such that \( |z_{0} - z_{*}| \lesssim_{d} \epsilon^{1/k} \).

We begin with the following lemma.

**Lemma 12.1.** Let \( \mathcal{P}_{d} \subseteq \mathbb{C}[X] \) denote the space of complex polynomials of degree at most \( d \). For \( P(z) = c_{d}z^{d} + \cdots + c_{1}z + c_{0} \in \mathcal{P}_{d} \), define \( \| P \| = \max_{j} |c_{j}| \). Then there exists a constant \( c_{d} > 0 \) depending only on \( d \) such that for every \( P \in \mathcal{P}_{d} \),

\[
|P^{(n)}(z)| \geq c_{d} \| P \| \quad \text{for all } z \in \mathbb{D}
\]

(45)
holds for some $0 \leq n \leq d$.

**Proof.** Consider the “norm”

$$|||P||| := \max_{j \geq 0} \min_{z \in \mathbb{D}} |P^{(j)}(z)|$$

which satisfies (1) if $|||P||| = 0$, then $P = 0$ and (2) $|||\lambda P||| = |\lambda|||P|||$ but it does not satisfy the triangle inequality. Nevertheless one can run the usual *equivalence of norms* argument with this “norm”.

Let $S = \{P \in \mathcal{P}_d : |||P||| = 1\}$ denote the unit sphere in $\mathcal{P}_d$ with respect to the norm $|||P||| = \max_j |c_j|$. We will show that

$$c_d := \inf_{P \in S} |||P||| > 0$$

which implies $|||P||| \geq c_d \|P\|$ for all $P \in \mathcal{P}_d$ and this establishes [43] by the definition of this triple “norm” $||| \cdot |||$. If $c_d = 0$, then there exists a sequence $P_j \in S$ such that $|||P_j||| \to 0$ and $|||P_j - P||| \to 0$ for some $P \in S$. Although the triangle inequality does not hold for $||| \cdot |||$ we do have the inequality

$$|||P||| \leq |||P_j||| + \max_n \|P^{(n)}_j - P^{(n)}\|_{L^\infty(\mathbb{D})} \leq |||P_j||| + c_d \|P_j - P\| \to 0,$$

implying $|||P||| = 0$ and so $P = 0$ but this is impossible since $P \in S$. \qed

We turn our attention to our polynomial $Q(z) = c_0 + c_1 z + \cdots + c_d z^d$ in Proposition 9.1 satisfying $|Q^{(k)}(z_0)| \geq 1$ and $|Q(z_0)| \leq \epsilon$ for some $z_0 \in \mathbb{D}$. Set $\lambda = \max_j |c_j|$ and note that $1 \leq d |Q^{(k)}(z_0)| \leq d \lambda$ implies $1 \leq d \lambda$. We will use Lemma 9.3 to prove Proposition 9.1. Since $|Q|_{L^\infty(\mathbb{D})} \leq C_d \lambda$, it suffices to use $\lambda$ in place of $M$ when verifying the conditions of Lemma 9.3.

By Lemma 12.1 there exists an $0 \leq n \leq d$ such that $|Q^{(n)}(z)/n!| \geq c_d \lambda$. Since $\lambda \leq d$ and $\epsilon \ll d$ is small, then in fact it must be the case that $1 \leq n \leq d$. Furthermore $|Q^{(j)}(z)| \lesssim d |Q^{(n)}(z)|$ for every $0 \leq j \leq d$ and every $z \in \mathbb{D}$. In particular when $k > n$, we have

$$1 \leq \left[\epsilon^{-1} |Q^{(k)}(z)|\right]^{1/k} \lesssim |Q^{(n)}(z)|^{1/n}, \quad (46)$$

a bound we will use in the proof of Proposition 9.1.

We fix small constants $c_1, c_2, \ldots, c_{n-1}$ (we define $c_n = 1$), depending only on $d$, satisfying the relationships

$$c_j^2 \ll c_{j-1} c_{j+1} \quad (47)$$

for every $2 \leq j \leq n - 1$. For each $2 \leq j \leq n - 1$, we will introduce small parameters $\rho_\ell = \rho_\ell(j), 2 \leq \ell \leq j$ below satisfying [49]. We first choose the small parameter $c_j$’s satisfying [47], then we choose the parameters $\rho_\ell$ satisfying [49] and finally we choose $\epsilon_d$, depending on all these other parameters, and assume $\epsilon < \epsilon_d$. 

We set
\[ K_j(z) := \left[ c_{n-j} \epsilon^{-1} |Q^{(n-j)}(z)|^{1/(n-j)} \right] \] and \( K(z) := \max_{0 \leq j \leq n-1} K_j(z) \)
and note that
\[ \epsilon^{-1/k} \leq \left[ \epsilon^{-1} |Q^{(k)}(z)| \right]^{1/k} \lesssim_d K(z). \] (48)
This is clearly true if \( k \leq n \) and when \( k > n \), this follows from (46).

We consider several cases.

**Case 0**: \( K(z_0) = K_0(z_0) = \left[ \epsilon^{-1} |Q^{(n)}(z_0)| \right]^{1/n} \).

In this case, we apply Lemma 9.3 with \( L = 1 \) to \( \phi = Q^{(n-1)} \) and so \( \delta = |Q^{(n-1)}(z_0)Q^{(n)}(z_0)|^{-2} \). Hence
\[ \lambda \delta \lesssim_d |Q^{(n-1)}(z_0)||Q^{(n)}(z_0)|^{-1} \leq c_{n-1}^{-1} \left[ \epsilon^{-1} |Q^{(n)}(z_0)| \right]^{-1/n} \]
and so by (48), we have
\[ \lambda \delta \lesssim_d \left[ \epsilon^{-1} |Q^{(n)}(z_0)|^{-1/n} \right] \lesssim_d \epsilon^{1/k} |Q^{(k)}(z_0)|^{-1/k} \ll 1 \]
since \( |Q^{(k)}(z_0)| \geq 1 \) and \( \epsilon < \epsilon_d \) is small. Hence Lemma 9.3 shows there exists a zero \( z_* \) of \( Q^{(n-1)} \) such that
\[ |z_0 - z_*| \leq 2 |Q^{(n-1)}(z_0)Q^{(n)}(z_0)| \leq 2 c_{n-1}^{-1} \left[ \epsilon^{-1} |Q^{(n)}(z_0)| \right]^{-1/n} \]
and so by (48), we have
\[ |z_0 - z_*| \lesssim_d \left[ \epsilon^{-1} |Q^{(n)}(z_0)|^{-1/n} \right] \lesssim_d \epsilon^{1/k} |Q^{(k)}(z_0)|^{-1/k} \leq \epsilon^{1/k} \]
since \( |Q^{(k)}(z_0)| \geq 1 \). This completes the proof in Case 0.

**Case 1**: \( K(z_0) = K_1(z_0) = \left[ c_{n-1} \epsilon^{-1} |Q^{(n-1)}(z_0)| \right]^{1/(n-1)} \).

We apply Lemma 9.3 with \( L = 2 \) to \( \phi = Q^{(n-2)} \) so that
\[ \delta = |Q^{(n-2)}(z_0)Q^{(n-1)}(z_0)|^{-1} |Q^{(n)}(z_0)|^{-1}. \]

Hence
\[ \lambda \delta \lesssim_d \left[ \epsilon^{-1} |Q^{(n-1)}(z_0)|^{-1/(n-1)} \right] \lesssim_d \epsilon^{1/k} |Q^{(k)}(z_0)|^{-1/k} \]
by (48). Hence \( \lambda \delta \ll 1 \) since \( |Q^{(k)}(z_0)| \geq 1 \) and \( \epsilon < \epsilon_d \) is small. For Lemma 9.3,
we also need to verify that \( \delta_1 \ll 1 \) where \( \delta_1 = |Q^{(n-2)}(z_0)Q^{(n-1)}(z_0)|^{-2} |Q^{(n)}(z_0)| \). We have
\[ \delta_1 \leq \frac{c_{2n-1}}{c_{n-2}} \frac{\epsilon}{\epsilon C^{n-2} C^{n-1}} |Q^{(n-1)}(z_0)|^{\frac{n-2}{n-1} + \frac{n-2}{n-1}} = \frac{c_{n-1}^{-1}}{c_{n-2}^{-1}} \ll 1 \]
by (47) applied to \( j = n - 1 \) (recall \( c_n = 1 \)). Therefore Lemma 9.3 implies there is a zero \( z_* \) of \( Q^{(n-2)} \) such that
\[ |z_0 - z_*| \leq 2 |Q^{(n-2)}(z_0)Q^{(n-1)}(z_0)|^{-1} \lesssim_d \left[ \epsilon^{-1} |Q^{(n-1)}(z_0)| \right]^{-1/(n-1)} \]
and so by (48),
\[ |z_0 - z_*| \lesssim_d \left[ \epsilon^{-1} |Q^{(k)}(z_0)| \right]^{-1/k} \lesssim_d \epsilon^{1/k} \]
since \( |Q^{(k)}(z_0)| \geq 1 \). The completes the proof in Case 1.
General case $j$: $K(z_0) = K_j(z_0)$. Here $2 \leq j \leq n$.

We split the general case $j$ into $j$ subcases. To define these subcases, we introduce the following conditions:

$$\left| Q^{(n-j+\ell)}(z_0) \right| \left| Q^{(n-j+\ell-1)}(z_0) \right|^{-1} \leq \rho^{j-1}_\ell K_j(z_0)$$

for $2 \leq \ell \leq j$ where $\rho_\ell$ are small constants satisfying the relationships

$$\rho_\ell \ll \rho_{\ell+1}$$

for every $2 \leq \ell \leq j - 1$.

**subcase 1**: $(I_1)$: $K_j(z_0) \left| Q^{(n-1)}(z_0) \right| \leq \rho_j \left| Q^{(n)}(z_0) \right|$. If $z_0$ does not satisfy $(I_1)$, then the property $S_j$ holds. Inductively the other subcases for $2 \leq \ell \leq j - 1$ are defined by

**subcase $\ell$:** $(I_{\ell-1})$ does not hold but

$$(I_\ell) : \quad K_j(z_0) \left| Q^{(n-\ell)}(z_0) \right| \leq \rho_{j-\ell+1} \left| Q^{(n-\ell+1)}(z_0) \right|$$

holds.

If $(I_\ell)$ does not hold, then the properties $S_{j-\ell}$ hold for every $0 \leq r \leq \ell - 1$. The final subcase is

**subcase $j$:** $(I_{j-1})$ does not hold.

In this last subcase $j$, all the properties $S_\ell$, for $2 \leq \ell \leq j$ hold.

Let us consider this final subcase. We apply Lemma 9.3 with $L = (j + 1)$ to $\phi = Q^{(n-j-1)}$ so that $\delta = \left| Q^{(n-j-1)}(z_0) \right| Q^{(n-j)}(z_0)^{-1} \left| Q^{(n)}(z_0) \right|^{-1}$. Also for $1 \leq \ell \leq j$, we have

$$\delta_\ell := \left| Q^{(n-j-1)}(z_0) \right| Q^{(n-j)}(z_0)^{-1} \left| Q^{(n-j+\ell-1)}(z_0) \right|^{-1} Q^{(n-j+\ell)}(z_0).$$

We need to verify $\lambda \delta, \delta_1 \ll 1$ and for $2 \leq \ell \leq j$, $\delta_\ell \ll_d 1$.

First, we have

$$\lambda \delta \ll_d \left( \epsilon^{-1} \left| Q^{(n-j)}(z_0) \right| \right)^{-1/(n-j)} \ll_d \epsilon^{1/k} \left| Q^{(k)}(z_0) \right|^{-1/k}$$

by (48) and so

$$\lambda \delta \ll_d \epsilon^{1/k} \ll 1$$

since $\left| Q^{(k)}(z_0) \right| \geq 1$ and $\epsilon < \epsilon_d$ is small. Also

$$\delta_1 \leq c_{n-j-1}^{n-j} c_{n-j}^{n-j} \epsilon^{1/(n-j)} \epsilon^{-1/(n-j)} \left| Q^{(n-j)}(z_0) \right|^{n-j-1} c_{n-j+1}^{n-j+1} - 2$$

$$= c_{n-j-1}^{n-j} c_{n-j+1}^{n-j+1} c_{n-j}^{2} \ll 1$$

by (47). Finally for every $2 \leq \ell \leq j$, since the properties $S_\ell$ for for every $2 \leq \ell \leq j$ hold, we have

$$\delta_\ell \ll_d \left| Q^{(n-j-1)}(z_0) \right| Q^{(n-j)}(z_0)^{-1} K_j(z_0) \ll_d 1.$$
Hence there exists a zero $z_*$ of $Q^{(n-j-1)}$ such that
\[ |z_0 - z_*| \leq 2|Q^{(n-j-1)}(z_0)Q^{(n-j)}(z_0)^{-1}| \lesssim_d \left[ \epsilon^{-1}|Q^{(n-j)}(z_0)| \right]^{-1/(n-j)} \]
\[ \lesssim_d \epsilon^{1/k}|Q^{(k)}(z_0)|^{-1/k} \lesssim_d \epsilon^{1/k} \]
by (48) and the fact that $|Q^{(k)}(z_0)| \geq 1$.

Finally we consider the subcases $\ell$ with $2 \leq \ell \leq j - 1$. In these subcases, we see that the properties $S_{j-\ell}$ hold for $0 \leq r \leq \ell - 2$. Here we apply Lemma 9.3 with $L = \ell$ to $\phi = Q^{(n-\ell)}$ so that
\[ \delta = |Q^{(n-\ell)}(z_0)Q^{(n-\ell+1)}(z_0)^{-1}Q^{(n)}(z_0)^{-1}|. \]
Also for $1 \leq t \leq \ell - 1$,
\[ \delta_t = |Q^{(n-\ell)}(z_0)Q^{(n-\ell+1)}(z_0)^{-1}Q^{(n-\ell+t)}(z_0)^{-1}Q^{(n-\ell+t+1)}(z_0)|. \]
We need to verify $\lambda \delta, \delta_1 \ll 1$ and for $2 \leq t \leq \ell - 1$, $\delta_t \lesssim_d 1$. We have
\[ \lambda \delta \lesssim_d \left[ \epsilon^{-1}|Q^{(n-j)}(z_0)| \right]^{-1/(n-j)} \lesssim_d \epsilon^{1/k}|Q^{(k)}(z_0)|^{-1/k} \ll 1 \]
by (48), our hypothesis $|Q^{(k)}(z_0)| \geq 1$ and since $\epsilon < \epsilon_d$ is small. Also since $S_{j-\ell+2}$ holds, we have
\[ \delta_1 \leq \rho_{j-\ell+1} K_j(z_0)^{-1}|Q^{(n-\ell+2)}(z_0)Q^{(n-\ell+1)}(z_0)^{-1}| \leq \rho_{j-\ell+1} \frac{K_j(z_0)^{-1}K_j(z_0)}{\rho_{j-\ell+2}} \ll 1 \]
by (49). Also since $S_{j-\ell+t+1}$ holds for $2 \leq t \leq \ell - 1$,
\[ \delta_t \lesssim_d K_j(z_0)^{-1}|Q^{(n-\ell+t+1)}(z_0)Q^{(n-\ell+t)}(z_0)^{-1}| \lesssim_d K_j(z_0)^{-1}K_j(z_0) = 1. \]
Hence there exists a zero $z_*$ of $Q^{(n-\ell)}$ such that
\[ |z_0 - z_*| \leq 2|Q^{(n-\ell)}(z_0)Q^{(n-\ell+1)}(z_0)^{-1}| \lesssim_d K_j(z_0)^{-1} \leq \epsilon^{1/k}|Q^{(k)}(z_0)|^{-1/k} \leq \epsilon^{1/k} \]
by (48) and from our hypothesis $|Q^{(k)}(z_0)| \geq 1$.

This completes the proof of Proposition 9.1.

13. PROOF OF LEMMA 9.3

It remains to prove Lemma 9.3 which we restate for convenience.

Lemma 9.3 Let $\phi \in \mathcal{H}(D_2)$ with $M = M_\phi := \max_{z \in \mathbb{D}} |\phi(z)|$. Fix $L \geq 1$.
Suppose $z_0 \in \mathbb{D}$ is a point where $\phi^{(k)}(z_0) \neq 0$ for each $1 \leq k \leq L$. Set
\[ \delta := |\phi(z_0)\phi'(z_0)^{-1}\phi^{(L)}(z_0)^{-1}| \text{ and for } 1 \leq k \leq L - 1, \text{ set} \]
\[ \delta_k := |\phi^{(k+1)}(z_0)\phi^{(k)}(z_0)^{-1}|. \]
Suppose $\delta_k \lesssim L$, $2 \leq k \leq L - 1$ and suppose $|\phi(z_0)\phi'(z_0)^{-1}| \leq 1/8$. Then if $\delta M, \delta_1 \ll L$, there is a $z \in \mathbb{D}_{3/4}$ such that
\[ (a) \phi(z) = 0 \quad \text{and} \quad (b) |z - z_0| \leq 2|\phi(z_0)\phi'(z_0)^{-1}|. \]

We will only give the proof in the case $L \geq 2$. The case $L = 1$ is easier.
We define a sequence \( \{z_n\} \) recursively by
\[
z_n = z_{n-1} - \phi(z_{n-1})\phi'(z_{n-1})^{-1}
\tag{50}
\]
and set \( \Lambda := c_L\delta_1 \) for some large constant \( c_L \) chosen later. Then our condition \( \delta_1 \ll L \) will imply in particular \( \Lambda \leq 1/2 \). We make the following claim.

**Claim:** For every \( n \geq 1 \),
\[
\begin{align*}
(1)_{n} & \quad |z_n - z_{n-1}| \leq |\phi(z_0)\phi'(z_0)^{-1}|\Lambda^{2^n-1-1}, \\
(2)_{n} & \quad |\phi(z_{n-1})| \leq |\phi(z_0)|\Lambda^{2^n-1-1}, \\
(3)_{n} & \quad |\phi'(z_{n-1})| \geq (1 - \epsilon_{n-1})|\phi'(z_0)|, \\
(4)_{n} & \quad \text{for } 2 \leq k \leq L, |\phi^{(k)}(z_{n-1})| \leq (1 + \epsilon_{n-1})|\phi^{(k)}(z_0)|
\end{align*}
\]
where for \( n \geq 2 \), \( \epsilon_{n-1} = \sum_{j=2}^{n}2^{-j} \) and \( \epsilon_{n-1} = \sum_{j=1}^{n-1}2^{-j} \). When \( n = 1 \), we set \( \epsilon_0 = \epsilon_0 = 0 \). Note the claim implies that for every \( n \geq 1 \),
\[
|z_n - z_0| \leq \sum_{j=1}^{n} |z_j - z_{j-1}| \leq |\phi(z_0)\phi'(z_0)^{-1}| \sum_{j=0}^{2^{-j}} \leq 2/8 
\tag{51}
\]
and hence \( z_n \in \mathbb{D}_{5/4} \).

Before we start the proof of the claim, we review Taylor series with remainder for a \( \psi \in \mathcal{H} (\mathbb{D}_2) \): for any \( m \geq 0 \),
\[
\psi(z_n) = \psi(z_{n-1}) + \sum_{j=1}^{m} \frac{\psi^{(j)}(z_{n-1})}{j!} (z_n - z_{n-1})^j + R_{m, \psi}
\tag{52}
\]
where \( z_t := z_{n-1} + t(z_n - z_{n-1}) \) and
\[
R_{m, \psi} := \frac{1}{m!} \int_{0}^{1} \psi^{(m+1)}(z_t) dt \ (z_n - z_{n-1})^{m+1}.
\]
The sum in (52) does not appear when \( m = 0 \). We will have need to apply (52) for \( \psi(z) = \phi^{(k)}(z) \) where \( |\phi(z)| \leq M \) for \( z \in \mathbb{D}_{7/4} \). Since \( z_n \in \mathbb{D}_{5/4} \) for all \( n \), then \( z_t \in \mathbb{D}_{5/4} \) and so by Cauchy’s integral formula,
\[
\psi^{(m)}(z_t) = \frac{(m+k)!}{2\pi i} \int_{C_{1/2}(z_t)} \frac{\phi(w)}{(w - z_t)^{m+k+1}} dw,
\]
implying \( |\psi^{(m)}(z_t)| = |\phi^{(m+k)}(z_t)| \lesssim_{m,k} M \) and hence
\[
|R_{m, \psi}| \lesssim_{m,k} M |z_n - z_{n-1}|^{m+1} \tag{53}
\]
where \( R_{m, k} = R_{m, \psi} \) with \( \psi = \phi^{(k)} \).

We now proceed with the proof of the claim. The claim is true for \( n = 1 \) and so suppose \( (1)_j, (2)_j \) and \( (3)_j \) holds for all \( 1 \leq j \leq n \). Note that \( (3)_n \) implies
for $1 \leq \delta$ by (4) establishing (4)

Furthermore by (1), there is a similar bound for the right hand side of (58) and using this and (59) in (53),

where by the remainder estimate in (53),

$|R_{L-k,k}| \lesssim M|\phi_{n-1}|^{L-k+1} \lesssim M|\phi(z_0)\phi'(z_0)^{-1}|A^{2n-1-1}|z_n-z_{n-1}|^{L-k}$.

By the definition of the $\delta$, $2 \leq \ell \leq L-1$ in the statement of Lemma 3.3, we have

$|\phi(z_0)\phi'(z_0)^{-1}|, |\phi(k)(z_0)| \lesssim |\phi'(z_0)|$ for $1 \leq j \leq L-k$ and hence

$|\phi(k+j)(z_n-1)(z_n-z_{n-1})^j| \lesssim L^{2n-1-1}|\phi'(z_0)|$.

Plugging (55) and (56) into (53), we see that

$|\phi(k)(z_n)| \leq |\phi'(z_0)|[1 + \varepsilon_{n-1} + b_LA^{2n-1-1}]$ by (4). But $b_LA^{2n-1-1} \ll 2^{-2(n-1)} \leq 2^{-n}$ and so

$|\phi(k)(z_n)| \leq |\phi'(z_0)|[1 + \varepsilon_{n-1} + 2^{-n}] = (1 + \varepsilon_n)|\phi(k)(z_0)|$,

establishing (4)$_{n+1}$. We now turn to (3)$_{n+1}$. Applying (52) to $\psi = \phi'$ and $m = L-1$, we see

$\phi'(z_n) = \phi'(z_{n-1}) + \sum_{j=1}^{L-1} \phi'(z_{n-1})j!(z_n-z_{n-1})^j + R_{L,k}$

where by the remainder estimate in (53),

$|R_{L,k}| \lesssim M|\phi_{n-1}|^{L} \lesssim M|\phi(z_0)\phi'(z_0)^{-1}|A^{2n-1-1}|z_n-z_{n-1}|^{L-1}$.

Furthermore by (1) and (4), for each $1 \leq j \leq L-1$,

$|\phi(j+1)(z_n-1)(z_n-z_{n-1})^j| \lesssim L^{2n-1-1}|\phi(z_0)\phi'(z_0)^{-1}|\phi(j+1)(z_0)|.

Again using the definition of the $\delta$, $2 \leq \ell \leq L-1$,

$|\phi(z_0)\phi'(z_0)^{-1}|, |\phi'(z_0)| \lesssim |\phi'(z_0)|$ for $1 \leq j \leq L-1$ and hence

$|\phi'(z_n)| \geq |\phi'(z_0)|[1 - \varepsilon_{n-1} - b_L\delta_1A^{2n-1-1}]$.
by (3)$_n$. But $b_L\delta_1A^{2n-1-1} \leq b_L\delta_1(1/2)^n-1 \leq 2^{-n-1}$ since we’ll take $\delta_1$ small so that $b_L\delta_1 \leq 1/4$. Therefore

$$|\phi(z_n)| \geq |\phi'(z_0)|[1 - \epsilon_n - 2^{-n-1}] = |\phi'(z_0)|[1 - \epsilon_n],$$

completing the proof of (3)$_{n+1}$.

For (2)$_{n+1}$, we apply (52) with $\psi = \phi$ and $m = L$ to conclude

$$\phi(z_n) = \phi(z_{n-1}) + \phi'(z_{n-1})(z_n - z_{n-1}) + \sum_{j=2}^L \phi^{(j)}(z_{n-1}) j! (z_n - z_{n-1})^j + R_L$$

and since $\phi(z_{n-1}) + \phi'(z_{n-1})(z_n - z_{n-1}) = 0$ by definition of $z_n$, we have

$$\phi(z_n) = \sum_{j=2}^L \phi^{(j)}(z_{n-1}) j! (z_n - z_{n-1})^j + R_L$$

where

$$|R_L| \lesssim_L M|z_n - z_{n-1}|^{L+1} \lesssim_L M\delta A^{2n-1} - 1|\phi^{(L)}(z_0)||z_n - z_{n-1}|^L$$

as before. For $2 \leq j \leq L$, we have by (4)$_n$, 

$$|\phi^{(j)}(z_{n-1})||z_n - z_{n-1}|^j \leq 2|\phi^{(j)}(z_0)||\phi(z_0)\phi'(z_0)^{-1} - 1 A^{2n-2}|\phi(z_0)\phi'(z_0)^{-1}|.$$

Proceeding as above, using the definition of the $\delta_i$’s, we have

$$|\phi^{(j)}(z_0)||\phi(z_0)\phi'(z_0)^{-1}|^j - 1 = (\delta_2 \cdots \delta_{j-1})|\phi''(z_0)\phi(z_0)\phi'(z_0)^{-1}|,$$

implying

$$|\phi^{(j)}(z_{n-1})||z_n - z_{n-1}|^j \lesssim_L \delta_1 A^{2n-2}|\phi(z_0)|.$$

We have a similar estimate for the right hand side of (60) and so, altogether, we have

$$|\phi(z_n)| \leq b_L\delta_1 A^{2n-2}|\phi(z_0)| \leq A^{2n-1}|\phi(z_0)|,$$

completing the proof of (2)$_{n+1}$ by ensuring $b_L \leq c_L$ (recall $\Lambda = c_L\delta_1$).

Finally for (1)$_{n+1}$, we use (2)$_{n+1}, (3)$_{n+1}$ and the inequality (*) above to see

$$|z_{n+1} - z_n| = |\phi(z_n)\phi'(z_n)^{-1}| \leq 2|\phi'(z_0)^{-1} \phi(z_n)| \leq 2 b_L\delta_1 A^{2n-2}|\phi(z_0)|.$$

By taking $c_L \geq 2 b_L$, we see that $|z_{n+1} - z_n| \leq |\phi(z_0)|A^{2n-1}$ which establishes (1)$_{n+1}$, completing the proof of the claim.

Statement (1)$_n$ of the claim implies that for $m \leq n$,

$$|z_n - z_m| \leq \sum_{j=m+1}^n |z_j - z_{j-1}| \leq \delta |\phi'(z_0)| \sum_{j=m}^n (1/2)^{j-1} - 1 \to 0$$

as $m \leq n \to \infty$ and hence $\{z_n\}$ forms a Cauchy sequence of complex numbers and hence $z_n \to z$ for some $z \in \mathbb{D}_{5/4}$. The statement (2)$_n$ then implies $\phi(z) = 0$ and [51] shows that $z \in \mathbb{D}_{5/4}$ and $|z - z_0| \leq 2|\phi(z_0)\phi'(z_0)^{-1}|$. This completes the proof of Lemma 9.3.
Some applications

We give two applications of the main estimate in Theorem 1.1 which are contained in Propositions 1.3 and 1.4. The first is a complex version of an oscillatory integral bound for polynomial phases due to Phong and Stein in [11]. Second, we prove a complex version of a result of Arkhipov, Chubarikov and Karatsuba [2] on the convergence exponent for the singular integral in Tarry’s problem which is equivalent to the $L^b$ integrability of the Fourier extension operator $Eb$ on the function $b = 1$ with respect to complex curves.

14. Proof of Proposition 1.3

Let $f \in \mathbb{C}[X]$ have degree at most $d$. Consider the derivative $f'(z) = a \prod_{j=0}^m (z - \xi_j)^{e_j}$ where $\{\xi_j\}$ are the distinct roots of $f'$ with multiplicities $\{e_j\}$. Here we prove the complex version of a stable bound for oscillatory integrals with polynomial phases due to Phong and Stein.

**Proposition 14.1.** For any $f \in \mathbb{C}[X]$ and $\phi \in C^\infty_c(\mathbb{C})$, we have

$$\left| \int_{\mathbb{C}} e(f(z)) \phi(z) \, dz \right| \leq C_{d, \phi} \max_{\xi} \min_{C \ni \xi} \left[ a \prod_{\xi_j \notin C} (\xi - \xi_j)^{e_j} \right]^{-1/(S(C)+1)}.$$  \hspace{1cm} (61)

**Proof.** By Theorem 1.1, it suffices to prove

$$\min_{\xi} \max_{C \ni \xi} \left[ a \prod_{\xi_j \notin C} (\xi - \xi_j)^{e_j} \right]^{-1/(S(C)+1)} \leq C_d H_f$$

where $H_f := \inf_{z \in \mathbb{C}} H_f(z)$ (note that $H_f, \phi \leq H_f$). The above bound in turn is implied by the following: for every $z_* \in \mathbb{C}$, there is a root $\xi$ of $f'$ such that

$$\left[ a \prod_{\xi_j \notin C} (\xi - \xi_j)^{e_j} \right]^{-1/(S(C)+1)} \leq C_d H_f(z_*).$$  \hspace{1cm} (62)

holds for every root cluster $C$ containing $\xi$.

We fix $z_* \in \mathbb{C}$ and choose any root $\xi$ of $f'$ such that $|z_* - \xi| = \min_j |z_* - \xi_j|$. Let $C$ be any root cluster containing $\xi$. Without loss of generality, suppose that $\xi = \xi_0$ and that the rest of roots are ordered so that

$$|z_* - \xi| \leq |z_* - \xi_1| \leq |z_* - \xi_2| \leq \cdots \leq |z_* - \xi_m|.$$

We fix a large constant $A > 0$ to be determined later and set $j_0 = 0$. Let $j_1 \geq 1$ be the smallest integer such that $A|z_* - \xi_{j_1-1}| \leq |z_* - \xi_j|$. Next let $j_2 \geq j_1 + 1$ be the smallest integer such that $A|z_* - \xi_{j_2-1}| \leq |z_* - \xi_{j_2}|$. And so on,..., producing a sequence $0 = j_0 < j_1 < \cdots < j_t$ of integers with

$$A|z_* - \xi_{j_{k-1}}| \leq |z_* - \xi_{j_k}| \text{ for every } 1 \leq k \leq t,$$

and $|z_* - \xi_j| \leq A|z_* - \xi_{j_{t}}|$ for every $t + 1 \leq j \leq m$. Here $0 \leq t \leq m$ where the $t = 0$ case means $|z_* - \xi_j| \leq A|z_* - \xi_j|$ for every $0 \leq j \leq m$. 

We split our root cluster $C$ as
\[ C = C_0 \cup C_1 \cup \cdots \cup C_t \] where $C_k = \{ \xi_{j_k}, \ldots, \xi_{j_k+1-1} \} \cap C$
for $0 \leq k \leq t - 1$ and $C_t = \{ \xi_{j_t}, \ldots, \xi_{n} \} \cap C$. We set $f_k = \sum_{\xi \in C_k} e_j$ so that $S(C) = f_0 + \cdots + f_t$. Note that if $z_* = \xi$, we have $C_0 = \{ \xi \}$ and so $f_0 = e_0$.

For each $0 \leq k \leq t$, set
\[
F_k(z) = \prod_{j=j_k}^{j_k+1-1} (z - \xi_j)^{e_j} =: F_k^1(z) F_k^2(z)
\]
where
\[
F_k^1(z) = \prod_{\xi_j \in C_k} (z - \xi_j)^{e_j} \quad \text{and} \quad F_k^2(z) = \prod_{\xi_j \not\in C_k} (z - \xi_j)^{e_j}.
\]

Therefore $f'(z) = a \prod_{k=0}^{t} F_k(z)$ and if $Q_C := |a \prod_{k=0}^{t} F_k^2(\xi)|$, then our goal is to prove
\[
Q_C^{1/(S(C)+1)} \leq C_d H_f(z_*)
\]
which will establish (63).

By the formula above for $f'$, we have
\[
H_f(z_*) \geq |f'(z_*)| = |a \prod_{k=0}^{t} F_k^1(z_*) F_k^2(z_*)| \gtrsim_d |z_* - \xi|^{S(C)} Q_C
\]
since for any root $\eta$ of $f'$, $|\xi - \eta| \leq |z_* - \xi| + |z_* - \eta| \leq 2|z_* - \eta|$.

To derive other lower bounds for $H_f(z_*)$ in terms of $Q_C$, we consider the derivatives $f^{(1+\rho_k)}$ of $f'$ where $\rho_k := \sigma_0 + \cdots + \sigma_k$ and $\sigma_k = \sum_{j = j_k}^{j_k+1-1} e_j$. To do this, set
\[
F_k(z) = \prod_{\ell=k+1}^{t} F_\ell(z)
\]
for each $0 \leq k \leq t - 1$ and note that
\[
f^{(1+\rho_k)}(z)/\rho_k! = aF_k(z) + aH_k(z)
\]
where both $F_k(z)$ and $H_k(z)$ are homogeneous functions of degree $d - 1 - \rho_k$ ($d = \deg(f)$) in the variables $z - \eta$ as $\eta$ runs over the distinct roots of $f'$. When $z_* = \xi$, we have $\rho_0 = \sigma_0 = e_0$ and so $f^{(1+\rho_0)}(z)/\rho_0! = aF_0(z)$ and $H_0(z) = 0$.

When $z_* \neq \xi$, each term in $H_k(z_*)$ has a factor $z_* - \xi_j$ for some $1 \leq j \leq j_{k+1} - 1$ and so $A|z_* - \xi_j| \leq |z_* - \xi_{j_{k+1}}|$ which implies $|aH_k(z_*)| \leq (1/2)|aF_k(z_*)|$ if $A$ is chosen large enough. Therefore $|f^{(1+\rho_k)}(z_*)/\rho_k!| \geq (1/2)|aF_k(z_*)|$.

Hence
\[
H_f(z_*)^{1+\rho_k} \gtrsim_d |f^{(1+\rho_k)}(z_*)/\rho_k!| \gtrsim_d |aF_k(z_*)|
\]
and if $f^k := \sigma_k - f_k$, then for all $0 \leq k \leq t - 1$,
\[
Q_C \gtrsim_d |z_* - \xi_{j_k}|^{f^0 + \cdots + f^k} \frac{1}{|\prod_{\ell=k+1}^{t} F_\ell(z_*)|} \cdot |aF_k(z_*)|
\]
\[ \lesssim_d \frac{1}{|z_s - \xi_{j_k}|^{f^0 + \cdots + f_k}} H_f(z_s)^{1+\rho_k}. \]

The first inequality follows from the fact that \(|\xi - \xi_j| \leq 2|z_s - \xi_j|\) for every \(j \geq 0\) and \(|z_s - \xi_j| \leq A|z_s - \xi_{j_k}|\) for every \(j_k \leq j \leq j_{k+1} - 1\). The second inequality follows from (63). If \(z_s = \xi\), then \(f^0 = 0\) and we interpret \(|z_s - \xi|^{f^0} = 1\) in the \(k = 0\) case.

Therefore since \(|z_s - \xi_{j_k}| \leq |z_s - \xi_{j_{k+1}}|\), we apply the above inequality for \(Q_c\) for \(k\) and \(k+1\) to conclude that

\[ Q_c \lesssim_d B_{k+1}^{\rho_{k+1} - S(\mathcal{C})} H_{f}(z_s)^{1+\rho_{k+1}} \]

where \(B_{k+1} = |z_s - \xi_{j_{k+1}}|\). The first inequality with \(k = -1\) incorporates (64) if we interpret \(\rho = 0\).

We now divide the analysis into cases depending on the size \(S(\mathcal{C})\) of \(\mathcal{C}\). Suppose \(\rho_k < S(\mathcal{C}) \leq \rho_{k+1}\) for some \(-1 \leq k \leq t - 1\). Again with the interpretation that \(\rho = 0\), we see that any cluster of roots \(\mathcal{C}\) must have a size lying in one of these intervals. With \(\rho_k < S(\mathcal{C}) \leq \rho_{k+1}\), we see that the first inequality in (66) implies

\[ B_{k+1}^{\rho_{k} - S(\mathcal{C})} Q_c \lesssim_d H_{f}(z_s)^{1+\rho_k} \]

and this implies (63) when \(Q_c^{-1/(S(\mathcal{C})+1)} \leq B_{k+1}\) and therefore we may assume

\[ B_{k+1} \leq Q_c^{-1/(S(\mathcal{C})+1)}. \]

When \(z_s = \xi\), the reduction to (67) when \(k = -1\) is automatic.

The second inequality in (66), together with (67), implies

\[ Q_c \leq B_{k+1}^{\rho_{k+1} - S(\mathcal{C})} H_{f}(z_s)^{1+\rho_{k+1}} \leq Q_c^{-\rho_{k+1} - S(\mathcal{C})/(S(\mathcal{C})+1)} H_{f}(z_s)^{1+\rho_{k+1}} \]

and this unravels to (63), completing the proof of (62).

\[ \square \]

15. THE FOURIER TRANSFORM OF MEASURES IN \(\mathbb{C}^d\)

Recall that \(t \to e^{2\pi i t}\) gives the basic character on \(\mathbb{R}\). All other characters on \(\mathbb{R}\) arise from elements \(s \in \mathbb{R}\), \(\chi_s(t) = e^{2\pi i s t}\). In the same way, starting with the basic character \(e(z) = e^{2\pi i x} e^{2\pi i y}\) where \(z = x + iy\), the other characters on \(\mathbb{C} \simeq \mathbb{R}^2\) arise from elements \(w = (u, v) \in \mathbb{R}^2\), \(e_w(z) = e^{2\pi i (xu + yv)}\).

The fourier transform \(\mathcal{F}(\sigma) = \mathcal{\hat{\sigma}}\) of a Borel measure \(\sigma\) on \(\mathbb{C}\) is defined as

\[ \mathcal{\hat{\sigma}}(\xi, \eta) = \int_{\mathbb{R}^2} e^{2\pi i [(\xi, \eta) \cdot (x, y)]} \, d\sigma(x, y) \]

where \((x, y) \cdot (\xi, \eta) = x\xi + y\eta\). Now we write this in complex notation using the transformation \(T : \mathbb{C} \to \mathbb{R}^2\) defined by \(T(w) = (u + v, u - v)\) where \(w = u + iv\). Note that if \(z = x + iy\), then \(zw = (xu - yv) + i(xv + yu)\) and so \(\text{Re}(zw) + \text{Im}(zw) = \)
It will be convenient for us to think of $F \circ T$ as the more appropriate notion of the Fourier transform.

This discussion readily extends to $C^d \simeq \mathbb{R}^{2d}$. A complex vector $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{C}^d$ can be viewed as a real vector in $\mathbb{R}^{2d}$ by writing out the real and imaginary parts of each $\xi_j = \gamma_j + i\eta_j$ so that $\xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{R}^{2d}$. If $\sigma$ now denotes a Borel measure on $\mathbb{C}^d$, then for $\omega = (z_1, \ldots, z_d) \in \mathbb{C}^d$ (or $\omega = (x_1, y_1, \ldots, x_d, y_d) \in \mathbb{R}^{2d}$),

$$\hat{\sigma}(\omega) = \int_{\mathbb{R}^{2d}} e^{2\pi i \langle \xi, \omega \rangle} d\sigma(\omega, y)$$

and this again can be written in complex notation (using complex multiplication). For $\omega, \nu \in \mathbb{C}^d$, we write $\langle \omega, \nu \rangle = \sum_{j=1}^d z_j w_j$ which is a slight variant of the usual hermitian inner product on $\mathbb{C}^d$ (the form being symmetric instead of being skew-symmetric). Extend the transformation $T$ above to $T : \mathbb{C}^d \to \mathbb{R}^{2d}$ by defining

$$T(\omega) = (Tw_1, \ldots, Tw_d) \in \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \simeq \mathbb{R}^{2d}.$$

We have

$$\hat{\sigma}(T\omega) = \int_{\mathbb{R}^{2d}} e^{2\pi i \langle \sum_{j=1}^d |\text{Re}(w_j z_j) + \text{Im}(w_j z_j)\rangle \rangle} d\sigma(x, y) = \int_{\mathbb{C}^d} e(\langle w, z \rangle) d\sigma(z)$$

and again, it will be more convenient to think of $\hat{\sigma} \circ T$ as the Fourier transform of $\sigma$.

An important class of Borel measures $\sigma$ which arise in euclidean harmonic analysis is surface measure on some polynomially parameterised $n$-dimensional surface $\varphi \in \mathbb{R}^n \to \Phi(\varphi) \in \mathbb{R}^N$ where $\Phi(\varphi) = (Q_1(\varphi), \ldots, Q_N(\varphi))$ is an $N$-tuple of polynomials $Q_j \in \mathbb{R}[X_1, \ldots, X_n]$.

One could consider problems over $\mathbb{R}$ and examine analogues over $\mathbb{C}$. For instance, the Fourier restriction problem with respect to the moment curve $\gamma_r : \mathbb{R} \to \mathbb{R}^d$ defined by $\gamma_r(t) = (t, t^2, \ldots, t^d)$ was solved (with a remarkable argument) by Drury in the 1980s [5]. One could try to see if arguments for the Fourier restriction problem with respect to $\gamma_r$ extend to establishing Fourier restriction estimates for the complex moment curve $\gamma_c : \mathbb{C} \to \mathbb{C}^d$ defined by $\gamma_c(z) = (z, z^2, \ldots, z^d)$, considered as a 2-surface in $\mathbb{R}^{2d} \simeq \mathbb{C}^d$. If $z = x + iy$, this 2-surface is parameterised by $\Gamma(x, y) = (\phi_1(x, y), \psi_1(x, y), \ldots, \phi_d(x, y), \psi_d(x, y))$ where

$$\phi_1(x, y) = x, \phi_2(x, y) = y, \phi_3(x, y) = x^2 - y^2, \psi_2(x, y) = 2xy, \phi_3(x, y) = x^3 - 3xy^2, \ldots$$

So this is a problem in real harmonic analysis.

The problem of Fourier restriction to complex curves has been studied by a number of authors; see for example, [5], [6] and more recently [7] and [10] where positive results have been obtained. The question arises whether the results are sharp.
In the real case, it is easy to determine by a scaling argument what is the necessary relationship between the Lebesgue exponents $p$ and $q$ when the restriction operator $R : L^p \rightarrow L^q$ to the curve $\gamma_r$ is bounded. To determine the necessary range in $p$ where there is some restriction estimate, one usually looks at the extension operator (the dual operator to $R$)

$$E b(x) := \int_0^1 e^{2\pi i [x \cdot \gamma_r(t)]} b(t) \, dt$$

(so that a bound $R : L^p \rightarrow L^q$ is equivalent to a bound $E : L^q' \rightarrow L^p'$) and applies it to $b \equiv 1$ which is in every $L^q'$. Hence we want to determine the range of $p'$ such that $E 1(x)$ is in $L^{p'}$.

Note that

$$E 1(x) = \hat{\sigma}(x) = \int_0^1 e^{2\pi i P(t)} \, dt$$

where $\sigma$ is (more or less) arclength measure on a compact piece of $\gamma_r$ and the phase $P(t) = P_x(t) = x_1 t + x_2 t^2 + \cdots + x_d t^d$ is a real polynomial whose coefficients give us a function in $x \in \mathbb{R}^d$. The bound $|\hat{\sigma}(x)| \leq C_d \|x\|^{-1/d}$ follows from the classical van der Corput estimates and the exponent $1/d$ is best possible if we measure the decay in terms of the isotropic norm $\|\cdot\|$ on $\mathbb{R}^d$. This estimate alone does not give us the correct $L^{p'}$ range of integrability. In fact the precise range in $p'$ where $\hat{\sigma} \in L^{p'}$ is not a straightforward problem to resolve.

It turns out that the Lebesgue norm $\|\hat{\sigma}\|_{L^{p'}(\mathbb{R}^d)}$ arises as a constant (the singular integral) in an asymptotic formula for the number of solutions to a system of diophantine equations. Determining a precise count for various systems of diophantine equations became known as Tarry’s problem which have been studied since the 1920s. Ever since then, number theorists have been interested in finding the exact range of $L^{p'}$ integrability for $\hat{\sigma}$ with some partial progress given by Vinogradov and Hua in the 1930s. In the late 1970s, Arkhipov, Chubarikov and Karatsuba (see also [1] and [3]) solved this problem with an elegant argument using a very clever application of van der Corput estimates for oscillatory integrals. As a consequence we now know that Drury’s result is sharp.

So one might ask whether the argument can extend to the complex case. For the complex moment curve, the extension operator applied to $b \equiv 1$ can be written as

$$E 1(\xi) = \int_{\mathbb{R}^2} e^{2\pi i [\xi \cdot \Gamma(x,y)]} \hat{\phi}(x,y) \, dx \, dy = \int_{\mathbb{R}^2} e^{2\pi i [\sum_{j=1}^d \xi_j \phi_j(x,y) + \sum_{j=1}^d \eta_j \psi_j(x,y)]} \hat{\phi}(x,y) \, dx \, dy$$

where (as above) the complex vector $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^d$ with $\xi_j = \gamma_j + i \eta_j$ can be viewed a real vector $\xi_r = (\gamma_1, \eta_1, \ldots, \gamma_d, \eta_d)$ in $\mathbb{R}^{2d}$. Here $\phi \in C^\infty_c(B)$ where $B$ is the unit ball in $\mathbb{R}^2$.

Using the notation developed above, we have

$$E 1(T_w) = \int_{\mathbb{C}} e(\langle w, \gamma_c(z) \rangle) \hat{\phi}(z) \, dz$$
where $\phi \in C^\infty_c(D)$. The problem of determining the $L^p$ integrability of $E_1$ is the same as the one for $E_1 \circ T$. Note that the phase $\langle w, \gamma_c(z) \rangle = \sum_{j=1}^d w_j z^j$ is a general complex polynomial of degree $d$. Hence we arrive at studying the two dimensional oscillatory integral

$$I(w) := \int_C e(P_w(z)) \phi(z) \, dz$$

where $P_w(z) = w_1 z + w_2 z^2 + \cdots + w_d z^d$ is a complex polynomial.

Also relevant for Tarry’s problem as well as for the Fourier restriction problem is the case of sparse polynomials $P_{\text{sparse}}(z) = w_1 z_{k_1} + \cdots + w_d z_{k_d}$ where $k_1 < k_2 < \cdots < k_d$ when $K := k_1 + \cdots + k_d < (0.5)k_d(k_d + 1)$. The interest is in determining when

$$I_{\text{sparse}}(w) := \int_C e(P_{\text{sparse}}(z)) \phi(z) \, dz$$

belongs to the space $L^q(C^d)$.

Proposition 1.4 determines the $L^q$ integrability of $I(w)$ and $I_{\text{sparse}}(w)$.

16. Proposition 1.4 — some preliminaries

For $w = (w_1, \ldots, w_d) \in C^d$, let

$$P(z) = P_w(z) = w_d z^d + w_{d-1} z^{d-1} + \cdots + w_1 z$$

be a complex polynomial of degree $d$ with coefficients $w$. Consider the oscillatory integral

$$I(w) = \int_C e(P_w(z)) \phi(z) \, dz$$

where $\phi \in C^\infty_c(D)$ satisfies $\phi(z) \equiv 1$ in $D_{1/2}$. For this particular oscillatory integral, we have changed notation from $I_\phi(P_w)$ to $I(w)$ to highlight that we want to view this oscillatory integral as a function of the coefficients of $P_w(z)$. Similarly we define $I_{\text{sparse}}(w)$ with respect to a polynomials with sparse powers as in the previous section.

Recall the statement of Proposition 1.4 we have

$$I := \int_{C^d} |I(w)|^q \, d\mu < \infty \quad (68)$$

if and only if $q > 0.05(d^2 + d) + 1$. Furthermore, $\|I_{\text{sparse}}\|_{L^q} < \infty$ if and only if $q > K$.

We follow the argument in [1], complexifying two key results. As in the real case, (68) relies on van der Corput bounds for oscillatory integrals as in Theorem 1.1 when $n = 1$, together with a sublevel set bound for

$$H(w) := \inf_{z \in \text{supp}(\phi)} \max(|P'(z)|, |P''(z)/2|^{1/2}, \ldots, |P^{(d)}(z)/d|^{1/d}).$$

This is the case whether we are examining the $L^q$ norm of $I$ or $I_{\text{sparse}}$. Here we will only give the details establishing (68) since the argument for $I_{\text{sparse}}$ runs in the same way as in the real case.
For convenience we restate the refined version of Theorem 1.1 in the case \( n = 1 \) as detailed in Section 11; see (37) and (38).

**Proposition 16.1.** For any polynomial \( Q \in \mathbb{C}[X] \) of degree \( d \), we have

\[
I_\phi(Q)| \leq C_{d,\phi} H_Q^{-2}
\]

where

\[
C_{d,\phi} \leq C_{d,k} [1 + J^{-k} \| \phi \|_{C^k}]
\]

for any large \( k \). In particular, for \( I(w) \) appearing in (68), we have

\[
|I(w)| \leq C_{d,\phi} H(w)^{-2}.
\]

The sublevel set bound for \( H(w) \) is the following.

**Proposition 16.2.** For any \( Q \gg 1 \), we have

\[
|\{ w \in \mathbb{C}^d : H(w) \leq Q \}| \lesssim d Q^{2[0.5(d^2+d)+1]}.
\]

The proof of Proposition 16.2 is a straightforward complex extension of the real version due to Arkhipov, Chuburakov and Karatsuba, see [1].

### 17. Proof of Proposition 1.4 - the sufficiency

We decompose

\[
\mathcal{I} = \int_{\{H(w) \leq 1\}} |I(w)|^q \, dw + \sum_{r \geq 0} \int_{\{2^r < H(w) \leq 2^{r+1}\}} |I(w)|^q \, dw.
\]

Of course we have the trivial bound \( |I(w)| \lesssim 1 \) and this shows that the first term is at most \( |\{H(w) \leq 1\}| \) which in turn is finite by (71) in Proposition 16.2. Let us denote the \( r \)th term in the sum defining the second term by \( \mathcal{I}_r \); that is,

\[
\mathcal{I}_r = \int_{\{2^r < H(w) \leq 2^{r+1}\}} |I(w)|^q \, dw =: \int_{E_r} |I(w)|^q \, dw.
\]

Proposition 16.1 implies that for \( w \in E_r \), \( |I(w)| \lesssim_d 2^{-2r} \) and so \( \mathcal{I}_r \lesssim_d 2^{-2rq} |E_r| \). By Proposition 16.2 we have \( |E_r| \lesssim_d 2^{2r[0.5(d^2+d)+1]} \) and so

\[
\sum_{r \geq 0} \int_{E_r} |I(w)|^q \, dw \lesssim_d \sum_{r \geq 0} 2^{-2r[q-0.5(d^2+d)-1]}
\]

which is finite when \( q > q_d \). This completes the proof of the sufficiency part of Proposition 1.4.
18. Proof of Proposition 16.2

Let
\[ S_Q := \{ w \in \mathbb{C}^d : H(w) \leq Q \} \]

so that (71) says \( |S_Q| \lesssim dQ^{2(0.5(d^2+d)+1)} \).

For \((r, s) \in \mathbb{Z}^2\) satisfying \(-Q/2 \leq r, s \leq Q/2\), let \( z_{r,s} = r/Q + ir/Q \) and define
\[ S_{r,s}^r := \{ w \in \mathbb{C}^d : |P_k^{(k)}(z_{r,s})| \leq k!c_k Q^k, \ 1 \leq k \leq d \} \]

for some appropriate large constants \( c_k \). Our basic claim is
\[ S_Q \subseteq \bigcup_{-Q/2 \leq r,s \leq Q/2} S_{r,s}^r. \] (72)

We write
\[ |S_{r,s}^r| = \int_{S_{r,s}^r} dw \]

and we will compute the integral defining \( |S_{r,s}^r| \) by making a certain change of variables. Note that
\[ P_k^{(k)}(z_{r,s}) = w_k + g_{k+1}w_{k+1}z + \cdots + g_{d+1}w_d z^{d-k} \]

for some combinatorial numbers \( g_j \). We make the linear change of variables \( y = Tw \) where for each \( 1 \leq k \leq d \),
\[ y_k := w_k + g_{k+1}w_{k+1}z_{r,s} + \cdots + g_{d+1}w_d z_{r,s}^{d-k}. \]

Note that \( \det T = 1 \) and \( |y_k| \leq c_k Q^k \) for \( w \in S_{r,s}^r \). Hence
\[ |S_{r,s}^r| = \int_{|y_1| \leq c_1 Q} \cdots \int_{|y_d| \leq c_d Q^d} dy = C_d Q^{2(0.5(d^2+d))} \]

and so the claim above implies \( |S_Q| \lesssim dQ^{2Q^{2(0.5(d^2+d))}} \), completing the proof of Proposition 16.2.

To prove the claim, fix \( w \in S_Q \) so that \( H(w) \leq Q \). By the definition of \( H \), we see that there is a \( z = u + iv \in \mathbb{D} \) such that \( |P_k^{(k)}(z)| \leq Q^k \) for all \( 1 \leq k \leq d \). Let \( r := \lfloor uQ \rfloor, s := \lfloor vQ \rfloor \) and set \( w = z_{r,s} - z \). Note that \( w = (r/Q - u) + i(s/Q - v) \) and so \( |w| \leq \sqrt{2Q}^{-1} \).

For \( 1 \leq k \leq d \), we Taylor expand
\[ P_k^{(k)}(z_{r,s}) = P_k^{(k)}(z) + P_k^{(k+1)}(z)w + \frac{1}{2!}P_k^{(k+2)}(z)w^2 + \cdots + \frac{1}{(d-k)!}P_k^{(n)}(z)w^{d-k} \]

so that
\[ |P_k^{(k)}(z_{r,s})| \lesssim \sum_{\ell=0}^{d-k} Q^{k+\ell}|w|^\ell \leq k!c_k Q^k \]

for appropriate constants \( c_k \). This implies that \( w \in S_{r,s}^r \), completing the proof of the claim and hence the proof of the proposition.
Here we follow [1] in the real case. We bound

$$I \geq \int_R |I(\underline{x})|^q \, d\underline{x},$$

where $R \subset \mathbb{C}^d$ is a region where we will be able to estimate $I(\underline{x})$ from below. The region $R$ will be a disjoint union of subregions $R_m, m \geq m_0$ for some large integer $m_0 = m_0(d)$. To define $R_m$, we fix a lacunary sequence $Q_m = A^m$ where $A = A_d > 2$ and define the planar sets

$$E_m := \{ z = x + iy \in \mathbb{C} : Q_m^d \leq x \leq (2Q_m)^d, Q_m^d \leq y \leq (2Q_m)^d \}.$$ 

These are disjoint since $A > 2$. Writing $\underline{x} = (\underline{x}', x_d) \in \mathbb{C}^d$, the sets $R_m$ will be of the form

$$R_m := \{ \underline{z} \in \mathbb{C}^d : x_d \in E_m, \underline{z}' \in R_m'(x_d) \}$$

where for each $x_d \in E_m$, the slice $R_m'(x_d) \subset \mathbb{C}^{d-1}$ will be specified momentarily. Note that the sets $\{R_m\}$ are pairwise disjoint whatever our choice for $R_m'(x_d)$.

Set $w_0 = 0.25(1 + i)$ and define, for each $m \gg 1$,

$$P_m := \{ (r, s) \in \mathbb{N}^2 : 1 \leq r, s \leq Q_m/10 \}.$$ 

For each $(r, s) \in P_m$, let $z_{r,s} = w_0 + (r/Q_m + is/Q_m) \in \mathbb{D}_{1/2}$. Note that if $(r, s) \neq (r', s')$, then $|z_{r,s} - z_{r', s'}| \geq Q_m^{-1}$. For each $x_d \in E_m$, we define

$$R_m'(x_d) := \bigcup_{(r, s) \in P_m} R_m^{r,s}(x_d)$$

where $R_m^{r,s}(x_d) = T_{x_d}^{r,s} B$ and

$$B := \{ y' = (y_1, \ldots, y_{d-1}) \in \mathbb{C}^{d-1} : |y_j| \leq (c_1 Q_m)^j, 1 \leq j \leq d - 1 \},$$

c_1 a small constant depending on $d$, and $T_{x_d}^{r,s}$ an affine transformation $T_{x_d}^{r,s} \underline{y}' = \underline{z}$

defined by the relationship

$$x_d z_d^d + \cdots + x_1 z = x_d (z - z_{r,s})^d + y_{d-1} (z - z_{r,s})^{d-1} + \cdots + y_1 (z - z_{r,s});$$

in other words,

$$x_{d-1} = y_{d-1} - dx_d z_{r,s}, \quad x_{d-2} = y_{d-2} - (d - 1) y_{d-1} z_{r,s} + \left(\frac{d}{2}\right) x_d z_{r,s}^{2}, \quad \text{etc...}$$

and generally, for $1 \leq k \leq d - 1$,

$$x_{d-k} = y_{d-k} + \sum_{\ell=1}^{k} (-1)^{\ell} \left(\frac{d - k + \ell}{\ell}\right) y_{d-k+\ell} z_{r,s}^{\ell}. \tag{73}$$

Here we write $x_d = y_d$ for convenience. Hence the linear part of $T_{x_d}^{r,s}$ is upper triangular with 1’s down the diagonal and so $\det T_{x_d}^{r,s} = 1$. In particular $T_{x_d}^{r,s}$ is a bijection.

For each $\underline{z}' \in R_m^{r,s}(x_d)$, we see that $x_{d-1} \in \mathbb{D}(c_1 Q_m)^{d-1}(-dx_d z_{r,s})$ and these discs are disjoint as we vary over $(r, s) \in P_m$. Indeed if $(r, s), (r', s') \in P_m$ are two distinct elements, then $|z_{r,s} - z_{r', s'}| \geq Q_m^{-1}$ and so the centres of these discs

$$|dx_d z_{r,s} - dx_d z_{r', s'}| = d|x_d| |z_{r,s} - z_{r', s'}| \geq Q_m^d Q_m^{-1} = Q_m^{d-1}.$$
are separated by more than twice the radius \((c_1 Q_m)^{d-1}\) if \(c_1\) is small. This shows that the sets \(\{R_m^{r,s}(x_d)\}_{(r,s) \in \mathcal{P}_m}\) are pairwise disjoint for each \(m\) and each \(x_d \in E_m\).

Hence
\[
\mathcal{I} \geq \sum_{m \geq m_0} \int_{R_m} |I(x)|^q \, dx = \sum_{m \geq m_0} \sum_{(r,s) \in \mathcal{P}_m} \int_{x_d \in E_m} \int_{z \in R_m^{r,s}(x_d)} |I(x)|^q \, dx.
\]

We perform the change of variables \(z' = T_{x_d}^{r,s} y'\) in the inner integral to write
\[
\int_{R_m^{r,s}(x_d)} |I(x)|^q \, dx' = \int_{|y_{d-1}| \leq (c_1 Q_m)^{d-1}} \cdots \int_{|y_1| \leq c_1 Q_m} |II_{r,s}(y)|^q \, dy'
\]
where
\[
II_{r,s}(y) = \int_{C} e(y_d z - z_{r,s})^q + \cdots + y_1 (z - z_{r,s})) \phi(z) \, dz.
\]
Recall we are writing \(y_d = x_d\) for convenience.

We make the change of variables \(z \to z - z_{r,s}\) in \(II_{r,s}\) and write
\[
II_{r,s}(y) = \int_{C} e(y_d z^d + \cdots + y_1 z) \phi(z + z_{r,s}) \, dz =: \int_{C} e(P(z; y)) \phi(z + z_{r,s}) \, dz.
\]
Fix a nonnegative \(\psi \in C_c^\infty(D_2)\) with \(\psi \equiv 1\) on \(D\) and define \(\psi_m(z) := \psi((Q_m/a) z)\) where \(a = a_d\) is a large constant. We split \(II_{r,s}(y) = II_{r,s}^1(y) + II_{r,s}^2(y)\) where
\[
II_{r,s}^1(y) = \int_{C} e(P(z; y)) \phi(z + z_{r,s}) \psi_m(z) \, dz
\]
and
\[
II_{r,s}^2(y) = \int_{C} e(P(z; y)) \phi(z + z_{r,s})(1 - \psi_m(z)) \, dz.
\]
Our goal is to prove the following:

\[
\text{for } y \text{ satisfying } y_d \in E_m, \quad |y_{d-1}| \leq (c_1 Q_m)^{d-1}, \quad \ldots, \quad |y_1| \leq c_1 Q_m, \quad (74)
\]
\[
|II_{r,s}^1(y)| \geq B Q_m^{-2} \quad \text{and} \quad |II_{r,s}^2(y)| \leq (B/2) Q_m^{-2} \quad (75)
\]
for some \(B = B_d > 0\).

The measure of those \(y\) satisfying (74) is \(C Q_m^{2(d+(d-1) + \cdots + 1)}\) or \(C Q_m^{2(0.5(d^2 + d))}\). So if (75) holds, then
\[
\mathcal{I} \gtrsim \sum_{m \geq m_0} \sum_{(r,s) \in \mathcal{P}_m} Q_m^{-2 q Q_m^{2(0.5(d^2 + d))}} \gtrsim \sum_{m \geq m_0} Q_m^{-2 q - 0.5(d^2 + d) - 1}
\]
which shows that \(\mathcal{I} = \infty\) if \(q \leq q_d\), establishing the necessity of Proposition 1.4.

We first establish the bound for \(II_{r,s}^2\) in (75). Set
\[
\rho(z) = \rho_{m,r,s}(z) := \phi(z + z_{r,s})(1 - \psi_m(z)).
\]
For \(z \in \text{supp}(\rho)\) and \(y\) satisfying (74), we have
\[
|P^{(d-1)}(z)| = |d! y_d z + (d-1)! y_{d-1}| \geq d! [a |y_d| Q_m^{-1} - (c_1 Q_m)^{d-1}] \geq a Q_m^{d-1}
\]
since \( a \) is large and \( c_1 \) is small. Therefore

\[
H_P = \inf_{z \in \text{supp}(\phi)} \max(|P^{(d)}(z)|^{1/d}, \ldots, |P(z)|) \geq a^{1/(d-1)} Q_m^2 \tag{76}
\]

and so we are in a position to apply Proposition 16.1 and conclude

\[
|II_{r,s}^2(y)| \leq C_{d,\rho} H_1^{-2} \leq C_{d,\rho} a^{-2/(d-1)} Q_m^{-2}
\]

for \( y \) satisfying (74). Since \( \rho \) depends on \( m \) and \( r, s \), we need to understand how \( C_{d,\rho} \) depends on \( \rho \). We apply (69) from Proposition 16.1 to conclude

\[
C_{d,\rho} \leq C_d \left[ 1 + J^{-k} \|\rho\|_{C^k} \right]
\]

for some large \( k = k(d) \). Here \( J \geq a^{1/(d-1)} Q_m^2 \) and \( \|\rho\|_{C^k} \lesssim_d [Q_m/a]^k \). Therefore \( C_{d,\rho} \leq C_d a^{-kd/(d-1)} \) implying

\[
|II_{r,s}^2(y)| \leq C_d a^{-2/(d-1)} Q_m^{-2} \text{ for } y \text{ satisfying (74)}.
\]

We note here that we could have considered \( |P^{(d)}(z)| = d!|z| \geq d!Q_m^d \) and bounded \( H_P \) in (76) below by \( (d!)^{1/d} Q_m \). But \( (d!)^{1/d} \leq d \) and we would not have gained the very large constant \( a^{1/(d-1)} \) in (76) and hence the small constant in (77).

We now turn to \( II_{r,s}^1(y) \) when \( y \) satisfies (74). First we note that \( \phi(z + z_{r,s}) \psi_m(z) = \psi_m(z) \) since when \( \psi_m(z) \neq 0 \), then \( |z| \leq 2aQ_m^{-1} \) and so \( z + z_{r,s} \in \mathbb{D}_{1/2} \), implying \( \phi(z + z_{r,s}) = 1 \). We can therefore write

\[
II_{r,s}^1(y) = \int_C e(ydz^d) \psi_m(z) \, dz + E
\]

where

\[
|E| = \left| \int_C \left[ e(P(z; y)) - e(ydz^d) \right] \psi_m(z) \, dz \right| \leq \int_{\{|z| \leq 2aQ_m^{-1}\}} |P(z; y) - ydz^d| \, dz.
\]

For \( |z| \leq 2aQ_m^{-1} \) we have

\[
|P(z; y) - ydz^d| \leq \sum_{j=1}^{d-1} |y_j z^j| \leq \sum_{j=1}^{d-1} (c_1 Q_m)^j (2aQ_m^{-1})^j \leq 2d c_1 a
\]

if we chose \( a \) and \( c_1 \) so that \( 2c_1 a \leq 1 \). Therefore \( |E| \lesssim_d (c_1 a)(2aQ_m^{-1})^2 \lesssim_d a^3 c_1 Q_m^{-2} \). This implies

\[
|II_{r,s}^1(y)| \geq \left| \int_C e(ydz^d) \psi_m(z) \, dz \right| \geq C_d a^3 c_1 Q_m^{-2}, \tag{78}
\]

leaving us to analyse the main term

\[
II_{r,s}^{\text{main}}(y_d) := \int_C e(ydz^d) \psi_m(z) \, dz = \int_C e(ydz^d) \psi((Q_m/a)z) \, dz.
\]

Finally we write

\[
II_{r,s}^{\text{main}}(y_d) = \int_C e(ydz^d) \, dz + \int_C e(ydz^d) \left[ 1 - \psi((Q_m/a)z) \right] \, dz =: A + B
\]

where these two improper integrals are interpreted as limits of truncated integrals. For example, for \( B \) we mean \( B = \lim_{R \to \infty} B_R \) where

\[
B_R := \int_C e(ydz^d) \left[ 1 - \psi((Q_m/a)z) \right] \Psi(R^{-1}z) \, dz
\]
and $\Psi \in C^\infty(\mathbb{D})$ with $\Psi(z) \equiv 1$ on $\mathbb{D}_{1/2}$. It is fairly straightforward to see that these limits exist (in fact, the analysis below will show this as a consequence). To bound $|B|$ from above, it suffices to bound $|B_R|$ from above, uniformly in $R$.

We change variable $w = R^{-1}z$ to write

$$B_R = R^2 \int_{\mathbb{C}} e(y_d R^d w^d) \Psi(w) \left[ 1 - \psi((Q_m R/a)w) \right] dw =: \int_{\mathbb{C}} e(y_d R^d w^d) \rho(w) dw.$$ 

We will apply Proposition 16.1 to $B_R$ with $P(w) = y_d R^d w^d$. Note that $H_P \geq \inf_{w \in \text{supp}(\rho)} |P'(w)| \geq d|y_d| R^d (a/(RQ_m))^{d-1} \geq da^{d-1} RQ_m$.

Hence

$$|B_R| \lesssim d^{-2(d-1)} [1 + J^{-k} \|\rho\|_{C^k}] Q_m^{-2},$$

where $\|\rho\|_{C^k} \lesssim (RQ_m/a)^k$ and $J = \inf_{w \in \text{supp}(\rho)} \max(|P^{(d)}(w)|^{1/d}, \ldots, |P''(w)|^{1/2}) \geq \sqrt{d(d-1)(RQ_m)^{d/2}}((a/RQ_m)^{(d-2)/2},$

implying

$$J \geq \sqrt{d(d-1)a^{(d-2)/2}} RQ_m.$$ 

Therefore $|B_R| \lesssim d^{-2(d-1)} Q_m^{-2}$ and so

$$|B| \lesssim d^{-2(d-1)} Q_m^{-2}. \quad (79)$$

It remains to treat the first integral

$$A = \int_{\mathbb{C}} e(y_d z^d) dz$$

of the main term $II_{r,s}^{\text{main}}(y)$ when $y_d \in E_m$. Fix $y_d = re^{i\theta} \in E_m$ so that $r = |y_d|$.

We make the change of variables $w = uz$ where $u = r^{1/d}e^{i\theta/d}$ so that $w^d = y_d z^d$. The Jacobian of this change of variables is $r^{2/d} = |y_d|^{2/d}$ and so

$$A = c_d \frac{1}{|y_d|^{2/d}} \int_{\mathbb{C}} e(u^d) du$$

is nonzero. This implies $|A| \geq |c_d| Q_m^{-2}$. This bound, together with the bound for $B$ in (79), implies

$$|II_{r,s}^{\text{main}}(y)| \geq (|c_d|/2) Q_m^{-2} \quad (80)$$

if we choose the constant $a = a_d$ large enough.

Putting (80) together with (77) and (78) (choosing first the constant $a$ large and then choosing the $c_1$ small so that $c_1 a^3 \ll 1$) establishes the desired bound (75), completing the necessity part and hence the proof Proposition 1.4.
20. Appendix: proofs of Propositions 3.1 and 3.2

In this appendix we give the proofs of Proposition 3.1 and Proposition 3.2 which rely on Lemma 9.3.

First let us see how Lemma 9.3 implies Proposition 3.1. In this case we have $|f'(z)| \geq 1$ on $D$ and we will apply Lemma 9.3 with $L = 1$ to

$$\phi(z) := f(z) \quad \text{and} \quad z_0 = \{z \in \mathbb{D} : |f(z)| \leq \epsilon \}.$$ 

Note that $\delta = |\phi(z_0)\phi'(z_0)^{-2}| \leq \epsilon$ and since $\delta M \leq \epsilon M < 1/64$, we see that there exists a $z_* \in \mathbb{D}_{5/4}$ with $f(z_*) = 0$ and $|z_0 - z_*| \leq 2|\phi(z_0)\phi'(z_0)^{-1}| \leq 2\epsilon$. Hence

$$\{z \in \mathbb{D} : |f(z)| \leq \epsilon \} \subseteq \bigcup_{z_* \in Z_0} \mathbb{D}_{2\sqrt{\epsilon}}(z_*),$$

(81)

where $Z_0 := \{z \in \mathbb{D}_{5/4} : f(z) = 0\}$. This completes the proof of Proposition 3.1.

We turn to the proof of Proposition 3.2 and we begin with the case $k = 2$ so that $|f''(z)| \geq 1$ on $D$. We split the sublevel set $S := \{z \in \mathbb{D} : |f(z)| \leq \epsilon\} = S_1 \cup S_2$ into two sets where

$$S_1 := \{z \in S : |f'(z)| \leq A\sqrt{\epsilon}\} \quad \text{and} \quad S_2 := \{z \in S : |f'(z)| > A\sqrt{\epsilon}\}$$

where $A = c_0\sqrt{M}$ for some large but absolute $c_0 \gg 1$. For $S_1$, we apply Lemma 9.3 with $L = 1$ to $\phi(z) := f'(z)$ and $z_0 \in S_1$. Note that $\delta = |\phi(z_0)\phi'(z_0)^{-2}| \leq A\sqrt{\epsilon}$ and $\delta M \leq c_0M^{3/2}\sqrt{\epsilon} < 1/64$ since $\epsilon \ll M^{-3}$. Hence we see that there exists a $z_* \in \mathbb{D}_{5/4}$ with $f'(z_*) = 0$ and $|z_0 - z_*| \leq 2|\phi(z_0)\phi'(z_0)^{-1}| \leq 2A\sqrt{\epsilon}$ and so

$$S_1 \subseteq \bigcup_{z_* \in Z_1} \mathbb{D}_{2A\sqrt{\epsilon}}(z_*)$$

(82)

where $Z_1 := \{z \in \mathbb{D}_{5/4} : f'(z) = 0\}$.

For $S_2$, we apply Lemma 9.3 with $L = 1$ to $\phi(z) := f(z)$ and $z_0 \in S_2$. Note that $\delta = |\phi(z_0)\phi'(z_0)^{-2}| \leq A^{-2}$ and $\delta M \leq (c_0)^{-1} < 1/64$ since $c_0 \gg 1$ is large. Hence we see that there exists a $z_* \in \mathbb{D}_{5/4}$ with $f(z_*) = 0$ and $|z_0 - z_*| \leq 2|\phi(z_0)\phi'(z_0)^{-1}| \leq 2A^{-1}\sqrt{\epsilon}$ and so

$$S_2 \subseteq \bigcup_{z_* \in Z_0} \mathbb{D}_{\sqrt{\epsilon}}(z_*),$$

(83)

completing the proof of Proposition 3.2 when $k = 2$.

For $k \geq 3$, we split the sublevel set $S := \{z \in \mathbb{D} : |f(z)| \leq \epsilon\} = S_1 \cup \cdots \cup S_k$ into $k$ sets where

$$S_1 := \{z \in S : |f^{(k-1)}(z)| \leq A\epsilon^{1/k}\}, \quad \text{and}$$

$$S_2 := \{z \in S : |f^{(k-1)}(z)| > A\epsilon^{1/k}, |f^{(k-2)}(z)| \leq d_2\epsilon^{1/k} |f^{(k-1)}(z)|\}.$$ 

The remaining sets $S_j$, $3 \leq j \leq n$, are defined with respect to a series of inequalities

$$I_j(z) := |f^{(k-j+1)}(z)| > d_{j-1}\epsilon^{1/k} |f^{(k-j+2)}(z)| > \cdots > (d_{j-1} \cdots d_2)\epsilon^{(j-2)/k} |f^{(k-1)}(z)| > A(d_{j-1} \cdots d_2)\epsilon^{(j-1)/k}$$

being satisfied. For $3 \leq j \leq k - 1$, we define

$$S_j := \{z \in S : I_j(z) \text{ is satisfied, } |f^{(k-j)}(z)| \leq d_j\epsilon^{1/k} |f^{(k-j+1)}(z)|\}.$$
Here $d_2 = M^{-1/k}$ and $d_j = \eta d_{j-1}, 3 \leq j \leq k-1$ for an appropriate small $\eta = \eta_k > 0$. Also $A = c_0 M^{(k-1)/k}$ where $c_0 \gg 1$ will be chosen large enough, depending on $\eta$ (and $k$). Finally we have

$$S_k = \{ z \in S : I_k is \text{ satisfied} \}.$$  

Of course we see that the definition of $S_2$ is incorporated in the definitions of $S_j$ above when $j = 2$ if we interpret the product $d_{j-1} \cdots d_2$ as the empty product when $j = 2$ and so equal to 1.

We will apply the Lemma 9.3 with various choices of $L$ to various derivatives $\phi(z) = f^{(j)}(z)$. For $z \in \mathbb{D}_{7/4}$, Cauchy’s integral formula gives us

$$\phi(z) = \frac{j!}{2\pi i} \int_{C_{1/4}(z)} \frac{f(w)}{(w - z)^{j+1}} \, dw$$

where $C_r(z) = \{ w : |w - z| = r \}$ denotes the circle of radius $r$ centred at $z$. Hence $M_j := M_{\phi} = \sup_{z \in \mathbb{D}_{7/4}} |f^{(j)}(z)| \leq j! \epsilon / M$.

For $z_0 \in S_1$, we apply Lemma 9.3 with $L = 1$ to $\phi(z) = f^{(k-1)}(z)$ and $M_{k-1} \leq 4^{-k}(k - 1)!M$. Here

$$\delta = |\phi(z_0)\phi'(z_0)^{-2}| = |f^{(k-1)}(z_0)f^{(k)}(z_0)^{-2}| \leq Ae^{1/k}$$

and so

$$\delta M_{k-1} \lesssim_k AMe^{1/k} \ll_k M^{(k-1)/k}Me^{1/k} \ll_k 1$$

since $\epsilon \ll_k M^{-k}$ and

$$2|\phi(z_0)\phi'(z_0)^{-1}| \leq 2Ae^{1/k},$$

we find a zero $z_* \in \mathbb{D}_{5/4}$ of $\phi = f^{(k-1)}$ in $\mathbb{D}_{5/4}$ such that $|z_* - z_0| \leq 2Ae^{1/k}$ and hence

$$z_* \in \mathbb{D}_{2Ae^{1/k}}(z_0)$$

where $Z_{k-1} := \{ z \in \mathbb{D}_{5/4} : f^{(k-1)}(z) = 0 \}$. In general we define

$$Z_j := \{ z \in \mathbb{D}_{5/4} : f^{(j)}(z) = 0 \}.$$

Now for $z_0 \in S_2$, we apply Lemma 9.3 with $L = 1$ to $\phi(z) = f^{(k-2)}(z)$ and $M_{k-2} \lesssim_k M$. Here

$$\delta = |\phi(z_0)\phi'(z_0)^{-2}| = |f^{(k-2)}(z_0)f^{(k-1)}(z_0)^{-2}| \leq d_2 A^{-1} = c_0^{-1} M^{-1}$$

and so

$$\delta M_{k-2} \lesssim_k c_0^{-1} M^{-1}M \ll_k 1$$

since $c_0 \gg 1$. Since $2|\phi(z_0)\phi'(z_0)^{-1}| \leq 2d_2 \epsilon^{1/k}$, we find a zero $z_* \in \mathbb{D}_{5/4}$ of $\phi = f^{(k-2)}$ in $\mathbb{D}_{5/4}$ such that $|z_* - z_0| \leq 2d_2 \epsilon^{1/k}$ and hence

$$S_2 \subseteq \bigcup_{z_* \in Z_{k-2}} \mathbb{D}_{2d_2 \epsilon^{1/k}}(z_*).$$

Next consider $z_0 \in S_j$ for $3 \leq j \leq k$. Here we will apply Lemma 9.3 to $\phi(z) = f^{(k-j)}(z)$ with $L = k-1$ and $M_{k-j} \lesssim_k M$. Hence

$$\delta = |\phi(z_0)\phi'(z_0)^{-1}\phi^{(L)}(z_0)^{-1}| = |f^{(k-j)}(z_0)f^{(k-j+1)}(z_0)^{-1}f^{(k-1)}(z_0)^{-1}|$$

and

$$\delta M_{k-j} \lesssim_k c_0^{-1} M^{-1}M \ll_k 1$$

since $c_0 \gg 1$. Since $2|\phi(z_0)\phi'(z_0)^{-1}| \leq 2d_2 \epsilon^{1/k}$, we find a zero $z_* \in \mathbb{D}_{5/4}$ of $\phi = f^{(k-j)}$ in $\mathbb{D}_{5/4}$ such that $|z_* - z_0| \leq 2d_2 \epsilon^{1/k}$ and hence

$$S_j \subseteq \bigcup_{z_* \in Z_{k-j}} \mathbb{D}_{2d_2 \epsilon^{1/k}}(z_*).$$
\[ d_j \epsilon^{1/k} |f^{(k-1)}(z_0)|^{-1} \leq d_j \epsilon^{1/k} A^{-1} \epsilon^{-1/k} = \eta^{j-2} M^{-1/k} c_0^{-1} M^{-(k-1)/k}\]

and so
\[ \delta M_{k-j} \lesssim_k \eta^{j-2} c_0^{-1} M^{-1} M \ll_k 1.\]

Also
\[ \delta_1 = |\phi(z_0) \phi''(z_0) \phi'(z_0) |^{-2} = |f^{(k-j)}(z_0) f^{(k-j+2)}(z_0) f^{(k-j+1)}(z_0) |^{-2} \leq d_j/d_{j-1} = \eta \]

and so \( \delta M_{k-j}, \delta_1 \ll_k 1. \) These bounds need to modified in the case \( j = k; \) here
\[ \delta = |f(z_0) f''(z_0) f^{(k-1)}(z_0)|^{-1} \leq \epsilon \frac{1}{AD_k \epsilon^{(k-1)/k}} \frac{1}{A \epsilon^{(k-2)/k}} = \frac{1}{A^2 D_k} \]

where \( D_k = d_{k-1} \cdots d_2. \) Hence (for some \( N = N_k \in \mathbb{N} \) which can be computed)
\[ \delta M \leq c_0^{-1} \eta^{-N_k} M^{-2(k-1)/k} M^{(k-2)/k} M = c_0^{-1} \eta^{-N_k} \ll_k 1 \]

since we choose \( c_0 \) large depending on \( \eta. \) Also
\[ \delta_1 = |f(z_0) f''(z_0) f'(z_0)|^{-2} \leq \epsilon \frac{1}{d_{k-1} \epsilon^{1/k}} \frac{1}{AD_k \epsilon^{(k-1)/k}} = c_0^{-1} \eta^{-N_k} \ll_k 1. \]

For \( 2 \leq r \leq j - 2, \) we have
\[ \delta_r = |f^{(k-j)}(z_0) f^{k-j+1}(z_0) f^{(k-j+r+1)}(z_0) f^{(k-j+r)}(z_0)|^{-1} \]
\[ \leq d_j \epsilon^{1/k} \frac{\epsilon^{(r-1)/k} (d_{j-1} \cdots d_{j-r+1})}{\epsilon^{r/k} (d_{j-1} \cdots d_{j-r})} = d_j d_{j-r} = \eta^r \ll_k 1. \]

Finally we see that for \( j \leq k - 1, \)
\[ |\phi(z_0) \phi'(z_0)|^{-1} = |f^{(k-j)}(z_0) f^{(k-j+1)}(z_0)|^{-1} \leq d_j \epsilon^{1/k} \leq 1/8 \]

and so Lemma 9.3 implies there is a zero \( z_* \) of \( f^{(k-j)} \) in \( \mathbb{D}_{5/4} \) such that \( |z_* - z_0| \leq 2|\phi(z_0) \phi'(z_0)|^{-1} | \leq 2d_j \epsilon^{1/k}. \) Hence for \( 3 \leq j < k, \)
\[ S_j \subseteq \bigcup_{z_* \in Z_{k-j}} \mathbb{D}_{2d_j \epsilon^{1/k}}(z_*). \quad (86) \]

For \( j = k, \)
\[ |\phi(z_0) \phi'(z_0)|^{-1} = |f(z_0) f'(z_0)|^{-1} \leq \epsilon \frac{1}{AD_k \epsilon^{(k-1)/k}} = c_0^{-1} \eta^{-N_k} M^{-1/k} \epsilon^{1/k} \]

and so Lemma 9.3 implies there is a zero \( z_* \) of \( f \) in \( \mathbb{D}_{5/4} \) such that \( |z_* - z_0| \leq 2|f(z_0) f'(z_0)|^{-1} | \ll_k \epsilon^{1/k}. \) Hence
\[ S_k \subseteq \bigcup_{z_* \in Z_0} \mathbb{D}_{b_k \epsilon^{1/k}}(z_*). \quad (87) \]

Hence (81), (82), (83), (84), (85), (86) and (87) imply the desired bound (8), completing the proof of Proposition 3.2.
References

[1] G.I. Arkhipov, V.N. Chubarikov and A.A. Karatsuba, *Trigonometric sums in Number Theory and Analysis*, de Gruyter Expositions in Mathematics 39, Walter de Gruyter, 2004.

[2] G. I. Arkhipov, V.N. Chubarikov and A. A. Karatsuba, *The convergence exponent of the singular integral in Tarry’s problem*, Dokl. Akad. Nauk SSSR 248 (1979), no. 2, 268–272; English transl.: Soviet Math. Dokl. 20 (1979), no. 5, 978–981.

[3] G. I. Arkhipov, V.N. Chubarikov and A. A. Karatsuba, *Trigonometric integrals*, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 5, 971–1003; English transl.: Math. USSR Izv. 15 (1980), 211–239.

[4] V.I. Arnold, S.N. Gusein-Zade and A.N. Varchenko, *Singularities of Differentiable Maps*, Monographs in Mathematics 83, Birkhauser, 1988.

[5] J. Bak and S. Ham, *Restriction of the Fourier transform to some complex curves*, J. Math. Anal. Appl. 409 (2014), 1107-1127.

[6] H. Chung and S. Ham, *Convolution estimates for measures on some complex curves*, Annali di Matematica 198 (2019), 837-867.

[7] J. de Dios Pont, *A geometric lemma for complex polynomial curves with applications in Fourier restriction theory*, available at https://arxiv.org/abs/2003.14140.

[8] S. W. Drury, *Restrictions of Fourier transforms to curves*, Annales de l’Institut Fourier, 35 (1985), no. 1, 117-123.

[9] M.W. Kowalski and J. Wright, *Elementary inequalities involving the roots of a polynomial with applications in harmonic analysis and number theory*, J. London Math. Soc. 86 (2012), 835-851.

[10] C. Meade, *Uniform Convolution and Fourier Restriction estimates for complex polynomial curves in \( \mathbb{C}^3 \)*, available at https://arxiv.org/abs/2012.09651

[11] D.H. Phong and E.M. Stein, *Oscillatory integrals with polynomial phases*, Inventiones Math. 110 (1992), 39-62.

[12] D.H. Phong, E.M. Stein and J.A. Sturm, *On the growth and stability of real analytic functions*, Amer. J. Math 121 (1999), 519-554.

[13] E. M. Stein, *Beijing Lectures in Harmonic Analysis* (ed.) Annals of Math. Studies 112, Princeton University Press, 1986.

[14] ———, *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series 43, Princeton University Press, 1993.

[15] J. Wright, *From oscillatory integrals and sublevel sets to polynomial congruences and character sums*, J. Geom. Anal. 21 (2011), 224-240.

[16] ———, *From oscillatory integrals to complete exponential sums*, Math. Res. Letters 18 (2011), 231-250.

Maxwell Institute of Mathematical Sciences and the School of Mathematics, University of Edinburgh, JCMB, The King’s Buildings, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, Scotland

Email address: j.r.wright@ed.ac.uk