**ON THE BEHAVIOR OF QUASI-LOCAL MASS AT THE INFINITY ALONG NEARLY ROUND SURFACES**

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**Abstract.** In this paper, we study the limiting behavior of the Brown-York mass and Hawking mass along nearly round surfaces at infinity of an asymptotically flat manifold. Nearly round surfaces can be defined in an intrinsic way. Our results show that the ADM mass of an asymptotically flat 3-manifold can be approximated by some geometric invariants of a family of nearly round surfaces which approach to infinity of the manifold.

1. **Introduction**

The ADM mass of an asymptotically flat (AF) manifold is a basic conserved quantity in General relativity. To state its explicit definition, we need the following:

**Definition 1.1.** A complete three manifold \((M, g)\) is said to be asymptotically flat (AF) of order \(\tau\) (with one end) if there is a compact subset \(K\) such that \(M \setminus K\) is diffeomorphic to \(\mathbb{R}^3 \setminus B_R(0)\) for some \(R > 0\) and in the standard coordinates in \(\mathbb{R}^3\), the metric \(g\) satisfies:

\[
g_{ij} = \delta_{ij} + \sigma_{ij}
\]

with

\[
|\sigma_{ij}| + r|\partial \sigma_{ij}| + r^2|\partial^2 \sigma_{ij}| = O(r^{-\tau}),
\]

for some constant \(1 \geq \tau > \frac{1}{2}\), where \(r\) and \(\partial\) denote the Euclidean distance and standard derivative operator on \(\mathbb{R}^3\) respectively.

A coordinate system of \(M\) near infinity so that the metric tensor in these coordinates satisfies the decay conditions in the definition is said to be admissible, in such a coordinate system, we have

**Definition 1.2.** The Arnowitt-Deser-Misner (ADM) mass (see [1]) of an asymptotically flat manifold \(M\) is defined as:

\[
m_{\text{ADM}}(M) = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} (g_{ij,i} - g_{ii,j}) \nu^j d\sigma^0,
\]

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where \( S_r \) is the Euclidean sphere, \( d\sigma^0 \) is the volume element induced by the Euclidean metric, \( \nu \) is the outward unit normal of \( S_r \) in \( \mathbb{R}^3 \) and the derivative is the ordinary partial derivative.

We always assume that the scalar curvature is in \( L^1(M) \) so that the limit exists in the definition. Under the decay conditions in the definition of AF manifold, the definition of the ADM mass is independent of the choice of admissible coordinates by the result of Bartnik [2]. Indeed \( S_r \) in the above definition does not need to be the Euclidean sphere in some admissible coordinates, it could be a connected boundary of an exhausting domain with its area growth like \( r^2 \). Here \( r = \min_{x \in \Sigma} r(x) \), \( r(x) \) is the distance function to some fixed point (see Proposition 4.1, [2]). Hence the ADM mass of an AF manifold is actually a geometric quantity. With these facts in mind and in the view point of geometry, one may intend to ask: Whether or not one can define certain geometric invariants on a family of surfaces defined in an intrinsic way, i.e. are independent of the choice of admissible coordinates, that tends to the ADM mass as the surfaces approach to the infinity of the manifold? In this paper, we will investigate this problem and give an affirmative answer to it.

Intuitively, the ADM mass is a kind of total mass of \((M, g)\). In many cases, we want to measure how much mass is contained in a bounded domain. For this purpose, the notion of quasi-local energy (mass) is needed. The Brown-York mass and the Hawking mass are two of them which are used frequently in literature and both of them are geometric invariants of the surfaces (see the definitions below). Physically, one natural property of quasi-local mass need to have is: the limit of quasi-local mass of the boundary of exhausting domains of an AF manifold should approach the ADM mass (see [10]). Many people have studied this problem, they verified that for boundary of certain exhausting domains this property is true for the Brown-York mass and the Hawking mass, see [3, 4, 7, 13, 18], see also [11, 23]. However, the definitions of these boundaries considered in above mentioned papers depend on some special coordinates. So, these surfaces are not intrinsic.

In this paper, we will discuss the problem mentioned above in the case that surfaces are nearly round at infinity of an AF manifold \((M, g)\). Let us begin with the following definition

**Definition 1.3.** Let \( \{\Sigma_r\} \) be a family of surfaces which are topological sphere in \((M, g)\), \( r = \min_{x \in \Sigma} r(x) \), then we call \( \Sigma_r \) as nearly round when \( r \) tends to infinity if it satisfies:

1. \( |\hat{\nabla} A| + r|\nabla \hat{\nabla} A| \leq Cr^{-1-\gamma} \),
2. \( \max_{x \in \Sigma_r} r(x) \leq C \min_{x \in \Sigma_r} r(x) + C \),
3. \( \text{diam}(\Sigma_r) \leq Cr \),
4. \( \text{Area}(\Sigma_r) \leq Cr^2 \).

Here \( C \) is a constant independent of \( r \). \( \text{diam}(\Sigma_r), \nabla, \text{ and } |\cdot| \), denote diameter of the surface, covariant derivatives and the norm with respect to
the induced metric of \( g \) respectively, \( r(x) \) is the distance of \( x \) to some fixed point in \((M, g)\), \( A \) is the second fundamental forms of \( \Sigma_r \) in \((M, g)\) and \( \tilde{A} \) is the trace free part of \( A \).

**Remark 1.4.**

(1) It is easy to see that the above definition of nearly round surface is intrinsic, i.e. it does not depend on any coordinates.

(2) We suspect that the third and the fourth assumptions are superfluous, since both of them can be derived from the first and second assumptions in the Euclidean space case.

(3) It is not difficult to see that the third assumption implies the second one.

One of very important and also quite natural surfaces in AF manifolds are those with constant mean curvature and approach to infinity of the manifolds, the existence of these surfaces was proved by [24], and [15] many years ago, and later the uniqueness was obtained by [21] (see also [13]). It is not so difficult to see that these constant mean curvature surfaces are nearly round (see the discussion at the beginning of Section 2). Besides this, all the surfaces considered in [3, 4, 7, 14, 18, 11, 23] are nearly round.

Now, let us move to the definition of the Brown-York mass and the Hawking mass.

Let \((\Omega, g)\) be a compact three manifold with smooth boundary \( \partial \Omega \). Suppose the Gauss curvature of \( \partial \Omega \) is positive, then the Brown-York quasi local mass of \( \partial \Omega \) is defined as (see [6, 7]):

**Definition 1.5.**

\[
(1.4) \quad m_{BY}(\partial \Omega) = \frac{1}{8\pi} \int_{\partial \Omega} (H_0 - H) d\sigma
\]

where \( H \) is the mean curvature of \( \partial \Omega \) with respect to the outward unit normal and the metric \( g, d\sigma \) is the volume element induced on \( \partial \Omega \) by \( g \) and \( H_0 \) is the mean curvature of \( \partial \Omega \) when embedded in \( \mathbb{R}^3 \).

The Brown-York mass is well-defined because by the result of Nirenberg [19], \( \partial \Omega \) can be isometrically embedded in \( \mathbb{R}^3 \) and the embedding is unique by [12, 22, 20]. In particular, \( H_0 \) is completely determined by the metric on \( \partial \Omega \). However, this is a global property. In contrast, the norm of the mean curvature vector of an embedding of \( \partial \Omega \) into the light cone in the Minkowski space can be expressed explicitly in terms of the Gauss curvature, see [5]. Hence in the the study of Brown-York mass, one of the difficulties is to estimate \( \int_{\partial \Omega} H_0 d\sigma \). We will use the Minkowski formulae [16] and the estimates of Nirenberg [19] to deal with this problem.

The Hawking quasi local mass is defined as (see [13]):

**Definition 1.6.**

\[
(1.5) \quad m_H(\partial \Omega) = \frac{|\partial \Omega|^{1/2}}{(16\pi)^{3/2}} \left( 16\pi - \int_{\partial \Omega} H^2 d\sigma \right)
\]
where $d\Sigma$ is the volume element induced on $\partial \Omega$ by $g$ and $|\partial \Omega|$ is the area of $\partial \Omega$.

Our main results in this paper are:

**Theorem 1.** Let $(M, g)$ be an AF manifold, $\{\Sigma_r\}$ be nearly round surfaces of $(M, g)$ when $r$ tends to infinity, then
$$\lim_{r \to \infty} m_{BY}(\Sigma_r) = m_{ADM}(M).$$

**Theorem 2.** Let $(M, g)$ be an AF manifold, $\{\Sigma_r\}$ be nearly round surfaces of $(M, g)$ when $r$ tends to infinity then
$$\lim_{r \to \infty} m_H(\Sigma_r) = m_{ADM}(M).$$

The remaining of this paper is organized in the following way. In Section 2 we will discuss the geometry of nearly round surfaces, show that many interesting surfaces are nearly round and present some useful formulae. In Section 3 we will show some estimates of isometric embedding and in Section 4 we will prove the main theorems.

### 2. Geometry of nearly round surfaces

In this section, we want to give some examples of nearly round surfaces and to investigate their geometric properties, and also we will derive some basic formulae which will be used later. Let us begin with some interesting examples.

**Example 2.1.** Constant mean curvature (CMC) surfaces constructed in [15] are nearly round at the infinity of the manifold.

Note that the CMC surfaces constructed in [15] are convex at infinity, and then by Propositions 3.5, 3.9 and 3.12 in [15] we see that they are nearly round at infinity of the manifolds.

Also, by a direct computations, it is not difficult to see that

**Example 2.2.** Let $(M, g, x^i), 1 \leq i \leq 3$, be an AF manifold with admissible coordinates $x^i$, then the coordinate sphere $S_r = \{(x^1, x^2, x^3) | (x^1)^2 + (x^2)^2 + (x^3)^2 = r^2\}$ is nearly round when $r$ tends to infinity.

Our next example relates to the surfaces in Kerr solution to vacuum Einstein equations.

**Example 2.3.** The well-known Kerr metric is given by

$$ds^2 = -(1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta})dt^2 - \frac{2mar \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} (dtd\phi + d\phi dt)$$

$$+ \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2mr + a^2} dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2$$

$$+ \frac{\sin^2 \theta}{(r^2 + a^2 \cos^2 \theta)^2} [(r^2 + a^2)^2 - a^2(r^2 - 2mr + a^2) \sin^2 \theta] d\phi^2.$$
For instance, see page 261 in [8]. Let \((M, g)\) be the slice with \(t = \text{const.}\), then it can be shown directly that \((M, g)\) is an AF manifold. Let \(\Sigma_\tau\) be the surface in \((M, g)\) with \(r = \tau\), then one can verify that \(\Sigma_\tau\) is nearly round in \((M, g)\) when \(\tau\) goes to infinity. Indeed, if we let \(e_1 = (\tau^2 + a^2 \cos^2 \theta)^{-\frac{1}{2}} \frac{\partial}{\partial \theta}\), \(e_2 = \tau^2 + a^2 \cos^2 \theta [(\tau^2 + a^2)^2 - a^2(\tau^2 - 2m\tau + a^2)\sin^2 \theta]^{-\frac{1}{2}} \frac{\partial}{\partial \phi}\), then it is easy to see that \(e_1, e_2\) is an orthonormal frame of \(\Sigma_\tau\), and in this frame the second fundamental form of \(\Sigma_\tau\) with respect to outward unit normal vector in \((M, g)\) is

\[
A_{11} = \left(1 - \frac{2m}{\tau} + \frac{a^2}{\tau^2}\right)^{-\frac{1}{2}} \left(1 + \frac{a^2}{\tau^2} \cos^2 \theta\right)^{-1}, \quad A_{12} = 0,
\]

\[
A_{22} = \frac{2}{\tau^2} \left(1 - \frac{2m}{\tau} + \frac{a^2}{\tau^2}\right)^{-\frac{1}{2}} \frac{1}{\left(1 + \frac{a^2}{\tau^2}\right)^2 - \tau^2 (1 - \frac{2m}{\tau} + \frac{a^2}{\tau^2})a^2 \sin^2 \theta} \cdot \frac{1}{(\tau^2 + a^2 \cos^2 \theta)^2}\left\{\tau^6 + 2a^2 \tau^4 \cos^2 \theta - a^2 \tau^2 m \sin^2 \theta + a^4 \tau (\cos 2\theta + \sin^4 \theta) + ma^4 \sin^2 \theta \cos^2 \theta\right\}.
\]

When \(\tau\) tends to infinity, we have

\[
A_{11} = 2 - \frac{2m}{\tau^2} + O(\tau^{-3}), \quad A_{12} = 0,
\]

\[
A_{22} = \frac{2}{\tau} - \frac{2m}{\tau^2} + O(\tau^{-3}).
\]

Hence

\[
|A| \leq C \tau^{-3}.
\]

Similarly we have

\[
|\nabla A| \leq C \tau^{-4}.
\]

Here \(C\) is a constant independent of \(\tau\). Hence, \(\Sigma_\tau\) is a nearly round surface when \(\tau\) tends to infinity.

**Lemma 2.4.** Let \(\Sigma_r\) be nearly round surfaces in \((M, g)\) when \(r\) goes to infinity, then there is a positive constant \(\Lambda\) which is independent of \(r\), and with \(\Lambda^{-1}r^2 \leq \text{Area}(\Sigma_r) \leq \Lambda r^2\).

**Proof.** It suffices to show the lower bound of the area. Since \(g\) is an AF metric, without loss of generality we may assume \(g\) and \(\hat{g}\) are equivalent on \(M \setminus K\). Here and in the sequel, \(\hat{g}\) is background Euclidean metric on \(M \setminus K\). Thus, we only need to show \(\text{Area}(\Sigma_r, \hat{g}) \geq C r^2\), where \(\text{Area}(\Sigma_r, \hat{g})\) is the area of \(\Sigma_r\) with respect to metric \(\hat{g}\). From the second assumption of nearly round sphere surfaces, we know that the standard sphere with radius \(\frac{r}{2}\), denoted by \(S^2_{\frac{r}{2}}\), is in the domain enclosed by \(\Sigma_r\). Let \(\Omega\) be the domain
enclosed by $\Sigma_r$ and $S^2_\gamma$, $X = (X^1, X^2, X^3)$ be the Euclidean coordinates. In view of
\[
\text{div}_g\left(\frac{X}{|X|}\right) = \frac{2}{|X|} > 0,
\]
we have
\[
\int_{\Sigma_r} \frac{X}{|X|} \cdot n - \int_{S^2_\gamma} \frac{X}{|X|} \cdot n = \int_{\Omega} \text{div}_g\left(\frac{X}{|X|}\right)
\geq 0,
\]
where $n$ is the outward unit normal vector of on one part of the boundary of $\Omega$, $\Sigma_r$ and $-n$ is the outward unit normal vector of on another part of the boundary of $\Omega$, $S^2_\gamma$. Therefore, we see that
\[
\text{Area}(\Sigma_r, \tilde{g}) \geq \pi r^2.
\]
This finishes the proof of the Lemma. 

Our next lemma is on the estimation of the decay of the second fundamental forms of nearly round sphere surfaces.

**Lemma 2.5.** Let $\Sigma_r$ be nearly round surfaces in $(M, g)$ when $r$ goes to infinity, then we have
\[
|A| \leq C r^{-1},
\]
where $C$ is a positive constant independent of $r$.

**Proof.** Let $e_0$, $e_1$ and $e_2$ be the orthonormal frame of $(M, g)$ at any fixed point of $\Sigma_r$, $e_1$ and $e_2$ be the tangential vectors of $\Sigma_r$. Let $\nabla_k A_{ij}$ be the components of $\nabla A$, then by a direct computation for $1 \leq i, j, k \leq 2$, we have
\[
\nabla_i A_{jk} = E_{ijk} + F_{ijk},
\]
where
\[
E_{ijk} = \frac{1}{4} (\nabla_i H \tilde{g}_{jk} + \nabla_j H \tilde{g}_{ik} + \nabla_k H \tilde{g}_{ij}) - \frac{1}{2} w_i \tilde{g}_{jk} + \frac{1}{2} (w_j \tilde{g}_{ik} + w_k \tilde{g}_{ij}).
\]
Here $w_i = R_{0kij} \tilde{g}^{kl}$ and $\tilde{g}_{ij}$ is the induced metric on $\Sigma_r$. By the Codazzi equations, we have
\[
\langle E_{ijk}, F_{ijk} \rangle = 0,
\]
and
\[
|E_{ijk}|^2 = \frac{3}{5} |\nabla H|^2 + |w|^2 - \langle w_i, \nabla_i H \rangle.
\]
thus combining these equalities we have
\[
|\nabla^\circ A|^2 = |\nabla A|^2 - \frac{1}{2} |\nabla H|^2 \geq |E|^2 - \frac{1}{2} |\nabla H|^2 \\
\geq \frac{1}{20} |\nabla H|^2 - C |w|^2 \geq \frac{1}{200} (|\nabla A|^2 - |\nabla^\circ A|^2) - C |w|^2,
\]
where $C$ is a constant. Thus, we see that

$$|\nabla A| \leq C(|\nabla \circ A| + r^{-2-\tau}).$$

Combining this with the first assumption of nearly round sphere surfaces we have

$$|\nabla H| \leq |\nabla A| \leq Cr^{-2-\tau}.$$ 

Due to the assumption $\text{diam}(\Sigma_r) \leq Cr$, we see that for any $x \in \Sigma_r$ we have

$$|H(x) - H(x_0)| \leq |\nabla H| \cdot \text{diam}(\Sigma_r) \leq Cr^{-1-\tau}.$$ 

Here $x_0$ is a fixed point on $\Sigma_r$. Setting

$$r_1 = \frac{2}{H(x_0)},$$

we have

$$H = \frac{2}{r_1} + O(r^{-1-\tau}).$$

We now claim that there is a constant $C > 1$ independent of $r$ with

$$C^{-1}r \leq r_1 \leq Cr.$$ 

Indeed, due to the assumption $|\hat{A}| \leq Cr^{-1-\tau}$ and the Gauss equations, we have

$$K = \frac{1}{r_1^2} + r_1^{-1}O(r^{-1-\tau}) + O(r^{-2-\tau}),$$

where $K$ is the Gauss curvature of $\Sigma_r$ with respect to the metric induced from $g$. Then by the Gauss-Bonnet formula we get

$$\frac{\text{Area}(\Sigma_r)}{r_1^2} + r_1^{-1}O(r^{-1-\tau}) + O(r^{-\tau}) = 4\pi.$$

Together with Lemma 2.4, it follows that the claim is true. Note that

$$\hat{A}_{ij} = \hat{A}_{ij} + \frac{H}{2} \hat{g}_{ij},$$

where $\hat{g}_{ij}$ is the metric on $\Sigma_r$ induced from $g$. The first assumption and the claim imply

$$|A| \leq Cr^{-1},$$

which finishes the proof of the lemma.

As mentioned before, we may regard $\Sigma_r$ as a surface in $M \setminus K$ with the Euclidean metric $\hat{g}$. Our next lemma is about the relationship between $A$ and $\hat{A}$, here and in the sequel, $\hat{A}$ is the second fundamental forms of $\Sigma_r$ with respect to outward unit normal vector and metric $\hat{g}$.
Lemma 2.6. Let $\rho$ be the Euclidean distance function to $\Sigma_r$, $\frac{\partial}{\partial x^i}$, $1 \leq i \leq 3$, be the standard coordinate frame in $\mathbb{R}^3$, $\Gamma^k_{ij}$ is the Christoffel symbols of metric $g$ with respect to $\frac{\partial}{\partial x^i}$, then

$$\hat{A}(X, Y) = |\nabla_g \rho| \cdot A(X, Y) + X^i Y^j \Gamma^k_{ij} \frac{\partial \rho}{\partial x^k},$$

where $X = X^i \cdot \frac{\partial}{\partial x^i}$ and $Y = Y^i \cdot \frac{\partial}{\partial x^i}$ are tangential vectors of $\Sigma_r$ and $\nabla_g$ is the Livi-Civita connection with respect to $g$.

Proof. By the definition of $\rho$, on $\Sigma_r$ we have

$$\nabla^2 \rho (X, Y) = XY(\rho) - \nabla_X Y(\rho) = -\langle \nabla_X Y, v \rangle v(\rho) = A(X, Y) v(\rho),$$

where $v$ is the outward unit normal vector of $\Sigma_r$. Again, a direct computations gives

$$v = |\nabla_g \rho|^{-1} g^{ij} \frac{\partial \rho}{\partial x^i} \frac{\partial}{\partial x^j},$$

Hence, we have

$$v(\rho) = |\nabla_g \rho|,$$

$$A(X, Y) = \frac{1}{|\nabla_g \rho|} \nabla^2 \rho (X, Y)$$

and

$$\nabla^2 \rho (X, Y) = X^i Y^j \frac{\partial^2 \rho}{\partial x^i \partial x^j} - X^i Y^j \Gamma^k_{ij} \frac{\partial \rho}{\partial x^k}.$$

Similarly we have

$$\hat{A}(X, Y) = X^i Y^j \frac{\partial \rho}{\partial x^i \partial x^j},$$

Combining these formulas together, we have

$$\hat{A}(X, Y) = |\nabla_g \rho| \cdot A(X, Y) + X^i Y^j \Gamma^k_{ij} \frac{\partial \rho}{\partial x^k}.$$

□

Let $v = \sum_{i=1}^3 v^i \frac{\partial}{\partial x^i}$, then $h_{ij} = g_{ij} - v_i v_j$, $1 \leq i, j \leq 3$. Then $h_{ij} dx_i dx_j$ is the induced metric on $\Sigma_r$ in $(M, g)$. Define $h^{ij} := g^{is} g^{jt} h_{st}$. We have

Lemma 2.7. Let $\rho$ be the Euclidean distance function to $\Sigma_r$, then on $\Sigma_r$, we have

$$\frac{\partial^2 \rho}{\partial x^i \partial x^j} = B_{ij} + \frac{\hat{H}}{2} h_{ij},$$

where $B_{ij} = \hat{A}((\frac{\partial}{\partial x^i})^T, (\frac{\partial}{\partial x^j})^T)$, and $(\frac{\partial}{\partial x^i})^T$ is the tangential part of $\frac{\partial}{\partial x^i}$. 
Proof. Let $\hat{\nabla}$ be the covariant derivatives with respect to $\hat{g}$, then by a direct computation gives

$$\frac{\partial^2 \rho}{\partial x^i \partial x^j} = \hat{\nabla}^2 \rho((\frac{\partial}{\partial x^i})^T, (\frac{\partial}{\partial x^j})^T) = \hat{A}((\frac{\partial}{\partial x^i})^T, (\frac{\partial}{\partial x^j})^T) = \hat{\circ}A((\frac{\partial}{\partial x^i})^T, (\frac{\partial}{\partial x^j})^T) + \frac{\hat{H}}{2} h_{ij}. \quad (2.3)$$

Combining Lemma (2.5), Lemma (2.6) and Lemma (2.7), we obtain

**Corollary 2.8.** Let $\Sigma_r$ be nearly round surfaces in $(M, g)$ as $r$ goes to infinity, then on $\Sigma_r$, we have:

$$|\frac{\partial^2 \rho}{\partial x^i \partial x^j}| \leq Cr^{-1},$$

where $C$ is a constant independent of $r$.

As a corollary, we have

**Corollary 2.9.** Let $\hat{A}$ be the trace free part of $\hat{A}$, $\bar{D}$ be the covariant derivatives of $\Sigma_r$ with respect to induced metric from $(M \setminus K, \hat{g})$, then we have

$$|\hat{\circ}A| + r|\bar{D}\hat{\circ}A| \leq Cr^{-1-\tau},$$

where $C$ is a positive constant independent of $r$.

**Proof.** By Lemma 2.6 and direct computations, we have

$$|H - \hat{H}| \leq Cr^{-1-\tau},$$

where $C$ is a positive constant independent of $r$, and hence

$$|\hat{A}| \leq Cr^{-1-\tau}.$$

Hence it suffices to show the second part estimate of the corollary is true. Let $p$ be any point of $\Sigma_r$, $e_1$ and $e_2$ be the orthonormal frame at $p$ with $\bar{D} e_i e_j = 0$, $1 \leq i, j \leq 2$. Let $X_i$, $1 \leq i \leq 3$, be one of $e_k$. Then, at $p$, we have

$$(\bar{D}_{X_3} \hat{A})(X_1, X_2) = X_3(\hat{A}(X_1, X_2))$$

$$= X_3(\nabla_g \rho) \cdot A(X_1, X_2) + |\nabla_g \rho| X_3(A(X_1, X_2))$$

$$+ (X_3(X_i)) X_i + X_i X_3(X_2)(\Gamma^k_{ij} \frac{\partial \rho}{\partial x^k}) + X_1 X_2 X_3(\Gamma^k_{ij}) \frac{\partial \rho}{\partial x^k}$$

$$+ X_1 X_2 \Gamma^k_{ij} X_3(\frac{\partial \rho}{\partial x^k}). \quad (2.4)$$
Here we assume $X_i = X^k_i \frac{\partial}{\partial x^k}$. Since $\rho$ is Euclidean distance to $\Sigma_r$, we have
\[
\sum_{i=1}^{3} (\frac{\partial \rho}{\partial x^i})^2 = 1.
\]
Thus,
\[
\left| \sum_{i=1}^{3} g^{ij} X_3 (\frac{\partial \rho}{\partial x^i}) \frac{\partial \rho}{\partial x^j} \right| \leq \left| \sum_{i=1}^{3} \sigma^{ij} X_3 (\frac{\partial \rho}{\partial x^i}) \frac{\partial \rho}{\partial x^j} \right| + C r^{-1-\tau} \leq C r^{-1-\tau},
\]
where $C$ is a constant independent of $r$ and $p$ and the orthonormal frames that we choose. Note that Corollary 2.8 was used in the last equality.

Due to the asymptotically flatness of manifold $(M, g)$, by Lemma (2.5) and Corollary (2.8) we have
\[
|X_3(\nabla g \rho) \cdot A(X_1, X_2)| + |X_i X_j X_3 (\Gamma^{k}_{ij} \frac{\partial \rho}{\partial x^k})| \leq Cr^{-2-\tau},
\]
where $C$ is a positive constant independent of $r$ and $p$ and the orthonormal frames that we choose. In order to get the estimates of the remaining part, we need to estimate covariant derivatives of $X_i$ at $p$ first. Note that by Lemma 2.6 we see that there is a constant $C$ which is independent of $r$ and $p$.

On the other hand, by the fundamental equations of the surface, for $i = 1, 2$, we have
\[
\bar{\nabla} X_3 X_i - \bar{D} X_3 X_i = (\nabla X_3 X_i - D X_3 X_i) + (\hat{A} - A)(X_3, X_i) v + \hat{A}(X_3, X_i)(\hat{v} - v) = X^k_i \Gamma^{kl}_{ik} \frac{\partial}{\partial x^l} + (\hat{A} - A)(X_3, X_i) v + \hat{A}(X_3, X_i)(\hat{v} - v),
\]
where $\nabla, \bar{\nabla}$ is covariant derivatives with respect to metric $g$ and its induced metric on $\Sigma_r$ respectively. By a direct computations, it is not difficult to see that
\[
|\hat{v} - v| \leq Cr^{-\tau},
\]
where $C$ is a positive universal constant independent of $r$. Hence, by the choice of $X_i$ and decay of $|A - \hat{A}|$, we get
\[
(2.5) \quad |\bar{\nabla} X_3 X_i| \leq Cr^{-1-\tau},
\]
at $p$, where $C$ is a constant independent of $r$ and $p$ and the orthonormal frames that we choose. Together this with decay of $|\nabla A|$ and the equality
\[
X_3(A(X_1, X_2)) = (\bar{\nabla} X_3 A)(X_i, X_j) + A(\bar{\nabla} X_3 X_i, X_j) + A(\bar{\nabla} X_3 X_j, X_i),
\]
i, $j = 1, 2$, we get
\[
|X_3(A(X_1, X_j))| \leq Cr^{-2-\tau},
\]
where $C$ is a constant independent of $r$ and $p$ and the orthonormal frames that we choose. By the choice of $X_i$, we have for $i = 1, 2$,

$$\bar{D}X_3X_i = DX_3X_i - \hat{A}(X_3, X_i)\hat{v} = 0,$$

at $p$, and hence

$$X_3(X_i^k) = \hat{g}(\hat{A}(X_3, X_i)\hat{v}, \frac{\partial}{\partial x^k})$$

at $p$ which implies

$$|X_3(X_i^k)| \leq Cr^{-1}\gamma$$

for a constant $C$ independent of $r$ and $p$ and the orthonormal frames that we choose. Combining the above estimates, we have

$$|\bar{D}\hat{A}| \leq Cr^{-2}\gamma$$

at $p$ for some universal constant independent of $r$ and $p$ and the orthonormal frames that we choose. Since $p$ is arbitrary, we complete to prove the corollary.

Let $H$ and $\hat{H}$ be the mean curvature of $\Sigma_r$ in $(\mathbb{R}^3 \setminus K, g)$ and $(\mathbb{R}^3 \setminus K, \hat{g})$ respectively. Both of them are with respect to outward unit normal vector.

**Lemma 2.10.** Let $\rho$ be the Euclidean distance to $\Sigma_r$ in $\mathbb{R}^3$, then

$$H = \hat{H} + \frac{H}{2} \sigma_{ij} \frac{\partial^2 \rho}{\partial x^i \partial x^j} + \frac{1}{2} \sigma_{st,i} \frac{\partial \rho}{\partial x^i} \frac{\partial \rho}{\partial x^s}$$

(2.6)

$$- \sigma_{ij} \frac{\partial^2 \rho}{\partial x^i \partial x^j} - g_{ij,i} \frac{\partial \rho}{\partial x^i} + \frac{1}{2} g_{ij,i} \frac{\partial \rho}{\partial x^i} + O(r^{-1-2\gamma}).$$

Here and in the sequel, $\sigma_{ij,k} = \frac{\partial \sigma_{ij}}{\partial x^k}$ and $g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k}$.

**Proof.** We first note that on $\Sigma$ we have

$$\Delta_0 \rho = \hat{H},$$

where $\Delta_0$ is Laplacian with respect to $\mathbb{R}^3$. The unit normal vector of $\Sigma_r$ in $\mathbb{R}^3$ is $\nabla_0 \rho = \frac{\partial \rho}{\partial x^i} \frac{\partial}{\partial x^i}$, denoted by $\hat{v}$. Let $\{e_1, e_2, v\}$ be the orthonormal frame in $(M \setminus K, g)$ and $\{e_1, e_2\}$ the tangential vector of $\Sigma$. We have

$$\Delta \rho = \nabla^2 \rho(e_1, e_1) + \nabla^2 \rho(e_2, e_2) + \nabla^2 \rho(v, v)$$

and

$$\nabla^2 \rho(e_1, e_1) + \nabla^2 \rho(e_2, e_2) = e_1e_1\rho - \nabla e_1 e_1\rho + e_2 e_2\rho - \nabla e_2 e_2\rho$$

$$= - (\nabla e_1 e_1 + \nabla e_2 e_2)\rho$$

$$= (\nabla e_1 e_1 + \nabla e_2 e_2, v)\rho$$

$$= H \cdot v(\rho).$$

(2.7)

On the other hand, it is clear that

$$\hat{v} = \langle \hat{v}, v \rangle g v + T,$$
where, $T$ is the tangential part of $\hat{v}$ on $\Sigma_r$ and $\langle \cdot, \cdot \rangle_g$ is the inner product with respect to the metric $g$. Thus we have

$$\hat{v}(\rho) = \langle \hat{v}, v \rangle_g v(\rho).$$

Since $\hat{v}(\rho) = 1$ on $\Sigma_r$, we get

$$v(\rho) = \langle \hat{v}, v \rangle_g^{-1}.$$

It is easy to see that

$$v = \frac{\nabla \rho}{|\nabla \rho|} = |\nabla \rho|^{-1} g^{ij} \frac{\partial \rho}{\partial x^i} \frac{\partial}{\partial x^j}$$

and

$$\langle \hat{v}, v \rangle_g = |\nabla \rho|^{-1} g^{ij} \frac{\partial \rho}{\partial x^i} \langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \rangle_g \frac{\partial \rho}{\partial x^k}$$

(2.8)

$$= |\nabla \rho|^{-1} g^{ij} \frac{\partial \rho}{\partial x^i} \sigma_{jk} \frac{\partial \rho}{\partial x^k}$$

$$= |\nabla \rho|^{-1}.$$

Hence, we have

$$v(\rho) = |\nabla \rho|, \quad \langle \hat{v}, v \rangle_g = |\nabla \rho|^{-1}.$$

Since $v$ is the unit norm vector along $\Sigma_r$, it is clear that $\langle \nabla_v v, v \rangle_g = 0$, which implies $\nabla_v v$ is a tangential vector on $\Sigma_r$. Therefore, we get

$$\nabla_v v(\rho) = 0.$$

Combining these facts we obtain

$$\nabla^2 \rho(v, v) = v(|\nabla \rho|)$$

and

(2.9) $\Delta \rho = H |\nabla \rho| + v(|\nabla \rho|)$.

Now we compute the second term in (2.9).

$$v(|\nabla \rho|^2) = v(g^{st} \frac{\partial \rho}{\partial x^s} \frac{\partial \rho}{\partial x^t})$$

(2.10)

$$= |\nabla \rho|^{-1} g^{ij} \frac{\partial \rho}{\partial x^i} \frac{\partial \rho}{\partial x^j} (g^{st} \frac{\partial \rho}{\partial x^s} \frac{\partial \rho}{\partial x^t})$$

$$= |\nabla \rho|^{-1} g^{ij} \frac{\partial \rho}{\partial x^i} g^{kl} \frac{\partial \rho}{\partial x^k} + 2 |\nabla \rho|^{-1} g^{ij} \frac{\partial \rho}{\partial x^i} g^{st} \frac{\partial^2 \rho}{\partial x^s \partial x^t} \frac{\partial \rho}{\partial x^j},$$

where

$$g^{st} = \frac{\partial g^{st}}{\partial x^j}$$

and

$$g^{ij} = \delta^{ij} - \sigma_{ij} + O(r^{-2r}).$$
Putting these things together, we get

\begin{equation}
(2.11)\quad v(|\nabla \rho|^2) = \frac{\partial \rho}{\partial x^i} g^{st}_{ij} \frac{\partial \rho}{\partial x^s} + 2 |\nabla \rho|^{-1} \frac{\partial \rho}{\partial x^i} \frac{\partial^2 \rho}{\partial x^i \partial x^j} \frac{\partial \rho}{\partial x^j} \\
- 2 |\nabla \rho|^{-1} \sigma_{ij} \frac{\partial \rho}{\partial x^i} \frac{\partial^2 \rho}{\partial x^i \partial x^j} - 2 |\nabla \rho|^{-1} \frac{\partial \rho}{\partial x^i} \frac{\partial^2 \rho}{\partial x^i \partial x^s} \frac{\partial \rho}{\partial x^s} \\
+ O(r^{-1-2\tau}).
\end{equation}

From

\begin{equation}
\sum_i \frac{\partial \rho}{\partial x^i} \frac{\partial^2 \rho}{\partial x^i \partial x^s} = \frac{1}{2} \sum_i \frac{\partial}{\partial x^s} (\frac{\partial \rho}{\partial x^i})^2 = 0
\end{equation}

and

\begin{equation}
v(|\nabla \rho|^2) = 2 |\nabla \rho| v(|\nabla \rho|),
\end{equation}

we get

\begin{equation}
v(|\nabla \rho|) = \frac{1}{2} g^{st}_{ij} \frac{\partial \rho}{\partial x^i} \frac{\partial \rho}{\partial x^j} + O(r^{-1-2\tau}).
\end{equation}

Thus, we have

\begin{equation}
\Delta \rho = H |\nabla \rho| + \frac{1}{2} g^{st}_{ij} \frac{\partial \rho}{\partial x^i} \frac{\partial \rho}{\partial x^j} + O(r^{-1-2\tau}).
\end{equation}

On the other hand, by the definition of Laplacian operator we have

\begin{equation}
(2.12)\quad \Delta \rho = g^{ij} \frac{\partial^2 \rho}{\partial x^i \partial x^j} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left( \sqrt{g} g^{ij} \frac{\partial \rho}{\partial x^j} \right) \\
= \hat{H} - \sigma_{ij} \frac{\partial^2 \rho}{\partial x^i \partial x^j} + g^{ij} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left( \sqrt{g} \right) \frac{\partial \rho}{\partial x^j}.
\end{equation}

Noticing that

\begin{equation}
\frac{\partial}{\partial x^i} (\sqrt{g}) = \frac{1}{2} g_{jj,i} + O(r^{-1-2\tau})
\end{equation}

and

\begin{equation}
g^{ij}_{,i} = -g_{ij,i},
\end{equation}

we get

\begin{equation}
(2.13)\quad H = \hat{H} + \frac{H}{2} \sigma_{ij} \frac{\partial \rho}{\partial x^i} \frac{\partial \rho}{\partial x^j} + \frac{1}{2} \sigma_{st,ij} \frac{\partial \rho}{\partial x^i} \frac{\partial \rho}{\partial x^j} \frac{\partial \rho}{\partial x^s} \\
- \sigma_{ij} \frac{\partial^2 \rho}{\partial x^i \partial x^j} - g_{ij,i} \frac{\partial \rho}{\partial x^j} + \frac{1}{2} g_{jj,i} \frac{\partial \rho}{\partial x^j} + O(r^{-1-2\tau})
\end{equation}

\begin{equation}
= \hat{H} + O(r^{-1-\tau}) + O(r^{-1-2\tau}).
\end{equation}
In the sequel, we want to calculate the integral of $H - \hat{H}$ on $\Sigma_r$. Let us begin with

\begin{equation}
\int_{\Sigma} \sigma_{st,i} \frac{\partial \rho}{\partial x^t} \frac{\partial \rho}{\partial x^s} d\sigma^0 = \int_{\Sigma} \frac{\partial}{\partial x^i} (\sigma_{st} \frac{\partial \rho}{\partial x^s}) \frac{\partial \rho}{\partial x^t} d\sigma^0 - \int_{\Sigma} \sigma_{st} \frac{\partial^2 \rho}{\partial x^s \partial x^t} \frac{\partial \rho}{\partial x^i} d\sigma^0
\end{equation}

where $d\sigma^0$ is the area element with respect to the Euclidean induced metric, and we have used the divergence theorem in the last equality. By Lemma \ref{lem:divergence}, we have

\begin{equation}
\int_{\Sigma} (H - \hat{H}) d\sigma = \int_{\Sigma} (H - \hat{H}) d\sigma^0 + O(r^{-2\tau})
\end{equation}

\begin{equation}
= \frac{1}{2} \int_{\Sigma} (H - \hat{H}) \sigma_{st} \frac{\partial \rho}{\partial x^s} \frac{\partial \rho}{\partial x^t} d\sigma^0 + \frac{1}{2} \int_{\Sigma} (g_{ii,j} - g_{ij,i}) \frac{\partial \rho}{\partial x^j} d\sigma^0
\end{equation}

\begin{equation}
- \frac{1}{2} \int_{\Sigma} \sigma_{st} \frac{\partial^2 \rho}{\partial x^s \partial x^t} d\sigma^0 + O(r^{-2 \tau}).
\end{equation}

Noticing that

$$H - \hat{H} = O(r^{-1-\tau})$$

we have

\begin{equation}
\int_{\Sigma} (H - \hat{H}) d\sigma = \frac{1}{2} \int_{\Sigma} (g_{ii,j} - g_{ij,i}) \frac{\partial \rho}{\partial x^j} d\sigma^0 - \frac{1}{2} \int_{\Sigma} \sigma_{st} \frac{\partial^2 \rho}{\partial x^s \partial x^t} d\sigma^0 + O(r^{-2 \tau}).
\end{equation}

### 3. Estimations of Isometric Embedding of Nearly Round Surfaces

In this section, we study the isometric embedding of nearly round surfaces, and the main purpose is to get the expansion of the mean curvature nearly round surfaces at the infinity of $(M, g)$.

Let $(S^2, g_0)$ be the standard unit sphere and $i_0$ an isometric embedding of $(S^2, g_0)$ into $\mathbb{R}^3$. Let $K_g$ denote the Gauss curvature of the metric $g$. We want to show the following

**Theorem 3.** There exists a positive constant $\epsilon_0 > 0$ such that for any metric $g$ on $S^2$ with

$$\|K_g - 1\|_{C^1} \leq \epsilon_0,$$

there exists an isometric embedding

$$i : (S^2, g) \to \mathbb{R}^3$$
and a conformal transformation \( \Psi_1 \) of \((S^2, g_0)\) with

\[
\| i \circ \Psi_1^{-1} - i_0 \|_{C^{2,\alpha}} \leq C_0 \| K - 1 \|_{C^\alpha},
\]

for some \( 0 < \alpha < 1 \). Here \( C_0 \) is a positive constant only depending on \( \alpha \), and \( \| \cdot \|_{C^{2,\alpha}}, \| \cdot \|_{C^\alpha} \) are taken with respect to \( g_0 \).

**Proof.** The key point of the proof of above theorem is to show that there is a conformal transformation \( \Psi_1 \) of the standard unit sphere with

\[
\| \Psi_1^*(g) - g_0 \|_{C^{2,\alpha}} \leq C_1 \| K - 1 \|_{C^{\alpha}},
\]

where \( C_1 \) is a constant only depending on \( \alpha \). Once (3.1) is verified then by the arguments in P353 of [19], we see that the conclusion of Theorem is true. Due to the Uniformization Theorem, we see that there is conformal diffeomorphism \( \Phi: (S^2, g_0) \mapsto (S^2, g) \) with \( \Phi^*(g) = e^{2u}g_0 \) and

\[
\Delta_{S^2} u + Ke^{2u} = 1.
\]

Without loss of generality, we may assume for \( 1 \leq i \leq 3 \)

\[
\int_{S^2} e^{2u} x_i = 0,
\]

where \( x_i, 1 \leq i \leq 3, \) is the coordinate function of the standard sphere in \( \mathbb{R}^3 \), integral is taken on the standard sphere \((S^2, g_0)\). Otherwise, by Lemma 2, part 3 in [9] we can find a conformal transformation \( \Phi_1: (S^2, g_0) \mapsto (S^2, g_0) \) so that \( u \circ \Phi_1 \) satisfying (3.2), and (3.3). Due to \( a' \) in the proof of Theorem 1, Part 7 in [9], we know that there a constant \( C \) only depending on \( w \) and \( W \) (here \( 0 < w \leq K \leq W \) ) with

\[
|u|_{C^0(S^2)} \leq C.
\]

We now claim that for any \( \eta > 0 \), there is \( \delta = \delta(\eta) \) so that

\[
|u|_{C^0(S^2)} \leq \eta,
\]

provided \( |K - 1|_{C^0(S^2)} \leq \delta. \)

Suppose the claim fails, then we may find a constant \( \eta_0 > 0 \), and a sequence of \( K_j \) and \( u_j \) satisfying (3.2) and (3.3), and \( |K_j - 1| \) tends to zero while \( |u_j|_{C^0(S^2)} \geq \eta_0 \). By (3.4) and the standard estimates in elliptic PDE, we may take a subsequence still denoted by \( u_j \) which converges to \( u_\infty \) in the sense of \( C^1(S^2) \), and \( u_\infty \) satisfying (3.3) and

\[
\Delta_{S^2} u_\infty + e^{2u_\infty} = 1.
\]

By this we see that \( u_\infty = 0 \) which is contradiction to the choice of \( u_j \). Thus, claim is true.

Let

\[
u = u_0 + u_1 + u_2
\]

with

\[
u_0 = \frac{1}{4\pi} \int_{S^2} u, \quad u_1 = a_1 x_1 + a_2 x_2 + a_3 x_3,
\]
where

\[ a_i = \frac{3}{4\pi} \int_{S^2} u \cdot x_i \]

for \(1 \leq i \leq 3\). Taking integral on (3.2) on \((S^2, g_0)\), we get

\[ \int_{S^2} Ke^{2u} = 4\pi. \]

Together this with the assumption of \(K\), we get

\[ |u_0| \leq C(\|K - 1\|_{C^0(S^2)} + \|u\|^2_{L^2(S^2)}). \]

Here and in the sequel, \(C\) is always a constant independent of \(u\). Let

\[ L(u) := \Delta_{S^2} u + 2u, \]

By equation (3.2) and the definition of \(u^2\), we get

\[ L(u^2) = (1 - K)e^{2u} + (1 + 2u - e^{2u}) - 2u_0, \]

from which we have

\[ \int_{S^2} (|\nabla u_2|^2 - 2|u_2|^2) \leq C(\|K - 1\|_{C^0(S^2)}\|u_2\|_{L^2(S^2)}) \]

(3.7)

\[ \text{where } C \text{ is a positive universal constant independent of } u. \]

By (3.5), we get

\[ \|1 + 2u - e^{2u}\|_{L^2(S^2)} \|u_2\|_{L^2(S^2)} \]

and

\[ \|u_0\|_{L^2(S^2)} \]

From the definition of \(u_2\) we see that

\[ \int_{S^2} (|\nabla u_2|^2 - 2|u_2|^2) \geq C\|u_2\|^2_{L^2(S^2)} \]

where \(C\) is a positive universal constant independent of \(u\). By (3.5) we get

\[ \|1 + 2u - e^{2u}\|_{L^2(S^2)} \leq C\|u\|^2_{L^2(S^2)} \]

(3.9)

Combining (3.7), (3.8), (3.9) and (3.6) we obtain

\[ \|u_2\|_{L^2(S^2)} \leq C(\|K - 1\|_{C^0(S^2)} + \|u\|^2_{L^2(S^2)}). \]

(3.10)

On the other hand, by the (3.3) and direct computations we obtain, for each \(i\),

\[ |a_i| \leq C\|u\|^2_{L^2(S^2)}, \]

which implies

\[ \|u_1\|_{L^2(S^2)} \leq C\|u\|^2_{L^2(S^2)}. \]

All the constants \(C\) above are independent of \(u\). Putting this estimate with (3.6), (3.10), and (3.5), we get

\[ \|u\|_{L^2(S^2)} \leq C\|K - 1\|_{C^0(S^2)}. \]

Together this with (3.2), we get

\[ \|u\|_{W^{2,2}(S^2)} \leq C\|K - 1\|_{C^0(S^2)}, \]

which implies

\[ \|u\|_{C^0(S^2)} \leq C\|K - 1\|_{C^0(S^2)}, \]
for some $\alpha > 0$, here $C$ is a constant that only depends on $\alpha$ and is independent of $u$. Then by the Schauder theory in partial differential equations, we get

$$\|u\|_{C^{2,\alpha}(\mathbb{S}^2)} \leq C\|K - 1\|_{C^0(\mathbb{S}^2)}.$$  

Thus, we see that

$$\|\Psi_1^1(\alpha) - g_0\|_{C^{2,\alpha}} \leq C_1\|K - 1\|_{C^0(\mathbb{S}^2)},$$  

where $C_1$ is a constant depending only on $\alpha$. It implies the conclusion of the Theorem is true. Thus, we finish to prove the Theorem.

Let $X$, $n$ be the position vector and the outward unit normal vector of $i(S^2)$ in $\mathbb{R}^3$ respectively, $H_0$ be its mean curvature with respect to $n$, then as a corollary, we have

**Corollary 3.1.** Let $(S^2, g)$ be a two dimensional Riemannian manifold, $K$ be its Gauss curvature. Then there exist a positive constant $\epsilon_0$ which is independent of $g$ such that if

$$\|K - 1\|_{C^1} \leq \epsilon_0,$$

then

$$|X \cdot n - 1| \leq C\|K - 1\|_{C^0(S^2)},$$

and

$$|H_0 - 2| \leq C\|K - 1\|_{C^0(S^2)},$$

where $C$ is a constant that independent of $g$.

**Proof.** By Theorem 3, we see that the statement of the theorem is true for $i(\Psi_1^1(S^2))$ in $\mathbb{R}^3$. It is also true for the case of $i(S^2)$, for the support function $X \cdot n$ and the mean curvature $H_0$ are independent of a parametrization of the domain manifold.

On the other hand, by Corollary 2.9 and the similar arguments as in Prop 2.1 in [15], we may prove the following

**Proposition 3.2.** Let $\hat{\lambda}_i$ be the principal curvature of $\Sigma$ in $(M \setminus K, \hat{g})$. If $|\hat{A}| = O(r^{-1-\tau})$, $|\nabla \hat{A}| = O(r^{-2-\tau})$, then there is a number $r_0 \in \mathbb{R}$ and a vector $\overrightarrow{a} \in \mathbb{R}^3$ such that

$$\hat{\lambda}_i - r_0^{-1} = O(r^{-1-\tau}),$$

$$|y - \overrightarrow{a} - r_0\hat{n}| = O(r^{1-\tau}).$$

Here $y$ is the position vector of $\Sigma$ in $(M \setminus K, \hat{g})$ (which is regarded as a subdomain of $\mathbb{R}^3$), $\hat{n}$ is the outward unit normal vector of $\Sigma_r$.

By the Gauss-Bonnet formula, Lemma 2.4 and the arguments we used before, we know that there is a constant $C > 0$ which is independent of $r$ and $r_0$ and with $C^{-1}r \leq r_0 \leq Cr$.

Now, we are in the position to study the isometric embedding of nearly round surfaces. Let $(M, g)$ be the AF manifolds and $\Sigma_r$ the nearly round surface in $(M, g)$ as $r$ goes to infinity. Then we have
**Theorem 4.** Let \( \Sigma_r \subset (M, g) \) be a nearly round surfaces in \( (M, g) \) as \( r \) goes to infinity, and \( r_0 \) defined as in Proposition 3.2. Then there is isometrically embedding \( X_r \) of \( \Sigma_r \) into \( \mathbb{R}^3 \) such that

\[
X_r \cdot n_0 = r_0 + O(r_0^{1-\tau})
\]

and

\[
\|H_0 - \frac{2}{r_0}\|_{C^0} \leq C_3 r_0^{-1-\tau},
\]

provided that \( r_0 \) is large enough. Here \( n_0 \) and \( H_0 \) are the unit outward normal vector and mean curvature of \( X_r(\Sigma_r) \) in \( \mathbb{R}^3 \) respectively. \( C_3 \) is a constant that is independent of \( r \).

**Proof.** By the assumption on \( A \) and \( \nabla A \), we see that

\[
K = \frac{1}{r_0^2} + O(r^{-2-\tau})
\]

and

\[
\|\nabla K\|_{C^0} \leq C r^{-3-\tau}.
\]

Then by a rescaling, we see that the resulting surface satisfying the assumptions of Theorem 3. Then combining Theorem 3 with direct computations we get the conclusion. \( \square \)

4. **Brown-York mass and Hawking mass of nearly round surfaces at infinity**

In this section, we prove our main results.

**Theorem 5.** Let \( (M, g) \) be an AF manifold, \( \Sigma_r \) be a nearly round surface of \( (M, g) \) as \( r \) goes to infinity, then

\[
\lim_{r \to \infty} m_{BY}(\Sigma_r) = m_{ADM}(M).
\]

**Theorem 6.** Let \( (M, g) \) be an AF manifold, \( \Sigma_r \) be a nearly round surface of \( (M, g) \) as \( r \) goes to infinity, then

\[
\lim_{r \to \infty} m_{H}(\Sigma_r) = m_{ADM}(M).
\]

**Proof of Theorem 5.** Let \( H_0 \) be the mean curvature of the isometric embedding image of \( (\Sigma_r, g) \) in \( \mathbb{R}^3 \). Then by Theorem 4, we have

\[
H_0 = \frac{2}{r_0} + O(r^{-1-\tau}), \quad X_r \cdot n_0 = r_0 + O(r^{1-\tau}), \quad K = \frac{1}{r_0^2} + O(r^{-2-\tau}).
\]

Then, by the same arguments as that at page 11 in [III], we claim that

\[
\int_{\Sigma_r} H_0 d\sigma = 4\pi r_0 + \frac{\text{Area}(\Sigma_r)}{r_0} + O(r^{1-2\tau}).
\]
In fact, let $K = \frac{1}{r_0^2} + \tilde{K}$, by one of the Minkowski integral formulae \[16, \text{Lemma 6.2.9}], we have

\[
\int_{\Sigma_r} H_0 d\sigma = 2 \int_{\Sigma_r} K X_r \cdot n_0 d\sigma
\]

\[
= 2 \int_{\Sigma_r} \left( \frac{1}{r_0^2} + \tilde{K} \right) X_r \cdot n_0 d\sigma
\]

\[
= \frac{2}{r_0^2} \int_{\Sigma_r} X_r \cdot n_0 d\sigma + 2 \int_{\Sigma_r} \tilde{K} X_r \cdot n_0 d\sigma
\]

\[
= \frac{6V(r)}{r_0^2} + 2 \int_{\Sigma_r} \tilde{K} \left( r_0 + O \left( r^{1-\tau} \right) \right) d\sigma
\]

\[
= \frac{6V(r)}{r_0^2} + 2 r_0 \int_{\Sigma_r} \tilde{K} + O(r^{1-2\tau})
\]

\[
= \frac{6V(r)}{r_0^2} + 2 r_0 \int_{\Sigma_r} \left( K - \frac{1}{r_0^2} \right) + O(r^{1-2\tau})
\]

\[
= \frac{6V(r)}{r_0^2} + 8\pi r_0 - \frac{2\text{Area}(\Sigma_r)}{r_0} + O(r^{1-2\tau}),
\]

where $V(r)$ is the volume of the interior of the surface $X_r(\Sigma_r)$ in $\mathbb{R}^3$. Here and in the sequel, $\text{Area}(\Sigma_r)$ is the area of $(\Sigma, g)$. On the other hand, by the above estimate we know that $H_0 = \frac{2}{r_0^2} + H_1$ with $H_1 = O \left( r^{1-\tau} \right)$ and by another Minkowski integral formula we have

\[
2\text{Area}(\Sigma_r) = \int_{\Sigma_r} H_0 X_r \cdot n_0 d\sigma
\]

\[
= \frac{6V(r)}{r_0} + \int_{\Sigma_r} H_1 X_r \cdot n_0 d\sigma
\]

\[
= \frac{6V(r)}{r_0} + r \int_{\Sigma_r} H_1 d\sigma + O \left( r^{2-2\tau} \right)
\]

\[
= \frac{6V(r)}{r_0} - 2\text{Area}(\Sigma_r) + r_0 \int_{\Sigma_r} H_0 d\sigma + O \left( r^{2-2\tau} \right),
\]

which implies

\[
\int_{\Sigma_r} H_0 d\sigma = -\frac{6V(r)}{r_0^2} + 4\text{Area}(\Sigma_r) + O(r^{1-2\tau}).
\]

The claim follows from (4.1) and (4.3). From the claim and (2.16) we get

\[
\int_{\Sigma_r} (H_0 - H) d\sigma = 4\pi r_0 + \frac{\text{Area}(\Sigma_r)}{r_0} - \int_{\Sigma_r} H d\sigma
\]

\[
+ \frac{1}{2} \int_{\Sigma_r} (g_{ij,i} - g_{ii,j}) \frac{\partial \rho}{\partial x^j} d\sigma_0 + \frac{1}{2} \int_{\Sigma_r} \sigma_{st} \frac{\partial^2 \rho}{\partial x^s \partial x^t} d\sigma_0 + O(r^{1-2\tau}).
\]
By the definition, we see that
\[
\frac{\text{Area}(\Sigma_r)}{r_0} = \frac{\text{Area}_0(\Sigma_r)}{r_0} + \frac{1}{2} r_0^{-1} \int_{\Sigma} h^{st} \sigma_{st} d\sigma_0 + O(r^{-2})
\]
where \(\text{Area}_0(\Sigma_r)\) is the area of \(\Sigma_r\) with respect to induce metric from \(\hat{g}\). We also have
\[
\int_{\Sigma} \hat{H} d\sigma = \int_{\Sigma} \hat{H} d\sigma_0 + \frac{1}{2} \int_{\Sigma} \hat{H} h^{st} \sigma_{st} d\sigma_0 + O(r^{-2})
\]
By Lemma 2.7 we have
\[
\frac{1}{2} \int_{\Sigma} \sigma_{st} \frac{\partial^2 \rho}{\partial x^s \partial x^t} d\sigma = \frac{1}{2} \int_{\Sigma} \sigma_{st} \hat{A}_{st} + \frac{1}{4} \int_{\Sigma} \hat{H} h^{st} \sigma_{st} d\sigma_0 + O(r^{-2})
\]
Combining these things together we get
\[
(4.5)
\int_{\Sigma} (H_0 - H) d\sigma = 4\pi r_0 + \frac{\text{Area}_0(\Sigma_r)}{r_0} - \int_{\Sigma} \hat{H} d\sigma_0 + \int_{\Sigma} \left(\frac{1}{2} r_0 - \frac{\hat{H}}{4}\right) h^{st} \sigma_{st} d\sigma_0
+ \frac{1}{2} \int_{\Sigma} (g_{ij,i} - g_{ii,j}) \frac{\partial \rho}{\partial x^j} d\sigma_0 + \frac{1}{2} \int_{\Sigma} \sigma_{st} B_{st} d\sigma.
\]
Set \(\hat{X} = y - a\). It is an isometric embedding of \((\Sigma_r, \hat{g})\) into \(\mathbb{R}^3\). By Proposition 3.2 we see that
\[
\hat{X} \cdot n = r_0 + O(r^{-1}), \quad \hat{H} = \frac{2}{r_0} + O(r^{-1}), \quad \hat{K} = \frac{1}{r_0^2} + O(r^{-2}),
\]
where \(\hat{K}\) is the Gauss curvature of \(\Sigma_r\) in \((M \setminus K, \hat{g})\). Using the same arguments as that at page 11 in [11] to \((\Sigma, \hat{g})\) we get
\[
4\pi r_0 + \frac{\text{Area}_0(\Sigma_r)}{r_0} - \int_{\Sigma} \hat{H} d\sigma_0 = O(r^{-1}).
\]
By Corollary 2.9 and Proposition 3.2 we see that
\[
\int_{\Sigma} \left(\frac{1}{2} r_0 - \frac{\hat{H}}{4}\right) h^{st} \sigma_{st} d\sigma_0 + \frac{1}{2} \int_{\Sigma} \sigma_{st} B_{st} d\sigma = O(r^{-1}).
\]
Thus we have
\[
\int_{\Sigma} (H_0 - H) d\sigma = \frac{1}{2} \int_{\Sigma} (g_{ij,i} - g_{ii,j}) \frac{\partial \rho}{\partial x^j} d\sigma_0 + O(r^{-2}).
\]
The first term in the right hand side of the above equality is the ADM mass of \((M, g)\). Thus, we finish to prove Theorem 5. □
Proof of Theorem 6. By Lemma 2.10 and (2.14) we have

\[
\int_{\Sigma_r} H^2 d\sigma = \int_{\Sigma_r} \tilde{H}^2 d\sigma + \int_{\Sigma_r} H \cdot \tilde{H} \frac{\partial \rho}{\partial x^i} \frac{\partial \rho}{\partial x^j} \sigma_{ij} d\sigma
\]

\[
+ \tilde{H} \int_{\Sigma_r} \sigma_{st,i} \frac{\partial \rho}{\partial x^i} \frac{\partial \rho}{\partial x^s} d\sigma^0 - 2 \tilde{H} \int_{\Sigma_r} \sigma_{ij} \frac{\partial^2 \rho}{\partial x^i \partial x^j} d\sigma^0
\]

\[
- 2 \tilde{H} \int_{\Sigma_r} g_{ij,i} \frac{\partial \rho}{\partial x^j} d\sigma^0 + \tilde{H} \int_{\Sigma_r} g_{jj,j} \frac{\partial \rho}{\partial x^l} d\sigma^0 + O(r^{-2\tau})
\]

(4.6)

\[
= \int_{\Sigma_r} \tilde{H}^2 d\sigma + \int_{\Sigma_r} H \cdot \tilde{H} \frac{\partial \rho}{\partial x^i} \frac{\partial \rho}{\partial x^j} \sigma_{ij} d\sigma^0
\]

Note that

(4.7) \[d\sigma = (1 + h^{ij} \sigma_{ij} + O(r^{-2\tau})) \frac{1}{2} d\sigma_0 = d\sigma^0 + \frac{1}{2} h^{ij} \sigma_{ij} d\sigma_0 + O(r^{-2\tau}).\]

Combining above equalities with Lemma 2.7 we get

\[
\int_{\Sigma_r} H^2 d\sigma = \int_{\Sigma_r} \tilde{H}^2 d\sigma_0 + H \int_{\Sigma_r} (g_{jj,i} - g_{jj,j}) \frac{\partial \rho}{\partial x^i} d\sigma^0 + O(r^{-2\tau}).
\]

Hence,

(4.8)

\[
m_H(\Sigma_r) = \frac{\text{Area}(\Sigma_r)^{\frac{1}{2}}}{(16\pi)^{\frac{3}{2}}} \left[ (16\pi)^{\frac{1}{2}} - \int_{\Sigma_r} \tilde{H}^2 d\sigma_0 \right]
\]

\[
- \frac{\text{Area}(\Sigma_r)^{\frac{1}{2}}}{(16\pi)^{\frac{3}{2}}} \cdot \tilde{H} \int_{\Sigma_r} (g_{jj,i} - g_{jj,j}) \frac{\partial \rho}{\partial x^i} d\sigma^0 + O(r^{1-2\tau})
\]

\[
= -2 \frac{\text{Area}(\Sigma_r)^{\frac{1}{2}}}{(16\pi)^{\frac{3}{2}}} \int_{\Sigma_r} |\tilde{A}|^2 d\sigma^0 - \frac{\text{Area}(\Sigma_r)^{\frac{1}{2}}}{(16\pi)^{\frac{3}{2}}} \cdot \tilde{H} \int_{\Sigma_r} (g_{jj,i} - g_{jj,j}) \frac{\partial \rho}{\partial x^i} d\sigma^0
\]

\[
+ O(r^{-1-2\tau})
\]

\[
= \frac{\text{Area}(\Sigma_r)^{\frac{1}{2}}}{(16\pi)^{\frac{3}{2}}} \cdot \tilde{H} \int_{\Sigma_r} (g_{jj,i} - g_{jj,j}) \frac{\partial \rho}{\partial x^i} d\sigma^0 + O(r^{1-2\tau}),
\]
where we have used estimate
\[ |\hat{A}| \leq Cr^{-1-\tau} \]
in the last equality. On the other hand, we have
\[ K = \frac{1}{r_0^2} + O(r^{-2-\tau}) \]
By the Gauss-Bonnet formula, we get
\[ \text{Area}(\Sigma_r) = 4\pi r_0^2 + O(r^{2-\tau}), \]
From Proposition 3.2 we see that
\[ \hat{H} = \frac{2}{r_0} + O(r^{-1-\tau}), \]
Combining these formulas we obtain
\[ m_H(\Sigma_r) = \frac{1}{16\pi} \int_{\Sigma_r} \left( g_{ij,j} - g_{jj,i} \right) \frac{\partial \rho}{\partial x^j} d\sigma^0 + O(r^{1-2\tau}). \]
Thus, we finish to prove the Theorem. \(\square\)

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