The vibration analysis of the elastically restrained functionally graded Timoshenko beam with arbitrary cross sections

Guofang Li, Gang Wang, Junfang Ni and Liang Li

Abstract
In this study, an investigation on the free vibration of the beam with material properties and cross section varying arbitrarily along the axis direction is studied based on the so-called Spectro-Geometric Method. The cross-section area and second moment of area of the beam are both expanded into Fourier cosine series, which are mathematically capable of representing any variable cross section. The Young's modulus, the mass density and the shear modulus varying along the lengthwise direction of the beam, are also expanded into Fourier cosine series. The translational displacement and rotation of cross section are expressed into the Fourier series by adding some polynomial functions which are used to handle the elastic boundary conditions with more accuracy and high convergence rate. According to Hamilton's principle, the eigenvalues and the coefficients of the Fourier series can be obtained. Some examples are presented to validate the accuracy of this method and study the influence of the parameters on the vibration of the beam. The results show that the first four natural frequencies gradually decrease as the coefficient of the radius \( b \) increases, and decreases as the gradient parameter \( n \) increases under clamped–clamped end supports. The stiffness of the functionally Timoshenko beam with arbitrary cross sections is variable compared with the uniform beam, which makes the vibration amplitude of the beam have different changes.

Keywords
Functionally graded material, Timoshenko beam, vibration, variable cross section, Reyleigh-Ritz method

Introduction
Functionally graded material (FGM), a new type of composite materials which has continuous gradient changing in compositions and structures, was first proposed by materials scientists\(^1\)--\(^3\) to solve the problem of materials under harsh conditions in technical fields such as aeronautics and astronautics. The mechanical performance of plates, shells and beams made from FGM can be improved obviously. The material properties of FGM have shown different features in any desired spatial orientation by different theories such as exponential law,\(^4\) power law,\(^5\) and sigmoid law.\(^6\) FGM have great potential in many fields due to its superior performance than ordinary materials. The numerical method is used to study the free vibration of FGM beam in previous studies. In this paper, the exact solution is obtained to analyze the free vibration of FGM Timoshenko beam with arbitrary cross sections. Great efforts have been made for the vibration of the FGM beam by many engineers and researchers. Moreover, most of the researchers studied the type of the beam with material properties varying thicknesswise\(^7\)--\(^16\) and only a few researchers studied the FGM beam with material properties varying lengthwise.\(^17\)--\(^20\) Wu et al.\(^17\)

School of Mechanical and Electric Engineering, Soochow University, Suzhou, China

Corresponding author:
Gang Wang, School of Mechanical and Electric Engineering, Soochow University Suzhou, Jiangsu 215006, China.

Email: wanggang81@suda.edu.cn

Creative Commons CC BY: This article is distributed under the terms of the Creative Commons Attribution 4.0 License (https://creativecommons.org/licenses/by/4.0/) which permits any use, reproduction and distribution of the work without further permission provided the original work is attributed as specified on the SAGE and Open Access pages (https://us.sagepub.com/en-us/nam/open-access-at-sage).
studied the vibration of axially functionally graded (AFG) beams through semi-inverse method. Huang$^{21}$ studied the free vibration of AFG beams with variable cross sections by the finite element method.

For the beam with non-uniform cross section, it is widely investigated by many researchers due to its better mechanical performance. A closed-form solution about non-uniform deflection beams was obtained by Romano.$^{22}$ Fertis and Zobel$^{23}$ studied this problem by the equivalent system method. Ece et al.$^{24}$ investigated the vibration of the non-uniform cross-section beam with exponentially varying width. Numerical methods such as Frobenius method,$^{25,26}$ Rayleigh-Ritz method,$^{27}$ differential transform method,$^{28}$ the differential quadrature method,$^{29}$ semi-inverse method$^{30}$ and finite element method$^{31-33}$ have also been used to study the free vibration of the non-uniform cross-section beam.

It is well known that the influence of transverse shear deformations and rotary inertia in Euler-Bernoulli beams$^{34,35}$ is ignored while it is not neglected by Timoshenko beams. The methods of differential transform and differential quadrature are adopted to analyze the free vibration of the FGM beam.$^{36,37}$ The power series method and dynamic stiffness method are used in Leung et al.$^{38}$ for analyzing the free vibration of non-uniform Timoshenko beams. The method that transforms complex couple functions into a single government function via introducing an additional function is adopted by Huang et al.$^{39}$ to discuss the free vibration of AFG Timoshenko beam with variable cross section. Recently, many researchers are focusing on solving the vibration of plates, shells and beams with different end supports. A unified solution for the vibration of plates with holes under elastic boundary conditions is given in Wang et al.$^{40}$ Wang et al.$^{41}$ made a prediction of break-out sound from a rectangular cavity via an elastically mounted panel. The free vibration of FGM plates and doubly curved shells with elastic ends supports are studied in literature.$^{42-44}$ The vibration problems of a FGM beam with thermo-elastic environment have been investigated by Chen et al.$^{45}$ according to a higher-order shear deformation theory. A numerical method of viewing a shifting as a multi-curved beam has been addressed for solving the vibration of the curved shifting in Wang et al.$^{46}$

As far as the authors know, there is no unified method which was proposed to study the vibration of the functional graded Timoshenko beam with arbitrary cross sections. In this paper, an unified procedure is proposed to study the elastically restrained FGMT beam with cross sections varying arbitrarily based on the spectro-geometric method (SGM)$^{47}$ in which the translational displacement and the rotation of cross section are both expressed into the Fourier series with some polynomial functions. The cross-sectional area, the second moment of area, the Young’s modulus, the mass density and the shear modulus have been expanded into Fourier cosine series. Several examples are introduced to validate the accuracy and reliability of this method. This method is generally applicable to any kind of beams with material properties and cross-sectional area varying arbitrarily along the axis direction under any elastic boundary conditions.

Theoretical formulation

Structural model

Figure 1 shows the FGMT beam with variable cross section. The beam is elastically restrained at the ends by translational and rotational springs, $k_0$, $k_1$, $K_0$ and $K_1$.

The coupled governing equations of the FGMT beam are expressed as$^{48}$

$$E(x)I(x) \frac{\partial^2 \theta}{\partial x^2} + kG(x)A(x) \left( \frac{\partial w}{\partial x} - \theta \right) - \rho(x)I(x) \frac{\partial^2 w}{\partial t^2} = 0$$

(1)

$$\rho(x)A(x) \frac{\partial^2 w}{\partial t^2} - kG(x)A(x) \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial \theta}{\partial x} \right) = 0$$

(2)

where $w$, $\theta$, $I(x)$, $A(x)$, $E(x)$, $G(x)$, $k$ and $\rho(x)$ represent the transverse displacement of the beam, the rotation of cross section, second moment of area, cross-sectional area, Young’s modulus, shear modulus, shear coefficient and mass density, respectively.

The end supports of the FGMT beams with elastic constraints are given as

$$k_0w = kA(0)G(0) \left( \frac{dw}{dx} - \theta \right) \quad K_0\theta = E(0)I(0) \frac{d\theta}{dx} \quad x = 0$$

(3)
where \( k_0 \) and \( k_1 \) represent the linear spring constants, and \( K_0 \) and \( K_1 \) denote the rotational spring constants at \( x = 0 \) and \( x = L \), respectively. Equations (3) and (4) represent a group of boundary conditions, and one can obtain different end supports by setting spring constants to be of different values.

The translational displacement and rotation of cross section of the beam can be expressed as

\[
  w(x) = \sum_{m=0}^{N} A_m \cos \lambda_m x + \sum_{i=1}^{4} C_i \tilde{\xi}_i(x) \\
  \theta(x) = \sum_{m=0}^{N} B_m \cos \lambda_m x + \sum_{i=1}^{4} D_i \tilde{\xi}_i(x)
\]

where \( \lambda_m = m\pi/L \). The additional polynomials, \( \tilde{\xi}(x) \), are introduced to overcome the discontinuity of the initial displacements and its derivatives when they are extended periodically to the end of the beam. In the Timoshenko beam, the four additional polynomials of displacement \( w(x) \) are introduced to calculate the vibration of Timoshenko beam considering the second moment of area \( I(x) \) and shear modulus \( G(x) \). However, \( I(x) \) and \( G(x) \) are neglected for Euler beam.

The selected additional polynomials should satisfy any subsequent differential operations and should be sufficiently smooth over \([0, L]\). Mathematically, the term “sufficiently smooth” means that the first and third derivative of supplementary functions exist and keep continuous on the entire beam. In this paper, the selected additional polynomials are as follows

\[
  \tilde{\xi}_1(x) = \frac{9L}{4\pi} \sin \left( \frac{\pi x}{2L} \right) - \frac{L}{12\pi} \sin \left( \frac{3\pi x}{2L} \right) \\
  \tilde{\xi}_2(x) = -\frac{9L}{4\pi} \cos \left( \frac{\pi x}{2L} \right) - \frac{L}{12\pi} \cos \left( \frac{3\pi x}{2L} \right) \\
  \tilde{\xi}_3(x) = \frac{L^3}{\pi^3} \sin \left( \frac{\pi x}{2L} \right) - \frac{L^3}{3\pi^3} \sin \left( \frac{3\pi x}{2L} \right) \\
  \tilde{\xi}_4(x) = -\frac{L^3}{\pi^3} \cos \left( \frac{\pi x}{2L} \right) - \frac{L^3}{3\pi^3} \cos \left( \frac{3\pi x}{2L} \right)
\]

It is obvious from equations (7) to (10) that

\[
  \tilde{\xi}_1'(0) = \tilde{\xi}_3'''(0) = \tilde{\xi}_2'(L) = \tilde{\xi}_4'''(L) = 1
\]
Solution for the system

For solving the vibration of the FGMT beam, the Rayleigh-Ritz method will be introduced in the following. The Lagrange function of the free vibration of the beam is

$$L = T - U$$

(12)

where $T$ and $U$ denote the total kinetic energy and the total potential energy of the beam. $U$ and $T$ can be expressed as

$$U = \frac{1}{2} \int_0^L \left[ E(x) I(x) \left( \frac{d\theta}{dx} \right)^2 + kA(x)G(x) \left( \frac{dw}{dx} - \theta \right)^2 \right] dx + \frac{1}{2} \left( k_0 w^2 + K_0 \theta^2 \right)_{x=0} + \frac{1}{2} \left( k_1 w^2 + K_1 \theta^2 \right)_{x=L}$$

(13)

$$T = \frac{1}{2} \int_0^L \left[ \rho(x) I(x) \left( \frac{d\theta}{dt} \right)^2 + \rho(x) A(x) \left( \frac{dw}{dt} \right)^2 \right] dx$$

(14)

It should be noted that the potential energy of the springs at boundaries are included in the total potential energy, and the stiffness of these elastic springs can be any value satisfying one’s requirements.

It is not easy to solve the integral in equations (13) and (14) with $A(x), I(x), E(x), G(x)$ and $\rho(x)$ varying arbitrarily along the beam. In order to overcome this problem, $A(x), I(x), E(x), G(x)$ and $\rho(x)$ are all expanded into the Fourier cosine series, as shown

$$A(x) = \sum_{m=0}^{\infty} S_m \cos \lambda_m x$$

(15)

$$I(x) = \sum_{m=0}^{\infty} I_m \cos \lambda_m x$$

(16)

$$E(x) = \sum_{m=0}^{\infty} H_m \cos \lambda_m x$$

(17)

$$\rho(x) = \sum_{m=0}^{\infty} Q_m \cos \lambda_m x$$

(18)

$$G(x) = \sum_{m=0}^{\infty} G_m \cos \lambda_m x$$

(19)

where $\lambda_m = m\pi/L$, $S_m$, $I_m$, $H_m$, $Q_m$ and $G_m$ are Fourier coefficients which are actually obtained by Discrete Cosine Transform (DCT).

Substituting equations (5), (6), (13), (14) into equation (12) and minimizing all the unknown coefficients, the governing equations of the FGMT beam can be obtained as

$$(K - \omega^2 M)X = 0$$

(20)

where $K$, $M$ and $X$ are the stiffness matrix, the mass matrix and the column vector containing all the unknown coefficients in equations (5) and (6), respectively. By solving equation (20), one can derive the natural frequencies and the mode shapes of the beam.
Results and discussion

To validate the accuracy of the proposed method, several numerical examples will be first considered. The influence of the parameters of the beam on the vibration behaviors is also analyzed.

Validation of the present method

First, the free vibration of a uniform beam is considered in this part. In this case, the parameters of the beam are set as the same as those in literature\textsuperscript{38,39} in which the parameters are $E_0 = 200$ GPa, $\rho_0 = 5700$ kg/m$^3$, $I_0/A_0 = 0.01$, $k = 5/6$ and $\Omega = \omega L^2 \sqrt{\rho_0 A_0/E_0 I_0}$. In literature,\textsuperscript{38,39} the vibration of the beam is derived based on the power series method and the transforms complex couple functions method, respectively. The influence of the truncation coefficient $N$ on the dimensionless natural frequency which is introduced to study the convergence of the present method is presented in Table 1. It is observed that the natural frequencies gradually converge by increasing $N$ and it is almost unchanged when $N$ is larger than 9. So $N = 15$ is efficient to study the vibration characteristics of the beam in the following parts. The dimensionless natural frequencies of the beam with different boundary conditions are shown in Table 2. It is observed that the results derived by the present method match well with those in literature.\textsuperscript{38,39}

To further validate the accuracy of the proposed method, the free vibration of a beam with variable cross section varies are studied. The parameters of the beam are set as the same with those in Huang et al.,\textsuperscript{39} as $E_0 = 200$ GPa, $\rho_0 = 5700$ kg/m$^3$, $I_0/A_0 = 0.01$, respectively. The cross-sectional area $A(x)$ and the second moment of area $I(x)$ are denoted as

$$A(x) = A_0 \left(1 - \frac{cx}{L}\right) \quad (21)$$

$$I(x) = I_0 \left(1 - \frac{cx}{L}\right)^3 \quad (22)$$

where $c$ is the geometrical parameter. The first five dimensionless natural frequencies of the beam with different boundary conditions are listed as in Table 3 and compared with those in Huang et al.\textsuperscript{39} It could be seen that these two results agree with each other very well.

Table 1. The first four dimensionless natural frequencies with different $N$.

| $\Omega$ | $N = 7$ | $N = 9$ | $N = 11$ | $N = 13$ | $N = 15$ |
|---------|--------|--------|--------|--------|--------|
| 1       | 13.8347| 13.8347| 13.8347| 13.8347| 13.8347|
| 2       | 28.5179| 28.5179| 28.5179| 28.5179| 28.5179|
| 3       | 45.6659| 45.6659| 45.6659| 45.6659| 45.6659|
| 4       | 61.8620| 61.8620| 61.8620| 61.8620| 61.8620|

Table 2. The first five dimensionless natural frequencies of the uniform beam.

| Boundary conditions | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---------------------|-----------|-----------|-----------|-----------|-----------|
| C-F                 | 3.2271    | 14.4689   | 31.5025   | 47.9090   | 62.3470   |
| Ref. [40]           | 3.227128  | 14.468930 | 31.502970 | 48.082740 | 62.62921  |
| Ref. [39]           | 3.23      | 14.47     | 31.50     | 47.91     | 62.35     |
| C-C                 | 13.8347   | 28.5179   | 45.6659   | 61.8620   | 68.2836   |
| Ref. [40]           | 13.834758 | 28.517926 | 45.667237 | 61.867699 | 68.292529 |
| Ref. [39]           | 13.84     | 28.54     | 45.67     | 61.86     | 68.28     |
| F-F                 | 16.7920   | 33.8149   | 51.5214   | 58.9920   | 73.7397   |
| Ref. [40]           | 16.791957 | 33.814869 | 51.526943 | 58.993336 | 73.963612 |
| Ref. [39]           | 16.79     | 33.82     | 51.52     | 58.99     | 73.74     |
Boundary conditions

| C-C | Present | Ref. [40] |
|-----|---------|-----------|
| C-F | Present | Ref. [40] |
| S-S |         |           |
| S-F |         |           |
| C-S |         |           |
| F-F |         |           |

| Boundary conditions | $\Omega_i$ | Present | Ref. [40] |
|---------------------|-----------|---------|-----------|
| C-C                 | 1         | 3.9443  | 3.94636  |
|                     | 2         | 14.9224 | 14.93640 |
|                     | 3         | 30.5315 | 30.57273 |
|                     | 4         | 46.3363 | 46.40879 |
| C-F                 | 1         | 12.6572 | 12.68157 |
|                     | 2         | 26.4155 | 26.49100 |
|                     | 3         | 42.5640 | 42.64203 |
|                     | 4         | 58.5490 | 58.66849 |

In this part, the vibration of a FGMT beam with non-uniform cross section will be taken into consideration to further validate the accuracy of the present method. The cross-sectional area $A(x)$ and the second moment of area $I(x)$ vary according to equation (21) and (22). The Young’s modulus $E(x)$ and the mass density $\rho(x)$ of the beam vary as

$$E(x) = (E_1 - E_0) \left( \frac{x}{L} \right)^n + E_0$$

$$\rho(x) = (\rho_1 - \rho_0) \left( \frac{x}{L} \right)^n + \rho_0$$

where $n$ is the material gradient parameter. The material properties of the beam are set as those in literature. $E_0 = 200$ GPa, $\rho_0 = 5700$ kg/m$^3$, $E_1 = 70$ GPa, $\rho_1 = 2702$ kg/m$^3$, $I_0/A_0 = 0.01$. The gradient parameter $n$ is 2 and $c$ is 0.1 in this section. The first four dimensionless natural frequencies of this beam with different boundary conditions are shown in Table 4. It is observed that the results derived by the present method match very well with those in literature.

### Free vibration of FGMT beams with variable circular cross section

In this part, the free vibration of a FGMT beam with circular cross section, shown in Figure 2, is studied. The parameters of the beam are $\rho_0 = 7800$ kg/m$^3$, $E_0 = 202$ GPa, $\rho_1 = 2700$ kg/m$^3$, $E_1 = 70$ GPa and $\nu=0.3$. The material properties and cross sections are

$$E(x) = (E_1 - E_0) \left( \frac{x}{L} \right)^n + E_0$$

$$I(x) = \pi R(x)^4/4$$
\[ A(x) = \pi R(x)^2 \]  \hspace{1cm} (27)\\
\[ \rho(x) = (\rho_1 - \rho_0) \left(\frac{x}{L}\right)^n + \rho_0 \]  \hspace{1cm} (28)\\
\[ R(x) = \beta \left( x - \frac{L}{2} \right)^2 + R_{\text{center}} \]  \hspace{1cm} (29)\\
\[ G(x) = \frac{E(x)}{(1 + \nu)} \]  \hspace{1cm} (30)

where \( R(x) \) is the radius of the beam, \( E_1, \rho_1 \) and \( E_0, \rho_0 \) are Young’s modulus and the mass density at \( x = L \) and \( x = 0 \), respectively, \( n \) is the material gradient parameter, \( \beta \) is the radius coefficient, \( R_{\text{center}} \) is the radius at the center of the beam.

The dimensionless natural frequency parameter \( \Omega \) is introduced to effectively study the influence of the parameters on the vibrational behaviors.

\[ \Omega = \omega L^2 \sqrt{\frac{\rho_0 A_0}{E_0 A_0}} \]  \hspace{1cm} (31)

The radius of the beam at the center is set as 0.05 m and the length is 1 m. The first four dimensionless natural frequencies of the beam with different \( \beta \) and \( n \) under clamped–clamped boundary conditions are given in Table 5. It could be seen that the natural frequencies gradually decrease as \( \beta \) increases and increases as \( n \) increases. The first four dimensionless natural frequencies with \( \beta = 0.1 \) are listed in Table 6. It could be observed that the natural frequencies with clamped-clamped end support decrease as the increasing of \( n \).

Figures 3 and 4 show the first four transverse displacement and the rotation of cross section mode shapes of the uniform beam with \( \beta = 0.2 \) and \( n = 2 \) compared with the uniform Timoshenko beam with radius 0.05 m under clamped–clamped boundary conditions. It is observed from Figures 3 and 4 that the maximum amplitude of the mode shapes appears at the right side of the beam for the transverse displacement mode shapes and the rotation of cross section mode shapes. For the higher order mode shapes, the amplitude of the right side of the beam is slightly larger than that of the left side. It could be explained that the average stiffness of the left side of the beam is larger than that of the right side while the uniform beam has same stiffness on both sides.

**Effects of material parameters and spring stiffness on the vibration of beams**

**Effects of the coefficient of the radius \( \beta \).** In this part, the effects of \( \beta \) on the natural frequencies and mode shapes with \( n = 2 \) with clamped–clamped boundary condition of the beam are discussed. The ratio of the non-dimensional natural frequencies \( \frac{u_i}{u_{i0}} \) with different \( \beta \) are plotted in Figure 5, in which \( u_i \) is the non-dimensional natural frequency.
corresponding to different $\beta$, $\omega_0$ is the non-dimensional natural frequency at $\beta = 0$. Figure 5 implies that the natural frequencies decrease with the increasing of $\beta$, and this trend is more obvious on higher order natural frequency.

The first four mode shapes of the transverse displacement and the rotation of cross section by different $\beta$ are shown in Figures 6 and 7, respectively. It can be seen that the amplitude of the first-order transverse displacement mode shape reaches the maximum on the middle right of the beam and the transverse displacement decreases with the increasing of $\beta$. The vibration amplitudes of the position around the center of the beam increase but the first-order transverse displacement and the second-order rotation of cross-section mode shape show decreasing trend with increasing $\beta$.

### Table 5. The first four dimensionless natural frequencies with different $\beta$ and different $n$ under clamped-clamped boundary conditions.

| $n$ | $\Omega$ | $\beta = -0.15$ | $\beta = -0.1$ | $\beta = 0$ | $\beta = 0.1$ | $\beta = 0.2$ | $\beta = 0.3$ |
|-----|---------|----------------|----------------|------------|-------------|-------------|-------------|
| 0   |         | 36.5495        | 25.7483        | 21.3771    | 20.4000     | 19.9452     | 19.5802     |
| 1   |         | 115.4109       | 78.6468        | 55.8060    | 46.4450     | 40.9382     | 37.0959     |
| 2   |         | 263.60         | 160.5796       | 102.6743   | 79.9726     | 67.0003     | 58.2666     |
| 3   |         | 447.6878       | 261.5574       | 158.4153   | 118.7607    | 96.6086     | 82.0775     |
| 4   |         | 35.4838        | 25.1709        | 20.9458    | 19.9815     | 19.5271     | 19.1641     |
| 1   |         | 114.4444       | 77.9257        | 55.1937    | 45.8655     | 40.3882     | 36.5768     |
| 2   |         | 263.7313       | 160.2628       | 101.9908   | 79.3142     | 66.3864     | 57.6986     |
| 3   |         | 448.3317       | 261.0738       | 157.6949   | 118.0639    | 95.9651     | 81.4883     |
| 4   |         | 34.6395        | 24.4125        | 20.0854    | 19.1230     | 18.7170     | 18.4140     |

### Table 6. The first four dimensionless natural frequencies with $\beta = 0.1$.

| Boundary conditions | $\Omega$ | $n = 0$ | $n = 1$ | $n = 2$ | $n = 3$ |
|---------------------|---------|--------|--------|--------|--------|
| C-C                 | 1       | 20.4000| 19.9185| 19.1230| 18.7765|
|                     | 2       | 46.4450| 45.8655| 44.7408| 44.2552|
|                     | 3       | 79.9726| 79.3142| 78.0625| 77.4790|
|                     | 4       | 118.7607| 118.0639| 116.7557| 116.1060|
| C-F                 | 1       | 2.5968 | 3.5198 | 3.5163 | 3.4201 |
|                     | 2       | 16.1178| 17.8649| 17.7166| 17.6181|
|                     | 3       | 42.5372| 44.1997| 44.1692| 44.2973|
|                     | 4       | 77.2404| 78.8305| 78.8983| 79.1572|
| C-S                 | 1       | 12.6203| 13.6815| 13.2787| 13.0646|
|                     | 2       | 37.1641| 37.9636| 37.4569| 37.2956|
|                     | 3       | 70.3177| 70.9606| 70.4022| 70.2475|
|                     | 4       | 109.6585| 110.1772| 109.5892| 109.4255|
| S-S                 | 1       | 6.7763 | 6.7239 | 6.7656 | 6.7903 |
|                     | 2       | 27.6868| 27.8181| 27.8990| 27.8943|
|                     | 3       | 60.1835| 60.2387| 60.2037| 60.1517|
|                     | 4       | 99.9577| 99.9512| 99.8401| 99.7485|
| S-F                 | 1       | 9.5858 | 10.2955| 10.5825| 10.6658|
|                     | 2       | 32.7368| 33.7635| 34.2769| 34.5051|
|                     | 3       | 67.0539| 68.1010| 68.6724| 69.0217|
|                     | 4       | 108.1090| 109.1420| 109.7502| 110.1819|
| F-F                 | 1       | 12.8119| 13.1952| 13.7336| 13.9382|
|                     | 2       | 38.1784| 38.8016| 39.6385| 39.9865|
|                     | 3       | 74.2521| 74.8806| 75.8030| 76.2443|
|                     | 4       | 116.4693| 117.0830| 118.0544| 118.5572|
Figure 3. The first four transverse displacement mode shape compared with uniform Timoshenko beam. (a): first order; (b) second order; (c) third order; (d) fourth order.

Figure 4. The first four rotations of cross section mode shapes compared with uniform Timoshenko beam. (a): first order; (b) second order; (c) third order; (d) fourth order.
Effects of the gradient parameter $n$. Furthermore, the influence of the gradient parameter $n$ on the dimensionless natural frequencies and mode shapes is investigated in this part. Here $R_{center}$ is set as 0.05 m, and $\beta = 0.2$, and the boundary condition is set as clamped-clamped end support. The ratio of the non-dimensional natural frequencies $u_i/u_0$ with different $n$ is displayed in Figure 8. It implies an important information that the higher-order natural frequencies decrease when $n$ is less than 20. However, the natural frequencies almost keep unchanged when $n$ is more than 20.

Figures 9 and 10 plot the first four mode shapes of the transverse displacement and the rotation of cross section with different $n$, respectively. It could be seen that the rotational mode shapes have one more vibration extremum compared with the transverse displacement ones for the same order mode shape.
Effects of elastic supports. The influence of elastic boundary conditions of the vibration of the beam is studied. The coefficient of radius $\beta$ is 0.2, and the gradient parameter $n$ is 2 in this part. The first four frequency parameters with one side of the beam is set as clamped end support and the other side is fixed by an infinite translational spring $k_1$ and a variable rotational spring $K_1$ from 0 to infinite is displayed in Figure 11. The frequency parameters with one end of the beam clamped and the other end restrained by a variable translational spring $k_1$ is shown in Figure 12. From Figures 11 and 12, it takes a conclusion that different spring values have different effects on the natural frequencies and the higher order natural frequencies change with larger spring values. And then, the first four dimensionless natural frequencies which are influenced by a variable translational spring $k_0$ and a variable rotational spring $K_0$ are displayed in Figure 13. Supposing translational spring $k_0$ and the rotational spring $K_0$ is changing from 0 to $10^{13}$, the other end of the beam is set as Clamped. One can see from Figure 13 that the
Figure 9. Transverse displacement mode shape with different \( n \). (a): first order; (b) second order; (c) third order; (d) fourth order.

Figure 10. Rotational mode shapes with different \( n \). (a): first order; (b) second order; (c) third order; (d) fourth order.
dimensionless natural frequencies are increasing gradually with $k_0$ and $K_0$ becoming bigger, and the dimensionless natural frequency is almost unchanged when the spring value is big enough. And the natural frequencies of the beam are more sensitive to the translational springs than the rotational springs.

**Effects of a variable translational spring $k_0$ and different $\beta$.** In this part, the effects of a variable translational spring $k_0$ and different $\beta$ on the first four non-dimensional natural frequencies and mode shapes with $n = 2$ with clamped-clamped boundary condition of the beam in Figure 14 are discussed. Supposing translational spring $k_0$ is changing from 0 to $10^{13}$, the other end of the beam is set as clamped, and $\beta$ is continuously changing from 0 to 0.3. One can see from Figure 14 that the non-dimensional natural frequency keeps unchanged when $k_0$ is relatively small, then it rises gradually when $k_0$ is bigger than $10^6$, and finally it almost unchanged when $k_0$ is close to infinity. On the other hand, the non-dimensional natural frequency decreases gradually by increasing $\beta$ and setting $k_0$ as a constant value. Additionally, the efforts of $k_0$ and $\beta$ on the higher order non-dimensional natural frequency are much higher than the first order one.

**Effects of the gradient parameter $n$ and coefficient of the radius $\beta$.** The influence of the gradient parameter $n$ and coefficient of the radius $\beta$ on the dimensionless natural frequencies is considered in this part. Supposing $n$ is changed from 0 to 30 and $\beta$ is continuously changed from 0 to 0.3, the end support is C-C. Figure 15 shows the first four dimensionless natural frequencies with different $n$ and variable $\beta$. As shown in Figure 15, there is a steady decrease with increasing $\beta$. The first-order non-dimensional natural frequency jumps when $n$ is relatively small, and then it rises slowly when $n$ is increasing gradually. The second, third and forth order non-dimensional natural frequencies show the same trend as the first order.
Figure 14. First four non-dimensional natural frequencies with variable $k_0$ and different $\beta (n = 2)$. (a): first order; (b) second order; (c) third order; (d) fourth order.

Figure 13. First four non-dimensional natural frequencies with variable $k_0$ and $K_0 (\beta = 0.2, n = 2)$. (a): first order; (b) second order; (c) third order; (d) fourth order.
This paper proposed a unified procedure to study the free vibration of an elastically restrained FGMT beam with any variable cross section. The translational displacement and the rotation of cross section are both expressed into the Fourier series with some polynomial functions based on the so-called spectro-geometric method. The cross-sectional area, the second moment of area, the Young's modulus, the mass density and the shear modulus have also been expanded into Fourier cosine series so that this method could be adopted to analyze any kind of FGMT beams with any variable cross sections. Several examples are introduced to validate the accuracy and reliability of the method. The natural frequencies and the mode shapes of the beams with different ends supports are studied. This method is generally applicable to any kinds of FGM beams with smoothly variable cross section under elastic boundary conditions.

Declaration of conflicting interests
The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding
The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This work is supported by the National Natural Science Foundation of China (grant no. 51805341), the Natural Science Foundation of Jiangsu Province (grant no. BK20180843), China Postdoctoral Science Foundation (grant no. 2020M671412) and Jiangsu Planned Projects for Postdoctoral Research Funds (grant no. 2020Z055).

ORCID iD
Guofang Li https://orcid.org/0000-0002-3783-4258

References
1. Yamanouchi M, Koizumi M, Hirai T, et al. Proceedings of the first international symposium on functionally gradient materials. FGM forum, Sendai, Japan, 1990.
2. Koizumi M. FGM activities in Japan. Compos Part B: Eng 1997; 28: 1–4.
3. Loy CT, Lam KY and Reddy JN. Vibration of functionally graded cylindrical shells. *Int J Mech Sci* 1999; 41: 309–324.
4. Benatta MA, Tounsi A, Mechab I, et al. Mathematical solution for bending of short hybrid composite beams with variable fibers spacing. *Appl Math Computat* 2009; 212: 337–348.
5. Alshorbagy AE, Eltaher MA and Mahmoud FF. Free vibration characteristics of a functionally graded beam by finite element method. *Appl Math Model* 2011; 35: 412–425.
6. Mahi A, Bedia EAA, Tounsi A, et al. An analytical method for temperature-dependent free vibration analysis of functionally graded beams with general boundary conditions. *Compos Struct* 2010; 92: 1877–1887.
7. Li XF. A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler-Bernoulli beams. *J Sound Vib* 2008; 318: 1210–1229.
8. Kang YA and Li XF. Large deflections of a non-linear cantilever functionally graded beam. *J Reinforced Plastics Compos* 2010; 29: 1761–1774.
9. Sina SA, Navazi HM and Haddadpour H. An analytical method for free vibration analysis of functionally graded beams. *Mater Des* 2009; 30: 741–747.
10. Giunta G, Crisafulli D, Belouettar S, et al. Hierarchical theories for the free vibration analysis of functionally graded beams. *Compos Struct* 2011; 94: 68–74.
11. Mohanty SC, Dash RR and Rout T. Parametric instability of a functionally graded Timoshenko beam on Winkler's elastic foundation. *Nucl Eng Des* 2011; 241: 2698–2715.
12. Wei D, Liu Y and Xiang Z. An analytical method for free vibration analysis of functionally graded beams with edge cracks. *J Sound Vib* 2012; 331: 1686–1700.
13. Thai HT and Vo TP. Bending and free vibration of functionally graded beams using various higher-order shear deformation beam theories. *Int J Mech Sci* 2012; 62: 57–66.
14. Li SR, Cao DF and Wan ZQ. Bending solutions of FGM Timoshenko beams from those of the homogenous Euler–Bernoulli beams. *Appl Math Model* 2013; 37: 7077–7085.
15. Şimşek M and Reddy JN. Bending and vibration of functionally graded microbeams using a new higher order beam theory and the modified couple stress theory. *Int J Eng Sci* 2013; 64: 37–53.
16. Aydogdu M and Taskin V. Free vibration analysis of functionally graded beams with simply supported edges. *Mater Des* 2007; 28: 1651–1656.
17. Wu L, Wang Q and Elishakoff I. Semi-inverse method for axially functionally graded beams with an anti-symmetric vibration mode. *J Sound Vib* 2005; 284: 1190–1202.
18. Elishakoff I and Candan S. Apparently first closed-form solution for vibrating: inhomogeneous beams. *Int J Solids Struct* 2001; 38: 3411–3441.
19. Elishakoff I and Guede Z. Analytical polynomial solutions for vibrating axially graded beams. *Mech Adv Mater Struct* 2004; 11: 517–533.
20. Ahmad S, Reza A and Shahin H. Free vibration and stability of axially functionally graded tapered Euler-Bernoulli beams. *Shock Vib* 2011; 18: 683–696.
21. Huang Y and Li XF. A new approach for free vibration of axially functionally graded beams with non-uniform cross-section. *J Sound Vib* 2010; 329: 2291–2303.
22. Romano F. Deflections of Timoshenko beam with varying cross-section. *Int J Mech Sci* 1996; 38: 1017–1035.
23. Fertis DG and Zobel EC. Equivalent systems for the deflection of variable stiffness members. *J Struct Div* 1958; 84: 1820.
24. Ece MC, Aydogdu M and Taskin V. Vibration of a variable cross-section beam. *Mech Res Commun* 2007; 34: 78–84.
25. Banerjee JR. Free vibration of centrifugally stiffened uniform and tapered beams using the dynamic stiffness method. *J Sound Vib* 2000; 233: 857–875.
26. Banerjee JR, Su H and Jackson DR. Free vibration of rotating tapered beams using the dynamic stiffness method. *J Sound Vib* 2006; 298: 1034–1054.
27. Abdel-Jaber MS, Al-Qaisia AA, Abdel-Jaber M, et al. Nonlinear natural frequencies of an elastically restrained tapered beam. *J Sound Vib* 2008; 313: 772–783.
28. Attarnejad R and Shabba A. Application of differential transform method in free vibration analysis of rotating non-prismatic beams. *World Appl Sci J* 2008; 5: 441–448.
29. Sherbourne AN and Pandey MD. Differential quadrature method in the buckling analysis of beams and composite plates. *Comput Struct* 1991; 40: 903–913.
30. Aydogdu M. Semi-inverse method for vibration and buckling of axially functionally graded beams. *J Reinf Plastics Compos* 2008; 27: 683–691.
31. Attarnejad R. Basic displacement functions in analysis of nonprismatic beams. *Eng Computat* 2010; 27: 733–745.
32. Shoshtari A and Khajavi R. An efficient procedure to find shape functions and stiffness matrices of nonprismatic Euler-Bernoulli and Timoshenko beam elements. *Eur J Mech-A/Solids* 2010; 29: 826–836.
33. Shahba A, Attarnejad R, Marvi MT, et al. Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions. *Compos Part B: Eng* 2011; 42: 801–808.
34. Attarnejad R and Shahba A. Basic displacement functions for centrifugally stiffened tapered beams. *Int J Num Meth Biomed Eng* 2011; 27: 1385–1397.
Appendix 1. Detailed expressions for the matrices $K$, $M$ and $X$

The detailed expressions for the mass matrix $M$, stiffness matrix $K$ and expansion coefficient vector $X$ in equation (20) are given as follows

$$K = \begin{bmatrix} M_A & M_{AC} & 0 & 0 \\ M_{AC}^T & M_{CC} & 0 & 0 \\ 0 & 0 & M_B & M_{BD} \\ 0 & 0 & M_{BD}^T & M_{DD} \end{bmatrix}$$

$$M = \begin{bmatrix} K_A & K_{AC} & K_{AB} & K_{AD} \\ K_{AC}^T & K_{CC} & K_{CB} & K_{CD} \\ K_{AB}^T & K_{CB}^T & K_B & K_{BD} \\ K_{AD}^T & K_{CD}^T & K_{BD} & K_{DD} \end{bmatrix}$$

$$X = \begin{bmatrix} A \\ C \\ B \\ D \end{bmatrix}$$

$$\{M_A\}_{N+1,N+1} = \int_0^L \sum_{m=0}^{\frac{Q}{2\pi}} Q_m \cos \lambda m x \sum_{m=0}^{\frac{S}{2\pi}} S_m \cos \lambda m x \left(-\lambda m \sum_{m=0}^{\frac{N}{2\pi}} A_m \sin(\lambda m x)\right)^2 dx$$

$$\{M_{AC}\}_{N+1,N+1} = \int_0^L \sum_{m=0}^{\frac{Q}{2\pi}} Q_m \cos \lambda m x \sum_{m=0}^{\frac{S}{2\pi}} S_m \cos \lambda m x \left[-\lambda m \sum_{m=0}^{\frac{N}{2\pi}} A_m \sin(\lambda m x)\right] \left(\frac{9}{8} \cos \frac{\pi x}{2L} - \frac{1}{8} \cos \frac{3\pi x}{2L}\right) dx$$

$$\{M_{2AC}\}_{N+1,N+1} = \int_0^L \sum_{m=0}^{\frac{Q}{2\pi}} Q_m \cos \lambda m x \sum_{m=0}^{\frac{S}{2\pi}} S_m \cos \lambda m x \left[-\lambda m \sum_{m=0}^{\frac{N}{2\pi}} A_m \sin(\lambda m x)\right] \left(\frac{9}{8} \sin \frac{\pi x}{2L} + \frac{1}{8} \sin \frac{3\pi x}{2L}\right) dx$$

$$\{M_{3AC}\}_{N+1,N+1} = \int_0^L \sum_{m=0}^{\frac{Q}{2\pi}} Q_m \cos \lambda m x \sum_{m=0}^{\frac{S}{2\pi}} S_m \cos \lambda m x \left[-\lambda m \sum_{m=0}^{\frac{N}{2\pi}} A_m \sin(\lambda m x)\right] \left(\frac{L^2}{2\pi^2} \cos \frac{\pi x}{2L} - \frac{L^2}{2\pi^2} \cos \frac{3\pi x}{2L}\right) dx$$
\[
\{M^{4}_{AC}\}_{N+1,1} = \int_0^L \sum_{m=0}^Q Q_m \cos \lambda_m x \sum_{m=0}^S S_m \cos \lambda_m x \left[ -\lambda_m \sum_{m=0}^N A_m \sin (\lambda_m x) \right] \left( \frac{L^2}{2\pi^2} \sin \frac{\pi x}{2L} + \frac{L^2}{2\pi^2} \sin \frac{3\pi x}{2L} \right) dx
\]

\[
\{M^{11}_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^Q Q_m \cos \lambda_m x \sum_{m=0}^S S_m \cos \lambda_m x \left( \frac{9}{8} \cos \frac{\pi x}{2L} - \frac{1}{8} \cos \frac{3\pi x}{2L} \right) \left( \frac{9}{8} \cos \frac{\pi x}{2L} + \frac{1}{8} \cos \frac{3\pi x}{2L} \right) dx
\]

\[
\{M^{12}_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^Q Q_m \cos \lambda_m x \sum_{m=0}^S S_m \cos \lambda_m x \left( \frac{9}{8} \cos \frac{\pi x}{2L} - \frac{1}{8} \cos \frac{3\pi x}{2L} \right) \left( \frac{9}{8} \cos \frac{\pi x}{2L} + \frac{1}{8} \cos \frac{3\pi x}{2L} \right) dx
\]

\[
\{M^{13}_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^Q Q_m \cos \lambda_m x \sum_{m=0}^S S_m \cos \lambda_m x \left( \frac{9}{8} \cos \frac{\pi x}{2L} - \frac{1}{8} \cos \frac{3\pi x}{2L} \right) \left( \frac{L^2}{2\pi^2} \sin \frac{\pi x}{2L} + \frac{L^2}{2\pi^2} \sin \frac{3\pi x}{2L} \right) dx
\]

\[
\{M^{14}_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^Q Q_m \cos \lambda_m x \sum_{m=0}^S S_m \cos \lambda_m x \left( \frac{9}{8} \cos \frac{\pi x}{2L} - \frac{1}{8} \cos \frac{3\pi x}{2L} \right) \left( \frac{L^2}{2\pi^2} \sin \frac{\pi x}{2L} + \frac{L^2}{2\pi^2} \sin \frac{3\pi x}{2L} \right) dx
\]

\[
\{M^{21}_{CC}\}_{1,1} = \{M^{12}_{CC}\}_{1,1}
\]

\[
\{M^{22}_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^Q Q_m \cos \lambda_m x \sum_{m=0}^S S_m \cos \lambda_m x \left( \frac{9}{8} \sin \frac{\pi x}{2L} + \frac{1}{8} \sin \frac{3\pi x}{2L} \right) \left( \frac{9}{8} \sin \frac{\pi x}{2L} + \frac{1}{8} \sin \frac{3\pi x}{2L} \right) dx
\]

\[
\{M^{23}_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^Q Q_m \cos \lambda_m x \sum_{m=0}^S S_m \cos \lambda_m x \left( \frac{9}{8} \sin \frac{\pi x}{2L} - \frac{1}{8} \sin \frac{3\pi x}{2L} \right) \left( \frac{L^2}{2\pi^2} \cos \frac{\pi x}{2L} + \frac{L^2}{2\pi^2} \cos \frac{3\pi x}{2L} \right) dx
\]

\[
\{M^{24}_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^Q Q_m \cos \lambda_m x \sum_{m=0}^S S_m \cos \lambda_m x \left( \frac{9}{8} \sin \frac{\pi x}{2L} + \frac{1}{8} \sin \frac{3\pi x}{2L} \right) \left( \frac{L^2}{2\pi^2} \cos \frac{\pi x}{2L} + \frac{L^2}{2\pi^2} \cos \frac{3\pi x}{2L} \right) dx
\]

\[
\{M^{31}_{CC}\}_{1,1} = \{M^{13}_{CC}\}_{1,1}
\]

\[
\{M^{32}_{CC}\}_{1,1} = \{M^{23}_{CC}\}_{1,1}
\]

\[
\{M^{33}_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^Q Q_m \cos \lambda_m x \sum_{m=0}^S S_m \cos \lambda_m x \left( \frac{L^2}{2\pi^2} \cos \frac{\pi x}{2L} - \frac{L^2}{2\pi^2} \cos \frac{3\pi x}{2L} \right) \left( \frac{L^2}{2\pi^2} \cos \frac{\pi x}{2L} + \frac{L^2}{2\pi^2} \cos \frac{3\pi x}{2L} \right) dx
\]

\[
\{M^{34}_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^Q Q_m \cos \lambda_m x \sum_{m=0}^S S_m \cos \lambda_m x \left( \frac{L^2}{2\pi^2} \cos \frac{\pi x}{2L} - \frac{L^2}{2\pi^2} \cos \frac{3\pi x}{2L} \right) \left( \frac{L^2}{2\pi^2} \sin \frac{\pi x}{2L} + \frac{L^2}{2\pi^2} \sin \frac{3\pi x}{2L} \right) dx
\]

\[
\{M^{41}_{CC}\}_{1,1} = \{M^{14}_{CC}\}_{1,1}
\]

\[
\{M^{42}_{CC}\}_{1,1} = \{M^{24}_{CC}\}_{1,1}
\]

\[
\{M^{43}_{CC}\}_{1,1} = \{M^{34}_{CC}\}_{1,1}
\]
\[
\{M^4_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^{G} \sum_{m=0}^{S} S_m \cos \lambda_m x \left[ \frac{L^2}{2 \pi^2} \sin \frac{\pi x}{2L} + \frac{L^2}{2 \pi^2} \sin \frac{3 \pi x}{2L} \right] dx
\]

\[
M_B = M_f
\]

\[
\{M^4_{BD}\}_{1,1} = \{M^4_{AC}\}_{1,1}
\]

\[
\{M^4_{BD}\}_{1,1} = \{M^4_{AC}\}_{1,1}
\]

\[
\{M^4_{BD}\}_{1,1} = \{M^4_{AC}\}_{1,1}
\]

\[
\{K_A\}_{N+1, N+1} = \int_0^L \sum_{m=0}^{G} \sum_{m=0}^{S} A_m \cos \lambda_m x \left[ \frac{L^2}{2 \pi^2} \sin \frac{\pi x}{2L} + \frac{L^2}{2 \pi^2} \sin \frac{3 \pi x}{2L} \right] dx
\]

\[
\{K^4_{AC}\}_{N+1, N+1} = \int_0^L \sum_{m=0}^{G} \sum_{m=0}^{S} A_m \cos \lambda_m x \left[ \frac{L^2}{2 \pi^2} \sin \frac{\pi x}{2L} + \frac{L^2}{2 \pi^2} \sin \frac{3 \pi x}{2L} \right] dx
\]

\[
\{K^4_{AC}\}_{N+1, N+1} = \int_0^L \sum_{m=0}^{G} \sum_{m=0}^{S} A_m \cos \lambda_m x \left[ \frac{L^2}{2 \pi^2} \sin \frac{\pi x}{2L} + \frac{L^2}{2 \pi^2} \sin \frac{3 \pi x}{2L} \right] dx
\]

\[
K_{AC} = \{K^4_{AC}\}_{N+1, N+1}
\]

\[
K_{AC} = \{K^4_{AC}\}_{N+1, N+1}
\]

\[
K_{AC} = \{K^4_{AC}\}_{N+1, N+1}
\]

\[
K_{AC} = \{K^4_{AC}\}_{N+1, N+1}
\]

\[
K_{AC} = \{K^4_{AC}\}_{N+1, N+1}
\]

\[
K_{AC} = \{K^4_{AC}\}_{N+1, N+1}
\]

\[
K_{AC} = \{K^4_{AC}\}_{N+1, N+1}
\]
\[
\{K_{AB}\}_{N+1,N+1} = \int_0^L k \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left[ -\lambda_m \sum_{m=0}^N A_m \sin (\lambda_m x) \right] \left[ \sum_{m=0}^N B_m \cos (\lambda_m x) \right] dx
\]

\[
\{K^1_{AD}\}_{N+1} = \int_0^L k \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left[ -\lambda_m \sum_{m=0}^N A_m \sin (\lambda_m x) \right] \left( \frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right) dx
\]

\[
\{K^2_{AD}\}_{N+1} = \int_0^L k \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left[ -\lambda_m \sum_{m=0}^N A_m \sin (\lambda_m x) \right] \left( -\frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) dx
\]

\[
K_{AD} = \left\{ \{K^1_{AD}\}_{N+1,1} \quad \{K^2_{AD}\}_{N+1,1} \right\}
\]

\[
\{K^{11}_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{9}{8} \cos \frac{\pi x}{2L} - \frac{1}{8} \cos \frac{3\pi x}{2L} \right)^2 dx + k_0 \int_0^L \left( \frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right)^2 dx \bigg|_{x=0} + k_L \int_0^L \left( \frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right)^2 dx \bigg|_{x=L}
\]

\[
\{K^{12}_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{9}{8} \cos \frac{\pi x}{2L} - \frac{1}{8} \cos \frac{3\pi x}{2L} \right) \left( \frac{9}{8} \sin \frac{\pi x}{2L} + \frac{1}{8} \sin \frac{3\pi x}{2L} \right) dx + k_0 \int_0^L \left( \frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) \left( \frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) dx \bigg|_{x=0} + k_L \int_0^L \left( \frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) \left( \frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) dx \bigg|_{x=L}
\]

\[
\{K^{13}_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{9}{8} \cos \frac{\pi x}{2L} - \frac{1}{8} \cos \frac{3\pi x}{2L} \right) \left( \frac{L^3}{\pi^3} \sin \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \sin \frac{3\pi x}{2L} \right) dx + k_0 \int_0^L \left( \frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right) \left( \frac{L^3}{\pi^3} \sin \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \sin \frac{3\pi x}{2L} \right) dx \bigg|_{x=0} + k_L \int_0^L \left( \frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right) \left( \frac{L^3}{\pi^3} \sin \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \sin \frac{3\pi x}{2L} \right) dx \bigg|_{x=L}
\]

\[
\{K^{14}_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{9}{8} \cos \frac{\pi x}{2L} - \frac{1}{8} \cos \frac{3\pi x}{2L} \right) \left( \frac{L^3}{\pi^3} \cos \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \cos \frac{3\pi x}{2L} \right) dx + k_0 \int_0^L \left( \frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) \left( \frac{L^3}{\pi^3} \cos \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \cos \frac{3\pi x}{2L} \right) dx \bigg|_{x=0} + k_L \int_0^L \left( \frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) \left( \frac{L^3}{\pi^3} \cos \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \cos \frac{3\pi x}{2L} \right) dx \bigg|_{x=L}
\]

\[
\{K^{21}_{CC}\}_{1,1} = \{K^{12}_{CC}\}_{1,1}
\]

\[
\{K^{22}_{CC}\}_{1,1} = \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{9}{8} \sin \frac{\pi x}{2L} + \frac{1}{8} \sin \frac{3\pi x}{2L} \right)^2 dx + k_0 \int_0^L \left( -\frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right)^2 dx \bigg|_{x=0} + k_L \int_0^L \left( -\frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right)^2 dx \bigg|_{x=L}
\]
\begin{align*}
\{K^{23}_{CC}\}_{1,1} &= \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{9}{8} \sin \frac{\pi x}{2L} + \frac{1}{8} \sin \frac{3\pi x}{2L} \right) \left( \frac{L^2}{2\pi^2} \cos \frac{\pi x}{2L} - \frac{L^2}{2\pi^2} \cos \frac{3\pi x}{2L} \right) dx \\
&+ k_0 \int_0^L \left( - \frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) \left( \frac{L^3}{\pi^3} \sin \frac{\pi x}{2L} - \frac{L^3}{\pi^3} \sin \frac{3\pi x}{2L} \right) dx |_{x=0} \\
&+ k_L \int_0^L \left( - \frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) \left( \frac{L^3}{\pi^3} \sin \frac{\pi x}{2L} - \frac{L^3}{\pi^3} \sin \frac{3\pi x}{2L} \right) dx |_{x=L}
\end{align*}

\begin{align*}
\{K^{24}_{CC}\}_{1,1} &= \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{9}{8} \sin \frac{3\pi x}{2L} + \frac{1}{8} \sin \frac{\pi x}{2L} \right) \left( \frac{L^2}{2\pi^2} \sin \frac{\pi x}{2L} + \frac{L^2}{2\pi^2} \sin \frac{3\pi x}{2L} \right) dx \\
&+ k_0 \int_0^L \left( - \frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) \left( \frac{L^3}{\pi^3} \cos \frac{\pi x}{2L} - \frac{L^3}{\pi^3} \cos \frac{3\pi x}{2L} \right) dx |_{x=0} \\
&+ k_L \int_0^L \left( - \frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) \left( \frac{L^3}{\pi^3} \cos \frac{\pi x}{2L} - \frac{L^3}{\pi^3} \cos \frac{3\pi x}{2L} \right) dx |_{x=L}
\end{align*}

\begin{align*}
\{K^{31}_{CC}\}_{1,1} &= \{K^{23}_{CC}\}_{1,1} \\
\{K^{32}_{CC}\}_{1,1} &= \{K^{23}_{CC}\}_{1,1}
\end{align*}

\begin{align*}
\{K^{33}_{CC}\}_{1,1} &= \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{L^2}{2\pi^2} \cos \frac{\pi x}{2L} + \frac{L^2}{2\pi^2} \cos \frac{3\pi x}{2L} \right)^2 dx \\
&+ k_0 \int_0^L \left( \frac{L^3}{\pi^3} \sin \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \sin \frac{3\pi x}{2L} \right)^2 dx |_{x=0} \\
&+ k_L \int_0^L \left( \frac{L^3}{\pi^3} \sin \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \sin \frac{3\pi x}{2L} \right)^2 dx |_{x=L}
\end{align*}

\begin{align*}
\{K^{34}_{CC}\}_{1,1} &= \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{L^2}{2\pi^2} \cos \frac{\pi x}{2L} - \frac{L^2}{2\pi^2} \cos \frac{3\pi x}{2L} \right) \left( - \frac{L^3}{\pi^3} \cos \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \cos \frac{3\pi x}{2L} \right) dx \\
&+ k_0 \int_0^L \left( \frac{L^3}{\pi^3} \sin \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \sin \frac{3\pi x}{2L} \right) \left( - \frac{L^3}{\pi^3} \cos \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \cos \frac{3\pi x}{2L} \right) dx |_{x=0} \\
&+ k_L \int_0^L \left( \frac{L^3}{\pi^3} \sin \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \sin \frac{3\pi x}{2L} \right) \left( - \frac{L^3}{\pi^3} \cos \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \cos \frac{3\pi x}{2L} \right) dx |_{x=L}
\end{align*}

\begin{align*}
\{K^{41}_{CC}\}_{1,1} &= \{K^{34}_{CC}\}_{1,1} \\
\{K^{42}_{CC}\}_{1,1} &= \{K^{34}_{CC}\}_{1,1}
\end{align*}

\begin{align*}
\{K^{43}_{CC}\}_{1,1} &= \{K^{34}_{CC}\}_{1,1}
\end{align*}

\begin{align*}
\{K^{44}_{CC}\}_{1,1} &= \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{L^3}{2\pi^2} \sin \frac{\pi x}{2L} + \frac{L^3}{2\pi^2} \sin \frac{3\pi x}{2L} \right)^2 dx \\
&+ k_0 \int_0^L \left( - \frac{L^3}{\pi^3} \cos \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \cos \frac{3\pi x}{2L} \right)^2 dx |_{x=0} + k_L \int_0^L \left( - \frac{L^3}{\pi^3} \cos \frac{\pi x}{2L} - \frac{L^3}{3\pi^3} \cos \frac{3\pi x}{2L} \right)^2 dx |_{x=L}
\end{align*}

\begin{align*}
\{K^1_{CB}\}_{1,N+1} &= \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{9}{8} \cos \frac{\pi x}{2L} - \frac{1}{8} \cos \frac{3\pi x}{2L} \right) \left( \sum_{m=0}^B B_m \cos \lambda_m x \right) dx
\end{align*}
\[
\{K^2_{CB}\}_{1,N+1} = \int_0^L \sum_{m=0}^S \sum_{n=0}^G S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{9}{8} \sin \frac{\pi x}{2L} + \frac{1}{8} \sin \frac{3\pi x}{2L} \right) \left( \sum_{n=0}^B B_n \cos \lambda_n x \right) \, dx
\]

\[
\{K^3_{CB}\}_{1,N+1} = \int_0^L \sum_{m=0}^S \sum_{n=0}^G S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{L^2}{2\pi^2} \cos \frac{\pi x}{2L} - \frac{L^2}{2\pi^2} \cos \frac{3\pi x}{2L} \right) \left( \sum_{n=0}^B B_n \cos \lambda_n x \right) \, dx
\]

\[
\{K^4_{CB}\}_{1,N+1} = \int_0^L \sum_{m=0}^S \sum_{n=0}^G S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{L^2}{2\pi^2} \sin \frac{\pi x}{2L} + \frac{L^2}{2\pi^2} \sin \frac{3\pi x}{2L} \right) \left( \sum_{n=0}^B B_n \cos \lambda_n x \right) \, dx
\]

\[
K_{CB} = \begin{bmatrix}
\{K^1_{CB}\}_{1,N+1} \\
\{K^2_{CB}\}_{1,N+1} \\
\{K^3_{CB}\}_{1,N+1} \\
\{K^4_{CB}\}_{1,N+1}
\end{bmatrix}
\]

\[
\{K^{11}_{CD}\}_{1,1} = \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{9}{8} \sin \frac{\pi x}{2L} - \frac{1}{8} \sin \frac{3\pi x}{2L} \right) \left( \frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right) \, dx
\]

\[
\{K^{12}_{CD}\}_{1,1} = \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{9}{8} \sin \frac{\pi x}{2L} + \frac{1}{8} \sin \frac{3\pi x}{2L} \right) \left( - \frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right) \, dx
\]

\[
\{K^{11}_{CD}\}_{1,1} = \{K^{12}_{CD}\}_{1,1}
\]

\[
\{K^{22}_{CD}\}_{1,1} = \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{9}{8} \sin \frac{\pi x}{2L} + \frac{1}{8} \sin \frac{3\pi x}{2L} \right) \left( - \frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right) \, dx
\]

\[
\{K_B\}_{N+1,N+1} = \int_0^L k \sum_{m=0}^S \sum_{n=0}^G S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \sum_{n=0}^B B_n \cos \lambda_n x \right)^2 \, dx
\]

\[
+ \int_0^L \sum_{m=0}^H H_m \cos \lambda_m x \sum_{m=0}^G L_m \cos \lambda_m x \left( - \lambda_m \sum_{n=0}^N B_n \sin \lambda_n x \right)^2 \, dx
\]

\[
+ K_0 \int_0^L \left( \sum_{m=0}^N B_m \cos \lambda_m x \right)^2 \, dx \bigg|_{x=0} + K_L \int_0^L \left( \sum_{m=0}^N B_m \cos \lambda_m x \right)^2 \, dx \bigg|_{x=L}
\]

\[
\{K^{11}_{BD}\}_{N+1,1} = \int_0^L k \sum_{m=0}^S \sum_{n=0}^G S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \sum_{n=0}^B B_n \cos \lambda_n x \right) \left( \frac{9}{8} \sin \frac{\pi x}{2L} - \frac{1}{8} \sin \frac{3\pi x}{2L} \right) \, dx
\]

\[
+ \int_0^L k \sum_{m=0}^S \sum_{n=0}^G S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \sum_{n=0}^B B_n \cos \lambda_n x \right) \left( \frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right) \, dx
\]

\[
+ \int_0^L k \sum_{m=0}^S \sum_{n=0}^G S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( - \lambda_m \sum_{n=0}^N B_n \sin \lambda_n x \right) \left( \frac{9}{8} \sin \frac{\pi x}{2L} - \frac{1}{8} \sin \frac{3\pi x}{2L} \right) \, dx
\]

\[
+ K_0 \int_0^L \left( \sum_{m=0}^N B_m \cos \lambda_m x \right) \left( \frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right) \, dx \bigg|_{x=0}
\]

\[
+ K_L \int_0^L \left( \sum_{m=0}^N B_m \cos \lambda_m x \right) \left( \frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right) \, dx \bigg|_{x=L}
\]
\[
\{K_{BD}\}^{N+1,1} = \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left[ \sum_{m=0}^B B_m \cos (\lambda_m x) \right] \left( \frac{9}{8} \sin \frac{\pi x}{2L} + \frac{1}{8} \sin \frac{3\pi x}{2L} \right) dx
\]
\[
+ \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left[ \sum_{m=0}^B B_m \cos (\lambda_m x) \right] \left( -\frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) dx
\]
\[
+ \sum_{m=0}^H H_m \cos \lambda_m x \sum_{m=0}^I I_m \cos \lambda_m x \left[ -\lambda_m \sum_{m=0}^B B_m \sin (\lambda_m x) \right] \left( \frac{9}{8} \sin \frac{\pi x}{2L} + \frac{1}{8} \sin \frac{3\pi x}{2L} \right) dx
\]
\[
+ K_0 \int_0^L \left( \sum_{m=0}^N B_m \cos (\lambda_m x) \right) \left( -\frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) dx|_{x=0} + K_L \int_0^L \left( \sum_{m=0}^N B_m \cos (\lambda_m x) \right) \left( -\frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) dx|_{x=L}
\]

\[
K_{BD} = \left[ \{K_1^{BD}\}^{N+1,1} - \{K_2^{BD}\}^{N+1,1} \right]
\]

\[
\{K_{11}^{DD}\}^{1,1} = \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right)^2 dx
\]
\[
+ \int_0^L \sum_{m=0}^H H_m \cos \lambda_m x \sum_{m=0}^I I_m \cos \lambda_m x \left( \frac{9}{8} \sin \frac{\pi x}{2L} - \frac{1}{8} \sin \frac{3\pi x}{2L} \right)^2 dx
\]
\[
+ K_0 \int_0^L \left( \sum_{m=0}^N B_m \cos (\lambda_m x) \right) \left( -\frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right)^2 dx|_{x=0} + K_L \int_0^L \left( \sum_{m=0}^N B_m \cos (\lambda_m x) \right) \left( -\frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right)^2 dx|_{x=L}
\]

\[
\{K_{12}^{DD}\}^{1,1} = \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left( \frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right) \left( -\frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) dx
\]
\[
+ \sum_{m=0}^H H_m \cos \lambda_m x \sum_{m=0}^I I_m \cos \lambda_m x \left( \frac{9}{8} \cos \frac{\pi x}{2L} - \frac{1}{8} \cos \frac{3\pi x}{2L} \right) \left( \frac{9}{8} \cos \frac{\pi x}{2L} + \frac{1}{8} \cos \frac{3\pi x}{2L} \right) dx
\]
\[
+ K_0 \int_0^L \left( \sum_{m=0}^N B_m \cos (\lambda_m x) \right) \left( -\frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right) \left( -\frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) dx|_{x=0} + K_L \int_0^L \left( \sum_{m=0}^N B_m \cos (\lambda_m x) \right) \left( -\frac{9L}{4\pi} \sin \frac{\pi x}{2L} - \frac{L}{12\pi} \sin \frac{3\pi x}{2L} \right) \left( -\frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right) dx|_{x=L}
\]

\[
\{K_{21}^{DD}\}^{1,1} = \{K_{12}^{DD}\}^{1,1}
\]

\[
\{K_{22}^{DD}\}^{1,1} = \int_0^L \sum_{m=0}^S S_m \cos \lambda_m x \sum_{m=0}^G G_m \cos \lambda_m x \left(-\frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right)^2 dx
\]
\[
+ \int_0^L \sum_{m=0}^H H_m \cos \lambda_m x \sum_{m=0}^I I_m \cos \lambda_m x \left( \frac{9}{8} \sin \frac{\pi x}{2L} + \frac{1}{8} \sin \frac{3\pi x}{2L} \right)^2 dx
\]
\[
+ K_0 \int_0^L \left(-\frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right)^2 dx|_{x=0} + K_L \int_0^L \left(-\frac{9L}{4\pi} \cos \frac{\pi x}{2L} - \frac{L}{12\pi} \cos \frac{3\pi x}{2L} \right)^2 dx|_{x=L}
\]

\[
A = \begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_N
\end{bmatrix},
B = \begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_N
\end{bmatrix},
C = \begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{bmatrix},
D = \begin{bmatrix}
D_1 \\
D_2
\end{bmatrix}
\]