Report on the Detailed Calculation of the Effective Potential in Spacetimes with $S^1 \times R^d$ Topology and at Finite Temperature

V.K.Oikonomou*
Dept. of Theoretical Physics Aristotle University of Thessaloniki, Thessaloniki 541 24 Greece
August 27, 2009

Abstract

In this paper we review the calculations that are needed to obtain the bosonic and fermionic effective potential at finite temperature and volume (at one loop). The calculations at finite volume correspond to $S^1 \times R^d$ topology. These calculations appear in the calculation of the Casimir energy and of the effective potential of extra dimensional theories. In the case of finite volume corrections we impose twisted boundary conditions and obtain semi-analytic results. We mainly focus in the details and validity of the results. The zeta function regularization method is used to regularize the infinite summations. Also the dimensional regularization method is used in order to renormalize the UV singularities of the integrations over momentum space. The approximations and expansions are carried out within the perturbative limits. After the end of each section we briefly present applications associated to the calculations. Particularly the calculation of the effective potential at finite temperature for the Standard Model fields, the effective potential for warped and large extra dimensions and the topological mass creation. In the end we discuss on the convergence and validity of one of the obtained semi-analytic results.

Keywords: Effective potential, zeta regularization, Casimir energy, finite temperature, extra dimensions

1 Introduction

During the development of Quantum Field Theory, many quantitative methods have been developed. Some of the most frequently used techniques are one-dimensional infinite lattice sums [3, 34]. In this article we shall review the calculations associated with these

*voiko@physics.auth.gr
summations, that appear in many important branches of Quantum Field Theory, three of which are, the physics of extra dimensions \[81, 65, 67, 68, 66, 55, 88\], the Casimir effect. \[4, 57, 31, 76, 82, 75, 61, 91, 89\] and finally in field theories at finite temperature \[60, 58, 73, 69, 3, 4, 6, 34, 54\]. In both three cases we shall compute the effective potential. The method we shall use involves the expansion of the potential in Bessel series and zeta regularization \[3, 4, 34, 11\]. We focus on the details of the calculation and we thing the paper will be a useful tool for the ones that want to study these theories.

1.1 Effective Potential in Theories with Large Extra Dimensions

In theories with large extra dimensions \[81, 65, 67, 68, 66, 55, 88\], the fields entering the Lagrangian are expanded in the eigenfunctions of the extra dimensions. Let us focus on theories with one extra dimension with the topology of a circle, namely of the type \(S^1 \times M_4\) (\(M_4\) stands for the 4-dimensional Minkowski space). In the following we shall also discuss the orbifold compactification apart from the circle compactification we describe here. For circle compactifications, the harmonic expansion of the fields reads,

\[
\phi(x, y) = \sum_{n=-\infty}^{\infty} \phi_n(x)e^{\frac{2\pi ny}{L}},
\]

where \(x\) stands for the 4-dimensional Minkowski space coordinates, \(y\) for the extra dimension and \(L\) the radius of the extra dimension. We note that fields are periodic in the extra dimension \(y\) namely, \(\phi(x, y) = \phi(x, y + 2\pi R)\). One of the ways to break supersymmetry is the Scherk-Schwarz compactification mechanism. This is based on the introduction of a phase \(q\). For fermions we denote it \(q_F\) and for bosons \(q_B\). Now the harmonic expansions for fermion and bosons fields read,

\[
\phi(x, y) = \sum_{n=-\infty}^{\infty} \phi_n(x)e^{\frac{2\pi (n+q_F)y}{L}},
\]

for fermions and,

\[
\phi(x, y) = \sum_{n=-\infty}^{\infty} \phi_n(x)e^{\frac{2\pi (n+q_B)y}{L}},
\]

for bosons. We can observe that the initial periodicity condition is changed. Using equations (1) and (2) we can find that the effective potential at one loop is equal to,

\[
V(\phi) = \frac{1}{2} \text{Tr} \sum_{n=-\infty}^{\infty} \int \frac{d^4p}{(2\pi)^4} \ln \left[ \frac{p^2 + \frac{(n+q_F)^2}{L^2} + M^2(\phi)}{p^2 + \frac{(n+q_B)^2}{L^2} + M^2(\phi)} \right]
\]

Note that fermions and bosons contribute to the effective potential with opposite signs. This is due to the fact that fermions are described by anti-commuting Grassmann fields. Also \(M^2(\phi)\) is a \(n\) independent term and depends on the way that spontaneous symmetry breaking occurs. We shall not care for the particular form of this and we focus on the general calculation of terms like the one in equation (4).
1.2 The Casimir Energy

One of the most interesting phenomena in Quantum Field Theory is the Casimir effect (for a review see [3, 4, 10, 22, 34, 30]). It expresses the quantum fluctuations of the vacuum of a quantum field. It originates from the “confinement” of a field in finite volume. Many studies have been done since H. Casimir’s original work [2]. The Casimir energy, usually calculated in these studies, is closely related to the boundary conditions of the fields under consideration [26, 29, 13, 14, 3, 4, 40, 41]. Boundary conditions influence the nature of the so-called Casimir force, which is generated from the vacuum energy.

In this paper we shall concentrate on the computation of the effective potential (Casimir Energy) of bosonic and fermionic fields in a space time with the topology $S^1 \times R^d$ [3, 4, 10, 21, 25, 27, 28, 34]. Fermionic and bosonic fields in spaces with non trivial topology are allowed to be either periodic or anti-periodic in the compact dimension. The forms of the potential to be studied are,

$$\frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \sum_{n=-\infty}^{\infty} \ln \left[ \frac{4\pi^2 n^2}{L^2} + k^2 + m^2 \right],$$

and the fermionic one,

$$\frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \sum_{n=-\infty}^{\infty} \ln \left[ \frac{(2n+1)^2 \pi^2}{L^2} + k^2 + m^2 \right].$$

We shall study them also in the cases $d = 2$ and $d = 3$, which are of particular importance in physics since they correspond to three and four total dimensions. Both have many applications in solid state physics and cosmology [10, 3]. Also we shall generalize to the case with fermions and bosons obeying general boundary conditions also in $d + 1$ dimensions. This is identical from a calculational aspect with the effective potential of theories with extra dimensions [55, 67]. So computing one of the two gives simultaneously the other. The expression that is going to be studied thoroughly is,

$$\frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \ln[(n + \omega) \frac{2\pi}{L} + k^2 + m^2] =$$

$$\int \frac{dk^{d+1}}{(2\pi)^{d+1}} \ln[k^2 + a^2]$$

$$+ \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \ln[1 - e^{-2(\frac{\omega}{L} + i\pi\omega})] + \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \ln[1 - e^{-2(\frac{\omega}{L} + i\pi\omega)}].$$

The calculations shall be done in $d + 1$ dimensions, quite general, and the application to every dimension we wish, can be done easily. The only constraint shall be if $d$ is even or odd. We shall make that clear in the corresponding sections and treat both cases in detail.

1.3 Field Theories at Finite Temperature

The calculations used in finite temperature field theories are based on the imaginary time formalism [58, 60, 3, 34, 4]:

$$t \rightarrow i\beta,$$
with $\beta = \frac{1}{T}$. The eigenfrequencies of the fields that appear to the propagators are discrete and are summed in the partition function. These are affected from the boundary conditions used for fermions and bosons \[3, 4\]. Bosons obey only periodic and fermions antiperiodic boundary conditions at finite temperature, as we shall see (this is restricted and dictated by the KMS relations \[60\]). Indeed for bosons the boundary conditions are:

$$\varphi(x, 0) = \varphi(x, \beta),$$  \hspace{1cm} (9)

where $x$ stands for space coordinates, and the fermionic boundary conditions are,

$$\psi(x, 0) = -\psi(x, \beta),$$  \hspace{1cm} (10)

In most calculations involving bosons, we are confronted with the following expression:

$$T \int \frac{dk^3}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \ln[4\pi^2 n^2 T^2 + k^2 + m^2],$$ \hspace{1cm} (11)

while the fermionic contribution is,

$$T \int \frac{dk^3}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \ln[(2n + 1)^2 \pi^2 T^2 + k^2 + m^2],$$ \hspace{1cm} (12)

and $k$ stands for the Euclidean momentum:

$$k^2 = k_1^2 + k_2^2 + k_3^2,$$  \hspace{1cm} (13)

while $m$ is the field mass. In the next sections we deal with the two above contributions in $d + 1$ dimensions and we specify the results for $d = 3$ and $d = 2$.

## 2 Bosonic Contribution at Finite Temperature

We will compute the following expression,

$$S_1 = T \int \frac{dk^3}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \ln[4\pi^2 n^2 T^2 + k^2 + m^2]$$  \hspace{1cm} (14)

In the following we generalize in $d$ dimensions. This will give us the opportunity to deal other cases apart from the $d = 4$. Consider the sum:

$$S_o = \sum_{n=-\infty}^{\infty} \frac{1}{4\pi^2 n^2 T^2 + a^2} = \frac{1}{4\pi^2 T^2} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + \frac{a^2}{4\pi^2 T^2}},$$ \hspace{1cm} (15)

where,

$$a^2 = k^2 + m^2$$ \hspace{1cm} (16)
Integrating over $a^2$,
\[ \sum_{n=-\infty}^{\infty} \frac{1}{4\pi^2 n^2 T^2 + a^2}, \quad (17) \]
we get:
\[ \int \sum_{n=-\infty}^{\infty} \frac{da^2}{4\pi^2 n^2 T^2 + a^2} = \sum \ln[4\pi^2 n^2 T^2 + a^2]. \quad (18) \]

Now,
\[ \sum_{n=-\infty}^{\infty} \frac{1}{4\pi^2 n^2 T^2 + a^2} = \frac{2}{4aT} \coth \left( \frac{a}{2T} \right), \quad (19) \]
thus equation (18) becomes,
\[ \int \sum_{n=-\infty}^{\infty} \frac{da^2}{4\pi^2 n^2 T^2 + a^2} = \int \frac{2}{4aT} \coth \left( \frac{a}{2T} \right) da^2 \]
\[ = 2 \ln(\sinh \left( \frac{a}{2T} \right)). \quad (20) \]

Using the relation [1],
\[ \ln(\sinh x) = \ln \left( \frac{1}{2} [e^x - e^{-x}] \right) = x + \ln[1 - e^{-2x}] - \ln[2], \quad (21) \]
and upon summation,
\[ \ln(\sinh \frac{a}{2T}) = \frac{a}{2T} + \ln[1 - e^{-\frac{a}{T}}] - \ln[2], \quad (22) \]
and,
\[ \ln(\sinh \frac{a}{2T}) = \frac{a}{2T} + \ln[1 - e^{-\frac{a}{T}}] - \ln[2]. \quad (23) \]

Summing equations (22) and (23) we obtain,
\[ \int \sum_{n=-\infty}^{\infty} \frac{da^2}{4\pi^2 n^2 T^2 + a^2} = 2 \ln(\sinh \frac{a}{2T}) = \frac{a}{T} + 2 \ln[1 - e^{-\frac{a}{T}}] - 2 \ln[2]. \quad (24) \]

Finally the result is [58] [60] [3] [34]:
\[ \sum_{n=-\infty}^{\infty} \ln[4\pi^2 n^2 T^2 + a^2] = \frac{a}{T} + 2 \ln[1 - e^{-\frac{a}{T}}] - 2 \ln[2]. \quad (25) \]

Upon using,
\[ \sum \ln \left( \frac{(n + \omega)^2 4\pi^2 T^2 + a^2}{(n + \omega)^2 4\pi^2 T^2 + b^2} \right) = 2(a - b), \quad (26) \]
equation (25) becomes,
\[ \sum \ln[4\pi^2 n^2 T^2 + a^2] = \frac{1}{2\pi T} \int_{-\infty}^{\infty} dx \ln[x^2 + a^2] + 2 \ln[1 - e^{-\frac{a}{T}}]. \quad (27) \]
Finally we have,
\[
T \int \frac{dk^3}{(2\pi)^3} \sum \ln[(2\pi n T)^2 + k^2 + m^2] = \int \frac{dk^3}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dx}{2\pi} \ln[x^2 + a^2] \quad (28)
\]
\[
+ 2T \int \frac{dk^3}{(2\pi)^3} \ln[1 - e^{-a T}] .
\]

Remembering that,
\[
a^2 = k^2 + m^2 , \quad (29)
\]
the first integral of equation (28) is the one loop contribution to the effective potential at zero temperature. The 4-momentum is:
\[
K^2 = k^2 + x^2 . \quad (30)
\]
Writing the above in \(d + 1\) dimensions (in the end we take \(d = 3\) to come back to four dimensions) we get,
\[
T \int \frac{dk^d}{(2\pi)^d} \sum \ln[4\pi^2 n^2 T^2 + k^2 + m^2] = \int \frac{dk^{d+1}}{(2\pi)^{d+1}} \ln[k^2 + a^2] \quad (31)
\]
\[
+ 2T \int \frac{dk^d}{(2\pi)^d} \ln[1 - e^{-a T}] .
\]

The temperature dependent part has singularities stemming from the infinite summations. These singularities are poles of the form \(\epsilon, 3, 34, 4\):
\[
\frac{1}{\epsilon} , \quad (32)
\]
where \(\epsilon \to 0\) the dimensional regularization variable \((d = 4 + \epsilon)\). As we shall see, by using the zeta regularization \(3, 4, 82, 34, 11\) these will be erased. In the following of this section we focus on the calculation of the temperature dependent part. Let,
\[
V_{\text{boson}} = 2T \int \frac{dk^d}{(2\pi)^d} \ln[1 - e^{-a T}] = 2T \int \frac{dk^d}{(2\pi)^d} \ln[1 - \frac{1}{e} - T]. \quad (33)
\]
By using \(\Pi\),
\[
\ln[1 - \frac{1}{e}] = - \sum_{q=1}^{\infty} \frac{e^{-\frac{a T q}{q}}}{q} , \quad (34)
\]
we obtain,
\[
V_{\text{boson}} = 2T \int \frac{dk^d}{(2\pi)^d} \ln[1 - e^{-\frac{a T}{q}}] = -2T \int \frac{dk^d}{(2\pi)^d} \sum_{q=1}^{\infty} \frac{e^{-\frac{a T}{q}}}{q} . \quad (35)
\]
and remembering,

\[ a = \sqrt{k^2 + m^2}, \]  

(36)

by integrating over the angles we get,

\[ V_{\text{boson}} = -2 \sum_{q=1}^{\infty} T \int \frac{dk^d}{(2\pi)^d} \int \frac{d^d q}{q} \int e^{-\frac{\sqrt{k^2 + m^2}}{T}} q \]

(37)

\[ = -2 \sum_{q=1}^{\infty} T \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^d} k^{d-1} (2\pi)^{\frac{d}{2}} e^{-\frac{\sqrt{k^2 + m^2}}{T} q} \]

\[ = -2 \sum_{q=1}^{\infty} T \int_{-\infty}^{\infty} \frac{(2\pi)^{\frac{d}{2}}}{\Gamma(d) q^{d} \pi^{d-1}} \int e^{-\sqrt{k^2 + m^2}}. \]

The integral,

\[ \int_{-\infty}^{\infty} dk q^{d-1} e^{-\frac{\sqrt{k^2 + m^2}}{T}} = 2^{d-1} \pi^{\frac{d}{2}} m^{d+1} \frac{\Gamma(d)}{\Gamma(d+1)} (\frac{m}{T}) \frac{d+1}{2}. \]

(38)

So \( V_{\text{boson}} \) can be written:

\[ V_{\text{boson}} = -2 \sum_{q=1}^{\infty} \frac{(2\pi)^{d-1}}{d^2 m^{d+1}} K_{\frac{d+1}{2}} \left( \frac{mq}{T^2} \right) \]

(40)

\[ = -2 \sum_{q=1}^{\infty} \frac{(2\pi)^{d-1}}{d^2 m^{d+1}} K_{\frac{d+1}{2}} \left( \frac{mq}{T^2} \right) \]

The function \( K_{\nu}(z) \)

\[ \frac{K_{\nu}(z)}{(i)^\nu} = \frac{1}{2} \int_{0}^{\infty} e^{-t - \frac{z^2}{4t}} t^{v+1} dt, \]

(41)

is even under the transformation \( z \rightarrow -z \). Thus equation (40) becomes:

\[ V_{\text{boson}} = -\sum_{q=1}^{\infty} \frac{1}{(2\pi)^d} m^{d+1} K_{\frac{d+1}{2}} \left( \frac{mq}{T^2} \right) \]

(42)

\[ = -\sum_{q=-\infty}^{\infty} \frac{1}{(2\pi)^d} m^{d+1} K_{\frac{d+1}{2}} \left( \frac{mq}{T^2} \right) \]

(The symbol ‘ in the summation denotes omission of the zero mode term \( q = 0 \)). By using,

\[ \frac{K_{\nu}(z)}{(i)^\nu} = \frac{1}{2} \int_{0}^{\infty} e^{-t - \frac{z^2}{4t}} t^{v+1} dt, \]

(43)
we get,

\[ V_{\text{boson}} = -\frac{1}{4} \frac{1}{(2\pi)^d} (2\pi)^{\frac{d-1}{2}} m^{d+1} \int_0^\infty dt e^{-t} \sum_{q=-\infty}^{\infty} e^{-\frac{m a^2}{4 t}}. \]  

(44)

Let \( \lambda = \frac{(\nu)^2}{a} \). Using the Poisson summation formula \([11, 34, 3, 4]\) we have,

\[ \sum_{q=-\infty}^{\infty} e^{-\lambda q^2} = \sqrt{\frac{\pi}{\lambda}} \sum_{k=-\infty}^{\infty} e^{-\frac{4\lambda k^2}{4\pi}} , \]  

(45)

and omitting the zero modes we obtain:

\[ 1 + \sum_{q=-\infty}^{\infty} e^{-\lambda q^2} = \sqrt{\frac{\pi}{\lambda}} \left( 1 + \sum_{k=-\infty}^{\infty} e^{-\frac{4\lambda k^2}{4\pi}} \right). \]  

(46)

Finally,

\[ \sum_{q=-\infty}^{\infty} e^{-\lambda q^2} = \sqrt{\frac{\pi}{\lambda}} \left( 1 + \sum_{k=-\infty}^{\infty} e^{-\frac{4\lambda k^2}{4\pi}} \right) - 1, \]  

(47)

and replacing in \( V_{\text{boson}} \) we take,

\[ V_{\text{boson}} = \]

\[ -\frac{1}{4} \frac{1}{(2\pi)^d} (2\pi)^{\frac{d-1}{2}} m^{d+1} \int_0^\infty dt e^{-t} \left( \sqrt{\frac{\pi}{\lambda}} \left( 1 + \sum_{k=-\infty}^{\infty} e^{-\frac{4\lambda k^2}{4\pi}} \right) - 1 \right). \]  

Set,

\[ \nu = \frac{d+1}{2}. \]  

(49)

and equation (48) reads,

\[ V_{\text{boson}} = \]

\[ -\frac{1}{4} \frac{1}{(2\pi)^d} (2\pi)^{\frac{d-1}{2}} m^{d+1} \int_0^\infty dt e^{-t} \left( \sqrt{\frac{\pi}{\lambda}} \left( 1 + \sum_{k=-\infty}^{\infty} e^{-\frac{4\lambda k^2}{4\pi}} \right) - 1 \right). \]  

(50)

Also by setting,

\[ a = \frac{m}{T}, \]  

(51)
equation (50) becomes (with $\lambda = \frac{d}{2}t$),

$$V_{\text{boson}} = -\frac{1}{4} (2\pi)^{d} a^{d-\frac{1}{2}} m^{d+1} \int_{0}^{\infty} dt e^{-t \frac{\sqrt{\pi} t^2}{a t^{\nu+1}}}$$

$$- \frac{1}{4} (2\pi)^{d} a^{d-\frac{1}{2}} m^{d+1} \int_{0}^{\infty} dt e^{-t \frac{\sqrt{\pi} t^2}{a t^{\nu+1}} (\sum_{k=\infty}^{\infty} e^{-\frac{4\pi^2 k^2}{a t^2}})}$$

$$+ \frac{1}{4} (2\pi)^{d} a^{d-\frac{1}{2}} m^{d+1} \int_{0}^{\infty} dt e^{-t \left( \frac{1}{\nu+1} \right)}.$$  

(52)

From this, after some calculations we obtain:

$$V_{\text{boson}} = -\frac{1}{2} \sqrt{\pi} (2\pi)^{d} a^{d+1} \int_{0}^{\infty} dt e^{t^{-\nu} - \frac{1}{2}}$$

$$- \frac{1}{2} \sqrt{\pi} (2\pi)^{d} a^{d+1} \int_{0}^{\infty} dt e^{t^{-\nu} - \frac{1}{2}} (\sum_{k=\infty}^{\infty} e^{-\frac{4\pi^2 k^2}{a t^2}})$$

$$+ \frac{1}{4} (2\pi)^{d} a^{d+1} \int_{0}^{\infty} dt e^{t^{-\nu} - \frac{1}{2}} \left( \frac{1}{\nu+1} \right).$$

(53)

By using [1],

$$\frac{1}{(x^2 + a^2)\mu + 1} = \frac{1}{\Gamma(\mu + 1)} \int_{0}^{\infty} dt e^{-(x^2+a^2)t} t^{\mu},$$

we finally have:

$$V_{\text{boson}} = -\frac{1}{2} \sqrt{\pi} (2\pi)^{d} a^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)$$

$$- \frac{1}{2} \sqrt{\pi} (2\pi)^{d} a^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)$$

$$\times \left[ \sum_{k=-\infty}^{\infty} \left( \frac{2\pi k}{a} \right)^{2\nu+\frac{d}{2} - 1} \right]$$

$$+ \frac{1}{4} (2\pi)^{d} a^{d+1} \Gamma(-\nu).$$

(54)

(55)

The sum,

$$\sum_{k=-\infty}^{\infty} \left( \frac{2\pi k}{a} \right)^{2\nu+\frac{d}{2} - 1},$$

is invariant under the transformation $k \rightarrow -k$. Thus we change the summation to,

$$2 \sum_{k=1}^{\infty} \left( \frac{2\pi k}{a} \right)^{2\nu+\frac{d}{2} - 1}.$$

(56)

(57)

9
Replacing the above to $V_{\text{boson}}$ after some calculations we get:

$$V_{\text{boson}} = -\frac{1}{2} \frac{\sqrt{\pi}}{(2\pi)^d a} (2\pi)^{\frac{d-1}{2}} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)$$

$$(2\pi)^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)(a^2)^{\frac{1}{2} - \nu}$$

$$\times \left[ \sum_{k=1}^{\infty} (a^2 + 4\pi^2 k^2)^{\nu + \frac{1}{2} - 1} \right]$$

$$+ \frac{1}{4} \frac{1}{(2\pi)^d a} (2\pi)^{\frac{d+1}{2}} m^{d+1} \Gamma(-\nu).$$

We use the binomial expansion (in the case that $d$ is even) or the Taylor expansion (in the case $d$ odd) [1]:

$$(a^2 + b^2)^{\nu - \frac{1}{2}} = \sum_{l=0}^{\sigma} \frac{(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!!} (a^2)^l (b^2)^{\nu - \frac{1}{2} - l}. \quad (59)$$

If $d$ is even, then $\sigma$ equals to,

$$\sigma = \nu - \frac{1}{2}. \quad (60)$$

If $d$ is odd then $\sigma \in \mathbb{N}^*$. We shall deal both cases. Replacing the sum into $V_{\text{boson}},$

$$V_{\text{boson}} = -\frac{1}{2} \frac{\sqrt{\pi}}{(2\pi)^d a} (2\pi)^{\frac{d-1}{2}} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)$$

$$+ \frac{1}{4} \frac{1}{(2\pi)^d a} (2\pi)^{\frac{d+1}{2}} m^{d+1} \Gamma(-\nu)$$

$$- \frac{\sqrt{\pi}}{(2\pi)^d a} (2\pi)^{\frac{d-1}{2}} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)(a^2)^{\frac{1}{2} - \nu}$$

$$\times \left[ \sum_{k=1}^{\sigma} \sum_{l=0}^{\sigma} \frac{((2\pi)^2)^{\nu - \frac{1}{2} - l}(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!!} (a^2)^l (k^2)^{\nu - \frac{1}{2} - l}. \right].$$

The last expression shall be the initial point for the following two subsections.

A much more elegant computation involves the analytic continuation of the Epstein-zeta function [8, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. In a following section we shall present the Epstein zeta functions in much more detail. In our case, relation (58) can be written in a much more elegant way, using the one dimensional Epstein zeta function,

$$Z^m_1(\nu, w, \alpha) = \sum_{n=1}^{\infty} \left[ w(n + \alpha)^2 + m^2 \right]^{-\nu}, \quad (62)$$

In our case, $\alpha = 0$. Particularly one can make the relevant substitutions in the sum,

$$\sum_{k=1}^{\infty} (a^2 + 4\pi^2 k^2)^{\nu + \frac{1}{2} - 1}],$$

in terms of the one dimensional Epstein zeta function, (62).
2.0.1 The Chowla-Selberg Formula

It worths mentioning at this point a very important formula related with the Bessel sums \[3, 4, 34\] of relation,

\[
V_{\text{boson}} = -\sum_{q=1}^{\infty} \frac{1}{(2\pi)^d} \frac{K_{d+1}(\frac{mq}{T})}{(\frac{mq}{2\pi})^{d+1}}
\]

Apart from the inhomogeneous Epstein zeta \[3, 11, 4, 34, 57, 56, 76, 72, 78, 77\], there exist in the literature a generalization of the inhomogeneous Epstein zeta function, namely the extended Chowla-Selberg formula \[3\], which we briefly describe at this point. We start with a two dimensional generalization of the Epstein zeta function,

\[
E(s; a, b, c; q) = \sum_{n, m \lor Z} = (am^2 + bmn + cn^2 + q)^{-q}.
\]

In the following \(Q\) is equal to,

\[
Q(m, n) = am^2 + bmn + cn^2,
\]

and also \(\Delta\) is,

\[
\Delta = 4ac - b^2.
\]

Following \[3\], relation (65), can be written as,

\[
E(s; a, b, c; q) = 2\zeta_{EH}(s, q/a)a^{-s} + \frac{2^s \sqrt{\pi} a^{s-1}}{\Gamma(s)\sqrt{a}} \sum_{n=1}^{\infty} \Gamma(s - 1/2)\zeta_{EH}(s - 1/2, 4aq/\Delta) + 2^s \sqrt{\pi} a^{s-1} \sum_{n=1}^{\infty} n^{s-1/2} \cos(n\pi b/a) \sum_{d/n} d^{1-2s} \left(\Delta + \frac{4aq}{d^2}\right)^{1/4-s/2} K_{s-1/2}\left(\frac{\pi n}{a} \sqrt{\Delta + \frac{4aq}{d^2}}\right).
\]

In the above relation, the summation \(\sum_{d/n}\) is over the \(1 - 2s\) powers of the divisors of \(n\). Also \(\zeta_{EH}\) stands for,

\[
\zeta_{EH}(s; p) = -\frac{p^{-s}}{2} + \sqrt{\pi \Gamma(s - 1/2)} p^{-s+1/2} + \frac{2\pi^s p^{-s/2+1/4}}{\Gamma(s)} \sum_{n=1}^{\infty} n^{-s-1/2} K_{s-1/2}(2\pi n \sqrt{p}).
\]

Relation (68) has very attractive features. Most importantly the exponential convergence. We just mention this here for completeness and because (68) is very important. For more details see the detailed description of \[3\]. Our case is a special case of the extended Chowla-Selberg formula.
2.0.2  The Case $d$ odd

As stated before in the $d$ odd case, $\sigma \epsilon N^*$. Then $V_{\text{boson}}$ is:

$$V_{\text{boson}} = -\frac{1}{2} \sqrt{\pi} (2\pi)^{d+1} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)$$ (70)

$$+ \frac{1}{4} \frac{1}{(2\pi)^{d+1} a} (2\pi)^{d} m^{d+1} \Gamma(-\nu)$$

$$- \frac{\sqrt{\pi}}{(2\pi)^{d} a} (2\pi)^{d} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1) (a^2)^{\frac{1}{2} - \nu}$$

$$\times \sum_{k=1}^{\infty} \sum_{l=0}^{\sigma} ((2\pi)^2)^{\nu - \frac{1}{2} - l} (\nu - \frac{1}{2})^! (a^2)^l (k^2)^{n - \frac{1}{2} - l} \zeta(n).$$

Using the analytic continuation of the Riemann zeta function [3, 4, 34, 59, 11],

$$\zeta(n) = \sum_{n=1}^{\infty} n^{-s},$$ (71)

to negative integers, $V_{\text{boson}}$ becomes:

$$V_{\text{boson}} = -\frac{1}{2} \sqrt{\pi} (2\pi)^{d+1} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)$$ (72)

$$+ \frac{1}{4} \frac{1}{(2\pi)^{d+1} a} (2\pi)^{d} m^{d+1} \Gamma(-\nu)$$

$$- \frac{\sqrt{\pi}}{(2\pi)^{d} a} (2\pi)^{d} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1) (a^2)^{\frac{1}{2} - \nu}$$

$$\times \sum_{k=1}^{\infty} \sum_{l=0}^{\sigma} ((2\pi)^2)^{\nu - \frac{1}{2} - l} (\nu - \frac{1}{2})^! (a^2)^l (k^2)^{n - \frac{1}{2} - l} \zeta(n + 2 - 2l).$$

This is the final form of the bosonic contribution to the effective potential for $d$ odd. In the following we compute the above in the case $d = 3$. This will be done by Taylor expanding the last expression in powers of $\varepsilon$ (with $d = 3 + \varepsilon$) as $\varepsilon \to 0$.

Let us explicitly show how the poles are erased. In the case $d = 3$ two terms of $V_{\text{boson}}$ have poles. The first pole appears in $\Gamma(-\nu)$ (remember $\nu = \frac{d+1}{2}$) and the other is contained in $\zeta(-2\nu + 1 + 2l)$ for the value $l = 2$ that gives the pole of $\zeta(n)$ for $s = 1$. These terms expanded around $d = 3 + \varepsilon$, in the limit $\varepsilon \to 0$ are written:

$$\frac{1}{4} \frac{1}{(2\pi)^{d+1} a} (2\pi)^{d} m^{d+1} \Gamma(-\nu) = \frac{-m^4}{16 \pi^2 \varepsilon} + \frac{3}{64 \pi^2} - \frac{\gamma m^4}{32 \pi^2}$$

$$+ \frac{m^4 \ln(2)}{32 \pi^2} - \frac{m^4 \ln(m)}{16 \pi^2}$$

$$+ \frac{m^4 \ln(\pi)}{32 \pi^2} + O(\varepsilon)$$

12
(where \( \gamma \) the Euler-Mascheroni constant) in which a pole appears,

\[
\frac{-m^4}{16 \pi^2 \varepsilon}.
\]  

(74)

Regarding the other pole containing term (for \( d = 3 + \epsilon, \epsilon \to 0 \)),

\[
-\frac{\sqrt{\pi}}{(2\pi)^{d}\alpha} (2\pi)^{d+1} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)(a^2)^{\frac{d-\nu}{2}} \\
\times \left[ \frac{(2\pi^2)^{\nu - \frac{1}{2} - 2}(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - 2)!} \right] (a^2)^{2}\zeta(-2\nu + 1 + 4) = \\
\frac{m^4}{16 \pi^2 \varepsilon} - \frac{\gamma m^4}{16 \pi^2} + \frac{m^4 \ln(2)}{32 \pi^2} \\
+ \frac{m^4 \ln(m)}{16 \pi^2} + \frac{m^4 \ln(\pi)}{32 \pi^2} - \\
\frac{m^4 \ln(a^2)}{32 \pi^2} - \frac{m^4 \psi(\frac{1}{2})}{32 \pi^2} \\
+ \frac{m^4 \psi(\frac{5}{2})}{32 \pi^2} + O(\varepsilon),
\]  

(75)

with \( \psi \) the digamma function. Summing the above expressions we observe that the poles are naturally erased as a consequence of the zeta regularization method. We expand \( V_{\text{boson}} \) keeping the most dominant terms in the high temperature limit \[3, 34, 58, 60\]:

\[
V_{\text{boson}} = \frac{-m^4}{16 \pi^2 \varepsilon} + \left( \frac{3 m^4}{64 \pi^2} - \frac{\gamma m^4}{32 \pi^2} - \frac{m^4 \ln(2)}{16 \pi^2} + \frac{m^4 \ln(\pi)}{16 \pi^2} - \frac{m^4 \ln(\alpha^2)}{32 \pi^2} - \frac{m^4 \psi(\frac{1}{2})}{32 \pi^2} \right) + O(\varepsilon)
\]  

(76)

and substituting \( \alpha = \frac{m}{T} \) we get:

\[
V_{\text{boson}} = \frac{-m^4}{16 \pi^2 \varepsilon} + \left( \frac{3 m^4}{64 \pi^2} - \frac{\gamma m^4}{32 \pi^2} - \frac{m^4 T}{16 \pi^2} + \frac{\gamma m^4}{16 \pi^2} + \frac{m^4 \ln(\alpha^2)}{32 \pi^2} - \frac{m^4 \psi(\frac{1}{2})}{32 \pi^2} \right) + O(\varepsilon).
\]  

(77)
In equation (77) we kept terms of order $\sim T$. For $\sigma = 8$ we have additionally,

\[
\begin{align*}
- \left( m^7 \frac{m}{\pi^6} \zeta(5) \right) & + \frac{m^9}{\pi^8} \zeta(7) - \frac{7m^{11}}{\pi^{10}} \zeta(9) \\
& + \frac{3m^{13}}{\pi^{12}} \zeta(11) - \frac{33m^{15}}{\pi^{14}} \zeta(13) \\
& = \frac{-1}{4096 \pi^6 T^3} + \frac{32768 \pi^8 T^5}{1572864 \pi^{10} T^7} + \frac{4194304 \pi^{12} T^9}{268435456 \pi^{14} T^{11}}.
\end{align*}
\]  

(78)

2.0.3 The Case $d$ even

In the case $d$ even, $\sigma$ takes a limited number of values. Particularly all the integer values up to the number $\sigma > d$. Before proceeding we comment on the values that $d$ can take. If it takes values $d > 2$ that is $4, 6, \ldots$, the theory ceases to be renormalizable and UV regulators must be used in order to cure UV singularities [60, 58, 10]. We shall not deal with these problems that usually appear in extra dimensional models. Now $V_{boson}$ in the $d$ even case becomes:

\[
V_{boson} = -\frac{1}{2} (2\pi)^{d+1} \ln((2\pi)^{d+1} \Gamma(-\nu + \frac{1}{2} + 1))\]

(79)

\[
+ \frac{1}{4} \frac{1}{(2\pi)^d} (2\pi)^{d-1} m^{d+1} \Gamma(-\nu)
\]

\[
- \frac{\sqrt{\pi}}{(2\pi)^d} (2\pi)^{d-1} m^{d+1} \Gamma(-\nu + \frac{1}{2} + 1) (a^2)^{-\nu}
\]

\[
\times \left[ \sum_{k=1}^{\infty} \sum_{l=0}^{\nu-\frac{1}{2}} \frac{((2\pi)^2)^{-l} (\nu - \frac{1}{2})^l}{(\nu - \frac{1}{2} - l)(2\pi)^l (a^2)^l (k^2)^{-\nu}} \right],
\]

and using the zeta regularization [31, 44, 34, 11] we get:

\[
V_{boson} = -\frac{1}{2} (2\pi)^{d+1} \ln((2\pi)^{d+1} \Gamma(-\nu + \frac{1}{2} + 1))\]

(80)

\[
+ \frac{1}{4} \frac{1}{(2\pi)^d} (2\pi)^{d-1} m^{d+1} \Gamma(-\nu)
\]

\[
- \frac{\sqrt{\pi}}{(2\pi)^d} (2\pi)^{d-1} m^{d+1} \Gamma(-\nu + \frac{1}{2} + 1) (a^2)^{-\nu}
\]

\[
\times \left[ \sum_{l=0}^{\nu-\frac{1}{2}} \frac{((2\pi)^2)^{-l} (\nu - \frac{1}{2})^l}{(\nu - \frac{1}{2} - l)(2\pi)^l (a^2)^l (k^2)^{-\nu}} \right].
\]

We compute for example the above in the case $d = 2$. We can easily see that the poles are contained in the terms $\Gamma(-\nu + \frac{1}{2} + 1)$ and $\Gamma(-\nu + \frac{1}{2} + 1)$. Expanding for $\nu \rightarrow 0$ ($d = 2 + \varepsilon$) the first pole containing term is:

\[
-\frac{1}{2} \frac{\sqrt{\pi}}{(2\pi)^d} (2\pi)^{d-1} m^{d+1} \Gamma(-\nu + \frac{1}{2} + 1)
= -\frac{(m^2 T)}{2 \sqrt{2} \pi \varepsilon} + \frac{m^2 T}{4 \sqrt{2} \pi} - \frac{\gamma m^2 T}{4 \sqrt{2} \pi} + \frac{m^2 T \ln(2)}{4 \sqrt{2} \pi} - \frac{m^2 T \ln(m)}{2 \sqrt{2} \pi} + \frac{m^2 T \ln(\pi)}{4 \sqrt{2} \pi}.
\]  

(81)
and the other one reads:

\[
- \sqrt{\pi} \frac{(2\pi)^{d-1}}{(2\pi)^d a} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)(a^2)^{\frac{1}{2} - \nu}
\]

\[
\times \left[ \sum_{l=0}^{\nu - \frac{1}{2}} \frac{((2\pi)^2)^{\nu - \frac{1}{2} - l}(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!} (a^2)^{\frac{1}{2} - \nu}(2\nu + 1 + 2l) \right] =
\]

\[
\frac{m^2 T}{2 \sqrt{2} \pi e} + \frac{\gamma m^2 T}{4 \sqrt{2} \pi} + \frac{m^2 T \ln(2)}{4 \sqrt{2} \pi} + \frac{m^2 T \ln(m)}{2 \sqrt{2} \pi}
\]

\[
+ \frac{m^2 T \ln(\pi)}{2 \sqrt{2} \pi} - \frac{m^2 T \ln(2\pi)}{2 \sqrt{2} \pi} - \frac{m^2 T \ln\left(\frac{m^2}{\pi^2}\right)}{4 \sqrt{2} \pi} + 2 \sqrt{2} \pi T^3 \zeta'(-2).
\]

Adding equation (81) and (82) we can see that the poles are erased naturally and \(V_{\text{boson}}\) becomes \((d = 2)\):

\[
V_{\text{boson}} = \left( \frac{m^3}{6 \sqrt{2} \pi} + \frac{m^2 T}{4 \sqrt{2} \pi} + \frac{m^2 T \ln(2)}{2 \sqrt{2} \pi} \right) +
\]

\[
\frac{m^2 T \ln(\pi)}{2 \sqrt{2} \pi} - \frac{m^2 T \ln(2\pi)}{2 \sqrt{2} \pi} - \frac{m^2 T \ln\left(\frac{m^2}{\pi^2}\right)}{4 \sqrt{2} \pi} + 2 \sqrt{2} \pi T^3 \zeta'(-2).
\]

### 2.1 Fermionic Contribution at Finite Temperature

In this section we will compute the fermionic contribution to the effective potential:

\[
T \int \frac{dk^3}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \ln[(2n+1)^2 \pi^2 T^2 + k^2 + m^2].
\]

Following the same procedures as in the bosonic case we obtain [34, 34, 60, 58]:

\[
T \int \frac{dk^3}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \ln[(2n+1)^2 \pi^2 T^2 + k^2 + m^2] =
\]

\[
\int \frac{dk^3}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dx}{2\pi} \ln[x^2 + a^2] + 2T \int \frac{dk^3}{(2\pi)^3} \ln[1 + e^{-\frac{k^2}{T^2}}].
\]

As before, the first term to the left hand side is the effective potential at zero temperature. We shall dwell on the temperature dependent contribution, which in \(d + 1\) dimensions is written,

\[
T \int \frac{dk^d}{(2\pi)^d} \sum_{n=-\infty}^{\infty} \ln[4\pi^2 n^2 T^2 + k^2 + m^2] =
\]

\[
\int \frac{dk^{d+1}}{(2\pi)^{d+1}} \ln[k^2 + a^2] + 2T \int \frac{dk^d}{(2\pi)^d} \ln[1 + e^{-\frac{k^2}{T^2}}].
\]
Let,
\[ V_{\text{fermion}} = 2T \int \frac{dk^d}{(2\pi)^d} \ln[1 + e^{-\frac{a}{2T}}] = 2T \int \frac{dk^d}{(2\pi)^d} \ln[1 + e^{-\frac{a}{2T}}]. \] (87)

By using \[II\],
\[ \ln[1 + e^{-\frac{a}{2T}}] = -\sum_{q=1}^{\infty} \frac{(-1)^q e^{-\frac{a}{2T}q}}{q}, \] (88)

\[ V_{\text{fermion}} \] becomes,
\[ V_{\text{fermion}} = 2T \int \frac{dk^d}{(2\pi)^d} \ln[1 + e^{-\frac{a}{2T}}] \] (89)
\[ = -2T \int \frac{dk^d}{(2\pi)^d} \sum_{q=1}^{\infty} \frac{(-1)^q e^{-\frac{a}{2T}q}}{q} \]
\[ = -2 \sum_{q=1}^{\infty} T \int \frac{dk^d}{(2\pi)^d} \frac{(-1)^q e^{-\frac{a}{2T}q}}{q}. \]

Recall that,
\[ a = \sqrt{k^2 + m^2}, \] (90)

and so,
\[ V_{\text{fermion}} = -2 \sum_{q=1}^{\infty} T \int \frac{dk^d}{(2\pi)^d} \frac{(-1)^q e^{-\frac{\sqrt{k^2 + m^2}}{2T}}}{q} \]
\[ = -2 \sum_{q=1}^{\infty} T \int \frac{dk^d}{(2\pi)^d} \frac{(-1)^q e^{-\frac{\sqrt{k^2 + m^2}}{2T}}}{\Gamma\left(\frac{d}{2}\right) q} \] (91)
\[ = -2 \sum_{q=1}^{\infty} \frac{T (2\pi)^{\frac{d}{2}} (-1)^q}{\Gamma\left(\frac{d}{2}\right) q (2\pi)^d} \int_{-\infty}^{\infty} dk k^{d-1} e^{-\frac{\sqrt{k^2 + m^2}}{2T}}. \]

The integral,
\[ \int_{-\infty}^{\infty} dk k^{d-1} e^{-\frac{\sqrt{k^2 + m^2}}{2T}}, \] (92)

equals to \[III\],
\[ \int_{-\infty}^{\infty} dk k^{d-1} e^{-\frac{\sqrt{k^2 + m^2}}{2T}} = 2^{\frac{d}{2}-1} (\sqrt{\pi})^{-1} \frac{q}{2T} \frac{\Gamma\left(\frac{d}{2}\right)}{m^{\frac{d-1}{2}} \Gamma\left(\frac{d}{2}\right)} \frac{K_{d+1} \left(\frac{mq}{2T}\right)}{2T}. \] (93)

So \[ V_{\text{fermion}} \] reads,
\[ V_{\text{fermion}} = -2 \sum_{q=1}^{\infty} \frac{2^{\frac{d}{2}-1} (-1)^q}{(2\pi)^d} (2\pi)^{\frac{d+1}{2}} m^{d+1} \frac{K_{d+1} \left(\frac{mq}{2T}\right)}{1} \frac{2T}{mq} \frac{d+1}{2} \]
\[ = -\sum_{q=1}^{\infty} \frac{(-1)^q}{(2\pi)^d} (2\pi)^{\frac{d-1}{2}} m^{d+1} \frac{K_{d+1} \left(\frac{mq}{2T}\right)}{\left(\frac{mq}{2T}\right)^{\frac{d-1}{2}}} \] (94)
Using the relation \[1, 3, 4, 34]:
\[
\sum_{q=1}^{\infty} (-1)^q f(r) = 2 \sum_{q=1}^{\infty} f(2r) - \sum_{q=1}^{\infty} f(r),
\]
we get,
\[
\sum_{q=1}^{\infty} (-1)^q K_{d+1} \left( \frac{mq}{2T} \right) \frac{d+1}{2} = 2 \sum_{q=1}^{\infty} K_{d+1} \left( \frac{mq}{2T} \right) \frac{d+1}{2} - \sum_{q=1}^{\infty} K_{d+1} \left( \frac{mq}{2T} \right) \frac{d+1}{2}.
\]
and upon replacing to \( V_{\text{fermion}} \) we obtain:
\[
V_{\text{fermion}} = - \sum_{q=1}^{\infty} (-1)^q \left( \frac{2\pi}{2\pi} \right)^d m^{d+1} \frac{K_{d+1} \left( \frac{mq}{2T} \right)}{\frac{d+1}{2}}
\]
\[
= - \frac{(2\pi)^d}{(2\pi)^d} \sum_{q=-\infty}^{\infty} \frac{K_{d+1} \left( \frac{mq}{2T} \right)}{\frac{d+1}{2}} - \sum_{q=-\infty}^{\infty} \frac{K_{d+1} \left( \frac{mq}{2T} \right)}{\frac{d+1}{2}}.
\]
The function,
\[
K_{\nu}(z) = \frac{1}{2} \int_0^{\infty} e^{-\frac{t-z^2}{4t}} \frac{dt}{\sqrt{\nu+1}},
\]
is even under the transformation \( z \to -z \). Thus the above becomes:
\[
V_{\text{fermion}} = - \frac{(2\pi)^d}{(2\pi)^d} \sum_{q=-\infty}^{\infty} \frac{K_{d+1} \left( \frac{mq}{2T} \right)}{\frac{d+1}{2}}
\]
\[
- \frac{1}{2} \sum_{q=-\infty}^{\infty} \frac{K_{d+1} \left( \frac{mq}{2T} \right)}{\frac{d+1}{2}}
\]
\[
= - \frac{(2\pi)^d}{(2\pi)^d} \sum_{q=-\infty}^{\infty} \frac{K_{d+1} \left( \frac{mq}{2T} \right)}{\frac{d+1}{2}} - \frac{1}{2} \sum_{q=-\infty}^{\infty} \frac{K_{d+1} \left( \frac{mq}{2T} \right)}{\frac{d+1}{2}},
\]
where the symbol \('\) denotes omission of the zero modes in the summation. Using,
\[
K_{\nu}(z) = \frac{1}{2} \int_0^{\infty} e^{-\frac{t-z^2}{4t}} \frac{dt}{\sqrt{\nu+1}},
\]
the two Bessel sums are written:
\[
- \frac{(2\pi)^d}{(2\pi)^d} \sum_{q=-\infty}^{\infty} \frac{K_{d+1} \left( \frac{mq}{2T} \right)}{\frac{d+1}{2}} =
\]
\[
- \frac{1}{2} \frac{1}{(2\pi)^d} \int_0^{\infty} dt e^{-t} \sum_{q=-\infty}^{\infty} e^{-\frac{\left( \frac{mq}{4t} \right)^2}{2t}}
\]
\[
- \frac{1}{2} \frac{1}{(2\pi)^d} \int_0^{\infty} dt e^{-t} \sum_{q=-\infty}^{\infty} e^{-\frac{\left( \frac{mq}{4t} \right)^2}{2t}}.
\]
Set $\lambda = \left(\frac{mT}{d}\right)^2$ and using the Poisson summation formula \[3, 4, 34\] we obtain:

$$
\sum_{q=-\infty}^{\infty} e^{-\lambda q^2} = \sqrt{\frac{\pi}{\lambda}} \left( 1 + \sum_{k=-\infty}^{\infty} e^{-\frac{4\pi^2k^2}{4\lambda}} \right) - 1.
$$

(102)

Upon replacing we get:

$$
- \frac{(2\pi)^{d-1}}{(2\pi)^d} \frac{m^{d+1}}{d+1} \sum_{q=-\infty}^{\infty} \frac{K_{d+1}(mq)}{(mq^2)^{d+1}} =
$$

(103)

$$
- \frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d-1} m^{d+1} \int_0^\infty dt e^{-t} \left( \sqrt{\frac{\pi}{\lambda}} \left( 1 + \sum_{k=-\infty}^{\infty} e^{-\frac{4\pi^2k^2}{4\lambda}} \right) - 1 \right).
$$

Set,

$$
\nu = \frac{d + 1}{2}.
$$

(104)

and thus,

$$
- \frac{(2\pi)^{d-1}}{(2\pi)^d} \frac{m^{d+1}}{d+1} \sum_{q=-\infty}^{\infty} \frac{K_{d+1}(mq)}{(mq^2)^{d+1}} =
$$

(105)

$$
- \frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d-1} m^{d+1} \int_0^\infty dt e^{-t} \left( \sqrt{\frac{\pi}{\lambda}} \left( 1 + \sum_{k=-\infty}^{\infty} e^{-\frac{4\pi^2k^2}{4\lambda}} \right) - 1 \right).
$$

Also,

$$
a = \frac{m}{T},
$$

(106)

and finally (with $\lambda = \frac{a^2}{4\tau}$),

$$
- \frac{(2\pi)^{d-1}}{(2\pi)^d} \frac{m^{d+1}}{d+1} \sum_{q=-\infty}^{\infty} \frac{K_{d+1}(mq)}{(mq^2)^{d+1}} =
$$

(107)

$$
- \sqrt{\frac{\pi}{(2\pi)^d a}} (2\pi)^{d-1} m^{d+1} \int_0^\infty dt e^{-t} \left( 1 - e^{-\frac{t}{\nu+\frac{1}{2}}} \right)
$$

$$
- \sqrt{\frac{\pi}{(2\pi)^d a}} (2\pi)^{d-1} m^{d+1} \int_0^\infty dt e^{-t} \left( \sqrt{\frac{\pi}{\lambda}} \left( 1 + \sum_{k=-\infty}^{\infty} e^{-\frac{4\pi^2k^2}{4\lambda}} \right) - 1 \right)
$$

$$
+ \frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d-1} m^{d+1} \int_0^\infty dt e^{-t} \left( \frac{1}{\nu+\frac{1}{2}} \right).
$$
By using \[\Pi\],
\[
\frac{1}{(x^2 + a^2)^{\mu+1}} = \frac{1}{\Gamma(\mu + 1)} \int_0^\infty dt e^{-(x^2 + a^2) t / \mu},
\] (108)
we obtain the equation:
\[
- \frac{(2\pi)^{d-1}}{(2\pi)^d} \frac{m^{d+1}}{K_{d+1}(\frac{ma}{2\pi})} \sum_{q=-\infty}^\infty \frac{K_{d+1}(\frac{mq}{2\pi})}{(mq)^{d+1}} = - \frac{\sqrt{\pi}}{(2\pi)^{d+1} a^{d+1}} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1) (109)
\]
\[
- \frac{\sqrt{\pi}}{(2\pi)^d a} (2\pi)^{d+1} \Gamma(-\nu - \frac{1}{2} + 1) \left[ \sum_{k=-\infty}^{\infty} \left(1 + \left(\frac{2\pi k}{a}\right)^2\right)^{\nu+\frac{1}{2} - 1}\right]
\]
\[
+ \frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d+1} m^{d+1} \Gamma(-\nu).
\]
The sum,
\[
\sum_{k=-\infty}^{\infty} \left(1 + \left(\frac{2\pi k}{a}\right)^2\right)^{\nu+\frac{1}{2} - 1},
\] (110)
is invariant under the transformation $k \to -k$. Thus we change the sum to,
\[
2 \sum_{k=1}^{\infty} \left(1 + \left(\frac{2\pi k}{a}\right)^2\right)^{\nu+\frac{1}{2} - 1}.\] (111)
Replacing again we get:
\[
- \frac{(2\pi)^{d-1}}{(2\pi)^d} \frac{m^{d+1}}{K_{d+1}(\frac{ma}{2\pi})} \sum_{q=-\infty}^\infty \frac{K_{d+1}(\frac{mq}{2\pi})}{(mq)^{d+1}} = (112)
\]
\[
- \frac{\sqrt{\pi}}{(2\pi)^d a} (2\pi)^{d+1} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)
\]
\[
- \frac{2\sqrt{\pi}}{(2\pi)^d a} (2\pi)^{d+1} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1) (a^2)^{\frac{1}{2} - \nu} \left[ \sum_{k=1}^{\infty} (a^2 + 4\pi^2 k^2)^{\nu+\frac{1}{2} - 1}\right]
\]
\[
+ \frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d+1} m^{d+1} \Gamma(-\nu).
\]
Using the binomial expansion (in the case $d$ even) or Taylor expansion (in the case $d$ odd) \[\Pi\]:
\[
(a^2 + b^2)^{\nu-\frac{1}{2}} = \sum_{l=0}^\sigma \frac{(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!!} (a^2)^l (b^2)^{\nu-\frac{1}{2} - l}.\] (113)
For $d$ even, $\sigma$ equals,
\[
\sigma = \nu - \frac{1}{2},\] (114)
If \( d \) is odd then \( \sigma \) is a positive integer. By Taylor expanding we obtain:

\[
- \frac{(2\pi)^{\frac{d-1}{2}} m^{d+1}}{(2\pi)^d} \sum_{q=-\infty}^{\infty} K_{\frac{d+1}{2}}(\frac{mq}{T}) \frac{d+1}{2} = (115)
\]

\[
- \sqrt{\pi} \frac{(2\pi)^{\frac{d-1}{2}} m^{d+1}}{(2\pi)^d a} \Gamma(-\nu - \frac{1}{2} + 1) + \frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{\frac{d-1}{2}} m^{d+1} \Gamma(-\nu)
\]

\[
- \frac{2\sqrt{\pi}}{(2\pi)^d a} (2\pi)^{\frac{d-1}{2}} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)(a^2)^{\frac{1}{2} - \nu}
\]

\[
\times \left[ \sum_{k=1}^{\infty} \sum_{l=0}^{\sigma} \frac{(2\pi)^{\nu - \frac{1}{2} - l}(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!} (a^2)^l (k^2)^{\nu - \frac{1}{2} - l} \right].
\]

Following the previous techniques we get for the second sum of equation (101):

\[
\frac{1}{2} \frac{(2\pi)^{\frac{d-1}{2}} m^{d+1}}{(2\pi)^d} \sum_{q=-\infty}^{\infty} K_{\frac{d+1}{2}}(\frac{mq}{T}) \frac{d+1}{2} = (116)
\]

\[
+ \frac{1}{2} \frac{\sqrt{\pi}}{(2\pi)^d a_1} (2\pi)^{\frac{d-1}{2}} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)
\]

\[
- \frac{1}{4} \frac{1}{(2\pi)^d} (2\pi)^{\frac{d-1}{2}} m^{d+1} \Gamma(-\nu)
\]

\[
+ \frac{1}{2} \frac{\sqrt{\pi}}{(2\pi)^d a_1} (2\pi)^{\frac{d-1}{2}} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)(a_1^2)^{\frac{1}{2} - \nu}
\]

\[
\left[ \sum_{k=1}^{\infty} \sum_{l=0}^{\sigma} \frac{(2\pi)^{\nu - \frac{1}{2} - l}(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!} (a_1^2)^l (k^2)^{\nu - \frac{1}{2} - l} \right],
\]

with,

\[
\alpha_1 = \frac{m}{2T}.
\]

(117)
Finally adding the resulting expressions we get:

\[
V_{\text{fermion}} = -\frac{(2\pi)^{d-\frac{1}{2}} m^{d+1}}{(2\pi)^d} \left( \sum_{q=-\infty}^{\infty} K_{d+1}(\frac{mq}{m^q}) - \frac{1}{2} \sum_{q=-\infty}^{\infty} K_{d+1}(\frac{mq}{m^q}) \right) - \frac{1}{2} \sum_{q=-\infty}^{\infty} \frac{K_{d+1}(\frac{mq}{m^q})}{(\frac{mq}{m^q})^{d+1/2}}
\]

\[
= -\frac{\sqrt{\pi}}{(2\pi)^d a} (2\pi)^{d-\frac{1}{2}} m^{d+1} \Gamma(-\nu - 1) + \frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d-\frac{1}{2}} m^{d+1} \Gamma(-\nu) - \frac{2\sqrt{\pi}}{(2\pi)^d a} (2\pi)^{d-\frac{1}{2}} m^{d+1} \Gamma(-\nu - 1) + \frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d-\frac{1}{2}} m^{d+1} \Gamma(-\nu)\]

\[
\times \left[ \sum_{k=1}^{\infty} \sum_{l=0}^{\sigma} \frac{((2\pi)^{2\nu-\frac{1}{2}I}(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!} (a^2)^l (k^2)^{2\nu-\frac{1}{2}I} \right]
\]

with \(a = \frac{m}{T}\) and \(a_1 = \frac{m}{2T}\). Using the zeta regularization technique \[3, 4, 11, 34, 48\] we obtain,

\[
V_{\text{fermion}} = -\frac{(2\pi)^{d-\frac{1}{2}} m^{d+1}}{(2\pi)^d} \left( \sum_{q=-\infty}^{\infty} K_{d+1}(\frac{mq}{m^q}) - \frac{1}{2} \sum_{q=-\infty}^{\infty} K_{d+1}(\frac{mq}{m^q}) \right) - \frac{1}{2} \sum_{q=-\infty}^{\infty} \frac{K_{d+1}(\frac{mq}{m^q})}{(\frac{mq}{m^q})^{d+1/2}}
\]  

\[
= -\frac{\sqrt{\pi}}{(2\pi)^d a} (2\pi)^{d-\frac{1}{2}} m^{d+1} \Gamma(-\nu - 1) + \frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d-\frac{1}{2}} m^{d+1} \Gamma(-\nu) - \frac{2\sqrt{\pi}}{(2\pi)^d a} (2\pi)^{d-\frac{1}{2}} m^{d+1} \Gamma(-\nu - 1) + \frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d-\frac{1}{2}} m^{d+1} \Gamma(-\nu)\]

\[
\times \left[ \sum_{k=1}^{\infty} \sum_{l=0}^{\sigma} \frac{((2\pi)^{2\nu-\frac{1}{2}I}(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!} (a^2)^l (k^2)^{2\nu-\frac{1}{2}I} \right]
\]

We kept the above expression without simplifying in order to have a clear picture of the terms appearing (compare with the bosonic case). In the case \(d = 3\) appear the poles we discussed in the bosonic case. Again we Taylor expand around \(d = 3 + \epsilon\) for \(\epsilon \to 0\).
As in the bosonic case, we can write the fermionic contribution at finite temperature more elegantly using the analytic continuation of the Epstein-zeta function. In this case the sums of the form,

\[ \sum_{k=1}^{\infty} \left( a^2 + 4\pi^2 (2k+1)^2 \right)^{\nu+\frac{1}{2} - 1}, \]  

(120)
can be written in terms of the one dimensional Epstein zeta function,

\[ Z^m_{n}^1(\nu, w, \alpha) = \sum_{n=1}^{\infty} \left[ w(n + \alpha)^2 + m^2 \right]^{-\nu}, \]  

(121)
with \( \alpha = \frac{1}{2} \) and so on. We postpone the detailed presentation of the Epstein zeta functions in the section in which we study the twisted boundary conditions effective potential.

2.1.1 Case \( d \) odd

For the case \( d = 3 \), keeping terms \( \sim T \) we have:

\[
V_{\text{fermion}} = \frac{m^4}{16\pi^2} + \frac{m^4}{16\pi^2} + \left( \frac{3m^4}{64\pi^2} - \frac{3\gamma m^4}{32\pi^2} \right) \\
- \frac{m^2 T^2}{6} + \frac{14}{45} - \frac{m^4 \ln(\pi)}{16\pi^2} \\
- \frac{m^4 \ln(m^2 T^2)}{32\pi^2} - \frac{m^4 \psi(-\left(\frac{3}{2}\right))}{32\pi^2} - \frac{m^4 \psi(\frac{1}{2})}{32\pi^2} \\
+ \frac{m^4 \psi(\frac{5}{2})}{32\pi^2} + \frac{7m^6 \zeta(3)}{1536\pi^4 T^2} - \frac{31m^8 \zeta(5)}{65536\pi^6 T^4},
\]  

(122)
There are terms which are inverse powers of the temperature which in the high temperature limit (which we use) are negligible.

2.1.2 Case \( d \) Even

The calculation is the same as in the bosonic case. We only quote the case \( d = 2 \)

\[
V_{\text{fermion}} = \left( \frac{m^3}{6\sqrt{2}\pi} - \frac{m^3 T \ln(2)}{\sqrt{2}\pi} \right) - 12\sqrt{2} \pi T^3 \zeta'(-2),
\]  

(123)
We observe that the results contain a finite number of terms and is not an infinite sum as in the case \( d \) odd.
2.2 Some Applications on Finite Temperature Field Theories

2.2.1 The Standard Model at Finite Temperature

Let us now present the 1-loop correction for the effective potential of standard model fields [69]. The calculations of the final results are based on relations (119) and (61), of the previous sections. We start with a scalar boson described by the Lagrangian,

\[ L = \frac{1}{2} \partial^{\mu} \phi \partial_{\mu} \phi - V_{0}(\phi), \]

with tree level potential,

\[ V_{0} = \frac{1}{2} m^{2} \phi^{2} + \frac{\lambda}{4!} \phi^{4}, \]

or in the case of \( N \) complex scalar fields,

\[ L = \frac{1}{2} \partial^{\mu} \phi^{\alpha} \partial_{\mu} \phi^{\dagger}_{\alpha} - V_{0}(\phi^{\alpha}, \phi^{\dagger}_{\alpha}), \]

and in the following,

\[ (M^{2}_{s})_{b}^{\alpha} = V_{b}^{\alpha} = \frac{\partial^{2}V}{\partial \phi_{\alpha} \partial \phi^{b}}. \]

Mention that \( \text{Tr} M^{2}_{s} = 2V_{0}^{\alpha} \), where 2 comes from the two degrees of freedom that every complex scalar field has. Also \( \text{Tr} I = 2N_{s} \). Now regarding the fermion fields we have,

\[ L = i \bar{\psi}_{\alpha} \gamma^{\mu} \partial_{\mu} \psi_{\alpha} - \bar{\psi}_{\alpha} (M^{f})_{b}^{\alpha} \psi_{b}, \]

where the mass matrix \( (M^{f})_{b}^{\alpha}(\phi_{c}) \), is a function of scalar fields linear in \( \phi_{c}^{i} \):

\[ (M^{f})_{b}^{\alpha} = \Gamma_{bi}^{\alpha} \phi_{c}^{i}. \]

It is assumed that a Higgs mechanism gives mass to fermions. Finally consider the \( SU(N) \) gauge invariant Lagrangian,

\[ L = -\frac{1}{4} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2} \text{Tr}(D_{\mu} \phi_{\alpha}) \dagger (D^{\mu} \phi_{\alpha})..., \]

describing the gauge bosons-Higgs interactions. In the following,

\[ (M_{gb})_{\alpha\beta}^{2}(\phi_{c}) = g_{\alpha} g_{\beta} \text{Tr} \left[(T_{\alpha i}^{a}) \dagger T_{\beta j}^{a} \phi_{j}\right], \]

are the gauge bosons masses, and \( T_{\alpha} \) are the \( SU(N) \) generators in the adjoint representation. For the case of scalar bosons the 1-loop correction to the effective potential is,

\[ V_{eff}^{\beta}(\phi_{c}) = V_{0}(\phi_{c}) + V_{1}^{\beta}(\phi_{c}), \]

with \( V_{0}(\phi_{c}) \) the tree order effective potential and the loop correction,

\[ V_{1}^{\beta}(\phi_{c}) = \frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^{3}p}{(2\pi)^{3}} \ln \left[ \omega_{n}^{2} + \omega^{2}(\phi_{c}) \right], \]
where:
\[ \omega_n = 2n\pi\beta^{-1}, \]  
(134)
and also
\[ \omega^2 = p^2 + m^2(\phi_c). \]  
(135)
In the above, \( m^2(\phi_c) \) is given in relation (127). Relation (133) was the starting point of the our calculation for the boson case, see relation (14). Now in the fermion case,
\[ V^\beta_{\text{eff}}(\phi_c) = V_0(\phi_c) + V^\beta_1(\phi_c). \]  
(136)
where as before \( V^\beta_0(\phi_c) \) the tree level potential and \( V^\beta_1(\phi_c) \) the 1-loop correction. The last equals to
\[ V^\beta_1(\phi_c) = -\frac{2\lambda}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \ln \left[ \omega_n^2 + \omega^2(\phi_c) \right], \]  
(137)
with \( \omega_n \) the fermionic Matsubara frequencies:
\[ \omega_n = (2n + 1)\pi\beta^{-1}. \]  
(138)
Also,
\[ \omega^2 = p^2 + M^2_f(\phi_c). \]  
(139)
Relation (137) was the starting point for the fermion effective potential calculation, relation (84). Finally for the gauge bosons case the tree effective potential with the 1-loop correction reads,
\[ V^\beta_1(\phi_c) = Tr\Delta \left( \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \ln \left[ p^2 + M^2_{gb}(\phi_c) \right] + \frac{1}{2\pi^2\beta^4} J_B[M^2_{gb}(\phi_c)\beta^2] \right), \]  
(140)
where \( Tr\Delta = 3 \). Notice that:
\[ J_B[m^2\beta^2] = \int_0^\infty dx x^2 \ln[1 - e^{-\sqrt{x^2 + \beta^2m^2}}], \]  
(141)
and as before:
\[ (M^2_{gb})_{\alpha\beta}(\phi_c) = g_\alpha g_\beta Tr \left[ (T^i_\alpha \phi_i)^\dagger T^j_\beta \phi_j \right]. \]  
(142)
Relation (141) was obtained from relation (14).

### 2.3 Supersymmetric Effective Potential at Finite Temperature

It is very useful to extend our analysis for scalar bosons, fermions and gauge bosons in the supersymmetric case. Consider an \( N = 1, d = 4 \) supersymmetric Lagrangian with an \( SU(N) \) gauge symmetry. After that we give a general formula for the supersymmetric potential at finite temperature. We shall use the \( \overline{DR} \) renormalization scheme [87]. The chiral superfield in components reads,
\[ \Phi(x, \theta, \bar{\theta}) = A(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu A - \frac{1}{4}\theta^2\bar{\theta}^2\Box A \]  
\[ + \sqrt{2}\theta \psi(x) - \frac{i}{\sqrt{2}}\theta\bar{\theta}\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta\theta F(x), \]  
(143)
and the vector hypermultiplet is described by the chiral superfield,
\[ W_a = T^\alpha ( - \lambda^\alpha_a + \theta_a D^\alpha - i 2 (\sigma^\mu \bar{\sigma}^\nu \theta)_{a} F^\alpha_{\mu \nu} + \theta^2 \sigma^\mu D^\alpha \bar{\lambda}^\alpha ), \]  
(144)
with,
\[ F^\alpha_{\mu \nu} = \partial^\mu A^\alpha_\nu - \partial^\nu A^\alpha_\mu + f^{abc} A^b_\mu A^c_\nu, \]  
(145)
and also,
\[ D^\mu \bar{\lambda}^\alpha = \partial^\mu \bar{\lambda}^\alpha + f^{abc} A^b_\mu \bar{\lambda}^\alpha, \]  
(146)
The \( N = 1 \) Lagrangian is,
\[ L = \frac{1}{8 \pi} \Im \left( \tau Tr \int d^2 \theta W^\alpha W_\alpha \right) + \int d^2 \theta d^2 \bar{\theta} e^{-2V} \Phi + \int d^2 \theta W + \int d^2 \bar{\theta} W. \]  
(147)
which in components is written,
\[ L = -\frac{1}{4g^2} F^\alpha_{\mu \nu} F^{\alpha \mu \nu} + \frac{\theta}{32 \pi^2} F^\alpha_{\mu \nu} \bar{F}^{\alpha \mu \nu} - \frac{i}{g^2} \bar{\lambda}^\alpha \sigma^\mu D^\mu \bar{\lambda}^\alpha + \frac{1}{2g^2} D^\alpha D^\alpha + (\partial^\mu A - i A^\alpha A^\alpha A^\alpha A) (\partial^\mu A - i A^\alpha A^\alpha A)
- i \bar{\psi} \bar{\sigma}^\mu (\partial^\mu \psi - i A^\alpha A^\alpha A^\alpha A) - D^\alpha A^1 T^\alpha A - i \sqrt{2} A^1 T^\alpha A
+ i \sqrt{2} \bar{\psi} T^\alpha A \bar{\lambda}^\alpha + F^\dagger_i F_i + \frac{\partial W}{\partial A_i} F_i + \frac{\partial W}{\partial A^\dagger_i} F^\dagger_i - \frac{1}{2} \frac{\partial W}{\partial A_i} \frac{\partial W}{\partial A^\dagger_j} \psi_i \psi_j - \frac{1}{2} \frac{\partial W}{\partial A^\dagger_i} \frac{\partial W}{\partial A^\dagger_j} \bar{\psi}_i \bar{\psi}_j. \]  
(148)
The computation of the finite temperature effective potential can be done easily. The general potential up to one loop at finite temperature is \[ V = V_0 + \frac{1}{64 \pi^2} (V_{T=0} + V_{T\neq 0}), \]  
(149)
In the above, \( V_0 \) is the tree order potential (appearing in the Lagrangian). Also \( V_{T=0} \) is the one loop effective potential at \( T = 0 \). It is given by:
\[ V_{T=0} = \sum_i \left( \ln \left( \frac{m_i^2}{Q^2} \right) - \frac{3}{2} \right) \]  
(150)
\[ + 3 \sum_j \left( \ln \left( \frac{M_j^2}{Q^2} \right) - \frac{3}{2} \right) - 2 \sum_k \left( \ln \left( \frac{M_k^2}{Q^2} \right) - \frac{3}{2} \right). \]
Finally, \( V_{T\neq 0} \), is given by:
\[ V_{T\neq 0} = \sum_i \int \frac{d^3 k}{(2\pi)^3} 2 T \ln \left( 1 - e^{-\frac{\sqrt{k^2 + m^2}}{T}} \right) \]  
(151)
\[ + 3 \sum_j \int \frac{d^3 k}{(2\pi)^3} 2 T \ln \left( 1 - e^{-\frac{\sqrt{k^2 + M_j^2}}{T}} \right) \]
\[ - 2 \sum_k \int \frac{d^3 k}{(2\pi)^3} 2 T \ln \left( 1 + e^{\frac{\sqrt{k^2 + M_k^2}}{2T}} \right). \]

25
The above is our final formula. Notice that relation (151) contains integrals we computed in the previous sections, both for bosons and for fermions, see for example relations (31) and (85). Also the first term corresponds to the scalar bosons part, the second to the gauge bosons and the third to the fermion part. The same correspondence applies to relation (150). The masses that appear in relations (151) and (150) are model dependent and can be found in the same way as in (127), (129) and (131).

All the above are invaluable to the theories of phase transitions at finite temperature. See for example reference [69] and references therein.

In conclusion the generalization of the above to any dimensions is straightforward. In general, apart from the phase transition application, a theory at finite temperature offers the possibility to connect a d dimensional theory with the d + 1 dimensional theory at finite temperature. Let us discuss a little on this. One could say that the calculations we obtained actually correspond to a three dimensional theory in the case of initial d = 4 theory. However one should be really cautious since the argument that a d dimensional field theory correspond to the same theory in d − 1 dimensions has been proven true [83] only for the φ4 theory (always within the limits of perturbation theory). Also this also holds true for supersymmetric theories. On the contrary this does not hold for QCD and Yang-Mills theories. Actually QCD3 resembles more QCD4 and not QCD4 at finite temperature! It would be more correct to say that a d dimensional theory at finite temperature resembles more the same theory with one dimension compactified to a circle and in the limit $R \to 0$, where $R$ the magnitude of the compact dimension. We shall report on these issues somewhere else [86].

3 Calculation of Effective Potential in Spacetime Topology $S^1 \times R^d$

In this section we will compute the fermionic and bosonic contributions to the effective potential of field theories quantized in spacetime topologies $S^1 \times R^d$ [11, 12, 52, 4, 3, 34, 25, 26, 27, 28]. The calculations are done in Euclidean time by making a Wick rotation in the time coordinate. By this we have static-time independent results. In space times with non trivial topology the fields can have periodic or antiperiodic boundary conditions without the restrictions that we had in the temperature case [3, 57] (that is bosons must obey only periodic and fermions only antiperiodic boundary conditions). We shall deal with periodic bosons and antiperiodic fermions.

The boundary conditions for bosons are,

$$\varphi(x, 0) = \varphi(x, L),$$

(152)

$L$ denoting the compact (circle) dimension, while the fermion boundary conditions,

$$\psi(x, 0) = -\psi(x, L).$$

(153)

Another more general set of boundary conditions that can be used is the so called twisted boundary conditions of the form:

$$\varphi(x, 0) = e^{-iw} \varphi(x, L),$$

(154)
for bosons and,
\[ \psi(x,0) = -e^{i\varphi} \psi(x, L), \]  \hspace{1cm} (155)

for fermions.

### 3.1 Periodic Bosons and Antiperiodic Fermions

Using,
\[ \varphi(x,0) = \varphi(x, L), \]  \hspace{1cm} (156)

for bosons and,
\[ \psi(x,0) = -\psi(x, L), \]  \hspace{1cm} (157)

for fermions, we shall compute the bosonic contribution,
\[ \frac{1}{L} \int \frac{dk^3}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \ln\left[ \frac{4\pi^2 n^2}{L^2} + k^2 + m^2 \right], \]  \hspace{1cm} (158)

and also the fermionic one,
\[ \frac{1}{L} \int \frac{dk^3}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \ln\left[ \frac{(2n+1)^2 \pi^2}{L^2} + k^2 + m^2 \right]. \]  \hspace{1cm} (159)

Following the techniques developed in the previous sections (roughly we substitute \( T \to \frac{1}{L} \)),
\[ \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \sum_{n=-\infty}^{\infty} \ln\left[ \frac{4\pi^2 n^2}{L^2} + k^2 + m^2 \right] = \]  \hspace{1cm} (160)

\[ -\frac{1}{2} \frac{\sqrt{\pi}}{(2\pi)^{d-1} a} (2\pi)^{d} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1) + \frac{1}{4} \frac{1}{(2\pi)^{d} a} (2\pi)^{d-1} m^{d+1} \Gamma(-\nu) \]

\[ -\frac{\sqrt{\pi}}{(2\pi)^{d} a} (2\pi)^{d} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1) (a^2)^{\frac{1}{2} - \nu} \]

\[ \times \sum_{l=0}^{\nu - \frac{1}{2}} \frac{((2\pi)^2)^{\nu - \frac{1}{2} - l} (\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!} (a^2)^l \zeta(-2\nu + 1 + 2l), \]
for the boson case, with \( \alpha = mL \) and,

\[
\frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \sum_{n=-\infty}^{\infty} \ln \left( \frac{(2n+1)^2 \pi^2}{L^2} + k^2 + m^2 \right) =
\]

\[
= \frac{(2\pi)^{d-1}}{(2\pi)^d} m^{d+1} \frac{K_{d+1}(mqL)}{(mqL)^{\frac{d+1}{2}}} \left( \frac{1}{2} \sum_{q=-\infty}^{\infty} \frac{K_{d+1}((mqL)^2)}{((mqL)^2)^{\frac{d+1}{2}}} \right) =
\]

\[
= \frac{\sqrt{\pi}}{(2\pi)^d a_2} (2\pi)^{d-1} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1) + \frac{1}{2} \frac{\sqrt{\pi}}{(2\pi)^d a_1} (2\pi)^{d-1} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)(a_1^2)^{\frac{1}{2} - \nu}
\]

\[
\times \left[ \sum_{l=0}^{\infty} \frac{((2\pi)^2)^{\nu - \frac{1}{2} - l} (\nu - \frac{1}{2})!(a_2^2)^{l}}{(\nu - \frac{1}{2} - l)!!((a_2^2)^{\nu - \frac{1}{2} - l})! \zeta(-2\nu + 1 + 2l)} \right] + \frac{1}{2} \frac{\sqrt{\pi}}{(2\pi)^d a_1} (2\pi)^{d-1} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)(a_1^2)^{\frac{1}{2} - \nu}
\]

\[
\times \left[ \sum_{l=0}^{\infty} \frac{((2\pi)^2)^{\nu - \frac{1}{2} - l} (\nu - \frac{1}{2})!(a_1^2)^{l}}{(\nu - \frac{1}{2} - l)!!((a_1^2)^{\nu - \frac{1}{2} - l})! \zeta(-2\nu + 1 + 2l)} \right] ,
\]

for the fermion case, with \( \alpha_2 = mL \) and \( \alpha_1 = \frac{mL}{2} \).

For the case \( d = 3 \) the bosonic contribution is:

\[
\frac{1}{L} \int \frac{dk^3}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \ln \left( 4\pi^2 n^2 + k^2 + m^2 \right) =
\]

\[
= \frac{m^4}{16 \pi^2} + \frac{m^4}{16 \pi^2} + \frac{m^4}{12 L^2} + \frac{m^4}{64 \pi^2} \frac{m^4}{32 \pi^2} - \frac{\gamma m^4}{16 L \pi^2} - \frac{m^4}{6 L \pi} - \frac{\pi^2}{45 L^3} + \frac{m^4 \ln(2)}{32 \pi^2} +
\]

\[
- \frac{m^4 \ln(2)}{32 \pi^2} - \frac{m^4 \ln(m)}{16 L \pi^2} - \frac{m^4 \ln(m)}{16 L \pi^2} - \frac{m^4 \ln(L^2 m^2)}{32 \pi^2} + \frac{m^4 \ln(m)}{32 \pi^2} + \frac{m^4 \ln(m)}{32 \pi^2} +
\]

\[
\frac{L^2 m^6 \zeta(3)}{384 \pi^4} - \frac{L^4 m^8 \zeta(5)}{4096 \pi^6},
\]

In equation (162) we omitted terms of higher order in \( L \). This is because we are interested in the limit \( L \to 0 \).
The fermionic contribution for $d = 3$ is:

$$\frac{1}{L} \int \frac{dk^3}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \ln \left( \frac{(2n+1)^2 \pi^2}{L^2} + k^2 + m^2 \right) = \left(163\right)$$

$$\frac{-m^4}{16 \pi^2} + \frac{m^4}{16 \pi^2} e^{\frac{-m^2}{6L^2}} + \left( \frac{m^2}{64 \pi^2} - \frac{3 m^4}{32 \pi^2} - \frac{\gamma m^4}{16 L \pi^2} \right) + \frac{14 \pi^2}{45 L^4} \frac{m^4 \ln(L^2 m^2)}{32 \pi^2} + \frac{m^4 \ln(\pi)}{16 \pi^2}$$

$$- \frac{m^4 \psi\left(-\frac{3}{2}\right)}{32 \pi^2} - \frac{m^4 \psi\left(\frac{3}{2}\right)}{32 \pi^2} + \frac{m^4 \psi\left(\frac{3}{2}\right)}{32 \pi^2}$$

$$+ \frac{7 m^6 L^2 \zeta(3)}{1536 \pi^4} - \frac{31 L^4 m^8 \zeta(5)}{65536 \pi^6}.$$ 

In the case $d = 2$ the bosonic contribution reads:

$$\frac{1}{L} \int \frac{dk^2}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \ln \left( \frac{4\pi^2 n^2}{L^2} + k^2 + m^2 \right) = \left(164\right)$$

$$\frac{m^2}{4 \sqrt{2} L \pi} + \frac{m^3}{6 \sqrt{2} \pi} + \frac{m^2 \ln(2)}{2 \sqrt{2} L \pi} - \frac{m^2 \ln(L^2 m^2)}{4 \sqrt{2} L \pi}$$

$$+ \frac{m^2 \ln(\pi)}{2 \sqrt{2} L \pi} - \frac{m^2 \ln(2 \pi)}{2 \sqrt{2} L \pi} + \frac{\zeta'(-2)}{L^3},$$

and the fermionic contribution:

$$\frac{1}{L} \int \frac{dk^2}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \ln \left( \frac{(2n+1)^2 \pi^2}{L^2} + k^2 + m^2 \right) = \left(165\right)$$

$$\frac{m^3}{6 \sqrt{2} \pi} - \frac{m^2 \ln(2)}{\sqrt{2} L \pi} + \frac{\zeta'(-2)}{L^3}.$$

### 3.2 Some Applications I

#### 3.2.1 Topological Symmetry Breaking in Self Interacting Field Theories

We now discuss some applications of the periodic bosons and anti-periodic fermions effective potential at finite volume. It is well known that field theory at finite volume plays an important role to topological symmetry breaking or restoration and topological mass generation \[41, 42, 52, 3, 34, 25, 27, 28, 21, 61, 75, 91, 89\]. Apart from the known influence of the topology to the boundary conditions of the sections of the fiber bundles studied, the effective mass and on particle creation \[3\] the need for studying field theories at finite volume is that the universe might exhibit non trivial topology as a whole \[76, 40, 41, 26, 3\].

Now we briefly present the topological mass generation. When spacetime has non-trivial topology then a massless field with periodic boundary conditions, can acquire mass through
loop corrections, in a dynamical way. Indeed, the one loop potential reads,

\[ V^1(\phi) = \frac{1}{\text{vol}(M)} \sum_n \ln(a_n/\mu^2), \]  

(166)

with \( \text{vol}(M) \) is the volume of the spacetime under study and \( a_n \) are the eigenvalues of the Laplace operator on this spacetime. A regularized form of the above involves the zeta function \([34]\),

\[ \zeta(s) = \sum_n a_n^{-s}. \]  

(167)

The potential at loop is written as,

\[ V^1(\phi) = \frac{1}{\text{vol}(M)} [\zeta'(0) + \zeta(0) \ln \mu^2], \]  

(168)

with \( \mu \) a dimensional regularization parameter that can be removed in the renormalization process. The topological mass is equal to,

\[ m^2 = \frac{d^2V(\phi)}{d\phi^2}, \]  

(169)

at \( \phi = 0 \). In the above relation, \( V(\phi) \) is equal to,

\[ V(\phi) = \frac{\lambda}{4!} \phi^4 - \frac{1}{\text{vol}(M)} [\zeta'(0) + \zeta(0) \ln \mu^2], \]  

(170)

Now for the spacetime \( S^1 \times R^3 \) the eigenvalues \( a_n \) are,

\[ a_n = \frac{\lambda}{2} \phi^2 + \left( \frac{2\pi n}{L} + k_1^2 + k_2^2 + k_3^2 \right), \]  

(171)

Also the zeta function \( \zeta(s) \) reads,

\[ \zeta(s) = \frac{L_1}{2\pi} \int d^3k_i \sum_{n=-\infty}^{\infty} \left[ \frac{\lambda}{2} \phi^2 + \left( \frac{4\pi^2 n^2}{L^2} + k_1^2 + k_2^2 + k_3^2 \right) \right], \]  

(172)

The calculation of the above can be done with the techniques we presented in the previous sections. Now at \( \phi = 0 \) the potential is,

\[ V(\phi = 0) = -\frac{\pi^2}{90L_1^4}, \]  

(173)

The above is just the Casimir energy for a real scalar field that satisfies periodic boundary conditions instead of Dirichlet. The topologically generated mass in this case is,

\[ m^2 = \frac{\lambda}{24L_1^2}. \]  

(174)
These techniques can be useful to determine the vacuum stability of the theory under consideration [40, 41, 91, 3, 75]. In the case of the periodic scalar field, the mass is positive, thus the $\phi = 0$ vacuum is stable. Let us now study the same setup in $S^1 \times R^3$ but with the scalar field satisfying anti-periodic boundary conditions along the compact dimension. This case resembles the calculations of a fermion field at finite volume we presented previously. The only vacuum expectation value that is allowed is $\phi = 0$ [74]. The zeta function now reads,

$$
\zeta(s) = \frac{L_1}{2\pi} \int d^3k_i \sum_{n=-\infty}^{\infty} \left[ \frac{\lambda}{2} \phi^2 + \left( \frac{\pi^2(2n+1)^2}{L^2} + k_1^2 + k_2^2 + k_3^2 \right) \right],
$$

(175)

and in this case, at $\phi = 0$ the potential is,

$$
V(\phi = 0) = \frac{7\pi^2}{720L_1^4},
$$

(176)

The above is just the Casimir energy for a real scalar field that satisfies periodic boundary conditions instead of Dirichlet. The topologically generated mass now reads,

$$
m^2 = -\frac{\lambda}{48L_1^2}.
$$

(177)

The negative sign indicates an instability in this theory [75, 3, 40].

3.2.2 Casimir Effect the Effective Potential and Extra Dimensions

The calculations for finite volume field theories with a toroidal compact dimension are useful for field theories with one compact extra dimension. We shall present some cases here. Also these are special cases of the effective potential with a twist in the fields boundary conditions that we describe in the next section.

Let us start with a scalar field in the Randall-Sundrum1 (RS1) model [92]. The line element is given by,

$$
ds^2 = e^{-2kr_0} \eta_{\mu\nu} dx^\mu dx^\nu - r_0^2 d\phi^2,
$$

(178)

The theory is quantized on the orbifold $S^1/Z_2$ and thus the points $(x^\mu, \phi)$ and $(x^\mu, -\phi)$ are identified. The exponential factor is the most appealing feature of the RS1 model. Actually the hierarchy problem can be solved within this scenario since a Tev mass scale can be produced from a Plank mass scale [92]. One of the most interesting problems appearing in models with extra compact dimensions is related with the size and stability of the compact dimension. Particularly the problem is two fold. First one must find a way to shrink the extra dimensions. This is a very serious feature since the visible spatial dimensions of our world inflated in the past. Also their size exponentially increased during inflation. So firstly, the extra dimensions must shrink. Second the extra dimensions must be stabilized and not to collapse to the Plank scale. One indicator to solve the first problem is the existence of negative energy in the bulk, that is the Casimir energy of the bulk scalar field must be negative. In the context of string theory there are setups such
us orientifolds planes and other structures \[23, 23\]. In some cases field theory corrections can be supplemented by string structures but we shall not discuss this here.

Consider a free scalar in the bulk, with Lagrangian density,

\[ L = G_{AB} \partial_A \Phi \partial_B \Phi - m^2 \Phi^2, \]  

The harmonic expansion of the scalar field is,

\[ \Phi(x^\mu, \phi) = \sum_n \psi_n(x^\mu) y_n(\phi), \]

Solving the equations of motion for the RS metric one obtains obtain,

\[ y_n(\phi) \sim e^{2kR \phi} \left[ J_\nu \left( \frac{M_n e^{kR \phi}}{k} \right) + Y_\nu \left( \frac{M_n e^{kR \phi}}{k} \right) \right], \]

and in order the field satisfies the orbifold boundary conditions, \( M_n \) must satisfy,

\[ M_n e^{kR \phi} k \sim \pi (N + \frac{1}{4}), \]

It is clear that the Casimir energy is significant due to the extra dimensions quantum fluctuations. For the bulk scalar field we obtain,

\[ V^+ = \frac{1}{2} \sum_{\nu = -\infty}^{\infty} \frac{d^4 k}{(2\pi)^4} \ln(k^2 + \left( \frac{n\pi}{r_c} \right)^2 + M_n^2), \]

with \( r_c \) the compact dimension radius. Notice that relation (183) is identical with relation (158) for the case of five dimensions. The calculation and generalization is straightforward, and we can find the result in closed form, in terms of the polylogarithm functions. This calculation is similar to the finite temperature one for \( d \) even, see relations (115) and (80).

For a more general calculation see the next section. In the case of a massless scalar relation (183) is modified to,

\[ V^{+1} = \frac{1}{2} \sum_{\nu = -\infty}^{\infty} \frac{d^4 k}{(2\pi)^4} \ln(k^2 + \left( \frac{n\pi}{r_c} \right)^2), \]

which is calculated to be,

\[ V^{+1} = \frac{3\zeta(5)}{64\pi^4 r_c^4}, \]

which is clearly negative, and thus this results to a shrinking of the compact dimension. Also the Casimir force in terms of the compact dimensions is repulsive which leads to a stabilization of the extra dimension. The calculations for fermions are straightforward. Also the existence of a minimum in the effective potential is an indicator of stabilization of the extra dimensions.

Finally let us mention that Casimir calculations have been done for de Sitter and anti-de Sitter brane worlds, see \[72, 49, 50, 47\]. Additionally same results hold for other 5-dimensional setups, such us large extra dimensions and universal extra dimensions. We shall briefly present some applications in relation to them after the next section.
3.3 The Case of Twisted Boundary Conditions

We shall study only the twisted boson case since the other case is similar \cite{3,34,4}. The twisted boundary conditions for bosons are:

\[ \varphi(x,0) = e^{-iw} \varphi(x,L), \]  
(186)

while for fermions:

\[ \psi(x,0) = -e^{i\rho} \psi(x,L), \]  
(187)

or equivalently,

\[ \psi(x,0) = e^{i(\rho+\pi)} \psi(x,L). \]  
(188)

We Fourier expand \( \varphi \):

\[ \sum_n \int dp^3 e^{ipx} = e^{iw} \sum_n \int dp^3 e^{ipx+iw_nL}, \]  
(189)

from which we obtain,

\[ w_nL = 2\pi n + w \rightarrow w_n = (2\pi n + w) \frac{1}{L}, \]  
(190)

with, \( G = \frac{1}{w_n^2 + k^2 + m^2} \).

Doing the same as in the previous with the difference:

\[ w_n = (2\pi n + w) \frac{1}{L} = (n + \omega) \frac{2\pi}{L}, \]  
(191)

with, \( \omega = \frac{w}{2\pi} \), we will compute \cite{3,34,4},

\[ \frac{1}{L} \int \frac{dk^3}{(2\pi)^3} \sum \ln[(n + \omega) \frac{2\pi}{L})^2 + k^2 + m^2]. \]  
(192)

Consider the sum:

\[ \sum_{n=-\infty}^{\infty} \frac{1}{(n + \omega)^2 \frac{4\pi}{L^2} + a^2} = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \frac{1}{(n + \omega)^2 \frac{4\pi}{L^2} + a^2}, \]  
(193)

with,

\[ a^2 = k^2 + m^2. \]  
(194)

Integrating,

\[ \sum_{n=-\infty}^{\infty} \frac{1}{(n + \omega)^2 \frac{4\pi}{L^2} + a^2}, \]  
(195)

over \( a^2 \), we get,

\[ \int \sum_{n=-\infty}^{\infty} \frac{da^2}{(n + \omega)^2 \frac{4\pi}{L^2} + a^2} = \sum_{n=-\infty}^{\infty} \ln[(n + \omega)^2 \frac{4\pi}{L^2} + a^2]. \]  
(196)
Also,
\[
\sum_{n=-\infty}^{\infty} \frac{da^2}{(n+\omega)^2(\frac{2\pi}{L})^2 + a^2} = \frac{L}{4a} \left( \coth\left( \frac{aL}{2} - i\pi\omega \right) + \coth\left( \frac{aL}{2} + i\pi\omega \right) \right),
\]
(197)

and consequently,
\[
\int \sum_{n=-\infty}^{\infty} \frac{da^2}{(n+\omega)^2(\frac{2\pi}{L})^2 + a^2} =
\]
(198)

\[
\int \frac{L}{4a} (\coth\left( \frac{aL}{2} - i\pi\omega \right) + \coth\left( \frac{aL}{2} + i\pi\omega \right) da^2 =
\]
(199)

\ln(\sinh(\frac{aL}{2} - i\pi\omega)) + \ln(\sinh(\frac{aL}{2} + i\pi\omega)).

Using \[[1]\],
\[
\ln(\sinh x) = \ln\left( \frac{1}{2} (e^x - e^{-x}) \right) = x + \ln(1 - e^{-2x}) - \ln[2],
\]
(200)

and summing,
\[
\ln(\sinh(\frac{aL}{2} - i\pi\omega)) = \frac{aL}{2} - i\pi\omega + \ln[1 - e^{-2(\frac{aL}{2} - i\pi\omega})] - \ln[2],
\]
(201)

and,
\[
\ln(\sinh(\frac{aL}{2} + i\pi\omega)) = \frac{aL}{2} + i\pi\omega + \ln[1 - e^{-2(\frac{aL}{2} + i\pi\omega})] - \ln[2].
\]
(202)

we get,
\[
\int \sum_{n=-\infty}^{\infty} \frac{da^2}{(n+\omega)^2(\frac{2\pi}{L})^2 + a^2} =
\]
(203)

\ln(\sinh(\frac{aL}{2} - i\pi\omega)) + \ln(\sinh(\frac{aL}{2} + i\pi\omega)) =
aL + \ln[1 - e^{-2(\frac{aL}{2} - i\pi\omega})] + \ln[1 - e^{-2(\frac{aL}{2} + i\pi\omega})] - 2 \ln[2].

After some calculations \[[3, 34, 4, 13]\]:
\[
\sum_{n=-\infty}^{\infty} \ln[(n + \omega)^2(\frac{2\pi}{L})^2 + a^2] =
\]
(204)

\alpha L + \ln[1 - e^{-2(\frac{aL}{2} - i\pi\omega})] + \ln[1 - e^{-2(\frac{aL}{2} + i\pi\omega})] - 2 \ln[2].

Using the identity \[[1]\],
\[
\sum \ln[(n + \omega)^24\pi^2T^2 + a^2] = 2(a - b),
\]
(205)
the relation (204) becomes,

$$\sum \ln[(n + \omega)^2(\frac{2\pi}{L})^2 + a^2] =$$

$$\frac{L}{2\pi} \int_{-\infty}^{\infty} dx \ln[x^2 + a^2] + \ln[1 - e^{-2(\frac{2\pi}{L} - i\pi\omega})] + \ln[1 - e^{-2(\frac{2\pi}{L} + i\pi\omega})].$$

Thus,

$$\frac{1}{L} \int \frac{dk^3}{(2\pi)^3} \sum \ln[((n + \omega)\frac{2\pi}{L})^2 + k^2 + m^2] =$$

$$\int \frac{dk^3}{(2\pi)^3} \int_{-\infty}^{\infty} dx \frac{2\pi}{2\pi} \ln[x^2 + a^2]$$

$$+ \frac{1}{L} \int \frac{dk^3}{(2\pi)^3} \ln[1 - e^{-2(\frac{2\pi}{L} - i\pi\omega})]$$

$$+ \frac{1}{L} \int \frac{dk^3}{(2\pi)^3} \ln[1 - e^{-2(\frac{2\pi}{L} + i\pi\omega})],$$

with,

$$a^2 = k^2 + m^2,$$

The first integral is the one loop correction to the effective potential for $L = 0$. In $d + 1$ dimensions relation (207) reads [8, 4, 34, 27, 32, 36, 37]:

$$\frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \sum \ln[((n + \omega)\frac{2\pi}{L})^2 + k^2 + m^2] =$$

$$\int \frac{dk^{d+1}}{(2\pi)^{d+1}} \ln[k^2 + a^2]$$

$$+ \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \ln[1 - e^{-2(\frac{2\pi}{L} - i\pi\omega})] + \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \ln[1 - e^{-2(\frac{2\pi}{L} + i\pi\omega})].$$

In the following we consider only the $L$ dependent part,

$$V_{twisted} = \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \ln[1 - e^{-2(\frac{2\pi}{L} - i\pi\omega})] + \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \ln[1 - e^{-2(\frac{2\pi}{L} + i\pi\omega})].$$

Let,

$$V_1 = \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \ln[1 - e^{-2(\frac{2\pi}{L} - i\pi\omega})],$$

and

$$V_2 = \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \ln[1 - e^{-2(\frac{2\pi}{L} + i\pi\omega})].$$

so relation (210) reads,
\[ V_{twisted} = V_1 + V_2, \] (213)

The calculation of \( V_1 \) and of \( V_2 \) is equivalent. Their analytic properties are the same. So we calculate only \( V_2 \). We have,

\[
V_2 = \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \ln[1 - e^{-2\left(\frac{aL}{\pi} + i\pi\omega\right)}] = \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \ln[1 - e^{-aL - 2i\pi\omega}].
\] (214)

Using,

\[
\ln[1 - e^{-aL - 2i\pi\omega}] = -\sum_{q=1}^{\infty} \frac{e^{-aLq - 2\pi\omega q}}{q}.
\] (215)

Now \( V_2 \) becomes,

\[
V_2 = \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \ln[1 - e^{-aL - 2i\pi\omega}]
\]

\[
= -\sum_{q=1}^{\infty} \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \frac{e^{-aLq - 2\pi\omega q}}{q},
\]

\[
= -\sum_{q=1}^{\infty} \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \frac{e^{-\sqrt{k^2 + m^2 qL} - 2\pi\omega q}}{q}
\]

\[
= -\sum_{q=1}^{\infty} \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \frac{(2\pi)^d \frac{e^{-\sqrt{k^2 + m^2 qL} - 2\pi\omega q}}{q}}{\Gamma(\frac{d}{2})} - \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \frac{e^{-\sqrt{k^2 + m^2 qL} - 2\pi\omega q}}{q}
\]

\[
= -\sum_{q=1}^{\infty} \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \frac{(2\pi)^d \frac{e^{-\sqrt{k^2 + m^2 qL} - 2\pi\omega q}}{q}}{\Gamma(\frac{d}{2})} - \frac{1}{L} \int \frac{dk^d}{(2\pi)^d} \frac{e^{-\sqrt{k^2 + m^2 qL} - 2\pi\omega q}}{q}
\]

we used \((a = \sqrt{k^2 + m^2})\). The integral,

\[
\int_{-\infty}^{\infty} dk^d e^{-\sqrt{k^2 + m^2 qL}},
\]

equals to \( \Pi \),

\[
\int_{-\infty}^{\infty} dk^d e^{-\sqrt{k^2 + m^2 qL}} = 2^{\frac{d}{2} - 1}(\sqrt{\pi})^{-1}(qL)^{\frac{d}{2} - \frac{3}{2}} m^{\frac{d+1}{2}} \Gamma(\frac{d}{2}) K_{\frac{d+1}{2}}(mqL).
\] (218)

thus \( V_2 \) is written:

\[
V_2 = -\sum_{q=1}^{\infty} \frac{2^{\frac{d}{2} - 1}}{(2\pi)^d} \frac{m^{\frac{d+1}{2}}}{\frac{d+1}{2} (\frac{d}{2})^{\frac{d+1}{2}}} \frac{K_{\frac{d+1}{2}}(mqL)}{mqL} \frac{1}{(\frac{d}{2})^{\frac{d+1}{2}}} e^{-2\pi\omega q}.
\] (219)
Equivalently $V_1$ equals to:

$$V_1 = -\frac{1}{2} \sum_{q=1}^{\infty} \frac{1}{(2\pi)^d} (2\pi)^{d+1} m^{d+1} \frac{K_{d+1}(mqL)}{(mqL)^{d+1}} e^{2\pi i\omega q}. \quad (220)$$

Summing $V_1$ and $V_2$

$$V_1 + V_2 = -\frac{1}{2} \sum_{q=1}^{\infty} \frac{1}{(2\pi)^d} (2\pi)^{d+1} m^{d+1} \frac{K_{d+1}(mqL)}{(mqL)^{d+1}} (e^{+2\pi i\omega q} + e^{-2\pi i\omega q}). \quad (221)$$

and using,

$$\cos x = \frac{1}{2}(e^{-ix} + e^{ix}), \quad (222)$$

we get:

$$V_1 + V_2 = -\sum_{q=1}^{\infty} \frac{1}{(2\pi)^d} (2\pi)^{d+1} m^{d+1} \frac{K_{d+1}(mqL)}{(mqL)^{d+1}} \cos(2\pi \omega q). \quad (223)$$

The function,

$$\frac{K_{d+1}(mqL)}{(mqL)^{d+1}} \cos(2\pi \omega q), \quad (224)$$

is invariant under the transformation $q \rightarrow -q$ and relation (223) is written,

$$V_1 + V_2 = -\sum_{q=1}^{\infty} \frac{1}{(2\pi)^d} (2\pi)^{d+1} m^{d+1} \frac{K_{d+1}(mqL)}{(mqL)^{d+1}} \cos(2\pi \omega q), \quad (225)$$

and finally,

$$V_1 + V_2 = -\frac{1}{2} \sum_{q=-\infty}^{\infty} \frac{1}{(2\pi)^d} (2\pi)^{d+1} m^{d+1} \frac{K_{d+1}(mqL)}{(mqL)^{d+1}} \cos(2\pi \omega q). \quad (226)$$

Again the symbol ${}'$ means omission of the zero modes.

By breaking the cosine function to exponentials, we introduce $F_1$ and $F_2$ with $V_{\text{twist}} = F_1 + F_2$, where,

$$F_1 = -\frac{1}{4} \sum_{q=-\infty}^{\infty} \frac{1}{(2\pi)^d} (2\pi)^{d+1} m^{d+1} \frac{K_{d+1}(mqL)}{(mqL)^{d+1}} e^{-2\pi i\omega q}, \quad (227)$$

and,

$$F_2 = -\frac{1}{4} \sum_{q=-\infty}^{\infty} \frac{1}{(2\pi)^d} (2\pi)^{d+1} m^{d+1} \frac{K_{d+1}(mqL)}{(mqL)^{d+1}} e^{2\pi i\omega q}. \quad (228)$$

We compute $F_1$ only, since the computation of the other is similar. We have:

$$\frac{K_{\nu}(z)}{(z)^\nu} = \frac{1}{2} \int_0^{\infty} e^{-t-z^2/t} \frac{e^{-t}}{t^{\nu+1}} dt, \quad (229)$$
and $F_1$ becomes:

$$F_1 = -\frac{1}{8} (2\pi)^d (2\pi)^{-d/2} m^{d+1} \int_0^\infty e^{-i q x} e^{-2\pi i\omega q} \sum_{l=0}^{\infty} e^{-\frac{(mqL)^2}{4t}} t^{d+1}.$$  \hspace{1cm} (230)

Using the Poisson identity \cite{3, 4, 34},

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1)e^{-2\pi i k x_1} dx_1,$$  \hspace{1cm} (231)

with,

$$f(x) = e^{-\frac{(mL)^2}{4t}} e^{-2\pi i\omega x},$$  \hspace{1cm} (232)

and $\lambda = \frac{(mL)^2}{4t}$, $\beta = 2$, $\pi \omega$, we get \cite{53}:

$$\sum_{q=-\infty}^{\infty} e^{-\lambda q^2} e^{-i\beta q} =$$  \hspace{1cm} (233)

$$\sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\lambda x^2} e^{-i\beta x} e^{-2\pi i k x} dx =$$

$$\sqrt{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\lambda x^2} e^{-i\beta x} e^{-2\pi i k x} dx =$$

$$\sqrt{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\lambda x^2} e^{ix(-\beta-2\pi k)} dx.$$  \hspace{1cm} (234)

The Fourier transformation of the function $e^{-\lambda x^2}$ is:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\lambda x^2} e^{ix(-\beta-2\pi k)} dx = \frac{e^{-\frac{(\beta+2\pi k)^2}{4\lambda}}}{\sqrt{2\sqrt{\lambda}}},$$  \hspace{1cm} (235)

and finally,

$$\sum_{q=-\infty}^{\infty} e^{-\lambda q^2} e^{-i\beta q} = \sum_{k=-\infty}^{\infty} \sqrt{2\pi} e^{-\frac{(\beta+2\pi k)^2}{4\lambda}} \frac{1}{\sqrt{2\sqrt{\lambda}}} =$$

$$\sum_{k=-\infty}^{\infty} \sqrt{\frac{\pi}{\lambda}} \frac{e^{-\frac{(\beta+2\pi k)^2}{4\lambda}}}{\sqrt{\lambda}} = \sqrt{\frac{\pi}{\lambda}} \sum_{k=-\infty}^{\infty} e^{-\frac{(\beta+2\pi k)^2}{4\lambda}}.$$  \hspace{1cm} (236)

Neglecting the zero modes we get:

$$\sum_{q=-\infty}^{\infty} e^{-\lambda q^2} e^{-i\beta q} = \sqrt{\frac{\pi}{\lambda}} \sum_{k=-\infty}^{\infty} e^{-\frac{(\beta+2\pi k)^2}{4\lambda}},$$  \hspace{1cm} (237)
from which,

\[ 1 + \sum_{q=-\infty}^{\infty} e^{-\lambda q^2} e^{-i\beta q} = \sqrt{\frac{\pi}{\lambda}} \left( e^{-\frac{\beta^2}{4\lambda}} + \sum_{k=-\infty}^{\infty} e^{-\frac{(\beta + 2\pi k)^2}{4\lambda}} \right), \] (237)

or equivalently,

\[ \sum_{q=-\infty}^{\infty} e^{-\lambda q^2} e^{-i\beta q} = \sqrt{\frac{\pi}{\lambda}} \left( e^{-\frac{\beta^2}{4\lambda}} + \sum_{k=-\infty}^{\infty} e^{-\frac{(\beta + 2\pi k)^2}{4\lambda}} \right) - 1. \] (238)

Replacing in \( F_1 \) we obtain,

\[ F_1 = -\frac{1}{8} \frac{1}{(2\pi)^d} \frac{d-1}{2} m^{d+1} \int_0^\infty dt e^{-t} \left( \sqrt{\frac{\pi}{\lambda}} \frac{e^{-\frac{\beta^2}{4\lambda}}}{t^{\nu+1}} \left( \sum_{k=-\infty}^{\infty} e^{-\frac{(\beta + 2\pi k)^2}{4\lambda}} \right) - 1 \right). \] (239)

Setting,

\[ v = \frac{d + 1}{2}, \] (240)

and the above becomes,

\[ F_1 = -\frac{1}{8} \frac{1}{(2\pi)^d} \frac{d-1}{2} m^{d+1} \int_0^\infty dt e^{-t} \left( \sqrt{\frac{\pi}{\lambda}} \frac{e^{-\frac{\beta^2}{4\lambda}}}{t^{\nu+1}} \right) \] (241)

\[ -\left( \frac{1}{8} \frac{1}{(2\pi)^d} \frac{d-1}{2} m^{d+1} \int_0^\infty dt e^{-t} \left( \frac{1}{t^{\nu+1}} \right) \right). \]

Substitute \( a = mL \) and the above relation is written \( (\lambda = \frac{a^2}{4\pi}) \),

\[ F_1 = -\frac{1}{8} \frac{1}{(2\pi)^d} \frac{d-1}{2} m^{d+1} \int_0^\infty dt e^{-t} \left( \sqrt{\frac{\pi}{\lambda}} \frac{e^{-\frac{\beta^2}{4\lambda}}}{at^{\nu+1}} \right) \] (242)

\[ -\left( \frac{1}{8} \frac{1}{(2\pi)^d} \frac{d-1}{2} m^{d+1} \int_0^\infty dt e^{-t} \left( \frac{1}{at^{\nu+1}} \right) \right). \]

After some calculations we get:

\[ F_1 = -\frac{1}{4} \frac{\sqrt{\pi}}{(2\pi)^d a} \frac{d-1}{2} m^{d+1} \int_0^\infty dt e^{-t \frac{\beta^2}{a^2+1}} t^{-\nu-\frac{1}{2}} \] (243)

\[ -\left( \frac{1}{4} \frac{\sqrt{\pi}}{(2\pi)^d a} \frac{d-1}{2} m^{d+1} \int_0^\infty dt e^{-t \left( \frac{\beta^2}{a^2+1} \right)} t^{-\nu+\frac{1}{2}} \right). \]

\[ +\left( \frac{1}{8} \frac{1}{(2\pi)^d} \frac{d-1}{2} m^{d+1} \int_0^\infty dt e^{-t} \left( \frac{1}{t^{\nu+1}} \right) \right). \]
Finally using the following,

\[
\frac{1}{(x^2 + a^2)^{\mu + 1}} = \frac{1}{\Gamma(\mu + 1)} \int_0^{\infty} dt e^{-(x^2 + a^2)t} t^\mu,
\]

we have:

\[
F_1 = -\frac{1}{4} \sqrt{\frac{\pi}{2(2\pi)^d}} (2\pi)^{\frac{d-1}{2}} m^{\nu+1} \Gamma(-\nu - \frac{1}{2} + 1)(\frac{\beta^2}{a^2} + 1)^{\nu + \frac{1}{2} - 1}
\]

\[
-\frac{1}{4} \sqrt{\frac{\pi}{2(2\pi)^d}} (2\pi)^{\frac{d-1}{2}} m^{\nu+1} \Gamma(-\nu - \frac{1}{2} + 1) \left[ \sum_{k=-\infty}^{\infty'} (1 + \left(\frac{\beta + 2\pi k}{a}\right)^2)^{\nu + \frac{1}{2} - 1} \right]
\]

\[
+ \frac{1}{8} \frac{1}{(2\pi)^d} (2\pi)^{\frac{d-1}{2}} m^{\nu+1} \Gamma(-\nu). \tag{245}
\]

Adding \(F_2\) (with \(-\beta + 2\pi k\)) we have,

\[
V_{\text{twist}} = -\frac{1}{2} \sqrt{\frac{\pi}{2(2\pi)^d}} (2\pi)^{\frac{d-1}{2}} m^{\nu+1} \Gamma(-\nu - \frac{1}{2} + 1)(\frac{\beta^2}{a^2} + 1)^{\nu + \frac{1}{2} - 1}
\]

\[
-\frac{1}{4} \sqrt{\frac{\pi}{2(2\pi)^d}} (2\pi)^{\frac{d-1}{2}} m^{\nu+1} \Gamma(-\nu - \frac{1}{2} + 1) \times \left[ \sum_{k=-\infty}^{\infty'} (1 + \left(\frac{\beta + 2\pi k}{a}\right)^2)^{\nu + \frac{1}{2} - 1} + (1 + \left(\frac{-\beta + 2\pi k}{a}\right)^2)^{\nu + \frac{1}{2} - 1} \right]
\]

\[
+ \frac{1}{4} \frac{1}{(2\pi)^d} (2\pi)^{\frac{d-1}{2}} m^{\nu+1} \Gamma(-\nu). \tag{246}
\]

The sum,

\[
\sum_{k=-\infty}^{\infty'} (1 + \left(\frac{\beta + 2\pi k}{a}\right)^2)^{\nu + \frac{1}{2} - 1} + (1 + \left(\frac{-\beta + 2\pi k}{a}\right)^2)^{\nu + \frac{1}{2} - 1}, \tag{247}
\]

is invariant under \(k \rightarrow -k\), thus:

\[
2 \sum_{k=1}^{\infty} (1 + \left(\frac{\beta + 2\pi k}{a}\right)^2)^{\nu + \frac{1}{2} - 1} + (1 + \left(\frac{-\beta + 2\pi k}{a}\right)^2)^{\nu + \frac{1}{2} - 1}, \tag{248}
\]

So we obtain:

\[
V_{\text{twist}} = -\frac{1}{2} \sqrt{\frac{\pi}{2(2\pi)^d}} (2\pi)^{\frac{d-1}{2}} m^{\nu+1} \Gamma(-\nu - \frac{1}{2} + 1)(\frac{\beta^2}{a^2} + 1)^{\nu + \frac{1}{2} - 1}
\]

\[
-\frac{1}{4} \sqrt{\frac{\pi}{2(2\pi)^d}} (2\pi)^{\frac{d-1}{2}} m^{\nu+1} \Gamma(-\nu - \frac{1}{2} + 1)(a^2)^{\frac{1}{2} - \nu}
\]

\[
\times \left[ \sum_{k=1}^{\infty} (a^2 + \left(\frac{\beta + 2\pi k}{a}\right)^2)^{\nu + \frac{1}{2} - 1} + (a^2 + \left(\beta - 2\pi k\right)^2)^{\nu + \frac{1}{2} - 1} \right]
\]

\[
+ \frac{1}{4} \frac{1}{(2\pi)^d} (2\pi)^{\frac{d-1}{2}} m^{\nu+1} \Gamma(-\nu). \tag{249}
\]

40
Depending on whether $d$ is even or odd we can Taylor expand or use the binomial expansion for the sum [1]:

$$(a^2 + b^2)^{\nu - \frac{1}{2}} = \sum_{l=0}^{\sigma} \frac{(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!l!} (a^2)^l(b^2)^{\nu - \frac{1}{2} - l}. \quad (250)$$

If $d$ is even then $\sigma = \nu - \frac{1}{2}$. If $d$ is odd, then $\sigma$ is a positive integer.

For $d$ odd, we Taylor expand:

$$V_{\text{twist}} = -\frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d-1} m^{d+1}\Gamma(-\nu + \frac{1}{2} + 1) \left(\frac{\beta^2}{a^2} + 1\right)^{\nu + \frac{1}{2} - 1} +$$

$$\frac{1}{4(2\pi)^d} (2\pi)^{d-1} m^{d+1}\Gamma(-\nu) - \frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d-1} m^{d+1}\Gamma(-\nu + \frac{1}{2} + 1) \left(\frac{\beta^2}{a^2} + 1\right)^{\nu + \frac{1}{2} - 1}$$

$$\times \left[ \sum_{k=1}^{\infty} \sum_{l=0}^{\sigma} \frac{(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!l!} (2\pi)^l((\beta + 2\pi k)^2)^{\nu - \frac{1}{2} - l} \right] +$$

$$\frac{1}{4(2\pi)^d} (2\pi)^{d-1} m^{d+1}\Gamma(-\nu) - \frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d-1} m^{d+1}\Gamma(-\nu + \frac{1}{2} + 1) \left(\frac{\beta^2}{a^2} + 1\right)^{\nu + \frac{1}{2} - 1}$$

$$\times \left[ \sum_{k=1}^{\infty} \sum_{l=0}^{\sigma} \frac{(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!l!} (2\pi)^l((\beta + 2\pi k)^2)^{\nu - \frac{1}{2} - l} \right],$$

and after calculations,

$$V_{\text{twist}} = -\frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d-1} m^{d+1}\Gamma(-\nu + \frac{1}{2} + 1) \left(\frac{\beta^2}{a^2} + 1\right)^{\nu + \frac{1}{2} - 1} +$$

$$\frac{1}{4(2\pi)^d} (2\pi)^{d-1} m^{d+1}\Gamma(-\nu) - \frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d-1} m^{d+1}\Gamma(-\nu + \frac{1}{2} + 1) \left(\frac{\beta^2}{a^2} + 1\right)^{\nu + \frac{1}{2} - 1}$$

$$\times \left[ \sum_{k=1}^{\infty} \sum_{l=0}^{\sigma} \frac{(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!l!} (2\pi)^l((\beta + 2\pi k)^2)^{\nu - \frac{1}{2} - l} \right] +$$

$$\frac{1}{4(2\pi)^d} (2\pi)^{d-1} m^{d+1}\Gamma(-\nu) - \frac{1}{2} \frac{1}{(2\pi)^d} (2\pi)^{d-1} m^{d+1}\Gamma(-\nu + \frac{1}{2} + 1) \left(\frac{\beta^2}{a^2} + 1\right)^{\nu + \frac{1}{2} - 1}$$

$$\times \left[ \sum_{k=1}^{\infty} \sum_{l=0}^{\sigma} \frac{(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!l!} (2\pi)^l((\beta + 2\pi k)^2)^{\nu - \frac{1}{2} - l} \right].$$

We use zeta regularization, expressed in terms of the Hurwitz zeta [3] [4] [34] [59] [11] [57]:

$$\zeta(s, v) = \sum_{k=0}^{\infty} \frac{1}{(k + v)^s} - \sum_{k=1}^{\infty} \frac{1}{(k + v)^s} = \zeta(s, v) - \frac{1}{v^s}. \quad (253)$$

which is defined for $0 < v \leq 1$ and the term $k + v = 0$ is omitted. In our case $v = \beta$ which contains the phase appearing in the boundary conditions. So $\omega$ must be positive ($\beta = \frac{\alpha}{2\pi}$).
Using Hurwitz zeta \[3, 4, 34, 59, 11, 57\]:

\[
V_{\text{twist}} = -\frac{1}{2} \frac{\sqrt{\pi}}{d^d \pi} (2\pi)^{d+1} m^{d+1} \Gamma(-\nu - 1 + 1)(\frac{\beta}{a^2} + 1)^{\nu + \frac{1}{2} - 1} + \frac{1}{4} \frac{1}{(2\pi)^d} \frac{\sqrt{\pi}}{d^d \pi} (2\pi)^{d+1} m^{d+1} \Gamma(-\nu)
\]

\[
- \frac{1}{2} \frac{\sqrt{\pi}}{d^d \pi} (2\pi)^{d+1} m^{d+1} \Gamma(-\nu - 1 + 1)(a^2)^{\nu + \frac{1}{2} - \frac{1}{2} - l}
\]

\[
\times \left( \sum_{k=1}^{\infty} \sum_{l=0}^{\sigma} \frac{(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!!} (a^2)^l (\zeta(-2\nu + 1 + 2l, \frac{\beta}{2\pi}) - (\frac{\beta}{2\pi})^{2\nu - l - 2l})
\right)
\]

\[
+ \sum_{k=1}^{\infty} \sum_{l=0}^{\sigma} \frac{(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!!} (a^2)^l (\zeta(-2\nu + 1 + 2l, -\frac{\beta}{2\pi}) - (-\frac{\beta}{2\pi})^{2\nu - l - 2l}) \right].
\]

The objective now is to make the $\beta$ dependence clear. For this we use the expansion of Hurwitz zeta \[1\]:

\[
\zeta(z, q) = \frac{2\Gamma(1 - z)}{(2\pi)^{1 - z}} \left( \sin \left[ \frac{\pi z}{2} \right] \sum_{n=1}^{\infty} \cos \left[ \frac{\pi q}{n^{1 - z}} \right] + \cos \left[ \frac{\pi z}{2} \right] \sum_{n=1}^{\infty} \sin \left[ \frac{2\pi q}{n^{1 - z}} \right] \right).
\]

Also the $\zeta(z, -q)$ expansion, can be found using \[34\],

\[
\zeta(1 - s, a) = \frac{\Gamma(s)}{(2\pi)^s} (e^{-i\pi s} F(s, a) + e^{i\pi s} F(s, -a)),
\]

where

\[
F(s, a) = \sum_{n=1}^{\infty} \frac{e^{2\pi i na}}{n^s}.
\]

which is valid if $Re z < 0$ and $0 < q \leq 1$

In our case $z = -2\nu + 1 + 2l$. Note that for $d = 3$, we have $-2\nu = -4$ and $-2\nu + 1 + 2l$ is negative for $l = 0, 1$. For $l = 2$ we use the Hurwitz zeta expansion, $\zeta(s, a)$, around $s = 1$, where a pole exists,

\[
\lim_{s \to 1} (\zeta(s, a) - \frac{1}{s - 1}) = -\psi_0(a).
\]

Thus we can compute $V_{\text{twist}}$ as an expansion up to order $L^{-2}$. By using dimensional regularization we Taylor expand the $d$ dependent terms around $d + \varepsilon$, $\varepsilon \to 0$ as before. Also for $d = 3$ the expression $-2\nu + 1 + 2l$ is always an odd number for all $l$. So the terms

42
are omitted. Below we quote the terms for $l = 0, 1, 2$:

\[
V_{\text{twist}} = \frac{1}{2} \sqrt{\pi} (2\pi)^{d-1} \pi m^{d+1} \Gamma(-\nu + \frac{1}{2} + 1) \left( \frac{\beta^2}{a^2} + 1 \right)^{\nu + \frac{1}{2} - 1} + \quad (259)
\]

\[
1 \frac{1}{4} \frac{1}{(2\pi)^d a} (2\pi)^{d-1} \pi m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1) (a^2)^{\frac{1}{2} - \nu}
\]

\[
\times \left[ (2\pi)^2 \nu^{-\frac{1}{2}} \frac{2\Gamma(2\nu)}{(2\pi)^{2\nu}} \right.
\]

\[
\times \left( \sin \frac{\pi(1-2\nu)}{2} \sum_{n=1}^{\infty} \cos \left[ \frac{\beta n}{n^{2\nu}} \right] + \cos \left[ \frac{\pi(1-2\nu)}{2} \right] \sum_{n=1}^{\infty} \sin \left[ \frac{\beta n}{n^{2\nu}} \right] \right.
\]

\[
+ \sin \left[ \frac{\pi(1-2\nu)}{2} \right] \sum_{n=1}^{\infty} \cos \left[ \frac{\beta n}{n^{2\nu}} \right] - \cos \left[ \frac{\pi(1-2\nu)}{2} \right] \sum_{n=1}^{\infty} \sin \left[ \frac{\beta n}{n^{2\nu}} \right] \right.
\]

\[
+ \left. \frac{(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - 1)!} a^2 \times \left( (2\pi)^2 \nu^{-\frac{1}{2}} \frac{2\Gamma(2\nu - 2)}{(2\pi)^{2\nu-2}} \sin \left[ \frac{\pi(3-2\nu)}{2} \right] \sum_{n=1}^{\infty} \cos \left[ \frac{\beta n}{n^{2\nu-2}} \right] \right.
\]

\[
+ \cos \left[ \frac{\pi(3-2\nu)}{2} \right] \sum_{n=1}^{\infty} \sin \left[ \frac{\beta n}{n^{2\nu-2}} \right] \right.
\]

\[
+ \sin \left[ \frac{\pi(3-2\nu)}{2} \right] \sum_{n=1}^{\infty} \cos \left[ \frac{\beta n}{n^{2\nu-2}} \right] - \cos \left[ \frac{\pi(3-2\nu)}{2} \right] \sum_{n=1}^{\infty} \sin \left[ \frac{\beta n}{n^{2\nu-2}} \right] \right)
\]

\[
\left. + \frac{(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - 2)!} a^4 ((2\pi)^2 \nu^{-\frac{1}{2}} - 2 \frac{\pi}{\xi} + \psi_o \left( \frac{\beta}{2\pi} \right) + \psi_o \left( -\frac{\beta}{2\pi} \right)) \right),
\]

43
poles is contained to the Hurwitz, and is of the form 

\[ V_{\text{twist}} = -\frac{1}{2} \frac{\sqrt{\pi}}{(2\pi)^d a}(2\pi)^{d-1} m^{d+1} \Gamma(-\nu - \frac{1}{2}) + 1)(\frac{\beta^2}{a^2} + 1)^{\nu + \frac{1}{2}} - 1 \]

(260)

\[ + \frac{1}{4} \frac{1}{(2\pi)^d a}(2\pi)^{d-1} m^{d+1} \Gamma(-\nu) \]

\[ - \frac{1}{2} \frac{1}{(2\pi)^d a}(2\pi)^{d-1} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)(a^2)^{\frac{1}{2} - \nu} \frac{2\Gamma(2\nu)}{(2\pi)^{2\nu - 2}} ((2\pi)^{2\nu - 2})^{\nu - \frac{1}{2}} \]

\[ \times [2\sin(\frac{\pi(1 - 2\nu)}{2}) \sum_{n=1}^{\infty} \cos(\frac{\beta n}{n^{2\nu}})] \]

\[ + \frac{(\nu - \frac{1}{2})^2}{(\nu - \frac{1}{2} - 1)!} a^2 ((2\pi)^{2\nu - 2} 2\sin(\frac{\pi(3 - 2\nu)}{2}) \sum_{n=1}^{\infty} \cos(\frac{\beta n}{n^{2\nu - 2}}) \]

\[ + \frac{(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - 2)!} a^4 ((2\pi)^{2\nu - 2} 2\sin(\frac{\pi(3 - 2\nu)}{2}) \sum_{n=1}^{\infty} \cos(\frac{\beta n}{n^{2\nu - 2}}) \]

\[ + O(\epsilon, \epsilon^2 \text{and higher}), \]

with \( \beta = 2\pi \omega, \nu = \frac{d+1}{2}, \ a = mL \). The sums appearing above are:

\[ \sum_{n=1}^{\infty} \cos(\frac{\beta n}{n^{2\nu}}) = \frac{1}{2} (Li_{2\nu}(e^{-i\beta}) + Li_{2\nu}(e^{i\beta})), \]

(261)

and

\[ \sum_{n=1}^{\infty} \cos(\frac{\beta n}{n^{2\nu - 2}}) = \frac{1}{2} (Li_{2\nu - 2}(e^{-i\beta}) + Li_{2\nu - 2}(e^{i\beta})). \]

(262)

Let us see how the poles cancel in the above expressions. In the case \( d = 3 \) one of the poles is contained to the Hurwitz, and is of the form \( \frac{\pi}{\epsilon} \) with \( \epsilon \to 0 \). The other pole is contained to the expression \( \frac{1}{4} \frac{1}{(2\pi)^d a}(2\pi)^{d-1} m^{d+1} \Gamma(-\nu) \). Thus we have:

\[ V_{\text{twist}} = -\frac{m^4}{16 \pi^2} + \frac{m^4}{16 \pi^2} + \left(3 \frac{m^4}{64 \pi^2} - \frac{m^4}{32 \pi^2}\right) \]

\[ + \frac{m^4 \left( 1 + \frac{\beta^2}{\pi^2} \right)}{6 \pi \alpha} + 2 \frac{m^4 \sqrt{\alpha^2 \cos(\beta)}}{\pi^2 \alpha^5} + \frac{m^4 \ln(2)}{16 \pi^2} \]

\[ + \frac{m^4 \ln(\pi)}{32 \pi^2} + \frac{m^4 \ln(\pi)}{32 \pi^2} \]

\[ - \frac{m^4 \ln(\alpha^2)}{32 \pi^2} - \frac{m^4 \psi(-\frac{3}{2})}{32 \pi^2} - \frac{m^4 \psi(-\frac{3}{2})}{32 \pi^2} \]

\[ + \frac{m^4 \psi(-\frac{1}{2})}{32 \pi^2} + \frac{m^4 \psi(-\frac{1}{2})}{32 \pi^2} + \frac{m^4 \psi(-\frac{1}{2})}{32 \pi^2}. \]

(263)

We can see how the poles cancel. The last expression is the vacuum energy in the case that arbitrary phases appear.
3.4 Some Applications II

3.4.1 Extra Dimensional Models with Twisted Boundary Conditions

Let us now briefly present an application of the twisted potential case we computed above. In models with large extra dimensions, supersymmetry can be broken in the bulk by the Scherk-Schwarz mechanism, as we described briefly in the introduction. Consider the immediate extra dimensional extension of the MSSM in five dimensions on the orbifold $S^1/Z_2$ [61, 64, 67, 68]. Assume that supersymmetry breaking occurs in the bulk through the Scherk-Schwarz mechanism [66]. Thus the fields have the following boundary conditions,

$$\Phi(x^\mu, y + 2\pi R) = e^{2\pi i q_\Phi} \Phi(x^\mu, y).$$

The Scherk-Schwarz mechanism consists in using different parameters $q_\Phi$ for fermions and bosons belonging to the same hypermultiplet. The harmonic expansion of the fields for circle compactification is,

$$\Phi(x^\mu, y) = \sum_{n=-\infty}^{\infty} \Phi_n(x) e^{i2\pi(n + q_\Phi) y / R}.$$  \hfill (265)

In the case of the $S^1/Z_2$ orbifold compactification, the $Z_2$ even fields have harmonic expansion,

$$\Phi(x^\mu, y) = \sum_{n=-\infty}^{\infty} \Phi_n(x) \cos \frac{2\pi(n + q_\Phi) y}{R},$$

while the $Z_2$ odd fields,

$$\Phi(x^\mu, y) = \sum_{n=-\infty}^{\infty} \Phi_n(x) \sin \frac{2\pi(n + q_\Phi) y}{R}.$$  \hfill (267)

The $Z_2$ even fields have zero modes and produce the 4 dimensional MSSM, while the $Z_2$ odd don’t have zero modes. The Kaluza-Klein modes within each hypermultiplet have masses,

$$m^2_B = \frac{(n + q_B)^2}{R^2},$$  \hfill (268)

for the boson case, and for the fermion case the mass reads,

$$m^2_F = \frac{(n + q_F)^2}{R^2}.$$  \hfill (269)

In the orbifold extra dimensional extension, the electroweak symmetry breaking occurs through radiative corrections to the Higgs mass. So it is necessary to include one loop corrections to the appropriate mass eigenstate Higgs scalar field mass (for more details see [68, 67]). The one loop corrected mass is induced by a tower of KK states and is equal to,

$$m^2_\phi(\phi = 0) = \frac{d^2 V(\phi)}{d\phi^2}.$$  \hfill (270)

45
with \( V(\phi) \) given by,

\[
V(\phi) = \frac{1}{2} \text{Tr} \sum_{n=-\infty}^{\infty} \int \frac{d^4p}{(2\pi)^4} \ln \left[ \frac{p^2 + \frac{(n+q)^2}{R^2} + M^2(\phi)}{p^2 + \frac{(n+q)^2}{R^2} + M^2(\phi)} \right].
\]

(271)

In the above, \( M^2(\phi) \) is the \( \phi \)-dependent mass of the KK states which are model dependent. It is obvious that the effective potential (271) is identical to (192) which was computed in the previous section. Thus the Scherk-Schwarz phases are like twists in the boundary conditions. The calculation follows as we described above. See also [34, 3].

### 3.5 An Alternative Elegant Approach. Epstein Zeta Functions

In this section we briefly present a much more elegant and more elegant computation method for the effective potential. Consider a massive scalar field quantized in \( T^N \times R^n \) with periodic boundary conditions in each of the torii, that is,

\[
\phi(x_i) = \phi(x_i + L_i),
\]

(272)

with \( x_i \) the coordinates describing the torii and \( L_i \) the torii radii. The zeta function corresponding to this setup is [34, 3, 53, 54, 77, 78],

\[
\zeta(s, L_i) = (2\pi)^{-n} \sum_{n_1,...,n_N=-\infty}^{\infty} \int d^nk \left[ \left( \frac{2\pi n_1}{L_1} \right) + \cdots + \left( \frac{2\pi n_N}{L_N} \right) + k^2 + M^2 \right]^{-s},
\]

(273)

The general summations can be written in terms of the Epstein zeta function. Indeed after performing the integration in relation (273), we obtain,

\[
\zeta(s, w_i) = \left( \frac{\sqrt{\pi}}{L_1} \right)^n \frac{\Gamma(s - n/2)}{\Gamma(s)} \left( \frac{L_1}{2\pi} \right)^{2s} Z^2_N \left( s - n/2; w_1, ..., w_N \right),
\]

(274)

with \( w_i = (L_1/L_i)^2 \). In the above we used the generalized Epstein zeta function,

\[
Z^2_N \left( s; w_1, ..., w_N \right) = \sum_{n_1,...,n_N=-\infty}^{\infty} [w_1n_1^2 + \cdots + w_Nn_N^2 + v^2]^{-s}.
\]

(275)

The interested reader can consult the references [77, 78, 3, 34], where the subject is developed in great detail.

### 3.6 Twisted Sections and Non Trivial Topology

One question that one might ask is if there a criterion or more correctly a way to know which are the allowed boundary conditions for a field in a specific topology. The answer can be given in terms of the allowed sections of the fibre bundles that the spacetime topology corresponds to.

Non trivial topology affects the fields entering the Lagrangian (twisted fields) (see for example [74, 76, 75, 61]). In our case, the topological properties of \( S^1 \times R^3 \) are classified
by the first Stieffel class $H^1(S^1 \times R^3, Z_2)$ which is isomorphic to the singular (simplicial) cohomology group $H_1(S^1 \times R^3, Z_2)$ because of the triviality of the $Z_2$ sheaf. It is known that $H^1(S^1 \times R^3, Z_2) = Z_2$ classifies the twisting of a bundle. Specifically, it describes and classifies the orientability of a bundle globally. In our case, the classification group is $Z_2$ and, we have two locally equivalent bundles, which are however different globally (like in the case of the cylinder and that of the moebius strip where both locally resemble $S^1 \times R$). The mathematical lying behind, is to find the sections that correspond to these two fibre bundles, and which are classified by $Z_2$ [74]. The sections we used are real scalar fields and Majorana or Dirac spinor fields. These carry a topological number called moebiosity (twist), which distinguishes between twisted and untwisted fields. The twisted fields obey anti-periodic boundary conditions, while untwisted fields periodic in the compact dimension. In the finite temperature case one takes scalar fields to obey periodic and fermion fields anti-periodic boundary conditions, disregarding all other configurations that may arise from non trivial topology. We shall consider all these configurations. Let $\varphi_u, \varphi_t$ and $\psi_t, \psi_u$ denote the untwisted and twisted scalar and twisted and untwisted spinor fields respectively. The boundary conditions in the $S^1$ dimension read,

$$\varphi_u(x, 0) = \varphi_u(x, L), \quad (276)$$

and

$$\varphi_t(x, 0) = -\varphi_t(x, L), \quad (277)$$

for scalar fields and

$$\psi_u(x, 0) = \psi_u(x, L), \quad (278)$$

and

$$\psi_t(x, 0) = -\psi_t(x, L), \quad (279)$$

for fermion fields, where $x$ stands for the remaining two spatial and one time dimension which are not affected by the boundary conditions. Spinors (both Dirac and Majorana), still remain Grassmann quantities. The untwisted fields are assigned twist $h_0$ (the trivial element of $Z_2$) and the twisted fields twist $h_1$ (the non trivial element of $Z_2$). Recall that $h_0 + h_0 = h_0 \ (0 + 0 = 0), \ h_1 + h_1 = h_0 \ (1 + 1 = 0), \ h_0 + h_0 = h_1 \ (1 + 0 = 1)$. We require the Lagrangian to be scalar under $Z_2$ thus to have $h_0$ moebiosity. Thus the topological charges flowing at the interaction vertices must sum to $h_0$ under $H^1(S^1 \times R^3, Z_2)$. For supersymmetric models, supersymmetry transformations impose some restrictions on the twist assignments of the superfield component fields [76].

No other field configuration is allowed to take non zero vev but the untwisted scalars. This is due to Grassmann nature of the vacuum or space dependent vacuum solutions that other configurations imply.

In the general case when the spacetime has topology $(S^1)^q \times R^{4-q}$, then the topologically allowed field configurations are classified by the representations of $H^1((S^1)^q \times R^{4-q}, Z_2) = Z_2^q$. Thus the different inequivalent twists that can be assigned are $2^q$. This means that we can have $2^q$ topologically inequivalent spin 0 scalars, spin 1/2 Majorana fermions and spin 3/2 Majorana fermions (this for supergravity). For our case $q = 1$.

47
It worths mentioning equivalent mathematical setups that exist in the literature. Twisted fields have frequently been used, for example as we seen in the Scherk-Schwarz mechanism [66] for supersymmetry breaking in our 4-dimensional world, where the harmonic expansion of the fields is of the form:

\[
\phi(x, y) = e^{imy} \sum_{n=-\infty}^{\infty} \phi_n(x) e^{i2\pi ny/L},
\]

(280)

The "m" parameter incorporates the twist mentioned above. This treatment is closely related to automorphic field theory [90] in more than 4 dimensions (which is an alternative to the one used by us).

Concerning the automorphic field theory, due to the compact dimension we can use generic boundary conditions for bosons and fermions in the compact dimension which are,

\[
\varphi_i(x_2, x_3, \tau, x_1) = e^{i\pi n_1 \alpha} \varphi_i(x_2, x_3, \tau, x_1 + L)
\]

(281)

\[
\Psi(x_2, x_3, \tau, x_1) = e^{i\pi n_1 \delta} \Psi(x_2, x_3, \tau, x_1 + L),
\]

with, \(0 < \alpha, \delta < 1, i = 1, 2, n_1 = 1, 2, 3,...\). The values \(\alpha = 0, 1\) correspond to periodic and antiperiodic bosons respectively while \(\delta = 0, 1\) corresponds to periodic and anti-periodic fermions [90].

### 3.7 The Validity of Approximations. Numerical Tests

Let us check numerically one of our results. We focus on the bosonic contribution at high temperature. We shall study the convergence properties of our approximation and how

\[
V_{\text{boson}} = -\sum_{q=1}^{\infty} \frac{1}{(2\pi)^d} \frac{1}{2} m^{d+1} \frac{K_{d+1}(\frac{mq}{T})}{\left(\frac{mq}{2T}\right)^{d+1}}.
\]

(282)

Figure 1: Plot of the dependence of \(V_{\text{boson}}/m^{d+1}\) as a function of \(m/T\). Numerical approximation of Bessel sum. 5-dimensional bosonic theory at finite temperature.

the semi-analytic results behave in comparison to the numerical evaluation of the potential. As we seen, before the high temperature limit was taken, the bosonic contribution is given by:

\[
V_{\text{boson}} = \frac{1}{(2\pi)^d} \frac{1}{2} m^{d+1} \frac{K_{d+1}(\frac{mq}{T})}{\left(\frac{mq}{2T}\right)^{d+1}}.
\]

(282)
Figure 2: Plot of the dependence of $V_{\text{boson}}/m^{d+1}$ as a function of $m/T$. Semi-analytic approximation. 5-dimensional bosonic theory at finite temperature.

Figure 3: Comparison of numerical and corresponding semi-analytic approximation. 5-dimensional bosonic theory at finite temperature.

After the high temperature limit was taken, the effective potential is given by the semi-analytic approximation:

$$V_{\text{boson}} = -\frac{1}{2} \frac{\sqrt{a}}{(2\pi)^d} \frac{2\pi}{d} m^{d+1} \Gamma(-\nu + \frac{1}{2} + 1)$$

$$+ \frac{1}{4} \frac{(2\pi)^{d+1}}{(2\pi)^d} m^{d+1} \Gamma(-\nu)$$

$$- \frac{\sqrt{a}}{(2\pi)^d} \frac{2\pi}{d} m^{d+1} \Gamma(-\nu - \frac{1}{2} + 1)(a^2)^{\nu^2 - \nu}$$

$$\times \sum_{l=0}^{\sigma} \frac{(2\pi)^{\nu - \frac{1}{2} - l}(\nu - \frac{1}{2})!}{(\nu - \frac{1}{2} - l)!} (a^2)^{l} \zeta(-2\nu + 1 + 2l).$$
The converge of (283) and (282) is quite fast. Also the two relations describe the same physics and are identical as can be checked. Particularly this holds even if we keep only a few terms of (283). We have checked this for values of $m/T$ that our approximation is valid, that is $m/T < 1$. Also this holds for several dimensions. Let us study the finite temperature limit of a 5-dimensional theory, that is for $d = 4$. In figure 1 we plot the dependence of $V_{\text{boson}}/m^{d+1}$ as a function of $m/T$, where $V_{\text{boson}}$ is given by the Bessel sum of relation (282). A numerical calculation is done for the sum over the Bessel functions. Also in figure 2 we plot the dependence of $V_{\text{boson}}/m^{d+1}$ as a function of $m/T$, with $V_{\text{boson}}$ given by the semi-analytic approximation of relation (283). In addition, in figure 3 we compare the above results. As we can see the two results are identical for a large range of the expansion parameter $m/T$. This shows us that in the high temperature limit ($m/T < 1$)

Figure 4: Plot of the dependence of $V_{\text{boson}}/m^{d+1}$ as a function of $m/T$. Numerical approximation of Bessel sum. 4-dimensional bosonic theory at finite temperature.

Figure 5: Plot of the dependence of $V_{\text{boson}}/m^{d+1}$ as a function of $m/T$. Semi-analytic approximation. 4-dimensional bosonic theory at finite temperature.

the semi-analytic expressions we obtained are in complete agreement to the numerical values. This holds regardless the number of terms of the semi-analytic expansion we keep. Thus the expansion is perturbative and valid. The same analysis can be done for the $d = 4$ case. We present the results in figures 4, 5 and 6. Thus within the perturbative limits the
semi-analytic approximation is valid and exponentially converging as expected (see also [3]).
Acknowledgements

The author would like to thank the referee of Reviews in Mathematical Physics for invaluable comments and suggestions that improved significantly the quality and appearance of the paper.
References

[1] I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals Series and Products (Academic Press, 1965).

[2] H. Casimir, Proc. Kon. Nederl. Akad. Wet. 51 793 (1948)

[3] E. Elizalde, A Romeo, Rev. Math. Phys. 1, 113 (1989); E. Elizalde, J. Phys. A39, 6299, 2006; E. Elizalde, "Ten physical applications of spectral zeta functions", Springer (1995); E. Elizalde, J. Math. Phys. 35,6100 (1994); E. Elizalde, A. Romeo, J. Math. Phys. 30, 1133 (1989); E. Elizalde, A. Romeo, Phys. Rev. D40, 436 (1989)

[4] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, "Zeta regularization techniques and applications", World Scientific (1994)

[5] M. Bordag, K. Kirsten Phys. Rev. D53, 5753 (1996)

[6] M. Bordag, Klaus Kirsten, J.S. Dowker, Commun. Math. Phys. 182, 371 (1996)

[7] M. Bordag, B. Geyer, K. Kirsten, E. Elizalde, Commun. Math. Phys. 179, 215 (1996)

[8] M. Bordag, E. Elizalde, K. Kirsten, J. Math. Phys. 37, 895 (1996)

[9] G. Lambiase, V.V. Nesterenko, Michael Bordag, J. Math. Phys. 40, 6254 (1999)

[10] M. Bordag, U. Mohideen, V.M. Mostepanenko, Phys. Rept. 353, 1 (2001)

[11] E. Elizalde, J. Phys. A41, 304040 (2008)

[12] E. Elizalde, J. Phys. A39, 6299 (2006)

[13] E. Elizalde, J. Phys. A39, 6725 (2006)

[14] E. Elizalde, A.C. Tort, Mod. Phys. Lett. A19, 111 (2004)

[15] E. Elizalde, F.C. Santos, A.C. Tort, J. Phys. A35, 7403 (2002)

[16] E. Elizalde, A.C. Tort, Phys. Rev. D66, 045033 (2002)

[17] G. Cognola, E. Elizalde, K. Kirsten, J. Phys. A34, 7311 (2001)

[18] E. Elizalde, Michael Bordag, K. Kirsten, J. Phys. A31, 1743 (1998)

[19] E. Elizalde, Commun. Math. Phys. 198, 83 (1998)

[20] M. Bordag, E. Elizalde, K. Kirsten , S. Leseduarte, Phys. Rev. D56, 4896 (1997)

[21] K. Kirsten, E. Elizalde, Phys. Lett. B365, 72 (1996)

[22] G. Plunien, B. Muller, W. Greiner, Phys. Rept. 134, 87 (1986)

[23] R. Obousy, G. Cleaver, arXiv:0810.1096
[24] E. Ponton, E. Poppitz, JHEP, 06, 019 (2001)
[25] E. Elizalde, K. Kirsten, Yu. Kubyshin, Z. Phys. C70, 159 (1996)
[26] E. Elizalde, J. Math. Phys. 35, 3308 (1994)
[27] E. Elizalde, Klaus Kirsten, J. Math. Phys. 35, 1260 (1994)
[28] E. Elizalde, Z. Phys. C44, 471 (1989)
[29] E. Elizalde, S. Nojiri, Sergei D. Odintsov, S. Ogushi Phys. Rev. D67, 063515 (2003)
[30] K. A. Milton, J. Phys. A37, R209 (2004)
[31] K. A. Milton, Phys. Rev. D68, 065020 (2003)
[32] Iver H. Brevik, K.A. Milton, S.D. Odintsov, K.E. Osetrin, Phys. Rev. D62, 064005 (2000)
[33] R. Kantowski, K.A. Milton, Phys. Rev. D36, 3712 (1987)
[34] Spectral functions in mathematics and physics, Klaus Kirsten (2001) Chapman Hall/CRC, Boca Raton, FL, 2001.
[35] M. Bordag, E. Elizalde and K. Kirsten, J. Math. Phys. 37, 895 (1996)
[36] M. Bordag, E. Elizalde, K. Kirsten and S. Leseduarte, Phys. Rev. D56, 4896 (1997)
[37] E. Elizalde, M. Bordag and K. Kirsten, J. Phys. A31, 1743 (1998)
[38] E. Elizalde, S. Naftulin, S.D. Odintsov, Phys. Rev. D49, 2852 (1994)
[39] E. Elizalde, S. Nojiri, S.D. Odintsov and S. Ogushi, Phys. Rev. D67, 063515 (2003)
[40] I L Buchbinder and S D Odintsov, Int. J. Mod. Phys. A4, 4337 (1989); Fortshrt. Phys. 37, 225 (1989)
[41] S. D. Odintsov, Sov. Phys. J. 31, 695 (1988)
[42] E. Elizalde, S. D. Odintsov and S. Leseduarte, Phys. Rev D49, 5551 (1994)
[43] I. brevik, K Milton, S. Nojiri and S. D. Odintsov, Nucl. Phys. B599, 305 (2001)
[44] S. D. Odintsov, Sov. Phys. J. 27, 554 (1984)
[45] I. L. Buchbinder, S.D. Odintsov Sov. Phys. J. 26, 359 (1983)
[46] S. D. Odintsov, Mod. Phys. Lett. A3, 1391 (1988)
[47] S. D. Odintsov, Phys. Lett. B306, 233 (1993)
[48] E. Elizalde, S. D. Odintsov, A. Romeo, J. Math. Phys. 37, 1128 (1996)
[49] E. Elizalde, S. D. Odintsov, A. Romeo, Phys. Rev. D54, 4152 (1996)
[50] E. Elizalde, S. Nojiri, S. D. Odintsov, S. Ogushi, Phys. Rev. D67, 063515 (2003)
[51] S. D. Odintsov, Sov. J. Nucl. Phys. 46, 1080 (1987)
[52] K. Kirsten, J. Phys. A26, 2421 (1993)
[53] K. Kirsten, J. Phys. A25, 6297 (1992)
[54] K. Kirsten, J. Phys. A24, 3281 (1991)
[55] Vicente Di Clemente, Yuri A. Kubyshin, Nucl. Phys. B636, 115 (2002)
[56] K. Kirsten, J. Math. Phys. 35, 459 (1994)
[57] Klaus Kirsten, J. Math. Phys. 32, 3008 (1991)
[58] Joseph. I. Kapusta, Finite Temperature Field Theory, Cambridge Monographs on Mathematical Physics, (1989)
[59] E. C. Titchmarsh, The Theory of the Riemann Function, Oxford At the Clarendon Press, (1951)
[60] Ashok Das, Finite Temperature Field Theory, World Scientific, (1997)
[61] G. Denardo and E. Spallucci, Nucl. Phys. B169, 514 (1980)
[62] L. Van. Hove, Phys. Rep. 137, 11 (1988), Nucl. Phys. B207, 15 (1982); D. Bailin and A. Love, Supersymmetric Gauge Field Theory and String Theory, Institute of Physics Publishing 2003
[63] M. Quiros, hep-ph/0606153; hep-ph/0302189; hep-ph/9901312
[64] I. Antoniadis, Phys. Lett. B246, 377, (1990)
[65] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G. R. Dvali, Phys. Lett. B436, 257, (1998)
[66] J. Scherk, J. H. Schwarz, Phys. Lett. B82, 60, (1979); Nucl. Phys. B153, 61 (1979)
[67] A. Pomarol, M. Quiros, Phys. Lett. B438, 255 (1998)
[68] A. Delgado, A. Pomarol, M. Quiros, Phys. Rev. D60, 095008 (1999)
[69] M. Quiros, hep-ph/9901312
[70] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G. R. Dvali, Phys. Lett. B436, 257, (1998)
[71] G. F. R. Ellis, Gen. Rel. Grav. 2, 7 (1971)
[72] E. Elizalde, S. Nojiri, S. D. Odintsov, S. Ogushi, Phys. Rev. D67, 063515 (2003)

[73] C. W. Bernard, Phys. Rev. D9, 3312 (1974); L. Dolan and R. Jackiw, Phys. Rev. D9, 3320 (1974)

[74] S. J. Avis, C. J. Isham, Commun. Math. Phys. 72, 103 (1980); C. J. Isham, Proc. R. Soc. London. A362, 383 (1978), A364, 591 (1978), A363, 581 (1978)

[75] L. H. Ford, Phys. Rev. D21, 933 (1980); D. J. Toms, Phys. Rev. D21, 2805 (1980); Phys. Rev. D21, 928 (1980); Annals. Phys. 129, 334 (1980); Phys. Lett. A77, 303 (1980)

[76] Yu. P. Goncharov, A. A. Bytsenko, Phys. Lett. B163, 155 (1985); Phys. Lett. B168, 239 (1986); Phys. Lett. B169, 171 (1986); Phys. Lett. B160, 385 (1985); Class. Quant. Grav. 8:L211, 1991; Class. Quant. Grav. 8:2269, 1991; Class. Quant. Grav. 4:555, 1987; Nucl. Phys. B271, 726 (1986)

[77] E. Elizalde, K. Kirsten, J. Math. Phys. 35, 1260 (1994)

[78] K. Kirsten, J. Phys. A24, 3281 (1991)

[79] J. S. Dowker, R. Banach, J. Phys. A11, 2255 (1978)

[80] J. S. Dowker, R. Banach, J. Phys. A12, 2527 (1979)

[81] I. Antoniadis, Phys. Lett. B246, 377, (1990)

[82] A. A. Bytsenko, E. Elizalde, S. Zerbini, Phys. Rev. D64, 105024, (2001)

[83] N. P. Landsman, Nucl. Phys. B322, 498 (1989)

[84] V. K. Oikonomou, J. Phys. A40, 5725, 2007

[85] V. K. Oikonomou, J. Phys. A40, 9929, 2007

[86] V. K. Oikonomou, work in preparation

[87] S. P. Martin, A supersymmetry primer, hep-ph/9709356; Phys. Rev. D65, 116003(2002)

[88] Graham. D. Kribs, Tasi 2004 Lectures on the Phenomenology of Extra Dimensions, hep-ph/0605325

[89] B. Alles, J. Soto, J. Taron, Z. Phys. C39, 489 (1988)

[90] J. S. Dowker, R. Banach, J. Phys. A11, 2255 (1978)

[91] E.J. Ferrer, V. de la Incera, A. Romeo, Phys. Lett. B515, 341 (2001)

[92] L. Randall, R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999); Phys. Rev. Lett. 83, 4690 (1999)