On Čech-completeness of the space of order-preserving functionals
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Abstract

In this paper we establish that if a Tychonoff space $X$ is Čech-complete then the space $O_\tau(X)$ of all $\tau$-smooth order-preserving, weakly additive and normed functionals is also Čech-complete.

Key words: order-preserving functional, Čech-complete space.

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1 Introduction

Nowadays the study of nonlinear functionals becomes one of attractive fields in topology and functional analysis. For example, in the papers [1] – [3] so called max-plus or idempotent linear functionals have been considered. Such functionals have various applications in mathematical physics, economics, dynamical optimization and other fields. Another example of nonlinear functionals is given by the notion of capacity on compacta introduced by Choquet [4]. In [5] various classes of nonlinear functionals have been considered and investigated. Order-preserving, weakly additive functionals on the algebra of continuous functions on a given compact Hausdorff space were considered in [6]. Note that nonnegative linear functionals, max-plus functionals and capacities are order-preserving and weakly additive.

In this paper we consider the functor $O_\tau$ of order-preserving $\tau$-smooth functionals in the category $\text{Tych}$ of Tychonoff spaces and their continuous mappings, and for a given Tychonoff space $X$ we establish that if $X$ is Čech-complete, then the space $O_\tau(X)$ of $\tau$-smooth order-preserving functionals is also Čech-complete.

Earlier proven the Baire category theorem for the class of all completely metrizable spaces has been generalized for the case of Čech-complete spaces. Therefore one of the important questions in the theory of covariant functors is whether a functor preserves Čech-completeness of spaces.

We denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{N}$ the set of all nonnegative integers, spaces mean topological spaces, $\overline{A}$ is the closure of $A$ in the considering space, compact is compact Hausdorff topological space, maps (or mappings) are continuous ones.

2 On properties of order-preserving functionals

Let $X$ be a compact and let $C(X)$ be the Banach algebra of all continuous real-valued functions with the usual algebraic operations and with the sup-norm. For functions $\varphi, \psi \in C(X)$ we shall write $\varphi \leq \psi$ if $\varphi(x) \leq \psi(x)$ for all $x \in X$. If $c \in \mathbb{R}$ then by $c_X$ we denote the constant function which is identically equal to $c$.

We recall that a functional $\mu : C(X) \rightarrow \mathbb{R}$ is said [6] to be:

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1) order-preserving if for any pair \( \varphi, \psi \in C(X) \) of functions the inequality \( \varphi \leq \psi \) implies \( \mu(\varphi) \leq \mu(\psi) \);
2) weakly additive if \( \mu(\varphi + cX) = \mu(\varphi) + c\mu(1_X) \) for all \( \varphi \in C(X) \) and \( c \in \mathbb{R} \);
3) normed if \( \mu(1_X) = 1 \).

For a compact \( X \) we denote by \( W(X) \) the set of all order-preserving weakly additive functionals \( \mu : C(X) \to \mathbb{R} \), and by \( O(X) \) we denote the set of all normed functionals from \( W(X) \). It is clearly that \( W(X) \) is a subset in the product \( \mathbb{R}^{C(X)} \). Equip \( W(X) \) with the induced topology, which coincides with the pointwise convergence topology on \( W(X) \). Note that in this topology the space \( O(X) \) is compact, for any given compact \( X \).

For a given continuous map \( f : X \to Y \) between compacta by

\[
O(f)(\mu)(\varphi) = \mu(\varphi \circ f), \quad \varphi \in C(Y),
\]

we define the map \( O(f) : O(X) \to O(Y) \). Thus we get the functor \( O \) acting in the category \( Comp \).

Order-preserving, weakly additive and normed functionals for brevity we will call as order-preserving functionals.

Note that according to proposition 1 in [7] each order-preserving functional is continuous.

Let \( A \) be a closed subset of a given compact \( X \). Since the functor \( O : Comp \to Comp \) is monomorphic [6], i.e. preserves embeddings of closed subsets of compacta, we may assume that \( O(A) \) is a subset of the compact \( O(X) \). Moreover the embedding of \( O(A) \) into \( O(X) \) is given by means of the mapping \( O(i_A) : O(A) \to O(X) \), where \( i_A : A \to X \) is the identical embedding. Therefore the following notion is well defined.

Given a closed subset \( A \) of the compact \( X \), an order-preserving functional \( \mu \in O(X) \) is said to be supported on \( A \) if \( \mu \in O(A) \). The set

\[
\text{supp}\mu = \cap \{ A : \mu \in O(A) \text{ and } A \text{ is closed in } X \}
\]

is called the support of the order-preserving functional \( \mu \).

Let \( X \) be a compact. For a closed \( F \subset X \), an open \( U \subset X \), real numbers \( \alpha \) and \( c \) put

if \( \alpha \geq 0 \), then \( \mu(\alpha X_F + cX) = \inf \{ \mu(\varphi) : \varphi \in C(X), \varphi \geq \alpha X_F \} + c\mu(1_X) \), \hspace{1cm} (1^+) 
if \( \alpha < 0 \), then \( \mu(\alpha X_F + cX) = \sup \{ \mu(\varphi) : \varphi \in C(X), \varphi \leq \alpha X_F \} + c\mu(1_X) \), \hspace{1cm} (1^-) 
if \( \alpha \geq 0 \), then \( \mu(\alpha X_U + cX) = \sup \{ \mu(\varphi) : \varphi \in C(X), \varphi \leq \alpha X_U \} + c\mu(1_X) \), \hspace{1cm} (2^+) 
if \( \alpha < 0 \), then \( \mu(\alpha X_U + cX) = \inf \{ \mu(\varphi) : \varphi \in C(X), \varphi \geq \alpha X_U \} + c\mu(1_X) \). \hspace{1cm} (2^-)

In this manner the functional \( \mu \) given on \( C(X) \) can be extended onto the following set

\[
\mathcal{X}(X) = C(X) \cup \{ \alpha X_A + cX : \alpha, c \in \mathbb{R}, A \text{ is open or closed in } X \}.
\]

Obviously the set \( \mathcal{X}(X) \) is an \( A \)-subspace of the space \( B(X) \) of all bounded functions on \( X \), i.e. \( 0_X \in \mathcal{X}(X) \) and for any \( \zeta \in \mathcal{X}(X) \) and each \( c \in \mathbb{R} \) we have \( \zeta + cX \in \mathcal{X}(X) \).

Hence by the version of the Hahn-Banach theorem [9] it follows that \( \mu \) has an extension \( \tilde{\mu} \) on \( B(X) \) such that \( \tilde{\mu}|\mathcal{X}(X) = \mu \).

The following result has been proved in [8, Th. 1.3].
Theorem 1. Let $A$ be an arbitrary subspace of the given compact $X$, $\mu \in O(X)$. Then for each extension $\tilde{\mu}$ satisfying $(1^\pm) - (2^\pm)$, the following conditions are equivalent:

(i) $\tilde{\mu}(n\chi_A) = 0$ for all integers $n$.
(ii) $\tilde{\mu}(\alpha\chi_A) = 0$ for all $\alpha \in \mathbb{R}$.
(iii) $\tilde{\mu}(\alpha\chi_{X\setminus A}) = \alpha$ for all $\alpha \in \mathbb{R}$.
(iv) $\tilde{\mu}(\varphi\chi_A) = \mu(\varphi)$ for all $\varphi \in C(X)$.
(v) $\tilde{\mu}(\varphi\chi_{X\setminus A}) = \mu(\varphi)$ for all $\varphi \in C(X)$.
(vi) $\text{supp}\mu \subset X\setminus A$.

We need the following statement.

Proposition 1. For each open subset $U$ of a given compact $X$ and for an arbitrary $\varepsilon > 0$ the sets

$$\langle U; \varepsilon \rangle = \{ \mu \in O(X) : \mu(\chi_U) > \varepsilon \} \text{ and } \langle U; -\varepsilon \rangle = \{ \mu \in O(X) : \mu(-\chi_U) < -\varepsilon \}$$

are open in $O(X)$ with respect to pointwise convergence topology.

Proof. For each $\varphi \in C(X)$ and $\varepsilon > 0$ put

$$\langle \varphi; \varepsilon \rangle = \{ \mu \in O(X) : \mu(\varphi) > \varepsilon \}.$$

From the definition of pointwise convergence topology follows that the set $\langle \varphi; \varepsilon \rangle$ is open in $O(X)$.

Consider the following set

$$\Phi = \{ \varphi \in C(X) : 0_X \leq \varphi \leq \chi_U, \text{ there exists } \mu \in O(X) \text{ such that } \mu(\varphi) > \varepsilon \}.$$

It is clear that $\Phi \neq \emptyset$.

We have

$$\langle U; \varepsilon \rangle = \bigcup_{\varphi \in \Phi} \langle \varphi; \varepsilon \rangle,$$

i. e. the set $\langle U; \varepsilon \rangle$ is open in $O(X)$. Analogously, the set $\langle U; -\varepsilon \rangle$ is also open in $O(X)$. Proposition 1 is proved.

We note that the sets

$$\langle U; \varepsilon \rangle_\lambda = \{ \mu \in O(X) : \mu(\lambda\chi_U) > \varepsilon \}, \quad \lambda > 0,$$

and

$$\langle U; -\varepsilon \rangle_\lambda = \{ \mu \in O(X) : \mu(\lambda\chi_U) < -\varepsilon \}, \quad \lambda < 0,$$

are also open in $O(X)$.

3 Order-preserving $\tau$-smooth functionals

Let $X$ be a Tychonoff space and let $C_b(X)$ be the algebra of all bounded continuous real-valued functions with the pointwise algebraic operations. For a function $\varphi \in C_b(X)$ put $\|\varphi\| = \sup\{|\varphi(x)| : x \in X\}$. $C_b(X)$ with this norm is a Banach algebra. For a net $\{\varphi_\alpha\} \subset C_b(X)$ the notation $\varphi_\alpha \downarrow 0_X$ means that for every point $x \in X$ at $\beta \succ \alpha$ implies the inequality $\varphi_\alpha(x) \geq \varphi_\beta(x)$, and $\lim_\alpha \varphi_\alpha(x) = 0_X$. In this case we say that $\{\varphi_\alpha\}$ is a monotone decreasing net pointwise convergent to zero.
For a Tychonoff space $X$ we denote by $\beta X$ its Stone-Čech compact extension. Given any function $\varphi \in C_b(X)$ consider its continuous extension $\tilde{\varphi} \in C(\beta X)$. This gives an isomorphism between the spaces $C_b(X)$ and $C(\beta X)$. Moreover, $\|\tilde{\varphi}\| = \|\varphi\|$, i.e. this isomorphism is an isometry, and the topological properties of the above spaces coincide. Therefore one may consider any function from $C_b(X)$ as an element of $C(\beta X)$.

**Definition 1.** An order-preserving functional $\mu \in W(\beta X)$ is said to be strongly $\tau$-smooth if $\mu(\pm \varphi_\alpha) \to 0$ for each monotone net $\{\varphi_\alpha\} \subset C(\beta X)$ decreasing to zero on $X$.

For brevity strongly $\tau$-smooth functionals we will call $\tau$-smooth functionals.

For a Tychonoff space $X$ we denote by $W_\tau(X)$ the set of all $\tau$-smooth order-preserving functionals from $W(\beta X)$. The set $W_\tau(X)$ is equipped with the pointwise convergence topology. The base of neighborhoods of a functional $\mu \in W_\tau(X)$ in the pointwise convergence topology consists of the sets

$$\langle \mu; \varphi_1, \ldots, \varphi_k; \varepsilon \rangle = \{ \nu \in W(\beta X) : |\nu(\varphi_i) - \mu(\varphi_i)| < \varepsilon \} \cap W_\tau(X)$$

where $\varphi_i \in C(\beta X)$, $i = 1, \ldots, k$ and $\varepsilon > 0$.

Put

$$O_\tau(X) = \{ \mu \in W_\tau(X) : \mu(1_X) = 1 \},$$

It is easy to see that equipped with pointwise convergence topology $O_\tau(X)$ is Tychonoff space.

Let $X$ and $Y$ be Tychonoff spaces and let $f : X \to Y$ be a continuous map. Put

$$O_\tau(f) = O(\beta f)|O_\tau(X),$$

where $\beta f : \beta X \to \beta Y$ is the Stone-Čech extension of $f$.

Note that the map $O_\tau(f) : O_\tau(X) \to O_\tau(Y)$ is defined correctly and it is also continuous. Thus we obtain the operation $O_\tau$ as a functor acting in the category $\text{Tych}$ of Tychonoff spaces and their maps [8].

The following theorem, which gives an equivalent definition of $\tau$-smoothness of order-preserving functionals, was proved in [8, Th. 2.2].

**Theorem 2.** A functional $\mu \in O(\beta X)$ is $\tau$-smooth if and only if $\mu(\varphi_\alpha) \to \mu(\varphi)$ and $\mu(-\varphi_\alpha) \to \mu(-\varphi)$ for any monotone increasing net of continuous functions $\{\varphi_\alpha\} \subset C(\beta X)$, pointwise converging to $\varphi \in C(\beta X)$ on $X$.

Now we shall prove a result which can be considered as another equivalent definition of $\tau$-smoothness of order-preserving functionals.

**Theorem 3.** Functional $\mu \in O(\beta X)$ is $\tau$-smooth if and only if $\mu(\lambda \chi_K) = 0$ for each compact $K \subset \beta X \setminus X$ and for every $\lambda \in \mathbb{R}$.

**Proof.** Let $\mu \in O(\beta X)$ be an arbitrary $\tau$-smooth functional and let $K \subset \beta X \setminus X$ be a compact. Consider a monotone decreasing net $\Phi = \{\varphi_\alpha\} \subset C(\beta X)$ such that $\varphi_\alpha \geq \lambda \chi_K$ and $\inf\{\mu(\varphi_\alpha) : \varphi_\alpha \in \Phi\} = \inf\{\mu(\varphi) : \varphi \in C(\beta X), \varphi \geq \lambda \chi_K\} \equiv \mu(\lambda \chi_K)$, where $\lambda > 0$. Then $\mu(\varphi_\alpha) \to 0$. Hence, $\mu(\lambda \chi_K) = 0$. From the construction immediately follows that for every compact $K \subset \beta X \setminus X$ for all $\lambda \in \mathbb{R}$ the equality $\mu(\lambda \chi_K) = 0$ takes place.

Conversely, let $\mu \in O(\beta X)$ be a functional such that $\mu(\lambda \chi_K) = 0$ for any compact $K \subset \beta X \setminus X$ and for an arbitrary $\lambda \in \mathbb{R}$. Consider an arbitrary net $\Phi = \{\varphi_\alpha\} \subset$
$C(\beta X)$, $\varphi_\alpha \downarrow 0_X$. It is sufficient to check the value $\lambda = 1$. Without loss of generality we may assume that $\varphi_\alpha \leq 1_\beta X$ for all $\alpha$.

For an arbitrary real number $\varepsilon > 0$ consider the set

$$Z_\alpha = \{x \in \beta X : \varphi_\alpha(x) \geq \varepsilon \}.$$  It is clear that $K_\varepsilon := \bigcap_\alpha Z_\alpha \subset \beta X \setminus X$. Moreover it is easy to see that for $\alpha < \beta$ the inclusion $Z_\alpha \supset Z_\beta$ is valid. Therefore $\chi_{Z_\alpha} \geq \chi_{Z_\beta}$ for $\alpha < \beta$. Hence, $\mu(\chi_{Z_\alpha}) \geq \mu(\chi_{Z_\beta})$. In other words the net $\{\mu(\chi_{Z_\alpha})\}$ is decreasing. From the equality $K_\varepsilon = \bigcap_\alpha Z_\alpha$ it follows that the net $\{\mu(\chi_{Z_\alpha})\}$ is bounded from below by $\mu(\chi_{K_\varepsilon})$. Moreover, $\mu(\chi_{Z_\alpha}) \to \mu(\chi_{K_\varepsilon})$. Hence one has

$$\mu(\varphi_\alpha) = \tilde{\mu}(\varphi_\alpha) = \tilde{\mu}(\varphi_\alpha \chi_{Z_\alpha} + \varphi_\alpha \chi_{\beta X \setminus Z_\alpha}) \leq (\text{since } \tilde{\mu} \text{ is order-preserving}) \leq$$

$$\leq \tilde{\mu}(\chi_{Z_\alpha} + \varepsilon \beta X) = (\text{by the weak additivity of } \tilde{\mu}) = \tilde{\mu}(\chi_{Z_\alpha}) + \varepsilon \mu(1_\beta X) =$$

$$= (\text{by definition of } \tilde{\mu}) = \mu(\chi_{Z_\alpha}) + \varepsilon \mu(1_\beta X) \to \mu(\chi_{K_\varepsilon}) + \varepsilon \mu(1_\beta X) = \varepsilon \mu(1_\beta X).$$

Now taking $\varepsilon$ convergent to zero we have $\mu(\varphi_\alpha) \to 0$.

One can similarly establish the convergence $\mu(-\varphi_\alpha) \to 0$. Theorem 3 is proved.

According to Theorems 1 and 3 we can write

$$O_\tau(X) = \{\mu \in O(\beta X) : \mu(\chi_K) = 0 \text{ for each compact } K \subset \beta X \text{ such that } K \cap X = \emptyset \text{, and for all integers } n\}. \quad (3)$$

4 The main result

Recall that a topological space $X$ is called to be Čech-complete if $X$ is a Tychonoff space and for some (or, equivalently, for every) compactification $bX$ of the space $X$ the remainder $bX \setminus X$ is an $F_\sigma$-set in $bX$ [10, P. 196].

The following statement is the main result of the paper.

**Theorem 4.** If a Tychonoff space $X$ is Čech-complete then the space $O_\tau(X)$ is also Čech-complete.

**Proof.** Let a Tychonoff space $X$ be Čech-complete. Then the remainder $\beta X \setminus X$ is an $F_\sigma$-set in $\beta X$, i.e. in $\beta X$ there exist closed subsets $K_n$, $n \in \mathbb{N}$ such that $\beta X \setminus X = \bigcup_{n=1}^\infty K_n$. Hence $X = \bigcap_{n=1}^\infty U_n$ is a $G_\delta$-set in $\beta X$, where $U_n = \beta X \setminus K_n$ are open in $\beta X$ sets.

Put $F_n = \bigcup_{i=1}^n K_n$, $V_n = \beta X \setminus F_n$. Then $F_n \subset F_{n+1}$ for all $n \in \mathbb{N}$ and $\beta X \setminus X = \bigcup_{n=1}^\infty F_n$.

According to (3) for each $\mu \in O_\tau(X)$ we have $\mu(\lambda \chi_{F_n}) = 0$ for all $n = 1, 2, \ldots$ and $\lambda \in \mathbb{N}$. From this by Theorem 1 it follows that $\mu(\lambda \chi_{V_n}) = \lambda$. Consider decreasing sequence $\{r_n\}$ of positive numbers such that $\lim_{n \to \infty} r_n = 0$. Then for any $\lambda \in \mathbb{N}$ we have $\mu \in (V_n; (\lambda - r_n)\lambda) \cap (V_n; -(\lambda - r_n)\lambda)$ (here without loss of generality we assumed that $r_n < \lambda$ for all $n \in \mathbb{N}$). From this we have the following inclusion

$$O_\tau(X) \subset \bigcap_{\lambda=1}^\infty \bigcap_{n=1}^\infty ((V_n; (\lambda - r_n)\lambda) \cap (V_n; -(\lambda - r_n)\lambda)). \quad (4)$$
Let now \( \mu \in \bigcap_{\lambda=1}^{\infty} \bigcap_{n=1}^{\infty} (\langle V_n; (\lambda - r_n) \rangle_{\lambda} \cap \langle V_n; - (\lambda - r_n) \rangle_{\lambda}) \). Then for all positive integers \( \lambda \) and \( n \) we will have \( \mu(\lambda \chi_{V_n}) > \lambda - r_n \) and \( \mu(-\lambda \chi_{V_n}) < -\lambda + r_n \). Hence \( \mu(\lambda \chi_{\bigcap_{n=1}^{\infty} V_n}) \geq \lambda \) and \( \mu(-\lambda \chi_{\bigcap_{n=1}^{\infty} V_n}) \leq -\lambda \), because \( \{V_n\} \) is a decreasing (with respect to inclusion) sequence. But then \( \mu(\lambda \chi_{\bigcap_{n=1}^{\infty} V_n}) = \lambda \) for all integers \( \lambda \), since \( \mu \) is order-preserving and normed. Hence \( \mu(\lambda \chi_{\bigcap_{n=1}^{\infty} F_n}) = 0 \).

If \( K \subset \beta X \setminus X \) is an arbitrary compact then \( \mu(\lambda \chi_K) = 0 \) for all integers \( \lambda \), since \( K \subset \bigcup_{n=1}^{\infty} F_n \). Therefore (3) implies that \( \mu \in O_{\tau}(X) \). This means that

\[
\bigcap_{\lambda=1}^{\infty} \bigcap_{n=1}^{\infty} (\langle V_n; (\lambda - r_n) \rangle_{\lambda} \cap \langle V_n; - (\lambda - r_n) \rangle_{\lambda}) \subset O_{\tau}(X). \quad (5)
\]

The inclusions (4) and (5) imply the equality

\[
O_{\tau}(X) = \bigcap_{\lambda=1}^{\infty} \bigcap_{n=1}^{\infty} (\langle V_n; (\lambda - r_n) \rangle_{\lambda} \cap \langle V_n; - (\lambda - r_n) \rangle_{\lambda}). \quad (6)
\]

The equality (6) means that \( O_{\tau}(X) \) is a \( G_{\delta} \)-set in \( O(\beta X) \). Therefore the remainder \( O(\beta X) \setminus O_{\tau}(X) \) is an \( F_{\sigma} \)-set in the compactification \( bO_{\tau}(X) \) defined as \( bO_{\tau}(X) = O(\beta X) \). Theorem 4 is proved.

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