THE INTEGRAL CHOW RINGS OF MODULI OF WEIERSTRASS FIBRATIONS

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Abstract. We compute the Chow rings with integral coefficients of moduli stacks of minimal Weierstrass fibrations over the projective line. For each integer $N \geq 1$, there is a moduli stack $W^\text{min}_N$ parametrizing minimal Weierstrass fibrations with fundamental invariant $N$. Following work of Miranda and Park–Schmitt, we give a quotient stack presentation for each $W^\text{min}_N$. Using these presentations and equivariant intersection theory, we determine a complete set of generators and relations for each of the Chow rings. For the cases $N = 1$ (respectively, $N = 2$), parametrizing rational (respectively, K3) elliptic surfaces, we give a more explicit computation of the relations.

1. Introduction

The study of the Chow rings of moduli spaces has played a central role in algebraic geometry ever since Mumford’s introduction of an intersection product for the moduli space of curves $\mathcal{M}_g$ and its compactification by stable curves. Mumford’s intersection product requires the use of rational coefficients, but Totaro [Tot99] and Edidin–Graham [EG98a] developed an intersection theory for quotient stacks that works with integral coefficients. Many moduli stacks of interest in algebraic geometry, including the moduli stacks of curves, are quotient stacks.

Chow rings with integral coefficients are often quite difficult to compute, but in turn they have a much richer structure than their rational counterparts: for instance, rational Chow rings of moduli of hyperelliptic curves are trivial, but the integral ones are not (see [Vis98, EF09, DL21]).

Only a few examples have been computed in full for moduli stacks of curves $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ with $g$ and $n$ small (see [DLFV21, Lar21, DLV21, DLPV21, Inc22]), and even less is known for moduli stacks parametrizing higher dimensional varieties.

In this paper, we study integral Chow rings of certain moduli stacks of surfaces that we denote $W^\text{min}_N$, indexed by an integer $N \geq 1$. The stacks $W^\text{min}_N$ parametrize surfaces called minimal Weierstrass fibrations over $\mathbb{P}^1$. The arithmetic and geometry of moduli spaces of minimal Weierstrass fibrations over $\mathbb{P}^1$ has already been the subject of investigation of several works (see for instance [Mir81, HP19, PS21, CK23]). Moreover, the stack $W^\text{min}_2$ is of particular interest, as it can be regarded as the moduli stack of elliptic K3 surfaces with a section (equivalently, K3 surfaces polarized by a hyperbolic lattice).

Minimal Weierstrass fibrations over $\mathbb{P}^1$ are flat, proper morphisms $p : X \to \mathbb{P}^1$ together with a section $s : \mathbb{P}^1 \to X$ satisfying the following conditions:

1. $X$ is normal, irreducible, with at most ADE singularities;
2. every fiber of $p$ is isomorphic to an elliptic curve, a rational curve with a node, or a rational curve with a cusp;
3. the section does not intersect the singular point of any of the fibers.

These fibrations arise naturally from contracting the components of the fibers of a smooth elliptic surface over $\mathbb{P}^1$ that do not meet the section. Associated to a minimal Weierstrass fibration is a fundamental invariant $N = \deg(R^1p_*\mathcal{O}_X) \geq 0$. For each $N \geq 1$, we consider moduli stacks $W^\text{min}_N$ parametrizing minimal Weierstrass fibrations with fundamental invariant $N$. 

1
Our main result is the following. For a more precise formulation, see Theorem 5.5.

**Theorem 1.1.** Suppose that the ground field has characteristic $\neq 2, 3$ and let $N \geq 1$ be an integer. Then

1. for $N$ odd, we have
   \[ CH^*(W_N^{\min}) \simeq \mathbb{Z}[c_1, c_2]/I_N \]
   where the generators are Chern classes of a certain rank two vector bundle $E_N$ and the ideal of relations $I_N$ is generated by $(N+2)$ relations, of which one has degree $8N+1$ and the others have degree $9k+m$ for $1 \leq k \leq N$ and $0 \leq m \leq k$. Explicit formulas for these relations are given in (14).

2. for $N$ even, we have
   \[ CH^*(W_N^{\min}) \simeq \mathbb{Z}[\tau_1, c_2, c_3]/(2c_3, I_N) \]
   where the generators are Chern classes of certain vector bundles $L_N$ and $E_N$ and the ideal of relations $I_N$ is generated by $(N+2)$ relations, of which one has degree $8N+1$ and the others have degree $9k+m$ for $1 \leq k \leq N$ and $0 \leq m \leq k$. Explicit formulas for these relations are given in (15) and (16).

Perhaps the most interesting cases are when $N = 1$ and $N = 2$. Minimal Weierstrass fibrations with fundamental invariant 1 are rational. They arise from the elliptic fibrations obtained by blowing up the base points of a pencil of cubics in $\mathbb{P}^2$. Moduli spaces of rational elliptic fibrations are well studied, and in particular are closely related to several other interesting moduli problems [Vak1]. Minimal Weierstrass fibrations with fundamental invariant 2 come from elliptic K3 surfaces. The intersection theory with rational coefficients of moduli spaces of K3 surfaces has been the subject of much recent research and is expected to behave analogously to that of the moduli space of curves [CK23, MOP17, PY20].

Specializing our Theorem 1.1 to $N = 1, 2$, we obtain the following completely explicit result.

**Theorem 1.2.** Suppose that the ground field has characteristic $\neq 2, 3$. Then

1. the integral Chow ring of the moduli stack $W_1^{\min}$ of rational elliptic surfaces with a section is
   \[ \mathbb{Z}[c_1, c_2]/(6c_1c_2r_6, c_1^2r_6, c_1^2c_2r_6) \]
   where the generators are Chern classes of a certain rank two vector bundle $E_1$ and
   \[ r_6 = 576(30c_1^6 + 151c_1c_2^2 + 196c_1^2c_2^2 + 64c_2^3); \]

2. the integral Chow ring of the moduli stack $W_2^{\min}$ of elliptic K3 surfaces with a section is
   \[ \mathbb{Z}[\tau_1, c_2, c_3]/(2c_3, r_9, r_{10}, r_{18}, r_{19}) \]
   where the generators are Chern classes of certain vector bundles $L_2$ and $E_2$ and
   \[ r_9 = 1152(691c_2^4 \tau_1 - 38005c_2^3 \tau_1^3 + 309568c_2^2 \tau_1^5 - 497520c_2 \tau_1^7 + 124416 \tau_1^9), \]
   \[ r_{10} = 1152(30c_2^3 - 6811c_2^2 \tau_1^2 + 133495c_2 \tau_1^4 - 481528c_2 \tau_1^6 + 327600c_2 \tau_1^8 - 20736 \tau_1^{10}), \]
   \[ r_{18} = 1152c_2^4(108314154642930c_2^4 + 1045672c_2^3 \tau_1^2 - 89483c_2^2 \tau_1^4 + 35c_2 \tau_1^6 - 4 \tau_1^8), \]
   \[ r_{19} = 2304c_2^5 \tau_1(118203201c_2^3 + 180502c_2^2 \tau_1^2 - 7c_2 \tau_1^4 + 4 \tau_1^6). \]

When $N \geq 2$, the rational Chow rings $CH^*(W_N^{\min}) \otimes \mathbb{Q}$ have been computed by the first author and Kong [CK23]: in particular, they proved that only $r_9$ and $r_{10}$ are needed in order to generate the ideal of relations with rational coefficients. This implies that there are many torsion classes in $CH^*(W_N^{\min})$, so the theory with integral coefficients is genuinely different from that with rational coefficients and contains much more information.
Structure of the paper. In Section 2, we construct the moduli stacks $\mathcal{W}^\text{min}_N$ as quotient stacks, following the work of Miranda [Mir81] who constructed coarse spaces for $\mathcal{W}^\text{min}_N$ and Park–Schmitt [PS21], who constructed $\mathcal{W}^\text{min}_N$ as quotients of a weighted projective stack by $\text{PGL}_2$. Our approach is slightly different from that of Park–Schmitt, but we will show that the two approaches coincide.

In Section 3, we discuss the equivariant intersection theory of projective spaces, which is a key tool in the proof of Theorem 1.1. In particular, we obtain generators for $\text{CH}^*(\mathcal{W}^\text{min}_N)$.

In Sections 4 and 5, we compute relations among the generators of the Chow ring that result from excising the locus of non-minimal Weierstrass fibrations, finishing the proof of Theorem 1.1.

Finally, in Section 6 we make explicit calculations of the relations in the cases $N = 1$ and $N = 2$, proving Theorem 1.2.

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2. MODULI OF MINIMAL WEIERSTRASS FIBRATIONS

In this Section, after recalling some basic definitions, we give in Theorem 2.9 a presentation as a quotient stack of $\mathcal{W}^\text{min}_N$, the moduli stack of minimal Weierstrass fibrations with fundamental invariant $N$. We also introduce some vector bundles $\mathcal{L}_N$ and $\mathcal{E}_N$ (see Definition 2.11 and Definition 2.13) which will be relevant for our computations.

We adopt the following convention: given a locally free sheaf $\mathcal{E}$ over a scheme $X$, the associated vector bundle $E$ is given by $\text{Spec}_{\mathcal{O}_X}(\text{Sym}(\mathcal{E}^*))$ and the projectivization of this vector bundle $\mathbb{P}(E)$ is therefore $\text{Proj}_{\mathcal{O}_X}(\text{Sym}(\mathcal{E}^*))$. Observe that with this convention, given $\pi : \mathbb{P}(E) \to X$, we have $\pi_*\mathcal{O}_{\mathbb{P}(E)}(1) \simeq \mathcal{E}^*$.

2.1. Basic notation. We collect here some basic notation that is used throughout the paper; in this section, we work over $\text{Spec}(\mathbb{Z})$. Let $G_N := \text{Aut}(\mathbb{P}^1, \mathcal{O}(N))$ be the automorphism group of the pair $(\mathbb{P}^1, \mathcal{O}(N))$. More precisely, an automorphism $\phi = (\phi_0, \phi_1)$ consists of a pair where $\phi_0 : \mathbb{P}^1 \to \mathbb{P}^1$ is an automorphism of $\mathbb{P}^1$ and $\phi_1 : \mathcal{O}(N) \xrightarrow{\sim} \phi_0^*\mathcal{O}(N)$ is an automorphism of line bundles.

We make the group $G_N$ act linearly on the $\mathbb{Z}$-module $V^N_N := H^0(\mathbb{P}^1, \mathcal{O}(N))$ as follows: given a global section $\sigma : \mathcal{O} \to \mathcal{O}(N)$, we define $\phi \cdot \sigma = (\phi_0, \phi_1) \cdot \sigma$ to be the composition

$$\mathcal{O} \xrightarrow{\text{can}} \phi_0^*\mathcal{O} \xrightarrow{\phi_0^*\sigma} \phi_0^*\mathcal{O}(N) \xrightarrow{\phi_1^{-1}} \mathcal{O}(N).$$

Given any positive integer $r$, we have a surjective homomorphism

$$p^N_{N,r} : G_N \to G_{N,r}, \quad \phi = (\phi_0, \phi_1) \mapsto (\phi_0, \phi_1^r).$$

We can use this homomorphism to define an action of $G_N$ on $V^N_{N,r}$ as

$$\sigma \mapsto (\phi_0, \phi_1^r) \cdot \sigma.$$

In what follows, we will denote this $G_N$-module as $V^N_{N,r}$. We will also use the notation $V^N_{N,r_1, N,r_2}$ to denote $V^N_{N,r_1} \oplus V^N_{N,r_2}$.

The $G_N$-modules that we just introduced can be made more explicit as follows. First, observe that $G_1 \simeq \text{GL}_2$, and that there is a surjective morphism

$$G_1 \simeq \text{GL}_2 \to G_N, \quad (\phi_0, \phi_1) \mapsto (\phi_0, \phi_1^r).$$
where the subgroup consists of the root of unity $\mu_N \subset GL_2$, embedded diagonally. The action of $G_N$ on $V^N_N$, after identifying $G_N$ with $GL_2/\mu_N$, can then be written as

$$[A] \cdot f((x,y)) := f(A^{-1}(x,y)).$$

Moreover, by [AV04, Proposition 4.4] we have isomorphisms

$$GL_2 \xrightarrow{\sim} GL_2 / \mu_N, \quad A \mapsto [\det(A)^{-\frac{N-1}{2}} A]$$

for $N$ odd, and

$$\mathbb{G}_m \times PGL_2 \xrightarrow{\sim} GL_2 / \mu_N, \quad (\alpha, A) \mapsto [\alpha^{\frac{1}{N}} \det(A)^{-\frac{1}{N}} A]$$

for $N$ even.

We can use these isomorphisms to describe the $G_N$-modules $V^N_N$ as $GL_2$-modules (resp. $\mathbb{G}_m \times PGL_2$) for $N$ odd (resp. for $N$ even). Denoting by $E$ the standard $GL_2$-module, and by $L$ the standard $\mathbb{G}_m$-module with trivial $PGL_2$-action, we obtain:

$$V^N_N \simeq \det(E)^{\frac{N-1}{2}} \otimes \text{Sym}^N(E^r),$$

for $N$ odd,

$$V^N_N \simeq L^{-\frac{r}{2}} \otimes V^N_{N^r} \simeq L^{-\frac{r}{2}} \otimes H^0(\mathbb{P}^1, \mathcal{O}(Nr)),$$

for $N$ even,

where in the last line we are endowing $H^0(\mathbb{P}^1, \mathcal{O}(Nr))$ with the trivial $\mathbb{G}_m$-action and the obvious $PGL_2$-action.

2.2. Stacks of conics with sections. In this section, we keep working over Spec($\mathbb{Z}$). We start by introducing a stack, denoted $\mathcal{F}_N^{\text{min}}$, which admits a natural presentation as a quotient stack (Proposition 2.3). The reason for introducing this stack is that, once restricted to Spec($\mathbb{Z}[\frac{1}{N}]$), it will turn out to be isomorphic to $W^\text{min}_N$, the moduli stack of minimal Weierstrass fibrations.

**Definition 2.1.** We denote by $\mathcal{F}_N^{\text{min}}$ the following fibered category over $\mathcal{S}ch/\mathbb{Z}$, the category of schemes over Spec($\mathbb{Z}$).

**Objects:** The objects over a scheme $T$ are tuples $(f : \mathcal{P} \to T, \mathcal{L}, A, B)$ consisting of a flat, proper morphism of finite presentation $\mathcal{P} \to T$ with geometric fibers isomorphic to $\mathbb{P}^1$, a line bundle $\mathcal{L}$ over $\mathcal{P}$ of degree $N$ along each fiber of $f$, and two sections $A, B$ of $H^0(\mathcal{P}, \mathcal{L}^\otimes 1)$ and $H^0(\mathcal{P}, \mathcal{L}^\otimes 6)$ respectively. We require the sections $A$ and $B$ to satisfy the following two conditions:

1. for each geometric point $s$ of $T$, the global section $4A^3 + 27B^2$ of $\mathcal{L}^{\otimes 12}$ is not zero once restricted to $\mathcal{P}_s$, and
2. for each geometric point $s$ of $T$, there is no point $p$ of $\mathbb{P}^1 \simeq \mathcal{P}_s$ such that $A_s$ (resp. $B_s$) vanishes in $p$ with order $\geq 4$ (resp. with order $\geq 6$).

**Morphisms:** A morphism $(f : \mathcal{P} \to T, \mathcal{L}, A, B) \to (f' : \mathcal{P}' \to T', \mathcal{L}', A', B')$ consists of a morphism $T \to T'$, together with two isomorphisms $\phi : \mathcal{P} \to \mathcal{P}' \times_T T$ and $\psi : \mathcal{L} \to \phi^* \mathcal{L}'$, such that $\psi$ sends $A$ (resp. $B$) to $A'$ (resp. $B'$).

**Definition 2.2.** We define the $G_N$-invariant closed subscheme $\Delta_N$ in $V^N_{4N^6N}$ as the union of $\Delta^1_N$ and $\Delta^2_N$, where

- the subscheme $\Delta^1_N$ is the locus of pairs $(A, B)$ such that $4A^3 + 27B^2 = 0$, and
- the subscheme $\Delta^2_N$ is the locus of pairs $(A, B)$ such that there exists a point $p \in \mathbb{P}^1$ such that $A$ (resp. $B$) vanishes in $p$ with order $\geq 4$ (resp. with order $\geq 6$).

The following Proposition gives a presentation of $\mathcal{F}_N^{\text{min}}$ as a quotient stack.

**Proposition 2.3.** There is an isomorphism $\mathcal{F}_N^{\text{min}} \cong [(V^N_{4N^6N} \setminus \Delta_N)/G_N]$.
Proof. Our argument follows [AV04, Theorem 4.1]. It suffices to construct a map \((V_{4n,6n}^N \setminus \Delta_N) \to \mathcal{T}^\min_N\) which is a \(G_N\)-torsor.

The data of a map \(T \to V_{4n,6n}^N\) is equivalent to a section of the projection \(\pi_2 : V_{4n,6n}^N \times T \to T\). Let \(p : \mathbb{P}^1 \times T \to T\) be the second projection. Since \(\pi_2\) is affine, a section of \(\pi_2\) induces a morphism

\[\text{Sym}^\bullet_{\mathcal{O}_T}(p_*(\mathcal{O}_{\mathbb{P}^1 \times T}(4N) \oplus \mathcal{O}_{\mathbb{P}^1 \times T}(6N)))' \to \mathcal{O}_T.\]

This is the same as a map \(p_*(\mathcal{O}_{\mathbb{P}^1 \times T}(4N) \oplus \mathcal{O}_{\mathbb{P}^1 \times T}(6N))' \to \mathcal{O}_T\) that in turn is equivalent to a map \(\mathcal{O}_T \to p_*(\mathcal{O}_{\mathbb{P}^1 \times T}(4N) \oplus \mathcal{O}_{\mathbb{P}^1 \times T}(6N))\), namely a choice of a pair of sections \(A \in H^0(T, p_\ast \mathcal{O}_{\mathbb{P}^1 \times T}(4N))\) and \(B \in H^0(T, p_\ast \mathcal{O}_{\mathbb{P}^1 \times T}(6N))\), equivalently a pair of sections of \(H^0(\mathbb{P}^1 \times T, \mathcal{O}_{\mathbb{P}^1 \times T}(4N))\) and \(H^0(\mathbb{P}^1 \times T, \mathcal{O}_{\mathbb{P}^1 \times T}(6N))\).

In particular, the data of a morphism \(T \to V_{4n,6n}^N \setminus \Delta_N\) is equivalent to the data of two sections \((A, B)\) of \(H^0(\mathbb{P}^1 \times T, \mathcal{O}_{\mathbb{P}^1 \times T}(4N))\) and \(H^0(\mathbb{P}^1 \times T, \mathcal{O}_{\mathbb{P}^1 \times T}(6N))\) such that for each geometric point \(s\) of \(T\), the restrictions of \(A\) and \(B\) over \(\mathbb{P}^1_s\) do not verify the two conditions given in Definition 2.2.

There is a natural transformation \(\Phi : (V_{4n,6n}^N \setminus \Delta_N) \to \mathcal{T}^\min_N\), that on objects is defined as follows. Given a map \(T \to V_{4n,6n}^N \setminus \Delta_N\), which corresponds to two sections \(A, B\) as above, we can associate the object of \(\mathcal{T}^\min_N\) given by \((\mathbb{P}^1 \times T, \mathcal{O}_{\mathbb{P}^1 \times T}(N), A, B)\).

Let \(\sigma : G_N \times (V_{4n,6n}^N \setminus \Delta_N) \to (V_{4n,6n}^N \setminus \Delta_N)\) be the map that defines the action of \(G_N\) on \(V_{4n,6n}^N \setminus \Delta_N\), and denote \(pr_2 : (V_{4n,6n}^N \setminus \Delta_N) \to (V_{4n,6n}^N \setminus N)\) the projection on the second factor.

We claim that \(\Phi\) is a \(G_N\)-torsor. We need to show that:

1. The two arrows \(\Phi \circ \sigma\) and \(\Phi \circ pr_2\) are isomorphic,
2. For every scheme \(T\) and every object \((f : \mathcal{P} \to T, \mathcal{L}, A, B)\) of \(\mathcal{T}^\min_N\)(\(T\)), there is an étale cover \(T' \to T\) such that the pull-back \((f' : \mathcal{P}' \to T', \mathcal{L}', A', B')\) is isomorphic to an object of \((V_{4n,6n}^N \setminus \Delta_N)(T')(\text{i.e. it is in the essential image of } \Phi)\), and
3. If \(\alpha := (f' : \mathcal{P}' \to T', \mathcal{L}', A', B')\) is in the essential image of \(\Phi\), the action of \(G_N\) on its essential fiber (i.e. the pairs \((\beta, \gamma)\) consisting of an element \(\beta \in (V_{4n,6n}^N \setminus \Delta_N)(T)\) and an isomorphism \(\Phi(\beta) \to \alpha\) is simply transitive.

To check point (1), we construct explicitly the isomorphism: for every \((\phi, A, B)\) in \(G_N(T) \times (V_{4n,6n}^N \setminus \Delta_N)\) we define the 2-morphism

\[\Phi \circ pr_2((\phi, A, B)) = (\mathbb{P}^1_T \to T, \mathcal{O}(N), A, B) \mapsto (\mathbb{P}^1_T \to T, \mathcal{O}(N), \phi \cdot A, \phi \cdot B) = (\Phi \circ \sigma)((\phi, A, B)).\]

to be exactly \((\phi \circ \alpha, \phi \circ \beta)\), where \(\phi \cdot A\) (resp. \(\phi \cdot B\)) is the action introduced in Section 2.1. To check (2), observe that \(f : \mathcal{P} \to T\) is a Severi-Brauer scheme [Gro66, Corollaire 8.3], hence there is an étale cover \(T' \to T\) and an isomorphism \(\mathcal{P} \times_T T' \cong \mathbb{P}^1 \times T'\) over \(T'\). Then if we denote by \(\mathcal{L}'\) the pull-back of \(\mathcal{L}\) to \(\mathbb{P}^1 \times T'\) we have two line bundles, \(\mathcal{L}'\) and \(\mathcal{O}_{\mathbb{P}^1 \times T'}(N)\) that are isomorphic along each fiber: in particular, for every point \(s\) of \(T\) we have \(H^1(\mathbb{P}^1 \times \{s\}, \mathcal{L}'(-N)) = 0\), hence [Har13, Theorem III.12.11] we obtain that \(\mathcal{G} := pr_2(\mathcal{L}' \otimes \mathcal{O}_{\mathbb{P}^1 \times T'}(N))\) is a line bundle, from which we immediately deduce that, up to replacing \(T'\) with a covering that trivializes \(\mathcal{G}\), we can assume that \(\mathcal{L}' \cong \mathcal{O}_{\mathbb{P}^1 \times T'}(N)\). This proves point (2).

To check point (3) it suffices to recall that the functor sending a scheme \(T\) to \(\text{Aut}_T(\mathbb{P}^1_T, \mathcal{O}_{\mathbb{P}^1}(N))\) is represented by \(G_N\) (see [AV04, Proof of Theorem 4.1]). Indeed, we need to check that:

- The action of \(G_N\) is transitive on the fibers of \((V_{4n,6n}^N \setminus \Delta_N) \to \mathcal{T}^\min_N\), and
- The action is simply transitive (this is analogous to the representability of \([\mathcal{T}^\min_N\]

To check the first bullet point, we need to check that if two objects of \(\mathcal{T}^\min_N\) that belong to the image of \(\Phi(T)\) are isomorphic, then there is an element of \(G_N(T)\) which sends the first one to the second one. To check the second bullet point, we need to check that such an element is unique. Both bullet points follow since \(\text{Aut}_T(\mathbb{P}^1_T, \mathcal{O}_{\mathbb{P}^1}(N))\) is represented by \(G_N\).
2.3. Moduli of Weierstrass fibrations. In this section, we work over Spec(ℤ[1/4]). Let \( \mathcal{W}^{\text{min}}_N \) be the moduli stack of minimal Weierstrass fibrations, as defined in [PS21, Section 4.2]. We will prove in Proposition 2.8 that \( \mathcal{W}^{\text{min}}_N \) is isomorphic to the stack \( \mathcal{F}^{\text{min}}_N \) that we introduced before. We start by recalling the relevant definitions from loc. cit..

**Definition 2.4.**

- A Weierstrass fibration over an algebraically closed field \( k \) is a proper, flat morphism \( f : X \to \mathbb{P}^1_k \) with geometrically integral fibers from an integral scheme \( X \) together with a section \( s : \mathbb{P}^1_k \to X \) such that every geometric fiber is either an elliptic curve, a rational curve with a node or a rational curve with a cusp, the generic fiber is smooth and the section \( s([\mathbb{P}^1_k]) \) does not contain any singular point of the fibers.
- A Weierstrass fibration has degree \( N \) if the line bundle \((R^1f_*\mathcal{O}_X)^\vee\) has degree \( N \).
- A Weierstrass fibration is minimal if it is a Weierstrass model of a smooth elliptic surface over \( \mathbb{P}^1_k \) with a section (see [Mir89, Section 1] for more details).

**Definition 2.5.** A family of minimal Weierstrass fibrations of degree \( N \) over a scheme \( T \) is the data of:

1. a flat, proper morphism of finite presentation \( \mathcal{P} \to T \) with geometric fibers isomorphic to \( \mathbb{P}^1 \), and
2. a flat, proper morphism of finite presentation \( f : X \to \mathcal{P} \) with a section \( S \subseteq X \).

We require that for every geometric point \( p \in T \), the fiber \((X_p, S_p) \to \mathcal{P}_p \) is a minimal Weierstrass fibration of degree \( N \), and we refer the reader to [Mir89] for a more detailed exposition on Weierstrass fibrations.

Given two families \((\mathcal{X}, S) \to \mathcal{P} \to T \) and \((\mathcal{X}', S') \to \mathcal{P}' \to T' \), a morphism from the latter to the former consists of a morphism \( g : T' \to T \) and isomorphisms \( \mathcal{X}' \cong X \times_T T' \) and \( \mathcal{P}' \cong \mathcal{P} \times_T T' \) which preserve the section and make the obvious square commutative.

It is shown in [PS21, Theorem 1.2] that there is an algebraic stack, which we denote by \( \mathcal{W}^{\text{min}}_N \), that parametrizes families of minimal Weierstrass fibrations. Our goal is to prove that \( \mathcal{W}^{\text{min}}_N \cong \mathcal{F}^{\text{min}}_N \). We need the following preparatory Lemma, which is proved in [Mir89, pages 22-24].

**Lemma 2.6.** Let \((\mathcal{X}, S) \xrightarrow{\phi} \mathcal{P} \to T \) be a family of minimal Weierstrass fibrations over \( T \). Then:

- \( R^1f_*\mathcal{O}_X \) is a line bundle, the dual of which will be denoted by \( \mathcal{L} \).
- the inclusion \( \mathcal{O}_X(-S) \subset \mathcal{O}_X \) induces an isomorphism \( f_*\mathcal{O}_X(S) \cong f_*\mathcal{O}_X \), and
- for every \( n \geq 2 \) we have an exact sequence

\[
0 \to f_*\mathcal{O}_X((n-1)S) \to f_*\mathcal{O}_X(nS) \to f_*\mathcal{O}_S(nS) \to 0.
\]

Moreover, the sequence above splits and we have \( f_*\mathcal{O}_X(nS) = \mathcal{O}_P \oplus \mathcal{L}^\otimes -2 \oplus \cdots \oplus \mathcal{L}^\otimes -n \).

The following is just a relative version of the arguments in [Mir89, II.5]. We report them below for convenience of the reader.

Consider \((\mathcal{X}, S) \xrightarrow{\phi} \mathcal{P} \to T \) be a family of minimal Weierstrass fibrations over a scheme \( T \). First, we choose a covering of \( \mathcal{P} \) which trivializes \( \mathcal{L} \), and we choose a generator \( e_1 \) for \( \mathcal{L}^{-1} \). In particular, \( e_n := e_1^\otimes n \) will be a generator for \( \mathcal{L}^{-n} \). From Lemma 2.6 this covering also trivializes \( \phi_*\mathcal{O}_X(nS) \), and we choose an element \( f \) of \( \phi_*\mathcal{O}_X(2S) \) (resp. \( g \) of \( \phi_*\mathcal{O}_X(3S) \)) that via the projection to \( \phi_*\mathcal{O}_S(2S) \) (resp. \( \phi_*\mathcal{O}_S(3S) \)) of Lemma 2.6 maps to \( e_2 \) (resp. \( e_3 \)). Then \( g^2 \) and \( f^3 \) are sections of \( \phi_*\mathcal{O}_X(6S) \), and \( g^2 = f^3 + h \) where \( h \) maps to 0 via the projection \( \phi_*\mathcal{O}_X(6S) \to \phi_*\mathcal{O}_S(6S) \).

Proceeding as in [Mir89, II.5] (i.e. completing the square and the cube), locally in \( \mathcal{P} \) there exists unique regular functions \( a \) and \( b \) such that we can (still locally) choose \( f \) and \( g \) with \( g^2 = f^3 + af + b \). If we pick another trivialization \( e'_1 = \lambda e_1 \) for \( \mathcal{L}^{-1} \), the regular functions \( a \) and \( b \) change into \( a' = \lambda^2a \) and \( b' = \lambda^6b \); in particular, we have that \( a' \cdot e'_{-4} = a \cdot e_{-4} \) (resp. \( b' \cdot e'_{-6} = b \cdot e_{-6} \)), hence we obtain a well defined global section \( A \) of \( \mathcal{L}^\otimes 4 \) (resp. a global section \( B \) of \( \mathcal{L}^\otimes 6 \)).
For every point \( x \) in \( T \), the smoothness of the generic fiber of \( \mathcal{X} \to \mathcal{P} \) is equivalent to imposing that the global section \( 4A_x^3 + 27B_x^2 \) is not zero. Moreover, from [Mir81, Corollary 2.5], there is no point \( p \) in a fiber of \( \mathcal{P} \to T \) where the order of vanishing of \( A \), at \( p \) is greater than 4 and the order of vanishing of \( B \) is greater than 6 (as \( \phi \) is a family of minimal Weierstrass fibrations). Note that here, for \( A \) and \( B \) we intend the restriction of the sections to the fiber of \( \mathcal{P} \to T \) containing \( p \).

Combining the previous paragraph with Lemma 2.6, we have

**Corollary 2.7.** Consider \( ((\mathcal{X}, S) \stackrel{\phi}{\to} \mathcal{P} \to T) \) a family of minimal Weierstrass fibrations over \( T \). Then:

1. The sheaf \( \mathcal{L} = (R^1\phi_*\mathcal{O}_\mathcal{X})^\vee \) is a line bundle.
2. From the data above we can canonically construct two sections \( A, B \) of \( H^0(\mathcal{P}, \mathcal{L}^\otimes 4) \) and \( H^0(\mathcal{P}, \mathcal{L}^\otimes 6) \).
3. For every \( x \in T \), the section \( 4A_x^3 + 27B_x^2 \) is not zero on \( \mathcal{P}_x \), and
4. For every \( x \in T \), there is no point \( y \in \mathcal{P}_x \) such that the sections \( A_x \) and \( B_x \) of \( H^0(\mathcal{P}_x, \mathcal{L}^\otimes 4) \) and \( H^0(\mathcal{P}_x, \mathcal{L}^\otimes 6) \) vanish at \( y \) with order \( \geq 4 \) and \( \geq 6 \), respectively.

Corollary 2.7 gives a map \( \mathcal{W}_{\text{min}}^N \to \mathcal{F}_{\text{min}}^N \). We show that this map is an isomorphism, by producing an inverse.

Given a family \( (f : \mathcal{P} \to T, \mathcal{L}, A, B) \), let \( E \) be the vector bundle associated to the the locally free sheaf \( \mathcal{L}^\otimes 2 \oplus \mathcal{L}^\otimes 3 \oplus \mathcal{O}_T \) then we can construct a family of Weierstrass fibrations by taking a closed subscheme of \( \mathbb{P}(E) \) as follows.

First consider a covering \( \mathcal{U} \to \mathcal{P} \) which trivializes \( \mathcal{L} \), and let \( s \) be the trivializing section of \( \mathcal{L} \). We can therefore write the pullback of \( A \) (respectively \( B \)) as \( a \cdot s^4 \) (respectively \( b \cdot s^6 \)). Then consider the closed subscheme of \( \mathbb{P}(E|_U) \) given by those lines generated by \( (x \cdot s^2, y \cdot s^3, z) \) with \( x, y, z \in \mathcal{O}_U \) such that

\[
(y \cdot s^3)^2 z = (x \cdot s^3)^3 + (a \cdot s^4)(x \cdot s^2)z^2 + (b \cdot s^6)z^3 \quad \text{(recall that with our convention we have } \mathbb{P}(E) = \text{Proj}_{\mathcal{O}_\mathcal{U}}(\text{Sym}(\mathcal{L}^\otimes 2|_U \oplus \mathcal{L}^\otimes 3|_U \oplus \mathcal{O}_U)))\]

One can check that these closed subspaces descend to a closed subscheme \( \mathcal{X} \subseteq \mathcal{P}(E) \). The map \( \mathcal{X} \to \mathcal{P} \) has a section \( \mathcal{S} \subseteq \mathcal{X} \), that over \( \mathcal{U} \) is given by \( z = x \cdot s^2 = 0 \). To check that this is a family in \( \mathcal{W}_{\text{min}}^N \) we need to check that when \( T = \text{Spec}(k) \) for an algebraically closed field \( k \), the resulting surface \( \mathcal{X} \) with section \( \mathcal{S} \) is a minimal Weierstrass fibration. The fact that it is a Weierstrass fibration follows from [Mir89, pg. 26], whereas minimaly follows from [Mir81, Corollary 2.5]. We have proven the following.

**Proposition 2.8.** We have an isomorphism \( \mathcal{W}_{\text{min}}^N \simeq \mathcal{F}_{\text{min}}^N \).

Combining the result above with Proposition 2.3, we obtain the main result of this Section.

**Theorem 2.9.** The following isomorphism of stacks holds over \( \text{Spec}(\mathbb{Z}[\frac{1}{6}]) \):

\[
\mathcal{W}_{\text{min}}^N \simeq [(V_{4N,6N}^N \setminus \Delta_N)/(\text{GL}_2 / \mu_N)]
\]

**Remark 2.10.** The presentation above specializes to the two following cases depending on the parity of \( N \), that is:

- if \( N \) is odd, then \( \mathcal{W}_{\text{min}}^N \simeq [(V_{4N,6N}^N \setminus \Delta_N)/\text{GL}_2] \);
- if \( N \) is even, then \( \mathcal{W}_{\text{min}}^N \simeq [(V_{4N,6N}^N \setminus \Delta_N)/\text{PGL}_2 \times \mathbb{G}_m] \).

The actions of these two groups are the ones explained in the Notation section.

### 2.4. Vector bundles on \( \mathcal{W}_{\text{min}}^N \) when \( N \) is odd

Let us suppose \( N \) odd. As observed in Remark 2.10, the stack \( \mathcal{W}_{\text{min}}^N \) has a presentation as a quotient by the action of \( \text{GL}_2 \). In particular, the \( \text{GL}_2 \)-equivariant morphism \( V_{4N,6N}^N \setminus \Delta_N \to \text{Spec}(\mathbb{Z}[\frac{1}{6}]) \) induces a morphism of quotient stacks \( \mathcal{W}_{\text{min}}^N \to \mathcal{B}_{\text{GL}_2} \). This should correspond to a rank two vector bundle on \( \mathcal{W}_{\text{min}}^N \).
**Definition 2.11.** For $N$ odd, we define the rank two vector bundle $E_N$ on $W^\text{min}_N$ as follows:

\[ E_N((X, S) \xymatrix{ \ar[r]^{f} & P \ar[r]^{p} & T}) := p_*((R^1f_*O)^\vee \otimes \omega_P^\otimes N-1). \]

**Proposition 2.12.** The map $W^\text{min}_N \to B\text{GL}_2$ is given by

\[ ((X, S) \xymatrix{ \ar[r]^{f} & P \ar[r]^{p} & T}) \mapsto E_N((X, S) \xymatrix{ \ar[r]^{f} & P \ar[r]^{p} & T}). \]

**Proof.** First we claim that the isomorphism $\mathcal{B}(\text{GL}_2/\mu_N) \simeq \mathcal{B}(\text{GL}_2)$ sends a pair $(\mathcal{P} \xymatrix{ \ar[r]^{p} & T, \mathcal{L}}$ to $p_*(\mathcal{L} \otimes \omega_P^\otimes N-1)$. Indeed, consider first the homomorphism $\text{GL}_2 \to \text{GL}_2$ that sends $A$ to $\det(A)^{\otimes N-1};$ this descends to the isomorphism $\text{GL}_2/\mu_N \to \text{GL}_2$.

The induced morphism $\mathcal{B}\text{GL}_2 \to \mathcal{B}\text{GL}_2$ sends a rank two vector bundle $E \to T$ to $\det(E)^{\otimes N-1} \otimes E$. On the other hand, the morphism $\mathcal{B}\text{GL}_2 \to \mathcal{B}(\text{GL}_2/\mu_N)$ sends $E$ to $(\mathcal{P}(E^\vee), \mathcal{O}_{\mathcal{P}(E^\vee)}(N))$, from which we deduce that $\mathcal{B}(\text{GL}_2/\mu_N) \to \mathcal{B}\text{GL}_2$ sends $(\mathcal{P}(E^\vee), \mathcal{L} = \mathcal{O}_{\mathcal{P}(E^\vee)}(N))$ to $\det(E)^{\otimes N-1} \otimes E$: with a straightforward computation involving the Euler short exact sequence on $\mathcal{P}(E^\vee)$, we see that the latter vector bundle is isomorphic to $p_*(\mathcal{L} \otimes \omega_P^\otimes N-1)$. The claimed description follows then by descent.

By construction, the map $W^\text{min}_N \to B(\text{GL}_2/\mu_N)$ is as follows:

\[ ((X, S) \xymatrix{ \ar[r]^{f} & P \ar[r]^{p} & T}) \mapsto (\mathcal{P} \xymatrix{ \ar[r]^{p} & B, (R^1f_*O)^\vee). \]

The composition $W^\text{min}_N \to \mathcal{B}(\text{GL}_2/\mu_N) \to \mathcal{B}\text{GL}_2$ corresponds then to $p_*((R^1f_*O)^\vee \otimes \omega_P^\otimes N-1). \]

**2.5. Vector bundles on $W^\text{min}_N$ when $N$ is even.** In this case, the presentation of the stack $W^\text{min}_N$ given in Theorem 2.9 can be recast in terms of the group $\text{PGL}_2 \times \mathbb{G}_m$. In particular, this shows that there is a map $W^\text{min}_N \to B\text{PGL}_2 \times B\mathbb{G}_m$, which then must be induced by a Severi-Brauer stack on $W^\text{min}_N$ together with a line bundle.

**Definition 2.13.** For $N$ even, we define the rank three vector bundle $E_N$ and the line bundle $L_N$ on $W^\text{min}_N$ as follows:

\[ E_N((X, S) \xymatrix{ \ar[r]^{f} & P \ar[r]^{p} & T}) := p_*((\omega_P^\vee)^\otimes 1), \]

\[ L_N((X, S) \xymatrix{ \ar[r]^{f} & P \ar[r]^{p} & T}) := p_*((R^1f_*O_X)^\vee \otimes \omega_P^\otimes N). \]

The vector bundle $E_N$ actually plays no role here, but it will be relevant later on.

**Proposition 2.14.** The map $W^\text{min}_N \to B\text{PGL}_2 \times B\mathbb{G}_m$ is given by

\[ ((X, S) \xymatrix{ \ar[r]^{f} & P \ar[r]^{p} & T}) \mapsto (\mathcal{P} \xymatrix{ \ar[r]^{p} & T, \mathcal{L} \xymatrix{ \ar[r]^{f} & P \ar[r]^{p} & T}) \]

**Proof.** The stack $B(\text{GL}_2/\mu_N)$ classifies pairs $(\mathcal{P} \xymatrix{ \ar[r]^{p} & T, \mathcal{L})$ where $\mathcal{P} \xymatrix{ \ar[r]^{p} & T}$ is a Severi-Brauer variety and $\mathcal{L}$ is a line bundle on $\mathcal{P}$ whose restriction to the geometric fibers of $\mathcal{P} \xymatrix{ \ar[r]^{p} & T}$ has degree $N$. We claim that the isomorphism $B(\text{GL}_2/\mu_N) \simeq B(\text{PGL}_2 \times \mathbb{G}_m)$ sends a pair $(\mathcal{P} \xymatrix{ \ar[r]^{p} & T, \mathcal{L})$ to the pairs $(\mathcal{P} \xymatrix{ \ar[r]^{p} & T, p_*(\mathcal{L} \otimes \omega_P^\otimes N)).$

Indeed, consider the homomorphism $\text{GL}_2 \to \text{PGL}_2 \times \mathbb{G}_m$ which sends $A \mapsto ([A], \det(A)^{\otimes N});$ this homomorphism descends to the isomorphism $\text{GL}_2/\mu_N \simeq \text{PGL}_2 \times \mathbb{G}_m$. The induced morphism $B\text{GL}_2 \to B(\text{PGL}_2 \times \mathbb{G}_m)$ then works as follows: a rank two vector bundle $E \to T$ with associated cocycles $\{A_{ij}\}$ is sent to the torsor whose associated cocycles are $\{[A_{ij}], \det(A_{ij})^{\otimes N}\}$, i.e. the object $(\mathcal{P}(E^\vee) \to T, \det(E)^{\otimes N}).$

Observe now that the morphism $B\text{GL}_2 \to B(\text{GL}_2/\mu_N)$ sends a rank two vector bundle $E \to T$ to $(\mathcal{P}(E^\vee) \to T, \mathcal{L} = \mathcal{O}_{\mathcal{P}(E^\vee)}(N));$ we deduce that $B(\text{GL}_2/\mu_N) \to B(\text{PGL}_2 \times \mathbb{G}_m)$ must send $(\mathcal{P}(E^\vee) \xymatrix{ \ar[r]^{p} & T, \mathcal{L} = \mathcal{O}_{\mathcal{P}(E^\vee)}(N)).$
$O_{\mathbb{P}(E')} (N)$ to $(\mathbb{P}(E') \to T, \det(E)^{\otimes \frac{N}{2}})$. A straightforward computation with the Euler sequence of $\mathbb{P}(E') \to T$ shows that $\det(E)^{\otimes \frac{N}{2}} \simeq p_* (\mathcal{L} \otimes \omega_{\mathbb{P}(E')/T})$.

As every object $(\mathcal{P} \to T, \mathcal{L})$ is étale locally isomorphic to an object of the form $(\mathbb{P}(E') \to T, O_{\mathbb{P}(E')} (N))$, we obtain by descent the claimed description of the isomorphism $\mathcal{B}(G_2 / \mu_N) \simeq \mathcal{B}(\text{PGL}_2 \times \mathbb{G}_m)$.

The map $\mathcal{W}_N^{\text{min}} \to \mathcal{B}(\text{PGL}_2 \times \mathbb{G}_m)$ can be factored as

$$\mathcal{W}_N^{\text{min}} \to \mathcal{T}_N^{\text{min}} \to \mathcal{B}(G_2 / \mu_N) \to \mathcal{B}(\text{PGL}_2 \times \mathbb{G}_m),$$

where the composition of the first two maps sends an object $((X, \mathcal{S}) \xrightarrow{f} \mathcal{P} \xrightarrow{p} T)$ to the pair $((\mathcal{P} \xrightarrow{p} T, (R^1 f_* \mathcal{O}_X)^\vee), \text{from which we deduce that the composition sends a family of minimal Weierstrass fibrations} \overline{T}$ to the pair

$$(\mathcal{P} \xrightarrow{p} T, p_* (f_* \mathcal{O}_X)^\vee \otimes \omega_{\mathcal{P} / T})).$$

\[\square\]

3. Equivariant intersection theory on projective spaces

In this section, we work over a ground field of any characteristic. Set $V_k := H^0(\mathbb{P}^1, \mathcal{O}(k))$. The projective space $\mathbb{P}V_k$ can be naturally identified with the Hilbert scheme of $k$ points on $\mathbb{P}^1$. In this section we consider two actions on $\mathbb{P}V_k$, namely:

- the PGL$_2$-action inherited from the natural action of PGL$_2$ on $\mathbb{P}^1$, that is $[A] : [f(x, y)] := [f(A^{-1} (x, y))];$
- the GL$_2$-action induced by the PGL$_2$-action above via the homomorphism GL$_2 \to$ PGL$_2$.

The aim of this Section is to collect some basic facts on the integral Chow ring of $[\mathbb{P}V_k / G]$, where $G$ is either GL$_2$ or PGL$_2$. We will divide our analysis in two parts, depending on whether $V_k$ is a $G$-representation or not. The reason for this is that, given a projective linear action of a group $G$ over a projective space $\mathbb{P}(V)$, the resulting quotient stack $[\mathbb{P}(V) / G]$ is a projective bundle over $BG$ if and only if the action of $G$ lifts to a linear action on $V$, i.e. if $V$ is a $G$-representation.

3.1. First case. As $V_k$ is a GL$_2$-representation, the stack $[\mathbb{P}V_k / \text{GL}_2]$ is a projective bundle over $BG$. Similarly, for $k$ even, the vector space $V_k$ is a PGL$_2$-representation, where the action is defined as

$$A \cdot f(x, y) := \det(A)^\frac{k}{2} f(A^{-1} (x, y)).$$

Equivalently, the representation above is obtained by taking the GL$_2 / \mu_k$-representation $V_k^k$ of Section 2.1 and endowing it with a PGL$_2$ action via the homomorphism $\text{PGL}_2 \to \text{GL}_2 / \mu_k$ defined as $[A] \mapsto [\det(A)^\frac{1}{2} A]$. Therefore, for $G = \text{GL}_2$ or $G = \text{PGL}_2$ and $k$ even, we have that $\pi : [\mathbb{P}V_k / G] \to BG$ is a projective bundle.

Let $h$ be the hyperplane class. From an equivariant point of view, we can regard $h$ as the class of the $G$-equivariant line bundle $O_{\mathbb{P}V_k}(1)$. The following Proposition is just the usual projective bundle formula.

**Proposition 3.1.** Assume that either $G = \text{GL}_2$ or $G = \text{PGL}_2$ and $k$ is even. Then:

1. The integral Chow ring of $[\mathbb{P}V_k / G]$ is generated as $\text{CH}^* (BG)$-module by $h^m$ for $m \leq k$.
2. We have $\pi_* (h^m) = s^G_{m - k} (V_k)$, where the latter denotes the $G$-equivariant Segre class of degree $m - k$ of $V_k$. 
3.2. Second case. For \( k \) odd, the vector space \( V_k \) is not a \( \text{PGL}_2 \)-representation: indeed, any lift of the \( \text{PGL}_2 \)-action on \( \mathbb{P}(V_k) \) to \( V_k \) should be of the form \([A] \cdot f(x, y) = \det(A)^{d} f(A^{-1}(x, y))\), and there is no choice of \( d \) which makes the formula above well defined; picking a different representative \( AA \) for \([A]\) makes the right hand side equal to \( \lambda^{2d-k} \det(A)^{d} f(A^{-1}(x, y))\). This implies that the quotient stack \([\mathbb{P}(V_k)/\text{PGL}_2]\) is not a projective bundle over \( \text{PGL}_2 \). We have to treat this second case differently.

Let \( \Sigma_k \subset \mathbb{P}(V_k) \times \mathbb{P}^1 \) be the \( \text{PGL}_2 \)-invariant subscheme defined as

\[
\Sigma_k = \{(f, x) \text{ such that } f(x) = 0\}.
\]

The line bundle \( \mathcal{O}(\Sigma_k) \) is isomorphic to \( \text{pr}_1^* \mathcal{O}_{\mathbb{P}(V_k)}(1) \otimes \text{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(k) \), hence the latter admits a \( \text{PGL}_2 \)-linearization. The canonical line bundle \( \omega_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \) admits a \( \text{PGL}_2 \)-linearization as well. Then the isomorphism

\[
\mathcal{O}_{\mathbb{P}V_k}(1) \otimes V_1 \simeq \text{pr}_1^*(\text{pr}_1^* \mathcal{O}_{\mathbb{P}V_k}(1) \otimes \text{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(k) \otimes \text{pr}_2^* \omega_{\mathbb{P}^1}^{k-1}),
\]

gives a \( \text{PGL}_2 \)-linearization to the rank two vector bundle \( \mathcal{O}_{\mathbb{P}V_k}(1) \otimes V_1 \).

Before going on, recall \([\text{Pan98}]\) that \( \text{CH}^*(\mathbb{B} \text{PGL}_2) \simeq \mathbb{Z}[c_2, c_3]/(2c_3) \), where \( c_i \) is the \( i \)th Chern class of the vector bundle \([\mathbb{V}_2/\text{PGL}_2] \rightarrow \mathbb{B} \text{PGL}_2\). In what follows we will use the following standard convention for Chern classes: as the projection morphism \( \pi : [\mathbb{V}_k/\text{PGL}_2] \rightarrow \mathbb{B} \text{PGL}_2 \) gives to \( \text{CH}^*([\mathbb{V}_k/\text{PGL}_2]) \) the structure of a \( \text{CH}^*(\mathbb{B} \text{PGL}_2) \)-module via the pullback homomorphism, we will use the notation \( c_i \) for the classes \( \pi^* c_i \).

With this convention, the projection formula \( \pi_*(\pi^* c_i^d \cdot \eta) = c_i^d \cdot \pi_* \eta \) simply reads as \( \pi_*(c_i^d \cdot \eta) = c_i^d \cdot \pi_* \eta \).

**Proposition 3.2.** For \( k \geq 0 \) odd, let \( \gamma_1, \gamma_2 \) be the \( \text{PGL}_2 \)-equivariant Chern classes of \( \mathcal{O}_{\mathbb{P}V_k}(1) \otimes V_1 \). Then we have

\[
\text{CH}^*([\mathbb{P}V_k/\text{PGL}_2]) \simeq \mathbb{Z}[\gamma_1, \gamma_2]/\left(\prod_{i=0}^{k-1} \left(\gamma_2 + \frac{(k-2i)^2 - 1}{4} c_2 \right)\right),
\]

where \( c_2 = -\gamma_1^2 + 4\gamma_2 \), and the generators of this ring as a module over \( \text{CH}^*(\mathbb{B} \text{PGL}_2) \) are \( \gamma_1^i \gamma_2^j \), where \( i = 0, 1 \) and \( j = 0, 1, \ldots, \frac{k-1}{2} \).

**Proof.** For \( k \) odd, in \([\text{ST21}, \text{Proposition 3.7}]\) it is proved that the Chow ring of \( \text{CH}^*([\mathbb{P}V_k/\text{PGL}_2]) \) is isomorphic to

\[
\mathbb{Z}[u, v]S_k / \left(\prod_{i=0}^{k} ((\frac{k+1}{2} - i)u + (\frac{-k+1}{2} + i)v)\right),
\]

where \( u, v \) are the Chern roots of \( \mathcal{O}_{\mathbb{P}V_k}(1) \otimes V_1 \), so that \( u + v = \gamma_1 \) and \( uv = \gamma_2 \).

Moreover, we know from \([\text{ST21}]\) that the pullback homomorphism \( \text{CH}^*(\mathbb{B} \text{PGL}_2) \rightarrow \text{CH}^*([\mathbb{P}V_k/\text{PGL}_2]) \) sends \( c_2 \mapsto -(u - v)^2 \) and \( c_3 \mapsto 0 \). This implies that

\[
\gamma_1^2 = (u + v)^2 = (u - v)^2 + 4uv = 4uv - c_2 = 4\gamma_2 - c_2.
\]

In particular \( \text{CH}^*([\mathbb{P}V_k/\text{PGL}_2]) \) is generated as a module by monomials of the form \((u + v)^i(1)^j\), where \( i \) is either 0 or 1.
We can then rewrite the relation as follows:

\[
\prod_{i=0}^{k-1} \left( \frac{k+1}{2} - i \right) u + \left( \frac{-k+1}{2} + i \right) v \left( \frac{-k+1}{2} + i \right) u + \left( \frac{k+1}{2} - i \right) v
\]

\[
= \prod_{i=0}^{k-1} \left( uv + \left( \frac{k+1}{2} - i \right) \left( \frac{-k+1}{2} + i \right) (u-v)^2 \right)
\]

This shows that the monomials \((u+v)^i(uv)^j\) for \(i \leq 1\) and \(j \leq \frac{k-1}{2}\) actually generate \(\text{CH}^*(\mathbb{P}V_k/\text{PGL}_2)\) as a module over \(\text{CH}^*(B\text{PGL}_2)\), as claimed.

\[\square\]

**Remark 3.3.** For \(k = 1\), we can see that the Chow ring of \(\mathbb{P}V_1/\text{PGL}_2\) is generated by the first Chern class of the normal bundle of the universal section which, coherently with the usual definition of psi classes, we denote \(\psi_1\). In fact, on the universal conic \(p: \mathbb{P} \to \mathbb{P}V_1/\text{PGL}_2\) we have a short exact sequence

\[
0 \to \mathcal{O}_p \to \mathcal{O}(\sigma) \to \sigma^* \mathcal{O}(\sigma) \to 0
\]

where \(\sigma\) is the universal section. By pushing forward along \(p\), we get an exact sequence of locally free sheaves

\[
0 \to \mathcal{O} \to p_* \mathcal{O}(\sigma) \to \sigma^* \mathcal{O}(\sigma) \to 0
\]

which shows that the rank two bundle in the middle is an extension of the normal bundle of the universal section by the trivial line bundle. This implies that \(\gamma_1 = \psi_1\) and \(\gamma_2 = 0\).

Next we give an explicit description of the pushforward morphism along \(\pi: \mathbb{P}V_k/\text{PGL}_2 \to B\text{PGL}_2\). For this, set

\[
E_{n,m}(q) := (-1)^q \sum_{a=0}^{m} \sum_{b=0}^{n} 2^{m-a} \binom{m}{a} \binom{n}{b}.
\]

**Lemma 3.4.** We have

\[
\pi_* (\gamma_1^{m} \gamma_2^{n}) = k^{-1} \sum_{0 \leq q \leq n + m} E_{n,m}(q) \cdot s^{\text{PGL}_2}_{2(n-q)+m-k} (V_{k-1}) \cdot 2c_2^q.
\]

Observe that the sum above is actually a scalar multiple of \(c_2\): this is because every monomial containing \(c_3\) that appears in a Segre class is killed by the multiplication by 2, hence the polynomial above lives in the ring \(\mathbb{Z}[c_2]\). In this way the multiplication by the inverse of \(k\) can be understood literally, i.e. as the division of the scalar coefficient by \(k\).

In particular, we are implying that such scalar coefficient is a multiple of \(k\), because the whole expression belongs to the integral Chow ring.

**Proof.** Consider the commutative diagram of quotient stacks

\[
\begin{array}{ccc}
[\mathbb{P}V_1 \times \mathbb{P}V_{k-1}/\text{PGL}_2] & \xrightarrow{\varrho} & [\mathbb{P}V_k/\text{PGL}_2] \\
\text{pr}_1 \downarrow & & \downarrow \pi \\
[\mathbb{P}V_1/\text{PGL}_2] & \xrightarrow{\pi'} & B\text{PGL}_2
\end{array}
\]
where the top horizontal arrow is induced by the multiplication map. This map is finite of degree \( k \), hence \( \rho_{\ast}\rho_{\ast}^\ast\xi = k\xi \) and
\[
k \cdot \pi_{\ast}\xi = \pi_{\ast}\rho_{\ast}\rho_{\ast}^\ast\xi = \pi_{\ast}'\rho_{\ast}^\ast\rho_{\ast}\rho_{\ast}^\ast\xi = \pi_{\ast}'\rho_{\ast}^\ast_{\ast}^\ast\xi.
\]
As \( k \) is odd, multiplication by \( k \) is an injective group endomorphism of \( \text{CH}^\ast(\mathcal{B}\operatorname{PGL}_2) \). This argument shows that once we understand how the pullback homomorphism \( \rho_{\ast}^\ast \) and the composition \( \pi_{\ast}'\rho_{\ast}^\ast\rho_{\ast}^\ast \) works, we also have an explicit formula for \( \pi_{\ast}^\ast \).

We first have to compute the pullback of \( \gamma_1 \) and \( \gamma_2 \) to the Chow ring of \([\mathbb{P}V_1 \times \mathbb{P}V_{k-1}/\operatorname{PGL}_2] \). For this, observe that \( \rho_{\ast}^\ast(V_1 \otimes \mathcal{O}_{\mathbb{P}V_1}(1)) = V_1 \otimes \mathcal{O}_{\mathbb{P}V_1}(1) \otimes \mathcal{O}_{\mathbb{P}V_{k-1}}(1) \). Recall from Proposition 3.1 and Remark 3.3 that \( h = c_1^\ast\rho_{\ast}^\ast(\mathcal{O}_{\mathbb{P}V_{k-1}}(1)) \) and \( \psi_1 = c_1^\ast\rho_{\ast}^\ast(V_1 \otimes \mathcal{O}_{\mathbb{P}V_1}(1)) \). Applying the splitting principle and the additivity of the total Chern class, we deduce
\[
\rho_{\ast}\gamma_1 = c_1^\rho_{\ast}^\ast \rho_{\ast}^\ast(V_1 \otimes \mathcal{O}_{\mathbb{P}V_1}(1) \otimes \mathcal{O}_{\mathbb{P}V_{k-1}}(1)) = \psi_1 + 2h
\]
\[
\rho_{\ast}\gamma_2 = c_2^\rho_{\ast}^\ast \rho_{\ast}^\ast(V_1 \otimes \mathcal{O}_{\mathbb{P}V_1}(1) \otimes \mathcal{O}_{\mathbb{P}V_{k-1}}(1)) = h(h + \psi_1).
\]
This implies that
\[
\pi_{\ast}\gamma_1^m \gamma_2^n = k^{-1} \pi_{\ast}'\rho_{\ast}^\ast((2h + \psi_1)^m h^n(h + \psi_1)^n).
\]
The computation of pushforwards along \( \operatorname{pr}_1 : [\mathbb{P}V_1 \times \mathbb{P}V_{k-1}/\operatorname{PGL}_2] \to [\mathbb{P}V_1/\operatorname{PGL}_2] \) is easy because \( k - 1 \) is even, hence this map is the projection from a projective bundle. We deduce
\[
\operatorname{pr}_1(h^i\psi_1^j) = s_{i-k+1}^\rho_{\ast}^\ast(V_{k-1})\psi_1^j.
\]
Also the pushforward along \( \pi' : [\mathbb{P}V_1/\operatorname{PGL}_2] \to \mathcal{B}\operatorname{PGL}_2 \) is not hard to determine: consider the cartesian diagram
\[
\begin{array}{ccc}
\mathbb{P}V_1 & \xrightarrow{\rho'} & \text{Spec } k \\
\downarrow{g} & & \downarrow{f} \\
[\mathbb{P}V_1/\operatorname{PGL}_2] & \xrightarrow{\pi'} & \mathcal{B}\operatorname{PGL}_2.
\end{array}
\]
The compatibility formula implies that for every element \( \xi \) in the Chow ring of \([\mathbb{P}V_1/\operatorname{PGL}_2] \) we have \( f^\ast \pi'_{\ast}(\xi) = \rho'_{\ast}g^\ast(\xi) \). To compute \( g^\ast(\psi_1) \), observe that \( g^\ast(V_1 \otimes \mathcal{O}_{\mathbb{P}V_1}(1)) = \mathcal{O}_{\mathbb{P}V_1}(1)^{\oplus 2} \), hence \( g^\ast(\psi_1) = 2c_1^\rho_{\ast}^\ast(\mathcal{O}_{\mathbb{P}V_1}(1)) \); this implies \( f^\ast \pi'_{\ast}(\psi_1) = \rho'_{\ast}g^\ast(\psi_1) = 2 \). In degree zero the pullback \( f^\ast : \text{CH}^0(\mathcal{B}\operatorname{PGL}_2) \to \text{CH}^0(\text{Spec } k) \) is an isomorphism, so we can conclude \( \pi'_{\ast}(\psi_1) = 2 \).

The relation \( \psi_1^2 = -c_2 \) implies that \( \pi'_{\ast}(\psi_1^2) = 0 \) and \( \pi'_{\ast}(\psi_1^{2j+1}) = (-1)^j 2c_2^j \). We deduce
\[
\pi'_{\ast}\rho_{\ast}^\ast_{\ast}^\ast(h^i\psi_1^j) = 0, \quad \pi'_{\ast}\rho_{\ast}^\ast_{\ast}^\ast(h^i\psi_1^{2j+1}) = (-1)^j s_{i-k+1}^\rho_{\ast}^\ast(V_{k-1})2c_2^j.
\]
Putting all together, we obtain the claimed formulas for the pushforward along \( \pi : [\mathbb{P}V_k/\operatorname{PGL}_2] \to \mathcal{B}\operatorname{PGL}_2 \).

\hfill \Box

3.3. Chern classes of representations. Here we outline how to explicitly compute the Chern classes of the representations that appeared before. This also gives formulas for the Segre classes by formally inverting the total Chern class.

First, let us consider the case \( G = \operatorname{GL}_2 \). The integral Chow ring of \( \mathcal{B}\operatorname{GL}_2 \) is isomorphic to \( \mathbb{Z}[c_1, c_2] \), where \( c_1 \) and \( c_2 \) are the Chern classes of the standard \( \operatorname{GL}_2 \)-representation \( E \). Therefore, if \( \ell_1 \) and \( \ell_2 \) are the Chern roots of \( E' \), we have that \( c_1 = -(\ell_1 + \ell_2) \) and \( c_2 = \ell_1\ell_2 \).
We have $V_m = \text{Sym}^m E^\vee$, hence the Chern roots of this symmetric power are given by $j\ell_1 + (m - j)\ell_2$, where $0 \leq j \leq m$. From this we deduce that the total Chern class of $V_m$ for $m$ even is equal to

$$c^{GL_2}(V_m) = \prod_{j=0}^{m} (1 + (j\ell_1 + (m - j)\ell_2)t)$$

$$= (1 + \frac{m}{2}(\ell_1 + \ell_2)t) \prod_{j<\frac{m}{2}} (1 + (j\ell_1 + (m - j)\ell_2)t)(1 + ((m - j)\ell_1 + j\ell_2)t)$$

$$= (1 - \frac{m}{2}c_1t) \prod_{j<\frac{m}{2}} (1 - mc_1t + (j(m - j)c_1^2 + (2j - m)^2c_2)t^2).$$

whether for $m$ odd the same argument gives us

$$c^{GL_2}(V_m) = \prod_{j\leq\frac{m}{2}} (1 - mc_1t + (j(m - j)c_1^2 + (2j - m)^2c_2)t^2).$$

Let $\langle p(t) \rangle_d$ denote the coefficient in front of $t^d$ in $p(t)$. Then we have proved the following:

**Proposition 3.5.**

$$c_{d}^{GL_2}(V_m) = \begin{cases} 
\langle (1 - \frac{m}{2}c_1t) \prod_{j<\frac{m}{2}} (1 - mc_1t + (j(m - j)c_1^2 + (2j - m)^2c_2)t^2) \rangle_d & \text{if } m \text{ is even}, \\
\langle \prod_{j\leq\frac{m}{2}} (1 - mc_1t + (j(m - j)c_1^2 + (2j - m)^2c_2)t^2) \rangle_d & \text{if } m \text{ is odd}.
\end{cases}$$

Next, we consider the case $G = \text{PGL}_2$. The vector space $V_{2m} = H^0(\mathbb{P}^1, \mathcal{O}(2m))$ is a $\text{PGL}_2$-representation of rank $2m + 1$, where the action is defined as $A \cdot f(x, y) := \det(A)^m f(A^{-1}(x, y))$. In what follows we will need explicit formulas for the $\text{PGL}_2$-equivariant Chern classes of $V_{2m}$. These have been computed by Fulghesu and Viviani in [FV11, Section 6].

Recall ([Pan98]) that $\text{CH}^*(B \text{PGL}_2)$ is isomorphic to $\mathbb{Z}[c_2, c_3]/(2c_3)$, where $c_i^{\text{PGL}_2}(V_2) = c_i$, and

**Proposition 3.6** ([FV11, Corollary 6.3]).

$$c_{d}^{\text{PGL}_2}(V_{2m}) = \begin{cases} 
\langle t \prod_{j=1}^{m}(t^2 + j^2c_2) + t^{m+1} \sum_{j=1}^{m} \left( \frac{m}{j} \right) (t^3 + c_2t)^{\left( \frac{m-j}{j} \right)}c_3 \rangle_{2m+1-d} & \text{if } m \text{ is even}, \\
\langle t \prod_{j=1}^{m}(t^2 + j^2c_2) + t^{\frac{m+1}{2}} \sum_{j=1}^{m} \left( \frac{m+1}{j} \right) (t^3 + c_2t)^{\left( \frac{m+1-j}{j} \right)}c_3 \rangle_{2m+1-d} & \text{if } m \text{ is odd}.
\end{cases}$$

The $\text{PGL}_2$-equivariant Segre classes of $V_{2m}$ can then be computed by formally inverting the total Chern classes of the $\text{PGL}_2$-representation.

4. Relations coming from $[\Delta_N^1/G_N]$

In this Section, we compute relations in the Chow ring of $W_N^m$ obtained excising $[\Delta_N^1/G_N]$. More precisely, we show that the ideal of relations obtained by excising this locus has a single generator (Lemma 4.1) and we give a recipe for computing it (see Proposition 4.2 and Proposition 4.4). We work over a ground field of characteristic different from 2 or 3.
4.1. Excision of $[\Delta^1_N/G_N]$. Consider the localization exact sequence

$$\text{CH}^*_{G_N}(\Delta^1_N) \to \text{CH}^*_{G_N}(V^N_{4N,6N}) \to \text{CH}^*_{G_N}(V^N_{4N,6N} \setminus \Delta^1_N) \to 0.$$  

We want to find generators for the ideal given by the image of the first map on the left. To do so, we construct an equivariant envelope of $\Delta^1_N$, in the sense of [EG98a, Page 603], i.e. a proper morphism $Z \to \Delta^1_N$ whose induced pushforward homomorphism between Chow groups is surjective. Let

$$\phi : V^N_{2N} \to V^N_{4N,6N}$$

be the map defined by $\phi(P) = (-3P^2, 2P^3)$. Observe that the image of $\phi$ lies in $\Delta^1_N$.

**Lemma 4.1.** The following hold true:

1. the map $\phi$ defines a $G_N$-equivariant bijective birational morphism $V^N_{2N} \to \Delta^1_N$ that is an isomorphism away from the origin;
2. the pushforward morphism $\text{CH}^*_{G_N}(V^N_{2N}) \to \text{CH}^*_{G_N}(\Delta^1_N)$ is surjective;
3. the image of $\phi_*$ is the ideal generated by $[\Delta^1_N]_{G_N}$.

**Proof.** Away from the origin, the map $(A, B) \mapsto -3B/2A$ defines an equivariant inverse to $V^N_{2N} \to \Delta^1_N$, so $\phi$ is bijective and is an isomorphism away from the origin. This also implies the surjectivity of the induced pushforward.

To prove the last point, observe that there is a well defined pullback morphism $\phi^*$ because $V^N_{4N,6N}$ is smooth, and $\phi^*$ is clearly surjective because both $[V^N_{2N}/G_N]$ and $[V^N_{4N,6N}/G_N]$ are vector bundles over $BG_N$.

Therefore, for every cycle $\zeta$ in $\text{CH}^*_{G_N}(V^N_{2N})$, we have $\phi_*(\zeta) = \phi_*\phi^*(\zeta') = \phi_*(1) \cdot \zeta'$. This proves the last point. \hfill \Box

4.2. The case $G_N = \text{GL}_2$. When $N$ is odd, the group $G_N$ is isomorphic to $\text{GL}_2$. To compute $[\Delta^1_N]_{G_N}$ we can apply the localization formula ([EG98b, Theorem 2]). In general, this formula only gives an expression which is true up to cycles that are zero divisor. In our case we are lucky, as the equivariant Chow ring of $V^N_{4N,6N}$ is a polynomial ring in the two variables $c_1$ and $c_2$, so the expression we obtain in the end holds true unconditionally.

**Proposition 4.2.** For $N$ odd, the image of $\text{CH}^*_{G_N}(\Delta^1_N) \to \text{CH}^*_{G_N}(V^N_{4N,6N})$ is generated as an ideal by

$$[\Delta^1_N]_{G_N} = \frac{c^G_{10N+2}(V^N_{4N,6N})}{c^G_{2N+1}(V^N_{2N})}.$$  

**Proof.** Let $T \subset \text{GL}_2$ be the maximal subtorus of diagonal matrices. The point in $V^N_{2N}$ fixed by the $T$-action is the origin, whose tangent space is isomorphic to $V^N_{2N}$ itself. Applying localization formula ([EG98b, Theorem 2]), we deduce that

$$\phi_*(1) = \frac{\phi_*([0]_T)}{c^G_{2N+1}(TV^N_{2N} \setminus \{0\})} = \frac{[0]_T}{c^G_{2N+1}(TV^N_{2N} \setminus \{0\})} = \frac{c^G_{10N+2}(V^N_{4N,6N})}{c^G_{2N+1}(V^N_{2N})}.$$  

As the $T$-equivariant Chern classes of a $\text{GL}_2$-equivariant vector bundle are equal to the $\text{GL}_2$-equivariant ones, we obtain an expression for $[\Delta^1_N]_{G_N}$. By Lemma 4.1, this class generates the ideal $\text{im}(\phi_*)$, and from the same Lemma we know that this ideal coincides with the image of $\text{CH}^*_{G_N}(\Delta^1_N) \to \text{CH}^*_{G_N}(V^N_{4N,6N})$. \hfill \Box
4.3. The case $G_N = \text{PGL}_2 \times \mathbb{G}_m$. For $N$ even, the group that we have to consider is $\text{PGL}_2 \times \mathbb{G}_m$. Because of the fact that in $\text{CH}^*(\mathcal{B}(\text{PGL}_2 \times \mathbb{G}_m))$ there are zero divisors, e.g. the integer 2, we cannot apply the localization formula directly. To overcome this obstacle, we will use a trick introduced in [DL21].

Let $f : X' \to X$ be a $\text{PGL}_2 \times \mathbb{G}_m$-equivariant morphism between $\text{PGL}_2 \times \mathbb{G}_m$-equivariant schemes. Then by [DL21, Theorem 2.11] there exist $\text{GL}_3 \times \mathbb{G}_m$-schemes $Y$ and $Y'$ with an equivariant morphism $Y' \to Y$ and a commutative diagram

$$
\begin{array}{ccc}
[X'/\text{PGL}_2 \times \mathbb{G}_m] & \xrightarrow{\sim} & [Y'/\text{GL}_3 \times \mathbb{G}_m] \\
\downarrow & & \downarrow \\
[X/\text{PGL}_2 \times \mathbb{G}_m] & \xrightarrow{\sim} & [Y/\text{GL}_3 \times \mathbb{G}_m].
\end{array}
$$

We refer to the $\text{GL}_3 \times \mathbb{G}_m$-scheme $Y$ (resp. $Y'$) as the $\text{GL}_3 \times \mathbb{G}_m$-counterpart of $X$ (resp. $X'$), as in [DL21, Definition 2.9]. Recall that $V^N_{2dN}$ is the $\text{PGL}_2 \times \mathbb{G}_m$-representation $H^0(\mathbb{P}^1, \mathcal{O}(2dN)) \otimes L^{\otimes (-2d)}$, where $L$ is the standard rank one representation of $\mathbb{G}_m$: our aim is to describe explicitly the $\text{GL}_3 \times \mathbb{G}_m$-counterpart of $V^N_{2dN}$.

The affine space $\mathbb{A}^6$ is the parameter space of quadratic forms in three variables, and let $D$ be the discriminant divisor, i.e. the divisor that parametrizes quadratic forms of rank $\leq 2$. We regard $\mathbb{A}^6$ as a $\text{GL}_3 \times \mathbb{G}_m$-scheme, where $\mathbb{G}_m$ acts trivially and $\text{GL}_3$ acts as $A \cdot q(x, y, z) = \det(A)q(A^{-1}(x, y, z))$. Observe that $D$ is invariant with respect to this action.

Over $\mathbb{A}^6 \setminus \{0\}$ we have an injective morphism of $\text{GL}_3 \times \mathbb{G}_m$-equivariant free sheaves

$$H^0(\mathbb{P}^2, \mathcal{O}(dm - 2)) \otimes L^{\otimes (-2d)} \otimes \mathcal{O}_{\mathbb{A}^6 \setminus \{0\}} \to H^0(\mathbb{P}^2, \mathcal{O}(dm)) \otimes L^{\otimes (-2d)} \otimes \mathcal{O}_{\mathbb{A}^6 \setminus \{0\}},$$

where the $\text{GL}_3$-action on these sheaves is inherited from the natural action of $\text{GL}_3$ on $\mathbb{P}^2$, the latter regarded as the projectivization of the standard $\text{GL}_3$-representation.

The quotient of the map in (5) is denoted $W^m_{2dm}$ and by [DL21, Proposition 3.4] the restriction of $W^m_{2dN}$ to $\mathbb{A}^6 \setminus D$ is the $\text{GL}_3 \times \mathbb{G}_m$-counterpart of $V^m_{2dN}$. Moreover, we adopt the notation $W^m_{2dm, 2em}$ for the direct sum of $W^m_{2dm}$ and $W^m_{2em}$. The $K$-points in the total space of $W^m_{2dm}$ should be thought as pairs $(q, [f])$ where $q$ is a non-zero ternary quadratic form on $K$, the polynomial $f$ is a homogeneous form in three variables of degree $dm$ and $[f] = [f']$ if and only if $q$ divides the difference $f - f'$.

In this way we can also describe the counterpart of the equivariant map $\psi : V^N_{2dN} \to V^N_{4dN, 6N}$ introduced in (3), which is the restriction to $\mathbb{A}^6 \setminus D$ of the morphism

$$\psi : W^N_{2dN} \to W^N_{4dN, 6N}, \quad (q, [f]) \mapsto (q, [-3f^2, [2f^3]).$$

In particular, this shows that the $\text{GL}_3 \times \mathbb{G}_m$-equivariant fundamental class of $\psi(W^N_{2dN})$ is equal to the $\text{PGL}_2 \times \mathbb{G}_m$-fundamental class of $\Delta^3_1$.

As in the case where $N$ is odd, we plan to use the localization formula to compute the image of $\text{CH}^*_N(\Delta^3_1) \to \text{CH}^*_N(V^N_{4dN, 6N})$. To pass to integral coefficients however, it is convenient to work in an ambient space $X$ such that $\text{CH}^*(X/\text{GL}_3 \times \mathbb{G}_m)$ is a free $\text{CH}^*(\mathcal{B}(\text{GL}_3 \times \mathbb{G}_m))$-module. Therefore, for our purposes, we set $X = \mathbb{P}^5$ to be the projectivization of the $\text{GL}_3 \times \mathbb{G}_m$-scheme $\mathbb{A}^6$. From [DL21, Definition 3.1] we know that there exists a locally free sheaf $\overline{\mathcal{W}}_{2dm}$ whose pullback along $\mathbb{A}^6 \setminus \{0\} \to \mathbb{P}^5$ is isomorphic to $W^m_{2dm}$. Points in the total space of $\overline{\mathcal{W}}_{2dm}$ are pairs $([q], [f])$, where $[q] = [q']$ if and only if $q = \lambda q'$ for some invertible scalar $\lambda$. We also have an equivariant map $\overline{\psi} : \overline{\mathcal{W}}_{2dN} \to \overline{\mathcal{W}}_{4dN, 6N}$, whose pullback along $\mathbb{A}^6 \setminus \{0\} \to \mathbb{P}^5$ is isomorphic to $\psi$. Recall from [EF08, 4.1] that

$$\text{CH}^*_\text{GL}_3 \times \mathbb{G}_m(\mathbb{P}^5) \cong \bigoplus_{i=0}^5 \text{CH}^*_\text{GL}_3 \times \mathbb{G}_m(\text{Spec}(k)) \cdot h^i \cong \mathbb{Z}[\tau_1, h, c_1, c_2, c_3]/((h^3 - 2c_1 h^2 + 4c_2 h - 8c_3)(h^3 - 2c_1 h^2 + (c_1^2 + c_2)h + c_3 - c_1 c_2))$$
where \( h = c_1^{GL_3 \times G_m}(O(1)) \). Observe in particular that this Chow ring is free as \( CH^*_{GL_3 \times G_m}(\text{Spec}(k)) \)-module.

Now we explain how to compute the Chern classes of \( W_{2m}^m \). For this, the basic ingredient is the short exact sequence

\[
0 \to \mathcal{O}_F^m(-1) \otimes \text{Sym}^{m-2} E^v \otimes L^{m-1} \to \text{Sym}^m E^v \otimes L^{m-1} \otimes \mathcal{O}_F^m \to W_{2m}^m \to 0
\]

of locally free sheaves on \( F \) (see [DL21, 2.3]), where \( E \) is the standard \( GL_3 \)-representation and \( L \) is the standard \( G_m \)-representation of weight one. This implies

\[
(6) \quad c_{2m+1}^{GL_3 \times G_m}(W_{2m}^m) = \begin{cases} 
\left( c^{GL_3 \times G_m}(\text{Sym}^m E^v \otimes L^{m-1} \otimes \mathcal{O}(-1)) \right)_{2m+1} \\
\left( \prod_{i+j \leq m} (1 + x(i\ell_1 + j\ell_2 + (m - i - j)\ell_3 - m\tau_1)) \right)
\end{cases}
\]

This expression in brackets should be interpreted as a formal series in \( x \), from which we are extracting the coefficient in front of \( x^{2m+1} \). Moreover, the symbols \( \ell_1, \ell_2 \) and \( \ell_3 \) stands for the Chern roots of \( E^v \), so that the elementary symmetric polynomial in \( \ell_1, \ell_2 \) and \( \ell_3 \) of degree \( d \) is equal to \((-1)^d \ell_d \).

**Remark 4.3.** Observe that \( c_{2m+1}^{GL_3 \times G_m}(W_{2m}^m) \) is not a zero divisor. Indeed, if it was a zero divisor, this would imply that there exists a non zero element \( \xi \) in the equivariant Chow ring of \( F \) such that

\[
\xi \cdot c_{2m+1}^{GL_3 \times G_m}(W_{2m}^m) \cdot c^{GL_3 \times G_m}(\mathcal{O}(-1) \otimes \text{Sym}^{m-2} E^v \otimes L^{m-1}) = 0,
\]

which would contradict the fact that \( CH^*_{GL_3 \times G_m}(F) \) is a free \( CH^*_{GL_3 \times G_m}(\text{Spec}(k)) \)-module.

**Proposition 4.4.** For \( N \) even, the image of \( CH^*_{GL_3 \times G_m}(\Delta_N^1) \to CH^*_{GL_3 \times G_m}(V_{4N,6N}^N) \) is generated as an ideal by

\[
\left[ c_{10N+2}^{GL_3 \times G_m}(W_{4N,6N}^N) \right]_c_{2N+1}^{GL_3 \times G_m}(W_{2N}^N)
\]

The expression above is a polynomial and after evaluation at \( h = c_1 = 0 \), it should be viewed as an element in \( \mathbb{Z}[c_1, c_2, c_3]/(2c_1) \), the \( G_N \)-equivariant Chow ring of \( V_{4N,6N}^N \).

**Proof.** The fact that \( [\Delta_N^1]_{G_N} \) generates as an ideal the image of \( CH^*_{GL_3 \times G_m}(\Delta_N^1) \to CH^*_{GL_3 \times G_m}(V_{4N,6N}^N) \) has already been proved in Lemma 4.1. Moreover, the previous discussion shows that \( [\Delta_N^1]_{G_N} = [\psi(W_{2N}^N)]_{GL_3 \times G_m} \).

Let \( T \subset GL_3 \times G_m \) be the maximal subtorus of pairs formed by diagonal matrices and an invertible scalar. The fixed points for the action of \( T \) on \( W_{2dm}^m \) are of the form \( ([q], [0]) \) where \( q \) is a monomial. Observe that the tangent space of \( W_{2dm}^m \) at \( p \) is isomorphic to the direct sum \( T P_{\mathbb{P}^2}^m \oplus W_{2dm,m}^m \). Moreover, the fundamental class of \( ([q], [0]) \) in the equivariant Chow ring of \( W_{2dm}^m \) is equal to the product \( [\psi(T) \cdot c_{2N+1}^{GL_3 \times G_m}(W_{2dm,p}^m) \cdot c_{10N+2}^{GL_3 \times G_m}(W_{4N,6N}^N)] \). The localization formula ([EG98b, Theorem 2]) then gives us the equality

\[
(7) \quad \psi_*(1) = \sum_{q=x,x,i \neq j} \left( [([q], [0])]_T c_{2N+6}^T(\mathcal{O}_{T W_{2N+1},n}^m) \right)
\]

\[
= \sum_{q=x,x,i \neq j} \left( [([q], [0])]_T c^T_5(T \mathbb{P}^2_\mathbb{P}^2)^m c_{2N+1}(W_{2N,0}^m) \right)
\]

\[
= \sum_{q=x,x,i \neq j} \left( [([q], [0])]_T \cdot c_{10N+2}^{T}(W_{4N,6N}^N) \right) c^T_5(T \mathbb{P}^2_\mathbb{P}^2)^m c_{2N+1}(W_{2N,0}^m)
\]

\[
= c^T_5(T \mathbb{P}^2_\mathbb{P}^2)^m c_{2N+1}(W_{2N,0}^m).
\]
where in the last equality we applied again the localization formula to obtain an expression in the equivariant Chow ring of $\overline{W}_{4N,6N}$. Observe that a priori the localization formulas would only give an equality in the ring $\text{CH}_T(\mathbb{P}^5) \otimes (\text{CH}_T(\text{Spec}(k))^+)^{-1}$ obtained by inverting the positive degree elements in $\text{CH}_T(\text{Spec}(k))$; nevertheless, as $\text{CH}_T(\mathbb{P}^5)$ is a free $\text{CH}_T(\text{Spec}(k))$-module, the natural homomorphism $\text{CH}_T(\mathbb{P}^5) \rightarrow \text{CH}_T(\mathbb{P}^5) \otimes (\text{CH}_T(\text{Spec}(k))^+)^{-1}$ is injective: this proves that (1) the last term is not just a rational function but a polynomial, and that (2) it coincides with $\overline{\psi}_a(1)$. Observe moreover that as $c_{2N+1}^T(\overline{W}_{2N})$ is not a zero divisor (Remark 4.3), the expression

$$\frac{c_{10N+2}^T(\overline{W}_{4N,6N})}{c_{2N+1}^T(\overline{W}_{2N})}$$

is well defined, in the sense that it coincides with the unique element $\xi$ such that $\xi \cdot c_{2N+1}^T(\overline{W}_{2N}) = c_{10N+2}^T(\overline{W}_{4N,6N})$.

The $T$-equivariant top Chern classes of $\overline{W}_{2dm}$ are equal to the $\text{GL}_3 \times \mathbb{G}_m$-ones. Observe also that the last term in (7) can be regarded as a polynomial in $h$, the hyperplane class of $\mathbb{P}^5$, so that the element

$$\frac{c_{10N+2}^T(\overline{W}_{4N,6N})}{c_{2N+1}^T(\overline{W}_{2N})} \bigg|_{h=c_1}$$

is well defined, and it coincides with the pullback of $\overline{\psi}_a(1)$ along the $\mathbb{G}_m$-torsor $A^6 \setminus \{0\} \rightarrow \mathbb{P}^5$, which in turn is equal to $\overline{\psi}_a(1)$. If we further restrict this cycle to $A^6 \setminus D$ (observe that this operation sends $c_1$ to zero), we get an explicit expression for the $\text{GL}_3 \times \mathbb{G}_m$-fundamental class of $\psi(\overline{W}_{2N})$, which we already observed to be equal to the $\text{PGL}_2 \times \mathbb{G}_m$-equivariant fundamental class of $\Delta_N^1$. \hfill \Box

5. Relations coming from $[\Delta_N^2/G_N]$

In this Section we compute the relations in the Chow ring of $W_{2dm}^m$ coming from the excision of $[\Delta_N^2/G_N]$. We first define an equivariant stratification for $\Delta_N^1$, which we leverage to compute the generators of the ideal of the relations. The final result is summarized in Proposition 5.4.

In the last part of the Section, we prove the first main result of the paper (Theorem 5.5). In this section we work over a ground field of any characteristic, with the only exception of Theorem 5.5.

5.1. An equivariant stratification of $\Delta_N^1$. First, we recall the definition of equivariant stratification in general.

**Definition 5.1.** Let $X$ be a $G$-scheme. An equivariant stratification of $X$ is a finite family $\{Z_\tau\}_{\tau \in J}$ of locally closed, pairwise disjoint, and equivariant subschemes of $X$ such that $\bigcup_{\tau \in J} Z_\tau = X$ and $Z_\tau \setminus Z_\tau' = \bigcup Z_{\tau'}$.

We can endow $\mathbb{P}V_k$ with a $G_N$ action as follows:

- for $N$ odd, we can regard $V_k$ as a $\text{GL}_2$-representation with the action defined at the beginning of Section 3; we can then use the isomorphism $G_N \rightarrow \text{GL}_2$ in order to give to $V_k$ the structure of a $G_N$-representation. This of course induces an action of $G_N$ on $\mathbb{P}V_k$. Observe that for $k = rN$, the two $G_N$-representations $V_k$ and $V_k^N$ are not the same (which also motivates the difference in the notation).
- For $N$ even, we can regard $\mathbb{P}V_k$ as a $\text{PGL}_2 \times \mathbb{G}_m$-scheme using the $\text{PGL}_2$-action defined at the beginning of Section 3, and letting $\mathbb{G}_m$ act trivially. Therefore, we can endow $\mathbb{P}V_k$ with the structure
of a $G_N$-scheme via the isomorphism $G_N \to \text{PGL}_2 \times \mathbb{G}_m$. If $k$ is even, we can also endow $V_k$ with a $G_N$-action, exactly in the same way.

Let $\Sigma_k^{(m+1)}$ denote the $(m+1)$-thickening of the subscheme $\Sigma_k \subset \mathbb{P}V_k \times \mathbb{P}^1$ defined in Section 3.2, i.e. the subscheme defined by the ideal sheaf $I_{\Sigma_k}^{m+1} \simeq \mathcal{O}_{\mathbb{P}V_k}(-m-1) \boxminus \mathcal{O}_{\mathbb{P}^1}(-(m+1)k)$. We then have a short exact sequence of $G_N$-equivariant sheaves

$$0 \to \mathcal{O}_{\mathbb{P}V_k}(-m-1) \boxminus \mathcal{O}_{\mathbb{P}^1}(-(m+1)k) \to \mathcal{O}_{\mathbb{P}(\mathbb{P}V_k \times \mathbb{P}^1)} \to i_* \mathcal{O}_{\Sigma_k^{(m+1)}} \to 0.$$ 

We can twist the sequence above by $\text{pr}^*_2 \mathcal{O}_{\mathbb{P}^1}(2d)$ and push everything down on $\mathbb{P}V_k$; if we further assume that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d- (m+1)k) = 0$, by cohomology and base change we obtain the following short exact sequence of $G_N$-equivariant locally free sheaves on $\mathbb{P}V_k$:

$$(9) \quad 0 \to \mathcal{O}_{\mathbb{P}V_k}(-m-1) \boxplus V_{2d- (m+1)k} \to V_{2d} \otimes \mathcal{O}_{\mathbb{P}V_k} \to \mathcal{P}_k^m(\mathcal{O}_{\mathbb{P}^1}(2d)) \to 0$$

where we define $\mathcal{P}_k^m(\mathcal{O}_{\mathbb{P}^1}(2d))$ as the locally free sheaf $\text{pr}_1^* (\text{pr}^*_2 \mathcal{O}_{\mathbb{P}^1}(2d)|_{\Sigma_k^{(m+1)}})$. This bundle coincides with the bundle of principal parts considered in [CK23].

In particular, if we specialize this short exact sequence to the cases $(d, m) = (2N, 3), (3N, 5)$, we get short exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}V_k}(-4) \boxplus V_{4(N-k)} \to V_{4N} \otimes \mathcal{O}_{\mathbb{P}V_k} \to \mathcal{P}_k^3(\mathcal{O}_{\mathbb{P}^1}(4N)) \to 0$$
$$0 \to \mathcal{O}_{\mathbb{P}V_k}(-6) \boxplus V_{6(N-k)} \to V_{6N} \otimes \mathcal{O}_{\mathbb{P}V_k} \to \mathcal{P}_k^5(\mathcal{O}_{\mathbb{P}^1}(6N)) \to 0$$

Define $L$ as follows:

- for $N$ odd, it is defined as $\text{det}(E)^{2N-1}$, where $E$ is the standard representation of $\text{GL}_2$;
- for $N$ even, it is defined as $L^{|0|}$, the rank one representation of $\mathbb{G}_m$ of weight $-1$.

There is an action of $\text{GL}_2 / \mathbb{C} \subset \mathbb{P}V_k$: for $N$ odd, we have $\text{GL}_2 / \mathbb{C} \simeq \text{GL}_2$, and for $N$ even we have $\text{GL}_2 / \mathbb{C} \simeq \text{PGL}_2 \times \mathbb{G}_m$; the action of these two groups on $\mathbb{P}V_k$ coincide with the ones mentioned at the beginning of Section 3.

We have then short exact sequences of $\text{GL}_2 / \mathbb{C}$-equivariant locally free sheaves

$$0 \to \mathcal{O}_{\mathbb{P}V_k}(-4) \boxplus V_{4(N-k)} \boxplus L^{|0|} \to V_{4N} \otimes L^{|0|} \otimes \mathcal{O}_{\mathbb{P}V_k} \to \mathcal{P}_k^3(\mathcal{O}_{\mathbb{P}^1}(4N)) \boxplus L^{|0|} \to 0$$
$$0 \to \mathcal{O}_{\mathbb{P}V_k}(-6) \boxplus V_{6(N-k)} \boxplus L^{|0|} \to V_{6N} \otimes L^{|0|} \otimes \mathcal{O}_{\mathbb{P}V_k} \to \mathcal{P}_k^5(\mathcal{O}_{\mathbb{P}^1}(6N)) \boxplus L^{|0|} \to 0$$

Let $Z_k$ denote the total space of the locally free sheaf

$$(\mathcal{O}_{\mathbb{P}V_k}(-4) \boxplus V_{4(N-k)} \boxplus L^{|0|}) \oplus (\mathcal{O}_{\mathbb{P}V_k}(-6) \boxplus V_{6(N-k)} \boxplus L^{|0|}).$$

Then $Z_k$ is a $G_N$-equivariant vector subbundle of $V^N_{4N,6N} \times \mathbb{P}V_k$ and we have an equivariant morphism

$$p_k : Z_k \to V^N_{4N,6N}$$

whose image corresponds to the invariant subscheme of pair of forms $(A, B)$ such that there exists a form $H$ of degree $k$ with $A$ vanishing with order $\geq 4$ along $H = 0$ and $B$ vanishing with order $\geq 6$ along $H = 0$.

Moreover, this morphism is one-to-one on the locally closed subscheme of pairs $(A, B)$ which satisfy the previous condition together with the further restraint that there exists no form $H'$ of degree $k+1$ such that $A$ (resp. $B$) vanish with order 4 (resp. 6) along $H' = 0$.

**Lemma 5.2.** The image of the pushforward $\text{CH}^{*+1}([\Delta_k^N/G_N]) \to \text{CH}^k([V^N_{4N,6N}/G_N])$ is equal to the sum of the images of the equivariant pushforwards $p_{k*}$, for $k = 1, \ldots, N$. 

Proof. Set $\Delta^2_{N,k} := \text{im}(p_k)$, so that we have an equivariant stratification of $\Delta^2_N$ given by

$$\Delta^2_N = \Delta^2_{N,1} \supset \Delta^2_{N,2} \supset \ldots \supset \Delta^2_{N,N-1} \supset \Delta^2_{N,N}.$$  

Observe that the induced maps $Z_k \setminus p_k^{-1}(\Delta^2_{N,k+1}) \to (\Delta^2_{N,k} \setminus \Delta^2_{N,k+1})$ are equivariant Chow envelopes of the strata. We can then apply [DLFV21, Lemma 3.3] and conclude the proof.

We have reduced the problem of computing the relations coming from $\Delta^2_N$ to determining the images of several pushforwards. The generators of the Chow groups of $[Z_k/G_N]$ are easier to compute, and so are their pushforwards. Indeed, consider the diagram

$$\begin{align*}
\left[ V^N_{4N,6N} \times \mathbb{P}V_k/G_N \right] & \xrightarrow{\text{pr}_1} \left[ V^N_{4N,6N}/G_N \right] \\
& \xrightarrow{\text{pr}_2} \left[ \mathbb{P}V_k/G_N \right].
\end{align*}$$

Then we have the following.

**Lemma 5.3.** The image of $p_{k*}$ is generated as an ideal by all the cycles of the form $\text{pr}_{1*}([Z_k]/G_N \cdot \text{pr}_2^*\eta)$, where $\eta$ ranges among all the generator of $\text{CH}^*(\mathbb{P}V_k/G_N)$ as $\text{CH}^*(BG_N)$-module.

**Proof.** Write $p_k$ as the composition of the closed embedding $i : Z_k \to V^N_{4N,6N} \times \mathbb{P}V_k$ followed by the projection $\text{pr}_1 : V^N_{4N,6N} \times \mathbb{P}V_k \to V^N_{4N,6N}$. Observe that the Chow ring of $[Z_k/G]$ is generated as a module over $\text{CH}^*(BG)$ by the pullback of generators of $\text{CH}^*(\mathbb{P}V_k)$, i.e. by elements of the form $i^*\text{pr}_2^*\eta$. We deduce that the image of $p_{k*}$ is generated as an ideal by

$$p_{k*}(i^*\text{pr}_2^*\eta) = \text{pr}_{1*}i_{1*}(i^*\text{pr}_2^*\eta) = \text{pr}_{1*}([Z_k]/G \cdot \text{pr}_2^*\eta),$$

as claimed. \qed

### 5.2. Computation of the fundamental class of $Z_k$.

The subvariety $Z_k \subset V^N_{4N,6N} \times \mathbb{P}V_k$ has codimension $10k$ and its equivariant fundamental class is equal to the equivariant top Chern class of the vector bundle

$$(\mathcal{P}^6_k(\mathcal{O}_{\mathbb{P}^1}(4N)) \otimes \mathbb{L}^\otimes 4) \oplus (\mathcal{P}^6_k(\mathcal{O}_{\mathbb{P}^1}(6N)) \otimes \mathbb{L}^\otimes 6),$$

which is equal to the product of the top Chern classes of the two factors. We write

$$c_{2dk}^G(\mathcal{P}^{2d-1}_k(\mathcal{O}_{\mathbb{P}^1}(2dN))) \otimes \mathbb{L}^\otimes 2d = \sum_{i=0}^{2dk} c_i^G(\mathcal{P}^{2d-1}_k(\mathcal{O}_{\mathbb{P}^1}(2dN)))(2d\zeta_1)^{2dk-i},$$

where we set $\zeta_1 = -\tau_1$ when $N$ is even and $\zeta_1 = (N-1)c_1/2$ when $N$ is odd. In this way, we have reduced our computation of the fundamental class of $Z_k$ to determining the $G_N$-equivariant Chern classes of $\mathcal{P}^{2d-1}_k(\mathcal{O}_{\mathbb{P}^1}(2dN))$. For this we use (9), which tells us that

$$c_i^G(\mathcal{P}^{2d-1}_k(\mathcal{O}_{\mathbb{P}^1}(2dN))) = \sum_{j=0}^{i} c_j^G(V_{2dN})s_{i-j}(V_{2d(N-k)}) \otimes \mathcal{O}_{\mathbb{P}^V_k}(-2d)).$$

Define $\xi_1$ as $c_1^G(\mathcal{O}_{\mathbb{P}^V_k}(1))$ for $N$ odd or $N$ even and $k$ even, and as $\frac{1}{2}c_1^G(\mathcal{O}_{\mathbb{P}^V_k}(2))\gamma_1/2$ for $N$ even and $k$ odd. Observe that $\mathcal{O}_{\mathbb{P}^V_k}(2)$ is indeed an equivariant $G_N \simeq \text{PGL}_2 \times \mathbb{G}_m$-line bundle: this follows from the fact that the square map $\mathbb{P}V_k \to \mathbb{P}V_{2k}$ is $\text{PGL}_2$-equivariant, and the restriction of the $\text{PGL}_2$-equivariant line bundle $\mathcal{O}_{\mathbb{P}^V_{2k}}(1)$ (which is equivariant because $\mathbb{P}V_{2k}$ is the projectivization of the $\text{PGL}_2$-representation $V_{2k}$) along this map coincides with $\mathcal{O}_{\mathbb{P}^V_k}(2)$. \qed
Applying the formula for the Segre classes of tensor products, we obtain
\[ s_{i-j}^{G_N}(V_{2d(N-k)} \otimes \mathcal{O}_{PV_k}(-2d)) = \sum_{\ell=0}^{i-j} (-1)^{i-j-\ell} \binom{2d(N-k)+i-j}{2d(N-k)+\ell} s_\ell^{G_N}(V_{2d(N-k)})(-2d\xi_1)^{i-j-\ell} \]
\[ = \sum_{\ell=0}^{i-j} \binom{2d(N-k)+i-j}{2d(N-k)+\ell} s_\ell^{G_N}(V_{2d(N-k)})(2d\xi_1)^{i-j-\ell} \]

Putting everything together, we get the following expression for the \(G_N\)-equivariant fundamental class of \(Z_k\):
\[ (10) \quad \prod_{d=2}^{3} \sum \binom{2d(N-k)+i-d-jd}{2d(N-k)+\ell_d}(2d)^{2dk-jd-\ell_d}c_{s_d}^{G_N}(V_{2dN})s_{\ell_d}^{G_N}(V_{2d(N-k)})\xi_1^{i-d-\ell_d}\xi_1^{2dk-jd} \]
where the sum index runs over all the triples \((i_d,j_d,\ell_d)\) such that \(j_d+\ell_d \leq i_d \leq 2dk\), for \(d=2,3\).

5.3. **Relations from \(\Delta^2_N\).** We are going to compute generators as an ideal for the image of
\[ (11) \quad \text{CH}^{*+9}([\Delta^2_N/G_N]) \rightarrow \text{CH}^{*}([V_{4N,6N}/G_N]). \]
Consider again the diagram
\[ \begin{align*}
[V_{4N,6N} \times PV_k/G_N] & \xrightarrow{pr_1} [V_{4N,6N}/G_N] \\
[V_{4N,6N}/G_N] & \xrightarrow{\pi} BG_N.
\end{align*} \]

From Lemma 5.3 we know that the image of (11) is generated by the cycles \(pr_{1*}([Z_k]_{G_N} \cdot pr_2^*\eta)\), where \(\eta\) ranges among all the generators of \(\text{CH}^{*}([PV_k/G_N])\) as \(\text{CH}^{*}(BG_N)\)-module.

Let us rewrite the formula for \([Z_k]_{G_N}\) contained in (10) as
\[ \sum C_k(i,j,\ell)\xi_1^{i-j-\ell} \]
where \(j + \ell \leq i \leq 10k\), and the coefficients are
\[ C_k(i,j,\ell) = \xi_1^{10k-i} \left( \prod_{d=2}^{3} \binom{2d(N-k)+i-d-jd}{2d(N-k)+\ell_d}(2d)^{2dk-jd-\ell_d}c_{s_d}^{G_N}(V_{2dN})s_{\ell_d}^{G_N}(V_{2d(N-k)}) \right) . \]
The sum above is taken over all the triples \((i_d,j_d,\ell_d), d=2,3\), such that
\[ j_d + \ell_d \leq i_d \leq 2dk, \quad \sum_{d=2}^{3}(i_d,j_d,\ell_d) = (i,j,\ell). \]

Pullbacks along the vertical arrows of the diagram (12) induce isomorphism of Chow rings. Thus, after identifying the Chow rings on the top of the diagram with the respective ones on the bottom, we have
\[ (13) \quad pr_{1*}([Z_k]_{G_N} \cdot pr_2^*\eta) = \sum C_k(i,j,\ell)\pi_*\xi_1^{i-j-\ell} \cdot \eta \]
Note that, in the equality above, we are allowed to apply the projection formula because the coefficients \(C_k(i,j,\ell)\) are cycles pulled back from \(BG_N\).
For $G_N \simeq \text{GL}_2$ we know from Proposition 3.1 that the Chow ring of $[\mathbb{P}V_k/\text{GL}_2]$ is generated by powers of the hyperplane class $h$ and we have $\zeta_1 = (N-1)c_1/2$ and $\xi_1 = h$. Therefore, applying Proposition 3.1 we get the following explicit expression for $f_{k,m}$ when $\eta = h^m$:

$$f_{k,m} := \sum_{i,j,\ell} \left((N-1)c_1/2\right)^{10k-i} s_{i-j-\ell-(k-m)}(V_k)$$

(14)

$$\cdot \left(\sum_{d=2}^3 \left(\frac{2d(N-k) + i_d - j_d}{2d(N-k) + \ell_d}\right)(2d)^{2d-j-d-\ell_d} s_{i_d}^{\text{GL}_2}(V_{2dN})\right)$$

where $(i, j, \ell) := (i_2, j_2, \ell_2) + (i_3, j_3, \ell_3)$ and the sum is taken over all the pairs of triples of positive numbers $\{(i_d, j_d, \ell_d)\}_{d=2,3}$ such that $j_d + \ell_d \leq i_d \leq 2d$.

For $G_N \simeq \text{PGL}_2 \times \mathbb{Z}_m$ and $k$ odd, we have a similar picture: the only difference is that $\zeta_1 = -\tau_1$, hence an explicit expression for $f_{k,m}$ when $\eta = h^{m'}$ is given by

$$g_{k,m'} := \sum_{i,j,\ell} \left(-\tau_1\right)^{10k-i} s_{i-j-\ell-(k-m')}$$

(15)

$$\cdot \left(\sum_{d=2}^3 \left(\frac{2d(N-k) + i_d - j_d}{2d(N-k) + \ell_d}\right)(2d)^{2d-j-d-\ell_d} s_{i_d}^{\text{PGL}_2}(V_{2dN})\right)$$

where again $(i, j, \ell) := (i_2, j_2, \ell_2) + (i_3, j_3, \ell_3)$ and the sum is taken over all the pairs of triples of positive numbers $\{(i_d, j_d, \ell_d)\}_{d=2,3}$ such that $j_d + \ell_d \leq i_d \leq 2d$.

Finally, for $G_N \simeq \text{PGL}_2 \times \mathbb{Z}_m$ and $k$ odd, we know from Proposition 3.2 that the Chow ring of the stack $[\mathbb{P}V_k/\text{PGL}_2 \times \mathbb{Z}_m]$ is generated as a module over $\text{CH}^*(\mathcal{B}(\text{PGL}_2 \times \mathbb{Z}_m))$ by monomials of the form $\gamma^m_1 \gamma^n_2$, where $m \in \{0, 1\}$ and $n \leq \frac{k-1}{2}$. Moreover, for $k$ odd we have $\xi_1 = \frac{2k}{h}$, hence

$$\text{pr}_{1,*}(\mathbb{Z}[G_N] \cdot \gamma^m_1 \gamma^n_2) = \sum 2^{-(i-j-\ell)} C_k(i, j, \ell) \gamma^m_{\ell+i-j-\ell} \gamma^n_2$$

Write $m' = 2n + m$, where $m$ is either $0$ or $1$. Applying Lemma 3.4, we get the following explicit expression for the pushforwards:

$$g_{k,m'} := \sum_{i,j,\ell} \left(-\tau_1\right)^{10k-i}$$

(16)

$$\cdot \left(\sum_{d=2}^3 \left(\frac{2d(N-k) + i_d - j_d}{2d(N-k) + \ell_d}\right)(2d)^{2d-j-d-\ell_d} s_{i_d}^{\text{PGL}_2}(V_{2dN})\right)$$

$$\cdot \left(\sum_{q \leq n+\frac{k-1}{2}} E_{n,m+i-j-\ell}(q) s^{\text{PGL}_2}(V_{2dN})\right)$$

where, as before, we set $(i, j, \ell) := (i_2, j_2, \ell_2) + (i_3, j_3, \ell_3)$ and the sum is taken over all the pairs of triples of positive numbers $\{(i_d, j_d, \ell_d)\}_{d=2,3}$ such that $j_d + \ell_d \leq i_d \leq 2d$. The quantity $E_{n,m}(q)$ is the one defined just before Lemma 3.4.

Putting all together, we deduce the following.

**Proposition 5.4.** The image of the pushforward $\text{CH}^{*+9}(\Delta^2_N/G_N) \to \text{CH}^*(\mathbb{V}^N_{4N,6N}/G_N)$ is generated by:

1. when $N$ is odd, by the cycles $f_{k,m}$ described in (14) for $1 \leq k \leq N$ and $0 \leq m \leq k$;
2. when $N$ is even, by the cycles $g_{k,m'}$ described in (15) and (16) for $1 \leq k \leq N$ and $0 \leq m' \leq k$.

**5.4. Proof of the main result.** We have all the ingredients necessary to prove our main result. Indeed, we know from Proposition 2.3 that the stack $\mathbb{W}^\text{min}_N$ is isomorphic to $[\mathbb{V}^N_{4N,6N} \times (\Delta^2_N \cup \Delta^2_N)/G_N]$, hence we have a localization exact sequence

$$\text{CH}((\Delta^2_N \cup \Delta^2_N)/G_N) \to \text{CH}([\mathbb{V}^N_{4N,6N}/G_N]) \to \text{CH}(\mathbb{W}^\text{min}_N) \to 0.$$
The image of the map on the left is equal to the sum of the images of the maps $CH_\ast([\Delta_N^N/G_N]) \to CH_\ast([V_{4N,6N}^N/G_N])$ for $i = 1, 2$, which have been computed in Lemma 4.1 and Proposition 5.4. The integral Chow ring of $[V_{4N,6N}^N/G_N]$ is isomorphic to the one of $BG_N$, where the isomorphism is induced by the pullback morphism along the map $[V_{4N,6N}^N/G_N] \to BG_N$. When $N$ is odd, we have $G_N \simeq GL_2$ and $CH^\ast(BGL_2) \simeq \mathbb{Z}[c_1, c_2]$, with $c_1$ and $c_2$ the Chern classes of the universal rank two vector bundle.

Therefore, the generators $c_1$ and $c_2$ of $CH^\ast(W_N^\text{min})$ are by construction the Chern classes of the pullback of the universal rank two vector bundle on $BGL_2$. The map $W_N^\text{min} \to BGL_2$ is induced by the rank two vector bundle $E_N$ of Definition 2.13 (see Proposition 2.12), hence the pullback of the universal vector bundle is equal to $E_N$.

Similarly, for $N$ even we have $G_N \simeq PGL_2 \times G_m$ and the integral Chow ring of the associated classifying stack is isomorphic to $\mathbb{Z}[\tau_1, c_2, c_3]/(2c_3)$. The generator $\tau_1$ is the first Chern class of the pullback of the universal line bundle on $BG_m$, which by Proposition 2.14 is equal to $L_N$. The other two generators $c_2$ and $c_3$ are by definition the pullback of the generators of $CH^\ast(BPGL_2)$, which are the Chern classes of the rank three vector bundle $(P \overrightarrow{E} B) \to p_2(\omega_B^\ast/B)$. The pullback of the latter is by definition the rank three vector bundle $E_N$ of Definition 2.13.

Putting all together, we obtain our first main result.

**Theorem 5.5.** Suppose that the ground field has characteristic $\neq 2, 3$. Then

1. for $N$ odd we have

$$CH^\ast(W_N^\text{min}) \simeq \mathbb{Z}[c_1, c_2]/I_N$$

where the ideal of relations $I_N$ is generated by the polynomials $f_{k,m}$ described in (14) for $1 \leq k \leq N$ and $0 \leq m \leq k$, together with the fundamental class $[\Delta_N^N]_{GL_2}$. The degree of $f_{k,m}$ is $9k + m$ and the degree of $[\Delta_N^N]_{GL_2}$ is $8N + 1$. The generators $c_1$ and $c_2$ are the Chern classes of the rank two vector bundle $E_N$ introduced in Definition 2.11.

2. for $N$ even, we have

$$CH^\ast(W_N^\text{min}) \simeq \mathbb{Z}[\tau_1, c_2, c_3]/(2c_3, I_N)$$

where the ideal of relations $I_N$ is generated by the polynomials $g_{k,m'}$ described in (15) and (16) for $1 \leq k \leq N$ and $0 \leq m' \leq k$, together with the fundamental class $[\Delta_N^N]_{PGL_2 \times G_m}$. The degree of $g_{k,m'}$ is $9k + m'$ and the degree of $[\Delta_N^N]_{PGL_2 \times G_m}$ is $8N + 1$. The generator $\tau_1$ is the first Chern class of the line bundle $L_N$ introduced in Definition 2.13, and the generators $c_2$ and $c_3$ are the Chern classes of the rank three vector bundle $E_N$ introduced in Definition 2.13.

Note the relations appearing in the Theorem above can be made fully explicit: one can apply Proposition 4.2 and Proposition 4.4 for computing the fundamental class of $\Delta_N^N$, and Proposition 3.5 and Proposition 3.6 to obtain explicit expressions for the Chern and Segre classes of the representations appearing in $f_{k,m}$ and $g_{k,m'}$. Plugging these formulas into the relations, one get the desired description. This is exactly what we will do in the next Section for $N = 1, 2$.

6. **Integral Chow rings of stacks of rational elliptic surfaces and elliptic K3 surfaces**

In this Section we compute the integral Chow ring of $W_1^\text{min}$, the moduli stack of rational elliptic surfaces, and of $W_2^\text{min}$, the moduli stack of elliptic K3 surfaces. The two main results are Theorem 6.1 and Theorem 6.2.

6.1. **The case $N = 1$.** A Weierstrass fibration $X \to \mathbb{P}^1$ with fundamental invariant $N = 1$ is a rational surface, obtained by blowing up $\mathbb{P}^2$ along the base locus of a pencil of cubics. Equivalently, we can think of $X$ as the blow-up of a Del Pezzo surface of degree 1 along the anticanonical divisor.

The stack $W_1^\text{min}$ is not Deligne-Mumford because of the presence of objects with infinite dimensional automorphism group [PS21, Remark 4.5].
Theorem 6.1. Suppose that the ground field \( k \) has characteristic \( \neq 2, 3 \) and set \( r_6 = 576(30c_1^6 + 151c_1^4c_2 + 196c_1^2c_2^2 + 64c_2^3) \). Then we have

\[
\text{CH}^*(W_{1\text{min}}) \cong \mathbb{Z}[c_1, c_2]/(6c_1c_2r_6, c_1^3r_6, c_1^2c_2r_6),
\]

where \( c_1 \) and \( c_2 \) are the Chern classes of the rank two vector bundle \( E_1 \) introduced in Definition 2.11.

Proof. This is a straightforward application of Theorem 5.5. To compute explicitly the Chern classes of the representations involved, one can use Proposition 3.5. The Segre classes are then obtained by formally inverting the total Chern classes. Then one can plug in these expressions into the formulas given in (14) and into the formula given in Proposition 4.2. After performing these computations with Mathematica, we obtain:

\[
[\Delta_1]_{\text{GL}_2} = -3456c_1c_2(30c_1^6 + 151c_1^4c_2 + 196c_1^2c_2^2 + 64c_2^3); \\
f_{1,0} = -576c_1^3(30c_1^6 + 151c_1^4c_2 + 196c_1^2c_2^2 + 64c_2^3); \\
f_{1,1} = -576c_1^2c_2(30c_1^6 + 151c_1^4c_2 + 196c_1^2c_2^2 + 64c_2^3).
\]

This concludes the proof. \( \square \)

6.2. The case \( N = 2 \). The stack \( W_{2\text{min}} \) can be regarded as the stack of lattice-polarized elliptic K3 surfaces, as explained in the Introduction of [CK23]. The coarse space of this moduli stack is particularly interesting and it has been the subject of much work (see for instance [MOP17, PY20]). Here we determine its integral Chow ring.

Theorem 6.2. Suppose that the ground field has characteristic \( \neq 2, 3 \). Then we have

\[
\text{CH}^*(W_{2\text{min}}) \cong \mathbb{Z}[\tau_1, c_2, c_3]/(2c_3, r_9, r_{10}, r_{18}, r_{19})
\]

where

\[
\begin{align*}
r_9 &= 1152(691c_1^2\tau_1 - 38056c_2^2\tau_1^3 + 309568c_2^2\tau_1^5 - 497520c_2\tau_1^7 + 124416\tau_1^9), \\
r_{10} &= 1152(30c_2^3 - 6811c_1\tau_1^2 + 133495c_1^2\tau_1^4 - 481528c_2\tau_1^6 + 327600c_2^2\tau_1^8 - 20736\tau_1^{10}), \\
r_{18} &= 1152c_2(108314154642930c_4 + 1045672c_2^2\tau_1^4 - 89483c_2^2\tau_1^6 + 35c_2\tau_1^8 - 4\tau_1^9), \\
r_{19} &= 2304c_2\tau_1(118203201c_2^2 + 180502c_2^2\tau_1^4 - 7c_2\tau_1^6 + 4\tau_1^8).
\end{align*}
\]

The generator \( \tau_1 \) is the first Chern class of the line bundle \( L_2 \) (see Definition 2.13), the other generators \( c_2 \) and \( c_3 \) are Chern classes of the rank three vector bundle \( E_2 \) (see Definition 2.13), whose first Chern class vanishes.

We will prove this Theorem by applying Theorem 5.5 and by explicitly computing the relations in terms of the generators \( \tau_1, c_2 \) and \( c_3 \).

Lemma 6.3.

\[
[\Delta_1]_{\text{PGL}_2 \times \mathbb{G}_m} = -995328\tau_1(9c_2^2 + 160c_2^2\tau_1^4 + 256\tau_1^6)(100c_2^2 + 5369c_2^2\tau_1^2 + 74074c_2^2\tau_1^4 + 400257c_2^2\tau_1^6 + 972972c_2^2\tau_1^8 + 1061424c_2\tau_1^{10} + 419904\tau_1^{12})
\]

Proof. Instead of applying directly the formula of Proposition 4.4, we first compute \([\Delta_2]_G \bmod c_3\), and then we conclude the computation modulo 2. This trick is inspired by [FV11].

The homomorphism of algebraic groups \( \text{SL}_2 \times \mathbb{G}_m \to \text{PGL}_2 \times \mathbb{G}_m \) given on the first factor by quotienting by \( \mu_2 \) and the second factor by the identity induces a morphism of stacks

\[
\mathcal{B}(\text{SL}_2 \times \mathbb{G}_m) \to \mathcal{B}(\text{PGL}_2 \times \mathbb{G}_m).
\]
By taking the pullback along this map we get a homomorphism of rings

$$\text{CH}^*(\mathcal{B}(\text{PGL}_2 \times \mathbb{G}_m)) \simeq \mathbb{Z}[c_2, c_3, \tau_1]/(2c_3) \rightarrow \text{CH}^*(\mathcal{B}(\text{SL}_2 \times \mathbb{G}_m)) \simeq \mathbb{Z}[c_2, \tau_1],$$

that sends $\tau_1$ to $\tau_1$, the class $c_2$ to $4c_2$ and $c_3$ is sent to zero (see [FV11, Proof of Lemma 5.1]). The pullback of $[\Delta_2^1]_{\mathbb{G}_2}$ along this map is equal to $[\Delta_2^1]_{\text{SL}_2 \times \mathbb{G}_m}$, hence if we compute this last class and we substitute $c_2$ with $c_2/4$ we get an expression of $[\Delta_2^1]_{\mathbb{G}_2}$ that holds up to multiples of $c_3$.

The same argument of Proposition 4.2 shows that

$$[\Delta_2^1]_{\text{SL}_2 \times \mathbb{G}_m} = \frac{c_{22}^\text{SL}_2 \times \mathbb{G}_m(V_{8,12}^2)}{c^\text{SL}_2 \times \mathbb{G}_m(V_4^2)} = \frac{c_{9}^\text{SL}_2 \times \mathbb{G}_m(V_8^2)c_{12}^\text{SL}_2 \times \mathbb{G}_m(V_{12}^2)}{c_{5}^\text{SL}_2 \times \mathbb{G}_m(V_4^2)}.$$

The representation $V_{2m}^2$ is equal to $\text{Sym}^m E^\vee \otimes L^{\otimes(-m)}$, where $E$ is the standard $\text{SL}_2$-representation and $L$ is the standard $\mathbb{G}_m$-representation (of weight one). If $\ell_1$ and $\ell_2$ denote the Chern roots of $E^\vee$ and $\tau_1$ is the first Chern class of $L$, we see that the Chern roots of $V_{2m}^2$ are of the form $i\ell_1 + (2m - i)\ell_2 - m\tau_1$, for $i = 0, \ldots, 2m$. As the product of the Chern roots is equal to the top Chern class, after some computations and after plugging in the relations $\ell_1 + \ell_2 = 0$ and $\ell_1\ell_2 = c_2$, we get

$$[\Delta_2^1]_{\text{SL}_2 \times \mathbb{G}_m} = -1019215872(9c_2^2 + 40c_2\tau_1^2 + 16\tau_1^4)(6400c_2^5\tau_1 + 85904c_2^2\tau_1^3 + 296296c_2^3\tau_1^2 + 243243c_2^4\tau_1 + 66339c_2\tau_1^{11} + 6561\tau_1^{13}).$$

We replace $c_2$ with $\frac{1}{4}c_2$, thus obtaining

$$-995328\tau_1(9c_2^2 + 160c_2\tau_1^2 + 256\tau_1^4)(100c_2^2 + 5369c_2^5\tau_1^2 + 74074c_2^3\tau_1^4 + 972972c_2^2\tau_1^6 + 1061424c_2\tau_1^{10} + 419904\tau_1^{12}).$$

We deduce that $[\Delta_2^1]_{\mathbb{G}_2}$ must be equal to the expression in (18) plus an element of the form $c_3\eta$, where $\eta$ belongs to $\mathbb{Z}[\tau_1, c_2, c_3]/(2)$. In particular, the class of $\Delta_2^1$ modulo 2 is equal to $c_3\eta$. For computing the class of $\Delta_2^1$ modulo 2, we first find an element $\xi$ such that

$$\xi \cdot c_5^{\text{GL}_3 \times \mathbb{G}_m}(\underline{W}_4^2) = c_5^{\text{GL}_3 \times \mathbb{G}_m}(\underline{W}_{8, 12}^2).$$

This task is accomplished by direct computations of the top Chern classes using (6), and then reduction modulo 2: we find a polynomial $\xi$ such that $\xi = \eta^{h^3}$ satisfies the condition (19), where $h = c_1^{\text{GL}_3 \times \mathbb{G}_m}(O(1))$. We are still not done, because by [EF08, page 8] the ring $\text{CH}^*(\mathbb{P}^5/\text{GL}_3 \times \mathbb{G}_m) \otimes \mathbb{Z}/2$ is isomorphic to

$$\mathbb{Z}[\tau_1, c_1, c_2, c_3, h]/(2, h^3(c_1c_2 + c_3 + c_2^2h + c_2h + h^3)),$$

hence the reduction modulo 2 of

$$\frac{c_{22}^{\text{GL}_3 \times \mathbb{G}_m}(\underline{W}_{8, 12}^2)}{c_{5}^{\text{GL}_3 \times \mathbb{G}_m}(\underline{W}_4^2)}$$

is equal to $\xi$ only up to annihilators of $c_5^{\text{GL}_3 \times \mathbb{G}_m}(\underline{W}_4^2)$. This top Chern class is equal modulo 2 to $h^2(c_1c_2 + c_3 + c_2^2h + c_2h + h^3)$, so if $\xi''$ is an annihilator of this element, it must be a multiple of $h$ (this can also be checked directly using the tautological exact sequence on $\mathbb{P}^5$). This shows that the reduction modulo 2 of (20) is divisible by $h$. As the reduction modulo 2 of $[\Delta_2^1]_{\mathbb{G}_2}$ is equal to (20) evaluated at $h = 0$ (see Proposition 4.4), we deduce that this reduction is zero, hence $[\Delta_2^1]_{\mathbb{G}_2}$ is equal to the expression in (18).

According to Theorem 5.5, we need to compute five other relations. The first two are obtained as follows: let $Z_1 \subset V_{8,12}^2 \times \mathbb{P}V_1$ be the subscheme of triples $(A, B, p)$ where $p$ is a point of $\mathbb{P}^1$ and the form $A$ (resp. the
form $B$) vanishes in $p$ with order $\geq 4$ (resp. $\geq 6$). Let $\gamma_1$ be the generator of the $\text{PGL}_2 \times G_m$-equivariant Chow ring of $\mathbb{P}V_1$ as a module over $\text{CH}^* (\mathcal{E}(\text{PGL}_2 \times G_m))$. Then the first two relations are given by
\[
g_{1,0} := \text{pr}_1^* [Z_1]_{\text{PGL}_2 \times G_m}, \quad g_{1,1} := \text{pr}_1^* ([Z_1]_{\text{PGL}_2 \times G_m} \cdot \text{pr}_2^* \gamma_1),
\]
where $\text{pr}_1$ (resp. $\text{pr}_2$) is the projection on the first (resp. second) factor.

Formulas for these two relations are given by (16) with $N = 2$, $k = 1$, $m \in \{0, 1\}$ and $n = 0$. To make these expressions completely explicit we have to plug in the formulas for Chern classes and Segre classes of $V_1$, $V_8$, and $V_{12}$, which can be extracted from Proposition 3.6. After some computations with Mathematica, we get
\[
(21) \quad g_{1,0} = -1152(691c_2^4 \tau_1 - 38005c_2^3 \tau_1^3 + 309568c_2^2 \tau_1^5 - 497520c_2 \tau_1^7 + 124416\tau_1^9)
\]
\[
(22) \quad g_{1,1} = -1152(30c_2^5 + 6811c_2^4 \tau_1^3 + 133495c_2^3 \tau_1^4 + 481528c_2^2 \tau_1^5 + 327600c_2 \tau_1^6 - 20736\tau_1^8)
\]
Let us recall how the other three relations are obtained: let $Z_2 \subset V_{8,12} \times \mathbb{P}V_2$ be the subscheme of triples $(p_1 + p_2, A, B)$ such that $p_1 + p_2$ is a dimension zero subscheme of $\mathbb{P}^1$ of length two and $A$ (resp. $B$) vanishes along $p_1 + p_2$ with order $\geq 4$ (resp. 6). If $h$ denotes the hyperplane section of $\mathbb{P}V_2$ and $\text{pr}_1$, the projection on the $i^{th}$-factor, then the cycles
\[
g_{2,0} := \text{pr}_1^* [Z_2]_{\text{PGL}_2 \times G_m}, \quad g_{2,1} := \text{pr}_1^* ([Z_2]_{\text{PGL}_2 \times G_m} \cdot \text{pr}_2^* h), \quad g_{2,2} := \text{pr}_1^* ([Z_2]_{\text{PGL}_2 \times G_m} \cdot \text{pr}_2^* h^2)
\]
are the three relations we are looking for.

Formulas for these relations are given in Proposition 5.4: they correspond to the cases $N = 2$, $k = 2$ and $0 \leq m \leq 2$. Observe that in this case the representation $V_{2d(N-k)}$ is trivial, hence the only non-zero Segre class is the one of degree zero, which is equal to one. This means that in the summation we can impose $\ell_d = 0$ for $d = 2, 3$.

To make the formulas completely explicit, we only need to plug in the values of the Chern classes of $V_8$ and $V_{12}$ and of the Segre classes of $V_2$, which are computed as before using Proposition 3.6. After some computations with Mathematica, we get
\[
g_{2,0} = -11943936(38562300c_2^3 - 109363770c_2^2 \tau_1 + 134699250c_2^1 \tau_1^3 - 303690446c_2^0 \tau_1^5 + 312766535c_2^1 \tau_1^8
\]
\[- 259047756c_2^1 \tau_1^6 + 192326864c_2^1 \tau_1^4 - 128471616c_2^1 \tau_1^4 + 87091200c_2^1 \tau_1^4 - 11943936\tau_1^{18}),
\]
\[
g_{2,1} = 23887872c_2(37514745c_2^5 - 64489645c_2^4 \tau_1 + 97093545c_2^3 \tau_1^4 - 170981502c_2^2 \tau_1^6 + 142583080c_2^1 \tau_1^8
\]
\[- 114176800c_2^1 \tau_1^6 + 78779520c_2^1 \tau_1^4 - 54743040c_2^1 \tau_1^4 + 23887872\tau_1^{18}),
\]
\[
g_{2,2} = -c_2 \cdot g_{2,0}.
\]
These five relations, together with the fundamental class $[\Delta_2]_{\text{PGL}_2 \times G_m}$ computed in Lemma 6.3, are all we need to compute the integral Chow ring of $W_{2m}^{\text{min}}$.

A quick computation with Mathematica shows that $[\Delta_2]_{\text{PGL}_2 \times G_m}$ belongs to the ideal generated by (21) and (22). After further simplifying it via Mathematica, we obtain the presentation given in Theorem 6.2.

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