Teleparallel gravity on the lattice.

M.A. Zubkov

ITEP, B.Cheremushkinskaya 25, Moscow, 117259, Russia

Abstract

We consider quantum gravity model with the squared curvature action. We construct lattice discretization of the model (both on hypercubic and simplicial lattices) starting from its teleparallel equivalent. The resulting lattice models have the actions that are bounded from below while Einstein equations (without matter) appear in their classical limit.

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Renormalizable asymptotic free theories are, doubtless, well-defined at small distances. We cannot claim the same in the case of renormalizable theories that are not asymptotic free and, of course, in the case of the non-renormalizable ones. Therefore, it would be quite natural to perform an attempt to construct quantum theory of gravity requiring that it must be asymptotic free.

It was recognized long time ago that ultraviolet divergences in the quantum gravity with the action that contains squared curvature term can be absorbed by an appropriate renormalization of the coupling constants [1]. This seems to be a good sign although certain problems are still present. The model considered in [1] and related publications has the following action (being rotated to Euclidean signature):

\[ S = \int \{ \alpha (R_{AB}R_{AB} - \frac{1}{3}R^2) + \beta R^2 - \gamma m_p^2 R + \lambda m_p^4 \} |E| d^4x, \]

where \( |E| = \det E^A_{\mu}, \) \( E^A_{\mu} \) is the inverse vierbein, the tetrad components of Ricci tensor are denoted by \( R_{AB}, \) and \( R \) is the scalar curvature. The coupling constants \( \alpha, \beta, \gamma \) and \( \lambda \) are dimensionless while \( m_p \) is a dimensional parameter. Linearized theory (around flat background) contains graviton together with additional tensor and scalar excitations. The propagator behaves like \( \frac{1}{q^4} \) in ultraviolet while \( \alpha, \beta \neq 0. \) Tensor excitation is a ghost, which leads to loss of unitarity. A complete, nonperturbative, consideration of the asymptotic states could possibly become a solution of this problem [2]. Probably, a traditional approach does not work here just because the interaction between quantum excitations (and their formation as well) is much more complicated than it is implied when usual perturbation methods are used. Nevertheless, the perturbation expansion for the Green functions could still contain an important information. We expect that due to the renormalizability of the theory this expansion appears to be self-consistent and could be, in principle, used as an approximation scheme.

However, the unitarity problem is not the only one that is encountered when we consider the theory with the action (1). Namely, the requirement that the action (1) is bounded from below leads to the appearance of a tachyon. This indicates that flat space is not the real vacuum of the model. The tachyon would disappear if we construct the perturbation expansion around the background that minimizes (1) [5]. In addition to the ultraviolet divergences the perturbation expansion may also contain infrared divergences. In order to separate their consideration from the consideration of
the ultraviolet ones we have to use an additional regularization. This can be done, for example, if the invariant volume $V$ of the space-time manyfold is kept constant. Then after the usual regularization (say, the dimensional or lattice regularization) is removed and all the ultraviolet divergences are absorbed by the redefinition of the coupling constants, each term of the perturbation expansion appears to be finite. In the theory with fixed invariant volume the cosmological constant does not influence the dynamics and the action is bounded from below if $(\alpha \geq 0, \beta > 0, \gamma \neq 0)$ or $(\alpha \geq 0, \beta \geq 0, \gamma = 0)$. The renormalization group analysis shows [5] that at $\alpha, \beta > 0$ there exists a region of couplings such that the theory is asymptotic free in $\alpha$ and $\beta$ while $\gamma$ can be made constant (up to one-loop approximation). We do not discuss in this paper the possible divergences that may appear in the limit $V \to \infty$. Let us mention, however, that similar divergences do appear in QED but they are compensated by the ejection of soft photons. Probably, the same mechanism may work here as well.

Unfortunately the classical Newtonian limit cannot be obtained directly from the action (1) unless it is not bounded from below. However, if we start from the pure gravity model with the action (1) (with $\lambda = 0$) and rotate it back to Minkowski signature, the solutions of Einstein equations would satisfy the appeared classical equations of motion $^1$. Then massive point-like objects could be treated, in principle, as space-time singularities [4] and the Newtonian limit appears as an asymptotic of black hole solutions. Suppose that the line-like singularity is embedded into the space-time. Then the Einstein equations in empty space lead to the Einstein equations in the presence of a particle moving along the mentioned singularity. Its mass is not fixed by the field equations but it is proved to be constant along the world trajectory [4]. This indicates that matter can be introduced into the quantum gravity theory with the action (1) in such a way that it reproduces the results of general relativity at $\alpha > 0, \beta > 0, \lambda = 0$.

In this paper we suggest the way to construct lattice theory (both on hypercubic and simplicial lattices) starting from the action (1) with $\alpha > 0, \beta > 0$. A traditional approach to the discretization of gravity is based upon its geometrical interpretation in terms of Riemannian geometry. Therefore the main concepts are metrics and $SO(3,1)$ (or $SO(4)$) connection. In several

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$^1$They are not the only solutions of the equations of motion. However, at $\gamma = \lambda = 0$ Einstein spaces minimize the (Euclidean) action. Therefore it could be interesting to consider the theory with the action (1) such that at some scale the renormalized couplings $\gamma$ and $\lambda$ vanish.
schemes the larger group (say, Poincare or de Sitter groups) are considered. Vierbein together with the $SO(3,1)$ gauge field appear as a part of the correspondent connection, and the metrics is composed of the vierbein$^6$ $^7$. Without the zero torsion constraint the correspondent geometry can be easily transferred to the lattice. However, it is exactly this constraint that makes the geometry Riemannian. Its implementation on the lattice is difficult (although possible $^2$ $^7$) and it is quite easy to understand the reason. The matter is that in Riemannian geometry the space-time indices are allowed to be mixed with the internal ones. Hypercubic discretization obviously breaks the $SO(3,1)$ ($SO(4)$) symmetry of space-time. This should lead, consequently, to breaking of the internal $SO(3,1)$ ($SO(4)$) symmetry. The theory that is a gauge theory in continuum looses its main symmetry while being transferred to lattice! Therefore, the $SO(3,1)$ symmetry (like the chiral one) should be treated on the lattice approximately. Unfortunately, the correspondent models $^2$ $^7$ have not yet been investigated numerically.

An alternative approach to the discretization of Riemannian geometry is given by the Regge calculus $^8$ $^9$ $^{10}$, where space-time is approximated by the set of simplices glued together. Each simplex is assumed to be flat. The geometry is defined by the sizes of the simplices. The correspondent numerical research has been performed extensively for last 20 years. The theory with the squared curvature action has been investigated in this approach (see $^9$). An essential shortcoming of the Regge approach is the uncertainty of the choice of the measure over the link lengths $^{10}$.

In the Dynamical Triangulation (DT) variant of Regge calculus $^{11}$ the sizes of simplices are fixed and the way to glue them together is the dynamical variable. Unfortunately, except for the case $D = 2$ the simplicial manyfold cannot reproduce flat space. It is usually implied that flat space appears approximately. But in this case, say, the local lattice squared curvature action can not tend to the action $^{11}$ at $\gamma = \lambda = 0$ in the continuum limit: the term linear in $R$ should always appear$^{12}$.

Our approach$^2$ is based upon the teleparallel formulation of general rel-
ativity [15]. The correspondent geometrical construction is the so-called Weitzenbock space that appears as a limiting case of a more general concept - Riemann - Cartan space. The latter is a tangent bundle equipped with the connection from Poincare algebra. Poincare group consists of the Lorentz transformations and translations. Translational part of connection can be identified with the inverse vierbein and defines space-time metrics. The correspondent part of the curvature becomes the torsion of the Lorentz part of the connection. The Riemannian geometry appears when the torsion is set to zero. Weitzenbock geometry is an opposite limit: Lorentz part of Poincare curvature is set to zero while the torsion remains arbitrary. Teleparallel gravity is the theory of Weitzenbock geometry, i.e. a translational gauge theory.

If the space-time manifold is parallelizable (as it is implied in the present paper), the zero curvature Lorentz connection can be chosen equal to zero. Therefore the only dynamical variable is the inverse vierbein, treated as a translational connection. Usually the action in teleparallel gravity is expressed through the translational curvature (torsion); the space-time and internal indices can be contracted by the vierbein. The equivalence between continuum theories of Riemannian and Weitzenbock geometries can be set up if everything in Riemannian geometry is expressed through the inverse vierbein and the latter is identified with the translational connection. For example, there exists a special choice of classical teleparallel gravity with the action quadratic in torsion that is identical to usual general relativity.

It is worth mentioning that the concept of torsion has already been transferred to lattice in the ranges of the Poincare gravity as the dislocation on the simplicial lattice [16]. Lattice discretization of teleparallel gravity may, in principle, be constructed starting from [16] if the simplicial manifold is kept correspondent to flat space. In the resulting model one would in addition specify definition of the vierbein, the functional measure and the lattice derivative of torsion. Another representation of lattice teleparallel gravity was considered in [17], where the basic construction is, in essence, the usual simplicial manifold used in Regge calculus. Riemann tensor of the simplicial manifold can be expressed through the squared derivatives of the inverse vierbein, that are identified with the lattice Weitzenbock torsion. As a result in [17] the component of lattice torsion relevant for the teleparallel equivalent of the Regge theory with Einstein action is defined. This approach allows simplified version ($\alpha = 0$) that is not renormalizable.
to treat the simplicial Riemannian manifold as an approximation of the lattice Weitzenbock space and to express classical Einstein - Regge equations in terms of torsion. Unfortunately, the extension of the formalism to higher derivative gravity has not been made.

In the present paper we suggest an alternative way to discretize teleparallel gravity that is equivalent to the usual quantum gravity theory with the action (1). In our approach the Weitzenbock geometry is transferred to lattice directly, without use of any additional structures (such as the simplicial Riemannian manifold of the mentioned above approaches of [16, 17]). Our main variable is the lattice connection of Abelian gauge group of translations and everything is expressed through it.

Before proceed with the description of the lattice construction let us remind briefly the continuum definition of the model under consideration. The inverse vierbein is denoted by $E^A_\mu$ (everywhere space - time indices are denoted by Greek letters contrary to the tetrad ones) and is regarded as the translational connection. In the teleparallel formulation Riemann tensor can be expressed through the translational curvature (torsion) $T_{ABC} = E^A_{[\mu,\nu]} E^\mu_B E^\nu_C$:

$$R_{ABCD} = \gamma_{AB[C,D]} + \gamma_{ABF} \gamma_{F[CD]} + \gamma_{ABC} \gamma_{FBD} - \gamma_{AFD} \gamma_{FBC}.$$  

where $\gamma_{ABC} = \frac{1}{2}(T_{ABC} + T_{BCA} - T_{CAB})$. Quantum theory is defined by the functional integral for the partition function:

$$Z = \int DE \exp(-S[E])$$  

The measure over $E$ is defined now as the measure over the translational gauge group.

The theory under consideration is diffeomorphism invariant ($x^\mu \rightarrow \tilde{x}^\mu; E^A_\mu \rightarrow \frac{\partial \tilde{x}^\mu}{\partial x^\nu} E^A_\nu$). Substituting Faddeev - Popov unity we therefore are able to fix partially the gauge, in which $|E| = \text{const}$. This freezes the fluctuations of the overall invariant volume. In this paper we disregard these fluctuations as well as the other global properties of the space - time manifold. The correspondent Faddeev - Popov determinant is independent of $E$. So, we have

$$Z = \int DE \exp(-S[E]) \delta(|E| - \text{const}),$$  

As it was mentioned, continuum version of the model was shown to be renormalizable and asymptotic free for the appropriate choice of couplings.$^3$

$^3$The choice of the measure $DE$ is known not to affect this result.
For these values of couplings the action is bounded from below (if the fixed invariant volume is kept constant). So we hope that the correspondent lattice Euclidean functional integral can define a self-consistent theory.

We consider either simplicial or hypercubic lattice. For the further convenience we refer to simplices (hypercubes) as to elements of the lattice. Space inside each lattice element is supposed to be flat. Form of the lattice elements is fixed by the set of vectors \( e_\mu \) connecting the center of the element with the centers of the sides of its boundary. The expression of \( e_\mu \) through elements of the orthonormal frame \( f_A \) \((A = 1, 2, 3, 4)\) (common for all lattice elements) is the basic variable of the construction. So we have

\[
e_\mu(x) = \sum_A E^A_\mu(x)f_A
\]

We imply that not all of the vectors \( e_\mu \) are independent. Namely, in hypercubic case \( e_\mu + e_{-\mu} = 0 \) \((\mu = \pm 4, \pm 3, \pm 2, \pm 1)\), where \( e_\mu \) and \( e_{-\mu} \) connect the center of the hypercube with the opposite sides of its boundary. In the case of simplicial lattice \((\mu = 1, 2, 3, 4, 5)\) we have \( \sum_\mu e_\mu = 0 \). Also we imply that all sides of element's boundary are ordered in some way. Independent variables in both cases are denoted by \( E^A_\mu \) with \( \mu = 1, 2, 3, 4 \).

Translational curvature (torsion) is attached to the bones (that are 2-dimensional subsimplices in simplicial case and plaquettes in hypercubic case). We consider the closed path connecting the centers of the lattice elements that contain the given bone. The piece of the path connecting the centers \( x \) and \( y \) of neighbor lattice elements consists of two vectors \( e_{M_{xy}}(x) \) and \( e_{M_{yx}}(y) \) that connect \( x \) and \( y \) with the center of the side common for the correspondent lattice elements. Thus integer-valued function \( M_{xy} \) is defined on the pairs of centers of neighbor lattice elements. We denote \( e_{xy} = e_{M_{xy}}(x) - e_{M_{yx}}(y) \) and \( e_{yx} = e_{M_{yx}}(y) - e_{M_{xy}}(x) \). For the closed path \( x \rightarrow y \rightarrow z \rightarrow ... \rightarrow w \rightarrow x \) around the given bone (consisted of \( O \) points) the lattice torsion is

\[
T_{xyz...w} = \frac{1}{s}(e_{xy} + e_{yz} + ... + e_{wx})
\]

where \( s \) is the area inside the path. If we set the distance between the centers of neighbor elements equal to unity, then \( s = 1 \) in the hypercubic case and \( s = O\sqrt{\frac{D+1}{D-1}} \) with \( D = 4 \) in the simplicial case. We also denote \( T_{xyz...w} = T_A^{M_{xy}M_{yw}}(x)f_A \), where summation over \( A \) is assumed.

Now let us express basis vectors \( f \) through \( e \). We denote

\[
f_A = \sum_\mu F^A_\mu(x)e_\mu(x)
\]
Here the sum is over all vectors $\mathbf{e}$ (not only over the four independent ones). For the hypercubic case we denote by $\mathbf{E}$ the $4 \times 4$ matrix consisted of $E^A_\mu$ with positive $\mu$. Then $F$ can be chosen in the form $F^\mu_A = \frac{\text{sign}(\mu)}{2} (\mathbf{E}^{-1})^\mu_A$. For the simplicial case the situation is a little more complicated. In order to obtain the symmetric expression let us denote $\bar{E}^\mu_A = \begin{bmatrix} E_1^\mu & \cdots & E_4^\mu \end{bmatrix}^T$ and $E^\mu = [\bar{E}_1^\mu \cdots \bar{E}_{\mu-1}^\mu \bar{E}_{\mu+1}^\mu \cdots \bar{E}_{5}^\mu]$. Therefore $F$ can be taken in the form: $F^\mu_A = \frac{1}{5} \sum_{\nu \neq \mu} (E^{-1})^\rho_{\nu,\mu} A$, where $\rho(\nu, \mu) = \mu$ if $\nu > \mu$ and $\rho(\nu, \mu) = \mu - 1$ if $\nu < \mu$.

Now tetrad components of the torsion can be easily defined as

$$T_{ABC}(x) = \sum_{\mu,\nu} F^\mu_B F^\nu_C T^A_{\mu\nu}(x) \tag{8}$$

We also define tetrad components of the lattice derivative of $T$: $T_{ABC,D}(x) = \sum_{\mu} F^\mu_B T_{ABC,\mu}(x)$, where $T_{ABC,\mu}(x) = T_{ABC}(y) - T_{ABC}(x)$ for $\mu = M_{xy}$.

Formally applying expression (2) to the defined lattice variables we obtain the definition of lattice Riemann tensor. Then Ricci tensor and the scalar curvature are defined as usual. Lattice action therefore is constructed as follows:

$$S = \sum_x \left\{ \alpha (R_{AB}(x)R_{AB}(x) - \frac{1}{3} R(x)^2) + \beta R(x)^2 - \gamma m_p^2 R \right\} \tag{9}$$

We omit here the volume factor $|E|$ and the term proportional to $\lambda$ since the lattice analogue of the gauge $|E| = \text{const}$ is implied. It is necessary to consider the model in this gauge if we want to consider the theory in constant invariant volume. Lattice analogue of the continuous gauge condition $|E| = \text{const}$ is the condition of constant invariant volume of each lattice element. If we denote $\mathbf{E} = \mathbf{E}_5$ for the simplicial case, the correspondent condition can be chosen in the form $\det \mathbf{E} - v = 0$ (for both lattices), where $v$ is the constant that is proportional to the overall volume of the lattice.

Thus $\mathbf{E}$ is the dynamical variable, all $E^A_\mu$ are expressed through it, and the partition function of the model has the form

$$Z = \int \Pi_x D\mathbf{E}(x) \exp(-S[\mathbf{E}]), \tag{10}$$

where

$$D\mathbf{E} = (\Pi_{A,B} dE^A_B) \delta(\det \mathbf{E} - v) \tag{11}$$

In order to complete the construction global properties of the discretized manifold $\mathcal{M}$ should be set up. First of all we should mention that to consider an open space - time manifold is impossible since we have not specified
boundary terms of the action. Therefore instead of performing a correspond-
dent construction (that should be based on rather serious theoretical consid-
erations) we restrict ourselves to the case of a closed manyfold. The natural
restriction on its topology is that it should be parallelizable. Otherwise our
variables do not describe Weitzenbock geometry.

It is much more useful to describe a case of complicated topology by
the simplicial manyfold than by a hypercubic lattice. On the other hand,
the theory defined on the latter one is well - suited for the investigations in
the case of simple topology. The hypercubic lattice is supposed to be used
mainly for the investigations in the case of torus $T^4$ that corresponds to the
usual periodic boundary conditions. Of course, the simplicial construction
described in this Letter can be used in this case as well.

The theory defined on the simplicial lattice can, in principle, also be used
for the investigation of fixed complicated topology case and the topology -
changing processes\textsuperscript{4}. The description of these cases is, however, out of the
scope of the present paper. In this respect we would like to mention that
in general case the topologies of the 4 - dimensional manyfolds cannot be
classified. In order to deal with the classified topologies a strong constraint
on $M$ should be imposed. A usually implied constraint is the existence of
a spinor structure [3]. In this case the signature $\tau$ and Euler characteristics
$\chi$ characterize manyfolds up to a homotopy (if $|\tau| \neq \chi - 2$). In order to
investigate topology changing processes the simplicial manyfold has to be
dynamical (like in the DT models). In addition to the topology - preserving
$(p, q)$ moves the topology changing moves should be defined (see, for example,
[18]). An alternative way to investigate dynamical topology is to find Matrix
model that is equivalent to the simplicial theory [19].

Finally, we would like to summarize our expectations. First of all, asymp-
totic freedom means that at small distances perturbation theory could be ap-
plied and the dynamics in weak coupling is described through the exchange
of virtual gravitons, tensor and scalar particles. Bare particles cannot, how-
ever, be treated as real quantum states because the virtual tensor particle

\textsuperscript{4}Strictly speaking, our simplicial construction is applicable to the case of parallelizable
manyfolds only. In the case of complicated topology the requirement that the manyfold
is parallelizable is not acceptable. Therefore in this case we should supplement the con-
struction described in this Letter with taking into account the fixed zero curvature $SO(4)$
connection (living on the 3 - simplices) that allows to joint different pieces of the given
simplicial manyfold (each piece has a fixed frame). Correspondingly, the derivatives (and
their discretizations) must be substituted by the covariant ones.
is a ghost. The complete treatment of the bound states should appear as a result of the nonperturbative investigation. A possible analytical approach to the nonperturbative dynamics is the investigation of the topological excitations. Nontrivial solutions of equations of motion for the action \[ \Pi \] (at \( \alpha, \beta > 0 \)) exist for any topology\(^5\). They can, in principle, play the same role as instantons in quantum chromodynamics. In their numerical investigation the simplicial formulation of the model could be useful. The main method of investigation of the constructed lattice models is, of course, the numerical simulation. The main expectation here is that (similarly to the asymptotic free Yang-Mills theory) the continuum limit may appear at \( \alpha, \beta \rightarrow \infty \).

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