A REACTION-DIFFUSION SYSTEM ARISING IN GAME THEORY: EXISTENCE OF SOLUTIONS AND SPATIAL DOMINANCE

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Abstract. The initial value problem for a reaction-diffusion system with discontinuous nonlinearities proposed by Hofbauer in 1999 as an equilibrium selection model in game theory is studied from the viewpoint of the existence and stability of solutions. An equilibrium selection result using the stability of a constant stationary solution is obtained for finite symmetric 2 person games with a $1/2$-dominant equilibrium.

1. Introduction. The concept of Nash equilibrium has played a central role as a solution concept in game theory. However, when a game has multiple Nash equilibria, the players face a problem which equilibrium they should play. As an important concept of equilibrium selection for $2 \times 2$ games with two strict Nash equilibria, there is the concept of risk-dominance by Harsanyi and Selten [6]. That a strict Nash equilibrium is risk-dominant means that it is the equilibrium which minimizes players’ shared risk that neither strict Nash equilibrium may be played. On the other hand, Hofbauer [9] introduced the concept of spatial dominance using the stability of a constant stationary solution, which corresponds to a Nash equilibrium, to a reaction-diffusion system. That a Nash equilibrium is spatially dominant means that if it prevails initially on a large enough finite part of the space, then it eventually takes over the whole space. According to Hofbauer [8, 9], the selection criterion of spatial dominance agrees with that of risk-dominance for $2 \times 2$ games with two strict Nash equilibria. This agreement of completely different approaches to equilibrium selection is interesting. Spatially dominant equilibria have been studied for supermodular games by Oyama, Takahashi and Hofbauer [13] and Takahashi [14]. However, the selection criteria of the spatial dominance concept for games without supermodularity are unknown. In addition, the problem of the existence of weak solutions of the reaction-diffusion system used is open, except the case of particular games, see [13, 14, 3, 4]. There is no uniqueness of weak solutions in general, see [1]. Asymptotic behavior of weak solutions in particular cases has been studied in Terman [15], McKean and Moll [11, 2] and [4].

The first purpose of this paper is to establish the existence of weak solutions of the reaction-diffusion system arising in the case of finite symmetric 2 person games.
The second is to investigate the selection criteria of spatial dominance for such games.

The rest of this paper is organized as follows. In Section 2, we first explain the approach of Hofbauer [9] using a reaction-diffusion system to equilibrium selection for finite symmetric 2-person games. Next, we give the definition of a weak solution of the reaction-diffusion system, and prove its existence (Theorem 2.3). Section 3 is devoted to the spatial dominance of a Nash equilibrium. It is shown that, if a Nash equilibrium is 1/2-dominant, then it is spatially dominant (Theorem 3.2). The concept of 1/2–dominance is a generalization of that of risk-dominance for finite symmetric 2-person games. Furthermore, a condition for a Nash equilibrium to be not spatially dominant is given (Proposition 3).

2. The model. In this section we first describe the approach of Hofbauer [9] using a reaction-diffusion system for the problem of equilibrium selection. We consider finite symmetric 2-person games. Such a game is characterized by an payoff matrix $A = (a_{ij})$. The real number $a_{ij}$ denotes the payoff which the pure strategy $i$ receives when playing against $j$. We assume that there is a large population of players distributed on $\mathbb{R}$, each of which is playing a pure strategy. Let $u_i = u_i(t, x) \geq 0$ denote the relative frequency of the pure strategy $i$ played in the population at time $t$ and spot $x \in \mathbb{R}$. The probability simplex $\Delta$ (of dimension $n - 1$) is then the set of all possible population states $u = (u_1, \ldots, u_n)^T$ (with $u_i \geq 0$ and $\sum_{i=1}^n u_i = 1$). Assume that the local interaction among the players is described by the best response dynamics. This dynamics was introduced by Gilboa and Matsui [5] and further studied by Hofbauer [7]. It models the situation that a certain proportion of players at each spot $x$ switches to locally best replies at any time $t$. Assume further that the random migration of the players can be modeled by diffusion. These assumptions yield the initial value problem

\[
\begin{cases}
  u_t - u_{xx} + u \in BR(u), & 0 < t < T, \quad x \in \mathbb{R}, \\
  u|_{t=0} = u_0, & x \in \mathbb{R},
\end{cases}
\]

where $BR(u(t, x))$ is the set of all (mixed) best replies to $u(t, x)$, namely,

$$BR(u(t, x)) := \{v \in \Delta \mid v \cdot Au(t, x) = \max_{w \in \Delta} w \cdot Au(t, x)\} \subseteq \Delta.$$ 

The best reply correspondence $u \mapsto BR(u)$ has the following two properties:

(i) $BR(u)$ is a non-empty convex subset of $\Delta$ for any $u \in \Delta$;

(ii) $u \mapsto BR(u)$ is upper-semi-continuous, i.e., for any $(u_m)_{m \in \mathbb{N}} \subseteq \Delta$ and any $(v_m)_{m \in \mathbb{N}} \subseteq \Delta$, if $v_m \in BR(u_m)$ for $m \in \mathbb{N}$ and if $u_m \to u$ and $v_m \to v$ as $m \uparrow \infty$, then $v \in BR(u)$.

Recall the notion of spatial dominance introduced by Hofbauer [9].

**Definition 2.1.** A Nash equilibrium $N \in \Delta$ is said to be spatially dominant if the corresponding constant stationary solution $u_N$ of problem (1) is asymptotically stable in the compact-open topology, that is, there exist two constants $r > 0$ and $\varepsilon > 0$ such that, for any initial data satisfying $|u_0(x) - u_N| < \varepsilon$ for $x \in [-r, r]$, the solution $u$ of problem (1) converges to $u_N$ as $t \uparrow \infty$, uniformly on any compact subset of $\mathbb{R}$.

In other words, if a spatially dominant equilibrium prevails initially on a large enough finite part of the space, then it takes over the whole space in the long run. By definition, at most one equilibrium can be spatially dominant. Hence, if
existence can be shown, spatial dominance can be used as an equilibrium selection criterion.

As stated above, we want to treat bounded solutions taking values in $\Delta$ of problem (1), so we define weak solutions of problem (1) as follows. Let $C_B([0,T])$ and $C_B([0,T] \times \mathbb{R})$ denote the space of bounded continuous functions on $\mathbb{R}$ and $[0,T] \times \mathbb{R}$, respectively. Let $C^{0,1}([0,T] \times \mathbb{R})$ be the space of continuous functions on $[0,T] \times \mathbb{R}$ that are continuously differentiable with respect to $x \in \mathbb{R}$.

**Definition 2.2.** For initial data $u_0$ taking values in $\Delta$ with $u_{0,i} \in C_B(\mathbb{R})$, a vector function $u$ taking values in $\Delta$ with $u_i \in C_B([0,T] \times \mathbb{R}) \cap C^{0,1}([0,T] \times \mathbb{R})$ is called a \textit{weak solution} of problem (1) if it satisfies the following two conditions:

(i) there exist $\beta_i \in L^\infty((0,T) \times \mathbb{R})$ such that $(u_i)_t = (u_i)_{xx} - u_i + \beta_i$ in $\mathcal{D}'((0,T) \times \mathbb{R})$, that is, for all $\varphi \in C_0^\infty((0,T) \times \mathbb{R})$,

$$
\int_0^T \int_\mathbb{R} (u_i \varphi_t - (u_i)_{xx} \varphi_x + (-u_i + \beta_i) \varphi) \, dx \, dt = 0, \quad i = 1, \ldots, n,
$$

and that

$$(\beta_1(t,x), \ldots, \beta_n(t,x))^T \in BR(u(t,x)) \tag{3}$$

a.e. in $(0,T) \times \mathbb{R}$;

(ii) $u_i(0,x) = u_{0,i}(x)$ on $\mathbb{R}$ for $i = 1, \ldots, n$.

**Remark 1.** By a similar argument to the proof of Lemma 2.5(a) of [1], we see that, if there exists a solution $u$ taking values in $\Delta$ with $u_i \in C_B([0,T] \times \mathbb{R})$ of problem (1), then $u_i$ satisfies the integral equation

$$
u_i(t,x) = e^{-t} \int_\mathbb{R} K(t,x-y)u_{0,i}(y) \, dy + \int_0^t \int_\mathbb{R} K(t-s,x-y)\beta_i(s,y)e^{-(t-s)} \, dy \, ds \tag{4}$$

in $(0,T) \times \mathbb{R}$, where $K(t,x) = (4\pi t)^{-1/2}e^{-x^2/(4t)}$. We find from this integral equation that $u_i$ belongs to $C^{0,1}((0,T) \times \mathbb{R})$. We thus study solutions $u$ taking values in $\Delta$ with $u_i \in C_B([0,T] \times \mathbb{R}) \cap C^{0,1}((0,T) \times \mathbb{R})$ of problem (1).

We now prove the existence of weak solutions of problem (1).

**Theorem 2.3.** For any initial data $u_0$, there exists a global weak solution $u$ of problem (1).

**Proof.** We use an idea of Hofbauer [7], where a simple proof of the existence of solutions independent of $x$ of problem (1) has been given.

For $k \in \mathbb{N}_0$ and $\varepsilon > 0$, we put $\tau_k^\varepsilon = k\varepsilon$ and consider the initial value problem

$$
\begin{cases}
    u^\varepsilon_t(\tau_k^\varepsilon + t, x) - u^\varepsilon_{xx}(\tau_k^\varepsilon + t, x) + u^\varepsilon(\tau_k^\varepsilon + t, x) \in BR(u^\varepsilon(\tau_k^\varepsilon, x)), & 0 < t < \varepsilon, \quad x \in \mathbb{R}, \\
    u^\varepsilon(0,x) = u_0(x), & x \in \mathbb{R}.
\end{cases} \tag{5}
$$

Using Lemma 2.4 of [1], we see that, for any $T > 0$, there exists a weak solution $u^\varepsilon$ with $u^\varepsilon_i \in C_B([0,T] \times \mathbb{R}) \cap C^{0,1}((0,T) \times \mathbb{R})$ of problem (5), and further that $u^\varepsilon_i$ can be expressed in the form

$$
u^\varepsilon_i(t,x) = e^{-t} \int_\mathbb{R} K(t,x-y)u_{0,i}(y) \, dy + \int_0^t \int_\mathbb{R} K(t-s,x-y)\beta^\varepsilon_i(s,y)e^{-(t-s)} \, dy \, ds \tag{6}$$
for \(i = 1, \ldots, n\), where
\[
B^\epsilon(s, y) := (\beta^\epsilon_1(s, y), \ldots, \beta^\epsilon_n(s, y))^T
\]
\[
= (\beta^\epsilon_1(\tau^\epsilon_k, y), \ldots, \beta^\epsilon_n(\tau^\epsilon_k, y))^T \in BR(\mathbf{u}^\epsilon(\tau^\epsilon_k, y))
\]
for \(\tau^\epsilon_k \leq s < \tau^\epsilon_{k+1}\) with \(k \in \mathbb{N}_0\). We will prove that \((\mathbf{u}^\epsilon)_\epsilon > 0\) has a convergent subsequence whose limit is a weak solution of problem \([1]\).

The uniform boundedness of \((B^\epsilon)_\epsilon > 0\) and Lemma 2.4 of \([1]\) show that \((\mathbf{u}^\epsilon)_\epsilon > 0\) is equicontinuous on any compact subset of \((0, T) \times \mathbb{R}\). Hence by the Arzelà–Ascoli theorem and a diagonal method, there exists a subsequence (denoted also by \((\mathbf{u}^\epsilon)_\epsilon > 0\)) converging to a function \(\mathbf{u}^0\) with \(u^0_i \in C_B((0, T) \times \mathbb{R})\). Note that \(L^\infty((0, T) \times \mathbb{R})\) is the dual space of \(L^1((0, T) \times \mathbb{R})\). Then we can apply the Banach–Alaoglu theorem to find that \((B^\epsilon)_\epsilon > 0\) has a weak–* convergent subsequence (denoted also by \((B^\epsilon)_\epsilon > 0\)). Denote by \(B\) with \(\beta_i \in L^\infty((0, T) \times \mathbb{R})\) the weak–* limit of this subsequence. The limit \(\mathbf{u}\) of \((\mathbf{u}^\epsilon)_\epsilon > 0\) on \([0, T) \times \mathbb{R}\) is given by
\[
\mathbf{u}(t, x) = \left\{ \begin{array}{ll}
\mathbf{u}^0(t, x), & \text{if } 0 < t < T \text{ and } x \in \mathbb{R}, \\
\mathbf{u}_0(x), & \text{if } t = 0 \text{ and } x \in \mathbb{R}.
\end{array} \right.
\]

We see that \(\mathbf{u}(t, x) \in \Delta\) in \([0, T) \times \mathbb{R}\), using the fact that \(\mathbf{u}^\epsilon(t, x) \in \Delta\) in \([0, T) \times \mathbb{R}\). Taking the limit as \(\epsilon \downarrow 0\) in equation \((\ref{eq:u})\) we get
\[
u_i(t, x) = e^{-t} \int_{\mathbb{R}} K(t, x - y) u_{0,i}(y) dy
\]
\[
+ \int_0^t \int_{\mathbb{R}} K(t - s, x - y) \beta_i(s, y) e^{-(t-s)} dy ds,
\]

since \((s, y) \mapsto K(t - s, x - y)\) belongs to \(L^1((0, t) \times \mathbb{R})\). It follows from \((\ref{eq:beta})\) that \(u_i\) belongs to \(C_B((0, T) \times \mathbb{R}) \cap C^{0,1}((0, T) \times \mathbb{R})\). Furthermore, it is seen that \(u_i\) satisfies equation \((\ref{eq:u})\). Thus it remains to prove that \(B\) obtained above satisfies \((\ref{eq:B})\).

Let \(h\) be any fixed element of \(L^1((0, T) \times \mathbb{R})\) such that \(h \geq 0\) and \(\operatorname{supp}\ h = C\) with compact set \(C \subset (0, T) \times \mathbb{R}\). Consider the limit of \(\int_0^T \int_{\mathbb{R}} \beta^\epsilon_i(t, x) h(t, x) dx dt\) as \(\epsilon \downarrow 0\). We apply Fatou’s lemma to get
\[
\int_0^T \int_{\mathbb{R}} \liminf_{\epsilon \downarrow 0} \beta^\epsilon_i(t, x) h(t, x) dx dt
\]
\[
\leq \int_0^T \int_{\mathbb{R}} \beta_i(t, x) h(t, x) dx dt \leq \int_0^T \int_{\mathbb{R}} \limsup_{\epsilon \downarrow 0} \beta^\epsilon_i(t, x) h(t, x) dx dt.
\]

By choosing suitable indicator functions as \(h\), we have
\[
\liminf_{\epsilon \downarrow 0} \beta^\epsilon_i(t, x) \leq \beta_i(t, x) \leq \limsup_{\epsilon \downarrow 0} \beta^\epsilon_i(t, x)
\]
a.e. in \((0, T) \times \mathbb{R}\) for \(i = 1, \ldots, n\), whence
\[
B(t, x) = \prod_{i=1}^n \left[ \liminf_{\epsilon \downarrow 0} \beta^\epsilon_i(t, x), \limsup_{\epsilon \downarrow 0} \beta^\epsilon_i(t, x) \right]
\]
a.e. in \((0, T) \times \mathbb{R}\). Using a similar argument and noting that \(B^\epsilon(t, x) \in \Delta\) in \((0, T) \times \mathbb{R}\), we obtain that
\[
B(t, x) \in \Delta
\]
a.e. in \((0, T) \times \mathbb{R}\). Now, for any \(t > 0\) and any \(\epsilon > 0\), we take \(k_{t, \epsilon} \in \mathbb{N}_0\) such that \(\tau^\epsilon_{k_{t, \epsilon}} \leq t < \tau^\epsilon_{k_{t, \epsilon} + 1}\). Then \(\tau^\epsilon_{k_{t, \epsilon}} \to t\) as \(\epsilon \downarrow 0\), and \(B^\epsilon(t, x) = B^\epsilon(\tau^\epsilon_{k_{t, \epsilon}}, x) \in \Delta\) a.e. in \((0, T) \times \mathbb{R}\).
BR(ue(τεi,ε,x)) for x ∈ ℝ. By the upper semicontinuity of the best reply correspondence and the uniform convergence of (ue(εT))ε>0 on any compact subset of (0, T) × ℝ, all accumulation points of (BR(t,x))ε>0 belong to BR(ue(x)) in (0, T) × ℝ. From this and the fact that BR(ue(x)) is a convex subset of Δ in (0, T) × ℝ, we find that
\[ \prod_{i=1}^{n} \left( \liminf_{\varepsilon \downarrow 0} \beta_{i}^{\varepsilon}(t,x), \limsup_{\varepsilon \downarrow 0} \beta_{i}^{\varepsilon}(t,x) \right) \cap \Delta \subseteq BR(ue(x)) \] (10)
in (0, T) × ℝ. Together with (8) and (9), (10) implies that \( BR(t,x) \in BR(ue(x)) \) a.e. in (0, T) × ℝ. The proof of Theorem 2.3 is now complete.

3. Spatial dominance. In this section we investigate the selection criteria of spatial dominance for finite symmetric 2 person games. Let us first recall the concept of 1/2-dominance for finite symmetric 2 person games. This is a special case of p-dominance, which was introduced by Morris, Rob and Shin [12] for an equilibrium selection criterion based on the incomplete information approach.

Definition 3.1. We say that the ith unit vector \( e_{i} \in \Delta \) is 1/2-dominant if for any \( j \in \{1, \ldots, n\} \) with \( j \neq i \) and any \( v \in \Delta \) with \( v_{i} \geq 1/2 \),
\[ e_{i} \cdot Av > e_{j} \cdot Av. \]

Namely, the pure strategy \( i \) is the unique best reply for a player if the other player plays \( i \) with probability at least 1/2. In particular, \( e_{i} \) is then a strict Nash equilibrium. We mention that the concept of 1/2-dominance is a generalization of that of risk-dominance for finite symmetric 2 person games.

Theorem 3.2. In finite symmetric 2 person games, a 1/2-dominant equilibrium \( e_{i} \) is spatially dominant.

As a preparation for the proof of Theorem 3.2 we explain some notation and definitions that we will use. Let
\[ H(u) = 0 \) in \((-\infty, 0)\), \( H(u) = 1 \) in \((0, \infty)\), and \( H(0) = [0, 1]\),
\[ \tilde{H}(u) = 0 \) in \((-\infty, 0)\) and \( \tilde{H}(u) = 1 \) in \([0, \infty)\),
\[ \hat{H}(u) = 0 \) in \((-\infty, 0)\) and \( \hat{H}(u) = 1 \) in \((0, \infty).\)

If \( u, v \in C_{R}([0, T] × ℝ) \cap C^{0,1}((0, T) × ℝ) \) and \( u \leq v \) in \([0, T] × ℝ) \), then \([u, v]\) denotes the order interval
\[ \{w \in C_{R}([0, T] × ℝ) \cap C^{0,1}((0, T) × ℝ) \mid u \leq w \leq v \text{ on } [0, T] × ℝ\}. \]

Let \( \alpha \in (0, 1) \), and consider the initial value problem
\[ \begin{cases} v_{t} - v_{xx} + v = H(v - \alpha), & 0 < t < T, \quad x \in ℝ, \\ v|_{t=0} = v_{0}, & x \in ℝ. \end{cases} \] (11)

Similarly to Definition 2.2 we define a weak solution of problem (11).

Definition 3.3. For initial data \( v_{0} \in C_{R}(ℝ) \), a function \( v \in C_{R}([0, T] × ℝ) \cap C^{0,1}((0, T) × ℝ) \) is said to be a weak solution of problem (11) if it satisfies the following two conditions:

(i) there exists \( \beta \in L^{\infty}((0, T) × ℝ) \) such that \( v_{t} = v_{xx} - v + \beta \) in \( \mathcal{D}'((0, T) × ℝ) \),

that is, for all \( \varphi \in C_{0}^{\infty}((0, T) × ℝ) \),
\[ \int_{0}^{T} \int_{ℝ} (v_{t} \varphi - v_{xx} \varphi + (-v + \beta) \varphi) \, dx \, dt = 0, \]
and that
\[ \hat{H}(v(t, x) - \alpha) \leq \beta(t, x) \leq \tilde{H}(v(t, x) - \alpha) \]
a.e. in \((0, T) \times \mathbb{R}\);
(ii) \(v(0, x) = v_0(x)\) on \(\mathbb{R}\).

The definitions of a weak upper solution and a weak lower solution of problem \((11)\) are given as follows.

**Definition 3.4.** For initial data \(v_0 \in C_B(\mathbb{R})\), a function \(v \in C_B([0, T) \times \mathbb{R}) \cap C^{0,1}((0, T) \times \mathbb{R})\) is called a weak upper solution of problem \((11)\) if it satisfies the following two conditions:

(i′) there exists \(\beta \in L^\infty((0, T) \times \mathbb{R})\) such that, for all non-negative functions \(\varphi \in C_0^\infty((0, T) \times \mathbb{R})\),
\[
\int_0^T \int_{\mathbb{R}} (v \varphi_t - v_x \varphi_x + (-v + \beta) \varphi) \, dx \, dt \leq 0
\]
and that
\[ \hat{H}(v(t, x) - \alpha) \leq \beta(t, x) \leq \tilde{H}(v(t, x) - \alpha) \]
a.e. in \((0, T) \times \mathbb{R}\);
(ii′) \(v(0, x) \geq v_0(x)\) on \(\mathbb{R}\).

A weak lower solution of problem \((11)\) is defined by reversing the inequalities in conditions (i′) and (ii′).

We now cite some results from [3]. Consider the following conditions:

(A1) problem \((11)\) has a weak upper solution \(u\) and a weak lower solution \(v\) such that \(u \leq v\) in \((0, T) \times \mathbb{R}\). Furthermore, there exist a non-positive function \(v_1 \in L^\infty((0, T) \times \mathbb{R})\) and a non-negative function \(v_2 \in L^\infty((0, T) \times \mathbb{R})\) such that, for all non-negative functions \(\varphi \in C_0^\infty((0, T) \times \mathbb{R})\),
\[
\int_0^T \int_{\mathbb{R}} (v \varphi_t - v_x \varphi_x + (-v + \hat{H}(v - \alpha) - v_1) \varphi) \, dx \, dt = 0,
\]
\[
\int_0^T \int_{\mathbb{R}} (u \varphi_t - u_x \varphi_x + (-u + \hat{H}(u - \alpha) - v_2) \varphi) \, dx \, dt = 0.
\]

(B1) \(0 \leq v_0(x) \leq 1\) on \(\mathbb{R}\).

The following existence result is a special case of Proposition 5.2 of [3].

**Proposition 1.** Assume that conditions \([A1]\) and \([B1]\) are satisfied. Then problem \((11)\) has the global maximal weak solution \(V\) and the global minimal weak solution \(v\) in the order interval \([u, v]\).

**Remark 2.** There exist the maximal and minimal weak solutions \(V, v \in [0, 1]\) of problem \((11)\) if condition \([B1]\) is satisfied, since \(\overline{v} = 1\) and \(\underline{v} = 0\) are a weak upper solution and a weak lower solution, respectively, of problem \((11)\) as in condition \((A1)\).

To state a uniqueness result, we require the following condition on \(v_0\):

(B2) the function \(v_0\) satisfies (see Figure 1)
\[
\begin{cases}
   v_0 \in C^1(\mathbb{R}), & v_0 \text{ is even on } \mathbb{R}, \quad v_0'(x) \leq 0 \text{ in } (0, \infty), \\
   v_0(x_0) = \alpha \quad \text{and} \quad v_0'(x_0) < 0 \text{ for some } x_0 > 0, & v_0(\infty) = 0.
\end{cases}
\]
Figure 1. Condition \([\text{B2}]\)

**Proposition 2.** Choose \(0 < \alpha < 1/2\), and assume that conditions \([\text{B1}]\) and \([\text{B2}]\) are satisfied. Then there exist two constants \(r > 0\) and \(\varepsilon > 0\) such that, if \(|v_0(x) - 1| < \varepsilon\) for \(x \in [-r, r]\), then the weak solution of problem (11) is globally unique in time.

The proof of Proposition 2 is essentially the same as that of Theorem 3.1 of [3] with the use of Propositions 5.2 and 5.3 of [3] and the following proposition.

**Proposition 3.** Choose \(0 < \alpha < 1/2\), and assume that conditions \([\text{B1}]\) and \([\text{B2}]\) are satisfied. Then there exist two constants \(r > 0\) and \(\varepsilon > 0\) such that, if \(|v_0(x) - 1| < \varepsilon\) for \(x \in [-r, r]\), then problem (11) has a weak lower solution \(v \in [0, 1]\) with the three properties that

(i) \(v\) is as in condition \([\text{A1}]\);

(ii) there exist two constants \(p > 0\) and \(\theta > 0\) such that \(v(t, x) > \alpha\) on \((0, T) \times [-p\sqrt{t} - \theta, p\sqrt{t} + \theta]\) for each \(T > 0\);

(iii) \(v(t, x)\) converges to 1 as \(t \uparrow \infty\), uniformly on any compact subset of \(\mathbb{R}\).

**Proof.** We can choose a constant \(p > 0\) small enough so that

\[
\int_{\frac{\pi}{2}}^{\infty} e^{-x^2} d\xi > \alpha.
\]

Put

\[
F(z) := \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \int_{\frac{\pi}{2} + \frac{\pi z}{p \sqrt{t}}}^{\infty} e^{-\xi^2 - \tau} d\xi d\tau
\]

for \(z \geq 0\). Then \(F\) is continuous and strictly decreasing in \((0, \infty)\). Furthermore, \(F(0) > \alpha\) and \(\lim_{z \uparrow \infty} F(z) = 0\). Hence there exist two constants \(\delta > 0\) and \(\theta > 0\) such that \(F(0) - F(2\theta) - \delta \geq \alpha\). By choosing suitably two constants \(r > 0\) and \(\varepsilon > 0\), we may assume that

\[
v_0(x) \geq 1 - \frac{\delta(x - \theta)^2}{p^2 + 2}
\]

for \(x \in \mathbb{R}\). Now, we define the function

\[
h(t, x) := \begin{cases} 1, & \text{if } |x| \leq p\sqrt{t} + \theta, \\ 0, & \text{otherwise} \end{cases}
\]

in \((0, T) \times \mathbb{R}\), and consider the initial value problem

\[
\begin{align*}
v_t &= v_{xx} - v + h(t, x), & 0 < t < T, & x \in \mathbb{R}, \\
v|_{t=0} &= v_0, & x \in \mathbb{R}.
\end{align*}
\]
Using Lemma 2.4 of [1], we see that problem (12) is uniquely solved in \( C_{\mathbb{R}}(\{0, T\} \times \mathbb{R}) \cap C^{0,1}(\{0, T\} \times \mathbb{R}) \) and \( v \) can be expressed in the form

\[
v(t, x) = e^{-t} \int_{\mathbb{R}} K(t, x - y)v_0(y) \, dy + \int_0^t \int_{\mathbb{R}} K(t - s, x - y)h(s, y)e^{-(t-s)} \, dy \, ds.
\]

Then \( v \) is a desired weak lower solution of problem (11). This can be shown as in Steps 1–6 in the proof of Proposition 3.2 of [3].

We next prove the following proposition on behavior of a weak upper solution of problem (11).

**Proposition 4.** Choose \( 0 < \alpha < 1/2 \) and let \( v \in [0, 1] \) be the weak lower solution constructed in Proposition 3 of problem (11). Furthermore, let \( \bar{v} \in [0, 1] \) be a weak upper solution of problem (11) as in condition (A1). Then, if \( \bar{v}(0, x) \geq v(0, x) \) in \( \mathbb{R} \), we have \( \bar{v}(t, x) \geq v(t, x) \) in \( (0, T) \times \mathbb{R} \) for each \( T > 0 \). Hence, for each \( T > 0 \), the weak upper solution \( \bar{v} \) satisfies

\[
\bar{v}(t, x) > \alpha \text{ on } (0, T) \times [-p\sqrt{t} - \theta, p\sqrt{t} + \theta]
\]

with some two constants \( p > 0 \) and \( \theta > 0 \), and further converges to 1 as \( t \uparrow \infty \), uniformly on any compact subset of \( \mathbb{R} \).

**Proof.** Let \( v, \bar{v} \in [0, 1] \) be as in the statement. By Proposition 1 there exists the minimal weak solution \( w \in [v, 1] \) of problem (11) with \( v_0(x) = v(0, x) \). Furthermore, by Proposition 2 the weak solution \( w \in [v, 1] \) is unique. Hence, by Proposition 1 again, if \( \bar{v}(0, x) \geq v(0, x) \) in \( \mathbb{R} \), then \( w \in [0, \bar{v}] \) must hold. Together with \( w \in [v, 1] \), this implies that, if \( \bar{v}(0, x) \geq v(0, x) \) in \( \mathbb{R} \), then \( \bar{v}(t, x) \geq v(t, x) \) in \( (0, T) \times \mathbb{R} \) for each \( T > 0 \). Thus the assertion follows.

We now turn to the proof of Theorem 3.2.

**Proof of Theorem 3.2.** Assume that \( e_i \) is 1/2-dominant. Let \( u = (u_1, \ldots, u_n)^T \) be a weak solution of problem (1) with initial data \( u_0 = (u_{0,1}, \ldots, u_{0,n})^T \). Then \( u_i \) satisfies the differential equation \( \partial_t u_i = \partial_{xx} u_i - u_i + \beta_i \) and initial data \( u_{0,i} \). The 1/2-dominance assumption on \( e_i \) gives \( \alpha \in (0, 1/2) \) such that \( BR(v) = \{e_i\} \) for any \( v \in \Delta \) with \( v_i \geq \alpha \). This implies that \( \beta_i(t, x) \geq H(u_i(t, x) - \alpha) \) in \( (0, T) \times \mathbb{R} \), whence \( u_i \) is a weak upper solution of problem (11) with \( v_0 = u_{0,i} \) as in condition (A1). By Proposition 4 if \( u_{0,i} \) is chosen in such a way that \( u_{0,i}(x) \geq v(0, x) \) in \( \mathbb{R} \) for the weak lower solution \( v \) constructed in Proposition 3 then \( u_i(t, x) \) converges to 1 as \( t \uparrow \infty \), uniformly on any compact subset of \( \mathbb{R} \). This proves the spatial dominance of \( e_i \). Thus the assertion follows.

**Example 1.** Consider a game with payoff matrix

\[
A = \begin{pmatrix}
a_{11} & 0 & 0 \\
0 & a_{22} & 0 \\
0 & 0 & a_{33}
\end{pmatrix}, \quad 0 < a_{11} < a_{22} < a_{33}.
\]

The game has three strict Nash equilibria \( e_1, e_2 \) and \( e_3 \). As may be seen from Figure 2 the equilibrium \( e_3 \) is 1/2-dominant and so is spatially dominant by Theorem 3.2.
We finally discuss under what condition a Nash equilibrium is not spatially dominant in finite symmetric 2 person games. The following proposition is a straightforward consequence of Theorem 1 of [8].

**Proposition 5.** Let \( e_i \) be a Nash equilibrium. Assume that there exists \( j \in \{1, \ldots, n\} \) with \( j \neq i \) such that

(i) \( e_j \in BR((e_i + e_j)/2) \);

(ii) for any \( a \in [0, 1] \), \( BR(a e_i + (1-a)e_j) \cap \{e_i, e_j\} \neq \emptyset \), i.e., for any \( a \in [0, 1] \), there exists \( k \in \{i, j\} \) such that

\[
e_k \cdot A(a e_i + (1-a)e_j) \geq \max_{\ell \in \{1, \ldots, n\}} e_\ell \cdot A(a e_i + (1-a)e_j)\]

Then \( e_i \) is not spatially dominant.

**Proof.** Since \( e_i \) is a Nash equilibrium, \( e_i \in BR(e_i) \). Together with this, assumptions (i) and (ii) give a constant \( b \in [1/2, 1] \) such that \( e_i \in BR(a e_i + (1-a)e_j) \) for \( a \in [b, 1] \) and \( e_j \in BR(a e_i + (1-a)e_j) \) for \( a \in [0, b] \), since \( e_k \cdot A(a e_i + (1-a)e_j) \) is linear in \( a \). Therefore as in Hofbauer [8] we can construct a monotone traveling wave solution of problem (1) that takes values in the edge \( i-j \) and connects \( e_i \) and \( e_j \). The wave converges to \( e_j \) as \( t \to \infty \) for each \( x \in \mathbb{R} \), or is a stationary solution. This shows that \( e_i \) is not spatially dominant.

**Example 2.** Let

\[
A = \begin{pmatrix} 0 & -1 & -2 \\ -2 & 0 & -1 \\ -1 & -2 & 0 \end{pmatrix}
\]

This game also has three strict Nash equilibria \( e_1, e_2 \) and \( e_3 \). Figure 3 shows that, for any \( i \in \{1, 2, 3\} \), the assumptions of Proposition 5 are satisfied with \( j = i - 1 \), where \( i - 1 \) are counted cyclically modulo 3. Hence none of \( e_1, e_2 \) and \( e_3 \) are spatially dominant.

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Figure 3. $\circ$ is the set of $u \in \Delta$ to which the pure strategy $i$ (i.e., $e_i$) is the best reply in the game (14) for $i = 1, 2, 3$. The three dots denote the middle points of the edges.

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