Experimental research of iterated dynamics for the complex exponentials with linear term

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Abstract. The research of the orbit of the point zero, fixed points, Julia and Fatou sets for the iterated complex-valued exponential is carried by means of computer experiment. The object of study is three one-parameter families based on $\exp(\imath z)$:

- $f: z \rightarrow (1 + \mu) \exp(\imath z)$,
- $g: z \rightarrow (1 + \mu |z - z^*|) \exp(\imath z)$,
- $h: z \rightarrow (1 + \mu (z - z^*)) \exp(\imath z)$.

Here $\mu$. For the first family 17- and 2-periodic regimes are detected when passing near the bifurcation value $\mu \approx 2.475i$, while the multiplicator equals 1. The second family shows a more interesting behavior: (i) three-valley structure of isolines of the convergence rate near fixpoint $z^*$ at $\mu = 0 + 1 \cdot i$; (ii) saddle-node transition when the parameter moves along a straight line $\text{Re} \mu = 0$, leading to the appearance of a second fixpoint and loss of stability by the old fixpoint at $\text{Im} \mu = 2.1682$; (iii) the nontrivial nature of the orbits of points in the vicinity of the new fixpoint and the presence of false fixpoints in the portrait of the Julia set; (iv) second phase transition leading to a radical change in the form of the Julia and Fatou sets at $\mu \approx 2.5i$. The dynamics of the third family during movement at $\text{Re} \mu = 0$ is similar to the first case, but 17th and 2nd periodic modes are replaced by 39th and 3rd modes. Transitions $17 \rightarrow 2$ and $39 \rightarrow 3$ seem to be rapid and discreet while their geometric interpretation matches the ratios $17=1+2\cdot8$, $39=13\cdot3$. At $\mu = |z^*|^{-1}$ for the $h$-family Julia set fills the entire complex plane.

1. Introduction

As is well known, the topology of the Mandelbrot and Julia sets of exponential maps of the type \cite{1,2} essentially differs from the classical case of quadratic one, showing the Cantor bouquet elements \cite{3,4} (the Cantors set analog for two dimensions). Although it is difficult to find a physical system modeled by an iterated exponential, nevertheless, its analogue in modular arithmetic traces its links to cryptography \cite{5}. For exponential maps a number of theoretical results was obtained, one of which \cite{1} emphasizes the point (0,0) in particular: 1) if there is an attractor, the orbit of point 0 tends to it; 2) if the orbit of zero goes to infinity, then the whole complex plane will be the Julia set.

This paper belongs to the area of experimental mathematics \cite{6}, and the phenomenological description serves to attract attention to the previously unnoticed features of exponential maps. Earlier \cite{7} we investigated the base map $e^{\imath z}$, easily reducible to the standard form by substitution, and:

- found a single stable fixpoint $z^* \approx 0.576 + 0.375i$, $|z^*| \approx 0.687$ and unstable multiple-periodic orbits;
• made sure that the point 0 is indeed a hidden fixpoint (a neighborhood of zero passes in a small neighborhood of \( z^* \) in approximately 15-20 iterations);

• portrayed many Julia and Fatou sets, in the structures of which the attention was paid to the local increase in the convergence rate regions;

• portrayed Mandelbrot set for the family (starting point is zero).

**Figure 1.** The topology of Cantor’s bouquets on Julia and Fatou sets for the map \( f(z) = \mu \exp(iz) \), \( \mu = 0 \). It shows the unique attractive point \( z^* \). Here and further on the axes \( \text{abs} : \text{Re}(z^0) \), \( \text{ord} : \text{Im}(z^0) \) are indicated. The image is rotated by 900 compared to the standard case, which facilitates perception.

Here we focus on parametric dependencies for three similar families: \( f : z \rightarrow (1 + \mu)\exp(iz), \mu \in \mathbb{C} \), and \( g : z \rightarrow (1 + \mu |z - z^*|)\exp(iz), \ z^* : g(z^*) = z^* \), and also \( h : z \rightarrow (1 + \mu (z - z^*))\exp(iz) \). As before, Fatou and Julia sets as well as the fixed points stability and orbit of zero will be the subject of our study. The second family is caused by the search for homogeneous equilibria in an ensemble of linearly coupled exponentials, i.e.:

\[
z_{s+1}^t = \lambda(t) \exp(iz_s^t), \lambda(t) = \mu_0 + \mu \left| \frac{1}{n} \sum_{k=1}^{n} z_k^t - z^*(\mu_0) \right|, \quad \mu_0 = 1
\]

Here \( t \) is discrete time, \( s \) is a spatial index of map instance, \( k \) is a cell index of the neighborhood template of size \( n \). It is appropriate to use the terminology of cellular automata (CA); if you allow the cell state to take value in the complex plane, then there is no difference with the coupled map lattice or ensemble [8,9].

2. Experimental method

The calculations were performed in MATLAB R2012 environment with double precision (double, 64 bits). No special measures to improve accuracy have been made; the change to algebraic
computation (command \texttt{vpa}) would lead to a significant slowdown in computations by two orders of magnitude or more. As soon as the value of \(z(t)\) reached the constant \(1.79769 \times 10^{308}\), it was assumed that the orbit goes to infinity, and the calculation passed to the next point. Note that the truth of the calculations is questioned much earlier - upon reaching \(\approx 10^{15}\), because to correctly calculate the expression \(\exp(iz)\) it is required to store at least a second digit after the comma of a large number (an error of 0.1 gives an error in the phase by 360). Taking into account the iterativity the requirements for accuracy is probably even tougher. We believe, however, that the quality of the dynamics does not affect the features of computer computation.

Figure 2. The orbit of the point \((0,0)\). The entire orbit range \(0..T\) is divided into several parts according to the colors of the rainbow, and the size of the marker-circle decreases with increasing point number \(t\). Two almost linear sections are distinguished on the period of the orbit. The differences between \(z(971)\) and \(z(988)\) suggest that convergence has not been reached; perhaps it is a case of quasiperiodicity.

By default, the depth of the iterations (time) was \(T = 1000\), and the spatial grid in constructing the Julia set was \(X = Y = 1000\) of original size of the region. A standard "window" for the region window of \([-5; 5]^2\). Sometimes smaller values were taken, primarily for CA. The orbit of the point \(z\) for the map \(f\) was assumed to be the finite sequence \(\{z^0 = z, z^1 = f(z), \ldots, z^T = f(z^{T-1})\}\). The Julia set was considered equivalent to the set of escape points, i.e. \(J \simeq J_{\text{esc}} = \{z : (z^0 = z) \Rightarrow (\exists t \leq T) (z^t = \text{Inf})\}\). In theory, Julia set is introduced in more complex ways, but given the limitation of the computer here we follow this simple definition. The standard imaging technique was used: the color of the points was indicated according to the speed of convergence. The Julia set is displayed in grey color scheme, a dark tone corresponds to small values of \(t\) and bright colors to a slow escaping into infinity.

For the Fatou set two cases \(F1\) and \(F2\) were distinguished: 1) the basin of the focus; 2) other cases: periodic, quasi-periodic or chaotic orbit. The stop condition in the first case is
Figure 3. Julia and Fatou sets for the case $\mu = 2.5i$ in two "windows": (a) [-5..5, -5..5]; (b) [0..0.4; 0.8..1.2]. In the center of the spiral arms there is, apparently, a Siegel disk with a diameter of no more than 0.05 and fixpoint $z^*_{2.5} \approx 0.1902 + 0.9862i$. The violet color corresponds to longer times (up to $T = 1000$).

either $|z^{t+1} - z^t| < \varepsilon_1 = 10^{-6}$, or $|z^{T+1} - z^T| < \varepsilon_2 = 10^{-2}$, if the convergence is slow. If the last condition is not satisfied, then the point comes from F2. By default, F1-points are marked with colors from blue (fast convergence) to green, and F2-points - in yellow-red scale (here the color tone has no meaning).

While rendering the command MATLAB `pcolor` was used instead of the standard `contourf` to suppress undesirable color interpolation effects. We refer the rhomboid-like patterns in Fig.1 to the discreteness of the grid, where the color is attributed to the entire patch, based on the color of one calculated point.

3. The map $z \to (1 + \mu) \exp(iz)$
At $\mu = 0$ we have basic map with simple dynamics: the points either escape to infinity, or are attracted by the point $(0,0)$, and then by the fixpoint $z^*$, or, passing zero, directly fall
into the neighborhood of $z^*$. For this case the Julia and Fatou sets are already known (and, obviously, 2-periodic in Re $z$), Fig.1. The orbit of zero represents the vertices of concentric triangles that slightly rotate around the center (that is, for three iterations the rotation is almost 2). Visualized by small areas of increased convergence, subsets of Fatou set are treated as multiplication (copies), possibly with the rotation of one wide subset near the fixpoint.

Figure 4. Julia (monochrome) and Fatou (red) sets for the case $\mu = 2.7i$ in 1x1 size window near the fixpoint.

The Mandelbrot set [7] of family $f_\lambda = \lambda \exp (iz)$ shows the presence near the zero of the basin of attraction restricted by a curve resembling a cardioid rotated by 900. Areas of periodic cycles and a network of regions with a chaotic regime lie outside it. The boundaries of this basin on a straight line Re$\lambda = 1$ lie within $\approx -0.9 \leq \text{Im}\mu \leq 2.5$. Let’s move in the direction of Re$\mu = 0$, Im$\mu > 0$, $\mu \searrow$. At $\mu = 1.5i$ no qualitative change in dynamics (in comparison with the base map) is observed: 1) ”bushes” are shifted to the left only, keeping distance between themselves equal to 2) the fixpoint coordinates change; 3) convergence slows down (in the worst case: from 30 to 80 iterations). At $\mu = 2.0i$ and $\mu = 2.25i$ these changes increase, while the orbit of zero acquires a two-arm spiral shape, and the number of iterations to convergence increases to 192 and 384, respectively. Note that at $\mu = 2.475i$ the multiplicator $|f'(\mu, z^*(\mu))|$ becomes 1.

At $\mu = 2.5i$ the convergence regime is replaced by a multiple-periodic one (Table 1). In Fig.2, the orbit of zero characterized by a cycle with period 17 is considered. The orbit of the point (0, -2) or (2, -2) is similar to it - its period is also 17 (or a multiple of 17). In this case, the Julia set has a more complex structure (Fig. 3): the sequences of Cantor’s bouquets that stretch from each ”bush” join each other and transform into a twisting multi-arm spiral. There is fixpoint $z^{*,2,5} \approx 0.1902+0.9862i$ at its center (the multiplicator is 1.0045, i.e. the point is almost neutral), around which Fatou set component is located in the shape of a small disk identified by us as a Siegel disk. It is not possible to distinguish experimentally Siegel’s disk from extremely slow escaping to infinity, as in the case of fixpoint $z^{*,2,5}$ or, more accurately, of a point in its small
neighborhood (Fig. 3b). The spirality of the sleeves, which is also characteristic of the orbit of
fixpoint vicinity, confirms it, but the distance from the center contradicts it. Visually on the
outer circumference of the Siegel disk 17 spiral arms, the components of the Julia set, begin (for
the near-to-disk points, escaping to infinity is slower than in the remote ones). Let us recall the
special role of 17 in mathematics (the discovery of young Gauss).

Figure 5. The neighborhood of fixpoint \( z^{*1}, \mu = 0.25 \) (Table 1). Such points exist that do
not have time to converge neither to infinity nor to \( z^{*} \). This again proves Julia and Fatou
sets fractality. The pixelization effect is connected with a limited number of digits used in
computations, and not with rendering or grid granularity.

At \( \mu = 2.7i \) the shape of the Julia set changes again, the spiral becomes a two-arm (Fig. 4),
and the multiplicity of the period for the points of Fatou set becomes 2 (such is the orbit of
zero). As an example we give the orbit of the point \( 0 + 1i \): during the first 60 iterations the
point "sways" between the two fixpoints, gradually moving away from them, then at the 62nd
iteration \( |z_{62}| > 2000 \), and at the 63rd iteration the point escapes to infinity.

At \( \mu = 3i \) the portrait of Julia and Fatou sets is simplified: for the most points, including zero,
there is a two-period mode with stable fixpoints \(-0.262 + 0.073i \) and \(1.621 + 2.452i \). The unstable
fixpoint \( z^{*3} \approx (0.1634, 1.0702) \) with the multiplicator 1.08 is fairly fast, for 200 iterations, goes
into this periodic mode. At \( \mu = 6i \) the orbit of the formal fixpoint converges to a cycle in about
20 iterations.

4. Proposition 1:
If the map of \( z \rightarrow \lambda \exp (iz) \), \( \Re\lambda = 1 \) kind has a periodic mode, then for all such points of the
complex plane its multiplicity \( n \) is uniquely determined by the parameter, i.e. \( n = n (\lambda) < \infty \).
The periods multiplicity \( n \) is assumed to be only 2 and 17 (or multiples of 17).
5. The map $z \rightarrow (1 + |z - z^*|) \exp(iz)$
The sign of the module instead of the brackets in the expression $g(z, \mu)$ makes the analysis more difficult, but simplifies the understanding. In Table 2 for selected $\mu$ values some of the equilibria closest to $z^*$ are listed; they are all unstable, except for the case $\mu = 2.5i$, $z^*3$. Let us discuss the appearance of such a fixpoint in more detail.

![Figure 6. Julia and Fatou sets for the $\mu = 0 + 1i$ case](image)

**Table 1.** Cycle multiplicities after bifurcation

| Parameter Im$\mu >$ Im$\mu^*$ | Cycle multiplicity n | Bifurcation value |
|---------------------------------|----------------------|------------------|
| 2.48                            | 49                   | Im$\mu^*$=2.474440 |
| 2.49                            | 51                   |                  |
| 2.50                            | 17                   |                  |
| 2.51                            | 17                   |                  |

Note: Some points of the $n$-periodic orbit are located near each other and clusterize on the plane. For example, Fig. 2 shows three clusters: 1, 1+7, 1+7. The last two clusters exhibit the geometric resemblance

Let us suggest in case of $\mu = \mu^{\text{double}}$: Re$\mu = 0$ the simultaneous existence of two equilibria $z^*, z^* + \varepsilon$ next to each other, and, relatively speaking, the root of the transcendental equation $z = g(\mu, z)$ has multiplicity 2: $z^* = g(\mu, z^*), \quad z^* + \varepsilon = g(\mu, z^* + \varepsilon), \quad \varepsilon \rightarrow 0$. If we neglect
terms of second order, we get:

\[ z^* + \varepsilon \approx z^* \left( 1 + \mu |\delta z| \right) (1 + i\varepsilon) \Rightarrow \varepsilon \left( \frac{1}{z^*} - i \right) = \mu |\varepsilon| \]

\[ \mu^{\text{double}} = \frac{\varepsilon}{|\varepsilon|} \left( \frac{1}{z^*} - i \right) \Rightarrow \theta = \frac{\pi}{2} - \text{angle} \left( \frac{1}{z^*} - i \right) \]  

\[ |\mu^{\text{double}}| = 2.1682, \quad \theta = 2.5442 \approx 146^0 \]  

\[ \mu \]

**Figure 7.** Julia and Fatou sets for the \( \mu = 1.5i \) case: fragment. The strong deformation of Cantor's bouquets is evident.

**Table 2.** Some fixpoints of the \( g(z, \mu) \) map for different \( \mu \)

| \( \mu \)   | 0 (base) | 0.25   | 0.25i   | -0.9i   | 2.5i          |
|------------|---------|--------|--------|--------|---------------|
| \( z^1 \)  | -2.76636-1.08965i | -2.98998 - 0.45682i | -3.74118 - 0.93942i | -2.07187 + 0.21721i | -4.84608 + 1.01677i |
| \( z^2 \)  | -9.18489 - 2.24661i | -9.319401 - 0.98569i | -10.53346 - 1.26968i | -7.97970 - 0.02618i | -11.04836 + 0.96522i |
| \( z^3 \)  | 5.98583 - 1.83427i | 6.13132 - 0.93833i | 5.16985 - 1.21627i | 7.67724 - 0.16871i | 0.37322 + 0.58971i |

It is easy to see that the multiplicator \( |g'(\mu, z^*)| \) is equal to 1 at \( \text{Im}\mu = \mu^{\text{double}} \). In comparison with the base map at \( \mu = 1i \) the Fatou set is more complicated: the Julia set bush is more asymmetric relative to its central axis; three curvilinear areas (valleys) gather at the point \( z^* \),
where the convergence rate is increased, as well as several areas in the form of the Cantor’s bouquets halo, where the convergence is locally reduced (Fig. 5, Fig. 6). This ”three-valley” effect is absent for quadratic maps and is observed for the first time. At $\mu = 1.5i$ the changes are more striking, despite the presence of two convergence rate maxima instead of the ”valley” (plus two more maxima to the right) inside the Fatou set, nevertheless the fixpoint $z^*$ remains unique (and the second point is a false fixpoint). Three ”valleys” disappear, but one of them turns into a dumbbell shape. We can point out deformation and ”vorticity” of the outer left and right bouquets of a bush (Fig. 7).

![Figure 8. Julia and Fatou sets before the bifurcation at $\mu = 2.16i$.](image)

Near the bifurcation $\text{Im}\mu = 2.16 < |\mu_{\text{double}}|$ in addition to significantly slowing down of convergence (300-400 iterations) two fixpoints are clearly observed (Fig.8): true ($z^*$) and false $z_{\mu}^* \approx -0.765 + 1.875i$. $g(\mu, z_{\mu}^*) = z^*$ false fixpoint passes in the true for the next iteration. The bifurcation itself is analogous to the saddle-node transition because of doubling and divarication the fixpoints from each other in the direction of the angle $\approx 135^0$, as was assumed in the derivation of Eq.(1).

Let us consider the neighborhood of the fixpoints immediately after the bifurcation via Fatou set (Fig.9). Because of the second order terms the old fixpoint $z^*$ remains stable in a very small vicinity; for $z^0 = z^* + 10^{-4} (1 + i)$ orbit tends to $z^*$, but the orbit of

$$z^0 = z^* + 10^{-3} (1 + i)$$

converges to a new fixpoint directly along a straight line connecting both fixpoints without any helical patterns. The number of iterations level-lines look quite unusual (Fig.9): for fast
convergence it has the shape of an irregular ring, and the attractor is located in the surrounding area, and not in the center.

![Figure 9. Fatou set after bifurcation at $\mu = 2.20$](image)

(a) (b)

Figure 9. Fatou set after bifurcation at $\mu = 2.20$ linear false (a) and true (b) fixpoint. The true fixpoint has new coordinates $(0.545 + 0.398i)$ and is near $z^*$.

At $\text{Im}\mu = 2.5 > |\mu^{\text{double}}|$ the transition of the neighborhoods $U(z^*) \rightarrow U(z^{*3})$ requires about 200 iterations, the orbit of zero goes to infinity. However, the Fatou and Julia sets themselves differ substantially (Fig. 10), even in comparison with the previous case $\mu = 2.2i$. The area between the two "bushes" became predominantly the part of Julia set. The Fatou set loses its simple connectedness and resembles "tadpoles"; some components of Julia set show variations of Cantors set in the form of "lunettes". On the selected window (Fig. 10a), the areas of these sets make about 80% (Julia) and 20% (Fatou). Obviously another bifurcation has occurred, but it is difficult for us to specify a concrete value.

In particular, at $\text{Im}\mu = 2.4$ the orbit of zero has an interesting 14-periodic mode, which consists of 7-member cycles of increasing amplitude, the latter of which takes the point $(-983.25+268.05i)$ in a vicinity of zero. Points $(0-2i)$ and $(-3+1i)$ also have 14-periodic mode. The same points at $\text{Im}\mu = 2.6$ gave 11-periodic mode, but whereas the penultimate point of the period showed a second order of magnitude, the latter point was 1049 to start a new period from $(0,0)$ vicinity. This order of magnitude casts doubt on the computer calculation. At $\text{Im}\mu = 3$ the orbit of zero is 8-periodic, and in general within Fatou set periodic modes dominates over the share of convergence to single fixpoint. While inside "tadpoles" the mode returns to the valley pattern, although the valley is one (not three, as previously). At $\text{Im}\mu = 4$ the orbit of zero is 32-periodic, and the component of Fatou set with convergence to fixpoint ("tadpoles") disappears, and its place is occupied by a chaotic mix of escaping points and points with periodic orbits. At periphery of this region there is an alternation of the bands of periodicity and, probably, of the deformed Cantor bouquets for escaping points, but closer to its center we find no signs of orderliness.

6. Proposition 2

Let $g(z) = (1 + \mu|z - z^*|)\exp(iz)$ map be given for any $\mu : \text{Re}\mu = 0$. Then: 1) if the orbit of zero goes to infinity, then there are no n-periodic ($n > 1$) regimes; 2) if the orbit of some point goes to the n-periodic regime, then the point $(0,0)$ belongs to this period, i.e.

$(\forall \varepsilon < 0)(\exists t \in \mathbb{N})(|z_t| < \varepsilon)$.

As we can see, the Sullivans results in a somewhat weakened form are also fair for our maps. They can be generalized, probably, either to linear-exponential maps $z \rightarrow (\mu_0 + \mu z)\exp(z)$or to quasi-polynomials $P(z)\exp(z)$.
Figure 10. Julia and Fatou sets at $\mu = 2.5i$: General view (a) and Detailed view (b).
7. The map \( z \rightarrow (1 + \mu(z - z^*)) \exp(iz) \)

The dynamics of this view is more simplified than the previous one and resembles more a classical
exponent, albeit in shape \( f(z) \). But unlike the exponent of Fatou and Julia sets they lose their
periodicity along the real axis, and the neighboring "bushes" seem to stretch out and begin to
hang over the "bush" close to zero. Tables 3 and 4 summarize the main features of dynamics
and Julia / Fatou sets when the parameter is varied, mainly along the imaginary axis, i.e.
\( \text{Re} \mu = 0, \text{Im} \mu > 0 \). For this mapping figures 3 and 39 instead of 2 or 17 have value. In addition,
examples have been identified when Fatou set disappears.

Table 3. Summary mapping table \( h(z) \) (Part 1)

| \( \text{Im}(\mu) \) | Quality description |
|----------------|---------------------|
| **During movement along imaginary axis** |                     |
| 0.1 | Convergence according to the trefoil scheme (analogous to Figure 11a), a fix point is a focus (See Note) |
| 0.25 | The same |
| 0.5 | A three-period cycle (the fix points are removed from zero), each fix point is a focus (one-arm spiral), Julia set: adjacent three-arm spiral - two sleeves from one "bush", one from the other |
| 0.75 | A three-period cycle (one of the fix point is shifted by 0.1 from zero), convergence to each point is fast in type of node; touching of the three arms (Fig. 11a) |
| 1 | The same |
| 1.5 | 3-periodic cycle, one fix point is in the neighborhood of zero; three "bush" converge at a single fix point (Fig. 11b) |
| 2 | 3-periodic cycle, one fix point is at a distance of no more than \( 2 \times 10^{-4} \) from zero |
| 3 | The same, with two fix points - 0 and "big" in module |
| 6 | Very rapid escape to infinity, there is no visualization of Fatou and Julia sets |
| 9 | Julia set fills the entire plane, a rapid escape to infinity through 0 |
| 12 | The same |
| 20 | 3-period mode on the verge of loss of accuracy, i.e. escaping from it to infinity: \( -27394509.8784864 + 9868306.50217099i \rightarrow 0,00000000000000+ 0,00000000000000i \leftrightarrow 8,49398041474234 - 11,5282544606287i \) |

Note: We refer the terms focus and node to visual characteristics of Julia set which is close to the standard graphic definitions of the focus and node as of the characteristics of fix point (trajectories of convergence). The difference is nevertheless substantial: the visual node can match the fixpoint-focus.

It is interesting to find such a value \( \text{Im} \mu = \mu^b \) when the multiplicator turns to unity at the very fixpoint. Analysis of formal critical points does not provide useful information; these values are:
\[
\mu = \frac{-i}{1+i(z^c-r)} \Leftrightarrow z^{cr} = z^* + \left(i - \frac{1}{\mu}\right), \\
\mu^{bf} = |z^*|^{-1} - 1 = 0.454553558270876, \quad z^{cr,bf} \equiv z^{cr} \left(\mu^{bf}\right) = z^* + 3.2000i
\]

Figure 11. Julia and Fatou sets for \(\text{Im} \mu = 0.75 \rightarrow 1.5\) (from (a) to (b)). The formation of the trefoil joint near the fixpoint \(z^*\).
Table 4. Summary mapping table $h(z)$ (Part 2)

| $Im(\mu)$ | Quality description |
|------------|---------------------|
| **At the bifurcation value** | |
| 0.33 | Convergence according to the trefoil scheme |
| 0.45 | Slow convergence (more than 3800) iterations, but convergence is achieved (at 10000 iteration deepth) |
| 0.46 | 39-fold cycle - two elements are deleted, the others merge into one strip, zero belongs to the cycle; the orbit of the point in the neighborhood of the fix point is $z^* + 0.05 \times (1 + i)$ at first shows a quasiperiodicity with a period of 3, then infinity is broken at the 720th iteration; if the shift is by 0.01 $(1 + i)$, then at the 1156th iteration the value of 10128 is reached as well as then going to zero and the 39-member cycle; chaos near the fix point, formed by points with periodic and runaway orbits. |
| $\mu_{bf} = 0.454553558270876$ | The orbit of zero is an attractor in the form of a (semi) closed tape, there are no large values, the period is not detected (Fig. 12), zero enters the attractor; Orbit $z^* + s (1 + i)$ is attractor in the form of a ring of size $(2 \times 3) \times s$, the period is not detected, $s = 0.001, 0.01, 0.1$; a similar pattern appears when shifting by 0.01, ring size $(2 \times 3) \times 0.01$; When $s = 1$, the ring resembles a tape for the orbit of zero; Orbit of the critical point: an attractor, there is no period, the neighborhood of zero enters the attractor, the part of the attractor (southern) is dense, the part contains discontinuities (northern). Julia Set: the spirals pass at some distance from fix point. |
| 0.47 | 3-cycle, point 0 does not enter the cycle |
| 0.465 | 39-cycle for the orbit of zero, while the fix points cluster into three self-similar groups: in each group there is a "leader" standing at a distance from the semicircle of his twelve brothers; all the fix points are concentrated on the "window" [-1.5..2, -14]; point (0,0) is next to the cycle at a distance of 0.01; 3-arm spiral on Julia set. |
| 0.4675 | 3-cycle, 13 arms are stretched to each fix points, i.e. convergence according to 13-leaf scheme (Fig. 13); it is easy to see the origin of the previous 39-fold cycle, because 39 = 3 * 13. |
| 0.4545 | The orbit of zero - convergence to zero over a complex closed curve (possibly of fractional dimension) for 323422 iterations. |
| 0.4546 | The orbit of zero - the orbit returns to zero after leaving to large values of the order of 10^{12}; For the first time this occurs at the 2149th iteration, i.e. at 2149th cycle |
Figure 12. The zero orbit at the bifurcation value of the parameter $\mu^{bf}$. The designations of the axes, colors and dimensions of the marker are the same as in Fig. 2. The depth of calculation is 3000 iterations. The northern part of the orbit is notable: is this portion of the orbit a dense set if the number of iterations is unlimited?

Figure 13. Degeneration of 39-member cycle into the triple cycle with 13 arms near bifurcation.

8. Conclusion

The main content of the paper is presented in the figures, which is a feature of experimental mathematics in application to nonlinear dynamics. Its goal was once again to draw attention to the iterated exponential (and its modifications) on the complex plane. We have seen at least two bifurcation transitions. For the first one, the critical value of the parameter was derived, but for the second one it was not.
In conclusion we formulate the questions that need to be solved by theoretical mathematicians:

- Within Fatou set there are regions of local increase of the convergence rate in the form of small circles (Fig. 2) or valleys (Fig. 6). This phenomenon we call "yellow spots" is absent for quadratic mappings. Can it be given a strictly formal definition avoiding reference to quantities?
- Is there a Siegel disk for some value of the parameter in the three studied variants of linear-exponential mappings? (Figure 3b leaves us in doubt).
- For the given families Proposition 1 needs to be proved with respect to the following: multiplicity of a cycle in a periodic mode, if such cycle exists, does not depend on the choice of the starting point.
- For these families Proposition 1 needs to be proved with respect to the following: at a continuous variation of the parameter near the bifurcation the cycle multiplicity varies discretely running through some special "magic" values (for the family f: 51, 17, 2).
- For these families under what conditions does the orbit contain a cycle after the quasiperiodic section?
- For the given families Proposition 2 needs to be proved, i.e. if the orbit of the point 0 does not contain a cycle, then periodic orbits are absent for any initial point.
- What are topological properties (is a set dense everywhere?) of an attractor for h-family for a parameter in the neighborhood of bifurcation value?
**Figure 15.** A fragment of Julia set at the real\(\mu = 1/|z^*|\). Its complex internal structure in the window can be seen \([-3..0;0..3]\), resolution 10001000, depth of calculation - 3000.

### 9. References

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