Async-RED: A Provably Convergent Asynchronous Block Parallel Stochastic Method using Deep Denoising Priors

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Abstract

Regularization by denoising (RED) is a recently developed framework for solving inverse problems by integrating advanced denoisers as image priors. Recent work has shown its state-of-the-art performance when combined with pre-trained deep denoisers. However, current RED algorithms are inadequate for parallel processing on multicore systems. We address this issue by proposing a new asynchronous RED (ASYNC-RED) algorithm that enables asynchronous parallel processing of data, making it significantly faster than its serial counterparts for large-scale inverse problems. The computational complexity of ASYNC-RED is further reduced by using a random subset of measurements at every iteration. We present complete theoretical analysis of the algorithm by establishing its convergence under explicit assumptions on the data-fidelity and the denoiser. We validate ASYNC-RED on image recovery using pre-trained deep denoisers as priors.

1 Introduction

Imaging inverse problems seek to recover an unknown image \( x \in \mathbb{R}^n \) from its noisy measurements \( y \in \mathbb{R}^m \). Such problems arise in many fields, ranging from low-level computer vision to biomedical imaging. Since many imaging inverse problems are ill-posed, it is common to regularize the solution by using prior information on the unknown image. Widely-adopted image priors include total variation, low-rank penalties, and transform-domain sparsity [1–5].

There has been considerable recent interest in plug-and-play priors (PnP) [6, 7] and regularization by denoising (RED) [8], as frameworks for exploiting image denoisers as priors for image recovery. The popularity of deep learning has led to a wide adoption of deep denoisers within PnP/RED, leading to their state-of-the-art performance in a variety of applications, including image restoration [9], phase retrieval [10], and tomographic imaging [11]. Their empirical success has also prompted a follow-up theoretical work clarifying the existence of explicit regularizers [12], providing new interpretations based on fixed-point projections [13], and analyzing their coordinate/online variants [11,14]. Nonetheless, current PnP/RED algorithms are inherently serial, which makes them suboptimal for large-scale inverse problems on multicore systems (see Fig.1 for an illustration).

We address this gap by proposing a novel asynchronous RED (ASYNC-RED) algorithm. The algorithm decomposes the inference problem into a sequence of partial (block-coordinate) updates on \( x \) executed asynchronously in parallel over a multicore system. ASYNC-RED leads to a more efficient usage of available cores by avoiding synchronization of partial updates. ASYNC-RED is also scalable in terms of the number of measurements, since it processes only a small random subset of \( y \) at every iteration. We present two new theoretical results on the convergence of ASYNC-RED based on a unified set of explicit assumptions on the data-fidelity and the denoiser. Specifically, we establish its fixed-point convergence in the batch setting and extend this analysis to the randomized minibatch scenario. Our results extend recent work on serial block-coordinate RED [14] and are fully consistent with the traditional asynchronous parallel
optimization methods [15],[16]. We numerically validate ASYNC-RED on image recovery from linear and noisy measurements using pre-trained deep denoisers as image priors.

2 Background

Inverse problems. Inverse problems are traditionally formulated as a composite optimization problem

$$\hat{x} = \arg \min_{x \in \mathbb{R}^n} g(x) + h(x),$$

(1)

where $g$ is the data-fidelity term that ensures consistency of $x$ with the measured data $y$ and $h$ is the regularizer that infuses the prior knowledge on $x$. For example, consider the smooth $\ell_2$-norm data-fidelity term $g(x) = \|y - Ax\|_2^2$, which assumes a linear observation model $y = Ax + e$, and the nonsmooth TV regularizer $h(x) = \tau \|Dx\|_1$, where $\tau > 0$ is the regularization parameter and $D$ is the image gradient [1].

Regularization by denoising (RED). RED is a recent methodology for imaging inverse problems that seeks vectors $x^* \in \mathbb{R}^n$ satisfying

$$G(x^*) = \nabla g(x^*) + \tau (x^* - D_\sigma(x^*)) = 0 \iff x^* \in \text{zer}(G) := \{x \in \mathbb{R}^n : G(x) = 0\}$$

(2)

where $\nabla g$ denotes the gradient of the data-fidelity term and $D_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is an image denoiser parameterized by $\sigma > 0$. Under additional technical assumptions, the solutions $x^* \in \text{zer}(G)$ can be associated with an explicit objective function of form [1]. Specifically, when $D_\sigma$ is locally homogeneous and has a symmetric Jacobian satisfying strong passivity [8],[12], $H(x)$ corresponds to the gradient of a convex regularizer

$$h(x) = \frac{1}{2} x^T (x - D_\sigma(x)).$$

(3)

A simple strategy for computing $x^* \in \text{zer}(G)$ is based on the following first-order fixed-point iteration

$$x^t = x^{t-1} - \gamma G(x^{t-1}), \quad \text{with} \quad G := \nabla g + \tau (I - D_\sigma), \quad G : \mathbb{R}^n \to \mathbb{R}^n,$$

(4)

where $\gamma > 0$ denotes the stepsize. In this paper, we extend this first-order RED algorithm to design ASYNC-RED. Since many denoisers do not satisfy the assumptions necessary for having an explicit objective [12], our theoretical analysis considers a broader setting where $D_\sigma$ does not necessarily correspond to any explicit
regularizer. The benefit of our analysis is that it accommodates powerful deep denoisers (such as DnCNN [17]) that have been shown to achieve the state-of-the-art performance [11,13,14].

Plug-and-play priors (PnP) and other related work. There are other lines of works that combine the iterative methods with advanced denoisers. One closely-related framework is known as the deep mean-shift priors [18]. It develops an implicit regularizer whose gradient is specified by a denoising autoencoder. Another well-known framework is PnP, which generalizes proximal methods by replacing the proximal map with an AMP-based algorithms are known to be nearly-optimal for random measurement matrices, but are generally unstable for general $A \in \mathbb{R}^{m \times n}$ [32–34]. The AMP-based algorithms are known to be nearly-optimal for random measurement matrices, but are generally unstable for general $A \in \mathbb{R}^{m \times n}$ [32–34].

Asynchronous parallel optimization. There are two main lines of work in asynchronous parallel optimization, the one involving the asynchrony in coordinate updates [16,33–40], and the other focusing on the study of various asynchronous stochastic gradient methods [15,41–44].

Our work contributes to the area by developing a novel deep-regularized asynchronous parallel method with provable convergence guarantees.

3 Asynchronous RED

Async-RED addresses the computational bottleneck by simultaneously considering the asynchronous partial updates of image $x$ and the randomized usage of measurements $y$. In this section, we introduce the algorithmic details of our method. We start with the basic batch formulation of Async-RED (Async-RED-BG) followed by its minibatch variant (Async-RED-SG).

3.1 Async-RED using Batch Gradient

When the gradient uses all the measurements $y \in \mathbb{R}^m$, Async-RED-BG is the asynchronous extension of the recent block-coordinate RED (BC-RED) algorithm [14]. Consider the decomposition of the variable space $\mathbb{R}^n$ into $b \geq 1$ blocks

$$x = (x_1, \ldots, x_b) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_b} = \mathbb{R}^n \text{ with } n = n_1 + n_2 + \cdots + n_b.$$ 

For each $i \in \{1, \ldots, b\}$, we introduce the operator $U_i : \mathbb{R}^{n_i} \to \mathbb{R}^n$ that injects a vector in $\mathbb{R}^{n_i}$ into $\mathbb{R}^n$ and its transpose $U_i^T$ that extracts the $i$th block from a vector in $\mathbb{R}^n$. This directly implies that

$$1 = U_1 U_1^T + \cdots + U_b U_b^T \quad \text{and} \quad \|x\|_2^2 = \|x_1\|_2^2 + \cdots + \|x_b\|_2^2 \quad \text{with} \quad x_i = U_i^T x. \quad (5)$$ 

In analogy to the RED operator $G$ in (2), we define the block-coordinate operator $G_i$ as

$$G_i(x) := U_i U_i^T G(x), \quad \text{with} \quad x \in \mathbb{R}^n \quad \text{and} \quad G_i : \mathbb{R}^n \to \mathbb{R}^n. \quad (6)$$ 

Due to the asynchrony in the block updates, the iterate might be updated several times by different cores during a single update cycle of a core, which means that the evaluation of $x^{k+1}$ relies on a stale iterate $\tilde{x}^k$

$$x^{k+1} \leftarrow x^k - \gamma G_i(\tilde{x}^k), \quad \text{with} \quad \tilde{x}^k = x^k + \sum_{s=k-\Delta_k}^{k-1} (x^s - x^{s+1}), \quad \Delta_k \leq \lambda. \quad (7)$$ 

Here, we assume that the stale iterate $\tilde{x}^k$ exits as a state of $x$ in the shared memory, and the delay between them is bounded by a finite number $\lambda \in \mathbb{Z}^+$. These two assumptions are often referred to as the consistent read [41] and the bounded delay [45] in the traditional asynchronous block coordinate optimization. Although
we implement the consistent read in ASYNC-RED, the algorithm never imposes a global lock on \( x^k \). We refer to Supplement [A] for the related discussion.

We now introduce the first variant, ASYNC-RED-BG.

**Algorithm 1 ASYNC-RED-BG**

1: **input:** \( x^0 \in \mathbb{R}^n, \gamma > 0, \tau > 0 \).
2: **setup:** A multicore system with one shared memory storing \( x \) and global iteration \( k \).
3: **for** global \( k = 1, 2, 3, \ldots \) **do**
4: \( \bar{x}^k \leftarrow \text{read}(x) \)
5: \( G_{ik}(\bar{x}^k) \leftarrow U_{ik} U_k^T G(\bar{x}^k) \) with random \( i_k \in \{1, \ldots, b\} \) \hspace{1cm} \triangleright \text{Block Operation}
6: \( x^k \leftarrow \text{read}(x) \)
7: \( x^{k+1} \leftarrow x^k - \gamma G_{ik}(\bar{x}^k) \)
8: update \( x \) in the shared memory using \( x^{k+1} \)
9: **end for**

When the algorithm is run on a single core system without parallelization (that is to say \( \bar{x}^k = x^k \)), it reduces to the normal BC-RED algorithm. Hence, our analysis is also applicable to BC-RED.

We specifically consider the random block selection strategy in ASYNC-RED-BG, namely that every block index \( i_k \) is selected as an i.i.d random variable uniformly distributed over \( \{1, \ldots, b\} \). Such a strategy is commonly adopted for simplifying the convergence analysis. Nevertheless, our method and analysis can be generalized to the scenario where \( i_k \) follows some arbitrary probability \( P(i_k = i) = p_i \) specified by the user.

Compared with serial RED algorithms, ASYNC-RED-BG enjoys considerable scalability by dividing the computation of the full operator \( G \) into \( b \) parallel evaluation of \( G_i \) distributed across all cores. Thus, without any modification to the algorithmic design, one can easily improve the performance of the algorithm by simply integrating more cores into the system. In Section 5 we experimentally demonstrate the significant speed-up and scale-up in solving the context of image recovery.

### 3.2 Async-RED using Stochastic Gradient

The scale of measurements is another important factor influencing the computational complexity in the large-scale inference tasks. ASYNC-RED-SG improves the applicability of ASYNC-RED to these cases by further considering the decomposition of the measurement space \( \mathbb{R}^m \) into \( \ell \geq 1 \) blocks

\[
y = (y_1, \ldots, y_\ell) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_\ell} = \mathbb{R}^m \quad \text{with} \quad m = m_1 + m_2 + \cdots + m_\ell.
\]

Hence, ASYNC-RED-SG considers the following data-fidelity \( g \) and its gradient \( \nabla g \)

\[
g(x) = \frac{1}{\ell} \sum_{j=1}^{\ell} g_j(x) \quad \Rightarrow \quad \nabla g(x) = \frac{1}{\ell} \sum_{j=1}^{\ell} \nabla g_j(x), \tag{8}
\]

where each \( g_j \) is evaluated on the subset \( y_j \in \mathbb{R}^{m_j} \) of the full \( y \). From (8), we know that the computation of \( \nabla g(x) \) is proportional to the total number \( \ell \). To reduce the per-iteration cost, we follow the idea of stochastic optimization to approximate the batch gradient by using the stochastic gradient that relies on a minibatch of \( w \ll \ell \) measurements

\[
\hat{\nabla} g(x) = \frac{1}{w} \sum_{s=1}^{w} \nabla g_{j_s}(x), \tag{9}
\]

where \( j_s \) is picked from the set \( \{1, \ldots, \ell\} \) as i.i.d uniform random variable. Based on the minibatch gradient, we define the block stochastic operator \( \tilde{G}_i : \mathbb{R}^n \to \mathbb{R}^n \) as

\[
\tilde{G}_i := U_i U_i^T \tilde{G}(x), \quad \text{with} \quad \tilde{G} := \hat{\nabla} g(x) + \tau (x - D_\sigma(x)), \quad \tilde{G} : \mathbb{R}^n \to \mathbb{R}^n. \tag{10}
\]
Note that the computation of $\tilde{G}$ is now dependent on the minibatch size $w$ that is adjustable to cope with the computation resources at hand. ASYNC-RED-SG is summarized in Algorithm 2

Algorithm 2 ASYNC-RED-SG

1: input: $x^0 \in \mathbb{R}^n$, $\gamma > 0$, $\tau > 0$.
2: setup: A multicore system with one shared memory storing $x$ and global iteration $k$.
3: for global $k = 1, 2, 3, \ldots$ do
4:   $\tilde{x}^k \leftarrow \text{read}(x)$
5:   $\tilde{G}(\tilde{x}^k) \leftarrow \text{minibatch}(\tilde{x}^k, w)$ with random $j_w \in \{1, \ldots, \ell\}$ \quad $\triangleright$ Minibatch Gradient
6:   $\tilde{G}_{ik}(\tilde{x}^k) \leftarrow U_{ik} U_{ik}^T \tilde{G}(\tilde{x}^k)$ with random $i_k \in \{1, \ldots, b\}$ \quad $\triangleright$ Block Operation
7:   $x^k \leftarrow \text{read}(x)$
8:   $x^{k+1} \leftarrow x^k - \gamma \tilde{G}_{ik}(\tilde{x}^k)$
9:   update $x$ in the shared memory using $x^{k+1}$
10: end for

We clarify the difference between ASYNC-RED-SG and ASYNC-RED-BG via a specific example. Consider the least-squares $g$ with a block-friendly operator $A$ and a block-efficient denoiser $D_\sigma$. We can write the update of ASYNC-RED-BG regarding a single iteration as

$$G_i(x) = A_i^T(A_i x - y_i) + \tau(x_i - D(x_i)), \quad (11)$$

where $x$ is the delayed iterate for $x$, and $A_i \in \mathbb{R}^{m \times n_i}$ is a submatrix of $A$ consisting of columns corresponding to the $i$th blocks. Although the per-iteration complexity is reduced by roughly $b = n/n_i$ times by working with $A_i$ instead of $A$, ASYNC-RED-BG still needs to work with all the measurements $y_i$ related to the $i$th block at every iteration. Consider the corresponding update of ASYNC-RED-SG with one measurement used at a time

$$\hat{G}_i(x) = A_{ij}^T(A_{ij} x - y_{ij}) + \tau(x_i - D(x_i)), \quad (12)$$

where $y_{ij}$ denotes the $j$th measurement of $x_{ij}$, and $A_{ij} \in \mathbb{R}^{m_j \times n_i}$ is the submatrix crossed by the rows and columns corresponding to the $j$th measurement and the $i$th blocks. This indicates that the reduction of the per-iteration complexity from ASYNC-RED-BG to ASYNC-RED-SG can be up to $\ell = m/m_j$ times. In the practice, it is common to use $w > 1$ measurements at a time to optimize the total runtime. Note that if $U = U^T = I$, ASYNC-RED-SG becomes the asynchronous stochastic RED algorithm. In the next section, we will present a complete analysis of ASYNC-RED and theoretically discuss its connection to the related algorithms.

4 Convergence Analysis of ASYNC-RED

The proposed analysis is based on the following explicit assumptions. Note that these assumptions serve as sufficient conditions for the convergence.

Assumption 1. We assume bounded maximal delay $\lambda < \infty$. Hence, during any update cycle of an agent, the estimate $x$ in the shared memory is updated at most $\lambda \in \mathbb{Z}_+$ times by other cores.

The value of $\lambda$ is often dependent on the number of cores involved in the computation [46]. If every core takes a similar amount of time to compute its update, $\lambda$ is expected to be a multiple of the number of cores. Related work has investigated the convergence with unbounded maximal delays in the context of traditional optimization [39],[43],[47].

Assumption 2. The operator $G$ is such that $\text{zer}(G) \neq \emptyset$, and the distance of the initial $x^0 \in \mathbb{R}^n$ to any element in $\text{zer}(G)$ is bounded, that is $\|x^0 - x^*\| \leq R_0$ for all $x^* \in \text{zer}(G)$ with $R_0 < \infty$.

This assumption ensures the existence of a solution for the RED problem and is related to the existence of minimizers in traditional coordinate minimization [48],[49].
Assumption 3. (a) Every component function $g_i$ is convex differentiable and has a Lipschitz continuous gradient of constant $L_i > 0$. (b) At every update, the stochastic gradient is unbiased estimator of $\nabla g$ that has a bounded variance:

$$\mathbb{E} \left[ \nabla g(x) \right] = g(x), \quad \mathbb{E} \left[ \| \nabla g(x) - \nabla g(x) \|^2 \right] \leq \frac{\nu^2}{w}, \quad x \in \mathbb{R}^n, \quad \nu > 0.$$ 

The first part of the assumption implies that $g$ is also convex and has Lipschitz continuous gradient with constant $L = \max\{L_1, \ldots, L_d\}$. The second part is a standard assumption on the unbiasedness and variance of the stochastic gradient [15][50]. Our final assumption is related to the deep denoiser used in Async-RED.

Assumption 4. The denoiser $D_\sigma$ is a nonexpansive operator $\|D_\sigma(x) - D_\sigma(y)\| \leq \|x - y\|$.

Compared with the conditions stated in Section 2 (namely, that it is locally homogeneous with a symmetric Jacobian), our requirement on the denoiser is milder. One can train a nonexpansive $D_\sigma$ by constraining the Lipschitz constant of $D_\sigma$ via the spectral normalization, which is an active area of research in deep learning [51][53].

We can now state the theorems on Async-RED.

**Theorem 1.** Let Assumptions [14] hold true. Run Async-RED-BG for $t > 0$ iterations with uniform i.i.d block selection using a fixed step-size $\gamma \in (0, 1/(1 + 2\lambda)(L + 2\tau))]$. Then, the iterates of the algorithm satisfy

$$\min_{0 \leq k \leq t-1} \mathbb{E} \left[ \|G(x^k)\|^2 \right] \leq \left[ \frac{D}{b} + 2 \right] \frac{(L + 2\tau)b}{\gamma t} R_0^2. \quad (13)$$

where $D = 2\lambda^2/(1 + \lambda)^2$ is a constant.

**Theorem 2.** Let Assumptions [14] hold true. Run Async-RED-SG for $t > 0$ iterations with uniform i.i.d selections of blocks and measurements using a fixed step-size $\gamma \in (0, 1/(1 + (2\lambda)(L + 2\tau))]$. Then, the iterates of the algorithm satisfy

$$\min_{0 \leq k \leq t-1} \mathbb{E} \left[ \|G(x^k)\|^2 \right] \leq \left[ \frac{D}{b} + 2 \right] \frac{(L + 2\tau)b}{\gamma t} R_0^2 + \left[ \frac{2D}{b} + 2 \right] \frac{\gamma}{w} C. \quad (14)$$

where $C = (L + 2\tau)(1 + \lambda)\nu^2$ and $D = 2\lambda^2/(1 + \lambda)^2$ are constants.

**Theorem 3.** Async-RED-SG approximates the solution obtained by Async-RED-BG up to a finite error that decreases for larger values of the minibatch size $w$. This relationship is consistent with the recent theoretical results on the online PnP and RED algorithms [11][27]. In practice, the selection of $w$ must balance the actual memory capacity of the system and the desired runtime for obtaining a reasonable solution. Our numerical evaluation in Section 5 demonstrates the excellent approximation of Async-RED-SG to the batch-gradient solution by using a small subset of data.

By carefully choosing the stepsize $\gamma$, we can state the following remark on Theorem 2.

**Remark 1.** Set the stepsize to be $\gamma = 1/\sqrt{wt}$. If the maximal delay satisfies $\lambda \leq (1/2)(\sqrt{w}/(L + 2\tau) - 1)$, then after $t > 0$ iterations we have

$$\min_{0 \leq k \leq t-1} \mathbb{E} \left[ \|G(x^k)\|^2 \right] \leq \left[ \frac{D}{b} + 2 \right] \frac{(L + 2\tau)b}{\sqrt{wt}} R_0^2 + \left[ \frac{2D}{b} + 2 \right] \frac{C}{\sqrt{wt}}. \quad (15)$$

This establishes the fixed-point convergence to the set $\text{zer}(G)$ at the rate of $O(1/\sqrt{wt})$ under specific conditions. If we treat entire $x$ as a block, namely that $U = U^T = I$ and $b = 1$, Async-RED-SG then becomes the asynchronous stochastic RED algorithm. Hence, the proposed remark immediately holds true for the
Figure 2: Convergence of ASYNC-RED-BG for different numbers of accessible cores \( n_c \in \{2, 4, 6, 8\} \). The left figure plots the average normalized distance to \( \text{zer}(G) \) against the iteration number; the middle and right figures plot these values, as well as SNR, plotted against the actual runtime in seconds. The shaded areas represent the range of values attained over the test images.

Figure 3: Left: Evolution of the convergence accuracy of ASYNC-RED-SG as the minibatch size \( w \) increases. The average distance is plotted against the number of iterations with the shaded areas representing the range of values attained over the test images. Middle & Right: Comparison of convergence speed between ASYNC-RED-BG/SG and other baselines. The right table summarizes the total runtime and the speed-up compared with GM-RED for all algorithms.

later. Note that our convergence rate \( O(1/\sqrt{wt}) \) is consistent with the rate proved for the serial \([54]\) and parallel \([15,55]\) stochastic gradient methods.

All the proofs are presented in the supplement. Our analysis never assumes the existence of an explicit regularizer, and hence applicable to advanced denoisers that are not associated with any regularizer.

5 Numerical Validation

We now present a numerical validation of ASYNC-RED. Our goals are first to validate the proposed theorems in Section 4 and then to demonstrate the effectiveness and the efficiency of our algorithm on the large-scale problem. We consider two image recovery tasks that have the form \( y = Ax + e \), where the measurement matrix \( A \) corresponds to either the random matrix in compressive sensing (CS) or the Radon transform in computed tomography (CT), and the noise \( e \) is assumed to be additive white Gaussian (AWGN). In particular, the random matrix is implemented with the block-diagonal structure \( A = \text{diag}(A_1, ..., A_b) \) for fast validation, while the Radon transform is used as its full matrix form to demonstrate the effectiveness of ASYNC-RED for overcoming the computation bottleneck. Our deep neural net prior adapts the DnCNN architecture \([17]\). We used the signal-to-noise ratio (dB) to quantify the quality of the reconstructed images. For each experiments, we selected the denoiser that achieves the best SNR performance from the ones corresponding to five noise levels \( \sigma \in \{5, 10, 15, 20, 25\} \). Supplement D provides additional technical details.

5.1 Convergence Behavior

We validate our theorems on the CS task with 6 test images selected from the Set 12 dataset \([17]\). Each test image is rescaled to the size of 240 \( \times \) 240 pixels (see Fig. 6 in the supplement for the visualization). The block-diagonal matrix \( A \) is set to consist of 9 submatrices, corresponding to a 3 \( \times \) 3 grid of blocks with the
size of 80 × 80 pixels in every image. The elements in \( A \) are i.i.d zero-mean Gaussian random variables of variance of \( 1/m \), and the compression ratio is set to be \( m/n = 0.7 \), which indicates that the total number of measurements is 4480 for each block. We obtain the measurements by multiplying \( A \) with each vectorized image and adding additional noise corresponding to the input SNR of 30 dB. Finally, we use the normalized distance \( \|G(x_k)\|^2_2/\|G(x_0)\|^2_2 \) to quantify the fixed-point convergence, with \( b \) block updates grouped as one iteration. The distance is expected to approach zero as the algorithm converges to a fixed point.

Theorem 1 establishes the convergence of Async-RED-BG to the fixed point set \( \text{zer}(G) \). This is illustrated in Fig. 2 for four different numbers of accessible cores \( n_c \in \{2, 4, 6, 8\} \). In the left figure, the average normalized distance is plotted against the iteration number, while the middle and right figures plot the corresponding distance and SNR values against the actual runtime in seconds. The shaded areas representing the range of values attained across all test images. We also plot the results of serial BC-RED using the dashed line as reference. Async-RED-BG is implemented to be run asynchronously on multiple cores, while BC-RED can only use one core to perform the computation. The left figure highlights the fixed-point convergence of Async-RED-BG in iteration for different \( n_c \), with all variants agreeing with the serial BC-RED. Since Async-RED-BG uses more cores, the middle and right figures demonstrate the significantly faster in-time convergence of Async-RED-BG than BC-RED to the same SNR value. Specifically, BC-RED takes 1.8 hours to achieve 29.00 dB, while Async-RED-BG (\( n_c = 8 \)) takes only 17.9 minutes to obtain the same value, corresponding to a \( 6 \times \) improvement in computation time.

Theorem 2 establishes the convergence of Async-RED-SG to \( \text{zer}(G) \) up to some error term, which is inversely proportional to the minibatch size \( w \). This is illustrated in Fig. 3 (left) for three different minibatch sizes \( w \in \{1120, 2240, 3360\} \). As before, we plotted the average distance against the iteration number with the shading area representing the variance. Note that the log-scale of y-axis highlights the change for smaller values. Fig. 3 demonstrates the improved convergence of Async-RED-SG to \( \text{zer}(G) \) for larger \( w \), which is consistent with our theoretical analysis. Fig. 3 (middle) compares the convergence speed between Async-RED-BG/SG, gradient-method RED (GM-RED), and synchronous parallel RED (Sync-RED). For Async-RED-SG, we use \( w = 1120 \). In particular, Async-RED-SG takes fewer total runtime (from 17.9 min to 13.0 min) to obtain the similar result (29.01 dB and 28.03 dB) and achieves \( 8.4 \times \) speedup compared with GM-RED. The table in Fig. 3 summarizes the detailed results.

### 5.2 Effectiveness for Computational Imaging

We additionally demonstrate the effectiveness of our algorithm by reconstructing a 800 × 800 CT image from its 180 projections. For block parallel updates, the image is decomposed into 16 blocks, each having the size of 200 × 200 pixels. The Radon matrix used in the experiment corresponds to 180 angles with 1131
detectors, and the noise level is set to 70 dB. We refer to Supplement D.2 for additional technical details. Fig. 4 shows the visual illustration of the reconstructed images by ASYNC-RED-BG/SG and GM-RED. Each algorithm starts from the filtered back-projection (FBP) of the measurements and runs for 1 hour. Here, ASYNC-RED-SG randomly uses one-third of the total measurements at every iteration. Given the same amount of time, ASYNC-RED-BG/SG successfully mitigates the noise-artifacts, while the result of GM-RED is still noisy. In particular, the per-iteration time cost of ASYNC-RED-BG/SG and GM-RED is 5.23, 3.21, and 19.19 seconds, respectively. This experiment clearly illustrates the fast processing speed of the asynchronous procedure.

6 Conclusion

Asynchronous parallel methods have gained increasing importance in optimization for solving large-scale imaging inverse problems. We have introduced ASYNC-RED as an extension of the recent RED framework and theoretically analyze its convergence in batch and stochastic settings. We have validated its convergence guarantees and demonstrated its effectiveness in CT image reconstruction. Future work will investigate theoretical limits of ASYNC-RED in the unbounded maximal delay setting and explore its applicability to various inference problems in other data-intensive fields.

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Supplementary Material

Our unified analysis of ASYNC-RED is based on the monotone operator theory [56]. In Supplement A, we first clarify our setting for the access of the shared memory. In Supplement B, we present the proof of Theorem 1 and Theorem 2, proving the fixed-point convergence of ASYNC-RED to zer(G) in both batch and stochastic settings. In Supplement C, we provide a brief review of the related knowledge on monotone operators. In Supplement D, we include additional technical details and experiments omitted from the main paper due to space.

A Memory Access without Global Lock

In the setting of ASYNC-RED, multiple cores may simultaneously read and update the blocks x_i in shared memory. We coordinate the memory access of different cores by imposing certain local locks. For example, consider one work cycle of core c_i for updating the block x_i. First, a local read lock is imposed to x_i such that only read operations (by c_i or others) can be performed on x_i. If, at the same time, other cores want to write x_i, then they have to wait until the read lock is released by the last one who finishes reading the block. However, if they want to write other blocks, their operations will not be blocked. Secondly, core c_i evaluates the RED update on x_i, while other cores continuously update x. Here, we assume that the number of updates by cores other than c_i is bounded by some positive integer, which is exactly what Assumption 1 refers to. After the evaluation finishes, core c_i imposes a local write lock, which prevents both read and write by other cores, on x_i and writes the block with the computed update. Similarly, other cores have to wait until the lock is released before operating on x_i. Finally, when the update finishes, the local lock will be released and core c_i will restart a new cycle. Note that x is never locked globally during the full update cycle, and the reads of each block are always consistent.

In order to ensure the consistent read of x, we leverage the dual-memory strategy for block coordinate settings proposed in [38] (see section 1.2.1 'Block coordinate'). Its key idea is that, before every write to a block x_i, a copy of the old version of the block is kept for reading. In this way, there always exists some state of x in the memory for the cores to access.

B Proof of Analysis

In this section, we first present the proof of Theorem 1 then followed by the proof of Theorem 2. For a review of monotone operators, we refer to Supplement C.

Throughout the proof, we consider the probability space (Ω, F, P), where Ω denotes the sample space, F the σ-algebra, and P the probability measure. x_k is a random variable defined in \( \mathbb{R}^n \). We use \( \| \cdot \| \) to denote the \( \ell_2 \)-norm. We define the sequence of sub σ-algebra \( \{ \mathcal{X}_k \}_{k \in \mathbb{N}} \) of F as

\[ \mathcal{X}_k := \sigma(x^0, \ldots, x^k, \Delta_0, \ldots, \Delta_k), \]

where σ generates the filtration (smallest σ-algebra) from \( x^0, \ldots, x^k, \Delta_0, \ldots, \Delta_k \). Note that the sequence \( \{ \mathcal{X}_k \}_{k \in \mathbb{N}} \) is such that \( \mathcal{X}_k \subseteq \mathcal{X}_{k+1} \) for any \( k \in \mathbb{N} \). We use \( x^* \) to denote some fixed point in the set zer(G).

B.1 Proof of Theorem 1

Our proof needs the following lemma on the RED operator.

Lemma 1. Let Assumption 3 and 4 hold for g and D_σ. The composite operator G is \( 1/(L + 2\tau) \)-cocoercive, that is

\[ (G(x) - G(y))^T(x - y) \geq \frac{1}{L + 2\tau} \| G(x) - G(y) \|^2. \]
Proof. This lemma is adapted from Lemma 3 in [14]. Consider the following decomposition

\[
1 - \frac{2}{L + 2\tau} G = \left( \frac{2}{L + 2\tau} \cdot \frac{L}{2} \right) \left[ 1 - \frac{2}{L} \nabla g \right] + \left( \frac{2}{L + 2\tau} \cdot \frac{2\tau}{2} \right) \left[ 1 - \frac{1}{\tau} H \right],
\]

where we recall $H = \tau (I - D_x)$. According to Assumption 3 $g$ is convex and $\nabla g$ is $L$-Lipschitz continuous. By Proposition 1 in Supplement C $\nabla g$ is $1/L$-coercive. Hence, by Proposition 2 in Supplement C $1 - (2/L) \nabla g$ is nonexpansive. Since $D_x = I - (1/\tau) H$, this means that $1 - (1/\tau) H$ is nonexpansive. From Proposition 3 in Supplement C we know that the convex combination of two nonexpansive operators is nonexpansive. Thus, $1 - (2/(L + 2\tau)) G$ is nonexpansive, which also means that $G$ is $1/(L + 2\tau)$-coercive according to Proposition 2 in Supplement C.

Now we can start the main proof. Under the fixed stepsize $\gamma > 0$, we begin with the following equations regarding the fixed point $x^* \in \text{zer}(G)$

\[
\mathbb{E} \left[ \|x^{k+1} - x^*\|_\mathcal{X}^2 \right] = \mathbb{E} \left[ \|x^k - \gamma G_i(x^k) - x^*\|_\mathcal{X}^2 \right] = \mathbb{E} \left[ \|x^k - x^*\|_\mathcal{X}^2 \right] + \gamma^2 \mathbb{E} \left[ \|G_i(x^k)\|_\mathcal{X}^2 \right] + 2\gamma \mathbb{E} \left[ (G_i(x^k))^T (x^* - x^k) \right],
\]

Since $G_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is evaluated on a random block of $x_i$, we have the following conditional expectations

\[
\mathbb{E} \left[ (G_i(x^k))^T (x^* - x^k) \right] = \frac{1}{b} \sum_{i=1}^{b} (G_i(x^k))^T (x^* - x^k) = \frac{1}{b} (G(x^k))^T (x^* - x^k),
\]

and

\[
\mathbb{E} \left[ \|G_i(x^k)\|_\mathcal{X}^2 \right] = \frac{1}{b} \sum_{i=1}^{b} \|G_i(x^k)\|^2 = \frac{1}{b} \|G(x^k)\|^2.
\]

Thus, plugging the above results into (17)

\[
\mathbb{E} \left[ \|x^{k+1} - x^*\|_\mathcal{X}^2 \right] \leq \|x^k - x^*\|^2 + \frac{\gamma^2}{b} \|G(x^k)\|^2 + \frac{2\gamma}{b} (G(x^k))^T (x^* - x^k).
\]

The term $(i)$ can be expressed as

\[
\frac{2\gamma}{b} (G(x^k))^T (x^* - x^k)
= \frac{2\gamma}{b} (G(x^k))^T (x^* - \tilde{x}^k + \sum_{s=k-\Delta_k}^{k-1} (x^s - x^{s+1}))
= \frac{2\gamma}{b} (G(x^k) - G(x^*))^T (x^* - \tilde{x}^k) + \frac{2\gamma}{b} (G(x^k))^T \left( \sum_{s=k-\Delta_k}^{k-1} (x^s - x^{s+1}) \right)
= \frac{2\gamma}{b} (G(x^k) - G(x^*))^T (x^* - \tilde{x}^k) + \frac{2\gamma^2}{b} \sum_{s=k-\Delta_k}^{k-1} G(x^k)^T G_i(x^s),
\]

where in the second line we used the definition of the stale iterate $x^{s+1} = x^s - \gamma G_i(x^k)$, and in the third line the fact that $G(x^*) = 0$. By using Lemma 1 we obtain the upper bound for the first term in equation (21)

\[
\frac{2\gamma}{b} (G(x^k) - G(x^*))^T (x^* - \tilde{x}^k) \leq \frac{2\gamma \|G(x^k)\|^2}{b(L + 2\tau)}.
\]
For the second term in (21), we have
\[ \frac{2\gamma^2}{b} \sum_{s=b-\Delta_k}^{b-1} \| G(\bar{x})^T G_{x_s}(\bar{x}) \| \leq \frac{\lambda \gamma^2 \| G(\bar{x})^k \|^2}{b} + \sum_{s=b-\Delta_k}^{b-1} \gamma^2 \| G_{x_s}(\bar{x}) \|^2, \]
\[ \leq \frac{\lambda \gamma^2 \| G(\bar{x})^k \|^2}{b} + \sum_{s=b-\lambda}^{b-1} \gamma^2 \| G(\bar{x})^s \|^2, \]
(23)
where in the first inequality we used the Young’s inequality
\[ x_1^T x_2 \leq \frac{1}{2} \| x_1 \|^2 + \| x_2 \|^2, \]
(24)
and in the second inequality we use
\[ \sum_{s=b-\Delta_k}^{b-1} \gamma^2 \| G_{x_s}(\bar{x}) \|^2 = \sum_{s=b-\Delta_k}^{b-1} \| \bar{x}^s - \bar{x}^{s+1} \|^2 \leq \sum_{s=b-\Delta_k}^{b-1} \| \bar{x}^s - \bar{x}^{s+1} \|^2 = \sum_{s=b-\lambda}^{b-1} \gamma^2 \| G(\bar{x})^s \|^2. \]
Applying (22) and (23) in (21) yields the overall upper bound for the term (\dag)
\[ \frac{2\gamma}{b} (G(\bar{x})^k)^T(\bar{x}^* - \bar{x}^k) \leq \frac{(L + 2\tau)\lambda \gamma^2 - 2\gamma}{(L + 2\tau)b} \| G(\bar{x})^k \|^2 + \sum_{s=b-\lambda}^{b-1} \gamma^2 \| G(\bar{x})^s \|^2. \]
(25)
Next, by plugging (25) into (17) and re-arranging the terms, we obtain the following inequality
\[ \begin{align*}
    & \mathbb{E} \left[ \| x^{k+1} - x^* \|^2 | \mathcal{X}^k \right] \\
    & \leq \| x^k - x^* \|^2 + \sum_{s=b-\lambda}^{b-1} \gamma^2 \mathbb{E} \left[ \| G(\bar{x})^s \|^2 \right] \\
    & \quad + \frac{(L + 2\tau)(1 + \lambda)\gamma^2 - 2\gamma}{(L + 2\tau)b} \mathbb{E} \left[ \| G(\bar{x})^k \|^2 \right].
\end{align*} \]
(26)
Taking the total expectation of equation (26) and re-arranging the terms yields that
\[ \begin{align*}
    & \frac{2\gamma - (L + 2\tau)(1 + \lambda)\gamma^2}{(L + 2\tau)b} \mathbb{E} \left[ \| G(\bar{x})^k \|^2 \right] \\
    & \quad \leq \mathbb{E} \left[ \| x^0 - x^* \|^2 \right] - \mathbb{E} \left[ \| x^k - x^* \|^2 \right] + \gamma^2 \mathbb{E} \left[ \| G(\bar{x})^k \|^2 \right] \\
    & \quad + \sum_{s=b-\lambda}^{b-1} \mathbb{E} \left[ \| G(\bar{x})^s \|^2 \right].
\end{align*} \]
(27)
We then telescope-sum equation (27) over \( t > 0 \) iterations to have
\[ \begin{align*}
    & \sum_{k=0}^{t-1} \frac{2\gamma - (L + 2\tau)(1 + \lambda)\gamma^2}{(L + 2\tau)b} \mathbb{E} \left[ \| G(\bar{x})^k \|^2 \right] \\
    & \quad \leq \mathbb{E} \left[ \| x^0 - x^* \|^2 \right] - \mathbb{E} \left[ \| x^t - x^* \|^2 \right] + \gamma^2 \sum_{k=0}^{t-1} \sum_{s=b-\lambda}^{b-1} \mathbb{E} \left[ \| G(\bar{x})^s \|^2 \right].
\end{align*} \]
(28)
where the index \( s \) always start at 0. Under the assumption of consistent read, it is true that
\[ \sum_{k=0}^{t-1} \sum_{s=b-\lambda}^{b-1} \mathbb{E} \left[ \| G(\bar{x})^s \|^2 \right] \leq \lambda \sum_{k=0}^{t-1} \mathbb{E} \left[ \| G(\bar{x})^k \|^2 \right]. \]
(29)
In the case of inconsistent read, the above inequality does not always hold. We refer to [38] for a comprehensive analysis for asynchronous block-coordinate methods with inconsistent reads. Now, we rewrite equation (28) as

\[
\sum_{k=0}^{t-1} \frac{2\gamma - (L + 2\tau)(1 + 2\lambda)\gamma^2}{(L + 2\tau)b} \mathbb{E} \left[ \|G(\bar{x}^k)\|^2 \right] \leq \mathbb{E} \left[ \|x^0 - x^*\|^2 \right] - \mathbb{E} \left[ \|x^t - x^*\|^2 \right].
\]

(30)

In order to ensure the convergence, we need the coefficient of \( \mathbb{E} \left[ \|G(\bar{x}^k)\|^2 \right] \) to be positive. From basic algebra, one feasible range for the stepsize \( \gamma \) is

\[
0 < \gamma \leq \frac{1}{(L + 2\tau)(1 + 2\lambda)},
\]

which directly implies that

\[
0 < \frac{\gamma}{(L + 2\tau)b} \leq \frac{2\gamma - (L + 2\tau)(1 + 2\lambda)\gamma^2}{(L + 2\tau)b}.
\]

By simplifying (30) with the above result and dropping the negative term, we can derive the following bound for the \( \mathbb{E} \left[ \|G(\bar{x}^k)\|^2 \right] \) averaged over \( t \) iterations

\[
\frac{1}{t} \sum_{k=0}^{t-1} \frac{(L + 2\tau)b}{\gamma t} \mathbb{E} \left[ \|x^0 - x^*\|^2 \right] \leq \frac{(L + 2\tau)b}{\gamma t} R_0^2.
\]

(31)

The above inequality establishes that the change of the stale iterate \( \bar{x}^k \) converges to zero as \( t \) increases. Next, we will use the bound to establish the similar result for the actual iterate \( x^k \). We know that \( \|G(x^k)\|^2 \) can be bounded by

\[
\|G(x^k)\|^2 \leq (\|G(x^k) - G(\bar{x}^k)\| + \|G(\bar{x}^k)\|)^2 \\
= \|G(x^k) - G(\bar{x}^k)\|^2 + \|G(\bar{x}^k)\|^2 + 2\|G(x^k) - G(\bar{x}^k)\|\|G(\bar{x}^k)\| \\
\leq 2\|G(x^k) - G(\bar{x}^k)\|^2 + 2\|G(\bar{x}^k)\|^2 \\
\leq 2(L + 2\tau)^2\|x^k - \bar{x}^k\|^2 + 2\|G(\bar{x}^k)\|^2.
\]

(32)

where in the second inequality we used the Young’s inequality (24), and in the third inequality we used the following result implied by Lemma [1]

\[
(L + 2\tau)\|x - y\| \geq \|G(x) - G(y)\|.
\]

By expressing the stale iterate \( \bar{x}^k \), we can write equation (32) as

\[
\|G(x^k)\|^2 \leq 2(L + 2\tau)^2 \sum_{s=k-\lambda}^{s=k-1} \gamma_{s, i} \bar{x}^s - \bar{x}^k\|^2 + 2\|G(\bar{x}^k)\|^2.
\]

\[
\leq 2\lambda(L + 2\tau)^2 \sum_{s=k-\lambda}^{s=k-1} \gamma^2 \|G_{s, i} \bar{x}^s\|^2 + 2\|G(\bar{x}^k)\|^2.
\]

(33)

where we use the fact

\[
\| \sum_{i=1}^{n} x_i \|^2 = \sum_{i=1}^{n} \| x_i \|^2 + \sum_{a \neq b} x_a^T x_b \leq \sum_{i=1}^{n} \| x_i \|^2 + \frac{1}{2} \sum_{a \neq b} \left[ \| x_a \|^2 + \| x_b \|^2 \right] = n \sum_{i=1}^{n} \| x_i \|^2
\]
We prove Theorem 2 by following the procedure in the proof of Theorem 1 with the adaptation to the block stochastic operator $G_i$. In the key steps, we will highlight the difference between the two proofs. In addition to Lemma 1, our second proof requires the following lemma related to the statistical properties of $\tilde{G}$.

\textbf{B.2 Prove of Theorem 2}

We prove Theorem 2 by following the procedure in the proof of Theorem 1 with the adaptation to the block stochastic operator $G_i$. In the key steps, we will highlight the difference between the two proofs. In addition to Lemma 1, our second proof requires the following lemma related to the statistical properties of $\tilde{G}$. 

Taking the expectation of equation (33) leads to
\[\mathbb{E}[\|G(x^k)\|^2]\]
\[\leq 2\lambda(L + 2\tau)^2 \sum_{s=k-\lambda}^{k-1} \gamma^2 \mathbb{E}[\|G_s(\bar{x})\|^2] + 2\mathbb{E}\left[\|G(\bar{x})\|^2\right]\]
\[\leq 2\lambda(L + 2\tau)^2 \sum_{s=k-\lambda}^{k-1} \gamma^2 \mathbb{E}\left[\frac{\|G(\bar{x})\|^2}{b}\right] + 2\mathbb{E}\left[\|G(\bar{x})\|^2\right], \quad (34)\]

By averaging (34) over $t > 0$ iterations, we obtain that
\[\frac{1}{t} \sum_{k=0}^{t-1} \mathbb{E}\left[\|G(x^k)\|^2\right]\]
\[\leq \frac{2\lambda(L + 2\tau)^2}{t} \sum_{k=0}^{t-1} \sum_{s=k-\lambda}^{k-1} \frac{\gamma^2 \mathbb{E}[\|G(\bar{x})\|^2]}{b} + \frac{2}{t} \sum_{k=0}^{t-1} \mathbb{E}\left[\|G(\bar{x})\|^2\right]\]
\[\leq \frac{2\lambda^2(L + 2\tau)^2}{t} \sum_{k=0}^{t-1} \frac{\gamma^2 \mathbb{E}[\|G(\bar{x})\|^2]}{b} + \frac{2}{t} \sum_{k=0}^{t-1} \mathbb{E}\left[\|G(\bar{x})\|^2\right] \quad (35)\]

where we again used result in (29) in the last inequality. Re-arranging the terms in (35) yields
\[\frac{1}{t} \sum_{k=0}^{t-1} \mathbb{E}\left[\|G(x^k)\|^2\right] \leq \left[\frac{2\lambda^2(L + 2\tau)^2}{b} \gamma^2 + 2 \right] \frac{1}{t} \sum_{k=0}^{t-1} \mathbb{E}\left[\|G(\bar{x})\|^2\right] \quad (36)\]

We plug the result in (31) into (36) and obtain
\[\frac{1}{t} \sum_{k=0}^{t-1} \mathbb{E}\left[\|G(x^k)\|^2\right] \leq \left[\frac{2\lambda^2(L + 2\tau)^2}{b} \gamma^2 + 2 \right] \frac{(L + 2\tau)b}{\gamma t} R_0^2, \quad (37)\]

Since it is always true that
\[\gamma \leq \frac{1}{(L + 2\tau)(1 + 2\lambda)} \leq \frac{1}{(L + 2\tau)(1 + \lambda)},\]
we can simplify the bound by using the above inequality related to the stepsize $\gamma$
\[\frac{1}{t} \sum_{k=0}^{t-1} \mathbb{E}\left[\|G(x^k)\|^2\right] \leq \left[\frac{2\lambda^2}{(1 + \lambda)^2} + 2 \right] \frac{(L + 2\tau)b}{\gamma t} R_0^2. \quad (38)\]

Let $D = 2\lambda^2/(1 + \lambda)^2$, and we derive the desired result.

\[\min_{0 \leq k \leq t-1} \mathbb{E}\left[\|G(x^k)\|^2\right] \leq \left[\frac{D}{b} + 2 \right] \frac{(L + 2\tau)b}{\gamma t} R_0^2. \quad (39)\]
Lemma 2. Let Assumptions 3 and 4 hold for \( g \) and \( D_\sigma \). Then, we can establish the following statements for operator \( \hat{G} \)

\[
\mathbb{E} \left[ \hat{G}(x) \right] = G(x), \quad \mathbb{E} \left[ \| \hat{G}(x) - G(x) \|^2 \right] \leq \frac{\nu^2}{w},
\]

which further implies that

\[
\mathbb{E} \left[ \| \hat{G}(x) \|^2 \right] \leq \frac{\nu^2}{w} + \| G(x) \|^2.
\]

Proof. Since the stochasticity happens only in the evaluation of the gradient, it is straightforward to see that

\[
\mathbb{E} \left[ \hat{G}(x) \right] = \mathbb{E}[\nabla g(x)] + D_\sigma(x) = G(x),
\]

Similarly, we have that

\[
\mathbb{E} \left[ \| \hat{G}(x) - G(x) \|^2 \right] = \mathbb{E} \left[ \| \hat{\nabla} g(x) - \nabla g(x) \|^2 \right] \leq \frac{\nu^2}{w}.
\]

Given that \( \text{Tr}(\mathbb{E} [X^T X]) = \text{Tr}(\text{Cov} [X]) + \text{Tr}(\mathbb{E} [X]^2) \), we obtain that

\[
\mathbb{E} \left[ \| \hat{G}(x) \|^2 \right] = \mathbb{E} \left[ \| \hat{G}(x) - G(x) \|^2 \right] + \mathbb{E} \left[ \| G(x) \|^2 \right] \leq \frac{\nu^2}{w} + \| G(x) \|^2,
\]

where we let \( \mathbb{E} \left[ \hat{G}(x) \right]^2 := \mathbb{E} \left[ \hat{G}(x) \right]^T \mathbb{E} \left[ \hat{G}(x) \right] \). Note that \( \text{Tr}(\cdot) \) and \( \text{Cov}(\cdot) \) denote the computation of the trace and covariance of a matrix and a vector, respectively.

Now we start the proof. Similar as (17), we write that

\[
\mathbb{E} \left[ \| x^{k+1} - x^* \|^2 | \mathcal{X}^k \right] = \mathbb{E} \left[ \| x^k - \gamma \hat{G}_i(\bar{x}^k) - x^* \|^2 | \mathcal{X}^k \right] = \mathbb{E} \left[ \| x^k - x^* \|^2 | \mathcal{X}^k \right] + \gamma^2 \mathbb{E} \left[ \| \hat{G}_i(\bar{x}^k) \|^2 | \mathcal{X}^k \right] + 2\gamma \mathbb{E} \left[ \hat{G}_i(\bar{x}^k)^T (x^* - x^k) | \mathcal{X}^k \right] \tag{40}
\]

Here, the conditional expectation is taken for \( \hat{G}_i(x) = U_i U_i^T \hat{G}(x) \). By using Lemma 2, we can compute conditional expectations as

\[
\mathbb{E} \left[ (\hat{G}_i(\bar{x}^k))^T (x^* - x^k) | \mathcal{X}^k \right] = \frac{1}{b} \mathbb{E} \left[ (\hat{G}(\bar{x}^k))^T (x^* - x^k) | \mathcal{X}^k \right] = \frac{1}{b} (G(\bar{x}^k))^T (x^* - x^k) \tag{41}
\]

and

\[
\mathbb{E} \left[ \| \hat{G}_i(\bar{x}^k) \|^2 | \mathcal{X}^k \right] = \frac{1}{b} \mathbb{E} \left[ \| \hat{G}(\bar{x}^k) \|^2 | \mathcal{X}^k \right] \leq \frac{\nu^2}{wb} + \frac{\| G(\bar{x}^k) \|^2}{b} \tag{42}
\]

where we first compute the expectation corresponding to the randomized block and then the expectation for the stochastic measurements. We note that the expectation of the cross term (41) remains the same as the result in (18), while the expectation in (42) has one extra term related to the norm variance of the stochastic operator compared with (19). As we shall see in the future steps, the difference in the expectation of the operator’s squared norm leads to the most modifications. Using the above results in equation (40) yields that

\[
\mathbb{E} \left[ \| x^{k+1} - x^* \|^2 | \mathcal{X}^k \right] \leq \| x^k - x^* \|^2 + \frac{\gamma^2}{b} \| G(\bar{x}^k) \|^2 + \frac{\gamma^2 \nu^2}{wb} + \frac{2\gamma}{b} (G(\bar{x}^k))^T (x^* - x^k). \tag{43}
\]
By following (21), we can express the term \( \langle i \rangle \) as
\[
\frac{2\gamma}{b} (G(\bar{x}^k))^T (x^* - x^k)
= \frac{2\gamma}{b} (G(\bar{x}^k) - G(x^*))^T (x^* - \bar{x}^k) + \frac{2\gamma^2}{b} \sum_{s=k-\Delta_k}^{k-1} G(\bar{x}^k)^T G_{s-1}(\bar{x}^*),
\]
(44)

The upper bound of the first term is the same as shown in (22), which is
\[
\frac{2\gamma}{b} (G(\bar{x}^k) - G(x^*))^T (x^* - \bar{x}^k) \leq - \frac{2\gamma \|G(\bar{x}^k)\|^2}{b(L+2\tau)}.
\]
(45)

Similarly, our second term is bounded by
\[
\frac{2\gamma^2}{b} \sum_{s=k-\Delta_k}^{k-1} G(\bar{x}^k)^T \hat{G}_{s-1}(\bar{x}^*) \leq \frac{\lambda \gamma^2 \|G(\bar{x}^k)\|^2}{b} + \sum_{s=k-\lambda}^{k-1} \frac{\gamma^2 \|\hat{G}(\bar{x}^*)\|^2}{b},
\]
where we used the Young’s inequality (24) together with the fact that
\[
\sum_{s=k-\Delta_k}^{k-1} \|\hat{G}_{s-1}(\bar{x}^k)\|^2 \leq \sum_{s=k-\lambda}^{k-1} \|\hat{G}_{s-1}(\bar{x}^k)\|^2 \leq \sum_{s=k-\lambda}^{k-1} \|\hat{G}(\bar{x}^k)\|^2.
\]

Equation (45) and (46) together establish the overall upper bound for the term \( \langle i \rangle \)
\[
\frac{2\gamma}{b} (G(\bar{x}^k))^T (x^* - x^k) \leq \frac{(L+2\tau)\lambda \gamma^2 - 2\gamma^2}{(L+2\tau) b} \|G(\bar{x}^k)\|^2 + \sum_{s=k-\lambda}^{k-1} \frac{\gamma^2 \|\hat{G}(\bar{x}^*)\|^2}{b}.
\]
(47)

By plugging (47) into (40) and re-arranging the terms, we obtain
\[
E \left[ \|x^{k+1} - x^*\|^2 | \mathcal{B}^k \right] 
\leq \|x^k - x^*\|^2 + \frac{\gamma^2 \nu^2}{wb} + \sum_{s=k-\lambda}^{k-1} \frac{\gamma^2 \|\hat{G}(\bar{x}^*)\|^2}{b} + \frac{(L+2\tau)(1+\lambda)\gamma^2 - 2\gamma^2}{(L+2\tau)b} \|G(\bar{x}^k)\|^2.
\]
(48)

Taking the total expectation of equation (48) and re-arranging the terms yields that
\[
\frac{2\gamma - (L+2\tau)(1+\lambda)\gamma^2}{(L+2\tau)b} E \left[ \|G(\bar{x}^k)\|^2 \right]
\leq E \left[ \|x^k - x^*\|^2 \right] - E \left[ \|x^{k+1} - x^*\|^2 \right] + \frac{\gamma^2 \nu^2}{wb} + \sum_{s=k-\lambda}^{k-1} \left[ \frac{\nu^2}{wb} + \frac{E \left[ \|G(\bar{x}^*)\|^2 \right]}{b} \right]
\]
(49)

where we use the following inequality derived by using the law of total expectation and Lemma 2
\[
E \left[ \|\hat{G}(\bar{x}^*)\|^2 \right] = E \left[ E \left[ \|\hat{G}(\bar{x}^*)\|^2 | \mathcal{B}^* \right] \right] \leq \frac{\nu^2}{w} + E \left[ \|G(\bar{x}^*)\|^2 \right].
\]
(50)

We telescope-sum equation (49) over \( t > 0 \) iterations to obtain
\[
\sum_{k=0}^{t-1} \frac{2\gamma - (L+2\tau)(1+\lambda)\gamma^2}{(L+2\tau)b} E \left[ \|G(\bar{x}^k)\|^2 \right]
\leq E \left[ \|x^0 - x^*\|^2 \right] - E \left[ \|x^t - x^*\|^2 \right] + \sum_{k=0}^{t-1} \frac{\gamma^2 \nu^2}{wb} + \gamma^2 \sum_{k=0}^{t-1} \sum_{s=k-\lambda}^{k-1} \left[ \frac{\nu^2}{wb} + \frac{E \left[ \|G(\bar{x}^*)\|^2 \right]}{b} \right]
\]
(51)
where we used Assumption 2 and let
\[ \sum_{k=0}^{t-1} \sum_{s=k-\lambda}^{k-1} \left[ \frac{\nu^2}{w_b} + \frac{\mathbb{E} \left[ \| \tilde{G}(\bar{x}^i) \|^2 \right]}{b} \right] \leq \lambda \sum_{k=0}^{t-1} \left[ \frac{\nu^2}{w_b} + \frac{\mathbb{E} \left[ \| \tilde{G}(\bar{x}^k) \|^2 \right]}{b} \right], \tag{52} \]
we then have
\[ \sum_{k=0}^{t-1} 2\gamma - (L + 2\tau)(1 + 2\lambda)\gamma^2 \frac{\mathbb{E} \left[ \| \tilde{G}(\bar{x}^k) \|^2 \right]}{(L + 2\tau)b} \leq \mathbb{E} \left[ \| x^0 - x^* \|^2 \right] + \frac{(1 + \lambda)\gamma^2 \nu^2}{w_b} \cdot t, \tag{53} \]
where we dropped the negative term. Recall that if \( \gamma \) is in the range \( \gamma \in (0, 1/(L + 2\tau)(1 + 2\lambda)) \), we have the inequality
\[ \frac{\gamma}{(L + 2\tau)b} \leq \frac{2\gamma - (L + 2\tau)(1 + 2\lambda)\gamma^2}{(L + 2\tau)b}. \]
By relaxing the coefficient in the lefthand side, dividing the inequality by \( t \), and re-arranging the terms, we obtain the convergence in terms of the stale iterate \( \bar{x}^k \)
\begin{equation}
\frac{1}{t} \sum_{k=0}^{t-1} \mathbb{E} \left[ \| \tilde{G}(\bar{x}^k) \|^2 \right] \leq \frac{(L + 2\tau)b}{\gamma t} \left[ \mathbb{E} \left[ \| x^0 - x^* \|^2 \right] + \frac{(1 + \lambda)\gamma^2 \nu^2}{w_b} \cdot t \right] \\
\leq \frac{(L + 2\tau)b}{\gamma t} \mathbb{E} \left[ \| \tilde{G}(\bar{x}^k) \|^2 \right] + \frac{\gamma^2}{w \mathbb{E}} R_0^2 + \frac{\gamma}{w} C \tag{54} \end{equation}
where we used Assumption 2 and let \( C = (L + 2\tau)(1 + \lambda)\nu^2 \). Compared with the result in equation (31), equation (54) has the extra term related to the variance of \( \tilde{G}_i(x) \). Next, we establish the convergence in terms of actual iterate \( x^k \). Following the steps from (32) to (34), we directly obtain the inequality related to \( \tilde{G}_i(\bar{x}) \)
\begin{equation}
\mathbb{E} \left[ \| \tilde{G}(\bar{x}^k) \|^2 \right] \leq 2\lambda(L + 2\tau)^2 \sum_{s=k-\lambda}^{k-1} \gamma^2 \mathbb{E} \left[ \| \tilde{G}_i(\bar{x}^s) \|^2 \right] + 2\mathbb{E} \left[ \| G(\bar{x}^k) \|^2 \right] \tag{55} \end{equation}
By using the result in (56), we derive from (55) that
\begin{equation}
\mathbb{E} \left[ \| G(\bar{x}^k) \|^2 \right] \leq 2\lambda(L + 2\tau)^2 \sum_{s=k-\lambda}^{k-1} \gamma^2 \left[ \frac{\nu^2}{w_b} + \frac{\mathbb{E} \left[ \| \tilde{G}(\bar{x}^s) \|^2 \right]}{b} \right] + 2\mathbb{E} \left[ \| G(\bar{x}^k) \|^2 \right]. \tag{56} \end{equation}
By averaging (55) over \( t > 0 \) iterations, we obtain that
\begin{align}
\frac{1}{t} \sum_{k=0}^{t-1} \mathbb{E} \left[ \| G(\bar{x}^k) \|^2 \right] & \leq \frac{2\lambda(L + 2\tau)^2}{t} \sum_{k=0}^{t-1} \sum_{s=k-\lambda}^{k-1} \gamma^2 \left[ \frac{\nu^2}{w_b} + \frac{\mathbb{E} \left[ \| \tilde{G}(\bar{x}^s) \|^2 \right]}{b} \right] + \frac{2}{t} \sum_{k=0}^{t-1} \mathbb{E} \left[ \| G(\bar{x}^k) \|^2 \right] \\
& \leq \frac{2\lambda^2(L + 2\tau)^2}{t} \sum_{k=0}^{t-1} \gamma^2 \left[ \frac{\nu^2}{w_b} + \frac{\mathbb{E} \left[ \| \tilde{G}(\bar{x}^k) \|^2 \right]}{b} \right] + \frac{2}{t} \sum_{k=0}^{t-1} \mathbb{E} \left[ \| G(\bar{x}^k) \|^2 \right] \tag{57} \end{align}
where we again used the relaxation (52) in the last inequality. Re-arranging the terms in (57) yields
\begin{align}
\frac{1}{t} \sum_{k=0}^{t-1} \mathbb{E} \left[ \| G(\bar{x}^k) \|^2 \right] & \leq \frac{2\lambda^2(L + 2\tau)^2}{w_b} \cdot \gamma^2 + \frac{2\lambda^2(L + 2\tau)^2}{w_b} \gamma^2 + 2 \frac{1}{t} \sum_{k=0}^{t-1} \mathbb{E} \left[ \| G(\bar{x}^k) \|^2 \right] \tag{58} \end{align}
We plug the result in (54) into (58) and obtain

\[
\frac{1}{t} \sum_{k=0}^{t-1} \mathbb{E} \left[ \|G(x^k)\|^2 \right] \leq \frac{2\lambda^2(L + 2\tau)^2 \cdot \nu^2}{wb} \gamma + \left[ \frac{2\lambda^2(L + 2\tau)^2}{b} \gamma^2 + 2 \right] \frac{(L + 2\tau)bR_0^2 + \gamma C}{\gamma t} \tag{59}
\]

Similarly, we can use the fact

\[
\gamma \leq \frac{1}{(L + 2\tau)(1 + \lambda)}.
\]

to simplify the bound in (59)

\[
\frac{1}{t} \sum_{k=0}^{t-1} \mathbb{E} \left[ \|G(x^k)\|^2 \right] \leq \frac{2\lambda^2(L + 2\tau)^2 \cdot \nu^2}{wb} \cdot \mathbb{E} \left[ \|G(x^0)\|^2 \right] \cdot \frac{1}{(L + 2\tau)(1 + \lambda)} \cdot \gamma + \left[ \frac{2\lambda^2(L + 2\tau)^2}{b} \gamma^2 + 2 \right] \frac{(L + 2\tau)bR_0^2 + \gamma C}{\gamma t} \tag{60}
\]

where we recall \( C = (L + 2\tau)(1 + \lambda)\nu^2 \). Let \( D = 2\lambda^2/(1 + \lambda)^2 \) and we can derive the result of Theorem 2

\[
\min_{0 \leq k \leq t-1} \mathbb{E} \left[ \|G(x^k)\|^2 \right] \leq \frac{1}{t} \sum_{k=0}^{t-1} \mathbb{E} \left[ \|G(x^k)\|^2 \right] \leq \left[ \frac{D}{b} + 2 \right] \frac{(L + 2\tau)bR_0^2 + \gamma C}{\gamma t} \tag{61}
\]

which immediately implies the result in remark 1 by setting \( \gamma = 1/\sqrt{wt} \)

\[
\min_{0 \leq k \leq t-1} \mathbb{E} \left[ \|G(x^k)\|^2 \right] \leq \frac{1}{t} \sum_{k=0}^{t-1} \mathbb{E} \left[ \|G(x^k)\|^2 \right] \leq \left[ \frac{D}{b} + 2 \right] \frac{(L + 2\tau)bR_0^2 + \gamma C}{\sqrt{wt}} \tag{62}
\]

From basic algebra, we can derive the condition for \( \lambda \)

\[
\frac{1}{\sqrt{wt}} \leq \frac{1}{(L + 2\tau)(1 + 2\lambda)} \quad \Rightarrow \quad \lambda \leq \frac{1}{2} \left[ \frac{\sqrt{wt}}{L + 2\tau} - 1 \right].
\]

C Background on Monotone Operators

The results in our review can be found in different forms in standard textbooks \([57–60]\), and we include these results for completeness.

**Definition 1.** An operator \( T \) is Lipschitz continuous with constant \( L > 0 \) if

\[
\|Tx - Ty\| \leq L\|x - y\|, \quad x, y \in \mathbb{R}^n.
\]

When \( L = 1 \), we say that \( T \) is nonexpansive. When \( L < 1 \), we say that \( T \) is a contraction.
Figure 5: Illustration of the architecture of DnCNN used in all experiments. The neural net is trained to remove the AWGN from its noisy input image. We also constrain the Lipschitz constant of $R_\sigma$ to be smaller than 2 by using the spectral normalization technique in [52]. This provides a necessary condition for the satisfaction of Assumption 4.

**Definition 2.** $T$ is monotone if

$$(T(x) - T(y))^T(x - y) \geq 0, \quad x, y \in \mathbb{R}^n.$$ 

We say that it is strongly monotone or coercive with parameter $\mu > 0$ if

$$(T(x) - T(y))^T(x - y) \geq \mu \|x - y\|^2, \quad x, y \in \mathbb{R}^n.$$ 

**Definition 3.** $T$ is cocoercive with constant $\beta > 0$ if

$$(T(x) - T(y))^T(x - y) \geq \beta \|T x - T y\|^2, \quad x, y \in \mathbb{R}^n.$$ 

When $\beta = 1$, we say that $T$ is firmly nonexpansive.

The following results are derived from the definition above.

**Proposition 1.** For a convex and continuously differentiable function $f$, we have

$\nabla f$ is $L$-Lipschitz continuous $\iff$ $\nabla f$ is $(1/L)$-cocoercive.

**Proof.** The proof is a minor variation of the one presented as Theorem 2.1.5 in Section 2.1 of [59]. □

**Proposition 2.** Consider $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\beta > 0$. Then, the following are equivalent

$T$ is $\beta$-cocoercive $\iff$ $I - 2\beta T$ is nonexpansive.

**Proof.** Let $R := 1 - 2\beta T$, then $T = 1/(2\beta)(1 - R)$. First suppose that $T$ is $\beta$-cocoercive. Let $h := x - y$ for any $x, y \in \mathbb{R}^n$. We then have

$$\beta \|T(x) - T(y)\|^2 \leq (T(x) - T(y))^T h = \frac{1}{2\beta} \|h\|^2 - \frac{1}{2\beta} (R(x) - R(y))^T h.$$ 

We also have that

$$\beta \|T(x) - T(y)\|^2 = \frac{1}{4\beta} \|h\|^2 - \frac{1}{2\beta} (R(x) - R(y))^T h + \frac{1}{4\beta} \|R(x) - R(y)\|^2.$$ 

By combining these two and simplifying the expression

$$\|R(x) - R(y)\| \leq \|h\|.$$ 

The converse can be proved by following this logic in reverse. □
Figure 6: Six test images used in the experiments on CS. From the left to right, there are *cameraman*, *house*, *pepper*, *starfish*, *butterfly*, and *jet*.

The following characterization is also convenient.

**Proposition 3.** For nonexpansive operators $T_1$ and $T_2$ with a constant $\alpha \in (0, 1)$, then the convex combination of the two operators $(1 - \alpha)T_1 + \alpha T_2$ is nonexpansive.

**Proof.** Let $T := (1 - \alpha)T_1 + \alpha T_2$. For any $x, y \in \mathbb{R}^n$, we can write

$$
\|T(x) - T(y)\| \leq (1 - \alpha)\|T_1(x) - T_1(y)\| + \alpha\|T_2(x) - T_2(y)\| \leq \|x - y\|
$$

\[\square\]

### D Additional Technical Details

This section presents several technical details that were omitted from the main paper for space. Section [D.1](#D.1) presents the architecture and training of our DnCNN prior. Section [D.2](#D.2) provides extra details and validations that compliment the experiments in Section 5 of the main paper.

#### D.1 Architecture and Training of the DnCNN Prior

Our denoiser follows the standard architecture of DnCNN [17]. Fig. 5 visualizes the architectural details of the DnCNN prior used in our experiments. Similar priors are extensively used in various PnP and RED algorithms [14, 19, 30]. In total, the network contains 7 layers, of which the first 6 layers consist of a convolutional layer and a rectified linear unit (ReLU), while the last layer contains only a convolutional operation. A skip connection from the input to the output is used to enforce the residual network $R_\sigma$ to predict the noise residual. The output images of the first 6 layers have 64 feature maps, while that of the last layer is a single-channel image. We set all convolutional kernels to be $3 \times 3$ with stride 1, which indicates that intermediate images have the same spatial size as the input image. We generated 44700 training examples by adding AWGN to 400 images from the BSD400 dataset [61] and extracting small patches of $128 \times 128$ pixels with stride 30. Our DnCNN denoiser is trained to optimize the mean squared error by using the Adam optimizer [62].

Different approaches have been used to constrain the Lipschitz constant (LC) of the denoising prior [14, 30]. We adopt the spectral normalization technique in [52] to control the LC of our DnCNN prior. In the training, we constrain the residual network $R_\sigma$ such that its LC is smaller than 2. Since the non-expansiveness of $D_\sigma$ implies that $R_\sigma$ has $LC \leq 2$, this provides a necessary condition for $D_\sigma$ to satisfy Assumption 4 [14].

#### D.2 Extra Details and Validations

All experiments are run on the server equipped with 32 Intel(R) Xeon(R) CPU E5-2620 v4 processors of 3.2 GHz and 264 GBs of DDR memory. We trained all neural nets using NVIDIA RTX 2080 GPUs. We define the SNR (dB) used in the experiments as

$$
SNR(\hat{x}, x) \triangleq 20 \log_{10} \left( \frac{\|x\|_2}{\|x - \hat{x}\|_2} \right)
$$
where $\hat{x}$ represents the reconstructed image and $x$ denotes the ground truth.

Fig. 6 shows the six test images used in the experiments of CS. They are resized to the size of $240 \times 240$ pixels by using the Matlab function `imresize`. As demonstrated in the middle figure in Fig. 3, Async-RED-SG converges faster than Async-RED-BG given a fixed amount of time. This is further visualized in Fig. 7, where each algorithm is run for roughly 700 seconds. Since Async-RED-SG uses only one-fourth of the total measurements, the per-iteration complexity is lower than Async-RED-BG, leading to the faster convergence speed. In particular, the final SNR value obtained by Async-RED-SG is roughly 2 dB higher than Async-RED-BG. Additionally, both Async-RED-BG/SG achieve significantly better results than Sync-RED and Gm-RED due to their adoption of asynchronous updates.

The test image used in the experiment of CT is selected from the dataset of human protein atlas [63]. We download 51 images that have the size of $3000 \times 3000$ pixels. We select one image for test, which is cropped to $800 \times 800$ pixels. We extract 39000 patches from the rest 50 images to train five specific DnCNN denoisers for the removal of AWGN with $\sigma \in \{5, 10, 15, 20, 25\}$. We report the result that has the highest SNR values. The Radon matrix used in the experiments corresponds to 180 angles with 1131 detectors. We synthesize the measurements by multiplying the Radon matrix with the vectorized image and add AWGN corresponding to 70 dB input SNR. In all tests, Async-RED-SG randomly uses the measurements of 60 angles at each iteration, while Async-RED-BG uses the entire measurement set. Fig. 8 provides a complete comparison between Async-RED-BG/SG, Sync-RED, and Gm-RED. As reference, we also include the proximal gradient method with total variation regularizer (PGM-TV). The visual result of each method is obtained by running the algorithm with a time budget of 1 hour. Specifically, the per-iteration time cost of Async-RED-BG/SG, Sync-RED, Gm-RED, PGM-TV are 5.23, 3.21, and 13.13, 19.19, and 44.74 seconds, respectively. The results clearly demonstrate that Async-RED are indeed effective and efficient for a realistic, nontrivial imaging task on a large-scale image.
Figure 8: Visualization of the reconstructed CT images by PGM-TV, Gm-RED, Sync-RED, and Async-RED-BG/SG. Each algorithm is run with a time budget of 1 hour. The colormap is adjusted for the best visual quality.