COMPUTER ALGEBRA AND LANCZOS POTENTIAL

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ABSTRACT
We found in 2016 a few results on the mathematical structure of the conformal Killing differential sequence in arbitrary dimension \( n \), in particular the rank and order of the successive differential operators for \( n = 3, n = 4 \) or \( n \geq 5 \). They were so striking that we did not dare to publish them before our former PhD student A. Quadrat (INRIA) could confirm them while using new computer algebra packages that he developed for studying extension modules in differential homological algebra. In the meantime, we found in 2017 the "missing link" justifying the doubts we had since a long time on the origin and existence of Gravitational Waves in General Relativity. We also provide an example showing how these extension modules are depending on the structure constants appearing in the Vessiot structure equations (1903), still not acknowledged after one century even if they generalize the constant Riemannian curvature integrability condition of L.P. Eisenhart (1926). The present paper is made from the transparencies we provided during a lecture at the recent 24th conference on Applications of Computer Algebra (ACA 2018) held in Santiago de Compostela, Spain, June 18-22, 2018.

KEY WORDS
Differential sequence; Variational calculus; Differential constraint; Control theory; Killing operator; Riemann tensor; Bianchi identity; Weyl tensor; Lanczos tensor; Contact transformations; Vessiot structure equations. Differential homological algebra; Extension modules.

REFERENCES
[1] Pommaret, J.-F.: The Mathematical Foundations of General Relativity Revisited, Journal of Modern Physics, 4 (2013) 223-239.  
http://dx.doi.org/10.4236/jmp.2013.48A022  
[2] Pommaret, J.-F.: The Mathematical Foundations of Gauge Theory Revisited, Journal of Modern Physics, 5 (2014) 157-170.  
http://dx.doi.org/10.4236/jmp.2014.55026  
[3] Pommaret, J.-F.: Deformation Theory of Algebraic and Geometric Structures, Lambert Academic Publisher, (LAP), Saarbrucken, Germany, 2016.  
http://arxiv.org/abs/1207.1964  
[4] Pommaret, J.-F.: Why Gravitational Waves Cannot Exist, Journal of Modern Physics, 8,13 (2017) 2122-2158.  
http://dx.doi.org/10.4236/jmp.2017.813130  
[5] Pommaret, J.-F.: Homological Solution of the Riemann/Lanczos and Weyl/Lanczos Problems in Arbitrary Dimension,  
http://arxiv.org/abs/1803.09610  
[6] Pommaret, J.-F.: New Mathematical Methods for Physics, NOVA Science Publisher, New York, 2018.  
[7] Quadrat, A., Robertz, R. (2014) A Constructive Study of the Module Structure of Rings of Partial Differential Operators, Acta Applicandae Mathematicae, 133, 187-234 (2014) 187-234.
GENERAL RELATIVITY ⇒ GRAVITATIONAL WAVES

1. **METRIC**: \( \omega = (\omega_{ij}) = (\omega_{ji}) \in S_2T^*, det(\omega) \neq 0 \) ⇒ \( \Omega \in S_2T^* \)

2. **CHRISTOFFEL SYMBOLS**: \( \gamma^k_{ij} = \frac{1}{2} \omega^{kr}(\partial_i \omega_{rj} + \partial_j \omega_{ir} - \partial_r \omega_{ij}) \) \( \lim \) \( \Gamma \in S_2T^* \otimes T \)

3. **RIEMANN TENSOR**: \( \rho^k_{i,j} = \partial_i \gamma^k_{lj} - \partial_j \gamma^k_{li} + \gamma^r_{li} \gamma^k_{jr} - \gamma^r_{lj} \gamma^k_{ri} \) \( \lim \) \( R \in \Lambda^2 T^* \otimes T^* \otimes T \)

   \[ \omega_k \rho^r_{i,j} = \rho_{kl,i,j} = -\rho_{lk,i,j} = -\rho_{kl,j} = \rho_{ij,k} \Rightarrow \rho^r_{i,j} = 0 \]

   \[ \rho^k_{i,j} + \rho^k_{i,j,l} + \rho^k_{j,i,l} = 0 \Rightarrow \rho^r_{i,j} = \rho^r_{j,i} \]

4. **KILLING OPERATOR**: \( D : T \rightarrow S_2T^* : \xi \rightarrow L(\xi)\omega = \Omega \)

   \( D\xi = 0, D\eta = 0 \Rightarrow D[\xi,\eta] = 0 \) (Lie operator)

5. **KILLING EQUATIONS**: \( (L(\xi)\omega)_{ij} \equiv \Omega_{ij} \equiv \omega_{rj}\partial_i \xi^r + \omega_{ir}\partial_j \xi^r + \xi^r \partial_i \omega_{ij} = 0 \)

   \[ \omega_{rj}v^r_i + \omega_{ir}v^r_j = 0 \) (symbol \( g \in T^* \otimes T \))

6. **KILLING SEQUENCE**:

   \[ 0 \rightarrow \Theta \rightarrow T \xrightarrow{D_1} S_2T^* \xrightarrow{D_1} F_1 \xrightarrow{D_2} F_2 \]

   \[ n \xrightarrow{Killing} \frac{n(n+1)}{2} \xrightarrow{Riemann} \frac{n^2(n^2-1)}{12} \xrightarrow{Bianchi} \frac{n^2(n^2-1)(n-2)}{24} \]

7. **RICCI TENSOR**: \( \rho_{ij} = \rho^r_{i,rj} = \rho^r_{ji} \) \( \lim \) \( (R_{ij}) \in S_2T^* \)

8. **EINSTEIN TENSOR**: \( \epsilon_{ij} = \rho_{ij} - \frac{1}{2} \omega_{ij} \omega^{rs} \rho_{rs} \) \( \lim \) \( (E_{ij}) \in S_2T^* \)

9. **EINSTEIN EQUATIONS**: \( (Einstein) \ div(E) = 0 \Rightarrow E_{ij} \sim \Sigma_{ij} \Rightarrow div(\Sigma) = 0 \) (Cauchy)

10. **WAVE EQUATIONS**: \( \bar{\Omega}_{ij} = \Omega_{ij} - \frac{1}{2} \omega_{ij} \omega^{rs} \Omega_{rs} \Rightarrow \Box \Omega_{ij} + ... \sim \Sigma_{ij} \)

**MATHEMATICALLY CORRECT BUT CONCEPTUALLY WRONG**
DIFFERENTIAL SEQUENCE

\[ \xi \xrightarrow{\mathcal{D}} \eta \xrightarrow{\mathcal{D}_1} \zeta \]

DIRECT PROBLEM \[ \mathcal{D} \text{ given, find } \mathcal{D}_1 : M. \text{ Janet (1920), D.C. Spencer (1970)} \]

\[ \downarrow \]

INVERSE PROBLEM \[ \mathcal{D}_1 \text{ given, find } \mathcal{D} : \text{ not always possible : Wheeler’s challenge (1970)} \]

THEOREM: A classical control system is controllable if and only if it is parametrizable, that is if and only if it generates the compatibility conditions (CC) of a previous operator describing therefore a parametrization.

EXAMPLE: DOUBLE PENDULUM

Rigid bar of length \( L \) moving along the left to right horizontal axis 0x, 0y downwards vertical axis parallel to gravity \( g \), first pendulum made by a mass \( m_1 \), having length \( l_1 \) and moving by an angle \( \theta_1 \) with respect to the vertical, second pendulum made by a mass \( m_2 \), having length \( l_2 \) and moving by an angle \( \theta_2 \) with respect to the vertical.

Control system:

\[
\begin{align*}
\ddot{x} + l_1 \dot{\theta}_1 + g \theta_1 &= 0, \\
\ddot{x} + l_2 \dot{\theta}_2 + g \theta_2 &= 0
\end{align*}
\]

Parametrization:

* \( l_1 \neq l_2 \):

\[
\begin{align*}
-l_1 l_2 d^4 \phi - g (l_1 + l_2) d^2 \phi - g^2 \phi &= x \\
l_2 d^4 \phi + g d^2 \phi &= \theta_1 \\
l_1 d^4 \phi + g d^2 \phi &= \theta_2
\end{align*}
\]

* \( l_1 = l_2 = l, \theta = \theta_1 - \theta_2 \Rightarrow l \ddot{\theta} + g \theta = 0 \)

\[ \theta(0) = 0, \dot{\theta}(0) = 0 \Rightarrow \theta(t) = 0. \]

COMPUTER ALGEBRA ABSOLUTELY NEEDED for \( l_1 = \text{cst}, l_2 = l_2(t) \).

COROLLARY: A classical control system defined over an ordinary differential field \( K \) by equations linearly independent over the ring \( \mathcal{D} = K[d] \) of differential operators with coefficients in \( K \) is controllable if and only if the formal adjoint of the corresponding differential operator is injective, even when \( \mathcal{D} \) is non-commutative.

COUNTEREXAMPLE: EINSTEIN EQUATIONS CANNOT BE PARAMETRIZED
DOUBLE DUALITY TEST

\[ \begin{array}{c}
\xi \xrightarrow{\mathcal{D}} \eta \xrightarrow{\mathcal{D}_1} \zeta \\
\xi' \xrightarrow{\mathcal{D}_1'} \zeta'
\end{array} \]

3 \quad \nu \xleftarrow{\text{ad} (\mathcal{D})} \mu \xleftarrow{\text{ad} (\mathcal{D}_1)} \lambda

\text{ad}(\text{ad}(\mathcal{D})) = \mathcal{D}, \quad \text{ad}(\mathcal{D}) \circ \text{ad}(\mathcal{D}_1) = \text{ad}(\mathcal{D}_1 \circ \mathcal{D}) = 0

\Rightarrow \mathcal{D}_1 \circ \mathcal{D} = 0 \Rightarrow \mathcal{D}_1 \text{ AMONG the CC of } \mathcal{D}

\text{Step 5} \Rightarrow \mathcal{D}_1' \text{ GENERATES the CC of } \mathcal{D} \quad \Rightarrow \quad \mathcal{D}_1 \leq \mathcal{D}_1'

THEOREM: \quad \mathcal{D}_1 \text{ parametrized by } \mathcal{D} \quad \Leftrightarrow \quad \mathcal{D}_1 = \mathcal{D}_1'

COUNTEREXAMPLE: Contrary to the Ricci operator, the Einstein operator is SELF-ADJOINT, the sixth terms being exchanged between themselves under \text{ad}:

\[ \lambda^{ij}(\omega^{rs}d_{ij}\Omega_{rs}) \xleftarrow{\text{ad}} (\omega^{rs}d_{ij}\lambda^{ij})\Omega_{rs} = (\omega_{ij}d_{rs}\lambda^{rs})\Omega^{ij} \]

Riemann 20

4 \quad \text{Killing} \quad 10 \quad \text{Einstein} \quad 10

4 \quad \text{Cauchy} \quad 10 \quad \text{Einstein} \quad 10
VARIATIONAL CALCULUS WITH CONSTRAINTS

MOTIVATION:
Suppose that $D_1$ generates the CC of $D$ AND that $\text{ad}(D)$ generates the CC of $\text{ad}(D_1)$.

$$D\xi = \eta$$

$$\Phi = \int_V \varphi(\eta) dx \Rightarrow \delta \Phi = \int \frac{\partial \varphi}{\partial \eta} \delta \eta dx$$

$$\Rightarrow \text{Cauchy}$$

$$D_1\eta = 0$$

$$\Phi = \int_V (\varphi(\eta) - \lambda D_1 \eta) dx \Rightarrow \delta \Phi = \int \left( \frac{\partial \varphi}{\partial \eta} - \lambda D_1 \delta \eta \right) dx$$

$$\Rightarrow \mu = \frac{\partial \varphi}{\partial \eta} = \text{ad}(D_1)\lambda $$\text{(parametrization by $\lambda$)} \xrightarrow{\text{ad}(D)} \text{ad}(D)\mu = 0$$\text{(elimination of $\lambda$)}$

EXAMPLE: $n = 2$ Airy parametrization (1863)

$$2 \xrightarrow{\text{Killing}} 3 \xrightarrow{\text{Riemann}} 1 \rightarrow 0$$

$$2 \xleftarrow{\text{Cauchy}} 3 \xleftarrow{\text{Airy}} 1$$

KEY RESULT

$$\text{Cauchy} = \text{ad}(\text{Killing}), \quad \text{Airy} = \text{ad}(\text{Riemann})$$

$$\lambda (d_{22}\Omega_{11} - 2d_{12}\Omega_{12} + d_{11}\Omega_{22}) = (d_{22}\lambda \Omega_{11} - 2d_{12}\lambda \Omega_{12} + d_{11}\lambda \Omega_{22}) + ...$$

$$\sigma^{ij}\Omega_{ij} = \sigma^{11}\Omega_{11} + 2\sigma^{12}\Omega_{12} + \sigma^{22}\Omega_{22}$$

Cauchy

$$d_1\sigma^{11} + d_2\sigma^{12} = f^1, \quad d_1\sigma^{21} + d_2\sigma^{22} = f^2$$

Airy

$$\sigma^{11} = d_{22}\lambda, \quad \sigma^{12} = \sigma^{21} = -d_{12}\lambda, \quad \sigma^{22} = d_{11}\lambda$$
SELF-ADJOINT OPERATOR

BELTRAMI PARAMETRIZATION (1892)

\[
\begin{pmatrix}
\sigma^{11} \\
\sigma^{12} \\
\sigma^{12} \\
\sigma^{13} \\
\sigma^{23} \\
\sigma^{33}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & 0 & d_{33} & -2d_{23} & d_{22} \\
0 & -d_{33} & 2d_{33} & 0 & d_{13} & -d_{12} \\
0 & d_{23} & -d_{22} & -d_{13} & d_{12} & 0 \\
d_{33} & 0 & -2d_{13} & 0 & 0 & d_{11} \\
d_{23} & d_{13} & d_{12} & 0 & d_{11} & 0 \\
d_{22} & -2d_{12} & 0 & d_{11} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\phi_{11} \\
\phi_{12} \\
\phi_{13} \\
\phi_{22} \\
\phi_{23} \\
\phi_{33}
\end{pmatrix}
\]

NOT SELF–ADJOINT

\[d_r \sigma^{ir} = 0 \text{ (Cauchy)}\]

PARAMETRIZATION: \(\text{ad(Riemann)} = \text{Beltrami}\)

\[
\begin{pmatrix}
\sigma^{11} \\
2\sigma^{12} \\
2\sigma^{13} \\
\sigma^{22} \\
2\sigma^{23} \\
\sigma^{33}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & 0 & d_{33} & -2d_{23} & d_{22} \\
0 & -2d_{33} & 2d_{23} & 0 & 2d_{13} & -2d_{12} \\
0 & 2d_{23} & -2d_{22} & -2d_{13} & 2d_{12} & 0 \\
d_{33} & 0 & -2d_{13} & 0 & 0 & d_{11} \\
-2d_{23} & d_{13} & d_{12} & 0 & d_{11} & 0 \\
d_{22} & -2d_{12} & 0 & d_{11} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\phi_{11} \\
\phi_{12} \\
\phi_{13} \\
\phi_{22} \\
\phi_{23} \\
\phi_{33}
\end{pmatrix}
\]

SELF–ADJOINT

TEXTBOOKS \(\sigma^{ij} = 2 \frac{\partial \phi^i}{\partial \Omega^j} \Rightarrow \sigma^{ij} = \sigma^{ji}\) WRONG
**FIRST CONTRADICTION**

\[ n=4 \]

\[ \text{ad}(\text{Killing}) = \text{Cauchy}, \quad \text{ad}(\text{Riemann}) = \text{Beltrami} \]

\[
\begin{array}{cccccc}
\text{Killing} & \rightarrow & 10 & \text{Riemann} & \rightarrow & 20 \\
\parallel & \downarrow & & & \downarrow \\
10 & \rightarrow & 10 & \text{div} & \rightarrow & 4 \\
\downarrow & & & \downarrow & & \\
0 & & 0 & & & \\
\end{array}
\]

\[
\begin{array}{cccccc}
\text{Cauchy} & \leftarrow & 10 & \text{Beltrami} & \leftarrow & 20 \\
\parallel & \uparrow & & & \uparrow \\
10 & \leftarrow & 10 & \text{ad}(\text{Bianchi}) & \leftarrow & 20 \\
\uparrow & & & \uparrow & & \\
0 & & 0 & & & \\
\end{array}
\]

**SECOND CONTRADICTION**

\[ E_{ij} = R_{ij} - \frac{1}{2} \omega_{ij} \omega^{rs} R_{rs} \Rightarrow E = C \circ R \]

EINSTEIN OPERATOR IS SELF-ADJOINT  
RICCI OPERATOR IS NOT SELF-ADJOINT

\[ E : \Omega \xrightarrow{C} \bar{\Omega} = \Omega - \frac{1}{2} \omega_{tr}(\Omega) \xrightarrow{X} S_2 T^* \]

*Einstein* operator \( E \) (6 terms) \( \rightarrow \) *Wave* operator \( X \) (4 terms only)  
\[ E = X \circ C \Rightarrow E = \text{ad}(E) = \text{ad}(C) \circ \text{ad}(X) = C \circ \text{ad}(X) = C \circ R \]

\[ \Rightarrow \text{ad}(X) = \text{Ricci} \Rightarrow X = \text{ad}(\text{Ricci}) \]

EINSTEIN OPERATOR IS USELESS  
ONLY RICCI OPERATOR IS USEFUL
POINCARE SEQUENCE IS SELF ADJOINT UP TO SIGN

\[ d : \wedge^0 T^* \rightarrow \wedge^1 T^* \rightarrow \wedge^2 T^* \rightarrow \ldots \rightarrow \wedge^n T^* \rightarrow 0 \]

**EXAMPLE:** \( n=3 \)

\[ \wedge^0 T^* \xrightarrow{\text{grad}} \wedge^1 T^* \xrightarrow{\text{curl}} \wedge^2 T^* \xrightarrow{\text{div}} \wedge^3 T^* \rightarrow 0 \]

\( ad(\text{grad}) = -\text{div}, \; ad(\text{curl}) = \text{curl}, \; ad(\text{div}) = -\text{grad} \)

**EXTENSION MODULES:**

\[ \begin{align*}
\xi & \rightarrow \begin{cases}
d_{22} \xi = \eta^2 \\
d_{12} \xi = \eta^1
\end{cases} \rightarrow d_1 \eta^2 - d_2 \eta^1 = \zeta \\
\xi & \xrightarrow{\mathcal{D}} \eta \xrightarrow{\mathcal{D}_1} \zeta \\
nu & \xleftarrow{ad(D)} \mu \xleftarrow{ad(D_1)} \lambda \\
\end{align*} \]

\[ d_{12} \mu^1 + d_{22} \mu^2 = \nu \leftrightarrow \begin{cases}
-d_1 \lambda = \mu^2 \\
d_2 \lambda = \mu^1
\end{cases} \leftrightarrow \lambda \]

\[ d_1 \mu^1 + d_2 \mu^2 = \nu' \]

**THEOREM:** If \( M \) is the differential module defined by \( \mathcal{D} \), the extension modules \( ext^i(M) \) do not depend on the sequence used for their computation.

**APPLICATION:** The **Spencer sequence** for any Lie operator \( \mathcal{D} \) which is coming from a Lie group of transformations is (locally) isomorphic to the tensor product of the Poincaré sequence by the corresponding finite Lie algebra \( \mathcal{G} \).

**COROLLARY:** \( ext^1(M) = 0, \; ext^2(M) = 0, \ldots \Rightarrow \text{NO GAP} \)

**CONFUSION:** Lanczos has been trying in vain to do for the **Bianchi** operator what he did for the **Riemann** operator, a useless but possible

**SHIFT BY ONE STEP**
Vertical down arrows are $\delta$-maps of Spencer

\[ g_1 \simeq \wedge^2 T^* \subset T^* \otimes T, \quad g_2 = 0 \Rightarrow g_3 = 0 \Rightarrow g_4 = 0 \]

**Snake Chase**  \Rightarrow  \[ F_2 = H^3(g_1) \quad \text{SPENCER COHOMOLOGY} \]

\[
0 \to F_2 \to \wedge^3 T^* \otimes g_1 \xrightarrow{\delta} \wedge^4 T^* \otimes T \to 0
\]

\[
0 \to 20 \to 24 \xrightarrow{\delta} 4 \to 0
\]

\[
\begin{align*}
B_{1,234}^i - B_{2,341}^i + B_{3,412}^i - B_{4,123}^i &= 0 \\
B_{i1,1} - B_{i2,2} + B_{i3,3} - B_{i4,4} &= 0 \\
i = 4 \Rightarrow B_{41,1} - B_{42,2} + B_{43,3} &= 0 \\
L_{23,1} + L_{31,2} + L_{12,3} &= 0
\end{align*}
\]

**Lanczos**  \[ L_{ij,k} + L_{ji,k} = 0, \quad L_{ij,k} + L_{jk,i} + L_{ki,j} = 0 \quad (24-4=20) \]
CLASSICAL / CONFORMAL

CLASSICAL: \[ \mathcal{L}(\xi)\omega = 0 \]

\[
\begin{align*}
n = 2 & \quad 2 \xrightarrow{1} 3 \xrightarrow{2} 1 \xrightarrow{3} 0 \\
n = 3 & \quad 3 \xrightarrow{1} 6 \xrightarrow{2} 6 \xrightarrow{3} 3 \xrightarrow{4} 0 \\
n = 4 & \quad 4 \xrightarrow{1} 10 \xrightarrow{2} 20 \xrightarrow{3} 20 \xrightarrow{4} 6 \xrightarrow{5} 0 
\end{align*}
\]

Euler-Poincaré: \[ 4 - 10 + 20 - 20 + 6 = 0 \]

CONFORMAL: \[ \mathcal{L}(\xi)\omega = A(x)\omega \]

\[ \Leftrightarrow \hat{\omega}_{ij} = \omega_{ij}\left| \det(\omega) \right|^{-\frac{1}{n}}, \quad \mathcal{L}(\xi)\hat{\omega} = 0 \]

\[
\begin{align*}
n = 3 & \quad 3 \xrightarrow{1} 5 \xrightarrow{2} 5 \xrightarrow{3} 3 \xrightarrow{4} 0 \\
n = 4 & \quad 4 \xrightarrow{1} 9 \xrightarrow{2} 10 \xrightarrow{3} 9 \xrightarrow{4} 4 \xrightarrow{5} 0 \\
n = 5 & \quad 5 \xrightarrow{1} 14 \xrightarrow{2} 35 \xrightarrow{3} 35 \xrightarrow{4} 14 \xrightarrow{5} 5 \xrightarrow{6} 0 
\end{align*}
\]

Euler-Poincaré: \[ 5 - 14 + 35 - 35 + 14 - 5 = 0 \]

\[ \textit{COMPUTER ALGEBRA WANTED} \]
VESSIOT STRUCTURE CONSTANTS

LIE PEUDOGroup:
\[ y = f(x) \in \text{aut}(\mathbb{R}^2), \Delta(x) = \text{det}\left(\partial_i f^k(x)\right) \neq 0 \]
\[ \Gamma = \{y^1 = f(x^1), y^2 = x^2/\partial_1 f(x^1)\} \]

SYSTEM: \[ y^2 dy^1 = x^2 dx^1 \Rightarrow dy^1 \wedge dy^2 = dx^1 \wedge dx^2 \Leftrightarrow \Delta = 1 \]

GENERAL OBJECT: \[ \omega = (\alpha, \beta) \in \mathcal{F} = \wedge^1 T^* \times X \wedge^2 T^* \]

LIE OPERATOR: \[ \mathcal{D}\xi = \mathcal{L}(\xi)\omega = 0 \Leftrightarrow \{\mathcal{L}(\xi)\alpha = 0, \mathcal{L}(\xi)\beta = 0\} \]

MEDOLAGHI EQUATIONS:
\[ \{\alpha_r \partial_i \xi^r + \xi^r \partial_i \alpha_i = 0, \beta \partial_r \xi^r + \xi^r \partial_r \beta = 0\} \]

SPECIAL OBJECT: \, \alpha = x^2dx^1, \beta = dx^1 \wedge dx^2 \Rightarrow \omega = (x^2, 0, 1)

VESSIOT STRUCTURE EQUATIONS:
\[ d\alpha = c\beta, \, c = \text{cst} \]

DIFFERENTIAL SEQUENCE:
\[ \xi \xrightarrow{\mathcal{D}} \eta \xrightarrow{\mathcal{D}_1} \zeta \]
\[ \left\{ \begin{array} {c} \xi^1 \\ \xi^2 \end{array} \right\} \xrightarrow{\mathcal{D}} \left\{ \begin{array} {c} \alpha_r \partial_i \xi^r + \xi^r \partial_i \alpha_i = \eta^i \\ \beta \partial_r \xi^r + \xi^r \partial_r \beta = \eta^3 \end{array} \right\} \xrightarrow{\mathcal{D}_1} \partial_1 \eta^2 - \partial_2 \eta^1 - c \eta^3 = \zeta \]

ADJOINT SEQUENCE:
\[ \nu \xleftarrow{\text{ad}(\mathcal{D})} \mu \xleftarrow{\text{ad}(\mathcal{D}_1)} \lambda \]
\[ \lambda \mid \partial_1 \eta^2 - \partial_2 \eta^1 - c \eta^3 \]

\[ \text{ad}(\mathcal{D}_1) \left\{ \begin{array} {c} \eta^1 \rightarrow \partial_2 \lambda = \mu^1 \\ \eta^2 \rightarrow -\partial_1 \lambda = \mu^2 \\ \eta^3 \rightarrow -c \lambda = \mu^3 \end{array} \right\} \]

\[ \text{INJECTIVE} \Leftrightarrow c \neq 0 \]

\[ \text{ad}(\mathcal{D}) \left\{ \begin{array} {c} \xi^1 \rightarrow -\alpha_1 \partial_r \mu^r + \beta (c \mu^2 - \partial_1 \mu^3) = \nu^1 \\ \xi^2 \rightarrow -\alpha_2 \partial_r \mu^r + \beta (-c \mu^1 - \partial_2 \mu^3) = \nu^2 \end{array} \right\} \]

\[ \left\{ \begin{array} {c} c = 0 \Rightarrow \partial_1 \mu^1 + \partial_2 \mu^2 = 0, \mu^3 = 0 \\ c \neq 0 \Rightarrow \partial_1 \mu^1 + \partial_2 \mu^2 = 0, \partial_1 \mu^3 - c \mu^2 = 0, \partial_2 \mu^3 + c \mu^1 = 0 \end{array} \right\} \]