Killing tensors and symmetries

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Abstract
A new method is presented for finding Killing tensors in spacetimes with symmetries. The method is used to find all the Killing tensors of Melvin’s magnetic universe and the Schwarzschild vacuum. We show that they are all trivial. The method requires less computation than solving the full Killing tensor equations directly, and it can be used even when the spacetime is not algebraically special.

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1. Introduction
Killing tensors are useful because, like Killing vectors, they provide conserved quantities for geodesic motion, most famously in the Kerr metric where the Killing tensor gives rise to the Carter constant [1]. However, it is much more difficult to find Killing tensors than Killing vectors. In 4 spacetime dimensions, the equation for a Killing tensor becomes 20 partial differential equations for 10 functions of 4 variables. Most of the known results on Killing tensors come from the fact that the Killing tensor equation simplifies in certain classes of algebraically special spacetimes [2, 3]. However, equations in general relativity also often simplify in the presence of symmetry [4]. We will show that the Killing tensor equation simplifies when a spacetime possesses a hypersurface orthogonal Killing vector, and that this simplification provides an effective method for finding Killing tensors. We will apply this method to Melvin’s magnetic universe [5] and the Schwarzschild vacuum solution. It is shown that all of their Killing tensors are trivial in the sense that they are either the metric or the symmetrized product of Killing vectors. The method is presented in section 2. It is applied to Melvin’s magnetic universe in section 3 and the Schwarzschild vacuum in section 4. Conclusions are given in section 5.

Notation: Lower case Latin indices, \( B^a \), range over \( n \) dimensions. Greek indices, \( B^\mu \), range over \( (n - 1) \) dimensions. For the Killing vector \( \xi^a \) an overdot will denote a Lie derivative, \( \dot{A} := \mathcal{L}_\xi A \).
2. The Killing tensor method

A Killing tensor (of order 2) is a symmetric tensor $X_{ab}$ that satisfies
\[ \nabla (a X_{bc}) = 0. \tag{1} \]

Now suppose that the spacetime has a hypersurface orthogonal Killing vector $\xi^a$. Define $V$ such that
\[ \xi^a \xi_a = \epsilon V^2 \tag{2} \]
where $\epsilon = \pm 1$. Then the metric in directions orthogonal to $\xi^a$ is given by
\[ h_{ab} = g_{ab} - \epsilon V^{-2} \xi_a \xi_b. \tag{3} \]

One can use $h_{a b}$ as a projection operator to project any tensor in directions orthogonal to $\xi^a$.

In particular, the Killing tensor can be decomposed as
\[ X_{ab} = A \xi_a \xi_b + 2 B(a \xi_b) + C_{ab} \tag{4} \]
where $B_a$ and $C_{ab}$ are orthogonal to $\xi^a$. The Killing tensor equation for this decomposition is
\[ \nabla (a A \xi_b \xi_c) + 2 \nabla (a B b \xi_c) + \nabla (a C_{bc}) = 0. \tag{5} \]

Projecting the Killing tensor equation using all combinations of $h_{a b}$ and $\xi a$ yields the following:
\[ D(a C_{bc}) = 0, \tag{6} \]
\[ \mathcal{L}_\xi A = -2 V^{-1} B^a D_a V, \tag{7} \]
\[ \mathcal{L}_\xi B_a = -V^{-1} C_{ab} D^b V - \frac{1}{2} \epsilon V^2 D_a A, \tag{8} \]
\[ \mathcal{L}_\xi C_{ab} = -2 \epsilon V^2 D(a B_b). \tag{9} \]

Here $D_a$ is the derivative operator associated with the metric $h_{ab}$ and $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the Killing vector $\xi^a$. Solving equation (8) for $D_a A$ yields
\[ D_a A = -\left(\frac{2}{\epsilon}\right)[V^2 \mathcal{L}_\xi B_a + V^{-1} C_{ab} D^b V]. \tag{10} \]

The left-hand side of equation (10) is curl-free and thus the curl on the right-hand side must vanish. Hence,\n\[ D_a (V^{-2} \mathcal{L}_\xi B_a) + D_a (V^{-1} C_{b a} D^b V) = 0. \tag{11} \]

Furthermore, since $\mathcal{L}_\xi$ commutes with $D_a$ it follows that $\mathcal{L}_\xi$ on the right-hand side of equation (10) equals $D_a$ on the right-hand side of equation (7), and therefore
\[ \mathcal{L}_\xi C_{ab} + V^{-1} (D^a V) \mathcal{L}_\xi C_{ab} - \epsilon V^2 D_a (V^{-1} B^b D_b V) = 0. \tag{12} \]

Equations (11) and (12) provide the integrability conditions for equations (7) and (10).

These equations are most easily implemented in a coordinate system adapted to the Killing vector. Choose a coordinate system $(y, x^\mu)$ such that $x^\mu$ are coordinates on the surface orthogonal to the Killing vector and $\mathcal{L}_\xi$ is simply a partial derivative with respect to $y$. Use $\partial_y$ or a comma to denote a derivative with respect to the $x^\mu$ coordinates. The Latin indices in this section are $n$-dimensional, and the method below projects objects and equations down to $(n - 1)$ dimensions with Greek indices.

Equation (9) becomes
\[ \dot{C}_{\mu \nu} = -\epsilon V^2 (B^a \partial_a h_{\mu \nu} + h_{a \nu} \partial_\mu B^a + h_{\mu \nu} \partial_a B^a) \tag{13} \]
while integrability conditions (11) and (12) become
\[ \partial_{\mu} (V^{-2} B_{\nu j}) + \partial_{\nu} (V^{-3} C_{\nu j} \partial^a V) = 0 \tag{14} \]
\[ B_\mu + V^{-1} C_{\mu\nu} \partial^\nu V - \epsilon V^2 \partial_\mu (V^{-1} B^\nu \partial_\nu V) = 0. \]  

Equations (7) and (10) for \( A \) are written as
\[ \dot{A} = -2 V^{-1} B_\mu \partial_\mu V \]  
\[ \partial_\mu A = -(2/\epsilon)(V^{-2} \dot{B}_\mu + V^{-3} C_{\mu\nu} \partial_\nu V). \]

Note that equations (13)–(17) can be evaluated without ever having to calculate a Christoffel symbol.

The complete set of Killing tensors can be found as follows: first solve equation (6) for the most general \( C_{\mu\nu} \). Note that this general solution will contain arbitrary 'constants' that are really functions of the Killing coordinate. Now using the general \( C_{\mu\nu} \), find the most general \( B_\mu \) satisfying equation (13). Then restrict this general solution by demanding that equations (14) and (15) also be satisfied. Finally, solve equations (16) and (17) for \( A \).

3. Killing tensors of the Melvin metric

Melvin’s magnetic universe is a static, cylindrically symmetric, Petrov-type D solution of the Einstein–Maxwell equations. Its metric is usually written as
\[ d\tilde{s}_\text{Mel}^2 = a^2 (-dt^2 + dr^2 + dz^2) + (\rho^2/a^2) d\phi^2 \]  

where the function \( a \) is
\[ a = 1 + \frac{1}{4} B_0^2 \rho^2. \]

The constant \( B_0 \) is the value of the magnetic field on the \( \rho = 0 \) axis. Define coordinates \( t = (B_0/2)\tilde{t}, r = (B_0/2)\rho, z = (B_0/2)\tilde{z}, \phi = \tilde{\phi} \). In terms of these coordinates we have
\[ a = 1 + r^2 \]
while the metric becomes
\[ d\tilde{s}_\text{Mel}^2 = (4/B_0^2) dx_\text{Mel}^2 \]  

with
\[ dx^2_\text{Mel} = a^2 (-dt^2 + dr^2 + dz^2) + (r^2/a^2) d\phi^2. \]

Since \( d\tilde{s}_\text{Mel}^2 \) and \( dx_\text{Mel}^2 \) differ only by an overall constant scale, they have the same Killing vectors and Killing tensors. For simplicity, we will work with metric \( dx_\text{Mel}^2 \). Since the metric components are independent of \( t, \phi \) and \( z \), it follows that there are Killing vectors for each of these coordinate directions. They are denoted by \( (t, \phi, z) \rightarrow (\tau^\alpha, \eta^\rho, \lambda^\eta) \) respectively. Each of these Killing vectors is hypersurface orthogonal. In addition, the metric has boost symmetry in the \( tz \) plane, with the corresponding Killing field \( t \lambda^\alpha + z \tau^\alpha \).

We will use the method of the previous section to work out the Killing tensors of Melvin’s magnetic universe. First we will find the Killing tensors of the two-dimensional \( rz \) surface, then use these to find the Killing tensors of the three-dimensional \( rz\phi \) surface and finally find the Killing tensors of the four-dimensional Melvin metric.

We will use \( c_1, c_2 \) etc to denote constants, and \( k_1, k_2 \) etc to denote quantities that depend only on the coordinate associated with the Killing vector.

3.1. An \( rz \) surface

The \( rz \) 2-surface has a metric of the form
\[ g_{ab} dx^a dx^b = a^2 (dr^2 + dz^2). \]
We have a z coordinate Killing vector \( \lambda^\mu \) for which \( V = a \) and \( \epsilon = 1 \), and a one-dimensional metric \( h_{\mu \nu} = a^2 r_{\mu} r_{\nu} \) orthogonal to the Killing vector. Since the \( r \)-direction is a one-dimensional line, it follows that the Killing tensor must take the form \( C_{\mu \nu} = F h_{\mu \nu} \) for some scalar \( F \). It then follows from equation (6) that \( F \) is independent of \( r \). We therefore have
\[
C_{\mu \nu} = k_1 h_{\mu \nu} = k_1 a^2 r_{\mu} r_{\nu},
\]
for some \( k_1(z) \). Equation (13) then becomes, with \( B^a \rightarrow B^r \) only,
\[
C_{rr} = k_1 a^2 = - a^2 \left( \frac{da^2}{dr} B^r + 2a^2 \partial_r B^r \right)
\]
or
\[
\dot{k}_1 = - 2a^2 \left( \frac{dB^r}{dr} + \frac{1}{a} \frac{da}{dr} B^r \right).
\]
The general solution for \( B^r \) is
\[
B^r = a^{-1} \left( k_2 - \frac{k_1}{2} \arctan r \right)
\]
Equation (14) is automatically satisfied, while equation (15) becomes
\[
0 = \left( - \frac{k_1}{2} k_2 - 2k_1 \right) a^2 \arctan r + 3k_1 a \arctan r + (k_2 + 4k_1) a^2 + 3k_1 r - 6k_2 a.
\]
Here we have grouped terms so that each term is a coefficient independent of \( r \) multiplied by a function of \( r \) and the functions of \( r \) are linearly independent. Thus, each coefficient must vanish, which yields \( k_2 = 0 \) and \( k_1 = 0 \), from which it follows that \( k_1 = c_1 \) and \( B^r = 0 \). It then follows from equation (7) that \( A \) is independent of \( z \). From equation (17) it follows that
\[
A(r) = c_2 + c_1 a^{-2}.
\]
Using equation (4) we find that the general Killing tensor of the \( rz \) surface is
\[
X_{ab} = c_1 g_{ab} + c_2 \lambda_a \lambda_b.
\]
3.2. An \( rz\phi \) surface
We now consider an \( rz\phi \) surface with metric
\[
g_{ab} dx^a dx^b = a^2 (dr^2 + dz^2) + (r^2 / a^2) d\phi^2.
\]
The Killing field is \( \eta^a \), with \( \epsilon = 1 \) and \( V = r/a \). Here \( h_{\mu \nu} \) is the \( g_{ab} \) of the previous subsection, while \( C_{\mu \nu} \) is the \( X_{ab} \) of the previous subsection. We have
\[
h_{\mu \nu} = a^2 (r_{\mu} r_{\nu} + z_{\mu} z_{\nu})
\]
\[
C_{\mu \nu} = k_1 h_{\mu \nu} + k_2 \lambda_\mu \lambda_\nu.
\]
With \( B^a \rightarrow (B^r, B^r) \), the \( rr \), \( zz \) and \( rz \) components of equation (13) are then, respectively,
\[
k_1 a^2 = - \frac{r^2}{a^2} \left( \frac{da}{dr} B^r + a^2 \partial_r B^r \right)
\]
\[
\dot{k}_1 a^2 + k_2 a^4 = - \frac{r^2}{a^2} \left( \frac{da}{dr} B^r + a^2 \partial_r B^r \right)
\]
\[
0 = \partial_r B^r + \partial_\phi B^r.
\]
From equation (33) we find that \(B'\) must take the form
\[
B' = \frac{1}{a} \left[ F(z, \phi) - \frac{k_1}{2} \left( -\frac{1}{r} + 3r + r^3 + \frac{r^5}{5} \right) \right]
\] (36)
for some function \(F(z, \phi)\). Then using equation (36) and integrating equation (35) we find
\[
B^z = G(z, \phi) - \left( \frac{\partial F}{\partial z} \right) \arctan \frac{r}{a^2}
\] (37)
for some function \(G(z, \phi)\). Finally, substituting the expressions in equations (36) and (37) into equation (34) we find
\[
-\frac{1}{2r^2}(k_1a^2 + k_2a^4) = \frac{\partial G}{\partial z} - \left( \frac{\partial^2 F}{\partial z^2} \right) \arctan \frac{r}{a^2} - \frac{k_1}{a^2} \left( -1 + 3r^2 + r^4 + \frac{r^6}{5} \right).
\] (38)

The only odd functions of \(r\) in this equation are \(\arctan \frac{r}{a^2}\) and \(\frac{r}{a^2}\) and these functions are linearly independent, so the coefficient of each must vanish. This implies that \(F = 0\). Furthermore, in order that the left-hand side not diverge as \(r \to 0\) we must have \(k_2 = -k_1\). Equation (38) then becomes
\[
\frac{\partial G}{\partial z} = k_1 \left[ \frac{a^2(1 + a)}{2} + \frac{1}{a^2} \left( -1 + 3r^2 + r^4 + \frac{r^6}{5} \right) \right].
\] (39)

It then follows that both \(k_1\) and \(\frac{\partial G}{\partial z}\) must vanish. Thus, we have found that \(C_{\mu\nu}\) and \(B^\mu\) take the form
\[
C_{\mu\nu} = c_1 h_{\mu\nu} + c_2 \lambda_{\mu} \lambda_{\nu}
\] (40)
\[
B^\mu = k_4(\phi) \lambda^\mu
\] (41)
for some function \(k_4(\phi)\). We now impose the integrability condition (14) which forces \(k_4\) to vanish. This implies \(k_4 = c_4\) and thus \(B^\mu = c_4 \lambda^\mu\). It then follows from equation (16) that \(A\) is independent of \(\phi\). Equation (17) then becomes
\[
\partial_\mu A = c_1 \partial_\mu (r/a)^{-2}
\] (42)
for which the solution is
\[
A(r) = c_3 + c_1 \frac{a^2}{r^2}.
\] (43)
Thus, the general Killing tensor of the \(r z \phi\) surface is
\[
X_{ab} = c_1 g_{ab} + c_2 \lambda_a \lambda_b + c_3 \eta_a \eta_b + 2c_4 \lambda_{(a} \eta_{b)}.
\] (44)

### 3.3. The Melvin metric

We are now ready to treat the full Melvin metric by adding the \(r^a\) Killing field to the metric of the previous subsection. We have
\[
h_{\mu\nu} = a^2(r_{\mu\nu} + c_{\mu\nu} z_{\nu}) + (r^2/a^2) \phi_{\nu} \phi_{\mu}
\] (45)
\[
C_{\mu\nu} = k_1 h_{\mu\nu} + k_2 \lambda_{\mu} \lambda_{\nu} + k_3 \eta_{\mu} \eta_{\nu} + 2k_4 \lambda_{(a} \eta_{b)}.
\] (46)
The $\tau^a$ Killing vector has $\epsilon = -1$ and $V = a$. Equation (13) for $C_{\mu\nu}$ becomes the following:

$$k_1 = 2a^2 \partial_r B^r + 2a \frac{da}{dr} B^r$$

(47)

$$k_1 + k_2 a^2 = 2a \frac{da}{dr} B^r + 2a^2 \partial_r B^r$$

(48)

$$a^{-2} k_1 + \frac{r^2}{a^2} k_3 = 2\partial_a B^\phi + 2 \left( \frac{1}{r} - \frac{1}{a} \frac{da}{dr} \right) B^r$$

(49)

$$0 = \partial_r B^z + \partial_r B^r$$

(50)

$$0 = a^2 \partial_a B^r + \frac{r^2}{a^2} \partial_a B^\phi$$

(51)

$$k_4 = \frac{a^4}{r^2} \partial_a B^z + \partial_z B^\phi.$$  

(52)

Solving equation (47) we find

$$B^r = \frac{1}{a} \left[ k_1 \frac{r}{2} \arctan r + F(z, \phi, t) \right]$$

(53)

for some function $F(z, \phi, t)$. Now, using this result in equation (50) we find

$$B^z = -\frac{\partial F}{\partial z} \arctan r + G(z, \phi, t)$$

(54)

for some function $G(z, \phi, t)$. Using equations (53) and (54) in equation (48), we obtain

$$0 = -k_1 + \left( 2 \frac{\partial G}{\partial z} - k_2 \right) a^2 + 4Fr + 2k_1 r \arctan r - 2 \left( \frac{\partial^2 F}{\partial z^2} \right) a^2 \arctan r.$$  

(55)

Here we have grouped terms so that each term consists of a function of $r$ multiplied by a coefficient that is independent of $r$. Since the functions of $r$ are linearly independent, it follows that each coefficient vanishes. We then find that $F = 0$, that $k_1 = c_1$ and that $G = k_2 z/2 + h(\phi, t)$ for some function $h(\phi, t)$. That is, $B^r$ vanishes, and $B^z$ takes the form

$$B^z = \frac{1}{2} k_2 z + h(\phi, t).$$  

(56)

Equations (49), (51) and (52) then become

$$\left( \frac{r^2}{a^2} \right) k_3 = 2\partial_a B^\phi$$

(57)

$$0 = \partial_r B^\phi$$

(58)

$$k_4 = \left( \frac{a^4}{r^2} \right) \frac{\partial h}{\partial \phi} + \partial_z B^\phi.$$  

(59)

Differentiating equations (57) and (59) with respect to $r$, and using equation (58), it follows that $k_3 = 0$ and $\partial h/\partial \phi = 0$. Thus, we have $k_3 = c_3$ for the constant $c_3$, $h = k_5$ for some function $k_5(t)$ and $B^\phi = k_4 z + k_6$ for function $k_6(t)$. $B^\mu$ takes the form

$$B^\mu = \left( \frac{1}{2} k_2 z + k_5 \right) \lambda^\mu + (k_4 z + k_6) \eta^\mu.$$  

(60)
We therefore have
\[ a^{-2} \dot{B}_\mu = \left( \frac{1}{2} \dot{k}_2 z + \dot{k}_5 \right) \partial_\mu z + \frac{r^2}{a^2} (\ddot{k}_4 z + \dot{k}_6) \partial_\mu \phi. \] (61)

Using the expression in equation (61), we find that equation (14) becomes
\[ 0 = \dot{k}_4 z + \dot{k}_6. \] (62)

From equation (62) we find that \( k_6 = c_6 \) and \( k_4 = c_4 t + c_7 \) for the constants \( c_4, c_6 \) and \( c_7 \).

Using the expression for \( B' \) of equation (60) in equation (15), we find
\[ 0 = a^2 \left( \frac{1}{2} \ddot{k}_2 z + \dot{k}_5 \right) \partial_\mu z + a^{-1} \left( \frac{d}{dr} \right) \dot{k}_1 \partial_\mu r \] (63)
from which it follows that \( \dot{k}_1, \dot{k}_5 \) and \( \ddot{k}_2 \) all vanish. Thus, we have \( k_1 = c_1, k_5 = c_5 t + c_8 \) and \( k_2 = c_2 t^2 + c_9 t + c_{10} \) for the constants \( c_1, c_2, c_5, c_8, c_9 \) and \( c_{10} \).

To summarize, we have found that the general solution for \( C_{\mu \nu} \) and \( B_\mu \) takes the form
\[ C_{\mu \nu} = c_1 \eta_{\mu \nu} + (c_2 t^2 + c_9 t + c_{10}) \lambda_\mu \lambda_\nu + c_3 \eta_\mu \eta_\nu + (c_4 t + c_7) 2 \lambda_\mu \eta_\nu \] (64)
\[ B_\mu = \left[ (c_2 t + c_3) z + (c_5 t + c_8) \right] \lambda_\mu + (c_6 z + c_9) \eta_\mu. \] (65)

It remains to find \( A \). Since \( B' = 0 \), it follows from equation (16) that \( \dot{A} = 0 \). Using the expressions of equations (64) and (65) in equation (17) for \( A \), we find
\[ \partial_\mu A = 2(c_2 z + c_3) \partial_\mu z - c_1 \partial_\mu (a^{-2}). \] (66)

The general solution of equation (66) is
\[ A = c_2 z^2 + 2 c_3 z - c_1 a^{-2} + c_{11}. \] (67)

Finally, using the expressions of equations (64)–(67) in equation (4) we find that the most general Killing tensor of Melvin’s magnetic universe takes the four-dimensional form
\[ X_{ab} = c_1 \delta_{ab} + c_2 [z^2 \tau_a \tau_b + z t \lambda_\alpha \tau_b + i^2 \lambda_a \lambda_b] + c_3 \eta_a \eta_b \]
\[ + c_4 [2 \tau_a \eta_b + 2 \tau_\eta_\alpha \lambda_\beta] + c_5 [2 \tau_a \tau_b + 2 \lambda_\alpha \tau_b] + c_6 2 \eta_\alpha \tau_b \]
\[ + c_7 \lambda_\alpha \eta_b + c_8 2 \lambda_\alpha \tau_b + c_9 [z \lambda_\alpha \tau_b + i \lambda_\alpha \lambda_b] + c_{10} \lambda_\alpha \lambda_b + c_{11} \tau_a \tau_b. \] (68)

This expression can be simplified by noting that the Melvin boost Killing vector is given by \( \psi^\alpha = i \lambda^\alpha + z \tau^\alpha \). The Killing tensor is then
\[ X_{ab} = c_1 \delta_{ab} + c_2 \psi_a \psi_b + c_3 \eta_a \eta_b + 2 c_4 \eta_\alpha \psi_b + 2 c_5 \psi_\alpha \tau_b + 2 c_6 \eta_\alpha \tau_b \]
\[ + 2 c_7 \lambda_\alpha \eta_b + c_8 2 \lambda_\alpha \tau_b + c_9 \psi_\alpha \lambda_b + c_{10} \lambda_\alpha \lambda_b + c_{11} \tau_a \tau_b. \] (69)

Now each term in the sum is either the metric or the symmetrized product of two Killing vectors. Thus, all the Killing tensors of Melvin’s magnetic universe are trivial.

4. Killing tensors of the Schwarzschild metric

The Schwarzschild vacuum solution is given by
\[ ds^2_{Sch} = -F dt^2 + F^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \] (70)
where the function $F$ is $F = 1 - (2m/r)$. The metric admits four Killing vectors $(\tau^a, \alpha^a, \beta^a, \gamma^a)$; a timelike Killing vector $\tau^a \partial_a = \partial_t$ and three spacelike vectors which comprise the SO(3) rotations

$$
\begin{align*}
\alpha^a \partial_a &= \sin \varphi \partial_\vartheta + \cot \vartheta \cos \varphi \partial_\varphi \\
\beta^a \partial_a &= -\cos \varphi \partial_\vartheta + \cot \vartheta \sin \varphi \partial_\varphi \\
\gamma^a \partial_a &= \partial_\varphi \\
2r^2 &= \alpha^a \alpha_a + \beta^a \beta_a + \gamma^a \gamma_a.
\end{align*}
$$

Each of the four Killing vectors is hypersurface orthogonal.

First we will find the Killing tensors of the two-dimensional $r\vartheta$ surface. As before, we will use $c_1, c_2, \ldots$ to denote constants, and $k_1, k_2, \ldots$ to denote quantities that depend only on the coordinate associated with the Killing vector.

4.1. An $r\vartheta$ surface

The $r\vartheta$ 2-surface has metric

$$g_{ab} dx^a dx^b = F^{-1} dr^2 + r^2 d\vartheta^2. \quad (71)$$

We have the $\vartheta$ coordinate Killing vector $\partial_\vartheta = \vartheta^a \partial_a$ for which $\epsilon = 1$ and $V = r$ (note that $\vartheta^a$ is a symmetry of the 2-surface but not of the spacetime). The metric on the space orthogonal to the Killing vector is $h_{\mu\nu} = F^{-1} r^\mu r^\nu$. Since the $r$-direction is a one-dimensional line, it follows as before that the Killing tensor on that one-dimensional space takes the form

$$C_{\mu\nu} = k_1 h_{\mu\nu} = k_1 F^{-1} r^\mu r^\nu. \quad (72)$$

for some $k_1(\vartheta)$. Equation (13) then becomes, with $B^a \to B'$ only,

$$\dot{C}_{rr} = k_1 F^{-1} = -r^2 \left[ -\left( \frac{2m}{r^2} \right) F^{-2} B' + 2 F^{-1} \frac{dB'}{dr} \right]. \quad (73)$$

or

$$k_1 = 2m F^{-1} B' - 2r^2 \frac{dB'}{dr}. \quad (74)$$

The general solution for $B'$ is

$$B' = -\left( \frac{k_1}{2m} \right) F + k_2 F^{1/2}. \quad (75)$$

Equation (14) is automatically satisfied, while equation (15) becomes

$$\left( \frac{k_2}{2m} \right)^{1/2} F^{-1/2} + \left( \frac{k_1}{2m} \right) \left( \frac{\dot{k_1}}{k_1} - \frac{k_1}{2m} \right) \frac{1}{2} \frac{k_1}{2m} F^{1/2} = \frac{3}{2m} k_1 F. \quad (76)$$

Here we have grouped terms so that each term is a coefficient independent of $r$ multiplied by a function of $r$, and the functions of $r$ are linearly independent. Thus, each coefficient must vanish, which yields $k_2 = 0$ and $k_1 = 0$, from which it follows that $k_1 = c_1$ and $B' = 0$. It then follows from equation (7) that $A$ is independent of $\vartheta$. From equation (17) integration provides

$$A(r) = \frac{c_1}{r^2} + c_2. \quad (77)$$

Using equation (4) we find that the general Killing tensor of the $r\vartheta$ surface is

$$X_{ab} = c_1 g_{ab} + c_2 \partial_a \vartheta_b. \quad (78)$$
4.2. A trϑ surface

We now consider a trϑ surface with metric
\[ g_{ab} dx^a dx^b = -F dt^2 + F^{-1} dr^2 + r^2 dϑ^2. \]  
(79)

The Killing field is \( τ^a \partial_a = \partial_t \), with \( ϵ = -1 \) and \( V = F^{1/2} \). Here \( h_{μν} \) is the \( g_{ab} \) of the previous subsection, while \( C_{μν} \) is the \( X_{ab} \) of the previous subsection. We have
\[ h_{μν} = F^{-1} r_{μ} r_{ν} + r^2 \partial_{μ} \partial_{ν}, \]  
(80)
\[ C_{μν} = k_1 h_{μν} + k_2 ϑ_{μ} ϑ_{ν}. \]  
(81)

With \( B^μ \to (B^r, B^ϑ) \), the \( rr \), \( ϑϑ \) and \( rϑ \) components of equation (13) are then, respectively,
\[ \dot{k}_1 F^{-1} = 2 \partial_r B^r - \frac{2m}{r} F^{-1} B^r \]  
(82)
\[ \dot{k}_1 r^2 + \dot{k}_2 r^4 = 2r^2 F \partial_ϑ B^ϑ + 2r F B^r \]  
(83)
\[ 0 = r^2 \partial_ϑ B^ϑ + F^{-1} \partial_r B^r. \]  
(84)

From equation (82) we find that \( B^r \) must take the form
\[ B^r = F^{1/2} H(t, ϑ) + \frac{1}{2} k_1 \left[ (r - 6m) + 3m F^{1/2} \ln \left( \frac{r}{m} (1 + F^{1/2}) - 1 \right) \right] \]  
(85)
for some function \( H(t, ϑ) \). Using the expression of equation (85) in equation (84) we find
\[ \partial_r B^ϑ = -r^{-2} F^{-1/2} \partial_ϑ H. \]  
(86)

Upon integration, it follows that \( B^ϑ \) is
\[ B^ϑ = Q(t, ϑ) - \frac{1}{m} F^{1/2} \partial_ϑ H \]  
(87)
for the integration function \( Q(t, ϑ) \). Substituting the expressions for \( B^r \) and \( B^ϑ \) from equations (85) and (87) into equation (83) provides
\[ \dot{k}_1 + \dot{k}_2 r^2 = \frac{2F}{r} \left[ F^{1/2} H(t, ϑ) + \frac{1}{2} k_1 \left[ (r - 6m) + 3m F^{1/2} \ln \left( \frac{r}{m} (1 + F^{1/2}) - 1 \right) \right] \right] \]
\[ + 2F \left[ \partial_ϑ Q - \frac{1}{m} F^{1/2} \frac{∂^2 H}{∂ϑ^2} \right]. \]  
(88)

Note that each term in equation (88) is a function of \( r \) multiplied by a coefficient that is independent of \( r \). Since the function of \( r \) that has the logarithmic term in equation (88) is linearly independent of the other functions of \( r \), its coefficient must vanish. This implies that \( k_1 = 0 \) and that \( k_1 = c_1 \) for some constant \( c_1 \). Equation (88) then simplifies to
\[ 0 = -\frac{1}{2} k_2 \left( \frac{r^2}{F} \right) + \dot{H} \left( \frac{F^{1/2}}{r} \right) + \partial_ϑ Q - \frac{1}{m} \frac{∂^2 H}{∂ϑ^2} \left( F^{1/2} \right). \]  
(89)

Here terms are grouped so that each term is a coefficient independent of \( r \) multiplied by a function of \( r \), and so that the functions of \( r \) are linearly independent. It therefore follows that each of the coefficients vanishes. We have \( k_2 = 0, H = 0 \) and \( \partial_ϑ Q = 0 \). Therefore, \( k_2 = c_2 \) for some constant \( c_2 \), and the components of the vector field \( B^μ \) are
\[ B^r = 0, \quad B^ϑ = k_3(t) \]  
(90)
for some function \( k_3(t) \). Equivalently
\[ B_μ = k_3 r^2 \partial_μ ϑ. \]  
(91)
Since \( k_1 = c_1 \) and \( k_2 = c_2 \), it follows from equation (81) that the tensor \( C_{\mu \nu} \) takes the form

\[
C_{\mu \nu} = c_1 h_{\mu \nu} + c_2 r^4 \bar{\theta}_{\mu} \bar{\theta}_{\nu}.
\]

Upon using equations (91) and (92) in equation (14) we find that

\[
\partial_{[\mu}(F^{-1} r^2 k_3 \partial_{\nu]} \bar{\theta}) = 0
\]

from which it follows that \( k_3 = 0 \) and therefore that \( k_3 = c_3 \) for some constant \( c_3 \). Thus, from equation (91) we have

\[
B_{\mu} = c_3 r^2 \partial_{\mu} \bar{\theta}.
\]

Equation (15) is identically satisfied by equations (92) and (94).

Since \( B' = 0 \) it follows from equation (16) that \( A' = 0 \). Using equations (92) and (94) in equation (17) we obtain

\[
\partial_{\mu} A = -c_1 \partial_{\mu} (F^{-1}).
\]

It then follows that

\[
A = c_4 - c_1 F^{-1}
\]

for some constant \( c_4 \). Finally, using the results of equations (92), (94) and (96) in equation (4), we find that the general Killing tensor of a \( tr\theta \) surface is

\[
X_{ab} = c_1 g_{ab} + c_2 \bar{\theta}_a \bar{\theta}_b + 2 c_3 \bar{\theta}_a \tau_b + c_4 \tau_a \tau_b.
\]

4.3. The Schwarzschild metric

We are now ready to treat the full Schwarzschild metric by adding the axial \( \gamma^a \) Killing vector to the metric of the previous subsection. We have

\[
h_{\mu \nu} = -F_{t,\mu} t_{,\nu} + F^{-1} r_{,\mu} r_{,\nu} + r^2 \bar{\theta}_{,\mu} \bar{\theta}_{,\nu}
\]

\[
C_{\mu \nu} = k_1 h_{\mu \nu} + k_2 \bar{\theta}_{,\mu} \bar{\theta}_{,\nu} + 2 k_3 \bar{\theta}_{(\mu} r_{,\nu)} + k_4 r_{,\mu} r_{,\nu}.
\]

The \( \gamma^a \) Killing vector has \( \epsilon = 1 \) and \( V = r \sin \theta \). Equation (13) for \( \dot{C}_{\mu \nu} \) then becomes

\[
-\dot{k}_1 F + \dot{k}_4 F^2 = -r^2 \sin^2 \theta (B' \partial_{\theta} F' + 2 F \partial_{\phi} B')
\]

(100)

\[
k_1 F^{-1} = r^2 \sin^2 \theta (F^{-2} B' \partial_{\theta} F' - 2 F^{-1} \partial_{\phi} B')
\]

(101)

\[
\dot{k}_1 r^2 + \dot{k}_2 r^4 = -2 r^3 \sin^2 \theta (B' + r \partial_{\theta} B^\theta)
\]

(102)

\[
0 = F^{-1} \partial_{\phi} B' - F \partial_{\phi} B'
\]

(103)

\[
-\dot{k}_3 r^2 F = r^2 \sin^2 \theta (F \partial_{\theta} B' - r^2 \partial_{\phi} B^\theta)
\]

(104)

\[
0 = r^2 \partial_{\phi} B^\theta + F^{-1} \partial_{\phi} B'.
\]

(105)

Equation (101) can be rewritten as

\[
\partial_{\phi} (F^{-1/2} B') = -\frac{\dot{k}_3}{2 F^{1/2} r^2 \sin^2 \theta}
\]

(106)

with integral

\[
B' = G(t, \bar{\theta}, \phi) F^{1/2} - \frac{k_1}{2 m \sin^2 \theta} F
\]

(107)
for the integration function $G(t, \vartheta, \varphi)$. Using the result of equation (107) in equation (105), we find

$$\partial_r B^\vartheta = -\frac{1}{r^2 F^{1/2}} \frac{k_1 \cos \vartheta}{m r^2 \sin^3 \vartheta}. \quad (108)$$

Integration provides

$$B^\vartheta = H(t, \vartheta, \varphi) - \left(\frac{\partial_{\vartheta} G}{m}\right) \frac{1}{m} \left[ F^{1/2} + \frac{k_1}{m} \frac{\cos \vartheta}{\sin^3 \vartheta}\right] \quad (109)$$

with the integration function $H(t, \vartheta, \varphi)$. Now using equations (107) and (109) in equation (102) we obtain

$$0 = -3\dot{k}_1 - (\dot{k}_2 + \partial_{\vartheta} H) 2\sin^2 \vartheta r^2 + \frac{k_1}{m} \left( \frac{6}{\sin^2 \vartheta} - 3 \right) r - 2\sin^2 \vartheta G(r F^{1/2})$$

$$+ \frac{2}{m} \sin^2 \vartheta \left( \frac{\partial^2 G}{\partial \vartheta^2} \right) r^2 F^{1/2}. \quad (110)$$

Here we have grouped terms so that each term is a coefficient independent of $r$ multiplied by a function of $r$, and so that the functions of $r$ are linearly independent. It therefore follows that each coefficient vanishes. Thus, $G = k_1 = 0$ and

$$\partial_{\vartheta} H = -\frac{\dot{k}_2}{2\sin^2 \vartheta}. \quad (111)$$

From the vanishing of $G$ and $k_1$ it follows that $B^\vartheta$ vanishes, that $k_1 = c_1$ for some constant $c_1$ and that $B^\vartheta = H$. From equation (111) it follows that $B^\vartheta$ takes the form

$$B^\vartheta = I(t, \varphi) + \frac{k_2}{2} \cot \vartheta \quad (112)$$

for some function $I(t, \varphi)$. Using equation (112), along with $B^r = 0$ and $k_1 = 0$, reduces equations (100)–(105) to the following:

$$\partial_r B^t = \frac{k_4 F}{2r^2 \sin^2 \vartheta} \quad (113)$$

$$\partial_r B^r = 0 \quad (114)$$

$$\partial_{\vartheta} B^r = r^2 F^{-1} \partial_{\vartheta} I - \frac{k_3}{\sin^2 \vartheta}. \quad (115)$$

Applying $\partial_r$ to equation (113) and using equation (114) yields $k_4 = 0$. Therefore, $B^t$ is independent of $r$ and $k_4 = c_2$ for some constant $c_2$. Now applying $\partial_r$ to equation (115) and using equation (114) it follows that $\partial_r I = 0$ and $I = k_5$ for some function $k_5(\varphi)$. Thus, $B^\vartheta$ becomes

$$B^\vartheta = k_5 + \frac{k_2}{2} \cot \vartheta. \quad (116)$$

Integrating equation (115) yields

$$B^t = k_6 + k_3 \cot \vartheta \quad (117)$$

for some function $k_6(\varphi)$. In summary, we have found that $B_\mu$ and $C_{\mu\nu}$ take the form

$$B_\mu = (k_6 + k_3 \cot \vartheta) \tau_\mu + \left( k_5 + \frac{k_2}{2} \cot \vartheta \right) \vartheta_\mu \quad (118)$$

$$C_{\mu\nu} = c_1 h_{\mu\nu} + k_2 \vartheta_\mu \vartheta_\nu + 2k_3 \partial_{(\mu} \tau_{\nu)} + c_2 \tau_\mu \tau_\nu. \quad (119)$$
We now impose the integrability conditions of equations (14) and (15) on the expressions above for $B_\mu$ and $C_{\mu\nu}$. From equations (118) and (119), with $\epsilon = 1$ and $V = r \sin \vartheta$, we find

\[
V^{-3}B_\mu + V^{-3}C_{\mu\nu} \partial^{\nu}V = c_1 V^{-3} \partial_\mu V - \frac{F}{r^2 \sin^2 \vartheta} [k_6 + (k_3 + k_5) \cot \vartheta] \partial_\mu t \\
+ \frac{1}{\sin^2 \vartheta} \left[ \dot{k}_5 + \left( \frac{\dot{k}_2}{2} + k_2 \right) \cot \vartheta \right] \partial_\mu \vartheta.
\]  

(120)

The integrability condition given in equation (14) is the statement that the right-hand side of equation (120) is curl-free. From this it follows that, for the term in equation (120) multiplying $\partial_\mu t$, the quantity in square brackets vanishes. That is, we have $\dot{k}_6 = 0$ and $\dot{k}_3 + k_5 = 0$. Thus,

\[
k_6 = c_3
\]  

(121)

\[
k_3 = c_4 \cos \varphi + c_5 \sin \varphi
\]  

(122)

for the constants $c_3$, $c_4$ and $c_5$.

From equations (118), (120)–(122) it follows that

\[
\dot{B}_\mu + V^{-1} \dot{C}_{\mu\nu} \partial^{\nu}V - \epsilon V^{-2} \partial_\mu (V^{-1} B^\nu \partial_\nu V) = \left[ \dot{k}_5 + k_5 + \left( \frac{\dot{k}_2}{2} + 2k_2 \right) \cot \vartheta \right] r^2 \partial_\mu \vartheta.
\]  

(123)

The integrability condition of equation (15) states that the right-hand side of equation (123) vanishes. Therefore, $\ddot{k}_5 + k_5 = 0$ and $\ddot{k}_2 + 4\dot{k}_2 = 0$, and thus

\[
k_5 = c_6 \cos \varphi + c_7 \sin \varphi
\]  

(124)

\[
k_2 = c_8 \cos 2\varphi + c_9 \sin 2\varphi + c_{10}
\]  

(125)

for the constants $c_6$, $c_7$, $c_8$, $c_9$ and $c_{10}$. We find that the general solution for $B_\mu$ and $C_{\mu\nu}$ is

\[
B_\mu = [c_3 + (-c_4 \sin \varphi + c_5 \cos \varphi) \cot \vartheta] \tau_\mu + \left[ c_6 \cos \varphi + c_7 \sin \varphi + (-c_8 \sin 2\varphi + c_9 \cos 2\varphi) \cot \vartheta \right] \partial_\mu
\]  

(126)

\[
C_{\mu\nu} = c_1 \delta_{\mu\nu} + (c_8 \cos 2\varphi + c_9 \sin 2\varphi + c_{10}) \partial_\mu \vartheta \delta_\nu + 2(c_4 \cos \varphi + c_5 \sin \varphi) \partial_\mu \tau_\nu + c_3 \tau_\mu \tau_\nu.
\]  

(127)

It remains to find $A$. Using equations (126) and (127) in equation (17) we obtain

\[
\partial_\mu A = -2c_1 V^{-3} \partial_\mu V \\
- \frac{2}{\sin^2 \vartheta} [-c_6 \cos \varphi + c_7 \cos \varphi + (-c_8 \cos 2\varphi - c_9 \sin 2\varphi + c_{10}) \cot \vartheta] \partial_\mu \vartheta,
\]  

(128)

with the general solution

\[
A = c_1 V^{-2} + 2(-c_6 \sin \varphi + c_7 \cos \varphi) \cot \vartheta \\
+ (-c_8 \cos 2\varphi - c_9 \sin 2\varphi + c_{10}) \cot^2 \vartheta + k_7
\]  

(129)

for some function $k_7(\varphi)$. Imposing equation (16) on equation (129) we find that $\dot{k}_7 = 0$ and therefore that $k_7 = c_{11}$ for some constant $c_{11}$. The solution for $A$ becomes

\[
A = c_1 V^{-2} + 2(-c_6 \sin \varphi + c_7 \cos \varphi) \cot \vartheta \\
+ (-c_8 \cos 2\varphi - c_9 \sin 2\varphi + c_{10}) \cot^2 \vartheta + c_{11}.
\]  

(130)
Using equations (126), (127) and (130) in equation (4), and grouping terms according to their constant coefficient, we find that the general Schwarzschild Killing tensor is

\[ X_{ab} = c_1(h_{ab} + V^{-2} \gamma_a \gamma_b) + c_2 \tau_a \tau_b + c_3 \tau_a \gamma_b + c_4 [2 \cos \varphi \partial_{(a} \tau_{b)} - 2 \sin \varphi \cot \theta \gamma_a \gamma_b] \]

\[ + c_5 [2 \sin \varphi \partial_{(a} \gamma_{b)} + 2 \cos \varphi \cot \theta \gamma_a \gamma_b] + c_6 [2 \cos \varphi \partial_{(a} \gamma_{b)} - 2 \sin \varphi \cot \theta \gamma_a \gamma_b] \]

\[ + c_7 [2 \sin \varphi \partial_{(a} \gamma_{b)} + 2 \cos \varphi \cot \theta \gamma_a \gamma_b] \]

\[ + c_8 [2 \sin \varphi \partial_{(a} \gamma_{b)} + 2 \cos \varphi \cot \theta \gamma_a \gamma_b] \]

\[ + c_9 [2 \sin \varphi \partial_{(a} \gamma_{b)} + 2 \cos \varphi \cot \theta \gamma_a \gamma_b] \]

\[ + c_{10} \partial_a \partial_b + \cos^2 \varphi \gamma_a \gamma_b + c_{11} \gamma_a \gamma_b. \] 

We now rewrite this expression for the Killing tensor in terms of Killing vectors and the metric. We have

\[ g_{ab} = h_{ab} + V^{-2} \gamma_a \gamma_b, \] 

\[ \alpha_a = \sin \varphi \partial_a + (\cot \theta \cos \varphi) \gamma_a, \] 

\[ \beta_a = -\cos \varphi \partial_a + (\cot \theta \sin \varphi) \gamma_a, \]

which yields

\[ \alpha_a \alpha_b + \beta_a \beta_b = \partial_a \partial_b + \cos^2 \varphi \gamma_a \gamma_b \]

\[ \alpha_a \alpha_b - \beta_a \beta_b = \cos 2\varphi (\partial_a \partial_b + \cos^2 \varphi \gamma_a \gamma_b) + 2 \cot \theta \sin 2\varphi \partial_{(a} \gamma_{b)} \]

\[ 2 \alpha_a \beta_b = \sin 2\varphi (\partial_a \partial_b + \cos^2 \varphi \gamma_a \gamma_b) - 2 \cot \theta \cos 2\varphi \partial_{(a} \gamma_{b)}. \]

Using the three equations above, we can rewrite \( X_{ab} \) as

\[ X_{ab} = c_1 g_{ab} + c_2 \tau_a \tau_b + c_3 \tau_a \gamma_b + c_4 [2 \cos \varphi \partial_{(a} \tau_{b)} - 2 \sin \varphi \cot \theta \gamma_a \gamma_b] \]

\[ - c_5 (\alpha_a \partial_b - \beta_a \beta_b) - 2 c_9 (\alpha_a \beta_b + \beta_a \alpha_b) + c_{10} (\alpha_a \alpha_b + \beta_a \beta_b) + c_{11} \gamma_a \gamma_b. \]

Finally, regrouping terms we have

\[ X_{ab} = c_1 g_{ab} + c_2 \tau_a \tau_b + c_3 \tau_a \gamma_b + c_4 [2 \cos \varphi \partial_{(a} \tau_{b)} + 2 \cos \varphi \cot \theta \gamma_a \gamma_b] \]

\[ - c_5 (\alpha_a \partial_b - \beta_a \beta_b) - 2 c_9 (\alpha_a \beta_b + \beta_a \alpha_b) + c_{10} (\alpha_a \alpha_b + \beta_a \beta_b) + c_{11} \gamma_a \gamma_b. \]

Thus, we have written the general Schwarzschild Killing tensor as a sum of terms where each term is either the metric or a product of Killing vectors. Therefore, all Killing tensors of the Schwarzschild spacetime are trivial.

5. Conclusions

The method used above consists of applying equations (13)–(17) to find the Killing tensor in \( n \) dimensions, using the equations in \((n-1)\) dimensions. For the Melvin metric and the Schwarzschild metric this is done three times, going from a one-dimensional space to the four-dimensional spacetime.

The method developed here for finding Killing tensors could be used on a wide variety of spacetimes where there are symmetries. It requires less computation than an attack on the full Killing tensor equations, and it can be used even when the spacetime is not algebraically special. The method could also be generalized in various ways. The equations for a Killing–Yano tensor [6] could be treated in an analogous way and should result in a method simpler
than a straightforward attempt to solve the Killing–Yano equations. Finally, the method might have a useful generalization to the case where the Killing vector is not hypersurface orthogonal. In that case one would expect to get more complicated equations that involve not only the norm of the Killing field but also the twist. However, it is in just such spacetimes (i.e. Kerr) that known examples of nontrivial Killing tensors exist. So an investigation along those lines might be useful.

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