Study of the Localization Transition on a Cayley-tree via Spectral Statistics

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The spectral statistics of a Cayley-tree is numerically studied. The statistics are non-universal due to the high ratio of boundary sites. Once the boundary sites are connected to each other in a way that preserves the local structure of the tree the universal statistics of the spectra is recovered. A clear localization transition is observed as function of on-site disorder strength, with a critical disorder $W_c = 11.44 \pm 0.04$ and critical index $\nu = 0.51 \pm 0.05$. The value of $\nu$ fits nicely to its mean field value, while the value of $W_c$ is puzzling.

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The properties of the Anderson transition have generated much interest since it was first predicted\(^1\). The transition is characterized by a lower critical dimension, believed to be equal to two, below which all states are localized for any amount of disorder. Above the lower critical dimension a transition between extended and localized states appears at some critical value of disorder (or energy). An upper critical dimension, above which the transition may be described by a mean field theory, is not well established.

The Anderson transition is usually characterized by two parameters. The critical disorder $W_c$ at the middle of the band (where $W$ is the width of the distribution from which the on-site energies are drawn in the canonical Anderson model defined in Eq. (1)) and the critical index $\nu$. The dependence of $W_c$ and $\nu$ on the dimensionality $d$ has been the subject of many recent numerical studies. For $d = 3, 4$ the values are well established - $W_c \sim 16.5, \nu \sim 1.5$ for $d = 3$\(^2\),\(^3\) and $W_c \sim 35, \nu \sim 1$ for $d = 4$\(^4\),\(^5\). The mean field value of the critical exponent is equal to $1/2$. Assuming that the upper critical dimension is equal to infinity, an extrapolation equation for $\nu \sim 0.8/(d - 2) + .5$ was proposed\(^6\). Verification of this extrapolation was obtained by studying the Anderson transition for bifractal system, where it was demonstrated that $d$ should be replaced by the spectral dimension $d_s$. Similarly, the critical disorder is also extrapolated by $W_c \sim 16.5(d - 2)$ (again for bifractals $d$ is replaced by $d_s$)\(^7\).

Our main goal in this study is to identify the metal-insulator transition in a disordered Cayley-tree, and to study its properties numerically by spectral statistics. Although many studies were performed for Cayley-tree structures, to the best of our knowledge, no studies were performed using spectral statistics \(^8\). Moreover, it is not trivial to extend the above extrapolations for $\nu$ and $W_c$ to the Cayley-tree. From analytical calculations it is known that for the Cayley-tree $\nu = 0.5$\(^6\),\(^7\),\(^8\),\(^9\), i.e., $d = \infty$. On the other hand, a mobility edge is predicted at some finite disorder, which is hard to conciliate with the extrapolation formula for $W_c$ which gives $W_c = \infty$ if $d = \infty$ is plugged in, and $W_c < 0$ if the spectral dimension of a Cayley-tree $d_s = 4/3$\(^10\) is inserted.

We based our calculations on the usual tight-binding Hamiltonian,

$$H = \sum_i \varepsilon_i a_i^\dagger a_i - \sum_{<i,j>} a_j^\dagger a_i,$$  (1)

where the left part of $H$ stands for the disordered on-site potential. The on-site energies, $\varepsilon_i$ are uniformly distributed over the range $-W/2 \leq \varepsilon_i \leq W/2$. The right part is the hopping element which is set to 1, and $<i,j>$ denotes nearest neighbors. Here we considered a tree where each site is connected to two sites below it. We diagonalize the Hamiltonian exactly, and obtain $N$ eigenvalues $E_i$ (where $N$ is the number of sites in the tree) and eigenvectors $\psi_i$. The calculations are made for $K$ different realizations $-K = 4000, 2000, 1000, \ldots, 125, 64$ for the corresponding tree sizes: $N = 63, 127, 255, \ldots, 2047, 4095$ or $L = 6, 7, 8, \ldots, 11, 12$ (where $L$ is the number of “generations” in the tree).
We have calculated the distribution \( P(s) \) of adjacent level spacings \( s \), where \( s = (E_{i+1} - E_i)/\sqrt{E_{i+1} - E_i} \). Typical results are presented in Fig. 1. It can be clearly seen that there is almost no change in the distribution as function of the disorder once the disorder is above \( W = 1 \).

This unusual behavior of the nearest-neighbor level spacing can be attributed to the special form of the Cayley-tree. Half of the sites in the tree are boundary "leaves"- sites at the boundary of the tree which are not connected any further. This peculiar structure of the tree is known to lead to unusual behavior such as a jump in the participation ratio at the mobility edge \[\eta \]. In Ref. 12 this peculiarity is remedied by connecting each of the boundary leaves randomly to two other leaf sites. Thus, the local structure of the Cayley-tree is preserved, while there are no boundary leaves. \( P(s) \) for such a tree is depicted in Fig. 2. As expected the distribution is shifting from the Wigner surmise distribution (characteristic of extended states),

\[
P_W(s) = \frac{\pi}{2} s \exp[-\frac{\pi}{4} s^2],
\]

to a Poisson distribution (localized states),

\[
P_P(s) = e^{-s}.
\]

We can recognize the Anderson transition also by noting that all curves intersect at \( s \sim 2 \) and the peak of the distribution "climbs" along the Poisson curve for larger values of \( W \). The transition point can be established more accurately, as shown in Ref. \[\eta \], from calculating \( \gamma \):

\[
\gamma = \frac{\int_0^\infty P(s) - \int_2^\infty P_W(s)}{\int_2^\infty P_P(s) - \int_2^\infty P_W(s)}.
\]

\[\gamma \rightarrow 0 \] as the distribution tends towards the Wigner distribution, while \( \gamma \rightarrow 1 \) if the distribution approaches the Poisson distribution. One expects that as the system size increases, the finite size corrections will become small resulting in a distribution closer to Wigner distribution in the metallic regime and to Poisson in the localized one.

At the transition point the distribution should be independent of the system size. Indeed, this is the behavior seen in Fig. 3 in which \( \gamma \) decreases with system size for small values of \( W \) while it increases with size for large values of \( W \). All curves seem to cross at a particular value of disorder signifying the critical disorder.

From finite size scaling arguments \[\eta \] one expects that around the critical disorder \( \gamma \) will depend on the the disorder and tree size in the following way:

\[
\gamma(W, L) = \gamma(W_c, L) + C \left( \frac{W}{W_c} - 1 \right) L^{1/\nu},
\]

where \( C \) is a constant. This relation enables us to extract the critical disorder \( W_c = 11.44^{+0.08}_{-0.04} \) and the critical index \( \nu = 0.51^{+0.05}_{-0.04} \). The scaling of the numerical data according to Eq. \[\eta \] is depicted in Fig. 4. Two branches corresponding to the metallic and localized regimes are clearly seen. The critical index \( \nu \) fits rather well the mean field results mentioned above. On the other hand, the critical disorder for the Cayley-tree is lower than in the three dimensional case. Thus, while from the extrapolation equation of \( \nu \) one concludes that the dimensionality of the Cayley-tree is infinity (as expected on geometrical grounds), from the extrapolation equation of the critical disorder one concludes that \( d = 2.7 \). This value does not correspond neither to the geometric dimensionality nor to the spectral dimensionality \( d_s = 4/3 \).

It is also interesting to check the behavior of the inverse
FIG. 4: The scaling of $\gamma$ according to Eq. (5) for different tree sizes $L$. Two branches, corresponding to the metallic and localized regimes, appear.

FIG. 5: The distribution of the IPR for different values of disorder $W$ for an $L = 10$ tree.

The participation ratio (IPR) defined as:

$$I = \sum_r |\psi_r(r)|^4.$$  \hspace{1cm} (6)

In the metallic regime $I \sim 1/N$, while in the localized regime $I \sim 1$. The distribution of $I$ as function of the disorder is depicted in Fig. 4. For small values of $W$ the distribution is peaked at small values of $I$, as expected in the metallic regime, while for larger values of $W$ the distribution is very wide. The transition of the distribution between the metallic regime and the localized one is, as expected, smooth.

In conclusion, the spectral statistics of a Cayley-tree depends strongly on the boundary condition. When the boundary leaves are connected to each other in a way that preserves the local structure of the tree, a clear Anderson transition is observed. As expected the critical index $\nu$ corresponds to its mean field value. On the other hand, the critical disorder for which the localization transition occurs does not fit into the usual extrapolation formulas when either the geometric or the spectral dimension of the Cayley-tree are used.

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