ON THE CENTER OF THE GROUP OF QUASI-ISOMETRIES OF THE REAL LINE

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Let $QI(\mathbb{R})$ denote the group of all quasi-isometries $f : \mathbb{R} \to \mathbb{R}$. Let $Q_+(\text{and } Q_-)$ denote the subgroup of $QI(\mathbb{R})$ consisting of elements which are identity near $-\infty$ (resp. $+\infty$). We denote by $QI_+^+(\mathbb{R})$ the index 2 subgroup of $QI(\mathbb{R})$ that fixes the ends $+\infty, -\infty$. We show that $QI_+^+(\mathbb{R}) \cong Q_+ \times Q_-$. Using this we show that the center of the group $QI(\mathbb{R})$ is trivial.

Key words: PL-homeomorphisms; quasi-isometry; center of group.

1. INTRODUCTION

We begin by recalling the notion of quasi-isometry.

Let $f : (X, d_X) \to (Y, d_Y)$ be a map between two metric spaces. We say that $f$ is a $K$-quasi-isometric embedding if there exists a $K > 1$ such that

$$\frac{1}{K} d_X(x_1, x_2) - K \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + K \forall x_1, x_2 \in X.$$

Again, if for any given point $y \in Y$, there is a point $x \in X$ such that $d_Y(f(x), y) < K$, then $f$ is said to be a $K$-quasi-isometry. If $f$ is a quasi-isometry (for some $K > 1$), then there exists a $K'$-quasi-isometry $f' : Y \to X$ such that $f' \circ f$ (resp. $f \circ f'$) is quasi-isometry, equivalent to the identity map of $X$ (resp. $Y$) (two maps $f, g : X \to Y$ are said to be quasi-isometrically equivalent if there exists a constant $M > 0$ such that $d_Y(f(x), g(x)) < M \forall x \in X$.) Let $[f]$ denote the equivalence class of a quasi-isometry $f : X \to X$. We denote the set of all equivalence classes of self-quasi-isometries of $X$ by $QI(X)$.

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It turns out that one has a well-defined notion of composition of isometry classes, where \([f] \cdot [g] := [f \circ g]\) for \([f], [g] \in QI(X)\). This makes \(QI(X)\) a group, referred to as the group of quasi-isometries of \((X, d_X)\). If \(\phi : X' \to X\), \(\phi' : X \to X'\) are a pair of inverse quasi-isometries, then \([f] \mapsto [\phi \circ f \circ \phi']\) defines an isomorphism of groups \(QI(X') \to QI(X)\). For example, \(t \to [t]\) is a quasi-isometry \(\mathbb{R} \to \mathbb{Z}\), which induces an isomorphism \(QI(\mathbb{R}) \cong QI(\mathbb{Z})\). We refer the reader to [1, Chapter I.8] for basic facts concerning quasi-isometry.

It is known that \(QI(\mathbb{R})\) is a rather large group; see [5, 3.3.B]. Several well-known groups can be embedded in \(QI(\mathbb{R})\). We first need to describe the following groups: (i) Let \(Diff(S^1)\) and \(PL(S^1)\) be the groups of all diffeomorphisms and piecewise linear homeomorphisms of \(S^1\) respectively. Note that any homeomorphism \(f\) of \(S^1\) can be lifted to obtain a homeomorphism \(\tilde{f}\) of \(\mathbb{R}\) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\tilde{f}} & \mathbb{R} \\
\downarrow{p} & & \downarrow{p} \\
S^1 & \xrightarrow{f} & S^1
\end{array}
\]

where \(p : \mathbb{R} \to S^1\) is defined as \(t \to e^{2\pi it}\). We denote the groups of all lifts of elements of \(Diff(S^1)\) and \(PL(S^1)\) by \(\widetilde{Diff}(S^1)\) and \(\widetilde{PL}(S^1)\) respectively. (ii) We denote the group of all piecewise linear homeomorphisms of \(\mathbb{R}\) with compact support by \(PL_\kappa(\mathbb{R})\).

It is known that the following groups can be embedded in \(QI(\mathbb{R})\): (i) \(\widetilde{Diff}(S^1)\) and \(\widetilde{PL}(S^1)\), (ii) \(PL_\kappa(\mathbb{R})\), (iii) the Thompson’s group \(F\), (iv) the free group of rank \(c\), the continuum. The group \(PL_\kappa(\mathbb{R})\) is simple (see [2] and Theorem 3.1 of [4]). The group \(\widetilde{Diff}(S^1)\) contains a free group of rank the continuum (see [6]). The Thompson’s group \(F\) has many remarkable properties and arises in many different contexts in several branches of mathematics. We list below some properties of \(F\): (i) the commutator subgroup \([F, F]\) is a simple group, (ii) every proper quotient group of \(F\) is abelian, (iii) \(F\) does not contain a non-abelian free subgroup (see [3]). Thus we see that the group \(QI(\mathbb{R})\) has a rich collection of subgroups having remarkable properties.

However, the lattice of normal subgroups of \(QI(\mathbb{R})\) does not seem to have been studied. As a first step in that direction we prove the following, which is the main result of this note.

**Theorem 1.1** — The center of the group \(QI(\mathbb{R})\) is trivial.

Our proof uses the description of \(QI(\mathbb{R})\) as a quotient of a certain subgroup of the group of all PL-homoemorphisms of \(\mathbb{R}\) due to Sankaran [7].
2. PL-homeomorphisms with Bounded Slopes

Let \( f : \mathbb{R} \to \mathbb{R} \) be any homeomorphism of \( \mathbb{R} \). We denote by \( B(f) \) the set of break points of \( f \), i.e. points where \( f \) fails to have derivatives, and, by \( \Lambda(f) \) the set of slopes of \( f \), i.e.

\[
\Lambda(f) = \{ f'(t) : t \in \mathbb{R} - B(f) \}.
\]

Note that \( B(f) \subset \mathbb{R} \) is discrete if \( f \) is piecewise differentiable.

**Definition 2.1** — We say that a subset \( \Lambda \) of \( \mathbb{R}^n \) (the set of non-zero real numbers) is bounded if there exists a \( M > 1 \) such that \( M^{-1} < |\lambda| < M \) for all \( \lambda \in \Lambda \).

We denote the set of all piecewise linear homeomorphisms \( f \) of \( \mathbb{R} \) such that \( \Lambda(f) \) is bounded, by \( PL_{\delta}(\mathbb{R}) \). This forms a subgroup of the group \( PL(\mathbb{R}) \) of all piecewise linear homeomorphisms of \( \mathbb{R} \).

The subgroup of \( PL_{\delta}(\mathbb{R}) \) consisting of orientation preserving PL-homeomorphisms will be denoted by \( PL_{\delta}^+(\mathbb{R}) \). We have \( PL_{\delta}(\mathbb{R}) = PL_{\delta}^+(\mathbb{R}) \ltimes \langle \rho \rangle \) where \( \rho \) is the reflection of \( \mathbb{R} \) about the origin. The following theorem is due to Sankaran [7].

**Theorem 2.2** — *The natural homomorphism \( \phi : PL_{\delta}(\mathbb{R}) \to QI(\mathbb{R}) \), defined as \( f \to [f] \) is surjective.*

The kernel of \( \phi \) in the above theorem equals the subgroup of all \( f \in PL_{\delta}(\mathbb{R}) \) such that \( ||f - id|| < \infty \). In particular \( \ker(\phi) \) contains all translations. It follows that the restriction of \( \phi \) to the subgroup \( PL_{\delta,0} := \{ f \in PL_{\delta}(\mathbb{R}) \mid f(0) = 0 \} \) is surjective.

**Notations** : We shall denote \( PL_{\delta,0}(\mathbb{R}) \) by \( P \) and \( QI(\mathbb{R}) \) by \( Q \). We denote by \( P_+ \) (resp. \( P_- \)) the subgroup of \( P \) consisting of homeomorphisms which are identity near \( -\infty \) (resp. \( +\infty \)). Note that \( P_{\delta} := P_+ \cap P_- \) is the group of compactly supported homeomorphisms in \( P \). Similarly, \( Q_+ \) (resp. \( Q_- \)) denote the subgroups of \( QI(\mathbb{R}) \) consisting of elements which are identity near \( -\infty \) and \( +\infty \) respectively. The subgroup of \( P \) consisting of orientation preserving homeomorphisms will be denoted by \( P^+ \). Similarly, \( Q^+ \) denotes the subgroup of \( Q \) whose elements can be represented by orientation preserving homeomorphisms and we have \( \phi(P^+) = Q^+ \), \( \phi(P_{\pm}) = Q_{\pm} \).

We shall denote by the same symbol \( \phi \) the restriction of \( \phi : PL_{\delta}(\mathbb{R}) \to QI(\mathbb{R}) \) to \( P \). The group \( P = PL_{\delta,0} \) contains no non-trivial translations and, as already noted, we have \( P = P^+ \ltimes \langle \rho \rangle \). Similarly the group \( Q = QI(\mathbb{R}) \) is a semi-direct product \( Q = Q^+ \ltimes \langle [\rho] \rangle \). Then \( Q_+ \cap Q_- = \phi(P_+ \cap P_-) = \phi(P_{\delta}) \) is trivial and so we have \( Q^+ = Q_+ \times Q_- \). Conjugation by \( [\rho] \) interchanges \( Q_+ \) and \( Q_- \). It follows that \( N \subset Q^+ \) is normal if and only if its projections \( N_+, N_- \) to the factors \( Q_+, Q_- \) are normal. Also \( N \) is a normal subgroup of \( Q \) if and only if \( N \) is normalized by \( Q^+ \) and \( [\rho]N[\rho]^{-1} = N \).
Remark 2.3: It may be an interesting problem to decide whether the groups $Q_+$ or $Q_-$ are simple.

3. CENTER OF THE GROUP $QI(\mathbb{R})$

We begin by making a few preliminary observations concerning central elements in $Q = QI(\mathbb{R})$. We first observe that any element $[f] \in Q$ in the center must fix the ends $+\infty, -\infty$. Indeed we may assume that $f(0) = 0$; that $f(t) > 0$ if and only if $t > 0$. Suppose that $f$ is orientation reversing. Let $h \in P = PL_{\delta,0}(\mathbb{R})$ be the orientation preserving PL-homeomorphism of $\mathbb{R}$ defined as

$$h(t) = \begin{cases} t, & t \leq 0 \\ 2t, & t \geq 0 \end{cases}$$

By a straightforward computation $h(f(t)) - f(h(t)) = 2f(t) - f(t) = f(t)$ for $t < 0$. So $||h \circ f - f \circ h||$ is unbounded and we conclude that $f$ is not in the center.

Any element $f \in P^+ = PL^+_{\delta,0}(\mathbb{R})$ may be decomposed as a composition $f = f_+ \circ f_-$ with $f_+ \in P_+, f_- \in P_-$, where $f_+(t) = t, f_-(s) = s$ for $s \leq 0 \leq t$. It follows that $[f] = ([f_+], [f_-]) \in Q_+ \times Q_- = Q^+$ is in the center if and only if $f_\pm$ is in the center of $Q_\pm$ and $[\rho \circ f_+ \circ \rho] = [f_-]$. We will show that the center of $Q_+$ is trivial. We need the following lemma.

Lemma 3.1 — Let $f \in P_+$ and $[f] \neq [id]$. Then there exists a strictly monotone divergent sequence $\{b_n\}$ of real numbers such that at least one of the following holds:

1. $b_n \to +\infty, b_{n+1} > 3f(b_n) \forall n$ and $f(b_n) - b_n \to +\infty$,

2. $b_n \to +\infty, b_{n+1} > 3f^{-1}(b_n) \forall n$ and $f^{-1}(b_n) - b_n \to +\infty$,

Proof: Since $f(t) = t$ for $t \leq n_1$ (for some $n_1 < 0$) and since $||f - id||$ is unbounded, we can find a strictly monotone divergent sequence $\{a_n\}$ of positive real numbers such that $|f(a_{n+1}) - a_{n+1}| > |f(a_n) - a_n| \forall n, |f(a_n) - a_n| \to \infty$. We obtain a subsequence of $\{a_n\}$ as follows: Set $a_{n_1} = a_1$. Choose $n_2 \geq 2$ such that $f(a_{n_2}) > \max\{a_1, 3f(a_1)\}$. Having chosen $a_{n_j}, 1 \leq j \leq k$, we choose $n_{k+1} > n_k$ such that $f(a_{n_{k+1}}) > \max\{a_{n_k}, 3f(a_{n_k})\}$. We set $c_k := a_{n_k}$. Since $|f(c_k) - c_k| \to \infty$ as $k \to \infty$, there has to be infinitely many values of $k$ for which $f(c_k) - c_k$ has the same sign. If this sign is positive, we have a subsequence $\{b_k\}$ of $\{c_k\}$ which satisfies (1). If this sign is negative, then we apply the above consideration to $f^{-1} \in P_+$ which yields a sequence $\{b_k\}$ satisfying (2).

The previous lemma will be the main tool to study the center of the group $QI(\mathbb{R})$.

Proof of Theorem 1.1: As observed already, it suffices to show that the center of $Q_+$ is trivial. If possible, suppose that $[f] \neq 1$ is in the center of $Q_+$ with $f \in P_+$. Then $[f^{-1}]$ is also in the center.
and so, without loss of generality we may assume the existence of a sequence \( \{b_n\} \) satisfying (1) of the above lemma. One has a PL-homeomorphism \( g \) such that

(i) \( g(t) = t, \ t \leq b_1 \),

(ii) \( g(J_k) = J_k \) where \( J_k := [b_k, b_{k+1}] \), and,

(iii) \( g \) has exactly one break point at \( f(b_k) \in J_k \) and \( g(f(b_k)) = (b_k + f(b_k))/2 \).

We claim that \( g \in P_+ \) and that \([g] \) and \([f] \) do not commute.

First, we compute the slopes of \( g \). Since \( g(J_k) = J_k \) and since \( g \) has exactly one break point in \( J_k \) at \( f(b_k) \), straightforward computation shows that \( g'(t) = 1/2 \) if \( b_k < t < f(b_k) \), and, when \( f(b_k) < t < b_{k+1} \), we have \( g'(t) = (b_{k+1} - f(b_k)) / 2(b_k - f(b_k)) + (f(b_k) - b_k) / 2(f(b_k) - f(b_k)) \) < 1 + \( (f(b_k) - b_k) / 2(f(b_k)) < 1 + 1/4 \). Since \( f(b_k) > b_k \) we also see that \( g'(t) > 1 \). So we conclude that for any \( t \) at which \( g \) is differentiable, we have \( g'(t) \in [1/2, 5/4] \), showing that \( g \in P_+ \).

Finally, \( f(g(b_k)) = f(b_k) \) whereas \( g(f(b_k)) = (f(b_k) + b_k)/2 \). So \( f(g(b_k)) - g(f(b_k)) = (f(b_k) - b_k)/2 \to \infty \) as \( k \to \infty \) and so \([f][g] \neq [g][f] \). This completes the proof.

\[ \square \]

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