Abstract.

We review some recent results on the calculation of renormalization constants in Yang-Mills theory using open bosonic strings. The technology of string amplitudes, supplemented with an appropriate continuation off the mass shell, can be used to compute the ultraviolet divergences of dimensionally regularized gauge theories. The results show that the infinite tension limit of string amplitudes corresponds to the background field method in field theory. (Proceedings of the Workshop “Strings, Gravity and Physics at the Planck scale”, Erice (Italy), September 1995. Preprint DFTT 82/95)

1. Introduction

In the past few years it has become clear that string theory is not only a good candidate for a unified theory of all interactions, but also a useful tool to understand the structure of perturbative field theories. Field-theoretical results can be recovered from string theory by decoupling the infinite tower of massive string states, that is by taking the limit of infinite string tension, or equivalently of vanishing Regge slope $\alpha'$. 
Since in the limit $\alpha' \to 0$ string theories reduce to non-abelian gauge theories, unified with gravity, order by order in perturbation theory, in this limit we may expect to reproduce, order by order, scattering amplitudes, ultraviolet divergences, and other physical quantities that one computes perturbatively in non-abelian gauge theories.

A very useful feature of string theory for this purpose is the fact that, at each order of string perturbation theory, one does not get the large number of diagrams characteristic of field theories, which makes it very difficult to perform high order calculations. Using closed strings, one gets only one diagram at each order, while with open strings the number of diagrams remains limited. Furthermore, compact expressions for these diagrams are known explicitly for an arbitrary perturbative order [1], in contrast with the situation in field theory, where no such all-loop formula is known. Finally, string amplitudes are naturally written in a way that takes maximal advantage of gauge invariance: the color decomposition is automatically performed, and so are integrations over loop momenta, so that the helicity formalism is readily implemented.

The combination of these different features of string theory has led several authors [2, 3, 4, 5, 6, 7] to use string theory as an efficient conceptual and computational tool in different areas of perturbative field theory. In particular, because of the compactness of the multiloop string expression, it is in some cases easier to calculate non-abelian gauge theory amplitudes by starting from a string theory, and performing the zero slope limit, rather than using traditional techniques. In this way the one-loop amplitude involving four external gluons has been computed, reproducing the known field-theoretical result with much less computational cost [8]. Following the same approach, also the one-loop five gluon amplitude has been computed for the first time [9].

The aim of this talk is to summarize the results obtained in Refs. [10] and [11]. There it was shown that, provided a simple off-shell continuation is performed, string theory also contains information on the ultraviolet divergences of Yang-Mills theory, and the information can be consistently extracted in the language of dimensional regularization. In particular, starting from the one-loop two, three and four-gluon amplitudes in the open bosonic string, we performed the field theory limit, and we showed that in this limit the renormalization constants $Z_A$, $Z_3$ and $Z_4$ of non-abelian gauge theories can be consistently recovered. String theory leads unambiguously to the background field method, as suggested by the on-shell analysis of Ref. [12].

Before going into the details of the calculation, let us first recall how field theory amplitudes are obtained from string theory, and how we expect those amplitudes to be renormalized.

In field theory one normally computes either connected Green functions, denoted here by $W_M(p_1 \ldots p_M)$, or one-particle irreducible (1PI) Green functions, $\Gamma_M(p_1 \ldots p_M)$. In both cases, in general, an off-shell continuation is performed, in order to avoid possible infrared divergences.

In string theory, on the other hand, one computes $S$-matrix elements involving gluon states, which are connected, via the reduction formulas, to on-shell connected Green functions, truncated with free propagators.
Taking the field theory limit, the natural ultraviolet regulator of string theory, $1/\alpha'$, is removed, so that the usual divergences are recovered. The Green functions one computes are thus unrenormalized, and a new regulator must be introduced, in our case dimensional continuation. We will see that also in this case an off-shell extrapolation is necessary in order to avoid infrared problems.

Once the field theory limit is taken, it is possible to isolate 1PI contributions, which lead to the 1PI Green functions $\Gamma_M$, or to compute the full amplitudes, which lead to the Green functions $W_M$. From the knowledge on how they renormalize we can then extract the renormalization constants. For example,

$$
\Gamma_2(g) = Z_A^{-1} \Gamma_2(R)(g), \quad \Gamma_3(g) = Z_3^{-1} \Gamma_3(R)(g), \quad \Gamma_4(g) = Z_4^{-1} \Gamma_4(R)(g),
$$

while

$$
W_3(g) = Z_3^{-1} Z_A W_3(R)(g),
$$

where $g$ is the renormalized coupling constant.

The talk is organized as follows. In Section 2 we consider the open bosonic string, and we write the explicit expression of the $M$-gluon amplitude at $h$ loops, including the overall normalization. In Section 3 we give the relevant amplitudes for the tree and one-loop diagrams. In Section 4 we sketch the calculation of the one-loop two gluon amplitude, already presented in [10], and we extract the gluon wave function renormalization constant $Z_A$. In Section 5 we present an alternative method, that allows one to exactly integrate over the punctures, and we use it to extract the renormalization constants $Z_A$, $Z_3$ and $Z_4$. Finally, in Section 6 we extend the calculation of Section 4 to the one-loop three gluon amplitude, and we discuss how to extract the contribution of the one-particle reducible diagrams, that were neglected in Section 5. Section 7 contains concluding remarks.

2. The $M$-gluon $h$-loop amplitude

In string theory the $M$-gluon scattering amplitude can be computed perturbatively and is given by

$$
A(p_1, \ldots, p_M) = \sum_{h=0}^{\infty} A^{(h)}(p_1, \ldots, p_M) = \sum_{h=0}^{\infty} g_s^{2h-2} \hat{A}^{(h)}(p_1, \ldots, p_M),
$$

where $g_s$ is a dimensionless string coupling constant, which is introduced to formally control the perturbative expansion. In Eq. (2.1), $A^{(h)}$ represents the $h$-loop contribution. For the closed string $A^{(h)}$ is given by only one diagram, while for the open string the number of diagrams is small in comparison with the large number of diagrams encountered in field theory.

Let us consider the open bosonic string, and let us restrict ourselves only to planar diagrams. For such diagrams the $M$-gluon $h$-loop amplitude,
including the appropriate Chan-Paton factor, is given by

\[
A_P^{(h)}(p_1, \ldots, p_M) = N^h \text{Tr}(\lambda^{a_1} \cdots \lambda^{a_M}) C_h N_0^M \\
\times \int [dm]_h^M \left\{ \prod_{i<j} \left[ \frac{\exp \left( G^{(h)}(z_i, z_j) \right)}{\sqrt{V_i'(0) V_j'(0)}} \right] \right\}^{2\alpha^f p_i p_j} \times \exp \left[ \sum_{i\neq j} \sqrt{2\alpha^f} p_j \cdot \varepsilon_i \partial_{z_i} G^{(h)}(z_i, z_j) \right. \right.
\left. + \frac{1}{2} \sum_{i\neq j} \varepsilon_i \cdot \varepsilon_j \partial_{z_i} \partial_{z_j} G^{(h)}(z_i, z_j) \right] \Bigg\} \text{m.l.,}
\]

where the subscript “m.l.” stands for multilinear, meaning that only terms linear in each polarization should be kept. Eq. (2.2) is written for transverse gluons, satisfying the condition \( \varepsilon_i \cdot p_i = 0 \), whereas the mass-shell condition \( p_i^2 = 0 \), though necessary for conformal invariance of the amplitude, has not been enforced yet.

The main ingredient in Eq. (2.2) is the \( h \)-loop world-sheet bosonic Green function \( G^{(h)}(z_i, z_j) \), which plays a key role in the field theory limit. \([dm]_h^M\) is the measure of integration on moduli space for an open Riemann surface of genus \( h \) with \( M \) operator insertions on the boundary [1]. The Green function \( G^{(h)}(z_i, z_j) \) can be expressed as

\[
G^{(h)}(z_i, z_j) = \log E^{(h)}(z_i, z_j) - \frac{1}{2} \int_{z_i}^{z_j} \omega^\mu (2\pi \text{Im} \tau_{\mu\nu})^{-1} \int_{z_i}^{z_j} \omega^\nu,
\]

where \( E^{(h)}(z_i, z_j) \) is the prime-form, \( \omega^\mu (\mu = 1, \ldots, h) \) the abelian differentials and \( \tau_{\mu\nu} \) the period matrix of an open Riemann surface of genus \( h \). All these objects, as well as the measure on moduli space \([dm]_h^M\), can be explicitly written down in the Schottky parametrization of the Riemann surface, and their expressions for arbitrary \( h \) can be found for example in Ref. [13]. Here we will only write the explicit expression for the measure, to give a flavor of the ingredients that enter the full string theoretic calculations. It is

\[
[dm]_h^M = \prod_{i=1}^M dz_i \prod_{\mu=1}^h \left[ \frac{d\xi_\mu d\eta_\mu}{k_\mu^2 (\xi_\mu - \eta_\mu)^2 (1 - k_\mu)^2} \right]^{d/2} \prod_{\alpha=1}^d \left[ \prod_{n=1}^{\infty} (1 - k_\alpha^n)^{-d} \prod_{n=2}^{\infty} (1 - k_\alpha^n)^2 \right].
\]

Here \( \tau_{\mu\nu} \) is the period matrix, while \( k_\mu \) are the multipliers and \( \xi_\mu \) and \( \eta_\mu \) the fixed points of the generators of the Schottky group; \( dV_{abc} \) is the projective invariant volume element

\[
dV_{abc} = \frac{d\rho_a \ d\rho_b \ d\rho_c}{(\rho_a - \rho_b) (\rho_a - \rho_c) (\rho_b - \rho_c)},
\]

where \( \rho_a, \rho_b, \rho_c \) are any three of the \( M \) Koba-Nielsen variables, or of the \( 2h \) fixed points of the generators of the Schottky group, which can be fixed.
at will; finally, the primed product over $\alpha$ denotes a product over classes of elements of the Schottky group [13].

Notice that in the open string the Koba-Nielsen variables must be cyclically ordered, for example according to
\[ z_1 \geq z_2 \cdots \geq z_M \quad , \]
and the ordering of Koba-Nielsen variables automatically prescribes the ordering of color indices.

The amplitude in Eq. (2.2) contains two normalization constants which were calculated in Ref. [11], and are given by
\[ C_h = \frac{1}{(2\pi)^d h} g_s^{2h-2} \frac{1}{(2\alpha')^{d/2}} \quad \mathcal{N}_0 = g_d \sqrt{2\alpha'} \quad , \]
where the string coupling $g_s$ and the $d$-dimensional gauge coupling $g_d$ are related by
\[ g_s = \frac{g_d}{2} (2\alpha')^{1-d/4} \quad . \]

An efficient way to explicitly obtain $A^{(h)}(p_1, \ldots, p_M)$ is to use the $M$-point $h$-loop vertex $V_{M,h}$ of the operator formalism. The explicit expression of $V_{M,h}$ for the planar diagrams of the open bosonic string can be found in Ref. [1]. The vertex $V_{M,h}$ depends on $M$ real Koba-Nielsen variables $z_i$ through $M$ projective transformations $V_i(z)$, which define local coordinate systems vanishing around each $z_i$, i.e. such that
\[ V_i^{-1}(z_i) = 0 \quad . \]

When $V_{M,h}$ is saturated with $M$ physical string states satisfying the mass-shell condition, the corresponding amplitude does not depend on the $V_i$’s. However, as we discussed in Ref. [10], to extract informations about the ultraviolet divergences that arise when the field theory limit is taken, it is necessary to relax the mass-shell condition, so that also the amplitudes $A^{(h)}$ will depend on the choice of projective transformations $V_i$’s, just like the vertex $V_{M,h}$. This is the reason of the appearence of $V_i$ in Eq. (2.2).

3. Tree and one-loop diagrams

For tree-level amplitudes, corresponding to $h = 0$, the situation is particularly simple. The Green function in Eq. (2.3) reduces to
\[ G^{(0)}(z_i, z_j) = \log(z_i - z_j) \quad , \]
while the measure $[dm]_0^M$ is simply
\[ [dm]_0^M = \frac{M}{\prod_{i=1}^M dz_i} dV_{abc} \quad . \]
Inserting Eqs. (3.1) and (3.2) into Eq. (2.2), and writing explicitly all the normalization coefficients, we obtain the color ordered, planar, on-shell $M$ gluon amplitude at tree level

\[
A_P^{(0)}(p_1, \ldots, p_M) = 4 \text{Tr}(\lambda^{a_1} \cdots \lambda^{a_M}) g_d^{M-2} (2\alpha')^{M/2-2} \times \int_{\Gamma_0} \frac{\prod_{i=1}^{M} dz_i}{dV_{abc}} \left\{ \prod_{i<j} (z_i - z_j)^{2\alpha' p_i \cdot p_j} \times \exp \left[ \sum_{i<j} \left( \sqrt{2\alpha'} \frac{p_j \cdot \varepsilon_i - p_i \cdot \varepsilon_j}{(z_i - z_j)} + \frac{\varepsilon_i \cdot \varepsilon_j}{(z_i - z_j)^2} \right) \right] \right\}_{m.l.},
\]

where $\Gamma_0$ is the region identified by Eq. (2.6). Notice that any dependence on the local coordinates $V_i(z)$ drops out in the amplitude after enforcing the mass-shell condition. Notice also that Eq. (3.3) is valid only for $M \geq 3$, since the measure given by Eq. (3.2) is ill-defined for $M \leq 2$.

We readily derive the three-gluon amplitude

\[
A^{(0)}(p_1, p_2, p_3) = -4 g_d \text{Tr}(\lambda^a \lambda^b \lambda^c) \left( \varepsilon_1 \cdot \varepsilon_2 p_2 \cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 p_3 \cdot \varepsilon_1 + \varepsilon_3 \cdot \varepsilon_1 p_1 \cdot \varepsilon_2 + O(\alpha') \right),
\]

and the four-gluon amplitude

\[
A_4^{(0)}(p_1, p_2, p_3, p_4) = 4g_d^2 \text{Tr}(\lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4}) \frac{\Gamma(1 - \alpha' s)\Gamma(1 - \alpha' t)}{\Gamma(1 + \alpha' u) st} \times \left[ (\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4) t u + (\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4) t s + (\varepsilon_1 \cdot \varepsilon_4)(\varepsilon_2 \cdot \varepsilon_3) s u + \ldots \right],
\]

where we have not written explicitly terms of the form $(\varepsilon \cdot \varepsilon)(\varepsilon \cdot p)(\varepsilon \cdot p)$ and higher orders in $\alpha'$.

At one loop ($h = 1$) we keep the gluons off the mass shell, and Eq. (2.2) gives, for $M \geq 2$ transverse gluons,

\[
A_P^{(1)}(p_1, \ldots, p_M) = N \text{Tr}(\lambda^{a_1} \cdots \lambda^{a_M}) g_d^M \frac{M!}{(4\pi)^{d/2}} (2\alpha')^{(M-d)/2}(-1)^M \times \int_0^\infty d\tau e^{2\tau} \tau^{-d/2} \prod_{n=1}^{\infty} \left( 1 - e^{-2\pi \tau} \right)^{2-d} \int_0^\tau dv_M \int_0^{v_M} dv_{M-1} \ldots \int_0^{v_3} dv_2 \times \left\{ \prod_{i<j} \left[ \frac{z_i z_j}{V_i(0) V_j(0)} \exp(G(\nu_{ij})) \right]^{2\alpha' p_i \cdot p_j} \times \exp \left[ \sum_{i \neq j} \left( \sqrt{2\alpha'} p_j \cdot \varepsilon_i \partial_i G(\nu_{ij}) + \frac{1}{2} \varepsilon_i \cdot \varepsilon_j \partial_i \partial_j G(\nu_{ij}) \right) \right] \right\}_{m.l.},
\]

where $\nu_{ij} \equiv \nu_j - \nu_i$, $\partial_i \equiv \partial / \partial \nu_i$ and $\tau$ is related to the period $\tilde{\tau}$ of the annulus by the relation

\[
\tau = -i\pi \tilde{\tau}.
\]
Instead of the Koba-Nielsen variables $z_i$, we have used the real variables

$$\nu_i = -\frac{1}{2} \log z_i ,$$

while the Green function $G(\nu_{ij})$ is given by

$$G(\nu_{ji}) = \log \left[ -2\pi i \frac{\theta_1 \left( \frac{1}{\pi} (\nu_j - \nu_i) \frac{1}{\pi \tau} \right)}{\theta_1' \left( \frac{1}{\pi \tau} \right)} \right] - \frac{(\nu_j - \nu_i)^2}{\tau} ,$$

where $\theta_1$ is the first Jacobi $\theta$ function.

If we enforce the mass-shell condition $p_i^2 = 0$, any dependence on the local coordinates $V_i$'s drops out. However, in order to avoid infrared divergences, we will continue the gluon momenta off shell, in an appropriate way to be discussed later. Then, following Ref. [11], we will regard the freedom of choosing $V_i$ as a gauge freedom. We make the simple choice

$$V_i'(0) = z_i ,$$

which will lead, in the field theory limit, to the background field Feynman gauge. The conditions (2.9) and (3.10) are easily satisfied by choosing for example

$$V_i(z) = z_i z + z_i .$$

4. The two-gluon amplitude

The one-loop two-gluon amplitude is given by

$$A^{(1)}(p_1, p_2) = N \text{Tr}(\lambda^a \lambda^b) \frac{g^2}{(4\pi)^{d/2}} (2\alpha')^{2-d/2} \varepsilon_1 \cdot \varepsilon_2 p_1 \cdot p_2 R(p_1 \cdot p_2) ,$$

where

$$R(s) = \int_0^\infty d\tau e^{2\tau} \tau^{-d/2} \prod_{n=1}^\infty \left( 1 - e^{-2n\tau} \right)^{2-d} \int_0^\tau d\nu e^{2\alpha' s G(\nu)} [\partial_\nu G(\nu)]^2 .$$

Notice that if the two gluons are on mass shell, the two-gluon amplitude becomes ill defined, as the kinematical prefactor vanishes, while the integral diverges. In order to avoid this problem we keep the two gluons off shell.

To take the field theory limit, we must remember that the modular parameter $\tau$ and the coordinate $\nu$ are related to proper-time Schwinger parameters for the Feynman diagrams contributing to the two point function. In particular, $t \sim \alpha' \tau$ and $t_1 \sim \alpha' \nu$, where $t_1$ is the proper time associated with one of the two internal gluon propagators, while $t$ is the total proper time around the loop. In the field theory limit these proper times have to remain finite, and thus the limit $\alpha' \to 0$ must correspond to the limit $\{\tau, \nu\} \to \infty$ in the integrand. The field theory limit is then determined by the asymptotic behavior of the Green function for large $\tau$, namely

$$G(\nu, \tau) = -\frac{\nu^2}{\tau} + \log (2 \sinh(\nu)) - 4 e^{-2\tau} \sinh^2(\nu) + O(e^{-4\tau}) .$$
where \( \nu \) must also be taken to be large, so that \( \hat{\nu} \) remains finite; in this region, we may use

\[
G(\nu, \tau) \sim (\hat{\nu} - \hat{\nu}^2) \tau - e^{-2\hat{\nu} \tau} - e^{-2\tau (1-\hat{\nu})} + 2e^{-2\tau},
\]

so that

\[
\frac{\partial G}{\partial \nu} \sim 1 - 2\hat{\nu} + 2e^{-2\hat{\nu} \tau} - 2e^{-2\tau (1-\hat{\nu})}.\tag{4.5}
\]

We now substitute these results into Eq. (4.1), and keep only terms that remain finite when \( k = e^{-2\tau} \to 0 \). Divergent terms must be discarded by hand, since they correspond to the propagation of the tachyon in the loop. The next-to-leading term corresponds to gluon exchange, and while it is also divergent in the field theory limit, the corresponding divergence is regularized by dimensional regularization. Finally, higher order terms \( e^{-2n\tau} \) with \( n > 0 \) are vanishing in the field theory limit.

Notice that by taking the large \( \tau \) and \( \nu \) limit we have discarded two singular regions of integration that potentially contribute in the field theory limit, namely \( \nu \to 0 \) and \( \nu \to \tau \). In these regions (often referred to as “pinching” regions) the Green function has a logarithmic singularity corresponding to the insertion of the two external states very close to each other, and this singularity in general gives non-vanishing contributions in the field theory limit. However, in the case of the two gluon amplitude, these regions correspond to Feynman diagrams with a loop consisting of a single propagator, i.e. a “tadpole”. Massless tadpoles are defined to vanish in dimensional regularization, and thus we are justified in discarding these contributions as well.

Replacing the variable \( \nu \) with \( \hat{\nu} \equiv \nu/\tau \), Eq. (4.2) becomes

\[
R(s) = \int_0^\infty d\tau \int_0^1 d\hat{\nu} \tau^{1-d/2} e^{2\alpha' s (\hat{\nu} - \hat{\nu}^2) \tau} \left[ (1 - 2\hat{\nu})^2 (d - 2) - 8 \right], \tag{4.6}
\]

so that the integral is now elementary, and yields

\[
R(s) = -\Gamma \left( 2 - \frac{d}{2} \right) (-2\alpha' s)^{d/2 - 2} \frac{6 - 7d}{1 - d} B \left( \frac{d}{2} - 1, \frac{d}{2} - 1 \right), \tag{4.7}
\]

where \( B \) is the Euler beta function.

If we substitute Eq. (4.7) into Eq. (4.1), we see that the \( \alpha' \) dependence cancels, as it must. The ultraviolet finite string amplitude, Eq. (4.1), has been replaced by a field theory amplitude which diverges in four space-time dimensions, because of the pole in the \( \Gamma \) function in Eq. (4.7). Defining as usual a dimensionless coupling constant \( g_d = g \mu^2 \), with \( \mu \) an arbitrary mass scale, and having set \( d = 4 - 2\epsilon \), we find

\[
A^{(1)}(p_1, p_2) = -N_\epsilon \frac{g^2}{(4\pi)^2} \left( \frac{4\pi \mu^2}{-p_1 \cdot p_2} \right)^\epsilon \Gamma(\epsilon) \frac{11 - 7\epsilon}{3 - 2\epsilon} B(1 - \epsilon, 1 - \epsilon) A^{(0)}(p_1, p_2) \tag{4.8}
\]

Eq. (4.8)) is exactly equal to the gluon vacuum polarization of the \( SU(N) \) gauge field theory that one computes with the background field method, in
Feynman gauge, with dimensional regularization, provided we use for the tree-level two-gluon amplitude the expression

\[ A^{(0)}(p_1, p_2) = \delta^{ab} [\varepsilon_1 \cdot \varepsilon_2 p_1 \cdot p_2 - \varepsilon_1 \cdot p_2 \varepsilon_2 \cdot p_2] \quad (4.9) \]

Comparing Eq. (4.8) with the equation for \( \Gamma_2 \) in Eq. (1.1) we can extract the minimal subtraction wave function renormalization constant

\[ Z_A = 1 + N \frac{g^2}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} \quad (4.10) \]

While this result is what we expected, it relies on our prescription to continue the string amplitude off shell, and on our choice of the projective transformations \( V_i \). To make sure that our prescription is consistent we need to compute the three and four point renormalizations as well, and verify that gauge invariance is preserved.

5. An alternative computation of proper vertices

In the previous section we have computed the 1PI two-gluon amplitude and we have extracted the wave function renormalization constant. In this section we present an alternative method, introduced by Metsaev and Tseytlin [2]. This method isolates the 1PI part of the amplitude, and is thus particularly suited to the evaluation of renormalization constants. It is based on the following alternative expression for the bosonic Green function [14]

\[ G(\nu_i, \nu_j) = -\sum_{n=1}^{\infty} \frac{1 + q^{2n}}{n(1 - q^{2n})} \cos 2\pi n \left( \frac{\nu_j - \nu_i}{\tau} \right) + \ldots , \quad (5.1) \]

where \( q = e^{-\pi^2/\tau} \) and the dots stand for terms independent of \( \nu_i \) and \( \nu_j \), that will not be important in our discussion.

An important advantage of this approach is that, at least at one loop, it allows to integrate exactly over the punctures before the field theory limit is taken. The result does not present pinching singularities, that are regularized directly in the Green function. As a consequence, for the two gluon amplitude, we will get the same expression that we derived in Section 4, while for the three and four gluon amplitudes we will get only the contributions that do not include pinchings and are therefore one-particle irreducible.

As a first step, we rewrite the one-loop \( M \)-gluon planar amplitude as

\[ A_p^{(1)}(p_1, \ldots, p_M) = N \operatorname{Tr}(\lambda^{a_1} \cdots \lambda^{a_M}) \frac{g_d^M}{(4\pi)^{d/2}} (2\alpha')^{2-d/2} \times (-1)^M \int_0^\infty d\tau e^{2\tau} \tau^{-d/2} \prod_{n=1}^\infty \left( 1 - e^{-2n\tau} \right)^{2-d} I^{(1)}_M(\tau) , \quad (5.2) \]

where \( I^{(1)}_M(\tau) \) is the integral over the punctures \( \nu_i \), and can be read off from Eq. (3.6).
For $M = 2$, after a partial integration with vanishing surface term, we get

$$I_2^{(1)}(\tau) = p_1 \cdot p_2 \varepsilon_1 \cdot \varepsilon_2 \int_0^\tau d\nu (\partial_\nu G(\nu))^2 \left( e^{G(\nu)} \right)^{2\alpha' p_1 \cdot p_2} . \quad (5.3)$$

Since we are only interested in divergent renormalizations, and since the overall power of $\alpha'$ is already appropriate to the field theory limit, as it vanishes when $d \rightarrow 4$, we can now neglect the exponential, which would contribute $1 + O(\alpha')$. Using the expression in Eq. (5.1) for the Green function, we can easily perform exactly the integral over the puncture, and we get

$$I_2^{(1)}(\tau) = \frac{2\pi^2}{\tau} p_1 \cdot p_2 \varepsilon_1 \cdot \varepsilon_2 \sum_{n=1}^\infty \left( \frac{1 + q^{2n}}{1 - q^{2n}} \right)^2 , \quad (5.4)$$

so that we can write

$$A^{(1)}(p_1, p_2) = \frac{N}{2} \text{Tr} (\lambda^a \lambda^a) \frac{g_4^2}{(4\pi)^{d/2}} (2\alpha')^{2-d/2} p_1 \cdot p_2 \varepsilon_1 \cdot \varepsilon_2 Z(d)$$

$$= \frac{N}{4} \frac{g_4^2}{(4\pi)^{d/2}} (2\alpha')^{2-d/2} Z(d) A^{(0)}(p_1, p_2) . \quad (5.5)$$

Here

$$Z(d) \equiv (2\pi)^2 \int_0^\infty d\tau e^{2\tau} \tau^{-1-d/2} \prod_{n=1}^\infty \left( 1 - e^{-2n\tau} \right)^{2-d} \sum_{m=1}^\infty \left( \frac{1 + q^{2m}}{1 - q^{2m}} \right)^2$$

is the string integral that generates the renormalization constants as $\alpha' \rightarrow 0$.

With three gluons we get

$$I_3^{(1)}(\tau) = \int_0^\tau d\nu_3 \int_0^{\nu_3} d\nu_2 \{ \varepsilon_1 \cdot \varepsilon_2 \partial_1 \partial_2 G(\nu_{21})$$

$$\times [ p_1 \cdot \varepsilon_3 \partial_3 G(\nu_{31}) + p_2 \cdot \varepsilon_3 \partial_3 G(\nu_{32}) ] + \ldots \} , \quad (5.7)$$

where terms needed for cyclic symmetry and terms of order $\alpha'$ are not written explicitly, and we discarded the exponentials of the Green functions, that are not contributing since the external gluons are on shell.

The integrals over $\nu_2$ and $\nu_3$ can be done by using the expression in Eq. (5.1) for the Green function. The result is

$$I_3^{(1)}(\tau) = \frac{(2\pi)^2}{\tau} [ \varepsilon_1 \cdot \varepsilon_2 p_2 \cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 p_3 \cdot \varepsilon_1 + \varepsilon_1 \cdot \varepsilon_3 p_1 \cdot \varepsilon_2 ]$$

$$\times \sum_{n=1}^\infty \left( \frac{1 + q^{2n}}{1 - q^{2n}} \right)^2 + O(\alpha') , \quad (5.8)$$

so that the three gluon amplitude is given by

$$A^{(1)}(p_1, p_2, p_3) = \frac{N}{4} \frac{g_4^2}{(4\pi)^{d/2}} (2\alpha')^{2-d/2} Z(d) A^{(0)}(p_1, p_2, p_3) + O(\alpha') . \quad (5.9)$$
Finally, the same calculation can be done for the four-gluon amplitude, where we can concentrate on the terms whose kinematical prefactor has no powers of the external momenta (and thus is of the form $\varepsilon_1 \cdot \varepsilon_j \varepsilon_k \cdot \varepsilon_k$). Other terms are suppressed by powers of $\alpha'$. Then we need to consider the expression

$$I_4^{(1)}(\tau) = \int_0^\tau d\nu_4 \int_0^{\nu_4} d\nu_2 \int_0^{\nu_2} d\nu_3 \left[ \varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4 \partial_1 \partial_2 G(\nu_21) \partial_3 \partial_4 G(\nu_43) \right.$$

$$+ \varepsilon_1 \cdot \varepsilon_3 \varepsilon_2 \cdot \varepsilon_4 \partial_1 \partial_3 G(\nu_31) \partial_2 \partial_4 G(\nu_42)$$

$$+ \varepsilon_1 \cdot \varepsilon_4 \varepsilon_3 \cdot \varepsilon_2 \partial_1 \partial_4 G(\nu_41) \partial_3 \partial_2 G(\nu_32) \right]. \quad (5.10)$$

Using again Eq. (5.1), we can perform the integrals over the punctures, and we get

$$I_4^{(1)}(\tau) = \frac{(2\pi)^2}{\tau} \sum_{n=1}^{\infty} \left( \frac{1 + q^{2n}}{1 - q^{2n}} \right)^2$$

$$\times \left[ \varepsilon_1 \cdot \varepsilon_3 \varepsilon_2 \cdot \varepsilon_4 - \frac{1}{2} \varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4 - \frac{1}{2} \varepsilon_2 \cdot \varepsilon_3 \varepsilon_1 \cdot \varepsilon_4 \right]. \quad (5.11)$$

The amplitude becomes then

$$A^{(1)}(p_1, p_2, p_3, p_4) = \frac{N}{4} \frac{g_4^2}{(4\pi)^{d/2}} (2\alpha')^{2-d/2} Z(d) A^{(0)}(p_1, p_2, p_3, p_4) + O(\alpha') \ , \quad (5.12)$$

where the 1PI part of the four-gluon amplitude at tree level is given by

$$A^{(0)}(p_1, p_2, p_3, p_4) = 4 g_4^2 \text{Tr}(\lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4})$$

$$\times \left[ \varepsilon_1 \cdot \varepsilon_3 \varepsilon_2 \cdot \varepsilon_4 - \frac{1}{2} \varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4 - \frac{1}{2} \varepsilon_2 \cdot \varepsilon_3 \varepsilon_1 \cdot \varepsilon_4 \right]. \quad (5.13)$$

Defining the factor

$$K(d) = \frac{N}{4} \frac{g_4^2}{(4\pi)^{d/2}} (2\alpha')^{2-d/2} Z(d) \ , \quad (5.14)$$

we can now perform the limit $\alpha' \to 0$, keeping the ultraviolet cutoff $\epsilon \equiv 2 - d/2$ small but positive, and eliminating by hand the tachyon contribution. The calculation of the integral $Z(d)$ in this limit is described in detail in Ref. [11]. The result is

$$K(4 - 2\epsilon) \to -\frac{11}{3} N \frac{g_4^2}{(4\pi)^2} \frac{1}{\epsilon} + O(\epsilon^0) \ . \quad (5.15)$$

If we finally compare Eqs. (1.1) with Eqs. (5.5), (5.9) and (5.12) we can determine the renormalization constants. They are given by

$$Z_A = Z_3 = Z_4 = 1 + \frac{11}{3} N \frac{g_4^2}{(4\pi)^2} \frac{1}{\epsilon} \ , \quad (5.16)$$

in agreement with the result of the previous section for $Z_A$, and as dictated by the background field method Ward identities.
6. The three-gluon amplitude

The methods described in the previous two sections are both adequate to compute one-particle irreducible contributions to the Green functions. Reducible diagrams, on the other hand, correspond to regions in moduli space where the gluons are inserted on the string world sheet very close to each other (pinching regions). These regions were excluded by hand in Section 5, since the corresponding logarithmic singularity in the world-sheet Green function was regularized by a \( \zeta \)-function regularization [11]. If we wish to include them along the lines of Section 4, we have to perform the field theory limit in a slightly different way. To see this, let us consider the simplest case in which these contributions arise, namely the three-gluon amplitude.

The one-loop correction to Eq. (3.4) can be written as

\[
A^{(1)}(p_1, p_2, p_3) = -N \text{Tr}(\lambda^a \lambda^b \lambda^c) \frac{g_d^3}{(4\pi)^{d/2}} (2\alpha')^{2-d/2} \times \int_0^\infty D\tau \int_0^\tau d\nu_3 \int_0^{\nu_3} d\nu_2 f_3(\nu_2, \nu_3, \tau) ,
\]

where

\[
f_3(\nu_2, \nu_3, \tau) \equiv e^{2\alpha' p_1 \cdot p_2 G(\nu_2)} e^{2\alpha' p_2 \cdot p_3 G(\nu_3)} e^{2\alpha' p_3 \cdot p_1 G(\nu_3)} \times \left\{ -\varepsilon_1 \cdot \varepsilon_2 \partial_2^2 G(\nu_2) (p_1 \cdot \varepsilon_3 \partial_3 G(\nu_3) + p_2 \cdot \varepsilon_3 \partial_3 G(\nu_3)) + \varepsilon_2 \cdot \varepsilon_3 \partial_3^2 G(\nu_2) (p_2 \cdot \varepsilon_1 \partial_1 G(\nu_2) + p_3 \cdot \varepsilon_1 \partial_1 G(\nu_3)) + \varepsilon_1 \cdot \varepsilon_3 \partial_3^2 G(\nu_3) (p_3 \cdot \varepsilon_2 \partial_2 G(\nu_3) - p_1 \cdot \varepsilon_2 \partial_2 G(\nu_2)) \right\} + O(\alpha'),
\]

and

\[
D\tau \equiv d\tau \ e^{2\tau} \tau^{-d/2} \prod_{n=1}^\infty \left( 1 - e^{-2n\tau} \right)^{2-d}.
\]

One-particle irreducible contributions can be calculated along the lines of Section 4, expanding the bosonic world-sheet Green function for large values of \( \tau \) as in Eq. (4.3). The calculation is described in some detail in Ref. [11], and gives

\[
A^{(1)}(p_1, p_2, p_3) \bigg|_{1\text{PI}} = \left( -\frac{11}{3} \right) N \left( \frac{g}{4\pi} \right)^2 \frac{1}{\epsilon} A^{(0)}(p_1, p_2, p_3) + O(\epsilon^0) ,
\]

which agrees with the results of Section 5.

Next, we turn to the analysis of the pinching regions. There are clearly three such regions, corresponding to \( \nu_2 \to 0, \nu_2 \to \nu_3 \) and \( \nu_3 \to \tau \), as dictated by cyclic symmetry and periodicity on the annulus.

Let us consider, for example, the first region, \( \nu_2 \to 0 \). Since this pinching contribution is localized in a neighbourhood of 0, we can replace the integral \( \int_0^{\nu_3} d\nu_2 \) with an integral \( \int_0^\eta d\nu_2 \), where \( \eta \) is an arbitrary small number.
Further, we can use for the bosonic Green function the approximation
\[ G(\nu) \sim \log(2\nu) \quad . \] (6.5)

After this is done, in \( f_3(\nu_2, \nu_3, \tau) \) we can expand \( G(\nu_2) \) in powers of \( \nu_2 \), which turns the amplitude \( A^{(1)}(p_1, p_2, p_3) \) into an infinite series. The \( n \)-th term of this series is proportional to an integral of the form
\[ C_n \equiv \int_0^\eta d\nu_2 \, \nu_2^{n-2+2\alpha' p_1 \cdot p_2} \quad , \] (6.6)
with \( n \geq 0 \). After a suitable analytic continuation in the momenta to insure convergence, we get
\[ C_n = \frac{\eta^{n-1+2\alpha' p_1 \cdot p_2}}{n-1+2\alpha' p_1 \cdot p_2} \quad . \] (6.7)

We see that, when the pinching \( \nu_2 \to 0 \) is performed, the amplitude becomes an infinite sum over all possible string states that are exchanged in the \( (12) \)-channel, \( n = 0 \) corresponding to the tachyon, \( n = 1 \) to the gluon and so on.

In the case of the three-gluon amplitude, one can verify that the exchange of a tachyon does not give any contribution: in fact the coefficient of the quadratic divergence \( \frac{1}{\nu_2^2} \) is zero because of the transversality of the externals states. The gluon contribution, on the other hand, survives in the field theory limit, and contributes to the ultraviolet divergence, as expected: the single pole in \( \nu_2 \) in fact generates, through the change of variable to \( \tilde{\nu}_2 \), the negative power of \( \tau \) needed for the integral to diverge. All other terms in the series, corresponding to \( n \geq 2 \), and to states whose mass becomes infinite as \( \alpha' \to 0 \), vanish in the field theory limit. Notice also that for \( n = 1 \) the dependence on the cutoff \( \eta \) in Eq. (6.7) disappears as \( \alpha' \to 0 \).

Keeping this in mind, and collecting all relevant factors, we find that the pinching contribution to the three gluon amplitude that we are considering is
\[ A^{(1)}(p_1, p_2, p_3) \big|_{\nu_2 \to 0} = -N \text{Tr}(\lambda^a \lambda^b \lambda^c) \frac{g_3^2}{(4\pi)^{d/2}} (2\alpha')^{2-d/2} \] (6.8)
\[ \times \frac{(p_1 + p_2) \cdot p_3}{p_1 \cdot p_2} \quad R \left[ (p_1 + p_2) \cdot p_3 \right] \varepsilon_1 \cdot \varepsilon_2 p_2 \cdot \varepsilon_3 \quad , \]
where \( R \) is the integral defined in Eq. (4.2). Notice that Eq. (6.8) contains a ratio of momentum invariants which are vanishing on shell. The appearance of such ratios in string amplitudes, in the corners of moduli space corresponding to loops isolated on external legs, is a well-known fact, which for example motivated the work of Ref. [15]. As we already remarked in Ref. [10], this \( \frac{0}{0} \) ambiguity is similar to the one that appears in the unrenormalized connected Green functions of a massless field theory, if the external legs are kept on the mass-shell and divergences are regularized with dimensional regularization. Our prescription to deal with this ambiguity is to continue off shell the momentum of the gluon attached to the loop, according to
\[ p_3^2 = (p_1 + p_2)^2 = m^2 \quad . \] (6.9)
The other gluon momenta, $p_1$ and $p_2$, on the other hand, are kept on shell. Here we rely on the assumption, substantiated by the results obtained so far, that string amplitudes lead to field theory amplitudes calculated with the background field method. As was shown in Ref. [16], $S$-matrix elements are obtained in this method by first calculating one-particle irreducible vertices to the desired order, and then gluing them together with propagators that are defined by fixing the gauge for the background field. This leads us to interpret Eq. (6.8) as a one-loop, one-particle irreducible two point function, whose momentum must be continued off shell according to Eq. (6.9), glued to a tree-level three point vertex, for which no off-shell continuation is necessary. We thus keep $p_1^2 = p_2^2 = 0$, which, using momentum conservation, implies

$$p_1 \cdot p_2 = \frac{m^2}{2}. \tag{6.10}$$

Then, comparing Eq. (6.8) with Eq. (3.4), and including a factor of three to account for the three pinching regions, we can write

$$A^{(1)}(p_1, p_2, p_3) \bigg|_{\text{pinch.}} = -\frac{3}{2} N \frac{g_2^2}{(4\pi)^{d/2}} (2\alpha')^{2-d/2} R(-m^2) A^{(0)}(p_1, p_2, p_3). \tag{6.11}$$

Extracting the ultraviolet divergence of Eq. (6.11), and adding Eq. (6.4), we find the total divergence of the unrenormalized, connected, three gluon Green function,

$$A^{(1)}(p_1, p_2, p_3) \bigg|_{\text{div}} = 2 \left(\frac{11}{3}\right) N \left(\frac{g}{4\pi}\right)^2 \frac{1}{\epsilon} A^{(0)}(p_1, p_2, p_3), \tag{6.12}$$

which leads again to the background field Ward identity (see Eq. (1.2))

$$Z_3 = Z_A = 1 + N \left(\frac{g}{4\pi}\right)^2 \frac{11}{3} \frac{1}{\epsilon}. \tag{6.13}$$

The same analysis can be carried out for the four-point amplitude, as described in Ref. [11], and no surprises arise.

7. Concluding remarks

We have shown that it is possible to calculate renormalization constants in Yang-Mills theories using the simplest of string theories, the open bosonic string. To do so it is necessary to continue off shell some of the external momenta, but this can be done consistently in the field theory limit, and the results coincide with the ones obtained using the background field method and dimensional regularization. Since bosonic string amplitudes are well understood at all orders in perturbation theory, this technique may be useful beyond one loop.
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