HERMITIAN SOLUTIONS TO THE EQUATION $AXA^* + BYB^* = C$, FOR HILBERT SPACE OPERATORS

Amina Boussaid and Farida Lombarkia

Faculty of Mathematics and informatics, Department of Mathematics, University of Batna 2, 05078, Batna, Algeria

Abstract. In this paper, by using generalized inverses we have given some necessary and sufficient conditions for the existence of solutions and Hermitian solutions to some operator equations, and derived a new representation of the general solutions to these operator equations. As a consequence, we have obtained a well-known result of Dajić and Koliha.

Keywords: Hilbert space, operator equations, inner inverse, Hermitian solution.

1. Introduction and basic definitions

Let $H$ and $K$ be infinite complex Hilbert spaces, and $\mathbb{B}(H, K)$ the set of all bounded linear operators from $H$ to $K$. Throughout this paper, the range and the null space of $A \in \mathbb{B}(H, K)$ are denoted by $\mathcal{R}(A)$ and $\mathcal{N}(A)$ respectively. An operator $B \in \mathbb{B}(K, H)$ is said to be the inner inverse of $A \in \mathbb{B}(H, K)$ if it satisfies the equation $ABA = A$, we denote the inner inverse by $A^{-}$. An operator $A$ is called regular if $A^{-}$ exists. It is well known that $A \in \mathbb{B}(H, K)$ is regular if and only if $A$ has closed range. There are many papers in which the basic aim is to find necessary and sufficient conditions for the existence of a solution or Hermitian solution to some matrix or operator equations using generalized inverses. In [15, 16, 18], Mitra and Navarra et al. established necessary and sufficient conditions for the existence of a common solution and gave a representation of the general common solution to the pair of matrix equations

(1.1) $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$. 

Received November 11, 2019; accepted January 7, 2021.
Corresponding Author: Amina Boussaid, Faculty of Mathematics and informatics, Department of Mathematics, University of Batna 2, 05078, Batna, Algeria | E-mail: boussaid1990@gmail.com

2010 Mathematics Subject Classification. 47A05; 47A62; 15A09

© 2021 by University of Niš, Serbia | Creative Commons License: CC BY-NC-ND
In [23], Wang considered the same problem for matrices over regular rings with identity. Furthermore, in [13, 16] Khatri and Mitra determined the conditions for the existence of a Hermitian solution and gave the expression of the general Hermitian solution to the matrix equation

\[(1.2) \quad AXB = C,\]

In [8] J. Groß gave the general Hermitian solution to matrix equation (1.2), where \(B = A^*\).

Quaternion matrix equations and its general Hermitian solutions have attracted more attention in recent years. The reason for this is a large number of applications in control theory and many other fields, see [9, 10, 11, 12, 14, 24] and the references therein. Among them, the matrix equation

\[(1.3) \quad AXA^* + BYB^* = C,\]

has been studied by Chang and Wang in [1]. They used the generalized singular value decomposition to find necessary and sufficient conditions for the existence of real symmetric solutions. Also in [27, Corollary 3.1], Xu et al found necessary and sufficient conditions for the equation (1.3) to have a Hermitian solution.

Recently several operator equations have been extended from matrices to infinite dimensional Hilbert space, Banach space and Hilbert \(C^\ast\)-modules, see [3, 4, 21], [6, 17, 22, 25, 26] and the references therein.

In this paper, our main objective is to give necessary and sufficient conditions for the existence of a Hermitian solution to the operator equation \(AXA^* + BYB^* = C\). After section one where several basic definitions are assembled, in section 2, we give necessary and sufficient conditions for the existence of a common solution to the operator equations

\[A_1XB_1 = C_1 \text{ and } A_2XB_2 = C_2.\]

In section 3, we apply the result of section 2 to determine new necessary and sufficient conditions for the existence of a Hermitian solution and give a representation of the general Hermitian solution to the operator equation \(AXB = C\). Finally, we give some necessary and sufficient condition for the existence of a Hermitian solution to the operator equation \(AXA^* + BYB^* = C\).

2. Common solutions to the operator equations \(A_1XB_1 = C\) and \(A_2XB_2 = C_2\)

In this section, we give necessary and sufficient conditions for the existence of a common solution to the pair of equations

\[A_1XB_1 = C_1, \quad A_2XB_2 = C_2,\]

with \(A_1, A_2, B_1, B_2, C_1\) and \(C_2\) are linear bounded operators defined on Hilbert spaces \(H, K, E, L, N\) and \(G\). Before enunciating our main results, we recall the following lemmas
Lemma 2.1. [2] Let $A, B \in \mathcal{B}(H, K)$ are regular operators and $C \in \mathcal{B}(H, K)$. Then the operator equation

$$AXB = C$$

has a solution if and only if $AA^{-}CB^{-}B = C$, or equivalently

$$\mathcal{R}(C) \subset \mathcal{R}(A) \text{ and } \mathcal{R}(C^*) \subset \mathcal{R}(B^*).$$

A representation of the general solution is

$$X = A^{-}CB^{-} + U - A^{-}AUBB^{-},$$

where $U \in \mathcal{B}(K, H)$ is an arbitrary operator.

Lemma 2.2. [2] Let $A, B \in \mathcal{B}(H, K)$ are regular operators and $C, D \in \mathcal{B}(H, K)$. Then the pair of operators equations

$$AX = C \quad \text{and} \quad XB = D$$

has a common solution if and only if

$$AA^{-}C = C, \quad DB^{-}B = D \quad \text{and} \quad AD = CB,$$

or equivalently

$$\mathcal{R}(C) \subset \mathcal{R}(A), \quad \mathcal{R}(D^*) \subset \mathcal{R}(B^*) \quad \text{and} \quad AD = CB.$$

A representation of the general solution is

$$X = A^{-}C + DB^{-} - A^{-}ADB + (I_H - A^{-}A)V(I_H - BB^{-}),$$

where $V \in \mathcal{B}(H)$ is an arbitrary operator.

The following two lemmas can be deduced from a result of Patrício and Puystjens [20] originally formulated for matrix with entries in an associative ring. A simple modification shows that it applies equally well to Hilbert space operators.

Lemma 2.3. [20] Let $A \in \mathcal{B}(H, K)$ and $B \in \mathcal{B}(E, K)$ be regular operators. Then $(A \ B) \in \mathcal{B}(H \times E, K)$ is regular if and only if $S = (I_K - AA^{-})B$ is regular. In this case, the inner inverse of $(A \ B)$ is given by

$$(A \ B)^{-} = \left( \begin{array}{c} A^{-} - A^{-}BS^{-}(I_K - AA^{-}) \\ S^{-}(I_K - AA^{-}) \end{array} \right).$$

Lemma 2.4. [3] Let $A \in \mathcal{B}(H, K)$ and $B \in \mathcal{B}(H, E)$ be regular operators. Then the regularity of any one of the following operators implies the regularity of the remaining three operators

$$X = A^{-}C + DB^{-} - A^{-}ADB + (I_H - A^{-}A)V(I_H - BB^{-}),$$

where $V \in \mathcal{B}(H)$ is an arbitrary operator.
A. Boussaid and F. Lombarkia

\[ D = B(I_H - A^*A), \quad M = A(I_H - B^*B), \quad \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B \\ A \end{pmatrix}. \]

In this case, the inner inverse of \( \begin{pmatrix} A \\ B \end{pmatrix} \) is given by

\[ \left( \begin{pmatrix} A \\ B \end{pmatrix} \right)^{-1} = (I_H - B^*B)M^* \quad B^* - (I_H - B^*B)M^*AB^* \]

**Lemma 2.5.** [2] Suppose that \( A_1 \in \mathbb{B}(H, K), \quad A_2 \in \mathbb{B}(H, E), \quad B_1 \in \mathbb{B}(L, G), \quad B_2 \in \mathbb{B}(N, G), \quad S_1 = A_2(I_H - A_1^*A_1) \quad \text{and} \quad M_1 = (I_G - B_1B_1^*)B_2 \) are regular operators. Then

\[ T_1 = (I_E - S_1S_1^*)A_2A_1^* \quad \text{and} \quad D_1 = B_1B_2(I_N - M_1^*M_1), \]

are regular with inner inverses \( T_1^* = A_1A_2^* \) and \( D_1^* = B_2^*B_1 \).

In the following theorem, we give necessary and sufficient conditions for the existence of a common solution of the operator equations

\[ A_1XB_1 = C_1, \quad A_2XB_2 = C_2 \]

**Theorem 2.1.** Suppose that \( A_1 \in \mathbb{B}(H, K), \quad A_2 \in \mathbb{B}(H, E), \quad B_1 \in \mathbb{B}(L, G), \quad B_2 \in \mathbb{B}(N, G), \quad M_1 = (I_G - B_1B_1^*)B_2 \) and \( S_1 = A_2(I_H - A_1^*A_1) \) are regular operators and \( C_1 \in \mathbb{B}(L, K), \quad C_2 \in \mathbb{B}(N, E). \) Then the following statements are equivalent

1. The pair of equations (1.1) have a common solution \( X \in \mathbb{B}(G, H). \)
2. There exists two operators \( U \in \mathbb{B}(N, K) \) and \( V \in \mathbb{B}(L, E) \), such that the operator equation \( AXB = C \) is solvable, where

\[ A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\ V \end{pmatrix}, \]

3. For \( i = 1, 2 \), \( \mathcal{R}(C_i) \subset \mathcal{R}(A_i), \quad \mathcal{R}(C_i^*) \subset \mathcal{R}(B_i^*) \) and

\[ T_1C_1D_1 = T_2C_2D_2, \]

where \( T_1 = (I_E - S_1S_1^*)A_2A_1^*, \quad T_2 = (I_E - S_1S_1^*), \quad D_1 = B_1B_2(I_N - M_1^*M_1) \)

and \( D_2 = (I_N - M_1^*M_1). \)

**Proof.**

(1) \( \Leftrightarrow \) (2) The equivalence is easily established.

(2) \( \Rightarrow \) (3) According to Lemma 2.1, the operator equation \( AXB = C \) has a solution if and only if

\[ \mathcal{R}(C) \subset \mathcal{R}(A) \quad \text{and} \quad \mathcal{R}(C^*) \subset \mathcal{R}(B^*), \]
Hermitian Solutions to the Equation $AXA^* + BYB^* = C$

then, we deduce that

\begin{equation}
(2.1) \quad \text{for } i = 1, 2, \quad \mathcal{R}(C_i) \subset \mathcal{R}(A_i) \quad \text{and} \quad \mathcal{R}(C_i^*) \subset \mathcal{R}(B_i^*).
\end{equation}

On the other hand, we have

\begin{equation}
T_1 C_1 D_1 = (I_E - S_1 S_1^*) A_2 A_1 C_1 B_1^* B_2 (I_N - M_1^* M_1)
\end{equation}

\begin{equation}
= (I_E - S_1 S_1^*) A_2 A_1^* A_1 X_0 B_1 B_1^* B_2 (I_N - M_1^* M_1),
\end{equation}

where $X_0$ is the common solution to the pair of equations (1.1).

Let

\begin{equation}
S_1 = A_2 (I_H - A_1^* A_1) \quad \text{and} \quad M_1 = (I_G - B_1 B_1^*) B_2.
\end{equation}

This implies that

\begin{equation}
A_2 A_1^* A_1 = A_2 - S_1 \quad \text{and} \quad B_1 B_1^* B_2 = B_2 - M_1.
\end{equation}

We insert (2.3) in (2.2) to obtain

\begin{equation}
T_1 C_1 D_1 = T_2 C_2 D_2.
\end{equation}

From (2.1) and (2.4), we deduce that (2) $\Rightarrow$ (3).

Conversely, since

\begin{equation}
T_1 C_1 D_1 = T_2 C_2 D_2.
\end{equation}

Then

\begin{equation}
\mathcal{R}(T_2 C_2) \subset \mathcal{R}(T_1) \quad \text{and} \quad \mathcal{R}(D_1^* C_1^*) \subset \mathcal{R}(D_2^*).
\end{equation}

By applying Lemma 2.2, there exist $U \in \mathcal{B}(N, K)$ which is the common solution to the pair of equations

\begin{equation}
\begin{cases}
T_1 U = T_2 C_2 \\
UD_2 = C_1 D_1,
\end{cases}
\end{equation}

given by

\begin{equation}
U = C_1 D_1 + T_1^* (I_E - S_1 S_1^*) C_2 M_1^* M_1 + (A_1 A_1^* - T_1^* T_1) Z M_1^* M_1,
\end{equation}

where $Z \in \mathcal{B}(N, K)$ is an arbitrary operator.

On other hand, since

\begin{equation}
T_1 C_1 D_1 = T_2 C_2 D_2.
\end{equation}

Then

\begin{equation}
\mathcal{R}(T_1 C_1) \subset \mathcal{R}(T_2) \quad \text{and} \quad \mathcal{R}(D_2^* C_2^*) \subset \mathcal{R}(D_1^*).
\end{equation}

It follows from Lemma 2.2 that there exist $V \in \mathcal{B}(L, E)$ which is the common solution to the pair of equations

\begin{equation}
\begin{cases}
T_2 V = T_1 C_1 \\
VD_1 = C_2 D_2,
\end{cases}
\end{equation}
given by

\begin{equation}
(2.8) \quad V = T_1C_1 + S_1S_1^i C_2 (I_N - M_1^- M_1) D_1^- + S_1 S_1^i Z' (B_1^- B_1 - D_1 D_1^-),
\end{equation}

where $Z' \in \mathbb{B}(L, E)$ is an arbitrary operator.

Thus, there exists $U \in \mathbb{B}(N, K)$ and $V \in \mathbb{B}(L, E)$ solutions to the pair of equations (2.5), (2.7) and as for $i = 1, 2$, we have $A_i A_i^- C_i = C_i$ and $C_i B_i^- B_i = C_i$, we obtain

$$AA^- CB^- B =$$

$$= \begin{pmatrix} A_1 A_1^- C_1 B_1^- B_1 & A_1 A_1^- (C_1 D_1 + UM_1^- M_1) \\ (T_1 C_1 + S_1 S_1^i V) B_1^- B_1 & T_1 (C_1 D_1 + U M_1^- M_1) + S_1 S_1^i (V D_1 + C_2 M_1 M_1^-) \end{pmatrix}$$

$$= C.$$

So that, the operator equation $AXB = C$ is solvable and (3) $\Rightarrow$ (2). $\blacksquare$

**Theorem 2.2.** Suppose that $A_1 \in \mathbb{B}(H, K)$, $A_2 \in \mathbb{B}(H, E)$, $B_1 \in \mathbb{B}(L, G)$, $B_2 \in \mathbb{B}(N, G)$, $M_1 = (I_G - B_1 B_1^-) B_2$ and $S_1 = A_2 (I_H - A_1^- A_1)$ are regular operators and $C_i \in \mathbb{B}(L, K)$, $C_2 \in \mathbb{B}(N, E)$, when any one of the conditions (2), (3) of Theorem 2.1 holds, a general common solution to the pair of equations (1.1) is given by

\begin{equation}
X = (A_1^- C_1 + (I_H - A_1^- A_1) S_1^- (V - A_2 A_1^- C_1)) B_1^- (I_G - B_2 M_1^- (I_G - B_1 B_1^-))
+ (A_1^- U + (I_H - A_1^- A_1) S_1^- (C_2 - A_2 A_1^- U)) M_1^- (I_G - B_1 B_1^-) + F (2.9) - (A_1^- A_1 + (I_H - A_1^- A_1) S_1^- S_1) F (B_1 B_1^- + M_1 M_1^- (I_G - B_1 B_1^-)),
\end{equation}

where $F \in \mathbb{B}(G, H)$ is an arbitrary operator and $U, V$ are given by

\begin{align*}
U &= C_1 B_1^- B_2 (I_N - M_1^- M_1) + A_1 A_2^- (I_E - S_1 S_1^-) C_2 M_1^- M_1 + A_1 A_2^- Z M_1^- M_1 \\
&\quad - A_1 A_2^- (I_E - S_1 S_1^-) A_2 A_1^- Z M_1^- M_1,
\end{align*}
and

\begin{align*}
V &= (I_E - S_1 S_1^-) A_2 A_1^- C_1 + S_1 S_1^- C_2 (I_N - M_1^- M_1) B_2^- B_1 + S_1 S_1^- Z' B_1^- B_1 \\
&\quad - S_1 S_1^- Z' B_1^- B_2 (I_N - M_1^- M_1) B_2^- B_1,
\end{align*}

where $Z \in \mathbb{B}(N, K)$ and $Z' \in \mathbb{B}(L, E)$ are arbitrary operators.

**Proof.** From Theorem 2.1, we get that the pair of equations (1.1) has a common solution equivalently the two conditions (2) and (3) holds.

On the other hand, since the pair of equations (1.1) is equivalent to

\begin{equation}
(2.10) \quad \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} X (B_1 \ B_2) = \begin{pmatrix} C_1 \ U \\ V \ C_2 \end{pmatrix},
\end{equation}

According to Lemma 2.3 and Lemma 2.4, we have

\begin{align*}
\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \in \mathbb{B}(H, K \times E) \quad \text{and} \quad \begin{pmatrix} B_1 \ B_2 \end{pmatrix} \in \mathbb{B}(L \times N, G)
\end{align*}
are regular with inner inverses

\[(2.11) \quad \left( \begin{array}{c} A_1 \\ A_2 \end{array} \right)^- = \left( (I_E - A_2^* A_2)S_1^* - (I_E - A_2^* A_2)S_1 A_1 A_2^* \right), \]

and

\[(2.12) \quad \left( \begin{array}{cc} B_1 & B_2 \end{array} \right)^- = \left( \begin{array}{c} B_1^* - B_1 M_1^* (I_G - B_1 B_1^*) \\ M_1^* (I_G - B_1 B_1^*) \end{array} \right), \]

respectively.

Using Lemma 2.1, we deduce that the general solution of (2.10) is given by

\[(2.13) \quad X = \left( A_1 \atop A_2 \right)^- \left( \begin{array}{cc} C_1 & U \\ V & C_2 \end{array} \right) \left( \begin{array}{cc} B_1 & B_2 \end{array} \right)^- + F \left( \begin{array}{cc} A_1 \\ A_2 \end{array} \right)^- \left( \begin{array}{cc} A_1 \\ A_2 \end{array} \right) F \left( \begin{array}{cc} B_1 & B_2 \end{array} \right)^-.

By substituting (2.11) and (2.12) in (2.13), we get the solution \(X\) as defined in (2.9) such that \(U, V\) are given in (2.6) and (2.8) respectively and \(F \in \mathcal{B}(G, H)\) is an arbitrary operator. □

3. Hermitian solutions to the operator equations \(AXB = C\) and \(AXA^* + BYB^* = C\)

Based on Theorem 2.1 and Theorem 2.2, in this section we give necessary and sufficient conditions for the existence of Hermitian solutions to the operator equations

\[AXB = C \quad \text{and} \quad AXA^* + BYB^* = C \]

and obtain the general Hermitian solution to those operator equations respectively. Before enunciating our main results we have the following lemma

Lemma 3.1. Let \(A \in \mathcal{B}(H, K)\) and \(B \in \mathcal{B}(K, H)\), such that \(A, B, S_1 = B^*(I_H - A^* A)\) and \(M_1 = (I_H - BB^*)A^*\) are regular. Then the operator equation

\[AXB = C,\]

has a Hermitian solution if and only if the pair of operator equations

\[(3.1) \quad AXB = C \quad \text{and} \quad B^*XA^* = C^* \]

has a common solution, a representation of the general Hermitian solution to \(AXB = C\) is of the form

\[X_H = \frac{X + X^*}{2}, \]

where \(X\) is the representation of the general common solution to the equations (3.1).
Proof. From Theorem 2.1 the pair of operator equations (3.1) has a common solution if and only if
\[ \mathcal{R}(C) \subset \mathcal{R}(A) \quad \text{and} \quad \mathcal{R}(C^*) \subset \mathcal{R}(B^*), \]
and
\[ (I_K - S_1 S_1^*) B^* A^- C B^- A^* (I_K - M_1^- M_1) = (I_K - S_1 S_1^*) C^* (I_K - M_1^- M_1). \]
A representation of the general common solution to equations (3.1) is given by (2.9) in Theorem 2.2, where \( A_1 = A, B_1 = B, C_1 = C, A_2 = B^*, B_2 = A^* \) and \( C_2 = C^* \). Clearly \( X_H \) is a Hermitian solution to (1.2). \( \Box \)

From the above proof and Theorem 2.2, we obtain the following corollary.

**Corollary 3.1.** Let \( A \in \mathbb{B}(H, K), B \in \mathbb{B}(K, H), M_1 = (I_H - BB^-) A^* \) and \( S_1 = B^* (I_H - A^- A) \) are regular operators and \( C \in \mathbb{B}(K) \). Then the operator equation
\[ AXB = C, \]
has a Hermitian solution if and only if
\[ 1. \mathcal{R}(C) \subset \mathcal{R}(A) \quad \text{and} \quad \mathcal{R}(C^*) \subset \mathcal{R}(B^*) \]
\[ 2. (I_K - S_1 S_1^*) B^* A^- C B^- A^* (I_K - M_1^- M_1) = (I_K - S_1 S_1^*) C^* (I_K - M_1^- M_1). \]

In this case, a representation of the general Hermitian solution is of the form
\[ X_H = \frac{X + X^*}{2}, \]
where
\[
X = (A^- C + (I_H - A^- A) S_1^* (V - B^* A^- C)) B^- (I_H - A^* M_1^- (I_H - BB^-)) \\
+ (A^- U + (I_H - A^- A) S_1^* (C^* - B^* A^- U)) M_1^- (I_H - BB^-) + F
\]
and
\[
(3.2) \quad (A^- A + (I_H - A^- A) S_1^* S_1) F (BB^- + M_1 M_1^- (I_H - BB^-)),
\]
where \( F \in \mathbb{B}(H) \) is an arbitrary operator and \( U, V \) are given by
\[
\begin{align*}
U &= CB^- A^* (I_K - M_1^- M_1) + A (B^*)^- (I_K - S_1 S_1^-) C^* M_1^* M_1 + AA^- Z M_1^- M_1 \\
&\quad - A (B^*)^- (I_K - S_1 S_1^-) B^* A^- Z M_1^- M_1 \quad \text{and}
\end{align*}
\]
\[
V = (I_K - S_1 S_1^-) B^* A^- C + S_1 S_1^- C^* (I_K - M_1^- M_1) (A^*)^- B + S_1 S_1^- Z B^- B \\
- S_1 S_1^- Z B^- A^* (I_K - M_1^- M_1) (A^*)^- B,
\]
where \( Z, Z' \in \mathbb{B}(K) \) are arbitrary operators.
Corollary 3.2. Let $A \in \mathcal{B}(H, K)$, $C \in \mathcal{B}(K)$ such that $A$ is regular and $C^* = C$. Then the operator equation

$$AXA^* = C$$

has a Hermitian solution $X \in \mathcal{B}(H)$ if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A)$$

In this case, a representation of the general Hermitian solution is

$$X_H = A^- C(A^-)^* + F - A^- AF(A^- A)^*,$$

where $F \in \mathcal{B}(H)$ is an arbitrary Hermitian operator.

Proof. We put $B = A^*$ in Corollary 3.1 we get the result. \qed

As a consequence of Corollary 3.1 we obtain the well-known Theorem of Alegra Dajić and J.J. Koliha [3, Theorem 3.1].

Corollary 3.3. [3, Theorem 3.1] Let $A, C \in \mathcal{B}(H, K)$ such that $A$ is a regular operator. Then the operator equation

$$AX = C$$

has a Hermitian solution $X \in \mathcal{B}(H)$ if and only if

$$AA^- C = C \quad \text{and} \quad AC^* \text{ is Hermitian.}$$

The general Hermitian solution is of the form

$$X_H = A^- C(A^-)^* + (I_H - A^- A)(A^- C)^* + (I_H - A^- A)Z'(I_H - A^- A)^*,$$

where $Z' \in \mathcal{B}(H)$ is an arbitrary Hermitian operator.

Proof. By applying Corollary 3.1, the operator equation $AX = C$ has a Hermitian solution if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A),$$

which is equivalent to

$$AA^- C = C,$$

and

$$(I_H - I_H + A^- A)A^- CA^* = (I_H - I_H + A^- A)C^*,$$

this implies that

$$CA^* = AC^*.$$

Hence, $AC^*$ is Hermitian. In this case,

$$X = [A^- C + (I_H - A^- A)(A^- C + (I_H - A^- A)C^*(A^*)^- +
+ (I_H - A^- A)Z'(I_H - A^- A)^* - A^- C)],
= A^- C + (I_H - A^- A)(A^- C)^* + (I_H - A^- A)Z'(I_H - A^- A)^*. $$
It follows that,

\[
X = \frac{X + X^*}{2} = A^+C + (I_H - A^-A)(A^+C)^* + (I_H - A^-A)Z'(I_H - A^-A)^*.
\]

\[\Box\]

**Theorem 3.1.** Let \(A, B \in \mathcal{B}(H, K)\) and \(A_1 = (I_K - AA^-)B, C_1 = (I_K - AA^-)C\) and \(S_2 = B(I_H - A^- A_1)\) be all regular and \(C \in \mathcal{B}(K)\) is Hermitian. Then the operator equation

\[
AXA^* + BYB^* = C,
\]

has a Hermitian solution if and only if

1. \(A_1 A_1^-(I_K - AA^-)(B^*)^{-1}B^* = (I_K - AA^-)C\)
2. \((I_K - S_2 S_2^-)[C - BA_1^-(I_K - AA^-)(B^*)^{-1}B^*](I_K - (A^-)^*A^*) = 0\).

In this case, a representation of the general Hermitian solution is of the form

\[
(X_H, Y_H) = \left(\frac{X + X^*}{2}, \frac{Y + Y^*}{2}\right),
\]

where \(X\) and \(Y\) are given by

\[
\begin{aligned}
X &= A^-(C - BYB^*)(A^+)^- + F - A^- A F (A^- A)^*
\end{aligned}
\]

and

\[
\begin{aligned}
Y &= A_1^- (I_K - AA^-)C(B^*)^- + (I_H - A^- A_1)S_2^- [V - BA_1^-(I_K - AA^-)C](B^*)^- + U
\end{aligned}
\]

\[
- [A_1^- A_1 + (I_H - A^- A_1)S_2^- S_2] UB^*(B^*)^-,
\]

and

\[
V = (I_K - S_2 S_2^-) BA_1^- (I_K - AA^-)C + S_2 S_2^- C(I_K - (A^-)^*A^*)(A_1^-)^{-1}B^*
\]

\[
+ S_2 S_2^- Z(B^*)^- (I_H - A_1^- (A_1^-)^*)B^*,
\]

with \(F \in \mathcal{B}(H), U \in \mathcal{B}(H)\) and \(Z \in \mathcal{B}(K)\) are arbitrary Hermitian operators.

**Proof.** The operator equation (1.3) is equivalent to

\[
(3.4) \quad AXA^* = C - BYB^*.
\]

Applying Corollary 3.2, the operator equation (3.4) has a Hermitian solution if and only if

\[
\mathcal{R}(C - BYB^*) \subset \mathcal{R}(A) \iff AA^-(C - BYB^*) = (C - BYB^*),
\]

\[
(3.5) \quad (I - AA^-)(C - BYB^*) = 0.
\]
Then, (3.5) is equivalent to the operator equation

\[ A_1 Y B^* = C_1, \]

with \( A_1 = (I_K - AA^\perp)B, \) and \( C_1 = (I_K - AA^\perp)C. \)

From Corollary 3.1, the operator equation (3.6) has a Hermitian solution if and only if

\[ \mathcal{R}(C_1) \subset \mathcal{R}(A_1) \iff A_1 A_1^\perp C_1 = C_1, \]

and

\[ \mathcal{R}(C_1^\perp) \subset \mathcal{R}(B) \iff C_1 (B^*)^* B^* = C_1, \]

From (3.7) and (3.8), we get

\[ A_1 A_1^\perp (I_K - AA^\perp) C(B^*)^* B^* = (I_K - AA^\perp) C. \]

On the other hand, we have

\[ (I_K - S_2 S_2^\perp) BA_1^\perp (I_K - AA^\perp) C(B^*)^* A_1^\perp = (I_K - S_2 S_2^\perp) C(I_K - (A^\perp)^* A^*). \]

This implies that

\[ (I_K - S_2 S_2^\perp) [C - BA_1^\perp (I_K - AA^\perp) C(B^*)^* B^*] (I_K - (A^\perp)^* A^*) = 0. \]

A representation of the general Hermitian solution to the operator equation (3.6) is of the form

\[ Y_H = \frac{Y + Y^*}{2}, \]

where \( Y \) is given by (3.2) in Corollary 3.1 such that \( A = A_1, B = B^* \) and \( C = C_1 \)

\[ Y = A_1^\perp (I_K - AA^\perp) C(B^*)^* + (I_H - A_1^\perp A_1) S_2^\perp [V - BA_1^\perp (I_K - AA^\perp) C(B^*)^* + U - \{ A_1^\perp A_1 + (I_H - A_1^\perp A_1) S_2^\perp S_2 \} U B^* (B^*)^*], \]

and

\[ V = (I_K - S_2 S_2^\perp) BA_1^\perp (I_K - AA^\perp) C + S_2 S_2^\perp C (I_K - (A^\perp)^* A^*) (A_1^\perp)^* B^* + S_2 S_2^\perp Z (B^*)^* (I_H - A_1^\perp (A_1^\perp)^*) B^*, \]

with \( U \in \mathbb{B}(H) \) and \( Z \in \mathbb{B}(K) \) are arbitrary Hermitian operators.

We return to the operator equation

\[ AXA^* = C - BY B^*, \]

in order to find the Hermitian solution \( X. \)
By Corollary 3.2, the operator equation (3.4) has a Hermitian solution if and only if
\[ \mathcal{R}(C - BYB^*) \subseteq \mathcal{R}(A). \]
So the operator equation (3.4) has a Hermitian solution \( X_H \) given by
\[ X_H = A^-(C - BYB^*)(A^*)^* + F - A^-AF(A^-A)^*, \]
with \( F \in \mathcal{B}(H) \) is an arbitrary Hermitian operator.

4. Conclusions

This paper gives necessary and sufficient conditions for the existence of a common solution to the pair of equations
\[ A_1XB_1 = C_1 \quad \text{and} \quad A_2XB_2 = C_2; \]
We have applied this result to determine new necessary and sufficient conditions for the existence of Hermitian solution and given a representation of the general Hermitian solution to the operator equation
\[ AXB = C. \]
Then, we have given necessary and sufficient conditions for the existence of Hermitian solution and a representation of the general Hermitian solution to the operator equation
\[ AXA^* + BYB^* = C. \]

Acknowledgments

The authors are grateful to the referee for several helpful remarks and suggestions concerning this paper.

REFERENCES

1. X. W. CHANG and J. WANG: The symmetric solutions of the matrix equations \( AX + YA = C \), \( AXA^T + BYB^T = C \) and \( (A^TXA, B^TXB) = (C, D) \). Linear Algebra and Appl. 179 (1993), 171–189.
2. A. Dajić: Common solution of linear equations in ring, with application. Electronic Journal of Linear Algebra, 30 (2015), 66–79.
3. A. Dajić and J. J. Koliha: Positive solution to the equation \( AX = C \) and \( XB = D \) for hilbert space operators. J. Math. Anal. Appl. 333 (2007), 567–576.
4. A. Dajić and J. J. Koliha: Equations \( ax = c \) and \( xb = d \) in rings and rings with involution with applications to Hilbert space operators. Linear Algebra and its Appl. 429 (2008), 1779–1809.
Hermitian Solutions to the Equation $AXA^* + BYB^* = C$

5. Y. Deng and X. Hu: On solutions of matrix equation $AXA^T + BYB^T = C$. Journal of Computational Math. 23 (2005), 17–26.
6. F. O. Farid, M. S. Moslehian, et al.: On the Hermitian solutions to a system of adjointable operator equations. Linear Algebra and its Appl. 437 (2012), 1854–1891.
7. P. A. Fillmore and J. P. Williams: On operator ranges. Advances in Math. 7 (1971), 244–281.
8. J. Groß: A note on the general Hermitian solution to $AXA^* = B$. Bull. Malaysian Math. Soc. (Second Series), 21 (1998), 57–62.
9. Z. H. He: Some new results on a system of Sylvester-type quaternion matrix equations. Journal of Computational Math. 23 (2005), 17–26.
10. Z. H. He, M. Wang and X. Liu: On the general solutions to some systems of quaternion matrix equations. RACSAM (2020), 114:95.
11. Z. H. He: Some quaternion matrix equations involving $\phi$-Hermicity. Filomat, 33 (2019), 5097–5112.
12. Z. H. He: A system of coupled quaternion matrix equations with seven unknowns and its applications. SIAM J. Appl. Math. 74(4) (2006), 213–216.
13. S. K. Mitra: Common solution to a pair of linear matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$. Mathematical Proceedings of the Cambridge Philosophical Society, 74 (1973), 213–216.
14. A. B. Özgüler and N. A. Akar: A common solution to a pair of linear matrix equations over a principal domain. Linear Algebra and its Appl. 144 (1991), 85–99.
15. P. Patrício and R. Puydtiens: About the von Neumann regularity of triangular block matrices. Linear Algebra and its Appl. 332-334 (2001), 485–502.
16. M. Vosough and M. S. Moslehian: Solvability of the matrix inequality. Linear and Multilinear Algebra, 66(9) (2017), 1799–1818.
17. Q. W. Wang: A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity. Linear Algebra and its Appl. 384 (2004), 43–54.
18. Q. W. Wang and Z. H. He: Some matrix equations with applications. Linear Multilinear Algebra, 60:11-12 (2012), 1327–1353.
19. Q. Xu: Common Hermitian and positive solutions to the adjointable operator equations $AX = C$, $XB = D$. Linear Algebra and its Appl. 429 (2008), 1–11.
26. Q. Xu, L. Sheng and Y. Gu: *The solutions to some operator equations.* Linear Algebra and its Appl. **429** (2008), 1997–2024.
27. G. Xu, M. Wei and D. Zheng: *On solutions of matrix equation* $AXB + CYD = F$. Linear Algebra and its Appl. **279** (1998), 93–109.