CHARACTERIZATIONS OF TRIVIAL MAPS IN 3-DIMENSIONAL REAL MILNOR FIBERS

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Abstract. In this paper we extend the characterization of trivial map-germs for the real Milnor fibrations started by Church and Lamotke in [CL]. Our main result cover all cases on the three dimensional real Milnor fibers.

1. Introduction

In [Mi] John W. Milnor showed that given a representative of a holomorphic function germ \( \psi : U \subset \mathbb{C}^{n+1} \to \mathbb{C} \) with, \( \psi(0) = 0 \), there exists a small enough real number \( \epsilon_0 > 0 \), such that for all \( 0 < \epsilon \leq \epsilon_0 \)

\[
\frac{\psi}{\|\psi\|} : S^{2n+1}_\epsilon \setminus K_\epsilon \to S^1
\]

is a smooth projection of a locally trivial fiber bundle, where \( K_\epsilon = \psi^{-1}(0) \cap S^{2n+1}_\epsilon \) is called the link of the singularity at origin. It is well known that in the complex setting and \( n \geq 1 \) the link \( K_\epsilon \) is never empty. Denote \( F_\theta = \psi^{-1}(e^{i\theta}) \) the fiber of the fibration above, where \( e^{i\theta} \in S^1 \). Using tools of Morse theory, Milnor proved that \( F_\theta \) is a parallelizable \( 2n \)-dimensional real manifold and has the homotopy type of a finite \( CW \)—complex of dimension \( n \). Moreover, its topological closure is given by \( \overline{F_\theta} = F_\theta \cup K_\epsilon \) and the topological space \( K_\epsilon \) is \( (n - 2) \)—connected, i.e. \( \pi_i(K_\epsilon) = 0 \) for all \( i = 0, \ldots, n - 2 \). It means that, for \( n = 2 \) the link is connected and for \( n \geq 3 \) it is simple connected.

In the case where \( 0 \in \mathbb{C}^{n+1} \) is an isolated singular point of \( \psi \), Milnor gave more details about the topology of the fiber and the link. He proved that the fiber \( F_\theta \) has the homotopy type of a wedge or bouquet.

2000 Mathematics Subject Classification. 58K15, 57Q45, 32S55, 32C40.

Key words and phrases. topology of the real fibers, real Milnor fibration, trivial maps, fibered links, topology of link, topology of real singularities.
of $n-$dimensional spheres $\bigvee_{\mu} S^n$, where the number of spheres $\mu$ is given by the topological degree, $\deg_0 \left( \frac{\nabla \psi}{\| \nabla \psi \|} \right)$, of the mapping $\frac{\nabla \psi}{\| \nabla \psi \|} : S^{2n+1} \rightarrow S^{2n+1}$.

From this follow easily that the Euler-Poincaré characteristic of the fiber is given by $\chi(F_\theta) = 1 + (-1)^n \mu$. Milnor also proved that for $\epsilon > 0$ small enough, the manifold $(f^{-1}(0) - \{0\}) \cap B^{2n+2}_\epsilon$ intersect transversally all spheres $S^{2n+1}_\epsilon$ and so by Tranversality Theorem the space $K_\epsilon$ is a $(2n-1)-dimensional smooth manifold. Furthermore, for each $\theta$ the space $\overline{F_\theta} = F_\theta \cup K_\epsilon$ is a compact manifold whose boundary is the link $K_\epsilon$.

In the real settings, Milnor considered a real polynomial mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$, $f(0) = 0$, $n \geq p \geq 2$, and assumed that in some open neighbourhood $U$ of the origin $0 \in \mathbb{R}^n$ we have $\Sigma(f) \cap U \subseteq \{0\}$, where $\Sigma(f) = \{ x \in U : \text{rank} (Jf)(x) \text{ fails to be maximal} \}$, i.e., $0$ is an isolated singular point of $f$. Next result follow from [Mi], Theorem 11.2, page 97:

**Theorem 1.1.** There exists $\epsilon_0 > 0$ small enough such that, for all $0 < \epsilon \leq \epsilon_0$, there exists $\eta$, $0 < \eta \ll \epsilon$, such that

(1) $f_\eta : f^{-1}(S^{p-1}_\eta) \cap B^n_\epsilon \rightarrow S^{p-1}_\eta$

is a smooth projection of a locally trivial fiber bundle, where $B^n_\epsilon$ stand for the $n-$dimensional closed ball centered at origin.

It follows from (1) above that the fiber is a smooth $(n-p)-dimensional compact manifold with boundary given by $(f_\eta)^{-1}(y) \cap S^{n-1}_\epsilon$. Moreover, this fibration induces the following smooth fiber bundle

(2) $f_\eta : f^{-1}(S^{p-1}_\eta) \cap B^n_\epsilon \rightarrow S^{p-1}_\eta$

where $B^n_\epsilon$ denote the interior of the closed ball, i.e. the $n-$dimensional open ball of radius $\epsilon$. In this case, the fiber is an open manifold.

If the link $K_\epsilon$ is not empty, then it is isotopic to the boundary of the fiber of fibration (1). In fact, since $0 \in \mathbb{R}^n$ is an isolated singular point of $f$, then we have that for each $\epsilon > 0$ small enough, there exists $\eta$, $0 < \eta \ll \epsilon$, such that the manifold $(f^{-1}(0) - \{0\}) \cap B^n_\epsilon$ intersect
transversally all spheres $S^n_{\epsilon}$. Since the base space is contractible, the mapping

$$f_1 : f^{-1}(B^p_\eta) \cap S^n_{\epsilon} \rightarrow B^p_\eta$$

is a smooth trivial fiber bundle. Furthermore, as an application of the Ehresmann Theorem for manifolds with boundary it is not difficult to show that the mapping

$$f_1 : f^{-1}(B^p_\eta - \{0\}) \cap B^n_\epsilon \rightarrow B^p_\eta - \{0\}$$

is a smooth projection of a locally trivial fiber bundle, where $B^p_\eta - \{0\}$ denotes the punched $p-$dimensional closed ball in $\mathbb{R}^p$ of radius $\eta$. Hence, since the inclusion $S^{p-1}_{\eta} \subset B^p_\eta - \{0\}$ is a homotopy equivalence then, up to homotopy, the topological information contained in the fibrations (1) and (4) are the same. In addition, using the fact that a compact manifold $M$ with no empty boundary $\partial M$ is homotopy equivalent to its interior $M = M - \partial M$ then, up to homotopy, the information contained in the fibers of fibrations (1) and (2) are the same. For these reasons, we can concentrate the topological study on the fibration (1).

**Remark.** In recent years the real Milnor fibrations has been extended in several directions in the settings of isolated and non-isolated singularities. Some results can be found for instance in [AT, TYA, RS] and references.

Milnor also provided information about the topology of the fiber of the fibration (1), see Proposition 2.5, section 2. In fact, it follows from Lemma 11.4, page 100, that if the link $K_\epsilon$ is not empty, then the fiber is $(p - 2)-$connected. That is, for $p = 2$ it is connected and for $p \geq 3$ it is simple connected.

More recently in [ADN], using tools from singularity theory, Morse theory and differential geometry, the authors proved an extended Khin- shiasvilli’s formula [Kh] for the Euler-Poincaré characteristic of an isolated singularity of a polynomial map germ $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$, $f(x) = (f_1(x), \ldots, f_p(x))$, $n \geq p \geq 2$. Denote by $F_f$ the fiber of the associated Milnor fibration (1). The main result in [ADN] is:

**Theorem 1.2 ([ADN]).** Given a polynomial map germ $f$ as above, the following holds:
a) If $n$ is even, then $\chi(F_f) = 1 - \deg_0(\nabla f_1)$, where $\deg_0(\nabla f_1)$ is the topological degree of the mapping $\frac{\nabla f_1}{\|\nabla f_1\|} : S^{n-1}_\epsilon \to S^{n-1}_1$. Moreover, $\deg_0(\nabla f_1) = \deg_0(\nabla f_2) = \cdots = \deg_0(\nabla f_p)$.

b) If $n$ is odd, then $\chi(F_f) = 1$. Moreover, $\deg_0(\nabla f_i) = 0$, for all $i = 1, \cdots, n$.

In [Mi], page 100, Milnor posed the following question:

“For which dimensions $n \geq p \geq 2$ do non-trivial example exist?”

Indeed Milnor did not specify what a “trivial map” should mean, but in a certain sense he proposed to call an isolated singularity polynomial mapping $f$ “trivial” if the fiber $F_f$ of its associated fiber bundle (1) is diffeomorphic to the $(n - p)$-dimensional closed disk.

Using this concept as a definition, Church and Lamotke in [CL] answered the above question in the following way.

**Theorem 1.3 ([CL], pags. 149–150).**

a) For $0 \leq n - p \leq 2$, non-trivial examples occur precisely for the dimension $(n, p) = (2, 2)$, $(4, 3)$ and $(4, 2)$.

b) For $n - p \geq 4$ non-trivial examples occur for all $(n, p)$.

c) For $n - p = 3$ non-trivial examples occur for $(5, 2)$ and $(8, 5)$. Moreover, if the (3-dimensional) Poincaré Conjecture is false, then there are non-trivial examples for all $(n, p)$. If the Poincaré Conjecture is true, all examples are trivial except $(5, 2)$, $(8, 5)$ and possibly $(6, 3)$.

Furthermore, in [CL] page 151, the authors proved an alternative characterization of trivial map-germs (see below) using the branch set $B_f$ of a map-germ $f$, i.e., the set where the map-germ fails to be locally topologically equivalent to a projection.

**Proposition 1.4 ([CL], page 151).** For $n - p \neq 4, 5$ $f$ is trivial if, and only if, $0 \notin B_f$.

Also in [CL] the authors found the pairs of dimensions $(n, p)$ for which $0$ is isolated in $f^{-1}(0)$, i.e., locally the level set $\{f^{-1}(0)\} = \{0\}$. This is equivalent to the link $K_\epsilon = f^{-1}(0) \cap S^{n-1}_\epsilon$ being empty.

**Lemma 1.5 ([CL], Lemma 1, page 151).** If $0$ is an isolated point of $f^{-1}(0)$, then $n = p$, or $(n, p) = (4, 3), (8, 5), (16, 9)$. If $n = p$ the singularity is trivial unless $k = 2$. The other cases are never trivial.

The aim of this paper is to use tools from algebraic topology, singularity theory and a formula for the Euler-Poincaré characteristic of
the real Milnor fiber proved in [ADN] (Theorem 1.2 above) to find conditions under which provide a (new) characterization of trivial map-germs. The main Theorem cover all cases when the dimension of the Milnor fiber is \( n - p = 3 \), i.e., the item (c) of Theorem 1.3 above. Our main result is:

**Theorem 1.6.** Let \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \), \( f(x) = (f_1(x), \ldots, f_p(x)) \) be a polynomial map-germ with an isolated singularity, and suppose that \( n - p = 3 \). Then, the following holds:

1) If the pair \((n, p) = (6, 3)\), then \( f \) is trivial if and only if \( \deg_0(\nabla f_1) = 0 \);
2) If the pair \((n, p) = (8, 5)\), then \( f \) is trivial if and only if the link \( K_e \) is not empty;
3) If the pair \((n, p) = (5, 2)\), then \( f \) is trivial if and only if \( \pi_1(F_f) = 0 \) (i.e. the respective Milnor fiber \( F_f \) is simply connected).

It is worth pointing out that in [ADN], section 4, the authors proved some formulae in order to describe geometrically/topologically the real Milnor fiber and a characterization of trivial map-germ for the pair \((4, 2)\). For this they used a condition analogous to that in the Theorem 1.6 (1), (see [ADN], Corollary 4.5). They proved that an isolated singularity polynomial map-germ \( f = (f_1, f_2) : (\mathbb{R}^4, 0) \to (\mathbb{R}^2, 0) \) is trivial if and only if \( \deg_0(\nabla f_1) = 0 \).

**2. Set up and preliminary results**

In this section we state and prove some results from algebraic topology in order to prove our main result. We focus on results related to the topology of three dimensional compact manifolds.

Next result is a very well known fact from algebraic topology, and we include it here for the sake of completeness. The prove follows from the additive property of Euler-Poincaré characteristic, (see also [V], Theorem 6.38, p. 180).

**Theorem 2.1.** If \( M \) is a compact topological \( n \)-manifold with boundary \( \partial M \), then the Euler characteristic of \( \partial M \) is even. More precisely,

\[
\chi(\partial M) = \begin{cases} 
2\chi(M) & \text{if } n \text{ is odd}, \\
0 & \text{if } n \text{ is even}.
\end{cases}
\]

**Proposition 2.2.** If \( M \) is a connected, compact and orientable \( n \)-manifold with non empty boundary \( \partial M \), then \( H_i(M; \mathbb{Z}) = 0 \), for \( i \geq n \).

**Proof.** Since \( M \) is a connected non-compact \( n \)-manifold without boundary and the inclusion map \( i : \overset{\circ}{M} \hookrightarrow M \) is an homotopy equivalence, by
Proposition 3.29, p. 239], we have

\[ H_i(M; \mathbb{Z}) \cong H_i(M; \mathbb{Z}) = 0, \quad \text{for } i \geq n. \]

\[ \square \]

Lemma 2.3. Let \( M \) be a compact, connected and orientable 3-manifold with boundary \( \partial M \). If \( H_1(M, \mathbb{Z}) = 0 \), then \( \partial M \) is a disjoint union of 2-spheres.

Proof. Consider the following piece of the long exact sequence in cohomology for the pair \((M, \partial M)\):

\[ \cdots \rightarrow H^1(M, \mathbb{Z}) \rightarrow H^1(\partial M; \mathbb{Z}) \rightarrow H^2(M, \partial M; \mathbb{Z}) \rightarrow \cdots \]

Since \( M \) is compact and orientable, by Poincaré-Lefschetz duality theorem (see [V], Theorem 6.25, p. 171) we have

\[ H^2(M, \partial M; \mathbb{Z}) \cong H_1(M; \mathbb{Z}) = 0. \]

On the other hand, \( H^1(M; \mathbb{Z}) \cong \text{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z}) = 0 \), therefore, from \([5]\) it follows that \( H^1(\partial M; \mathbb{Z}) = 0 \) and by using the Poincaré duality for the orientable closed 2-manifold \( \partial M \),

\[ 0 = H^1(\partial M; \mathbb{Z}) \cong H_1(\partial M; \mathbb{Z}). \]

By the standard classification of the compact 2-manifolds, we conclude that \( \partial M \) is a disjoint union of 2-spheres.

\[ \square \]

Lemma 2.4. Let \( M \) be a compact, connected and orientable 3-manifold with boundary \( \partial M \). Then the following statements are equivalent:

(i) \( M \) is contractible.

(ii) \( M \) is simply-connected and has the same homology of a point.

(iii) \( M \) is simply-connected and \( \partial M \) is a 2-sphere.

(iv) \( M \) is diffeomorphic to a 3-disk.

Proof. It is follow from \([B, \text{Corollary 10.11, p. 479}]\) that (i) is equivalent to (ii).

For (ii) \( \Leftrightarrow \) (iii), since \( M \) is simply-connected, from Lemma 2.3, \( \partial M \) is a disjoint union of 2-spheres. Therefore, if \( M \) has the same homology of a point, we have \( \chi(M) = 1 \), and by Theorem 2.1,

\[ \chi(\partial M) = 2\chi(M) = 2, \]

that is, \( \partial M \) is a 2-sphere.
Conversely, since $M$ is a simply-connected space (in particular, $H_1(M; \mathbb{Z}) = 0$) and from Proposition 2.2, $H_j(M; \mathbb{Z}) = 0$ for $j \geq 3$, it is sufficient to show that $H_2(M; \mathbb{Z}) = 0$. Indeed, we have $H^1(M; \mathbb{Z}) \cong \text{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z}) = 0$ and by Poincaré-Lefschetz duality theorem

$$0 \cong H^1(M; \mathbb{Z}) \cong H_2(M, \partial M; \mathbb{Z}).$$

Now, by [V, Theorem 6.24, p. 169], there is a unique fundamental class $z \in H_3(M, \partial M; \mathbb{Z})$ such that $\Delta(z)$ is a fundamental class of $\partial M$, where $\Delta : H_3(M, \partial M; \mathbb{Z}) \to H_2(\partial M; \mathbb{Z})$ is the connecting homomorphism. By assumptions $\partial M$ is a 2-sphere, we conclude that $\Delta$ is onto and considering a particular part of the long exact sequence in homology for the pair $(M, \partial M)$

$$H_3(M, \partial M) \xrightarrow{\Delta} H_2(\partial M) \to H_2(M) \to H_2(M, \partial M) = 0,$$

it follows that $H_2(M; \mathbb{Z}) = 0$.

For (i)$\Leftrightarrow$ (iv), let $N = M \cup_{\partial M} M$ be the “double” of $M$ and let $M_1$ and $M_2$ be two copies of $M$ in $N$. We have that $N$ is a closed orientable 3-manifold. By [H, Proposition 3.42, p. 253] we can consider $U$ the union of $M_1$ and a collar neighborhood of $M_2$; similarly we can consider $V$ the union of $M_2$ and a collar neighborhood of $M_1$. Since the pathwise connected spaces $U, V$ and $U \cap V$ are homotopy equivalents to $M_1, M_2$ and $\partial M$, respectively and $\partial M$ is simply connected, by Van Kampen Theorem (see [V, Corollary 4.27, p. 116]), it follows that

$$\pi_1(N) \cong \pi_1(U) \ast \pi_1(V) \cong \pi_1(M_1) \ast \pi_1(M_2).$$

Therefore, since by assumptions $M_1 = M_2 = M$ are contractible, we have $\pi_1(N) = 0$. Thus, by Poincaré Theorem we conclude that $N$ is diffeomorphic to a 3-sphere and consequently, $M$ is diffeomorphic to a 3-disk. The converse it is obvious.

Next proposition follow from [Mi], Lemma 11.4, page 100, which gives an important information about the connectedness of the real Milnor fibre.

**Proposition 2.5.** Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $f(x) = (f_1(x), \ldots, f_p(x))$ be an isolated singularity polynomial map-germ, with $n \geq p \geq 2$. If the link $K_\epsilon$ is not empty, then the fiber $F_f$ of the fibration $(1)$ is $(p - 2)$-connected.

**Proof.** See the first two paragraphs on the proof of Lemma 11.4, pages 100–101, in [Mi].
3. Proof of the main result

In this section we are assuming that \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \), \( f(x) = (f_1(x), \ldots, f_p(x)) \) is an isolated singularity polynomial map-germ, with \( n - p = 3 \). For simplicity, we split the prove of Theorem 1.6 into three parts. Below we start with the proof of the item (1). Recall that, in the case \((6, 3)\) it follows from Lemma 1.5 that the link \( K_\varepsilon \), which is diffeomorphic to the boundary of the Milnor fiber \( \partial F_f \), is not empty.

3.1. **The (6,3) case.** Let \( f : (\mathbb{R}^6, 0) \to (\mathbb{R}^3, 0) \) be with isolated singularity at origin.

**Proof.** (Proof of Theorem 1.6, item (1)) : If \( f \) is trivial, \( F_f \) is diffeomorphic to a 3-disk, so the Euler Characteristic of \( F_f \) satisfies \( \chi(F_f) = 1 \). As \( \chi(F_f) = 1 - \deg_0 \nabla f_1 \) (Theorem 1.2 (a)) we conclude that \( \deg_0 \nabla f_1 = 0 \).

Conversely, since \( p = 3 \) the Milnor fiber \( F_f \) is simply-connected, so in particular \( H_1(F_f; \mathbb{Z}) = 0 \). Now, from Lemma 2.3 we have that \( \partial F_f \), which is diffeomorphic to \( K_\varepsilon \), is a disjoint union of 2-spheres. Therefore, if \( \deg_0 \nabla f = 0 \), then \( \chi(F_f) = 1 \), and

\[
\chi(K_\varepsilon) = \chi(\partial F_f) = 2 \chi(F_f) = 2.
\]

Hence, the link \( K_\varepsilon \) is diffeomorphic to one 2-sphere and by Lemma 2.4 we have that \( F_f \) must be diffeomorphic to a 3-disk. \( \square \)

**Proposition 3.1.** For the pair \((n, p) = (6, 3)\), the following are equivalent:

1. \( f \) is trivial;
2. The link \( K_\varepsilon \) is connected.

**Proof.** The implication (1) \( \Rightarrow \) (2) is trivial. In fact, if we assume that \( f \) is trivial, the boundary of the fiber \( \partial F_f \) is diffeomorphic to the 2-sphere, and so is connected.

For the converse, if we assume that the link is connected, we know that \( K_\varepsilon \) must be diffeomorphic to one single copy of a 2-sphere and again by Lemma 2.4 (iii), \( F_f \) is diffeomorphic to a 3-disk. \( \square \)

We can now summarize all these results in the following way.

**Corollary 3.2.** Given \( f : (\mathbb{R}^6, 0) \to (\mathbb{R}^3, 0) \), \( f(x) = (f_1(x), f_2(x), f_3(x)) \), an isolated singularity polynomial map-germ, the following statements are equivalent:

1. \( f \) is trivial;
2. The link \( K_\varepsilon \) is connected;
3. \( \deg_0(\nabla f_1) = 0 \).
Remark. The statement (2) of Corollary 3.2 does not characterize in general the trivial fibration. For example, in the case of the pair (4, 2), if we consider $f(x, y) = x^2 + y^3$, then the link is the (2, 3)–torus knot and so it is connected, and the open fiber is diffeomorphic to the torus without an open disc removed.

Proposition 3.3. Given $f : (\mathbb{R}^6, 0) \to (\mathbb{R}^3, 0)$, an isolated singularity polynomial map-germ, then the Euler-Poincaré characteristic of the fiber $\chi(F_f)$ is precisely the number of spheres on its boundary $\partial F_f$.

Proof. Since $F_f$ is a 3–dimensional, simply connected and compact manifold with boundary given by $d$–disjoint copies of 2–dimensional spheres, then each sphere on $\partial F_f$ contributes with 2 to its Euler characteristic. Hence,

$$2.d = \chi(\partial F_f) = 2\chi(F_f).$$

Therefore, the Euler characteristic of the fiber $F_f$ is the number of spheres in the boundary.

3.2. The (8,5) case. Consider $f : (\mathbb{R}^8, 0) \to (\mathbb{R}^5, 0)$ a polynomial map-germ with an isolated singularity at origin. From Lemma 1.5 we know that for the pair (8, 5) the link may be empty. Of course that, for the purpose of characterization of trivial mapping, a necessary condition is that the link be not empty.

Proof. (Proof of Theorem 1.6, item (2)): The first implication is obvious. Conversely, since the link $K_\epsilon \neq \emptyset$ and $p = 5$, the Milnor fiber $F_f$ is a 3-connected (and so is simple connected), compact 3-manifold with boundary. From Hurewicz Theorem, $F_f$ must have the homology type of a point, and so by Lemma 2.4 (ii), $F_f$ is diffeomorphic to a 3-disk.

Remark. By a similar argument to that used in the proof of Theorem 1.6 (1), it follows that the condition $\deg_0(\nabla f_1) = 0$ also characterizes the triviality of $f$ in this case. Hence, we have:

Corollary 3.4. For the pair $(n, p) = (8, 5)$, the following statements are equivalent:

1. $f$ is trivial;
2. The link $K_\epsilon \neq \emptyset$;
3. $\deg_0(\nabla f_1) = 0$. 
3.3. The case (5,2). Now consider $f : (\mathbb{R}^5, 0) \to (\mathbb{R}^2, 0)$ be a polynomial map-germ with an isolated singularity at origin. In this case, we only have that the Milnor fiber $F_f$ is connected and the link is not empty.

Proof. (Proof of Theorem 1.6, item (3)) : Again, for the first implication there is nothing to prove, since the closed disk is simply connected. For the converse, if we assume that $F_f$ is simply-connected, i.e., $\pi_1(F_f) = 0$, then we know from Lemma 2.3 that $\partial F_f \approx K_\epsilon$ is diffeomorphic to a disjoint union of finite many 2-spheres. In another hand, since the source space have odd dimension, then by Theorem 1.2 (b), see [ADN], we have $\chi(K_\epsilon) = \chi(\partial F_f) = 2$.

Therefore, the link $K_\epsilon$ must be a 2-sphere and by Lemma 2.4 we have that $F_f$ is diffeomorphic to a 3-disk. \hfill \Box

Remark. In [Lo], page 421, E. Looijenga remarked that we can use the example $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, $f(x, y) = x^2 + y^3$, which is not trivial (see for instance [ADN]), as a real polynomial map-germ from $(\mathbb{R}^4, 0) \to (\mathbb{R}^2, 0)$ to guarantee the existence of non-trivial polynomial map-germ from $(\mathbb{R}^5, 0) \to (\mathbb{R}^2, 0)$. Therefore, as a by-product of our characterization above, we can conclude that its respective Milnor fiber can not be simply connected.

Remark. In the case (5, 2), if we suppose that $H_1(F_f; \mathbb{Z}) = 0$, from Lemma 2.3 and Theorem 1.2 (b), then the link $K_\epsilon$ is a 2-sphere. However, we cannot ensure that $F_f$ is diffeomorphic to a 3-disk, as the following example shows: by [B], Theorem 8.10 (Poincaré), page 353, there is a compact 3-manifold $W$ having the homology groups of the sphere $S^3$ but which is not simply-connected. So, $W \setminus D^3$ is a compact 3-manifold whose boundary is a 2-sphere, which is not simply-connected.

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