Joint State Estimation and Communication over a State-Dependent Gaussian Multiple Access Channel

Viswanathan Ramachandran, Sibi Raj B. Pillai, Member, IEEE, and Vinod M. Prabhakaran, Member, IEEE

Abstract—A hybrid communication network with a common analog signal and an independent digital data stream as input to each node in a multiple access network is considered. The receiver/base-station has to estimate the analog signal with a given fidelity, and decode the digital streams with a low error probability. Treating the analog signal as a common state process, we set up a joint state estimation and communication problem in a Gaussian multiple access channel (MAC) with additive state. The transmitters have non-causal knowledge of the state process, and need to communicate independent data streams in addition to facilitating state estimation at the receiver. We first provide a complete characterization of the optimal trade-off between mean squared error distortion performance in estimating the state and the data rates for the message streams from two transmitting nodes. This is then generalized to an $N$-sender MAC. To this end, we show a natural connection between the state-dependent MAC model and a hybrid multi-sensor network in which a common source phenomenon is observed at $N$ transmitting nodes. Each node encodes the source observations as well as an independent message stream over a Gaussian MAC without any state process. The receiver is interested estimating the source and all the messages. Again the distortion-rate performance is characterized.

Index Terms—Multiple Access Channel, Gelfand-Pinsker, Dirty Paper coding, State Amplification, MMSE Estimation, Uncoded Communication.

I. INTRODUCTION

Hybrid digital radio systems [1], [2], involving analog and digital information superposed in the same communication signal, are getting increasingly popular nowadays. In these systems, the receiver must estimate the analog signal, while also decoding the digital information. Such systems can be modelled as state-dependent channels [3], where the transmitter aids the receiver in estimating the channel state while also conveying a stream of messages [4]. This is a simultaneous estimation and communication problem. For an additive channel, when the channel noise and the state are independent Gaussian processes, Sutivong et al. [4] established the optimal trade-off between the mean squared error distortion in estimating the state and the communication rate in a point-to-point setting. Joint state estimation and communication is also relevant in the context of multi-user networks with several sensor nodes observing a common phenomenon. The base station/receiver is interested not only in the source process but also in the data from each node, this is the topic of this work.

Channels with state are used to model situations in which the channel statistics are controlled by an external random process, known as the state process. The state process may be known either at the encoder, decoder, or both. The encoder state information can be either causal or non-causal (i.e., the entire state sequence is known a priori). Seminal papers by Shannon [5] (causal case) and Gelfand & Pinsker [6] (non-causal case) introduced state-dependent models. The latter model was motivated by coding for memory with defects, first studied in [7]. Costa [8] introduced the notion of dirty paper coding (DPC) for a state-dependent AWGN channel with non-causal state knowledge at the encoder, wherein the surprising conclusion that the capacity is unchanged by the presence of the state was arrived at. DPC later found extensive applications in broadcast settings, leading to the solution of the MIMO broadcast capacity region [9].

In certain state dependent channels, the transmitter may wish to aid the receiver in estimating the channel state, in addition to communicating messages. For a point-to-point AWGN channel with additive state, splitting the available average power between uncoded transmission of the state and DPC for the message was found to be optimal for the mean squared error distortion measure [4]. There is only limited success in extending this result to other models. For example, the discrete memoryless counterpart of the state estimation problem was analyzed by Kim et al. [10], for a restricted setting where the distortion is measured in terms of the state uncertainty reduction rate. Under non-causal state knowledge at the encoders, there is no known network setting where the joint state estimation and communication trade-off is completely available, to the best of our knowledge. The current paper solves this for the AWGN MAC setting. We now briefly mention some of the relevant contributions in the literature.

Following [4], the idea of channel state amplification has been studied in several network information theoretic settings. In [11], the problem of communicating a common source and two independent messages over a Gaussian broadcast channel (BC) without state was analyzed, and it was shown that a power splitting strategy to meet the two different goals is not optimal. The problem of communicating channel state information over a state dependent discrete memoryless channel with causal state information at the encoder was analyzed in [12]. More recently, it was shown in [13] that for simultaneous...
message and state communication over memoryless channels with memoryless states, feedback can improve the optimal trade-off region both for causal and strictly-causal encoder side information. The dual problem of [4], known as state masking, in which the transmitter tries to conceal the state from the receiver was studied in [14]. In [15], a state-dependent Gaussian BC was considered with the goal of amplifying the channel state at one of the receivers while masking it from the other receiver, with no message transmissions. [16] gave inner bounds for simultaneous message transmission and state estimation over a state-dependent Gaussian BC. For message communication and state masking over a discrete memoryless BC, inner and outer bounds were derived in [17].

The main concern of this paper is state estimation and communication in a scalar Gaussian multiple access setting. We will present a model with two senders first. For the model shown in Fig. 1 a common additive independent and identically distributed (IID) Gaussian state-process affects the transmissions from both senders. The transmitters know the state-process in a non-causal fashion. Our first objective is to obtain an estimate of the state process at the receiver to within a prescribed distortion bound. In addition, there is a message stream from each encoder to the receiver. Given a rate pair for their respective private messages, the transmitters attempt to minimize the distortion incurred in state estimation at the receiver. Under individual average transmit power constraints at the encoders, the Gaussian state dependent MAC with state estimation requirement leads to interesting trade-offs between the achievable distortion and the rates. We name this model as the dirty paper MAC with state estimation.

In the absence of a state process, the AWGN MAC capacity can be seen as a natural extension of the point to point AWGN model [18]. When the state is present, it is tempting to look for such an extension of the joint state estimation and communication tradeoff, using the single user results in [4]. However, notice that the former connection is greatly aided by the polymatroidal capacity region of a Gaussian MAC (GMAC). Essentially three inequalities suffice to establish the converse result for a two user MAC [18]. On the other hand, for joint estimation and communication in a two user MAC, even the cross-section of the optimal rate-region under a given distortion is not always a polytope. This explains why single user techniques are not enough in our setup. Our main contributions are summarized below.

- We provide a complete characterization of the optimal trade-off between joint state estimation and communication over a two user dirty paper MAC with state estimation.
- For a multi-sensor network of nodes observing a common source phenomenon, with each node possibly having an additional independent message stream, we characterize the optimal distortion-rate performance in joint state-estimation and communication over a Gaussian MAC without state. The model is sufficiently general to include cases where the source symbols also act as additive state, which is known non-causally at the transmitters.

The transmission of correlated sources through a MAC is a very important open problem in literature [19], [20]. Our problem is related, but an extreme case called the cooperative MAC [19], where the source observation is common to all the transmitters. In our model, we are only constraining the reconstruction fidelity, whereas [19] considers the lossless case. In some sense, the problem which comes closest to the one here is the source estimation problem in [21]. Here N transmitters in a Gaussian MAC observe independent noisy versions of a single source. The transmitters are assumed to be symmetric, i.e. they have identical power constraints, and the same noise variance in the source observations. The channel state process is completely absent, but uncoded transmissions turn out optimal. Some relaxations on symmetric users are provided in [22], [23]. In fact [22] also considers the scalar single user model with additive state non-causally known at the transmitter, and shows that uncoded transmission is optimal for state estimation in the Gaussian setting. However, even for the point to point system, optimal communication schemes in presence of additional messages are unknown when the source observations are noisy [22]. Other relevant studies regarding communication of sources over a MAC include [24] (distributed correlated sources over an orthogonal MAC), [25] (bivariate Gaussian source over GMAC with individual distortion constraints), [26] (reliable function computation over MAC) and [27] (linear functions of correlated sources over GMAC). Notice that none of these models consider additional data streams along with source communications over MACs. For noiseless source observations, [4] characterizes the optimal tradeoff for message as well as rate, the current paper extends this to a N-sender GMAC, with or without state.

Notations: We use $\mathbb{P}(\cdot)$ to denote the probability of an event, and $\mathbb{E}[\cdot]$ to denote the expected value of a random variable. All logarithms in this paper are to the base $\log_2$. Unless specified otherwise. We denote random vectors as $U^n := U_1, \ldots, U_n$ and $U_i^n := U_{1i}, \ldots, U_{ni}$. Calligraphic letters represent the alphabets. $\| \cdot \|$ denotes the Euclidean norm of a vector.

The paper is organized as follows: we introduce the system model and main results in Section II. Sections III and IV respectively contain the achievable coding scheme and converse to the optimal region. Section VI considers the generalization to N transmitters over a GMAC, with and without state. Concluding remarks are given in Section VII.
II. SYSTEM MODEL AND RESULTS

The dirty paper MAC with state estimation is shown in Fig. 1. Here $S \sim \mathcal{N}(0,Q)$ is the channel state and $Z$ is the channel noise, with $S \perp Z$. The state and noise processes are i.i.d., with the state being non-causally available at both encoders. The receiver observes (for a single channel use)

$$Y = X_1 + X_2 + S + Z,$$

where $Z \sim \mathcal{N}(0,\sigma_Z^2)$. After $n$ observations, the decoder estimates $\hat{S}^n = \phi(Y^n)$ using a reconstruction map $\phi(\cdot) : Y^n \to \mathbb{R}^n$, and also decodes the independent messages $(W_1, W_2)$, which are assumed be independent of $S^n$. We also take $W_j$ to be uniformly drawn from $\{1, \cdots, 2^{nR_j}\}$ for $j = 1, 2$.

Our objective here is to maintain the distortion below a prescribed value, while ensuring that the average error probability of decoding the messages is small enough, i.e.

$$\frac{1}{n} \mathbb{E}[\|S^n - \phi(Y^n)\|^2] \leq D + \epsilon,$$  \hspace{1cm} (2)

$$\mathbb{P}(\psi(Y^n) \neq (W_1, W_2)) \leq \epsilon.$$  \hspace{1cm} (3)

Here $\psi : Y^n \to \{1, \cdots, 2^{nR_1}\} \times \{1, \cdots, 2^{nR_2}\}$ is the decoding map, $D$ represents the distortion target, and $\epsilon > 0$ is the probability of error target.

**Definition 1.** A scheme achieving (2) – (3) using the encoder maps $E_j : \{1, \cdots, 2^{nR_j}\} \times S^n \to X^n$ such that $\mathbb{E}[\|X^n\|^2] \leq nP_j$, $j = 1, 2$, along with two maps $\phi(\cdot)$ and $\psi(\cdot)$ at the receiver is called an $(n, R_1, R_2, D, \epsilon)$ communication scheme.

We say that a triple $(R_1, R_2, D)$ is achievable if $(n, R_1, R_2, D, \epsilon)$ communication scheme exists for every $\epsilon > 0$, possibly by taking $n$ large enough. Let $C_{est}^{mac}(P_1, P_2)$ be the closure of the set of all achievable $(R_1, R_2, D)$ triples, with $0 \leq D \leq Q$. Our main result is stated below.

**Theorem 2.** For the dirty-paper MAC with state, the optimal trade-off region $C_{est}^{mac}(P_1, P_2)$ is given by the convex closure of all $(R_1, R_2, D) \in \mathbb{R}_+^3$ such that

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{\gamma P_1}{\sigma_Z^2}\right),$$  \hspace{1cm} (4)

$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{\beta P_2}{\sigma_Z^2}\right),$$  \hspace{1cm} (5)

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{\gamma P_1 + \beta P_2}{\sigma_Z^2}\right),$$  \hspace{1cm} (6)

$$D \geq \frac{Q(\sigma_Z^2 + \gamma P_1 + \beta P_2)}{P_1 + P_2 + Q + \sigma_Z^2 + 2\sqrt{\gamma P_1 Q} + 2\sqrt{\beta P_2 Q} + 2\sqrt{\gamma \beta P_1 P_2}}.$$  \hspace{1cm} (7)

for some $\gamma \in [0, 1]$ and $\beta \in [0, 1]$, with $\bar{\gamma} = 1 - \gamma$ and $\bar{\beta} = 1 - \beta$.

Before we prove this result, notice that the rate region is not in general a polytope for any given distortion value, unlike the case where state estimation is not required [28]. Nevertheless, the region admits a compact representation as given in (4) – (7).

**Proof.** In Section III, we present a communication scheme to achieve tuples satisfying the constraints (4) – (7). Then in Section IV we show a converse result which bounds the distortion-rate performance for any successful communication scheme. We further show that the tradeoff cannot be better than the ones defined by (4) – (7) for some values of $\gamma, \beta \in [0, 1]$, this is given in Section V. The main novelty of the proof is in the converse result.

**Remark 3.** Notice that on setting $\gamma = \beta = 1$, which amounts to no state estimation requirements, we recover the multiuser writing on dirty paper result in [28], which proves that the capacity region of the dirty paper MAC with state is not affected by the presence of state.

One of the motivations of our model comes from sensor networks employed in on-board platforms. We now make this connection more explicit.

A. Connection to Multi-sensor models

Let us introduce a more general $N$-sender Gaussian MAC framework as in Figure 2 where the receiver observes

$$Y = \sum_{i=1}^{N} X_i + \alpha S + Z.$$  \hspace{1cm} (8)

When the parameter $\alpha = 1$ we recover the state dependent model, and $\alpha = 0$ corresponds to a source estimation and message communication problem over a GMAC without state. The transmissions $X_i$ of user $i$ is subjected to an average power constraint of $P_i$. Each transmitter observes the source process $S$, which is assumed to be non-causally available to them. The receiver should estimate the source, as well as an independent message stream from each transmitter. We term the model as the *source-message communication* problem. Notice that some of the nodes may not have any messages, they simply help in the estimation of source. It turns out that having uncoded transmission at each node devoid of any messages is indeed the optimal strategy in such set ups, marking the importance of the results presented for state-dependent models in the previous sub-section. A brief literature review on source and message communication is in order.

Goblick [29] showed that for transmission of Gaussian sources over Gaussian channels, an uncoded strategy of sending a scaled version of the source to meet the power constraint and then MMSE estimation at the receiver is optimal.
The same approach can be used to communicate Gaussian sources over Gaussian broadcast channels, as was observed in Prabhakaran et. al [20]. We already mentioned that the uncoded approach is optimal for source estimation in some symmetric GMAC models, in the absence of messages [21]. In the presence of independent private message stream at each node in an $N$-sender Gaussian MAC, we show that uncoded source transmissions are indeed optimal while communicating a common source observation.

Our main result for the $N$-sender Gaussian MAC with source-message communication is stated in the following theorem.

**Theorem 4.** For an $N$-sender GMAC with message and state communication, the optimal trade-off region is given by the convex closure of the set $(R_1, R_2, \cdots, R_N, D)$ such that

$$\sum_{j \in \mathcal{J}} R_j \leq \frac{1}{2} \log \left( 1 + \sum_{j \in \mathcal{J}} \gamma_j P_j \right) \quad \forall \mathcal{J} \subseteq \{1 : N\},$$

$$D \geq \frac{Q \left( \sigma^2 + \sum_{j=1}^{N} \gamma_j P_j \right)}{\sigma^2 + \sum_{j=1}^{N} \gamma_j P_j + \left( \alpha \sqrt{Q} + \sum_{j=1}^{N} \sqrt{(1 - \gamma_j)P_j} \right)^2},$$

for some $(\gamma_1, \cdots, \gamma_N) \in [0, 1]^N$.

**Proof:** This is given in Section [VI] Let us specialize our results to a three sender Gaussian MAC, as shown in Fig. 3. The channel model is

$$Y = X_1 + X_2 + X_3 + Z,$$

with $Z \sim N(0, \sigma^2_z)$, and the power constraints $E[X_j]^2 \leq P_j, j = 1, 2, 3$. Suppose the third terminal is only interested

in conveying the source process under a distortion constraint, and it follows an uncoded strategy by sending

$$X_3 = \sqrt{\frac{P_3}{Q}} S.$$  \hfill (12)

Then the overall model becomes

$$Y = X_1 + X_2 + \sqrt{\frac{P_3}{Q}} S + Z$$  \hfill (13)

$$= X_1 + X_2 + S' + Z,$$  \hfill (14)

where $S' = \sqrt{\frac{P_3}{Q}} S$ is non-causally known to both the encoders. Notice that this is indeed the joint state estimation and communication model for a dirty paper MAC with state estimation. The following corollary is of interest when the third user has no message.

**Corollary 5.** For the three sender MAC with source and message communication, the optimal trade-off region $C_{sue}(r_1, r_2, r_3)$ when $\alpha = r_3 = 0$ is given by the convex closure of all $(r_1, r_2, D) \in \mathbb{R}_+^3$ such that (4) - (6) holds and

$$D \geq \frac{Q \left( \sigma^2 + \gamma P_1 + \beta P_2 \right)}{\sum_{i=1}^{3} P_i + \sigma^2 + 2\sqrt{\gamma P_1 P_3} + 2\sqrt{\beta P_2 P_3} + 2\sqrt{\gamma \beta P_1 P_2}},$$

for some $\gamma, \beta \in [0, 1]$ with $\gamma = 1 - \gamma$ and $\beta = 1 - \beta$.

**III. Achievability for the Dirty Paper MAC**

We employ a suitable power splitting strategy along with dirty paper coding to prove the achievability. This is rather straightforward, but the details are given for completeness. The available power $P_1$ at encoder 1 is split into two parts: namely $\gamma P_1$ for message transmission and $\beta P_1$ for state amplification, for some $\gamma \in [0, 1]$. Likewise, the power $P_2$ available at the second encoder is split into $\beta P_2$ (message transmission) and $\beta P_2$ (state amplification) for some $\beta \in [0, 1]$. Then generate the state amplification signals

$$X_{1s} = \sqrt{\frac{\gamma P_1}{Q}} S_j$$ and $X_{2s} = \sqrt{\frac{\beta P_2}{Q}} S_j, 1 \leq j \leq n$ (16)

at the respective encoders. Now the system model in (1) can be rewritten as

$$Y = X_{1m} + X_{1s} + X_{2m} + X_{2s} + S + Z$$

$$= X_{1m} + X_{2m} + \left( 1 + \sqrt{\frac{\gamma P_1}{Q}} + \sqrt{\frac{\beta P_2}{Q}} \right) S + Z. \hfill (17)$$

Here the index $m$ in the subscript indicates that the corresponding signals are intended for message transmission, while the subscript $s$ indicates state amplification signals. Now in order to communicate the messages across to the receiver, we employ the writing on dirty paper result for a Gaussian MAC [28].

Recall that a known dirt over an AWGN channel can be completely cancelled by dirty paper coding [8]. More generally, a rate $R$ satisfying

$$R \leq I(U_1; Y) - I(U_1; S),$$

when evaluated for some feasible distribution $p(u_1, x, y)p(y|x, s)$, can be achieved by Gelfand-Pinsker coding [6] for a point-to-point channel with noncausally known state. In order to achieve (4) - (6), we first consider a dirty paper channel with input $X_{1m}$, known state $S' = (1 + \sqrt{\gamma P_1/Q} + \sqrt{\beta P_2/Q}) S$ and unknown noise $X_{2m} + Z$. We choose $U_1 = X_{1m} + \alpha_1 S'$, $X_{1m} \perp S$ with
The above facts will be extensively used in our proofs. For the dirty paper MAC with state as well, however we now consider the distortion rate pair is also achievable (for the dirty paper MAC with state as well, however we now consider the distortion rate pair is also achievable).

The MMSE can be readily calculated to be the RHS of expression (24). In addition, we will use the empirical covariance matrix $K_i$ of the combined vector $(X_{1i}, X_{2i}, S_i)$, with $K_i(m, n)$ denoting its entries. Let us denote

$$K_i(1, 1) = E[X_{1i}]^2 = P_{1i};$$
$$K_i(2, 2) = E[X_{2i}]^2 = P_{2i}.$$

Now, let us introduce two parameters $\gamma_i, \beta_i \in [0, 1]$ for each $i \in \{1, \cdots, n\}$ such that

$$K_i(1, 3) = E[X_{1i}S_i] = \eta_{1i} \sqrt{(1 - \gamma_i)P_{1i}Q}$$
$$K_i(2, 3) = E[X_{2i}S_i] = \eta_{2i} \sqrt{(1 - \beta_i)P_{2i}Q},$$

where $\eta_{ji} \in \{-1, +1\}, j = 1, 2$ is the sign of the correlation. The remaining terms of $K_i$ can be evaluated using (22) and

$$Var\ [X_{1i}|S_i] = \min_a E [X_{1i} - aS_i]^2 = \gamma_i P_{1i}$$
$$Var\ [X_{2i}|S_i] = \min_a E [X_{2i} - aS_i]^2 = \beta_i P_{2i}.$$

Let us also define two parameters $\gamma \in [0, 1]$ and $\beta \in [0, 1]$:

$$\gamma = \frac{1}{nP_{1i}} \sum_{i=1}^{n} \gamma_i P_{1i}$$
$$\beta = \frac{1}{nP_{2i}} \sum_{i=1}^{n} \beta_i P_{2i}. \quad (28)$$

With this, we are all set to prove (25). First of all, considering $\mu_1 \geq \mu_2$ is sufficient, as a simple renaming of the indices will give us the opposite case. For $\mu_2 > 0$, since $\lambda$ is an arbitrary positive number, we can equivalently maximize $\mu_1 R_1 + \mu_2 R_2 + \mu_2 \lambda \frac{1}{2} \log \frac{Q}{D_n}$. Dividing by $\mu_2$, and then renaming $\frac{\mu_1}{\mu_2}$ as $\mu$, the maximization becomes $\forall \mu \geq 1, \lambda \geq 0$,

$$\max \mu R_1 + R_2 + \frac{\lambda}{2} \log \frac{Q}{D_n}. \quad (29)$$

For a given $\mu > 1$, three regimes of $\lambda$ are of interest, as depicted in Figure 4. These regimes can be identified in the $\lambda - \mu$ plane as the three cases marked in Figure 5.

We give slightly different proofs for the three cases marked above. We begin with some discussion common to all the
cases. Let $R_1(\gamma), R_2(\beta), R_{sum}(\gamma, \beta)$ and $D(\gamma, \beta)$, respectively, denote the RHS of equations (4) – (7). The following two lemmas play a key role in our proofs for the various regimes.

Lemma 7. For $\lambda \leq 1$, and $\gamma, \beta$ defined in (28), we have

$$\mu R_1 + R_2 + \frac{\lambda}{2} \log \frac{Q}{D_n} \leq (\mu - 1) R_1(\gamma) + R_{sum}(\gamma, \beta) + \frac{\lambda}{2} \log \frac{Q}{D(\gamma, \beta)} + o(1).$$

Proof: The proof is given in Appendix A

Lemma 8. For $\lambda > 1$, the function $f(\gamma, \beta) := R_{sum}(\gamma, \beta) + \frac{1}{2} \log \frac{Q}{D(\gamma, \beta)}$ is a non-increasing function in each of the arguments, i.e., for $\gamma \in [0, 1]$ and $\beta \in [0, 1]$.

Proof: Notice that $D(\gamma, \beta)$ increases with $\gamma$ (or $\beta$), see (7). Furthermore, a simple inspection shows that the function $f(\gamma, \beta)$ is decreasing in each of the arguments.

Let us now consider the different regimes for $\lambda$ as in Figure 5.

Case 1 ($\lambda \leq 1$ and $\mu \geq 1$): In this regime, Lemma 7 directly gives a bound on the weighted sum-rate.

Case 2 ($\lambda \geq \mu$ and $\mu \geq 1$): This requires a slightly different approach than above. Since $\mu \geq 1$, we write

$$\mu R_1 + R_2 + \frac{\lambda}{2} \log \frac{Q}{D_n} \leq \mu R_1 + \mu R_2 + \frac{\lambda}{2} \log \frac{Q}{D_n}$$

$$= \mu (R_1 + R_2) + \frac{\lambda}{2} \log \frac{Q}{D_n}$$

$$= \mu (R_1 + R_2 + \frac{1}{2} \log \frac{Q}{D_n}) + \frac{\lambda - \mu}{2} \log \frac{Q}{D_n}.$$

In step (a) we used Lemma 7 followed by Lemma 8 and (b) follows from the fact that the minimal distortion possible is obtained by uncoded transmission of the state by the two users acting as a super-user with power $(\sqrt{P_1} + \sqrt{P_2})^2$ [4]. In other words, an equivalent point-to-point channel results in the absence of messages at both the transmitters.

Case 3 ($1 \leq \lambda \leq \mu$ and $\mu \geq 1$): Here also we modify an appropriate hyperplane by changing the weight on $R_2$, but this time with respect to the weight $\lambda$ on the distortion. More specifically, since $\lambda \geq 1$

$$\mu R_1 + R_2 + \frac{\lambda}{2} \log \frac{Q}{D_n} \leq \mu R_1 + \lambda R_2 + \frac{\lambda}{2} \log \frac{Q}{D_n}$$

$$= (\mu - \lambda) R_1 + \lambda (R_1 + R_2 + \frac{1}{2} \log \frac{Q}{D_n})$$

$$\leq (\mu - \lambda) R_1 + \lambda \left( R_{sum}(\gamma, 0) + \frac{1}{2} \log \frac{Q}{D(\gamma, 0)} \right),$$

where the last step used Lemmas 7 and 8. From (49), we can infer that $R_1$ is at most $\frac{1}{2} \log (1 + \gamma P/\sigma^2)$. Thus,

$$\mu R_1 + R_2 + \frac{\lambda}{2} \log \frac{Q}{D_n} \leq \mu R_{sum}(\gamma, 0) + \frac{\lambda}{2} \log \frac{Q}{D(\gamma, 0)}.$$

(32)

Let us now show that bounds (30) – (32) indeed define the region given in Theorem 2.

V. EQUIVALENCE OF INNER AND OUTER BOUNDS

In this section, we prove that the respective regions defined by the inner and outer bounds in Sections III and IV coincide, thereby establishing the capacity region. As before, for each value of the weight $\mu \geq 1$, we will consider three different regimes for $\lambda \geq 0$, and show that the maximal value of $\mu R_1 + R_2 + \frac{\lambda}{2} \log \frac{Q}{D_n}$ in the outerbound can be achieved. Before we embark on this, a numerical example is in order.

Let us take $P_1 = 2, P_2 = 2, \sigma^2 = 1, Q = 1$. The optimal trade-off is plotted in Figure 6 where $(x, y, z)$-axis have $(R_1, R_2, \log \frac{Q}{D})$ values. In the first quadrant of $\mathbb{R}^3$, 6 distinct faces to the region can be identified. Notice the oblique face along $z$–axis in the plot. This face is a pentagon, corresponding to the maximal distortion, however this does not coincide with $(x-y)$ plane in the plot shown (the plot begins at $z = z_m$). This is to emphasize the fact that even if we care only about optimizing the transmission rates, still some reduction of distortion from its maximal possible value of $Q$ can be achieved. This can also be seen from the employed DPC scheme. Essentially the maximal distortion $D_{max} < Q$ can be achieved while operating at the maximal sum-rate. In principle, the extreme pentagon at $z = \log \frac{Q}{D_{max}}$ in the current plot can be extended all the way to $z = 0$. Notice that any other cross-section along $z$–axis in the interior of the plot is not even a polytope, see Figure 7.
While maximizing $\gamma, \beta$ we already observed that the region $\mu R_1 + R_2 + \frac{1}{2} \log \sigma_n^2 \leq f(\gamma, \beta)$ for some value of $(\gamma, \beta) \in [0, 1]^2$, the strict concavity of $f(\cdot)$ suggests that for the given $\mu > 1$ and $0 \leq \lambda \leq 1$, there is a unique $(\gamma, \beta)$ for which $\mu R_1 + R_2 + \frac{1}{2} \log \sigma_n^2$ is maximized. Clearly, choosing the maximizing parameters $\gamma, \beta$ in our achievable theorem will give us the same operating point. Reversing the roles of $R_1$ and $R_2$, we have covered the whole region, except when $\mu = 1$. Anticipating the end-result, we will call the extremal surface for $\mu = 1$ as the dominant face of the plot, somewhat abusing the term face. Clearly $R_{\text{sum}}(\gamma, \beta) + \frac{1}{2} \log \frac{Q}{D(\gamma, \beta)}$ is a strictly concave function, and hence maximized at a unique value of $(\gamma, \beta)$. Thus for a given value of distortion, the dominant line simply connects the points $A_1$ and $A_2$ given by

$$A_1 = \left( \frac{1}{2} \log \left( 1 + \frac{\gamma P_1}{\sigma_Z^2} \right), \frac{1}{2} \log \left( 1 + \frac{\beta P_2}{\gamma P_1 + \sigma_Z^2} \right) \right)$$

$$A_2 = \left( \frac{1}{2} \log \left( 1 + \frac{\gamma P_1}{\beta P_2 + \sigma_Z^2} \right), \frac{1}{2} \log \left( 1 + \frac{\beta P_2}{\gamma P_1 + \sigma_Z^2} \right) \right)$$

for appropriate $(\gamma, \beta) \in [0, 1]^2$. Evidently, each rate-pair in the dominant line is achievable by our communication scheme. This completes the proof of Theorem 2.

Let us now turn our attention towards a multi-user MAC with or without state.

VI. MESSAGE AND SOURCE COMMUNICATION FOR A $N-$SENDER GMAC

In the $N-$sender model all the transmitting nodes observe the same source process. They should help the receiver estimate the state process. In addition each node may have an independent stream of messages to be communicated to the base station. Theorem 4 gives the optimal trade-off region. We prove this theorem below.

A. Achievable Scheme

The proof of achievability follows the same lines as before, via power sharing, dirty paper coding and MMSE estimation. In particular we choose

$$X_{ij} = \sqrt{\frac{(1 - \gamma_i)P_i}{Q}} S_j , \quad 1 \leq i \leq N, \quad 1 \leq j \leq n,$$

and choose $X_{ij} = X_{ij} + X_{imj}$, where $X_{imj} \sim N(0, \gamma_i P_i)$ is dirty paper coded. Suppose we employ successive cancellation at the decoder. Then user $i$ does DPC to generate $X_{im}^n$, treating $\sum_{j=1}^n X_{imj}^n + \alpha S_j^n$ as the known dirt. The effective interference for user $i$ is $\sum_{j \neq i} X_{imj}^n + Z_i^n$, where $S_j$ is the set of users decoded after user $i$ by the successive cancellation decoder.
Once all the messages are decoded, these signals are removed from the received symbols, and state estimation is done using a linear MMSE estimate. Taking different user permutations for successive cancellation, and further time-sharing will give the rate region given in Theorem 4. These straightforward computations are omitted here.

B. Converse Bound

We now prove the converse. We are interested in maximizing
\[
\max \sum_{i=1}^{N} \mu_i R_i + \frac{\lambda}{2} \log \frac{Q}{D_n},
\]
for \( \mu_i \geq 0 \) and \( \lambda \geq 0 \). Without loss of generality, assume that \( \mu_i \leq \mu_{i-1}, \forall i \). As we did for the two user case, since \( \lambda \geq 0 \) is arbitrary, by suitable scaling, we can take \( \mu_N = 1 \), and \( \mu_i \geq 1, 1 \leq i \leq N-1 \).

Then the following three regimes arise.

\begin{itemize}
\item \( 0 \leq \lambda \leq 1 \)
\item \( \lambda \geq \mu_1 \)
\item \( \mu_j \leq \lambda \leq \mu_{j-1} \) for some \( j \in \{2, \ldots, N\} \).
\end{itemize}

Let us first extend Lemma 7 to \( N \)– senders. To this end, define for \( i = 1, \ldots, N \),
\[
R_{\text{sum}}(\gamma_1, \ldots, \gamma_i) := \frac{1}{2} \log(1 + \sum_{j=1}^{i} \gamma_j P_j).
\]

Also let \( D(\gamma_1, \ldots, \gamma_N) \) denote the RHS of (10).

Lemma 10. For \( \lambda \leq 1 \), \( \mu_{N+1} = 0 \), and \( 1 = \mu_N \leq \mu_{N-1} \leq \ldots \), \( \lambda \leq \mu_1 \),
\[
\sum_{i=1}^{N} \mu_i R_i + \frac{\lambda}{2} \log \frac{Q}{D_n} \leq \sum_{i=2}^{N+1} (\mu_{i-1} - \mu_i) R_{\text{sum}}(\gamma_1, \ldots, \gamma_i) + \frac{\lambda}{2} \log \frac{Q}{D(\gamma_1, \ldots, \gamma_N)} + o(1).
\]

Proof: The proof is relegated to Appendix C.

The following lemma can be found true by inspection.

Lemma 11. For \( \nu \in [0, 1]^N \), the function \( f(\nu) \triangleq R_{\text{sum}}(\nu) + \frac{1}{2} \log \frac{Q}{D(\nu)} \) is a non-increasing function in each of the arguments \( \nu_i, 1 \leq i \leq N \).

Now we consider the various regimes involving \( \lambda \).

Case 1 (\( \lambda \leq 1 \)): In this regime, Lemma 10 directly gives a bound on the weighted sum-rate.

Case 2 (\( \lambda \geq \mu_1 \)): Since \( \lambda \geq \mu_i, \forall i \) in this case, we can write
\[
\sum_{i=1}^{N} \mu_i R_i + \frac{\lambda}{2} \log \frac{Q}{D_n} \leq \mu_1 \sum_{i=1}^{N} R_i + \frac{\lambda}{2} \log \frac{Q}{D_n} = \mu_1 \left( \sum_{i=1}^{N} R_i + \frac{1}{2} \log \frac{Q}{D_n} \right) + \frac{\lambda - \mu_1}{2} \log \frac{Q}{D_n},
\]
where \( \bar{0} \) is a vector of \( N \) zeros. Observe that (a) is implied by Lemmas 10 – 11 and (b) follows since \( D(\bar{0}) \) is the minimal distortion possible. Notice that \( D(\bar{0}) \) can be achieved by uncoded communication of the state by all transmitters.

Case 3 (\( \mu_j \leq \lambda \leq \mu_{j-1} \)): Since \( \lambda \geq \mu_j, \mu_{j+1}, \ldots \) let us bound
\[
\sum_{i=1}^{N} \mu_i R_i + \frac{\lambda}{2} \log \frac{Q}{D_n} \leq \sum_{i=1}^{j-1} \mu_i R_i + \sum_{i=j}^{N} \lambda R_i + \frac{\lambda}{2} \log \frac{Q}{D_n} = \sum_{i=1}^{j-1} (\mu_i - \lambda) R_i + \lambda \left( \sum_{i=1}^{N} R_i + \frac{1}{2} \log \frac{Q}{D_n} \right) \leq \sum_{i=1}^{j-1} (\mu_i - \lambda) R_i + \lambda \left( \sum_{i=1}^{N} R_i + \frac{1}{2} \log \frac{Q}{D(\gamma_1, \ldots, \gamma_N)} \right) \leq \sum_{i=1}^{j-1} (\mu_i - \lambda) R_i + \lambda R_{\text{sum}}(\gamma_1, \ldots, \gamma_{j-1}) + \frac{1}{2} \log \frac{Q}{D(\gamma_1, \ldots, \gamma_{j-1}, 0)},
\]
where \( \bar{0} \) is a vector of \( N-j+1 \) zeros, (a) follows from Lemma 10 with \( \mu_1 = \cdots = \mu_N = \lambda \) and (b) follows from Lemma 11 and equation (36). Expression (36) will imply that
\[
\sum_{i=1}^{j-1} (\mu_i - \lambda) R_i \leq \sum_{i=2}^{j-1} (\mu_i - \lambda) R_{\text{sum}}(\gamma_1, \ldots, \gamma_{i-1}) + \mu_{j-1} R_{\text{sum}}(\gamma_1, \ldots, \gamma_{j-1}).
\]
From the last two expressions, we get
\[
\sum_{i=1}^{N} \mu_i R_i + \frac{\lambda}{2} \log \frac{Q}{D_n} \leq \mu_{j-1} R_{\text{sum}}(\gamma_1, \ldots, \gamma_{j-1}) + \sum_{i=2}^{j-1} (\mu_i - \lambda) R_{\text{sum}}(\gamma_1, \ldots, \gamma_{i-1})
\]
\[ C. \text{ Equivalence of Inner and Outer bounds} \]

Let us now show that the outerbound defines the same region as that can be achieved by our communication scheme. We give an inductive argument. The base case of \( N = 1 \) follows from [3]. Suppose the inner and outer bounds given in the previous subsections are equivalent for \( N - 1 \) users or lower. We will then show that the equivalence extends to \( N \) users also. Again, let us consider different regimes for \( \lambda > \mu \), notice that the outer bound corresponds to messages of zero rate and a distortion of \( D(0) \). Clearly, this can be achieved by any user sending a scaled source process, i.e., for \( 1 \leq k \leq N \),

\[ X_{ki} = \sqrt{\frac{P_{ki}}{Q}} S_i, \quad (42) \]

If \( \mu_j \leq \lambda \leq \mu_{j-1} \), then the outerbound has terms of the form \( R_{sum}^{\lambda} (\gamma_1, \ldots, \gamma_i) \) with \( l < j \). A natural communication choice is to set [42] for users \( j, \ldots, N \). Notice that the model now is effectively a \( j - 1 < N \) user problem, with a modified state process. By induction, the inner and outerbounds coincide here. Thus, we are left with the case \( 0 \leq \lambda < 1 \).

Assume for simplicity that \( \lambda > 0 \) and \( \mu_i > 0 \), \( 1 \leq i \leq N \). Using Lemma [17] (Appendix B), we can find a set of parameters \( \gamma_1, \ldots, \gamma_N \), as the unique maximizer of the RHS in (37), for a given \( \lambda \), \( \mu_1, \ldots, \mu_N \). Clearly the achievable scheme can get these rates by using a power split of \((1-\gamma_k)P\) and \(\gamma_kP\) respectively between the uncoded source transmission and message rate at user \( k \). This solves the \( N \)-sender problem.

\[ \text{VII. Conclusion} \]

In this paper, we considered joint message transmission and state estimation in a state dependent Gaussian multiple access channel. The optimal trade-off between the rates of the messages at two encoders and state estimation distortion was completely characterized. It was also shown that for source and message communication over a GMAC without state, a strategy of uncoded communication at the nodes without any messages and power sharing between message transmission (using DPC) and state amplification at terminals that have a message to transmit in addition, turns out to be optimal.

The discrete memoryless MAC counterpart of the current model would be an interesting open problem for further investigations.

\[ \text{APPENDIX A} \]

\[ \text{Proof of Lemma [7]} \]

By Fano’s inequality [18], we can write for any \( \epsilon > 0 \) for large enough \( n \)

\[ H(W_1, W_2|Y^n) \leq n \epsilon. \quad (43) \]

Since the \( n \epsilon \) terms do not affect our end results, we will neglect these in the sequel.

\[ n\mu R_1 + n R_2 + n \frac{\lambda}{2} \log \left( \frac{Q}{D_n} \right) \]

\[ = n(\mu - 1) R_1 + n \sum_{i=1}^2 R_i + n \frac{\lambda}{2} \log \left( \frac{Q}{D_n} \right) \]

\[ \leq (\mu - 1) H(W_1) + H(W_1, W_2) + \lambda I(S^n; Y^n) \]

\[ \leq (\mu - 1) I(W_1; X_2^n, S_i^n) + H(W_1, W_2|S_i^n) + \lambda I(S^n; Y^n) \]

\[ \leq (\mu - 1)(h(Y^n|X_2^n, S_i^n) - h(Y^n|W_1, X_2^n, S_i^n)) \]

\[ + \lambda h(Y^n) - h(Y^n|W_1, S_i^n)) \]

\[ + (1 - \lambda)(h(Y^n|S_i^n) - h(Y^n|W_1, W_2, S_i^n)) \]

\[ \leq \sum_{i=1}^n \{ (\mu - 1)(h(Y_i|X_2, S_i) - h(Z_i))) \]

\[ + \lambda h(Y_i) + (1 - \lambda) h(Y_i|S_i) - h(Z_i)), \quad (44) \]

where (a) uses Lemma [8], (b) follows since \( W_1, W_2 \perp \perp S_i^n \) and \( W_1 \perp \perp X_2^n \), and (c) follows from Fano’s inequality. Let us now upper bound the term \( \lambda h(Y_i) + (1 - \lambda) h(Y_i|S_i) \) in (44). Notice that \( \lambda h(Y_i) + (1 - \lambda) h(Y_i|S_i) \) is simultaneously maximized (for fixed \( K_i(X_{1i}, X_{2i}, S_i) \)) when \( (X_{1i} + X_{2i}) \) is jointly Gaussian with \( S_i \) [4]. Without loss of generality, for the purposes of finding an upper bound on \( \mu R_1 + R_2 + \frac{\lambda}{2} \log \frac{Q}{D_n} \), we can express, using (27)

\[ \sum_{j=1}^2 X_{ji} = V_i + \left( \eta_{1i} \sqrt{1 - \gamma_i} \frac{P_{1i}}{Q} + \eta_{2i} \sqrt{1 - \beta_i} \frac{P_{2i}}{Q} \right) S_i, \]

where \( V_i \) is zero mean Gaussian, independent of \( S_i \). The second term on the RHS can be understood as the linear estimate of \( (X_{1i} + X_{2i}) \) given \( S_i \). Since \( V_i \) and \( S_i \) are independent,

\[ \text{Var}[V_i] = \text{Var}(X_{1i} + X_{2i})|S_i^n) \]

\[ \leq \text{Var}(X_{1i}|S_i) + \text{Var}(X_{2i}|S_i) \]

\[ = \gamma_i P_{1i} + \beta_i P_{2i}. \]

Using this

\[ \mathbb{E}[(X_{1i} + X_{2i})^2] \leq P_{1i} + P_{2i} + 2 \sqrt{(1 - \gamma_i)(1 - \beta_i) P_{1i} P_{2i}}, \]

where we have taken \( \eta_{1i} = \eta_{2i} = 1 \) as the sign of correlation in (27), as negative correlation can only be detrimental for the RHS. Now on denoting \( g(x) = (1/2) \log(2\pi e x) \) and using the differential entropy maximizing property of Gaussian random variables for a given variance, we can write

\[ h(Y_i|X_{2i}, S_i) \leq h(X_{1i} + Z_i|S_i) \]

\[ \leq g(\text{Var}[X_{1i}|S_i] + \sigma_Z^2) = g(\gamma_i P_{1i} + \sigma_Z^2). \quad (45) \]
\( h(Y_i | S_i) = h(V_i + Z_i) \leq g(\gamma_i P_{1i} + \beta_i P_{2i} + \sigma_Z^2), \) \tag{46} \\
h(Y_i) \leq g(Q + \sigma_Z^2 + \mathbb{E}[X_{1i} + X_{2i}]^2 + 2\mathbb{E}[(X_{1i} + X_{2i}) S_i]) \leq \left( g(\gamma_i P_{1i} + P_{2i} + Q + \sigma_Z^2) + 2\sqrt{\gamma_i} P_{1i} Q + 2\sqrt{\gamma_i} \beta_i P_{3i} P_{2i} \right), \tag{47} \\
where \([46]\) is under the choice of \( X_{1i} + X_{2i} \) which maximizes \( \lambda h(Y_i) + (1 - \lambda) h(Y_i | S_i) \), for all \( \lambda \in [0, 1] \). Continuing the chain of inequalities from \([44]\):

\[
\begin{align*}
n \mu R_1 + n R_2 + \frac{n \lambda}{2} \log \left( \frac{Q}{D_n} \right) \\
\leq \sum_{i=1}^{n} (\mu - 1) \frac{1}{2} \log \left( \frac{\gamma_i P_{1i} + \sigma_Z^2}{\sigma_Z^2} \right) \\
+ \sum_{i=1}^{n} \frac{\lambda}{2} \log \left( \frac{P_{1i} + P_{2i} + Q + \sigma_Z^2 + 2\sqrt{\gamma_i} P_{1i} Q}{\sigma_Z^2} \right) \\
+ \sum_{i=1}^{n} \frac{(1 - \lambda)}{2} \log \left( \frac{\gamma_i P_{1i} + \beta_i P_{2i} + \sigma_Z^2}{\sigma_Z^2} \right) \\
= (\mu - 1) \frac{n}{2} \log \left( \frac{\gamma P_1 + \sigma_Z^2}{\sigma_Z^2} \right) \\
+ \frac{\lambda n}{2} \log \left( \frac{P_1 + P_2 + Q + \sigma_Z^2 + 2\sqrt{\gamma P_1 Q}}{\sigma_Z^2} \right) \\
+ \frac{n(1 - \lambda)}{2} \log \left( \frac{\gamma P_1 + \beta P_2 + \sigma_Z^2}{\sigma_Z^2} \right) \\
\leq (\mu - 1) \frac{n}{2} \log \left( \frac{Q}{D_{(\gamma, \beta)}} \right) \\
+ n R_{sum}(\gamma, \beta) + \frac{\lambda n}{2} \log \left( \frac{Q}{D_{(\gamma, \beta)}} \right), \tag{48}
\end{align*}
\]

where (a) follows from the fact that both \( \lambda \) and \( (1 - \lambda) \) are non-negative for \( \lambda \in [0, 1] \) and expression \([46]\) and \([47]\), (b) follows from Jensen’s Inequality, and (c) follows from the definitions of \( R_{sum}(\gamma, \beta), D(\gamma, \beta) \) from \([6]\) and \([7]\). Thus the lemma is proved for all \( \lambda \in [0, 1] \). From \([45]\), we also get

\[
R_1 \leq \frac{1}{2} \log(1 + \gamma P_1). \tag{49}
\]

\section*{Appendix B}

\subsection*{Proof of Lemma 9}

Here we prove a slightly more general result, which turns out useful for the remaining sections. Consider a concave function \( L(\nu), \nu \in [0, 1]^N \), and let us define

\[
f(\nu) := \sum_{i=1}^{N} \alpha_i \log(1 + \nu_j P_j) + \frac{\lambda}{2} \log L(\nu), \tag{50}
\]

where \( \alpha_i, 1 \leq i \leq N \) and \( \lambda \) are non-negative constants.

\begin{lemma}
For \( 0 < \lambda \leq 1 \), \( f(\cdot) \) is strictly concave in \( \nu \in (0, 1)^N \), whenever \( \alpha_i, 1 \leq i \leq N \) are not identically zero.
\end{lemma}

\begin{proof}
The first term, being the linear combination of logarithms, is strictly concave. Let us consider the second term. Let \( x_1 \) and \( x_2 \) be two \( N \)- dimensional vectors in \( \mathbb{R}^N \). Notice that for \( \zeta \in [0, 1] \),

\[
\zeta \log L(x_1) + (1 - \zeta) \log L(x_2) \leq \log(\zeta L(x_1) + (1 - \zeta) L(x_2)) \leq \log L(\zeta x_1 + \zeta x_2), \tag{51}
\]

since \( L(\cdot) \) itself is concave by assumption. This proves the lemma.
\end{proof}

Let us proceed to show Lemma 9. Denote

\[
L(\gamma, \beta) = P_1 + P_2 + Q + \sigma_Z^2 + 2\sqrt{\gamma P_1 Q} + 2\sqrt{\beta P_2 Q} + 2\sqrt{\gamma \beta P_1 P_2}, \tag{52}
\]

for convenience. Rewriting the function given in lemma,

\[
f(\gamma, \beta) = \frac{(\mu - 1)}{2} \log \left( 1 + \frac{\gamma P_1}{\sigma_Z^2} \right) + \frac{(1 - \lambda)}{2} \log \left( 1 + \frac{\gamma P_2}{\sigma_Z^2} \right) + \frac{\lambda}{2} \log \left( \frac{L(\gamma, \beta)}{\sigma_Z^2} \right). \tag{53}
\]

Thus \( f(\gamma, \beta) \) is a sum similar to \([50]\). Our proof will be complete by showing \( L(\cdot) \) in \([52]\) to be a concave function. For \( c_0 > 0 \), and non-negative constants \( c_1, \cdots, c_N \), the function

\[
L(\nu) = c_0 + \sum_{i=1}^{N} \sum_{j=1}^{N} c_i c_j \sqrt{(1 - \nu_i)(1 - \nu_j)}
\]

is concave. To see this, notice that \( \sqrt{x} \) is strictly concave in \( x \geq 0 \). Also, \( \sqrt{xy} \) is jointly concave in \( (x, y) \in [0, 1]^2 \), making \( L(\nu) \) a concave function. Notice that concavity in the range of interest is maintained by replacing each and every variable \( x \in [0, 1] \) by \( 1 - x \). The proof of the lemma is now complete.

\section*{Appendix C}

\subsection*{Proof of Lemma 10}

Recall that \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_N = 1 \). We need to show \([57]\) for \( \lambda \leq 1 \). Let us define \( \mu_{N+1} := 0 \). Now

\[
\begin{align*}
n \sum_{j=1}^{N} \mu_j R_j + \frac{n \lambda}{2} \log \left( \frac{Q}{D_n} \right) \\
= \sum_{j=2}^{N+1} (\mu_{j-1} - \mu_j) \sum_{i=1}^{j-1} R_i + \frac{n \lambda}{2} \log \left( \frac{Q}{D_n} \right) \\
= \sum_{j=2}^{N+1} (\mu_{j-1} - \mu_j) H(W^{j-1}) + \frac{n \lambda}{2} \log \left( \frac{Q}{D_n} \right) \\
= \sum_{j=2}^{N} (\mu_{j-1} - \mu_j) H(W^{j-1}) + H(W^{N}) + \frac{n \lambda}{2} \log \left( \frac{Q}{D_n} \right) \\
\leq \sum_{j=2}^{N} (\mu_{j-1} - \mu_j) I(W^{j-1}; Y^n | S^n, X^n_j, \cdots, X^n_N), \tag{a}
\end{align*}
\]
where (a) follows since $W^N \leq (1 + \lambda) I(W^N; Y^n) + \lambda I(S^n; Y^n)$

\[
\sum_{j=2}^N (\mu_{j-1} - \mu_j) I(W^{j-1}; Y^n|S^n, X^n_j, \ldots, X^n_N) \leq \sum_{j=2}^N (\mu_{j-1} - \mu_j) I(W^{j-1}; Y^n|S^n, X^n_j, \ldots, X^n_N) + (1 - \lambda) I(W^N; Y^n) + \lambda I(S^nW^N; Y^n)
\]

\[
= \sum_{j=2}^N (\mu_{j-1} - \mu_j) I(Y^n|S^n, X^n_j, \ldots, X^n_N) - h(Z^n)) + (1 - \lambda) (h(Y^n|S^n) - h(Z^n)) + \lambda (h(Y^n) - h(Z^n))
\]

\[
= \sum_{j=2}^N (\mu_{j-1} - \mu_j) \sum_{i=1}^n (h(Y_i|S^n, X^n_j, \ldots, X^n_N) - h(Z_i)) + (1 - \lambda) \sum_{i=1}^n h(Y_i|S^n) + \lambda \sum_{i=1}^n h(Y_i) - \sum_{i=1}^n h(Z_i)
\]

where (a) follows since $W^N \perp S^n$, $W^{j-1} \perp (S^n, X^n_j, \ldots, X^n_N)$ and Fano’s inequality.

Let us now consider the covariance matrix $K_i$ of $(X_{1i}, \ldots, X_{Ni}, S_i)$. For some $\gamma_{ki} \in [0,1]$ we can write

\[
\mathbb{E} X_{ki} S_i = \gamma_{ki} \sqrt{(1 - \gamma_{ki}) P_k \lambda Q},
\]

where $P_k = K_i(k,k), 1 \leq k \leq N$ is the empirical average power of user $k$ for transmission index $i$ in a block, and $\gamma_{ki} \in \{-1, +1\}$ is the sign of the correlation. From this, we also get

\[
\text{Var}(X_{ki}|S_i) = \gamma_{ki} P_k.
\]

Let us now bound the entropy terms in (54).

\[
h(Y_i|S^n, X_j, \ldots, X_{Ni}) \leq g(\text{Var}(\sum_{k=1}^{j-1} X_{ki} + Z_i|S^n)))
\]

\[
\leq g(\sum_{k=1}^{j-1} \text{Var}(X_{ki} + Z_i|S^n))
\]

\[
\leq g(\sum_{k=1}^{j-1} \text{Var}(X_{ki}|S_i) + \sigma_{ki}^2)
\]

\[
\leq g(\sigma_{ki}^2 + \sum_{k=1}^{j-1} \gamma_{ki} P_k).
\]

Putting these altogether

\[
h(Y_i|S_i) = h(V_i + Z_i) \leq g \left( \sum_{j=1}^N \gamma_{ji} P_{ji} + \sigma_{Z_i}^2 \right),
\]

\[
h(Y_i) \leq g \left( \sum_{j=1}^N P_{ji} + \alpha^2 Q + \sigma_{Z_i}^2 + 2\alpha \sum_{j=1}^N \sqrt{\gamma_{ji} P_{ji} Q} \right)
\]

\[
+ 2 \sum_{j=1}^N \sum_{k \neq j} \sqrt{\gamma_{ji} \gamma_{ki} P_{ji} P_{ki}}.
\]

Defining $\gamma_k = \frac{1}{\sqrt{\lambda_k}} \sum_{i=1}^n \gamma_{ki} P_k$, for $\lambda \in (0, 1)$, we get from (54), (56) and (59)

\[
\sum_{j=1}^N \mu_j R_j + \frac{\lambda}{2} \log \left( \frac{Q}{D_n} \right)
\]

\[
\leq \sum_{j=2}^N (\mu_{j-1} - \mu_j) g(\sigma_{Z_i}^2 + \sum_{k=1}^{j-1} \gamma_{ki} P_k)
\]

\[
+ (1 - \lambda) g(\sigma_{Z_i}^2 + \sum_{j=1}^N \gamma_{ji} P_j)
\]

\[
+ \lambda g \left( \sum_{j=1}^N P_{ji} + \alpha^2 Q + \sigma_{Z_i}^2 + 2\alpha \sum_{j=1}^N \sqrt{\gamma_{ji} P_{ji} Q} \right)
\]

\[
+ 2 \sum_{j=1}^N \sum_{k \neq j} \sqrt{\gamma_{ji} \gamma_{ki} P_{ji} P_{ki}} - \mu_1 g(\sigma_{Z_i}^2).
\]

Here we used Jensen’s inequality on (56), (58) and (59) to get a single letter form independent of the transmission index. This proves Lemma 10. Note that for $S \subseteq \{1, \ldots, N\}$, by giving $(S^n, X^n_{j\in S^c})$ to the receiver and using Jensen’s inequality, we obtain (by following similar steps as (56)),

\[
n \sum_{j \in S} R_j \leq I(X^n_{j\in S}; Y^n|S^n, X^n_{j\in S^c})
\]

\[
= h(Y^n|S^n, X^n_{j\in S^c}) - h(Z^n)
\]

\[
\leq \frac{n}{2} \log(1 + \frac{\sum_{j \in S} \gamma_{ji} P_j}{\sigma_{Z_i}^2}), \forall S \subseteq \{1, \ldots, N\}.
\]

(61)

REFERENCES

[1] “HD Radio — Wikipedia, the free encyclopedia,” https://en.wikipedia.org/wiki/HD_Radio&oldid=861909787 2018, [Online; accessed 6-November-2018].

[2] F. Wang, J. Huang, X. Chen, G. Zhang, and A. Men, “Simultaneous broadcasting of analog fm and digital signals by separating co-channel fm signals,” IEEE Communications Letters, vol. 20, no. 11, pp. 2197–2200, 2016.

[3] A. El Gamal and Y.-H. Kim, Network information theory. Cambridge University Press, 2011.

[4] A. Sutivong, M. Chiang, T. M. Cover, and Y.-H. Kim, “Channel capacity and state estimation for state-dependent Gaussian channels,” Information Theory, IEEE Transactions on, vol. 51, no. 4, pp. 1486–1495, 2005.

[5] C. E. Shannon, “Channels with side information at the transmitter,” IBM journal of Research and Development, vol. 2, no. 4, pp. 289–293, 1958.

[6] S. Gelfand and M. Pinsker, “Coding for channels with random parameters,” Probl. Contr. Inform. Theory, vol. 9, no. 1, pp. 19–31, 1980.

[7] A. V. Kuznetsov and B. S. Tsybakov, “Coding in a memory with defective cells,” Problemy peredachi informatsii, vol. 10, no. 2, pp. 52–60, 1974.
[8] M. H. Costa, “Writing on dirty paper (corresp.),” *Information Theory, IEEE Transactions on*, vol. 29, no. 3, pp. 439–441, 1983.

[9] H. Weingarten, Y. Steinberg, and S. Shamai, “The capacity region of the gaussian multiple-input multiple-output broadcast channel,” *Information Theory, IEEE Transactions on*, vol. 52, no. 9, pp. 3936–3964, 2006.

[10] Y.-H. Kim, A. Sutivong, and T. M. Cover, “State amplification,” *IEEE Transactions on Information Theory*, vol. 54, no. 5, pp. 1850–1859, 2008.

[11] Y. Zhao and B. Chen, “Capacity theorems for multi-functioning radios,” in *Information Theory (ISIT), 2014 IEEE International Symposium on*. IEEE, 2014, pp. 2406–2410.

[12] C. Choudhuri, Y.-H. Kim, and U. Mitra, “Causal state communication,” *IEEE Transactions on Information Theory*, vol. 59, no. 6, pp. 3709–3719, 2013.

[13] S. I. Bross and A. Lapidoth, “Conveying data and state with feedback,” in *Information Theory (ISIT), 2016 IEEE International Symposium on*. IEEE, 2016, pp. 1277–1281.

[14] N. Merhav and S. Shamai, “Information rates subject to state masking,” *IEEE Transactions on Information Theory*, vol. 53, no. 6, pp. 2254–2261, 2007.

[15] O. O. Koyluoglu, R. Soundararajan, and S. Vishwanath, “State amplification subject to masking constraints,” *IEEE Transactions on Information Theory*, vol. 62, no. 11, pp. 6233–6250, 2016.

[16] W. Liu and B. Chen, “Message transmission and state estimation over Gaussian broadcast channels,” in *Information Sciences and Systems, 2009. CISS 2009. 43rd Annual Conference on*. IEEE, 2009, pp. 147–151.

[17] M. Dikshtein and S. Shamai, “Broadcasting information subject to state masking,” *arXiv preprint arXiv:1810.11781*, 2018.

[18] T. M. Cover and J. A. Thomas, *Elements of information theory*. John Wiley & Sons, 2012.

[19] T. Cover, A. E. Gamal, and M. Salehi, “Multiple access channels with arbitrarily correlated sources,” *IEEE Transactions on Information theory*, vol. 26, no. 6, pp. 648–657, 1980.

[20] G. Dueck, “A note on the multiple access channel with correlated sources (corresp.),” *IEEE Transactions on Information Theory*, vol. 27, no. 2, pp. 232–235, 1981.

[21] M. Gastpar, “Uncoded transmission is exactly optimal for a simple Gaussian sensor network,” *IEEE Transactions on Information Theory*, vol. 54, no. 11, pp. 5247–5251, 2008.

[22] C. Tian, B. Bandemer, and S. S. Shitz, “Gaussian state amplification with noisy observations,” *IEEE Transactions on Information Theory*, vol. 61, no. 9, pp. 4587–4597, 2015.

[23] C. Tian, J. Chen, S. N. Diggavi, and S. S. Shitz, “Matched multiuser gaussian source channel communications via uncoded schemes,” *IEEE Transactions on Information Theory*, vol. 63, no. 7, pp. 4155–4171, 2017.

[24] J.-J. Xiao and Z.-Q. Luo, “Multiterminal source–channel communication over an orthogonal multiple-access channel,” *IEEE Transactions on Information Theory*, vol. 53, no. 9, pp. 3255–3264, 2007.

[25] A. Lapidoth and S. Tinguely, “Sending a bivariate Gaussian over a Gaussian MAC,” *IEEE Transactions on Information Theory*, vol. 56, no. 6, pp. 2714–2725, 2010.

[26] B. Nazer and M. Gastpar, “Computation over multiple-access channels,” *IEEE Transactions on information theory*, vol. 53, no. 10, pp. 3498–3516, 2007.

[27] R. Soundararajan and S. Vishwanath, “Communicating linear functions of correlated Gaussian sources over a MAC,” *IEEE Transactions on Information Theory*, vol. 58, no. 3, pp. 1853–1860, 2012.

[28] Y.-H. Kim, A. Sutivong, and S. Sigurjonsson, “Multiple user writing on dirty paper,” in *Information Theory, 2004. ISIT 2004. Proceedings. International Symposium on*. IEEE, 2004, p. 534.

[29] T. Goblick, “Theoretical limitations on the transmission of data from analog sources,” *IEEE Transactions on Information Theory*, vol. 11, no. 4, pp. 558–567, 1965.

[30] V. M. Prabhakaran, R. Puri, and K. Ramchandran, “Hybrid digital-analog codes for source-channel broadcast of gaussian sources over gaussian channels,” *IEEE Transactions on Information Theory*, vol. 57, no. 7, pp. 4573–4588, 2011.