A basis construction of the extended Catalan and Shi arrangements of the type $A_2$

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Abstract

In [9], Terao proved the freeness of multi-Coxeter arrangements with constant multiplicities by giving an explicit construction of bases. Combining it with algebro-geometric method, Yoshinaga proved the freeness of the extended Catalan and Shi arrangements in [11]. However, there have been no explicit constructions of the bases for the logarithmic derivation modules of the extended Catalan and Shi arrangements. In this paper, we give the first explicit construction of them when the root system is of the type $A_2$.

Keywords: Hyperplane arrangement, Shi arrangement, Catalan arrangement, Free arrangement, Weyl group, Affine Weyl group, Logarithmic derivations

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1 Introduction

Let $V$ be an $ℓ$-dimensional vector space over a field $\mathbb{K}$. Let $S = S(V^*)$ be the symmetric algebra of the dual space $V^*$ and $\text{Der}(S)$ the module of derivations

$$\text{Der}(S) = \{ \theta : S \to S \mid \theta \text{ is } \mathbb{K}\text{-linear}, \quad \theta(fg) = \theta(f)g + f\theta(g) \ (f, g \in S) \}. $$

An (affine) arrangement (of hyperplanes) $\mathcal{A}$ is the finite set of affine hyperplanes in $V$. An arrangement is central if every hyperplane in $\mathcal{A}$ is linear.

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For a central arrangement $A$, the logarithmic derivation module $D(A)$ is defined by
\[
D(A) = \{ \theta \in \text{Der}(S) \mid \theta(Q(A)) \in Q(A) S \}
\]
\[
= \{ \theta \in \text{Der}(S) \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in A \},
\]
where $\alpha_H (H \in A)$ is a linear form such that $\ker(\alpha_H) = H$ and $Q(A)$ is the defining polynomial of $A$, that is, $Q(A) = \prod_{H \in A} \alpha_H$. When $D(A)$ is a free $S$-module, $A$ is called a free arrangement. Then there exists a homogeneous basis $\{\theta_1, \ldots, \theta_\ell\}$ for $D(A)$ and $\exp A := (\deg \theta_1, \ldots, \deg \theta_\ell)$ is called the exponents of $A$. For an affine arrangement $A$ in $V$, $cA$ denotes the cone [4, Definition 1.15] over $A$. The cone $cA$ is a central arrangement in an $(\ell + 1)$-dimensional vector space $U$. We may regard $U^* = V^* \oplus \langle z \rangle$ by using a new coordinate $z$, and let $S_z$ denote the symmetric algebra $S(U^*)$ of the dual space $U^*$.

Let $E$ be an $\ell$-dimensional Euclidean space and $\Phi$ be a crystallographic irreducible root system in the dual space $E^*$. Let $\Phi^+$ be a positive system of $\Phi$. For $\alpha \in \Phi^+ \text{ and } i \in \mathbb{Z}$, define the affine hyperplane $H_{\alpha,i}$ by
\[
H_{\alpha,i} := \{ v \in V \mid \alpha(v) = i \}.
\]
Then the arrangement $A_\Phi = \{ H_{\alpha,0} \mid \alpha \in \Phi^+ \}$ is called the Weyl arrangement of the type $\Phi$.

Definition 1.1. Let $k \in \mathbb{Z}_{\geq 0}$. Then the extended Shi arrangement $\text{Shi}^k$ of the type $\Phi$ and the extended Catalan arrangement $\text{Cat}^k$ of the type $\Phi$ are affine arrangements defined by
\[
\text{Shi}^k := \{ H_{\alpha,i} \mid \alpha \in \Phi^+, -k + 1 \leq i \leq k \},
\]
\[
\text{Cat}^k := \{ H_{\alpha,i} \mid \alpha \in \Phi^+, -k \leq i \leq k \}.
\]
In particular, the arrangement $\text{Shi}^1$ is called the Shi arrangement which was introduced by J.-Y. Shi in [6] in the study of the Kazhdan-Lusztig representation theory of the affine Weyl groups.

There are a lot of researches on the freeness of the cones over the extended Catalan and Shi arrangements. The first breakthrough was the proof of the freeness of multi-Coxeter arrangements with constant multiplicities by Terao in [9]. Combining this result with algebro-geometric methods, Yoshinaga proved the freeness of the cones over the extended Catalan and Shi arrangements in [11]. However, there have been few researches how to construct their explicit bases. Recently, in the case of types $A_\ell, B_\ell, C_\ell, D_\ell$, explicit bases for the cone $c\text{Shi}^1$ over the Shi arrangements were constructed ([7], [8], [3]). Also, a nice basis for the extended Shi arrangements was determined in [2].
In this paper, we give the first explicit construction of a series of bases for the extended Catalan and Shi arrangements when the corresponding root system is of the type $A_2$. Namely, we construct bases for the logarithmic modules of these arrangements as follows:

**Theorem 1.2.** Let $\Phi$ be the root system of the type $A_2$, $\{\alpha_1, \alpha_2\}$ a simple system and $\{\partial_1, \partial_2\}$ its dual basis for $\text{Der}(S)$. For $i \in \mathbb{Z}_{\geq 0}$, define

$$M_n = \begin{pmatrix} \alpha_1 + nz & (2\alpha_1 + 4\alpha_2 + 3nz)(\alpha_1 + nz) \\ \alpha_2 + nz & -(4\alpha_1 + 2\alpha_2 + 3nz)(\alpha_2 + nz) \end{pmatrix},$$

$$N_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (M_n^T)|_{z \rightarrow -z} = \begin{pmatrix} (2\alpha_1 + 4\alpha_2 - 3nz)(\alpha_1 - nz) & -(4\alpha_1 + 2\alpha_2 - 3nz)(\alpha_2 - nz) \\ \alpha_1 - nz & \alpha_2 - nz \end{pmatrix},$$

$$T_n = \begin{pmatrix} 1 & 0 \\ 3n + 1 & 1 \\ 0 & 3n + 2 \end{pmatrix},$$

$$A = [I^*(\alpha_i, \alpha_j)]_{1 \leq i, j \leq 2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

where $I^*$ is the natural inner product on $E^*$ induced from the $W$-invariant inner product $I$ on $E$. Then the Euler derivation and

$$[\partial_1, \partial_2] \prod_{i=0}^{k-1} (M_i T_i N_{i+1} A^{-1})$$

form a basis for $D(c\text{Shi}^k)$, and

$$[\partial_1, \partial_2] \prod_{i=0}^{k-1} (M_i T_i N_{i+1} A^{-1}))M_k$$

a $W$-invariant basis for $D(c\text{Cat}^k)$.

The idea to prove Theorem 1.2 is to use the simple-root bases ([2]) and Terao’s matrix $B^{(k)}$ ([9], [1]) with the invariant theory. Namely, if we fix a simple system and a primitive derivation, then we know the existence of a family of nice bases (the simple-root basis plus/minus) for the logarithmic modules of $c\text{Shi}^k$ for all $k \in \mathbb{Z}_{\geq 0}$. In general, we cannot compute the explicit
form of simple-root bases. However, by computations based on invariant theory and Weyl group actions, we can find a way to construct the bases for that of $c\text{Cat}^k$ from simple-root bases. by restricting them onto the infinite hyperplane and applying the invariant theoretic method, we may connect these new bases. Hence starting from the empty arrangement, we can construct the required bases for the extended Catalan and Shi arrangements inductively. In that invariant theory, Terao’s matrix $B^{(k)}$ plays the essential role.

The organization of this paper is as follows. In section 2, we review the simple-root bases for the extended Shi arrangements introduced in [2], which play key roles in our construction of bases. In section 3, we give an explicit construction of bases for the extended Catalan and Shi arrangements of the type $A_2$ in Theorem 3.1 for a certain primitive derivation. Using Theorem 3.1 we prove Theorem 1.2.

2 Preliminaries

In this section we review the definition and properties of multiarrangements and the simple-root bases for the extended Shi arrangements.

First, let $\mathcal{A}$ be a central arrangement in $V = \mathbb{K}^\ell$, $\{x_1, \ldots, x_\ell\}$ a basis for $V^*$ and fix $H \in \mathcal{A}$. Define

$$D_0(\mathcal{A}) := \{ \theta \in D(\mathcal{A}) \mid \theta(\alpha_H) = 0 \}.$$ 

Then it is known (e.g., see [4, Proposition 4.27]) that

$$D(\mathcal{A}) = S\theta_E \oplus D_0(\mathcal{A})$$

for the Euler derivation $\theta_E := \sum_{i=1}^\ell x_i \frac{\partial}{\partial x_i}$. Hence $\mathcal{A}$ is free if and only if $D_0(\mathcal{A})$ is a free $S$-module, and $\theta_2, \ldots, \theta_\ell$ form a basis for $D_0(\mathcal{A})$ if and only if $\theta_E, \theta_2, \ldots, \theta_\ell$ form a basis for $D(\mathcal{A})$. To check the freeness, the following is the most convenient.

Proposition 2.1. (Saito’s criterion, [5]) Let $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A})$ and $M := (\theta_i(x_j))$. Then $D(\mathcal{A})$ is a free $S$-module with basis $\theta_1, \ldots, \theta_\ell$ if and only if

$$\det M = c \prod_{H \in \mathcal{A}} \alpha_H$$

for some non-zero $c \in \mathbb{K}$.

For a fixed $H \in \mathcal{A}$, let $\mathcal{A}^H := \{ K \cap H \mid K \in \mathcal{A} \setminus \{H\} \}$ and define a map $m_H : \mathcal{A}^H \to \mathbb{Z}_{>0}$ by

$$m_H(K \cap H) := |\{ L \in \mathcal{A} \setminus \{H\} \mid L \cap H = K \cap H \}|.$$
Then for a logarithmic module
\[ D(A^H, m_H) := \{ \theta \in \text{Der}(S/(\alpha_H)) \mid \theta(\alpha_K) \in (S/(\alpha_H))(\alpha_K)^{m_H(K)} \text{ for any } K \in A^H \}, \]
the Ziegler restriction map \( \text{res} : D_0(A) \to D(A^H, m_H) \) is defined by \( \text{res}(\theta) := \theta|_{\alpha_H=0} \). For details, see [12].

**Proposition 2.2** ([12]). Assume that \( A \neq \emptyset \) is free with \( \exp(A) = (1, d_2, \ldots, d_\ell) \). Then \( D_0(A^H, m_H) \) is also free with basis \( \varphi_2, \ldots, \varphi_\ell \) such that \( \deg(\varphi_i) = d_i (i = 2, \ldots, \ell) \). Moreover, the Ziegler restriction map is surjective.

For the rest of this section, let \( E \) be an \( \ell \)-dimensional Euclidean space, and we recall the simple-root bases introduced in [2]. Let \( W \) be a finite irreducible reflection group corresponding to an irreducible root system \( \Phi \) in \( E^* \). Then by the famous theorem of Chevalley, there are homogeneous basic invariants \( P_1, \ldots, P_\ell \) generating the \( W \)-invariant ring \( S^W \) of \( S \) as \( \mathbb{R} \)-algebra such that
\[ \deg P_1 < \deg P_2 \leq \cdots \leq \deg P_{\ell-1} < \deg P_\ell. \]
Let \( F \) be the quotient field of \( S \). Then the primitive derivation \( D = \frac{\partial}{\partial x_\ell} \in \text{Der}(F) \) is characterized by
\[ D(P_i) = \begin{cases} c \in \mathbb{R}^\times & (i = \ell) \\ 0 & (1 \leq i \leq \ell - 1) \end{cases}. \]

The primitive derivation \( D \) is \( W \)-invariant and uniquely determined up to nonzero constant multiple \( c \) independent of the choice of the basic invariants. Define an affine connection \( \nabla : \text{Der}(F) \times \text{Der}(F) \to \text{Der}(F) \) by
\[ \nabla_{\theta_1, \theta_2} = \sum_{i=1}^{\ell} \theta_1(f_i) \frac{\partial}{\partial x_i} \]
for \( \theta_1, \theta_2 \in \text{Der}(F) \) with \( \theta_2 = \sum_{i=1}^{\ell} f_i \frac{\partial}{\partial x_i} \). For \( m \in \mathbb{Z}_{>0} \), define an \( S \)-module \( D(A_\Phi, m) \) by
\[ D(A_\Phi, m) = \{ \theta \in \text{Der}(S) \mid \theta(\alpha_H) \in \alpha_H^m S \text{ for any } H \in A_\Phi \}. \]

Note that the action of \( W \) onto \( E \) canonically extends to those onto \( E^* \), \( S \), \( \text{Der}(S) \) and \( D(A_\Phi, m) \). Let \( D(A_\Phi, m)^W \) denote the \( W \)-invariant set of \( D(A_\Phi, m) \).
Lemma 2.3. ([10], Lemma 9) For the derivations $\frac{\partial}{\partial x_i} \in \text{Der}(S^W)$ $(1 \leq i \leq \ell)$, 
\[ \nabla_{\frac{\partial}{\partial x_i}} D(A_\Phi, 2k+1)^W \subset D(A_\Phi, 2k-1)^W \quad (k > 0). \]

In particular, as shown in [1], the connection $\nabla_D$ induces an $\mathbb{R}[P_1, \ldots, P_{\ell-1}]$-isomorphism 
\[ \nabla_D : D(A_\Phi, 2k+1)^W \cong D(A_\Phi, 2k-1)^W \quad (k > 0). \]

So we can consider the inverse map 
\[ \nabla_D^{-1} : D(A_\Phi, 2k-1)^W \cong D(A_\Phi, 2k+1)^W. \]

Proposition 2.4. ([10], Theorem 7) Define $\partial_v (v \in E)$ by $\partial_v (\alpha) := \alpha(v)$ for $\alpha \in E^*$ which is canonically extended to a derivation $\partial_v : S \rightarrow S$. Define $\Xi : E \rightarrow D(A_\Phi, 2k)$ by $\Xi(v) = \nabla_{\partial_v} \nabla_D^{-1} \theta_E$. Then $\Xi$ is a $W$-isomorphism.

Proposition 2.5. ([11], Theorem 1.2) Let $D_0(c\text{Shi}^k) = \{ \theta \in D(c\text{Shi}^k) | \theta(z) = 0 \}$. Then the Ziegler restriction map $\text{res} : D_0(c\text{Shi}^k) \rightarrow D(A_\Phi, 2k)$ is surjective. In particular, the morphism $\text{res} : D_0(c\text{Shi}^k)_{kh} \rightarrow D(A_\Phi, 2k)_{kh}$ is an $\mathbb{R}$-linear isomorphism, where $D_0(c\text{Shi}^k)_{kh}$ and $D(A_\Phi, 2k)_{kh}$ are the homogeneous parts of degree $kh$ of $D_0(c\text{Shi}^k)$ and $D(A_\Phi, 2k)$ respectively, and $h$ is the Coxeter number of $\Phi$.

Definition 2.6. ([2], Definition 1.2) Fix $k \in \mathbb{Z}_{\geq 0}$. Define a linear isomorphism $\Theta : E \rightarrow D_0(c\text{Shi}^k)$ by $\Theta = \text{res}^{-1} \circ \Xi$. Let $\{\alpha_1, \ldots, \alpha_\ell\} \subset E^*$ be a simple system of $\Phi^*$ and $\{\alpha_1^*, \ldots, \alpha_\ell^*\} \subset E$ be its dual basis. Then the derivations 
\[ \varphi^{(k)}_i = \Theta(\alpha_i^*) \quad (1 \leq i \leq \ell) \]
are called a simple-root basis plus (SRB$_+$) of $D_0(c\text{Shi}^k)$ and the derivations 
\[ \psi^{(k)}_i = \sum_{p=1}^\ell I^*(\alpha_i, \alpha_p) \varphi^{(k)}_p \quad (1 \leq i \leq \ell) \]
are called a simple-root basis minus (SRB$_-$) of $D_0(c\text{Shi}^k)$. Here $I^*$ is the natural inner product on $E^*$ induced from the inner product $I$ on $E$.

Remark 2.7. Let $\Omega_F$ denote the dual $S$-module of $\text{Der}(F)$. The inner product $I^* : E^* \times E^* \rightarrow \mathbb{R}$ can be extended to a nondegenerate $F$-bilinear form $I^* : \Omega_F \times \Omega_F \rightarrow F$. Define an $F$-linear isomorphism $I^* : \Omega_F \rightarrow \text{Der}(F)$.
by $I^*(\omega)(f) := I^*(\omega, df)$ for $\omega \in \Omega_F, f \in F$. Then the restriction of $\text{SRB}_- \text{res}(\psi_i^{(k)})$ can be expressed as follows:

$$\text{res}(\psi_i^{(k)}) = \sum_{p=1}^\ell I^*(\alpha_i, \alpha_p) \nabla_{\alpha_p} \nabla^{-k}_D \theta_E$$

$$= \nabla(\sum_{p=1}^\ell I^*(\alpha_i, \alpha_p) \nabla^{-k}_D \theta_E)$$

$$= \nabla I^*(d\alpha_i \nabla^{-k}_D \theta_E).$$

Remark 2.8. Let $A = [I^*(\alpha_i, \alpha_j)]_{1 \leq i,j \leq \ell}$ be the inner product matrix. Then by definition, an $\text{SRB}_+$ $\{\varphi_1^{(k)}, \ldots, \varphi_\ell^{(k)}\}$ and an $\text{SRB}_-$ $\{\psi_1^{(k)}, \ldots, \psi_\ell^{(k)}\}$ are related as follows:

$$[\varphi_1^{(k)}, \ldots, \varphi_\ell^{(k)}] = [\psi_1^{(k)}, \ldots, \psi_\ell^{(k)}] A^{-1}.$$

It follows from Schur’s lemma that these bases are uniquely determined if we fix a simple system and a primitive derivation $D$. These bases can be characterized by the following conditions:

Proposition 2.9. ([2], Theorem 1.3)

1. Let $\varphi_1^{(k)}, \ldots, \varphi_\ell^{(k)}$ be an $\text{SRB}_+$ of $D_0(c\text{Shi}^k)$. Then $\varphi_1^{(k)}, \ldots, \varphi_\ell^{(k)}$ satisfy

$$\varphi_i^{(k)}(\alpha_j + k z) \in (\alpha_j + k z)S_z \quad (i \neq j).$$

2. Let $\psi_1^{(k)}, \ldots, \psi_\ell^{(k)}$ be an $\text{SRB}_-$ of $D_0(c\text{Shi}^k)$. Then $\psi_1^{(k)}, \ldots, \psi_\ell^{(k)}$ satisfy

$$\psi_i^{(k)} \in (\alpha_i - k z)\text{Der}(S_z) \quad (1 \leq i \leq \ell).$$

Now we introduce some propositions concerning the action of $W$ to these bases.

Proposition 2.10. ([2], Theorem 4.3) The derivation

$$\sum_{i=1}^\ell (\alpha_i + k z)\varphi_i^{(k)}$$

is called the $k$-Euler derivation. The $k$-Euler derivation is $W$-invariant and belongs to $D_0(c\text{Cat}^k)_{kh+1}$.

Proposition 2.11. ([2], Theorem 3.5) Let $s_i \in W$ be the reflection corresponding to $\alpha_i$ for $1 \leq i \leq \ell$. Then

1. $s_i \varphi_j^{(k)} = \varphi_j^{(k)}$ whenever $i \neq j$, and

2. $s_i \left( \frac{\psi_i^{(k)}}{(\alpha_i - k z)} \right) = \frac{\psi_i^{(k)}}{(\alpha_i - k z)}$ for $1 \leq i \leq \ell$. 

3 Construction of bases of the type $A_2$

For the rest of this paper, we assume that the root system $\Phi$ is of the type $A_2$. Hence the Coxeter number $h = 3$ and Yoshinaga’s result in [11] tells us that $c\text{Shi}^k$ and $c\text{Cat}^k$ are free with exponents

$$\exp(c\text{Shi}^k) = (1, 3k, 3k), \quad \exp(c\text{Cat}^k) = (1, 3k + 1, 3k + 2).$$

Let $\{\alpha_1, \alpha_2\} \subset E^*$ be a simple system. For $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$, let $H_{\alpha-kz} := \{\alpha - kz = 0\}$. Then the results in [2] show that $c\text{Shi}^k \setminus \{H_{\alpha_i-kz}\}$ are both free with exponents

$$\exp(c\text{Shi}^k \setminus \{H_{\alpha_i-kz}\}) = (1, 3k - 1, 3k),$$
$$\exp(c\text{Shi}^k \setminus \{H_{\alpha_1-kz}, H_{\alpha_2-kz}\}) = (1, 3k - 1, 3k - 1)$$

for $i = 1, 2$. Now we prove the key result to show Theorem 1.2.

**Theorem 3.1.** Let us fix basic invariants

$$P_1 := \alpha_1^2 + \alpha_1 \alpha_2 + \alpha_2^2, \quad P_2 := \frac{2}{27}(\alpha_1 - \alpha_2)(\alpha_1 + 2\alpha_2)(2\alpha_1 + \alpha_2)$$

of the Weyl group $W$ and choose the primitive derivation $D$ in such a way that $D(P_2) = \frac{1}{3}$. For $k \in \mathbb{Z}_{\geq 0}$, let $M_k$, $N_k$ and $T_k$ be the same as in Theorem 1.2.

Let $\varphi_1^{(k)}, \varphi_2^{(k)}$ be an $\text{SRB}_+$ of $D_0(c\text{Shi}^k)$. Then

$$[\varphi_1^{(k)}, \varphi_2^{(k)}]M_k$$

form a $W$-invariant basis for $D_0(c\text{Cat}^k)$, and

$$[\varphi_1^{(k)}, \varphi_2^{(k)}]M_kT_kN_{k+1}$$

form an $\text{SRB}_-$ of $D_0(c\text{Shi}^{k+1})$.

We prove Theorem 3.1 by using following propositions.

**Proposition 3.2.** Let $[\theta_1^{(k)}, \theta_2^{(k)}] = [\varphi_1^{(k)}, \varphi_2^{(k)}]M_k$. Then $\theta_1^{(k)}, \theta_2^{(k)}$ form a $W$-invariant basis for $D_0(c\text{Cat}^k)$.

**Proof.** Since $\theta_1^{(k)} = (\alpha_1 + kz)\varphi_1^{(k)} + (\alpha_2 + kz)\varphi_2^{(k)}$ is the $k$-Euler derivation, it follows from Proposition 2.10 that $\theta_1^{(k)} \in D_0(c\text{Cat}^k)^W$. Let us show $\theta_2^{(k)} \in$
\( D_0(\text{cCat}^k)^W \). By Proposition 2.11(1), it is clear that \( \theta_2^{(k)}(\alpha_i + kz) \in (\alpha_i + kz)S_z \) (\( i = 1, 2 \)). Since

\[
\theta_2^{(k)} = (2\alpha_1 + 4\alpha_2 + 3kz)(\alpha_1 + kz)\varphi_1^{(k)} - (4\alpha_1 + 2\alpha_2 + 3kz)(\alpha_2 + kz)\varphi_2^{(k)} \\
= (2\alpha_1 + 4\alpha_2 + 3kz)\{\theta_1^{(k)} - (\alpha_2 + kz)\varphi_2^{(k)}\} \\
- (4\alpha_1 + 2\alpha_2 + 3kz)(\alpha_2 + kz)\varphi_2^{(k)} \\
= (2\alpha_1 + 4\alpha_2 + 3kz)\theta_1^{(k)} - 6(\alpha_1 + \alpha_2 + kz)(\alpha_2 + kz)\varphi_2^{(k)},
\]

it holds that \( \theta_2^{(k)}(\alpha_1 + \alpha_2 + kz) \in (\alpha_1 + \alpha_2 + kz)S_z \). So \( \theta_2^{(k)} \in D_0(\text{cCat}^k) \). Moreover, since \( s_i\varphi_j^{(k)} = \varphi_j^{(k)} \) (\( i \neq j \)) for the reflection \( s_i \) corresponding to \( \alpha_i \), because of Proposition 2.11(1),

\[
s_1\theta_2^{(k)} = (2\alpha_1 + 4\alpha_2 + 3kz)s_1\theta_1^{(k)} - 6(\alpha_2 + kz)(\alpha_1 + \alpha_2 + kz)s_1\varphi_2^{(k)} = \theta_2^{(k)}.
\]

Similarly, we can express \( \theta_2^{(k)} \) in terms of \( \theta_1^{(k)} \) and \( \varphi_1^{(k)} \). Then the same argument as the above shows that \( s_2\theta_2^{(k)} = \theta_2^{(k)} \). Hence \( \theta_2^{(k)} \) is \( W \)-invariant. Finally, since

\[
\det(M_k) = -6(\alpha_1 + kz)(\alpha_2 + k\alpha)(\alpha_1 + \alpha_2 + kz),
\]

and \( \varphi_1^{(k)}, \varphi_2^{(k)} \) form a basis for \( D_0(\text{cShi}^k) \), Proposition 2.11 shows that \( \theta_1^{(k)}, \theta_2^{(k)} \) form a basis for \( D_0(\text{cCat}^k) \).

**Lemma 3.3.** Let \( \Omega^1(A_\Phi) \) denote the module of logarithmic differential forms of \( A_\Phi \) (i.e., the dual \( S \)-module of \( D(A_\Phi) \)). If \( \omega \in \Omega^1(A_\Phi) \), then \( \nabla_{I^*(\omega)}\nabla D^{-k}\theta_E \in D(A_\Phi, 2k - 1) \).

**Proof.** By Definition 3.1 and Theorem 3.3 in [1], it follows that

\[
I^*(\Omega^1(A_\Phi)) \subset \bigoplus_{i=1}^\ell S \frac{\partial}{\partial P_i}.
\]

Since \( \nabla D^{-k}\theta_E \in D(A_\Phi, 2k - 1) \) by Lemma 2.3, we conclude that \( \nabla_{I^*(\omega)}\nabla D^{-k}\theta_E \in D(A_\Phi, 2k - 1) \).

**Proposition 3.4.** Let \( \psi_1^{(k)}, \psi_2^{(k)} \) be an SRB \(_-\) of \( D_0(\text{cShi}^k) \). Then \([\eta_1^{(k-1)}, \eta_2^{(k-1)}] := [\psi_1^{(k)}, \psi_2^{(k)}]N_k^{-1} \) form a \( W \)-invariant basis for \( D_0(\text{cCat}^{k-1}) \).
Proof. First we will show that $\eta_{k-1}^{(k-1)} \in D_0(\text{cCat}^{k-1})^W$. Since

$$N_{k-1} = \begin{pmatrix}
1 & 4\alpha_1 + 2\alpha_2 - 3kz \\
6(\alpha_1 - k)(\alpha_1 + \alpha_2 - k) & 6(\alpha_1 - k)(\alpha_1 + \alpha_2 - k)
\end{pmatrix}
\begin{pmatrix}
1 & 2\alpha_1 + 4\alpha_2 - 3kz \\
6(\alpha_2 - k)(\alpha_1 + \alpha_2 - k) & 6(\alpha_2 - k)(\alpha_1 + \alpha_2 - k)
\end{pmatrix},$$

we have

$$\eta_{k-1}^{(k-1)} = \frac{1}{6(\alpha_1 + \alpha_2 - k)} \left( \frac{\psi_1^{(k)}}{\alpha_1 - k} - \frac{\psi_2^{(k)}}{\alpha_2 - k} \right).$$

Consider a commutative diagram

$$D_0(\text{cShi}^k \setminus \{H_{\alpha_1-k}, H_{\alpha_2-k}\})_{3k-1} \xrightarrow{\text{res}} D(\mathcal{A}_\Phi, 2k - m)_{3k-1}$$

$$\cup$$

$$D(\mathcal{A}_\Phi, 2k - m)_{3k-1} \xrightarrow{\text{res}} (\alpha_1 + \alpha_2) D(\mathcal{A}_\Phi, 2k - 1)_{3k-2};$$

where $m : \mathcal{A}_\Phi \to \{0, 1\}$ is a multiplicity defined by

$$m(H) = \begin{cases} 
1 & H \in \{H_{\alpha_1}, H_{\alpha_2}\} \\
0 & H = H_{\alpha_1 + \alpha_2}
\end{cases} \quad (H \in \mathcal{A}_\Phi).$$

Let

$$\eta := 6(\alpha_1 + \alpha_2 - k)\eta_{k-1}^{(k-1)} = \frac{\psi_1^{(k)}}{\alpha_1 - k} - \frac{\psi_2^{(k)}}{\alpha_2 - k}.$$}

It follows from Proposition 2.9 (2) that $\eta \in D_0(\text{cShi}^k \setminus \{H_{\alpha_1-k}, H_{\alpha_2-k}\})_{3k-1}$. By the definition of SRB_ and Remark 2.7, we have

$$\frac{1}{\alpha_1 + \alpha_2} \text{res}(\eta) = \frac{1}{\alpha_1 + \alpha_2} \text{res}\left( \frac{\psi_1^{(k)}}{\alpha_1 - k} - \frac{\psi_2^{(k)}}{\alpha_2 - k} \right)$$

$$= \frac{1}{\alpha_1 + \alpha_2} \left( \frac{\nabla_{I^*}(d\alpha_1)}{\alpha_1} \nabla_{D}^{-k}\theta_E - \frac{\nabla_{I^*}(d\alpha_2)}{\alpha_2} \nabla_{D}^{-k}\theta_E \right)$$

$$= \nabla_{I^*} \left( \frac{d\alpha_1}{\alpha_1} - \frac{d\alpha_2}{\alpha_2} \right) \nabla_{D}^{-k}\theta_E.$$
Hence

\[ \text{res}(\eta) \in (\alpha_1 + \alpha_2)D(\mathcal{A}_\Phi, 2k-1)_{3k-2}. \]

Then we can see that \( \eta \in (\alpha_1 + \alpha_2 - k\cdot z)D_0(\text{cCat}^{k-1})_{3k-2} \) by chasing the diagram above. Thus we may conclude that \( \eta_1^{(k-1)} \in D_0(\text{cCat}^{k-1})_{3k-2} \). Since \( D_0(\text{cCat}^{k-1})_{3k-2} = D_0(\text{cCat}^{k-1})_W^{3k-2} \) is a one-dimensional \( \mathbb{R} \)-vector space generated by \((k-1)\)-Euler derivation by Proposition 2.11 and \( \exp(c\text{Cat}^{k-1}) = (1, 3k-2, 3k-1) \), we obtain \( \eta_1^{(k-1)} \in D_0(\text{cCat}^{k-1})_W \). Next we will prove that \( \eta_2^{(k-1)} \in D_0(\text{cCat}^{k-1})_W \). We compute

\[
\eta_2^{(k-1)} = \frac{4\alpha_1 + 2\alpha_2 - 3k\cdot z}{6(\alpha_1 - k\cdot z)(\alpha_1 + \alpha_2 - k\cdot z)} \psi_1^{(k)} + \frac{2\alpha_1 + 4\alpha_2 - 3k\cdot z}{6(\alpha_2 - k\cdot z)(\alpha_1 + \alpha_2 - k\cdot z)} \psi_2^{(k)}
\]

\[
= (4\alpha_1 + 2\alpha_2 - 3k\cdot z) \left( \eta_1^{(k-1)} + \frac{\psi_2^{(k)}}{6(\alpha_2 - k\cdot z)(\alpha_1 + \alpha_2 - k\cdot z)} \right) + \frac{2\alpha_1 + 4\alpha_2 - 3k\cdot z}{6(\alpha_2 - k\cdot z)(\alpha_1 + \alpha_2 - k\cdot z)} \psi_2^{(k)}
\]

\[
= (4\alpha_1 + 2\alpha_2 - 3k\cdot z) \eta_1^{(k-1)} + \frac{\psi_2^{(k)}}{\alpha_2 - k\cdot z}.
\]

Since \( \psi_2^{(k)}/(\alpha_2 - k\cdot z) \in D_0(\text{cSh}^k) \), it holds that \( \eta_2^{(k-1)} \in D_0(\text{cCat}^{k-1}) \). Moreover, since \( s_i(\psi_i^{(k)}/(\alpha_i - k\cdot z)) = (\psi_i^{(k)}/(\alpha_i - k\cdot z)) \) for the reflection \( s_i \) corresponding to \( \alpha_i \) because of Proposition 2.11 (2),

\[
s_2 \eta_2^{(k-1)} = s_2(4\alpha_1 + 2\alpha_2 - 3k\cdot z) \cdot s_2 \eta_1^{(k-1)} + s_2 \left( \frac{\psi_2^{(k)}}{\alpha_2 - k\cdot z} \right)
\]

\[
= (4\alpha_1 + 2\alpha_2 - 3k\cdot z) \eta_1^{(k-1)} + \frac{\psi_2^{(k)}}{\alpha_2 - k\cdot z} = \eta_2^{(k-1)}.
\]

Similarly, we can express \( \eta_2^{(k-1)} \) in terms of \( \eta_1^{(k-1)} \) and \( \psi_1^{(k)}/(\alpha_1 - k\cdot z) \). Then the same argument as the above shows that \( s_1 \eta_2^{(k-1)} = \eta_2^{(k-1)} \). Hence \( \eta_2^{(k-1)} \) is \( W \)-invariant. Finally, since

\[
det(N^{-1}) = \frac{1}{6(\alpha_1 - k\cdot z)(\alpha_2 - k\cdot z)(\alpha_1 + \alpha_2 - k\cdot z)},
\]

and \( \psi_1^{(k)}, \psi_2^{(k)} \) form a basis for \( D_0(\text{cSh}^k) \), Proposition 2.4 shows that \( \eta_1^{(k-1), \eta_2^{(k-1)} \) form a basis for \( D_0(\text{cCat}^{k-1}) \).

It follows from Proposition 3.2 and Proposition 3.4 that both \([\varphi_1^{(k)}, \varphi_2^{(k)}]M_k \) and \([\psi_1^{(k+1)}, \psi_2^{(k+1)}]N_{k+1}^{-1} \) are bases for \( D_0(\text{cCat}^k)_W \) and their degrees are equal.
to \((3k + 1, 3k + 2)\). Therefore, there exists a matrix \(T_k \in M_2(\mathbb{R}[\alpha_1, \alpha_2, z])\) such that \([\varphi_1^{(k)}, \varphi_2^{(k)}]M_k \cdot T_k = [\psi_1^{(k+1)}, \psi_2^{(k+1)}]N_{k+1}^{-1} = [\varphi_1^{(k+1)}, \varphi_2^{(k+1)}]AN_{k+1}^{-1} \).

To study this matrix \(T_k\), let us show the following lemmas and proposition.

**Lemma 3.5.** Let \(\tau\) be the reflection corresponding to \(z\) and \(s_0\) that to \(\alpha_1 + \alpha_2\). Let \(\psi_1^{(k)}, \psi_2^{(k)}\) be an SRB_ of \(D_0(c\text{Shi}^k)\). Then \(\tau s_0(\psi_1^{(k)}) = -\psi_1^{(k)}\) and \(\tau s_0(\psi_2^{(k)}) = -\psi_2^{(k)}\).

**Proof.** First, note that \(s_0(\alpha_1) = -\alpha_2, s_0(\alpha_2) = -\alpha_1\) and \(s_0(\alpha_1 + \alpha_2) = -(\alpha_1 + \alpha_2)\). Also, \(\tau s_0 = s_0\tau\) since \(\tau\) acts on \(E\) and \(S\) trivially. Since \(\tau s_0\) preserves the Shi arrangement \(c\text{Shi}^k\), it holds that \(\tau s_0(D_0(c\text{Shi}^k)) = D_0(c\text{Shi}^k)\). Therefore, \(\tau s_0(\psi_1^{(k)}) \in D_0(c\text{Shi}^k)\). Moreover,

\[
\frac{\tau s_0(\psi_1^k)}{\alpha_2 - kz} = -\tau s_0 \left( \frac{\psi_1^k}{\alpha_2 - kz} \right) \in \text{Der}(S_2).
\]

Hence Proposition 2.9 (2) shows that \(\tau s_0(\psi_1^{(k)}) = \psi_2^{(k)}\) for some non-zero \(c \in \mathbb{R}^x\). Since

\[
eq c \nabla I^* (d_{a2}) \nabla D^{-k} \theta_E = c \psi_2^{(k)}|_{z=0} = \tau s_0(\psi_1^{(k)})|_{z=0}
\]

\[
= \tau s_0(\nabla I^* (d_{a1}) \nabla D^{-k} \theta_E)
\]

\[
= \nabla s_0(I^* (d_{a1})) \nabla D^{-k} \theta_E
\]

\[
= -\nabla I^* (d_{a2}) \nabla D^{-k} \theta_E,
\]

hence Proposition 2.4 shows that \(c = -1\), which implies that \(\tau s_0(\psi_1^{(k)}) = -\psi_2^{(k)}\). Since \(\tau s_0 = s_0\tau\) is a reflection, we obtain \(\tau s_0(\psi_2^{(k)}) = -\psi_1^{(k)}\).

**Lemma 3.6.** Let \(\varphi_1^{(k)}, \varphi_2^{(k)}\) be an SRB_ of \(D_0(c\text{Shi}^k)\). Then \(\tau s_0(\varphi_1^{(k)}) = -\varphi_2^{(k)}\) and \(\tau s_0(\varphi_2^{(k)}) = -\varphi_1^{(k)}\).

**Proof.** By Remark 2.8 and Lemma 3.5, we may compute

\[
\tau s_0[\varphi_1^{(k)}, \varphi_2^{(k)}] = \tau s_0[\psi_1^{(k)}, \psi_2^{(k)}]A^{-1}
\]

\[
= [\psi_1^{(k)}, \psi_2^{(k)}] \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} A^{-1}
\]

\[
= [\varphi_1^{(k)}, \varphi_2^{(k)}]A \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} A^{-1}
\]

\[
= [\varphi_1^{(k)}, \varphi_2^{(k)}] \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},
\]

which completes the proof.
Proposition 3.7. The bases \( \theta_1^{(k)}, \theta_2^{(k)} \) for \( D_0(\text{cCat}^k) \) and \( \eta_1^{(k-1)}, \eta_2^{(k-1)} \) for \( D_0(\text{cCat}^{k-1}) \) are \( W \) and \( \tau \)-invariant.

Proof. The \( W \)-invariance is checked in Propositions 3.2 and 3.4. First we show the \( \tau \)-invariance of \( \theta_1^{(k)} \) and \( \eta_1^{(k-1)} \). Note that the action of \( \tau \) preserves \( \text{cCat}^k \). Hence \( \tau \) acts on \( D_0(\text{cCat}^k) \) with the degree preserving. By Propositions 3.2 and 3.4, we know that \( \dim \mathbb{R} D_0(\text{cCat}^k)_{3k+1} = \dim \mathbb{R} D_0(\text{cCat}^{k-1})_{3k-2} = 1 \), and they are generated by \( \theta_1^{(k)} \) and \( \eta_1^{(k-1)} \) respectively. Hence \( \tau \theta_1^{(k)} = c_1 \theta_1^{(k)} \) and \( \tau \eta_1^{(k-1)} = c_2 \eta_1^{(k-1)} \) for some non-zero \( c_1, c_2 \in \mathbb{R}^\times \). Since \( \theta_1^{(k)}|_{z=0} \) and \( \eta_1^{(k-1)}|_{z=0} \) are \( \tau \)-invariant by Propositions 3.2 and 3.4, it holds that \( c_1 = c_2 = 1 \). Hence \( \theta_1^{(k)} \) and \( \eta_1^{(k-1)} \) are \( \tau \)-invariant. To show the \( \tau \)-invariance of \( \theta_2^{(k)} \) and \( \eta_2^{(k-1)} \), it suffices to show that \( \tau s_0(\theta_2^{(k)}) = \theta_2^{(k)}, \tau s_0(\eta_2^{(k)}) = \eta_2^{(k)} \) because \( \theta_2^{(k)}, \eta_2^{(k-1)} \) are \( W \)-invariant. By using Lemma 3.5 and 3.6 we may compute

\[
\begin{align*}
\tau s_0(\theta_2^{(k)}) &= \tau s_0 \left( (2\alpha_1 + 4\alpha_2 + 3kz)(\alpha_1 + kz)\varphi_1^{(k)} - (4\alpha_1 + 2\alpha_2 + 3kz)(\alpha_2 + kz)\varphi_2^{(k)} \right) \\
&= (-2\alpha_2 - 4\alpha_1 - 3kz)(-\alpha_2 - k)(-\varphi_2^{(k)}) - (-4\alpha_2 - 2\alpha_1 - 3kz)(-\alpha_1 - k)(-\varphi_1^{(k)}) \\
&= \theta_2^{(k)},
\end{align*}
\]

\[
\tau s_0(\eta_2^{(k)}) = \tau s_0 \left( \frac{4\alpha_1 + 2\alpha_2 - 3kz}{6(\alpha_1 - kz)(\alpha_1 + \alpha_2 - k)(\alpha_1 + \alpha_2 - k)} \psi_1^{(k)} + \frac{2\alpha_1 + 4\alpha_2 - 3kz}{6(\alpha_2 - k)(\alpha_1 + \alpha_2 - k)} \psi_2^{(k)} \right) \\
&= \frac{-4\alpha_2 - 2\alpha_1 + 3kz}{6(\alpha_2 + k)(-\alpha_2 - \alpha_1 + k)} (-\psi_2^{(k)}) + \frac{-2\alpha_2 - 4\alpha_1 + 3kz}{6(\alpha_1 + k)(-\alpha_2 - \alpha_1 + k)} (-\psi_1^{(k)}) \\
&= \eta_2^{(k)}.
\]

Hence \( \theta_2^{(k)} \) and \( \eta_2^{(k)} \) are \( \tau \)-invariant. \( \square \)

Now let us study the entries of \( T_k \). Note that every entry of \( T_k \) is \( W \)-invariant since \( T_k \) gives a transformation between the \( W \)-invariant bases in \( D_0(\text{cCat}^k)^W \). Comparing the degrees of both sides, we can see that the \( (2,1) \)-entry of \( T_k \) is 0, the \( (1,1) \)-entry and the \( (2,2) \)-entry of \( T_k \) are constants, and the \( (1,2) \)-entry of \( T_k \) is a \( W \)-invariant polynomial of degree 1. However, \( \mathbb{R}[\alpha_1, \alpha_2, z]^W \) is generated by \( z \). Hence the \( (1,2) \)-entry is \( cz \) for \( c \in \mathbb{R} \). Now apply Proposition 3.7 to conclude that \( c = 0 \).
Hence we may assume that 

\[ T_k = \begin{pmatrix} a_k & 0 \\ 0 & b_k \end{pmatrix} \] (\(a_k, b_k \in \mathbb{R}\)).

Thus \(T_k|_{z=0} = T_k\) and \([\varphi_1^{(k)}, \varphi_2^{(k)}]|_{z=0}M_k|_{z=0}T_k = [\varphi_1^{(k+1)}, \varphi_2^{(k+1)}]|_{z=0}AN_{k+1}|_{z=0}\).

Now recall the following:

**Theorem 3.8.** ([1], Proposition 4.2) Define

\[ R_{2k} := (-1)^k J(D^k(\alpha_1), D^k(\alpha_2))^{-1}, \]

where \(J(f_1, f_2)\) denotes the Jacobian matrix of \(f_1, f_2 \in S\) with respect to the simple system \(\alpha_1, \alpha_2\), i.e., \(J(f_1, f_2) = (\partial f_j/\partial \alpha_i)\). Then

\[ [\varphi_1^{(k)}|_{z=0}, \varphi_2^{(k)}|_{z=0}] = [\nabla_{\partial_1} \nabla_D^{-k}\theta_E, \nabla_{\partial_2} \nabla_D^{-k}\theta_E] = [\partial_1, \partial_2]AR_{2k}A^{-1}. \]

By using these two, let us compute \(T_k\) directly in terms of \(D(A_\Phi, 2k+1)\). For that purpose, let us rewrite several polynomials and matrices in [2] in terms of \(\alpha_1\) and \(\alpha_2\). First, it is easy to check that

\[ P_1 = \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2, \quad P_2 = \frac{2}{27} (\alpha_1 - \alpha_2)(\alpha_1 + 2\alpha_2)(2\alpha_1 + \alpha_2). \]

are basic invariants of the type \(A_2\). Then the Jacobian matrix \(J = J(P_1, P_2)\) is

\[ J = \begin{pmatrix} 2\alpha_1 + \alpha_2 & \frac{2}{9}(2\alpha_1^2 + 2\alpha_1\alpha_2 - \alpha_2^2) \\ \alpha_1 + 2\alpha_2 & \frac{2}{9}(\alpha_1^2 - 2\alpha_1\alpha_2 - 2\alpha_2^2) \end{pmatrix}. \]

Hence the primitive derivation \(D\) is expressed as

\[ D = \frac{1}{Q} \begin{vmatrix} \partial_1(P_1) & \partial_1 \\ \partial_2(P_1) & \partial_2 \end{vmatrix} \]

\[ = \frac{1}{6\alpha_1\alpha_2(\alpha_1 + \alpha_2)}[(\alpha_1 + 2\alpha_2)\partial_1 - (2\alpha_1 + \alpha_2)\partial_2], \]

where \(Q = \alpha_1\alpha_2(\alpha_1 + \alpha_2)\) is the defining polynomial of the Weyl arrangement of the type \(A_2\). Also in the above, we multiplied \(-1/6\) to \(D\) to satisfy the condition \(D(P_2) = 1/3\) in Theorem 3.1. For a matrix \(M = (m_{ij})\), let \(D[M] := (D(m_{ij}))\). Then we can compute

\[ D[J] = \frac{1}{18\alpha_1\alpha_2(\alpha_1 + \alpha_2)} \begin{pmatrix} 9\alpha_2 & 4\alpha_2(2\alpha_1 + \alpha_2) \\ -9\alpha_1 & 4\alpha_1(\alpha_1 + 2\alpha_2) \end{pmatrix}. \]
Moreover, the matrix \( B := J^TAD[J] \) and \( B^{(k)} := kB + (k - 1)B^T \) are also computed as follows:

\[
B = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad B^{(k)} = \begin{pmatrix} 0 & 3k - 1 \\ 3k - 2 & 0 \end{pmatrix}
\]

Hence

\[
(B^{(k)})^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{3k - 2} \end{pmatrix} \begin{pmatrix} 1 \\ 3k - 1 \end{pmatrix}.
\]

Now by using Theorem 3.8 we can determine the matrix \( T_k \).

**Proposition 3.9.**

\[
T_k = \begin{pmatrix} \frac{1}{3k + 1} & 0 \\ 0 & \frac{1}{3k + 2} \end{pmatrix}.
\]

**Proof.** First recall that

\[
[\varphi_1^{(k)} : \varphi_2^{(k)}] M_k T_k = [\varphi_1^{(k+1)} : \varphi_2^{(k+1)}] A N^{-1}_{k+1}.
\]

Restricting the equality above onto \( z = 0 \) and applying Theorem 3.8 we obtain

\[
AR_{2k}A^{-1}(M_k|_{z=0})(T_k|_{z=0}) = AR_{2k+2}A^{-1}A(N_{k+1}|_{z=0})^{-1}.
\]

Therefore,

\[
T_k = T_k|_{z=0} = (M_k|_{z=0})^{-1}AR_{2k}^{-1}R_{2k+2}(N_{k+1}|_{z=0})^{-1}.
\]

By Proposition 2.6 in [1],

\[
R_{2k}^{-1}R_{2k+2} = J(B^{(k+1)})^{-1}J^TA.
\]

Now we can compute \( T_k = T_k|_{z=0} \) directly as follows:

\[
T_k = T_k|_{z=0} = (M_k|_{z=0})^{-1}AJ(B^{(k+1)})^{-1}J^TA(N_{k+1}|_{z=0})^{-1}
= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3k + 2} \end{pmatrix}.
\]

\[\square\]
Proof of Theorem 3.1. Combine Propositions 3.2, 3.4 and 3.9.

Proof of Theorem 1.2. First, note that $P_1$ and $P_2$ are unique up to nonzero-constant when $\Phi$ is of the type $A_2$ since there is no $W$-invariant polynomial of degree one. Therefore, the construction in Theorem 3.1 shows that for any choice of $P_1$, $P_2$ and $D$, the bases constructed by them are unique up to nonzero constants. Moreover, we can connect the SRB$_+$ and SRB$_-$ using the inner product matrix $A$ as Remark 2.8. Hence we may apply Theorem 3.1 starting from $[\partial_1, \partial_2]$ inductively to obtain the bases stated in Theorem 1.2 which completes the proof.

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