Extreme Type-II Superconductors in a Magnetic Field: A Theory of Critical Fluctuations

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A theory of critical fluctuations in extreme type-II superconductors subjected to a finite but weak external magnetic field is presented. It is shown that the standard Ginzburg-Landau representation of this problem can be recast, with help of a novel mapping, as a theory of a new “superconductor”, in an effective magnetic field whose overall value is zero, consisting of the original uniform field and a set of neutralizing unit fluxes attached to $N_4$ fluctuating vortex lines. The long distance behavior of this theory is governed by a phase transition line in the ($H, T$) plane, along which the new “superconducting” order parameter, $\Phi(r)$, attains long range order. Physically, this novel phase transition arises through the proliferation, or “expansion”, of thermally-generated infinite vortex loops in the background of field-induced vortex lines. Simultaneously, the field-induced vortex lines lose their effective line tension relative to the field direction. It is suggested that the critical behavior at $T_b(H)$ belongs to the universality class of the anisotropic Higgs-Abelian gauge theory, with the original magnetic field playing the role of “charge” in this fictitious “electrodynamics”. At zero field, $\Phi(r)$ and the familiar superconducting order parameter, $\Psi(r)$, are equivalent, and the effective line tension of large loops and the helicity modulus vanish simultaneously, at $T = T_c$. In a finite field, however, these two forms of “superconducting” order are not the same and the “superconducting” transition is generally split in two branches: the helicity modulus typically vanishes at the vortex lattice melting line, $T_{m}(H)$, while the line tension and associated $\Phi$-order disappear only at $T_b(H)$. We expect $T_b(H) > T_{m}(H)$ at lower fields and $T_b(H) = T_{m}(H)$ for higher fields. Both $\Phi$- and $\Psi$-order are present in the Abrikosov vortex lattice ($T < T_{m}(H)$) while both are absent in the true normal state ($T > T_b(H)$). The intermediate $\Phi$-ordered phase, between $T_{m}(H)$ and $T_b(H)$, contains precisely $N_4$ field-induced vortices having a finite line tension relative to $H$ and could be viewed as a “line liquid” in the long wavelength limit. The consequences of this “gauge theory” scenario for the critical behavior in high temperature and other extreme type-II superconductors are explored in detail, with particular emphasis on the questions of 3D XY versus Landau level scaling, physical nature of the vortex “line liquid” and the true normal state, and fluctuation thermodynamics and transport. It is suggested that the empirically established “decoupling transition” may be associated with the loss of integrity of field-induced vortex lines as their effective line tension disappears at $T_b(H)$. A “minimal” set of requirements for the theory of vortex lattice melting in the critical region is also proposed and discussed. The mean-field based description of the melting transition, containing only field-induced London vortices, is shown to be in violation of such requirements.

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I. INTRODUCTION

Recent intense activity in the area of superconducting fluctuations has brought into sharp focus the following fundamental questions: What is the relationship between the Landau level-based and the 3D XY-based descriptions of superconducting fluctuations in magnetic field? Can the mean-field based London model containing only magnetic field-induced vortices describe the vortex lattice melting transition in the region of strong (critical) fluctuations? What is the nature of the normal phase and can it be usefully represented as a “line liquid” of field-induced vortices? What role is played at finite fields by thermally-generated vortex loops which are responsible for the zero-field transition in extreme type-II superconductors? Particular importance and urgency has been attached to these questions following the ground-breaking experiments on the thermodynamics of vortex lattice melting transition which clearly indicate that the low-field end of the melting line is entering the critical regime of high temperature superconductors.

In this paper, precise answers to these questions are provided within a theoretical framework which allows for a systematic solution to the problem of critical fluctuations in an extreme type-II superconductor subjected to a finite, but weak magnetic field. This framework is built around the “gauge theory” scenario proposed earlier. Two main predictions follow from this scenario: first, there is a new transition line in the $H - T$ phase diagram, $T_b(H)$, along which a thermally-generated vortex loop “expansion” takes place, reminiscent of the zero field transition. At $T_b(H)$, well defined field-induced vortex lines are formed, having a finite line tension relative.
to the field direction. Initially, these lines are in a liquid state and solidify only at some lower temperature $T_m(H)$ (Fig. 1). This is different from the Abrikosov’s theory, where vortices and their (Abrikosov) lattice are formed simultaneously, at $H_c2(T)$; second, in contrast to the 3D XY behavior at zero field, the description of the critical behavior along $T_\Phi(H)$ requires the combination of a complex “superconducting” order parameter $\Phi$ associated with vortex loops and a fictitious gauge field $S$, describing fluctuations in the background system of field-induced vortex lines. The magnetic field determines the “charge” which couples $\Phi$ and $S$. The physical picture arising from the “gauge theory” is remarkably detailed and compelling, and so entirely distinct from the “standard” approach\cite{10} that a concentrated effort should be directed at exploring its consequences. The main purpose of this paper is to provide a novel explicit model for critical fluctuations, to examine its main ramifications in some detail, and to advance a set of specific predictions which can help establish the value of the “gauge theory” description though experiments and numerical simulations.

The essential feature of our description is that it contains, on equal footing, both the field-induced vortex lines and thermally-generated critical fluctuations of the superconducting order parameter $\Psi$, associated primarily with vortex loops. It is the latter that dominate the entropy in the critical region\cite{11}. This is in fundamental contrast to other approaches which include only the field-induced vortices: the Landau level (LL) description\cite{12} (where other fluctuations become irrelevant at sufficiently high fields) and the mean-field based picture of London vortex\cite{13} (where other fluctuations can be ignored at sufficiently low temperatures, far below critical regime). Following the prescription proposed earlier\cite{14}, which seeks to conveniently isolate the background of field-induced from thermally-generated degrees of freedom, we derive the following results: First, it is shown in Sec. II that the familiar and frequently used “helium” or “London model” of extreme type-II superconductors, in which the amplitude fluctuations are suppressed, allows for a direct mapping of the original problem to that of a new “superconductor”, whose order parameter $\Phi$ experiences an overall magnetic field composed of the uniform external field, $H$, and the set of $N_4$ neutralizing “fluxes” attached to fluctuating vortex lines. This novel mapping constitutes an explicit and transparent realization of the general connection proposed in Ref.\cite{15}

The “helium model” is then a candidate to, in addition to the familiar vortex lattice melting line, $T_m(H)$, exhibit the conjectured “$\Phi$-transition” (or the vortex loop “expansion” transition in a finite field), the universality class of which is defined by an anisotropic Higgs-Abelian gauge theory\cite{16}. Physically, this $\Phi$-transition corresponds to vanishing of the effective line tension for very large thermally-generated vortex loops: at $T_\Phi(H)$, the energy-entropy balance in the free energy shifts in favor of large loops and spontaneously created infinite vortex-antivortex paths proliferate across the system. The ensuing change in the topology of the vortex paths results in the breaking of a “dual” $U(1)$ symmetry and a thermodynamic phase transition. Simultaneously, the field-induced vortex lines lose their line tension relative to the field direction and the “line liquid” description breaks down. As our second result, it is shown that the fictitious gauge theory passes a crucial test, allowing us to connect its “charge” to the original magnetic field. Third, we use this connection in Sec. IV to construct scaling functions for the critical thermodynamics of extreme type-II superconductors. Furthermore, the much-debated difference between the 3D XY-like description at low fields and the LL description appropriate at high fields is closely linked here to the difference between the extreme “type-II” and the extreme “type-I” behavior of the gauge theory (Sec. III). Fourth, it is shown in Sec. V that a vortex loop “expansion” leads to an abrupt drop in the coefficient of the $q^2$-term in the helicity modulus, from which one can extract the thermodynamic exponent ($\nu$) of the $\Phi$-transition. Related criteria are also proposed which test for the presence/absence of effective “diffusion” of vortex lines along the field and demonstrate close relation between the “$\Phi$-order” and viability of the vortex “line liquid” description. These predictions, based only on global topological properties of loops and lines, can be used to efficiently identify the vortex loop “expansion” line, $T_\Phi(H)$, in numerical simulations of the weakly frustrated 3D XY and related models. Fifth, assuming widely used form of dynamical scaling, the explicit expression for the fluctuation conductivity, $\sigma(T, H)$, is derived in Sec. VI in the vicinity of the $T_\Phi(H)$ line. At $T_\Phi(H)$ there is an experimentally detectable rapid onset of additional dissipation, caused by thermally expanding vortex loops whose size is reaching sample boundaries. It is tempting to associate this onset at $T_\Phi(H)$ with what is empirically known as the “decoupling” transition\cite{17} although the physical origin of such additional dissipation in our theory is entirely unrelated to any “decoupling” of any “layers”\cite{18}. Instead, it signifies the loss of integrity of field-induced vortex lines as their effective line tension disappears at $T_\Phi(H)$. At this point, one also expects a distinct change in the pinning properties of the liquid state: there is no pinning in the true normal state above $T_\Phi(H)$ just like there is no pinning above $T_d$. In this sense $T_\Phi(H)$ represents an upper boundary for pinning and could be viewed loosely as “renormalized” $H_c2(T)$\cite{19}. Finally, in Sec. VII, it is demonstrated that the vortex lattice melting transition in the critical region involves simultaneous ordering of the field-induced and thermally-generated degrees of freedom and thus cannot be faithfully represented by a mean-field based London model\cite{20} which includes only the former. Actually, as the melting line tends toward $T_d$ in the limit of vanishing magnetic field, the entropy change involved in ordering of thermally-generated loops overwhelms the configurational entropy of the field-induced vortex lines. This provides direct theoretical support for the fundamental
nature and significance of the experiments by Zeldov et al., Schilling et al., and Roulin et al., and new numerical simulations of Nguyen and Sudbø.

II. FROM THE GINZBURG-LANDAU THEORY TO THE GAUGE THEORY

The starting point is the anisotropic Ginzburg-Landau (GL) theory $Z = \int \mathcal{D}\Psi \exp\{-\int d^4r \mathcal{F}/T\}$, where

$$
\mathcal{F} = \alpha |\Psi|^2 + \sum_{\mu = \parallel, \perp} \gamma_{\mu}(|\nabla_{\mu} + \frac{2ei}{c} A_{\mu}|^2 + \frac{\beta}{2} |\Psi|^4), \quad (1)
$$

and $\alpha = a_0(T - T_c)$, $\gamma_{\mu}$, and $\beta$ are the GL coefficients. Free (periodic) boundary conditions are imposed in $\parallel$ ($\perp$) direction. The limit $\kappa \to \infty$ is considered, which is particularly appropriate for high temperature superconductors (HTS). In this limit, the external magnetic field, $\mathbf{H} = \nabla \times \mathbf{A}$, acts as a constraint, forcing every allowed configuration of the system to have the overall vorticity $\alpha$.

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The overall vorticity along $\mathbf{H}$ is defined as a line integral $\int \, d\mathbf{r} \cdot \nabla \varphi/2\pi$, where the contour of integration goes around perimeter of the system in the xy-plane and $\varphi(r)$ is the phase of $\Psi$. $N_{\Phi}$, the number of elementary flux quanta $\phi_0$, is given by $L_{\parallel}^2/2\pi\ell^2$, where $\ell = \sqrt{c/2eH}$ is the magnetic length. It is assumed that this constraint is enforced by $N_{\Phi}$ vortex paths, meandering from one end of the system to another, along $\mathbf{H}$.

In this paper, a new method is introduced to enforce the constraint explicitly, by considering a different partition function: $Z' = \int \mathcal{D}\Phi \int \prod_{i=1}^{N_{\Phi}} (d\mathbf{r}_i/s)/N_{\Phi}! \exp\{-\int d^4r \mathcal{F}'/T\}$, with

$$
\mathcal{F}' = \alpha |\Phi|^2 + \gamma_{\mu}(|\nabla_{\mu} + iU_{\mu} + \frac{2ei}{c} A_{\mu}|^2 + \frac{\beta}{2} |\Phi|^4). \quad (2)
$$

$Z'$ describes the system of $N_{\Phi}$ “shadow”, or s, vortices $\{\mathbf{r}_i,s_i\}$ in thermal equilibrium with a complex field $\Phi(r)$. These s vortices sample arbitrary paths that originate (terminate) at $z = 0$ ($z = L_{\parallel}$) and differ from the ones introduced in Ref. by the full inclusion of “overhang” configurations. The effective magnetic field $\mathbf{H}'$ experienced by $\Phi$ consists of the uniform external field $\mathbf{H}$ and the collection of unit “fluxes” attached to s vortices: $\nabla \times \mathbf{U} = 2\pi n_s(r)$; $\nabla \cdot \mathbf{U} = 0$, where $n_s(r)$ is the flux density associated with a given configuration of s vortices, $\{\mathbf{r}_i,s_i\}$.

$$
n_s(r) = \sum_{i}^{N_{\Phi}} \int_{\mathcal{L}} d\mathbf{r} \delta(\mathbf{r} - \mathbf{r}_i,s_i), \quad (3)
$$

with $\mathcal{L}$ denoting the line integral. The net value of $\mathbf{H}'$ averaged over the system vanishes.

The superconductors and are equivalent within the familiar “helium model” of extreme type-II behavior; they are just two different representations of the same physical problem. To show this we recall the main features of the “helium model”: the true transition temperature $T_{\alpha}$ (Fig. 1) is assumed to be sufficiently below the mean-field $T_c$ for amplitude fluctuations to have effectively subsided. Around $T_{\alpha}$, the relevant fluctuations are considered to be those of London-type vortex loops/lines with steric repulsion and well-defined, tight cores of size $a \ll \ell$. Consider now a single configuration of these loops and lines. First, we extract the singular part of $\nabla \varphi(r)$ by solving two equations: $\nabla \times \nabla \varphi = 2\pi n(r)$ and $\nabla \cdot \nabla \varphi = 0$, where $n(r)$ is defined by the same expression as $n_s(r)$ but with the summation running over all vortex loops and lines. After this “vortex” part has been extracted, the rest of $\Phi(r)$ is assumed to take the form which minimizes $\mathcal{F}$ for a given configuration of these line singularities. We then integrate over all regular (“spin-wave”) fluctuations in $\varphi(r)$. Finally, all such distinct configurations of vortex loops and lines are summed over to produce $Z$. This same procedure is imposed on $\Phi(r)$: first we extract the part of its phase, $\nabla \varphi(r)$, due to vortex loop/line singularities in $\Phi(r)$ and then determine the rest of $\Phi(r)$ by minimizing $\mathcal{F}'(2)$ for a given configuration of these line defects and s vortices. Again, we integrate over all “spin-wave” fluctuations of $\phi$ relative to this given configuration of loops and lines. By direct comparison of these “helium model” expressions obtained from (1) and (2), it is evident that all configurations contributing to the original $Z$ are reproduced in $Z'$ and have the same energy. However, some of these configurations are counted more than once in $Z'$. This overcounting of configurations in $Z'$ relative to $Z$, given by $(N_{\Phi} + N_{\alpha})!/N_{\Phi}!N_{\alpha}!$ with $N_{\alpha}$ being the number of vortex lines in $\Phi$ which traverse the sample along $\mathbf{H}$, is a surface effect in 3D and should be unimportant in the thermodynamic limit, $L_{\parallel} \to \infty$. Moreover, within the conjectured $\Phi$-ordered phase (Fig. 1), the configurations with $N_{\alpha} \neq 0$ are irrelevant in the thermodynamic limit and there is no overcounting at all. We conclude that, within the “helium” model, the free energy evaluated from $Z'$ coincides with the free energy of the original problem and the two superconductors have identical thermodynamics. Consequently, Eq. (3) accomplishes a straightforward and transparent reformulation of the original problem, in the spirit of Ref. While avoiding more cumbersome gauge transformation method employed there. More details on the “helium model” are presented in Appendix A.

If we relax the above minimization condition on the amplitude of our order parameters, $\Psi$ and $\Phi$, and permit weak amplitude fluctuations, we expect that the above close relation between $Z$ and $Z'$ still holds, as long as the parameters of the GL theory keep us in the extreme type-II limit. This requires the average core size $a$ to be smaller than the average spacing between vortex segments, so that vortex excitations remain well-defined. It is precisely this same requirement that is invoked to justify the frequent use of the “helium model” to emulate fluctuation behavior of extreme type-II superconductors in zero field. It is natural to expect that, if such re-
quirement is satisfied at zero field, it will remain so at low fields, such that \( a \ll \ell \). Based on this, on the equivalence of representations (1) and (2) in the “helium model” limit, and on our general expectation that the extreme type-II behavior with only weak amplitude fluctuations is effectively equivalent to the “helium model”, for the rest of this paper I consider (2) to be simply an alternative formulation of the original problem. This new reformulation (2) can now be used instead of (1) to compute various fluctuation properties and, most importantly, its critical behavior should coincide with that of the original GL theory (1).

The advantage of \( Z' \) (2) is that, by isolating the background of field-induced degrees of freedom (s vortices), it focuses our attention on the new “superconducting” order parameter, \( \Phi(r) \), and its spatial correlations, measured by \( \langle \Phi(r)\Phi^*(r') \rangle \), where \( \{ \cdots \} \) denotes thermal average over \( Z' \). All excitations of \( \Phi(r) \) are thermally generated, in the following precise sense: every configuration of \( \Phi(r) \), contributing a finite weight to \( Z' \) in the thermodynamic limit, has the overall vorticity along \( \mathbf{H} \) equal to zero. In particular, \( \Phi(r) \) contains vortex loop excitations, whose “expansion” across the system is the mechanism behind the \( \mathbf{H} = 0 \) superconducting transition (Fig. 2). By focusing on \( \Phi \), we can fashion a theory of the strongly interacting Wilson-Fisher (3D XY) critical point, “perturbed” by a weak field. This is precisely the opposite of the classic approach (3) where the Gaussian theory in a finite field is perturbed by weak interaction. Such approach starts with the LL structure from the outset and its critical behavior is always dominated by the lowest LL (2). In the new formulation (3) we had built in from the start our expectation that the weak field modifies zero field configurations only by introducing a low density of s vortex lines, the cores of which are well-defined by virtue of strong amplitude correlations at the 3D XY critical point. This is a “low-field” approach by design and offers a better prospect of constructing the desired theory.

To extract such a theory from (2) we must resort to approximations. We construct the long wavelength (\( \gg \ell \)) limit of (2) by coarse-graining vortex fluctuations produced by s vortices. The “hydrodynamic” vorticity \( \mathbf{V}(r) \) is defined as the coarse-grained version of the “microscopic” flux density \( \Delta n_i(r) = n_i(r) - \left( 2\pi \ell^2 \right)^{-1} \mathbf{z} \). Upon inserting \( \int \mathbf{D} \mathbf{V} \delta[\mathbf{V}(r) - \Delta n_i(r)] \) in (2), integrating over \( \{ \mathbf{r}, [s] \} \), and after introducing the fictitious vector potential \( \nabla \times \mathbf{S} = 2\pi \mathbf{V}; \nabla \cdot \mathbf{S} = 0 \), the effective long wavelength theory becomes \( \sim \int \mathcal{D} \Phi \int \mathcal{D} \mathbf{S} \exp \left\{ - \int d^3 r F_{\text{eff}} / T \right\} \),

\[
F_{\text{eff}} = \alpha |\Phi|^2 + \gamma_\mu |D_\mu \Phi|^2 + \frac{\beta}{2} |\Phi|^4 + \frac{K_\mu}{2} (\nabla \times \mathbf{S})_\mu^2, \tag{4} \]

where \( D_\mu = \nabla_\mu + i \mathbf{S}_\mu \) and \( K_{\perp}(T, H) = c_{\perp} \Gamma^{-1} T \ell \), \( K_{\parallel}(T, H) = c_{\parallel} \Gamma T \ell \). Higher powers and derivatives of \( (\nabla \times \mathbf{S})_\mu \), essential for the description of the vortex lattice melting, also appear in (4), but are unimportant at \( T_\Phi(H) \). Detailed derivation of the “gauge theory” (3) is given in Appendix A. \( c_{\perp, \parallel} \sim O(1) \) are dimensionless and have a relatively weak \( H, T \) dependence in that portion of the critical region which is well described by Eq. (3). A close relation between \( K_{\perp, \parallel} \) and the components of the helicity modulus tensor of the GL theory (1) is discussed later in the text (see Sec. V and the Appendix B). \( \Gamma \) is the anisotropy at \( T_\Phi \). Small \( H \)-dependent corrections to GL coefficients that also should appear in (4) are ignored, since they are not important for our present purposes.

The following assumptions have been used in going from (2) to (4) (see also Appendix A):

i) The correlation length, \( \xi_\Phi \), associated with the new order parameter \( \Phi \), is not limited by \( \ell \) and can be much longer than the original superconducting correlation length, \( \xi_{\text{sec}} \), associated with \( \Psi \). Of course, this is the basic reason why we are interested in the reformulation (2) in the first place. When \( \xi_\Phi \gg \xi_{\text{sec}} \), this assumption enables us to drop as irrelevant at long distances, terms containing higher derivatives and powers of \( \mathbf{S} \) from Eq. (4). Note, however, that such higher order terms in (4), particularly those reflecting the absence of up-down symmetry along \( \mathbf{H} ((\nabla \times \mathbf{S})_3 \) and the like), must be restored when discussing vortex lattice melting (Sec. VII).

The gauge theory (4) offers in this case \( \xi_\Phi \gg \ell, \xi_{\text{sec}} \) a direct access to the deeply non-perturbative regime of the original GL theory (1), characterized by \( \xi / \ell \gg 1 \), where \( \xi \) is the \( H = 0 \) correlation length. In the opposite case, \( \xi_\Phi \ll \ell \), we are in the perturbative regime, \( \xi / \ell \ll 1 \), of the original theory. The long wavelength expansion that led from (2) to (4) is then not justified and the new reformulation (2) is not particularly useful.

ii) The system (4) is not in its superconducting state in the vicinity of the putative \( \Phi \)-transition. This assumption fixes the form of the last two terms in (4) (see Sec. V and the Appendix B). Physically, it means that the system of s vortices contains configurations that wind from one end of a sample to another in the \( \perp \) directions (xy-plane). The presence of such windings allows complete “screening” of an arbitrary infinitesimal field \( \mathbf{h}(r) \) added to \( \mathbf{H} \) and the helicity modulus tensor vanishes along all of its principal axes (see Sec. V and the Appendix B). This assumption has a strong theoretical justification. It must be emphasized, however, that, to my knowledge, there is no rigorous argument which could rule out another possibility, that of the state right below the \( \Phi \)-transition being an extremely anisotropic “superconducting” liquid, containing no windings in the xy-plane, with a finite helicity modulus \( \text{along} \) the field and zero perpendicular to it. Indeed, some numerical studies are suggestive of this second possibility. However, other numerical simulations\( \tilde{\text{\footnotesize{3,4}}} \), as well as the available experimental data, favor our original non-superconducting liquid assumption. Both alternatives can be described within the framework of the gauge theory, with \( K_{\perp} / K_{\parallel} \rightarrow \infty \) and potentially finite “mass terms” below \( T_\Phi(H) \) added.
to (1) in the extreme anisotropy case. On general physical grounds, I have chosen to explore in this paper the case of finite anisotropy ratio $K_\perp/K_\parallel$ and vanishing mass terms but the reader should be aware that the extreme anisotropy alternative remains a possibility and would lead to results which, while similar on general level, differ in details.

iii) There are two relevant lengthscales controlling the critical behavior: $\xi_\parallel$, which characterizes both the spatial correlations of $\Phi$ and the size of “overhangs” in the system of $s$ vortices, and $\ell$, which characterizes the long wavelength fluctuations of the background field-induced vorticity.

iv) The core effects can be ignored. Clearly, the “helium model” itself is perfectly well defined in the limit $a \to 0$. For a small but finite, there is a small correction to the core line energy $\varepsilon_c \to \varepsilon_c + w_c |H| \cdot \mathbf{v}$, where $\mathbf{v} = dr/ds$ is the “velocity” of a vortex segment and $w_c |H|/\varepsilon_c \sim a^2/\ell^2 \ll 1$. Similarly, there are “velocity” dependent corrections to the short range repulsion between vortex cores. Such terms are irrelevant since they result in higher order derivatives in (4). For example, it is easy to see that the correction to the core energy cancels out for any finite vortex loop and can be factored out for $s$ vortices.

What is the physics behind gauge theory (4)? The external field has been eliminated from the gradient terms in (4) ($\mathbf{H}' = 0$) but, of course, it has not vanished: it reappears through the $H$-dependence of $K_\perp,\parallel$. The gauge theory (4) can be viewed as fictitious, anisotropic “electrodynamics” with “magnetic permeability” $\mu_0 = 1/(4\pi T)$. The “vector potential” $\mathbf{S}$ is coupled to the “matter” field $\Phi$ via “electrical charge”

$$e_{\perp,\parallel}^2 = \frac{\Gamma^{1/3}}{c_\perp c_\parallel \ell} \propto \sqrt{H}.$$  

The above “charge” and $K_\perp,\parallel$ describe the “polarizability” of the medium composed of $s$ vortices and are directly related to the long wavelength components of the helicity modulus tensor (Sec. V and the Appendix B). This picture embodies the physical idea that the dominant effect of weak magnetic field in (4), once $\xi_\parallel$ has saturated to $\sim \ell$, arises through “screening” of large thermally-generated loops by the background of field-induced vorticity, at distances $\gg \ell$. Such “screening” reduces the effective line tension of these large loops relative to its value at the $H = 0$ ($e_{\perp,\parallel} = 0$) transition. The strength of the “screening” is measured by the fictitious “Ginzburg parameter” of (4),

$$\kappa_s^2 \sim \frac{\ell}{\beta a_0 |T| c_{GL}} \propto \frac{1}{\sqrt{H}},$$  

where $c = (\xi_\parallel^2 \xi_\perp^2)^{1/3}$ and $\xi_\parallel = (\xi_{GL,\perp} \xi_{GL,\parallel})^{1/3}$, with $\xi_{GL,\perp,\parallel} = \sqrt{1/\xi_\perp/\xi_\parallel a_0^2 e}$ being the GL coherence lengths. $b = \beta a_0^2 |T| c_{GL}$ and $q_0^2 = e^2 \xi_{GL}$ are the dimensionless quartic coupling and “charge”, respectively. As $H \to 0$, the fictitious “charge” vanishes and we recover the zero-field 3D XY critical point. For $H$ finite but weak, the “screening” is weak ($\kappa_s \gg 1$), indicating that the effects of finite $\varepsilon$ are small compared to strong amplitude correlations produced by the quartic term in the GL theory (4). We have therefore manufactured a critical theory describing the strongly-interacting Wilson-Fisher (3D XY) critical point weakly “perturbed” by a finite magnetic field (finite “charge” $e_{\perp,\parallel}$).

The gauge theory scenario is clearly different from what takes place in spin systems, where the external field couples paramagnetically to the order parameter. In an extreme type-II superconductor (4), the diamagnetic coupling of $H$ to $\Psi$ does not explicitly break the $U(1)$ symmetry, which was spontaneously broken at the zero-field 3D XY critical point. The high temperature phase (true normal state) still retains the full $U(1)$ symmetry. This symmetry can be broken at low temperatures, either in a “simple” way, with $\Psi$ acting as the order parameter, as is the case in the “vortex solid” state, or in a more subtle fashion, with $\Phi$ assuming the role of the new order parameter. Similarly, the gauge theory (4) differs from a frequently used “dimensional reduction” approach, where the behavior of (4) at finite fields is related to that at zero field but in a finite system, the size of which is set by the magnetic length, $\ell$. A typical dimensional reduction ($D \to D - 2$) approach leads to the superconducting correlation length which is limited by $\ell$, i.e. the “system size”. This agrees with the gauge theory scenario since “electrodynamics” (4) also predicts $\xi_\parallel(H) \sim \ell$ in the critical region (see Sec. V). However, a dimensional reduction approach also predicts that all other correlations are limited by $\ell$, and, consequently, eliminates the possibility of any true thermodynamic phase transition in the GL theory (4). This is in contradiction with the overwhelming experimental and numerical evidence indicating some form of a “vortex liquid” to “vortex solid” transition at low temperatures. In sharp contrast, gauge theory (4) and reformulation (2) are fully three-dimensional theories, just like (4). They naturally lead to two basic types of correlations that can extend over distances $\gg \ell$ and produce phase transition(s) at low temperatures (Fig. 1): those associated with positional order of s vortices and the familiar superconducting order parameter $\Psi(r)$ and those associated with the new “superconducting” order parameter $\Phi(r)$.

The conjectured that connects the critical behavior of an extreme type-II superconductor (4) to a fictitious superconductor in zero field (4); the “charge” of which is set by the original external field $H$, must pass the following test: the way $H$ enters in $F_{\text{eff}}$ must be consistent with its being a relevant operator of scaling dimension 2 in the renormalization group (RG) sense at the 3D XY critical point. This scaling dimension is suggested by dimensional analysis (relevant effects of $H$ enter through the dimensionless ratio $\xi_{GL}(H = 0)/\varepsilon^2 \propto H \xi_{GL}(H = 0)$), is correct to two-loop order and is likely an exact property of the original GL theory (4) by virtue of gauge
independence on the nature of the zero-field critical point (i.e., whether it is 3D XY or Gaussian). On the other hand, as the “charge” $\tilde{e}$ is turned on in the gauge theory (4), the RG analysis indicates that first, the finite charge anisotropy ($\tilde{e}_{\perp} \neq \tilde{e}_0$) is marginally irrelevant, and second, the scaling dimension of “charge” at the 3D XY critical point is $1/2$, i.e., the relevant dimensionless operator is $\tilde{e}\sqrt{s_{sc}}(\tilde{e} = 0)$. The second statement is exact to all orders in perturbative RG and is also independent on the nature of the neutral critical point. Since, in $F_{\text{eff}}$ (4), $\tilde{e}^2 \propto 1/\ell \propto \sqrt{H}$ this translates immediately to the scaling dimension of $H$ being 2, as required. More generally, for dimension $D < 4$, $\tilde{e}^2 \propto 1/\ell^{D-4}$, while the scaling dimension of $\tilde{e}$ is $2 - (D/2)$, again consistent with the scaling dimension of $H$ being 2. Note that in both formulations, Eqs. (1) and (4), the corresponding relevant operators $H$ and $\tilde{e}$ ($\propto H^{1/4}$) are protected against acquiring anomalous dimensions by the same symmetry, the gauge invariance. These results demonstrate internal consistency of the coarse-graining procedure leading to (4) and strongly support the conjecture that $F_{\text{eff}}$ captures the long wavelength (critical) behavior of (4) and (1). In what follows, I promote this conjecture to a fact and examine its consequences.

III. HIGH FIELDS VERSUS LOW FIELDS

Two key consequences for the physics of the present problem follow from $F_{\text{eff}}$. First, the gauge theory (4) has two distinct regimes of behavior: the weak “screening” limit ($\kappa_s \gg 1$) corresponding to the extreme “type-II” limit of the fictitious “electrodynamics” and the strong “screening” limit ($\kappa_s \ll 1$) corresponding to the extreme “type-I” behavior. The extreme “type-II” behavior of (4) is precisely the low-field regime of the original theory (4) which exhibits the 3D XY-like critical fluctuations. In this low-field regime, the “screening” provided by the background of field-induced vorticity is weak and the dominant fluctuations are still London-type vortex loops and lines. The core size $a$ remains small and well-defined, kept in check by strong amplitude correlations coming from the quartic term in (4), just as was the case at the zero-field 3D XY critical point. It is in this sense ($\kappa_s \gg 1$) that we can think of a 3D XY critical point weakly “perturbed” by a finite field. In the extreme “type-I” limit, the situation is entirely different. There, the “screening” is strong ($\kappa_s \ll 1$) and the amplitude fluctuations run rampant. It is not possible any longer to think of relevant fluctuations in the gauge theory (4) nor in (2) and (1), as being London-like vortices. Rather, amplitude fluctuations are now of essential importance and individual vortex cores are ill-defined. In the gauge theory (4), the two regimes are separated by the condition $\kappa_s \sim 1$. However, a word of caution must be inserted here since, once we are in the “type-I” regime of (4), our original line of reasoning that lead from Eq. (4) to the gauge theory (4) via reformulation (1), is itself compromised and it is not clear whether there is a useful connection between the extreme “type-I” limit of (4) and our original problem (1). Instead, we must return back to the beginning (1) and start from scratch. It is natural to identify this extreme “type-I” behavior at high fields, characterized by strong amplitude fluctuations, as the regime in which the Landau level structure of the original GL theory (4) becomes important. The condition $\kappa_s \sim 0.4/\sqrt{H}$ separating “type-II” from “type-I” “electrodynamics” in Eq. (4), translates to the criterion for the external magnetic field, $H \sim H_s$, telling us whether $H$ is “low” or “high”. From Eq. (3) one gets:

$$H_s \approx \left(\frac{c}{0.16}\right)^2 b^2 H_{c2}^{GL}(0).$$

If $H \ll H_s$ then the field is “low” and the use of a 3D XY-like description is justified. In the opposite limit, $H \gg H_s$, the field is “high” and a 3D XY-like description falls apart (Fig. 1). If this is the case, we must abandon our zero- and low-field imagery of the “helium model” and use as a starting point an approach that is explicitly designed to deal with a high field behavior, an example being the GL-LLL theory. Note that $H_s$, within factors of order unity, coincides with $H_{b0}$, the field below which the high-field, Landau level-broadened description breaks down, due to strong LL mixing. Since this criterion is derived from entirely different arguments, we briefly reproduce it here for completeness. Going back to the GL theory (4), we expand $\Psi(r) = \sum_{j=0}^{c} \psi_j(r)$ in the set of LL manifolds, $\psi_j(r)$. Recast in terms of dimensionless variables, the GL free energy functional becomes:

$$\int d^3r \{ \sum_{j=0}^{c} [t + (2j + 1)h] |\psi_j|^2 + |\nabla|\psi_j|^2 + b \frac{1}{2} |\psi|^4 \},$$

where $t = (T/T_c) - 1$, $h = H/H_{c2}^{GL}(0)$ and $b$ is defined below Eq. (4). After rescaling $\Psi$ and $r$ by $b$ in such a way that the coefficients of quartic and gradient terms in (4) are set to $1/2$ and 1, respectively, the “mass term” for $\psi_j$ becomes

$$\frac{t}{b^2} + (2j + 1)\frac{h}{b^2}.$$

As we reduce the field $H$, for $T$ in the critical region ($T \approx T_c$), the mixing of LLs becomes strong when $h/b^2$ becomes some number of order unity. We can view this as a “Ginzburg criterion” along the $H$-axis. It implies that the high-field, Landau level description becomes inadequate for fields less than:

$$H_b \sim b^2 H_{c2}^{GL}(0) \sim \kappa H_{c2}^{GL}(0),$$

where we have used a close relation between $b$ and a conventional Ginzburg fluctuation parameter, $\kappa$. While
the definition and meaning of $G_i$ exhibit wide variations in the literature, typically $G_i \sim b^2$. As advertised, $H_b \sim H_s$. The same situation is encountered in 2D, except now $H_s \sim H_b \sim bH_{GL}^2(0)$. The fact that the criterion for the breakdown of the Landau level-based theory derived from the high-field side agrees with the region of validity of our 3D XY-like approach derived from the opposite, low-field side, is another argument in favor of the gauge theory scenario.

While it is the GL theory (1) that provides a realistic description of fluctuation behavior in extreme type-II superconductors, many numerical studies are performed on the 3D XY model. The ultimate low-field critical behavior should be the same and computational effort is much reduced. It is therefore useful to discuss here the physical meaning of the “high” and “low” field regimes in the context of the frustrated 3D XY model. There is an immediate difference between this model and the GL theory (1) regarding the high field behavior. In the GL theory this regime is dominated by Landau levels and is characterized by strong amplitude fluctuations. In contrast, in the 3D XY model, the amplitude fluctuations are frozen at the “microscopic” level of a single XY spin. As a result, there is no Landau level formation in this model. Instead, the high field behavior of a uniformly frustrated 3D XY model, as one approaches the “mean-field” $H_{c2}(0)$, is entirely determined by the pinning of field-induced vortices by the underlying lattice. We can think of this situation, to some extent, as having the LL structure of Eq. (7) thoroughly “mixed” by a very strong external periodic potential. There is, however, a relationship between the low-field critical behavior of the 3D XY model and the GL theory. It derives from our concept of “screening” of large thermally-generated vortex loops by the background of field-induced vorticity. In the weakly frustrated 3D XY model such “screening” is measured by a parameter $\kappa_{sXY}$, which is the XY model counterpart of $\kappa_s$ in the GL theory (1):

$$\kappa_{sXY}^2 \sim \frac{f^T(T, H)}{f_\Phi} \propto \frac{1}{\sqrt{H_c}} ,$$

where $f_\Phi$ measures the uniform frustration and is the fraction of the elementary flux quantum $\phi_0$ per plaquette, while $f^T(T, H)$ is the average density per plaquette of vortex and antivortex segments $|H$ piercing the xy-plane. Note that $f^T(T, H)$ includes all such vortex segments, not just those connected to infinite vortex loops. In order for the system to be in the low-field critical regime of a weakly frustrated 3D XY model we need $\kappa_{sXY} \gg 1$ or $f_\Phi \ll f^T(T, H)$.

The actual value of $H_s$ (or $H_b$) in high temperature superconductors is of considerable importance. There are numerous estimates in the literature, based both on Eq. (8) and on the analysis of various experimentally measured quantities in terms of either the GL-LLL theory or the so-called “3D XY scaling” (see Sec. IV). A direct estimate from Eq. (8), in a moderately anisotropic HTS system like optimally-doped YBCO, uses $G_i \approx 0.01$ and $H_{GL}^2(0) \approx 160$ Tesla, leading to $H_s \sim 1 - 2$ Tesla. This estimate is subject to an irksome uncertainty, both intrinsic (due to our inability to theoretically determine $H_s$ or $H_b$ with a precision better than within factors of order unity) and extrinsic (due to difficulties in extracting precise values of the GL parameters entering Eq. (8), although the situation here is rapidly improving). The estimates of $H_b$ based on the fits of fluctuation thermodynamics to the GL-LLL theory are in general agreement with the above value of 1-2 Tesla and seem to give an upper limit $H_b < 8$ Tesla. Similar analyses, based on the fits to a low-field “3D XY scaling”, generally produce results which seem consistent with the 3D XY-like behavior to much higher fields, 14 Tesla or even higher.

An important difference between the two approaches is that, within the GL-LLL theory, not only the scaling law but the scaling function and explicit expressions for thermodynamic quantities are known with considerable accuracy. In the 3D XY approach only the scaling law itself is known but the actual scaling function and, more importantly, the physics behind it are not. The gauge theory scenario should help remedy this situation.

### IV. CRITICAL THERMODYNAMICS AND $\Phi$-TRANSITION

This brings us to the second important consequence of description (1) which has bearing on the nature of critical behavior in the low-field ($H \ll H_s$), extreme “type-II” limit of the gauge theory. The most significant property in this regime is that, for $\varepsilon$ small but finite, there is a true thermodynamic phase transition separating the high and low temperature phases of the theory, the “normal” and the “Meissner” state, respectively. For $\varepsilon = 0$ this is the standard $H = 0$ phase transition of the Ginzburg-Landau theory. This phase transition is continuous and in the universality class of the 3D XY model. The actual mechanism of the phase transition is directly tied to the expansion of thermally-generated vortex loops, as depicted in Fig. 2. In the ordered state below $T_{c0}$, there is a finite average size for such loops, $\Lambda_\Phi$, and configurations which contain infinite loops, “percolating” from one end of the system to another, do not contribute to the partition function in the thermodynamic limit. At distances much larger than $\Lambda_\Phi$, there is nothing to disturb the long range correlations in $\langle \Phi(0)\Phi^*(\mathbf{r}) \rangle$; it is not possible to “polarize” closed loops at such large distances and they behave as bound “dipoles”. This is what enables the long range phase order that characterizes the superconducting state. Above $T_{c0}$, as more and more vortex segments are created by thermal excitation, the loops connect, in the sense that now there is finite contribution to the partition function from configurations having infinite loops, “percolating” across the system. This implies that $\Lambda_\Phi \to \infty$ and it is now possible to “polarize” the system of loops over arbitrary large distances. Such infinite
vortex loops act as “free charges” and produce “metallic screening” of small external magnetic fields, resulting in vanishing of the helicity modulus, as discussed in the next section. This picture of the 3D XY phase transition as vortex loop “expansion” has its origins in the works by Onsager and Feynman in the context of superfluid helium, but should equally well apply to high-temperature superconductors (HTS) with their short BCS coherence lengths and extremely large \( \kappa \sim 100 \).

As finite \( \tilde{c} \) (finite \( H \) in (1)) is turned on in Eq. (1), we are facing potentially dramatic change in this picture. In the neutral-superfluid picture described previously, vortex loops have long range London-Biot-Savart interactions. Once \( \tilde{c} \) is finite, these interactions are “screened” by the vector potential \( S \) and, at distances much longer than the “penetration depth” \( \lambda_s \sim 1/\tilde{c} \), all the interactions are short-ranged. The simplest and best known example of this is just an ordinary superconductor at zero external field. There \( \tilde{c} \) is the real electrical charge \( e \), while \( S \) turns into the ordinary Maxwell’s vector potential \( A \). This charged-superfluid problem has been studied extensively, starting with Ref.[14] and is presently thought to have the following properties[15] as already indicated in Sec. II, the charge \( e \) is a relevant operator in the RG sense, with scaling dimension equal to 1/2. This immediately destabilizes the neutral-superfluid, zero-charge 3D XY critical point. There are, however, two new critical points, characterized by finite charge. The behavior of strongly type-II superconductors (\( \kappa \gg 1 \)) is determined by the stable critical point and describes the continuous phase transition between the normal state and the Meissner phase in real superconductors. Another critical point is tricritical and unstable in one RG direction, in addition to temperature. This tricritical point defines the transition between type-II (small charge) and type-I (large charge) behavior and takes place for \( \kappa \sim 0.4/\sqrt{\tilde{c}} \).

In a type-I superconductor, the phase transition is expected to be discontinuous, as originally argued in Ref. [17]. In the type-II regime, where the transition is continuous, the universality class for the charged-superfluid appears to be the “inverted 3D XY” [14,15,16] or very “close” to it [17,18] (see Appendix A for further details).

What is the connection between these general properties of charged-superfluid model and our problem? It stems from the gauge theory (4). This theory looks just like the theory for charged-superfluid, except for the charge anisotropy, which should be irrelevant [14,16,17]. The underlying physics, of course, is very different. There is no fluctuating electrodynamical vector potential in our case, since we are in the \( \kappa \to \infty \) limit. Instead, our fictitious vector potential \( S \) describes the long wavelength vorticity fluctuations in the background s vortex system and our “charge” \( \tilde{c} \) is the original magnetic field \( H \) in disguise \( \tilde{c}^2 \propto \sqrt{H} \). Despite this difference in physical meaning, the long distance behavior of (4) should still be closely related to the electrodynamics of charged-superfluid. In particular, we expect two different thermodynamic phases of (4): the high temperature phase with only short range correlations in \( \Phi(0)\Phi^* (r) \) (\( \langle \Phi \rangle = 0 \)) and the low temperature phase, in which \( \langle \Phi(0)\Phi^* (r) \rangle \) develops long range order (\( \langle \Phi \rangle \neq 0 \)). The “Meissner phase” (or the \( \Phi \)-ordered state) of (4) corresponds to the state of the original GL theory (1) in which only \( N_s \) field-induced vortex loops cross the system from one end to another along \( H \). All other vortex excitations form either closed thermally-generated loops of finite size or finite “overhang” configurations decorating field-induced lines as they make their way meandering from bottom to top of the sample. These field-induced vortex lines, or s vortices in reformulation (5), have a finite line-tension relative to the field direction and undergo effective “diffusion” along the z-axis (this is discussed in greater detail in the next section). In the high temperature, “normal metal” phase of (4), the \( \Phi \)-order is destroyed by expansion of thermally-generated vortex loops and “overhangs” decorating s (field-induced) vortices. We now have new, thermally generated infinite loops “percolating” all the way through the system in all directions. These new infinite loops come on top of the always present background of \( N_s \) s vortices. This is the nature of the \( \Phi \)-transition in the gauge theory (4) and in the reformulation (5). The \( \Phi \)-transition is the finite field version of the zero-field superconducting transition, its thermodynamics, however, belongs to a different universality class: charged-superfluid (“inverted 3D XY”) as opposed to neutral-superfluid (3D XY) at \( H = 0 \), with finite \( H \) playing the role of finite charge \( \tilde{c}^2 \propto \sqrt{H} \) in the gauge theory (4).

The \( \Phi \)-transition line, \( T_\Phi(H) \), plays a pivotal role in the gauge theory scenario. Since, on general grounds, we do not expect any true criticality associated with the first-order vortex lattice melting line in 3D, \( T_\Phi(H) \) is the only critical line in the \( H - T \) phase diagram of the original GL theory (1) and controls fluctuation thermodynamics and transport at weak magnetic fields. It decides the issues of relevance or irrelevance of various terms that can be added to (1) (point or columnar disorder, true electromagnetic screening with finite \( \kappa \), etc.) and provides the foundation on which one can build a meaningful phenomenology of extreme type-II superconductors. In this respect, \( \Phi(r) \), the new “superconducting” order parameter characterizing the “line liquid” state, is “more fundamental” than the original \( \Psi(r) \). This will now be amply illustrated.

To start building such phenomenology, we first need a reasonable estimate of \( T_\Phi(H) \) (Fig. 1). It starts at the zero-field superconducting transition \( T_{\Phi 0} \), where \( \Phi(r) \) and \( \Psi(r) \) are one and the same: the \( H \to 0 \) limit of (4) coincides with the \( N_s \to 0 \) limit in Eq. (1) and with the \( \tilde{c} \to 0 \) limit of the gauge theory (4). At finite, but weak field, we are in the “extreme type-II” regime (\( \kappa, \tilde{c} \gg 1 \)) of the gauge theory (4) and we expect that the \( \Phi \)-transition is continuous and immediately becomes “inverted”. The transition temperature \( T_\Phi(H) \) is gradually reduced as function of \( H \) (or, equivalently, \( \tilde{c} \) in (4)), due primar-
ily to the reduction in the effective line tension of very large ($\gg \ell$) vortex loops caused by “screening” generated by the “medium” of field-induced vorticity. At these low fields, $T_\Phi(H)$ can be evaluated directly from (2). As $H$ increases, however, numerous additional terms present in Eq. (4), but not included in the gauge theory (3), on the grounds of their RG irrelevance at long distances ($\gg \ell$), start affecting $T_\Phi(H)$. Among such terms none are more important than short distance ($\sim \ell$) positional correlations which eventually lead to s vortex lattice formation at low temperatures.

In general, the $\Phi$-transition and the vortex lattice melting are two completely different phase transitions, with two different order parameters, driven by two different mechanisms. One is a $q \to 0$, another $q \sim 1/\ell$ transition. They are not entirely unrelated, however, since they arise in the same theory, (3) or (4). For instance, as $H \to 0$, we must have $T_\Phi(H) \geq T_m(H)$ (2). This is so because only in the $\Phi$-ordered state do s vortices in (2) have finite long range interactions, $\propto \langle \Phi \rangle^2$. Without such long range interactions the s vortex system would remain in a liquid state as $H \to 0$. Similarly, in the solid phase, s vortices form a lattice and cannot screen large thermally-generated vortex loops, i.e. $\ell$ becomes effectively zero even for $H \neq 0$. All vortex loops will then remain small and bound, just as they were at $H = 0$. This is discussed in more detail in Sec. VII. The problem is that the melting transition is always first-order and thus, in principle, we could have $T_\Phi(H) = T_m(H)$ over most of the $H - T$ phase diagram or even for all $H$. This would mean that melting is so strongly discontinuous that it always “jumps” over the intermediate, $\Phi$-ordered phase, straight into the true normal state. I know of no argument to rule out this possibility.

This being said, the most likely outcome is the one depicted in Fig. 1. At higher fields, as we approach the “type-I” regime of (3) ($H \sim H_s$), the gauge theory suggests that $\Phi$-transition itself converts to first-order. In this situation, it seems justified to assume that $T_\Phi(H) = T_m(H)$, as shown in Fig. 1. For low fields, $H \ll H_s$, where the melting transition becomes weakly first-order and (4) predicts a strong “type-II” behavior and continuous $\Phi$-transition, it is natural to expect $T_\Phi(H) > T_m(H)$. At fixed low field, as we increase the temperature in Fig. 1, both the effective strength of the Biot-Savart interaction between s vortices (Sec. VII) and their effective “mass” (Sec. V) decrease. As interactions and line tension go down, a natural progression of thermodynamic phases follows: a solid (Abrikosov lattice), a “massive” liquid ($\Phi$-ordered phase or “line liquid”) and, finally, a “massless” fluid of unbound loops (a true normal state).

A mean-field calculation, performed in Ref. 9, indeed leads to such results. I propose here a simple criterion which summarizes the results of such calculations and can be used to determine $T_\Phi(H)$ and $T_m(H)$ at low fields, $H \ll H_s$: the s vortex lattice melts when the average size of thermally-generated loops, $\Lambda_0(T, H)$, reaches a fraction $d_m$ of the average distance between field-induced (s) vortices: $
abla_0(T, H) = d_m \sqrt{2\pi} (\xi_{||}/\xi_{\perp})^{1/3} \ell$. $d_m \sim 0.2 - 0.3$ and $\Lambda_0(T, 0) \approx (\xi_{||}^{1/3})^{1/3}$ seems a reasonable estimate. Here $\xi_{⊥} = \xi_{0⊥} |t|^{-\nu}$ are the true superconducting correlation lengths at $H = 0$. This results in the expression for the vortex lattice transition temperature in the critical region:

$$t_m(h) = \frac{-1}{(\ell_m \sqrt{2\pi})^{3/2}} \left( \frac{\xi_0⊥}{\xi_{GL,⊥}} \right)^{3/2} h^{3/4} \quad (10)$$

where the temperature is measured relative to the true zero-field superconducting transition, $t = (T/T_{c0}) - 1$, $h$ is defined below Eq. (7) and $\nu_{gy}$ was set to 2.3. The ratio $\xi_0⊥/\xi_{GL,⊥}$ should be $\sim 1$. As argued above, we expect $d_m \sim 0.2 - 0.3$. The $\Phi$-transition, on the other hand, takes place when the size of thermally-generated loops, at finite $H$, reaches the sample dimensions, $\Lambda_0(T, H) \to \infty$. This should take place along the line where the average loop size for $H = 0$, $\Lambda_0(T, 0)$, becomes of the order of average distance between s vortices, i.e. $\Lambda_0(T, 0) = d_m \sqrt{2\pi} (\xi_{||}/\xi_{\perp})^{1/3} \ell$, with $\Lambda_0 \sim 1$. This determines the vortex loop “expansion” line, or $T_\Phi(H)$:

$$t_\Phi(h) = \frac{-1}{(\ell_\Phi \sqrt{2\pi})^{3/2}} \left( \frac{\xi_0⊥}{\xi_{GL,⊥}} \right)^{3/2} h^{3/4} \quad (11)$$

with $d_\Phi \sim 1.8$. Obviously, $T_\Phi(H) > T_m(H)$ since $d_m < d_\Phi$. Eqs. (10,11) are valid only in the limit of low fields, $H \ll H_s$ (4). At higher fields, $H \sim H_s$, $T_\Phi(H)$ and $T_m(H)$ merge together and both vortex loop “expansion” and vortex lattice melting occur simultaneously when $\Lambda_0(T, H)$ reaches $\sim \sqrt{2\pi h}$ from within the solid phase (Fig. 1). The above expressions (10,11), with $d_m$ and $d_\Phi$ serving as numerical parameters, can be viewed as a “Lindemann criterion” for vortex loops and should provide reasonable estimates of $T_\Phi(H)$ and $T_m(H)$. The reader should bear in mind that the above expressions for $T_m(H)$ and $T_\Phi(H)$ are only estimates, however reasonable. Only the first-principle calculation, starting from (3) or (4), and computing accurately free energies of different phases (solid, $\Phi$-ordered liquid, and the true normal state) can settle this question within the theoretical framework proposed in this paper. Unfortunately, such calculation is not feasible at the present level of analytical sophistication. Additional careful experiments on HTS and numerical simulations of 3D XY and related models are far more promising in this regard.

I now proceed to further investigate the phase diagram represented by Fig. 1 and Eqs. (10,11). As the field is turned on in Eq. (1) we can immediately write down the scaling expression for the dimensionless singular part of the free energy, $f$, associated with critical fluctuations:

$$f = |t|^{2-\alpha} \phi_{±} \left( \frac{H}{H_s |t|^\Delta} \right) \quad (12)$$

where $t = (T/T_{c0}) - 1$ and $H_s$ depends on material parameters. This expression is completely general and as
such conveys little information. It is based only on the existence of the zero-field critical point. The same expression can be written for spin systems or any other system exhibiting a critical point which is then perturbed by a generalized “field” \( e \). In our case, the \( H = 0 \) critical point is in the 3D XY universality class and we should have \( \alpha = \alpha_{xy} \). Furthermore, based on dimensional analysis and general physical arguments, it was proposed in Ref. 12 that \( \Delta = 2\nu_{xy} \). This result holds to two-loop order in RG \( 3 \) and is likely exact, as emphasized in Sec. II. \( \phi_{\pm}(H_{\mathrm{GL}}) \) is a universal function of its argument inside the 3D XY critical region of the GL theory. At present, its form is not known. Note that Eq. (12), while completely general, is written in the form which implicitly suggests that the finite-field critical behavior is governed by the zero-field critical point, as is frequently the case in spin systems.

In the gauge theory scenario, the situation is different and we can be more specific. First, as already emphasized, within this scenario the critical fluctuations are governed by a critical line and not a critical point. This means that we immediately learn something about the function \( \phi_{\pm} \) defined in Eq. (12): \( \phi_{\pm} \) is non-analytic along the \( \Phi \)-transition line, \( T_{\Phi}(H) \). (11). This line singularity should be explicitly incorporated into the expression for the free energy. To devise such a new scaling function, based on the gauge theory scenario, we start by observing that we can eliminate the “trivial” part of the charge anisotropy from (4) by rescaling all lengths and fictitious vector potential \( S \) with an appropriate superconducting correlation length, \( \xi_{\perp||} \), in the way that makes the \( \Phi \)-dependent part of (4) isotropic. This rescaling procedure is a variation on the familiar rescaling of anisotropy at the \( \Phi \)-transition line, \( T_{\Phi}(H) \). (11). This line singularity should be explicitly incorporated into the expression for the free energy. To devise such a new scaling function, based on the gauge theory scenario, we start by observing that we can eliminate the “trivial” part of the charge anisotropy from (4) by rescaling all lengths and fictitious vector potential \( S \) with an appropriate superconducting correlation length, \( \xi_{\perp||} \), in the way that makes the \( \Phi \)-dependent part of (4) isotropic. This rescaling procedure is a variation on the familiar rescaling of anisotropy at the \( \Phi \)-transition line, \( T_{\Phi}(H) \). (11). This line singularity should be explicitly incorporated into the expression for the free energy. To devise such a new scaling function, based on the gauge theory scenario, we start by observing that we can eliminate the “trivial” part of the charge anisotropy from (4) by rescaling all lengths and fictitious vector potential \( S \) with an appropriate superconducting correlation length, \( \xi_{\perp||} \), in the way that makes the \( \Phi \)-dependent part of (4) isotropic. This rescaling procedure is a variation on the familiar rescaling of anisotropy at the \( \Phi \)-transition line, \( T_{\Phi}(H) \). (11). This line singularity should be explicitly incorporated into the expression for the free energy.

The following quantities appear in Eq. (13): \( \xi_{\perp||} = \xi_{0\perp||} |t|^{-\nu} \) and \( \xi = (\xi_{\perp}^2 \xi_{||}^2)^{1/3} = \xi_0 |t|^{-\nu} \) are the true diverging superconducting correlation lengths at \( T_{\Phi} \), defined by the eigenvalues of the helicity modulus tensor (see the Appendix B). Accordingly, \( \Gamma = \xi_{\perp||}/\xi_{||} \) is the true anisotropy ratio at the \( H = 0 \) critical point (\( T_{\Phi} \)) and \( \Gamma_{\Phi} \), the GL anisotropy, \( \Gamma_{GL} = \xi_{GL\perp||}/\xi_{GL||} \). It now becomes clear why \( K_{\perp||} \) have been defined in Eq. (4) with the anisotropy \( \Gamma \) explicitly factored out: new rescaled coupling constants can simply be written as: \( K_{\perp||} = c_{\perp||}^0 T_{\Phi} \ell \),

\[
K_{\perp||} \to K_{\perp||}' = c_{\perp||}^0 (\xi_{\perp||}/\xi_{\perp\perp})^{1/3} T_{\Phi} \ell .
\]

The dimensionless ratio \( \xi/\ell \) is common to both charges and (non-trivial) anisotropy is stored in \( c_{\perp||} \) and \( c_{\perp\perp} \) and \( c_{\parallel\parallel} \), however, are also functions of \( \xi/\ell \) only. Therefore, there is only a single relevant scaling variable, the dimensionless charge:

\[
q_{\Phi}^0 = \frac{\xi}{\ell} = \sqrt{\frac{2\pi e_{\Phi}(H)}{\phi_0}} \xi_0^{1/3} \frac{T_{\Phi}}{H} \ell^{1/2} .
\]

which is precisely the original scaling variable \( \xi/\ell \) of the GL theory, \( \xi/\ell \), since \( \xi/\ell = \xi_{\perp}/\ell \). The functions \( c_{\perp||}(\xi/\ell) \) are discussed further in the next section and Appendix B.

We are now in position to write down the scaling function for the free energy within the gauge theory scenario, with the non-analytic part associated with the \( \Phi \)-transition explicitly factored out:

\[
f_\Phi = |t - t_\Phi(h)|^{2 - \alpha} \Omega_{\Phi}^{\perp\perp}(x) (t - t_\Phi(h)) .
\]

Note that \( t_\Phi(h) \), defined in Eq. (11), also follows from \( q_{\Phi}^0 = \sqrt{2\pi e_{\Phi} \phi_0} \). \( \phi_0 \) is therefore a universal number of the GL theory, \( \phi_0 \), as is \( d_m \) and \( \Omega^{\perp\perp}(x) \) is a universal and regular function of its argument. The subscripts \( \pm \) refer to \( x > 0 \) \( (x < 0) \), while the superscripts \( L \) and \( S \) indicate the “vortex liquid” and “vortex solid” branches of \( \Omega \), respectively. For example, below \( T_m(H) \), we should use \( \Omega^S(x) \). In writing down (12) I have assumed that the correlation length exponent of the gauge theory \( \nu_{GL} \sim \nu_{xy} \sim 2/3 \) and that the hyperscaling relation holds, resulting in \( \alpha \sim \alpha_{xy} \).

How do we evaluate the crossover function, \( \Omega(x) \)? I alert the reader to the following important point: the gauge theory scenario explored in this paper allows one to, in principle, determine all the branches of the crossover function \( \Omega(x) \) by using a combination of perturbation theory and RG techniques. Such analytic calculation is extremely laborious and far beyond the scope of this paper. Well informed reader will immediately realize...
that many aspects of this calculation are computationally extremely demanding, and actually have not been accomplished in the published literature even for the ordinary \( H = 0 \) situation. Indeed, the technical difficulties involved are of the same general nature. This, however, does not detract from the main message of this section: the underlying physical picture of the gauge theory scenario provides a systematic, conceptually straightforward way to compute the \( H \neq 0 \) 3D XY critical thermodynamics at the same level of analytical accuracy as is presently feasible for the \( H = 0 \) case.

Faced with such odds, I assume, for the purposes of this paper, that \( \Omega(x) \) in Eq. (18) is some unknown universal crossover function, to be determined either from numerical simulations or directly from experiments. With the free energy thus specified, we can proceed to evaluate the singular part of all thermodynamic functions, simply by taking requisite derivatives.

V. HELICITY MODULUS, LINE “DIFFUSION”, TOPOLOGICAL WINDINGS, AND PHYSICAL NATURE OF \( \Phi \)-ORDER

I now turn to physical properties which allow more direct look at actual configurations of loops and lines that characterize the state of a superconductor above and below \( T_\Phi(H) \). A useful measure of a degree of superconducting order is a \( q \)-dependent helicity modulus tensor, \( \Upsilon(q) \), whose components are defined as:

\[
\Upsilon_{\mu\nu}(q) = V \frac{\delta^2 F}{\delta a_\mu(q) \delta a_\nu(-q)},
\]

(19)

where \( \mu, \nu = x, y, z \), \( V \) is the free energy of the GL theory and \( a(\mathbf{r}) \) is a small (infinitesimal) vector potential added to the external \( \mathbf{A} \). The uniform component of the associated magnetic field, \( \mathbf{h}(\mathbf{r}) = \nabla \times \mathbf{a} \), vanishes. The above second derivative is evaluated in the \( \mathbf{a} \to 0 \) limit.

\( \Upsilon(q) \) measures the ability of a system to “screen” out tiny external fields. In the superconducting phase \( \lim_{q \to 0} \Upsilon(q) \) is finite and the system is said to exhibit a differential Meissner effect. In a normal phase, \( \Upsilon(q) \sim q^2 \) and vanishes in the \( q \to 0 \) limit. Within our “helium model”, the way \( \Upsilon \) is reduced to zero in the long wavelength limit is through proliferation of infinite vortex loops/lines which go the way across the system and can act as “free charges”, screening a weak external perturbation. In this intuitive sense, we can think of a normal state as a vortex “metal”, while the superconducting state is a vortex “dielectric”, with only vortex loops of finite size present as thermal excitations.

To compute the helicity modulus of our original GL theory one adds an infinitesimal \( \mathbf{a}_\mu \) to \( \mathbf{A}_\mu \) in Eq. (4). If we now go to the reformulation and finally, through the coarse-graining procedure of Appendix A, end up with our fictitious gauge theory, \( \mathbf{a}_\mu \) appears as a small addition to the “vector potential” \( \mathbf{S}_\mu \) in the second (gradient) term of Eq. (3). This implies that the long wavelength (\( q \ll 1/\ell \)) form of the helicity modulus of the gauge theory coincides with that of the original GL theory (4). Using the gauge theory and ignoring the anisotropy, we obtain that below \( T_\Phi(H) \) (see Appendix B for details):

\[
\Upsilon(q) = K q^2 - \frac{K^2 q^4}{T} (\mathbf{S}(q) \cdot \mathbf{S}(-q)) = K q^2 + O(q^4). 
\]

(20)

In the \( \Phi \)-ordered state our fictitious gauge field is “massive”, i.e. exhibits a Meissner effect, and \( \lim_{q \to 0} (\mathbf{S}(q) \cdot \mathbf{S}(-q)) \propto |\langle \Phi \rangle|^2 \) goes to a finite value. The simple physics behind this is that thermally-generated vortex loops in \( \Phi(\mathbf{r}) \) have average size that is finite and do not contribute at all to \( \Upsilon(q) \) in the \( q \to 0 \) limit. Furthermore, Eq. (20) tells us that, if \( K_{\perp} \) are finite, the \( \Phi \)-ordered phase is not a superconductor and has a finite superconducting correlation length, both perpendicular and parallel to the external field (see Appendix B for details),

\[
\xi_{\parallel} \sim K_{\perp}/T \sim c_{\parallel} \ell, \quad \xi_{\perp}/\xi_{\parallel} \sim K_{\perp}/T \sim c_{\parallel} \ell.
\]

(21)

This result is easily understood: with \( H \) finite, there now must be \( N_\Phi \) field-induced vortex lines moving about in the sample. Unless these (s) vortices are pinned down, as it happens in the vortex-solid phase, they will be available to “screen” weak (infinitesimal) external fields and the system always a vortex “metal” with the “screening length” \( K/T \). In particular, according to our assumption ii), the system of field-induced vortex lines also contains windings in the xy plane and such “screening”, although anisotropic, is finite in all directions. Below \( T_\Phi(H) \), where s vortices are exclusively responsible for the vanishing \( \lim_{q \to 0} \Upsilon_{\mu\nu}(q), \xi_{\perp} \sim q_0^2 \) determine the \( \perp \) and \( \parallel \) “screening lengths” in the s vortex system, in units of magnetic length \( \ell \) (Eq. (23) in Appendix B).

Above \( T_\Phi(H) \) the situation changes and thermally generated vortex loops “expand” across the system. Obviously, the helicity modulus still vanishes, but now there is an abrupt drop in the coefficient of the \( q^2 \) term:

\[
\Upsilon(q) = (K - \frac{K^2}{T_\Phi}) q^2 = K(1 - C T^{\nu_{GT}}/\sqrt{H}) q^2, \tag{22}
\]

where \( \tau(T, H) = (T - T_\Phi(H))/T_\Phi \) and \( C \) and \( G \) are constants. \( \nu_{GT} \) is the thermodynamic exponent of the Meissner transition in our fictitious electrodynamics, and \( \nu_{GT} \sim \nu_{xy} \sim 2/3 \), as argued in Sec. IV. The second term in the above equation arises from \( \lim_{q \to 0} (\mathbf{S}(q) \cdot \mathbf{S}(-q)) = G/\xi_0 q^2 \) in Eq. (23), right above \( T_\Phi(H) \). This implies that the original superconducting correlations, measured by \( \langle \Psi(0) \Psi(\mathbf{r}) \rangle \), remain finite in all directions on both sides of \( T_\Phi(H) \), but there is a non-analytic drop in the superconducting correlation length...
at the \( \Phi \)-transition, as thermally generated loops proliferate through the system and additional infinite vortices become “free charges” and available to screen. The new order parameter, \( \Phi(r) \), however, does attain a true long range order below \( T_\Phi(H) \), i.e. \( \xi_\Phi \to \infty \) as \( T \to T_\Phi(H) \) from above. It is unfortunately rather difficult to measure the \( \Phi \)-correlations directly, by probing some suitably defined “helicity modulus” associated with the \( \Phi \)-order. This would require defining quantities which are configuration-dependent and highly non-local, a rather time-consuming proposition in a typical numerical simulation of a 3D XY or related model.

Still, the situation is far from hopeless. We can devise another set of criteria that are relatively easy to implement in numerical simulations and yet allow for a rather intimate look at the \( \Phi \)-order and what precisely takes place as we cross the \( \Phi \)-transition line. Below \( T_\Phi(H) \), thermally generated vortex loops are bound and field-induced vortices execute an effective “diffusive” motion along the field direction. An average transverse displacement of a single field-induced vortex line from the point where it starts at \( z = 0 \) to its ending point at \( z = L_z \) goes as

\[
\sqrt{\langle r_z^2 \rangle} \sim \sqrt{D_s L_z^p}, \quad p \approx \frac{1}{2},
\]

where \( D_s \) is the effective “diffusion” constant. This is shown in Fig. 3. The cutting/reconnecting of vortex lines does not affect this diffusion process except by renormalizing \( D_s \), as long as we are in the \( \Phi \)-ordered phase. For example, in the 3D XY model, where the identification of an individual field-induced line is not unique, we should simply look at the distribution of distinguishable vortex paths, obtained by randomly resolving all the crossings, and average over all distinct configurations. Such distribution will be “diffusive”, with the average rms displacement given by Eq. (23). We can use this effective “diffusion” constant \( D_s \) to define an effective “mass” in the elegant non-relativistic boson analogy of Nelson. Note that the worldlines of such flux-bosons do not correspond to (s) vortex lines in individual configurations of an extreme type-II superconductor. This is clear since non-relativistic bosons describe strictly directed lines, i.e. contain no “overhang” configurations as they advance from bottom to top of the system along \( z \)-axis (the “time” axis in the boson analogy). Such “overhang” configurations, plus numerous vortex loop excitations floating around, describe worldlines of “particle-antiparticle” creation processes and cannot be accommodated within the non-relativistic quantum boson analogy. Still, as long as we are in the \( \Phi \)-ordered phase, it is only the \( N_\Phi \) field-induced vortex lines that go all the way across the system. We can then define an effective system of \( N_\Phi \) flux-bosons in the boson analogy, with suitably adjusted bare mass \( m_s \) and effective interactions, so that its long distance \( (\gg l \text{ and } \gg \Lambda_\Phi(T, H)) \) behavior faithfully emulates an extreme type-II superconductor (Appendix A). Above \( T_\Phi(H) \), as infinite tangles of field-induced and thermally-generated vortices proliferate across the sample in all directions, \( D_s \to \infty \) \( (m_s \to 0) \) and we get hyperdiffusion:

\[
\sqrt{\langle r_z^2 \rangle} \sim L_z^{p'}, \quad p' \sim 1.
\]

This hyperdiffusion arises through processes depicted in Fig. 3, where a vortex line winding along the field direction simultaneously winds all the way in the xy-plane by connecting itself to thermally-generated tangles, which are naturally present in the \( \Phi \)-disordered phase. In the 3D XY model, this implies that the distribution of transverse displacements of individual field-induced vortex paths is no longer “diffusive” and has rms displacement limited only by the system size (24); more precisely, the distribution of \( r_z^2 \) acquires a power-law tail above \( T_\Phi(H) \). Such windings in the xy-plane are plainly in evidence in the recent numerical simulations of Nguyen and Sudbø right above their melting line. With such additional xy windings present with a finite weight in the partition function, the “effective mass” \( m_s \) of non-relativistic flux-bosons vanishes since the vortex line tension relative to the field direction has gone to zero. Above \( T_\Phi(H) \), infinite vortex paths of length \( \sim L^2 \) (assuming \( L_\perp = L_\parallel = L \) and a simple random walk) crossing the system in all directions contain a finite fraction of all vortex segments: these are the “massless excitations”. In this respect, the \( \Phi \)-transition corresponds to the restoration of “relativistic invariance” in a dual system of quantum particles whose worldlines are our original vortex loops and lines. To wit, the ground state of such quantum system, containing only “vortex matter” below \( T_\Phi(H) \), explodes with “vortex matter”, “vortex antimatter” and “vortex tachyons” (Fig. 3), as the “vacuum” becomes unstable at \( T_\Phi(H) \) to spontaneous creation of “particles” (Appendix A).

The above connection between the “\( \Phi \)-order” and the effective line tension of field-induced (s) vortex lines reveals directly the physical content of the gauge theory and permits us to construct a purely geometrical picture of the \( \Phi \)-transition. To do so, consider once again the normal state of an extreme type-II superconductor or a 3D XY model at \( H = 0 \). Above \( T_{c0} \), we can find vortex paths that go all the way from one end of a sample to another, in any direction (Fig. 2). If we work with periodic boundary conditions in all directions, this statement means that we have some windings along x-, y-, and z-axes. To understand what is precisely meant by such windings, we wrap our system on a three-dimensional generalized torus, embedded in a four-dimensional space—this is just a geometrical way of representing periodic boundary conditions. Now, the number of windings along, say, x-axis, \( N_x \), is the total number of continuous vortex paths in the whole system that wind all around the torus in the x-direction, irrespective of their orientation, i.e., whether they are “vortex” or “antivortex” paths relative to the x-axis. Such paths are topologically distinct from finite closed vortex loops: the latter can be continuously

\[
\sqrt{\langle r_z^2 \rangle} \sim L_z^{p'}, \quad p' \sim 1.
\]
shrunk to a point while the former can not. \( N_z \) is different from the winding number, \( W_z \); \( W_z \) also counts all the windings along x-axis but with a single “vortex” path contributing +1 while an “antivortex” path counts as −1. In the widely recognized vocabulary of the 2D XY model, \( N_z \) would correspond to the total number of vortices plus antivortices in the yz-plane, while \( W_z \) would be the total number of vortices minus the total number of antivortices. Back in 3D, in the superconducting state below \( T_{c0} \), all thermally-generated vortices come in the form of finite closed loops and both \( N_z = 0 \) and \( W_z = 0 \). Above \( T_{c0} \), \( W_z \) must remain equal to zero due to the “vortex neutrality” of the GL theory (2) or the 3D XY model, but \( N_z \) is now finite and \( N_z \propto L^{1+u} \), where \( L \) is the linear size of the system (we are assuming \( L_{\perp} = L_{\parallel} = L \) and \( u \) is the “anomalous dimension” of such infinite paths. This implies that there is a finite fraction of all vortex segments contributing to such windings along x in the yz-plane. The same holds for the winding number and the total number of windings along y- and z-axes, \( W_{y(z)} = 0 \) and \( N_{y(z)} \propto L^{1+u} \). This can be summarized as:

\[
N_{x,y,z} = 0 \quad \text{for} \quad T < T_{c0},
\]
\[
N_{x,y,z} = 2n_x^T L^{1+u} \quad \text{for} \quad T > T_{c0},
\]

where \( n_x^T \) is the “density” in the yz-plane of thermally-generated infinite vortex-antivortex winding paths traversing the system along the x-axis, and so on. In the isotropic case \( n_x^T = n_y^T = n_z^T \). Of course, the presence of such windings in all directions is the reason why the material is not in the superconducting state above \( T_{c0} \): these infinite vortex paths can now move to “screen” weak external fields, driving the helicity modulus to zero in the long wavelength limit and producing finite dissipation.

As we turn on a finite field in (1), we still have \( W_{x(y)} = 0 \) but \( W_z = N_\Phi \) and consequently \( N_z \) must be at least \( N_\Phi \) in every configuration of the system. Imagine now how the state of the system evolves along a small circle in the \( H - T \) phase diagram (Fig. 1), surrounding \( T_{c0} \). Our circular path starts at \( H = 0 \) and at some temperature \( T \) slightly above \( T_{c0} \), and evolves in the counter clockwise direction toward its end point at \( H = 0 \) and some temperature slightly below \( T_{c0} \). Initially, we are very close to the \( H = 0 \) normal state and it is safe to assume that \( N_\Phi L \ll N_z(T, H = 0) L^{2-u} \) at some temperature \( T \) slightly above \( T_{c0} \). In this case it is natural to expect that, in a finite field:

\[
N_{x,y} = 2n_x^T L^{1+u}, \quad N_z = N_\Phi + 2n_x^T L^{1+u},
\]

(26)

where \( N_\Phi = N_\Phi / L^2 = 1/2 \pi t^2 \) is the density of field-induced \((s)\) vortex lines and \( n_{x,y,z}^T(T, H) \) are “densities” of thermally-generated vortex-antivortex windings. Note that now \( 2n_x^T \neq N_\Phi + 2n_x^T \) and \( n_x^T \neq n_x^T \) even in the isotropic case (although the gauge theory scenario strongly suggests \( u_\perp = u_\parallel \)). There is a finite “density” of thermally generated infinite vortex (or antivortex) winding paths in any direction. Eq. (26) describes how the normal state of the system (2) has changed after the application of a finite, but weak external field.

On the opposite side of our imaginary circle, near its end point at \( H = 0 \) and \( T < T_{c0} \), the situation is completely different. Now, the zero-field state is a superconductor and \( N_{x,y,z} = 0 \). Any finite field, no matter how small, has a drastic effect. For very low fields, it is natural to expect that there are no thermally-generated infinite vortex loops and only those windings associated with the field-induced vortex lines are present in the system:

\[
N_{x,y} = 0, \quad N_z = N_\Phi = n_\Phi L^2.
\]

(27)

This is just the \((s)\) vortex lattice state in Fig. 1. Note that in these general geometrical terms there is no difference between the \((s)\) vortex lattice state and the “anisotropic superconducting liquid” of Feigel’man et al. (2) Due to the absence of windings in the xy-plane both have superconducting response along the field direction, i.e. the \( \mathbf{Y}_{zz} \) component is finite in the \( q \to 0 \) limit. Also, in both cases, \( \mathbf{Y}_{xx(yy)} \) vanish as \( q \to 0 \). The only difference is that it takes arbitrary weak pinning to restore superconductivity in all directions in the vortex lattice state. On this basis, I have assumed in the phase diagram of Fig. 1 that such an “anisotropic superconducting liquid” phase is preempted by the first-order transition at \( T_m(H) \). (2)

In our proposed phase diagram depicted in Fig. 1, the intermediate \( \Phi \)-ordered phase is inserted between the true normal state (2) and the vortex lattice phase (2). What is its nature in simple geometrical terms used to describe the other two phases in Eqs. (26) and (27)? In the \( \Phi \)-ordered state all thermally-generated vortex loops are bound and only \( N_\Phi \) field-induced \((s)\) vortex lines go from one end of the sample to another, along \( \mathbf{H} \), resulting in precisely \( N_\Phi \) windings along the \( z \)-axis. However, these \((s)\) vortex lines are in a liquid state, characterized by some finite effective line tension, \( \mathcal{T} \), and they “diffuse” in the xy-plane while winding along \( \mathbf{H} \). This translates into a non-vanishing total number of windings in the xy-plane:

\[
N_{x,y} = \mathcal{L}_{x,y} L^{2p}, \quad N_z = N_\Phi = n_\Phi L^2.
\]

(28)

We expect \( 1 > p \approx 1/2 \) \( \mathcal{L}_{x,y} \) are some finite quantities having dimension of \((length)^{-2p}\) and we use Eq. (28) as their definition. Eq. (28) describes the \( \Phi \)-ordered state using a simple geometrical language of this section. The fact that \( N_{x,y} \neq 0 \) has nothing to do with the thermal “expansion” of vortex loops; these are still all of finite size. Rather, it is due to the lateral “diffusion” of field-induced lines, as they wind along the \( z \)-axis. The field-induced vortices tend to form infinite “clusters”, which manage to wind a finite number of times, \( N_z \), along \( x(y) \) by winding \emph{infinitely} many times, \( N_z \sim L \), along the field direction (1 now set \( p = 1/2 \). For such infinite
clusters, we can define a long distance, effective “diffusion” constant, $D$, and the associated line tension, $T$, both relative to the field direction:

$$D_{x,y} = \frac{N_{x,y}^T L}{N^2} , \ T_{x,y} \sim D_{x,y}^{-1} .$$

Defined in this fashion, $D$ and $T$ are truly global quantities, detached from complicated individual configurations of interacting vortex loops and lines and dependent only on thermodynamic state of the system as a whole. In the $\Phi$-ordered state both are finite and have a well defined average. In the true normal state, however, the infinite clusters appear for which $D \to \infty$ and $T \to 0$. Finite vortex paths inside such clusters manage to wind in the $x(y)$ direction after only a finite number of windings along the field. In this sense, $T$ can serve as a probe for the presence/absence of the $\Phi$-order. Above $T_\Phi(H)$, such clusters with $T = 0$ contain a finite fraction of all vortex segments. Again, in the $\Phi$-ordered state of the 3D XY model, where there is no unique identification of individual vortex paths due to their crossings, every configuration that contributes to the thermodynamic limit has precisely $N_\Phi$ vortex lines going from bottom to top in every distinguishable assignment of such paths. All other vortex paths form finite closed loops. The distribution of the total number of windings, obtained by randomly resolving all the crossings, still satisfies (28).

Once the system makes this phase transition to the normal state (20) and the $\Phi$ order is lost, there is only a smooth evolution. The “densities” of thermally-generated windings $n_{x,y,z}^T$, as well as the way these windings are realized in individual configurations of vortex loops and lines, can change considerably, depending on where we are in the $H-T$ phase diagram relative to $T_\Phi(H)$, but we do not expect to cross any additional phase boundaries.

The geometrical picture presented here, based only on global topological properties of loops and lines, gives a clear insight into two forms of “superconducting” order, described by two different order parameters, $\Psi(r)$ and $\Phi(r)$. The familiar superconducting order, measured by $\Psi(r)$, reflects the system’s ability to expel tiny external fields and is manifested by a finite helicity modulus and absence of linear dissipation. This leads to the well-known spectacular experimental consequences and is naturally of great practical importance. A more subtle form of “superconducting” order, associated with the new order parameter $\Phi(r)$ introduced in Ref. 9, describes the presence of finite line tension at all length scales and is manifested by suppression of large thermally-generated vortex loops in the partition function. At $H = 0$, or in an infinitesimal field with only a single field-induced line, $\Psi$ and $\Phi$ are equivalent. The helicity modulus and the effective line tension vanish simultaneously at a single superconducting transition, $T = T_{c0}$. In a finite field, with a finite density of field-induced lines, the situation is different: the “superconducting” transition can now be viewed as split in two branches, $T_m(H)$ and $T_\Phi(H)$. At $T_m(H)$ the standard superconducting order is lost as the vortex lattice melts into a liquid of field-induced (s) vortex lines. Even though the total number of windings along $H$ is still locked at $N_\Phi$, just like in the vortex lattice state, the effective “diffusion” of (s) vortex lines leads to windings in the $xy$-plane, $N_{x,y} \propto L^{2p}$. This amount of winding suffices to cause vanishing of the helicity modulus (21) and drives $\Psi(r)$ to zero (i.e., $\langle \Psi(0) \Psi^*(r) \rangle$ is short-ranged). In contrast, the long distance line tension $T \propto 1/D$ (29) is still finite below $T_\Phi(H)$, same as in the Meissner state of the $H = 0$ superconductor. This state differs from the normal state by the presence of long range $\Phi$-order, measured by $\langle \Phi(0) \Phi^*(r) \rangle$. It can be considered isomorphic, in the long wavelength limit, to a liquid (or solid, as shown in Fig. 1) of $N_\Phi$ field-induced vortex lines with some finite effective line tension relative to the field direction, and interacting via a long range, London-Biot-Savart-type interactions, whose overall strength is set by $\sim |\langle \Phi(0) \Phi^*(r) \rangle|$. Only at some higher temperature $T_\Phi(H)$, does $T$ vanish and infinite vortex loops proliferate across the system. This signals the destruction of the $\Phi$-order as the system finally makes transition to the true normal state. Above $T_\Phi(H)$, both $\Psi$ and $\Phi$-order are absent. Consequently, the true normal state of the GL theory in a finite magnetic field should not be viewed as a “line-liquid” in the commonly accepted sense. The general geometrical arguments of this section preclude such identification and point to a direct connection between the presence (absence) of the $\Phi$-order and our ability (inability) to emulate the long wavelength behavior of the system in terms of a conventional “line-liquid” (or a “line-solid”) of field-induced vortices. In the language of the non-relativistic “boson analogy”, the mass of the bosons vanishes at $T_\Phi(H)$ and such analogy breaks down in the true normal state.

The above criteria should allow one to clearly distinguish the high and the low temperature states of the system in numerical simulations and to establish whether they are separated by a true thermodynamic phase transition or a sharp crossover. Such procedure should be superior to measurements of specific heat which possesses at most nearly-logarithmic singularity and is severely limited by the finite size effects.

VI. FLUCTUATION CONDUCTIVITY

Electrical conductivity (or resistivity), is a quantity easily measured experimentally. Unfortunately, fluctuation conductivity, while in principle a very useful probe of a degree of superconducting order, is not purely thermodynamic quantity and cannot be evaluated from the GL theory (1) unless additional assumptions are made concerning the time-dependence of the fluctuating superconducting order parameter. We adopt here a frequently used and empirically successful assumption of dynamical
scaling, which connects the decay of spatial correlations with that of time correlations: $\tau_{sc} \approx \xi_{sc}^z$, where $\tau_{sc}$ is the relaxation time associated with the superconducting order parameter and $z$ is the dynamical critical exponent. At zero field, this assumption leads to the expression for the fluctuation conductivity:

$$\sigma \propto \xi_{sc}^{z+2-D} \sim t^{\nu_{xy}(D-2-z)} ,$$

(30)

with $z \sim 1.5$ from numerical simulations. For simplicity, we have suppressed anisotropy in the above expression. When a finite external field is turned on, and we are inside the critical region near $T_{c0}$, where $\xi_{sc}$ is very long, we can still write:

$$\sigma \propto \left[ \xi_{sc}(H \neq 0) \right]^{z+2-D} .$$

(31)

Strictly speaking, the dynamical exponent could be different in the finite field case but I will ignore such possibility. I also concentrate on dissipative transport and do not consider Hall conductivity. Right below $T_\Phi(H)$, the gauge theory (4) suggests that $\xi_{sc}$ is finite in all directions and $\xi_{sc} \propto K/T \sim c \ell$, as discussed in the Appendix B. Above $T_\Phi(H)$, the screening length $\lambda$, defined in the Appendix B by Eq. (34), drops abruptly (32) and we assume that the superconducting correlation length $\xi_{sc}$ does the same,

$$\xi_{sc} \propto c \ell \left( 1 - G \frac{\ell}{\xi_\Phi} \right) = c \ell \left( 1 - C \frac{\tau_{VGT} \sqrt{H}}{\ell} \right) .$$

(32)

This assumption seems perfectly justified on physical grounds, since the drop in the screening length arises through appearance above $T_\Phi(H)$ of thermally-generated infinite vortex loops which lead to additional screening. It is natural to expect that these same loops suddenly increase dissipation and produce a non-analytic drop in the conductivity.

After restoring the anisotropy, the relative change in the conductivity, $\delta\sigma_{\mu}$, as one crosses over from the $\Phi$-order to the true normal state is:

$$\delta\sigma_{\mu} = \sigma_{\mu,\lessgtr} - \sigma_{\mu,\lless} \propto (z + 2 - D) C_{\mu} \frac{\tau_{VGT} \sqrt{H}}{\ell} ,$$

(33)

where $\mu = (\perp, \parallel)$, $\sigma_{\mu,\lessgtr}$ is the fluctuation conductivity below (above) $T_\Phi(H)$, $C_{\mu}$ are constants depending on material parameters, and $\nu_{VGT} \approx \nu_{xy} \sim 2/3$. The vortex loop “expansion” that takes place at $T_\Phi$ leads to a non-analytic increase in dissipation and a corresponding drop in conductivity.

VII. VORTEX LATTICE MELTING IN THE CRITICAL REGION

A theory of the vortex lattice melting in the critical region is an elaborate theory and its detailed discussion will be presented elsewhere. There are, however, several important consequences of the present gauge theory scenario that concern the very nature of the melting transition. We therefore discuss here the “minimal” set of requirements that should be satisfied by any theory of melting consistent with the gauge theory scenario. We should first observe that the reformulation (3) obviously has a (s) vortex lattice as its ground state at low temperatures, just as the original GL theory (1). The gauge theory (4), however, does not, for the simple reason: up to this point we were interested in the long wavelength, $q \ll 1/\ell$, behavior. The $\Phi$-transition, for example, is clearly the $q \to 0$ transition. On this basis, we have dropped a large number of terms from Eq. (6), by arguing that they are irrelevant at very long distances. The melting transition, in contrast, is a finite $q$-transition ($q \sim 1/\ell$), and it requires additional terms and modifications to the coarse-graining procedure applied on the way from (2) to (6). For instance, higher powers of $S$, particularly the odd ones, reflecting the up-down asymmetry along $H$ manifest in Eqs. (6) and (7), where unimportant in the long wavelength limit but are essential for the transition to a non-uniform (s) vortex lattice state.

This being so, the $\Phi$-transition casts a long shadow on the melting transition. This is clear from (6) and, by inference, from (7). In the $\Phi$-ordered phase, there is an effective long range interaction between (s) vortex lines which, in the $q \ll 1/\ell$ limit, takes the London-Biot-Savart form. To see this, note that, in the $\Phi$-ordered “Meissner” phase, our fictitious “photon” is “massive”, i.e., the second (gradient) term in Eq. (6) becomes:

$$\gamma_\perp \left| \langle \Phi \rangle \right|^2 S_\perp^2 + \gamma_\parallel \left| \langle \Phi \rangle \right|^2 S_\parallel^2 ,$$

(34)

where $\langle \Phi \rangle$ is the order parameter associated with the $\Phi$-order. As seen in Sec. V, we expect that, as long as $\langle \Phi \rangle$ is finite, we can write some effective description of such a state in terms of a “line liquid” of field-induced vortices, in the sense of Nelson (1). Long distance physics can be described through fluctuations in the density and “currents” of vortex lines, $\delta \rho(r)$ and $\vec{j}(r) = (\delta j_x, \delta j_y)$, respectively. The connection with our fictitious “gauge” potential $S(r)$ (8) is (Appendix A):

$$\langle \nabla \times S \rangle_\parallel \to 2\pi \delta \rho \ , \ \langle \nabla \times S \rangle_\perp \to 2\pi \vec{j} .$$

(35)

If we now reexpress (34) in terms of $\delta \rho$ and $\vec{j}$ we get precisely the long distance part of the London-Biot-Savart interaction between field-induced vortex lines:

$$4\pi^2 \gamma \left| \langle \Phi \rangle \right|^2 \sum_q \frac{\delta \rho(q) \delta \rho(-q) + \vec{j}(q) \cdot \vec{j}(-q)}{q^2 + q^2 ||} ,$$

(36)

where the continuity condition, $\vec{q} \cdot \vec{j}(q) = -q_0 \delta \rho(q)$, is assumed. The anisotropy, suppressed in the above expression for simplicity, can be straightforwardly restored. This expression (36) for the effective interaction at long distances is just what is obtained in the mean-field based approach (6) but with one crucial difference. The overall
strength of the interaction is not given by the mean-field amplitude squared of the superconducting order parameter, but by $|\langle \Phi \rangle|^2 \propto n_{\Phi}$, where $n_{\Phi}$ is the $\Phi$-superfluid density, whose physical meaning is apparent from reformulation \(3\) and gauge theory \(4\).

The immediate consequence of \(3\) is that the melting line, $T_m(H)$, goes into $T_{c0}$, the true zero-field superconducting transition, as $H \to 0$. This result is strongly suggested by all available numerical simulations on the 3D XY model and arises naturally in the gauge theory scenario. For all its apparent simplicity, this result is not trivial: the mean-field based theories of melting including only field-induced London vortices\(2\) naturally lead to $T_m(H) \to T_c$, the mean-field transition temperature, as $H \to 0$. Furthermore, the exponent $\nu$ has its mean-field value $\nu_{mf} = 1/2$ and is not equal to $\nu_{xy} \approx 2/3$. Therefore, the thermodynamics of the melting transition resulting from such theories cannot satisfy the 3D XY scaling properties of Sec. IV. This is a direct consequence of ignoring those very degrees of freedom (vortex loops) which are primarily responsible for moving the true zero-field superconducting transition temperature from $T_c$ to $T_{c0}$ and changing $\nu_{mf}$ to $\nu_{xy}$ in the first place. This is a serious flaw and must be rectified in a proper theory of vortex lattice melting in the critical region.

As a first step, we attempt to remedy the situation by simply replacing, by hand, the mean-field amplitude squared with the true superfluid density at zero field. This amounts to installing $|\langle \Phi \rangle|_{H=0}^2$ instead of $|\langle \Phi \rangle|^2$ in Eq. \(3\). With the interaction fixed in this fashion, we can then proceed to analyze the same, finite field model of Ref.\(2\), still including only the field-induced vortices. This is equivalent to having $T_g(H) = T_g(H = 0) \equiv T_{c0}$, as represented by the dashed vertical line in Fig. 1. This procedure, arbitrary as it is within the framework of mean-field based theories, seems to get us out of the above difficulties, since now evidently $T_m(H) \to T_{c0}$ for $H \to 0$ and the exponent $\nu$ takes its true $H = 0$ value, $\nu_{xy} \approx 2/3$.

However, things are not that simple: to understand why note that the above remedial procedure is in fact exact, but only for a single field-induced line. For a finite density of lines, the physical state of thermally-generated vortex loops and other fluctuations described by $\Phi(r)$, which controls the effective interaction between field-induced vortex lines through $|\langle \Phi \rangle|^2$ in Eq. \(3\), is itself strongly affected by interactions with those same field-induced lines. The effective coupling of these two interpenetrating systems, vortex loops and lines in reformulation \(3\), must be solved self-consistently at finite $H$: this is precisely what is accomplished in the gauge theory scenario \(4\) for the long wavelength ($q \ll 1/\ell$) behavior. Clearly, a “minimal” theory of vortex lattice melting in the critical region must involve both the positional order parameter of the vortex lattice $\rho_G$ (or the original $\Psi$) and the new “superconducting” order parameter $\Phi$. The coupled equations governing the $(T, H)$ dependence of these two order parameters must be solved simultaneously and self-consistently near $T_m(H)$.

An important physical feature is expected to emerge from such a solution: the formation of vortex lattice is a phase transition involving simultaneous ordering of both field-induced and thermally-generated degrees of freedom. This is illustrated with two qualitative points. First, at low fields within the $\Phi$-ordered state (Fig. 1), we can consider some effective “line-liquid” description \(3\). Right above $T_m(H)$, in the liquid phase, the self-consistent solution gives $\langle \Phi \rangle = \langle \Phi \rangle_L$ to be inserted in the effective interaction \(3\). Similarly, just below $T_m(H)$, we have $\langle \Phi \rangle = \langle \Phi \rangle_S$. In general, however, $\langle \Phi \rangle_L \neq \langle \Phi \rangle_S$! This result follows immediately from the gauge theory scenario \(4\) since positional correlations of s vortices strongly influence $\langle \Phi \rangle$. While both $\langle \Phi \rangle_L$ and $\langle \Phi \rangle_S$ are finite, as we cross $T_m(H)$, there is a discontinuous change in the average density and size of vortex loop and “overhang” excitations, resulting in different effective interaction \(3\) on two sides of the melting line (Fig. 4). The entropy jump at melting, $\Delta S$, will receive significant contribution from such excitations. In fact, while $\Delta S \to 0$ as $H \to 0$, the critical fluctuations greatly enhance $\Delta S$ over the configurational entropy of field-induced lines. This provides natural explanation for the excess entropy at $T_m(H)$ observed in low field thermodynamics.\(3\) It should be stressed that this effect is different from the “microscopic entropy” contribution discussed by Hu and MacDonald\(2\) (see also Ref.\(1\)). Such entropy arises from the electronic degrees of freedom and is reflected in the $T$-dependence of our GL coefficients \(\lambda\). This contribution is important in the high-field, LL regime.\(4\) In the 3D XY critical region, however, such $T$-dependence is a minor effect since it involves $T_c$ and not the true transition temperature $T_{c0}$. The entropy contribution discussed here, arising from $T$-dependence of $\langle \Phi \rangle^2$ in Eq. \(3\) and $\langle \Phi \rangle_L \neq \langle \Phi \rangle_S$, is due to degrees of freedom of the superconducting order parameter itself and must be part of any proper theory of melting.

Second, at higher fields, the self-consistent solution for $\rho_G$ (or $\Psi$) and $\Phi$ leads to a rapid suppression of $T_g(H)$ far below $T_{c0}$, and its subsequent merging with $T_m(H)$ for $H > H_Z$ (Fig. 1). This exposes a large region of the phase diagram where the Abrikosov vortex lattice melts directly into the true normal state. In this case $\langle \Phi \rangle_L = 0$, while $\langle \Phi \rangle_S$ might still be close to its mean-field value. Such dramatic difference in the nature of these two phases, with the solid being rather unremarkably mean-field like, and the liquid right above the melting line exhibiting very strong fluctuations even at rather low fields, with vortex lines winding both along the field and in the perpendicular directions, is evident in new simulations by Nguyen and Sudbø\(2\)\(3\). Note that such situation never arises in the mean-field based theories with only field-induced vortices\(4\) even after the application of our remedial procedure, since we would still have $|\langle \Phi \rangle|^2 = |\langle \Phi \rangle_L|^2 = |\langle \Phi \rangle_{H=0}|^2$ in Eq. \(3\). As argued in Sec. V, it does not appear possible to write an effective description of the true normal state in terms
of field-induced degrees of freedom only. For example, we could start again with our remedial procedure and argue that, even though it fails for $q \ll 1/\ell$, it still describes the effective interaction of field-induced vortices for $q \sim 1/\ell$, which is what matters most at $T_m(H)$. However, within such a "line liquid" description, I do not see any simple way by which one could account for the part of $\Delta S$ associated with a discontinuous change in windings across $T_m(H)$. In this region of higher fields, where $T_m(H) = T_m(H)$, our gauge theory is becoming less and less "type-II" and amplitude fluctuations are becoming stronger in the GL theory. It is likely that the low-field description discussed here and the LL theories now have an equal chance of providing reliable theory of the melting transition: both routes, however, are certain to be most challenging.

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25 $U(r)$ is the vector potential associated with a given configuration of $s$ vortices, with unit fluxes attached. In notation of Ref. 12 would correspond to $\nabla \Theta$, where $\Theta$ is the phase of the gauge transformation used in that paper. An explicit form of $\Theta$ is, however, not uniquely determined by the positions of $s$ vortices: one must also specify the positions of branch cuts across which the phase changes from 0 to $2\pi$. The present formulation eliminates this unnecessary complication since $U$ itself is uniquely determined for any given configuration of $s$ vortices.

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There is, however, no simple correspondence at the level of correlation functions. The equivalence between the two representations, \( [1] \) and \( [2] \), works only for the full partition functions and not at the level of individual configurations. This is clear since, for a given configuration of \( Z = \{1\} \) with \( N_\lambda + N_\perp \) vortex lines “percolating” across the system along \( \mathbf{H} \), we cannot tell which ones are field-induced and which are thermally-generated. All we know is that each configuration must have at least \( N_\Phi \) of such vortex lines. For example, this precludes direct connection between \( \Phi \) and the original \( \Psi \).

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As will be discussed later in the text, the \( \Phi \)-transition in gauge theory \( [1] \) is governed by an isotropic critical point. However, with \( K_\perp/K_\parallel \to \infty \) and, potentially, finite \( M^2_{\Phi S_\parallel} \) and/or Chern-Simons terms, the same transition could be governed by an anisotropic critical point, with a different set of critical exponents in \( \| \) and \( \perp \) directions and correspondingly different details of critical thermodynamics and transport.

“Perturbed” comes with quotation marks since, even though the magnetic field enters Eq. \( [1] \) through a small parameter \( \hat{e} \), this perturbation is strongly relevant and it immediately destabilizes the zero-field ("neutral") 3D XY critical point.

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Note that \( T_\Phi(H) < T_{\Phi}(H) \) is not excluded at higher fields.

Unlike the mean-field based approach including only the field-induced London vortex lines, where the strength of this interaction is set by the mean-field amplitude squared of \( \Psi(r) \).

Eq. \( [1] \) for \( t_\Phi(H) \) differs from the estimate reported in Ref. [1] in two ways: first, the previous estimate was made for the weakly-coupled layered case and, second, it was based on the mean-field approximation to (\( \Gamma = 1 \)). In contrast, the present expression \( [1] \) is exact, provided \( d_\Phi \) is specified.

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This fundamental anisotropy is present even for \( \xi_\perp = \xi_\parallel (\Gamma = 1) \) and is set by \( c_\perp/c_\parallel \). The value of \( c_\perp/c_\parallel \) along
the $T_B(H)$ line (more precisely, for $T_B(H) - T \to 0^+$) is a univ-

eral ratio of the gauge theory scenario.

While the rescaling procedure described here is a straight-

forward generalization of the familiar $H = 0$ rescaling of

anisotropy, it does contain one salient point: the rescaled
gauge theory looks just like the original Eq. (4) but with the

Φ-dependent part isotropic, $K_{\perp, r} \to K_{\perp, r}'$, $r \to r' =

(\Gamma^{1/3} r_{\perp}, \Gamma^{-2/3} r_{\parallel})$, and $S \to S' = (\Gamma^{-1/3} S_{\perp}, \Gamma^{2/3} S_{\parallel})$. However,

data deal with by introducing a gauge transformation:

$S' \to S'' = S' - \nabla \Pi$ and $\Phi \to \Phi'' = \Phi \exp \Pi$, where

$\Pi(r')$ is a regular phase defined so that the new gauge

theory condition, i.e.

$\nabla' \cdot S'' = \nabla' \cdot S' - \nabla' \cdot \nabla \Pi = 0$. Such gauge transfor-
mation is both perfectly justified and desirable. It is jus-
tified because, even though our fictitious electrodynamics

is always defined in a specific gauge and does not pos-

sess the local gauge freedom of its real namesake, its free

energy is still invariant under gauge transformations. It is
desirable since the Coulomb gauge insures that $\Phi$ (or, $\Phi''$,
in the anisotropic case) can be used as a true physical or-

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volving gauge transformations are discussed in Ref.[1]. In

the isotropic case ($\gamma_{\parallel} = \gamma_{\perp}$ in Eq. (1)) none of these niceties are

necessary and we can stay in the Coulomb gauge through-

out. The fundamental anisotropy of the gauge theory Eq.

(1), expressed as $K_{\perp} \neq K_{\parallel}$ ($c_{\parallel} \neq c_{\perp}$), always remains.

One naively expects the range of fields over which $d_m$ is uni-

eral to be smaller than the corresponding range for $d_k$: the

melting line is first-order and involves a non-uniform

low temperature state.

The exponent $p$ is assumed to have the same value, $1 > p \equiv 1/2$, throughout the $\Phi$-ordered phase. We can show that this is the case in the regime where the average size

of vortex-loops and $s$ vortex overhangs is $\ll \ell$ and we as-

ume that it remains true everywhere below $T_B(H)$. This

is consistent with our physical expectation that the long

wavelength limit of the $\Phi$-ordered phase can be related to

a suitably defined “line liquid”, in the sense of Nelson[4]. An

alternative possibility, that of $p$ being a continuously vary-
ing exponent, changing from $p \equiv 1/2$ far below $T_B(H)$ to

$p \to 1$ as we approach $T_B(H)$, appears less likely in view

of the overall picture painted by the gauge theory scenario.

For example, the $\Phi$-ordered state of (4) has a true long

range order and is not a critical phase with continuously vary-
ing exponents.

G. W. Crabtree and D. R. Nelson, Phys. Today 50, 38

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“Vortex tachyons” denote infinite vortex paths that tra-

verse the system along $x$ or $y$ direction without wind-

ing along the $z$-axis. Thus, in the boson analogy, they

travel longer intervals across “space” (xy-plane) than along

“time” ($z$-axis $|| H$).

In principle, however, such phase could result in a new

transition line between $T_B(H)$ and $T_m(H)$, at least in a

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therein.

The same would be true for the LL-based theories, if un-

critically extended to the $H \to 0$ limit.

Eventually, for extremely low fields, the effect of electro-

magnetic screening (i.e., finite $\kappa$) becomes important and

$T_m(B) \to 0$ as $B \to 0$.

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munication. The author is grateful to Prof. A. Sudbo and

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APPENDIX A: DERIVATION OF THE GAUGE THEORY

I present here a derivation of the gauge theory (1). An abbrevi-

ated version is found in Ref.[1]. For simplicity, I con-

sider the isotropic case, $\gamma_{\parallel} = \gamma_{\perp} = \gamma$.

Within the “helium model”[2] the partition function of the superconductor (1) can be written as:

\[ Z = \sum_{N_{\phi}} \cdots \sum_{N_{\omega}} \prod_{\omega} \frac{1}{N_{\omega}!} \prod_{l_{\omega}} \int D\xi_{l_{\omega}} [s_{l_{\omega}}] \exp \left( -\frac{E_{\omega}}{T} \right); F_{\omega} = \gamma |\Phi|^2 \int d^3r |\nabla \phi (r)|^2 \frac{2e}{c} A^2 + \sum_{\omega} \sum_{l_{\omega}} \int ds_{l_{\omega}} E^{(1)}(\{|\xi_{l_{\omega}}[s_{l_{\omega}}]\}|) \]

\[ + \frac{1}{2} \sum_{\omega} \sum_{l_{\omega}} \sum_{l_{\omega}'} \int d\xi_{l_{\omega}} \int d\xi_{l_{\omega}'} \{ V^{(2)}(|\xi_{l_{\omega}}[s_{l_{\omega}}] - \xi_{l_{\omega}'}[s_{l_{\omega}'}]|) + \frac{d\xi_{l_{\omega}}}{d\xi_{l_{\omega}'}}, \frac{d\xi_{l_{\omega}'}^*}{d\xi_{l_{\omega}}} V^{(2)}(|\xi_{l_{\omega}'}[s_{l_{\omega}'}] - \xi_{l_{\omega}}[s_{l_{\omega}}]|) \} \quad \text{ , (37)} \]

where

\[ \nabla \times A = H ; \quad \nabla \cdot \nabla \phi (r) = 0 ; \quad \nabla \times \nabla \phi (r) = 2\pi \sum_{\omega} \sum_{l_{\omega}} \int d\xi_{l_{\omega}} \delta (r - \xi_{l_{\omega}}[s_{l_{\omega}}]) \quad \text{ . (38)} \]
The partition function \( \langle \Phi \rangle \) is a 3D counterpart of the familiar representation of the (continuum) 2D XY model in terms of its point-like topological excitations, vortices and anti-vortices. In 3D, the relevant excitations are loops/lines of vortices, classified by their global topology. Only vortex paths of unit vortex “charge” are considered since they are the important excitations in the critical region. The lattice regularization of \( \langle \Phi \rangle \) is a gas of non-intersecting oriented paths on a lattice which are either closed or can start/end only on sample surfaces. The lattice spacing is set by the characteristic “bending length” of vortex lines. Each individual step along a path takes a given energy to create and can be either up or down along \( x, y \) or \( z \) axis. These steps represent vortex segments and have a long range “directional” Coulomb interaction, operating only between the steps going along the same axis. This is the lattice version of the “Biot-Savart” interaction between vortex loops/lines of the continuum model. The background free energy, composed of the uniform condensation energy and “spin-wave” contributions, is not included explicitly.

The summation in \( \langle \Phi \rangle \) runs over all distinct configurations of vortex line excitations of arbitrary length and shape. The index \( \omega \) denotes different classes of oriented loops/lines which are distinguished by their global topology. For example, for periodic boundary conditions, when only closed loops are present, \( \omega = (m_x^Z; m_y^Z; m_z^Z) \), where \( m_x^Z, m_y^Z, m_z^Z = 0, \pm 1, \pm 2, \ldots \) denote the winding numbers of a given loop around \( x, y, z \) directions. Finite closed loops correspond to \( m_x^Z = m_y^Z = m_z^Z = 0 \). Similarly, for free (periodic) boundary conditions along \( z \) \((x, y)\) direction, \( \omega = (m_x; m_y; 0, 0) \) denotes loops that wind in the \( x \) \((y)\) direction while \( \omega = (0, 0, 0; 0, 0, 0) \) again denotes finite closed loops. In addition, there are vortex (antivortex) paths that traverse the system from \( z = 0 \) to \( z = L_z \) and “half-loops” which originate and terminate at the same \( z = 0 \) or \( z = L_z \) surface. \[ \langle \Phi \rangle = \exp \left( \sum_{\omega} \omega \right) \]

\[ \langle \Phi \rangle \text{ represents summation over all configurations of a given loop/line } \omega \text{ consistent with its global topology. } \]

Consequently, in the thermodynamic limit, \( \Delta S \) scales as \( L_x L_y L_z \), in contrast to the full entropy which goes as \( L_x L_y L_z \). Consequently, in the thermodynamic limit, \( \Delta S \) does not affect the free energy per unit volume. In particular, within the \( \Phi \)-ordered state, \( Z \) and \( Z' \) have identical expansions:

\[ Z = \sum_{N^{(0)}=0}^{\infty} \frac{1}{N^{(0)!}} \frac{1}{N_{\Phi}!} \prod_{l=1}^{N^{(0)}} \int \mathcal{D} \phi_l \int \mathcal{D} x_{l_0} \int \mathcal{D} x_{i} \exp \left( -\frac{F_v}{T} \right) ; \]

\[ F_v = \gamma \| \phi \|^2 + \int \frac{d^3 r}{c} \int \mathcal{D} r' \left\{ \int \frac{d^3 r}{c} \int \mathcal{D} r' \left[ \{ d_0(r) + d_a(r) \} \{ d_0(r') + d_a(r') \} V_0^{(2)}(|r - r'|) + [n_0(r) + n_a(r)] \cdot [n_0(r') + n_a(r')] V_1^{(2)}(|r - r'|) \right] \right\} . \]

Here \( \{ x_{l_0}[s_{l_0}] \} \) denote finite closed loops of arbitrary length and shape. Half-loops attached to \( z = 0 \) and \( z = L_z \) surfaces are not included since they do not matter for the bulk thermodynamics. The symbol \( \mathcal{D} \phi \) indicates that the “single-particle” bending energy \( (E_b(\{ x_{l_0}[s_{l_0}] \})) \) has been absorbed into the measure of the path integral. “Densities” and “currents” \( d_{0,a} \) and \( n_{0,a} \) are defined as:

\[ \{ d_{0}, n_{0} \}(r) = \sum_{l_{0}=1}^{N^{(0)}} \int ds_{l_{0}} \{ 1, \frac{d x_{l_{0}}}{d s_{l_{0}}} \} \delta(r - x_{l_{0}}[s_{l_{0}}]) ; \}

\[ \{ d_{a}, n_{a} \}(r) = \sum_{i=1}^{N_{\Phi}} \int ds_{i} \{ 1, \frac{d r_{i}}{d s_{i}} \} \delta(r - r_{i}[s_{i}]) . \]
To understand the problem at low fields, we first consider the $H = 0$ situation. The superconducting (Meissner) state is described by the $H \to 0$, $N_\Psi \to 0$ limit of (43):

$$Z(H = 0) = \sum_{N^{(0)}=0}^{\infty} \frac{1}{N^{(0)!}} \prod_{\mu=1}^{N^{(0)}} \int D\delta x_\mu[s_\mu]\left(-\frac{F_\mu(0)}{T}\right) ; \quad F_\mu(0) = \gamma\langle|\Psi|^2\rangle \int_C d^3 r |\nabla \phi(r)|^2 + \sum_{\mu=1}^{N^{(0)}} E_\mu \int ds_\mu +$$

$$+ \frac{1}{2} \int d^3 r \int d^3 r' \left\{ d_\mu(r)d_\mu(r')V_1^{(2)}(|r-r'|) + n_\mu(r)\cdot n_\mu(r')V_1^{(2)}(|r-r'|) \right\}. \tag{42}$$

The gradient term in $F_\mu(0)$ can be decoupled by a dual gauge field $A_d(r)$, in the Coulomb gauge $\nabla \cdot A_d = 0$:

$$\exp\left[-\frac{\gamma\langle|\Psi|^2\rangle}{T} \int_C d^3 r |\nabla \phi(r)|^2 \right] \to \int D A_d(r) \exp\left[\int d^3 r \left( -i n_0 \cdot A_d - \frac{1}{2e_d^2} (\nabla \times A_d)^2 \right) \right]. \tag{43}$$

$\Psi_d(r)$ is a field operator of these relativistic bosons. The gradient term in the action has been rescaled into dimensionless form so that short-range repulsion and dual charge assume their canonical dimensions:

$$e_d^2 = 8\pi^2 \gamma \langle|\Psi|^2\rangle/T$$

is a dual charge. Eqs. (43) and (42) have an appearance of a path integral over trajectories of relativistic charged quantum particles in a $2 + 1$ dimensional Euclidean space ($z$ being the imaginary time) coupled to the “electromagnetic” gauge potential $A_d$ and interacting via short-range “density-density” and “current-current” interactions. $Z(H = 0)$ describes the vacuum structure of such “electrodynamics”, with our vortex loops corresponding to worldlines of relativistic quantum bosons and describing virtual particle-antiparticle creation and annihilation processes in the vacuum. This similarity can be exploited further by using the particle-field duality to define the field-theory version of $Z(H = 0)$ (43, 42):

$$\int D \Psi_d(r) \int D A_d(r) \exp\left\{ -\int d^3 r \left[ m_d^2 \Psi_d^2 + |(\nabla - i A_d)\Psi_d|^2 + \frac{1}{2} g_0 |\Psi_d|^4 + \frac{1}{2 q_d^2} (\nabla \times A_d)^2 + \frac{M_d^2}{2} A_d^2 \right] \right\}. \tag{44}$$

$\Psi_d(r)$ is a field operator of these relativistic bosons. The gradient term in the action has been rescaled into dimensionless form so that short-range repulsion and dual charge assume their canonical dimensions: $V_1\delta(r - r') \to g_0\delta(r - r')$, $g_0 = (\text{length})^{-1}$: $e_d^2 \to q_d^2$, $q_d^2 = (\text{length})^{-1}$. $V_1$ has been dropped since it is irrelevant in the long wavelength limit. Its effect on critical behavior can be incorporated into the bare values of $g_0$, $q_d^2$, and $M_d^2$. Finally, the “bare mass” of $A_d$, $M_d$, is absent in our problem ($M_d = 0$). Finite $M_d$ reflects the presence of a gauge field minimally coupled to the original, superconducting order parameter. For example, if the condition $\kappa \to \infty$ is relaxed and the real electromagnetic screening is restored in (4), $M_d^2 \to \mu_0 e^2/\pi$, where $\kappa$ and $\mu_0$ are the real charge and magnetic permeability, respectively.

The expression (44) forms the basis for the “dual” picture of the 3D XY critical behavior. In this picture, we are viewing vortices as primary objects and their field-operator, $\Psi_d(r)$, as our order parameter, instead of the original $\Psi(r)$. We can think of $\Psi(r)$ in Eq. (41) as being the field-operator describing creation and annihilation of Cooper pairs. In the GL theory, with $H = 0$ (4), Cooper pairs have only short range interactions and it suffices to keep only the quartic term, describing the point-like repulsion, since the rest is irrelevant for the critical behavior. In contrast, the vortex excitations of $\Psi$ interact via long range London-Biot-Savart forces, mediated by massless ($M_d = 0$) $A_d$ (44). Next, we can convert our neutral GL theory (4) into the one with finite real charge, $e$, by introducing the fluctuating vector potential, $A$, as well as the electromagnetic field energy, $(1/8\pi \mu_0 e^2)(\nabla \times A)^2$. Now, it is the Cooper pairs that have long range interactions, mediated by $A$, but the vortices in $\Psi$ interact only through short range forces, due to the electromagnetic screening inducing a finite $M_d = \mu_0 e^2/\pi$ in (44). Precisely in 3D, there is an exact duality, at least on a lattice (4), between the form of this interaction for vortices and for Cooper pairs (44). This is what “inverted” stands for in the “inverted 3D XY” model. In the dual language of $\Psi_d$, it is the low temperature Meissner phase of the original superconductor that is symmetric ($\langle\Psi_d\rangle = 0$), while the high temperature normal metal is the “broken symmetry” state of the dual theory ($\langle\Psi_d\rangle \neq 0$). So, there is an inversion of the temperature axis. However, it is still the same symmetry, $U(1)$, that is being broken and this implies that the thermodynamic exponent should be the same, $\nu = \nu_{xy}$, both for $e = 0$ and for $e \neq 0$. It is important to stress that the “inverted 3D XY” behavior of a charged-superfluid remains different from the 3D XY critical behavior of a neutral-superfluid, described by (44) with $H = 0$, since they are associated with two different critical points. For example, the anomalous dimension exponent, $\eta_{\Psi}$, of $\langle\Psi(0)\Psi^*(r)\rangle$, will be different in two cases, $e = 0$ and $e \neq 0$. 

Eq. (44) describes the “true vacuum” state $|0\rangle$ of a Euclidean relativistic field theory. Particle (antiparticle) excitations are massive, $m_\Psi \sim 1/\xi_d \sim 1/\Lambda_\Psi$, where $\xi_d$ is the dual correlation length associated with $\Psi_d$ and $\Lambda_\Psi$
measures the average loop size. The actual value of $m_\Phi$ reflects both the cost in energy and gain in entropy arising from large thermally-generated vortex loops in the original problem. As we approach $T_c$ from below, $m_\Phi \to 0$, and we enter the “false vacuum” state $|f\rangle$ of the theory. Particle (antiparticle) excitations are now massless and infinite vertex paths proliferate across the system, as depicted in Fig. 2. This “false vacuum” is just the normal metallic state. If we introduce the particle (antiparticle) number operators, $\hat{N}_{P,A}$, $|0\rangle$ is an eigenstate of $\hat{N}_{P,A}$, and $\hat{N}_{P}(0) = 0|0\rangle $. In contrast, $|f\rangle$ is not an eigenstate of $\hat{N}_{P,A}$, and contains a finite average number of (anti)particles, $\langle f| \hat{N}_{P,A} |f\rangle = \hat{N}_{P,A} \neq 0$. Both $|0\rangle$ and $|f\rangle$, however, are eigenstates of the total vorticity operator $\hat{N}_P - \hat{N}_A$ with eigenvalue 0, which insures $N_P = N_A$. For $H$ small but finite, the ground state of (44) must still be an eigenstate of $\hat{N}_P - \hat{N}_A$ but now the eigenvalue is $\Phi$. Starting from $|0\rangle$, such ground state $|\Phi\rangle$ is naturally constructed by introducing $N_\Phi$ massive particles into the true, stable vacuum. We then have $\hat{N}_P|\Phi\rangle = N_\Phi|\Phi\rangle$ and $\hat{N}_A|\Phi\rangle = 0|\Phi\rangle$. On the other hand, starting from the “false vacuum” $|f\rangle$, the ground state $|n\rangle$ is formed by having additional $N_\Phi$ particles added to an already present finite average number of massless particle-antiparticle pairs. $|n\rangle$ is not an eigenstate of $\hat{N}_{P,A}$ and satisfies $\langle n| \hat{N}_{P,A} |n\rangle = N_{P,A} \neq 0$, where now $N_P - N_A = N_\Phi$. At $T_\Phi(H)$, a phase transition takes place between these two different types of ground state, $|\Phi\rangle$ and $|n\rangle$, driven by the change in $U(1)$ symmetry of the vortex system.

We now return to Eq. (43). A finite density of $s$ vortices produces two main effects on the loop “expansion” as we approach $T_\Phi(H)$ from below. First, there is a long-range interaction between loops and $s$ vortices which will influence the long-range correlations among the loops themselves. Second, there is a short-range effect of $s$ vortices suppressing certain configurations of loops, through mutual contact interactions (an “excluded volume” effect). Intuitively, one expects the long-range effect to be essential for the critical behavior at low fields. Furthermore, the short range effect should be weak since the total number of vortex segments connected to $s$ vortices forms a minority of all vortex segments. Based on these observations, we devise the following strategy: according to our basic assumption ii) and results of Sec. V, the system of $s$ vortices below $T_\Phi(H)$ can be viewed as equivalent, in the long wavelength limit, to an effective system of non-relativistic 2D quantum bosons in its superfluid state, in the sense of Nelson [11]. The collective modes of such a system are its “density” and “current” fluctuations, which is precisely how the loops couple to $s$ vortices. The long wavelength effective action of these collective modes is described by two coupling constants, $m_s/n_s$ and $c_s^2$, where, in this boson analogy, $n_s$, $m_s$ and $c_s$ are the superfluid density, mass and speed of “sound”, respectively. We could compute these quantities explicitly, by starting sufficiently below $T_\Phi(H)$. Here, however, we treat them as general parameters which characterize the long wavelength fluctuations of $s$ vortices and whose ultimate values can be determined through their direct connection with the components of the helicity modulus tensor (Sec. V and Appendix B). We assume that the effect of mutual contact interaction between loop and $s$ vortex subsystems, measured by some interaction strength $V_\rho$, can be fully included into these ultimate values of $m_s/n_s$ and $c_s^2$, as well as into a renormalized loop-loop contact interaction, $\tilde{V}_0$ and the loop core line energy $\tilde{E}_c$. This amounts to reexpressing the short-range interaction in (43) as:

$$\frac{1}{2} \int d^3r \left\{ \tilde{V}_0 d_0^2(r) + 2V_\rho \delta d_0(r) \delta d_s(r) + V_\rho \delta^2_s(r) \right\} ; \quad \tilde{E}_c = E_c + V_0(d_0(r)) , \quad E_s = E_c + \tilde{V}_0(d_0(r)) ,$$

while the core line energies of loops and $s$ vortices are also renormalized to include the average “excluded volume” effect. $E_c$ and $V_\rho d_0$ will shortly be subsumed into $m_s/n_s$ and $c_s^2$. We set $V_\rho \to 0$ in (45) and proceed to derive the long distance description of (29). Indeed, we find that the long-range interactions lead to a major change in the behavior of the system once $H$ is finite: most importantly, the long-range interactions between vortex loops are “screened” by fluctuations of $s$ vortex lines. The amount of “screening” is determined by $m_s/n_s$ and $c_s^2$. Once the critical behavior associated with this “screening” has been understood, we reinstate the residual contact interaction between loop and $s$ vortex subsystems and test for consistency. We find, both within the $\epsilon$ expansion and perturbative RG in fixed dimension $D = 3$, that such residual coupling is irrelevant for $3 \leq D < 4$, i.e., it does not lead to any new relevant terms in the effective action, apart from those already present. Our procedure is therefore self-consistent.

The first step is to decouple the gradient term in (43) by using $A_{\alpha\beta}$, except that now $n_0 \to n_0 + \Delta n_s$ in Eq. (43) ($\Delta n_s$ is defined above Eq. (4)). The $s$ vortex part of (43) becomes:

$$\frac{1}{N_\Phi} \prod_{i=1}^{N_\Phi} \int D_\Phi r_i|s_i| \exp \left\{ - \sum_{i=1}^{N_\Phi} \frac{E_s}{T} \int ds_i - \frac{V_\rho d_0}{2T} \int d^3r d_s^2(r) - i \int d^3r \Delta n_s(r) \cdot A_{\alpha\beta}(r) \right\} .$$

This part is now reexpressed in terms of a vortex “density” and “current” fluctuations. Sufficiently below $T_\Phi(H)$, the overhangs are small and $s$ vortex lines are almost directed. We can use a non-relativistic boson analogy of Nelson [11] and replace $\int ds_i \approx \int_{\mathbb{R}} dz \left( 1 + \frac{1}{2} \left( \frac{\partial r_i}{\partial x} \right)^2 \right)$, where $\tilde{r}_i(z) = (x_i(z), y_i(z))$, while absorbing the effect of small overhangs into the effective mass, $m_s \sim E_s/T_s$. Similarly, $d_s(r) \to n_{s\parallel}(r) = \sum_i \delta(\tilde{r} - \tilde{r}_i(z))$ and $n_{s\perp}(r) = \sum_i \langle d\tilde{r}_i/dz \rangle \delta(\tilde{r} - \tilde{r}_i(z))$.
We then introduce a field operator of these non-relativistic bosons, $\Psi_s(\vec{r}, z)$, replace the path integral over $\Psi_s$ (periodic or free boundary conditions give the same result in the thermodynamic limit), and follow the standard procedure for deriving the “hydrodynamic” action of s vortices. If the phase $\varphi_s(\vec{r}, z)$ and amplitude $\pi_s(\vec{r}, z)$ are introduced via $\Psi_s = (\Psi_s^\dagger \exp(i\varphi_s))$ and $\pi_s = |\Psi_s|^2 - \langle |\Psi_s|^2 \rangle$, where $\langle |\Psi_s|^2 \rangle$ is evaluated self-consistently. After integration over $\pi_s$, the “hydrodynamic” action for the phase becomes:

$$
\int d^3r \left\{ \frac{n_s}{2m_s c_s^2} (\nabla \varphi_s - A_d)^2 + \frac{n_s}{2m_s} (\nabla_\perp \varphi_s - A_{d\perp})^2 + (\cdots) \right\},
$$

where $(\cdots)$ denotes higher powers of $\nabla \varphi_s - A_d$ and higher order derivatives. The “mean-field” part of the action is absorbed into the background. The action (47) is decoupled using real fields $\delta\rho(\vec{r})$ and $\vec{j}(\vec{r}) = (j_x, j_y)$:

$$
\int D\delta\rho D\vec{j} \exp \left\{ \int d^3r \left[ -i\delta\rho A_{d\parallel} - i\vec{j} \cdot A_{d\perp} + i\delta\rho (\nabla_\parallel \varphi_s + \vec{j} \cdot \nabla_\perp \varphi_s - \frac{m_s c_s^2}{2m_s} \delta\rho - \frac{m_s}{2m_s} \right] \right\},
$$

which finally results in Eq. (49) expressed in terms of the true collective modes of the superfluid s vortex system, its “density” $\delta\rho$ and “current” $\vec{j}$ fluctuations:

$$
Z \to \sum_{N^{(0)}=0} \frac{1}{N^{(0)}} \prod_{l_{0}=1}^{N^{(0)}} D_{s \chi_{l_{0}}} [s_{l_{0}}] \int D\delta\rho D\vec{j} \exp(-S) ; \quad S = \sum_{l_{0}=1}^{N^{(0)}} \frac{E_{c\ell}}{T} \int ds_{l_{0}} + \int d^3r \left\{ \frac{\hat{V}_0}{2T} \delta\rho^2(\vec{r}) + \bar{m}_{s} \cdot A_{d} + i\delta\rho A_{d\parallel} + i\vec{j} \cdot A_{d\perp} + \frac{m_s c_s^2}{2m_s} \delta\rho^2 + \frac{m_s}{2m_s} \vec{j}^2 + W(\rho, \vec{j}) + \frac{1}{2c_s^2} (\nabla \times A_d)^2 \right\},
$$

where the functional integral must be appended by the continuity condition, $\nabla_\parallel \delta\rho + \nabla_\perp \cdot \vec{j} = 0$, which follows from the integration over $\varphi_s$ in (48). $W$ arises from $(\cdots)$ terms in (47) and contains powers higher than quadratic in $\delta\rho$ and $\vec{j}$, as well as assorted derivatives. In particular, odd powers of $\delta\rho$ are present, i.e., $\sim \delta\rho^{2}$, reflecting the $\Delta n_{d\parallel} \to -\Delta n_{d\parallel}$ asymmetry of the original problem (43). Such higher order asymmetric terms are essential for description of the melting transition but, as shown below, are irrelevant at the $\Phi$-transition, provided the latter is continuous. Note that the above relatively simple dependence of (49) on $\delta\rho$ and $\vec{j}$ holds only at distances $\ell$. This is precisely what we are interested in as we approach $T_{\Phi}(H)$. Eq. (49) captures an essential effect of a finite field on the loop “expansion”: fluctuations of field-induced s vortex lines result in the “screening” of the “Biot-Savart” interaction between the loops. This “screening” is manifested by $A_{d}$ gaining a finite “mass”, $M_s \sim n_s / m_s c_s^2$, $M_s^2 \sim n_s / m_s$, after integration over $\delta\rho$ and $\vec{j}$. The effect of finite magnetic field in the original problem (3) is now stored in the finite values of $M_s$ and $M_s^2$. As one attempts to create a small density of very large loops in (3), upon approaching $T_{\Phi}(H)$ from below, their effective line tension and mutual interactions will be essentially influenced by such “screening”.

The importance of this “screening” mechanism is particularly apparent in the dual representation (44). The action (49) becomes:

$$
\int d^3r [m_{d\parallel}^2 |\Phi_d|^2 + (|\nabla - iA_d|\Phi_d|^2 + \frac{\hat{g}_0}{2} |\Phi_d|^4 + i\delta\rho A_{d\parallel} + i\vec{j} \cdot A_{d\perp} + \frac{m_s c_s^2}{2m_s} \delta\rho^2 + \frac{m_s}{2m_s} \vec{j}^2 + W + \frac{1}{2c_s^2} (\nabla \times A_{d})^2],
$$

where the meaning of $\Phi_d(\vec{r})$, $m_{d\parallel}$, and $\hat{g}_0$ is evident in light of the discussion surrounding Eq. (44). For simplicity, I suppress the anisotropy in the second (gradient) term of (50) which generally arises (even if $\gamma = \gamma_{\perp}$ in (3)) from the interaction of loops with s vortices. The finite mass of $A_{d\parallel}$, generated by the integration over $\delta\rho$ and $\vec{j}$, “cuts off” the long-range “Biot-Savart” interactions present in zero field (44). This “screening” decouples the dual gauge field and transforms the critical behavior from a “charged” (44) to a “neutral” (45) dual superfluid. This is the inverted (anisotropic) 3D XY behavior of the dual theory, hinting at the presence of a massless gauge field in the original “superconducting” formulation (3), as discussed below Eq. (45).

To investigate the critical behavior of (49) in more detail, we generalize (50) to arbitrary dimension $D$. Simultaneously, we restore in the action the residual part $V_{\rho}$ (44) of the contact interaction between loops and s vortex lines, i.e., the part not already incorporated into the values of coupling constants appearing in (46):

$$
\int d^3r [m_{d\parallel}^2 |\Phi_d|^2 + (|\partial_\mu - iA_\mu|\Phi_d|^2 + \frac{1}{2} g_0 |\Phi_d|^4 + g_0 |\Phi_d|^2 J_0 + i\mu J^- + \frac{1}{2M_\mu^2} J_\mu J^\mu + W(J_\mu) + \frac{1}{2g_{d\mu}} F_{\mu\nu} F^{\mu\nu}],
$$

Here $g_\rho$ denotes $V_{\rho}$ rescaled to its canonical dimension, $\mu = 0, 1, 2, \ldots, D - 1$, $W(J_\mu)$ is the generalization of $W(\rho, \vec{j})$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The functional integration runs over fields $\Phi_d$, $A_\mu$, and $J_\mu$ and includes a constraint
\[ \partial_{\mu} J^\mu = 0. \] The integration over \( J_\mu \) generates a finite mass \( M_{\mu=0} = M_\parallel \), \( M_{\mu\neq0} = M_\perp \) for the dual gauge field \( A_\mu \).

The theory, however, retains gauge invariance implying that the combination \( g_0 A_\mu \) must be an RG invariant. This in turn sets the canonical dimension of \( g_0 \) to \( [g_0] = (\text{length})^{D-3} \). Therefore, the term \( g_0 \langle \Phi_d^2 J_0 \rangle \) is irrelevant within \( \epsilon \) expansion around the upper critical dimension \( D = 4 \). Similarly, all higher powers and derivatives of \( J_\mu \) appearing in \( W \) are irrelevant as well. For example, the canonical dimension of the \( J_0^2 \) coupling constant is \( (\text{length})^{2D-3} \). The relevant couplings below \( D = 4 \) are \( g_0 \), \( g_0^2 \) and \( M_{\parallel,\perp} \). Due to finite \( M_{\parallel,\perp} \), \( A_\mu \) decouples and the \( \beta \) function for \( g_0 \) is the same as that of the neutral \( (g_0 = 0) \) complex \( \Phi^1 \)-theory. In this case the \( \epsilon \) expansion is expected to hold down to, and include, \( D = 3 \), where the critical behavior of the \( \Phi^3 \)-theory should be that of a (inverted) 3D XY model, in agreement with our earlier assertion. The same conclusions can also be reached within the perturbative RG in fixed dimension \( D = 3 \). Here \( g_0 \) is marginal at the “engineering” level, the gauge field again decouples due to finite \( M_{\parallel,\perp} \), and we can compute the relevant \( \beta \) functions at the one-loop order and to the leading order in \( g_\rho \):

\[
\beta_0(\hat{g}_0, \hat{g}_\rho) = \frac{dg_0}{d\log(p)} = -\hat{g}_0 + C_1 \hat{g}_0^2 \quad ; \quad \beta_\rho(\hat{g}_0, \hat{g}_\rho) = \frac{dg_\rho}{d\log(p)} = C_2 \hat{g}_0 \hat{g}_\rho \quad ,
\]

where \( \hat{g}_0, \hat{g}_\rho(p) \) are the dimensionless running coupling constants and \( C_{1,2} \) are (regularization dependent) numerical constants which are both positive, \( C_{1,2} > 0 \). At the (inverted) 3D XY critical point \( \hat{g}_0 = 1/C_1 \) and therefore \( \beta_\rho > 0 \), indicating stability of our assumed \( g_0 = 0 \) fixed point against residual \( g_\rho \) perturbation. The above results allow us to conclude that the critical theory \( \Phi^3 \) (with \( W = 0 \)) remains valid and that the effects of \( V_\rho \) can be included by a proper choice of relevant couplings, as originally assumed. The presence of long-range interactions between vortices, mediated by \( A_\mu \), is essential for the validity of this argument.

One step remains: as \( T \to T_\Phi(H) \), some overhangs attached to \( s \) vortex lines become very large and we might doubt the accuracy of the straightforward non-relativistic boson analogy approximations below Eq. \( \Phi^3 \). However, throughout the \( \Phi \)-ordered state, the \( s \) vortices remain “massive” and there should always exist a suitably defined quantum system of non-relativistic 2D bosons whose long “distance” \( (x,y) \) and “imaginary time” \( (z) \) behavior faithfully emulates that of \( s \) vortices. We therefore expect that the overall symmetry of \( \Phi^3 \) remains preserved at \( T_\Phi(H) \).

This leads to the generalization of \( [13] \):

\[
Z(H) \to \sum_{N(0) = 0}^{N(0)} \frac{1}{N(0)!} \prod_{l_0 = 1}^{N(0)} \int D_b x_{l_0}[s_{l_0}] \int D V(r) \int D A_d(r) \exp(-S) \quad ; \quad S = \sum_{l_0 = 1}^{N(0)} \frac{\tilde{E}_c}{T} \int ds_{l_0} + \\
+ \int d^3r \left\{ \frac{\tilde{V}_0}{2T} r_0^2(r) + i m_0 \cdot A_d + i V \cdot A_d + \frac{2\pi^2 K \parallel V^2}{T} + \frac{2\pi^2 K \perp V^2}{T} + \frac{1}{2 e_d^2} (\nabla \times A_d)^2 \right\} ,
\]

where \( V(r) \) describes long distance \( (\gg \ell) \) fluctuations of \( s \) vortex “currents” \( n_s(r) \) \( [11,12,13] \) and satisfies \( \nabla \cdot V = 0 \). \( K \parallel, K \perp \) now play the role of \( m_s^2/n_s \) and \( m_s/n_s \) in \( [13] \) and fully include the effect of overhang configurations as \( T \to T_\Phi(H) \). At present, we cannot compute \( K \parallel, K \perp (T,H) \) (nor \( \tilde{E}_c \) and \( \tilde{V}_0 \)) from first principles. This would require an analytic solution to the problem of large overhangs, something far beyond the scope of this paper. However, if we start with the general form \( [13] \), we can determine various parameters appearing there by connecting them self-consistently to directly (numerically or experimentally) measurable physical quantities. For example, \( K \parallel (T,H) \) can be extracted from the components of the helicity modulus tensor (Appendix B) or the fluctuation conductivity (Sec. VI). We should therefore consider \( [13] \) a self-consistent, perturbative RG description of the \( \Phi \)-transition.

We can now enforce the constraint \( \nabla \cdot V = 0 \) by introducing a gauge field \( S(r) \): \( 2\pi V \to \nabla \times S, \nabla \cdot S = 0 \). Alternatively, we can integrate over \( V \), obtain the mass term for \( A_d \), and then decouple it by introducing \( S \). The final result is:

\[
Z(H) \to \sum_{N(0) = 0}^{N(0)} \frac{1}{N(0)!} \prod_{l_0 = 1}^{N(0)} \int D_b x_{l_0}[s_{l_0}] \int D S(r) \int D A_d(r) \exp(-S) \quad ; \quad S = \sum_{l_0 = 1}^{N(0)} \frac{\tilde{E}_c}{T} \int ds_{l_0} + \\
+ \int d^3r \left\{ \frac{\tilde{V}_0}{2T} r_0^2(r) + i m_0 \cdot A_d + \frac{i}{2\pi} (\nabla \times S) \cdot A_d + \frac{K \parallel}{2T} (\nabla \times S)^2 + \frac{K \perp}{2T} (\nabla \times S)^2 + \frac{1}{2 e_d^2} (\nabla \times A_d)^2 \right\} .
\]

This is just the vortex loop expansion \( [12,13] \) of the Meissner phase of a “superconductor” described by an order parameter \( \Phi(r) \) and coupled to the gauge field \( S (\nabla \cdot S = 0) \). The GL functional of such a superconductor is

\[
\mathcal{F}_{\text{eff}} = \tilde{\alpha} |\Phi|^2 + \tilde{\gamma}_\mu |(\nabla_\mu + i S_\mu) \Phi|^2 + \frac{\tilde{\gamma}}{2} |\Phi|^4 + \frac{K \parallel}{2T} (\nabla \times S)^2 + \frac{K \perp}{2T} (\nabla \times S)^2_\perp \quad ,
\]

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which is precisely the gauge theory (I): \( \bar{\alpha}, \bar{\beta} \) and \( \bar{\gamma}_\mu \) are some suitably renormalized GL coefficients which can be determined phenomenologically. Note that we have now restored the anisotropy in \( \bar{\gamma}_\mu \), arising both from the bare anisotropy \( \gamma_\parallel \neq \gamma_\perp \) in Eq. (I) and the anisotropy induced by the interaction of loops with the s vortex background.

APPENDIX B: GAUGE THEORY AND THE HELICITY MODULUS

Here we consider the connection between \( K_{\perp,\parallel} \) appearing in Eq. (I) and the helicity modulus tensor \( \Upsilon(\mathbf{q}) \). The conventional definition of the components of \( \Upsilon(\mathbf{q}) \) can be found in Ref. [30].

\[
\Upsilon_{\mu\nu}(\mathbf{q}) = V \frac{\delta^2F}{\delta a_\nu(\mathbf{q})\delta a_\mu(-\mathbf{q})}. \tag{56}
\]

All quantities appearing in (56) are defined in Sec. V, below Eq. (19). We will limit our attention to the isotropic situation \( \gamma_\parallel = \gamma_\perp \) in Eq. (I) whenever we consider the \( H \neq 0 \) situation. The anisotropic case with finite field requires far more extensive algebra and can be reconstructed by combining the discussion below with illuminating presentation in Ref. [30].

We first evaluate \( \Upsilon(\mathbf{q}) \) from the original GL theory (I) and start with the \( H = 0 \) (\( \mathbf{A} = 0 \)) case. We consider the situation right above \( T_{c0} \), so there is no superconducting long range order. After adding small \( \mathbf{a} \) to the gradient term, we can expand the free energy up to second order in \( \mathbf{a} \):

\[
F[\nabla \times \mathbf{a}] = F[0] + \int d^3r \left[ \frac{e^2}{2c^2} \chi_\perp(\nabla \times \mathbf{a})_\perp^2 + \frac{e^2}{2c^2} \chi_\parallel(\nabla \times \mathbf{a})_\parallel^2 \right] + \ldots . \tag{57}
\]

This equation requires a brief explanation: \( e \) is the real electric charge and \( c \) is the real speed of light, appearing in (I). \( F[0] \) is just the original free energy of the GL theory with \( H = 0 \) and \( \mathbf{a} = 0 \). The full free energy with small \( \mathbf{a}(r) \) is written as a functional of \( \nabla \times \mathbf{a} \) only. This is required by the gauge invariance of (I). Two additional terms, proportional to the perpendicular and parallel components of \( (\nabla \times \mathbf{a})^2 \), represent the leading corrections in powers and derivatives of \( \nabla \times \mathbf{a} \). The subleading contributions are denoted by dots. These subleading terms are unimportant in the long wavelength limit.

\( \chi_\perp \) and \( \chi_\parallel \) are some functions of \( T \) and are different from each other in the anisotropic case, \( \gamma_\parallel \neq \gamma_\perp \), while \( \chi_\perp = \chi_\parallel = \chi \) if the superconductor is isotropic. As \( T \) approaches \( T_{c0} \) from above, these functions take the following form:

\[
\chi_\perp = C T \xi_\parallel , \quad \chi_\parallel = C T \frac{\xi_\perp}{\xi_\parallel} . \tag{58}
\]

\( \xi_{\perp,\parallel} = \xi_{0,\perp,\parallel} e^{-\nu} \) are the superconducting correlation lengths and \( C \) is an unknown universal constant, intrinsic to the GL theory (I). Note that \( e^2 \chi_{\perp,\parallel}/c^2 \) is just the perpendicular (parallel) magnetic susceptibility.

The components of the helicity modulus tensor in the long wavelength limit \( (q \rightarrow 0) \) are uniquely determined by \( \chi_{\perp,\parallel} \). Conversely, the measurement of the long wavelength “tilt” and “compression” helicity modulus \( \Upsilon_{\perp,\parallel} \) determines \( \chi_{\perp,\parallel} \). In general, from definition (58), we find:

\[
\frac{e^2}{c^2} \Upsilon_{\mu\nu}(\mathbf{q}) = \chi_{\rho\mu\alpha\beta\gamma} q_\alpha q_\beta q_\gamma , \tag{59}
\]

for the isotropic case, while for the anisotropic situation \( \chi_\perp \neq \chi_\parallel \):

\[
\frac{e^2}{c^2} \Upsilon_{\mu\nu}(\mathbf{q}) = (\chi_\parallel - \chi_\perp) \epsilon_{\rho\mu\alpha} \epsilon_{\rho\beta\nu} q_\alpha q_\beta + \chi_\perp \epsilon_{\rho\mu\alpha} \epsilon_{\rho\beta\nu} q_\alpha q_\gamma . \tag{60}
\]

\( \epsilon_{\alpha\beta\gamma} \) is the Levi-Civita symbol and summation over repeated indices is understood. \( e^2/c^2 \) is factored out for later convenience.

After this preliminary discussion, we go to the case of interest, finite \( H \) in Eq. (I), and limit our consideration to the isotropic situation \( \gamma_\perp = \gamma_\parallel \). We introduce small \( \mathbf{a} \) into Eq. (I) and expand to second order in \( \nabla \times \mathbf{a} \):

\[
F[H, \nabla \times \mathbf{a}] = F[H, 0] + \int d^3r \left[ \frac{e^2}{2c^2} \chi_\perp(\nabla \times \mathbf{a})_\perp^2 + \frac{e^2}{2c^2} \chi_\parallel(\nabla \times \mathbf{a})_\parallel^2 \right] + \ldots , \tag{61}
\]
where $F[H,0]$ is now the free energy of the GL theory at finite field $H$ and $a = 0$. We are again focusing on the “normal”, i.e. not superconducting state, in accordance with assumption ii) of Sec. II. The above expression looks very much like \((27)\) but there are following significant differences: the expansion is anisotropic, $\tilde{x}_+ \neq \tilde{x}_\parallel$, even though our superconductor is isotropic. The reason for this is finite field $H$ along the $z$-axis which reduces spherical symmetry of the $H = 0$ situation down to cylindrical. The finite field also breaks the “up-down” symmetry along the $z$-axis. This is manifested by the subleading corrections, denoted by dots in \((61)\), containing in general odd powers of $\nabla \times a$ (the leading such term is cubic); such terms were prohibited by symmetry in the $H = 0$ case. Again, by combining definition \((24)\) and Eq. \((22)\), we arrive at the expression for the long wavelength limit of the helicity modulus:

$$\frac{e^2}{c^2} Y_{\mu\nu}(q) = (\tilde{x}_\parallel - \tilde{x}_\perp) \xi_{\perp} \xi_{\parallel} \xi_{\perp} \xi_{\parallel}.$$  

(62)

The components of the helicity modulus tensor for the finite field (isotropic) case are determined by $\tilde{x}_\perp, \tilde{x}_\parallel$ which are some functions of $T$ and $H$. It is tempting to conclude that:

$$\tilde{x}_\perp \rightarrow CT \xi_\parallel (T, H), \quad \tilde{x}_\parallel \rightarrow CT \xi^2_\parallel (T, H).$$  

(63)

where $\xi_\perp, \xi_\parallel (T, H)$ are now superconducting correlation lengths at finite field. This result is plausible on physical grounds, expressing the fact that, with $H \neq 0$, the superconducting correlation lengths are now finite in the “liquid” phase, limited by magnetic length $\ell$ and consequently the helicity moduli vanish in the $q \rightarrow 0$ limit. We are simply making the assumption that the same length that limits the range of superconducting correlations appears in the coefficient of the $q^2$-term in the helicity modulus; this assumption is known to be correct for the $H = 0$ case \((63)\). Unfortunately, I am unable to provide a mathematical proof that the conjectured result \((63)\) is exact. Instead, I define perpendicular and parallel “screening” lengths $\Lambda_\perp(T, H)$ and $\Lambda_\parallel(T, H)$ by:

$$\tilde{x}_\perp = CT \Lambda_\parallel (T, H), \quad \tilde{x}_\parallel = CT \Lambda^2_\parallel (T, H).$$  

(64)

Note that there is a one-to-one correspondence between these “screening” lengths $\Lambda_\perp, \Lambda_\parallel$ and $\chi_\perp, \chi_\parallel$, and in turn, between $\Lambda_\perp, \Lambda_\parallel$ and the long wavelength helicity moduli \((62)\). Since $\Lambda_\perp, \Lambda_\parallel$ are purely thermodynamic quantities they satisfy scaling laws of Sec. IV, just like the superconducting correlation lengths:

$$\Lambda_\parallel (T, H) = \ell \mathcal{R}_\parallel (q^2_0), \quad \xi_\parallel (T, H) = \ell \mathcal{R}_\parallel^\xi (q^2_0),$$  

(65)

where “dimensionless charge” $q^2_0 = \xi(T, H = 0)/\ell \propto H^{1/2}/|\mu|$ is our scaling variable of Eq. \((17)\) and $\mathcal{R}_\parallel, \mathcal{R}_\perp$ are the screening length and correlation length scaling functions, respectively. In the $H \rightarrow 0$ ($q^2_0 \rightarrow 0$) limit, all these scaling functions, $\mathcal{R}_\parallel (q^2_0)$ and $\mathcal{R}_\perp^\xi (q^2_0)$, go as $q^2_0$. We now make the following assumption: around $T_\Phi(H)$, the ratios

$$\frac{\Lambda_\perp (T, H)}{\Lambda_\parallel (T, H)} = \frac{\mathcal{R}_\perp (q^2_0)}{\mathcal{R}_\parallel (q^2_0)}, \quad \frac{\Lambda_\parallel (T, H)}{\Lambda_\parallel (T, H)} = \frac{\mathcal{R}_\parallel^\xi (q^2_0)}{\mathcal{R}_\perp^\xi (q^2_0)}.$$  

(66)

are some unremarkable smooth functions of the scaling variable $q^2_0$. In particular, the non-analytic drop in the coefficient of the $q^2$-term in the helicity modulus, which takes place as we cross the $T_\Phi(H)$ transition line \((21)\) and which is directly reflected as a non-analytic decrease in the screening lengths $\Lambda_\perp, \Lambda_\parallel (T, H)$, is manifested also in the superconducting correlation lengths $\xi_\perp, \xi_\parallel (T, H)$. This assumption, which appears justified physical grounds, was used in the discussion of fluctuation conductivity (Sec. VI).

Finally, we are in position to discuss our anisotropic gauge theory of Eq. \((4)\). Small $a$ added to the external vector potential $A$ in the original GL theory \((1)\) translates into a small vector potential $(e/c)a$ added to our fictitious gauge field $S$ in Eq. \((4)\). Since we are integrating over $S$, it is useful to define new gauge field $S_n = S + (e/c)a$ and integrate over $S_n$ in the partition function. The effect of this is to move $(e/c)a$ from the covariant gradient terms, $|D_n \Phi|^2$, to the “gauge field energy” $K_\mu |\nabla \times (S - (e/c)a)|^2$ in Eq. \((4)\). We now expand the free energy of the gauge theory, $F_{\text{eff}}$, to second order in $\nabla \times a$, following the same philosophy as in Eq. \((61)\):

$$F_{\text{eff}}[\tilde{\xi}_\perp, \tilde{\xi}_\parallel, \nabla \times a] = F_{\text{eff}}[\tilde{\xi}_\perp, \tilde{\xi}_\parallel, 0] + \int d^3 r \left[ \frac{e^2}{2c^2} K_\perp |\nabla \times a|^2 + \frac{e^2}{2c^2} K_\parallel |\nabla \times a|^2 \right] + \cdots,$$  

(67)

where
\[ \mathcal{K}_{\perp,\parallel} = K_{\perp,\parallel} - \frac{K_{\perp,\parallel}^2}{T} \lim_{q \to 0^+} \int d^3(r - r') e^{i q (r - r')} \langle (\nabla \times \mathbf{S}(r))_{\perp,\parallel} \langle (\nabla' \times \mathbf{S}(r'))_{\perp,\parallel} \rangle \rangle. \] (68)

The anisotropic charges \( \tilde{e}_{\perp,\parallel}(T, H) \) and coupling constants \( K_{\perp,\parallel}(T, H) \) are defined in Eqs. (15) and (4), respectively. The thermal average \( \langle \cdots \rangle \) is over the gauge theory defined by the free energy functional \( F_{\text{eff}}(4) \). Combining (56) and (68), we get, as before:

\[ c_2^2 \mathcal{E}^{\mu\nu}(q) = (K_{\parallel} - K_{\perp}) \epsilon_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} q_\alpha q_\beta + K_{\perp} \epsilon_{\rho\alpha\mu} \epsilon_{\rho\beta\nu} q_\alpha q_\beta. \] (69)

Comparing this to the general expression for the helicity modulus of the GL theory at finite field, given by Eq. (62), we conclude that \( \tilde{\chi}_{\perp,\parallel} = K_{\perp,\parallel} \).

A very important point: below \( T_{\Phi}(H) \) we have \( K_{\perp,\parallel} = K_{\perp,\parallel} \). This is a mathematical consequence of the following physical picture (Sec. V). In the \( \Phi \)-ordered state, only the field-induced (s) vortex lines can “screen” the test vector potential \( a(r) \). All thermally generated vortex loops are of finite size and cannot contribute to the “screening” in the long wavelength limit, described by \( K_{\perp,\parallel} \) in Eq. (67). This implies that the new order parameter \( \Phi \) is finite and therefore \( \langle (\nabla \times \mathbf{S})^2_{\perp,\parallel} \rangle \) vanishes in the long wavelength limit, as \( \sim q^2/|\langle \Phi \rangle|^2 \). Therefore, we can uniquely fix the coupling constants \( K_{\perp,\parallel}(T, H) \) (and corresponding anisotropic charges \( \tilde{e}_{\perp,\parallel}(T, H) \)) that enter the gauge theory description (4), by connecting them directly to the components of the helicity modulus tensor right below \( T_{\Phi}(H) \) (or, more precisely, for \( T_{\Phi}(H) - T \to 0^+ \)). In particular, the functions \( c_{\perp,\parallel}(T, H) \), introduced below Eq. (4) and referred to at various points in the main text, follow from the general expression (64):

\[ c_{\perp}(T, H) = CR_{\perp}(q_0^2), \quad c_{\parallel}(T, H) = CR_{\parallel}(q_0^2)/R_{\parallel}(q_0^2) \] (70)

where \( R_{\perp,\parallel}(q_0^2) \) are the scaling functions introduced in Eq. (15) and are to be evaluated below \( T_{\Phi}(H) \). The gauge theory scenario predicts that the fundamental anisotropy ratio \( c_{\parallel}/c_{\perp} \) takes on a universal value along \( T_{\Phi}(H) \).

Above \( T_{\Phi}(H) \), in the true normal state, infinite vortex loops proliferate across the system and can contribute to screening. \( \langle (\nabla \times \mathbf{S})^2_{\perp,\parallel} \rangle \) in Eq. (68) becomes finite and \( \sim 1/\xi_{\Phi} \). This causes the non-analytic drop in the \( q^2 \)-term of the helicity modulus and the corresponding screening lengths (64), just as discussed in Sec. V (see Eq. (21)).

The reader should note that the set of results presented in this Appendix, connecting the properties of the gauge theory description (4) around \( T_{\Phi}(H) \) to the general long distance form of the helicity modulus and screening lengths of the GL theory at finite field (1), is not only physically transparent and appealing but also exact, provided our assumptions i)-iv) (Sec. II), are satisfied.
FIG. 1. Proposed $H - T$ phase diagram for the critical region of extreme type-II superconductors. The dashed region denotes a crossover from the Gaussian regime, where amplitude fluctuations are strong, to the critical 3D XY-like regime, where amplitude fluctuations are suppressed. Within the 3D XY-like critical regime, the London-type vortex loops and lines with tight cores are well-defined excitations. Along the temperature axis, this critical region is bounded by the mean-field $T_c$. Along the field axis, the critical region is bounded by $H_s (\sim H_b)$. Above $H_s$, the physics of GL theory is dominated by the formation of Landau levels for Cooper pairs. The $\Phi$-transition, or the vortex-loop “expansion” transition, $T_\Phi(H)$, and the vortex lattice melting, $T_m(H)$, occur simultaneously for $H > H_Z \sim H_s$. This is a first-order transition from the Abrikosov vortex lattice (VS) directly to the normal state (N). Below $H_Z$, $T_\Phi(H)$ and $T_m(H)$ split into two separate transitions and merge again only at the true zero-field superconducting transition, $T_{c0}$, as $H \to 0$. For $H \ll H_s$, the transition at $T_\Phi(H)$ is continuous, while the vortex lattice melting transition remains first-order. The intermediate phase ($\Phi$), below $T_\Phi(H)$ but above $T_m(H)$, is not a superconductor ($\langle \Psi \rangle = 0$), but it differs from the true normal state (N) by a new type of long range order, characterized by the “superconducting” order parameter $\Phi(r)$. Only $N_b$ field-induced vortex lines traverse the system along the field direction in this $\Phi$-ordered state, while the average size of thermally-generated vortex loops is finite. In the true normal state (N), the $\Phi$-order is destroyed as numerous additional vortex paths “expand” across the system in all directions.

FIG. 2. A schematic representation of the $H = 0$ transition in an extreme type-II superconductor. The low temperature Meissner phase ($T < T_{c0}$) contains only finite vortex loops. In the high temperature normal state ($T > T_{c0}$) these loops connect and “expand” across the system, leading to loss of phase coherence and finite dissipation. Two forms of superconducting order, described by $\Psi$ and $\Phi$, are equivalent here. For clarity, the vortex paths are drawn far smoother than they actually are near $T_{c0}$.

FIG. 3. Characteristic configurations of the system a) below and b) above $T_\Phi(H)$ (but always above $T_m(H)$). a) Field-induced vortices (depicted in blue) wind all the way across the system along the field direction but only undergo effective “diffusion” in the transverse direction. b) After the loop “expansion” at $T_\Phi(H)$ this effective transverse “diffusion” is destroyed, as field-induced vortices can “hitch” a ride all the way across the system in the xy-plane by connecting to thermally-generated infinite loops present in the true normal state. Note the presence of “vortex tachyons” (depicted in red) which wind only in the xy-plane. Again, for illustrative purposes, the vortex paths are drawn far smoother than they actually are near $T_{c0}$.

FIG. 4. Characteristic configurations of the system a) below and b) above $T_m(H)$ (but always below $T_\Phi(H)$). a) Field-induced vortices (blue) execute small oscillations around their equilibrium positions. Thermally-generated vortex-loops (red) are small and rare. b) Above the melting transition thermally-generated loops discontinuously grow larger and more numerous, although they still remain of finite size. This discontinuous change in the state of the loops accounts for $\langle \Phi \rangle_S \neq \langle \Phi \rangle_L$ at the simplest mean-field level.

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