Spectral Properties of the Canonical Solution Operator to $\delta$

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SPECTRAL PROPERTIES OF THE CANONICAL SOLUTION OPERATOR TO $\bar{\partial}$

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ABSTRACT. We study boundedness, compactness, and Schatten-class membership of the canonical solution operator to $\bar{\partial}$, restricted to $(0,1)$-forms with holomorphic coefficients, on $L^2(d\mu)$ where $\mu$ is a measure with the property that the monomials form an orthogonal family in $L^2(d\mu)$. The characterizations are formulated in terms of moment properties of $\mu$. Our results generalize the results of the first author to several variables, contain some known results for several variables, and also cover new ground.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper, we study spectral properties of the canonical solution operator to $\bar{\partial}$ acting on spaces of $(0,1)$-forms with holomorphic coefficients in $L^2(d\mu)$ for measures $\mu$ with the property that the monomials $z^\alpha$, $\alpha \in \mathbb{N}^n$, are orthogonal in $L^2(d\mu)$. This situation covers a number of basic examples:

- Lebesgue measure on bounded domains in $\mathbb{C}^n$ which are invariant under the torus action
  $$(\theta_1, \ldots, \theta_n)(z_1, \ldots, z_n) \mapsto (e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n)$$
  (i.e. Reinhardt domains).
- Weighted $L^2$ spaces with radially symmetric weights (e.g., generalized Fock spaces).
- Weighted $L^2$ spaces with decoupled radial weights, that is,
  $$d\mu = e^{\sum_j \varphi_j(|z_j|^2)}dV,$$
  where $\varphi_j : \mathbb{R} \to \mathbb{R}$ is a weight function.

Sufficient conditions for the weight in order for the Fock space to be infinite dimensional are known from the work of Shigekawa [12]. Some of these examples have been studied previously; our approach has the advantage of unifying these previous result as well as of being applicable in new situations as well. Our main focus in this paper is the case $n > 1$; indeed, we generalize results of the first author (see [6], [4], [5]) to this setting.

The behaviour of the canonical solution operator $S$ is interesting from many points of view. First, there is a close connection between properties of $S$ and properties of the $\bar{\partial}$-Neumann operator $N$; indeed, $S = \bar{\partial}^* N$. In particular, noncompactness of $S$ prohibits compactness of $N$. As is well known, $S$ behaves quite nicely on spaces of $(0,1)$-forms with holomorphic coefficients, and we shall exploit this connection. On the other hand, for convex domains, a result of Fu and Straube [2] shows that
compactness of $S$ on forms with holomorphic coefficients is also sufficient for compactness on all of $L^2$.

There is also an intriguing connection between the canonical solution operator $S$ and the theory of magnetic Schrödinger operators (see [3] and [7]); this connection has been exploited in the recent paper of the first author and Helffer [8] in order to study compactness of $S$ on general (not rotation-invariant) weighted $L^2$-spaces on $\mathbb{C}^n$.

Let us introduce the notation used in this paper. We denote by

$$A^2(d\mu) = \{z^\alpha : \alpha \in \mathbb{N}^n\},$$

the closure of the monomials in $L^2(d\mu)$, and write

$$m_\alpha = c_\alpha^{-1} = \int |z^\alpha|^2 d\mu.$$

We will give necessary and sufficient conditions in terms of these multimoments of the measure $\mu$ for the canonical solution operator to $\bar{\partial}$, when restricted to $(0,1)$-forms with coefficients in $A^2(d\mu)$ to be bounded, compact, and to belong to the Schatten class $S_p$. This is accomplished by presenting a complete diagonalization of the solution operator by orthonormal bases with corresponding estimates. In the case of radially symmetric measures our results specializes to the results of [10] applied to this specific case; we are also able to characterize membership in $S_p$ for all positive $p$ in some cases (a question left open in [10]).

As usual, for a given function space $F$, $F_{(0,1)}$ denotes the space of $(0,1)$-forms with coefficients in $F$, that is, expressions of the form

$$\sum_{j=0}^n f_j d\bar{z}_j, \quad f_j \in F.$$

The $\bar{\partial}$ operator is the densely defined operator

$$\bar{\partial}f = \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

The canonical solution operator $S$ assigns to each $\omega \in L^2_{(0,1)}(d\mu)$ the solution to the $\bar{\partial}$ equation which is orthogonal to $A^2(d\mu)$; this solution need not exist, but if the $\bar{\partial}$ equation for $\omega$ can be solved, then $S\omega$ is defined, and is given by the unique $f \in L^2(d\mu)$ which satisfies

$$\bar{\partial}f = \omega \text{ in the sense of distributions and } f \perp A^2(d\mu).$$

Our main interest in this paper is the spectral behaviour of the map $S$ restricted to $A^2_{(0,1)}(d\mu)$. We first give a criterion for $S$ to be a bounded operator. We will frequently encounter multiindices $\gamma$ which might have one (but not more than one) entry equal to $-1$: in that case, we define $c_\gamma = 0$. We will denote the set of these multiindices by $\Gamma$. We let $e_j = (0, \cdots, 1, \cdots, 0)$ be the multiindex with a 1 in the $j$th spot and 0 elsewhere.

**Theorem 1.** $S : A^2_{(0,1)}(d\mu) \to L^2(d\mu)$ is bounded if and only if there exists a constant $C$ such that

$$\frac{c_\gamma + e_p}{c_\gamma + 2e_p} \leq \frac{c_\gamma}{c_\gamma + e_p} < C$$

for all multiindices $\gamma \in \Gamma$. 
We have a similar criterion for compactness:

**Theorem 2.** \( S: A^2_{(0,1)}(d\mu) \to L^2(d\mu) \) is compact if and only if

\[
\lim_{\gamma} \left( \frac{c_{\gamma+ep}}{c_{\gamma+2ep}} - \frac{c_{\gamma}}{c_{\gamma+ep}} \right) = 0
\]

for all \( p = 1, \ldots, n \).

In particular, the only if implication of Theorem 2 implies several known noncompactness statements for \( S \), e.g. of Knirsch and Schneider [9], Schneider [11], as well as the noncompactness of \( S \) on the polydisc. The main interest in these noncompactness statements is that if \( S \) fails to be compact, so does the \( \partial \)-Neumann operator \( N \).

The multimoments also lend themselves to characterizing the finer spectral property of being in the Schatten class \( S_p \). Let us recall that an operator \( T: H_1 \to H_2 \) belongs to the Schatten class \( S_p \) if the self-adjoint operator \( T^*T \) has a sequence of eigenvalues belonging to \( \ell^p \).

**Theorem 3.** Let \( p > 0 \). Then \( S: A^2_{(0,1)}(d\mu) \to A^2(d\mu) \) is in the Schatten-\( p \)-class \( S_p \) if and only if

\[
\sum_{\gamma \in \Gamma} \left( \sum_{j} \frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}} - \frac{c_{\gamma}}{c_{\gamma+e_j}} \right)^p < \infty
\]

The condition above is substantially easier to check if \( p = 2 \) (we will show that the sum is actually a telescoping sum then), i.e. for the case of the Hilbert-Schmidt class; we state this as a Theorem:

**Theorem 4.** The canonical solution operator \( S \) is in the Hilbert-Schmidt class if and only if

\[
\lim_{k \to \infty} \sum_{\gamma \in [n], |\gamma| = k} \frac{c_{\gamma}}{c_{\gamma+e_p}} < \infty.
\]

1.1. **Application in the case of decoupled weights.** Let us apply Theorem 1 to the case of decoupled weights, or more generally, of product measures \( d\mu = d\mu_1 \times \cdots \times d\mu_n \), where each \( d\mu_j \) is a (circle-invariant) measure on \( \mathbb{C} \). Note that for such measures, there is definitely no compactness by Theorem 2. If we denote by

\[
c_{\gamma} = \left( \int_{\mathbb{C}} |z|^{2\gamma}d\mu_k \right)^{-1},
\]

we have that

\[
c_{(\gamma_1, \ldots, \gamma_n)} = \prod_{k=1}^{n} c_{\gamma_k}.
\]

We thus obtain the following corollary.

**Corollary 5.** For a product measure \( d\mu = d\mu_1 \times \cdots \times d\mu_n \) as above, the canonical solution operator \( S: A^2_{(0,1)}(d\mu) \to L^2(d\mu) \) is bounded if and only if there exists a constant \( C \) such that

\[
\frac{c_{\gamma+ep}}{c_{\gamma+2ep}} - \frac{c_{\gamma}}{c_{\gamma+ep}} < C
\]
for all \( j \in \mathbb{N} \) and for all \( k = 1, \cdots, n \). Equivalently, \( S \) is bounded if and only if the canonical solution operator \( S_j : A^2(d\mu_j) \to L^2(d\mu_j) \) is bounded for every \( j = 1, \cdots, n \).

1.2. Application in the case of rotation-invariant measures. In the case of a rotation-invariant measure \( \mu \), we write

\[
m_d = \int_{\mathbb{C}^n} |z|^{2d} d\mu;
\]
a computation (see [10, Lemma 2.1]) implies that

\[
c_\gamma = \frac{(n + |\gamma| - 1)!}{(n - 1)! |\gamma|!} \frac{1}{m_{|\gamma|}}.
\]

In order to express the conditions of our Theorems, we compute (setting \( d = |\gamma| + 1 \))

\[
\sum_p \left( \frac{c_{\gamma+ep}}{c_{\gamma+2ep}} - \frac{c_\gamma}{c_{\gamma+ep}} \right) = \left\{ \begin{array}{ll}
\frac{d+2n-1}{d+n} \frac{m_{d+1}}{m_d} - \frac{m_d}{m_{d-1}} & \gamma_p \neq -1 \text{ for all } p \\
\frac{1}{d+n} \frac{m_{d+1}}{m_d} & \text{else}.
\end{array} \right.
\]

Note that the Cauchy-Schwarz inequality implies that for large enough \( d \), the first case in (5) always dominates the second case; using this observation and some trivial inequalities, we get the following Corollaries, which should be compared to the results of the first author in the one-dimensional case [6] and the results of Lovera-Youssfi [10].

**Corollary 6.** Let \( \mu \) be a rotation invariant measure on \( \mathbb{C}^n \). Then the canonical solution operator to \( \bar{\partial} \) is bounded on \( A^2_{(0,1)}(d\mu) \) if and only if

\[
\sup_{d \in \mathbb{N}} \left( \frac{(2n+d-1)m_{d+1}}{(n+d)m_d} - \frac{m_d}{m_{d-1}} \right) < \infty.
\]

**Corollary 7.** Let \( \mu \) be a rotation invariant measure on \( \mathbb{C}^n \). Then the canonical solution operator to \( \bar{\partial} \) is compact on \( A^2_{(0,1)}(d\mu) \) if and only if

\[
\lim_{d \to \infty} \left( \frac{(2n+d-1)m_{d+1}}{(n+d)m_d} - \frac{m_d}{m_{d-1}} \right) = 0.
\]

**Corollary 8.** Let \( \mu \) be a rotation invariant measure on \( \mathbb{C}^n \). Then the canonical solution operator to \( \bar{\partial} \) is a Hilbert-Schmidt operator on \( A^2_{(0,1)}(d\mu) \) if and only if

\[
\lim_{d \to \infty} \left( \frac{n+d-1}{n-1} \right) \frac{m_{d+1}}{m_d} < \infty.
\]

**Corollary 9.** Let \( \mu \) be a rotation invariant measure on \( \mathbb{C}^n \), \( p > 0 \). Then the canonical solution operator to \( \bar{\partial} \) is in the Schatten class \( \mathcal{S}_p \), as an operator from \( A^2_{(0,1)}(d\mu) \) to \( L^2(d\mu) \) if and only if

\[
\sum_{d=1}^{\infty} \left( \frac{n+d-2}{n-1} \right) \left( \frac{(2n+d-1)m_{d+1}}{(n+d)m_d} - \frac{m_d}{m_{d-1}} \right)^{\frac{p}{2}} < \infty.
\]

In particular, Corollary 9 improves Theorem C of [10] in the sense that it also covers the case \( 0 < p < 2 \). We would like to note that our techniques can be adapted to the setting of [10] by considering the canonical solution operator on a Hilbert space \( \mathcal{H} \) of holomorphic functions endowed with a norm which is comparable to the \( L^2 \)-norm on each subspace generated by monomials of a fixed degree \( d \), if in
addition to the requirements in [10] we also assume that the monomials belong to \( \mathcal{H} \); this introduces the additional weights found by [10] in the formulas, as the reader can check. In our setting, the formulas are somewhat “cleaner” by working with \( A^2(d\mu) \) (in particular, Corollary 8 only holds in this setting).

2. Monomial bases and diagonalization

In what follows, we will denote by

\[ u_{\alpha} = \sqrt{c_{\alpha}} z^\alpha \]

the orthonormal basis of monomials for the space \( A^2(d\mu) \), and by \( U_{\alpha,j} = u_{\alpha} d\bar{z}_j \) the corresponding basis of \( A^2_{(0,1)}(d\mu) \). We first note that it is always possible to solve the \( \bar{\partial} \)-equation for the elements of this basis; indeed, \( \bar{\partial} \bar{z}_j u_{\alpha} = U_{\alpha,j} \). The canonical solution operator is also easily determined for forms with monomial coefficients:

**Lemma 10.** The canonical solution \( Sz^\alpha \bar{z}_j \) for monomial forms is given by

\[ Sz^\alpha \bar{z}_j = \bar{z}_j z^\alpha - \frac{c_{\alpha-e_j}}{c_{\alpha}} z^{\alpha-e_j}, \quad \alpha \in \mathbb{N}^n. \]  

**Proof.** We have \( \langle \bar{z}_j z^\alpha, z^\beta \rangle = \langle z^\alpha, z^{\beta+e_j} \rangle \); so this expression is nonzero only if \( \beta = \alpha - e_j \) (in particular, if this implies (10) for multiindices \( \alpha \) with \( \alpha_j = 0 \); recall our convention that \( c_\gamma = 0 \) if one of the entries of \( \gamma \) is negative). Thus \( Sz^\alpha \bar{z}_j = \bar{z}_j z^\alpha + cz^{\alpha-e_j} \), and \( c \) is computed by

\[ 0 = \langle \bar{z}_j z^\alpha + cz^{\alpha-e_j}, z^{\alpha-e_j} \rangle = c^{-1}_\alpha + cc_{\alpha-e_j}, \]

which gives \( c = -c_{\alpha-e_j}/c_\alpha \). \( \square \)

We are going to introduce an orthogonal decomposition

\[ A^2_{(0,1)}(d\mu) = \bigoplus_{\gamma \in \Gamma} E_\gamma \]

of \( A^2_{(0,1)}(d\mu) \) into at most \( n \)-dimensional subspaces \( E_\gamma \) indexed by multiindices \( \gamma \in \Gamma \) (we will describe the index set below), and a corresponding sequence of mutually orthogonal finite-dimensional subspaces \( F_\gamma \subset L^2(d\mu) \) which diagonalizes \( S \) (by this we mean that \( SE_\gamma = F_\gamma \)). To motivate the definition of \( E_\gamma \), note that

\[ \langle Sz^\alpha \bar{z}_k, Sz^\beta \bar{z}_\ell \rangle = \begin{cases} 0 & \beta \neq \alpha + e_\ell - e_k, \\ \frac{1}{c_\alpha} \left( \frac{c_\alpha}{c_{\alpha+e_\ell}} - \frac{c_{\alpha-e_k}}{c_{\alpha+e_\ell-e_k}} \right) & \beta = \alpha + e_\ell - e_k, \end{cases} \]

so that \( \langle Sz^\alpha \bar{z}_k, Sz^\beta \bar{z}_\ell \rangle \neq 0 \) if and only if there exists a multiindex \( \gamma \) such that \( \alpha = \gamma + e_\ell \) and \( \beta = \gamma + e_k \). We thus define

\[ E_\gamma = \text{span} \{ U_{\gamma+e_\ell,j} : 1 \leq j \leq n \} = \text{span} \{ z^{\gamma+e_\ell} \bar{z}_j : 1 \leq j \leq n \}, \]

and likewise \( F_\gamma = SE_\gamma \). Recall that \( \Gamma \) is defined to be the set of all multiindices whose entries are greater or equal to \(-1\) and at most one negative entry. Note that \( E_\gamma \) is \( 1 \)-dimensional if exactly one entry in \( \gamma \) equals \(-1\), and \( n \)-dimensional otherwise. We have already observed that \( F_\gamma \) are mutually orthogonal subspaces of \( L^2(d\mu) \).

Whenever we use multiindices \( \gamma \) and integers \( p \in \{1, \cdots, n\} \) as indices, we use the convention that the \( p \) run over all \( p \) such that \( \gamma + e_p \geq 0 \); that is, for a fixed multiindex \( \gamma \in \Gamma \), either the indices are either all \( p \in \{1, \cdots, n\} \) or there is exactly one \( p \) such that \( \gamma_p = -1 \), in which case the index is exactly this one \( p \).
We next observe that we can find an orthonormal basis of \( E_\gamma \) and an orthonormal basis of \( F_\gamma \) such that in these bases \( S_\gamma = S|_{E_\gamma} : E_\gamma \rightarrow F_\gamma \) acts diagonally. First note that it is enough to do this if \( \dim E_\gamma = n \) (since an operator between one-dimensional spaces is automatically diagonal). Fixing \( \gamma \), the functions \( U_j := U_{\gamma + e_j,j} \) are an orthonormal basis of \( E_\gamma \). The operator \( S_\gamma \) is clearly nonsingular on this space, so the functions \( SU_j = \Psi_j \) constitute a basis of \( F_\gamma \). For a basis \( B \) of vectors \( v^j = (v^1_j, \ldots, v^n_j), \ j = 1, \ldots, n \) of \( \mathbb{C}^n \) we consider the new basis

\[
V_k = \sum_{j=1}^n v^j_k U_j;
\]

since the basis given by the \( U_j \) is orthonormal, the basis given by the \( V_k \) is also orthonormal provided that the vectors \( v_k = (v^1_k, \ldots, v^n_k) \) constitute an orthonormal basis for \( \mathbb{C}^n \) with the standard hermitian product. Let us write

\[
\Phi_k = SV_k = \sum_j v^j_k SU_j.
\]

The inner product \( \langle \Phi_p, \Phi_q \rangle \) is then given by \( \sum_{j,k} v^j_p \overline{v}^k_q \langle SU_j, SU_k \rangle \). We therefore have

\[
\begin{pmatrix}
\langle \Phi_1, \Phi_1 \rangle & \cdots & \langle \Phi_1, \Phi_n \rangle \\
\vdots & \ddots & \vdots \\
\langle \Phi_n, \Phi_1 \rangle & \cdots & \langle \Phi_n, \Phi_n \rangle
\end{pmatrix} = 
\begin{pmatrix}
v^1_1 & \cdots & v^n_1 \\
\vdots & \ddots & \vdots \\
v^1_n & \cdots & v^n_n
\end{pmatrix}
\begin{pmatrix}
\langle \Psi_1, \Psi_1 \rangle & \cdots & \langle \Psi_1, \Psi_n \rangle \\
\vdots & \ddots & \vdots \\
\langle \Psi_n, \Psi_1 \rangle & \cdots & \langle \Psi_n, \Psi_n \rangle
\end{pmatrix} 
\begin{pmatrix}
\overline{v}^1_1 & \cdots & \overline{v}^n_1 \\
\vdots & \ddots & \vdots \\
\overline{v}^1_n & \cdots & \overline{v}^n_n
\end{pmatrix}.
\]

Since the matrix \( (\langle \Psi_j, \Psi_k \rangle)_{j,k} \) is hermitian, we can unitarily diagonalize it; that is, we can choose an orthonormal basis \( B \) of \( \mathbb{C}^n \) such that with this choice of \( B \) the vectors \( \varphi_{\gamma,k} = V_k = \sum_j v^j_k U_{\gamma + e_j,j} \) of \( E_\gamma \) are orthonormal, and their images \( \Phi_k = SV_k \) are orthogonal in \( F_\gamma \). Therefore, \( \Phi_k / \| \Phi_k \| \) is an orthonormal basis of \( F_\gamma \) such that \( S_\gamma : E_\gamma \rightarrow F_\gamma \) is diagonal when expressed in terms of the bases \( \{ V_1, \ldots, V_n \} \subset E_\gamma \) and \( \{ \Phi_1, \ldots, \Phi_n \} \subset F_\gamma \), with entries \( \| \Phi_k \| \).

Furthermore, the \( \| \Phi_k \| \) are exactly the square roots of the eigenvalues of the matrix \( (\langle \Psi_p, \Psi_q \rangle) \) which by (11) is given by

\[
\langle \Psi_p, \Psi_q \rangle = \langle SU_{\gamma + e_p,p}, SU_{\gamma + e_q,q} \rangle
\]

\[
= \sqrt{c_{\gamma + e_p}} \sqrt{c_{\gamma + e_q}} \langle S z^{\gamma + e_p} d\tilde{z}_p, S z^{\gamma + e_q} d\tilde{z}_q \rangle
\]

\[
= \sqrt{c_{\gamma + e_p} c_{\gamma + e_q}} \frac{1}{c_{\gamma + e_p} + c_{\gamma + e_q}} \left( \frac{c_{\gamma + e_p}}{c_{\gamma + e_p} + c_{\gamma + e_q}} - \frac{c_{\gamma}}{c_{\gamma + e_q}} \right)
\]

\[
= \frac{c_{\gamma + e_p} c_{\gamma + e_q} - c_{\gamma} c_{\gamma + e_p} + e_q}{c_{\gamma + e_p} + e_q \sqrt{c_{\gamma + e_p} c_{\gamma + e_q}}}
\]

Summarizing, we have the following Proposition.

**Proposition 11.** With \( \mu \) as above, the canonical solution operator \( S : A^2_{(0,1)}(d\mu) \rightarrow L^2_{(0,1)}(d\mu) \) admits a diagonalization by orthonormal bases. In fact, we have a decomposition \( A^2_{(0,1)} = \bigoplus \gamma E_\gamma \) into mutually orthogonal finite dimensional subspaces \( E_\gamma \), indexed by the multiindices \( \gamma \) with at most one negative entry (equal to \(-1\)), which are of dimension 1 or \( n \), and orthonormal bases \( \varphi_{\gamma,j} \) of \( E_\gamma \), such that \( S \varphi_{\gamma,j} \) is a set.
of mutually orthogonal vectors in $L^2(d\mu)$. For fixed $\gamma$, the norms $\|S\varphi_{\gamma,j}\|$ are the square roots of the eigenvalues of the matrix $C_\gamma = (C_{\gamma,p,q})_{p,q}$ given by

$$C_{\gamma,p,q} = \frac{c_{\gamma+p}c_{\gamma+q} - c_{\gamma}c_{\gamma+p+q}}{c_{\gamma+p+q}\sqrt{c_{\gamma+p}c_{\gamma+q}}}.$$  

In particular, we have that

$$\sum_{j=1}^{n} \|S\varphi_{\gamma,j}\|^2 = \text{trace}(C_{\gamma,p,q})_{p,q} = \sum_{p=1}^{n} \left( \frac{c_{\gamma+p}}{c_{\gamma+2p}} - \frac{c_{\gamma}}{c_{\gamma+e}} \right).$$

3. Boundedness: Proof of Theorem 1

In order to prove Theorem 1, we are using Proposition 11. We have seen that we have an orthonormal basis $\varphi_{\gamma,j}$, $\gamma \in \Gamma$, $j \in \{1, \cdots, n\}$, such that the images $S\varphi_{\gamma,j}$ are mutually orthogonal. Thus, $S$ is bounded if and only if there exists a constant $C$ such that

$$\|S\varphi_{\gamma,j}\|^2 \leq C$$

for all $\gamma \in \Gamma$ and $j \in \{1, \cdots, \dim E_{\gamma}\}$. If $\dim E_{\gamma} = 1$, then $\gamma$ has exactly one entry (say the $j$th one) equal to $-1$; in that case, let us write $\varphi_{\gamma} = U_{\gamma+e}d\bar{z}_j$. We have $S\varphi_{\gamma} = \sqrt{c_{\gamma+e}}z_\gamma d\bar{z}_j$, and so

$$\|S\varphi_{\gamma}\|^2 = \frac{c_{\gamma+e}}{c_{\gamma+2e}}.$$  

On the other hand, if $\dim E_{\gamma} = n$, we argue as follows: Writing $\|S\varphi_{\gamma,j}\|^2 = \lambda_{\gamma,j}^2$ with $\lambda_{\gamma,j} > 0$, from (15) we find that

$$\sum_{j=1}^{n} \lambda_{\gamma,j}^2 = \sum_{j=1}^{n} \left( \frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}} - \frac{c_{\gamma}}{c_{\gamma+e_j}} \right).$$

The last 2 equations complete the proof of Theorem 1.

4. Compactness

In order to prove Theorem 2, we use the following elementary Lemma (which is for example contained in [1]):

Lemma 12. Let $H_1$ and $H_2$ be Hilbert spaces, and assume that $S: H_1 \to H_2$ is a bounded linear operator. Then $S$ is compact if and only if for every $\varepsilon > 0$ there exists a compact operator $T_\varepsilon: H_1 \to H_2$ such that the following inequality holds:

$$\|Sv\|_{H_2}^2 \leq \|T_\varepsilon v\|_{H_2}^2 + \varepsilon \|v\|^2_{H_1}.$$  

Proof of Theorem 2. We first show that (1) implies compactness. We will use the notation which was already used in the proof of Theorem 1; that is, we write $\|S\varphi_{\gamma,j}\|^2 = \lambda_{\gamma,j}^2$. Let $\varepsilon > 0$. There exists a finite set $A_\varepsilon$ of multiindices $\gamma \in \Gamma$ such that for all $\gamma \notin A_\varepsilon$,

$$\sum_{j=1}^{n} \lambda_{\gamma,j}^2 = \sum_{j=1}^{n} \left( \frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}} - \frac{c_{\gamma}}{c_{\gamma+e_j}} \right) < \varepsilon.$$
Hence, if we consider the finite dimensional (and thus, compact) operator $T_\varepsilon$ defined by

$$T_\varepsilon \sum a_{\gamma,j} \varphi_{\gamma,j} = \sum_{\gamma \in A_\varepsilon} a_{\gamma,j} S \varphi_{\gamma,j},$$

for any $v = \sum a_{\gamma,j} \varphi_{\gamma,j} \in A^2_{(0, 1)}(d\mu)$ we obtain

$$\|Sv\|^2 = \|T_\varepsilon v\|^2 + \left\| S \sum_{\gamma \in A_\varepsilon} a_{\gamma,j} \varphi_{\gamma,j} \right\|^2 = \|T_\varepsilon v\|^2 + \sum_{\gamma \notin A_\varepsilon} |a_{\gamma,j}|^2 \|S \varphi_{\gamma,j}\|^2 = \|T_\varepsilon v\|^2 + \sum_{\gamma \notin A_\varepsilon} |a_{\gamma,j}|^2 \lambda^2_{\gamma,j} \leq \|T_\varepsilon v\|^2 + \varepsilon \sum_{\gamma \notin A_\varepsilon} |a_{\gamma,j}|^2 \leq \|T_\varepsilon v\|^2 + \varepsilon \|v\|^2.$$

Hence, (16) holds and we have proved the first implication in Theorem 2.

We now turn to the other direction. Assume that (1) is not satisfied. Then there exists a $K > 0$ and an infinite family $A$ of multiindices $\gamma$ such that for all $\gamma \in A$,

$$\sum_{j=1}^n \lambda^2_{\gamma,j} = \sum_{j=1}^n \left( \frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}} - \frac{c_{\gamma}}{c_{\gamma+e_j}} \right) > nK.$$

In particular, for each $\gamma \in A$, there exists a $j_0(\gamma)$ such that $\lambda^2_{\gamma,j_0(\gamma)} > K$. Thus, we have an infinite orthonormal family $\{\varphi_{\gamma,j_0(\gamma)} : \gamma \in A\}$ of vectors such that their images $S \varphi_{\gamma,j_0(\gamma)}$ are orthogonal and have norm bounded from below by $\sqrt{K}$, which contradicts compactness.

5. Membership in the Schatten classes $S_p$ and in the Hilbert-Schmidt class

We keep the notation introduced in the previous sections. We will also need to introduce the usual grading on the index set $\Gamma$, that is, we write

$$\Gamma_k = \{ \gamma \in \Gamma : |\gamma| = k \}, \quad k \geq -1.$$

In order to study the membership in the Schatten class, we need the following elementary Lemma:

**Lemma 13.** Assume that $p(x)$ and $q(x)$ are continuous, real-valued functions on $\mathbb{R}^N$ which are homogeneous of degree 1 (i.e. $p(tx) = tp(x)$ and $q(tx) = tq(x)$ for $t \in \mathbb{R}$), and $q(x) = 0$ as well as $p(x) = 0$ implies $x = 0$. Then there exists a constant $C$ such that

$$\frac{1}{C} |q(x)| \leq |p(x)| \leq C |q(x)|.$$

**Proof.** Note that the set $B_q = \{x : q(x) = 1\}$ is compact: it’s closed since $q$ is continuous, and since $|q|$ is bounded from below on $S^N$ by some $m > 0$, it is necessarily
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contained in the closed ball of radius $1/m$. Now, the function $|p|$ is bounded on the compact set $B_q$; say, by $1/C$ from below and $C$ from above. Thus for all $x \in \mathbb{R}^N$,

$$
\frac{1}{C} \leq \left| p \left( \frac{x}{q(x)} \right) \right| \leq C,
$$

which proves (18).

Proof of Theorem 3. Note that $S$ is in the Schatten class $\mathcal{S}_p$ if and only if

$$
\sum_{\gamma \in \Gamma, j} \lambda_{\gamma,j}^p < \infty.
$$

We rewrite this sum as

$$
\sum_{\gamma \in \Gamma} \left( \sum_j \lambda_{\gamma,j}^p \right) =: M \in \mathbb{R} \cup \{\infty\}.
$$

Lemma 13 implies that there exists a constant $C$ such that for every $\gamma \in \Gamma$,

$$
\frac{1}{C} \left( \sum_j \lambda_{\gamma,j}^2 \right)^{p/2} \leq \sum_j \lambda_{\gamma,j}^p \leq C \left( \sum_j \lambda_{\gamma,j}^2 \right)^{p/2}.
$$

Hence, $M < \infty$ if and only if

$$
\sum_{\gamma} \left( \sum_j \lambda_{\gamma,j}^2 \right)^{p/2} < \infty,
$$

which after applying (15) becomes the condition (2) claimed in Theorem 3.

Proof of Theorem 4. $S$ is in the Hilbert-Schmidt class if and only if

$$
\sum_{\gamma \in \Gamma, j} \lambda_{\gamma,j}^2 < \infty.
$$

We will prove that

$$
\sum_{\ell=-1}^{k} \sum_{\gamma \in \Gamma_{\ell,j}} \lambda_{\gamma,j}^2 = \sum_{\alpha \in \mathbb{N}^n, |\alpha|=k+1} \frac{c_{\alpha}}{c_{\alpha+e_p}},
$$

which immediately implies Theorem 4. The proof is by induction over $k$. For $k = -1$, the left hand side of (21) is

$$
\sum_{j=1}^{n} \lambda_{-e_j,j}^2 = \sum_{j=1}^{n} \|z_j\|^2 c_0 = \sum_{j=1}^{n} \frac{c_{e_j}}{c_{e_p}},
$$

which is equal to the right hand side. Now assume that the (21) holds for $k = K-1$; we will show that this implies it holds for $k = K$. We write

$$
\sum_{\ell=-1}^{K} \sum_{\gamma \in \Gamma_{\ell,j}} \lambda_{\gamma,j}^2 = \sum_{\alpha \in \mathbb{N}^n, |\alpha|=K-1} \frac{c_{\alpha}}{c_{\alpha+e_p}} + \sum_{\gamma \in \Gamma_{K,j}} \left( \frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}} - \frac{c_{\gamma}}{c_{\gamma+e_j}} \right)
\sum_{\alpha \in \mathbb{N}^n, |\alpha|=K} \frac{c_{\alpha}}{c_{\alpha+e_p}}.
$$
This finishes the proof of Theorem 4.

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