A Distributed Observer for a Time-Invariant Linear System

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Abstract—A time-invariant, linear, distributed observer is described for estimating the state of an m > 0 channel, m-dimensional continuous-time linear system of the form \( \dot{x} = Ax, \ y_i = C_i x, \ i \in \{1, 2, \ldots, m\} \). The state \( x \) is simultaneously estimated by \( m \) agents assuming each agent \( i \) senses \( y_i \) and receives the state \( z_j \) of each of its neighbors’ estimators. Neighbor relations are characterized by a constant directed graph \( N \) whose vertices correspond to agents and whose arcs depict neighbor relations. For the case when the neighbor graph is strongly connected, the overall distributed observer consists of \( m \) linear estimators, one for each agent; \( m - 1 \) of the estimators are of dimension \( n \) and one estimator is of dimension \( n + m - 1 \). Using results from classical decentralized control theory, it is shown that subject to the assumptions that (i) none of the results from classical decentralized control theory, it is shown that (ii) the neighbor graph \( N \) is strongly connected, (iii) the system whose state is to be estimated is jointly observable, and nothing more, it is possible to freely assign the spectrum of the overall distributed observer. For the more general case when \( N \) has \( q > 1 \) strongly connected components, it is explained how to construct a family of \( q \) distributed observers, one for each component, which can estimate \( x \) at a preassigned convergence rate.

I. INTRODUCTION

State estimators such as Kalman filters and observers have had a huge impact on the entire field of estimation and control. This paper deals with observers for time-invariant linear systems. An observer for a process modeled by a continuous-time, time-invariant linear system with state \( x \), measured output \( y = Cx \) and state-dynamics \( \dot{x} = Ax \), is a time-invariant linear system with input \( y \) which is capable of generating an asymptotically correct estimate of \( x \) exponentially fast at a pre-assigned but arbitrarily large convergence rate. As is well known, the only requirement on the system \( y = Cx, \dot{x} = Ax \) for such an estimator to exist is that the matrix pair \((C, A)\) be observable. In this paper we will be interested in the natural generalization of this concept appropriate to a network of \( m \) agents. We now make precise what we mean by this.

A. The Problem

We are interested in a fixed network of \( m > 0 \) autonomous agents labeled \( 1, 2, \ldots, m \) which are able to receive information from their neighbors where by the neighbor of agent \( i \) is meant any other agent in agent \( i \)’s reception range. We write \( \mathcal{N}_i \) for the set of labels of agent \( i \)’s neighbors and we take agent \( i \) to be a neighbor of itself. Neighbor relations between distinct pairs of agents are characterized by a directed graph \( N \) with \( m \) vertices and a set of arcs defined so that there is an arc from vertex \( j \) to vertex \( i \) if whenever agent \( j \) is a distinct neighbor of agent \( i \). Each agent \( i \) can sense a signal \( y_i \in \mathbb{R}^{n_i}, i \in \{1, 2, \ldots, m\}, \) where

\[
\begin{align*}
y_i &= C_ix, \quad i \in m \\
\dot{x} &= Ax
\end{align*}
\]

and \( x \in \mathbb{R}^n \).

Agent \( i \) estimates \( x \) using an \( n_i \) dimensional linear system with state vector \( z_i \) and we assume the information agent \( i \) can receive from neighbor \( j \in \mathcal{N}_i \) is \( z_j(t) \) and \( y_j(t) \). The problem of interest is to construct a suitably defined family of linear systems

\[
\begin{align*}
\dot{z}_i &= \sum_{j \in \mathcal{N}_i} (H_{ij}z_j + K_{ij}y_j), \quad i \in m \\
x_i &= \sum_{j \in \mathcal{N}_i} (M_{ij}z_j + N_{ij}y_j), \quad i \in m
\end{align*}
\]

in such a way so that no matter what the initializations of \( \mathbf{1} \) and \( \mathbf{3} \), each signal \( x_i(t) \) is an asymptotically correct estimate of \( x(t) \) in the sense that each estimation error \( e_i = x_i(t) - x(t) \) converges to zero as \( t \to \infty \) at a preassigned, but arbitrarily fast convergence rate. We call such a family a distributed (state) observer.

We assume throughout that \( C_i \neq 0, \ i \in m \), and that the system defined by \( \mathbf{1} \), \( \mathbf{2} \) is jointly observable; i.e., with \( C = [C_1, C_2, \ldots, C_m] \), the matrix pair \((C, A)\) is observable. Generalizing the results which follow to the case when \((C, A)\) is only detectable is quite straightforward and can be accomplished using well-known ideas. However the commonly made assumption that each pair \((C_i, A)\), \( i \in m \), is observable, or even just detectable, is very restrictive, grossly simplifies the problem and is as unnecessary. It is precisely the exclusion of this assumption which distinguished the problem posed here from almost all of the distributed estimator problems addressed in the literature. The one exception we are aware of is the recent paper \( \mathbf{1} \) which has provided the main motivation for this work.

B. Background

There is a huge literature which seeks to deal with distributed Kalman filters or distributed observers; see, for example \( \mathbf{1} - \mathbf{6} \) and the many references cited therein. Many result are only partial and most problem formulations are different in detail than the problem posed here. The problem we have posed was prompted specifically by the work in \( \mathbf{1} \) which
seeks to devise a time-invariant distributed observer for the discrete-time analog of (1), (2). Two particularly important contributions are made in [1]. First it is recognized that the problem of crafting a 'stable' distributed observer is more or less equivalent to devising a stabilizing decentralized control in [7], [8]. Second, it is demonstrated that under suitable conditions, it is only necessary for the dimension of one of the agent subsystems in [1] to be larger than \( n \), and that the enlarged dimension need not exceed \( n + m - 1 \).

The work reported in this paper clarifies and expand on the results of [1] in several ways. First we outline a construction for systems with strongly connected neighbor graphs which enables one to freely adjust the observer's spectrum. Second, the results obtained here apply whether \( A \) is singular or not; the implication of this generalization is that the construction proposed can be used to craft observers for continuous time processes whereas the construction proposed in [1] cannot unless \( A \) is nonsingular.

II. OBSERVER DESIGN EQUATIONS

We now develop the interrelationships between the matrices appearing in (3) and (4) which must hold for each \( x_i \) to be an asymptotically correct estimate of \( x \). Note first that because (4) must hold even when all estimates are correct, for each \( i \in \mathbb{m} \) it is necessary that the equation

\[
x_i = \sum_{j \in \mathbb{N}_i} (M_{ij}z_j + N_{ij}C_j)x_i, \quad i \in \mathbb{m}
\]

has a solution \( z_i \). Thus if we define

\[
V_i = \begin{bmatrix} \z_1^1 & \z_2^1 & \cdots & \z_n^1 \end{bmatrix} M_{i,n}, \quad i \in \mathbb{m}, \quad u_k \quad \text{is the} \quad k\text{th unit vector in } \mathbb{R}^n,
\]

then

\[
I = \sum_{j \in \mathbb{N}_i} (M_{ij}V_j + N_{ij}C_j), \quad i \in \mathbb{m}
\]

(5)

This and (4) imply that the \( m \) estimation errors satisfy

\[
x_i - x = \sum_{j \in \mathbb{N}_i} M_{ij}\epsilon_j, \quad i \in \mathbb{m}
\]

(6)

where

\[
\epsilon_i = z_i - V_ix, \quad i \in \mathbb{m}
\]

(7)

Moreover, as a direct consequence of (1), (2), and (5),

\[
\dot{\epsilon}_i = \sum_{j \in \mathbb{N}_i} H_{ij}\epsilon_j + \left[ \sum_{j \in \mathbb{N}_i} (H_{ij}V_j + K_{ij}C_j) - V_iA \right]x, \quad i \in \mathbb{m}
\]

Thus if we stipulate that

\[
V_iA = \sum_{j \in \mathbb{N}_i} (H_{ij}V_j + K_{ij}C_j), \quad i \in \mathbb{m}
\]

(8)

then

\[
\dot{\epsilon}_i = \sum_{j \in \mathbb{N}_i} H_{ij}\epsilon_j, \quad i \in \mathbb{m}
\]

(9)

We shall refer to (5) and (8) as the observer design equations. These equations are quite general. They apply to all time-invariant continuous and discrete time state observers whether they are distributed or not.

It is clear from (6) that if the \( V_i, H_{ij}, M_{ij}, N_{ij} \) and \( K_{ij} \) can be chosen so that the observer design equations (5), (8) hold and the system defined by (9) is exponentially stable, then each \( x_i \) will be an asymptotically correct estimate of \( x \). The distributed observer design problem is to develop constructive conditions which ensure that the \( V_i, H_{ij}, M_{ij}, N_{ij} \) and \( K_{ij} \) can be so chosen.

III. CENTRALIZED OBSERVERS

The purpose of this section is to review the well-known concept of a (centralized) observer with the aim summarizing certain less well know ideas which will play a role in the construction of a distributed observer. In the centralized case \( m = 1 \) and a state observer is a \( n_1 \)-dimensional linear system with input \( y = Cx \), state \( z \in \mathbb{R}^{n_1} \) and output \( x_1 \) of the form \( \dot{z} = Hz + Ky, x_1 = Mz + Ny \). In this case the observer design equations are \( I = MV + NC \) and \( VA = HV + KC \) and the observer design problem is to determine matrices \( H, K, M, N \) and \( V \) so that the observer design equations hold and \( H \) is a stability matrix. Observers fall into three broad categories depending on the dimension \( n_1 \): full state observers, minimal state observers, and extended state observers. Each type is briefly reviewed below.

Full-State Observers: Just about the easiest solution to the observer design problem that one can think of, is the one for which \( M \triangleq I, N = 0, V = I_{n_1 \times n} \) and \( H \triangleq A - KC \). Any observer of this type is called a full-state observer because in this case \( z_1 \) is an asymptotic estimate of \( x \). Of course it is necessary that \( K \) be chosen so that \( A - KC \) is a stable matrix. One way to accomplish this is to exploit duality and use spectrum assignment, as is well known. No matter how one goes about defining \( K \), the definitions of \( H, M, N \), and \( V \) given above show that a full-state observer is modeled by equations of the form \( \dot{z}_1 = (A - KC)z_1 + Ky, x_1 = z_1 \).

Reduced State Observers: By a minimal state observer is meant an observer of least dimension which can generate an asymptotic estimate of \( x \). Minimal dimensional observers are obtained by exploiting the fact that \( y = Cx \) is a “partial” measurement of \( x \). Note that the observer design equation \( I = MV + NC \) implies that the number of linearly independent rows of \( V_{n_1 \times n} \) must be at least equal to the dimension of \( \ker C \). Thus the dimension of any observer must be at least equal to dimension \( \ker C \). Techniques for constructing minimal state observers are well known [9], [10].

Extended State Observers: Much less well known are what might be called 'extended state observers.' A observer of this type would be of dimension \( n_1 = n + \bar{n} \) where \( \bar{n} \) is a nonnegative integer chosen by the designer. With \( \bar{n} \) fixed, an extended observer can be obtained by first picking \( M = [I \quad 0]_{n \times (n + \bar{n})}, V' = [I \quad 0]_{n \times (n + \bar{n})} \) and \( N = 0 \) thereby ensuring that observer design equation \( I = MV + NC \) is satisfied. With \( V \) so chosen, \( z_1 \) must be of the corresponding form \( z_1 = \begin{bmatrix} x_1' & \bar{x}_1' \end{bmatrix} \). Accordingly, the partitioned matrices

\[
H = \begin{bmatrix} A + \bar{D}C & \bar{C} \\ BC & A \end{bmatrix}_{(n + \bar{n}) \times (n + \bar{n})}, \quad K = \begin{bmatrix} \bar{D} \\ \bar{B} \end{bmatrix}
\]

satisfy the observer design equation \( VA = HV + KC \) for any
values of the matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ and
\[
\dot{x}_1 = (A + DC)x_1 + \tilde{C}z_1 - Dy
\]
\[
\dot{z}_1 = \tilde{B}Cx_1 + \tilde{A}z_1 - By
\]
Moreover the estimation error $e = x_1 - x$ satisfies
\[
\dot{e} = (A + \tilde{D}C)e + \tilde{C}z_1
\]
\[
\dot{z}_1 = \tilde{B}Ce + \tilde{A}z_1
\]
These equations suggest the following feedback diagram.

![Feedback Diagram](image)

Thus the design of an extended state observer amounts to picking the coefficient matrices $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$ of the lower subsystem in the block diagram to at least stabilize the loop. Of course if $n = 0$, this subsystem is just the constant matrix $\tilde{D}$ and one has again a classical full-state observer of dimension $n$. Exactly what might be gained by picking $n$ greater than zero is not clear in the case of a centralized observer. However, for the decentralized observer we describe next, the flexibility of a dynamic lower loop will become self-evident.

IV. DISTRIBUTED OBSERVERS

The primary goal of distributed observer design is to choose the matrices $V_i, H_{ij}, M_{ij}, N_{ij}$ and $K_{ij}$ so that the observer design equations (5) and (8) hold and the system defined by (9) is exponentially stable. Another goal might be to choose these matrices to reduce the information which needs to be transmitted between neighboring agents. Still another goal might be to choose these matrices so that the dimensions of the individual estimators are as small as possible. In this paper we will consider the case when the only information transmitted between neighboring agents is estimator states $z_i$ and we will make no attempt to construct estimators of least dimension. This means that we will set all $K_{ij} = 0$ except for $K_{ii}$ in (8) and all $N_{ij} = 0$ in (4). The easiest way to satisfy the observer design equations is to set $V_i = I_{m \times n}$ for $i \in m$ and to pick the $M_{ij}$ so that $I = \sum_{j \in N_i} M_{ij}, i \in m$. With the $V_i$ so chosen, observer design equation (8) simplifies to

\[
A - K_{ii}C_i = \sum_{j \in N_i} H_{ij}, \quad i \in m
\]

(10)

where we have adopted the notation $K_i = K_{ii}$ In view of (6) and (9), the observer design problem for this type of an observer is to try to choose the $K_i$ and $H_{ij}$ so that (10) holds and in addition so that $H = [H_{ij}]$ is a stability matrix where $H_{ij} = 0$ if $j \notin N_i$. It is possible to express $H$ in a more explicit form which takes into account the constraints on the $H_{ij}$ imposed by (10). For this let $\tilde{A}$ denote the block diagonal matrix $\tilde{A} = I_{m \times m} \otimes A$ where $\otimes$ is the Kronecker product. Set $B_{ij} = b_i \otimes I_{n \times n}$ $i \in m$ where $b_i$ is the $i$th unit vector in $\mathbb{R}^m$; in addition, let $C_{ij} = C_i B_{ij}^T, i \in m$, and $C_{ij} = c_{ij} \otimes I_{n \times n}, j \in N_i, j \neq i, i \in m$ where $c_{ij}$ is the row in the transpose of the incidence matrix of $N$ corresponding to the arc from $j$ to $i$. It is then possible to express $H$ in the compact form

\[
H = \tilde{A} + \sum_{i \in m} \sum_{j \in N_i} B_{ij} F_{ij} C_{ij}
\]

(11)

where $F_{ij} = -K_{ij}, i \in m$ and $F_{ij} = H_{ij}, j \in N_i, j \neq i, i \in m$. Note that there are no constraints on the $F_{ij}$. In this form it is clear that $H$ is what results when output feedback laws $u_{ij} = F_{ij} y_{ij}$ are applied to the system

\[
\dot{e} = \tilde{A}e + \sum_{i \in m} \sum_{j \in N_i} B_{ij} u_{ij}
\]

(12)

\[
y_{ij} = C_{ij} e, \quad i j \in I
\]

(13)

where $I \subset m \times m$ is the set of double indices $I = \{ij : i \in m, j \in N_i\}$. The problem of constructing a distributed observer of this type thus reduces to trying to choose the $F_{ij}$ to at least stabilize $H$ if such matrices exist. Of course, one also wants control over rate of convergence, so stabilization of $H$ alone is not all that is of interest. Whether the goal is just stabilization of $H$ or control over convergence rate, choosing the $F_{ij}$ to accomplish this will typically not be possible except under special conditions. In fact the problem trying to stabilize $H$ by appropriately choosing the $F_{ij}$ is mathematically the same as the classical decentralized stabilization problem for which there is a substantial literature [7], [8].

A. Strongly Connected Neighbor Graph $N$

One approach is to decentralized stabilization problem is to try to choose the $F_{ij}$ so that for given $p \in m$ and $q \in N_p$, the matrix pairs $(H, B_p)$ and $(C_{pq}, H)$ are controllable and observable respectively. Having accomplished this, stabilization can then be achieved by applying standard centralized feedback techniques such as those in [11] to the resulting controllable observable system. This is the approach taken in this paper. The following proposition provides the key technical result which we need.

Proposition 1: Suppose that the neighbor graph $N$ is strongly connected. There exist gain matrices $F_{ij}, i j \in I$ such that the matrix pairs $(H, B_p)$ and $(C_{pq}, H)$ are controllable and observable respectively for all $p \in m$ and all $q \in N_p$. Moreover, for any such pair, $m$ is the controllability index of $(H, B_p)$.

The proof of this proposition will be given is Section VI.

In the light of Proposition 1 the way to construct a distributed observer is clear. As a first step, choose matrices $M_{ij}, i \in m, j \in N_i$ so that $I = \sum_{j \in N_i} M_{ij}, i \in m$. Next choose the $F_{pq}$ so that the conclusions of the Proposition 1 hold. Having so chosen the $F_{ij}$ or equivalently the $H_{ij}$ and the $K_i$, fix values of $p \in m$ and $q \in N_p$. Next set $\bar{n} = m - 1$ and use a standard construction technique such as that given in [11] to pick matrices $A_{n \times \bar{n}}, B_{n \times \omega}, C_{n \times \bar{n}}$ and $D_{n \times \omega}$ to assign a desirable spectrum to the matrix

\[
\tilde{H} = \begin{bmatrix} H + B_{\bar{n}} DC_{pq} & B_{\bar{n}} C \end{bmatrix} \begin{bmatrix} A \end{bmatrix}^{(n+\bar{n}) \times (n m + \bar{n})}
\]
where \( \omega = s_p \) if \( q = p \) or \( \omega = n \) if \( p \neq q \). This can be done because \((C_{pq}, H)\) is an observable pair and because \((H, B_p)\) is a controllable pair with controllability index \( m \).

The corresponding distributed observer equations are

\[
\begin{aligned}
\dot{z}_i &= \sum_{k \in \mathcal{N}_i} H_{ik} \dot{x}_k + K_i y_i, \quad i \in \mathcal{m}, i \neq p \\
\dot{z}_p &= \sum_{k \in \mathcal{N}_p} H_{pk} \dot{x}_k + K_p y_p + \bar{C} \dot{z} + \bar{D} \dot{v} \\
\dot{\bar{z}} &= \bar{A} \bar{z} + \bar{B} \bar{v} \\
x_i &= \sum_{k \in \mathcal{N}_i} M_{ik} \dot{x}_k, \quad i \in \mathcal{m}
\end{aligned}
\]

where \( v = y_p \) if \( q = p \) or \( v = x_p - x_q \) if \( p \neq q \).

It is possible to verify that the observer design equations hold. For simplicity, assume that \( p = m \) and redefine \( V_m \) to be \([1, 0]^{\times (n+\bar{n})} \). For \( k \in \mathcal{m} \), redefine \( M_{km} \) to be \([-K_m D]^{\times (n+\bar{n})} \) thereby ensuring that observer design equation (5) holds. To ensure that observer design equation (5) holds, first replace \( H_{km} \) with \([H_{km} 0]^{\times (n+\bar{n})} \) for \( k \in \{1, 2, \ldots, m-1\} \). If \( q = m \) replace \( H_{mi}, i \in \mathcal{m}, i \neq m \), and \( H_{mm} \) with the matrices

\[
\begin{bmatrix}
H_{mi} \\
0
\end{bmatrix}, i \in \{1, 2, \ldots, m-1\}
\]

and

\[
\begin{bmatrix}
H_{mm} + \bar{D} C_m & \bar{C} \\
BC_m & \bar{A}
\end{bmatrix}
\]

respectively; in addition, replace \( K_m \) with the matrix \([- \bar{D} \bar{C}_m]^{\times (n+\bar{n})} \). If on the other hand, \( m \neq q \), replace \( H_{mi}, i \in \mathcal{m}, i \neq p, m, H_{mp} \) and \( H_{mm} \) with the matrices

\[
\begin{bmatrix}
H_{mi} \\
0
\end{bmatrix}, i \in \{1, 2, \ldots, m-1\}, i \neq q,
\]

\[
\begin{bmatrix}
H_{mq} - \bar{D} \\
-\bar{B}
\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}
H_{mm} + \bar{D} C \\
\bar{B} & \bar{A}
\end{bmatrix}
\]

respectively; in addition, replace \( K_m \) with matrix \([- \bar{D} \bar{C}_m]\). In either case, observer design equation (5) holds. Thus the error \( \bar{\varepsilon} = [\varepsilon'_1 \varepsilon'_2 \ldots \varepsilon'_{m-1}] \) satisfies \( \dot{\bar{\varepsilon}} = H \bar{\varepsilon} \) where as before, \( x_p - x = \sum_{k \in \mathcal{N}_p} M_{pk} \varepsilon_p, \ p \in \mathcal{m} \).

We are led to the main result of this paper.

**Theorem 1:** Suppose that (1), (2) is a jointly observable system and that \( C_i \neq 0, i \in \mathcal{m} \). If the neighbor graph \( \mathcal{N} \) is strongly connected, then for each symmetric set of \( mn+m-1 \) complex numbers \( \Lambda \) there is a distributed observer (3), (4) for which the spectrum of the \((mn+m-1) \times (mn+m-1)\) matrix \( H \) is \( \Lambda \). Moreover, the observer’s \( m \) outputs \([x_i(t), i \in \mathcal{m}] \), all asymptotically correctly estimate \( x(t) \) in the sense that each estimation error \( e_i = x_i(t) - x(t) \) converges to zero as \( t \to \infty \) as fast \( e^{Ht} \) converges to zero, no matter what the initializations of (2) and (3) are.

**B. Non-Strongly Connected Neighbor Graph \( \mathcal{N} \)**

We now turn briefly to the problem of developing a distributed observer for the case when \( \mathcal{N} \) is not strongly connected. We will assume for simplicity and without loss of generality that \( \mathcal{N} \) is weakly connected. For if it is not, the ideas which follow can be applied to each maximally weakly connected subgraph of \( \mathcal{N} \), since each such subgraph is isolated from the rest. As before, the goal is to devise \( m \) estimators whose estimates converge to \( x \) exponentially fast at arbitrary, pre-assigned rates. We suppose that \( \mathcal{N} \) has \( q \) strongly connected components \( \mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_q \) and for each \( i \in \mathcal{q} \) we write \( \mathcal{N}_i \) for the \( m_i \) channel component subsystem \( \dot{x} = Ax \)

\[
y_j = C_j x, \quad j \in \mathcal{V}_i \]

where \( \mathcal{V}_i \) is the set of labels of the vertices of \( \mathcal{N}_i \) and \( m_i \) is the number of labels in \( \mathcal{V}_i \). We say that there is a directed path (resp. arc) from strongly connected component \( \mathcal{N}_i \) to strongly connected component \( \mathcal{N}_j \) if there is a directed path (resp. arc) from \( \mathcal{N}_i \) to at least one vertex in \( \mathcal{N}_j \) to at least one vertex in \( \mathcal{N}_j \).

### Following (1)

We say that \( \mathcal{N}_j \) is a source component of \( \mathcal{N} \) if \( \mathcal{N}_j \) has no incoming arcs from any other strongly connected component of \( \mathcal{N} \). It is clear that \( \mathcal{N} \) must contain at least one source component. Moreover, since \( \mathcal{N} \) is weakly connected, it is also clear that for any strongly connected component of \( \mathcal{N} \), which is not a source, there must be at least one directed path from at least one source \( \mathcal{N}_j \) to \( \mathcal{N}_i \).

Let \( \mathcal{N}_j \) be a source component and \( \mathcal{S}_j \) be its associated component subsystem. Note that there cannot be any signal flow to any channel in \( \mathcal{S}_j \) from any channel of any other component subsystem. It follows that for there to exist estimators for each channel in \( \mathcal{S}_j \) which are capable of estimating \( x \) at a preassigned convergence rate, it is necessary that \( \mathcal{S}_j \) be a jointly observable subsystem. In view of Theorem 1 joint observability of \( \mathcal{S}_j \) is also sufficient for such a distributed observer to exist because \( \mathcal{N}_j \) is strongly connected. Suppose therefore that for each source component \( \mathcal{N}_j \), the associated component subsystem \( \mathcal{S}_j \) is jointly observable and that a distributed observer has been constructed with preassigned converge rate for each such \( \mathcal{S}_j \). If all strongly connected components of \( \mathcal{N} \) are sources, then these observers solve the distribute observer design problem. Suppose therefore that there is at least one strongly connected component which is not a source. Then there must be at least one strongly connected component \( \mathcal{N}_i \) which is not a source for which there is a source \( \mathcal{N}_j \) with an arc to \( \mathcal{N}_i \). This implies that there must be a channel \( k \in \mathcal{V}_j \) of \( \mathcal{S}_j \) whose estimator state \( z_k \) is available to at least one channel - say channel \( i \) of component subsystem \( \mathcal{S}_i \). But \( e_k = z_k - V_k \dot{x} \). Moreover, for the full-state observers we are considering, \( V_k^T \) is a left inverse of \( V_k \) so \( V_k e_k = C_l \dot{x} + V_k^T e_k \) where \( C_l = I_{\times \times n} \). Therefore \( V_k \bar{e}_k \) can be regarded as a measurement of \( x \) with exponentially decaying additive measurement noise \( V_k^T e_k \).

Thus if the readout equation

\[
y_i = \begin{bmatrix} C_l \\ C_k \end{bmatrix} x + \begin{bmatrix} 0 \\ V_k^T e_k \end{bmatrix}
\]

then the resulting subsystem, denoted by \( \bar{\mathcal{S}}_i \) will be jointly observable with unmeasurable but exponentially decaying
measurement noise. Since $N_i$ is strongly connected, a distributed observer with the same convergent rate as that of $\epsilon_k$, can therefore be constructed for $\Sigma_i$. If $N_i$ is the only strongly connected component of $\mathbb{N}$ which is not a source, then construction is complete. If, on the other hand, $\mathbb{N}$ has other strongly connected components which are not sources, the same ideas as just described, can be applied to each corresponding component subsystem in a sequential manner.

We are led to the following

**Corollary 1**: Suppose that $C_i \neq 0$, $i \in \mathbb{m}$ and that neighbor graph $\mathbb{N}$ has $q$ strongly connected components $\mathbb{N}_i$, $i \in q$. Let $\Sigma_i$ be the component subsystem of $\Sigma$ corresponding to strongly connected component $i$. In order for there to exist distributed observers for each of the component subsystems which are a capable of estimating $x$ at an arbitrary but preassigned convergence rate, it is necessary and sufficient that each of the component subsystems whose graphs are sources, are jointly observable.

V. DECENTRALIZED CONTROL THEORY

The aim of this section is to summarize the concepts and results from [8] and [12] which we will make use of to justify Proposition 1. We do this for a $k$ channel, $n$-dimensional linear system of the form

$$\dot{x} = Ax + \sum_{i \in \mathcal{I}} B_i u_i \quad y_i = C_i x, \quad i \in \mathcal{I}$$

(14)

where $\mathcal{I} = \{1,2,\ldots,k\}$ and $C_i \neq 0$, $i \in \mathcal{I}$. Application of decentralized feedback laws of the form $u_i = F_i y_i$, $i \in \mathcal{I}$ to this system yields the equation $\dot{x} = Hx$ where $H = A + \sum_{i \in \mathcal{I}} B_i F_i C_i$. For given $p \in \mathcal{I}$, explicit necessary and sufficient conditions under which there exist $F_i$ which make $(C_p, H, B_p)$ controllable and observable are given in [12] and [8]. There are two conditions. First, (14) must be jointly controllable and jointly observable. Second, each “complementary subsystem” of (14) must be “complete,” [cf. Theorem 3, [8]]. There are as many complementary subsystems of (14) as there are strictly proper subsets of $\mathcal{I}$. By the complementary subsystem of (14) corresponding to a nonempty proper subset $\mathcal{C} \subset \mathcal{I}$, is meant a subsystem with input matrix $B(\mathcal{C}) = \text{block row}\{B_i : i \in \mathcal{C}\}$, state matrix $A$ and readout matrix $C(\mathcal{C}) = \text{block column}\{C_i : i \in \mathcal{C}\}$ where $\mathcal{C}$ is the complement of $C$ in $\mathcal{I}$. The complementary subsystem determined by $\mathcal{C}$ is uniquely determined up to the orderings of the block rows and block columns of $B(\mathcal{C})$ and $C(\mathcal{C})$ respectively; as will become clear in a moment, the properties which characterize completeness do not depend on these orderings.

For a given complementary subsystem $(C(\mathcal{C}), A, B(\mathcal{C}))$ to be complete, its transfer matrix $C(\mathcal{C}) (sI - A)^{-1} B(\mathcal{C})$ must be nonzero and the matrix pencil

$$\pi(\mathcal{C}) = \begin{bmatrix} A - A & B(\mathcal{C}) \\ C(\mathcal{C}) & 0 \end{bmatrix}$$

(15)

must have rank no less than $n$ for all real and complex $\lambda$ [See [13] or Corollary 4 of [12]]. The requirement that the transfer matrix of each complementary subsystem be nonzero, can be established in terms of the connectivity of the “graph” of (14). By the graph of (14), written $\mathbb{G}$, is meant that $k$-vertex directed graph with labels in $\mathcal{I}$ and arcs defined so that there is an arc from vertex $j$ to $i$ if $C_i (sI - A)^{-1} B_j \neq 0$ for all labels $i,j \in \mathcal{I}$. For the transfer matrices of all complementary subsystems of (14) to be nonzero, it is necessary and sufficient that $\mathbb{G}$ be a strongly connected graph [Lemma 8, [8]].

VI. ANALYSIS

The aim of this section is to prove Proposition 1. To do this it is useful to first establish certain properties of the sub-system of [12], [13] defined by the equations

$$\dot{x} = \dot{\epsilon} + \sum_{i \in \mathbb{m}} \sum_{j \in \mathbb{N}_i} B_i u_{ij}$$

(16)

$$y_{ij} = C_{ij} \epsilon, \quad i,j \in \mathcal{J}$$

(17)

where $\mathcal{J}$ is the complement of the set $\{ii : i \in \mathbb{m}\}$ in $\mathcal{I}$ and for $i \in \mathbb{m}, \mathbb{N}_i$ is the complement of the set $\{i\}$ in $\mathbb{N}_i$. This subsystem is what results when outputs $y_{ii}$, $i \in \mathbb{m}$, are deleted from (13). Our goal here is to show that with suitable scalars $f_{ij}$, the matrix pairs $(H, B_p)$, $p \in \mathbb{m}$, are all controllable with controllability index $m$ where

$$\tilde{H} = \tilde{A} + \sum_{i \in \mathbb{m}} \sum_{j \in \mathbb{N}_i} B_i F_{ij} C_{ij}$$

(18)

and $F_{ij} = f_{ij} L_n$. Note that for any $f_{ij}$ and any $p \in \mathbb{m}$ the submatrix $[B_p \ H B_p \ \cdots \ H^{m-1} B_p]$ has exactly $nm$ columns. Since $nm$ is the dimension of the system (16), (17), $m$ is the smallest possible controllability index which the pair $(H, B_p)$ might attain as the $f_{ij}$ range over all possible values. From this it is obvious that if for each $p \in \mathbb{m}$, there exist $f_{ij}$ for which $(H, B_p)$ has controllability index $m$, then there must be $f_{ij}$ for which $(H, B_p)$ has controllability index $m$ for all $p \in \mathbb{m}$, and moreover the set of $f_{ij}$ for which this is true is the complement of a proper algebraic set in the linear space in which the vector of $f_{ij}$ takes values.

To proceed we will first show that with the $f_{ij}$ chosen properly, the matrix pair $(F, b_m)$ is controllable, where $F$ is the $m \times m$ matrix

$$F = \sum_{i \in \mathbb{m}} \sum_{j \in \mathbb{N}_i} b_i f_{ij} c_{ij}$$

(19)

and for $i \in \mathbb{m}$, $b_i$ is the $i$th unit vector in $\mathbb{R}^m$. Note that $F$ is what results when the feedback laws $v_{ij} = f_{ij} w_{ij}$ are applied to the system

$$\dot{z} = \sum_{i \in \mathbb{m}} \sum_{j \in \mathbb{N}_i} b_i v_{ij}$$

(20)

$$w_{ij} = c_{ij} z, \quad i,j \in \mathcal{J}$$

(21)

where as before, $c_{ij}$ is the row in the transpose of the incidence matrix of $\mathbb{N}$ corresponding to the arc from $j$ to $i$. Note that (20), (21) can be viewed as a $m^*$ channel system where $m^*$ is the number of labels in $\mathcal{J}$. In view of the fact that $\text{span}\{b_1, b_2, \ldots, b_m\} = \mathbb{R}^m$, it is obvious that (20) is jointly

\[\text{The symbols used in this section such as } x, C_i, A, I, \text{ are generic and do not have the same meanings as the same symbols do when used elsewhere in the paper.}\]
controllable. Let \( G \) denote that \( m^* \)-vertex directed graph with vertex labels in \( J \) and arcs defined so that there is an arc from vertex \( ij \) to \( kq \) if \( c_{ij}(sI)^{-1}b_i \neq 0 \) for \( j \in \bar{N}_i \).

**Lemma 1:** If the neighbor graph \( N \) is strongly connected, then \( G \) is strongly connected.

**Proof of Lemma 1** Note that for each \( j \in \bar{N}_i \), \( c_{ij}(sI)^{-1}b_i = -\frac{1}{2} \) and \( c_{ij}(sI)^{-1}b_j = \frac{1}{2} \). From these expressions it follows that \( c_{ij}(sI)^{-1}b_i \neq 0 \) and \( c_{ij}(sI)^{-1}b_j \neq 0 \) for \( i \in \textbf{m}, \ j \in \bar{N}_i \). Therefore for each \( i \in \textbf{m} \), the subgraph \( G_i \) induced by vertices \( ij, j \in \bar{N}_i \) is complete. By the quotient graph of \( G \), written \( Q \), it is meant that directed graph with arcs labeled \( 1, 2, \ldots, m \) and an arc from \( i \) to \( k \) if there is an arc in \( G \) from a vertex in the set \( \{ij : j \in \bar{N}_i\} \) to a vertex in the set \( \{kq : q \in \bar{N}_k\} \). Because each of the subgraphs \( G_i \) is complete, \( G \) will be strongly connected if \( Q \) is strongly connected. But \( Q = N \) so \( Q \) is strongly connected. Therefore \( G \) is strongly connected.

**Lemma 2:** If the neighbor graph \( N \) is strongly connected, then each complementary subsystem of \( [20], [21] \) is complete.

**Proof of Lemma 2** Let \( C \subset J \) be a nonempty subset and let \((C, 0_{m \times m}, B)\) be the coefficient matrices of the complementary subsystem determined by \( C \). Thus \( B = \text{block row}\{b_i : ij \in C\} \), and \( C = \text{block column}\{c_{ij} : ij \in C\} \) where \( C \) is the complement of \( C \) in \( J \). To prove the lemma, it is enough to show that the coefficient matrix triple \((C, 0_{m \times m}, B)\) is complete. To establish completeness the transfer matrix \( C(sI)^{-1}B \) must be nonzero and the matrix pencil \( p(C) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \) (22)

must have rank no less than \( m \) for all real and complex \( \lambda \) \{cf. Corollary 4, [24]\}. In view of Lemma 1 and the assumption that \( N \) is strongly connected, \( G \) is strongly connected. Therefore by Lemma 8 of [18], \( C(sI)^{-1}B \neq 0 \).

To complete the proof it is enough to show that for all complex numbers \( \lambda \), rank \( p(C) \geq m \). In view of the structure of \( p(C) \) in (22), it is clear that for all such \( \lambda \), rank \( p(C) \geq \) rank \( C \) + rank \( B \). To establish completeness, it is therefore sufficient to show that

\[
\text{rank } C + \text{rank } B \geq m \tag{23}
\]

Let \( q \in \textbf{m} \) denote the number of distinct integers \( i \) such that \( ij \in C \). In view of the definition of \( B \), rank \( B = q \). If \( q = m \), rank \( B = m \) and (23) holds. Suppose next that \( q < m \). Let \( C^* \) denote the submatrix of \( C \) which results when all rows \( c_{ij} \) in \( C \) for which \( ik \in C \) for some \( k \), are deleted. Since rank \( C \geq \text{rank } C^* \) and rank \( B = q \), (23) will hold if

\[
\text{rank } C^* \geq (m - q) \tag{24}
\]

Corresponding to the definition of \( C^* \), let \( N^* \) denote the spanning subgraph of \( N \) which results when any arc in \( N \) from \( i \) to \( j \) for which there is a \( k \) such that \( ik \in C \) is removed. There are exactly \( q \) distinct values of \( i \) for which \( ik \in C \) for some \( k \). Moreover, for any such \( i \) the corresponding vertex in \( N^* \) cannot have any outgoing arcs. Since \( N \) is strongly connected, any other vertex \( k \) in \( N^* \) must have at least one outgoing arc not incident on vertex \( k \). This means that the un-oriented version of \( N^* \) must have at least \( q \) connected components. Thus if \( M_{N^*} \) is the incidence matrix of \( N^* \), then as a consequence of Theorem 8.3.1 of [14],

\[
\text{rank } M_{N^*} \geq m - q \tag{25}
\]

But for any \( ij \in J \) such that \( ik \not\in C \) for some \( k \), \( c_{ij} \) is the row in the transpose of the incidence matrix of \( N^* \) corresponding to the arc from \( j \) to \( i \). Therefore, up to a possible re-ordering of rows, \( C^* = M_{N^*} \). From this and (25) it follows that (24) holds. Therefore the lemma is true.

**Lemma 3:** Let \( A_{m \times n}, F_{m \times m}, \) and \( g_{m \times 1} \) be any given real-valued matrices. There is a \( mn \times mn \) nonsingular matrix \( T \) such that

\[
\begin{bmatrix} G & H & \cdots & H^{m-1}G \end{bmatrix} = [g \otimes I_n (Fg) \otimes I_n \cdots (F^{m-1}g) \otimes I_n]T \tag{26}
\]

where \( G = g \otimes I_n \) and \( H = I_m \otimes A + F \otimes I_n \).

**Proof of Lemma 3** Since \((I_m \otimes A)(F \otimes I_n) = (F \otimes I_n)(I_m \otimes A)\), for \( k \geq 1 \)

\[
H^k = (I_m \otimes A + F \otimes I_n)^k = \sum_{i=0}^{k} \binom{k}{i} F^i g \otimes A^{k-i} \tag{27}
\]

where \( \binom{k}{i} \) is the binomial coefficient. Thus

\[
H^k G = (I_m \otimes A + F \otimes I_n)^{k}(g \otimes I_n) = \sum_{i=0}^{k} \binom{k}{i} F^i g \otimes A^{k-i}, \ k \geq 1
\]

Define \( T_1 = I_{mn} \) and for \( k \in \{2, 3, \ldots, m\} \) let \( T_k \) be that \( mn \times mn \) matrix composed of \( m^2 \) \( n \times n \) submatrices \( T_{ij}(k) \) defined so that \( T_{ii}(k) = I_n, i \in \textbf{m}. \ T_{i(i+1), k}(k) = (k^{-1}) A^{k-i-1}. \ i \in \{0, 1, \ldots, k - 1\} \), and all remaining \( T_{ij}(k) = 0 \).

Let \( X(k) = [g \otimes I_n \cdots (F^{k-1}g) \otimes I_n, H^k G \cdots H^{m-1}G] \) for \( k \in \textbf{m} \). Obviously, \( X(1) = \begin{bmatrix} G & H & \cdots & H^{m-1}G \end{bmatrix} \), and \( X(m) = [g \otimes I_n (Fg) \otimes I_n \cdots (F^{m-1}g) \otimes I_n] \).

The definition of \( T_k \) and (27) imply that

\[
X(k) T_k = X(k - 1), \ k \geq 1. \tag{28}
\]

We claim that \( T = T_m T_{m-1} \cdots T_1 \) has the required properties. Note first that each of the \( T_i \) is an upper triangular matrix with ones on the main diagonal. Thus each \( T_i \) is nonsingular which implies that \( T \) is nonsingular. According to (25)

\[
[g \otimes I_n (Fg) \otimes I_n \cdots (F^{m-1}g) \otimes I_n]T = X(m)T_m T_{m-1} \cdots T_1 = X(m-1)T_{m-1} T_{m-2} \cdots T_1 \cdots = X(1)T_1.
\]

Since \( T_1 = I_{mn} \), (25) is true.

**Lemma 4:** Suppose \( N \) is strongly connected. The \( m^* + m \) channel system \((12), (13)\) is jointly controllable and jointly observable.

**Proof of Lemma 4** In view of the definitions of the \( B_i \), it is clear that \( B_1 + B_2 + \cdots B_m = \mathbb{R}^{nm} \) where \( B_i \) is
the column span of $B_i$. It follows at once that (12), (13) is jointly controllable. To establish joint observability it is enough to show that $0$ is the only vector $x \in \mathbb{R}^{m \times n}$ for which $C_{ij} x = 0$, $ij \in \mathcal{I}$ and $\dot{x} = \lambda x$ for some complex number $\lambda$. Suppose $\dot{x} = \lambda x$ in which case $\dot{x}_{ij} = \lambda x_{ij}$ where $x = [x_1 \ x_2 \ \cdots \ x_m]'$ and $x_i \in \mathbb{R}^m$, $i \in \mathcal{M}$. Moreover, if $C_{ij} x = 0$, $ij \in \mathcal{I}$, then $C_{ij} x_i = 0$, $i \in \mathcal{M}$ and $M_j x = 0$ where $M_j$ is the transpose of the incidence matrix of $\mathcal{N}$. Since $\mathcal{N}$ is strongly connected, $M_j x = 0$ implies that $x_i = x_1$, $i \in \mathcal{M}$. Thus $C_{ij} x_1 = 0$, $i \in \mathcal{M}$. But $(C_i A)$ is observable by assumption where $C = [C_1 \ C_2 \ \cdots \ C_m]'$. Therefore $x_1 = 0$. This implies that $x = 0$ and thus that (12), (13) is jointly observable.

**Proof of Proposition** 1: Since $\text{span \{b}_1, b_2, \ldots, b_m\} = \mathbb{R}^m$, the subsystem defined by (20) (21) is jointly controllable. From this and Theorem 1 of [8] it follows that for each $p \in \mathcal{M}$, there exist $f_{ij}$ such that $(F, b_p)$ is a controllable pair where $F$ is as defined (19). Since the set of $f_{ij}$ for which this is true, is the complement of a proper algebraic set in the space in which the $f_{ij}$ takes values, there also exist $f_{ij}$ for which $(F, b_p)$ is a controllable pair for all $p \in \mathcal{M}$. Fix such a set of $f_{ij}$.

By definition $B_i = b_i \otimes I_n$, $i \in \mathcal{M}$, $C_{ij} = c_{ij} \otimes I_n$, $ij \in \mathcal{I}$ and $A = I_m \otimes A$. In view of the definition of $\tilde{H}$ in (13), $\tilde{H} = I_m \otimes A + F \otimes I_n$. From this and Lemma 3 it follows that for each $p \in \mathcal{M}$ there is a non-singular matrix $T_p$ such that $[B_p \ H B_p \ \cdots \ H^{m-1} B_p] = ([b_p \ F b_p \ \cdots \ F^{m-1} b_p] \otimes I_n) T_p$. Since each $T_p$ is non-singular and each $(F, b_p)$ is a controllable pair, $\text{rank \{B_p \ H B_p \ \cdots \ H^{m-1} B_p\}} = mn$

Therefore for each $p \in \mathcal{M}$, $(\tilde{H}, B_p)$ is a controllable pair with controllability index $m$. Note that if we define $F_{ii} = 0$, $i \in \mathcal{M}$, then in view of (11), $\tilde{H} = \tilde{H}$. Therefore, for each $p \in \mathcal{M}$, $(\tilde{H}, B_p)$ is a controllable pair with controllability index $m$. Clearly this must be true generically, for almost all $F_{ij}, ij \in \mathcal{I}$.

In view of Theorem 1 of [8], the complementary subsystems of (12) and (13) must all be complete. But by Lemma 4 (12) and (13) is a jointly controllable, jointly observable system.

From this and Corollary 1 of [8], it follows that there exist $F_{ij}$, $ij \in \mathcal{I}$ such that for all $p \in \mathcal{M}$ and all $q \in N_p$, the matrix pairs $(H, B_p)$ and $(C_{pq}, H)$ controllable and observable respectively. Since this also must be true generically for almost all $F_{ij}$ the proposition is true.

**VIII. Concluding Remarks**

In this paper we have explained how to construct a family of distributed observers for a given neighbor graph $\mathcal{N}$ which are capable of estimating the state of the system (1) (2) at an pre-assigned but arbitrarily fast convergence rate. There are many additional issues to be addressed. For example, how might one construct distributed observers of least dimension which can estimate $x$? Accomplishing this will almost certainly require the transmission to each agent $i$ from each neighbors $j$, the signal $y_j$ which agent $j$ measures. This of course comes at a price, so there is a trade-off to be studied between required observer dimension on the one hand and the amount of information to be transferred across the network on the other. Another issue of importance would be to try to construct a distributed observer for the case when $\mathcal{N}$ changes over time; of course this problem will call for a different type of mathematics since the equations involved will be time-varying systems. Finally it would be useful to try to determine how to construct distributed observers when in place of (2), one has $\dot{x} = Ax + \sum_{i=2}^{m} B_i u_i$ where $u_i$ is an input signal which can be measured by agent $i$. Some of these problems will be addressed in the future.

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