The development of composite technologies contributes to their being widely introduced into the practice of designing modern different-purpose structures. Reliable prediction of the stress-strain state of composite elements is one of the conditions for creating reliable structures with optimal parameters. Analytical theories for determining the stress-strain state of multilayer rods (bars, beams) are significantly inferior in development to those for composite plates and shells, although their core structural elements are most common. The purpose of this paper is to design an analytical model for bending double support multilayer beams under concentrated load based on the previously obtained solution of the elasticity theory for a multi-layer cantilever. The first part of the article includes a statement of the problem, accepted prerequisites and main stages of constructing a model for bending a double-support multi-layer beam with a concentrated load (normal, tangential force and moment) and general-view supports in the extreme cross-sections. When building the model, the double support beam was divided across the loaded cross-section and presented in the form of two separate sections with equivalent loads on the ends. Using the general solution of the elasticity theory for a multilayer cantilever with a load on the ends, the main stress-strain state of the design sections was described without taking into account the local effects of changing the stress state near the concentrated load application points and supports. The obtained relations contain 12 unknown initial parameters. To determine them on the basis of the conditions of joint deformation (static and kinematic) of design sectors, a system of algebraic equations has been constructed. The constructed model allows one to determine the components of the main stress-strain state of double support beams each consisting of an arbitrary number of orthotropic layers, taking into account the amenability of their materials to lateral shear deformations and compression.

**Keywords:** multilayer beam, orthotropic layer, concentrated load, stresses, displacements.

**Introduction**

With the development of technology, composite materials are increasingly being used in various-purpose constructions. Reliable determination of the stress-strain state (SSS) of composite elements of various types is one of the conditions for creating durable and reliable structures. Modern numerical methods and software systems built on their basis have a large arsenal of tools for solving this problem when performing checking calculations. However, at the stage of designing and solving optimization problems it is more convenient to use analytical methods for determining the SSS of composite elements.

A significant number of scientific papers [1–8] are devoted to separate types of composite, in particular, multilayer elements such as plates and shells, in which various analytical, numerical, and analytical methods for determining SSS are constructed.

The deformation of composite rods (bars, beams) has not been studied well enough, although such structural elements are most common. As it is difficult to take into account the inhomogeneous structure of multilayer composite rods when constructing analytical theories of their deformation, approximate methods for solving problems in the theory of elasticity, in particular, the iterative method [9–11], are very common. Despite the introduction of simplifications, the deformation models constructed using this method remains cumbersome and complex for practical application.

Exact solutions to elasticity theory problems have been obtained only for those of bending narrow cantilevers with separate types of loads [12, 13]. Such solutions are quite limited in terms of accounting for various types of supports and loads. However, on their basis it is possible to build relatively simple, but fairly accurate applied solutions to typical problems of bending beams.

The purpose of this paper is to build an analytical bending model for double-support composite multilayer beams under the action of a concentrated load, based on the general solution of the theory of elasticity for a multilayer cantilever with a load on the free end [12].

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Main part

Consider the general case of the plane transverse bending of a straight multi-layer beam under the action of a concentrated load, taking into account that, in the general case, the beam has rigid supports excluding all the displacements of the extreme cross-sections (Fig. 1).

The beam consists of \( m \) parallel layers \( P_k \) \( (k = 1, m) \), made of various materials and rigidly connected on the contact surfaces. The cross-sections (Fig. 1, b) along the beam axis have a uniform structure and dimensions that meet the condition \( b = h = l \).

The beam is related to the Cartesian coordinate system \( xyz \), whose origin \( O \) coincides with the leftmost section rigidity center. The \( Ox \) axis coincides with the longitudinal axis of beam stiffness, and the plane \( xOz \) coincides with the beam symmetry plane and the external load plane.

Beam layers are made of homogeneous orthotropic materials with elastic symmetry planes parallel to the coordinate planes. The elastic material properties of all the beam layers are known. For an arbitrary \( k \)-st layer they are represented by a set of constants $a_k = E_k, E_{yz}, E_{xz}, G_{xy}, G_{xz}, G_{yz}, \nu_{xy}, \nu_{xz}, \nu_{yz}$, where $E_k, E_{yz}, E_{xz}$ are the elastic moduli along the coordinate axes of the system $xyz$; $G_{xy}, G_{xz}, G_{yz}$ are the shear moduli in the planes parallel to the coordinate planes; $\nu_{xy}, \nu_{xz}, \nu_{yz}$ are Poisson’s ratios.

For an entire multi-layer beam, the elastic material properties will be piecewise constant functions $\mu^S_a(z)$, which, by analogy with [12] and [13], will be represented using the Heaviside functions $H(z)$

$$\mu^S_a(z) = \sum_{k=1}^m \left( S^a_k \frac{H(z - z_{bd,k-1}) - H(z - z_{bd,k})}{l_k} \right).$$

Consider SSS of such a beam during the elastic work of the materials of its layers, neglecting the local distortions of the stress and displacement distribution near the concentrated load application points and near the fixing supports. Such SSS, by analogy with the theory of shells [14], will be called the main one.

We present the concentrated load to the beam stiffness axis and represent it in the form of components $F_x$, $F_z$ and $M$, applied to the section stiffness center (Fig. 2). Next, we divide the beam across the loaded section \( (x = l_i) \) into two design segments. In so doing, we replace the support and the discarded part of the beam with the corresponding internal force factors for each section and introduce our own reference system of the section coordinates \( (i = 1, 2 - \text{section number}) \).

If we consider the beam segments separately, then at some distance from the extreme sections their stress state (SS) will be similar to that of the cantilever beam with the load on the end, for which an exact solution was obtained in [12]. Using this solution, we write (for the \( i \)-th design segment) the relations for the SSS components

\[ \text{Fig. 1. Double support multi-layer beam:} \\
\text{a – load and support diagram; b – cross-section diagram} \]

\[ \text{Fig. 2. Beam design diagram} \]
For the entire beam, the expressions for the SSS components and internal force factors can be combined using the Heaviside function

\[ f = f^{(1)} l_{x=0} (1 - H(x - l_i)) + f^{(2)} l_{x=l_i} H(x - l_i), \]

where \( f^{(1)}, f^{(2)} \) is the SSS component distribution in the first and second design segments, respectively.

Relations (2) – (4) contain 3 unknown internal force factors (static initial parameters) and 3 unknown displacements of the points in the initial section (kinematic initial parameters). With the help of these unknowns, one can specify the displacements of 4 points of the beam design segments, making it possible to simulate various restrictions on the displacements of its extreme sections.

For the two design segments of the beam under consideration, in the general case, we will have 12 unknown constants. Such a number of the unknowns is not sufficient both for the exact fulfillment of the compatibility conditions for the displacement of the adjacent design segments of said beam

\[ \begin{align*}
(u^{(1)} |_{x=0} = u^{(2)} |_{x=1} = 0, & \quad w^{(1)} |_{x=0} = w^{(2)} |_{x=1} = 0, \\
(u^{(1)} |_{x=0} = 0, & \quad w^{(1)} |_{x=0} = 0) \end{align*} \]

and absolutely rigid fixation of its ends

\[ \begin{align*}
(u^{(1)} |_{x=0} = 0, & \quad w^{(1)} |_{x=1} = 0) \end{align*} \]
Therefore, the kinematic conditions on the boundary between the beam design segments will be described in a simplified way, combining the displacement of only the extreme cross-section points according to the diagram in Fig. 3, b.

In this case, we will have the following system of kinematic conditions for the beam design segment joint deformation:

\[ u^{(1)} |_{x_0=\ell_i} = u^{(2)} |_{x_1=0}, \]
\[ w^{(1)} |_{x_0=\ell_i} = w^{(2)} |_{x_1=0}, \]
\[ u^{(1)} |_{x_1=\ell_i} = u^{(2)} |_{x_2=0}, \]

or, taking into account the accepted conditions

\[ u^{(1)}_{21} = u^{(2)}_{12}, \]
\[ w^{(1)}_{21} = w^{(2)}_{12}, \]
\[ u^{(1)}_{21} = u^{(2)}_{11}. \] (7)

In addition to the compatibility of displacements of beam segments, we will integrally ensure the compatibility of the SSS components. To do this, on the boundary between the segments, we will require that the internal force factors be equal, taking into account the acting concentrated load

\[ N^{(2)}_{x_1} = N^{(1)}_{x_1} - F_x, \]
\[ \dot{Q}^{(2)}_{x_1} = \dot{Q}^{(1)}_{x_1} + F_z, \]
\[ M^{(2)}_{x_1} = l_1Q^{(1)}_{x_1} + M^{(1)}_{x_1} + M. \] (8)

Using conditions (7), (8) and relations (3) − (5), we construct a system of equations to determine all the unknown constants in solution (2) − (4).

For this, we replace the variables in relations (3) − (5) with the values: \( x_i = \ell_i, \ z = \ell_i, \ell_{i+1} \)

\[ N^{(i)}_{x_1} = N^{(i)}_{x_1} - F_x, \]
\[ \dot{Q}^{(i)}_{x_1} = \dot{Q}^{(i)}_{x_1} + F_z, \]
\[ M^{(i)}_{x_1} = l_1Q^{(i)}_{x_1} + M^{(i)}_{x_1} + M; \]
\[ u^{(i)}_{21} = \frac{l}{bB_0}N^{(i)}_{x_1} + \frac{z_1^2}{2bB_2}Q^{(i)}_{x_1} + \frac{z_1}{bB_2}M^{(i)}_{x_1} + u^{(i)}_{12}, \]
\[ u^{(i)}_{22} = \frac{l}{bB_0}N^{(i)}_{x_1} + \frac{z_1^2}{2bB_2}Q^{(i)}_{x_1} + \frac{z_1}{bB_2}M^{(i)}_{x_1} + u^{(i)}_{11}, \]
\[ w^{(i)}_{21} = -\frac{hl^3}{6bB_2}Q^{(i)}_{x_1} - \frac{l^2}{2bB_2}M^{(i)}_{x_1} + \frac{l}{h}u^{(i)}_{11} - \frac{l}{h}u^{(i)}_{12} + w^{(i)}_{11}, \]

where \( u^{(i)}_{21} = u^{(i)} |_{x_0=\ell_i, \ z=\ell_i}, \ u^{(i)}_{22} = u^{(i)} |_{x_1=\ell_i, \ z=\ell_{i+1}}, \ w^{(i)}_{21} = w^{(i)} |_{x_0=\ell_i, \ z=\ell_i}, \ i = 1, 2. \)

Substituting (7) and (8) into (9), upon transformations, we obtain such a system of relations between the initial (\( N^{(1)}_{x_1}, \dot{Q}^{(1)}_{x_1}, M^{(1)}_{x_1}, u^{(1)}_{12}, w^{(1)}_{12} \)) and final (\( N^{(2)}_{x_1}, \dot{Q}^{(2)}_{x_1}, M^{(2)}_{x_1}, u^{(2)}_{21}, w^{(2)}_{21} \)) parameters of the beam design segments:

\[ N^{(2)}_{x_1} = N^{(1)}_{x_1} - F_x; \]
\[ \dot{Q}^{(2)}_{x_1} = \dot{Q}^{(1)}_{x_1} + F_z; \]
\[ M^{(2)}_{x_1} = l_1Q^{(1)}_{x_1} + M^{(1)}_{x_1} + M; \]
\[ u^{(2)}_{21} = \frac{l}{bB_0}N^{(1)}_{x_1} + \frac{z_1^2}{2bB_2}Q^{(1)}_{x_1} + \frac{z_1}{bB_2}M^{(1)}_{x_1} + u^{(1)}_{12}, \]
\[ u^{(2)}_{22} = \frac{l}{bB_0}N^{(1)}_{x_1} + \frac{z_1^2}{2bB_2}Q^{(1)}_{x_1} + \frac{z_1}{bB_2}M^{(1)}_{x_1} + u^{(1)}_{11}, \]
\[ w^{(2)}_{21} = -\frac{hl^3 + 6Dl_x}{6bB_2}Q^{(1)}_{x_1} - \frac{l^2}{2bB_2}M^{(1)}_{x_1} + \frac{l}{h}u^{(1)}_{11} - \frac{l}{h}u^{(1)}_{12} + w^{(1)}_{11}. \] (10)
Discussion of the results

System (10) consists of 6 equations, which together contain 12 static and kinematic parameters. For each particular type of supports, the values of 6 parameters will be known or can be expressed in terms of other parameters and known values. This allows one to bring the original system (10) to the correct form and determine the remaining static and kinematic parameters.

The first segment initial parameters obtained by the solution of system (10) are the input data for determining the initial parameters of the second segment, using relations (7) − (9)

\[ N_{x_1}^{(2)} = N_{x_1}^{(1)} - F_z, \quad Q_{z_1}^{(2)} = Q_{z_1}^{(1)} + F_z, \quad M_{y_1}^{(2)} = l_1Q_{z_1}^{(1)} + M_{y_1}^{(1)} + M, \]

\[ u_{11}^{(2)} = \frac{l_1}{bB_0}N_{x_1}^{(1)} + \frac{z_1l_1^2}{2bB_2}Q_{z_1}^{(1)} + \frac{z_2l_1}{bB_2}M_{y_1}^{(1)} + u_{11}, \]

\[ u_{12}^{(2)} = \frac{l_1}{bB_0}N_{x_1}^{(1)} + \frac{z_2l_1^2}{2bB_2}Q_{z_1}^{(1)} + \frac{z_2l_1}{bB_2}M_{y_1}^{(1)} + u_{12}, \]

\[ w_{11}^{(2)} = -\frac{hl_1^3}{6hbB_2}Q_{z_1}^{(1)} - \frac{l_1^2}{2bB_2}M_{y_1}^{(1)} + \frac{l_1}{h}u_{11}^{(1)} - \frac{l_1}{h}u_{12}^{(1)} + w_{11}. \]

Substituting the initial parameters known and defined using (10) and (11) into the initial relations (2) – (4) allows one to obtain expressions for the components of the main SSS of all the design segments of the beam. In the future, the solutions for design segments using (6) can be combined into general expressions for an entire multi-layer beam.

Conclusions

Thus, there has been constructed an analytical model of flat bending of double-support multi-layer beams under the action of a concentrated load, which are represented by relations (2) − (4), (10) and (11). The model allows determining the components of the main SSS of multilayer beams each consisting of an arbitrary number of orthotropic layers, taking into account the amenability of their materials to transverse shear deformations and compression.

The obtained relations can be used to solve the problems of deforming multilayer beams with different types of supports on the extreme cross-sections.

The approach used to construct the model can be generalized and extended to the case of multi-span beams with an arbitrary number of concentrated forces and intermediate supports, as well as beams with different rigidity.

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Основний напруженого-деформований стан двохопорних багатошарових балок під дією зосередженого навантаження. Частина 1. Побудова моделі

Ковальчук С. Б., Горик О. В.

Полтавська державна аграрна академія, 36003, Україна, м. Полтава, вул. Сквороди, 1/3

Розвиток технологій композитів сприяє їх широкому впровадженню в практику проектування сучасних конструкцій різного призначення. Достовірне прогнозування напруженого-деформованого стану композитних елементів відносно із умов створення надійних конструкцій з оптимальними параметрами. Аналітичні теорії визначення напруженого-деформованого стану багатошарових стержнів (брусів, балок) значно поступаються у розвитку теорій для композитних плит і оболонок, хоча стрижневі елементи конструкцій є найпоширенишими. Метою цієї роботи є побудова аналітичної моделі викликом двохопорних багатошарових балок під дією зосередженого навантаження на основі отриманого раніше розв’язку теорії пружності для багатошарової консолі. У першій частині статті наведено постановку задачі, прийняття передумов і основні етапи побудови моделі згідно багатошарової двохопорної балики із зосередженим навантаженням (нормальна, дотична сила і момент) і закріпленням загального вигляду в крайніх перетинах. Під час побудови моделі двохопорна балка була розділена по навантаженному перерізу і подана у вигляді двох окремих ділянок з еквівалентними навантаженнями на торцях. З використанням загального розв’язку теорії пружності для багатошарової консолі з навантаженням на торцях був описані основний напруженого-деформований стан розрахункових ділянок без урахування локальних ефектів зміни напруженого стану поблизу точок прикладення зосередженого навантаження і закріплення. Отримані співвідношення містять 12 невідомих початкових параметрів, для визначення яких з умов стислого деформування (статичних і кинематичних) розрахункових ділянок побудована система алгебраїчних рівнянь. Побудована модель дозволяє визначити компоненти основного напруженого-деформованого стану двохопорних балок, що складаються з довільної кількості ортотропних шарів, з урахуванням податливості їх матеріалів деформації поперечного зсуву і обімнення.

Ключові слова: багатошарова балка, ортотропний шар, зосереджене навантаження, напруження, переміщення.

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