Regularity of dissipative differential operators

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Abstract

S.G. Krein’s conjecture concerning Birkhoff-regularity of dissipative differential operators has been proved in the even order case. As a byproduct an existence of the limit of characteristic matrix as \( \lambda \to \infty \) in the lower half-plane has been established. Up to multiplication by a nonvanishing matrix \( D \) this limit coincides with the ratio

\[
\Theta(b^0, b^1)^{-1} \cdot \Theta(b^1, b^0) \cdot D
\]

of the matrices of regularity determinants.
1 Introduction

1.1 History

Spectral theory of boundary value problems in a finite interval is one of the most elaborated parts of the theory of linear nonselfadjoint operators. Just at the beginning of the century G.D.Birkhoff discovered a class of the so-called regular boundary conditions (further on, Birkhoff-regular) with a lot of remarkable spectral properties: estimate of the Green’s function, asymptotics of eigenvalues and eigenfunctions, point convergence similar to that of trigonometric Fourier series and so on [16].

However, further investigations revealed absence of concrete classes of boundary conditions with the same list of properties if Birkhoff-regularity is violated. In this case the resolvent admits a polynomial and even an exponential growth. More precisely, for separated boundary conditions the Green’s function grows exponentially at least in one of the triangles $0 < x < t < 1$ or $0 < t < x < 1$, and in 1991 A.P.Khromov described a class of third order irregular problems [7] where it has an exponential growth in both triangles.

From the other hand the general theory of linear operators provides such good subsets as self-adjoint, normal or dissipative ones. Of course, the list may be increased but we stop here. Quite naturally there raises a question of an intersection between them and the concrete class of differential operators generated by Birkhoff-regular boundary value
problems. Recall first that regularity of self-adjoint boundary conditions has been proved by S.Salaff in 1968 for even order operators [18], see also work of H.Fiedler [5]. In the odd case this fact was established in 1977 [9]. Later these results were generalized in [10].

At the beginning of nineties we supposed validity of the following conjecture.

**Conjecture 1.1** Dissipative differential operators are Birkhoff-regular.

Recently V.A.II’in has kindly informed us that it belongs to S.G.Krein, to one of his talks during the Voronez mathematical schools in seventies-eighties (personal communication).

Our main result is as follows.

**Theorem 1.1** Even order dissipative differential operators are Birkhoff-regular.

In the odd order case our demonstration provides nonvanishing of only one regularity determinant instead of the two ones involved in the definition (2.6) of Birkhoff-regularity.

### 1.2 Abstract approach

Before passing to the proof it should be pointed out that the spectral theory of abstract dissipative operators is deeply explored by means of the functional model theory [15, 17]. For instance, S.R.Treil’ has found a strong criterion of unconditional basicity of eigenfunctions [20, theorem 14.1]. Observe that it requires their uniform minimality, which seems to be unattainable unless \( L \in (R) \). Hence, the abstract approach turns out to be ineffective for boundary value problems.

Instead, an almost orthogonality property of the Birkhoff’s fundamental system of solutions serves below as a main tool. It has been proved in [11] for ordinary differential
expressions, in [12] for the quasidifferential ones with a summable coefficient by the \((n-1)\)-st derivative and asserts that

\[
\left\| \sum_{k=0}^{n-1} c_k \cdot y_k(x, \varrho) \right\|_{L^2(0,1)}^2 \asymp \sum_{k=0}^{n-1} |c_k|^2 \cdot \|y_k(x, \varrho)\|_{L^2(0,1)}^2
\]  

(1.1)

for any constants \(c_k\) which may vary with \(\varrho\). Other applications of (1.1) may be found in [14].

1.3 Notations

Throughout, matrices with block entries are written in boldface, for instance, \(\Delta = [\Delta_{jk}]\) stands for the matrix of a determinant \(\Delta\). Components of matrices and vectors are enumerated beginning from zero, for instance, '0-th' row, 'k-th' column etc. We use few abbreviations:

- \(a := b\) or \(d =: c\) means that \(a\) equals \(b\) or \(c\) equals \(d\) by definition;

- \([a] := a + O(1/\varrho)\) stands for the Birkhoff’s symbol;

- \(A \asymp B\) means a double-sided estimate \(C_1 \cdot |A| \leq |B| \leq C_2 \cdot |A|\) with some absolute constants \(C_{1,2}\), which don’t depend on the variables \(A\) and \(B\). In this case we shall say that \(A\) is equivalent to \(B\);

- \(\mathbb{C}_\pm\) - upper/lower half-plane, \(\mathbb{R}\) stands for the real axis.

Different constants are denoted \(C, C_1, c\) and so on. They may vary even during a single computation.
1.4 General remarks

The paper is organized as follows. In §2 we recall the definition of Birkhoff-regularity and build a suitable form of the Green’s function. In §3 boundedness of a characteristic matrix is established. The proof is completed in §4. Then at the end of the paper we give some remarks concerning applications and possible generalizations.

2 Preliminaries

2.1 Birkhoff-regular problems

Consider a differential operator $L$ in $L^2(0,1)$ defined by a two-point boundary value problem ($(D = -id/dx)$):

$$l(y) \equiv D^n y + \sum_{k=0}^{n-2} p_k(x) D^k y = \lambda y, \quad 0 \leq x \leq 1, \quad p_k \in L(0,1) \quad (2.1)$$

and $n$ linearly independent normalized boundary conditions [16, p.65–66]:

$$U_j(y) \equiv b_0^j D^j y(0) + b_1^j D^j y(1) + \ldots = 0, \quad j = 0, \ldots, n - 1. \quad (2.2)$$

Here the ellipsis takes place of lower order terms at 0 and at 1. Further $b_0^j, b_1^j$ are column vectors of length $r_j$, where

$$0 \leq r_j \leq 2, \quad \sum_{k=0}^{n-1} r_j = n, \quad \text{rank } (b_0^j b_1^j) = r_j.$$

Such form of normalized boundary conditions was introduced by S.Salaff [18, p.356–357].

It is evident that $r_j = 0$ implies the absence of order $j$ conditions. In the case $r_j = 2$ we
merely put

\[(b^0_j b^1_j) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\]

**Definition 2.1** Let \( q = \text{Entier}(n/2) \), \( \varepsilon_j := \exp(2\pi ij/n) \), \( k = 0, \ldots, n - 1; \)

\[b^i = (b^i_j)_{j=0}^{n-1} = \begin{pmatrix} b_0^i \\ \vdots \\ b_{n-1}^i \end{pmatrix}, \quad B^i_k = (b^i_j \cdot \varepsilon_k^j)_{j=0}^{n-1}, \quad i = 0, 1; \quad (2.3)\]

\[\Theta_p(b^0, b^1) = |B^0_k, k = 0, \ldots, p - 1| |B^1_k, k = p, \ldots, n - 1|, \quad (2.4)\]

\[\Theta(b^0, b^1) := \Theta_q(b^0, b^1) \quad (2.5)\]

The vertical line \( | \) separates columns with superscripts 0 and 1. The determinant \( \Theta_p(b^1, b^0) \) is defined like (2.5) by interchanging the superscripts 0 and 1. We shall call boundary conditions (2.2) and the corresponding operator \( L \) Birkhoff-regular and write \( L \in (R) \) if

\[
\begin{cases}
\Theta(b^0, b^1) \neq 0, & n = 2q, \\
\Theta(b^0, b^1) \neq 0 \text{ and } \Theta(b^1, b^0) \neq 0, & n = 2q + 1.
\end{cases} \tag{2.6}
\]

**Definition 2.2** Birkhoff-regular boundary value problem is strongly regular if either \( n \) is odd or if it is even, \( n = 2q \), and the polynomial

\[F(s) := |B^0_0 + s \cdot B^0_0, B^0_k, k = 1, \ldots, q - 1 | B^1_q + s \cdot B^0_q, B^1_k, k = q + 1, \ldots, n - 1|\]

has two simple roots.

This form of Birkhoff-regularity was invented by S.Salaff [18, p.361] who has done a first serious investigation of the nature of the regularity determinants. The reader must keep
in mind that our determinants differ slightly from those in [IS p.361], namely $\Theta(b^1, b^0)$ coincides with $\Theta(b^0, b^1)$ from [IS] but this does not affect further considerations.

2.2 Particular solution

Set $\varrho = \lambda^{1/n}$, $|\varrho| = |\lambda|^{1/n}$ and

$$\arg \varrho = \arg \lambda / n, \quad 0 \leq \arg \lambda < 2\pi.$$ (2.7)

Then $\varrho \in S_0 \cup S_1$, where

$$S_k = \{ \varrho \mid \pi k / n \leq \arg \varrho < \pi (k + 1) / n \}.$$ 

Note that in every sector $S_k$ there exists a fundamental system of solutions $\{y_j(x, \varrho)\}_{j=0}^{n-1}$ with an exponential asymptotics:

$$D^k y_j(x, \varrho) = (\varrho \varepsilon_j)^k \cdot \exp(i \varrho \varepsilon_j x)[1], \quad j, k = 0, \ldots, n - 1.$$ (2.8)

In the sequel it will be convenient to introduce a number $p$ such that solutions $y_j(x, \varrho)$ decay as $j < p$ and exponentially grow otherwise. Clearly, $p$ depends upon the sector’s choice and its values are presented in the table

| $\varrho \setminus n$ | 2q   | 2q+1 |
|----------------------|------|------|
| $\in S_0$            | $q - 1$ | $q$ |
| $\in S_1$            | $q - 1$ | $q - 1$ |

Table 1: Values of $p$

Introduce a wronskinian

$$W(x, \varrho) = |D^k y_j|_{j,k=0}^{n-1}$$
and let $W_j(x, \varrho)$ be the algebraic complement of the element $D^{n-1}y_j$. Set $\tilde{y}_j(x, \varrho) := W_j/W$.

It is easy to check that

$$\tilde{y}_j(x, \varrho) = \frac{1}{n(\varrho \varepsilon_j)^{n-1}} \exp(-i \varrho \varepsilon_j x)[1].$$ \hfill (2.9)

Introducing the kernel

$$g_0(x, \xi, \varrho) = \begin{cases} \\
\sum_{k=0}^{p-1} \varepsilon_k^{-(n-1)} y_k(x, \varrho) \tilde{y}_k(\xi, \varrho), & x > \xi \\
- \sum_{k=p}^{n-1} \varepsilon_k^{-(n-1)} y_k(x, \varrho) \tilde{y}_k(\xi, \varrho), & x > \xi 
\end{cases}$$

we get a particular solution $g_0(f)$ of the equation $l(y) = \lambda y + f$,

$$g_0(f) := \int_0^1 g_0(x, \xi, \varrho) f(\xi) d\xi.$$ \hfill (2.10)

### 2.3 New fundamental system of solutions

In the sequel it will be more convenient to use another fundamental system of solutions

$\{z_k\}_{k=0}^{n-1}$, where

$$z_k(x, \varrho) := \begin{cases} \\
y_k(x, \varrho), & k = 0, \ldots, p - 1, \\
y_k(x, \varrho)/\exp(i \varrho \varepsilon_k), & k = p, \ldots, n - 1.
\end{cases}$$ \hfill (2.11)

This choice of a fundamental system of solutions is natural due to the fact that

$$z_k = O(1), k = 0, \ldots, n - 1; 0 \leq x, \xi \leq 1$$ \hfill (2.12)

for $\varrho \in S_0, S_1$. 

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2.4 Green’s function representation

Recall that the Green’s function admits a representation as a ratio of two determinants

\[
G(x, \xi, \varrho) = \frac{(-1)^n \Delta(x, \xi, \varrho)}{n \varrho^{n-1} \Delta(\varrho)}. \tag{2.13}
\]

Its denominator

\[
\Delta(\varrho) = \left| \varrho^{-j} U_j(z_k) \right|_{j,k=0}^{n-1} \tag{2.14}
\]

is usually referred to as the characteristic determinant. The nominator has the form:

\[
\Delta(x, \xi, \varrho) := i \cdot \begin{vmatrix} z^T & g(x, \xi, \varrho) \\ \Delta(\varrho) & V(\xi, \varrho) \end{vmatrix}, \tag{2.15}
\]

\(z^T\) stands for the row \((z_0(x, \varrho), \ldots, z_0(x, \varrho))\),

\[
g(x, \xi, \varrho) := g_0(x, \xi, \varrho) \cdot (n \varrho^{n-1})/i \tag{2.16}
\]

and

\[
V(\xi, \varrho) = (\varrho^{-j} U_{jx}(g(x, \xi, \varrho)))_{j=0}^{n-1}. \tag{2.17}
\]

Here the subscript \(x\) means that the boundary form \(U_j\) acts on the kernel \(g(x, \xi, \varrho)\) over the first argument.

Changing a little bit notation from [4, p.1185] we use an abbreviation:

\[
[[a]] := a + O(1/\varrho) + O \left( \exp \left( - \min_{m=0, \ldots, n-1} |\text{Im}(\varrho \varepsilon_m)| \right) \right),
\]

where \(\varrho\) lies in a fixed sector \(S_k, \ 0 \leq k \leq n-1\) off some sufficiently large circle \(|\varrho| \leq R_0\).

Obviously \(\Delta = [\Delta_{jk}]_{j,k=0}^{n-1}\) is a block-matrix with \(r_j \times 1\) entries

\[
\Delta_{jk} := \varepsilon^j_k \cdot (b^0_j \cdot [z_k(0)] + b^1_j \cdot [z_k(1)]). \tag{2.18}
\]
But
\[ z_k(1) = [0], \quad z_k(0) = [1], \quad k < p; \quad z_k(1) = [1], \quad z_k(0) = [0], \quad k \geq p. \]

Therefore
\[ \Delta(\varrho) = \left[ \Theta_p \left( b^0, b^1 \right) \right]. \quad (2.18) \]

Next, setting
\[
\begin{align*}
\epsilon_k \cdot n(\varrho \xi_k)^{n-1} \cdot \tilde{y}_k(\xi, \varrho) &= e^{i \varrho \xi_k (1-\xi)} \cdot [1], \\
\tilde{y}_k(\xi, \varrho) \cdot n(\varrho \xi_k)^{n-1} &= e^{i \varrho \xi_k (-\xi)} \cdot [1], \\
\end{align*}
\]
we come to the following relation
\[
\begin{align*}
V(\xi, \varrho) &= i \sum_{k<p} [B^1_k] \cdot \xi_k u_k(\xi, \varrho) \\
&\quad - i \sum_{k \geq p} [B^0_k] \cdot \xi_k u_k(\xi, \varrho). \\
\end{align*}
\]

At last, expanding \( \Delta(x, \xi, \varrho) \) along the 0-th row we obtain an important representation:
\[
G(x, \xi, \varrho) = g_0(x, \xi, \varrho) + \frac{-2\pi i}{n \varrho^{n-1}} \sum_{t,k=0}^{n-1} a_{tk}(\varrho) \cdot z_k(x, \varrho) \cdot u_t(\xi, \varrho), \quad (2.21)
\]
where
\[
a_{tk} := \begin{cases} 
\frac{\xi}{2\pi} \cdot \frac{\Delta \left. \left[ B^1_k \right] \right/ \Delta}{}, & t < p \\
-\frac{\xi}{2\pi} \cdot \frac{\Delta \left. \left[ B^0_k \right] \right/ \Delta}{}, & t \geq p.
\end{cases} \quad (2.22)
\]

Here
\[
\left| \Delta \left. \left[ B^1_k \right] \right/ \Delta \right|
\]
stands for the determinant \( \Delta(\varrho) \) with the \( k \)-th column replaced by a vector \( d \). Earlier
the coefficients \( a_{tk} \) were introduced in [13]. Therefore we inserted the factor \(-2\pi i\) in
in order to preserve notations from that paper as well as checked a misprint there in the sign.

It would be quite natural to call the matrix \( A = A(\varrho) = [a_{tk}]_{t,k=0}^{n-1} \) a characteristic matrix of the boundary value problem (2.1)-(2.2) because it differs from the analogous object from [16, p.276] or [1, p.309] by another choice of the fundamental system of solutions. Namely, in these books the latter is taken analytic in \( \lambda \) with a unit Cauchy data matrix at the end-point 0 of the main interval \([0,1]\).

3 Boundedness of the characteristic matrix

Further we shall need the following estimate for the resolvent in the lower half-plane

\[
\| R_\lambda \| \leq |\text{Im}\lambda|^{-1}, \text{Im}\lambda < 0
\]  

(3.1)

which stems from the dissipativity of operator \( L \). A ray \( \{\arg \lambda = \alpha \}, \pi < \alpha < 2\pi \) in \( \mathbb{C}_- \) corresponds to the ray \( \{\arg \varrho = \alpha/n\} \) lying in the sector \( S_1 \) in the \( \varrho \)-plane. Then the estimate (3.1) turns into

\[
\| R_\lambda \| \leq C \cdot |\varrho|^{-n}, \quad \arg \varrho = \alpha/n.
\]  

(3.2)

**Lemma 3.1** The integral operator \( g_0 \) in \( L^2(0,1) \) (see (2.11)) admits an estimate

\[
\| g_0 \| \leq C \cdot r^{-n}, \quad \arg \varrho = \alpha/n, \quad r := |\varrho| \geq R_0,
\]  

(3.3)

where \( R_0 \) is some fixed positive number.
of Removing square brackets from the asymptotic expressions for the functions $y_k(x, \varrho)$ and $\tilde{y}_j(\xi, \varrho)$ we obtain a kernel $G_0(x, \xi, \varrho)$ which is naturally extended to $\mathbb{R}$. Obviously
the latter coincides with the Green’s function of the self-adjoint operator $D^n$ in $L^2(\mathbb{R})$. Then

$$g_0(x, \xi, \varrho) = G_0(x, \xi, \varrho) + O \left( \frac{1}{\varrho^n} \right).$$

Clearly operator $G_0$ in $L^2(\mathbb{R})$ with the kernel $G_0(x, \xi, \varrho)$ obeys estimate (3.3) which completes the proof. ■

**Lemma 3.2** Let $P = P(\varrho)$ be a finite dimensional operator in $L^2(0, 1)$ with the kernel

$$P(x, \xi, \varrho) := \sum_{t, k=0}^{n-1} a_{tk}(\varrho) z_k(x, \varrho) u_t(\xi, \varrho).$$

Then (3.3) and representation (2.21) yield an estimate:

$$\|P\|_{L^2(0, 1) \to L^2(0, 1)} \leq C/r, \ r \geq R_0, \ \arg \varrho = \alpha. \quad (3.4)$$

In addition, a double-sided estimate holds

$$\|P\|_{L^2(0, 1) \to L^2(0, 1)} \asymp \left( \sum_{t, k=0}^{n-1} |a_{tk}(\varrho)|^2 \cdot \frac{1}{r^2} \right)^{1/2} \quad (3.5)$$

uniformly with respect to $\varrho$, $r = |\varrho| \geq R_0, \ \arg \varrho = \alpha/n$.

of First observe that (3.4) stems readily from (3.2),(3.3) and (2.21). Further, let

$f \in L^2(0, 1)$. Then $P f = \sum_{k=0}^{n-1} d_k z_k(x, \varrho)$ with

$$d_k := \sum_{t=0}^{n-1} a_{tk}(\varrho) \int_0^1 f(\xi) u_t(\xi, \varrho) d\xi.$$
Invoking *almost orthogonality* of the f.s.s. (2.11), we arrive at the relation:

\[ \| Pf \|_{L^2(0,1)} \approx \sum_{k=0}^{n-1} |d_k|^2 \| z_k \|_{L^2(0,1)}^2. \]

Next a direct calculation demonstrates that

\[ \| z_k \| \approx \frac{1}{r}, \quad r \geq R_0, \quad \arg \varrho = \alpha. \]

Hence it suffices to prove that

\[ \sup_{\| f \| \leq 1} \sum_{k=0}^{n-1} |d_k|^2 \approx \frac{1}{r} \sum_{t,j=0}^{n-1} |a_{tj}(\varrho)|^2, \quad \forall k = 0, \ldots, n-1. \]  \[ (3.6) \]

Fix \( \varrho \) and suppose that the sum \( (k = 0, \ldots, n-1) \)

\[ \sum_{t=0}^{n-1} |a_{tk}(\varrho)|^2 \]

attains its maximum for \( k = k_0(\varrho) \). Then it suffices to check validity of (3.6) when \( k = k_0 \).

The left-hand side of (3.5) with \( k = k_0 \) equals

\[ \left\| \sum_{t=0}^{n-1} a_{t+k_0} u_t(\xi, \varrho) \right\|_{L^2(0,1)} \]

which is equivalent to \( \sum_{t=0}^{n-1} |a_{tk_0}|^2 \| u_t \|^2 \). Here we applied the *almost orthogonality* of the system (2.19). But the latter has just an exponential asymptotics and this is the unique ingredient needed for this property (see [12]). At last, a direct calculation shows that

\[ \| u_t \|^2 \approx \frac{1}{r}, \quad r \geq R_0, \quad \arg \varrho = \alpha/n \]

which completes the proof. \( \blacksquare \)

**Corollary 3.1** The characteristic matrix is bounded

\[ \sum_{t,k=0}^{n-1} |a_{tk}(\varrho)|^2 = O(1), \quad \arg \varrho = \alpha/n, \quad |\varrho| \geq R_0. \]  \[ (3.7) \]

Indeed, one should compare (3.4) and (3.5).
4 End of the proof

Let \( A_t \) be the \( t \)-th column of the matrix \( A = (A_0, \ldots, A_{n-1}) \). Then \( A_t \) satisfies an equation

\[
\Delta \cdot A_t = \pm \frac{\varepsilon_t}{2\pi} \begin{bmatrix} B_t^{\#} \end{bmatrix}, \quad 0 \leq t \leq n - 1; \quad \# = \begin{cases} 1, & t < p \\ 0, & t \geq p. \end{cases}
\]  

(4.1)

Here and in what follows the sign ‘+’ corresponds to the case \( t < p \), ‘−’ — to the case \( t \geq p \).

**Lemma 4.1** For every \( t \in \{0, \ldots, n - 1\} \) there exists a vector \( \eta_t \in \mathbb{C}^n \) such that

\[
\Theta (b^0, b^1) : \eta_t = \pm \frac{\varepsilon_t}{2\pi} \begin{bmatrix} B_t^{\#} \end{bmatrix}.
\]

(4.2)

offFix \( t \in \{0, \ldots, n - 1\} \). Using the compactness of the set of vectors

\[
A_t(\varrho), \quad \arg \varrho = \alpha/n, \quad |\varrho| \geq R_0
\]

we obtain an existence of a vector \( \eta_t \) such that

\[
\eta_t = \lim_{m \to \infty} A_t(\varrho_m)
\]

for some sequence \( \varrho_m \to \infty \). In the meantime the formula

\[
\lim_{m \to \infty} \Delta(\varrho_m) = \Theta_p (b^0, b^1)
\]

(4.3)

stems immediately from (2.18). Combine the two formulas above and we are done.

**Lemma 4.2** Denote \( R(A) \) the image of the matrix \( A \). Then

\[
R \left( \Theta (b^0, b^1) \right) \supset \text{span} \left( B_0^0, \ldots, B_{p-1}^0, B_p^1, \ldots, B_{n-1}^1 \right). 
\]

(4.4)

\[
R \left( \Theta (b^0, b^1) \right) \supset \text{span} \left( B_0^1, \ldots, B_{p-1}^1, B_p^0, \ldots, B_{n-1}^0 \right).
\]

(4.5)
ofFirstly, apply the matrix $\Theta_p(b^0, b^1)$ to the standard basis in $\mathbb{C}^n$ and get (4.4). Second-
ly, (4.5) stems from (4.2) when $t$ runs over $0, \ldots, n-1$. ■

**Lemma 4.3** The following final relation is valid:

$$R \left( \Theta_p \left( b^0, b^1 \right) \right) = \mathbb{C}^n.$$  

(4.6)

ofInclusions (4.4)-(4.5) yield that

$$R \left( \Theta_p \left( b^0, b^1 \right) \right) \supset \text{span} \left( B^0_j, j = 0, \ldots, n-1; B^1_j, j = 0, \ldots, n-1 \right).$$

Set

$$\Psi = (\mathbf{e}_k)_{jk}^{n-1}, \quad Q = \left( Q^0, Q^1 \right),$$

where $Q^i := (B^i_t, t = 0, \ldots, n-1)$ is an $n \times n$ matrix. Then a $(j, k)$-entry of the product $Q^i \cdot \Psi^*$ will be a $r_j \times 1$ vector

$$b^i_j \sum_{t=0}^{n-1} \mathbf{e}_t \cdot \mathbf{e}_t^* = b^i_j \cdot n \cdot \delta_{jk}, \quad i = 0, 1; \quad j, k = 0, \ldots, n-1,$$

whence

$$\frac{1}{n} Q \cdot \Psi^* = B := \begin{pmatrix}
    b^0_0 & b^1_0 \\
    \vdots & \ddots & \vdots \\
    b^0_{n-1} & b^1_{n-1}
\end{pmatrix}.$$  

(4.7)

Clearly, $\text{rank } B = \sum_{j=0}^{n-1} r_j = n$. Therefore $R \left( Q \Psi^* \right) = R(B) = \mathbb{C}^n$. ■

The proof of theorem 1.1 is completed. Indeed, we proved that $Q$ is a full range matrix since $\Psi$ is invertible. Hence $Q$ is itself invertible. ■
5 Concluding remarks

5.1 Dissipative boundary conditions

Consider an operator $L_{ess}$, generated by the simplest expression $D^n$ and the boundary conditions

$$b^0_j D^j y(0) + b^1_j D^j y(1) = 0, \quad j = 0, \ldots, n - 1.$$

It is natural to call $L_{ess}$ an essential part of the differential operator $L$. Obviously, $L \in (R) \iff L_{ess} \in (R)$.

In [18] it was shown that $L$ is self-adjoint if and only if so is $L_{ess}$. Therefore we are able to speak about self-adjointness of the boundary conditions themselves. Of course, it would be desirable to establish the same facts in the case of dissipative differential operators and therefore to derive regularity of dissipative boundary conditions themselves from the theorem 1.1.

However, the present description of dissipative boundary conditions is rather abstract [21] and requires further clarification.

5.2 A limit of the characteristic matrix

As a byproduct of previous considerations we also established a statement which seems to be of independent interest.

Theorem 5.1 Let $T^\pm_\varepsilon$ be sectors in $\mathbb{C}_\pm$ ($0 < \varepsilon < \pi$):

$$T^\pm_\varepsilon := \{\lambda \mid \varepsilon \leq \arg \lambda \leq \pi - \varepsilon\},$$
\[ T_\varepsilon^- := \{ \lambda \mid \pi + \varepsilon \leq \text{arg} \lambda \leq 2\pi - \varepsilon \} \]

and \( L \) be an \( n \)-th order differential operator (not necessarily dissipative), defined by boundary value problem (2.1)-(2.2).

1. Given a sequence \( \{ \lambda_m \}_1^\infty \subset T_\varepsilon^- \), such that estimate (3.2) is fulfilled for \( \lambda = \lambda_m \), we have

\[
\exists \lim_{m \to \infty} \Delta(\lambda_m) =: \Delta_\infty = \Theta_p \left( b^0, b^1 \right),
\]

\[
\exists \lim_{m \to \infty} A(\lambda_m) =: A_\infty = A_\infty(L) = \Theta_p \left( b^0, b^1 \right)^{-1} \Theta_p \left( b^1, b^0 \right) \cdot D.
\]

Here \( D := \text{diag} (\varepsilon_0, \ldots, \varepsilon_{p-1}, -\varepsilon_p, \ldots, -\varepsilon_{n-1}) / (2\pi) \); all the matrices don’t vanish.

2. The same relations are valid if such a sequence exists in the sector \( T_\varepsilon^+ \).

Remark 5.1 In the odd order case, \( n = 2q + 1 \), take \( \varrho \in S_1 \). Then theorem 5.1 yields

\[
\Theta \left( b^0, b^1 \right) \neq 0
\]

because \( p = q - 1 \) (see table 7). However, we are not able to assert the same with respect to another regularity determinant \( \Theta \left( b^1, b^0 \right) \) because there is no information concerning the resolvent’s estimate in the upper half-plane \( \mathbb{C}_+ \), equivalently, \( \varrho \in S_0 \).

Note that if (3.2) were valid than theorem 5.1 would imply

\[
0 \neq \Theta_p \left( b^0, b^1 \right) \equiv \Theta_q \left( b^0, b^1 \right),
\]

but the determinant from the right coincides with \( \Theta \left( b^1, b^0 \right) \) up to some nonzero multiplicative constant [18].
Add here that (5.1) implies half-regularity of the boundary conditions (2.2). This notion has been recently introduced in [14] and it was shown there that it yields certain information on the eigenfunctions’ and eigenvalues’ behaviour.

5.3 N.Dunford-J.Schwartz’ spectrality

Remark 5.2 In the case, when \( L \) is spectral (in N.Dunford-J.Schwartz’ sense [3]) instead of dissipativity, we recently established existence of appropriate sequences

\[
\{\lambda_m^\pm\}_{1}^{\infty} \subset T_{\epsilon}^\pm,
\]

such that estimate (3.2) is fulfilled for \( \lambda = \lambda_m^\pm \). Then theorem 5.1 implies \( L \in (R) \). However, the construction of these sequences is rather cumbersome and we shall consider it elsewhere.

Note that the converse was proved in works of G.M.Kesel’man, V.P.Mikhailov and A.A.Shkalikov (see [6, 8, 19] or [16, p.98–99]). More precisely, Birkhoff-regularity yields spectrality if the boundary conditions are strongly regular or unconditional basicity of eigenfunctions with brackets otherwise.

Thus remark 5.2 finishes classification of spectral two point boundary value problems up to a gap between Birkhoff- and strong regularity.

5.4 Abstract Birkhoff-regularity

Perhaps the most striking fact established in the proof of theorem 1.1 is existence of the limit of the characteristic matrix \( A(\varphi) \) as \( \lambda \to \infty \) in any sector lying strictly in the lower
half-plane. It is helpful to note here that this limit is an invariant of Birkhoff-regular problems. Namely, the following statement has been recently obtained by our student T.Mizrova. The proof is purely algebraic and will be published elsewhere.

**Theorem 5.2 (T.Mizrova,1997)** Given two differential operators $L_1, L_2 \in (R)$, assume that the limits of their characteristic matrices coincide: $A_\infty(L_1) = A_\infty(L_2)$. Then their essential parts also coincide: $L_{1\text{ess}} \equiv L_{2\text{ess}}$.

Further, we think that there is a close connection between $A(\varrho)$ and the B.S.Nagy-C.Foiaş’ characteristic function of the dissipative operator $L$ [15]. Moreover, we suppose validity of the following

**Conjecture 5.1** Consider an abstract completely continuous operator $G$. Let $\Lambda := \{\lambda_j\}$ be the set of its eigenvalues and let

$$K_\lambda := \{|z - \lambda| \leq \delta \cdot (1 + |Im\lambda|)\}$$

be a circle centered at the point $\lambda$, where $\delta > 0$ is some fixed number. Then existence of an invertible limit of the characteristic function of $G$ in the sectors $T^\pm_\varepsilon$ off the circles $K_{\lambda_j}, \overline{K_{\lambda_j}}, \lambda_j \in \Lambda$ is a correct reformulation of the Birkhoff-regularity condition and yields unconditional basicity of eigenvectors of $G$, perhaps, with brackets.

Recall that in this and more general situations characteristic function was defined by M.S.Livšic, see, for instance, [2].
5.5 Quasidifferential expressions

All results of the paper may be applied as well for a general quasidifferential expression

\[ l(y) \equiv y^{[n]} \]

of the form

\[ y^{[0]} = y, \quad y^{[j]} = Dy^{[j-1]} + \sum_{k=0}^{j-1} p_{j-1,k}(x)y^{[k]}, \quad j = 1, \ldots, n \]  

(5.2)

with summable coefficients \( p_{j-1,k} \) because the main tool—property of almost orthogonality— is valid for such expressions [12].
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