q-log-convexity from linear transformations and polynomials with only real zeros *

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Abstract

In this paper, we mainly study the stability of iterated polynomials and linear transformations preserving the strong $q$-log-convexity of polynomials.

Let $[T_{n,k}]_{n,k \geq 0}$ be an array of nonnegative numbers. We give some criteria for the linear transformation

$$y_n(q) = \sum_{k=0}^{n} T_{n,k} x_k(q)$$

preserving the strong $q$-log-convexity (resp. log-convexity). As applications, we derive that some linear transformations (for instance, the Stirling transformations of two kinds, the Jacobi-Stirling transformations of two kinds, the Legendre-Stirling transformations of two kinds, the central factorial transformations, and so on) preserve the strong $q$-log-convexity (resp. log-convexity) in a unified manner. In particular, we confirm a conjecture of Lin and Zeng, and extend some results of Chen et al., and Zhu for strong $q$-log-convexity of polynomials, and some results of Liu and Wang for transformations preserving the log-convexity.

The stability property of iterated polynomials implies the $q$-log-convexity. By applying the method of interlacing of zeros, we also present two criteria for the stability of the iterated Sturm sequences and $q$-log-convexity of polynomials. As consequences, we get the stabilities of iterated Eulerian polynomials of type $A$ and $B$, and their $q$-analogs. In addition, we also prove that the generating functions of alternating runs of type $A$ and $B$, the longest alternating subsequence and up-down runs of permutations form a $q$-log-convex sequence, respectively.

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1 Introduction

The main objective of this paper is twofold: one is to study the linear transformations preserving the strong $q$-log-convexity of the sequences of polynomials and the other is to study one strong property of $q$-log-convexity called the stability of polynomials.

The Jacobi-Stirling numbers $JS_k^2(z)$ of the second kind, which were introduced in [17], are the coefficients of the integral composite powers of the Jacobi differential operator

$$\ell_{\alpha,\beta}[y](t) = \frac{1}{(1-t)^{\alpha}(1+t)^{\beta}} \left(-(1-t)^{\alpha+1}(1+t)^{\beta+1}y/(t)\right)' ,$$

with fixed real parameters $\alpha, \beta \geq -1$. They also satisfy the following recurrence relation:

$$\left\{ \begin{array}{ll}
JS_0^0(z) = 1, & JS_n^k(z) = 0, \text{ if } k \not\in \{1, \ldots, n\}, \\
JS_n^k(z) = JS_n^{k-1}(z) + k(k+z)JS_{n-1}^k(z), & n, k \geq 1,
\end{array} \right.$$

where $z = \alpha + \beta + 1$. Similarly, the Jacobi-Stirling numbers $Jc_n^k(z)$ of the first kind are defined by

$$\left\{ \begin{array}{ll}
Jc_0^0(z) = 1, & Jc_n^k(z) = 0, \text{ if } k \not\in \{1, \ldots, n\}, \\
Jc_n^k(z) = Jc_{n-1}^{k-1}(z) + (n-1)(n-1+z)Jc_{n-1}^k(z), & n, k \geq 1,
\end{array} \right.$$

where $z = \alpha + \beta + 1$. Actually, these numbers are a generalization of the Legendre-Stirling numbers of two kinds: it suffices to choose $\alpha = \beta = 0$. Recently, the Jacobi-Stirling numbers and Legendre-Stirling numbers have generated a significant amount of interest from some researchers in combinatorics, see Andrews et al. [1, 2], Everitt et al. [17], Mongelli [33], Lin and Zeng [24] and Zhu [46] for instance. In [24], Lin and Zeng proposed the next conjecture.

**Conjecture 1.1.** [24] *The Jacobi-Stirling transformations of two kinds*

$$y_n = \sum_{k=0}^{n} JS_n^k(z)x_k \quad \text{and} \quad w_n = \sum_{k=0}^{n} Jc_n^k(z)x_k$$

*preserve the log-convexity for $z = 0, 1$.*

Recall some notation and definitions. Let $\{a_n\}_{n \geq 0}$ be a sequence of nonnegative real numbers. It is called *log-convex* (resp. *log-concave*) if for all $k \geq 1$, $a_{k-1}a_{k+1} \geq a_k^2$ (resp. $a_{k-1}a_{k+1} \leq a_k^2$), which is equivalent to that $a_{n-1}a_{m+1} \geq a_na_m$ (resp. $a_{n-1}a_{m+1} \leq a_na_m$) for all $1 \leq n \leq m$. The log-concave and log-convex sequences arise often in combinatorics, algebra, geometry, analysis, probability and statistics and have been extensively investigated, see Stanley [39], Brenti [8], Liu and Wang [27] and Zhu [45] for details.

For two polynomials with real coefficients $f(q)$ and $g(q)$, denote $f(q) \geq_q g(q)$ if the difference $f(q) - g(q)$ has only nonnegative coefficients. For a polynomial sequence $\{f_n(q)\}_{n \geq 0}$, it is called *$q$-log-concave* first suggested by Stanley, if

$$f_n(q)^2 - f_{n+1}(q)f_{n-1}(q) \geq_q 0$$

1.1 The Jacobi-Stirling transformations of two kinds

$$y_n = \sum_{k=0}^{n} JS_n^k(z)x_k \quad \text{and} \quad w_n = \sum_{k=0}^{n} Jc_n^k(z)x_k$$

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$$f_n(q)^2 - f_{n+1}(q)f_{n-1}(q) \geq_q 0$$

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for \( n \geq 1 \) and is called \textit{strongly \( q \)-log-concave} introduced by Sagan, if
\[
f_n(q)f_m(q) - f_{n+1}(q)f_{m-1}(q) \geq_0 0
\]
for any \( n \geq m \geq 1 \). Obviously, the strong \( q \)-log-concavity of polynomials implies the \( q \)-log-concavity. However, the converse does not hold. The \( q \)-log-concavity of polynomials have been extensively studied, see Butler \[11\], Leroux \[23\], and Sagan \[35\] for instance.

For the polynomial sequence \( \{f_n(q)\}_{n \geq 0} \), it is called \textit{\( q \)-log-convex} introduced by Liu and Wang, if
\[
f_{n+1}(q)f_{n-1}(q) - f_n(q)^2 \geq_0 0
\]
for \( n \geq 1 \) and is called \textit{strongly \( q \)-log-convex} defined by Chen et al., if
\[
f_{n+1}(q)f_{m-1}(q) - f_n(q)f_m(q) \geq_0 0
\]
for any \( n \geq m \geq 1 \). Clearly, strong \( q \)-log-convexity of polynomials implies the \( q \)-log-convexity. However, the converse does not hold.

The operator theory often is used to study the log-concavity or log-convexity. For example, the log-convexity and log-concavity are preserved under the binomial convolution respectively, see Davenport and Pólya \[16\] and Wang and Yeh \[43\]. Brändén \[5\] studied some linear transformations preserving the Pólya frequency property of sequences. Brenti \[7\] obtained some transformations preserving the log-concavity. Liu and Wang \[27\] also studied linear transformations preserving the log-convexity. In \[47\] \[48\], we strengthened partial results for the linear transformations preserving the log-convexity to the strong \( q \)-log-convexity. However, there are fewer results about the linear transformations preserving the strong \( q \)-log-convexity. One of the aims of this paper is to continue studying linear transformations preserving the strong \( q \)-log-convexity.

Given an array \( [T_{n,k}]_{n,k \geq 0} \) of nonnegative real numbers and a sequence of polynomials \( \{x_n(q)\}_{n \geq 0} \), define the polynomials
\[
y_n(q) = \sum_{k \geq 0} T_{n,k}x_k(q)
\]
for \( n \geq 0 \). If we take \( x_k(q) = q^k \), then it was demonstrated that the corresponding sequence \( \{y_n(q)\}_{n \geq 0} \) has the \( q \)-log-convexity or strong \( q \)-log-convexity for many famous triangles \( [T_{n,k}]_{n,k \geq 0} \), including the Stirling triangle of the second kind, the Jacobi-Stirling triangle of the second kind, the Legendre-Stirling triangle of the second kind, the Eulerian triangles of type \( A \) and \( B \), the Narayana triangles of type \( A \) and \( B \), and so on, see \[12\] \[13\] \[14\] \[27\] \[28\] \[45\] \[46\] for instance. Thus it is natural to consider the strong \( q \)-log-convexity of the linear transformation \( y_n(q) \) by that of \( x_n(q) \). On the other hand, note that a log-convex sequence is one special case of the strongly \( q \)-log-convex sequence since the real number sequence \( \{a_n\}_{n \geq 0} \) is log-convex if and only if \( a_{n-1}a_{m+1} \geq a_na_m \) for all
$1 \leq n \leq m$. So it is easy to see that the linear transformation preserving the strong $q$-log-convexity also preserves the log-convexity.

Let $[T_{n,k}]_{n,k \geq 0}$ be an array of nonnegative numbers satisfying the recurrence relation:

$$T_{n,k} = (a_0n + a_2k + a_3)T_{n-1,k} + (b_0n + b_2k + b_3)T_{n-1,k-1}$$

(1.1)

with $T_{n,k} = 0$ unless $0 \leq k \leq n$ and $T_{0,0} = 1$. Chen et al. [14] proved the strong $q$-log-convexity of the row generating functions $T_n(q) = \sum_{k=0}^{n} T_{n,k} q^k$ when all $a_0, a_2, b_0$ and $b_2$ are nonnegative real numbers.

In [46], we considered another array of nonnegative numbers $[T_{n,k}]_{n,k \geq 0}$ satisfying the recurrence relation:

$$T_{n,k} = (a_1k^2 + a_2k + a_3)T_{n-1,k} + (b_1k^2 + b_2k + b_3)T_{n-1,k-1}$$

(1.2)

where $T_{0,0} = 1$ and $T_{0,k} = 0$. We also proved the strong $q$-log-convexity of its row generating functions when all $a_0, a_1, b_0$ and $b_1$ are nonnegative real numbers.

In this paper, we consider a more generalized array of nonnegative numbers $[T_{n,k}]_{n,k \geq 0}$ satisfying the recurrence relation:

$$T_{n,k} = [r(n) + f(k)]T_{n-1,k} + [s(n) + g(k)]T_{n-1,k-1} + [t(n) + h(k)]T_{n-1,k-2}$$

(1.3)

with $T_{n,k} = 0$ unless $0 \leq k \leq n$ and $T_{0,0} = 1$. For $n \geq 0$, let $T_n(q) = \sum_{k=0}^{n} T_{n,k} q^k$ be the row generating functions.

Recall that a matrix $M = (m_{ij})_{i,j \geq 0}$ of nonnegative numbers is said to be $r$-order totally positive (TP$_r$ for short) if its all minors of order at most $r$ are nonnegative. Total positivity of matrices has been extensively studied and is very useful, see Karlin [21] for more details. By total positivity of matrices, we have the next extensive result for linear transformations preserving the strong $q$-log-convexity.

**Theorem 1.2.** Let $[T_{n,k}]_{n,k \geq 0}$ be the nonnegative array satisfying the recurrence (1.3). Assume that the matrix $[T_{n,k}]_{n,k \geq 0}$ is TP$_2$ and all $r(n), s(n), t(n), f(n), g(n)$ and $h(n)$ are nonnegative and increasing in $n$ for $n \geq 0$. If $\{x_n(q)\}_{n \geq 0}$ is strongly $q$-log-convex, then so is $y_n(q) = \sum_{k=0}^{\infty} T_{n,k} x_k(q)$. In particular, if $\{x_n\}_{n \geq 0}$ is log-convex, then so is $y_n = \sum_{k=0}^{\infty} T_{n,k} x_k$.

For $a_2 \geq 0$ and $b_2 \geq 0$, Chen et al. [14] proved that $[T_{n,k}]_{n,k \geq 0}$ satisfying the recurrence (1.1) is TP$_2$. For $a_2 \geq 0$ and $b_2 \geq 0$, we [46] also showed that $[T_{n,k}]_{n,k \geq 0}$ satisfying the recurrence (1.2) is TP. Thus the following result is immediate from Theorem 1.2.

**Theorem 1.3.** Let $[T_{n,k}]_{n,k \geq 0}$ be an array of nonnegative numbers satisfying the recurrence relation:

$$T_{n,k} = (a_0n + a_1k^2 + a_2k + a_3)T_{n-1,k} + (b_0n + b_1k^2 + b_2k + b_3)T_{n-1,k-1}$$

(1.4)
with $T_{n,k} = 0$ unless $0 \leq k \leq n$ and $T_{0,0} = 1$. Assume that $a_2 \geq 0, b_2 \geq 0$ and $a_1 = b_1 = 0$, or $a_2 \geq 0, b_2 \geq 0$ and $a_0 = b_0 = 0$. If $\{x_n(q)\}_{n \geq 0}$ is strongly $q$-log-convex, then so is $y_n(q) = \sum_{k=0}^{n} T_{n,k} x_k(q)$. In particular, if $\{x_n\}_{n \geq 0}$ is log-convex, then so is $y_n = \sum_{k=0}^{n} T_{n,k} x_k$.

**Remark 1.4.** For $a_2 \geq 0, b_2 \geq 0$ and $a_1 = b_1 = 0$ in Theorem 1.3, recently Liu and Li [25] independently proved the result.

Twenty five years ago Gian-Carlo Rota said “The one contribution of mine that I hope will be remembered has consisted in just pointing out that all sorts of problems of combinatorics can be viewed as problems of the locations of zeros of certain polynomials...”, see the end of the introduction of [4]. In fact, polynomials with only real zeros play an important role in attacking log-concavity of sequences. One classical result is that if the polynomial $\sum_{i=0}^{n} a_i x^i$ with nonnegative coefficients has only real zeros, then the sequence $a_0, a_1, \ldots, a_n$ is log-concave. In addition, many log-concave sequences arising in combinatorics have the stronger property, see Liu and Wang [26] and Wang and Yeh [42] for instance. Using the algebraical method, Liu and Wang [27] found that many polynomials with real zeros have the $q$-log-convexity. Thus at the end of their paper, they proposed the problem to research this relation between the $q$-log-convexity and real zeros. This is our another motivation.

One of the classical problems of the theory of equations is to find relations between the zeros and coefficients of a polynomial. A real polynomial is (Hurwitz) stable if all of its zeros lie in the open left half of the complex plane. A well-known necessary condition for a real polynomial with positive leading coefficient to be stable is that all its coefficients are positive. Polynomial stability problems of various types arise in a number of problems in mathematics and engineering. We refer to [32, Chapter 9] for deep surveys on the stability theory. Clearly, the stability property of iterated polynomials implies the $q$-log-convexity. Thus it is natural to consider the following stronger problem.

**Problem 1.5.** Given a sequence $\{f_n(q)\}_{n \geq 0}$ of polynomials with only real zeros, under which conditions can we obtain that $f_{n+1}(q)f_{n-1}(q) - f_n^2(q)$ is stable for $n \geq 1$ ?

We say a polynomial is a generalized stable polynomial if all of its zeros excluding 0 lie in the open left half of the complex plane. The following result gives an answer to Problem 1.5.

**Theorem 1.6.** Let $\{f_n(q)\}_{n \geq 0}$ be a sequence of polynomials with nonnegative coefficients, where $\deg(f_n(q)) = \deg(f_{n-1}(q)) + 1$ for $n \geq 1$. Assume that the sequence $\{f_n(q)\}_{n \geq 0}$ satisfies the recurrence relation

$$f_n(q) = [a_1 n + a_2 + (b_1 n + b_2) q + (c_1 n + c_2) q^2] f_{n-1}(q) + q (a_3 + b_3 q + c_3 q^2) f'_{n-1}(q),$$
where $a_1, b_1, c_1, a_1 + a_3, b_1 + b_3, c_1 + c_3$ are all nonnegative. If $\{f_n(q)\}_{n\geq 0}$ is a generalized Sturm sequence, then it is $q$-log-convex. Furthermore, assume that $c_1 = c_3 = 0$. If $a_1 + 2a_3 \geq 0$ and $b_1 \geq b_3$, then $f_{n+1}(q)f_{n-1}(q) - f_n^2(q)$ is a generalized stable polynomial for $n \geq 1$.

The generalized Sturm sequences arise often in combinatorics. In addition, the following result given by Liu and Wang [26] provides an approach to the generalized Sturm sequences.

**Proposition 1.7.** Let $\{P_n(x)\}$ be a sequence of polynomials with nonnegative coefficients and $	ext{deg } P_n = 	ext{deg } P_{n-1} + 1$. Suppose that

$$P_n(x) = (a_n x + b_n)P_{n-1}(x) + x(c_n x + d_n)P'_{n-1}(x)$$

where $a_n, b_n \in \mathbb{R}$ and $c_n \leq 0, d_n \geq 0$. Then $\{P_n(x)\}_{n\geq 0}$ is a generalized Sturm sequence.

It is well-known that many classical combinatorial sequences of polynomials arising in certain triangular arrays, e.g., Pascal triangle, Stirling triangle, Eulerian triangle and so on, satisfy the recurrence relation (1.1). For its row generating functions $T_n(q) = \sum_{k=0}^{n} T_{n,k}q^k$, by the recurrence relation (1.1), we have

$$T_n(q) = [a_1n + a_3 + (b_1n + b_2 + b_3)q]T_{n-1}(q) + (a_2 + b_2q)qT'_{n-1}(q).$$

By Proposition 1.7, we know that if $a_2 \geq 0 \geq b_2$ then the polynomials $T_n(q)$ form a generalized Sturm sequence. Thus, the next result follows from Theorem 1.6

**Proposition 1.8.** Let $[T_{n,k}]_{n,k\geq 0}$ be the nonnegative array defined in (1.7) and the row generating functions $T_n(q) = \sum_{k=0}^{n} T_{n,k}q^k$. If $a_2 \geq 0 \geq b_2$, then $\{T_{n+1}(q)T_{n-1}(q) - T_n^2(q)\}_{n\geq 1}$ is a sequence of generalized stable polynomials.

The remainder of this paper is structured as follows. In Section 2, we will present the proofs of Theorem 1.2. In Section 3, we give the proof of Theorem 1.6. In Section 4, we apply Theorem 1.2 to some famous triangular arrays in a unified manner, including Stirling triangles of two kinds, the Jacobi-Stirling triangles of two kinds, the Legendre-Stirling triangles of two kinds, the central factorial numbers triangle, the Ramanujan transformation, and so on. In particular, we solve the Conjecture 1.1. Finally, we also apply Proposition 1.8 to Eulerian polynomials of type A and B, and their $q$-analogs. Using Theorem 1.6, we also obtain the $q$-log-convexity of the generating functions of alternating runs, the longest alternating subsequence and up-down runs of permutations, respectively. In the Section 5, we give some remarks about linear transformations preserving the strong $q$-log-convexity. In addition, we also present a criterion for the strong $q$-log-convexity.
2 Proof of Theorem 1.2

The next lemma plays an important role in our proof.

Lemma 2.1. [17] Given four sequences \(\{a_i\}_{i=0}^{n}, \{b_i\}_{i=0}^{n}, \{c_i\}_{i=0}^{n}\) and \(\{d_i\}_{i=0}^{n}\), we have

\[
\sum_{i=0}^{n} a_i c_i \sum_{i=0}^{n} b_i d_i - \sum_{i=0}^{n} a_i d_i \sum_{i=0}^{n} b_i c_i = \sum_{0 \leq i < j \leq n} (a_i b_j - a_j b_i)(c_i d_j - c_j d_i).
\]

Proof of Theorem 1.2: In the following proof, we simply write \(x_k\) for \(x_k(q)\).

In order to prove the strong \(q\)-log-convexity of \(\{y_n(q)\}_{n \geq 0}\), it suffices to show for \(n \geq m \geq 1\) that

\[
y_{n+1}(q)y_{m-1}(q) - y_{n}(q)y_{m}(q) \geq_q 0.
\]

Then, for \(n \geq m \geq 1\), by the recurrence relation (1.3), we have

\[
y_{n+1}(q)y_{m-1}(q) - y_{n}(q)y_{m}(q) = \sum_{k \geq 0} r(n+1) + f(k) T_{n,k} x_k \sum_{k \geq 0} T_{m-1,k} x_k - \sum_{k \geq 0} T_{n,k} x_k \sum_{k \geq 0} [r(m) + f(k)] T_{m-1,k} x_k +
\]

\[
\sum_{k \geq 1} [s(n+1) + g(k)] T_{n,k-1} x_k \sum_{k \geq 0} T_{m-1,k} x_k - \sum_{k \geq 0} T_{n,k} x_k \sum_{k \geq 1} [s(m) + g(k)] T_{m-1,k-1} x_k +
\]

\[
\sum_{k \geq 2} [t(n+1) + h(k)] T_{n,k-2} x_k \sum_{k \geq 0} T_{m-1,k} x_k - \sum_{k \geq 0} T_{n,k} x_k \sum_{k \geq 2} [t(m) + h(k)] T_{m-1,k-2} x_k
\]

\[
= \sum_{k \geq 0} r(n+1) T_{n,k} x_k \sum_{k \geq 0} T_{m-1,k} x_k - \sum_{k \geq 0} T_{n,k} x_k \sum_{k \geq 0} r(m) T_{m-1,k} x_k \quad + \quad (2.1)
\]

\[
\sum_{k \geq 0} f(k) T_{n,k} x_k \sum_{k \geq 0} T_{m-1,k} x_k - \sum_{k \geq 0} T_{n,k} x_k \sum_{k \geq 0} f(k) T_{m-1,k} x_k \quad + \quad (2.2)
\]

\[
\sum_{k \geq 1} s(n+1) T_{n,k-1} x_k \sum_{k \geq 0} T_{m-1,k} x_k - \sum_{k \geq 0} T_{n,k} x_k \sum_{k \geq 1} s(m) T_{m-1,k-1} x_k \quad + \quad (2.3)
\]

\[
\sum_{k \geq 1} g(k) T_{n,k-1} x_k \sum_{k \geq 0} T_{m-1,k} x_k - \sum_{k \geq 0} T_{n,k} x_k \sum_{k \geq 1} g(k) T_{m-1,k-1} x_k \quad + \quad (2.4)
\]

\[
\sum_{k \geq 2} t(n+1) T_{n,k-2} x_k \sum_{k \geq 0} T_{m-1,k} x_k - \sum_{k \geq 0} T_{n,k} x_k \sum_{k \geq 0} t(m) T_{m-1,k-2} x_k \quad + \quad (2.5)
\]

\[
\sum_{k \geq 2} h(k) T_{n,k-2} x_k \sum_{k \geq 0} T_{m-1,k} x_k - \sum_{k \geq 0} T_{n,k} x_k \sum_{k \geq 0} h(k) T_{m-1,k-2} x_k \quad . \quad (2.6)
\]

In what follows we will prove that every difference in (2.1)-(2.6) is \(q\)-nonnegative.
Obviously, for (2.1), we have
\[
\sum_{k \geq 0} r(n + 1)T_{n,k}x_k \sum_{k \geq 0} T_{m-1,k}x_k - \sum_{k \geq 0} T_{n,k}x_k \sum_{k \geq 0} r(m)T_{m-1,k}x_k
\]
\[
= \left[ r(n + 1) - r(m) \right] \sum_{k \geq 0} T_{n,k}x_k \sum_{k \geq 0} T_{m-1,k}x_k
\]
\[
\geq_q 0
\]
since \( r(n) \) is nonnegative and increasing in \( n \).

For (2.2), if we view \( T_{m-1,k}, T_{n,k}, x_k \) and \( f(k)x_k \) as \( a_k, b_k, c_k \) and \( d_k \) in Lemma 2.1 respectively, then we obtain that
\[
\sum_{k \geq 0} f(k)T_{n,k}x_k \sum_{k \geq 0} T_{m-1,k}x_k - \sum_{k \geq 0} T_{n,k}x_k \sum_{k \geq 0} f(k)T_{m-1,k}x_k
\]
\[
= \sum_{k=0}^p f(k)T_{n,k}x_k \sum_{k \geq 0} T_{m-1,k}x_k - \sum_{k=0}^p T_{n,k}x_k \sum_{k \geq 0} f(k)T_{m-1,k}x_k
\]
\[
= \sum_{0 \leq i < j \leq p} [f(j) - f(i)](T_{m-1,i}T_{n,j} - T_{m-1,j}T_{n,i})x_ix_j
\]
\[
\geq_q 0
\]
since \( f(k) \) is increasing and the matrix \( [T_{n,k}]_{n,k \geq 0} \) is TP_2, where
\[
p = \max\{k : T_{n,k} \neq 0 \text{ or } T_{m-1,k} \neq 0\}.
\]

Similarly by Lemma 2.1, for (2.3) and (2.4), we derive that
\[
\sum_{k \geq 1} T_{n,k-1}x_k \sum_{k \geq 0} T_{m-1,k}x_k - \sum_{k \geq 0} T_{n,k}x_k \sum_{k \geq 1} T_{m-1,k-1}x_k
\]
\[
= \sum_{k \geq 0} T_{n,k}x_{k+1} \sum_{k \geq 0} T_{m-1,k}x_k - \sum_{k \geq 0} T_{n,k}x_k \sum_{k \geq 0} T_{m-1,k}x_{k+1}
\]
\[
= \sum_{k=0}^p T_{n,k}x_{k+1} \sum_{k \geq 0} T_{m-1,k}x_k - \sum_{k=0}^p T_{n,k}x_k \sum_{k \geq 0} T_{m-1,k}x_{k+1}
\]
\[
= \sum_{0 \leq i < j \leq p} [T_{m-1,i}T_{n,j} - T_{m-1,j}T_{n,i}] [x_ix_{j+1} - x_{i+1}x_j]
\]
\[
\geq_q 0
\]
and
\[
\sum_{k \geq 1} g(k)T_{n,k-1}x_k \sum_{k \geq 0} T_{m-1,k}x_k - \sum_{k \geq 0} T_{n,k}x_k \sum_{k \geq 1} g(k)T_{m-1,k-1}x_k
\]
\[
= \sum_{k \geq 0} g(k + 1)T_{n,k}x_{k+1} \sum_{k \geq 0} T_{m-1,k}x_k - \sum_{k \geq 0} T_{n,k}x_k \sum_{k \geq 0} g(k + 1)T_{m-1,k}x_{k+1}
\]
\[
= \sum_{k=0}^p g(k + 1)T_{n,k}x_{k+1} \sum_{k \geq 0} T_{m-1,k}x_k - \sum_{k=0}^p T_{n,k}x_k \sum_{k \geq 0} g(k + 1)T_{m-1,k}x_{k+1}
\]
\[
= \sum_{0 \leq i < j \leq p} [T_{m-1,i}T_{n,j} - T_{m-1,j}T_{n,i}] [g(j + 1)x_ix_{j+1} - g(i + 1)x_{i+1}x_j]
\]
This completes the proof.

For (2.5) and (2.6), we obtain that

\[ \geq \sum_{0 \leq i < j \leq n} [T_{m-1,i}T_{n,j} - T_{m-1,j}T_{n,i}] g(j + 1) [x_i x_{j+1} - x_{i+1} x_j] \]

\[ \geq q \ 0. \]

and

\[ \sum_{k \geq 2} h(k) T_{n,k-2} x_k \sum_{k \geq 0} T_{m-1,k} x_k - \sum_{k \geq 0} T_{n,k} x_k \sum_{k \geq 2} h(k) T_{m-1,k-2} x_k \]

\[ = \sum_{k \geq 0} h(k + 2) T_{n,k} x_{k+2} \sum_{k \geq 0} T_{m-1,k} x_k - \sum_{k \geq 0} T_{n,k} x_k \sum_{k \geq 0} h(k + 2) T_{m-1,k} x_{k+2} \]

\[ = \sum_{k = 0}^p h(k + 2) T_{n,k} x_{k+2} \sum_{k = 0}^p T_{m-1,k} x_k - \sum_{k = 0}^p T_{n,k} x_k \sum_{k = 0}^p h(k + 2) T_{m-1,k} x_{k+2} \]

\[ = \sum_{0 \leq i < j \leq p} [T_{m-1,i}T_{n,j} - T_{m-1,j}T_{n,i}] [h(j + 2)x_i x_{j+2} - h(i + 2)x_{i+2} x_j] \]

\[ \geq \sum_{0 \leq i < j \leq p} [T_{m-1,i}T_{n,j} - T_{m-1,j}T_{n,i}] h(i + 2) [x_i x_{j+2} - x_{i+2} x_j] \]

\[ \geq q \ 0. \]

This completes the proof.

**3 Proof of Theorem 1.6**

Following Wagner [40], a real polynomial is said to be standard if either it is identically zero or its leading coefficient is positive. Suppose that both \( f \) and \( g \) only have real zeros. Let \( \{r_i\} \) and \( \{s_j\} \) be all zeros of \( f \) and \( g \) in nondecreasing order respectively. We say that \( g \) interlaces \( f \) if \( \deg f = \deg g + 1 = n \) and

\[ r_n \leq s_{n-1} \leq \cdots \leq s_2 \leq r_2 \leq s_1 \leq r_1. \] (3.1)

By \( g \preceq f \) we denote “\( g \) interlaces \( f \)”. For notational convenience, let \( a \preceq bx + c \) for any real constants \( a, b, c \) and \( f \preceq 0, 0 \preceq f \) for all real polynomial \( f \) with only real zeros.
Let \(\{P_n(x)\}_{n \geq 0}\) be a sequence of standard polynomials. Recall that \(\{P_n(x)\}\) is a Sturm sequence if \(\deg P_n = n\) and \(P_{n-1}(r)P_{n+1}(r) < 0\) whenever \(P_n(r) = 0\) and \(n \geq 1\). We say that \(\{P_n(x)\}\) is a generalized Sturm sequence if \(P_n \in \text{RZ}\) and \(P_0 \preceq P_1 \preceq \cdots \preceq P_{n-1} \preceq P_n \preceq \cdots\). For example, if \(P\) is a standard polynomial with only real zeros and \(\deg P = n\), then \(P^{(n)}, P^{(n-1)}, \ldots, P', P\) form a generalized Sturm sequence by Rolle’s theorem.

In order to simplify our proof, we need the following lemma.

**Lemma 3.1.** \([18] \text{Lemma 1.20}\) Let both \(f(x)\) and \(g(x)\) be standard real polynomials with only real zeros. Assume that \(\deg(f(x)) = n\) and all real zeros of \(f(x)\) are \(s_1, \ldots, s_n\). If \(\deg(g) = n - 1\) and we write

\[
g(x) = \sum_{i=1}^{n} \frac{c_i f(x)}{x - s_i},
\]

then \(g(x)\) interlaces \(f(x)\) if and only if all \(c_i\) are positive.

**Proof of Theorem 1.6:**

Since

\[
f_n(q) = [a_1 n + a_2 + (b_1 n + b_2) q + (c_1 n + c_2) q^2] f_{n-1}(q) + q(a_3 + b_3 q + c_3 q^2) f'_{n-1}(q),
\]

it follows that

\[
f_{n+1}(q) f_{n-1}(q) - f_n^2(q) = [a_1 n + a_2 + (b_1 n + b_2 + b_1) q + (c_1 n + c_2 + c_1) q^2] f_n(q) f_{n-1}(q) + q(a_3 + b_3 q + c_3 q^2) f'_n(q) f_{n-1}(q) - [a_1 n + a_2 + (b_1 n + b_2 + b_1) q + (c_1 n + c_2 + c_1) q^2] f_{n-1}(q) f_n(q)
\]

\[
= (a_1 + b_1 q + c_1 q^2) f_n(q) f_{n-1}(q) + q(a_3 + b_3 q + c_3 q^2) [f'_n(q) f_{n-1}(q) - f_n(q) f'_{n-1}(q)]
\]

\[
= f_n^2(q) \left[ (a_1 + b_1 q + c_1 q^2) \frac{f_{n-1}(q)}{f_n(q)} - q(a_3 + b_3 q + c_3 q^2) \left( \frac{f_{n-1}(q)}{f_n(q)} \right)' \right].
\]

By assumption, \(\{f_n(q)\}_{n \geq 0}\) is a generalized Sturm sequence. Thus, if we assume that the all non-positive zeros of \(f_n(q)\) are ordered as \(r_1 \geq r_2 \geq \cdots \geq r_n\), then \(f_{n-1}(q) = f_n(q) \sum_{i=1}^{n} \frac{s_i}{q - r_i}\) by Lemma 3.1 where \(s_i > 0\) for \(1 \leq i \leq n\). Hence,

\[
f_{n+1}(q) f_{n-1}(q) - f_n^2(q) = f_n^2(q) \left[ (a_1 + b_1 q + c_1 q^2) \frac{f_{n-1}(q)}{f_n(q)} - q(a_3 + b_3 q + c_3 q^2) \left( \frac{f_{n-1}(q)}{f_n(q)} \right)' \right]
\]

\[
= f_n^2(q) \left[ (a_1 + b_1 q + c_1 q^2) \sum_{i=1}^{n} \frac{s_i}{q - r_i} + q(a_3 + b_3 q + c_3 q^2) \sum_{i=1}^{n} \frac{s_i}{(q - r_i)^2} \right]
\]

\[
= f_n^2(q) \sum_{i=1}^{n} s_i \left[ (a_1 + b_1 q + c_1 q^2)(q - r_i) + q(a_3 + b_3 q + c_3 q^2) \right] \frac{s_i}{(q - r_i)^2}
\]

\[
= \sum_{i=1}^{n} s_i \left[ (c_1 + c_3) q^3 + (b_1 + b_3 - c_1 r_i) q^2 + (a_1 + a_3 - b_1 r_i) q - a_1 r_i \right] \left( \frac{f_n(q)}{q - r_i} \right)^2,
\]
which is a polynomial with nonnegative coefficients since
\[ (c_1 + c_3)q^3 + (b_1 + b_3 - c_1r_i)q^2 + (a_1 + a_3 - b_1r_i)q - a_1r_i, \frac{f_n(q)}{q - r_i} \]
are all polynomials with nonnegative coefficients for \(1 \leq i \leq n\). Thus \(\{f_n(q)\}_{n \geq 0}\) is \(q\)-log-convex.

In the following, we proceed to demonstrate the second part that
\[ f_{n+1}(q)f_{n-1}(q) - f_n^2(q) \]
is a generalized stable polynomial for each \(n \geq 1\). Note that
\[
\begin{align*}
 f_{n+1}(q)f_{n-1}(q) - f_n^2(q) &= f_n^2(q) \sum_{i=1}^n s_i \left[ (a_1 + b_1q + c_1q^2)(q - r_i) + q(a_3 + b_3q + c_3q^2) \right] \left/ (q - r_i)^2 \right.
\end{align*}
\]
Thus we only need to show that
\[ \sum_{i=1}^n s_i \left[ (b_1 + b_3)q^2 + (a_1 + a_3 - b_1r_i)q - a_1r_i \right] \left/ (q - r_i)^2 \right. \tag{3.3} \]
has no zeros in the right half plane since \(c_1 = c_3 = 0\). Let \(q = x + yI\), where \(I\) is the imaginary number unit. Then, for \(x \geq 0\) and \(r \leq 0\), it follows from \(a_1 + 2a_3 \geq 0\) and \(b_1 \geq b_3\) that we have
\[
\begin{align*}
[(x - r)^2 + y^2]^2 Re \left( \frac{(b_1 + b_3)q^2 + (a_1 + a_3 - b_1r)q - a_1r}{(q - r)^2} \right)
&= Re \left( [(b_1 + b_3)(x + yI)^2 + (a_1 + a_3 - b_1r)(x + yI) - a_1r](x - r - yI)^2 \right)
&= B(x^2 - y^2)^2 + Bx^2(r^2 - 2xr) + (xA - xb_1r - a_1r)(x - r)^2 + \\
y^2 \{4x^2B + xA + (2B + b_1)xr - (a_1 + 2a_3)r + (b_1 - b_3)r^2 \}
&\geq 0,
\end{align*}
\]
where \(A = a_1 + a_3\) and \(B = b_1 + b_3\). Thus,
\[
Re \left( \sum_{k=1}^n c_k \left[ (b_1 + b_2)q^2 + (a_1 + a_2 - b_1r_k)q - a_1r_k \right] \right) \geq 0
\]
with the equality if and only if \(q = 0\). This completes the proof.

\[ \square \]

4 Applications

In this section, we give some applications of the main results.
4.1 Stirling transformations of two kinds

Let $S_{n,k}$ denote the number of partitions of a set with $n$ elements consisting of $k$ disjoint nonempty sets. It is well known that $S_{n,k}$ is called the Stirling number of the second kind. In addition, the Stirling numbers of the second kind satisfy the recurrence

$$S_{n+1,k} = kS_{n,k} + S_{n,k-1}.$$  

Let $c_{n,k}$ be the signless Stirling number of the first kind, i.e., the number of permutations of $[n]$ which contain exactly $k$ permutation cycles. Similarly, signless Stirling numbers of the first kind $c_{n,k}$ satisfy the recurrence

$$c_{n,k} = (n-1)c_{n-1,k} + c_{n-1,k-1}.$$  

The $q$-log-convexity and strong $q$-log-convexity of the row generating functions $S_n(q) = \sum_{k=0}^{n} S_{n,k}q^k$ have been proved, see Liu and Wang [27], Chen et al. [14] and Zhu [45, 46] for instance. Liu and Wang [27] also proved that both the Stirling transformations of two kinds preserve the log-convexity. By Theorem 1.3, we can extend above results to the strong $q$-log-convexity as follows.

**Proposition 4.1.** The linear transformation $y_n(q) = \sum_{k=0}^{n} S_{n,k}x_k(q)$ preserves the strong $q$-log-convexity.

**Proposition 4.2.** The linear transformation $y_n(q) = \sum_{k=0}^{n} c_{n,k}x_k(q)$ preserves the strong $q$-log-convexity.

4.2 Jacobi-Stirling transformation of the second kind

The Jacobi-Stirling numbers $JS_n^k(z)$ of the second kind satisfy the following recurrence relation:

$$JS_n^0(z) = 1, \quad JS_n^k(z) = 0, \quad \text{if } k \not\in \{1, \ldots, n\},$$

$$JS_n^k(z) = JS_{n-1}^{k-1}(z) + k(k + z) JS_{n-1}^k(z), \quad n, k \geq 1,$$

(4.1)

where $z = \alpha + \beta + 1$. Lin and Zeng [24] and Zhu [46] independently proved the strong $q$-log-convexity of the row generating functions of Jacobi-Stirling numbers. By Theorem 1.3, we have the following generalization, which in particular confirms Conjecture 1.1 for Jacobi-Stirling numbers of the second kind.

**Proposition 4.3.** The Jacobi-Stirling transformation of the second kind

$$y_n(q) = \sum_{k=0}^{n} JS_n^k(z)x_k(q)$$
preserves the strong $q$-log-convexity for $z \geq 0$. In particular,

$$y_n = \sum_{k=0}^{n} J^k_n(z) x_k$$

preserves the log-convexity for $z \geq 0$.

The Jacobi-Stirling numbers $J^k_n(z)$ of the first kind are defined by

$$\begin{align*}
J^0_n(z) &= 1, \\
J^k_n(z) &= 0, \quad \text{if } k \notin \{1, \ldots, n\}, \\
J^k_n(z) &= J^{k-1}_n(z) + (n-1)(n-1+z)J^{k-1}_n(z), \quad n, k \geq 1,
\end{align*}$$

where $z = \alpha + \beta + 1$. It is well known that the unsigned inverse of a totally positive matrix is also totally positive. Thus by the total positivity of $[J^k_n(z)]_{n,k}$, we derive that $[J^k_n(z)]_{n,k}$ is totally positive, also see [33]. Thus by Theorem 1.2 we also have the following result, which in particular confirms Conjecture 1.1 for Jacobi-Stirling numbers of the first kind.

**Proposition 4.4.** The Jacobi-Stirling transformation of the first kind

$$w_n(q) = \sum_{k=0}^{n} J^k_n(z) x_k(q)$$

preserves the strong $q$-log-convexity for $z \geq 0$. In particular,

$$w_n = \sum_{k=0}^{n} J^k_n(z) x_k$$

preserves the log-convexity for $z \geq 0$.

**Remark 4.5.** If $z = 1$, then $J^k_n(1)$ and $J^k_n(1)$ are the Legendre-Stirling numbers of two kinds, respectively.

### 4.3 Central factorial transformations

The *central factorial numbers* of the second kind $T(n,k)$ are defined in Riordan’s book [34, p. 213-217] by

$$x^n = \sum_{k=0}^{n} T(n,k) x^{k-1} \left( x + \frac{k}{2} - i \right). \quad (4.2)$$

Therefore, if let $U(n,k) = T(2n,2k)$ and $V(n,k) = 4^{n-k} T(2n+1,2k+1)$, then

$$U(n,k) = U(n-1,k-1) + k^2 U(n-1,k),$$

$$V(n,k) = V(n-1,k-1) + (2k+1)^2 V(n-1,k).$$

Zhu [46] proved that the row generating functions of $U(n,k)$ (respectively, $V(n,k)$) form a strongly $q$-log-convex sequence. In view of Theorem 1.3 these can be extended to the following result.
Proposition 4.6. The linear transformation \( y_n(q) = \sum_{k=0}^{n} U(n,k)x_k(q) \) preserves the strong \( q \)-log-convexity. In particular, \( y_n = \sum_{k=0}^{n} U(n,k)x_k \) preserves the log-convexity.

Proposition 4.7. The linear transformation \( y_n(q) = \sum_{k=0}^{n} V(n,k)x_k(q) \) preserves the strong \( q \)-log-convexity. In particular, \( y_n = \sum_{k=0}^{n} V(n,k)x_k \) preserves the log-convexity.

4.4 Ramanujan transformation

Let \( r_{n,k} \) be the number of rooted labeled trees on \( n \) vertices with \( k \) improper edges. Then numbers \( r_{n,k} \) satisfy the following recurrence relation:

\[
r_{n,k} = (n - 1)r_{n-1,k} + (n + k - 2)r_{n-1,k-1}
\]

where \( r_{1,0} = 1 \), \( n \geq 1 \), \( k \leq n - 1 \), and \( r_{n,k} = 0 \) otherwise, see Shor [36]. It was proved that the row generating functions of \( [r_{n,k}]_{n,k \geq 0} \) are the famous Ramanujan polynomials \( r_n(y) \), which are defined by the recurrence relation

\[
r_1(y) = 1, \quad r_{n+1} = n(1 + y)r_n(y) + y^2r'_n(y).
\]

The first values of the polynomials \( r_n(y) \) are

\[
r_2(y) = 1 + y, \quad r_3(y) = 2 + 4y + 3y^2, \quad r_4(y) = 6 + 18y + 25y^2 + 15y^3.
\]

Chen et al. [14] proved that the polynomials \( r_n(y) \) form a strongly \( q \)-log-convex sequence, which can be extended to the following result by Theorem 1.3.

Proposition 4.8. If \( \{x_n(q)\}_{n \geq 0} \) is strongly \( q \)-log-convex, then so is \( \{y_n(q)\}_{n \geq 0} \) defined by \( y_n(q) = \sum_{k=0}^{n} r_{n,k}x_k(q) \). In particular, \( y_n = \sum_{k=0}^{n} r_{n,k}x_k \) preserves the log-convexity.

4.5 Associated Lah transformation

The associated Lah numbers defined by

\[
L_m(n, k) = \frac{(n!/k!)}{m!} \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} \binom{n + mi - 1}{n}
\]

satisfy the recurrence

\[
L_m(n, k) = (mk + n - 1)L_m(n - 1, k) + mL_m(n - 1, k - 1).
\]

Let \( L_n(q) = \sum_{k=0}^{n} L_m(n, k)q^k \). By virtue of Theorem 1.3 and Proposition 1.8, we have the following two results, respectively.

Proposition 4.9. The linear transformation \( z_n(q) = \sum_{k=0}^{n} L_m(n, k)x_k(q) \) preserves the strong \( q \)-log-convexity. In particular, \( z_n = \sum_{k=0}^{n} L_m(n, k)x_k \) preserves the log-convexity.

Proposition 4.10. \( L_{n+1}(q)L_{n-1}(q) - L_n^2(q) \) is a stable polynomial for each \( n \geq 1 \).
4.6 Eulerian polynomials of types A and B

Let $\pi = a_1a_2 \cdots a_n$ be a permutation of $[n]$. An element $i \in [n - 1]$ is called a descent of $\pi$ if $a_i > a_{i+1}$. The number of permutations of $[n]$ having $k - 1$ descents is called the Eulerian number, denoted by $A_{n,k}$, and its row-generating function $A_n(q) = \sum_{k=0}^{n} A_{n,k}q^k$ is called the classical Eulerian polynomial. It is known that

$$A_n(q) = nqA_{n-1}(q) + q(1 - q)A'_{n-1}(q).$$

We refer reader to Comtet [15] for further properties about Eulerian polynomials. Let $B_{n,k}$ be the Eulerian number of type B counting the elements of $B_n$ with $k$ $B$-descents. It is known that the Eulerian numbers of type $B$ satisfy the recurrence

$$B_{n,k} = (2k + 1)B_{n-1,k} + (2n - 2k + 1)B_{n-1,k-1}.$$  \hspace{1cm} (4.3)

Assume that $B_n(q) = \sum_{k=0}^{n} B_{n,k}q^k$ is the Eulerian polynomial of type B. Then we have

$$B_n(q) = (1 + q)B_{n-1}(q) + 2x(1 - x)B'_{n-1}(q).$$

It was proved that polynomials $A_n(q)$ (respectively $B_n(q)$) form a strongly $q$-log-convex sequence, respectively, see [46, 28]. By Theorem 1.6, the following result is immediate.

**Proposition 4.11.** Both $A_{n+1}(q)A_{n-1}(q) - A^2_n(q)$ and $B_{n+1}(q)B_{n-1}(q) - B^2_n(q)$ are generalized stable polynomials in $q$ for $n \geq 1$.

**Remark 4.12.** The polynomial $A_{n+1}(q)A_{n-1}(q) - A^2_n(q)$ is stable, which was proved by Fisk [18, Lemma 21.92].

4.7 q-Eulerian polynomials

For a finite Coxeter group $W$, let $d_W(\pi)$ denote the number of $W$-descents of $\pi$. Then the Eulerian polynomial of $W$ is defined by

$$P(W, x) = \sum_{\pi \in W} x^{d_W(\pi)},$$

see Björner and Brenti [3] for instance.

For Coxeter groups of type $A$, $P(A_n, x) = A_n(x)/x$, where $A_n(x)$ is the classical Eulerian polynomial. Let $\text{exc}(\pi)$ and $\text{c}(\pi)$ denote the numbers of excedances and cycles in $\pi$, respectively. In [20], Foata and Schützenberger defined a $q$-analog of the classical Eulerian polynomials by

$$A_n(x; q) = \sum_{\pi \in S_n} x^{\text{exc}(\pi)+1}q^\text{c}(\pi).$$
Obviously, for $q = 1$, $A_n(x; q)$ reduces to the classical Eulerian polynomial $A_n(x)$. In addition, in [10] Proposition 7.2, Brenti demonstrated the recurrence relation

$$A_n(x; q) = (nx + q - 1)A_{n-1}(x; q) + x(1-x)\frac{\partial}{\partial x}A_{n-1}(x; q),$$

with the initial condition $A_0(x; q) = x$.

For Coxeter groups of type $B$, letting $N(\pi) = |\{i \in [n] : \pi(i) < 0\}|$, Brenti introduced a $q$-analogue of $P(B_n, x)$ by

$$B_n(x; q) = \sum_{\pi \in B_n} q^{N(\pi)}x^{d_B(\pi)}.$$ 

In fact, $B_n(x; q)$ reduces to $A_n(x)$ for $q = 0$ and to $P(B_n, x)$ for $q = 1$. It is also known that $\{B_n(x; q)\}_{n \geq 0}$ satisfies the recurrence relation

$$B_n(x; q) = \{1 + [(1 + q)n - 1]x\}B_{n-1}(x; q) + (1 + q)x(1-x)\frac{\partial}{\partial x}B_{n-1}(x; q),$$

with $B_0(x; q) = 1$, see [9] Theorem 3.4 (i)]. Thus the following result follows from Proposition 1.8.

**Proposition 4.13.** Both $A_{n+1}(x; q)A_{n-1}(x; q) - A_n^2(x; q)$ and $B_{n+1}(x; q)B_{n-1}(x; q) - B_n^2(x; q)$ are generalized stable polynomials in $x$ for any fixed $q \geq 0$.

### 4.8 Alternating runs

Suppose that $S_n$ denotes the symmetric group of all permutations of $\{1, 2, \ldots, n\}$. For $\pi = \pi(1)\pi(2)\cdots\pi(n) \in S_n$, we say that $\pi$ changes direction at position $i$ if either $\pi(i - 1) < \pi(i)$ or $\pi(i - 1) > \pi(i) < \pi(i + 1)$. We say that $\pi$ has $k$ alternating runs if there are $k - 1$ indices $i$ such that $\pi$ changes direction at these positions. Denote by $R(n, k)$ the number of permutations in $S_n$ having $k$ alternating runs. Then we have

$$R(n, k) = kR(n-1, k) + 2R(n-1, k-1) + (n-k)R(n-1, k-2) \quad (4.4)$$

for $n, k \geq 1$, where $R(1, 0) = 1$ and $R(1, k) = 0$ for $k \geq 1$, see Bóna [6] for instance. For $n \geq 1$, define the alternating runs polynomials $R_n(x) = \sum_{k=1}^{n-1} R(n, k)x^k$. It follows from the recurrence (4.4) that

$$R_{n+2}(x) = x(nx + 2)R_{n+1}(x) + x(1-x^2)R'_{n+1}(x)$$

with initial conditions $R_1(x) = 1$ and $R_2(x) = 2x$. Moreover, the polynomial $R_n(x)$ has a close connection with the classical Eulerian polynomial $A_n(x)$ by

$$R_n(x) = \left(\frac{1 + x}{2}\right)^{n-1}(1 + w)^{n+1}A_n\left(\frac{1 - w}{1 + w}\right), \quad w = \sqrt{\frac{1 - x}{1 + x}},$$
see Knuth [22]. The polynomials $R_n(x)$ also have only non-positive real zeros and $R_n(x) \preceq R_{n+1}(x)$, see Ma and Wang [31]. It follows from Theorem 1.6 that the next result is immediate.

**Proposition 4.14.** The alternating runs polynomials $R_n(q)$ form a $q$-log-convex sequence.

### 4.9 The longest alternating subsequence and up-down runs of permutations

For a subsequence $\pi(i_1) \cdots \pi(i_k)$ of $\pi$, it is called an **alternating subsequence** if

$$\pi(i_1) > \pi(i_2) < \pi(i_3) > \cdots \pi(i_k).$$

Let $a_k(n)$ denote the length of the longest alternating subsequence of $\pi$ and the number of permutations in $S_n$ with $a(\pi) = k$. Define its ordinary generating function $t_n(x) = \sum_{k=1}^{n} a_k(n)x^k$. For $n \geq 2$, Bóna [6, Section 1.3.2] derived the following identity:

$$t_n(x) = \frac{1}{2}(1 + x)R_n(x).$$

Ma [29] also proved that the polynomials $t_n(x)$ satisfy the recurrence relation

$$t_{n+1}(x) = x(nx + 1)t_n(x) + x(1 - x^2)t'_n(x),$$

with initial conditions $t_0(x) = 1$ and $t_1(x) = x$. We refer reader to Stanley [38] for more properties about the longest alternating subsequences.

On the other hand, $a_k(n)$ is also the number of permutations in $S_n$ with $k$ up-down runs. The **up-down runs** of a permutation $\pi$ are defined to the alternating runs of $\pi$ endowed with a $0$ in the front, see [37, A186370]. In addition, the up-down runs of a permutation have a close connection with interior peaks and left peaks. Based on the interior peaks and left peaks, Ma [30] defined polynomials $M_n(x)$, which satisfy the recurrence relation

$$M_{n+1}(x) = (1 + nx^2)M_n(x) + x(1 - x^2)M'_n(x),$$

with initial conditions $M_1(x) = 1 + x$ and $M_2(x) = 1 + 2x + x^2$, see [30, Section 2]. In addition, $M_n(x)$ has only non-positive real zeros and $M_n(x) \preceq M_{n+1}(x)$, see Ma [30]. So, we have the following result by Theorem 1.6

**Proposition 4.15.** Both $\{t_n(q)\}_{n \geq 0}$ and $\{M_n(q)\}_{n \geq 0}$ are $q$-log-convex sequences.
4.10 Alternating runs of type $B_n$

A run of a signed permutation $\pi \in B_n$ is defined as a maximal interval of consecutive elements on which the elements of $\pi$ are monotonic in the order $\cdots < 2 < 1 < 0 < 1 < 2 < \cdots$. Let $T(n, k)$ denote the number of signed permutations in $B_n$ with $k$ alternating runs and $\pi(1) > 0$. In [44, Theorem 4.2.1], it was shown that the array $[T(n, k)]_{n,k}$ satisfies the recurrence relation

$$T(n, k) = (2k - 1)T(n - 1, k) + 3T(n - 1, k - 1) + (2n - 2k + 2)T(n - 1, k - 2) \quad (4.5)$$

for $n \geq 2$ and $1 \leq k \leq n$, where $T(1, 1) = 1$ and $T(1, k) = 0$ for $k > 1$. Let $T_n(x) = \sum_{k=1}^{n} T(n, k)x^k$ denote the alternating run polynomials of type $B$. It follows from (4.5) that we have the recurrence relation

$$T_n(x) = [2(n - 1)x^2 + 3x - 1]T_{n-1}(x) + 2x(1 - x^2)T'_{n-1}(x).$$

Zhao proved that polynomials $T_n(x)$ form a generalized Sturm sequence by using Theorem 2 of Ma and Wang [31]. Thus we get the following result from Theorem 1.6.

**Proposition 4.16.** For $n \geq 1$, the alternating run polynomials $T_n(q)$ form a $q$-log-convex sequence.

5 Remarks

There are also many famous triangular arrays, including the Motzkin triangle, the Bell triangle, the Catalan triangle, the large Schröder triangle, and so on, satisfying such recurrence relation

$$T_{n,k} = f_kT_{n-1,k-1} + g_kT_{n-1,k} + h_kT_{n-1,k+1} \quad (5.1)$$

with $T_{0,0} = 1$ and $T_{n,k} = 0$ unless $0 \leq k \leq n$ and the nonnegative array $[T_{n,k}]_{n,k}$ has a general combinatorial interpretation from the weighted Motzkin path, see [19]. In [45], we proved that the array $[T_{n,k}]_{n,k}$ is TP$_2$ and the first column $\{T_{n,0}\}_{n\geq0}$ is log-convex if $g_{k+1}g_k \geq h_kf_{k+1}$ for $k \geq 0$. For this array $[T_{n,k}]_{n,k}$ in (5.1), by Theorem 1.2, it is natural to ask whether we have a similar result. However, for the general case, the answer is not. In the following, we give a simple example.

**Example 5.1.** Let a triangular array $[M_{n,k}]_{n,k}$ satisfy the recurrence relation

$$M_{n,k} = M_{n-1,k-1} + M_{n-1,k} + M_{n-1,k+1}$$

with $M_{0,0} = 1$ and $M_{n,k} = 0$ unless $0 \leq k \leq n$. This array is called the Motzkin triangle and $M_{n,0}$ is the Motzkin number. Assume that $x_0(q) = 1$ and $x_n(q) = 2^{n-1}q^n$ for $n \geq 1$. It
is obvious that \( \{ x_n(q) \} \) is strongly \( q \)-log-convex. Let \( y_n = \sum_{k \geq 0} M_{n,k} x_k(q) \) for \( n \geq 0 \). It is easy to get
\[
y_3y_1 - y_2^2 = q - q^2 + 2q^3,
\]
which is not \( q \)-nonnegative. Thus the transformation \( y_n = \sum_{k \geq 0} M_{n,k} x_k(q) \) does not preserve the strong \( q \)-log-convexity.

If all \( f_k \), \( g_k \) and \( h_k \) in the \([5.1]\) are constants, then in \([18]\), we gave a result for the strong \( q \)-log-convexity of its row-generating functions. In fact, using Lemma \([21]\), similar to the proof of Theorem \([12]\) we can get the following generalized criterion, whose proof is omitted for brevity.

**Theorem 5.2.** Let \( \{ f_n \} \), \( \{ g_n \} \) and \( \{ h_n \} \) be nonnegative and increasing sequences, respectively. Define a triangular array \([ T_{n,k} ]_{n,k \geq 0} \) by
\[
T_{n,k} = f_k T_{n-1,k-1} + g_k T_{n-1,k} + h_k T_{n-1,k+1}
\]
for \( n \geq 1 \) and \( k \geq 0 \), where \( T_{0,0} = 1 \), \( T_{0,k} = T_{k,-1} = 0 \) for \( k > 0 \). If
\[
g_k g_{k+1} - h_k f_{k+1} \geq 0
\]
for all \( k \geq 0 \), then its row generating functions \( T_n(q) \) form a strongly \( q \)-log-convex sequence.

Let \([ T_{n,k} ]_{n,k} \) be a triangle. Assume that \( m \geq n \). For \( 0 \leq t \leq m+n \), define
\[
T_k(m,n,t) = T_{n-1,k} T_{m+1,t-k} + T_{m+1,k} T_{n-1,t-k} - T_{m,k} T_{n,t-k} - T_{n,k} T_{m,t-k}
\]
if \( 0 \leq k < t/2 \), and
\[
T_k(m,n,t) = T_{n-1,k} T_{m+1,k} - T_{n,k} T_{m,k}
\]
if \( t \) is even and \( k = t/2 \). In \([17]\) Theorem 2.1, we proved the next result.

**Theorem 5.3.** The transformation \( y_n = \sum_{k \geq 0} T_{n,k} x_k(q) \) preserves the strong \( q \)-log-convexity if the following two conditions hold:

(C1) Its row generating functions form a strongly \( q \)-log-convex sequence;

(C2) There exists an index \( r = r(m,n,t) \) such that \( T_k(m,n,t) \geq 0 \) for \( k \leq r \) and \( T_k(m,n,t) < 0 \) for \( k > r \).

Thus our Example \([5.1]\) also indicates that the C2 condition is necessary.
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