Comment on “Formation of primordial black holes by cosmic strings”

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Abstract

We show that in a pioneering paper by Polnarev and Zembowicz, some conclusions concerning the characteristics of the Turok-strings are generally not correct. In addition we show that the probability of string collapse given there, is off by a large prefactor ($\sim 10^3$).
In one of the pioneering and often cited papers on the probability of cosmic string collapse \cite{1}, Polnarev and Zembowicz analyzed the 2-parameter Turok-strings \cite{2}:

\[
X(\tau, \sigma) = \frac{A}{2} \left[ (1 - \alpha) \sin(\sigma - \tau) + \frac{\alpha}{3} \sin 3(\sigma - \tau) + \sin(\sigma + \tau) \right]
\]

\[
Y(\tau, \sigma) = \frac{A}{2} \left[ (1 - \alpha) \cos(\sigma - \tau) + \frac{\alpha}{3} \cos 3(\sigma - \tau) + (1 - 2\beta) \cos(\sigma + \tau) \right]
\]

\[
Z(\tau, \sigma) = \frac{A}{2} \left[ 2\sqrt{\alpha(1 - \alpha)} \cos(\sigma - \tau) + 2\sqrt{\beta(1 - \beta)} \cos(\sigma + \tau) \right] \quad (1)
\]

(We included a dimensionfull parameter \(A\) to keep \(\tau\) and \(\sigma\) dimensionless).

It was concluded \cite{1}, among other things, that:

- The strings have their minimal size \(R\) at
  \[
  \tau = \frac{\pi}{2} \quad (2)
  \]

- For generic parameters \((\alpha, \beta)\):
  \[
  \frac{R^2}{A^2} = \left( \sqrt{\alpha(1 - \alpha)} - \sqrt{\beta(1 - \beta)} \right)^2 + \left( \frac{\alpha}{3} - \beta \right)^2 \quad (3)
  \]

We now give two simple explicit examples showing that the two conclusions \cite{2}, \cite{1} cannot generally be correct.

**A.** Consider first the case \(\alpha = 1, \beta = 0\). Besides \(Z = 0\), this corresponds to:

\[
X(\tau, \sigma) = \frac{A}{2} \left[ \frac{1}{3} \sin 3(\sigma - \tau) + \sin(\sigma + \tau) \right]
\]

\[
Y(\tau, \sigma) = \frac{A}{2} \left[ \frac{1}{3} \cos 3(\sigma - \tau) + \cos(\sigma + \tau) \right] \quad (4)
\]

This is in fact a rigidly rotating string:

\[
\begin{pmatrix}
X(\tau, \sigma) \\
Y(\tau, \sigma)
\end{pmatrix} = \begin{pmatrix}
\cos(3\tau) & \sin(3\tau) \\
-\sin(3\tau) & \cos(3\tau)
\end{pmatrix} \begin{pmatrix}
X(0, \hat{\sigma}) \\
Y(0, \hat{\sigma})
\end{pmatrix} \quad (5)
\]
where $\tilde{\sigma} \equiv \sigma - 2\tau$. It follows that the minimal string size $R$ (the radius of the minimal sphere that can ever enclose the string completely) is independent of time. Thus it can be computed at any time, say $\tau = 0$:

$$R = \max_{\sigma \in [0, 2\pi]} \left[ \sqrt{X^2(0, \sigma) + Y^2(0, \sigma)} \right] = \frac{2A}{3} \tag{6}$$

Notice that the minimal sphere is found by maximization over $\sigma$. Thus the result (3) is not correct in this case. In fact, it gives the minimal distance from origo to the string (namely $A/3$), but to completely enclose the string, one needs a sphere with radius corresponding to the maximal distance (namely $2A/3$).

**B.** Now consider the case $\alpha = 1/2$, $\beta = 1$. Let us consider the distance from origo to the string as a function of $\sigma$ at two different times, namely $\tau = 0$ and $\tau = \pi/2$. It is straightforward to show that

$$\max_{\sigma \in [0, 2\pi]} \left[ \sqrt{X^2(\pi/2, \sigma) + Y^2(\pi/2, \sigma) + Z^2(\pi/2, \sigma)} \right] < \max_{\sigma \in [0, 2\pi]} \left[ \sqrt{X^2(0, \sigma) + Y^2(0, \sigma) + Z^2(0, \sigma)} \right] \tag{7}$$

Thus the string does not have its minimal size at $\tau = \pi/2$; at $\tau = 0$ it can be enclosed in a much smaller sphere. More precisely, at $\tau = 0$, the string can be enclosed in a sphere of radius $\sqrt{155/288}A$ while at $\tau = \pi/2$, a sphere of radius $\sqrt{17/18}A$ is needed. Therefore, the result (2) is not correct in this case.

On the other hand, for some other particular examples, it seemed that the conclusions (2)-(3) were indeed correct. Thus to clarify the situation, we did a complete re-analysis of the problem (see [4] for the details) using both analytical and numerical methods. This led to a precise classification of the Turok-strings, and a subsequent subdivision into 3 different families (see Fig. 1):

**I.** These strings have their minimal size at $\tau = \pi/2$. That is, starting from their original size at $\tau = 0$, they generally contract to their minimal size at $\tau = \pi/2$, and then generally expand back to their original size at $\tau = \pi$.  


II. These strings start from their minimal size at \( \tau = 0 \). Then they generally expand towards their maximal size and then recontract towards their minimal size at \( \tau = \pi \).

III. These strings have their minimal size at two values of \( \tau \) symmetrically around \( \pi/2 \). That is, they first generally contract and reach the minimal size at some \( \tau_0 \in [0; \pi/2] \). Then they expand for a while, and then recontract and reach the minimal size again at \( \tau = \pi - \tau_0 \). Then they expand again towards the original size at \( \tau = \pi \). In this family of strings, the value of \( \tau_0 \) depends on \((\alpha, \beta)\).

Then by comparison, we see that the conclusion (2) is correct in the region I of parameter-space, but incorrect in regions II and III.

As for the conclusion (3), let us restrict ourselves to the region I of parameter-space. This is the most relevant region for string collapse since it includes the circular string \((\alpha = \beta = 0)\), and string collapse is only to be expected for low angular momentum near-circular strings. In any case, in the region I, it is easy to derive the exact analytical expression for the minimal string size \([4]\):

\[
R^2 = \text{Max}\left(R^2_1, R^2_2\right) \tag{8}
\]

where

\[
\frac{R^2_1}{A^2} = \frac{4\alpha^2}{9} \tag{9}
\]

and

\[
\frac{R^2_2}{A^2} = \left(\sqrt{\alpha (1 - \alpha)} - \sqrt{\beta (1 - \beta)}\right)^2 + \left(\frac{\alpha}{3} - \beta\right)^2 \tag{10}
\]

Notice that Eq. (10) is precisely the result (3) of Polnarev and Zembowicz \([1]\). However, in Ref. [1], the other solution (9) was completely missed, and this is actually the relevant solution in Eq. (8) in approximately half of the parameter-space \((\alpha, \beta)\).

Finally, let us also compute the probability \( f \) of string collapse in the region I of parameter space:

\[
f = \int_{R \leq R_s} d\alpha d\beta \tag{11}
\]
where \( R_S = 4\pi AG\mu \) is the Schwarzschild radius of the string. Using Eqs. (8)-(11), and assuming that \( G\mu << 1 \) \([3]\), one finds \([4]\):

\[
 f = \frac{12\sqrt{6}}{5} (4\pi G\mu)^{\frac{5}{2}} \int_0^1 \frac{t^2 dt}{\sqrt{1 - t^4}} + \mathcal{O}\left((G\mu)^{\frac{7}{2}}\right) 
\]

\[
 = \frac{3\frac{7}{2}(4\pi)^4}{5 \Gamma^2 \left(\frac{1}{4}\right)} (G\mu)^{\frac{5}{2}} + \mathcal{O}\left((G\mu)^{\frac{7}{2}}\right)
\]

The result \((12)\) is a very good approximation for \( G\mu < 10^{-2} \), thus for any “realistic” cosmic strings we conclude:

\[
 f \approx 2 \cdot 10^3 \cdot (G\mu)^{\frac{5}{2}}
\]

Our result \((13)\) partly agrees with that of Ref. \([1]\) in the sense that \( f \propto (G\mu)^{5/2} \). However, we find that there is in addition a large numerical prefactor in the relation. This factor is of the order \( 10^3 \).

To conclude, simple explicit examples show that the conclusions of \([1]\) concerning the minimal string size of the Turok-strings are generally not correct. In this comment we re-analyzed the problem and performed a classification of the Turok-strings, to clarify the situation. We also computed the probability of string collapse again, and found that the original result \([1]\) is off by approximately 3 orders of magnitude.

References

[1] A. Polnarev and R. Zembowicz, Phys. Rev. D43, 1106 (1991).

[2] N. Turok, Nucl. Phys. B242, 520 (1984).

[3] A. Vilenkin and E.P.S. Shellard, “Cosmic Strings and other Topological Defects” (Cambridge University Press, 1994).

[4] R.N. Hansen, M. Christensen and A.L. Larsen, “Cosmic String Loops Collapsing to Black Holes”, gr-qc/9902048 (unpublished).
Figure 1: The considered strings fall into three families. The ones that reach their minimal size $R$ at $\tau = \pi/2$ (I), at $\tau = 0$, $\pi$ (II) and at $\tau = \tau_0$, $\pi - \tau_0$ for $\tau_0 \in ]0, \pi/2[$ (III).