The Baker-Akhiezer function and factorization of the Chebotarev-Khrapkov matrix

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Abstract

A new technique is proposed for the solution of the Riemann-Hilbert problem with the Chebotarev-Khrapkov matrix coefficient

\[ G(t) = \alpha_1(t)I + \alpha_2(t)Q(t), \quad \alpha_1(t), \alpha_2(t) \in H(L), \quad I = \text{diag}\{1,1\}, \quad Q(t) \text{ is a } 2 \times 2 \text{ zero-trace polynomial matrix.} \]

This problem has numerous applications in elasticity and diffraction theory. The main feature of the method is the removal of the essential singularities of the solution to the associated homogeneous scalar Riemann-Hilbert problem on the hyperelliptic surface of an algebraic function by means of the Baker-Akhiezer function. The consequent application of this function for the derivation of the general solution to the vector Riemann-Hilbert problem requires the finding of the \( \rho \) zeros of the Baker-Akhiezer function (\( \rho \) is the genus of the surface). These zeros are recovered through the solution to the associated Jacobi problem of inversion of abelian integrals or, equivalently, the determination of the zeros of the associated degree-\( \rho \) polynomial and solution of a certain linear algebraic system of \( \rho \) equations.

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1 Introduction

Many problems of elasticity \([19, 24, 5, 1, 2]\), electromagnetic diffraction \([12, 16, 9, 21, 7, 8, 3, 4]\), and acoustic diffraction \([18, 25, 6]\) require the solution of the vector Riemann-Hilbert problem (RHP) of the theory of analytic functions \([26]\)

\[ \Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in L, \]

where \( L \) is either the whole real axis, or a finite segment, when the matrix \( G(t) \) has the Chebotarev-Khrapkov (also known as Daniele-Khrapkov) structure \([10, 19, 12]\).

\[ G(t) = \alpha_0(t)I + \alpha_1(t)Q(t). \]  \hspace{1cm} (1.1)

Here, \( \alpha_0(t) \) and \( \alpha_1(t) \) are Hölder functions on \( L \), \( I = \text{diag}\{1,1\} \), and \( Q(t) \) is a \( 2 \times 2 \) zero-trace polynomial matrix. In the case \( n = \deg f_0(z) \leq 2 \) (\( \det Q(z) = h^2(z)f_0(z) \), and \( f_0(z) \) has simple zeros only) the problem was solved in \([19]\). For a particular case of the matrix \((1.1)\) and when \( n = 4 \), the exact solution was derived in \([12]\). For any finite \( n \), the vector problem is reduced \([23]\) to a scalar RHP on a hyperelliptic surface of genus \( \rho = [(n-1)/2] \). A theory of the RHP on compact Riemann surfaces and a constructive procedure for the solution of the associated Jacobi inversion problem was proposed in \([27]\). This technique was further developed and adjusted to specific needs of the RHPs on hyperelliptic surfaces arising in elasticity \([24, 5]\), diffraction theory in \([6, 7, 8]\) and for symmetric vector RHPs in \([3, 4]\). The method for the vector RHP with the coefficient \((1.1)\) in the elliptic and hyperelliptic cases first factorizes the coefficient of the associated scalar RHP using...
the Weierstrass analogue of the Cauchy kernel. In general, that solution has an essential singularity at the infinite points of the surface due to unavoidable poles of the Weierstrass kernel. The next step of the procedure, the removal of the essential singularities, leads to the classical problem of the inversion of abelian integrals and, eventually, to the finding of the zeros of a certain degree-$\rho$ polynomial.

The Baker-Akhiezer function plays an important role in the study of analytic properties of eigenfunctions of ordinary differential operators with periodic coefficients [13, 17, 15, 20, 14]. The representation of the Baker-Akhiezer function on a genus-$\rho$ hyperelliptic surface $R$

$$F(P) = e^{\Omega(P)}\frac{\theta(u_1(P) - \epsilon_1 + V_1^\circ, \ldots, u_\rho(P) - \epsilon_\rho + V_\rho^\circ)}{\theta(u_1(P) - \epsilon_1, \ldots, u_\rho(P) - \epsilon_\rho)}$$ (1.2)

that we employ for the solution of the Wiener-Hopf matrix factorization problem was first written by Its in context of the finite gap solutions of the KdV equation [22]. Here, $P \in \mathcal{R}$, $\Omega(P)$ is an abelian integral of the second kind with zero $A$-periods and a certain prescribed polynomial growth at the two infinite points of the surface $\mathcal{R}$, $\theta$ is the theta Riemann function, $u_1, \ldots, u_\rho$ form the canonical basis of abelian integrals of the first kind, $e_j = k_j + u_1(Q_1) + \ldots + u_j(Q_\rho)$, $Q_j$ are simple poles of the Baker-Akhiezer function, $k_j$ are the Riemann constants associated with the homology basis $a_j$, $b_j$, and $V_j^\circ = (2\pi i)^{-1}\int_{\gamma_j} d\Omega$, $j = 1, \ldots, \rho$.

The main goal of this paper was to develop a new factorization procedure for matrices of the form (1.1) based on the use of the Baker-Akhiezer function. In section 2 we state the vector RHP in the real axis with the matrix coefficient (1.1) and reduce it to a scalar RHP on a hyperelliptic surface $\mathcal{R}$ of the algebraic function $w^2 = f(z)$. We derive a particular solution, $\Psi_0(z, w)$, to the scalar RHP in section 3. This solution satisfies the boundary condition but has inadmissible essential singularities at the two infinite points $\infty_1$ and $\infty_2$ of the surface. In section 4 we construct the Baker-Akhiezer function (1.2) of the surface $\mathcal{R}$. This function is associated with an abelian integral of the second type with zero-$A$-periods used to remove the essential singularities and two Riemann $\theta$-functions which serve to make the solution continuous through the $B$-cross-sections. We find the Wiener-Hopf factors in terms of the functions $\Psi_0(z, w)$ and $F(P)$ in section 5. In section 6 we derive the general solution to the vector RHP.

2 Scalar RHP on a Riemann surface associated with the Chebotarev-Khrapkov matrix

Motivated by numerous applications in acoustics, electromagnetic theory, fluid mechanics and elasticity we assume that the Riemann-Hilbert contour, $L$, is the whole real axis which splits the plane of a complex variable $z$ into two half-planes, $D^+: \text{Im } z > 0$ and $D^-: \text{Im } z < 0$. Let $G(t)$ be a $2 \times 2$ matrix which is nonsingular in $L$ and whose structure is

$$G(t) = \begin{pmatrix} \alpha_1(t) + \alpha_2(t)l_0(t) & \alpha_2(t)l_1(t) \\ \alpha_2(t)l_2(t) & \alpha_1(t) - \alpha_2(t)l_0(t) \end{pmatrix},$$ (2.1)

where $\alpha_1(t), \alpha_2(t) \in H(L)$ (Hölder functions), $l_0(t), l_1(t)$ and $l_2(t)$ are polynomials. Assume that $g(t)$ is an order-2 Hölder vector-function on $L$. Consider the following RHP.

Find two vectors, $\Phi^+(z)$ and $\Phi^-(z)$, analytic in the domains $D^+$ and $D^-$, respectively, bounded at infinity, $H$-continuous up to the contour $L$ and satisfying the boundary condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in L.$$ (2.2)
Denote \( f_0(z) = l_0^2(z) + l_1(z)l_2(z) \) and assume that \( \deg l_j(z) \leq \deg f_0(z), \, j = 0, 1, 2 \). Let \( f_0(z) = h^2(z)f(z) \), where \( h(z) \) and \( f(z) \) are polynomials, and \( f(z) \) has simple zeros \( r_1, r_2, \ldots, r_n \) only. For the sake of simplicity, it is assumed that none of them lie in the contour \( L \) (we refer to \[1\] in the case when some or all zeros lie in the contour). Let \( \rho = [(n - 1)/2] \), where \([x]\) is the integer part of \( x \). Choose a single branch of \( f^{1/2}(z) \) in the plane cut along simple smooth curves \( \gamma_1 = r_1r_2, \gamma_2 = r_3r_4, \ldots, \gamma_{\rho+1} = r_{2\rho+1}r_{2\rho+2} \) (if \( n = 2\rho + 1 \), then it is set \( r_{2\rho+2} = \infty \), and \( \gamma_{\rho+1} \) is the semi-infinite segment \( \arg z = \phi_{2\rho+1} \in (0, \pi) \) joining the points \( r_{2\rho+1} \) and \( \infty, \phi_{2\rho+1} = \arg r_{2\rho+2} \)). The branch is fixed by the condition \( g^{1/2}(z) \sim \varepsilon_0z^{\rho+1}, \, z \to \infty, \varepsilon_0 = \text{const}, \) if \( r_{2\rho+2} \) is finite. In the case \( r_{2\rho+2} = \infty, \, f^{1/2}(z) \sim \varepsilon_0z^{\rho+1/2}, \, z \to \infty, \arg z \in (\phi_{2\rho+1}, 2\pi + \phi_{2\rho+1}) \).

The functions \( \alpha_1(t) + \alpha_2(t)h(t)\sqrt{f(t)} \) and \( \alpha_1(t) - \alpha_2(t)h(t)\sqrt{f(t)} \) are the eigenvalues of the matrix \( G(t) \), and their product \( \delta(t) = \alpha_1^2(t) - \alpha_2^2(t)h^2(t)f(t) \) is the determinant of \( G(t) \). Fix a branch \( \sqrt{\delta(t)} \) of the two-valued function \( \delta^{1/2}(t) \). Since \( \det G(t) \neq 0 \) on \( L \), the function \( \delta^{1/2}(t) \) does not have branch points on the contour \( L \). It will be convenient to split the matrix \( G(t) \) as

\[
G(t) = \sqrt{\delta(t)}G_0(t), \quad G_0(t) = \begin{pmatrix} a_1(t) + a_2(t)l_0(t) & a_2(t)l_1(t) \\ a_2(t)l_2(t) & a_1(t) - a_2(t)l_0(t) \end{pmatrix},
\]

where \( a_j(t) = \alpha_j(t)/\sqrt{\delta(t)}, \, j = 1, 2 \). It is directly verified that \( \det G_0(t) = 1 \), and

\[
\lambda_1(t) = a_1(t) + a_2(t)h(t)\sqrt{f(t)}, \quad \lambda_2(t) = a_1(t) - a_2(t)h(t)\sqrt{f(t)} = \frac{1}{\lambda_1(t)}
\]

are the eigenvalues of the matrix \( G_0(t) \). To pursue the Wiener-Hopf factorization of \( G_0(t) \), we split it as

\[
G_0(t) = T(t)\Lambda(t)[T(t)]^{-1},
\]

where \( \Lambda(t) = \text{diag}\{\lambda_1(t), \lambda_2(t)\} \),

\[
T(t) = \begin{pmatrix} 1 \\ -[l_0(t) - h(t)\sqrt{f(t)}/l_1(t)] \end{pmatrix} / \left( 1 - [l_0(t) + h(t)\sqrt{f(t)}/l_1(t)] \right),
\]

and reduce the problem of matrix factorization to a scalar RHP on a Riemann surface (Moiseev, 1989). First we introduce two new vectors \( \psi(z) = (\psi_1(z), \psi_2(z)) \) and \( \mathbf{g}^{\circ}(t) = (g_1^\circ(t), g_2^\circ(t)) \)

\[
\psi(z) = [T(z)]^{-1}\Phi(z), \quad \mathbf{g}^{\circ}(t) = [T(t)]^{-1}\mathbf{g}(t).
\]

Due to continuity of the vector \( \Phi(z) \) through the branch cuts \( \gamma_j \) \( (j = 1, 2, \ldots, \rho + 1) \), we have \( T^+(t)\psi^+(t) = T^-(t)\psi^-(t), \, t \in \gamma_j \). This implies that the components of the vector \( \psi(z) \) satisfy the following Riemann-Hilbert boundary conditions:

\[
\psi_1^+(t) = \psi_2^-(t), \quad \psi_2^+(t) = \psi_1^-(t), \quad t \in \gamma_j, \quad j = 1, 2, \ldots, \rho + 1,
\]

\[
\psi_j^+(t) = \lambda_j(t)\psi_j^-(t) + g_j^\circ(t), \quad t \in L, \quad j = 1, 2.
\]

We wish to reformulate (2.5) as a scalar RHP on a Riemann surface. Let \( \mathcal{R} \) be the two-sheeted Riemann surface of the algebraic function \( w^2 = f(z) \) formed by gluing two copies, \( \mathbb{C}_1 \) and \( \mathbb{C}_2 \), of the extended complex plane \( \mathbb{C} \cup \infty \) along the cuts \( \gamma_j \) \( (j = 1, 2, \ldots, \rho + 1) \) such that the function

\[
w = \begin{cases} \sqrt{f(z)}, & z \in \mathbb{C}_1 \\ -\sqrt{f(z)}, & z \in \mathbb{C}_2 \end{cases}
\]

is a single-valued function on the surface \( \mathcal{R} \). Here, \( \sqrt{f(z)} \) is the branch chosen before. Let \( a_j, b_j \) \( (j = 1, 2, \ldots, \rho) \) be a homology basis of the genus-\( \rho \) surface \( \mathcal{R} \) (Figure 1). Denote \( \mathcal{L} = L_1 \cup L_2 \) the
contour on the surface $R$ with $L_j \subset \mathbb{C}_j$ ($j = 1, 2$) being two copies of the contour $L$. With each pair of functions $(\psi_1, \psi_2)$, $(\lambda_1, \lambda_2)$ and $(g_1^0, g_2^0)$ we associate the following functions on the surface $R$:

$$\Psi(z, w) = \psi_j(z), \quad (z, w) \in \mathbb{C}_j,$$

$$\lambda(t, \xi) = \lambda_j(t), \quad g^*(t, \xi) = g_j^0(t), \quad (t, \xi) \in L_j, \quad j = 1, 2, \quad \xi = w(t). \quad (2.10)$$

We assert that the function $\Psi(z, w)$ is continuous through the contours $\gamma_j$ ($j = 1, 2, \ldots, \rho + 1$), and therefore the vector RHP (2.2) on the plane is equivalent to the following scalar RHP on the surface $R$:

Find a piece-wise analytic function $\Psi(z, w)$ with the discontinuity contour $L$, bounded at the two infinite points $\infty_j = (\infty, (-1)^{j-1} \infty)$ ($j = 1, 2$) of the surface $R$, $H$-continuous up to the contour $L$ and satisfying the boundary condition

$$\Psi^+(t, \xi) = \lambda(t, \xi) \Psi^-(t, \xi) + g^*(t, \xi), \quad (t, \xi) \in L. \quad (2.11)$$

3 Solution with an essential singularity at the infinite points

We begin with factorization of the function $\lambda(t, \xi)$. For an analogue of the Cauchy kernel we choose the Weierstrass kernel

$$dW = \frac{w + \xi}{2\xi} \frac{dt}{t - z} \quad (3.1)$$

and analyze the integral

$$\frac{1}{2\pi i} \int_L \log \lambda(t, \xi) dW = \frac{1}{4\pi i} \int_L [\log \lambda_1(t) + \log \lambda_2(t)] \frac{dt}{t - z}$$

$$+ \frac{w}{4\pi i} \int_L [\log \lambda_1(t) - \log \lambda_2(t)] \frac{dt}{\sqrt{f(t)(t - z)}}. \quad (3.2)$$

Pick a point on $L$, $z_0$, and treat it as the starting point $z_0^+$ of the contour $L$ (it is convenient to take $z_0 = 0$). Let

$$\kappa_1 = \text{ind} \lambda_1(t) = \frac{1}{2\pi} [\arg \lambda_1(t)]|_L, \quad (3.3)$$

where $[F(t)]|_L$ is the increment of the function $F(t)$ as $t$ traverses the contour $L$ in the positive direction with $z_0^+$ being the starting point. Because of the continuity $\kappa_1$ is an integer. Recall that $\lambda_2(t) = [\lambda_1(t)]^{-1}$ and fix branches of the logarithmic functions $\log \lambda_1(t)$ and $\log \lambda_2(t)$ by

$$\arg \lambda_1(z_0^+) = \phi, \quad \arg \lambda_2(z_0^+) = -\phi, \quad 0 \leq \phi < 2\pi. \quad (3.4)$$
Then at the terminal point $z_0$ of the contour $L$ (to distinguish the terminal and starting points we denote the former point as $z_0^-$)

$$\arg \lambda_1(z^-_0) = \phi + 2\pi \kappa_1, \quad \arg \lambda_2(z^-_0) = -\phi - 2\pi \kappa_1,$$
(3.5)

and therefore

$$\log \lambda_1(t) + \log \lambda_2(t) = 0, \quad \log \lambda_1(t) - \log \lambda_2(t) = 2\log \lambda_1(t), \quad t \in L.$$ (3.6)

In these circumstances (3.2) implies that

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \log \lambda(t, \xi) dW = \frac{w}{2\pi i} \int_{\mathcal{L}} \log \lambda_1(t) \frac{dt}{\sqrt{f(t)(t - z)}},$$
(3.7)

Since the function $\lambda_1(t)$ is discontinuous at the point $z_0$, the integral in (3.7) has a logarithmic singularity

$$\frac{1}{2\pi i} \int_{\mathcal{L}} \log \lambda(t, \xi) dW \sim (-1)^{m-1} \kappa_1 \log(z - z_0), \quad z \to z_0, \quad (z, w) \in \mathbb{C}_m, \quad m = 1, 2.$$ (3.8)

It is an easy matter to move the singularity from the contour to the surface $\mathcal{R} \setminus \mathcal{L}$ by adding the extra term

$$I(z, w) = \text{sgn} \kappa_1 \sum_{m=1}^2 (-1)^{m-1} \sum_{j=1}^{\lvert \kappa_1 \rvert} \int_{q_{m0}}^{q_{mj}} dW,$$
(3.9)

(\exp\{I(z, w)\} is continuous through the contour $\mathcal{L}$). Here, $q_{mj} = (z_j, (-1)^{m-1} \sqrt{f(z_j)}) \in \mathbb{C}_m \setminus L_m$, $j = 1, 2, \ldots, \lvert \kappa_1 \rvert$, $q_{m0} = (z_0, (-1)^{m-1} \sqrt{f(z_0)}) \in L_m$, $m = 1, 2$. We can transform (3.9) as

$$I(z, w) = w \text{sgn} \kappa_1 \sum_{j=1}^{\lvert \kappa_1 \rvert} \int_{z_0}^{z_j} \frac{dt}{\sqrt{f(t)(t - z)}}$$
(3.10)

It is seen that the function $I(z, w)$ has the logarithmic singularities at the points $(z_0, \pm \sqrt{f(z_0)}) \in \mathcal{L}$, $I(z, w) \sim (-1)^{m} \kappa_1 \log(z - z_0)$, $(z, w) \in \mathbb{C}_m$, and also extra logarithmic singularities at the internal points $q_{m1}, q_{m2}, \ldots, q_{m\lvert \kappa_1 \rvert} \in \mathbb{C}_m \setminus L_m$, $m = 1, 2$. At the same time, the sum of the integral (3.7) and $I(z, w)$ does not have this singularity.

Now, to factorize the function $\lambda(t, \xi)$, we consider the function

$$\Psi_0(z, w) = e^{w\beta(z)},$$
(3.11)

where

$$\beta(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \log \lambda_1(t) \frac{dt}{\sqrt{f(t)(t - z)}} + \text{sgn} \kappa_1 \sum_{j=1}^{\lvert \kappa_1 \rvert} \int_{z_0}^{z_j} \frac{dt}{\sqrt{f(t)(t - z)}}.$$ (3.12)

The function $\Psi_0(z, w)$ satisfies the homogeneous boundary condition

$$\Psi_0^+(t, \xi) = \lambda(t, \xi) \Psi_0(t, \xi), \quad (t, \xi) \in \mathcal{L},$$
(3.13)

but has inadmissible essential singularities at the points $\infty_1$ and $\infty_2$. Also, if $\kappa_1 > 0$, then it has simple zeros $z_j$ on $\mathbb{C}_1$ and simple poles $z_j$ on $\mathbb{C}_2$. If $\kappa_1 < 0$, then $z_j$ are simple poles on $\mathbb{C}_1$ and simple zeros on $\mathbb{C}_2$ ($j = 1, 2, \ldots, \lvert \kappa_1 \rvert$).
4 Baker-Akhiezer function

Our aim is to quench the essential singularities at the infinite points of the function $\Psi_0(z, w)$ by employing the Baker-Akhiezer function, $F(z, w)$, on the genus-$\rho$ surface $R$ associated with the function $\Psi_0(z, w)$. The function $F(z, w)$ has to satisfy the following two conditions:

(i) it is meromorphic everywhere on $R$ except at the points $\infty_1$ and $\infty_2$,

(ii) the function $\Psi_0(z, w)F(z, w)$ is bounded at the points $\infty_1$ and $\infty_2$.

First, we study the behavior of the function $\Psi_0(z, w)$ at the infinite points. Let $\nu = s + \rho + 1$ ($s = -1/2$ if $r_{2\rho+2} = \infty$ and $s = 0$ if $r_{2\rho+2}$ is finite). For the branch $\sqrt{f(z)}$ chosen

$$\sqrt{f(z)} = \varepsilon_0 \prod_{j=1}^{2\nu} (z - r_j) = z^\nu \sum_{m=0}^{\infty} c_m z^{-m}, \quad (4.1)$$

(in the case $r_{2\rho+2} = \infty$, arg $z \in (\phi_{2\rho+1}, 2\pi + \phi_{2\rho+1})$). Here,

$$c_0 = \varepsilon_0, \quad c_1 = \varepsilon_0 \frac{(-1/2)_1}{1!} \sum_{j=1}^{2\nu} r_j,$$

$$c_2 = \varepsilon_0 \left[ \frac{(-1/2)_2}{2!} \sum_{j=1}^{2\nu} r_j^2 + \frac{(-1/2)_1^2}{(1!)^2} \sum_{j=1}^{2\nu} r_j \sum_{m=1, m \neq j}^{2\nu} r_m \right], \quad c_3 = \varepsilon_0 \left[ \frac{(-1/2)_3}{3!} \sum_{j=1}^{2\nu} r_j^3 + \frac{(-1/2)_1(-1/2)_2}{1!2!} \sum_{j=1}^{2\nu} r_j^2 \sum_{m=1, m \neq j}^{2\nu} r_m \right], \ldots, \quad (4.2)$$

where $(a)_m = a(a + 1) \ldots (a + m - 1)$. By virtue of \textbf{(3.12)}

$$\beta(z) = \sum_{j=0}^{\infty} \frac{\tilde{c}_j}{z^{j+1}}, \quad (4.3)$$

where

$$\tilde{c}_j = -\frac{1}{2\pi i} \int_L \log \lambda_1(t) \frac{t^j dt}{\sqrt{f(t)}} - \text{sgn} \kappa_1 \sum_{m=1}^{\kappa_1} \int_{z_0}^{z_m} \frac{t^j dt}{\sqrt{f(t)}}, \quad (4.4)$$

and therefore, as $z \to \infty$,

$$\sqrt{f(z)}\beta(z) = z^{s+\rho} \sum_{m=0}^{\infty} \frac{d_m}{z^m}, \quad d_m = \varepsilon_0 \sum_{k=0}^{m} c_k \tilde{c}_{m-k}. \quad (4.5)$$

This brings us to the expansion of the function $\Psi_0(z, w)$ at the infinite points

$$\Psi_0(z, w) = \exp\{(-1)^{j-1}[z^sM(z) + O(1)]\}, \quad (z, w) \to \infty_j, \quad (z, w) \in C_j, \quad j = 1, 2, \quad (4.6)$$

where

$$M(z) = d_0 z^\rho + d_1 z^{\rho-1} + \ldots + d_{\rho-1} z,$$

and arg $z \in (\phi_{2\rho+1}, 2\pi + \phi_{\rho+1})$ if $r_{2\rho+2} = \infty$.

Our next step is to construct a special abelian integral of the second kind,

$$\Omega(P) = \int_{P_0}^{P} d\Omega, \quad P_0 = (r_1, 0), \quad P = (z, w). \quad (4.8)$$
Determine $\Omega(P)$ by the following properties:

(a) $\Omega(P) \sim (-1)^j z^s M(z)$, $P \to \infty_j \in \mathbb{C}_j$, $j = 1, 2$,

(b) $\int_{a_j} d\Omega = 0$, $j = 1, 2, \ldots, \rho$.

We seek the abelian differential $d\Omega$ in the form

$$d\Omega = \frac{e_0 z^{2\rho} + e_1 z^{2\rho-2} + \ldots + e_2 \rho}{w} dz,$$  \hspace{1cm} (4.9)

where the coefficients $e_j$ are to be determined. We wish to exploit this formula in order to study the behavior of the integral $\Omega(P)$ at the infinite points. Because of (4.11) we have

$$d\Omega = (-1)^{j-1} \left( \tilde{e}_0 z^{\rho-1} + \tilde{e}_1 z^{\rho-2} + \ldots + \tilde{e}_{\rho-1} z + \tilde{e}_\rho \log z - \frac{\tilde{e}_{\rho+1}}{z} - \ldots \right) dz,$$  \hspace{1cm} (4.10)

where $\tilde{e}_m$ are defined recursively by

$$\tilde{e}_m = \frac{-1}{\varepsilon_0} \tilde{e}_{m-k} c_k + \frac{e_m}{\varepsilon_0}, \hspace{0.5cm} m = 0, 1, \ldots, \rho.$$  \hspace{1cm} (4.11)

By integrating (4.10) we determine the asymptotic expansion of the abelian integral $\Omega(P)$. If $s = 0$ ($r_{2\rho+2}$ is finite), then

$$\Omega(P) = (-1)^{j-1} \left( \tilde{e}_0 z^{\rho-1} + \tilde{e}_1 z^{\rho-2} + \ldots + \tilde{e}_{\rho-1} z + \tilde{e}_\rho \log z - \frac{\tilde{e}_{\rho+1}}{z} - \ldots \right) + C,$$  \hspace{1cm} (4.12)

where $C$ is a constant. Similarly, in the case $s = -1/2$ ($r_{2\rho+2} = \infty$), we have

$$\Omega(P) = (-1)^{j-1} \left( \tilde{e}_0 z^{\rho+1/2} + \tilde{e}_1 z^{\rho-1/2} + \ldots + \frac{\tilde{e}_{\rho+1} z^{-1/2}}{1/2} - \frac{\tilde{e}_{\rho+1} z^{1/2}}{1/2} - \ldots \right) + C,$$  \hspace{1cm} (4.13)

where $C$ is a constant. Similarly, in the case $s = -1/2$ ($r_{2\rho+2} = \infty$), we have

$$\Omega(P) = (-1)^{j-1} \left( \tilde{e}_0 z^{\rho+1/2} + \tilde{e}_1 z^{\rho-1/2} + \ldots + \frac{\tilde{e}_{\rho+1} z^{-1/2}}{1/2} - \frac{\tilde{e}_{\rho+1} z^{1/2}}{1/2} - \ldots \right) + C,$$  \hspace{1cm} (4.14)

On satifying the property (a) of the integral $\Omega(P)$ we find the coefficients $\tilde{e}_0, \ldots, \tilde{e}_\rho$. If $s = 0$, then

$$\tilde{e}_0 = -\rho d_0, \hspace{0.5cm} \tilde{e}_1 = -(\rho - 1) d_1, \hspace{0.5cm} \tilde{e}_2 = -(\rho - 2) d_2, \ldots, \tilde{e}_{\rho-1} = -d_{\rho-1}, \hspace{0.5cm} \tilde{e}_\rho = 0.$$  \hspace{1cm} (4.15)

In the case $s = -1/2$, these coefficients become

$$\tilde{e}_0 = 0, \hspace{0.5cm} \tilde{e}_1 = -\left( \rho - \frac{1}{2} \right) d_0, \hspace{0.5cm} \tilde{e}_2 = -\left( \rho - \frac{3}{2} \right) d_1, \ldots, \tilde{e}_{\rho-1} = -\frac{3}{2} d_{\rho-2}, \hspace{0.5cm} \tilde{e}_\rho = -\frac{1}{2} d_{\rho-1}.$$  \hspace{1cm} (4.16)

Due to (4.11) we can express the coefficients $e_m$ ($m = 0, 1, \ldots, \rho$) through $\tilde{e}_m$

$$e_m = \varepsilon_0 \tilde{e}_m + \sum_{k=1}^{m} \tilde{e}_{m-k} c_k, \hspace{0.5cm} m = 0, 1, \ldots, \rho.$$  \hspace{1cm} (4.16)

The remaining coefficients $e_{\rho+1}, e_{\rho+2}, \ldots, e_{2\rho}$ in the representation (4.9) of the abelian differential

$$\sum_{m=\rho+1}^{2\rho} U_{jm} e_m = d_j, \hspace{1cm} j = 1, 2, \ldots, \rho,$$  \hspace{1cm} (4.17)
which follows from the property (b) of the integral $\Omega(P)$. Here,

$$d_j = -\sum_{m=0}^{\rho} U_{jm} e_m, \quad U_{jm} = \int_{a_j} z^{2\rho-m} w(z) dz. \quad (4.18)$$

This completes the construction of the abelian integral $\Omega(P)$.

It becomes evident that the product $\Psi_0(z, w) \exp\{\Omega(P)\}$ is bounded as $P \to \infty, j \in \mathbb{C}, j = 1, 2$. This function is continuous through the cross-sections $a_j$ of the surface $\mathcal{R}$ because of the zero $A$-periods and discontinuous through the cross-sections $b_j$ ($j = 1, 2, \ldots, \rho$) due to the non-zero $B$-periods of the integral $\Omega(P)$. Our efforts will now be directed towards annihilating the jumps $\exp\{V_m\}$,

$$V_m = \int_{b_m} d\Omega, \quad m = 1, 2, \ldots, \rho, \quad (4.19)$$

of the function $\exp\{\Omega(P)\}$ through the cross-sections $b_m$, $m = 1, 2, \ldots, \rho$.

Let $d\omega_j$ ($j = 1, 2, \ldots, \rho$) be the canonical basis of Abelian differentials of the first kind

$$d\omega_j = \frac{c_j^{(1)} z^{\rho-1} + c_j^{(2)} z^{\rho-2} + \ldots + c_j^{(\rho)}}{w} dz, \quad (4.20)$$

where the constants $c_j^{(k)}$ ($k, j = 1, 2, \ldots, \rho$) are chosen such that

$$\int_{a_k} d\omega_j = \delta_{jk}. \quad (4.21)$$

Denote the $B-$periods of the basis $d\omega_j$ by

$$B_{jk} = \int_{b_k} d\omega_j. \quad (4.22)$$

The matrix $B = (B_{jk})$ ($j, k = 1, 2, \ldots, \rho$) is symmetric and Im $B$ is positive definite. The principal tool we shall use to suppress the discontinuities of $\exp\{\Omega(P)\}$ is the Riemann $\theta$-function

$$\theta(s(P)) = \theta(s_1(P), s_2(P), \ldots, s_\rho(P)) \quad (4.23)$$

defined by

$$\theta(s(P)) = \sum_{m_1, \ldots, m_\rho = -\infty}^{\infty} \exp \left\{ \sum_{j=1}^{\rho} \sum_{k=1}^{\rho} B_{jk} m_j m_k + 2\pi i \sum_{j=1}^{\rho} m_j s_j(P) \right\}. \quad (4.24)$$

Because of the positive definiteness of the matrix Im $B$ the series converges for all $s(P)$. The $\theta$-function has periods $m = (m_1, m_2, \ldots, m_\rho)$, $m_j$ are integers, and quasiperiods $B_j = (B_{j1}, B_{j2}, \ldots, B_{j\rho})$, $j = 1, 2, \ldots, \rho$,

$$\theta(s_1 + m_1, \ldots, s_\rho + m_\rho) = \theta(s_1, \ldots, s_\rho),$$

$$\theta(s_1 + B_{j1}, \ldots, s_\rho + B_{j\rho}) = \exp\{-\pi i B_{jj} - 2\pi i s_j\} \theta(s_1, \ldots, s_\rho). \quad (4.25)$$

Introduce next the function

$$F_0(P) = \frac{\theta(u_1(P) - e_1 + V^\circ, \ldots, u_\rho(P) - e_\rho + V^\circ)}{\theta(u_1(P) - e_1, \ldots, u_\rho(P) - e_\rho)}. \quad (4.26)$$

Here, $V^\circ_j = (2\pi i)^{-1} V_j$ and $u_j(P)$ are the integrals

$$u_j(P) = \int_{P_0}^{P} d\omega_j, \quad j = 1, 2, \ldots, \rho, \quad (4.27)$$
which form the canonical basis of abelian integrals of the first kind. The numbers $e_j$ are chosen to be

$$e_j = \sum_{m=1}^{\rho} u_j(Q_m) + k_j,$$

where $Q_m$ $(m = 1, 2, \ldots, \rho)$ are some arbitrary distinct fixed points on $\mathcal{R}$ say, on $\mathbb{C}_1$, $Q_m = (\zeta_m, \sqrt{f(\zeta_m)})$, such that the $\theta$-functions in (4.26) are not identically zero. The parameters $k_j$ in (4.28) are the Riemann constants which, for the hyperelliptic surface $\mathcal{R}$ and for the homology basis chosen, can be taken as (see for example [8])

$$k_j = -\frac{j}{2} + \frac{1}{2} \sum_{m=1}^{\rho} B_{jm}.$$

The function $F_0(P)$ has $\rho$ simple poles $Q_1, Q_2, \ldots, Q_\rho$ [11, p. 303], $Q_j \in \mathbb{C}_1$, and $\rho$ simple zeros which may lie on either sheet of the surface. This function is continuous through the cross-sections $a_j$ and discontinuous through the cross-sections $b_j$, $j = 1, \ldots, \rho$. Due to (4.25) its jumps are $exp\{-V_j\}$. This implies that the function $F(z, w) = exp\{\Omega(P)\}F_0(P)$ is meromorphic on $\mathcal{R}$ (it is continuous through the loops $b_j$). The set of singularities of the function $F(z, w)$ comprises the two infinite points $\infty_1$ and $\infty_2$ and $\rho$ simple poles $Q_1, Q_2, \ldots, Q_\rho \in \mathbb{C}_1$ whose affixes are prescribed. Therefore,

$$F(P) = e^{\Omega(P)} \left( \frac{\theta(u_1(P) - e_1 + V^o_1, \ldots, u_\rho(P) - e_\rho + V^o_\rho)}{\theta(u_1(P) - e_1, \ldots, u_\rho(P) - e_\rho)} \right)$$

is the Baker-Akhiezer function of the surface $\mathcal{R}$ with the homology basis $a_j$, $b_j$ $(j = 1, \ldots, \rho)$ associated with the abelian integral $\Omega(P)$ and the poles $Q_1, \ldots, Q_\rho$.

## 5 Matrix factorization in terms of the Baker-Akhiezer function

We are interested in factorizing the matrix $G(t)$ in terms of the function $F(z, w)$. In other words, we wish to express two matrices $X^+(t)$ and $X^-(t)$ through the Baker-Akhiezer function such that

$$G(t) = X^+(t)[X^-(t)]^{-1}, \quad t \in L,$$

where $X(z) = X^\pm(z)$, $z \in D^\pm$, and $X^+(z)$ and $X^-(z)$ are analytic and nonsingular everywhere in $D^+$ and $D^-$, respectively, apart from at most a finite number of points where they may have poles or where $\det X(z) = 0$. The preliminary splitting (2.3) implies the need to factorize the function $\sqrt{\delta(t)}$ first. It is apparent that we are allowed considerable freedom in our selection of the factors associated with $\sqrt{\delta(t)}$. Let

$$\text{ind} \delta(t) = \text{ind} \det G(t) = 2\kappa_2 + \eta,$$

where $\kappa_2$ is an integer, and $\eta$ equals either 0, or 1. In the case $\eta = 1$, we consider the auxiliary function $\varphi_0(t) = \tanh \pi(t + iy)$, $0 < y < 1/2$. Its argument, arg $\varphi_0(t)$, has the increment $-\pi$ as $t$ traverses the contour $L$ in the positive direction. The function $\varphi_0(t)$ can be factorized in terms of the $\Gamma$-functions

$$\varphi_0(t) = \frac{\delta^-_1(t)}{\delta^+_1(t)}, \quad t \in L,$$

where

$$\delta^+_1(z) = \frac{\Gamma(-iz + y)}{\Gamma(1/2 - iz + y)}, \quad \delta^-_1(z) = \frac{i\Gamma(1/2 + iz - y)}{\Gamma(1 + iz - y)}$$

are functions analytic in $D^+$ and $D^-$, respectively. On using these functions we can write

$$\sqrt{\delta(t)} = \delta^+_1(t) \frac{\left( \frac{t - i}{t + i} \right)^{\kappa_2} \delta^+_1(t)}{\delta^-_1(t)}, \quad t \in L, \quad \eta = 0, 1.$$
where in the case $\eta = 0$ we set $\delta^{\pm}_0(t) = 1$, and
\[
\tilde{\delta}(t) = \frac{1}{\delta_0^{\pm}} \left( \frac{t - i}{t + i} \right)^{-\kappa_2} \tanh^\eta \pi(t + iy) \sqrt{\delta(t)},
\]
\[
\delta^{\pm}_\eta(t) = \lim_{|t| \to \infty} \tanh^\eta \pi(t + iy) \sqrt{\delta(t)}.
\] (5.6)

Thus, $\tilde{\delta}(t) \to 1$ as $t \to \pm \infty$, and $\text{ind} \tilde{\delta}(t) = 0$. These properties of $\tilde{\delta}(t)$ permit us to factorize it in terms of the Cauchy integrals
\[
\tilde{\delta}(t) = \frac{\tilde{\delta}^+(t)}{\tilde{\delta}^-(t)}, \quad t \in L,
\]
\[
\tilde{\delta}^{\pm}(z) = \exp \left[ \frac{1}{2\pi i} \int_L \log \tilde{\delta}(t) dt \right], \quad z \in D^{\pm},
\] (5.7)
and the factorization of the function $\sqrt{\delta(t)}$ is complete, $\sqrt{\delta(t)} = \Delta^+(t)[\Delta^-(t)]^{-1}, t \in L$, where
\[
\Delta^+(z) = \delta^{\pm}_\eta \tilde{\delta}^+(z) \delta^{\pm}_\eta(z), \quad z \in D^+, \quad \Delta^-(z) = \left( \frac{z - i}{z + i} \right)^{-\kappa_2} \tilde{\delta}^-(z) \delta^{\pm}_\eta(z), \quad z \in D^-.
\] (5.8)

Notice that the function $\Delta^+(z)$ is analytic and nonzero everywhere in $D^+$, while the function $\Delta^-(z)$ has a multiplicity $-\kappa_2$ pole at the point $z = -i$ if $\kappa_2 < 0$ and a zero at this point if $\kappa_2 > 0$.

Now we turn our attention to factorization of the matrix $G_0(t)$. Let $\chi(z, w)$ be a nontrivial solution to the following homogeneous RHP problem on the surface $\mathcal{R}$:

Find a piece-wise meromorphic function $\Psi(z, w)$ with the discontinuity contour $\mathcal{L}$, $H$-continuous up to the contour $\mathcal{L}$ except for a finite number of poles and satisfying the boundary condition
\[
\chi^+(t, \xi) = \lambda(t, \xi) \chi^-(t, \xi), \quad (t, \xi) \in \mathcal{L}.
\] (5.9)

Then the matrix of factorization $X_0(z)$ can be expressed exclusively through the function $\chi(z, w)$ and the matrix $Y(z, w)$ given by
\[
Y(z, w) = \frac{1}{2} \left[ I + \frac{1}{\mathcal{h}(z) w} Q(z) \right], \quad Q(z) = \begin{pmatrix} l_0(z) & l_1(z) \\ l_2(z) & -l_0(z) \end{pmatrix}, \quad I = \text{diag}\{1, 1\},
\] (5.10)
in the form [23], [6]
\[
X_0(z) = \chi(z, w) Y(z, w) + \chi(z, -w) Y(z, -w).
\] (5.11)

It is a simple matter to verify that
\[
[X_0(z)]^{-1} = \frac{Y(z, w)}{\chi(z, w)} + \frac{Y(z, -w)}{\chi(z, -w)}.
\] (5.12)

We assert that the function $\mathcal{F}(z, w) \exp\{w\beta(z)\}$ meets the boundary condition [5.9], and it is bounded at the infinite points $\infty_1$ and $\infty_2$ (the Baker-Akhiezer function $\mathcal{F}(z, w)$ annihilates the essential singularities of the function $\exp\{w\beta(z)\}$ at the infinite points). Thus, the function
\[
\chi(z, w) = \mathcal{F}(z, w) \exp\{w\beta(z)\}
\]
\[
= e^{w\beta(z) + \Omega(P)} \frac{\theta(u_1(P) - e_1 + V_\rho \ldots, u_\rho(P) - e_\rho + V_\rho)^\circ}{\theta(u_1(P) - e_1, \ldots, u_\rho(P) - e_\rho)}
\] (5.13)
is a meromorphic solution to the scalar RHP (5.9) on the surface $\mathcal{R}$. 

10
6 Vector RHP

6.1 General solution

The factorization of the matrix \( G(t) \)

\[
G(t) = \Delta^+(t) X^+_0(t)[\Delta^-(t)X^-_0(t)]^{-1}, \quad t \in L,
\]

enables us to rewrite the boundary condition (2.2) of the vector RHP as

\[
\frac{1}{\Delta^+(t)}[X^+_0(t)]^{-1}\Phi^+(t) - S^+(t) = \frac{1}{\Delta^-(t)}[X^-_0(t)]^{-1}\Phi^-(t) - S^-(t), \quad t \in L,
\]

where

\[
S(z) = \frac{1}{2\pi i} \int_L [X^+_0(t)]^{-1} g(t) \frac{dt}{\Delta^+(t)(t - z)}.
\]

By means of the continuity principle and the generalized Liouville theorem, the left and right hand-sides of equation (6.2) continue analytically each other through the contour \( L \) and equal a rational vector-function \( R(z) \) everywhere in the complex plane. In terms of \( R(z) \), the vector-functions \( \Phi^\pm(z) \)

admit the representation

\[
\Phi^\pm(z) = \Delta^\pm(z) X^+_0(z)[S^\pm(z) + R(z)], \quad z \in D^\pm.
\]

Analysis of equation (6.2) shows that the rational vector-function \( R(z) \)

(i) is bounded at the infinite point,

(ii) has a pole of order \( \kappa_2 \) at the point \( z = -i \) if \( \kappa_2 > 0 \) and is bounded otherwise,

(iii) has simple poles at the affixes say, \( t_j \), of the zeros \( P_j = (t_j, w(t_j)) \in R, \) \( (j = 1, 2, \ldots, \rho) \) of the Baker-Akhiezer function \( F(z, w) \) (these zeros are to be determined),

(iv) has simple poles at the affixes \( z_j \) of the points \( q_{mj} \in C_m \) \( (m = 1, 2, j = 1, 2, \ldots, |\kappa_1|) \) if \( \kappa_1 \neq 0 \) and is bounded otherwise.

Note that if \( \eta = 1 \) and \( z \to \infty \), then \( \delta^+_j(z) \sim (-iz)^{-1/2}, z \in D^+, \) and \( \delta^-_j(z) \sim i(iz)^{-1/2}, z \in D^- \).

Therefore, \( \Phi^\pm(z) = O(z^{-1/2}), z \to \infty, z \in D^\pm \). Otherwise, if \( \eta = 0 \), then the vectors \( \Phi^\pm(z) \) are bounded at infinity.

In view of these properties, the most general form of the vector \( R(z) \) is

\[
R(z) = r(z) + \sum_{j=1}^{\rho} \frac{D_j}{z - t_j} + \sum_{j=1}^{\kappa_1} \frac{E_j}{z - z_j}.
\]

Here, \( D_j \) and \( E_j \) \( (j = 1, \ldots, \rho) \) are constant order-2 vectors. If \( \kappa_2 > 0 \), then \( r(z) \) is an order-2 rational vectors whose components, \( r_1(z) \) and \( r_2(z) \), are

\[
r_m(z) = \frac{1}{(z + i)^{\kappa_2}} \sum_{j=0}^{\kappa_2} C_{mj} z^j,
\]

where \( C_{mj} \) are arbitrary constants. If \( \kappa_2 \leq 0 \), then \( r(z) \) is a constant vector \( C_0 = (C_{01}, C_{02}) \). In the case \( \kappa_2 < 0 \) the vector \( \Phi^-(z) \) has an inadmissible pole of order \( -\kappa_2 \) at the point \( z = -i \). To quench this pole, we require that the following \(-2\kappa_2 \) conditions are satisfied:

\[
C_0 \delta_{m0} + S^-_m - \sum_{j=1}^{\rho} \frac{D_j}{t_j + i} - \sum_{j=1}^{\kappa_1} \frac{E_j}{z_j + i} = 0, \quad m = 0, 1, \ldots, -\kappa_2 - 1.
\]

Here, \( \delta_{m0} = 1 \) if \( m = 0 \) and 0 otherwise, and

\[
S^-_m = \frac{1}{2\pi i} \int_L [X^+_0(t)]^{-1} g(t) \frac{(t + i)^{-m-1} dt}{\Delta^+(t)}, \quad m = 0, 1, \ldots, -\kappa_2 - 1.
\]
6.2 Additional conditions

At the affixes \( z = t_m \) of the zeros \( P_m \) of the Baker-Akhiezer function the solution has inadmissible simple poles. Since rank \( Y(z,w) = 1 \), the vector \([X(z)]^{-1} \Phi(z)\) has the following representation in a neighborhood of the points \( z = t_m \):

\[
[X(z)]^{-1} \Phi(z) = \frac{\hat{C}_m}{z - t_m} \left( \begin{array}{c} 1 \\ \tau_m \end{array} \right), \quad m = 1, \ldots, \rho, \tag{6.9}
\]

where \( \hat{C}_m \) are nonzero constants,

\[
\tau_m = \frac{\ell_2(t_m)}{\ell_0(t_m) + (-1)^j \sqrt{f(t_m)}/h(t_m)}, \quad P_m \in \mathbb{C}_j, \quad m = 1, \ldots, \rho, \quad j = 1, 2. \tag{6.10}
\]

This implies that the components of the vectors \( D_m \) are connected by the conditions

\[
D_{2m} = \tau_mD_{1m}, \quad m = 1, \ldots, \rho. \tag{6.11}
\]

We recall that if \( \kappa_1 > 0 \), then the function \( \chi(z,w) \) has simple zeros \( z_m \) \( (m = 1, 2, \ldots, |\kappa_1|) \) on \( \mathbb{C}_1 \) and simple poles \( z_m \) on \( \mathbb{C}_2 \). It is bounded and nonzero at these points if \( \kappa_1 = 0 \). Otherwise, if \( \kappa_1 < 0 \), then \( z_m \) are simple poles of \( \chi(z,w) \) on \( \mathbb{C}_1 \) and simple zeros on \( \mathbb{C}_2 \). In general, the solution given by (6.4) has order-2 poles at the points \( z = z_m \). On employing the fact that the rank of \( Y(z,w) \) at the points \( (z_j, \pm \sqrt{f(z_j)}) \) is equal to 1 we establish that if

\[
E_{2m} = \sigma_jE_{1m}, \quad m = 1, \ldots, |\kappa_1|, \tag{6.12}
\]

then the points \( z = z_m \) become simple poles. Here,

\[
\sigma_m = \frac{\ell_2(z_j)}{\ell_0(z_m) + \text{sgn} \kappa_1 \sqrt{f(z_m)}h(z_m)}, \quad m = 1, \ldots, |\kappa_1|. \tag{6.13}
\]

To remove these poles, we expand the matrix \( Y(z,w) \) in a neighborhood of the points \( (z_m, \pm \sqrt{f(z_m)}) \)

\[
Y(z, \pm \sqrt{f(z)}) \sim Y_{\pm}^z + (z - z_m) \tilde{Y}_{\pm}^z, \quad z \to z_m, \tag{6.14}
\]

where

\[
Y_{\pm}^z = Y(z_m, \pm \sqrt{f(z_m)}), \quad \tilde{Y}_{\pm}^z = \pm \frac{1}{2} \frac{d}{dz} \frac{Q(z)}{\sqrt{f(z)}h(z)} \bigg|_{z=z_m}. \tag{6.15}
\]

There is no problem now to show that the points \( z = z_m \) are removable singularities of the vectors \( \Phi(z) \) provided the free constants meet \( 2|\kappa_1| \) conditions. These conditions written in the vector form are

\[
Y_m R_m^\circ + \tilde{Y}_m E_m = 0, \quad m = 1, \ldots, |\kappa_1| \text{ if } \kappa_1 > 0, \tag{6.16}
\]

and

\[
Y_m^+ R_m^\circ + \tilde{Y}_m^+ E_m = 0, \quad m = 1, \ldots, |\kappa_1| \text{ if } \kappa_1 < 0, \tag{6.17}
\]

where \( R_m^\circ = S(z_m) + R(z_m) \). Indeed, by substituting the relations (6.15) into (6.14) and using the relations \( Y_\pm^z E_m = 0, \quad m = 1, \ldots, |\kappa_1|, \quad \pm \kappa_1 > 0, \) we eventually obtain (6.16) and (6.17).

Analysis of formula (6.4) indicates that because of the simple poles of the matrix \( X(z) \) at the points \( z = \zeta_j \) \( (j = 1, \ldots, \rho) \) (due to the poles of the Baker-Akhiezer function), the vector \( \Phi(z) = \Phi^\pm(z), \quad z \in D^\pm, \) has inadmissible poles at \( z = \zeta_j \). Note that on the contrary to the zeros of the Baker-Akhiezer function, its poles are known and they are chosen to lie in the first sheet of the surface. Since rank \( Y(z,w) = 1 \), to remove these singularities, it is necessary and sufficient that

\[
Y_{11}(Q_m)[R_1(\zeta_m) + S_1(\zeta_m)] + Y_{12}(Q_m)[R_2(\zeta_m) + S_2(\zeta_m)] = 0, \quad m = 1, 2, \ldots, \rho. \tag{6.18}
\]

Here, \( Q_m = (\zeta_m, \sqrt{f(\zeta_m)}), \quad Y_{11}, \quad Y_{12} \) form the first row of the matrix \( Y, \quad S = (S_1, S_2)^T \) and \( R = (R_1, R_2)^T \).
6.3 Zeros of the Baker-Akhiezer function

To complete the procedure presented we have to determine the points $P_j$ ($j = 1, \ldots, \rho$), the zeros of the Baker-Akhiezer function (4.30), or, equivalently, the zeros of the $\theta$-function (without loss of generality we may assume that it is not identically zero)

$$\theta(u_1(P) - e_1 + V^o_1, \ldots, u_\rho(P) - e_\rho + V^o_\rho).$$

(6.19)

The affixes of these zeros, $t_j$, are needed for the rational vector $R(s)$ given by (6.5), while the identification of the sheet in which they locate is required for the determination of the parameters $\tau_j$ in (6.10). By setting

$$e_m - V^o_m = \sum_{j=1}^{\rho} u_m(P_j) + k_m, \quad \text{(modulo the periods)} \quad m = 1, 2, \ldots, \rho,$$

(6.20)

we obtain that the points $P_j$ are the zeros of the function $F(P)$ indeed. The system (6.20) is the Jacobi problem of inversion of abelian integrals:

Find points $P_1, P_2, \ldots, P_\rho \in \mathbb{R}$ and integers $\mu_1, \mu_2, \ldots, \mu_\rho$ and $\nu_1, \nu_2, \ldots, \nu_\rho$ such that

$$\frac{\rho}{\rho - 1} \sum_{j=1}^{\rho} \int_{P_0}^{P_j} d\omega_m + \sum_{j=1}^{\rho} \nu_j B_{mj} + \mu_m = \hat{e}_m - k_m, \quad m = 1, 2, \ldots, \rho,$$

(6.21)

where $\hat{e}_m = e_m - V^o_m$.

This problem reduces [27] to the system of symmetric algebraic equations

$$t_1^m + t_2^m + \ldots + t_\rho^m = T_m, \quad m = 1, 2, \ldots, \rho,$$

(6.22)

where $T_m$ are known and given in terms of the residues at the infinite points [27] or the two zeros of the surface [8] of functions expressible in terms of the $\theta$-function. The system may be converted into the problem of determination of $\rho$ zeros of an associated order-$\rho$ polynomial. The integers $\nu_m$ are found by solving the linear system [8]

$$\sum_{j=1}^{\rho} \text{Im}(B_{mj} \nu_j) = \text{Im} v_m, \quad m = 1, 2, \ldots, \rho,$$

(6.23)

while the integers $\mu_m$ are defined by

$$\mu_m = \text{Re} v_m - \sum_{j=1}^{\rho} \text{Re}(B_{mj}) \nu_j$$

(6.24)

explicitly. Here,

$$v_m = \hat{e}_m - k_m - \sum_{j=1}^{\rho} u_m(P_j).$$

(6.25)

There are $2^\rho$ points on the surface $\rho$ which have affixes defined by the $\rho$ zeros of the polynomial associated with the system (6.22). However, there is one and only one set of points $P_1, P_2, \ldots, P_\rho$ which have the affixes $t_1, t_2, \ldots, t_\rho$, respectively, and in addition, lead to integer solutions $v_m, \mu_m$ ($m = 1, 2, \ldots, \rho$) of (6.23) and (6.24).

We now summarize the results.

**Theorem.** Let $G(t)$ be a nonsingular $2 \times 2$ matrix

$$G(t) = \begin{pmatrix} \alpha_1(t) + \alpha_2(t) l_0(t) & \alpha_2(t) l_1(t) \\ \alpha_2(t) l_2(t) & \alpha_1(t) - \alpha_2(t) l_0(t) \end{pmatrix},$$

(6.26)
where \( \alpha_1, \alpha_2 \in H(L) \), \( L \) is the real axis, and \( l_0, l_1, l_2 \) are polynomials. Denote the polynomial \( l_0^2 + l_1 l_2 \) by \( h^2 f \) (\( f(z) \) has simple zeros only) and the eigenvalues of \( G \) by \( \lambda_j = \alpha_1 - (-1)^j \alpha_2 h \sqrt{f} \), \( j = 1, 2 \). Let \( \kappa_1 \) and \( \kappa_2 \) be the integers defined by

\[
2\kappa_1 = \text{ind} \frac{\lambda_1(t)}{\lambda_2(t)}, \quad 2\kappa_2 + \eta = \text{ind} \lambda_1(t) \lambda_2(t),
\]

where \( \eta \in \{0, 1\} \). Then the solution to the vector RHP (2.2) in the class of bounded at infinity functions is given by (6.4). Furthermore, if the number of free constants in the solution and the number of equations for their determination are denoted by \( n \) and \( n' \), respectively, then

\[
n - n' = 2\kappa_2 - |\kappa_1| + 2.
\]

Conclusions

We have proposed a new technique for deriving the Wiener-Hopf factors of the Chebotarev-Khrapkov matrix \( G(t) = \alpha_1(t) I + \alpha_2(t) Q(t) \), \( \alpha_j(t) \in H(L) \), \( Q(t) \) is a \( 2 \times 2 \) zero-trace polynomial matrix, and solving the vector RHP \( \Phi^+(t) = G(t) \Phi^-(t) + g(t), \ t \in L \). The known technique [23], [6] first reduces the vector problem to a scalar RHP on the Riemann surface \( \mathcal{R} \) of the algebraic function \( w^2 = f(z) \), \( \det Q(z) = h^2(z) f(z) \). Then it finds a function \( \Psi_0(z,w) \) which factorizes the coefficient of the RHP on the surface and allows for essential singularities at each infinite points of \( \mathcal{R} \). These singularities are removed by solving a certain Jacobi problem of inversion of hyperelliptic integrals. At this stage a meromorphic solution is derived. The inadmissible poles due to the technique applied are removed afterwards. In contrast with this method, the technique we have developed hinges on the derivation of the Baker-Akhiezer function widely used in the theory of integrable systems. This procedure quenches the essential singularities by constructing a special abelian integral of the second type \( \Omega(P) \). It has zero \( A \)-periods, and the principal part of the function \( \exp\{\Omega(P)\} \) at the infinite points is derived according to the behavior of the function \( \Psi_0(z,w) \) at the infinite points. The consequent use of the quotient of two Riemann \( \theta \)-functions serves to annihilate the discontinuity of the function \( \exp\{\Omega(P)\} \) due to the nonzero \( B \)-periods of the integral \( \Omega(P) \). The product of the function \( \exp\{\Omega(P)\} \) and the quotient of the two \( \theta \)-functions forms the Baker-Akhiezer function \( \mathcal{F}(P) \), while the product of \( \Psi_0(z,w) \) and \( \mathcal{F}(P) \) forms a solution of the homogeneous scalar RHP on the surface. It does not have essential singularities and is a meromorphic function in \( \mathcal{R} \) with a finite number of prescribed poles. This gives the Wiener-Hopf factors of \( G(t) \) and does not require the solution to a Jacobi problem. For the general solution of the vector RHP, however, the solution of the associated Jacobi inversion problem is unavoidable. This is because the Baker-Akhiezer function has \( \rho \) zeros (\( \rho \) is the genus of the surface \( \mathcal{R} \)) and their location cannot be prescribed. At the stage of application of the generalized Liouville theorem, the affixes of the zeros are needed for determination of the rational vector in the general solution. In addition, to remove inadmissible poles due to the method applied, we also need to know on which sheet of the surface each zero of the Baker-Akhiezer function lies. This information can be recovered by stating and solving the corresponding Jacobi inversion problem.

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