UNIVERSAL FAMILIES OF GUSHEL-MUKAI FOURFOLDS

HANINE AWADA AND MICHELE BOLOGNESI

Abstract. We define universal Gushel-Mukai fourfolds over certain Noether-Lefschetz loci in the moduli stack of Gushel-Mukai fourfolds $\mathcal{M}^4_{GM}$. Using the relation between these fourfolds and K3 surfaces, we relate moduli of K3 surfaces to universal Gushel-Mukai varieties, and their birational geometry. This allows us to prove the unirationality or rationality of some of these universal families.

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1. Introduction

In the last few years a lot of attention has been attracted by the so-called Fano varieties of K3 type. These Fano varieties are characterized by the fact that a part of their cohomology, or of their derived category of coherent sheaves, shares many proprieties with the same object, computed on a K3 surface. For example, the rank of certain cohomology groups, the Serre functor of certain (sub-)categories, etc. In some cases it actually happens that the cohomology contains a part of the cohomology of an honest K3 surface, or the derived category contains a full subcategory equivalent to the derived category of a K3 surface. In the archetypal case of cubic fourfolds this coincidence has also been related to the rationality of the variety.
A similar situation has emerged in recent years with respect to Gushel-Mukai varieties, and notably GM fourfolds. A four-dimensional GM variety $X \subset \mathbb{P}^8$, is a (complex) smooth dimensionally tranverse intersection of a cone in $\mathbb{P}^{10}$ over the Grassmannian $G(1,4) \subset \mathbb{P}^9$ with a linear space isomorphic to $\mathbb{P}^8$ and a quadric hypersurface. Equivalently, it is a prime Fano fourfold of degree 10 and index 2 with $Pic(X) = \mathbb{Z}H$ for an ample divisor $H$ such that $H^4 = 10$ and $K_X \cong -2H$.

These fourfolds are isomorphic to either a quadric section of a linear section of $G(1,4) \subset \mathbb{P}^9$ (called ordinary fourfolds) or a double cover of a linear section of $G(1,4)$ branched along a quadric section (called Gushel-type fourfolds) (see [5], [4]). They are parametrized by a moduli stack $M^{4}_{GM}$ of dimension 24 (see [16, Proposition A.2.]). The interest in these fourfolds is related to their rationality problems and similarities with cubic fourfolds. In a fashion very similar to cubic fourfolds, all GM fourfolds are unirational, and moreover some examples of rational GM fourfolds are known. Nevertheless, one expects that a very general GM fourfold should be irrational, but no example has yet been proven to be irrational.

Once again, similarly to the case of cubic fourfolds (see [12], [13]), via Hodge theory and the period map, Debarre, Iliev and Manivel [4] define Hodge-special GM fourfolds, that is, fourfolds that contain a surface whose cohomology class does not come from the tautological classes of the Grassmannian $G(1,4)$. They show that these families of fourfolds in $M^{4}_{GM}$, called Noether-Lefschetz loci, correspond to a countable union of hypersurfaces in the period domain.

In this paper we consider ordinary Hodge-special GM fourfolds. In [4, Sect. 7] and [14] several families of fourfolds in $M^{4}_{GM}$ were studied and explicit descriptions of the general member of these families were given. One can define these loci in the moduli space $M^{4}_{GM}$ in terms of the surfaces contained by the generic member. The class of these surfaces in $G(1,4)$ is a linear combination of both Schubert cycles $\sigma_{2,2}$ and $\sigma_{3,1}$, corresponding to the two types of linear subspaces in $G(1,4)$. More recently, Staglianò [25] gives an explicit and alternative description of these families and of some new ones. They are described as the closure in $M^{4}_{GM}$ of families of fourfolds containing certain rational surfaces that can be embedded in a smooth del Pezzo fivefold (that is, a five dimensional linear section of $G(1,4)$). These families are proved to be uniruled, and hence of negative Kodaira dimension.

When GM fourfolds are rational, a geometric construction is very often provided by means of K3 surfaces (for example seen as base loci of linear systems) or cubic fourfolds (via birational maps).

Lately, in [1] we defined universal cubic fourfolds $C_{d,1} \to C_d$ over divisors in the moduli space of cubic fourfolds for $8 \leq d \leq 42$. These universal families were proved to be unirational. In particular for $d = 2(n^2 + n + 1)$, $n \geq 2$ (more precisely for $C_{14,1}$, $C_{26,1}$ and $C_{42,1}$), the proof of the unirationality is based on the construction of moduli quotients parametrizing families of scrolls inside cubic fourfolds. Notably, these families are birational to certain moduli spaces $F_{y,1}$ of pointed K3 surfaces (see [10], [9], [22], [23]).

One can, in the same spirit, consider universal fourfolds over some specific families of GM fourfolds. The birational geometry of these universal families once again depends very much on the birational geometry of certain moduli spaces $F_{g,n}$ of $n$-pointed K3 surfaces of genus $g$ or on universal families $C_{d,1}$ of pointed cubic fourfolds (see [17], [2]). Of course this also gives information about the birational
geometry of the base of the family inside $\mathcal{M}_{GM}^4$. Notably we will consider five
universal families of $GM$ fourfolds, defined by the kind of surface contained in the
general member of the family. Two of these families, $\mathcal{M}_{\sigma \text{-plane}}^4$ and $\mathcal{M}_{\rho \text{-plane}}^4$, are
made up by fourfolds containing some special projective planes (see Sect. 4.1, 4.4);
a third one is $\mathcal{M}_{\tau \text{-quadric}}^4$, the family of fourfolds containing a $\tau$–quadric (see Sect. 4.2); we consider also $\mathcal{M}_{DP5}^4$, given by fourfolds containing a del Pezzo quintic (see Sect. 4.3), and a last one, $\mathcal{M}_{W_9,2}^4$, is made by fourfolds that contain a special degree 9 and sectional genus 2 surface (Sect. 4.5).

**Theorem:** The universal families of $GM$ fourfolds over the loci $\mathcal{M}_{\sigma \text{-plane}}^4$,
$\mathcal{M}_{\rho \text{-plane}}^4$, $\mathcal{M}_{\tau \text{-quadric}}^4$, $\mathcal{M}_{DP5}^4$ and $\mathcal{M}_{W_9,2}^4$ are unirational.

On the way to our Theorem, we also show the following result.

**Proposition 3.7:** $\mathcal{F}_{11,7}$, the moduli space of $7$–pointed $K3$ surfaces of genus 11, is unirational.

As far as the authors know, the unirationality of $\mathcal{F}_{11,n}$ is proved only for $n \leq 6$
(see [2]).

Moreover, in the last section we introduce an 18-dimensional Noether-Lefschetz
locus $\mathcal{V}^{nod}$ of $K3$ surfaces of genus 11, which is involved in the construction of $GM$
fourfolds (see Sect. 4.6 for more details). We show that it is rational, and that also
the families $\mathcal{V}^{nod}_n$ of $n$–pointed $K3$ surfaces over $\mathcal{V}^{nod}$ are rational if $n \leq 9$. Then by
considering $\mathcal{V}^{nod}_2$ we construct a new, rational, universal family, dubbed $\mathcal{M}_{W_9,2}^{4nod}$ of
$GM$ fourfolds, over a 22-dimensional base $\mathcal{M}_{W_9,2}^{4nod} \subset \mathcal{M}_{W_9,2}^4$.

The structure of this note is as follows. First, in §2 we mention some generalities
on $GM$ fourfolds. Then, in §3 we define and discuss the existence of universal fourfolds over families in $\mathcal{M}_{GM}^4$ and mention some useful results on the unirationality of certain universal families over the moduli space of $K3$ surfaces and of cubic fourfolds. In §4 we recall the explicit descriptions of certain families of fourfolds and prove the unirationality of their universal families deducing also their unirationality.

Finally, in §4.6 we study the moduli space of certain nodal genus 11 pointed
$K3$ surfaces $\mathcal{V}^{nod}_n$, the birational geometry of $\mathcal{V}^{nod}_n$ and of the universal families of the associated $GM$ fourfolds.

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2. SOME GENERAL FACTS ABOUT GUSHEL-MUKAI FOURFOLDS

We briefly recall some general facts about Gushel-Mukai fourfolds. For a more
detailed reference, see [4]. We will denote by $\mathcal{M}_{GM}^4$ the fine moduli stack of $GM$
fourfolds. In addition, our fourfolds have a coarse moduli space which is obtained as a GIT quotient [6, Theorem 5.15].

Let $\rho : \mathcal{M}_{GM}^4 \to \mathcal{D}$ be the period map from the moduli stack onto the 20–
dimensional quasi-projective period domain $\mathcal{D}$. Each $GM$ fourfold $X$ determines a period point $\rho(X) \in \mathcal{D}$. The map $\rho$ is dominant with smooth fibers of dimension 4
(see [4] or [3]).

Let $X \in \mathcal{M}_{GM}^4$ be a smooth $GM$ fourfold. The Hodge diamond of $X$ is as follows
(see [15] Lemma 4.1):
In order to focus on the middle cohomology of $X$, containing the most non-trivial Hodge theoretic informations, we define a positive definite lattice $A(X) = H^{2,2}(X) \cap H^4(X, \mathbb{Z}) \supseteq A(\mathbb{G}(1,4))$ containing the rank 2 lattice $\Gamma_2 := H^4(\mathbb{G}(1,4), \mathbb{Z})|_X$ defined as follows

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\sigma_{1,1|X} & \sigma_{2|X} - \sigma_{1,1|X} \\
\sigma_{2|X} - \sigma_{1,1|X} & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
2 & 0 & b \\
0 & 2 & a - b \\
b & a - b & S_X^2 \\
\end{pmatrix}
\]

Note that the Schubert cycles $\sigma_1$ and $\sigma_2$ form a basis for $\Gamma_2$ (see [11]).

Recall that $X$ is said to be Hodge-special (see Definition in §1) if, equivalently, the rank of $A(X)$ is at least 3.

One can associate to a very general Hodge-special Gushel-Mukai fourfold a labeling of discriminant $d$ which is a primitive positive-definite rank 3 sublattice $K_d \subset H^4(X, \mathbb{Z})$ containing the rank 2 lattice $\Gamma_2$ and the smooth surface $S$ such that $[S] \not\in A(\mathbb{G}(1,4))$ with the following intersection matrix of determinant $d$:

\[
\begin{pmatrix}
2 & 0 & b \\
0 & 2 & a - b \\
b & a - b & S_X^2 \\
\end{pmatrix}
\]

in the basis $(\sigma_{1,1|X}, \sigma_{2|X} - \sigma_{1,1|X}, [S])$ where $[S] = a\sigma_{3,1} + b\sigma_{2,2}$ in terms of Schubert cycles in $\mathbb{G}(1,4)$ for some integers $a$ and $b$ (see [13], [25] for more details).

These special fourfolds are parametrized by a countable union of hypersurfaces $D_d \notin D$ labelled by the integers $d \equiv 0, 2$ or $4[8]$ (see [4, Lemma 6.1]). The divisor $D_d$ is irreducible if $d \equiv 0[4]$ and has two irreducible components $D'_d$ and $D''_d$ if $d \equiv 2[8]$ (see [4, Corollary 6.3]).

3. Existence of universal families

3.1. Universal families of Gushel-Mukai fourfolds. Since the moduli stack $\mathcal{M}^4_{GM}$ of smooth Gushel-Mukai fourfolds is a fine moduli stack, there exists then a universal family of Gushel-Mukai fourfolds over $\mathcal{M}^4_{GM}$. More generally:

**Definition 3.1.** Let $\mathcal{M}^4_{GM} \subset \mathcal{M}^4_{GM}$ be a sub-stack. By the universal Gushel-Mukai fourfold over $\mathcal{M}^4_{GM}$ we mean the moduli stack $\mathcal{M}^4_{*,1}$ of 1-pointed fourfolds.

**Remark 3.2.** By induction, one can define the moduli stack $\mathcal{M}^4_{*,n}$ of $n$-marked Gushel-Mukai fourfolds, and there exists a tower of forgetful maps:

\[
\mathcal{M}^4_{*,n} \longrightarrow \mathcal{M}^4_{*,1} \longrightarrow \mathcal{M}^4_{*}
\]

While the existence of a universal family over the moduli stack (which is the one we will be interested in) is straightforward, the construction of universal fourfolds over the coarse moduli space needs more justification. The following Proposition is a direct consequence of [5, Proposition 3.21].
Proposition 3.3. For a very general ordinary smooth Gushel-Mukai fourfold \(X\), the group of automorphisms \(\text{Aut}(X)\) is trivial.

The upshot is the following:

Corollary 3.4. The universal family of ordinary Gushel-Mukai fourfolds exists over an open subset of their coarse moduli space.

3.2. Universal families of K3 surfaces. Universal families over K3 surfaces are well studied and several results about these objects have appeared recently in \([10], [9], [2]\) and \([17]\). In the following, we mention some facts about the birational geometry of universal families of K3 surfaces, which we will need in the next section. We will denote by \(F_{g,n}\) the moduli space of \(n\)-pointed K3 surfaces of genus \(g\).

Recall that the generic K3 surface in \(F_6\) has no automorphism, hence universal families exist (see \([18], [10]\)) over an open set of the coarse space. Since we will be working in the birational category, this is enough for our goals.

Proposition 3.5. \([17, \text{Proposition 5.5}]\) \(F_{6,22}\), the moduli space of \(22\)-pointed K3 surfaces of genus 6, is unirational.

Corollary 3.6. For \(n < 22\), \(F_{6,n}\) is unirational.

Though probably already known to the specialists, the following result does not seem to have appeared elsewhere.

Proposition 3.7. The moduli space \(F_{11,7}\) is unirational.

Proof. By \([2, \text{Theorem 0.4}]\), we have that \(F_{11,7}\) is birational to a \(\mathbb{Z}^2\)-quotient of \(M_{9,9}\), the moduli space of curves of genus 9 with 9-marked points. Since \(M_{9,9}\) is unirational (see \([8, \text{Proposition 5.7}]\)), the quotient \(M_{9,9}/\mathbb{Z}^2\) is unirational as well. Therefore, \(F_{11,7}\) is unirational.

\(\square\)

3.3. Universal families of certain cubic fourfolds. Universal families of cubic fourfolds exist in the sense of \([1]\) since the generic smooth cubic fourfold in the divisors \(C_d\) does not have projective automorphisms. In particular, we have the following result concerning the locus of cubic fourfolds containing a cubic scroll:

Proposition 3.8. \([1, \text{Theorem 4.2}]\) The universal family \(C_{12,1}\) is unirational.

4. Unirationality of universal families of Gushel-Mukai fourfolds

We recall, in this section, the explicit description of some specific fourfolds. Then, using the unirationality of some moduli spaces of pointed K3 surfaces or cubic fourfolds mentioned in \([3]\) we prove the unirationality of universal families of these GM fourfolds.

4.1. GM fourfolds containing a \(\sigma\)-plane. A \(\sigma\)-plane \(P\) is a 2-dimensional linear section of \(G(1,4)\); its class is in \(G(1,4)\) is \(\sigma_{3,1}\) (see \([4, \text{Proposition 7.4}]\)). The family \(M_{4,\tau=\sigma}\) of fourfolds containing a \(\sigma\)-plane is irreducible of codimension 2 in \(M_{GM}\).

Let \(X\) be a general fourfold in \(M_{4,\tau=\sigma}\). Its intersection matrix in the basis \((\sigma_{1,1}|X, \sigma_{2}|X - \sigma_{1,1}|X, [P])\) is the following:

\[
\begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 1 \\
0 & 1 & 3
\end{pmatrix},
\]
of discriminant 10, and $X$ belongs to $\rho^{-1}(D'_{10})$ (see [4] Corollary 6.3)).

In [4], an explicit geometric construction of $X$ is given by starting from a general degree 10 K3 surface $S \subset \mathbb{P}^6$. One projects off a general point $p \in S$ and obtain a birational model $\tilde{S} := Bl_p(S)$, which is a smooth degree 9 surface in $\mathbb{P}^5$. Let $Y$ be a smooth quadric in $\mathbb{P}^5$ containing $\tilde{S}$. The full linear system of cubics through $\tilde{S}$ defines a birational map from $Y$ to a general GM fourfold in $\mathcal{M}_{\sigma\text{-plane}}^4$, whose inverse map is the projection $\Pi_P : X \to \mathbb{P}^5$ of $X$ from the $\sigma$-plane $P$ as follows:

\[
\begin{array}{c}
\mathbb{P}^5 \ni Y \xleftarrow{-} \Pi_P \xrightarrow{-} X \\
\tilde{S} = Bl_p(S) \downarrow \rho \\
p \in S \subset \mathbb{P}^6 \quad \rho(X) \in D'_{10}
\end{array}
\]

In the diagram above, we abuse slightly of notation by denoting by $X$ both the variety and its moduli point in $\mathcal{M}_{GM}^4$.

**Theorem 4.1.** The universal family $\mathcal{M}_{\sigma\text{-plane},1}^4$ is unirational.

**Proof.** Let $\mathcal{M}_{\sigma\text{-plane},1}^4$ be the universal Gushel-Mukai fourfold over $\mathcal{M}_{\sigma\text{-plane}}^4$. We consider $\mathcal{F}_{6,3}$, the moduli space of polarized 3–pointed K3 surfaces of genus 6 with a polarisation $H$, whose full linear system has dimension 6. The moduli space $\mathcal{F}_{6,3}$ is the universal K3 surface over $\mathcal{F}_{6,2}$, endowed with a forgetful map $\pi : \mathcal{F}_{6,3} \to \mathcal{F}_{6,2}$ having two tautological sections $\delta_1, \delta_2 : \mathcal{F}_{6,2} \to \mathcal{F}_{6,3}$. Note that $\mathcal{F}_{6,3}$ is contained in a $\mathbb{P}^6$–bundle over $\mathcal{F}_{6,2}$. By projecting $\mathcal{F}_{6,3}$ from $\delta_1$, we obtain a $\mathbb{P}^5$–bundle $\mathbb{P}(V)$ containing the blow-up of $\mathcal{F}_{6,3}$ along $\delta_1$, which we will denote by $\mathcal{F}_{6,3}$.

Now let us consider the relative linear system $\mathbb{P}H^0(I_{\mathcal{F}_{6,3}}(2))$ over $\mathcal{F}_{6,2}$. This is generically a $\mathbb{P}^1$–bundle over $\mathcal{F}_{6,2}$, since generically these blown up K3 surfaces are contained in a pencil of quadrics. Hence, thanks to Cor. 3.4 it is a unirational variety. Let us denote by
the natural projection. Then there exists a universal quadric fourfold $Y_1$ containing $\omega^*\widetilde{F}_{6,3}$ inside the pull-back $\omega^*(\mathbb{P}(V))$ of the $\mathbb{P}^5$-bundle to $\mathbb{P}H^0(I_{\widetilde{F}_{6,3}}(2))$.

$$
\omega^*\widetilde{F}_{6,3} \subset Y_1 \subset \omega^*\mathbb{P}(V) \xrightarrow{\omega} \widetilde{F}_{6,3} \subset \mathbb{P}(V)
$$

The variety $Y_1$ has a point over the function field of $\mathbb{P}H^0(I_{\widetilde{F}_{6,3}}(2))$ since all surfaces of the family $\omega^*\widetilde{F}_{6,3}$ come with a point defined by $\omega^*\delta_2$ (notably, the strict transform of $\delta_2$ after the blow up of $\delta_1$). Hence $Y_1$ is rational over $\mathbb{P}H^0(I_{\widetilde{F}_{6,3}}(2))$, and unirational over $\mathbb{C}$. Since, as we have seen, the generic GM fourfold in $\mathcal{M}_{4\text{-plane}}$ is birational to one of these quadric fourfolds [4, Section 7.1], $Y_1$ dominates the universal family $\mathcal{M}_{4\text{-quadric}}$ via the relative linear system of cubics in $\omega^*\mathbb{P}(V)$ passing through $\widetilde{F}_{6,3}$, concluding the proof. 

\[ \square \]

**Corollary 4.2.** $\mathcal{M}_{4\text{-plane}}$ is unirational.

4.2. **GM fourfolds containing a $\tau$-quadric.** Using K3 surfaces once again, one can prove the unirationality of the universal family of another class of GM fourfolds, which is also irreducible of codimension 1 in $\mathcal{M}_{GM}^4$. Let $\Sigma$ be a smooth quadric surface in $G(1, 4)$ of class $\sigma_1^T = \sigma_{1,1}^T + \sigma_{2,2}^T$. We will call this a $\tau$-quadric and denote by $\mathcal{M}_{4\text{-quadric}}$ the family of GM fourfolds containing a $\tau$-quadric. Let $X$ be a fourfold in $\mathcal{M}_{4\text{-quadric}}$. The intersection matrix of $X$ in the basis $(\sigma_{1,1}|X, \sigma_{2}|X - \sigma_{1,1}|X; |\Sigma|)$ is the following:

$$
\begin{pmatrix}
2 & 0 & 1 \\
0 & 2 & 0 \\
1 & 0 & 3
\end{pmatrix},
$$

it has discriminant 10, and $X$ belongs then to $\rho^{-1}(D_{10})$, more precisely to $\rho^{-1}(D'_{10})$ (see [4] Corollary 6.3).

In [4] Section 7.3, an explicit geometric construction of $X$ is given by starting from a general K3 surface $S \subset \mathbb{P}^6$ of degree 10 and two general points $p_1, p_2$ on $S$. Let $S'$ be the projection of $S$ from these two points. A birational map $\mathbb{P}^4 \dashrightarrow X$ is given by the linear system of quartics containing $S'$ (see [4] Proposition 7.3). This is displayed in the diagram below, where once again we abuse of the notation of the period map.
**Theorem 4.3.** The universal family of fourfolds $\mathcal{M}_{\tau - \text{quadric},1}^4$ is unirational.

**Proof.** Let us consider the universal family $\mathcal{F}_{6,3}$ over $\mathcal{F}_{6,2}$. This comes equipped with an embedding inside a $\mathbb{P}^6$-bundle over $\mathcal{F}_{6,2}$, and two sections $\delta_1, \delta_2 : \mathcal{F}_{6,2} \to \mathcal{F}_{6,3}$. Projecting from $\delta_1$ and $\delta_2$, we obtain $\pi : \mathcal{F}_{6,3} \to \mathcal{F}_{6,2}$, that is, a family of K3 surfaces, each blown-up in two points, contained in a $\mathbb{P}^4$-bundle over $\mathcal{F}_{6,2}$. Call $\mathbb{P}(U)$ this $\mathbb{P}^4$-bundle, and observe that it is unirational, since $\mathcal{F}_{6,2}$ itself is unirational (see Corollary 3.6). Consider now $\mathbb{P}H^0(I_{\mathcal{F}_{6,3}}(4))$, the relative linear system of quartics inside the $\mathbb{P}^4$-bundle through $\mathcal{F}_{6,3}$. This relative linear system yields a birational map between $\mathbb{P}(U)$ and $\mathcal{M}_{\tau - \text{quadric},1}^4$ (by the geometric construction), hence $\mathcal{M}_{\tau - \text{quadric},1}^4$ is unirational.

**Corollary 4.4.** $\mathcal{M}_{\tau - \text{quadric}}^4$ is unirational.

4.3. GM fourfolds containing a quintic del Pezzo surface. A quintic del Pezzo surface $Z$ is a 2-dimensional linear section of $\mathbb{G}(1,4)$ with $\mathbb{P}^5$. Its class inside $\mathbb{G}(1,4)$ is $\sigma_1^2 = 3\sigma_{3,1} + 2\sigma_{2,2}$ (see [4, Proposition 7.5]). The family $\mathcal{M}_{DP5}^4$ of fourfolds containing a quintic del Pezzo surface is irreducible of codimension 1 in $\mathcal{M}_{GM}^4$.

Let $X$ be a general fourfold in $\mathcal{M}_{DP5}^4$. Its intersection matrix in the basis $(\sigma_{1,1}|X, \sigma_2|X - \sigma_{1,1}|X, [Z])$ is the following:

$$
\begin{pmatrix}
2 & 0 & 2 \\
0 & 2 & 1 \\
2 & 1 & 5
\end{pmatrix}.
$$

This matrix has discriminant 10, hence the corresponding irreducible family of fourfolds is $\rho^{-1}(D_{10}^\prime)$.

Recall that the projection $X \to \mathbb{P}^2$ from the quintic del Pezzo surface $Z$ yields a quintic del Pezzo surface fibration (see [21]).

**Theorem 4.5.** The universal family $\mathcal{M}_{DP5,1}^4$ and $\mathcal{M}_{DP5}^4$ are unirational.
Proof. Since all Grassmannians $G(1,4)$ are projectively equivalent, the Hilbert scheme $\mathcal{H}$ of Grassmannians $G(1,4)$ in $\mathbb{P}^9$ is dominated by the projective general linear group of $10 \times 10$-matrix $PGL(10)$, hence $\mathcal{H}$ is unirational.

We have a natural $\mathbb{P}^8$-bundle over $\mathcal{H}$, containing the universal $G(1,4)$, denoted by $\mathbb{P}(V)$. Consider now the relative flag-variety bundle $F(5,8) \to \mathcal{H}$ that parametrizes flags of projective bundles $\mathbb{P}^5 \subset \mathbb{P}^8 \subset \mathbb{P}(V)$. By construction, $F(5,8)$ is rational over $\mathbb{C}(\mathcal{H})$ hence unirational over $\mathbb{C}$.

Let $Z := \mathbb{P}^5 \cap G(1,4)$ be the quintic del Pezzo surface over $F(5,8)$ contained in the relative del Pezzo fivefold $\mathcal{Y} := \mathbb{P}^8 \cap G(1,4)$ over $F(5,8)$. These of course are contained inside a flag of rank 5 and 8 projective bundles over $F(5,8)$. Then consider the projective bundle $P \to F(5,8)$ given by the relative linear system $\mathcal{I}_Z(2)$ of relative quadrics in the $\mathbb{P}^8$-bundle, passing through $Z$. Our GM fourfolds are quadric sections of the del Pezzo fivefolds from the family $\mathcal{Y}$, that are in turn obtained fiberwise as $\mathbb{P}^8 \cap G(1,4)$. Hence, $P$ is a projective bundle over $F(5,8)$, parameterizing GM fourfolds containing quintic del Pezzo surface. It is straightforward then to see that $P$, and thus $\mathcal{M}_{DP5}^4$, is unirational.

Now, let us consider the universal family $\varphi : \mathcal{M}_{DP5,1}^4 \to \mathcal{M}_{DP5}^4 \cong P$. By an argument of Roth [21], projecting off the quintic del Pezzo, we have that $\varphi$ factors through a $\mathbb{P}^2$-bundle $W \to P$ and the universal Gushel-Mukai fourfold $\mathcal{M}_{DP5,1}^4$ is birational to a quintic del Pezzo surface fibration over $W$. By the unirationality of $P$, we find that $W$ is unirational. By a celebrated result of Enriques [7], a quintic del Pezzo surface over an infinite field $k$ is $k$-rational, hence in particular our del Pezzo fibration has a rational section over $W$ and it is rational over the function field $\mathbb{C}(W)$. This in turn implies that $\mathcal{M}_{DP5,1}^4$ is unirational over $\mathbb{C}$. \hfill $\square$

**Corollary 4.6.** $\mathcal{M}_{DP5,n}^4$ is unirational, $n \in \mathbb{N}^*$.

Proof. This can be proved inductively. We proved that $\mathcal{M}_{DP5,1}^4$ is rational over $\mathbb{C}(W)$ and then unirational over $\mathbb{C}$. Now $\mathcal{M}_{DP5,n+1}^4$ is a del Pezzo quintic fibration, with a rational section, over a $\mathbb{P}^2$-bundle over $\mathcal{M}_{DP5,n}^4$. If $\mathcal{M}_{DP5,n}^4$ is unirational, then the $\mathbb{P}^2$-bundle is unirational as well, and by applying Enriques Theorem as above, we find that $\mathcal{M}_{DP5,n+1}^4$ is unirational. \hfill $\square$

4.4. An easy remark using cubic fourfolds: fourfolds containing a $\rho$-plane.

A $\rho$-plane $P'$ is a 2-dimensional linear section of $G(1,4)$; its class is $\sigma_{2,2}$ (see [4, Proposition 7.2]). The family $\mathcal{M}_{\rho-plane}^4$ of fourfolds containing a $\rho$–plane is irreducible of codimension 3 in $\mathcal{M}_{GM}^4$.

Let $X$ be a general fourfold in $\mathcal{M}_{\rho-plane}^4$. Its intersection matrix in the basis $(\sigma_{1,1}|X, \sigma_{2}|X - \sigma_{1,1}|X, [P'])$ is the following:

$$
\begin{pmatrix}
2 & 0 & 1 \\
0 & 2 & -1 \\
1 & -1 & 4
\end{pmatrix}
$$

of discriminant 12, and $X$ belongs then to $\rho^{-1}(D_{12})$. From [4, Section 7.2], we know that $X$ is birationally isomorphic to a cubic fourfold containing a smooth cubic surface scroll in $\mathcal{C}_{12}$. 


Theorem 4.7. The universal family $\mathcal{M}^4_{\rho\text{-plane},1}$ is unirational. As a consequence, $\mathcal{M}^4_{\rho\text{-plane}}$ is unirational.

Proof. This can be easily seen from Proposition 3.8 and the birational isomorphism between $\mathcal{M}^4_{\rho\text{-plane}}$ and $C_{12}$. □

4.5. GM fourfolds containing a rational surface of degree 9 and sectional genus 2. Let $\mathcal{M}^4_{W,9,2}$ be the irreducible family of GM fourfolds in $\mathcal{M}^4_{GM}$, containing a rational surface $W$ of degree 9 and sectional genus 2. Such a surface can be obtained by applying to $\mathbb{P}^2$ the full linear system of quartics through three simple points and one double point in general position.

This codimension 1 (in $\mathcal{M}^4_{GM}$) family of fourfolds has been recently studied in [14]. As we have done before, we consider these surfaces as embedded in a smooth del Pezzo fivefold in $\mathbb{P}^8$. In this sense, we observe that it has a class $[W] = 6\sigma_3 + 3\sigma_2$ inside $G(1,4)$.

Let us denote by $X$ a GM-fourfold belonging to this family. The intersection matrix of $X$ in the basis $(\sigma_1|X, \sigma_2|X - \sigma_1|X, [W])$ is the following:

\[
\begin{pmatrix}
2 & 0 & 3 \\
0 & 2 & 3 \\
3 & 3 & 14
\end{pmatrix}
\]

Its discriminant is 20 and $X$ belongs then to $\rho^{-1}(D_{20})$.

Let us recall shortly the geometric construction of a general fourfold belonging to this family ([14, 23]). There exists a birational map $\mathbb{P}^4 \dashrightarrow X$, defined by the linear system of hypersurfaces of degree 9 having double points along $S \subset \mathbb{P}^4$ a singular surface of degree 10 and sectional genus 8. The construction of $S$ is rather involved so it is worth recalling some details.

We start from a K3 surface $Z \subset \mathbb{P}^{11}$ of degree 20 and sectional genus 11. We take two points $p, q \in Z$, and perform first a triple projection from $p$ to $\mathbb{P}^5$, then a simple projection off $q$ to $\mathbb{P}^4$. The image is the required surface $S$.

\[
\begin{array}{c}
S \subset \mathbb{P}^4 \dashrightarrow X \\
Z \subset \mathbb{P}^{11}
\end{array}
\]

\[\rho(X) \in D_{20}\]

Theorem 4.8. The universal family $\mathcal{M}^4_{W,9,2,1}$ is unirational. As a consequence, the family itself $\mathcal{M}^4_{W,9,2}$ is unirational.

Proof. Once again, the philosophy is to do the rationality construction in families. In order to do this in families we need then to consider the moduli space $\mathcal{F}_{11,2}$ of polarized K3 surface of genus 11 with two marked points. It is of dimension 23. The moduli space $\mathcal{F}_{11,3}$ is the universal K3 surface over $\mathcal{F}_{11,2}$, it comes equipped with an embedding inside a $\mathbb{P}^{11}$-bundle over $\mathcal{F}_{11,2}$ and with two sections $\delta_1, \delta_2 : \mathcal{F}_{11,2} \to \mathcal{F}_{11,3}$. Performing a relative triple projection from the image of $\delta_1$ and a
simple one from the image of \( \delta_2 \) we obtain a \( \mathbb{P}^4 \)-bundle \( \mathbb{P}(U) \) over \( \mathcal{F}_{11,2} \) containing the family \( \mathcal{S} \) of degree 10 surfaces.

\[
\mathcal{F}_{11,3} \subset \mathbb{P}^{11} \quad \ldots \quad \ldots \quad \ldots \quad \rightarrow \mathcal{S} \subset \mathbb{P}(U)
\]

Since \( \mathcal{F}_{11,2} \) is unirational [2, Theorem 0.1], the projective bundle \( \mathbb{P}(U) \) is unirational as well, and it is clear that the relative linear system of degree 9 hypersurfaces, with multiplicity two along \( \mathcal{S} \) gives a birational map between \( \mathcal{M}^4_{\text{W}_{9,2,1}} \) and the \( \mathbb{P}^4 \)-bundle \( \mathbb{P}(U) \). This concludes the proof. \( \square \)

### 4.6. A rational Noether-Lefschetz divisor of genus 11 K3 surfaces, and their associated GM fourfolds.

In [14], Hoff and Staglianó also consider a codimension one subfamily of genus 11 K3 surfaces, that forms a Noether-Lefschetz divisor inside \( \mathcal{F}_{11} \). This divisor seems particularly interesting under our point of view, since the wealth of geometry going on here allows us to strengthen our rationality results concerning the universal families of GM fourfolds related to these K3s. But let us give a couple more details about these surfaces.

We start from a smooth Fano threefold \( Y \) of type \( X_{22} \subset \mathbb{P}^{13} \). It is well known that the generic tangent hyperplane sections of \( Y \) are one-nodal (a double point) K3 surfaces (see [19], [24]). The projection off the node of such a K3 surface gives a K3 surface in \( \mathbb{P}^{11} \), of degree 20 and sectional genus 11, containing a further conic (the exceptional divisor over the node). In fact, such a construction gives a Noether-Lefschetz divisor inside the 19-dimensional moduli space of K3 surfaces of genus 11. It is easy to check the dimension: the moduli space of \( X_{22} \)-Fano varieties has dimension 6 (and is rational, [20]), then we need to choose a tangent point of \( Y \) - and this adds 3 dimensions - and finally an hyperplane in \( \mathbb{P}^{13} \) containing the tangent space to \( Y \) in \( p \), and this gives a 9-dimensional linear system. These dimensions sum up to 18, and the intersection lattice of these K3 surfaces contains a sublattice of type

\[
\begin{pmatrix}
20 & 2 \\
2 & -2
\end{pmatrix}
\]

Before studying the universal family of GM fourfolds obtained from these special K3 surfaces, we need to show some results on the birational geometry of their Noether-Lefschetz locus. We will denote by \( \mathcal{V}^\text{nod}_n \) the moduli space of \( n \)-pointed one nodal K3 surfaces of sectional genus 12, obtained by cutting a \( X_{22} \)-type 3fold with tangent hyperplanes as above. The generic element of \( \mathcal{V}^\text{nod}_n \) is represented by a vector \( (Y, p, H, q_1, \ldots, q_n) \), where \( Y \) is a Fano threefold of type \( X_{22} \), \( p \) is a point of \( Y \), \( H \) is a hyperplane tangent to \( Y \) in \( p \), and \( q_1, \ldots, q_n \) are \( n \) points on the surface \( S_H := Y \cap H \). We will also denote by \( \mathcal{X} \) the rational moduli space of
Fano threefolds of type $X_{22}$. All these threefolds are rational and birational among them, we will need to fix one $\bar{X}_{22} \in \mathcal{X}$.

**Theorem 4.9.** The moduli space $\mathcal{V}_n^{\text{nod}}$ is rational if $n \leq 9$.

**Proof.** Let us consider the rational map

\begin{align}
\varphi : \mathcal{V}_n^{\text{nod}} & \to \mathcal{X} \times \bar{X}_{22}^{n+1} \\
(Y,p,H,q_1,\ldots,q_n) & \mapsto (Y,p,q_1,\ldots,q_n).
\end{align}

Remark that $\mathcal{X} \times \bar{X}_{22}^{n+1}$ is rational (and of dimension $3n + 9$) since it is the product of rational varieties. Then, the fiber of $\varphi$ over $(Y,p,q_1,\ldots,q_n)$ is exactly the linear system of hyperplanes in $\mathbb{P}^{13}$ that are tangent to $Y$ in $p$, and pass through $q_1,\ldots,q_n$. It is straightforward to check that this shows that $\mathcal{V}_n^{\text{nod}}$ is birational to a $\mathbb{P}^{9-n}$-projective bundle over $\mathcal{X} \times \bar{X}_{22}^{n+1}$, and hence is rational if $n \leq 9$. $\square$

Now we want to apply the theorem above, to GM fourfolds inside $\mathcal{M}^{4}_{W_{9,2}}$. We recall that the projection off the node sends birationally $\mathcal{V}_n^{\text{nod}}$ onto a $(18 + 2n)$-dimensional NL locus inside $\mathcal{F}_{11,n}$. Let us denote by $\mathcal{M}^{4}_{W_{9,2}}$, the moduli space of GM fourfolds obtained from the NL K3 surfaces described above, and by $\mathcal{M}^{4}_{W_{9,2}}$ the universal family above. The moduli space $\mathcal{M}^{4}_{W_{9,2}}$ is of dimension 22, and is contained in $\mathcal{M}^{4}_{W_{9,2}}$.

**Corollary 4.10.** The universal family $\mathcal{M}^{4}_{W_{9,2}}$ is rational. The moduli space $\mathcal{M}^{4}_{W_{9,2}}$ is rational.

**Proof.** The moduli space $\mathcal{V}_2^{\text{nod}}$ of nodal, 3-pointed K3 surfaces can be embedded in a $\mathbb{P}^{12}$-bundle, and endowed with two sections $\delta_1, \delta_2 : \mathcal{V}_2^{\text{nod}} \to \mathcal{V}_3^{\text{nod}}$, over $\mathcal{V}_2^{\text{nod}}$. Since we are working in the birational category, we can even consider (at least an open subset of) $\mathcal{V}_2^{\text{nod}}$ as contained in $\mathcal{F}_{11,2}$. Now, we project fiberwise off the node, obtaining a family of NL K3 surfaces in a $\mathbb{P}^{11}$-bundle, with two sections, over $\mathcal{V}_2^{\text{nod}}$. Then, as we did in Sect. 4.5, we perform as usual the two projections off the sections and we obtain a $\mathbb{P}^{4}$-bundle over $\mathcal{V}_2^{\text{nod}}$, containing a family $\mathcal{T}$ of degree 10 surfaces. The moduli space $\mathcal{V}_2^{\text{nod}}$ is rational, hence the $\mathbb{P}^{4}$-bundle is rational as well. Then, by applying the relative linear system of degree 9 hypersurfaces through $\mathcal{T}$ as in Theorem 4.8, we obtain a rational family of GM fourfolds over $\mathcal{V}_2^{\text{nod}}$, hence $\mathcal{M}^{4}_{W_{9,2}}$ is rational. By construction $\mathcal{M}^{4}_{W_{9,2}}$ is birational to $\mathcal{V}_2^{\text{nod}}$ and hence rational. $\square$

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Institut Montpellierain Alexander Grothendieck, CNRS, Université de Montpellier, Case Courrier 051 - Place Eugène Bataillon, 34095 Montpellier Cedex 5, France

E-mail address: hanine.awada@umontpellier.fr

Institut Montpellierain Alexander Grothendieck, Université de Montpellier, Case Courrier 051 - Place Eugène Bataillon, 34095 Montpellier Cedex 5, France

E-mail address: michele.bolognesi@umontpellier.fr