Equivalence between Scalar-Tensor theories and \( f(R) \)-gravity: from the action to cosmological perturbations

Joel Velásquez and Leonardo Castañeda
Grupo de Gravitación y Cosmología, Observatorio Astronómico Nacional Universidad Nacional de Colombia, cra 45 #26-85, Ed.Uriel Gutiérrez, Bogotá D.C., Colombia
E-mail: jvelasquez@unal.edu.co and lcastanedac@unal.edu.co

Abstract

In this paper we calculate the field equations for Scalar-Tensor from a variational principle, taking into account the Gibbons-York-Hawking type boundary term. We do the same for the theories \( f(R) \), following (Guarnizo (2010), Gen. Rel. Grav. 42, 2713–2728). Then, we review the equivalences between both theories in the metric formalism. Thus, starting from the perturbations for Scalar-Tensor theories, we find the perturbations for \( f(R) \) gravity under the equivalences. Working with two specific models of \( f(R) \), we explore the equivalences between the theories under conformal-Newtonian gauge. Further, we show the perturbations for both theories under the sub-horizon approach.

1. Introduction

Recent observations of the CMB show that the Universe is in accelerated expansion [1, 2]. The broadly used model is the Λ-Cold-Dark-Matter (ΛCDM). However, this model introduces an exotic term of energy, called Dark Energy (DE), associated to the cosmological constant term \( \Lambda \). Assuming that the theory of general relativity (GR) is not entirely correct at cosmological scales, it is possible that a cosmological constant term is not necessary to explain the accelerated expansion of the Universe. The alternative theories to the Einstein’s proposal are known as modified gravity theories (MG). One set of these theories is known as Scalar-Tensor gravity theories (ST) [3–5], where the gravitational action in these theories, in addition to the metric, to contain a scalar field which intervenes in the generation of the space-time curvature, associated to the metric. This scalar field is not directly coupled to the matter and, therefore, the matter responds only to the metric. It should be noted that the Brans-Dicke theory (BD), [6] proposed by C.H. Brans y R.H. Dicke in 1961, is a particular case of theories ST, where the parameter \( \omega(\phi) \) is independent of the scalar field.

Another type of generalization to GR are the theories of gravity \( f(R) \) [7–9], where the lagrangian of Einstein-Hilbert is generalized, replacing the scalar curvature \( R \) by a more general function of it, \( f(R) \). The gravitational field in this theory is represented by the metric like GR does.

The equivalence between theories ST and \( f(R) \) has been studied e.g., in [10–15]. It is, starting from the ST action, without the kinetic term of the scalar field, we arrive at the action of the gravity theories \( f(R) \). In this paper, in addition to the above, we show these equivalences for the field equations, the Friedmann equations of the homogeneous and isotropic universe and the Friedmann’s perturbations in any gauge. Further, we show two specific examples of theories \( f(R) \) under the conformal-Newtonian gauge.

The paper is organized as following: in the section 2 we get the field equations for RG, ST and \( f(R) \) theories starting from the variational principle, taking into account the Gibbons-York-Hawking (GYH) boundary term type, for every of the above theories. It is found that the consideration to obtain the field equations for ST, under the equivalence of the theories, to coincide to the \( f(R) \) condition. In the section 3 the equivalence between ST and \( f(R) \) for the actions and the field equations of the theories is shown. In the section 4 the Friedmann equations for the background universe (homogeneous and isotropic) are calculated. Besides, we calculate the perturbed Friedmann equations, for ST, and the ones for \( f(R) \), using the equivalence between the theories. Then, we show
how to construct the potential for the Hu-Sawicki and Starobinsky $f(R)$ models, in order to calculate the Friedmann equations for the background and perturbed universe in these models for the two formalisms under the conformal-Newtonian gauge. Immediately, we perform the sub-horizon approach to the perturbations, for both theories, and we show that they can not be calculated using the equivalences, due to the parameter $\omega = cte$ for ST. Finally, in the section 5 we show the conclusions. In the appendix B we show how the perturbations were calculated under the package $xPand$ from software Mathematica.

Throughout the review, we adopt natural units $8\pi G = c = 1$, here $G$ is Newton’s gravitational constant and $c$ is speed of light. Have a metric signature $(-+++)$.

Small latin indices $a, b, K$ assume the values 0 to 3, while greek indices $\alpha, \beta, K$ assume the values 1, 2, 3.

2. Field equations and variational principles

This section shows how the field equations, through a variational principle for the theories GR, ST and $f(R)$ are found; taking into account in all of these theories the boundary term type GYH.

2.1. Field equations in GR

The Einstein field equations (EFEs) can be deduced through a variational principle. We give a detailed review following $[16–18]$. The action for GR is

$$S^{(GR)} = \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} R + S^{(m)}(g_{ab}, \psi),$$

(1)

where the first term is known as the Einstein-Hilbert action, $d^4x \sqrt{-g}$ is the element of invariant volume and $R$ is the Ricci scalar.

The second term is the matter action defined by

$$S^{(m)} = \int_{\mathcal{M}} d^4x \sqrt{-g} \mathcal{L}^{(m)}(g_{ab}, \psi),$$

(2)

where $\psi$ denotes the matter fields.

The variation of the action (1) with respect to $g^{ab}$ takes the form

$$\delta S^{(GR)} = \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} \delta R + \delta S^{(m)}.$$

(3)

Given the variation of the Ricci scalar

$$\delta R = \delta g^{ab}R_{ab} + \nabla (g^{ab}\delta \Gamma^c_{ab}) - \nabla_l (g^{ab}\delta \Gamma^c_{al}),$$

(4)

we get

$$\delta S^{(GR)} = \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} \left( R_{ab} - \frac{1}{2} g_{ab} R \right) \delta g^{ab} + \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} \nabla_a V^d + \delta S^{(m)},$$

(5)

where

$$V^d = g^{ab}\delta \Gamma^d_{ab} - g^{ad}\delta \Gamma^c_{ac}.$$  

(6)

The second integral of the equation (5) is a divergence term. Thus, we can use the Gauss-Stokes theorem

$$\int_{\partial\mathcal{M}} d^3x \sqrt{|h|} \nabla_a A^d = \oint_{\partial\mathcal{M}} d^3y \sqrt{|h|} n_d A^d,$$

(7)

where $\partial\mathcal{M}$ its the boundary of a hypervolume on $\mathcal{M}$, $h$ is the determinant of the induced metric, $n_d$ is the unit normal vector to $\partial\mathcal{M}$, $\epsilon$ is +1 if $\partial\mathcal{M}$ is timelike and −1 if $\partial\mathcal{M}$ is spacelike (it is assumed that $\partial\mathcal{M}$ is nowhere null). Coordinates $x^a$ are used for the finite region $\mathcal{M}$ and $y^a$ for the boundary $\partial\mathcal{M}$.

In the equation (6) the variations of the Christoffel symbols are present. Calculating these variations in the boundary, we have

$$\delta \Gamma^a_{bc}(x)|_{\partial\mathcal{M}} = \frac{1}{2} g^{ad} (\partial_b \delta g_{dc} + \partial_c \delta g_{bd} - \partial_d \delta g_{bc}),$$

(8)

where it has been imposed that the variation of the metric tensor is null in the boundary, i.e.,

$$\delta g_{ab}(x)|_{\partial\mathcal{M}} = 0.$$

(9)

Found the equation (8), the vector $V_d = g_{ad} V^c$ is calculated at the boundary

$$V_d(x)|_{\partial\mathcal{M}} = g^{ab} (\partial_b \delta g_{da} - \partial_a \delta g_{db}).$$

(10)
Now we evaluate the term \( n^d V_{\partial M} \), using for this
\[
g^{ab} = h^{ab} + \epsilon n^a n^b, \tag{11}\]
then
\[
n^d V_{\partial M} = n^d h^{ab} (\partial_b \delta g_{ab} - \partial_a \delta g_{ba}), \tag{12}\]
where we use the antisymmetric part of \( \varepsilon n^a n^b \), with \( \varepsilon = n^a n^b = \pm 1 \). To the fact \( \delta g_{ab} = 0 \) in the boundary we have \( h^{ab} \partial_b \delta g_{ba} = 0 \), we get
\[
n^d V_{\partial M} = - n^d h^{ab} \partial_a \delta g_{ba}. \tag{13}\]
The variation of the action (5) takes the form
\[
\delta S^{(RG)} = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} \left(R_{ab} - \frac{1}{2} g_{ab} R\right) \delta g^{ab} - \frac{1}{16\pi} \oint_{\partial M} d^3y e \sqrt{|h|} n^d h^{ab} \partial_b \delta g_{ba} + \delta S^{(m)} \tag{14}.\]
The above equation shows that fixing \( \delta g_{ab} = 0 \) on \( \partial M \) there is an additional boundary term. It could be argued that both the variation of the metric and its first derivative vanish in the boundary, i.e., \( \delta g_{ab} = 0 \) and \( \partial_a \delta g_{ba} = 0 \) in \( \partial M \). Although this last argument leads directly to Einstein field equations, it implies to fix two conditions in the boundary. To avoid this, a boundary term is introduced, the Gibbons-York-Hawking (GYH) boundary term, that allows to have a well defined variational problem only fixing the variation of the metric in the boundary, \( \delta g_{ab} |_{\partial M} = 0 \) \cite{19, 20}. This term is
\[
S_{GYH}^{(RG)} = \frac{1}{8\pi} \oint_{\partial M} d^3y e \sqrt{|h|} K, \tag{15}\]
where \( K \) is the trace of extrinsic curvature. The variation of the GYH action is
\[
\delta S_{GYH}^{(RG)} = \frac{1}{8\pi} \oint_{\partial M} d^3y e \sqrt{|h|} \delta K, \tag{16}\]
where \( \delta h^{ab} = 0 \) in the boundary \( \partial M \).
Using the definition of the extrinsic curvature \cite{17}\n\[
K_{ab} = h^{a'} \nabla_{n_b} n_{a'}, \tag{17}\]
the trace is given by
\[
K = \nabla_{n} n^{a} g^{ab} \nabla_{n_b} n_a = h^{ab} (\partial_b n_a - \Gamma^{c}_{ba} n_c), \tag{18}\]
where we have used the equation (11). Taking into account (8), \( \delta K \) is calculated on the boundary
\[
\delta K = - h^{ab} \Gamma^{c}_{ba} n_c = \frac{1}{2} h^{ab} \partial_a \delta g_{ba} n^d, \tag{19}\]
The variation (16) gives
\[
\delta S_{GYH}^{(RG)} = \frac{1}{16\pi} \oint_{\partial M} d^3y e \sqrt{|h|} h^{ab} \partial_b \delta g_{ba} n^d, \tag{20}\]
This term to cancel with the second integral of (14) (the boundary term contribution). Hence we have
\[
\delta S_{GYH}^{(RG)} = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} \left(R_{ab} - \frac{1}{2} g_{ab} R\right) \delta g^{ab} + \delta S^{(m)}. \tag{21}\]
The variation of the action (2) takes the form
\[
\delta S^{(m)} = \int_M d^4x \delta \left(\frac{\sqrt{-g} L^{(m)}}{g^{ab}}\right) = \int_M d^4x \sqrt{-g} \left(\frac{\partial L^{(m)}}{\partial g^{ab}} - \frac{1}{2} L^{(m)} g^{ab}\right) \delta g^{ab}, \tag{22}\]
Defining the stress-energy tensor by
\[
T_{ab} = -2 \frac{\partial L^{(m)}}{\partial g^{ab}} + L^{(m)} g_{ab} = - \frac{2}{\sqrt{-g}} \frac{\delta S^{(m)}}{\delta g^{ab}}, \tag{23}\]
then
\[
\delta S^{(m)} = - \frac{1}{2} \int_M d^4x \sqrt{-g} T_{ab} \delta g^{ab}, \tag{24}\]
Imposing that the total variations to remain invariant with respect to \( \delta g^{ab} \), i.e.,
\[
\frac{1}{\sqrt{-g}} \delta g^{(RG)}_{ab} = 0. \tag{25}\]
Finally, we get

\[ R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}, \tag{26} \]

which are the Einstein field equations.

### 2.2. Field equations in ST gravity

Scalar-Tensor theories of gravity belong to the MG theories, where a function of scalar field \( \phi \) is non-minimal coupling to the Ricci scalar \( R \). The action in the so-called Jordan Frame is \(^{[21]}\)

\[ S^{(ST)} = \int_M d^4x \sqrt{-g} \left[ \frac{f(\phi)}{2} R - \frac{\omega(\phi)}{2} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) \right] + S^{(m)}, \tag{27} \]

where \( S^{(m)} \) is the action \(^{(2)}\) describing ordinary matter (any form of matter different from the scalar field \( \phi \)), \( \omega \) is a parameter that is a function of the scalar field \( \phi \). Notice that the matter is not directly coupled to \( \phi \), in the sense that the Lagrangian density \( \mathcal{L}^{(m)} \) does not depend on \( \phi \), but the scalar field is directly coupled to the Ricci scalar \( R \). The scalar field potential \( V(\phi) \) constitutes a natural generalization of the cosmological constant \(^{[4]}\).

From the action of ST theories of gravity, the BD’s action can be gotten by \(^{[21]}\)

\[ f(\phi) = \frac{\phi}{8\pi}, \quad \omega(\phi) = \frac{\omega_0}{8\pi \phi} \tag{28} \]

where \( \omega_0 \) is a constant, and the potential is rescaled by a factor \( 16\pi \).

The ST field equations can be obtained from a variational principle. The variation of the action \(^{(27)}\) with respect to \( d g_{ab} \) gives

\[ \delta S^{(ST)} = \int_M d^4x \sqrt{-g} \left[ \frac{f(\phi)}{2} \left(R_{ab} - \frac{1}{2} g_{ab} R \right) + \frac{1}{2} g_{ab} V(\phi) \right] \]

\[ - \frac{\omega(\phi)}{2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla_c \phi \nabla_d \phi \right) \delta g^{ab} \]

\[ + \int_M d^4x \sqrt{-g} \left[ \frac{f(\phi)}{2} \left[ \nabla_c (g^{ab} \delta \Gamma^c_{\ ab}) - \nabla_b (g^{ab} \delta \Gamma^c_{\ ac}) \right] + \delta S^{(m)} \right]. \tag{29} \]

Let us write the second integral in the following way

\[ \delta S_B^{(ST)} = \int_M d^4x \sqrt{-g} \frac{f(\phi)}{2} \nabla_a (g^{ab} \delta \Gamma^d_{\ ab} - g^{ad} \delta \Gamma^c_{\ ac}). \tag{31} \]

The term in parentheses is given by (e.g. see \(^{[22]}\))

\[ g^{ab} \delta \Gamma^d_{\ ab} - g^{ad} \delta \Gamma^c_{\ ac} = g_{df} \nabla^d \delta g^{df} - \nabla_d \delta g^{df}. \tag{32} \]

Using the above relation and the fact about the metric compatibility \( (\nabla_i g_{ab} = 0) \), the term \(^{(31)}\) yields

\[ \delta S_B^{(ST)} = \int_M d^4x \sqrt{-g} \frac{f(\phi)}{2} \left( g_{df} \Box \delta g^{df} - \nabla_d \nabla_f \delta g^{df} \right), \tag{33} \]

where the D’Alembert operator definition has been used, i.e. \( \Box \equiv \nabla^d \nabla_d \). It allow us to define the next quantities to express the integral above in a different way

\[ M = \frac{f(\phi)}{2} g_{df} \nabla_d (\delta g^{df}) - \frac{1}{2} (\delta g^{df}) g_{df} \nabla_d f(\phi) \tag{34} \]

\[ N^c = \frac{f(\phi)}{2} \nabla_f (\delta g^{df}) - \frac{1}{2} (\delta g^{df}) \nabla_f f(\phi). \tag{35} \]

The quantities \( M \), and \( N^c \) allow us to write the equation \(^{(33)}\) as (for details view appendix A)

\[ \delta S_B^{(ST)} = \frac{1}{2} \int_M d^4x \sqrt{-g} \delta g^{df} (g_{df} \Box f(\phi) - \nabla_d \nabla_f f(\phi)) + \int_M d^4x \sqrt{-g} \left( \nabla^d M - \nabla_d N^c \right). \tag{36} \]
Thus, the variation of the action (30) takes the form

\[
\delta S^{(ST)} = \int_M d^4x \sqrt{-g} \left[ \frac{f(\phi)}{2} \left( R_{ab} - \frac{1}{2} g_{ab} R \right) + \frac{1}{2} g_{ab} V(\phi) \right] + \frac{\omega(\phi)}{2} (\nabla_a \phi)_{\nabla_b \phi} - \frac{1}{2} g_{ab} \varepsilon_{cdef} \nabla_c \phi \nabla_d \phi \right) + \frac{1}{2} (g_{ab} \Box f(\phi) - \nabla_a \nabla_b f(\phi)) \right] \delta g^{ab} \\
+ \oint_{\partial M} d^3y \sqrt{|h|} \epsilon n^a h^{bc} \partial_b \left( \delta g_{ac} \right) + \delta S^{(m)},
\]  

(37)

where the Gauss-Stokes theorem (7) has been used in the boundary term. Evaluating the terms \( M_c \) and \( N^c \) at the boundary, we have

\[
M_c \big|_{\partial M} = \frac{f(\phi)}{2} g^{ef} \nabla_c \delta g^{cf} = - \frac{f(\phi)}{2} g^{ef} \delta g^{cf} \partial_c \delta g_{ab}
\]

(38)

and

\[
N^c \big|_{\partial M} = - \frac{f(\phi)}{2} g^{ac} \epsilon_{bcdf} \partial_d \delta g_{ab}.
\]

(39)

Using (11) we compute the following terms that appear in the integrals (37)

\[
n^a M_c \big|_{\partial M} = - \frac{f(\phi)}{2} n^a (h^{bc} + \epsilon n^b n^c \partial_c \delta g_{ab})
\]

(40)

and

\[
n_c N^c \big|_{\partial M} = - \frac{f(\phi)}{2} n^b h^{cf} \partial_f (\delta g_{ab}) = 0,
\]

(41)

where we have used the facts that \( n_c h^{ac} = 0, \epsilon^2 = 1 \) and the tangential derivative \( h^{cf} \partial_f (\delta g_{ab}) \) to vanish (e.g., see [17]).

The variation of the action (37) takes the form

\[
\delta S^{(ST)} = \int_M d^4x \sqrt{-g} \left[ \frac{f(\phi)}{2} \left( R_{ab} - \frac{1}{2} g_{ab} R \right) + \frac{1}{2} g_{ab} V(\phi) \right] + \frac{\omega(\phi)}{2} (\nabla_a \phi)_{\nabla_b \phi} - \frac{1}{2} g_{ab} \varepsilon_{cdef} \nabla_c \phi \nabla_d \phi \right) + \frac{1}{2} (g_{ab} \Box f(\phi) - \nabla_a \nabla_b f(\phi)) \right] \delta g^{ab} \\
- \oint_{\partial M} d^3y \sqrt{|h|} \epsilon f(\phi) n^a h^{bc} \partial_b \left( \delta g_{ac} \right) + \delta S^{(m)}.
\]

(42)

As previously mentioned for GR, the last integral can be vanished arguing that, in addition to the variation of the metric \( \delta g^{ab} \), its first derivative \( \partial_c \delta g_{ab} \) to vanish in the boundary. Instead of, we use the boundary term type GYH for ST theories [23, 24]

\[
S^{(ST)}_{GYH} = 2 \oint_{\partial M} d^3y \sqrt{|h|} \epsilon f(\phi) \delta K.
\]

(43)

The variation of this term with respect to \( \delta g^{ab} \) is

\[
\delta S^{(ST)}_{GYH} = \oint_{\partial M} d^3y \sqrt{|h|} \epsilon f(\phi) \delta K.
\]

(44)

Taking into account (19), the above equation gives

\[
\delta S^{(ST)}_{GYH} = \frac{1}{2} \oint_{\partial M} d^3y \sqrt{|h|} \epsilon f(\phi) n^a h^{bc} \partial_b (\delta g_{ac})
\]

(45)

Thus, we can see that the term type GYH cancels with the second integral of the equation (42).
Finally, using (24), the variation of the action of ST theories yields

\[
\delta S^{(ST)} = \int_M d^4x \sqrt{-g} \left[ \frac{f(\phi)}{2} \left( R_{ab} - \frac{1}{2} g_{ab} R \right) + \frac{1}{2} g_{ab} V(\phi) \right] - \frac{\omega(\phi)}{2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \right) + \frac{1}{2} \left( \nabla_a \phi \nabla_b \phi - \nabla_a \phi \nabla_b \phi \right) - \frac{1}{2} T^{(m)}_{ab} \delta g^{ab}.
\]

(46)

Imposing that this variation becomes stationary

\[
\frac{1}{\sqrt{-g}} \frac{\delta S^{(ST)}}{\delta g^{ab}} = 0,
\]

(47)

we get

\[
f(\phi) G_{ab} = T^{(m)}_{ab} + \omega(\phi) \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \right) + \left( \nabla_a \nabla_b f(\phi) - g_{ab} \Box f(\phi) - g_{ab} V(\phi) \right),
\]

(48)

which are the field equations in the metric formalism of ST theories of gravity.

Since the action (27) it depends on the metric as the scalar field \( \phi \), the variation of the action (27) with respect to \( \delta \phi \) is calculated

\[
\delta S^{(ST)} = \int_M d^4x \sqrt{-g} \left[ \frac{1}{2} R \delta f(\phi) - \frac{1}{2} \delta (\omega(\phi) \nabla^c \phi \nabla_c \phi) - \delta V(\phi) \right].
\]

(49)

Allow us to write, \( \delta f(\phi) = \frac{d f(\phi)}{d \phi} \delta \phi = f'_\phi \delta \phi \).

Now, the second term in the integral we can write it as

\[
\delta (\omega(\phi) \nabla^c \phi \nabla_c \phi) = \nabla^c \phi \nabla_c \phi \delta \phi + \omega(\phi) \delta (\nabla^c \phi \nabla^c \phi) = \nabla^c \phi \nabla_c \phi \delta \phi + 2\omega(\phi) \nabla^c \phi \nabla_c \delta \phi.
\]

(50)

Thus, the variation gives

\[
\delta S^{(ST)} = \int_M d^4x \sqrt{-g} \left[ \frac{1}{2} R f'_\phi - \frac{1}{2} \omega_\phi \nabla^c \phi \nabla_c \phi - V_\phi \right] \delta \phi - \int_M d^4x \sqrt{-g} \omega(\phi) \nabla^c \phi \nabla_c \delta \phi.
\]

(51)

we define the following quantity for can be expressed differently the above integral

\[ L^c = \omega(\phi) \nabla^c \phi \delta \phi. \]

The covariant derivative of \( L^c \) is

\[ \nabla_c L^c = \nabla_c (\omega(\phi) \nabla^c \phi \delta \phi) + \omega(\phi) \nabla_c (\nabla^c \phi \delta \phi) = \omega_\phi \nabla_c \phi \nabla^c \phi \delta \phi + \omega(\phi) \nabla_c (\nabla^c \phi \delta \phi). \]

Because

\[ \nabla_c \phi \nabla^c (\delta \phi) = \nabla_c (\delta \phi \nabla^c \phi) = \delta \phi \Box \phi, \]

(53)

the second term in (51) takes the form

\[
\delta S^{(ST)} = \int_M d^4x \sqrt{-g} \left[ \frac{1}{2} R f'_\phi + \frac{1}{2} \omega_\phi \nabla^c \phi \nabla_c \phi + \omega(\phi) \Box \phi - V_\phi \right] \delta \phi - \int_M d^4x \sqrt{-g} \nabla_c L^c.
\]

(54)

Using the Gauss-Stokes theorem (7) at the divergence term, we have

\[
\int_M d^4x \sqrt{-g} \nabla_c L^c = \oint_{\partial M} d^3y \sqrt{|h|} \epsilon n_c L^c = \oint_{\partial M} d^3y \sqrt{|h|} \epsilon n_c \omega(\phi) \nabla^c \delta \phi.
\]

(55)

Imposing that the variation of the scalar field in the boundary vanishes

\[ \delta \phi |_{\partial M} = 0, \]

(56)

we can see that the Gauss-Stokes term cancels-off.

Now, the variation of the term type GYH for ST theories (43) with respect to \( \delta \phi \) yields

\[
\delta S_{GYH}^{(ST)} = \oint_{\partial M} d^3y \sqrt{|h|} \epsilon f'_\phi K \delta \phi,
\]

(57)

because the imposition (56), the above term vanishes. Wherewith, the variation of the action (54) gives

\[
\delta S^{(ST)} = \int_M d^4x \sqrt{-g} \left[ \frac{1}{2} R f'_\phi + \frac{1}{2} \omega_\phi \nabla^c \phi \nabla_c \phi + \omega(\phi) \Box \phi - V_\phi \right] \delta \phi.
\]

(58)
Imposing that this variation become stationary
\[ \frac{1}{\sqrt{-g}} \delta S^{(ST)} = 0, \]
we have
\[ \omega(\phi) \Box \phi + \frac{1}{2} R \phi + \frac{1}{2} \omega_\phi \nabla^\mu \nabla_\mu \phi - V_\phi = 0. \]
which are the field equation for the scalar field in ST theories of gravity.

### 2.3. Field equations in f(R) theories

As a natural extension of GR and higher order theories, f(R) theories emerge, which consider an arbitrary function of the Ricci scalar. The action \( f(R) \) is
\[ S^{(f(R))} = \frac{1}{2} \int_M d^4 x \sqrt{-g} f(R) + S^{(m)}, \]
where \( f(R) \) is a non-linear analytical function of the Ricci scalar and \( S^{(m)} \) is given by (2). In the paper [22], shows how the field equations are obtained taking into account the boundary term type GYH for \( f(R) \). Here show the main results found there.

The variation of the action with respect to \( \delta g^{ab} \) is
\[ \delta S^{(f(R))} = \frac{1}{2} \int_M d^4 x \sqrt{-g} \left( f_R R_{ab} - \frac{1}{2} g_{ab} f(R) + g_{ab} \Box f_R - \nabla_a \nabla_b f_R \right) \delta g^{ab} \]
\[ + \frac{1}{2} \int_M d^4 x (\nabla^a H_e - \nabla_a I^e) + \delta S^{(m)}, \]
where the terms \( H_e \) and \( I^e \) are given by
\[ H_e = f_R g_{ab} \nabla^a \delta g^{ab} - \delta g^{ab} \nabla_a f_R \]
and
\[ I^e = f_R \nabla^a \delta g'^{a} - \delta g'^{a} \nabla^a f_R. \]
Here \( f_R = \frac{df}{dR} \). Using the Gauss-Stokes theorem to the divergence term in the variation and evaluating the terms \( n^e H_e \) and \( n^e I^e \) at the boundary, we have
\[ \delta S^{(f(R))} = \int_M d^4 x \sqrt{-g} \left( f_R R_{ab} - \frac{1}{2} g_{ab} f(R) + g_{ab} \Box f_R - \nabla_a \nabla_b f_R \right) \delta g^{ab} \]
\[ - \oint_{\partial M} d^3 y \sqrt{|h|} \epsilon f_R n^a h^{ab} \partial_b \delta g_{ab} + \delta S^{(m)}. \]
The boundary term type GYH for \( f(R) \) is [24]
\[ S_{GYH}^{(f(R))} = \oint_{\partial M} d^3 y \sqrt{|h|} \epsilon f_R K, \]
The variation of the above action gives
\[ \delta S_{GYH}^{(f(R))} = \oint_{\partial M} d^3 y \sqrt{|h|} \epsilon f_R K e^{R} + \frac{1}{2} \oint_{\partial M} d^3 y \sqrt{|h|} \epsilon n^{a} h^{ab} \partial_{b} \delta g_{ab}. \]
The second term of the above equation cancels the boundary term of the equation (65), but in addition needs to impose \( \delta R = 0 \) in the boundary to obtain the field equations [22, 25].

Taking into account the variation of the matter action (24) and imposing that the variation for \( f(R) \) theories becomes stationary
\[ \frac{1}{\sqrt{-g}} \frac{\delta S^{(f(R))}}{\delta g^{ab}} = 0, \]
thus, we have
\[ f_R R_{ab} - \frac{1}{2} g_{ab} f(R) + g_{ab} \Box f_R - \nabla_a \nabla_b f_R = T^{(m)}_{ab}. \]
which are the field equations for \( f(R) \) theories.

In this section we recover in the variational approach the set of field equations for GR, ST and \( f(R) \) theories emphasizing the boundary problem. We explore directly the equivalence between ST and \( f(R) \) theories at the GYH boundary term, and it is clear that the boundary term makes the theory well defined mathematical
problem. It is important to notice that in the literature the equivalence problem has been widely studied [10–15], but in this paper it was shown how the field equations were obtained for ST theories with the GYH boundary term, in complete agreement with previous work [9, 14, 15, 25], but connecting a previous work [22] through the equivalence in the important issue of the boundary for both theories. Also, the condition to get the equation to \( \phi \), it had to be imposed on the boundary that the variation \( \delta \phi \) be equal to zero. The variational approach in \( f(R) \) gravity brings the condition \( \delta R = 0 \) at the boundary in total agreement with the equivalence between both theories, showing the mathematical power of the equivalence.

A more detailed analysis of the equivalences will be discussed in the next section.

3. Equivalence between ST and \( f(R) \) theories

The equivalence between ST and \( f(R) \) theories has been broadly studied at the classical level, e.g., in [10–15], but also a quantum level [26, 27]. In this paper shows the equivalence between the actions and the field equations, but as we will see in the next section, in addition we will show them in the cosmological perturbations.

We start from the following ST action without a kinetic term in the scalar field

\[
S = \int d^4x \sqrt{-g} \left( \psi(\phi)R - V(\phi) \right),
\]

(70)

donde \( \phi \) has been included as an auxiliary field.

when \( f_{\phi\phi} \neq 0 \) in the above action, we can set

\[
\psi = f_\phi
\]

(71)

\[
V(\phi) = \phi f_\phi - f(\phi) = \phi \psi^2 - f(\phi),
\]

(72)

Thus, the action (70) takes the form

\[
S = \int d^4x \sqrt{-g} \left( f_\phi(R - \phi) + f(\phi) \right).
\]

(73)

If \( \phi = R \), we have

\[
\psi = f_R
\]

(74)

and we recover the action (61). Moreover, the variation with respect to \( \phi \) of the above action gives

\[
f_{\phi\phi}(R - \phi) = 0,
\]

(75)

if \( f_{\phi\phi} \neq 0 \) it implies that

\[
\phi = R.
\]

(76)

The action (70) corresponds to the action (27) of ST theories with the parameter \( \omega(\phi) = 0 \).

If we start with the field equations \( f(R) \) (rewriting the equations (69) for to include the Einstein tensor \( G_{ab} \))

\[
G_{ab} f_R = T^{(m)}_{ab} + \nabla_a \nabla_b f_R - g_{ab} \Box f_R + \frac{1}{2} g_{ab} (f - R f_R),
\]

(77)

Taking into account (74) in the above field equations, we get

\[
G_{ab} \psi = T^{(m)}_{ab} + \nabla_a \nabla_b \psi - g_{ab} \Box \psi + \frac{1}{2} g_{ab} (f(\phi) - \phi \psi)
\]

\[
= T_{ab} + \nabla_a \nabla_b \psi - g_{ab} \Box \psi - g_{ab} V(\phi),
\]

where it has been used (72), with the potential rescaled by \( \frac{1}{2} \). The above equations are the field equations (48) for ST theories with the parameter \( \omega(\phi) = 0 \).

4. Cosmological perturbations

In this section we study the Friedmann equations in a homogeneous and isotropic universe with the metric Friedmann-Lematre-Robertson-Walker (FLRW) as the background metric for the ST and \( f(R) \) theories. Then we calculate the linear cosmological perturbations under conformal-Newtonian gauge for the theories above mentioned. Note that the equations found by \( f(R) \) theories for both the background and the perturbed ones were found under the equivalence relations with the ST theories.

4.1. Background universe

Consider a statistically spatially homogeneous and isotropic universe with the spatially flat FLRW metric as background.
The energy conservation is
\[ \nabla^b T_{ab} = 0, \]  
(79)
where
\[ T_{ab} = p g_{ab} + (p + \rho) u_a u_b \]  
(80)
is the stress-energy tensor for perfect fluid. With this, the energy conservation gives
\[ \rho + 3H(p + \rho) = 0. \]  
(81)
Here, \( p \) is the fluid pressure, \( \rho \) the energy density, \( u^\alpha \) is the four-velocity of the fundamental observers.

The Friedmann equations for the evolution of the background in ST theories are [21]
\[ 3H^2f = \rho a^2 + \frac{\omega}{2} \phi'^2 + Va^2 - 3Hf' \]  
(82)
and
\[ -(2H' + H^2)f = pa^2 + \frac{1}{2} \omega \phi'^2 + Hff' + f'' - Va^2, \]  
(83)
where \( H \equiv \frac{d \ln a}{d \eta} \). The equation for the evolution of the scalar field is
\[ \omega(\phi'' + 2H\phi') = 3f_0(H' + H^2) - \frac{1}{2} \omega \phi'^2 - V_0 a^2. \]  
(84)

To obtain the Friedmann equations for BD theory, must be taking into account the relations (28) in the friedmann equation for ST theories.

From the equivalence relation (74), we have
\[ \psi_0 = f_{RR}, \]  
(85)
\[ \psi_{\phi 0} = f^{(3)}_R, \]  
(86)
where \( f^{(3)}_R = \frac{df}{dR} \). Replacing (72) and the above relations in the equations (82) and (83) with the parameter \( \omega (\phi) = 0 \), we come to Friedmann equations for the \( f(R) \) theories
\[ 3H^2f_R = \rho a^2 + \frac{a^2}{2}(Rf_R - f(R)) - 3Hf_{R R}R', \]  
(87)
and
\[ -(2H' + H^2)f_R = pa^2 + Hf_Rf_{R R} + \frac{a^2}{2}(f(R) - Rf_R) + R'f_{R R} + R'^2f^{(3)}_R. \]  
(88)

As mentioned above, one of the motivations for MG theories, is to explain the accelerating expansion of the Universe. For ST theories, given a potential \( V(\phi) \) [28–30] we can get a universe in accelerating expansion, while for \( f(R) \) theories, the same function is responsible for achieve it [31–33].

Through the equivalence we have found the Friedmann equations for the theories \( f(R) \) starting from the ST equations, taking the parameter \( \omega = 0 \). Next we will find the Friedmann equations perturbed for both theories in a complete general framework.

### 4.2. Equivalence between cosmological perturbations in ST and \( f(R) \) gravity

The line element of the perturbed universe is
\[ ds^2 = a^2(\eta)(-d\eta^2 + \delta_{\mu\nu}dx^\mu dx^\nu), \]  
(89)
where \( A, B_\mu, D \) and \( E_{\mu\nu} \) are metric perturbations. Now, we can decomposed the 0 − i and the i − j components of the metric tensor into, scalar, vector and tensor parts
\[ B_\mu = -B_\mu + B_\mu^V, \quad \text{where} \quad \delta^{\mu\nu}B_\mu^V = 0 \]  
(90)
and
\[ E_{\mu\nu} = E_{\mu\nu}^S + E_{\mu\nu}^V + E_{\mu\nu}^T, \]  
(91)
here
\[ E_{\mu\nu}^S = \left( \partial_\mu \partial_\nu - \frac{1}{3} \delta_{\mu\nu} \nabla^2 \right) E, \]  
(92)
\[ E_{\mu\nu}^V = -\frac{1}{2} (\nabla_\mu E_\nu + \nabla_\nu E_\mu), \quad \text{where} \quad \delta^{\mu\nu}E_{\mu\nu} = 0, \]  
(93)
Due to this division perturbation to first order, we can study the scalar, vector and tensor perturbations separately. In the following we show the perturbations for ST theories and the equivalences with $f(R)$ gravity into the components above mentioned

### 4.2.1. Scalar perturbations
Scalar metric perturbations are described by the line element

$$\delta \mu \nu E_{\mu \nu}^E = 0, \quad \delta \mu \nu E_{\mu \nu}^F = 0. \quad (94)$$

where $A, B, \psi$ and $E$ are scalar perturbations. The curvature perturbation $\psi$ is defined by $\psi \equiv D + \frac{1}{2} \nabla^2 E$. To find the linear perturbations of ST theories, the field equations (48) are perturbed, taking into account the metric (95). Here, $\delta \phi$ represents the perturbation of the scalar field. The perturbed Friedmann equations are

$$\begin{align*}
-2\nabla^2 \psi + 6H \psi' + 6H^2 A + 2H \nabla^2 B - 2H \nabla^2 E' \right)^2 (\phi) - 3H^2 \delta \phi = -a^2 \delta \rho \\
+ \omega(\phi)(\tilde{\omega}'^2 A - \tilde{\omega}' \delta \phi) - \frac{1}{2} \tilde{\omega}(\tilde{\omega}'^2 \delta \phi + 3H(\tilde{\omega}' \delta \phi)' - 6H\tilde{\omega}' A - \tilde{\omega}' \theta \delta \phi) \\
- \tilde{\omega}' \theta (\delta^2 B + 3 \psi' - \delta^2 E') - a^2 \tilde{\nabla} \delta \phi, 
\end{align*} \quad (96)$$

which is the $0 - 0$ perturbed component,

$$-2(\psi' + \mathcal{H} \lambda) f(\phi) = -a^2 (p + \tilde{p})(v_{,\mu} - B_{,\mu}) - \omega(\phi) \tilde{\omega}' \delta \phi - \tilde{\omega}' \delta \phi' - \tilde{\omega}' \theta \delta \phi + A_{,\mu} \tilde{\omega}' \theta + H\tilde{\omega}' \delta \phi, \quad (97)$$

which is the $0 - \mu$ perturbed component,

$$\begin{align*}
-2(\mathcal{H}' + \mathcal{H}^2) \tilde{\omega}' \delta \phi \delta \omega &+ [2\psi'' - \nabla^2(\psi - A) + \mathcal{H}(2A' + 4\psi') + (4\mathcal{H}' + 2\mathcal{H}^2) A \\
+ \nabla^2 B' + 2H \nabla^2 B \tilde{f}(\phi) \delta \omega - (\nabla^2 E'' + 2H \nabla^2 E') \tilde{f}(\phi) \delta \omega &+ (\psi - A' - B - 2HB + E'' + 2HE) \delta \omega \tilde{f}(\phi) = \delta p a^2 \delta \mu + \tilde{p} a^2 \left( \Pi_{,\mu} - \frac{3}{3} \delta_{\mu \nu} \nabla^2 \Pi \right) \\
+ \omega(\phi)(\tilde{\omega}'^2 \delta \phi - \tilde{\omega}' \phi' \delta \phi) - \frac{1}{2} \tilde{\omega}(\tilde{\omega}'^2 \delta \phi + \tilde{\omega}' \delta \phi' \delta \phi - \tilde{\omega}' \theta \delta \phi &+ H(\tilde{\omega}' \delta \phi)' - 2\tilde{\omega}' \delta \phi) \delta \omega + \tilde{\omega}' \theta (B_{,\mu} - 2\psi' \delta \mu - E_{,\mu}') + (\tilde{\omega}' \delta \phi)' \delta \omega \\
- 2(\tilde{\omega}' \theta)' A \delta \mu + \tilde{\omega}' \theta (\delta^2 B - A') \delta \mu - a^2 \tilde{\nabla} \delta \phi \delta \mu, \quad (99) 
\end{align*}$$

and finally, the $\mu - \nu$ perturbed component,}

$$\begin{align*}
\tilde{f}(\phi)(\Psi - \Phi) &= a^2 \tilde{p} \Pi + \tilde{f}_0 \delta \phi + \tilde{f}_0 \delta \phi'(B - E'), \quad (100) 
\end{align*}$$

where has been used the so-called Bardeen potentials, $\Phi$ and $\Psi$ [35], which are defined by

$$\begin{align*}
\Phi &\equiv A + (B - E)' + \mathcal{H}(B - E') \quad (101) \\
\Psi &\equiv \psi - \mathcal{H}(B - E'). \quad (102) 
\end{align*}$$

Now, if there is no anisotropic pressure, i.e., if $\Pi = 0$, the two potentials can be related to each other as

$$\begin{align*}
\Psi &= \Phi + \frac{\tilde{f}_0}{f} \delta \phi + \frac{\tilde{f}_0}{f} \delta \phi'(B - E'), \quad (103) 
\end{align*}$$

For $f = 1$, it implies that $\Phi = \Psi$, which corresponds to the case of GR in the absence of anisotropic pressure. We can see, that if we work in the Newtonian gauge, i.e., $(E = B = 0)$, we obtain of the equation (100), the following

$$\begin{align*}
\tilde{f}(\phi)(\Psi - \Phi) &= a^2 \tilde{p} \Pi + \tilde{f}_0 \delta \phi, \quad (104) 
\end{align*}$$

or in absence of anisotropic pressure

$$\begin{align*}
\Psi &= \Phi + \frac{\tilde{f}_0}{f} \delta \phi. \quad (105) 
\end{align*}$$
The perturbed equation of the evolution of the scalar field (60) is (see appendix B)
\[
\omega [-\ddot{\phi} + 2\dot{\phi}'A + \nabla^2 \phi - 2H\dot{\phi}' + 4H\dot{\phi}'A + (A' + \nabla^2 B + 3\psi - \nabla^2 E')\dot{\phi}]
- \omega_0 (\ddot{\phi}' + 2H\dot{\phi}')] + \frac{1}{2}a^2 \dot{\phi}' R + 3(H' + H^2)\dot{\phi} + \frac{1}{2}a_0 (-2\dot{\phi}' + 2\dot{\phi}'A) - \frac{1}{2}a_0 \ddot{\phi}'\delta \phi' = 0,
\]
(106)
where $\delta R$ is (for more details, e.g., see [36])
\[
\delta R = a^{-2} (-6\Psi'' + 2\nabla^2 (2\psi - A) - 6H(A' + 3\psi') + 12(H' + H^2)A - 2\nabla^2 B' - 6H\nabla^2 B + 2\nabla^2 E' + 6H\nabla^2 E').
\]
(107)
Now, we find the perturbed Friedmann equation for $f(R)$ theories, starting from the equations (96)-(99), which are the perturbed Friedmann equations for ST theories with the parameter $\omega(\phi) = 0$, under the equivalence relations between both theories.

From the equation (72), we take
\[
\tilde{V}_0 = \tilde{\phi}^\mu \tilde{\psi}_\mu.
\]
(108)
To relate the above equation in terms of $R$ and $f_R$, the equivalence relations (76) and (85) are taken, wherewith we get
\[
\tilde{V}_0 = \tilde{R}^{\frac{\nu}{f_R}},
\]
(109)
where the potential has been rescaled by $\frac{1}{f'}$.

Thus, the perturbed Friedmann equations of $f(R)$ theories take the form
\[
[-2\nabla^2 \psi + 6H\psi' + 6\nabla^2 A + 2H\nabla^2 B - 2H\nabla^2 E]\tilde{f}_R - 3\nabla^2 R\delta R = -a^3 \dot{\phi}' - \frac{1}{2}a^2 \tilde{R}^2 R_{\delta R} \delta R + 3H(\tilde{f}_R R' + f^{(3)}_R R' \delta R) - 6\tilde{f}_R R' A - \tilde{f}_R \delta^2 R - \tilde{f}_R R'(\delta^2 B + 3\psi' - \delta^2 E'),
\]
(110)
which is the $0 - 0$ perturbed component,
\[
-2(\psi' + H\Lambda)_{R R} = -a^3 (\hat{\phi} + \hat{\chi})_{\nu, \mu} - \tilde{f}_R \delta_{\nu, \mu} R' - f^{(3)}_R R' \delta_{\nu, \mu} R,
\]
(111)
which is the $0 - \mu$ perturbed component,
\[
2(\psi' + H\Lambda - H' B + H^2 B)_{R R} = a^3 (\hat{\phi} + \hat{\chi})_{\nu, \mu} + B_{\mu, \nu}, \tilde{f}_R R' + f^{(3)}_R R' \delta_{\nu, \mu} R - a_0 \tilde{f}_R R' - H\delta_{\nu, \mu} R,
\]
(112)
which is the $\mu - 0$ perturbed component, and
\[
-2(\psi' + H^2)\tilde{f}_R R' \delta_{\nu, \mu} + [2\psi'' - \nabla^2 (\psi - A) + H(2A' + 4\psi') + (4H'^2 + 2H^2) A + \nabla^2 B' + 2H\nabla^2 B] \tilde{f}_R \delta_{\nu, \mu} - (\nabla^2 E' + 2H\nabla^2 E') \tilde{f}_R \delta_{\nu, \mu} + (\psi - A - B' - 2H B + E' \}_{\nu, \mu}) \tilde{f}_R = \delta a^3 \delta_{\nu, \mu} + \hat{\phi} a^3 \left(\Pi_{\nu, \mu} - \frac{1}{3} \delta_{\nu, \mu} \nabla^2 \Pi\right)
\]
\[
- \frac{1}{a^2} a^2 \tilde{R}^2 R_{\delta R} \delta_{\nu, \mu} + \tilde{f}_R (\partial_{\nu} \partial_{\mu} - \delta_{\nu, \mu} \partial^2) \delta R + H(\tilde{f}_R R' + f^{(3)}_R R' \delta R - 2\tilde{f}_R R'(A) \delta_{\nu, \mu}) \delta_{\nu, \mu}
\]
\[
+ \tilde{f}_R R'(B_{\nu, \mu} - 2\psi' \delta_{\nu, \mu} - E'_{\nu, \mu}) + f^{(3)}_R R' \delta_{\nu, \mu} + 2f^{(3)}_R R' \delta_{\nu, \mu} + f^{(3)}_R R' \delta_{\nu, \mu} \delta_{\nu, \mu} + 2f^{(3)}_R R' \delta_{\nu, \mu} + f^{(3)}_R R' \delta_{\nu, \mu} \delta_{\nu, \mu},
\]
(113)
is $\mu - \nu$ perturbed component.

The off-diagonal component, after having calculated the trace of the above equation
\[
\tilde{f}_R (\Phi - \Phi) = a^2 \phi \Pi + \tilde{f}_R R \delta R + \tilde{f}_R R'(B - E'),
\]
(114)
For $\Pi = 0$, we take the relations between potentials
\[
\Psi = \Phi + \tilde{f}_R R \delta R + \tilde{f}_R R'(B - E'),
\]
(115)
Taking $f(R) = R$, we get the relations of GR $\Phi = \Psi$ is absence of anisotropic pressure.
4.2.2. Vector perturbations

Vector perturbations are described by the line element
\[ ds^2 = a^2(\eta)[ -dt^2 + 2B_{\mu}^{(V)}dy^\mu + (\delta_{\mu\nu} - 2E_{\mu\nu}^{(V)})dx^\mu dx^\nu]. \] (116)

Following the same procedure as the one used to find the scalar perturbations, we obtain the vector perturbations,
\[ \frac{1}{2} \nabla^2(E'_\mu - B_\mu)\tilde{f}(\phi) = a^2(\ddot{\rho} + \ddot{\varphi})(v_\mu - B_\mu), \] (117)
which is the 0 − \mu perturbed component,
\[ \left( \frac{1}{2} \nabla^2(B_\mu - E'_\mu) + 2a^{-2}(H'_\mu - H_\mu^2)B_\mu \right)\tilde{f}(\phi) = -a^2(\ddot{\rho} + \ddot{\varphi})v_\mu - (\ddot{f}_R(\ddot{\varphi}))^2B_\mu + (f_0\ddot{\varphi})^2B_\mu + 2f_0\ddot{\varphi}E'_\mu, \] (119)
which is the \mu − 0 perturbed component,
\[ (B'_\mu + 2H(B_\mu - E'_\mu) - 2HE'_\mu)\tilde{f}(\phi) = -a^2\Pi_{\mu\nu} - f_0\ddot{\varphi}B_{\mu\nu} + f_0\ddot{\varphi}E'_\mu, \] (120)
which is the \mu − \nu perturbed component. The 0 − 0 component does not contribute to the vector perturbations.

Following the same procedure for the equivalence between both theories shown in the scalar perturbations, we obtain
\[ \frac{1}{2} \nabla^2(E'_\mu - B_\mu)\tilde{f}_R = a^2(\ddot{\rho} + \ddot{\varphi})(v_\mu - B_\mu), \] (121)
which is the 0 − \mu perturbed component,
\[ \left( \frac{1}{2} \nabla^2(B_\mu - E'_\mu) + 2a^{-2}(H'_\mu - H_\mu^2)B_\mu \right)\tilde{f}_R = -a^2(\ddot{\rho} + \ddot{\varphi})v_\mu - (f_R(\ddot{\varphi}))^2B_\mu + 2f_0\ddot{\varphi}E'_\mu, \] (122)
which is the \mu − \nu component.

4.2.3. Tensor perturbations

The procedure to get to the tensor perturbations is the same as the scalar and vector perturbations. The perturbed equation for the \mu − \nu perturbed component is
\[ (E''_{\mu\nu} - \nabla^2E_{\mu\nu} + 2\mathcal{H}E'_{\mu\nu})\tilde{f}(\phi) = a^2\Pi_{\mu\nu} - f_0\ddot{\varphi}E_{\mu\nu}. \] (123)
The other components don’t contribute to this perturbation type. Using the equivalences between ST and f(R) theories, we get the perturbed equation for f(R) gravity,
\[ (E''_{\mu\nu} - \nabla^2E_{\mu\nu} + 2\mathcal{H}E'_{\mu\nu})\tilde{f}_R = a^2\Pi_{\mu\nu} - f_R\ddot{\varphi}E_{\mu\nu}. \] (124)
Perturbed Friedmann equations were found to both theories. It is important to write down that the equations to f(R) theories were obtained starting from the equations for ST theories, with the parameter \( \omega = 0 \). Our analysis is general since it is not restricted to a specific gauge. Once the general form for the equivalence was founded, we will concentrate in two examples but now in a specific gauge, in our case we choose the Newtonian-gauge. Henceforth, we are going to use the gauge above mentioned for all our calculations.

4.3. Hu-sawicki and starobinsky models

Below are shown two examples from the equivalences between both f(R) and ST theories under conformal-Newtonian gauge. The first is Hu-Sawicki model \[28\], which is important since it is able of reproducing the accelerated expansion of the Universe \[37–39\] besides to satisfy the tests of the solar system \[28\]. Although the reconstruction of the potential has already been studied, we show the equivalences, in the Friedmann equations of the background and the scalar perturbed ones. Now, this model is given by \[40\]
\[ f(R) = -\frac{2\Lambda}{1 + 2\epsilon \left( \frac{\Lambda}{3\epsilon} \right)^\pi}, \] (125)
where \( \Lambda \) is a constant energy scale whose value coincides with the measured value \( \Lambda = \Lambda_{obs} = 3H_0^2\Omega_\Lambda \) and \( \epsilon \ll 1 \) is a small positive deformation parameter. We note that the derivative of f(R) is
\[ f_R = -\epsilon \left( \frac{4\Lambda}{R} \right)^{n+1}. \]  

From the equivalence (74), we have
\[ R = 4\Lambda \left( \frac{\left| \phi \right|}{\epsilon} \right)^{\frac{n}{n+1}}. \]

Thus, rewriting the function (125) in terms of the scalar field, gives
\[ f = -2\Lambda + 4\Lambda \frac{\epsilon}{n} \left( \frac{\left| \phi \right|}{\epsilon} \right)^{\frac{n}{n+1}}. \]

Using the equivalence in the potential (72). The potential for the Hu-Sawicki model is
\[ V(\phi) = 2\Lambda \left( 1 - 2\epsilon \frac{n+1}{n} \left( \frac{\left| \phi \right|}{\epsilon} \right)^{\frac{n}{n+1}} \right). \]

Once the potential is obtained, we calculate the Friedmann equations in the background in terms of the scalar field, which are
\[ 3H^2\phi = \rho a^2 - 3H\phi' + 2\Lambda \left( 1 - 2\epsilon \frac{n+1}{n} \left( \frac{\left| \phi \right|}{\epsilon} \right)^{\frac{n}{n+1}} \right) a^2 \]  

and
\[ -(2H' + \mathcal{H}^2)\phi = \rho a^2 + \phi'' + \mathcal{H}\phi' - 2\Lambda \left( 1 - 2\epsilon \frac{n+1}{n} \left( \frac{\left| \phi \right|}{\epsilon} \right)^{\frac{n}{n+1}} \right) a^2. \]

From the relation (74), the Friedmann equations in the formalism \( f(R) \) take the form
\[ -3H^2 \epsilon \left( \frac{4\Lambda}{R} \right)^{n+1} = \rho a^2 + 2a^2\Lambda \left( 1 - 2\epsilon \frac{n+1}{n} \left( \frac{4\Lambda}{R} \right)^n \right) - 3\epsilon (n+1) \mathcal{H} \left( \frac{R}{R} \right)^{n+1}, \]

and
\[ (2\mathcal{H} + \mathcal{H}^2)\epsilon \left( \frac{4\Lambda}{R} \right)^{n+1} = \rho a^2 + \epsilon (n+1) \left( \frac{R'}{R} \right) \mathcal{H} \left( \frac{4\Lambda}{R} \right)^{n+1} - 2\Lambda \left( 1 - 2\epsilon \frac{n+1}{n} \left( \frac{4\Lambda}{R} \right)^n \right) \]
\[ + \epsilon (n+1) \left( \frac{R''}{R} \right) \left( \frac{4\Lambda}{R} \right)^{n+1} - \epsilon (n+1) (n+2) \left( \frac{R'^2}{2} \right) \left( \frac{4\Lambda}{R} \right)^{n+1}. \]

The perturbed Friedmann equations (time-time and space-space components) in terms of the scalar field are
\[ [-2\nabla^2\Psi + 6\mathcal{H}\Psi' + 6\mathcal{H}^2\phi] \tilde{\phi} - 3\mathcal{H}^2 \delta \phi = -a^2\delta \rho + 3\mathcal{H}\delta \phi' - 6\mathcal{H}\delta \phi' \Phi - \nabla^2 \delta \phi \]
\[ - 3\delta \phi' - 4a^2\Lambda \left( \frac{\left| \phi \right|}{\epsilon} \right)^{\frac{n}{n+1}} \delta \phi \]  

and
\[ [(2\Psi'' + \nabla^2(\Phi - \Psi) + \mathcal{H}(2\Psi' + 4\Psi') + (4\mathcal{H}' + 2\mathcal{H}^2)\Phi)] \delta_{\mu\nu} + (\Psi - \Phi)_{\mu\nu} \tilde{\phi} \]
\[ + (-2\mathcal{H} - \mathcal{H}^2) \delta \phi \delta_{\mu\nu} = a^2 \left( \delta_{\Phi} \delta_{\mu\nu} + \tilde{\phi} \left( \Pi_{\mu\nu} - \frac{1}{3} \delta_{\mu\nu} \nabla^2 \Pi \right) \right) \delta_{\mu\nu} + \delta_{\phi} \delta_{\mu\nu} \delta \phi + \nabla^2 \delta \phi \delta_{\mu\nu} \]
\[ + \left( \delta \phi'' - 2\tilde{\phi}'' \Phi - 2\mathcal{H}\tilde{\phi}' \Phi + \mathcal{H} \delta \phi' - (2\Psi' + \Phi) \tilde{\phi}' - 4a^2\Lambda \left( \frac{\left| \phi \right|}{\epsilon} \right)^{\frac{n}{n+1}} \delta \phi \right) \delta_{\mu\nu} \]

where we have used
\[ V_\phi = 4\Lambda \left( \frac{\left| \phi \right|}{\epsilon} \right)^{\frac{n}{n+1}}. \]
Now, the Friedmann equations in the formalism \( f(R) \), using the equivalence relations, take the form

\[
-\left(-2\nabla^2 \Psi + 6\mathcal{H}\Psi' + 6\mathcal{H}^2 \Phi\right) e^\left(\frac{4\mathcal{A}}{R}\right)^{n+1} - 3\mathcal{H}^2 e^\left(\frac{n+1}{R}\right)\left(\frac{4\mathcal{A}}{R}\right)^{n+1} \frac{\delta R}{R} = -a^2\delta \rho
\]

\[
- \frac{a^2 \mathcal{R}}{2} \left(\frac{4\mathcal{A}}{R}\right)^{n+1} \frac{\delta R}{R} - \frac{\mathcal{H}^2 e^\left(\frac{n+1}{R}\right)}{\left(\frac{4\mathcal{A}}{R}\right)^{n+1}} \nabla^2 \delta R
\]

\[
+ 3\mathcal{H} \left(\frac{e^\left(\frac{n+1}{R}\right)}{\left(\frac{4\mathcal{A}}{R}\right)^{n+1}} \delta R' - \mathcal{R}' e^\left(\frac{n+1}{R}\right)\left(\frac{4\mathcal{A}}{R}\right)^{n+1} \frac{R^2}{R^2} \delta R \right)
\]

\[
- 6\mathcal{H} \frac{e^\left(\frac{n+1}{R}\right)}{\left(\frac{4\mathcal{A}}{R}\right)^{n+1}} R^2 \Phi - 3\frac{e^\left(\frac{n+1}{R}\right)}{\left(\frac{4\mathcal{A}}{R}\right)^{n+1}} \frac{R}{R} \delta \Psi'
\]  

(137)

and

\[
- \left[2\Psi'' + \nabla^2 (\Phi - \Psi) + \mathcal{H} (2\Phi' + 4\Psi') + (4\mathcal{H}' + 2\mathcal{H}^2) \Phi\right] \delta_{\mu\nu} + (\Psi - \Phi)_{,\mu\nu} \right] e^\left(\frac{4\mathcal{A}}{R}\right)^{n+1}
\]

\[
- (2\mathcal{H} + \mathcal{H}') \frac{e^\left(\frac{n+1}{R}\right)}{\left(\frac{4\mathcal{A}}{R}\right)^{n+1}} \frac{\delta R}{R} \delta_{\mu\nu} = a^2 (\delta \rho \delta_{\mu\nu} + \bar{p} (\Pi_{,\mu\nu} - \frac{1}{3} \delta_{\mu\nu} \nabla^2 \Pi))
\]

\[
- \frac{a^2 \mathcal{R}}{2} \frac{e^\left(\frac{n+1}{R}\right)}{\left(\frac{4\mathcal{A}}{R}\right)^{n+1}} \frac{\delta R}{R} \delta_{\mu\nu} + \frac{e^\left(\frac{n+1}{R}\right)}{\left(\frac{4\mathcal{A}}{R}\right)^{n+1}} \left(\partial_{\mu}\partial_{\nu} - \delta_{\mu\nu} \nabla^2\right) \delta R
\]

\[
+ \mathcal{H} \left(\frac{e^\left(\frac{n+1}{R}\right)}{\left(\frac{4\mathcal{A}}{R}\right)^{n+1}} \delta R' - \mathcal{R}' e^\left(\frac{n+1}{R}\right)\left(\frac{4\mathcal{A}}{R}\right)^{n+1} \delta R \right)
\]

\[
- 2\mathcal{H} \mathcal{R}' \frac{e^\left(\frac{n+1}{R}\right)}{\left(\frac{4\mathcal{A}}{R}\right)^{n+1}} \Phi \delta_{\mu\nu} + \left(-2\mathcal{R} e^\left(\frac{n+1}{R}\right)\left(\frac{4\mathcal{A}}{R}\right)^{n+1} \delta R'
\]

\[
+ R^2 e^\left(\frac{n+1}{R}\right)\left(\frac{4\mathcal{A}}{R}\right)^{n+1} \frac{\delta R}{R} + \frac{e^\left(\frac{n+1}{R}\right)}{\left(\frac{4\mathcal{A}}{R}\right)^{n+1}} \frac{R}{R} \delta R
\]

\[
- R e^\left(\frac{n+1}{R}\right)\left(\frac{4\mathcal{A}}{R}\right)^{n+1} \delta R - 2\Phi \left(-R e^\left(\frac{n+1}{R}\right)\left(\frac{4\mathcal{A}}{R}\right)^{n+1} \delta R
\]

(138)

where has been used the following relation

\[
\delta \phi = \frac{e^\left(\frac{n+1}{R}\right)}{\left(\frac{4\mathcal{A}}{R}\right)^{n+1}} \frac{\delta R}{R}.
\]  

(139)

The second model to discuss is the Starobinsky model [41], which is a cosmic inflation model. Whose perturbations in the inflationary era were first discussed by Mukhanok and Starobinsky himself [42, 43]. His predictions agree with the recent CMB data [44]. For more discussions on this model, see e.g., [45].

The Starobinsky model is given by

\[
f(R) = R + \frac{R^2}{6M^2},
\]  

(140)

where the constant \( M \) has mass dimension. Performing the same above procedure for to construct the potential, we start from

\[
f_k = 1 + \frac{R}{3M^2}.
\]  

(141)

From the equivalence (74), we have

\[
R = 3M^2(\phi - 1).
\]  

(142)

Thus, the potential gives

\[
V(\phi) = \frac{3}{4}M^2(\phi - 1)^2,
\]  

(143)

where the potential has been rescaled by \( \frac{1}{2} \). Found the potential, we calculate the Friedmann equations in terms of the scalar field
\[ 3\mathcal{H}^2\phi = \rho a^2 - 3\mathcal{H}\phi' + \frac{3}{4}M^2(\phi - 1)a^2 \]

and

\[ -(2\mathcal{H}' + \mathcal{H}^2)\phi = \rho a^2 + \phi'' + \mathcal{H}\phi' - \frac{3}{4}M^2(\phi - 1)a^2. \]

From the relation (74), we obtain the Friedmann equations for the \( f(R) \) formalism

\[ 3\mathcal{H}^2\left(1 + \frac{R}{3M^2}\right) = \rho a^2 + a^2\frac{R^2}{12M^2} - \frac{\mathcal{H}R'}{M^2} \]

\[ -(2\mathcal{H} + \mathcal{H}^2)\left(1 + \frac{R}{3M^2}\right) = \rho a^2 + \mathcal{H}\mathcal{R}' + a^2\frac{R^2}{12M^2} + \frac{R''}{M^2}. \]

Now, the perturbed Friedmann equations (time-time and space-space components) in terms of the scalar field are

\[ -2\nabla^2\Psi + 6\mathcal{H}\Psi' + 6\mathcal{H}^2\Phi\tilde{\phi} - 3\mathcal{H}^2\delta\phi = -a^2\delta\rho + 3\mathcal{H}\delta\phi' - 6\mathcal{H}\tilde{\phi}' \Phi - \nabla^2\delta\phi \]

\[ -3\tilde{\Psi}' \frac{\delta a^2}{2M^2(\phi - 1)}a^2\delta\phi, \]

where

\[ \tilde{\Psi}' \frac{\delta a^2}{2M^2(\phi - 1)}a^2\delta\phi, \]

and

\[ (\nabla^2\Psi + \mathcal{H}\Psi' + \mathcal{H}^2\Phi)[\tilde{\phi} - 3\mathcal{H}^2\delta\phi = -a^2\delta\rho + 3\mathcal{H}\delta\phi' - 6\mathcal{H}\tilde{\phi}' \Phi - \nabla^2\delta\phi \]

\[ -3\tilde{\Psi}' \frac{\delta a^2}{2M^2(\phi - 1)}a^2\delta\phi. \]

We show how to obtain the perturbed Friedmann equations for Hu-Sawicki and Starobinsky models for \( f(R) \) theories from the perturbed equations for ST theories. In the next subsection, the linear evolution of matter density perturbations under the sub-horizon approximation in the conformal-Newtonian gauge for each of the theories will be calculated.

### 4.4. Sub-horizon approximation in ST and \( f(R) \) theories under conformal-newtonian gauge

The perturbed energy conservation equations are [36]

\[ \frac{\delta'}{(1 + w)}(\nabla^2\nu + 3\Psi') + 3\mathcal{H}\left(w\delta - \frac{\delta p}{\rho}\right) \]

\[ \delta' = (1 + w)(\nabla^2\nu + 3\Psi') + 3\mathcal{H}\left(w\delta - \frac{\delta p}{\rho}\right) \]

and

\[ \nu' = -\mathcal{H}(1 - 3w)\nu - \frac{w'}{1 + w}\nu + \frac{\delta p}{\rho(1 + w)} + \frac{2}{3}\frac{w}{1 + w}\nabla^2\nu + \Phi, \]

where \( \delta = \delta \rho / \rho_p \) is the perturbation of the relative energy density and \( \nu \) is the perturbation of velocity.

The above equations in the matter domain, i.e., \( w = 0 \) (do not confuse with the parameter \( \omega \) of ST theories) and taking \( \Pi = 0 \), takes the form in the space Fourier as
\[ \delta^i_m + \mathcal{H} \delta^i_{m'} + k^2 \Phi - 3 \mathcal{H} \Psi' - 3 \Psi'' = 0, \]

\[ \delta^0_m + \mathcal{H} \delta^0_{m'} + k^2 \Phi = 0. \]

To obtain the Poisson type equation in the sub-horizon approach for ST theories, the perturbed time-time component (96) in the Fourier space is taken

\[ 2k^2 \Psi f = -a^2 \delta \rho + k^2 f_0 \delta \phi. \]

To find \( \delta \phi \) of the above equation, we take \( \bar{\omega} \) constant in the equation (106), thus

\[ -\bar{\omega} k^2 \delta \phi + \frac{1}{2} a^2 f_\phi \delta \mathcal{R} = 0. \]

Applying the sub-horizon approach to the term \( \delta R \) (107), we get

\[ \delta R = -2a^{-1} k^2 (2 \Psi - \Phi), \]

replacing \( \delta R \) in the equation (157), gives

\[ \delta \phi = \frac{\bar{\omega}}{\bar{\omega} + 2 f_\phi} (\Phi - 2 \Psi). \]

Using the relation (105) in the above equation

\[ \delta \phi = -\frac{\bar{\omega}}{\bar{\omega} + 2 f_\phi} \frac{1}{\bar{\omega} + 2 f_\phi} \Phi. \]

It can be seen that the perturbations of the scalar field in ST gravity theories do not depend on the wave number \( k \) in the sub-horizon approach. Replacing the above expression and using (105) in (156), we get Poisson type equation

\[ k^2 \Phi = -4\pi G_{\text{eff}}^{\text{ST}} a^2 \rho_m \delta m, \]

where

\[ G_{\text{eff}}^{\text{ST}} = \frac{1}{8\pi f} \left( \frac{2 \bar{\omega} + 4 f_\phi}{2 \bar{\omega} + 3 f_\phi} \right), \]

is the gravitational effective constant for ST theories.

The linear evolution of matter density perturbations and scalar field perturbations in sub-horizon approach in the framework of ST theory of gravity can be written as follows

\[ \delta^i_m + \mathcal{H} \delta^i_{m'} - 4\pi G_{\text{eff}}^{\text{ST}} a^2 \rho_m \delta m = 0, \]

where it has been used (155). For more details about the density linear perturbations see [46–48].

To calculate the perturbations in \( f(R) \) theory under the same approach, we start of the perturbed time-time component (110)

\[ 2k^2 \Psi f_R = -a^2 \delta \rho_m + k^2 f_{R\phi} \delta \mathcal{R}, \]

replacing the relation (115) in the above equation, we take

\[ k^2 \Phi = -\frac{a^2}{2 f_R} \delta \rho_m - \frac{k^2 \bar{\omega}}{2 f_R} \delta \mathcal{R}. \]

Using the equation (158), we get

\[ k^2 \Phi = -4\pi G_{\text{eff}}^{(R)} a^2 \rho_m \delta m. \]

This is the Poisson equation in the Fourier space in \( f(R) \) theories, where

\[ G_{\text{eff}}^{(R)} = \frac{1}{8\pi f_R} \left( \frac{1 + 4 f_{R\phi}}{1 + 3 f_{R\phi}} \right), \]

is the gravitational effective constant for \( f(R) \) theories. The linear evolution of matter density perturbations in sub-horizon approach for \( f(R) \) theories of gravity is
\[ \delta_m'' + \mathcal{H}\delta_m' - 4\pi G_{\text{eff}} f''(R) a^2 \delta_m = 0, \]

where has been used (155). For more detail about the linear density perturbations in \( f(R) \) theories see [49–51].

To see the equivalences in both theories, we take \( \bar{\omega} = 0 \) in the effective gravitational constant for ST theories and let’s use the equation (74) to obtain

\[ G_{\text{eff}} = \frac{4}{3} \frac{1}{8\pi f_R}, \]

if we use \( \frac{k^2 f_g}{a^3 f_R} \gg 1 \) in (167), we obtain the same effective gravitational constant [51]. In this way, we show the equivalence between \( f(R) \) and ST theories under the sub-horizon approximation using the above limit. Finally, if we take the variation of \( \delta R \) (158) and we use the relation (115), we obtain

\[ \delta R = -\frac{\frac{k^2}{a^3}}{1 + \frac{k^2 f_g}{a^3 f_R}}. \]

Now, using the above limit, the expression yields

\[ \delta R = -\frac{f_R}{2f_R} \Phi. \]

If we apply the equivalence in (160), i.e., taking \( \bar{\omega} = 0 \) and using (76), (74) we obtain the same above expression. Evidencing once again the equivalence between both theories.

5. Summary and conclusions

In this section we present the results found in this paper

- Using the variational principle to obtain the field equations in the metric formalism we have used the Gibbons-York-Hawking boundary term type to make no further assumptions about variations of the metric \( \delta g_{ab} \) at the boundary. Furthermore, to obtain the field equation for the scalar field, it is necessary to impose, \( \delta \phi = 0 \) at the boundary. Following [22], where they found the field equations for \( f(R) \) and they showed that, in addition to \( \delta g_{ab} \), it is necessary to impose that \( \delta R = 0 \) at the boundary. It allows us to reinforced equivalences between both theories.

- We found the scalar, vector and tensor cosmological perturbations for ST theories. Then, using the equivalences between both theories we obtained the perturbed ones, for \( f(R) \) gravity in any gauge.

- We showed how to obtain the potential for the Hu-Sawicki and Starobinsky \( f(R) \) models. Then, we calculated the Friedmann equations for the background and perturbed universe in terms of the scalar field for these models under the conformal-Newtonian gauge, and taking into account the equivalence between theories, we find the Friedmann equations for the \( f(R) \) formalism.

- The Poisson type equations and linear evolution of matter density perturbations for both ST (with the parameter \( \omega \) constant) and \( f(R) \) theories were obtained under the conformal-Newtonian gauge. We showed the equivalences between effective gravitational constant for both theories, using for ST the parameter \( \omega = 0 \) and using the limit \( \frac{k^2 f_g}{a^3 f_R} \gg 1 \) for \( f(R) \) theories.

- We showed how to obtain the perturbations for the ST theories in the package \textit{xPAnd} under the software Mathematica.

Appendix A. terms with \( M_c \) and \( N^c \)

As already mentioned, the quantities \( M_c \) and \( N^c \) are defined by

\[ M_c = \frac{f(\phi)}{2} \nabla_c (\delta g^c) - \frac{1}{2} (\delta g^c) \nabla_c f(\phi), \quad (A.1) \]

and

\[ N^c = \frac{f(\phi)}{2} \nabla_c (\delta g^c) - \frac{1}{2} (\delta g^c) \nabla_c f(\phi). \quad (A.2) \]
The covariant derivative of $M_c$ is

$$\nabla^c M_c = \frac{1}{2} \nabla^c (f(\phi)g_{\alpha\beta} \nabla \delta g^{\alpha\beta}) - \frac{1}{2} \nabla^c (\delta g^{\alpha\beta} g_{\alpha\beta} \nabla f(\phi))$$

$$= \frac{1}{2} (\nabla^c f(\phi))g_{\alpha\beta} \nabla \delta g^{\alpha\beta} + \frac{1}{2} f(\phi)g_{\alpha\beta} \nabla \nabla \delta g^{\alpha\beta}$$

$$- \frac{1}{2} (\nabla^c \delta g^{\alpha\beta})g_{\alpha\beta} \nabla f(\phi) - \frac{1}{2} \delta g^{\alpha\beta} g_{\alpha\beta} \nabla^c \nabla f(\phi),$$

where the first and third term are canceled, with which

$$\nabla^c M_c = \frac{1}{2} f(\phi)g_{\alpha\beta} \nabla \delta g^{\alpha\beta} - \frac{1}{2} \delta g^{\alpha\beta} g_{\alpha\beta} \nabla^c f(\phi),$$

(A.3)

so

$$\frac{1}{2} f(\phi)g_{\alpha\beta} \nabla \delta g^{\alpha\beta} = \nabla^c M_c + \frac{1}{2} \delta g^{\alpha\beta} g_{\alpha\beta} \nabla^c f(\phi).$$

(A.4)

The covariant derivative of $N^c$ is

$$\nabla_\alpha N^c = \frac{1}{2} \nabla_\alpha (f(\phi) \nabla \delta g^{\alpha\beta}) - \frac{1}{2} \nabla_\alpha (\delta g^{\alpha\beta} \nabla_\beta f(\phi))$$

$$= \frac{1}{2} (\nabla_\alpha f(\phi)) \nabla_\beta \delta g^{\alpha\beta} + \frac{1}{2} f(\phi) \nabla_\beta \nabla \delta g^{\alpha\beta}$$

$$- \frac{1}{2} (\nabla_\alpha \delta g^{\alpha\beta}) \nabla_\beta f(\phi) - \frac{1}{2} \delta g^{\alpha\beta} \nabla_\alpha \nabla_\beta f(\phi),$$

where the first and third term are canceled, thus

$$\nabla_\alpha N^c = \frac{1}{2} f(\phi) \nabla_\beta \delta g^{\alpha\beta} - \frac{1}{2} \delta g^{\alpha\beta} \nabla_\alpha \nabla_\beta f(\phi)$$

(A.5)

getting

$$\frac{1}{2} f(\phi) \nabla_\alpha \delta g^{\alpha\beta} = \nabla^\alpha N^c + \frac{1}{2} \delta g^{\alpha\beta} \nabla_\alpha \nabla_\beta f(\phi).$$

(A.6)

Subtracting (A.4) with (A.6), we get

$$\frac{f(\phi)}{2} (g_{\alpha\beta} \nabla \delta g^{\alpha\beta} - \nabla_\alpha \nabla_\beta f(\phi)) = \frac{1}{2} \delta g^{\alpha\beta} (g_{\alpha\beta} \nabla f(\phi) - \nabla_\beta \nabla_\alpha f(\phi)) + (\nabla^c M_c - \nabla^c N^c),$$

(A.7)

**Appendix B. Cosmological perturbations in xPAnd**

To obtain the cosmological perturbations, the package xPAnd was used (for more details of the package see [52]).

We upload the notebook on https://github.com/JoelVelasquez/Cosmological-perturbations, where we show how to calculate the cosmological perturbations in ST and f(R) gravity.

**ORCID iDs**

Joel Velásquez. © https://orcid.org/0000-0002-5826-1508

**References**

[1] Riess A G et al 1998 Observational evidence from supernovae for an accelerating universe and a cosmological constant Astron. J. 116 1009–38
[2] Perlmutter S et al 1999 Measurements of omega and lambda from 42 high redshift supernovae Astrophys. J. 517 565–86
[3] Fujii Y and Maeda K-I 2003 The scalar-tensor theory of gravitation (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press) (https://doi.org/10.1017/CBO9780511535093)
[4] Capozziello S and Faraoni V 2010 Beyond einstein gravity: a survey of gravitational theories for cosmology and astrophysics (Fundamental Theories of Physics) (Netherlands: Springer) (https://doi.org/10.1007/978-94-007-0165-6)
[5] Cooper F and Venturi G 1981 Cosmology and broken scale invariance Phys. Rev. D 24 3338–40
[6] Brans C and Dicke R H 1961 Mach’s principle and a relativistic theory of gravitation Phys. Rev. 124 925–35
[7] Sotiriou T and Faraoni V 2010 f(R) theories Of gravity Rev. Mod. Phys. 82 451–97
[8] de Felice A and Tsujikawa S 2010 f(r) theories Living Rev. Relativ. 13 06
[9] Nojiri S and Odintsov S D 2011 Unified cosmic history in modified gravity: from f(R) theory to lorentz non-invariant models Phys. Rept. 505 59–144
[10] Belenchia M, Letizia M, Libarati S and di Casola E 2018 Higher-order theories of gravity: diagnosis, extraction and reformulation via non-local extra degrees of freedom—a review Rept. Prog. Phys. 81 036001
[11] Capone M and Ruggiero M L 2010 Jumping from metric f(R) to scalar–tensor theories and the relations between their post–newtonian parameters Class. Quant. Grav. 27 125006
[12] Nistahmpagaza J, Abebe A and Mbonye M 2017 On f (R) gravity in scalar–tensor theories Int. J. Geom. Meth. Mod. Phys. 14 1750107
[13] Sotiriou T P 2006 f(R) gravity and scalar–tensor theory Class. Quant. Grav. 23 5117–28
[14] Capozziello S, Nojiri S and Odintsov S D 2006 Dark energy: the equation of state description versus scalar–tensor or modified gravity Phys. Lett. B 634 93–100
[15] Nojiri S, Odintsov S D and Oikonomou V K 2017 Modified gravity theories on a nutshell: inflation, bounce and late-time evolution Phys. Rept. 692 1–104
[16] Wald R M 1984 General Relativity (Chicago, IL: University of Chicago Press) (https://doi.org/10.7280/chicago/9780226870373.001.0001)
[17] Poisson E 2004 A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics (Cambridge: Cambridge University Press) (https://doi.org/10.1017/CBO9780511606601)
[18] Carroll SM 2019 Spacetime and Geometry: An Introduction to General Relativity (Cambridge: Cambridge University Press) (https://doi.org/10.1017/9781108770385)
[19] Gibbons G W and Hawking S W 1977 Action integrals and partition functions in quantum gravity Phys. Rev. D 15 2752–6
[20] Hawking S W and Horowitz G T 1996 The gravitational hamiltonian, action, entropy and surface terms Class. Quant. Grav. 13 1487–98
[21] Faraoni V 2004 Cosmology in scalar–tensor gravity (Fundamental Theories of Physics) (Netherlands: Springer) (https://doi.org/10.1007/978-1-4020-1989-0)
[22] Guarinno A, Castanela L and Tejiero J M 2010 Boundary term in metric f(R) gravity: field equations in the metric formalism Gen. Rel. Grav. 42 2713–28
[23] Padilla A and Sivasesan V 2012 Boundary terms and junction conditions for generalized scalar–tensor theories JHEP 08 122
[24] Dyer E and Hinterbichler K 2009 Boundary terms, variational principles and higher derivative modifications Phys. Rev. D 79 024028
[25] Sultas ID and Hindmarsh M 2011 The dynamical equivalence of modified gravity revisited Class. Quant. Grav. 28 035002
[26] Ohta N 2018 Quantum equivalence of (R) gravity and scalar–tensor theories in the jordan and einstein frames PTEP 2018 033B02
[27] Ruš M S and Steinwachs C F 2018 Quantum equivalence of (R) gravity and scalar–tensor theories Phys. Rev. D 97 044050
[28] Hu W and Sawicki I 2007 Models of R gravity acceleration that evade solar-system tests Phys. Rev. 76 064004
[29] Fujii Y 2012 Accelerating universe and the scalar–tensor theory Entropy 14 1997–2035
[30] Gannouji R, Polarski D, Ranquet A and Starobinsky A A 2007 Scalar–tensor dark energy models Recent developments in theoretical and experimental general relativity, gravitation and relativistic field theories. Proceedings, 11th Marcel Grossmann Meeting (MG11, Berlin, Germany, July 23–29, 2006) 1794–6 Pt. A-C
[31] Clifton T and Dunsky P K S 2015 On the emergence of accelerating cosmic expansion in f (R) theories of gravity Phys. Rev. D 91 103528
[32] Morita M and Takahashi H 2014 Reconstructing f(R) modified gravity with dark energy parametrization Journal of Physics: Conference Series 490 012087
[33] Bahamonde S, Odintsov S D, Oikonomou V K and Tretyakov P V 2017 Deceleration versus acceleration universe in different frames of f(R) gravity Phys. Lett. B 766 225–30
[34] Mukhanov V 2005 Physical Foundations of Cosmology (Cambridge: Cambridge University Press) (https://doi.org/10.1017/CBO9780511790553)
[35] Bardeen J M 1980 Gauge-invariant cosmological perturbations Phys. Rev. D 22 1882–905
[36] Kurki-Suonio H 2015 Cosmological perturbation theory, part 1 Lecture Notes for a Course of Cosmological Perturbation Theory given at the University of Helsinki 2015 1–51
[37] Kowalski M et al 2008 Improved cosmological constraints from new, old and combined supernova datasets Astrophys. J. 686 749–78
[38] Simon J, Verde L and Jimenez R 2005 Constraints on the redshift dependence of the dark energy potential Phys. Rev. D 71 123001
[39] Eisenstein D J, Seo H-J and White M J 2007 On the robustness of the acoustic scale in the low-redshift clustering of matter Astrophys. J. 660 660–74
[40] Kopp M, Appleby S A, Achitouv I and Wellier J 2013 Spherical collapse and halo mass function in f (R) theories Phys. Rev. D 88 084015
[41] Starobinsky A A 1980 A new type of isotropic cosmological models without singularity Phys. Lett. 91B 99–102
[42] Mukhanov V F and Chibisov G V 1981 Quantum fluctuations and a nonsingular universe JETP Lett. 33 532–5
[43] Mukhanov V F and Chibisov G V 1981 Pisma Zh. Eksp. Teor. Fiz. 33 549
[44] Starobinski A A 1983 The perturbation spectrum evolving from a nonsingular initially de-sitter cosmos and the microwave background anisotropy Soviet Astronomy Letters 9 302–4
[45] Ade P A R et al 2014 Planck 2013 results. XXI. constraints on inflation Astron. Astrophys. 571 A22
[46] Artymowski M, Lak Z and Lewicki M 2015 Inflationary scenarios in Starobinsky model with higher order corrections JCAP 1506 032
[47] Boisseau B, Esposito-Farese G, Polarski D and Starobinsky A A 2000 Reconstruction of a scalar tensor theory of gravity in an accelerating universe Phys. Rev. Lett. 85 2236
[48] Nazari-Pooya N, Maleknejad M, Pace F and Jassur D M-Z 2016 Growth of spherical overdensities in scalar–tensor cosmologies Mon. Not. Roy. Astron. Soc. 458 7395–807
[49] Sanchez J C B and Perivolaropoulos L 2010 Evolution of dark energy perturbations in scalar–tensor cosmologies Phys. Rev. D 81 103505
[50] de la Cruz-Dombriz A, Dobado A and Maroto A L 2008 On the evolution of density perturbations in (R) theories of gravity Phys. Rev. D 77 123515
[51] Chiu M-C, Taylor A, Shu C and Tu H 2015 Cosmological perturbations and quasistatic assumption in f(R) theories Phys. Rev. D 92 103514
[52] Tsujikawa S 2007 Matter density perturbations and effective gravitational constant in modified gravity models of dark energy Phys. Rev. D 76 023514
[53] Pitrou C, Roy X and Umeo O 2013 xPand: An algorithm for perturbing homogeneous cosmologies Class. Quant. Grav. 30 165002