On Characters of Weyl Groups

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Abstract

In this note a combinatorial character formula related to the symmetric group is generalized to an arbitrary finite Weyl group.

1 The Case of the Symmetric Group

The length $\ell(\pi)$ of a permutation $\pi \in S_n$ is the number of inversions of $\pi$, i.e., the number of pairs $(i, j)$ with $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$.

For any permutation $\pi \in S_n$ let $m(\pi)$ be defined as

$$m(\pi) := \begin{cases} (-1)^m, & \text{if there exists } 0 \leq m < n \text{ so that } \\
0, & \text{otherwise.} \end{cases}$$

Let $\mu = (\mu_1, \ldots, \mu_t)$ be a partition of $n$, and let $S_\mu = S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_t}$ be the corresponding Young subgroup of $S_n$. For any permutation $\pi = r \cdot (\pi_1 \times \cdots \times \pi_t)$, where $\pi_i \in S_{\mu_i}$ ($1 \leq i \leq t$) and $r$ is a representative of minimal length of a left coset of $S_\mu$ in $S_n$, define

$$\text{weight}_\mu(\pi) := \prod_{i=1}^t m(\pi_i),$$
where $m(\pi_i)$ is defined in $S_\mu$ by (1).

Denote by $\chi^k_\mu$ the value, at a conjugacy class of type $\mu$, of the character of the natural $S_n$-action on the $k$-th homogeneous component of the coinvariant algebra. The following combinatorial character formula was proved in [Ro2, Theorem 1].

**Theorem.** With the above notations

$$
\chi^k_\mu = \sum_{\{\pi \in S_n : \ell(\pi) = k\}} \text{weight}_\mu(\pi).
$$

2 Arbitrary Weyl Group

Let $W$ be an arbitrary finite Weyl group. Denote the set of positive roots by $\Phi_+$. Let $t_\alpha$ be the reflection corresponding to $\alpha \in \Phi_+$, and let $\check{\alpha}$ be the corresponding coroot. Let $\alpha_i$ be the simple root corresponding to the simple reflection $s_i$. Denote by $S_w$ the Schubert polynomial indexed by $w \in W$.

The following theorem describes the action of the simple reflections on the coinvariant algebra. This theorem is a reformulation of [BGG, Theorem 3.14 (iii)].

**Theorem 1.** For any simple reflection $s_i$ in $W$ and any $w \in W$,

$$
s_i(S_w) = \begin{cases} 
S_w, & \text{if } \ell(ws_i) > \ell(w); \\
- S_w + \sum_{\{\alpha \in \Phi_+ : \alpha \neq \alpha_i \wedge \ell(wsi_t) = \ell(w)\}} \alpha_i(\check{\alpha})S_{ws_i t}, & \text{if } \ell(ws_i) < \ell(w).
\end{cases}
$$

**Proof.** In the above notations, [BGG, Theorem 3.14 (iii)] states that

$$
s_i(S_w) = \begin{cases} 
S_w, & \text{if } \ell(ws_i) > \ell(w); \\
\sum_{\gamma \in \Phi_+ : \ell(ws_i t) = \ell(w)} w(\alpha_i)(\check{\gamma})S_{t ws_i}, & \text{if } \ell(ws_i) < \ell(w).
\end{cases}
$$

Obviously, for any $\gamma \in \Phi_+$ there exists a unique $\alpha \in \Phi_+$ such that $t_\gamma ws_i = ws_i t_\alpha$. In this case $s_i w^{-1}(\gamma) = \alpha$. If $\ell(ws_i) < \ell(w)$ and $t_\gamma ws_i \neq w$ then the coefficient of $S_{ws_i t_\alpha} = S_{t ws_i}$ in $s_i(S_w)$ is equal to

$$
-w(\alpha_i)(\check{\gamma}) = -\alpha_i(w^{-1}(\check{\gamma})) = -\alpha_i(s_i(\check{\alpha})) = \alpha_i(\check{\alpha}).
$$

If $t_\gamma ws_i = w$ then $t_\alpha = s_i$. Hence, the coefficient of $S_w$ in $s_i(S_w)$ is $1 - w(\alpha_i)(\check{\gamma}) = 1 - \alpha_i(\check{\alpha}) = -1$ if $\ell(s_i w) < \ell(w)$, and 1 otherwise. \qed
Let \( \langle \cdot, \cdot \rangle \) be the inner product on the coinvariant algebra defined by \( \langle S_v, S_w \rangle = \delta_{v,w} \) (the Kronecker delta). Theorem 1 implies

**Corollary 2.** Let \( s_i \) be a simple reflection in \( W \), and let \( z \in W \) such that \( \ell(zs_i) < \ell(z) \). Then for any \( w \in W \)

\[
\langle s_i(S_w), S_z \rangle = \begin{cases} 
0, & \text{if } z \neq w \\
-1, & \text{if } z = w 
\end{cases}
\]

**Proof.** For \( z = w \) this follows from the second case of Theorem 1. For \( z \neq w \) if \( \langle s_i(S_w), S_z \rangle \neq 0 \) then (by Theorem 1) \( z = ws_i t_\alpha \) for some \( \alpha \in \Phi_+ \) such that \( \alpha \neq \alpha_i \) and \( \ell(ws_i t_\alpha) = \ell(w) > \ell(ws_i) \). Now, for \( \alpha \in \Phi_+ \), \( \ell(ws_i t_\alpha) > \ell(ws_i) \) if and only if \( ws_i(\alpha) \in \Phi_+ \). On the other hand, \( \alpha_i \neq \alpha \in \Phi_+ \Rightarrow s_i(\alpha) \in \Phi_+ \). Since \( ws_i(\alpha) \in \Phi_+ \) it follows that \( \ell(w t_{s_i(\alpha)}) > \ell(w) \). But \( wt_{s_i(\alpha)} = ws_i t_\alpha s_i \). Hence, \( \ell(zs_i) = \ell(ws_i t_\alpha s_i) > \ell(w) = \ell(ws_i t_\alpha) = \ell(z) \). \( \square \)

The following is, surprisingly, an exact Schubert analogue of a useful vanishing condition for Kazhdan-Lusztig coefficients [Ro1, Lemma 4.3].

**Corollary 3.** Let \( s_i, s_j \) be commuting simple reflections in \( W \), and let \( w, z \in W \) such that \( \ell(ws_i) > \ell(w) \) and \( \ell(zs_i) < \ell(z) \). Then

\[
\langle s_j(S_w), S_z \rangle = 0.
\]

**Proof.** Obviously, \( z \neq w \). If \( \ell(ws_j) > \ell(w) \) then our claim is an immediate consequence of Theorem 1. Assume that \( \ell(ws_j) < \ell(w) \), and denote \( \langle s_j(S_w), S_z \rangle \) by \( b^{(j)}_z(w) \). By Corollary 2

\[
s_i(1 + s_j)(S_w) = s_i \left( \sum_{\ell(zs_j) > \ell(z)} b^{(j)}_z(w) S_z \right) = \sum_{\ell(zs_j) > \ell(z)} b^{(j)}_z(w) s_i(S_z).
\]

On the other hand, by Theorem 1, \( S_w \) is an invariant under \( s_i \). Thus,

\[
s_i(1 + s_j)(S_w) = (1 + s_j)s_i(S_w) = (1 + s_j)(S_w) = \sum_{\ell(zs_j) > \ell(z)} b^{(j)}_z(w) S_z.
\]

We conclude that

\[
\sum_{\ell(zs_j) > \ell(z)} b^{(j)}_z(w)(1 - s_i) S_z = 0.
\]
But
\[\sum_{\ell(zs_j) > \ell(z)} b^{(j)}_z(w)(1 - s_i)(\mathfrak{S}_z) = \sum_{\ell(zs_j) > \ell(z) \land \ell(zs_i) < \ell(z)} b^{(j)}_z(w)(1 - s_i)(\mathfrak{S}_z) = \sum_{\ell(zs_j) > \ell(z) \land \ell(zs_i) < \ell(z)} b^{(j)}_z(w)[2\mathfrak{S}_z - \sum_{\ell(ts_i) > \ell(t)} b^{(i)}_t(z)\mathfrak{S}_t].\]

This sum is equal to zero if and only if \(b^{(j)}_z(w) = 0\) for all \(z\) with \(\ell(zs_j) > \ell(z)\) and \(\ell(zs_i) < \ell(z)\).

It remains to check the case in which \(\ell(zs_j) < \ell(z)\). By assumption \(\ell(zs_i) < \ell(z) \Rightarrow z \neq w\). Corollary 2 completes the proof. \(\square\)

Let \(H\) be a parabolic subgroup of \(W\), which is isomorphic to a direct product of symmetric groups. In the following definition we refer to cycle type and weight \(\mu\) of elements in \(H\) under the natural isomorphism, sending simple reflections of \(H\) to simple reflections of \(W\).

**Definition.** Let \(\mu\) be a cycle type of an element in \(H\). For any element \(w = r \cdot \pi \in W\), where \(\pi \in H\) and \(r\) is the representative of minimal length of the left coset of \(wH\) in \(W\), define

\[\text{weight}_{\mu}(w) := \text{weight}_{\mu}(\pi).\]

Here \(\text{weight}_{\mu}(\pi)\) is defined as in Section 1.

Note that \(\text{weight}_{\mu}\) is independent of the choice of \(H\), provided that \(H\) is isomorphic to a direct product of symmetric groups and that \(\mu\) is the cycle type of some element in \(H\).

Let \(R^k\) be the \(k\)-th homogeneous component of the coinvariant algebra of \(W\). Denote by \(\chi^k\) the \(W\)-character of \(R^k\). Let \(v_{\mu} \in H\) have cycle type \(\mu\). Then

**Theorem 4.** With the above notations

\[\chi^k(v_{\mu}) = \sum_{\{w \in W : \ell(w) = k\}} \text{weight}_{\mu}(w).\]

**Proof.** Imitate the proof of [Ro2, Theorem 1]. Here Corollary 2 plays the role of [Ro2, Corollary 3.2] and implies an analogue of [Ro2, Corollary 3.3]. Alternatively, one can prove Theorem 4 by imitating the proof of [Ro1, Theorems 1-2], where Corollary 3 plays the role of [Ro1, Lemma 4.3]. \(\square\)
Note: A formally similar result appears also in Kazhdan-Lusztig theory. The Kazhdan-Lusztig characters of $W$ at $v_\mu$ may be represented as sums of exactly the same weights, but over Kazhdan-Lusztig cells instead of Bruhat levels [Ro1, Corollary 3]. This curious analogy seems to deserve further study. For a $q$-analogue of the result for the symmetric group see [APR]. A $q$-analogue of Theorem 4 is desired.

References

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