Finitely semisimple spherical categories and modular categories are self-dual

Hendryk Pfeiffer*

Department of Mathematics, The University of British Columbia,
1984 Mathematics Road, Vancouver, BC, V2T 1Z2, Canada
February 3, 2009

Abstract

We show that every essentially small finitely semisimple \(k\)-linear additive spherical category for which \(k = \text{End}(\mathbf{1})\) is a field, is equivalent to its dual over the long canonical forgetful functor. This includes the special case of modular categories. In order to prove this result, we show that the universal coend of the spherical category, with respect to the long forgetful functor, is self-dual as a Weak Hopf Algebra.

Mathematics Subject Classification (2000): 16W30, 18D10
keywords: Modular category, spherical category, Weak Hopf Algebra, Tannaka–Krein reconstruction, Dual of a monoidal category

1 Introduction

1.1 Spherical categories

A spherical category is a monoidal category in which each object \(X\) has a specified left-dual \(X^*\), in which there are canonical isomorphisms \(X \to X^{**}\) that can be used to equip each object with a right-dual as well, and in which the two ways of forming the trace of a morphism coincide. There is a coherence theorem for spherical categories generalizing MacLane’s coherence theorem for monoidal categories: it states that the morphisms of a spherical category can be represented by string diagrams in the sphere \(S^2\) up to isotopy, hence the name “spherical”. In particular, every ribbon category is spherical.

The notion of a spherical category was invented by Barrett and Westbury [1] when they developed their version [2] of the Turaev–Viro invariant [3]. Finitely semisimple spherical categories form the most general categories for which this invariant can be still be formulated. Neither a braiding, a ribbon structure nor a non-degenerate \(S\)-matrix are required.

1.2 Duals of monoidal categories

In the abstract, we claim that certain finitely semisimple spherical categories are equivalent to their duals over some functor. What is the dual of a monoidal category?

*E-mail: pfeiffer@math.ubc.ca
Assume for the moment that the monoidal category is the category $\mathcal{M}_H$ of finite-dimensional modules of a Hopf algebra $H$ over some field $k$. Then the algebra structure of $H$ determines the category underlying $\mathcal{M}_H$, while the coalgebra structure of $H$ is responsible for the monoidal structure of $\mathcal{M}_H$. Under certain conditions, the coalgebra structure can be reconstructed from $\mathcal{M}_H$. The coalgebra structure of $H$ also determines the category $\mathcal{M}^H$ of finite-dimensional comodules of $H$. This raises a question: can we characterize the category $\mathcal{M}_H$ in terms of the monoidal structure of the category $\mathcal{M}_H$? For Hopf algebras, Majid’s notion of the dual of a monoidal category over a strong monoidal functor [4, 5] answers this question: under suitable conditions, the dual of $\mathcal{M}_H$ over the forgetful functor $\mathcal{M}_H \to \text{Vect}_k$ is equivalent to $\mathcal{M}^H$. In particular, if $H$ is finite-dimensional, then the dual of $\mathcal{M}_H$ over the forgetful functor $\mathcal{M}_H \to \text{Vect}_k$ is precisely $\mathcal{M}^H \simeq \mathcal{M}_{\widehat{H}}$, the category of finite-dimensional modules of the dual Hopf algebra $\widehat{H}$. This justifies the term ‘dual of a monoidal category’.

The following special case of this notion of the dual of a monoidal category is much more widely known: the dual of a monoidal category $\mathcal{C}$ over the identity functor on itself is the double of $\mathcal{C}$ [4, 6], also called the categorical center.

### 1.3 Tannaka–Kreın reconstruction

The most interesting finitely semisimple spherical categories, however, do not admit any strong monoidal functor to $\text{Vect}_k$, and they are therefore not the categories of modules of any Hopf algebra. Nevertheless, we can show that the relevant spherical categories are equivalent to the categories of comodules of Weak Hopf Algebras (WHAs). Such a WHA is obtained as a universal coend using a generalization of Tannaka–Kreın reconstruction:

**Theorem 1.1.** Let $\mathcal{C}$ be an essentially small finitely semisimple $k$-linear additive spherical category for which $k = \text{End}(1)$ is a field, and let $\omega: \mathcal{C} \to \text{Vect}_k$ be the long forgetful functor. Then $H = \text{coend}(\mathcal{C}, \omega)$ is a finite-dimensional split cosemisimple cospherical WHA for which $H_t \cap H_s \cong k$. Furthermore, $\mathcal{C}$ is equivalent as a $k$-linear additive spherical category to the category $\mathcal{M}^H$ of finite-dimensional right $H$-comodules.

The special case of modular categories was addressed in the article [7]. The functor $\omega$ is the long version of Hayashi’s canonical forgetful functor [8]. We proceed by showing that the reconstructed WHA is self-dual, and then generalizing Majid’s work on duals of monoidal categories to the case of WHAs. Thus equipped, we can finally prove our claim from the abstract.

### 1.4 Double triangle algebras and Depth-2 Frobenius extensions

Why can we expect the reconstructed WHA to be self-dual? Ocneanu’s idea of a ‘double triangle algebra’ gives an early idea of how to reconstruct some algebra-like structure from a semisimple monoidal category, see, for example, the diagrams in [9]. In hindsight, the precise definition that captures the idea of a double triangle algebra is the notion of a Weak Hopf Algebra [10, 11]. If we replace white vertices with black vertices in the double triangle diagrams of, say, [9], we see immediately that we ought to get a self-dual WHA. In particular, the very first examples of WHAs [12] whose categories of modules are modular categories with objects of non-integer Frobenius–Perron dimension, have been constructed by this method and are

---

1This argument uses a generalization of Tannaka–Kreın reconstruction, and one usually reconstructs the universal coacting coalgebra.
indeed self-dual. It remains to generalize the argument to work for the WHAs reconstructed from our spherical categories.

In fact, Böhm and Szlachányi [13] have implemented the duality relation that leads to the self-duality of Ocneanu’s double triangle algebra in the context of abstract depth-2 Frobenius extensions. For every such extension, they construct a pair of dual Hopf algebroids.

It turns out that the WHA that arises as our universal coend $H = \text{coend}(C, \omega)$ coincides with one of these Hopf algebroids. Since the base of these Hopf algebroids is the endomorphism algebra $R = \text{End}(\tilde{V})$ of the universal object

$$\tilde{V} = \bigoplus_{j \in I} V_j,$$

the direct sum of one representative $V_j$ for each isomorphism class of simple objects of $C$, and since $R$ is finite-dimensional, commutative and separable, both Hopf algebroids of [13] are in fact WHAs. Furthermore, since $C$ is semisimple as a pivotal category (Definition A.15), one can swap $\tilde{V}$ and its dual $\tilde{V}^*$ and show that these dually paired WHAs are isomorphic.

**Theorem 1.2.** The WHA $H$ of Theorem 1.1 is self-dual as a pivotal WHA.

**Corollary 1.3.** Every essentially small finitely semisimple $k$-linear additive spherical category for which $k = \text{End}(\mathbb{1})$ is a field, is equivalent as a $k$-linear additive spherical category to its dual over the long forgetful functor $\omega: C \to \text{Vect}_k$.

We can view the isomorphism $H \to \hat{H}$ used to find a pair of dual bases of $H$ and its dual $\hat{H}$, as a generalized Fourier transform. For modular categories, it is known in a somewhat different context [14] that the $S$-matrix provides a Fourier transform. In our case, however, the Fourier transform is implemented by a generalized $6j$-symbol. It is therefore no surprise that our result is already available for spherical categories and even in the absence of any braiding.

### 1.5 TQFTs and state sum invariants

We are interested in spherical categories because they form the most general categories for which the Turaev–Viro invariant is still available. Not only do modular categories form a subclass of finitely semisimple spherical categories, but also the doubles of a large class of finitely semisimple spherical categories are modular [15], and so the finitely semisimple spherical categories can be viewed as more basic than the modular ones.

Since their doubles are modular, finitely semisimple spherical categories form the most plausible candidate for the structure underlying some as yet unknown 3-dimensional extended Topological Quantum Field Theories (TQFTs) that would specialize to the familiar TQFTs based on modular categories when the manifolds have no corners. This construction would categorify the TQFTs of [16], based on the idea that the double of a monoidal category categorifies the notion of the center of an algebra.

We are interested in the self-duality of our spherical categories because of the following observation. While the Turaev–Viro invariant can be constructed using a finitely semisimple spherical category, Kuperberg’s invariant [17] is based on a certain involutory Hopf algebra. Barrett and Westbury have shown [18] that Kuperberg’s invariant for some Hopf algebra agrees with the Turaev–Viro invariant for its category of modules (without taking any quotient of this category modulo ‘negligible morphisms’!). This observation raises a number of questions.
First, all known examples of finitely semisimple spherical categories for which the Turaev–Viro invariant is interesting (i.e., stronger than an invariant of homotopy type), are not the categories of modules of any Hopf algebra. We know, however, from Theorem 1.1 that they are nevertheless the categories of comodules of some WHAs. The obvious question is: can Kuperberg’s invariant be generalized to WHAs so as to capture all interesting examples of the Turaev–Viro invariant? Then, in order to pass from Kuperberg to Turaev–Viro, we would take the category of comodules (no quotient!), and in order to pass from Turaev–Viro back to Kuperberg, we would Tannaka–Krein reconstruct.

Second, Kuperberg’s invariant for some involutory Hopf algebra $H$ and some Heegaard splitting agrees by construction with the invariant for the dual Hopf algebra $\hat{H}$ and the Poincaré dual Heegaard splitting. Since the Poincaré dual Heegaard splitting characterizes the same 3-manifold, the Kuperberg invariants for $H$ and $\hat{H}$ always agree for any given Heegaard splitting. How can this be explained? Once Kuperberg’s invariant has been generalized to WHAs, Theorem 1.2 will provide the answer.

Third, just as the Kuperberg invariant for some Hopf algebra $H$ and for a given Heegaard splitting agrees with the invariant for the dual Hopf algebra $\hat{H}$ and the Poincaré dual Heegaard splitting, we can ask the analogous question about the Turaev–Viro invariant. We would need the generalization of the Turaev–Viro invariant to cellular complexes [19] and study the following problem. Given the Turaev–Viro invariant for some finitely semisimple spherical category and some cellular complex, which category do we need in order to make the Turaev–Viro invariant for the dual cellular complex agree with the invariant for the original complex? Corollary 1.3 will provide the answer.

Using the long forgetful functor, we can therefore improve a number of results by Müger on the relationship between the invariants of Kuperberg and Turaev–Viro for the original and for the dual Hopf algebra. We refer in particular to [20, Section 7].

Crane and Frenkel [21] originally proposed to categorify Kuperberg’s 3-manifold invariant in order to get access to combinatorial 4-manifold invariants. Settling the open questions about spherical categories and their reconstructed WHAs is a crucial step in the process of deciding which invariant it is that we want to categorify.

1.6 Overview

The article is organized as follows. In Section 2, we summarize the definitions and basic properties of WHAs and of the forgetful functors of their categories of comodules. These are functors with a separable Frobenius structure. In Section 3, we reconstruct WHAs from our spherical categories and prove Theorem 1.1. We show the self-duality of the reconstructed WHAs (Theorem 1.2) in Section 4. In Section 5, we introduce the notion of a dual of a monoidal category over a functor with separable Frobenius structure and show that the category of modules of the reconstructed WHA is equivalent as a spherical category to the dual of the original spherical category over the long forgetful functor. We then combine all these ingredients and show that our spherical categories are self-dual (Corollary 1.3). Finally, in Section 6, we briefly sketch all these constructions for the special case of the modular category associated with the quantum group $U_q(\mathfrak{sl}_2)$, $q$ a root of unity, using the familiar diagrams. The reader may wish to take a quick look at this example before reading the other sections. In the appendix, we have collected the basic definitions and results on monoidal categories with duals and on abelian and semisimple categories.
2 Preliminaries

2.1 Functors with Frobenius structure

We use the following notation. If \( \mathcal{C} \) is a category, we write \( X \in |\mathcal{C}| \) for the objects \( X \) of \( \mathcal{C} \), \( \text{Hom}(X, Y) \) for the collection of all morphisms \( f: X \to Y \) and \( \text{End}(X) = \text{Hom}(X, X) \). We ignore all set theoretic issues and tacitly assume that the \( \text{Hom}(X, Y) \) are all sets. We denote the identity morphism of \( X \) by \( \text{id}_X: X \to X \) and the composition of morphisms \( f: X \to Y \) and \( g: Y \to Z \) by \( g \circ f: X \to Z \). If two objects \( X, Y \in |\mathcal{C}| \) are isomorphic, we write \( X \cong Y \). If two categories \( \mathcal{C} \) and \( \mathcal{D} \) are equivalent, we write \( \mathcal{C} \cong \mathcal{D} \). The identity functor on \( \mathcal{C} \) is denoted by \( 1_{\mathcal{C}} \), and \( \mathcal{C}^{\text{op}} \) is the opposite category of \( \mathcal{C} \). The category of vector spaces over a field \( k \) is denoted by \( \text{Vect}_k \) and its full subcategory of finite-dimensional vector spaces by \( \text{fdVect}_k \). Both are \( k \)-linear abelian and symmetric monoidal, while \( \text{fdVect}_k \) is in addition spherical.

Appendix A.2 gives background on monoidal categories with duals.

The forgetful functor of the category of finite-dimensional comodules of a WHA is not necessarily strong monoidal, but it satisfies the following more general conditions of a functor with separable Frobenius structure as defined by Szlachányi [22].

Definition 2.1. Let \( \mathcal{C} \) and \( \mathcal{C}' \) be monoidal categories. A functor with Frobenius structure \( (F, F_{X,Y}, F_0, F^{X,Y}, F^0): \mathcal{C} \to \mathcal{C}' \) is a functor \( F: \mathcal{C} \to \mathcal{C}' \) that is lax monoidal as \( (F, F_{X,Y}, F_0) \) and oplax monoidal as \( (F, F_{X,Y}, F^0) \) (see Definition A.2) and that satisfies the following compatibility conditions,

\[
F(X \otimes Y) \otimes' FZ \xrightarrow{F_{X,Y,Z}} F((X \otimes Y) \otimes Z) \xrightarrow{F_{0,X,Y,Z}} F(X \otimes (Y \otimes Z))
\]

\[
(FX \otimes' FY) \otimes' FZ \xrightarrow{\alpha_{FX,FY,FZ}} FX \otimes' (FY \otimes' FZ) \xrightarrow{id_{FX} \otimes' F_{Y,Z}} FX \otimes' F(Y \otimes Z),
\]

for all \( X, Y, Z \in |\mathcal{C}| \). It is called a functor with separable Frobenius structure if in addition

\[
F_{X,Y} \circ F^{X,Y} = \text{id}_{F(X \otimes Y)},
\]

for all \( X, Y \in |\mathcal{C}| \).

Note that every strong monoidal functor between monoidal categories is a functor with separable Frobenius structure, and that in this case, the conditions (2.1) and (2.2) both follow from the hexagon axiom.
2.2 Weak Hopf Algebras and their comodules

In this section, we summarize the relevant definitions and properties of Weak Bialgebras (WBAs) and Weak Hopf Algebras (WHAs) following [10, 11, 23] and of their categories of comodules following [7].

Definition 2.2. A Weak Bialgebra \((H, \mu, \eta, \Delta, \varepsilon)\) over a field \(k\) is a \(k\)-vector space \(H\) with linear maps \(\mu: H \otimes H \to H\) (multiplication), \(\eta: k \to H\) (unit), \(\Delta: H \to H \otimes H\) (comultiplication), and \(\varepsilon: H \to k\) (counit) such that the following conditions hold:

1. \((H, \mu, \eta)\) is an associative unital algebra, i.e. \(\mu \circ (\mu \otimes \text{id}_H) = \mu \circ (\text{id}_H \otimes \mu)\) and \(\mu \circ (\eta \otimes \text{id}_H) = \text{id}_H = \mu \circ (\text{id}_H \otimes \eta)\).
2. \((H, \Delta, \varepsilon)\) is a coassociative counital coalgebra, i.e. \((\Delta \otimes \text{id}_H) \circ \Delta = (\text{id}_H \otimes \Delta) \circ \Delta\) and \((\varepsilon \otimes \text{id}_H) \circ \Delta = \text{id}_H = (\text{id}_H \otimes \varepsilon) \circ \Delta\).
3. The following compatibility conditions hold:

\[
\Delta \circ \mu = (\mu \otimes \mu) \circ (\text{id}_H \otimes \sigma_{H,H} \otimes \text{id}_H) \circ (\Delta \otimes \Delta),
\]

\[
\varepsilon \circ \mu \circ (\mu \otimes \text{id}_H) = (\varepsilon \otimes \varepsilon) \circ (\mu \otimes \mu) \circ (\text{id}_H \otimes \Delta \otimes \text{id}_H) = (\varepsilon \otimes \varepsilon) \circ (\mu \otimes \mu) \circ (\text{id}_H \otimes \Delta^{\text{op}} \otimes \text{id}_H),
\]

\[
(\Delta \otimes \text{id}_H) \circ \Delta \circ \eta = (\text{id}_H \otimes \mu \otimes \text{id}_H) \circ (\Delta \otimes \Delta) \circ (\eta \otimes \eta) = (\text{id}_H \otimes \mu^{\text{op}} \otimes \text{id}_H) \circ (\Delta \otimes \Delta) \circ (\eta \otimes \eta).
\]

Here \(\sigma_{V,W}: V \otimes W \to W \otimes V, v \otimes w \mapsto w \otimes v\) is the transposition of the tensor factors, i.e. the symmetric braiding of \(\text{Vect}_k\), and by \(\Delta^{\text{op}} = \sigma_{H,H} \circ \Delta\) and \(\mu^{\text{op}} = \mu \circ \sigma_{H,H}\), we denote the opposite comultiplication and opposite multiplication, respectively. We tacitly identify the vector spaces \((V \otimes W) \otimes U \cong V \otimes (W \otimes U)\) and \(V \otimes k \cong V \cong k \otimes V\), exploiting the coherence theorem for the monoidal category \(\text{Vect}_k\).

We use the term \textit{comultiplication} for the operation \(\Delta\) in a coalgebra, whereas \textit{coproduct} always refers to a colimit in a category.

Definition 2.3. A homomorphism \(\varphi: H \to H'\) of WBAs \((H, \mu, \eta, \Delta, \varepsilon)\) and \((H', \mu', \Delta', \varepsilon')\) over the same field \(k\) is a \(k\)-linear map that is a homomorphism of unital algebras, i.e. \(\varphi \circ \eta = \eta'\) and \(\varphi \circ \mu = \mu' \circ (\varphi \otimes \varphi)\), as well as a homomorphism of counital coalgebras, i.e. \(\varepsilon' \circ \varphi = \varepsilon\) and \(\Delta' \circ \varphi = (\varphi \otimes \varphi) \circ \Delta\).

Definition 2.4. Let \((H, \mu, \eta, \Delta, \varepsilon)\) be a WBA. The linear maps \(\varepsilon_t: H \to H\) (target counital map) and \(\varepsilon_s: H \to H\) (source counital map) are defined by

\[
\varepsilon_t := (\varepsilon \otimes \text{id}_H) \circ (\mu \otimes \text{id}_H) \circ (\text{id}_H \otimes \sigma_{H,H} \otimes \text{id}_H) \circ (\Delta \otimes \text{id}_H),
\]

\[
\varepsilon_s := (\text{id}_H \otimes \varepsilon) \circ (\text{id}_H \otimes \mu) \circ (\sigma_{H,H} \otimes \text{id}_H) \circ (\text{id}_H \otimes \Delta) \circ (\text{id}_H \otimes \eta).
\]

Both \(\varepsilon_t\) and \(\varepsilon_s\) are idempotents. A WBA \((H, \mu, \eta, \Delta, \varepsilon)\) is a bialgebra if and only if \(\Delta \circ \eta = \eta \otimes \eta\), and only if \(\varepsilon \circ \mu = \varepsilon \otimes \varepsilon\), and if and only if \(\varepsilon_s = \eta \otimes \varepsilon\) and if and only if \(\varepsilon_t = \eta \circ \varepsilon\).

Proposition 2.5. Let \((H, \mu, \eta, \Delta, \varepsilon)\) be a WBA.

1. The subspace \(H_t := \varepsilon_t(H)\) (target base algebra) forms a unital subalgebra and a left coideal, i.e.

\[
\Delta(H_t) \subseteq H \otimes H_t.
\]
2. The subspace \( H_s := \varepsilon_s(H) \) (source base algebra) forms a unital subalgebra and a right coideal, i.e.
\[
\Delta(H_s) \subseteq H_s \otimes H. \tag{2.10}
\]

3. The subalgebras \( H_s \) and \( H_k \) commute, i.e. \( xy = yx \) for all \( x \in H_t \) and \( y \in H_s \).

**Definition 2.6.** A Weak Hopf Algebra \((H, \mu, \eta, \Delta, \varepsilon, S)\) is a Weak Bialgebra \((H, \mu, \eta, \Delta, \varepsilon)\) with a linear map \( S: H \to H \) (antipode) that satisfies the following conditions:
\[
\begin{align*}
\mu \circ (\id_H \otimes S) \circ \Delta &= \varepsilon_t, \\
\mu \circ (S \otimes \id_H) \circ \Delta &= \varepsilon_s, \\
\mu \circ (\mu \otimes \id_H) \circ (S \otimes \id_H \otimes S) \circ (\Delta \otimes \id_H) \circ \Delta &= S. \tag{2.13}
\end{align*}
\]

For convenience, we write \( 1 = \eta(1) \) and omit parentheses in products, exploiting associativity. We also use Sweedler’s notation and write \( \Delta(x) = x' \otimes x'' \) for the comultiplication of \( x \in H \) as an abbreviation of the expression \( \Delta(x) = \sum_k a_k \otimes b_k \) with some \( a_k, b_k \in H \). Similarly, we write \( (\Delta \otimes \id_H)(x) = x' \otimes x'' \otimes x''' \), exploiting coassociativity.

**Definition 2.7.** A homomorphism \( \varphi: H \to H' \) of WHAs is a homomorphism of WBAs for which \( \varphi \circ S = S' \circ \varphi \).

The antipode of a WHA is an algebra antihomomorphism, i.e. \( S \circ \mu = \mu^{\op} \circ (S \otimes S) \) and \( S \circ \eta = \eta \), as well as a coalgebra antihomomorphism, i.e. \( (S \otimes S) \circ \Delta = \Delta^{\op} \circ S \) and \( \varepsilon \circ S = \varepsilon \).

We extend Sweedler’s notation to the right \( H \)-comodules and write \( \beta(v) = v_V \otimes v_H \) for the coaction \( \beta: V \to V \otimes H \) of \( H \) on some vector space \( V \).

**Proposition 2.8** (see [7,23]). Let \( H \) be a WBA. Then the category \( \mathcal{M}^H \) of finite-dimensional right \( H \)-comodules is a monoidal category \( (\mathcal{M}^H, \otimes, H_s, \alpha, \lambda, \rho) \). Here the monoidal unit object is the source base algebra \( H_s \) with the coaction
\[
\beta_{H_s}: H_s \to H_s \otimes H, \quad x \mapsto x' \otimes x''. \tag{2.14}
\]

The tensor product \( V \overset{\otimes}{W} \) of two right \( H \)-comodules is the truncated tensor product, which is the vector space
\[
V \overset{\otimes}{W} := \{ v \otimes w \in V \otimes W \mid v \otimes w = P_{V,W}(v \otimes w) \} \tag{2.15}
\]
where \( P_{V,W} \) is the \( k \)-linear idempotent
\[
P_{V,W}: V \otimes W \to V \otimes W, \quad v \otimes w \mapsto (v_V \otimes w_W)\varepsilon(v_H w_H). \tag{2.16}
\]

The coaction on \( V \overset{\otimes}{W} \) is given by
\[
\beta_{V \overset{\otimes}{W}}: V \overset{\otimes}{W} \to (V \overset{\otimes}{W}) \otimes H, \quad v \otimes w \mapsto (v_V \otimes w_W) \otimes (v_H w_H). \tag{2.17}
\]

The unit constraints of the monoidal category are
\[
\lambda_V: H_s \overset{\otimes}{V} \to V, \quad x \otimes v \mapsto v_V \varepsilon(xv_H), \tag{2.18}
\]
\[
\rho_V: V \overset{\otimes}{H_s} \to V, \quad v \otimes x \mapsto v_V \varepsilon(v_H \varepsilon_s(x)), \tag{2.19}
\]
and the associator is induced from that of \( \text{Vect}_k \).
Finitely semisimple spherical categories are self-dual

Proposition 2.9 (see [7, Proposition 5.9]). Let \((H, \mu, \eta, \Delta, \varepsilon)\) be a WBA and \(U: \mathcal{M}^H \to \text{Vect}_k\) be the obvious forgetful functor. Then \((U, U_{X,Y}, U_0, U^{X,Y}, U^0)\) is a \(k\)-linear faithful functor with separable Frobenius structure and it takes values in \(\text{fdVect}_k\). The Frobenius structure is given by

\[
U_{X,Y} = \text{coim} P_{X,Y}: UX \otimes UY \to P_{X,Y}(UX \otimes UY),
\]

\[
U_0 = \eta: k \to H_s,
\]

\[
U^{X,Y} = \text{im} P_{X,Y}: P_{X,Y}(UX \otimes UY) \to UX \otimes UY,
\]

\[
U^0 = \varepsilon|_{H_s}: H_s \to k
\]

(2.20 - 2.23)

Here \(P_{X,Y}\) denotes the idempotent of (2.16) with its image factorization \(P_{X,Y} = \text{im} P_{X,Y} \circ \text{coim} P_{X,Y}\). Its image \(P_{X,Y}(UX \otimes UY) = U(X \hat{\otimes} Y)\) is the vector space underlying the truncated tensor product. Finally, \(H_s = U1\) is the vector space underlying the monoidal unit.

Proposition 2.10. Let \(H\) be a WHA. Then \(\mathcal{M}^H\) is left-autonomous if the left-dual of every object \(V \in |\mathcal{M}^H|\) is chosen to be \((V^*, \text{ev}_V, \text{coev}_V)\), where the dual vector space \(V^*\) is equipped with the coaction

\[
\beta_{V^*}: V^* \to V^* \otimes H, \quad \vartheta \mapsto (v \mapsto \vartheta(v) \otimes S(v_H)),
\]

and the evaluation and coevaluation maps are given by

\[
\text{ev}_V: V^* \hat{\otimes} V \to H_s, \quad \vartheta \otimes v \mapsto \vartheta(v)\varepsilon_s(v_H),
\]

\[
\text{coev}_V: H_s \to V \hat{\otimes} V^*, \quad x \mapsto \sum_j ((v_j)_V \otimes \vartheta^j)\varepsilon(x(v_j)_H).
\]

(2.24 - 2.26)

Here we have used the evaluation and coevaluation maps that turn \(V^*\) into a left-dual of \(V\) in the category \(\text{Vect}_k\):

\[
\text{ev}^{(\text{Vect}_k)}_V: V^* \otimes V \to k, \quad \vartheta \otimes v \mapsto \vartheta(v),
\]

\[
\text{coev}^{(\text{Vect}_k)}_V: k \to V \otimes V^*, \quad 1 \mapsto \sum_j v_j \otimes \vartheta^j.
\]

(2.27 - 2.28)

3 Tannaka–Kreǐn reconstruction

In this section, we define the notion of a cospherical WHA. We show that the universal coend, \(H = \text{coend}(C, \omega)\), of our spherical category \(C\) with respect to the long forgetful functor \(\omega: C \to \text{Vect}_k\) is cospherical and that the category \(C\) is equivalent as a spherical category to the category of finite-dimensional right \(H\)-comodules. This generalizes [7] to our class of spherical categories. In order to construct \(\text{coend}(C, \omega)\), we need \(C\) to be small. We assume this from now on.

3.1 The long forgetful functor

First, we study the properties of the long forgetful functor \(\omega: C \to \text{Vect}_k\) associated with our spherical category \(C\). It turns out that Section 3 of [7] applies to the spherical case without any significant change, and so we keep this section brief.
Definition 3.1. Let \( C \) be a finitely semisimple \( k \)-linear additive spherical category such that \( k = \text{End}(1) \) is a field. The long forgetful functor is the functor
\[
\omega : C \to \text{Vect}_k, \quad X \mapsto \text{Hom}(\tilde{V}, \tilde{V} \otimes X),
\]
\[
f \mapsto (\text{id}_{\tilde{V}} \otimes f) \circ -.
\]
Here \( \tilde{V} \) denotes the universal object of \( C \),
\[
\tilde{V} = \bigoplus_{j \in I} V_j,
\]
where the sum ranges over a set of representatives of the equivalence classes of simple objects \( V_j, j \in I, \) of \( C \).

Proposition 3.2. Let \( C \) be a finitely semisimple \( k \)-linear additive spherical category, \( k = \text{End}(1) \) be a field and \( \omega : C \to \text{Vect}_k \) be the long forgetful functor. Then for each \( X \in |C| \), \( \omega(X) \) has a left-dual \( (\omega(X)^*, \text{ev}_{\omega(X)}, \text{coev}_{\omega(X)}) \) where \( \omega(X)^* = \text{Hom}(\tilde{V} \otimes X, \tilde{V}) \),
\[
\text{ev}_{\omega(X)} : \omega(X)^* \otimes \omega(X) \to k, \quad \vartheta \otimes v \mapsto \text{tr}_{\tilde{V}}(D_V \circ \vartheta \circ v),
\]
\[
\text{coev}_{\omega(X)} : k \to \omega(X) \otimes \omega(X)^*, \quad 1 \mapsto \sum_j e_j^{(X)} \otimes e_j'^{(X)}.
\]
By \( D : 1_C \Rightarrow 1_C \) we denote the natural equivalence
\[
D_X : X \to X, \quad D_X := n^X_X \sum_{\ell=1}^{n^X} t_\ell X \circ \pi_\ell X (\dim V_{j_\ell X})^{-1}.
\]

Theorem 3.3. Let \( C \) be a finitely semisimple \( k \)-linear additive spherical category such that \( k = \text{End}(1) \) is a field. The long forgetful functor is \( k \)-linear, faithful, takes values in \( \text{fdVect}_k \) and has a separable Frobenius structure \((\omega, \omega_{X,Y}, \omega_0, \omega_{X,Y}^0, \omega^0)\) with
\[
\omega_{X,Y} : \omega(X) \otimes \omega(Y) \to \omega(X \otimes Y), \quad f \otimes g \mapsto \alpha_{\tilde{V},X,Y} \circ (f \otimes \text{id}_Y) \circ g,
\]
\[
\omega_0 : k \to \omega(1), \quad 1 \mapsto \rho_{\tilde{V}}^{-1}.
\]
and
\[
\omega^{X,Y} : \omega(X \otimes Y) \to \omega(X) \otimes \omega(Y), \quad h \mapsto \sum_{j, \ell} e_{\omega(X \otimes Y)}(e^{j}_{\ell}(X) \otimes h) e^{(Y)}_{j}, \tag{3.10}
\]
\[
\omega^{0} : \omega(1) \to k, \quad v \mapsto e_{\omega(1)}(\rho_{\tilde{V}} \otimes v). \tag{3.11}
\]
where we have written
\[
e^{j}_{\ell}(X) := e^{j}_{\ell} \circ (e^{j}_{\ell}(X) \otimes \text{id}_{Y}) \circ \alpha_{X,Y}^{-1}.	ag{3.12}
\]

**Lemma 3.4.** Let \(C\) be a finitely semisimple \(k\)-linear additive spherical category, \(k = \text{End}(1)\) be a field and \(\omega : C \to \text{Vect}_{k}\) be the long forgetful functor. Then there are natural isomorphisms for all \(X \in |C|\),
\[
\Phi_{X} : \omega(X) \to \omega(X^\ast),
\]
\[
v \mapsto D_{\tilde{V}}^{-1} \circ \rho_{\tilde{V}} \circ (\text{id}_{\tilde{V}} \otimes \text{ev}_X) \circ \alpha_{X,X}^{-1} \circ (v \otimes \text{id}_{X^\ast}) \circ (D_{\tilde{V}} \otimes \text{id}_{X^\ast}), \tag{3.13}
\]
\[
\Psi_{X} : \omega(X)^\ast \to \omega(X^\ast),
\]
\[
\vartheta \mapsto (\vartheta \otimes \text{id}_{X^\ast}) \circ \alpha_{X,X}^{-1} \circ (\text{id}_{\tilde{V}} \otimes \text{coev}_X) \circ \rho_{\tilde{V}}^{-1}. \tag{3.14}
\]

Their composites are given by
\[
\Psi_{X^\ast} \circ \Phi_{X} : \omega(X) \to \omega(X^{\ast \ast}),
\]
\[
v \mapsto (D_{\tilde{V}}^{-1} \otimes \tau_{X}) \circ v \circ D_{\tilde{V}}, \tag{3.15}
\]
\[
\Phi_{X^\ast} \circ \Psi_{X} : \omega(X)^\ast \to \omega(X^{\ast \ast}),
\]
\[
\vartheta \mapsto D_{\tilde{V}}^{-1} \circ \vartheta \circ (D_{\tilde{V}} \otimes \tau_{X}^{-1}). \tag{3.16}
\]

Here \(\tau_{X} : X \to X^{\ast \ast}\) denotes the pivotal structure of \(C\) (see (A.20)).

The following diagrams illustrate the maps \(\Phi_{X}\) and \(\Psi_{X}\):}

\[
\Phi_{X} \left( \begin{array}{c}
\tilde{V} \\
X
\end{array} \right) := \begin{array}{c}
\tilde{V} \\
\vdots \\
X
\end{array} 
\]
\[
\Psi_{X} \left( \begin{array}{c}
\tilde{V} \\
\vartheta
\end{array} \right) := \begin{array}{c}
\tilde{V} \\
\vdots \\
X
\end{array}
\]

These diagrams specify well-defined morphisms of \(C\) due to the coherence theorem of [1].

**Proposition 3.5.** Let \(C\) be a finitely semisimple \(k\)-linear additive spherical category, \(k = \text{End}(1)\) be a field and \(\omega : C \to \text{Vect}_{k}\) be the long forgetful functor. Let \(\{e^{(X)}_{j}\}_{j}\) and \(\{e^{j}_{(X)}\}_{j}\) form a pair of dual bases of \(\omega(X)\) and \(\omega(X)^\ast\) with respect to (3.3). Then
\[
\sum_{j} e^{(X)}_{j} \circ e^{j}_{(X)} = \text{id}_{\tilde{V} \otimes X}, \tag{3.18}
\]
and
\[
\sum_{j} \Psi(e^{j}_{(X)}) \circ \Phi(e^{(X)}_{j}) = \text{id}_{\tilde{V} \otimes X^\ast}. \tag{3.19}
\]
3.2 Cospherical Weak Hopf Algebras

In this section, we define the notion of a cospherical WHA and show that its category of finite-dimensional comodules is spherical. The special case of cospherical Hopf algebras reduces to the definitions given in [24]. For background on autonomous, pivotal and spherical categories, we refer to Appendix A.2.

Definition 3.6. Let \((H, \mu, \eta, \Delta, \varepsilon, S)\) be a WHA. A linear form \(f: H \to k\) is called:

1. convolution invertible if there exists some linear \(\overline{f}: H \to k\) such that \(f(x') \overline{f}(x'') = \varepsilon(x) = \overline{f}(x') \overline{f}(x'')\) for all \(x \in H\),

2. dual central if \(f(x')x'' = x'f(x'')\) for all \(x \in H\),

3. dual group-like if it is convolution invertible and if for all \(x, y \in H\),

\[
\omega(x')\omega(y')\varepsilon(x''y'') = \omega(xy) = \varepsilon(x')\omega(x'')\omega(y'').
\] (3.20)

Note that \(\overline{f}\) in (1) is uniquely determined by \(f\). Every dual group-like linear form \(f: H \to k\) also satisfies \(f(S(x)) = f(x)\) and \(\omega(\varepsilon_s(x)) = \varepsilon(x) = \omega(\varepsilon_t(x))\) for all \(x \in H\).

Definition 3.7. A copivotal WHA \((H, \mu, \eta, \Delta, \varepsilon, S, w)\) is a WHA \((H, \mu, \eta, \Delta, \varepsilon, S)\) with a dual group-like linear form \(w: H \to k\) that satisfies

\[
S^2(x) = w(x')x''\overline{w}(x'')
\] (3.21)

for all \(x \in H\).

Proposition 3.8. Let \((H, \mu, \eta, \Delta, \varepsilon, S, w)\) be a copivotal WHA. Then the category \(\mathcal{M}^H\) is pivotal with \(\tau_V: V \to V^{**}\) given by

\[
\tau_V(v) = \tau_V^{(\text{Vect}_k)}(v_V)w(v_H)
\] (3.22)

for all finite-dimensional right \(H\)-comodules \(V \in |\mathcal{M}^H|\) and \(v \in V\). Here we denote by \(\tau_V^{(\text{Vect}_k)}: V \to V^{**}\) the pivotal structure of \(\text{Vect}_k\) which is just the usual canonical identification \(V \cong V^{**}\).

Proof. The category \(\mathcal{M}^H\) is left-autonomous by Proposition 2.10. The map \(\tau_V\) is obviously \(k\)-linear. Its inverse is given by

\[
\tau_V^{-1}(\tau_V^{(\text{Vect}_k)}(v)) = v_V\overline{w}(v_H)
\] (3.23)

for all \(\tau_V^{(\text{Vect}_k)}(v) \in V^{**}\). In order to show that \(\tau_V\) is a morphism of right \(H\)-comodules, we need (3.21) and the fact that \(H\) coacts on \(V^{**}\) by

\[
(\tau_V^{(\text{Vect}_k)}(v))_V \otimes (\tau_V^{(\text{Vect}_k)}(v))_H = \tau_V^{(\text{Vect}_k)}(v_V) \otimes S^2(v_H)
\] (3.24)

for all \(v \in V\). Naturality follows from the properties of a comodule and from the naturality of \(\tau_V^{(\text{Vect}_k)}\). In order to verify (A.20), we compute the dual \(\tau_V^{**}\) in \(\mathcal{M}^H\), using (A.16), Propositions 2.8 and 2.10 (3.21), [7, equation (5.13)], (3.24) and the fact that \(\text{Vect}_k\) is pivotal. \(\square\)
Recall that in a pivotal category, the left- and right-duals are related, and we can define traces. In general, however, left- and right-traces $\text{tr}_V^{(L)}(f)$ and $\text{tr}_V^{(R)}(f)$ of some morphism $f : V \to V$ need not agree (Appendix A.2).

**Definition 3.9.** A cospherical WHA $H$ is a copivotal WHA for which

$$\text{tr}_V^{(L)}(f) = \text{tr}_V^{(R)}(f)$$

(3.25)

for all finite-dimensional right $H$-comodules $V \in |\mathcal{M}^H|$ and all morphisms $f : V \to V$. Recall that (3.25) is an identity between morphisms $H_s \to H_s$.

If $H$ is a cospherical WHA, then $\mathcal{M}^H$ is therefore not only pivotal, but also spherical.

**Example 3.10.** Every coribbon WHA [7, Definition 4.17] is cospherical.

**Proof.** Every coribbon WHA is copivotal because of [7, Lemma 5.4]. Its category of finite-dimensional comodules is ribbon [7, Proposition 5.13], and so left- and right-traces agree by Example A.12.

The traces in (3.25) are the left- and right-traces in the pivotal category $\mathcal{M}^H$. They can be computed as follows.

**Proposition 3.11.** Let $(H, \mu, \eta, \Delta, \varepsilon, S, w)$ be a copivotal WHA, $V \in |\mathcal{M}^H|$ and $f : V \to V$. Then

$$\text{tr}_V^{(L)}(f)[h] = \varepsilon(hS(c'_f)))\varepsilon_s(c'_f)\overline{\varepsilon}(c''_f) \in H_s,$$

(3.26)

$$\text{tr}_V^{(R)}(f)[h] = w(c'_f)\varepsilon(hc''_f)\varepsilon_s(S(c''_f)) \in H_s,$$

(3.27)

for all $h \in H_s$. Here $c_f = \sum_{j,\ell=1}^n f_{j\ell} c_{V_j}^{(V)}$ where $f_{j\ell} \in k$ are the coefficients of $f$, i.e. $f(e_j) = \sum_{\ell=1}^n e_\ell f_{j\ell}$, and $c_{V_j}^{(V)}$ are the matrix elements of the right $H$-comodule $V$, i.e. $\beta_V(e_j) = \sum_{\ell=1}^n e_\ell \otimes c_{V_j}^{(V)}$, for some basis $(e_j)_{1 \leq j \leq n}$ of $V$.

**Proof.** Compute the traces of Definition A.10 in the pivotal category $\mathcal{M}^H$, using the left-autonomous structure of Proposition 2.10 and the pivotal structure of Proposition 3.8.

In the following, we study the left- and right-traces for WHAs that satisfy a number of additional conditions. We will see below that the universal coend of our spherical categories over the long forgetful functor satisfies these conditions.

**Proposition 3.12.** Let $(H, \mu, \eta, \Delta, \varepsilon, S, w)$ be a copivotal WHA over $k$ with $H_l \cap H_s \cong k$. Then there are elements $t_L^{(V)}, t_R^{(V)} \in k$ for each $V \in |\mathcal{M}^H|$ such that

$$\sum_{j,\ell=1}^n \varepsilon_s(c'_{j\ell}^{(V)})\overline{\varepsilon}(c'_{j\ell}^{(V)}) = t_L^{(V)} 1,$$

(3.28)

$$\sum_{j,\ell=1}^n w(c'_{j\ell}^{(V)})\varepsilon_s(S(c'_{j\ell}^{(V)})) = t_R^{(V)} 1.$$

(3.29)
Proof. Since $H_t \cap H_s \cong k$, the monoidal unit object $H_s$ is simple, i.e. $\text{End}(H_s) \cong k$ in $\mathcal{M}^H$ [7, Lemma 5.16]. Evaluate (3.26) and (3.27) for $h = 1$ and $f = \text{id}_V$, i.e. $f_{ij} = \delta_{ij}$ and $c_f = \sum_{j=1}^n c^{(V)}_{jj}$.

Theorem 3.13. Let $H$ be a finite-dimensional split cosemisimple copivotal WHA over $k$ with $H_t \cap H_s \cong k$. $H$ is cospherical if, and only if, $t_V^{(L)} = t_V^{(R)}$ in Proposition 3.12 for all simple $V \in |\mathcal{M}^H|$.

Proof. Since $H$ is finite-dimensional split cosemisimple and $H_t \cap H_s \cong k$, the category $\mathcal{M}^H$ is finitely semisimple as a pivotal category [7, Corollary 5.17 and Proposition 5.18], c.f. Definition A.15, in particular $H_s$ is simple in $\mathcal{M}^H$.

According to the definition, $H$ is cospherical if, and only if, $t_V^{(L)}(f) = t_V^{(R)}(f)$ for all $V \in |\mathcal{M}^H|$ and all morphisms $f : V \to V$. Since $\mathcal{M}^H$ is finitely semisimple, this holds if and only if the condition is satisfied for all simple $V$. If $V$ is simple, however, then $f = \lambda \text{id}_V$ for some $\lambda \in k$. Now we take the traces (3.26) and (3.27) for $f = \text{id}_V$. As $H_s$ is simple, we can evaluate at $h = 1$ and obtain (3.28) and (3.29). $H$ is cospherical if, and only if, the two traces are equal for each simple $V$. 

3.3 Tannaka–Kreın reconstruction

In this section, we show that the universal coend $H = \text{coend}(\mathcal{C}, \omega)$ of a finitely semisimple $k$-linear additive spherical category $\mathcal{C}$ for which $k = \text{End}(\mathbb{1})$ is a field, with respect to the long forgetful functor $\omega : \mathcal{C} \to \text{Vect}_k$, forms a cospherical WHA.

The following theorem was shown in [7] for modular categories. Its proof is exactly the same for our spherical categories.

Theorem 3.14 (see [7, Section 4]). Let $\mathcal{C}$ be a finitely semisimple $k$-linear additive spherical category, $k = \text{End}(\mathbb{1})$ a field, and $\omega : \mathcal{C} \to \text{Vect}_k$ be the long forgetful functor. Then the universal coend $H = \text{coend}(\mathcal{C}, \omega)$ forms a finite-dimensional split cosemisimple WHA $(H, \mu, \eta, \Delta, \varepsilon, S)$ with $H_t \cap H_s \cong k$.

The operations of $H$ are defined in [7] using the universal property of the coend. Here we just review how to compute them. As a vector space,

$$H \cong \bigoplus_{j \in I} \omega(V_j)^* \otimes \omega(V_j) \quad (3.30)$$

where the sum is over a set of representatives $V_j \in |\mathcal{C}|$, $j \in I$, of the isomorphism classes of simple objects. We write for the homogeneous elements of $H$,

$$[\vartheta|v]_X = \vartheta \otimes v \in \omega(X)^* \otimes \omega(X). \quad (3.31)$$

We use this notation for all objects $X \in |\mathcal{C}|$. For all $X$ that do not equal the chosen representatives of the simple objects, this becomes an element of (3.30) upon identifying $[\eta \circ (\text{id}_V \otimes f)]_X = [\eta(\text{id}_V \otimes f) \circ v]_Y$ for all $v \in \omega(X) = \text{Hom}(\bar{V}, V \otimes X)$, $\eta \in \omega(Y)^* = \text{Hom}(V \otimes Y, \bar{V})$ and for all morphisms $f : X \to Y$ of $\mathcal{C}$. 
The operations of \( H \) are given as follows.

\[
\Delta([\vartheta|v]_X) = \sum_j [\vartheta|e_j^{(X)}]_X \otimes [e_j^{(X)}|v]_X, \tag{3.32}
\]

\[
\varepsilon([\vartheta|v]_X) = \text{ev}_{\omega(X)}(\vartheta \otimes v), \tag{3.33}
\]

\[
\mu([\vartheta|v]_X \otimes [\zeta|w]_Y) = [\zeta \circ (\vartheta \otimes \text{id}_Y) \circ \alpha^{-1}_{V,X,Y} \circ \alpha^{-1}_{V,Y,X} \circ (v \otimes \text{id}_Y) \circ w]_X \otimes_Y, \tag{3.34}
\]

\[
\eta(1) = [\rho_{\tilde{V}}|\rho_{\tilde{V}}^{-1}]_1, \tag{3.35}
\]

\[
S([\vartheta|v]_X) = [\Phi_X(v)|\Psi_X(\vartheta)]_X, \tag{3.36}
\]

where \( v \in \omega(X) \), \( \vartheta \in \omega(X)^* \), \( w \in \omega(Y) \) and \( \zeta \in \omega(Y)^* \), \( X, Y \in |\mathcal{C}| \), and \( (e_j^{(X)})_j \) and \( (e_j^{(X)})_j \) denote a pair of dual bases of \( \omega(X) \) and \( \omega(X)^* \). The maps \( \Phi_X \) and \( \Psi_X \) are as in Lemma 3.4.

For convenience, we also give the target and source counital maps and the square of the antipode,

\[
\varepsilon_t([\vartheta|v]_X) = [\Phi_X(v) \circ \Psi_X(\vartheta) \circ \rho_{\tilde{V}}|\rho_{\tilde{V}}^{-1}]_1, \tag{3.37}
\]

\[
\varepsilon_s([\vartheta|v]_X) = [\rho_{\tilde{V}}|\rho_{\tilde{V}}^{-1}]_1 \circ \vartheta \circ v_1, \tag{3.38}
\]

\[
S^2([\vartheta|v]_X) = [D_{\tilde{V}}^{-1} \circ \vartheta \circ (D_{\tilde{V}} \otimes X)|(D_{\tilde{V}}^{-1} \otimes \text{id}_X) \circ v \circ D_{\tilde{V}}], \tag{3.39}
\]

[7] showed that if \( \mathcal{C} \) is modular, then the WHA \( H \) is coribbon. We now generalize this result to our spherical categories.

**Proposition 3.15.** Under the assumptions of Theorem 3.14, the reconstructed WHA \( H = \text{coend}(\mathcal{C}, \omega) \) is cospherical, and its copivotal structure is given by \( w : H \to k \) and its convolution inverse \( \overline{w} \) as follows.

\[
w([\vartheta|v]_X) = \text{ev}_{\omega(X)}((D_{\tilde{V}}^{-1} \circ \vartheta \circ (D_{\tilde{V}} \otimes \text{id}_X)) \otimes v), \tag{3.40}
\]

\[
\overline{w}([\vartheta|v]_X) = \text{ev}_{\omega(X)}(\vartheta \otimes ((D_{\tilde{V}}^{-1} \otimes \text{id}_X) \circ v \circ D_{\tilde{V}})). \tag{3.41}
\]

**Proof.** The triangle identities for the left-duals of Proposition 3.2 suffice to show that \( w \) and \( \overline{w} \) are mutually convolution inverse and that \( w \) satisfies (3.20) and (3.21). Thus \( H \) is copivotal.

In order to show that \( H \) is cospherical, we note that in the usual bases of \( \omega(X) \) and \( \omega(X)^* \) for arbitrary simple \( X \), the matrix elements of the coaction read

\[
e^{(X)}_{ij} = [e_i^{(X)}|e_j^{(X)}]_X. \tag{3.42}
\]

We compute the left-hand sides of (3.28) and (3.29), using \( w \) and \( \overline{w} \) of (3.40) and (3.41), \( \varepsilon_s \) of (3.38), the triangle identities for \( \text{ev}_{\omega(X)} \) and \( \text{coev}_{\omega(X)} \), and the conditions (3.18) and (3.19) and find \( t^{(L)}_X = \dim X = t^{(R)}_X \), and so \( H \) is cospherical by Theorem 3.13.

### 3.4 Pivotal functors and traces

In this section, we define the notion of a pivotal functor, i.e., a functor that preserves the pivotal structure, and see how it behaves with respect to traces.

The following proposition is well known for strong monoidal functors, but it cannot even be formulated for functors that are merely lax or merely oplax monoidal. It nevertheless holds for functors with a Frobenius structure.
Proposition 3.16. Let \( \mathcal{C} \) be a left-autonomous and \( \mathcal{C}' \) be a monoidal category and
\[
(F, F_{X,Y}, F_0, F^{X,Y}, F^0): \mathcal{C} \to \mathcal{C}'
\]
be a functor with Frobenius structure. Then for every \( X \in |\mathcal{C}| \), the object \( F X \in |\mathcal{C}'| \) has a left-dual \((F(X^*), ev_{F(X^*)}, coev_{F(X^*)})\) where
\[
ev_{F(X^*)} = F^0 \circ F ev_X \circ F_{X^*,X} : F(X^*) \otimes' FX \to 1',
\]
\[
coev_{F(X^*)} = F^{X,X^*} \circ F coev_X \circ F_0 : 1' \to FX \otimes F(X^*).
\]

Proof. In order to verify the triangle identity \((A.12)\), we use \((2.1)\), naturality of \(F\) and the image under \(F\) of the triangle identity in \(\mathcal{C}\). For the other triangle identity \((A.13)\), we need \((2.2)\), naturality of \(F_{X^*,X}^{-}\) and \(F_{-X^*}^{-}\) and the image under \(F\) of the triangle identity in \(\mathcal{C}\).

If both \( \mathcal{C} \) and \( \mathcal{C}' \) are left-autonomous, then we can use a standard result (Proposition \([A.7]\)) to show that the left-duals in \( \mathcal{C}' \) and those obtained from the duals in \( \mathcal{C} \) by using Proposition 3.16 are canonically isomorphic.

Corollary 3.17. Let \( \mathcal{C} \) and \( \mathcal{C}' \) be left-autonomous categories and \((F, F_{X,Y}, F_0, F^{X,Y}, F^0): \mathcal{C} \to \mathcal{C}'\) be a functor with Frobenius structure. Then there are natural isomorphisms \( u_X : F(X^*) \to (FX)^* \) given by
\[
u_X = \lambda_{(FX)^*}^\prime \circ (F^0 \otimes' id_{(FX)^*}) \circ (F ev_X \otimes' id_{(FX)^*}) \circ (F_{X^*,X} \otimes' id_{(FX)^*})
\]
\[
\circ \alpha_{(FX)^*,F_0,F_0}^{-1} \circ (id_{F(X^*)} \otimes' coev_{FX}) \circ \rho_{F(X^*)}^{-1}
\]
with inverse
\[
u_X^{-1} = \lambda_{(FX^*)}^\prime \circ (ev_{F_0} \otimes' id_{(FX^*)}) \circ \alpha_{(FX^*)}^{-1} \circ (id_{(FX^*)} \otimes' id_{FX^*})
\]
\[
\circ (id_{FX^*} \otimes' coev_{FX}) \circ (id_{(FX^*)} \otimes' F_0) \circ \rho_{(FX^*)}^{-1}.
\]
They satisfy
\[
\begin{array}{ccc}
F(X^*) \otimes' FX & \xrightarrow{u_X \otimes id_{FX}} & (FX)^* \otimes' FX \\
F_{X^*,X} \downarrow & & \downarrow ev_{FX} \\
F(X^* \otimes X) \xrightarrow{Fev_X} F1 \xrightarrow{F0} 1',
\end{array}
\]
and
\[
\begin{array}{ccc}
1' \xrightarrow{coev_{FX}} FX \otimes' (FX)^* \\
\downarrow id_{FX} \otimes u_X^{-1} & & \\
F1 \xrightarrow{F_{coev_X}} F(X \otimes X^*) \xrightarrow{F_{X,X^*}} FX \otimes' F(X^*).
\end{array}
\]
Note that $\Psi_X$ of (3.14) is precisely the $u_X^{-1}$ associated with the long forgetful functor $\omega$.

**Definition 3.18.** Let $\mathcal{C}$ and $\mathcal{C}'$ be pivotal categories. A strong monoidal functor

$$(F, F_{X,Y}, F_0): \mathcal{C} \to \mathcal{C}'$$

(3.50)

is called **pivotal** if

$$
\begin{array}{ccc}
FX & \xrightarrow{ \ F\tau_X \ } & F(X^{**}) \\
\downarrow \tau'_{FX} & & \downarrow u_X^* \\
(FX)^{**} & \xrightarrow{(u_X)^*} & (F(X^*))^*
\end{array}
$$

(3.51)

for all $X \in |\mathcal{C}|$. Here, $u_X : F(X^*) \to (FX)^*$ are the natural isomorphisms of Corollary 3.17 associated with $F$.

**Example 3.19.** Every strong monoidal ribbon functor (see, for example [7, Definition A.14]) between ribbon categories is pivotal.

**Proof.** By [7, Definition A.9, Definition A.14, equation (A.24)], the axioms of a strong monoidal functor and equations (3.48) and (3.49).

The following is a refinement of Proposition A.4 in the case of a pivotal functor.

**Proposition 3.20.** Let $\mathcal{C}$ and $\mathcal{C}'$ be left-autonomous categories and $F \dashv G : \mathcal{C} \to \mathcal{C}'$ be an adjoint equivalence. If $F$ is pivotal strong monoidal, then $G$ is pivotal.

**Proof.** First, we observe that if the isomorphisms $u_X$ are defined as in (3.46) for the functor $F$, the analogues of $u_X$ associated with $G$ turn out to be equal to

$$v_Y = \eta_{(GY)^*}^{-1} \circ Gu_Y^{-1} \circ G\varepsilon_Y : G(Y^*) \to (GY)^*,$$

(3.52)

$Y \in |\mathcal{C}'|$, where $\eta : 1_\mathcal{C} \Rightarrow G \circ F$ and $\varepsilon : F \circ G \Rightarrow 1_\mathcal{C}$ are the unit and counit of the adjunction, respectively.

For the proof, we need Definition 3.18 the triangle identities of the adjunction and $v_Y$- and $(v_Y)^*$ from (3.52).

By an equivalence of pivotal categories we mean an equivalence of categories in which one functor is pivotal strong monoidal. Pivotal strong monoidal functors $F : \mathcal{C} \to \mathcal{C}'$ relate the traces of $\mathcal{C}$ and $\mathcal{C}'$ as follows.

**Proposition 3.21.** Let $\mathcal{C}$ and $\mathcal{C}'$ be pivotal categories and $(F, F_{X,Y}, F_0) : \mathcal{C} \to \mathcal{C}'$ be a pivotal strong monoidal functor. Then for every morphism $f : X \to X$ of $\mathcal{C}$,

$$
\begin{array}{ccc}
1' & \xrightarrow{F_0} & F1 \\
\downarrow u_{(F_{l})}^{(l)}(f) & & \downarrow F\tau_{X}^{(l)}(f) \\
1' & \xrightarrow{F_0} & F1
\end{array}
$$

(3.53)

and similarly for the right-trace.
Proof. By Definition A.10, Definition 3.18, equations (3.48) and (3.49).

Proposition 3.22 (analogous to [7, Proposition A.28]). Let $\mathcal{C}$ and $\mathcal{C}'$ be semisimple $k$-linear pivotal categories, $k = \text{End}(1)$ be a field, and $(F, F_{X,Y}, F_0): \mathcal{C} \to \mathcal{C}'$ be a pivotal strong monoidal $k$-linear functor. Then for each morphism $f: X \to X$ of $\mathcal{C}$,

$$
\text{tr}^{(L)}_X(f) = \text{tr}^{(L)}_{FX}(Ff) \quad \text{and} \quad \text{tr}^{(R)}_X(f) = \text{tr}^{(R)}_{FX}(Ff).
$$

(3.54)

In particular, if $F$ is essentially surjective and full and $\mathcal{C}$ is spherical, then $\mathcal{C}'$ is spherical, too.

By an equivalence of spherical categories, we therefore mean an equivalence of categories, one functor of which is pivotal strong monoidal.

3.5 Equivalence of categories

Here we show that the original spherical category $\mathcal{C}$ is equivalent as a spherical category to $\mathcal{M}^H$, the category of finite-dimensional comodules over the universal coend $H = \text{coend}(\mathcal{C}, \omega)$ with respect to the long forgetful functor.

The proof of the following theorem (shown in [7] for modular categories) is exactly identical in the spherical case.

Theorem 3.23 (see [7, Theorem 6.1]). Let $\mathcal{C}$ be a finitely semisimple $k$-linear additive spherical category, $k = \text{End}(1)$ be a field, $\omega: \mathcal{C} \to \text{Vect}_k$ be the long forgetful functor and $H = \text{coend}(\mathcal{C}, \omega)$ be the reconstructed WHA.

1. The long forgetful functor factors through $\mathcal{M}^H$, i.e., the diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{M}^H \\
\omega \downarrow & & \downarrow U \\
\text{Vect}_k & \xrightarrow{V} & 
\end{array}
$$

(3.55)

commutes. Here $U: \mathcal{M}^H \to \text{Vect}_k$ is the forgetful functor of Proposition 2.9.

2. The functor $F$ is $k$-linear, essentially surjective and fully faithful.

3. $(F, F_{X,Y}, F_0)$ forms a strong monoidal functor with

$$
F_{X,Y}: FX \otimes FY \to F(X \otimes Y), \quad f \otimes g \mapsto \alpha^{-1}_{V,X,Y} \circ (f \otimes \text{id}_Y) \circ g, \quad (3.56)
$$

$$
F_0: H_s \mapsto F1, \quad [\rho_{\bar{V}}[v]]_1 \mapsto v. \quad (3.57)
$$

If $\mathcal{C}$ is modular, then the functor $F$ forms part of an equivalence of ribbon categories. We now generalize this result to our spherical categories.

Lemma 3.24. Let $\mathcal{C}$ be a finitely semisimple $k$-linear additive spherical category, $k = \text{End}(1)$ a field, and $\omega: \mathcal{C} \to \text{Vect}_k$ be the long forgetful functor. Then the natural isomorphisms of (3.36) associated with $F$ are given by

$$
u_X: F(X^*) \to (FX)^*, \quad \nu \mapsto \rho_{\bar{V}} \circ (\text{id}_V \otimes \text{ev}_X) \circ \alpha^{-1}_{\bar{V},X,X^*} \circ (\nu \otimes \text{id}_X), \quad (3.58)
$$

$$
u_X^{-1}: (FX)^* \to F(X^*), \quad \nu \mapsto (\vartheta \otimes \text{id}_{X^*}) \circ \alpha^{-1}_{\bar{V},X,X^*} \circ (\text{id}_V \otimes \text{coev}_X) \circ \rho_{\bar{V}}^{-1}. \quad (3.59)
$$
The following diagrams illustrate the maps $u_X$ and $u_X^{-1}$:

\[
\begin{align*}
    u_X \left( \begin{array}{c}
        \tilde{V} \\
        \tilde{v} \\
        V \\
        X
    \end{array} \right) := \begin{array}{c}
        \tilde{V} \\
        \tilde{v} \\
        \tilde{V} \\
        X
    \end{array}, & \quad u_X^{-1} \left( \begin{array}{c}
        \vartheta \\
        \vartheta \\
        \tilde{V} \\
        \tilde{V} \\
        X
    \end{array} \right) := \begin{array}{c}
        \vartheta \\
        \vartheta \\
        \tilde{V} \\
        \tilde{V} \\
        X
    \end{array}
\end{align*}
\]

\[ (3.60) \]

**Proposition 3.25.** Let $\mathcal{C}$ be a finitely semisimple $k$-linear additive spherical category, $k = \text{End}(1)$ a field. Then the functor $F$ of Theorem 3.23 is pivotal.

**Proof.** Let $H = \text{coend}(\mathcal{C}, \omega)$ be the reconstructed WHA. We first compute $\tau_{FX}: FX \to (FX)^{**}$ (c.f. (3.22)) in $\mathcal{M}^H$ for the copivotal form of (3.40):

\[ \tau_{FX}(v) = \tau_{FX}^{(\text{Vect}_k)}((D_{\tilde{V}} \otimes \text{id}_X) \circ v \circ D_{\tilde{V}}^{-1}) \]  

(3.61)

for all $v \in FX$. We also need the dual morphism $(u_X)^* \text{ of (3.58)}$ in $\mathcal{M}^H$. Therefore, we compute (A.16) in $\mathcal{M}^H$ and find

\[ (u_X)^*(\tau_{FX}^{(\text{Vect}_k)}(v)) = D_{\tilde{V}}^{-1} \circ \rho_{\tilde{V}} \circ (\text{id}_{\tilde{V}} \otimes \text{ev}_{X^*}) \circ (\text{id}_{\tilde{V}} \otimes (\tau_X \otimes \text{id}_{X^*})) \circ \alpha_{\tilde{V}, X, X^*} \circ (v \otimes \text{id}_{X^*}) \circ (D_{\tilde{V}} \otimes \text{id}_{X^*}) \in (F(X^*))^* \]  

(3.62)

for all $v \in FX$. Then we can verify that

\[ u_X^* (F\tau_X(v)) = \rho_{\tilde{V}} \circ (\text{id}_{\tilde{V}} \otimes \text{ev}_{X^*}) \circ (\text{id}_{\tilde{V}} \otimes (\tau_X \otimes \text{id}_{X^*})) \circ \alpha_{\tilde{V}, X, X^*} \circ (v \otimes \text{id}_{X^*}) \]  

(3.63)

for all $v \in FX$, and so $F$ is pivotal. \qed

Since the functor $F$ is strong monoidal, this implies Theorem 1.1. Note that this theorem also shows that every finitely semisimple $k$-linear additive spherical category $\mathcal{C}$ for which $k = \text{End}(1)$ is a field, is abelian and that the long forgetful functor $\omega: \mathcal{C} \to \text{Vect}_k$ is exact.

## 4 Self-duality

### 4.1 The coend as $\text{End}(\tilde{V}^* \otimes \tilde{V})$

Thanks to the finite semisimplicity of our spherical categories $\mathcal{C}$, the vector space underlying the coend

\[ H = \text{coend}(\mathcal{C}, \omega) = \bigoplus_{j \in I} \omega(V_j)^* \otimes \omega(V_j) \]  

(4.1)

is just $\text{End}(\tilde{V}^* \otimes \tilde{V})$. Here, $\omega(X) = \text{Hom}(\tilde{V}, \tilde{V} \otimes X)$ and $\omega(X)^* = \text{Hom}(\tilde{V} \otimes X, \tilde{V})$. It is instructive to see how our pair of dual bases of $\omega(X)$ and $\omega(X)^*$ for simple $X$ provides a decomposition of $\tilde{V}^* \otimes \tilde{V}$ into a finite biproduct of simple objects.
Proposition 4.1. Let \( C \) be a finitely semisimple \( k \)-linear additive spherical category, \( k = \text{End}(1) \) a field and \( \omega: C \to \text{Vect}_k \) be the long forgetful functor. Define maps \( \pi^{(X)}_\ell: \widetilde{V}^* \otimes \widetilde{V} \to X \) and \( \iota^{(X)}_\ell: X \to \widetilde{V}^* \otimes \widetilde{V} \) by

\[
\begin{align*}
\pi^{(X)}_\ell & := \lambda_X \circ (\text{ev}_{\widetilde{V}} \otimes \text{id}_X) \circ \alpha^{-1}_{\widetilde{V}, \widetilde{V}, X} \circ (\text{id}_{\widetilde{V}}, \otimes e^{(X)}_\ell), \\
\iota^{(X)}_\ell & := \dim X \cdot (\text{id}_{\widetilde{V}, \otimes D_{\widetilde{V}}}) \circ (\text{id}_{\widetilde{V}, \otimes e^{(X)}_\ell}) \circ \alpha_{\widetilde{V}, \widetilde{V}, X} \\
& \circ (\text{coev}_{\widetilde{V}} \otimes \text{id}_X) \circ \lambda^{-1}_X
\end{align*}
\]  

(4.2)

(4.3)

where \( (e^{(X)}_\ell)_\ell \) and \( (e^{(X)}_\ell)_\ell \) form a pair of dual bases of \( \omega(X) \) and \( \omega(X)^* \), \( X \in |C| \), with respect to (3.3). By \( \text{coev}_{\widetilde{V}} \), we denote the coevaluation map associated with the right-dual of \( \widetilde{V} \) as in (A.22). Then these maps satisfy

1. The domination property,

\[
\sum_{j \in I} \sum_{\ell = 1}^{\dim \omega(V_j)} \iota^{(V_j)}_\ell \circ \pi^{(V_j)}_\ell = \text{id}_{\widetilde{V}^* \otimes \widetilde{V}}.
\]  

(4.4)

2. If \( X, Y \in |C| \) are simple and \( X \not\cong Y \), then for all \( m, \ell \),

\[
\pi^{(Y)}_m \circ \iota^{(X)}_\ell = 0.
\]  

(4.5)

3. If \( X \in |C| \) is simple, then

\[
\pi^{(X)}_m \circ \iota^{(X)}_\ell = \delta_{m \ell} \text{id}_X.
\]  

(4.6)

4. Every morphism \( f: \widetilde{V}^* \otimes \widetilde{V} \to \widetilde{V}^* \otimes \widetilde{V} \) is of the form

\[
f = \sum_{j \in I} \sum_{\ell, m = 1}^{\dim \omega(V_j)} f^{(V_j)}_{\ell m} \cdot (\iota^{(V_j)}_\ell \circ \pi^{(V_j)}_m),
\]  

(4.7)

where

\[
f^{(X)}_{\ell m} = \text{tr}_X (\pi^{(X)}_\ell \circ f \circ \iota^{(X)}_m) / \dim X.
\]  

(4.8)

Proof. We need (A.28) and (A.29) for \( \widetilde{V}^* \otimes \widetilde{V} \), the Schur axiom of Definition A.13(3b), the fact that any \( g: X \to X \) for simple \( X \) is of the form \( g = (\text{tr}_X (g) / \dim X) \text{id}_X \), and the triangle identities for the duals given in Proposition 3.2.

The following diagrams illustrate the above definitions:

\[
\begin{align*}
\begin{tikzpicture}
  \node (A) at (0,0) {$\widetilde{V}$};
  \node (B) at (1,0) {$\widetilde{V}$};
  \node (C) at (0,1) {$\widetilde{V}$};
  \node (D) at (1,1) {$\widetilde{V}$};
  \node (E) at (0,2) {$X$};
  \node (F) at (1,2) {$X$};
  \draw[->] (A) to node {$\pi^{(X)}_\ell$} (E);
  \draw[->] (B) to node {$\iota^{(X)}_\ell$} (F);
  \draw[->] (A) to node {} (B);
  \draw[->] (C) to node {} (D);
\end{tikzpicture}
\end{align*}
\]  

and

\[
\begin{align*}
\begin{tikzpicture}
  \node (A) at (0,0) {$\widetilde{V}$};
  \node (B) at (1,0) {$\widetilde{V}$};
  \node (C) at (0,1) {$\widetilde{V}$};
  \node (D) at (1,1) {$\widetilde{V}$};
  \node (E) at (0,2) {$X$};
  \node (F) at (1,2) {$X$};
  \draw[->] (A) to node {$\iota^{(X)}_\ell$} (E);
  \draw[->] (B) to node {$\pi^{(X)}_\ell$} (F);
  \draw[->] (A) to node {} (B);
  \draw[->] (C) to node {} (D);
  \draw[->] (E) to node {} (F);
\end{tikzpicture}
\end{align*}
\]  

(4.9)
**Definition 4.2.** Under the assumptions of Proposition [4.1] we write $H = \text{End}(\tilde{V}^* \otimes \tilde{V})$ and $\tilde{H} = \text{End}(\tilde{V} \otimes \tilde{V}^*)$ and define linear isomorphisms $\varphi_L, \varphi_R : H \to \tilde{H}$ (left- and right-Fourier transform) by

\[
\varphi_L(f) = (\lambda_{\tilde{V}} \otimes \text{id}_{\tilde{V}}^*) \circ ((\overline{\epsilon_{\tilde{V}}} \otimes \text{id}_{\tilde{V}}^*) \otimes \text{id}_{\tilde{V}}^*) \circ (\alpha_{\tilde{V}, \tilde{V}}^{-1} \otimes \text{id}_{\tilde{V}}^*) \circ (D_{\tilde{V}}^{-1} \otimes \text{id}_{\tilde{V}}^*) \circ \alpha_{\tilde{V}, \tilde{V}^*}^{-1} \otimes \text{id}_{\tilde{V}}^*) \circ (\text{id}_{\tilde{V}} \otimes \text{id}_{\tilde{V}}^*),
\]

and

\[
\varphi_L(f) = (\alpha_{\tilde{V}, \tilde{V}^*} \otimes \text{id}_{\tilde{V}}^*) \circ (\text{id}_{\tilde{V}} \otimes (\overline{\epsilon_{\tilde{V}}} \otimes \text{id}_{\tilde{V}}^*)) \circ (\text{id}_{\tilde{V}} \otimes \alpha_{\tilde{V}, \tilde{V}^*}) \circ ((\text{id}_{\tilde{V}} \otimes f) \otimes D_{\tilde{V}}^{-2}) \circ (\alpha_{\tilde{V}, \tilde{V}^*} \otimes \text{id}_{\tilde{V}}^*) \circ (\text{id}_{\tilde{V}} \otimes \text{id}_{\tilde{V}}^*),
\]

(4.10)

The maps $\varphi_L$ and $\varphi_R$ are identical to the maps of [13] except for the factors $D_{\tilde{V}}$. Although you might suspect that these factors are merely a consequence of the choice of the bilinear form (3.3), they are actually present because of the pivotal form (3.40) and cannot be avoided by simple redefinitions. Diagrammatically, the Fourier transforms read

(4.12)

\[
\varphi_L( \begin{array}{c} f \end{array} ) = \begin{array}{c} f \end{array} \begin{array}{c} D_{\tilde{V}}^{-1} \end{array}
\]

(4.13)

\[
\varphi_R( \begin{array}{c} f \end{array} ) = \begin{array}{c} f \end{array} \begin{array}{c} D_{\tilde{V}}^{-2} \end{array}
\]

where all arrows are labeled by $\tilde{V}$.

**Proposition 4.3.** Under the assumptions of Proposition [4.1] the unital algebra underlying $\text{coend}(\mathcal{C}, \omega)$ is isomorphic to $H = \text{End}(\tilde{V}^* \otimes \tilde{V})$ with the convolution product

\[
f * g = (\rho_{\tilde{V}}^{-1} \otimes \text{id}_{\tilde{V}}) \circ ((\text{id}_{\tilde{V}} \otimes \overline{\epsilon_{\tilde{V}}} \otimes \text{id}_{\tilde{V}}^*) \circ (\alpha_{\tilde{V}, \tilde{V}, \tilde{V}^*} \otimes \text{id}_{\tilde{V}}^*) \circ (\text{id}_{\tilde{V}} \otimes D_{\tilde{V}}^{-2} \otimes \text{id}_{\tilde{V}}^*) \circ (\text{id}_{\tilde{V}} \otimes f) \circ (\text{id}_{\tilde{V}} \otimes \text{id}_{\tilde{V}}^*) \circ (\text{id}_{\tilde{V}} \otimes \text{id}_{\tilde{V}}^*) \circ (\text{id}_{\tilde{V}} \otimes \text{id}_{\tilde{V}}^*) \circ (\rho_{\tilde{V}} \otimes \alpha_{\tilde{V}}),(4.14)
\]

and unit

\[
1_\ast = (D_{\tilde{V}} \otimes \text{id}_{\tilde{V}}) \circ \overline{\text{coev}}_\ast \circ \epsilon_{\tilde{V}}^\ast.
\]

(4.15)

The coalgebra structure of the coend reads in terms of $\text{End}(\tilde{V}^* \otimes \tilde{V})$ as follows,

\[
\Delta(e^{(X)}_{m \ell}) = \sum_{p=1}^{\dim \omega(X)} e^{(X)}_{p \ell} \otimes e^{(X)}_{mp},
\]

(4.16)

\[
\varepsilon(e^{(X)}_{m \ell}) = \delta_{\ell m},
\]

(4.17)

(4.18)

for all simple $X \in |\mathcal{C}|$. Here we have written,

\[
e^{(X)}_{m \ell} := (\iota^{(X)}_{\ell} \circ \pi^{(X)}_{m}) / \dim X,
\]

(4.19)

with $\iota^{(X)}_{\ell}$ and $\pi^{(X)}_{m}$ as in (4.2) and (4.3). The antipode is given by $S = \varphi_L^{-1} \circ \varphi_R$. 


Proof. By a direct computation using the linear isomorphism
\[
\text{coend}(\mathcal{C}, \omega) \rightarrow \text{End}(\check{V}^* \otimes \check{V}), \quad [e^\ell_{(X)} | e^m_{(X)}]_X \mapsto e^m_{(X)}. \tag{4.20}
\]

Diagrammatically, the convolution algebra structure of \( H = \text{End}(\check{V}^* \otimes \check{V}) \) reads,
\[
\begin{align*}
\begin{array}{ccc}
f & * & g \\
\downarrow & & \downarrow \\
\check{V} & = & \check{V} \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
f & \check{} & g \\
\downarrow & \check{} & \downarrow \\
\check{V} & \check{} & \check{V} \\
\end{array}
\end{align*}
\tag{4.21}
\]

where all arrows are labeled with \( \check{V} \). For our purposes, we do not need the structure of \( \text{End}(\check{V}^* \otimes \check{V}) \) as a unital associative algebra with respect to composition. Note that the left-Fourier transform \( \varphi_L \) maps composition of \( H \) to convolution of \( \hat{H} \) and convolution of \( H \) to opposite composition of \( \hat{H} \), whereas the right-Fourier transform \( \varphi_R \) maps them to opposite convolution and composition, respectively.

4.2 A non-degenerate pairing of WHAs

In this section, we use Fourier transform in order to establish a dual pairing of WHAs between \( H \) and \( \hat{H} \).

First, observe that all the constructions of Section 3, from the definition of the long forgetful functor (Definition 3.1) to the WHA structure of the universal coend (Theorem 3.14) can be done with \( \check{V}^* \) rather than \( \check{V} \) as the universal object. This is possible because taking the dual is an involution on the set of isomorphism classes of simple objects (Definition A.15). Similarly, Proposition 4.3 is available for \( \check{V}^* \) rather than \( \check{V} \), and so we get a WHA structure on
\[
\hat{H} = \text{End}(\check{V} \otimes \check{V}^*) \cong \text{End}(\check{V}^* \otimes \check{V}^*). \tag{4.22}
\]

If we denote by \((\check{e}_m^{(X)})_{\ell}\) and \((\check{e}_m^{(X)})_{\ell}\) a pair of dual bases of \( \hat{\omega}(X) := \text{End}(\check{V}^* \otimes X, \check{V}^*) \) and \( \hat{\omega}(X)^* := \text{End}(\check{V} \otimes X, \check{V}^*) \), respectively, we have the (non-canonical) isomorphism of WHAs
\[
H \rightarrow \hat{H}, \quad [e^\ell_{(X)} | e^m_{(X)}]_X \mapsto [\check{e}^\ell_{(X)} | \check{e}^m_{(X)}]_{\check{X}}, \tag{4.23}
\]

just because the construction of both WHAs in Section 3 proceeds identically except for using \( \check{V}^* \) instead of \( \check{V} \). For all steps in the construction of \( \hat{H} \), we put a hat on the corresponding symbol that we use for \( H \).

Proposition 4.4. Under the assumptions of Proposition 4.1, there are non-degenerate bilinear maps
\[
\langle -; \check{-}\rangle : H \otimes \hat{H} \rightarrow k, \quad f \otimes \check{g} \mapsto \text{tr}_{\check{V} \otimes \check{V}^*}(\check{g} \circ \varphi_L(f)), \tag{4.24}
\]
\[
\langle \check{-}; -\rangle : \hat{H} \otimes H \rightarrow k, \quad \check{f} \otimes g \mapsto \text{tr}_{\check{V}^* \otimes \check{V}}(g \circ \varphi^{-1}_L(\check{f})). \tag{4.25}
\]
Finitely semisimple spherical categories are self-dual

Proof. Choose the basis \( (e_{j\ell}^{(X)}) \) of \( H \) as in (4.19) and, analogously, a basis \( \hat{e}_{j\ell}^{(X)} \) of \( \hat{H} \). Compute \( \varphi_L \) and \( \varphi_{L^{-1}} \) in these bases, using Proposition 4.1(4). Then the conditions \( \varphi_{L^{-1}} \circ \varphi_L = \text{id}_H \) and \( \varphi_L \circ \varphi_{L^{-1}} = \text{id}_\hat{H} \) imply the following orthogonality relations:

\[
\dim Y \sum_{j \in I} \text{dim } V_j \sum_{p,q=1}^{\text{dim } \omega(V_j)} \langle e_{\ell m}^{(X)}; e_{qp}^{(V_j)} \rangle \langle \hat{e}_{pq}^{(V_j)}; e_{sr}^{(Y)} \rangle = \delta_{XY} \delta_{re} \delta_{sm}, \tag{4.26}
\]

\[
\dim Y \sum_{j \in I} \text{dim } V_j \sum_{p,q=1}^{\text{dim } \omega(V_j)} \langle \hat{e}_{\ell m}^{(X)}; e_{qp}^{(V_j)} \rangle \langle e_{pq}^{(V_j)}; \hat{e}_{sr}^{(Y)} \rangle = \delta_{XY} \delta_{re} \delta_{sm}, \tag{4.27}
\]

for all \( \ell, m, r, s \), upon comparing coefficients, for any simple \( X, Y \in |C| \). By \( \delta_{XY} \), we mean that \( \delta_{XY} = 1 \) if \( X \cong Y \) and \( \delta_{XY} = 0 \) otherwise. We finally get the canonical element

\[
G: k \to \hat{H} \otimes H,
\]

\[
1 \mapsto \sum_{i,j \in I} \sum_{p,q=1}^{\text{dim } \omega(V_i)} \sum_{r,s=1}^{\text{dim } \omega(V_j)} \hat{e}_{pq}^{(V_i)} \otimes e_{rs}^{(V_j)} \langle \hat{e}_{pq}^{(V_i)}; e_{sr}^{(Y)} \rangle \dim V_i \dim V_j, \tag{4.28}
\]

that satisfies the triangle identities together with \( \langle -; \hat{=} \rangle \), and a similar one for \( \langle \hat{=}; - \rangle \), establishing non-degeneracy.

The following diagrams illustrate the bilinear maps:
Finitely semisimple spherical categories are self-dual

\[ \langle e_{j \ell}^{(X)} : e_{pq}^{(Y)} \rangle = \frac{1}{(\dim X \dim Y)}. \]  

(4.30)

Here, all unlabeled arrows refer to the object \( \hat{V} \). Expanding all occurrences of \( \hat{V} \) and \( \hat{V}^* \) as direct sums of simple objects shows that both bilinear forms are just sums of generalized 6j-symbols associated with the category \( \mathcal{C} \).

**Definition 4.5.** Let \( H \) and \( L \) be WHAs over some field \( k \). A dual pairing of \( H \) and \( L \) is a non-degenerate \( k \)-bilinear map \( g : H \otimes L \rightarrow k, h \otimes \ell \mapsto g(h; \ell) \) such that

\[ g(\Delta h; \ell_1 \otimes \ell_2) = g(h; \ell_1 \ell_2), \]  

(4.31)

\[ \varepsilon(h) = g(h; 1), \]  

(4.32)

\[ g(h_1 h_2; \ell) = g(h_1 \otimes h_2; \Delta \ell), \]  

(4.33)

\[ g(1; \ell) = \varepsilon(\ell), \]  

(4.34)

\[ g(S h; \ell) = g(h; S \ell), \]  

(4.35)

for all \( h, h_1, h_2 \in H \) and \( \ell, \ell_1, \ell_2 \in L \).

Note that we extend \( g \) to tensor products by \( g(h_1 \otimes h_2; \ell_1 \otimes \ell_2) := g(h_1; \ell_1) g(h_2; \ell_2) \).

**Theorem 4.6.** Under the assumptions of Proposition 4.1, the map \( \langle -; \hat{-} \rangle \) of (4.24) forms a dual pairing of WHAs.

**Proof.** Verify the conditions for the bases \( e_{j \ell}^{(X)} \) of \( H \) and \( \hat{e}_{j \ell}^{(X)} \) of \( \hat{H} \), c.f. (4.19), and use the triangle identities for evaluation and coevaluation of Proposition 3.2.

Combining Theorem 4.6 with the trivial fact that there exists the isomorphism (4.23), we see that the WHA \( H \) is self-dual. It is instructive to give the basis of \( \hat{H} \) dual to our basis \( (e_{j \ell}^{(X)})_{X \ell} \) of \( H \) with respect to \( \langle -; \hat{-} \rangle \):

\[ \hat{E}_{j \ell}^{(X)} := \dim X \sum_{j \in I} \dim V_j \sum_{p,q=1}^{\dim \omega(V_j)} e_{pq}^{(V_j)} \langle e_{qp}^{(V_j)} : e_{lj}^{(X)} \rangle \in \hat{H}, \]  

(4.36)

where \( X \in |\mathcal{C}| \) is simple. It can be read off the canonical element (4.28) and satisfies \( \langle e_{r s}^{(Z)} : \hat{E}_{j \ell}^{(X)} \rangle = \delta_{XZ} \delta_{r j} \delta_{s \ell} \) for all simple \( X, Z \in |\mathcal{C}| \) and all \( j, \ell, r, s \).
5 Self-duality of spherical categories

5.1 Duals of monoidal categories

We first generalize the notion of the dual of a monoidal category of \([4,5]\) from strong monoidal functors to functors with a separable Frobenius structure.

**Definition 5.1.** Let \(C\) and \(V\) be monoidal categories and \((F, F_{X,Y}, F_0, F^{X,Y}, F^0) : C \to V\) be a functor with separable Frobenius structure.

1. A right \((C, F)\)-module \((V, c_V)\) consists of an object \(V \in |V|\) and a natural transformation \(c_V : V \otimes F \Rightarrow F \otimes V\) such that

\[
\begin{align*}
V \otimes 1 & \xrightarrow{id_V \otimes F_0} V \otimes F 1 \\
\rho_V & \downarrow \quad (c_V)_1 \\
V & \quad (c_V)_1 \\
1 \otimes V & \xleftarrow{F_0 \otimes id_V} F 1 \otimes V \\
\end{align*}
\]

and

\[
\begin{align*}
V \otimes (FX \otimes FY) & \xrightarrow{id_V \otimes F_{X,Y}} V \otimes F(X \otimes Y) \xrightarrow{(c_V)_X \otimes V} F(X \otimes Y) \otimes V \\
\alpha_{V,FX,FY}^{-1} & \downarrow \quad F^{X,Y} \otimes id_V \\
(V \otimes FX) \otimes FY & \quad (FX \otimes FY) \otimes V \\
(c_V)_X \otimes id_{FY} & \downarrow \quad \alpha_{FX,FY,V} \\
(FX \otimes V) \otimes FY & \xleftarrow{\alpha_{FX,XY,FY}} FX \otimes (V \otimes FY) \xrightarrow{id_{FX} \otimes (c_V)_Y} FX \otimes (FY \otimes V) \\
\end{align*}
\]

commute for all \(X, Y \in |C|\).

2. A morphism \(\varphi : (V, c_V) \to (W, c_W)\) of right \((C, F)\)-modules is a morphism \(\varphi : V \to W\) of \(V\) such that

\[
\begin{align*}
V \otimes FX & \xrightarrow{\varphi \otimes id_{FX}} W \otimes FX \\
(c_V)_X & \downarrow \quad (c_W)_X \\
FX \otimes V & \xleftarrow{id_{FX} \otimes \varphi} FX \otimes W \\
\end{align*}
\]

commutes for all \(X \in |C|\).
3. The category \((C, F)^*\) whose objects and morphisms are the right \((C, F)\)-modules and their morphisms, is called the full right-dual of \(C\) over \(F\).

4. The full subcategory \((C, F)^\circ\) of \((C, F)^*\) whose objects are those right \((C, F)\)-modules \((V, c_V)\) for which \(c_V\) is a natural equivalence, is called the right-dual of \(C\) over \(F\).

5. If \(V = \text{Vect}_k\) and \(k\) is a field, we denote by \(M_{(C, F)}\) the full subcategory of \((C, F)^*\) whose objects are those right \((C, F)\)-modules \((V, c_V)\) for which \(V\) is finite-dimensional.

In order to keep this section brief, we do not develop the abstract theory of the dual category nor do we say how \(\tilde{C}\) and \(M_{(C, F)}\) are related. We merely show for the case of our spherical categories and the reconstructed WHAs \(H\) that the categories \(M_{(M^H, U)}\) and \(M_H\) are isomorphic as categories (without extra structure). Here, \(U : \mathcal{M}_H \to \text{Vect}_k\) is the forgetful functor of Proposition 2.9. Since we know the WHA \(H\) in detail, it is not difficult to see in a second step how the extra structure of \(M_H\) as a pivotal category equips \(M_{(M^H, U)}\) with the structure of a pivotal category.

**Proposition 5.2.** Let \(H\) be a WBA over the field \(k\) and \(U : \mathcal{M}_H \to \text{Vect}_k\) be the forgetful functor of the category \(\mathcal{M}_H\) of finite-dimensional right \(H\)-comodules (Proposition 2.9).

1. There is a functor

\[
\Phi : M_{(M^H, U)} \to \mathcal{M}_H
\]

given as follows. \(\Phi\) assigns to every right \((\mathcal{M}_H, U)\)-module \((V, c_V)\) the right \(H\)-module whose underlying vector space is \(V\) with the action \(\gamma_V : V \otimes UH \to V\) defined by commutativity of

\[
\begin{array}{c}
V \otimes UH \\
\downarrow \gamma_V \\
V
\end{array}
\begin{array}{c}
\overset{(c_V)_H}{\longrightarrow}
\downarrow \varepsilon \otimes \text{id}_V \\
UH \otimes V
\end{array}
\begin{array}{c}
\overset{\lambda_V}{\longrightarrow}
\downarrow \\
k \otimes V
\end{array} \tag{5.5}
\]

Here we write \(UH\) for the vector space underlying the WHA \(H\) where \(H\) is viewed as the regular right \(H\)-comodule. The functor \(\Phi\) assigns to each morphism \(\varphi : (V, c_V) \to (W, c_W)\) of right-(\(\mathcal{M}_H, U)\)-modules the underlying \(k\)-linear map \(\varphi : V \to W\) that forms a morphism of right \(H\)-modules.

2. There is a functor

\[
\Psi : \mathcal{M}_H \to M_{(M^H, U)}
\]

given as follows. \(\Psi\) assigns to each right \(H\)-module \(\gamma_V : V \otimes UH \to V\) the right \((\mathcal{M}_H, U)\)-module with the same underlying vector space \(V\) and \(c_V\) defined by commutativity of

\[
\begin{array}{c}
V \otimes UX \\
\downarrow (c_V)_X \\
UX \otimes V
\end{array}
\begin{array}{c}
\overset{\text{id}_V \otimes \beta_X}{\longrightarrow}
\downarrow \\
V \otimes (UX \otimes UH)
\end{array}
\begin{array}{c}
\overset{\alpha_{VUX,UV}}{\longrightarrow}
\downarrow \\
(V \otimes UX) \otimes UH
\end{array}
\begin{array}{c}
\overset{\tau_{VUX,UX} \otimes \text{id}_{UH}}{\longrightarrow}
\downarrow \\
(UX \otimes V) \otimes UH
\end{array} \tag{5.7}
\]

Here we write \(UX\) for the vector space underlying the WHA \(H\) where \(H\) is viewed as the regular right \(X\)-comodule.
for all $X \in |\mathcal{M}^H|$. Here, $\beta_X : U_X \to U_X \otimes U_H$ denotes the comodule structure of $X$. The functor $\Psi$ assigns to each morphism $\varphi : V \to W$ of right $H$-modules the morphism $\varphi : (V, c_V) \to (W, c_W)$ of right $(\mathcal{M}^H, U)$-modules with the same underlying $k$-linear map.

3. The composition $\Phi \circ \Psi = 1_{\mathcal{M}_H}$ is the identity functor.

Proof. For the separable Frobenius structure of $U$, see Proposition 2.9.

1. We claim that if $(V, c_V)$ is a right $(\mathcal{M}^H, U)$-module, then $V$ forms a right $H$-module with the action $\gamma_V$ of (5.5).

In order to show that $\gamma_V \circ (\text{id}_V \otimes \eta) = \rho_V$, we use the fact that $\eta = U_1 \circ U_0$ where $\eta : \mathbb{1} = H_1 \to H$ is the inclusion; the definition (5.5) of $\gamma_V$; the fact that $c_V$ is natural and the inclusion $\iota : H_1 \to H$ is a morphism of right $H$-comodules; the identity $\varepsilon \circ U_1 = U_0$; and the axiom (5.1).

In order to show that $\gamma_V \circ (\text{id}_V \otimes \mu) \circ \alpha_{V,U,H,U,H} = \gamma_V \circ (\gamma_V \otimes \text{id}_{U,H})$, we start with the left-hand side and use the definition (5.5); the identity $\mu = \tilde{\mu} \circ U_{H,H}$ where $\tilde{\mu} = \mu \circ U^{H,H}$ and $H \otimes H$ denotes the tensor product of two copies of the regular right $H$-comodule in $\mathcal{M}^H$; the fact that $c_V$ is natural and $\tilde{\mu} : H \otimes H \to H$ is a morphism of right $H$-comodules; the identity $\varepsilon \circ \tilde{\mu} = (\varepsilon \otimes \varepsilon) \circ U^{H,H}$; the axiom (5.2); and twice the definition (5.5) of $\gamma_V$ again.

Furthermore, if $\varphi : (V, c_V) \to (W, c_W)$ is a morphism of right $(\mathcal{M}^H, U)$-modules, then $\varphi : V \to W$ is a morphism of right $H$-modules. In order to see this, we use the axiom (5.2) and the definition (5.3) for $\gamma_V$ and $\gamma_W$.

2. We claim that if $\gamma_V : V \otimes U_H \to V$ is a right $H$-module structure on $V$, then $(V, c_V)$ with $c_V$ as in (5.7) forms a right $(\mathcal{M}^H, U)$-module.

In order to see that $c_V$ is natural for some morphism $f : X \to Y$ of $\mathcal{M}^H$, we need the condition that $f$ is a morphism and the definition (5.7) for $c_V$ and $c_W$.

In order to verify (5.1), we need the definition (5.7) for $X = \mathbb{1}$; the identity $\lambda_{U,H} \circ (U^0 \otimes \text{id}_{U,H}) \circ \beta_1 \circ U_0 = \eta$; and the condition $\gamma_V \circ (\text{id}_V \otimes \eta) = \rho_V$.

In order to verify (5.2), we need the definition (5.7) for both $(c_V)_X$ and $(c_V)_Y$ and the condition $\gamma_V \circ (\text{id}_V \otimes \mu) \circ \alpha_{V,U,H,U,H} = \gamma_V \circ (\gamma_V \otimes \text{id}_{U,H})$.

Furthermore, if $\varphi : V \to W$ is a morphism of right $H$-modules, then $\varphi : (V, c_V) \to (W, c_W)$ is a morphism of right $(\mathcal{M}^H, U)$-modules. In order to see this, we need the condition that $\varphi$ is a morphism of right $H$-modules and the definition (5.7) for $c_V$ and $c_W$.

3. In order to show that $\Phi \circ \Psi = 1_{\mathcal{M}_H}$, let $\gamma_V : V \otimes U_H \to V$ define a right $H$-module. Then $c_V$ of (5.7) is a right $(\mathcal{M}^H, U)$-module, and (5.5) defines another right $H$-module structure which we now call $\gamma_V : V \otimes U_H \to V$. We verify that $\gamma_V = \gamma_V$ by using the definition of the regular right $H$-comodule structure, i.e., $\beta_H = \Delta$, and the fact that $H$ forms a counital coalgebra.

Theorem 5.3. Let $C$ be a finitely semisimple $k$-linear additive spherical category, $k = \text{End}(\mathbb{1})$ be a field, $\omega : C \to \text{Vec}_k$ be the long forgetful functor, $H = \text{coend}(C, \omega)$ and $U : \mathcal{M}^H \to \text{Vec}_k$ be the forgetful functor. Then the functors $\Phi$ and $\Psi$ of Proposition 5.2 satisfy in addition that $\Psi \circ \Phi = 1_{\mathcal{M}(\mathcal{M}^H,U)}$, i.e. the categories $\mathcal{M}_H \cong \mathcal{M}(\mathcal{M}^H,U)$ are isomorphic.
Proof. Whereas Proposition 5.2 uses only the abstract properties of WBAs, the present theorem requires some knowledge of how to reconstruct H from \( \mathcal{M}^H \). Since H is split cosemisimple,

\[
H = \bigotimes_{j \in I} (UV_j)^* \otimes UV_j
\]

is a direct sum of matrix coalgebras \((UV_j)^* \otimes UV_j\). For each simple \( X \in |C| \), there are therefore homomorphisms of coalgebras \( \iota^X : (UX)^* \otimes UX \to H \) and \( \pi^X : H \to (UX)^* \otimes UX \) such that

\[
\text{id}_H = \sum_{j \in I} \iota^X_j \circ \pi^X_j \quad \text{and} \quad \pi^X_j \circ \iota^X_\ell = \delta_{j,\ell} \text{id}_{(UV_j)^* \otimes UV_j}.
\]

If H is viewed as the regular right H-comodule, we have as vector spaces:

\[
H = \bigotimes_{j \in I} k^{\dim UV_j} \otimes UV_j,
\]

and both \( \iota^X \) and \( \pi^X \) are morphisms of right H-comodules. Since H coacts only on the right tensor factor,

\[
(V \otimes ((UX)^* \otimes UX) \xrightarrow{\alpha^{-1}} (V \otimes (UX)^*) \otimes UX \xrightarrow{\tau \otimes \text{id}} ((UX)^* \otimes V) \otimes UX
\]

\[
((UX)^* \otimes UX) \otimes V \xrightarrow{\alpha^{-1}} (UX)^* \otimes (UX \otimes V) \xrightarrow{\text{id} \otimes (c_V)_X} (UX)^* \otimes (V \otimes UX)
\]

commutes for each simple \( X \in |C| \). We can therefore compute

\[
(c_V)_H = \sum_{j \in I} (U_i^V_j \otimes \text{id}_V) \circ (c_V)_{(UV_j)^* \otimes UV_j} \circ (\text{id}_V \otimes U \pi^V_j)
\]

in terms of the \((c_V)_X\) for the simple \( X \in |C| \).

Given some right \((\mathcal{M}^H, U)\)-module \((V, c_V)\), there is a right H-module \( \gamma_V : V \otimes UH \to V \) given by \( (5.3) \) and another right \((\mathcal{M}^H, U)\)-module from \( (5.7) \) which we now call \((V, \bar{c}_V)\). Expressing \((\bar{c}_V)_Y\) first in terms of \( \gamma_Y \), then in terms of \((c_V)_H\), and finally in terms of the \((c_V)_Y\) for the simple \( Y \in |C| \) using \( (5.12) \) shows that \( c_V = \bar{c}_V \).

5.2 Pivotal structure

So far, we have an isomorphism \( \mathcal{M}_H \cong \mathcal{M}_{(\mathcal{M}^H, U)} \) of categories (without extra structure). Putting the structure of a pivotal category on \( \mathcal{M}_{(\mathcal{M}^H, U)} \) is straightforward because we can show that \( \mathcal{M}_H \) is a pivotal category. The subsequent constructions are dual to those for \( \mathcal{M}^H \).

**Proposition 5.4.** Let H be a WBA. Then the category \( \mathcal{M}_H \) of finite-dimensional right H-modules is a monoidal category \((\mathcal{M}_H, \otimes, H_s, \alpha, \lambda, \rho)\). The monoidal unit object is the source base algebra \( H_s \) with the action

\[
\gamma_{H_s} : H_s \otimes H \to H_s, \quad x \otimes h \mapsto x \triangleleft h := \varepsilon_s(xh).
\]
The tensor product $V \hat{\otimes} W$ of two right $H$-modules is the vector space
\[ V \hat{\otimes} W := \{ v \otimes w \in V \otimes W \mid v \otimes w = (v \triangleleft 1') \otimes (w \triangleleft 1'') \} \] (5.14)
with the action
\[ (v \otimes w) \triangleleft h := (v \triangleleft h',) \otimes (w \triangleleft h''). \] (5.15)
The unit constraints are given by
\[ \lambda_V : H_s \hat{\otimes} V \rightarrow V, \quad h \otimes v \mapsto \varepsilon(h1') (v \triangleleft 1''), \] (5.16)
\[ \rho_V : V \hat{\otimes} H_s \rightarrow V, \quad v \otimes h \mapsto v \triangleleft h, \] (5.17)
and the associator is induced from that of Vect$_k$.

**Proposition 5.5.** Let $H$ be a WHA. Then the category $\mathcal{M}_H$ is left-autonomous if the left-dual of every object $V \in |\mathcal{M}_H|$ is chosen to be $(V^*, ev_V, coev_V)$, where the dual vector space $V^*$ is equipped with the action
\[ \gamma_{V^*} : V^* \otimes H \rightarrow V^*, \quad \vartheta \otimes h \mapsto \vartheta(v \triangleleft (S'h)), \] (5.18)
and evaluation and coevaluation are given by
\[ ev_V : V^* \hat{\otimes} V \rightarrow H_s, \quad \vartheta \otimes v \mapsto 1' \vartheta(v \triangleleft (S1'')), \] (5.19)
\[ coev_V : H_s \rightarrow V \hat{\otimes} V^*, \quad h \mapsto \sum_j (v_j \triangleleft (S'h)) \otimes \vartheta_j. \] (5.20)

**Definition 5.6.** Let $H$ be a WHA. An element $m \in H$ is called group-like if it has a multiplicative inverse and
\[ (m1') \otimes (m1'') = m' \otimes m'' = (1'm) \otimes (1''m). \] (5.21)
Note that every group-like element $m \in H$ also satisfies $m = Sm$ and $\varepsilon_s(m) = 1 = \varepsilon_t(m)$.

**Definition 5.7.** A pivotal WHA $(H, \mu, \eta, \Delta, \varepsilon, S, m)$ is a WHA $(H, \mu, \eta, \Delta, \varepsilon, S)$ with a group-like element $m \in H$, called the pivotal element, that satisfies
\[ S^2(x) = m xm^{-1} \] (5.22)
for all $x \in H$.

**Proposition 5.8.** Let $(H, \mu, \eta, \Delta, \varepsilon, S, m)$ be a pivotal WHA. Then the category $\mathcal{M}_H$ is pivotal with $\tau_V : V \rightarrow V^{**}$ given by
\[ \tau_V (v) = \tau_{V^*}^{\text{Vect}_k} (v \triangleleft m) \] (5.23)
for all finite-dimensional right $H$-modules $V \in |\mathcal{M}_H|$ and all $v \in V$.

**Definition 5.9.** A spherical WHA $H$ is a pivotal WHA for which $\text{tr}_V^L (f) = \text{tr}_V^R (f)$ for all finite-dimensional right $H$-modules $V \in |\mathcal{M}_H|$ and all morphisms $f : V \rightarrow V$. 

Finitely semisimple spherical categories are self-dual

**Proposition 5.10.** Let \( C \) be a finitely semisimple \( k \)-linear additive spherical category, \( k = \text{End}(1) \) be a field and \( \omega: C \to \text{Vect}_k \) be the long forgetful functor. Then \( H = \text{coend}(C, \omega) \) is a pivotal WHA with pivotal element

\[
m = [D_V \circ \rho_V | \rho_V^{-1} \circ D_V^{-1}]_1.
\]

**Proof.** Direct computation. \( \square \)

Note that both of the functors, \( \Phi \) and \( \Psi \), that form the isomorphism \( M_H \cong M_{(M_H, U)} \), leave the vector spaces underlying the objects and the linear maps underlying the morphisms unchanged. We can therefore use the monoidal, left-autonomous and pivotal structure of \( M_H \) to equip \( M_{(M_H, U)} \) with the structure of a pivotal category in such a way that the functor \( \Phi \) becomes pivotal and strict monoidal. Finally, \( \Phi \) is \( k \)-linear. We therefore get the following theorem:

**Theorem 5.11.** Let \( C \) be a finitely semisimple \( k \)-linear additive spherical category, \( k = \text{End}(1) \) be a field, \( \omega: C \to \text{Vect}_k \) be the long forgetful functor, \( H = \text{coend}(C, \omega) \) and \( U: M_H \to \text{Vect}_k \) be the forgetful functor. Then

\[
M_H \cong M_{(M_H, U)}
\]

are equivalent as \( k \)-linear additive pivotal categories.

**5.3 Self-duality of spherical categories**

Before we can combine all our results in order to prove Corollary 5.3 we need to relate \( M_H \) with \( M_{\hat{H}} \).

**Proposition 5.12.** Under the assumptions of Theorem 5.11 the categories

\[
M_H \cong M_{\hat{H}}
\]

are equivalent as \( k \)-linear additive pivotal categories.

**Proof.** We use the fact that the pairing \( \langle -; \hat{m}_\beta \rangle \) and the canonical element \( G \) of Proposition 4.4 satisfy the triangle identities. Then, \( M_H \cong M_{\hat{H}} \) are isomorphic as \( k \)-linear additive categories using the functor that turns every right \( H \)-comodule \( \beta_V: V \to V \otimes H \) into a right \( \hat{H} \)-module

\[
\gamma_V: V \otimes \hat{H} \to V, \quad v \otimes \hat{h} \mapsto v_V \langle v_V; \hat{h} \rangle,
\]

and the functor that turns every right \( \hat{H} \)-module \( \gamma_V: V \otimes \hat{H} \to V \) into a right \( H \)-comodule

\[
\beta_V = (\gamma_V \otimes \text{id}_H) \circ \alpha^{-1}_{V, \hat{H}, H} \circ (\text{id}_V \otimes G) \circ \rho^{-1}_V: V \to V \otimes H.
\]

The categories \( M_H \cong M_{\hat{H}} \) are also equivalent as pivotal categories, because the pairing also satisfies

\[
\langle h; \hat{m} \rangle = w(h)
\]

for all \( h \in H \). Here \( w \) denotes the pivotal form \( \langle -; \hat{m} \rangle \) of \( H \) and \( \hat{m} \) the analogue of the pivotal element \( (5.24) \) in \( \hat{H} \). \( \square \)
Proof of Corollary 5.13. Under the assumptions of Theorem 5.11 we have the following equivalences of $k$-linear additive pivotal categories

$$
\mathcal{C} \simeq \mathcal{M}^H \simeq \mathcal{M}_{\tilde{H}} \simeq \mathcal{M}_H \simeq \mathcal{M}(\mathcal{M}_H, U) \simeq \mathcal{M}((\mathcal{C}, \omega)).
$$

(5.30)

The first one is from Theorem 1.1, the second is from Proposition 5.12, the third follows from the isomorphism (4.23), the fourth is from Theorem 5.11 and the fifth from Theorem 3.23. Since $\mathcal{C}$ is spherical, all other categories are spherical, too, and the equivalence is an equivalence of spherical categories. In addition, $H$ and $\tilde{H}$ are both spherical and cospherical.

Remark 5.13. In order to see how the objects $X \in |\mathcal{C}|$ give rise to right $(\mathcal{C}, \omega)$-modules in (5.30), we combine the relevant functors as follows. For each simple $X \in |\mathcal{C}|$, there is a right $H$-comodule $\omega(X) = \text{Hom}(\tilde{V}, \tilde{V} \otimes X)$, given by

$$
\beta_X : \omega(X) \rightarrow \omega(X) \otimes H,
$$

$$
e_j^{(X)} \rightarrow \sum_{\ell} e_\ell^{(X)} \otimes [e_\ell^{(X)}|e_j^{(X)}]_X.
$$

(5.31)

It can be turned into a right $H$-module

$$
\gamma_{\omega(X)} : \omega(X) \otimes H \rightarrow \omega(X),
$$

$$
e_j^{(X)} \otimes [e_p^{(Y)}|e_\ell^{(Y)}]_Y \rightarrow \sum_q e_q^{(X)} \left\langle e_j^{(X)}; e_p^{(Y)} \right\rangle.
$$

(5.32)

Here the $e_{jq}^{(X)}$ are as in (4.19). Note that the above expression involves the non-canonical isomorphism (4.23), putting hats on all expressions and replacing $\tilde{V}$ by $\tilde{V}^*$ everywhere. Finally, $\omega(X)$ forms a right $(\mathcal{C}, \omega)$-module with

$$
(c_{\omega(X)})_Y : \omega(X) \otimes \omega(Y) \rightarrow \omega(Y) \otimes \omega(X),
$$

$$
e_j^{(X)} \otimes e_\ell^{(Y)} \rightarrow \sum_{p,q} e_q^{(Y)} \otimes e_p^{(X)} \left\langle e_j^{(X)}; e_\ell^{(Y)} \right\rangle.
$$

(5.33)

for all simple $Y \in |\mathcal{C}|$. Note that in our construction of $(c_{\omega(X)})_Y$, $\omega(X)$ forms a right $H$-module in $\text{Vect}_k$, but not in general a right $H$-module in $\mathcal{M}^H$ and that the tensor product $\omega(X) \otimes \omega(Y)$ above is in $\text{Vect}_k$ as opposed to $\mathcal{M}^H$.

5.4 The modular case

Every modular category is finitely semisimple $k$-linear additive spherical, and so if $\mathcal{C}$ is modular, we know that (5.30) is an equivalence of $k$-linear additive spherical categories. In order to show that it is actually an equivalence of additive ribbon categories, we can proceed as follows. First, we explain the additional structure.

1. $\mathcal{C} \simeq \mathcal{M}^H$ are equivalent as additive ribbon categories by [7]. $H = \text{coend}(\mathcal{C}, \omega)$ is now a finite-dimensional split-cosemisimple coribbon WHA that is weakly cofactorizable and for which $H_s \cap H_t \cong k$. The new structure is the coquasitriangular structure and the universal ribbon form of $H$. 
2. In order to obtain an equivalence $\mathcal{M}^H \simeq \mathcal{M}_H$ of additive ribbon categories, we define
the notion of a quasitriangular and ribbon WHA in such a way that the pairing relates the coquasitriangular and coribbon structure of $H$ with the quasitriangular and ribbon structure of $\hat{H}$.

3. The equivalence $\mathcal{M}_H \simeq \mathcal{M}_H$ is automatically an equivalence of additive ribbon categories.

4. For the equivalences $\mathcal{M}_H \simeq \mathcal{M}(\mathcal{M}_H, U) \simeq \mathcal{M}(C, \omega)$, one defines the braiding and ribbon twist of $\mathcal{M}(\mathcal{M}_H, U)$ and $\mathcal{M}(C, \omega)$ accordingly.

The additional property, namely weak cofactorizability of $H$, weak factorizability (to be defined accordingly) of $\hat{H}$, and the non-degeneracy condition of $\mathcal{M}_H$, $\mathcal{M}_H$, $\mathcal{M}(\mathcal{M}_H, U)$ and $\mathcal{M}(C, \omega)$ then follows from the non-degeneracy of $C$.

6 Example

In this section, we specialize the key expressions used in the present article to the case of the modular category $C$ associated with the quantum group $U_q(\mathfrak{sl}_2)$, $q$ a root of unity. We use the diagrammatic description of [25] and precisely follow their notation.

Let $r \in \{2, 3, 4, \ldots\}$ and $A$ be a primitive $4r$-th root of unity, $q = A^2$. For simplicity, we work over the complex numbers $k = \mathbb{C}$. The morphisms of $C$ are represented by plane projections of oriented framed tangles, drawn in blackboard framing. The coherence theorem for ribbon categories [26] ensures that each diagram defines a morphism of $C$. Since $C$ is $k$-linear, we can take formal linear combinations of diagrams with coefficients in $k$. All our diagrams are read from top to bottom.

The braiding of $C$ is such that a crossing in our plane projections can be resolved using the recursion relation for the Kauffman bracket

\[
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\end{array} & = & A \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} + A^{-1} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} , \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} & = & - (q + q^{-1}) ,
\end{array}
\]

(6.1)

ignoring the orientations for now. The Jones–Wenzl idempotents $P_n$, $n \in \mathbb{N}_0$, are formal linear combinations of planar $(n, n)$-tangles that can be defined recursively by

\[
P_1 := , \quad P_{n+1} := P_n + \left[ \frac{n+1}{q} \right] \frac{[n+2]}{[n]} P_n
\]

(6.2)

where $[n]_q = (q^n - q^{-n})/(q - q^{-1})$, $n \in \mathbb{Z}$, are the quantum integers. The isomorphism classes of simple objects of $C$ are indexed by the set $I = \{0, 1, \ldots, r - 2\}$. The identity morphism of the object $n \in I$ is the identity $(n, n)$-tangle with the idempotent $P_n$ inserted somewhere (anywhere). As a shortcut, we write a single line labeled by $n$, $n := P_n$.

(6.3)
The object indexed by $0 \in I$ is the monoidal unit and can be made invisible in our diagrams thanks to the coherence theorem. The categorical dimension of the simple objects is given by

$$\Delta_n := \bigodot^n = (-1)^n [n + 1]_q,$$

which is non-zero for all $n \in I$.

Two special features of $U_q(sl_2)$ are exploited. First, the simple objects are isomorphic to their duals, and the choice of representatives $V_j$, $j \in I$, of the simple objects is such that $(V_j)^* = V_j$ are equal rather than merely isomorphic. This allows us to omit any arrows from the diagrams that would indicate the orientation of the ribbon tangle.

Second, there are no higher multiplicities, i.e., for all $a, b, c \in I$, we have $\dim_k \text{Hom}(V_a \otimes V_b, V_c) \in \{0, 1\}$. More precisely, $\text{Hom}(V_a \otimes V_b, V_c) \cong k$ if and only if the triple $(a, b, c)$ is admissible. Otherwise, $\text{Hom}(V_a \otimes V_b, V_c) = \{0\}$.

**Definition 6.1.** A triple $(a, b, c) \in I^3$ is called admissible if the following conditions hold.

1. $a + b + c \equiv 0 \mod 2$ (parity),
2. $a + b - c \geq 0$ and $b + c - a \geq 0$ and $c + a - b \geq 0$ (quantum triangle inequality),
3. $a + b + c \leq 2r - 4$ (non-negligibility).

A special choice of basis vector of $\text{Hom}(V_a, V_b \otimes V_c)$ is denoted by a trivalent vertex:

$$\nu(a, b, c) := \begin{array}{c}
  a \\
  b \\
  c
\end{array},$$

where $i = (a + b - c)/2$, $j = (a + c - b)/2$ and $k = (b + c - a)/2$. If we draw such a diagram for a triple $(a, b, c) \in I^3$ that is not admissible, then by convention, we multiply the entire diagram by zero. We also need the theta graph

$$\vartheta(a, b, c) := \begin{array}{c}
  a \\
  b \\
  c
\end{array},$$

which is non-zero for all admissible triples $(a, b, c)$. When we compose the morphisms associated with such diagrams, the composition is zero unless the labels at the open ends of the tangles match, i.e., putting

$$\begin{align*}
  \begin{array}{c}
    r \\
    j \\
    s
  \end{array} & \text{ below } \\
  \begin{array}{c}
    q \\
    k
  \end{array} & \text{ gives } \\
  \begin{array}{c}
    p \\
    j
  \end{array}
\end{align*}$$

We use the following pair of dual bases of $\omega(V_j) = \text{Hom}(\tilde{V}, \tilde{V} \otimes V_j)$ and $\omega(V_j)^* = \text{Hom}(\tilde{V} \otimes V_j, \tilde{V})$, $j \in I$,

$$e_{pq}^{(V_j)} = \begin{array}{c}
  p \\
  q \\
  j
\end{array} \quad \text{and} \quad e_{pq}^{(V_j)^*} = \frac{\Delta_p}{\vartheta(q, p, j)} \begin{array}{c}
  p \\
  q \\
  j
\end{array}.$$
Finitely semisimple spherical categories are self-dual

where \( p, q \in I \) are such that \((p, q, j)\) is admissible. Then the basis \((4.19)\) of \( H = \text{coend}(C, \omega) \) is given by

\[
\langle V_j \rangle_{pq,rs} = \frac{1}{\vartheta(j,r,s)} q \bigg\uparrow p \,
\bigg\downarrow j \quad r \bigg\downarrow s.
\]  

(6.9)

The quantum 6j-symbol is defined as

\[
\{a \ b \ i \ c \ d \ j\}_q := \frac{\Delta_i}{\vartheta(a,d,i)\vartheta(b,c,i)} b \bigg\downarrow c \ 
\bigg\uparrow j \quad a \bigg\downarrow d \quad i .
\]  

(6.10)

It is used in the recoupling identity,

\[
\sum_i \{a \ b \ i \ c \ d \ j\}_q b \bigg\downarrow c \ 
\bigg\uparrow j \quad a \bigg\downarrow d \quad i = \delta_{pb}\delta_{sc}\vartheta(j,b,q) \Delta_c \Delta_q \sum_{u \in I} \{r \ j \ u \ c \ \ell d \ a \ b \ \ell \ q\}_q \{u \ j \ q \ b \ a \ \ell \ p\}_q \bigg\downarrow u \bigg\uparrow \bigg\downarrow d .
\]  

(6.11)

Diagrammatically, the WHA structure of \( H \) is:

\[
\eta(1) = \sum_{p,q} \frac{1}{\Delta_q} p \bigg\uparrow q \bigg\downarrow r .
\]  

(6.12)

\[
\mu \left( \begin{array}{ccc}
q & p & b \\
r & j & a \\
s & c & d
\end{array} \bigg\uparrow \bigg\downarrow \bigg\downarrow \bigg\uparrow \right) \otimes \left( \begin{array}{ccc}
\ell & u & \ell \\
d & c & d
\end{array} \bigg\uparrow \bigg\downarrow \bigg\downarrow \bigg\uparrow \right) = \delta_{pb}\delta_{sc}\vartheta(j,b,q) \frac{\Delta_q}{\vartheta(j,u,t)} \sum_{r,u,t} \{r \ j \ u \ c \ \ell d \ a \ b \ \ell \ q\}_q \{u \ j \ q \ b \ a \ \ell \ p\}_q \bigg\downarrow u \bigg\uparrow \bigg\downarrow d .
\]  

(6.13)

\[
\Delta \left( \begin{array}{ccc}
q & p & j \\
r & j & s \\
s & r & j
\end{array} \bigg\uparrow \bigg\downarrow \bigg\downarrow \bigg\uparrow \right) = \sum_{t,u} \frac{1}{\vartheta(j,u,t)} t \bigg\downarrow j \quad q \bigg\uparrow p \,
\bigg\downarrow r \quad s \bigg\downarrow u \quad t,
\]  

(6.14)

\[
\varepsilon \left( \begin{array}{ccc}
q & p & j \\
r & j & s \\
s & r & j
\end{array} \bigg\uparrow \bigg\downarrow \bigg\downarrow \bigg\uparrow \right) = \delta_{qr}\delta_{ps}\vartheta(j,r,s),
\]  

(6.15)

\[
S \left( \begin{array}{ccc}
q & p & j \\
r & j & s \\
s & r & j
\end{array} \bigg\uparrow \bigg\downarrow \bigg\downarrow \bigg\uparrow \right) = \frac{\Delta_q}{\Delta_p} \frac{\vartheta(j,r,s)}{\vartheta(j,p,q)} \frac{\Delta_p}{\Delta_q} .
\]  

(6.16)

The pivotal form of \( H \) is given by

\[
w \left( \begin{array}{ccc}
q & p & j \\
r & j & s \\
s & r & j
\end{array} \bigg\uparrow \bigg\downarrow \bigg\downarrow \bigg\uparrow \right) = \delta_{ps}\delta_{qr}\vartheta(j,r,s) \frac{\Delta_p}{\Delta_q}. \]  

(6.17)
Since $\tilde{V} = \tilde{V}^*$, the dual WHA $\tilde{H}$ has precisely the same description as $H$. The pairing $\langle - | - \rangle$ of (4.24) reads:

$$\langle \begin{array}{cccc}
 b & a & q & p \\
 j & d & r & s \\
 c & \end{array} \rangle = \delta_{pa} \delta_{qd} \delta_{rc} \delta_{sb} \vartheta(j, c, d) \vartheta(\ell, r, s) \left\{ \begin{array}{ccc}
 \ell & d & c \\
 j & b & a \\
 \end{array} \right\}_q.$$ (6.18)

The other pairing $\langle - | - \rangle$ is different. The canonical element $G(1) \in \hat{H} \otimes H$ of (4.28) is given by

$$G(1) = \sum_{j,a,b,c,d} \frac{\Delta_j}{\Delta_c} \vartheta(j, a, b) \vartheta(j, c, d) \left\{ \begin{array}{ccc}
 a & b & c \\
 j & d & c \\
 \end{array} \right\} \left\{ \begin{array}{ccc}
 a & b & c \\
 j & d & c \\
 \end{array} \right\}.$$ (6.19)

from which we can read off a pair of dual bases of $\hat{H}$ and $H$ with respect to $\langle - | - \rangle$. The coaction (5.31) of $H$ on $\omega(V_j)$ and the action (5.32) of $H$ on $\omega(V_j)$ are as follows:

$$\beta_{V_j} \left( \begin{array}{c}
 p \\
 j \end{array} \right) = \sum_{r,s} \frac{1}{\vartheta(j, r, s)} \left. \left\{ \begin{array}{ccc}
 s & q & p \\
 j & r & s \\
 \end{array} \right\} \left\{ \begin{array}{ccc}
 j & b & c \\
 \ell & d & c \\
 \end{array} \right\}_q \right. \left\{ \begin{array}{ccc}
 j & b & c \\
 \ell & d & c \\
 \end{array} \right\}_q.$$ (6.20)

$$\gamma_{\omega(V_j)} \left( \begin{array}{c}
 p \\
 j \end{array} \right) \otimes \left( \begin{array}{c}
 c \\
 d \end{array} \right) = \delta_{ap} \delta_{dq} \vartheta(\ell, c, d) \left\{ \begin{array}{ccc}
 c & q & p \\
 j & a & s \\
 \ell \end{array} \right\}_q.$$ (6.21)

Finally, we have the structure of $\omega(V_j)$ as a right $(\mathcal{C}, \omega)$-module,

$$\left. (c_{\omega(V_j)})_{V_j} \left( \begin{array}{c}
 p \\
 j \end{array} \right) \otimes \left( \begin{array}{c}
 r \\
 \ell \end{array} \right) \right. = \delta_{rp} \sum_{a} \left\{ \begin{array}{ccc}
 s & q & a \\
 j & p & a \\
 \ell \end{array} \right\}_q \left\{ \begin{array}{ccc}
 s & q & a \\
 j & p & a \\
 \ell \end{array} \right\}_q.$$ (6.22)

A Background on tensor categories

In this appendix, we collect the relevant definitions and properties of autonomous, pivotal and spherical categories, following Schauenburg [27], Freyd–Yetter [28], Barrett–Westbury [1] and Turaev [29], and of abelian categories following MacLane [30].

A.1 Monoidal categories

Definition A.1. A monoidal category $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ is a category $\mathcal{C}$ with a bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (tensor product), an object $1 \in |\mathcal{C}|$ (monoidal unit) and natural isomorphisms $\alpha_{X,Y,Z}: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ (associator), $\lambda_X: 1 \otimes X \to X$ (left-unit constraint) and $\rho_X: X \otimes 1 \to X$ (right-unit constraint) for all $X, Y, Z \in |\mathcal{C}|$, subject to the pentagon axiom

$$\alpha_{X,Y,Z} \circ \alpha_{X \otimes Y,Z,W} = (\lambda_X \otimes \alpha_{Y,Z,W}) \circ (\alpha_{X,Y,Z} \otimes \lambda_W)$$ (A.1)

and the triangle axiom

$$\rho_X \otimes \lambda_Y = (\lambda_X \otimes \rho_Y) \circ (\alpha_{X,Y} \otimes \operatorname{id}_Y)$$ (A.2)

for all $X, Y, Z, W \in |\mathcal{C}|$. 

Definition A.2. Let \((\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)\) and \((\mathcal{C}', \otimes', 1', \alpha', \lambda', \rho')\) be monoidal categories.

1. A lax monoidal functor \((F, F_{X,Y}, F_0): \mathcal{C} \rightarrow \mathcal{C}'\) consists of a functor \(F: \mathcal{C} \rightarrow \mathcal{C}'\), morphisms \(F_{X,Y}: F(X \otimes Y) \rightarrow F(X') \otimes' F(Y')\) that are natural in \(X, Y \in \mathcal{C}\), and of a morphism \(F_0: 1' \rightarrow F1\), subject to the hexagon axiom

\[
F_{X,Y} \circ (\text{id}_F \otimes' F_{Y,Z}) \circ \alpha_{F_{X,Y},F_{Y,Z}} = F \alpha_{X,Y,Z} \circ F_{X \otimes Y,Z} \circ (F_{X,Y} \otimes' \text{id}_{FZ})
\]

(A.3) and the two squares

\[
\begin{align*}
\lambda_{F,X}' &= F \lambda_X \circ F_{1,X} \circ (F_0 \otimes' \text{id}_F), \\
\rho_{F,X}' &= F \rho_X \circ F_{X,1} \circ (\text{id}_F \otimes' F_0)
\end{align*}
\]

(A.4, A.5)

for all \(X, Y, Z \in |\mathcal{C}|\).

2. An oplax monoidal functor \((F, F^{X,Y}, F^0): \mathcal{C} \rightarrow \mathcal{C}'\) consists of a functor \(F: \mathcal{C} \rightarrow \mathcal{C}'\), morphisms \(F^{X,Y}: F(X \otimes Y) \rightarrow F(X) \otimes' F(Y)\) that are natural in \(X, Y \in \mathcal{C}\), and of a morphism \(F^0: F1 \rightarrow 1'\), subject to the hexagon axiom

\[
(\text{id}_F \otimes' F^{Y,Z}) \circ F^{X,Y} \circ \gamma_{F,X,Y,Z} = \alpha_{F_{X,Y},F_{Y,Z}}' \circ (F_{X,Y} \otimes' \text{id}_{FZ}) \circ F^{X \otimes Y,Z}
\]

(A.6) and the two squares

\[
\begin{align*}
F \lambda_X &= \lambda_{F,X}' \circ (F^0 \otimes' \text{id}_F) \circ F_{1,X}, \\
F \rho_X &= \rho_{F,X}' \circ (\text{id}_F \otimes' F^0) \circ F_{X,1}
\end{align*}
\]

(A.7, A.8)

for all \(X, Y, Z \in |\mathcal{C}|\).

3. A strong monoidal functor \((F, F_{X,Y}, F_0): \mathcal{C} \rightarrow \mathcal{C}'\) is a lax monoidal functor such that \(F_0\) and all \(F_{X,Y}, X, Y \in |\mathcal{C}|\), are isomorphisms.

4. A strict monoidal functor \((F, F_{X,Y}, F_0): \mathcal{C} \rightarrow \mathcal{C}'\) is a strong monoidal functor for which \(F_0\) and all \(F_{X,Y}, X, Y \in |\mathcal{C}|\), are identity morphisms.

Definition A.3. Let \((F, F_{X,Y}, F_0): \mathcal{C} \rightarrow \mathcal{C}'\) and \((G, G_{X,Y}, G_0): \mathcal{C} \rightarrow \mathcal{C}'\) be lax monoidal functors between monoidal categories \(\mathcal{C}\) and \(\mathcal{C}'\). A monoidal natural transformation \(\eta: F \Rightarrow G\) is a natural transformation such that

\[
\eta_{X \otimes Y} \circ F_{X,Y} = G_{X,Y} \circ (\eta_X \otimes' \eta_Y)
\]

(A.9)

for all \(X, Y \in \mathcal{C}\).

There is a similar notion of monoidal natural transformation if the functors are oplax rather than lax monoidal. Compositions of \([\text{lax, oplax, strong}]\) monoidal functors are again \([\text{lax, oplax, strong}]\) monoidal. The following result is well known, but quite laborious to verify.

Proposition A.4. Let \(\mathcal{C}\) and \(\mathcal{C}'\) be monoidal categories and \(F \dashv G: \mathcal{C}' \rightarrow \mathcal{C}\) be an adjunction with unit \(\eta: 1_\mathcal{C} \Rightarrow G \circ F\) and counit \(\varepsilon: F \circ G \Rightarrow 1_{\mathcal{C}'}\).

1. If \(F\) has an oplax monoidal structure \((F, F^{C_1,C_2}, F^0)\), then \(G\) has a lax monoidal structure \((G, G_{D_1,D_2}, G_0)\) as follows,

\[
\begin{align*}
G_{D_1,D_2} &= G(\varepsilon_{D_1} \otimes \varepsilon_{D_2}) \circ G(F^{G(D_1),G(D_2)}) \circ \eta_{G(D_1) \otimes G(D_2)}, \\
G_0 &= G(F^0) \circ \eta_1.
\end{align*}
\]

(A.10, A.11)
2. If $F$ is strong monoidal, then both $\eta$ and $\varepsilon$ are monoidal natural transformations.

3. If $F$ is strong monoidal and the adjunction is an equivalence, then $G$ is strong monoidal.

By an equivalence of monoidal categories, we mean an equivalence of categories such that one of the functors is strong monoidal and write $\mathcal{C} \simeq \mathcal{D}$ in this case.

### A.2 Duality

**Definition A.5.** Let $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ be a monoidal category.

1. A left-dual $(X^*, \text{ev}_X, \text{coev}_X)$ of an object $X \in |\mathcal{C}|$ consists of an object $X^* \in |\mathcal{C}|$ and morphisms $\text{ev}_X : X^* \otimes X \to 1 \text{ (left-evaluation)}$ and $\text{coev}_X : 1 \to X \otimes X^* \text{ (left-coevaluation)}$ that satisfy the triangle identities

\[
\rho_X \circ (\text{id}_X \otimes \text{ev}_X) \circ \alpha_{X, X^*}^{-1} \circ (\text{coev}_X \otimes \text{id}_X) \circ \lambda_X^{-1} = \text{id}_X, \quad (A.12)
\]

\[
\lambda_{X^*} \circ (\text{ev}_X \otimes \text{id}_{X^*}) \circ \alpha_{X^*, X}^{-1} \circ (\text{id}_{X^*} \otimes \text{coev}_X) \circ \rho_{X^*}^{-1} = \text{id}_{X^*}. \quad (A.13)
\]

2. A right-dual $(\overline{X}, \overline{\text{ev}}_X, \overline{\text{coev}}_X)$ of an object $X \in |\mathcal{C}|$ consists of an object $\overline{X} \in |\mathcal{C}|$ and morphisms $\overline{\text{ev}}_X : X \otimes \overline{X} \to 1 \text{ (right-evaluation)}$ and $\overline{\text{coev}}_X : 1 \to \overline{X} \otimes X \text{ (right-coevaluation)}$ that satisfy the triangle identities

\[
\lambda_X \circ (\overline{\text{ev}}_X \otimes \text{id}_X) \circ \alpha_{X, \overline{X}}^{-1} \circ (\overline{\text{coev}}_X \otimes \text{id}_X) \circ \rho_X^{-1} = \text{id}_X, \quad (A.14)
\]

\[
\rho_X \circ (\text{id}_X \otimes \overline{\text{ev}}_X) \circ \alpha_{\overline{X}, X}^{-1} \circ (\overline{\text{coev}}_X \otimes \text{id}_X) \circ \lambda_X^{-1} = \text{id}_{\overline{X}}. \quad (A.15)
\]

**Definition A.6.** Let $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ be a monoidal category and $f : X \to Y$ be a morphism of $\mathcal{C}$.

1. If both $X$ and $Y$ have left-duals, the left-dual of $f$ is defined as

\[
f^* := \lambda_{Y^*} \circ (\text{ev}_Y \otimes \text{id}_{Y^*}) \circ \alpha_{Y^*, Y, X}^{-1} \circ (\text{id}_X \otimes \text{ev}_X) \circ (\text{id}_Y \otimes \text{coev}_X) \circ \rho_{Y^*}^{-1}. \quad (A.16)
\]

2. If both $X$ and $Y$ have right-duals, the right-dual of $f$ is defined as

\[
\overline{f} := \rho_X \circ (\text{id}_X \otimes \overline{\text{ev}}_Y) \circ \alpha_{X, \overline{Y}, \overline{X}} \circ ((\text{id}_X \otimes f) \otimes \text{id}_{\overline{X}}) \circ (\overline{\text{coev}}_X \otimes \text{id}_\overline{X}) \circ \lambda_{\overline{X}}. \quad (A.17)
\]

**Proposition A.7.** Let $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ be a monoidal category and $(X^*, \text{ev}_X, \text{coev}_X)$ and $(X', \text{ev}'_X, \text{coev}'_X)$ both be left-duals of $X \in |\mathcal{C}|$. Then there is a natural isomorphism

\[
u_X = \lambda_{X^*} \circ (\text{ev}_X \otimes \text{id}_{X^*}) \circ \alpha_{X^*, X, X}^{-1} \circ (\text{id}_{X^*} \otimes \text{coev}_X) \circ \rho_{X^*}^{-1} : X^* \to X' \quad (A.18)
\]

with inverse

\[
u_X^{-1} = \lambda_{X^*} \circ (\text{ev}_X \otimes \text{id}_{X^*}) \circ \alpha_{X^*, X, X}^{-1} \circ (\text{id}_{X^*} \otimes \text{coev}_X) \circ \rho_{X^*}^{-1} : X' \to X^* \quad (A.19)
\]

They satisfy $\text{ev}_X = \text{ev}'_X \circ (u_X \otimes \text{id}_X)$ and $\text{coev}_X = (\text{id}_X \otimes u_X^{-1}) \circ \text{coev}'_X$.

**Definition A.8.** A \textit{[left-, right-]autonomous category} is a monoidal category in which each object is equipped with a specified [left-, right-]dual. An autonomous category is a monoidal category that is both left- and right-autonomous.
**Definition A.9.** A pivotal category \((\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho, (-)^*, \text{ev}, \text{coev}, \tau)\) is a left-autonomous category \((\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho, (-)^*, \text{ev}, \text{coev})\) with natural isomorphisms \(\tau_X: X \to X^{**}\) such that
\[
(\tau_X)^* = \tau_X^{-1}.
\] (A.20)
for all \(X \in |\mathcal{C}|\).

Note that every pivotal category is also right-autonomous with \(\overline{X} = X^*\) and
\[
\overline{\text{ev}}_X = \text{ev}_{X^*} \circ (\tau_X \otimes \text{id}_{X^*}),
\]
\[
\overline{\text{coev}}_X = (\text{id}_{X^*} \otimes \tau_X^{-1}) \circ \text{coev}_{X^*}
\] (A.21)
(A.22)
for all \(X \in |\mathcal{C}|\). In a pivotal category, \(\overline{f} = f^*\) for all morphisms \(f: X \to Y\).

**Definition A.10.** Let \((\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho, (-)^*, \text{ev}, \text{coev}, \tau)\) be a pivotal category and \(f: X \to X\) be a morphism of \(\mathcal{C}\). We define the left-trace of \(f\)
\[
\text{tr}^{(L)}_X(f) = \text{ev}_X \circ (\text{id}_{X^*} \otimes f) \circ \overline{\text{coev}}_X : 1 \to 1
\] (A.23)
and the right-trace of \(f\)
\[
\text{tr}^{(R)}_X(f) = \overline{\text{ev}}_X \circ (f \otimes \text{id}_{X^*}) \circ \text{coev}_X : 1 \to 1.
\] (A.24)

Note that in a pivotal category, both left- and right-traces are cyclic, i.e. \(\text{tr}^{(L)}_X(g \circ f) = \text{tr}^{(L)}_Y(f \circ g)\) for all \(f: X \to Y\) and \(g: Y \to X\) and similarly for the right-trace.

**Definition A.11.** A spherical category \((\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho, (-)^*, \text{ev}, \text{coev}, \tau)\) is a pivotal category in which
\[
\text{tr}^{(L)}_X(f) = \text{tr}^{(R)}_X(f)
\] (A.25)
for all morphisms \(f: X \to X\) in \(\mathcal{C}\). In this case, the above expression is just called the trace of \(f\) and denoted by \(\text{tr}_X(f)\), and
\[
\dim(X) = \text{tr}_X(\text{id}_X)
\] (A.26)
is called the dimension of \(X\).

Note that in a spherical category, \(\text{tr}_X(f) = \text{tr}_{X^*}(f^*)\) for every morphism \(f: X \to X\) and thus \(\dim(X) = \dim(X^*)\).

**Example A.12.** Every ribbon category (see, for example [7, Appendix A.3]) is spherical.

*Proof.* Every ribbon category is a pivotal category because of [7, eq. (A.24)]. The spherical property follows from [7, eq. (A.25) and eq. (A.26)] and from the naturality of the braiding. □
A.3 Abelian and semisimple categories

A category $\mathcal{C}$ is called $\textbf{Ab}$-enriched if it is enriched in the category $\textbf{Ab}$ of abelian groups, i.e. if $\text{Hom}(X,Y)$ is an abelian group for all objects $X,Y \in |\mathcal{C}|$ and if the composition of morphisms is $\mathbb{Z}$-bilinear. If $k$ is a commutative ring, a category $\mathcal{C}$ is called $k$-linear if it is enriched in $k\text{-Mod}$, the category of $k$-modules, i.e. if $\text{Hom}(X,Y)$ is a $k$-module for all $X,Y \in |\mathcal{C}|$ and if the composition of morphisms is $k$-bilinear.

A functor $F : \mathcal{C} \to \mathcal{C}'$ between [$\textbf{Ab}$-enriched, $k$-linear] categories is called additive, $k$-linear if it induces homomorphisms of additive groups, $k$-modules

$$\text{Hom}(X,Y) \to \text{Hom}(FX, FY) \quad (A.27)$$

for all $X,Y \in |\mathcal{C}|$.

An additive category is an $\textbf{Ab}$-enriched category that has a terminal object and all binary products. A preabelian category is an $\textbf{Ab}$-enriched category that has all finite limits. An abelian category is a preabelian category in which every monomorphism is a kernel and in which every epimorphism is a cokernel. A functor $F : \mathcal{C} \to \mathcal{C}'$ between preabelian categories is called exact if it preserves all finite limits. An equivalence of [$\textbf{Ab}$-enriched, $k$-linear] categories is an equivalence of categories, one functor of which is additive, $k$-linear.

Definition A.13. Let $\mathcal{C}$ be a $k$-linear category and $k$ a commutative ring.

1. An object $X \in |\mathcal{C}|$ is called simple if $\text{End}(X) \cong k$ are isomorphic as $k$-modules.
2. An object $X \in |\mathcal{C}|$ is called null if $\text{End}(X) \cong \{0\}$.
3. The category $\mathcal{C}$ is called semisimple if there exists a family $\{V_j\}_{j \in I}$ of objects $V_j \in |\mathcal{C}|$, $I$ some index set, such that
   (a) $V_j$ is simple for all $j \in I$.
   (b) $\text{Hom}(V_j, V_\ell) = \{0\}$ for all $j, \ell \in I$ for which $j \neq \ell$.
   (c) For each object $X \in |\mathcal{C}|$, there is a finite sequence $j_1(X), \ldots, j_n(X) \in I$, $n^X \in \mathbb{N}_0$, and morphisms $\iota^{X}_\ell : V_{j_\ell} \to X$ and $\pi^{X}_\ell : X \to V_{j_\ell}$ such that

$$\text{id}_X = \sum_{\ell=1}^{n^X} \iota^{X}_\ell \circ \pi^{X}_\ell. \quad (A.28)$$

and

$$\pi^{X}_\ell \circ \iota^{X}_m = \begin{cases} \text{id}_{V_{j_m}^X}, & \text{if } \ell = m, \\ 0, & \text{else} \end{cases} \quad (A.29)$$

4. The category is called finitely semisimple (also Artinian semisimple) if it is semisimple with a finite index set $I$ in condition (3).

Proposition A.14 (see [29, Lemma II.4.2.2]). Let $\mathcal{C}$ be a $k$-linear category and $k$ a commutative ring. If $\mathcal{C}$ is finitely semisimple, then there is a finite set $J \subseteq |\mathcal{C}|$ of non-null objects such that

$$\Phi : \bigoplus_{J \in J} \text{Hom}(X,J) \otimes \text{Hom}(J,Y) \to \text{Hom}(X,Y),$$

$$f \otimes g \mapsto g \circ f, \quad (A.30)$$

is an isomorphism for all $X,Y \in |\mathcal{C}|$. 
If $\mathcal{C}$ is a semisimple $k$-linear category, then by [29, Proposition II.4.2.1], $\text{Hom}(X, Y)$ is a finitely generated projective $k$-module for all $X, Y \in |\mathcal{C}|$. If $\mathcal{C}$ is a semisimple $k$-linear category with family $\{V_j\}_{j \in I}$ of simple objects as in Definition A.13(3) and $k$ is a field, then for each simple $X \in |\mathcal{C}|$, there is some $j \in I$ such that $X \cong V_j$.

### A.4 Additive and non-degenerate spherical categories

A monoidal category is called [Ab-enriched, $k$-linear] if $\mathcal{C}$ is [Ab-enriched, $k$-linear] and if the tensor product of morphisms is [Z-bilinear, $k$-bilinear].

In a monoidal category $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$, the set $k := \text{End}(1)$ forms a commutative monoid with respect to composition. If $\mathcal{C}$ is Ab-enriched as a monoidal category, then $k$ is a unital commutative ring and $\mathcal{C}$ is $k$-linear as an ordinary category, but not necessarily as a monoidal category.

In a $k$-linear pivotal category, the left- and right-traces

$$\text{tr}^{(L)}_X : \text{End}(X) \to k \quad \text{and} \quad \text{tr}^{(R)}_X : \text{End}(X) \to k \quad (A.31)$$

are $k$-linear for all $X \in |\mathcal{C}|$.

If we work with semisimple pivotal categories, we also require the set of representatives of the simple objects to contain the monoidal unit and to be closed under duality as follows.

**Definition A.15.** A $k$-linear pivotal category $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho, (-)^*, \text{ev}, \text{coev}, \tau)$, $k = \text{End}(1)$, is called [finitely] semisimple if the underlying $k$-linear category is [finitely] semisimple and the family $\{V_j\}_{j \in I}$ of Definition A.13(3) satisfies the following conditions:

1. There is an element $0 \in I$ such that $V_0 \cong 1$.
2. For each $j \in I$, there is some $j^* \in I$ such that $V_j^* \cong V_{j^*}$.

**Definition A.16.** A $k$-linear spherical category $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho, (-)^*, \text{ev}, \text{coev}, \tau)$, $k = \text{End}(1)$, is called non-degenerate if the $k$-bilinear maps

$$\text{Hom}(Y, X) \otimes \text{Hom}(X, Y) \to k, \quad f \otimes g \mapsto \text{tr}_X(f \circ g) \quad (A.32)$$

are non-degenerate for all objects $X, Y \in |\mathcal{C}|$, i.e. if $\text{tr}_X(f \circ g) = 0$ for all $g : X \to Y$ implies $f = 0$.

In a $k$-linear spherical category, the trace is multiplicative with respect to the tensor product, i.e.

$$\text{tr}_{X_1 \otimes X_2}(h_1 \otimes h_2) = \text{tr}_{X_1}(h_1) \cdot \text{tr}_{X_2}(h_2) \text{ for all } h_1 : X_1 \to X_1 \text{ and } h_2 : X_2 \to X_2.$$

**Proposition A.17** (analogous to [29, Lemma II.4.2.3]). Every semisimple $k$-linear spherical category is non-degenerate.

**Proposition A.18** (analogous to [29, Lemma II.4.2.4]). Let $\mathcal{C}$ be a semisimple $k$-linear spherical category with family $\{V_j\}_{j \in I}$ as in Definition A.13(3). Then for all $j \in I$, $\dim V_j$ is invertible in $k$.

**Proposition A.19** (see [7, Proposition A.29]). Let $\mathcal{C}$ be a $k$-linear spherical category and $k = \text{End}(1)$ be a field. If $\mathcal{C}$ satisfies all conditions of a finitely semisimple category of Definition A.13(3) except maybe for (A.29), then the $t_k^X$ and $\pi_k^X$ can be chosen in such a way that (A.29) holds as well.
Finitely semisimple spherical categories are self-dual

Acknowledgements
The author would like to thank Gabriella Böhm, Shahn Majid and Kornél Szlachányi for valuable discussions and everyone at RMKI Budapest for their hospitality.

References

[1] J. W. Barrett and B. W. Westbury: Spherical categories. *Adv. Math.* 143, No. 2 (1999) 357–375, arxiv:hep-th/9310164 [math-ph], MR 1686423.

[2] J. W. Barrett and B. W. Westbury: Invariants of piecewise-linear 3-manifolds. *Trans. AMS* 348, No. 10 (1996) 3997–4022, arxiv:hep-th/9311155, MR 1357878

[3] V. G. Turaev and O. Y. Viro: State sum invariants of 3-manifolds and quantum 6j-symbols. *Topology* 31, No. 4 (1992) 865–902, MR 1191386.

[4] S. Majid: Representations, duals and quantum doubles of monoidal categories. *Rend. Circ. Mat. Palermo (2) Suppl.* 26 (1991) 197–206, MR 1151906.

[5] S. Majid: Braided groups and duals of monoidal categories. In *Category theory 1991 (Montreal, PQ, 1991)*, Canadian Mathematical Society Conference Proceedings 13. Amsterdam, Providence, 1992, pp. 329–343, MR 1192156.

[6] A. Joyal and R. H. Street: Tortile Yang-Baxter operators in tensor categories. *J. Pure Appl. Alg.* 71, No. 1 (1991) 43–51, MR 1107651.

[7] H. Pfeiffer: Tannaka–Krein reconstruction and a characterization of modular tensor categories (2007). Preprint arxiv:0711.1402 [math.QA].

[8] T. Hayashi: A canonical Tannaka duality for finite semisimple tensor categories (1999). Preprint arxiv:math/9904073 [math.QA].

[9] G. Böhm and K. Szlachányi: Weak C*-Hopf algebras: the coassociative symmetry of non-integral dimensions. In *Quantum groups and quantum spaces (Warsaw, 1995)*, Banach Center Publications 40. Polish Academy of Sciences, Warsaw, 1997, pp. 9–19, MR 1481730.

[10] G. Böhm, F. Nill and K. Szlachányi: Weak Hopf algebras I. Integral theory and C*-structure. *J. Algebra* 221, No. 2 (1999) 385–438, arxiv:math/9805116 [math.QA], MR 1726707.

[11] G. Böhm and K. Szlachányi: Weak Hopf algebras II. Representation theory, dimensions, and the Markov trace. *J. Algebra* 233, No. 1 (2000) 156–212, arxiv:math/9906045 [math.QA], MR 1793595.

[12] G. Böhm: Weak C*-Hopf algebras and their application to spin models. PhD thesis, Research Institute for Particle and Nuclear Physics, Budapest (1997).

[13] G. Böhm and K. Szlachányi: Hopf algebroid symmetry of abstract Frobenius extensions of depth 2. *Comm. Alg.* 32, No. 11 (2004) 4433–4464, arxiv:math/0305136 [math.QA], MR 2102458.
Finitely semisimple spherical categories are self-dual

[14] V. Lyubashenko and S. Majid: Braided groups and quantum Fourier transform. *J. Algebra* **166**, No. 3 (1994) 506–528, [MR 1280590](http://cid-3251f42e0a615792 ipv4:80 MathSciNet).

[15] M. Müger: From subfactors to categories and topology. II. The quantum double of tensor categories and subfactors. *J. Pure Appl. Alg.* **180**, No. 1-2 (2003) 159–219, [arxiv:math/0111205 [math.CT]](http://arxiv.org/abs/math/0111205) [MR 1966525](http://mathscinet.ams.org/mathscinet-getitem?mr=1966525).

[16] A. D. Lauda and H. Pfeiffer: State sum construction of two-dimensional open-closed Topological Quantum Field Theories. *J. Knot Th. Ramif.* **16**, No. 9 (2007) 1121–1163, [arxiv:math/0602047 [math.QA]](http://arxiv.org/abs/math/0602047) [MR 2375819](http://mathscinet.ams.org/mathscinet-getitem?mr=2375819).

[17] G. Kuperberg: Involutory Hopf algebras and 3-manifold invariants. *Int. J. Math.* **2**, No. 1 (1991) 41–66, [arxiv:math/9201301 [math.QA]](http://arxiv.org/abs/math/9201301) [MR 1082836](http://mathscinet.ams.org/mathscinet-getitem?mr=1082836).

[18] J. W. Barrett and B. W. Westbury: The equality of 3-manifold invariants. *Math. Proc. Cam. Phil. Soc.* **118** (9 1995) 503–510, [arxiv:hep-th/9406019](http://arxiv.org/abs/hep-th/9406019) [MR 1342967](http://mathscinet.ams.org/mathscinet-getitem?mr=1342967).

[19] F. Girelli, R. Oeckl and A. Perez: Spin foam diagrammatics and topological invariance. *Class. Quant. Grav.* **19** (2002) 1093–1108, [arxiv:gr-qc/0111022](http://arxiv.org/abs/gr-qc/0111022) [MR 1894581](http://mathscinet.ams.org/mathscinet-getitem?mr=1894581).

[20] M. Müger: From subfactors to categories and topology. I. Frobenius algebras in and Morita equivalence of tensor categories. *J. Pure Appl. Alg.* **180**, No. 1-2 (2003) 81–157, [arxiv:math/0111204 [math.CT]](http://arxiv.org/abs/math/0111204) [MR 1966524](http://mathscinet.ams.org/mathscinet-getitem?mr=1966524).

[21] L. Crane and I. B. Frenkel: Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases. *J. Math. Phys.* **35**, No. 10 (1994) 5136–5154, [arxiv:hep-th/9405183](http://arxiv.org/abs/hep-th/9405183) [MR 1295461](http://mathscinet.ams.org/mathscinet-getitem?mr=1295461).

[22] K. Szlachányi: Adjointable monoidal functors and quantum groupoids. In *Hopf algebras in noncommutative geometry and physics*, Lecture Notes in Pure and Applied Mathematics 239. Marcel Dekker, New York, 2005, pp. 291–307, [arxiv:math/0301253 [math.QA]](http://arxiv.org/abs/math/0301253) [MR 2106937](http://mathscinet.ams.org/mathscinet-getitem?mr=2106937).

[23] F. Nill: Axioms for weak bialgebras (1998). Preprint [arxiv:math/9805104 [math.QA]](http://arxiv.org/abs/math/9805104).

[24] R. Oeckl: Generalized lattice gauge theory, spin foams and state sum invariants. *J. Geom. Phys.* **46**, No. 3-4 (2003) 308–354, [arxiv:hep-th/010259](http://arxiv.org/abs/hep-th/010259) [MR 1976954](http://mathscinet.ams.org/mathscinet-getitem?mr=1976954).

[25] L. H. Kauffman and S. L. Lins: Temperly-Lieb recoupling theory and invariants of 3-manifolds. Annals of Mathematics Studies 134. Princeton University Press, Princeton, 1994. [MR 1280463](http://mathscinet.ams.org/mathscinet-getitem?mr=1280463).

[26] N. Reshetikhin and V. G. Turaev: Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.* **127**, No. 1 (1990) 1–26, [MR 1036112](http://mathscinet.ams.org/mathscinet-getitem?mr=1036112).

[27] P. Schauenburg: Tannaka duality for arbitrary Hopf algebras. Algebra Berichte 66. Verlag Reinhard Fischer, München, 1992. [MR 1623637](http://mathscinet.ams.org/mathscinet-getitem?mr=1623637).

[28] P. Freyd and D. N. Yetter: Coherence theorems via knot theory. *J. Pure Appl. Alg.* **78**, No. 1 (1992) 49–76, [MR 1154897](http://mathscinet.ams.org/mathscinet-getitem?mr=1154897).
[29] V. G. Turaev: Quantum invariants of knots and 3-manifolds. Walter de Gruyter, Berlin, 1994. [MR 1292673].

[30] S. MacLane: Categories for the working mathematician. Graduate Texts in Mathematics 5. Springer, Berlin, 1973. [MR 1712872].