Fedosov Star-Products and 1-Differentiable Deformations

Philippe Bonneau *

Abstract
We show that every star product on a symplectic manifold defines uniquely a 1-differentiable deformation of the Poisson bracket. Explicit formulas are given. As a corollary we can identify the characteristic class of any star product as a part of its explicit (Fedosov) expression.

1 Introduction

The 1-differentiable deformations of the Poisson bracket Lie algebra of differentiable functions on a Poisson manifold $M$ are usual formal deformations (in the sense of [3]) built using exclusively (1,1)-bidifferential operators as cochains. They define a formal Poisson structure on $M$ starting with the initial Poisson structure (contravariant 2-tensor) of $M$. In 1974, in one of the first papers of what became the deformation quantization theory (for a review see [9]), M. Flato, A. Lichnerowicz and D. Sternheimer [3] studied these deformations for a symplectic manifold $M$. They showed in particular that the infinitesimal 1-differentiable deformations (with cochains vanishing on constants) are exactly classified by the second de Rham cohomology space of $M$.

As is now well-known, there is a similar classification for star products, given by a sequence of de Rham 2-cocycles. Comparing both results one suspects that the “difference” (the “difference of what is added on Moyal”) between two star products on $M$ is made of a sequence of complete 1-differentiable deformations of the Poisson bracket associated with the symplectic form. This is what we prove in the present paper. In addition we are able to give explicit formulas for these deformations. As a corollary we show that the characteristic class of a star product (in the formal Poisson bivector form, as in [3] ) is explicitly written in the expression of the star product.

Our ultimate goal is to show that one can reconstruct any star product on a Poisson manifold from any other by adding (in a sense to define) some

*Département de Mathématiques, Université de Bourgogne, BP 400, F-21011 Dijon Cedex, France. E-mail: bonneau@u-bourgogne.fr
1-differentiable deformations. To find a precise definition of this “addition” is related with the hope that, for any given star product on a Poisson manifold, the method used here can give a way to identify the formal Poisson bivector type characteristic class (see [8]) of the star product by looking solely at its explicit formula.

The paper is constructed as follows: in Section 2 we define an algebraic operation (a contraction) and give some useful formulas related to it. Section 3 is devoted to the statement of our main result describing the impact of a 1-differentiable modification of a cochain at any given level on subsequent levels, and developing the above mentioned consequences. In Section 4 we give the proofs, relying for clarity of the exposition on some intermediary lemmas which we also prove, omitting details of straightforward computations. The last Section gives an idea about the forms and the occurrences of the 1-differentiable terms appearing in the formula of a Fedosov star-product. In the appendix we give the complete details of the proofs.

2 Definitions

Throughout the article $M$ will be a symplectic manifold of dimension $2d$, $\omega$ its non-degenerate closed 2-form and $\mu : \otimes^2 T^* M \to \otimes^2 T M$ the canonical isomorphism given by $\omega$. For $\alpha \in \Gamma(M, \otimes^2 T^* M)$ we note $\bar{\alpha} = \mu(\alpha)$.

If $\alpha \in \Gamma(M, \wedge^2 T^* M) \subset \Gamma(M, \otimes^2 T^* M)$, we write (in local coordinates on an arbitrary chart) $\alpha = \alpha_{ij} dx^i \otimes dx^j = \frac{1}{2} \alpha_{ij} dx^i \wedge dx^j$ and the same for skew-symmetric bivectors. For the symplectic form $\omega = \omega_{ij} dx^i \otimes dx^j = \frac{1}{2} \omega_{ij} dx^i \wedge dx^j$ we define $\bar{\omega} = \bar{\omega}^{ij} \partial_i \otimes \partial_j = \frac{1}{2} \bar{\omega}^{ij} \partial_i \wedge \partial_j$ by $\omega_{ij} \bar{\omega}^{jk} = \delta^k_j$.

We have $\bar{\alpha} = \mu(\alpha) = -\omega^{ij} \bar{\omega}^{jk} \alpha_{rs} \partial_i \otimes \partial_j$ for $\alpha \in \Gamma(M, \otimes^2 T^* M)$. As in [3, 4] we use the Einstein convention on repeated indices $i, j = 1, \ldots, 2d$.

Definition 1 The “diamond” contraction.

$\diamond$ is the following contraction operation:

$$\Gamma(M, \otimes^2 T^* M) \otimes \Gamma(M, \otimes^2 T^* M) \to \Gamma(M, \otimes^2 T^* M)$$

$$\alpha \otimes \beta \mapsto \bar{\alpha} \otimes \bar{\beta} = \bar{\omega}^{rs} \alpha_{ri} \beta_{sj} dx^i \otimes dx^j$$

and

$$\Gamma(M, \otimes^2 TM) \otimes \Gamma(M, \otimes^2 TM) \to \Gamma(M, \otimes^2 TM)$$

$$A \otimes B \mapsto A \diamond B = \omega_{rs} A^{ri} B^{sj} \partial_i \otimes \partial_j$$

We define $\alpha^{on} = \alpha \diamond \alpha^{o(n-1)}$ and $A^{on} = A \diamond A^{o(n-1)}$.

Proposition 1 Let $\alpha, \beta \in \Gamma(M, \wedge^2 T^* M)$ and $A, B \in \Gamma(M, \wedge^2 TM)$.

(i) The isomorphism $\mu$ acts multiplicatively with respect to the diamond contraction: $\mu(\alpha \diamond \beta) = \mu(\alpha) \diamond \mu(\beta)$; $\mu^{-1}(A \diamond B) = \mu^{-1}(A) \diamond \mu^{-1}(B)$

(ii) In this case (values in skew-symmetric tensors) we have also
\[\alpha^{on} \in \Gamma(M, \wedge^2 T^* M) \text{ and } A^{on} \in \Gamma(M, \wedge^2 TM)\].
Moreover, for \(l + m = n\), \(\alpha^{on} = \alpha^{ol} \diamond \alpha^{om}\) and the same for \(A\).

Proof:
(i) direct computation.
(ii) By induction. We use that, for \(\alpha, \beta, \gamma \in \Gamma(M, \wedge^2 T^* M)\), we have:
\[\alpha \diamond (\beta \diamond \gamma) = (\beta \diamond \alpha) \diamond \gamma\]
and for \(\tau \in \Gamma(M, \otimes^2 T^* M)\),
\[\omega \diamond \tau = -\tau \diamond \omega; \quad (\tau \diamond \omega)_{ij} = \tau_{ji}.\]
The contravariant part is deduced by (i).

One could also remark that, with the notations used in [3] [4], in the Weyl bundle framework we have:
\[\alpha^{o2} = \frac{\hbar}{16}(\delta^{-1} \alpha \diamond \delta^{-1} \alpha).\]
For \(\alpha \in \Gamma(M, \wedge^2 T^* M)\), we define the coefficients \(\alpha^{on}_{ij}\) by:
\[\alpha^{on} = \alpha^{on}_{ij} dx^i \otimes dx^j = \frac{1}{2} \alpha^{on}_{ij} dx^i \wedge dx^j\]

3 Results

Let \((M, \omega)\) be a symplectic manifold, \(\nabla\) a symplectic connexion on \(M\) and \(R\) its curvature. Let \(*\) be a Fedosov star product on \(M\): for \(f, g \in C^\infty(M)\),
\[f * g = f.g - i\hbar \frac{\omega(f, g)}{2} + \sum_{n \geq 1} \hbar^n C_n(f, g).\]  

(1)

Let \(\Omega \in \omega + hZ^2_{DR}(M)[[\hbar]]\) be the Weyl curvature of \(*\) (actually Fedosov [3] [4] takes \(-\Omega\)), where \(Z^2_{DR}(M)\) is the space of de Rham 2-cocycles on \(M\). We denote by \(H^2_{DR}(M)\) the second de Rham cohomology space. The characteristic class of \(*\) is the class of \(\Omega\) in \([\omega] + hH^2_{DR}(M)[[\hbar]]\).

Let \(\tilde{*}\) be the Fedosov star product of Weyl curvature \(\tilde{\Omega} = \Omega + h^k \alpha\), with \(\alpha \in Z^2_{DR}(M)\)
\[f \tilde{*} g = f.g - i\hbar \frac{\tilde{\omega}(f, g)}{2} + \sum_{n \geq 1} \hbar^n \tilde{C}_n(f, g).\]

(2)

We know by [4] that \(C_n = \tilde{C}_n\) for all \(n \leq k\) and \(\tilde{C}_{k+1} = C_{k+1} + \frac{i}{2} \tilde{\alpha}\). What happens for \(n > k + 1\)? This is the subject of the following proposition.

Proposition 2 The change \(\Omega \rightarrow \Omega + h^k \alpha\) adds the series \(\sum_{p \geq 1} h^{p+1} \tilde{\alpha}^{op}\) to the explicit expression of the star-product \(*\). The series \(\tilde{\omega} - \sum_{p \geq 1} h^p \tilde{\alpha}^{op}\) is a formal Poisson bracket and contains all the 1-differentiable terms of \(\tilde{*}\) not depending explicitly on \(R\), \(\Omega - \omega\) and derivatives of \(\alpha\).

Remarks:
1. Let us choose \( \Omega = \omega + h^k \alpha \). We define the skew-symmetric bivector \( \bar{\Omega} \) by \( \Omega_{ij} \bar{\Omega}^{jk} = \delta_i^k \) and we obtain by easy computations

\[
\bar{\Omega} = \bar{\omega} - \sum_{p \geq 1} (h^k \bar{\alpha})^{op} = \bar{\omega} - \sum_{p \geq 1} h^p \bar{\alpha}^{op} \tag{3}
\]

Since \( d_{DR} \Omega = 0 \) we have \([\bar{\Omega}, \bar{\Omega}] = 0\) for the Schouten bracket. Thus \( \bar{\Omega} \) is a formal Poisson bivector.

2. Formula (3) is valid for a formal \( \alpha \), i.e. \( \bar{\Omega} = \bar{\omega} - \sum_{p \geq 1} (\bar{\alpha} h)^{op} \), with \( \alpha h = h^1 \alpha_1 + h^2 \alpha_2 + \cdots + h^m \alpha_m + \cdots, \) \( \Omega = \omega + \alpha h \). This gives all the 1-differentiable deformations of \( \bar{\omega} \). If \( [\alpha h] = [\beta h] \) in \( H^2(M)[[h]] \) the two resulting formal Poisson brackets are equivalent (see [8]).

Proposition 2 can be reformulated in order to make more explicit the relation between this formal Poisson bracket and the characteristic class of the star product:

**Corollary 1** Let * be the Fedosov star product of trivial Weyl curvature \( \omega \) and \( \ast \) the one with curvature \( \Omega = \omega + \alpha h \). \( \Omega \) appears explicitly in the formal Poisson form \( \bar{\omega} \) as a part of the formula for \( \ast \). \( \bar{\Omega} \) can be seen as all the 1-differentiable terms of \( \ast \) not containing \( R \) or derivatives of \( \alpha h \). We have:

\[
f \ast g = f * g + \frac{i}{2} h \sum_{p \geq 1} (\bar{\alpha}_h)^{op}(f, g) + \rho(f, g)
= f \cdot g - \frac{i}{2} h \left( \bar{\omega}(f, g) - \sum_{p \geq 1} (\bar{\alpha}_h)^{op}(f, g) \right) + \sum_{n \geq 2} h^n C_n(f, g) + \rho(f, g)
= f \cdot g - \frac{i}{2} h \bar{\Omega}(f, g) + \sum_{n \geq 2} h^n C_n(f, g) + \rho(f, g) \tag{4}
\]

where the terms occurring in the remainder \( \rho \) either depend explicitly on the curvature \( R \) or on derivatives of \( \alpha h \), or are not 1-differentiable.

**Remark:** Expression (4) shows that on \( (M, \omega) \), for a star-product \( \ast \) of characteristic class \( \Omega \), the corresponding bracket \( \{ f, g \}_\ast = \frac{i}{h} (f \ast g - g \ast f) \) can be seen not only as a deformation of the Lie algebra \( (C(M), \bar{\omega}) \) but also as a deformation of the “formal” Lie algebra \( (C(M)[[h]], \bar{\Omega}) \). Thus \( \ast \) can be viewed as the star-product of trivial characteristic class on the “formal” symplectic manifold \( (M, \Omega) \), i.e. \( M \) endowed with the formal symplectic structure given by \( \Omega \). We can also consider \( \ast \) as a deformation of * with \( \frac{i}{2} \sum_{p \geq 1} (\bar{\alpha}_h)^{op} \) as infinitesimal deformation.
4 Proofs

4.1 Fedosov notations

We use the notations of Fedosov [3, 4]: we choose a symplectic connexion \( \nabla \) on the symplectic manifold \( M \) and we denote by \( \partial \) the covariant exterior derivative associated to \( \nabla \). Let \( \circ \), \( [\cdot, \cdot] \) be, respectively, the (Moyal) product and the bracket on \( W = \Gamma(M, W) \), the sections of the Weyl bundle \( W \) associated to \( M \), and \( \delta, \delta^{-1} \) the operators on \( W \) defined in [3, 4]. We construct on \( W \) an Abelian connexion \( D = \partial - \delta^{-1}Q \) where \( Q = R + (\Omega - \omega) \). Defining \( \lambda = -1 \partial r + \frac{i}{\hbar} [r, \cdot] \), \( \Omega \) is the curvature of \( D \). Defining \( Q = R + (\Omega - \omega) \), \( r \) is the unique solution, under suitable conditions, of the equation:

\[
\lambda = \delta^{-1}Q + \delta^{-1}(\partial r + \frac{i}{\hbar} \lambda^2)
\]

Fedosov shows that the space \( W_D = \{ a = a(x, y, \hbar) \in W \mid Da = 0 \} \) is isomorphic to \( C^\infty(M)[[\hbar]] \) by the isomorphism \( \sigma_D \) given by the equation

\[
\sigma^{-1}_D(f) = f + \delta^{-1}(\partial a + \frac{i}{\hbar} [r, a]) \quad f \in C^\infty(M)[[\hbar]].
\]

\( \sigma_D \) is the restriction to \( W_D \) of the projection \( \sigma, \sigma(x) = a(x, 0, \hbar) \) (\( \sigma \) replaces the \( y \)'s by 0). So the star product corresponding to \( \Omega \) is given by \( f \ast g = \sigma_D(\sigma^{-1}_D(f) \circ \sigma^{-1}_D(g)) \). For \( a \in W, a^{(n)} \) will denote the part of degree \( n \) of \( a \) in the usual filtration of the Weyl bundle while \( a_n \) will be defined as \( a \) modulo the terms of degree \( > n \), i.e. \( a_n = a^{(0)} + a^{(1)} + \cdots + a^{(n)} \).

4.2 Equation for \( \tilde{r} \)

We are now looking for an \( \tilde{r} \) giving an Abelian connexion \( \tilde{D} \) of Weyl curvature \( \tilde{\Omega} = \Omega + \hbar k \alpha = \frac{1}{2} \Omega_{ij} dx^i \wedge dx^j + \hbar k \frac{1}{2} \alpha_{ij} dx^i \wedge dx^j \). So \( \tilde{r} \) satisfies

\[
\tilde{r} = \delta^{-1}Q + \delta^{-1}(\hbar k \alpha) + \delta^{-1}(\partial \tilde{r} + \frac{i}{\hbar} \tilde{r}^2).
\]

This modification gives some additional terms of interest:

- the first appears in degree \( 2k + 1 \). We have \( r^{(2k+1)} = r^{(2k+1)} + \hbar^k s_1 \) with \( s_1 = \delta^{-1} \alpha = \frac{1}{2} \alpha_{ij} y^i dx^j = \sigma_1 \alpha_{ij} y^i dx^j \)

- in degree \( 4k + 1 \):

\[
\hbar^k s_2 = \hbar^k \delta^{-1}(\frac{i}{\hbar} \delta^{-1} \alpha \circ \delta^{-1} \alpha) = \hbar^k \delta^{-1}(\frac{i}{\hbar} (s_1 \circ s_1)) = \hbar^k \frac{1}{8} \alpha_{ij}^2 y^i dx^j = \hbar^k \sigma_2 \alpha_{ij}^2 y^i dx^j
\]

Actually we consider only the terms depending exclusively on the \( s_i \)'s. So the next degrees to consider are (using \( 2\tilde{r} = [\tilde{r}, \tilde{r}] \)):
• degree $6k+1$:

\[ h^{3k}s_3 = h^{3k} \delta^{-1} \frac{i}{\hbar} \left( [s_1, s_2] + [s_2, s_1] \right) = h^{3k} \sigma_3 \alpha_{ij}^3 y^i dx^j \]

• ... 

• degree $2pk+1$:

\[ h^{pk} s_p = h^{pk} \delta^{-1} \frac{i}{\hbar} \sum_{l+m=p; \ l, m \geq 1} [s_l, s_m] = h^{pk} \sigma_p \alpha_{ij}^p y^i dx^j. \]

The $\sigma_p$’s will be described in the proof of Lemma 4.

In these computations we have used the following straightforward lemma:

Lemma 1 \[ \delta^{-1} \left( \frac{i}{\hbar} [\alpha_{ij}^l y^i dx^j, \alpha_{ij}^m y^i dx^j] \right) = \frac{1}{2} \alpha_{ij}^{(l+m)} y^i dx^j. \]

4.3 Equation for $\tilde{a}$

We now describe the consequences of these changes in the computations of $\sigma_D^{-1}(f) = \tilde{a} = f + \delta^{-1}(\partial \tilde{a} + \frac{i}{\hbar}[\tilde{r}, \tilde{a}]), \ f \in C^\infty(M)$. Recall that for any Weyl curvature, $a^{(1)} = \partial_j f y^j$. We then obtain:

• in degree $2k+1$:

\[ \tilde{a}^{(2k+1)} = a^{(2k+1)} + h^k \delta^{-1} \left( \frac{i}{\hbar} [s_1, a^{(1)}] \right) = a^{(2k+1)} + h^k x_1 \]

\[ = a^{(2k+1)} + \frac{1}{2} h^k \omega^l \alpha_{ij} \partial_l f y^j \]

• among other additional terms, there is (with $x_2 = \frac{3}{2}$):

\[ h^{2k} x_2 = h^{2k} \delta^{-1} \left( \frac{i}{\hbar} [s_2, a^{(1)}] + [s_1, x_1] \right) \]

\[ = h^{2k} x_2 \omega^l (\alpha_{ij}^{\omega 2}) \partial_l f y^j. \]

• ... 

• in degree $2pk+1$ (with $x_0 = a^{(1)}$ , $x_0 = 1$):

\[ h^{pk} x_p = h^{pk} \delta^{-1} \left( \frac{i}{\hbar} \sum_{l+m=p; \ l \geq 1, m \geq 0} [s_l, x_m] \right) \]

\[ = h^{pk} x_p \omega^l (\alpha_{ij}^{\omega p}) \partial_l f y^j. \]

The $x_p$’s will be described in the proof of Lemma 4.

In these computations we have used the following lemma:

Lemma 2 \[ \delta^{-1} \left( \frac{i}{\hbar} [\alpha_{ij}^l y^i dx^j, \omega^l \alpha_{ij}^m \partial_l f y^j] \right) = \omega^l \alpha_{ij}^{(l+m)} \partial_l f y^j. \]
4.4 End of the proof of Proposition 2

Finally, from the formula
\[
\tilde{C}^n(f, g) = \sigma \left( (\tilde{a} \circ \tilde{b})^{(2n)} \right) = \sigma \left( \sum_{l+m=2n} \tilde{a}^{(l)} \circ \tilde{b}^{(m)} \right),
\]
with \( \tilde{a} = \sigma^{-1}_D(f) \), \( \tilde{b} = \sigma^{-1}_D(g) \), \( f, g \in C^\infty(M) \) and taking into account (straightforward computations)

**Lemma 3**

\[
\sigma \left( x_{l}^{(a)} \circ x_{m}^{(b)} \right) = \frac{i \hbar}{2} x_{n} x_{l} x_{m} (\tilde{\alpha}^2)^{(m+l)} \partial_i f \partial_j g = \frac{i \hbar}{2} x_{n} x_{l} x_{m} \tilde{\alpha}^{(m+l)}(f, g)
\]

we obtain:

- \( \tilde{C}^{k+1}(f, g) = C^{k+1}(f, g) + \frac{1}{2} \tilde{a}(f, g) \)
- \( \tilde{C}^{2k+1}(f, g) = C^{2k+1}(f, g) + i c_1 \tilde{\alpha}^2(f, g) + \rho_2(f, g) \)
- \ldots
- \( \tilde{C}^{pk+1}(f, g) = C^{pk+1}(f, g) + i c_p \tilde{\alpha}^p(f, g) + \rho_p(f, g) \)

**Lemma 4** \( c_p = \frac{1}{2} \)

**Proof of Lemma 4:**

Define \( S(x) = \sum_{n \geq 1} \sigma_n x^n \) and \( X(x) = \sum_{n \geq 0} \kappa_n x^n \).

Since \( \sigma_n = \frac{1}{2} \sum_{l+m=n; \ l, m \geq 1} \sigma_l \sigma_m \), we find \( \frac{1}{2} S^2(x) = S(x) - \frac{1}{2} x \) and therefore \( S(x) = -\sqrt{1-x} + 1 \).

In the same way, since \( \kappa_n = \sum_{l+m=n; \ l \geq 0, m \geq 1} \kappa_l \kappa_m \), we have \( S(x)X(x) = X(x) - 1 \) and so \( X(x) = \frac{1}{\sqrt{1-x}} \).

Finally, since \( c_n = \frac{1}{2} \sum_{l+m=n; \ l, m \geq 0} \kappa_l \kappa_m \), \( c_n \) is the \( n^{th} \) coefficient of the Taylor expansion of \( \frac{1}{2} X^2(x) = \frac{1}{2(1-x)} \).

This completes the proof of Lemma 4. \( \blacksquare \)

**Specificity of these terms:**

In Equation (5) for \( r \), we considered all the terms involving solely \( \delta^{-1} \alpha \). The other terms always depend at least on \( \delta^{-1}Q = \delta^{-1}(R + \Omega - \omega) \) or \( (\delta^{-1} \partial)^n \delta^{-1} \alpha \), \( n \geq 1 \), and therefore involve \( Q \) or derivatives of \( \alpha \).

At the next step, the only 1-differentiable term of \( a = \sigma^{-1}_D(f) \) constructed without \( \delta^{-1}Q \) is \( a^{(1)} = y^{i} \partial_i f \). So we have considered all the terms obtained...
inductively with $\frac{i}{\hbar}\delta^{-1}[\hat{r}, \hat{a}]$ mixing the ones found in the first step and $a^{(1)}$. The terms coming from the part "$\delta^{-1}\partial \hat{a}$" of the equation on $\hat{a}$ will depend on derivatives of $\alpha$ or won’t be 1-differentiable anymore.

The last step just contracts these selected terms with $\bar{\omega}$.

Some ideas about the form and the propagation of the other 1-d differentiable terms are given in the next section.

4.5 Proof of the Corollary

The corollary is straightforward for $\Omega = \omega + \hbar \alpha$, $\alpha \in \Gamma(M, \bigwedge^2 T^*M)$. For $\alpha^h \in \Gamma(M, \bigwedge^2 T^*M)[[\hbar]]$ the proof of Proposition 2 is easily adaptable.

Indeed, let us take $\alpha^h = \hbar k_1 \alpha_1 + \hbar k_2 \alpha_2 + \cdots + \hbar k_n \alpha_n$ with $\alpha_q = \frac{1}{2}(\alpha_q)_{ij} dx^i \wedge dx^j$. We are looking for a $\tilde{r}$ such that

$$\tilde{r} = \delta^{-1} R + \delta^{-1} \alpha^h + \delta^{-1} (\partial \tilde{r} + \frac{i}{\hbar} \tilde{r}^2)$$

(6)

So $\tilde{r}_{2k_n+1} = r_{2k_n+1} + \delta^{-1} \alpha^h + \cdots$. We can also write $\tilde{r} = r + \delta^{-1} \alpha^h + \cdots$ so that another application of (6) gives:

$$\tilde{r} = \delta^{-1} R + \delta^{-1} \alpha^h + \delta^{-1} \left( \partial \tilde{r} + \frac{i}{\hbar} (r + \delta^{-1} \alpha^h + \cdots)^2 \right),$$

and we see that $\tilde{r}$ contains $\delta^{-1}(\frac{i}{\hbar}(\delta^{-1} \alpha^h \circ \delta^{-1} \alpha^h)) = \frac{1}{\hbar}(\alpha^h)_{ij}^2 y^i dx^j$ (actually this is true from $\tilde{r}_{4k_n+1}$). Iteration of this process gives way to the same computations as before.

The same argument can be used for solving the equation

$$\hat{a} = f + \delta^{-1}(\partial \hat{a} + \frac{i}{\hbar}[\hat{r}, \hat{a}]).$$

(7)

We find $\hat{a} = a + \sum_{p \geq 1} \kappa_p(\alpha^h)^{\circ p} + \cdots$ with $(\alpha^h)^{\circ p} = \omega^{it}(\alpha^h)_{ij} \partial_i f y^j$.

Then

$$f* g = \sigma(\hat{a} \circ \hat{b})$$

$$= \sigma(a \circ b) + \sigma \left( \sum_{l \geq 1} \kappa_l(\alpha^h)^{\circ l} \circ \sum_{m \geq 1} \kappa_m(\alpha^h)^{\circ m} \right) + \rho(f, g)$$

$$= f* g + \frac{i\hbar}{2} \sum_{p \geq 1} (\alpha^h)^{\circ p}(f, g) + \rho(f, g)$$

(8)

by lemmas 3 and 4.

So the corollary is proved for $\alpha^h \in \Gamma(M, \bigwedge^2 T^*M)[[\hbar]]$ and by induction for $\alpha^h \in \Gamma(M, \bigwedge^2 T^*M)[[\hbar]]$. 

8
5 Ideas about the form and the propagation of 1-differentiable terms

In this section we want to give an idea about the occurences and the forms of the other 1-differentiable terms that can appear in the explicit expression of a Fedosov star-product. We use the notations of Section 4.

5.1 In the star-product of Weyl curvature $\Omega = \omega$

$\delta^{-1}R \circ \delta^{-1}R$ contains a term without any "y", so it is a 2-form on $M$. Let's denote it $\beta_0$. $h^2\delta^{-1}\beta_0$ appears in $r^{(5)}$. More generally let's denote by $\beta_n$ the 2-form (part without any "y") appearing in $((\delta^{-1}\partial)^n\delta^{-1}R)^2$, i.e. we have $\frac{i}{h}((\delta^{-1}\partial)^n\delta^{-1}R)^2|_{y=0} = h^{n+2}\beta_n$ which appears in $r^{(2n+5)}$. For $n$ odd, $\beta_n = 0$. Then

$$\sigma\left(\delta^{-1}\frac{i}{h}[h^{n+2}\delta^{-1}\beta_n, a^{(1)}] \circ b^{(1)} + a^{(1)} \circ \delta^{-1}\frac{i}{h}[h^{n+2}\delta^{-1}\beta_n, b^{(1)}]\right) \quad (9)$$

gives a 1-differentiable term in every $C^{3+n}$, $n \in 2\mathbb{N}$. Since, between the part "in a" and the one "in b", it uses the product $\circ$ only at the first order in $\hbar$, it is skewsymmetric.

$$\sigma\left(\delta^{-1}\frac{i}{h}[(\delta^{-1}\partial)^n\delta^{-1}R, a^{(1)}] \circ \delta^{-1}\frac{i}{h}[(\delta^{-1}\partial)^n\delta^{-1}R, b^{(1)}]\right) \quad (10)$$

gives a symmetric (resp. skewsymmetric) 1-differentiable term in $C^{3+n}$ for $n$ odd (resp. for $n$ even).

So, at worse, each $C_l$ contains a 1-differentiable part for $l \geq 3$.

But the above term (10) might be cancelled because it might appear under other forms. For example, in the case $\beta_0$, three kinds of 1-differentiable terms appear in $C_3$:

1) $\sigma\left(\delta^{-1}\frac{i}{h}[[\delta^{-1}R, a^{(1)}] \circ \delta^{-1}\frac{i}{h}[[\delta^{-1}R, b^{(1)}]]\right)$

2) $\sigma\left(\delta^{-1}\frac{i}{h}[[\delta^{-1}R, \delta^{-1}\frac{i}{h}[[\delta^{-1}R, a^{(1)}]]] \circ b^{(1)} + a^{(1)} \circ \delta^{-1}\frac{i}{h}[[\delta^{-1}R, \delta^{-1}\frac{i}{h}[[\delta^{-1}R, b^{(1)}]]]]\right)$

3) $\sigma\left(\delta^{-1}\frac{i}{h}[h^2\delta^{-1}\beta_0, a^{(1)}] \circ b^{(1)} + a^{(1)} \circ \delta^{-1}\frac{i}{h}[h^2\delta^{-1}\beta_0, b^{(1)}]\right)$

In this case these three terms are the same, up to a positive coefficient, so they cannot cancel. I do not know if this always happens. This kind of phenomena can occur for all the terms we consider in these sections.

One can observe an interesting phenomenon of propagation: since $h^{n+2}\delta^{-1}\beta_n$ appears in $r$, it propagates exactly in the same way as $h^k\delta^{-1}\alpha$ in the proof of Proposition 2, so a series $\sum_{p \geq 1} h^{p(n+2)+1}\beta_n \delta^{op}(df, dg)$ appears.
5.2 Effects of the change $\Omega \to \Omega + h^k \alpha$

5.2.1 Mixed terms

(by “mixed terms” we mean the terms involving both $R$ and $\alpha$).

It is not difficult to see that in $\bar{C}_{k+2}$ there are no supplementary 1-differentiable terms compared to $C_{k+2}$ ($k \geq 2$).

In $\bar{C}_{k+3}$, there is one. It is the case $n = 0$ of the following fact: the term

$$
\sigma \left( \delta^{-1} \frac{i}{\hbar} [\delta^{-1} h^k \alpha, \delta^{-1} \frac{i}{\hbar} [\delta^{-1} h^{n+2} \beta_n, a^{(1)}]] \circ b^{(1)} \right.
+ a^{(1)} \circ \delta^{-1} \frac{i}{\hbar} [\delta^{-1} h^k \alpha, \delta^{-1} \frac{i}{\hbar} [\delta^{-1} h^{n+2} \beta_n, b^{(1)}]]
$$

is 1-differentiable, skewsymmetric and part of $\bar{C}_{k+3+n}$, $\forall n$ even.

And

$$
\sigma \left( \delta^{-1} \frac{i}{\hbar} [\delta^{-1} h^k \alpha, \delta^{-1} \frac{i}{\hbar} ((\delta^{-1} \partial)^n \delta^{-1} R, a^{(1)}]) \circ \delta^{-1} \frac{i}{\hbar} [\delta^{-1} h^k \alpha, \delta^{-1} \frac{i}{\hbar} ((\delta^{-1} \partial)^n \delta^{-1} R, b^{(1)}]) \right)
$$

is 1-differentiable, symmetric for $n$ odd, skewsymmetric for $n$ even. So the change $\Omega \to \Omega + h^k \alpha$ can give a supplementary 1-differentiable term in every $\bar{C}_l$ for $l \geq k + 3$.

Another phenomenon of propagation can be observed: denote $X_n(f) = \frac{i}{\hbar} [\delta^{-1} \beta_n, a^{(1)}] = \bar{\omega}^{il} \beta_{n,ij} \partial_i f dx^j$. It is a 1-form on $M$. In the proof of Proposition 4, it is possible to replace $a^{(1)} = \delta^{-1} \partial f$ by $\delta^{-1} X_n(f)$. Thus the characteristic class can appear again in the form $\sum_{n \geq 1} h^{k+2n} \bar{\alpha}^{op}(X_n(f), X_n(g))$.

5.2.2 Terms purely in $\alpha$

The first 1-differentiable terms not involving $R$ and depending on derivatives of $\alpha$ can appear in $\bar{C}_{2k+2}$: in the same way than in the preceding subsection, for $n \geq 1$,

$$
\sigma \left( \delta^{-1} \frac{i}{\hbar} [h^k (\delta^{-1} \partial)^n \delta^{-1} \alpha, a^{(1)}] \circ \delta^{-1} \frac{i}{\hbar} [h^k (\delta^{-1} \partial)^n \delta^{-1} \alpha, b^{(1)}] \right)
$$

is 1-differentiable, symmetric for $n$ odd, skewsymmetric for $n$ even.

$$
\frac{i}{\hbar} ((\delta^{-1} \partial)^n \delta^{-1} \alpha)^2 |_{y=0} = h^{2k+n} \gamma_n
$$

where $\gamma_n$ is a 2-form on $M$. $\gamma_n = 0$ for $n$ odd. As with $\beta_n$ (eq. 9) one can construct a skewsymmetric 1-differentiable term with $\gamma_n$.

So one can find 1-differentiable terms of these types in every $\bar{C}_{2k+1+n}$, $n \geq 1$.

Defining $Y_n(f) = \frac{i}{\hbar} [\delta^{-1} h^{2k+n} \gamma_n, a^{(1)}], \sum_{p \geq 1} h^{p(2k+n)+1} \bar{\alpha}^{op}(df, dg)$ and $\sum_{p \geq 1} h^{p+4k+2n+1} \bar{\alpha}^{op}(Y_n f, Y_n g)$ appear in the formula.

Remarks:
1. For simplicity in 5.2.1 we have considered 1-differentiable “mixed” terms not involving derivatives of $\alpha$, but there exist, for example, terms like $\beta_n(\alpha m(f), \alpha m(g))$.

2. It is also possible to have an idea of the propagation of all the terms in $\alpha$, not necessarily 1-differentiable, which do not contain $R$. Let $u$ be the solution of $u = \hbar^k \delta^{-1} \alpha + \delta^{-1} (\partial u + \frac{i}{\hbar} u^2)$ in $\Gamma(M, W \otimes T^*M)$ and $a_u$ the solution of $a_u = f + \delta^{-1} (\partial a_u + \frac{i}{\hbar} [u, a_u])$ in $\Gamma(M, W)$. These solutions exist and are unique because $\delta^{-1} (\partial + \frac{i}{\hbar} \cdot \circ \cdot)$ and $\delta^{-1} (\partial + \frac{i}{\hbar} [u, \cdot])$ raise degree (see [4]). Putting $R = 0$ in the expression of $\delta$ there is only $\sigma(a_u \circ b_u)$ left. Formally, this is the expression of the Fedosov star-product on $\mathbb{R}^{2n}$ of Weyl curvature $\omega + \hbar^k \alpha$.

For a Poisson bivector field $\pi$ we denote

$$\pi^n(f, g) = \pi^{i_1j_1} \pi^{i_2j_2} \cdots \pi^{injn} (\partial_{i_1} \cdots \partial_{i_n} f)(\partial_{i_1} \cdots \partial_{i_n} g)$$

So $\sigma(a_u \circ b_u)$ contains $\exp(\frac{-i}{\hbar} \omega)(f, g)$ [10] and the other terms form the part “purely” in $\alpha$ that is added to the formula of the star-product when we change $\Omega = \omega$ in $\Omega = \omega + \hbar^k \alpha$. The conjecture is that this part contains $\exp(\frac{-i}{\hbar} \Omega)(f, g)$ with $\Omega = \omega + \hbar^k \alpha$. Actually Proposition 2 shows that it is true at order 1 of differentiation.

Acknowledgements

I want to thank M. Flato, P. Gautheron and D. Sternheimer for asking me questions which push me to do this work (and especially D.S. for constant disponibility throughout the elaboration). I also want to thank F. Bidegain, P. Bieliavsky and participants at the Warwick symposium in December 97 for numerous comments.
APPENDIX

We give here the complete details of the proofs of the above results. After this paper was completed, we received [7] and noticed (S. Gutt, private communication) that, in a nonexplicit form, results similar to ours can be derived from there.

A Definitions

Let $W$ be the “Weyl” bundle i.e. the bundle of formal Weyl algebras defined in [3, 4]. A section $a$ of $W$ is a sum of “monomials” of the form $\hbar k a_{i_1, \ldots, i_p} (x, y, \hbar) y^i_1 \ldots y^i_p$. We give to it the degree $2k + p$ and this gives a filtration on $W = \Gamma(M, W)$. For $a, b \in W$ we have the following product:

$$a \circ b = \sum_{k=0}^{\infty} \left( \frac{-i\hbar}{2} \right)^k \frac{1}{k!} \tilde{\omega}^{i_1j_1} \ldots \tilde{\omega}^{i_kj_k} \frac{\partial^k a}{\partial y^{i_1} \ldots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{j_1} \ldots \partial y^{j_k}}$$

$$= \sum_{k=0}^{\infty} a \circ_k b$$

This product can be extended to the differential forms with values in $W$ by means of the exterior product on the “$dx^i$’s”.

A graded commutator is defined by $[a, b] = a \circ b - (-1)^{q_1q_2} b \circ a$, for $a \in \Gamma(M, W \otimes \bigwedge^{q_1} T^* M)$ and $b \in \Gamma(M, W \otimes \bigwedge^{q_2} T^* M)$.

We use the following two operators on the forms:

$$\delta a = dx^k \wedge \frac{\partial a}{\partial y^k}, \quad \delta^{-1} a = \frac{1}{p + q} y^k i(\frac{\partial}{\partial x^k}) a$$

for $a \in \Gamma(M, W \otimes \bigwedge^p T^* M)$ and of degree $q$ in the filtration of $W$.

Lemma A.1 $a, b \in \Gamma(M, W), \alpha, \beta \in \Gamma(M, W \otimes T^* M)$

(i) $a \circ_k b = (-1)^k b \circ_k a; \alpha \circ_k b = (-1)^k b \circ_k \alpha; \alpha \circ_k \beta = (-1)^{k+1} \beta \circ_k \alpha$

(ii) $[a, b] = 2 \sum_{p \geq 0} a \circ_{2p+1} b; [\alpha, b] = 2 \sum_{p \geq 0} \alpha \circ_{2p+1} b; [\alpha, \beta] = 2 \sum_{p \geq 0} \alpha \circ_{2p+1} \beta$

In particular, $[a, b] = 2a \circ_1 b, [\alpha, b] = 2\alpha \circ_1 b, [\alpha, \beta] = 2\alpha \circ_1 \beta$ for $\alpha, \beta, b$ of degree 1 in $y$.

Proof:

(i) $a \circ_k b = \left( \frac{-i\hbar}{2} \right)^k \frac{1}{k!} \tilde{\omega}^{i_1j_1} \ldots \tilde{\omega}^{i_kj_k} \frac{\partial^k a}{\partial y^{i_1} \ldots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{j_1} \ldots \partial y^{j_k}}$
\[
\frac{\mathrm{i} h}{2} \frac{1}{k!} \bar{\omega}_{j_1 j_2} \ldots \bar{\omega}_{j_k j_1} \partial^k a \partial^k b \\
= \frac{\mathrm{i} h}{2} \frac{1}{k!} (-1)^k \bar{\omega}_{j_1 j_2} \ldots \bar{\omega}_{j_k j_1} \partial^k b \partial^k a
\]

\[
\alpha, \beta \in \Gamma(M, W \otimes T^* M) \text{ so we can write } \alpha = \alpha_i (x, y, h) dx^i \text{ and } \beta = \beta_j (x, y, h) dx^j.
\]

Then \(\alpha \circ_k b = \alpha_i \circ_k b \, dx^i = (-1)^k b \circ_k \alpha_i \, dx^i = (-1)^k b \circ_k \alpha\)

and \(\alpha \circ_k \beta = \alpha_i \circ_k \beta_j \, dx^i \wedge dx^j = (-1)^k \beta_j \circ_k \alpha_i \, dx^i \wedge dx^j = (-1)^{k+1} \beta_j \circ_k \alpha_i \, dx^i \wedge dx^j = (-1)^{k+1} b \circ_k \alpha\)

(ii) \([a, b] = \alpha \circ b - b \circ \alpha\) and \([\alpha, \beta] = \alpha \circ \beta + \beta \circ \alpha\) and the computations are the same.

### About the diamond product:

Let \(\alpha, \beta, \gamma \in \Gamma(M, \Lambda^2 T^* M), \tau \in \Gamma(M, \otimes^2 T^* M)\). We will show

1. \(\mu(\alpha \circ \beta) = \mu(\alpha) \circ \mu(\beta)\)
2. \(\alpha^{\circ n} \in \Gamma(M, \Lambda^2 T^* M)\) and for \(l + m = n\), \(\alpha^{\circ n} = \alpha^{\circ l} \circ \alpha^{\circ m}\) with the help of
   
   (a) \(\alpha \circ \beta \circ \gamma = \beta \circ (\alpha \circ \gamma)\)
   
   (b) \((\tau \circ \omega)_{ij} = \tau_{ji}\)

**Proofs:**

1. 
\[
(\mu(\alpha) \circ \mu(\beta))^{ij} = \omega_{rs} \mu(\alpha)^s r \mu(\beta)^{s j} \\
= \omega_{rs} (-\bar{\omega}^{kr} \omega^{li} \alpha_{kl}) (-\bar{\omega}^{ms} \omega^{nj} \beta_{mn}) \\
= \delta^k_s \omega^{ts} \omega^{ms} \omega^{nj} \alpha_{kl} \beta_{mn} \\
= \bar{\omega}^{ij} \omega^{nj} \omega^{mk} \alpha_{kl} \beta_{mn} \\
= \bar{\omega}^{ij} \omega^{nj} (-\bar{\omega}^{km} \alpha_{kl} \beta_{mn}) \\
= -\bar{\omega}^{ij} \omega^{nj} (\alpha \circ \beta)_{tn} = \mu(\alpha \circ \beta)^{ij}
\]
2.
2(a)
\[(\alpha \circ \beta) \circ \gamma)_{jl} = \bar{\omega}^{r_{2s_2}} \bar{\omega}^{r_{1s_1}} \alpha_{r_{1r_2}} \beta_{s_1} \gamma_{s_2} \]
\[(\beta \circ (\alpha \circ \gamma))_{jl} = \bar{\omega}^{r_{1s_1}} \bar{\omega}^{r_{2s_2}} \beta_{r_{1j}} \alpha_{r_{2s_1}} \gamma_{s_2} \]
\[(r_1 \leftrightarrow s_1) = \bar{\omega}^{r_{1s_1}} \bar{\omega}^{r_{2s_2}} \beta_{s_1} \alpha_{r_{2r_1}} \gamma_{s_2} = ((\alpha \circ \beta) \circ \gamma)_{jl} \]

2(b)
\[(\tau \circ \omega)_{ij} = \bar{\omega}^{r_{s}} \tau_{r_{ij}} \omega_{s_{ij}} = \delta_{r_{ij}}^{r_{i} \tau_{i}} = \tau_{ji} \]

Let us suppose, by induction, that, \(\forall p \leq n\), \(\alpha \circ p\) is skewsymmetric and \(\alpha \circ n = \alpha \circ l \circ \alpha \circ m\), \(\forall l, m \geq 1\) s.t. \(l + m = n\).

We take now \(l, m \geq 1\) s.t. \(l + m = n + 1\). We have, using 2(a), 2(b) and the induction hypothesis:
\[\alpha^{(n+1)} = \alpha \circ \alpha^{n} = \alpha \circ (\alpha^{(l-1)} \circ \alpha^{m}) = (\alpha^{(l-1)} \circ \alpha) \circ \alpha^{m} = \alpha^{l} \circ \alpha^{m} \]
and
\[\alpha \circ (\alpha^{n} \circ \omega)_{ij} = (\alpha \circ (\alpha^{n})_{ij} = -(\alpha^{(n+1)})_{ij} \]
\[\alpha \circ (\alpha^{n} \circ \omega)_{ij} = ((\alpha^{n} \circ \alpha) \circ \omega)_{ij} = (\alpha^{(n+1)} \circ \omega) = (\alpha^{(n+1)})_{ji} \]

B About Section 4

Lemma B.1 (Lemma 1)
\[\delta^{-1}\left(\frac{i}{2\hbar}[\alpha^{om}_{ij} y^j dx^j, \alpha^{on}_{kl} y^k dx^l]\right) = \frac{1}{2} \alpha^{(m+n)}_{il} y^i dx^l \]

Proof:
\[\frac{i}{2\hbar}[\alpha^{om}_{ij} y^j dx^j, \alpha^{on}_{kl} y^k dx^l] = 2 \frac{i}{2\hbar}(\alpha^{om}_{ij} y^j dx^j \circ_1 \alpha^{on}_{kl} y^k dx^l) \quad \text{(Lemma A.1)}\]
\[= -i\hbar \frac{i}{\hbar} \bar{\omega}^{ik} \alpha^{om}_{ij} \alpha^{on}_{kl} dx^j \wedge dx^l \]
\[= \frac{1}{2} \alpha^{(m+n)}_{il} dx^j \wedge dx^l \]
and \[\delta^{-1}\left(\frac{1}{2} \alpha^{(m+n)}_{il} dx^j \wedge dx^l\right) = \frac{1}{2} \alpha^{(m+n)}_{il} y^i dx^l \]

\[\blacksquare\]
Lemma B.2 (Lemma 2)

\[ \delta^{-1}(\frac{i}{\hbar}[\alpha_{ij}^m y^j dx^i, \bar{\omega}^{kr} \alpha_{kl}^m \partial_r f y^j]) = \bar{\omega}^{kr} \alpha_{kj}^{(m+n)} \partial_r f y^j \]

Proof:

\[
\delta^{-1}(\frac{i}{\hbar}[\alpha_{ij}^m y^j dx^i, \bar{\omega}^{kr} \alpha_{kl}^m \partial_r f y^j]) = 2 \frac{i}{\hbar} (\alpha_{ij}^m y^j dx^i \circ_1 \bar{\omega}^{kr} \alpha_{kl}^m \partial_r f y^j) \\
= -\frac{i\hbar}{2} 2\frac{i}{\hbar} \bar{\omega}^{il} \alpha_{ij}^m \alpha_{kl}^m \partial_f dx^j \\
= \bar{\omega}^{kr} \bar{\omega}^{il} \alpha_{ij}^m (-\alpha_{lk}^m) \partial_f dx^j \\
= \bar{\omega}^{kr} (-\alpha_{jk}^m) \partial_f dx^j \\
= \bar{\omega}^{kr} \alpha_{kj}^{(m+n)} \partial_f dx^j
\]

and \( \delta^{-1}(\bar{\omega}^{kr} \alpha_{kj}^{(m+n)} \partial_f dx^j) = \bar{\omega}^{kr} \alpha_{kj}^{(m+n)} \partial_f y^j \)

Lemma B.3 (Lemma 3)

\[ \sigma(\bar{\omega}^{i_1 l_1} \alpha_{i_1 j_1}^m \partial_{i_1} f y^{j_1} \circ \bar{\omega}^{i_2 l_2} \alpha_{i_2 j_2}^m \partial_{i_2} g y^{j_2}) = \frac{i\hbar}{2} (\bar{\alpha}^{(m+n)})^l_1 \partial_{i_1} f \partial_{i_2} g \]

Proof: \( \bar{\omega}^{i_1 l_1} \alpha_{i_1 j_1}^m \partial_{i_1} f y^{j_1} \circ \bar{\omega}^{i_2 l_2} \alpha_{i_2 j_2}^m \partial_{i_2} g y^{j_2} \) has a term given by \( \alpha_0 \) which contains some "y" and one given by \( \alpha_1 \) which does not contain any. So

\[ \sigma(\bar{\omega}^{i_1 l_1} \alpha_{i_1 j_1}^m \partial_{i_1} f y^{j_1} \circ \bar{\omega}^{i_2 l_2} \alpha_{i_2 j_2}^m \partial_{i_2} g y^{j_2}) = \]

(Lemma A.1)

\[ = -\frac{i\hbar}{2} (\bar{\omega}^{i_1 j_2} \bar{\omega}^{i_1 l_1} \omega^{i_2 l_2} \alpha_{i_1 j_1}^m \alpha_{i_2 j_2}^m \partial_{i_1} f \partial_{i_2} g) \\
= -\frac{i\hbar}{2} (\bar{\omega}^{i_1 j_2} \bar{\omega}^{i_1 l_1} \omega^{i_2 l_2} \alpha_{i_1 j_1}^m \alpha_{i_2 j_2}^m \partial_{i_1} f \partial_{i_2} g) \\
= \frac{i\hbar}{2} (\bar{\omega}^{i_1 j_2} \bar{\omega}^{i_1 l_1} \omega^{i_2 l_2} \alpha_{i_1 j_1}^m \alpha_{i_2 j_2}^m \partial_{i_1} f \partial_{i_2} g) \\
= \frac{i\hbar}{2} (\bar{\alpha}^{(m+n)})^l_1 \partial_{i_1} f \partial_{i_2} g \]

C About Section 5

We need the following notations:

\[ R = \frac{1}{4} R_{ijkl} y^i y^j dx^k \wedge dx^l \text{ so } \delta^{-1} R = \frac{1}{8} R_{ijkl} y^i y^j dx^l \]
\[
\delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, a^{(1)}] = \delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, y^m \partial_m f] = \frac{-1}{24} \tilde{\omega}^{lm} R_{ijkl} y^i y^j y^k \partial_m f \tag{11}
\]

Proof of (11):

\[
\delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, a^{(1)}] = 2 \delta^{-1} R \circ_1 a^{(1)}
\]

\[
= -\frac{i\hbar}{2} \frac{1}{8} (\tilde{\omega}^{im} R_{ijkl} y^j y^k + \tilde{\omega}^{jm} R_{ijkl} y^j y^k + \tilde{\omega}^{km} R_{ijkl} y^j y^k) \partial_m f dx^l
\]

\[
= \frac{-i\hbar}{8} \tilde{\omega}^{im} (R_{ijkl} + R_{ijkl} + R_{kjil}) y^j y^k \partial_m f dx^l
\]

\[
= \frac{-i\hbar}{8} \tilde{\omega}^{im} (2R_{ijkl} + R_{kjil}) y^j y^k \partial_m f dx^l
\]

So

\[
\delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, a^{(1)}] = \frac{1}{3} \frac{1}{8} \tilde{\omega}^{im} (2R_{ijkl} + R_{kjil}) y^j y^k \partial_m f dx^l
\]

By the symmetry properties of \( R \) we have \( R_{ijkl} = -R_{kjil} + R_{ijkl} \) but \( R_{kjil} y^j y^k = -R_{ijkl} y^j y^k \) so \( R_{ijkl} y^j y^k = 0 \) and then

\[
\delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, a^{(1)}] = \frac{1}{24} \tilde{\omega}^{im} R_{kjil} y^j y^k \partial_m f dx^l
\]

\[
= \frac{1}{24} \tilde{\omega}^{im} (-R_{kjil}) y^j y^k \partial_m f dx^l
\]

\[
= \frac{-1}{24} \tilde{\omega}^{im} R_{ijkl} y^j y^k \partial_m f dx^l
\]

by renaming the indices. \(\blacksquare\)

We define \( \mathcal{R}_{l_1 l_2} \) by

\[
\sigma (\delta^{-1} R \circ \delta^{-1} R) = \delta^{-1} R \circ_3 \delta^{-1} R
\]

\[
= \left( \frac{1}{8} \right)^2 \mathcal{R}_{l_1 l_2} dx^{l_1} \wedge dx^{l_2}
\]

i.e.

\[
\mathcal{R}_{l_1 l_2} = R_{r_1 l_1 k_1 l_1} y^{r_1} y^{k_1} \circ_3 R_{r_2 l_2 k_2 l_2} y^{r_2} y^{k_2}
\]

In particular \( \mathcal{R}_{l_1 l_2} = -\mathcal{R}_{l_2 l_1} \) (Lemma A.1)

We denote \( \mathcal{R}_{m_1 m_2} = -\tilde{\omega}^{i m_1} \tilde{\omega}^{j m_2} \mathcal{R}_{l_1 l_2} \)

In Section 5 we defined \( \beta_n = \frac{i}{\hbar^{n+1}} \left( \delta^{-1} \partial \right)^n \delta^{-1} R \circ_{n+3} (\delta^{-1} \partial)^n \delta^{-1} R \).
Since \((\delta^{-1}\partial)^n\delta^{-1}R\) is a 1-form, for \(n + 3\) even \((n\ odd)\), we have, by Lemma A.1:

\[
(\delta^{-1}\partial)^n\delta^{-1}R \circ_{n+3} (\delta^{-1}\partial)^n\delta^{-1}R = (-1)^{n+4} (\delta^{-1}\partial)^n\delta^{-1}R \circ_{n+3} (\delta^{-1}\partial)^n\delta^{-1}R
\]

so \(\beta_n = 0\) for \(n\ odd\).

We will now show:

(1)

\[
\sigma \left( \delta^{-1}\frac{i}{\hbar}\delta^{-1}R, a^{(1)} \right) \circ \delta^{-1}\frac{i}{\hbar}\left[\delta^{-1}R, b^{(1)}\right] = -\frac{1}{9.26} R^{m_1m_2} \partial_{m_1} f \partial_{m_2} g
\]

(2)

\[
\sigma \left( \delta^{-1}\frac{i}{\hbar}\delta^{-1}R, \delta^{-1}\frac{i}{\hbar}\left[\delta^{-1}R, a^{(1)}\right] \right) \circ \delta^{-1}\frac{i}{\hbar}\left[\delta^{-1}R, b^{(1)}\right] = -\frac{1}{3.26} R^{m_1m_2} \partial_{m_1} f \partial_{m_2} g
\]

(3)

\[
\sigma \left( \delta^{-1}\frac{i}{\hbar}\left[\h^2\delta^{-1}\beta_0, a^{(1)}\right] \circ b^{(1)} + a^{(1)} \circ \delta^{-1}\frac{i}{\hbar}\left[\h^2\delta^{-1}\beta_0, b^{(1)}\right] \right) = -\frac{1}{26} R^{m_1m_2} \partial_{m_1} f \partial_{m_2} g
\]

Proofs:

Proof of (1):

\[
\sigma \left( \delta^{-1}\frac{i}{\hbar}\left[\delta^{-1}R, a^{(1)}\right] \circ \delta^{-1}\frac{i}{\hbar}\left[\delta^{-1}R, b^{(1)}\right] \right)
= \left(-\frac{1}{24}\right)^2 \delta m_1 R_{i_1j_1k_1l_1} y^1 y^2 y^3 y^4 \partial_{m_1} f \circ_3 \omega^{l_2m_2} R_{i_2j_2k_2l_2} y^2 y^3 y^4 \partial_{m_2} g
= -\frac{1}{9.26} R^{m_1m_2} \partial_{m_1} f \partial_{m_2} g
= -\frac{1}{9.26} R^{m_1m_2} \partial_{m_1} f \partial_{m_2} g
\]

Proof of (2): We have

\[
\sigma \left( \delta^{-1}\frac{i}{\hbar}\left[\delta^{-1}R, a^{(1)}\right] \circ b^{(1)} + a^{(1)} \circ \delta^{-1}\frac{i}{\hbar}\left[\delta^{-1}R, b^{(1)}\right] \right)
= \delta^{-1}\frac{i}{\hbar}\left[\delta^{-1}R, \delta^{-1}\frac{i}{\hbar}\left[\delta^{-1}R, a^{(1)}\right] \partial_{m_1} f \partial_{m_2} g
\]

17
where
\[
[\delta^{-1} R, \delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, a^{(1)}]]_3 = \delta^{-1} R \circ_3 \delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, a^{(1)}] \circ_3 \delta^{-1} R
\]
\[
= 2\delta^{-1} R \circ_3 \delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, a^{(1)}]
\]
\[
= 2\left( \frac{1}{24} R_{l_1 j_1 k_1 l_1} y^{i_1} y^{j_1} y^{k_1} dx^{l_1} \circ_3 \bar{\omega}^{l_2 m_2} R_{l_2 j_2 k_2 l_2} y^{i_2} y^{j_2} y^{k_2} \partial_{m_2} f \right)
\]
\[
= -\frac{1}{3.25} \bar{\omega}^{l_2 m_2} \mathcal{R}_{l_1 l_2} \partial_{m_2} f dx^{l_1}
\]
then
\[
\delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, \delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, a^{(1)}]]_3 = -\frac{1}{3.25} \frac{i}{\hbar} \bar{\omega}^{l_2 m_2} \mathcal{R}_{l_1 l_2} \partial_{m_2} f y^{l_1}
\]
So,
\[
\sigma \left( \delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, \delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, a^{(1)}]] \circ b^{(1)} \right)
\]
\[
= \frac{1}{3.25} \frac{i}{\hbar} \bar{\omega}^{l_2 m_2} \mathcal{R}_{l_1 l_2} \partial_{m_2} f y^{l_1} \circ_1 y^{m_1} \partial_{m_1} g
\]
\[
= -i\hbar \left( \frac{1}{3.25} \frac{i}{\hbar} \bar{\omega}^{l_1 m_1} \bar{\omega}^{l_2 m_2} \mathcal{R}_{l_1 l_2} \partial_{m_2} f \partial_{m_1} g \right)
\]
\[
(l_1 \leftrightarrow l_2; m_1 \leftrightarrow m_2)
\]
\[
\sigma \left( a^{(1)} \circ \delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, \delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, b^{(1)}]] \right)
\]
\[
= \frac{1}{2} \left( \frac{1}{3.25} \mathcal{R}^{m_1 m_2} \partial_{m_1} f \partial_{m_2} g \right)
\]
In the same way
\[
\sigma \left( a^{(1)} \circ \delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, \delta^{-1} \frac{i}{\hbar} [\delta^{-1} R, b^{(1)}]] \right)
\]
\[
= \frac{1}{2} \left( \frac{1}{3.25} \mathcal{R}^{m_1 m_2} \partial_{m_1} f \partial_{m_2} g \right)
\]
So we have (2).

Proof of (3): We have \( h^2 \delta^{-1} \beta_0 = \frac{1}{2 \pi \hbar} \mathcal{R}_{l_1 l_2} y^{l_1} dx^{l_2} \). Then
\[
[h^2 \delta^{-1} \beta_0, a^{(1)}] = 2 h^2 \delta^{-1} \beta_0 \circ_1 a^{(1)}
\]
\[
= \left( \frac{-i\hbar}{2} \right) 2 \frac{1}{24} \frac{i}{\hbar} \bar{\omega}^{l_1 m_1} \mathcal{R}_{l_1 l_2} \partial_{m_1} f dx^{l_2}
\]
\[
= \frac{1}{24} \bar{\omega}^{l_1 m_1} \mathcal{R}_{l_1 l_2} \partial_{m_1} f dx^{l_2}
\]
\[ \delta^{-1} \frac{i}{\hbar} [h^2 \delta^{-1} \beta_0, a^{(1)}] = \frac{i}{\hbar} \frac{1}{2\delta} \omega_l^1 m_1 R_l^1 R_l^2 \partial_1 \frac{f}{y^2} \] and
\[
\sigma \left( \delta^{-1} \frac{i}{\hbar} [h^2 \delta^{-1} \beta_0, a^{(1)}] \circ b^{(1)} \right) = \frac{i}{\hbar} \frac{1}{2\delta} \omega_l^1 m_1 R_l^1 R_l^2 \partial_1 \frac{f}{y^2} \circ_1 b^{(1)}
\]
\[
= \left( \frac{-i\hbar}{2} \right) \frac{i}{\hbar} \frac{1}{2\delta} \omega_l^2 \omega_l^1 \omega_l^1 \omega_l^1 R_l^1 \partial_1 f \partial m_2 g
\]
\[
= \frac{1}{2} \left( -\frac{1}{2\delta} \right) R^1 m_1 m_2 \partial m_1 f \partial m_2 g
\]

In the same way
\[
\sigma \left( a^{(1)} \circ \delta^{-1} \frac{i}{\hbar} [h^2 \delta^{-1} \beta_0, b^{(1)}] \right) = \frac{1}{2} \left( -\frac{1}{2\delta} \right) R^1 m_1 m_2 \partial m_1 f \partial m_2 g
\]
So we have (3).

About \( \tilde{C}_{k+2} \):

The assertion is that there are no 1-differentiable terms in \( \tilde{C}_{k+2} \), for \( k \geq 2 \) (for \( k = 1 \), Proposition 3 shows that \( \tilde{C}_3 \) contains \( \frac{1}{2} \tilde{a}^{(2)} \):

\[
\tilde{C}_{k+2} = C_{k+2} + \sigma \left( \underbrace{a^{(1)} \circ \rho_3^a + \rho_1^a \circ b^{(1)}}_{(i)} \right.
\]
\[
+ \underbrace{a^{(2)} \circ \rho_2^a + \rho_2^a \circ b^{(2)}}_{(ii)}
\]
\[
+ \underbrace{a^{(3)} \circ \rho_1^a + \rho_1^a \circ b^{(3)}}_{(iii)} \right)
\]

with
\[
\tilde{a}^{(2k+1)} = a^{(2k+1)} + \rho_1^a
\]
so
\[
\rho_1^a = \frac{i}{\hbar} \delta^{-1} [h^k \delta^{-1} \alpha, a^{(1)}]
\]
\[
\tilde{a}^{(2k+2)} = a^{(2k+2)} + \rho_2^a
\]
so
\[
\rho_2^a = \delta^{-1} \partial_1 \rho_1^a
\]
\[
= \frac{i}{\hbar} \delta^{-1} \left( [h^k \delta^{-1} \alpha, a^{(1)}] + [h^k \delta^{-1} \alpha, a^{(2)}] \right)
\]
\[
\tilde{a}^{(2k+3)} = a^{(2k+3)} + \rho_3^a
\]
so
\[
\rho_3^a = \delta^{-1} \partial_1 \rho_2^a
\]
\[
= \frac{i}{\hbar} \delta^{-1} \left( [h^k \delta^{-1} \alpha, a^{(1)}] + [h^k \delta^{-1} \alpha, a^{(2)}] \right.
\]
\[
+ [h^k \delta^{-1} \alpha, a^{(3)}] + \left[ \frac{i}{\hbar} \delta^{-1} [\delta^{-1} R, h^k \delta^{-1} \alpha], a^{(1)} \right]
\]
\[
+ [\delta^{-1} R, \frac{i}{\hbar} \delta^{-1} [h^k \delta^{-1} \alpha, a^{(1)}]] \right)
\]
• Since each term of $a^{(3)}$ contains three $y$'s and $\rho_1^a$ just one, $\sigma$ yields (iii) to zero.

• $a^{(2)} = \frac{1}{2} y_i y_j \partial_i \partial_j f$ so $a^{(2)}$ and $b^{(2)}$ are 2-differentiable. Then (ii) is at least 2-differentiable in one argument.

• Except a part of $\frac{1}{\hbar} \delta^{-1} [\hbar^k \delta^{-1} \partial \delta^{-1} \alpha, a^{(2)}]$ which is 2-differentiable, each term of $\rho_3^a$ contains three $y$'s. Their product with $b^{(1)}$ contain two $y$'s. So they cancel applying $\sigma$. Thus (i) gives no 1-differentiable terms.

References

[1] Bertelson M., Cahen M. and Gutt S.: “Equivalence of star products”, Classical and Quantum Gravity 14 (1997), A93-A107.

[2] Bayen F., Flato M., Fronsdal C., Lichnerowicz A. and Sternheimer D.: “Deformation theory and quantization I and II” Ann. Phys. 111 (1978), 61-151.

[3] Fedosov B.V.: “A simple geometrical construction of deformation quantization”, J. Diff. Geom. 40 (1994), 213–238.

[4] Fedosov B.V.: Deformation Quantization and Index Theory, Akademie Verlag, Berlin, 1996.

[5] Flato M., Lichnerowicz A. and Sternheimer D.: “Déformations 1-différentiables des algèbres de Lie attachées à une variété symplectique ou de contact”, Comp. Math. 31 fasc.1 (1975), 47-82.

[6] Gerstenhaber M.: “On the deformation of rings and algebras”, Ann. of Math. 79 (1964), 59-103.

[7] Gutt S. and Rawnsley J.: “Equivalence of star products on a symplectic manifold”, Preprint University of Warwick, June 1998.

[8] Kontsevich M.: “Deformation quantization of Poisson manifolds I”, q-alg/9709040.

[9] Weinstein A.: “Deformation quantization”, Exposé No. 789 in Séminaire Bourbaki, Vol. 1993/94, Astérisque vol. 227, Soc. Math. Fr. (1995).

[10] Xu P.: “Fedosov *-products and quantum momentum maps”, q-alg/9608006.