Two-loop representations of low-energy pion form factors and $\pi\pi$ scattering phases in the presence of isospin breaking

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Dispersive representations of the $\pi\pi$ scattering amplitudes and pion form factors, valid at two-loop accuracy in the low-energy expansion, are constructed in the presence of isospin-breaking effects induced by the difference between the charged and neutral pion masses. Analytical expressions for the corresponding phases of the scalar and vector pion form factors are computed. It is shown that each of these phases consists of the sum of a “universal” part and a form-factor dependent contribution. The first one is entirely determined in terms of the $\pi\pi$ scattering amplitudes alone, and reduces to the phase satisfying Watson’s theorem in the isospin limit. The second one can be sizeable, although it vanishes in the same limit. The dependence of these isospin corrections with respect to the parameters of the subthreshold expansion of the $\pi\pi$ amplitude is studied, and an equivalent representation in terms of the $S$-wave scattering lengths is also briefly presented and discussed. In addition, partially analytical expressions for the two-loop form factors and $\pi\pi$ scattering amplitudes in the presence of isospin breaking are provided.

I. INTRODUCTION

In recent years, our knowledge of low-energy pion-pion scattering has improved in a very significant way and in several respects. Firstly, the high precision $K^{\pm}\ell^{4}$ experiments performed at the BNL AGS by the E865 experiment [1, 2] and, more recently, by the NA48/2 collaboration [3, 4] at the CERN SPS, have provided very accurate determinations of the difference $\delta_0^0 - \delta_1^1$ of the pion-pion phase shifts in the $S$ and $P$ waves in the energy range between threshold and the kaon mass. Next, one should mention the measurement of the invariant mass distribution in $K^{\pm}\rightarrow\pi^{\pm}\pi^0\pi^0$ decays [5, 6], that gives information on the $S$-wave $\pi\pi$ scattering lengths $a_0^0$ (see also [8–10]). Finally, forthcoming analyses of the data collected by the NA48/2 experiment on the $K^{+}_{e4}$ decay channel into a pair of neutral pions (for preliminary reports, see [11, 12]), or on the $K^{0}_{e4}\rightarrow\pi^0\pi^0\pi^0$ decay mode [13], together with the measurement of the pionium lifetime by the DIRAC collaboration [14], should provide additional information, and might sharpen the picture even more. In the meantime, the accuracy obtained on $K^{+}_{e4}$ decays from NA48/2 implies that these data clearly drive the current determination of the difference between the $S$ and $P$ phase shifts at low energies, and in particular of the two scattering lengths $a_0^0$ and $a_2^0$, for which very accurate predictions are available [15]. This provides a particularly stringent test of two-flavour chiral perturbation theory [16], and its underlying assumptions [17].

In order to extract relevant information on low-energy pion-pion scattering from the above processes, it has become mandatory to take isospin violations into account. This is certainly quite easy to understand in the case of the $K^{\pm}\rightarrow\pi^{\pm}\pi^0\pi^0$ decay, where one exploits the presence of a unitarity cusp in the invariant $\pi^0\pi^0$ mass distribution, which occurs only if the masses of the charged and neutral pions differ [2, 18]. Perhaps somewhat more unexpectedly, isospin-violating corrections proved also of importance [19] in the analysis of the $K^{+}_{e4}$ data, in order to account for the high precision reached by the recent NA48/2 experiment, and to make comparison...
with theory meaningful [20, 21]. Actually, once isospin corrections are applied also to the E865 data, there remains a disagreement with NA48/2 [20], whose origin seems to lie in the original analysis performed by the E865 collaboration (for details, see the errata quoted under refs. [1, 2]). Anyway, the analysis of the full data set collected by NA48/2 has by now completely superseded the E865 results, and one should focus on the former to study pion-pion scattering from $K^+_{e4}$ decays.

In the present paper, we propose to address the issue of isospin-violating effects in low-energy pion-pion interactions using an approach based on a dispersive construction of the various $\pi\pi$ scattering amplitudes and pion form factors in the presence of isospin breaking. Ultimately, we wish to extend this program [22] to the $K^+_{e4}$ form factors analysed in the NA48/2 experiment. Before undertaking this enterprise and hitting the full complexity of this four-body decay, we want to demonstrate its feasibility and exhibit the general features of such a method by considering the somewhat simpler setting provided by the scalar and vector form factors of the pion.

As far as the amplitude for elastic $\pi\pi$ scattering in the isospin limit is concerned, the general framework has been laid down in ref. [23], and the explicit construction of the two-loop amplitude has subsequently been performed along these lines in detail in ref. [24]. Concerning the pion form factors, the corresponding dispersive representations in the framework of the chiral expansion have been studied in ref. [25] in the isospin limit, but only one-loop expressions were given in analytical form. Full two-loop expressions of the vector form factors have been obtained by integrating the corresponding dispersive integrals in ref. [26]. In ref. [27] the two-loop expressions of the vector and scalar form-factors have also been obtained in the absence of isospin violation by the direct evaluation of Feynman graphs generated from the effective chiral Lagrangian at next-to-next-to-leading order. A similar calculation for the pion-pion scattering amplitude in the isospin limit has been achieved in ref. [28]. Finally, let us also mention that the reconstruction theorem for elastic $\pi\pi$ scattering in the isospin limit of ref. [23] was extended by the authors of ref. [29] to the whole set of scattering amplitudes involving the mesons of the lightest pseudoscalar octet. Applications of this framework to the decay modes $P \to \pi\pi\pi$, with $P = K, \eta$, have also been considered [30, 31].

These dispersive constructions generate subtraction polynomials with unspecified coefficients. The latter are in one-to-one correspondence with the appropriate combinations of low-energy constants and chiral logarithms that would be encountered in a calculation of the corresponding Feynman diagrams generated by the chiral lagrangian. In the case of the form factors, these coefficients may be identified with their slopes and curvatures. In the case of the $\pi\pi$ scattering amplitudes, they can be expressed in terms of the subthreshold parameters occurring in the expansions of these amplitudes as Taylor series around the center of the Mandelstam triangle. This was the option considered in the isospin-symmetric case in ref. [24]. By no means, however, is this choice a necessity. It has, for instance, become customary to rather let the scattering lengths play a prominent role. They have a more direct physical interpretation than the subthreshold parameters, and are thus considered as more “experimentalist friendly”. We will therefore also provide expressions where the subtraction polynomials are given in terms of the two $S$-wave $I = 0$ and $I = 2$ scattering lengths, $a_0^0$ and $a_0^2$, in the isospin limit. In the isospin-symmetric situation, this provides an alternative to the choice made in ref. [24]. Of course, taking the expressions at two-loop order provided in the latter reference, one could convert the expressions for the $\pi\pi$ scattering amplitude given there to the one presented here in terms of the scattering lengths. The two formulations are equivalent, up to corrections that are of higher order. In the situation where isospin is broken, this allows us to discuss the size of the corresponding corrections to the phases of the form factors in terms of $a_0^0$ and $a_0^2$. This second option is of course the most interesting in the present context, where these scattering lengths are the quantities one would eventually like to determine from the data. It is thus important that the corrections due to isospin breaking are not studied for a fixed $a$ priori value for them. Indeed, given the precision reached by the latest experiments, one ought to perform a quantitative evaluation of the possible bias introduced if isospin corrections are evaluated for fixed values of these scattering lengths. This provides another motivation for the present work.

Here we will mainly concentrate on the phases of the pion form factors. Full two-loop expressions for the scattering amplitudes and form factors themselves require the evaluation of dispersion integrals corresponding to specific topologies of two-loop three-point Feynman diagrams of the non-factorizing type (“acnode” or “fish” diagrams, cf. fig. 3). Explicit analytical expressions for them do not seem to be available in the literature in
the cases where several distinct masses are present. We therefore present only partially analytical expressions for the scattering amplitudes and form factors. Note that a similar situation arises in the evaluation of the $SU(3)$ vector and scalar form factors at two loops without isospin breaking, but where the difference between the pion and kaon masses has to be dealt with. These difficulties do not show up in the computation of the phases of the two-loop form factors, where the technically most demanding step is the computation of the projections of the one-loop amplitudes on the $S$ and $P$ partial waves, which can be done analytically.

Coming now to the outline of this paper, our first purpose will be to extend the frameworks of refs. [22, 23] to the situation where the difference between the masses of charged and neutral pions is taken into account. The general framework leading to two-loop representations for form factors and scattering amplitudes when isospin symmetry no longer holds is thus described in section II In section III we implement the program of constructing the corresponding form factors and scattering amplitudes at the one-loop level and provide explicit expressions for them. The second iteration, leading to two-loop representations of the form factors and the scattering amplitudes, is discussed in section IV. The issue of isospin breaking in the phases of the two-loop form factors is addressed in section V. Section VI is devoted to the numerical analysis of the isospin-breaking contributions in the phases of the form factors. Finally, a summary of this study and our conclusions are to be found in section VII. This main body of the text is supplemented with six appendices, where details concerning more computational or technical aspects have been gathered.

II. GENERAL FRAMEWORK AND PRELIMINARY REMARKS

The objects of our study are the scalar and vector form factors of the pion, defined through the following matrix elements

\[\langle \pi^0(p_1)\pi^0(p_2)|\hat{m}(\pi u + \bar{d}d)(0)|\Omega\rangle = +F_S^0(s),\]

\[\langle \pi^+(p_+)|\pi^-(p_-)|\hat{m}(\pi u + \bar{d}d)(0)|\Omega\rangle = -F_S^0(s),\]

with \(\hat{m} \equiv (m_u + m_d)/2\), and

\[\frac{1}{2}\langle \pi^+\pi^-|(\pi\gamma_\mu u - \bar{d}\gamma_\mu d)(0)|\Omega\rangle = (p_+ - p_-)\mu F_V^0(s),\]

respectively, in the presence of isospin-breaking corrections induced by the difference in the masses of the charged and neutral pions. In each case, \(s\) denotes the squared invariant mass of the dipion system, \(s = (p_1 + p_2)^2\) or \(s = (p_- + p_+)^2\), with \(p_{1,2}^2 = M_{\pi^0}^2\), \(p_{\pm}^2 = M_{\pi^\pm}^2\), and \(|\Omega\rangle\) stands for the QCD vacuum state. The mass of the neutral pion is denoted by \(M_{\pi^0}\), while \(M_{\pi}\) stands for the mass of the charged pion. We will define the isospin limit as the case when the neutral pion mass tends to the charged pion mass, \(M_{\pi^0} \rightarrow M_{\pi}\), while keeping the latter fixed. This explains the convention that we follow in this paper, namely that all quantities without superscript refer to the charged pion case (default case), and that we refer to quantities involving neutral pions by an explicit 0 superscript. The minus sign in the definition of \(F_S^0(s)\) reflects a choice of phase for the charged-pion states. In addition, we choose the crossing phases to be \(-1\) for charged pions and \(+1\) for neutral pions. These choices are compatible with the Condon and Shortley phase convention in the isospin-symmetric situation. We further assume throughout that symmetry under charge conjugation holds.

These form factors, while being perfectly well-defined observable quantities in QCD, are however not observables from a strictly experimental point of view: they can only be measured indirectly, and should thus at best be considered as pseudo-observables. For instance, in the Standard Model, the vector form factor \(F_V^0(s)\) appears in the physical process \(e^+e^- \rightarrow \pi^+\pi^-\) through the exchange of a single neutral spin-one gauge boson, which in practice reduces to only photon exchange at low energies. As far as the scalar form factors \(F_S^{\pi^0}(s)\) and \(F_S^0(s)\) are concerned, a similar statement can in principle also be made, but is of little use in practice, since the Standard Model contributions to the processes \(e^+e^- \rightarrow \pi^+\pi^-\), \(\pi^0\pi^0\), arising from the exchange of a Higgs particle, are well below the level of sensitivity that one could expect for any experiment of this type in the foreseeable future. Despite these limitations on the experimental side, these form factors prove useful as a theoretical laboratory. They allow us to discuss and to illustrate several issues related to the structure
of isospin-breaking contributions within a rather simple context. The full complexity of experimentally more interesting situations, like the $K_{44}$ form factors or the decay amplitudes of light pseudoscalar mesons (eta or kaons) into three pions, can then be addressed on the basis of these considerations and the general framework developed here, see [22] and the forthcoming publication [30] in the former or the latter case, respectively.

In the present section, we aim at tackling two issues, namely discussing precisely the isospin contributions that we intend to deal with, and describing our general theoretical framework. Then we can focus on the specific pion form factors that we use as an illustration.

A. Electromagnetic corrections

At the fundamental level, isospin violations have two origins within the Standard Model: the electroweak interactions mediated by the gauge bosons, and the quark mass difference $m_u - m_d$, arising through the coupling of the light $u$ and $d$ quark flavours to the Higgs boson. Both effects contribute to the mass difference between charged and neutral pions, although the second one turns out to be marginal: the pion mass difference is mainly an electromagnetic effect [34, 35].

Chiral perturbation theory [16, 36] including electromagnetism [37–42] provides in principle a suitable framework to deal with these isospin-breaking contributions in the low-energy domain. It has been applied to the computation of several quantities at the one-loop level, including the $\pi\pi$ scattering amplitudes [40, 41, 43] and pion form factors [44] in the two-flavour case. Unfortunately, from a practical point of view, this is not a level of accuracy able to match the experimental one in several cases of interest (low-energy pion-pion scattering or $K_{44}$ decay, for instance). One might of course contemplate the extension of this effective Lagrangian framework to next-to-next-to-leading order, but this is a more ambitious program, the interest of which might be limited eventually by the proliferation of low-energy constants. We will therefore not pursue this issue here.

Instead, we will rather consider the point of view described in refs. [19, 21]: we thus assume a situation, as is, actually, often the case in the analyses of experimental data (such as for E865 and NA48/2), where part of the radiative corrections due to real and virtual corrections have already been dealt with in some manner, while those that may remain are supposed to be negligible. In this kind of procedure, radiative corrections of the type shown on the left-hand side of fig. 1 for instance, together with emission of soft photons, are subtracted away, but other photonic effects, like the one shown on the right-hand side of the same figure, are not taken into account, but are considered to be negligible. Notice that the latter would be included in a genuine two-loop calculation in the framework of the QCD+QED effective theory. One might also think of considering the possibility of treating them in the dispersive framework that we are using here, for instance upon including also photons among the possible intermediate states in the unitarity conditions for the relevant partial waves. While this remains an interesting issue, it would however lead us beyond the purposes of the present work. Within the framework assumed here, this leaves the difference in the pion masses as the only remaining source of isospin breaking that we have to consider. In practice, it means that the charged and neutral pion masses will be kept at their experimental values, but the origin of the difference in their masses will not be addressed. In other words, we assume that general properties like analyticity, unitarity, and crossing, together with chiral counting, can be used to describe a world where the charged and neutral pion masses differ, even though the interaction at the origin of this difference is not explicitly accounted for.
B. Dispersive construction of the form factors

The starting point of our study is provided by the dispersion relations satisfied by the pion form factors and scattering amplitudes. Here we will only be interested in the structure of these quantities in the low-energy region. In order to obtain dispersive representations that agree with their analytic structures up to two loops in the low-energy expansion, it is convenient to consider thrice subtracted dispersion relations (for a discussion of this issue in the isospin limit, see e.g. ref. 22 for the pion form factors, and ref. 23 for the \( \pi\pi \) scattering amplitude). Let us start with the form factors, for which these dispersive representations read

\[
F_S^{\pi^0}(s) = F_S^{\pi^0}(0) \left[ 1 + \frac{1}{6} (r^2)^S s + c_S^0 s^2 + U_S^{\pi^0}(s) \right]
\]

\[
F_S^\pi(s) = F_S^\pi(0) \left[ 1 + \frac{1}{6} (r^2)^S s + c_S^\pi s^2 + U_S^\pi(s) \right]
\]

\[
F_V^\pi(s) = 1 + \frac{1}{6} (r^2)^V s + c_V^\pi s^2 + U_V^\pi(s).
\]

In the last of these relations, the condition \( F_V^\pi(0) = 1 \), due to the conservation of the electromagnetic current and thus valid to all orders, has been used. Through crossing, \( F_S^{\pi^0}(0) \) and \( F_S^\pi(0) \) are equal to the corresponding sigma-term type form-factors, \( \langle \pi^0(p)\bar{m}(\vec{u} + \vec{d})|0|\pi^0(p) \rangle \) and \( \langle \pi^\pm(p)\bar{m}(\vec{u} + \vec{d})|0|\pi^{\pm}(p) \rangle \), respectively, for which there also exist relations [45] valid to all orders, that follow from the Feynman-Hellmann theorem [46],

\[
F_S^{\pi^0}(0) = \frac{\partial M_{\pi^0}^2}{\partial m}, \quad F_S^\pi(0) = \frac{\partial M_{\pi}^2}{\partial m}.
\]

Since the dominant contribution to the pion mass difference is purely of electromagnetic origin and independent of the quark masses [47], one has

\[
\frac{F_S^{\pi^0}(0)}{F_S^\pi(0)} = 1 + \ldots,
\]

where the ellipsis denotes higher order terms. The unitarity parts are given in terms of dispersion integrals,

\[
U_S^{\pi^0}(s) = \frac{s^3}{\pi} \int \frac{dx}{x^3} \frac{\text{Im} F_S^{\pi^0}(x) / F_S^{\pi^0}(0)}{x - s - i0}
\]

\[
U_S^\pi(s) = \frac{s^3}{\pi} \int \frac{dx}{x^3} \frac{\text{Im} F_S^\pi(x) / F_S^\pi(0)}{x - s - i0}
\]

\[
U_V^\pi(s) = \frac{s^3}{\pi} \int \frac{dx}{x^3} \frac{\text{Im} F_V^\pi(x)}{x - s - i0}.
\]

In the low-energy region, the form factors are analytical functions in the complex \( s \)-plane, except for cut singularities on the positive real axis, starting at \( s = 4M_{\pi^0}^2 \) in the case of \( F_V^\pi(s) \), and at \( s = 4M_\pi^2 \) in the cases of \( F_S^{\pi^0}(s) \) and \( F_S^{\pi^\pm}(s) \). For \( s \) real and below these cuts, the form factors are real. In the chiral expansion, the form factors behave dominantly as

\[
\text{Re} F_S^{\pi^0}(s) \sim \mathcal{O}(E^2), \quad \text{Im} F_S^{\pi^0}(s) \sim \mathcal{O}(E^4),
\]

\[
\text{Re} F_V^\pi(s) \sim \mathcal{O}(E^0), \quad \text{Im} F_V^\pi(s) \sim \mathcal{O}(E^2),
\]

where \( E \) denotes either a pion momentum or a pion mass. Furthermore, intermediate states with more than two pions contribute only from the three-loop level onwards. Therefore, below the thresholds involving other states than the pions, and up to and including two loops in the two-flavour chiral expansion, only discontinuities arising from two-pion intermediate states need to be retained [24], as illustrated in fig. 2. In order to distinguish among the different \( \pi\pi \) scattering channels, we use the following superscripts: 00 for \( \pi^0\pi^0 \to \pi^0\pi^0 \), + for \( \pi^+\pi^- \to \pi^+\pi^- \), ++ for \( \pi^+\pi^- \to \pi^+\pi^- \), ++- for \( \pi^+\pi^- \to \pi^+\pi^- \), +0 for \( \pi^+\pi^- \to \pi^+\pi^- \), 00 for \( \pi^0\pi^0 \to \pi^+\pi^- \), and \( x \) for the inelastic channels \( \pi^0\pi^0 \to \pi^+\pi^- \) and \( \pi^+\pi^- \to \pi^0\pi^0 \). We have

\[
\text{Im} F_S^{00}(s) = \text{Re} \left\{ \frac{1}{2} \sigma_0(s) f_0^{00}(s) F_S^{00}(s) \theta(s - 4M_{\pi^0}^2) - \sigma(s) f_0^+(s) F_S^+(s) \theta(s - 4M_{\pi}^2) \right\} + \mathcal{O}(E^8)
\]
processes involving four particles with the same mass, it is simply given by
\[ \text{Im} F^{\pi}(s) = \text{Re} \left\{ \sigma(s) f_0(s) F^{\pi}(s) \theta(s - 4M^2_{\pi}) - \frac{1}{2} \sigma_0(s) f_0(s) F^{\pi}(s) \theta(s - 4M^2_{\pi}) \right\} + \mathcal{O}(E^8) \]
\[ \text{Im} F^{\pi}(s) = \text{Re} \left\{ \sigma(s) f_0(s) F^{\pi}(s) \theta(s - 4M^2_{\pi}) \right\} + \mathcal{O}(E^6), \]
where we define the phase-space functions
\[ \sigma_0(s) = \sqrt{1 - \frac{4M^2_{\pi}}{s}}, \quad \sigma(s) = \sqrt{1 - \frac{4M^2_{\pi}}{s}}. \]

In these expressions, \( f_0(s) \) and \( f_1(s) \) denote the \( S \) and \( P \) partial waves, respectively, of the \( \pi\pi \) scattering amplitudes \( A(s, t) \) in the corresponding channels. These partial waves have been defined as usual by projections of the corresponding amplitudes,
\[ f_\ell(s) = \frac{1}{32\pi} \int_{-1}^{+1} dz A(s, t) P_\ell(z), \]
with \( P_0(z) = 1 \) and \( P_1(z) = z \) the appropriate Legendre polynomials, and \( z = \cos \theta \), where \( \theta \) denotes the scattering angle in the center-of-mass frame. The relation with the Mandelstam variables \( s, t, u \) (summing up to the squared masses of the incoming and outgoing particles) depends on the process under consideration. For processes involving four particles with the same mass, it is simply given by
\[ t = \frac{s - 4M^2_{\pi}}{2} (1 - z). \]

This is, in particular, the case for \( A^{00}(s, t) \) (with \( M = M_{\pi^0} \)), as well as for \( A^{+-}(s, t) \) and for \( A^{--}(s, t) \) (with now \( M = M_{\pi} \)). In the reactions involving both charged and neutral pions, it becomes
\[ t = -\frac{1}{2} (s - 2M^2_{\pi^0} - 2M^2_{\pi}) + \frac{z}{2} \sqrt{(s - 4M^2_{\pi^0})(s - 4M^2_{\pi})} \]
for \( A^\pi(s, t) \), and
\[ t = -\frac{\lambda(s)}{2s} (1 - z), \quad \lambda(s) = [s - (M_{\pi} + M_{\pi^0})^2][s - (M_{\pi} - M_{\pi^0})^2]. \]

for \( A^0(s, t) \). These expressions hold above the kinematical threshold, \( s \geq 4M^2_{\pi} \) for \( A^\pi(s, t) \), and \( s \geq (M_{\pi} + M_{\pi^0})^2 \) for \( A^0(s, t) \), for instance. Let us point out that the normalization in (II.10) differs from the usual definition of the \( \pi\pi \) partial-wave amplitudes by a factor of 2: it would correspond to a decomposition of the scattering amplitudes given by
\[ A(s, t) = 16\pi \sum_{\ell \geq 0} (2\ell + 1) P_\ell(\cos \theta) f_\ell(s), \]
i.e. with the normalization factor 16\( \pi \) instead of the usual 32\( \pi \). This modification is motivated by the fact that, in the presence of isospin breaking, the two-pion states do no longer obey (generalized) Bose symmetry, except in the cases of two neutral pions, or of two identically charged pions. If these cases, the appropriate symmetry factor has been included instead in the expressions of the corresponding phase spaces, cf. eq. (II.8). In the chiral expansion, the \( \pi\pi \) partial waves behave as
\[ \text{Re} f_\ell(s) \sim \mathcal{O}(E^2), \quad \text{Im} f_\ell(s) \sim \mathcal{O}(E^4), \quad \ell = 0, 1, \]
\[ \text{Re} f_\ell(s) \sim \mathcal{O}(E^4), \quad \text{Im} f_\ell(s) \sim \mathcal{O}(E^8), \quad \ell \geq 2. \]
This is in perfect agreement with the chiral counting (II.7) of the form factors, together with the expressions (II.8) of their low-energy discontinuities.

C. Dispersive construction of $\pi\pi$ scattering amplitudes

It is actually possible to obtain dispersive representations for the two-loop $\pi\pi$ scattering amplitudes themselves, following the same procedure as for the form factors. As in the isospin-symmetric case [22], they follow from fixed-$t$ dispersion relations, combined with very general properties like relativistic invariance, unitarity, analyticity, crossing, and from the chiral counting properties that have just been recalled. Isospin breaking is not expected to modify the asymptotic high-energy behaviour of the amplitudes, so that two subtractions should be enough in order to obtain convergent dispersion relations. We start with three subtractions in order to construct low-energy expressions of the scattering amplitudes that are valid up to and including two loops in the chiral expansion.

Whereas the various channels can all be described in terms of a single amplitude $A(s,t,u)$ as long as isospin symmetry holds, several independent amplitudes, not related by crossing, are necessary in order to deal with all the different available channels once isospin is broken. Otherwise, the derivation proceeds as in the isospin-symmetric case [48], up to the kinematical peculiarities due to the presence of particles with different masses. The relevant features can be inferred from the discussion of ref. [29], devoted to the extension of the isospin-symmetric case [48], up to the kinematical peculiarities due to the presence of particles with different masses. We will therefore directly write down the resulting expressions, and then provide a few additional comments on their structure.

i) Elastic scattering involving only neutral pions remains the simplest case, with a single, fully crossing invariant amplitude, which has the following two-loop structure (for convenience, we display, from now on, the dependence on the three Mandelstam variables $s,t,u$, although they are not independent)

$$A^{00}(s,t,u) = P^{00}(s,t,u) + W^{00}_0(s) + W^{00}_0(t) + W^{00}_0(u) + \mathcal{O}(E^8).$$  \hfill (II.16)

It involves a polynomial $P^{00}(s,t,u)$ of third order in $s,t,u$, symmetric under any permutation of its variables, together with a dispersive integral $W^{00}(s)$. This function has a discontinuity on the positive real $s$-axis starting at $s = 4M_{\pi^0}^2$, and specified by the $\ell = 0$ partial-wave amplitude $f^{00}_0(s)$,

$$\text{Im}W^{00}_0(s) = 16\pi \text{Im}f^{00}_0(s) \theta(s - 4M_{\pi^0}^2).$$  \hfill (II.17)

Again, at the order under consideration, this discontinuity is provided by the unitarity condition, in terms of the $\ell = 0$ partial waves in the relevant channels, $f^{00}_0(s)$ and $f^{00}_0(s)$,

$$\frac{1}{16\pi} \text{Im}W^{00}_0(s) = \frac{1}{2} \sigma_0(s) |f^{00}_0(s)|^2 \theta(s - 4M_{\pi^0}^2) + \sigma(s) |f^{00}_0(s)|^2 \theta(s - 4M_{\pi^0}^2) + \mathcal{O}(E^8).$$  \hfill (II.18)

ii) The processes involving exactly two neutral pions, i.e. $\pi^+\pi^- \rightarrow \pi^+\pi^0$ and $\pi^+\pi^- \rightarrow \pi^0\pi^0$, provide the next family of amplitudes that are related under crossing. They display the following structure at two loops in the chiral expansion:

$$A^\pi(s,t,u) = -P^\pi(s,t,u) - W^-_0(s) - [W^+^0_0(t) + 3(s-u)W^+_1^0(t)] - [W^+^0_0(u) + 3(s-t)W^+_1^0(u)] + \mathcal{O}(E^8),$$  \hfill (II.19)

whereas $A^{+0}(s,t,u) = -A^\pi(t,s,u)$ through crossing. In the above expression, $P^\pi(s,t,u)$ represents a polynomial of third order in the Mandelstam variables, symmetric under exchange of $t$ and $u$ (Bose symmetry). The functions $W^-_0(s)$ and $W^-_1(s)$ have discontinuities on the positive real $s$-axis, starting at $s = (M_{\pi^+} + M_{\pi^0})^2$ and at $s = 4M_{\pi^0}^2$, respectively. These discontinuities are again given in terms of the appropriate lowest ($S$ and $P$) $\pi\pi$ partial waves

$$\text{Im}W^{+0}_0(s) = 16\pi \left[ \text{Im}f^{+0}_0(s) + \frac{3(M_{\pi^+}^2 - M_{\pi^0}^2)^2}{\lambda(s)} \text{Im}f^{+0}_1(s) \right] \theta(s - (M_{\pi^+} + M_{\pi^0})^2)$$

$$\text{Im}W^{+0}_1(s) = \frac{s}{\lambda(s)} \text{Im}f^{+0}_1(s) \theta(s - (M_{\pi^+} + M_{\pi^0})^2)$$

$$\text{Im}W^-_0(s) = -16\pi \text{Im}f^{00}_0(s) \theta(s - 4M_{\pi^0}^2).$$  \hfill (II.20)
Up to higher-order contributions, the unitarity condition allows to rewrite these expressions in terms of the same lowest partial waves. In the elastic channel, there is only one contribution, arising from the $\pi^+\pi^0$ intermediate state, whereas the inelastic channel involves two contributions:

\[
\frac{1}{16\pi} \Im W_0^{+0}(s) = \left[ \frac{\lambda_1/2(s)}{s} \right] f_0^{+0}(s)^2 + 3 \frac{(M_2^2 - M_\pi^0)^2}{s\lambda_1/2(s)} f_1^{+0}(s)^2 \theta(s - (M_\pi + M_\pi^0)^2) + \mathcal{O}(E^8)
\]

\[
\frac{1}{16\pi} \Im W_1^{+0}(s) = \frac{1}{\lambda_1/2(s)} f_1^{+0}(s)^2 \theta(s - (M_\pi + M_\pi^0)^2) + \mathcal{O}(E^8)
\]

\[
\frac{1}{16\pi} \Im W_0^{++}(s) = -\frac{1}{2} \sigma_0(s) f_0^{++}(s) [f_0^{+0}(s)]^* \theta(s - 4M_\pi^2) - \sigma(s) f_0^{++}(s) [f_0^{+0}(s)]^* \theta(s - 4M_\pi^2) + \mathcal{O}(E^8). \tag{II.21}
\]

\[\text{ii) Finally, the subset of elastic scattering processes involving only charged pions remains to be considered. Their amplitudes being all related by crossing, it is enough to display explicitly one of them, for instance,}
\]

\[A^{+-}(s,t,u) = P^{+-}(s,t,u) + [W_0^{+}(s) + 3(t-u)W_1^{+}(s)] + [W_0^{-}(t) + 3(s-u)W_1^{-}(t)] + W_0^{++}(u) + \mathcal{O}(E^8), \tag{II.22}
\]

where the third order polynomial $P^{+-}(s,t,u)$ is symmetric under exchange of $s$ and $t$ (Bose symmetry in the crossed $u$-channel). The three functions $W_{0,1}^{+-}(s)$ and $W_0^{++}(s)$ have cut singularities along the real $s$ axis, starting at $s = 4M_\pi^2$ or at $s = 4M_\pi^2$. The corresponding discontinuities read

\[\Im W_0^{+}(s) = 16\pi \Im f_0^{+0}(s) \theta(s - 4M_\pi^2)\]

\[\Im W_1^{+}(s) = 16\pi \Im f_1^{+0}(s) \theta(s - 4M_\pi^2)\]

\[\Im W_0^{++}(s) = 16\pi \Im f_0^{++}(s) \theta(s - 4M_\pi^2). \tag{II.23}\]

The unitarity condition for the three $\pi\pi$ partial waves involved then leads to

\[\frac{1}{16\pi} \Im W_0^{-}(s) = \sigma(s) |f_0^{-0}(s)|^2 \theta(s - 4M_\pi^2) + \frac{1}{2} \sigma_0(s) |f_0^{+0}(s)|^2 \theta(s - 4M_\pi^2) + \mathcal{O}(E^8)\]

\[\frac{1}{16\pi} \Im W_1^{-}(s) = \sigma(s) |f_1^{-0}(s)|^2 \theta(s - 4M_\pi^2) + \mathcal{O}(E^8)\]

\[\frac{1}{16\pi} \Im W_0^{++}(s) = \frac{1}{2} \sigma(s) |f_0^{++}(s)|^2 \theta(s - 4M_\pi^2) + \mathcal{O}(E^8). \tag{II.24}\]

The reason why only the lowest $S$ and $P$ partial waves play a role in these expressions follows again from the chiral counting [II.15] for the partial waves. In the following, we will make use of the chiral expansion for the real parts of the $\ell = 0, 1$ partial waves, for values of $s$ corresponding to the cut along the positive real axis,

\[\text{Re} f_\ell(s) = \varphi_\ell(s) + \psi_\ell(s) + \mathcal{O}(E^6), \tag{II.25}\]

with $\varphi_\ell(s) \sim \mathcal{O}(E^2)$ and $\psi_\ell(s) \sim \mathcal{O}(E^4)$, so that

\[|f_\ell(s)|^2 = |\text{Re} f_\ell(s)|^2 + \mathcal{O}(E^8) = |\varphi_\ell(s)|^2 + 2\varphi_\ell(s)\psi_\ell(s) + \mathcal{O}(E^8), \quad \ell = 0, 1. \tag{II.26}\]

Let us also emphasize that the functions $W(s)$ only have a right-hand cut, that coincides with the right-hand cut of the corresponding $S$ and $P$ $\pi\pi$ partial-wave projections [24]. This structure is in agreement with the analyticity properties of the $\pi\pi$ scattering amplitudes $A(s,t,u)$ required by unitarity and crossing. The decompositions (II.16, II.19), and (II.22) satisfy these constraints, to the given order in the low-energy expansion. Of course, the partial-wave amplitudes have a more complicated analytical structure, coming from the projection in eq. (II.10), with also a left-hand cut, and even a circular cut in the case of the $\pi^+\pi^0 \rightarrow \pi^+\pi^0$ channel (for a description of the analytic structure of partial-wave amplitudes in a general context, see e.g. [49]). At this stage we should also stress that a full partial-wave decomposition (II.14) of the $\pi\pi$ amplitudes is actually not required. For our purposes, it is sufficient to know that the discontinuity of the latter in the complex $s$-plane can be written, in the low-energy region of interest here, as

\[\text{Im} A(s,t) = 16\pi [\text{Im} f_0(s) + 3z\text{Im} f_1(s)] + \Phi_{\ell \geq 2}(s,t), \tag{II.27}\]

with $\Phi_{\ell \geq 2}(s,t) \sim \mathcal{O}(E^8)$ as its dominant chiral behaviour.
The very general features and the results that have just been presented are at the basis of the construction of two-loop representations of the pion form factors and scattering amplitudes in the low-energy regime. This construction is achieved through a two-step recursive process of which we now give a short outline, summarised also in fig. [3]. Chiral counting provides the initial information, namely that at lowest order the form factors reduce to real constants, to be identified with their values at $s = 0$, while the $\pi\pi$ scattering amplitudes consist of $O(E^2)$ polynomials of at most first order in the Mandelstam variables. This initial input, together with unitarity, fixes the discontinuities of the form factors and of the amplitudes, through the expressions of the functions $\varphi_\ell(s)$ at next-to-leading order. The complete one-loop expressions are then recovered up to a subtraction polynomial of at most first order (second order) in $s$ (in the Mandelstam variables) in the case of the form factors (of the scattering amplitudes). In turn, these one-loop expressions then provide the discontinuities at next-to-next-to-leading order, and thus the form factors (and amplitudes) themselves at order $O(E^6)$, up to a polynomial ambiguity of second order in $s$ (third order in the Mandelstam variables). In the case of the $\pi\pi$ scattering amplitudes, crossing imposes further restrictions on the possible terms that may appear in these polynomials. Notice that the presence of these polynomials reflect the fact that the functions $W_0^0(s)$, $W_{0,1}^0(s)$, etc. are only specified by their analytical properties, in particular as far as their discontinuities are concerned. This leaves room for polynomial ambiguities in the expressions of these functions, and the maximal degree of the polynomials is then limited by chiral power counting.

This second iteration relies on the possibility to obtain analytical expressions for the $O(E^4)$ pieces $\psi_\ell(s)$ of the real parts of the lowest partial waves from the one-loop $\pi\pi$ amplitudes, whose structures are no longer polynomial. This represents the technically most demanding step. The following sections are therefore devoted to the detailed implementation of this program. Since our main interest lies in discussing the effects of isospin breaking on the phases of the pion form factors, we will provide explicit two-loop expressions for the latter only. This means that we will stop in the middle of the second iteration of the recursive procedure. Completing this second iteration would provide full two-loop expressions for the form factors, and not only that of their imaginary part as needed for the phase shifts. This in turn would require one to obtain analytical expressions for the corresponding dispersion integrals, a daunting task as already explained in the introduction. We therefore defer this remaining step to future work.

D. The subtraction polynomials: subthreshold parameters vs. scattering lengths

The two-loop dispersive construction provides representations of the $\pi\pi$ scattering amplitudes that involve subtraction polynomials of at most third order in the Mandelstam variables. These polynomials depend on a certain number of parameters that are not fixed by the general properties, listed at the beginning of subsection [II C], on which the representations for the $\pi\pi$ scattering amplitudes rest. Furthermore, beyond general constraints coming, for instance, from the crossing property, there is, of course, a certain degree of arbitrariness in the form
of these polynomials, and thus on the physical meaning of the corresponding coefficients.

In the isospin-symmetric case treated in refs. \cite{23,24}, the form of the subtraction polynomial \( P(s|t,u) \) was chosen so that some of its coefficients were identified with the subthreshold parameters of the amplitudes (the coefficients of its Taylor expansion around the center of the Mandelstam triangle). Among other reasons, this was motivated by the fact that the chiral expansions of these quantities show better convergence properties than, for instance, the scattering lengths. The latter can then be obtained from their expressions in terms of these subthreshold parameters (see, e.g., the corresponding two-loop expressions in Appendix B of ref. \cite{24} and the discussion in ref. \cite{17}). Along the same line of thought, in the presence of isospin breaking the most general subtraction polynomials of third order in the Mandelstam variables, and compatible with the symmetries of the amplitudes under crossing, can then be written as

\[
P^{00}(s,t,u) = \frac{\alpha_{00} M_{a}^{2}}{F_{a}^{2}} \left[ (s-2M_{a}^{2}) + (t-2M_{a}^{2}) + (u-2M_{a}^{2}) \right]
\]

\[
+ \frac{3\lambda_{00}^{(1)}}{F_{a}^{4}} \left[ (s-2M_{a}^{2}) + (t-2M_{a}^{2}) + (u-2M_{a}^{2}) \right]
\]

\[
+ \frac{3\lambda_{00}^{(2)}}{F_{a}^{6}} \left[ (s-2M_{a}^{2}) + (t-2M_{a}^{2}) + (u-2M_{a}^{2}) \right]
\]

\[
P^{2}(s,t,u) = \frac{\beta_{s}}{F_{a}^{2}} \left[ s - \frac{2}{3} M_{a}^{2} - \frac{2}{3} M_{a}^{2} \right] + \frac{\alpha_{s} M_{a}^{2}}{3F_{a}^{2}}
\]

\[
+ \frac{\lambda_{s}^{(1)}}{F_{a}^{4}} \left[ (s-2M_{a}^{2}) + (t-2M_{a}^{2}) + (u-2M_{a}^{2}) \right]
\]

\[
+ \frac{\lambda_{s}^{(2)}}{F_{a}^{6}} \left[ (s-2M_{a}^{2}) + (t-2M_{a}^{2}) + (u-2M_{a}^{2}) \right]
\]

\[
P^{+-}(s,t,u) = \frac{\beta_{s+} M_{a}^{2}}{F_{a}^{2}} \left[ s + t - \frac{8}{3} M_{a}^{2} \right] + \frac{2\alpha_{s+} M_{a}^{2}}{3F_{a}^{2}}
\]

\[
+ \frac{\lambda_{s+}^{(1)}}{F_{a}^{4}} \left[ (s-2M_{a}^{2}) + (t-2M_{a}^{2}) + (u-2M_{a}^{2}) \right]
\]

\[
+ \frac{\lambda_{s+}^{(2)}}{F_{a}^{6}} \left[ (s-2M_{a}^{2}) + (t-2M_{a}^{2}) + (u-2M_{a}^{2}) \right]
\]

\[
+ \frac{\lambda_{s+}^{(3)}}{F_{a}^{6}} \left[ (s-2M_{a}^{2}) + (t-2M_{a}^{2}) + (u-2M_{a}^{2}) \right]
\]

\[
+ \frac{\lambda_{s+}^{(4)}}{F_{a}^{6}} \left[ (s-2M_{a}^{2}) + (t-2M_{a}^{2}) + (u-2M_{a}^{2}) \right].
\]

(II.28)

In each of these polynomials, the first line is of at most first order in \( s, t, \) and \( u, \) and corresponds to the tree-level amplitudes. The second line corresponds to the subtraction terms at one-loop level: to construct the scattering amplitudes at one-loop precision, it is enough to consider only twice-subtracted dispersion relations \cite{23,24}.

On the other hand, it has become customary to rather let the subtraction lengths play a prominent role, since they are usually considered to have a more direct physical interpretation than subthreshold parameters. The point we would like to stress here is that the framework developed in refs. \cite{23,24}, and that we extend in the present work to the isospin-violating situation, is rather flexible from this point of view, and can accommodate several choices of parameters, according to one’s needs or purposes. It is simply a matter of appropriately choosing the forms of the lowest-order amplitudes and of the subtraction polynomials introduced at each of the two iterations. Different sets of parameters can be related, order by order in the chiral expansion, and the corresponding two-loop amplitudes differ only by higher-order terms. In the present article, we will use the representation in terms of subthreshold parameters, with the subtraction polynomials given in eq. (II.28) above. In appendix \( \text{F} \), we provide the corresponding expressions in terms of the scattering lengths, and give an outline of how the present analysis can be reformulated in terms of the latter quantities. For more details, we refer the reader to our forthcoming work \cite{23}.

Whatever choice is eventually considered, these polynomials altogether depend on fifteen independent subtraction constants. In the isospin limit, only six independent subtraction constants are required: isospin symmetry induces nine linear relations among these constants. These relations can be summarised by the
statements that, as \( M_{\pi^0} \rightarrow M_{\pi} \), one has

\[
P^{00}(s, t, u) \rightarrow P(s|t, u) + P(t|s, u) + P(u|s, t)
\]

\[
P^x(s, t, u) \rightarrow P(s|t, u)
\]

\[
P^{+-}(s, t, u) \rightarrow P(s|t, u) + P(t|s, u)
\]

(II.29)

where

\[
P(s|t, u) = \frac{\beta}{F^x_\pi} \left( s - \frac{4}{3} M^2_\pi \right) + \frac{\alpha M^2_\pi}{3 F^x_\pi} + \frac{\lambda_1}{F^x_\pi} (s - 2M^2_\pi)^2 + \frac{\lambda_2}{F^x_\pi} \left[ (t - 2M^2_\pi)^2 + (u - 2M^2_\pi)^2 \right]
\]

\[
+ \frac{\lambda_3}{F^x_\pi} (s - 2M^2_\pi)^3 + \frac{\lambda_4}{F^x_\pi} \left[ (t - 2M^2_\pi)^3 + (u - 2M^2_\pi)^3 \right]
\]

(II.30)

is the subtraction polynomial for the isospin-symmetric scattering amplitude \( A(s|t, u) \), cf. reference [24]. Some relations between these subtraction constants are given explicitly below [see, for instance, section III A and the end of section III C].

### III. FIRST ITERATION: ONE-LOOP EXPRESSIONS

In this section, we discuss the pion form factors and the \( \pi\pi \) scattering amplitudes at leading order, and then proceed with the construction of the corresponding one-loop expressions along the lines described above.

#### A. Leading-order form factors and \( \pi\pi \) amplitudes

At lowest order in the chiral expansion, the form factors are constants, that may be identified with their values at \( s = 0 \), \( F^x_\pi(0) \), \( F^y_0(0) \), and \( F^y_1(0) = 1 \).

At the same order, the \( \pi\pi \) scattering amplitudes in the relevant channels read [cf. eq. (II.28)]

\[
A^x(s, t) = -\frac{\beta_x}{F_\pi} \left( s - \frac{2}{3} M^2_\pi - \frac{2}{3} M^2_{\pi^0} \right) - \frac{\alpha_x M^2_{\pi^0}}{3 F^x_\pi}
\]

\[
A^{+-}(s, t) = \frac{\beta_+}{F^x_\pi} \left( s + t - \frac{8}{3} M^2_\pi \right) + \frac{2\alpha_+ M^2_{\pi^0}}{3 F^x_\pi}
\]

\[
A^{00}(s, t) = \frac{\alpha_{00} M^2_{\pi^0}}{F^x_\pi}.
\]

(III.1)

From these amplitudes, the partial wave projections are obtained as \( \text{Re} f_{\ell}(s) = \varphi_{\ell}(s) + \mathcal{O}(E^4) \), \( \ell = 0, 1 \), with

\[
\varphi_{\ell}^x(s) = -\frac{\beta_x}{16\pi F_\pi} \left( s - \frac{2}{3} M^2_\pi - \frac{2}{3} M^2_{\pi^0} \right) - \frac{\alpha_x M^2_{\pi^0}}{48\pi F^x_\pi}
\]

\[
\varphi_{\ell}^{+-}(s) = \frac{\beta_+}{32\pi F^x_\pi} \left( s - \frac{4}{3} M^2_\pi \right) + \frac{\alpha_+ M^2_{\pi^0}}{24\pi F^x_\pi}
\]

\[
\varphi_{\ell}^{++}(s) = \frac{\beta_+}{96\pi F^x_\pi} (s - 4M^2_\pi)
\]

\[
\varphi_{\ell}^{00}(s) = \frac{\alpha_{00} M^2_{\pi^0}}{16\pi F^x_\pi}.
\]

(III.2)

The various parameters, like \( \beta_x \) or \( \alpha_{00} \), that occur in these expressions are free, in the sense that they are not fixed by the general principles (analyticity, unitarity, crossing, and chiral symmetry). The occurrence, in these expressions, of \( M^2_{\pi^0} \) rather than \( M^2_\pi \) in the terms proportional to \( \alpha_x, \alpha_{+-} \), or \( \alpha_{00} \) is a pure matter of choice, and can be considered as part of the definition of these parameters. The presence, in the denominator, of \( F_\pi \), the pion decay constant in the isospin limit, is likewise a matter of convention.
As discussed in subsection (II.D) in the isospin limit \( \alpha_x, \alpha_{+-}, \) and \( \alpha_{00} \) take a common value \( \alpha \). Similarly, the parameters \( \beta_x \) and \( \beta_- \) become equal to the same quantity \( \beta \) in this limit. Let us notice that there is no analogous quantity \( \beta_{00} \) in the case of elastic \( \pi^0\pi^0 \) scattering, due to the Bose symmetry and the identity \( s + t + u = 4M^2_{\pi} \). Both \( \alpha \) and \( \beta \) were introduced in ref. [23]. They remain finite in the chiral limit, and describe the \( I = 0 \) and \( I = 2 \) \( S \)-wave scattering lengths \( a^0_0 \) in the isospin limit [23, 24] at lowest order,

\[
a^0_0 = \frac{M^2_\pi}{96\pi F^2_\pi} (5\alpha + 16\beta), \quad a^0_3 = \frac{M^2_\pi}{48\pi F^2_\pi} (\alpha - 4\beta).
\]  

On the other hand, the lowest-order \( S \)-wave scattering lengths corresponding to the amplitudes \( A^x(s,t), A^{+-}(s,t), \) and \( A^{00}(s,t) \) were computed in ref. [40] and read

\[
a^x_0 = \frac{2}{3} (-a^0_0 + a^0_2) - (4\beta - \alpha) \frac{M^2_\pi - M^2_{\pi^0}}{48\pi F^2_\pi},
\]

\[
a^{+-}_0 = \frac{1}{3} (2a^0_0 + a^0_2) + (4\beta - \alpha) \frac{M^2_\pi - M^2_{\pi^0}}{24\pi F^2_\pi},
\]

\[
a^{00}_0 = \frac{2}{3} (a^0_0 + 2a^0_2) - \alpha \frac{M^2_\pi - M^2_{\pi^0}}{16\pi F^2_\pi}.
\]

As compared to ref. [40], where \( \beta = 1 \) was taken at lowest order, we have kept the dependence with respect to \( \beta \) in the isospin-violating correction terms. These expressions also account for the difference in the normalization of the partial-wave amplitudes as compared to [40], see eq. (II.14). Upon comparing these formulae with the expressions of the scattering lengths computed directly from the amplitudes displayed in (III.1), we obtain the following identifications:

\[
\beta_x = \beta_- = \beta = \alpha + 2\beta \frac{M^2_\pi - M^2_{\pi^0}}{M^2_{\pi^0}}, \quad \alpha_{+-} = \alpha + \frac{4\beta}{\alpha} \frac{M^2_\pi - M^2_{\pi^0}}{M^2_{\pi^0}}, \quad \alpha_{00} = \alpha.
\]

At higher orders, and in the absence of isospin symmetry, all these coefficients become independent, and these simple expressions receive additional contributions. The computation of the corresponding isospin-breaking corrections at next-to-leading order will be addressed below, see subsection (VI.A) and appendix (F).

Finally, let us recall, from ref. [40], that the lowest-order \( \pi\pi \) amplitudes (III.1) take the form

\[
A^x(s,t) = -A(s|t,u) \quad [s + t + u = 2M^2_{\pi^0} + 2M^2_\pi],
\]

\[
A^{+-}(s,t) = A(s|t,u) + A(t|s,u) \quad [s + t + u = 4M^2_\pi],
\]

\[
A^{00}(s,t) = A(s|t,u) + A(t|u,s) + A(u|s,t) \quad [s + t + u = 4M^2_{\pi^0}],
\]

with

\[
A(s|t,u) = \frac{s - 2\hat{n}B}{F^2},
\]

and one has to be aware that the variable \( u \) that appears in \( A(s|t,u) \) takes a different meaning in each case, as indicated between brackets. Identifying these expressions with the ones in eq. (III.1) then gives

\[
\alpha = \frac{F^2_\pi}{F^2} \left( 4 - 3 \frac{2\hat{n}B}{M^2_{\pi^0}} \right), \quad \beta = \frac{F^2_\pi}{F^2}
\]

at this order. Beyond leading order, the expressions (III.8) involve the low-energy constants \( \hat{\ell}_3 \) and \( \hat{\ell}_4 \) of ref. [24]. At this point, it may be useful to make briefly contact with the discussion towards the end of subsection (II.D) after eq. (II.28). Indeed, one might actually consider three sets of independent quantities, \((\alpha, \beta), (a^0_0, a^0_2), (\hat{\ell}_3, \hat{\ell}_4)\) as the unknowns of the problem. From a theoretical point of view, they are, to some extent, interchangeable. The last set naturally arises in the quark mass expansion that is implemented through the calculation of Feynman graphs generated by the effective chiral lagrangian. The two first sets are better suited for addressing phenomenological issues related to the analysis of experimental data. Here, we choose to organize the discussion in terms of the set \((\alpha, \beta)\). The transcription in terms of the two \( S \)-wave scattering lengths can be found, as already mentioned, in appendix (F) and in a forthcoming article [22].
B. Pion form factors at one loop

We can now start the procedure described in figure 3. At this stage, the unitarity conditions take then the following form:

\[
\text{Im} F^0_S(s) = \frac{1}{2} \sigma_0(s) \varphi^{00}_0(s) F^0_S(0) \theta(s - 4M^2_\pi) - \sigma(s) \varphi^0_0(s) F^0_S(0) \theta(s - 4M^2_\pi) + \mathcal{O}(E^6)
\]

\[
\text{Im} F^0_S(s) = \sigma(s) \varphi^0_0(s) F^0_S(0) \theta(s - 4M^2_\pi) - \frac{1}{2} \sigma_0(s) \varphi^0_0(s) F^0_S(0) \theta(s - 4M^2_\pi) + \mathcal{O}(E^6)
\]

\[
\text{Im} F^0_V(s) = \sigma(s) \varphi^0_1(s) \theta(s - 4M^2_\pi) + \mathcal{O}(E^4).
\]

We have now to determine the full form factors, exploiting the fact that we know their analytic structure, i.e., a cut along the positive real axis, and the value of the discontinuity along this cut. We introduce the well-known polynomials of the (one-loop) radii and a similar expression for \( \bar{J}(s) \) defined by the following dispersion integrals

\[
\bar{J}_0(s) = \frac{s}{16\pi^2} \int_{4M^2_\pi}^{\infty} dx \frac{1}{x - s - i0} \sigma_0(x)
\]

\[
\bar{J}(s) = \frac{s}{16\pi^2} \int_{4M^2_\pi}^{\infty} dx \frac{1}{x - s - i0} \sigma(x).
\]

These functions correspond to the standard one-loop integrals subtracted at \( s = 0 \), and through a change of variable the integrals can be brought into the more familiar form

\[
\bar{J}(s) = \frac{-1}{16\pi^2} \int_0^1 dx \ln \left[ 1 - x(1 - x) \frac{s}{M^2_\pi} \right],
\]

and a similar expression for \( \bar{J}_0(s) \), with \( M_\pi \) replaced by \( M_\pi^0 \). In the latter form, the integration is easy to perform for, say, \( s < 0 \). The expression of \( \bar{J}(s) \) for the remaining values of \( s \) is obtained through analytic continuation, with the \( s + i0 \) prescription on the cut, as made evident in the representation (III.10). The result is well known and reads

\[
\bar{J}(s) = \frac{1}{16\pi^2} \left[ 2 + \sigma(s) L(s) + i\pi \sigma(s) \theta(s - 4M^2_\pi) \right]
\]

(III.12)

Here, the function \( L(s) \) is defined by

\[
L(s) = \begin{cases} 
\ln \left( \frac{1 - \sigma(s)}{1 + \sigma(s)} \right) & [s \geq 4M^2_\pi] \\
\ln \left( \frac{\sigma(s) - 1}{\sigma(s) + 1} \right) & [s \leq 4M^2_\pi]
\end{cases}
\]

with \( \sigma(s) = \) \begin{cases} \sqrt{1 - \frac{4M^2_\pi}{s}} & [s \leq 0 \text{ or } s \geq 4M^2_\pi] \\
i\sqrt{\frac{4M^2_\pi}{s} - 1} & [0 \leq s \leq 4M^2_\pi]
\end{cases}

(III.13)

where we have performed a similar analytical continuation of the phase-space function \( \sigma(s) \), whose cut extends over the interval \( 0 \leq s \leq 4M^2_\pi \). The function \( \ln(s) \) is defined as usual with its cut on the negative real axis. Analogous functions \( L_0(s) \) and \( \sigma_0(s) \) are defined upon replacing \( M_\pi \) by \( M_\pi^0 \) in the above expressions. Then one can easily find functions with the appropriate discontinuities (III.9) to represent the form factors following the dispersive representation eq. (II.3):

\[
U^0_S(s) = P^0_S(s) + 16\pi \frac{1}{2} \varphi^{00}_0(s) \bar{J}_0(s) - 16\pi \varphi^0_0(s) \frac{F^0_S(0)}{F^0_S(0)} \bar{J}(s) + \mathcal{O}(E^6)
\]

\[
U^0_V(s) = P^0_V(s) + 16\pi \varphi^0_0(s) \bar{J}(s) - 16\pi \frac{1}{2} \varphi^0_0(s) \frac{F^0_S(0)}{F^0_S(0)} \bar{J}_0(s) + \mathcal{O}(E^6)
\]

\[
U^V_V(s) = P^V_V(s) + 16\pi \varphi^0_1(s) \bar{J}(s) + \mathcal{O}(E^6).
\]

(III.14)

\( P^0_S(s) \), \( P^0_V(s) \), and \( P^V_V(s) \) represent calculable polynomials at most quadratic in \( s \), which are determined by the property that the functions \( U^0_{S,V}(s) \) and \( U^V_S(s) \) have vanishing first and second derivatives at \( s = 0 \). These polynomials can be reabsorbed into the subtraction constants such as to build up the (one-loop) radii and
curvatures. At one loop, only one subtraction constant is required for each form factor. The corresponding expressions can therefore be rewritten as

\[ F_{\pi^0_S}(s) = F_{\pi^0_S}(0) \left[ 1 + a_{\pi^0}^s s + 16\pi \frac{\varphi_{\pi^0}(s)}{2} \bar{J}_0(s) \right] - 16\pi F_{\pi^0_S}(0) \varphi_{\pi^0}(s) \bar{J}(s) \]

\[ F_{\pi^0_S}(s) = F_{\pi^0_S}(0) \left[ 1 + a_{\pi^0}^s s + 16\varphi_{\pi^0}(s) \bar{J}(s) \right] - 16\pi F_{\pi^0_S}(0) \frac{1}{2} \varphi_{\pi^0}(s) \bar{J}_0(s) \]

\[ F_{\pi^0_V}(s) = 1 + a_{\pi^0}^v s + 16\varphi_{\pi^0}(s) \bar{J}(s). \quad (\text{III.15}) \]

At this order, the subtraction constants \(a_{\pi^0}^s\), \(a_{\pi^0}^\pi\), and \(a_{\pi^0}^\gamma\) are then related to the radii through

\[ \langle r^2 \rangle_{\pi^0_S} = 6 a_{\pi^0}^s - \frac{1}{48\pi^2 F^2_{\pi^0}} \left[ \frac{F_{\pi^0_S}^2(0)}{F_{\pi^0_S}^2(0)} \left( 2\beta_x M^2_{\pi^0} + M^2_{\pi^0} - \alpha_x M^2_{\pi^0} \right) - \frac{3}{2} \alpha_{\pi^0} \right] \]

\[ \langle r^2 \rangle_{\pi^0_S} = 6 a_{\pi^0}^\gamma - \frac{1}{96\pi^2 F^2_{\pi^0}} \left[ \frac{F_{\pi^0_S}^2(0)}{F_{\pi^0_S}^2(0)} \left( 2\beta_x M^2_{\pi^0} + M^2_{\pi^0} - \alpha_x M^2_{\pi^0} \right) + 4\beta_{\pi^0} - 4\alpha_{\pi^0} \right] \]

\[ \langle r^2 \rangle_{\pi^0_V} = 6 a_{\pi^0}^\gamma - \frac{1}{24\pi^2 F^2_{\pi^0}} \beta_{\pi^0}. \quad (\text{III.16}) \]

while the curvatures are given by

\[ c_{\pi^0_S} = \frac{1}{2880\pi^2 F^2_{\pi^0} M^2_{\pi^0}} \left[ \frac{F_{\pi^0_S}^2(0)}{F_{\pi^0_S}^2(0)} \left( 28\beta_x - 2\beta_x M^2_{\pi^0} + \alpha_x M^2_{\pi^0} \right) + \frac{3}{2} \alpha_{\pi^0} - 2\alpha_{\pi^0} \right] \]

\[ c_{\pi^0_S} = \frac{1}{5760\pi^2 F^2_{\pi^0} M^2_{\pi^0}} \left[ \frac{F_{\pi^0_S}^2(0)}{F_{\pi^0_S}^2(0)} \left( 28\beta_x - 2\beta_x M^2_{\pi^0} + \alpha_x M^2_{\pi^0} \right) + 26\beta_{\pi^0} + 4\alpha_{\pi^0} \right] \]

\[ c_{\pi^0_V} = \frac{1}{960\pi^2 F^2_{\pi^0} M^2_{\pi^0}} \beta_{\pi^0}. \quad (\text{III.17}) \]

C. One-loop representation of \(\pi\pi\) scattering amplitudes

The form factors at two loops are obtained once we know the real parts of the one-loop \(S\) and \(P\) \(\pi\pi\) partial wave projections, as shown in eq. (III.9). With this aim in mind, we now undertake the construction of the \(\pi\pi\) scattering amplitudes to one-loop in the presence of isospin breaking. The starting point is provided by the lowest-order expressions (III.1) of these amplitudes, supplemented with two amplitudes that are deduced from the former ones by crossing, and that are needed to express unitarity in the crossed channels,

\[ A^{+0}(s, t) = \frac{\beta_x}{F^2_{\pi}} \left( \frac{t}{3} - \frac{2}{3} M^2_{\pi} \right) + \frac{\alpha_x M^2_{\pi^0}}{3 F^2_{\pi}} \]

\[ A^{++}(s, t) = \frac{-\beta_{\pi^0}}{F^2_{\pi}} \left( \frac{s}{3} - \frac{2}{3} M^2_{\pi} \right) + \frac{2\alpha_{\pi^0} M^2_{\pi^0}}{3 F^2_{\pi}}, \quad (\text{III.18}) \]

together with the corresponding lowest-order partial wave projections,

\[ \varphi^{+0}_{0}(s) = -\frac{\beta_x}{16\pi F^2_{\pi}} \left( \frac{\lambda(s)}{2s} + \frac{2}{3} M^2_{\pi} + \frac{2}{3} M^2_{\pi^0} \right) + \frac{\alpha_x M^2_{\pi^0}}{48\pi F^2_{\pi}} \]

\[ \varphi^{+0}_{1}(s) = \frac{\beta_x}{48\pi F^2_{\pi}} \frac{\lambda(s)}{2s} \]

\[ \varphi^{+0}_{0}(s) = -\frac{\beta_{\pi^0}}{16\pi F^2_{\pi}} \left( \frac{\lambda(s)}{2s} + \frac{2}{3} M^2_{\pi} \right) + \frac{\alpha_{\pi^0} M^2_{\pi^0}}{24\pi F^2_{\pi}}. \quad (\text{III.19}) \]

Applying the formulae given in sec. III.C and recalling that at the one-loop order \(|f_\ell(s)|^2 = |\varphi_\ell(s)|^2 + O(E^6)|\), one easily obtains the expressions for the unitarity parts of the various amplitudes, up to a polynomial ambiguity that can be reabsorbed into the corresponding subtraction polynomials \(P(s, t, u)\). For the amplitudes corresponding
to the elastic channels, with either only neutral (II.18) or only charged pions (II.24), these expressions read

\[ W_0^{00}(s) = \frac{1}{2} [16\pi \varphi_0^{00}(s)]^2 \tilde{J}_0(s) + [16\pi \varphi_0^0(s)]^2 \tilde{J}(s) \]
\[ W_0^{-}(s) = [16\pi \varphi_0^{-}(s)]^2 \tilde{J}(s) + \frac{1}{2} [16\pi \varphi_0^0(s)]^2 \tilde{J}_0(s) \]
\[ W_1^{+}(s) = \frac{\beta^2_s}{36F_4^2} (s - 4M_\pi^2) \tilde{J}(s) \]
\[ W_0^{++}(s) = \frac{1}{2} [16\pi \varphi_0^{++}(s)]^2 \tilde{J}(s). \] (III.20)

For the amplitudes corresponding to the processes involving both charged and neutral pions (II.20), one obtains

\[ W_0^{+0}(s) = \frac{1}{2} (16\pi)^2 \varphi_0^+(s) \varphi_0^0(s) \tilde{J}_0(s) - (16\pi)^2 \varphi_0^-(s) \varphi_0^0(s) \tilde{J}(s) \]
\[ = \left\{ \frac{\beta_s^2}{12F_4^2} \left[ \frac{M_\pi^2 - M_\pi^4}{s} \right] \left[ s - 6(M_\pi^2 + M_\pi^4) \right] \right. \]
\[ + \frac{\beta_s^2}{F_4^2} \left[ \frac{s^2}{4} - s \left( \frac{M_\pi^2 + M_\pi^4}{3} \right) + \frac{11M_\pi^4 - 14M_\pi^2M_\pi^4}{18} \right] \]
\[ - \frac{\beta_x \alpha_x M_\pi^2}{3F_4^2} \left[ s - \frac{2}{3}(M_\pi^2 + M_\pi^4) \right] + \frac{\alpha_x^2 M_\pi^4}{9F_4^2} \right\} \tilde{J}_0(s) \]
\[ + \frac{\beta_s^2}{36F_4^2} \left( \frac{(M_\pi^2 - M_\pi^4)^2}{s^2} \right) \tilde{J}_{-0}(s) \]
\[ W_1^{00}(s) = \frac{\beta_s^2}{36F_4^2} \left[ s - 2(M_\pi^2 + M_\pi^4) + \frac{(M_\pi^2 - M_\pi^4)^2}{s} \right] \tilde{J}_{+0}(s). \] (III.21)

In these last expressions, we have introduced another one-loop integral \([s_{,0} \equiv (M_\pi + M_{\pi^0})^2]\),

\[ \tilde{J}_{-0}(s) = \frac{s}{16\pi^2} \int_{s_{,0}}^\infty dx \frac{1}{x - s - i0} \lambda^{1/2}(x) \]
\[ \lambda^{1/2}(x) = \frac{\lambda^{1/2}(x)}{x}. \] (III.22)

together with the subtracted integral \(s_{+0}(s) = \tilde{J}_0(s) - s\tilde{J}_{-0}(0)\), i.e.

\[ \tilde{J}_{+0}(s) = \frac{s^2}{16\pi^2} \int_{s_{+0}}^\infty dx \frac{1}{x^2} \frac{\lambda^{1/2}(x)}{x}. \] (III.23)

The expression for \(\tilde{J}_{+0}(s)\) can again be brought into the more familiar form of an integral over a Feynman parameter,

\[ \tilde{J}_{+0}(s) = -\frac{1}{16\pi^2} \int_0^1 dx \ln \left[ 1 - \frac{x(1-x)s}{M_\pi^2 - x(M_\pi^2 - M_{\pi^0}^2)} \right], \] (III.24)

through an appropriate change of variable.

The functions \(W_0^{00}(s), W_0^{-}(s), \text{etc.}\) are defined by their discontinuities up to polynomial ambiguities. At one-loop order, these polynomials need only be of at most second order in the variables \(s, t, u\). Taking into account the symmetry properties of the corresponding amplitudes, they may therefore be written as [see also eq. (II.28) and the discussion following it]

\[ P^{00}(s, t, u) = \frac{\alpha_{00} M_\pi^2}{F_4^2} + \frac{3\lambda_{00}^1}{F_4^2} \left[ (s - 2M_{\pi^0}^2)^2 + (t - 2M_{\pi^0}^2)^2 + (u - 2M_{\pi^0}^2)^2 \right] \]
\[ P^x(s, t, u) = \frac{\beta_x}{F_4^2} \left( s - \frac{2}{3}M_\pi^2 - \frac{2}{3}M_{\pi^0}^2 \right) + \frac{\alpha_x M_\pi^2}{3F_4^2} \]
\[ + \frac{\lambda_{x}^1(M_\pi^2 - 2M_{\pi^0}^2)(s - 2M_{\pi^0}^2) + \lambda_{x}^2 (t - M_\pi^2 - M_{\pi^0}^2)^2 + (u - M_\pi^2 - M_{\pi^0}^2)^2} {F_4^2} \]
\[ P^{++}(s, t, u) = \frac{\beta_s}{F_4^2} \left( s - \frac{8}{3}M_\pi^2 \right) + \frac{\alpha_x M_\pi^2}{3F_4^2} \]
\[ + \frac{\lambda_{s}^1 + \lambda_{s}^2 (t - 2M_\pi^2)^2 + (u - 2M_\pi^2)^2} {F_4^2} \] (III.25)
The five subtraction constants $\lambda_{00}^{(1)}, \lambda_x^{(i)}, \lambda_{+}^{(i)}, i = 1, 2$, are new free parameters. In the isospin limit, they are given by

$$
\lambda_x^{(i)} \rightarrow \lambda_1, \quad \lambda_+^{(i)} \rightarrow \lambda_2, \quad \lambda_{00}^{(1)} \rightarrow \frac{\lambda_1 + 2\lambda_2}{3}, \quad (III.26)
$$

where, at this order, $\lambda_{1,2}$ can be expressed in terms of the low-energy constants $\bar{\ell}_1$ and $\bar{\ell}_2$ of [16], cf. [24] and equation (E.8) below.

The expressions for the $\pi\pi$ scattering amplitudes that follow from the above results agree with those derived in refs. [40, 41, 43] from a Feynman graph calculation based on the low-energy effective lagrangian for QCD+QED, provided that one drops the contributions coming from virtual photons, while keeping the difference between charged and neutral pion masses [this procedure is discussed in greater detail in subsection [VIA] and in appendix [E]]

D. Infrared behaviour of one-loop amplitudes and form factors

In the isospin-symmetric situation, the $\pi\pi$ scattering amplitude $A(s|t, u)$, the vector form factor $F_V^\pi(s)$ and the scalar form factor $F_S^\pi(s)$ are well behaved in the limit where the pion mass $M_{\pi}$ vanishes. For the pion form factors, this has been shown explicitly at the two loop level in [25]. This good behaviour in the chiral limit requires that some of the parameters that appear in these quantities develop themselves logarithmically singular terms in the limit where the pion mass vanishes. This is necessary in order to compensate for the singularities coming from the unitarity part, for, as $M_{\pi} \rightarrow 0$,

$$
\tilde{J}(s) \equiv \frac{1}{16\pi^2} \ln \left( \frac{M_{\pi}^2}{-s} \right) + \frac{1}{8\pi^2}, \quad (III.27)
$$

Taking into account the pion mass difference offers a wider range of possibilities from this point of view. Indeed, besides the path just described, reaching the isospin limit first, then letting the common pion mass vanish, two additional options might be considered, where one lets, say, the neutral pion mass tend to zero, keeping the charged pion mass fixed, or the other way around. In the first case, singular contributions in the unitarity part come from the function $J_0(s)$, which behaves as in (III.27), but with $M_{\pi}$ replaced by $M_{\pi^0}$. In the second case, eq. (III.27) applies directly. Notice that the function $\tilde{J}_{+0}(s)$ remains finite as either of these two limits is taken. It requires that both pion masses vanish for it to develop an infrared singular behaviour.

Let us first consider the $\pi\pi$ scattering amplitudes obtained in the preceding sub-section. In order for the one-loop amplitudes $A^{00}$, $A^x$, and $A^{+-}$ to remain finite as $M_{\pi^0} \rightarrow 0$, with $M_{\pi}$ fixed, we must have

$$
\alpha_{00} M_{\pi^0} \rightarrow 0, \quad \alpha_x M_{\pi^0} \rightarrow \tilde{\alpha}_x M_{\pi^0}, \quad \beta_x \rightarrow \tilde{\beta}_x, \quad (III.28)
$$

and [for the sake of simplicity, we keep the notation $F_{\pi}$ for the pion decay constant, which has a regular behaviour in either limit]

$$
\alpha_+ M_{\pi^0}^2 \sim -\frac{M_{\pi}^4}{96\pi^2 F_{\pi}^2} \tilde{\alpha}_x (\tilde{\alpha}_x + 4\tilde{\beta}_x) \ln M_{\pi^0}^2 + \text{finite},
$$

$$
\beta_+ \sim -\frac{M_{\pi}^2}{48\pi^2 F_{\pi}^2} \tilde{\beta}_x (\tilde{\alpha}_x + 4\tilde{\beta}_x) \ln M_{\pi^0}^2 + \text{finite},
$$

$$
\lambda_{+}^{(1)} \sim -\frac{1}{32\pi^2} \tilde{\beta}_x^2 \ln M_{\pi^0}^2 + \text{finite}, \quad (III.29)
$$

whereas the remaining coefficients, $\lambda_{00}^{(1)}$, $\lambda_{+}^{(2)}$, $\lambda_x^{(1)}$, and $\lambda_x^{(2)}$ remain finite in this limit. Likewise, the coefficients $\tilde{\alpha}_x$ and $\tilde{\beta}_x$ are free of infrared singularities. In order to avoid any possible confusion, we remind the reader that the coefficients $\alpha_{00}$, $\alpha_x$, $\alpha_+$ appear in the amplitudes multiplied by $M_{\pi^0}$ as a pure matter of convention [see the remark after eq. (III.1)]. Therefore, $\alpha_x M_{\pi^0}^2$ and $\alpha_+ M_{\pi^0}^2$ need not vanish as $M_{\pi^0}$ tends to zero with $M_{\pi}$ fixed. This feature is actually exhibited already by the lowest-order expressions given in eq. (III.3). Furthermore, the quantities which appear on the right-hand sides of the above equations have to be understood as taking their lowest-order values. We have not distinguished them from their values at next-to-leading order, which appear
on the left-hand sides, in order not to overburden the notation. The appearance of infrared singular behaviours is a loop effect, and higher-order corrections will induce new singularities. At the next order, these may involve log-squared terms \( \log^2 \), with an additional \( 1/(4\pi F_\pi)^2 \) loop suppression factor. Concretely, from eq. (III.5) one obtains the lowest-order values

\[
\tilde{\alpha}_x = 2\beta, \quad \tilde{\beta}_x = \beta.
\]

Taking now the second limit, \( M_\pi \to 0 \) with \( M_\pi^0 \) fixed, we find that the finiteness of the one-loop amplitudes \( A^{00}, A^x, \) and \( A^+ \) makes the various coefficients [we denote their lowest-order, infrared finite, limiting values with a tilde on top of them, except for \( F_\pi \)] behave as follows:

\[
\begin{align*}
\alpha_{00} &\sim -\frac{M_{\pi^0}^2}{48\pi^2 F_\pi^2} \tilde{\alpha}_x (\tilde{\alpha}_x + 4\tilde{\beta}_x) \ln M_\pi^2 + \text{finite} \\
\lambda_{00}^{(1)} &\sim -\frac{1}{32\pi^2} \tilde{\beta}_x \ln M_\pi^2 + \text{finite} \\
\alpha_x &\sim -\frac{1}{48\pi^2} \frac{M_{\pi^0}^2}{F_\pi^2} \left[ 2\tilde{\alpha}_x \tilde{\alpha}_x + \tilde{\beta}_+ (\tilde{\alpha}_x + 4\tilde{\beta}_x) \right] \ln M_\pi^2 + \text{finite} \\
\beta_x &\sim -\frac{1}{96\pi^2} \frac{M_{\pi^0}^2}{F_\pi^2} \left[ 4\tilde{\alpha}_x \tilde{\beta}_+ + \tilde{\beta}_+ (\tilde{\alpha}_x + 4\tilde{\beta}_x) \right] \ln M_\pi^2 + \text{finite} \\
\lambda_x^{(1)} &\sim -\frac{1}{32\pi^2} \tilde{\beta}_x \tilde{\beta}_+ \ln M_\pi^2 + \text{finite} \\
\alpha_{+-} &\sim -\frac{5}{48\pi^2} \frac{M_{\pi^0}^2}{F_\pi^2} \tilde{\beta}_+ \ln M_\pi^2 + \text{finite} \\
\beta_{+-} &\sim -\frac{1}{12\pi^2} \frac{M_{\pi^0}^2}{F_\pi^2} \tilde{\alpha}_x \tilde{\beta}_+ \ln M_\pi^2 + \text{finite} \\
\lambda_{+-}^{(1)} &\sim -\frac{1}{96\pi^2} \tilde{\beta}_+ \ln M_\pi^2 + \text{finite} \\
\lambda_{+-}^{(2)} &\sim -\frac{1}{48\pi^2} \tilde{\beta}_+ \ln M_\pi^2 + \text{finite}.
\end{align*}
\]

(III.31)

Now the lowest-order values inferred from eq. (III.5) read:

\[
\tilde{\alpha}_x = \alpha - 2\beta, \quad \tilde{\alpha}_{+-} = \alpha - 4\beta, \quad \tilde{\beta}_x = \tilde{\beta}_{+-} = \beta.
\]

(III.32)

In appendix \( \text{E} \) we determine the various subtraction constants that appear in the expressions of the \( \pi \pi \) scattering amplitudes obtained after the first iteration from the corresponding expressions obtained from a one-loop calculation. One may check that the formulae given in appendix \( \text{E} \) indeed exhibit the expected infrared behaviour.

Let us now turn towards the form factors, and consider the same two limits. As \( M_\pi^0 \) vanishes while \( M_\pi \) is kept fixed, the form factors remain free of infrared singularities provided

\[
\begin{align*}
a_{S}^{\pi} &\sim -\frac{1}{32\pi^2 F_\pi^2} \frac{F_S^{\pi}(0)}{F_\pi^2} \tilde{\beta}_x \ln M_\pi^2 + \text{finite} \\
F_S^{\pi}(0) &\sim -\frac{1}{96\pi^2 F_\pi^2} \frac{M_{\pi^0}^2}{F_\pi^2} F_S^{\pi}(0)(\tilde{\alpha}_x - 2\tilde{\beta}_x) \ln M_\pi^2 + \text{finite},
\end{align*}
\]

(III.33)

while \( a_{S}^{\pi} \) and \( F_S^{\pi}(0) \) remain finite [at this order]. Actually, in view of (III.30), this is also true for \( F_S^{\pi}(0) \). In the case of the second limit, \( M_\pi \to 0 \) and \( M_\pi^0 \) fixed, infrared finite form factors require that

\[
\begin{align*}
a_{V}^{\pi} &\sim -\frac{1}{96\pi^2 F_\pi^2} \tilde{\beta}_+ \ln M_\pi^2 + \text{finite} \\
a_{S}^{\pi} &\sim -\frac{1}{16\pi^2 F_\pi^2} \frac{F_S^{\pi}(0)}{F_\pi^2} \tilde{\beta}_x \ln M_\pi^2 + \text{finite} \\
F_S^{\pi}(0) &\sim -\frac{1}{48\pi^2 F_\pi^2} \frac{M_{\pi^0}^2}{F_\pi^2} F_S^{\pi}(0)(\tilde{\alpha}_x - 2\tilde{\beta}_x) \ln M_\pi^2 + \text{finite}
\end{align*}
\]
for wave projections from the one-loop scattering amplitudes is less straightforward, and represents the next issue obtained in the preceding section, obtained here from quite general arguments.

Again, one may check that the explicit one-loop expressions given in appendix \[E\] reproduce the infrared behaviour obtained here from quite general arguments.

To conclude this short discussion, we may observe that many of these infrared singularities disappear once the second mass also tends to zero, thus restoring the infrared features of the isospin-symmetric chiral limit. Upon studying the expressions (III.10) and (III.17) in the light of the present discussion, a similar worsening of the infrared behaviour, as compared to the isospin-symmetric limit, can also be brought forward in the radii and curvatures of the scalar form factors. We thus can conclude that, to a certain extent, isospin symmetry tames the infrared behaviour of the soft pion clouds.

IV. SECOND ITERATION: TWO-LOOP EXPRESSIONS

So far, we have completed the first cycle of the procedure sketched in figure \[3\]. The one-loop expressions of the form factors and scattering amplitudes obtained in the preceding section can now be used in order to construct the two-loop representations of the form factors. At next-to-leading order in the chiral expansion, the structure of the form factors is now as follows

\[
\text{Re} F_S^0(s) = F_S^0(0) \left[ 1 + \Gamma_S^0(s) \right] + \mathcal{O}(E^6)
\]

\[
\text{Re} F_S^\pi(s) = F_S^\pi(0) \left[ 1 + \Gamma_S^\pi(s) \right] + \mathcal{O}(E^6),
\]  

(IV.1)

and

\[
\text{Re} F_V(s) = 1 + \Gamma_V(s) + \mathcal{O}(E^4).
\]  

(IV.2)

The one-loop corrections \(\Gamma_S^0(s), \Gamma_S^\pi(s),\) and \(\Gamma_V(s)\) are easy to extract from the expressions of the form factors obtained in the preceding section,

\[
\Gamma_S^0(s) = a_S^0 s + 16\pi \frac{1}{2} \varphi_0^{00}(s) \text{Re} J_0(s) - 16\pi \varphi_0^s(s) \frac{F_S^0(0)}{F_S^0(0)} \text{Re} \bar{J}(s)
\]

\[
\Gamma_S^\pi(s) = a_S^\pi s + 16\pi \varphi_0^{+-}(s) \text{Re} J(s) - 16\pi \frac{1}{2} \varphi_0^s(s) \frac{F_S^\pi(0)}{F_S^0(0)} \text{Re} \bar{J}(s)
\]

\[
\Gamma_V(s) = a_V s + 16\pi \varphi_0^{-+}(s) \text{Re} \bar{J}(s).
\]  

(IV.3)

As far as the discontinuities are concerned, they start at \(\mathcal{O}(E^4)\) for \(F_S^0(s)\) and \(F_S^\pi(s),\) and at \(\mathcal{O}(E^2)\) for \(F_V(s).\) Using this power counting and eq. (II.25), one obtains the discontinuities of the form factors at next-to-next-to-leading order

\[
\text{Im} F_S^0(s) = \frac{1}{2} \sigma_0(s) F_S^0(0) \left\{ \varphi_0^{00}(s) \left[ 1 + \Gamma_S^0(s) \right] + \psi_0^{00}(s) \right\} \theta(s - 4M_{\pi^0}^2)
\]

\[- \sigma(s) \varphi_0^s(s) \left[ 1 + \Gamma_S^0(s) + \psi_0^s(s) \right] \theta(s - 4M_{\pi}^2) + \mathcal{O}(E^6)\]

\[
\text{Im} F_S^\pi(s) = \sigma(s) \varphi_0^{-+}(s) \left[ 1 + \Gamma_S^\pi(s) + \psi_0^{-+}(s) \right] \theta(s - 4M_{\pi}^2) + \mathcal{O}(E^6)
\]

\[- \frac{1}{2} \sigma_0(s) F_S^\pi(0) \left\{ \varphi_0^{+-}(s) \left[ 1 + \Gamma_S^\pi(s) \right] + \psi_0^{+-}(s) \right\} \theta(s - 4M_{\pi}^2) + \mathcal{O}(E^6)\]

\[
\text{Im} F_V(s) = \sigma(s) \left\{ \varphi_0^{-+}(s) \left[ 1 + \Gamma_V(s) \right] + \psi_0^{-+}(s) \right\} \theta(s - 4M_{\pi}^2) + \mathcal{O}(E^4).
\]  

(IV.4)

A. Partial-wave projections from the one-loop amplitudes

The computation of the one-loop corrections \(\psi_0^{00}(s), \psi_0^s(s), \psi_0^{-+}(s),\) and \(\psi_0^{-+}(s)\) to the \(\pi\pi\) \(S\) and \(P\) partial wave projections from the one-loop scattering amplitudes is less straightforward, and represents the next issue
to be addressed. In order to illustrate how this difficulty can be handled, let us start with the elastic scattering of neutral pions, i.e. the quantity \( \psi_0^{(0)}(s) \) describing the next-to-leading-order correction to the real part of the \( S \)-wave projection for \( A^{(0)}(s, t) \) in the range \( s \geq 4M_\pi^2 \). Trading the integration over the scattering angle for an integration over the variable \( t \), with \( t_{-}(s) \equiv -(s - 4M_\pi^2) \leq t \leq 0 \), we obtain (the functions \( \tilde{J}(t) \) and \( \tilde{J}_0(t) \) are real for \( t \leq 0 \))

\[
\psi_0^{(0)}(s) = \frac{\lambda_{(1)}^{(s)}}{16\pi F_\pi^2} (5s^2 - 16sM_\pi^2 + 20M_\pi^4) + \frac{1}{32\pi} [16\pi\varphi_0^{(0)}(s)]^2 \text{Re} \tilde{J}_0(s) + \frac{1}{16\pi} [16\pi\varphi_0^{(0)}(s)]^2 \text{Re} \tilde{J}(s)
\]

\[
+ \frac{1}{16\pi} \frac{1}{s - 4M_\pi^2} \frac{M_\pi^2}{F_\pi^2} \int_{t_{-}(s)}^{0} dt \tilde{J}_0(t) + \frac{1}{16\pi} \frac{1}{s - 4M_\pi^2} \int_{t_{-}(s)}^{0} dt [16\pi\varphi_0^{(0)}(t)]^2 \tilde{J}(t).
\]  

(IV.5)

It turns out that the remaining integrals can be performed analytically. The relevant formulae can be found in appendix \[\text{A}\]. The resulting expression can be written as

\[
16\pi \psi_0^{(0)}(s) = 2 \frac{M_\pi^2}{F_\pi^2} \sqrt{s} \left\{ \xi_0^{(0)}(s)\sigma_0(s) + 2\xi_{00}^{(1,0)}(s)L_0(s) + 2\xi_{00}^{(1,0)}(s) \frac{\sigma_0(s)}{\sigma_\psi(s)}L_\psi(s) + 2\xi_{00}^{(2,0)}(s) \left[ 1 - \frac{4M_\pi^2}{s} \right] L_0(s) + 3\xi_{00}^{(3,0)}(s) \frac{M_\pi^2}{s - 4M_\pi^2} L_0^2(s) + 3\xi_{00}^{(3,0)}(s) \frac{M_\pi^2}{s - 4M_\pi^2} \frac{M_\pi^2}{s - 4M_\pi^2} L_\psi^2(s) \right\},
\]  

(IV.6)

with \( \sigma_\psi(s) = \sigma(s - 4M_\pi^2 + 4M_\pi^2) \) and \( L_\psi(s) = L(s - 4M_\pi^2 + 4M_\pi^2) \). The various functions \( \xi_0^{(0)}(s), \xi_{00}^{(1,0)}(s), \) etc. that enter this expression of \( \psi_0^{(0)}(s) \) are polynomials in \( s \) and in the subthreshold parameters, which are given in appendix \[\text{B}\]. We have written the result in a way that allows for a straightforward connection with the similar expressions for the isospin-symmetric case, as displayed in ref. \[\text{[24]}\]. Indeed in the limit \( M_{\pi^0} \rightarrow M_\pi \) (and \( \alpha_0, \alpha_x \rightarrow \alpha, \beta_x, \beta_x \rightarrow \beta \)) one obtains the expected combination of \( I = 0 \) and \( I = 2 \) contributions, weighted by the corresponding \( SU(2) \) Clebsch-Gordan coefficients,

\[
\psi_0^{(0)}(s) \rightarrow \tilde{\psi}_0^{(0)}(s) = 2 \frac{M_\pi^2}{F_\pi^2} \sqrt{s} \left\{ \xi_0^{(0)}(s)\sigma_0(s) + 2\xi_{00}^{(1,0)}(s)L_0(s) + 2\xi_{00}^{(1,0)}(s) \frac{\sigma_0(s)}{\sigma_\psi(s)}L_\psi(s) + 2\xi_{00}^{(2,0)}(s) \left[ 1 - \frac{4M_\pi^2}{s} \right] L_0(s) + 3\xi_{00}^{(3,0)}(s) \frac{M_\pi^2}{s - 4M_\pi^2} L_0^2(s) + 3\xi_{00}^{(3,0)}(s) \frac{M_\pi^2}{s - 4M_\pi^2} \frac{M_\pi^2}{s - 4M_\pi^2} L_\psi^2(s) \right\},
\]  

(IV.7)

where, for the reader’s convenience, we reproduce the expressions of the functions \( k_n(s) \) of ref. \[\text{[24]}\] (the function \( k_4(s) \) appears only in the \( P \) wave component \( \psi_1^{(0)}(s) \), to be discussed below),

\[
k_0(s) = \frac{1}{16\pi} \sqrt{\frac{s - 4M_\pi^2}{s}}, \quad k_1(s) = \frac{1}{8\pi} L(s),
\]

\[
k_2(s) = \frac{1}{8\pi} \left( 1 - \frac{4M_\pi^2}{s} \right) L(s), \quad k_3(s) = \frac{3}{16\pi} \frac{M_\pi^2}{\sqrt{s(s - 4M_\pi^2)}} L^2(s),
\]

\[
k_4(s) = \frac{1}{16\pi} \sqrt{\frac{s - 4M_\pi^2}{s}} \left\{ 1 + \sqrt{\frac{s - 4M_\pi^2}{s}} L(s) + \frac{M_\pi^2}{s - 4M_\pi^2} L^2(s) \right\}.
\]  

(IV.8)

For the remaining notation we refer the reader to \[\text{[24]}\] (see in particular eqs. (3.36), (3.37) and the Appendix B therein). Let us, however, point out that the factor of 2 in \( \text{[IV.4]} \) takes care of the difference in normalization in the partial waves as compared to that reference, see eq. \[\text{[II.14]}\] and the comment preceding it.

For the elastic scattering of charged pions, the computation of \( \psi_0^+(s) \) and of \( \psi_1^+(s) \), now in the range \( s \geq M_\pi^2 \), proceeds along similar lines. The starting point is provided by the following formulae,

\[
\psi_0^+(s) = \frac{\lambda_{(1)}^{(1)} + \lambda_{(2)}^{(2)}}{F_\pi^4} \left( s - 2M_\pi^2 \right)^2 + \frac{\lambda_{(1)}^{(1)} + 3\lambda_{(2)}^{(2)}}{3F_\pi^4} \left( s^2 - 2sM_\pi^2 + 4M_\pi^4 \right) + \frac{1}{32\pi} [16\pi\varphi_0^+(s)]^2 \text{Re} \tilde{J}_0(s)
\]

\[
+ \frac{1}{16\pi} \frac{1}{s - 4M_\pi^2} \frac{M_\pi^2}{F_\pi^2} \int_{t_{-}(s)}^{0} dt [16\pi\varphi_0^+(t)]^2 \tilde{J}_0(t)
\]

\[
+ \frac{1}{32\pi} \frac{1}{s - 4M_\pi^2} \int_{t_{-}(s)}^{0} dt \left\{ \frac{2}{16\pi} [16\pi\varphi_0^+(t)]^2 + [16\pi\varphi_0^+(t)]^2 \right\} \tilde{J}(t)
\]

\[
+ \frac{1}{16\pi} \frac{1}{s - 4M_\pi^2} \int_{t_{-}(s)}^{0} dt \frac{\beta^2}{12F_\pi^2} (t - 4M_\pi^2)(2s + t - 4M_\pi^2) \tilde{J}(t),
\]  

(IV.9)
and

\[
\psi^-(s) = -\frac{\lambda(1) - \lambda(2)}{96\pi F^2} s(s - 4M^2) + \frac{1}{16\pi} \frac{\beta^2}{36F^2} (s - 4M^2)^2 \text{Re} \tilde{J}(s)
\]

\[
+ \frac{1}{32\pi s - 4M^2} \int_{t_-}^0 dt \left[ 16\pi \varphi^2_0(t) \right] (1 + \frac{2t}{s - 4M^2}) \tilde{J}_0(t)
\]

\[
+ \frac{1}{32\pi s - 4M^2} \int_{t_-}^0 dt \left\{ 2 \left[ 16\pi \varphi^+_0(t) \right]^2 - \left[ 16\pi \varphi^+_0(t) \right]^2 \right\} \left( 1 + \frac{2t}{s - 4M^2} \right) \tilde{J}(t)
\]

\[
+ \frac{1}{16\pi s - 4M^2} \int_{t_-}^0 dt \frac{\beta^2}{12F^2} (t - 4M^2) (2s + t - 4M^2) \left( 1 + \frac{2t}{s - 4M^2} \right) \tilde{J}(t),
\] (IV.9)

with now \(t_- = -(s - 4M^2)\). Performing the remaining integrations with the help of the formulae given in appendix A leads then to

\[
16\pi \psi^0_0(s) = \frac{M^4}{F^2} \sqrt{\frac{s}{s - 4M^2}} \left\{ \xi^{(0)}_{2;3} \sigma(s) + 2\xi^{(1)}_{1;2} \sigma(s) + 2\xi^{(1;3)} \sigma(s) \frac{\sigma(s)}{\sigma_\Delta(s)} L_\Delta(s) \right\},
\] (IV.10)

\[
16\pi \psi^1_0(s) = \frac{M^4}{F^2} \sqrt{\frac{s}{s - 4M^2}} \left\{ \xi^{(0)}_{2;3} \sigma(s) + 2\xi^{(1)}_{1;2} \sigma(s) + 2\xi^{(1;3)} \sigma(s) \frac{\sigma(s)}{\sigma_\Delta(s)} L_\Delta(s) \right\},
\] (IV.11)

with \(\sigma_\Delta(s) = \sigma_0(s + 4M^2 - 4M^2)\) and \(L_\Delta(s) = L_0(s + 4M^2 - 4M^2)\). The various polynomials that enter the expression of \(\psi^0_0(s)\) and \(\psi^1_0(s)\) have been gathered in appendix B. Taking the isospin limit, as described above in the case of \(\psi^0_0(s)\), one recovers, for the S wave, the expected combination of \(I = 0\) and \(I = 2\) contributions, and, for the P wave, the corresponding I = 1 component,

\[
\psi^0_0(s) \rightarrow \psi^0_0(s) \equiv 2 \frac{M^4}{F^2} \sqrt{\frac{s}{s - 4M^2}} \sum_{n=0}^3 \left[ \frac{1}{6} \xi^{(n)}_2(s) + \frac{1}{3} \xi^{(n)}_3(s) \right] k_n(s),
\]

\[
\psi^1_0(s) \rightarrow \psi^1_0(s) \equiv 2 \frac{M^4}{F^2} \sqrt{\frac{s}{s - 4M^2}} \sum_{n=0}^4 \frac{1}{2} \xi^{(n)}_1(s) k_n(s),
\] (IV.12)

in full agreement with the results of [24].

Turning eventually towards the inelastic scattering \(\pi^+\pi^- \rightarrow \pi^0\pi^0\), i.e. \(\psi_0(s)\), the range of integration corresponding to \(-1 \leq z \equiv \cos \theta \leq +1\) is \(t_- (s) \leq t \leq t_+ (s)\), with

\[
t_\pm(s) = -\frac{1}{2} (s - 2M^2) \pm \frac{1}{2} \sqrt{(s - 4M^2)(s - 4M^2)}.
\] (IV.13)

For \(s \geq 4M^2\), one has \(t \leq 0\) and \(u \leq 0\). In terms of an integration over \(t\), one thus obtains

\[
\psi^0_0(s) = -\frac{\lambda(1)}{16\pi F^2} (s - 2M^2) (s - 2M^2) - \frac{\lambda(2)}{24\pi F^2} s^2 - s (M^2 + M^2) + 4M^2 M^2
\]
Let us point out that the expression (IV.15) for $\psi_0^T(s)$ holds in the range $s \geq 4M_\pi^2$, where the functions $t_\pm(s)$ are real. An analytical continuation is necessary in order to describe $\psi_0^T(s)$ in, say, the range $4M_\pi^2 \leq s \leq 4M_\pi^2$, as required, for instance, for $\text{Im} F_π^T(s)$, cf. eq. (IV.3). For the applications that will be discussed in the following sections, we need not deal with this aspect, and the expression (IV.15) is sufficient. It is also useful to notice that in the isospin limit $t_\pm(s)$, $\lambda^{1/2}(t_+(s))\mathcal{L}_+(s)$, and $\mathcal{L}_+^2(s)$ all behave as $\mathcal{O}((M_\pi^2 - M_\rho^2)^2)$. We keep however the contributions involving $\mathcal{L}_+(s)$ as indicated in equation (IV.15), so that each of the three pieces, when taken together...
separately, displays a regular behaviour as \( s \) approaches \( 4M^2 \) (from above). Finally, in the limit where the value of the mass \( M_{\pi^0} \) tends to \( M_\pi \), one recovers the result of [24],

\[
\psi_0^\pi(s) \rightarrow \psi_0^\pi(s) \equiv 2 \frac{M^4}{F_\pi^2} \sqrt{\frac{s}{s - 4M^2}} \sum_{n=0}^{3} \left[ \frac{1}{3} \xi_2(n)(s) - \frac{1}{3} \xi_0(n)(s) \right] k_n(s). \tag{IV.18}
\]

B. Two-loop representation of the form factors and scattering amplitudes

Having obtained the partial wave projections in the \( S \) and \( P \) waves from the relevant one-loop \( \pi\pi \) scattering amplitudes, we may now proceed towards obtaining the two-loop expressions of the form factors and scattering amplitudes. This requires one to evaluate the dispersive integrals in terms of which they are expressed. As mentioned previously, we will not be able to work out analytical expressions for all the integrals involved. Closed expressions will be obtained only for the contributions corresponding to so-called factorizing two-loop diagrams, see fig. [4] They will involve the functions \( \bar{K}_n(s) \), defined as [24]

\[
\bar{K}_n(s) = \frac{s}{\pi} \int_{4M^2}^{\infty} \frac{dx}{x} k_n(s), \tag{IV.19}
\]

with the functions \( k_n(s) \) given in eq. [IV.8], and the understanding that \( K_0(s) \) remains denoted by \( \bar{J}(s) \). Explicit expressions of the functions \( \bar{K}_n(s) \) in terms of \( J(s) \) can be found in ref. [24]. There will also appear functions \( \bar{K}_n^0(s) \), which are defined in the same way in terms of functions \( k_n^0(s) \), identical to the functions \( \bar{K}_n(s) \), but with the charged pion mass \( M_\pi \) replaced by \( M_{\pi^0} \). Similarly, we will keep the notation \( \bar{J}_0(s) \) for \( \bar{K}_n^0(s) \).

Starting with the form factors, we obtain the two-loop representations

\[
F_S^0(s) = F_S^0(0) \left( 1 + a_S^0 s + b_S^0 s^2 \right) \nonumber
+ 8\pi F_S^0(0) \varphi_0^0(0) \left[ 1 + a_S^0 s + \frac{1}{\pi} \varphi_0^0(s) - \frac{2}{\pi} \frac{F_S^0(0)}{F_S^0(0)} \varphi_0^0(s) \right] \bar{J}_0(s) \nonumber
- 16\pi F_S^0(0) \varphi_0^0(0) \left[ 1 + a_S^0 s + \frac{2}{\pi} \varphi_0^0(s) - \frac{1}{\pi} \frac{F_S^0(0)}{F_S^0(0)} \varphi_0^0(s) \right] \bar{J}(s) \nonumber
+ \frac{M^4}{F_\pi^2} F_S^0(0) \left[ \xi_{00}^0(s) \bar{J}_0(s) + \xi_{10}^0(s) \bar{K}_1(s) + 2\xi_{00}^0(s) \bar{K}_0(s) + \xi_{00}^{(2;0)}(s) \bar{K}_3(s) \right] \nonumber
- \frac{M^4}{F_\pi^2} F_S^0(0) \left[ 2\xi_0^0(s) \bar{J}(s) + 4\xi_2^2(s) \bar{K}_2(s) \right] \nonumber
+ \frac{2M^4}{F_\pi^2} \left[ F_S^0(0) \xi_{00}^0(s) \bar{J}_0(s) - 2F_S^0(0) \xi_{00}^{(2;0)}(s) \right] \left[ 16\pi^2 \bar{J}(s) \bar{J}_0(s) - 2\bar{J}(s) - 2\bar{J}_0(s) \right] \nonumber
+ \Delta_{nf} F_S^0(s) + \mathcal{O}(E^6), \tag{IV.20}
\]

and

\[
F_S^\pi(s) = F_S^\pi(0) \left( 1 + a_S^\pi s + b_S^\pi s^2 \right) \nonumber
+ 16\pi F_S^\pi(0) \varphi_0^-(s) \left[ 1 + a_S^\pi s + \frac{2}{\pi} \varphi_0^-(s) - \frac{1}{\pi} \frac{F_S^\pi(0)}{F_S^\pi(0)} \varphi_0^-(s) \right] \bar{J}(s) \nonumber
- 8\pi F_S^\pi(0) \varphi_0^-(0) \left[ 1 + a_S^\pi s + \frac{1}{\pi} \varphi_0^-(s) - \frac{2}{\pi} \frac{F_S^\pi(0)}{F_S^\pi(0)} \varphi_0^-(s) \right] \bar{J}_0(s) \nonumber
+ 2\frac{M^4}{F_\pi^2} F_S^\pi(0) \left[ \xi_{00}^0(s) \bar{J}_0(s) + \xi_{10}^0(s) \bar{K}_1(s) + 2\xi_{00}^0(s) \bar{K}_0(s) + \xi_{00}^{(2;0)}(s) \bar{K}_3(s) \right] \nonumber
- \frac{M^4}{F_\pi^2} F_S^\pi(0) \left[ \xi_0^0(s) \bar{J}_0(s) + 2\xi_2^2(s) \bar{K}_2(s) \right] \nonumber
+ 2\frac{M^4}{F_\pi^2} \left[ 2F_S^\pi(0) \xi_{00}^{(2;0)}(s) - F_S^\pi(0) \xi_{22}^0(s) \right] \left[ 16\pi^2 \bar{J}(s) \bar{J}_0(s) - 2\bar{J}(s) - 2\bar{J}_0(s) \right] \nonumber
+ \Delta_{nf} F_S^\pi(s) + \mathcal{O}(E^6), \tag{IV.21}
\]
for the scalar form factors, whereas the vector form factor reads

\[
F_V^\pi(s) = 1 + a_0^\pi s + b_0^\pi s^2
+ 16\pi \varphi_1^+ (s) \left[ 1 + a_0^\pi s + \frac{2}{\pi} \varphi_1^+ (s) \right] \bar{J}(s)
+ 2 M_1^4 \left[ \xi_{+,0}^1 (s) \bar{J}(s) + \xi_{+,1}^1 (s) \bar{J}(s) + 2 \xi_{+,2}^1 (s) \bar{J}(s) + \xi_{+,3}^1 (s) \bar{J}(s) + \xi_{+,4}^1 (s) \bar{J}(s) \right]
+ \Delta_{NF} F_V^\pi (s) + \mathcal{O}(E^6).
\]

(IV.22)

The contributions $\Delta_{NF} F_S^0 (s)$, $\Delta_{NF} F_S^\pi (s)$, and $\Delta_{NF} F_V^\pi (s)$ stemming from non-factorizing two-loop graphs, see fig. H, are expressed as dispersive integrals,

\[
\Delta_{NF} F_S^0 (s) = \frac{M_1^4}{F_1^2} \int \frac{dx}{x} x - s - i0 \left[ \xi_{+,0}^1 (s) \frac{\sigma_0 (x)}{8\pi \sigma (x)} L_0 (x) + \xi_{+,1}^1 (s) \frac{3 M_2^2}{16 \pi x \sigma (x)} L_0^2 (x) \right]
\]

(IV.23)

\[
\Delta_{NF} F_S^\pi (s) = \frac{M_1^4}{F_1^2} \int \frac{dx}{x} x - s - i0 \left[ \xi_{+,0}^1 (s) \frac{\sigma_0 (x)}{8\pi \sigma (x)} L_0 (x) + \xi_{+,1}^1 (s) \frac{3 M_2^2}{16 \pi x \sigma (x)} L_0^2 (x) \right]
\]

(IV.24)

\[
\Delta_{NF} F_V^\pi (s) = \frac{M_1^4}{F_1^2} \int \frac{dx}{x} x - s - i0 \left[ \xi_{+,0}^1 (s) \frac{\sigma_0 (x)}{8\pi \sigma (x)} L_0 (x) + \xi_{+,1}^1 (s) \frac{3 M_2^2}{16 \pi x \sigma (x)} L_0^2 (x) \right]
\]

(IV.25)

Their evaluation has to be performed numerically. Notice, however, that these representations are not necessarily best suited for this purpose, due to possible numerical stability problems. These functions are actually often expressed as two-dimensional integrals, which can be computed more efficiently. Examples of such representations can be found in the articles quoted under [51], and we refer the reader to them and to the papers quoted therein for further discussions on these aspects.

We could proceed in a similar way in order to write down two-loop representations for the $\pi\pi$ amplitudes, but this is not very useful for the applications of interest here. For the sake of illustration, let us consider the function $W_0^{00} (s)$, involved in the amplitude for elastic scattering of two neutral pions, as an example. With the results already at our disposal, we obtain

\[
W_0^{00} (s) = \frac{1}{2} \left[ 16 \pi \varphi_0^{00} (s) \right]^2 \bar{J}_0 (s) + \left[ 16 \pi \varphi_0^\pi (s) \right]^2 \bar{J}(s)
+ 32 \pi \frac{M_1^4}{F_1^2} \varphi_0^{00} (s) \left[ \xi_{0,0}^1 (s) \bar{J}_0 (s) + \xi_{0,1}^1 (s) \bar{J}_1^0 (s) + \xi_{0,2}^1 (s) \bar{J}_2^0 (s) + \xi_{0,3}^1 (s) \bar{J}_3^0 (s) \right]
+ 64 \pi \frac{M_1^4}{F_1^2} \varphi_0^\pi (s) \left[ \xi_{0,0}^1 (s) \bar{J}(s) + \xi_{0,1}^1 (s) \bar{J}_1^0 (s) + \xi_{0,2}^1 (s) \bar{J}_2^0 (s) + \xi_{0,3}^1 (s) \bar{J}_3^0 (s) \right]
+ \Delta_{NF} W_0^{00} (s),
\]

(IV.26)
As for the corresponding subtraction polynomials, they were already given in eq. (II.28).

V. ISOSPIN BREAKING IN PHASE SHIFTS

In this section we turn to the issue of isospin breaking in the phases of the form factors, making use of the results obtained so far. We begin with a discussion of some general aspects of the phases of the form factors in the low-energy regime, and consider the lowest-order isospin-breaking corrections. The corrections at next order are then discussed in a framework where only contributions of first order in the difference of the pion masses are kept.

A. General discussion and leading-order results

The phases of the form factors are defined generically as

$$ F(s + i0) = e^{i\delta(s)} F(s - i0), $$  \hspace{1cm} (V.1)

where the phases will be denoted $\delta_0^\alpha(s)$, $\delta_s^\alpha(s)$, and $\delta_1^\alpha(s)$ for the form factors $F_0^\alpha(s)$, $F_s^\alpha(s)$, and $F_2^\alpha(s)$, respectively. For a discussion of the analyticity properties of the form factors, we refer the reader to ref. [21]. Each of these phases $\delta_\ell(s)$ has itself a low-energy expansion, $\delta_\ell(s) = \delta_{\ell,2}(s) + \delta_{\ell,4}(s) + \mathcal{O}(E^6)$. Our aim is to address the issue of isospin-breaking corrections in the phases order by order in this expansion, i.e. our interest lies in the differences

$$ \Delta \delta_\ell(s) \equiv \delta_\ell(s) - \delta_{\ell,2}(s) = \Delta \delta_\ell(s) + \mathcal{O}(E^6) $$  \hspace{1cm} (V.2)

between the phases $\delta_\ell(s)$ in the presence of isospin breaking and the phases $\delta_{\ell,2}(s)$ in the isospin limit, with $\Delta_\ell \delta_{\ell,2}(s) \equiv \delta_{\ell,n}(s) - \delta_{\ell,n}(s)$. For the cases under consideration, we have

$$ \delta_0^\alpha(s) = \frac{1}{2} \sigma_0(s) \left[ \varphi_0^\alpha(s) + \psi_0^\alpha(s) \right] \theta(s - 4M_{\pi^0}^2) $$

$$ - \sigma(s) \frac{F_0^\alpha(0)}{F_2^\alpha(0)} \left[ \varphi_0^\alpha(s) \frac{1 + \Gamma_0^\alpha(s)}{1 + \Gamma_2^\alpha(s)} + \psi_0^\alpha(s) \right] \theta(s - 4M_\pi^2) + \mathcal{O}(E^6) $$

$$ \delta_s^\alpha(s) = \sigma(s) \left[ \varphi_s(s) + \psi_s(s) \right] \theta(s - 4M_s^2) $$

$$ - \frac{1}{2} \sigma_0(s) \frac{F_s^\alpha(0)}{F_2^\alpha(0)} \left[ \varphi_s(s) \frac{1 + \Gamma_s(s)}{1 + \Gamma_2^s(s)} + \psi_s(s) \right] \theta(s - 4M_{\pi^0}^2) + \mathcal{O}(E^6) $$

$$ \delta_1^\alpha(s) = \sigma(s) \left[ \varphi_1(s) + \psi_1(s) \right] \theta(s - 4M_\pi^2) + \mathcal{O}(E^6). $$  \hspace{1cm} (V.3)

We thus deduce that

$$ \delta_{0,2}^\alpha(s) = \frac{1}{2} \sigma_0(s) \varphi_0^\alpha(s) \theta(s - 4M_{\pi^0}^2) - \sigma(s) \psi_0^\alpha(s) \theta(s - 4M_\pi^2) $$

$$ \delta_{0,2}^\beta(s) = \sigma(s) \varphi_1(s) \theta(s - 4M_\pi^2) - \frac{1}{2} \sigma_0(s) \psi_0^\alpha(s) \theta(s - 4M_{\pi^0}^2) $$

$$ \delta_{1,2}^\alpha(s) = \sigma(s) \varphi_1(s) \theta(s - 4M_\pi^2), $$  \hspace{1cm} (V.4)
at leading order, while, at next-to-leading order,

\[
\delta^\pi_{0,4}(s) = \frac{1}{2} \sigma_0(s) \psi_0^{00}(s) \theta(s - 4M_{\pi^0}^2) - \sigma(s) \psi_0^{\pi}(s) \theta(s - 4M_\pi^2) - \sigma(s) \phi^\pi_0(s) \left[ \left( \frac{F_S^c(0)}{F_S^c(0)} - 1 \right) + \left( \Gamma_S^c(s) - \Gamma_S^{\pi^0}(s) \right) \right] \theta(s - 4M_\pi^2)
\]

\[
\delta^\pi_{0,4}(s) = \sigma(s) \psi_0^{\pi^+}(s) \theta(s - 4M_{\pi^+}^2) - \frac{1}{2} \sigma_0(s) \psi_0^{\pi}(s) \theta(s - 4M_{\pi^0}^2) - \frac{1}{2} \sigma_0(s) \phi^\pi_0(s) \left[ \left( \frac{F_S^c(0)}{F_S^c(0)} - 1 \right) - \left( \Gamma_S^c(s) - \Gamma_S^{\pi^0}(s) \right) \right] \theta(s - 4M_\pi^2)
\]

\[
\delta^\pi_{1,4}(s) = \sigma(s) \psi_1^{\pi^+}(s) \theta(s - 4M_{\pi^+}^2) - \frac{1}{2} \sigma_0(s) \phi^\pi_0(s) \left[ \left( \frac{F_S^c(0)}{F_S^c(0)} - 1 \right) - \left( \Gamma_S^c(s) - \Gamma_S^{\pi^0}(s) \right) \right] \theta(s - 4M_\pi^2).
\]

(5.5)

Here we have used the property that the quantities \(F_S^c(0)/F_S^c(0) - 1, \Gamma_S^c(s), \text{ and } \Gamma_S^{\pi^0}(s)\) are all of order \(O(E^2)\). At order \(O(E^3)\), the isospin-symmetric phases \(\delta_{\ell,n}(s)\) are obtained from the expressions (5.4) and (5.5) by setting \(M_{\pi^0}^0\) equal to \(M_\pi\), and by replacing the lowest order expressions of the \(\pi\pi\) waves by their counterparts in the isospin limit, \(\phi^\pi_0(s)\) and \(\phi^\pi_1(s)\), respectively. In this limit, the differences \(F_S^{\pi^0}(0)/F_S^c(0) - 1\) and \(\Gamma_S^c(s) - \Gamma_S^{\pi^0}(s)\) vanish.

Before proceeding with the actual calculation, let us make a few comments. First, we should point out an important aspect that emerges from the above expressions, and that has already been observed in ref. [21]. At order \(O(E^2)\), the phases of the form factors are entirely determined by the \(\pi\pi\) scattering data. In the case of the vector form factor where, due to Bose symmetry, the \(\pi^+\pi^-\) channel alone contributes, Watson’s theorem is still operative: the phase of \(F_V^c(s)\) coincides with the phase of the \(P\)-wave projection of the corresponding scattering amplitude \(A^{++}(s, t, u)\). For the scalar form factors, the situation is different, due to the mixing between the two channels that contribute to the unitarity sum in the \(S\) wave. Nevertheless, the phase has still a “universal” character, in the sense that its expression involves only the partial waves of the \(\pi\pi\) scattering amplitudes, and makes no explicit reference to the form factors themselves. This property, however, rests entirely on the fact that \(F_S^{\pi^0}(0)/F_S^c(0) = 1\) at this order. Hence, this situation does no longer survive at the next order in the case of the scalar form factors. In addition to the universal parts, \(\Delta^T_1 \delta_0^\pi(s)\) and \(\Delta^T_1 \delta_0^\pi(s)\), provided by the \(O(E^4)\) partial waves of the \(\pi\pi\) scattering amplitudes, there now appear contributions \(\Delta^T_1 \delta_0^{\pi^0}(s)\) and \(\Delta^T_1 \delta_0^{\pi^0}(s)\) that

FIG. 5: The lowest-order phase difference \(\Delta^T_1 \delta_0^\pi(s)\) (solid line), defined in eq. (V.9), as a function of the energy \(s\) in MeV, for \(\alpha = 1.40\) and \(\beta = 1.08\). Also shown is the quantity \(\Delta^T_1 \delta_0^{\pi^0}(s)\) (long-dashed line), defined in eq. (V.8), as well as the approximation to first order in \(\Delta_\pi\) (short-dashed line), cf. eq. (V.15).
The lowest-order phase difference $\Delta_2 \delta_0^0 (s)$ (solid line), defined in eq. (V.3), as a function of the energy $\sqrt{s}$ in MeV, for $\alpha = 1.40$ and $\beta = 1.08$. Also shown is the quantity $\Delta_2 \delta_0^0 (s)$ (long-dashed line), defined in eq. (V.8), as well as the approximation to first order in $\Delta_\rho$ (short-dashed line), cf. eq. (V.15).

depends explicitly on the form factors considered:

$$
\Delta_4 \delta_0^0 (s) = \Delta_4^U \delta_0^0 (s) + \Delta_4^F \delta_0^0 (s)
$$

$$
\Delta_4 \delta_0^0 (s) = \Delta_4^U \delta_0^0 (s) + \Delta_4^F \delta_0^0 (s).
$$

(V.6)

The universal parts $\Delta_4^U \delta_0^0 (s)$ and $\Delta_4^F \delta_0^0 (s)$ correspond to the first lines of the expressions of $\delta_0^0 (s)$ and of $\delta_{0,4} (s)$ given in eq. (V.5), respectively, from which the isospin-symmetric contributions are subtracted. The second lines in these same expressions correspond to the form-factor dependent contributions $\Delta_4^F \delta_0^0 (s)$ and $\Delta_4^F \delta_0^0 (s)$. Finally, we also note that for the scalar form factors some contributions in the expressions (V.4) and (V.5) start at $s = M_{\pi}^2$, while others appear only for $s \geq M_{\pi}^2$. This is of course the manifestation of the unitarity cusp in the phases themselves.

At order $O(E^2)$, the isospin-symmetric phases $\phi_{0,2}(s)$ are obtained from the expressions (V.4), putting $M_{\pi,0}$ equal to $M_{\pi}$, and replacing the lowest order expressions of the $\pi\pi$ $S$ and $P$ waves $\varphi_{0}(s)$ by their counterparts in the isospin limit, $\tilde{\varphi}_{0}(s)$, which read

$$
\phi_{0,0}^0 (s) = \frac{\alpha M_{\pi}^2}{16\pi F_{\pi}^2}
$$

$$
\phi_{0,0}^0 (s) = \frac{-\beta}{16\pi F_{\pi}^2} \left( s - \frac{4}{3} M_{\pi}^2 \right) - \frac{\alpha M_{\pi}^2}{48\pi F_{\pi}^2}
$$

$$
\phi_{0}^+ (s) = \frac{\beta}{32\pi F_{\pi}^2} \left( s - \frac{4}{3} M_{\pi}^2 \right) + \frac{\alpha M_{\pi}^2}{24\pi F_{\pi}^2}
$$

$$
\phi_{0}^- (s) = \frac{\beta}{96\pi F_{\pi}^2} \left( s - 4 M_{\pi}^2 \right).
$$

(V.7)

We then obtain

$$
\Delta_2 \delta_0^0 (s) = \frac{\alpha_{00} M_{\pi}^2}{32\pi F_{\pi}^2} \left[ \sigma_0(s)(s - 4 M_{\pi}^2) - \sigma(s)(s - 4 M_{\pi}^2) \right] + \frac{4\beta_{x} - 2\alpha_{x} - 3\alpha_{00}}{96\pi} \frac{\Delta_\rho}{F_{\pi}^2}
$$

$$
\Delta_2 \delta_0^0 (s) = \frac{\beta_{x} - \beta}{16\pi F_{\pi}^2} \left( s - \frac{4}{3} M_{\pi}^2 \right) + \frac{\alpha_{x} - \alpha M_{\pi}^2}{48\pi F_{\pi}^2} + \frac{\alpha_{00} - \alpha M_{\pi}^2}{32\pi F_{\pi}^2} \sigma(s)(s - 4 M_{\pi}^2)
$$

$$
\Delta_2 \delta_0^0 (s) = \frac{1}{32\pi F_{\pi}^2} \left[ \beta_{x} \left( s - \frac{2}{3} M_{\pi}^2 - \frac{2}{3} M_{\pi}^2 \right) + \frac{\alpha_{x} M_{\pi}^2}{3} \right] \left[ \sigma_0(s)(s - 4 M_{\pi}^2) - \sigma(s)(s - 4 M_{\pi}^2) \right]
$$
larger values of $\alpha$, the contribution proportional to $\alpha$ being suppressed by a factor of three as compared to $\Delta^{\text{LO}}_2 \delta^0_0(s)$. However, the relative variation in $\beta$ only covers a restricted range, approximatively $\pm 5\%$, so that the variations in $\alpha$ account for the largest part of the effect. As one leaves the cusp region towards larger values of $s$, the contribution proportional to $\alpha$ looses weight, and the variations are less ample. This behaviour is conveyed in a simple manner by the high-energy asymptotic expressions of $\Delta^{\text{LO}}_2 \delta^0_0(s)$, and, even more strongly, of $\Delta^{\text{LO}}_2 \delta^+_0(s)$:

\[
\Delta^{\text{LO}}_2 \delta^0_0(s) \sim \frac{8 \beta - 5 \alpha}{96 \pi} \frac{\Delta^s}{F^s} + \frac{\alpha - \beta}{6 \pi} \frac{\Delta^s}{F^s} \frac{M^2}{s} + \ldots
\]

\[
\Delta^{\text{LO}}_2 \delta^+_0(s) \sim \frac{26 \beta - 5 \alpha}{96 \pi} \frac{\Delta^s}{F^s} + \frac{\alpha - 4 \beta}{8 \pi} \frac{\Delta^s}{F^s} \frac{M^2}{s} + \ldots
\]
B. Isospin-breaking corrections at next-to-leading order

The evaluation of $\Delta_4 \delta_{\ell}(s)$, the isospin-breaking effects in the phases at next-to-leading order, relies on the results obtained in the preceding sections. The corresponding numerical analysis will be the subject of section VI below. Here, we wish to proceed for a while at the analytical level, but for simplicity, and since the next-to-leading order isospin-breaking corrections are expected to be small, we will restrict ourselves to the first order in $\Delta_{\pi}$. For this purpose, we expand the various quantities of interest with respect to $\Delta_{\pi}$, and neglect contributions beyond the linear terms. In the case of the phase-space factor for the neutral two-pion state, an expansion like:

$$\sigma_0(s) = \sigma(s) \left[ 1 + \frac{2}{s - 4M_{\pi}^2} \Delta_{\pi} + O(\Delta_{\pi}^2) \right],$$

will not make sense when $s$ remains close to $4M_{\pi}^2$. This means that our expansion to first order in isospin breaking will only provide an adequate description in regions of phase space sufficiently away from the $\pi^0\pi^0$ and $\pi^+\pi^-$ thresholds. From an experimental point of view, this needs not constitute a serious drawback, since the vicinity of the two-pion thresholds is usually part of the regions of phase space where the acceptance is low, as can be seen, for instance, from figures 5 and 6 in the case of the $K^+\pi^-$ decay (see, however, the discussion on the $K^\pm \rightarrow \pi^0\pi^0 e^\pm \nu_e$ decay mode in [12]). From a practical point of view, we gain the advantage of having to deal with expressions which remain tractable. In the rest of this section, we will therefore remain within the framework set up by these two conditions – staying away from the two-pion thresholds, and considering only first-order isospin-violating effects. For illustration, let us consider the lowest-order corrections in the $S$-wave phases, $\Delta_2^{LO} \delta_{\pi_0}(s)$ and $\Delta_2^{LO} \delta_{\pi^0}(s)$ that we have discussed in the preceding subsection. Applying the procedure that we have just described to the expressions (V.9) yields, for $s > 4M_{\pi}^2$,

$$\Delta_2^{LO} \delta_{\pi_0}(s) = \sigma(s) \left[ \frac{\alpha}{16\pi} \frac{M_{\pi}^2}{s - 4M_{\pi}^2} + \frac{8\beta - 5\alpha}{96\pi} \right] \frac{\Delta_{\pi}}{F_{\pi}^2} + O(\Delta_{\pi}^2)$$

$$\Delta_2^{LO} \delta_{\pi^0}(s) = \sigma(s) \left[ \frac{8\beta + \alpha}{48\pi} \frac{M_{\pi}^2}{s - 4M_{\pi}^2} + \frac{26\beta - 5\alpha}{96\pi} \right] \frac{\Delta_{\pi}}{F_{\pi}^2} + O(\Delta_{\pi}^2).$$

On figures 5 and 6, respectively, these expressions for $\Delta_2^{LO} \delta_{\pi_0}(s)$ and $\Delta_2^{LO} \delta_{\pi^0}(s)$ are shown together with the exact expressions given in equations (V.8) and (V.9). We see that the curve corresponding to the approximate

FIG. 7: The lowest-order phase difference $\Delta_2^{LO} \delta_{\pi_0}(s)$ for different values of $\alpha$ and $\beta$. The solid curve corresponds to $(\alpha, \beta) = (1.4, 1.08)$, while the cases $(\alpha, \beta) = (1.0, 1.04)$ and $(\alpha, \beta) = (1.8, 1.12)$ are represented by the long-dashed and short-dashed curves, respectively.
expression overshoots the exact formula by about 25% at $\sqrt{s} = 285$ MeV, while the difference drops to a level around 5% for $\sqrt{s}$ above $\sim 325$ MeV.

The expressions of the next-to-leading partial waves $\psi_0^{(0)}(s)$, $\psi_1^{-}(s)$, and $\psi_0^{-}(s)$ in the isospin limit have been given in eqs. (IV.7), (IV.12), and (IV.18), respectively. In order to evaluate $\Delta_0^\pi$, we will proceed as follows. First, we replace, in the polynomials $\xi_0^{(n)}(s)$, $\xi_{+;S, P}^{(n)}$ and $\xi_{x}^{(n)}(s)$, each occurrence of $M_0^2$ by $M_0^2 - \Delta_\pi$.

Keeping only the terms of first order in $\Delta_\pi$, we thus obtain

$$\xi^{(n)}_0(s) = \overline{\xi}^{(n)}_0(s) + \frac{\Delta_\pi}{M_0^2} \delta \xi^{(n)}_0(s) + O(\Delta_\pi^2),$$

and so on. Next, we have to expand the functions that multiply the polynomials $\xi^{(n)}_0(s)$, $\xi_{+;S, P}^{(n)}$ and $\xi_{x}^{(n)}(s)$, see eqs. (IV.6), (IV.10), (IV.11), and (IV.15), to first order in $\Delta_\pi$. Finally, when subtracting from the functions $\xi^{(n)}_0(s)$, $\xi^{(n)}_{+;S, P}$, $\xi^{(n)}_{x}(s)$ the corresponding combinations of polynomials $\xi^{(n)}_0(s)$ arising in the isospin limit, as given in [(IV.7), (IV.12), and (IV.18)], isospin breaking only occurs through differences like $\alpha_x - \alpha$, or $\beta_+ - \beta$, for instance. Collecting these three contributions provides us with an expression for $\Delta_0^\pi(s)$ accurate at first order in $\Delta_\pi$.

As an illustration, consider the case of $\psi_0^{(0)}(s)$, cf. eq. (IV.6). The first step is a straightforward algebraic exercise. It produces functions $\overline{\xi}^{(n)}_0(s)$, $\xi^{(0)}_0(s) = 0, 1, 2, 3$, with

$$\overline{\xi}^{(1)}_0(s) = \overline{\xi}^{(1;0)}_0(s) + 4\frac{\Delta_\pi}{M_0^2} [4k_0(s) + k_1(s)] \quad + O(\Delta_\pi^2).$$

For the second step, we expand the various polynomials that appear in the expression for $\psi_0^{(0)}(s)$ to first order in $\Delta_\pi$ [the functions $k_0(s)$ are defined in equation (IV.5)]:

$$2 \frac{\sigma(s)}{\sigma_0(s)} L_0(s) = 16\pi k_1(s) + 32\pi \frac{\Delta_\pi}{M_0^2} \left\{ k_0(s) + \frac{M_0^2}{s - 4M_0^2} [4k_0(s) + k_1(s)] \right\} + O(\Delta_\pi^2)$$

$$2 \frac{\sigma(s)}{\sigma_0(s)} L_1(s) = 16\pi k_1(s) + 8\pi \frac{\Delta_\pi}{M_0^2} \left\{ k_1(s) - k_2(s) - \frac{4M_0^2}{s - 4M_0^2} [4k_0(s) + k_1(s)] \right\} + O(\Delta_\pi^2)$$

$$2 \sigma(s) \sigma_0(s) L_2(s) = 16\pi k_1(s) + 8\pi \frac{\Delta_\pi}{M_0^2} \left\{ k_1(s) - k_2(s) - 4k_0(s) \right\} + O(\Delta_\pi^2)$$

$$3M_0^2\frac{\sigma(s)}{s - 4M_0^2} L_3(s) = 16\pi k_1(s) - 16\pi \frac{\Delta_\pi}{M_0^2} \left\{ k_3(s) + \frac{M_0^2}{s - 4M_0^2} [3k_1(s) + 4k_0(s)] \right\} + O(\Delta_\pi^2)$$

![Graph showing the lowest-order phase difference $\Delta_2^\pi$ for different values of $s$. The solid curve corresponds to $(\alpha, \beta) = (1.4, 1.08)$, while the cases $(\alpha, \beta) = (1.0, 1.04)$ and $(\alpha, \beta) = (1.8, 1.12)$ are represented by the long-dashed and short-dashed curves, respectively.](image)
Concerning the form-factor dependent parts, the one-loop result (IV.3) gives, at the same level of accuracy, 

\[ \psi_0^{00}(s) = 2 \frac{M_4^4}{F_\pi} \sqrt{\frac{s}{s - 4M_\pi^2}} \sum_{n=0}^{3} \left[ \xi_0^{(n)}(s) + \frac{\Delta_\pi}{M_\pi^2} \Delta_\pi^{(n)}(s) \right] k_n(s) + O(\Delta^2_\pi), \]  

(V.19)

where \( \Delta_\pi^{(n)}(s) \) is the sum of \( \delta_\pi^{(n)}(s) \) in eq. (V.10) and of the contribution generated by the expansions (V.18) to first order in \( \Delta_\pi \). The expressions of the functions \( \Delta_\pi^{(n)}(s) \) are displayed in appendix D. Let us briefly comment on the appearance of contributions involving the factor \( M_\pi^2/(s - 4M_\pi^2) \) in equation (V.18). Upon closer inspection, one finds that the combinations \( (4k_0(s) + k_1(s))/(s - 4M_\pi^2) \), \( [k_1(s) + 4k_3(s)]/(s - 4M_\pi^2) \), and \( k_2(s)/(s - 4M_\pi^2) \), become actually proportional to \( \sigma(s) \) as \( s \) approaches \( 4M_\pi^2 \) from above.

The extraction of the first-order isospin-breaking contributions from the remaining one-loop partial waves proceeds along similar lines, and we merely quote the resulting formulae:

\[ \psi_0^{+-}(s) = 2 \frac{M_4^4}{F_\pi} \sqrt{\frac{s}{s - 4M_\pi^2}} \sum_{n=0}^{3} \left[ \xi_0^{(n)}(s) + \frac{\Delta_\pi}{M_\pi^2} \Delta_\pi^{(n)}(s) \right] k_n(s) + O(\Delta^2_\pi), \]  

(V.20)

More details, as well as expressions of the functions \( \Delta_\pi^{(n)}(s) \), \( \Delta_\pi^{(n)}(p,s) \), and \( \Delta_\pi^{(n)}(x) \) are given in appendix D. Working at first order in \( \Delta_\pi \) has allowed us to cast the functions \( \psi_0^{00}(s) \), \( \psi_0^{+-}(s) \), and \( \psi_0^{0}(s) \) into a form that makes the comparison with their expressions in the isospin limit straightforward.

At next-to-leading order, the isospin-breaking contributions to the P-wave phase are thus simply proportional to the difference \( \psi_0^{+-}(s) - \psi_0^{+-}(s) \):

\[ \Delta_4^\pi \delta_0(s) = 2 \frac{M_4^4}{F_\pi} \sum_{n=0}^{4} \left( \xi_0^{(n)}(s) - \frac{1}{2} \xi_0^{(n)}(s) + \frac{\Delta_\pi}{M_\pi^2} \Delta_\pi^{(n)}(s) \right) k_n(s) + O(\Delta^2_\pi). \]  

(V.21)

In the case of the S-wave phases, the corresponding corrections are naturally split into a universal contribution \( \Delta_4^\pi \delta_0(s) \) and a form-factor dependent piece \( \Delta_4^\pi \delta_0(s) \), cf. eq. (V.6). Keeping only the first-order isospin-breaking contributions, the two universal pieces read, again for \( s > 4M_\pi^2 \),

\[ \Delta_4^\pi \delta_0^\pi(s) = \sigma(s) \left\{ \frac{1}{2} \psi_0^{00}(s) - \frac{1}{2} \psi_0^{-+}(s) + \frac{1}{2} \left( \frac{\sigma_0(s)}{\sigma(s)} - 1 \right) \psi_0^{00}(s) \right\} \]

\[ = 2 \frac{M_4^4}{F_\pi} \sum_{n=0}^{3} \left( \frac{1}{2} \xi_0^{(n)}(s) - \xi_0^{(n)}(s) - \frac{1}{2} \xi_0^{(n)}(s) \right) k_n(s) + O(\Delta^2_\pi), \]

\[ \Delta_4^\pi \delta_0^\pi(s) = \sigma(s) \left\{ \psi_0^{00}(s) - \psi_0^{-+}(s) - \frac{1}{2} \psi_0^{00}(s) + \frac{1}{2} \psi_0^{-+}(s) - \frac{1}{2} \left( \frac{\sigma_0(s)}{\sigma(s)} - 1 \right) \psi_0^{00}(s) \right\} \]

\[ = 2 \frac{M_4^4}{F_\pi} \sum_{n=0}^{3} \left( \xi_0^{(n)}(s) - \frac{1}{2} \xi_0^{(n)}(s) - \frac{1}{2} \xi_0^{(n)}(s) \right) \]

\[ + \frac{\Delta_\pi}{M_\pi^2} \left[ \frac{1}{2} \Delta_\pi^{(n)}(s) - \frac{1}{2} \Delta_\pi^{(n)}(s) - \frac{1}{2} \Delta_\pi^{(n)}(s) \right] k_n(s) + O(\Delta^2_\pi). \]  

(V.22)

Concerning the form-factor dependent parts, the one-loop result (IV.3) gives, at the same level of accuracy,

\[ \Gamma_4^\pi(s) - \Gamma_4^\pi(s) = s(\alpha_\pi^\pi - \alpha_\pi^\pi) + \frac{\Delta_\pi}{32\pi^2 F_\pi^2} \left[ -\beta \frac{s}{M_\pi^2} + \frac{28\beta + 2\alpha}{3} \right] + \frac{\Delta_\pi}{96\pi^2 F_\pi^2} (4\beta - \alpha) L(s) \sigma(s) \]

\[ + \Delta_\pi \frac{\Delta_\pi}{96\pi^2 F_\pi^2} (14\beta + \alpha) \sigma(s) L(s) + O(\Delta^2_\pi). \]

(V.23)
VI. NUMERICAL EVALUATION

This section is devoted to the numerical evaluation of the isospin breaking corrections $\Delta_4 \delta_i(s)$ keeping the full dependence on $\Delta_\pi$. For this, we first need to know how the subtraction parameters that appear in the amplitudes and form factors after the first iteration are related to the corresponding ones in the isospin limit.

A. Determination of the subtraction parameters

From the dispersive representations of the form factors and scattering amplitudes, we have obtained the isospin-breaking corrections in the phases of the pion form factors beyond leading order. These expressions involve the normalizations $F_\pi^0(0)$ and $F_\pi^0(0)$ and the two subtraction parameters $a_\pi^0$ and $a_\pi^s$ in the one-loop expressions of the form factors, and only a subset of the 15 subtraction constants that appear in the $\pi\pi$ amplitudes, namely $\alpha_{00}$, $\alpha_x$, $\alpha_+$, $\beta_x$, and $\beta_+$ on the one hand, $\lambda_{00}^{(1)}$, $\lambda_x^{(i)}$, and $\lambda_+^{(i)}$, $i = 1, 2$, on the other hand. In the isospin limit, the latter are given, as indicated in equation (III.2), in terms of the constants $\lambda_0$ discussed and evaluated in refs. 24, 50. More accurate determinations have appeared since then 15, 17, see below. We thus merely need to evaluate the size of the isospin-breaking deviations like, say, $\lambda_0^{(i)} - \lambda_i$. The subset $\alpha_{00}...\beta_+$ is likewise related to the subthreshold parameters $\alpha$ and $\beta$ in the isospin limit. At lowest order, these relations were given in eq. (III.5), but in order to evaluate the phases at next-to-leading order, it is necessary to go beyond this approximation. Again, we only need to know the size of the deviations from the isospin-limit quantities $\alpha$ and $\beta$. As discussed at the end of subsection (III.A), $\alpha$ and $\beta$ represent the observables that we eventually would like to pin down from a phenomenological analysis of experimental data, so we have to trace down the dependence on these parameters beyond the lowest-order expressions.

Let us now explain how we proceed with these tasks. For this purpose, we briefly come back to the discussion in subsection (II.A). The framework that we have presented there can be described by an “effective” lagrangian, whose form is similar to the chiral lagrangian used to treat electromagnetic corrections, but without including photons as dynamical degrees of freedom, as their effect is supposed to be treated by other means or otherwise to be negligible. The leading-order (strong) lagrangian $L_2$ is then supplemented with a contribution of the form [21]:

$$L_2 \rightarrow L_2 + \hat{C}(QUQU^\dagger), \ Q = \text{diag}(2e/3,-e/3), \quad (VI.1)$$

where $\hat{C}$ is a low-energy constant that breaks isospin symmetry among the pion masses. The last term, through its transformation properties under chiral symmetry, encodes the information about the electromagnetic origin of the pion mass difference. Although we could have absorbed it into the definition of $\hat{C}$, we have left the electric charge $e$ apparent, in order to make the comparison with the usual effective theory in presence of electromagnetic interactions more convenient. We call (VI.1) an “effective” lagrangian since there is no identifiable fundamental theory of which it would constitute the effective theory in the usual sense [68], the quotation marks serving as a reminder of this limitation. Nevertheless, eq. (VI.1) constitutes a suitable starting point for a low-energy expansion, with a well-defined and consistent power counting, which reproduces the features of the framework adopted here as far as isospin-violating corrections are concerned. Thus, the “effective” lagrangian at next-to-leading order is supplemented with the terms described in ref. 40, but with the corresponding low-energy constants denoted with a hat, to distinguish them from those obtained in the theory with virtual photons included. Indeed, the absence of virtual photons modifies the structure of the one-loop divergences, and the scale dependence of the renormalized low-energy constants $\hat{k}_i^\mu(\mu)$ is given by

$$e^2 \mu \frac{d}{d\mu} \hat{k}_i^\mu(\mu) = -\frac{1}{16\pi^2} \hat{\sigma}_i, \quad (VI.2)$$

with (for the low-energy constants of interest in our case)

$$\hat{\sigma}_1 = \hat{\sigma}_5 = -\frac{1}{10} \frac{\Delta_\pi}{F^2}, \quad \hat{\sigma}_8 = -\frac{1}{2} \frac{\Delta_\pi}{F^2}$$

$$\hat{\sigma}_2 = \hat{\sigma}_4 = \hat{\sigma}_6 = \frac{A_\pi^2}{F^2}, \quad \hat{\sigma}_3 = \hat{\sigma}_7 = 0. \quad (VI.3)$$
One has also a contribution quadratic in the difference $M_{\pi}^2 - M_{\pi^0}^2$ from the low-energy constant $\hat{k}_{14}(\mu)$:

$$e^2 \mu \frac{d}{dp} \hat{k}_{14}(\mu) = -\frac{3}{16\pi^2} \frac{\Delta^2}{F_4}.$$  \hspace{1cm} (VI.4)

We emphasize that these scale dependences are different from the ones of the equivalent low-energy counterterms discussed in ref. [40], since we have considered a theory where no virtual photons are included. Furthermore, they follow in a straightforward manner from the expressions given in eqs. (3.9)-(3.11) of [40] upon dropping the terms that do not contain the parameter $Z = C/F_4$, with $M_{\pi}^2 - M_{\pi^0}^2 = 2e^2F_2Z$ at lowest order, in the notation of that reference [in the case of $\tilde{\sigma}_{14}$, only the terms in $Z^2$ must be retained, the constant $\hat{k}_{14}$ being multiplied by $e^4$].

The relevant subtraction constants are then obtained upon matching the one-loop expressions obtained within the framework we have just described with the representations obtained in section III. The outcome of this exercise is displayed in appendix E. Let us recall here that there exist explicit one-loop calculations of the various $\pi \pi$ amplitudes [40, 41, 43] and form factors [44] considered here, obtained within the full QCD+QED effective theory [37–42]. As mentioned in subsection II A, these calculations also include isospin-violating contributions arising from the exchanges of virtual low-energy photons, which are however not considered here. In order to make a comparison with these one-loop calculations, we must therefore remove the contributions of virtual photons from the expressions given in these references, and only keep the effects due to the difference of the pion masses [21]. From a practical point of view, this can be done as described above: contributions proportional to $e^2$, but without the appropriate $Z$ factor, arise from the exchange of virtual photons and are discarded, while at the same time the low-energy constants $k_i^r(\mu)$ are replaced by $\hat{k}_i^r(\mu)$. Furthermore, since we want to display the dependence on the two independent parameters $\alpha$ and $\beta$, we have, in the computation of the loop contributions, explicitly kept the quantities $F$ and $\tilde{m}B$ defining the leading-order amplitude (III.7), for which we have then substituted the lowest-order expressions given in (III.8). This brings in another difference with the one-loop calculations available in the literature.

**B. Numerical input values**

We wish to investigate the size of the isospin-breaking corrections as functions of $\alpha$ and $\beta$, for fixed values of $\lambda_{1,2}$ and of the $\hat{k}_{14}^r$'s. As mentioned above, the former parameters have been evaluated before [15, 17, 24, 50] using sum rules and medium-energy $\pi \pi$ data. For the numerical evaluations below, we use the values from the “extended fit” of [17]:

$$\lambda_1 = (-4.18 \pm 0.63) \cdot 10^{-3}, \quad \lambda_2 = (8.96 \pm 0.12) \cdot 10^{-3}.$$  \hspace{1cm} (VI.5)

Let us notice that the sum rules that lead to these determinations of $\lambda_1$ and $\lambda_2$ also exhibit a mild dependence with respect to $\alpha$ and $\beta$ [24, 51]. This dependence is, however, covered by the quoted uncertainties, and we will therefore not consider it further.

As far as the constants $\hat{k}_{14}^r$ are concerned, we will assume that they take the same numerical values as the low-energy constants $k_{14}^r$. Even though this identification constitutes an approximation whose precision is difficult to assess, it is not obvious to consider simple alternatives to this choice at the time being. Incidentally, this is also the option that was retained in ref. [19].

To obtain numerical estimates of the $k_{14}^r$’s, we proceed in several steps. First, we make use of the relation between these two-flavour constants and their three-flavour counterparts $K_i^r$ [37], as worked out at one-loop level in ref. [52]. For the constants $K_{14}^r$...$K_{10}^r$ we then use the evaluation of ref. [53], as given by the last line of Table 1 in this reference. These determinations rely on a set of sum-rules [54] involving QCD four-point functions, that are saturated by the lowest-mass resonances in the corresponding channels. This kind of minimal hadronic ansatz, which finds some justification in the limit of a large number of colors $N_C$ [35, 56], is usually a good approximation [41]. We therefore endow the numbers of ref. [53] with a relative error of 33% ($1/N_C$ for $N_C = 3$) accounting for neglected subleading effects in the $1/N_C$ expansion. We assign the same relative error to the constants $K_{10}^r$ and $K_{14}^r$ that were estimated along similar lines in ref. [54]. Next, the relation between $k_8^r$ and $K_{11}^r$ also involves [52] the $SU(3)$ low-energy constants $L_4$ and $L_6$. For the latter we have taken the
\( O(p^4) \) determination \( 10^3 \cdot L_5^\prime(M_\rho) = 1.46 \pm 0.15 \) from ref. 58. \( L_5^\prime \) is not so well determined, and can induce significant differences between the patterns of chiral symmetry breaking for two and three massless flavours 54, 62. For our present purposes, we take \( 10^3 \cdot L_5^\prime(M_\rho) = 0 \pm 0.5 \), a value which was advocated on the basis of the Zweig rule 57, 58, even though later fits and discussions favour larger values 63, 64. In any case, it turns out that \( k_8 \) plays only a minor role in the numerical evaluation of isospin breaking in the phases. Finally, the relation between the \( k_i^\prime \)'s of interest to us and the \( K_i^\prime \)'s also involve four constants, \( K_{7,8,9,15}^\prime \) for which there exist no reliable determinations. We have assigned an overall uncertainty of \( \pm 1/(16\pi^2) = \pm 6.3 \cdot 10^{-3} \), based on naive dimensional analysis, to the values of \( k_{5,6,7,14}^\prime \), where these constants occur. The results of this analysis are displayed in table I. Our values reproduce those given in ref. 52, where however an overall uncertainty of \( \pm 6.3 \cdot 10^{-3} \) was assigned uniformly to all the constants \( k_i^\prime(M_\rho) \).

For the sake of illustration, we let the parameters \( \alpha \) and \( \beta \) vary within the intervals \( 1.0 \leq \alpha \leq 1.8 \), \( 1.04 \leq \beta \leq 1.12 \), suggested by the analysis of ref. 17. These intervals cover a reasonable range of possibilities, but of course, if necessary, other values can be considered.

To summarize, for the numerical analysis that follows we thus use as inputs the values of the constants \( \lambda_{1,2} \) in (VI.3), the values of the constants \( \hat{k}_i^\prime(\mu) \) as given in table I together with 66,

\[
M_\pi = 139.57 \text{ MeV}, \quad M_{\pi^0} = 134.98 \text{ MeV}, \quad F_\pi = 92.2 \text{ MeV}. \quad (VI.6)
\]

C. Size of isospin corrections to the phases

Let us start with the phase of \( F_S^\pi(s) \). We will from now on restrict ourselves to the range of energies above the cusp. On figure 9 we show the isospin-violating correction \( \Delta \delta_0^\pi(s) = \Delta_2 \delta_0^\pi(s) + \Delta_4 \delta_0^\pi(s) \), defined as explained after equation (VI.6), i.e., for \( s > 4M_\pi^2 \),

\[
\Delta \delta_0^\pi(s) = \sigma(s) \left[ \phi_0^+(s) - \hat{\phi}_0^+(s) \right] - \frac{1}{2} \sigma_0(s) \left[ \psi_0^0(s) - \hat{\psi}_0^0(s) \right] \\
+ \sigma(s) \left[ \psi_0^+(s) - \hat{\psi}_0^+(s) \right] - \frac{1}{2} \sigma_0(s) \left[ \psi_0^0(s) - \hat{\psi}_0^0(s) \right] \\
- \frac{1}{2} \sigma_0(s) \phi_0^0(s) \left[ \frac{F_S^0(0)}{F_S^0(0)} - 1 \right] - \left[ \Gamma_S^\pi(0) - \Gamma_S^\pi(0) \right] \\
= \Delta_2 \delta_0^\pi(s) + \Delta_4 \delta_0^\pi(s) + \Delta_4 \delta_0^\pi(s).
\]

The three terms of the decomposition in the second equality correspond, in succession, to the three lines of the first one, see the discussion after equation (VI.6). In the cusp region, \( \Delta \delta_0^\pi(s) \) is rather well described by \( \Delta_2 \delta_0^\pi(s) + \Delta_4 \delta_0^\pi(s) \), the contribution of \( \Delta_4 \delta_0^\pi(s) \) is only marginal. At energies above \( \sim 300 \text{ MeV} \), \( \Delta_4 \delta_0^\pi(s) \) starts to provide a sizeable negative contribution that more and more compensates for \( \Delta_4 \delta_0^\pi(s) \), so that eventually \( \Delta_4 \delta_0^\pi(s) \sim \Delta_2 \delta_0^\pi(s) \). This situation is reproduced if we take values of \( \alpha \) and \( \beta \) different from the ones adopted for figure 9, but in the range considered here. We stress that it is \( \Delta_2 \delta_0^\pi(s) \), and not \( \Delta_2 \delta_0^\pi(s) \), that provides a good description of the total effect in the region of higher energies. In this region \( \Delta_2 \delta_0^\pi(s) \) and \( \Delta_2 \delta_0^\pi(s) \) differ by more than 2 milliradians, see figure 9.

Turning next to the phase of \( F_S^\pi(s) \), the definition of the different relevant quantities reads

\[
\Delta \delta_0^\pi(s) = \frac{1}{2} \sigma_0(s) \left[ \phi_0^0(s) - \hat{\phi}_0^0(s) \right] - \sigma(s) \left[ \phi_0^\pi(s) - \hat{\phi}_0^\pi(s) \right].
\]
We find a rather different situation than in the previous case. As can be seen on figure 10, the correction $\Delta_2\delta_0^0(s)$ is large, with $\Delta_1^U\delta_0^0(s)$ and $\Delta_1^F\delta_0^0(s)$ both contributing in a substantial way. Here, $\Delta_2\delta_0^0(s)$, and even less so $\Delta_1^U\delta_0^0(s)$, do not provide a decent representation of the full isospin-violating contribution. Again, the situation shown on figure 10 for specific values of $\alpha$ and $\beta$ is actually generic for all values of these parameters in the ranges considered.

Both $\Delta\delta_0^\pi(s)$ and $\Delta\delta_0^{\pi^0}(s)$ receive contributions from $\psi_0^\pi(s)$, which contains a piece $\Delta_2\psi_0^0(s)$ of order $O(\Delta_2^\pi)$. We have checked that its numerical value is indeed tiny, in the range between $-3 \cdot 10^{-3}$ milliradian and $-2 \cdot 10^{-3}$ milliradian for all values of $s$ and of the parameters $\alpha$ and $\beta$ considered here.

In the case of $F_1^\pi(s)$, the form factor effects are absent from the isospin-violating contribution to the phase, which reads simply

$$
\Delta\delta_1^\pi(s) = \sigma(s) \left[ \varphi_1^{-}(s) - \varphi_1^{+}(s) \right] - \sigma(s) \left[ \psi_1^{-}(s) - \psi_1^{+}(s) \right]
$$

$$
= \Delta_2\delta_1^\pi(s) + \Delta_4\delta_1^\pi(s). \tag{VI.9}
$$

As can be inferred from figure 11 the overall effect is small in this case, but the unitarity correction gives a substantial decrease of the lowest-order correction.

In table II we have also summarized our result in numerical form, for some values of the energy in the range between the $2M_\pi$ threshold and the kaon mass. The presence of the form-factor dependent contributions in $\Delta_4\delta_0^\pi(s)$ and in $\Delta_4\delta^{\pi^0}_0(s)$ however precludes a direct application of our results to the experimentally more interesting case of the $K^{\pm}\to\pi^+\pi^-e^\pm\bar{\nu}_e$ decay modes. This necessitates a dedicated study, which will be reported on elsewhere. Nevertheless, it is interesting to investigate the kind of conclusions that such a study might lead to from the analysis presented so far for the scalar and vector form factors of the pion. The quantity that comes closest to the observable of interest in the context of the decay mode $K^{\pm}\to\pi^+\pi^-e^\pm\bar{\nu}_e$ is the difference between the $S$ and $P$ phases, $\delta_0^\pi(s) - \delta_1^\pi(s)$, for which the total isospin-breaking correction reads

$$
\Delta\delta_{\text{tot}}^\pi(s) \equiv \Delta\delta_0^\pi(s) - \Delta\delta_1^\pi(s)
$$
The total phase difference $\Delta \delta_0^{\pi^0}(s)$ at next-to-leading order (solid line) for $\alpha = 1.40$ and $\beta = 1.08$, as a function of energy (in MeV). Also shown are the lowest-order contribution $\Delta \delta_0^{\pi^0}(s)$ (lower dashed curve), and the combination $\Delta_2 \delta_0^{\pi^0}(s) + \Delta_4^{U} \delta_0^{\pi^0}(s) = \Delta_0^{\pi^0}(s) - \Delta_4^{E} \delta_0^{\pi^0}(s)$ (upper dashed curve).

\[
\Delta \delta_0^{\pi^0}(s) = \Delta_2 \delta_0^{\pi^0}(s) + \Delta_4^{U} \delta_0^{\pi^0}(s) + \Delta_4^{E} \delta_0^{\pi^0}(s) - \Delta_2 \delta_0^{\pi^0}(s) - \Delta_4 \delta_1^{\pi}(s).
\] (VI.10)

This correction in the phase difference is shown on figure 12 together with the error band induced by the uncertainties on the various input parameters, for fixed values of $\alpha$ and $\beta$. The main contributors to these error bars are the low-energy constants $\tilde{k}_i$, and in particular $\tilde{k}_{1,2,3,4}$, which enter in the correction $\Delta_4^{U} \delta_0^{\pi^0}(s)$, through the isospin-breaking differences such as $\alpha_{00} - \alpha$, $\alpha_x - \alpha$, and so on. Similar error bands have to be associated to the curves for $\Delta_4 \delta_0^{\pi^0}(s)$ and $\Delta_4 \delta_0^{\pi^0}(s)$ shown on figs. 9 and 10, respectively. Except in the vicinity of the cusp, the correction is thus relatively constant, around 11 milliradians for $(\alpha, \beta) = (1.4, 1.08)$, with an uncertainty that is somewhat less than ±1 milliradian. We show the correction, with the associated error band, for three sets of values for $(\alpha, \beta)$. Despite the uncertainties, there remains a sensitivity with respect to these parameters.

We have also compared the exact results obtained in this section with the approximation where only corrections of first order in $\Delta_r$ are retained, as discussed in subsection VI.3. For the range of parameters considered here, we find that using the approximate expressions for $\Delta_4 \delta_0^{\pi^0}(s)$, $\Delta_4 \delta_0^{\pi^0}(s)$, and $\Delta_4 \delta_1^{\pi}(s)$ does not

| $\sqrt{s}$ (MeV) | $\Delta_2 \delta_0^{\pi^0}(s)$ | $\Delta_4^{U} \delta_0^{\pi^0}(s)$ | $\Delta_4^{E} \delta_0^{\pi^0}(s)$ | $\Delta_0^{\pi^0}(s)$ | $\Delta_4^{U} \delta_0^{\pi^0}(s)$ | $\Delta_4^{E} \delta_0^{\pi^0}(s)$ | $\Delta_0^{\pi^0}(s)$ | $\Delta_2 \delta_0^{\pi^0}(s)$ | $\Delta_4 \delta_1^{\pi}(s)$ | $\Delta_4 \delta_1^{\pi}(s)$ |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| 280.07           | 11.10            | 1.42             | -0.29            | 12.23            | 3.53             | 2.37             | 0.38             | 6.29             | -0.007           | -0.001           | 0.006           |
| 290.97           | 9.80             | 1.24             | -0.38            | 10.66            | 2.70             | 2.08             | 0.61             | 5.40             | -0.027           | -0.005           | 0.022           |
| 304.89           | 9.54             | 1.19             | -0.45            | 10.28            | 2.41             | 2.04             | 0.78             | 5.23             | -0.050           | -0.009           | 0.041           |
| 313.47           | 9.53             | 1.17             | -0.51            | 10.19            | 2.27             | 2.06             | 0.91             | 5.25             | -0.077           | -0.015           | 0.062           |
| 322.02           | 9.64             | 1.15             | -0.57            | 10.23            | 2.21             | 2.11             | 1.04             | 5.36             | -0.107           | -0.022           | 0.085           |
| 330.78           | 9.82             | 1.13             | -0.62            | 10.33            | 2.20             | 2.18             | 1.16             | 5.54             | -0.141           | -0.030           | 0.111           |
| 340.17           | 10.05            | 1.11             | -0.68            | 10.48            | 2.23             | 2.25             | 1.28             | 5.76             | -0.180           | -0.041           | 0.139           |
| 350.92           | 10.34            | 1.06             | -0.74            | 10.67            | 2.29             | 2.34             | 1.41             | 6.04             | -0.228           | -0.055           | 0.173           |
| 364.52           | 10.75            | 1.00             | -0.82            | 10.93            | 2.41             | 2.45             | 1.57             | 6.42             | -0.295           | -0.076           | 0.218           |
| 389.71           | 11.55            | 0.80             | -0.95            | 11.40            | 2.70             | 2.61             | 1.85             | 7.16             | -0.430           | -0.126           | 0.304           |

TABLE II: The isospin-breaking corrections at order $O(E^4)$ to the phases (in milliradians) of the scalar and vector form factors for several values of the energy in the range between the $2M_\pi$ threshold and the kaon mass. The break-up into the various contributions discussed in the text is also given.
modify the values of $\Delta \delta_0^\pi(s)$, $\Delta \delta_0^\pi(s)$, and $\Delta \delta_1^\pi(s)$ by more than a few percents, as soon as the energy is more than $\sim 20$ MeV higher than the $2M\pi$ threshold, i.e. $\sqrt{s} \geq 300$ MeV. For practical purposes, one possible option consists therefore in keeping the exact expressions for $\Delta \delta_0^\pi(s)$, $\Delta \delta_0^\pi(s)$, and $\Delta \delta_1^\pi(s)$, and in using the expressions truncated at first order in $\Delta \pi$ for the next-to-leading contributions.

FIG. 11: The total phase difference $\Delta \delta_1^\pi(s)$ at next-to-leading order (solid line) for $\alpha = 1.40$ and $\beta = 1.08$, as a function of energy (in MeV). The lowest-order contribution $\Delta_2 \delta_1^\pi(s)$ (dashed line) is also shown.

FIG. 12: The isospin-breaking correction $\Delta_\text{tot}^\pi$ as a function of energy, with the error band corresponding to the uncertainties on the input parameters, for $(\alpha, \beta) = (1.8, 1.04)$ (lower band, green), $(1.4, 1.08)$ (middle band, blue), $(1.0, 1.12)$ (upper band, brown).
VII. SUMMARY AND CONCLUSIONS

In this article, we have addressed the issue of isospin breaking due to the difference between the charged and neutral pion masses in the pion form factors and \( \pi \pi \) scattering amplitudes in the low-energy domain. We have implemented a dispersive approach to obtain representations of the various \( \pi \pi \) scattering amplitudes and pion form factors that are valid at next-to-next-to-leading order in the low-energy expansion. These representations rely on general properties such as relativistic invariance, analyticity, crossing, and unitarity, combined with the chiral counting for the form factors and partial waves, and provide an extension of the general frameworks developed previously in the isospin-symmetric case to the situation where isospin-breaking effects are taken into account. This construction needs as inputs the lowest-order expressions of the pion form factors and \( \pi \pi \) \( S \) and \( P \) partial waves, and proceeds through a two-step iterative process, the partial wave projections obtained from the one-loop representation after the first step serving as inputs for the second step. At the two-loop level, we have obtained partially analytical expressions only, due to the difficulty of performing the dispersive integrals related to contributions of the non-factorizing type. We have nevertheless shown that in the limit where the pion mass difference vanishes, we reproduce known two-loop results for the scattering amplitudes and form factors in the isospin limit. On the other hand, we have obtained explicit analytical expressions for the phases of the two-loop form factors, on which we have focused. We have also provided somewhat more tractable expressions of the isospin-breaking corrections in the phases valid at first order in the difference \( M_\pi^2 - M_\pi^0 \). This approximation provides a very good description of isospin-breaking effects at next-to-leading order in the phases, for all energies lying between \( \sim 300 \text{ MeV} \) and the kaon mass.

The dispersive representations involve a limited number of subtraction constants, which have to be fixed from experimental data or theoretical sources. We have related the subtraction constants involved in the phases of the two-loop form factors to their counterparts in the isospin limit, and we have provided a numerical evaluation of the isospin-breaking corrections in these subtraction constants. This has allowed us to perform a quantitative study of the size of isospin-violating effects in the phases of the form factors.

We have displayed our results in terms of the subthreshold parameters \( \alpha \) and \( \beta \) of the pion scattering amplitude in the isospin limit. These parameters represent the unknown quantities that have to be extracted from low-energy data. Equivalently, one may take the two \( S \)-wave scattering lengths \( a_0^0 \) and \( a_2^0 \) as unknowns, and we have provided the formulae necessary in order to operate the translation between the two formulations.

As far as the phases of the form factors are concerned, the main difference with the isospin-symmetric situation is that Watson’s theorem is no longer operative when the various \( \pi \pi \) intermediate states that contribute to the unitarity sum become distinguishable, as a consequence of the explicit breaking of isospin symmetry through the pion mass-difference. The phase is still given by a universal contribution, expressed entirely in terms of data related to the \( \pi \pi \) scattering amplitudes, but at next-to-leading order there appears a second contribution, that explicitly depends on the form factors under consideration. The numerical size of this form-factor dependent part is relatively small in the case of the \( \pi^+\pi^- \) scalar form factor, but definitely more important in the \( \pi^0\pi^0 \) case.

We have also investigated the sensitivity of the isospin-breaking correction to two subthreshold parameters \( \alpha \) and \( \beta \), or equivalently to the two \( S \)-wave scattering lengths \( a_0^0 \) and \( a_2^0 \), which represent the quantities that have to be determined from data. Despite the uncertainties induced by the various other parameters, we find that the correction remains sensitive to the values of the subthreshold parameters. For the range of values that we have considered, and depending on the value of the energy, the effect in the total correction can represent up to 5 milliradians. The issue raised in the introduction about the possibility of a bias if isospin-breaking corrections are evaluated for fixed values of these scattering lengths remains therefore, in our opinion, open. Obviously, the situation that prevails after the present study devoted to the scalar form factors of the pions need not be representative of the one encountered in the case of the \( K_{\ell 4} \) form factors. To settle the issue, a dedicated study of this experimentally more interesting case is needed. This will be the subject of a forthcoming article [22].
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Appendix A: Indefinite integrals of the scalar loop function

In this appendix, we list the integrals involving the scalar loop function \( \bar{J}(t) \) that are used for the computation of the one-loop partial wave projections in section [IV]. For unequal masses \( m_1 \neq m_2 \) [when necessary, we assume that \( m_1 > m_2 \)], the loop function \( \bar{J}(t) \) is given by

\[
\bar{J}(t) = \frac{1}{16\pi^2} \left\{ 1 + \frac{m_1^2 - m_2^2}{t} \ln \frac{m_2}{m_1} - \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} \ln \frac{m_2}{m_1} + \frac{\lambda^{1/2}(t)}{2t} \ln \left[ \frac{[t - \lambda^{1/2}(t)]^2 - (m_1^2 - m_2^2)^2}{[t + \lambda^{1/2}(t)]^2 - (m_1^2 - m_2^2)^2} \right] \right\} \tag{A.1}
\]

where \( \lambda(t) = t^2 - 2t(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 \). The above expression holds for \( t < (m_1 - m_2)^2 \). For \( m_1 = m_2 = M_\pi \), or \( m_1 = m_2 = M_{\pi^0} \), this expression corresponds to the functions \( \bar{J}_1(t) \) and \( \bar{J}_0(t) \), respectively, defined in eqs. (III.10) and (III.11). Finally, the function \( \bar{J}_{\pi\pi}(s) \) defined in eqs. (III.22) and (III.24) corresponds to the choice \( m_1 = M_\pi \) and \( m_2 = M_{\pi^0} \).

The partial wave projections of the one-loop \( \pi\pi \) amplitudes discussed require the computation of integrals of the type

\[
\int dt^n \bar{J}(t) \tag{A.2}
\]

where \( n \) can take negative or positive integer values. In the present context, the range \(-2 \leq n \leq 3\) is particularly relevant. It proves convenient to define the variable \( \chi(t) \) through

\[
t = m_1^2 + m_2^2 - m_1 m_2 \left( \chi + \frac{1}{\chi} \right), \tag{A.3}
\]

so that \( 0 \leq \chi \leq 1 \) when \(-\infty < t < (m_1 - m_2)^2\). The expression of \( \chi \) in terms of \( t \) is given in eq. (IV.17), upon replacing \( M_\pi \) by \( m_1 \), and \( M_{\pi^0} \) by \( m_2 \),

\[
\chi(t) = \frac{\sqrt{(m_1 + m_2)^2 - t} - \sqrt{(m_1 - m_2)^2 - t}}{\sqrt{(m_1 + m_2)^2 - t} + \sqrt{(m_1 - m_2)^2 - t}}. \tag{A.4}
\]

For strictly positive values of the integer \( n \), the corresponding (indefinite) integrals take a relatively simple form, even when the masses \( m_1 \) and \( m_2 \) are not equal (irrelevant integration constants have been discarded),

\[
16\pi^2 \int dt \bar{J}(t) = \left[ \frac{3}{2} - \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} \ln \frac{m_2}{m_1} \right] \times \frac{t^2}{2} + \left[ (m_1^2 - m_2^2) \ln \frac{m_2}{m_1} - \frac{m_1^2 + m_2^2}{2} \right] \times t \tag{A.5}
\]

\[
16\pi^2 \int dt^2 \bar{J}(t) = \left[ \frac{4}{3} - \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} \ln \frac{m_2}{m_1} \right] \times \frac{t^3}{3} + \left[ (m_1^2 - m_2^2) \ln \frac{m_2}{m_1} - \frac{m_1^2 + m_2^2}{6} \right] \times \frac{t^2}{2} - \frac{t}{6} (m_1^4 + m_2^4 + 10m_1^2 m_2^2) \nonumber \]

\[
- \frac{1}{6} \left[ 2t^2 - (m_1^2 + m_2^2)t - (m_1^4 + m_2^4 + 10m_1^2 m_2^2) \right] \lambda^{1/2}(t) \ln \chi(t) \nonumber \]

\[
+ m_1^2 m_2^2 (m_1^2 + m_2^2) \ln^2 \chi(t) \tag{A.6}
\]

\[
16\pi^2 \int dt^3 \bar{J}(t) = \left[ \frac{5}{4} - \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} \ln \frac{m_2}{m_1} \right] \times \frac{t^4}{4} + \left[ (m_1^2 - m_2^2) \ln \frac{m_2}{m_1} - \frac{m_1^2 + m_2^2}{12} \right] \times \frac{t^3}{3} \nonumber
\]
\[-\frac{t^2}{24} (m^4 + m_1^2 + 8m_1^2 m_2^2) - \frac{t}{12} (m_1^2 + m_2^2)(m_1^2 + m_2^2 + 28m_1^2 m_2^2)\]
\[-\frac{1}{12} [3t^3 - (m_2^2 + m_2^2)t^2 - (m_1^2 + m_2^2 + 8m_1^2 m_2^2)t \quad -(m_2^2 + m_2^2)(m_1^2 + m_2^2 + 28m_1^2 m_2^2)] \chi^{1/2}(t) \ln(t)\]
\[+ m_1^2 m_2^2 (m_1^2 + m_2^2 + 3m_1^2 m_2^2) \ln^2(t). \quad (A.7)\]

For \( n \leq 0 \), the corresponding expressions are more complicated, at least when the masses are different, but can be given a simpler form when expressed in terms of the function \( H_{1,0}(x) = -Li_2(x) - \ln(x) \ln(1 - x) \), which belongs to the family of harmonic polylogarithms [67]:

\[
16\pi^2 \int dt \bar{J}(t) = \left[ 2 - \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} \ln \frac{m_2}{m_1} \right] \times t - \chi^{1/2}(t) \ln(t) + \frac{m_1^2 + m_2^2}{2} \ln^2(t)
\[+ (m_1^2 - m_2^2) \left[ H_{1,0} \left( \frac{m_1}{m_2} \chi(t) \right) - H_{1,0} \left( \frac{m_2}{m_1} \chi(t) \right) + \ln \frac{m_1}{m_2} \ln(t) \right] \quad (A.8)\]

\[
16\pi^2 \int \frac{dt}{t} \bar{J}(t) = \left[ m_1^2 + m_2^2 - 2m_1 m_2 \chi(t) \right] \ln \frac{\chi(t)}{t} - \frac{1}{2} \ln^2(t) \ln(\chi(t)) - (m_1^2 - m_2^2) \ln \frac{m_2}{m_1} \times \frac{1}{t}
\]\[- \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} \left[ H_{1,0} \left( \frac{m_1}{m_2} \chi(t) \right) - H_{1,0} \left( \frac{m_2}{m_1} \chi(t) \right) + \ln \frac{m_1}{m_2} \ln(t) \right] \quad (A.9)\]

\[
16\pi^2 \int \frac{dt}{t^2} \bar{J}(t) = \left[ -\frac{1}{2} - \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} \ln \frac{m_2}{m_1} \right] \times t - (m_1^2 - m_2^2) \ln \frac{m_2}{m_1} \times \frac{1}{2t^2}
\]\[- \frac{\chi^{1/2}(t)}{2t^2} \left[ \frac{m_1^2 + m_2^2}{m_1^2 - m_2^2} t - 1 \right] \ln(t)
\]\[- 2\frac{m_1^2 m_2^2}{(m_1^2 - m_2^2)^2} \left[ H_{1,0} \left( \frac{m_1}{m_2} \chi(t) \right) - H_{1,0} \left( \frac{m_2}{m_1} \chi(t) \right) + \ln \frac{m_1}{m_2} \ln(t) \right] \quad (A.10)\]

The range of integration, \( t_- (s) \leq t \leq t_+ (s) \), with

\[
t_\pm (s) = -\frac{1}{2} (s - 2m_1^2 + 2m_2^2) \pm \frac{1}{2} \sqrt{(s - 4m_1^2)(s - 4m_2^2)}, \quad (A.11)\]

will depend on the process under consideration. As the masses become equal, \( m_2 \rightarrow m_1 \), one has \( |\Delta_{12} = m_1^2 - m_2^2| \)

\[
t_-(s) = -(s - 4m_1^2) - 2\Delta_{12} = O(\Delta_{12}^2)
\]\[- t_+(s) = -\frac{4\Delta_{12}^2}{s - 4m_1^2} + O(\Delta_{12}^4). \quad (A.12)\]

Then

\[
\chi_- (s) = \frac{1 - \sigma(s)}{1 + \sigma(s)} + O(\Delta_{12})
\]\[- \chi_+ (s) = 1 - \frac{\Delta_{12}}{2 m_1^2} \frac{1}{\sigma(s)} + O(\Delta_{12}^2) \]
\[
\lambda(t_- (s)) = s(s - 4m_1^2) + O(\Delta_{12})
\]\[- \lambda(t_+ (s)) = \frac{s}{s - 4m_1^2} \Delta_{12}^4 + O(\Delta_{12}^4), \quad (A.13)\]

where \( \chi_{\pm} (s) = \chi(t_{\pm} (s)) \).

**Appendix B: Polynomials of the next-to-leading-order \( \pi \pi \) partial waves**

The expressions of the one-loop the partial-wave projections displayed in eqs. (IV.6), (IV.10), (IV.11), and (IV.15) involve a certain number of polynomials whose expressions are given in this appendix. In the case of \( \phi_{00}^0 (s) \), these polynomials read

\[
\xi_{s_0}^{(0)} (s) = \frac{\lambda_{00} (1)}{2 M_{\pi}} \left( 5s^2 - 16M_{\pi}^2 s + 20M_{\pi}^4 \right)
\]
\[
\psi(x) = \psi_0(x) + \psi_1(x) + \psi_2(x) + \psi_3(x) + \psi_4(x)
\]

The polynomials involved in the expression for \(\psi_0(s)\) read

\[
\psi_0 = \alpha_0^2 M_\pi^4 / 64\pi^2 M_\pi^2
\]

\[
\psi_1 = \frac{1}{32\pi^2 M_\pi^4} \left[ \frac{1}{3} \beta_x^2 s^2 + \frac{1}{3} \beta_x s \left[ \beta_x (3M_\pi^2 - 6M_\pi^2) - \alpha_x M_\pi^2 \right] \right]
\]

\[
\psi_2 = \frac{1}{64\pi^2 M_\pi^4} \left[ \beta_x \left( s - \frac{2}{3} M_\pi^2 - \frac{2}{3} M_\pi^2 \right) + \frac{1}{3} \alpha_x M_\pi^2 \right]^2
\]

\[
\psi_3 = \frac{1}{432\pi^2 M_\pi^4} \left[ 2 \beta_x^2 (5M_\pi^4 - 2M_\pi^2 M_\pi^2 + 2M_\pi^4) + 2 \beta_x \alpha_x M_\pi^2 M_\pi^2 (2M_\pi^2 - 2M_\pi^2) + \alpha_x^2 M_\pi^2 \right]
\]

while for \(\psi_1(s)\) we obtain

\[
\psi_1(s) = \frac{1}{1152\pi^2 M_\pi^4} \left[ 3 \beta_x^2 s^2 - 9 \beta_x^2 s M_\pi^2 + 2 \beta_x^2 (2M_\pi^4 - 8M_\pi^2 M_\pi^2 - 2M_\pi^4) \right]
\]

\[
\psi_2(s) = \frac{1}{128\pi^2 M_\pi^4} \left[ \beta_x \left( s - \frac{2}{3} M_\pi^2 - \frac{2}{3} M_\pi^2 \right) + \frac{1}{3} \alpha_x M_\pi^2 \right]^2
\]

\[
\psi_3(s) = \frac{1}{288\pi^2 M_\pi^4} \left[ 2 \beta_x^2 (2M_\pi^4 + 2M_\pi^2 M_\pi^2 - 5M_\pi^4) + 2 \beta_x \alpha_x M_\pi^2 (2M_\pi^2 - 2M_\pi^2) - \alpha_x^2 M_\pi^2 \right]
\]
\[ \xi_{\pm, P}^{(1, \Delta)}(s) = \frac{1}{576 \pi^2 M_\pi^4} \left[ \frac{3}{4} \beta_x^2 s^2 + \beta_x^2 s (5 M_\pi^2 - M_\pi^2) - \beta_x^2 \left( 8 M_\pi^4 - 6 M_\pi^2 M_\pi^2 - \frac{1}{2} M_\pi^4 \right) \right] + \frac{1}{2} \beta_x \alpha_s M_\pi^2 - \beta_x \alpha_s M_\pi^2 \left( 2 M_\pi^2 + M_\pi^2 \right) \]

\[ \xi_{\pm, P}^{(2, \xi)}(s) = - \frac{1}{2304 \pi^2 M_\pi^4} \beta_x^2 \left( s - 4 M_\pi^2 \right)^2 \]

\[ \xi_{\pm, P}^{(3, \Delta)}(s) = \frac{1}{864 \pi^2 M_\pi^4} \left[ \beta_x^2 M_\pi^4 + 4 \beta_x \alpha_+ M_\pi^2 M_\pi^2 - 2 \alpha_+ M_\pi^4 + \frac{9}{2} \beta_x^2 s M_\pi^2 \right] \]

\[ \xi_{\pm, P}^{(3, \Delta)}(s) = \frac{1}{864 \pi^2 M_\pi^4} \left[ \beta_x^2 M_\pi^4 + 4 \beta_x \alpha_+ M_\pi^2 M_\pi^2 - 5 M_\pi^4 + \beta_x \alpha_s M_\pi^2 \left( 2 M_\pi^2 - M_\pi^2 \right) - \alpha_+ M_\pi^4 \right] \]

\[ \xi_{\pm, P}^{(4, \xi)}(s) = \frac{1}{288 \pi^2 M_\pi^4} \left[ 23 \beta_x^2 M_\pi^4 - 16 \beta_+ \alpha_+ M_\pi^2 M_\pi^2 - 4 \alpha_+^2 M_\pi^4 \right] \]

\[ \xi_{\pm, P}^{(4, \Delta)}(s) = \frac{1}{288 \pi^2 M_\pi^4} \left[ \beta_x^2 \left( -4 M_\pi^4 + 16 \beta_+ \alpha_+ M_\pi^2 M_\pi^2 - 25 M_\pi^4 \right) - 4 \beta_x \alpha_s M_\pi^2 \left( 2 M_\pi^2 - M_\pi^2 \right) - \alpha_+^2 M_\pi^4 \right]. \quad (B.3) \]

Finally, for \( \psi_0(s) \) the polynomials are given by

\[ \xi_x^{(0)}(s) = - \frac{\lambda^{(1)}}{2M_\pi^2} \left( s - 2 M_\pi^2 \right) \left( s - 2 M_\pi^2 \right) - \frac{\lambda^{(2)}}{3M_\pi^2} \left( s^2 - s M_\pi^2 - s M_\pi^2 + 4 M_\pi^2 M_\pi^2 \right) \]

\[ - \frac{1}{864 \pi^2 M_\pi^4} \beta_x \left( 108 \beta_+ + 11 \beta_x \right) + s \beta_x M_\pi^2 \left( 36 \alpha_+ + \frac{45}{2} \alpha_x + 27 \alpha_0 \right) \]

\[ - s \beta_x \left( 18 \beta_+ \left( 3 M_\pi^2 + M_\pi^2 \right) + 9 \beta_x \left( M_\pi^2 + M_\pi^2 \right) \right) + 9 s \beta_x \alpha_+ M_\pi^2 \]

\[ + \beta_x \left( 144 \beta_+ + 144 \beta_x - 112 M_\pi^2 M_\pi^2 \right) + 24 \beta_x \alpha_+ \left( M_\pi^2 + M_\pi^2 \right) \]

\[ - 6 \beta_+ M_\pi^2 \left( 4 \alpha_+ + 3 \alpha_0 \right) \left( M_\pi^2 + M_\pi^2 \right) - 12 \beta_+ \alpha_+ M_\pi^2 \left( M_\pi^2 + M_\pi^2 \right) + 3 \alpha_+ M_\pi^2 \left( 3 \alpha_0 + 4 \alpha_+ + 6 \alpha_x \right) \}

\[ \xi_x^{(1)}(s) = - \frac{1}{288 \pi^2 M_\pi^4} \left\{ \frac{1}{4} \beta_x^2 s^2 - \beta_x s \left[ \beta_x \left( 3 M_\pi^2 + \frac{5}{2} M_\pi^2 \right) - \frac{3}{2} \alpha_x M_\pi^2 \right] \right. \]

\[ + \alpha_x^2 M_\pi^2 + \beta_x \alpha_s M_\pi^2 \left( 2 M_\pi^2 - M_\pi^2 \right) + \beta_x^2 \left( 7 M_\pi^2 + 10 M_\pi^2 - 7 M_\pi^2 M_\pi^2 \right) \}

\[ \xi_x^{(2; \Delta)}(s) = - \frac{1}{128 \pi^2 M_\pi^4} \beta_+ s - \frac{2}{3} \beta_x \left( M_\pi^2 + M_\pi^2 \right) + \frac{1}{3} \alpha_+ M_\pi^2 \left( s - \frac{4}{3} M_\pi^2 \right) + \frac{4}{3} \alpha_+ M_\pi^2 \]

\[ \xi_x^{(3)}(s) = \frac{1}{864 \pi^2 M_\pi^4} \left[ -3 \beta_x^2 \frac{s}{M_\pi^2} \left( M_\pi^2 + M_\pi^4 \right) + 2 \beta_x \alpha_s M_\pi^2 \left( M_\pi^2 - M_\pi^2 + M_\pi^2 \right) + 2 \beta_x \alpha_+ M_\pi^2 \left( M_\pi^2 \right) \right. \]

\[ + 10 \beta_x^2 \left( 1 + \frac{M_\pi^2}{M_\pi^2} \right) \left( M_\pi^4 + M_\pi^2 - M_\pi^2 M_\pi^2 \right) \]

\[ \xi_x^{(3; \Delta)}(s) = \frac{1}{128 \pi^2 M_\pi^4} \left( s - M_\pi^2 \right) \left( s - M_\pi^2 \right) \left( s - M_\pi^2 \right) \left( s - M_\pi^2 \right) \}

In addition, eq. (15,15) involves two other contributions,

\[ 16 \pi \Delta_1 \psi_0(s) = \frac{1}{96 \pi^2 F_\pi^4} \left[ \frac{1}{6} \beta_x^2 s + \beta_x \alpha_s M_\pi^2 + \beta_x^2 \left( M_\pi^2 + M_\pi^2 \right) \right] \]

\[ \times \left[ \left( \frac{s - 4 M_\pi^2}{s - 4 M_\pi^2} - 1 \right) \lambda^{1/2}(t_+(s)) \mathcal{L}_+(s) - \left( \frac{s - 4 M_\pi^2}{s - 4 M_\pi^2} + 1 \right) \lambda^{1/2}(t_+(s)) \mathcal{L}_+(s) \right], \quad (B.5) \]

and

\[ 16 \pi \Delta_2 \psi_0(s) = \frac{1}{144 \pi^2 F_\pi^4} \left[ 1 - \frac{M_\pi^2 + M_\pi^2}{M_\pi^2 - M_\pi^2} \ln \frac{M_\pi^2}{M_\pi^2} \right] \times \left[ \frac{1}{2} \beta_x^2 s^2 - 5 \beta_x^2 s \left( M_\pi^2 + M_\pi^2 \right) \right] \]
\[\begin{align*}
&+ 2 \beta_2^2 \left( 7 M_\pi^4 + 7 M_\pi^2 M_\pi^2 - 6 M_\pi^2 M_\pi^2 \right) + 3 \beta_2 \alpha_2 M_\pi^2 s - 2 \beta_2 \alpha_x M_\pi^2 (M_\pi^2 + M_\pi^2) + 2 \alpha_2^2 M_\pi^2 \\
&\quad - \frac{1}{144 \pi^2 F_\pi^2} \frac{1}{\sqrt{(s-4 M_\pi^2)(s-4 M_\pi^2)}} \left\{ [4 \beta_2 (5 M_\pi^4 - 2 M_\pi^2 M_\pi^2 + 5 M_\pi^4) - 6 \beta_2 s (M_\pi^2 + M_\pi^2) \right. \\
&\quad + 4 \beta_2 \alpha_x M_\pi^2 (M_\pi^2 + M_\pi^2) + 2 \alpha_x M_\pi^2] [\mathcal{F}_+ (s) - \mathcal{F}_- (s)] \\
&\quad + 6 \beta_2 \left[ 2 \beta_2 (M_\pi^4 + M_\pi^2) + \alpha_x M_\pi^2 - \frac{8 \beta_2}{3} \right] [\mathcal{G}_+ (s) - \mathcal{G}_- (s)] \\
&\quad + 3 \beta_2^2 [\mathcal{H}_+ (s) - \mathcal{H}_- (s)] \right\}, \quad (B.6)
\end{align*}\]

where \(\mathcal{F}_\pm (s) \equiv \mathcal{F}(t_\pm (s))\), and similar definitions for \(\mathcal{G}_\pm (s)\) and \(\mathcal{H}_\pm (s)\), with

\[t_\pm (s) = - \frac{1}{2} (s - 2 M_\pi^2 - 2 M_\pi^2) \pm \frac{1}{2} \sqrt{(s - 4 M_\pi^2)(s - 4 M_\pi^2)}, \quad (B.7)\]

and

\[\begin{align*}
\mathcal{F}(t) & = (M_\pi^2 - M_\pi^2) \left[ H_{1,0} \left( \frac{M_\pi}{M_\pi^2} \chi(t) \right) - H_{1,0} \left( \frac{M_\pi^2}{M_\pi^2} \chi(t) \right) + \ln \left( \frac{M_\pi^2}{M_\pi^2} \right) \ln \chi(t) \right] \\
\mathcal{G}(t) & = (M_\pi^2 + M_\pi^2) \mathcal{F}(t) + (M_\pi^2 - M_\pi^2) t \ln \left( \frac{M_\pi}{M_\pi^2} \right) \\
&\quad - (M_\pi^2 - M_\pi^2)^2 \left( M_\pi^2 + M_\pi^2 - 2 M_\pi M_\pi \chi(t) \right) \frac{\ln \chi(t)}{t} - \frac{1}{2} \ln^2 \chi(t) - \ln \chi(t) + \frac{M_\pi^2 - M_\pi^2}{t} \ln \left( \frac{M_\pi^2}{M_\pi^2} \right) \\
\mathcal{H}(t) & = - 4 M_\pi^2 M_\pi^2 \mathcal{F}(t) - (M_\pi^2 - M_\pi^2) t^2 \ln \left( \frac{M_\pi}{M_\pi^2} \right) \\
&\quad - (M_\pi^2 - M_\pi^2)^2 \left[ \frac{1}{2} \left( 1 + \frac{M_\pi^2 + M_\pi^2}{M_\pi^2 - M_\pi^2} \ln \left( \frac{M_\pi}{M_\pi^2} \right) \right) \frac{1}{t} - \frac{M_\pi^2 - M_\pi^2}{t^2} \ln \left( \frac{M_\pi^2}{M_\pi^2} \right) \\
&\quad + \frac{1}{t^2} \left( \frac{M_\pi^2 + M_\pi^2}{M_\pi^2 - M_\pi^2} \right)^2 (1 - \frac{1}{t}) \chi^{1/2} (t) \ln \chi(t) \right] \right). \quad (B.8)
\end{align*}\]

Notice that individual terms in \(\Delta_2 \psi_0^0(s)\) may behave as \(\mathcal{O}[\Delta_2 \times \ln(\Delta_2/M_\pi^2)]\) in the isospin limit, but cancellations occur between these terms, so that overall \(\Delta_2 \psi_0^0(s) = \mathcal{O}(\Delta_2^2)\).

**Appendix C: Two-loop form factors in the isospin limit**

In this appendix, we discuss the scalar and vector form factors at two loops in the isospin limit, where the complications due to the mass difference between neutral and charged pions are absent, and the dispersive integrals can all be expressed in terms of the known function \(\tilde{J}(s)\). The two form factors \(F_S^\mp\) and \(F_V^\mp\) then become identical, since in both cases the two-pion states are projected on their \(I = 0\), S-wave components, with identical Clebsch-Gordan coefficients. Furthermore, this exercise will provide a check of the calculation in the general case, which has to reduce to the expressions to be found below in the limit \(M_\pi^2 \to M_\pi^2\). Let us recall that ref. \[22\] did not give analytical expressions for the form factors at two loops. Analytical two-loop expressions for the scalar and the vector form factors were given in \[22\] in the isospin limit. The analytical expression for the vector form factor only had also been given earlier in ref. \[20\].

For the first iteration, the discontinuities of the form factors reduce to \([\text{in this appendix, we omit the superscript } \pi \text{ most of the time}]

\[\begin{align*}
\text{Im} F_S(s) & = \sigma(s) F_S(0) \varphi_0(s) \theta(s - 4 M_\pi^2) + \mathcal{O}(E^0) \\
\text{Im} F_V(s) & = \sigma(s) \varphi_1(s) \theta(s - 4 M_\pi^2) + \mathcal{O}(E^4), \quad (C.1)
\end{align*}\]

with

\[\begin{align*}
\varphi_0(s) & = \varphi_0^+ (s) - \frac{1}{2} \varphi_0^- (s) = \frac{1}{2} \varphi_0^0 (s) - \varphi_0^0 (s) = \frac{1}{16 \pi F_\pi^2} \left[ \beta \left( s - \frac{4}{3} M_\pi^2 \right) + \frac{5}{6} \alpha M_\pi^2 \right] \\
\varphi_1(s) & = \varphi_1^+ (s) - \frac{1}{96 \pi F_\pi^2} \beta (s - 4 M_\pi^2). \quad (C.2)
\end{align*}\]
The one-loop expressions of the form factors in the isospin limit read

\[ F_S(s) = F_S(0) \left[ 1 + \frac{1}{6} (r^2)^s s + c_S^s s^2 + U_S(s) \right] \]  
(3.3)

\[ F_V = 1 + \frac{1}{6} (r^2)^V s + c_V^s s^2 + U_V(s), \]  
(4.4)

with

\[ U_S(s) = 16\pi\varphi_0(s)\bar{J}(s) + \frac{M^2}{16\pi^2F^2}\left\{ \frac{1}{36} \frac{s}{M^2} (8\beta - 5\alpha) - \frac{1}{360} \left( \frac{s}{M^2} \right)^2 (52\beta + 5\alpha) \right\} \]  
(5.5)

\[ U_V(s) = \frac{M^2}{16\pi^2F^2} \beta \left\{ \frac{1}{9} \frac{s}{M^2} \left[ 1 + 24\pi^2\sigma^2(s)\bar{J}(s) \right] - \frac{1}{60} \left( \frac{s}{M^2} \right)^2 \right\}. \]  
(6.6)

In order to implement the second iteration, it is necessary to include the next-to-leading contributions to the discontinuities of the form factors,

\[ \text{Im} F_S(s) = \sigma(s) F_S(0) \left\{ \varphi_0(s) \left[ 1 + \Gamma_S(s) \right] + \psi_0(s) \right\} \theta(s - 4M^2) + \mathcal{O}(E^6) \]

\[ \text{Im} F_V(s) = \sigma(s) \left\{ \varphi_1(s) \left[ 1 + \Gamma_V(s) \right] + \psi_1(s) \right\} \theta(s - 4M^2) + \mathcal{O}(E^6). \]

The relevant one-loop corrections \( \Gamma_S(s) \) and \( \Gamma_V(s) \) to the real parts of the form factors are easy to obtain from their expressions given above,

\[ \Gamma_S(s) = \frac{s}{6} \left\{ (r^2)^s + \frac{1}{96\pi^2F^2} (8\beta - 5\alpha) \right\} + \frac{1}{\pi} \varphi_0(s) \left[ 2 + \sigma(s)L(s) \right] \]  
(7.7)

\[ \Gamma_V(s) = \frac{s}{6} \left\{ (r^2)^V + \frac{1}{24\pi^2F^2} \beta \right\} + \frac{1}{\pi} \varphi_1(s) \left[ 2 + \sigma(s)L(s) \right]. \]

As for the one-loop contributions to the real parts of the \( S \) and \( P \) partial waves, they are conveniently expressed as

\[ \psi_0(s) = \psi_0^+(s) - \frac{1}{2} \psi_0^0(s) = \frac{1}{2} \psi_0^{00}(s) - \psi_0^0(s) \]

\[ \psi_1(s) = \psi_1^-(s), \]

where the expressions for \( \psi_0^{00}(s), \psi_0^+(s), \) and \( \psi_1^0(s) \) are given in eqs. [IV.7], [IV.12], and [IV.18], respectively. They involve the polynomials \( \xi_{01}^{(n)}(s) \) of reference [24]. We reproduce them in a slightly different notation, in terms of the variable \( s \) instead of the relative momentum \( q = \sqrt{s/4M^2} - 1 \) in the center-of-mass frame,

\[ \xi_0^{(0)}(s) = \frac{1}{432\pi^2} (105\alpha^2 - 120\alpha\beta + 392\beta^2) + \frac{2}{3} (11\lambda_1 + 14\lambda_2) \]

\[ + \left\{ \frac{1}{864\pi^2} (180\alpha - 617\beta) - \frac{20}{3} (\lambda_1 + \lambda_2) \right\} \frac{s}{M^2} \]

\[ + \left\{ \frac{311}{1728\pi^2} \beta^2 + \frac{1}{6} (11\lambda_1 + 14\lambda_2) \right\} \left( \frac{s}{M^2} \right)^2 \]

\[ \xi_0^{(1)}(s) = \frac{5}{192\pi^2} (\alpha^2 + 4\beta^2) - \frac{5}{72\pi^2} \beta^2 \frac{s}{M^2} + \frac{7}{576\pi^2} \beta^2 \left( \frac{s}{M^2} \right)^2 \]

\[ \xi_0^{(2)}(s) = \frac{1}{1152\pi^2} (25\alpha^2 - 80\alpha\beta + 64\beta^2) + \frac{1}{96\pi^2} \beta(5\alpha - 8\beta) \left( \frac{s}{M^2} \right) + \frac{2}{32\pi^2} \beta^2 \left( \frac{s}{M^2} \right)^2 \]

\[ \xi_0^{(3)}(s) = -\frac{5}{288\pi^2} (\alpha^2 + 4\beta^2) + \frac{1}{48\pi^2} \beta^2 \frac{s}{M^2} \]

\[ \xi_0^{(0)}(s) = \frac{1}{864\pi^2} (93\alpha^2 + 48\alpha\beta + 112\beta^2) + \frac{4}{3} (\lambda_1 + 4\lambda_2) \]

\[ - \left\{ \frac{1}{1728\pi^2} (207\alpha + 256\beta) + \frac{2}{3} (\lambda_1 + 7\lambda_2) \right\} \frac{s}{M^2} \]
Performing the dispersive integrals gives

\[\xi_2^{(1)}(s) = \frac{1}{192\pi^2}(3\alpha^2 - 2\alpha\beta) - \frac{1}{576\pi^2}\beta (9\alpha + 7\beta) \frac{s}{M^2} + \frac{11}{1152\pi^2}\beta^2 \left(\frac{s}{M^2}\right)^2\]

\[\xi_2^{(2)}(s) = \frac{1}{288\pi^2}(\alpha^2 + 4\alpha\beta + 4\beta^2) - \frac{1}{96\pi^2}\beta (\alpha + 2\beta) \frac{s}{M^2} + \frac{1}{128\pi^2}\beta^2 \left(\frac{s}{M^2}\right)^2\]

\[\xi_2^{(3)}(s) = \frac{1}{288\pi^2}(-3\alpha^2 + 2\alpha\beta) - \frac{1}{96\pi^2}\beta^2 \frac{s}{M^2}\]

\[\xi_1^{(0)}(s) = \frac{1}{1728\pi^2}(15\alpha^2 - 460\alpha\beta + 286\beta^2)\]

\[+ \left\{ \frac{5}{1728\pi^2}(11\alpha - 12\beta)\beta + \frac{2}{3}(\lambda_1 - \lambda_2) \right\} \frac{s}{M^2}\]

\[= \frac{1}{96\pi^2}\beta (5\alpha - 3\beta) + \frac{1}{576\pi^2}\beta (5\alpha + \beta) \frac{s}{M^2} - \frac{1}{1152\pi^2}\beta^2 \left(\frac{s}{M^2}\right)^2\]

Putting all elements together leads to

\[\frac{1}{F_S(0)} \text{ disc } F_S(s) = \sum_{n=0}^{3} S_n(s) k_n(s) \times \theta(s - 4M^2)\]

\[\text{disc } F_V(s) = \sum_{n=0}^{4} V_n(s) k_n(s) \times \theta(s - 4M^2)\]

with

\[S_n(s) = 16\pi \varphi_0(s) \delta_{n,0}\]

\[+ \left\{ \frac{8\pi s}{3} \left(r^2\right)_S + \frac{1}{96\pi^2 F^2} (8\beta - 5\alpha) \right\} \varphi_0(s) \delta_{n,0}\]

\[+ 8 |\varphi_0(s)|^2 \delta_{n,2} + \frac{M^4}{F^2} \xi_0^{(n)}(s)\]

\[V_n(s) = 16\pi \varphi_1(s) \delta_{n,0}\]

\[+ \left\{ \frac{8\pi s}{3} \left(r^2\right)_V + \frac{1}{24\pi^2 F^2} \beta \right\} \varphi_1(s) \delta_{n,0}\]

\[+ 8 |\varphi_1(s)|^2 \delta_{n,2} + \frac{M^4}{F^2} \xi_1^{(n)}(s)\]

Performing the dispersive integrals gives

\[U_S(s) = \sum_{n=0}^{3} S_n(s) \tilde{K}_n(s) + P_S(s)\]

\[U_V(s) = \sum_{n=0}^{4} V_n(s) \tilde{K}_n(s) + P_V(s)\]
The two second-order polynomials $P_S(s)$ and $P_V(s)$ are obtained as follows. First write

$$\bar{K}_n(s) = \kappa_n^{(1)} \frac{s}{M^2} + \kappa_n^{(2)} \left( \frac{s}{M^2} \right)^2 + \frac{s^3}{\pi^2} \int_4 \frac{dx}{x^3} \frac{\kappa_n(x)}{x - s - i0}, \quad (C.13)$$

and, next, expand the polynomials $S_n(s)$ and $V_n(s)$,

$$S_n(s) = S_n^{(0)} + S_n^{(1)} \frac{s}{M^2} + S_n^{(2)} \left( \frac{s}{M^2} \right)^2,$$

$$V_n(s) = V_n^{(0)} + V_n^{(1)} \frac{s}{M^2} + V_n^{(2)} \left( \frac{s}{M^2} \right)^2.$$

The polynomials $P_S(s)$ and $P_V(s)$ are then given by

$$P_S(s) = -\frac{s^2}{M^2} \sum_{n=0}^{3} \left[ \kappa_n^{(2)} S_n^{(0)} + \kappa_n^{(1)} S_n^{(1)} \right] - \frac{s}{M^2} \sum_{n=0}^{3} \kappa_n^{(1)} S_n^{(0)} ,$$

$$P_V(s) = -\frac{s^2}{M^2} \sum_{n=0}^{4} \left[ \kappa_n^{(2)} V_n^{(0)} + \kappa_n^{(1)} V_n^{(1)} \right] - \frac{s}{M^2} \sum_{n=0}^{4} \kappa_n^{(1)} V_n^{(0)} . \quad (C.14)$$

Using the coefficients $\kappa_n^{(1)}$ and $\kappa_n^{(2)}$ displayed in the following table:

| $n$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|-----|-----|-----|-----|-----|-----|
| $\pi^2 \kappa_n^{(1)}$ | $\frac{\pi^2}{2}$ | $-\frac{1}{16}$ | $\frac{\pi^2}{27}$ | $-\frac{1}{32}$ | $-\frac{\pi^2}{48} + \frac{1}{64}$ |
| $\pi^2 \kappa_n^{(2)}$ | $\frac{1}{36}$ | $\frac{1}{12}$ | $\frac{\pi^2}{2880}$ | $\frac{1}{128}$ | $1/1920 + \frac{1}{3840}$ |

we obtain

$$P_S(s) = \frac{M^2}{16 \pi^2 F^2} \left[ \frac{8\beta - 5\alpha}{36} \left( \frac{s}{M^2} \right) - \frac{52\beta + 5\alpha}{360} \left( \frac{s}{M^2} \right)^2 \right]$$

$$+ \left( \frac{M^2}{16 \pi^2 F^2} \right)^2 \left\{ \left( \frac{s}{M^2} \right) \left[ \frac{1}{324} (45\alpha^2 + 232\beta^2) + \frac{5\pi^2}{216} (\alpha^2 + 4\beta^2) - \frac{16\pi^2}{9} (11\lambda_1 + 14\lambda_2) \right] \right\} .$$

$$P_V(s) = \frac{M^2}{16 \pi^2 F^2} \left[ \frac{\beta}{9} \left( \frac{s}{M^2} \right) - \frac{\beta}{60} \left( \frac{s}{M^2} \right)^2 \right]$$

$$+ \left( \frac{M^2}{16 \pi^2 F^2} \right)^2 \left\{ \left( \frac{s}{M^2} \right) \left[ \frac{1}{648} (45\alpha^2 + 340\alpha\beta + 94\beta^2) - \frac{\pi^2}{648} (5\alpha^2 + 50\alpha\beta - 28\beta^2) \right] \right\}.$$
allow us to eliminate $\bar{K}_2(s)$ from $F_S(s)$, and both $\bar{K}_2(s)$ and $\bar{K}_3(s)$ from $F_V(s)$, thus leading to the following expressions of the form factors

\[
F_S(s)/F_S(0) = 1 + \frac{1}{6}(r^2)_{Ss} s + c_{Ss} s^2 + \frac{M_S^2}{16\pi^2 F_\pi^2} \left\{ 16\pi^2 \left( \frac{s}{M_S^2} \beta - \frac{1}{6}(8\beta - 5\alpha) \right) \bar{J}(s) + \frac{1}{36} \frac{s}{M_S^2} \left( 8\beta - 5\alpha \right) - \frac{1}{360} \left( \frac{s}{M_S^2} \right)^2 \left( 52\beta - 5\alpha \right) \right\} 
+ \left( \frac{M_S^2}{16\pi^2 F_\pi^2} \right)^2 \left\{ \frac{8\pi^2}{9} \bar{J}(s) \left( \frac{s}{M_S^2} \right)^2 \left( 48\pi^2(11\lambda_1 + 14\lambda_2) + 48\pi^2 F_\pi^2 \beta(r^2)_{Ss} + \frac{1}{6} \beta(551\beta - 15\alpha) \right) - \left( \frac{s}{M_S^2} \right)(1920\pi^2(\lambda_1 + \lambda_2) + 8\pi^2 F_\pi^2(8\beta - 5\alpha)(r^2)_{Ss} \right. 
+ \frac{1}{12} \left( 3684\beta^2 - 1520\alpha\beta + 25\alpha^2 \right) \right) 
+ 192\pi^2(11\lambda_1 + 14\lambda_2) + \frac{1}{3}(976\beta^2 - 480\alpha\beta + 285\alpha^2) \right\} 
+ 12\pi^2 \bar{K}_1(s) \left[ \frac{43}{27} \left( \frac{s}{M_S^2} \right)^2 \beta^2 - \frac{20}{27} \left( \frac{s}{M_S^2} \right) \beta(14\beta - 3\alpha) + \frac{4}{27} (127\beta^2 - 80\alpha\beta + 10\alpha^2) \right. 
- \frac{4}{27} \left( \frac{M_S^2}{s} \right) (64\beta^2 - 80\alpha\beta + 25\alpha^2) \right] 
+ \frac{16\pi^2}{3} \bar{K}_3(s) \left[ \left( \frac{s}{M_S^2} \right) \beta^2 - \frac{5}{6} \left( 4\beta^2 + \alpha^2 \right) \right] 
+ \left( \frac{s}{M_S^2} \right)^2 \left[ \frac{8\pi^2}{45} (89\lambda_1 + 86\lambda_2) + \frac{1}{27} (8\beta - 5\alpha) F_\pi^2 \beta(r^2)_{Ss} - \frac{1}{3240} (1322\beta^2 + 135\alpha^2) \right. 
\left. - \frac{\pi^2}{216} (2\beta^2 - \alpha^2) \right] 
+ \left( \frac{s}{M_S^2} \right) \left[ \frac{16\pi^2}{9} (11\lambda_1 + 14\lambda_2) + \frac{1}{324} (3224\beta^2 - 2160\alpha\beta - 45\alpha^2) + \frac{5\pi^2}{216} (4\beta^2 + \alpha^2) \right] 
- \frac{1}{9} (8\beta - 5\alpha)^2 \right),
\]

\[
(C.17)
\]

\[
F_V(s) = 1 + \frac{1}{6}(r^2)_{Vs} s + c_{Vs} s^2 + \frac{M_V^2}{16\pi^2 F_\pi^2} \left\{ \frac{8\pi^2}{3} \beta \left( \frac{s}{M_V^2} \beta - 4 \right) \bar{J}(s) + \frac{1}{9} \frac{s}{M_V^2} \beta \left( \frac{s}{M_V^2} \beta - 4 \right) \bar{J}(s) + \frac{1}{60} \frac{s}{M_V^2} \beta \left( \frac{s}{M_V^2} \beta - 4 \right) \bar{J}(s) \right\} 
+ \left( \frac{M_V^2}{16\pi^2 F_\pi^2} \right)^2 \left\{ \frac{8\pi^2}{9} \bar{J}(s) \left( \frac{s}{M_V^2} \beta^2 - 10 \left( \frac{s}{M_V^2} \beta - \frac{1}{2} \right) \left( 4\beta - \alpha \right) + 2(74\beta^2 - 40\alpha\beta + 5\alpha^2) - 128 \left( \frac{M_V^2}{s} \beta \right) \right) \right\} 
+ \frac{2\pi^2}{9} \bar{K}_1(s) \left[ \left( \frac{s}{M_V^2} \right)^2 \beta^2 - 10 \left( \frac{s}{M_V^2} \beta - \frac{1}{2} \right) \left( 4\beta - \alpha \right) + 2(74\beta^2 - 40\alpha\beta + 5\alpha^2) - 128 \left( \frac{M_V^2}{s} \beta \right) \right] 
+ \frac{8\pi^2}{9} \bar{K}_1(s) \left[ \left( \frac{s}{M_V^2} \right)^2 \beta^2 - 10 \left( \frac{s}{M_V^2} \beta - \frac{1}{2} \right) \left( 4\beta - \alpha \right) + 2(74\beta^2 - 40\alpha\beta + 5\alpha^2) - 128 \left( \frac{M_V^2}{s} \beta \right) \right] 
+ \left( \frac{s}{M_V^2} \right)^2 \left[ \frac{16\pi^2}{9} (\lambda_2 - \lambda_1) + \frac{8\pi^2}{27} \beta F_\pi^2 \beta(r^2)_{Vs} + \frac{1}{6480} (1130\beta^2 + 1650\alpha\beta + 195\alpha^2) \right. 
\left. - \frac{\pi^2}{1620} (71\beta^2 + 35\alpha\beta + 5\alpha^2) \right] 
+ \left( \frac{s}{M_V^2} \right) \left[ \frac{1}{648} (338\beta^2 + 520\alpha\beta - 45\alpha^2) + \frac{\pi^2}{648} (52\beta^2 - 80\alpha\beta + 10\alpha^2) \right] - \frac{16}{9} \beta^2 \right). 
\]

\[
(C.18)
\]
In order to recover the expressions of \[26\] from these formulae, one simply needs to replace the various quantities by their expressions at leading or at next-to-leading order, as they can be found in refs. \[16, 24\].

\[
\begin{align*}
\alpha &= 1 + \frac{1}{32\pi^2} \frac{M^2}{F^2} \left(4\tilde{\ell}_4 - 3\tilde{\ell}_3 - 1\right) \\
\beta &= 1 + \frac{1}{8\pi^2} \frac{M^2}{F^2} \left(\tilde{\ell}_4 - 1\right) \\
\lambda_1 &= \frac{1}{48\pi^2} \left(\tilde{\ell}_1 - \frac{4}{3}\right) \\
\lambda_2 &= \frac{1}{48\pi^2} \left(\tilde{\ell}_2 - \frac{5}{6}\right) \\
\langle r^2 \rangle_S^\pi &= \frac{3}{8\pi^2} \left(\tilde{\ell}_4 - \frac{13}{12}\right) \\
\langle r^2 \rangle_V^\pi &= \frac{1}{16\pi^2} \left(\tilde{\ell}_6 - 1\right).
\end{align*}
\]

At the end of this process, we then obtain perfect agreement with ref. \[26\].

**Appendix D: First-order isospin-breaking corrections to the one-loop partial waves**

The purpose of this appendix is to provide the explicit expressions of the functions that describe the isospin-breaking corrections to the one-loop partial waves, as given in eqs. \[(V.19)\] and \[(V.20)\].

In the case of \(\psi^0_{00}(s)\) the corrections that appear in eq. \[(V.19)\] read

\[
\begin{align*}
\Delta \xi_{00}^{(0)}(s) &= \frac{4}{3} (\lambda_1 + 2\lambda_2) \left(\frac{2}{M^2} \frac{s^2}{\pi} - 5\right) + \frac{1}{576\pi^2} \left(160\beta^2 \frac{s}{M^2} + 6\alpha\beta \frac{s}{M^2} + 24\alpha^2 - 504\beta^2 - 203\alpha^2\right) \\
&- \frac{1}{72\pi^2} \frac{M^2}{s - 4M^2} \left(44\beta^2 - 28\alpha\beta + 11\alpha^2\right) \\
\Delta \xi_{00}^{(1)}(s) &= \frac{1}{2304\pi^2} \left(12\beta^2 \frac{s^2}{M^2} + 60\beta^2 \frac{s}{M^2} + 12\alpha\beta \frac{s}{M^2} - 79\alpha^2 - 368\beta^2\right) \\
&- \frac{1}{12\pi^2} \frac{M^2}{s - 4M^2} (\beta - \alpha) \beta \\
\Delta \xi_{00}^{(2)}(s) &= \frac{1}{2304\pi^2} \left(-12\beta^2 \frac{s^2}{M^2} + 84\beta^2 \frac{s}{M^2} - 12\alpha\beta \frac{s}{M^2} - 53\alpha^2 + 48\alpha\beta - 64\beta^2\right) \\
\Delta \xi_{00}^{(3)}(s) &= \frac{1}{864\pi^2} \left(8\beta^2 - 12\alpha\beta + 31\alpha^2\right) + \frac{1}{216\pi^2} \frac{M^2}{s - 4M^2} (20\beta^2 - 4\alpha\beta + 11\alpha^2).
\end{align*}
\]

In the case of \(\pi^+\pi^-\) scattering, we proceed as described in subsection \[V.B\] one first obtains the functions [\(X = S, P\)]

\[
\begin{align*}
\bar{\xi}_{+\pi^-}^{(1)}(s) &= \bar{\xi}_{+\pi^-}^{(1;+)}(s) + \bar{\xi}_{+\pi^-}^{(1;\lambda)}(s) \\
\bar{\xi}_{+\pi^-}^{(2)}(s) &= \bar{\xi}_{+\pi^-}^{(2;+)}(s) + \bar{\xi}_{+\pi^-}^{(2;\lambda)}(s) \\
\bar{\xi}_{+\pi^-}^{(3;+)}(s) &= \bar{\xi}_{+\pi^-}^{(3;+;\lambda)}(s) + \bar{\xi}_{+\pi^-}^{(3;\lambda)}(s) \\
\bar{\xi}_{+\pi^-}^{(4)}(s) &= \bar{\xi}_{+\pi^-}^{(4;+;\lambda)}(s) + \bar{\xi}_{+\pi^-}^{(4;\lambda)}(s).
\end{align*}
\]

The expansion of the remaining functions gives

\[
\begin{align*}
2 \frac{\sigma(s)}{\sigma_{\lambda}(s)} L_\Delta(s) &= 16\pi k_1(s) - 8\pi \frac{\Delta \pi}{M^2} \left[4k_0(s) + k_1(s) - k_2(s)\right] + \mathcal{O}(\Delta^2) \\
3 \frac{M^2}{\sqrt{s(s - 4M^2)}} L^2_\Delta(s) &= 16\pi k_3(s) - 8\pi \frac{\Delta \pi}{M^2} \left[3 \frac{s}{2} k_1(s) + \frac{3}{2} k_2(s) + 2k_3(s)\right] + \mathcal{O}(\Delta^2),
\end{align*}
\]

and

\[
\begin{align*}
\frac{M^2}{\sqrt{s(s - 4M^2)}} \left[1 + \frac{1}{\sigma_{\lambda}(s)} L_\Delta(s) + \frac{M^2}{s - 4M^2} L^2_\Delta(s)\right] &= 16\pi k_4(s) \\
&- 8\pi \frac{\Delta \pi}{M^2} \left[4k_4(s) - \frac{M^2}{s - 4M^2} k_2(s)\right] + \mathcal{O}(\Delta^2).
\end{align*}
\]
For the $S$ and $P$ partial-wave projections $\psi_0^-(s)$ and $\psi_1^-(s)$, this then leads to the expression (W.20), with

\begin{align*}
\Delta \xi_x^{(0)}(s) &= \frac{1}{1728\pi^2} \left( -12\beta^2 \frac{s^2}{M^2} + 81\beta^2 \frac{s}{M^2} - 3\beta^2 \frac{s}{M^2} - 12\beta \frac{s}{M^2} - 48\alpha^2 + 16\alpha^2 \right) \\
\Delta \xi_x^{(1)}(s) &= \frac{1}{2304\pi^2} \left( 6\beta^2 \frac{s^2}{M^2} - 33\beta^2 \frac{s}{M^2} + 15\alpha^2 \frac{s}{M^2} - 51\alpha^2 + 16\alpha^2 + 56\beta^2 \right) \\
\Delta \xi_x^{(2)}(s) &= \frac{1}{2304\pi^2} \left( -6\beta^2 \frac{s^2}{M^2} + 39\beta^2 \frac{s}{M^2} - 45\alpha^2 \frac{s}{M^2} - 37\alpha^2 + 64\alpha^2 - 48\beta^2 \right) \\
\Delta \xi_x^{(3)}(s) &= \frac{1}{1728\pi^2} (26\beta^2 - 2\alpha + 27\alpha^2) \\
\Delta \xi_x^{(4)}(s) &= \frac{1}{24\pi^2} (5\beta^2 + 3\alpha^2 - \alpha^2).
\end{align*}

Turning eventually to the inelastic $\pi^+\pi^- \rightarrow \pi^0\pi^0$ channel, one first rewrites the polynomials in eq. (V.15) as $\xi^{(0)}(s) = \xi_x^{(0)}(s) + (\Delta \pi/F_2)\delta \xi^{(0)}(s) + O(\Delta_x^2)$, with $\xi^{(2)}(s) = \xi_x^{(2;0)}(s) + \xi_x^{(2;0)}(s)$. Next, one proceeds with the expansion of the remaining functions,

\begin{align*}
2 \frac{\lambda^{1/2}(t_-(s))}{\sqrt{s(s - 4M^2_{\pi^0})}} L_-(s) &= 16\pi k_1(s) - 16\pi \frac{M^2_{\pi^0}}{M^2_{\pi^+}} \left\{ k_0(s) + \frac{M^2_{\pi^0}}{s - 4M^2_{\pi^0}} [4k_0(s) + k_1(s)] \right\} + O(\Delta_x^2) \\
3 \frac{M^2_{\pi^0}}{\sqrt{s(s - 4M^2_{\pi^0})}} L^2(s) &= 16\pi k_3(s) - 16\pi \frac{\Delta x}{s - 4M^2_{\pi^0}} \left\{ \frac{3}{2} k_1(s) + 2k_3(s) \right\} + O(\Delta_x^2).
\end{align*}

Similar expressions with $t_-(s)$ replaced by $t_+(s)$ are of order $O(\Delta_x^2)$, cf. equation (A.13), and thus need not be retained in the present context. Finally, there are the two additional pieces to consider,

\begin{align*}
\Delta_1 \psi_0^-(s) &= 2 \frac{M^4}{F_\pi} \sqrt{\frac{s}{s - 4M^2_{\pi^0}}} \times \frac{\Delta x}{M^2_{\pi^0}} \left( -\frac{1}{6} \frac{\beta^2}{M^2_{\pi^0}} + \alpha^2 + 2\beta^2 \right) k_1(s) + O(\Delta_x^2), \\
\Delta_2 \psi_0^-(s) &= O(\Delta_x^2),
\end{align*}

Putting the various parts together then leads to the expression given in (W.20), with

\begin{align*}
\Delta \xi_x^{(0)}(s) &= -\lambda_1 \left( \frac{s}{M^2_{\pi^0}} - 2 \right) - \lambda_2 \left( \frac{s}{M^2_{\pi^0}} - 4 \right) \\
&\quad + \frac{1}{1728\pi^2} \left( \frac{3}{2} \beta^2 \frac{s^2}{M^2_{\pi^0}} - 154\beta^2 \frac{s}{M^2_{\pi^0}} + 225\alpha^2 \frac{s}{M^2_{\pi^0}} + 352\beta^2 - 270\alpha^2 + 171\alpha^2 \right) \\
&\quad - \frac{1}{72\pi^2} \frac{M^2_{\pi^0}}{s - 4M^2_{\pi^0}} (8\beta^2 - 7\alpha^2 - \alpha^2) \\
\Delta \xi_x^{(1)}(s) &= \frac{1}{2304\pi^2} \left( -20\beta^2 \frac{s}{M^2_{\pi^0}} + 3\alpha^2 \frac{s}{M^2_{\pi^0}} + 80\beta^2 + 36\alpha^2 - 13\alpha^2 \right) \\
&\quad + \frac{1}{48\pi^2} \frac{M^2_{\pi^0}}{s - 4M^2_{\pi^0}} \alpha \beta \\
\Delta \xi_x^{(2)}(s) &= \frac{1}{2304\pi^2} \left( -12\beta^2 \frac{s}{M^2_{\pi^0}} + 57\alpha \beta \frac{s}{M^2_{\pi^0}} + 16\beta^2 - 104\alpha^2 + 31\alpha^2 \right) \\
\Delta \xi_x^{(3)}(s) &= \frac{1}{864\pi^2} (9\beta^2 \frac{s}{M^2_{\pi^0}} - 12\beta^2 - 4\alpha^2 - 5\alpha^2) + \frac{1}{216\pi^2} \frac{M^2_{\pi^0}}{s - 4M^2_{\pi^0}} (8\beta^2 - \alpha^2 - \alpha^2).
\end{align*}
Let us close this appendix with a remark concerning the occurrence of contributions proportional to $M^2_\pi/(s - 4M^2_\pi)$ in the expressions of the functions $\Delta \xi^{(i)}(s)$. When summed together into the functions $\psi^{00}_0(s)$, $\psi^0_1(s)$, and $\psi^+_1(s)$ [they are absent in $\psi^{0-}_0(s)$], these singularities combine to give a regular behaviour as $s \to 4M^2_\pi$. Just like their lowest-order counterparts $\varphi^{00}_0(s)$, $\varphi^{0}_0(s)$, $\varphi^{+-}_0(s)$, and $\varphi^{+}_1(s)$, the real parts of the partial-wave projections at next-to-leading order are regular at $s = 4M^2_\pi$, and the expansion in powers of $\Delta_\pi$ preserves this regularity, see also the remark following eq. (V.19).

Appendix E: Expressions of the subtraction constants

The phases of the form factors discussed in section [V] involve a certain number of subtraction constants, whose values are not fixed by the general properties underlying the dispersive relations that form the starting point of our construction. Two sets of parameters, $\alpha_{00}$, $\alpha_x$, and $\alpha_{+-}$ on the one hand, and $\beta_x$ and $\beta_+$ on the other hand, are directly related to the parameters $\alpha$ and $\beta$ of the isospin-symmetric $\pi\pi$ amplitude, that are themselves related to the two scattering lengths in the $S$ wave, cf. eq. (III.3). They represent the quantities to be extracted from experiment. What we need to know, however, is what becomes of the relations (III.5) at next-to-leading order. For the remaining set of parameters, the $\lambda$’s, the isospin-breaking corrections to their values in the isospin limit, given in eqs. (III.20), also need to be worked out. In order to obtain this information, we have performed a one-loop calculation of the form factors and scattering amplitudes using the “effective” lagrangian approach described in subsection (VI.A). The results of this calculation are shown in this appendix.

First, at next-to-leading order, the expressions (III.5) become [the definition of the constants $\hat{\mathcal{K}}^{00}$ and $\hat{\mathcal{K}}^{00}$ in terms of low-energy constants introduced in [40] is given in eq. (E.3) below]

$$\frac{F^2}{F^2} \left( 4 - 3 \frac{2\hat{m}_B}{M^2_\pi} \right) - \alpha = \frac{\Delta_\pi}{M^2_\pi} (\beta - \alpha) + \frac{1}{96\pi^2} \frac{M^2_\pi}{F^2_\pi} (11\alpha^2 - 8\beta^2) + \frac{1}{48\pi^2} \frac{M^2_\pi}{F^2_\pi} (4\alpha^2 - 7\alpha\beta + 6\beta^2) L_\pi - 3\beta \frac{\alpha e^2}{32\pi^2} (\hat{\mathcal{K}}^{00} + \hat{\mathcal{K}}^{00})$$

$$\frac{F^2}{F^2} - \beta = \frac{1}{48\pi^2} \frac{M^2_\pi}{F^2_\pi} \beta (\beta + 5\alpha), \quad \text{(E.1)}$$

with $L_\pi \equiv \ln(M^2_\pi/M^2_{\pi^0})$. Notice the occurrence of the term $(\beta - \alpha) \frac{\Delta_\pi}{M^2_\pi}$ in the first expression. Since $\beta - \alpha \sim \mathcal{O}(M^2_\pi \times \ln M^2_\pi)$, this term reveals a logarithmic singularity (at most) in the chiral limit. Actually, it is finite as $M^2_{\pi^0} \to 0$. However, as $M^2_\pi \to 0$, it develops an infrared singular behaviour,

$$\frac{(\beta - \alpha) \Delta_\pi}{M^2_\pi} \sim \frac{1}{32\pi^2} \frac{M^2_\pi}{F^2_\pi} (7\beta - 4\alpha) \alpha \ln M^2_\pi. \quad \text{(E.2)}$$

We then obtain the following identification, at one-loop precision, with the various parameters involved in the polynomial part of these amplitudes:

$$\alpha_{00} = \alpha + (\beta - \alpha) \frac{\Delta_\pi}{M^2_\pi} + \frac{\alpha}{48\pi^2} \frac{M^2_\pi}{F^2_\pi} (5\alpha + \beta) \beta + \frac{1}{96\pi^2} \frac{M^2_\pi}{F^2_\pi} (2\beta^2 - 3\alpha\beta + 10\alpha^2) L_\pi$$

$$+ \frac{\beta e^2}{32\pi^2} (\hat{\mathcal{K}}^{00} + \hat{\mathcal{K}}^{00}) - \frac{\alpha e^2}{32\pi^2} \hat{\mathcal{K}}^{00}$$

$$\alpha_x = \alpha + 2\beta \frac{\Delta_\pi}{M^2_\pi} + \frac{\Delta_\pi}{M^2_\pi} + \frac{\Delta_\pi}{M^2_\pi} \left[ \beta \left( 11 - 18 \frac{\Delta_\pi}{M^2_{\pi^0}} \right) - 17\alpha \right] \beta$$

$$- \frac{1}{96\pi^2} \frac{M^2_\pi}{F^2_\pi} \left( 6\beta^2 \left( 9 + 2 \frac{\Delta_\pi}{M^2_{\pi^0}} \right) - \alpha \beta \left( 47 + 6 \frac{\Delta_\pi}{M^2_{\pi^0}} \right) - 4\alpha^2 \right) L_\pi$$

$$- \frac{1}{24\pi^2} \frac{M^2_\pi}{F^2_\pi} \left[ \frac{M^2_\pi}{\Delta_\pi} L_\pi - 1 \right] (\alpha - \beta) \left[ \beta \left( 4 \frac{\Delta_\pi}{M^2_{\pi^0}} + 1 \right) + \alpha \right]$$

$$+ \frac{\Delta_\pi}{M^2_{\pi^0}} \frac{\beta e^2}{32\pi^2} \hat{\mathcal{K}}^{00} + \frac{\beta e^2}{32\pi^2} \left( 2 \hat{\mathcal{K}}^{00} + 4 \hat{\mathcal{K}}^{00} - 3K^{00} - 4 \hat{\mathcal{K}}^{00} \right) - \frac{\alpha e^2}{32\pi^2} \hat{\mathcal{K}}^{00}$$

$$\alpha_{+-} = \alpha + 4\beta \frac{\Delta_\pi}{M^2_\pi} + \frac{\alpha}{16\pi^2} \frac{M^2_\pi}{F^2_\pi} \left[ 3 - 28 \frac{\Delta_\pi}{M^2_{\pi^0}} - 13\alpha \right] \beta$$

$$+ \frac{\beta e^2}{32\pi^2} \hat{\mathcal{K}}^{00}$$

$$\alpha_{+-} = \alpha + (\beta - \alpha) \frac{\Delta_\pi}{M^2_\pi} + \frac{\alpha}{16\pi^2} \frac{M^2_\pi}{F^2_\pi} \left[ 3 - 28 \frac{\Delta_\pi}{M^2_{\pi^0}} - 13\alpha \right] \beta$$

$$+ \frac{\beta e^2}{32\pi^2} \hat{\mathcal{K}}^{00}.$$
\[ + \frac{1}{96\pi^2} \frac{M_\pi^2}{F_\pi^2} \left[ 2\beta^2 \left( 6 \frac{\Delta_\pi^2}{M_\pi^2} - 16 \frac{\Delta_\pi}{M_\pi^2} - 45 \right) + \alpha \beta \left( 97 + 8 \frac{\Delta_\pi^2}{M_\pi^2} \right) + 2\alpha^2 \right] L_\pi \\
+ \beta \frac{e^2}{32\pi^2} \left( \tilde{K}_1^{+} + \tilde{K}_2^{+} + 4\tilde{K}_3^{+} - 3\tilde{K}_0^{00} - 3\tilde{K}_0^{0}\right) - \alpha \frac{e^2}{32\pi^2} \tilde{K}_3^{-} \\
+ \frac{\Delta_\pi}{M_\pi^2} \beta \frac{e^2}{32\pi^2} \left( \tilde{K}_1^{+} + \tilde{K}_2^{+} \right) + \frac{F_\pi}{M_\pi^2} \left[ 24\alpha^2 \tilde{K}_4^0(\mu) - \frac{9}{4\pi^2} \frac{\Delta_\pi^2}{M_\pi^2} \ln \frac{M_\pi^2}{\mu^2} \right] \]

\[ \beta_x = \beta + \frac{1}{16\pi^2} \frac{\Delta_\pi}{M_\pi^2} (10\alpha - 19\beta) \beta + \frac{1}{16\pi^2} \frac{M_\pi^2}{F_\pi^2} (13\alpha - 10\beta) \beta L_\pi + \frac{1}{48\pi^2} \frac{M_\pi^2}{F_\pi^2} \left[ \frac{M_\pi^2}{\Delta_\pi} - 1 \right] (4\beta - \alpha) \beta \\
+ \beta \frac{e^2}{32\pi^2} \tilde{K}_1^{+} \\
\beta_{+-} = \beta + \frac{1}{16\pi^2} \frac{\Delta_\pi}{M_\pi^2} (5\alpha - 20\beta) \beta + \frac{1}{16\pi^2} \frac{M_\pi^2}{F_\pi^2} \left[ \beta \left( 3\frac{\Delta_\pi}{M_\pi^2} + 2 \right) - 2\alpha \right] \beta L_\pi + \beta \frac{e^2}{32\pi^2} \tilde{K}_1^{-} \]

\begin{align*}
\lambda_{00}^{(1)} &= \frac{1}{3} (\lambda_1 + 2\lambda_2) \\
\lambda_{00}^{(1)} &= \lambda_1 + \frac{1}{16\pi^2} \frac{M_\pi^2}{\Delta_\pi} \left[ L_\pi - 1 \right] \beta^2 \\
\lambda_x^{(2)} &= \lambda_2 - \frac{1}{16\pi^2} \frac{M_\pi^2}{\Delta_\pi} \left[ L_\pi - 1 \right] \beta^2 \\
\lambda_{+-}^{(1)} &= \lambda_1 + \frac{1}{32\pi^2} L_\pi \beta^2 \\
\lambda_{+-}^{(2)} &= \lambda_2,
\end{align*}

(E.3)

\begin{align*}
\frac{e^2}{32\pi^2} \tilde{K}_1^{00} &= e^2 \left[ \frac{20}{9} \tilde{K}_1^{00}(\mu) - \frac{20}{9} \tilde{K}_2^{00}(\mu) + 4\tilde{K}_3^{0} - 2\tilde{K}_4^{0}(\mu) \right] + \beta \frac{\Delta_\pi}{8\pi^2} \frac{M_\pi^2}{F_\pi^2} \ln \frac{M_\pi^2}{\mu^2} \\
\frac{e^2}{32\pi^2} \tilde{K}_1^{0} &= e^2 \left[ \frac{40}{9} \tilde{K}_2^{00}(\mu) + 4\tilde{K}_2^{00}(\mu) - 8\tilde{K}_3^{0} + 4\tilde{K}_4^{0}(\mu) \right] - 20 \frac{\tilde{K}_2^{00}(\mu)}{9} - 20 \frac{\tilde{K}_2^{00}(\mu)}{9} + 4 \frac{\tilde{K}_7^{0}}{9} \\
\frac{e^2}{32\pi^2} \tilde{K}_1^{0} &= e^2 \left[ \frac{40}{9} \tilde{K}_1^{00}(\mu) - \frac{20}{9} \tilde{K}_2^{00}(\mu) + 4\tilde{K}_3^{0} + 4\tilde{K}_4^{0}(\mu) \right] - \frac{\beta}{4\pi^2} \frac{\Delta_\pi}{F_\pi^2} \ln \frac{M_\pi^2}{\mu^2} \\
\frac{e^2}{32\pi^2} \tilde{K}_2^{00} &= e^2 \left[ -\frac{24}{9} \tilde{K}_2^{00}(\mu) + 24 \tilde{K}_3^{0} - 12 \tilde{K}_4^{0}(\mu) \right] + \frac{9}{4\pi^2} \frac{\Delta_\pi}{F_\pi^2} \ln \frac{M_\pi^2}{\mu^2} \\
\frac{e^2}{32\pi^2} \tilde{K}_2^{0} &= e^2 \left[ \frac{40}{9} \tilde{K}_2^{00}(\mu) + 4\tilde{K}_2^{00}(\mu) - 8\tilde{K}_3^{0} + 4\tilde{K}_4^{0}(\mu) \right] - \frac{\beta}{4\pi^2} \frac{\Delta_\pi}{F_\pi^2} \ln \frac{M_\pi^2}{\mu^2} \\
\frac{e^2}{32\pi^2} \tilde{K}_3^{0} &= e^2 \left[ \frac{20}{9} \tilde{K}_1^{00}(\mu) + 52 \tilde{K}_2^{00}(\mu) + 12 \tilde{K}_3^{0} + 6\tilde{K}_4^{0}(\mu) \right] + \frac{3\beta}{8\pi^2} \frac{\Delta_\pi}{F_\pi^2} \ln \frac{M_\pi^2}{\mu^2} \\
\frac{e^2}{32\pi^2} \tilde{K}_4^{0} &= e^2 \left[ \frac{20}{3} \tilde{K}_1^{00}(\mu) + \frac{92}{3} \tilde{K}_2^{00}(\mu) - 12 \tilde{K}_3^{0} - 6\tilde{K}_4^{0}(\mu) \right] + \frac{9}{8\pi^2} \frac{\Delta_\pi}{F_\pi^2} \ln \frac{M_\pi^2}{\mu^2} \\
\frac{e^2}{32\pi^2} \tilde{K}_5^{0} &= e^2 \left[ \frac{40}{9} \tilde{K}_1^{00}(\mu) + 4\tilde{K}_2^{00}(\mu) - 20 \tilde{K}_3^{0} + 12 \tilde{K}_4^{0}(\mu) + \frac{12}{9} \tilde{K}_5^{00}(\mu) - \frac{4}{9} \tilde{K}_7^{0} + 16 \tilde{K}_6^{0}(\mu) \right] - \frac{5\beta}{16\pi^2} \frac{\Delta_\pi}{F_\pi^2} \ln \frac{M_\pi^2}{\mu^2}.
\end{align*}

(E.4)

From the one-loop expressions of the form factors, we obtain the following information on the subtraction constants in eq. (E.10): at this level of accuracy, \( a_S^{\alpha} \) and \( a_V^{\alpha} \) are unchanged as compared to the isospin limit, while

\[ a_S^{\alpha} - a_S^{\alpha} = \frac{\beta}{32\pi^2 F_\pi^2} L_\pi. \]

(E.5)

Finally [we have discarded contributions proportional to \( e^2(m_u - m_d) \) or to \((m_u - m_d)^2\)]

\[ F_S(0) = F_S^{\pi}(0) \left[ 1 + \frac{e^2}{32\pi^2} \left( \tilde{K}_1^{+} + \frac{1}{3} \tilde{K}_2^{+} + \tilde{K}_3^{+} - 2\tilde{K}_1^{00} - \tilde{K}_2^{00} \right) - \frac{1}{8\pi^2} \frac{\Delta_\pi}{F_\pi^2} \beta - \frac{1}{48\pi^2} \frac{M_\pi^2}{F_\pi^2} (5\beta + \alpha) L_\pi \right]. \]

(E.6)

As far as comparison is possible, we find agreement with the existing results in the literature quoted at the beginning of this appendix, except in two instances. The expressions for charged pion scattering given in ref. [43] only included corrections of first order in isospin breaking, with which we agree. The formulae we give here
are not limited to this approximation. Furthermore, we found a slight disagreement with the result of [44] for $F_V(s)$: the radius $\langle r \rangle_V$ exhibits an infrared divergence proportional to $\ln M_{\pi^0}^2$ as $M_{\pi^0}$ goes to zero, whereas we find that $\langle r \rangle_V$ remains finite in this limit, but diverges as $\sim \ln M_{\pi^0}^2$ if we send the charged pion mass to zero, keeping $M_{\pi^0}$ fixed. This is also what follows from our analysis in subsection III.D [69]. In this context, it is important to stress that in the expressions given above, the scale-independent low-energy constants $\tilde{\ell}_i$ are defined as

$$\tilde{\ell}_i(\mu) = \frac{\gamma_i}{32\pi^2} \left( \tilde{\ell}_i + \ln \frac{M_{\pi}^2}{\mu^2} \right).$$

(E.7)

i.e. the normalization of the logarithm is provided by the charged pion mass. We have also checked that the results given in this appendix display infrared behaviours in agreement with the ones obtained in section III.D provided one takes

$$\lambda_1 = \frac{1}{48\pi^2} \left( \tilde{\ell}_1 - \frac{4}{3} \right) \beta^2, \quad \lambda_2 = \frac{1}{48\pi^2} \left( \tilde{\ell}_2 - \frac{5}{6} \right) \beta^2.$$

(E.8)

Notice that these expressions differ from the ones given at the end of appendix C, see eq. (C.19), by the factor $\beta^2$. Both are compatible at one-loop order, where one would take $\beta = 1$ in the above formula.

**Appendix F: Two-loop phases in terms of scattering lengths**

It is perfectly possible, within the framework adopted in this article, to write down expressions that involve the scattering lengths instead of the subthreshold parameters. This is achieved by choosing a parameterization of the lowest-order amplitudes in terms of the scattering lengths, i.e. the value of the amplitudes at their respective thresholds, rather than in terms of their values at the center of the Mandelstam triangle, as done in the rest of the present article. The expressions (III.11) and (III.18) are thus replaced by

$$A^x(s, t) = 16\pi \left[ a_x + b_x \frac{s - 4M_{\pi}^2}{F_{\pi}^2} \right]$$

$$A^{+-}(s, t) = 16\pi \left[ a_{+-} + b_{+-} \frac{s - 4M_{\pi}^2}{F_{\pi}^2} + c_{+-} \frac{t - u}{F_{\pi}^2} \right]$$

$$A^{00}(s, t) = 16\pi a_{00}$$

$$A^{+0}(s, t) = 16\pi \left[ a_{+0} + b_{+0} \frac{s - (M_{\pi} + M_{\pi^0})^2}{F_{\pi}^2} + c_{+0} \frac{t - u + (M_{\pi} - M_{\pi^0})^2}{F_{\pi}^2} \right]$$

$$A^{++}(s, t) = 16\pi \left[ a_{++} + b_{++} \frac{s - 4M_{\pi}^2}{F_{\pi}^2} \right].$$

(F.1)

At this order, the relation between the two sets of parameters is simple,

$$a_{00} = \frac{\alpha_{00} M_{\pi^0}^2}{16\pi F_{\pi}^2}, \quad a_x = \frac{\beta_x}{24\pi F_{\pi}^2} (M_{\pi^0}^2 - 5M_{\pi}^2) - \frac{\alpha_{x} M_{\pi^0}^2}{48\pi F_{\pi}^2}, \quad b_x = -\frac{\beta_x}{16\pi}$$

$$a_{+0} = -\frac{\beta_x}{24\pi F_{\pi}^2} (M_{\pi^0}^2 + M_{\pi}^2) + \frac{\alpha_{x} M_{\pi^0}^2}{48\pi F_{\pi}^2}, \quad b_{+0} = -c_{+0} = -\frac{\beta_x}{32\pi}$$

$$a_{+-} = \frac{\beta_+}{12\pi F_{\pi}^2} M_{\pi}^2 + \frac{\alpha_{+} M_{\pi^0}^2}{24\pi F_{\pi}^2}, \quad b_{+-} = c_{-} = \frac{\beta_+}{32\pi}$$

$$a_{++} = -\frac{\beta_+}{6\pi F_{\pi}^2} M_{\pi}^2 + \frac{\alpha_{+} M_{\pi^0}^2}{24\pi F_{\pi}^2}, \quad b_{++} = -\frac{\beta_+}{16\pi}.$$  

(F.2)

The quantities $a = a_x, a_{+-}, a_{00}$ etc., are scattering lengths to the extent that the tree-level amplitudes (F.1) satisfy

$$\text{Re} \, A(s, t, u) \big|_{\text{thr}} = 16\pi a.$$

(F.3)
The parameters $a$ will keep their meaning up to next-to-next-to-leading order if the above relation still holds for the two-loop amplitudes. This can be achieved upon adjusting the subtraction polynomials accordingly. In practice, this is done through the following choice:

$$P^{00}(s, t, u) = 16\pi a_{00} - w_{00} + \frac{3\lambda^{(1)}_{00}}{F_{\pi}^2} \left[ s(s - 4M_{\pi^0}^2) + t(t - 4M_{\pi^0}^2) + u(u - 4M_{\pi^0}^2) \right]$$

$$+ \frac{3\lambda^{(2)}_{00}}{F_{\pi}^2} \left[ s(s - 4M_{\pi^0}^2)(s - 2M_{\pi^0}^2) + t(t - 4M_{\pi^0}^2)(t - 2M_{\pi^0}^2) + u(u - 4M_{\pi^0}^2)(u - 2M_{\pi^0}^2) \right]$$

$$P^{a}(s, t, u) = 16\pi a_x + w_x + 16\pi b_x \frac{s - 4M_{\pi^0}^2}{F_{\pi}^2} - \frac{\lambda^{(1)}_{a}}{F_{\pi}^4} s(s - 4M_{\pi}^2)$$

$$- \frac{\lambda^{(2)}_{a}}{F_{\pi}^4} \left[ (t + M_{\pi^0}^2 - M_{\pi^0}^2)(t - 3M_{\pi^0}^2 - M_{\pi^0}^2) + (u + M_{\pi^0}^2 - M_{\pi^0}^2)(u - 3M_{\pi^0}^2 - M_{\pi^0}^2) \right]$$

$$- \frac{\lambda^{(3)}_{a}}{F_{\pi}^6} 2s(s - 4M_{\pi}^2)(s - M_{\pi}^2 - M_{\pi^0}^2)$$

$$- \frac{\lambda^{(4)}_{a}}{F_{\pi}^6} \left[ (t + M_{\pi}^2 - M_{\pi^0}^2)\lambda(t) + (u + M_{\pi}^2 - M_{\pi^0}^2)\lambda(u) \right]$$

$$P^{+0}(s, t, u) = 16\pi a_{+0} - w_{+0} + 16\pi b_{+0} \frac{s - (M_{\pi} + M_{\pi^0})^2}{F_{\pi}^2} + 16\pi c_{+0} \frac{t - u + (M_{\pi} - M_{\pi^0})^2}{F_{\pi}^2}$$

$$+ \frac{\lambda^{(1)}_{+}}{F_{\pi}^4} t(t - 4M_{\pi}^2) + \frac{\lambda^{(2)}_{+}}{F_{\pi}^4} \left[ \lambda(s) + \lambda(u) \right]$$

$$+ \frac{\lambda^{(3)}_{+}}{F_{\pi}^6} 2t(t - 4M_{\pi}^2)(t - M_{\pi}^2 - M_{\pi^0}^2) + \frac{\lambda^{(4)}_{+}}{F_{\pi}^6} \left[ (s + M_{\pi}^2 - M_{\pi^0}^2)\lambda(s) + (u + M_{\pi}^2 - M_{\pi^0}^2)\lambda(u) \right]$$

$$P^{+-}(s, t, u) = 16\pi a_{+-} - w_{+-} + 16\pi b_{+-} \frac{s - 4M_{\pi}^2}{F_{\pi}^2} + 16\pi c_{+-} \frac{t - u}{F_{\pi}^2}$$

$$+ \frac{\lambda^{(1)}_{+-}}{F_{\pi}^4} \left[ s(s - 4M_{\pi}^2) + t(t - 4M_{\pi}^2) \right] + \frac{2\lambda^{(2)}_{+-}}{F_{\pi}^4} u(u - 4M_{\pi}^2)$$

$$+ \frac{\lambda^{(3)}_{+-}}{F_{\pi}^6} \left[ s(s - 4M_{\pi}^2)(s - 2M_{\pi}^2) + t(t - 4M_{\pi}^2)(t - 2M_{\pi}^2) \right] + \frac{2\lambda^{(4)}_{+-}}{F_{\pi}^6} u(u - 4M_{\pi}^2)(u - 2M_{\pi}^2)$$

$$P^{++}(s, t, u) = 16\pi a_{++} - w_{++} + 16\pi b_{++} \frac{s - 4M_{\pi}^2}{F_{\pi}^2}$$

$$+ \frac{\lambda^{(1)}_{++}}{F_{\pi}^4} \left[ t(t - 4M_{\pi}^2) + u(u - 4M_{\pi}^2) \right] + \frac{2\lambda^{(2)}_{++}}{F_{\pi}^4} s(s - 4M_{\pi}^2)$$

$$+ \frac{\lambda^{(3)}_{++}}{F_{\pi}^6} \left[ t(t - 4M_{\pi}^2)(t - 2M_{\pi}^2) + u(u - 4M_{\pi}^2)(u - 2M_{\pi}^2) \right] + \frac{2\lambda^{(4)}_{++}}{F_{\pi}^6} s(s - 4M_{\pi}^2)(s - 2M_{\pi}^2)$$

where we have subtracted the values of the one-loop integrals at the appropriate kinematical points to ensure eq. (F.3)

$$w_{00} = \text{Re} \left[ W^0_{00}(4M_{\pi^0}^2) + W^0_{00}(0) + W^0_{00}(0) \right]$$

$$w_x = \text{Re} \left[ W^0_{0}(4M_{\pi}^2) + 2W^0_{0}(4M_{\pi^0}^2 - M_{\pi^0}^2) + 6(5M_{\pi}^2 - M_{\pi^0}^2)W^0_{1}(0)(M_{\pi}^2 - M_{\pi^0}^2) \right]$$

$$w_{+0} = \text{Re} \left[ W^0_{+0}(s + M_{\pi}^2 + M_{\pi^0}^2)^2 - 3(M_{\pi}^2 - M_{\pi^0}^2)^2W^0_{1^+}(0)(M_{\pi}^2 + M_{\pi^0}^2) \right]$$

$$+ \text{Re} \left[ W^0_{0}(4M_{\pi}^2 - M_{\pi}^2)^2 - 3(M_{\pi}^2 + M_{\pi^0}^2)^2W^0_{1^+}(0)(M_{\pi}^2 - M_{\pi^0}^2) \right] + \text{Re} W_{0}^{-}(0)$$

$$w_{+-} = \text{Re} \left[ W^0_{0}(4M_{\pi}^2 - M_{\pi}^2)^2 - 3(M_{\pi}^2 + M_{\pi^0}^2)^2W^0_{1^+}(0)(M_{\pi}^2 - M_{\pi^0}^2) \right] + \text{Re} W_{0}^{-}(0)$$

$$w_{++} = \text{Re} \left[ 2W^0_{+0} + W^0_{0}(4M_{\pi}^2 - 24M_{\pi}^2W_{1^+}(0) \right].$$

(F.5)
These expressions should involve the same number (fifteen) of independent subtraction constants (among them now the scattering lengths) as the ones given in eqs. (IV.28). This means that there exist six relations between the twenty-one parameters occurring in the above polynomials, which stem from crossing symmetry [by construction, the unitarity parts satisfy separately the crossing relations], \( P^z(t,s,u) + P^{+0}(s,t,u) = 0 \) and \( P^{--}(u,t,s) - P^{++}(s,t,u) = 0 \). This yields

\[
b_{+0} + c_{+0} = 0, \quad b_x - 2b_{+0} = 0, \quad b_{+-} - c_{+-} = 0, \quad b_{++} + 2b_{+} = 0,
\]

and

\[
16\pi \left( a_x + a_{+0} - 4 \frac{M^2}{F_\pi^2} b_x \right) = w_{+0} - w_x - 8 \frac{M^2_\pi}{F_\pi^2} (M^2_\pi - M^2_{\pi\pi}) \lambda_{+0} \quad \text{(F.6)}
\]

\[
16\pi \left( a_{+-} - a_{++} + 4 \frac{M^2}{F_\pi^2} b_{++} \right) = w_{+-} - w_{++}.
\]

For the time being, it is convenient not to make use of the two last relations, and to treat all the \( s \)-wave scattering lengths as independent. Notice also that in the chiral counting the scattering lengths are of order \( \mathcal{O}(E^2) \), whereas \( b_x, b_{++} \) and \( b_{+-} \) are of order \( \mathcal{O}(E^0) \). One may now repeat the computation of the relevant partial-wave projections starting with the expressions of the lowest-order amplitudes in terms of the scattering lengths \( a_i \) and effective range parameters \( b_i \) \((i = 00, \pm 0, x, \pm +, \pm +)\), considered as independent quantities, following the procedure outline in sec. [II] and fig. [3]. The results can still be brought into the representations (IV.6), (IV.10), (IV.11), or (IV.15), but the expressions of the polynomials involved are different from the ones given in appendix [E]. For the scattering of neutral pions, the polynomials for \( \psi^{00}_0(s) \) now read

\[
\xi^{(0)}_{00}(s) = \frac{\lambda^{(0)}_{00}}{2M^2_\pi} (5s + 4M^2_\pi - s - 4M^2_{\pi\pi}) - 128\pi^2 \frac{F^4_\pi}{M^4_\pi} \left( a_x - 4b_x \Delta_x \right)^2 \text{Re} J_0(4M^2_{\pi\pi}) + \frac{8b^2_\pi}{9M^2_\pi} (32s^2 - 112sM^2_{\pi\pi} + 39sM^2_\pi + 224M^4_\pi - 732M^2_{\pi\pi} M^2_\pi + 1260M^4_\pi)
\]

\[
- \frac{8a_x b_x F^2_\pi}{M^2_\pi} (s - 20M^2_{\pi\pi} + 68M^2_\pi) + 8 \frac{F^4_\pi}{M^4_\pi} (3a^2_{00} + 8a^2_x)
\]

\[
\xi^{(1;0)}_{00}(s) = \frac{4a^2_{00} F^4_\pi}{M^4_\pi}
\]

\[
\xi^{(1;1)}_{00}(s) = \frac{8b^2_\pi}{3M^2_\pi} (s^2 - 2sM^2_{\pi\pi} + 13sM^2_\pi + 16M^4_{\pi\pi} - 52M^2_{\pi\pi} M^2_\pi + 66M^4_\pi)
\]

\[
- \frac{8a_x b_x F^2_\pi}{M^2_\pi} (s - 4M^2_{\pi\pi} + 10M^2_\pi) + 8a^2_x F^4_\pi / M^4_\pi
\]

\[
\xi^{(2;0)}_{00}(s) = \frac{8a^2_{00} F^4_\pi}{M^4_\pi}
\]

\[
\xi^{(2;\pm)}_{00}(s) = \frac{4}{3} \frac{F^4_\pi}{M^4_\pi} \left[ b_x s - 4M^2_\pi + a_x \right]^2
\]

\[
\xi^{(3;0)}_{00}(s) = - \frac{8}{3} a^2_{00} \frac{F^4_\pi}{M^4_\pi}
\]

\[
\xi^{(3;\pm)}_{00}(s) = - \frac{160}{3} b^2_x + 32a_x b_x \frac{F^2_\pi}{M^2_\pi} - 16 \frac{a^2_x F^4_\pi}{M^4_\pi}.
\]

For the scattering of charged pions, the polynomials involved in the expression for \( \psi^{+0}_0(s) \) (S-wave) read

\[
\xi^{(0)}_{+-,S}(s) = \frac{\lambda^{(1)}_{+-,S}}{3M^2_\pi} (s - 4M^2_\pi)(2s + M^2_\pi) + \frac{\lambda^{(2)}_{+-,S}}{M^2_\pi} (s - 4M^2_\pi)(s + M^2_\pi) - 64\pi^2 \frac{F^4_\pi}{M^4_\pi} a^2_x \text{Re} J_0(4M^2_\pi)
\]

\[
+ \frac{48}{9M^2_\pi} \left( 25b^2_\pi + 7b^2_{++} + \frac{119}{3} b^2_{+-} \right) + \frac{2sM^2_\pi}{3M^2_\pi} b^2_x + \frac{2s}{9M^2_\pi} \left( 71b^2_{++} - 220b^2_{+-} - \frac{1736}{3} b^2_{--} \right)
\]

\[
+ \frac{2sF^2_\pi}{M^2_\pi} (3a_x b_x - 5b_{++} a_{++} + 6b_{+-} a_{+-}) + \frac{32}{9} \left( 29b^2_{++} + 59b^2_x + \frac{448}{3} b^2_{+-} \right) - \frac{8M^2_{\pi\pi}}{M^2_\pi} b^2_x + \frac{88M^2_{\pi\pi}}{3M^2_\pi} b^2_{--}
\]

\[
- \frac{8F^2_\pi}{M^2_\pi} \left( 8a_{++} b_{++} + 15a_x b_x + 32a_{+-} b_{+-} \right) - \frac{8M^2_{\pi\pi} F^2_\pi}{M^2_\pi} a_x b_x + 4 \left( 3a^2_{++} + 5a^2_x + 6a^2_{+-} \right) \frac{F^4_\pi}{M^4_\pi}.
\]
\[ \xi_{i \rightarrow \pi}^i (s) = \frac{2s^2}{9M^4_\pi} (3b^2_{++} + 2b^2_{+-}) + \frac{2s}{9M^2_\pi} (15b^2_{++} + 4b^2_{+-}) - \frac{2sF^2_s}{M^2_\pi} (b_{++} + 2b_{+-}) + \frac{20}{3} (3b^2_{++} + 8b^2_{+-}) - \frac{12F^2_s}{M^2_\pi} (a_{++} + 2a_{+-}) + 2 (a^2_{++} + 2a^2_{+-}) \]

\[ \xi_{i \rightarrow \pi}^i (s) = \frac{4s^2}{3M^4_\pi} b^2_{xx} + \frac{2s(M^2_{\pi^0} + 4M^2_s)}{3M^4_s} b^2_{xx} - \frac{2sF^2_s}{M^2_\pi} a_x b_x + \frac{4(8M^2_\pi + 10M^2_sM^2_{\pi^0} - 3M^4_{\pi^0})}{3M^4_s} b^2_{xx} - \frac{4F^2_s(2M^2_{\pi^0} + M^2_{\pi^0})}{M^4_\pi} a_x b_x + 2a^2_F \]

\[ \xi_{i \rightarrow \pi}^{(2)} (s) = 2 \left( \frac{F^4_s}{M^4_\pi} \right) \left[ b_{x - s - 4M^2_\pi} - a_x \right]^2 \]

\[ \xi_{i \rightarrow \pi}^{(3)} (s) = \frac{16b^4_{xx}}{3M^4_\pi} + \frac{40}{9} (3b^2_{++} + 8b^2_{+-}) + \frac{8F^2_s}{M^2_\pi} (a_{++} + 2a_{+-}) - \frac{4}{3} (a^2_{++} + 2a^2_{+-}) \]

\[ \xi_{i \rightarrow \pi}^{(4)} (s) = \left[ \frac{2}{3} M^4_{\pi^0} - \frac{4M^2_{\pi^0}M^2_s + M^4_s}{3M^4_s} \right] \left( b^2_{xx} + 2 \left( \frac{M^2_{\pi^0} + M^2_s}{M^4_\pi} \right) a_x b_x + \frac{4F^4_s}{M^4_\pi} \right), \]  

while for the P-wave contribution, \( \psi_{i -}^i (s) \), we obtain

\[ \xi_{i \rightarrow \pi}^{(0)} (s) = \frac{\lambda^{(1)}_{-} - \lambda^{(2)}_{-}}{12M^2_\pi} s \left( s - 4M^2_\pi \right) + \frac{s^2}{18M^4_\pi} \left( 25b^2_{++} - 25b^2_{+-} - 16b^2_{-+} \right) + \frac{4s}{3M^2_\pi} \left( \frac{7}{6} b^2_{xx} - \frac{208}{9} b^2_{++} \right) + \frac{2sM^2_{\pi^0}}{9M^4_\pi} b^2_{xx} + \frac{22sF^2_s}{9M^4_\pi} (a_x b_x - b_{++} + 2a_{+-} a_{+-}) + \frac{8}{9} \left( 55b^2_{xx} + 25b^2_{+-} - 291b^2_{-+} \right) - \frac{14M^4_s}{3M^4_\pi} b^2_{xx} + \frac{184M^2_{\pi^0}}{9M^4_\pi} b^2_{xx} + \frac{8F^2_s}{9M^2_\pi} (35a_{++} + 29a_{+-} b_{xx} - 70a_{+-} b_{+-}) - \frac{16M^4_{\pi^0} F^2_s}{3M^4_\pi} a_x b_x - 2 (a^2_{++} - a^2_{+-} - a^2_{-+} - a^2_{-++}) \]

\[ \xi_{i \rightarrow \pi}^{(1)} (s) = \frac{s^2}{3M^4_\pi} b^2_{++} + \frac{2sF^2_s}{3M^2_\pi} a_x b_x + 2b^2_{xx} 8M^2_s + 8M^2_s M^2_{\pi^0} - M^4_s \frac{3M^4_s}{3M^4_\pi} - 4a_x b_x \left( \frac{2M^2_{\pi^0} + M^2_s}{M^4_\pi} \right) F^2_s \]

\[ \xi_{i \rightarrow \pi}^{(2)} (s) = \frac{4}{9} \left( \frac{s}{M^2_\pi} - 4 \right) b^2_{xx} \]

\[ \xi_{i \rightarrow \pi}^{(3)} (s) = \frac{16s}{3M^4_\pi} b^2_{xx} + \frac{40}{9} b^2_{++} - \frac{256}{9} b^2_{+-} + \frac{8F^2_s}{M^2_\pi} (2a_{++} - a_{+-} a_{+-}) + \frac{4}{3} (a^2_{++} - 2a^2_{+-}) \]

\[ \xi_{i \rightarrow \pi}^{(4)} (s) = -8 \left( \frac{8M^4_s - 4M^2_{\pi^0} M^2_s + M^4_s}{3M^4_\pi} \right) b^2_{xx} + 8 \left( \frac{4M^2_{\pi^0} - M^4_s}{M^2_\pi} \right) a_x b - \frac{4}{3} F^4 \]

Finally, in the case of a scattering involving two neutral pions and two charged pions, we obtain the expression for the polynomials describing \( \psi_0 \)

\[ \xi_{i \pi}^{(0)} (s) = -\frac{\lambda^{(1)}_{\pi} - \lambda^{(2)}_{\pi}}{2M^2_\pi} (s - 4M^2_\pi) + \frac{\lambda^{(2)}_{\pi} (s - 4M^2_\pi) (s + 3M^2_\pi - M^2_{\pi^0}) - 64\pi^2 F^4 s}{M^4_\pi} a_x \text{Re} \bar{J}_0 (4M^2_\pi) + (16\pi^2) \left[ 4 \left( \frac{7M^4_s + 2M^2_{\pi^0} M^2_s - M^4_s}{3M^4_\pi} \right) b^2_{xx} - 8a_x b_{xx} F^2_s + a^2 \frac{4F^4_s}{M^4_\pi} \text{Re} \bar{J}_0 (-\Delta_x) + \frac{4}{3} \left( \frac{32}{M^2_\pi} - 11b^2_{xx} \right) \right] \]
In addition, eq. (IV.15) involves two other contributions, one of order ∆ can be neglected for practical purposes, as indicated in section VI [22].

\[ a_{x} = \frac{\beta_{x}}{24 \pi F^{2}_{\pi}} (M_{x}^{2} - 5 M_{x}^{2}) - \frac{\alpha_{x} M_{x}^{2}}{48 \pi F^{2}_{\pi}} - \frac{\lambda_{x}^{(1)}}{4 \pi} \frac{M_{x}^{2} (2 M_{x}^{2} - M_{x}^{2})}{F^{2}_{\pi}} - \frac{\lambda_{x}^{(2)}}{2 \pi} \frac{M_{x}^{4}}{F^{2}_{\pi}} \]

\[ a_{o_{0}} = \frac{\alpha_{00} M_{o_{0}}^{2}}{16 \pi F^{2}_{\pi}} + \frac{9}{4 \pi} \lambda_{00}^{(1)} \frac{M_{o_{0}}^{2}}{F^{2}_{\pi}} + \frac{1}{32 \pi} \left( \frac{\alpha_{00} M_{o_{0}}^{2}}{F^{2}_{\pi}} \right)^{2} J_{0}(4 M_{x}^{2}) \]

\[ a_{x} = \frac{\beta_{x}}{24 \pi F^{2}_{\pi}} (M_{x}^{2} - 5 M_{x}^{2}) - \frac{\alpha_{x} M_{x}^{2}}{48 \pi F^{2}_{\pi}} - \frac{\lambda_{x}^{(1)}}{4 \pi} \frac{M_{x}^{2} (2 M_{x}^{2} - M_{x}^{2})}{F^{2}_{\pi}} - \frac{\lambda_{x}^{(2)}}{2 \pi} \frac{M_{x}^{4}}{F^{2}_{\pi}} \]

In addition, eq. (IV.15) involves two other contributions, one of order O(Δπ) which reads

\[ \xi_{x}^{(1)}(s) = -\frac{8 s + 2 M_{x}^{2}}{9 M_{x}^{2}} b_{x}^{2} + \frac{8 s F^{2}_{\pi}}{M_{x}^{2}} a_{x,0_{0}} b_{x} + 15 M_{x}^{2} + 4 M_{x}^{2} M_{x}^{2} + 11 M_{x}^{2} b_{x}^{2} \]

\[ \xi_{x}^{(2:0)}(s) = 2 a_{00} \frac{F^{4}_{\pi}}{M_{x}^{2}} \left[ \frac{s - 4 M_{x}^{2}}{F^{2}_{\pi}} b_{x} + a_{x} \right] \]

\[ \xi_{x}^{(2:0)}(s) = 4 \frac{F^{2}_{\pi}}{M_{x}^{2}} \left[ \frac{s - 4 M_{x}^{2}}{F^{2}_{\pi}} b_{x} + a_{x} \right] \]

\[ \xi_{x}^{(3)}(s) = -\frac{32 s}{9 M_{x}^{2}} b_{x}^{2} + \frac{32 F^{2}_{\pi}}{3 M_{x}^{2}} \left( 1 + \frac{M_{x}^{2}}{M_{x}^{2}} + \frac{M_{x}^{4}}{M_{x}^{2}} \right) + 8 a_{x,0_{0}} \left( 1 + \frac{M_{x}^{2}}{M_{x}^{2}} \right) \frac{F^{4}_{\pi}}{M_{x}^{2}} \]

We do not give the explicit expression of Δπψ_{0}^{0}, since it represents a tiny contribution of order O(Δπ^{2}) which can be neglected for practical purposes, as indicated in section [VII [22]].

At this stage, one can follow the discussion of section [V] and determine the isospin-breaking differences Δδ_{R}, Δδ_{L}, Δδ_{T} in terms of the different scattering lengths and effective range parameters. Even though one might hope to determine all these parameters from high-precision data on the different channels involved, it seems more realistic to express them in terms of the subthreshold parameters α_{i}, β_{i}, λ_{i}^{(n)} with i = 0, ±, ±, ±, +, +.
\[ + \frac{1}{144 \pi F^2_\pi} \left[ \beta^2 \left( M^4 + M^2 \pi^0 \right) - 10M^2 M^2 \pi^0 - 12M^2 \pi^0 \right] - 4 \beta_x \alpha_x M^2 \pi^0 + \alpha_x M^2 \pi^0 \right] J_{1/0} \left( M^2 - M^2 \pi^0 \right) \]
\[ + \frac{\beta^2}{48 \pi F^2_\pi} \left( M^2 - M^2 \pi^0 \right)^2 J_{1/0} \left( M^2 + M^2 \pi^0 \right) \]
\[ + \frac{\beta^2}{48 \pi F^2_\pi} \left( M^2 + M^2 \pi^0 \right)^2 J_{1/0} \left( M^2 - M^2 \pi^0 \right) \]
\[ a_+ = \frac{\beta_+}{12 \pi F^2_\pi} M^2 + \frac{\alpha_+ M^2 \pi^0}{24 \pi F^2_\pi} + \frac{1}{2 \pi F^2_\pi} \left( \lambda_{+1}^{(1)} + 2 \lambda_{+2}^{(2)} \right) M^4 \]
\[ + \frac{1}{36 \pi F^2_\pi} \left( 2 \beta_+ M^2 + \alpha_+ M^2 \pi^0 \right)^2 J_4 \left( 4M^2 \pi^0 \right) + \frac{1}{288 \pi F^2_\pi} \left( 8 \beta_+ M^2 + \alpha_x M^2 \pi^0 \right)^2 \text{Re} J_0 \left( 4M^2 \pi^0 \right) \]
\[ a_+ = -\frac{\beta_+}{6 \pi F^2_\pi} M^2 + \frac{\alpha_+ M^2 \pi^0}{24 \pi F^2_\pi} + \frac{1}{2 \pi F^2_\pi} \left( \lambda_{+1}^{(1)} + 2 \lambda_{+2}^{(2)} \right) M^4 + \frac{1}{72 \pi F^2_\pi} \left( 4 \beta_+ M^2 - \alpha_+ M^2 \pi^0 \right)^2 J (4M^2 \pi^0) \quad (F.13) \]

These expressions can be exploited, by relying on appendix E and expressing the subthreshold parameters \(\alpha_i, \beta_i, \lambda_i^{(n)}\) with \(i = 0, \pm 0, \pm \) in terms of the isospin-limit parameters \(\alpha, \beta, \lambda^{(n)}\). The latter could be taken as the fundamental parameters of the analysis, but they can be also traded for the two \(\pi \pi\) scattering lengths \(a^2_0\) and \(a^2_\pi\) (up to higher-order corrections that can be estimated using Chiral Perturbation Theory). This series of matching will be indeed the point of view adopted for the analysis of \(K_{4\pi}\) decays, allowing us to reexpress the isospin-breaking correction to be applied to the phase-shift difference in terms of the two scattering lengths \(a_0^2\) and \(a_\pi^2\). [22]
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