On a Muckenhoupt-type condition for Morrey spaces

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Abstract

As is known, the class of weights for Morrey type spaces $L^{p,\lambda}(\mathbb{R}^n)$ for which the maximal and/or singular operators are bounded, is different from the known Muckenhoupt class $A_p$ of such weights for the Lebesgue spaces $L^p(\Omega)$. For instance, in the case of power weights $|x-a|^\nu$, $a \in \mathbb{R}^1$, the singular operator (Hilbert transform) is bounded in $L^p(\mathbb{R})$, if and only if $-1 < \nu < p - 1$, while it is bounded in the Morrey space $L^{p,\lambda}(\mathbb{R})$, $0 \leq \lambda < 1$, if and only if the exponent $\alpha$ runs the shifted interval $\lambda - 1 < \nu < \lambda + p - 1$. A description of all the admissible weights similar to the Muckenhoupt class $A_p$ is an open problem. In this paper, for the one-dimensional case, we introduce the class $A^{p,\lambda}$ of weights, which turns into the Muckenhoupt class $A_p$ when $\lambda = 0$ and show that the belongness of a weight to $A^{p,\lambda}$ is necessary for the boundedness of the Hilbert transform in the one-dimensional case. In the case $n > 1$ we also provide some $\lambda$-dependent $a$ priori assumptions on weights and give some estimates of weighted norms $\|\chi_B\|_{p,\lambda;w}$ of the characteristic functions of balls.

Key Words: Morrey spaces; weights; maximal operator; singular operators;

1 Introduction

The well known Morrey spaces $L^{p,\lambda}$ introduced in [16] in relation to the study of partial differential equations, and presented in various books, see [11], [15], [31], as well as their various generalizations, were widely studied during last decades, including the study of classical operators of harmonic analysis - maximal, singular and potential operators - in these spaces; we refer for instance to papers [2, 3, 4, 5], [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30], where Morrey spaces on metric measure spaces may be also found. Surprisingly, weighted estimates of these classical operators, in fact, almost were not studied. Recently, in [26] there were proved weighted $p \to p$-estimates in Morrey spaces for Hardy operators on $\mathbb{R}_+$ and one-dimensional singular operators (on $\mathbb{R}$ or on Carleson curves in the complex plane). More general weighted estimates may be found in [25] and [23]. In some papers there were considered special weighted situations when the weight was a power of the function $\varphi$ defining the generalized Morrey space, as for instance, in [14], [27].

Let $\Omega \subseteq \mathbb{R}^n$ and $L^{p,\lambda}(\Omega, w)$ denote the classical Morrey space with weight:

$$L^{p,\lambda}(\Omega, w) := \{f : \|f\|_{p,\lambda;w} < \infty\}, \quad 1 \leq p < \infty, \quad 0 \leq \lambda \leq 1,$$

by the norm

$$\|f\|_{p,\lambda;w} := \sup_{x,r} \left( \frac{1}{|B(x,r)|^{\lambda}} \int_{B(x,r)} |f(y)|^p w(y) \, dy \right)^{\frac{1}{p}},$$

by the norm

$$\|f\|_{p,\lambda;w} := \sup_{x,r} \left( \frac{1}{|B(x,r)|^{\lambda}} \int_{B(x,r)} |f(y)|^p w(y) \, dy \right)^{\frac{1}{p}},$$

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where we always assume that a function $f$ is continues beyond $\Omega$ as zero whenever necessary. We write $\|f\|_{p,\lambda}$ in the non-weighted case $w \equiv 1$.

As shown in [26], the Muckenhoupt class $A_p$ may not be an appropriate class of weights for the case of Morrey spaces. The appropriate "Muckenhoupt-type" class for the Morrey spaces must depend on the parameter $\lambda$. As proved in [26] for the one-dimensional case, the singular integral operator

$$Sf(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{t-x} dt$$

is bounded in the space $L^{p,\lambda}(\mathbb{R}^1, w)$ with the power weight $w(x) = |x-a|^\nu$, $a \in \Omega$, if and only if

$$\lambda - 1 < \nu < \lambda + p - 1$$  \hspace{1cm} (1.1)

which is a shifted interval in comparison with the Muckenhoupt condition $-1 < \nu < p - 1$. Thus, condition (1.1) partially deletes Muckenhoupt power weights, but on the other hand, adds new ones.

As is known, a description of all the admissible weights for Morrey spaces, similar to the Muckenhoupt class $A_p$ is an open problem. Since the $A_p$-condition

$$A_p : \sup_B \left( \frac{1}{|B|} \int_B w(x) \, dx \right) \left( \frac{1}{|B|} \int_B w^{1-p'}(y) \, dy \right)^{p-1} < \infty$$  \hspace{1cm} (1.2)

has the form

$$\sup_B \frac{1}{|B|} \|v\|_{L^p(B)} \left( \frac{1}{|B|} \right)^{\frac{1}{p'}} \|v\|_{L^{p'}(B)} < \infty, \quad v = w^\frac{1}{p'},$$  \hspace{1cm} (1.3)

where $\sup$ is taken with respect to all balls in $\mathbb{R}^n$, one can expect that the corresponding Muckenhoupt-type class $A_{p,\lambda}$ may be defined by the condition

$$A_{p,\lambda} : \sup_B \frac{1}{|B|} \|v\|_{L^p(B)} \left( \frac{1}{|B|} \right)^{\frac{1}{p'}} \|v\|_{[L^p,\lambda]'}(B) < \infty, \quad v = w^\frac{1}{p'},$$  \hspace{1cm} (1.4)

where $[L^p,\lambda]'$ may stand for the dual (or predual ?) of the Morrey space. The preduals of Morrey spaces were studied in [1], [3], [12] and [32]. Their characterizations are known to be given in capacitory terms and/or in terms of the so called $(q, \lambda)$-atomic decompositions, which makes them uneasy in concrete applications.

We introduce a certain class $A_{p,\lambda} = A_{p,\lambda}(\mathbb{R}^n)$ of weights, which might be conditionally called called a pre-Muckenhoupt class for Morrey spaces. It turns into the Muckenhoupt class $A_p$ when $\lambda = 0$ and we show that the belonging of a weight to this class is necessary for the one-dimensional singular integral operator (Hilbert transform) to be bounded in the Morrey space. In the case $n > 1$ we also provide some $\lambda$-dependent à priori assumptions on weights and give some estimates of weighted norms $\|\chi_B\|_{p,\lambda;w}$ of the characteristic functions of balls.

Problems in proving the sufficiency of the introduced $A_{p,\lambda}$-condition are caused by difficulties of transferring various known properties of the class $A_p$, such as for instance its openness with respect to $p$, to the class $A_{p,\lambda}$. We hope to have advances in this relation in another publication.

## 2 Some à priori assumptions and the class $A_{p,\lambda}$

The definition (1.2) of the Muckenhoupt class $A_p$ for the spaces $L^p$ (the case $\lambda = 0$) preassumes that

$$\text{the functions } w \text{ and } w^{1-p'} \text{ are locally in } L^p.$$  \hspace{1cm} (2.1)
What should be similar à priori assumptions for the Morrey spaces? In the case of the power weight \( w = |x - a|^p \) the conditions in (2.1) mean that \( \nu > -n \) and \( \nu < n(p - 1) \), respectively. In the case of Morrey spaces, the corresponding interval \((-n, n(p - 1))\) should be shifted to \((n\lambda - n, n\lambda + n(p - 1))\), as noted in (1.1) in the one-dimensional case \( n = 1 \). Thus for general weights we expect that the à priori assumption \( w \in L^1_{\text{loc}} \) must be replaced by some more restrictive condition, while the condition \( w^{1-p'} \in L^1_{\text{loc}} \) is expected to be weakened, both in dependence on the parameter \( \lambda \).

As a substitution of the first assumption in (2.1) we will use now the following natural condition on the weight \( w \):
\[
\chi_B \in L^{p,\lambda}(\Omega, w) \iff \chi_B w^{\frac{1}{1-p}} \in L^{p,\lambda}(\Omega) \quad (2.2)
\]
for all the balls \( B \), where \( \chi_E \) denotes the characteristic function of an open set \( E \subset \Omega \). As a substitution of another condition \( w^{1-p'} \in L^1_{\text{loc}} \) we introduce the condition
\[
\chi_B \in L^{p,\lambda}(\Omega, w^{-\frac{1}{1-p'}}) \iff \chi_B w^{-\frac{1}{1-p'}} \in L^{p,\lambda}(\Omega, w), \quad (2.3)
\]
which turns into \( w^{1-p'} \in L^1_{\text{loc}} \) when \( \lambda = 0 \). With the notation \( w(E) := \int_E w(x) \, dx \), the conditions (2.2) and (2.3) have the form
\[
\sup_B \frac{w(B \cap B(x_0, r))}{|B|^{\lambda}} < \infty, \quad \sup_B \frac{w^{-\frac{1}{1-p'}}(B \cap B(x_0, r))}{|B|^{\lambda}} < \infty, \quad (2.4)
\]
respectively, where the sup is taken with respect to all balls \( B \subset \Omega \).

**Definition 2.1.** A weight function \( w \) is called \((p, \lambda)\)-admissible weight, if it satisfies the assumptions (2.2), (2.3).

The condition (2.2) of belongingness of functions \( \chi_B \) to the weighted space \( L^{p,\lambda} \) is quite natural. As for the exponent \(-\frac{1}{1-p'}\) in the condition (2.3), its choice originated in particular from the upper bound in the conditions (1.1) known to be necessary and sufficient for power weights.

Now we introduce the class \( A_{p,\lambda} \) by the following definition.

**Definition 2.2.** By \( A_{p,\lambda} \) we denote the class of \((p, \lambda)\)-admissible weights satisfying the condition
\[
A_{p,\lambda} := \sup_B \left( \frac{\|\chi_B\|_{p,\lambda;w}}{\|\chi_B\|_{p,\lambda;w}} \left( \frac{1}{|B|} \int_B w^{-\frac{1}{1-p'}} \, dy \right) \right) < \infty, \quad w_w = w^{-\frac{1}{1-p'}}, \quad (2.5)
\]
where sup is taken with respect to all balls. Obviously we obtain the Muckenhoupt class \( A_p \) when \( \lambda = 0 \).

**Remark 2.3.** While \( A_{p,\lambda} = A_{p,0} = A_p \) in the case \( \lambda = 0 \), a comparison of the classes \( A_{p,\lambda} \) and \( A_{p,\lambda} \) in the case \( \lambda > 0 \) is an open question. Note that
\[
A_{p,\lambda} \subseteq A_{p,\lambda} \iff \int_B v^{-\frac{p}{1-p'}} \, dy \leq C \left\| u^{-\frac{1}{1-p'}} \right\|_{L^{p,\lambda}(B)} \left\| \frac{1}{v} \right\|_{L^{p,\lambda}(B)}, \quad (2.6)
\]
the latter inequality may be also rewritten in the form
\[
\int_B u \, dy \leq C \left\| u^{-\frac{1}{1-p}} \right\|_{L^{p,\lambda}(B)} \left\| u^{-\frac{1}{1-p'}} \right\|_{L^{p,\lambda}(B)}, \quad u = v^{-\frac{p}{1-p'}}; \quad (2.7)
\]
(a Hölder-type looking inequality).
3 Necessity of the $A_{p,\lambda}$-condition for the Hilbert transform

We pass to the one-dimensional case and consider the singular operator $S$ (Hilbert transform). We follow the known approach to prove the necessity of the $A_{p,\lambda}$-conditions for the boundedness of the singular operator known for the Lebesgue spaces, as presented for instance in [13].

We find it convenient to use the notation

$$I = I(x, r) = \{ y : x - r < y < x + r \}$$

for the one-dimensional balls. We assume that the weight $w$ is $(p, \lambda)$-admissible in the sense of Definition 2.1 which now means that

$$\chi_I \in L^{p,\lambda}(\mathbb{R}, w) \quad (3.1)$$

and

$$\chi_I \in L^{p,\lambda}\left(\mathbb{R}, w^{-\frac{1}{p+\lambda}}\right), \quad (3.2)$$

for all intervals $I \subset \mathbb{R}$.

In the sequel, by $I'$ and $I''$ we denote two arbitrary adjoint intervals

$$I' = I(x', r'), \; \; \; I'' = I(x'', r'')$$

that is, we suppose that either $x' + r' = x'' + r''$ or $x' - r' = x'' + r''$. We always have the pointwise estimate

$$(S\chi_I')(x) \geq \frac{1}{2} \text{ for all } x \in I'' \quad (3.3)$$

and all $I'$ and $I''$ with $|I'| = |I''| \leq 1$.

Suppose that the singular operator $S$ is bounded in the weighted Morrey space:

$$\|Sf\|_{p,\lambda,w} \leq k\|f\|_{p,\lambda,w}. \quad (3.4)$$

Note that $k \geq 1$, which follows from the fact that $S^2 = -I$.

**Remark 3.1.** In the case of Lebesgue spaces ($\lambda = 0$) it is known that the boundedness (3.4) implies both the conditions (3.1) and (3.2). A similar direct proof of the validity of (3.1), for instance, from (3.4) does not hold, because it is based on the use of duality arguments, which fails in the case of Morrey spaces. Instead we suppose that (3.1) and (3.2) à priori hold.

**Lemma 3.2.** Assume that (3.4) holds and the weight $w$ has the property (3.1). Then

$$\frac{1}{2k}\|\chi_{I'}\|_{p,\lambda,w} \leq \|\chi_{I''}\|_{p,\lambda,w} \leq 2k\|\chi_{I'}\|_{p,\lambda,w} \quad (3.5)$$

for all adjoint intervals $I'$ and $I''$ with equal lengths $|I'| = |I''| \leq 1$.

**Proof.** We substitute the function $f = \chi_{I'}$ into (3.4), which is possible by (3.1), and obtain

$$\sup_{x,r} \frac{1}{p\lambda} \int_{I(x,r)} |S\chi_{I'}(y)|^p w(y) \, dy \leq k^p \sup_{x,r} \frac{1}{p\lambda} \int_{I(x,r)} |\chi_{I'}(y)| w(y) \, dy.$$

Then moreover

$$\sup_{x,r} \frac{1}{p\lambda} \int_{I(x,r) \cap I''} |S\chi_{I'}(y)|^p w(y) \, dy \leq k^p \sup_{x,r} \frac{1}{p\lambda} \int_{I(x,r)} |\chi_{I'}(y)| w(y) \, dy.$$
and consequently
\[
\sup_{x,r} \frac{1}{I(x,r)^{\gamma}} \int_{I(x,r) \cap I''} w(y) \, dy \leq (2k)^p \sup_{x,r} \frac{1}{I(x,r)^{\gamma}} \int_{I(x,r)} |\chi_{I'}(y)| w(y) \, dy
\]
by (3.3), i.e. we arrive at the right-hand side inequality in (3.5). Similarly the left-hand side inequality is proved.

Theorem 3.3. Let the assumption (3.4) hold with a \((p, \lambda)\)-admissible weight \(w\). Then
\[
\sup_{I: |I| \leq 1} \frac{1}{|I|} \int_{I} \frac{1}{w^{1/\lambda}} \, dy \leq 2k < \infty, \quad w_* = w^{-\frac{1}{\lambda}}
\]
with \(k = \|S\|_{L^{p,\lambda}(\mathbb{R}) \to L^{p,\lambda}(\mathbb{R})}\).

Proof. Let \(I'\) and \(I''\) be two adjoin intervals with \(|I'| = |I''| \leq 1\). Now we substitute \(f = \chi_{I'} w^{-\beta}\), where \(\beta = \frac{1}{\lambda + p - 1}\) into (3.4), which is possible by the assumption (2.3):
\[
\sup_{B} \frac{1}{|B|} \int_{B} \left| S \left( \chi_{I'} w^{-\beta} \right)(y) \right|^p w(y) \, dy \leq k^p \sup_{B} \frac{1}{|B|^{\lambda}} \int_{B} \left| \chi_{I'}(y) w^{-\beta}(y) \right|^p w(y) \, dy
\]
\[
= k^p \sup_{B} \frac{1}{|B|^{\lambda}} \int_{B} \chi_{I'}(y) w^{1-\beta p}(y) \, dy
\]
where \(B = (x - r, x + r)\) is an arbitrary interval. Hence, moreover
\[
\sup_{B} \frac{1}{|B|^{\lambda}} \int_{B \cap I''} \left| S \left( \chi_{I'} w^{-\beta} \right)(y) \right|^p w(y) \, dy \leq k^p \sup_{B} \frac{1}{|B|^{\lambda}} \int_{B} \chi_{I'}(y) w^{1-\beta p}(y) \, dy.
\]
Similarly to (3.3) we have
\[
S \left( \chi_{I'} w^{-\beta} \right)(y) \geq \frac{1}{2|I'|} \int_{I'} w^{-\beta}(t) \, dt \quad \text{for} \quad y \in I''.
\]
Consequently,
\[
\frac{1}{|I'|^p} \left( \int_{I'} w^{-\beta}(t) \, dt \right)^p \sup_{B} \frac{1}{|B|^{\lambda}} \int_{B \cap I''} w(y) \, dy \leq (2k)^p \sup_{B} \frac{1}{|B|^{\lambda}} \int_{B} \chi_{I'}(y) w^{1-\beta p}(y) \, dy,
\]
i.e.
\[
\frac{1}{|I'|} \int_{I'} w^{-\beta}(t) \, dt \|\chi_{I''}\|_{p,\lambda;w} \leq 2k \|\chi_{I'}\|_{p,\lambda;w^{1-\beta p}}.
\]
Since \(\|\chi_{I''}\|_{p,\lambda;w} \sim \|\chi_{I'}\|_{p,\lambda;w}\) by Lemma 3.2 we arrive at the condition (3.6) with \(I'\) redenoted by \(I\).

Corollary 3.4. Let \(w\) be a \((p, \lambda)\)-admissible weight. The condition \(w \in A_{p,\lambda}\) is necessary for the boundedness of the singular operator in the weighted Morrey space \(L^{p,\lambda}(\mathbb{R}, w)\).
4 Norms of characteristic functions of balls in Morrey spaces

In relation to the weighted Morrey-norms of functions $\chi_B(x,r)$ appearing in (2.5), in this section we give some details on estimation of such norms.

Note that every simple function belongs to non-weighted Morrey spaces, while it is not the case in general for weighted Morrey spaces. Any such beloneness for functions $\chi_B$ imposes conditions on the weight, which were already discussed in Section 2. The aim of this section is to shed more light on such beloneness and to give some estimations of the norms $\|\chi_B\|_{p,\lambda;w}$ involved in the $A_{p,\lambda}$-condition (2.5) in the case $\Omega = \mathbb{R}^n$.

4.1 The non-weighted case

Let $B(x,r) := \{y \in \mathbb{R}^n : |y-x| < r\}$. Fix a ball $B(x_0, r_0)$. The following lemma holds.

**Lemma 4.1.** Let $1 \leq p < \infty$, $0 \leq \lambda \leq 1$. The formula

$$\|\chi_{B(x_0,r_0)}\|_{p,\lambda} = |B(x_0, r_0)|^{\frac{n(1-\lambda)}{p}} = (\omega_n r_0^n)^{\frac{1-\lambda}{p}}.$$  \hspace{1cm} (4.1)

is valid, where $\omega_n = |\mathbb{S}^{n-1}|$.

**Proof.** By the definition of the norm we have

$$\|\chi_{B(x_0,r_0)}\|_{p,\lambda} = \sup_{x,r} \left(\frac{1}{(\omega_n r^n)^\lambda} |B(x,r) \cap B(x_0,r_0)| \right)^\frac{1}{p} = \max\{A, B\},$$  \hspace{1cm} (4.2)

where

$$A = \sup_{0 < r < r_0} \left(\frac{1}{(\omega_n r^n)^\lambda} |B(x,r) \cap B(x_0,r_0)| \right)^\frac{1}{p}, \quad B = \sup_{r > r_0} \left(\frac{1}{(\omega_n r^n)^\lambda} |B(x,r) \cap B(x_0,r_0)| \right)^\frac{1}{p}.$$  

We have

$$A \leq \sup_{0 < r < r_0} \left(\frac{|B(x_0,r_0)|}{\omega_n r_0^n} \right)^\frac{1}{p} = \sup_{0 < r < r_0} (\omega_n r_0^n)^{\frac{1-\lambda}{p}} = (\omega_n r_0^n)^{\frac{1-\lambda}{p}}.$$  

The same estimate $B \leq \left(\frac{|B(x_0,r_0)|}{\omega_n r_0^n} \right)^\frac{1}{p}$ for $B$ is obvious, so that

$$\|\chi_{B(x_0,r_0)}\|_{p,\lambda} \leq |B(x_0, r_0)|^{\frac{n(1-\lambda)}{p}} = (\omega_n r_0^n)^{\frac{1-\lambda}{p}}.$$ \hspace{1cm} (4.3)

To obtain the inequality inverse to (4.3) we observe that

$$\|\chi_{B(x_0,r_0)}\|_{p,\lambda} \geq \sup_r \left(\frac{1}{(\omega_n r^n)^\lambda} |B(x_0,r) \cap B(x_0,r_0)| \right)^\frac{1}{p} = \sup_r \left(\frac{1}{(\omega_n r^n)^\lambda} |B(x_0, \min(r,r_0))| \right)^\frac{1}{p}$$

$$= \max \left\{ \sup_{0 < r < r_0} \left(\frac{1}{(\omega_n r^n)^\lambda} |B(x_0,r)| \right)^\frac{1}{p}, \sup_{r > r_0} \left(\frac{1}{(\omega_n r^n)^\lambda} |B(x_0,r_0)| \right)^\frac{1}{p} \right\}.$$  

Hence

$$\|\chi_{B(x_0,r_0)}\|_{p,\lambda} \geq \left(\frac{|B(x_0,r_0)|}{(\omega_n r_0^n)^\lambda} \right)^\frac{1}{p}. \hspace{1cm} (4.4)$$

Combining this with (4.3), we arrive at (4.1).
4.2 The weighted case

In the weighted case we cannot already write a precise formula of type (4.1) localized to the point \( x_0 \), since the values of the weight \( w \) at the points \( x \) different from \( x_0 \) may already heavily influence on the value of the norm \( \| \chi_{B(x_0,r_0)} \|_{p,\lambda,w} \).

With the usual notation \( w(E) = \int_E w(x) \, dx \) we can write
\[
\| \chi_{B(x_0,r_0)} \|_{p,\lambda,w} = \sup_{x \in \mathbb{R}^n, r > 0; \ |x-x_0| < r + r_0} \left( \frac{w(B(x,r) \cap B(x_0,r_0))}{(\omega_n r^n)^\lambda} \right)^{\frac{1}{p}},
\]
where we took into account that \( w(B(x,r) \cap B(x_0,r_0)) = \emptyset \) when \( |x-x_0| > r + r_0 \); however, (4.5) is just a direct usage of the definition of the norm. From (4.5) we can derive the following statement.

Lemma 4.2. The norm \( \| \chi_{B(x_0,r_0)} \|_{p,\lambda,w} \) admits the estimate
\[
\frac{1}{\omega_n} \sup_{0 < r < r_0} \left( \frac{w(B(x_0,r))}{r^n} \right)^{\frac{1}{p}} \leq \| \chi_{B(x_0,r_0)} \|_{p,\lambda,w} \leq \frac{1}{\omega_n} \sup_{0 < r < r_0} \left( \frac{w(B(x,r))}{r^n} \right)^{\frac{1}{p}}.
\]

Proof. The proof is similar to that of Lemma 4.1; the left-hand side inequality is proved exactly in the same way as the lower bound (4.3) in Lemma 4.1, while the validity of the right-hand side one becomes obvious from the line
\[
\| \chi_{B(x_0,r_0)} \|_{p,\lambda,w} \leq \max \{ \mathcal{A}, \mathcal{B} \}
\]
with
\[
\mathcal{A} = \sup_{0 < r < r_0} \left( \frac{w(B(x,r))}{(\omega_n r^n)^\lambda} \right)^{\frac{1}{p}}
\]
and
\[
\mathcal{B} = \sup_{r > r_0} \left( \frac{w(B(x_0,r))}{(\omega_n r^n)^\lambda} \right)^{\frac{1}{p}} = \left( \frac{w(B(x_0,r_0))}{(\omega_n r_0^n)^\lambda} \right)^{\frac{1}{p}} \leq \mathcal{A}.
\]

The following corollary provides a sufficient condition on the weight function \( w \) for which bounded functions with a compact support belong to the weighted space \( L^{p,\lambda}(\mathbb{R}^n, w) \).

Corollary 4.3. For the characteristic function \( \chi_{B(x_0,r_0)} \) of a ball \( B(x_0,r_0) \) to belong to the space \( L^{p,\lambda}(\mathbb{R}^n, w) \), the condition
\[
\sup_{0 < r < r_0} \frac{w(B(x,r))}{r^n} < \infty
\]
(4.7)
is sufficient, and the condition
\[
\sup_{0 < r < r_0} \frac{w(B(x_0,r))}{r^n} < \infty
\]
(4.8)
is necessary.

Remark 4.4. Let \( w(x) = |x-a|^\nu g(x) \), where \( a \in \mathbb{R}^n \), \( g \) is a bounded function with compact support, and \( \nu > -n \). When \( a \in B(x_0,r_0) \), then
\[
\chi_{B(x_0,r_0)} \in L^{p,\lambda}(\mathbb{R}^n, w)
\]
(4.9)
if and only if
\[ \nu \geq n\lambda - n, \]  
(4.10)
and
\[ \| \chi_B \|_{p,\lambda; w} \sim \frac{|B|^{n+\nu-n\lambda}}{np} \]  
(4.11)
in this case. When \(|x_0 - a|\) is large enough, \(|x_0 - a| > 2r_0\), the inclusion (4.9) holds for any \(\nu > -n\).

Proof. Direct estimations via the passage to polar coordinates, dilation change of variables and rotation yield
\[
\frac{w(B(x,r))}{(\omega_n r^n)^\lambda} = \frac{1}{r^{n\lambda}} \int_{|y-x|<r} |y-a|^{\nu} \, dy
\]
where \(e_1 = (1, 0, \ldots, 0)\). The remaining integral \(I(t) = \int_{|y|<t} |y-e_1|^{\nu} \, dy\) is estimated by standard means:
\[
I(t) \sim \begin{cases} 
\frac{t^n}{t^{n+\nu}}, & 0 < t < 1 \\
\frac{t^n}{t^{n+\nu}}, & t > 1
\end{cases},
\]
where \(I(t) \sim \text{R.H.S.}\) means that \(c_1\text{R.H.S.} \leq I(t) \leq c_2\text{R.H.S.}\) with \(c_1\) and \(c_2\) not depending on \(t\). Then
\[
\frac{w(B(x,r))}{r^{n\lambda}} \sim \begin{cases} 
\frac{r^{n-n\lambda}|x-a|^{\nu}}{r^{n+\nu-n\lambda}}, & r \leq |x-a| \\
\frac{r^{n-n\lambda}|x-a|^{\nu}}{r^{n+\nu-n\lambda}}, & r \geq |x-a|
\end{cases}
\]
from where the statement of the remark follows, with the necessity statement checked directly at the point \(x = x_0\). \(\square\)

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