Robust Equilibria in Concurrent Games

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Abstract We study the problem of finding robust equilibria in multiplayer concurrent games. A \((k, t)\)-robust equilibrium is a strategy profile such that no coalition of size \(k\) can improve the payoff of one its member by deviating, and no coalition of size \(t\) can decrease the payoff of other players. We are interested in pure equilibria, that is, solutions that can be implemented using non-randomized strategies. We suggest a general transformation from multiplayer games to two-player games such that pure equilibria in the first game correspond to winning strategies in the second one. We then devise from this transformation, an algorithm which computes equilibria in regular and mean-payoff games. Robust equilibria in regular games reduce to winning strategies in two-player Muller games and equilibria in mean-payoff games reduce to winning strategies in multidimensional mean-payoff games. In both cases, the obtained algorithms work in polynomial space. PSPACE-hardness of the decision problem already holds for Büchi objectives.

1 Introduction

Games are intensively used in computer science to model interactions in computerized systems. Two player antagonistic games have been successfully used for the synthesis of reactive systems. In this context, the opponent acts as a hostile environment, and winning strategies provide controllers that ensure correctness of the system under any scenario. In order to model complex systems in which several rational entities interact, multiplayer concurrent games come into the picture. Correctness of the strategies can rely on different solution concepts, which describe formally what is a “good” strategy. In game theory, the fundamental solution concept is Nash equilibrium [14] and other ones have been proposed to refine or relax it, such as subgame perfect equilibrium [17], iterative admissibility [15], and robust equilibria [1]. The notion of robust equilibria refines Nash equilibria in two ways:

- a robust equilibrium is resilient, i.e. when a “small” coalition of player change its strategy, it can not improve the payoff of one of its players;
- it is immune, i.e. when a “small” coalition changes its strategy, it will not lower the payoff of the non-deviating players.

The size of small coalitions is determined by some bounds written \(k\) for resilience and \(t\) for immunity. When a strategy is both \(k\)-resilient and \(t\)-immune, it is called a \((k, t)\)-robust equilibrium. We also generalize this concept to the notion \((k, t, r)\)-robust equilibrium, where if \(t\) players are deviating, the others should not have their payoff decrease by more than \(r\).
Example  In network design, when many users are interacting, coalitions can easily be formed and resilient strategies are necessary to avoid deviation. It is also likely that some devices are faulty and start behave unexpectedly, hence the need for immune strategies. As an example, consider the problem of medium access control. It was first given a game-theoretic model in [12]. Several users share the access to a wireless channel. These users are selfish and try to maximize the number of packets they transmit. During each slot, they can choose to either transmit or wait for the next slot. If too many users are emitting in the same slot, then they all fail to send data.

However Nash equilibria can present two weaknesses:

– there is no guarantee when two (or more) users deviate together;
– when a deviation occurs, the strategies of the equilibrium can punish the deviating user without any regard for payoffs of the others. This can result in a situation where, because of a faulty device, nobody can emit.

By comparison, finding resilient equilibria with $k$ greater than 1, ensures that players have no interest in forming coalitions (up to size $k$), and finding immune equilibria with $t$ greater than 0 ensures that other players will not suffer from some agents (up to $t$) behaving differently from what was expected.

Contribution In this paper, we study the problem of finding robust equilibria in multiplayer concurrent games. In Section 3, we describe a generic transformation from multiplayer games to two-player games. The resulting two-player game is called the deviator game. We show that equilibria in the original game correspond to winning strategies in the second one. In Section 4, we show that for games with regular preferences, the game obtained by the transformation can be expressed as a two-player Muller game. Making use of this transformation, we present an algorithm to find equilibria in regular games. We show that this algorithm works in polynomial space for Muller objectives. This is matched by a PSPACE-hardness lower bound for Büchi objectives. In Section 5, we study quantitative games with mean-payoff objectives. We show that the game obtained by our transformation is equivalent to a multidimensional mean-payoff game and provide a polynomial space algorithm.

Related works Other solution concepts have been studied on game on graph, in particular Nash equilibrium [19,20,4], subgame perfect equilibria [18,6], regret minimization [11], secure equilibria [8]. Note that in this paper we only consider pure strategies: in the general case of randomized strategies, existence of a Nash equilibrium is undecidable [20]. The central construction of the deviator game, is inspired by the suspect game of [4], which was developed for Nash equilibria. However, the analysis of quantitative games such as mean-payoff games is more elaborate than for Nash equilibria (see [20]). To solve quantitative games, we rely on the analysis of multidimensional mean-payoff games [21,5]. Note that the concept of robust equilibria for games with LTL objectives is expressible in logics such as strategy logic [9] or ATL* [2]. However, deciding satisfiability in these logic is difficult: it is $2^\text{EXPTIME}$-complete for ATL* and undecidable for
strategy logic in general ($\exists$EXPTIME-complete fragments exist [13]). Moreover, these logics cannot express equilibria in games with quantitative games such as mean-payoff.

Nash equilibria correspond to the special case of $(1,0)$-robust equilibria. Secure equilibria [8], do not exactly correspond to a class of robust equilibria, however we can note that $(1,1)$-robust $\implies$ secure $\implies$ $(1,0)$-robust. Although we do not directly solve the problem of finding secure equilibria in this paper, the techniques presented here could be adapted.

2 Definitions

2.1 Concurrent game structures

We study concurrent game structures (CGS) as defined in [2].

Definition 1 (Concurrent game structure). A concurrent game structure $G$ is a tuple $(\text{Stat}, s_0, \text{Agt}, \text{Act}, \text{Mov}, \text{Tab})$, where:

- $\text{Stat}$ is a finite set of states and $s_0 \in \text{Stat}$ is the initial state;
- $\text{Agt}$ is a finite set of players;
- $\text{Act}$ is a finite set of actions; a tuple $(m_A)_{A \in \text{Agt}}$ containing one action for each player is called a move;
- $\text{Mov} : \text{Stat} \times \text{Agt} \to \mathcal{P}(\text{Act}) \setminus \{\emptyset\}$ indicates the actions available to a given player in a given state; if an action $m$ belongs to $\text{Mov}(s,A)$ it is said legal in state $s$ for player $A$; a move $(m_A)_{A \in \text{Agt}}$ is said legal in state $s$ if for each $A \in \text{Agt}$, $m_A$ is legal in $s$ for $A$;
- $\text{Tab} : \text{Stat} \times \text{Act}^{\text{Agt}} \to \text{Stat}$ is the transition function, it associates with a given state and a given move, the resulting state; it is only defined for pairs $(s, (m_A)_{A \in \text{Agt}})$ where $(m_A)_{A \in \text{Agt}}$ is legal in $s$.

In a CGS $G$, whenever we arrive at a state $s$, the players simultaneously select an action, which results in a move $m_{A\text{gt}}$; the next state of the game is then $\text{Tab}(s, m_{A\text{gt}})$. This process starts from $s_0$ and is repeated ad infinitum to form an infinite sequence of states. An example of a CGS is given in Figure ??.

Definition 2 (History and plays). A history of the game $G$ is a finite sequence of states and moves ending with a state, i.e. an element of $(\text{Stat} \cdot \text{Act}^{\text{Agt}})^*$. A history $h$ is the $i$-th state of $h$, starting from 0, and $m_i$ its $i$-th move, thus $h = h_0 \cdot m_0^h \cdot h_1 \cdots m_{n-1}^h \cdot h_n$. The length $|h|$ of such a history is $n + 1$. We write $\text{last}(h)$ the last state of $h$, i.e. $h_{|h|-1}$. A play $\rho$ is an infinite sequence of states and moves, i.e. an element of $(\text{Stat} \cdot \text{Act}^{\text{Agt}})^\omega$. We write $\rho_{\leq n}$ for the prefix of $\rho$ of length $n + 1$, i.e. $\rho_0 \cdot m_0^\rho \cdots m_{n-1}^\rho \cdot \rho_n$.

1 if $S$ is a set, then $\mathcal{P}(S)$ is the set of subsets of $S$, i.e. $\mathcal{P}(S) = \{T \mid T \subseteq S\}$
Definition 3 (Strategies). Let $\mathcal{G}$ be a CGS, and $A \in \text{Agt}$. A strategy for player $A$ maps histories to available actions. Formally, a strategy is a function $\sigma_A : (\text{Stat} \cdot \text{Act}^{\text{Agt}})^* \cdot \text{Stat} \rightarrow \text{Act}$ such that for all $h \in (\text{Stat} \cdot \text{Act}^{\text{Agt}})^* \cdot \text{Stat}$, $\sigma_A(h) \in \text{Mov}(\text{last}(h), A)$. A coalition $C \subseteq \text{Agt}$ is a set of players, its size is written $|C|$. A strategy $\sigma_C$ for a coalition $C \subseteq \text{Agt}$ is a tuple of strategies, one for each player in $C$. We write $\sigma_C = (\sigma_A)_{A \in C}$ for such a strategy. A strategy profile is a strategy for $\text{Agt}$. We will write $(\sigma_{-C}, \sigma_C')$ for the strategy profile $\sigma''_{\text{Agt}}$ such that if $A \in C$ then $\sigma''_A = \sigma'_A$ and otherwise $\sigma''_A = \sigma_A$. We write $\text{Strat}_C(C)$ for the set of strategies of coalition $C$.

Definition 4 (Outcomes). Let $\mathcal{G}$ be a CGS, $C$ a coalition, and $\sigma_C$ a strategy for $C$. A play $\rho$ is compatible with the strategy $\sigma_C$ if, for all $k < |\rho| - 1$, there exists a move $(m_A)_{A \in \text{Agt}}$ s.t. $m_A = \sigma_A(\rho_{\leq k})$ for all $A \in C$, and $\text{Tab}(\rho_{= k}, (m_A)_{A \in \text{Agt}}) = \rho_{= k+1}$. We write $\text{Out}_C(s, \sigma_C)$ for the set of paths in $\mathcal{G}$ that are compatible with strategy $\sigma_C$ of $C$ and have initial state $s$, these paths are called outcomes of $\sigma_C$ from $s$. We simply write $\text{Out}_C(\sigma_C)$ when $s = s_0$. Note that when the coalition $C$ is composed of all the players the outcome is unique.

2.2 Preferences

In this work, we will give techniques to study games with a general notion of preferences, and deeper analyze the algorithmic aspects for Muller objectives and mean payoff preferences. In general, we will assume that preferences of each player are given by payoff functions that assign a real value to each run.

Definition 5. A concurrent game is a CGS equipped with payoff functions for each player. Formally, a concurrent game is a tuple $\langle \mathcal{G}, (\text{payoff}_A)_{A \in \text{Agt}} \rangle$, where $\mathcal{G}$ is a concurrent game structure and for each $A \in \text{Agt}$, $\text{payoff}_A : (\text{Stat} \cdot \text{Act}^{\text{Agt}})^\omega \rightarrow \mathbb{R}$. It is called the payoff of player $A$. When $\text{payoff}_A(\rho) < \text{payoff}_A(\rho')$, we say that $A$ prefers $\rho'$ over $\rho$. The tuple $u \in \mathbb{R}^{\text{Agt}}$ such that $u_A = \text{payoff}_A(\rho)$ for each player $A \in \text{Agt}$ is called the payoff vector of $\rho$.

In the case of purely qualitative preferences, $\text{payoff}_A$ can be given by a set $\Omega_A$, so that $\text{payoff}(\rho) = 1$ if $\rho' \in \Omega_A$ and $0$ otherwise. In that case $\Omega_A$ is called an objective. A play $\rho \in \Omega_A$ is said winning for $A$, and a play $\rho \notin \Omega_A$ is said losing for $A$. An example of purely qualitative preference is that of Muller objectives that we now define.

Definition 6 (Muller objectives). We denote the set of states occurring infinitely often in a play $\rho$ by $\text{Inf}(\rho) = \{ s \in \text{Stat} | \forall j \in \mathbb{N} \exists i > j, \rho_i = s \}$. A Muller objective is given by $F \subseteq \mathcal{P}(\text{Stat})$. It requires that the set of states seen infinitely often equals one of $F$. Formally, $\Omega_M(F) = \{ \rho \mid \text{Inf}(\rho) \in F \}$.

Remark 1. Representing the set $F$ explicitly can be costly, as its size can be exponential in the number of states in the game. Instead, we will assume that the objective are represented by Boolean circuits, which can be exponentially more succinct [10]. A $P$ is a Boolean circuits with $|\text{Stat}|$ inputs defines an objective $\Omega_P$: it is the set of run $\rho$ for which the circuit $P$ evaluates $\text{Inf}(\rho)$. 
We now present mean payoffs which are given by the long term average of the weights.

**Definition 7.** A mean payoff is given by a weight function \( w : \text{Stat} \rightarrow \mathbb{Z} \) which assigns to each state an integer weight. The payoff associated to a run is: 
\[
\text{MP}_w(\rho) = \lim_{n \to \infty} \frac{1}{n} \sum_{0 \leq k \leq n} w(\rho_k).
\]

**2.3 Equilibria notions**

In this section, we describe the different solution concepts we will study. Solution concepts are formal descriptions of “good” strategy profiles. The most famous of them is Nash equilibrium \([14]\), in which no player can improve the outcome for her preference relation, by only changing her strategy. This notion can be generalized to consider coalitions of players, it is then called a resilient strategy profile. Nash equilibria correspond to the special case of 1-resilient strategy profiles.

**Resilience** \([3]\) Given a coalition \( C \), a strategy profile \( \sigma_{\text{Agt}} \) is \( C \)-resilient if for all player \( A \) in \( C \), \( A \) cannot improve her payoff even if all players in \( C \) change their strategies, i.e. \( \sigma_{\text{Agt}} \) is said \( C \)-resilient when:
\[
\forall \sigma'_C \in \text{Strat}_G(C), \forall A \in C. \text{payoff}_A(\text{Out}_G(\sigma_A, \sigma'_C)) \leq \text{payoff}_A(\text{Out}_G(\sigma_{\text{Agt}}))
\]

Given an integer \( k \), we say that a strategy profile is \( k \)-resilient if it is \( C \)-resilient for every coalition \( C \) of size \( k \).

**Immunity** \([1]\) To ensure that players not deviating are not too much affected by deviation, we consider immune strategies. A strategy profile \( \sigma_{\text{Agt}} \) is \((C, r)\)-immune if all players not in \( C \), are not worse off by more than \( r \) if players in \( C \) deviates, i.e. when:
\[
\forall \sigma'_C \in \text{Strat}_G(C), \forall A \in \text{Agt}\setminus C. \text{payoff}_A(\text{Out}_G(\sigma_{\text{Agt}})) - r \leq \text{payoff}_A(\text{Out}_G(\sigma_A, \sigma'_C))
\]

Given an integer \( t \), a strategy profile is said \((t, r)\)-immune if it is \((C, r)\)-immune for every coalition \( C \) of size \( t \). Note that \( t \)-immunity as defined in \([1]\) corresponds to \((t, 0)\)-immunity.

**Robust Equilibrium** \([1]\) Combining resilience and immunity, gives the notion of robust equilibrium. A strategy profile is a \((k, t, r)\)-robust equilibrium if it is both \( k \)-resilient and \((t, r)\)-immune.

We are interested in the following decision problems and in computing the corresponding equilibria.

**Robustness Decision Problem:** Given a concurrent game \( G \), integers \( k \), \( t \), rational \( r \), and payoff vector \( v \in \mathbb{Q}^{\text{Agt}} \), does there exist a \((k, t, r)\)-robust equilibrium \( \sigma_{\text{Agt}} \) in \( G \) such that for all \( A \in \text{Agt} \), \( \text{payoff}_A(\text{Out}_G(\sigma_{\text{Agt}})) \geq v_A ? \)
2.4 A logic for equilibria

In order to generalize these concepts of equilibria, we propose a logic for describing equilibria notions. The syntax of this logic is given by the following grammar:

\[ \phi ::= \phi \land \phi \mid [C]. \psi \quad \psi ::= \psi \land \psi \mid \neg \psi \mid \text{payoff}_i \bowtie q \]

where \( C \subseteq \text{Agt} \) represents a coalition of players, \( i \in \text{Agt}, \bowtie \in \{<, \leq, =, \geq, >\} \) and \( q \in \mathbb{Q} \). We call such a formula \( \phi \) an equilibrium formula.

The intuitive meaning of the \([C]\) is that we quantify over all possible strategies of \( C \). The precise semantics of this logic is that the strategy profile \((\sigma_1, \ldots, \sigma_n)\) satisfies:

- \( \phi = \phi_1 \land \phi_2 \) if \( (\sigma_1, \ldots, \sigma_n) \) satisfies both \( \phi_1 \) and \( \phi_2 \);
- \( \phi = [C].\psi \) if for every strategy profile \((\sigma'_1, \ldots, \sigma'_n)\) with for all \( i \notin C, \sigma'_i = \sigma_i \), \((\sigma'_1, \ldots, \sigma'_n)\) satisfies \( \psi \);
- \( \psi = \text{payoff}_i \bowtie z \) if \( \text{payoff}_i(\text{Out}_G(\sigma_1, \ldots, \sigma_n)) \bowtie z \);
- \( \psi = \psi_1 \land \psi_2 \) if \( (\sigma_1, \ldots, \sigma_n) \) satisfies both \( \psi_1 \) and \( \psi_2 \);
- \( \psi = \neg \psi_1 \) if \( (\sigma_1, \ldots, \sigma_n) \) does not satisfy \( \psi_1 \).

As we will see this logic is expressive enough for the equilibria notions we defined and we will see efficient algorithms for checking its satisfiability.

**Example 1.** We give an example of a simple three-player turn-based game in Figure 1. In this game, there is no 2-resilient strategy profile. If \( A_1 \) wins (state \( s_3 \)) then \( A_2 \) and \( A_3 \) can form a coalition so that \( A_2 \) wins by both playing action \( b \) in state \( s_1 \). Similarly, if \( A_2 \) or \( A_3 \) wins then the two others can form a coalition so that \( A_1 \) wins. However, there is a (1, 1)-robust equilibrium. Consider the strategy for \( A_1 \) that always plays \( a \), and the one for \( A_2 \) and \( A_3 \) that plays \( a \) in \( s_1 \). The outcome of this strategy profile ends in \( s_3 \) with payoff \((1, 0, 0)\). Players \( A_2 \) cannot change her strategy alone in order to win, and \( A_3 \) does not either, so this is a Nash equilibrium (i.e. 1-resilient). If the strategy of \( A_1 \) changes, it is harmless for other players, since they were already losing. If the strategy of \( A_2 \) changes, then because of \( A_3 \) the run still ends up in \( s_3 \). The situation is similar when the strategy of \( A_3 \) changes. Therefore this strategy is also 1-immune.

3 Deviator Game

In order to obtain simple algorithms for the robustness problem, we use a correspondence with winning strategies. The concept of winning strategy has been well studied in computer science and we can make use of existing algorithms. We present the deviator game, which is a transformation of multiplayer game into a turn-based zero-sum game, such that there are strong links between robust equilibria in the first one and winning strategies in the second one. This is inspired by the suspect game construction [4].
The basic notion we use to solve the robustness problem is that of deviators. It identifies players that cause the current deviation from the expected outcome.

A deviator from \(m'_{\text{Agt}}\) to \(m_{\text{Agt}}'\) is a player \(D\) such that \(m_D \neq m_D'\). We write this set of deviators: \(\text{Dev}(m_{\text{Agt}}, m'_{\text{Agt}}) = \{ A \in \text{Agt} \mid m_A \neq m'_A \}\). We extend the definition to histories and strategies by taking the union of deviator sets, formally \(\text{Dev}(h, \sigma_{\text{Agt}}) = \bigcup_{0 \leq i < |h|} \text{Dev}(m^i_h, \sigma_{\text{Agt}}(h_{\xi_i}))\). It naturally extends to plays: if \(\rho\) is a play, then \(\text{Dev}(\rho, \sigma_{\text{Agt}}) = \bigcup_{i \in \mathbb{N}} \text{Dev}(m^i_\rho, \sigma_{\text{Agt}}(\rho_{\leq i}))\).

Intuitively, given an play \(\rho\) and a strategy profile \(\sigma_{\text{Agt}}\), deviators represent the players that need to change their strategies from \(\sigma_{\text{Agt}}\) in order to obtain the play \(\rho\). This intuition is formalized in the following lemma.

**Lemma 1.** Given a play \(\rho\), a coalition \(C\) contains \(\text{Dev}(\rho, \sigma_{\text{Agt}})\) if, and only if, there exists \(\sigma'_C\) such that \(\rho \in \text{Out}_C(\rho_0, \sigma_C, \sigma'_C)\).

**Proof.** Let \(\rho\) be a play, \(\sigma_{\text{Agt}}\) a strategy profile, and \(C\) a coalition which contains \(\text{Dev}(\rho, \sigma_{\text{Agt}})\). We define \(\sigma'_C\) to be such that for all \(i\), \(\sigma'_C(\rho_{\leq i}) = (m^i_\rho)_C\). We have that for all index \(i\), \(\text{Dev}(\rho_{i+1}, \sigma_{\text{Agt}}(\rho_{\leq i})) \subseteq C\). Therefore for all agent \(A \notin C\), \(\sigma_A(\rho_{\leq i}) = (m^i_\rho)_A\). Then \(\sigma(\text{Agt}\setminus C)(\rho_{\leq i}), \sigma_C(\rho_{\leq i}) = \rho_{i+1}\). Hence \(\rho\) is the outcome of the profile \((\sigma_C, \sigma'_C)\).

\(\Rightarrow\) Let \(\sigma_{\text{Agt}}\) be a strategy profile, \(\sigma'_C\) a strategy for coalition \(C\), and \(\rho \in \text{Out}_C(\rho_0, \sigma_C, \sigma'_C)\). We have for all index \(i\) that \(m^i_\rho = (\sigma_C(\rho_{\leq i}), \sigma'_C(\rho_{\leq i}))\). Therefore for all agent \(A \notin C\), \((m^i_\rho)_A = \sigma_A(\rho_{\leq i})\). Then \(\text{Dev}(m^i_\rho, \sigma_{\text{Agt}}(\rho_{\leq i})) \subseteq C\). Hence \(\text{Dev}(\rho, \sigma_{\text{Agt}}) \subseteq C\).
3.2 Deviator Arena

We now use the notion of deviators to draw a link between multiplayer games and a two-player game structure that we will use to solve the robustness problem. Given a concurrent game structure \( G \), we define the deviator arena \( D(G) \) between two players called Eve and Adam. Intuitively Eve needs to play according to an equilibrium, while Adam tries to find a deviation of a coalition which will profit one of its player or harm one of the others. The states are in \( \text{Stat}' = \text{Stat} \times 2^{\text{Agt}} \); the second component records the deviators of the current history. The game starts in \((s_0, \emptyset)\) and then proceeds as follows: from a state \((s, D)\), Eve chooses an action profile \(m_\text{Agt}\) and Adam chooses another one \(m'_\text{Agt}\); then the next state is \((\text{Tab}(s, m'_\text{Agt}), D \cup \text{Dev}(m_\text{Agt}, m'_\text{Agt}))\). In other words, Adam can chose any move he wants to, but this can be at the price of adding players to the \( D \) component.

The construction of the deviator arena is illustrated in Figure 2.

We define projections \( \pi_{\text{Stat}}, \pi_{\text{Dev}} \) and \( \pi_{\text{Act}} \) from \( \text{Stat}' \) to \( \text{Stat} \), from \( \text{Stat}' \) to \( 2^{\text{Agt}} \) and from \( 2^{\text{Agt}} \times 2^{\text{Agt}} \) to \( 2^{\text{Agt}} \) respectively. This is given by \( \pi_{\text{Stat}}(s, D) = s \), \( \pi_{\text{Dev}}(s, D) = D \) and \( \pi_{\text{Act}}(m_\text{Agt}, m'_\text{Agt}) = m'_\text{Agt} \). We extend these projections to plays in a natural way; letting \( \pi_{\text{Stat}}(\rho) = \pi_{\text{Stat}}(\rho_0) \cdot \pi_{\text{Act}}(m_{\rho_0}') \cdot \pi_{\text{Stat}}(\rho_1) \cdot \pi_{\text{Act}}(m_{\rho_1}') \cdot \ldots \) and \( \pi_{\text{Dev}}(\rho) = \pi_{\text{Dev}}(\rho_0) \cdot \pi_{\text{Dev}}(\rho_1) \cdot \ldots \). For any play \( \rho \), and any index \( i \), \( \pi_{\text{Dev}}(\rho_i) \subseteq \pi_{\text{Dev}}(\rho_{i+1}) \). Therefore \( \pi_{\text{Dev}}(\rho) \) seen as a sequence of sets of coalitions is increasing and bounded by \( \text{Agt} \), its limit \( \delta(\rho) = \cup_{i \in N} \pi_{\text{Dev}}(\rho_i) \) is well defined. To a strategy profile \( \sigma_{\text{Agt}} \) in \( G \), we can naturally associate a strategy \( \pi(\sigma_{\text{Agt}}) \) for Eve in \( D(G) \) such that for all history \( h \) by \( \pi(\sigma_{\text{Agt}})(h) = \sigma_{\text{Agt}}(\pi_{\text{Stat}}(h)) \).

The following proposition shows the link between strategies of the original game and associated strategies of the deviator game.

**Proposition 1.** Let \( G \) be a game and \( \sigma_{\text{Agt}} \) be a strategy profile and \( \sigma_\exists = \pi(\sigma_{\text{Agt}}) \) the associated strategy in the deviator game.

1. If \( \rho \in \text{Out}_{D(G)}(\sigma_\exists) \), then \( \text{Dev}(\pi_{\text{Stat}}(\rho), \sigma_{\text{Agt}}) = \delta(\rho) \).
2. If \( \rho \in \text{Out}_{G}(\sigma_{\text{Agt}}) \) and \( \rho' = (\rho_i, \text{Dev}(\rho_{\leq i}, \sigma_{\text{Agt}}))_{i \in N} \) then \( \rho' \in \text{Out}_{D(G)}(\sigma_\exists) \).

**Proof (Proof of (1)).** We prove that for all \( i \), \( \text{Dev}(\pi_{\text{Stat}}(\rho_{\leq i}, \sigma_{\text{Agt}})) = \pi_{\text{Dev}}(\rho_{\leq i}) \), which implies the property. The property holds for \( i = 0 \), since initially both sets are empty. Assume now that it holds for \( i \geq 0 \).

\[
\begin{align*}
\text{Dev}(\pi_{\text{Act}}(\rho_{\leq i+1}, \sigma_{\text{Agt}})) &= \text{Dev}(\pi_{\text{Act}}(\rho_{\leq i}, \sigma_{\text{Agt}})) \cup \text{Dev}(\sigma_{\text{Agt}}(\pi_{\text{Act}}(\rho_{\leq i}), \pi_{\text{Act}}(\rho_{= i+1}))) \\
&= \pi_{\text{Dev}}(\rho_{\leq i}) \cup \text{Dev}(\sigma_{\text{Agt}}(\pi_{\text{Act}}(\rho_{\leq i}), \pi_{\text{Act}}(\rho_{= i+1}))) \quad \text{(by induction hypothesis)} \\
&= \pi_{\text{Dev}}(\rho_{\leq i}) \cup \text{Dev}(\sigma_{\exists}(\rho_{\leq i}), \pi_{\text{Act}}(\rho_{= i+1})) \quad \text{(by def. of } \sigma_\exists) \\
&= \pi_{\text{Dev}}(\rho_{\leq i+1}) \quad \text{(by construction of } D(G))
\end{align*}
\]

Which concludes the induction.
Figure 2. Example of the deviator game construction for the game of Figure 1. States with rectangle shapes are controlled by Eve, and circles by Adam.

Proof (Proof of (2)). The property is shown by induction. It holds for the initial state. Assume it is true until index $i$, then

$$\text{Tab}'(\rho_i, \sigma_3(\rho_i^{\leq i}), \rho_{i+1})$$

$$= (\rho_{i+1}, \text{Dev}(\rho_i^{\leq i}, \sigma_{\text{Agt}}) \cup \text{Dev}(\sigma_3(\rho_i^{\leq i}), \rho_{i+1}))$$

$$= (\rho_{i+1}, \text{Dev}(\rho_i^{\leq i}, \sigma_{\text{Agt}}) \cup \text{Dev}(\sigma_{\text{Agt}}(\rho_i^{\leq i}), \rho_{i+1}))$$ (by construction of $\sigma_3$)

$$= (\rho_{i+1}, \text{Dev}(\rho^{\leq i+1}, \sigma_{\text{Agt}})) = \rho_{i+1}'$$

This shows that $\rho'$ is an outcome of $\sigma_3$.

3.3 Objectives of the deviator game

We now show how to transform equilibria notions into objectives of the deviator game. These objectives are defined so that winning strategies correspond to solution for these equilibria notions. Let $\phi$ be an equilibrium formula, we define
indirectly the associated objective \( \tau(\phi) \):

\[
\tau(\phi_1 \land \phi_2) = \tau(\phi_1) \cap \tau(\phi_2)
\]

\[
\tau([C]\psi) = \{ \rho \mid \delta(\rho) \subseteq C \implies \rho \in \tau(\psi) \}
\]

\[
\tau(\psi_1 \land \psi_2) = \tau(\psi_1) \cap \tau(\psi_2)
\]

\[
\tau(\neg \psi) = \text{Stat}^\tau \setminus \tau(\psi)
\]

\[
\tau(\text{payoff}_A \triangleright r_j) = \{ \rho \mid \text{payoff}_A(\pi_{\text{Stat}}(\rho)) \triangleright r_j \}
\]

**Example 2.** As an illustration of the expressiveness of this fragment considered, it is easy to specify secure equilibria. Following the definition of [8], the strategy profile \((\sigma_A, \sigma_A')\) is a secure equilibria in \(G\) when it is a Nash equilibrium and:

\[
\forall \sigma_A' \in \text{Stat}_G(\sigma_A). \text{payoff}_A(\text{Out}_G(\sigma_A, \sigma_A')) \geq r_1 \implies \text{payoff}_A(\text{Out}_G(\sigma_A, \sigma_A')) \geq r_2
\]

\[
\land \forall \sigma_A' \in \text{Stat}_G(\sigma_A). \text{payoff}_A(\text{Out}_G(\sigma_A, \sigma_A')) \geq r_2 \implies \text{payoff}_A(\text{Out}_G(\sigma_A, \sigma_A')) \geq r_1
\]

Where \(r_1 = \text{payoff}_A(\text{Out}_G(\sigma_A, \sigma_A))\) and \(r_2 = \text{payoff}_A(\text{Out}_G(\sigma_A, \sigma_A))\). There is a secure equilibria with payoff \(r_1\), \(r_2\) if the following equilibrium formula is satisfiable:

\[
[\emptyset] \text{payoff}_A = r_1 \land \text{payoff}_A = r_2
\]

\[
\land [(A_1)] \text{payoff}_A \geq r_1 \implies \text{payoff}_A \geq r_2
\]

\[
\land [(A_2)] \text{payoff}_A \geq r_2 \implies \text{payoff}_A \geq r_1
\]

We then have the following property:

**Proof.** Let \(\rho\) be an outcome of \(\sigma_3 = \pi(\sigma_{\text{Ag}})\). By Prop. 1, we have that \(\text{Dev}(\pi_{\text{Stat}}(\rho), \sigma_{\text{Ag}}) = \delta(\rho)\). The objective \(\tau(\phi)\) is of the form \(\bigcap_{i \in [1,n]} \{ \rho \mid \delta(\rho) \subseteq C_i \implies \tau(\psi_i) \}\) where \(\phi\) is the conjunction of the formulas \([C_i]\psi_i\) for \(i \in [1,n]\). Let \(i \in [1,n]\), by Lem. 1. \(\pi_{\text{Stat}}(\rho')\) is the outcome of \((\sigma_{\text{Ag}}(\rho'), \sigma_{\text{Ag}}')\) for some \(\sigma_{\text{Ag}}(\rho')\). If \(\delta(\rho') \subseteq C_i\), then since \(\sigma_{\text{Ag}}\) satisfies \(\phi\), \(\psi_i\) is satisfied by the pair \((\sigma_{\text{Ag}}(\rho'), \sigma_{\text{Stat}}(\rho'))\). Therefore it satisfies \(\psi_i\) where we replace all \(\text{payoff}_A \triangleright r_j\) by \(\text{payoff}_A(\pi_{\text{Stat}}(\rho')) \triangleright r_j\), which gives that \(\rho'\) satisfies \(\tau(\phi_i)\). This holds for all \(i \in [1,n]\) and all outcome of \(\sigma_3\), thus \(\sigma_3\) is a winning strategy for \(\tau(\psi)\).

\(\Longleftarrow\) Assume \(\sigma_3 = \pi(\sigma_{\text{Ag}})\) is a winning strategy in \(D(G)\) for \(\tau(\phi)\). The equilibrium formula \(\phi\) is the conjunction of formulas of the form \([C_i]\psi_i\) for \(i \in [1,n]\).

Let \(i \in [1,n]\), \(\sigma_{C_i}^j\) be strategy of \(C_i\) and \(\rho\) the outcome of \((\sigma_{C_i}^j, \sigma_{-C_i})\). By Lem. 1. \(\text{Dev}(\rho, \sigma_{\text{Ag}}) \subseteq C_i\). By Prop. 1, \(\rho' = (\rho_j, \text{Dev}(\rho_{\leq j}, \sigma_{\text{Ag}}))_{j \in N}\) is an outcome of \(\sigma_3\). We have that \(\delta(\rho') = \text{Dev}(\rho, \sigma_{\text{Ag}}) \subseteq C_i\). Since \(\sigma_3\) is winning, \(\rho\) satisfies \(\tau(\phi_i)\). Since \(\text{payoff}_A(\pi_{\text{Stat}}(\rho')) = \text{payoff}_A(\rho)\) for all player \(A_i\), \((\sigma_{C_i}^j, \sigma_{-C_i})\) satisfies \(\phi_i\). This is true for all \(i\), thus \(\sigma_{\text{Ag}}\) satisfies \(\phi\).

**Lemma 2.** Let \(G\) be a concurrent game and \(\sigma_{\text{Ag}}\) a strategy profile in \(G\). The strategy profile \(\sigma_{\text{Ag}}\) satisfies \(\phi\) if, and only if, strategy \(\pi(\sigma_{\text{Ag}})\) is winning in \(D(G)\) for objective \(\tau(\phi)\).
We now use the transformation for the equilibria notions we defined. For instance, concerning resilience, this gives a resilience objective for Eve where, informally, if there are more than \( k \) deviators then Eve has nothing to do; if the deviators are exactly \( k \) then she has to show that none of them gain anything; and if there are less than \( k \) then no player at all should gain anything. This is because if a new player joins the coalition, its size remains smaller or equal to \( k \).

**Theorem 1.** Let \( G \) be a concurrent game, \( \sigma_{Agt} \) a strategy profile in \( G \), \( \rho \) a path, \( k, t \) integers, and \( r \) a rational.

- The strategy profile \( \sigma_{Agt} \) is \( k \)-resilient if, and only if, strategy \( \pi(\sigma_{Agt}) \) is winning in \( D(G) \) for the resilience objective \( \mathcal{R}(k, \text{Out}(\sigma_{Agt})) \) defined by:

\[
\mathcal{R}(k, \rho) = \{ \rho' \mid |\delta(\rho')| > k \} \\
\cup \{ \rho' \mid |\delta(\rho')| = k \land \forall B \in \delta(\rho'). \text{payoff}_E(\pi_{\text{Stat}}(\rho')) \leq \text{payoff}_E(\rho) \} \\
\cup \{ \rho' \mid |\delta(\rho')| < k \land \forall B \in \text{Agt}. \text{payoff}_E(\pi_{\text{Stat}}(\rho')) \leq \text{payoff}_E(\rho) \}
\]

- The strategy profile \( \sigma_{Agt} \) is \((t, r)\)-immune if, and only if, strategy \( \pi(\sigma_{Agt}) \) is winning for the immunity objective \( I(t, r, \text{Out}(\sigma_{Agt})) \) defined by: Where the of Eve is:

\[
I(t, r, \rho) = \{ \rho' \mid |\delta(\rho')| \leq t \implies \forall B \in \text{Agt}\backslash \delta(\rho'). \text{payoff}_E(\rho) - r \leq \text{payoff}_E(\pi_{\text{Stat}}(\rho')) \}
\]

- The strategy profile \( \sigma_{Agt} \) is a \((k, t, r)\)-robust profile in \( G \) if, and only if, \( \pi(\sigma_{Agt}) \) is winning for the robustness objective:

\[
\mathcal{R}(k, t, r, \text{Out}(\sigma_{Agt})) = \mathcal{R}(k, \text{Out}(\sigma_{Agt})) \cap I(t, r, \text{Out}(\sigma_{Agt})).
\]

### 3.4 Proof of Thm. 1

**Lemma 3.** Let \( G \) be a concurrent game and \( \sigma_{Agt} \) a strategy profile in \( G \). The strategy profile \( \sigma_{Agt} \) is \( k \)-resilient if, and only if, strategy \( \pi(\sigma_{Agt}) \) is winning in \( D(G) \) for objective \( \mathcal{R}(k, \text{Out}(\sigma_{Agt})) \). Where \( \mathcal{R} \) is defined by:

\[
\mathcal{R}(k, \rho) = \{ \rho' \mid |\delta(\rho')| > k \} \cup \{ \rho' \mid |\delta(\rho')| = k \land \forall B \in \delta(\rho'). \text{payoff}_E(\pi_{\text{Stat}}(\rho')) \leq \text{payoff}_E(\rho) \} \\
\cup \{ \rho' \mid |\delta(\rho')| < k \land \forall B \in \text{Agt}. \text{payoff}_E(\pi_{\text{Stat}}(\rho')) \leq \text{payoff}_E(\rho) \}
\]

**Proof.** Let \( \rho \) be the outcome of \( \sigma_{Agt} \), and for each player \( A \), \( q_A = \text{payoff}_A(\rho) \), resilience of \( \sigma_{Agt} \) can by expressed by the formula:

\[
\phi = \bigwedge_{C \subseteq \text{Agt} : |C| \leq k} [C] \bigwedge_{A \in C} \text{payoff}_A \leq q_A.
\]

By Lem. 2, \( \sigma_{Agt} \) is then resilient if, and only if, \( \pi(\sigma_{Agt}) \) is winning for \( \tau(\phi) \). We now show that \( \tau(\phi) \) is equivalent to \( \mathcal{R}(k, \rho) \).

\[
\tau(\phi) \subseteq \mathcal{R}(k, \rho)
\]

Let \( \rho' \in \tau(\phi) \):

- if \( |\delta(\rho')| > k \), then \( \rho' \) is in \( \mathcal{R}(k, \text{Out}(\sigma_{Agt})) \) by definition;
Let $\tau$ now show that

\begin{align*}
\text{Lemma 4.} \quad \text{The strategy profile $\sigma_{\text{Agt}}$ is $(t, r)$-immune if, and only if, strategy } \pi(\sigma_{\text{Agt}}) \text{ is winning for objective } \mathcal{I}(t, r, \text{Out}(\sigma_{\text{Agt}})). \text{ Where, given } t, r \text{ and a path } \rho, \text{ the immunity objective of Eve is:}

\mathcal{I}(t, r, \rho) = \{ \rho' \mid |\delta(\rho')| \leq t \implies \forall B \in \text{Agt}\setminus\delta(\rho'). \text{ payoff}_B(\rho') - r \leq \text{payoff}_B(\pi(\sigma_{\text{Agt}})) \}

\text{Proof. Let } \rho \text{ be the outcome of } \sigma_{\text{Agt}}, \text{ and for each player } A, q_A = \text{payoff}_A(\rho), \text{ (t,r)-immunity of } \sigma_{\text{Agt}} \text{ can be expressed by the formula:}

\psi = \bigwedge_{C \subseteq \text{Agt}, |C| \leq t} \left[ \bigwedge_{A \not\in C} \text{payoff}_A \geq q_A - r. \right]

\text{By Lem. 2, } \sigma_{\text{Agt}} \text{ is then resilient if, and only if, } \pi(\sigma_{\text{Agt}}) \text{ is winning for } \tau(\phi). \text{ We now show that } \tau(\phi) \text{ is equivalent to } \mathcal{I}(t, r, \rho). \text{ Hence } \rho' \text{ satisfies the immunity objective } \mathcal{I}(t, r, \text{Out}(\sigma_{\text{Agt}})).}

\text{Lemma 5.} \quad \text{The strategy profile } \sigma_{\text{Agt}} \text{ is a $(k, t, r)$-robust profile in } \mathcal{G} \text{ if, and only if, the associated strategy of Eve is winning for the objective } \mathcal{R}(k, t, r, \text{Out}(\sigma_{\text{Agt}})) = \mathcal{R}e(k, \text{Out}(\sigma_{\text{Agt}})) \cap \mathcal{I}(t, r, \text{Out}(\sigma_{\text{Agt}})).

\text{Proof. This is a simple consequence of Lem. 3 and Lem. 4. Let } \sigma_{\text{Agt}} \text{ be a $(k, t)$-robust strategy profile. It is } k\text{-resilient, so } \pi(\sigma_{\text{Agt}}) \text{ is winning the resilience objective. It is also } t\text{-immune, so } \pi(\sigma_{\text{Agt}}) \text{ is winning the immunity objective. Therefore } \sigma_{\text{Agt}} \text{ wins the robustness objective.}
In the other direction, assume $\pi(\sigma_{\text{Agt}})$ wins the robustness objective. Then $\sigma_{\text{Agt}}$ wins both the $k$-resilience objective and the $t$-immunity objective. Using lemmas 3 and 4, $\sigma_{\text{Agt}}$ is $k$-resilient and $t$-immune. Therefore, $\sigma_{\text{Agt}}$ is $(k, t)$-robust.

4 Algorithm for Muller objectives

In this section, we use the previous construction to devise an algorithm for robust equilibria in games with Muller objectives. Note that for qualitative objective the value of $r$ is of little importance and we talk about $(k, t)$-robust equilibria for $(k, t, 0)$-robust equilibria. The algorithm we describe works in polynomial space. This is because once a payoff is fixed, staying above this payoff can be reduced to a Muller objective whose description in terms of circuits is succinct. The complexity of the algorithm is matched by a PSPACE lower bound for resilience in Büchi games, which is one of the simplest regular preferences.

Reduction to Muller objective To describe the robustness objective $R(k, t, \rho)$, we need to characterize paths whose payoff is greater of equal to $\rho$ and paths whose payoff is smaller or equal to $\rho$. We show that this can be done with succinct Muller conditions. Note that we assume the Muller conditions to be given by Boolean circuits in order to keep succinctness. The winner of a game with such conditions can be decided in PSPACE [10].

Proposition 2. If the payoff of each player is given by a Muller condition, then $R(k, t, \rho)$ is expressible by a Muller condition of polynomial size.

Proof. This is because in the qualitative case, $R(k, t, \rho)$ is a Boolean combination of the objectives of the players and the size of the deviator set. The fact that the number of deviators is not greater than $k$ or $t$ can be checked by a Boolean circuit of polynomial size. Since Muller condition described by Boolean circuit are easy to combine by Boolean operations by just adding one gate, the global size of the circuit is polynomial.

Computation of the strategies This reduction to Muller games allows to define a naive algorithm that would construct the deviator game $D$ and compute winning strategies of the Muller game defined by $D$ if there exists one. Once such a winning strategy $\sigma_3$ has been constructed, it is easy to construct a strategy profile that constitutes a robust equilibria in the original game by encoding in the memory of the strategy of each player the current state of the $D$ component. Such a procedure would be in doubly exponential time since a Muller games are known to be PSPACE-complete [10] and the game $D$ is of exponential size with respect to the number of players. However we will see in the following that we can decide in polynomial space whether such a strategy profile exists by exploiting the particular structure of the deviator game.
**Fixed coalition** Although the deviator game may be of exponential size, it presents a particular structure, that we will use to obtain a polynomial space algorithm. As the set of deviators only increases during any run, the game can be seen as the product of the original game with a directed acyclic graph (DAG). This DAG is of exponential size but polynomial degree and depth. We take advantage of this structure by remembering which states are winning only if they are in a component corresponding to a node of the DAG immediately below the current one. We now present the details of the construction. For a fixed set of deviator $D$, the possible successors of the component $\text{Stat} \times D$ are the states in:

$$\text{Succ}(D) = \{ (\text{Tab}_D((s,D),(m_{\text{Agt}},m'_{\text{Agt}})) \mid s \in \text{Stat}, m_{\text{Agt}}, m'_{\text{Agt}} \in \text{Mov}(s)) \} \setminus \text{Stat} \times D.$$  

Notice that $\text{Succ}(D)$ is of size bounded by $|\text{Stat}| \times |\text{Tab}|$, hence of polynomial size. A winning path $\rho$ from a state in $\text{Stat} \times D$ is either: 1) such that $\delta(\rho) = D$; 2) or it reaches a state in $\text{Succ}(D)$ and follow a winning path from there. Assume we have computed all the states in $\text{Succ}(D)$ which are winning. We can stop the game as soon as $\text{Succ}(D)$ is reached, and declare that Eve the winner if the state that is reached is a winning state of $\mathcal{G}$. This can be seen as a game $M(D,\rho)$, called the fixed-coalition game, where: the states are those of $(\text{Stat} \times D) \cup \text{Succ}(D)$; the game is played normally on the states of $\text{Stat} \times D$ and the states of $\text{Succ}(D)$ have only self loops. The winning condition is given by $\mathcal{R}(k,t,\rho)$ on $(\text{Stat} \times D)$, with the addition for $\text{Succ}(D)$ of all the $(s',D') \in \text{Succ}(D)$ such that Eve has a winning strategy from $(s',D')$.

**Muller objectives the fixed coalition game** When players have Muller objectives, the winning condition we defined can still be expressed as a Muller objective. We write $F_B(\rho) = \{ \inf(\rho') \mid \pi_{\text{Stat}}(\rho') \preceq_B \rho \}$ and $F'_B(\rho) = \{ \inf(\rho') \mid \rho \preceq_B \pi_{\text{Stat}}(\rho') \}$. Let $F_{\text{Succ}}$ be given by $F_{\text{Succ}} = \{ \{ s \mid s \in \text{Succ}(D) \} \}$. Let $F_{R_e}$ and $F_I$ be given by:

- if $|D| > k$ then $F_{R_e} = \mathcal{P}(\text{Stat} \times D)$;
- if $|D| = k$ then $F_{R_e} = \bigcap_{B \in D} F_B(\rho)$;
- if $|D| < k$ then $F_{R_e} = \bigcap_{B \in \text{Agt}} F_B(\rho)$;
- if $|D| > t$ then $F_I = \mathcal{P}(\text{Stat} \times D)$;
- if $|D| \leq t$ then $F_I = \bigcap_{B \in \text{Agt} \setminus D} F'_B(\rho)$;

The objective of Eve in $M(D,\rho)$ is given by the Muller condition: $F_{\text{Succ}} \cup (F_{R_e} \cap F_I)$. The next lemma shows that this transformation preserves winning states.

**Lemma 6.** Let $\mathcal{G}$ be a concurrent game with Muller objective and $(s,D)$ a state of $\mathcal{D}(\mathcal{G})$. Eve has a winning strategy from $(s,D)$ in $\mathcal{D}(\mathcal{G})$ for $\mathcal{R}(k,t,\rho)$ if, and only if, she has a winning strategy from $s$ in $M(D,\rho)$.

**Proof.** Let $\sigma_\exists$ be a winning strategy from $(s,D)$, we define $\sigma'_\exists$ in $M(D)$ by following the same strategy until we end up in a state of $\text{Succ}(D)$. We now prove that $\sigma'_\exists$ is winning. Let $\rho$ be a possible outcome of $\sigma'_\exists$. If it stays in the $\text{Stat} \times D$ component then $\sigma'_\exists$ follows $\sigma_\exists$ and therefore the path is winning for
Eve. Otherwise, ρ ends up in a state \((t, D')\) of Succ\((D)\) after some history \(h\). Since \(h\) is compatible with \(σ_3\) and \(σ_3\) is a winning strategy, \(σ_3 \circ h\) is a winning for Eve from last\((h) = (t, D')\). Therefore \((t, D')\) is a winning state of Succ\((D)\) and ρ is winning for Eve.

\[\leq\]

For the other direction, let \(σ'_3\) be a winning strategy from \(s\) in \(M(D)\). We define the strategy \(σ_3\) in the original game \(D\) by following \(σ'_3\) as long as we do not get out of the Stat \(× D\) component and then follow a winning strategy if there is one. Let ρ be a possible outcome of \(σ_3\). If it never reaches Succ\((D)\), then \(σ_3\) follows \(σ'_3\) and therefore the path is winning for Eve. Otherwise it reaches a state \((t, D')\) in Succ\((D)\). As the history \(h\) until this state is compatible with \(σ'_3\) which is a winning strategy, \((t, D')\) is a winning state. Therefore \(σ_3\) follows a winning strategy for Eve from \((t, D')\). As the objective is prefix independent this means that ρ is winning for Eve.

\[\square\]

Computing the winning states of the deviator game Using the fixed-coalition game, we define the following algorithm which determines if a particular state of the deviator game is a winning for Eve. Formally, we say that a state \((s, D)\) of \(D(G)\) is winning, if there is a strategy of Eve such that all outcomes of this strategy from \((s, D)\) are winning. The procedure win\((D, D)\), in Algorithm 1, tells if \((s, D)\) is winning for Eve in \(D(G)\).

| Algorithm 1: \(\text{win}(D, D)\): computes winning states of Stat \(× \{D\}\) in \(D\). |
|-------------------------------------------------------------|
| compute Succ\((D)\); \(F_{\text{Succ}} := \emptyset\); |
| forall \((s', D') \in \text{Succ}(D)\) |
| | if \(s' \in \text{win}(D, D')\) then \(F_{\text{Succ}} := F_{\text{Succ}} \cup \{(s', D')\}\) |
| \(W := \text{winning states of } M(D, ρ) \text{ for Muller condition } F_{\text{Succ}} \cup (F_{Rc} \cap F_{I})\); |
| return \(W\); |

Global algorithm In order to define a PSPACE algorithm we first need to restrict our search to paths of a particular form.

Lemma 7. Let \(G\) be a concurrent game with Muller objectives. If there is a \((k,t)\)-robust equilibrium, then there is one with outcome of the form \(π \cdot τ^ω\) with \(|π| ≤ |\text{Stat}|^2\) and \(|τ| ≤ |\text{Stat}|^2\).

The idea of the proof is similar to the one used for Nash equilibria [4].

Proof. Let \(σ_{\text{Ag}}\) be a \((k,t)\)-robust equilibrium, and ρ be its outcome from \(s\). We define a new strategy profile \(σ'_{\text{Ag}}\), whose outcome from \(s\) is ultimately periodic, and then show that \(σ'_{\text{Ag}}\) is \((k,t)\)-robust from \(s\).

To begin with, we inductively construct a history \(π = π_0π_1...π_n\) that is not too long and visits precisely those states that are visited by ρ.

The initial state is \(π_0 = ρ_0 = s\). Then we assume we have constructed \(π_{≤ k} = π_0...π_k\) which visits exactly the same states as \(ρ_{≤ k'}\) for some \(k'\). If all
the states of \( \rho \) have been visited in \( \pi_{\leq k} \) then the construction is over. Otherwise there is an index \( i \) such that \( \rho_i \) does not appear in \( \pi_{\leq k} \). We therefore define our next target as the smallest such \( i \): we let \( t(\pi_{\leq k}) = \min\{i \mid \forall j \leq k. \, \pi_j \neq \rho_i\} \).

We then look at the occurrence of the current state \( \pi_k \) that is the closest to the target in \( \rho \): we let \( c(\pi_{\leq k}) = \max\{i < t(\pi_{\leq k}) \mid \pi_k = \rho_i\} \). Then we emulate what happens at that position by choosing \( \pi_{i+1} = \rho_{c(\pi_{\leq i})+1} \). Then \( \pi_{i+1} \) is either the target, or a state that has already been seen before in \( \pi_{\leq k} \), in which case the resulting \( \pi_{\leq k+1} \) visits exactly the same states as \( \rho_{\leq c(\pi_{\leq i})+1} \).

At each step, either the number of remaining targets strictly decreases, or the number of remaining targets is constant but the distance to the next target strictly decreases. Therefore the construction terminates. Moreover, notice that between two targets we do not visit the same state twice, and we visit only states that have already been visited, plus the target. As the number of targets is bounded by \(|\text{Stat}|\), we get that the length of the path \( \pi \) constructed thus far is bounded by \( 1 + |\text{Stat}| \cdot (|\text{Stat}| - 1)/2 \).

Using similar ideas, we now inductively construct \( \tau = \tau_0 \tau_1 \ldots \tau_m \), which visits precisely those states which are seen infinitely often along \( \rho \), and which is not too long. Let \( l \) be the least index after which the states visited by \( \rho \) are visited infinitely often, i.e. \( l = \min\{i \in \mathbb{N} \mid \forall j \geq i. \, \rho_j \in \text{Inf}(\rho)\} \). The run \( \rho_{\geq l} \) is such that its set of visited states and its set of states visited infinitely often coincide. We therefore define \( \tau \) in the same way we have defined \( \pi \) above, but for play \( \rho_{\geq l} \). As a by-product, we also get \( c(\tau_{\leq k}) \), for \( k < m \).

We now need to glue \( \pi \) and \( \tau \) together, and to ensure that \( \tau \) can be glued to itself, so that \( \pi \cdot \tau^\omega \) is a real run. We therefore need to link the last state of \( \pi \) with the first state of \( \tau \) (and similarly the last state of \( \tau \) with its first state). This possibly requires appending some more states to \( \pi \) and \( \tau \): we fix the target of \( \pi \) and \( \tau \) to be \( \tau_0 \), and apply the same construction as previously. The total length of the resulting paths \( \pi \) and \( \tau \) is bounded by \( 1 + (|\text{Stat}| - 1) \cdot (|\text{Stat}| + 2)/2 \) which less than \( |\text{Stat}|^2 \).

We let \( \rho' = \pi \cdot \tau^\omega \), and abusively write \( c(\rho'_{\leq k}) \) for \( c(\pi_{\leq k}) \) if \( k \leq |\pi| \) and \( c(\tau_{\leq k'}) \) with \( k' = (k-1-|\pi|) \mod |\tau| \) otherwise. We now define our new strategy profile, having \( \rho' \) as outcome from \( s \). Given a history \( h \):

- if \( h \) followed the expected path, i.e. \( h = \rho'_{\leq k} \) for some \( k \), we mimic the strategy at \( c(h) \): \( \sigma'_{\text{Agt}}(h) = \sigma_{\text{Agt}}(\rho_{c(h)}) \). This way, \( \rho' \) is the outcome of \( \sigma'_{\text{Agt}} \) from \( s \).
- otherwise we take the longest prefix \( h_{\leq k} \) that is a prefix of \( \rho' \), and define \( \sigma'_{\text{Agt}}(h) = \sigma_{\text{Agt}}(\rho_{c(h_{\leq k})} \cdot h_{\geq k+1}) \).

We now show that \( \sigma'_{\text{Agt}} \) is a \((k,t)\)-robust equilibrium. Assume that a coalition \( C \) of size \( k \) (resp. size \( t \)) changes its strategy while playing according to \( \sigma'_{\text{Agt}} \): either the resulting outcome does not deviate from \( \pi \cdot \tau^\omega \), in which case the payoff of the players is not changed; or it deviates at some point, and from that point on, \( \sigma'_{\text{Agt}} \) follows the same strategies as in \( \sigma_{\text{Agt}} \). Assume that the resulting outcome is an improvement over \( \rho' \) for one player in \( C \) (resp. not as good as \( \rho' \) for one player in \( \text{Agt} \setminus C \)). The suffix of the play after the deviation is the suffix
of a play of $\sigma_{\text{Agt}}$ after a deviation by the same coalition. By construction, both plays have the same visited and infinitely-visited sets. Hence we have found an advantageous deviation from $\sigma_{\text{Agt}}$ for a player in $C$ (resp. disadvantageous for a player in $\text{Agt} \setminus C$), contradicting that $\sigma_{\text{Agt}}$ is $k$-resilient (resp. $t$-immune).

We now use Algorithm 1 and the result of Thm. 1 to devise an algorithm for the robustness problem.

**Theorem 2.** The robustness problem is PSPACE-complete for Muller games.

**Proof (Sketch).** The algorithm proceeds by trying all path $\rho = \pi \cdot \tau \omega$ where $|\tau|$ and $|\pi|$ bounded by $|\text{Stat}|^2$, computing which states are winning in the deviator game for objective $R(k, t, \rho)$, and then checking that $\rho$ stays in these winning states. Correctness of this algorithm holds thanks to Thm. 1 and Lem. 7. The space needed by the algorithm is polynomial in the size of the input. This is because the sizes of the lasso paths we try are polynomial (Lem. 7), and Algorithm 1 that we use to compute winning states, works in polynomial space. Prop. 2 ensures that we call it on a Muller game of polynomial size. Hardness of the problem already holds for turn-based Büchi games (proof in the Appendix).

| Algorithm 2: deciding the robustness problem in Muller games |
|---------------------------------------------------------------|
| forall path $\rho = \pi \cdot \tau \omega$ where $|\tau|$ and $|\pi|$ bounded by $|\text{Stat}|^2$ |
| compute which states of $\text{Stat} \times \{\emptyset\}$ are winning for $R(k, t, \rho)$; |
| if $\rho$ stays in the winning region then return true |

4.1 Proof of Thm. 2

We now give more details concerning the proof of Thm. 2.

**Space usage**

The space usage depends on the type of preferences, since the Muller game $\mathcal{M}(D)$ is possibly exponential. However in the case of Muller objectives, we can show that the algorithm works in polynomial space. Indeed, the size of the stack of recursive calls is bounded by the number of players, as the set of deviators can only be increasing. The only information we remember about this computation is whether each state of $\text{Succ}(D)$ is winning, this only requires polynomial space. Computing the winner of a Muller game [10] and all the other operations inside the functions are done in polynomial space. Hence the procedure we describe is executed in polynomial space.
**Correctness**

**Lemma 8.** The algorithm return true if, and only if, there is a \((k,t)\)-robust equilibrium.

**Proof.** Assume the algorithm return true, then let \(\rho\) be such that \(\rho\) stays in the winning region. We define a strategy \(\sigma_3\) of \(D(G)\) which follows \(\rho\) when possible and revert to a winning strategy otherwise. Formally:

- if \(\pi_{\text{Stat}}(h)\) is a prefix of \(\rho\) then \(\sigma_3(h) = m_{|h|+1}^\rho\);
- otherwise, let \(i\) be the first index such that \(\pi_{\text{Stat}}(h_i) \neq \rho_i\) or \(\pi_{\text{Act}}(m_{i-1}^h) \neq m_{i-1}^\rho\), then \(\sigma_3(h) = \sigma_3^{h_i}(h_{\geq i})\), where \(\sigma_3^{h_i}\) is a winning strategy for Eve. Such a strategy exists because \(\rho\) stays in the winning region.

There is an outcome of \(\sigma_3\) whose projection is \(\rho\) and any outcome whose projection is different from \(\rho\) is winning for Eve. By definition of the objective \(R(k,t,\rho)\), the outcome whose projection is \(\rho\) is also winning. By Theorem 1, the strategy \(\sigma_{\text{Agt}}\) whose projection is \(\sigma_3\) is a \((k,t)\)-robust equilibrium.

**⇐** In the other direction, assume there is a \((k,t)\)-robust equilibrium. By Lem. 7, there is one \(\sigma_{\text{Agt}}\) whose outcome \(\rho\) is of the form \(\pi \cdot \tau^\omega\) with \(|\pi|\) and \(|\tau|\) bounded by \(|\text{Stat}|^2\). This ensures that \(\rho\) will be considered at some point in the forall loop. By Theorem 1, the strategy of Eve associated to \(\sigma_{\text{Agt}}\) is a winning strategy, therefore it has to be winning from all states of \(\rho\), which shows that \(\rho\) stays in the winning region. Hence the algorithm returns true.

**Lemma 9 (space usage).** The space needed by the algorithm is polynomial in the size of the input.

**Proof.** This is because the size of the lasso paths we have to try is polynomial, so it is possible to enumerate all of them using polynomial space. Then Algorithm 1 that we use to compute winning states, works in PSPACE.

**Hardness**

We show that the problem is PSPACE-hard, even in the restricted case of resilience for Büchi objectives.

**Theorem 3.** The resilience problem is PSPACE-hard, even if all objectives are Büchi objectives.

**Proof.** We encode QSAT formulae with \(n\) variable into a game with \(2 \cdot n + 2\) players, such that the formula is valid if, and only if, there is \(n\)-resilient equilibria.

We can assume that we are given a formula of the form \(\phi = \forall x_1. \exists x_2. \forall x_3. \ldots \exists x_n. C_1 \wedge \ldots \wedge C_k\), where each \(C_k\) is of the form \(\ell_{1,k} \vee \ell_{2,k} \wedge \ell_{3,k}\) and each \(\ell_{j,k}\) is a literal (i.e. \(x_m\) or \(\neg x_m\) for some \(m\)). We define the game \(G_\phi\) as illustrated by an example in Figure 3. It has a player \(A_m\) for each positive literal \(x_m\), and a player \(B_m\) for each negative literal \(\neg x_m\). We add two extra players Eve and
Adam. Eve is making choices for the existential quantification and Adam for the universal ones. When they chose a literal, the corresponding player can either go to a sink state ⊥ or continue the game to the next quantification. Once a literal has been chosen for all the variables, Eve needs to chose a literal for each clause.

The objective for Eve and the litteral players is to reach ⊥. The objective for Adam is to reach T. We ask whether there is a \((n + 1)\)-resilient equilibrium.

If the outcome is going to the state winning for Adam, it is possible for a \(A_i\) to change its strategy and go to ⊥, thus improving its payoff. Therefore a \((n + 1)\)-resilient equilibrium is necessarily losing for Adam and winning for all the others.

To a history \(h = \text{Adam}_1 \cdot X_1 \cdot \text{Eve}_2 \cdot X_2 \cdot \text{Adam}_3 \cdots \cdot \text{Eve}_m \cdot X_m\) with \(X_i \in \{A_i, B_i\}\), we associate a valuation \(v_h\), such that \(v_h(x_i) = \text{true}\) if \(X_i = B_i\) and \(v_h(x_i) = \text{false}\) if \(X_i = A_i\).

\[
\text{Validity} \implies \text{equilibrium.} \quad \text{Assume that } \phi \text{ is valid, we will show that there is a } (n + 1)\text{-resilient equilibrium. We define a strategy of Eve such that if } v_h \text{ makes } \exists x_m, \forall x_{m+1} \cdots \exists x_n, C_1 \land \cdots \land C_k \text{ valid, then } \sigma_3(h) = X_m \text{ such that } v_{h \cdot X_m} \text{ makes } \forall x_{m+1} \cdots \exists x_n, C_1 \land \cdots \land C_k \text{ valid. As } \phi \text{ is valid, we know that for all outcome } h \text{ of } \sigma_3 \text{ of the form } \text{Adam}_1 \cdot X_1 \cdot \text{Eve}_k \cdot X_k, v_h \text{ makes } C_1 \land \cdots \land C_k \text{ valid. Then from } X_m, \text{Eve can choose for each close a state } Y \text{ that is different from all } X_1 \ldots X_m. \text{ We also fix the strategy of all players } A_i, B_i, \text{ and Adam to go to the state } \perp. \text{ This defines a strategy profile that we will write } \sigma_{\text{Agt}.}
\]

Consider a strategy profile \(\sigma'_{\text{Agt}}\) where at most \((n + 1)\) strategies are different from the ones in \(\sigma_{\text{Agt}}\). Assume \(\sigma'_{\text{Agt}}\) reaches Eve\(_{m}\). We know that in \(\sigma'_{\text{Agt}}\) at least \(n + 1\) strategies are different from the ones \(\sigma_{\text{Agt}}\) and \(\{A \in \text{Agt} \mid \sigma'_{A} \neq \sigma_{A}\} = \{\text{Adam}, X_1, \ldots, X_m\}\). Then, by the choice of the strategy for Eve, the states that are seen in the following are controlled by players that are different from \(X_1, \ldots, X_m\). Thus the run ends in \(\perp\).

\[
\text{Equilibrium} \implies \text{validity.} \quad \text{Assume that } \sigma_3 \text{ is part of a } (n + 1)\text{-resilient equilibrium, we will show that } \phi \text{ is valid. Given a partial valuation } v_m: \{x_1, \ldots, x_m\} \mapsto \{\text{true}, \text{false}\}, \text{ we define the function } f(v_m) \text{ such that:}
\]

\[
f(v_m) \Leftrightarrow \sigma_3(\text{Adam}_1 \cdot X_1 \cdots \text{Adam}_m \cdot X_m) = B_{m+1}.
\]

We will show that every valuation \(v\), such that \(v(x_{2k}) = f(v_{2k-1})\), makes the formula \(C_1 \land \cdots \land C_k\) valid, which shows that the formula \(\phi\) is valid.

For all such valuation \(v\), we can define strategies of Adam and players \(X_i\) such that \(X_i = A_i\) if \(v(x_i) = \text{false}\) and \(X_i = B_i\) otherwise, such that keeping all other strategies similar to \(\sigma_{\text{Agt}}\), the state Eve\(_m\) is reached. Then, if we see a state belonging to one of the \(X_i\), we can make the strategy go to the \(T\) state. Since the profile is \((n + 1)\)-resilient, this is impossible. Which shows that Eve choses for each clause a litteral such that \(v(\ell) = \text{true}\). Therefore \(v\) makes the formula \(C_1 \land \cdots \land C_k\) valid.
Figure 3. Encoding of a formula $\phi = \forall x_1. \exists x_2. \exists x_3. \exists x_4. (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_2 \lor x_3 \lor x_4)$. The dashed edges represent the strategies in the equilibrium of the players other than Eve.

5 Mean-payoff preferences

We now turn our attention to mean-payoff games. We first show that the deviator game reduces the robustness problem to a winning strategy problem in multidimensional mean-payoff games. Similarly to Muller objectives, the algorithm looks for a path which stays in the winning region corresponding to its payoff. This is done by requests to the polyhedron value problem of [5].

Reduction to multidimensional mean-payoff objectives Our goal is to describe in a known formalism the objectives of Eve in the deviator game. For that we define multidimensional mean-payoff objectives [21,5] and then show how to encode the objective of Eve as a multidimensional mean-payoff objective.

Definition 8. Given a game $G$, two weight functions $w, \overline{w}: \text{Stat} \to \mathbb{R}^d$, we say that Eve can ensure threshold $(u, u') \in \mathbb{R}^d \times \mathbb{R}^d$ if she has a strategy $\sigma_3$ such that all outcome $\rho$ of $\sigma_3$ is such that for all $i \in [1, d]$, $\overline{\text{MP}}_{w_i}(\rho) \geq u_i$ and for all $j \in [1, d]$, $\overline{\text{MP}}_{\overline{w}_j}(\rho) \geq u'_j$, where $\overline{\text{MP}}_{w_i}(\rho) = \lim \sup_{n \to \infty} \frac{1}{n} \sum_{0 \leq k < n} w_i(\rho_k)$. That is, for all dimension $i$, the limit inferior of the average of $w_i$ is greater than $u_i$ and the limit superior of $\overline{w}_i$ is greater than $u'_i$.

For us the number of dimension $d$ will be equal to $|\text{Agt}|$, we then number players so that $\text{Agt} = \{A_1, \ldots, A_d\}$. Let $W = \max\{|w_{A_i}(s)| \mid A_i \in \text{Agt}, s \in \text{Stat}\}$ be the maximum constant occurring in the game, notice that for all player $A_i$ and play $\rho$, $-W - 1 < w_{A_i}(\rho) \leq W$. We fix parameters $k$, $t$ and define two weight functions $v, \overline{v}: \text{Stat} \to \mathbb{Z}^n$ given by:

1. if $|D| \leq t$ and $A_i \not\in D$, then $v_i(s) = w_{A_i}(s)$
2. if $|D| > t$ or $A_i \in D$, then $v_i(s) = W$.
3. if $|D| < k$, then for all $A_i \in \text{Agt}$, $\overline{v}_i(s) = -w_{A_i}(s)$;
4. if $|D| = k$ and $A_i \in D$, then $\overline{v}_i(s) = -w_{A_i}(s)$
5. if $|D| > k$ or $A_i \not\in D$, then $\overline{v}_i(s) = W$. 

The following lemma shows that with these weights we obtain a multidimensional mean-payoff objective that is equivalent to the objective of Eve.

**Proof.** We now show the immunity part. 

- If $|D| \leq t \land A \not\in D$, then $v_i(s) = w_{A_i}(s)$ and $\text{payoff}_{A_i}(\rho) - r \leq \text{payoff}_{A_i}(\pi) \Leftrightarrow u_i - r \leq \liminf_{n \to \infty} \frac{1}{n} \sum_{0 \leq j \leq n} w_{A_i}(\pi_j) \Leftrightarrow u_i - r \leq MP(v_i(\pi))$.

- Otherwise there is some index $j$ such that either $|\text{Dev}(\pi_j)| > t$ or $A \in \text{Dev}(\pi_j)$. Then, by monotonicity of $\text{Dev}$ along $\pi$, for all $j' \geq j$, $v_i(\pi_{j'}) = W \geq u_i$. Hence $MP(v_i(\pi)) \geq u_i$.

This shows that $\pi$ verifies the two conditions: $MP(v(\pi)) \geq u - r$ and $\text{MP}_\pi(\pi) \geq -u$.

$\Leftarrow$ Assume $MP(v(\pi)) \geq u - r$ and let $A_i$ be a player. If $|\delta(\pi)| \leq t$ and $A_i \not\in \delta(\pi)$, then for all $j$, $|\text{Dev}(\pi_j)| \leq t$ and $A \not\in \text{Dev}(\pi_j)$. Therefore for all $j$ we have that $v_i(\pi_j) = w_{A_i}(\pi_j)$. Then, as showed for the implication $MP(v(\pi)) \geq u_i - r \Leftrightarrow \text{payoff}_{A_i}(\rho) - r \leq \text{payoff}_{A_i}(\pi)$. This shows that $\pi$ is winning in $D(\mathcal{G})$.

**Lemma 10.** Let $\mathcal{G}$ be a concurrent game, $\rho$ a path of $\mathcal{G}$ and $\pi$ a path of $D(\mathcal{G}, \rho)$. Let $u = (MP_{A_i}(\rho))_{A_i \in Acc}$. Play $\pi$ is winning for Eve in $D(\mathcal{G}, \rho)$ if, and only if, $MP(v(\pi)) \geq u - r$ and $\text{MP}_\pi(\pi) \geq -u$.

**Proof.** We prove here the part of the proof that concerns resilience and the immunity part is similar and can be found in the appendix. For the direction of the implication, assume $\pi \in \text{Re}(k, \rho)$ and let $A_i$ be a player.

- If $|\delta(\pi)| < k$ or $|\delta(\pi)| = k \land A_i \in \delta(\pi)$, then there is a $j$ such that for all $j' \geq j$, either $|\text{Dev}(\pi_{j'})| < k$ or $A_i \in \text{Dev}(\pi_{j'})$. Therefore for all $j' \geq j$ we have that $v_i(\pi_{j'}) = -w_{A_i}(\pi_{j'})$ (item 3 and 4 of the definition). Then as $\pi$ is winning in $D(\mathcal{G})$, $\text{payoff}_{A_i}(\pi) \leq \text{payoff}_{A_i}(\rho)$. $\text{payoff}_{A_i}(\pi) \leq \text{payoff}_{A_i}(\rho) \Leftrightarrow \liminf_{n \to \infty} \frac{w_{A_i}(\pi_{k+n})}{n} \leq u_i \Leftrightarrow \limsup_{n \to \infty} - \frac{w_{A_i}(\pi_{k+n})}{n} \geq -u_i \Leftrightarrow MP(v(\pi)) \geq u_i$.

- Otherwise there is some index $j$ such that either $|\text{Dev}(\pi_j)| > k$ or $|\text{Dev}(\pi_j)| = k \land A \not\in \text{Dev}(\pi_j)$. Then, by monotonicity of $\text{Dev}$ along $\pi$, for all $j' \geq j$, $v_i(\pi_{j'}) = W$ (item 5). Since $u_i \leq W$, $\text{MP}_\pi(\pi) \geq u_i$.

Now in the other direction. Assume $\text{MP}_\pi(\pi) \geq -u$ and let $A_i$ be a player. If $|\delta(\pi)| < k$ or $|\delta(\pi)| = k \land A_i \in \delta(\pi)$, then for all $j$, either $|\text{Dev}(\pi_{j'})| < k$ or $A \in \text{Dev}(\pi_j)$. Therefore for all $j$ we have that $v_i(\pi_{j'}) = -w_{A_i}(\pi_{j'})$ and as previously mentioned $\text{MP}_\pi(\pi) \geq u \Leftrightarrow \text{payoff}_{A_i}(\pi) \leq \text{payoff}_{A_i}(\rho)$.

Putting together this lemma and the correspondence between the deviator game and robust equilibria of Thm. 1 we obtain the following proposition.

**Proof.** From a robust equilibrium, using Thm. 1 and Lem. 10 we deduce a strategy of Eve in $D(\mathcal{G})$ which ensures $(u + r, -u)$. We let $\rho$ be its outcome when no player deviates, we then have that Eve can ensure $(u + r, -u)$ from all states occurring in $\rho$.

In the other direction, if such a path exists, we can define a strategy of Eve that follows $\rho$ and is winning the game $D(\mathcal{G})$. Using Thm. 1, this strategy defines an equilibrium in $\mathcal{G}$ with the payoff $u$. 

Lemma 11. Let $G$ be a concurrent game with mean-payoff objectives and $u$ a payoff vector. There is a $(k, t, r)$-robust equilibrium in $G$ whose outcome has payoff $u$ if, and only if, there is a path $\rho$ in $G$ of mean-payoff $u$ and Eve can ensure $(u+r, -u)$ from all states $(\rho_i, \emptyset)$ for the multidimensional weight functions $v, \overline{v}$.

**Fixed coalition game** Our goal here is to describe a PSPACE algorithm to obtain a robust equilibrium with a given payoff using the fixed-coalition game (see section 4). We will see how to obtain from this a solution for the robust equilibrium problem when the payoff is not fixed. We first describes objective of the fixed-coalition game $M(D, \rho)$ as a multidimensional mean-payoff game as we did for the deviator game. In the fixed coalition game we keep the weights previously defined for states of $\text{Stat} \times D$, and fix it for the states that are not in the same $D$ component, by giving a high payoff on states that are winning and a low one on the losing ones:

- if $s \in \text{Succ}(D)$ and $s$ is winning, then $v_i(s) = W$ and $\overline{v}_i(s) = W$;
- if $s \in \text{Succ}(D)$ and $s$ is losing, then $v_i(s) = -W - 1$ and $\overline{v}_i(s) = -W - 1$.

With this definition and Lem.10 we easily obtain the following:

**Proof.** This is a consequence of Lem.10 if $\pi$ does not get out of the $D$ component and otherwise it is clear by the definition of the weight functions $v$ and $\overline{v}$.

Lemma 12. Let $G$ be a concurrent game, $\rho$ a path of $G$ and $\pi$ a path of $M(D, \rho)$. Let $u = (\text{MP}_{A_i}(\rho))_{A_i \in \text{Agt}}$. Play $\pi$ is winning for Eve in $M(D, \rho)$ if, and only if, $\text{MP}_v(\pi) \geq u - r$ and $\overline{\text{MP}}_v(\pi) \geq -u$.

Using this correspondence, there is an algorithm working in the same way than for Muller objectives (see Sect. 4).

**Proof.** As for Muller objectives, we can compute for each state of $M(D, \rho)$ whether it is winning by recursive calls on the successors $\text{Succ}(D)$ (note that to do that we only need to now the payoff vector of $\rho$). Then we compute whether we can ensure the payoff $(u - r, -u)$ in this game from each states occurring in the projection of $\rho$ on the deviator game (note that we can deduce which states occur in the projection from the states and moves that are occurring in $\rho$). This last step can be done in coNP [21] and as the size of the stack of recursive calls is bounded by $|\text{Agt}|$, the algorithm uses polynomial space.

**Proposition 3.** There is a polynomial space algorithm, that given a concurrent game $G$ and the payoff vector and occurring states and moves of some path $\rho$, tells if $\rho$ is a robust equilibrium.

Algorithm for Robust Equilibria Thanks to the characterization of Lem. 11, to solve the robust equilibrium problem is equivalent to find a path which stays in the winning region corresponding to its payoff. We already have a polynomial space algorithm to check if a candidate path is a solution in Prop. 3. The problem is to know how many paths are to be tried. In the next lemma, using techniques
Lemma 13. There is a polynomial function $P$ such that given a game $G$, if there is a $(k, t, r)$-robust equilibrium in $G$ then there is one whose outcome has payoff vector $u$ where for all $i \in \{1, d\}$, $||u_i|| \leq P(||W||, ||\text{Stat}||, ||\text{Agt}||)$. Moreover checking that there exists a path of $G$ which has payoff vector $u$ can be done in $\Sigma_2$-P.

Theorem 4. Robustness is $\text{PSPACE}$-complete for mean-payoff games.

Proof. The algorithm proceeds by trying all payoff vectors $u$ of size bounded by $P(||W||, ||\text{Stat}||, ||\text{Agt}||)$; checking that there is a path which realizes this payoff (we saw in Lem. 13 that this can be done $\Sigma_2$-P); use algorithm of Prop. 3 to check
that Eve can ensure \((u + r, -u)\) in the deviator game. The proof of PSPACE-hardness is the same than for Büchi objectives (see proof of Thm. 2).

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