Sine-Squared pulse approximation using generalized bessel polynomials

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Abstract. This paper presents the approximation of sine-squared pulse based on the generalized Bessel polynomials. For designing a circuit to synthesize a sine-squared pulse test signal. The generalized Bessel polynomials have more parameter than classical Bessel polynomials that have alpha and beta parameters for adjusting the dominator of the transfer function to approximate the sine-squared pulse that closes to the ideal pulse. The simulation results show that the generalized Bessel polynomial can adjust the approximation response close to the ideal response. The orders of the transfer function are decreased that confirm a better performance than the previous works.

1. Introduction
Sine-squared pulse is impulse response signals that can be applied to many fields of communication system[1]. For example, using to detect deficiency of linear chrominance distortion of colour television transmission system [2]. It can be revealed by the modulated 20T sine-squared pulse test signal. Moreover, an approximation of sine-squared pulse can be used to design matched filters for matching a data between transmitter and receiver[3]. However, the sine-squared pulse cannot synthesis into an electrical circuit. Because it has a format of trigonometry mathematical equation.

In the fields of an analog filter, the filter can be approximated by using mathematical polynomials. It used to synthesize a network function and electrical circuit from a polynomial. Also, the sine-squared pulse signal can be approximated by using some polynomials that have similar some mathematical characteristic. The Bessel filter [4-5] is the all-pole filter having a maximally flat delay and linear phase. The Bessel polynomials have similar features based on the theory of the Bessel Thomson filter. In the previous works, it can be written in form of generalized Bessel polynomials (GBP). The generalized Bessel polynomials have a parameter alpha and beta for adjusting the responses and have some characteristic that similar to the sine-squared pulse signal. For the sine-squared pulse approximation, we introduced the generalized Bessel polynomials as the dominator of the transfer function. The transmission zero pairs are used as a nominator of the transfer function. Unfortunately, the transfer function has high-order degree n of polynomials. It has the complexity to synthesize electrical circuits or networks. Moreover, sine-squared pulse power spectrum in previous works are still far from ideal sine-squared pulse [6-7].

This paper presents a method to solve a problem of power spectrum magnitude and attenuation response of sine-squared pulse by using the generalized Bessel polynomials as a dominator of the
transfer function. It has a parameter to adjust the power spectrum close to the ideal sine-squared pulse. In addition, a polynomials order in a transfer function will be decreased.

2. Generalized Bessel polynomials

The generalized-Bessel polynomials \( B_n(s, \alpha, \beta) \) was defined by [8] and the explicit expression of the GBP degree \( n \) defined by

\[
B_n(s, \alpha, \beta) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{(n+k+\alpha-2)^k}{\beta^k} \right) e^{-s}
\]  

(1)

where \( \binom{n}{k} \) is a binomial coefficient and \( (z)^k = z(z-1)(z-2)...(z-k+1) \), \( z = n+k+\alpha-2 \) is the backward factorial function of order \( k \), \( \alpha \) and \( \beta \) are real parameters and \( \beta \neq 0 \). In the previous literature, L. Storch [9] and J. Martinez [10] presented a procedure for Bessel approximation of \( e^{-s} \) that the transfer function as

\[
h(s) = e^{-\alpha s}
\]  

(2)

From Eq. (2), fractional this equation, can be expressed as the exponential form and convert to hyperbolic function as follows

\[
h(s) = \frac{1}{\sinh \left( \frac{\alpha s}{2} \right)}
\]  

(3)

From Eq. (3), see that the dominator contains the cotangent-hyperbolic function. This characteristic has similar to the characteristic in hyperbolic function of the sine-squared pulse that will describe in the next section.

3. Sine-Squared pulse testing signal

Sine-squared pulse is used for signal testing in baseband communication. Besides, it is a mathematical equation that cannot be used to create matched filter circuits. By equation, sine-squared pulses represent ideal equations. Therefore, it requires specific features of the sine-squared pulse that similar with Bessel polynomials. The ideal of the sine-squared pulse is given as

\[
h(t) = \begin{cases} 
\sin^2 \left( \frac{\pi t}{2} \right), & |t| \leq \tau \\
0, & |t| \geq \tau
\end{cases}
\]  

(4)

From Eq. (4), using the Laplace’s Transform and then given as

\[
h(s) = \frac{4}{s^2 + 4} \left( \frac{1}{1 + \text{coth} \left( \frac{s \tau}{2} \right)} \right)
\]  

(5)

From Eq. (5), see that the result of Laplace’s transform of the sin-squared pulse has a hyperbolic mathematical function. It cannot synthesize to a network function or electrical. From section 2, Eq. (3), it has dominator in cotangent-hyperbolic mathematical equations similar characteristic as the ideal sine-squared pulse in Eq. (5). From this similar characteristic, the sine-squared pulse can be approximated and synthesized into network function or electrical circuits. Thus, the GBP has the appropriate characteristic to synthesize the sine-squared pulse signal. From Eq. (4) using Fourier’s Transform. Hence, the frequency spectrum of the sine-squared pulse as
Eq. (6) is the equation of the sine-squared pulse frequency spectrum. The magnitude response of the power spectrum of the sine-squared pulse shown in figure 1 and figure 2 shows the attenuation response.

\[ h(f) = \tau \frac{\sin(2\pi f \tau)}{2\pi f \tau} \left( \frac{1}{1-(2f \tau)^2} \right) \]  

(6)

4. Sine-Squared pulse approximation method

For an approximation of sine-squared pulse, the transfer function can be written as polynomials to synthesize the electrical circuit for sine-squared pulse test signal generating. The transfer function has numerator with transmission zero pairs \( \pm jk\pi, k = 2, 3, 4, ..., m \). The overall system has minimum stopband attenuation. Therefore, the transfer function \( H(s) \) of the filter can be written as Eq. (7)

\[ H(s) = \prod_{k=1}^{m/2} (s^2 + \omega_k^2) \left( \prod_{k=1}^{n} (s - p_k) \right) \]  

(7)

where \( m \) is the degree of the numerator of polynomial, \( n \) is the degree of the denominator polynomial, \( \omega_k \) is the transmission zero pairs \( \pm jk\pi \) and \( p_k \) is the poles in the left-half of s-plane.

In this paper, the approximation of a sine-squared pulse is considered with the main lobe and first side lobe. The numerator of the transfer function is used two transmissions zero pairs at \( \pm j2\pi \) and \( \pm j3\pi \). In addition, dominator of the transfer function is used in the GBP due to the Bessel polynomials have a stability with all poles in left-half of s-plane and similar characteristic with sine-squared pulse equation. Therefore, this paper proposes a method to reduce the attenuation by using the GBP as the dominator of the transfer function and varies the parameters alpha and beta in the function.

5. Simulation results

In the simulation results, we define the GBP dominator of transfer function firstly and set two variables of GBP: \( \alpha = 2 \) and \( \beta = 2 \) that call as classical Bessel polynomials. The results from previous works have high-order of polynomials. It will cause to complexity synthesize to an electrical circuit. Next, for the approximation of sine-squared pulse, we set the order of GBP are order 3 and set parameters \( \alpha \) and \( \beta \) for adjusting the GBP dominator of the transfer function. First, we define \( \beta = 1 \) and adjust parameter \( \alpha = 0.001, 0.1, 1, 10 \) and 100. The magnitude response and attenuation response are compared, as shown in figure 2.
Figure 2. The comparison of magnitude and attenuation response of sine-squared pulse power spectrum with $\beta = 1$ and vary $\alpha = 0.001, 0.1, 1, 10$ and 100.

Figure 3. The comparison of magnitude and attenuation response of sine-squared pulse power spectrum with $\alpha = 0.1$ and vary $\beta = 0.92, 1, 1.2, 1.3$ and 1.5.

Figure 4. The comparison of the magnitude response of sine-squared pulse power spectrum with classical Bessel polynomials orders 3 and GBP method.
To sum up, the simulation results show the GBP order 3 with $\alpha = 0.1$ and $\beta = 0.95$ that have the power spectrum of sine-squared pulse close to the ideal sine-squared pulse response that summarizes as figure 4.

6. Conclusion
In this paper, sine-squared pulse approximation using the transfer function with generalized Bessel polynomials has been proposed. The generalized Bessel polynomials have a parameter with $\alpha$ and $\beta$ for adjusting the power spectrum close to ideal sine-squared pulse. The results of the simulations show the generalized Bessel polynomials can control the peak of approximation response by adjusting the $\alpha$ and $\beta$ parameter. It can decrease the order of the Bessel polynomial to order 3 with generalized Bessel polynomials with $\alpha = 0.1$ and $\beta = 0.95$. Future works, we will consider in the second, third, and next side lobe with the algorithm to define the best $\alpha$ and $\beta$ parameter and synthesize to the electrical circuit for testing and confirm the performance of approximation method.

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