A Lyapunov Function for Robust Stability of Moving Horizon Estimation

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Abstract—We provide a novel robust stability analysis for moving horizon estimation (MHE) using a Lyapunov function. In addition, we introduce linear matrix inequalities (LMIs) to verify the necessary incremental input/output-to-state stability (δ-IOSS) detectability condition. We consider an MHE formulation with time-discounted quadratic objective for nonlinear systems admitting an exponential δ-IOSS Lyapunov function. We show that with a suitable parameterization of the MHE objective, the δ-IOSS Lyapunov function serves as an M-step Lyapunov function for MHE. Provided that the estimation horizon is chosen large enough, this directly implies exponential stability of MHE. The stability analysis is also applicable to full information estimation, where the restriction to exponential δ-IOSS can be relaxed. Moreover, we provide simple LMI conditions to systematically derive δ-IOSS Lyapunov functions, which allows us to easily verify δ-IOSS for a large class of nonlinear detectable systems. This is useful in the context of MHE in general, since most of the existing nonlinear (robust) stability results for MHE depend on the system being δ-IOSS (detectable). In combination, we thus provide a framework for designing MHE schemes with guaranteed robust exponential stability. The applicability of the proposed methods is demonstrated with a nonlinear chemical reactor process and a 12-state quadrotor model.

Index Terms—Incremental system properties, moving horizon estimation (MHE), state estimation.

I. INTRODUCTION

STATE estimation for nonlinear systems based on noisy output measurements is a challenging problem of high practical relevance. The design of corresponding state observers is an active field of research, with recent results using differential dynamics and reduced coordinates [1], [2], and observers for constrained nonlinear systems with a quadratic Lyapunov function [3]. An optimization-based approach to nonlinear state estimation is moving horizon estimation (MHE) [4], [5, Ch. 4]. Our main contribution is twofold: We provide a robust stability analysis for MHE using a Lyapunov function, and we provide simple linear matrix inequality (LMI) conditions to verify the underlying incremental input/output-to-state stability (δ-IOSS) detectability assumption.

Related work: Based on an observability assumption, nominal stability of MHE without prior weighting and with a constant quadratic form as prior weighting was shown in [6] and [7], respectively. In [8], an approximation of the arrival cost was used as prior weighting to mimic the effect of the neglected past measurements. Alternative approaches to stability of MHE are based on the inclusion of a robustly stabilizing observer [9], [10].

More general robust stability results of MHE are based on δ-IOSS as a notion of detectability [11]. In particular, in [12], this detectability assumption was used to show suitable stability properties for full information estimation (FIE) in the case of convergent disturbances. This restriction of convergent disturbances has been relaxed in [13] and [14] for MHE, by introducing an additional max-term penalizing the largest stage cost into the MHE objective. In [4] and [15], robust stability was shown without the additional max-term, allowing for standard quadratic objective functions. However, in both [4] and [15], the resulting estimation error bounds become worse for larger MHE horizons and the derived stability properties do not hold globally (cf. [4, Rem. 2]). A first step toward a generalized stability analysis of MHE based on Lyapunov theory was presented in [16] and [17] by introducing a Lyapunov-like function for the stability analysis of FIE, using the fact that robust exponential stability of FIE also implies stability of MHE for a sufficiently large horizon, cf. [17] and compare also [14], [18], and [19]. Finally, the use of δ-IOSS and further a time-discounted objective function allowed to show robust stability of FIE and MHE [18], [20].

Contribution: In this article, we present an MHE scheme with exponential discounting, but otherwise standard quadratic stage cost and prior weighting, for systems admitting an exponential δ-IOSS Lyapunov function (Section III-A). Provided the time-discounting factor in the MHE objective satisfies a certain condition based on the δ-IOSS Lyapunov function, any quadratic objective can be considered in the MHE problem. For
the stated MHE scheme, we present a robust stability analysis based on a Lyapunov function for MHE (Section III-B), resulting in theoretical guarantees which improve as the estimation horizon increases. In particular, we show that the current δ-IOSS Lyapunov function is bounded by a past δ-IOSS Lyapunov function, the current value function, and a bound depending on the disturbances within the current estimation horizon. This result is partially motivated by a similar bound derived in [21] in the context of output-feedback model predictive control (MPC).

Based on the abovementioned bound, we show that the δ-IOSS Lyapunov function is an $M$-step Lyapunov function for the MHE scheme (Theorem 1), where $M$ is the horizon length of the MHE problem. Thereby, a sufficient lower bound on the horizon length $M$ is obtained, which ensures a decrease in the Lyapunov function over $M$ time steps and, thus, directly ensures robust stability (Corollary 1). The proposed analysis is directly applicable to FIE, where we can additionally relax the restriction to asymptotic (instead of exponential) δ-IOSS (Section III-C). Section III-D, we provide a detailed discussion of our proposed stability analysis with respect to recent MHE and FIE stability results [4], [14], [15], [16], [17], [18], [19], [20], [22]. In particular, compared with existing Lyapunov-like techniques [16], [17], our framework allows for a much simpler Lyapunov function and corresponding robust stability analysis. Moreover, we show that the proposed condition on the horizon length for guaranteed robust stability of MHE is (significantly) less conservative than the corresponding conditions required by recent robust stability results from the literature on nonlinear MHE, in particular, [4], [17], [18], compare Table I in Section III-D.

As second contribution, we provide a systematic approach to verify exponential δ-IOSS for a large class of nonlinear detectable systems based on their differential dynamics (Section IV). More precisely, we provide sufficient conditions for a quadratically bounded δ-IOSS Lyapunov function that can be easily cast in terms of LMIs (Theorem 2). This contribution is of particular relevance for all MHE schemes that rely on δ-IOSS, e.g., [15], [16], [17], [18], [19], [20], [23], [24], but also for the verification of more general system properties, such as incremental dissipativity.

Overall, the proposed δ-IOSS verification, and in particular the resulting δ-IOSS Lyapunov function, allows us to choose a beneficial quadratic prior weighting such that the MHE scheme from Section III-A is robustly exponentially stable with relatively short horizon. The applicability of the proposed framework is demonstrated in Section V with a nonlinear chemical reactor process from [5] and [25], and a 12-state quadrator model. For these examples, we can rigorously show satisfaction of the posed conditions and derive a short (practical) horizon bound that guarantees robust exponential stability of MHE.

**Notations:** Let the nonnegative real numbers be denoted by $\mathbb{R}_{\geq 0}$, the set of integers by $\mathbb{Z}$, the set of all integers greater than or equal to $a$ for some $a \in \mathbb{R}$ by $\mathbb{Z}_{\geq a}$, and the set of integers in the interval $[a, b]$ for some $a, b \in \mathbb{R}$ with $a \leq b$ by $\mathbb{Z}_{[a, b]}$. Let $\|x\|$ denote the Euclidean norm of the vector $x \in \mathbb{R}^n$. The quadratic norm with respect to a positive definite matrix $Q = Q^\top$ is denoted by $\|x\|^2_Q = x^\top Q x$, and the minimal and maximal eigenvalues of $Q$ are denoted by $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$, respectively. The maximum generalized eigenvalue of positive definite matrices $A = A^\top$ and $B = B^\top$ is denoted as $\lambda_{\max}(A, B)$, i.e., the largest scalar $\lambda$ satisfying $\det(A - \lambda B) = 0$. The identity matrix is denoted by $I_n \in \mathbb{R}^{n \times n}$. A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class $K$ if it is continuous, strictly increasing, and satisfies $\alpha(0) = 0$. If $\alpha$ is additionally unbounded, it is of class $K_\infty$. We denote the class of functions $\theta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ that are continuous, nonincreasing, and satisfy $\lim_{t \rightarrow \infty} \theta(t) = 0$ by $\mathcal{L}$. By $\mathcal{K}_\infty$, we denote the functions $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with $\beta(\cdot, t) \in \mathcal{K}$ and $\beta(r, \cdot) \in \mathcal{L}$ for any fixed $t \in \mathbb{R}_{\geq 0}$ and $r \in \mathbb{R}_{\geq 0}$.

**II. Problem Setup and Preliminaries**

We consider the discrete-time, nonlinear perturbed system

$$\begin{align*}
x_{t+1} &= f(x_t, u_t, w_t), \\
y_t &= h(x_t, u_t)
\end{align*}$$

with state $x_t \in \mathbb{R}^n$, control input $u_t \in \mathbb{R}^m$, disturbance $w_t \in \mathbb{R}^q$, noisy output measurement $y_t \in \mathbb{R}^p$, and time $t \in \mathbb{R}_{\geq 0}$. The nonlinear continuous functions $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \rightarrow \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \rightarrow \mathbb{R}^p$ represent the system dynamics and the output equation, respectively. Note that the generalized disturbance $w$ accounts for both the process disturbance $(1a)$ and the measurement noise $(1b)$ to keep the presentation concise. This general formulation also covers the standard setting of independent process disturbance and measurement noise as a special case, cf. the numerical examples in Section V. Since we only consider the estimation problem, the control input $u$ is treated as a known external variable.

Given some initial guess $x_0$ of the true state $x_0$, the main objective is to obtain, at each time $t \in \mathbb{R}_{\geq 0}$, an estimate $\hat{x}_t$ of the current state $x_t$. We consider the general case where we may know some additional information of the form

$$\begin{align*}
x_{t}, u_{t}, w_{t}, y_{t} &\in \mathbb{X} \times \mathbb{U} \times \mathbb{W} \times \mathbb{Y} =: \mathbb{Z}, & t \in \mathbb{R}_{\geq 0} \\
\end{align*}$$

with sets $\mathbb{X} \subseteq \mathbb{R}^n$, $\mathbb{U} \subseteq \mathbb{R}^m$, $\mathbb{W} \subseteq \mathbb{R}^q$, and $\mathbb{Y} \subseteq \mathbb{R}^p$. Note that $\mathbb{Z}$ does not represent a set of constraints in the sense of a control problem, but rather the domain of real system trajectories. This typically arises from physical nature, e.g., mechanically imposed limits on joint angles or a measurement device, or nonnegativity of the absolute temperature, partial pressures, or concentrations of species in a chemical reaction. Taking such information into account can often significantly improve the estimation results, cf. [5, Sec. 4.4].

In order to establish robust stability of the proposed MHE scheme, an appropriate detectability assumption is required. To this end, we consider the following δ-IOSS Lyapunov function.

**Definition 1 (δ-IOSS Lyapunov function [23, Def. 2.9]):** A function $W_\delta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is a δ-IOSS Lyapunov function if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\sigma_u \in \mathbb{R}_+$, and $\eta \in [0, 1]$ such that

$$\begin{align*}
\alpha_1 (\|x - \hat{x}\|) &\leq W_\delta(x, \hat{x}) \leq \alpha_2 (\|x - \hat{x}\|), \\
\end{align*}$$

$$\begin{align*}
W_\delta (f(x, u, w), f(\hat{x}, \hat{u}, \hat{w})) &\leq \eta W_\delta(x, \hat{x}) + \sigma_u (\|w - \hat{w}\|) + \sigma_y (\|y - \hat{y}\|) \\
\end{align*}$$

To simplify the notation, we restrict our analysis to decoupled constraint sets here. The stability properties for MHE and FIE presented in Section III remain valid also for coupled constraints, i.e., $(x_{t}, u_{t}, w_{t}, y_{t}) \in \mathbb{Z}$. Note that in the unconstrained case, i.e., when no additional information of the form (2) is available, the results presented in Sections III and IV remain valid with $\mathbb{X} = \mathbb{R}^n$, $\mathbb{U} = \mathbb{R}^m$, $\mathbb{W} = \mathbb{R}^q$, and $\mathbb{Y} = \mathbb{R}^p$.

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for all \((x, u, w, y), (\hat{x}, u, \hat{w}, \hat{y}) \in \mathbb{Z}_n\), where \(y = h(x, u, w)\) and \(\hat{y} = h(\hat{x}, u, \hat{w}).\)

Definition 1 is equivalent to a corresponding \(\delta\)-IOSS property involving general \(\mathcal{K}\mathcal{L}\)-functions (cf. [23, Th. 3.2])\(^2\), which became standard as a description of nonlinear detectability in the context of MHE in recent years [15], [16], [17], [18], [19], [20], [23], [24]. Note that \(\delta\)-IOSS is both necessary (cf. [24, Prop. 3], [23, Prop. 2.6]) and sufficient (cf. [18, Th. 1]) for the existence of robustly stable state estimators as characterized below by Definition 2.

In order to provide stability guarantees with a finite-horizon, we assume quadratic bounds and supply rates, which implies an \textit{exponential} \(\delta\)-IOSS condition, compare also the discussion in Section III-D below.

\textbf{Assumption 1 (Exponential \(\delta\)-IOSS):} The system (1) admits a \(\delta\)-IOSS Lyapunov function \(W_\delta\) according to Definition 1 with quadratic bounds and supply rates, i.e., there exist \(P_1, P_2 \succ 0\) and \(Q, R \succeq 0\) such that

\[
\|x - \hat{x}\|^2_{P_1} \leq W_\delta(x, \hat{x}) \leq \|x - \hat{x}\|^2_{P_2},
\]

\[
W_\delta(f(x, u, w), (\hat{x}, u, \hat{w})) \leq \eta W_\delta(x, \hat{x}) + \|\hat{w} - w\|^2_Q + \|y - \hat{y}\|^2_R
\]

for all \((x, u, w, y), (\hat{x}, u, \hat{w}, \hat{y}) \in \mathbb{Z}_n\), where \(y = h(x, u, w)\) and \(\hat{y} = h(\hat{x}, u, \hat{w}).\)

The explicit computation of such a quadratically bounded \(\delta\)-IOSS Lyapunov function is discussed in detail in Section IV. In the following, we utilize \(\delta\)-IOSS to design an MHE framework that is robustly stable by means of the following notion.

\textbf{Definition 2 (RGAS [23, Def. 2.3], RGES [20, Def. 1]):} A state estimator for system (1) is robustly globally asymptotically stable (RGAS) if there exist \(\beta_1, \beta_2 \in \mathcal{K}\mathcal{L}\) such that the resulting state estimate \(\hat{x}_t\) satisfies

\[
\|x_t - \hat{x}_t\| \leq \max_j \left\{ \beta_1(\|x_0 - \hat{x}_0\|_t), \beta_2(\|w_j\|, t - j - 1) \right\}
\]

for all \(t \in \mathbb{N}_0\), all initial conditions \(x_0, \hat{x}_0 \in \mathbb{X}\), and every trajectory \((x_t, u_t, w_t, y_t)_{t=0}^T\) satisfying (1) and (2). If, additionally, \(\beta_1(r, t) = C_1\lambda_1^r\) and \(\beta_2(r, t) = C_2\lambda_2^r\) with \(\lambda_1, \lambda_2 \in [0, 1)\) and \(C_1, C_2 > 0\), then the state estimator is robustly globally exponentially stable (RGES).

This definition of robust stability is often used in the recent MHE literature and has already been adequately studied in, e.g., [5], [18], [22], [23], and [24]. Note that this characterization is particularly suitable for MHE and FIE, since indeed implies that the estimation error converges to zero if the disturbances vanish [5, Prop. 4.3], which would not immediately be the case using robust stability notions without time-discounting, compare [4], [14], and [15].

\textbf{III. LYAPUNOV FUNCTION FOR MHE}\n
In this section, we start by introducing the proposed MHE formulation with a cost function related to the \(\delta\)-IOSS Lyapunov function in Section III-A. We then present the stability analysis of the MHE scheme by showing that it admits a Lyapunov function in Section III-B. In particular, we establish a relation between the \(\delta\)-IOSS Lyapunov function and the value function of the MHE problem. Based on this relation, we show that the \(\delta\)-IOSS Lyapunov function serves as an \(M\)-step Lyapunov function for MHE, where \(M\) is the horizon length of the MHE. In Section III-C, an extension of the analysis to FIE is presented, which allows us to avoid Assumption 1, i.e., \textit{exponential} \(\delta\)-IOSS. Finally, a detailed discussion of the differences between the presented stability analysis with related approaches is presented in Section III-D.

\textbf{A. Moving Horizon Estimator Formulation}\n
At time \(t\), the MHE scheme considers past input and output data \((u, y)\) in a window of length \(M_t = \min\{t, M\}\), with \(M \in \mathbb{N}_0\), and the past estimate\(^3\) \(\hat{x}_{t-M_t}\). Thereby, the MHE optimizes over the initial estimate \(\hat{x}_{t-M_t}\) and a sequence of \(M_t\) disturbance estimates \(\hat{w}_{j=M_t} = \{\hat{w}_j\}_{j=M_t}^{t-1}\). Combined, the initial estimate and sequence of disturbance estimates define a sequence of state estimates \(\hat{x}_j = f(\hat{x}_{j-1}) + \hat{w}_j\) and a sequence of output estimates \(\hat{y}_j = \{\hat{y}_j\}_{j=M_t}^{t-1}\). The objective of this optimization-based state estimation problem is to minimize the following cost function:

\[
V_{\text{MHE}}(\hat{x}_{t-M_t}, \hat{w}_{j=M_t}, \hat{y}_{j=M_t}, t) = 2\eta M_t \|\hat{x}_{t-M_t} - \hat{x}_{t-M_t}\|^2_{P_2}
\]

\[
+ \sum_{j=M_t}^{t-1} \eta^{j-M_t-1}(2\|\hat{w}_{j-M_t}\|_Q + \|\hat{y}_{j-M_t} - y_{j-M_t}\|_R)^2,
\]

where \(\eta, Q, R, P_2\) are based on the exponential \(\delta\)-IOSS property and corresponding Lyapunov function \(W_\delta\) according to Assumption 1. In fact, provided the discount factor \(\gamma\) in the MHE cost (6) is chosen such that \(1 - \eta > 0\) is sufficiently small, the theoretical analysis in Section III-B remains valid for any positive definite matrices \(Q, R\), and \(P_2\) in the MHE cost (6), compare also Remark 1 in the following. A similar time-discounted MHE cost has been previously suggested in [20], compare also [18] for a more general asymptotic discounting. Except for the discount factor \(\eta\), the cost (6) allows for standard (quadratic) MHE stage costs. The use of a discount factor in the cost (6) allows us to obtain tighter upper bounds in (5), compared with, e.g., [4] and [15] where the bounds deteriorate with a larger horizon length \(M\) (compare also the discussion in Section III-D). The state estimate at time step \(t\) is then obtained by solving the following nonlinear program (NLP):

\[
\min_{\hat{x}_{t-M_t}, \hat{w}_{j=M_t}, \hat{y}_{j=M_t}} V_{\text{MHE}}(\hat{x}_{t-M_t}, \hat{w}_{j=M_t}, \hat{y}_{j=M_t}, t)
\]

\[
\text{s.t. } \hat{x}_{j+1} = f(\hat{x}_j, u_j, \hat{w}_j), j \in \mathbb{N}_{t-M_t, t-1},
\]

\[
\hat{y}_j = h(\hat{x}_j, u_j, \hat{w}_j), j \in \mathbb{N}_{t-M_t, t-1},
\]

\[
\hat{w}_j \in \mathbb{W}, \hat{y}_j \in \mathbb{Y}, j \in \mathbb{N}_{t-M_t, t-1},
\]

\[
\hat{x}_j \in \mathbb{X}, j \in \mathbb{N}_{t-M_t, t}.
\]

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\(^2\)We conjecture that the converse Lyapunov results from [23, Th. 3.2], where \(y = h(x)\) is assumed, remain valid for our more general nonlinear setup (1).

\(^3\)This choice is typically called filtering prior, cf. [4] and [5, Ch. 4].
We denote a minimizer⁴ to (7) by \( \hat{x}_{-M_t|t}^* \) and \( \hat{w}_{-j|t}^* \), and the corresponding estimated state and output trajectories as \( \hat{x}_{|t}^* \) and \( \hat{y}_{-j|t}^* \), respectively. The resulting state estimate at time step \( t \) is given by

\[
\hat{x}_t = \hat{x}_{|t}^* \tag{8}
\]

and the estimation error at time step \( t \), i.e., the difference between the true system state and the state estimate (8), as

\[
\hat{e}_t = x_t - \hat{x}_t. \tag{9}
\]

The MHE estimator (7) is then applied in a receding horizon fashion, i.e., at each time step \( t \), the current state estimate (8) is obtained by solving the MHE problem (7) based on the \( M_t \) most recent output measurements.

Remark 1 (Parameterization of the MHE objective): Assumption 1 is invariant with respect to scaling of the \( \delta \)-IOSS Lyapunov function \( W_\delta \). Specifically, if there exists a \( \delta \)-IOSS Lyapunov function with decay rate \( \eta \) satisfying Assumption 1, the assumption can be satisfied for any \( \bar{\eta} \geq \eta \), with \( \bar{\eta} < 1 \), and any positive definite matrices \( P_2, R, \) and \( Q \). Consequently, the cost (6) can be parameterized with any positive definite matrices \( P_2, R, \) and \( Q \), and any discount factor \( \bar{\eta} \) provided that \( 1 - \bar{\eta} > 0 \) is chosen sufficiently small (see also [10] and [18] for a similar discussion).

However, the choice of the matrix \( P_2 \) influences the minimal horizon length required for RGES and the resulting error bounds (cf. Section III-B and Theorem 1). If the ratio between \( P_1 \) and \( P_2 \) improves, i.e., the largest generalized eigenvalue \( \lambda_{\text{max}}(P_2, P_1) \) of \( P_2 \) and \( P_1 \) approaches 1, the horizon \( M \) can be chosen smaller and the resulting estimation error bounds are less conservative. Consequently, choosing the prior weighting similar to the \( \delta \)-IOSS Lyapunov function improves the ratio between \( P_1 \) and \( P_2 \). Similar considerations also apply to Definition 1 and the FIE cost function (22) used in Section III-C, where a scaling of the functions \( \alpha_2, \sigma_w, \) and \( \sigma_y \) can be considered.

B. Lyapunov-Based Stability Analysis

In the following, we elaborate how the specific choice of the cost function (6) based on the \( \delta \)-IOSS Lyapunov function \( W_\delta \) results in \( W_\delta \) being an \( M \)-step Lyapunov function for MHE. Given the horizon \( M \) satisfies some lower bound, it follows directly that the proposed MHE scheme is an RGES state estimator according to Definition 2, compare Theorem 1 in the following. We start by the relation of \( W_\delta(\hat{x}_t, x_t) \) and the value function \( V_{MHE}(\hat{x}_{-M_t|t}, \hat{w}_{-j|t}, \hat{y}_{-j|t}, t) \) of (7) in the following Proposition.

**Proposition 1:** Let Assumption 1 hold. Then, for all \( t \in \overline{1,0} \), the state estimate \( \hat{x}_t \) (8) satisfies

\[
W_\delta(\hat{x}_t, x_t) \leq 2\eta_{M_t}^M \lambda_{\text{max}}(P_2, P_1) W_\delta(\hat{x}_{-M_t|t}, x_{-M_t|t}) + V_{\text{MHE}}(\hat{x}_{-M_t|t}, \hat{w}_{-j|t}, \hat{y}_{-j|t}, t) + 2 \sum_{j=1}^{M_t} \eta_{M_t}^j \|w_{t-j}\|_Q^2. \tag{10}
\]

**Proof:** Due to the constraints (7b)–(7e) in the MHE optimization problem (7), we have at each time step \( t \) that \( (\hat{x}_{|t}^*, \hat{w}_{|t}^*, \hat{y}_{|t}^*) \in \mathbb{Z} \) for all \( j \in \overline{1, M_t} \) and \( \hat{x}_{|t}^* \in \mathbb{X} \), and the estimated trajectory satisfies (1). Thus, we can apply Inequality (4b) \( M_t \) times, which together with application of the upper bound (4a) leads to

\[
W_\delta(\hat{x}_t, x_t) \leq \sum_{j=1}^{M_t} \eta_{M_t}^{j-1} \left(2 \|\hat{w}_{-j|t}^*\|_Q^2 + 2 \|w_{t-j}\|_Q^2 + \|\hat{y}_{-j|t}^* - y_{t-j}\|_R^2\right) + \eta_{M_t}^M \|\hat{x}_{-M_t|t}^* - x_{-M_t|t}\|_{P_2}^2, \tag{11}
\]

where we also used the fact that \( \|\hat{w}_{-j|t}^* - w_{t-j}\|_Q^2 \leq 2 \|\hat{w}_{-j|t}^*\|_Q^2 + 2 \|w_{t-j}\|_Q^2 \) by Cauchy–Schwarz and Young’s inequality. Using the same inequalities, we have

\[
\|\hat{x}_{-M_t|t}^* - x_{-M_t|t}\|_{P_2}^2 = \|\hat{x}_{-M_t|t}^* - \hat{x}_{-M_t|t} + \hat{x}_{-M_t|t} - x_{-M_t|t}\|_{P_2}^2 \leq 2 \|\hat{x}_{-M_t|t} - \hat{x}_{-M_t|t}\|_{P_2}^2 + 2 \|\hat{x}_{-M_t|t} - x_{-M_t|t}\|_{P_2}^2. \tag{12}
\]

Inserting (12) into (11) results in

\[
W_\delta(\hat{x}_t, x_t) \leq 2\eta_{M_t}^M \|\hat{x}_{-M_t|t} - x_{-M_t|t}\|_{P_2}^2 + 2 \sum_{j=1}^{M_t} \eta_{M_t}^{j-1} \|w_{t-j}\|_Q^2 + V_{\text{MHE}}(\hat{x}_{-M_t|t}, \hat{w}_{-j|t}, \hat{y}_{-j|t}, t). \tag{13}
\]

As final step, we use the fact that

\[
\|\hat{x}_{-M_t|t} - x_{-M_t|t}\|_{P_2}^2 \leq \lambda_{\text{max}}(P_2, P_1) \|\hat{x}_{-M_t|t} - x_{-M_t|t}\|_{P_2}^2 \leq \lambda_{\text{max}}(P_2, P_1) W_\delta(\hat{x}_{-M_t|t}, x_{-M_t|t}). \tag{14}
\]

Application to (13) yields the desired bound (10).

Applying the bounds on \( W_\delta \) in (4a) and the definition of the estimation error in (9), Inequality (10) provides a bound on the estimation error \( \hat{e}_t \), dependent on the value function, the past estimation error \( \hat{e}_{-M_t} \), and the past disturbances \( w_{-j} \) with \( j \in \overline{1, M_t} \). Hence, if we have uniform bounds on the past estimation error \( \hat{e}_{-M_t} \), and disturbances \( w \), the value function provides a measure for the uncertainty in the state estimate, i.e., for a large value function, we have a large bound on the estimation error, and thus large uncertainty in the state estimate. Consequently, the value function can serve as a measure for the accuracy of the current state estimate. Related estimation error bounds were derived for Luenberger-like observers and MHE in [21, Prop. 1] and [21, Th. 3], respectively. We note that the result in [21] relies on a (local) continuity condition for \( W_\delta \), while the presented result exploits quadratic bounds in (4a) to derive a linear bound in (10). In the following, we use the bound derived in Proposition 1 to show that the \( \delta \)-IOSS Lyapunov function is an \( M \)-step Lyapunov function for MHE.

**Theorem 1 (M-step Lyapunov function for MHE):** Let Assumption 1 hold. Then, for all \( t \in \overline{1,0} \), the state estimate \( \hat{x}_t \) in (8) satisfies

\[
W_\delta(\hat{x}_t, x_t) \leq 4 \eta_{M_t}^M \lambda_{\text{max}}(P_2, P_1) W_\delta(\hat{x}_{-M_t|t}, x_{-M_t|t}) + 4 \sum_{j=1}^{M_t} \eta_{M_t}^{j-1} \|w_{t-j}\|_Q^2. \tag{14}
\]
Since we assume the true underlying system to satisfy the constraints \((2)\), the true disturbance, state, and output sequences are a feasible solution to the MHE problem \((7)\), i.e., \(V_{\text{MHE}}(\hat{x}_{t-M|t}, \hat{w}_{t-j|t}, \hat{y}_{t-j|t}) \leq V_{\text{MHE}}(x_{t-M|t}, w_{t-j|t}, y_{t-j|t})\) by optimality. Inserting this in Inequality \((3)\) and using \((6)\) results in

\[ V_{\delta}(t + M) \leq 4\eta M \lambda_{\max}(P_2, P_1) V_{\delta}(t, x_t) + \left(4 \sum_{j=1}^{M} \eta^{j-1} ||w_{t-M-j}||_Q^2 \right). \]

Using \((4a)\) to upper bound \(||\hat{x}_{t-M|t} - x_{t-M|t}||_2^2\) as in the proof of Proposition 1 leads to \((14)\), which concludes the proof.

In case the horizon length \(M\) of the MHE problem \((7)\) is chosen such that

\[ \rho^M := 4\eta M \lambda_{\max}(P_2, P_1) < 1 \]

with \(\rho \in [0, 1)\), then, for any \(t \geq M\), the bound \((14)\) in Theorem 1 results in

\[ W_{\delta}(\hat{x}_t, x_t) \leq \rho^M W_{\delta}(\hat{x}_{t-M}, x_{t-M}) + 4 \sum_{j=1}^{M} \eta^{j-1} ||w_{t-M-j}||_Q^2. \]

Consequently, \(W_{\delta}(\hat{x}_t, x_t)\) is an \(M\)-step (incremental) Lyapunov function for the estimation error \((9)\).

By its very nature, the MHE computes an estimate \(\hat{x}_t\) that depends on the past estimate \(\hat{x}_{t-M}\), and a sequence of \(M\) most recent measurements \(y_{t-j}\) for \(j \in [1, M]\). As such, it is not surprising that the resulting Lyapunov function is defined over \(M\) steps, as opposed to standard Lyapunov functions \([26]\).

We note that Ahmadi and Parrilo \([27]\), for autonomous systems \(x_{t+1} = f(x_t)\), provided a method to compute a standard Lyapunov function based on an \(M\)-step decrease condition. However, in the considered MHE case, the dynamics also depend on the past \(M\) measurements, and thus, a resulting Lyapunov function would also depend on those quantities. In conclusion, the \(M\)-step Lyapunov function for MHE presented previously follows naturally from the definition of the MHE, and is preferred for the following stability analysis, since a Lyapunov function with one-step decrease would not have a concise analytic expression.

Remark 2 (Alternative \(M\)-step Lyapunov-like function): We note that instead of the \(\delta\)-IOSS Lyapunov function \(W_{\delta}(\hat{x}_t, x_t)\), an alternative Lyapunov-like function is naturally given by the weighted sum of a past \(\delta\)-IOSS Lyapunov function, the value function, and a sum of the past \(M\) disturbance

\[ V_{\delta}(t) := 4\eta M \lambda_{\max}(P_2, P_1) W_{\delta}(\hat{x}_{t-M}, x_{t-M}) + \sum_{j=1}^{M} \eta^{j-1} ||w_{t-M-j}||_Q^2 \]

for \(t \geq M\). In particular, this function satisfies

\[ V_{\delta}(t + M) \leq 4\eta M \lambda_{\max}(P_2, P_1) V_{\delta}(t, x_t) + \left(4 \sum_{j=1}^{M} \eta^{j-1} ||w_{t-M-j}||_Q^2 \right). \]

where we used the fact that the true state, disturbance, and output sequences are a feasible solution of \((7)\) at time step \(t + M\), yielding an upper bound of the value function, compare also the proof of Theorem 1. Consequently, \(V_{\delta}(t)\) is an \(M\)-step Lyapunov-like function for MHE provided the horizon \(M\) is chosen such that \((16)\) holds. In the case where no disturbances act on the system model \((1)\), i.e., \(w_t = 0\) for all \(t \in \mathbb{I}_{\geq 0}\), the Lyapunov-like function \((17)\) reduces to a weighted sum of the \(\delta\)-IOSS Lyapunov function and the value function. This is particularly interesting due to the parallelism to output tracking MPC \([22\text{, Sec. 4.3]}\), where the Lyapunov function is given by a weighted sum of the \(\delta\)-IOSS Lyapunov function and the value function of the MPC. In particular, in \([22]\), it is shown that a nontrivial Lyapunov-like function based on a sequence of augmented infinite horizon control problems can be used to study the stability of output tracking MPC and MHE, respectively. In contrast, the stability results for output tracking MPC in \([28]\) and \([29\text{, Sec. 4.1]}\) directly provide a Lyapunov function given by a sum of the storage and value function based on a different analysis. In a similar spirit, the derived alternative Lyapunov-like function for MHE in \((17)\) follows naturally from the \(\delta\)-IOSS Lyapunov function \((1\text{ and the definition of the cost function}\ (6))\), see Section III-D for a detailed comparison of our stability analysis with the Lyapunov-like functions from \([17]\) and \([22]\).

In the following, we show how RGES of MHE follows directly from \((14)\).

Corollary 1 (MHE is RGES): Let Assumption 1 hold, and suppose the horizon \(M\) satisfies Inequality \((16)\). Then, for all \(t \in \mathbb{I}_{\geq 0}\), the estimation error \((9)\) satisfies

\[ ||\hat{e}_t||_{P_1} \leq \max \left\{ 4\sqrt{\rho} ||e_0||_{P_2}, \max_{j \in [0, t-1]} \left(4 \sqrt{1 - \frac{1}{\sqrt{\rho}}} \right)||w_{t-j-1}||_Q \right\} \]

with \(\rho\) as defined in \((16)\), i.e., the MHE estimator \((7)\) is an RGES estimator according to Definition 2.

Proof: Consider some time \(t = kM + l, \) with unique \(l \in \mathbb{I}_{[0, M-1]}\) and \(k \in \mathbb{I}_{\geq 0}\). Using Inequality \((15)\) we get

\[ W_{\delta}(\hat{x}_t, x_t) \leq 4\eta ||\hat{x}_t - x_t||_2^2 + 4 \sum_{j=1}^{M} \eta^{j-1} ||w_{t-M-j}||_Q^2. \]

Further, applying the bound \((14)\) \(k\) times with \(M_t = M\) and \(\rho\) as defined in \((16)\), we arrive at

\[ W_{\delta}(\hat{x}_t, x_t) \leq 4\eta ||\hat{x}_0 - x_0||_2^2 + 4 \sum_{j=1}^{M} \eta^{j-1} ||w_{t-M-j}||_Q^2. \]
\[
W_\delta (\hat{x}_t, x_t) \\
\leq \rho^k M W_\delta (\hat{x}_t, x_t) + 4 \sum_{i=0}^{k-1} \rho^M \sum_{j=1}^{M} \eta^{-1} \|w_{t-i-M-j}\|^2_Q
\]

\[
\leq \rho^k M \left( 4\eta^t \|\hat{e}_0\|^2_{\mathcal{P}_2} + 4 \sum_{j=1}^{t} \eta^{j-1} \|w_{t-j}\|^2_Q \right).
\]

The objective of the considered FIE problem is to minimize the following cost function:

\[
V_{\text{FIE}}(\hat{x}_{0:t}, \hat{\hat{x}}_{t}, \hat{\hat{y}}_{t}, t) = \eta^t \alpha_2 \left( 2\|\hat{x}_{0:t} - \hat{x}_0\| \right) + \sum_{j=1}^{t} \eta^{j-1} (\sigma_u(2\|\hat{x}_{j-1}; y_j\|) + \sigma_g(\|\hat{y}_{j-1}; y_j\|)).
\]

The main difference to the MHE objective (6) is the use of \(\sigma_u\), \(\sigma_g\), and \(\alpha_2\) of the \(\delta\)-IOSS Lyapunov function (3), instead of the quadratic functions as provided in Assumption 1. The FIE state estimate is then obtained by solving the following problem with \(M_t = t\) at each time step:

\[
\min_{\hat{x}_{0:t}, \hat{y}_{t}, \hat{\hat{y}}_{t}} V_{\text{FIE}}(\hat{x}_{0:t}, \hat{y}_{t}, \hat{\hat{y}}_{t}, t) \text{ s.t. (7b), (7c), (7d), (7e)}
\]

with the optimal state estimate denoted as \(\hat{x}_t = \hat{x}_{t|t}\).

In the case of MHE, Proposition 1 showed the relation of the \(\delta\)-IOSS Lyapunov function and the value function (7a), and Theorem 1 allowed us to conclude that \(W_\delta (\hat{x}_t, x_t)\) is an \(M\)-step Lyapunov function for MHE. In the case of FIE without Assumption 1, a similar analysis allows us to derive the bound on \(W_\delta (\hat{x}_t, x_t)\) as presented in the following proposition.

Proposition 2 (Bound on \(W_\delta (\hat{x}_t, x_t)\)): Let the system (1) admit a \(\delta\)-IOSS Lyapunov function according to Definition 1. Then, for all \(t \geq 1\), the state estimate \(\hat{x}_t\) satisfies

\[
W_\delta (\hat{x}_t, x_t) \leq 2\eta^t \alpha_2 \left( 2\eta^{t-1} (W_\delta (\hat{x}_0, x_0)) \right) + 2 \sum_{j=1}^{t} \eta^{j-1} \sigma_u(2\|w_{t-j}\|).
\]

Proof: Using the monotone increase property of \(K\)-functions and the weak triangular inequality [30], we obtain

\[
\alpha_2 \left( \|\hat{x}_{0:t} - x_0\| \right) \leq \alpha_2 \left( \|\hat{x}_{0:t} - \hat{x}_0\| + \|x_0 - \hat{x}_0\| \right) \leq \alpha_2 \left( 2\|\hat{x}_{0:t} - \hat{x}_0\| \right) + \alpha_2 \left( 2\|x_0 - \hat{x}_0\| \right).
\]

Applying the bound (3b) on \(W_\delta (\hat{x}_0, x_0)\) for \(t\) times and using this inequality, we arrive at

\[
W_\delta (\hat{x}_t, x_t) \leq \eta^t \alpha_2 \left( 2\|\hat{x}_0 - x_0\| \right) + V_{\text{FIE}}(\hat{x}_{0:t}, \hat{\hat{y}}_{t}, \hat{\hat{y}}_{t}, t) + \sum_{j=1}^{t} \eta^{j-1} \sigma_u(2\|w_{t-j}\|)
\]

analogous to Inequality (10) in the proof of Proposition 1. An upper bound for the value function can be computed by using the true sequence as a candidate, resulting in

\[
W_\delta (\hat{x}_t, x_t) \leq 2\eta^t \alpha_2 \left( 2\|\hat{x}_0 - x_0\| \right) + 2 \sum_{j=1}^{t} \eta^{j-1} \sigma_u(2\|w_{t-j}\|).
\]

Finally, using the lower bound in (3a), we have

\[
\|\hat{x}_0 - x_0\| \leq \alpha_1^{-1} (W_\delta (\hat{x}_0, x_0))
\]

which leads to (25) and concludes the proof.

A similar bound transferring sum-based into max-based expressions was recently used in the MHE literature, e.g., in [24, Eq. (25)] and [17, Prop. 3.13].
The bound presented in Proposition 2 allows us to show that the FIE scheme is an RGAS state estimator.

**Corollary 2 (FIE is RGAS):** Let the system (1) admit a δ-IOSS Lyapunov function according to Definition 1. Then, the FIE estimator (23) is an RGAS estimator according to Definition 2.

**Proof:** We start from (26). Using the lower bound in (3a) and the weak triangular inequality leads to
\[
\|\hat{x}_t - x_t\| \leq \alpha_1^{-1} \left( 4\eta^i \alpha_2 (2\|\hat{x}_0 - x_0\|) \right) + \alpha_1^{-1} \left( 4 \sum_{j=1}^{t} \eta^{j-1} \sigma_w (2\|w_{t-j}\|) \right).
\]

To bound the abovementioned second term, we use the same procedure as in the proof of Corollary 1 to transform the sum into a max-term, resulting in
\[
\alpha_1^{-1} \left( 4 \max_{i \in [0, t], j} \alpha_2^{-1} \left( \frac{4}{1 - \sqrt{\eta}} \sigma_w (2\|w_{t-j}\|) \right) \right).
\]

Inserting the abovementioned inequality, we obtain
\[
\|\hat{x}_t - x_t\| \leq \max_{i \in [0, t], j} \alpha_1^{-1} \left( 4 \max_{i \in [0, t], j} \alpha_2^{-1} \left( \frac{4}{1 - \sqrt{\eta}} \sigma_w (2\|w_{t-j}\|) \right) \right),
\]

which shows that the FIE estimator satisfies Definition 2, and thus concludes the proof.

**Remark 4:** By considering a quadratically bounded δ-IOSS Lyapunov function (Assumption 1), we could derive a simple condition (16) to provide an $M_t$-step Lyapunov function for MHE. In case the system admits a δ-IOSS Lyapunov function according to Definition 1, but Assumption (1) does not hold, i.e., the system (1) does not admit a quadratically bounded δ-IOSS Lyapunov function, and the MHE objective is replaced by (22) defined over the past $M_t$ time steps, Inequality (14) is replaced by
\[
W_{\hat{x}}(\hat{x}_t, x_t) \leq 2\eta^M t \alpha_2 (2\alpha_1^{-1} (W_\hat{x}(\hat{x}_{t-M_t}, x_{t-M_t}))) + 2 \sum_{j=1}^{M_t} \eta^{j-1} \sigma_w (2\|w_{t-j}\|),
\]

compare the proof of Proposition 2. For general $\alpha_1, \alpha_2 \in K_{\infty}$, no minimal horizon length $M$ ensuring a decrease can be found. Instead, typically, the minimal horizon depends explicitly on a bound on the estimation error at time $t - M_t$, resulting in nonglobal stability results. This issue was also discussed in more detail in [22, Sec. 5.5.3]. In [18, Th. 18], asymptotic stability of MHE is established under certain conditions, for which Assumption 1 is sufficient, using a nonlinear contraction.

**D. Discussion**

Developing MHE schemes for nonlinear systems generally always requires balancing practical designs against valid theoretical guarantees. For example, one may choose a very simple scheme involving a zero prior weighting and standard quadratic penalties, i.e., (6) with $\eta = 1$ and $P_2 = 0$, compare [6]. However, since past data are completely neglected in the design, the system must in general be observable to ensure stability of MHE, and furthermore, large estimation horizons may be required to obtain a performance comparable to FIE, cf. [5, Sec. 4.3.1]. Therefore, a nonzero prior weighting seems appropriate, which, on the other hand, may require intricate conditions to ensure stable estimation, cf. [8].

Establishing MHE for general detectable nonlinear systems that follow a practical design, provide good theoretical guarantees, and require conditions that can be easily verified has also emerged as a major problem in the more recent literature, cf. [4], [14], [15], and [20]. In particular, robust stability of MHE could be established in [4] and [15] based on a δ-IOSS using a general cost function that permits standard quadratic penalties; however, the robustness bounds deteriorate with an increasing estimation horizon. Such a behavior is counter-intuitive and undesired since one would naturally expect better estimation results if more information is taken into account. This issue could be avoided using a modified cost: either by adding a max-term that penalizes the largest single disturbance as in [14] and [15], or by using a specific cost structure satisfying the triangle inequality, cf. [20]. Thus, a tradeoff between a standard quadratic cost function and good performance guarantees for MHE has arisen. In contrast, we were able to resolve this conflict of objectives: we both consider standard quadratic penalties (except for time-discounting) and provide theoretical guarantees that improve as $M$ increases. We point out that comparable stability results were achieved earlier by using more general time-discounting with $K.L.$-functions [18] or without discounting using a Lyapunov-like function [17], [22]. The following discussion compares these two structurally different approaches to the framework presented in Section III.

1) **General Time-Discounting [18]:** The requirements of Theorem 1 for guaranteed robust stability of MHE are fundamentally the same as in [18], namely, that the detectability property of the system (given by δ-IOSS) must be suitably related to the cost function used for MHE by employing additional time-discounting. Consequently, the robustness bounds derived in Section III are qualitatively comparable to those from [18, Th. 14] for the special case of exponential stability. However, we point out that under certain conditions, [18, Th. 14] also implies an asymptotic stability result using a nonlinear contraction. In contrast, we require exponential detectability (i.e., a quadratically bounded δ-IOSS Lyapunov function (4a)) to achieve linear contraction in the proof of Proposition 1, cf. Remark 4. Note that this is in line with most of the recent results on nonlinear MHE [4], [14], [15], [19], [20], where (local) exponential detectability is applied to achieve linear contraction over the estimation horizon, see also [4, Prop. 1] and [19, Lemma 1].

While the authors in [4], [14], [15], [18], [19], and [20] build their analysis on properties of certain $K.L.$-functions, we employ a Lyapunov characterization in Section III. To the best of the authors’ knowledge, a Lyapunov function for nonlinear MHE has been missing in the literature and, therefore, is an interesting contribution on its own. Moreover, based on the proposed analysis and, in particular, a general reasoning in Lyapunov coordinates, further theoretical insights and practical improvements
arise. First, verifying the required detectability condition in order to guarantee stability of MHE is rather straightforward in our setup. This becomes immediately apparent in Section IV, where we give a simple condition for computing a $\delta$-IOSS Lyapunov function that satisfies Assumption 1. Even though this directly implies a traditional $\mathcal{KL}$-characterization of $\delta$-IOSS as mentioned in Remark 8, we have a stronger relationship between the $\delta$-IOSS Lyapunov function and the MHE cost function. Indeed, since we circumvent the additional step of calculating the respective $\mathcal{KL}$-functions, the tuning of the cost function becomes more easy and intuitive (cf. Remark 1). Moreover, note that arguing in Lyapunov coordinates generally allows for less restrictive conditions on the minimal horizon length for guaranteed RGES of MHE compared with, e.g., [18, Th. 14], which can be seen in Table I and the discussion in the following. Finally, a beneficial feature of Definition 1 is that even in the asymptotic case (i.e., where $\alpha_1$ and $\alpha_2$ are arbitrary $\mathcal{KL}$-functions in (3a)), we can still use an exponential decrease in (3b) (without loss of generality). This is exploited in Section III-C where we address asymptotic stability of FIE, which in the end allows for a much simpler and more intuitive tuning of the FIE cost function compared with, e.g., [18, Assumption 1] and [19, Assumption 3] using general $\mathcal{KL}$-function inequalities.

2) Lyapunov-Like Function Framework [17, 22]: We point out that a Lyapunov approach to stability of FIE already appeared in the literature; in particular, a Lyapunov-like function (termed a Q-function) was used in [16] to establish nominal stability of FIE, and the results were extended in [17] and [22, Ch. 5] to RGES and RGAS of FIE, respectively. However, the Q-function differs significantly from the Lyapunov function presented in this work, especially in its nontrivial structure utilizing two time arguments. The key ingredient in [17] and [22, Ch. 5] is a sequence of augmented infinite-horizon problems, each considering the first $t$ disturbances and zero disturbances thereafter. As a result, each of this infinite-horizon problems has finite disturbance sequences and, therefore, well-defined solutions. Then, at any time $t \in \mathbb{I}_{0,0}$, the respective infinite-horizon cost function is compared with the truncated finite-horizon cost function considering the partial time interval $\mathbb{I}_{0,j,-1}$ for some $j \in \mathbb{I}_{0,t}$. This procedure allows to establish one-step dissipation in $j$ for each $j \in \mathbb{I}_{0,t}$. However, since the resulting function is only semidefinite, it needs to be combined with the $\delta$-IOSS Lyapunov function to finally create the desired Q-function. Taking into account its different components, ("pessimistic") Lyapunov-like bounds on the Q-function are established in [17, Prop. 3.14] for exponential, and in [22, Sec. 5.3] for asymptotic stability. In this context, note that the lower bound of the Q-function is given by the lower bound of the $\delta$-IOSS Lyapunov function; the upper bound, however, requires an additional stabilizability assumption of certain structure, cf. [17, Assumption 3.6] and [22, Assumption 5.14].

Our analysis, on the other hand, is much simpler in many respects, simply enabled by an additional discount factor in the cost function (6), compare also Remark 2. Note that this corresponds to a fading memory design, a concept that has been widely used in the literature for decades in many research areas (see, e.g., [31] and [32]), and which was previously exploited also in the context of state estimation, e.g., to deal with model errors in Kalman filter applications, cf. [33]. Within our framework, this results in a strong connection between our notion of detectability and the cost such that the $\delta$-IOSS Lyapunov function immediately provides a valid bound for FIE, cf. Proposition 2, and consequently, directly serves as an $M$-step Lyapunov function for MHE, cf. Theorem 1. As a result, we are able to avoid many (potentially conservative) steps, excessive overapproximations, and additional conditions, such as stabilizability in the analysis. Interestingly, the conditions on the cost function in terms of compatibility with $\delta$-IOSS are fundamentally similar for all the results considered previously (except for the time-discounting), see Remark 1, [17, Assumption 3.5], [22, Assumption 5.13], and [18, Assumption 1].

3) Conditions on the Horizon Length for RGES of MHE: In the following, we compare the methods discussed previously by means of their respective conditions on the horizon length for guaranteed RGES of MHE, which illustrates the general benefit of arguing in Lyapunov coordinates. For a broader overview, we also consider [4, Th. 1], i.e., MHE based on a $\mathcal{KL}$-function characterization of $\delta$-IOSS, but without a time-discounted objective function as in [18]. For a fair comparison, we choose the cost functions for [4, 17], and [18] such that the smallest possible horizon follows in each case; more specifically, we consider $b(s, t) = \beta(2s, t)$ according to [18, Remark 6] and $V_b(\chi, \bar{x}) = \|\chi - \bar{x}\|^2$ in [4, Assumption 3]. Since the respective analysis is much more involved for [17], we consider only an ideal (strict) lower bound on the minimal horizon length, which follows from a vanishing prior weighting ($\omega_c = \omega_m = 0$ in [17, Assumption 3.4]) and under a perfect stabilizability condition ($\epsilon_c \rightarrow 0$ in [17, Assumption 3.6]). Provided that a $\delta$-IOSS Lyapunov function according to Assumption 1 is given, Table I shows for each case the resulting constants $C > 0$ and $\mu \in (0, 1)$ for the general contraction condition $C \mu^M < 1$ used to establish RGES of MHE. Solving the conditions for the minimal stabilizing horizon length by $M_{\text{min}} = \lceil -\log C / \log \mu \rceil$ and applying standard properties of the logarithmic function, we arrive at the following conclusions. First, under optimal choices of the cost functions in terms of the horizon length, the contraction conditions from [18, Th. 14] and [4, Th. 1] are (except for the additional factor 3) very similar to each other, despite a structurally different MHE design and proof technique.9 Second, we can generally conclude that our approach provides

9Note that RGES of MHE was shown in [17, Th. 4.2] assuming that the underlying FIE is RGES and provides a linear contraction over the horizon, i.e., without explicitly constructing a Q-function for MHE, compare [22, Sec. 5.5.3] and [18, Sec. 4].

9Wording according to the discussion in the following [17, Corollary 3.18]; it is not distinguished between the influences of long past or recent disturbances.
the least conservative estimate on the minimal horizon length \( M_{\min} \) for guaranteed RGES of MHE; to see this, recall that 
\[
\lambda_{\max}(P_2)/\lambda_{\min}(P_1) \geq \lambda_{\max}(P_2, P_1) \quad \text{for all } P_2 \geq P_1 > 0
\]
and observe that each constant \( C, \mu \) in Table I has its minimal value at \( \epsilon_1 = \epsilon_2 \). This general fact is also observed in the numerical example in Section V-A, where we compute the minimal stabilizing horizon length for each case (cf. Table II in Section V-A).

As a side remark, we note that a direct consequence of the choices made in the proof of [17, Prop. 3.15] is that generally no better contraction rate than \( \mu = \sqrt{3}/4 \approx 0.93 \) and, hence, no smaller horizon length than \( M = 10 \) can be obtained using [17, Th. 4.2], even in the case of \( \eta = 0 \) in (4b) and under the ideal setup considered here.

Overall, the proposed MHE framework employs a practical (fading memory) least-squares cost function and provides theoretical stability and robustness guarantees that improve as the horizon length \( M \) increases. In addition, the arguments directly extend to the nonexponential case if FIE is used. In the next section, we establish a simple condition to compute a quadratically bounded \( \delta \)-IOSS Lyapunov function according to Assumption 1 so that an MHE design with guaranteed robust exponential stability is directly obtained.

IV. SYNTHESIS OF \( \delta \)-IOSS LYAPUNOV FUNCTIONS BASED ON DIFFERENTIAL DYNAMICS

In the following, we provide a constructive and systematic approach to derive \( \delta \)-IOSS Lyapunov functions for system (1) based on its differential dynamics. This results in simple matrix inequality conditions involving the Jacobians of \( f \) and \( h \), which can be efficiently verified using standard tools, such as sum-of-squares (SOS) optimization or linear parameter-varying (LPV) embeddings. In combination with Section III, we hence provide a sufficient condition that directly yields an MHE scheme that is guaranteed to be RGES.

The differential analysis of nonlinear systems plays an important role as simple and intuitive tools from linear control theory become applicable; for example, by shifting the analysis of convergence between arbitrary system trajectories to the study of the linearizations along each trajectory, see, e.g., [34]. Global properties can then be inferred by using tools from differential geometry—typically based on a suitable Riemannian (or Finsler) metric under which the differential displacements decrease with the flow of the system. This universal concept offers wide applicability in the analysis (cf. [35], [36], and [37]) and control (cf. [38], [39], and [40]) of nonlinear systems. In addition, as pointed out in [34], it is also naturally suitable for characterizing the convergence of observers (see, e.g., [1], [41], and [42]), and consequently, also for characterizing \( \delta \)-IOSS.

To this end, throughout the following, we assume that \( f \) and \( h \) in (1) are continuously differentiable. The corresponding linearizations at a given point \((x, u, w) \in \mathbb{X} \times \mathbb{U} \times \mathbb{W}\) are then given by

\[
A = \frac{\partial f}{\partial x}(x, u, w), \quad B = \frac{\partial f}{\partial w}(x, u, w), \\
C = \frac{\partial h}{\partial x}(x, u, w), \quad D = \frac{\partial h}{\partial w}(x, u, w).
\]

TABLE I

| Result | \( C \) | \( \mu \) |
|--------|--------|--------|
| Proposed (16) | \(4\lambda_{\max}(P_2, P_1)\) | \(\eta\) |
| [17, Thm. 4.2] | \(\sqrt{\epsilon_2}/\epsilon_1\) | \(\eta\) |
| [18, Thm. 14] | \(8\epsilon_1/\epsilon_1\) | \(\eta\) |
| [4, Thm. 1] | \(3\sqrt{\epsilon_2}/\epsilon_1\) | \(\sqrt{\eta}\) |

Now consider an arbitrary change of coordinates \( \bar{x} = \phi(x) \) with \( \phi : \mathbb{R}^n \to \mathbb{R}^n \), which results in the equivalent system dynamics

\[
\bar{f}(\bar{x}, u, w) := f(\phi^{-1}(\bar{x}), u, w), \quad (28a)
\]
\[
\bar{h}(\bar{x}, u, w) := h(\phi^{-1}(\bar{x}), u, w). \quad (28b)
\]

Assumption 2 (Coordinate transformation): There exists a diffeomorphism \( \phi : \mathbb{R}^n \to \mathbb{R}^n \) such that \( \bar{h} \) is affine in \((\bar{x}, w)\), and it is affine in \((\bar{x}, w)\), and \( \partial \bar{h}/\partial \bar{x}_i = 0 \) for all \( i = 1, \ldots, n-p \).

Provided that Assumption 2 holds, the transformed dynamics (28) are such that the output \( \bar{h} \) depends affinely on a subset of the system state \( \bar{x} \), which is similar to the class of systems considered in [1]. Note that this is a fairly general setup covering several observability normal forms and, therefore, many physical models that admit a corresponding transformation, cf. [1, Remark 1] and compare also [43, Sec. 5.1] for further details. Moreover, as we show in the following remark, the design of \( \phi \) is particularly simple when a linear combination of the state is measured, which is the case in many practical applications, compare also the example systems in Section V.

Remark 5 (Coordinate transformation): In case the output function (1b) is given by \( h(x) = Cx \) for some \( C \), Assumption 2 can be trivially satisfied using a linear change of coordinates \( \bar{x} = \phi(x) = Tx \) with \( T \) being a suitable nonsingular transformation matrix. In particular, if \( T \) is chosen such that \( h(x) = Cx = CT^{-1}x = (0, C) \cdot \bar{x} = \bar{h} \), then it immediately follows that \( h \) is linear (and thus affine) in \( \bar{x} \) with \( \partial \bar{h}/\partial \bar{x}_i = 0 \) for all \( i = 1, \ldots, n-p \).

We partition the state \( \bar{x} \) into two parts \( \bar{x} = (\bar{x}_x, \bar{x}_y) \) with \( \bar{x}_x \in \mathbb{R}^{n-p} \) and \( \bar{x}_y \in \mathbb{R}^p \). Then, the following theorem yields a quadratically bounded \( \delta \)-IOSS Lyapunov function according to Assumption 1.

Theorem 2 (\( \delta \)-IOSS Lyapunov function): Let Assumption 2 hold and \( P : \mathbb{R}^n \to \mathbb{R}^{n \times m} \) be such that

\[
P(x) = \frac{\partial \phi}{\partial x}(x) \bar{P}(\phi(x)) \frac{\partial \phi}{\partial x}(x)
\]

with

\[
\bar{P}(\bar{x}) = \begin{pmatrix}
\bar{P}_x(\bar{x}_x) & 0 \\
0 & \bar{P}_y
\end{pmatrix}
\]

for some \( \bar{P}_x : \mathbb{R}^{n-p} \to \mathbb{R}^{(n-p) \times (n-p)} \) and \( \bar{P}_y \in \mathbb{R}^{p \times p} \). Let \( \mathbb{X} \) be weakly geodesically convex, and \( \mathbb{W} \) be convex. If there

\[10\]Geodesic convexity is a natural generalization of convexity for sets to Riemannian manifolds, which reduces to convexity for the special case of constant metrics. For a formal definition, see, e.g., [41, Def. 2.6].
exist \( \eta \in [0, 1) \) and symmetric matrices \( P_1, P_2 \geq 0 \) and \( Q, R \geq 0 \) such that
\[
\begin{pmatrix}
A^T P_1 A - \eta P - C^T R C & A^T P_1 B - C^T R D \\
B^T P_1 A - D^T R C & B^T P_1 B - Q - D^T R D
\end{pmatrix} \preceq 0
\]
and
\[
P_1 \preceq P(x) \preceq P_2
\]
hold for all \((x, u, w) \in \mathbb{X} \times \mathbb{U} \times \mathbb{W}\) with \( P_+ = P(x^+) \), then there exists a quadratically bounded \( \delta \)-IOSS Lyapunov function \( W_\delta \) that satisfies Assumption 1.

The proof of Theorem 2 employs several properties and arguments from Riemannian geometry and can be found in Appendix A.

We point out that for a fixed transformation \( \phi \), conditions (31) and (32) reduce to linear constraints that need to be verified over the full domain \( \mathbb{X} \times \mathbb{U} \times \mathbb{W} \). Computationally tractable sufficient conditions in terms of LMIs can then be obtained by using, e.g., LPV embeddings (see, e.g., [44]) or SOS relaxations, cf. [45] and [46]. In case \( \phi \) is treated as a decision variable (which may be less restrictive due to this additional degree of freedom), the conditions of Theorem 2 can be reformulated as a convex optimization problem in a similar manner as in [1].

The following corollary of Theorem 2 provides even simpler conditions for the case where \( h \) in (1b) is affine in \((x, w)\) and we restrict ourselves to a quadratic \( \delta \)-IOSS Lyapunov function.

**Corollary 3 (Quadratic \( \delta \)-IOSS Lyapunov function):** Let the output function \( h \in (1b) \) be affine in \((x, w)\) and let \( \mathbb{X} \) and \( \mathbb{W} \) be convex. If there exist \( \eta \in [0, 1) \) and symmetric matrices \( P > 0 \) and \( Q, R \succeq 0 \) such that
\[
\begin{pmatrix}
A^T P A - \eta P - C^T R C & A^T P B - C^T R D \\
B^T P A - D^T R C & B^T P B - Q - D^T R D
\end{pmatrix} \preceq 0
\]
holds for all \((x, u, w) \in \mathbb{X} \times \mathbb{U} \times \mathbb{W}\), then \( W(x, \tilde{x}) = \|x - \tilde{x}\|_P^2 \) is a \( \delta \)-IOSS Lyapunov function and satisfies Assumption 1 with \( P_1 = P_2 = P \).

The proof of Corollary 1 is provided in Appendix B. Some remarks are in order.

**Remark 6 (Relation to dissipativity):** The proof of Theorem 2 introduces a differential version of IOSS (compare Claim 1 in Appendix A for further details). This characterization is equivalent to the notion of differential (Q,S,R)-dissipativity [47, Def. 3] with \( S = 0 \), compare also [36] and [44]. However, as pointed out in [47, Remark 7], the corresponding works crucially rely on \( R \succeq 0 \) in order to derive incremental results by simply exhausting the Cauchy–Schwarz inequality, cf. [44, Lemma 16], compare also [38, Th. 1] and [46, Th. 2.4]. Note that in our case, this would restrict the results to open-loop stable systems (since (4b) would need to hold with \( R = 0 \), which directly implies \( \delta \)-ISS of system (1)). Moreover, this would result in the cost function (6) not being positive definite, which generally can lead to an ill-defined optimization problem (7). In contrast, we circumvent this technical condition by suitably relating the state and output manifolds as it was similarly done in [1] and [2] for observer design. More specifically, from Assumption 2, i.e., by imposing the existence of coordinates \( \tilde{x} \) in which the output function \( \tilde{h} \) is affine (which directly implies that \( \tilde{h} \) is totally geodesic by assumption, cf. [48]), and due to our choice of the metric \( \tilde{P}(\tilde{x}) \) according to Theorem 2 (or Corollary 3), we immediately obtain an equality relation between the integral of the differential supply rates and the incremental supply rates, compare (42) and (47) in Appendix A for details. Consequently, as a side result, we note that Theorem 2 (and Corollary 3) with \( \eta = 1 \) can be used to verify incremental dissipativity of system (1) subject to a positive definite supply rate, relaxing [47, Remark 7].

**Remark 7 (Extensions):** To further generalize the parameterization of \( \tilde{P}(\tilde{x}) \) with respect to \( \tilde{x} \), we note that the following minor extension of Theorem 2 is possible if, e.g., \( \tilde{h}(\tilde{x}) = \tilde{x}_y \) (neglecting \( u \) and \( w \) for ease of presentation). We could choose
\[
\begin{pmatrix}
P_x(\tilde{x}_x) & 0 \\
0 & P_y(\tilde{x}_y)
\end{pmatrix}
\]
with \( P_{y,1} \preceq P_y(\tilde{x}_y) \preceq P_{y,2} \) uniformly for all possible \( \tilde{x}_y \) and some constant matrices \( P_{y,1}, P_{y,2} \succeq 0 \), i.e., \( P \in \mathbb{P} \) in (30) with an additional dependency of \( P_y \) on \( \tilde{x}_y \). Then, by additionally imposing \( R \preceq P_y(\tilde{x}_y) \) in (31), one can derive a similar \( \delta \)-IOSS Lyapunov function as provided by Theorem 2 that satisfies Assumption 1; the technical details can be found in Remark 10 in Appendix A. Finally, we note that one may relax Assumption 2, i.e., affinity of \( \tilde{h} \), by imposing that \( \tilde{h} \) is a Riemannian submersion, cf. [2].

**Remark 8 (Closed-form expression):** Note that Theorem 2 yields only an implicit \( \delta \)-IOSS Lyapunov function \( W_\delta \), which is due to the fact that we have no analytical closed-form expression for the Riemannian energy of the minimizing geodesic, compare Appendix A for further details. However, note also that this is not needed for the particular MHE (or FIE) scheme, since we only require knowledge of the matrices \( P_2, Q, R \) and \( R \) to design the cost functions (6) and (22), and additionally \( P_1 \) to compute the minimal horizon length for guaranteed RGES of MHE, cf. Section III. Similar considerations apply if Theorem 2 is used to compute the KL-functions of the standard \( \delta \)-IOSS bound. Again, one only needs to know the matrices \( P_1, P_2, Q, R \) and \( R \) and use (4a) after repeated application of the dissipation inequality (4b) to obtain the desired result. If, nevertheless, an analytical expression for the \( \delta \)-IOSS Lyapunov function \( W_\delta \) is desired, Corollary 3 can be used to obtain a quadratic function.

**Remark 9 (Alternative derivation):** An alternative way to compute a quadratic \( \delta \)-IOSS Lyapunov function is to first design an RGES observer based on, e.g., [1], [3], and [42]. Then, under certain conditions, one can show that the corresponding Lyapunov function also serves as a \( \delta \)-IOSS Lyapunov function, cf. [21, Prop. 4], and compare also [10, Sec. VII]. However, these sufficient conditions are crucially limited to quadratic Lyapunov functions and additive disturbances in the dynamics (1a), and hence are only applicable to a smaller class of detectable systems (in comparison to Theorem 2).

**V. NUMERICAL EXAMPLES**

In order to illustrate our results, we apply the proposed methods to two examples from the literature: a chemical reactor...
process (cf. Section V-A) and a 12-state quadrotor model with flexible rotor blades (cf. Section V-B). The simulations were performed in MATLAB using CasADi [49] and the NLP solver IPOPT [50]; LMIs were verified using YALMIP [51] and the semidefinite programming solver MOSEK [52].

Overall, these examples demonstrate the practicability of the offline $\delta$-IOSS verification (Theorem 2), the (significantly) shorter horizon bounds obtained through Theorem 1 compared with the literature (cf. Table II in the following), and the applicability of the proposed MHE framework—in particular, its ability to provide valid theoretical guarantees under practical conditions.

### A. Two-State Chemical Reaction

We consider the following system:

\[
\begin{align*}
    x_1^+ &= x_1 + t_\Delta (-2k_1x_1^2 + 2k_2x_2) + w_1, \\
    x_2^+ &= x_2 + t_\Delta (k_1x_1^2 - k_3x_2) + w_2, \\
    y &= x_1 + x_2 + w_3
\end{align*}
\]

with $k_1 = 0.16, k_2 = 0.0064$, and sampling time $t_\Delta = 0.1$. This corresponds to the chemical reaction $2A \rightarrow B$ taking place in a constant-volume batch reactor from [25, Sec. 5] using an Euler discretization and with additional disturbances $w \in \mathbb{R}^3$. In the following, we treat them as uniformly distributed random variables satisfying $|w_i| \leq 10^{-3}, i = 1, 2$ for the process disturbances and $|w_3| \leq 0.1$ for the measurement noise. As in [25, Sec. 5], we consider $x_0 = [3, 1]^\top$ and the poor initial estimate $\hat{x}_0 = [0.1, 4.5]^\top$. This setup poses a challenge for state estimators; in fact, simple estimators, such as the standard extended Kalman filter (EKF), fail to provide meaningful results, compare the simulation results in Fig. 1. This example is also frequently used in the related MHE literature (e.g., [5, Example 4.38]); however, $\delta$-IOSS has never been certified.

To this end, we assume that the prior knowledge $X = [0.1, 4.5] \times [0.1, 4.5]$ is available, which follows from the physical nature of the system under the abovementioned conditions (in particular, the initial conditions and boundedness of $\mathbb{W}$), compare also the simulation results in Fig. 1. For the considered system, we can even apply Corollary 3 in combination with SOS optimization to compute a quadratic Lyapunov Function $W_0 = \|x - \tilde{x}\|^2_2$ that satisfies Assumption 1 with

\[
P = \begin{bmatrix} 4.539 & 4.171 \\ 4.171 & 3.834 \end{bmatrix}, \quad Q = \begin{bmatrix} 10^3 & 0 & 0 \\ 0 & 10^4 & 0 \\ 0 & 0 & 10^5 \end{bmatrix}, \quad R = 10^3,
\]

and the decay rate $\eta = 0.91$. We point out that, to the best of the authors’ knowledge, this is the first time that $\delta$-IOSS has been explicitly verified for this example. It is also worth noting that the lack of such a method in the literature was generally considered a major problem in [17], since $\delta$-IOSS became a standard detectability assumption in the recent nonlinear MHE literature, compare Section III-D. Theorem 2 provides a useful tool to actually verify this crucial property in practice.

Based on the abovementioned $\delta$-IOSS Lyapunov function, we can now compute the minimum horizon length $M_{\text{min}}$ sufficient for robust stability of MHE according to condition (16) and Remark 3, and compare it to corresponding bounds from the recent nonlinear MHE literature, i.e., the Lyapunov-like function framework [17], MHE with general time-discounting [18], and without time-discounting [4], by resolving the respective conditions in Table I. As can be seen from Table II, the proposed Lyapunov approach yields a minimum horizon length that is (at least) one order of magnitude better (i.e., smaller) than those obtained from the literature.

For the following simulation, we choose $M = 30 > M_{\text{min}}$ to provide a small estimation error bound. The simulation results are depicted in Fig. 1, which shows robustly stable estimation as guaranteed by Theorem 1 due to satisfaction of condition (16). In order to compare the results, we also simulated the EKF. As can be seen in Fig. 1, however, the corresponding estimates exhibit a serious error compared with MHE, which is partly due to the fact that the physical constraints were not met. In summary, the overall simulation results are similar to [25, Sec. 5] and [5, Example 4.38], but with valid robustness guarantees for MHE.

### B. 12-State Quadrotor Model

We adapt the example from [53] and consider a quadrotor model involving four rotors with flexible blades. Let $\mathcal{I}$ denote the stationary inertial system with its vertical component pointing into the Earth, where position and velocity of the quadrotor are represented by $z = [z_1, z_2, z_3]^\top$ and $v = [v_1, v_2, v_3]^\top$, respectively. By $\mathcal{B}$, we denote the body-fixed frame attached to the
quadrotor, with the third component pointing in the opposite direction of thrust generation. The attitude of $B$ with respect to $I$ is captured by a rotation matrix $R$ (where we use $xyz$-convention), which involves the roll, pitch, and yaw angles of the quadrotor represented by $\xi = [\phi, \theta, \psi]^T$. The angular velocity of the quadrotor in $B$ with respect to $I$ is given by $\Omega = [\Omega_1, \Omega_2, \Omega_3]^T$. Assuming a wind-free environment, the overall dynamics can be described as

$$\dot{z} = v, \quad m\dot{v} = mgc_3 - TR(\xi)e_3 - R(\xi)B\Omega,$$

$$\dot{\xi} = \Gamma(\xi)\Omega, \quad J\dot{\Omega} = -\Omega^T J\Omega + \tau - D\Omega,$$

where $e_3 = [0, 0, 1]^T$ and $(\cdot)^T$ refers to the skew symmetric matrix associated with the cross product such that $u^Tv = u \times v$ for any $u, v \in \mathbb{R}^3$.

The thrust $T \in \mathbb{R}$ and the torque $\tau \in \mathbb{R}^3$ are generated by the four rotors by means of their angular velocities $\omega_i$ via

$$[T] = \begin{bmatrix} c_T & c_T & c_T & c_T \\ -lc_T & 0 & lc_T & 0 \\ lc_T & 0 & -lc_T & 0 \\ -cQ & cQ & -cQ & cQ \end{bmatrix} \begin{bmatrix} \omega_1^T \\ \omega_2^T \\ \omega_3^T \\ \omega_4^T \end{bmatrix},$$

and the matrix $\Gamma$ is defined as

$$\Gamma(\xi) = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{bmatrix},$$

compare [53] and [54] for further details on the model and its derivation. The parameters are chosen as $m = 1.9$, $J = \text{diag}(5.9, 5.9, 10.7) \cdot 10^{-3}$, $g = 9.8$, $l = 0.25$, $c_T = 10^{-5}$, $c_Q = 10^{-6}$, $B = 1.14 \cdot c_3$, and $D = 0.0297 \cdot c_3 e_3^T$. In summary, the overall model has the states $x = [z^T v^T \Omega^T]^T \in \mathbb{R}^{12}$ and the inputs $u = [\omega_1 \omega_2 \omega_3 \omega_4]^T \in \mathbb{R}^4$. We additionally assume that the dynamics of $\dot{x}_i$ is corrupted by an additive disturbance $d_i$, $i \in \{1, 12\}$, and that only noisy position and orientation measurements $y = [z^T \xi^T]^T + v$ with noise $v \in \mathbb{R}^6$ are available. In the following, we consider $d$ and $v$ uniformly distributed such that $|d_i| \leq 10^{-3}$, $i \in \{1, 12\}$, and $|v_i| \leq 0.1$, $i \in \{1, 6\}$, and define $w = [d^T v^T]^T \in \mathbb{R}^{18}$. The discrete-time model (1) is then obtained via Euler-discretization using the sampling time $t_\Delta = 0.05$.

We assume that some input/output sequences $(u, y)$ have been measured while performing a certain control scenario of the quadrotor that guarantees $x \in X = \{x : |x_i| \leq \pi/6, |\Omega_i| \leq 1, i \in \{1, 12\}\}$ and $u \in U = \{u : |u_i| \leq 1500, i \in \{1, 4\}\}$; the objective is to reconstruct the corresponding state trajectory using the proposed MHE framework. To this end, we verify condition (33) on $X \times U$ by suitably gridding the state space and, thus, compute a quadratic $\delta$-IOSS Lyapunov function with the decay rate $\eta = 0.87$. Choosing the horizon length $M = 30$ satisfies condition (16), so that the proposed MHE design (6) and (7) is guaranteed to be RGES according to Theorem 1.

Fig. 2 shows the real, measured, and estimated positions of the quadrotor (in the frame $I$) to illustrate the maneuver flown. The overall estimation error in Lyapunov coordinates is depicted in Fig. 3 and illustrates exponential convergence to a neighborhood around the origin, as guaranteed by Theorem 1.

**VI. Conclusion**

In this article, we have presented a novel robust stability analysis for MHE using Lyapunov functions. The analysis generally applies to nonlinear exponentially detectable ($\delta$-IOSS) systems admitting a corresponding $\delta$-IOSS Lyapunov function. Considering an MHE formulation with time-discounted quadratic objective, we have shown that an $M$-step Lyapunov function naturally arises, which directly implies robust exponential stability of MHE provided that the horizon length $M$ satisfies a posed lower bound.

The main feature of the proposed analysis is that, in contrast to most of the MHE literature, we argue entirely in Lyapunov coordinates; this is beneficial in several respects: First, tuning the MHE objective by suitably relating it to $\delta$-IOSS in order to achieve good theoretical guarantees (which typically yields general $KL$-function inequalities in the literature) becomes easy and intuitive, even when we show robust asymptotic stability for FIE; second, the proposed Lyapunov analysis generally allows for less conservative (i.e., shorter) horizon bounds compared with recently proposed MHE designs.

Nonlinear detectability ($\delta$-IOSS) is a common detectability assumption in most of the existing recent results on nonlinear MHE; however, there was no systematic method of verifying this.

**Fig. 2.** Comparison of the estimated (blue), true (red), and measured (green) positions of the quadrotor.

**Fig. 3.** Estimation error of the quadrotor in Lyapunov coordinates.
crucial condition so far, which was also considered a major problem in [17] to establish guarantees for MHE beyond conceptual nature. We were able to solve this issue by providing a systematic tool to verify δ-IOSS for a large class of nonlinear detectable systems based on their differential dynamics. The sufficient conditions were stated in terms of simple matrix inequalities that can be efficiently verified using, e.g., SOS optimization or LPV embeddings. In combination, these conditions directly yield an MHE design with guaranteed robust exponential stability.

The applicability of the overall framework was illustrated with two examples from the literature: a standard MHE benchmark example where we verified δ-IOSS for the first time, and a nonlinear 12-state quadrotor model. In the end, we were able to achieve guaranteed robustly stable estimation under practical conditions enabled by a significantly shorter bound on the horizon length compared with the literature.

An interesting question for future research is under which conditions the MHE stability analysis proposed in Section III-B are also applicable in the case of a (relaxed) asymptotic δ-IOSS condition or an objective without time-discounting. Scalability of the approach to verify the underlying δ-IOSS condition as presented in Section IV to higher dimensional systems is mainly limited by the tools applied to verify the underlying matrix inequalities.

APPENDIX

A. Technical Details of Theorem 2

In the following, we provide further technical details of Theorem 2, including the proof itself and the modifications required by Remark 7.

Proof of Theorem 2: The proof consists of three parts. First, we establish the dissipation inequality (4b) and then derive the bounds (4a), where we initially assume that the conditions (31) and (32) hold globally on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q$. Finally, we show that the corresponding results also hold if the conditions are enforced on the subset $X \times U \times W$ only.

Part 1: Consider two arbitrary points $(x, u, w, y)$ and $(\tilde{x}, u, \tilde{w}, \tilde{y})$ each of which is an element of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^p$. Define a smooth path $c : [0, 1] \to \mathbb{R}^p$ parameterized by $s$ joining $x$ to $\tilde{x}$ with $c(0) = x$ and $c(1) = \tilde{x}$. Define the smooth path of disturbances $\omega(s)$ joining $\omega(0) = w$ and $\omega(1) = \tilde{w}$ by the straight line

$$\omega(s) = w + s(\tilde{w} - w), \quad s \in [0, 1].$$

Note that this particular choice is valid since this disturbance can generally be treated as an external variable that does not depend on any dynamics; therefore, the path connecting $w$ and $\tilde{w}$ can be of arbitrary form. Given the tuple $(c(s), u, \omega(s))$, we can apply the dynamics (1) and obtain

$$c^+(s) = f(c(s), u, \omega(s)), \quad (36a)$$
$$\zeta(s) = h(c(s), u, \omega(s)), \quad (36b)$$

where the corresponding output $\zeta$ yields a smooth path joining $\zeta(0) = y$ and $\zeta(1) = \tilde{y}$. By differentiating (36) with respect to $s \in [0, 1]$, from the chain rule and the linearizations (27), we obtain the differential dynamics

$$\delta_y = C(c(s), u, \omega(s)) \delta_x + D(c(s), u, \omega(s)) \delta_w, \quad (37b)$$

where the path derivatives are defined as $\delta^+_x := dc^+/ds(s)$, $\delta_x := dc/ds(s)$, $\delta_w := dw/ds(s)$, and $\delta_y := dc/ds(s)$. Formally, each $\delta_i$ with $i \in \{x, w, y\}$ denotes a vector on the tangent space of the domain of $i$ at $i$, cf. [38] and [39]. We make the following claim.

Claim 1: Let (31) hold for some $(x, u, w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q$. Then, $V(x, \delta_x) = \delta_x^T P(x) \delta_x$ satisfies

$$V(x^+, \delta^+_x) \leq \eta V(x, \delta_x) + \|\delta_w\|_Q^2 + \|\delta_y\|_R^2. \quad (38)$$

Proof: By applying the definition of $V$ together with the differential dynamics (37a) and (37b) to (38), we obtain

$$\begin{pmatrix} \delta_x^T \\ \delta_w^T \end{pmatrix} \begin{pmatrix} A^T P_x A - \eta P & A^T P_x B \\ B^T P_x A & B^T P_x B \end{pmatrix} \begin{pmatrix} \delta_x \\ \delta_w \end{pmatrix} \leq \begin{pmatrix} \delta_x^T \\ \delta_w^T \end{pmatrix} \begin{pmatrix} C^T R C & C^T R D \\ D^T R C & Q + D^T R D \end{pmatrix} \begin{pmatrix} \delta_x \\ \delta_w \end{pmatrix},$$

which clearly is equivalent to (31).

Consequently, by the definition of the differential storage function $V$ and the path derivatives $(\delta_x, \delta_w, \delta_y)$, from (38) it follows that

$$\left\| \frac{dc^+}{ds}(s) \right\|^2_{P_x} \leq \eta \left\| \frac{dc}{ds}(s) \right\|^2_p + \left\| \frac{dw}{ds}(s) \right\|^2_q + \left\| \frac{dc}{ds}(s) \right\|^2_R. \quad (39)$$

This differential property can now be transformed into an incremental property by integration over $s \in [0, 1]$ and utilizing tools from Riemannian geometry. In particular, we treat $P$ as a Riemannian metric with which the manifold $\mathbb{R}^n$ is endowed. Let

$$E(c) := \int_0^1 \frac{dc}{ds}(s)^T P(c(s)) \frac{dc}{ds}(s) ds \quad (40)$$

denote the Riemannian energy associated with the path $c$. The minimizer of $E(c)$ over all possible smooth paths joining $c(0)$ to $c(1)$ is given by a (maybe nonunique) geodesic $\gamma$, existence of which is ensured by the uniformly boundedness of $P$ in (32), cf. [39, Lemma 1], compare also [41, Lemma A.1].

Now, consider (39) and choose $c = \gamma$; it, hence, follows that $c^+(s) = f(\gamma(s), u, \omega(s))$ by (36a). Let $\gamma^+$ denote the geodesic at the subsequent time instance joining the successor states $x^+$ and $\tilde{x}^+$, where we point out that in general $c^+ \neq \gamma^+$. However, by the integration of (39) over $s \in [0, 1]$ and the definition of the Riemannian energy (40), we obtain

$$E(\gamma^+) \leq E(c^+),$$

$$\leq \eta E(\gamma) + \int_0^1 \left\| \frac{dw}{ds}(s) \right\|^2_Q ds + \int_0^1 \left\| \frac{dc}{ds}(s) \right\|^2_R ds, \quad (41)$$

where the first inequality used the fact that $c^+$ is a feasible candidate curve providing an upper bound for the (minimal) energy $E(\gamma^+)$. In the following, we show that $W_{\delta i}(x, \tilde{x}) = E(\gamma)$ satisfies the dissipation inequality (4b). First, exploiting our
particular choice of \( \omega \) in (35) yields
\[
\int_0^1 \left\| \frac{d\omega}{ds}(s) \right\|^2_Q ds = \int_0^1 \left\| \ddot{w} - w \right\|^2_Q ds = \left\| \ddot{w} - w \right\|^2_Q. \tag{42}
\]
Now we focus on the output term in (41) and make the following claim.

Claim 2: The derivative \( d\zeta/ds(s) \) is constant in \( s \in [0, 1] \).

Proof: Given \( \gamma \) and \( \phi \), we can define the geodesic in transformed coordinates \( \tilde{\gamma} := \phi(\gamma) \). From (36b), we have that
\[
\tilde{\zeta}(s) = \tilde{h} (\tilde{\zeta}^{-1}(\tilde{\gamma}(s)), u, \omega(s)) = \ddot{h} (\tilde{\gamma}(s), u, \omega(s)) \tag{43}
\]
for all \( s \in [0, 1] \). Taking the derivative of (43) with respect to \( s \in [0, 1] \) using the chain rule yields
\[
\frac{d\tilde{\zeta}}{ds} = \frac{d\tilde{h}}{d\tilde{\zeta}} (\tilde{\gamma}(s), u, \omega(s)) \frac{d\tilde{\zeta}}{ds} = \frac{\partial h}{\partial \tilde{\zeta}} (\tilde{\gamma}(s), u, \omega(s)) \frac{d\tilde{\zeta}}{ds}. \tag{44}
\]
for all \( i = 1, \ldots, n \). Since, in addition, \( d\omega/ds(s) \) is constant in \( s \in [0, 1] \) due to (35), it remains to show that this is also the case for \( d\tilde{\zeta}/ds(s) \) for all \( i = n - p + 1, \ldots, n \).

To this end, recall that \( \tilde{\zeta} = \phi(\gamma) \). Hence, by the chain rule
\[
\frac{d\tilde{\zeta}}{ds} = \frac{d\phi}{dx}(\gamma(s)) \left. \frac{d\gamma}{ds} \right|_{s}. \tag{45}
\]
Due to our choice of \( P \) in (29), it, therefore, holds that
\[
\tilde{E}(\tilde{\gamma}) := \int_0^1 \left( \frac{d\gamma}{ds}(s) \right)^\top \tilde{P}(\gamma(s)) \left( \frac{d\gamma}{ds}(s) \right) ds = \int_0^1 \frac{d\gamma}{ds}(s)^\top P(\gamma(s)) \frac{d\gamma}{ds}(s) ds = E(\gamma). \tag{46}
\]
Thus, given a minimizing geodesic \( \gamma \) for \( E(\gamma) \), the curve \( \tilde{\gamma} \) is a minimizing geodesic for \( \tilde{E}(\tilde{\gamma}) \) (by contradiction). Consequently, we have that \( \tilde{\gamma}(s) = \phi(\gamma(s)) \) is a solution to the geodesic equation [55, Def. 2.77], i.e., to the differential system
\[
\frac{d^2\tilde{\gamma}_k}{ds^2}(s) - \sum_{i,j} \tilde{\Gamma}_{i,j}^k (\gamma(s)) \frac{d\tilde{\gamma}_i}{ds}(s) \frac{d\tilde{\gamma}_j}{ds}(s) = 0, \quad k = 1, \ldots, n. \tag{47}
\]
The objects \( \tilde{\Gamma}_{i,j}^k \) represent the Christoffel symbols associated with the metric \( \tilde{P} \), which are, following [55, Prop. 2.54] and [2, Appendix A1.1], defined by
\[
\tilde{\Gamma}_{i,j}^k (\tilde{x}) = \frac{1}{2} \sum_{a=1}^k \tilde{Y}_{k,a}(\tilde{x}) \left( \frac{\partial \tilde{P}_{a,i}}{\partial \tilde{x}_j}(\tilde{x}) + \frac{\partial \tilde{P}_{a,j}}{\partial \tilde{x}_i}(\tilde{x}) - \frac{\partial \tilde{P}_{i,j}}{\partial \tilde{x}_a}(\tilde{x}) \right) \tag{48}
\]
with the shorthand notation \( \tilde{Y}(\tilde{x}) = \tilde{P}(\tilde{x})^{-1} \) and \( \tilde{Y}_{k,a} \) the \((k, a)\)-element of \( \tilde{Y} \). Note that in (44), we are only interested in the states of the geodesic \( \tilde{\gamma} \) that appear in the output (43), i.e., \( \tilde{\gamma}_k \) for all \( k = n - p + 1, \ldots, n \). For ease of notation, let us define \( r := n - p + 1 \) for the rest of this proof. Calculating the respective Christoffel symbols reveals that
\[
\tilde{\Gamma}_{i,j}^k = 0, \quad k = r, \ldots, n. \tag{49}
\]
which is a direct consequence of the proposed block-diagonal structure of \( \tilde{P} \) in (30); to see this, note the following: First, the fact that \( \tilde{P} \) is block-diagonal implies that also \( \tilde{Y} = \tilde{P}^{-1} \) is block-diagonal, and thus \( \tilde{Y}_{i,j} = \tilde{P}_{i,j} = 0 \) for \( i < r \) and \( j \geq r \) (and vice versa); second, each derivative \( \partial \tilde{P}_{i,j}/\partial x_a \) is zero if \( a \geq r \) since \( \tilde{P} \) is independent of \( x \); third, each derivative \( \partial \tilde{P}_{a,i}/\partial x_j \) is zero if \( a, i \geq r \) and \( j < r \) since \( \tilde{F}_r \) is constant.

Consequently, from (46), we have that all the Christoffel symbols affecting the states \( \tilde{\gamma}_i, i = r, \ldots, n \), vanish, and hence, our special choice of \( \tilde{P} \) leads to a decoupling of the geodesic equation (44); in particular, we obtain the simple second-order homogeneous differential equation
\[
\frac{d^2\tilde{\gamma}_k}{ds^2}(s) = 0, \quad k = r, \ldots, n, \tag{50}
\]
which directly implies that \( d^2\tilde{\gamma}_i/ds^2(s) \) is constant in \( s \in [0, 1] \) for all \( i = r, \ldots, n \) and, hence, yields the desired result.

Consequently, the output term in (41) consists only of terms constant in \( s \in [0, 1] \). Hence, by the fundamental theorem of calculus, we obtain
\[
\int_0^1 \left\| \frac{d\zeta}{ds}(s) \right\|^2_R ds = \left( \zeta(1) - \zeta(0) \right)^\top R \left( \zeta(1) - \zeta(0) \right) = \left\| \tilde{y} - y \right\|^2_R. \tag{51}
\]
Applying (42) and (47) to (41) then yields
\[
E(\gamma^+) \leq \eta E(\gamma) + \left\| w - \bar{w} \right\|^2_Q + \left\| y - \bar{y} \right\|^2_R, \tag{52}
\]
which establishes the dissipation inequality (4b) with \( W(x, \dot{x}) = E(\gamma) \).

Part II: We now show satisfaction of (4a) and start with the upper bound. Note that since \( \gamma \) is the path of minimum energy joining \( x \) to \( \tilde{x} \), every other path yields a higher amount of energy, which clearly also applies to the straight line \( l(s) = x + s(\tilde{x} - x) \). Therefore,
\[
E(\gamma) \leq E(l) = \int_0^1 (x - \tilde{x})^\top P(l(s))(x - \tilde{x}) ds \leq \left\| x - \tilde{x} \right\|^2_{P_1}, \tag{53}
\]
where the last step follows from uniform boundedness of \( P \) (32). For the lower bound, again by uniform boundedness of \( P \), we have
\[
E(\gamma) \geq \int_0^1 \frac{\partial^2\gamma}{ds^2}(s) \tilde{P}_1 \frac{\partial^2\gamma}{ds^2}(s) ds \tag{54}
\]
where for the second inequality, we exploited the fact that the minimizer of the expression on the right-hand side of (50) is given by the straight line \( l \) since \( P_1 \) is constant. To verify this, recall that each minimizer of the Riemannian energy \( E \) solves the geodesic (44); now observe that all the Christoffel symbols (45) vanish if the underlying metric is constant. Therefore, (49) and (51) establish (4a). Together with Part I, we can, thus, conclude that \( W(x, \dot{x}) = E(\gamma) \) is a \( \delta \)-IOSS Lyapunov function satisfying (4a) and (4b) for all \( (x, u, u), (\tilde{x}, u, \tilde{u}, \tilde{y}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^p \).

Part III: Finally, we note that the results from Part I and Part II (including the Claims 1 and 2) can be easily restricted to any subset \( \tilde{\mathcal{X}} \times \tilde{\mathcal{U}} \times \tilde{\mathcal{W}} \times \tilde{\mathcal{Y}} \) if it is ensured that the minimizing geodesic...
connecting any two points on each of the subsets $\mathbb{X}$ and $\mathbb{W}$ stays in the respective subset for all $s \in [0, 1]$. This is indeed the case for $\mathbb{X}$ being weakly geodesically convex (cf. [41, Def. 2.6]) and $\mathbb{W}$ being convex (as long as $\omega$ is chosen according to (35)). Provided this applies, if conditions (31) and (32) are enforced on the subset $\mathbb{X} \times U \times \mathbb{W}$, we have that $W_1(x, \tilde{x}) = E(\gamma)$ is a quadratically bounded $\delta$-IOSS Lyapunov function satisfying Assumption 1 for all $(x, u, w, y), (\tilde{x}, u, \tilde{w}, \tilde{y}) \in \mathbb{X} \times U \times \mathbb{W} \times \mathbb{Y}$, which completes this proof.

We now discuss the modifications of the proof of Theorem 2 that are necessary to allow for the slightly more general metric $\bar{P}$ from Remark 7.

Remark 10: If $\bar{P}$ is chosen according to (34), i.e.,
\[
\bar{P}(\bar{x}) = \begin{pmatrix} \bar{P}_x(\bar{x}_x) & 0 \\ 0 & \bar{P}_y(\bar{x}_y) \end{pmatrix},
\]
we have that the geodesic $\bar{\gamma}$ minimizes the two independent functionals
\[
E(\bar{\gamma}) = \int_0^1 \frac{d^2y_\bar{\gamma}}{ds^2}(s) \bar{P}_y(\bar{\gamma}_y(s)) \, ds,
\]
\[
+ \int_0^1 \frac{d^2\bar{\gamma}_x}{ds^2}(s) \bar{P}_x(\bar{\gamma}_x(s)) \, ds.
\]
Note that a direct consequence of $\bar{P}_y(\bar{x}_y)$ not being constant is that Claim 2 does not hold in this case. However, since $R \leq \bar{P}_y(\bar{x}_y)$ by Remark 7, the output functional in (39) can be bounded by
\[
\int_0^1 \left\| \frac{d^2\bar{\gamma}_x}{ds^2}(s) \right\|^2_R \, ds \leq \int_0^1 \frac{d^2\bar{\gamma}_y}{ds^2}(s) \bar{P}_y(\bar{\gamma}_y(s)) \, ds,
\]
i.e., the same functional that also appears in (52) and, hence, is minimized by $\bar{\gamma}$. Then, by following similar arguments as in the second part of the proof of Theorem 2 (in particular, exploiting uniform boundedness of $\bar{P}$ according to Remark 7), one can show that $\int_0^1 \left\| \frac{d^2\bar{\gamma}}{ds^2}(s) \right\|^2_R \, ds \leq \left\| \bar{y} - \tilde{y} \right\|^2_{P_{x, y}}$, and subsequently derive a similar $\delta$-IOSS Lyapunov function as in Theorem 2 that satisfies Assumption 1.

B. Proof of Corollary 3

Proof: The result follows immediately by setting $\bar{x}_y = \bar{x} = \phi(x) = \bar{x}$ in the proof of Theorem 2. As a direct consequence, we obtain (48) with $E(\gamma) = \left\| x - \bar{x} \right\|_P^2$ and $E(\gamma^+) = \left\| x^+ - \bar{x}^+ \right\|_P^2$ since $P$ is constant (resulting in the geodesics being straight lines), which lets us conclude that $W_1(x, \tilde{x}) = \left\| x - \tilde{x} \right\|_P^2$ is a quadratic $\delta$-IOSS Lyapunov function that satisfies Assumption 1 with $P_1 = P_2 = P$ for all $(x, u, w, y), (\tilde{x}, u, \tilde{w}, \tilde{y}) \in \mathbb{X} \times U \times \mathbb{W} \times \mathbb{Y}$.

References

[1] B. Yi, R. Wang, and I. R. Manchester, “Reduced-order nonlinear observers via contraction analysis and convex optimization,” IEEE Trans. Autom. Control, vol. 67, no. 8, pp. 4045–4060, Aug. 2022.
[2] R. G. Sanfelice and L. Praly, “Convergence of nonlinear observers on $\mathbb{R}^n$ with a Riemannian metric (Part III),” 2021, arXiv:2102.08340.
[3] D. Astolfi, P. Bernard, R. Postoyan, and L. Marconi, “Constrained state estimation for nonlinear systems: A redesign approach based on convexity,” IEEE Trans. Autom. Control, vol. 67, no. 2, pp. 824–839, Feb. 2022.
[4] D. A. Allan and J. B. Rawlings, “Moving horizon estimation,” in Handbook of Model Predictive Control, S. V. Raković and W. S. Levine, Eds. Basel: Birkhäuser, 2019, pp. 99–124.
[5] J. B. Rawlings, D. Q. Mayne, and M. Diehl, Model Predictive Control: Theory, Computation, and Design, 2nd ed. Santa Barbara, CA, USA: Nob Hill Pub., LLC, 2020.
[6] H. Michalska and D. Q. Mayne, “Moving horizon observers and observer-based control,” IEEE Trans. Autom. Control, vol. 40, no. 6, pp. 995–1006, Jun. 1995.
[7] A. Alessandri, M. Baglietto, and G. Battistelli, “Moving-horizon state estimation for nonlinear discrete-time systems: New stability results and approximation schemes,” Automatica, vol. 44, no. 7, pp. 1753–1765, 2008.
[8] C. V. Rao, J. B. Rawlings, and D. Q. Mayne, “Constrained state estimation for nonlinear discrete-time systems: Stability and moving horizon approximations,” IEEE Trans. Autom. Control, vol. 48, no. 2, pp. 246–258, Feb. 2003.
[9] M. Gharbi, F. Bayer, and C. Ebenbauer, “Proximity moving horizon estimation for discrete-time nonlinear systems,” IEEE Control Syst. Lett., vol. 5, no. 6, pp. 2090–2095, Dec. 2021.
[10] J. D. Schiller and M. A. Müller, “Suboptimal nonlinear moving horizon estimation,” IEEE Trans. Autom. Control, vol. 68, no. 4, pp. 2199–2214, Apr. 2023.
[11] E. D. Sontag and Y. Wang, “Output-to-state detectability of nonlinear systems,” Syst. Control Lett., vol. 29, no. 5, pp. 279–290, 1997.
[12] J. B. Rawlings and L. Ji, “Optimization-based state estimation: Current status and some new results,” J. Process Control, vol. 22, pp. 1439–1444, 2012.
[13] M. A. Müller, “Nonlinear moving horizon estimation for systems with bounded disturbances,” in Proc. Amer. Control Conf., 2016, pp. 883–888.
[14] W. Hu, “Robust optimization of optimization-based state estimation,” 2017, arXiv:1702.01903v3.
[15] M. A. Müller, “Nonlinear moving horizon estimation in the presence of bounded disturbances,” Automatica, vol. 79, pp. 306–314, 2017.
[16] D. A. Allan and J. B. Rawlings, “A Lyapunov-like function for full information estimation,” in Proc. Amer. Control Conf., 2019, pp. 4497–4502.
[17] D. A. Allan and J. B. Rawlings, “Robust stability of full information estimation,” SIAM J. Control Optim., vol. 59, no. 5, pp. 3472–3497, 2021.
[18] S. Knüfer and M. A. Müller, “Nonlinear full information and moving horizon estimation: Robust global asymptotic stability,” Automatica, vol. 150, 2023, Art. no. 110603.
[19] W. Hu, “Generic stability implication from full information estimation to moving-horizon estimation,” in Proc. IEEE 57th Conf. Decis. Control, 2018, pp. 3477–3482.
[20] J. Köhler, M. A. Müller, and F. Allgöwer, “Robust output feedback model predictive control using online estimation bounds,” 2021, arXiv:2105.03427.
[21] D. A. Allan, “A Lyapunov-like function for analysis of model predictive control and moving horizon estimation,” Ph.D. dissertation, Univ. Wisconsin-Madison, Madison, WI, USA, 2020.
[22] D. A. Allan, J. B. Rawlings, and A. R. Teel, “Nonlinear detectability and incremental input/output-to-state stability,” SIAM J. Control Optim., vol. 59, no. 4, pp. 3017–3039, 2021.
[23] S. Knüfer and M. A. Müller, “Time-discounted incremental input/output-to-state stability,” in Proc. IEEE 59th Conf. Decis. Control, 2020, pp. 5394–5400.
[24] M. J. Tenny and J. B. Rawlings, “Efficient moving horizon estimation and nonlinear model predictive control,” in Proc. Amer. Control Conf., 2002, pp. 4475–4480.
[25] D. Angeli, “A Lyapunov approach to incremental stability properties,” IEEE Trans. Autom. Control, vol. 47, no. 3, pp. 410–421, Mar. 2002.
[26] A. A. Ahmadi and P. A. Parrilo, “Non-monotonic Lyapunov functions for stability of discrete time nonlinear and switched systems,” in Proc. IEEE 47th Conf. Decis. Control, 2008, pp. 614–621.
[27] J. D. Schiller and M. A. Müller, “Model predictive control: For want of a local control Lyapunov function, all is not lost,” IEEE Trans. Autom. Control, vol. 50, no. 5, pp. 546–558, May 2005.
[28] J. Köhler, “Analysis and design of MPC frameworks for dynamic operation of nonlinear constrained systems,” Ph.D. dissertation, Universität Stuttgart, Stuttgart, Germany, 2021.
[29] E. Sontag, “Smooth stabilization implies coprime factorization,” IEEE Trans. Autom. Control, vol. 34, no. 4, pp. 435–443, Apr. 1989.
[31] S. Boyd and L. Chua, “Fading memory and the problem of approximating nonlinear operators with volterra series,” IEEE Trans. Circuits Syst., vol. 32, no. 11, pp. 1150–1161, Nov. 1985.

[32] W. Maass and E. D. Sontag, “Neural systems as nonlinear filters,” Neural Comput., vol. 12, no. 8, pp. 1743–1772, 2000.

[33] H. Sorenson and J. Sacks, “Recursive fading memory filtering,” Inf. Sci., vol. 3, no. 2, pp. 101–119, 1971.

[34] W. Lohmiller and J.-J. E. Slotine, “On contraction analysis for non-linear systems,” Automatica, vol. 34, pp. 683–696, 1998.

[35] I. R. Manchester and J.-J. E. Slotine, “Transverse contraction criteria for existence, stability, and robustness of a limit cycle,” Syst. Control Lett., vol. 63, pp. 32–38, 2014.

[36] F. Forni and R. Sepulchre, “On differentially dissipative dynamical systems,” IFAC Proc. Volumes, vol. 46, no. 23, pp. 15–20, 2013.

[37] F. Forni and R. Sepulchre, “A differential Lyapunov framework for contraction analysis,” IEEE Trans. Autom. Control, vol. 59, no. 3, pp. 614–628, Mar. 2014.

[38] I. R. Manchester and J.-J. E. Slotine, “Robust control contraction metrics: A convex approach to nonlinear state-feedback $H_\infty$ control,” IEEE Control Syst. Lett., vol. 2, no. 3, pp. 333–338, Jul. 2018.

[39] I. R. Manchester and J.-J. E. Slotine, “Control contraction metrics: Convex and intrinsic criteria for nonlinear feedback design,” IEEE Trans. Autom. Control, vol. 62, pp. 3046–3053, Jun. 2017.

[40] P. J. Koelewijn, R. Töth, and S. Weiland, “Incremental dissipativity based control of discrete-time nonlinear systems via the LPV framework,” in Proc. IEEE 60th Conf. Decis. Control, 2021, pp. 3281–3286.

[41] R. G. Sanfelice and L. Praly, “Convergence of nonlinear observers on $\mathbb{R}^n$ with a Riemannian metric (Part I),” IEEE Trans. Autom. Control, vol. 57, no. 7, pp. 1709–1722, Jul. 2012.

[42] R. G. Sanfelice and L. Praly, “Convergence of nonlinear observers on $\mathbb{R}^n$ with a Riemannian metric (Part II),” IEEE Trans. Autom. Control, vol. 61, no. 10, pp. 2848–2860, Oct. 2016.

[43] H. Nijmeijer and A. Van der Schaft, Nonlinear Dynamical Control Systems. New York, NY, USA: Springer, 1990.

[44] P. J. Koelewijn and R. Töth, “Incremental stability and performance analysis of discrete-time nonlinear systems using the LPV framework,” IFAC-PapersOnLine, vol. 54, no. 8, pp. 75–82, 2021.

[45] P. A. Parrilo, “Semidefinite programming relaxations for semialgebraic problems,” Math. Prog., vol. 96, pp. 293–320, 2003.

[46] L. Wei, R. McCloy, and J. Bao, “Control contraction and control synthesis for discrete-time nonlinear processes,” J. Process Control, vol. 115, pp. 58–66, 2022.

[47] C. Verhoek, P. J. W. Koelewijn, S. Haesaert, and R. Töth, “Convex incremental dissipativity analysis of nonlinear systems,” Automatica, vol. 150, 2023, Art. no. 110859.

[48] J. Vilms, “Totally geodesic maps,” J. Differ. Geometry, vol. 4, no. 1, pp. 73–79, 1970.

[49] J. A. Andersson, J. Gillis, G. Horn, J. B. Rawlings, and M. Diehl, “CasADi: A software framework for nonlinear optimization and optimal control,” Math. Program. Comput., vol. 11, no. 1, pp. 1–36, 2019.

[50] A. Wächter and L. T. Biegler, “On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming,” Math. Prog., vol. 106, no. 1, pp. 25–57, 2005.

[51] J. Loßig, “Pre- and post-processing sum-of-squares programs in practice,” IEEE Trans. Autom. Control, vol. 54, no. 5, pp. 1007–1011, May 2009.

[52] The MOSEK Optimization Toolbox for MATLAB Manual. Version 9.0, Mosek ApS, Copenhagen, Denmark, 2019.

[53] J.-M. Kai, G. Allibert, M.-D. Hua, and T. Hanel, “Nonlinear feedback control of quadrotors exploiting first-order drag effects,” IFAC-PapersOnLine, vol. 50, no. 1, pp. 8189–8195, 2017.

[54] T. P. Nascimento and M. Saska, “Position and attitude control of multi-rotor aerial vehicles: A survey,” Ann. Rev. Control, vol. 48, pp. 129–146, 2019.

[55] S. Gallot, D. Hulin, and J. Lafontaine, Riemannian Geometry. Berlin, Germany: Springer, 2004.

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