Abstract. Let \( K/k \) be a finite Galois extension and \( \pi = \text{Gal}(K/k) \). An algebraic torus \( T \) defined over \( k \) is called a \( \pi \)-torus if \( T \times_{\text{Spec}(k)} \text{Spec}(K) \cong \mathbb{G}_m^n \times_{\mathbb{Z},n} K \) for some integer \( n \). The set of all algebraic \( \pi \)-tori defined over \( k \) under the stably isomorphism form a semigroup, denoted by \( T(\pi) \). We will give a complete proof of the following theorem due to Endo and Miyata [EM5].

Theorem. Let \( \pi \) be a finite group. Then \( T(\pi) \cong C(\Omega_{\mathbb{Z}(\pi)}) \), where \( \Omega_{\mathbb{Z}(\pi)} \) is a maximal \( \mathbb{Z} \)-order in \( \mathbb{Q} \pi \) containing \( \mathbb{Z} \pi \) and \( C(\Omega_{\mathbb{Z}(\pi)}) \) is the locally free class group of \( \Omega_{\mathbb{Z}(\pi)} \), provided that \( \pi \) is isomorphic to the following four types of groups:

1. \( C_n \) (\( n \) is any positive integer),
2. \( D_m \) (\( m \) is any odd integer \( \geq 3 \)),
3. \( q^f \times D_m \) (\( m \) is any odd integer \( \geq 3 \), \( q \) is an odd prime number not dividing \( m \), \( f \geq 1 \), and \( (\mathbb{Z}/q^f\mathbb{Z})^\times = \langle \bar{p} \rangle \) for any prime divisor \( p \) of \( m \)),
4. \( Q_{4m} \) (\( m \) is any odd integer \( \geq 3 \), \( p \equiv 3 \) (mod 4) for any prime divisor \( p \) of \( m \)).

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§1. Introduction

In [EM4] and [EM5], Endo and Miyata investigated the classification of the function fields of algebraic tori. An additional paper was planned, which would contain a complete proof of (1') \( \Rightarrow \) (2) of Theorem 3.3 in [EM4] (also see [EM5, page 187]). This article was announced in [EM5, p.189, line −14]. Unfortunately the plan didn’t materialize. The present article arose as a result of many e-mail exchanges between the author and Prof. Endo. I thank heartily Prof. Endo for his generosity and patience to communicate to me the ideas and the proofs in his joint papers with Miyata, although he rejected the invitation to be a coauthor of this paper. The present article may be regarded as a supplement to the papers [EM4], [EM5] and the paper [Sw6].

To begin with, we recall some definitions and terminology. Let \( k \) be a field, \( k \subset L \) be a field extension. The field \( L \) is said to be rational over \( k \) (in short, \( k \)-rational) if, for some \( n \), \( L \simeq k(X_1, \ldots, X_n) \) over \( k \) where \( k(X_1, \ldots, X_n) \) is the rational function field of \( n \) variables over \( k \). Two field extensions \( k \subset L_1, L_2 \) are stably isomorphic over \( k \) if, \( L_1(X_1, \ldots, X_m) \simeq L_2(Y_1, \ldots, Y_n) \) over \( k \) where \( X_1, \ldots, X_m \) are algebraically independent over \( L_1 \) and \( Y_1, \ldots, Y_n \) are algebraically independent over \( L_2 \). In particular, a field extension \( k \subset L \) is stably \( k \)-rational if \( L(X_1, \ldots, X_m) \) is \( k \)-rational where \( X_1, \ldots, X_m \) are some elements algebraically independent over \( L \). When \( V \) is an irreducible algebraic variety defined over \( k \), \( V \) is \( k \)-rational (resp. stably \( k \)-rational) if so is the function field \( k(V) \) over \( k \). If \( V_1 \) and \( V_2 \) are irreducible varieties over \( k \), \( V_1 \) and \( V_2 \) are stably isomorphic over \( k \) if so are their function fields over \( k \).

An algebraic torus \( T \) defined over a field \( k \) is an affine algebraic group defined over \( k \) such that \( T \otimes \text{Spec}(k) \simeq \mathbb{G}_{m,k}^n \) for some integer \( n \) where \( k \) is the algebraic closure of \( k \) and \( \mathbb{G}_{m,K} \) is the 1-dimensional multiplicative group defined over a field \( K \) (containing the base field \( k \)) [On; Sw5, page 36; Vo]. By [On, page 106, Proposition 1.2.1], for any algebraic torus \( T \) over \( k \), there is a finite separable extension field \( K \) of \( k \) satisfying that \( T \otimes \text{Spec}(k) \text{Spec}(K) \simeq \mathbb{G}_{m,K}^n \); such a field \( K \) is called a splitting field of \( T \).

Let \( k \) be a field, \( \pi \) be a finite group. We will say that the field \( k \) admits a \( \pi \)-extension if there is a Galois field extension \( K/k \) such that \( \pi = \text{Gal}(K/k) \).

**Definition 1.1** Let \( \pi \) be a finite group, \( k \) be field admitting a \( \pi \)-extension. An algebraic torus \( T \) over \( k \) is called a \( \pi \)-torus if it has a splitting field \( K \) which is Galois over \( k \) with \( \text{Gal}(K/k) = \pi \).

Let \( \pi \) be a finite group. Recall that a finitely generated \( \mathbb{Z}\pi \)-module \( M \) is called a \( \pi \)-lattice if it is torsion-free as an abelian group.

If \( T \) is a \( \pi \)-torus over a field \( k \) with \( \pi = \text{Gal}(K/k) \), then its character module \( \text{Hom}(T \otimes \text{Spec}(k) \text{Spec}(K), \mathbb{G}_{m,K}) \) is a \( \pi \)-lattice. Conversely, every \( \pi \)-lattice \( M \) is isomorphic to the character module of some algebraic \( \pi \)-torus \( T \) over \( k \) (as \( \mathbb{Z}\pi \)-modules) [On; Sw5, page 36].
Definition 1.2 Let $K/k$ be a finite Galois field extension with $\pi = \text{Gal}(K/k)$. Let $M = \bigoplus_{1 \leq i \leq n} \mathbb{Z} \cdot e_i$ be a $\pi$-lattice. We define an action of $\pi$ on $K(M) = K(x_1, \ldots, x_n)$, the rational function field of $n$ variables over $K$, by $\sigma \cdot x_j = \prod_{1 \leq i \leq n} x_i^{a_{ij}}$ if $\sigma \cdot e_j = \sum_{1 \leq i \leq n} a_{ij} e_i \in M$, for any $\sigma \in \pi$ (note that $\pi$ acts on $K$ also). The fixed field is denoted by $K(M)_{\pi}$.

Let $K/k$ be a finite Galois extension with $\pi = \text{Gal}(K/k)$. There is a duality between the category of algebraic $\pi$-tori defined over $k$ and the category of $\pi$-lattices. In fact, if $T$ is a $\pi$-torus and $M$ is its character module, then the function field of $T$ is isomorphic to $K(M)_{\pi}$ (see [On; Sw5, page 36]). Thus the study of rationality problems of $\pi$-tori is reduced to that of $\pi$-lattices.

Definition 1.3 ([EM4, page 86]) Let $\pi = \text{Gal}(K/k)$ be a finite group where $K/k$ is a Galois extension. Define an equivalence relation in the category of $\pi$-lattices: Two $\pi$-lattices $M$ and $N$ are equivalent, denoted by $M \sim N$, if the fields $K(M)_{\pi}$ and $K(N)_{\pi}$ are stably isomorphic over $k$, i.e. $K(M)_{\pi}(X_1, \ldots, X_m) \simeq K(N)_{\pi}(Y_1, \ldots, Y_n)$ for some algebraically independent elements $X_i, Y_j$.

Let $\pi$ be a finite group. Define a commutative monoid $T(\pi)$ as follows. As a set, $T(\pi)$ is the set of all equivalence classes $[M]$ under the equivalence relation “$\sim$” defined above (note that $[M]$ is the equivalence class containing the $\pi$-lattice $M$); the monoid operation is defined by $[M] + [N] = [M \oplus N]$.

Theorem 1.4 (Endo and Miyata [EM4, page 87 and page 95, Theorem 3.3; EM5, page 187]) Let $\pi$ be a finite group. Then the following statements are equivalent:

1. $\pi$ is isomorphic to
   (i) a cyclic group $C_n$ where $n$ is any positive integer, or
   (ii) a dihedral group $D_m$ of order $2m$ where $m$ is an odd integer $\geq 3$, or
   (iii) a direct product $C_{q^f} \times D_m$ where $q$ is an odd prime number, $f \geq 1$, $m$ an odd integer $\geq 3$, $\gcd(q, m) = 1$ and for any prime divisor $p$ of $m$, $(\mathbb{Z}/q^f\mathbb{Z})^\times = \langle p \rangle$, or
   (iv) $Q_{4m} = \langle \sigma, \tau : \sigma^{2m} = \tau^4 = 1, \sigma^m = \tau^2, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle$, the generalized quaternion group of order $4m$, where $m \geq 3$ is an odd integer and $p \equiv 3$ (mod 4) for any prime divisor $p$ of $m$.

2. $T(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) \simeq C(\Omega_{\mathbb{Z}\pi})$ where $\Omega_{\mathbb{Z}\pi}$ is a maximal $\mathbb{Z}$-order in $\mathbb{Q}\pi$ containing $\mathbb{Z}\pi$.

3. $T(\pi)$ is a finite group.

The purpose of this article is to give a proof of Theorem 1.4 supplementing the proof outlined in [EM4] and [EM5]. Note that the definition of the locally free class groups $C(\mathbb{Z}\pi), C(\Omega_{\mathbb{Z}\pi})$ and the associated group $C^q(\mathbb{Z}\pi)$ may be found in Definition 2.11.
and Definition 2.12; the isomorphisms $T(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$ and $C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) \simeq C(\Omega_{\mathbb{Z}\pi})$ are described in Definition 2.13.

The main ideas of the proof of $(1) \Rightarrow (2)$ in Theorem 1.4 will be explained at the beginning of Section 5.

The organization of this paper is as follows. In Section 2, the notions of flabby, coflabby, invertible, permutation $\pi$-lattices are recalled. Some fundamental results are summarized also. Section 3 contains the definitions of twisted group rings, denoted by $S \circ G$, and crossed-product orders, denoted by $(S \circ G)_f$ (where $f$ is a 2-cocycle of $G$). Some results of Rosen’s Ph.D. dissertation (unpublished in the journals) will be quoted for the convenience of the reader. Section 4 is a first step of the proof of $(1) \Rightarrow (2)$ of Theorem 1.4; this section contains a devissage theorem for $[M]^{fl}$ where $M$ is an invertible $\pi$-lattice. Section 5 is devoted to the proof of $(1) \Rightarrow (2)$ of Theorem 1.4. In the final section, we compute $C(\Omega_{\mathbb{Z}\pi})$ when $\pi$ are the groups in Theorem 1.4.

In fact, Swan determine all the maximal orders of $\mathbb{Z}\pi$ when $\pi$ is a dihedral group or a generalized quaternion group; we just use partial results of Swan’s investigation [Sw4]. As a consequence, we deduce a result about $D_{p^n}$-tori; the case of $D_p$-tori was proved by Hoshi, Kang and Yamasaki by a different method [HKY]. On the other hand, some properties $\pi$-lattices proved by Colliot-Thélène and Sansuc, when $\pi$ is the Klein four-group or the quaternion group of order 8 [CTS, pages 186-187; Lo, page 43], will be generalized to the situation when $\pi$ is the dihedral 2-group or quaternion groups of order 8, 16 or 32.

Terminology and notations. Throughout the paper, we denote by $k$ a field and by $\pi$ a finite group. We denote by $\mathbb{Z}\pi$ the integral group ring of the group $\pi$.

Denote by $C_n$ (resp. $D_n$) the cyclic group of order $n$ (resp. the dihedral group of order $2n$). The group $Q_{4n}$ denotes the generalized quaternion group of order $4n$ where $n \geq 2$, i.e. $Q_{4n} = \langle \sigma, \tau : \sigma^{2n} = \tau^4 = 1, \sigma^n = \tau^2, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle$.

If $q$ is a prime power, $\mathbb{F}_q$ denotes the finite field with $q$ elements. For any positive integer $n \geq 2$, $\zeta_n$ denotes a primitive $n$-th root of unity and $\Phi_n(X) \in \mathbb{Z}[X]$ the $n$-th cyclotomic polynomial. $(\mathbb{Z}/n\mathbb{Z})^\times$ is the group of units of the ring $\mathbb{Z}/n\mathbb{Z}$. When $\pi$ is a finite group and $\Lambda$ is a $\mathbb{Z}$-order satisfying that $\mathbb{Z}\pi \subset \Lambda \subset \mathbb{Q}\pi$ (see [CR1, page 524, Definition 23.2]), a projective $\Lambda$-module $A$ is called a projective ideal over $\Lambda$ if $\mathbb{Q}\Lambda \simeq \mathbb{Q}\Lambda$; a projective $\Lambda$-module $M$ is called locally free if there exists a non-negative integer $n$ such that $\mathbb{Q}M \simeq \mathbb{Q}\Lambda^n$ ($\mathbb{Q}\Lambda^n$ denotes the free $\mathbb{Q}\Lambda$-module of rank $n$) where $\mathbb{Q}\Lambda = \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$ and $\mathbb{Q}M = \mathbb{Q}\Lambda \otimes_{\Lambda} M$ (see [Sw1], page 573; Ri, page 463; CR2, page 218).

An commutative integral domain $R$ is called a DVR if it is a discrete rank-one valuation ring. If $R$ is a ring, $M_n(R)$ denotes the matrix ring of all $n \times n$ matrices over $R$.

We remind the reader that the definitions of $T(\pi)$, $T^q(\pi)$, $C(\Lambda)$ and $[M]^{fl}$, are given in Definition 1.3, Definition 2.8, Definition 2.11 and Definition 2.1; the monoid $F_\pi$ is defined in the paragraph before Definition 2.1.
§2. Preliminaries

In this section we recall some notions related to \( \pi \)-lattices and summarize several results in [EM4] and [Lo; Sw5].

A \( \pi \)-lattice \( M \) is called a permutation lattice if \( M \) has a \( \mathbb{Z} \)-basis permuted by \( \pi \); \( M \) is called an invertible lattice if it is a direct summand of some permutation lattice. A \( \pi \)-lattice \( M \) is called a flabby lattice if \( H^{-1}(\pi', M) = 0 \) for any subgroup \( \pi' \) of \( \pi \); it is called coflabby if \( H^1(\pi', M) = 0 \) for any subgroup \( \pi' \) of \( \pi \). For details, see [EM4; CTS; Sw5; Lo].

Two \( \pi \)-lattices \( M_1 \) and \( M_2 \) are similar, denoted by \( M_1 \sim M_2 \), if \( M_1 \oplus P_1 \simeq M_2 \oplus P_2 \) for some permutation \( \pi \)-lattices \( P_1 \) and \( P_2 \). The flabby class monoid \( F_\pi \) is the monoid whose elements consist of flabby \( \pi \)-lattices under the above the similarity relation. In particular, if \( M \) is a flabby \( \pi \)-lattice, a typical element in \( F_\pi \) is \([M] = [M]_{\text{fl}} \) where \([M]_{\text{fl}} \) denotes the equivalence class containing \( M \); we define \([M_1] + [M_2] = [M_1 \oplus M_2] \). Thus \( F_\pi \) becomes an abelian monoid and \([P] \) is the identity element of it where \( P \) is any permutation \( \pi \)-lattice [Sw5, page 33].

**Definition 2.1** Let \( \pi \) be a finite group, \( M \) be any \( \pi \)-lattice. Then \( M \) has a flabby resolution, i.e. there is an exact sequence of \( \pi \)-lattices: \( 0 \rightarrow M \rightarrow P \rightarrow E \rightarrow 0 \) where \( P \) is a permutation lattice and \( E \) is a flabby lattice [EM4, Lemma 1.1]. The class \([E] \in F_\pi \) is uniquely determined by the lattice \( M \) [Sw5, Lemma 8.7]. We define \([M]_{\text{fl}} = [E] \in F_\pi \), following the nomenclature in [Lo, page 38]. Sometimes we will say that \([M]_{\text{fl}} \) is permutation or invertible if the class \([E] \) contains a permutation or invertible lattice.

Be aware that the equivalence relation \( M \sim N \) in Definition 1.3 is different from the above similarity relation \( M_1 \sim M_2 \). The monoids \( T(\pi) \) in Definition 1.3 and the above monoid \( F_\pi \) are isomorphic through the following lemma.

**Lemma 2.2** Let \( \pi \) be a finite group.

1. If \( M \) and \( N \) are \( \pi \)-lattices. Then \( M \sim N \) if and only if \([M]_{\text{fl}} = [N]_{\text{fl}} \).
2. Define a monoid homomorphism \( \Phi : T(\pi) \rightarrow F_\pi \) by \( \Phi([M]) = [M]_{\text{fl}} \). Then \( \Phi \) is an isomorphism.

**Proof.** Let \( \pi = \text{Gal}(K/k) \). By the same idea in the proof of [Lo, Theorem 1.7], it is not difficult to show that \( K(M)^\pi \) and \( K(N)^\pi \) are stably isomorphic over \( k \) if and only if there exist exact sequences of \( \pi \)-lattices \( 0 \rightarrow M \rightarrow E \rightarrow P \rightarrow 0 \) and \( 0 \rightarrow N \rightarrow E \rightarrow Q \rightarrow 0 \) where \( E \) is some \( \pi \)-lattice and \( P, Q \) are permutation \( \pi \)-lattices. The latter condition is equivalent to \([M]_{\text{fl}} = [N]_{\text{fl}} \) by [Sw3, Lemma 8.8].

For the isomorphism of \( \Phi \), note that \( \Phi \) is well-defined by (1). If \( \Phi([M]) = 0 \), then \([M]_{\text{fl}} = 0 \). Thus \( K(M)^\pi \) is stably \( k \)-rational by the following Lemma 2.3. Hence \( M \sim Z \) where \( Z \) is the \( \pi \)-lattice with trivial \( \pi \) actions. Thus \( \Phi \) is injective. On the other hand, if \( E \) is any flabby \( \pi \)-lattice, let \( E^0 = \text{Hom}_\pi(E, \mathbb{Z}) \) be its dual lattice. Take a flabby resolution of \( E^0 \), \( 0 \rightarrow E^0 \rightarrow P \rightarrow F \rightarrow 0 \) as in Definition 2.1. We get an exact sequence \( 0 \rightarrow F^0 \rightarrow P^0 \rightarrow E \rightarrow 0 \). Thus \([E] = [F^0]_{\text{fl}} = \Phi([F^0]) \) and \( \Phi \) is surjective. \( \blacksquare \)
The following lemma is due to Endo and Miyata [EM2] Theorem 1.2 and Voskresenskii.

**Lemma 2.3** ([Le Theorem 1.7]) Let \( K/k \) be a finite Galois field extension, \( \pi = \text{Gal}(K/k) \), \( M \) be a \( \pi \)-lattice. Then \( K(M)^\pi \) is stably \( k \)-rational if and only if \( [M]^f = 0 \) in \( F_\pi \).

**Lemma 2.4** Let \( \pi \) be a finite group, \( 0 \to M' \to M \to M'' \to 0 \) be an exact sequence of \( \pi \)-lattices.

1. (Sw6 Lemma 3.1) If \( M'' \) is invertible, then \( [M] = [M'] + [M''] \).
2. (EM4 Lemma 2.2) If all the Sylow subgroups of \( \pi \) are cyclic and \( \tilde{H}^0(\pi', M') = 0 \) for any subgroup \( \pi' \subset \pi \), then \( [M] = [M'] + [M''] \). (Note that \( \tilde{H}^0(\pi', M') \) denotes the Tate cohomology.)

**Lemma 2.5** Let \( \pi \) be a finite group, \( M \) be a \( \pi \)-lattice.

1. (Sw6 Corollary 2.5) \( M \) is invertible if and only if, for any coflabby \( \pi \)-lattice \( C \), any short exact sequence \( 0 \to C \to E \to M \to 0 \) splits.
2. (EM6 Lemma 1.1) There is a short exact sequence of \( \pi \)-lattices \( 0 \to M \to C \to P \to 0 \) such that \( C \) is coflabby and \( P \) is permutation.

**Theorem 2.6** (Endo and Miyata [EM4] Theorem 1.5; Sw5, Theorem 4.4; Lo, 2.10.1) Let \( \pi \) be a finite group. Then all the flabby \( \pi \)-lattices (resp. all the coflabby \( \pi \)-lattices) are invertible if and only if all the Sylow subgroups of \( \pi \) are cyclic.

**Theorem 2.7** (Endo and Miyata [EM6] Theorem 2.1) Let \( \pi \) be a finite group. Then the \( \pi \)-lattices which are both flabby and coflabby are necessarily invertible if and only if all the \( p \)-Sylow subgroups of \( \pi \) are cyclic for odd prime \( p \), and all the \( 2 \)-Sylow subgroups of \( \pi \) are cyclic or dihedral (including the Klein four-group).

**Definition 2.8** ([EM4 page 86]) Let \( \pi \) be a finite group. Define \( T^g(\pi) = \{ [M] \in T(\pi) : \text{There exists } [N] \in T(\pi) \text{ such that } [M] + [N] = 0 \} \). It follows that \( T^g(\pi) \) is a subgroup of \( T(\pi) \); it is the maximal subgroup of \( T(\pi) \). By Jacobinski’s Theorem [Ja], Proposition 5.8, \( T^g(\pi) \) is finitely generated.

If \( M \) is a \( \pi \)-lattice and \( [M] \in T^g(\pi) \), then there is an invertible \( \pi \)-lattice \( E \) such that \( [M] = [E] \) in \( T^g(\pi) \) (see [EM4 Lemma 1.6]).

Now we turn to \( R \)-orders in a separable \( K \)-algebra \( \Sigma \) where \( R \) is a Dedekind domain with quotient field \( K \) [CR1 page 523]. Recall that a separable \( K \)-algebra \( \Sigma \) is a finite-dimensional semi-simple algebra over \( K \) such that the center \( K^{\prime} \) of \( \Sigma \) is an étale \( K \)-algebra, i.e. \( K^{\prime} = \prod_{1 \leq i \leq t} K_i \) where each \( K_i \) is a finite separable field extension of \( K \). An \( R \)-order \( \Lambda \) is a subring of \( \Sigma \), \( R \subset \Lambda \subset \Sigma \) satisfying that \( K\Lambda = \Sigma \) and \( \Lambda \) is a finitely generated \( R \)-module.

**Definition 2.9** Let \( R \) be a Dedekind domain with quotient field \( K \), \( \Lambda \) be an \( R \)-order in a separable \( K \)-algebra \( \Sigma \). A \( \Lambda \)-lattice \( M \) is a left \( \Lambda \)-module which is finitely generated and projective as an \( R \)-module [CR1 page 524]. Two \( \Lambda \)-lattices \( M \) and \( N \) are in the same genus if \( M_{\wp} \cong N_{\wp} \) for any prime ideal \( \wp \) of \( R \) where \( M_{\wp} = R_{\wp} \otimes_R M \) and \( R_{\wp} \) is the localization of \( R \) at the prime ideal \( \wp \) [CR1 pages 642–643].
Theorem 2.10 (Jacobinski, Roiter [CR2, page 660, Theorem 31.28]) Let $R$ be a Dedekind domain whose quotient field is a global field $K$. Let $\Sigma$ be a separable $K$-algebra and $\Lambda$ be an $R$-order in $\Sigma$. Let $M$ and $N$ be $\Lambda$-lattices in the same genus and $F$ be a faithful $\Lambda$-lattice. Then there is a $\Lambda$-lattice $F'$ such that $F$, $F'$ are in the same genus and $M \oplus F \simeq N \oplus F'$. In particular, if $M$ and $N$ are $\Lambda$-lattices in the same genus, then $M \oplus \Lambda \simeq N \oplus \Lambda$ for some projective ideal $\mathcal{A}$ over $\Lambda$.

Definition 2.11 Let $\pi$ be a finite group, let $\Lambda$ be a $\mathbb{Z}$-order with $\mathbb{Z}\pi \subset \Lambda \subset \mathbb{Q}\pi$. Let $K_0(\Lambda)$ be the Grothendieck group of the category of locally free $\Lambda$-modules (the definition of a locally free $\Lambda$-module is given at the end of Section 1). The locally free class group of $\Lambda$, denoted by $C(\Lambda)$, is a subgroup of $K_0(\Lambda)$ defined as $C(\Lambda) = \{[\mathcal{A}] - [\Lambda] \in K_0(\Lambda) : \mathcal{A} \text{ is a projective ideal over } \Lambda\}$. The group $C(\Lambda)$ is also called the projective class group of $\Lambda$.

Note that $C(\mathbb{Z}\pi)$ is a finite group [Sw1, page 573]. For details of $C(\Lambda)$, see [EM2, page 397; CR2, page 50, page 219, page 230].

Definition 2.12 Let $\pi$ be a finite group and $\Omega_{\mathbb{Z}\pi}$ be a maximal $\mathbb{Z}$-order in $\mathbb{Q}\pi$ satisfying $\mathbb{Z}\pi \subset \Omega_{\mathbb{Z}\pi} \subset \mathbb{Q}\pi$. It is known that the natural map $\varphi_1 : C(\mathbb{Z}\pi) \to C(\Lambda)$ defined by $\varphi_1([\mathcal{A}] - [\mathbb{Z}\pi]) = [\Omega_{\mathbb{Z}\pi} \otimes_{\mathbb{Z}\pi} \mathcal{A}] - [\Omega_{\mathbb{Z}\pi}]$ is surjective (see [CR2, page 290, Theorem 49.25]). Define $\tilde{C}(\mathbb{Z}\pi) = \ker(\varphi_1)$. From the definition, $\tilde{C}(\mathbb{Z}\pi) = \{[\mathcal{A}] - [\mathbb{Z}\pi] \in C(\mathbb{Z}\pi) : (\Omega_{\mathbb{Z}\pi} \otimes_{\mathbb{Z}\pi} \mathcal{A}) \oplus \Omega_{\mathbb{Z}\pi} \simeq \Omega_{\mathbb{Z}\pi} \oplus \Omega_{\mathbb{Z}\pi}\}$. 

In [CR2, page 234] and other literature, $\tilde{C}(\mathbb{Z}\pi)$ is written as $D(\mathbb{Z}\pi)$; but we will stick to the notation in [EM2, EM3, EM4].

On the other hand, a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ may be regarded as a $\pi$-lattice. Thus we may define $\varphi_2 : C(\mathbb{Z}\pi) \to T^q(\pi)$ defined by $\varphi_2([\mathcal{A}] - [\mathbb{Z}\pi]) = [\mathcal{A}] \in T^q(\pi)$. It is easy to check that $\varphi_2$ is a well-defined morphism of abelian groups. Define $C^q(\mathbb{Z}\pi) = \ker(\varphi_2)$. Clearly $C^q(\mathbb{Z}\pi) = \{[\mathcal{A}] - [\mathbb{Z}\pi] \in C(\mathbb{Z}\pi) : \mathcal{A} \text{ is a projective ideal over } \mathbb{Z}\pi \text{ satisfying } [\mathcal{A}]^{f_{\pi}} = 0 \text{ in } F_{\pi}\}.

Finally define $\tilde{C}^q(\mathbb{Z}\pi) = \{[\mathcal{A}] - [\mathbb{Z}\pi] \in C^q(\mathbb{Z}\pi) : \mathcal{A} \text{ is a projective ideal over } \mathbb{Z}\pi \text{ satisfying } \mathcal{A} \oplus P \simeq \mathbb{Z}\pi \oplus P \text{ for some permutation } \pi\text{-lattice } P\}$ (see [EM3, page 698]).

It can be shown that $\tilde{C}^q(\mathbb{Z}\pi)$ is a subgroup of $\tilde{C}(\mathbb{Z}\pi)$. In fact, Oliver proves that $\tilde{C}^q(\mathbb{Z}\pi) = \tilde{C}(\mathbb{Z}\pi)$ for any finite group $\pi$ [Ol, Theorem 5]. Hence $\tilde{C}(\mathbb{Z}\pi) \subset C^q(\mathbb{Z}\pi)$. It follows that there is always a surjective map $C(\Omega_{\mathbb{Z}\pi}) \to C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$.

In [EM2, EM3] it is shown that $\tilde{C}(\mathbb{Z}\pi) = C^q(\mathbb{Z}\pi)$ for many groups $\pi$, including those groups listed in Theorem 1.4 (1) (see Section 5).

Definition 2.13 Let $\pi$ be a finite group satisfying $C^q(\mathbb{Z}\pi) = \tilde{C}(\mathbb{Z}\pi)$.

Let $\varphi_1$ and $\varphi_2$ be the group homomorphisms in Definition 2.12. Since $0 \to C^q(\mathbb{Z}\pi) \to C(\mathbb{Z}\pi) \xrightarrow{\varphi_2} T^q(\pi)$ is left-exact, we will write $T^q(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$ when the canonical injection $C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) \xrightarrow{\varphi_2} T^q(\pi)$ is an isomorphism.

Similarly, when $C^q(\mathbb{Z}\pi) = \tilde{C}(\mathbb{Z}\pi)$ and $c : T^q(\pi) \to T(\pi)$ is the inclusion map, then the composite map $c \cdot \varphi_2 \cdot \varphi_1^{-1} : C(\Omega_{\mathbb{Z}\pi}) \to C(\mathbb{Z}\pi)/\tilde{C}(\mathbb{Z}\pi) = C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) \to T^q(\pi) \to T(\pi)$ is well-defined. We will write $T(\pi) \simeq C(\Omega_{\mathbb{Z}\pi})$ if the injection $c \cdot \varphi_2 \cdot \varphi_1^{-1}$ is surjective and hence is an isomorphism.
Theorem 2.14 Let $\pi$ be a finite group.

1. ([EM4, Proposition 3.1]) $T^q(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$ if and only if, for any invertible $\pi$-lattice $M$, there is a projective ideal $A$ over $\mathbb{Z}\pi$ such that $[M]^f = [A]^f$.

2. ([EM4, Theorem 3.2]) If $\pi$ is a $p$-group, then $T^q(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$.

§3. Twisted group rings

Definition 3.1 ([CR1 page 183, page 599–600; CR2, page 291]) Let $K$ be a field, $L/K$ be a finite Galois field extension with $G = \text{Gal}(L/K)$, and $f : G \times G \to L^\times$ be a 2-cocycle of $G$ where $L^\times = L\setminus\{0\}$ is the multiplicative group of $L$. The crossed product algebra, denoted by $(L \circ G)_f$, is defined by

$$(L \circ G)_f = \bigoplus_{\sigma \in G} L \cdot u_\sigma, \quad u_\sigma \cdot u_\tau = f(\sigma, \tau)u_{\sigma\tau}, \quad u_\sigma \cdot \alpha = \sigma(\alpha) \cdot u_\sigma$$

where $\alpha \in L$, $\sigma, \tau \in G$. The $K$-algebra $(L \circ G)_f$ is a central simple $K$-algebra.

Suppose that $R$ is a Dedekind domain with quotient field $K$ and $K$ is a number field. Let $S$ be the integral closure of $R$ in $L$. Let $h : G \times G \to U(S)$ be a 2-cocycle where $U(S)$ is the group of units in $S$. Then we may define the crossed-product order, denoted by $(S \circ G)_h$, as follows:

$$(S \circ G)_h = \bigoplus_{\sigma \in G} S \cdot u_\sigma, \quad u_\sigma \cdot u_\tau = h(\sigma, \tau)u_{\sigma\tau}, \quad u_\sigma \cdot \alpha = \sigma(\alpha) \cdot u_\sigma$$

where $\alpha \in S$, $\sigma, \tau \in G$. The $R$-algebra $(S \circ G)_h$ is an $R$-order in $(L \circ G)_h$.

Note that $(S \circ G)_h$ is not an Azumaya $R$-algebra in general (an Azumaya $R$-algebra is called a central separable $R$-algebra in [AG2]). When $S/R$ is unramified (i.e. $S$ is a Galois extension of $R$ relative to the group $G$ in the sense of [AG2, page 396]), then $(S \circ G)_h$ is an Azumaya $R$-algebra by [AG2, page 406, Theorem A.12].

Theorem 3.2 (Williamson, Harada [CR1 page 600, Theorem 28.12; Re, page 375, Theorem 40.15]) Let the notations be the same as in Definition 3.1. The crossed-product $R$-order $(S \circ G)_h$ is a hereditary order if and only if $S/R$ is tamely ramified, i.e. for any ramified prime ideal $Q$ of $S$ over $R$, if $\text{char} S/Q = p > 0$, then $p \nmid e(Q, L/K)$ where $e(Q, L/K)$ is the ramification index of $Q$ in $L/K$.

Definition 3.3 The twisted group algebra and the twisted group ring are special cases of the crossed-product algebra and the crossed-product order when the 2-cocycle $f, h$ are the trivial one, i.e. $f(\sigma, \tau) = h(\sigma, \tau) = 1$ for all $\sigma, \tau \in G$.

For emphasis, we repeat the definition of a twisted group ring and denote it by $S \circ G$ (see [CR1 page 589]). Recall that $S$ is a Dedekind domain whose quotient field
$L$ is an algebraic number field, $G$ is a finite subgroup of $\text{Aut}(S)$ with $R = S^G = \{a \in S : \sigma(a) = a \text{ for any } \sigma \in G\}$. Then
\[
S \circ G = \bigoplus_{\sigma \in G} S \cdot u_\sigma, \quad (au_\sigma) \cdot (bu_\tau) = (a \cdot \sigma(b)) \cdot u_{\sigma \tau}
\]
where $a, b \in S$, $\sigma, \tau \in G$.

**Theorem 3.4** (1) ([Rosen] page 374, Theorem 40.14; CR1, page 591, Theorem 28.5) The twisted group ring $S \circ G$ is a maximal $R$-order if and only if $S/R$ is unramified.

(2) (Rosen [Rosen] page 373, Theorem 40.13) The twisted group ring $S \circ G$ is a hereditary $R$-order if and only if $S/R$ is tamely ramified.

**Remark.** Theorem 3.4 (2) was obtained first. Then Theorem 3.2 came out as the generalization of Theorem 3.4 (2).

**Definition 3.5** Let $\Lambda = S \circ G$ be a twisted group ring and $L$ be the quotient field of $S$ with $R = S^G$. We will endow on $S$ a $\Lambda$-module structure by defining $(au_\sigma) \cdot \alpha = a \cdot \sigma(\alpha)$ for any $a, \alpha \in S$, any $\sigma \in G$. The field $L$ can be given a $\Lambda$-module structure by the same way. If $J$ is a fractional $S$-ideal of $L$ such that $\sigma(J) \subseteq J$ for any $\sigma \in G$, then $J$ becomes a $\Lambda$-submodule of $L$; such an ideal $J$ is called an ambiguous ideal [CR1] page 596; Ro].

**Theorem 3.6** (Rosen) Let $\Lambda = S \circ G$ be a twisted group ring and $R = S^G$.

(1) ([Rosen] Proposition 3; CR1, page 596) Let $Q_1, Q_2, \ldots, Q_t$ be all the ramified primes of $S$ over $R$ and $e_i = e_i(Q_i, S/R)$ be the ramification index of $Q_i$ for $1 \leq i \leq t$. For each $1 \leq i \leq t$, let $\{Q_i^{(j)} : 1 \leq j \leq g_i\}$ be the set of $G$-orbit of $Q_i$ (i.e. $Q_i^{(j)} = \sigma(Q_i)$ for some $\sigma \in G$, and $Q_i^{(1)}, \ldots, Q_i^{(g_i)}$ are distinct prime ideals of $S$). Define $J_i = \prod_{1 \leq j \leq g_i} Q_i^{(j)}$. As $\Lambda$-modules, any ambiguous ideal $J$ is isomorphic to, $J_1^{a_1} \cdots J_t^{a_t} I$ where $0 \leq a_i < e_i$ and $I$ is some ideal of $R$.

(2) ([Rosen] Theorem 3.2 and Theorem 3.3) Assume that $S/R$ is tamely ramified. Then $\Lambda$ is a left hereditary ring. The ambiguous ideals $J_1^{a_1} J_2^{a_2} \cdots J_t^{a_t} I$ in (1) are indecomposable projective $\Lambda$-module. If $M$ is a $\Lambda$-module such that $M$ is a finitely generated torsion-free $R$-module, then $M$ is isomorphic to a direct sum of these ambiguous ideals $J_1^{a_1} J_2^{a_2} \cdots J_t^{a_t} I$.

**Remark.** In the above theorem, it is possible that some of $J_i$ (where $1 \leq i \leq t$) are isomorphic to each other. See [Rosen] for the uniqueness statement.

§4. A devissage theorem

The following theorem is a generalization of Swan’s Theorem 5.1 in [Sw6].

If $M$ is a $\mathbb{Z}\pi$-module, we will write $(M)_0 = M/t(M)$ where $t(M)$ is the torsion submodule of $M$. 
Theorem 4.1 Let $\pi$ be a finite group, $\sigma \in \pi$ such that the cyclic subgroup $\langle \sigma \rangle \simeq C_m$ is a normal subgroup of $\pi$. Let $\phi$ be a map from the category of $\pi$-lattices to an abelian group satisfying the following property: If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $\pi$-lattices with $M''$ invertible, then $\phi(M) = \phi(M') + \phi(M'')$. Suppose that $n \mid m$ and $M$ is an invertible $\pi$-lattice. Then

$$\phi(M/(\sigma^n - 1)M) = \sum_{d \mid n} \phi((M/\Phi_d(\sigma)M)_0),$$

and

$$\phi((M/\Phi_n(\sigma)M)_0) = \sum_{d \mid n} \mu \left( \frac{n}{d} \right) \phi(M/(\sigma^d - 1)M)$$

where $\mu$ is the Möbius function.

Proof. The above two formulae are equivalent by Möbius inversion formula. It suffices to prove the second formula.

Step 1. Since $M$ is invertible, $M/(\sigma^n - 1)M$ is torsion-free by [Sw6, Lemma 5.3].

Step 2. We will generalize [Sw6, Lemma 5.5]. Let $\pi$ and $\sigma$ be the same as in Theorem 4.1 and $M$ be a $\mathbb{Z}\pi$-module. Let $n \mid m$ and $f(X) \cdot g(X) \mid X^n - 1$ where $f(X), g(X) \in \mathbb{Z}[X]$. If $M/g(\sigma)M$ is torsion-free, we will show that $0 \to (M/f(\sigma)M)_0 \to (M/f(\sigma)g(\sigma)M)_0 \to M/g(\sigma)M \to 0$ is a short exact sequence.

Let $f(\sigma) \cdot \mathbb{Z}\pi$ be the right ideal of $\mathbb{Z}\pi$ generated by $f(\sigma)$. For any $\lambda \in \pi$, then $\lambda \cdot \sigma \cdot \lambda^{-1} = \sigma^i$ for some $i \geq 1$ with $\gcd\{i, m\} = 1$; thus $\lambda \cdot f(\sigma) \cdot \lambda^{-1} = f(\sigma^i)$. If $d \mid m$, note the $\Phi_d(X^i) = \Phi_d(X) \cdot h(X)$ for some $h(X) \in \mathbb{Z}[X]$, because $\gcd\{i, d\} = 1$ (see, for examples, [Lan] page 280). Since $f(X)$ is a product of cyclotomic polynomials $\Phi_d(X)$ with $d \mid n$, it follows that $f(\sigma^i) \in f(\sigma) \cdot \mathbb{Z}\pi$.

In summary, the right ideal $f(\sigma) \cdot \mathbb{Z}\pi$ is also a left ideal. We write it simply as $\langle f(\sigma) \rangle$. Hence $\mathbb{Z}\pi/\langle f(\sigma) \rangle$, $\mathbb{Z}\pi/\langle f(\sigma)g(\sigma) \rangle$, $\mathbb{Z}\pi/\langle g(\sigma) \rangle$ may be regarded as two-sided $\mathbb{Z}\pi$-modules.

It is not difficult to verify that

$$0 \to \mathbb{Z}\pi/\langle f(\sigma) \rangle \to \mathbb{Z}\pi/\langle f(\sigma)g(\sigma) \rangle \to \mathbb{Z}\pi/\langle g(\sigma) \rangle \to 0$$

is an exact sequence of two-sided $\mathbb{Z}\pi$-modules.

Tensoring the above sequence with $M$ over $\mathbb{Z}\pi$, we get

$$\text{Tor}^\mathbb{Z}\pi_1(\mathbb{Z}\pi/\langle g(\sigma) \rangle, M) \to M/f(\sigma)M \to M/f(\sigma)g(\sigma)M \to M/g(\sigma)M \to 0.$$

Note that $\text{Tor}^\mathbb{Z}\pi_1(\mathbb{Z}\pi/\langle g(\sigma) \rangle, M) \otimes_{\mathbb{Z}\pi} \mathbb{Q} \simeq \text{Tor}^\mathbb{Q}_1(\mathbb{Q}\pi/\langle g(\sigma) \rangle, \mathbb{Q} \otimes_{\mathbb{Z}} M)$. Since $\mathbb{Q}\pi$ is a semi-simple algebra, it follows that $\text{Tor}^\mathbb{Q}_1(\mathbb{Q}\pi/\langle g(\sigma) \rangle, \mathbb{Q} \otimes_{\mathbb{Z}} M) = 0$. Thus the kernel of $\langle M/f(\sigma)M \to M/f(\sigma)g(\sigma)M \rangle$ is torsion. Now we may apply [Sw6, Lemma 5.4] because $M/g(\sigma)M$ is torsion-free by assumption.

Step 3. The remaining proof is the same as that given in [Sw6 pages 246–247]. Hence the result. \qed
**Theorem 4.2** Let $\pi$ be a finite group such that all the Sylow subgroups are cyclic, $\sigma \in \pi$ such that the subgroup $\langle \sigma \rangle \simeq C_m$ is normal in $\pi$. Suppose that $n \mid m$ and $M$ is an invertible $\pi$-lattice. Then

$$\left[ M/\sigma^n M \right] = \sum_{d|n} \left[ (M/\Phi_d M) \right]$$

and

$$\left[ (M/\Phi_n M) \right] = \sum_{d|n} \mu \left( \frac{n}{d} \right) \left[ (M/\sigma^d M) \right]$$

where $\mu$ is the Möbius function.

**Proof.** Note that $F_{\pi}$ is a group by Theorem 2.6.

In Theorem 4.1 define $\phi(M) = \left[ M \right]$. If $0 \to N' \to N \to N'' \to 0$ is an exact sequence of $\pi$-lattice such that $N''$ is invertible, then $\left[ N \right] = \left[ N' \right] + \left[ N'' \right]$ by Lemma 2.4 (1). Thus the required assumption about $\phi$ is fulfilled. Hence the result.

We will give another proof of this theorem by using a method of Lenstra [Le, page 308].

Let $E(n)$ be the set of all positive divisors of $n$, $G(n)$ be the set of all partitions (i.e. all equivalence relations) on the set $E(n)$.

For $u \in G(n)$, denote by $[u]$ the set of all disjoint subsets of the partition $u$ (i.e. the set of all non-empty equivalence classes of $u$).

Define a undirected graph on $G(n)$: Two vertices $u, v \in G(n)$ are connected by an edge (denoted by $u \sim v$), if there is some $d$ dividing $n$ and some $D \in [u]$ such that $E(d) \subseteq D$ and $[v] = \{E(d), D \setminus E(d), C : C \in [u] \setminus \{D\}\}$.

It is proved that the graph $G(n)$ defined above is connected [Le, page 308, Lemma 2.10].

Then we follow the arguments of [Le] pages 308–309. Let $M$ be an invertible lattice. If $D$ is a subset of $E(n)$, define

$$M_D = M/\left( \prod_{d \in D} \Phi_d \right) M.$$ 

Suppose $u, v \in G(n)$ and $u \sim v$. Then we get an exact sequence of $\mathbb{Z}\pi$-modules

$$0 \to (M_D \setminus E(d))_0 \to (M_D)_0 \to M_{E(d)} \to 0$$

because $M_{E(d)} = M/(\sigma^d - 1)M$ is invertible by [Sw6] Lemma 5.3] (alternatively, we may use [Le] Lemma 2.8] which is valid if $M$ is invertible).

Adding a summand

$$N = \bigoplus_{C \in [u] \setminus \{D\}} M_C$$

yields an exact sequence

$$0 \to (N)_0 \oplus (M_D \setminus E(d))_0 \to (N)_0 \oplus (M_D)_0 \to M_{E(d)} \to 0.$$
Let $K/k$ be a Galois extension with $\pi = \text{Gal}(K/k)$. Then $K((N)_0 \oplus (M_{E(d)})_0 \oplus (M_{E(d)})_0^n \simeq K((N)_0 \oplus (M_D)_0 \oplus (M_{E(d)})_0^n \simeq K((N)_0 \oplus (M_D)_0 \oplus (M_{E(d)})_0^n \simeq (M(u))_0^n \simeq K(M(u))_0^n$.

The remaining proof is similar to “Proof of (2.4)” in [EM4, page 96]. Since $G(m)$ is connected, it follows that $K(M/(\sigma^n - 1)M) \simeq K(\bigoplus_{d|n} (M/\Phi_d(\sigma)M)_0^n$ and therefore $[M/(\sigma^n - 1)M] = \sum_{d|n} [(M/\Phi_d(\sigma)M)_0^n]$ in $T(\pi)$ (by the definition of $T(\pi)$). But this is equivalent to $[M/(\sigma^n - 1)M]^{\ell} = \sum_{d|n} [(M/\Phi_d(\sigma)M)_0]^{\ell}$. 

Remark. An intuitive interpretation of the above proof is contained in [EM4, page 96]. For examples, when $\pi = \langle \sigma \rangle \simeq C_{p^t}$ for some $t \geq 1$. Define $\chi^{(0)}(X) = X^{p^t} - 1 \in \mathbb{Z}[X]$, $b = p^t$, $\Psi^{(1)}(X) = \Phi_{p^t}(X)$, $b_1 = p^{t-1}$ (in the notation of [EM4, page 96]). We get a short exact sequence of $\pi$-lattices

$$0 \rightarrow \mathbb{Z}\pi/\langle \Psi^{(1)}(\sigma) \rangle \rightarrow \mathbb{Z}\pi \rightarrow \mathbb{Z}\pi/\langle \sigma^{b_1} - 1 \rangle \rightarrow 0$$

where $\mathbb{Z}\pi/\langle \Psi^{(1)}(\sigma) \rangle \simeq \mathbb{Z}[\zeta_{p^t}]$.

Here is another example. Let $\pi = \langle \sigma \rangle \simeq C_{pq}$ where $p, q$ are distinct prime numbers. Define $\chi^{(0)}(X) = \Phi_{pq}(X)$, $b = pq$, $\Psi^{(1)}(X) = \Phi_{pq}(X)\Phi_q(X)$, $b_1 = p$. Then we have

$$0 \rightarrow \mathbb{Z}\pi/\langle \Psi^{(1)}(\sigma) \rangle \rightarrow \mathbb{Z}\pi \rightarrow \mathbb{Z}\pi/\langle \sigma^{b_1} - 1 \rangle \rightarrow 0.$$ 

Further, let $\chi^{(1)}(X) = \Phi_{pq}(X)\Phi_q(X)\Phi_1(X)$, $b_2 = 1$. Then we have

$$0 \rightarrow \mathbb{Z}\pi/\langle \chi^{(1)}(\sigma) \rangle \rightarrow \mathbb{Z}\pi/\langle \chi^{(1)}(\sigma) \rangle \rightarrow \mathbb{Z}\pi/\langle \sigma^{b_2} - 1 \rangle \rightarrow 0.$$ 

Finally, let $\Psi^{(2)}(X) = \Phi_{pq}(X)$, $b_3 = q$. We have

$$0 \rightarrow \mathbb{Z}\pi/\langle \Psi^{(2)}(\sigma) \rangle \rightarrow \mathbb{Z}\pi/\langle \chi^{(1)}(\sigma) \rangle \rightarrow \mathbb{Z}\pi/\langle \sigma^{b_3} - 1 \rangle \rightarrow 0$$

where $\mathbb{Z}\pi/\langle \Psi^{(2)}(\sigma) \rangle \simeq \mathbb{Z}[\zeta_{pq}]$.

The five short exact sequences in the middle of page 96 of [EM4] are just the general case of the above arguments. Note that $b_i \parallel b$ on [EM4, page 96] means that $b_i \mid b$ and $b_i \neq b$. Similar exact sequences appear in [EM1, page 12] and [EM2, pages 401–402].

§5. Proof of Theorem 1.4

We will devote this section to proving “(1) $\Rightarrow$ (2) of Theorem 1.4”. The goal is to show that $T(\pi) = T^3(\pi), T^9(\pi) \simeq C(\mathbb{Z}\pi)/C^{\Omega}(\mathbb{Z}\pi)$ and $C(\mathbb{Z}\pi)/C^{\Delta}(\mathbb{Z}\pi) \simeq C(\Omega_{E})$.

The key ideas for the proof of $T^g(\pi) \simeq C(\mathbb{Z}\pi)/C^{\Delta}(\mathbb{Z}\pi)$ is as follows. We use Theorem 2.11 (1) to prove $T^g(\pi) \simeq C(\mathbb{Z}\pi)/C^{\Delta}(\mathbb{Z}\pi)$, i.e. for any invertible $\pi$-lattice $M$, we will find a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ such that $[M]^f = [\mathcal{A}]^f$. For this purpose, we apply Theorem 1.2 and reduce the question to the situation of $(M/\Phi_d(\sigma)M)_0$ for
any \(d \mid n\) where \(\langle \sigma \rangle \simeq C_n\) and \(\sigma\) is some element in the group \(\pi\). Since \((M/\Phi_d(\sigma)M)\) is a torsion-free module over \(\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle\), it is important to understand the structure of modules over the \(\mathbb{Z}\)-order \(\Lambda_d := \mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle\). In most situations, \(\Lambda_d\) is a Dedekind domain, a twisted group ring or a maximal \(\mathbb{Z}\)-order. Thus the results in Section 3 are applicable. The final blow is to use Theorem 2.10, i.e. Jacobinski-Roiter’s Theorem, to find the projective ideal \(\mathcal{A}_d\) for \((M/\Phi_d(\sigma)M)\).

Before the proof, we recall some basic facts of maximal orders [CR1, Section 26; Re1].

**Definition 5.1** Let \(K\) be a field, \(\Sigma\) be a finite-dimensional separable algebra over \(K\). Let \(R\) be a Dedekind domain with quotient field \(K\), and \(\Lambda\) be a maximal \(R\)-order in \(\Sigma\). A finitely generated left \(\Lambda\)-module \(M\) is called a \(\Lambda\)-lattice if it is a projective \(\Lambda\)-module.

**Theorem 5.2** ([CR1, page 565]) Let the notations be the same as in Definition 5.1, let \(\Lambda\) be a maximal \(R\)-order and \(K\) be the quotient field of \(R\). Then

1. \(\Lambda\) is a left and right hereditary ring.
2. Every left \(\Lambda\)-lattice is \(\Lambda\)-projective.
3. A left \(\Lambda\)-lattice \(M\) is indecomposable if and only if \(KM\) is a simple module over \(K\Lambda\).

**Theorem 5.3** ([AG1, Theorem 2.3]) Let \(R\) be a DVR with quotient field \(K\), \(\Lambda\) be an \(R\)-order such that \(K\Lambda\) is a central simple algebra over a field containing \(K\). Then \(\Lambda\) is a maximal order if and only if \(\Lambda\) is a hereditary ring and \(\text{rad}(\Lambda)\) is a maximal two-sided ideal where \(\text{rad}(\Lambda)\) is the Jacobson radical of \(\Lambda\).

*Proof of (1) ⇒ (2) of Theorem 5.2.*

First we show that \(T(\pi) = T^q(\pi)\).

Note that \(\pi\) is a group such that all Sylow subgroups of \(\pi\) are cyclic. Hence any coflabby \(\pi\)-lattice is invertible by Theorem 2.6. On the other hand, if \(M\) is any \(\pi\)-lattice, by Lemma 2.5 (2), there is an exact sequence \(0 \to M \to C \to P \to 0\) where \(C\) is coflabby (and hence invertible) and \(P\) is permutation. Apply Lemma 2.4 (1). We get \([C]^{|\pi|} = [M]^{|\pi|} + [P]^{|\pi|} = [M]^{|\pi|}\).

In conclusion, for any \(\pi\)-lattice \(M\) with \([M] \in T(\pi)\), there is an invertible \(\pi\)-lattice \(C\) such that \([M] = [C]\) in \(T(\pi)\) by Lemma 2.2. Thus \(T(\pi) = T^q(\pi)\).

We will use [EM3, Theorem 4.2] to show that \(C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) \simeq C(\Omega_{\mathbb{Z}\pi})\), i.e. \(C^q(\mathbb{Z}\pi) = \tilde{C}^q(\mathbb{Z}\pi) = C(\mathbb{Z}\pi)\) (see Definition 2.13).

We remark first that [EM3, page 707, line 6] contains a misprint which should be corrected as follows: If \(\pi = C \rtimes P\) where \(C = \langle \sigma \rangle \simeq C_n\) is normal in \(\pi\) and \(m \mid n\), the natural map \(\mu_m : P \to \text{Aut}(C/\langle \sigma^m \rangle)\) is induced from the action of \(P\) on \(C\) and \(C/\langle \sigma^m \rangle\). Define \(D_m = \text{Ker}(\mu_m)\). In particular, \(P_n = \{\lambda \in P : \lambda \sigma \lambda^{-1} = \sigma\}\).

Return to the group \(\pi\) in (1) of Theorem 1.4. We may write \(\pi = C \rtimes P\) where \(C\) is cyclic, \(P = \{1\}, C_2\) or \(C_4\). Thus the assumptions (c) or (d) of [EM3, Theorem 4.2]
are fulfilled. We conclude that $C^q(\mathbb{Z}\pi) = \tilde{C}^q(\mathbb{Z}\pi) = \tilde{C}(\mathbb{Z}\pi)$. Hence $C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi) \simeq C(\Omega_{\mathbb{Z}\pi})$.

By Theorem 2.14 to show that $T^q(\pi) \simeq C(\mathbb{Z}\pi)/C^q(\mathbb{Z}\pi)$, it suffices to show that, for any invertible $\pi$-lattice $M$, there is a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ such that $[M]^{fl} = [\mathcal{A}]^{fl}$. This is what we will prove in the sequel. The ideas of the proof have been explained at the beginning of this section. Once it is proved, the proof of (1) $\Rightarrow$ (2) is finished.

Case 1. $\pi = \langle \sigma \rangle \simeq C_n$.

We remark that this case is proved also in [Sw6, Theorem 2.10] using some ideas in [EM4] and [EM5]. In the following, we give a slightly different proof.

Step 1. For any $d \mid n$, note that $\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle \simeq \mathbb{Z}[\zeta_d]$ where $\zeta_d$ is some primitive $d$-th root of unity.

Moreover, by Formula (2) of Theorem 4.2 we find that

$$[\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle]^{fl} = \sum_{e \mid d} \mu\left(\frac{d}{e}\right) [\mathbb{Z}\pi/\langle \sigma^e - 1 \rangle]^{fl}. $$

Since $\mathbb{Z}\pi/\langle \sigma^e - 1 \rangle \simeq \mathbb{Z}[\pi/\pi']$ where $\pi' = \langle \sigma^e \rangle$, it follows that $\mathbb{Z}\pi/\langle \sigma^e - 1 \rangle$ is a permutation $\pi$-lattice and therefore $[\mathbb{Z}\pi/\langle \sigma^e - 1 \rangle]^{fl} = 0$. Thus $[\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle]^{fl} = 0$.

Step 2. Suppose that $M$ is an invertible $\pi$-lattice.

Apply Formula (1) of Theorem 4.2. We get

$$[M]^{fl} = [M/\langle \sigma^n - 1 \rangle M]^{fl} = \sum_{d \mid n} [(M/\Phi_d(\sigma)M_0]^{fl}. $$

Since $(M/\Phi_d(\sigma)M_0)$ is a torsion-free module over $\mathbb{Z}/\Phi_d(\sigma) \simeq \mathbb{Z}[\zeta_d]$, it follows that $(M/\Phi_d(\sigma)M_0) \simeq F_d \oplus J_d$ where $F_d$ is a free module over $\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle$, $J_d$ is a projective ideal over $\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle$. Hence $[F_d]^{fl} = 0$ because $[\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle]^{fl} = 0$ by Step 1.

Thus we get $[M]^{fl} = \sum_{d \mid n} [J_d]^{fl}$.

Step 3. Each $J_d$ is locally isomorphic to $\mathbb{Z}\pi/\Phi_d(\sigma)$. Hence $J_d$ and $\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle$ belong to the same genus. Apply Theorem 2.10 to $J_d \oplus \mathbb{Z}\pi$ and $\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle$. We find a projective ideal $\mathcal{A}_d$ over $\mathbb{Z}\pi$ such that $J_d \oplus \mathbb{Z}\pi \simeq \mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle \oplus \mathcal{A}_d$. Hence $[J_d]^{fl} = [\mathcal{A}_d]^{fl}$ because $[\mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle]^{fl} = 0$ by Step 1.

Step 4. By [Sw1] page 552, $\bigoplus_{d \mid n} \mathcal{A}_d \simeq F \oplus \mathcal{A}$ for some projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ and some free $\mathbb{Z}\pi$-module $F$. Hence $[M]^{fl} = \sum_{d \mid n} [J_d]^{fl} = \sum_{d \mid n} [\mathcal{A}_d]^{fl} = [F \oplus \mathcal{A}]^{fl} = [\mathcal{A}]^{fl}$ as expected.

Step 5. We rephrase the above result as a form which will be used in the sequel.

Let $\pi \simeq C_n$, $M$ be a $\pi$-lattice. Then there is a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ such that $[M]^{fl} = [\mathcal{A}]^{fl}$. 

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As before, by Lemma 2.3 (2), find an exact sequence \( 0 \to M \to N \to P \to 0 \) where \( N \) is coflabby and \( P \) is permutation. Apply Lemma 2.4 (1). We get \([N]^{fl} = [M]^{fl} + [P]^{fl} = [M]^{fl}\).

By Theorem 2.6 any coflabby \( \pi \)-lattice is invertible; thus \( N \) is invertible. Since \( N \) is invertible, it is possible to find a projective ideal \( A \) over \( \mathbb{Z}_\pi \) such that \([N]^{fl} = [A]^{fl}\) from the above Steps 1,2,3 and 4. It follows that \([M]^{fl} = [A]^{fl}\).

Case 2. \( \pi = \langle \sigma, \tau : \sigma^m = \tau^2 = 1, \tau^{-1} \sigma \tau = \sigma^{-1} \rangle \cong D_m \) where \( m \) is an odd integer with \( m \geq 3 \).

Subcase 2.1 \( m = p^c \) where \( p \) is an odd prime number and \( c \geq 1 \).

Step 1. We use similar methods as in Case 1. For any \( d \mid p^c \), \( \mathbb{Z}_\pi / \langle \Phi_d(\sigma) \rangle \) is no longer isomorphic to \( \mathbb{Z}[\zeta_d] \).

When \( d = 1 \), \( \mathbb{Z}_\pi / \langle \Phi_d(\sigma) \rangle \) is isomorphic to \( \mathbb{Z}_{\pi'} \) where \( \pi' \cong C_2 \). Since \((M/\Phi_d(\sigma)M)_0\) is a torsion-free module over \( \mathbb{Z}_\pi / \langle \Phi_d(\sigma) \rangle \) (where \( M \) is an invertible \( \pi \)-lattice), we may apply Step 5 of Case 1. Thus there is a projective ideal \( A \) over \( \mathbb{Z}_{\pi'} \) such that \([M/\Phi_d(\sigma)M]^{fl} = [A]^{fl}\). Note that \( A \) and \( \mathbb{Z}_{\pi'} \) are in the same genus as \( \pi'-lattices \) and also as \( \pi \)-lattices. Apply Theorem 2.10 to find a projective ideal \( B \) over \( \mathbb{Z}_\pi \) such that \( A \oplus \mathbb{Z}_{\pi} \cong \mathbb{Z}_{\pi'} \oplus B \). Hence \([M/\Phi_d(\sigma)M]^{fl} = [B]^{fl}\). Done.

When \( d \mid p^c \) and \( d > 1 \), we will show that \( \mathbb{Z}_\pi / \langle \Phi_d(\sigma) \rangle \) is isomorphic to a twisted group ring.

Step 2. For any \( d \mid p^c \) such that \( d > 1 \), write \( \Lambda_d := \mathbb{Z}_\pi / \langle \Phi_d(\sigma) \rangle = S_d \circ H \) where \( S_d = \mathbb{Z}[\zeta_d] \), \( H = \langle \tau \rangle \) and \( \zeta_d \in S_d \) is the image of \( \sigma \) in \( \mathbb{Z}_\pi / \langle \Phi_d(\sigma) \rangle \). Define \( R_d = S_d(\tau) \); note that \( \tau \cdot \zeta_d = \zeta_d^{-1} \).

The only prime ideal which ramifies in \( \mathbb{Z}[\zeta_{p^c}] \) over \( \mathbb{Z} \) is the prime ideal lying over \( p \mathbb{Z} \) and \( p \mathbb{Z}[\zeta_{p^c}] = Q^{p^c-1}(p-1) \) where \( Q = (1 - \zeta_{p^c}) \mathbb{Z}[\zeta_{p^c}] \) (see, for examples, [CR1] page 96). Thus, for any \( d \mid p^c \) (with \( d > 1 \)), \( S_d \) is tamely ramified over \( R_d \).

By Theorem 3.6, \( \Lambda_d \) is a left hereditary ring and the ambiguous ideals are isomorphic to \( IS_d \) or \( IQ_d \) where \( I \) is an ideal in \( R_d \) and \( Q_d = Q \cap S_d \) where \( Q = (1 - \zeta_{p^c}) \mathbb{Z}[\zeta_{p^c}] \).

Note that \( IS_d \) and \( S_d \) belong to the same genus because they are locally isomorphic; similarly for \( IQ_d \) and \( Q_d \).

Step 3. For any \( d \mid p^c \), applying Formula (2) of Theorem 4.2, we find that \([\mathbb{Z}_\pi / \Phi_d(\sigma)]^{fl} = 0 \) because \( \mathbb{Z}_\pi / \langle \sigma^{p^c} - 1 \rangle \cong \mathbb{Z}[\pi / \pi'] \) where \( \pi' = \langle \sigma^{p^c} \rangle \) if \( e' \mid d \). In conclusion, \([\Lambda_d]^{fl} = 0 \).

On the other hand, if \( d \mid p^c \), define \( \sigma'' = \langle \sigma^d, \tau \rangle \) and \( N = \mathbb{Z}[\pi / \pi''] \). It is not difficult to verify that \((N/\Phi_d(\sigma)N)_0 \cong S_d \) as \( \pi \)-lattices.

By Theorem 4.2 to \( N = \mathbb{Z}[\pi / \pi'] \). For any \( e' \mid d \), since \( N / \langle \sigma^{p^c} - 1 \rangle N \cong \mathbb{Z}[\pi / \pi'] \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi / \pi''] \) (where \( \pi'' = \langle \sigma^{p^c} \rangle \)) is a permutation \( \pi \)-lattice by [Sw6] Lemma 5.3, it follows that \([S_d]^{fl} = 0 \).

Step 4. For any \( d \mid p^c \) (with \( d > 1 \)), we will prove in Step 6 that there is an ideal \( I \) of \( R_d \) such that \( \Lambda_d \cong S_d \oplus Q_dI \) as \( \Lambda_d \)-modules, and hence as \( \pi \)-lattices.

Assume the above claim. By Step 3, \([\Lambda_d]^{fl} = [S_d]^{fl} = 0 \), it follows that \([Q_dI]^{fl} = 0 \).
Let $M$ be an invertible $\pi$-lattice. Apply Theorem 4.2 as in Step 3 of Case 1. Since $(M/\Phi_d(\sigma)M)_0$ is a torsion-free module over $\mathbb{Z}\pi/(\Phi_d(\sigma)) = \Lambda_d$, we may apply results in Step 2. Thus $(M/\Phi_d(\sigma)M)_0$ is a direct sum of $S_d I_i$ $(1 \leq i \leq u)$ and $Q_d I_j$ $(1 \leq j \leq v)$ where $I_i$, $I_j$ are ideals in $R_d$. In particular, $(M/\Phi_d(\sigma)M)_0$ and $S_d^{(u)} \oplus (Q_d I)^{(v)}$ are in the same genus. Applying Theorem 2.10, we find a projective ideal $A_d$ over $\mathbb{Z}\pi$ such that $(M/\Phi_d(\sigma)M)_0 \oplus \mathbb{Z}\pi \simeq (S_d^{(u)} \oplus (Q_d I)^{(v)}) \oplus A_d$. Thus $[(M/\Phi_d(\sigma)M)_0]^{fl} = [A_d]^{fl}$.

Step 5. From Step 1 and Step 4, we find $[M]^{fl} = \sum_{d \mid n} [A_d]^{fl}$ by Theorem 4.2 (the reader may compare the present situation with Step 3 of Case 1).

Use the same argument as in Step 4 of Case 1. It is easy to find a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ satisfying $[M]^{fl} = [\mathcal{A}]^{fl}$.

Step 6. It remains to show that $\Lambda_d \simeq S_d \oplus Q_d I$ for some ideal $I$ of $R_d$.

Recall that $\Lambda_d = S_d \circ H = S_d + S_d u_\tau$ such that $u_\tau^2 = 1$, $u_\tau \cdot \alpha = \tau(\alpha) \cdot u_\tau$ for any $\alpha \in S_d$.

Define $f : S_d \to \Lambda_d$ by $f(\alpha) = \alpha(1 + u_\tau)$ for any $\alpha \in S_d$. Note that $f$ is an injective morphism of $\Lambda_d$-modules. Define $N = \Lambda_d / f(S_d)$.

Since $N$ is torsion-free, we find $N \simeq S_d I$ or $Q_d I$ for some ideal $I$ of $R_d$ by Theorem 3.6 Moreover, the sequence $0 \to S_d \to \Lambda_d \to N \to 0$ splits, because all the terms of the exact sequence are projective modules. We will prove that $N \simeq Q_d I$.

Let $\mathbb{Z}_{(p)}$ be the complete DVR of $p$-adic integers. Then $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \Lambda_d \simeq (\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} S_d) \oplus (\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} N)$. Because $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \Lambda_d$ is a semi-local ring, we find that $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} S_d I \simeq \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} S_d$ and $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} Q_d I \simeq \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} Q_d$. However, by [Ro, Theorem 4.6], we have $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \Lambda_d \simeq (\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} S_d) \oplus (\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} Q_d)$. Use Krull-Schmidt-Azumaya Theorem (see [CR11, page 128]) to cancel the factor $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} S_d$ (note that $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \Lambda_d$ is a finitely generated module over the complete local ring $\mathbb{Z}_{(p)}$). Hence $N \simeq Q_d I$.

Subcase 2.2 There exist distinct odd prime numbers $p_1$ and $p_2$ such that $p_1 p_2 \mid m$.

Step 1. Let $m'$ be an odd inter such that $p_1 p_2 \mid m'$ for two distinct prime numbers. Then $\mathbb{Z}[\zeta_{m'}]$ is unramified over $\mathbb{Z}[\zeta_{m'} + \zeta_{m'}^{-1}]$ by [Wa] page 16, Proposition 2.15; Sw4, page 153, Corollary B.10]. This observation will play a crucial role in the subsequent proof.

Step 2. Let $M$ be any invertible $\pi$-lattice. We will find a projective ideal $\mathcal{A}$ over $\mathbb{Z}\pi$ such that $[M]^{fl} = [\mathcal{A}]^{fl}$. The proof is similar to that of Subcase 2.1.

Apply Theorem 4.2. We may reduce the problem to the case $(M/\Phi_d(\sigma)M)_0$ for any $d \mid m$.

If $d = 1$, then Step 5 of Case 1 takes care of this situation (see Step 1 of Subcase 2.1).

If $d > 1$, note that $(M/\Phi_d(\sigma)M)_0$ is a torsion-free module over $\Lambda_d := \mathbb{Z}\pi/(\Phi_d(\sigma)) = S_d \circ H$ where $S_d = \mathbb{Z}[\zeta_d]$, $H = \langle \tau \rangle$ and $\zeta_d$ is the image of $\sigma$ in $\mathbb{Z}\pi/(\Phi_d(\sigma))$. The group $H$ acts on $S_d$ by $\tau \cdot \zeta_d = \zeta_d^{-1}$. Define $R_d = S_d^{(r)} = \mathbb{Z}[\zeta_d + \zeta_d^{-1}]$.

Suppose that $d$ has two distinct prime divisors. Then $S_d / R_d$ is unramified by Step 1. Thus $\Lambda_d$ is hereditary; moreover, $S_d I$ are the only ambiguous ideals of $S_d$ (where $I$ runs over some ideals in $R_d$) by Theorem 3.6. We conclude that $(M/\Phi_d(\sigma)M)_0$ and $S_d^{(u)}$...
belong to the same genus. It can be proved that $[S_d]^{fl} = 0$ as in Step 3 of Subcase 2.1. Thus we may find a projective ideal $A_d$ over $\mathbb{Z}\pi$ satisfying $[(M/\Phi_d(\sigma)M)_0]^{fl} = [A_d]^{fl}$ as Step 4 of Subcase 2.1.

On the other hand if $d = p^c$ for some odd prime number $p$ and some $c \geq 1$, then the ideal $(1 - \zeta_{p^c})S_d$ is the only ramified prime ideal of $S_d$ over $R_d$; moreover, $S_d$ is tamely ramified over $R_d$. The remaining proof is the same as that of Subcase 2.1. Done.

**Case 3.** $\pi = (\sigma', \rho, \tau : \sigma'^0 = \rho^m = \tau^2 = 1, \tau^{-1}\sigma'\tau = \sigma', \tau^{-1}\rho\tau = \rho\sigma') \simeq C_{q'} \times D_m$ where $q$ is an odd prime number, $f \geq 1$, $m$ is an odd integer $\geq 3$, $\gcd\{q, m\} = 1$, and $\langle \mathbb{Z}/q\mathbb{Z}\rangle^\times = \langle \bar{p} \rangle$ for any prime divisor $p$ of $m$.

The proof is similar to that of Case 2.

**Step 1.** Define $\sigma = \sigma'\rho$. The $\sigma$ is an element of order $q^f m$ and $\langle \sigma \rangle \leq \pi$.

Let $M$ be an invertible $\pi$-lattice.

For any $d \mid q^f m$, we will find a projective ideal $A_d$ over $\mathbb{Z}\pi$ such that $[(M/\Phi_d(\sigma)M)_0]^{fl} = [A_d]^{fl}$.

Note that $(M/\Phi_d(\sigma)M)_0$ is a module over $\Lambda_d$ where $\Lambda_d := \mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle$.

Write $d = d_1 d_2$ where $d_1 \mid q^f$, $d_2 \mid m$. Denote by $\zeta_d$, $\zeta_{d_1}$, $\zeta_{d_2}$ the images of $\sigma$, $\sigma'$, $\rho$ in $\Lambda_d$. Note that $\Lambda_d = \mathbb{Z}[\zeta_d, \tau] = \mathbb{Z}[\zeta_{d_1}, \zeta_{d_2}, \tau]$ and $\tau^{-1} \cdot \zeta_{d_1} \cdot \tau = \zeta_{d_1}$, $\tau^{-1} \zeta_{d_2} \tau = \zeta_{d_2}^{-1}$.

Moreover $\mathbb{Z}\zeta_d = \mathbb{Z}[\zeta_{d_1}] \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_{d_2}]$, because $\gcd\{d_1, d_2\} = 1$.

**Step 2.** Suppose $d_2 > 1$. Then $\zeta_{d_2} = 1$.

We find that $\Lambda_d = \mathbb{Z}[\zeta_{d_1}, \tau]$. Define $\pi' = (\sigma', \tau) \simeq \pi/\langle \rho \rangle$. Since $\mathbb{Z}\pi' \rightarrow \mathbb{Z}[\zeta_{d_1}, \tau] = \Lambda_d$ is surjective, we may regard $(M/\Phi_d(\sigma)M)_0$ as a $\mathbb{Z}\pi'$-module.

Since $\pi'$ is a cyclic group, we may apply Step 5 of Case 1. Thus there is a projective ideal $A$ over $\mathbb{Z}\pi'$ such that $[(M/\Phi_d(\sigma)M)_0]^{fl} = [A]^{fl}$. As in Step 1 of Subcase 2.1, we may find a projective ideal $B$ over $\mathbb{Z}\pi$ such that $[(M/\Phi_d(\sigma)M)_0]^{fl} = [B]^{fl}$.

**Step 3.** Now assume that $d_2 > 1$. Then $\Lambda_d = \mathbb{Z}\pi/\langle \Phi_d(\sigma) \rangle$ can be written as a twisted group ring: $\Lambda_d = S_d \underset{H}{\circ} H$ where $S_d = \mathbb{Z}[\zeta_{d_1}]$, $H = \langle \tau \rangle$. Note that $H$ acts on $S_d$ by $\tau \cdot \zeta_{d_1} = \zeta_{d_1}$, $\tau \cdot \zeta_{d_2} = \zeta_{d_2}^{-1}$. Define $R_d = S_d^{\langle \tau \rangle} = \mathbb{Z}[\zeta_{d_1}][\zeta_{d_2} + \zeta_{d_2}^{-1}]$.

Assume that $p_1 p_2 \mid d_2$ for two distinct odd prime numbers (the case $d_2 = p^c$ for an odd prime number will be considered in Step 4).

Then $S_d^u R_d$ is unramified by [Wa] page 16, Proposition 2.15. Hence $\Lambda_d$ is hereditary; moreover, $S_d^u I$ are the only ambiguous ideals of $S_d$ (where $I$ runs over some ideals in $R_d$) by Theorem 3.6. It follows that $(M/\Phi_d(\sigma)M)_0$ and $S_d^u$ are in the same genus. The remaining proof is the same as in the proof of Step 2 of Subcase 2.2.

**Step 4.** Suppose that $d_2 = p^c$ for some odd prime number $p$ and some $c \geq 1$.

Write $d_1 = q^{f'}$ for some $f' \leq f$. Remember the assumption that $(\mathbb{Z}/q^f\mathbb{Z})^\times = \langle \bar{p} \rangle$.

This assumption is equivalent to the fact that $p \cdot \mathbb{Z}[\zeta_{q^{f'}}]$ is a prime ideal in $\mathbb{Z}[\zeta_{q^{f'}}]$ (just think of the decomposition of $p \cdot \mathbb{Z}[\zeta_q]$ and the Frobenius automorphism associated to a prime ideal $Q$ of $\mathbb{Z}[\zeta_q]$ such that $p \in Q$). Since $p \cdot \mathbb{Z}[\zeta_q]$ is a prime ideal, it follows that $p \cdot \mathbb{Z}[\zeta_{q^{f'}}]$ is also a prime ideal in $\mathbb{Z}[\zeta_{q^{f'}}]$ for any $f' \leq f$.

Return to $\Lambda_d = S_d \circ H$ with $H = \langle \tau \rangle$, $R_d = S_d^{\langle \tau \rangle}$ and $d = q^{f'} p^c$. 

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We claim that the principal ideal \((1 - \zeta^r) = (1 - \zeta^p)S_d\) is a prime ideal. For, 
\[ S_d/(1 - \zeta^r) = \mathbb{Z}[[\zeta^r, \zeta^p]]/(1 - \zeta^p) \cong \mathbb{F}_p[X]/\Phi_q(X) \text{ is a field because } p \cdot \mathbb{Z}[\zeta^p] \text{ is a prime ideal (see, for examples, [Wa] page 15, Proposition 2.14)}. 

Note that \(\Lambda_d\) is tamely ramified. Define \(Q_d = (1 - \zeta^r)\). As in Subcase 2.1, it can be shown that \([\Lambda_d]^{\text{fl}} = [S_d]^{\text{fl}} = [Q_d \cdot I]^{\text{fl}} = 0\) for some ideal \(I\) in \(R_d\). The remaining proof is the same and is omitted.

Step 5. Finally we remark that we cannot replace the prime power \(q^f\) in the assumption by an odd integer \(n\) with \(\gcd\{n, m\} = 1\) in this situation, because \(\mathbb{F}_p[X]/\Phi_n(X)\) is not an integral domain if \(q_1q_2 \mid n\) for two distinct odd prime numbers \(q_1, q_2\) different from \(p\). Thus \((1 - \zeta^p)\) is not a prime ideal of \(S_d\) if \(\mathbb{F}_p[X]/\Phi_n(X)\) is not a field. See [EM5] page 188] for further investigation.

Case 4. \(\pi = \langle \sigma, \tau : \sigma^{2m} = \tau^4 = 1, \sigma^m = \tau^2 = 1, \tau^{-1} \sigma \tau = \sigma^{-1} \rangle \cong Q_{4m}\) where \(m\) is an odd integer \(\geq 3\) such that \(p \equiv -3 \pmod{4}\) for any prime divisor \(p\) of \(m\).

Step 1. We follow the method in Cases 1,2,3.

Let \(M\) be an invertible \(\pi\)-lattice.

For any \(d \mid 2m\), consider \((M/\Phi_d(\sigma)M)_0\) which is a module over \(\Lambda_d := \mathbb{Z}[\pi]/(\Phi_d(\sigma))\).

If \(d = 1\), then \(\Lambda_d \simeq \mathbb{Z}[\pi']\) where \(\pi' \simeq C_2\). If \(d = 2\), then \(\Lambda_d \simeq \mathbb{Z}\sqrt{-1};\) thus \(\mathbb{Z}[\pi]/(\sigma^2) \to \Lambda_d\) is surjective (note that \(\mathbb{Z}[\pi]/(\sigma^2) \cong \mathbb{Z}[\pi']\) where \(\pi' \simeq C_4\)). Apply the methods in Step 1 of Subcase 2.1 and Step 2 of Case 3 to settle these two cases.

Now consider the case \(d \geq 3\).

Write \(\zeta_d\) and \(u_\tau\) to be the images of \(\sigma\) and \(\tau\) in \(\Lambda_d\) respectively. Define \(S_d = \mathbb{Z}[\zeta_d]\).

If \(d \mid m\), from \(\tau^2 = \sigma^m\), we find \(u_\tau^2 = 1\) in \(S_d\). Thus \(\Lambda_d \simeq S_d \circ H\) where \(H = \langle \tau^r \rangle \simeq C_2\). The proof is the same as Case 2.

If \(d \mid 2m, d \nmid m\) and \(d \geq 3\), since \(\tau^2 = \sigma^m, (\sigma^m)^2 = \sigma^{2m} = 1\), we find \(u_\tau^2 = -1\) in \(S_d\).

Thus \(\Lambda_d = S_d + S_d \cdot u_\tau\) where \(u_\tau^2 = -1\). Note that \(\Lambda_d\) may be regarded as a crossed-product order \(\Lambda_d = (S_d \circ H)_f\) where \(H = \langle \tau^r \rangle \simeq C_2, f : H \times H \to U(S_d)\) is defined as \(f(1, 1) = f(1, \tau^r) = f(\tau^r, 1) = 1, f(\tau^r, \tau^r) = -1\). Define \(R_d = S_d^{(\tau^r)}\) where \(\tau^r \cdot \zeta_d = \zeta_d^{-1}\).

Step 2. From now on till the end of the proof, we assume \(d \mid 2m, d \nmid m\) and \(d \geq 3\). In Step 3 and Step 4, we will show that \(\Lambda_d\) is a maximal \(R_d\)-order in \((L \circ H)_f\) where \(L = \mathbb{Q}(\zeta_d)\).

Assume this fact. Then \(\Lambda_d\) is hereditary and any \(\Lambda_d\)-lattice is a direct sum of indecomposable projective \(\Lambda_d\)-module by Theorem 5.2.

Let \(K\) be the quotient field of \(R_d\). Since \(K\Lambda_d\) is a central simple \(K\)-algebra of degree 2, it is either a central division \(K\)-algebra or is isomorphic to \(M_2(K)\). We will show that \(K\Lambda_d\) is a central division \(K\)-algebra. Otherwise, the 2-cocycle \(f\) is a 2-coboundary. Equivalently, \(-1\) belongs to the image of the norm map from \(L\) to \(K\). This implies that there exist \(a, b \in K\) satisfying that \(-1 = a^2 + (\zeta_d + \zeta_d^{-1})ab + b^2\), which is impossible because \(K\) is a real field.

Since \(K\Lambda_d\) is a division ring, it is obvious that \(\Lambda_d\) is an indecomposable projective \(\Lambda_d\)-module by Theorem 5.2.
We may prove \([\Lambda_d]_f = 0\) as in Case 2. By Theorem 5.2, \((M/\Phi_d(\sigma)M)_0\) is in the same genus as \(\Lambda_d^{(u)}\). Apply Theorem 2.10 as in Case 2. Thus there is a projective ideal \(\mathcal{A}_d\) over \(\mathbb{Z}[\pi]\) such that \([\mathcal{A}_d]_f = [\Lambda_d]_f\). The remaining proof is the same as before.

Step 3. We will show that \(\Lambda_d\) is a maximal order if \(p_1p_2 \mid d\) for two distinct odd prime numbers \(p_1\) and \(p_2\). The situation when \(d = p^c\) or \(2p^c\) (where \(p\) is an odd prime number and \(c \geq 1\)) will be taken care of in Step 4.

Since \(p_1p_2 \mid d\), \(S_d\) is unramified over \(R_d\) by [Wa] page 16, Proposition 2.15. Thus \(S_d\) is a Galois extension of \(R_d\) relative to \(H = \langle \tau' \rangle \simeq C_2\) in the sense of [AG2] pages 395–402. Thus \(\Lambda_d = (S_d \circ H)_f\) is an Azumaya \(R_d\)-algebra by [AG2] page 402, Theorem A.12. Hence \(\Lambda_d\) is a maximal order in \(K\Lambda_d\) (where \(K\) is the quotient field of \(R_d\)) by [AG2] page 386, Proposition 7.1.

Step 4. When \(d = p^c\) or \(2p^c\) for some odd prime number \(p\), we will show that \(\Lambda_d\) is a maximal order.

Since \(\Lambda_{p^c} \simeq \Lambda_{2p^c}\), we can consider the case \(d = p^c\) only. For simplicity, we write \(\Lambda\) for \(\Lambda_{p^c}\) throughout this step, i.e. \(\Lambda = S + S \cdot u_r\), where \(S = \mathbb{Z}[\zeta_{p^c}], \tau \cdot \zeta_{p^c} = \zeta_{p^c}^{-1}, u_r^2 = -1\) and \(R = S[\tau] = \mathbb{Z}[[\zeta_{p^c} + \zeta_{p^c}^{-1}]]\).

Note that to be a maximal order is a local property [AG1] Proposition 1.4. We will check it at all localizations of \(R\).

Let \(P\) be a prime ideal of \(R\) such that \(p \notin P\). Write \(\Lambda_P, S_P, \text{ and } R_P\) for the localizations of \(\Lambda, S\) and \(R\) at \(P\). It follows that \(S_P\) is unramified over \(R_P\). Hence \(\Lambda_P\) is an Azumaya algebra as in Step 3. Thus \(\Lambda_P\) is a maximal order.

On the other hand, let \(P\) be the prime ideal of \(R\) with \(p \in P\). We will show that \(\Lambda_P\) is a maximal order also.

Since \(S_P\) is tamely ramified over \(R_P\), \(\Lambda_p\) is a hereditary order by Theorem 3.2.

We will apply Theorem 5.3 to show that \(\Lambda_P\) is a maximal order. Let \(K\) be the quotient field of \(R\). It is clear that \(K\Lambda = K\Lambda_P\) is a central simple \(K\)-algebra. It remains to verify that \(\text{rad}(\Lambda_P)\) is a maximal two-sided ideal of \(\Lambda_P\) where \(\text{rad}(\Lambda_P)\) is the Jacobson radical of \(\Lambda_P\).

Consider \(R_P \subset S_P \subset \Lambda_P\). Note that \(\text{rad}(S_P) = (1 - \zeta_{p^c})S_P\). Thus \((1 - \zeta_{p^c})(1 - \zeta_{p^c}^{-1}) \in \text{rad}(R_P)\), because \(S_P\) is integral over \(R_P\). By [Lam] page 74, Corollary 5.9, \((1 - \zeta_{p^c})(1 - \zeta_{p^c}^{-1}) \in \text{rad}(\Lambda_P)\). Note that \((1 - \zeta_{p^c})(1 - \zeta_{p^c}^{-1}) = -\zeta_{p^c}^{-1}(1 - \zeta_{p^c})^2\). We find that \((1 - \zeta_{p^c})^2 \in \text{rad}(\Lambda_P)\). It follows that \(1 - \zeta_{p^c} \in \text{rad}(\Lambda_P)\) (because, for any simple module \(M\) over \(\Lambda_P\), \((1 - \zeta_{p^c})^2M = 0\) implies \((1 - \zeta_{p^c})M = 0\)). Hence the two-sided ideal \((1 - \zeta_{p^c})\Lambda_P = (1 - \zeta_{p^c})\Lambda_P\) is contained in \(\text{rad}(\Lambda_P)\).

We will show that \(\Lambda_P/(1 - \zeta_{p^c})\) is a field. Once it is proved, then \((1 - \zeta_{p^c}) = \text{rad}(\Lambda_P)\) and \(\Lambda_P/\text{rad}(\Lambda_P)\) is a field. And therefore \(\text{rad}(\Lambda_P)\) is a maximal two-sided ideal of \(\Lambda_P\).

It remains to show that \(\Lambda_P/(1 - \zeta_{p^c})\) is a field. Note that \(p \equiv 3 \pmod{4}\) by assumption; this is equivalent to \(\mathbb{F}_p[X]/(X^2 + 1)\) is a field.

Note that \(p \in (1 - \zeta_{p^c})\). We have \(u_r \cdot \zeta_{p^c}^{-1}u_r^{-1} = \zeta_{p^c}^{-1}\). However, in \(\Lambda_d/(1 - \zeta_{p^c})\), we find that \(\bar{u}_r \cdot \bar{\zeta}_{p^c} = \bar{\zeta}_{p^c}^{-1}\bar{u}_r = \bar{\zeta}_{p^c}^{-1}u_r + (\bar{\zeta}_{p^c}^{-1} - \bar{\zeta}_{p^c})\bar{u}_r = \bar{\zeta}_{p^c}^{-1}\bar{u}_r\), because \(\zeta_{p^c}^{-1} - \zeta_{p^c} = \zeta_{p^c}^{-1}(1 + \zeta_{p^c}^{-1})\).
Thus \( \Lambda_P/(1-\zeta_p^r) \) is a commutative ring; furthermore, \( \Lambda_P/(1-\zeta_p^r) \simeq \mathbb{F}_p[\zeta_p, u_\tau]/(1-\zeta_p^r) \) which is a field by assumption.

A final remark. Since \( \Lambda_P/\text{rad}(\Lambda_P) \) is a field, it follows that \( K\Lambda = K\Lambda_P \) is a division ring by \textbf{[AG1, Theorem 3.11]}, which has been proved in Step 2.

We explain briefly the strategy of the proof of Theorem 1.4 In \textbf{[EM4]} and \textbf{[EM5]}, this theorem is proved by showing \( (2) \Rightarrow (3) \Rightarrow (1') \Rightarrow (1) \) where \( (1') \) is given as follows.

(1') \( \pi \) is isomorphic to

(i') a cyclic group, or

(ii') \( C \times H \) where \( C = \langle \sigma' \rangle \) is a cyclic group of order \( n, H = \langle \rho, \tau : \rho^m = \tau^d = 1, \tau^{-1}\rho \tau = \rho^{-1} \rangle \) such that \( n \) and \( m \) are odd positive integers, \( \gcd\{n, m\} = 1 \), \( d \geq 1 \), and, for any prime divisor \( p \) of \( m \), the principal ideal \( p\mathbb{Z}[\zeta_{n-2d}] \) is a prime ideal in \( \mathbb{Z}[\zeta_{n-2d}] \).

The implication of \( (2) \Rightarrow (3) \) is easy, because \( C(\mathbb{Z}[\pi]) \rightarrow C(\mathbb{Q}_{\pi}) \) is surjective \textbf{[?; CR2, page 230, Theorem (49.25)]} and \( C(\mathbb{Z}[\pi]) \) is a finite group by \textbf{[Sw1, page 573]}.

The proof of \( (3) \Rightarrow (1') \) was given in \textbf{[EM5, pages 188–189]} together with \textbf{[EM4, pages 97–98]}. We will not repeat the proof.

For the remaining part of this section, we will give a proof of \( (1) \Leftrightarrow (1') \).

Proof of \( (1) \Leftrightarrow (1') \).

For the proof of \( (1) \Rightarrow (1') \), if \( \pi \simeq C_n \) or \( D_m \), it is trivial to see that \( \pi \) belongs to the class described in \( (1') \). If \( \pi = Q_{4m} = \langle \sigma, \tau : \sigma^2 = 1, \tau^2 = 1, \sigma\tau = \sigma^{-1} \rangle \), define \( \rho = \sigma^2 \). Then \( \pi = \langle \rho, \tau : \rho^{2m} = \tau^d = 1, \tau^{-1}\rho \tau = \rho^{-1} \rangle \). The assumption on the prime divisor \( p \) of \( m \) in (iv) of \( (1) \) is equivalent to \( p \cdot \mathbb{Z}[\sqrt{-1}] \) is a prime ideal in \( \mathbb{Z}[\sqrt{-1}] \) (we have \( d = 2 \), \( n = 1 \) for (ii') of \( (1') \) in this situation).

Now consider the case \( \pi = C_{q^f} \times D_m \) in (iii) of \( (1) \). Take \( n = q^f \) in (ii') of \( (1') \). If \( p \) is a prime divisor of \( m \), since \( p \neq q \), the prime number \( p \) is unramified in \( \mathbb{Q}(\zeta_{q^f}) \). Let \( P \) be a prime ideal of \( \mathbb{Z}[\zeta_{q^f}] \) with \( p \in P \). Since \( (\mathbb{Z}/q^f\mathbb{Z})^\times = \langle \tilde{p} \rangle \), it follows that \( \text{Gal}(\mathbb{Q}(\zeta_{q^f})/\mathbb{Q}) = \langle \varphi \rangle \) where \( \varphi(\zeta_{q^f}) = (\zeta_{q^f})^p \). On the other hand, \( \varphi \) induces the Frobenius automorphism of \( \mathbb{Z}[\zeta_{q^f}]/P \) over \( \mathbb{Z}/p\mathbb{Z} \). It follows that \( P = p\mathbb{Z}[\zeta_{q^f}] \) and thus \( p \) remains prime in \( \mathbb{Z}[\zeta_{q^f}] \) as expected.

For the proof of \( (1') \Rightarrow (1) \), we first note that, if \( n \) is odd and \( d \geq 1 \), then \( (\mathbb{Z}/n \cdot 2^d\mathbb{Z})^\times \) is a cyclic group if and only if \( (n, 2^d) = (1, 2), (1, 4), \) or \( (q^f, 2) \) where \( q \) is some odd prime number.

On the other hand, note that \( \text{Gal}(\mathbb{Q}(\zeta_{n, 2^d})/\mathbb{Q}) \simeq (\mathbb{Z}/n \cdot 2^d\mathbb{Z})^\times \). If a prime number \( p \) remains prime in \( \mathbb{Z}[\zeta_{n, 2^d}] \), then \( [\mathbb{Z}[\zeta_{n, 2^d}]/p\mathbb{Z}[\zeta_{n, 2^d}] : \mathbb{Z}/p\mathbb{Z}] = |(\mathbb{Z}/n \cdot 2^d\mathbb{Z})^\times| \). Thus the Frobenius automorphism of \( p \) generates \( \text{Gal}(\mathbb{Q}(\zeta_{n, 2^d})/\mathbb{Q}) \). Hence \( (\mathbb{Z}/n \cdot 2^d\mathbb{Z})^\times \) is cyclic and \( (n, 2^d) = (1, 2), (1, 4), \) or \( (q^f, 2) \).
When \((n, 2^d) = (1, 2)\), the group \(\pi\) in (ii') of (1') is isomorphic to \(D_m\).
When \((n, 2^d) = (1, 4)\), the group \(\pi\) in (ii') of (1') is isomorphic to \(Q_{4m}\) such that \(p \equiv 3 \pmod{4}\) for any prime divisor \(p\) of \(m\).
When \((n, 2^d) = (q^f, 2)\), the group \(\pi\) in (ii') of (1') is isomorphic to \(C_{q^f} \times D_m\). Since every prime divisor \(p\) of \(m\) remains prime in \(\mathbb{Z}[\zeta_{q^f}]\), write \(P = p\mathbb{Z}[\zeta_{q^f}]\) the prime ideal of \(\mathbb{Z}[\zeta_{q^f}]\). Then the Frobenius automorphism of \(\mathbb{Z}[\zeta_{q^f}]/P\) generates \(\text{Gal}(\mathbb{Q}(\zeta_{q^f})/\mathbb{Q}) \simeq (\mathbb{Z}/q^f\mathbb{Z})^\times\). Thus \((\mathbb{Z}/q^f\mathbb{Z})^\times = \langle \bar{p} \rangle\).

\[\square\]

\section{The maximal orders}

Because of Theorem \ref{thm:main}, we will determine \(C(\Omega_{\mathbb{Z}\pi})\) when \(\pi = C_n, D_n, C_n \times D_m, Q_{4n}\). Given a finite group \(\pi\), there may be more than one maximal orders containing \(\mathbb{Z}\pi\). For our purpose, it is enough to select just one of them.

Maximal orders of \(\mathbb{Z}\pi\) when \(\pi = D_n\) or \(Q_{4n}\) are investigated by Swan \cite{Sw4} pages 75–80]. Our purpose is to compute \(C(\Omega_{\mathbb{Z}\pi})\) using a result of Swan \cite{Sw2}, which will be cited as Theorem \ref{thm:maximal-order}.

**Definition 6.1** Let \(K\) be an algebraic number field, \(A\) be a central simple \(K\)-algebra, \(v\) be a place of \(K\) (finite or infinite). We say that \(A\) ramifies at \(v\) if \([K_v \otimes_K A] \neq 0\) in the Brauer group of \(K_v\) where \(K_v\) is the completion of \(K\) at \(v\) \cite[page 272]{Re}.

**Definition 6.2** Let \(R\) be a Dedekind domain whose quotient field \(K\) is an algebraic number field, and let \(A\) be a central simple \(K\)-algebra. Define \(I(R)\) to be the multiplicative group of \(R\)-ideals in \(K\), and define \(P(R) = \{R\alpha : \alpha \in K \setminus \{0\}\}\). Recall that \(C(R) = I(R)/P(R)\).

Define \(S\) to be the set of all infinite places of \(K\) ramified in \(A\). Define \(P_A(R) = \{R\alpha : \alpha \in K \setminus \{0\}, \alpha_v > 0\text{ for all }v\in S\}\), the principal ray group (mod \(S\)). The ray class group (mod \(S\)), denoted by \(C_A(R)\), is defined as \(C_A(R) = I(R)/P_A(R)\) \cite[page 309]{Re}. If \(S\) is the empty set, then \(C(R) \simeq C_A(R)\).

In general, the kernel of the surjective map \(C_A(R) \rightarrow C(R)\) is the group \(P(R)/P_A(R)\) which may be computed by the exact sequence \(U(R) \rightarrow D \rightarrow P(R)/P_A(R) \rightarrow 0\), where \(U(R)\) is the group of units of \(R\), \(D = \prod_{v \in S} K_v^\times/(K_v^\times)^+\); note that \(K_v^\times\) is the multiplicative group of non-zero elements in \(K_v\), \((K_v^\times)^+\) is the group of positive elements of \(K_v^\times\), and \(K_v^\times/(K_v^\times)^+ \simeq \mathbb{Z}/2\mathbb{Z}\). The map \(U(R) \rightarrow D\) is define by \(\alpha \mapsto (\ldots, \bar{\alpha}_v, \ldots)\) where \(\alpha_v\) is the image of \(\alpha\) in \(K_v^\times\); the map of \(D\) to \(P(R)/P_A(R)\) can be found in \cite[page 139]{Sw3}.

**Theorem 6.3** (Swan \cite[Theorem 1; Re, page 313]{Sw2}) Let the notations be the same as in Definition \ref{def:maximal-order}. Let \(\Lambda\) be a maximal \(R\)-order in \(A\). Then \(C(\Lambda) \simeq C_A(R)\) under the reduced norm map.

**Theorem 6.4** Let \(\pi\) be a group, \(\Omega_{\mathbb{Z}\pi}\) be a maximal \(\mathbb{Z}\)-order in \(\mathbb{Q}\pi\) containing \(\mathbb{Z}\pi\).
(1) If $\pi = C_n$, then
\[ C(\Omega_{\mathbb{Z}[\pi]}) \simeq \oplus_{d|n} C(\mathbb{Z} \langle \zeta_d \rangle). \]

(2) If $\pi = D_n$ where $n$ is an integer $\geq 2$, then
\[ C(\Omega_{\mathbb{Z}[\pi]}) \simeq \oplus_{d|n} C(\mathbb{Z}[\zeta_d + \zeta_d^{-1}]). \]

(3) If $\pi = C_n \times D_m$ where $\gcd\{n, m\} = 1$ and $m$ is an integer $\geq 2$. For any $d \mid nm$, write $d = d_1d_2$ where $d_1 \mid n$, $d_2 \mid m$.

If $m$ is odd, then
\[ C(\Omega_{\mathbb{Z}[\pi]}) \simeq (\oplus_{d|n} C(\mathbb{Z}[\zeta_d])^{(2)}) \oplus (\oplus_{d|m} \{C(\mathbb{Z}[\zeta_d, \zeta_d + \zeta_d^{-1}])\}). \]

If $m$ is even, then
\[ C(\Omega_{\mathbb{Z}[\pi]}) \simeq (\oplus_{d|n} C(\mathbb{Z}[\zeta_d])^{(4)}) \oplus (\oplus_{d|m} \{C(\mathbb{Z}[\zeta_d, \zeta_d + \zeta_d^{-1}])\}). \]

(4) If $\pi = Q_{4n}$ where $n$ is an integer $\geq 2$, then
\[ C(\Omega_{\mathbb{Z}[\pi]}) \simeq \oplus_{d|n} (C(\mathbb{Z}[\zeta_d + \zeta_d^{-1}]) \oplus (\oplus_{d, \pi} C_{A_d}(R_d))). \]

where $C_{A_d}(R_d)$ is defined in Definition 6.2 with $R_d = \mathbb{Z}[\zeta_d + \zeta_d^{-1}]$, $K_d = \mathbb{Q}(\zeta_d + \zeta_d^{-1})$, $L_d = \mathbb{Q}(\zeta_d)$, and $A_d$ is the central simple $K_d$-algebra defined by $A_d \cong L_d \oplus L_d^\tau$ with $u^2 = -1$, $u\alpha = \tau(\alpha)u$ for any $\alpha \in L_d$ ($\tau$ acts on $L_d$ by $\tau(\zeta)_d = \zeta_d^{-1}$.

\textbf{Proof.} Case 1. $\pi = C_n$.

Write $\mathbb{Z}[\pi] = \mathbb{Z}[X]/\langle X^n - 1 \rangle$. Note that $\mathbb{Z}[X]/\langle X^n - 1 \rangle \hookrightarrow \prod_{d|n} \mathbb{Z}[X]/\langle \Phi_d(X) \rangle \hookrightarrow \mathbb{Q}[X]/\langle X^n - 1 \rangle$. Hence $\Omega_{\mathbb{Z}[\pi]} \simeq \prod_{d|n} \mathbb{Z}[\zeta_d]$.

Case 2. $\pi = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau^{-1} \sigma \tau = \sigma^{-1} \rangle \simeq D_n$.

For any monic polynomial $f(X) \mid X^n - 1$, the right ideal $f(\sigma) \cdot \mathbb{Z}[\pi]$ is a two-sided ideal in $\mathbb{Z}[\pi]$ (see Step 2 in the proof of Theorem 4.1); write it as $\langle f(\sigma) \rangle$. By abusing the notation, we will also write $\langle f(\sigma) \rangle$ as the two-sided ideal in $\mathbb{Q}[\pi]$ generated by $f(\sigma)$.

It is not difficult to verify that $\mathbb{Z}[\pi] \hookrightarrow \prod_{d|n} \mathbb{Z}[\pi]/\langle \Phi_d(\sigma) \rangle$ and $\mathbb{Q}[\pi] = \prod_{d|n} \mathbb{Q}[\pi]/\langle \Phi_d(\sigma) \rangle$.

When $d \geq 3$, $\mathbb{Z}[\pi]/\langle \Phi_d(\sigma) \rangle \simeq S_d \circ H$ where $S_d = \mathbb{Z}[\zeta_d]$, $H = \langle \tau \rangle \simeq C_2$ and $\tau(\zeta_d) = \zeta_d^{-1}$. Define $R_d = S_d^{(\tau)} = \mathbb{Z}[\zeta_d + \zeta_d^{-1}]$. Let $L_d = \mathbb{Q}(\zeta_d)$ and $K_d = \mathbb{Q}(\zeta_d + \zeta_d^{-1})$ be the quotient field of $S_d$ and $R_d$ respectively. Note that $S_d \circ H \simeq M_2(R_d)$ is a maximal order [Sw incredu page 75].

Similarly $L_d \circ H \simeq M_2(K_d)$ is a central simple $K_d$-algebra. Hence $C(S_d \circ H) \simeq C(R_d)$ by Theorem 6.3 because $M_2(K_d)$ has no ramified infinite place.

When $d = 1$, $\mathbb{Z}[\pi]/\langle \Phi_1(\sigma) \rangle \simeq \mathbb{Z}[\pi']$ where $\pi' \simeq C_2$. Since $C(\mathbb{Z}[\pi']) = \{0\}$, the factor $\mathbb{Z}[\pi]/\langle \Phi_1(\sigma) \rangle$ has no contribution to $C(\Omega_{\mathbb{Z}[\pi]})$. The situation when $d = 2$ may be treated similarly.
Case 3. \( \pi = \langle \sigma', \rho, \tau : \sigma^n = \rho^m = \tau^2 = 1, \tau^{-1} \sigma' \tau = \sigma', \tau^{-1} \rho \tau = \rho^{-1}, \sigma' \rho = \rho \sigma' \rangle \simeq C_n \times D_m. \)

Define \( \sigma = \sigma' \rho \) and consider \( \Lambda_d := \mathbb{Z}[\pi/\langle \Phi_d(\sigma) \rangle \) where \( d \mid nm \) as in Case 2. The proof is almost the same as that in Case 2. Write \( d = d_1d_2 \) where \( d_1 \mid n \) and \( d_2 \mid m. \)

Case 3.1 \( m \) is odd.

If \( d \mid n \), then \( \Lambda_d \) is isomorphic to the group ring of \( C_2 = \langle \tau \rangle \) over \( S_d = \mathbb{Z}[\zeta_d] \) (see Step 2 of Case 3 in the proof of Theorem 1.4). A maximal order containing \( \Lambda_d \) is \( S_d/(1+\tau)/2 \oplus S_d/(1-\tau)/2 \). Thus the contribution of \( \Lambda_d \) to \( C(\Omega_{\mathbb{Z}[\pi]}) \) is \( C(\mathbb{Z}[\zeta_d]) \oplus C(\mathbb{Z}[\zeta_d]). \)

Now consider \( \Lambda_d \) where \( d \nmid n \). Then \( \Lambda_d \simeq S_d \circ H \) where we keep the notation of \( S_d, \ H, \ L_d \) in Case 2, but \( R_d \) and \( K_d \) should be modified as \( R_d = S_d^{(\tau)} = \mathbb{Z}[\zeta_d, \zeta_d + \zeta_d^{-1}], \)

\( K_d = \mathbb{Q}(\zeta_d, \zeta_d + \zeta_d^{-1}). \)

As in Case 2, \( \Lambda_d \) is a maximal \( R_d \)-order and \( C(\Lambda_d) \simeq C(R_d). \)

Case 3.2 \( m \) is even.

The proof is almost the same as Case 3.1. Consider the situations \( d \mid n, \ d \nmid n \) (but \( d_2 \neq 2 \)) and \( d_2 = 2. \)

For the last situation \( d_2 = 2 \), then \( \Lambda_d \) is isomorphic to the group ring of \( C_2 = \langle \tau \rangle \) over \( S_d = \mathbb{Z}[\zeta_d] = \mathbb{Z}[\zeta_d/2] \). This explain the reason why there is an extra summand \( C(\mathbb{Z}[\zeta_d/2]) \).

Case 4. \( \pi = \langle \sigma, \tau : \sigma^{2n} = \tau^2 = 1, \sigma^n = \tau^2, \tau^{-1} \sigma \tau = \sigma^{-1} \rangle \simeq Q_{4n}. \)

For \( d \mid 2n \), consider \( \Lambda_d := \mathbb{Z}[\pi/\langle \Phi_d(\sigma) \rangle \) as before.

Case 4.1 \( n \) is odd.

If \( d = 1 \) or \( 2 \), it is easy to check that \( \Lambda_d \) has no contribution to \( C(\Omega_{\mathbb{Z}[\pi]}) \) (see Step 1 of Case 4 in the proof of Theorem 1.4).

Now consider \( \Lambda_d \) where \( d \geq 3 \). It may happen either \( d \mid n \) or \( d \nmid n. \)

If \( d \mid n \) and \( d \geq 3 \), then \( \Lambda_d \simeq S_d \circ H \) where we keep the same notation of \( S_d, H, R_d, \)

\( L_d, K_d \) as in Case 2. Since \( \Lambda_d \) is a maximal \( R_d \)-order, we conclude that \( C(\Lambda_d) \simeq C(R_d). \)

If \( d \nmid n \) and \( d \geq 3 \) then \( \Lambda_d = (S_d \circ H)_f \) where the notation is the same as in Step 1 of Case 4 in the proof of Theorem 1.4. Define \( R_d = S_d^{(\tau')} = \mathbb{Z}[\zeta_d + \zeta_d^{-1}] \) and \( K_d = \mathbb{Q}(\zeta_d + \zeta_d^{-1}). \) By Step 3 and Step 4 of Case 4 in the proof of Theorem 1.4, we find that \( \Lambda_d \) is a maximal \( R_d \)-order in the central simple \( K_d \)-algebra \( A_d := (L_d \circ H)_f. \) Note that \( A_d \) is a central division \( K_d \)-algebra with \( [A_d : K_d] = 4; \) in fact, it is a totally definite quaternion algebra by Step 2 of Case 4 in the proof of Theorem 1.4 (alternatively, see \( \text{Sw} \) Lemma 4.2). By Theorem 5.3 \( C(\Lambda) \simeq C_{A_d}(R_d). \)

Case 4.2 \( n \) is even.

Consider the situations \( d \mid n, \ d \nmid n. \) The situation \( d \mid n \) contributes the summand \( C(R_d). \) It is necessary that \( d \geq 3 \) for the situation \( d \nmid n \) (but \( d \mid 2n \); thus \( \tau \) acts on \( S_d \) faithfully. This situation contributes the summand \( C_{A_d}(R_d). \)

Let \( h_m \) be the class number of \( \mathbb{Q}(\zeta_m) \), \( h_m^+ \) be the class number of \( \mathbb{Q}(\zeta_m + \zeta_m^{-1}) \). It is known that \( h_m^+ \) is a divisor of \( h_m \) [Wa] page 40, Theorem 4.14. Recall the definition of \( \pi \)-tori in Definition 1.1.
Theorem 6.5 Let $p$ be an odd prime number, $c \geq 1$, and $K/k$ be a Galois extension with $\text{Gal}(K/k) = D_{p^c}$. Then all the $D_{p^c}$-tori defined over $k$ are stably rational over $k$ if and only if $h_{p^c}^+ = 1$.

Proof. Apply Theorem 6.4.

For any $c' \leq c$, since the extension $\mathbb{Z}[\zeta_{p^c} + \zeta_{p^c}^{-1}] \hookrightarrow \mathbb{Z}[\zeta_{p^c} + \zeta_{p^c}^{-1}]$ has one fully ramified prime divisor. Thus we may apply Iwasawa’s Theorem: $h_{p^c}^+ = 1$ implies $h_{p^c'}^+ = 1$ for any $c' \leq c$ [Iw].

Remark. (1) The case of $D_{p^c}$-tori in the above theorem is proved by Hoshi, Kang and Yamasaki by a different method [HKY, Theorem 1.5].

(2) According to Washington [Wa, p.420], the calculation of $h_m^+$ is rather sophisticated. It is known that $h_m^+ = 1$ if $m \leq 66$; if the generalized Riemann hypothesis is assumed, then $h_m^+ = 1$ if $m \leq 161$ [Wa, page 421].

We turn to the situation of 2-groups such as $D_n$ (the dihedral group of order $2n$ with $n \geq 2$) and $Q_{4n}$ (the generalized quaternion group of order $4n$ with $n \geq 2$).

The following proposition is an easy consequence of Endo and Miyata’s Theorems in [EM3], [EM4] and [EM6]. We record it just to keep the reader aware.

Proposition 6.6 Let $\pi = D_n$, the dihedral group of order $2n$ where $n = 2^t$ and $t \geq 1$. Then $C(\Omega_{\mathbb{Z}\pi}) \simeq T^g(\pi)$. Consequently, if $h_n^+ = 1$ (e.g. $1 \leq t \leq 6$) and $M$ is a $\pi$-lattice, then $M$ is both flabby and coflabby if and only if it is stably permutation, i.e. $M \oplus P$ is a permutation lattice where $P$ is some permutation lattice.

Proof. Since $\pi$ is a dihedral group, $C^g(\mathbb{Z}\pi) = \tilde{C}(\mathbb{Z}\pi)$ by [EM3, Theorem 4.6]. Thus $C(\Omega_{\mathbb{Z}\pi}) \simeq C(\mathbb{Z}\pi) / C^g(\mathbb{Z}\pi)$. On the other hand, $\pi$ is a 2-group, it follows that $C(\mathbb{Z}\pi) / C^g(\mathbb{Z}\pi) \simeq T^g(\pi)$ by Theorem 2.14. Hence the result.

Suppose that $h_n^+ = 1$. Then $h_{2^s}^+ = 1$ for all $1 \leq s \leq t$ by the same arguments as in the proof of Theorem 6.5. It follows that $C(\Omega_{\mathbb{Z}\pi}) = 0$ according to Theorem 6.4. Hence $T^g(\pi) = 0$.

The condition $T^g(\pi) = 0$ is equivalent to $[M]^f = 0$ for any invertible $\pi$-lattice $M$, which means that there is a short exact sequence $0 \to M \to P_1 \to P_2 \to 0$ for some permutation lattices $P_1$ and $P_2$. But this sequence splits because of Lemma 2.5; thus $M$ is stably permutation. By Theorem 2.7, $M$ is stably permutation if and only if it is flabby and coflabby.

Note that, by [Wa, page 421], $h_{2^t}^+ = 1$ if $1 \leq t \leq 6$.

Remark. Using a different method, Colliot-Thélène and Sansuc show that, if $\pi$ is the Klein-four group, a $\pi$ lattice is flabby and coflabby if and only if it is stably permutation [CTS, page 186, Proposition 4].

The following lemma is communicated to me by Prof. Endo.
Lemma 6.7 Let $\pi = Q_{4n}$ be the generalized quaternion group of order $4n$ where $n = 2^t$ with $t \geq 1$. If $\Omega_{2n}$ is a maximal order in $Q_{2n}$ containing $\mathbb{Z}[\pi]$, then $C(\Omega_{2n}) \simeq \bigoplus_d \mathbb{Z}[\zeta_4 + \zeta_4^{-1}]$. In particular, if $\pi = Q_8, Q_{16}, Q_{32}, Q_{64}$ or $Q_{128}$, then $C(\Omega_{2n}) = 0$ and $C(\Omega_{2n}) \simeq \mathbb{Z}/2\mathbb{Z}$.

Proof. Apply Theorem 6.4. It suffices to determine $C_A(R)$ where $R = \mathbb{Z}[\zeta_n + \zeta_n^{-1}]$, $K = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$ is the quotient field of $R$, and $A$ is the central simple $K$-algebra defined by $A = L + Lu$ with $u^2 = -1$, $u\alpha = \tau(\alpha)u$ for any $\alpha \in L$ ($\tau$ acts on $L = \mathbb{Q}(\zeta_n)$ by $\tau(\zeta_n) = \zeta_n^{-1}$).

Use the exact sequence $U(R) \to D \to P(R)/P_A(R) \to 0$ in Definition 6.2. By Weber’s Theorem [CR2 page 272] the map $U(R) \to D$ is surjective. Thus $P_A(R) = P(R)$ and $C_A(R) \simeq C(R) \simeq C(\mathbb{Z}[\zeta_n + \zeta_n^{-1}])$.

By [Wa page 421], $h^2_A = 1$ if $1 \leq s \leq 6$. Hence $C(\Omega_{2n}) = 0$ if $\pi = Q_8, Q_{16}, Q_{32}, Q_{64}$ or $Q_{128}$. Note that $C(\Omega_{2n}) \simeq C(\mathbb{Z}[\pi])/\tilde{C}(\mathbb{Z}[\pi])$ and $\tilde{C}(\mathbb{Z}[\pi]) \simeq \mathbb{Z}/2\mathbb{Z}$ by [CR2 page 272, Theorem 50.36]. We find that $C(\mathbb{Z}[\pi]) \simeq \mathbb{Z}/2\mathbb{Z}$.

We remark that it is already known that $C(\mathbb{Z}[\pi]) \simeq \mathbb{Z}/2\mathbb{Z}$ if $\pi = Q_8, Q_{16}, Q_{32}$ (see Sw4 page 67, Theorems III and IV)).

Proposition 6.8 If $\pi \simeq Q_8, Q_{16}, Q_{32}, Q_{64}$ or $Q_{128}$, then $C(\Omega_{2n}) = 0 = T^g(\pi)$. It follows that an invertible $\pi$-lattice is always stably permutation.

Proof. Since there is a surjection $C(\Omega_{2n}) \to C(\mathbb{Z}[\pi])/C^g(\mathbb{Z}[\pi])$ (see Definition 2.12), it follows that $C(\mathbb{Z}[\pi])/C^g(\mathbb{Z}[\pi]) = 0$ by Lemma 6.7.

Because of Theorem 2.14 $T^g(\pi) \simeq C(\mathbb{Z}[\pi])/C^g(\mathbb{Z}[\pi])$. Thus $T^g(\pi) = 0$. The remaining proof is similar to that of Proposition 6.6.

Remark. Colliot-Thélène and Sansuc show that, if $\pi$ is the quaternion group of order 8, an invertible $\pi$-lattice is always stably permutation [CTS, R5, page 187]. We indicate briefly their proof.

By [EM2, Proposition 3.8] $\tilde{C}(\mathbb{Z}[\pi]) = C^g(\mathbb{Z}[\pi])$. Thus $C(\Omega_{2n}) \simeq C(\mathbb{Z}[\pi])/C^g(\mathbb{Z}[\pi])$. By Theorem 2.14 $T^g(\pi) \simeq C(\mathbb{Z}[\pi])/C^g(\mathbb{Z}[\pi])$. Note that $\Omega_{2n} \simeq \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \Lambda_4$ where $\Lambda_4$ is the Hurwitz order [Sw4 Proposition 4.7]. It follows that $T^g(\pi) \simeq C(\Omega_{2n}) = 0$, because $\Lambda_4$ is a non-commutative principal ideal domain.

The same argument of the above proposition may be apply to the semi-dihedral groups and the modular groups also. Let $n = 2^t$ where $t \geq 3$, define $SD_2n = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1+n/2} \rangle$ (the semi-dihedral group of order $2n$), and define $M_2n = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{1+n/2} \rangle$ (the modular group of order $2n$). If $\pi$ is a 2-group of order $\geq 16$ and contains a cyclic normal subgroup of index 2, then $\pi$ is isomorphic to the dihedral group, the semi-dihedral group, the generalized quaternion group or the modular group [Sn1 page 107].

The proof of the following proposition is similar to that of Theorem 6.4 and is omitted.
Proposition 6.9 Let \( n = 2^t \) where \( t \geq 3 \).

1. If \( \pi = \text{SD}_{2n} \), then \( C(\Omega_{\mathbb{Z}[\pi]}) \simeq (\oplus_{0 \leq s \leq t-1} C(\mathbb{Z}[\zeta_{2^s} + \zeta_{2^s}^{-1}]))) \oplus C(\mathbb{Z}[\zeta_n - \zeta_n^{-1}]) \).

2. If \( \pi = \text{M}_{2n} \), then \( C(\Omega_{\mathbb{Z}[\pi]}) \simeq (\oplus_{0 \leq s \leq t-1} C(\mathbb{Z}[\zeta_{2^s}])^{(2)}) \oplus C(\mathbb{Z}[\zeta_{n/2}]) \).

Proposition 6.10 If \( \pi \simeq \text{M}_{16}, \text{M}_{32} \) or \( \text{M}_{64} \), then \( C(\Omega_{\mathbb{Z}[\pi]}) = 0 = T^g(\pi) \). It follows that an invertible \( \pi \)-lattice is always stably permutation.

Proof. By Proposition 6.9, \( C(\Omega_{\mathbb{Z}[\pi]}) = 0 \) because of [MM]. Thus \( T^g(\pi) = 0 \). ■

Remark. The situation of the group \( \text{SD}_{2n} \) is left open because we don’t know the class number of \( \mathbb{Q}(\zeta_n - \zeta_n^{-1}) \) when \( n \geq 16 \).

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