Hirzebruch Manifolds and Positive Holomorphic Sectional Curvature

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Abstract. This paper is the first step in a systematic project to study examples of Kähler manifolds with positive holomorphic sectional curvature ($H > 0$). Previously Hitchin proved that any compact Kähler surface with $H > 0$ must be rational and he constructed such examples on Hirzebruch surfaces $M_{2,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{C}P^1})$. We generalize Hitchin’s construction and prove that any Hirzebruch manifold $M_{n,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{C}P^{n-1}})$ admits a Kähler metric of $H > 0$ in each of its Kähler classes. We demonstrate that the pinching behaviors of holomorphic sectional curvatures of new examples differ from those of Hitchin’s which were studied in the recent work of Alvarez-Chaturvedi-Heier. Some connections to recent works on the Kähler-Ricci flow on Hirzebruch manifolds are also discussed.

It seems interesting to study the space of all Kähler metrics of $H > 0$ on a given Kähler manifold. We give higher dimensional examples such that some Kähler classes admit Kähler metrics with $H > 0$ and some do not.

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1. Introduction

Let $(M, J, g)$ be a Kähler manifold, then one can define the holomorphic sectional curvature of any $J$-invariant real 2-plane $\pi = \text{Span}\{X, JX\}$ by

$$H(\pi) = \frac{\langle R(X, JX, JX, X) \rangle}{\|X\|^4}.$$ 

It is the Riemannian sectional curvature restricted on any $J$-invariant real 2-plane (p165 [29]). Compact Kähler manifolds with positive holomorphic sectional ($H > 0$) form an interesting class of Kähler manifolds. For example, these manifolds are simply-connected, by the work of Tsukamoto [38]. An averaging argument due to Berger [6] showed that $H > 0$ implies positive scalar curvature, which further leads to the vanishing of its pluri-canonical ring by a Bochner-Kodaira type identity, see [26]. In 1975 Hitchin [24] proved that any Hirzebruch surface $M_{2,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{C}P^1})$ admits a Kähler metric with $H > 0$. Moreover, he proved that any rational surface admits a Kähler metric with positive scalar curvature.

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Recall that one can define another positive curvature condition on Kähler manifolds, the so-called (holomorphic) bisectional curvature, which is stronger than $H > 0$. Any compact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{C}P^n$ by the work of Mori [31] and Siu-Yau [35]. Motivated by these results and Hitchin’s example, Yau [41] asked if the positivity of holomorphic sectional curvature can be used to characterize the rationality of algebraic manifolds. More precisely, the following question was asked.

**Question 1.1** (Yau, Problem 67 and 68 in [42]). Consider a compact Kähler manifold with positive holomorphic sectional curvature, is it unirational? Is it projective? If a projective manifold is obtained by blowing up a compact manifold with positive holomorphic sectional curvature along a subvariety, does it still carry a metric with positive holomorphic sectional curvature? In general, can we find a geometric criterion to distinguish the concept of unirationality and rationality?

There are some recent progress on Question 1.1. For example, an important criterion on non-uniruledness of projective manifolds in terms of pseudoeffective canonical line bundles was established by Boucksom-Demailly-Păun-Peternell [9]. Heier-Wong [22] gave a nice application of this result to show that any projective manifold with a Kähler metric with positive total scalar curvature is uniruled, i.e., having a rational curve passing through every point. Very recently, they [23] proved that any projective manifold with a Kähler metric of $H > 0$ is rationally connected, i.e., any two points can be connected by a rational curve. The latter paper also contains analogous results for $H \geq 0$ and nonnegative Ricci curvatures. There are also some recent studies on Hermitian manifolds with $H \geq 0$ ([40]).

In this paper, we focus on examples of Kähler metrics with $H > 0$. More specifically, we want to carry out a detailed study of such metrics on any Hirzebruch manifold $M_{n,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{C}P^{n-1}})$.

One of our motivation is the work of Chen-Tian ([15] and [16]), where they studied Kähler-Ricci flow with positive bisectional curvature and proved that the space of all Kähler metrics with positive bisectional curvature on $\mathbb{C}P^n$ is path-connected. Similarly, we would like to understand the space of all Kähler metrics with $H > 0$ on any Hirzebruch manifold $M_{n,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{C}P^{n-1}})$.

**Question 1.2.** Given any Hirzebruch manifold $M_{n,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{C}P^{n-1}})$, what can we say about the space of all Kähler metrics with $H > 0$? Is it path-connected?

As a first step to answer Question 1.2, we prove the following result.

**Theorem 1.3.** Given any Hirzebruch manifold $M_{n,k}$, there exists a Kähler metric of $H > 0$ in each of its Kähler classes.

Let us explain the background of Theorem 1.3 and its relation to Hitchin’s examples in [24]. Recall any Hirzebruch surface $M_{2,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{C}P^{1}})$ can be obtained from $\mathbb{C}P^2$ by blowing up and down, i.e. is a rational surface. It is the projective bundle associated to the rank-2 vector bundle $H^k \oplus 1_{\mathbb{C}P^{1}}$, $H$ being the hyperplane bundle over $\mathbb{C}P^1$ and $1_{\mathbb{C}P^{1}}$ the trivial bundle. We call a surface minimal if it has no rational curve of self-intersection $-1$, then the only minimal rational surface are $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, and $M_{2,k}$ with $k > 1$. In other words, any rational surface is the blow up of $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, and $M_{2,k}$ with $k > 1$. We refer the reader to Chapter 4 of Griffiths-Harris [19] for a detailed description of rational surfaces.

Now we focus on the Kähler metrics on Hirzebruch manifolds $M_{n,k}$ where $\pi : M_{n,k} \rightarrow \mathbb{C}P^{n-1}$ for $n \geq 2$. Note that $M_{n,k}$ can also be described as $\mathbb{P}(H^{-k} \oplus 1_{\mathbb{C}P^{n-1}})$, the projective bundle associated to $H^{-k} \oplus 1_{\mathbb{C}P^{1}}$. Let $E_0$ denote the divisor in $M_{n,k}$ corresponding to the section $(0,1)$ of $H^{-k} \oplus 1_{\mathbb{C}P^{n-1}}$, $E_\infty$ the divisor in $M_{n,k}$ corresponding to the section $(0,1)$ of $H^k \oplus 1_{\mathbb{C}P^{n-1}}$, and $F$ the divisor corresponding to the pull-back line bundle $\pi^*H$ over $M_{n,k}$.

Then the Picard group of $M_{n,k}$ is generated by the divisors $E_0$ and $F$, while $E_\infty = E_0 + kF$. The integral cohomology ring of $M_{n,k}$ is $\mathbb{Z}[F, E_0]/(F^n, E_0^2 + kE_0F)$.

\footnotetext{1}{Note that we follow the notations of $E_0$ and $E_\infty$ as in Calabi [12], which is different with those in [19].}
The anti-canonical class of $M_{n,k}$ can be expressed as

$$K_{M_{n,k}}^{-1} = 2E_\infty - (k-n)F = \frac{n+k}{k}E_\infty - \frac{n-k}{k}E_0,$$

and every class $\alpha$ in the Kähler cone of $M_{n,k}$ can be expressed as

$$\alpha = \frac{b}{k}[E_\infty] - \frac{a}{k}[E_0].$$

for any $b > a > 0$.

In [24] Hitchin proved that each $M_{2,k}$ admits Kähler metrics with $H > 0$. His example was motivated by a natural choice of Kähler metric on any projective vector bundles over Kähler manifold. Namely let $\pi : (E, h) \to (M, g)$ be any Hermitian vector bundle over a compact Kähler manifold. The Chern curvature form $\Theta(\mathcal{O}(E))$ of $\mathcal{O}(E)$ has the fiber direction components given by the Fubini-Study form, hence is positive. Therefore

$$\tilde{\omega} = \pi^*\omega_g + s\sqrt{-1}\Theta(\mathcal{O}(E))$$

is a well-defined Kähler metric on $P(E)$ when $s > 0$ is sufficiently small. Fix a Hirzebruch surface $M_{2,k}$, one picks $(E, h) = (H^k \oplus \mathbb{CP}^1, h)$ and $(M, g)$ as $(\mathbb{CP}^3, g_{FS})$ where $g_{FS}$ is the standard Fubini-Study metric and $h$ the induced metric. Under this situation, $\tilde{\omega}$ has a natural $U(2)$ isometric action. Hitchin showed that $\tilde{\omega}$ has $H > 0$ if $0 < s(1 + ks)^2 < \frac{1}{k(2k-1)}$ by an explicit calculation. It was further observed by Alvarez-Chaturvedi-Heier [2] that Hitchin’s examples satisfy $H > 0$ if and only if $s < \frac{1}{\pi^2}$. We note that Hitchin’s examples can only exist on a proper open set of Kähler cone of $M_{2,k}$, for example, on $M_{2,1}$, by a scaling his examples lie in Kähler class $b[E_\infty] - aE_0$ where $a < b < 2a$. Now Theorem 1.3 implies the existence of Kähler metric of $H > 0$ in each of Kähler class on any $M_{n,k}$ where $n \geq 2$.

To prove Theorem 1.3, we follow Calabi’s ansatz ([11] and [12]). The crucial observation (pointed out in [12]) that the group of holomorphic transformations of $M_{n,k}$ contains $U(n)/\mathbb{Z}_k$ as its maximal compact group. Therefore it is natural to study Kähler metrics with $U(n)$-symmetry. Calabi’s method has been very fruitful to produce examples of special Kähler metrics in various settings, including Kähler-Einstein metrics, Kähler-Ricci solitons, Kähler metrics with constant scalar curvature, extremal Kähler metrics, etc., see for example [28], [30], [34], [27], [1], [10], [17], [25]. To be more precise, we follow the slightly general approach due to Koiso-Sakane [28]. It turns out that for $M_{n,k}$ this method is equivalent to Calabi’s ansatz.

Let us view $M_{n,k}$ as a compactification of the $\mathbb{C}^*$ bundle over $\mathbb{CP}^{n-1}$ obtained by $H^{-k} \setminus E_0$ where $E_0$ is its zero section. In general, given any $\mathbb{C}^*$ bundle $\pi : L^* \to M$ obtained by a Hermitian line bundle $(L, h)$, we may consider the following metric on the total space of $L^*$:

$$\tilde{g} = \pi^*g_t + dt^2 + (dt \circ J)^2,$$

where $g_t$ is a continuous family of Kähler metrics on the base $(M, J)$, $t$ is a function which only depends on the norm of Hermitian metric $h$, and $J$ the complex structure on the total space of $L$. Koiso-Sakane [28] gave a sufficient condition to ensure the resulting metric $\tilde{g}$ is Kähler and studied the compactification of such metrics. They were able to give new examples of non-homogeneous Kähler-Einstein metrics.

Now let us focus on $M_{n,k}$, where we will pick $L = H^{-k}$ in the approach of Koiso-Sakane. After a suitable reparametrization and some careful analysis, one can show that $\tilde{g}$ in (3) can be compactified to produce Kähler metrics on $M_{n,k}$ if a generating function $\phi(U)$ of a single variable $U$ satisfying suitable boundary conditions. For simplicity, let use still call such a metric $(M_{n,k}, \tilde{g})$. Making full use of the $U(n)$-isometric action, it can be shown that the curvature tensors of $(M_{n,k}, \tilde{g})$ are completely determined by three components, namely

1. $A$ which is the holomorphic sectional curvature along the fiber direction $F$,
2. $B$ which is the bisectional curvature along the fiber and any direction in the base $E_0$,
3. $C$ which is the holomorphic sectional curvature along $E_0$.

Thus we can reduce the problem of constructing Kähler metric with $H > 0$ in the form of $\tilde{g}$ by looking for a suitable generating function $\phi(U)$ satisfying some differential inequalities related
to $A, B, C$ defined above. In particular, we show that Hitchin’s example is canonical among all Kähler metrics with $H > 0$ on $M_{n,k}$ in the following sense:

**Proposition 1.4.** Hitchin’s examples can be uniquely characterized as $U(n)$-invariant Kähler metrics on $M_{n,k}$ which have the constant curvature component $A$.

In the level of the generating function $\phi(U)$ in the setting of Koiso-Sakane, each of Hitchin’s example corresponds to some quadratic even function defined on $[-c, c]$ with $0 < c < \frac{n}{\sqrt{2k+1}}$. Indeed, the boundary conditions of the generating function $\phi(U)$ where $U \in [-c, c]$ reflects the Kähler class of the resulting metric $\tilde{g}$, the volume of the zero section $E_0$ is $(1 - \frac{k}{\pi^2})V_{FS}$, where $V_{FS}$ denote the volume of $CP^n$ endowed with $Ric(g_{FS}) = g_{FS}$, and the volume of the infinity section $E_{\infty}$ is $(1 + \frac{k}{\pi^2})V_{FS}$. To produce examples in each of the Kähler class of $M_{n,k}$ is equivalent to produce examples of $\phi(U)$ for any $c \in (0, \frac{n}{\pi})$ and yet satisfying inequalities related to $H > 0$. We are able to construct such $\phi(U)$ by establishing some delicate estimates on even polynomials with large degrees.

Recently, Alvarez-Chaturvedi-Heier [2] studied the pinching constants of Hitchin’s examples on $M_{2,k}$.

**Theorem 1.5 (Alvarez-Chaturvedi-Heier [2]).** The local and global pinching constant of holomorphic sectional curvature are the same for any of the Hitchin’s examples on $M_{2,k}$. The maximum among them is $\frac{1}{2k+1}$ and the ray of the corresponding Kähler classes is $b[E_0] - aE_0$ of the slope $\frac{2k+2}{2k+1}$.

Recall that the local holomorphic pinching constant is the maximum of all $\lambda \in (0, 1]$ such that $0 < \lambda H(\pi) \leq H(\pi)$ for any $J$-invariant real 2-planes $\pi, \pi \subset T_\rho(M)$ at any $\rho \in M$, while the global holomorphic pinching constant is the maximum of all $\lambda \in (0, 1]$ such that there exists a positive constant $C$ so that $\lambda C \leq H(p, \pi) \leq C$ holds for any $p \in M$ and any $J$-invariant real 2-plane $\pi \subset T_\rho(M)$. Obviously the global holomorphic pinching constant is no larger than the local one. We show that the conclusion of Theorem 1.5 is not always true for other Kähler metrics with $U(n)$-symmetry and with $H > 0$, which again reflects the specialness of Hitchin’s examples.

**Proposition 1.6.** There exist Kähler metrics with $H > 0$ on $M_{n,k}$ whose local and global pinching constants for holomorphic sectional curvature are not equal.

In general, the local holomorphic pinching constant of any $U(n)$-invariant Kähler metric on $M_{n,k}$ is bounded from above by $\frac{1}{\pi}$. A direct calculation enables us to generalize the result of Alvarez-Chaturvedi-Heier [2] to $M_{n,k}$. It is interesting to see that the optimal pinching constant is dimension free. It is the same constant $\frac{1}{2k+1}$ discovered in [2], with the corresponding Kähler class on $M_{n,k}$ satisfies

$$b[E_\infty] - aE_0$$

where $b = \frac{2k+2}{2k+1} > 0$. It is unclear to us how to determine the optimal holomorphic pinching constant among all $U(n)$-invariant Kähler metrics on $M_{n,k}$, though we believe it is achieved among Hitchin’s examples. Motivated by the result of Alvarez-Chaturvedi-Heier [2], we would like to propose the following question:

**Question 1.7.** Is the following statement true? If a compact Kähler surface with $H > 0$ has its local pinching constant $\lambda > \frac{1}{9}$, then it must be biholomorphic to $CP^2$ or $CP^1 \times CP^1$.

As a partial evidence on Question 1.7, by using some previous results on positive orthogonal bisectional curvature, we give a complete classification of compact Kähler manifolds with local holomorphic pinching constant $\lambda \geq \frac{1}{2}$. In the case of Kähler surfaces, they are biholomorphic to $CP^2$ or $CP^1 \times CP^1$.

It is of course desirable to make further studies on Question 1.2. We are able to show that the proof of Theorem 1.3 can be used to establish the path-connectedness of all $U(n)$-invariant Kähler metrics of $H > 0$ on any Hirzebruch manifold $M_{n,k}$. 
Corollary 1.8. The space of all $U(n)$-invariant Kähler metrics of $H > 0$ on $M_{n,k}$ is path-connected.

At this moment, it is not clear to use how to prove the path-connectedness without the assumption of $U(n)$-symmetry. In the meantime it seems impossible to make use of the path constructed in Corollary 1.8 and improve the holomorphic pinching constants. In the other direction, as illustrated in the work of Chen-Tian [15] and [16], one may wonder if the Kähler-Ricci flow can be used to study the space of all Kähler metric of $H > 0$ and Question 1.7. To that end, we calculate the holomorphic sectional curvature for the Kähler-Ricci shrinking soliton on Fano Hirzebruch manifold $M_{n,k}$ with $n > k$ due to Koiso [27] and Cao [10], and also for the noncompact ones on the total space of $H^{-k} \to \mathbb{CP}^{n-1}$ with $n > k$ due to Feldman-Ilmanen-Knopf [17].

Proposition 1.9. The Cao-Koiso shrinking soliton on Hirzebruch manifold $M_{n,k}$ have $H > 0$ as the ratio $\frac{n}{k}$ is sufficiently large. If we fix $k = 1$, the first example with $H > 0$ is on $M_{3,1}$; if we fix $k = 2$, the first one is on $M_{7,2}$.

In the complete noncompact case, the Feldman-Ilmanen-Knopf shrinking solitons do not have $H > 0$ if $k < n \leq k^2 + 2k$.

In fact we expect that none of Feldman-Ilmanen-Knopf shrinking solitons will have $H > 0$. It would be interesting to know if any complete Kähler-Ricci soliton with $H > 0$ must be compact, in view of the recent work Munteanu-Wang [32] and the previous work of Ni [33].

As a corollary to the previous works of Zhu [43], Weinkove-Song [36], Fong [18], Guo-Song [21] on Kähler-Ricci flow on Hirzebruch manifolds with Calabi’s symmetry, we exhibit various pinching behaviors along the Kähler-Ricci flow when the initial metrics are chosen from examples constructed in Theorem 1.3, in particular we have the following:

Corollary 1.10. $H > 0$ is not preserved under the Kähler-Ricci flow.

The above corollary entails the following question: Can we construct a suitable one-parameter family of deformation of Kähler metrics $M_{n,k}$ so that the holomorphic pinching constant is monotone along the deformation?

We would like to point out another generalization of Hitchin’s examples. In a very recent work of Alvarez, Heier, and the second-named author [3], it was proved that the projectivization $\mathbb{P}(E)$ of any Hermitian vector bundle $E$ over a compact Kähler manifold with $H > 0$ also admits a Kähler metric of $H > 0$. The resulting metric on $\mathbb{P}(E)$ is of the form (2) for $s$ sufficiently small. Instead of working with the line bundle $H^{-k}$ on $\mathbb{CP}^{n-1}$, it is possible to apply the method of Koiso-Sakane developed in the proof of Theorem 1.3 to get more examples of Kähler metrics of $H > 0$ on some $\mathbb{CP}^k$ bundles. For example, consider $M = \mathbb{CP}^{n_1-1} \times \mathbb{CP}^{n_2}$ and $L = \pi_1^*H_1^{-1} \otimes \pi_2^*H_2^{-k_2}$ where $H_1$ and $H_2$ are hyperplanes bundles on $\mathbb{CP}^{n_1-1}$ and $\mathbb{CP}^{n_2}$. $\pi_1$ and $\pi_2$ are projections to its factors. Then we can produce a $\mathbb{CP}^{n_1}$-bundle over $\mathbb{CP}^{n_2}$ as a suitable compactification of $L^* \to M$. It is also interesting to study the space of all Kähler metrics of $H > 0$ on it.

In complex dimensions higher than one, it is highly desirable to find examples of compact Kähler manifold with $H > 0$. Among the known such examples are all compact Hermitian symmetric spaces and some Kähler C-spaces (rational homogeneous space). In the last section of the paper, we study the holomorphic pinching constant for the canonical Kähler-Einstein metric on the flag 3-fold, the only Kähler C-space which is not Hermitian symmetric in dimension 3. We also demonstrate a higher dimensional projective manifold such that some of its classes admit Kähler metric with $H > 0$ while some do not. More precisely, we prove:

Theorem 1.11. Let $M$ be the hypersurface in $\mathbb{CP}^n \times \mathbb{CP}^n$ defined by $\sum_{i=1}^{n+1} z_iz_i w_i = 0$ equipped with the restriction of the product of the Fubini-Study metric, where $z, w$ are the homogeneous coordinates. Then the holomorphic pinching constant of $M$ is $\frac{1}{4}$.

Consider $N$ which is a smooth bidegree $(p, 1)$ hypersurface in $\mathbb{CP}^r \times \mathbb{CP}^s$ where $r, s \geq 2, p \geq 1$, and $p > r + 1$, then some Kähler classes of $N$ admit Kähler metrics of $H > 0$ and some do not.
Similarly as in Question 1.7, we may ask

**Question 1.12.** Is it true that if a compact Kähler 3-fold with $H > 0$ has its local holomorphic pinching constant $\lambda > \frac{1}{4}$, then it must be biholomorphic to a compact Hermitian symmetric space?

In a sequel of this paper, we will study examples of Kähler metrics with $H > 0$ on other rational surfaces, other Kähler $C$-spaces, and higher dimensional projective manifolds.

This paper is organized as follows: In Section 2, we prove the classification theorem of compact Kähler manifolds with local holomorphic pinching constant $\lambda \geq \frac{4}{7}$. In Section 3, we prove the main Theorem 1.3 and study the relation between Kähler-Ricci flow and $H > 0$ on $M_{n,k}$. In the last Section 4, we study the canonical Kähler-Einstein metric on the flag 3-fold and prove Theorem 1.11. We end the paper with some discussions on $H > 0$ in the higher dimension case from the submanifold point of view.

## 2. Holomorphic sectional curvature: preliminary results

Let us begin the definition of various curvatures on a Kähler manifold.

**Definition 2.1.** Let $(M, g, J)$ be a Kähler manifold of complex dimension $n \geq 2$ with the Levi-Civita connection $\nabla$ and Riemannian curvature tensor $R$.

1. Sectional curvature for any real 2-plane $\pi \subset T_p(M)$ is defined by $K(\pi) = \frac{R(X, JY, JX, X)}{|X|^2|Y|^2g(X, Y)}$ where $\pi = \text{span}\{X, Y\}$.

2. Holomorphic sectional curvature $(H)$ for any $J$-invariant real 2-plane $\pi \subset T_p(M)$ is defined by $H(\pi) = \frac{R(X, JX, JY, Y)}{|X|^2|Y|^2}$ where $\pi = \text{span}\{X, JX\}$. For the sake of simplicity, we freely use $H(X)$, $H(X - \sqrt{-1}JX)$ or $H(\pi)$ for holomorphic sectional curvature.

3. (Holomorphic) bisectional curvature for any two $J$-invariant real 2-planes $\pi, \pi' \subset T_p(M)$ is defined by $B(\pi, \pi') = \frac{R(X, JX, JY, Y)}{|X|^2|Y|^2}$ where $\pi = \text{span}\{X, JX\}$ and $\pi' = \text{span}\{Y, JY\}$.

In the study of Kähler manifolds with positive curvature, it is useful to consider some pinching condition in either a local or a global sense.

**Definition 2.2 (Local pinching and global pinching).** Let $\lambda, \delta \in (0, 1)$, we define the following pinching conditions on a Kähler manifold $(M, g)$.

1. $\lambda \leq H \leq 1$ in the local sense if for any $p \in M$, $0 < \lambda H(p) \leq H(p) \leq \frac{1}{\lambda}H(\pi)$ for any $J$-invariant real 2-planes $\pi, \pi' \subset T_p(M)$. In other words, there exists a function $\varphi(p) > 0$ on $M^n$ such that $0 < \lambda \varphi(p) \leq H(p, \pi) \leq \frac{1}{\lambda} \varphi(p)$ for any $p$ and any holomorphic plane $\pi \subset T_p(M)$.

2. $\delta \leq K \leq 1$ in the local sense if for any $p \in M$, $0 < \delta K(p) \leq K(p) \leq \frac{1}{\delta}K(\pi)$ for any two real 2-planes $\pi, \pi' \subset T_p(M)$.

3. $\lambda \leq H \leq 1$ in the global sense if $\lambda < H(\pi) \leq 1$ for any $p \in M$ and any $J$-invariant real 2-plane $\pi \subset T_p(M)$.

Kähler manifolds with $H > 0$ are less understood and somewhat mysterious. For example, if one works with linear algebra aspects of curvature tensors, then $H > 0$ alone does not give any helpful information on the Ricci curvature. In fact, most of the Hirzebruch surfaces in Hitchin’s examples are not Fano, thus do not admit any Kähler metric with positive Ricci curvature. Nonetheless one may study Kähler manifolds with $H > 0$ pinched by a large constant. In this regard, the following results of Berger [5] and Bishop-Goldberg [7] are very interesting.

**Proposition 2.3 (Berger [5]).** Let $(M^n, g)$ be Kähler, then $0 < \lambda \leq H \leq 1$ in the local sense implies \( \frac{2 - \lambda}{\lambda} \leq K \leq \frac{4 - \lambda}{\lambda} \) in the local sense.

**Proposition 2.4 (Bishop-Goldberg [7]).** If $(M^n, g)$ is Kähler, then $0 < \lambda \leq H(p) \leq 1$ implies

$$\frac{1}{4} [3(1 + \cos^2 \theta)\lambda - 2] \leq K(X, Y) \leq 1 - \frac{3}{4} \lambda \sin^2 \theta$$

for any unit tangent vectors $X, Y$ at $p$ with $g(X, Y) = 0$ and $g(X, JY) = \cos \theta$. In particular, $\lambda$-holomorphic pinching implies \( \frac{1}{4}(3\lambda - 2) \)-pinching on sectional curvatures.
In the proof of the above Proposition 2.3, Berger discovered an interesting inequality.

**Lemma 2.5** (Berger [4] and [5]). Let \((M^n, g)\) be a Kähler manifold and \(0 < \lambda \leq H \leq 1\) in the local sense, then for any unit vector \(X, Y\) with \(g(X, Y) = 0\) and \(g(X, JY) = \cos \theta\), we have

\[
\lambda - \frac{1}{2} + \frac{\lambda}{2} \cos^2 \theta \leq R(X, JX, JY, Y) \leq 1 - \frac{\lambda}{2} + \frac{1}{2} \cos^2 \theta.
\]

For the convenience of the readers, we sketch Berger’s proof of Lemma 2.5, as it will be crucial in the proof of Proposition 2.6 below.

**Berger’s proof of Lemma 2.5.** Given any unit vector \(X, Y\) with \(g(X, Y) = 0\) and \(g(X, JY) = \cos \theta\), consider

\[
\lambda \leq \frac{1}{2} \left[ H(aX + bY) + H(aX - bY) \right] \leq 1
\]

By the left half of inequality (5), we conclude that

\[
(H(X) - \lambda)a^2 + (R(X, JX, JY, Y) + 2R(X, JY, JY, X) - \lambda)2a^2b^2 + (H(Y) - \lambda)b^4 \geq 0
\]

holds for any real numbers \(a, b\). Apply \(H(X), H(Y) \leq 1\), it follows from (6) that

\[
R(X, JX, JY, Y) + 2R(X, JY, JY, X) \geq 2\lambda - 1.
\]

Next consider

\[
\lambda \leq \frac{1}{2} \left[ H(aX + bJY) + H(aX - bJY) \right] \leq 1.
\]

Since

\[
\frac{1}{2} \{(a^2 + b^2 + 2ab \cos \theta)^2 + (a^2 + b^2 - 2ab \cos \theta)^2\} = (a^2 + b^2)^2 + 4a^2b^2 \cos^2 \theta,
\]

a similar argument as in (6) and (7) leads to

\[
3R(X, JX, JY, Y) - 2R(X, JY, JY, X) \geq 2\lambda + 2\lambda \cos^2 \theta - 1.
\]

By adding (7) and (9) we have

\[
R(X, JX, JY, Y) \geq \lambda + \frac{\lambda}{2} \cos^2 \theta - \frac{1}{2}.
\]

The right half of inequality (4) can be proved similarly if we work on the right halves of inequalities in both (5) and (8). \(\square\)

It is possible to get some characterization of Kähler manifolds with a large holomorphic pinching constant \(\lambda\). For example, Bishop-Goldberg [7] proved that if \(\frac{1}{3} < \lambda \leq H \leq 1\) holds in the local sense on a compact Kähler manifold \((M, g)\), then \(M\) has the homotopy type of \(\mathbb{C}P^n\). They also proved in [8] that \(\lambda > \frac{1}{2}\) implies \(b_2(M) = 1\). Note that a direct calculation shows that \(\mathbb{C}P^k \times \mathbb{C}P^l\) with the product of Fubini-Study metric has exactly \(\frac{1}{2} \leq H \leq 1\) (see [2] for a general result on holomorphic pinching of product metrics). In light of these results, it is natural to ask if \(\frac{1}{2} < \lambda \leq H \leq 1\) in the local sense implies that \(M^n\) is biholomorphic to \(\mathbb{C}P^n\). This is indeed the case and we have the following:

**Proposition 2.6.** Let \((M^n, g)\) be a compact Kähler manifold with \(0 < \lambda \leq H \leq 1\) in the local sense, then the following holds:

1. If \(\lambda > \frac{1}{2}\), then \(M^n\) is biholomorphic to \(\mathbb{C}P^n\).
2. If \(\lambda = \frac{1}{2}\), then \(M^n\) is one of the following
   2a. \(M^n\) is biholomorphic to \(\mathbb{C}P^n\).
   2b. \(M^n\) is holomorphically isometric to \(\mathbb{C}P^k \times \mathbb{C}P^{n-k}\) with a product of Fubini-Study metrics.
   2c. \(M^n\) is holomorphically isometric to an irreducible compact Hermitian symmetric space of rank 2 with its canonical Kähler-Einstein metric.
**Proof of Proposition 2.6.** Let us consider \( n \geq 2 \), the crucial observation is that \( \frac{1}{2} \leq \lambda \leq 1 \) in the local sense implies that \( (M^n, g) \) has nonnegative orthogonal holomorphic bisectional curvature. Namely for any two \( J \)-invariant planes \( \pi = \text{span}\{X, JX\} \) and \( \pi' = \text{span}\{Y, JY\} \) in \( T_p(M) \) which are orthogonal in the sense that \( g(X, Y) = g(X, JY) = 0 \), then

\[
R(X, JX, JY, Y) \geq 0.
\]

This follows from Berger’s inequality (4).

Nonnegative and positive orthogonal bisectional curvature is well studied in [14], [20], and [39].

If \( \lambda > \frac{1}{2} \) then \((M^n, g)\) has positive orthogonal bisectional curvature, it is proved in [14], [20], and [39] that the Kähler-Ricci flow evolves such a metric to positive bisectional curvature, which is biholomorphic to \( \mathbb{CP}^n \) by [31] and Siu-Yau [35].

If \( \lambda > \frac{1}{2} \) then \((M^n, g)\) has nonnegative orthogonal bisectional curvature, according to a classification result due to Gu-Zhang (Theorem in 1.3 in [20]), combining the fact \( H > 0 \), then the universal covering manifold \((\tilde{M}, \tilde{g})\) is holomorphically isometric to

\[
(\mathbb{CP}^{k_1}, g_{k_1}) \times \cdots \times (\mathbb{CP}^{k_r}, g_{k_r}) \times (N^{l_1}, h_{l_1}) \times \cdots \times (N^{l_k}, h_{l_k})
\]

Where each of \((N^{l_i}, h_{l_i})\) is a compact irreducible Hermitian symmetric spaces of rank \( \geq 2 \) with its canonical Kähler-Einstein metric. The holomorphic pinching constant of such a metric was well-studied and it is exactly the reciprocal of its rank, see for example [13]. On the other hand, the pinching of product Kähler metrics was studied by [2], the proved that the pinching constant \( H > 0 \) of a produce Kähler metric \((M_1 \times M_2, g_1 \times g_2)\) is \( \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} \) where \( \lambda_1 \) and \( \lambda_2 \) are pinching constants of \( M_1 \) and \( M_2 \) respectively. It is clearly that \( \lambda_1 = \lambda_2 = 1 \).

Therefore if \( \lambda = \frac{1}{2} \), The decomposition (11) reduces to either a single \( \mathbb{CP}^n \) or a single irreducible Hermitian symmetric space of rank \( 2 \), or a product of \( \mathbb{CP}^k \times \mathbb{CP}^{n-k} \) with product Fubini-Study metrics. Obviously this decomposition descends to the original manifold \((M^n, g)\).

A natural question following Proposition 2.6 is what is the next threshold, if any, for the holomorphic pinching constants for Kähler manifolds with \( H > 0 \). In general, the situation might be complicated. Note that the canonical Kähler-Einstein metric on a compact Hermitian symmetric space has holomorphic pinching constant determined by its rank ([13]). The Kähler-Einstein metrics on a lot of the Kähler C-spaces also have \( H > 0 \), and in general one has to work with the corresponding Lie algebra carefully to determine its holomorphic pinching constant. Nonetheless, in this paper we focus on the case of dimension 2 and 3, we will see in Section 3 and 4 that Hirzebruch surfaces and the flag 3-space might be the right objects to provide the next interesting threshold for the holomorphic pinching constant.

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\]

3. Kähler metrics with \( H > 0 \) on Hirzebruch manifolds

In this section we first review Hitchin’s examples on Hirzebruch surfaces \( M_{2,k} \), then we prove the main Theorem 1.3 and study the relation between the Kähler-Ricci flow and \( H > 0 \).

3.1. A review of Hitchin’s construction. Hitchin [24] proved that any compact Kähler surface with positive sectional curvature is rational. Any rational surface can be obtained by blowing up points on \( \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1 \), and Hirzebruch surfaces \( M_{2,k} \). The natural question is which rational surface admits Kähler metric with \( H > 0 \). In this regard, Hitchin proved that any Hirzebruch surface \( M_{2,k} \) admits a Hodge metric of \( H > 0 \). Moreover, he proved that the blow up of any compact Kähler manifold with positive scalar curvature admits a Kähler metric with positive scalar curvature when the complex dimension \( n \geq 2 \). As a corollary, he showed that any rational surface admits a Kähler metric with positive scalar curvature.
In general, given any Hermitian vector bundle \((E, h) \to (M, g)\) where \((M, g)\) is a compact Kähler manifold, the Chern curvature form \(\Theta(\mathcal{O}_{P(E)}(1))\) of \(\mathcal{O}_{P(E)}(1)\) over \(P(E)\) has the fiber direction component given by the Fubini-Study form, hence is positive. Therefore
\[
\tilde{\omega} = \pi^*\omega_g + s\sqrt{-1}\Theta(\mathcal{O}_{P(E)}(1))
\]
is a well-defined Kähler metric on \(P(E)\) when \(s > 0\) is sufficiently small.

Hitchin [24] studied Kähler metrics of the form (12) on Hirzebruch surfaces \(M_{2,k}\). Here we pick \((E, h) = (H^k \oplus 1_{\mathbb{C}P^1}, h)\) and \((M, g)\) as \((\mathbb{CP}^1, g_{FS})\) where \(g_{FS}\) is the standard Fubini-Study metric and \(h\) the induced metric. If we use the local parametrization \((z_1, (d_1)^{-\frac{k}{2}}, z_2)\) and write down the metric locally
\[
\tilde{\omega} = \sqrt{-1}\partial \bar{\partial} \log(1 + |z_1|^2) + s\sqrt{-1}\partial \bar{\partial} \log[(1 + |z_1|^2)^k + |z_2|^2]
\]
In this case, since the vector bundle \(H^k \oplus 1_{\mathbb{C}P^1}\) has nonnegative curvature, the component of the Chern curvature form \(\Theta(\mathcal{O}_{P(E)}(1))\) along the base direction is nonnegative, so \(\tilde{\omega}\) is in fact a Kähler metric for all \(s > 0\).

Hitchin [24] proved that \(\tilde{\omega}\) has \(H > 0\) if \(0 < s(1 + ks)^2 < \frac{1}{k(2k-1)}\). Alvarez-Chaturvedi-Heier [2] further proved that it suffices to assume \(s < \frac{1}{k}\) to guarantee \(H > 0\). Let us define the optimal local (global) holomorphic pinching constant to be the maximum value among all the pinching constants of Hitchin’s examples according to Definition 2.2. It was calculated in [2] that the optimal local and global holomorphic pinching constants are the same and equal to \(\frac{1}{(2k+2)^2}\), and the corresponding \(s = \frac{1}{2k+2}\). The corresponding Kähler class is \(bE_{\infty} - aE_0\) where \(b = \frac{2k+2}{2k+1} > 0\). In particular, if \(k = 1\), then \(s = \frac{1}{3}\), the corresponding Kähler metric \(\tilde{\omega}\) is not in the anti-canonical class of \(2\pi c_1(M_{2,1})\), note that \(M_{2,1}\) is the only Fano Hirzebruch surface.

Let us rephrase the question we proposed in Section 1 of this paper.

**Question 3.1.** Hitchin’s examples produce a family of Kähler metrics with \(H > 0\) whose Kähler classes only stay in a subset of the Kähler cone. The path does not approach both sides of the essential boundary of the Kähler cone of \(M_{2,k}\). Here by essential we mean that here the vertex of the cone is not counted as the boundary.

Are there Kähler metric with \(H > 0\) from each of the Kähler classes of \(M_{2,k}\)? In particular, since \(c_1(M_{2,1}) > 0\), it would be interesting to know there is any metric with \(H > 0\) from the anti-canonical class of \(M_{2,1}\).

What is the best holomorphic pinching constant \(\lambda_k\) among all Kähler metrics of \(H > 0\) on the Hirzebruch surfaces \(M_{2,k}\)? Note that Hitchin’s examples are of \(U(2)\)-symmetry, it seems reasonable to expect the optimal holomorphic pinching constant \(\lambda_k\) to be realized by some Kähler metric with a large symmetry.

Let \(\lambda_k\) denote the optimal holomorphic pinching constant among all Kähler metrics of \(H > 0\) on \(M_{2,k}\). Is it true that any compact Kähler surface with pinching constant strictly greater \(\lambda_1\) must be biholomorphic to \(\mathbb{CP}^2\) or \(\mathbb{CP}^1 \times \mathbb{CP}^1\)?

### 3.2. Hirzebruch manifolds by Calabi’s ansatz

Let us recall a powerful method to construct canonical metrics pioneered by Calabi (Calabi’s ansatz). Our exposition follows more closely from Koseki-Sakane [28]. As we shall see later, for Hirzebruch manifolds \(M_{n,k}\), Calabi’s ansatz can be applied to produce \(U(n)\)-invariant Kähler metrics which include Hitchin’s examples as a special case.

#### 3.2.1. Kähler metrics on \(\mathbb{C}^*\)-bundles reviewed

First we review some facts on the construction of a Kähler metric on a \(\mathbb{C}^*\)-bundle over a compact Kähler manifold where \(\mathbb{C}^* = \mathbb{C} - \{0\}\). Given a holomorphic line bundle \(L \to M\) on a complex manifold \(M\), where \(\pi\) is the natural projection, we consider the \(\mathbb{C}^*\)-action on \(L^* = L \setminus L_0\), where \(L_0\) is the zero section of \(L\). Denote by \(H\) and \(S\) the two holomorphic vector fields generated by the \(\mathbb{R}^+\) and \(\mathbb{S}^1\) action, respectively.

Let \(\pi : (L, h) \to (M, g)\) be a Hermitian line bundle over a compact Kähler manifold \((M, g)\). Denote by \(J\) the complex structure on \(L\). Assume \(t\) is a smooth function on \(L\) depending only
on the norm and is increasing in the norm of Hermitian metric $h$. Consider a Hermitian metric on $L^*$ of the form
\begin{equation}
g = \pi^* g_t + dt^2 + (dt \circ J)^2,
\end{equation}
where $g_t$ is a family of Riemannian metrics on $M$. Denote by $u(t)^2 = \tilde{g}(H, H)$. It can be checked that $u$ depends only on $t$.

The following results were proved in [28].

**Lemma 3.2** ([28]). The Hermitian metric $\tilde{g}$ defined by (14) is Kähler on $L^*$ if and only if each $g_t$ is Kähler on $M$, and $g_t = g_0 - U \Theta(L)$, where $U = \int_0^t u(r) dr$, and if the range of $t$ includes 0, then the corresponding value of $U$ is 0.

**Assumption 3.3.** We further assume the eigenvalues of the curvature $\Theta(L)$ with respect to $g_0$ are constant on $M$.

Let $z_1 \cdots z_{n-1}$ be local holomorphic coordinates on $M$ and $z_0 \cdots z_{n-1}$ be local coordinates on $L^*$ such that $\frac{\partial}{\partial z_0} = H - \sqrt{-1} S$.

**Lemma 3.4** ([28]). $\tilde{g}_{\alpha\beta} = 2u^2$, $\tilde{g}_{\alpha\bar{z}_\beta} = 2u \partial_{\alpha}t$, $\tilde{g}_{\bar{z}_\alpha \bar{z}_\beta} = g_{t\alpha\beta} + 2 \partial_{\alpha}t \partial_{\beta}t$. Define $p = det(g_0^{-1} \cdot g_t)$, then $det(\tilde{g}) = 2u^2 \cdot p \cdot det(g_0)$.

**Lemma 3.5** ([28]). If we assume that $\partial_{\alpha}t = \partial_{\beta}t = 0$ (1 $\leq$ $\alpha$ $\leq$ $n-1$) on a fiber, and if a function $f$ on $L^*$ depends only on $t$, then $\partial_{\alpha} \partial_{\beta}f = u \frac{dt}{u} (u \frac{dt}{u})$, $\partial_{\alpha} \partial_{\beta}f = 0$, $\partial_{\alpha} \partial_{\beta}f = -2(u \frac{dt}{u} \Theta(L))_{\alpha \beta}$. Moreover, the Ricci curvature of $\tilde{g}$ becomes: $R_{\alpha\beta\bar{z}_\alpha} = -u \cdot \frac{d^2}{du^2} (u^2 (\log(u^2)))$, $\tilde{R}_{\alpha\beta\bar{z}_\alpha} = 0$, $\tilde{R}_{\alpha\beta\bar{z}_\alpha} = R_{\alpha\beta\bar{z}_\alpha} + \frac{1}{2} u \cdot \frac{d^2}{du^2} (\log(u^2)) \cdot \Theta(L)_{\alpha \beta}$. It is convenient to introduce the new functions $\phi(U) = u^2(t)$ and $Q(U) = p$. Recall that $\sqrt{\tilde{h}}$ is the norm of Hermitian metric on $L$. Since $\frac{d\phi}{\sqrt{\phi(U)}} = dt$ and $\frac{dQ}{\sqrt{Q(U)}} = \frac{d\sqrt{\tilde{h}}}{\sqrt{\tilde{h}}}$, for any given $\phi(U)$ we can solve for $t$ with respect to $\sqrt{\tilde{h}}$, hence recover the metric $\tilde{g}$. The following lemma characterizes any $\phi(U)$ which corresponds to a well-defined (maybe incomplete) Kähler metric in the form of (14) on the total space of $L^*$.

**Lemma 3.6.** Given any hermitian line bundle $(L, h)$ over a compact Kähler manifold $(M, g_0)$ and Assumption 3.3 holds. Fix $-\infty < U_{\min} < U_{\max} \leq +\infty$ such that $g_t = g_0 - U \Theta(L)$ remains positive on $(U_{\min}, U_{\max})$.

Let $\phi(U)$ be a smooth positive function on $(U_{\min}, U_{\max})$ with $\phi(U_{\min}) = \phi(U_{\max}) = 0$. We further assume that $\int_{U_{\min}}^{U_{\max}} \frac{dt}{\sqrt{\phi(U)}} = +\infty$, $\int_{U_{\min}}^{U_{\max}} \frac{dt}{\sqrt{\phi(U)}} = +\infty$ and $\int_{U_{\min}}^{U_{\max}} \frac{dt}{\sqrt{\phi(U)}}$ is finite for all $U \in (U_{\min}, U_{\max})$.

Then we can solve for $t$ as a function of $\sqrt{\tilde{h}}$ which is strictly increasing on $\sqrt{\tilde{h}}$, and $t$ has the range $(t_{\min}, t_{\max})$ which contains 0. Therefore we can get a well-defined smooth Kähler metric $\tilde{g}$ in the form of (14) on $L^*$.

**3.2.2. Metric completion by compactification, Kähler metrics on $\mathbb{P}(L \oplus 1)$.** Koiso-Sakane [28] had a general discussion on when the Kähler metric $\tilde{g}$ on $L^*$ admits a compactification so that it can be extended onto $\mathbb{P}(L \oplus 1)$. We summarize their results below.

**Lemma 3.7** (Koiso-Sakane [28]). Let $(t_{\min}, t_{\max})$ be the range of function $t$ on $L^*$, and assume that $t$ extends to $\mathbb{P}(L \oplus 1)$ with the range $[t_{\min}, t_{\max}]$, where the subset $M_{\min}$ (or $M_{\max}$) defined by $t = t_{\min}$ (or $t = t_{\max}$) is a complex submanifold with codimension $D_{\min}$ (or $D_{\max}$). Moreover, assume the Kähler metric $\tilde{g}$ extends to $\mathbb{P}(L \oplus 1)$, which is also denoted by $\tilde{g}$.

Then it implies that near $U = U_{\min}$ the Taylor expansion of $\phi(U)$ has the first term $2(U - U_{\min})$, and near $U = U_{\max}$, it has the first term $2(U_{\max} - U)$. In other words, $t - t_{\min}$ gives the distance from $M_{\min}$ to points in $\mathbb{P}(L \oplus 1)$ and from $M_{\max}$ in $\mathbb{P}(L \oplus 1)$, and $t_{\max} - t_{\min}$ the distance from $M_{\min}$.

A standard example which satisfies the assumptions of Lemma 3.7 is Hirzebruch manifold $M_{n,k}$. Indeed We may view any $M_{n,k}$ as the compactification of the total space of $\mathbb{C}^*$-bundle
induced from $k$-th power of the tautological bundle $H^{-k} \to \mathbb{CP}^{n-1}$. Here we assume the base \(\mathbb{CP}^{n-1}\) is endowed with the Fubini-Study metric $\text{Ric}(g_0) = g_0$, hence $R_{ijkl}(g_0) = \frac{1}{n}(g_{ij}g_{kl} + g_{kj}g_{il})$. Then from the previous results due to Koiso-Sakane we get

$$\hat{g}_\alpha = (1 + \frac{k}{n}U)g_\alpha, \quad \Theta(H^{-k}) = -\frac{k}{n}g_\alpha, \quad t_\gamma = \frac{k}{2n}ug_\gamma.$$

Here we need to pick \([U_{\min}, U_{\max}]\) such that $U_{\min} > -\frac{k}{n}$ and $U_{\max} < \infty$.

From now on, we will focus our consideration on $M_{n,k}$. But before that, let us remark that there are other types of compactification covered in Lemma 3.7. For example, consider the tautological bundle $H^{-1} \to \mathbb{CP}^{n-1}$, if we pick $U_{\min} = -n$ and $U_{\max} < \infty$, the corresponding Kähler metric on $\mathbb{CP}^n$. Also it is clear that a similar discussion can be carried out on $C^*$-bundles obtained from a vector bundle other than a line bundle.

The following proposition gives the formulas of curvature tensors of $\tilde{\nabla}$ on $M_{n,k}$. Note that $U(n)$ acts isometrically on the base $\mathbb{CP}^{n-1}$, and it can be lifted as isometric actions on the total space $M_{n,k}$, so the local calculation along a fiber in Lemma 3.5 works on the total space $M_{n,k}$.

**Proposition 3.8** (Curvature tensors of $\hat{\nabla}$ on $M_{n,k}$). Let us assume that $\partial_{x^\alpha} = \partial_{\phi^\alpha} = 0$ ($1 \leq \alpha \leq n - 1$) on a fiber. Consider the unitary frame \(\{e_0, e_1, \cdots, e_{n-1}\}\) on $M_{n,k}$:

$$e_0 = \frac{1}{2\phi} \frac{\partial}{\partial z_0}, \quad e_i = \frac{1}{\sqrt{(1 + \frac{k}{n}U)g_{ii}}} \frac{\partial}{\partial z_i} (1 \leq i \leq n - 1),$$

where \(\{e_1, \cdots, e_{n-1}\}\) is a unitary frame on $(\mathbb{CP}^{n-1}, g_0)$. The only nonzero curvature components of $\hat{\nabla}$ on $M_{n,k}$ are:

$$A = \tilde{\mathcal{R}}_{0000} = -\frac{1}{2} \frac{d^2\phi}{dU^2},$$

$$B = \tilde{\mathcal{R}}_{00i} = \frac{k^2\phi - k(n + kU)\frac{d\phi}{dU}}{2(n + kU)^2},$$

$$C = \tilde{\mathcal{R}}_{ii} = 2\tilde{\mathcal{R}}_{ij} = \frac{[2(n + kU) - k^2\phi]}{(n + kU)^2},$$

where $1 \leq i, j \leq n - 1$ and $i \neq j$.

Proposition 3.8 leads to the following characterization of $U(n)$-invariant Kähler metrics with $H > 0$ on $M_{n,k}$.

**Proposition 3.9** (The generating function $\phi$). Any $U(n)$-invariant Kähler metric on $M_{n,k}$ has positive holomorphic sectional curvature if and only if

$$A > 0, \quad C > 0, \quad 2B > -\sqrt{AC}. \quad (15)$$

In other words, it is characterized by a smooth concave function $\phi(U)$ where $-\infty < U_{\min} \leq U \leq U_{\max} < +\infty$ with $1 + \frac{k}{n}U_{\min} > 0$ such that the following conditions hold:

1. $\phi > 0$ on $(U_{\min}, U_{\max})$, \(\phi(U_{\min}) = \phi(U_{\max}) = 0\), \(\phi'(U_{\min}) = 2\), and $\phi'(U_{\max}) = -2$.
2. $A > 0$, $B > 0$, $C > 0$, $\phi(U) < 0$, $\phi(U) > 2\frac{k\phi}{kU}$, $\frac{k\phi}{n + kU} - \phi' > -\sqrt{\phi'^2 - \frac{1}{k^2}(n + kU)}$

for any $U \in [U_{\min}, U_{\max}]$.

As a corollary of Proposition 3.8, we also have the following rough estimates on holomorphic pinching constants of $U(n)$-invariant Kähler metrics on $M_{n,k}$.

**Corollary 3.10** (A rough estimate on holomorphic pinching constants). Any $U(n)$-invariant Kähler metric on $M_{n,k}$ with $H > 0$ have its local holomorphic pinching constant bounded from above by $\frac{k}{n}$. Fix $0 < c < \frac{n}{k + 2}$, if we consider any $U(n)$-invariant Kähler metric with $H > 0$ whose corresponding Kähler class lies in the following ray $S$ in the Kähler cone

$$S = \{ b[E_\infty] - a[E_0] \mid a = b \frac{n + kc}{n} \},$$

where $a = b \frac{n - kc}{n}$,
Then its holomorphic pinching constant is bounded from above by \( \frac{2c}{n+k} \).

Another consequence of Proposition 3.8 is the path-connectedness of Kähler metrics of \( H > 0 \) in the same Kähler class on \( M_{n,k} \).

**Corollary 3.11** (A convexity property in a fixed Kähler class). If \( \phi_1 \) and \( \phi_2 \) are two generating functions of two Kähler metrics of \( H > 0 \) in the same Kähler class on \( M_{n,k} \), so is any convex combination \( t\phi_1 + (1-t)\phi_2 \) with \( 0 < t < 1 \).

### 3.2.3. Hitchin’s examples reformulated.

Hitchin’s construction gives a family of Kähler metrics with \( U(2) \) symmetry on \( M_{2,k} \). We observe a similar construction works for \( M_{n,k} \).

**Example 3.12** (Hitchin’s examples on \( M_{n,k} \)). Given \( s > 0, U_{\min} = 0, U_{\max} = ns \), define \( \phi_s(U) = -\frac{2}{ns}U^2 + 2U \). Now

\[
A = \frac{2}{ns}, \quad B = \frac{k^2U^2 + 4kU - kn}{(n+kU)^2}, \quad C = \frac{2k^2U^2 + (2k - 2k^2)U + 2n}{(n+kU)^2}.
\]

For all \( s > 0 \), \( \phi_s \) gives a family of Kähler metric on \( M_{n,k} \). Now assume \( n = 2 \), Hitchin proved that these metrics have \( H > 0 \) if \( s \) suitably small. Indeed it is observed by Alvarez-Chaturvedi-Heier [2] that it suffices to assume \( 0 < s < \frac{1}{2}\sqrt{s} \) to get \( H > 0 \). They also calculated the pinching constants of these metrics and concluded that the optimal value is \( \frac{1}{2k+1} \) when \( s = \frac{1}{2k+2} \).

When \( k = 1 \), the optimal metric lies in the Kähler class \( \frac{4}{3}[D_\infty] - [D_0] \).

In fact, we observe that Hitchin’s example is canonical in the following sense.

**Proposition 3.13.** Hitchin’s examples can be uniquely characterized as \( U(n) \)-invariant Kähler metrics on \( M_{n,k} \) with the constant radial curvature \( A \). In particular, the following example gives the unique form of \( \phi(U) \) up to a scaling and a translation of \( [U_{\min}, U_{\max}] \).

**Example 3.14** (Hitchin’s example in a canonical form). Let \( c > 0, U_{\min} = -c, U_{\max} = c \), define \( \phi_c(x) = c - \frac{x^2}{c} \) on \([-c, c]\). Since \( \phi'(-c) = 2, \phi'(c) = -2 \), and \( \phi(\pm c) = 0 \), we have a Kähler metric on \( M_{2,1} \). Now

\[
A = \frac{1}{c}, \quad B = \frac{1}{2(U+2)}U^2 + \frac{1}{2U}c, \quad C = \frac{1}{2(U+2)}U^2 + 2U + 4 - c.
\]

If we assume \( 0 < c < 2 \), then obviously \( 1 - \frac{1}{U} < 0, A > 0 \), and \( C > 0 \) on \([-c, c]\). Consider

\[
D = 2B + \sqrt{AC} = \frac{1}{2(U+2)}U^2 + \frac{1}{2U}c + \frac{1}{2}U^2 + 2cU + c(4 - c) \cdot (U + 2).
\]

Then \( D(-c) > 0 \) is equivalent to \( c < \frac{2}{3} \). Moreover, one can check that the numerator of \( D(U) \) is increasing on \( U \in (-c, c) \), hence \( D(U) > 0 \) for any \(-c < c < 0 < c < \frac{2}{3} \). Therefore \( \phi_c \) provides a family of Kähler metrics of \( H > 0 \).

Next let us find the pinching constant for \( \phi_c(x) = c - \frac{x^2}{c} \). For any given \( \phi_c \), the expression of the holomorphic sectional curvature is

\[
H(X) = A|x_1|^4 + 4B|x_1x_2|^2 + C|x_2|^4,
\]

where \( X = x_1e_1 + x_2e_2 \) with \( |x_1|^2 + |x_2|^2 = 1 \).

If we set \( t = |x_1|^2 \), then \( H(X) = (A + C - 4B)t^2 + t(4B - 2C) + C \) with \( t \in [0, 1] \), it is elementary to discuss its extremal values. In particular, we will show that for any \( c \in (0, \frac{2}{3}) \), the pinching constant \( \inf_{U \in (-c, c)} \max_{V \in [1]} \frac{H(U)}{H(V)} \) is always attained at \( U = -c \), i.e. along the zero section of \( M_{2,1} \).

Indeed

\[
\min_{||v||=1} H(U, v) = \frac{AC - 4B^2}{(A + C - 4B)} \max_{||v||=1} H(U, v) = A.
\]

Therefore, the local pinching constant equals

\[
\frac{AC - 4B^2}{(A + C - 4B)} = \frac{2U^3 + (6 - 3c)U^2 - 12cU - c^3 - 2c^2 - 8c}{(2U - 2 - 3c)(U + 2)^2}.
\]
It is direct to check that the above expression is increasing on $U \in [-c, c]$. If $U = -c$, it becomes $\frac{2c(c-3c)}{(2c-3c)(6c+2c)}$. When $c = \frac{1}{2}$, it attains the maximum $\frac{1}{4}$. The optimal pinching constant among the family $\phi_c$ agrees with the result of Alvarez-Chaturvedi-Heier \cite{2}, and the corresponding optimal Kähler metrics are just multiples of those in \cite{2}.

It is also straightforward to solve the optimal holomorphic pinching constant of Hitchin’s examples on any Hirzebruch manifold $M_{n,k}$. Given any $1 \leq k < n$, pick $c > 0$, $U_{\text{min}} = -c$, $U_{\text{max}} = c$, define $\phi_c(U) = c - \frac{c^2}{c}$ on $[-c, c]$. We claim that $\phi_c$ gives a Kähler metric on $M_{n,k}$ with $H > 0$ as long as $c < \frac{c_{\text{opt}}}{k(2k+1)}$. Note that

$$A = \frac{1}{c}, \quad B = \frac{k^2 U^2 + \frac{2nk}{c} U + ck^2}{2(n + kU)^2}, \quad C = \frac{k^2 U^2 + 2kU + 2n - ck^2}{(n + kU)^2}.$$ 

Similarly

$$D = 2B + \sqrt{AC} = \frac{k^2 U^2 + \frac{2nk}{c} U + ck^2 + \frac{n + kU}{c} \sqrt{k^2 U^2 + 2kU + (2n - ck^2)c}}{(n + kU)^2}.$$

First note that $0 < c < \frac{n}{k(2k+1)}$ is equivalent to $D(-c) > 0$, then similarly we could show under this condition on $c$ the numerator of $D$ is strictly increasing on $U \in (-c, c)$, therefore $D(U) > 0$ holds on $[-c, c]$.

Note that when $n \geq 4$, for any positive integer $k$ satisfying $k(2k+1) < n$, we may pick $c = 1$. In this case the Kähler class of the resulting metric is proportional to the anti-canonical class.

For the above metric on $M_{n,k}$ given by $\phi_c(x) = c - \frac{c^2}{c}$ on $[-c, c]$, where $0 < c < \frac{n}{k(2k+1)}$ is a constant, one can carry out a similar calculation to conclude that maximal local pinching constant achieves its maximum at $U = -c$, which is

$$\frac{2c(n-c(2k^2 + k))}{(n-ck)((3k+2)c+n)}.$$

It can be shown that it obtains its maximum value at $c = \frac{n}{4k+3k}$, and the optimal pinching constant is $\frac{1}{4k+3k}$. Note that the optimal pinching constant is dimension free.

3.3. New examples and the proof of Theorem 1.3. Next let us consider $M_{2,1}$ which is the only Fano Hirzebruch surface. Note that Kähler metrics of Hitchin’s examples on $M_{2,1}$ can not be proportional to the anti-canonical class. A natural question is whether there exists a Kähler metric with $H > 0$ in $2\pi c_1(M_{2,1})$. Note that the corresponding $\phi(U)$ of such a metric must satisfy $(2 + U_{\text{max}}) = 3(2 + U_{\text{min}})$ besides $\phi(U_{\text{min}}) = \phi(U_{\text{max}}) = 0$, $\phi'(U_{\text{min}}) = 2$, and $\phi'(U_{\text{max}}) = -2$ required by the smooth compactification.

In the following we exhibit such an example with different global and local holomorphic pinching constants.

**Proposition 3.15** (A new family of Kähler metrics of $H > 0$ on $M_{2,1}$). Given any real number $0 < c < \frac{1}{2}$, pick a real number $\mu \in (\frac{1}{2}, c)$, define $\phi_{c,\mu} : [-c, c] \to \mathbb{R}$ by

$$\phi_{c,\mu}(U) = \mu - \frac{1}{c^3} - \frac{\mu}{c^3} U^4 - \frac{2\mu}{c^2} - \frac{1}{c} U^2$$

Thus $\phi_{c,\mu}$ determines a family of Kähler metrics on $M_{2,1}$, and in particular when $c = 1$, the Kähler class of $\phi_{c,\mu}$ is proportional to the anti-canonical class of $M_{2,1}$.

There exists some $\delta \in (0, \frac{1}{2})$ which depends on $c$ such that for any $\frac{1}{2}c < \mu < (\frac{1}{2} + \delta)c$, $\phi_{c,\mu}(U)$ defines a Kähler metric on $M_{2,1}$ with $H > 0$. 

Proof of Proposition 3.15. We begin with the curvature tensors of $\phi_{c,\mu}(U)$:

$A = 6\left(\frac{1}{c^3} - \frac{\mu}{c^4}\right)U^2 + \left(\frac{2\mu}{c^2} - \frac{1}{c}\right)$,

$B = \frac{3\left(\frac{1}{c^3} - \frac{\mu}{c^4}\right)U^4 + 8\left(\frac{1}{c^3} - \frac{\mu}{c^4}\right)U^3 + \left(\frac{2\mu}{c^2} - \frac{1}{c}\right)U^2 + 4\left(\frac{2\mu}{c^3} - \frac{1}{c}\right)U + \mu}{2(U + 2)^2}$,

$C = \frac{\left(\frac{1}{c^3} - \frac{\mu}{c^4}\right)U^4 + \left(\frac{2\mu}{c^2} - \frac{1}{c}\right)U^2 + 2(U + 2) - \mu}{(U + 2)^2}$.

First note that for any $c \in (0, 2)$, $\mu \in \left(\frac{1}{4}c, c\right)$, we have $A > 0$ for any $U \in [-c, c]$, then since $C(-c) = \frac{2}{2-c}$ and $C(U + 2)^2$ is increasing on $[-c, c]$, we also have $C > 0$.

Next one can check that

$$2B + \sqrt{AC}|_{U = -c} = -\frac{2}{2-c} + \frac{1}{c} \sqrt{\frac{2(5c - 4\mu)}{2-c}}.$$  

From here, it is direct to see that given any $c \in (0, \frac{5}{2})$, there exists some $\delta > 0$ such that for any $\frac{1}{4}c < \mu < \left(\frac{1}{4} + \delta\right)c$, $2B + \sqrt{AC} > 0$ at $U = -c$.

From now on let us consider $T_\mu(U) = (U + 2)^2(2B + \sqrt{AC})$, it suffices to show that $T_\mu(U) > 0$ for any $U \in [-c, c]$ if $\mu$ is sufficiently close to $\frac{1}{4}c$. Note that

$$T_\mu(U) = P_\mu(U) + (U + 2)\sqrt{Q_\mu(U)}$$

where

$$P_\mu(U) = 3\left(\frac{1}{c^3} - \frac{\mu}{c^4}\right)U^4 + 8\left(\frac{1}{c^3} - \frac{\mu}{c^4}\right)U^3 + \left(\frac{2\mu}{c^2} - \frac{1}{c}\right)U^2 + 4\left(\frac{2\mu}{c^3} - \frac{1}{c}\right)U + \mu$$

and

$$Q_\mu(U) = 6\left(\frac{1}{c^3} - \frac{\mu}{c^4}\right)^2U^6 + 7\left(\frac{1}{c^3} - \frac{\mu}{c^4}\right)^2U^5 + 12\left(\frac{1}{c^3} - \frac{\mu}{c^4}\right)^3U^4 + 4\left(\frac{2\mu}{c^2} - \frac{1}{c}\right)U^3 + 2\left(\frac{2\mu}{c^2} - \frac{1}{c}\right)U + \left(-\frac{2}{c^2}\mu + \frac{8 + c}{c^2}\mu - \frac{4}{c}\right).$$

Claim 3.16. $T_\mu(U) > 0$ on $[-c, c]$ and in particular it has a positive lower bound at 0.

Proof of Claim 3.16. To see it is true, note that:

$$T_\mu(U) = \left(\frac{3}{2c^3}U^4 + \frac{4}{c^3}U^3 + \frac{6}{c} + (U + 2)\sqrt{\frac{3}{2c^3}U^6 + \frac{6}{c^3}U^5 + \frac{3(8-c)}{2c^3}U^4}\right).$$

It suffices to consider the interval $[-c, 0]$, we will show that $T_\mu(U)$ is strictly concave on $[-c, 0]$, thus it attains its minimum either at $U = -c$ or at $U = 0$, which are both positive.

$$\frac{d^2T_\mu(U)}{dU^2} = \frac{6}{c^3}U(3U + 4) + \frac{9}{2c^3}U^3 \cdot R(U) \left(\frac{1}{2c^3}U^6 + \frac{6}{c^3}U^5 + \frac{3(8-c)}{2c^3}U^4\right)^{\frac{3}{2}}$$

where

$$R(U) = 6U^8 + 6U^7 + 36c^3U^5 + (108c^3 - 9c^4)U^4 + (80c^3 - 10c^4)U^3 + 30c^6U^2 + (108c^6 - 12c^7)U + c^8 - 20c^7 + 96c^6.$$  

Note that $R(-c) = 4c^6(2 - c^2) > 0$, we will prove that $\frac{dR(U)}{dU} > 0$ on $[-c, 0]$, which leads to $R(U) > 0$ for any $0 < c < \frac{5}{2}$. Indeed,

$$\frac{dR(U)}{6dU} = 8U^7 + 7U^6 + 30c^3U^4 + 6(12c^3 - c^4)U^3 + 5(8c^3 - c^4)U^2 + 10c^6U + (18c^6 - 2c^7) = I_1 + I_2 + I_3.$$  

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where we have
\[ I_1 = 10e^6 U + \frac{72}{5} e^6 - 2c^7 \geq \frac{72}{5} e^6 - 12c^7 = 12(\frac{6}{5} - c) \geq 0, \]
\[ I_2 = 8U^7 + 7U^6 + \frac{13}{6} e^6 U^4 \geq U^6(8U + 7 + \frac{13}{6} e) \geq 7U^6(1 - \frac{5}{6} e) \geq 0, \]
\[ I_3 = \frac{167}{6} e^4 U^4 + \frac{18}{5} e^5 + 6(12e^3 - e^4)U^3 + 5(8e^3 - e^4)U^2, \]
for any \( U \in [-c, 0] \) where \( 0 < c < \frac{6}{5} \).

Next we prove \( I_3 > 0 \) on \([-c, 0]\).

\[ I_3 \geq U^2 \left[ \frac{167}{6} e^5 U^2 + 6(12e^3 - e^4)U + 5(8e^3 - e^4) + \frac{18}{5} e^4 \right]. \]

Let \( S(U) \) denote the quadratic function inside the bracket:
\[ S(U) = \frac{167}{6} e^5 U^2 + 6(12e^3 - e^4)U + 5(8e^3 - e^4) + \frac{18}{5} e^4. \]

Now it is straightforward to see that under the assumption \( 0 < c < \frac{6}{5} \), \( S(U) \) attains its minimum at \( U = -c \), and
\[ S(-c) = e^5 \left( \frac{203}{6} e^2 - \frac{367}{5} c + 40 \right) > 0. \]

Putting these together, we have proved that \( \frac{d^2 T_\mu(U)}{dU^2} \) is strictly negative on \([-c, 0]\), and therefore \( T_\mu(U) > 0 \) on \([-c, c]\).

Now let us continue with the proof of Proposition 3.15, note that as \( \mu \to (\frac{c}{2})^+ \), \( T_\mu(U) \) converges to \( T_\mu(U) \) uniformly on \([-1, -U_0] \cup [U_0, 1]\) for some fixed small number \( U_0 > 0 \),
\[ \frac{\partial T_\mu(U)}{\partial \mu} = \frac{\partial P}{\partial \mu} + \frac{(U + 2)}{2\sqrt{Q}} \frac{\partial Q}{\partial \mu} \]
\[ = \left( -\frac{3}{c^2} U^4 - \frac{8}{c^3} U^3 + \frac{2}{c^2} U^2 + \frac{8}{c^3} U + 1 \right) \]
\[ + \frac{(U + 2)}{2\sqrt{Q}} \left( -\frac{12}{c^4} U^6 - \frac{\mu}{c^5} U^5 + 7 \left[ \frac{2}{c^2} \left( \frac{1}{c^3} - \frac{\mu}{c^4} \right) - \frac{2\mu}{c^3} \right] U^4 \right) \]
\[ - \frac{12}{c^4} U^3 \left[ \frac{4}{c^3} \left( \frac{2\mu}{c^4} - \frac{1}{c} \right) - 6 \frac{\mu}{c^3} \right] + \frac{6}{c^3} (4 - \mu) U^2 + \frac{4}{c^2} U + \frac{8 + c - 4\mu}{c^2} \]

Take \( U_0 = \min \{ \frac{1}{100}, \frac{1}{100} e^2 \} \), we see that \( \frac{\partial T_\mu(U)}{\partial \mu} \) is strictly positive for any \( |U| < U_0 \) as long as \( \mu > \frac{c}{2} \) is small enough. In other words, we can find some \( \delta > 0 \) such that \( T_\mu(U) > 0 \) on \([-U_0, U_0]\) for any \( \frac{c}{2} < \mu < (\frac{c}{2} + \delta)c \). Moreover, \( T_\mu(U) \) converges to \( T_\mu(U) \) outside \([-U_0, U_0]\), hence we get \( T_\mu(U) > 0 \) for \( \frac{c}{2} < \mu < (\frac{c}{2} + \delta)c \). \( \square \)

**Example 3.17** (Pinching constants in the anti-canonical class). Let us now focus on the anti-canonical examples constructed in the previous proposition, namely, with \( c = 1 \). We expect that Proposition 3.15 is still true for any \( \frac{1}{2} < \mu < \frac{3}{4} \). Numerical tests suggest that \( 2B + \sqrt{AC} \) is indeed positive on \( U \in [-1, 1] \) for any \( \frac{1}{2} < \mu < \frac{3}{4} \). However, it seems rather tedious to prove it rigorously, as \( T_\mu(U) \) is not always increasing on \([-1, 1]\). We need some better estimates on the critical points of \( T_\mu(U) \) which lie in \([-1, 0]\).

The following table shows that if \( \mu \) is close to \( \frac{c}{2} \), then the local pinching constants of Kähler metric generated by \( \phi_\mu \) differs from the global one.

| Intervals of \( U \) | \([-1, U_1]\) | \([U_1, U_2]\) | \([U_2, U_3]\) | \([U_3, U_4]\) | \([U_4, 1]\) |
|----------------------|----------------|----------------|----------------|----------------|----------------|
| \( \min_{|U| = 1} H(U, v) \) | \( \frac{AC - 4B}{A^2} \) | \( \frac{AC - 4B}{A^2} \) | \( A \) | \( \frac{AC - 4B}{A^2} \) | \( \frac{AC - 4B}{A^2} \) |
| \( \max_{|U| = 1} H(U, v) \) | \( A \) | \( A \) | \( A \) | \( A \) | \( A \) |

In the above, \( U_1 < U_4 \) are values which corresponds to \( A = C \), and \( U_2 < U_3 \) are values which corresponds to \( A = 2B \).
For example along the zero section $U = -1$, $\min H = \frac{AC - B^2}{A + C - 4B} = \frac{6 - 8\mu}{11 - 4\mu}$ and $\max H = A(-1) = 5 - 4\mu$. Therefore the pinching constant along zero section is $\frac{6 - 8\mu}{(5 - 4\mu)(11 - 4\mu)}$, which is close to $\frac{2}{7}$ as $\mu$ goes close to $\frac{1}{2}$. It is clear that the global maximum of holomorph sectional curvature is attained at $U = -1$ by $A = 5 - 4\mu$ while the global minimum is attained at $U = 0$ by $A = 2\mu - 1$. Therefore, we conclude that the local pinching constants of Kähler metric generated by $\phi_\mu$ differs from the global one. Indeed, we have

$$\min_{U \in [-1, 1]} \frac{\min_{|v||v|=1} H(U, v)}{\max_{|v||v|=1} H(U, v)} = \frac{4(2\mu - 1)}{4 - \mu} \text{ and } \frac{\min_{|v||v|=1} H(U, v)}{\max_{|v||v|=1} H(U, v)} = \frac{2\mu - 1}{5 - 4\mu}.$$ 

Now we are ready to prove our main theorem.

**Theorem 3.18.** Given any Hirzebruch manifold $M_{n,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{C}^{n-1}})$, there exists a Kähler metric of $H > 0$ in each of its Kähler classes.

Theorem 3.18 is a corollary of the following proposition.

**Proposition 3.19.** Let $n \geq 2$ and $k \geq 1$ be any two integers, there exists some $p_0(n, k) \in \mathbb{N}$ and a sequence of positive real numbers $\{\epsilon_p\}_{p \geq p_0}$ with $\lim_{p \to \infty} \epsilon_p = 0$, such that for any $c \in (0, \frac{n}{k} - 2\epsilon_p)$, there exists $p_1(n, k) > p_0$ with the following property:

Given any $p \geq p_1$ there exists some $\delta_1 > 0$ and $\delta_2 > 0$ such that for any $\alpha_2 \in (0, \delta_1)$ and $\mu = \frac{c}{p} + \delta_2$, $\phi(x)$ defined on $[-c, c]$ by

$$\phi(x) = \mu - \alpha_2x^2 - \alpha_2(x - 2\epsilon_2x^2) - \alpha_2x^2.$$

where

$$\alpha_{2p-2} = \frac{p\mu - c - (p - 1)\alpha_2c^2}{c^{2p-2}}, \quad \alpha_{2p} = \frac{c - (p - 1)\mu + (p - 2)\alpha_2c^2}{c^2},$$

generates a Kähler metric with $H > 0$ on $M_{n,k}$.

**Proof of Proposition 3.19.** Let $\epsilon_p = \frac{2\mu}{2p + 2k - 1}$ and pick any $c \leq \frac{n}{k} - 2\epsilon_p$, we will determine constants $\delta_1$ and $\delta_2$ step by step. A quick observation is that in order to make sure both $\alpha_{2p-2}$ and $\alpha_{2p}$ positive, we need

$$\delta_2 < \frac{c}{p(p - 1)} \quad \text{and} \quad (p - 1)\delta_1^2 < p\delta_2.$$ 

Note that

$$A = p(2p - 1)\alpha_{2p}U^{2p-2} + (p - 1)(2p - 3)\alpha_{2p-2}U^{2p-4} + \alpha_2,$$

$$B = \frac{(2p - 1)\alpha_{2p}U^{2p} + 2p nk \alpha_{2p}U^{2p-1} + (2p - 3)k^2 \alpha_{2p-2}U^{2p-2}}{2(n + kU)^2}$$

$$+ \frac{(2p - 2)nk\alpha_{2p-2}U^{2p-3} + k^2 \alpha_2U^2 + 2nk\alpha_2U + k^2\mu}{2(n + kU)^2},$$

$$C = \frac{k^2 \alpha_{2p}U^{2p} + k^2 \alpha_{2p-2}U^{2p-2} + k^2 \alpha_2U^2 + 2kU + 2n - k^2\mu}{(n + kU)^2}.$$

Obviously $A > 0$ on $[-c, c]$, note that

$$C > \frac{2n - k^2\mu - 2kc}{(n + kU)^2} \quad \text{for any} \ U \in [-c, c].$$

By plugging $c \leq \frac{n}{k} - 2\epsilon_p$ and $\mu = \frac{c}{p} + \delta_2$ into the above one can conclude that is a sufficient condition for $C > 0$ is

$$\delta_2 < \frac{n(2p + 2k + 1)}{kp(2p + 2k - 1)}.$$

Note that $2B + \sqrt{AC|_{U=-c}} > 0$ is equivalent to $\phi'(c) < -\frac{4k^2}{n-kc}$. By a direct calculation we see that it is further equivalent to

$$\frac{k(2p - 1) + 2k^2}{c(n-kc)} \left[\frac{c}{k} - \epsilon_p - c\right] > \frac{2p(p - 1)}{c^2}\delta_2 - \alpha_2(2p^2 - 6p + 4).$$
In order that (22) holds, it suffices to pick $\delta_2$ so that the following holds:

\[
(23) \quad \frac{k(2p-1) + 2k^2}{n-kc} \left[ \frac{n}{k} - c_p - c \right] > \frac{2p(p-1)}{c} \delta_2.
\]

In other words, for any $c \in (0, \frac{2}{k} - 2c_p]$, it is easy to pick $\delta_1$ and $\delta_2$ such that all the inequalities in (17), (21), and (23) are satisfied.

It remains to show $2B + \sqrt{A} C$ is positive on $[-c, c]$. Motivated by the proof of Proposition 3.15 let us introduce

\[
T(\mu, \alpha_2, U) = P(\mu, \alpha_2, U) + (n + kU) \sqrt{Q(\mu, \alpha_2, U)},
\]

where

\[
P(\mu, \alpha_2, U) = 2(n + kU)^2 B(U), \quad Q(\mu, \alpha_2, U) = (n + kU)^2 A(U) \cdot C(U).
\]

where $A, B, C$ are given in (18), (19), and (20).

We need to show that $T(\mu, \alpha_2, U) > 0$ for any $U \in [-c, c]$ under the assumption $c \in (0, \frac{2}{k} - 2c_p]$, $\alpha_2 \in (0, \delta_1)$, and $\mu = \frac{p}{c} + \delta_2$, where $\delta_1$ and $\delta_2$ satisfy (17), (21), and (22). By a similar argument as in the proof of Proposition 3.15, one checks that for $\alpha_2, \delta_2$ small, there exists $U_0$ sufficiently small.

\[
\frac{\partial T(\mu, \alpha_2, U)}{\partial \mu} > 0, \quad \text{and} \quad \frac{\partial T(\mu, \alpha_2, U)}{\partial \alpha_2} > 0
\]

for any $|U| < U_0$.

For example, note that

\[
\frac{\partial \alpha_{2p-2}}{\partial \mu} = \frac{p}{c^{2p-2}} \frac{\partial \alpha_{2p}}{\partial \alpha_2} = -\frac{p-1}{c^{2p}}.
\]

Therefore,

\[
\frac{\partial P}{\partial \mu} = -\frac{(p-1)(2p-1)k^2}{c^{2p}} U^{2p} + \frac{k^2p}{c^{2p-2}} U^{2p-2} + \frac{2p-2}{c^{2p-2}} U^{2p-3} + k^2,
\]

\[
\frac{\partial Q}{\partial \mu} = \frac{\partial A}{\partial \mu} C + A \frac{\partial C}{\partial \mu} = \left( -\frac{p(p-1)(2p-1)}{c^{2p}} U^{2p} + \frac{p(p-1)(2p-3)}{c^{2p-2}} U^{2p-4} \right) C
\]

\[
+ \left( \frac{k^2(p-1)}{c^{2p}} U^{2p} + \frac{k^2p}{c^{2p-2}} U^{2p-2} - k^2 \right) A,
\]

\[
\geq \frac{p(p-1)(2p-3)}{2c^{2p-2}} U^{2p-4} (2n - k^2 U) - k^2 \left( \alpha_2 + (p-1)(2p-3) \alpha_{2p-2} U^{2p-4} \right).
\]

It follows that when $|U|, \alpha_2, \delta_2$ are small enough,

\[
(24) \quad \frac{\partial T(\mu, \alpha_2, U)}{\partial \mu} = \frac{\partial P}{\partial \mu} + \frac{(n + kU) \partial Q}{\partial \mu} \geq \frac{k^2}{2} - \frac{4nk^2 \alpha_2}{2\sqrt{\alpha_2} \cdot n} > 0.
\]

Therefore to prove Theorem 3.19 it suffices to show that $T(\frac{2}{p}, 0, U) > 0$ for $U \in [-c, c]$. Now we have

\[
\alpha_{2p-2} = 0, \quad \text{and} \quad \alpha_{2p} = \frac{1}{p c^{2p-1}}.
\]

Therefore,

\[
P_{\frac{2}{p}}(\frac{c}{p}, 0, U) = \frac{(2p-1)k^2}{pc^{2p-1}} U^{2p} + \frac{2nk}{c^{2p-1}} U^{2p-1} + \frac{c}{p} k^2,
\]

\[
Q_{\frac{2}{p}}(\frac{c}{p}, 0, U) = \frac{2p-1}{c^{2p-1}} U^{2p-2} \left( \frac{k^2}{pc^{2p-1}} U^{2p} + 2kU + 2n - \frac{c}{p} k^2 \right).
\]
Let us reparametrize $x = \frac{t}{c}$, then $T(x)$ is defined on $[-1, 1]$.

(27) \[ T\left(\frac{c}{p}, 0, x\right) = P(x) + (n + kcx)\sqrt{Q(x)} = \frac{(2p - 1)k^2c}{p}x^{2p} + 2nkx^{2p-1} + \frac{c}{p}k^2x + (n + kcx)\sqrt{(2p - 1)x^{2p-2}} + \frac{k^2}{p}x^{2p} + 2kx + \frac{2n}{c} - \frac{k^2}{p} \]

(28) 

Note that $P(x)$ is increasing on $[0, -1]$ and has a unique zero $x_0 \in (-1, 0)$. We already have $T(-1) > 0$ and $T(x_0) > 0$ from (17), (21), and (22). It suffices to show $T(x) > 0$ on $[-1, x_0]$. To that end, let us introduce

\[ W(x) = -(P(x))^2 + (n + kcx)^2Q(x). \]

Obviously we also have $W(-1) > 0$ and $W(x_0) > 0$. By Lemma 3.20 below we conclude that $W(x) > 0$ on $[-1, x_0]$, which implies $T(x) > 0$ for any $x \in [-1, 0]$, thus completing the proof of Proposition 3.19.

\[ \square \]

**Lemma 3.20.** Given any $n \geq 2$ and $k$ positive integers and $c \in (0, \frac{2}{k} - 2\epsilon_p]$ for $p \geq p_0(n, k)$, there exists $p_1(n, k, c)$ such that for any $p > p_1$, either there exists $-1 < x_1 < x_0$ such that $W(x)$ is increasing on $[-1, x_1]$ and decreasing on $[x_1, x_0]$, or $W(x)$ is increasing on $[-1, x_0]$.

**Proof of Lemma 3.20.** A straightforward calculation shows that

\[ \frac{dW}{dx} = -2P(x)P'(x) + 2(n + kcx)kQ(x) + (n + kcx)^2Q'(x) = (2p - 1)(n + kcx)x^{2p-3}J(x), \]

where

\[ J(x) = -4k^3c\frac{p-1}{p}x^{2p+1} - 2nk^2\frac{p+1}{p}x^{2p} + 2k^2c(2p + 1)x^2 + (8pk - 2kn - \frac{4k^3c}{p} - 2k^3c)x + (\frac{4kn}{c} - \frac{4n^2}{c} - 2k^2n + \frac{2nk^2}{p}). \]

We add a brief remark on $J(-1)$ when $c = \frac{2}{k} - 2\epsilon_p$. It follows that

\[ J(-1) = \frac{4p}{c}(n - k)^2 + 6k^3c - 6nk^2 + 2k^2c + 2kn - \frac{4n^2}{c} \]

\[ = \frac{4p}{k} - 2\epsilon_p - \frac{4p^2}{k}c + \epsilon_p(-12k^3 - 4k^2 - \frac{8nk}{k} - 2\epsilon_p) \]

\[ = \epsilon_p(-12k^3 - 4k^2 - \frac{8nk}{k} - 2\epsilon_p + \frac{16pk^2\epsilon_p}{k}). \]

Recall that $\epsilon_p = \frac{2n}{2p+2k-1}$. A tedious calculation will lead to the fact that for $k$ suitably large ($k \geq 5$), $J(-1) > 0$ for $p = p(n, k)$ large enough. On the other hand, if $k$ is small ($k \leq 2$ for example), then $J(-1) < 0$.

The crucial observation leads to the proof of the lemma is that, for any $0 < c \leq \frac{2}{k} - 2\epsilon_p$, we can find $p_1 = p(n, k, c)$ so that $J'(x) > 0$ for any $x \in [-1, x_0]$.

First let us estimate $x_0$, the unique zero of $P(x)$ on $(-1, 0)$. We have

\[ P(x_0) = \frac{(2p - 1)k^2c}{p}x_0^{2p} + 2nkx_0^{2p-1} + \frac{c}{p}k^2 \]

\[ = x_0^{2p-1}\left[(2 - \frac{1}{p})k^2cx + 2nk\right] + \frac{c}{p}k^2. \]

Note that (30) implies that $2nk(-x_0)^{2p-1} > \frac{2}{p}k^2$, while (31) implies that

\[ (-x_0)^{2p-1}\left[2nk - (2 - \frac{1}{p})k^2c\right] < \frac{c}{p}k^2. \]
To sum up, we have the following estimates on $x_0$

\[
\left(\frac{ck}{2pn}\right)^{\frac{1}{2p-1}} < x_0 < \left(\frac{ck}{n}\right)^{\frac{1}{2p-1}}.
\]

Next we compute $J'(x)$:

\[
\frac{1}{p} J'(x) = -4k^3c(1 - \frac{1}{p})(2 + \frac{1}{p})x^{2p} - 4nk^2(2 + \frac{1}{p})x^{2p-1}
\]

\[
+ 4k^2c(2 + \frac{1}{p})x + \left(8nk - \frac{2kn}{p} - \frac{4k^3c}{p^2} - \frac{2k^3c}{p}\right).
\]

In order to do calculation in the $O\left(\frac{1}{p}\right)$ order, we note that when $p = p(n, k)$ is large enough, we have

\[
\frac{2n}{3p} < \epsilon_p < \frac{n}{p}.
\]

Plugging into $c = \frac{n}{k} - s$ where $2\epsilon_p \leq s < \frac{n}{k}$, it follows from (33) that

\[
\frac{1}{p} J'(x) = -4k^3(\frac{n}{k} - s)(2 - \frac{1}{p} - \frac{1}{p^2})x^{2p} - 4nk^2(2 + \frac{1}{p})x^{2p-1}
\]

\[
+ 4k^2(\frac{n}{k} - s)(2 + \frac{1}{p})x + \left(8nk - \frac{2kn}{p} - \frac{4k^3s}{p^2} + \frac{4k^3s}{p} - \frac{2nk^2}{p^2} + \frac{2k^3s}{p}\right)
\]

\[
\geq -8nk^2x^{2p-1}(x + 1) + 8nk(1 + x)
\]

\[
+ 4k^3\frac{n}{kp} x^{2p} - 4\frac{nk^2}{p} x^{2p-1} + 4k^2(-2s + \frac{n}{kp})x - \frac{2kn}{p} - \frac{2nk^2}{p^2} + O\left(\frac{1}{p^2}\right)
\]

It follows from (32), (34), and (36) that

\[
\frac{1}{p} J'(x) \geq 4k^3\frac{5n}{3p}\left(\frac{ck}{2pn}\right)^{\frac{1}{2p-1}} - \frac{2kn}{p} - \frac{2nk^2}{p^2} + O\left(\frac{1}{p^2}\right) > \frac{2kn(2k-1)}{p} > 0
\]

for any $p > p_1(n, k)$ large enough and any $-1 \leq x \leq x_0$. Note that in (37), we have used

\[
\lim_{p \to +\infty} \left[\frac{ck}{2pn}\right]^{\frac{1}{2p-1}} = 1.
\]

This completes the proof of Lemma 3.20. \qed

Let us remark that if we consider $M_{2,1}$, the conclusion of Theorem 3.18 is not necessarily stronger than that of Proposition 3.15. The point is that the degree of the generating polynomial $\phi(U)$ in Proposition 3.19 might depend on the $c$ where $[-c, c]$ is the domain of $\phi(U)$. In particular, the degree $p$ goes to infinity as $c$ approaches to 0, while the generating function in Proposition 3.15 is quartic polynomial. However, we are able to show that the proof of Proposition 3.19 can be used to establish the path-connectedness of all $U(n)$-invariant Kähler metrics of $H > 0$ on any Hirzebruch manifold $M_{n,k}$.

**Corollary 3.21.** The space of all $U(n)$-invariant Kähler metrics of $H > 0$ on $M_{n,k}$ is path-connected.

**Proof.** In view of Corollary 3.11, it suffices to show that given any $0 < c_1 < c_2 < \frac{n}{k}$, we can construct a continuous family of generating functions $\phi_c(U)$ where $U \in [-c, c]$ for any $c_1 \leq c \leq c_2$.

Such a family can be constructed following the proof of Proposition 3.19. In particular, if we examine (17), (21), (23), (24), and (38), we conclude by the continuous dependence of parameters that for any $c \in [c_1, c_2]$ there exists an sufficiently large integer $p = p(n, k)$ which is independent of the choice of $c$, $\delta_1 = \delta_1(p, n, k, c)$, and $\delta_2 = \delta_2(p, n, k, c)$ such that $\phi_c(U)$ defined by (16) is a continuous path of Kähler metrics with $H > 0$. \qed
3.4. Complete Kähler-Ricci solitons revisited. In this subsection, we are interested in complete Kähler-Ricci solitons in the following two cases.

(1) Compact shrinking Kähler-Ricci soliton on Hirzebich manifolds $M_{n,k}(k < n)$, which is the compactification of the total space of $H^{-k} \to \mathbb{CP}^{n-1}$.

(2) Complete noncompact shrinking Kähler-Ricci soliton on the total space of $H^{-k} \to \mathbb{CP}^{n-1}$ when $k < n$.

These were constructed by Cao [10], Koiso [27], Feldman-Ilmanen-Knopf [17].

**Question 3.22.** (1) What can we say about compact shrinking Kähler-Ricci soliton with $H > 0$?

(2) Are there any complete noncompact shrinking Kähler-Ricci soliton with $H > 0$?

Before answering Question 3.22, let us review the construction of Cao, Koiso, Feldman-Ilmanen-Knopf. We follow Koiso’s approach in the compact case and it can be extended to the complete noncompact case. Recall the Kähler metric $\hat{g}$ on the compactification of the $\mathbb{CP}^{n-1}$-bundle obtained by $H^{-k} \to \mathbb{CP}^{n-1}$. In the compact case, We make the following assumption on $M_{n,k}$:

**Assumption 3.23.** Kähler metric $\hat{g}$ on $M_{n,k}$ is in $2\pi C_1(M_{n,k})$. i.e. there exists a function $f$ on $\hat{L}$ such that

$$\text{Ric}(\hat{g}) - \omega_{\hat{g}} = \sqrt{-1} \partial \bar{\partial} f.$$ 

**Proposition 3.24** (Consequence of Assumption 3.7 and Assumption 3.23). Under Assumption 3.7 and Assumption 3.23, if we further assume $U_{\text{min}} = -D_{\text{min}}$, then $f$ in Assumption 3.23 is given by

$$\frac{d\phi}{dU} + \frac{\phi}{Q} \frac{dQ}{dU} + 2U - \phi \frac{df}{dU} = 0.$$ 

Moreover, we have $U_{\text{max}} = D_{\text{max}}$ and $\text{Ric}(g_0) = g_0$ on $M$.

If we further assume that $f$ in Assumption 3.23 is give by $f = -\alpha U$ for some constant $\alpha$, then it follows that $\nabla f = -\frac{\alpha}{2} H$ is a holomorphic vector field. Equation (39) becomes the shrinking soliton equation.

$$\frac{d\phi}{dU} + \frac{\phi}{Q} \frac{dQ}{dU} + 2U - \alpha \phi = 0.$$ 

Note that $Q = (1 + \frac{k}{n} U)^{n-1}$, so Equation (40) takes the form

$$\frac{d\phi}{dU} + \frac{k(n-1)}{n+kU} \phi + 2U - \alpha \phi = 0.$$ 

Equation (41) can be solved explicitly:

$$\phi(U) = \frac{2\eta(U, \alpha)}{Q(U)} - \frac{2e^{\alpha (U - U_{\min})}}{Q(U)} \eta(U_{\min}, \alpha),$$

where $\eta(U, \alpha)$ is a polynomial of degree $n$ defined by

$$\int xe^{-\alpha x} Q(x) dx = -e^{-\alpha x} \eta(x, \alpha).$$

In the compact case it follows from Proposition 3.24 that $U_{\text{min}} = -1$ and $U_{\text{max}} = 1$, therefore $\phi > 0$ is a smooth function on $(-1,1)$ with $\frac{d\phi}{dU}|_{U=-1} = 2$ and $\frac{d\phi}{dU}|_{U=1} = -2$. While in the noncompact case we set $U_{\text{min}} = -1$ and $U_{\text{max}} = +\infty$, therefore $\phi > 0$ is a smooth function on $(-1, +\infty)$ with $\frac{d\phi}{dU}|_{U=1} = 2$.

**Theorem 3.25** (Koiso [27], Cao [10], Feldman-Ilmanen-Knopf [17]). For any integer $1 \leq k < n$, consider $(\mathbb{CP}^{n-1}, g_0)$ where $g_0$ is the Fubini-Study metric with $\text{Ric}(g_0) = g_0$.

(1) We assume that the shrinking soliton metric on $M_{n,k}$ is of the form (14), and satisfies Assumptions 3.3, 3.7, and 3.23. Then there exists a unique shrinking Kähler-Ricci soliton on each $M_{n,k}$ when $1 \leq k < n$. It is unique in the sense that the value $\alpha > 0$ in the associated holomorphic vector field $\nabla f = -\frac{\alpha}{2} H$ is determined by the unique solution of $\phi(1) = 0$. Cao proved that the Ricci curvature of the soliton metric is positive on $M_{n,k}$ if and only if $k = 1$. 
(2) There exists a unique complete shrinking Kähler-Ricci soliton on the total space of $L^k \to \mathbb{CP}^{n-1}$ whose value $\alpha > 0$ is determined by the unique solution to $\eta(-1, \alpha) = 0$.

Indeed, by (41) and Proposition 3.8, it is easy to write down the curvature tensors for shrinking solitons.

\[ A = \frac{1}{2}(\alpha^2 - 2\alpha k(n - 1) + k^2 n(n - 1)(n+kU)^2)\phi + (\alpha - \frac{k(n-1)}{n+kU})U + 1 \]
\[ B = \frac{[k^2 n - k(n+kU)]\phi + 2k(n+kU)U}{2(n+kU)^2} \]
\[ C = \frac{2(n+kU) - k^2 \phi}{(n+kU)^2} \]

It is easy to see that along the zero section $U = -1$, $2B + \sqrt{AC} > 0$ implies that
\[ \alpha < \alpha_0(n,k) = \frac{(n-2k)(k+1)}{n-k} \]

For example, when $n = 2, k = 1$, the necessary condition for $H > 0$ is $\alpha < \alpha_0(2,1) = 0$, however the corresponding $\alpha$ on $M_{2,1}$ is the unique positive root of the equation $e^{2\alpha}(-\alpha^2 + 2) - 3\alpha^2 - 4\alpha - 2 = 0$ ($\alpha \simeq 0.5276195199$). Therefore the Cao-Koiso shrinking soliton on $M_{2,1}$ does not satisfy $H > 0$.

Next we analyze the noncompact case in more details. Consider the following polynomial which is of degree $n$ in terms of $\alpha$.

\[ \alpha^{n+1} \eta(x,\alpha) = -\alpha^{n+1}e^{\alpha x} \int xe^{-\alpha x}(1 + \frac{k}{n}x)^{n-1}dx \]

Making use of the integration formula
\[ \int z^n e^z dz = (\sum_{l=0}^{n} (-1)^{n-l} \frac{n!}{l!} z^l)e^z + C, \]

Some routine calculation leads to
\[ \alpha^{n+1} \eta(U,\alpha) = \left(\frac{k}{n}\right)^{n-1} \left( \sum_{l=1}^{n} \frac{n!}{l!} \frac{(n+kU)^{l-1}(n+kU - l)}{k^l} \alpha^l + n! \right). \]

Therefore the value of $\alpha$, which solves the shrinking soliton equation, namely, the root of the polynomial $\eta(-1, \alpha)$, is reduced to root of the following polynomial of degree $n$

\[ \chi(\alpha) = \sum_{l=1}^{n} \frac{n!}{l!} \frac{(n-k)^{l-1}(n-k-l)}{k^l} \alpha^l + n!. \]

Similarly as in Feldman-Ilmanen-Knopf (see the equation $f$ in P197 [17]), since $\chi(\infty) < 0$ and $\chi(0) > 0$, Descartes' rule of signs implies there exists a unique positive root for $\chi(\alpha)$. Call it $\alpha_*$. For this choice $\alpha = \alpha_*, \phi(U)$ has the asymptotical behavior $\phi \sim \frac{1}{\alpha_*}U$ as $U \to \infty$, and it shows that the soliton metric is complete along infinity.

Now we are interested in a more precise estimate of $\alpha_*$. First note that $\alpha_* > k$. To see this, we check

\[ \chi(\alpha) = \sum_{l=1}^{n} \frac{n!}{l!} \frac{(n-k)^{l-1}(n-k-l)}{k^l} \alpha^l + n! \]
\[ = \left(\frac{\alpha(n-k)}{k}\right)^n + \sum_{l=0}^{n-1} \frac{n!}{l!} \left(\frac{\alpha(n-k)}{k}\right)^l (1 - \frac{\alpha}{k}). \]

We propose the following conjecture on a precise estimate on $\alpha_*$.

**Conjecture 3.26.** For any $1 \leq k < n$, $\alpha_*$, which is the unique positive root of of the polynomial $\chi(\alpha)$, satisfies $\alpha_0(n,k) < \alpha_* < k + 1$. If so, then none of Feldman-Ilmanen-Knopf shrinking solitons have positive holomorphic sectional curvature.
Argue similarly as in (45), we can show that Conjecture 3.26 is indeed true if $k < n \leq k^2 + 2k$. It is very likely that it is true in general, as numerical experiments suggest.

On the other hand, compact shrinking solitons on $M_{n,k}$ could have positive holomorphic sectional curvature as the ratio $\frac{H}{\delta}$ grows larger.

**Proposition 3.27.** (Some compact shrinking solitons on $M_{n,k}$ have $H > 0$). If we fix $k = 1$, then the lowest dimension example of Cao-Koiso shrinking solitons which have $H > 0$ is $n = 3$. On $M_{n,1}$ its local pinching constant is $\frac{1}{(2\alpha_0(4-\alpha))} \simeq 0.05587$ where $\alpha_0 \simeq 0.6820161326$. If $k = 2$, the first compact example with $H > 0$ is on $M_{7,2}$, where $\alpha \simeq 1.742423694$.

Let us explain the calculation in the case of $M_{3,1}$. In this case the corresponding $\alpha$ is the unique positive solution of the following equation

$$16\alpha^3 + 24\alpha^2 + 18\alpha + 6 - e^{2\alpha}(-4\alpha^3 + 6\alpha + 6) = 0.$$  

In particular, $\alpha \simeq 0.6820161326 < \alpha_0(3,1) = 1.5$ where $\alpha_0(n,k)$ defined in (44). One can show that that $(2B + \sqrt{AC})(n + kU)^2$ is increasing on $U \in [-1, 1]$ by a direct calculation, therefore we have $H > 0$. In general, for any $M_{n,1}$ with $n \geq 3$, one could expect to verify that $\alpha < \alpha_0(n, k)$ and $(2B + \sqrt{AC})(n + kU)^2$ is increasing on $U \in [-1, 1]$. Therefore the Cao-Koiso shrinking soliton on Hirzebruch manifold $M_{n,1}$ have $H > 0$ for any $n \geq 3$.

Let us calculate the local holomorphic pinching constant of Cao-Koiso soliton on $M_{3,1}$. A similar argument as in Example 3.15 shows that

$$\min_{\|v\|=1} H(U,v) \bigg/ \max_{\|v\|=1} H(U,v) = \begin{cases} \frac{AC - 4B^2}{A(A + C - 4B)}, & U \in [-1, U_*] \\ \frac{c}{\delta}, & U \in [U_*, 1] \end{cases}$$

Here $U_*$ is the solution of $2B = C$ on $[-1, 1]$, whose numerical value is about $-0.573003$. Therefore the pinching constant is obtained at $U = -1$, which is $\frac{1 - E - E'}{(2 - E)(\delta - E')^2} \simeq 0.05587$.

Let us remark that if we drop the assumption of the shrinking soliton, it is easy to write down examples of complete Kähler metrics of $H > 0$ on the total space of $H^{-k} \to \mathbb{C}P^1$. For instance, we have

**Example 3.28.** Define a function $\omega(U)$ on $[-\frac{1}{2}, +\infty)$ as follows:

$$\omega(U) = \begin{cases} \frac{1}{2} - 2U^2, & U \in [-\frac{1}{2}, -\frac{1}{4}] \\ c(U + 2) + \frac{1}{2} \ln(U + 2), & U \in [-\frac{1}{4}, \infty) \end{cases}$$

where $c = \frac{3}{4} - \frac{3}{6} \ln(\frac{3}{4}) \sim 0.05439$ so that $\omega$ is continuous at $U = -\frac{1}{4}$. Choose a small number $\delta > 0$ so that $\omega$ admits a convex smoothing $\phi$ which equals to $\omega$ except inside $(-\frac{1}{4} - \delta, -\frac{1}{4} + \delta)$. Note that for $\omega$, both $C > 0$ and $2B + \sqrt{AC}$ have positive lower bounds in $(-\frac{1}{4} - \delta, -\frac{1}{4} + \delta)$ and $(-\frac{1}{4}, -\frac{1}{4} + \delta)$. It guarantees the existence of a convex smoothing $\phi$ which in turns gives a complete Kähler metric with $H > 0$ on the total space of $L^{-1} \to \mathbb{C}P^1$. The metric actually has positive bisectional curvature outside a compact subset. Moreover, its bisectional curvatures decay quadratically along the infinity. Asymptotically such a metric has a conical end and there exist holomorphic functions with polynomial growth on it.

### 3.5. $H > 0$ is not preserved along the Kähler-Ricci flow

Recently there have been much progress on Kähler-Ricci flow with Calabi-symmetry on Hirzebruch manifolds $M_{n,k}$. See for example Zhu [43], Weinkove-Song [36], Fong [18], and Guo-Song [21]. In this subsection, we apply Hitchin’s example and new examples constructed in Theorem 3.18 to show that in general $H > 0$ is not preserved by the Kähler-Ricci flow.

These results imply that if the initial metric $g_0$ is of $U(n)$-symmetry and in the Kähler class $\frac{1}{n}[E_0] - \frac{1}{n}[E_0]$ where $b_0 > a_0 > 0$, then the flow always develops a singularity in finite time. Let $T < \infty$ denote the maximal existence time. Their results can be summarized as:

1. Suppose that the initial metric $g_0$ satisfies $\frac{|E_0|}{|E_0|} = \frac{b_0}{a_0} = \frac{n + k}{n-k}$, where $|E_0|$ denotes the volume of the divisor $E_0$ with respect to $g_0$. In this case the Kähler class of $g_0$ is proportional
to the anti-canonical class of $M_{n,k}$. Then the Kähler-Ricci flow shrinks the fiber and the base uniformly and collapses to a point. The rescaled flow converges to the Cao-Koiso soliton on $M_{n,k}$ \((18)\).

(2) If $g_0$ satisfies $\frac{b_0}{a_0} < \frac{n+k}{n-k}$, then the Kähler-Ricci flow shrinks the fiber first, and the flow collapses to the base $\mathbb{CP}^{n-1}$ \((36)\). The rescaled flow converges to $\mathbb{C}^{n-1} \times \mathbb{C}P^1$ \((18)\).

(3) If $g_0$ satisfies $\frac{b_0}{a_0} > \frac{n+k}{n-k}$, then the Kähler-Ricci flow shrinks the zero section $E_0$ first and hence ‘contracts the exceptional divisor’ \((36)\). The rescaled flow converges to the Feldman-Ilmanen-Knopf shrinking soliton on the total space of $H^{-k} \to \mathbb{CP}^{n-1}$ \((21)\).

(4) If $n \leq k$, then the Kähler-Ricci flow shrinks the fiber first, and the flow converges to the base $\mathbb{CP}^{n-1}$ in the Gromov-Hausdorff sense as $t \to T$ \((36)\).

Based on Hitchin’s examples reformulated in Example 3.14 and Proposition 3.19, we have the following examples of the Kähler-Ricci flow.

On $M_{2,1}$, let us take the metric in the anti-canonical class constructed in Example 3.15 as the initial metric. Then the normalized flow converges to the Cao-Koiso soliton. Unfortunately, the positivity of $H$ breaks down. Therefore, in general, $H > 0$ is not preserved along the Kähler-Ricci flow. If instead we start from initial metric corresponding to $\phi_c(U) = c - \frac{x^2}{4}$ on $[-c, c]$ in Example 3.14, where $0 < c < \frac{4}{3}$, then the limiting metric of the unnormalized flow is $(\mathbb{C}P^1, cg_{FS})$. In this case, it is not clear whether $H > 0$ is preserved, or how the holomorphic pinching constant of $g(t)$ evolves.

On $M_{3,1}$, if we pick initial metric as the Cao-Koiso shrinking soliton, it is a fixed point of the normalized flow, therefore the holomorphic pinching constant remains constant.

On $M_{4,1}$, if we pick initial metric by the examples $\phi_c(U) = c - \frac{x^2}{4}$ on $[-c, c]$ for any $0 < c < \frac{4}{3}$. Then all three cases mentioned above could occur. For $c = 1$, the normalized flow evolves $\phi_1(U)$ to the Cao-Koiso soliton. While the initial metric has the holomorphic pinching constant $2/27 \approx 0.074$, the limit metric has the holomorphic pinching constant $\approx 0.095$. It indeed improves after a long time, even though we do not know the short time effect of it. If $1 < c < \frac{4}{3}$, then the normalized flow evolves $\phi_c(U)$ to the limiting Feldman-Ilmanen-Knopf shrinking soliton on the total space of $L^{-1} \to \mathbb{CP}^3$. However the limiting soliton no longer has $H > 0$, and once again we see that $H > 0$ is not preserved under the Kähler-Ricci flow.

Therefore a natural question arises, namely, is there an effective way to construct a one-parameter family of deformation of Kähler metrics with $H > 0$? It would be ideal if the holomorphic pinching constant could enjoy some monotonicity properties along this deformation.

4. Kähler metrics of $H > 0$ from the submanifold point of view

In this section, we discuss holomorphic pinching of the canonical Kähler-Einstein metrics on some Kähler $C$-spaces. In general, we would like to discuss the question of constructing $H > 0$ metric from the submanifold point of view.

**Proposition 4.1.** Consider the flag threefold, or more generally, let $M$ be the hypersurface in $\mathbb{CP}^n \times \mathbb{CP}^n$ defined by

$$\sum_{i=1}^{n+1} z_i w_i = 0,$$

where $n \geq 2$ and $([z], [w])$ are the homogeneous coordinates. Let $g$ be the restriction on $M$ of the product of the Fubini-Study metrics (each of which has $H = 2$). Then the holomorphic sectional curvature of $g$ is between $2$ and $\frac{4}{3}$. So the holomorphic pinching constant is $\frac{4}{3}$, which is dimension free.

**Proof.** Let us work on the case $n = 2$ and in the inhomogeneous coordinates $[1, z_1, z_2]$ and $[w_1, 1, w_2]$. The hypersurface $M^3 \subset \mathbb{CP}^2 \times \mathbb{CP}^2$ is defined by $w_1 + z_1 + z_2 w_2 = 0$ and can be parametrized by

$$(t_1, t_2, t_3) \to [1, t_1, t_2] \times [-t_1 - t_2 t_3, 1, t_3].$$
It follows that 
\[ \frac{\partial}{\partial t_1} = \frac{\partial}{\partial z_1} - \frac{\partial}{\partial w_1}, \quad \frac{\partial}{\partial t_2} = \frac{\partial}{\partial z_2} - t_3 \frac{\partial}{\partial w_1}, \quad \frac{\partial}{\partial t_3} = -t_2 \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2}. \]

Therefore under \( \{ \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3} \} \) the induced metric \( \bar{g} \) has the form:
\[
\begin{pmatrix}
g_{\bar{1}\bar{1}} + h_{\bar{1}\bar{1}} & g_{\bar{1}\bar{3}} + \bar{t}_3 h_{\bar{1}\bar{1}} & \bar{t}_2 h_{\bar{1}\bar{1}} + h_{\bar{1}\bar{2}} \\
g_{\bar{2}\bar{1}} + \bar{t}_4 h_{\bar{1}\bar{1}} & g_{\bar{2}\bar{2}} + |t_3|^2 h_{\bar{1}\bar{1}} & \bar{t}_3 h_{\bar{2}\bar{1}} - t_3 h_{\bar{1}\bar{2}} \\
g_{\bar{3}\bar{1}} - h_{\bar{2}\bar{1}} & \bar{t}_2 h_{\bar{3}\bar{1}} - t_3 h_{\bar{1}\bar{2}} & |t_2|^2 h_{\bar{1}\bar{1}} + h_{\bar{2}\bar{2}} 
\end{pmatrix}
\]
where \( g_{ij} = g(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial w_j}) \) and \( h_{ij} = g(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial w_j}) \), where \( 1 \leq i, j \leq 2 \) are Fubini-Study metrics on two factors \( \mathbb{CP}^2 \) respectively:
\[
g_{ij} = \frac{\delta_{ij}}{1 + |z|^2} - \frac{z_i z_j}{(1 + |z|^2)^2}, \quad h_{ij} = \frac{\delta_{ij}}{1 + |w|^2} - \frac{w_i w_j}{(1 + |w|^2)^2}.
\]

Recall that the curvature tensor of \( \bar{g} \) is given by the formula
\[
R_{ijkl} = -\frac{\partial^2 \bar{g}_{kl}}{\partial z_i \partial z_j} + \bar{g}_{pq} \frac{\partial g_{kl}}{\partial z_p} \frac{\partial g_{ij}}{\partial z_q}.
\]
Note that there is a natural \( U(3) \)-action on \( M \) because of the defining equation (46), which acts transitively. Therefore it suffices to calculate the curvature tensor of the induced metric \( \bar{g} \) at the point \((t_1, t_2, t_3) = (0, 0, 0)\). Now pick an orthonormal frame \( \{ e_1, e_2, e_3 \} = \{ \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3} \} \).

Then a straightforward calculation shows that the only non-vanishing curvature components under \( \{ e_1, e_2, e_3 \} \) are the following:
\[
R_{1111} = 1, \quad R_{2222} = 2, \quad R_{3333} = 2, \quad R_{1122} = R_{1133} = \frac{1}{2}, \quad R_{2233} = -\frac{1}{2}.
\]

From this, we get that \( R_{ij} = 2h_{ij} \) for any \( 1 \leq i \neq j \leq 3 \), so \( \bar{g} \) is Kähler-Einstein. Once we have all the curvature components, it is direct to see that \( \min_{|X|=1} H(X) = H(\frac{Z_{2+2+2}}{\sqrt{2}}) = \frac{1}{2} \) and \( \max_{|X|=1} H(X) = 2 \).

The same argument works for the hypersurface defined by \( \sum_{i=1}^{n+1} z_i w_i = 0 \) in \( \mathbb{P}^n \times \mathbb{P}^n \). It gives the same pinching constant \( \frac{1}{2} \) for any \( n \geq 2 \).

However, if we try a similar calculation on other types of bidegree \((p, q)\) hypersurfaces in \( \mathbb{CP}^n \times \mathbb{CP}^n \), the argument breaks down.

As a simple example, consider the bidegree \((2, 1)\) hypersurface defined by \( \sum_{i=0}^2 z_i^2 w_i = 0 \) in \( \mathbb{CP}^2 \times \mathbb{CP}^2 \) given by homogeneous coordinates \( (|z|, |w|) \). Consider the following parametrization of the hypersurface:
\[
(t_1, t_2, t_3) \to [1, t_1, t_2, | -t_1^2 - t_2^2 + t_3, 1, t_3].
\]
and
\[
\frac{\partial}{\partial t_1} = \frac{\partial}{\partial z_1} - 2t_1 \frac{\partial}{\partial w_1}, \quad \frac{\partial}{\partial t_2} = \frac{\partial}{\partial z_2} - 2t_2 \frac{\partial}{\partial w_1}, \quad \frac{\partial}{\partial t_3} = -t_2 \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2}.
\]

The corresponding \( \bar{g} \) induced from the product of Fubini-Study metric on \( \mathbb{CP}^2 \times \mathbb{CP}^2 \) is
\[
\begin{pmatrix}
g_{11} + 4|t_1|^2 h_{11} & g_{13} + 4|t_1|^2 t_3 h_{11} & 2t_1 |t_2|^2 h_{11} - 2t_3 h_{12} \\
g_{21} + 4|t_2|^2 t_3 h_{11} & g_{22} + |t_3|^2 h_{11} & 2 |t_2|^2 t_3 h_{11} - 2t_3 h_{12} \\
g_{31} - 2t_1 h_{21} & 2 |t_2|^2 t_3 h_{11} - 2t_3 h_{12} & |t_2|^2 h_{11} + h_{22}
\end{pmatrix}
\]

If we only calculate the curvature at \((t_1, t_2, t_3) = (0, 0, 0)\), we already encounter some negativity of \( H \). First note that \( \bar{g}(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}) \) is orthonormal at \((0, 0, 0)\), then it follows that
\[
R_{1111} = -\frac{\partial^2 \bar{g}_{11}}{\partial t_1^2} = -4h_{11} - \frac{\partial^2 g_{11}}{\partial z_1^2} = -2.
\]

The same problem occurs for a general bidegree \((2, 1)\) hypersurface in \( \mathbb{CP}^n \times \mathbb{CP}^n \). Of course, this just means that for a bidegree \((2, 1)\) hypersurface in \( \mathbb{CP}^n \times \mathbb{CP}^n \), the restriction of the product of the Fubini-Study metric on the hypersurface does not have \( H > 0 \). But presumably
there could be other metrics on it with $H > 0$. This is indeed the case, and we have the following result:

**Proposition 4.2.** Let $M^n$ be any smooth bidegree $(p, 1)$ hypersurface in $\mathbb{CP}^r \times \mathbb{CP}^s$, where $n = r + s - 1$, $p \geq 1$, and $r, s \geq 2$. Then $M^n$ admits a Kähler metric with $H > 0$. Moreover, when $p > r + 1$, the Kähler classes of all the Kähler metrics with $H > 0$ form a proper subset of the Kähler cone of $M^n$.

**Proof of Proposition 4.2.** Let $[z]$ and $[w]$ be the homogeneous coordinates of $\mathbb{CP}^r$ and $\mathbb{CP}^s$, respectively. Let $\pi : \mathbb{CP}^r \times \mathbb{CP}^s \to \mathbb{CP}^r$ be the projection map. Suppose that $M^n$ is defined by

$$\sum_{i=1}^{s+1} f_i(z_1, \cdots, z_{r+1})w_i = 0,$$

where each $f_i$ is a homogeneous polynomial of degree $p$. Consider the sheaf map $h : \mathcal{O}(p) \to \mathcal{O}(p)$ on $\mathbb{CP}^s$ defined by

$$h(e_i) = f_i(z), \quad 1 \leq i \leq s + 1,$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ has 1 at the $i$-th position. Clearly, $h$ is surjective, and its kernel sheaf $E$ is locally free. Since $M^n = \mathbb{P}(E)$ over $\mathbb{CP}^r$, by the result of [3], we know that $M^n$ admits Kähler metrics with $H > 0$.

To see the second part of the statement, let us denote by $H_1, H_2$ the hyperplane section from the two factors restricted on $M$, then we have $c_1(M) = (r + 1 - p)H_1 + sH_2$. Clearly, $H_1^{r+1} = 0$, $H_2^{s+1} = 0$, and since $M^n \sim pH_1 + H_2$, we have $H_1^rH_2^{s-1} = 1$ and $H_1^{r-1}H_2^s = p$ on $M^n$. For any Kähler class $[\omega] = aH_1 + bH_2$ where $a > 0$ and $b > 0$, we have

$$c_1(M) \cdot [\omega]^{n-1} = a^{-2}b^{s-2} \left( \binom{n-1}{r} sa^2 + \binom{n-1}{r-1} (r+1+sp-p)ab + \binom{n-1}{r-2} (r+1-p)b^2 \right).$$

So when $p > r + 1$ and $b >> a$, we know that the total scalar curvature of $(M^n, \omega)$ is negative. Thus the Kähler classes of metrics with $H > 0$ can not fill in the entire Kähler cone. \qed

For a smooth bidegree $(p, 2)$ hypersurface $M^n$ in $\mathbb{CP}^r \times \mathbb{CP}^s$, where $n = r + s - 1$ and $p \geq 2$, one may raise the question of whether $M^n$ admits Kähler metrics with $H > 0$? The answer might be yes in view of Proposition 4.2. Note that if we project to $\mathbb{CP}^r$, then $M$ becomes a holomorphic fibration over $\mathbb{CP}^r$ whose generic fiber are smooth quadrics.

**Question 4.3.** If $M^3$ is a compact Kähler manifold with local holomorphic pinching constant strictly greater than $\frac{1}{4}$, then is it biholomorphic to a compact Hermitian symmetric space, i.e. $\mathbb{CP}^3$, $\mathbb{CP}^2 \times \mathbb{CP}^1$, $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, or $Q^3$ which is the smooth quadric in $\mathbb{CP}^4$?

Let us conclude the discussion here by giving a couple of general remarks. First, if we want to construct metrics with $H > 0$ from the submanifold point of view, in particular, as complete intersections. Then it seems difficult to find examples other than those already known (such as Hermitian symmetric spaces or Kähler C-spaces, or projectivized vector bundles covered in [3]). For instance, if we consider a cubic hypersurface $M^n \subset \mathbb{CP}^{n+1}$ and $g$ be the restriction on $M$ of the Fubini-Study metric. Then it is unclear if $(M^n, g)$ can have $H > 0$, though we expect the answer is no. As another example, if we consider the restriction of the ambient Fubini-Study metric onto a complete intersection, where typically we need to restrict to degree 1 or 2. Let us consider $M^n$ as the intersection of two quadrics, in the case of $n = 2$, it is the del Pezzo surface of degree 4, or $\mathbb{CP}^2$ blowing up 5 points. It is unlikely that the induced metric on $M^n$ could have $H > 0$. We plan to discuss these questions elsewhere.

Secondly, it is a general belief that the existence of a Kähler metric of $H > 0$ is a ‘large open’ condition, as illustrated in this paper on any Hirzebruch manifold. Therefore, it is reasonable to expect if a projective manifold $M$ admits a Kähler metric $g_0$ of $H > 0$, then there exists a small deformation of such a metric $g_1$ which lies in a Hodge class and still has $H > 0$. Hence one can conclude from a theorem of Tian ([37]) that $g_1$ can be approximated by pull backs of Fubini-Study metrics by a sequence of projective embeddings $\phi_k : M \to \mathbb{CP}^{N_k}$. However, it
seems difficult to construct examples of Kähler metrics of $H > 0$ in this way because of the implicit nature of $\phi_k$ and $N_k$.

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