Gaussian fluctuations for the directed polymer partition function for \(d \geq 3\) and in the whole \(L^2\)-region.

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Abstract

We consider the discrete directed polymer model with i.i.d. environment and we study the fluctuations of the tail \(n^{(d-2)/4}(W_\infty - W_n)\) of the normalized partition function. It was proven by Comets and Liu [8], that for sufficiently high temperature, the fluctuations converge in distribution towards the product of the limiting partition function and an independent Gaussian random variable. We extend the result to the whole \(L^2\)-region, which is predicted to be the maximal high-temperature region where the Gaussian fluctuations should occur under the considered scaling. To do so, we manage to avoid the heavy 4th-moment computation and instead rely on the local limit theorem for polymers [23, 25] and homogenization.

Keywords: Directed polymers, random environment, weak disorder, tail martingale, rate of convergence, local limit theorem for polymers, martingale central limit theorem.

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1 Introduction

1.1 The model

The directed polymer model was first introduced by Huse and Henly in the physics literature [12] and was reformulated in mathematics by Imbrie and Spencer [17]. The model is a description of a long chain of monomers, called a polymer, which interacts with impurities that it may encounter on its path. The reader is referred to [5] for a recent review of the model. In the discrete case, the model is defined as follows.

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The impurities, also called the environment, are modeled by a collection of non-constant, i.i.d. random variables \( \omega(i, x) \), \( i \in \mathbb{N}, x \in \mathbb{Z}^d \), defined under a probability measure \( P \) of expectation denoted by \( E \). We will assume that \( E[\exp(\beta \omega(i, x))] < \infty \) for all \( \beta \in \mathbb{R} \), and define:

\[
\lambda(\beta) = \log E \left[ e^{\beta \omega(i, x)} \right].
\]

Let \( \Omega = \{(S_k)_{k \geq 0}, S_k \in \mathbb{Z}^d\} \) be the state space of the trajectories, and \( P_x \) the probability measure on \( \Omega \), such that the canonical process \( (S_k)_{k \geq 0} \) is the simple random walk on \( \mathbb{Z}^d \) starting at position \( x \), i.e. under \( P_x \), \( S_1 - S_0, \ldots, S_{k+1} - S_k \) are independent and

\[
P_x(S_0 = x) = 1, \quad P_x(S_{k+1} - S_k = \pm e) = \frac{1}{2d},
\]

where \( e \) is any vector of the canonical basis of \( \mathbb{R}^d \). We denote by \( E_x \) the expectation under \( P_x \), and \( P = P_0, E = E_0 \).

Then, the Gibbs measure of the polymer \( P_{x, \beta, n} \) on \( \Omega \) is defined as:

\[
dP_{x, \beta, n}(S) = \frac{\exp \left\{ \sum_{i=1}^{n} \beta \omega(i, S_i) \right\}}{Z_n(\beta)} dP_x(S),
\]

where \( \beta \geq 0 \) stands for the inverse temperature of the polymer, and where \( Z_n(\beta) = E[\exp\{\sum_{i=1}^{n} \beta \omega(i, S_i)\}] \) is called the partition function.

A polymer path of horizon \( n \) is the realization of \( (S_k)_{0 \leq k \leq n} \) under the polymer measure \( P_{x, \beta, n} \). The parameter \( \beta \) models the strength of the interaction of the polymer with the environment: the higher \( \beta \), the more the polymer path is tempted to go through high values of the environment.

The normalized partition function:

\[
W_n = Z_n e^{-n\lambda(\beta)} = E[e_n], \quad \text{with: } e_n = e^{\sum_{i=1}^{n} \omega(i, S_i) - n\lambda(\beta)},
\]

is a mean 1, positive martingale with respect to the filtration \( F_n \) generated by the variables \( \omega(i, x), i \leq n, x \in \mathbb{Z}^d \). The martingale verifies the following dichotomy [10]: for \( d \geq 3 \) (which will be assumed from now), there exist some critical parameters \( \beta_c^+ \) and \( \beta_c^- \), such that

- For all \( \beta_c^- < \beta < \beta_c^+ \), \( W_n \to W_\infty \) a.s., with \( P(W_\infty > 0) = 1 \),
- For all \( \beta \in \mathbb{R} \setminus [\beta_c^-, \beta_c^+] \), \( W_n \to 0 \) a.s.

The region below \( (\beta_c^- , \beta_c^+) \) is called the weak disorder regime, while the region \( \mathbb{R} \setminus [\beta_c^-, \beta_c^+] \) is called the strong disorder regime. In the weak disorder region, the polymer path is diffusive (it was first proved in a more restrained region in [3, 17], then in the whole weak disorder region in [11]), while in the strong disorder regime, it is believed that the polymer path should be superdiffusive. Moreover, it was shown that for large enough \( \beta \), the polymer path localizes [9, 2, 4].
The subregion of the weak disorder, where \( W_n \to W_\infty \) in \( L^2 \), is called the \( L^2 \)-region. It corresponds to the \( \beta \)-region (see e.g. (9)-(11) in [8]):

\[
\text{(L2)} \quad \lambda_2(\beta) := \lambda(2\beta) - 2\lambda(\beta) < \log(1/\pi_d),
\]

where \( \pi_d \in (0,1) \) is the probability of return to 0 of the simple random walk:

\[
\pi_d = \mathbb{P}(\exists n \geq 1, S_n = 0).
\]

Moreover, since \( \pi_{d+1} < \pi_d \) for all \( d \geq 3 \) [22, Lemma 1] and \( \pi_3 = 0.3405 \ldots \) [24, page 103], condition (L2) is always verified for \( |\beta| \) small enough. As the function \( \lambda_2 \) is non-decreasing on \( \mathbb{R}_+ \) and non-increasing on \( \mathbb{R}_- \), this implies that

\[
\lambda_2(\beta) \leftrightarrow \beta \in (\beta^-_2, \beta^+_2),
\]

where \( \beta^-_2 = \beta^-_2(d) \in [-\infty, 0) \) and \( \beta^+_2 = \beta^+_2(d) \in (0, \infty] \).

Then, again by [3],

\[
\mathbb{E}[W_\infty^2] = \begin{cases} 
\left(1 - \frac{\pi_d}{1-\pi_d e^{\lambda_2(\beta)}}\right)^{\lambda_2(\beta)} & \text{if } \beta \in (\beta^-_2, \beta^+_2), \\
\infty & \text{else.}
\end{cases} \tag{2}
\]

In the following, we will always assume that \( d \geq 3 \) and \( \beta \in (\beta^-_2, \beta^+_2) \).

### 1.2 The results

We introduce two additional types of convergences, referring to [8].

**Definition 1.1.** Let \( Y_n \) be a family of random variables defined on a common probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Suppose that \( Y_n \) converges to some random variable \( Y \) in distribution.

- We say that this convergence is stable if for any \( B \in \mathcal{F} \) with \( \mathbb{P}(B) > 0 \), the law of \( Y_n \) under the condition \( B \) converges to some probability distribution, which might depend on \( B \).
- We say that this convergence is mixing if it is stable and the limit of conditional law is independent of given \( B \). Then this conditional limit is the law of \( Y \) itself.

**Theorem 1.1.** For all \( \beta \in (\beta^-_2, \beta^+_2) \), as \( n \to \infty \),

\[
n^{-\frac{d}{2}} (W_\infty - W_n) \xrightarrow{(d)} \sigma W_\infty G, \tag{3}
\]

and

\[
n^{-\frac{d}{2}} \frac{W_\infty - W_n}{W_n} \xrightarrow{(d)} \sigma G, \tag{4}
\]

where \( G \) is a standard centered Gaussian random variable which is independent of \( W_\infty \), and \( \sigma = \sigma(\beta) \) is defined in (24). Moreover, convergence (3) is stable and convergence (4) is mixing.
Corollary 1.2. For all \( \beta \in (\beta_2^-, \beta_2^+) \), as \( n \to \infty \),
\[
n^{d-2} (\log W_\infty - \log W_n) \xrightarrow{(d)} \sigma G,
\]
where \( G \) and \( \sigma \) are as above. Moreover, this convergence is mixing.

The proof of Theorem 4 is given in Section 2.1 and the proof of its corollary can be found in Section 3.6.

1.3 Comments and connections to other models

Our work is an extension of the results of Comets and Liu [8] to the whole \( L^2 \)-region. Although our proof partially relies on their method, we manage to avoid the 4th-moment computation which is not valid in the whole \( L^2 \)-region. Instead, we make a natural use of the local limit theorem for polymers [23, 25] and appeal to homogenization via a fine truncation method, rather than brute force moment computation. Moreover, since \( \sigma(\beta) \) from (24) blows up at \( \beta_2^\pm \), we predict that our results are optimal in the sense that another scaling, or other limiting laws should hold for (3)-(5) outside the \( L^2 \)-region.

In branching process literature, the study of the rate of convergence and the nature of the fluctuations of the tail of characteristic martingales is a common subject. For the Galton-Watson process, this has been studied in [13, 14]. We note that the rate of convergence is there exponential, while the rate is polynomial in our case. In the model of the branching random walk, the fluctuations of the tail of Biggins’ martingale are an active subject of research. In the sub-critical region, it was shown that the fluctuations are of Gaussian nature for small enough inverse temperature parameters [15, 26] and that they become of alpha-stable nature at criticality [20]. What happens close to criticality is still an open question. See also [16] for recent results including complex parameters, for which different types of scaling exponents and both Gaussian and stable laws are exhibited.

In recent works [21, 19, 6, 7], the question of defining the KPZ equation in higher dimension \( (d \geq 3) \) has been investigated through techniques coming from polymer models. The starting point of these studies is to consider at first the KPZ equation with mollified white noise. Then, the goal is to try to find a limit when the mollification is removed (see also [1], where this method was first applied to define the KPZ equation in dimension \( d = 1 \)). Using the interpretation of the mollified solution through the partition function of a polymer, it was shown that for small noise intensity (corresponding to the weak disorder region of the polymer model), the mollified solution converges in law towards the limiting partition function of the polymer. In [6], it was further shown that the difference, between the mollified solution and the rescaled partition function, vanishes at polynomial rate, and that the renormalized difference converges to a Gaussian in distribution. The result is based on the martingale technique from [8] and is valid in a restrained part of the \( L^2 \)-region of the polymer. We believe that our method could possibly apply to extend the result to the entire \( L^2 \)-region of the polymer.
2 Idea of the proof

2.1 A central limit theorem for martingales

As in [8], the main tool to prove Theorem 1.1 is the following theorem:

**Theorem 2.1** (Corollary 3.2. in [8]). Let $(M_n)_{n \geq 0}$ be a martingale defined on a probability space $(\Omega, \mathcal{F}, P)$, with adapted filtration $(\mathcal{F}_n)_{n \geq 0}$, $M_0 = 0$, which is bounded in $L^2$. Let $D_{k+1} = M_{k+1} - M_k$ for all $k \geq 0$ and let $M_\infty = \lim_{n \to \infty} M_n$ be the a.s. limit of $M_n$. Also define:

$$v_n^2 = \mathbb{E}[(M_\infty - M_n)^2] = \mathbb{E} \sum_{k=n}^{\infty} D_{k+1}^2. \quad (6)$$

Suppose that $v_n$ is always positive and that:

(a) There exists a non-negative and finite random variable $V$, such that

$$V_n^2 = \frac{1}{v_n^2} \sum_{k=n}^{\infty} \mathbb{E} [D_{k+1}^2 | \mathcal{F}_k] \xrightarrow{p} V^2;$$

(b) The following conditional Lindeberg condition holds:

$$\forall \epsilon > 0, \quad \frac{1}{v_n^2} \sum_{k=n}^{\infty} \mathbb{E} [D_{k+1}^2 1_{\{|D_{k+1}| > \epsilon v_n\}} | \mathcal{F}_k] \xrightarrow{p} 0.$$

Then,

$$\frac{M_\infty - M_n}{v_n} \xrightarrow{(d)} V \cdot G, \quad (7)$$

where $G$ is a standard Gaussian random variable which is independent of $V$. If, additionally, $V \neq 0$ a.s., then

$$\frac{M_\infty - M_n}{V_n} \xrightarrow{(d)} G. \quad (8)$$

Moreover, convergence (7) is stable and convergence (8) is mixing.

To prove Theorem 1.1, we show that condition (a) and (b) hold for $M_n = W_n$ and some suited $V$. The proof of condition (b) is delayed to Section 3.5. Our main focus will be condition (a): if we let

$$D_{k+1} = W_{k+1} - W_k,$$

then, by estimate (10), condition (a) follows from:

**Theorem 2.2.** For all $\beta \in (\beta_2^-, \beta_2^+)$, as $n \to \infty$,

$$s_n^2 := n^{(d-2)/2} \sum_{k \geq n} \mathbb{E} [D_{k+1}^2 | \mathcal{F}_k] \xrightarrow{L^1} \sigma^2 W_\infty^2. \quad (9)$$
The structure of the proof for Theorem 2.2 is described in Section 2.2. We now turn to the proof of the main theorem.

Proof of Theorem 1.1. It follows directly from Theorem 2.2 that as \( n \to \infty \),

\[
v_n^2 := \mathbb{E}[(W_\infty - W_n)^2] \sim \sigma^2 \mathbb{E}[W_\infty^2] n^{\frac{d-2}{2}},
\]

(10)

where \( v_n \) is as in (6) (we also refer the reader to Proposition 2.1 of [8] for a more direct argument). Hence, Theorem 2.2 implies condition (a) with limiting variable \( V \) given by

\[
V = \mathbb{E}[W_\infty^2]^{-1/2} W_\infty.
\]

Combined with condition (b), Theorem 2.1 implies convergence (7) which in turn gives (3). Then, to get (4) from (8), observe that \( V_n/V \) and \( W_\infty/W_n \) both converge in probability 1, so that by simple multiplication, one can replace \( V_n \) by \( V \) in (8), and then \( W_\infty \) by \( W_n \) to obtain (4). \( \square \)

2.2 Structure of the proof of Theorem 2.2

By a standard computation, the summand in (9) satisfies

\[
\mathbb{E}[D_{k+1}^2 \mid \mathcal{F}_k] = \kappa_2(\beta) \sum_{x \in \mathbb{Z}^d} \mathbb{E}[e_k \mathbf{1}_{\{S_{k+1} = x\}}]^2,
\]

(11)

where

\[
\kappa_2(\beta) = e^{\lambda_2(\beta)} - 1.
\]

In order to study the right-hand side of (11), we appeal to the following theorem:

Theorem 2.3 (Local limit theorem for polymers in the \( L^2 \)-region [23, 25]). Let \( \beta \in (\beta_2, \beta_3) \) and \( \alpha > 0 \). For any sequence \( (l_k)_{k \geq 0} \), verifying that \( l_k \to \infty \) and \( l_k = o(k^a) \) for some \( a < 1/2 \),

\[
\mathbb{E}[e_k \mid S_{k+1} = x] = W_{l_k} \tilde{W}_{k+1,l_k}^x + \delta_k^x,
\]

(12)

where \( \tilde{W}_{k,l}^y = P_y \left[ \exp \left( \sum_{i=1}^{l'} \omega(k-i, S_i) \right) \right] \) is the time-reversed partition function, and where, as \( k \to \infty \),

\[
\sup_{|x| \leq \alpha \sqrt{k}} \mathbb{E}\left[|\delta_k^x|^2\right] \to 0.
\]

(13)

Remark 2.1. Note that we have reformulated the result with endpoint distribution at time \( k+1 \), for a polymer measure of horizon \( k \), so that the time-reversed partition function \( \tilde{W}_{k+1,l_k}^x \) does not take into account the environment at time \( k+1 \).
By the local limit theorem for polymers,
\[ s_n^2 = \kappa_2(\beta)n^{(d-2)/2} \sum_{k \geq n} \sum_{x \in \mathbb{Z}^d} E \left[ e_k 1\{S_{k+1} = x\}\right]^2 \]
\[ =: A_n + B_n + C_n + F_n , \]  
where:
\[ A_n = \kappa_2(\beta)n^{(d-2)/2} \sum_{k \geq n} \sum_{|x| \leq \alpha \sqrt{k}} \left( W_{k}\overrightarrow{W}_{k+1,l_k} \right)^2 P(S_{k+1} = x)^2, \]
and:
\[ B_n = 2\kappa_2(\beta)n^{(d-2)/2} \sum_{k \geq n} \sum_{|x| \leq \alpha \sqrt{k}} \delta_k^x W_{k}\overrightarrow{W}_{k+1,l_k} P(S_{k+1} = x)^2, \]
\[ C_n = \kappa_2(\beta)n^{(d-2)/2} \sum_{k \geq n} \sum_{|x| \leq \alpha \sqrt{k}} (\delta_k^x)^2 P(S_{k+1} = x)^2, \]
\[ F_n = \kappa_2(\beta)n^{(d-2)/2} \sum_{k \geq n} \sum_{|x| > \alpha \sqrt{k}} E \left[ e_k 1\{S_{k+1} = x\}\right]^2. \]

Section 3.2 is dedicated to showing that \( B_n, C_n \) and \( F_n \) all vanish in \( L^1 \) norm. Turning to \( A_n \), we note that \( \overrightarrow{W}_{k+1,l_k} \) and \( \overrightarrow{W}_{k+1,l_k} \) are independent whenever \( |x - y| > l_k \), so that, by some homogenization argument, we can show that
\[ A_n \approx \kappa_2(\beta)n^{(d-2)/2} \sum_{k \geq n} \sum_{|x| \leq \alpha \sqrt{k}} W_k^2 E \left[ \left( \overrightarrow{W}_{k+1,l_k} \right)^2 \right] P(S_{k+1} = x)^2 \]
\[ \rightarrow \sigma^2 W_\infty^2, \]
as \( n \to \infty \) and \( \alpha \to \infty \) in this order. Approximation (16) is justified in Section 3.3, while, letting \( \overline{A}_n \) denote the RHS of (16), convergence in the second line is proved in Proposition 3.6.

3 Proof

Notations

- \( | \cdot | \) stands for the Euclidean norm on \( \mathbb{R} \) or \( \mathbb{R}^d \).
- \( | \cdot |_1 \) stands for the usual \( L^1 \)-norm on \( \mathbb{R}^d \).
- Let \( B(r) \) denote the closed ball of radius \( r \) in the Euclidean norm.
- \( E^{\otimes 2} \) and \( P^{\otimes 2} \) will stand for resp. the expectation and the probability measure for two independent simple random walks \( S \) and \( \tilde{S} \).
- We write \( E_k[ \cdot ] = E[ \cdot | F_k] \).
- Given two paths \( S \) and \( \tilde{S} \), we denote the overlap of \( S \) and \( \tilde{S} \), from time \( m \) to \( k \), as \( N_{m,k} = \sum_{i=m}^{k} 1\{S_i = \tilde{S}_i\} \). When \( m = 1 \), we simply write \( N_k \). This \( k \) can be taken to be infinity.
3.1 Some tools.

**Theorem 3.1** (Local central limit theorem for the simple random walk [18]).

For all $x$ such that $P(S_k = x) \neq 0$, as $k \to \infty$,

$$P(S_k = x) = 2 \left( \frac{d}{2\pi k} \right)^{d/2} e^{-d|x|^2/2k} + |x|^{-2} O \left( k^{-d/2} \right),$$

$$P(S_{2k} = 0) \sim 2 \left( \frac{d}{4\pi k} \right)^{d/2},$$

where the big $O$ term is uniform in $x$.

**Proposition 3.1.** There exists $Z_d > 0$ such that

$$n^{(d-2)/2} \sum_{k \geq n} \sum_{x \in \mathbb{Z}^d} P(S_{k+1} = x)^2 \to Z_d.$$  

**Proof.** For $x \in \mathbb{Z}^d$, let $G$ be the Green function :

$$G(x) = E_x \left[ \sum_{i=0}^{\infty} 1_{S_i = 0} \right].$$

Then, by strong Markov property, for any $x \in \mathbb{Z}^d$,

$$P_x(\exists n \in \mathbb{Z}_{\geq 0}, S_n = 0) = G(x)/G(0),$$

Since $S_k - \tilde{S}_k \overset{law}{=} S_{2k}$ and using again the strong Markov property,

$$n^{(d-2)/2} \sum_{k \geq n} \sum_{x \in \mathbb{Z}^d} P(S_{k+1} = x)^2$$

$$= n^{(d-2)/2} E^{\otimes 2}[N_{n+1, \infty}]$$

$$= E^{\otimes 2} \left[ n^{(d-2)/2} P_{S_{n+1} - \tilde{S}_{n+1}} (\exists n \in \mathbb{Z}_{\geq 0}, S_n = 0) \right] G(0).$$

On the other hand, using (20),

$$E^{\otimes 2} \left[ P_{S_{n+1} - \tilde{S}_{n+1}} (\exists n \in \mathbb{Z}_{\geq 0}, S_n = 0) \right] = G(0)^{-1} E^{\otimes 2}[G(S_{n+1} - \tilde{S}_{n+1})].$$

By (23) and (25) in [8], $n^{(d-2)/2} E^{\otimes 2}[G(S_{n+1} - \tilde{S}_{n+1})]$ converges to a positive constant, which completes the proof. □

We will also require the following technical proposition:

**Proposition 3.2.** Let $v$ be a non-negative bounded function on $\mathbb{Z}^d$ with $d \geq 3$, such that:

$$\sup_{x \in \mathbb{Z}^d} E_x \left[ e^{\sum_{k=1}^{\infty} v(S_{2k})} \right] < \infty.$$

Then, there exists $C \in (0, \infty)$, such that for all $n \geq 0$,

$$E_d \left[ e^{\sum_{k=1}^{n} v(S_{2k})} \left| S_{2(n+1)} = 0 \right. \right] \leq C.$$
To prove this proposition, we use an analogue of Lemma 3.3 in [25]:

**Lemma 3.1.** Under the assumptions of Proposition 3.2, there exists a constant $C \in (0, \infty)$, such that for all non-negative function $f$ on $\mathbb{Z}^d$,

$$
\sup_{x \in \mathbb{Z}^d} \mathbb{E}_x \left[ e^{\sum_{i=1}^{\infty} v(S_{2i})} f(S_{2n}) \right] \leq \frac{C}{n^{d/2}} \sum_{y \in \mathbb{Z}^d : |y|_1 \text{ even}} f(y).
$$

**Proof.** We repeat the argument of [25], which is a little simpler in our case. Let $\mathcal{A} = \{x \in \mathbb{Z}^d, |x|_1 \text{ is even} \}$ be the underlying graph of $(S_{2i})$. We also let $x \sim y$ denote the fact that $x$ and $y$ are nearest neighbors in $\mathcal{A}$, and $p^{(2)}(x, y)$ be the transition kernel $P(S_2 = y | S_0 = x)$. For all $x \in A$, define $h(x) = \mathbb{E}_x \left[ e^{\sum_{i=1}^{\infty} v(S_{2i})} \right]$; then $h$ satisfies

$$
h(x) = \sum_{y \sim x} p^{(2)}(x, y) e^{v(y)} h(y).
$$

Hence, similarly to Doob’s $h$-transform, the kernel

$$
K(x, y) = \frac{h(y)}{h(x)} e^{v(y)} p^{(2)}(x, y)
$$

defines a probability transition kernel of a Markov chain on $\mathcal{A}$, for which $m(x) = h(x)^2 e^{v(x)}$ is a reversible measure.

By assumption, $m$ is uniformly bounded and bounded away from 0, and there exists a constant $c \in (0, \infty)$, such that $K(x, y) \geq cp^{(2)}(x, y)$ for all $x, y \in \mathcal{A}$. As by Theorem 4.18 in [27], $S_{2n}$ satisfies the $d$-isoperimetric inequality (cf. pp. 39-40 therein) on $\mathcal{A}$, this implies that $K$ also satisfies it.

Therefore, we get from Corollary 14.5 in [27] that there exists a finite $C$, such that

$$
\frac{1}{h(x)} \mathbb{E}_x \left[ h(S_{2n}) e^{\sum_{i=1}^{\infty} v(S_{2i})} f(S_{2n}) \right] = \sum_{y \in A} K^{(n)}(x, y) f(y) \leq \frac{C}{n^{d/2}} \sum_{y \in A} f(y).
$$

This in turn implies the lemma by our assumptions. □

**Proof of Proposition (3.2).** Choose $f(y) = 1_{\{y = 0\}}$ and $x = 0$ in Lemma 3.1, and conclude using estimate (18) from the local CLT. □

### 3.2 Removing the negligible terms

The following proposition justifies that $F_n$ from Section 2.2 is negligible in $L^1$-norm.

**Proposition 3.3.** We have:

$$
\lim_{\alpha \to \infty} \limsup_{n \to \infty} \mathbb{E} \left[ n^{(d-2)/2} \sum_{k \geq n} \sum_{|x| \geq \alpha \sqrt{k}} \mathbb{E} \left[ e_k 1_{\{S_{k+1} = x\}} \right]^2 \right] = 0.
$$
Proof. The finite positive constants $C$ that will arise in the paper may change from line to line, but they will not depend on any varying parameter. We write:

$$E n^{(d-2)/2} \sum_{k \geq n, |x| \geq \alpha \sqrt{k}} E \left[ e_k 1_{\{S_{k+1} = x\}} \right]^2$$

$$= n^{(d-2)/2} \sum_{k \geq n} E^{\otimes 2} \left[ e^{\lambda_2 N_k} 1_{S_{k+1} = \tilde{S}_{k+1}} 1_{|S_{k+1}| \geq \alpha \sqrt{k}} \right]$$

$$= n^{(d-2)/2} \sum_{k \geq n} E^{\otimes 2} \left[ e^{\lambda_2 N_k} 1_{|S_{k+1}| \geq \alpha \sqrt{k}} \right] P(S_{k+1} = \tilde{S}_{k+1}).$$

As $E^{\otimes 2}[e^{\lambda_2 N_k}] = \mathbb{E}[W_{x_k}^2]$ is given by the RHS of (2), we can apply Hölder’s inequality, for $p^{-1} + q^{-1} = 1$ with the only constraint that $p \lambda_2(\beta) < \log(1/\pi_d)$, and use Lemma 3.1 (note that $S_k - \tilde{S}_k \Rightarrow S_{2k}$), to get that

$$E^{\otimes 2} \left[ e^{\lambda_2 N_k} 1_{|S_{k+1}| \geq \alpha \sqrt{k}} \right]_{S_{k+1} = \tilde{S}_{k+1}} \leq C E^{\otimes 2} \left[ 1_{|S_{k+1}| \geq \alpha \sqrt{k}} \right] P(S_{k+1} = \tilde{S}_{k+1})^{1/q}$$

Then, by the local central limit theorem (Theorem 3.1), there exists some positive constants $C$, such that for large enough $n$,

$$E^{\otimes 2} \left[ 1_{|S_{k+1}| \geq \alpha \sqrt{k}} \right] P(S_{k+1} = \tilde{S}_{k+1}) \leq \sum_{|x| \geq \alpha \sqrt{k}} \frac{P(S_{k+1} = x)}{P^{\otimes 2}(S_{k+1} = \tilde{S}_{k+1})} \leq \max_{|x| \geq \alpha \sqrt{k}} \frac{P(|S_{k+1}| = x)}{P^{\otimes 2}(S_{k+1} = \tilde{S}_{k+1})} \leq C \alpha^{-2}.$$ 

It follows from the local CLT that

$$E n^{(d-2)/2} \sum_{k \geq n, |x| \geq \alpha \sqrt{k}} E \left[ e_k 1_{\{S_{k+1} = x\}} \right]^2 \leq C \alpha^{-2/q} n^{(d-2)/2} \sum_{k \geq n} P(S_{k+1} = \tilde{S}_{k+1})$$

$$= C \alpha^{-2/q} n^{(d-2)/2} \sum_{k \geq n} P(S_{2k+1} = 0)$$

$$\leq C \alpha^{-2/q},$$

which vanishes as $\alpha \to \infty$. \qed

Proposition 3.4. As $n \to \infty$,

$$B_n = 2\kappa_2(\beta) 2^{n(d-2)/4} \sum_{k \geq n, |x| \leq \alpha \sqrt{k}} \delta_k^x W_{k_{k+1}} \tilde{W}_{k+1}^x P(S_{k+1} = x)^2 \overset{L^1}{\to} 0.$$ 

$$C_n = \kappa_2(\beta) 2^{n(d-2)/4} \sum_{k \geq n, |x| \leq \alpha \sqrt{k}} (\delta_k^x)^2 P(S_{k+1} = x)^2 \overset{L^1}{\to} 0.$$
Proof. From independence of $W_{l_k}$ and $\tilde{W}_{k+1,l_k}$, we get from Cauchy-Schwarz inequality:

$$
E \left[ |\delta_{l_k} W_{l_k} \tilde{W}_{k+1,l_k}^x| \right] \leq E \left[ W_{l_k}^2 \right] E \left[ (\delta_{l_k})^2 \right]^{1/2}.
$$

The first term of the right-hand side is bounded in the $L^2$-region, so the convergence for $B_n$ follows simply from (13) and (19). $C_n$ is treated in the same way. □

### 3.3 The homogenization result

This section is dedicated to proving the next proposition, which justifies approximation (16):

**Proposition 3.5.** Let $\overline{A}_n$ denote the right-hand side of equation (16). Then, for any $\alpha > 0$,

$$
\lim_{n \to \infty} E|A_n - \overline{A}_n| = 0.
$$

To show the result of the proposition, it is enough to prove that, as $k \to \infty$,

$$
M_k := k^{d/2} \sum_{|x| \leq \alpha \sqrt{k}} (Y_{k,x} - EY_{k,x})P(S_{k+1} = x)^2 \overset{L^1}{\to} 0,
$$

where $Y_{k,x} = (\tilde{W}_{k+1,l_k}^x)^2$. Indeed, for large $k$, we have $l_k < k/2$ and so $W_{l_k}$ and $\tilde{W}_{k+1,l_k}^x$ are independent. Hence:

$$
E|A_n - \overline{A}_n|
$$

$$
\leq \kappa_2 n^{(d-2)/2} \sum_{k \geq n} E\left[W_{l_k}^2\right] E\left[\sum_{|x| \leq \alpha \sqrt{k}} (Y_{k,x} - EY_{k,x})P(S_{k+1} = x)^2\right]
$$

$$
\leq \kappa_2 E\left[W_{\infty}^2\right] n^{(d-2)/2} \sum_{k \geq n} k^{-d/2} E[|M_k|],
$$

where the last term vanishes, as $n \to \infty$, if one assumes convergence (21). To prove this convergence, we rely on a truncation technique:

**Lemma 3.2.** Let $\tilde{Y}_{k,x} = Y_{k,x} \wedge (k^{d/2} l_k^{-d})$. We have,

$$
\lim_{k \to \infty} E\left(k^{d/2} \sum_{|x| \leq \alpha \sqrt{k}} (\tilde{Y}_{k,x} - E\tilde{Y}_{k,x})P(S_{k+1} = x)^2\right)^2 = 0.
$$
Proof. Since $Y_{k,x}$ and $Y_{k,y}$ are independent for $|x-y|/l_k \geq 1$, we have

$$E \left( k^{d/2} \sum_{|x| \leq \alpha \sqrt{k}} (\tilde{Y}_{k,x} - E\tilde{Y}_{k,x}) P(S_{k+1} = x)^2 \right)$$

$$= k^d \sum_{|x|,|y| \leq \alpha \sqrt{k}} E \left( \tilde{Y}_{k,x} - E\tilde{Y}_{k,x} \right) \left( \tilde{Y}_{k,y} - E\tilde{Y}_{k,y} \right) P(S_{k+1} = x)^2 P(S_{k+1} = y)^2$$

$$\leq Ck^{-d} \sum_{|x-y| \leq l_k, |x| \leq \alpha \sqrt{k}} E \left( \tilde{Y}_{k,x} - E\tilde{Y}_{k,x} \right) \left( \tilde{Y}_{k,y} - E\tilde{Y}_{k,y} \right)$$

$$\leq Ck^{-d} \sum_{|x-y| \leq l_k, |x| \leq \alpha \sqrt{k}} E \left( \tilde{Y}_{k,0} - E\tilde{Y}_{k,0} \right)^2,$$

where we have used the local central limit theorem in the first inequality; we used the Cauchy-Schwarz inequality and the fact that $\tilde{Y}_{k,x}$ are identically distributed with respect to $x$ in the last one. This is further bounded from above by

$$Ck^{-d/2} \int_0^{l_k} E \left( \tilde{Y}_{k,0} - E\tilde{Y}_{k,0} \right)^2 \to 0,$$

as $k \to \infty$, where, observing that the family $(Y_{k,0})_k$ is uniformly integrable since $W_k^2$ converges in $L^1$, the convergence in the second line is justified by the following lemma.

**Lemma 3.3.** Let $(X_k)_{k \in \mathbb{N}}$ be a non-negative, uniformly integrable family of random variables. Then, for any sequence $a_k \to \infty$, $a_k^{-1} E \left[ (X_k \wedge a_k)^2 \right] \to 0.$

**Proof.** By property: $x \mathbb{P}(X_k \geq x) \leq E[X_k 1_{\{X_k \geq x\}}]$, we have

$$a_k^{-1} E \left[ (X_k \wedge a_k)^2 \right] = a_k^{-1} \int_0^{a_k} 2x \mathbb{P}(X_k \geq x) \, dx$$

$$\leq 2a_k^{-1} \int_0^{a_k} \sup_{k \in \mathbb{N}} E[X_k 1_{\{X_k \geq x\}}] \, dx \to 0,$$

as $k \to \infty$, since $\sup_k E[X_k 1_{\{X_k \geq x\}}] \to 0$ as $x \to \infty$, by uniform integrability. \qed

The next lemma will be used in order to remove the truncation:

**Lemma 3.4.** We have:

$$k^{d/2} \sum_{|x| \leq \alpha \sqrt{k}} (Y_{k,x} - \tilde{Y}_{k,x}) P(S_{k+1} = x)^2 \overset{L^1}{\to} 0. \quad (22)$$
Moreover,
\[
\lim_{k \to \infty} k^{d/2} \sum_{|x| \leq \alpha \sqrt{k}} \left( EY_{k,x} - E\tilde{Y}_{k,x} \right) P(S_{k+1} = x)^2 = 0. \tag{23}
\]

**Proof.** Note that
\[
E[|Y_{k,0} - \tilde{Y}_{k,0}|] \leq E \left[ Y_{k,0}; \ Y_{k,0} > k^{d/2} l_k^{-d} \right] \to 0.
\]
Thus, combining with the local CLT, we safely get (22) and (23).

Finally, putting together Lemma 3.2 and Lemma 3.4, we get that \( M_k \xrightarrow{L^1} 0 \), as desired.

### 3.4 Proof of Theorem 2.2

Combined to propositions of the two last sections, the following theorem entails Theorem 2.2:

**Proposition 3.6.** With \( \sigma^2 \) defined as in (24),
\[
\lim_{\alpha \to \infty} \limsup_{n \to \infty} E \left| \mathcal{A}_n - \sigma^2 W^2_\infty \right| = 0.
\]

**Proof.** Note first that
\[
E \left| \mathcal{A}_n - \kappa_2(\beta) n^{(d-2)/2} \sum_{k \geq n} \sum_{|x| \leq \alpha \sqrt{k}} W^2_\infty E \left[ \left( \tilde{W}_{k+1,l_k} \right)^2 \right] P(S_{k+1} = x)^2 \right|
\]
\[
\leq C \sup_{k \geq n} E|W^2_\infty - W^2_{l_k}| \to 0,
\]
as \( n \to \infty \), and
\[
E \left| \kappa_2(\beta) n^{(d-2)/2} \sum_{k \geq n} \sum_{|x| \leq \alpha \sqrt{k}} W^2_\infty \left( E \left[ \left( \tilde{W}_{k+1,l_k} \right)^2 \right] - EW^2_\infty \right) P(S_{k+1} = x)^2 \right|
\]
\[
\leq C \sup_{k \geq n} E|W^2_\infty - W^2_{l_k}| \to 0.
\]
Moreover, by Proposition 3.3, we have
\[
\lim_{\alpha \to \infty} \limsup_{n \to \infty} E \left| \kappa_2(\beta) n^{(d-2)/2} \sum_{k \geq n} \sum_{|x| > \alpha \sqrt{k}} W^2_\infty EW^2_\infty P(S_{k+1} = x)^2 \right| = 0.
\]
Therefore, it suffices to show that as $n \to \infty$,
\[
\kappa_2(\beta) n^{(d-2)/2} \sum_{k \geq n} \sum_{x \in \mathbb{Z}^d} \mathbb{E} W_\infty^2 P(S_{k+1} = x)^2 \to \sigma^2(\beta),
\]
where
\[
\sigma^2(\beta) = \frac{(1 - \pi_d)(e^{\lambda_2(\beta)} - 1)}{1 - \pi_d e^{\lambda_2(\beta)}} Z_d.
\tag{24}
\]
Recalling $\kappa_2(\beta) = e^{\lambda_2(\beta)} - 1$ and $\mathbb{E} W_\infty^2 = \frac{1 - \pi_d}{1 - \pi_d e^{\lambda_2(\beta)}}$ (cf. (2)), this follows from convergence (19).

3.5 Proof of condition (b): the Lindeberg condition

Given the asymptotics of $v_n$ in (10), condition (b) of Theorem 2.1 follows from the following proposition:

**Proposition 3.7** (Lindeberg condition). For any $\epsilon > 0$,
\[
n^{(d-2)/2} \sum_{k \geq n} \mathbb{E}_k \left[ D_{k+1}^2 \mathbb{1}_{\{n \frac{d-2}{2} | D_{k+1} | > \epsilon \}} \right] \overset{L^1}{\to} 0.
\]

**Proof.** We first observe that it is enough to prove that
\[
\lim_{k \to \infty} k^{d/2} \mathbb{E} \left[ D_{k+1}^2 \mathbb{1}_{\{k \frac{d-2}{2} | D_{k+1} | > \epsilon \}} \right] = 0.
\tag{25}
\]

Indeed, since $n \frac{d-2}{2} | D_{k+1} | > \epsilon$ implies $k \frac{d-2}{2} | D_{k+1} | > \epsilon$ for $k \geq n$, we would have, assuming (25),
\[
\lim_{n \to \infty} \mathbb{E} \left[ n \frac{d-2}{2} \sum_{k \geq n} \mathbb{E}_k \left( D_{k+1}^2 \mathbb{1}_{\{n \frac{d-2}{2} | D_{k+1} | > \epsilon \}} \right) \right] \leq \limsup_{n \to \infty} n \frac{d-2}{2} \sum_{k \geq n} k^{-d/2} \mathbb{E} \left[ k^{d/2} D_{k+1}^2 \mathbb{1}_{\{k \frac{d-2}{2} | D_{k+1} | > \epsilon \}} \right] \\
\leq C \limsup_{k \to \infty} k^{d/2} \mathbb{E} \left[ D_{k+1}^2 \mathbb{1}_{\{k \frac{d-2}{2} | D_{k+1} | > \epsilon \}} \right] = 0,
\]

where the third inequality comes from the boundedness of $n \frac{d-2}{2} \sum_{k \geq n} k^{-d/2}$.

We now focus on showing (25).

Using $S_k - \tilde{S}_k \overset{\text{law}}{=} S_{2k}$, we may write
\[
\mathbb{E} D_{k+1}^2 = \kappa_2 \mathbb{E} \left[ e^{\lambda_2 N_k} \mathbb{1}_{\{S_{k+1} = \tilde{S}_{k+1} \}} \right] \\
= \kappa_2 \mathbb{E} \left[ e^{\lambda_2 N_k} | S_{k+1} = \tilde{S}_{k+1} \right] \mathbb{P}^\otimes 2(S_{k+1} = \tilde{S}_{k+1}) \\
= \kappa_2 \mathbb{E} \left[ e^{\lambda_2 \sum_{i=1}^{k} 1(S_{2i} = 0)} | S_{2(k+1)} = 0 \right] \mathbb{P}(S_{2(k+1)} = 0)
\]
By Proposition 3.2 and the local CLT, we have $k^{d/2} \mathbb{E} D_{k+1}^2 = O(1)$. Thus, applying Markov’s inequality, we get

$$\mathbb{P} \left( k^{d/2} |D_{k+1}| > \epsilon \right) = \mathbb{P} \left( k^{d/2} D_{k+1}^2 > \epsilon^2 k \right) \leq \frac{1}{\epsilon^2 k} k^{d/2} \mathbb{E} D_{k+1}^2 \to 0,$$

as $k \to \infty$.

In order to prove (25), we will rely on estimate (26) and uniform integrability properties. We will need the following simple lemma.

**Lemma 3.5.** Let $\{X_n\}, \{Y_n\}$ be independent uniformly integrable families of random variables. Then $\{X_n Y_n\}$ is also uniformly integrable.

**Proof.** Let us note that $|X_n Y_n| \geq t$, then $|X_n| \geq \sqrt{t}$ or $|Y_n| \geq \sqrt{t}$. Thus,

$$\mathbb{E}[|X_n Y_n|; |X_n| \geq \sqrt{t}] \leq \mathbb{E}[|X_n Y_n|; |X_n| \geq \sqrt{t}] + \mathbb{E}[|X_n Y_n|; |Y_n| \geq \sqrt{t}] = \mathbb{E}[|Y_n|||X_n|; |X_n| \geq \sqrt{t}] + \mathbb{E}[|X_n|||Y_n|; |Y_n| \geq \sqrt{t}],$$

which uniformly goes to 0 as $t \to \infty$. □

For all $k \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, we write $\eta_k(x) = e^{\beta \omega(k+1, x)} - \lambda(\beta) - 1$. Note that $\mathbb{E}\eta_k(x) = 0$ and $\mathbb{E}\eta_k(x)^2 = \kappa_2(\beta)$.

We decompose,

$$D_{k+1} = W_{k+1} - W_k = \sum_{|x| \leq \alpha \sqrt{k}} \mathbb{E}[e_k 1_{S_{k+1} = x}] \eta_k(x) + \sum_{|x| > \alpha \sqrt{k}} \mathbb{E}[e_k 1_{S_{k+1} = x}] \eta_k(x),$$

and first observe that by proposition 3.3, we have

$$\lim_{\alpha \to \infty} \limsup_{k \to \infty} k^{d/2} \mathbb{E} \left[ \left( \sum_{|x| > \alpha \sqrt{k}} \mathbb{E}[e_k 1_{S_{k+1} = x}] \eta_k(x) \right)^2 \right] = 0.$$

Then, if we let $(l_k)_k$ be any positive sequence satisfying the conditions of Proposition 2.3, we can write

$$\sum_{|x| \leq \alpha \sqrt{k}} \mathbb{E}[e_k 1_{S_{k+1} = x}] \eta_k(x) = \sum_{|x| \leq \alpha \sqrt{k}} W_{l_k} \hat{W}_{k+1, l_k} P(S_{k+1} = x) \eta_k(x) + \sum_{|x| \leq \alpha \sqrt{k}} \delta_{k,x} P(S_{k+1} = x) \eta_k(x).$$

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For the second term of the right hand side, it is easy to check as in Proposition 3.4 that
\[
\lim_{k \to \infty} k^{d/2} E \left( \sum_{|x| \leq \alpha \sqrt{k}} \delta_{k,x} P(S_{k+1} = x) \eta_k(x) \right)^2 = 0.
\]

For the first term, denoting by \(B(r)\) the closed ball of \(\mathbb{R}^d\) of radius \(r\), we compute,
\[
\begin{align*}
&k^{d/2} \left( \sum_{|x| \leq \alpha \sqrt{k}} W_k \tilde{W}_{k+1} W_{k+1} P(S_{k+1} = x) \eta_k(x) \right)^2 \\
&= k^{d/2} \sum_{x,y \in B(\alpha \sqrt{k})} W_k^2 \tilde{W}_{k+1} W_k \eta_k(x) \eta_k(y) P(S_{k+1} = x) P(S_{k+1} = y) \\
&= k^{d/2} \sum_{x,y \in B(\alpha \sqrt{k})} W_k^2 \tilde{W}_{k+1} W_k \eta_k(x) \eta_k(y) P(S_{k+1} = x) P(S_{k+1} = y) \\
&\quad + W_k^2 k^{d/2} \sum_{x,y \in B(\alpha \sqrt{k})} \tilde{W}_{k+1} \eta_k(x) \eta_k(y) P(S_{k+1} = x) P(S_{k+1} = y) \\
&=: D^{(1)}_k + W_k^2 D^{(2)}_k. \tag{27}
\end{align*}
\]

By Cauchy-Schwarz inequality and Theorem 3.1,
\[
\left| D^{(1)}_k \right| \leq C k^{-d/2} l_k^d \sum_{x \in B(\alpha \sqrt{k})} W_k^2 \left( \tilde{W}_{k+1} W_k \right)^2 \eta_k(x)^2.
\]

By (26), Lemma 3.5 and uniform integrability of \(W_m^2\) (note that \(W_m^2\) converges in \(L^1\)), we get that as \(k \to \infty\),
\[
a_k := \sup_{0 \leq m < k/2} \sup_{x \in \mathbb{R}^d} E \left[ W_m^2 \left( \tilde{W}_{k+1,m} \right)^2 \eta_k(x)^2 1_{\{k \leq m \leq D_k+1, |x| > \epsilon\}} \right] \\
\to 0,
\]

where, in order to use Lemma 3.5, we have restricted the supremum to \(m < k/2\), so that \(W_m^2 \tilde{W}_{k+1,m}^2\) are then independent from each other, and, by definition, independent of \(\eta_k(x)\). Then, we choose and fix a specific \((l_k)_k\), which satisfies both \(l_k^d a_k \to 0\) and the conditions of Proposition 2.3 (and hence \(l_k < k/2\) for large \(k\)). Thereby, as \(k \to \infty\),
\[
E \left[ k^{-d/2} l_k^d \sum_{x \in B(\alpha \sqrt{k})} W_k^2 \left( \tilde{W}_{k+1,l_k} \right)^2 \eta_k(x)^2 1_{\{k \leq m \leq D_k+1, |x| > \epsilon\}} \right] \leq C l_k^d a_k \to 0.
\]
As a consequence, we have

$$\lim_{k \to \infty} \mathbb{E} \left[ D_k^{(1)} 1_{\{ k \frac{d-2}{4} |D_{k+1}| > \epsilon \}} \right] = 0.$$ 

Finally, note that

$$\mathbb{E} \left[ (D_k^{(2)})^2 \right] = k^d \sum_{x,y \in B(\alpha \sqrt{k})} \sum_{|x-y| > l_k} \mathbb{E} \left[ \prod_{u \in \{x,y,w\}} \tilde{W}_{k+1,l_k}^u \eta_k(u) \right] P(S_{k+1} = u),$$

where, by independence of $\eta_k(u)$ and $\tilde{W}_{k+1,l_k}^u$, each term inside the sum vanishes unless either $x = z$, $y = w$ or $x = w$, $y = z$. Hence, by Theorem 3.1,

$$\mathbb{E} \left[ (D_k^{(2)})^2 \right] \leq C k^{-d} \sum_{x,y \in B(\alpha \sqrt{k})} \mathbb{E} \left[ (\tilde{W}_{k+1,l_k})^2 (\tilde{W}_{k+1,l_k}^y)^2 \eta_k(x)^2 \eta_k(y)^2 \right]$$

$$\leq C k^{-d} \sum_{x,y \in B(\alpha \sqrt{k})} \mathbb{E} \left[ (\tilde{W}_{k+1,l_k}^u)^2 \right]^2 \mathbb{E} \left[ \eta_k(0)^2 \right]^2$$

$$= O(1),$$

where the second inequality comes from the independence of $\tilde{W}_{k+1,l_k}^x$, $\tilde{W}_{k+1,l_k}^y$ whenever $|x-y| > l_k$. Therefore, $D_k^{(2)}$ is uniformly integrable, so by independence of $W_{l_k}^x$ and $D_k^{(2)}$ and Lemma 3.5,

$$\lim_{k \to \infty} \mathbb{E} \left[ W_{l_k}^x D_k^{(2)} 1_{\{ k \frac{d-2}{4} |D_{k+1}| > \epsilon \}} \right] = 0.$$

Putting things together, we have shown (25).

\[ 3.6 \text{ Proof of Corollary 1.2} \]

**Proof.** We write

$$\log W_\infty - \log W_n = \log \left( 1 + \frac{W_\infty - W_n}{W_n} \right).$$

By Taylor expansion, there exists a constant $M > 0$, such that for all $|x| < 1/2$, we have

$$|\log (1 + x) - x| \leq M x^2. \quad (28)$$
Then, we write $X_n = \frac{W_\infty - W_n}{W_n}$, so that by Theorem 1.1, $n^{d^{-2}}X_n \overset{(d)}{\to} \sigma G$ and this convergence is mixing. In particular $X_n \overset{p}{\to} 0$.

By the inequality in (28), we have

$$\mathbb{P}\left(n^{d^{-2}}|\log(1 + X_n) - X_n| > \epsilon; |X_n| < 1/2\right) \leq \mathbb{P}\left(Mn^{d^{-2}}|X_n|^2 > \epsilon\right),$$

which vanishes as $n \to \infty$. Moreover, $\mathbb{P}(|X_n| \geq 1/2) \to 0$, so that

$$n^{d^{-2}}(\log (1 + X_n) - X_n) \overset{p}{\to} 0.$$

**Lemma 3.6.** Suppose that $Y_n \overset{(d)}{\to} Y$ and $Z_n \overset{p}{\to} 0$, where $Y$ has a continuous cumulative distribution function. Then

$$Y_n + Z_n \overset{(d)}{\to} Y.$$ (29)

Moreover, if, in addition, the convergence $Y_n \overset{(d)}{\to} Y$ is mixing, the convergence (29) is also mixing.

**Proof.** Let us denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space. Recall that the property that $Y_n \overset{(d)}{\to} Y$ is mixing is equivalent to that for any $x \in \mathbb{R}$ and $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$,

$$\lim_{n \to \infty} \mathbb{P}(Y_n \leq x; B) = \mathbb{P}(Y \leq x)\mathbb{P}(B).$$

We fix $x \in \mathbb{R}$ and $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$. For any $\epsilon > 0$,

$$\limsup_{n \to \infty} \mathbb{P}(Y_n + Z_n \leq x; B) \leq \lim_{n \to \infty} \mathbb{P}(Y_n \leq x + \epsilon; B) + \lim_{n \to \infty} \mathbb{P}(Z_n < -\epsilon)$$

$$= \mathbb{P}(Y \leq x + \epsilon)\mathbb{P}(B).$$

Letting $\epsilon \downarrow 0$, since $Y$ has a continuous cumulative distribution function, we have

$$\limsup_{n \to \infty} \mathbb{P}(Y_n + Z_n \leq x; B) \leq \mathbb{P}(Y \leq x)\mathbb{P}(B).$$

Conversely, for any $\epsilon > 0$,

$$\liminf_{n \to \infty} \mathbb{P}(Y_n + Z_n \leq x; B) \geq \liminf_{n \to \infty} \mathbb{P}(Y_n + Z_n \leq x; Z_n \leq \epsilon; B)$$

$$\geq \lim_{n \to \infty} \mathbb{P}(Y_n \leq x - \epsilon; B) - \lim_{n \to \infty} \mathbb{P}(Z_n > \epsilon)$$

$$\geq \mathbb{P}(Y \leq x - \epsilon)\mathbb{P}(B).$$

Similarly, we have

$$\liminf_{n \to \infty} \mathbb{P}(Y_n + Z_n \leq x; B) \geq \mathbb{P}(Y \leq x)\mathbb{P}(B).$$

Using this lemma, we get

$$n^{d^{-2}} \log \left(1 + \frac{W_\infty - W_n}{W_n}\right) \overset{(d)}{\to} \sigma G,$$

and this convergence is mixing. 

\[\square\]
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