Inversion of Higher Dimensional Radon Transforms of Seismic-Type

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Abstract

We study integral transforms mapping a function on the Euclidean space to the family of its integration on some hypersurfaces, that is, a function of hypersurfaces. The hypersurfaces are given by the graphs of functions with fixed axes of the independent variables, and are imposed some symmetry with respect to the axes. These transforms are higher dimensional version of generalization of the parabolic Radon transform and the hyperbolic Radon transform arising from seismology. We prove the inversion formulas for these transforms under some vanishing and symmetry conditions of functions.

Keywords Radon transform · Inversion formula · Seismology

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1 Introduction

Let $n$ be a positive integer. Fix arbitrary $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ and $\alpha_1, \ldots, \alpha_n, \beta > 1$. Set $
abla = (\alpha_1, \ldots, \alpha_n)$ for short. Let $(x, y) = (x_1, \ldots, x_n, y), (s, u) = (s_1, \ldots, s_n, u) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ be independent variables of functions. We study the inversions of the integral transforms $P_\nabla f(s, u), Q_\nabla f(s, u)$ and $R_{\alpha, \beta} f(s, u)$ of a function $f(x, y)$. These are the integrations of $f(x, y)$ on some special families of hypersurfaces. We do not deal with the general hypersurfaces. The precise definitions of our transforms are the following.

Firstly, $P_\nabla f(s, u)$ is defined by

$$P_\nabla f(s, u) = \int_{\mathbb{R}^n} f\left(x, \sum_{i=1}^n s_i |x_i - c_i|^{\alpha_i} + u\right) dx$$

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which is
\[ \mathcal{P}_\alpha f( s, u ) = \int_{\mathbb{R}^n} f \left( x + c_i, \sum_{i=1}^{n} s_i |x_i|^{\alpha_i} + u \right) dx. \]
\( \mathcal{P}_\alpha f( s, u ) \) is the integration of \( f \) on a hypersurface
\[ \Gamma_{\mathcal{P}}(\alpha; s, u) = \left\{ (x, y) \in \mathbb{R}^{n+1} : y = \sum_{i=1}^{n} s_i |x_i - c_i|^{\alpha_i} + u \right\}, \]
and \( dx \) is not the standard volume of \( \Gamma_{\mathcal{P}}(\alpha; s, u) \) induced by the Euclidean metric of \( \mathbb{R}^{n+1} \).
\( \Gamma_{\mathcal{P}}(2, \ldots, 2; s, u) \) is a paraboloid if \( s_i \neq 0 \) for all \( i = 1, \ldots, n \). In particular, when \( n = 1 \), \( \mathcal{P}_2 f( s, u ) \) is an integration over a parabola in \( \mathbb{R}^2 \), and is called the parabolic Radon transform of \( f \) in seismology. We assume that
\[ f(x_1, \ldots, x_i-1, -x_i + c_i, x_{i+1}, \ldots, x_n, y) = f(x_1, \ldots, x_i-1, x_i + c_i, x_{i+1}, \ldots, x_n, y) \] (1)
for \( i = 1, \ldots, n \), that is, \( f(-x + c, y) = f(x + c, y) \). If we split \( f(x) \) into the even and odd parts in some \( x_i \) with respect to the hyperplane \( x_i = c_i \) in \( \mathbb{R}^n \), then the contribution of the odd part to \( \mathcal{P}_{\alpha f} \) becomes 0, and the injectivity of \( \mathcal{P}_\alpha \) fails to hold.

Secondly, \( \mathcal{Q}_\alpha f( s, u ) \) is defined by
\[ \mathcal{Q}_\alpha f( s, u ) = \int_{\mathbb{R}^n} f \left( x, \sum_{i=1}^{n} s_i (x_i - c_i)|x_i - c_i|^{\alpha_i-1} + u \right) dx \]
which is
\[ \mathcal{Q}_\alpha f( s, u ) = \int_{\mathbb{R}^n} f \left( x + c, \sum_{i=1}^{n} s_i x_i |x_i|^{\alpha_i-1} + u \right) dx. \]
\( \mathcal{Q}_\alpha f( s, u ) \) is the integration of \( f \) on a hypersurface
\[ \Gamma_{\mathcal{Q}}(\alpha; s, u) = \left\{ (x, y) \in \mathbb{R}^{n+1} : y = \sum_{i=1}^{n} s_i (x_i - c_i)|x_i - c_i|^{\alpha_i-1} + u \right\} \]
for \( f \) and the measure is not the standard volume of \( \Gamma_{\mathcal{Q}}(\alpha; s, u) \) induced by the Euclidean metric of \( \mathbb{R}^{n+1} \). We do not need some symmetry conditions like (1) for \( \mathcal{Q}_\alpha \). Indeed there is no specific relationship between the values of \( y = \sum s_i x_i |x_i|^{\alpha_i-1} + u \) for \( x_i \) and \(-x_i\) with some \( i = 1, \ldots, n \).

Thirdly, \( \mathcal{R}_{\alpha,\beta} f( s, u ) \) is defined by
\[ \mathcal{R}_{\alpha,\beta} f( s, u ) = \int_{\sum s_i |x_i - c_i|^{\alpha_i} + u > 0} f \left( x, \left( \sum_{i=1}^{n} s_i |x_i - c_i|^{\alpha_i} + u \right)^{1/\beta} \right) \left( \sum_{i=1}^{n} s_i |x_i - c_i|^{\alpha_i} + u \right)^{1/\beta} dx \]
which is
\[ \mathcal{R}_{\alpha,\beta} f( s, u ) = \int_{\sum s_i |x_i|^{\alpha_i} + u > 0} f \left( x + c, \left( \sum_{i=1}^{n} s_i |x_i|^{\alpha_i} + u \right)^{1/\beta} \right) \left( \sum_{i=1}^{n} s_i |x_i|^{\alpha_i} + u \right)^{1/\beta} dx. \]
\( \mathcal{R}_{\alpha,\beta} f( s, u ) \) is the integration of \( f \) on a hypersurface
\[ \Gamma_{\mathcal{R}}(\alpha, \beta; s, u) = \left\{ (x, y) \in \mathbb{R}^{n+1} : |y|^{\beta} = \sum_{i=1}^{n} s_i |x_i - c_i|^{\alpha_i} + u \right\} \]
for \( f \) satisfying (1) and

\[
f(x + c, -y) = f(x + c, y).
\] (2)

The condition (1) is required for the injectivity of \( \mathcal{R}_{\alpha, \beta} \) in the same way as \( \mathcal{P}_\alpha \). We need the condition (2) to obtain the inversion. Our method of the proof of the inversion is the reduction to the standard Radon transform on \( \mathbb{R}^{n+1} \), and we can only deal with functions \( f(x, y) \) defined on the whole space \( \mathbb{R}^{n+1} \). Note that the measure is not the standard volume of \( \Gamma_{\mathcal{R}}(\alpha, \beta; s, u) \) induced by the Euclidean metric of \( \mathbb{R}^{n+1} \). To resolve the singularity at \( \sum s_i |x_i - c_i|^{\alpha_i} + u = 0 \), it is natural to assume that \( f(x, 0) = 0 \). \( \Gamma_{\mathcal{R}}(2, \ldots, 2; s, u) \) is a quadratic hypersurface like hyperboloids if \( s_1, \ldots, s_n \neq 0 \). In particular, when \( n = 1 \), \( \mathcal{R}_{2,2} f(s, u) \) is called the hyperbolic Radon transform of \( f \) in seismology.

Here we recall the background of our transforms. We start with the transform on \( \mathbb{R}^2 \), that is, the integration on the plane curves. In the early 1980s, Cormack introduced the Radon transform of a family of plane curves and studied the basic properties in his pioneering works [3] and [4]. More than a decade later, Denecker, van Overloop and Sommen in [5] studied the parabolic Radon transform without fixed axis, in particular, the support theorem, higher dimensional generalization and etc. In 2011 which is more than ten years later, Jollivet, Nguyen and Truong in [8] studied some properties of the parabolic Radon transform with fixed axis, which is the exact contour integration. Recently, Moon established the inversion of the parabolic Radon transform \( \mathcal{P}_2 \) and the inversion of the hyperbolic Radon transform \( \mathcal{R}_{2,2} \) respectively in his interesting paper [14]. He introduced some change of variables in \((x, y) \in \mathbb{R}^2\) so that the Radon transform of a family of plane curves became so-called the X-ray transform, that is, the Radon transform of a family of lines. More recently, replacing \( x^2 \) by some function \( \varphi(x) \) in the parabolic Radon transform, Ustaoglu developed Moon’s idea to try to obtain the inversion of more general Radon transforms on the plane in [16]. More recently, the author studied \( \mathcal{P}_\alpha, \mathcal{Q}_\alpha \) and \( \mathcal{R}_{\alpha, \beta} \) on \( \mathbb{R}^2 \), and obtained the inversion formulas for them in [2]. Those are the mathematical background of our transforms on the plane. For the scientific background of the parabolic Radon transform \( \mathcal{P}_2 \) and the hyperbolic transform \( \mathcal{R}_{2,2} \), see [6] and the introduction of [14] for both of them, [10, 11] for the parabolic Radon transform, and [1] for the hyperbolic Radon transform respectively.

Here we turn to the higher dimensional case \( n \geq 2 \). There are no mathematical results on our transforms so far. Unfortunately, however, our transforms with \( n \geq 2 \) have no scientific background at this time. Here we quote some works on the transforms which are integrations on hypersurfaces. Moon studied integral transforms over ellipsoids in [12] and [13]. This arises in synthetic aperture radar (SAR), ultrasound reflection tomography (URT) and radio tomography. Recently, transforms on cones have been intensively studied. This models Compton cameras, and is sometimes called cone transform or Compton transform. See, e.g., [9] and the references therein.

The aim of the present paper is to establish the inversion formulas for \( \mathcal{P}_\alpha, \mathcal{Q}_\alpha \) and \( \mathcal{R}_{\alpha, \beta} \). Here we introduce some function spaces to state our results. These function spaces consists of Schwartz functions on \( \mathbb{R}^{n+1} \) satisfying some symmetries and vanishing conditions. The symmetries are natural for our transforms, and the vanishing conditions are used for justifying the change of variables for the reduction to the standard transform. We denote the set of all Schwartz functions on \( \mathbb{R}^{n+1} \) by \( \mathcal{S}(\mathbb{R}^{n+1}) \).

**Definition 1** Fix \( c = (c_1, \ldots, c_n) \in \mathbb{R}^n \). Let \( m = (m_1, \ldots, m_n) \) be a multi-index of non-negative integers. We define function spaces \( \mathcal{S}_{c,m}(\mathbb{R}^{n+1}), \mathcal{S}_{c,m}^P(\mathbb{R}^{n+1}) \) and \( \mathcal{S}_{c,m}^R(\mathbb{R}^{n+1}) \) as follows.
(i) \( \mathcal{S}_{c,m}(\mathbb{R}^{n+1}) \) is the set of all \( f(x, y) \in \mathcal{S}(\mathbb{R}^{n+1}) \) satisfying the following vanishing conditions
\[
\left. \frac{\partial^k f}{\partial x_i^k} (x + c, y) \right|_{x_i=0} = 0, \quad k = 0, 1, \ldots, m_i, \ i = 1, \ldots, n.
\]
(ii) \( \mathcal{S}_{c,m}^P(\mathbb{R}^{n+1}) \) is the set of all \( f(x, y) \in \mathcal{S}_{c,m}(\mathbb{R}^{n+1}) \) satisfying the symmetry (1).
(iii) \( \mathcal{S}_{c,m}^R(\mathbb{R}^{n+1}) \) is the set of all \( f(x, y) \in \mathcal{S}_{c,m}^P(\mathbb{R}^{n+1}) \) satisfying the symmetry (2) and the vanishing condition \( f(x + c, 0) = 0 \).

Recall \( \alpha_1, \ldots, \alpha_n, \beta > 1 \). Throughout the present paper we assume that the vanishing order \( m_i \) at \( x_i = c_i \) satisfies \( m_i \geq \alpha_i - 2 \) for all \( i = 1, \ldots, n \). This condition guarantees the reduction to the standard Radon transform. Our main results are the following.

**Theorem 2** Let \( c = (c_1, \ldots, c_n) \in \mathbb{R}^n \), and let \( \alpha_1, \ldots, \alpha_n, \beta > 1 \). Suppose that \( m_i \) is a nonnegative integer satisfying \( m_i \geq \alpha_i - 2 \) for all \( i = 1, \ldots, n \).

(i) For any \( f \in \mathcal{S}_{c,m}^P(\mathbb{R}^{n+1}) \), if \( n \) is odd, then
\[
f(x, y) = \frac{4 \cdot (-1)^{(n-1)/2}}{(4\pi)^{n+1}} \left( \prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1} \right)
\times \int_{\mathbb{R}^n} \left( \text{pv} \int_{-\infty}^{\infty} \frac{\partial u^n \mathcal{P}_\alpha f(s, u)}{y - \sum_{i=1}^{n} s_i |x_i - c_i|^{\alpha_i} - u} \right) ds,
\]
and if \( n \) is even, then
\[
f(x, y) = \frac{(-1)^{n/2}}{(4\pi)^n} \left( \prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1} \right) \int_{\mathbb{R}^n} (\partial_u^n \mathcal{P}_\alpha f) \left( s, y - \sum_{i=1}^{n} s_i |x_i - c_i|^{\alpha_i} \right) ds.
\]

(ii) For any \( f \in \mathcal{S}_{c,m}(\mathbb{R}^{n+1}) \), if \( n \) is odd, then
\[
f(x, y) = \frac{2 \cdot (-1)^{(n-1)/2}}{(2\pi)^{n+1}} \left( \prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1} \right)
\times \int_{\mathbb{R}^n} \left( \text{pv} \int_{-\infty}^{\infty} \frac{\partial u^n \mathcal{Q}_\alpha f(s, u)}{y - \sum_{i=1}^{n} s_i (x_i - c_i)|x_i - c_i|^{\alpha_i - 1} - u} \right) ds,
\]
and if \( n \) is even, then
\[
f(x, y) = \frac{(-1)^{n/2}}{(2\pi)^n} \left( \prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1} \right)
\times \int_{\mathbb{R}^n} (\partial_u^n \mathcal{Q}_\alpha f) \left( s, y - \sum_{i=1}^{n} s_i (x_i - c_i)|x_i - c_i|^{\alpha_i - 1} \right) ds.
\]

(iii) For any \( f \in \mathcal{S}_{c,m}^R(\mathbb{R}^{n+1}) \), if \( n \) is odd, then
\[
f(x, y) = \frac{4 \cdot (-1)^{(n-1)/2}}{(4\pi)^{n+1}} \left( \prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1} \right) |y|
\times \int_{\mathbb{R}^n} \left( \text{pv} \int_{-\infty}^{\infty} \frac{\partial u^n \mathcal{R}_\alpha f(s, u)}{|y|^\beta - \sum_{i=1}^{n} s_i |x_i - c_i|^{\alpha_i} - u} \right) ds,
\]
and if \( n \) is even, then
\[
f(x, y) = \frac{(-1)^{n/2}}{(4\pi)^n} \left( \prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1} \right) |y|^{\alpha_i - 1} \times \int_{\mathbb{R}^n} \left( \partial^n_{\alpha_\beta} f \right) \left( s, |y|^{\beta} - \sum_{i=1}^{n} s_i |x_i - c_i|^{\alpha_i} \right) ds.
\] (8)

We develop the method in [2] and prove Theorem 2. The basic idea is the reduction to the standard Radon transform due to Moon in [14]. We begin with the basic facts on the standard Radon transform in Section 2. Section 3 is devoted to studying the coordinatization of the upper hemisphere by \( s \in \mathbb{R}^n \). Next, we prepare some lemmas related with vanishing conditions in Section 4. Finally, we prove Theorem 2 in Section 5.

2 The Standard Radon Transform

We recall the definition of the standard Radon transform on \( \mathbb{R}^{n+1} \) and the inversion formula.

Let \( S^n \) be the unit sphere in \( \mathbb{R}^n \) defined by
\[
S^n = \{ \omega = (\omega_1, \ldots, \omega_{n+1}) \in \mathbb{R}^{n+1} : |\omega|^2 = \omega_1^2 + \cdots + \omega_{n+1}^2 = 1 \},
\]
and let \( S^n_+ \) be the upper hemisphere
\[
S^n_+ = \{ \omega = (\omega_1, \ldots, \omega_{n+1}) \in S^n : \omega_{n+1} > 0 \}.
\]
Set
\[
H(\omega, t) = \{ \zeta \in \mathbb{R}^{n+1} : \langle \zeta, \omega \rangle = t \}, \quad \omega \in S^n, \ t \in \mathbb{R},
\]
where \( \langle \cdot, \cdot \rangle \) is the standard inner product of the Euclidean space. Note that \( H(-\omega, -t) = H(\omega, t) \) and
\[
H(\omega, t) = \{ \zeta + t\omega : \zeta \in H(\omega, 0) \}
\]
since \( \langle \zeta + t\omega, \omega \rangle = t \) for \( \zeta \in H(\omega, 0) \). \( H(\omega, t) \) is a hyperplane in \( \mathbb{R}^{n+1} \) which is perpendicular to \( \omega \) and is passing through \( t\omega \). In particular, \( H(\omega, 0) \) is the orthogonal complement of \( \{ \omega \} \) in \( \mathbb{R}^{n+1} \). The Radon transform of a function \( F(\zeta) \) is defined by
\[
\mathcal{R} F(\omega, t) = \int_{H(\omega, t)} F(\zeta) dm(\zeta) = \int_{H(\omega, 0)} F(t\omega + \zeta) dm(\zeta),
\]
where \( dm \) is the induced measure on the hyperplane from the Lebesgue measure on \( \mathbb{R}^{n+1} \).

Note that \( \mathcal{R} F(-\omega, -t) = \mathcal{R} F(\omega, t) \). In the present paper, we deal with integrations on the graph of a function \( \langle s, \xi \rangle + u \) of the variables \( \xi \in \mathbb{R}^n \) with some constants \( (s, u) \in \mathbb{R}^n \times \mathbb{R} \). It is easy to see that
\[
\int_{\mathbb{R}^n} F(\xi, \langle s, \xi \rangle + u) d\xi = \frac{1}{\sqrt{1 + |s|^2}} \mathcal{R} F \left( \frac{-s, 1}{\sqrt{1 + |s|^2}}, \frac{u}{\sqrt{1 + |s|^2}} \right)
\]
(9)
since
\[
\{ \langle \xi, \langle s, \xi \rangle + u \rangle : \xi \in \mathbb{R}^n \} = H \left( \frac{-s, 1}{\sqrt{1 + |s|^2}}, \frac{u}{\sqrt{1 + |s|^2}} \right), \quad dm = \sqrt{1 + |s|^2} d\xi.
\]

The inversion formula is as follows.
Theorem 3. For $F(\xi, \eta) \in \mathcal{S}(\mathbb{R}^{n+1})$, if $n$ is odd, then

$$F(\xi, \eta) = \frac{2 \cdot (-1)^{(n-1)/2}}{(2\pi)^{n+1}} \int_{S^d_+} \left( \text{pv} \int_{-\infty}^{\infty} \frac{\partial^n_t XF(\omega, t)}{((\xi, \eta), \omega) - t} \right) d\omega,$$

and if $n$ is even, then

$$F(\xi, \eta) = \frac{(-1)^{n/2}}{(2\pi)^{n}} \int_{S^d_+} (\partial^n_t XF)(\omega, ((\xi, \eta), \omega)) d\omega,$$

where $d\omega$ is the induced measure on $S^n$ from $\mathbb{R}^{n+1}$.

For Theorem 3, see, e.g., Corollary 2.6 and the remark below in page 33 of Palamodov’s textbook [15]. It is important to mention that Theorem 3 holds for smooth functions $F(\xi, \eta)$ satisfying $F(\xi, \eta) = O((1 + |\xi| + |\eta|)^{-d})$ with some $d > n$, compactly supported distributions, rapidly decaying Lebesgue measurable functions, and etc. See, e.g., [7] for the detail.

3 Coordinatization of Hemisphere

We introduce a coordinatization $s \in \mathbb{R}^n$ of the upper hemisphere $S^n_+$. The polar coordinates $(\theta_1, \ldots, \theta_n) \in (0, \pi)^n$ for the point $\omega \in S^n_+$ is given by

$$\omega_1 = \cos \theta_1, \quad \omega_i = \sin \theta_1 \cdots \sin \theta_{i-1} \cos \theta_i \quad (i = 1, \ldots, n), \quad \omega_{n+1} = \sin \theta_1 \cdots \sin \theta_n,$$

and the volume form $d\omega$ given by

$$d\omega = d\theta_1 \quad (n = 1), \quad d\omega = \left( \prod_{i=1}^{n-1} \sin^{n-i} \theta_i \right) d\theta_1 \cdots d\theta_n \quad (n = 2, 3, 4, \ldots)$$

is well-known.

Since $\omega_1^2 + \cdots + \omega_{n+1}^2 = 1$, we introduce new coordinates $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ of $S^n_+$ defined by

$$\omega = \frac{(-s_1, 1)}{\sqrt{1 + |s|^2}}, \quad \text{i.e.,} \quad (\omega_1, \ldots, \omega_n, \omega_{n+1}) = \frac{(-s_1, \ldots, -s_n, 1)}{\sqrt{1 + |s|^2}}. \quad (10)$$

Note that $s$ moves in $\mathbb{R}^n$ if and only if $\theta$ moves $(0, \pi)^n$. Moreover we have

$$s_i = -\frac{\cot \theta_i}{\sin \theta_i \cdots \sin \theta_{n+1}} \quad (i = 1, \ldots, n - 1), \quad s_n = -\cot \theta_n.$$

Elementary calculus yields

$$\frac{\partial s_i}{\partial \theta_j} = 0 \quad (i = 2, \ldots, n, \ j < i),$$

$$\frac{\partial s_i}{\partial \theta_i} = \frac{1}{\sin^2 \theta_i \cdot \sin \theta_{i+1} \cdots \sin \theta_n} \quad (i = 1, \ldots, n - 1), \quad \frac{\partial s_n}{\partial \theta_n} = \frac{1}{\sin^2 \theta_n}.$$
Hence, we have

\[
\frac{\partial (s_1, \ldots, s_n)}{\partial (\theta_1, \ldots, \theta_n)} = \det \begin{bmatrix}
\frac{\partial s_1}{\partial \theta_1} & \cdots & \ast \\
0 & \ddots & \vdots \\
\vdots & \ddots & \ast \\
0 & \cdots & 0 \frac{\partial s_n}{\partial \theta_n}
\end{bmatrix} = \prod_{i=1}^{n} \frac{\partial s_i}{\partial \theta_i} = \left( \prod_{i=1}^{n} \sin^{i+1} \theta_i \right)^{-1},
\]

and

\[
d\omega = \left( \prod_{j=1}^{n-1} \sin^{n-j} \theta_j \right) \left| \frac{\partial (s_1, \ldots, s_n)}{\partial (\theta_1, \ldots, \theta_n)} \right|^{-1} ds
\]

\[
= \left( \prod_{j=1}^{n-1} \sin^{n-j} \theta_j \cdot \prod_{i=1}^{n} \sin^{i+1} \theta_i \right) ds
\]

\[
= \left( \prod_{i=1}^{n} \sin \theta_i \right)^{n+1} ds = \frac{1}{(1 + |s|^2)^{(n+1)/2}} d\omega.
\]

(11)

We also use change of variables

\[
(\omega, t) = \left( \frac{(-s, 1)}{\sqrt{1 + |s|^2}}, \frac{u}{\sqrt{1 + |s|^2}} \right).
\]

(12)

In this case we have

\[
\frac{\partial (\theta_1, \ldots, \theta_n, t)}{\partial (s_1, \ldots, s_n, u)} = \det \begin{bmatrix}
\frac{\partial \theta_i}{\partial s_j} 0 \\
0 & \frac{\partial t}{\partial u}
\end{bmatrix} = \frac{\partial (\theta_1, \ldots, \theta_n)}{\partial (s_1, \ldots, s_n)} \cdot \frac{\partial t}{\partial u}
\]

\[
= \left( \prod_{i=1}^{n} \sin^{i+1} \theta_i \right) \cdot \frac{1}{\sqrt{1 + |s|^2}},
\]

and

\[
d\omega dt = \frac{1}{(1 + |s|^2)^{(n+2)/2}} d\omega du.
\]

(13)

4 Vanishing Conditions

We prepare some lemmas related to vanishing conditions and symmetries. These lemmas are used for reducing our transforms to the standard Radon transform in the next section. We need the following lemma to make full use of the vanishing conditions.
Lemma 4

(i) For \( f(x, y) \in \mathcal{S}_{c,m}(\mathbb{R}^{n+1}) \),

\[
f(x + c, y) = \frac{x^{m_2+1} \cdots x^{m_n+1}}{m_2! \cdots m_n!} \int_0^1 \cdots \int_0^1 (1 - t_2)^{m_2} \cdots (1 - t_n)^{m_n} \]
\[
\times \frac{\partial^{m_2+\cdots+m_n+n-1} f}{\partial x_2^{m_2+1} \cdots \partial x_n^{m_n+1}} (x_1 + c_1, t_2 x_2 + c_2, \ldots, t_n x_n + c_n, y) dt_2 \cdots dt_n, 
\]

\[
= \frac{x_1^{m_1+1} \cdots x_n^{m_n+1}}{m_1! \cdots m_n!} \int_0^1 \cdots \int_0^1 (1 - t_1)^{m_1} \cdots (1 - t_n)^{m_n} \]
\[
\times \frac{\partial^{m_1+\cdots+m_n+n} f}{\partial x_1^{m_1+1} \cdots \partial x_n^{m_n+1}} (t_1 x_1 + c_1, \ldots, t_n x_n + c_n, y) dt_1 \cdots dt_n. 
\]

(ii) For \( f(x, y) \in \mathcal{P}_{c,m}(\mathbb{R}^{n+1}) \),

\[
f(x + c, y) = \frac{x_1^{m_1+1} \cdots x_n^{m_n+1} y}{m_1! \cdots m_n!} \int_0^1 \cdots \int_0^1 (1 - t_2)^{m_2} \cdots (1 - t_n)^{m_n} \]
\[
\times \frac{\partial^{m_2+\cdots+m_n+n} f}{\partial x_2^{m_2+1} \cdots \partial x_n^{m_n+1} \partial y} (x_1 + c_1, t_2 x_2 + c_2, \ldots, t_n x_n + c_n, \tau y) dt_2 \cdots dt_n d\tau, 
\]

\[
= \frac{x_1^{m_1+1} \cdots x_n^{m_n+1} y}{m_1! \cdots m_n!} \int_0^1 \cdots \int_0^1 (1 - t_1)^{m_1} \cdots (1 - t_n)^{m_n} \]
\[
\times \frac{\partial^{m_1+\cdots+m_n+n+1} f}{\partial x_1^{m_1+1} \cdots \partial x_n^{m_n+1} \partial y} (t_1 x_1 + c_1, \ldots, t_n x_n + c_n, \tau y) dt_1 \cdots dt_n d\tau. 
\]

We can replace the role of \( x_1 \) in (14) and (16) by the other \( x_i, i = 2, \ldots, n \).

Proof Here we prove (14) and (15) for \( n = 2 \). The other parts can be proved in the same way. We omit the detail. Suppose \( f(x_1, x_2, y) \in \mathcal{P}_{c,m}(\mathbb{R}^3) \). Since

\[
\frac{\partial^k f}{\partial x_2^k} (x_1 + c_1, c_2, y) = 0, \quad k = 0, 1, \ldots, m_2.
\]

Taylor’s formula gives

\[
f(x_1 + c_1, x_2 + c_2, y) = \sum_{k=0}^{m_2} \frac{x_2^k}{k!} \frac{\partial^k f}{\partial x_2^k} (x_1 + c_1, c_2, y)
\]
\[
+ \frac{x_2^{m_2+1}}{m_2!} \int_0^1 (1 - t_2)^{m_2} \frac{\partial^{m_2+1} f}{\partial x_2^{m_2+1}} (x_1 + c_1, t_2 x_2 + c_2, y) dt_2
\]
\[
= \frac{x_2^{m_2+1}}{m_2!} \int_0^1 (1 - t_2)^{m_2} \frac{\partial^{m_2+1} f}{\partial x_2^{m_2+1}} (x_1 + c_1, t_2 x_2 + c_2, y) dt_2. 
\]
This is (14) for \( n = 2 \). Since

\[
\frac{\partial^k f}{\partial x_1^k} (c_1, x_2 + c_2, y) = 0, \quad k = 0, 1, \ldots, m_1,
\]

\[
\frac{\partial^k+l f}{\partial x_1^k \partial x_2^l} (c_1, x_2 + c_2, y) = 0, \quad k = 0, 1, \ldots, m_1, \quad l = 0, 1, 2, 3, \ldots.
\]

If we use this with \( l = m_2 + 1 \), we have

\[
\frac{\partial^{m_2+1} f}{\partial x_2^{m_2+1}} (x_1 + c_1, t_2 x_2 + c_2, y)
\]

\[
= \frac{x_1^{m_1+1}}{m_1!} \int_0^1 \int_0^1 (1 - t_1)^{m_1} (1 - t_2)^{m_2}
\]

\[
\frac{\partial^{m_1+m_2+2} f}{\partial x_1^{m_1+1} \partial x_2^{m_2+1}} (t_1 x_1 + c_1, t_2 x_2 + c_2, y) dt_1 dt_2,
\]

which is (15) for \( n = 2 \).

Now we introduce functions defined by \( f \), which is used for reducing our transforms to the standard Radon transform. For \( f(x, y) \), set

\[
F^P_{\alpha} (\xi, \eta) = \begin{cases} 
2^n f \left( \xi_1^{1/\alpha_1} + \xi_2^{1/\alpha_2} + \cdots + \xi_m^{1/\alpha_m} + c_n, \eta \right) / \alpha_1 \cdots \alpha_m \xi_1^{(\alpha_1-1)/\alpha_1} \cdots \xi_m^{(\alpha_m-1)/\alpha_m} & (\xi_1, \ldots, \xi_n > 0), \\
0 & \text{(otherwise)},
\end{cases}
\]

\[
F^Q_{\alpha} (\xi, \eta) = f \left( \xi_1 |\xi_1|^{-1+1/\alpha_1} + \cdots + \xi_n |\xi_n|^{-1+1/\alpha_n} + c_n, \eta \right) / \alpha_1 \cdots \alpha_n \xi_1^{(\alpha_1-1)/\alpha_1} \cdots \xi_n^{(\alpha_n-1)/\alpha_n} (\xi_1, \ldots, \xi_n \neq 0),
\]

and

\[
F^R_{\alpha, \beta} (\xi, \eta) = \begin{cases} 
2^n f \left( \xi_1^{1/\alpha_1} + \cdots + \xi_n^{1/\alpha_n} + c_n, \eta^{1/\beta} \right) / \alpha_1 \cdots \alpha_n \xi_1^{(\alpha_1-1)/\alpha_1} \cdots \xi_n^{(\alpha_n-1)/\alpha_n} \eta^{1/\beta} & (\xi_1, \ldots, \xi_n, \eta > 0), \\
0 & \text{(otherwise)}.
\end{cases}
\]

Note that \( F^P_{\alpha} (\xi, \eta) = 2^n F^Q_{\alpha} (\xi, \eta) \) for \( \xi_1, \ldots, \xi_n > 0 \).

**Lemma 5** Suppose that \( \alpha_1 \geq m_i - 2 \) for all \( i = 1, \ldots, n \).

(i) For \( f(x, y) \in \mathcal{S}^P_{c,m} (\mathbb{R}^{n+1}) \),

\[
F^P_{\alpha} (\xi, \eta) = 2^n \left( \prod_{i=1}^n \alpha_i |x_i| - c_i \right) F^P_{\alpha} (|x_1 - c_1|^{\alpha_1}, \ldots, |x_n - c_n|^{\alpha_n}, y), \quad (19)
\]

and for any \( N > 0 \), there exists a constant \( C_N > 0 \) such that

\[
|F^P_{\alpha} (\xi, \eta)| \leq C_N (1 + |\xi| + |\eta|)^{-N}. \quad (20)
\]

Moreover, when \( \xi \in (0, \infty)^n \) tends to the boundary, \( F^P_{\alpha} (\xi, (s, \xi) + u) \) has a finite limit for any \( (s, u) \in \mathbb{R}^n \times \mathbb{R} \).
(ii) For a function $f(x, y) \in \mathcal{F}_{c,m}(\mathbb{R}^{n+1})$,

$$f(x, y) = \left(\prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1}\right)\times F^Q_{\alpha}(|x_1 - c_1|^{\alpha_1 - 1}, \ldots, |x_n - c_n|^{\alpha_n - 1}, y),$$

and for any $N > 0$, there exists a constant $C_N > 0$ such that

$$|F^Q_{\alpha}(\xi, \eta)| \leq C_N (1 + |\xi| + |\eta|)^{-N}.$$  \hspace{1cm} (22)

Moreover, when $\sigma \xi \in (0, \infty)^n$ tends to the boundary, $F^P_{\alpha}(\xi, \sigma \xi + u)$ has a finite limit for any $(s, u) \in \mathbb{R}^n \times \mathbb{R}$. Here, $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{\pm 1\}^n$ and $\sigma \xi = (\sigma_1 \xi_1, \ldots, \sigma_n \xi_n)$.

(iii) For $f(x, y) \in \mathcal{F}_{c,m}(\mathbb{R}^{n+1})$,

$$f(x, y) = \frac{1}{2^n} \left(\prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1}\right) |y| \cdot F^R_{\alpha, \beta}(|x_1 - c_1|^{\alpha_1}, \ldots, |x_n - c_n|^{\alpha_n}, |y|^\beta),$$

and for any $N > 0$, there exists a constant $C_N > 0$ such that

$$|F^R_{\alpha, \beta}(\xi, \eta)| \leq C_N (1 + |\xi| + |\eta|)^{-N}.$$  \hspace{1cm} (24)

Moreover, when $\xi \in (0, \infty)^n$ tends to the boundary, $F^R_{\alpha, \beta}(\xi, \xi + u)$ has a finite limit for any $(s, u) \in \mathbb{R}^n \times \mathbb{R}$.

Proof We prove only (i) here. (ii) and (iii) can be proved in the same way as (i). We omit the detail.

Suppose that $f \in \mathcal{F}^P_{c,m}(\mathbb{R}^{n+1})$. Direct computation gives (19) immediately. Set $C_0 = 2^n/\alpha_1 \cdots \alpha_n$ for short. It suffices to consider $F^P_{\alpha}(\xi, \eta)$ only in $\xi \in (0, \infty)^n$. Note that (15) implies that

$$F^P_{\alpha}(\xi, \eta) = \frac{C_0}{m_1! \cdots m_n!} \prod_{i=1}^{n} \xi_i^{(m_i + 2 - \alpha_i)/\alpha_i} \int_0^1 \cdots \int_0^1 (1 - t_1)^{m_1} \cdots (1 - t_n)^{m_n}$$

$$\times \frac{\partial^{m_1 + \cdots + m_n + n} f}{\partial x_1^{m_1 + 1} \cdots \partial x_n^{m_n + 1}} \left(t_1^{1/\alpha_1} + c_1, \ldots, t_n^{1/\alpha_n} + c_n, \eta\right) \, dt_1 \cdots dt_n. \hspace{1cm} (25)$$

This shows that $F^P_{\alpha}(\xi, \eta)$ is bounded for $|\xi| + |\eta| < 1$ since $m_i + 2 - \alpha_i \geq 0$ for $i = 1, \ldots, n$. When $|\xi| + |\eta| \geq 1$, it follow that $|\eta| \geq (|\xi| + |\eta|)/(\sqrt{n} + 1)$ or $\xi_i \geq (|\xi| + |\eta|)/(\sqrt{n} + 1)$ for some $i = 1, \ldots, n$, say $i = 1$. When $|\eta| \geq (|\xi| + |\eta|)/(\sqrt{n} + 1) \geq 1/(\sqrt{n} + 1)$, (25) shows (20). Note that (14) implies that

$$F^P_{\alpha}(\xi, \eta) = \frac{C_0}{\xi_1^{(\alpha_1 - 1)/\alpha_1} m_1! \cdots m_n!} \prod_{i=2}^{n} \xi_i^{(m_i + 2 - \alpha_i)/\alpha_i} \int_0^1 \cdots \int_0^1 (1 - t_2)^{m_2} \cdots (1 - t_n)^{m_n}$$

$$\times \frac{\partial^{m_2 + \cdots + m_n + n - 1} f}{\partial x_1^{m_1 + 1} \cdots \partial x_n^{m_n + 1}} \left(\xi_1^{1/\alpha_1} + c_1, t_2^{1/\alpha_2} + c_2, \ldots, t_n^{1/\alpha_n} + c_n, \eta\right) \, dt_2 \cdots dt_n. \hspace{1cm} (26)$$
When $\xi_1 \geq (|\xi| + |\eta|)/(|\sqrt{n} + 1|) \geq 1/(|\sqrt{n} + 1|)$, (26) shows (20). Combining the above all, we prove (20). We can prove the existence and the finiteness of the limit of $F^P_\alpha(\xi, (s, \xi) + u)$ at the boundary of $(0, \infty)^n$ by using (25) and (26). We omit the detail.

5 Proof of Main Theorem

We begin with computing $P_\alpha f(s, u)$, $Q_\alpha f(s, u)$ and $R_{\alpha, \beta} f(s, u)$.

Lemma 6

(i) For $f(x, y) \in \mathcal{S}^P_{c, m} (\mathbb{R}^{n+1})$,

$$P_\alpha f(s, u) = \frac{1}{\sqrt{1 + |s|^2}} \mathcal{X} F^P_\alpha \left( \frac{(-s, 1)}{\sqrt{1 + |s|^2}}, \frac{u}{\sqrt{1 + |s|^2}} \right),$$

$$\partial_u^n P_\alpha f(s, u) = \frac{1}{(1 + |s|^2)(n+1)/2} (\partial_u^n \mathcal{X} F^P_\alpha) \left( \frac{(-s, 1)}{\sqrt{1 + |s|^2}}, \frac{u}{\sqrt{1 + |s|^2}} \right).$$

(ii) For $f(x, y) \in \mathcal{S}_{c, m} (\mathbb{R}^{n+1})$,

$$Q_\alpha f(s, u) = \frac{1}{\sqrt{1 + |s|^2}} \mathcal{X} F^Q_\alpha \left( \frac{(-s, 1)}{\sqrt{1 + |s|^2}}, \frac{u}{\sqrt{1 + |s|^2}} \right),$$

$$\partial_u^n Q_\alpha f(s, u) = \frac{1}{(1 + |s|^2)(n+1)/2} (\partial_u^n \mathcal{X} F^Q_\alpha) \left( \frac{(-s, 1)}{\sqrt{1 + |s|^2}}, \frac{u}{\sqrt{1 + |s|^2}} \right).$$

(iii) For $f(x, y) \in \mathcal{S}^R_{c, m} (\mathbb{R}^{n+1})$,

$$R_{\alpha, \beta} f(s, u) = \frac{1}{\sqrt{1 + |s|^2}} \mathcal{X} F^R_{\alpha, \beta} \left( \frac{(-s, 1)}{\sqrt{1 + |s|^2}}, \frac{u}{\sqrt{1 + |s|^2}} \right),$$

$$\partial_u^n R_{\alpha, \beta} f(s, u) = \frac{1}{(1 + |s|^2)(n+1)/2} (\partial_u^n \mathcal{X} F^R_{\alpha, \beta}) \left( \frac{(-s, 1)}{\sqrt{1 + |s|^2}}, \frac{u}{\sqrt{1 + |s|^2}} \right).$$

Proof Firstly, we prove (i). It suffices to prove (27). Using the symmetry (1), we have

$$P_\alpha f(s, u) = 2^n \int_{(0, \infty)^n} f \left( x + c, \sum_{i=1}^n s_i x_i^{\alpha_i} + u \right) dx.$$ 

We use the change of variables $x_i = \xi_i^{1/\alpha_i}$, $i = 1, \ldots, n$. Note that

$$\frac{dx_i}{d\xi_i} = \frac{1}{\alpha_i} \cdot \frac{1}{\xi_i^{(\alpha_i-1)/\alpha_i}}, \quad x_i^{\alpha_i} = \xi_i.$$
By using this and (9), we deduce that

\[
\mathcal{P}_\alpha f(s, u) = \int_{(0, \infty)^n} \frac{2^n f(\xi_1^{\alpha_1} + c_1, \ldots, \xi_n^{\alpha_n} + c_n, \langle s, \xi \rangle + u)}{\alpha_1 \cdots \alpha_n \xi_1^{(\alpha_1-1)/\alpha_1} \cdots \xi_n^{(\alpha_n-1)/\alpha_n}} \, d\xi
\]

\[
= \int_{(0, \infty)^n} F_{\alpha}^P(\xi, \langle s, \xi \rangle + u) \, d\xi
\]

\[
= \int_{\mathbb{R}^n} F_{\alpha}^P(\xi, \langle s, \xi \rangle + u) \, d\xi
\]

\[
= \frac{1}{\sqrt{1 + |s|^2}} \mathcal{X}F_{\alpha}^P \left( \frac{(-s, 1)}{\sqrt{1 + |s|^2}}, \frac{u}{\sqrt{1 + |s|^2}} \right),
\]

which is (27).

(ii) can be proved in the same way as (i). We omit the detail.

Secondly, we prove (iii). It suffices to prove (31). Using the symmetry (1), we have

\[
\mathcal{R}_{\alpha, \beta} f(s, u) = 2^n \int_{x \in (0, \infty)^n} \frac{f(x + c, (\sum s_i x_i^{\alpha_i} + u)^{1/\beta})}{(\sum s_i x_i^{\alpha_i} + u)^{1/\beta}} \, dx.
\]

By using the change of variables \(x_i = \xi_i^{1/\alpha_i}, i = 1, \ldots, n\), and (9), we deduce that

\[
\mathcal{R}_{\alpha, \beta} f(s, u) = \int_{\xi \in (0, \infty)^n} \frac{2^n f(\xi_1^{1/\alpha_1} + c_1, \ldots, \xi_n^{1/\alpha_n} + c_n, \langle s, \xi \rangle + u)^{1/\beta}}{\alpha_1 \cdots \alpha_n \xi_1^{(\alpha_1-1)/\alpha_1} \cdots \xi_n^{(\alpha_n-1)/\alpha_n} \cdot \langle s, \xi \rangle + u)^{1/\beta}} \, d\xi
\]

\[
= \int_{(0, \infty)^n} F_{\alpha, \beta}^R(\xi, \langle s, \xi \rangle + u) \, d\xi
\]

\[
= \int_{\mathbb{R}^n} F_{\alpha, \beta}^R(\xi, \langle s, \xi \rangle + u) \, d\xi
\]

\[
= \frac{1}{\sqrt{1 + |s|^2}} \mathcal{X}F_{\alpha, \beta}^R \left( \frac{(-s, 1)}{\sqrt{1 + |s|^2}}, \frac{u}{\sqrt{1 + |s|^2}} \right),
\]

which is (28). \qed

Finally we prove Theorem 2.

Proof of Theorem 2 Firstly, we prove (i). Suppose that \(f(x, y) \in \mathcal{S}_{c, m}^P(\mathbb{R}^{n+1})\), and \(m_i \geq \alpha_i - 2\) for all \(i = 1, \ldots, n\). When \(n\) is odd, by using the identity (19), the inversion formula of
the standard Radon transform Theorem 3, the change of the variables (12), and the identity (28) in order, we deduce that

\[ f(x, y) = \frac{1}{2^n} \left( \prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1} \right) F^{p}_{\alpha} (|x_1 - c_1|^{\alpha_1}, \ldots, |x_n - c_n|^{\alpha_n}, y) \]

\[ = \frac{1}{2^n} \left( \prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1} \right) \]

\[ \times \int_{\mathbb{R}^n} (\partial_t^{n} \mathcal{X} F^p_{\alpha} (\omega, t)) \frac{\omega_{n+1} y - t}{(1 + |\omega|^2)^{n+1/2}} d\omega \]

\[ = \frac{4 \cdot (-1)^{(n-1)/2}}{(4\pi)^{n+1}} \left( \prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1} \right) \]

\[ \times \int_{\mathbb{R}^n} \left( \partial_u^{n} \mathcal{P}_\alpha f (s, u) \right) \frac{y - \sum s_i |x_i - c_i|^{\alpha_i}}{(1 + |s|^2)^{n/2}} du \] ds.

which is (3).

When \( n \) is even, by using the identity (19), the inversion formula of the standard Radon transform Theorem 3, the change of the variables (10), and the identity (28) in order, we deduce that

\[ f(x, y) = \frac{1}{2^n} \left( \prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1} \right) F^{p}_{\alpha} (|x_1 - c_1|^{\alpha_1}, \ldots, |x_n - c_n|^{\alpha_n}, y) \]

\[ = \frac{1}{2^n} \left( \prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1} \right) \]

\[ \times \int_{\mathbb{R}^n} (\partial_t^{n} \mathcal{X} F^p_{\alpha} (\omega, t)) \frac{\omega_{n+1} y - t}{(1 + |\omega|^2)^{n+1/2}} d\omega \]

\[ = \frac{(-1)^{n/2}}{(4\pi)^{n}} \left( \prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1} \right) \]

\[ \times \int_{\mathbb{R}^n} \frac{1}{(1 + |s|^2)^{(n+1)/2}} \cdot (\partial_u^{n} \mathcal{P}_\alpha f (s, u)) \frac{(-s, 1)}{\sqrt{1 + |s|^2}}, \frac{y - \sum s_i |x_i - c_i|^{\alpha_i}}{\sqrt{1 + |s|^2}} ds \]

\[ = \frac{(-1)^{n/2}}{(4\pi)^{n}} \left( \prod_{i=1}^{n} \alpha_i |x_i - c_i|^{\alpha_i - 1} \right) \]

\[ \times \int_{\mathbb{R}^n} (\partial_u^{n} \mathcal{P}_\alpha f) (s, y - \sum s_i |x_i - c_i|^{\alpha_i}) ds, \]
which is (4).

Secondly, we prove (ii). Suppose that $f(x, y) \in \mathcal{S}_{c,m}(\mathbb{R}^{n+1})$, and $m_i \geq \alpha_i - 2$ for all $i = 1, \ldots, n$. When $n$ is odd, applying the inversion formula of the standard Radon transform Theorem 3, the change of the variables (12) and the identity (30) in order, we obtain (5) in the exactly same way as (3). When $n$ is even, applying the inversion formula of the standard Radon transform Theorem 3, the change of the variables (10) and the identity (30) in order, we obtain (6) in the exactly same way as (4). We omit the detail.

Finally, we prove (iii). Suppose that $f(x, y) \in \mathcal{S}_{c,m}(\mathbb{R}^{n+1})$, and $m_i \geq \alpha_i - 2$ for all $i = 1, \ldots, n$. When $n$ is odd, applying the inversion formula of the standard Radon transform Theorem 3, the change of the variables (12) and the identity (32) in order, we obtain (7) in the exactly same way as (3). When $n$ is even, applying the inversion formula of the standard Radon transform Theorem 3, the change of the variables (10) and the identity (32) in order, we obtain (8) in the exactly same way as (4). We omit the detail.

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References

1. Bickel, S.H.: Focusing aspects of the hyperbolic Radon transform. Geophysics 65, 652–655 (2000)
2. Chihara, H.: Inversion of seismic-type Radon transforms on the plane. arXiv:1910.02645 (2019)
3. Cormack, A.M.: The Radon transform on a family of curves in the plane. Proc. Amer. Math. Soc. 83, 325–330 (1981)
4. Cormack, A.M.: The Radon transform on a family of curves in the plane II. Proc. Amer. Math. Soc. 86, 293–298 (1982)
5. Denecker, K., van Overloop, J., Sommen, F.: The general quadratic Radon transform. Inverse Probl. 14, 615–633 (1998)
6. Hampson, D.: Inverse velocity stacking for multiple elimination. J. Can. Soc. Explor. Geophys. 22, 44–55 (1986)
7. Helgason, S.: Integral Geometry and Radon Transforms. Springer, New York (2011)
8. Jollivet, A., Nguyen, M.K., Truong, T.T.: Properties and inversion of a new radon transform on parabolas with fixed axis direction in $\mathbb{R}^2$. Manuscript (2011)
9. Kuchment, P., Terzioglu, F.: Three-dimensional image reconstruction from Compton camera data. SIAM J. Imaging Sci. 9, 1708–1725 (2016)
10. Maeland, E.: Focusing aspects of the parabolic Radon transform. Geophysics 63, 1708–1715 (1998)
11. Maeland, E.: An overlooked aspect of the parabolic Radon transform. Geophysics 65, 1326–1329 (2000)
12. Moon, S.: On the determination of a function from an elliptical Radon transform. J. Math. Anal. Appl. 416, 724–734 (2014)
13. Moon, S., Heo, J.: Inversion of the elliptical Radon transform arising in migration imaging using the regular Radon transform. J. Math. Anal. Appl. 436, 138–148 (2016)
14. Moon, S.: Inversion of the seismic parabolic Radon transform and the seismic hyperbolic Radon transform. Inverse Probl. Sci. Eng. 24, 317–327 (2016)
15. Palamodov, V.: Reconstructive Integral Geometry. Birkhäuser, Basel (2004)
16. Ustaoglu, Z.: On the inversion of a generalized Radon transform of seismic type. J. Math. Anal. Appl. 453, 287–303 (2017)

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