Representations of the vertex algebra $\mathcal{W}_{1+\infty}$ with a negative integer central charge

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Abstract

Let $\mathcal{D}$ be the Lie algebra of regular differential operators on $\mathbb{C}\setminus\{0\}$, and $\hat{\mathcal{D}} = \mathcal{D} + \mathbb{C}C$ be the central extension of $\mathcal{D}$. Let $W_{1+\infty,-N}$ be the vertex algebra associated to the irreducible vacuum $\hat{\mathcal{D}}$–module with the central charge $c = -N$. We show that $W_{1+\infty,-N}$ is a subalgebra of the Heisenberg vertex algebra $M(1)$ with $2N$ generators, and construct $2N$–dimensional family of irreducible $W_{1+\infty,-N}$–modules. Considering these modules as $\mathcal{D}$–modules, we identify the corresponding highest weights.

1 Introduction

Let $\mathcal{D}$ be the Lie algebra of regular differential operators on $\mathbb{C}\setminus\{0\}$, and $\hat{\mathcal{D}} = \mathcal{D} + \mathbb{C}C$ be the central extension of $\mathcal{D}$. In the representation theory of the Lie algebra $\hat{\mathcal{D}}$ the most important are the irreducible quasi-finite highest weight modules. These modules were classified by V. Kac and A. Radul in [KR1]. In [FKRW] was shown that the language of vertex algebras is very useful in $\hat{\mathcal{D}}$–module theory. In particular, on the irreducible vacuum $\hat{\mathcal{D}}$–module $L(0, c, \hat{\mathcal{D}})$ exists a natural structure of a simple vertex algebra (cf. [FKRW], [K]). This vertex algebra is usually denoted by $W_{1+\infty,c}$. The results from [KR1] give that the representation theory of the vertex algebra $W_{1+\infty,c}$ is nontrivial only in the case $c \in \mathbb{Z}$. When $N \in \mathbb{N}$, the irreducible modules for $W_{1+\infty,N}$ were classified in [FKRW].

The structure of the vertex algebra $W_{1+\infty,-N}$ is much complicated. The representation theory of $W_{1+\infty,-N}$ was began by Kac and Radul in [KR2]. They realized $W_{1+\infty,-N}$ as a vertex subalgebra of the vertex algebra $V_N$ constructed from Weyl algebra $W_N$ (we recall this result in the Section 3). They also constructed a large class of irreducible $W_{1+\infty,-N}$–modules. The next step in this direction was made by Wang in [W1], [W2]. He considered the case $c = -1$, and proved that $W_{1+\infty,-1}$ is isomorphic to tensor product $W_{3,-2} \otimes H_0$, where $W_{3,-2}$ is a vertex algebra associated with $W_3$–algebra with the central charge $-2$ and $H_0$ is a Heisenberg vertex algebra. He also classified all irreducible modules for $W_{3,-2}$ and $W_{1+\infty,-1}$. The representations obtained in [W2] weren’t identified as a highest weight $\hat{\mathcal{D}}$–modules.
In the present paper we will construct $2N$-dimensional family of irreducible $W_{1+\infty,-N}$-modules. Our family includes the modules constructed in [KR2], and also coincides with the modules from [W2] in the case $c = -1$.

Let us explain our result in more details. We consider lattice vertex superalgebra $V_L$ (cf. 4), and for suitably chosen lattice $L$ we show that $V_N$ and $W_{1+\infty,-N}$ are vertex subalgebras of $V_L$ (cf. Section 5). This fact is in physical literature known as a bosonization of $\beta\gamma$ system (see [FMS], [W1]). The embedding of $W_{1+\infty,-N}$ into vertex superalgebra $V_L$ will imply that $W_{1+\infty,-N}$ can be realized as a vertex subalgebra of the Heisenberg vertex algebra $M(1)$ with $2N$-generators. We explicitly identify the generators of $W_{1+\infty,-N}$ in terms of Schur polynomials. Considering $M(1)$-modules $M(1,\lambda)$ we obtain $W_{1+\infty,-N}$-modules $V(\lambda,-N)$ as irreducible subquotients of $M(1,\lambda)$ (see Section 5). Considering $V(\lambda,-N)$ as a $\hat{D}$-module, we identify its highest weights. As a result we get that all modules $V(\lambda,-N)$ are quasi-finite $\hat{D}$-modules.

2 Vertex superalgebras

In this section we recall the definition of vertex (super)algebra and a few basic formulas. For more details about vertex (super)algebras its representations, and representation theory of certain examples of vertex (super)algebras see [3], [DL], [FLM], [K], [Li], [A].

Let $V$ be a vector space. A field is a series of the form

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1},$$

where $a_n \in \text{End}V$ are such that for each $v \in V$ one has: $a_nv = 0$ for $n \gg 0$. Here $z$ is a formal indeterminate.

For a $\mathbb{Z}_2$-graded vector space $W = W^{\text{even}} + W^{\text{odd}}$ we write $|u| \in \mathbb{Z}_2$, a degree of $u$, only for homogeneous elements: $|u| = 0$ for an even element $u \in W^0$ and $|u| = 1$ for an odd element $u \in W^1$. For any two $\mathbb{Z}_2$-homogeneous elements $u$ and $v$ we define $\epsilon_{u,v} = (-1)^{|u||v|} \in \mathbb{Z}$.

**Definition 2.1** A vertex superalgebra is a quadruple $(V, 1, D, Y)$, where $V = V^{\text{even}} \oplus V^{\text{odd}}$ is a $\mathbb{Z}_2$-graded vector space, $D$ is a $\mathbb{Z}_2$-endomorphism of $V$, 1 is a specified vector called the vacuum of $V$, and $Y$ is a linear map

$$Y(\cdot, z) : V \to (\text{End}V)[[z, z^{-1}]];$$

$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \quad (\text{where } a_n \in \text{End}V)$$

such that

(V1) For any $a, b \in V$, $a_nb = 0$ for $n$ sufficiently large;
(V2) \([D, Y(a, z)] = Y(D(a), z) = \partial_z Y(a, z)\) for any \(a \in V\);

(V3) \(Y(1, z) = Id_V\) (the identity operator of \(V\));

(V4) \(Y(a, z)1 \in (\text{End} V)[[z]]\) and \(\lim_{z \to 0} Y(a, z)1 = a\) for any \(a \in V\);

(V5) For \(\mathbb{Z}_2\) -homogeneous \(a, b \in V\), the following Jacobi identity holds:

\[
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(a, z_1)Y(b, z_2) - \varepsilon_{a,b} z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y(b, z_2)Y(a, z_1) = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(a, z_0)b, z_2) .
\]

In the case \(V = V^{\text{even}}\), i.e. when all vectors are even we say that \(V\) is vertex algebra.

From the Jacobi identity follows

\[
[a_m, b_n] = \sum_{j=0}^{\infty} \binom{m}{j} (a_j b)_{m+n-j}, \quad m, n \in \mathbb{Z},
\]

(2.1)

\[
Y(a_{-1} b, z) =: Y(a, z)Y(b, z) :,
\]

(2.2)

where the normally ordered product is defined, as usual, by

\[
:Y(a, z)Y(b, z) := Y(a, z)Y(b, z) + Y(b, z)Y(a, z)_+ + Y(a, z)_- = Y(a, z) - Y(a, z)_+ + Y(a, z)_-
\]

and

\[
Y(a, z)_+ = \sum_{j \in \mathbb{Z}_+} a_j z^{-j-1}, \quad Y(a, z)_- = Y(a, z) - Y(a, z)_+
\]

are the annihilation and the creation parts of \(Y(a, z)\).

3 The vertex algebra \(W_{1+\infty,-N}\)

In this section we recall some of the results from \([\text{KR}1, \text{KR}2, \text{FKRW}]\).

Let \(\mathcal{D}\) be the Lie algebra of complex regular differential operators on \(\mathbb{C}^\times\) with the usual bracket, in an indeterminate \(t\). The elements

\[
J^l(k) = -t^{l+k}(\partial_t)^l \quad (k \in \mathbb{Z}, l \in \mathbb{Z}_+)
\]

form a basis of \(\mathcal{D}\). The Lie algebra \(\mathcal{D}\) has the following 2-cocycle with values in \(\mathbb{C}\):

\[
\Psi(f(t)\partial_t^m, g(t)\partial_t^n) = \frac{m!n!}{(m+n+1)!} \text{Res}_{t=0} f^{(n+1)}(t)g^{(m)}(t)dt,
\]

where \(f^{(m)}(t) = \partial_t^m f(t)\). We denote by \(\hat{\mathcal{D}} = \mathcal{D} \oplus \mathbb{C}C\), where \(C\) is the central element, the corresponding central extension of the Lie algebra \(\mathcal{D}\).
Another important basis of $\mathcal{D}$ is
\[ L^j(k) = -t^k D^j \quad (k \in \mathbb{Z}, l \in \mathbb{Z}_+) \]
where $D = t \partial_t$. These two bases are related by the formula \([\text{KR1}]\):
\[ J^j(k) = -t^k D(D-1) \cdots (D-l+1). \tag{3.1} \]

Given a sequence of complex numbers $\lambda = (\lambda_j)_{j \in \mathbb{Z}_+}$ and a complex number $c$ there exists a unique irreducible $\hat{\mathcal{D}}$-module $L(\lambda, c; \hat{\mathcal{D}})$ which admits a non-zero vector $v_\lambda$ such that:
\[ L^j(k)v_\lambda = 0 \quad \text{for} \quad k > 0, \quad L^j(0)v_\lambda = \lambda_j v_\lambda, \quad C = cI. \]
This is called a highest weight module over $\hat{\mathcal{D}}$ with highest weight $\lambda$ and central charge $c$. The module $L(\lambda, c; \hat{\mathcal{D}})$ is called quasifinite if all eigenspaces of $D$ are finite-dimensional (note that $D$ is diagonalizable). It was proved in (\text{[KR1]} Theorem 4.2) that $L(\lambda, c; \hat{\mathcal{D}})$ is a quasi-finite module if and only if the generating series
\[ \Delta_\lambda(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \lambda_n \]
has the form
\[ \Delta_\lambda(x) = \frac{\phi(x)}{e^x - 1}, \]
where
\[ \phi(x) + c = \sum_{i} p_i(x)e^{r_i x} \quad (\text{a finite sum}), \]
$p_i(x)$ are non-zero polynomials in $x$ such that $\sum_i p_i(0) = c$ and $r_i$ are distinct complex numbers. The numbers $r_i$ are called the exponents of this module and the polynomials $p_i(x)$ are called their multiplicities.

Recall now that the $\mathcal{D}$-module $L(0, c; \mathcal{D})$ has a canonical structure of a vertex algebra with the vacuum vector $1 = v_0$ and generated by the fields $J^k(z) = \sum_{m \in \mathbb{Z}} J^k(m) z^{-m-k-1}$ \([\text{FKRW}]\).

In \([\text{KR2}]\), the vertex algebra $L(0, -N, \mathcal{D})$ was realized by using Weyl algebra $W_N$ and the corresponding vertex algebra $V_N$. The Weyl algebra $W_N$ is an associative algebra over $\mathbb{C}$ generated by $a_i(m), a_i^*(m)$ ($i = 1, \ldots, N; m \in \mathbb{Z}$) and $C$ with the following defining relations
\[ [a_i(m), a_j(n)] = [a_i^*(m), a_j^*(n)] = 0, \quad [a_i^*(m), a_j(n)] = \delta_{i,j} \delta_{m,-n} C \]
for all $i, j \in \{1, \ldots, N\}$, $n, m \in \mathbb{Z}$, and $C$ is central element.

The vacuum $W_N$-module $V_N$ is generated by one vector $1$, and the following relations
Let \( h \) be a lattice. Set \( \mathfrak{h} = \mathbb{C} \otimes_\mathbb{Z} \mathfrak{L} \) and extend the \( \mathbb{Z} \)-form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{L} \) to \( \mathfrak{h} \). Let \( \hat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}c \) be the affinization of \( \mathfrak{h} \). We also use the notation \( h(n) = t^n \otimes h \) for \( h \in \mathfrak{h}, n \in \mathbb{Z} \).

Set \( \hat{\mathfrak{h}}^+ = t\mathbb{C}[t] \otimes \mathfrak{h}; \hat{\mathfrak{h}}^- = t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{h} \). Then \( \hat{\mathfrak{h}}^+ \) and \( \hat{\mathfrak{h}}^- \) are abelian subalgebras of \( \hat{\mathfrak{h}} \). Let \( U(\hat{\mathfrak{h}}^-) = S(\hat{\mathfrak{h}}^-) \) be the universal enveloping algebra of \( \hat{\mathfrak{h}}^- \). Let \( \lambda \in \mathfrak{h} \). Consider the induced \( \mathfrak{h} \)-module

\[
M(1, \lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C}c)} \mathbb{C}_\lambda \simeq S(\hat{\mathfrak{h}}^-) \quad \text{(linearly),}
\]

where \( t\mathbb{C}[t] \otimes \mathfrak{h} \) acts trivially on \( \mathbb{C} \), \( h \) acting as \( \langle h, \lambda \rangle \) for \( h \in \mathfrak{h} \) and \( c \) acts on \( \mathbb{C} \) as multiplication by 1. We shall write \( M(1) \) for \( M(1, 0) \). For \( h \in \mathfrak{h} \) and \( n \in \mathbb{Z} \) write \( h(n) = t^n \otimes h \). Set \( h(z) = \sum_{n \in \mathbb{Z}} h(n)z^{-n-1} \).

Then \( M(1) \) is vertex algebra which is generated by the fields \( h(z) \), \( h \in \mathfrak{h} \), and \( M(1, \lambda) \), for \( \lambda \in \mathfrak{h} \), are irreducible modules for \( M(1) \).

Let \( \hat{\mathfrak{L}} \) be the canonical central extension of \( \mathfrak{L} \) by the cyclic group \( \langle \pm 1 \rangle \):

\[
1 \to \langle \pm 1 \rangle \to \hat{\mathfrak{L}} \to \mathfrak{L} \to 1 \quad (4.1)
\]

with the commutator map \( c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle} \) for \( \alpha, \beta \in \mathfrak{L} \). Let \( e : \mathfrak{L} \to \hat{\mathfrak{L}} \) be a section such that \( e_0 = 1 \) and \( e : \mathfrak{L} \times \mathfrak{L} \to \langle \pm 1 \rangle \) be the corresponding 2-cocycle. Then \( \epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{\langle \alpha, \beta \rangle} \),

\[
\epsilon(\alpha, \beta)\epsilon(\beta + \gamma, \gamma) = \epsilon(\beta, \gamma)\epsilon(\alpha, \beta + \gamma) \quad (4.2)
\]
and $e_\alpha e_\beta = \epsilon(\alpha, \beta)e_{\alpha+\beta}$ for $\alpha, \beta, \gamma \in L$. Form the induced $\hat{L}$-module

$$\mathbb{C}\{L\} = \mathbb{C}[\hat{L}] \otimes_{(z^1)} \mathbb{C} \simeq \mathbb{C}[L] \text{ (linearly)},$$

where $\mathbb{C}[\cdot]$ denotes the group algebra and $-1$ acts on $\mathbb{C}$ as multiplication by $-1$. For $a \in L$, write $\iota(a)$ for $a \otimes 1$ in $\mathbb{C}\{L\}$. Then the action of $\hat{L}$ on $\mathbb{C}\{L\}$ is given by: $a \cdot \iota(b) = \iota(ab)$ and $(-1) \cdot \iota(b) = -\iota(b)$ for $a, b \in \hat{L}$.

Furthermore we define an action of $\mathfrak{h}$ on $\mathbb{C}\{L\}$ by: $h \cdot \iota(a) = \langle h, \bar{a} \rangle \iota(a)$ for $h \in \mathfrak{h}, a \in \hat{L}$. Define $z^h \cdot \iota(a) = z^{(h, \bar{a})} \iota(a)$.

The untwisted space associated with $L$ is defined to be

$$V_L = \mathbb{C}\{L\} \otimes_{\mathbb{C}} \mathbb{M}(1) \simeq \mathbb{C}[L] \otimes S(\mathfrak{h}^-) \text{ (linearly)}.$$ 

Then $\hat{L}, \mathfrak{h}, z^h (h \in \mathfrak{h})$ act naturally on $V_L$ by acting on either $\mathbb{C}\{L\}$ or $\mathbb{M}(1)$ as indicated above. Define $1 = \iota(e_0) \in V_L$. We use a normal ordering procedure, indicated by open colons, which signify that in the enclosed expression, all creation operators $h(n) \ (n < 0), a \in \hat{L}$ are to be placed to the left of all annihilation operators $h(n), z^h \ (h \in \mathfrak{h}, n \geq 0)$. For $a \in \hat{L}$, set

$$Y(\iota(a), z) = :e^{\int(\bar{a}(z) \bar{a}(0) z^{-1})} a z^{\bar{a}} :.$$

Let $a \in \hat{L}; h_1, \ldots, h_k \in \mathfrak{h}; n_1, \ldots, n_k \in \mathbb{Z} \ (n_i > 0)$. Set

$$v = \iota(a) \otimes h_1(-n_1) \cdots h_k(-n_k) \in V_L.$$

Define vertex operator $Y(v, z)$ with

$$: \left( \frac{1}{(n_1 - 1)!} \frac{d}{dz}^{n_1 - 1} h_1(z) \right) \cdots \left( \frac{1}{(n_k - 1)!} \frac{d}{dz}^{n_k - 1} h_k(z) \right) Y(\iota(a), z) : \quad (4.3)$$

This gives us a well-defined linear map

$$Y(\cdot, z) : V_L \to (\text{End}V_L)[[z, z^{-1}]]$$

$$v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}, \ (v_n \in \text{End}V_L).$$

Let $\{ h_i \mid i = 1, \ldots, d \}$ be an orthonormal basis of $\mathfrak{h}$ and set

$$\omega = \frac{1}{2} \sum_{i=1}^{d} h_i(-1) h_i(-1) \in V_L.$$

Then $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ gives rise to a representation of the Virasoro algebra on $V_L$ with the central charged $d$ and

$$L_0 \ (\iota(a) \otimes h_1(-n_1) \cdots h_n(-n_k))$$

$$= \left( \frac{1}{2} (\bar{a}, \bar{a}) + n_1 + \cdots + n_k \right) (\iota(a) \otimes h_1(-n_1) \cdots h_k(-n_k)) \quad (4.4)$$

The following theorem was proved in [DI] and [K].
Theorem 4.1 \((V_L, 1, L_{-1}, Y)\) is vertex superalgebra.

Vertex algebra \(M(1)\) can be treated as a subalgebra of \(V_L\).

Define the Schur polynomials \(p_r(x_1, x_2, \cdots) (r \in \mathbb{Z}_+)\) in variables \(x_1, x_2, \cdots\) by the following equation:

\[
\exp \left( \sum_{n=1}^{\infty} \frac{x_n y^n}{n} \right) = \sum_{r=0}^{\infty} p_r(x_1, x_2, \cdots) y^r. \tag{4.5}
\]

For any monomial \(x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}\) we have an element \(h(-1)^{n_1} h(-2)^{n_2} \cdots h(-r)^{n_r} 1\) in both \(M(1)\) and \(V_L\) for \(h \in \mathfrak{h}\). Then for any polynomial \(f(x_1, x_2, \cdots)\), \(f(h(-1), h(-2), \cdots) 1\) is a well-defined element in \(M(1)\) and \(V_L\). In particular, \(p_r(h(-1), h(-2), \cdots) 1\) for \(r \in \mathbb{Z}_+\) are elements of \(M(1)\) and \(V_L\).

Suppose \(a, b \in \hat{L}\) such that \(\bar{a} = \alpha, \bar{b} = \beta\). Then

\[
Y(\iota(a), z)\iota(b) = z^{\langle \alpha, \beta \rangle} \exp \left( \sum_{n=1}^{\infty} \frac{\alpha(-n)}{n} z^n \right) \iota(ab) = \sum_{r=0}^{\infty} p_r(\alpha(-1), \alpha(-2), \cdots) \iota(ab) z^{r+\langle \alpha, \beta \rangle}. \tag{4.6}
\]

Thus

\[
\iota(a) \iota(b) = 0 \quad \text{for} \quad i \geq -\langle \alpha, \beta \rangle. \tag{4.7}
\]

Especially, if \(\langle \alpha, \beta \rangle \geq 0\), we have \(\iota(a) \iota(b) = 0\) for all \(i \in \mathbb{Z}_+\), and if \(\langle \alpha, \beta \rangle = -n < 0\), we get

\[
\iota(a) \iota(-1) \iota(b) = p_{n-i}(\alpha(-1), \alpha(-2), \cdots) \iota(ab) \quad \text{for} \quad i \in \mathbb{Z}_+. \tag{4.8}
\]

We will need one structural result on Heisenberg vertex algebras.

Element \(L_0\) of the Virasoro algebra defines a \(\mathbb{Z}_+\)-gradation on vertex algebra \(M(1) = \bigoplus_{n \in \mathbb{Z}_+} M(1)_n\). Let \(v_\lambda\) be the highest weight vector in the \(M(1)\)-module \(M(1, \lambda)\).

The following lemma can be proved by using standard calculation in the Heisenberg vertex algebras.

Lemma 4.1 Let \(h \in \mathfrak{h}\), and \(n_1, \ldots, n_r \in \mathbb{N}\). Let \(k = n_1 + \cdots + n_r\). Let \(u = h(-n_1) \cdots h(-n_r) 1\), and \(Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}\). Then \(u \in M(1)_k\), and we have

\[
\begin{align*}
(1) \quad & u_n v_\lambda = 0 \quad \text{for} \quad n > k - 1, \\
(2) \quad & u_{k-1} v_\lambda = (-1)^{n_1 + \cdots + n_r} \langle \lambda, h \rangle v_\lambda.
\end{align*}
\]
Proposition 4.1 Let $h \in \mathfrak{h}$, and $r \in \mathbb{Z}_+$. Let $u = p_r(h(-1), h(-2), \ldots)1$. Set $Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}$. Then $u \in M_r$, and we have

\begin{align*}
(1) & \quad u_n v_\lambda = 0 \quad \text{for } n > r - 1, \\
(2) & \quad u_{r-1} v_\lambda = \left(\langle \lambda, h \rangle \right) v_\lambda.
\end{align*}

Proof. Since $u \in M(1)_r$, we have that $u_n v_\lambda = 0$ for $n > r$. Set $x = \langle \lambda, h \rangle$. Using Lemma 4, one can easily see that

\[
u_{r-1} v_i = p_r(x, -x, x, -x, \cdots) v_\lambda.
\]

Since

\[
\exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} y^n \right) = (1 + y)^x = \sum_{r \geq 0} \left(\begin{array}{c} x \\ r \end{array}\right) y^r,
\]

we have that $p_r(x, -x, x, -x, \cdots) = \left(\begin{array}{c} x \\ r \end{array}\right)$, and we conclude that $u_{r-1} v_\lambda = \left(\begin{array}{c} x \\ r \end{array}\right)$. \(\square\)

Remark 4.1 Proposition 4.1 can be also proved by using Zhu’ s algebra theory (see [DLM], Section 3.)

5 Representations of the vertex algebra $W_{1+\infty, -N}$

In this section we will construct $2N$–dimensional family of irreducible modules for the vertex algebra $W_{1+\infty, -N}$. We will prove that $W_{1+\infty, -N}$ can be realized as a vertex subalgebra of Heisenberg vertex algebra $M(1)$.

First we will consider the following lattice:

$$L = \bigoplus_{i=1}^{N} (\mathbb{Z} \alpha_i + \mathbb{Z} \beta_i),$$

where

$$\langle \alpha_i, \alpha_j \rangle = \delta_{i,j}, \quad \langle \beta_i, \beta_j \rangle = -\delta_{i,j}, \quad \langle \alpha_i, \beta_j \rangle = \langle \beta_j, \alpha_i \rangle = 0,$$

for every $i, j \in \{1, \cdots, N\}$.

Let $\mathfrak{h}$, $M(1)$ and $V_L$ be defined as in the Section 4. Then $\mathfrak{h}$ is a Heisenberg algebra, $\dim \mathfrak{h} = 2N$, $M(1)$ is a Heisenberg vertex algebra, and $V_L$ is a lattice vertex superalgebra.

For every $i \in \{1, \cdots, N\}$, let $a^i, b^i \in \hat{L}$ such that

$$\overline{a^i} = \alpha_i + \beta_i, \quad \overline{b^i} = -(\alpha_i + \beta_i).$$

Then we define $e^i, f^i \in V_L$ with $e^i = \iota(a^i), f^i = \iota(b^i)$.

Set $Y(e^i, z) = \sum_{n \in \mathbb{Z}} e_n^i z^{-n-1}$, and $Y(f^i, z) = \sum_{n \in \mathbb{Z}} f_n^i z^{-n-1}$.

Recall the definition of Schur polynomial $p_h(h(-1), h(-2), \cdots) (h \in \mathfrak{h})$ from Section 4. Set $p_h := p_h(h(-1), h(-2), \cdots) \in S(\mathfrak{h}^*)$. Let $\gamma_i = \alpha_i + \beta_i$. Then $\langle \gamma_i, \gamma_i \rangle = 0$.

Now, relations (4.7) and (4.8) imply the following lemma.
Lemma 5.1 For all $i, j \in \{1, \cdots, N\}$, $k, n \in \mathbb{Z}$ we have:

1. $e_i^j e^j = f_i^j f^j = 0,$ for $n \geq 0,$
2. $e_{i-1}^j f^j = \delta_{i,j} 1,$ $f_{i-1}^j e^j = \delta_{i,j} 1,$
3. $e_{i-1}^j f^j = p_i(\gamma_i) 1,$ $f_{i-1}^j e^j = p_i(-\gamma_i) 1,$
4. $[\alpha_i(k), e_i^j] = \delta_{i,j} e_{n+k}^j,$ $[\alpha_i(k), f_i^j] = -\delta_{i,j} f_{n+k}^j.$ (5.1)

Define

$$A^i(z) = \sum_{n \in \mathbb{Z}} A^i(n)z^{-n-1} = Y(e^i \otimes \alpha_i(-1), z) = : \alpha_i(z)Y(e^i, z):,$$

$$\bar{A}^i(z) = \sum_{n \in \mathbb{Z}} \bar{A}^i(n)z^{-n} = Y(f^i, z).$$

Lemma 5.2 We have

$$[A^i(n), A^j(m)] = [\bar{A}^i(n), \bar{A}^j(m)] = 0,$$

$$[\bar{A}^i(n), A^j(m)] = \delta_{i,j}\delta_{m+m,0},$$

i.e. the components of the fields $A^i(z), \bar{A}^i(z), 1 \leq i \leq N,$ span an associative algebra isomorphic to the Weyl algebra $W_N$ with $C = 1.$

Proof. Using (2.2) we have that

$$A^i(m) = \sum_{n<0} \alpha_i(n)e_i^{m-n-1} + \sum_{n\geq 0} e_i^{m-n-1}\alpha_i(n)$$

for $1 \leq i \leq N.$ Using Lemma 5.1 we get

$$A^i(0)f^j = -\delta_{i,j}f^j,$$ $A^i(m)f^j = 0$ for $m > 0,$

$$A^i(m)\bar{A}^j(-1)1 = 0,$$ $f_m^i f^j = 0,$ for $m \geq 0.$

Now, the statment of the lemma follows from the commutator formulae (2.1). $\Box$

From Lemma 5.2 follows the following result.

Proposition 5.1 The subalgebra of the vertex superalgebra $V_L$ generated by the fields $A^i(z), \bar{A}^i(z)$ $i = 1, \ldots, N,$ is isomorphic to the vertex algebra $V_N.$ (Under this isomorphism the fields $a_i(z)$ corresponds to $A^i(z),$ and $\bar{a}_i(z)$ to $\bar{A}^i(z).$)

Proof. From the Lemma 5.2 follows that $V_L$ is a module for the Weyl algebra $W_N.$ The subalgebra of $V_L$ generated by the fields $A^i(z)$ and $\bar{A}^i(z)$ is exactly the $W_N$ submodule $W_N.1$ generated by 1. Then we have

$$A^i(m)1 = \bar{A}^i(n)1 = 0,$$ for $m \geq 0,$ $n > 0.$

Then the fact that $V_N$ is an irreducible $W_N$-module implies that $V_N \cong W_N.1.$ $\Box$
Remark 5.1  In the physical literature the vertex algebra $V_N$ is known as $\beta\gamma$ system, and the lattice construction of $\beta\gamma$ system is known as Friedan-Martinec-Shenker bosonization (cf. \[FMS\], \[W1\]).

With the respect to Proposition 5.1 we can identify vertex algebra $V_N$ with the subalgebra of $V_L$ generated by the fields $A^i(z)$, $\bar{A}^i(z)$. Since $W_{1+\infty,-N}$ is a vertex subalgebra of $V_N$ generated by the fields

$$J^k(z) = -\sum_{i=1}^{N} :A^i(z)\partial^k \bar{A}^i(z):$$

we have that $W_{1+\infty,-N}$ is also a vertex subalgebra of $V_L$. For every $i \in \{1, \ldots, N\}$, let $U^i_k = A^i(-1)f^i_{-k-1}\mathbf{1} \in V_L$, and $U_k = \sum_{i=1}^{N} U^i_k$. We have

$$J^k(z) = -k! \sum_{i=1}^{N} Y(A^i(-1)\bar{A}^i(-k)\mathbf{1}, z) = -k!Y(U_k, z).$$

Lemma 5.3 We have

$$U^i_k = A^i(-1)\bar{A}^i(-k)\mathbf{1} = \alpha_i(-1)p_k(-\gamma_i)\mathbf{1} + p_{k+1}(-\gamma_i)\mathbf{1}.$$ 

In particular, $U^i_k \in M(1)$.

Proof. Since $[A^i(-1), \bar{A}^i(-k)] = 0$, we have that

$$U^i_k = \bar{A}^i(-k)A^i(-1)\mathbf{1} = f^i_{-k-1}\alpha_i(-1)e^i = \alpha_i(-1)f^i_{-k-1}e^i + f^i_{-k-2}e^i.$$ 

Then Lemma 5.3 implies that

$$U^i_k = \alpha_i(-1)p_k(-\gamma_i)\mathbf{1} + p_{k+1}(-\gamma_i)\mathbf{1},$$

and lemma holds. \hfill \square

Theorem 5.1 Vertex algebra $W_{1+\infty,-N}$ is a subalgebra of the vertex algebra $M(1)$.

Proof. From Lemma 5.3 follows that $U_k = \sum_{i=1}^{N} U^i_k \in M(1)$, for every $k \in \mathbb{Z}_+$. Since $J^k(z) = -k!Y(U_k, z)$, $k \in \mathbb{Z}_+$, generate $W_{1+\infty,-N}$, we have that $W_{1+\infty,-N}$ is a subalgebra of the vertex algebra $M(1)$. \hfill \square

Lemma 5.4 For every $\lambda \in \mathfrak{h}$, let $v_\lambda$ be the highest weight vector in $M(1, \lambda)$. Then we have $J^k(n)v_\lambda = 0$ for $n > 0$, and

$$J^k(0)v_\lambda = -k! \sum_{i=1}^{N} \left( \left( -\frac{\langle \lambda, \gamma_i \rangle}{k+1} \right) + \frac{\langle \lambda, \alpha_i \rangle}{k} \left( -\frac{\langle \lambda, \gamma_i \rangle}{k} \right) \right) v_\lambda.$$
Proof. Set $Y(U^i_k, z) = \sum_{n \in \mathbb{Z}} U^i_k(n) z^{-n-k-1}$. Proposition 4.1 and Lemma 5.3 implies that

$$U^i_k(n)v_\lambda = 0 \quad \text{for } n > 0,$$

$$U^i_k(0)v_\lambda = \langle \lambda, \alpha_i \rangle \left(-\frac{\langle \lambda, \gamma_i \rangle}{k}\right) + \left(-\frac{\langle \lambda, \gamma_i \rangle}{k+1}\right).$$

Since $J^k(n) = -k! \sum_{i=1}^N U^i_k(n)$, we have that the lemma holds.

Since $W_{1+\infty,-N}$ is a subalgebra of VOA $M(1)$, then for every $\lambda \in \mathfrak{h}$, $M(1, \lambda)$ is a $W_{1+\infty,-N}$–module. Let $V(\lambda, -N)$ be the irreducible $W_{1+\infty,-N}$–subquotient of $M(1, \lambda)$ generated by the vector $v_\lambda$.

**Theorem 5.2** For every $\lambda \in \mathfrak{h}$, $V(\lambda, -N)$ is the irreducible module for the vertex algebra $W_{1+\infty,-N}$. As a $\hat{D}$–module, $V(\lambda, -N)$ is a irreducible quasi-finite highest weight module, and the corresponding generating series is

$$\Delta_\lambda(x) = -\sum_{i=1}^N \left(\frac{e^{s_ix} - 1}{e^x - 1} + t_ie^{s_ix}\right), \quad (5.2)$$

where

$$s_i = -\langle \lambda, \alpha_i + \beta_i \rangle, \quad t_i = \langle \lambda, \alpha_i \rangle, \quad i = 1, \ldots, N. \quad (5.3)$$

Proof. Lemma 5.4 implies that $V(\lambda, -N)$ is a highest weight $\hat{D}$–module. It remains to prove that $V(\lambda, -N)$ is a quasi-finite $\hat{D}$–module. In order to prove this we have to identify the generating series $\Delta_\lambda(x)$.

Using the relation $[3.1]$, it is straightforward to see the following formulae.

$$\Delta_\lambda(x) = \sum_{k=0}^\infty \frac{x^k}{k!} L^k(0)v_\lambda = \sum_{k=0}^\infty \frac{(e^x - 1)^k}{k!} J^k(0)v_\lambda.$$

Let $s_i, t_i$ be defined with $(5.3)$. Then Lemma 5.4 implies

$$\Delta_\lambda(x) = -\sum_{i=1}^N \sum_{k=0}^\infty \left(\frac{s_i}{k+1} + t_i\frac{s_i}{k}\right) (e^x - 1)^k$$

$$= -\sum_{i=1}^N \left(\frac{e^{s_ix} - 1}{e^x - 1} + t_ie^{s_ix}\right)$$

$$= \frac{\Phi(x)}{e^x - 1},$$

where

$$\Phi(x) - N = \sum_{i=1}^N \left((t_i - 1)e^{s_ix} - t_i e^{(s_i+1)x}\right).$$

This implies that $V(\lambda, -N)$ is a quasi-finite $\hat{D}$–module. \qed
Remark 5.2 Theorem 5.2 gives the existence of $2N$-dimensional family of irreducible $W_{1+\infty,-N}$-modules. If we take in (5.2) $t_i = 0$ for every $i = 1, \ldots, N$, we get exactly $W_{1+\infty,-N}$-modules constructed in [KR2].

We have the following conjecture.

Conjecture 5.1 The set $V(\lambda, -N)$, $\lambda \in \mathfrak{h}$, lists all the irreducible modules for the vertex algebra $W_{1+\infty,-N}$.

Remark 5.3 In Section 6 we will see that the Conjecture 5.1 is true for $c = -1$.

6 The case of $c = -1$

In this section we will compare our results with the results from [W1], [W2]. In [W1], Wang proved that the vertex algebra $W_{1+\infty,-1}$ is isomorphic to the tensor product $W_{3,-2} \otimes H_0$, where $W_{3,-2}$ is a simple vertex algebra associated to $W_3$-algebra with $c = -2$, and $H_0$ is a Heisenberg vertex algebra. Moreover, in [W2] Wang classified all the irreducible modules for $W_{3,-2}$ and $W_{1+\infty,-1}$. The methods used in [W1], [W2] didn’t imply the identification of $W_{1+\infty,-1}$-modules as a highest weight $\hat{D}$-modules (see Section 5 in [W2]). Our approach gives an explicit identification of two-dimensional family $W_{1+\infty,-1}$-modules in terms of highest weights.

Let $N = 1$, and set $\alpha = \alpha_1$, $\beta = \beta_1$. For $\lambda \in \mathfrak{h}$ set $\lambda_\alpha = \langle \lambda, \alpha \rangle$, $\lambda_\beta = \langle \lambda, \beta \rangle$. Let $M_\alpha(1, \lambda_\beta)$ (resp. $M_\beta(1, \lambda_\beta)$) be the submodules of $M(1, \lambda)$ generated by the highest weight vector $v_\lambda$ and $\alpha(n)$ (resp. $\beta(n)$). Set $M_\alpha(1) = M_\alpha(1, 0)$, $M_\beta(1) = M_\beta(1, 0)$. Then

$$M(1) = M_\alpha(1) \otimes M_\beta(1), \quad M(1, \lambda) = M_\alpha(1, \lambda_\alpha) \otimes M_\beta(1, \lambda_\beta). \quad (6.1)$$

As in [W2] we define

$$T(z) = \frac{1}{2} \left( : \alpha(z)^2 : + \partial \alpha(z) \right),$$
$$W(z) = \frac{1}{12} \left( 4 : \alpha(z)^3 : + 6 : \alpha(z) \partial \alpha(z) : + \partial^2 \alpha(z) \right).$$

Theorem 6.1 [W1], [W2]

(1) The fields $T(z)$, $W(z)$ span a subalgebra of $M_\alpha(1)$ isomorphic to $W_{3,-2}$, and $W_{1+\infty,-1} \cong W_{3,-2} \otimes M_\beta(1)$.

(2) Let $V_r$ be the irreducible subquotient of $W_{3,-2}$-module $M_\alpha(1, r)$. Then $V_r$, $r \in \mathbb{C}$, gives all the irreducible $W_{3,-2}$-modules.
Recall the definition of $W_{1+\infty,-1}$-modules $V(\lambda, -1)$ from Section 5. Then we have the following consequence of Theorem 5.2 and Theorem 6.1.

**Corollary 6.1** We have

1. $V(\lambda, -1) = \mathcal{V}_\lambda \otimes M_{\beta}(1, \lambda_\beta)$ for every $\lambda \in \mathfrak{h}$.
2. The set $V(\lambda, -1)$, $\lambda \in \mathfrak{h}$, gives all irreducible $W_{1+\infty,-1}$-modules.

$$\Delta_\lambda(x) = -\frac{e^{-(\lambda_\alpha+\lambda_\beta)x} - 1}{e^x - 1} - \lambda_\alpha e^{-(\lambda_\alpha+\lambda_\beta)x}.$$

**Proof.** (1) follows from the definition of $\mathcal{V}_\alpha$ and (6.1). Then theorem 6.1 implies that $V_r \otimes M_{\beta}(1, s)$, $r, s \in \mathbb{C}$, are all irreducible $W_{1+\infty,-1}$-modules. Since $V(\lambda, -1) = \mathcal{V}_\lambda \otimes M_{\beta}(1, \lambda_\beta)$, we see that $V(\lambda, -1)$, $\lambda \in \mathfrak{h}$ gives all irreducible $W_{1+\infty,-1}$-modules, and we get (2). The statement (3) follows from the Theorem 5.2. $\square$

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