Four-loop cusp anomalous dimension in QED

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ABSTRACT: The 4-loop $C_3^3 F_{T T F n l}$ and 5-loop $C_4^4 F_{T T F n l}$ terms in the HQET field anomalous dimension $\gamma_h$ are calculated analytically (the 4-loop one agrees with the recent numerical result [1]). The 4-loop $C_3^3 F_{T T F n l}$ and 5-loop $C_4^4 F_{T T F n l}$ terms in the cusp anomalous dimension $\Gamma(\varphi)$ are calculated analytically, exactly in $\varphi$ (the $\varphi \to \infty$ asymptotics of the 4-loop one agrees with the recent numerical result [2]). Combining these results with the recent 4-loop $d_{FFn l}$ contributions to $\gamma_h$ and to the small-$\varphi$ expansion of $\Gamma(\varphi)$ up to $\varphi^4$ [3] (recently extended to $\varphi^6$ [4]) we now have the complete analytical 4-loop result for the Bloch–Nordsieck field anomalous dimension in QED, and the small-$\varphi$ expansion of the 4-loop QED cusp anomalous dimension up to $\varphi^6$.

KEYWORDS: QCD, QED, Effective field theories
1 Introduction

QCD problems with a single heavy quark $Q$ having momentum $P = M_Q v + p$ (where $M_Q$ is the on-shell mass and $v$ is some vector with $v^2 = 1$) can be described by heavy quark effective theory (HQET) if characteristic heavy-quark residual momentum $p$, as well as characteristic gluon and light-quark momenta $k_i$, are $\ll M_Q$ (see, e.g., [5–7]). The heavy quark is described by the field

$$h_{v0} = Z_1^{1/2}(\alpha_s(\mu), a(\mu)) h_v(\mu), \quad (1.1)$$

where we use the $\overline{\text{MS}}$ scheme, and $Z_h$ is a minimal renormalization constant. We use the covariant gauge: $-(\partial_\mu A_0^{\text{aux}})^2/(2a_0)$ is added to the Lagrangian, the gauge-fixing parameter is renormalized by the same minimal constant as the gluon field: $a_0 = Z_A(\alpha_s(\mu), a(\mu)) a(\mu)$.

The HQET heavy-quark field anomalous dimension is defined as $\gamma_h = d \log Z_h/d \log \mu$. The $h_{v0}$ coordinate-space propagator in the $v$ rest frame has the form

$$S_h(x) = \delta^{(d-1)}(\vec{x}) \theta(x^0) W(x^0), \quad (1.2)$$

where $W(t)$ is the straight Wilson line along $v$ of length $t$. The heavy-quark field is QCD and HQET are related by the matching coefficient $z$ [8]:

$$Q_0 = z_0^{1/2} h_{v0} + \mathcal{O}(1/M_Q), \quad Q(\mu) = z^{1/2}(\mu) h_v(\mu) + \mathcal{O}(1/M_Q). \quad (1.3)$$

The HQET field anomalous dimension $\gamma_h$ is known up to three loops [9, 10]. In the first of these papers, it was obtained as a by-product of the three-loop calculation of the heavy-quark field renormalization constant in the on-shell scheme $Z_Q^{os}$, from the requirement that the renormalized matching coefficient $z(\mu)$ (1.3) must be finite; in the second paper, it was confirmed by a direct HQET calculation. Several color structures of the 4-loop result are also known: $C_F(T_F n_l)^3$ [11] ($n_l$ is the number of light flavors), $C_F^2(T_F n_l)^2$ [12, 13], $C_F C_A(T_F n_l)^2$ [1] (from the analytical $Z_Q^{os}$ result [14] using the finiteness of $z(\mu)$), $d_F n_l$ [3]. Here $C_R$ ($R = F, A$) are the standard quadratic Casimirs: $t_R^a t_R^b = C_R 1_R$ ($t_R^a$ are the generators in the representation $R$, $1_R$ is the corresponding unit matrix); $\text{Tr} t_F^a t_F^b = T_F \delta^{ab}$;
\[ d_{FF} = d_{FR}^{abcd} d_{FR}^{abcd} / N_c, \] where \( d_{FR}^{abcd} = \text{Tr} R^i R^j R^k R^l \) (the brackets mean symmetrization), and \( N_c = \text{Tr} 1_F \). The remaining terms are known numerically \cite{1}, from the numerical 4-loop \( Z_Q^{\text{os}} \) using the finiteness of \( z(\mu) \) \((1.3)\). Here I calculate the \( C_F^{L-1} T_{FNl} \alpha_s^L \) terms up to \( L = 5 \) analytically (sect. 2); the \( L = 4 \) term agrees with the numerical result \cite{1}.

If the heavy-quark velocity is substantially changed (e.g., a weak decay into another heavy quark), we have HQET with 2 unrelated fields \( h_{\nu}, h'_{\nu'} \). At the effective-theory level this is described by the current
\[ J_0 = h_{\nu'} h_{\nu} = Z_J (\alpha_s(\mu)) J(\mu). \] (1.4)

The minimal renormalization constant \( Z_J \) is gauge invariant (unlike \( Z_h \)) because the current \( J_0 \) is color singlet. The anomalous dimension of this current, also known as the cusp anomalous dimension, is defined as \( \Gamma(\varphi) = d \log Z_J / d \log \mu \), where \( \cosh \varphi = v \cdot v' \).

The QCD cusp anomalous dimension \( \Gamma(\varphi) \) is known up to three loops \cite{12, 15}. At \( \varphi \ll 1 \) it is a regular series in \( \varphi^2 \). At \( \varphi \gg 1 \) it is \( \Gamma_1 \varphi + O(\varphi^0) \) \cite{16}, where \( \Gamma_1 \) is the light-like cusp anomalous dimension. Several color structures of the 4-loop \( \Gamma(\varphi) \) are also known: \( C_F(T_{FNl})^3 \) \cite{17}, \( C_F^2(T_{FNl})^2 \) \cite{12, 13}. The \( d_{FFNl} \) term is known at \( \varphi \ll 1 \) up to \( \varphi^4 \) \cite{3}. For the \( \varphi \gg 1 \) asymptotics (i.e. \( \Gamma_1 \)), both \( n_2^0 \) terms are known from combining the \( C_F^2(T_{FNl})^2 \) result \cite{12, 13} and the large-\( N_c \) \( N_c^2 n_t^2 \) result \cite{19}. Large-\( N_c \) results for \( \Gamma_1 \) at \( n_t^1 \) \cite{2, 19} and \( n_t^0 \) \cite{2, 21} are also known analytically. Contributions of individual color structures of \( \Gamma_1 \) at \( n_t^1, n_t^0 \) are only known numerically \cite{2}. Here I calculate the \( C_F^{L-1} T_{FNl} \alpha_s^L \) terms up to \( L = 5 \) in \( \Gamma(\varphi) \) analytically, as exact functions of \( \varphi \) (sect. 3). In particular, I find their \( \varphi \gg 1 \) asymptotics; the analytical \( L = 4 \) result agrees with the numerical one \cite{2}.

In QED without light lepton flavors (\( n_t = 0 \)), as explained below, both \( \gamma_h \) and \( \Gamma(\varphi) \) are exactly given by the one-loop formulas. When \( n_t \neq 0 \), higher corrections appear.

Combining the 4-loop \( \gamma_h \) results for \( C_F(T_{FNl})^3 \) \cite{11}, \( C_F^2(T_{FNl})^2 \) \cite{12, 13}, \( C_F^3 T_{FNl} \) (sect. 2) and \( d_{FFNl} \) \cite{3} structures, I obtain the complete analytical 4-loop result for the Bloch–Nordsieck field anomalous dimension \( \gamma_h \) in QED (sect. 4). Combining the 4-loop \( \Gamma(\varphi) \) full results for \( C_F(T_{FNl})^3 \) \cite{17}, \( C_F^2(T_{FNl})^2 \) \cite{12, 13}, \( C_F^3 T_{FNl} \) (sect. 3) structures with the \( d_{FFNl} \) term \cite{3} (expansion up to \( \varphi^4 \)), I obtain the expansion of the 4-loop QED \( \Gamma(\varphi) \) up to \( \varphi^4 \) (sect. 4).

\section{HQET field anomalous dimension: the \( C_F^{L-1} T_{FNl} \alpha_s^L \) terms}

This is a QED problem. Due to exponentiation \cite{22, 23}, the coordinate-space propagator of the Bloch–Nordsieck field (i.e. the straight Wilson line \( W \)) is
\[ W = \exp \left( \sum w_i \right), \] (2.1)
where \( w_i \) are single-web diagrams. Due to \( C \) parity conservation in QED, webs have even numbers of legs (fig. 1). In QED with \( n_t = 0 \) there is only 1 single-web diagram: fig. 1a with the free photon propagators. Therefore, \( \log W \) is exactly 1-loop; the \( \beta \) function is 0, and hence \( \gamma_h \) is also exactly 1-loop. At \( n_t > 0 \) corrections to the photon propagator in

\footnote{Such expansion was first used in \cite{18} at 2 loops.}
fig. 1a appear. Webs with 4 legs (fig. 1b) first appear at 4 loops; they have been calculated in [3]. All contributions to \( \log W \) are gauge invariant except the 1-loop one, because proper vertex functions with any numbers of photon legs are gauge invariant and transverse with respect to each photon leg due to the QED Ward identities.

\[
D_{\mu\nu}(k) = -\frac{i}{k^2} \left( g_{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \frac{1}{1 - \Pi(k^2)} - i a_0 \frac{k^\mu k^\nu}{(k^2)^2},
\]

(2.2)

The full momentum-space photon propagator in the covariant gauge is

\[
\Pi = \sum_{L=1}^{\infty} \Pi_L A_L \left(-k^2\right)^{-L\varepsilon}, \quad A_0 = \frac{\epsilon_0^2}{(4\pi)^{d/2}} e^{-\gamma\varepsilon}
\]

(2.3)

\(\epsilon_0^2\) has dimensionality \(m^2\varepsilon\), so that the power of \(-k^2\) is obvious; \(\gamma\) is the Euler constant.

Only the 0-loop term in (2.2) is gauge dependent. Writing \(\Pi_L\) as \(\tilde{\Pi}_L + \left(n_l > 1\right)\) terms, we obtain in the Landau gauge \(a_0 = 0\)

\[
D_{\mu\nu}(k) = \tilde{D}_{0\mu\nu}(k) + \sum_{L=1}^{\infty} \tilde{\Pi}_L \tilde{D}_{L\mu\nu}(k) A_0 + \left(n_l > 1\right)\) terms,
\]

(2.4)

The MS charge renormalization is

\[
A_0 = \mu^{2\varepsilon} \frac{\alpha(\mu)}{4\pi} Z_\alpha(\alpha(\mu)),
\]

(2.5)

\[
\frac{d \log \alpha(\mu)}{d \log \mu} = -2\varepsilon - 2\beta(\alpha(\mu)), \quad \beta(\alpha) = \frac{1}{2} \frac{d \log Z_\alpha}{d \log \mu} = \sum_{L=1}^{\infty} \beta_L \left(\frac{\alpha}{4\pi}\right)^L
\]

(note that here we call the \(L\)-loop \(\beta\) function coefficient \(\beta_L\), not \(\beta_{L-1}\) as usually done; this makes subsequent formulas more logical). In QED \(\log \left(1 - \Pi(k^2)\right) = \log Z_\alpha + \) (finite); writing \(\beta_L = \tilde{\beta}_L n_l + \left(n_l > 1\right)\) terms, we see that \(1/\varepsilon\) terms in \(\tilde{\Pi}_L\) are related to \(\beta_L\):

\[
\tilde{\Pi}_L = \frac{\beta_L}{L \varepsilon} + \tilde{\Pi}_L + \mathcal{O}(\varepsilon)
\]

(2.6)

Here the \(\beta\) function coefficients are [24]

\[
\beta_1 = -\frac{4}{3}, \quad \beta_2 = -4, \quad \beta_3 = 2, \quad \beta_4 = 46,
\]

(2.7)
where the 1-loop HQET integral

\[ \Pi_1 = -\frac{20}{9}, \quad \Pi_2 = 16\zeta_3 - \frac{55}{3}, \quad \Pi_3 = -2\left(80\zeta_5 - \frac{148}{3}\zeta_3 - \frac{143}{9}\right), \]

\[ \Pi_4 = 2240\zeta_7 - 1960\zeta_5 - 104\zeta_3 + \frac{31}{3}. \]

(2.8)

The coordinate-space full photon propagator is the Fourier transform of (2.4):

\[
D^{\mu\nu}(x) = \frac{1}{(4\pi)^{d/2}} \left[ D_0^{\mu\nu}(x) + n_l \sum_{L=1}^{\infty} \Pi_L D_L^{\mu\nu}(x) A_0^L \right] + (n_l^\gamma \text{ terms}),
\]

\[
D_L^{\mu\nu}(x) = \frac{\Gamma(1-(L+1)\epsilon)}{\Gamma(1+L\epsilon)} \left( \frac{4}{-x^2} \right)^{1-(L+1)\epsilon} \times \left[ -g^{\mu\nu} + g^{\mu\nu} + 2(1-(L+1)\epsilon) x^\mu x^\nu/(x^2) \right].
\]

(2.9)

The sum of single-web diagrams (fig. 1) in the Landau gauge, analytically continued to Euclidean \( t = -i\tau \), is

\[
\log W = S_1 A + n_l \sum_{L=2}^{\infty} S_L \Pi_{L-1} A^L + (n_l^\gamma \text{ terms}) + (w>2 \text{ legs terms}), \quad A = A_0 \left( \frac{\tau}{2} \right)^{2\epsilon} e^{2\gamma\epsilon},
\]

(2.10)

where the 1-loop HQET integral

\[
S_L = \frac{3 - 2\epsilon}{L\epsilon(1-2L\epsilon)(1+(L-1)\epsilon)} \frac{\Gamma(1-L\epsilon)}{\Gamma(1+(L-1)\epsilon)} e^{-(2L-1)\gamma\epsilon} = \frac{3}{L\epsilon} + 3 + \frac{1}{L} + O(\epsilon)
\]

(2.11)
can be calculated in coordinate space (2.9), or as a Fourier transform of the momentum-space HQET propagator. Now we re-express \( \log W \) via the renormalized \( \alpha(\mu) \) at \( \mu_0 = 2e^{-\gamma/\tau} \) (it is sufficient to do this in the 1-loop term) and obtain

\[
\log W = S_1 \frac{\alpha}{4\pi} + n_l \sum_{L=2}^{\infty} \left[ S_L \left( \frac{\beta_{L-1}}{L-1} \frac{1}{\xi} + \Pi_{L-1} \right) - S_1 \frac{\beta_{L-1}}{L-1} \frac{1}{\xi} \right] \left( \frac{\alpha}{4\pi} \right)^L
+ (n_l^\gamma \text{ terms}) + (w>2 \text{ legs terms}) = \log Z_h + (\text{finite}).
\]

(2.12)

Extracting \( \log Z_h \) and differentiating it in \( \log \mu \), we obtain \( \gamma_h \). Restoring the color factors and adding the gauge dependent term\(^2\), we obtain

\[
\gamma_h = \frac{\alpha_s}{4\pi} \left[ 2(a-3)C_F + T_F n_l \sum_{L=1}^{\infty} (-6\Pi_L + 2\beta_L) \left( C_F \frac{\alpha_s}{4\pi} \right)^L \right] + (\text{other color structures})
\]

\(^2\)In the arbitrary covariant gauge the extra term to be added to \( w_1 \) (2.10) is \( \Gamma(-\epsilon) e^{-\gamma\epsilon} a_0 A = \Gamma(-\epsilon) e^{-\gamma\epsilon} a(\mu_0)\alpha(\mu_0)/(4\pi) \), because in QED \( Z_a = Z_a^{-1} \). Hence the extra term to be added to \( \log Z_h \) is purely 1-loop: \( -(a/\xi)\alpha/(4\pi) \). In QED \( \partial (\alpha(\mu)\alpha(\mu))/\partial \log \mu = -2\epsilon \) exactly, and hence the extra term in \( \gamma_h \) (2.13) is also purely 1-loop: \( 2a\alpha/(4\pi) \).

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\[ = 2(a - 3)C_F \alpha_s^{4} + T_F n_l C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left[ \frac{32}{3} - 6(16\zeta_3 - 17) \right] C_F \alpha_s^{4} + \frac{16}{3} (180\zeta_5 - 111\zeta_3 - 35) \left( C_F \frac{\alpha_s}{4\pi} \right)^2 \\
\quad - 6(2240\zeta_7 - 1960\zeta_5 - 104\zeta_3 - 5) \left( C_F \frac{\alpha_s}{4\pi} \right)^3 + O(\alpha_s^4) \]

\[ + (\text{other color structures}) \, . \] (2.13)

We have reproduced the $C_F^2 T_F n_l$ term in the 3-loop anomalous dimension \[9, 10\] by a simpler method. The coefficient of $C_F^3 T_F n_l (\alpha_s/\pi)^4$ in $\frac{1}{2} \gamma_h$ is

\[ \frac{180\zeta_5 - 111\zeta_3 - 35}{96} \approx 0.189778 \, , \]

in perfect agreement with the numerical result $0.1894 \pm 0.0030$ (Table III in [1]).

### 3 QCD cusp anomalous dimension: the $C_F^{L-1} T_F n_l \alpha_s^L$ terms

Now we consider the Green function $\langle T \{ h_+^\mu(x) J(0) h_\nu(x') \} \rangle$. Up to obvious $\delta$ functions similar to (1.2), it is the broken Wilson line $W(\varphi)$ from $x = -vt$ to 0 and then to $x' = v't'$. Renormalization constants cannot depend on kinematics of Green functions we choose to calculate, and so we choose $t' = t$ to have a single-scale problem. We have

\[ \log \frac{W(\varphi)}{W(0)} = \sum (w_i(\varphi) - w_i(0)) \, , \] (3.1)

where the sum runs over all single-web diagrams. Diagrams in which all photon vertices are to the left (or to the right) of the $J$ vertex cancel in $w_i(\varphi) - w_i(0)$. The remaining 2-leg webs are shown in fig. 2. At 4 loops 4-leg webs appear; they have been calculated (at $\varphi \ll 1$) in [3].

![Figure 2. Cusp: the 2-leg webs.](image)

The the $L$-loop $n_l^L$ contribution is ($L \geq 2$)

\[ w_L(\varphi) = -\tilde{\Pi}_{L-1} n_l A_L^L e^{L\gamma_E} \int_0^t dt_1 \int_0^{t_1} dt_2 v_\mu v_\nu \bar{D}_{L-1}^{\mu\nu} (vt_1 + v't_2) \, , \] (3.2)

where $\bar{D}_{L}^{\mu\nu}(x)$ is given by (2.9). We can write it, together with the 1-loop Landau-gauge contribution, in the form

\[ w_1(\varphi) = V_1(\varphi) A \, , \quad w_L(\varphi) = V_L(\varphi) \tilde{\Pi}_{L-1} n_l A_L^L \, , \] (3.3)
where
\[
V_L(\varphi) = 4 \frac{\Gamma(1 - u)}{\Gamma(1 + u - \varepsilon)} e^{-(2L-1)\gamma_e^L} \left[ -I_1(\varphi) \cosh \varphi + \frac{uI_1(\varphi) \cosh \varphi - (1-u)I_2(\varphi)}{2(1 + u - \varepsilon)} \right],
\]

\( u = L\varepsilon \). The integrals \( I_{1,2}(\varphi) \) are

\[
I_1(\varphi) = \int_0^1 dt_1 \int_0^1 dt_2 (e^{\varphi^2/2t_1} + e^{-\varphi^2/2t_2})^{-1+u}(e^{-\varphi^2/2t_1} + e^{\varphi^2/2t_2})^{-1+u} \]

\[
= \frac{e^{-2u\varphi}}{4u^2 \sinh \varphi} (g_1(\varphi) - g_2(\varphi)), \tag{3.5}
\]

\[
I_2(\varphi) = \int_0^1 dt_1 \int_0^1 dt_2 (e^{\varphi^2/2t_1} + e^{-\varphi^2/2t_2}) u(e^{-\varphi^2/2t_1} + e^{\varphi^2/2t_2})^{-2+u} \]

\[
= \frac{1}{2u(1-u)} \left[ 1 + \frac{e^{-2u\varphi}}{2 \sinh \varphi} (e^{-\varphi} g_1(\varphi) - e^\varphi g_2(\varphi)) \right], \tag{3.6}
\]

\[
I_1(0) = I_2(0) = \frac{2 - 2^2u}{2u(1 - 2u)}, \tag{3.7}
\]

g_1(\varphi) = (e^\varphi + 1)^2u f_1(1 - e^{2\varphi}) - f_1(1 - e^{-2\varphi}), \quad g_2(\varphi) = (e^\varphi + 1)^2u f_2(1 - e^{2\varphi}) - f_2(1 - e^{-2\varphi}),

\[
f_1(x) = 2F_1 \left( \begin{array}{c} -2u, -u \\ 1 - 2u \end{array} \bigg| x \right) = 1 + 2 \text{Li}_2(x) u^2 + \mathcal{O}(u^3), \tag{3.8}
\]

\[
f_2(x) = 2F_1 \left( \begin{array}{c} -2u, 1 - u \\ 1 - 2u \end{array} \bigg| x \right) = 1 + 2 \log(1-x) u + (\log^2(1-x) - 2 \text{Li}_2(x)) u^2 + \mathcal{O}(u^3). \tag{3.9}
\]

We obtain

\[
V_L(\varphi) - V_L(0) = -2 \frac{\varphi \coth \varphi - 1}{L\varepsilon} + \bar{V}(\varphi) + \mathcal{O}(\varepsilon), \tag{3.10}
\]

\[
\bar{V}(\varphi) = \coth \varphi [4 \text{Li}_2(1 - e^{2\varphi}) - 4 \text{Li}_2(1 - e^{-\varphi}) + \varphi (4 \log(e^\varphi + 1) + \varphi) + 2 \log(e^\varphi + 1) - \varphi - 6 \log 2 + 4 = \bar{V}(-\varphi). \]

Similarly to (2.12), we substitute (3.4) into

\[
\log \frac{W(\varphi)}{W(0)} = \log Z_J + \text{(finite)}
\]

and re-express it via \( \alpha(\mu_0) \) (it is sufficient to do this in the 1-loop term). Note that \( \bar{V}(\varphi) \) does not depend on \( L \); as a result, terms \( \tilde{\beta}_{L-1} \bar{V}(\varphi) \) cancel in \( \log Z_J \) (in contrast to the first line of (2.13) where they contributed because of the 1/\( L \) term in (2.11)). Differentiating \( \log Z_J \) we obtain

\[
\Gamma(\varphi) = 4(\varphi \coth \varphi - 1) \frac{\alpha_s}{4\pi} \left[ C_F + T_F n_l \sum_{L=1}^\infty \tilde{\Pi}_L \left( C_F \frac{\alpha_s}{4\pi} \right)^L \right] + \text{(other color structures)}
\]

\[
= 4(\varphi \coth \varphi - 1) C_F \frac{\alpha_s}{4\pi} \left( 1 + T_F n_l \frac{\alpha_s}{4\pi} \left[ \frac{20}{9} + \left( 16\zeta_3 - \frac{55}{3} \right) C_F \frac{\alpha_s}{4\pi} \right] \right).
\]

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\(-2 \left(80\zeta_5 - \frac{148}{3}\zeta_3 - \frac{143}{9}\right) \left(C_F \frac{\alpha_s}{4\pi}\right)^2 + \left(2240\zeta_7 - 1960\zeta_5 - 104\zeta_3 + \frac{31}{3}\right) \left(C_F \frac{\alpha_s}{4\pi}\right)^3 + \mathcal{O}(\alpha_s^4)\right) + \text{(other color structures)}.

Thus we have reproduced the 3-loop \(C_F^2 T_F n_l\) term in \([12, 15]\). The coefficient of \(2T_F n_l C_F^3 (\alpha_s/(4\pi))^4\) in the light-like cusp anomalous dimension \(\Gamma_l\) is

\[-4 \left(80\zeta_5 - \frac{148}{3}\zeta_3 - \frac{143}{9}\right) \approx -31.055431,\]

in perfect agreement with the numerical result \(-31.00 \pm 0.4\) (Table 2 in [2]).

The \(C_L^{-1} T_F n_l\) terms in the quark–antiquark potential in Coulomb gauge are given by a single Coulomb-gluon propagator:

\[
V(\vec{q}) = -\frac{4\pi\alpha_s}{\bar{q}^2} \left[C_F + T_F n_l \sum_{L=1}^{\infty} \bar{\Pi}_L \left(C_F \frac{\alpha_s}{4\pi}\right)^L\right] + \text{(other color structures)}
\]

\[
= -\frac{4\pi\alpha_s}{\bar{q}^2} \left[C_F + T_F n_l \frac{\alpha_s}{4\pi} \left[-\frac{20}{9} + \left(16\zeta_3 - \frac{55}{3}\right) \frac{\alpha_s}{4\pi}\right] - 2 \left(80\zeta_5 - \frac{148}{3}\zeta_3 - \frac{143}{9}\right) \left(C_F \frac{\alpha_s}{4\pi}\right)^2 + \left(2240\zeta_7 - 1960\zeta_5 - 104\zeta_3 + \frac{31}{3}\right) \left(C_F \frac{\alpha_s}{4\pi}\right)^3 + \mathcal{O}(\alpha_s^4)\right] + \text{(other color structures)},
\]

(3.12)

where \(\alpha_s\) is taken at \(\mu^2 = \bar{q}^2\). The terms up to \(\alpha_s^4\) agree with [26]. The cusp anomalous dimension at Euclidean angle \(\varphi_E = \pi - \delta, \delta \to 0\), is related to the quark–antiquark potential [27]

\[
\delta \Gamma(\pi - \delta) \bigg|_{\delta \to 0} = \frac{\bar{q}^2 V(\bar{q})}{4\pi};
\]

(3.13)

this relation follows from conformal invariance, and in QCD it is broken by extra terms proportional to coefficients of the \(\beta\) function [12, 15]. Comparing (3.11) with (3.12), we see that the relation (3.13) for the \(C_L^{-1} T_F n_l\) color structures is valid to all orders in \(\alpha_s\).

4 QED results

The 4-loop anomalous dimension of the QED Bloch–Nordsieck field is now known completely analytically. Adding terms with higher powers of \(n_l\) from [12, 13] and the 4-loop contribution of the webs with 4 legs [3], we obtain

\[
\gamma_h = 2(a - 3) \frac{\alpha}{4\pi} + \frac{32}{3} n_l \left(\frac{\alpha}{4\pi}\right)^2 + \left[-6(16\zeta_3 - 17) + \frac{160}{27} n_l\right] n_l \left(\frac{\alpha}{4\pi}\right)^3 + \left[16 \left(40\zeta_5 + \frac{32}{3}\pi^2 \zeta_3 - 21\zeta_3 - \frac{32}{3}\pi^2 - \frac{35}{3}\right) - 32 \left(\frac{\pi^4}{15} - 12\zeta_3 + \frac{103}{27}\right) n_l - \frac{256}{9} \left(\zeta_3 - \frac{1}{3}\right) n_l^2\right] n_l \left(\frac{\alpha}{4\pi}\right)^4.
\]

(4.1)
Adding terms with higher powers of \( n_l \) [12, 13] and the 4-legs webs contribution [3, 4] (known only up to \( \varphi^6 \)) to (3.11), we obtain the QED cusp anomalous dimension up to 4 loops

\[
\Gamma(\varphi) = 4(\varphi \coth \varphi - 1) \frac{\alpha}{4\pi} \left\{ 1 + n_l \frac{\alpha}{4\pi} \left[ -\frac{20}{9} + \left( 16\zeta_3 - \frac{55}{3} - \frac{16}{27} n_l \right) \frac{\alpha}{4\pi} + \frac{8}{3} \left( 2\zeta_3 - \frac{1}{3} \right) n_l \left( \frac{\alpha}{4\pi} \right)^2 \right] \right\} \\
- \frac{8}{3} \varphi^2 \left[ 80\zeta_5 + \frac{128}{3} \pi^2 \zeta_3 - \frac{40}{3} \zeta_3 - \frac{80}{9} \pi^2 - \frac{143}{9} \right] \\
+ \frac{1}{3} \left( 112\zeta_5 + \frac{512}{75} \pi^2 \zeta_3 - \frac{392}{225} \pi^4 - \frac{6076}{75} \zeta_3 + \frac{1256}{225} \pi^2 + \frac{2371}{225} \right) \varphi^2 \\
- \frac{2}{3} \left( \frac{304}{49} \zeta_5 + \frac{512}{1225} \pi^2 \zeta_3 - \frac{42004}{11025} \zeta_3 - \frac{3368}{33075} \pi^4 + \frac{10664}{33075} \pi^2 - \frac{9341}{33075} \right) \varphi^4 \\
+ O(\varphi^6) \right\} n_l \left( \frac{\alpha}{4\pi} \right)^4.
\] (4.2)

The \( n_l \alpha^4 \) term is known only up to \( \varphi^6 \);

\[
\varphi \coth \varphi - 1 = \frac{\varphi^2}{3} \left( 1 - \frac{\varphi^2}{15} + \frac{2}{315} \varphi^4 + O(\varphi^6) \right).
\]

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