Bayesian Optimal Two-sample Tests in High-dimension

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Abstract

We propose optimal Bayesian two-sample tests for testing equality of high-dimensional mean vectors and covariance matrices between two populations. In many applications including genomics and medical imaging, it is natural to assume that only a few entries of two mean vectors or covariance matrices are different. Many existing tests that rely on aggregating the difference between empirical means or covariance matrices are not optimal or yield low power under such setups. Motivated by this, we develop Bayesian two-sample tests employing a divide-and-conquer idea, which is powerful especially when the difference between two populations is sparse but large. The proposed two-sample tests manifest closed forms of Bayes factors and allow scalable computations even in high-dimensions. We prove that the proposed tests are consistent under relatively mild conditions compared to existing tests in the literature. Furthermore, the testable regions from the proposed tests turn out to be optimal in terms of rates. Simulation studies show clear advantages of the proposed tests over other state-of-the-art methods in various scenarios. Our tests are also applied to the analysis of the gene expression data of two cancer data sets.

Key words: Bayesian hypothesis test; Bayes factor consistency; high-dimensional covariance matrix; optimal high-dimensional tests.

1 Introduction

Consider two samples of observations from high-dimensional normal models

\begin{equation}
\begin{aligned}
X_1, \ldots, X_{n_1} \mid \mu_1, \Sigma_1 \overset{i.i.d.}{\sim} N_p(\mu_1, \Sigma_1), \\
Y_1, \ldots, Y_{n_2} \mid \mu_2, \Sigma_2 \overset{i.i.d.}{\sim} N_p(\mu_2, \Sigma_2),
\end{aligned}
\end{equation}

(1)
where $N_p(\mu, \Sigma)$ is the $p$-dimensional normal distribution with mean vector $\mu \in \mathbb{R}^p$ and covariance matrix $\Sigma \in \mathbb{R}^{p \times p}$, and the number of variables $p$ can increase to infinity as the sample sizes ($n_1$ and $n_2$) grow. Given two samples of such observations, there is abundant interest in testing the homogeneity between two populations through testing the equality of high-dimensional mean vectors or covariance matrices with applications in medical imaging, genetics and biology (Tsai and Chen; 2009; Shen et al.; 2011). Although there is an emerging literature on high-dimensional hypothesis testing, most of the literature has focused on proposing frequentist testing statistics with relatively little work on developing Bayesian hypothesis tests in particular for high-dimensional problems. Bayesian tests, which typically are based on Bayes factor with appropriate design of prior distributions for the model under the null and the alternative operate differently from their frequentist counterparts, and there is independent interest in developing Bayesian testing approaches. We add to the limited literature by developing powerful and scalable Bayesian high-dimensional tests for testing the equality of means and covariance matrices between two populations.

Our initial focus is on the two-sample mean test, where we assume $\Sigma_1 = \Sigma_2$ and test whether $\mu_1 = \mu_2$ in model (1). When $\mu_1 \neq \mu_2$, we call the nonzero elements in the mean difference vector $\mu_1 - \mu_2 \in \mathbb{R}^p$ the signals. It is well known that the types of tests with good power are different depending on the number and magnitude of the signals. From a frequentist perspective, Bai and Saranadasa (1996) and Srivastava and Du (2008) proposed high-dimensional two-sample mean tests based on estimators of $\|A(\mu_1 - \mu_2)\|^2_2$ for some positive definite matrix $A \in \mathbb{R}^{p \times p}$. We call these tests $L_2$-type tests because their test statistics involve the $L_2$-norm. It is known that $L_2$-type tests tend to have good power when there are dense signals, i.e., when a large portion of $\mu_1 - \mu_2$ is nonzero. When there are many but small signals, $L_2$-type tests tend to show better performance over other types of tests.

In many applications, however, it is more natural to assume rare signals, where only few entries of $\mu_1 - \mu_2 \in \mathbb{R}^p$ are nonzero. Under the presence of rare but significant signals, it is well known that maximum-type tests tend to outperform $L_2$-type tests. Here, a maximum-type test refers to a class of tests whose test statistic involves the maximum-norm. Cai et al. (2014) proposed a consistent maximum-type test for high-dimensional two-sample mean test. They standardized the difference between sample mean vectors using an estimated precision matrix based on either the constrained $\ell_1$-minimization for inverse matrix estimation (CLIME) (Cai et al.; 2011) or the inverse of the adaptive thresholding estimator for a covariance matrix (Cai and Liu; 2011). Because their test statistics depend on an estimated precision matrix, practical performance of the tests could be impacted by performance of the estimated precision matrix.

Besides the aforementioned papers, many other interesting studies have been conducted for the two-sample testing setup. Gregory et al. (2015) proposed a two-sample mean test which bypasses the needs of the estimation of precision matrix and is robust to highly unequal covariance matrices between two populations. Xu et al. (2016) proposed an adaptive two-sample mean test that retains
high power against a wide range of alternatives. Cao et al. (2018) developed a test for compositional data based on the centered log-ratio transformation. Recently, Wang et al. (2019) suggested a robust version of the maximum-type test for contaminated data.

Our second focus is the two-sample covariance test of whether $\Sigma_1 = \Sigma_2$ or not in model (1) under the assumption $\mu_1 = \mu_2 = 0$. In this case, we call the nonzero entries in $\Sigma_1 - \Sigma_2 \in \mathbb{R}^{p \times p}$ the signals. Some frequentist tests have also been suggested in the literature for two-sample covariance in high-dimensional settings. Schott (2007) and Li and Chen (2012) proposed to test equality of covariance matrices based on an estimator of $\|\Sigma_1 - \Sigma_2\|_F^2$, whereas Srivastava and Yanagihara (2010) suggested a test based on a consistent estimator of $\text{tr}(\Sigma_1^2)/\{\text{tr}(\Sigma_1)\}^2 - \text{tr}(\Sigma_2^2)/\{\text{tr}(\Sigma_2)\}^2$. These tests can be categorized as $L_2$-type tests. A two-sample covariance test based on super-diagonals was proposed by He and Chen (2018) whose test turned out to be more powerful than other existing tests when $\Sigma_1$ and $\Sigma_2$ have bandable structures. However, the aforementioned tests target dense signals, where most of components of $\Sigma_1 - \Sigma_2$ are nonzero. Thus, they might be less powerful under the rare signals setting, where only a few entries in $\Sigma_1 - \Sigma_2$ are nonzero. To obtain good power when there are rare signals, Cai et al. (2013) proposed a maximum-type test for two-sample covariance test. Similar to two-sample mean test in Cai et al. (2014), Cai et al. (2013) standardized the difference between sample covariances and took the maximum over the standardized sample covariances. Recently, Zheng et al. (2017) combined the two tests in Li and Chen (2012) and Cai et al. (2013) by taking weighted average to handle both dense and sparse alternatives.

Bayesian hypothesis testing has very different characteristics from those of its frequentist counterpart, thus it is important and of an independent interest to develop Bayesian tests for the above hypothesis testing problems. However, up to our knowledge, no theoretically supported Bayesian method has been proposed for high-dimensional two-sample tests, except a recent work of Zoh et al. (2018). They proposed a Bayesian test for high-dimensional two-sample mean test by reducing the dimension of data via random projections. They proved consistency of the proposed Bayesian test under the joint distribution of data and prior, where the true mean vector is a random variable from the prior distribution.

In this paper, we develop scalable Bayesian two-sample tests supported by theoretical guarantees. Since rare signals can be more realistic in many applications, our goal is to develop a consistent Bayesian test achieving good power when there are rare signals. To this end, we apply the maximum pairwise Bayes factor approach suggested by Lee et al. (2021), which is essentially a divide-and-conquer idea. Rather than comparing the whole mean vectors or covariance matrices at once, we divide them into smaller pieces and reformulate the original testing problem into a multiple testing problem. The proposed Bayesian tests turn out to be consistent under both null and alternative hypotheses, and especially, attain good power when there are rare but significant signals. We prove consistency of the Bayesian tests in different context where the true parameter is fixed unknown quantity, which differentiates our results from those in Zoh et al. (2018). Furthermore, the proposed
tests achieve theoretical and practical improvements compared to those in the existing tests, which will be stated later in more detail.

Although we employ the general idea of modularization by Lee et al. (2021), the former work however only focuses on one-sample testing of the structure of covariance matrices. Substantial new developments have been made in this work which differs in terms of problem setup, prior choice, theory development as well as computational approaches. The main contributions of this paper can be summarized as follows. The proposed Bayesian tests are scalable with simple implementations that can be readily used by practitioners. It accelerates the computation speed by circumventing computational issues such as inversion of a large matrix. Furthermore, up to our knowledge, these are the first results on Bayes factor consistency in high-dimensional two-sample testings. We prove that the proposed Bayesian tests are consistent under both null and alternative under mild conditions (Theorems 2.1 and 3.1). The proposed tests have the desired property of being much more powerful than $L_2$-type tests under rare signals settings. Besides the development of new Bayesian methods, our proposal also improves state-of-the-art methods theoretically and empirically. We show that the derived testable regions from the proposed tests are optimal in terms of rates (Theorem 3.2), and the required conditions for achieving the theoretical results are much weaker than those used in existing literature. Furthermore, although there are existing frequentist maximum-type tests (Cai et al.; 2014, 2013), the proposed tests in this paper outperform the contenders in various settings.

The rest of paper is organized as follows. Sections 2 and 3 present the proposed Bayesian two-sample tests for mean vectors and covariance matrices, respectively. In Section 4, the practical performance of the proposed methods is evaluated based on numerical study. Concluding remarks are given in Section 5, and proofs of the main results are included in the supplementary material.

## 2 Two-sample mean test

### 2.1 Notation

For any given constants $a$ and $b$, we denote the maximum and minimum between the two by $a \lor b$ and $a \land b$. For a vector $x = (x_1, \ldots, x_p)^T$ and a positive integer $q$, we denote the vector $\ell_q$-norm as $\|x\|_q = (\sum_{j=1}^p x_j^q)^{1/q}$. For any positive sequences $a_n$ and $b_n$, $a_n \ll b_n$, or equivalently $a_n = o(b_n)$, means that $a_n/b_n \rightarrow 0$ as $n \rightarrow \infty$. We denote $a_n = O(b_n)$ if there exists a constant $C > 0$ such that $a_n/b_n \leq C$ for all large $n$, and $a_n \asymp b_n$ means that $a_n = O(b_n)$ and $b_n = O(a_n)$. For a given matrix $A \in \mathbb{R}^{p \times p}$, we denote the Frobenius norm $\|A\|_F = \left(\sum_{i=1}^p \sum_{j=1}^p a_{ij}^2\right)^{1/2}$, the matrix $\ell_1$-norm $\|A\|_1 = \sup_{x \in \mathbb{R}^p, \|x\|_1 = 1} \|Ax\|_1$, the spectral norm $\|A\| = \sup_{x \in \mathbb{R}^p, \|x\|_2 = 1} \|Ax\|_2$, and the matrix maximum norm $\|A\|_{\max} = \max_{1 \leq i \leq j \leq p} |a_{ij}|$. The maximum and minimum eigenvalues of a matrix $A$ are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively. For given positive numbers $a$ and $b$,
IG$(a, b)$ denotes the inverse-gamma distribution with shape parameter $a$ and rate parameter $b$.

Throughout the paper, we assume that $0 < \epsilon_0 < 1$, $C_1 = 1 + \epsilon, C_2 = 2 + \epsilon$ and $C_3 = 3 + \epsilon$ for some small constant $\epsilon > 0$ in that $C_1, C_2$ and $C_3$ are arbitrarily close to 1, 2 and 3, respectively.

### 2.2 Maximum pairwise Bayes factor for two-sample mean test

Suppose that we observe the data from two populations

$$X_1, \ldots, X_{n_1} \mid \mu_1, \Sigma \stackrel{i.i.d.}{\sim} N_p(\mu_1, \Sigma),$$

$$Y_1, \ldots, Y_{n_2} \mid \mu_2, \Sigma \stackrel{i.i.d.}{\sim} N_p(\mu_2, \Sigma),$$

where $\mu_1, \mu_2 \in \mathbb{R}^p$ and $\Sigma$ is a $p \times p$ covariance matrix. Let $X_{n_1} = (X_1, \ldots, X_{n_1})^T \in \mathbb{R}^{n_1 \times p}$ and $Y_{n_2} = (Y_1, \ldots, Y_{n_2})^T \in \mathbb{R}^{n_2 \times p}$ be the data matrices for each population. We are interested in the testing problem

$$H_0: \mu_1 = \mu_2 \quad \text{versus} \quad H_1: \mu_1 \neq \mu_2.$$ (3)

The goal is to test the homogeneity between two populations based on underlying trends.

Bayesian hypothesis tests are typically based on Bayes factors. To construct a Bayes factor for two-sample mean test, marginal likelihoods should be calculated based on priors for each hypothesis. Using normal priors for mean vectors and the Jeffreys’ prior for a covariance matrix, which corresponds to a default choice, the resulting Bayes factor can be calculated in a closed form when $1 < p < n - 2$. See Zoh et al. (2018) for the details. However, the Bayes factor under such priors involves the inverse of a pooled sample covariance matrix, which prevents one from using when $p \geq n - 2$. Zoh et al. (2018) suggested projecting the data to a lower-dimensional subspace to reduce the dimensionality.

In this paper, we apply the maximum pairwise Bayes factor (mxPBF) approach suggested by Lee et al. (2021). Specifically, we compare two mean vectors by comparing them element-by-element. For a given integer $1 \leq j \leq p$, let $\tilde{X}_j = (X_{1j}, \ldots, X_{nj})^T$ and $\tilde{Y}_j = (Y_{1j}, \ldots, Y_{nj})^T$ be the $j$th columns of $X_{n_1}$ and $Y_{n_2}$, respectively. From model (2), we have the following marginal models

$$\tilde{X}_j \mid \mu_{1j}, \sigma_{jj} \sim N_{n_1}(\mu_{1j}1_{n_1}, \sigma_{jj}I_{n_1}),$$

$$\tilde{Y}_j \mid \mu_{2j}, \sigma_{jj} \sim N_{n_2}(\mu_{2j}1_{n_2}, \sigma_{jj}I_{n_2}),$$

where $\mu_k = (\mu_{k1}, \ldots, \mu_{kp})^T$ for $k = 1, 2$, $\Sigma = (\sigma_{jj})$ and $1_q = (1, \ldots, 1)^T \in \mathbb{R}^q$. The hypothesis testing problem (3) can be reformulated as

$$H_{0j}: \mu_{1j} = \mu_{2j} \quad \text{versus} \quad H_{1j}: \mu_{1j} \neq \mu_{2j},$$

in the sense that $H_0$ is true if and only if $H_{0j}$ is true for all $j = 1, \ldots, p$. Thus, we will first construct Bayesian tests for each testing problem $H_{0j}$ versus $H_{1j}$ and calculate pairwise Bayes factors (PBFs)
based on \((\tilde{X}_j, \tilde{Y}_j)\) for \(j = 1, \ldots, p\). For a given \(1 \leq j \leq p\), we suggest the following prior \(\pi_{0j}(\mu_j, \sigma_{jj})\) under \(H_{0j}\),

\[
\mu_j = \mu_{1j} = \mu_{2j} | \sigma_{jj} \sim N\left(\tilde{Z}_j, \frac{\sigma_{jj}}{n_j}\right),
\]

\[
\pi(\sigma_{jj}) \propto \sigma_{jj}^{-1},
\]

and the following prior \(\pi_{1j}(\mu_{1j}, \mu_{2j}, \sigma_{jj})\) under \(H_{1j}\),

\[
\mu_{1j} | \sigma_{jj} \sim N\left(\tilde{X}_j, \frac{\sigma_{jj}}{n_j}\right),
\]

\[
\mu_{2j} | \sigma_{jj} \sim N\left(\tilde{Y}_j, \frac{\sigma_{jj}}{n_j}\right),
\]

\[
\pi(\sigma_{jj}) \propto \sigma_{jj}^{-1},
\]

where \(\tilde{Z}_j = (\tilde{X}_j^T, \tilde{Y}_j^T)^T\), \(\tilde{Z}_j = n^{-1} \sum_{i=1}^n Z_{ij}\), \(\tilde{X}_j = n^{-1} \sum_{i=1}^{n_1} X_{ij}\), \(\tilde{Y}_j = n^{-1} \sum_{i=1}^{n_2} Y_{ij}\), \(n = n_1 + n_2\) and \(\gamma = (n \vee p)^{-\alpha}\). Throughout this paper, we consider \(\alpha\) as a fixed positive constant.

For any vector \(v\), define the projection matrix \(H_v = v(v^Tv)^{-1}v^T\). Let \(\hat{\sigma}_Z^2 = n^{-1} \tilde{Z}_j^T (I_{n_1} - H_{1n_1}) \tilde{Z}_j\), \(\hat{\sigma}_X^2 = n_1^{-1} \tilde{X}_j^T (I_{n_1} - H_{1n_1}) \tilde{X}_j\) and \(\hat{\sigma}_Y^2 = n_2^{-1} \tilde{Y}_j^T (I_{n_2} - H_{1n_2}) \tilde{Y}_j\). Then, the resulting log PBF is

\[
\log B_{10}(\tilde{X}_j, \tilde{Y}_j) := \log \frac{p(\tilde{X}_j, \tilde{Y}_j | H_{1j})}{p(\tilde{X}_j, \tilde{Y}_j | H_{0j})}
= \frac{1}{2} \log \left(\frac{\gamma}{1 + \gamma}\right) + \frac{n}{2} \log \left(\frac{n \hat{\sigma}_Z^2}{n_1 \hat{\sigma}_X^2 + n_2 \hat{\sigma}_Y^2}\right),
\]

where

\[
p(\tilde{X}_j, \tilde{Y}_j | H_{0j}) = \int \int p(\tilde{X}_j | \mu_j, \sigma_{jj}, H_{0j}) p(\tilde{Y}_j | \mu_j, \sigma_{jj}, H_{0j}) \pi_{0j}(\mu_j, \sigma_{jj}) d\mu_j d\sigma_{jj},
\]

\[
p(\tilde{X}_j, \tilde{Y}_j | H_{1j}) = \int \int \int \int p(\tilde{X}_j | \mu_{1j}, \sigma_{jj}, H_{1j}) p(\tilde{Y}_j | \mu_{2j}, \sigma_{jj}, H_{1j}) \pi_{1j}(\mu_{1j}, \mu_{2j}, \sigma_{jj}) d\mu_{1j} d\mu_{2j} d\sigma_{jj}.
\]

To aggregate PBFs for all \(j = 1, \ldots, p\), we define the mxPBF as

\[
B_{\text{max}, 10}^\mu(X_{n_1}, Y_{n_2}) := \max_{1 \leq j \leq p} B_{10}(\tilde{X}_j, \tilde{Y}_j).
\]

Then one can conduct a Bayesian test by considering the mxPBF as a usual Bayes factor: for a given threshold \(C_{\text{th}} > 0\), we support \(H_1 : \mu_1 \neq \mu_2\) if \(B_{\text{max}, 10}^\mu(X_{n_1}, Y_{n_2}) > C_{\text{th}}\). It is easy to see that \(B_{\text{max}, 10}^\mu(X_{n_1}, Y_{n_2}) > C_{\text{th}}\) if and only if \(B_{10}(\tilde{X}_j, \tilde{Y}_j) > C_{\text{th}}\) for some \(1 \leq j \leq p\). Thus, a Bayesian test based on the mxPBF supports \(H_1 : \mu_1 \neq \mu_2\) if and only if there is at least one strong evidence in favor of \(H_{1j} : \mu_{1j} \neq \mu_{2j}\).

### 2.3 Bayes factor consistency

A mxPBF is said to be consistent if it (i) converges to zero under \(H_0\) and (ii) diverges to infinity under \(H_1\) in probability. Let \(\mu_{0,1} \in \mathbb{R}^p\) and \(\mu_{0,2} \in \mathbb{R}^p\) be true mean vectors for each population,
respectively, and \( \Sigma_0 = (\sigma_{0,ij}) \in \mathbb{R}^{p \times p} \) be the true covariance matrix. Theorem 2.1 shows that the mxPBF is consistent under mild conditions.

**Theorem 2.1** Consider model (2) and the two-sample mean test \( H_0 : \mu_1 = \mu_2 \) versus \( H_1 : \mu_1 \neq \mu_2 \).

Assume that \( \log p \leq n\epsilon_0 \) and

\[
\alpha > \frac{2(1 + \epsilon_0)}{1 - 3\sqrt{C_1\epsilon_0}}
\]

for some \( C_1 > 1 \) and \( 0 < \epsilon_0 < 1 \). Then, the mxPBF (4) is consistent under \( H_0 \): for some constant \( c > 0 \),

\[
\{ B_{\text{max},10}^\mu(X_{n_1}, Y_{n_2}) \}^{-1} = O_p\{ (n \lor p)^{-c} \} \quad \text{under } H_0.
\]

When \( H_1 \) is true, assume that there is at least one of indices \( 1 \leq j \leq p \) satisfying

\[
\frac{n_1n_2(\mu_{0,1j} - \mu_{0,2j})^2}{n^2\sigma_{0,ij}} \geq \left[ \sqrt{2C_1} + \sqrt{2C_1} + \alpha C_1 \{ 1 + (1 + 8C_1)\epsilon_0 \} \right]^2 \frac{\log(n \lor p)}{n}.
\]

Then, the mxPBF is also consistent under \( H_1 \): for some constant \( c' > 0 \),

\[
\{ B_{\text{max},10}^\mu(X_{n_1}, Y_{n_2}) \}^{-1} = O_p\{ (n \lor p)^{-c'} \} \quad \text{under } H_1.
\]

It is worthwhile to compare our result to those of the existing literature. As mentioned earlier, the test statistic of Cai et al. (2014) depends on an estimated precision matrix that some conditions for consistent estimation of the precision matrix are required. For example, it was assumed that \( \Omega_0 \) has bounded eigenvalues and absolute correlations of \( X_i, Y_i, \Omega_0X_i \) and \( \Omega_0Y_i \) are bounded away from 1, where \( \Omega_0 \) is the true precision matrix. Furthermore, \( \Omega_0 \) is assumed to satisfy \( \| \Omega_0 \|_2 = o\left( \frac{\sqrt{n}}{\log p} \right)^3 \) or stronger sparsity assumption, which essentially means that a large amount of entries in \( \Omega_0 \) is sufficiently small. They also assumed that \( \mu_{0,1} - \mu_{0,2} \) has at most \( pr \) nonzero entries, where \( r \in [0, 1/4) \).

On the other hand, theoretical results in Theorem 2.1 do not require any condition on the true precision matrix and allow the number of nonzero entries in \( \mu_{0,1} - \mu_{0,2} \) to have the same order with \( p \). Therefore, we suspect that the mxPBF would perform better than the maximum-type test in Cai et al. (2014) when these conditions are violated. Indeed, we find empirical evidences for this conjecture in our simulation study in Section 4.1.

Recently, Zoh et al. (2018) proposed a Bayesian two-sample mean test and proved consistency of the Bayes factor in high-dimensional settings. They used random projections to reduce the dimensionality of the data and assumed that the reduced dimension has the same order with \( (n_1 \lor n_2) \). To conduct a Bayesian test, a single random projection matrix was considered, which can lead to different results depending on the generated projection matrix. Furthermore, under the true alternative, no lower bound condition of \( \mu_{0,1} - \mu_{0,2} \) was provided to ensure consistency, like...
condition (7). They assumed that $\mu_1 - \mu_2$ is a random vector under $H_1 : \mu_1 \neq \mu_2$ rather than considering a fixed true value $\mu_{0,1} - \mu_{0,2}$, which differentiates our results from those in Zoh et al. (2018).

We note here that, by Theorem 3 in Cai et al. (2014), condition (7) is rate-optimal to guarantee the existence of a consistent test when $\log n = O(\log p)$. Thus, the proposed mxPBF-based test provides an optimal testable region with respect to the maximum norm. Cai et al. (2014) assumed the condition $\max_j (\mu_{0,1j} - \mu_{0,2j})^2 \geq C \log p/n$ for some constant $C > 0$, which is similar to (7).

### 3 Two-sample covariance test

In this section, we propose Bayesian two-sample tests for testing the equity of high-dimensional covariance matrices and consider their theoretical properties in terms of Bayes factor consistency and optimality of the testing regions.

#### 3.1 Maximum pairwise Bayes factor for two-sample covariance test

Suppose that we observe the data from two populations

\[
X_1, \ldots, X_{n_1} \mid \Sigma_1 \overset{i.i.d.}{\sim} N_p(0, \Sigma_1), \\
Y_1, \ldots, Y_{n_2} \mid \Sigma_2 \overset{i.i.d.}{\sim} N_p(0, \Sigma_2),
\]

where $\Sigma_1 = (\sigma_{1,ij})$ and $\Sigma_2 = (\sigma_{2,ij})$ are $p \times p$ covariance matrices. In this section, we consider the testing problem

\[
H_0 : \Sigma_1 = \Sigma_2 \text{ versus } H_1 : \Sigma_1 \neq \Sigma_2.
\]

(9)

To apply the mxPBF approach, we need to divide the comparison of two covariance matrices into smaller problems. Among various options for that, we use the reparametrization trick used in Lee et al. (2021). Specifically, for a given pair $(i, j)$ with $1 \leq i \neq j \leq p$, (8) induces the conditional distributions

\[
\tilde{X}_i \mid \tilde{X}_j, a_{1,ij}, \tau_{1,ij} \sim N_{n_1}(a_{1,ij} \tilde{X}_j, \tau_{1,ij} I_{n_1}), \\
\tilde{Y}_i \mid \tilde{Y}_j, a_{2,ij}, \tau_{2,ij} \sim N_{n_2}(a_{2,ij} \tilde{Y}_j, \tau_{2,ij} I_{n_2}),
\]

where $a_{k,ij} = \sigma_{k,ij}/\sigma_{k,jj}$, $\tau_{k,ij} = \sigma_{k,ii}(1 - \rho_{k,ij}^2)$ and $\rho_{k,ij} = \sigma_{k,ij}/(\sigma_{k,ii} \sigma_{k,jj})^{1/2}$ for $k = 1, 2$. The hypothesis testing problem (9) can be reformulated as

\[
H_{0,ij} : a_{1,ij} = a_{2,ij} \text{ and } \tau_{1,ij} = \tau_{2,ij} \text{ versus } H_{1,ij} : \text{not } H_{0,ij},
\]

(11)

in the sense that $H_0$ is true if and only if $H_{0,ij}$ is true for all pairs $(i, j)$, $1 \leq i \neq j \leq p$. 
To construct a Bayesian test for testing (11), we suggest the following prior distribution \( \pi_{0,ij}(a_{ij}, \tau_{ij}) \) under \( H_{0,ij} \),

\[
\begin{align*}
    a_{ij} &= a_{1,ij} = a_{2,ij} \mid \tau_{ij} \sim N\left(\tilde{a}_{ij}, \frac{\tau_{ij}}{\gamma \|Z_j\|_2^2}\right), \\
    \tau_{ij} &= \tau_{1,ij} = \tau_{2,ij} \sim IG(a_0, b_{0,ij}),
\end{align*}
\]

and the prior \( \pi_{1,ij}(a_{1,ij}, a_{2,ij}, \tau_{1,ij}, \tau_{2,ij}) \) under \( H_{1,ij} \),

\[
\begin{align*}
    a_{1,ij} \mid \tau_{1,ij} &\sim N\left(\tilde{a}_{1,ij}, \frac{\tau_{1,ij}}{\gamma \|X_j\|_2^2}\right), \\
    a_{2,ij} \mid \tau_{2,ij} &\sim N\left(\tilde{a}_{2,ij}, \frac{\tau_{2,ij}}{\gamma \|Y_j\|_2^2}\right), \\
    \tau_{1,ij} &\sim IG(a_0, b_{0,1,ij}), \\
    \tau_{2,ij} &\sim IG(a_0, b_{0,2,ij}),
\end{align*}
\]

where \( a_0, b_{0,ij}, b_{0,1,ij} \) and \( b_{0,2,ij} \) are positive constants, \( \gamma = (n \vee p)^{-\alpha} \), \( \tilde{a}_{ij} = \tilde{Z}_j \tilde{Z}_j / \|\tilde{Z}_j\|_2^2 \), \( \tilde{a}_{1,ij} = \tilde{X}_j \tilde{X}_j / \|\tilde{X}_j\|_2^2 \) and \( \tilde{a}_{2,ij} = \tilde{Y}_j \tilde{Y}_j / \|\tilde{Y}_j\|_2^2 \). Let \( \tilde{\tau}_{ij} = n^{-1} \tilde{Z}_j^T (I_n - H_{\tilde{Z}_j}) \tilde{Z}_i \), \( \tilde{\tau}_{1,ij} = n_{1}^{-1} \tilde{X}_j^T (I_n - H_{\tilde{X}_j}) \tilde{X}_i \) and \( \tilde{\tau}_{2,ij} = n_{2}^{-1} \tilde{Y}_j^T (I_n - H_{\tilde{Y}_j}) \tilde{Y}_i \). The resulting log PBF is given by

\[
\log B_{10}(\tilde{X}_i, \tilde{Y}_i, \tilde{X}_j, \tilde{Y}_j) \tag{12}
\]

\[
:= \log \frac{p(\tilde{X}_i, \tilde{Y}_i \mid \tilde{X}_j, \tilde{Y}_j, H_{1,ij})}{p(\tilde{X}_i, \tilde{Y}_i \mid \tilde{X}_j, \tilde{Y}_j, H_{0,ij})}
\]

\[
= \frac{1}{2} \log \left( \frac{\gamma}{1 + \gamma} \right) + \log \Gamma \left( \frac{n_1}{2} + a_0 \right) + \log \Gamma \left( \frac{n_2}{2} + a_0 \right) - \log \Gamma \left( \frac{n}{2} + a_0 \right) + \log \frac{b_{0,1,ij}^{b_{0,1,ij}} b_{0,2,ij}^{b_{0,2,ij}}}{b_{0,ij}^{b_{0,ij}} \Gamma(a_0)}
\]

\[
- \left( \frac{n_1}{2} + a_0 \right) \log \left( b_{0,1,ij} + n_1 \tilde{\tau}_{1,ij} \right) - \left( \frac{n_2}{2} + a_0 \right) \log \left( b_{0,2,ij} + n_2 \tilde{\tau}_{2,ij} \right)
\]

\[
+ \left( \frac{n}{2} + a_0 \right) \log \left( b_{0,ij} + n \tilde{\tau}_{ij} \right), \tag{13}
\]

where

\[
p(\tilde{X}_i, \tilde{Y}_i \mid \tilde{X}_j, \tilde{Y}_j, H_{0,ij}) = \int \int p(\tilde{X}_i \mid \tilde{X}_j, a_{ij}, \tau_{ij}, H_{0,ij}) p(\tilde{Y}_i \mid \tilde{Y}_j, a_{ij}, \tau_{ij}, H_{0,ij}) \pi_{0,ij}(a_{ij}, \tau_{ij}) da_{ij} d\tau_{ij},
\]

\[
p(\tilde{X}_i, \tilde{Y}_i \mid \tilde{X}_j, \tilde{Y}_j, H_{1,ij}) = \int \int \int \int p(\tilde{X}_i \mid \tilde{X}_j, a_{1,ij}, \tau_{1,ij}, H_{1,ij}) p(\tilde{Y}_i \mid \tilde{Y}_j, a_{2,ij}, \tau_{2,ij}, H_{1,ij})
\]

\[
\times \pi_{1,ij}(a_{1,ij}, a_{2,ij}, \tau_{1,ij}, \tau_{2,ij}) da_{1,ij} da_{2,ij} d\tau_{1,ij} d\tau_{2,ij}.
\]

Then, the mxPBF for two-sample covariance test is given by

\[
B_{\max,10}(X_{n_1}, Y_{n_2}) := \max_{i \neq j} B_{10}(\tilde{X}_i, \tilde{Y}_i, \tilde{X}_j, \tilde{Y}_j). \tag{15}
\]

Similar to the two-sample mean test, one can conduct a Bayesian test by supporting \( H_1 : \Sigma_1 \neq \Sigma_2 \) if \( B_{\max,10}(X_{n_1}, Y_{n_2}) > C_{th} \) for a given threshold \( C_{th} > 0 \).

### 3.2 Bayes factor consistency

In this section, we show that the mxPBF in (15) is consistent for high-dimensional two-sample covariance test. We first introduce sufficient conditions that guarantee consistency of the mxPBF. The first condition, (A1), roughly means that \( p = O(\exp(n^c)) \) for some \( 0 < c < 1 \).
\((A1)\) \(\epsilon_{0k} := \log(n \lor p)/n_k = o(1)\) for \(k = 1, 2\).

When \(H_0 : \Sigma_1 = \Sigma_2\) is true, we denote \(\Sigma_0\) as the true covariance matrix. Furthermore, we define \(a_{0,ij} = \sigma_{0,ij}/\sigma_{0,jj}\), \(\tau_{0,ij} = \sigma_{0,ii}(1 - \rho_{0,ij}^2)\), and \(R_0 = (\rho_{0,ij})\) is a correlation matrix. Condition \((A2)\) is a sufficient condition for consistency under the null \(H_0 : \Sigma_1 = \Sigma_2\).

\((A2)\) \(\min_{i \neq j} \tau_{0,ij} \gg \{\log(n \lor p)\}^{-1}\).

Condition \((A2)\) is satisfied if \(\min_{1 \leq i \leq p} \sigma_{0,ii} > \epsilon\) and \(\max_{i \neq j} \rho_{0,ij}^2 < 1 - \epsilon\) for some small constant \(\epsilon > 0\). However, in fact, condition \((A2)\) allows more general cases where possibly \(\sigma_{0,ii} \rightarrow 0\) and \(\rho_{0,ij}^2 \rightarrow 1\) as \(p \rightarrow \infty\) at certain rates.

When \(H_1 : \Sigma_1 \neq \Sigma_2\) is true, we denote \(\Sigma_{01} = (\sigma_{01,ij})\) and \(\Sigma_{02} = (\sigma_{02,ij})\) as the true covariance matrices for each population. Furthermore, we define \(a_{01,ij} = \sigma_{01,ij}/\sigma_{01,jj}\), \(\tau_{01,ij} = \sigma_{01,ii}(1 - \rho_{01,ij}^2)\) and \(R_{01} = (\rho_{01,ij})\) is a correlation matrix for \(k = 1, 2\). Under the alternative \(H_1 : \Sigma_1 \neq \Sigma_2\), we assume that \((\Sigma_{01}, \Sigma_{02})\) satisfies condition \((A3)\) or \((A3^*)\):

\((A3)\) There exists a pair \((i, j)\) with \(i \neq j\) such that

\[
\{\log(n \lor p)\}^{-1} \leq \tau_{01,ij} \lor \tau_{02,ij} \leq \tau_{01,ij} \lor \tau_{02,ij} \ll (n \lor p),
\]

satisfying either

\[
\frac{\tau_{01,ij}}{\tau_{02,ij}} > \frac{1 + C_{bm}\sqrt{\epsilon_{01}}}{1 - 4\sqrt{C_1(\epsilon_{01} \lor \epsilon_{02})}},
\]

or

\[
\frac{\tau_{02,ij}}{\tau_{01,ij}} > \frac{1 + C_{bm}\sqrt{\epsilon_{02}}}{1 - 4\sqrt{C_1(\epsilon_{01} \lor \epsilon_{02})}},
\]

for some constants \(C_{bm}^2 > 8(\alpha + 1)\) and \(C_1 > 1\).

\((A3^*)\) There exists a pair \((i, j)\) with \(i \neq j\) such that \(\sigma_{01,ii} \lor \sigma_{02,ii} \ll (n \lor p)\) and

\[
(a_{01,ij} - a_{02,ij})^2 \geq \frac{25}{2} C_1 \sum_{k=1}^2 \left\{ \frac{\tau_{0k,ij}\epsilon_{0k}}{\sigma_{0k,ij}(1 - 2\sqrt{C_1\epsilon_{0k}})} \right\}, \tag{16}
\]

\[
(a_{01,ij} - a_{02,ij})^2 \geq \frac{10n}{n + 2a_0} \sum_{k=1}^2 \left\{ \frac{\epsilon_{0k}}{\sigma_{0k,ij}(1 - 2\sqrt{C_1\epsilon_{0k}})} \right\} \tag{17}
\times \left[ \frac{b_{0,ij}}{\log(n \lor p)} + \left\{ \sum_{k=1}^2 \sigma_{0k,ii}(1 + 4\sqrt{C_1\epsilon_{0k}}) + \frac{2b_{0,ij}}{n} \right\} C_{bm,a} \right]
\]

for some constants \(C_{bm,a} > \alpha + a_0 + 1\) and \(C_1 > 1\).
Conditions (A3) and (A3*) may seem complicated at first glance, but it can be transformed into simpler conditions. For given positive constants $\alpha, C_{bm}$ and $C_{bm,a}$ such that $C_{bm}^2 > 8(\alpha + 1)$ and $C_{bm,a} > \alpha + 1$, define a class of two covariance matrices

$$H_1(C_{bm}, C_{bm,a}) := \{(\Sigma_1, \Sigma_2) : (\Sigma_1, \Sigma_2) \text{satisfies condition (A3) or (A3*)}\}.$$ 

Conditions (A3) and (A3*) specify the minimum difference condition between $\Sigma_{01}$ and $\Sigma_{02}$ to consistently detect the alternative $H_1 : \Sigma_1 \neq \Sigma_2$ under the reparametrization using $\{a_{0k,ij}, \tau_{0k,ij} : k = 1, 2 \text{ and } 1 \leq i \neq j \leq p\}$. Suppose that

$$\max_{1 \leq k \leq 21} \max_{1 \leq i \neq j \leq p} \rho_{0k,ij}^2 \leq 1 - c_0,$$

$$\{\log(n \vee p)\}^{-1} \ll \min_{1 \leq k \leq 21} \min_{1 \leq i \leq p} \sigma_{0k,ii} \leq \max_{1 \leq k \leq 21} \max_{1 \leq i \leq p} \sigma_{0k,ii} \ll (n \vee p),$$

for some small constant $c_0 > 0$. If $\alpha > 1$, $n_1 \asymp n_2$ and

$$\tilde{H}_1(C_*, c_0) := \{(\Sigma_1, \Sigma_2) : \max_{1 \leq i \leq j \leq p} \frac{(\sigma_{1,ij} - \sigma_{1,ij})^2}{\sigma_{1,ii}\sigma_{1,jj} + \sigma_{2,ii}\sigma_{2,jj}} \geq C_* \frac{\log(n \vee p)}{n},$$

$$(\Sigma_1, \Sigma_2) \text{satisfies conditions in (18) with } c_0 \}.$$ 

then $\tilde{H}_1(C_*, c_0) \subset H_1(C_{bm}, C_{bm,a})$ for some large constant $C_* > 0$ by Lemma 3.1 in the supplementary material. Condition (19) characterizes the difference between $\Sigma_{01}$ and $\Sigma_{02}$ using the squared maximum standardized difference. Hence, conditions (A3) and (A3*) can essentially be understood as the squared maximum standardized difference condition given at (19). Cai et al. (2013) also used a similar difference measure between $\Sigma_{01}$ and $\Sigma_{02}$.

The following theorem shows consistency of the mxPBF (15). We note that the condition $\lim_{(n_1 \wedge n_2) \to \infty} n_1/n = 1/2$ in Theorem 3.1 can be relaxed to $n_1 \asymp n_2$, although constants in conditions (A2), (A3) and (A3*) should be changed accordingly.

**Theorem 3.1** Consider model (8) and the two-sample covariance test $H_0 : \Sigma_1 = \Sigma_2$ versus $H_1 : \Sigma_1 \neq \Sigma_2$. Assume that $\lim_{(n_1 \wedge n_2) \to \infty} n_1/n = 1/2$ and condition (A1) holds. Then, under $H_0$, if $\alpha > 6C_2$ and condition (A2) holds, for some constant $c > 0$,

$$B_{\max, 10}^\Sigma(\mathbf{X}_{n_1}, \mathbf{Y}_{n_2}) = O_p((n \vee p)^{-c}).$$

Under $H_1$, if $(\Sigma_{01}, \Sigma_{02}) \in H_1(C_{bm}, C_{bm,a})$, for some constant $c' > 0$,

$$\{B_{\max, 10}^\Sigma(\mathbf{X}_{n_1}, \mathbf{Y}_{n_2})\}^{-1} = O_p((n \vee p)^{-c'}).$$

Cai et al. (2013) considered a high-dimensional setting, $(\log p)^5 = o(n_k)$, while we assume a weaker condition, $\log p = o(n_k)$ for $k = 1, 2$ (condition (A1)). For given constants $C > 0$ and $0 < r < 1$, define $s_j(C) = \text{card}\{i : |\rho_{01,ij}| \geq (\log p)^{1-r} \}$ or $|\rho_{02,ij}| \geq (\log p)^{1-C}$ and $\Lambda(r) = \{i : \Sigma_1 \neq \Sigma_2 \}$.

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$|\rho_{01,ij}| > r$ or $|\rho_{02,ij}| > r$ for some $j \neq i$, where $\text{card}(A)$ means the cardinality of the set $A$. Cai et al. (2013) assumed that there exist $\Gamma \subset \{1, \ldots, p\}, C > 0$ and $0 < r < 1$ such that $\text{card}(\Gamma) = o(p)$, $\max_{j \in \Gamma} s_j(C) = o(p^c)$ for some constant $c > 0$, and $\text{card}(\Lambda(r)) = o(p)$. These conditions essentially restrict the number of highly correlated variables. They are satisfied if $\lambda_{\max}(R_{01}) \lor \lambda_{\max}(R_{02}) \leq C$ for some constant $C > 0$ and $\|R_{01}\|_{\max} \lor \|R_{02}\|_{\max} \leq r < 1$. The power of their test tends to one if $\lambda_{\max}(R_{01}) \lor \lambda_{\max}(R_{02}) \leq C'$ for some constant $C' > 0$ and $\|R_{01}\|_{\max} \lor \|R_{02}\|_{\max} \leq r < 1$.

One of the interesting findings from Theorem 3.1 is that the mxPBF does not require any standardization step. Cai et al. (2013) mentioned that the standardization of the test statistic is necessary to deal with a wide range of variability and heteroscedasticity of sample covariances.

However, the mxPBF (15) still enjoys consistency for the similar parameter space without standardization. Although we did not mention earlier, a similar phenomenon is observed for the two-sample mean test: the proposed mxPBF (5) does not require a standardization step while having similar properties with a standardized test.

Another important finding is that condition (A3) (or (A3*)) is rate-optimal to guarantee consistency under $H_0 : \Sigma_1 = \Sigma_2$ as well as $H_1 : \Sigma_1 \neq \Sigma_2$. Theorem 3.2 shows that, for some small constants $C_{bm}$ and $C_{bm,a} > 0$, there is no consistent test having power tending to one for any true alternative satisfying $(\Sigma_{01}, \Sigma_{02}) \in H_1(C_{bm}, C_{bm,a})$.

**Theorem 3.2** Let $E_{\Sigma_{01}, \Sigma_{02}}$ be the expectation corresponding to model (8) with $(\Sigma_{01}, \Sigma_{02})$. Suppose that $n_1 \sim n_2$ and $p \geq n^c$ for some constant $c > 0$. Then, there exists small constants $C_{bm}$ and $C_{bm,a} > 0$ such that for any $0 < \alpha_0 < 1$ and all large $n$ and $p$,

$$\inf_{(\Sigma_{01}, \Sigma_{02}) \in H_1(C_{bm}, C_{bm,a})} \sup_{\phi \in \mathcal{T}} E_{\Sigma_{01}, \Sigma_{02}} \phi \leq \alpha_0 + o(1),$$

where $\mathcal{T}$ is the set of consistent tests over the multivariate normal distributions such that $E_0 \phi \rightarrow 0$ as $n \rightarrow \infty$ for any $\phi \in \mathcal{T}$, and $E_0$ is the expectation corresponding to model (8) under $H_0 : \Sigma_1 = \Sigma_2$.

## 4 Numerical results

### 4.1 Simulation study: two-sample mean test

In this section, we illustrate performance of the mxPBF for two-sample mean test through simulation studies. We generate the data as follows: $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \mathcal{N}_p(\mu_{01}, \Sigma_0)$ and $Y_1, \ldots, Y_n \overset{i.i.d.}{\sim}$
\( N_p(\mu_{02}, \Sigma_0) \) with \( n = 100 \) and \( p \in \{100, 300\} \). Under the null hypothesis, \( H_0 : \mu_{01} = \mu_{02} \), we set \( \mu_{01} = \mu_{02} = 0 \in \mathbb{R}^p \). Under the alternative hypothesis, \( H_1 : \mu_{01} \neq \mu_{01} \), we set \( \mu_{01} = 0 \in \mathbb{R}^p \) and randomly choose \( n_0 \) entries in \( \mu_{02} \), say \( \{\mu_{02,j} : 1 \leq j_1 < \cdots < j_{n_0} \leq p\} \), and set \( \mu_{02,j} = \mu > 0 \) for all \( j = j_1, \ldots, j_{n_0} \) and \( \mu_{02,j} = 0 \) for the rest. Thus, \( n_0 \) and \( \mu \) are the number and magnitude of signals in the alternative, respectively. Here, signals mean nonzero elements in \( \mu_{02} - \mu_{01} \in \mathbb{R}^p \). In our simulation study, the following scenarios for alternatives are considered:

1. \((H_{1R}: \text{Rare signals})\) To demonstrate a situation where only a few signals exist, we set \( n_0 = 5 \) and consider various magnitudes of signals \( \mu \in \{0.2, 0.25, 0.3, 0.35, 0.4, 0.5, 0.6, 0.8, 1.0, 1.5\} \).

2. \((H_{1M}: \text{Many signals})\) To demonstrate a situation where a lot of signals exist, we set \( n_0 = p/2 \) and consider various magnitudes of signals \( \mu \in \{0.025, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.4, 0.5, 0.6\} \).

Note that relatively smaller signals are used compared to “rare signals” setting, due to the larger number of signals.

Furthermore, we consider the following two settings for the true covariance matrix \( \Sigma_0 \):

1. \((\text{Sparse} \ \Omega_0 = \Sigma_0^{-1})\) To demonstrate a situation where the true precision matrix is sparse, we randomly choose 1% of entries in \( \Omega_0 = (\omega_{0,ij}) \) and set their value to \( \omega_{0,ij} = 0.3 \). The rest of entries in \( \Omega_0 \) are set to 0. When the resulting \( \Omega_0 \) is not positive definite, we make it positive definite by adding \( \{-\lambda_{\min}(\Omega_0) + 0.13\} I_p \) to \( \Omega_0 \). Finally, we set \( \Sigma_0 = \Omega_0^{-1} \).

2. \((\text{Dense} \ \Omega_0 = \Sigma_0^{-1})\) To demonstrate a situation where the true precision matrix is dense, we randomly choose 40% of entries in \( \Omega_0 \) and set their value to \( \omega_{0,ij} = 0.3 \). The rest of the steps for constructing \( \Sigma_0 \) is the same as above.

In each setting and hypothesis, 50 simulated data sets are generated. For the proposed mxPBF-based two-sample mean test, the hyperparameter \( \alpha \) is set to \( \alpha = 2.01 \) to satisfy condition (6). As contenders, we consider the tests proposed by Bai and Saranadasa (1996), Srivastava and Du (2008) and Cai et al. (2014), which will be simply denoted as BS, SD and CLX, respectively. Here, CLX means the two-sample mean test based on the CLIME, while CLX.AT refers to the two-sample mean test based on the inverse of the adaptive thresholding estimator with the tuning parameter \( \delta = 2 \) as a default choice. Note that BS and SD are \( L_2 \)-type tests, while mxPBF, CLX and CLX.AT are maximum-type tests. It is expected that \( L_2 \)-type tests perform better (worse) than maximum-type tests in “many signals” (“rare signals”) setting. To illustrate performance of each test, receiver
operating characteristic (ROC) curves are drawn. Points of the curves are obtained by adjusting thresholds and significance levels for the mxPBF and frequentist tests, respectively.

Furthermore, we compare the performance of each test at a fixed threshold or significance level. Note that we need to fix threshold and significance level in practice. As default choices, threshold $C_{th} = 10$ and significance level 0.05 are used. Note that $C_{th} = 10$ corresponds to “strong evidence” for the alternative hypothesis based on the criteria suggested by Jeffreys (1998) and Kass and Raftery (1995).

Figure 1 shows ROC curves based on 50 simulated data sets for each hypothesis, $H_0 : \mu_{01} = \mu_{02}$ and $H_{1R} : \mu_{01} \neq \mu_{02}$, with $p = 100$. Here, $H_{1R}$ represents the “rare signals” scenario where $\mu_{02} - \mu_{01} \in \mathbb{R}^p$ has only five nonzero elements with size $\mu$. The dots in Figure 1 show the results with $C_{th} = 1$ for the mxPBF and significance level 0.05 for frequentist tests. When the true precision matrix $\Omega_0$ is sparse, the maximum-type tests overall slightly work better than the $L_2$-type tests as expected. However, when the true precision matrix $\Omega_0$ is dense, we find that performance of CLX is not satisfactory. We suspect this is because, as mentioned earlier, CLX relies on an estimated precision matrix by the CLIME. In fact, we confirmed that the performance of the CLIME is worse in the dense $\Omega_0$ setting than in the sparse $\Omega_0$ setting, which supports our conjecture. On the other hand, the mxPBF and CLX.AT outperform other tests in the dense $\Omega_0$ setting. When $\mu \leq 0.5$, CLX.AT works better than the mxPBF in terms of the area under the curve (AUC), while when $\mu \geq 0.8$, the two tests produce quite similar ROC curves. However, we find that CLX.AT with significance level 0.05 tends to have low specificity in the dense $\Omega_0$ setting. On the other hand, the mxPBF-based Bayesian test with $C_{th} = 1$ performs reasonably well. Especially in the dense $\Omega_0$ setting, when there are detectable signals ($\mu \geq 0.8$), its specificity and sensitivity are close to 1, while other tests suffer from low specificity or low sensitivity. This clearly shows the relative advantage of the mxPBF-based two-sample mean test over the existing maximum-type tests, CLX and CLX.AT.

Figure 2 shows ROC curves based on 50 simulated data sets for each hypothesis, $H_0 : \mu_{01} = \mu_{02}$ and $H_{1M} : \mu_{01} \neq \mu_{02}$, with $p = 100$. Here, $H_{1M}$ represents the “many signals” scenario where $\mu_{02} - \mu_{01} \in \mathbb{R}^p$ has $p/2 = 50$ nonzero elements with size $\mu$. When the true precision matrix $\Omega_0$ is sparse, overall, the $L_2$-type tests slightly work better than the maximum-type tests as expected. However, when the true precision matrix $\Omega_0$ is dense, somewhat surprisingly, the mxPBF outperforms the $L_2$-type tests. This observation can be partially explained by theoretical properties of the $L_2$-type tests: Bai and Saranadasa (1996) and Srivastava and Du (2008) showed that powers of their tests decrease as the Frobenius norm of the true covariance and correlation matrices increase, respectively. Indeed, in our simulations, we find that $\|\Sigma_0\|_F$ and $\|R_0\|_F$ are much larger in the dense $\Omega_0$ setting than in the sparse $\Omega_0$ setting. We further confirmed that, when $H_{1M}$ is true, the $L_2$-type tests tend to fail to reject $H_0$ even when the size of signals is large. Therefore, this observation suggests another advantage of the mxPBF that reasonable performance is maintained even when $\|\Sigma_0\|_F$ is large.
Figure 1: ROC curves for the two-sample mean tests based on 50 simulated data sets for each hypothesis, $H_0$ and $H_{1R}$, with $p = 100$. The mxPBF, SD and BS represent the test proposed in this paper, Srivastava and Du (2008) and Bai and Saranadasa (1996), respectively. The CLX and CLX.AT mean the tests proposed by Cai et al. (2014) based on the CLIME and the adaptive thresholding estimator, respectively. The dots show the results with $C_{th} = 1$ for the mxPBF and significance level 0.05 for frequentist tests.
Figure 2: ROC curves for the two-sample mean tests based on 50 simulated data sets for each hypothesis, $H_0$ and $H_{1M}$, with $p = 100$. 
Again, the dots in Figure 2 show the results with $C_{th} = 1$ for the mxPBF and significance level 0.05 for frequentist tests. The performance of the mxPBF-based Bayesian test with $C_{th} = 1$ seems reasonable although it is a bit conservative in the sparse $\Omega_0$ setting. In the dense $\Omega_0$ setting, however, the mxPBF clearly outperforms other tests when there are detectable signals ($\mu \geq 0.4$). Similar to $H_{1R}$ setting, the other tests suffer from low specificity or low sensitivity even when $\mu \geq 0.4$.

When $p = 300$, similar phenomena are observed, thus we omit it here for reasons of space. The results with $p = 300$ including ROC curves and descriptions are deferred to the Supplementary material.

4.2 Simulation study: two-sample covariance test

Now, we illustrate performance of the mxPBF for two-sample covariance test. We generate the data as follows: $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N_p(0, \Sigma_{01})$ and $Y_1, \ldots, Y_n \overset{i.i.d.}{\sim} N_p(0, \Sigma_{02})$ with $n = 100$ and $p \in \{100, 300\}$. Under the null hypothesis, $H_0 : \Sigma_{01} = \Sigma_{02}$, we set $\Sigma_{01} = \Sigma_{02} \equiv \Sigma_0 \in \mathbb{R}^{p \times p}$. Under the alternative hypothesis, $H_1 : \Sigma_{01} \neq \Sigma_{02}$, we set $\Sigma_{01} \equiv \Sigma_0$ and $\Sigma_{02} = \Sigma_0 + U$ for some matrix $U \in \mathbb{R}^{p \times p}$ containing signals. If $\Sigma_{01}$ or $\Sigma_{02}$ is not positive definite, we add a small diagonal matrix $\delta_1 I_p$ to them, where $\delta_1 = \lvert \min\{\lambda_{\min}(\Sigma_{01}), \lambda_{\min}(\Sigma_{02})\}\rvert + 0.05$. In our simulation study, the following two scenarios for alternatives are considered:

1. ($H_{1R}$: Rare signals) To demonstrate a situation where only few signals exist, we randomly select five entries in the lower triangular part of $U$ and generate their values from Unif(0, $\rho$) with $\rho \in \{0.5, 0.8, 1.5, 3, 6, 15\}$.

2. ($H_{1M}$: Many signals) To demonstrate a situation where a lot of signals exist, we generate $u = (u_1, \ldots, u_p)^T$ from $u_j \overset{i.i.d.}{\sim} \text{Unif}(0, \rho)$ for $\rho \in \{0.2, 0.3, 0.5, 0.7, 1, 1.5\}$.

Then, we set $U = uu^T$ that leads to $p(p + 1)/2$ signals in $U$ (except upper triangular part). Note that relatively smaller signals are used compared to “rare signals” setting, due to the larger number of signals.

Note that in the above, $\rho$ is the magnitude of signals. Furthermore, we consider the following two settings for $\Sigma_{01}$:

1. (Sparse $\Sigma_{01}$) To demonstrate a situation where $\Sigma_{01}$ is sparse, we randomly choose 5% of entries in $\Delta_1 = (\delta_{1,jk})$ and set their value to $\omega_{0,ij} = 0.5$. The rest of entries in $\Delta_1$ are set to 0. To make it positive definite, we set $\Delta = \Delta_1 + \delta I_p$, where $\delta = \lvert \lambda_{\min}(\Delta_1)\rvert + 0.05$. Finally, we set
\[ \Sigma_{01} = D^{1/2} \Delta D^{1/2}, \text{ where } D = diag(d_j) \text{ and } d_j \sim \text{Unif}(0.5, 2.5). \] This setting corresponds to Model 3 in Cai et al. (2013).

2. (Dense \( \Sigma_{01} \)) To demonstrate a situation where \( \Sigma_{01} \) is dense, we set \( \Sigma_{01} = O \Delta O \), where \( O = \text{diag}(\omega_j) \) and \( \omega_j \sim \text{Unif}(1, 5) \), \( \Delta = (\delta_{ij}) \) and \( \delta_{ij} = (-1)^{i+j}0.4|i-j|^{1/10} \). This setting corresponds to Model 4 in Cai et al. (2013).

In each setting and hypothesis, we generate 50 simulated data. The mxPBF (12) have hyperparameters \( a_0, b_{0,ij}, b_{01,ij}, b_{02,ij} \) and \( \alpha \). We suggest using \( a_0 = b_{0,ij} = b_{01,ij} = b_{02,ij} = 0.01 \) for all \( 1 \leq i \neq j \leq p \). Note that by the proof of Theorem 3.1, the leading terms affect the asymptotic behavior of the mxPBF (12) while including above hyperparameters are (13) and (14). Thus, it can be considered that the above choice leads to noninformative priors that have little effect on the mxPBF. By Theorem 3.1, \( \alpha > 6C_2 \) is required for consistency under the null. This choice roughly means \( \alpha \) is slightly larger than 12, but we found this to overly conservative in practice. Therefore, we suggest using \( \alpha = 2.01 \) similar to the two-sample mean test.

For comparison, we consider the tests proposed by Schott (2007), Li and Chen (2012) and Cai et al. (2013), which will be denoted as Sch, LC and CLX, respectively. Note that Sch and LC are \( L_2 \)-type tests, while mxPBF and CLX are maximum-type tests. Because the unbiased version of the test in Li and Chen (2012) is computationally expensive, we use the biased version as suggested by Li and Chen (2012). In our settings, we confirm that the biased version gives quite similar results to unbiased version. Similar to the simulation study for two-sample mean test, ROC curves are drawn to demonstrate performance of each test.

Figure 3 shows ROC curves based on 50 simulated data sets for each hypothesis, \( \mathcal{H}_0 : \Sigma_{01} = \Sigma_{02} \) and \( \mathcal{H}_{1R} : \Sigma_{01} \neq \Sigma_{02}, \) with \( p = 100 \). When \( \Sigma_{01} \) is sparse and signals are moderate (\( \rho \geq 1.5 \)), the maximum-type tests work better than the \( L_2 \)-type tests as expected. The performance of the \( L_2 \)-type tests are slowly improved as \( \rho \) gets larger. Similar phenomena are observed in the dense \( \Sigma_{01} \) setting, but in this case, the \( L_2 \)-type tests do not work well even when there are large signals (\( \rho = 15 \)). Overall, we find that the mxPBF shows better performance than CLX.

Figure 4 shows ROC curves based on 50 simulated data sets for each hypothesis, \( \mathcal{H}_0 : \Sigma_{01} = \Sigma_{02} \) and \( \mathcal{H}_{1M} : \Sigma_{01} \neq \Sigma_{02} \), with \( p = 100 \). As expected, the \( L_2 \)-type tests slightly work better than the maximum-type tests when \( \Sigma_{01} \) is sparse. Note that the performance of the maximum-type tests are also rapidly improved as the signal \( \rho \) gets larger. Somewhat surprisingly, when \( \Sigma_{01} \) is dense and signals are moderate (\( \rho \geq 1 \)), the maximum-type tests outperform the \( L_2 \)-type tests. We suspect that it is likely that the conditions for deriving the null distribution of Sch and LC are violated in the dense \( \Sigma_{01} \) setting. Schott (2007) and Li and Chen (2012) assumed that \( \lim_{p \to \infty} \text{tr}(\Sigma^{i}_0)/p = \gamma_i \in (0, \infty) \) for \( i = 1, \ldots, 8 \) and \( \text{tr}(\Sigma^{i}_0) = O\{\text{tr}(\Sigma^{2}_0)^2 \} \), respectively, to derive the null distribution. In our settings, we find that \( \text{tr}(\Sigma^{i}_0)/p \) and \( \text{tr}(\Sigma^{i}_0)/\text{tr}(\Sigma^{2}_0)^2 \) are much larger in the dense \( \Sigma_{01} \) setting than in the sparse \( \Sigma_{01} \) setting. This partially supports our conjecture, although more rigorous investigation
Figure 3: ROC curves for the two-sample covariance tests based on 50 simulated data sets for each hypothesis, $H_0$ and $H_{1R}$, with $p = 100$. The mxPBF, CLX, LC and Sch represent the test proposed in this paper, Cai et al. (2013), Li and Chen (2012) and Schott (2007), respectively. The dots show the results with $C_{th} = 1$ for the mxPBF and significance level 0.05 for frequentist tests.
Figure 4: ROC curves for the two-sample covariance tests based on 50 simulated data sets for each hypothesis, $H_0$ and $H_{1M}$, with $p = 100$. 

- Sparse $\Sigma_{G1} / H_{1M}$ with $\rho=0.2$
- Sparse $\Sigma_{G1} / H_{1M}$ with $\rho=0.3$
- Sparse $\Sigma_{G1} / H_{1M}$ with $\rho=0.5$
- Sparse $\Sigma_{G1} / H_{1M}$ with $\rho=0.7$
- Sparse $\Sigma_{G1} / H_{1M}$ with $\rho=1$
- Sparse $\Sigma_{G1} / H_{1M}$ with $\rho=1.5$
- Dense $\Sigma_{G1} / H_{1M}$ with $\rho=0.2$
- Dense $\Sigma_{G1} / H_{1M}$ with $\rho=0.3$
- Dense $\Sigma_{G1} / H_{1M}$ with $\rho=0.5$
- Dense $\Sigma_{G1} / H_{1M}$ with $\rho=0.7$
- Dense $\Sigma_{G1} / H_{1M}$ with $\rho=1$
- Dense $\Sigma_{G1} / H_{1M}$ with $\rho=1.5$
might be needed to determine the exact cause.

The dots in Figures 3 and 4 show the results with $C_{th} = 1$ for the mxPBF or significance level 0.05 for frequentist tests. The mxPBF and CLX with these default choices seem to work well if there is a reasonable amount of signals. On the other hand, the overall performances of LC and Sch with significance level 0.05 are not satisfactory, especially in the sparse $\Sigma_{01}$ setting.

Lastly, we note that the experiment for $p = 300$ showed similar phenomena whose results including ROC curves and descriptions are deferred to the Supplementary material due to lack of space.

4.3 Real data analysis

In this section, we apply the proposed two-sample mean and covariance tests to two real datasets, small round blue cell tumors (SRBCT) dataset and prostate cancer dataset, respectively. For both datasets, the sample sizes are quite small compared to the number of variables. Thus, based on this numerical study, we would like to illustrate the practical performance of mxPBF-based tests in “small $n$ large $p$” situations.

We first apply two-sample mean tests to the SRBCT dataset. The SRBCT dataset is available in the R package plsgenomics. This is a gene expression data having 83 samples with 2308 genes ($p = 2308$) from the microarray experiments in (Khan et al.; 2001). Among 83 samples, we focus on 11 cases of Burkitt lymphoma (BL) ($n_1 = 11$) and 18 cases of neuroblastoma (NB) ($n_2 = 18$). Our main interest is to test equality of mean vectors of the gene expressions between BL and NB tumors. We apply the mxPBF, CLX.AT, SD and BS to test equality of mean vectors. Note that CLX.AT is used because the lack of prior information about the sparsity of the covariance matrix. For this dataset, the value of the mxPBF is greater than $10^8$, and $p$-values of CLX.AT, SD and BS are less than $10^{-15}$. Therefore, all the tests reject the null hypothesis, $H_0 : \mu_1 = \mu_2$, if we use the default choices, threshold $C_{th} = 10$ an significance level 0.05.

The prostate cancer dataset is available in the R package SIS. This dataset contains 12600 gene expressions from 52 patients with prostate tumors ($n_1 = 52$) and 50 patients with normal prostate ($n_2 = 50$). As suggested by Cai et al. (2013), 5000 genes ($p = 5000$) with the largest absolute values of the $t$ statistics are selected. Data were centered prior to analysis. In this dataset, we would like to test equality of covariance matrices of the gene expressions between tumor and normal samples. We apply the mxPBF, CLX, LC and Sch to test equality of covariance matrices. For this dataset, the value of the mxPBF is greater than $10^{32}$, and $p$-values of CLX, LC and Sch are less than 0.0058, $10^{-15}$ and $10^{-15}$, respectively. Therefore, all the tests reject the null hypothesis, $H_0 : \Sigma_1 = \Sigma_2$, if we use the default choices, threshold $C_{th} = 10$ an significance level 0.05.
5 Discussion

In this paper, we propose a Bayesian two-sample mean test and a Bayesian two-sample covariance test in high-dimensional settings based on the idea of the maximum pairwise Bayes factor (Lee et al.; 2021). These tests are not only computationally scalable but also enjoy Bayes factor consistency under relatively weak or similar conditions compared to existing tests. The proposed methods can be applied to change point detection for mean vectors or covariance matrices, which is indeed one of our ongoing works. Note that from the first data point, using only a subset of data within a certain window, a two-sample test can be sequentially conducted to detect change points. Due to consistency of the proposed mxPBF-based two-sample tests, it is expected that the resulting change point detection procedures can consistently detect and estimate change points.
Supplementary Material

A Additional numerical results when \( p = 300 \)

A.1 Two-sample mean test

In this section, we provide additional numerical results for the two-sample mean tests when \( p = 300 \). Figures 5 and 6 show ROC curves based on 50 simulated data sets for each hypothesis where the alternative is \( H_{1R} : \mu_{01} \neq \mu_{02} \) and \( H_{1M} : \mu_{01} \neq \mu_{02} \), respectively.

Overall, in the rare signals setting \( H_{1R} \), the mxPBF outperforms the other tests. Note that CLX.AT produces similar ROC curves when \( \Omega_0 \) is sparse, but it does not seem to work well when \( \Omega_0 \) is dense. When there are many signals (\( H_{1M} \)) and \( \Omega_0 \) is sparse, the \( L_2 \)-type tests slightly work better than the maximum-type tests. However, when \( \Omega_0 \) is dense, the mxPBF and CLX.AT are better than the \( L_2 \)-type tests in terms of ROC curves. This is consistent with what we observed when \( p = 100 \).

A.2 Two-sample covariance test

In this section, we provide additional numerical results for the two-sample covariance tests when \( p = 300 \). Figures 7 and 8 show ROC curves based on 50 simulated data sets for each hypothesis where the alternative is \( H_{1R} : \Sigma_{01} \neq \Sigma_{02} \) and \( H_{1M} : \Sigma_{01} \neq \Sigma_{02} \), respectively.

In the rare signals setting \( H_{1R} \), the maximum-type tests outperform the \( L_2 \)-type tests as expected. Especially, the mxPBF seems to work better than CLX in terms of ROC curves. In the many signals setting \( H_{1M} \), the \( L_2 \)-type tests outperform the maximum-type tests. However, when \( \Sigma_{01} \) is dense and signals are moderate (\( \rho \geq 0.7 \)), the mxPBF and CLX outperform the other tests. This is consistent with our observation when \( p = 100 \).

B Proofs

Throughout Section B, we denote \( Z_n = (X_{n1}, Y_{n2}) \) for simplicity.

B.1 Proof of Theorem 2.1

**Lemma B.1** (Lemmas 6 and 8 in Kolar and Liu (2012)) Let \( \chi^2_k \) be a chi-squared random variable with degree of freedom \( k \), and \( \chi^2_k(\lambda) \) be a non-central chi-squared random variable with
Figure 5: ROC curves for the two-sample mean tests based on 50 simulated data sets for each hypothesis, $H_0$ and $H_{1R}$, with $p = 300$. 
Figure 6: ROC curves for the two-sample mean tests based on 50 simulated data sets for each hypothesis, $H_0$ and $H_{1M}$, with $p = 300$. 
Figure 7: ROC curves for the two-sample covariance tests based on 50 simulated data sets for each hypothesis, $H_0$ and $H_{1R}$, with $p = 300$. 
Figure 8: ROC curves for the two-sample covariance tests based on 50 simulated data sets for each hypothesis, $H_0$ and $H_{1M}$, with $p = 300$. 
degree of freedom \( k \) and non-centrality \( \lambda \). Then, for any \( x > 0 \),

\[
\Pr(\chi_k^2 \leq k + 2\sqrt{kx} + 2x) \geq 1 - \exp(-x),
\]

\[
\Pr(\chi_k^2 \geq k - 2\sqrt{kx}) \geq 1 - \exp(-x),
\]

\[
\Pr(\chi_k^2(\lambda) \leq k + \lambda + 2\sqrt{(k+2\lambda)x} + 2x) \geq 1 - \exp(-x),
\]

\[
\Pr(\chi_k^2(\lambda) \geq k + \lambda - 2\sqrt{(k+2\lambda)x}) \geq 1 - \exp(-x).
\]

**Proof of Theorem 2.1** We first assume the true null \( H_0 : \mu_1 = \mu_2 = \mu_0 \). For a given \( 1 \leq j \leq p \), the PBF is

\[
\log B_{10}(\hat{X}_j, \hat{Y}_j) = \frac{1}{2} \log \left( \frac{\gamma}{1+\gamma} \right) + \frac{n}{2} \log \left( \frac{n\hat{\sigma}_j^2}{n_1\hat{\sigma}_X^2 + n_2\hat{\sigma}_Y^2} \right).
\]

(20)

Since \( \log(\gamma/(1+\gamma)) \leq -\alpha \log(n \vee p) \), we focus on an upper bound for (20). We have

\[
\log \left( \frac{n\hat{\sigma}_j^2}{n_1\hat{\sigma}_X^2 + n_2\hat{\sigma}_Y^2} \right) \leq \frac{n\hat{\sigma}_j^2 - n_1\hat{\sigma}_X^2 - n_2\hat{\sigma}_Y^2}{n_1\hat{\sigma}_X^2 + n_2\hat{\sigma}_Y^2}
\]

because \( \log(1 + x) \leq x \) for any \( x \in \mathbb{R} \). It is easy to show that

\[
n_1\hat{\sigma}_X^2 + n_2\hat{\sigma}_Y^2 \overset{d}{=} \sigma_{0,jj} \cdot \chi_{n-2}^2.
\]

under \( H_{0j} : \mu_{1j} = \mu_{2j} = \mu_{0j} \). Furthermore,

\[
n\hat{\sigma}_j^2 - n_1\hat{\sigma}_X^2 - n_2\hat{\sigma}_Y^2
\]

\[
= \frac{1}{n} \left( \sqrt{\frac{n_2}{n_1}} \sum_{i=1}^{n_1} X_{ij} - \sqrt{\frac{n_1}{n_2}} \sum_{i=1}^{n_2} Y_{ij} \right)^2
\]

\[
= \frac{1}{n} \left( \sqrt{\frac{n_2}{n_1}} \left( \sum_{i=1}^{n_1} X_{ij} - n_1\mu_{0j} \right) - \sqrt{\frac{n_1}{n_2}} \left( \sum_{i=1}^{n_2} Y_{ij} - n_2\mu_{0j} \right) \right)^2 \overset{d}{=} \sigma_{0,jj} \cdot \chi_{1}^2.
\]

Define the events

\[
A_{1j} = \left\{ Z_n : \sigma_{0,jj}^{-1}(n\hat{\sigma}_X^2 + n_2\hat{\sigma}_Y^2) \geq n - 2 - 2\sqrt{(n - 2)C \log(n \vee p)} \right\},
\]

\[
A_{2j} = \left\{ Z_n : \sigma_{0,jj}^{-1}(n\hat{\sigma}_X^2 - n_1\hat{\sigma}_X^2 - n_2\hat{\sigma}_Y^2) \leq 1 + 2\sqrt{C \log(n \vee p)} + 2C \log(n \vee p) \right\}
\]

for any constant \( 1 < C < C_1 \). By Lemma B.1, \( \sum_{j=1}^{p} \sum_{l=1}^{2} P_0(A_{ij}^c) \leq 2(n \vee p)^{-C+1} = o(1) \). On the
event $A_{1j} \cap A_{2j} \cap A_{3j}$, (20) is bounded above by

$$\begin{align*}
n \log n &+ 2\sqrt{C \log(n \vee p)} + 2C \log(n \vee p) \\ \frac{1}{2} - n - 2 &- 2\sqrt{(n - 2)C \log(n \vee p)} \\
&= 1 \left\{ \frac{C \log(n \vee p)}{n - 1} + 2\sqrt{(n - 1)Cn^{-1} \log(n \vee p)} \right\} - 1 \left\{ \frac{C \log(n \vee p)}{n - 1} + 2\sqrt{(n - 1)Cn^{-1} \log(n \vee p)} \right\} \\
&= 1 \left\{ \frac{C \log(n \vee p)}{n - 1} + 2\sqrt{(n - 1)Cn^{-1} \log(n \vee p)} \right\} - 1 \left\{ \frac{C \log(n \vee p)}{n - 1} + 2\sqrt{(n - 1)Cn^{-1} \log(n \vee p)} \right\} \\
\leq & \frac{n + 2\sqrt{C \log(n \vee p)} + 2C \log(n \vee p)}{2(n - 1)} \cdot C \log(n \vee p) \\
=: & C_0 \log(n \vee p)
\end{align*}$$

for all large $n_1 \wedge n_2$. Since $\alpha > 2C_0$ by condition (6), by taking $C$ arbitrarily close to 1, we have $C_0 \alpha - 2 < 0$. Thus,

$$\begin{align*}
P_0 \left( \log B_{\max,10}^\mu(X_{n_1}, Y_{n_2}) \leq (C_0 \alpha - 2) \log(n \vee p) \right) \\
= 1 - P_0 \left( \max_{1 \leq j \leq p} \log B_{10}(\tilde{X}_j, \tilde{Y}_j) > (C_0 \alpha - 2) \log(n \vee p) \right) \\
\geq 1 - \sum_{1 \leq j \leq p} P_0 \left( \log B_{10}(\tilde{X}_j, \tilde{Y}_j) > (C_0 \alpha - 2) \log(n \vee p) \right) \\
\geq 1 - 2(n \vee p)^{-C+1}
\end{align*}$$

for some constant $1 < C < C_1$. It completes the proof under $H_0$.

Now, for a given $1 \leq j \leq p$, we assume $H_{1j}$ is true, i.e., $\mu_{0,1j} \neq \mu_{0,2j}$, and condition (7) holds. Note that

$$\log B_{10}(\tilde{X}_j, \tilde{Y}_j) \geq \frac{1}{2} \log \left( \frac{\gamma}{1 + \gamma} \right) + \frac{n}{2} \frac{n \hat{\sigma}_Z^2 - n_1 \hat{\sigma}_{X_j}^2 - n_2 \hat{\sigma}_{Y_j}^2}{n \hat{\sigma}_Z^2}$$

because $-\log(1 - x) \leq x/(1 - x)$ for any $x < 1$, and

$$n \hat{\sigma}_Z^2 = n_1 \hat{\sigma}_{X_j}^2 + n_2 \hat{\sigma}_{Y_j}^2 + \frac{1}{n} \left( \sqrt{\frac{n_1}{n_2} \sum_{i=1}^{n_1} X_{ij}} - \sqrt{\frac{n_1}{n_2} \sum_{i=1}^{n_2} Y_{ij}} \right)^2.$$

Under $H_{1j}$,

$$\begin{align*}
\frac{1}{n \sigma_{0,jj}} \left( \sqrt{\frac{n_1}{n_2} \sum_{i=1}^{n_1} X_{ij}} - \sqrt{\frac{n_1}{n_2} \sum_{i=1}^{n_2} Y_{ij}} \right)^2 &\Rightarrow \chi^2 \left( \frac{n_1 n_2}{n \sigma_{0,jj}} (\mu_{0,1j} - \mu_{0,2j})^2 \right), \\
n_1 \hat{\sigma}_{X_j}^2 + n_2 \hat{\sigma}_{Y_j}^2 &\Rightarrow \sigma_{0,jj}^2 \chi^2/n - 2.
\end{align*}$$
Define the sets
\[
A_{3j} = \left\{ \mathbf{z}_n : (n\sigma_{0,jj})^{-1}\left( \sqrt{\frac{n_2}{n_1}} \sum_{i=1}^{n_1} X_{ij} - \sqrt{\frac{n_1}{n_2}} \sum_{i=1}^{n_2} Y_{ij} \right)^2 \geq 1 + \frac{n_1 n_2}{n \sigma_{0,jj}} (\mu_{0,1j} - \mu_{0,2j})^2 \right\},
\]
\[
A_{4j} = \left\{ \mathbf{z}_n : \sigma_{0,jj}^{-1} (n_1 \hat{\sigma}_{X_j}^2 + n_2 \hat{\sigma}_{Y_j}^2) \leq n - 2 + 2 \sqrt{(n - 2)C \log(n \land p) + 2C \log(n \land p)} \right\},
\]
then \( \sum_{i=3}^{4} \mathbb{P}_0(A_{ij}^c) \leq 2(n \land p)^{-C} = o(1) \) by Lemma B.1 for any constant \( C \) such that \( 0 < C < C_1 \). On the event \( A_{3j} \cap A_{4j} \),
\[
n \hat{\sigma}_{X_j}^2 - n_1 \hat{\sigma}_{X_j}^2 - n_2 \hat{\sigma}_{Y_j}^2 \geq \sigma_{0,jj} \left\{ 1 + \frac{n_1 n_2}{n \sigma_{0,jj}} (\mu_{0,1j} - \mu_{0,2j})^2 - 2 \sqrt{(1 + 2 \frac{n_1 n_2}{n \sigma_{0,jj}} (\mu_{0,1j} - \mu_{0,2j})^2) C \log(n \land p)} \right\}.
\]
Let \( \lambda_n := \frac{n_1 n_2 (\mu_{0,1j} - \mu_{0,2j})^2 / (n \sigma_{0,jj})} \). Since \( \log B_10(\hat{X}_j, \hat{Y}_j) \to \infty \) as \( n_1 \land n_2 \to \infty \) on event \( A_{3j} \cap A_{4j} \) when \( \lambda_n \geq \epsilon_0 n \), we only need to consider the case \( \lambda_n \leq \epsilon_0 n \). Thus, on the event \( A_{3j} \cap A_{4j} \),
\[
\log B_{10}(\hat{X}_j, \hat{Y}_j) \geq \frac{1}{2} \log \left( \frac{\gamma}{1 + \gamma} \right) + \frac{\lambda_n \left( 1 + \lambda_n^{-1} - 2 \lambda_n^{-1} \sqrt{(1 + 2 \lambda_n) C \log(n \land p)} \right)}{2 \left\{ 1 + 2 \sqrt{C \epsilon_0} + (1 + 2C + 2\sqrt{3C}) \epsilon_0 \right\}}
\]
\[
= \frac{\alpha}{2} \log(n \land p) + \frac{1}{2} \log(1 + \gamma) + \frac{\lambda_n \left( 1 + \lambda_n^{-1} - 2 \lambda_n^{-1} \sqrt{(1 + 2 \lambda_n) C \log(n \land p)} \right)}{2 \left\{ 1 + 2 \sqrt{C \epsilon_0} + (1 + 2C + 2\sqrt{3C}) \epsilon_0 \right\}}
\]
for all large \( n_1 \land n_2 \). Let
\[
K = \left( \sqrt{2C_1} + \sqrt{2C_1} + \alpha C_1 \{ 1 + (1 + 8C_1) \epsilon_0 \} \right)^2
\]
then \( \lambda_n \geq K \log(n \land p) \) and
\[
\lambda_n^{-1} \sqrt{(1 + 2 \lambda_n) C \log(n \land p)} = \sqrt{(2 + \lambda_n^{-1} C \lambda_n^{-1} \log(n \land p)} \leq \sqrt{(2 + K^{-1} (\log(n \land p) - 1)) C K^{-1}} \leq \sqrt{2C_1 K^{-1}}
\]
for some constants \( 0 < C < C_1 \) and all large \( n_1 \land n_2 \), and
\[
\frac{\lambda_n \left( 1 + \lambda_n^{-1} - 2 \lambda_n^{-1} \sqrt{(1 + 2 \lambda_n) C \log(n \land p)} \right)}{2 \left\{ 1 + 2 \sqrt{C \epsilon_0} + (1 + 2C + 2\sqrt{3C}) \epsilon_0 \right\}} \geq \frac{K (1 - 2 \sqrt{C_1 K^{-1}} \log(n \land p)}{2 \left\{ 1 + 2 \sqrt{C \epsilon_0} + (1 + 2C + 2\sqrt{3C}) \epsilon_0 \right\}} \geq \frac{\alpha C_1}{2} \log(n \land p)
\]
because
\[
K \geq \left( \sqrt{2C_1} + \sqrt{2C_1} + \alpha C_1 \{ 1 + 2 \sqrt{C \epsilon_0} + (1 + 2C + 2\sqrt{3C}) \epsilon_0 \} \right)^2.
\]
It implies
\[ P_0 \left( \log B_{\text{max,10}}(X_{n_1}, Y_{n_2}) \geq \{\alpha(C_1 - 1)/2\} \log(n \lor p) \right) \]
\[ \geq \max_{1 \leq j \leq p} P_0 \left( \log B_{10}(\tilde{X}_j, \tilde{Y}_j) \geq \{\alpha(C_1 - 1)/2\} \log(n \lor p) \right) \]
\[ \geq 1 - 2(n \lor p)^{-C} \]
for some constants $0 < C < C_1$.

**B.2 Proof of Theorem 3.1**

**Lemma B.2** For any integer $n_1$ and $n_2$ such that $n_1 \land n_2 > 4$ and constant $a_0 > 0$,
\[ \sum_{l=1}^{2} \log \Gamma\left(\frac{n_l}{2} + a_0\right) - \log \Gamma\left(\frac{n}{2} + a_0\right) \]
\[ \leq 2(1 - \log 2) - a_0 + a_0 \log\left(\frac{n}{2} + a_0\right) + \sum_{l=1}^{2} \left(\frac{n_l - 1}{2} + a_0\right) \log\left(\frac{n_l + 2a_0}{n + 2a_0}\right), \]
\[ \sum_{l=1}^{2} \log \Gamma\left(\frac{n_l}{2} + a_0\right) - \log \Gamma\left(\frac{n}{2} + a_0\right) \]
\[ \geq 2(1 - \log 2) - a_0 + \left(\frac{1}{2} + a_0\right) \log\left(\frac{n}{2} + a_0\right) \]
\[ + \sum_{l=1}^{2} \left\{\left(\frac{n_l}{2} + a_0\right) \log\left(\frac{n_l + 2a_0}{n + 2a_0}\right) - \frac{1}{2} \log\left(\frac{n_l}{2} + a_0\right)\right\}, \]
where $n = n_1 + n_2$.

**Proof** By Theorem 1 of Kečkić and Vasić (1971),
\[ (x - 1) \log x + y - (y - 1) \log y - x \leq \log \Gamma(x) - \log \Gamma(y) \]
\[ \leq (x - \frac{1}{2}) \log x + y - (y - \frac{1}{2}) \log y - x \]
for any $x \geq y > 1$. Thus,
\[ \log \Gamma\left(\frac{n_l}{2} + a_0\right) - \log \Gamma(2) \leq \left(\frac{n_l - 1}{2} + a_0\right) \log\left(\frac{n_l}{2} + a_0\right) + 2 - \frac{3}{2} \log 2 - \left(\frac{n_l}{2} + a_0\right), \]
\[ - \log \Gamma\left(\frac{n_l}{2} + a_0\right) + \log \Gamma(2) \leq -\left(\frac{n}{2} + a_0\right) \log\left(\frac{n}{2} + a_0\right) - 2 + \log 2 + \left(\frac{n}{2} + a_0\right) \]
for $l = 1, 2$. This completes the proof for the first inequality. By Theorem 1 of Kečkić and Vasić (1971), we also obtain
\[ - \log \Gamma\left(\frac{n}{2} + a_0\right) + \log \Gamma\left(\frac{n_1}{2} + a_0\right) \geq -\left(\frac{n}{2} + a_0 - \frac{1}{2}\right) \log\left(\frac{n}{2} + a_0\right) - \left(\frac{n_1}{2} + a_0\right) \]
\[ + \left(\frac{n_1}{2} + a_0 - \frac{1}{2}\right) \log\left(\frac{n_1}{2} + a_0\right) + \left(\frac{n}{2} + a_0\right), \]
\[ \log \Gamma\left(\frac{n_1}{2} + a_0\right) \geq \left(\frac{n_2}{2} + a_0 - 1\right) \log\left(\frac{n_2}{2} + a_0\right) + 2 - \log 2 - \left(\frac{n_2}{2} + a_0\right), \]
which completes the proof for the second inequality.

**Lemma B.3** Suppose \( \Sigma_1 = \Sigma_2 = \Sigma_0 = (\sigma_{0,ij}) \) in model (8). Define

\[
A_{0ij} = \left\{ Z_n : 1 - 2\sqrt{C \log(n \vee p)} \leq D_{ij} \leq 1 + 2\sqrt{C \log(n \vee p)} + 2C \log(n \vee p) \right\},
\]

where

\[
D_{ij} = \frac{1}{\tau_{0,ij} \| Z_j \|^2} \left( \| \hat{Z}_j \| \frac{\hat{X}_i^T \hat{X}_j}{\| \hat{X}_j \|^2} - \| \hat{X}_j \|^2 \frac{\hat{Y}_i^T \hat{Y}_j}{\| \hat{Y}_j \|^2} \right)^2.
\]

Then,

\[
\sum_{i=1}^{p} \sum_{j=1}^{p} \mathbb{P}_0(A_{0ij}^c) \leq 2p^2(n \vee p)^{-C},
\]

for any constant \( C > 0 \).

**Proof** By (10), we have

\[
\begin{align*}
\hat{X}_i^T \hat{X}_j & \sim N(a_{0,ij} \| \hat{X}_j \|^2, \tau_{0,ij} \| \hat{X}_j \|^2), \\
\hat{Y}_i^T \hat{Y}_j & \sim N(a_{0,ij} \| \hat{Y}_j \|^2, \tau_{0,ij} \| \hat{Y}_j \|^2).
\end{align*}
\]

Then, it is easy to check that

\[
\| \hat{Y}_j \|^2 \frac{\hat{X}_i^T \hat{X}_j}{\| \hat{X}_j \|^2} - \| \hat{X}_j \|^2 \frac{\hat{Y}_i^T \hat{Y}_j}{\| \hat{Y}_j \|^2} \mid \hat{Z}_j \sim N \left( 0, \tau_{0,ij} \| \hat{Z}_j \|^2 \right),
\]

which implies \( D_{ij} \sim \chi_1^2 \). Thus, by Lemma B.1,

\[
\sum_{i=1}^{p} \sum_{j=1}^{p} \mathbb{P}_0(A_{0ij}^c) \leq \sum_{i=1}^{p} \sum_{j=1}^{p} 2 \exp \left( - C \log(n \vee p) \right) \leq 2p^2(n \vee p)^{-C}.
\]

**Lemma B.4** Consider model (8) with true covariances \( \Sigma_{01} = (\sigma_{01,ij}) \) and \( \Sigma_{02} = (\sigma_{02,ij}) \). Define the sets

\[
A_{1ij} = \left\{ Z_n : \left( 1 - 2 \sqrt{\frac{C\epsilon_{01}}{1 - n_1^{-1}}} \right) \leq \frac{\hat{\tau}_{1,ij}}{\tau_{01,ij}} \left( 1 - n_1^{-1} \right) \leq \left( 1 + 2 \sqrt{\frac{C\epsilon_{01}}{1 - n_1^{-1}}} + 2C\epsilon_{01} \right) \right\},
\]

where \( \tau_{0k,ij} = \sigma_{0k,ii}(1 - \rho_{0k,ij}^2) \) and \( \rho_{0k,ij} = \sigma_{0k,ij} / (\sigma_{0k,ii} \sigma_{0k,jj})^{1/2} \) for \( k = 1, 2 \). Similarly, define the sets

\[
A_{2ij} = \left\{ Z_n : \left( 1 - 2 \sqrt{\frac{C\epsilon_{02}}{1 - n_2^{-1}}} \right) \leq \frac{\hat{\tau}_{2,ij}}{\tau_{02,ij}} \left( 1 - n_2^{-1} \right) \leq \left( 1 + 2 \sqrt{\frac{C\epsilon_{02}}{1 - n_2^{-1}}} + 2C\epsilon_{02} \right) \right\},
\]

Then,

\[
\sum_{l=1}^{2} \mathbb{P}_0(A_{lij}^c) \leq 4(n \vee p)^{-C}
\]

for any constant \( C > 0 \).
Proof Since $\tilde{X}_i | \tilde{X}_j \sim N_{n_1}(a_{01,ij} \tilde{X}_j, \tau_{01,ij} I_{n_1})$, we have
\[ n_1 \tilde{\tau}_{1,ij}/\tau_{01,ij} = \tilde{X}_i^T(I_{n_1} - H \tilde{X}_j) \tilde{X}_i/\tau_{01,ij} \sim \chi^2_{n_1-1}. \]

Then,
\[
\frac{\tilde{\tau}_{1,ij}}{\tau_{01,ij}} \leq \frac{1}{n_1} \left\{ n_1 - 1 + 2 \sqrt{(n_1 - 1)C \log(n \vee p) + 2C \log(n \vee p)} \right\}
\leq (1 - n_1^{-1}) \left\{ 1 + 2 \sqrt{\frac{C \log(n \vee p)}{n_1 - 1}} + 2C \epsilon_01/(1 - n_1^{-1}) \right\}
= (1 - n_1^{-1}) \left\{ 1 + 2 \sqrt{\frac{C \epsilon_01/(1 - n_1^{-1})}{n_1 - 1}} + 2C \epsilon_01/(1 - n_1^{-1}) \right\}
\]
with probability at least $1 - (n \vee p)^{-C}$ and
\[
\frac{\tilde{\tau}_{1,ij}}{\tau_{01,ij}} \geq \frac{1}{n_1} \left\{ n_1 - 1 - 2 \sqrt{(n_1 - 1)C \log(n \vee p)} \right\}
= (1 - n_1^{-1})(1 - 2 \sqrt{\frac{C \epsilon_01/(1 - n_1^{-1})}{n_1 - 1}})
\]
with probability at least $1 - (n \vee p)^{-C}$, by Lemma B.1. Thus,
\[
\mathbb{P}_0(A_{1ij}^c) \leq 2(n \vee p)^{-C}.
\]
By similar arguments, it is easy to see that
\[
\mathbb{P}_0(A_{2ij}^c) \leq 2(n \vee p)^{-C},
\]
which completes the proof.

Proof of Theorem 3.1 We prove the consistency under the null and alternative in turn.

Consistency under $H_0 : \Sigma_1 = \Sigma_2$. We first assume the true null $H_0 : \Sigma_1 = \Sigma_2$, i.e., $\Sigma_0 \equiv \Sigma_{01} = \Sigma_{02}$, where $\Sigma_0 = (\sigma_{0,ij})$. For a given pair $(i, j), i \neq j$, the PBF is
\[
\log B_{10}(\tilde{X}_i, \tilde{Y}_i, \tilde{X}_j, \tilde{Y}_j)
= \frac{1}{2} \log \left( \frac{\gamma}{1 + \gamma} \right) + \log \Gamma \left( \frac{n_1}{2} + a_0 \right) + \log \Gamma \left( \frac{n_2}{2} + a_0 \right) - \log \Gamma \left( \frac{n}{2} + a_0 \right) + \log \left( \frac{b_{01,ij}^{\sigma_0}}{b_{01,ij}^{\sigma_0} \Gamma(a_0)} \right)
- \left( \frac{n_1}{2} + a_0 \right) \log \left( b_{01,ij} + \frac{n_1}{2} \tilde{\tau}_{1,ij} \right) - \left( \frac{n_2}{2} + a_0 \right) \log \left( b_{02,ij} + \frac{n_2}{2} \tilde{\tau}_{2,ij} \right)
+ \left( \frac{n}{2} + a_0 \right) \log \left( b_{0,ij} + \frac{n}{2} \tilde{\tau}_{ij} \right).
\]
By Lemma B.2,

\[
\log B_{10}(\tilde{X}_i, \tilde{Y}_i, \tilde{X}_j, \tilde{Y}_j) \\
\leq \frac{1}{2} \log \left( \frac{\gamma}{1 + \gamma} \right) + 2(1 - \log 2) - a_0 - \frac{1}{2} \left\{ \log \left( \frac{n_1 + 2a_0}{n + 2a_0} \right) + \log \left( \frac{n_2 + 2a_0}{n + 2a_0} \right) \right\} \\
- a_0 \log \left( \frac{n\tilde{\tau}_{ij} + 2b_{0,ij}}{n + 2a_0} \right) + a_0 \log \left( \frac{n\tilde{\tau}_{ij} + 2b_{0,ij}}{n\tilde{\tau}_{ij} + 2b_{0,ij}} \right) + a_0 \log \left( \frac{n\tilde{\tau}_{ij} + 2b_{0,ij}}{n\tilde{\tau}_{ij} + 2b_{0,ij}} \right) \\
+ a_0 \log \left( \frac{n_1 + 2a_0}{n + 2a_0} \right) + a_0 \log \left( \frac{n_2 + 2a_0}{n + 2a_0} \right) + \log \left( \frac{\tilde{b}_{0,ij}}{\tilde{b}_{0,ij}} \right) \\
+ a_0 \log \left( \frac{n_1 + 2a_0}{n + 2a_0} \right) + a_0 \log \left( \frac{n_2 + 2a_0}{n + 2a_0} \right) + \log \left( \frac{\tilde{b}_{0,ij}}{\tilde{b}_{0,ij}} \right) \\
+ \frac{n_1}{2} \log \left( \frac{n\tilde{\tau}_{ij} + 2b_{0,ij}}{n\tilde{\tau}_{ij} + 2b_{0,ij}} \right) + \frac{n_2}{2} \log \left( \frac{n\tilde{\tau}_{ij} + 2b_{0,ij}}{n\tilde{\tau}_{ij} + 2b_{0,ij}} \right) \\
+ \frac{n_1}{2} \log \left( \frac{n_1 + 2a_0}{n + 2a_0} \right) + \frac{n_2}{2} \log \left( \frac{n_2 + 2a_0}{n + 2a_0} \right) 
\]

We first focus on the last two lines. Note that

\[
n\tilde{\tau}_{ij} = n_1\tilde{\tau}_{1,ij} + n_2\tilde{\tau}_{2,ij} + \frac{1}{\|Z_j\|^2\|\tilde{Y}_j\|^2} \left( \|\tilde{Y}_j\|^2 \|\tilde{X}_j\|^2 - \|\tilde{X}_j\|^2 \|\tilde{Y}_j\|^2 \right) 
\approx n_1\tilde{\tau}_{1,ij} + n_2\tilde{\tau}_{2,ij} + \gamma_{0,ij}D_{ij},
\]

where \(D_{ij} \sim \chi^2 \) under \(H_0\). It implies

\[
\frac{n_1}{2} \log \left( \frac{n\tilde{\tau}_{ij} + 2b_{0,ij}}{n\tilde{\tau}_{ij} + 2b_{0,ij}} \right) + \frac{n_1}{2} \log \left( \frac{n_1 + 2a_0}{n + 2a_0} \right) + \frac{n_2}{2} \log \left( \frac{n\tilde{\tau}_{ij} + 2b_{0,ij}}{n\tilde{\tau}_{ij} + 2b_{0,ij}} \right) + \frac{n_2}{2} \log \left( \frac{n_2 + 2a_0}{n + 2a_0} \right) \\
\leq \frac{n_1}{2} \left[ \log \left( \frac{n^{-1}n_1\tilde{\tau}_{1,ij} + n^{-1}n_2\tilde{\tau}_{2,ij} + n^{-1}\tau_{0,ij}D_{ij} + 2b_{0,ij}n^{-1}}{\tilde{\tau}_{1,ij} + 2b_{0,ij}n^{-1}} \right) + \log \left( \frac{n(n_1 + 2a_0)}{n_1(n + 2a_0)} \right) \right] \\
+ \frac{n_2}{2} \left[ \log \left( \frac{n^{-1}n_1\tilde{\tau}_{2,ij} + n^{-1}n_2\tilde{\tau}_{2,ij} + n^{-1}\tau_{0,ij}D_{ij} + 2b_{0,ij}n^{-1}}{\tilde{\tau}_{2,ij} + 2b_{0,ij}n^{-1}} \right) + \log \left( \frac{n(n_2 + 2a_0)}{n_2(n + 2a_0)} \right) \right] \\
\leq \frac{n}{2} \left[ \log \left( \frac{n\tilde{\tau}_{1,ij} + n\tilde{\tau}_{2,ij} + \frac{\tau_{0,ij}}{n}D_{ij} + \frac{2b_{0,ij}}{n}}{\tilde{\tau}_{1,ij} + \frac{2b_{0,ij}}{n\tilde{\tau}_{2,ij}}} - \frac{n_1}{n} \log(\tilde{\tau}_{1,ij}) - \frac{n_2}{n} \log(\tilde{\tau}_{2,ij}) \right) \\
+ \frac{n_1}{2} \log \left( \frac{n(n_1 + 2a_0)}{n_1(n + 2a_0)} \right) + \frac{n_2}{2} \log \left( \frac{n(n_2 + 2a_0)}{n_2(n + 2a_0)} \right) \right] \\
= \frac{n}{2} \left[ \log \left( \frac{n\tilde{\tau}_{1,ij} + n\tilde{\tau}_{2,ij} + \frac{\tau_{0,ij}}{n\tilde{\tau}_{2,ij}}D_{ij} + \frac{2b_{0,ij}}{n\tilde{\tau}_{2,ij}}} - \frac{n_1}{n} \log \left( \frac{\tilde{\tau}_{1,ij}}{\tilde{\tau}_{2,ij}} \right) \right) \right] \\
+ \frac{n_1}{2} \log \left( \frac{n(n_1 + 2a_0)}{n_1(n + 2a_0)} \right) + \frac{n_2}{2} \log \left( \frac{n(n_2 + 2a_0)}{n_2(n + 2a_0)} \right) \right]. \tag{21}
\]

It is easy to see that (22) is bounded by a constant because

\[
\frac{n_1}{2} \log \left( \frac{n(n_1 + 2a_0)}{n_1(n + 2a_0)} \right) + \frac{n_2}{2} \log \left( \frac{n(n_2 + 2a_0)}{n_2(n + 2a_0)} \right) \leq a_0,
\]

so we only need to focus on (21).

Let \(\tilde{\tau}_{1,ij}/\tilde{\tau}_{2,ij} \equiv 1 + R_{ij}\) and

\[
R_{ij}' = R_{ij} + \frac{\tau_{0,ij}}{n\tilde{\tau}_{2,ij}}D_{ij} + \frac{2b_{0,ij}}{n\tilde{\tau}_{2,ij}}.
\]
Then, we can rewrite (21) as
\[
\frac{n}{2} \left[ \log \left( 1 + \frac{n_1}{n} R'_{ij} \right) - \frac{n_1}{n} \log(1 + R_{ij}) \right]
= \frac{n}{2} \left[ \log \left( 1 + \frac{n_1}{n} R'_{ij} \right) - \frac{n_1}{n} \log(1 + R_{ij}) + R_{ij} - \log(1 + R_{ij}) \right] + \frac{n_1}{n} \left( R'_{ij} - R_{ij} \right).
\]  
(23)

Define a set \( A_{ij} := \cap_{l=0}^2 A_{lij} \) with some constant \( C > 2 \), where \( A_{0ij}, A_{1ij} \) and \( A_{2ij} \) are defined at Lemmas B.3 and B.4. By Lemmas B.3 and B.4, it suffices to focus on the event \( \cap_{l=1}^p \cap_{j=1}^p A_{ij} \) to prove Theorem 3.1 because
\[
\sum_{i=1}^p \sum_{j=1}^p \mathbb{P}(A_{ij}^c) \leq 6(n \lor p)^{-C+2} = o(1)
\]
for any constant \( C > 2 \). On the event \( A_{ij} \), we have the following bounds:
\[
\frac{\hat{\tau}_{1,ij}}{\hat{\tau}_{2,ij}} \leq \frac{1 + 2\sqrt{C\epsilon_01 + 2C\epsilon_01}}{(1 - n_2^{-1})(1 - 2\sqrt{C\epsilon_02/(1 - n_2^{-1})})}
\]
\[
\leq (1 - n_2^{-1})^{-1} \left( 1 + 2\sqrt{C\epsilon_01 + 2\sqrt{C\epsilon_02/(1 - n_2^{-1})} + O(\epsilon_01 \lor \epsilon_02)} \right)
\]
\[
\leq (1 + 3n_2^{-1}) \left( 1 + 2\sqrt{C\epsilon_01 + 2\sqrt{C\epsilon_02(1 + 3n_2^{-1})} + O(\epsilon_01 \lor \epsilon_02)} \right)
\]
\[
\leq (1 + 3n_2^{-1}) \left( 1 + 2\sqrt{C\epsilon_01 + 2\sqrt{C\epsilon_02(1 + 2n_2^{-1})} + O(\epsilon_01 \lor \epsilon_02)} \right)
\]
\[
\leq 1 + 4\sqrt{C(\epsilon_01 \lor \epsilon_02)} + O(\epsilon_01 \lor \epsilon_02)
\]
\[
\frac{\hat{\tau}_{1,ij}}{\hat{\tau}_{2,ij}} \geq \frac{(1 - n_1^{-1})(1 - 2\sqrt{C\epsilon_01/(1 - n_1^{-1})})}{1 + 2\sqrt{C\epsilon_01 + 2C\epsilon_01}}
\]
\[
\geq (1 - n_1^{-1}) \left( 1 - 2\sqrt{C\epsilon_01/(1 - n_1^{-1})} - 2\sqrt{C\epsilon_02 - 2C\epsilon_01} \right)
\]
\[
\geq 1 - 4\sqrt{C(\epsilon_01 \lor \epsilon_02)} - O(\epsilon_01 \lor \epsilon_02)
\]

for all large \( n_1 \land n_2 \). Thus, on the event \( A_{ij} \),
\[
-4\sqrt{C(\epsilon_01 \lor \epsilon_02)} - O(\epsilon_01 \lor \epsilon_02) \leq R_{ij} \leq 4\sqrt{C(\epsilon_01 \lor \epsilon_02)} + O(\epsilon_01 \lor \epsilon_02)
\]
and
\[
R'_{ij} - R_{ij} = \frac{\tau_{0,ij} - D_{ij}}{n_1 \tau_{2,ij}} + \frac{2b_{0,ij}}{n_1 \tau_{2,ij}}
\]
\[
\leq \frac{1 + 2\sqrt{C\log(n \lor p) + 2C\log(n \lor p)}}{n_1(1 - n_2^{-1})(1 - 2\sqrt{C\epsilon_02(1 - n_2^{-1})^{-1}})}
\]
\[
+ \frac{2b_{0,ij}}{\tau_{0,ij} n_1(1 - n_2^{-1})(1 - 2\sqrt{C\epsilon_02(1 - n_2^{-1})^{-1}})}
\]
\[
\leq 2C\epsilon_01 + o(\epsilon_01 \lor \epsilon_02)
\]
for all sufficiently large $n_1 \land n_2$ because $\tau_{0,ij} \gg (\log(n \lor p))^{-1}$ by condition (A2). Thus $|R_{ij}|$ and $|R'_{ij}|$ can be regarded as small values close to 0 on $A_{ij}$.

Note that $x - \log(1 + x) \leq x^2/2 + |x|^3/(3(1 + x))$ and $x - \log(1 + x) \geq x^2/2 - |x|^3/3$ for any small $|x| > 0$. Then, on the event $A_{ij}$, the upper bound of (23) can be derived as

$$
\frac{n}{2} \left[ - \left\{ \frac{1}{2} \left( \frac{n_1}{n} R_{ij}^2 \right)^2 - \frac{1}{3} \left( \frac{n_1}{n} |R_{ij}| \right)^3 \right\} + \frac{n_1}{2n} R_{ij}^2 + \frac{n_1}{3n} |R_{ij}| + \frac{n_1}{n} (R'_{ij} - R_{ij}) \right]
$$

$$
= \frac{n_1}{4} R_{ij}^2 \left[ 1 - \frac{n_1}{n} \left( \frac{R'_{ij}}{R_{ij}} \right)^2 + \frac{2}{3} \left( \frac{n_1}{n} \right)^2 |R_{ij}| + \frac{2}{3} \left( \frac{|R_{ij}|}{1 + |R_{ij}|} \right) \right] + \frac{n_1}{2} (R'_{ij} - R_{ij})
$$

$$
= \frac{n_1}{4} R_{ij}^2 \left[ 1 - \frac{n_1}{n} + \frac{2}{3} \left( \frac{n_1}{n} \right)^2 |R_{ij}| + \frac{2}{3} \left( \frac{|R_{ij}|}{1 + |R_{ij}|} \right) \right] + \frac{n_1}{2} (R'_{ij} - R_{ij})
$$

$$
- \frac{n_1}{6n^2} |R_{ij}|^3 \left\{ 1 - \left( \frac{R_{ij}'}{R_{ij}} \right)^2 \right\} + \frac{n_1^2 R_{ij}^2}{4n} \left\{ 1 - \left( \frac{R_{ij}'}{R_{ij}} \right)^2 \right\}
$$

$$
\leq 4n_1 C_2 (\epsilon_0 \lor \epsilon_0) \left[ 1 - \frac{n_1}{n} \right] + \frac{n_1}{2} 2C_2 \epsilon_0 + o(n_1 (\epsilon_0 \lor \epsilon_0))
$$

$$
\leq \frac{n_1}{2} (\epsilon_0 \lor \epsilon_0) (8C_2 (1 - \frac{n_1}{n}) + 2C_2) + o(n_1 (\epsilon_0 \lor \epsilon_0))
$$

$$
= \frac{n_1}{2} (\epsilon_0 \lor \epsilon_0) (6C_2 + o(1))
$$

for some constant $C_2$ such that $C_2 > C > 2$. Also note that on the event $A_{ij}$, the rest part of $\log B_{10}(\tilde{X}_i, \tilde{Y}_i, \tilde{X}_j, \tilde{Y}_j)$, except $\log((\gamma/(1 + \gamma))/2)$, is bounded above by $\log(\log(n \lor p))$ due to the assumption $\tau_{0,ij} \gg (\log(n \lor p))^{-1}$. Since $\log((\gamma/(1 + \gamma))/2) \leq -\alpha \log(n \lor p)/2$, we have

$$
\log B_{10}(\tilde{X}_i, \tilde{Y}_i, \tilde{X}_j, \tilde{Y}_j) \leq -\frac{1}{2} \left( \alpha - 6C_2 + o(1) \right) \log(n \lor p)
$$

on the event $A_{ij}$. Thus,

$$
P_0 \left( \log B^\Sigma_{\max,10}(X_{n_1}, Y_{n_2}) < -\frac{1}{2} \left( \alpha - 6C_2 + o(1) \right) \log(n \lor p) \right)
$$

$$
= 1 - P_0 \left( \log B^\Sigma_{\max,10}(X_{n_1}, Y_{n_2}) > -\frac{1}{2} \left( \alpha - 6C_2 + o(1) \right) \log(n \lor p) \right)
$$

$$
\geq 1 - \sum_{(i,j):i \neq j} P_0 \left( \log B_{10}(\tilde{X}_i, \tilde{Y}_i, \tilde{X}_j, \tilde{Y}_j) > -\frac{1}{2} \left( \alpha - 6C_2 + o(1) \right) \log(n \lor p) \right)
$$

$$
\geq 1 - \sum_{(i,j):i \neq j} P_0 (A'_{ij})
$$

$$
\geq 1 - 6(n \lor p)^{-C+2} = 1 - o(1).
$$

It completes the proof under the true null $H_0 : \Sigma_1 = \Sigma_2$ because we assume $\alpha > 6C_2$.

**Consistency under $H_1 : \Sigma_1 \neq \Sigma_2$.** Specifically, assume that $H_{1,ij} : a_{1,ij} \neq a_{2,ij}$ or $\tau_{1,ij} \neq \tau_{2,ij}$ is true for some pair $(i,j)$. First, we focus on the case $\tau_{01,ij} \neq \tau_{02,ij}$ and $a_{01,ij} = a_{02,ij} \equiv a_{0,ij}$, and
suppose condition (A3) holds. By Lemma B.2,

\[ \log B_{10}(\tilde{X}_i, \tilde{Y}_i, \tilde{X}_j, \tilde{Y}_j) \geq \frac{1}{2} \log \left( \frac{\gamma}{1 + \gamma} \right) + 2 - \log 2 - a_0 + \log \left( \frac{\tilde{\tau}_{01,i,j}^a \tilde{\tau}_{02,i,j}^a}{\tilde{\tau}_{01,i,j} \tilde{\tau}_{02,i,j}} \right) \]

\[ - \frac{1}{2} \log \left( \frac{n_1}{2} + a_0 \right) - \log \left( \frac{n_2}{2} + a_0 \right) + \frac{1}{2} \log \left( \frac{n_1}{2} + a_0 \right) \]

\[ + \left\{ - a_0 \left[ \log \left( \frac{n_1 \tilde{\tau}_{ij} + b_{0,i,j}}{2} \right) - \log \left( \frac{n}{2} + a_0 \right) \right] \right\} \]

\[ + \left( \frac{n_1}{2} + a_0 \right) \left[ \log \left( \frac{n_1 \tilde{\tau}_{ij} + 2b_{0,i,j}}{n_1 \tilde{\tau}_{ij} + 2b_{0,i,j}} \right) + \log \left( \frac{n_1 + 2a_0}{n + 2a_0} \right) \right] \]  \hspace{1cm} (24)

\[ + \left( \frac{n_2}{2} + a_0 \right) \left[ \log \left( \frac{n_2 \tilde{\tau}_{2,i,j} + 2b_{0,i,j}}{n_2 \tilde{\tau}_{2,i,j} + 2b_{0,i,j}} \right) + \log \left( \frac{n_2 + 2a_0}{n + 2a_0} \right) \right] \} \right\}. \]  \hspace{1cm} (26)

Since the sum of three terms (24), (25) and (26) is increasing in \( \tilde{\tau}_{ij} \) and

\[ n \tilde{\tau}_{ij} = n_1 \tilde{\tau}_{1,i,j} + n_2 \tilde{\tau}_{2,i,j} + \frac{1}{\| \tilde{Y}_j \|_2^2} \left( \| \tilde{X}_j \|_2 \tilde{X}_i^T \tilde{X}_j - \| \tilde{X}_j \|_2 \| \tilde{Y}_j \|_2 \right)^2 \]

\[ \geq n_1 \tilde{\tau}_{1,i,j} + n_2 \tilde{\tau}_{2,i,j}, \]

a lower bound for the sum of three terms (24), (25) and (26) is given by

\[ \left\{ - a_0 \left[ \log \left( \frac{n_1 \tilde{\tau}_{1,i,j} + n_2 \tilde{\tau}_{2,i,j} + b_{0,i,j}}{2} \right) - \log \left( \frac{n}{2} + a_0 \right) \right] \right\} \]

\[ + \frac{n}{2} \left[ \log \left( \frac{n_1 \tilde{\tau}_{1,i,j} + n_2 \tilde{\tau}_{2,i,j} + 2b_{0,i,j}}{n_1 \tilde{\tau}_{1,i,j} + 2b_{0,i,j}} \right) - \frac{n_1}{n} \log \left( \frac{n_1 \tilde{\tau}_{1,i,j} + 2b_{0,i,j}}{n_1 \tilde{\tau}_{1,i,j}} \right) \right] \]

\[ - \frac{n_2}{n} \log \left( \frac{n_2 \tilde{\tau}_{2,i,j} + 2b_{0,i,j}}{n_2 \tilde{\tau}_{2,i,j}} \right) \]  \hspace{1cm} (28)

\[ + a_0 \log \left( \frac{n_1 + 2a_0}{n + 2a_0} \right) + a_0 \log \left( \frac{n_2 + 2a_0}{n + 2a_0} \right). \]  \hspace{1cm} (29)

Term (29) is bounded below by a constant because we assume \( n_1 / n \to 1/2 \) as \( n_1 \wedge n_2 \to \infty \).

We will first calculate a lower bound for (28), and then calculate a lower bound for (27). Since we are considering the case \( \tau_{01,i,j} \neq \tau_{02,i,j} \), without loss of generality, we assume that \( \tau_{01,i,j} > \tau_{02,i,j} \).

Note that (28) is bounded below by

\[ \frac{n}{2} \left[ \log \left( \frac{n_1 \tilde{\tau}_{1,i,j} + n_2 \tilde{\tau}_{2,i,j} + 2b_{0,i,j}}{n_1 \tilde{\tau}_{1,i,j} + 2b_{0,i,j}} \right) - \frac{n_1}{n} \log \left( \frac{n_1 \tilde{\tau}_{1,i,j} + 2b_{0,i,j}}{n_1 \tilde{\tau}_{1,i,j}} \right) \right] \]

\[ \geq \frac{n}{2} \left[ \log \left( \frac{n_1 \tilde{\tau}_{1,i,j} + n_2 \tilde{\tau}_{2,i,j} + 2b_{0,i,j}}{n_1 \tilde{\tau}_{1,i,j} + 2b_{0,i,j}} \right) - \frac{n_1}{n} \log \left( 1 + \frac{2b_{0,i,j}}{n_1 \tilde{\tau}_{1,i,j}} \right) \right] \]

\[ \geq \frac{n}{2} \left[ \log \left( \frac{n_1 \tilde{\tau}_{1,i,j} + n_2 \tilde{\tau}_{2,i,j}}{n_1 \tilde{\tau}_{1,i,j}} \right) - \frac{n_1}{n} \log \left( 1 + \frac{2b_{0,i,j}}{n_1 \tilde{\tau}_{1,i,j}} \right) \right] \]

\[ \equiv \frac{n}{2} \left[ \log \left( 1 + \frac{n_1}{n} R_{ij} \right) - \frac{n_1}{n} \log \left( 1 + R_{ij} \right) \right] = \frac{2(n_1 \tilde{\tau}_{1,i,j} + n_2 \tilde{\tau}_{2,i,j})}{n} \left( \tilde{\tau}_{1,i,j}^{-1} + \tilde{\tau}_{2,i,j}^{-1} \right) \]  \hspace{1cm} (30)

by the definition of \( R_{ij} \). Consider the sets \( \{A_{1,i,j}, A_{2,i,j}\} \), which are defined at Lemma B.4, with some constant \( 0 < C < C_1 \). Then by the similar arguments, on the event \( A_{0,i,j} = \tilde{\tau}_{02,i,j}^a A_{ij} \),

\[ 1 + R_{ij} = \frac{\tilde{\tau}_{1,i,j}}{\tilde{\tau}_{2,i,j}} \geq \frac{\tau_{01,i,j}}{\tau_{02,i,j}} \left( 1 - 4\sqrt{C_1 (\epsilon_{01} \vee \epsilon_{02})} - O(\epsilon_{01} \vee \epsilon_{02}) \right) \]

\[ \geq 1 + C b_{n1} \sqrt{\epsilon_{01} \vee \epsilon_{02}} + O(\epsilon_{01} \vee \epsilon_{02}) =: 1 + R_{\text{min}} \]

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for all sufficiently large \( n_1 \land n_2 \). The last inequality follows from condition (A3). It implies that \( R_{ij} > 0 \) on the event \( A_{-0,ij} \). Note that \( \log(1 + xn_1/n) - n_1 \log(1 + x)/n \) is increasing in \( x > 0 \). Thus, (30) is bounded below by

\[
\frac{n}{2} \left[ \log \left(1 + \frac{n_1}{n} R_{\min} \right) - \frac{n_1}{n} \log \left(1 + R_{\min} \right) - \frac{2(b_{01,ij} \vee b_{02,ij})}{n} \left( \tilde{r}_{1,ij}^{-1} + \tilde{r}_{2,ij}^{-1} \right) \right] \\
\geq \frac{n}{2} \left[ R_{\min}^2 n_1 - R_{\min}^3 n_1 \frac{1}{2} - R_{\min}^2 n_1 \frac{2}{3} \right] - \frac{2(b_{01,ij} \vee b_{02,ij})}{n} \left( \tilde{r}_{1,ij}^{-1} + \tilde{r}_{2,ij}^{-1} \right) \\
\geq \frac{n_1 R_{\min}^2}{4} \left[ 1 - \frac{n_1}{n} - \frac{2}{3} R_{\min} - \frac{2}{3} \left( \frac{n_1}{n} \right)^2 \right] - \frac{2(b_{01,ij} \vee b_{02,ij})}{n} \left( \tilde{r}_{1,ij}^{-1} + \tilde{r}_{2,ij}^{-1} \right) \\
\geq C_{bm}^2 \log(n \vee p) \left[ 1 - \frac{n_1}{n} - \frac{2}{3} (1 + \left( \frac{n_1}{n} \right)^2) R_{\min} \right] + o(\log(n \vee p)) \\
\geq C_{bm}^2 \log(n \vee p) + o(\log(n \vee p))
\]

for all sufficiently large \( n_1 \land n_2 \) because we assume that \( \tau_{01,ij} \land \tau_{02,ij} \gg (\log(n \vee p))^{-1} \) by condition (A3). Note that, on the event \( A_{-0,ij} \), (27) is negligible compared with \( \log(n \vee p) \) because we assume that \( \tau_{01,ij} \lor \tau_{02,ij} \ll (n \lor p) \) by condition (A3). Thus, the leading terms in the lower bound of \( \log B_{10}(\tilde{X}_i, \tilde{Y}_i, \tilde{X}_j, \tilde{Y}_j) \) is

\[
C_{bm}^2 \log(n \vee p) + \frac{1}{2} \log \left( \frac{\gamma}{1 + \gamma} \right) - \frac{1}{2} \log \left( \frac{n_1}{2} + a_0 \right) - \log \left( \frac{n_2}{2} + a_0 \right) + \frac{1}{2} \log \left( \frac{n}{2} + a_0 \right)
\]

on the event \( A_{-0,ij} \), which implies

\[
\mathbb{P}_0 \left( \log B_{10}^{\mathcal{E}}(X_{n_1}, Y_{n_2}) \geq \left( C_{bm}^2 / 8 - \alpha - 1 + o(1) \right) \log(n \lor p) \right) \\
\geq \max_{(i,j):i \neq j} \mathbb{P}_0 \left( \log B_{10}(\tilde{X}_i, \tilde{Y}_i, \tilde{X}_j, \tilde{Y}_j) \geq \left( C_{bm}^2 / 8 - \alpha - 1 + o(1) \right) \log(n \lor p) \right) \\
\geq \max_{(i,j):i \neq j} \mathbb{P}_0(A_{-0,ij}) \\
\geq 1 - 4(n \lor p)^{-C} = 1 - o(1)
\]

for some constant \( 0 < C < C_1 \). It completes the proof because we assume that \( C_{bm}^2 > 8(\alpha + 1) \). If \( \tau_{01,ij} < \tau_{02,ij} \), the same arguments hold by condition (A3).

Now, we consider the case \( a_{01,ij} \neq a_{02,ij} \), and suppose condition (A3*) holds. Define the sets

\[
A_{3ij} = \left\{ Z_n : \| \tilde{X}_i \|_2^2 \leq \sigma_{01,ij} n_1 (1 + 2\sqrt{C_0} + 2C_0) \right\}, \\
A_{4ij} = \left\{ Z_n : \| \tilde{Y}_i \|_2^2 \leq \sigma_{02,ij} n_2 (1 + 2\sqrt{C_0} + 2C_0) \right\}, \\
A_{5ij} = \left\{ Z_n : n_1 (1 - 2\sqrt{C_0}) \leq \sigma_{01,ij}^{-1} \| \tilde{X}_j \|_2^2 \leq n_1 (1 + 2\sqrt{C_0} + 2C_0) \right\}, \\
A_{6ij} = \left\{ Z_n : n_2 (1 - 2\sqrt{C_0}) \leq \sigma_{02,ij}^{-1} \| \tilde{Y}_j \|_2^2 \leq n_2 (1 + 2\sqrt{C_0} + 2C_0) \right\}
\]
for some constant $0 < C < C_1$, and let $A'_{ij} = \cap_{i=1}^{n} A_{ij}$. Then, $P_0((A'_{ij})^c) = o(1)$ by Lemma B.1, so we can focus on the set $A'_{ij}$. On the set $A'_{ij}$, (24) is bounded below by

$$-a_0 \left[ \log \left( \|\tilde{X}_i\|_2^2 + \|\tilde{Y}_i\|_2^2 + 2b_{0,ij} \right) - \log \left( n + 2a_0 \right) \right]$$

which is smaller than $-a_0 \log (n \vee p)$ because $\sigma_{01,ij} \vee \sigma_{02,ij} \ll (n \vee p)$ by condition (A3*). Note that (25) is bounded below by

$$\left( \frac{n_1}{2} + a_0 \right) \log \left( 1 - \frac{n_1 \hat{\tau}_{1,ij} + 2b_{0,ij}}{n_1 + 2a_0} \right)$$

and

$$\left( \frac{n_1}{2} + a_0 \right) \left( 2 \frac{\hat{\tau}_{ij}}{\hat{\tau}_{ij} + \frac{2b_{0,ij}}{n}} \left( \frac{b_{0,ij} (1 + \frac{2a_0}{n}) - b_{0,ij} \left( \frac{b_{0,ij}}{n} + \frac{2a_0}{n} \right)}{n_1 + 2a_0} - \hat{\tau}_{1,ij} \right) \right)$$

for all large $n_1 \wedge n_2$. Define

$$\lambda := \frac{(a_{01,ij} - a_{02,ij})^2 \|\tilde{X}_j\|_2^2 \|\tilde{Y}_j\|_2^2}{\tau_{01,ij} \|\tilde{Y}_j\|_2^2 + \tau_{02,ij} \|\tilde{X}_j\|_2^2}.$$

It is easy to see that

$$n \hat{\tau}_{ij} - n_1 \hat{\tau}_{1,ij} - n_2 \hat{\tau}_{2,ij} = \frac{1}{\|\tilde{Z}_j\|_2^2} \left( \|\tilde{Y}_j\|_2 \|\tilde{X}_j\|_2 - \|\tilde{X}_j\|_2 \|\tilde{Y}_j\|_2 \right)^2 \equiv \frac{\tau_{01,ij} \|\tilde{Y}_j\|_2^2 + \tau_{02,ij} \|\tilde{X}_j\|_2^2}{\|\tilde{Z}_j\|_2^2} \cdot \chi_1^2 (\lambda)$$

given $\tilde{Z}_j$ because

$$\frac{1}{\|\tilde{Z}_j\|_2^2} \left( \|\tilde{Y}_j\|_2 \|\tilde{X}_j\|_2 - \|\tilde{X}_j\|_2 \|\tilde{Y}_j\|_2 \right) | \tilde{Z}_j \sim N \left( \frac{(a_{01,ij} - a_{02,ij})^2 \|\tilde{X}_j\|_2 \|\tilde{Y}_j\|_2}{\|\tilde{Z}_j\|_2}, \frac{\tau_{01,ij} \|\tilde{Y}_j\|_2^2 + \tau_{02,ij} \|\tilde{X}_j\|_2^2}{\|\tilde{Z}_j\|_2} \right).$$
Then, on the set $A'_{ij}$,

\[
\lambda = \frac{(a_{01,ij} - a_{02,ij})^2}{\tau_{01,ij} \|X_j\|_2^{-2} + \tau_{02,ij} \|Y_j\|_2^{-2}} \geq \frac{(a_{01,ij} - a_{02,ij})^2}{\tau_{01,ij}(\sigma_{01,ij}n_1(1 - 2\sqrt{C\epsilon_0})^{-2} + \tau_{02,ij}(\sigma_{02,ij}n_2(1 - 2\sqrt{C\epsilon_0}))^{-2}} \geq \frac{25}{2} C_1 \log(n \lor p)
\]

by condition (A3*). By Lemma B.1, with probability at least $1 - (n \lor p)^{-C}$ for the constant $0 < C < C_1$ used to define the set $A'_{ij}$,

\[
n\widehat{r}_{ij} - n_1\widehat{r}_{1,ij} - n_2\widehat{r}_{2,ij} \geq \frac{\tau_{01,ij} \|\bar{Y}_j\|_2^2 + \tau_{02,ij} \|\bar{X}_j\|_2^2}{\|\bar{Z}_j\|_2^2} \left\{1 + \frac{\tau_{01,ij} \|\bar{Y}_j\|_2^2 + \tau_{02,ij} \|\bar{X}_j\|_2^2}{\|\bar{Z}_j\|_2^2} \left(1 + \frac{\lambda}{5}\right) \right\} \geq \frac{5}{2} \left(\sum_{k=1}^{2} \left(\sigma_{0k,ij}n_k(1 - 2\sqrt{C\epsilon_0k})^{-1}\right)^{-1}\right)^{-1}
\]

where the second inequality holds by (32). Furthermore, on the set $A'_{ij}$,

\[
\tau_{ij,\gamma} \leq \sigma_{01,ij}(1 + 4\sqrt{C\epsilon_01}) + \sigma_{02,ij}(1 + 4\sqrt{C\epsilon_02}).
\]

Then, (31) is bounded below by

\[
\frac{1}{\sigma_{01,ii}(1 + 4\sqrt{C\epsilon_01}) + \sigma_{02,ii}(1 + 4\sqrt{C\epsilon_02}) + \frac{2b_{0,ij}}{n}} \times \left\{\left(\frac{1}{2} + \frac{a_0}{n}\right) \frac{(a_{01,ij} - a_{02,ij})^2}{5} \left(\sum_{k=1}^{2} \left(\sigma_{0k,ij}n_k(1 - 2\sqrt{C\epsilon_0k})^{-1}\right)^{-1}\right)^{-1} - 2b_{0,ij}\right\} \geq C_{bm,a} \log(n \lor p)
\]

by condition (A3*). Thus, the leading terms in the lower bound of $\log B_{10}(\hat{X}_i, \hat{Y}_i, \hat{X}_j, \hat{Y}_j)$ is

\[
(C_{bm,a} - a_0) \log(n \lor p) + \frac{1}{2} \log \left(\frac{\gamma}{1 + \gamma}\right) - \frac{1}{2} \log \left(\frac{n_1}{2} + a_0\right) - \log \left(\frac{n_2}{2} + a_0\right) + \frac{1}{2} \log \left(\frac{n}{2} + a_0\right)
\]

\[
\geq \left(C_{bm,a} - a_0 - \alpha - 1\right) \log(n \lor p)
\]

on the set $A'_{ij}$, which implies

\[
\mathbb{P}_0 \left(\log B_{\max,10}(X_{n_1}, Y_{n_2}) \geq (C_{bm,a} - a_0 - \alpha - 1) \log(n \lor p)\right) \geq \max_{(i,j);i \neq j} \mathbb{P}_0 (A'_{ij}) \geq 1 - 14(n \lor p)^{-C} = 1 - o(1)
\]
for some constant $0 < C < C_1$. It completes the proof because we assume that $C_{bm,a} > a_0 + \alpha + 1$.

B.3 Proof of Theorem 3.2

Proof Let

$$U(C) := \{ (\Sigma_1, \Sigma_2) : \max_{1 \leq i \leq j \leq p} \frac{(\sigma_{1,ij} - \sigma_{1,ij})^2}{\theta_{1,ij}/n_1 + \theta_{2,ij}/n_2} \geq C \log p \},$$

where $\theta_{1,ij} = \sigma_{1,ii} \sigma_{1,jj} + \sigma_{1,ij}^2$ and $\theta_{2,ij} = \sigma_{2,ii} \sigma_{2,jj} + \sigma_{2,ij}^2$, then $\tilde{H}_1(C_*, c_0) \subset U(C)$ for some $C_*, c_0$ and $C > 0$. By Theorem 3, the proof of Theorem 4 in Cai et al. (2013) and arguments in Baraud et al. (2002) (p. 595), we have

$$\inf_{(\Sigma_01, \Sigma_02) \in U(C)} \sup_{\phi \in \mathcal{T}_{\alpha_0}} \mathbb{E}_{\Sigma_01, \Sigma_02} \phi_{\alpha_0} \leq \alpha + o(1)$$

for some constant $C > 0$, where $\mathcal{T}_{\alpha_0}$ is the set of $\alpha_0$-level tests over the multivariate normal distributions with $\alpha_0 > 0$, that is, $\mathbb{E}_{\phi_{\alpha_0}} \leq \alpha_0$. In the proof of Theorem 4 in Cai et al. (2013), the infimum of $(\Sigma_01, \Sigma_02)$ is essentially taken over a subset of $\tilde{H}_1(C_*, c_0)$ for some small $C_*>0$ and any $c_0 > 0$. Hence, because $p \geq n^c$ for some constant $c > 0$, we have

$$\inf_{(\Sigma_01, \Sigma_02) \in \tilde{H}_1(C_*, c_0)} \sup_{\phi \in \mathcal{T}_{\alpha_0}} \mathbb{E}_{\Sigma_01, \Sigma_02} \phi_{\alpha_0} \leq \alpha_0 + o(1)$$

for some small $C_*>0$. Note that $\tilde{H}_1(C_*, c_0) \subset H_1(C_{bm}, C_{bm,a})$ for some small constants $C_{bm}$ and $C_{bm,a} > 0$. Furthermore, we have $\mathcal{T} \subset \mathcal{T}_{\alpha_0}$ for any $\alpha_0 > 0$. Therefore, for any $\alpha_0 > 0$,

$$\inf_{(\Sigma_01, \Sigma_02) \in H_1(C_{bm}, C_{bm,a})} \sup_{\phi \in \mathcal{T}} \mathbb{E}_{\Sigma_01, \Sigma_02} \phi \leq \alpha_0 + o(1).$$
C Auxiliary results

Lemma C.1 Consider two different $p \times p$ covariances $\Sigma_1 = (\sigma_{01,ij})$ and $\Sigma_2 = (\sigma_{02,ij})$ such that
\[
\max_{1 \leq k \leq 2} \max_{1 \leq i \neq j \leq p} \rho_{0k,ij}^2 \leq 1 - c_0 \n \}
\end{align}
for some small constant $c_0 > 0$. If
\[
\max_{1 \leq i \leq p} \frac{(\sigma_{01,ij} - \sigma_{01,ij})^2}{\sigma_{01,ii} \sigma_{01,jj} + \sigma_{02,ii} \sigma_{02, jj}} \geq C_* \frac{\log(n \vee p)}{n}
\]
for some large constant $C_* > 0$, then condition (A3) (or $A3^*$) holds with $\alpha > 1$ and $n_1 \asymp n_2$.

Proof Throughout the proof, suppose $\alpha > 1$ and $n_1 \asymp n_2$. Note that if $\sigma_{01, jj}^2 / \sigma_{02, jj} \to \infty$ for some $j$ as $n \to \infty$, condition (A3) is met for some pair of indices because $\max_{1 \leq k \leq 2} \max_{1 \leq i \neq j \leq p} \rho_{0k,ij}^2 \leq 1 - c_0$ for some constant $c_0 > 0$. Since it trivially completes the proof, suppose $\sigma_{01, jj}^2 / \sigma_{02, jj} \vee \sigma_{02, jj} < C$ for all $j$ and some constant $C > 0$. Let
\[
\max_{1 \leq i \leq p} \frac{(\sigma_{01,ij} - \sigma_{01,ij})^2}{\sigma_{01,ii} \sigma_{01, jj} + \sigma_{02,ii} \sigma_{02, jj}} \equiv \frac{(\sigma_{01,00j} - \sigma_{01,00j})^2}{\sigma_{01,00j} \sigma_{01,00j} + \sigma_{02,00j} \sigma_{02,00j}} \geq C_* \frac{\log(n \vee p)}{n}
\]
for some $(i_0, j_0)$.

First, suppose that $i_0 = j_0$. Without loss of generality, assume that $\sigma_{01,i_0i_0} > \sigma_{02,i_0i_0}$. If condition $A3^*$ is satisfied for some $(i_0, j)$ or $(j, i_0)$, the proof is completed. Thus, now we assume that condition $A3^*$ does not hold for $(i_0, j)$ and $(j, i_0)$ for all $j \neq i_0$. Since $\sigma_{01,i_0i_0} \ll (n \vee p)$ always holds, at least one of inequalities (16) and (17) for $(i_0, j)$ and $(j, i_0)$ does not hold. Note that
\begin{align*}
\tau_{01,i_0j} &= \frac{\sigma_{01,i_0j}}{\sigma_{02,i_0j}} \frac{1 - a_{01,i_0j} a_{01,j_0j}}{1 - a_{02,i_0j} a_{02,j_0j}} \\
&= \frac{\sigma_{01,i_0j}}{\sigma_{02,i_0j}} \frac{1 - (a_{02,i_0j} + \Delta_1)(a_{02,j_0j} + \Delta_2)}{1 - a_{02,i_0j} a_{02,j_0j}} \\
&= \frac{\sigma_{01,i_0j}}{\sigma_{02,i_0j}} \left\{ 1 - \frac{\Delta_1 a_{02,j_0j} + \Delta_2 a_{02,i_0j}}{1 - a_{02,i_0j} a_{02,j_0j}} \right\}.
\end{align*}
Since we assume that at least one of inequalities (16) and (17) for $(i_0, j)$ and $(j, i_0)$ does not hold,
\[
|\Delta_1| \lesssim \left\{ \left( \sigma_{01,i_0j}^2 + \sigma_{02,i_0j}^2 \right) \log(n \vee p) \right\}^{\frac{1}{2}} \\left\{ \left( \frac{1}{\sigma_{01,i_0j}} + \frac{1}{\sigma_{02,i_0j}} \right) (\sigma_{01,i_0j} + \sigma_{02,i_0j}) \log(n \vee p) \right\}^{\frac{1}{2}},
\]
\[
|\Delta_2| \lesssim \left\{ \left( \sigma_{01,i_0j}^2 + \sigma_{02,i_0j}^2 \right) \log(n \vee p) \right\}^{\frac{1}{2}} \\left\{ \left( \frac{1}{\sigma_{01,i_0j}} + \frac{1}{\sigma_{02,i_0j}} \right) (\sigma_{01,i_0j} + \sigma_{02,i_0j}) \log(n \vee p) \right\}^{\frac{1}{2}}.
\]
Because we assume that $\frac{\sigma_{01,ij}}{\sigma_{02,ij}} \land \frac{\sigma_{02,ij}}{\sigma_{01,ij}} < C$ for all $j$, we have $|\Delta_1 \Delta_2| \lesssim \log(n \lor p)/n$. Furthermore,

$$\begin{align*}
|\Delta_1 a_{02,ji0}| & \leq \frac{|\sigma_{02,ji0}|}{\sigma_{02,ji0}} |\Delta_1| \\
& \leq \left( \frac{\sigma_{02,ji0}}{\sigma_{02,ji0}} \right)^{\frac{1}{2}} |\Delta_1| \\
& \lesssim \left( \frac{\sigma_{02,ji0}}{\sigma_{01,ji0}} \right)^{\frac{1}{2}} \left\{ \left[ \frac{1}{\sigma_{01,ji0}} + \frac{1}{\sigma_{02,ji0}} \right] \log(n \lor p) \right\}^{\frac{1}{2}} \left\{ \left[ \frac{1}{\sigma_{01,ji0}} + \frac{1}{\sigma_{02,ji0}} \right] \log(n \lor p) \right\}^{\frac{1}{2}} \\
& \lesssim \frac{1}{n} \left[ \log(n \lor p) \right]^{\frac{1}{2}} \left[ \log(n \lor p) \right]^{\frac{1}{2}} \\
& \lesssim \sqrt{\log(n \lor p)/n}.
\end{align*}$$

Similarly, we have $|\Delta_2 a_{02,ji0}| \lesssim \sqrt{\log(n \lor p)/n}$. Then,

$$\begin{align*}
\frac{|\Delta_1 a_{02,ji0} + \Delta_2 a_{02,ji0} + \Delta_1 \Delta_2|}{1 - a_{02,ji0} a_{02,ji0}} & \lesssim \sqrt{\frac{\log(n \lor p)}{n} + |\Delta_1 \Delta_2|} \lesssim \sqrt{\frac{\log(n \lor p)}{n}}.
\end{align*}$$

Since (33) implies

$$\frac{\sigma_{01,ji0}}{\sigma_{02,ji0}} \geq 1 + C \sigma_{01,ji0} \sqrt{\frac{\log(n \lor p)}{n}} > 1 + \sqrt{C} \sqrt{\frac{\log(n \lor p)}{n}},$$

it implies that condition (A3) holds for $(i_0, j)$ for some large $C_*>0$.

Now, suppose that $i_0 \neq j_0$. If $\sigma_{01,ji0} = \sigma_{02,ji0}$, we have

$$\begin{align*}
(a_{01,ji0} - a_{02,ji0})^2 = \frac{1}{\sigma_{01,ji0}^2} (\sigma_{01,ji0} - \sigma_{02,ji0})^2 \\
& \geq \frac{\sigma_{01,ji0} \sigma_{01,ji0} + \sigma_{01,ji0} \sigma_{02,ji0}}{\sigma_{01,ji0}^2} \frac{\log(n \lor p)}{n} \\
& = \frac{\sigma_{01,ji0} + \sigma_{02,ji0}}{\sigma_{01,ji0}} \frac{\log(n \lor p)}{n}.
\end{align*}$$

Then, condition (A3*) holds for $(j_0, i_0)$ and some large $C_*>0$. Similarly, if $\sigma_{01,ji0} = \sigma_{02,ji0}$, then condition (A3*) holds for $(i_0, j_0)$ and some large $C_*>0$. Thus, we only need to consider the case $\sigma_{01,ji0} \neq \sigma_{02,ji0}$ and $\sigma_{01,ji0} \neq \sigma_{02,ji0}$. Without loss of generality, suppose $\sigma_{01,ji0} > \sigma_{02,ji0}$. Note that if $\sigma_{01,ji0}/\sigma_{02,ji0} > 1 + C \sqrt{\log(n \lor p)/n}$ for some large constant $C > 0$, condition (A3) is met as shown in the previous paragraph. If $1 < \sigma_{01,ji0}/\sigma_{02,ji0} \leq 1 + C \sqrt{\log(n \lor p)/n}$ and
\( \sigma_{01,i_0j_0}/\sigma_{02,i_0j_0} \equiv 1 + \Delta_3 \) with \( \Delta_3 > 0 \), we have

\[
(a_{01,i_0j_0} - a_{02,i_0j_0})^2 = \left( \frac{\sigma_{01,i_0j_0}}{\sigma_{01,i_0j_0}} - \frac{\sigma_{02,i_0j_0}}{\sigma_{02,i_0j_0}} \right)^2
\]

\[
= \left( \frac{\sigma_{01,i_0j_0}}{\sigma_{01,i_0j_0}} - \frac{\sigma_{02,i_0j_0}}{\sigma_{01,i_0j_0}} \frac{\sigma_{01,i_0j_0}}{\sigma_{02,i_0j_0}} \right)^2
\]

\[
= \frac{1}{\sigma_{01,i_0j_0}^2} \left( \sigma_{01,i_0j_0} - \sigma_{02,i_0j_0} - \sigma_{02,i_0j_0} \Delta_3 \right)^2
\]

\[
\geq \frac{1}{\sigma_{01,i_0j_0}^2} \left( ||\sigma_{01,i_0j_0} - \sigma_{02,i_0j_0}|| - ||\sigma_{02,i_0j_0} \Delta_3|| \right)^2
\]

\[
\geq C_*(\frac{\sigma_{01,i_0j_0}}{\sigma_{01,i_0j_0}} + \frac{\sigma_{02,i_0j_0}}{\sigma_{02,i_0j_0}}) \frac{\log(n \lor p)}{\sigma_{02,i_0j_0}^2} \frac{\log(n \lor p)}{\sigma_{02,i_0j_0}^2}
\]

\[
\geq C_*(\frac{\sigma_{01,i_0j_0}}{\sigma_{01,i_0j_0}} + C') \frac{\sigma_{02,i_0j_0}}{\sigma_{02,i_0j_0}} \frac{\log(n \lor p)}{\sigma_{02,i_0j_0}^2}
\]

for some large constant \( C_* > 0 \) and constant \( C' > 0 \). Thus, condition \((A3^*)\) holds for \((j_0, i_0)\) and some large constant \( C_* > 0 \), and it completes the proof.

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