An Uniqueness Result on Spherically Stratified Media with Interior Transmission Eigenvalues

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Abstract

Given a set of transmission eigenvalues, we describe the connection between such a set and the indicator functions in entire function theory. The indicator functions control the asymptotic growth rate of the solution of the Sturm-Liouville problem which has an uniqueness in the inverse spectral theory. Accordingly, the set of transmission eigenvalues has an inverse spectral property.

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1 Introduction and the Main Result

In this paper, we consider the stationary scattering problem

\[
\begin{aligned}
\Delta u + k^2 n(x)u &= 0, & \text{in } \mathbb{R}^3; \\
u &= u^s + u^i; \\
\lim_{r \to \infty} r \left\{ \frac{\partial u^s}{\partial r} - ik u^s \right\} &= 0,
\end{aligned}
\]

\[
(1.1)
\]

where \(u^i\) is an entire solution of the Helmholtz equation, \(k\) is the wave number and \(n \in C^1(0,a) \cap H^2(0,a)\) is specified spherically stratified in \(B\), where \(B := \{ \mathbb{R}^3 | x < a \}\), such that \(n(x) = n(r) > 0\) and \(n(a) = 1\).

We may ask that is there any incident fields \(u^i\) such that the scattered field \(u^s\) is identically zero? The answer is positive provided that there exists a nontrivial solution to the following interior transmission problem:

\[
\begin{aligned}
\Delta w + k^2 n(r)w &= 0, & \text{in } B; \\
\Delta v + k^2 v &= 0, & \text{in } B; \\
w &= v, & \text{on } \partial B; \\
\frac{\partial w}{\partial r} = \frac{\partial v}{\partial r}, & \text{on } \partial B.
\end{aligned}
\]

\[
(1.2)
\]

For spherical perturbations as the one given by (1.1), we consider the spherical harmonics which we refer to Colton and Kress [5] for a theory in inverse problems. In particular, we let \(Y_l\) be the solution of following equation.

\[
Y_l'' + \frac{2}{r} Y_l' + \left\{ k^2 n(r) - \frac{l(l+1)}{r^2} \right\} Y_l = 0,
\]

\[
(1.3)
\]

such that

\[
\lim_{r \to 0} \{ Y_l(r) - j_l(kr) \} = 0,
\]

\[
(1.4)
\]

where \(j_l\) is a spherical Bessel function. The existence of the nontrivial solution of the interior transmission problem (1.2) is implied by the existence of the nontrivial constants \(a_l, b_l\) for some \(k, l\) to the following system of equations.

\[
\begin{aligned}
a_l Y_l(a) - b_l j_l(ka) &= 0; \\
a_l Y_l'(a) - b_l k j_l'(ka) &= 0.
\end{aligned}
\]

\[
(1.5)
\]
Equivalently, we are looking for the zeros of the following functional determinant.

$$d_l(k) := \det \begin{pmatrix} Y_l(a) & -j_l(ka) \\ Y_l'(a) & -k_j(ka) \end{pmatrix}. \quad (1.6)$$

These zeros of the functional determinant $d_l(k)$ are the interior transmission eigenvalues of $(1.2)$. In particular, when $l = 0$, we consider the zeros of

$$d_0(k) := \det \begin{pmatrix} Y_0(a) & -j_0(ka) \\ Y_0'(a) & -k_j(ka) \end{pmatrix}, \quad (1.7)$$

where $Y_0 = \frac{y(r)}{r}$ and $y(r)$ satisfies

$$y'' + k^2 n(r)y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad (1.8)$$

which is well-known that there exists an unique solution to $(1.8)$ to every $k \in \mathbb{C}$. Hence, we may see $y = y(x; k)$. $j_0(kr) = \frac{\sin(kr)}{kr}$. Moreover, we have the asymptotics of $y(r; k)$ and $y'(r; k)$:

$$y(x; k) = \frac{1}{[n(0)n(r)]^{1/4}} \sin(k \int_0^r [n(r)]^{1/2} dr) [1 + O(\frac{1}{|k|})], \quad \forall k \in \mathbb{C} \setminus \{0i + \mathbb{R}\}. \quad (1.9)$$

Similarly,

$$y'(x; k) = \frac{1}{[n(0)/n(r)]^{1/2}} \cos(k \int_0^r [n(r)]^{1/2} dr) [1 + O(\frac{1}{|k|})], \quad \forall k \in \mathbb{C} \setminus \{0i + \mathbb{R}\}. \quad (1.10)$$

We refer such asymptotics to [11 Proposition 2.3]. Such asymptotic expansions are classic in spectral theory. See Pöschel and Trubowitz [11] and Naimark [10].

**Definition 1.1** We define $s := a - \int_0^a [n(r)]^{1/2} dr$ and $b := \int_0^a [n(r)]^{1/2} dr$.

In this paper, we assume either

$$a > b \quad \text{or} \quad a < b, \quad (1.11)$$

simultaneously for each $n(r)$’s. The following asymptotic behavior holds.

$$d_0(k) = \frac{1}{a^2 k [n(0)]^{1/4}} \sin k(a - b) + O(\frac{1}{k^2}), \forall k \in \{0i + \mathbb{R}\}. \quad (1.12)$$

Such asymptotic expansion is classical in spectral theory. See Colton and Kress [5]. Let us consider its behavior in $\mathbb{C}$. From $(1.9)$ and $(1.10)$,

$$\frac{y(a)}{a}[-k_j(ka)] = \frac{B \sin k a \sin k b}{a^2 k^2} - \frac{B \cos k a \sin k b}{a^2 k} [1 + O(\frac{1}{|k|})], \quad B := \frac{1}{[n(0)n(a)]^{1/2}};$$

$$j_0(ka) \frac{(y(r))_r}{r}|_{r=a} = \frac{C \sin k a \cos k b}{a^2 k} - \frac{B \sin k a \sin k b}{a^2 k^2} [1 + O(\frac{1}{|k|})], \quad C := \frac{n(a)}{n(0)}^{1/2}. \quad (1.13)$$

Hence, we have the asymptotics for $d_0(k)$.

$$d_0(k) = j_0(ka) \frac{(y(r))_r}{r}|_{r=a} + \frac{y(a)}{a}[-k_j(ka)] \quad (1.14)$$

$$= \frac{1}{a^2 k [n(0)]^{1/2}} \sin k(a - b) [1 + O(\frac{1}{|k|})], \quad \forall k \in \mathbb{C} \setminus \{0i + \mathbb{R}\}. \quad (1.15)$$

In Aktosun, Gintides and Papanicolaou [11 p.5], they have shown if $\int_0^a [n(r)]^{1/2} dr < a$, then the transmission eigenvalues corresponding to spherically symmetric solutions of the interior transmission problem uniquely determine the $n(r)$. Furthermore, in [11], it is shown if $d_0(k) \equiv 0$ for $\lambda \in \mathbb{C}$, then $n(r) \equiv 1$ in $[0, a]$. It is also discussed that the signs of the quantity $a - b$ plays a role in the inverse spectral theory in [11]. Furthermore, in Cakoni, Colton and Gintides [2 Theorem 2.1], they have shown if $n(0)$ is given and $n(r) > 1$, then $n(r)$ is uniquely determined from some knowledge of the transmission eigenvalues. It
is expected among mathematicians, say, [6], that such an uniqueness holds for the condition \( n(r) > 1 \). In this paper, we show that only the interior transmission eigenvalues near the real axis are needed to determine the \( n(r) \). In addition, we propose another qualitative description on the counting function to the zeros of \( d_{l}(z) \). In the spectral theory of Sturm-Liouville, such a qualitative description on the growth rate of zeros of \( d_{l} \) is connected to the inverse problem on finding index \( n(r) \). For such a connection, we refer to McLaughlin and Polyakov [9] which is based on P"oschel and Trubowitz’s work in [11]. In [9], there is an argument on the qualitative description for the zeros of \( d_{0}(z) \).

Firstly, we need some vocabulary from entire function theory. We refer to the Levin’s book [7, 8].

Definition 1.2 Let \( f(z) \) be an entire function. Let \( M_{f}(r) := \max_{|z|=r} |f(z)| \). An entire function of \( f(z) \) is said to be a function of finite order if there exists a positive constant \( k \) such that the inequality

\[
M_{f}(r) < e^{r^{k}}
\]

is valid for all sufficiently large values of \( r \). The greatest lower bound of such numbers \( k \) is called the order of the entire function \( f(z) \). By the type \( \sigma \) of an entire function \( f(z) \) of order \( \rho \), we mean the greatest lower bound of positive number \( A \) for which asymptotically we have

\[
M_{f}(r) < e^{Ar^{\sigma}}.
\]

That is

\[
\sigma := \limsup_{r \to \infty} \frac{\ln M_{f}(r)}{r^{\rho}}.
\]

If \( 0 < \sigma < \infty \), then we say \( f(z) \) is of normal type or mean type.

Definition 1.3 If an entire function \( f(z) \) is of order one and of normal type, then we say it is an entire function of exponential type \( \sigma \).

Definition 1.4 Let \( \rho \in \mathbb{R} \) and \( \rho(r) : \mathbb{R}^{+} \to \mathbb{R}^{+} \). We say \( \rho(r) \) is a proximate order to \( \rho \) if

\[
\lim_{r \to \infty} \rho(r) = \rho \geq 0; \lim_{r \to \infty} r \rho'(r) \ln r = 0.
\]

Definition 1.5 Let \( f(z) \) be an integral function of finite order in the angle \([\theta_{1}, \theta_{2}]\), we call the following quantity as the generalized indicator of the function \( f(z) \).

\[
h_{f}(\theta) := \limsup_{r \to \infty} \frac{|f(re^{i\theta})|}{r^{\rho(r)}}, \theta_{1} \leq \theta \leq \theta_{2},
\]

where \( \rho(r) \) is some proximate order.

We review two inequalities for indicator functions.

\[
h_{f+g}(\theta) \leq h_{f}(\theta) + h_{g}(\theta); \quad (1.21)
\]

\[
h_{f+g} \leq \max\{h_{f}(\theta), h_{g}(\theta)\}. \quad (1.22)
\]

We can find these in [7, p.51].

The order and the type of an integral function in an angle can be defined similarly. The connection between the indicator \( h_{f}(\theta) \) and its type \( \sigma \) can be specified by the following theorem.

Lemma 1.6 (Levin [7, p.72]) The maximum value of the indicator \( h_{f}(\theta) \) of the function \( f(z) \) on the interval \( \alpha \leq \theta \leq \beta \) is equal to the type \( \sigma_{f} \) of this function inside the angle \( \alpha \leq \arg z \leq \beta \).

Lemma 1.7 Let \( f(z) = C \sin Az \), where \( A, C \) are constants. \( f(z) \) is an entire function of exponential type \( |A| \).

Proof It suffices to see \( \frac{\sin z}{z} = \frac{e^{iz} - e^{-iz}}{2iz} \) and

\[
h_{\text{max}}(\theta) = \limsup_{r \to \infty} \frac{\ln \sqrt[2r]{\sin^{2} \sin^{2} - 2 \cos 2r \cos \theta}}{r} = |\sin \theta|.
\]

\( \square \)
The foundation of this paper is the following theorem in Cartwright [3, p.538].

**Theorem 2.1** Suppose that $d$ is a non-negative integer. Furthermore, $j$ is a spherical Bessel function. We see that $h$ is defined as above. Then, the following asymptotics hold.

**Proof** Let $\eta := za - zb$. We compute that

$$|\sin \eta|^2 = \frac{1}{4} [e^{-\eta + \pi} - e^{-\eta - \pi} + e^{\eta + \pi} - e^{\eta - \pi}]$$

$$= \frac{1}{4} [e^{-2\eta + 2\pi y} - e^{2\eta + 2\pi y} - e^{-2\eta + 2\pi y} + e^{2\eta + 2\pi y}]$$

$$= \frac{1}{4} [e^{-2\eta + 2\pi y} + e^{2\eta + 2\pi y} - 2\cos(2\pi a - 2\pi b)].$$

Following from [14,15] that

$$h_{d_0}(\theta) = \limsup_{r \to \infty} \frac{\ln |\sin \eta|}{r} = \limsup_{r \to \infty} \frac{|\eta||a - b|}{r} = |\sin \theta||a - b|, \theta \neq 0, \pi. \quad (1.26)$$

We see that $h_{d_0}(\theta)$ is continuous in $[0, 2\pi]$, Levin [71, p.54], we conclude

$$h_{d_0}(\theta) = |a - b||\sin \theta|, \theta \in [0, 2\pi]. \quad (1.27)$$

Furthermore, $d_0(z)$ is an entire function of exponential growth due to the definition [16] and the fact that spherical Bessel function $j_0(z) = \sin \frac{z}{z}$ and $y(r)$ are entire functions of exponential type.

We see that $h_{d_0}(\theta)$ attains its nontrivial maximum $|a - b|$ at $\theta = \pm \frac{\eta}{2}$. □

The following is the main theorem of this paper.

**Theorem 1.9** Let the 0-th functional determinant $d_0(z)$ be defined as above. Then, the zeros of $d_0(z)$ inside the angular wedge

$$\Sigma_\epsilon := \{ z \in \mathbb{C} | -\epsilon \leq \arg k \leq \epsilon, \pi - \epsilon \leq \arg k \leq \pi + \epsilon \}, \forall \epsilon > 0, \quad (1.28)$$

uniquely determines between the indices $n(r)$’s provided they are of the same value at $r = 0$.

### 2 M.L. Cartwright’s Theorem

The foundation of this paper is the following theorem in Cartwright [3, p.538].

**Theorem 2.1** Suppose that $f(z)$ is an integral function of order $\rho = 1$ with the following Hadamard’s representation:

$$f(z) = z^m \exp \{c_0 + c_1 z\} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) \exp \{ \frac{z}{z_n} \} \quad (2.1)$$

and that the indicator

$$h(\theta) = \max \{ A \cos \theta, B \cos \theta \}, \quad (2.2)$$

where $B \geq A, B > 0$. Then, the following asymptotics hold.

$$n(r, \frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta) \sim 0; \quad (2.3)$$

$$n(r, \frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta) \sim 0; \quad (2.4)$$

$$n(r, \pm \frac{\pi}{2} - \delta, \pm \frac{\pi}{2} + \delta) \sim \frac{1}{2\pi} (B(r) - A(r)) r^\rho(r), \quad (2.5)$$

$$c_1 + \sum_{|z_n| \leq r} \frac{1}{z_n} \sim \frac{1}{2} (A(r) + B(r)) r^\rho(r) - 1, \quad (2.6)$$
where $A(r)$ and $B(r)$ are real functions such that $B(r) \geq A(r)$,

\[
\liminf_{r \to \infty} A(r) = A; \limsup_{r \to \infty} B(r) = B; \tag{2.7}
\]

\[
\lim_{r \to \infty} \{A(r) - A(qr)\} = 0; \lim_{r \to \infty} \{B(r) - B(qr)\} = 0, \forall q > 0, \tag{2.8}
\]

where $\rho(r)$ is a proximate order to $\rho$.

**Remark 2.2** We may choose $A = -B$ for this note. We also note that $d_0(z)$ is bounded on $0i + \mathbb{R}$. Cartwright’s paper may be hard to obtained. We consider the Levin’s book on functions of class $C$ in [8] for some modern reference and backup theory source. To answer some mathematicians’ question, rotating the functional $d(z)$ by $\frac{\pi}{2}$ won’t alter the nature of the distribution of the zeros. Such a picture is clear in the discussion in Levin [8, p.127]. Our functional $d(z)$ is trivially of class $C$ as already discussed in Chen [4]. Two conditions listed in [8, p.115] can be justified by the fact that $d(z)$ is bounded over the real axis. We compare with the indicator function appears in (2.2), the one in Lemma 1.8 and the one in Levin [8, p.126]. We see all these three functions are consistent with each other. Another examination on (3.5), (3.8), (2.6) and (3.17) yields the same indicator functions.

**Corollary 2.3** Let $n(d^j, r, \alpha, \beta)$ be the number of zeros of $d^j(z)$ that are located in the closed cone $\{z \in \mathbb{C}|\alpha \leq \arg z \leq \beta, |z| \leq r\}$ and

\[
\Delta^j(\alpha, \beta) := \lim_{r \to \infty} \frac{n(d^j, r, \alpha, \beta)}{r}, \quad \Delta^j(\beta) - \Delta(\alpha) := \Delta(\alpha, \beta) + C, \tag{2.9}
\]

where $j = 1, 2$, and $C$ is some constant. Then,

\[
\Delta^j(\delta, \pi - \delta) = 0; \tag{2.10}
\]

\[
\Delta^j(\pi + \delta, -\delta) = 0; \tag{2.11}
\]

\[
\Delta^j(-\delta, \delta) > 0; \tag{2.12}
\]

\[
\Delta^j(\pi - \delta, \pi + \delta) > 0; \tag{2.13}
\]

\[
\Delta^j(-\delta, \delta) = \Delta^2(-\delta, \delta) \neq 0; \tag{2.14}
\]

\[
\Delta^j(\pi - \delta, \pi + \delta) = \Delta^2(\pi - \delta, \pi + \delta) \neq 0. \tag{2.15}
\]

Accordingly, let $E$ be the set of points of discontinuity of the function $\Delta^j(\psi)$. Then, $E = \{0, \pi\}$.

**Proof** This is only Cartwright’s theory. The indicator function $h_d(\theta)$ is computed in Lemma 1.8. Combining with (2.2), we prove the corollary.  

**3 Proof of Theorem 1.9**

Let $d(z) := d_0(z)$. From (2.1), suppose that

\[
d(z) = z^n \exp\{c_0 + c_1 z\} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\{\frac{z}{z_n}\}. \tag{3.1}
\]

Letting $z := re^{i\theta}$,

\[
\delta(r) := c_1 + \sum_{|z_n| \leq r} \frac{1}{z_n}. \tag{3.2}
\]

\[
f_r(z) := \prod_{|z_n| \leq r} \left(1 - \frac{z}{z_n}\right) \prod_{|z_n| > r} \left(1 - \frac{z}{z_n}\right) e^{\frac{\pi}{4} i}. \tag{3.3}
\]

For each index $n^j(r)$, we use

\[
f_r^j(z) := \prod_{|z_n^j| \leq r} \left(1 - \frac{z}{z_n^j}\right) \prod_{|z_n^j| > r} \left(1 - \frac{z}{z_n^j}\right) e^{\frac{\pi}{4} i}. \tag{3.4}
\]
Hence, (5.1) becomes
\[ d(z) = z^n e^{\psi} e^{\delta(r)z} f_r \] \hspace{1cm} (3.5)

From (3.5), we have
\[ h_d(\theta) = \lim sup \frac{\ln |d(re^{i\theta})|}{r} = \lim sup \frac{|\delta(r)z|}{r} + \lim sup \frac{\ln |f_r(re^{i\theta})|}{r} \] \hspace{1cm} (3.6)

For the first limit in (3.6). We see from (2.8) that \( \delta(r) \sim 0 \). That is \( |\delta(r)| < \delta' \) for some large \( r \) for any given \( \delta' > 0 \). Hence,
\[ |e^{\delta(r)z}| < e^{\delta(r)|z|} = e^{\delta(r)r} < e^{\delta'r}, \forall \delta' > 0. \] \hspace{1cm} (3.7)

Most important of all, we approximate the infinite product \( f_r(z) \) by Levin [7, p.112. Lemma 5; p.92. Theorem 2]: out of some zero density set we have the following asymptotic behavior:
\[ \ln |f_r(re^{i\theta})| \sim H_1(\theta) r^\rho(\theta), \text{ where } \rho(\theta) \rightarrow 1, \] \hspace{1cm} (3.8)
in which
\[ H_1(\theta) = -\int_0^\theta \left[ (\theta - \psi - \pi) \sin(\theta - \psi) + \cos(\theta - \psi) \right] d\Delta(\psi). \] \hspace{1cm} (3.9)

That integral is approximated by the following sum:
\[ S_n(\theta) = -\sum_{l=0}^{n} [(\theta - \psi_l - \pi) \sin(\theta - \psi_l) + \cos(\theta - \psi_l)] [\Delta(\psi_{l+1}) - \Delta(\psi_l)], \] \hspace{1cm} (3.10)
where \( \psi_0 < \psi_1 < \cdots < \psi_n < \psi_{n+1} \), \( |\psi_{l+1} - \psi_l| < \delta, l = 0, 1, 2, \cdots, n. \psi_{n+1} = \psi_0 + 2\pi \). We can choose \( \delta \) small such that
\[ |H_1(\theta) - S_n(\theta)| < \epsilon/3. \] \hspace{1cm} (3.11)

Let \( h^1(\theta), H^1(\theta), \{\psi^1_l\}, f^1_r(z) \) and \( S^1_n(\theta) \) be the corresponding quantities with respect to index \( n^1 \). Let \( 0 \in [\psi^1_{l_0}, \psi^1_{l_0+1}], \pi \in [\psi^1_{l_1}, \psi^1_{l_1+1}], \pi \in [\psi^1_{l_2}, \psi^1_{l_2+1}] \) be the angular intervals containing nonzero density angles which are 0 and \( \pi \) by (2.14) and (2.15). Using (2.10) and (2.11), we conclude that, \( \forall n \),
\[ S^1_n(\theta) \equiv S^2_n(\theta). \] \hspace{1cm} (3.15)

Accordingly,
\[ H^1_1(\theta) \equiv H^2_1(\theta). \] \hspace{1cm} (3.16)

Similarly, (3.9) gives
\[ H^1_1(\theta) = \pi \Delta^1|\sin \theta|, \] \hspace{1cm} (3.17)
with its maximal value happening at \( \theta = \pm \frac{\pi}{2} \). Also,
\[ \ln |f^1_r(re^{i\theta})| \sim \ln |f^2_r(re^{i\theta})|, \text{ as } r \rightarrow \infty. \] \hspace{1cm} (3.18)

We conclude from (3.6), (3.7), (3.8), (3.10) and (3.13),
\[ h^1(\theta) \equiv h^2(\theta). \] \hspace{1cm} (3.19)

Hence, we proved the following theorem.

**Proposition 3.1** If zeros of \( \delta^1(z) \), \( j = 1, 2 \) inside the wedge \( \Sigma_\epsilon, \forall \epsilon > 0 \) shares the same density, then the indicator function \( h^1(\theta) \equiv h^2(\theta) \) in \( [0, \pi] \). In particular, \( d^1 \) and \( d^2 \) are entire functions of exponential type of the same type \( |a - \int_0^\infty \sqrt{n^1(r)dr}| = |a - \int_0^\infty \sqrt{n^2(r)dr}| \).
Proof Since they share the same indicator function, the related maximal value of the indicator functions is the same. Hence, \( d^1, d^2 \) are entire functions of the same type by Lemma 1.8. The type is just 
\[ |a - \int_0^a \sqrt{n^2(r)} dr| = |a - \int_0^a \sqrt{n^2(r)} dr|. \]
□

Definition 3.2 Let \( b^j := \int_0^a \sqrt{n^2(r)} dr. s^j := a - b^j, j = 1, 2. \)

Corollary 3.3 Let \( h_{d^1 - d^2}(\theta) \) be the indicator function related to the \( d^1(z) - d^2(z) \). Then,
\[ h_{d^1 - d^2}(z, \theta) = 0. \] (3.20)

Proof By definition and that \( n^1(0) = n^2(0), s^1 = s^2, (1.11) \) and \( (1.13) \) implies
\[ h_{d^1 - d^2}(\theta) = \lim_{r \to \infty} \sup \frac{|d^1(z) - d^2(z)|}{r} = 0. \] (3.21)
□

On the other hand, we consider the substraction in the form of 3.3. Let \( f^j(z), j = 1, 2, \) be the quantities corresponding to \( n^j(r) \) in 3.3. We set
\[
f^j(z) := \prod_{\{ |z| \leq r \}} \left( 1 - \frac{z}{z_n} \right) \prod_{\{ |z| > r \}} \left( 1 - \frac{z}{z_n} \right)e^{\frac{z}{z_n}}. \] (3.22)

Using the assumption that zeros inside \( \Sigma \), coincides as a set, we have
\[
d^1(z) - d^2(z) = \frac{z^{m_1}}{z_n} \prod_{\{ |z| \leq r; z \in \Sigma \}} \left( 1 - \frac{z}{z_n} \right) \prod_{\{ |z| > r; z \in \Sigma \}} \left( 1 - \frac{z}{z_n} \right)e^{\frac{z}{z_n}}
\times [e^{\delta(z)}(z_n) \prod_{\{ |z| \leq r; z \notin \Sigma \}} \left( 1 - \frac{z}{z_n} \right) \prod_{\{ |z| > r; z \notin \Sigma \}} \left( 1 - \frac{z}{z_n} \right)e^{\frac{z}{z_n}}
- \frac{z^{m_2}}{z_n} \prod_{\{ |z| \leq r; z \notin \Sigma \}} \left( 1 - \frac{z}{z_n} \right) \prod_{\{ |z| > r; z \notin \Sigma \}} \left( 1 - \frac{z}{z_n} \right)e^{\frac{z}{z_n}}]. \] (3.24)
Let us define
\[ Q_r(z) := \frac{z^{m_1}}{z_n} [e^{\delta(z)}(z_n) \prod_{\{ |z| \leq r; z \notin \Sigma \}} \left( 1 - \frac{z}{z_n} \right) \prod_{\{ |z| > r; z \notin \Sigma \}} \left( 1 - \frac{z}{z_n} \right)e^{\frac{z}{z_n}}
- \frac{z^{m_2}}{z_n} \prod_{\{ |z| \leq r; z \notin \Sigma \}} \left( 1 - \frac{z}{z_n} \right) \prod_{\{ |z| > r; z \notin \Sigma \}} \left( 1 - \frac{z}{z_n} \right)e^{\frac{z}{z_n}}], \] (3.25)
which has no zero along \( 0 + i \mathbb{R} \). Now we have
\[
d^1(z) - d^2(z) = \prod_{\{ |z| \leq r; z \notin \Sigma \}} \left( 1 - \frac{z}{z_n} \right) \prod_{\{ |z| > r; z \notin \Sigma \}} \left( 1 - \frac{z}{z_n} \right)e^{\frac{z}{z_n}}] Q_r(z). \] (3.26)
Using 2.6,
\[ |e^{\delta(z)}(z)| \lesssim C e^{\delta |z|}, \forall \delta > 0, j = 1, 2. \] (3.27)

Given \( \{ z_n \} \) of zero density outside \( \Sigma \), we compute the following quantity.
\[
\overline{F}_r(z) := \prod_{\{ |z| \leq r; z \notin \Sigma \}} \left( 1 - \frac{z}{z_n} \right) \prod_{\{ |z| > r; z \notin \Sigma \}} \left( 1 - \frac{z}{z_n} \right)e^{\frac{z}{z_n}}. \] (3.28)
We use formula 3.8 and 3.9.
\[ \ln |\overline{F}_r(re^{i\theta})| \sim \overline{F}_r(\theta)r, \text{ where } r \to \infty, \] (3.29)
in which
\[
\overline{P}_j^i(\theta) = - \int_{\theta-2\pi}^{\theta} [(\theta - \psi - \pi) \sin(\theta - \psi) + \cos(\theta - \psi)] d\Delta^1(\psi),
\]
outside certain exceptional set. We refer to [7, p.112 Lemma 5] for a complete introduction. For the application here, we obtain that \( \Delta^1(\psi) \equiv 0 \) for zeros outside \( \Sigma^* \). That is
\[
\overline{P}_j^i(\theta) \equiv 0, \quad j = 1, 2.
\] (3.31)
We note that \( \overline{P}_j^i(\theta) = h_{\overline{P}_j^i}(\theta) \), the indicator function of \( \overline{P}_j^i, j = 1, 2 \). We refer this to [7, p.498].

Using (1.21) and (1.22), we have
\[
h_{Q_j}(\theta) \leq \max\{h_{e^{i\arg(z)}}, r_j(z)\},
\]
Therefore, (3.27), (3.29) and (3.31) combine to yield \( h_{Q_j}(\theta) = 0 \) and
\[
|Q_j(z)| \lesssim Ce^{\delta|z|}, \quad \text{for some } C > 0, \forall \delta > 0.
\] (3.33)
Again, the infinite product \( \prod_{|z| < r, \ z \in \Sigma} (1 - \frac{1}{\pi} e^{r \frac{1}{r} z}) \prod_{|z| > r, \ z \in \Sigma} (1 - \frac{1}{\pi} e^{r \frac{1}{r} z}) \) in (3.20) can be computed similarly as the \( f(z) \) in the (3.30), (3.31) and (3.32). From (3.20), (3.30) and (3.32), and Lindelöf’s theorem [7, p.28], we have
\[
h_{d_1 - d_2}(\theta) = \pi \Delta^1(-\delta, \delta) |\sin\theta|.
\] (3.34)
However, Corollary 3.3 says that \( \Delta^1(-\delta, \delta) = 0 \). Hence, \( d_1(z) - d_2(z) \) is an exponential function of minimal type.

Therefore, (3.33) and (3.34) give
\[
|d_1(z) - d_2(z)| \lesssim Ce^{\delta|z|}, \quad \text{for some } C > 0, \forall \delta > 0.
\] (3.35)
In addition to that, using (1.12),
\[
d_1(x) - d_2(x) \to 0, \quad \text{as } x \to 0i \pm \infty.
\] (3.36)
Using Phragmén-Lindelöf theorem as in Titchmarsh [12, p.178], (3.35) and (3.36) imply that \( d_1(z) - d_2(z) \) is bounded both in lower and half complex planes. Any bounded entire function is a constant which is zero as suggested by (3.36). Hence, \( d_1(z) \equiv d_2(z) \).

\[
(3.37)
\]
Finally, following the argument of Aktosun, Gintides and Papanicolaou around (3.7), (3.8), (3.9), (3.10) in [7, Corollary 2.10], we see that
\[
d(z) = \frac{1}{a^2} \left\{ \sin \frac{za}{a} - y'(a) - \cos(za)y(a) \right\}.
\] (3.38)
Consider a substraction for two different \( n^j(r)\)’s, we obtain
\[
d_1(z) - d_2(z) = \frac{1}{a^2} \left\{ \sin \frac{za}{a} \left[ y_1(a) - y_2(a) \right] - \cos(za) \left[ y_1'(a) - y_2'(a) \right] \right\} \equiv 0.
\] (3.39)
Let \( za = n\pi, \ n \in \mathbb{Z} \). In this case,
\[
y_1'(a; \frac{n\pi}{a}) = y_2'(a; \frac{n\pi}{a}), \ n \in \mathbb{Z}.
\] (3.40)
Similarly, let \( za = \frac{n\pi}{2}, \ n \), odd. We obtain
\[
y_1'(a; \frac{n\pi}{2a}) = y_2'(a; \frac{n\pi}{2a}), \ n \text{ odd}.
\] (3.41)
Again, \( y_1(a; za)\)’s are entire functions of exponential type. We use a generalized Carlson’s theorem from Levin [7, p.190]. This is a substitute for Phragmén-Lindelöf theorem.
The last ingredient is applying the uniqueness for the following inverse Sturm-Liouville problem as did in [1, Corollary 2.10]. We conclude that

\[ h_F(\alpha) + h_F(\alpha + \pi / \rho) \geq H_N(\alpha) + H_N(\alpha + \pi / \rho). \]  

(3.42)

Now let

\[ F(z) := y^1(a; z) - y^2(a; z), \]  

(3.43)

\[ G(z) := y^3(a; z) - y^4(a; z) \]  

(3.44)

and \( \rho \equiv 1, \alpha = -\pi / 2 \).

We deal with \( F(z) \) firstly, it has a set of common zeros of density \( \Delta^N \) and supported only at \( \psi = 0, \pi \).

Hence,

\[ H_N(\theta) = \pi \int_0^\pi \sin(\psi - \theta)d\Delta_N(\psi) = \Delta^N \pi \sin \theta, \theta \in [-\pi, \pi/2]. \]  

(3.45)

where \( N \) is the set of common zeros as described by (3.40). Accordingly, we may compute from (3.40) that \( \Delta^N = \frac{\pi}{2} \). We refer to [11, p.91; Ch 2. Sec 3.] for a systematic and a step by step calculation. Besides that, we need to compute the indicator function \( h_F(\theta) \). Recalling from (1.9),

\[ y^j(a; z) = \frac{1}{|\epsilon^j_1(0)|^2 z} \int_0^a \sqrt{\epsilon^j_1(\rho)d\rho}[1 + O\left(\frac{1}{z}\right)], j = 1, 2. \]  

(3.46)

Therefore,

\[ \ln |y^1(a; z) - y^2(a; z)| = \ln \frac{1}{z|\epsilon^1_1(0)|^2 z} + \ln |\sin[z \int_0^a \sqrt{\epsilon^1_1(\rho)d\rho}] - \sin[z \int_0^a \sqrt{\epsilon^2_1(\rho)d\rho}]| + \ln |1 + O\left(\frac{1}{z}\right)|. \]

Using (1.9), (1.11), Proposition 3.1 and Definition 1.5, we obtain

\[ h_F(\theta) = 0, \theta \neq 0, \pi. \]  

(3.47)

Again, \( h_F(\theta) \) is a continuous function given \( F \) is entire. Hence, we have

\[ h_F(\theta) = 0, \theta \in [0, 2\pi]. \]  

(3.48)

Combining Theorem 3.4, (3.45) and (3.48), we obtain \( F(z) \equiv 0 \).

Similarly, we can prove \( G(z) \equiv 0 \). In particular, we have shown

\[ y^1(a; z) \equiv y^2(a; z); y^3(a; z) \equiv y^4(a; z). \]  

(3.49)

The last ingredient is applying the uniqueness for the following inverse Sturm-Liouville problem

\[ \psi''(x) + k^2 \rho(x)\psi(x) = 0, 0 < x < a; \]  

(3.50)

\[ \psi(0) = \psi(a) = 0. \]  

(3.51)

as did in [1 Corollary 2.10]. We conclude that

\[ \epsilon^1_1(r) \equiv \epsilon^1_1(r). \]  

(3.52)
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