NON-STANDARD NEUTRAL KAONS DYNAMICS
FROM D-BRANE STATISTICS

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Abstract
The neutral kaon system can be effectively described by non-unitary, dissipative, completely positive dynamics that extend the usual treatment. In the framework of open quantum systems, we show how the origin of these non-standard time evolutions can be traced to the interaction of the kaon system with a large environment. We find that D-branes, effectively described by a heat-bath of quanta obeying infinite statistics, could constitute a realistic example of such an environment.
1. INTRODUCTION

Recently, it has been proposed to describe the system of neutral kaons using effective dynamics that differ from the standard Weisskopf-Wigner one.[1, 2] These generalized time-evolutions produce loss of quantum coherence and lead to CP and CPT violating effects that can be in principle detected in the next generations of neutral kaon experiments.

The original physical motivation underlying such approaches come from quantum gravity that predicts incoherent quantum phenomena at Planck’s length due to the fluctuation of the gravitational field.[3-6] Actually, such non-standard dynamics could also be the result of much more general considerations, based on the observation that unstable systems can be viewed as specific examples of open quantum systems.[7-10]

These systems can be modeled as being small subsystems in “weak” interaction with large environments. Under mild physical assumptions, the reduced dynamics of the subsystem, obtained by eliminating the environment degrees of freedom, is realized by dissipative time-evolution maps, with forward in time composition law (semigroup property) and the additional characteristic of being completely positive. This set of transformations forms a so-called dynamical semigroup.[11-13]

In the case of the neutral kaon system, whose states can be conveniently described by $2 \times 2$ density matrices $\rho$, the above general considerations lead to the following effective time-evolution equation:[7]

$$\frac{\partial \rho(t)}{\partial t} = -i(H_{\text{eff}} \rho(t) - \rho(t) H_{\text{eff}}^\dagger) + D[\rho(t)]. \quad \text{(1.1)}$$

The effective Hamiltonian $H_{\text{eff}}$ (the Weisskopf-Wigner Hamiltonian) accounts for the kaon decays and includes a non-hermitian part, that characterizes the natural width of the kaon states. The non-standard piece $D[\rho]$ is a linear, trace-preserving map, transforming density matrices into density matrices. It has the general form:

$$D[\rho] = \frac{1}{2} \sum_{i,j=1}^{3} a_{ij} [2 \sigma_j \rho \sigma_i - \sigma_i \sigma_j \rho - \rho \sigma_i \sigma_j], \quad \text{(1.2)}$$

where $\sigma_i$, $i = 1, 2, 3$, are the Pauli matrices and the coefficients $a_{ij}$ form a real, symmetric, positive definite matrix $[a_{ij}]$. They can be expressed in terms of six phenomenological parameters, obeying certain inequalities.[7] The magnitude of these parameters is surely very small: the contribution $D[\rho]$ in (1.1) has yet to be directly detected and therefore it should be considered as subleading with respect to the standard $H_{\text{eff}}$ term.

The positive definiteness of $[a]$ guarantees the complete positivity of the map $D[\rho]$, and hence of the time-evolution produced by (1.1); this property assures not only the positivity for all times of the eigenvalues of $\rho(t)$, but also of any density matrix describing systems of correlated kaons. On the other hand, the reality of the coefficients $a_{ij}$ assures the increase in time of the von Neumann entropy: $-d/dt \text{Tr}[\rho(t) \ln \rho(t)] \geq 0$.[8]

The phenomenological consequences of the time-evolution (1.1) have been studied in detail, in particular in view of their possible experimental measure at $\phi$-factories.[9, 10] In the following, we shall devote our attention to the discussion about the origin of the
additional dissipative term $D[\rho]$ in (1.1). As mentioned before, this term can be considered as the consequence of the incoherent interaction of the kaon system with an environment. In the next sections, we shall study in detail all the steps and procedures that justify this result. The outcome of this analysis is that a heat-bath of D-branes obeying the so-called infinite statistics could be the ultimate origin of the non-standard contribution (1.2). The experimental study of the effects of this term could therefore provide important (although indirect) informations on the underlying dynamics of low energy string theory.

2. THE MASTER EQUATION

Let us consider a system $S$, later to be identified with the neutral kaon system, in interaction with a large environment $E$. At this stage, we shall try to be as general as possible, and therefore we leave the basic characteristics of $E$ unspecified. The hamiltonian $H$ of the global system $S + E$ can be in general decomposed as

$$H = H_S \otimes 1 + 1 \otimes H_E + H',$$

(2.1)

where $H_S$ describes the dynamics of $S$ in absence of the environment, while $H_E$ that of $E$; the term $H'$ takes care of the interaction between the two systems, that is assumed to be weak.

The system $S$ could be unstable, as indeed is the case for the kaons, and therefore the hamiltonian $H_S$ needs not be hermitian. On the contrary, the contribution of the environment to possible decay processes can be estimated to be negligible for any practical considerations,[15] specifically, the kaon decay is driven by the weak interactions and not by the effects of the environment. The description of the kaon self-dynamics $H_S$ by means of a Weisskopf-Wigner hamiltonian is therefore appropriate and will be adopted in the following. On the other hand, there is no restriction in taking the environment hamiltonian $H_E$ to be hermitian.

Moreover, since the kaons are produced via the strong interactions, it is natural to further assume that $S$ and $E$ be uncorrelated at the moment of the formation of the unstable system $S$. This implies that initially the density matrix $\rho_{S+E}$ describing the state of the global system $S + E$ is in factorized form: $\rho_{S+E} = \rho_S \otimes \rho_E$.

The time evolution of the total system $S + E$ is simply given by the equation:

$$\frac{\partial \rho_{S+E}(t)}{\partial t} = -i(H \rho_{S+E}(t) - \rho_{S+E}(t) H^\dagger),$$

(2.2)

or, in finite form,

$$\rho_{S+E}(t) = e^{-iHt} \left( \rho_S \otimes \rho_E \right) e^{iH^\dagger t}.$$  

(2.3)

However, we are interested in the dynamics of the subsystem $S$ only; this can be obtained by tracing over the environment degrees of freedom: $\rho_S \rightarrow \rho(t) \equiv \text{Tr}_E \left( \rho_{S+E}(t) \right)$. To obtain
the equation that describes this reduced time-evolution, it is convenient to introduce a vector notation, rewriting the density matrix $\rho_{S+E}$ for the global system as the vector \(|\rho_{S+E}\rangle\). The operation of tracing over the environment degrees of freedom can now be represented by a projector operator $P$, so that the state vector of the subsystem $S$ is simply:

\[
|\rho\rangle \equiv P|\rho_{S+E}\rangle = \text{Tr}_E(\rho_{S+E})
\]

with $P^2 = P$; the complement projector $1 - P$ will be denoted by $Q$. By introducing the generalized Liouville operator $L_H$, corresponding to the total Hamiltonian $H$,

\[
L_H|\rho_{S+E}\rangle \equiv |H\rho_{S+E} - \rho_{S+E}H^\dagger\rangle,
\]

one can write (2.2) as a Schrödinger-like equation:

\[
i\frac{\partial}{\partial t}|\rho_{S+E}(t)\rangle = L_H|\rho_{S+E}(t)\rangle.
\]

(2.6)

Notice that one can identify $P|\rho_{S+E}(0)\rangle$ again with $|\rho_{S+E}(0)\rangle$, since at $t = 0$ the state of $S + E$ is factorized; as a consequence, one has: $Q|\rho_{S+E}(0)\rangle = 0$. All these steps can be rigorously justified from the mathematical point of view; however, for sake of simplicity we shall keep all technical discussions to a minimum.

The equation describing the subdynamics for $S$ can now be easily obtained by projecting (2.6) on $P$ and $Q$. Using the Laplace transformed vector $|\tilde{\rho}\rangle$ of (2.4), one finds:

\[
\left[z + i\left(L_{PP} + L_{PQ}\frac{1}{iz - L_{QQ}}L_{QP}\right)\right]|\tilde{\rho}(z)\rangle = |\tilde{\rho}(0)\rangle \equiv |\tilde{\rho}_S\rangle,
\]

(2.7)

where $z$ is the Laplace variable and

\[
L_{PP} = PL_HP, \quad L_{QQ} = QL_HQ, \quad L_{PQ} = PL_QP, \quad L_{QP} = PL_LPQ.
\]

(2.8)

From this result, using the inverse transform, one immediately obtains the “master equation” describing the time evolution of the projected vector $|\rho(t)\rangle$, representing the state of the subsystem $S$:

\[
i\frac{\partial}{\partial t}|\rho(t)\rangle = L_{PP}|\rho(t)\rangle - i\int_0^t dt' L_{PQ}e^{-iL_{QQ}(t-t')}L_{QP}|\rho(t')\rangle.
\]

(2.9)

One can prove that the evolution map $\rho_S \rightarrow \rho(t)$ given by this equation is completely positive.$^{[11-13]}$ as the one produced by the equation (1.1). However, (2.9) is clearly not of the form (1.1): although the term $L_{PP}|\rho(t)\rangle \equiv (L_{HS})_{PP}|\rho(t)\rangle$ (see below) can be identified with the Weisskopf-Wigner part of (1.1), the second term in the r.h.s. of (2.9) is not of Markovian form; in fact, $|\rho(t)\rangle$ as given by (2.9) depends not only on the initial state $|\rho(0)\rangle$, but also on all the states $|\rho(t')\rangle$, with $t' < t$. This is a general result of any reduced dynamics: the form of the master equation (2.9) can be further simplified only on the basis of additional physical considerations. In the present case, these additional assumptions correspond to the requirement that the interaction between the system $S$ and the environment $E$ be weak. Indeed, as we shall see in the next section, this condition allows the suppression of all memory effects.
3. THE WEAK COUPLING LIMIT

In writing down the hamiltonian (2.1) for the total system $S + E$ we have assumed the interaction between $S$ and $E$ to be weak. There are essentially two different ways of implementing this physical requirement in the general master equation (2.9).[11-13] In the first approach, one multiplies the interaction hamiltonian $H'$ by a small dimensionless coupling constant $\lambda$; in this way, one treats the motion of the system $S$ of order one, while the dissipative part of (2.9) (the second term in the r.h.s.) turns out to by of order $\lambda^2$. In view of this, the evolution of the state $|\rho(t)\rangle$ has to be studied on a time scale $1/\lambda^2$. In order to consistently implement this condition, one rescales the time variable, $t \rightarrow t/\lambda^2$, and then takes the limit $\lambda \rightarrow 0$. The idea is to use these steps to extend the integral in (2.9) to infinity and to eliminate the dependence on $t'$. This procedure is known as Markov approximation; it requires some care because in general it is not uniquely defined.[16]

For a more complete discussion, it is useful to define two time scales: $\tau_S$, the typical variation time of $|\rho(t)\rangle$ given by the “free” evolution $L_{PP}$ in (2.9), and $\tau_E$, the decay time of the correlations in the environment $E$. On general grounds, one expects the memory effects in (2.9) to be negligible if $\tau_S$ is much longer than $\tau_E$, or more precisely when the ratio $\tau_S/\tau_E$ becomes very large. And indeed, the limit sketched above corresponds to sending $\tau_S$ to infinity, while keeping $\tau_E$ finite.

In the case of an unstable subsystem $S$, like the neutral kaon system, $\tau_S$ can be identified with the corresponding lifetime, which is necessarily finite. The limiting procedure that corresponds to letting $\tau_S \rightarrow \infty$ is clearly inappropriate and can not be considered in these situations.

However, in order to attain $\tau_S/\tau_E \rightarrow \infty$, one can also let $\tau_E$ to become small, while leaving $\tau_S$ finite. This situation corresponds to the second weak coupling limit approach, which is obtained by assuming that the uncoupled motion of the system $S$ and of the dissipative terms produced by the interaction with the environment are of the same order of magnitude. As we shall see in the next sections, this limiting procedure is realized when the typical time-correlations of the environment approach a $\delta$-function; this explains why in the literature this kind of weak coupling limit is also referred to as “singular coupling limit”.

Introducing again a dimensionless coupling constant $\lambda$, the total hamiltonian can now be written as:[13]

$$H = \lambda^2 H_S \otimes 1 + 1 \otimes H_E + \lambda H', \quad (3.1)$$

and the Liouville operator $L_H$ appearing in (2.6) can be correspondingly decomposed as:

$$L_H = \lambda^2 L_{HS} + L_{HE} + \lambda L_{H'}. \quad (3.2)$$

In this case, it is convenient to pass to a rescaled time variable, $t \rightarrow t/\lambda^2$, directly in the original equation (2.6), so that the master equation (2.9) becomes:

$$i \frac{\partial}{\partial t} |\rho(t)\rangle = \frac{1}{\lambda^2} L_{PP} |\rho(t)\rangle - \frac{i}{\lambda^4} \int_0^t dt' L_{PQ} e^{-i L_{QQ} t'/\lambda^2} L_{QP} |\rho(t-t')\rangle.$$

By letting $t' = \lambda^2 \tau$ and taking the limit $\lambda \rightarrow 0$, the previous equation reduces to:

$$i \frac{\partial}{\partial t} |\rho(t)\rangle = \left[ (L_{HS})_{PP} - i \int_0^\infty d\tau (L_{H'})_{PQ} e^{-i(L_{HE})_{QQ} \tau} (L_{H'})_{QP} \right] |\rho(t)\rangle, \quad (3.3)$$

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where the conditions $L_{PP} = (L_{H_E})_{PP}$ and $L_{PQ} = (L_{H'})_{PQ}$, $L_{QP} = (L_{H'})_{QP}$ have been used. These properties follow from

\[ P L_{H'} P = 0, \quad P L_{H_E} = L_{H_E} P = 0, \quad P L_{H_S} = L_{H_S} P. \]  

The first relation is equivalent to the assumption that the interaction hamiltonian $H'$ has no diagonal elements in the representation in which $H_E$ is diagonal; in practice, this is not really a restriction, since we can always redefine $H_S$ and $H'$ in such a way to absorb such diagonal terms in $H_S$. The second relation in (3.4) is a consequence of probability conservation in the environment (first half) and of the condition of thermal equilibrium, $L_{H_E} \rho_E \equiv [H_E, \rho_E] = 0$, to be discussed in the next section (second half). Finally, the third condition above holds since $P$ and $L_{H_S}$ act in different spaces.

The equation (3.3) can be further simplified by using $Q L_{H_E} Q = L_{H_E}$ which is a consequence of (3.4). Coming back to the standard density matrix notation, one finally finds:

\[ \frac{\partial \rho(t)}{\partial t} = -i(H_S \rho(t) - \rho(t) H_S^\dagger) + D[\rho(t)], \]  

with

\[ D[\rho(t)] = -\int_0^\infty d\tau \text{Tr}_E \left\{ \left[ e^{iH_E \tau} H' e^{-iH_E \tau}, [H', \rho(t) \otimes \rho_E] \right] \right\}. \]

This evolution equation is of markovian form: all memory effects have disappeared. The dissipative term $D[\rho]$ involve an integration over environment “time-correlations” of the interacting hamiltonian $H'$. Its explicit form will depend on the structure of $H'$ and of the characteristic properties of the environment. In the next sections we shall specialize the rather general evolution equation (3.5), (3.6) to the case of the neutral kaons.

4. THE ROLE OF THE ENVIRONMENT

As it was mentioned in the introductory remarks, in the case of elementary particle systems it is natural to associate the effects of the environment $E$ as being gravitational in origin. Despite the smallness of the gravitational coupling, the quantum fluctuations at Planck’s length, e.g. realized via the space-time foam, could indeed act as a weak coupled environment.[17]

From a more fundamental viewpoint, such gravitational effects are likely to originate from the dynamics of extended objects, strings or branes, and could in principle be deduced using low energy string theory. Although attempts in this direction have been presented in the literature,[18] for our considerations it will be sufficient to have an “effective” description of such an environment, taking into account the most fundamental characterizing properties of the underlying string dynamics.

To be more specific, we shall model the environment as a heat-bath, i.e. a gas of non-interacting bosonic quantum modes, in thermodynamical equilibrium at Planck’s temperature $\beta_P^{-1} \sim M_P$, the natural scale at which the dissipative effects can be assumed
to start being relevant. The environment initial state is then proportional to the Gibbs
distribution:
\[ \rho_E = \frac{e^{-\beta_F H_E}}{\text{Tr}_E (e^{-\beta_F H_E})} \tag{4.1} \]
Moreover, by assumption, the environment is very large so that its statistical properties
are essentially unaffected by the weak coupling to the system \( S \). In other terms, the
environment remains in equilibrium and its state is given by (4.1) also at later times.
Notice that this implies \([H_E, \rho_E] = 0\), as used in checking the second relation in (3.4).

The form of the hamiltonian \( H' \) that describes the interaction of \( S \), henceforth iden-
tified with the neutral kaon system, with such an environment is largely arbitrary. The
only physical requirement that needs to be taken into account is that \( H' \), when inserted
in (3.6), must produce “small” effects on the system \( S \), which is for the most part driven
by the effective Weisskopf-Wigner hamiltonian term in (3.5). In view of these considera-
tions, it is natural to choose an hamiltonian \( H' \) that is linear in the kaon and environment
observables.

As already observed, the kaon system can be effectively represented by means of a
two-dimensional Hilbert space.\[14\] With a suitable choice of basis vectors in this space,
the interaction \( H' \) can be written in the following generic form:
\[ H' = g \sum_{\mu = 0}^{3} \sigma_{\mu} \otimes B_{\mu} , \tag{4.2} \]
where as before \( \sigma_i, i = 1, 2, 3 \), are the Pauli matrices, with \( \sigma_0 \) the \( 2 \times 2 \) unit matrix, and
\( B_{\mu} \) is an hermitian operator describing the environment, whose explicit expression will be
discussed later; \( g \) is a dimensionless coupling constant, that should be expressible in terms
of the relevant mass scales, \( i.e. \) the kaon mass \( m_K \) and the Planck mass \( M_P \). Since \( g \) is
small, it must be of order \( (m_K/M_P)^{\delta} \), with \( \delta \) a fixed power; as a working assumption, in
the following we shall take \( \delta \sim 1 \).

At this point, one can insert (4.2) in (3.6) and simplify the expression of the dissipative
term \( D[\rho] \). Since the hamiltonian \( H_E \) acts only on the environment degrees of freedom,
one first observes that:
\[ e^{i H_E t} H' e^{-i H_E t} = g \sum_{\mu = 0}^{3} \sigma_{\mu} \otimes B_{\mu}(t) . \tag{4.3} \]
Then, expanding the double commutator in (3.6) and performing the trace operation over
the environment degrees of freedom, one finally obtains:
\[
D[\rho] = g^2 \sum_{\mu, \nu} \int_0^\infty dt \left[ -\sigma_\mu \sigma_\nu \rho \langle B_\mu(t) B_\nu \rangle + \sigma_\mu \rho \sigma_\nu \langle B_\nu B_\mu(t) \rangle \\
+ \sigma_\nu \rho \sigma_\mu \langle B_\mu(t) B_\nu \rangle - \rho \sigma_\nu \sigma_\mu \langle B_\nu B_\mu(t) \rangle \right] , \tag{4.4}
\]
\[\]
where
\[
\langle B_\mu(t) B_\nu \rangle = \text{Tr}_E \left[ B_\mu(t) B_\nu \rho_E \right], \tag{4.5}
\]
are the time-correlation functions of the environment operators. Using the definition (4.5), one can easily verify that the following properties hold:
\[
\langle B_\mu(t) B_\nu \rangle^* = \langle B_\nu B_\mu(t) \rangle = \langle B_\nu(-t) B_\mu \rangle . \tag{4.6}
\]
Further, by introducing the following $4 \times 4$ hermitian matrices, with entries:
\[
\alpha_{\mu\nu} = \int_0^\infty dt \langle B_\mu B_\nu(t) \rangle + \int_0^\infty dt \langle B_\mu B_\nu(t) \rangle \equiv \int_{-\infty}^{\infty} dt \langle B_\mu(t) B_\nu \rangle , \tag{4.7a}
\]
\[
\beta_{\mu\nu} = i \left( \int_0^\infty dt \langle B_\mu B_\nu(t) \rangle - \int_0^\infty dt \langle B_\nu B_\mu(t) \rangle \right) , \tag{4.7b}
\]
one can rewrite (4.4) as:
\[
D[\rho] = \frac{g^2}{2} \sum_{i,j=1}^{3} \alpha_{ij} \left[ 2 \sigma_j \rho \sigma_i - \sigma_i \sigma_j \rho - \rho \sigma_i \sigma_j \right] + \frac{ig^2}{2} [\rho , \tilde{H} ] , \tag{4.8}
\]
where the hermitian operator $\tilde{H}$ is explicitly given by the combination:
\[
\tilde{H} = \sum_{\mu, \nu=0}^{3} \beta_{\mu\nu} \sigma_\mu \sigma_\nu + i \sum_{i=1}^{3} \left( \alpha_{0i} - \alpha_{i0} \right) \sigma_i . \tag{4.9}
\]
The second term in (4.8) is hamiltonian in character and can be reabsorbed in a redefinition of the hermitian part of the kaon effective hamiltonian $H_S$ in (3.5). The remaining piece is the true dissipative term. The equation (3.5) that describes the time-evolution of the kaon $2 \times 2$ density matrix $\rho(t)$ has now the form (1.1), (1.2) presented in the Introduction, although the precise structure of the coefficients $\alpha_{\mu\nu}$ in (4.7a) has yet to be addressed.

We would like to stress that this evolution equation has been derived by means of standard techniques used in the analysis of open quantum systems,[11-13] more specifically in quantum optics:[23, 24] with the help of rather general assumptions, motivated by plausible physical arguments, we have been able to show that these universal tools can be effectively applied to describe also elementary particle systems, like the neutral kaons. As we shall see in the next section, this description could be motivated more precisely by certain general aspects of the low energy dynamics of string theory.
5. D-BRANES AS ENVIRONMENT

The structure of the integrated correlation functions $\alpha_{\mu\nu}$ in (4.7a) are essentially determined by the statistics obeyed by the quanta that constitute the heat-bath, which effectively describe the environment. In $n$ space-time dimensions, the bath operator $B_\mu$, that as explained before should be taken to be linear in the modes creation and annihilation operators, can be assumed to have the following very general structure:

$$B_\mu(t) = \frac{1}{M_P^{(n-4)/2}} \sum_a \int \frac{d^{n-1}k}{[2(2\pi)^{n-1}\omega(k)]^{1/2}} f^a(k) \left[ \chi^a_\mu A_a(k) e^{-i\omega(k)t} + (\chi^a_\mu)^* A^\dagger_a(k) e^{i\omega(k)t} \right].$$

(5.1)

The coefficients $\chi^a_\mu$ “embed” the bath modes into the two-dimensional kaon Hilbert space and can be taken to satisfy

$$\sum_a (\chi^a_\mu)^* \chi^a_\nu = \delta_{\mu\nu},$$

(5.2)

while $f^a(k)$ are appropriate test functions necessary to make the operator $B_\mu$ and its correlations well-defined; in general, it can be taken to be of the form $(|k|/M_P)^m/2 g^a(k)$, for some positive integer $m$, with $g^a(k)$ of Gaussian type. The function $\omega(k)$ gives the dispersion relation obeyed by the bath modes; in the following, for simplicity we shall take an ultrarelativistic law: $\omega(k) = |k| \equiv \omega$.† The powers of Planck’s mass, characterizing the energy scale of the bath, are necessary to give $B_\mu$ the right dimension of energy.

As implicit in the form of the time evolution in (5.1), the creation $A^\dagger_a$ and annihilation $A_a$ operators for the bath modes fulfill the following general commutation relations with the environment hamiltonian $H_E$:

$$[H_E, A^\dagger_a(k)] = \omega(k) A^\dagger_a(k), \quad [H_E, A_a(k)] = -\omega(k) A_a(k).$$

(5.3)

However, from these relations one can not infer any conclusion concerning the nature of the quanta in the bath, and in particular about their statistics. Indeed, it has been observed long ago that statistics other than that of Bose and Fermi could be envisaged within standard many-body quantum mechanics;[25] this fact has led to the study of the so called “parastatistics” (see [26, 27] and references therein). The modes of the elementary excitations obeying these generalized statistics are quantized using commutation relations that are trilinear in the creation and annihilation operators, and whose representations are characterized by an integer $p$, the so called order of parastatistics. This number basically corresponds to the number of quanta in a given symmetric or antisymmetric state.

A further generalization occurs in the case of the “infinite statistics”,[28] that correspond to the case where the number $p$ is left as a free parameter. They are realized by $q$-oscillator algebras, i.e. by one-parameter extensions of the standard oscillator algebra.

These generalized infinite statistics have recently attracted a lot of attention in string theory and in particular in $M$ (matrix) theory. This interest originates from the observation

† Possible infrared infinities can be avoided by a suitable choice of the integer $m$ in the test function $f^a(k)$. 

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that the thermodynamical properties of black holes can be obtained studying the statistics of effective string models, described in terms of D-branes.[29-33, 34-37] It has been recognized that these extended objects (and in particular D0-branes, the quanta of Matrix theory) satisfy quantum infinite statistics,[38, 39, 36] and this result further clarify the D-brane interpretation of neutral black holes thermodynamics.

More in general, D-branes are shown to capture in a non-perturbative way many low energy properties of $M$-theory,[40] and therefore should provide the right description for studying quantum gravity effects at Planck’s scale.

In view of these considerations, it seems plausible to assume that the heat-bath responsible for the appearance of the dissipative term (3.6) in the effective evolution equation for the $K^0-K^0$ system is actually given by an ensemble of D-branes; in this case, the creation and annihilation operators in (5.1) should obey an infinite statistics. As previously noted, this situation can be realized by a $q$-oscillator algebra, that in our case can be presented in the following general form:[41]

$$A^\dagger_a(k) A_b(k') - q A^\dagger_b(k') A_a(k) = \delta_{ab} \delta^{(n-1)}(k - k') , \quad (5.4)$$

where $q$ is the deformation parameter; to assure the reality of the operator $B_\mu$ in (5.1), $q$ must be real, and without loss of generality we can take $q \leq 1$. The case $q = 1$ corresponds to standard bosons, while for $q = 0$ one obtains the degenerate algebra discussed in [38, 39, 36], in connection with D0-branes and black holes. In the computations that follows we shall therefore assume $q < 1$. The single-mode hamiltonian is taken to be proportional to the corresponding number operator, so that the total hamiltonian $H_E$ indeed satisfies the relations (5.3).

We are interested in computing the correlation functions $\langle B_\mu(t) B_\nu \rangle$; using (5.1), these can be expressed in terms the $A^\dagger - A$ thermal correlations. A straightforward computation shows that:[41, 42]

$$\langle A^\dagger_a(k) A_b(k') \rangle = \delta_{ab} \delta^{(n-1)}(k - k') N_q(\omega(k)) , \quad (5.5)$$

where

$$N_q(\omega(k)) = \frac{1}{e^{\beta P \omega(k)} - q} ; \quad (5.6)$$

is the generalized ensemble distribution. Explicit use of (5.4) further gives:

$$\langle A_a(k) A^\dagger_b(k') \rangle = \delta_{ab} \delta^{(n-1)}(k - k') e^{\beta P \omega(k)} N_q(\omega(k)) , \quad (5.7)$$

while $\langle A_a A_b \rangle$ and $\langle A^\dagger_a A^\dagger_b \rangle$ vanish, as follows from the orthogonality of states with different occupation number.

Using these relations and introducing $(n-1)$-dimensional spherical coordinates, with the help of (5.1) one obtains:

$$\langle B_\mu(t) B_\nu \rangle = \frac{1}{2 M_P^{m+n-4}} \int_0^\infty d\omega \omega^{m+n-3} \left[ X^{*\mu}(\omega) e^{-i\omega(t+i\beta P)} + X^{\mu}(\omega) e^{i\omega t} \right] N_q(\omega) , \quad (5.8)$$
while the integration over the angle variables gives,

\[ X_{\mu\nu}(\omega) = \sum_a \left[ \int \frac{d\Omega}{(2\pi)^{n-1}} \left( g^a(k) \right)^2 \right] (\chi^a_\mu)^* \chi^a_\nu \equiv X^*_{\nu\mu}(\omega) . \] (5.9)

Notice that the matrix \([X_{\mu\nu}]\) is positive definite. Further, the expression (5.8) for the environment correlations makes it transparent that the following periodicity property holds:

\[ \langle B_\mu(t) B_\nu \rangle = \langle B_\nu B_\mu(t + i\beta P) \rangle = \langle B_\nu(-t - i\beta P) B_\mu \rangle . \] (5.10)

The first equality amounts to the so-called Kubo-Martin-Schwinger (KMS) condition,\[43\] that holds for our choice of the D-brane thermal state (4.1), while the second equality just expresses the time-invariance of the correlation functions in that equilibrium state.

The matrix coefficients \(\alpha_{ij}\), \(i, j = 1, 2, 3\), appearing in the dissipative term in (4.8) can now be written in a more explicit form; in fact, from the definition (4.7a) and (5.8) one obtains:

\[ \alpha_{ij} = \frac{1}{2M_P^{m+n-4}} \int_{-\infty}^{\infty} dt \int_0^{\infty} d\omega \omega^{m+n-3} \cos(\omega t) \left[ X^*_{ij}(\omega) e^{\beta P \omega} + X_{ij}(\omega) \right] N_q(\omega) . \] (5.11)

This double integral converges provided the function multiplying the cosine in the integrand satisfies some mild regularity conditions. To be specific, these technical conditions requires \(\omega^{m+n-3} \left[ X^*_{ij}(\omega) e^{\beta P \omega} + X_{ij}(\omega) \right] N_q(\omega)\) to be integrable and of bounded variation on the real half-line.\[44, 45\] One can check that, by suitably fixing the positive integer \(m\) so that \(m + n \geq 3\), these requirements are satisfied by our choice of the test function \(f^a\) in (5.1). In particular, general considerations in Fourier analysis allows us to conclude that \(\alpha_{ij}\) in (5.11) is non-vanishing only when \(m = 3 - n\), and in this case:

\[ \alpha_{ij} = \frac{\pi}{1 - q} M_P x_{ij} , \] (5.12)

where

\[ x_{ij} = \frac{1}{2} \left( X_{ij}(0) + X^*_{ij}(0) \right) , \] (5.13)

turns out to be a real, symmetric, positive definite matrix. From (5.9), it is clear that the structure of this matrix depends on the specific choice for the test function \(g^a\) and the coefficients \(\chi^a_\mu\). A more detailed analysis of the D-brane dynamics would certainly provide more informations on \(x_{ij}\) and we hope to come back to these issues in the future. However, for the scopes of the present investigation, it is sufficient to note that the matrix \([x_{ij}]\) satisfies the same properties required for the matrix \([a_{ij}]\) in (1.2), and that its entries are of order one.

At this point, we can summarize our results by writing down the explicit form that the matrix coefficients \(a_{ij}\) in (1.2) take as a consequence of our considerations. Indeed, comparing (5.12) and (4.8) with the expression in (1.2) and recalling that the coupling
constant \( g \) is of the order \( \sim (m_K/M_P)^\delta \), with \( \delta \sim 1 \), one can now identify the coefficient matrix \( [a_{ij}], i, j = 1, 2, 3 \) as:

\[
a_{ij} = \frac{\pi}{1 - q^2 M_P} g^2 x_{ij} = \frac{\pi}{1 - q^2 M_P} \left( \frac{m_K}{M_P} \right)^{2(\delta - 1)} x_{ij}.
\]

From this discussion, we can therefore predict that the coefficients that determine the magnitude of the dissipative term in the non-standard evolution equation (1.1) should be at most of the order \( m_K^2/M_P \sim 10^{-19} \) GeV.

An interesting outcome of the previous analysis is that the assumption of a D-brane origin for the dissipative effects in the kaon dynamics gives a constraint on the dimension of the space in which the quanta of the heat-bath, describing the incoherent effects of the environment, effectively move. Indeed, as discussed above, the non-vanishing result (5.12) directly implies the bound \( n \leq 3 \). Clearly, this result is tight to the statistical properties of those quanta, direct consequence of the underlying D-brane dynamics. By changing the statistics obeyed by the heat-bath modes, one also changes the constraint on the dimension of the space in which those modes live.

For the standard Bose statistics, obtained from (5.4) by setting \( q = 1 \), the evaluation of the integral in (5.11) would produce the result, \( \alpha_{ij} = \pi M_P x_{ij} \), with the bound \( n \leq 4 \). In this case, the dynamics of the D-branes heat-bath would be effectively described by scalar “gravitons” living in an effective space-time of at most dimension four. Finally, if one chooses to work with a different kind of infinite statistics, described by the algebra that for a single mode reads, \( A A^\dagger - q A^\dagger A = q^{-N} \) (\( N \) being the number operator), which is another possible realization of the \( q \)-oscillator algebra, then the constraint \( n \leq 2 \) would be selected by the evaluation of (5.11). In other terms, these arguments seem to suggest that the presence of the extra dissipative term (1.2) in the kaon evolution equation (1.1) is related to the dimension of the space where the elementary excitations of the D-brane heat-bath effectively move and to the statistics they obey.

These results are interesting in view of possible experimental measures of the non-standard term in (1.2). As discussed in detail in [9, 10], if the coefficients \( a_{ij} \) are really of order \( m_K^2/M_P \), then they should be in the reach of the next generation of \( K \) experiments, in particular those that study correlated kaons at \( \phi \)-factories. If these phenomenological expectations turn out to be actually confirmed by the experiment, they will be not only of great importance for the understanding of the kaon physics, but they could also provide, as the considerations above seem to indicate, informations on the effective dynamics of low energy string theory. We find this possibility one of the most intriguing results of our analysis.
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