WILLMORE SPHERES IN THE 3-SPHERE REVISITED

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Abstract. Bryant [1] classified all Willmore spheres in 3-space to be given by minimal surfaces in $\mathbb{R}^3$ with embedded planar ends. This note provides new explicit formulas for genus 0 minimal surfaces in $\mathbb{R}^3$ with $2k+1$ embedded planar ends for all $k \geq 4$. Peng and Xiao claimed these examples to exist in [6], but in the same paper they also claimed the existence of a minimal surface with 7 embedded planar ends, which was falsified by Bryant [2].

1. Surfaces

Let $\phi : \hat{\Sigma} \to S^3$ be a compact, smooth, conformally parametrised, and immersed surface such that for a suitable chosen point $p \in S^3$, with its stereographic projection $\pi_p : S^3 \setminus \{p\} \to \mathbb{R}^3$, the composition

$$f = \pi_p \circ \phi : \Sigma = \hat{\Sigma} \setminus \phi^{-1}\{p\} \to \mathbb{R}^3$$

is a minimal surface in $\mathbb{R}^3$. We call such minimal surfaces $f$ minimal surfaces with embedded planar ends. It was shown by Bryant [1] that all Willmore spheres in the 3-sphere are of this type. Conversely, every $\phi$ as above is a Willmore sphere. By definition, (immersed) Willmore surfaces $\phi : \hat{\Sigma} \to S^3$ are the critical points for the Willmore functional

$$W(\phi) = \int_\Sigma (H^2 - K + 1)dA,$$  \hspace{1cm} (1)

where 1 is the sectional curvature of the round 3-sphere, $H$ is the mean curvature, $dA$ and $K$ are the area form and the curvature of the induced metric of $\phi$. The Willmore energy of a Willmore sphere is given by $4\pi(n-1)$, with $n$ being the number of ends of $f = \pi_p \circ \phi$.

Remark. Peng and Xiao claim the existence of a minimal surface with 7 embedded planar ends in [6] and remark that the existence for $n = 2k+1 \geq 9$ follows by a long but straight forward computation. Though the $n = 7$ case was falsified by Bryant [2], we show that the surfaces predicted in [6] do exist for $n \geq 9$ by giving a simple and explicit parametrization.

Let $X$ be a Riemann surface, and $g : X \to \mathbb{K}^r$ be a smooth map, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $r \in \mathbb{N}$. We denote by

$$dg = \partial g + \bar{\partial} g$$

the decomposition of the differential $dg$ of $g$ into its complex linear part $\partial g$ and its complex antilinear part $\bar{\partial} g$. For a minimal surface $f : \Sigma \to \mathbb{R}^3$ there exists a holomorphic line bundle $S \to \Sigma$ with $S^2 = K_\Sigma$ and two holomorphic sections $s_1, s_2 \in H^0(\Sigma, S)$ such that

$$\partial f = (s_1^2 + s_2^2, is_1^2 - is_2^2, -2is_1s_2).$$  \hspace{1cm} (2)

This is called the Weierstrass representation, and the two spinors $(s_1, s_2)$ are the Weierstrass data of $f$. In the case a minimal surface $f : \Sigma \to \mathbb{R}^3$ with embedded planar ends its Weierstrass data are meromorphic spinors on $\hat{\Sigma}$ with first order poles at $\phi^{-1}(p)$, see for example [3] and the references therein. Moreover, $\partial f$ has no residues at the embedded planar ends.

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1.1. Existing examples in the literature. Peng and Xiao \[6\] considered the following ansatz
\[
s_1 = \frac{z^{k-2}}{(z^2 - a)(z^2 - b)} \sqrt{dz} \quad \text{and} \quad s_2 = \frac{(z^k - a)(z^k - b)}{2(z^2 - a)(z^2 - b)} \sqrt{dz}
\]
for the Weierstrass data of a minimal surface of the \((2k+1)\)-punctured sphere
\[
\Sigma = \mathbb{C}P^1 \setminus \{z \in \mathbb{C} \mid z(z^k - 1)(z^k - \lambda) = 0\},
\]
where \(a, b, c, \lambda \in \mathbb{C}\) are pairwise distinct and satisfy the following algebraic condition:
\[
0 = \text{res}_p s_1^2 = \text{res}_p s_1 s_2 = \text{res}_p s_2^2
\]
holds at every point \(p \in \{z \in \mathbb{C} \mid z(z^k - 1)(z^k - \lambda) = 0\}\). Peng and Xiao \[6\] claim that a solution \((a, b, c, \lambda)\) always exist implying via \((2)\) the existence of immersed minimal surface \(s\) with \(2n\) curve of degree \(k\) for the Weierstrass data of a minimal surface of the \((2k+1)\)-punctured sphere.

Moreover, \(\Psi\) is a holomorphic curve with simple poles at \(p\) and \(\partial f\) has no residues at the ends \(\partial F\) solve Equation \((3)\). We obtain a minimal surface \(f\) with embedded planar ends by reversing the above construction. Consider a genus 0 minimal surface \(\Psi\) rises to a minimal surface with embedded planar ends by reversing the above construction.

Moreover, \(\Psi\) is again a null curve, i.e., for every local holomorphic lift \(\hat{\Psi}\) of \(\Psi\) \(\partial \hat{\Psi}\) gives a local lift of a well-defined holomorphic curve
\[
\psi^2 : \Sigma \to \mathcal{Q}_\Omega \subset PW;
\]
the second associated curve of \(\psi\). It is a null curve with respect to \(\frac{1}{2} \Omega \wedge \Omega\). A curve into a projective space is called nondegenerate if it is not contained in any hyperplane. The Klein correspondence (see \[3\]) states

2. Curves

The following description of genus 0 minimal surfaces with embedded planar ends is due to Bryant \[2\], see also \[3\]. Consider a genus 0 minimal surface \(f : \mathbb{C}P^1 \setminus \{p_1, ..., p_n\} \to \mathbb{R}^3\) with \(n\) embedded planar ends. As \(\partial f\) has no residues at the ends \(p_1, ..., p_n\), the surface \(f\) is the real part of a meromorphic map \(F : \mathbb{C}P^1 \to \mathbb{C}^3\) with simple poles at \(p_1, ..., p_n\). Since \(f\) is conformally parametrised \(F\) is a null curve, i.e., with respect to the standard symmetric inner product \(\langle ., . \rangle\) on \(\mathbb{C}^3\) we have \(\langle \partial F, \partial F \rangle = 0\). Consider \(\mathbb{C}^5\) with the inner product
\[
\langle ., . \rangle = -e_0^* \otimes e_4^* - e_4^* \otimes e_0^* + e_1^* \otimes e_4^* + e_2^* \otimes e_3^* + e_3^* \otimes e_2^*,
\]
the 3-quadric
\[
Q^3 = P\{v \in \mathbb{C}^5 \setminus \{0\} \mid \langle v, v \rangle = 0\}
\]
and the conformal embedding
\[
\Psi : \mathbb{C}^3 \to Q^3; \quad (z_1, z_2) \mapsto [\frac{1}{2}(z_1^2 + z_2^2 + z_3^2), z_1, z_2, z_3, 1].
\]
For a minimal sphere \(f = \mathbb{R}(F)\) with \(n\) embedded planar ends, \(\Psi \circ F : \mathbb{C}P^1 \to Q^3\) is an unbranched rational curve of degree \(n\). Moreover, \(\Psi \circ F\) is again a null curve, i.e., for every local holomorphic lift \(\hat{\Psi}\) of \(\Psi \circ F\) the condition \(\langle \hat{\Psi}, \hat{\Psi} \rangle = 0 = \langle \partial \hat{\Psi}, \partial \hat{\Psi} \rangle\) holds. Conversely, every (nondegenerate) unbranched null curve gives rise to a minimal surface with embedded planar ends by reversing the above construction.

Let \(V = \mathbb{C}^4\) be equipped with the 2-form \(\Omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4\). Consider the 5-dimensional space
\[
W = \{\eta \in \Lambda^2 V \mid \Omega(\eta) = 0\} \text{ equipped with the non-degenerated symmetric inner product } \frac{1}{2} \Omega \wedge \Omega, \text{ and the corresponding 3-quadric } Q_\Omega \text{ of null lines in } PW. \text{ Identifying } (W, \frac{1}{2} \Omega \wedge \Omega) \cong (\mathbb{C}^5, \langle ., . \rangle) \text{ yields } Q_\Omega \cong Q^3. \text{ A holomorphic curve } \psi : \Sigma \to C \to \mathbb{C}^3 \text{ is a contact curve if } \Omega(\hat{\psi} \wedge \partial \hat{\psi}) = 0 \text{ holds for every local holomorphic lift } \hat{\psi} \text{ of } \psi. \text{ Then, with respect to a local holomorphic coordinate } z, \text{ the map } z \mapsto \hat{\psi} \wedge \frac{\partial \hat{\psi}}{\partial z} \text{ gives a local lift of a well-defined holomorphic curve}
\]
that every nondegenerate null curve is given by a nondegenerate contact curve in $\mathbb{C}P^3$. For $\Sigma = \mathbb{C}P^1$ and $\psi$ of degree $d$ its second associated curve $\psi^2$ is unbranched if and only if the total branch order of $\psi$ is $d - 3$. This is a direct consequence of the Plücker relations applied to the duality between $\psi$ and its third associated curve, see [4]. Hence, to construct genus 0 minimal surfaces with $2k + 1$ embedded planar ends, we have to construct rational contact curves of degree $2k$ with total branch order $2k - 3$.

2.1. Rational contact curves of degree $2k$ with total branch order $2k - 3$. For $k \in \mathbb{N}\setminus\{3\}$ consider the map $\psi: \mathbb{C}P^1 \to \mathbb{C}P^3$ defined via the lift

$$
\hat{\psi}(z) = \left(\frac{1}{6}(-6 + 13k - 9k^2 + 2k^3 + \frac{12(-3+2k)}{3-3k}z^k + 3k(-3+2k)z^k - 6z^{2k})}{z^2(-1 + k + z^k)} \right). \quad (4)
$$

This is a nondegenerate rational curve of degree $2k$ if $k \in \mathbb{N}\setminus\{3\}$. It can be directly verified that it is a contact curve, i.e., $\Omega(\psi \wedge \frac{\partial \psi}{\partial \bar{z}}) = 0$. For $k \in \mathbb{N}\setminus\{3\}$ its branch points are at the $k$-th roots of unity and at $z = \infty$. The branch order at the roots of unity is 1, and at $z = \infty$ the branch order is $k - 3$. Hence, the total branch order is $2k - 3$. The construction fails for $k \leq 3$ : for $k = 1$, the curve is of degree 2 and branched at $z = 0$ and $z = 1$. Thus, its second associate curve cannot be unbranched. For $k = 2$, the degree of the curve is $2 \neq 4$. For $k = 3$ and $k = 0$, the formula [1] gives a point in $\mathbb{C}P^3$. For $k = 4$ we obtain the first valid example

$$
\hat{\psi}(z) = (z^3, 5 + 20z^4 - z^8, 5z + 5z^5, 3z^2 + z^6).
$$

Together with the examples of unbranched null curves of even degree $d \geq 4$ and the nonexistence results of Bryant in [2], this shows:

**Theorem.** There exist a Willmore sphere $\phi_n: \mathbb{C}P^1 \to S^3$ with Willmore energy $4\pi(n-1)$ if and only if $n \in \mathbb{N}\setminus\{2, 3, 5, 7\}$.

For completeness, we state the formula for the genus 0 minimal surfaces $f = \Re(F)$ with $(2k + 1)$ embedded planar ends corresponding to the contact curve (4). Note that these surfaces are only determined up to Goursat transformations [4], see also [2]. We leave it as an exercise for the interested reader to verify that we have reobtained the surfaces of Peng and Xiao [5] up to a Goursat transformation and reparametrisation. The meromorphic map $F: \mathbb{C}P^1 \to \mathbb{C}^3$ is given by

$$
F(z) = \left(\frac{-12\sqrt{-1}(3k)z^2(2-3k+z^2)}{-12\sqrt{-1}(3k)(-1+k)z^2(-3-2k)(-6(-1+k)(-3+2k)z^2(-3-3k)z^2(-3+k)(-6+k+2k)^2z^{12}-12z^6))}{4z(-3+k)(-1+k)z^2(3-2k)(-6(-1+k)(-3-2k)z^2(-3-3k)z^2(-3+k)(-6+k+2k)^2z^{12}-12z^6))} \right).
$$

**References**

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