ASSOCIATIVE CONFORMAL ALGEBRAS WITH FINITE FAITHFUL REPRESENTATION

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Abstract. We describe irreducible conformal subalgebras of \( \mathfrak{c} \) and build the structure theory of associative conformal algebras with finite faithful representation.

1. Introduction

The notion of a conformal algebra appears as an algebraic language describing singular part of operator product expansion (OPE) in conformal field theory \([4]\). The explicit algebraic exposition of this theory (see, e.g., \([5, 15, 19]\) leads to the notion of a vertex (chiral) algebra. Roughly speaking, conformal algebras correspond to vertex algebras by the same way as Lie algebras correspond to their associative enveloping algebras (see \([21]\) for a detailed explanation).

In the recent years a great advance in the structure theory of associative and Lie conformal algebras of finite type has been obtained. In \([12, 13, 17, 18, 22]\), simple and semisimple Lie conformal (super)algebras of finite type were described (as well as associative ones). The main result of \([13]\) was generalized in \([1]\) for finite pseudoalgebras.

Some features of the structure theory of conformal algebras (and their representations) of infinite type were also considered in a series of works (see \([9, 10, 14, 22, 24, 25, 29, 30]\)). One of the most urgent problems in this field is to describe the structure of conformal algebras with faithful irreducible representation of finite type (these algebras could be of infinite type themselves). In \([22, 24, 29, 30]\), conjectures on the structure of such algebras (associative and Lie) were stated. The papers \([9, 14, 30]\) contain confirmations of these conjectures under some additional conditions.

Another problem is to classify simple and semisimple conformal algebras of linear growth (i.e., of Gel'fand–Kirillov dimension one) \([24]\). In the papers \([24, 25, 29, 30]\), this problem was solved for finitely generated associative conformal algebras which contain a unit \([24, 25]\), or at least an idempotent \([29, 30]\). The objects appearing from the consideration of conformal algebras with faithful irreducible representation of finite type are similar to those examples of conformal algebras stated in these papers.

A combinatorial aspect of the theory of conformal algebras requires the notion of a free conformal (and vertex) algebra. Actually, it was stated in \([8]\) what is to be the free vertex algebra. Studying of free conformal algebras was initiated in \([20]\). The basic notions of the combinatorial theory of associative conformal
algebras (Gröbner–Shirshov bases, Composition-Diamond lemma) were considered in [3, 6, 7].

The class of conformal algebras coincides with the class of pseudoalgebras [10] over the polynomial Hopf algebra $k[D]$, where $D$ is just a formal variable, $k$ is a field of zero characteristic. In general, a pseudoalgebra over a cocommutative Hopf algebra $H$ is just an algebra in the pseudotensor category associated with $H$ [3]. Therefore, the notion of a pseudoalgebra provides the common point of view to the theory of ordinary algebras ($H = k$) and to the theory of conformal algebras ($H = k[D]$). In particular, for every finitely generated $H$-module $V$ one can define a pseudoalgebra structure on the set of all $H$-linear (with respect to the second tensor factor) maps $V \rightarrow H \otimes H \otimes H V$ [3, Proposition 10.1]. The pseudoalgebra obtained is denoted by $\text{Cend}_V$ (conformal endomorphisms). This is a direct analogue of the ordinary algebra $\text{End}_U$ of all linear transformations of a vector space $U$.

From now on, let $k$ be an algebraically closed field of zero characteristic, and let $H = k[D]$ be the polynomial algebra in one variable $D$. By $\partial_x$ we denote the operator of formal derivation with respect to a variable $x$. We will also use the notation $x^{(n)} = x^n/n!$, $n \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ means the set of non-negative integers.

Our purpose is to develop the structure theory of associative conformal algebras with faithful representation of finite type (as a corollary, we obtain the classification of simple and semisimple associative conformal algebras of finite type [22]). The main point of this theory is an analogue of the classical Burnside theorem on irreducible subalgebras of the matrix algebra $M_N(k)$. The last statement says that, if a subalgebra $S \subseteq M_N(k)$ acts irreducibly on $k^n$ (i.e., there are no non-trivial submodules), then $S = M_N(k)$. A similar question for conformal algebras was raised by V. Kac [22]. In this context, the “classical” objects $k$, $k^n$, $M_N(k)$ should be replaced with $H$, $V_N$, $\text{Cend}_N$, respectively. Here $V_N$ is the free $N$-generated $H$-module, $\text{Cend}_N = \text{Cend}V_N$. The associative conformal algebra $\text{Cend}_N$ (and its adjoint Lie conformal algebra $\mathfrak{g}_{C_N}$) plays the same role in the theory of conformal algebras as $M_N(k)$ (respectively, $\mathfrak{gl}_N(k)$) does in the theory of ordinary algebras.

The paper [9] contains a systematical investigation of the algebra $\text{Cend}_N$: the structure of its left/right ideals and their modules of finite type, classification of (anti-) automorphisms and involutions, etc. Since the algebra $\text{Cend}_N$ is the main object of our study, we will also state some of these results.

In the paper [22] the following statement was conjectured: every irreducible conformal subalgebra $C \subseteq \text{Cend}_N$ is either a left ideal of $\text{Cend}_N$ or a conjugate of the current subalgebra (in the last case, $C$ is of finite type). In [9], the second part of this conjecture (i.e., finite type case) was derived from the well-developed representation theory of Lie conformal algebras of finite type [13]. Also, the conjecture was proved in [9] for $N = 1$ as well as for unital conformal subalgebras.

We prove this conjecture in general (Theorem 5.3.1). Moreover, although it follows from [13] [22] that any simple associative conformal algebra of finite type is isomorphic to the current conformal algebra over $M_n(k)$, we focus on the independent proof of this fact.

The paper is organized as follows. In Section 2 we state the definitions of conformal algebras and their representations. Also, we write down the general construction of an associative conformal algebra: given an (ordinary) associative conformal algebra $A$ with a locally nilpotent derivation $\partial$, the free $H$-module $H \otimes A$ could
be endowed with the structure of an associative conformal algebra denoted by Diff $A$.

In Section 3, we introduce the conformal algebra $C_{\text{end}} N = C_{\text{end}} V_N$ ($N$ is a positive integer) which is the main object of our study. This conformal algebra is isomorphic to $\text{Diff} M_N(k[v])$, where $v$ is a formal variable, and $\partial = \partial_v$ is the usual derivation with respect to $v$. Therefore, $C_{\text{end}} N$ could be identified with $M_N(k[D,v])$.

As a main tool of our investigation we use a correspondence between conformal subalgebras of $C_{\text{end}} N$ and subalgebras of $M_N(W)$, where $W = k\langle p, q \mid qp - pq = 1 \rangle$ is the 1st Weyl algebra. This correspondence is provided by a construction called operator algebra, which is very close to the annihilation algebra. Namely, for any conformal subalgebra $C \subseteq C_{\text{end}} N$ we consider the ordinary algebra $S(C) \subseteq M_N(W)$ which consists of linear operators on $V_N$ corresponding to all elements of $C$. If $C$ is a left (right) ideal then so is $S(C)$. Moreover, $C$ has a canonical structure of a (left) $S(C)$-module.

In Section 4, we prove some properties of operator algebras. First, we show that for any irreducible conformal subalgebra $C \subseteq C_{\text{end}} N$ its “expanded” operator algebra $k[p]S(C)$ is a dense subalgebra of $M_N(W)$ with respect to the finite topology. The rest of the section is devoted to the properties of “small” operator algebras (i.e., those of linear growth) corresponding to conformal subalgebras of finite type. Roughly speaking, we show that if $S = S(C) \subseteq M_N(W)$ is an operator algebra of linear growth corresponding to an irreducible conformal subalgebra $C \subseteq C_{\text{end}} N$, then $C$ is a conjugate of the current subalgebra of $C_{\text{end}} N$. In this way, we obtain an independent description of irreducible conformal subalgebras of finite type.

Section 5 contains the main results of the paper. The basic point is to describe all irreducible conformal subalgebras of $C_{\text{end}} N$. First, we show that if $C \subseteq C_{\text{end}} N$ is irreducible, then

$$\sum_{n \geq 0} v^n C = C_{\text{end}} N, Q,$$

where $C_{\text{end}} N, Q = M_N(k[D,v])Q(-D + v)$, $Q = Q(v)$ is a nondegenerate matrix with polynomial entries.

It is left to consider three cases:
1) the sum (1.0.1) is direct;
2) $C \cap vC \neq 0$;
3) the sum (1.0.1) is non-direct, but $C \cap vC = 0$.

The first case is considered in Section 5.2. Using the results of Sections 4.2 and 4.3 we show that $C$ is a conjugate of the current subalgebra of $C_{\text{end}} N$. In the second case, we use the method proposed in [9, Section 2] for $N = 1$: in Section 5.3 we reduce the computation to the one-dimensional case and obtain $C = C_{\text{end}} N, Q$. The third case is impossible, as we show in Section 5.4.

In Section 5.5 we complete the classification of irreducible subalgebras (Theorem 5.5.1) and proceed with the structure theory of associative conformal algebras with faithful representation of finite type: we describe simple and semisimple ones. Moreover, we show that any conformal algebra of this kind has a maximal nilpotent ideal. As a corollary, we obtain the classification theorem for associative conformal algebras of finite type.
2. Conformal algebras and their representations

In this section, we state necessary definitions and constructions from the theory of conformal algebras.

2.1. Definition and examples of conformal algebras.

Definition 2.1.1 ([21]). A vector space \( C \) endowed with a linear map \( D \) and with a family of bilinear operations \( \circ_n, n \in \mathbb{Z}_+ \), is said to be a conformal algebra, if

\[
\begin{align*}
(C1) & \quad a \circ_n b = 0, \quad \text{for } n \text{ sufficiently large, } a, b \in C; \\
(C2) & \quad Da \circ_n b = -na \circ_{n-1} b; \\
(C3) & \quad a \circ_n Db = D(a \circ_n b) + na \circ_{n-1} b.
\end{align*}
\]

The axiom (C1) allows to define so called locality function \( N : C \times C \to \mathbb{Z}_+ \).

Every conformal algebra could be considered as a unital left module over \( H = \mathbb{K}[D] \) endowed with a family of sesquilinear (i.e., satisfying (C2), (C3)) products \( \circ_n, n \in \mathbb{Z}_+ \), such that the locality axiom (C1) holds.

For any pair of \( D \)-invariant subspaces (\( H \)-submodules) \( X, Y \subseteq C \) the space \( X \circ_\omega Y = \sum_{n \in \mathbb{Z}_+} X \circ_n Y \) is also an \( H \)-submodule.

An ideal of a conformal algebra \( C \) is an \( H \)-submodule \( I \subseteq C \) which is closed under all multiplications by \( C \), i.e., \( C \circ_\omega I, I \circ_\omega C \subseteq I \). Left and right ideals are defined analogously.

A conformal algebra \( C \) is simple, if \( C \circ_\omega C \neq 0 \) and there are no non-trivial ideals.

For any ideal \( I \) of a conformal algebra \( C \) one can define descending sequences of ideals \( \{I^{(n)}\}_{n=1}^{\infty} \) and \( \{I^n\}_{n=1}^{\infty} \) by the usual rule: \( I^{(1)} = I^1 = I, I^{(n)} = I^{(n-1)} \circ_\omega I^{(n-1)}, I^n = \sum_{s=1}^{n-1} I^s \circ_\omega I^{n-s}, n > 1 \). An ideal \( I \) is said to be solvable (resp., nilpotent), if \( I^{(n)} = 0 \) (resp., \( I^n = 0 \)) for sufficiently large \( n \). If a conformal algebra \( C \) has no non-zero solvable ideals then \( C \) is called semisimple.

If a conformal algebra \( C \) is finitely generated as an \( H \)-module, then it is said to be of finite type (or finite conformal algebra).

A homomorphism \( \phi : C_1 \to C_2 \) of conformal algebras is a \( D \)-invariant linear map such that \( \phi(a \circ_n b) = \phi(a) \circ_n \phi(b), a, b \in C_1, n \in \mathbb{Z}_+ \).

For any conformal algebra \( C \) in the sense of Definition 2.1.1 there exists an ordinary algebra \( A \) such that \( C \) lies in the space of formal distributions \( A[[z, z^{-1}]] \), where \( D = \partial_z \) and the \( \circ_n \)-products are given by

\[
(a(z) \circ_n b(z) = \text{Res}_{w=0} a(w)b(z)(w-z)^n, \quad n \in \mathbb{Z}_+.
\]

Such an algebra \( A \) is not unique, but there exists a universal one denoted by Coeff \( C \). Namely, for any algebra \( A \) such that \( A[[z, z^{-1}]] \) contains \( C \) as above, there exists a homomorphism Coeff \( C \to A \) such that the natural expansion Coeff \( C[[z, z^{-1}]] \to A[[z, z^{-1}]] \) acts on \( C \) as an identity [22, 23]. The algebra Coeff \( C \) is called the coefficient algebra of \( C \).
There is a correspondence between identities on Coeff $C$ and systems of conformal identities on $C$. In particular, Coeff $C$ is associative if and only if $C$ satisfies

\[(2.1.3) \quad (a \circ_n b) \circ_m c = \sum_{s \geq 0} (-1)^s \binom{n}{s} a \circ_{n-s} (b \circ_{m+s} c)\]

or

\[(2.1.4) \quad a \circ_n (b \circ_m c) = \sum_{s \geq 0} \binom{n}{s} (a \circ_{n-s} b) \circ_{m+s} c,\]

for all $n, m \in \mathbb{Z}_+$. The systems of relations (2.1.3) and (2.1.4) are equivalent. Also, Coeff $C$ is a Lie algebra if and only if $C$ satisfies

\[(2.1.5) \quad a \circ_n b = -\sum_{s \geq 0} (-1)^{n+s} D(s)(b \circ_{n+s} a),\]

\[(2.1.6) \quad a \circ_n (b \circ_m c) - b \circ_m (a \circ_n c) = \sum_{s \geq 0} \binom{n}{s} (a \circ_{n-s} b) \circ_{m+s} c\]

for all $n, m \in \mathbb{Z}_+$.

A conformal algebra $C$ is called associative, if Coeff $C$ is associative, i.e., if $C$ satisfies (2.1.3) or (2.1.4). Analogously, $C$ is called Lie conformal algebra, if Coeff $C$ is a Lie algebra, i.e., if $C$ satisfies (2.1.5) and (2.1.6). It is easy to note that the notion of solvability coincides with nilpotency in the case of associative conformal algebras.

**Proposition 2.1.2** (see, e.g., [22]). Let $C$ be an associative conformal algebra. Then the same $H$-module $C$ endowed with new operations

\[(2.1.7) \quad [a \circ_n b] = a \circ_n b - \sum_{s \geq 0} (-1)^{n+s} D(s)(b \circ_{n+s} a), \quad a, b \in C, \quad n \in \mathbb{Z}_+,\]

is a Lie conformal algebra denoted by $C^{(-)}$. \hfill \Box

**Example 2.1.3.** Let $A$ be an (associative or Lie) algebra. Then formal distributions of the form

\[a(z) = \sum_{n \in \mathbb{Z}} at^n z^{-n-1} \in A[t, t^{-1}][[z, z^{-1}]], \quad a \in A,\]

together with all their derivatives span a conformal algebra in $A[t, t^{-1}][[z, z^{-1}]]$ called the current conformal algebra Cur $A$. The operations (2.1.2) are given by

\[a(z) \circ_n b(z) = \begin{cases} (ab)(z), & n = 0; \\ 0, & n > 0. \end{cases}\]

The conformal algebra Cur $A$ is associative or Lie if and only if so is $A$.

**Example 2.1.4.** Consider $A = k[t, t^{-1}, \partial | \partial t - t \partial = 1]$. Then formal distributions of the form

\[v_m(z) = \sum_{n \in \mathbb{Z}} t^n \partial^m z^{-n-1} \in A[[z, z^{-1}]], \quad m \in \mathbb{Z}_+,\]

together with all their derivatives span a conformal algebra in $A[[z, z^{-1}]]$ called the Weyl conformal algebra.
Example 2.1.5. Consider the element \( L = v_1 \) from the previous example. It is easy to compute that the one-generated \( H \)-submodule \( H \otimes kL \) of the Weyl conformal algebra is closed under operations \( [2.1.4] \):

\[
[L \circ_0 L] = -DL, \quad [L \circ_1 L] = -2L, \quad [L \circ_n L] = 0, \quad n \geq 2.
\]

The Lie conformal algebra obtained is called the Virasoro conformal algebra.

2.2. Representations of associative conformal algebras. Remind that \( H = k[D] \) is a Hopf algebra, so \( H \otimes H \) could be considered as the outer product of regular right \( H \)-modules (the action is given by \( (f \otimes g) \cdot D = fD \otimes g + f \otimes gD \)). If \( \{h_i \mid i \in I\} \) is a linear basis of \( H \), then \( \{h_i \otimes 1 \mid i \in I\} \) is an \( H \)-basis of the right \( H \)-module \( H \otimes H \) (see, e.g., [1, Lemma 2.3]).

Definition 2.2.1 ([2]). Let \( V \) be a unital left \( H \)-module. A linear map

\[
a : V \to (H \otimes H) \otimes H V
\]

is said to be a conformal endomorphism of \( V \), if \( afv = ((1 \otimes f) \otimes_H 1)a(v) \) for any \( f \in H, v \in V \).

Denote the set of all conformal endomorphisms of \( V \) by \( \text{Cend} V \). For any \( a \in \text{Cend} V \) there exists a uniquely defined sequence \( \{a_n\}_n \) of (ordinary) \( k \)-linear endomorphisms of \( V \) such that:

\[
\begin{align*}
(2.2.1) & \quad a_n(v) = 0 \text{ for } n \text{ sufficiently large, } v \in V; \\
(2.2.2) & \quad [a_n, D] = na_{n-1}, \quad n \in \mathbb{Z}_+; \\
(2.2.3) & \quad a(v) = \sum_{n \geq 0} ((-D)^{(n)} \otimes 1) \otimes_H a_n(v), \quad v \in V.
\end{align*}
\]

Conversely, any sequence \( \{a_n\}_n \subseteq \text{End}_k V \) satisfying \( [2.2.1] \) and \( [2.2.2] \) defines a conformal endomorphism \( a \in \text{Cend} V \) via \( [2.2.3] \).

For any \( a, b \in \text{Cend} V \) define the sequences \( \{(Da)_n\}_n \) and \( \{(a \circ_m b)_n\}_n \), \( m \in \mathbb{Z}_+ \), by the following rules (see, e.g., [13, Section 3]):

\[
\begin{align*}
(2.2.4) & \quad (Da)_n = [D, a_n] = -na_{n-1}, \\
(2.2.5) & \quad (a \circ_m b)_n = \sum_{s \geq 0} (-1)^s \binom{n}{s} a_{m-s} b_{n+s}.
\end{align*}
\]

It is clear that these sequences satisfy \( [2.2.1] \) and \( [2.2.2] \), so the conformal endomorphisms \( Da, a \circ_m b \) \( (m \in \mathbb{Z}_+) \) are defined.

For an arbitrary \( H \)-module \( V \), the vector space \( \text{Cend} V \) endowed with the operations \( D \) and \( \circ_m \) \( (m \in \mathbb{Z}_+) \) given by \( [2.2.1] \) and \( [2.2.2] \) satisfies (C2) and (C3). The locality property (C1) does not hold, in general. However, if \( V \) is a finitely generated \( H \)-module, then (C1) holds, so \( \text{Cend} V \) is an associative conformal algebra.

Let \( C \) be a conformal algebra, and let \( V \) be a left unital \( H \)-module. If \( \varrho : C \to \text{Cend} V \) is a linear map such that

\[
\varrho(Da) = D\varrho(a), \quad \varrho(a \circ_n b) = \varrho(a) \circ_n \varrho(b), \quad a, b \in C, \quad n \in \mathbb{Z}_+,
\]

then \( \varrho(C) \subseteq \text{Cend} V \) is a conformal algebra. By abuse of terminology, we will say that \( \varrho \) is a homomorphism of conformal algebras (although \( \text{Cend} V \) is not necessarily a conformal algebra).

Definition 2.2.2 ([2]). A representation of an associative conformal algebra \( C \) on a left unital \( H \)-module \( V \) is a homomorphism \( \varrho : C \to \text{Cend} V \) of
conformal algebras. If \( C \) has a representation on \( V \), then \( V \) is said to be a module over conformal algebra \( C \) (or \( C \)-module, if the context is clear).

The last definition is equivalent to the following one [11]: an \( H \)-module \( V \) is a module over an associative conformal algebra \( C \), if there is a family of \( \circ_n \)-products

\[
\circ_n : C \otimes V \to V
\]

such that

\[
\begin{align*}
(M1) \quad & a \circ_n v = 0, \quad \text{for } n \text{ sufficiently large, } a \in C, \ v \in V; \\
(M2) \quad & D a \circ_n v = -n a \circ_{n-1} v; \\
(M3) \quad & a \circ_n D v = D(a \circ_n v) + n a \circ_{n-1} v; \\
(M4) \quad & a \circ_n (b \circ_m v) = \sum_{s \geq 0} \binom{n}{s} (a \circ_{n-s} b) \circ_{m+s} v.
\end{align*}
\]

A representation \( \rho : C \to \text{Cend} V \) is called faithful, if \( \rho \) is injective. If \( V \) is a finitely generated \( H \)-module, then \( \rho \) is said to be a representation of finite type (or finite representation).

In the case of ordinary algebras, every algebra which has a faithful finite-dimensional representation is finite-dimensional itself. It is not the case for conformal algebras: there exist infinite conformal algebras with faithful finite representation.

2.3. **Differential conformal algebras.** Let \( A \) be an associative algebra with a locally nilpotent derivation \( \partial \). We consider the free \( H \)-module \( H \otimes A \) as an associative conformal algebra. Notice that by (C2), (C3) it is sufficient to define the family of \( \circ_n \)-products between elements of the form \( 1 \otimes a, a \in A \). Hereinafter we identify \( 1 \otimes a \) with \( a \in A \).

**Proposition 2.3.1** (see, e.g., [1, 2, 3, 24]). (i) The family of \( \circ_n \)-products

\[
(a \circ_n b) = a \partial^n(b), \quad a, b \in A,
\]

defines an associative conformal algebra structure on \( H \otimes A \). The conformal algebra obtained is denoted by \( \text{Diff} A \).

(ii) The same is true for another family of operations:

\[
(a \circ_{(n)} b) = \sum_{s \geq 0} D^{(s)} \otimes \partial^{n+s}(a)b, \quad a, b \in A.
\]

The conformal algebra obtained is denoted by \( \text{Diff}^\circ A \). \( \square \)

**Proposition 2.3.2.** \( \text{Diff} A \simeq \text{Diff}^\circ A \).

**Proof.** The isomorphism \( \varphi : \text{Diff} A \to \text{Diff}^\circ A \) is given by

\[
h \otimes a \mapsto \sum_{s \geq 0} D^{(s)} h \otimes \partial^s(a);
\]

the inverse \( \varphi^{-1} : \text{Diff}^\circ A \to \text{Diff} A \) could be defined via

\[
h \otimes a \mapsto \sum_{s \geq 0} (-D)^{(s)} h \otimes \partial^s(a).
\]

Direct computation shows that \( \varphi(a \circ_n b) = \varphi(a) \circ_{(n)} \varphi(b) \). \( \square \)
Example 2.3.3. Let \( \mathcal{A} \) be an (associative) algebra, and let \( \partial \) be the zero derivation: \( \partial(\mathcal{A}) = 0 \). Then \( \text{Diff} \mathcal{A} = \text{Diff}^0 \mathcal{A} \) is exactly the current conformal algebra \( \text{Cur} \mathcal{A} \) (Example 2.1.3).

Example 2.3.4. Consider \( \mathcal{A} = \mathbb{k}[v] \), and let \( \partial = \partial_v \) be the usual derivation. Then the conformal algebras \( \text{Diff} \mathcal{A} \) and \( \text{Diff}^0 \mathcal{A} \) are isomorphic to the Weyl conformal algebra (Example 2.1.4).

Example 2.3.5. Let \( X \) be a set of symbols (generators), \( v \notin X \), and let \( \mathcal{A} = \mathbb{k}(X,v) \) be the free associative algebra generated by \( X \cup \{v\} \) endowed with the derivation \( \partial = \partial_v \). Consider a conformal subalgebra of \( \text{Diff} \mathcal{A} \) generated by \( \{v^{(N-1)}x \mid x \in X\} \), \( N \geq 1 \). This subalgebra is isomorphic to the free associative conformal algebra generated by \( X \) with respect to the constant locality function \( N(x,y) \equiv N \), \( x,y \in X \) (see [5, 29]).

The last example shows that every finitely generated associative conformal algebra is a homomorphic image of a subalgebra of a differential conformal algebra.

It was conjectured in [25] that every annihilation-free (see [24] for rigorous explanation) associative conformal algebra is just a subalgebra of some differential conformal algebra. Example 2.3.5 confirms this conjecture for free associative conformal algebras.

3. Conformal algebra \( \text{Cend}_N \)

From now on, we will use the term “conformal algebra” for associative conformal algebra, unless stated otherwise.

3.1. Construction. Let us introduce the main object of our study: the conformal algebra \( \text{Cend}_N = \text{Cend} V_N \), where \( V_N \) is the free \( N \)-generated \( H \)-module. We will use the following presentation of \( \text{Cend}_N \).

Consider the algebra \( M_N(\mathbb{k}[v]) \) of all \( N \times N \)-matrices with polynomial entries endowed with the derivation \( \partial_v \), with respect to the variable \( v \). Denote the conformal algebras \( \text{Diff} M_N(\mathbb{k}[v]) \) and \( \text{Diff}^0 M_N(\mathbb{k}[v]) \) by \( \mathfrak{A}_N \) and \( \mathfrak{A}_N^0 \), respectively. Also, there is the usual multiplication on \( H \otimes M_N(\mathbb{k}[v]) \) as on \( M_N(\mathbb{k}[D,v]) \), and the following property of \( \mathfrak{A}_N \) is clear:

\[
(3.1.1) \quad v(a \circ_n b) = va \circ_n b, \quad a \circ_n vb = va \circ_n b + na \circ_{n-1} b.
\]

Let us fix an \( H \)-basis \( \{e_1, \ldots, e_N\} \) of \( V_N \), and let \( U \subset V_N \) be the finite-dimensional subspace spanned by this basis. Since \( U \cong \mathbb{k}^N \), the free \( H \)-module \( V_N \) could be identified with \( H \otimes \mathbb{k}^N = H^N \). For any element \( a = \sum_{s \geq 0} (-D)^{(s)} \otimes A_s(v) \in \mathfrak{A}_N \), \( A_s(v) \in M_N(\mathbb{k}[v]) \), consider the following sequence of \( \mathbb{k} \)-linear endomorphisms of \( V_N \):

\[
(3.1.2) \quad a_n : u \mapsto \sum_{s \geq 0} \binom{n}{s} A_s(D) \partial_D^{-s}(u), \quad u \in V_N, \ n \in \mathbb{Z}_+.
\]

It is easy to see that this sequence satisfies \( \text{Proposition 1.1} \) and \( \text{Proposition 2} \), so it defines a conformal endomorphism \( \tilde{a} \in \text{Cend}_N \).

Proposition 3.1.1 ([11, 13, 21, 24]). The map \( a \mapsto \tilde{a} \) is an isomorphism of conformal algebras \( \mathfrak{A}_N \) and \( \text{Cend}_N \). \( \square \)
Remark 3.1.2. It follows from Proposition 2.3.2 that
\[ \mathfrak{A}_N \simeq \mathfrak{A}_W \simeq \text{Cend}_N. \]
The isomorphism \( \varphi \) from Proposition 2.3.2 corresponds to the map \( \mathfrak{A}_N \to \mathfrak{A}_W \) given by \( D \mapsto D, \ v \mapsto D + v \). In [9, 22], the second construction 2.3.2 was used for \( \text{Cend}_N \). We will preferably follow [24] and use the \( \circ_n \)-products 2.3.1 for \( \text{Cend}_N \).

From now on, identify \( \text{Cend}_N \) with \( \mathfrak{A}_N = H \otimes M_N(\mathbb{k}[v]), \) where \( A(v) \circ_n B(v) = A(v)\partial^n_q (B(v)) \).

Definition 3.1.3. A conformal subalgebra \( C \subseteq \text{Cend}_N \) is called \emph{irreducible}, if there are no non-trivial \( C \)-submodules of \( V_N \).

3.2. Operator Algebras. Let us fix an \( H \)-linear basis of \( V_N \), and identify \( V_N \) with \( H \otimes \mathbb{k}^N \) as above. For every \( a \in \text{Cend}_N \) and for every \( n \in \mathbb{Z}_+ \) consider the \( k \)-linear map
\[ a(n) = a_n : V_N \to V_N, \quad a(n) : u \mapsto a \circ_n u, \quad u \in V_N, \]
given by 3.1.2. Introduce a new variable \( q \) instead of \( D \) in this context, and set \( q = \partial_q \). Thus, we get \( a(n) \in M_N(W) \), where \( W = \mathbb{k}[p, q \mid qp - pq = 1] \) is the (1st) Weyl algebra. The axiom (C2) implies \((Da)(n) = -na(n-1) = -\partial_q a(n)\), where \( \partial_q \) is the partial derivation with respect to \( q \).

In fact, the set \( S(\text{Cend}_N) = \{ a(n) \mid a \in \text{Cend}_N, \ n \in \mathbb{Z}_+ \} \) is a linear space with multiplication (composition) given by
\[ (3.2.1) \quad a(n)b(m) = \sum_{s \geq 0} \binom{n}{s} (a \circ_{n-s} b)(m + s). \]
It is easy to note that \( S(\text{Cend}_N) = M_N(W) \).

For every \( H \)-submodule \( X \) of \( \text{Cend}_N \) one may consider the subspace
\[ S(X) = \{ a(n) \mid a \in X, \ n \in \mathbb{Z}_+ \} \subseteq S(\text{Cend}_N). \]
If \( C \) is a conformal subalgebra of \( \text{Cend}_N \), then \( S(C) \) is a \( \partial_q \)-invariant subalgebra of \( M_N(W) \). The same is true for left and right ideals.

Definition 3.2.1. Let \( C \subseteq \text{Cend}_N \) be a conformal subalgebra. The subalgebra \( S(C) \) of \( M_N(W) \) is called the \emph{operator algebra} of \( C \).

Let us consider the topology (called \( q \)-topology, for short) on \( M_N(W) \) defined by the sequence of the left ideals generated by \( q^n, \ n \geq 0 \):
\[ M_N(W) \supset M_N(W)q \supset \cdots \supset M_N(W)q^n \supset \cdots \supset 0. \]
It is clear that this topology is equivalent to the finite topology 2.3.1 defined by the canonical action of \( M_N(W) \) on \( V_N \).

Definition 3.2.2. A sequence \( \{ a_n \}_{n=0}^\infty \), \( a_n \in M_N(W) \), is called \emph{differential}, if \( \partial_q(a_n) = n a_{n-1} \) (we mean \( \partial_q(a_0) = 0 \)) and \( \lim_{n \to \infty} a_n = 0 \) (in the sense of \( q \)-topology).

Lemma 3.2.3. Let \( \{ a_n \}_{n=0}^\infty \) be a differential sequence. Then there exist a finite number of matrices \( A_s \in M_N(\mathbb{k}[p]), \ s = 0, \ldots, m \), such that
\[ a_n = \sum_{s=0}^{m} \binom{n}{s} A_s q^{n-s} \]
for all \( n \in \mathbb{Z}_+ \).
Proof. By the definition of a differential sequence, there exists an integer $m \geq 0$ such that $a_n \in M_N(W)q$ for all $n > m$. Then we may represent the element $a_m$ in the following form:

$$a_m = \sum_{s=0}^{m} \binom{m}{s} A_s q^{m-s},$$

where $A_0, \ldots, A_m \in M_N(k[p])$. For every $n \geq 0$ we may write

$$a_n = \sum_{s=0}^{m} \binom{n}{s} A_s q^{n-s} + \sum_{k=m+1}^{\infty} \binom{n}{k} B_k q^{n-k}.$$

The conditions $a_n \in M_N(W)q$ (for $n > m$) and $\partial_q a_n = na_{n-1}$ imply $B_k = 0$ for all $k > m$.

Proposition 3.2.4. A subalgebra $S \subseteq M_N(W)$ is an operator algebra of some conformal subalgebra $C \subseteq \text{Cend}_N$ if and only if every element of $S$ lies in a differential sequence of elements from $S$.

Proof. If $S = S(C)$ for some $C \subseteq \text{Cend}_N$, then every $a(m) \in S$ ($a \in C, m \geq 0$) lies in the differential sequence $\{a(n)\}_{n=0}^{\infty}$.

Let us prove the converse. By Lemma 3.2.3 every differential sequence $\{a_n\}_{n=0}^{\infty}$ corresponds to a finite family of matrices $A_s \in M_N(k[p])$. Consider the elements

$$a = \sum_{s \geq 0} (-D)^{(s)} \otimes A_s(v) \in \text{Cend}_N$$

corresponding to all differential sequences in $S$. It is clear that these elements form a conformal subalgebra $C$ of $\text{Cend}_N$. Indeed, let $a, b \in \text{Cend}_N$ correspond to $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$, respectively. Then $Da$ corresponds to $\{-\partial_q(a_n)\}_{n=0}^{\infty}$, and $a \circ_n b$ corresponds to $\{c_n\}_{n=0}^{\infty}$, where $c_n = \sum_{s \geq 0} (-1)^s \binom{n}{s} a_{m-s} b_{n+s}$ (it is easy to see that these $c_n$s form a differential sequence). It follows from the construction that $S(C) = S$.

Definition 3.2.5. Let $S \subseteq M_N(W)$ be a subalgebra satisfying the conditions of Proposition 3.2.3. Then the conformal algebra $C$ constructed is called the reconstruction of $S$ (cf. [11 Section 11]). We will denote it by $R(S)$.

In particular, every operator algebra $S = S(C) \subseteq M_N(W)$ is $\partial_q$-invariant. Hence, the subspace $k[p]S = S + pS + p^2S + \ldots$ is also a subalgebra of $M_N(W)$.

Proposition 3.2.6. A conformal subalgebra $C \subseteq \text{Cend}_N$ is irreducible if and only if the subalgebra $S_1 = k[p]S(C)$ acts irreducibly on $V_N$.

Proof. Let $C$ be an irreducible conformal subalgebra. For every $0 \neq u \in V_N$ the following $H$-module is also a $C$-module:

$$V_u = H\{a \circ_n u \mid a \in C, n \in \mathbb{Z}_+\}.$$ 

If $V_u = 0$, then $\{u \in V_N \mid V_u = 0\}$ is a non-trivial $C$-submodule. Hence, $V_u = V_N$. Then for every $w \in V_N$ there exists a finite family $\{a_i, n_i, f_i\}$, $a_i \in C$, $n_i \in \mathbb{Z}_+$, $f_i \in H$, such that $w = \sum f_i(a_i \circ_{n_i} u)$. Then for the $S$-module $V_N$ we have

$$w = \sum f_i(p)a_i(n_i)u \in S_1u.$$

The converse statement has a similar proof.
Proposition 3.2.7. The algebra $M_N(W)$ acts on the vector space $\text{Cend}_N$. The action is provided by

\begin{equation}
(3.2.2) \quad a(n) \cdot b = a \circ_n b, \quad a, b \in \text{Cend}_N, \quad n \in \mathbb{Z}_+.
\end{equation}

Proof. The explicit expression for the action is

\[ A(p)q^n \cdot (D(s) \otimes B(v)) = \sum_{t \geq 0} \binom{n}{t} D^{(s-t)} \otimes A(v) \partial_v^{n-t} B(v). \]

It is straightforward to check the relation (3.2.2) and the associativity. \hfill $\square$

Corollary 3.2.8. For every conformal subalgebra $C \subseteq \text{Cend}_N$ we have $S(C) \cdot C \subseteq C$.

Corollary 3.2.9. If $C \subseteq \text{Cend}_N$ is a conformal subalgebra, then $C$ is a left ideal of $\mathcal{R}(S(C))$.

Proof. Let $x \in \mathcal{R}(S(C))$ be an element, corresponding to a differential sequence $\{b_n\}_{n=0}^{\infty}$, $b_n \in S(C)$. Then $x(n) = b_n$ for all $n \geq 0$. For every $a \in C$ we have $x \circ_a a = b_n \cdot a \in C$. \hfill $\square$

3.3. Automorphisms of $\text{Cend}_N$ and $M_N(W)$. In this section, we consider a relation between automorphisms of $\text{Cend}_N$ and $M_N(W)$. We will also obtain the full description of automorphisms of $\text{Cend}_N$, as in [9].

Proposition 3.3.1. Every automorphism $\Theta$ of the conformal algebra $\text{Cend}_N$ induces an automorphism $\theta$ of $M_N(W)$:

\begin{equation}
(3.3.1) \quad \theta : a(n) \mapsto \Theta(a)(n).
\end{equation}

This automorphism is $\partial_q$-invariant and continuous in the sense of $q$-topology.

Proof. First, let us show that the map (3.3.1) is well-defined. Suppose $a(n) = b(m) \in M_N(W)$. Without loss of generality we may assume $n = m$, so it is sufficient to show that $a(n) = 0$ implies $\Theta(a)(n) = 0$. If $a(n) = 0$, then $a = D^{n+1}x$ for some $x \in \text{Cend}_N$, and $\Theta(a) = D^{n+1}\Theta(x)$. Hence, $\Theta(a)(n) = 0$ and $\theta$ is well-defined.

It follows from (3.2.1) that $\theta$ is an automorphism of $M_N(W)$. Moreover, since $\{a(n)\}_{n=0}^{\infty}$ is a differential sequence, we may write $\theta(\partial_q(a(n))) = n\theta(a(n-1)) = n\Theta(a)(n-1) = \partial_q(\theta(a(n)))$. Hence, $\theta$ is $\partial_q$-invariant.

Let us consider the image of $e = 1 \otimes 1_N$ under $\Theta$. If $\Theta(e) = a \in \text{Cend}_N$, then $\theta(q^n) = a(n)$. The sequence $\{a(n)\}_{n=0}^{\infty}$ is differential, and $\theta(M_N(k[p])) = M_N(k[p])$ since $\theta$ is $\partial_q$-invariant. It is easy to note that, under these conditions, $\theta$ has to be continuous in the sense of $q$-topology. \hfill $\square$

Example 3.3.2. Let us consider the map $\Theta_{\alpha,Q} : \text{Cend}_N \to \text{Cend}_N$ such that

\begin{equation}
(3.3.2) \quad \Theta_{\alpha,Q}(a(D,v)) = (1 \otimes Q^{-1}(v))a(D,v + \alpha)Q(-D \otimes 1 + 1 \otimes v), \quad \alpha \in k, \quad Q, Q^{-1} \in M_N(k[v]).
\end{equation}

It is straightforward to check that $\Theta_{\alpha,Q}$ is an automorphism of $\text{Cend}_N$ (see [9]). The corresponding automorphism $\theta_{\alpha,Q}$ of $M_N(W)$ acts as follows:

\[ \theta_{\alpha,Q} : a(p,q) \mapsto Q^{-1}(p)a(p + \alpha,q)Q(p). \]

It is shown in [9] that $\Theta_{\alpha,Q}$ exhaust all automorphisms of $\text{Cend}_N$. Later we will also obtain this result.
Lemma 3.3.3. Let \( \theta \) be a \( \partial_q \)-invariant automorphism of \( M_N(W) \). Then \( \theta(p) = p + \alpha \), \( \theta(q) = q - B(p) \), where \( \alpha \in k \), \( B(p) \in M_N(k[p]) \).

Proof. Since \( \theta(M_N(k[p])) = M_N(k[p]) \), we have \( \theta(p) \in M_N(k[p]) \). The relation \([p, M_N(k[p])] = 0 \) implies \( \theta(p) \) to be a scalar matrix: \( \theta(p) = f(p)1_N \equiv f(p) \). Analogously, \( \partial_q(\theta(q)) = 1 \), so \( \theta(q) = q - B(p) \) for some \( B(p) \in M_N(k[p]) \). Then \( \theta((q, p)) = (q - B(p), f(p)) = f'(p) = 1 \), so, \( \deg f = 1 \).

Lemma 3.3.4. (i) Let \( \theta \) be an automorphism of \( M_N(k[p]) \) such that \( \theta(p) = p \). Then there exists a nondegenerate matrix \( Q \in M_N(k[p]) \) such that \( Q^{-1} \in M_N(k[p]) \) and \( \theta(X) = Q^{-1}XQ \) for all \( X \in M_N(k[p]) \).

(ii) Let \( \theta \) be an automorphism of \( M_N(W) \) such that \( \theta(p) = p \). Then \( \theta(q) = q - B(p) \), where \( B = h(p) - Q^{-1}Q' \) for some polynomial \( h(p) \in k[p] \) and invertible matrix \( Q \in M_N(k[p]) \), \( Q' = \partial_p(Q) \).

Proof. (i) Denote \( S_0 = \theta^{-1}(M_N(k)) \). It is easy to see that \( k[p]S_0 = M_N(k[p]) \) and the sum
\[
\sum_{n \geq 0} p^n S_0 = M_N(k[p])
\]
is direct.

Let us consider the canonical action of \( S_0 \) on \( V_N \). Relation \( \theta \) implies that \( S_0 u \neq 0 \) for every \( u \in V_N \). So there exist non-zero finite dimensional \( S_0 \)-submodules of \( V_N \), and we have a non-zero \( S_0 \)-submodule \( U_0 \) of minimal dimension over \( k \). Since \( S_0 \) is simple, the \( S_0 \)-module \( U_0 \) is faithful and \( \dim k U_0 = N \).

Consider \( u_1, \ldots, u_N \in V_N \) such that \( U_0 = k u_1 \oplus \cdots \oplus k u_N \). Then these vectors are linearly independent over the field of rational functions \( k(p) \). Indeed, let there exist \( f_1(p), \ldots, f_N(p) \in k(p) \) such that
\[
f_1(p)u_1 + \cdots + f_N(p)u_N = 0
\]
(it is sufficient to consider only \( f_i(p) \in k(p) \)). Then we may choose \( a_i \in S_0 = \text{End}_k(U_0) \) such that \( a_i u_j = \delta_{ij} u_j \). Since \( [S_0, k[p]] = 0 \), the relation \( \theta \) implies \( f_j(p)u_j = 0 \) for every \( j = 1, \ldots, N \).

In particular, the matrix \( P = (u_1, \ldots, u_N) \) has non-zero determinant. In general, \( P^{-1} \) does not lie in \( M_N(k[p]) \), but we still have \( U_0 = P k N, P^{-1}S_0P = M_N(k) \).

Then \( \theta \) allows to conclude that \( P^{-1}M_N(k[p])P = M_N(k[p]) \). The last relation implies \( P = g(p)R \), where \( R, R^{-1} \in M_N(k[p]) \), \( g(p) \in k[p] \). For the matrix \( R \) obtained we also have \( R^{-1}S_0R = M_N(k) \).

Let \( \tau \) be the automorphism of \( M_N(k[p]) \) defined as the conjugation by \( R \). Then the composition \( \theta_1 = \tau \circ \theta^{-1} \) preserves \( p \) and \( M_N(k) \). Hence, \( \theta_1 \) acts as the conjugation by a nondegenerate matrix, i.e., \( \theta_1 : x \mapsto TxT^{-1} \) for some \( T \in M_N(k) \).

Finally, \( \theta \) is the conjugation by invertible matrix \( Q = RT \in M_N(k[p]) \).

(ii) Since \( \partial_q(a) = [a, p] \) for every \( a \in M_N(W) \), we may conclude that \( \theta \) satisfies the conditions of Lemma 3.3.3. It follows from (i) that there exists an invertible matrix \( Q \in M_N(k[p]) \) such that the restriction \( \theta|_{M_N(k[p])} \) is just the conjugation by \( Q \).

If \( S_0 = \theta^{-1}(M_N(k)) \), then for \( q + A = \theta^{-1}(q) \) we have \([q + A, S_0] = 0 \). So \([Q^{-1}(q + A)Q, M_N(k)] = 0 \), and the matrix \( Q^{-1}(q + A)Q = q + Q^{-1}AQ + Q^{-1}Q' \) has to be scalar. In particular, \( Q^{-1}AQ + Q^{-1}Q' = h(p) \) for some polynomial \( h(p) \). Hence, \( A = h(p) - Q'Q^{-1} \). But \( \theta(A) = Q^{-1}AQ \) by (i), so \( \theta(q) = q - \theta(A) = q - (h - Q'Q^{-1}) \).
Theorem 3.3.5. Let \( \theta \) be a \( \partial_q \)-invariant automorphism of \( M_N(W) \). Then
\[
\theta = \theta_{\alpha,Q,h} : a(p,q) \mapsto Q^{-1}(p)a(p+\alpha,q-h(p))Q(p)
\]
for some \( \alpha \in \mathbb{k} \), \( h(p) \in \mathbb{k}[p] \), \( Q, Q^{-1} \in M_N(\mathbb{k}[p]) \).

The automorphism \( \theta_{\alpha,Q,h} \) is continuous if and only if \( h(p) = 0 \).

Proof. By Lemma 3.3.3 there exists \( \alpha \in \mathbb{k} \) such that \( \theta(p) = p + \alpha \). Consider \( \tau_1 = \theta_{\alpha,1N,0} \), then the composition \( \theta_1 = \tau_1^{-1} \circ \theta \) preserves \( p \): \( \theta_1(p) = p \).

Lemma 3.3.4 implies that there exist a polynomial \( h \in \mathbb{k}[p] \) and an invertible matrix \( Q \in M_N(\mathbb{k}[p]) \) such that \( \theta_1(q) = q - h(p) + Q^{-1}Q' \) and \( \theta_1|_{M_N(\mathbb{k}[p])} \) is the conjugation by \( Q \). Consider \( \tau_2 = \theta_{\alpha,Q,h} \), then \( \tau_2(q) = \theta_1(q) \). Since \( \tau_2 \) preserves \( p \), we may conclude that the composition \( \theta_2 = \tau_2^{-1} \circ \theta_1 \) is just the identity map. So, \( \theta = \tau_1 \circ \tau_2 = \theta_{\alpha,Q,h} \).

It is clear that \( \theta_{\alpha,Q,0} \) is continuous in the sense of \( q \)-topology. If \( h \neq 0 \), then \( \theta_{\alpha,Q,h} \) is not continuous since the sequence \( \theta_{\alpha,Q,h}(q^n) = Q^{-1}(q-h(p))^nQ \) does not converge to zero. \( \square \)

Corollary 3.3.6 ([9, Theorem 4.1]). Any automorphism of \( \text{Cend}_N \) is of the form \( \Theta_{\alpha,Q} \), as in (3.3.8).

Proof. If \( \Theta \) is an automorphism of \( \text{Cend}_N \), then the corresponding automorphism \( \theta \) is of the form \( \theta_{\alpha,Q,0} \) by Proposition 3.3.1 and Theorem 3.3.5. By the construction,
\[
\Theta(a)(n) = \theta_{\alpha,Q,0}(a(n)) = \Theta_{\alpha,Q}(a)(n),
\]
so \( \Theta = \Theta_{\alpha,Q} \). \( \square \)

Proposition 3.3.7. Let \( C \subseteq \text{Cend}_N \) be a conformal subalgebra, and let \( \theta : S(C) \to S \) be an isomorphism from \( S(C) \) onto a subalgebra \( S \subseteq M_N(W) \). If \( \theta \) is \( \partial_q \)-invariant and continuous, then there exists an injective homomorphism \( \Theta : C \to \mathcal{R}(S) \) of conformal subalgebras.

Proof. Since \( \theta \) is \( \partial_q \)-invariant and continuous, the subalgebra \( S \) satisfies the conditions of Proposition 3.2.1.

For every element \( a \in C \) the sequence \( \{\theta(a(n))\}_{n=0}^{\infty} \) is differential. By Lemma 3.2.3 there exists an element \( \hat{a} \in \text{Cend}_N \) (moreover, \( \hat{a} \in \mathcal{R}(S) \)) such that \( \hat{a}(n) = \theta(a(n)) \) for all \( n \geq 0 \). The map \( \Theta : a \mapsto \hat{a} \) is an injective homomorphism of conformal algebras. \( \square \)

3.4. Left and right ideals of \( \text{Cend}_N \). In this section, we state some results of [9] Section 1] on the structure of left and right ideals of \( \text{Cend}_N \).

The following description of one-sided ideals was obtained by B. Bakalov.

Proposition 3.4.1 ([9]). The conformal algebra \( \text{Cend}_N \) is simple. Every right ideal of \( \text{Cend}_N \) has the form \( \text{Cend}_{P,N} = P(v)\text{Cend}_N \), where \( P \in M_N(\mathbb{k}[v]) \). Every left ideal has the form \( \text{Cend}_{N,Q} = \text{Cend}_N \varphi^{-1}(Q) = \text{Cend}_N Q(-D \otimes 1 + 1 \otimes v) \), where \( Q \in M_N(\mathbb{k}[v]) \), \( \varphi \) is the isomorphism from Proposition 3.2.3. \( \square \)

Lemma 3.4.2 ([9]). Every \( \text{Cend}_{N,Q} \), det \( Q \neq 0 \), is a conjugate of \( \text{Cend}_{N,D} \), where \( D \) is the canonical diagonal form of \( Q \), i.e., \( D = \text{diag}(f_1, \ldots, f_N) \), every \( f_i \) is monic, and \( f_i | f_{i+1} \). \( \square \)

Proposition 3.4.3 ([9]). Conformal subalgebra \( \text{Cend}_{N,Q} \subseteq \text{Cend}_N \) is irreducible if and only if \( \text{det} \neq 0 \). \( \square \)
Example 3.4.4. Consider $C = M_N(k[D]) \subseteq \mathrm{Cend}_N$. This subalgebra is just the current conformal algebra $\mathrm{Cur}_N = \mathrm{Cur} M_N(k)$. It is clear (see also [9]) that $\mathrm{Cur}_N$ is irreducible since $S(\mathrm{Cur}_N) = M_N(k[q])$ and $k[p]S(\mathrm{Cur}_N) = M_N(W)$.

It was conjectured in [22] that $\Theta_{0,T}(\mathrm{Cur}_N), T, T^{-1} \in M_N(k[v])$, and $\mathrm{Cend}_{N,Q}, Q \in M_N(k[v]), \det Q \neq 0$, exhaust all irreducible subalgebras of $\mathrm{Cend}_N$. In Section 5 we will prove the conjecture.

Proposition 3.4.5 ([9]). If $\det Q \neq 0$, then:
(i) every left ideal of $\mathrm{Cend}_{N,Q}$ is of the form $\mathrm{Cend}_{N,TQ}, T \in M_N(k[v])$;
(ii) $\mathrm{Cend}_{N,Q}$ is a simple conformal algebra. \hfill \Box

Remark 3.4.6. The structure of right ideals of $\mathrm{Cend}_{N,Q}, \det Q \neq 0$, was also described in [9]. In fact, every right ideal of $\mathrm{Cend}_{N,Q}$ is of the form $PC_{\mathrm{Cend}_{N,Q}}, P \in M_N(k[v])$.

4. Properties of operator algebras

4.1. Density of irreducible subalgebras. Proposition 3.2.4 shows the relation between irreducible conformal subalgebras of $\mathrm{Cend}_N$ and irreducible subalgebras of $M_N(W)$: if $C \subseteq \mathrm{Cend}_N$ is irreducible, then $S_1 = k[p]S(C) \subseteq M_N(W)$ acts on $V_N$ irreducibly. The examples of irreducible conformal subalgebras provided by Proposition 3.2.4 and Example 3.4.4 correspond to dense (in the sense of $\vartheta$-topology) subalgebras of $M_N(W)$. In this section, we prove that the density of $k[p]S(C)$ is a necessary property of an irreducible conformal subalgebra $C$.

Let us remind the classical Density Theorem by N. Jacobson. Consider a primitive algebra $S$ over a field $k$ with a faithful irreducible left $S$-module $V$. One may identify $S$ with its image in $\mathrm{End}_k V$. The centralizer $D = \{\phi \in \mathrm{End}_k V \mid [\phi, S] = 0\}$ is a skew field over $k$. The module $V$ could be considered as a right vector space over $D^\text{op}$, where $D^\text{op}$ is anti-isomorphic to $D$. There exists the natural embedding of $S$ into $E = \mathrm{End}_{D^\text{op}} V$.

Theorem 4.1.1 ([20]). The algebra $S$ is a dense subalgebra of $E$ in the sense of finite topology, i.e., for any $D^\text{op}$-independent family $u_1, \ldots, u_n \in V$ and for any $w_1, \ldots, w_n \in V$ there exists $a \in S$ such that $au_i = w_i, i = 1, \ldots, n$. \hfill \Box

Theorem 4.1.2. Let $S \subseteq M_N(W)$ be a $\vartheta_q$-invariant subalgebra such that $pS \subseteq S$. If $S$ acts on $V_N$ irreducibly, then $S$ is a dense subalgebra of $M_N(W)$ in the sense of $\vartheta_q$-topology.

Proof. Let us fix $0 \neq a \in S$. By $D \subseteq \mathrm{End}_k V_N$ we denote the centralizer of $S$. For any $\phi \in D$ the relations $[\phi, a] = [\phi, pa] = 0$ imply

$$[\phi, p]a = 0. \tag{4.1.1}$$

But for every $b \in S$ we have

$$[[\phi, p], b] = [[\phi, b], p] + [\phi, [p, b]] = 0,$$

since $[p, b] = -\partial_q b \in S$. Hence, $[\phi, p] \in D$ is either invertible or zero. The first is not the case because of (4.1.1), so $[\phi, p] = 0$.

Let us fix an $H$-basis $\{e_1, \ldots, e_N\}$ of $V_N$. Since $[\phi, p] = 0$, the matrix of $\phi$ in the $k$-basis $\{e_1, \ldots, e_N, pe_1, \ldots, pe_N, p^2e_1, \ldots, p^2e_N, \ldots\}$ of $V_N$ has the block-triangular
form

\[ [\phi] = \begin{pmatrix} A_0 & 0 & 0 & \cdots \\ * & A_0 & 0 & \cdots \\ * & * & A_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots \end{pmatrix}, \]

where \( A_0 \) is a matrix from \( M_N(k) \). Let \( \lambda \in k \) be an eigenvalue of \( A_0 \). Then the transformation \( \phi - \lambda I_N \) lies in \( D \), and it is not invertible. Hence, \( \phi = \lambda I_N \), \( \lambda \in k \).

Thus, \( D = k \). Theorem \ref{thm:finite-topology-1} implies \( S \) to be a dense subalgebra of \( \text{End}_k V_N \) in the sense of finite topology. It is clear that the finite topology is equivalent to the \( q \)-topology on \( M_N(W) \), so \( S \) is dense in the sense of \( q \)-topology. \( \square \)

4.2. Operator algebras of linear growth. In this section, we prepare some additional facts about operator algebras of Gel’fand–Kirillov dimension one. These subalgebras correspond to conformal subalgebras of finite type. We will essentially use the following result of \cite{28}.

**Theorem 4.2.1** (\cite{28}). Let \( S \) be a finitely generated prime unital algebra such that \( \text{GKdim} \, S = 1 \). Then \( S \) is a finitely generated module over its center. \( \square \)

**Proposition 4.2.2.** Assume that \( S \subseteq M_N(W) \) is a \( \partial_q \)-invariant subalgebra such that \( S_1 = k[p]S \) is prime and satisfies ascending chain condition (a.c.c.) for right annihilation ideals. Then \( S \) is prime.

**Proof.** Suppose that \( S \) is not prime, i.e., there exist non-zero ideals \( I, J \triangleleft S \) such that \( IJ = 0 \).

If either \( I \) or \( J \) is \( \partial_q \)-invariant, then \((k[p]I)(k[p]J) = 0\), where \( k[p]I, k[p]J \triangleleft S_1 \).

Since \( S_1 \) is prime, either \( I \) or \( J \) is zero.

Consider the following ascending sequence of ideals of \( S \): \( J_0 = 0, J_1 = J, J_{n+1} = J_n + \partial_q(J_n) \) for \( n \geq 1 \). It is easy to show by induction on \( n \) that \( IJ_n \subseteq J_{n-1} \). In particular, \( I^n J_n = 0 \) for every \( n \geq 0 \).

Since \( S_1 \) satisfies a.c.c. for right annihilation ideals, there exists \( m \geq 1 \) such that

\[ \text{Ann}_{S_1}(I) \subseteq \text{Ann}_{S_1}(I^2) \subseteq \cdots \subseteq \text{Ann}_{S_1}(I^m) = \text{Ann}_{S_1}(I^{m+1}) = \ldots. \]

We have shown that \( J_n \subseteq \text{Ann}_{S_1}(I^m) \), so for every \( p \geq m \) the ideal \( J_p \) lies in \( \text{Ann}_{S_1}(I^m) \). In particular, \( I^m J_\emptyset = 0 \), where \( J_\emptyset = \bigcup_{n \geq 0} J_n \) is a \( \partial_q \)-invariant ideal. If \( I^m \neq 0 \), then we obtain a contradiction as it was shown above.

Therefore, it is sufficient to prove that \( S \) has no non-zero nilpotent ideals. By the very same reasons as stated above, \( S \) has no non-zero \( \partial_q \)-invariant nilpotent ideals.

It is easy to note that, if \( I^n = 0 \), then \( (I + \partial_q(I))^n = 0 \). Hence, for every nilpotent ideal \( I \triangleleft S \) and for every \( m \geq 0 \) the ideal \( I + \partial_q(I) + \cdots + \partial_q^m(I) \) is also nilpotent. If \( I \neq 0 \), then we may choose \( 0 \neq a \in I \) and consider the ideal \( I_a \) generated in \( S \) by all derivatives of \( a \): \( I_a = \langle a, \partial_q(a), \ldots, \partial_q^m(a) \rangle \triangleleft S \). This ideal is \( \partial_q \)-invariant and lies in \( I + \partial_q(I) + \cdots + \partial_q^m(I) \), so \( I_a \) is nilpotent. As it was shown above, \( I_a = 0 \) in contradiction with \( a \neq 0 \). \( \square \)

For a subalgebra \( S \subseteq M_N(W) \) denote by \( Z(S) \) the center of \( S \). If \( S_1 \) is a subalgebra of \( M_N(W) \) such that \( S_1 \supseteq S \), then by \( Z_{S_1}(S) \) we denote the centralizer of \( S \) in \( S_1 \). If \( S \) and \( S_1 \) are \( \partial_q \)-invariant, then so are \( Z_{S_1}(S) \) and \( Z(S) \).
Lemma 4.2.3. Let \( S \subseteq M_N(W) \) be a finitely generated prime \( \partial_q \)-invariant subalgebra such that \( \mathrm{GKdim} \ S = 1 \) and \( Z_{M_N(W)}(k[p])S = k \). Then there exists a matrix \( A \in M_N(k[p]) \) such that \( q + A \in Z(S) \).

Proof. Although \( S \) is not supposed to be unital, it is still possible to apply Theorem 4.2.1 since \( S + k \) also satisfies the conditions of the statement. By Theorem 4.2.1, \( S \) is a finitely generated module over its center \( Z(S) \). In particular, \( Z(S) \neq 0, k \).

Since \( Z(S) \) is \( \partial_q \)-invariant, there exists a non-zero element \( y \in Z(S) \cap M_N(k[p]) \).

But \( [y, p] = 0 \), so \( y \in k \) and \( k \subseteq Z(S) \). Now, let \( x \in Z(S) \setminus k \). We may assume that \( \partial_q(x) \in k \), hence, \( x = q + A \), where \( A \in M_N(k[p]) \).

Proposition 4.2.4. Let \( S \subseteq M_N(W) \) be a \( \partial_q \)-invariant subalgebra such that

\[
\bigoplus_{n \geq 0} p^n S = M_N(W).
\]

Then there exists a \( \partial_q \)-invariant automorphism \( \theta \) of \( M_N(W) \) such that \( \theta(p) = p \) and \( \theta(S) = M_N(k[q]) \).

Proof. It is clear that \( S \) is finitely generated and \( \mathrm{GKdim} S = 1 \) (see, e.g., [27]). Since \( M_N(W) \) is (left and right) Noetherian [16], it satisfies a.c.c. for right annihilation ideals. By Proposition 4.2.2, the algebra \( S \) is prime, and by Lemma 4.2.3 there exists \( x = q + A \in Z(S) \).

Moreover, for every \( \partial_q \)-invariant ideal \( I \triangleleft S \) its “envelope” \( k[p]I \) is an ideal of \( M_N(W) \). Thus, either \( I = 0 \) or \( k[p]I = M_N(W) \). The last relation implies \( I = S \) since the sum (4.2.1) is direct. Therefore, \( S \) has no non-trivial \( \partial_q \)-invariant ideals.

Denote \( S_0 = S \cap M_N(k[p]) = \mathrm{Ker} \partial_q|_S \). For every \( k \geq 1 \) let us define \( S_k = \{a \in S \mid \partial_q(a) \in S_{k-1}\} \). Since \( \partial_q \) is locally nilpotent, we have

\[
S = \bigcup_{k \geq 0} S_k, \quad S_k = \mathrm{Ker} \partial_q^{k+1}|_S.
\]

It is easy to show by induction on \( k \) that

\[
S_k = S_0 + S_0x + \cdots + S_0x^k.
\]

Indeed, it is clear for \( k = 0 \). For every \( k \geq 1 \) and for every \( a \in S_k \) we have

\[
\partial_q^k \left( a - \frac{1}{k} \partial_q(a)x \right) = 0,
\]

so \( a - \frac{1}{k} \partial_q(a)x \in S_{k-1} \). Since \( \partial_q(a) \in S_{k-1} \), we may use the inductive assumption. Hence, (4.2.2) is true.

It is also clear that for every \( k \geq 0 \) the sum (4.2.2) is direct. So, \( S = \bigoplus_{n \geq 0} S_0x^n \) and \( \mathrm{GKdim} S_0 = 0 \) [27]. The algebra \( S_0 \) has no non-trivial ideals since \( S \) is differentially simple. Moreover, for this simple finite-dimensional algebra \( S_0 \) we have

\[
k[p]S_0 = \bigoplus_{n \geq 0} p^n S_0 = M_N(k[p]).
\]

Hence, there exists an isomorphism \( \theta : S_0 \to M_N(k) \) which could be extended via \( \theta(x) = q \), \( \theta(p) = p \) to a \( \partial_q \)-invariant automorphism of \( M_N(W) \) such that \( \theta(S) = M_N(k[q]) \).
4.3. Operator algebras of current-type subalgebras. The main result of this section could be stated as follows: if there exists a \( \partial_s \)-invariant automorphism of \( M_N(W) \) such that an operator algebra \( S = S(C) \) goes onto \( M_N(\mathbb{k}[q]) \), then \( C \) is a conjugate of \( \text{Cur}_N \) by an automorphism of \( \text{Cend}_N \) (i.e., \( C \) is a current-type conformal subalgebra of \( \text{Cend}_N \)). To prove this, we will show that one may choose the isomorphism \( \theta \) in Proposition \ref{prop:main} to be continuous.

Let \( h \in \mathbb{k}[p] \) be a polynomial. Consider two sequences of polynomials in \( p \): \( \{h_{(n)}\}_{n \geq 0} \) and \( \{h_{[n]}\}_{n \geq 0} \) such that

\[
(4.3.1) \quad h_{(0)} = h_{[0]} = 1, \quad h_{(n)} = -hh_{(n-1)} + h'_{(n-1)}, \quad h_{[n]} = hh_{[n-1]} + h'_{[n-1]}.
\]

It is easy to see that these sequences are constructed by the following reason. For any \( n \geq 0 \) we may write

\[
(4.3.2) \quad (q - h)^n = a_n(p,q)q + h_{(n)}, \quad (q + h)^n = b_n(p,q)q + h_{[n]}
\]

in the Weyl algebra \( W \).

Lemma 4.3.1. For every \( k \geq 0 \) we have

\[
(4.3.3) \quad \sum_{s=\xi}^{k} \binom{k - \xi}{s - \xi} h_{[s-\xi]} h_{(k-s)} = \begin{cases} 0, & \xi < k, \\
1, & \xi = k. \end{cases}
\]

Proof. The case \( \xi = k \) is obvious, so we will concentrate our attention on the case \( \xi < k \). For \( k = 1, \xi = 0 \) the relation \ref{eq:lemma_4.3.1} is clear. Suppose that \ref{eq:lemma_4.3.1} holds for some integer \( k \) and for all \( \xi \leq k \). Then for \( k + 1 \) instead of \( k \) and for \( \xi = k \) the corresponding equality in \ref{eq:lemma_4.3.1} is also true. For \( \xi < k \) we have

\[
\sum_{s=\xi}^{k+1} \binom{k - \xi + 1}{s - \xi} h_{[s-\xi]} h_{(k+1-s)}
\]

\[
= \sum_{s=\xi}^{k} \binom{k - \xi}{s - \xi} h_{[s-\xi]} \left( -hh_{(k-s)} + h'_{(k-s)} \right)
\]

\[
+ \sum_{s=\xi+1}^{k+1} \binom{k - \xi}{s - \xi - 1} \left( hh'_{[s-1]-\xi} + h'_{[s-1]-\xi} \right) h_{(k-(s-1))}
\]

\[
= \sum_{s=\xi}^{k} \binom{k - \xi}{s - \xi} \left( h_{[s-\xi]} h'_{(k-s)} + h'_{[s-\xi]} h_{(k-s)} \right) = 0.
\]

\[\square\]

Corollary 4.3.2. For any \( \xi < k \) we have

\[
(4.3.4) \quad \sum_{s=\xi}^{k-1} \binom{k}{s} \binom{s}{\xi} h_{[s-\xi]} h_{(k-s)} = -\binom{k}{\xi} h_{[k-\xi]}.
\]

\[\square\]

Lemma 4.3.3. Let

\[
a(n) = \sum_{k \geq 0} \binom{n}{k} A_k(p) q^{n-k}
\]
be an element of \(M_N(W)\). Replace \(q\) with \(x - h\), where \(x = q + h\), and consider the presentation
\[
a(n) = \sum_{s \geq 0} \binom{n}{s} B_s(p)x^{n-s}.
\]
Then
\[
(4.3.5) \quad A_k = \sum_{s \geq 0} \binom{k}{s} B_s(p)h_{[k-s]}.
\]

Proof. Since \(\partial_x = \partial_q = [\cdot, p]\) on \(M_N(W)\), we may assume \(A_0 = B_0\), \(A_1 = B_0h + B_1\).

For greater \(k\), the relation (4.3.5) could be proved by induction. Indeed,
\[
B_k = A_k + \sum_{s=0}^{k-1} \binom{k}{s} A_s h_{[k-s]},
\]
so by the assumption
\[
A_k = B_k - \sum_{s=0}^{k-1} \binom{k}{s} \sum_{\xi=0}^{s-1} \binom{s}{\xi} B_{\xi} h_{[s-\xi]} h_{[k-s]},
\]
and it is sufficient to apply (4.3.4). □

**Lemma 4.3.4.** Let \(C \subset \text{Cend}_N\) be a conformal subalgebra such that \(S(C) = M_N(k[q + h(p)])\). Then \(\deg h = 0\) (or \(h = 0\)).

Proof. Consider an arbitrary element \(a \in C\):
\[
a = \sum_{k=0}^{n} (-D)^{(k)} \otimes A_k(v) \neq 0.
\]
Then
\[
(4.3.6) \quad a(n+1) = \sum_{k=0}^{n} \binom{n+1}{k} A_k q^{n+1-k} \in S(C) = M_N(k[q + h(p)]).
\]

We may assume that \(A_0 \neq 0\); otherwise, \(a = D^{(m-1)}b\), \(b \in \text{Cend}_N\), \(m \geq 2\) and it is sufficient to consider \(a(n+m)\) instead of \(a(n+1)\) (although the element \(b\) may not lie in \(C\), the operator \(b(n+1)\) lies in \(S(C) = M_N(k[q + h(p)])\)). Rewriting (4.3.6) by using \(q = x - h\) gives
\[
B_{n+1} = \sum_{k=0}^{n} \binom{n+1}{k} A_k h_{(n+1-k)} \in M_N(k).
\]

But Lemma 4.3.3 and Lemma 4.3.4 imply that
\[
B_{n+1} = \sum_{k,s=0}^{n} \binom{n+1}{k} \binom{k}{s} B_s h_{[k-s]} h_{(n+1-k)}
\]
\[
= \sum_{s=0}^{n} \binom{n+1}{s} B_s \sum_{k=s}^{n} \binom{n+1-s}{k-s} h_{[k-s]} h_{(n+1-k)}
\]
\[
= -\sum_{s=0}^{n} \binom{n+1}{s} B_s h_{[n+1-s]}.
\]
If $\deg h > 0$, then the term of highest degree in (5.3.7) corresponds to $s = 0$: 
$$B_{n+1} = -A_0 h_{[n+1]} + \text{terms of lower degree} \notin M_N(k).$$
Hence, either $h = 0$ or $\deg h = 0$. \hfill $\square$

**Theorem 4.3.5.** Let $S \subset M_N(W)$ be a subalgebra satisfying the conditions of Proposition 4.2.4 and Proposition 3.2.2, i.e., $S = S(C)$ for a conformal subalgebra $C \subset \text{Cend}_N$. Then there exists an automorphism $\Theta$ of the conformal algebra $\text{Cend}_N$ such that $\Theta(C) = \text{Cur}_N$.

**Proof.** By Proposition 4.2.4 there exists a $\partial_q$-invariant automorphism $\theta$ of $M_N(W)$ such that $\theta(S) = M_N([k[q]])$, $\theta(q + A(p)) = q$. The matrix $A(p)$ is defined up to an additive scalar, i.e., instead of $q + A(p)$ one may consider $q + A(p) + \alpha \in Z(S)$, for any $\alpha \in k$. Theorem 3.3.5 implies that $\theta = \theta_{0,Q,h}$, $A = h(p) - Q_{p}Q^{-1}$, and we may assume that $h(0) = 0$.

Consider the automorphism $\Theta_{0,Q}$ of the conformal algebra $\text{Cend}_N$, and denote $\Theta_{0,Q}(C) = C^Q$. Then the subalgebra
$$S^Q = S(C^Q) = Q^{-1}S(C)Q \subset M_N(W)$$
is also isomorphic to $M_N([k[q]])$, and the isomorphism is given by $\theta_{0,1_{N},h}$, where $h(0) = 0$.

By Lemma 3.3.4 $h = 0$, so $\theta = \theta_{0,Q,0}$ is continuous by Theorem 3.3.5. Proposition 3.3.4 and Corollary 3.2.2 imply $C^Q \cong R(M_N([k[q]])) = \text{Cur}_N$. It is well-known that every left ideal of $\text{Cur}_N$ is of the form $H \otimes I_0$, where $I_0 \trianglelefteq M_N(k)$. If $C^Q = H \otimes I_0$, then $S^Q = M_N([k[q]]) = I_0 \otimes k[q]$; hence, $I_0 = M_N(k)$. Thus, we obtain $C^Q = \text{Cur}_N$. \hfill $\square$

5. Irreducible Conformal Subalgebras

**5.1. Preliminary notes.**

**Theorem 5.1.1.** Let $C \subset \text{Cend}_N$ be an irreducible conformal subalgebra. Then

\[(5.1.1) \quad k[v]C = C + vC + v^2C + \cdots = \text{Cend}_NQ,\]

where $Q = Q(v) \in M_N([k[v]])$ is a nondegenerate matrix.

**Proof.** By Proposition 5.2.6 the subalgebra $S_1 = k[p]S(C)$ acts irreducibly on $V_N$. So Theorem 1.1.2 implies $S_1$ to be a dense subalgebra of $M_N(W)$ in the sense of $q$-topology. In particular, for every $A(p)q^n \in M_N([k[p]])q^n$, and for every integer $M \geq n + 1$ there exists $a_{A,n,M} \in S_1$ such that $a_{A,n,M} - A(p)q^n \in M_N(W)q^M$, i.e.,

$$a_{A,n,M} = A(p)q^n + x(p,q)q^M$$

for some $x(p,q) \in M_N(W)$.

The $H$-submodule $C_1 = k[v]C \subset \text{Cend}_N$ is also a conformal subalgebra of $\text{Cend}_N$ (see 3.2.4), and $S(C_1) = S_1$. So Corollary 3.2.8 implies $S_1 \cdot C_1 \subset C_1$.

Let us fix an element $b \in C_1$:

$$b = \sum_{s=0}^{m} D^{(s)} \otimes B_s.$$

For any $n \geq 0$, choose an integer $M > n + m + \max_{s=0,\ldots,m} \deg_v(B_s)$. Then by Proposition 3.2.8

$$a_{A,n,M} \cdot b = A(v) \circ_n b \in C_1.$$
for every \( A(p)q^n \in M_N(W) \). Hence, \( C_1 \) is a left ideal of \( \text{Cend}_N \), and by Proposition 3.4.1 it is of the form \( \text{Cend}_{N,Q} \) for some \( Q \in M_N(\mathbb{k}[v]) \). Proposition 3.4.3 implies \( \det Q \neq 0 \).

**Lemma 5.1.2.** (i) If \( I \) is a left (right) ideal of \( C \), then \( \mathbb{k}[v]I \) is a left (right) ideal of \( \mathbb{k}[v]C = \text{Cend}_{N,Q} \).

(ii) If \( 0 \neq I \triangleleft C \), then \( \mathbb{k}[v]I = \text{Cend}_{N,Q} \).

**Proof.** Statement (i) easily follows from (3.1.1).

To prove (ii), it is sufficient to note that \( \mathbb{k}[v]I \neq 0 \) is an ideal of \( \text{Cend}_{N,Q} \) for every non-zero ideal \( I \) of \( C \). By Proposition 3.4.3(ii), the conformal algebra \( \text{Cend}_{N,Q} \) is simple, so \( \mathbb{k}[v]I = \text{Cend}_{N,Q} \). □

**Proposition 5.1.3.** If there exists an element \( a \in C \) such that \( a \neq 0 \) and \( v^k a \in C \) for all \( k \geq 0 \), then \( C = \text{Cend}_{N,Q} \).

**Proof.** It is clear from (3.1.1) that \( I = \{ a \in C \mid v^k a \in C, k \geq 0 \} \) is an ideal of \( C \). In particular, \( \mathbb{k}[v]I = I \). If \( I \neq 0 \), then by Lemma 5.1.2(ii) we have \( C \supseteq I = \mathbb{k}[v]I = \text{Cend}_{N,Q} \). □

For every irreducible subalgebra \( C \subseteq \text{Cend}_N \) there exist three options as follows:

Case 1: the sum (5.1.1) is direct, i.e.,

\[
\text{Cend}_{N,Q} = \bigoplus_{n \geq 0} v^n C.
\]

Case 2: \( C \cap vC \neq 0 \).

Case 3: the sum (5.1.1) is non-direct, but \( C \cap vC = 0 \).

We will show that the first case corresponds to the current-type conformal subalgebras, the second one gives \( C = \text{Cend}_{N,Q} \), and the third one is impossible.

Without loss of generality we may assume \( Q \) to be in the canonical diagonal form:

\( Q = \text{diag}(f_1, \ldots, f_N) \), where \( f_i \) are monic polynomials and \( f_i \mid f_{i+1} \). Indeed, for an arbitrary \( Q \in M_N(\mathbb{k}[v]) \), \( \det Q \neq 0 \), there exist \( U, T \in M_N(\mathbb{k}[v]) \), \( \det U, \det T \in \mathbb{k} \setminus \{0\} \), such that \( TQU = D \), where \( D \) has the canonical diagonal form. Then for \( C^U = \Theta_{0,v}(C) \), we have \( \mathbb{k}[v]C^U = \text{Cend}_{N,D} \), and all the conditions described by Cases 1–3 hold.

**5.2. Finite type case.** Now, let us consider Case 1. Throughout this section \( C \) is a conformal subalgebra of \( \text{Cend}_N \) satisfying (5.1.2) for a fixed matrix \( Q = \text{diag}(f_1, \ldots, f_N) \), \( 0 \neq f_i \in \mathbb{k}[v] \), \( i = 1, \ldots, N \).

**Lemma 5.2.1.** (i) If \( h(D)a \in \text{Cend}_{N,Q} \) for some \( 0 \neq h \in H \), then \( a \in \text{Cend}_{N,Q} \).

(ii) For \( S = S(C) \) we have

\[
M_N(W)Q = \bigoplus_{n \geq 0} p^n S.
\]

**Proof.** (i) Consider \( U = \text{Cend}_N / \text{Cend}_{N,Q} \) as a module over the conformal algebra \( \text{Cend}_N \). If \( h(D)a \in \text{Cend}_{N,Q} \), then \( \bar{a} = a + \text{Cend}_{N,Q} \) lies in the torsion of \( U \). In particular, \( \text{Cend}_N \circ_w a \in \text{Cend}_{N,Q} \). But there exists \( e = 1 \otimes 1_N \in \text{Cend}_N \) such that \( e \circ_w a = a \in \text{Cend}_{N,Q} \).

(ii) Suppose that the sum (5.2.1) is non-direct. Since \( (Da)(n) = -na(n-1) \), we may consider

\[
0 = a_0(n) + \cdots + p^n a_m(n), \quad a_i \in C, \quad n \geq 0.
\]
Then for $a = a_0 + v a_1 + \cdots + v^m a_m \in \text{Cend}_{N,Q}$ we obtain $a(0) = \cdots = a(n) = 0$, i.e., $a = D^{n+1} b$. The statement (i) implies $b \in \text{Cend}_{N,Q}$, so $b = b_0 + v b_1 + \cdots + v^k b_k$, $b_k \in C$. Since (5.1.1) is direct, we have $k = m$ and $D^{n+1} b_i = a_i$, so $a_i(n) = 0$ for all $i = 0, \ldots, m$. \hfill $\square$

Therefore, the operator algebra $S = S(C)$ of the initial conformal subalgebra $C$ satisfies \ref{5.2.1}. Consider the presentations of all elements

$$q^m e_{ij} Q = \sum_{s \geq 0} p^s a_{ijm,s},$$

$$i, j = 1, \ldots, N, m = 0, \ldots, \max_j \deg f_j + 1, a_{ijm,s} \in S,$$

where $e_{ij}$ are the matrix units (so $e_{ij} Q = e_{ij} f_j$). The finite set of all $a_{ijm,s}$ together with all their derivatives $\partial^s a_{ijm,s}$ generates a $\partial_q$-invariant subalgebra $S_0$ of $S$. Denote $S_{01} = \mathbb{k}[p] S_0$, then $S_{01}$ is a $\partial_q$-invariant subalgebra of $S_1 = M_N(W) Q$.

**Lemma 5.2.2.** (i) $\text{GKdim } S_{01} = 2$;
(ii) $Z_{M_N(W)}(S_{01}) = \mathbb{k}$;
(iii) $S_{01}$ has no non-zero nilpotent ideals.

**Proof.** (i) Since $\text{GKdim } M_N(W) = 2$, we have $\text{GKdim } S_{01} \leq 2$. Let us consider the subalgebra $S_{00}$ generated by $f_1 e_{11}, q f_1 e_{11}$ in $S_{01}$. If $\deg f_1 > 0$, then $\text{GKdim } S_{00} = 2$, so $\text{GKdim } S_{01} = 2$. If $\deg f_1 = 0$, then we obtain that $W e_{11} \subseteq S_{01}$, hence, $\text{GKdim } S_{01} = 2$ as well.

(ii) Let $a \in Z = Z_{M_N(W)}(S_{01})$. Then, in particular, $[e_{ij} f_j, a] = 0$, so $a$ is a diagonal matrix. Moreover, $[pe_{ij} f_j, a] = 0 = [p, a] e_{ij} f_j$ for all $i, j = 1, \ldots, N$. Hence, $[p, a] = 0$. By the very same reasons $[a, q] = 0$. Since all $f_j$ are unital, we conclude that $a = a I_N, a \in \mathbb{k}$.

(iii) If $0 \neq I \subseteq S_{01}$, then for any $a \in I$ and for all $i, j, k, l = 1, \ldots, N$ the element $f_j e_{ij} a f_k e_{lk}$ lies in $I$. In particular, $I$ contains an element $g e_{11}$ for some $0 \neq g \in W$. Then $I^n \ni g^n e_{11} \neq 0$ for all $n \geq 1$. \hfill $\square$

**Lemma 5.2.3.** (i) $S_{01} [q] = M_N(W)$;
(ii) if $I \lhd S_{01}$, then $I q^k [q] \lhd M_N(W)$.

**Proof.** (i) If $Q = I_N$ (i.e., $\deg f_1 = \cdots = \deg f_N = 0$), then the statement is clear. If $\deg f_j > 0$ for some $j \in \{1, \ldots, N\}$, then consider

$$S_{01} \ni q^k f_j e_{ij} = f_j q^k e_{ij} + k \partial_p (f_j) q^{k-1} e_{ij} + \cdots + \partial^k_p (f_j) e_{ij},$$

$$k = 1, \ldots, \deg f_j.$$

It is clear that $\partial^k_p (f_j) e_{ij} \in S_{01} [q]$ for all $k = 1, \ldots, \deg f_j$, so, $e_{ij} \in S_{01} [q]$.

(ii) It follows from (i). \hfill $\square$

**Proposition 5.2.4.** The algebra $S_{01}$ is prime and satisfies a.c.c. for right annihilation ideals.

**Proof.** First, let us show that $S_{01}$ satisfies a.c.c. for right ideals of the form $\text{Ann}_{S_{01}}(X) = \{a \in S_{01} \mid X a = 0\}$, $X \subseteq S_{01}$.

Consider an ascending chain of right annihilation ideals of $S_{01}$:

$$I_1 \subseteq I_2 \subseteq \cdots, \quad I_k = \text{Ann}_{S_{01}}(X_k), \quad X_k \subseteq S_{01}.$$

By Lemma \ref{5.2.3} (ii), $I_k [q] \subseteq I_{k+1} [q] \subseteq \cdots$ is an ascending chain of right ideals of $M_N(W)$. Since $M_N(W)$ is a Noetherian algebra \cite{16}, we have $I_n [q] = I_{n+1} [q]$
for a sufficiently large number $n$. In particular, $I_{n+1} \subseteq I_n[k[q]$. Thus, $X_nI_{n+1} \subseteq X_nI_n[k[q] = 0$, so $I_{n+1} = I_n$.

Now, suppose that there exist two ideals $I, J \triangleleft S_{01}$ such that $IJ = 0$. If $I \neq 0$, then by Lemma 5.2.2(iii) the ideal $I$ is not nilpotent. By the very same reason as in the proof of Proposition 4.2.2, we may assume that $J$ is $\partial_\nu$-invariant.

If $J$ is a non-zero $\partial_\nu$-invariant ideal of $S_{01}$, then by Lemma 5.2.3(ii) $J_1 = Jk[q] \triangleleft_r M_N(W)$ is also $\partial_\nu$-invariant. Moreover, $S_{01}J_1 \subseteq J_1$. Then for every pair $(i,j)$, $i, j = 1, \ldots, N$, there exist $k, l \in \{1, \ldots, N\}$ such that $e_{ik}f_je_{ij}f_j \neq 0$. Thus, we have $J_{ij} = J_1 \cap e_{ij}W \neq 0$ for all $i, j = 1, \ldots, N$. It is clear that $A_{ij} = \{a \in W \mid ae_{ij} \in J_{ij}\} \triangleleft W$, and $\partial_\nu(A_{ij}) \subseteq A_{ij}$. Then $A_{ij} = g_{ij}(p)W$ for some polynomial $0 \neq g_{ij} \in k[p]$. In particular, $P = \sum_{i=1}^N g_{ii}e_{ii} \in J_1$ is a diagonal matrix such that $\det P \neq 0$.

The relation $IJ = 0$ implies $IJ_1 = (IJ)k[q] = 0$, so $IP = 0$. Then $I = 0$ since $\det P \neq 0$. \hfill $\Box$

Corollary 5.2.5. $Q = I_N$, $C_{n, q} = C_{n, 1}$.

Proof. By the construction, $S_{01} = \bigoplus_{n \geq 0} p^n S_0$, so $\text{GKdim } S_0 = 1$. Lemma 4.2.3 implies that there exists a matrix $A \in M_N(k[p])$ such that $q + A \in Z(S_0)$. Then $\partial_\nu(x) = [x, A]$ for every $x \in S_0$.

Let us consider

$$S_{01} \ni Q = a_0 + pa_1 + \cdots + p^n a_n, \quad a_i \in S_0.$$ 

Then

$$\partial_\nu(Q) = \partial_\nu(a_0) + p\partial_\nu(a_1) + \cdots + p^n\partial_\nu(a_n) + a_1 + \cdots + np^{n-1} a_n.$$ 

On the other hand,

$$[Q, A(p)] = \partial_\nu(a_0) + p\partial_\nu(a_1) + \cdots + p^n\partial_\nu(a_n).$$

So, $\partial_\nu(Q) - [Q, A(p)] \in S_{01} \subseteq M_N(W)Q$. It is easy to note that it is not possible, if $\deg f_j \neq 0$ for at least one $j \in \{1, \ldots, N\}$. \hfill $\Box$

Theorem 5.2.6 (\cite[Theorem 5.2]{9}). There exists an automorphism $\Theta = \Theta_{0, \nu}$ of $C_{n, \nu}$ such that $\Theta(C) = \text{Cur}_N$.

Proof. By Lemma 5.2.1(ii) and Corollary 5.2.5, the operator algebra $S = S(C)$ satisfies the conditions of Theorem 4.3.5. The last statement implies that $C$ is a conjugate of $C_{n, \nu}$. \hfill $\Box$

5.3. The case $C \cap vC \neq 0$. Let $C$ be an irreducible conformal subalgebra of $C_{n, \nu}$ such that $C \cap vC \neq 0$.

Remind the definition of the map $\varphi$ from Proposition 2.8.2. For any element $a = f(D) \otimes A(v) \in H \otimes M_N(k[v])$ we put

$$\varphi(a) = (f(D) \otimes 1)A(D \otimes 1 + 1 \otimes v),$$

$$\varphi^{-1}(a) = (f(D) \otimes 1)A(-D \otimes 1 + 1 \otimes v).$$

First, let us consider the main technical features of \cite{9} concerning the description of irreducible subalgebras for $N = 1$. 

**Lemma 5.3.1** (c.f. [9] Section 2]). The following relations hold in $\text{Cend}_N$:

\begin{align}
(1 \otimes A) \circ_n \varphi^{-1}(B) &= \begin{cases}
(1 \otimes A)\varphi^{-1}(B), & n = 0, \\
0, & n > 0,
\end{cases} \\
(5.3.2) &\quad (1 \otimes A)\varphi^{-1}(B) \circ_n (1 \otimes A_1) = 1 \otimes A\partial^m_n(BA_1), \\
(5.3.3) &\quad (1 \otimes A)\varphi^{-1}(B) \circ_n (1 \otimes A_1)\varphi^{-1}(B_1) = (1 \otimes A\partial^m_n(BA_1))\varphi^{-1}(B_1),
\end{align}

where $A, B, A_1, B_1 \in M_N(k[v])$.

**Proof.** Note that

\[ \varphi^{-1}(B) = \sum_{s > 0} (-D)^{(s)} \otimes \partial^s(B), \]

so (5.3.1) follows from the axiom (C3).

Relation (5.3.2) follows from (C2) and the Leibnitz formula. In order to obtain (5.3.3) one may use (5.3.1), (5.3.2) and the associativity relation (2.1.4). \(\square\)

**Theorem 5.3.2** ([9]). Let $C \subseteq \text{Cend}_1$ be an irreducible conformal subalgebra, and let $C \cap v \text{Cend}_1 \neq 0$. Then $C = \text{Cend}_{1,Q}$ for some $0 \neq Q \in k[v]$.

**Proof.** By Theorem 5.3.1 $k[v]C = \text{Cend}_{1,Q}$. Let $0 \neq a \in C$ be an arbitrary element. Let us present it as $a = \sum_{s=0}^{m} D^s \otimes a_s$. Consider $\varphi(a)$, where $\varphi$ is the isomorphism from Proposition 2.3.2. Present $b = \varphi(a) = \sum_{t=0}^{n} D^t \otimes b_t$. Here $a_m, b_n \in k[v] \setminus \{0\}$. Then

\[ a = \varphi^{-1}(b) = \sum_{t=0}^{n} (D^t \otimes 1)\varphi^{-1}(b_t). \]

It follows from (5.3.1), (C2), (C3) that $N(a, a) = n + m + 1$ and

\[ a \circ_{n+m} a = a \circ_{n+m} \varphi^{-1}(b) = \left( \sum_{s=0}^{m} D^s \otimes a_s \right) \circ_{n+m} \left( \sum_{t=0}^{n} (D^t \otimes 1)\varphi^{-1}(b_t) \right) = (-1)^m (m + n)!(1 \otimes a_m)\varphi^{-1}(b_n). \]

So, $(1 \otimes a_m)\varphi^{-1}(b_n) \in C$. Since $C \cap v \text{Cend}_1 \neq 0$, we may assume $\deg_a a_m > 0$. Therefore, $C$ contains an element of the form $(1 \otimes g)\varphi^{-1}(f), f, g \in k[v] \setminus \{0\}$, $\deg_a g > 0$. Consider two elements of this form in $C$ (they might be equal): $a = (1 \otimes g)\varphi^{-1}(f), b = (1 \otimes g)\varphi^{-1}(h)$. It follows from (5.3.3) that $N(a, b) = n + 1$, where $n = \deg_a f + \deg_a g$, and

\[ a \circ_n b = \gamma(1 \otimes g)\varphi^{-1}(h), \quad \gamma \in k \setminus \{0\}; \]

\[ a \circ_{n-1} b = \gamma'(1 \otimes (v + a)g)\varphi^{-1}(h), \quad \gamma' \in k \setminus \{0\}, a \in k. \]

So we may conclude that $(1 \otimes vg)\varphi^{-1}(h) \in C$ as well as $(1 \otimes v^kg)\varphi^{-1}(h) \in C$ for all $k \geq 0$. Hence, the conditions of Proposition 5.3.2 hold and $C = \text{Cend}_{1,Q}$. \(\square\)

**Theorem 5.3.3.** Let $C \subseteq \text{Cend}_N$ be an irreducible conformal subalgebra, and let $C \cap vC \neq 0$. Then $C = \text{Cend}_{N,Q}$ for some $Q \in M_N(k[v])$, $\det Q \neq 0$. 

We will use this technique in the general case.
Proof. By Theorem 5.1.1 \( k[v]C = \text{Cend}_{N,Q} \), det \( Q \neq 0 \). Let \( I_1 = \{ a \in C \mid va \in C \} \neq 0 \). It is clear that \( I_1 \) is a non-zero ideal of \( C \). Moreover, \( I_2 = \{ a \in I_1 \mid va \in I_1 \} \) is also non-zero: for example, \( I_1 \cap I_1 \subseteq I_2 \) (it follows from Lemma 5.1.2(ii)) that \( I_1 \cap I_1 \neq 0 \). One may define \( I_k = \{ a \in I_{k-1} \mid va \in I_{k-1} \} \) for every \( k > 1 \). By the same way, \( I_m \cap I_k \subseteq I_{k+m} \), so all \( I_k \) are non-zero ideals of \( C \).

By Theorem 5.1.1 there exist an integer \( m \geq 0 \) and some elements \( a_i \in C \), \( i = 0, \ldots, m \), such that the element \( E_{N,Q} = (1 \otimes e_{NN})\varphi^{-1}(Q) \in \text{Cend}_{N,Q} \) is presented as

\[
E_{N,Q} = a_0 + va_1 + \cdots + v^m a_m,
\]

where \( e_{NN} \) is the matrix unit.

Consider an element \( 0 \neq a \in I_{2m+1} \) such that \( e_{NN}ae_{NN} = f(D,v)e_{NN} \), where \( 0 \neq f(D,v) \in \text{Cend}_{1} \). (Note that \( e_{NN}I_{2m+1}e_{NN} = 0 \) is impossible by Lemma 5.1.2(ii).) It follows from (3.1.1) that

\[
E_{N,Q} \cap va \circ_n E_{N,Q} = (vf_N(v)f(D,v) \circ_n \varphi^{-1}(f_N(v)))(1 \otimes e_{NN})
\]

lies in \( C \) for any \( n \in \mathbb{Z}_+ \). Hence, the conformal algebra \( C \) contains matrices of the form

\[
0 \neq x = vg(D,v)(1 \otimes e_{NN}),
\]

where \( g(D,v) \in \text{Cend}_{1} \). By the very same reasons as in the proof of Theorem 5.3.2 we conclude that there exists an element \( y \in C \) such that \( y \neq 0 \) and \( v^ky \in C \) for all \( k \geq 0 \). Hence, by Proposition 5.1.3 we have \( C = \text{Cend}_{N,Q} \).

5.4. The case \( C \cap vC = 0 \). Let \( C \cap vC = 0 \) but the sum (5.1.1) is still non-direct. Then there exists a minimal \( n \geq 1 \) such that the sum \( C + vC + \cdots + v^n C \) is direct. Let us use the following notation: for a subspace \( X \subseteq \text{Cend}_N \) we denote \( X + vX + \cdots + v^kX \) by \( (v^k)x \).

The set \( J_1 = \{ a \in C \mid v^{n+1}a \in (v^n)C \} \) is a non-zero ideal of \( C \). If \( J_1 = C \), then \( k[v]C = C \oplus vC \oplus \cdots \oplus v^n C = \text{Cend}_{N,Q} \).

If \( J_1 \neq C \), then we may construct \( J_{k+1} = \{ a \in J_k \mid v^{n+1}a \in (v^n)J_k \} \) for \( k \geq 1 \). All these \( J_k \) are ideals of \( C \), and it is clear that

\[
J_1 \cap J_{m} \subseteq J_{m+1}, \quad (v^{n+k})J_k \subseteq (v^{n+k-j})J_{j-1}, \quad j = 1, \ldots, k.
\]

If \( J_k \neq 0 \) for some \( k \geq 1 \), then by Lemma 5.1.2(ii) we have \( k[v]J_k = \text{Cend}_{N,Q} \). If \( J_1 \cap J_{m} = 0 \), then \( k[v]J_1 \cap J_{m} = \text{Cend}_{N,Q} \cap J_{m} = 0 \). Since \( J_1 \neq 0 \), the relation (5.4.1) implies \( J_k \neq 0 \) for all \( k \geq 1 \). It follows from Lemma 5.1.2(ii) that \( k[v]J_k = \text{Cend}_{N,Q} \) for all \( k \geq 1 \). In particular, \( k[v]J_{n+1} = \text{Cend}_{N,Q} \). But (5.1.1) and (5.4.1) imply that \( C_0 = (v^n)J_{n+1} \) is a conformal subalgebra of \( \text{Cend}_{N,Q} \). If \( n \geq 1 \), then \( C_0 \) satisfies the conditions of Theorem 5.3.3 so \( C_0 = \text{Cend}_{N,Q} \).

In any case, we obtain that there exists a conformal subalgebra \( C \subseteq \text{Cend}_{N,Q} \) such that \( \text{Cend}_{N,Q} = C \oplus vC \oplus \cdots \oplus v^n C, n \geq 1 \).

Theorem 5.4.1. Let \( \text{Cend}_{N,Q} = C \oplus vC \oplus \cdots \oplus v^n C \). Then \( n = 0 \).

Proof. Suppose that \( n \geq 1 \). For every \( a \in C \) we may consider a unique presentation

\[
v^{(n+1)}a = a_0 + va_1 + \cdots + v^n a_n, \quad a_i \in C.
\]

Define the map \( \chi : C \to C \) by the rule \( \chi(a) = a_n \). This is an injective \( H \)-linear map satisfying the conditions

\[
\chi(a \circ_m x) = \chi(a) \circ_m x, \quad \chi(x \circ_m a) = x \circ_m \chi(a) = mx \circ_{m-1} a
\]
for $a, x \in C, m \geq 0$. Let us consider $\psi : C \to \text{Cend}_{N,Q}$ defined as follows:

$$\psi(a) = \chi(a) - va, \quad a \in C.$$ 

This is an injective $H$-linear map, and $\psi(C) \cap C = 0$. The map $\psi$ satisfies the following conditions:

$$\psi(a \circ_m x) = \psi(a) \circ_m x, \quad \psi(x \circ_m a) = x \circ_m \psi(a),$$

$$a, x \in C, \quad m \geq 0.$$ 

Since $\text{Cend}_{N,Q} = C \oplus vC \oplus \cdots \oplus v^nC$, we may extend $\psi$ to the map $\tilde{\psi} : \text{Cend}_{N,Q} \to \text{Cend}_{N,Q}$ by the rule $\tilde{\psi}(v^k a) = v^k \psi(a), k = 0, \ldots, n$. It is easy to check that this $\tilde{\psi}$ satisfies (5.4.3) for every $a \in C, x \in \text{Cend}_{N,Q}$:

$$\tilde{\psi}(a \circ_m x) = \tilde{\psi}(a) \circ_m x = a \circ_m \tilde{\psi}(x),$$

$$\tilde{\psi}(x \circ_m a) = \tilde{\psi}(x) \circ_m a = x \circ_m \tilde{\psi}(a),$$

In particular, let us put $x = \varphi^{-1}(Q), \tilde{\psi}(x) = \sum_{t \geq 0} (-D)^{t}) \otimes X_t$. Then for an arbitrary element $a \in C$ we have

$$\tilde{\psi}(x \circ_0 a) = x \circ_0 \psi(a) = \tilde{\psi}(x) \circ_0 a.$$ 

Hence, $Q\psi(a) = X_0 a$, and we may conclude that $\psi(a) = Q^{-1}X_0a$. Since $\tilde{\psi}$ is defined by $\tilde{\psi}(v^k a) = v^k \psi(a), k = 0, \ldots, n, a \in C$, we may assume that $\psi(x) = Q^{-1}X_0 x$ for every $x \in \text{Cend}_{N,Q}$. In particular, it follows that $\tilde{\psi}(vx) = v\psi(x)$ for every $x \in \text{Cend}_{N,Q}$. It is easy to conclude that

$$\tilde{\psi}(x \circ_m y) = \tilde{\psi}(x) \circ_m y = x \circ_m \tilde{\psi}(y), \quad x, y \in \text{Cend}_{N,Q}, \quad m \geq 0.$$ 

Also, $\tilde{\psi}$ is an injective $H$-linear map.

The only possibility for (5.4.3) is $\tilde{\psi} = \alpha \text{id}, \alpha \in \mathbb{k}$. Indeed, let us apply (5.4.3) for $y = \varphi^{-1}(Q)$ and $x = e_{ij}Q$. Denote $B = Q^{-1}X_0 \in M_N(\mathbb{k}[v])$, and consider

$$\tilde{\psi}(x) \circ_0 y = (1 \otimes Be_{ij}Q)\varphi^{-1}(Q) = x \circ_0 \tilde{\psi}(y) = (1 \otimes e_{ij}QB)\varphi^{-1}(Q).$$ 

In particular, we may conclude that

$$Be_{ij}Q = e_{ij}QB,$$ 

so $B$ is a scalar matrix.

Now, consider

$$\tilde{\psi}(x) \circ_1 y = (1 \otimes Be_{ij} \partial_\psi(Q))\varphi^{-1}(Q) = x \circ_1 \tilde{\psi}(y)$$

$$= (1 \otimes e_{ij} \partial_\psi(Q)B)\varphi^{-1}(Q) + (1 \otimes e_{ij}QB \partial_\psi(Q))\varphi^{-1}(Q).$$

It follows from (5.4.5), (5.4.6) that $Be_{ij} \partial_\psi(Q) = e_{ij} \partial_\psi(QB)$ for all $i, j = 1, \ldots, N$. Since $B$ is a scalar matrix in $M_N(\mathbb{k}[v])$, we obtain $B\partial_\psi(Q) = \partial_\psi(QB)$, so $B = \alpha I_N$, $\alpha \in \mathbb{k}$. Therefore, $\psi(C) \subseteq C$, and $C \cap vC \neq 0$ in contradiction with $n \geq 1$.  

5.5. Associative conformal algebras with finite faithful representation.

Let us compile the arguments stated above in order to prove the following

**Theorem 5.5.1.** Let $C \subseteq \text{Cend}_N$ be an irreducible conformal subalgebra. Then either $C$ is a conjugate of $\text{Cur}_N$ or $C = \text{Cend}_{N,Q}$ for some matrix $Q \in M_N(\mathbb{k}[v])$, $\det Q \neq 0$. 

Proof. It follows from Theorem 5.1.1 that \( k[v|C = \sum_{n \geq 0} v^n C = \text{Cend}_{N,Q} \) for a suitable matrix \( Q \). If the sum is direct, then by Theorem 5.2.4 \( C \) is a conjugate of \( \text{Cur}_N \). If the sum is not direct, then Theorem 5.4.1 implies \( C \cap vC \neq 0 \), so \( C = \text{Cend}_{N,Q} \) by Theorem 5.3.3. \( \square \)

Lemma 5.5.2. Let \( C \neq 0 \) be an associative conformal algebra with a faithful representation of finite type. If \( \{a \in C \mid C \circ_w a = 0\} = \{a \in C \mid a \circ_w C = 0\} = 0 \), then \( C \) could be embedded into \( \text{Cend}_N \) for some \( N \) in such a way that every proper \( C \)-submodule of \( V_N \) is not faithful.

Proof. It is clear that \( C \subseteq \text{Cend}_m \) for some \( m \geq 1 \) (see, e.g., [22]). Consider any descending chain of faithful \( C \)-submodules of \( V_m \):

\[
V_m \supseteq U_1 \supseteq U_2 \supseteq \ldots
\]

It is easy to note (see, e.g., [1] Lemma 2.1) that there exists \( n \geq 1 \) such that for every \( k \geq 0 \) we have \( U_{n+k} \supseteq f_k U_n \) for some \( f_k \in H \).

In particular, the \( H \)-module \( U_n/U_{n+k} \) coincides with its torsion, so \( C \circ_s U_n \subseteq U_{n+k} \). Since the \( C \)-module \( C \circ_u U_n \subseteq V_n \) is faithful, it is a lower bound of the initial chain \( (5.5.1) \) in the set of faithful \( C \)-submodules of \( V_m \). Hence, there exists a minimal faithful \( C \)-submodule \( U \) of \( V_m \). Since \( U \) is finitely generated over \( H \) and torsion-free, \( U \) is isomorphic to \( V_N \) for some \( N \geq 1 \). \( \square \)

Theorem 5.5.3. Let \( C \) be a conformal algebra with a faithful representation of finite type. Then:

(i) there exists a maximal nilpotent ideal \( \mathfrak{N} \) of \( C \);

(ii) if \( C \) is simple, then \( C \) is isomorphic to either \( \text{Cur}_N \) or \( \text{Cend}_{N,Q} \), \( \det Q \neq 0 \);

(iii) if \( C \) is semisimple, then \( C = \bigoplus_{s=1}^n I_s \), where \( I_s, s = 1, \ldots, n \), are simple ideals of \( C \), described in (ii).

Proof. (i) Let \( V \) be a faithful \( C \)-module of finite type. First, consider the ascending chain of submodules \( U_k = \{u \in V \mid C^k \circ_w u = 0\} \), \( k \geq 1 \). Since \( V \) is a Noetherian \( H \)-module, the \( H \)-module \( V/U_n \) is torsion-free for sufficiently large \( n \geq 1 \). In particular, \( V/U_n \) is isomorphic as an \( H \)-module to \( V_N \) for a suitable \( N \). The ideal \( I = \{a \in C \mid C^n \circ_w a = 0\} \triangleq C \) is nilpotent (or even zero), and \( C/I \) acts faithfully on \( V/U_n \).

Therefore, it is sufficient to show that for every ascending chain of nilpotent ideals

\[
I_1 \subseteq I_2 \subseteq \ldots, \quad I_j \triangleq C \subseteq \text{Cend}_N, \ j \geq 1,
\]

their upper bound \( \bigcup_{j \geq 0} I_j \) is also nilpotent.

Let us consider a chain \( (5.5.2) \) of nilpotent ideals. Denote by \( n_j \) \(( j \geq 1 \) the nilpotency index of \( I_j \). Assume that the union of this chain is not nilpotent, so it could be conjectured that \( n_j < n_{j+1} \) for all \( j \geq 1 \).
Since the $H$-module $V_N$ is Noetherian, any ascending chain of submodules becomes stable. In particular,

$$I_1 \circ \omega V_N \subseteq I_2 \circ \omega V_N \subseteq \cdots = W_1 = I_j \circ \omega V_N, \quad j \geq l_1,$$
$$I_1^2 \circ \omega V_N \subseteq I_2^2 \circ \omega V_N \subseteq \cdots = W_2 = I_j^2 \circ \omega V_N, \quad j \geq l_2,$$

$$\cdots$$
$$I_1^k \circ \omega V_N \subseteq I_2^k \circ \omega V_N \subseteq \cdots = W_k = I_j^k \circ \omega V_N, \quad j \geq l_k,$$

It is clear that $W_1 \supseteq W_2 \supseteq \ldots$, so there exists an integer $k \geq 1$ such that $W_k/W_{k+j}$ coincides with its torsion for every $j \geq 0$ (see, e.g., [11 Lemma 2.1]). Hence,

$$(5.5.3) \quad C \circ \omega W_k \subseteq W_m \quad \text{for all } m > k.$$  

Let us fix some $m > k + 1$. Since the sequence $\{n_j\}$ is assumed to be increasing, there exists a number $p \geq \max\{l_k, l_m\}$ such that $n_p > m$. Then $W_k = I_p^m \circ \omega V_N$, $W_m = I_p^n \circ \omega V_N$. The relation \((5.5.3)\) implies

$$I_p^{n_p-m} \circ \omega C \circ \omega I_p^k \circ \omega V_N \subseteq I_p^{n_p \circ \omega V_N} = 0,$$

so

$$0 = I_p^{n_p-m} \circ \omega C \circ \omega I_p^k \supseteq I_p^{n_p-m+k+1}.$$  

Therefore, we obtain the contradiction.

(ii) It follows from Lemma 5.5.2 that $C$ has an irreducible faithful representation of finite type. Theorem 5.5.1 implies that either $C \simeq \text{Cur}_N$ or $C \simeq \text{Cend}_N, Q$, det $Q \neq 0$.

(iii) By Lemma 5.5.2, we may assume that $C \subseteq \text{Cend}_N$ and every proper $C$-submodule of $V_N$ is not faithful.

Let $U \subset V_N$ be a maximal $C$-submodule. Then $I = \text{Ann}_C(U) = \{a \in C \mid a \circ \omega U = 0\} \neq 0$ is an ideal of $C$.

Denote by $J = \text{Ann}_C(I) = \{a \in C \mid I \circ \omega a = 0\}$ the annihilation ideal of $I$ in $C$. Since $C$ is semisimple, $I \cap J = 0$. It is clear that $J \circ \omega V_N \subseteq \text{Ann}_{V_N}(I) = U$ since $U$ is a maximal $C$-submodule.

Now, note that $V_N/U$ is an irreducible $C$-module, $V_N/U$ is a faithful $I$-module, $V_N/U$ is a faithful irreducible $C/J$-module. Since $I$ could be considered as an ideal of $C/J$, we have $C = I \oplus J$, and $I \simeq C/J$ has a faithful irreducible representation of finite type.

By (i), $I \simeq \text{Cur}_N$ or $I \simeq \text{Cend}_N, Q$, det $Q \neq 0$. It is clear that $J$ is also semisimple, and $J$ has a faithful representation of finite type.

Therefore, every semisimple conformal subalgebra $C$ of $\text{Cend}_m$ could be presented as $C = I \oplus J$, where $I$ is simple and $J$ is semisimple, $I = \text{Ann}_C(J)$. Let us proceed with the decomposition and write $C = I_1 \oplus \cdots \oplus I_n \oplus J_n$, where all $I_j$, $j = 1, \ldots, n$, are simple and $J_n$ is semisimple, $I_1 \oplus \cdots \oplus I_n = \text{Ann}_C(J_n)$. Since $\text{Cend}_m$ is a (left and right) Noetherian conformal algebra, $C$ satisfies a.c.c. for annihilation ideals. Hence, there exists a sufficiently large integer $n \geq 1$ such that $J_n = 0$. Then $C = I_1 \oplus \cdots \oplus I_n$.  

\begin{corollary}[c.f. [9][22]] Let $C$ be an associative conformal algebra of finite type. If $C$ is simple, then $C \simeq \text{Cur}_N$. If $C$ is semisimple, then $C$ is a direct sum of simple ideals.  
\end{corollary}
6. Open problems

Let us complete the paper with listing some open problems closely related with the considered one.

I. Describe all irreducible subalgebras of $M_N(W)$ with respect to the canonical action on $(k[p])^N$.

Theorems 4.1.2 and 5.1.1 describe the class of irreducible subalgebras of $M_N(W)$ related with conformal subalgebras of $Cend_N$. It is unclear, what would be the answer in general.

II. Classify irreducible infinite type Lie conformal subalgebras of the Lie conformal algebra $gc_N = Cend_N^{(-)}$.

Irreducible Lie conformal subalgebras of finite type were described in [13] (see also [22]). In [9], it is conjectured that the following Lie conformal subalgebras of $gc_N$ exhaust all (infinite type) irreducible Lie conformal subalgebras of $gc_N$ up to the conjugation by an automorphism of $Cend_N$ (det $Q 
eq 0$ everywhere):

$$gc_{N,Q} = Cend_{N,Q}^{(-)};$$
$$oc_{N,Q} = \{ a\varphi^{-1}(Q) \mid a \in Cend_N, \sigma(a) = -a \}, \quad Q^t(-v) = Q(v);$$
$$spc_{N,Q} = \{ a\varphi^{-1}(Q) \mid a \in Cend_N, \sigma(a) = a \}, \quad Q^t(-v) = -Q(v).$$

Here $\sigma$ is the anti-involution of $Cend_N$ defined by the rule [9]

$$\sigma(h \otimes A(v)) = \sum_{s \geq 0} (-D)^{(s)} h \otimes \partial_s^e A^{(s)}(v), \quad h \in H, \ A(v) \in M_N(k[v]).$$

It is proved in [13] that every irreducible Lie conformal subalgebra of infinite type which is an $sl_2$-module (with respect to the action of a Virasoro-like element of $gc_N$) is of the type $oc_{N,Q}$ or $spc_{N,Q}$. The result of [30] shows that the conjecture is true for simple irreducible Lie subalgebras that contain $Cur(sl_2)$.

III. Describe finitely generated simple and semisimple conformal algebras of Gel’fand–Kirillov dimension one.

This problem was partially solved in [24] [25] [30]. The conjecture stated in the last paper says that $Cend_{N,Q}$, det $Q \neq 0$, exhaust all (finitely generated) simple associative conformal algebras of Gel’fand–Kirillov dimension one. We believe it is true, although the similar statement does not hold for Lie conformal algebras: $gc_{N,Q}$, $oc_{N,Q}$, $spc_{N,Q}$, together with $Cur(g)$ (where $g$ is a simple finitely generated Lie algebra of Gel’fand–Kirillov dimension one) is not a complete list of simple Lie conformal algebras of Gel’fand–Kirillov dimension one.

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