Twisted $N = 1$, $d = 4$ supergravity and its symmetries

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Abstract

We display the construction of a twisted superalgebra for the $N = 1$ Euclidean supergravity on 4-manifolds with an almost complex structure. It acts on a representation of twisted supersymmetry made of forms with odd and even statistics and it is covariant under a $U(2) \subset SO(4)$ Lorentz invariance of the manifold’s tangent-space. It contains 4 twisted supersymmetry generators, one nilpotent scalar, one vector and one pseudo-scalar. The superalgebra closes on the twisted fields of supergravity in its new minimal set of auxiliary fields. Its couplings to the twisted Wess and Zumino and vector multiplets are also determined.

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1 Introduction

Twisting is an important tool in the study of supersymmetric theories and has given important new insights in those studies. It fundamentally means that one supercharge is singled out and used as the primary symmetry of the theory. The twist often allows for a splitting in the set of supersymmetric generators, which can be very useful. One can find in some cases a subset of the generators that is sufficient to constrain the Lagrangian to be invariant under the full supersymmetry, while it admits off-shell closed field representations.

The first examples have used non-trivial $R$-symmetries associated to extended supersymmetries to retain a full Lorentz invariance. However it has proved useful to consider the twist of $N = 1$ theories, even if it means that only part of the Lorentz symmetry is explicitly realized, a $Spin(7)$ or $U(4)$ symmetry in dimension eight, a $G_2$ symmetry in dimension seven.

Here we consider the case of the simplest four-dimensional supergravity, to illustrate the formalism of twisted symmetry in curved space. We work with an Euclidean signature, which allows us to retain a $U(2)$ subgroup of the rotational symmetry. This same twist has been previously considered in the theory with only global supersymmetry [1].

In the case of the $N = 1$, $d = 4$ Euclidean supergravity, only a subset of the rotational symmetry is explicitly realized after the twist, and spinors are no longer present in the theory. All fields transform as tensorial products of the fundamental representation of $U(2) \subset SO(4)$. The fermionic part of the symmetry algebra consists in four fermionic twisted generators, one scalar, one vector and one pseudo-scalar. The translations are part of the supersymmetry algebra and appear, in the twisted formalism, in the anticommutator of the vector supersymmetry generators and the scalar or pseudo-scalar generators. The twisted generators can be untwisted to recover the spinorial anticommuting generators of Poincaré supergravity.

The twisted superalgebra and the superalgebra of Poincaré supergravity are related in the fact that they define the same invariant action, modulo a twist. Twisted and untwisted supergravity transformation laws can be related by a linear mapping, in a way that generalizes the case of super-Yang–Mills theories [1].

The construction of the twisted superalgebra is done on a 4-manifold with an Euclidean signature and an almost complex structure. In this case, the Majorana spinors can be decomposed in holomorphic and antiholomorphic forms.

Among the twisted fermionic generators, the scalar nilpotent one is of main interest to us. It is formally similar to a BRST operator, and has an analogous interpretation as the twisted supersymmetry generator of topological Yang–Mills symmetry, 2d-quantum gravity or topological string [2]. The supergravity action in a twisted form is in fact determined by the invariance under this scalar supersymmetry, with an interesting decomposition occurring for both the Einstein and Rarita–Schwinger actions. Subtle phenomena arise when one requires the additional invariance under the full $SO(4)$ symmetry group.
Building the twisted superalgebra produces a new interesting framework. First, we mention that supersymmetric invariants exist as non-trivial local cocycles, a property that might be of significant importance if the twisted construction can be extended to supergravities of rank \( N \geq 2 \). Second, the fact that the invariance under the twisted scalar supersymmetry generator alone is enough to write down an action for the twisted fields might be of interest to bypass the issues raised by the lack of a system of auxiliary fields in theories such as higher dimensional supergravities. It could be that requiring the off-shell closure of the complete Poincaré superalgebra is just too demanding. Within this approach, the super-Poincaré symmetry is not postulated, but is an emergent property once the invariance under the twisted scalar supersymmetry is imposed.

In view of these hypothetical higher dimensional generalizations, we have computed the twisted formulation for the couplings of supergravity to scalar and vector multiplets. The results are less aesthetic than those obtained for the genuine supergravity multiplet, but their existence is a plausible four-dimensional signal that twisted formulations could also be obtained in \( 2n \geq 4 \) dimensions, with a corresponding \( U(2) \to U(n) \) generalization.

The scheme of the paper is as follows. In section 2 we recall some known facts about \( N = 1, d = 4 \) supergravity in the new minimal scheme, focusing on the BRST formulation of its symmetries. In section 3, we display a possible (anti)selfdual decomposition of the supergravity action by exploring some properties of the Einstein and Rarita–Schwinger Lagrangians. In sections 4 and 5, the twisted formalism is introduced through definitions of the twisted fields and the twisted operators corresponding to the symmetries of the supergravity action. The various curvatures needed to build the supergravity action are also displayed in twisted form. In section 6, we use the so-called 1.5 order formalism to build the twisted scalar symmetry generator for all fields but the spin-connection and give a primitive twisted form of the supergravity action. In section 7, we explore the consequences of requiring the invariance of the action under the twisted vector symmetry, which eventually yields the complete twisted supergravity action. In section 8, we compute the coupling to twisted supergravity of the twisted Wess–Zumino and vector multiplets. Finally, appendices give useful formulas.

2 \( N = 1, d = 4 \) supergravity in the new minimal scheme

The \( N = 1, d = 4 \) supergravity multiplet in the new minimal system of auxiliary fields \([3, 4, 5]\) is

\[
e^a, \lambda, \omega^{ab}, A, B_2
\]

Here \( e^a \) is the 1-form vielbein, the Majorana spinor \( \lambda = \lambda_\mu dx^\mu \) is the 1-form gravitino and \( \omega^{ab} \) is the spin-connection 1-form. \( A \) and \( B_2 \) are auxiliary fields, with gauge invariances, such that the multiplet has as many bosonic and fermionic degrees of freedom both on-shell and off-shell,
modulo the gauge invariances. The abelian 1-form gauge field $A \sim A + dc$ gauges chirality and $B_2 \sim B_2 + d\Lambda_1, \Lambda_1 \sim \Lambda_1 + d\Lambda_0$ is a gauge real 2-form.

The associated curvatures are

\[
R^{ab} = d\omega^{ab} + \frac{i}{2} [\omega, \omega]^{ab} \\
T^a = de^a + \omega^{ab} e_b + \frac{i}{2} \bar{\lambda} \gamma^a \lambda \\
\rho = d\lambda + \left( \frac{i}{2} \omega^{ab} \gamma_{ab} + A \gamma^5 \right) \lambda \\
G_3 = dB_2 + \frac{i}{2} \bar{\lambda} \gamma^a \lambda e_a \\
F = dA
\]

We will often use the covariant derivative notation $D \equiv d + \omega + A$. We use the following expression of the $N = 1$ supergravity action, as in [6]

\[
I = \int_{M_4} \left( \frac{1}{4} e^{abcd} e^a \wedge e^b \wedge R^{cd} \omega + i \bar{\lambda} \wedge e^a \wedge (\rho(\lambda, \omega, A) \wedge e_a - 2B_2 \wedge dA + *G_3 \wedge G_3) \right)
\]

The multiplet (1) is an off-shell balanced multiplet with 6 bosonic degrees of freedom defined modulo all gauge invariances, 12 fermionic ones, and 6 auxiliary ones, according to the following count:

$e^a : 6 = 16 - 6$ Lorentz - 4 reparametrizations

$\lambda : 12 = 16 - 4$ supersymmetries

$A : 3 = 4 - 1$ chiral

$B_2 : 3 = 6 - 4$ vector + 1 scalar

The spin-connection is not an independent field, but is fixed by the (super)covariant constraint

\[
T^a(e, \lambda) = -\frac{1}{2} G_{bc} e^b e^c
\]

so that $\omega^{ab} = \omega^{ab}(e, \lambda, B_2) \equiv \omega^{ab}(e, \lambda) + \frac{i}{2} G_{bc} e^b e^c$, where $\omega^{ab}(e, \lambda)$ is the usual spin-connection seen as a function of the vielbein and gravitino. This necessary constraint expresses the fact that no first-order formalism exists for getting an off-shell closed Poincaré supersymmetry and an invariant action.

The transformation laws of the various fields under supersymmetry can be expressed using a BRST symmetry operator $s$, where one replaces all parameters of supergravity infinitesimal transformations by local ghost fields with opposite statistics. All ghosts transform under the BRST symmetry, in such a way that $s$ is nilpotent. The nilpotence of $s$ is equivalent to the off-shell closure of the system of supergravity infinitesimal transformations, as shown in [6]. This BRST symmetry can be built directly (both in the minimal and new minimal set of auxiliary fields), as outlined below.
Call $\xi^\mu$ the vector ghost for reparametrization. The other ghosts are those of local SUSY (\(\chi\)), Lorentz symmetry (\(\Omega\)), the chiral \(U(1)\) symmetry (\(c\)) and the 2-form gauge symmetry (\(B_1^\lambda\)).

The $\xi^\mu$-dependent part of the supergravity BRST algebra decouples by redefining

$$\hat{s} = s - \mathcal{L}_\xi, \quad \hat{d} = d + \hat{s} + i\phi,$$

where the vector field $\phi$ is a bilinear in the supersymmetry ghost $\chi$

$$\phi^\mu = -\frac{i}{2}\bar{\chi}\gamma^\mu \chi = s\xi^\mu - \xi^\nu \partial_\nu \xi^\mu,$$

\(i_V\) is the interior derivative on the manifold for a given vector $V$ and $\mathcal{L}$ is the Lie derivative, $\mathcal{L}_V = i_V d + d i_V$. One has the important property

$$\hat{d} = \exp(-i\xi)(d + s) \exp(+i\xi)$$

which ensures that $(d + s)^2 = 0$ and $\hat{d}^2 = 0$ are equivalent, and $s^2 = 0 \Leftrightarrow \hat{s}^2 = \mathcal{L}_\phi$. The supergravity BRST transformations can be obtained by imposing constraints on the curvatures (2), in a way that merely generalizes the Yang–Mills case. Using ghost unification allows for a direct check of the off-shell closure by means of the Bianchi identities. In the end, one finds the following action of the BRST operator $\hat{s}$ on the fields:

$$\hat{s}e^a = -\Omega^{ab}e_b - i\bar{\chi}\gamma^a \lambda$$
$$\hat{s}\lambda = -D\chi - \Omega^{ab}\gamma_{ab} \lambda - c\gamma^5 \lambda$$
$$\hat{s}B_2 = -dB_1^b - i\bar{\chi}\gamma^a e_a$$
$$\hat{s}A = -dc - \frac{1}{2}i\bar{\chi}\gamma^5 \gamma^a X_a$$
$$\hat{s}\omega^{ab} = -(D\Omega)^{ab} - i\bar{\chi}\gamma^{[a} X^{b]}$$

where the spinor $X_a$ is

$$X_a = \rho_{abe} e^b - \left(\frac{1}{2}G_{abc}\gamma^{bc} + \frac{1}{12}\epsilon_{abcd}G^{bcd}\gamma^5\right) \lambda$$

The ghost transformation laws can be found in Appendix A. They are such that the closure relation $s^2 = 0 \Leftrightarrow \hat{s}^2 = \mathcal{L}_\phi$ is satisfied. The way the BRST symmetry transforms the supersymmetry ghost will have non-trivial consequences in the twisted formulation.

By using the twist formulas of Majorana spinors as in [1, 7, 8, 9, 10, 11], one could analytically continue and twist by brute force these transformations in Euclidean space.

We will rather try to obtain the twisted formulation in a more straightforward way, so as to unveil and better understand the mechanisms taking place in the twisted formalism. Therefore, we now proceed to our direct construction of the twisted superalgebra, keeping in mind that both untwisted and twisted formulations can be compared at any given stage.

As we will see, the whole information about supergravity is actually contained in the twisted scalar nilpotent generator that is hidden in the Poincaré supersymmetry algebra. To reach this result, we need to separate both the Einstein and Rarita–Schwinger Lagrangians in parts depending only on the selfdual or the antiselfdual parts of the spin-connection.
3 Selfdual decomposition of the supergravity action

Each of the Einstein and Rarita–Schwinger Lagrangians can be naturally split into two parts, one that only depends on the selfdual components of the spin-connection while the other one depends on the antiselfdual ones. These two parts are equal modulo suitable boundary terms. In the case of the Einstein Lagrangian, this property was already used for other types of twisting [11].

The Einstein Lagrangian can be written as

\[ L_E = \frac{1}{4} \epsilon_{abcd} e^a e^b R^{cd} = \frac{1}{2} e^a e^b (R^+_{ab} - R^-_{ab}) \] (10)

Since the so(4) Lie algebra splits into two parts, the selfdual components of the curvature \( R^{\pm ab} = d\omega^{\pm ab} + \omega^{\pm a}_c \omega^{\pm cb} \) only depend on the components of the spin-connection \( \omega^{\pm ab} \) with the same selfduality.

In supergravity, the torsion is often taken to be \( T_a = De_a + \frac{i}{2} \bar{\lambda} \gamma_a \lambda \), but to establish the equality between the two parts of the Einstein Lagrangian, it is simpler to also use the purely bosonic torsion \( t_a \equiv De_a \) which satisfies the Bianchi identity \( Dt_a = R_{ab} e^b \). Indeed, contracting this identity with \( e^a \), one has:

\[ e^a Dt_a = e^a e^b (R^+_{ab} + R^-_{ab}) \] (11)

while

\[ D(e^a t_a) = t^a t_a - e^a Dt_a \] (12)

One then gets:

\[ L_E = -e^a e^b R^-_{ab} + \frac{1}{2} t^a t_a - \frac{1}{2} d(e^a t_a) \]
\[ = -e^a e^b R^-_{ab} - \frac{i}{2} \bar{\lambda} \gamma^a \lambda T_a + \frac{1}{2} T^a T_a - \frac{1}{2} d(e^a T_a - \frac{i}{2} e^a \bar{\lambda} \gamma_a \lambda) \] (13)
\[ = +e^a e^b R^+_{ab} + \frac{i}{2} \bar{\lambda} \gamma^a \lambda T_a - \frac{1}{2} T^a T_a + \frac{1}{2} d(e^a T_a - \frac{i}{2} e^a \bar{\lambda} \gamma_a \lambda) \] (14)

The second line is obtained by expressing \( t_a \) in terms of \( T_a \), remembering that \( \bar{\lambda} \gamma^a \lambda \bar{\lambda} \gamma_a \lambda = 0 \)

when \( \lambda \) is a Majorana spinor.

Since \( T^a \) is constrained to be zero or a quantity independent of the spin-connection, the expressions obtained for the Einstein action only depend on the antiselfdual part \( \omega^{-ab} \) (in the case of Eq. (13)) or the selfdual part \( \omega^{+ab} \) (for Eq. (14)) of the spin-connection.

An analogous property holds true for the Rarita–Schwinger Lagrangian. One can derive it using the decomposition of the gravitino on its chiral components (which are not independent for a Majorana spinor). Defining \( \lambda = \lambda^+ + \lambda^- \) with \( \lambda^\pm = \frac{1}{2} (1 \pm i \gamma^5) \lambda \), one writes \(^2\)

\[ L_{RS} = i \bar{\lambda} \gamma^5 \gamma^a \rho e_a = \bar{\lambda}^+ \gamma^a \rho^- e_a - \bar{\lambda}^- \gamma^a \rho^+ e_a \] (15)

\(^1\)Our conventions for (anti)selfdual tensors are collected in Appendix B.

\(^2\)See Appendix B for the details of our chirality conventions.
using $\lambda^\pm \gamma^a \lambda^\pm = 0$. By adding a suitable total divergence, one gets

$$L_{RS} = 2\bar{\lambda}^+ \gamma^a \rho^- e_a - \bar{\lambda}^- \gamma^a \lambda^+ T_a + d(\bar{\lambda}^- \gamma^a \lambda^+)$$  \hspace{1cm} (16)

With anticommuting Majorana fermions, we have the identity $\bar{X}^- \gamma_a Y^+ = -Y^+ \gamma_a X^-$. Since the chiral projections commute with the generators of Lorentz transformations on spinors, we simply have $\rho^- = D(\lambda^-)$. Chiral fermions give the minimal representations of the subalgebras associated to the selfdual and antiselfdual parts of the rotation generators, so that $\rho^-$ only depends on the antiselfdual part of the spin-connection $\omega^{-ab}$:

$$\rho^- = \left( d + \frac{1}{2} \omega^{-ab} \gamma_{ab} + iA \right) \lambda^-$$ \hspace{1cm} (17)

The Rarita–Schwinger action can therefore be written as

$$I_{RS} = i \int \bar{\lambda}^+ \gamma^a D^{(\omega)} \lambda e_a = \int 2\bar{\lambda}^+ \gamma^a D^{(\omega^-)} \lambda^- e_a - \bar{\lambda}^- \gamma^a \lambda^+ T_a$$ \hspace{1cm} (18)

We succeeded in expressing $I_E + I_{RS}$ in a way that only depends on either the selfdual or the antiselfdual part of the spin-connection, whenever the constraint on the torsion is independent of the spin-connection. This condition is necessary for the closure of the supersymmetry algebra acting on the vielbein.

4 Twisted supergravity variables

In order to be able to twist the theory, we must work in a Euclidean space with an almost complex structure, \textit{i.e.} a map on each tangent space $J(x)$ with $J^2 = -1$, or more explicitly $J^\mu_i(x)J^\nu_j(x) = -\delta^\nu_j^\mu_i$.

Introducing complex coordinates $z_m, \bar{z}_{\bar{m}}$, where $m = 1, 2$, one can locally reduce the complex structure to a diagonal one, $J_m^n = i\delta_m^n, J_{\bar{m}}^{\bar{n}} = -i\delta_{\bar{m}}^{\bar{n}}$. Making use of a compatible metric to lower one of the indices in $J$, $J$ becomes an antisymmetric tensor with $J_{m\bar{n}}$ as the only non-vanishing components.

The tensor $J_{m\bar{n}}$ can be used instead of the metric to lower and raise indices in the tangent space, according to $X^m = -iJ^{m\bar{n}} \bar{X}_{\bar{n}}$ and $X^{\bar{n}} = iJ^{\bar{m}m} X_m$. In order to keep our formulas as uncluttered as possible, we will use a notation similar to Einstein’s notation for contracting antiholomorphic and holomorphic $SU(2)$-indices by means of the complex structure constant tensor, as follows

$$X^a Y_a = g^{ab} X_a Y_b = -iJ^{m\bar{n}} (X_m Y_{\bar{n}} + X_{\bar{n}} Y_m) \equiv X_m Y_{\bar{n}} - X_{\bar{n}} Y_m$$ \hspace{1cm} (19)

The antisymmetry of the tensor $J_{m\bar{n}}$ implies that one must be careful about the ordering of indices. It explains the minus sign appearing in the last term of Eq. (19).

\footnote{Care must be taken in Minkowski space where the conjugation changes chirality, so that for example $\bar{X}^+ = \bar{\lambda}^+$.}
Twisting must be done in Euclidean space where it is known that there are no Majorana spinors. We therefore forget the Majorana condition, the effect of which can be recovered afterwards from a careful consideration of the Wick rotation [12]. We associate to the spinor \((\lambda^\alpha, \lambda_{\dot{\alpha}})\) the following four quantities with only holomorphic or antiholomorphic indices:

\[(\Psi_m, \Psi_{\bar{m}n}, \Psi_0)\] (20)

The indices \(m\) and \(\bar{m}\) take two different values and the object \(\Psi_{\bar{m}n}\) is antisymmetric in its indices, so that it only has one non-zero component.

The twisted components of a spinor (20) are defined from the following linear mapping, which uses Pauli matrices elements [1, 7, 8, 9, 10, 11]:

\[
\begin{align*}
\Psi_m &= \lambda^\alpha (\sigma_m)_{\alpha 1} \\
\Psi_{\bar{m}n} &= \bar{\lambda}_{\dot{\alpha}} (\sigma_{\bar{m}n})^{\dot{\alpha} 2} \\
\Psi_0 &= \bar{\lambda}_{\dot{\alpha}} \delta^{\dot{\alpha} 2}
\end{align*}
\] (21)

In Appendix C, we give the expression of the twist of \(\Gamma \lambda\) as functions of the twisted components of \(\lambda\) for some elements \(\Gamma\) of the Clifford algebra.

This construction reduces the tangent space \(SO(4)\) symmetry into an \(SU(2) \times U(1) \subset SO(4)\) symmetry. With this change of variables, \(SO(4)\)-invariant expressions can be related to their twisted counterparts, which generally split into a sum of independently \(U(2)\)-invariant terms. For instance, the Rarita–Schwinger Lagrangian can be decomposed as follows

\[
\bar{\lambda} \gamma_5 \gamma^a p e_a = (\Psi_0 \rho_m + \Psi_m \rho_0) e_{\bar{m}} - (2\Psi_{\bar{m}n} \rho_n - \Psi_n \rho_{\bar{m}n}) e_m
\] (22)

The commuting Majorana ghost of local supersymmetry \(\chi\) is twisted as follows

\[
\chi \sim (\chi_m, \chi_{\bar{m}n}, \chi_0)
\] (23)

and the vector field in Eq. (6), \(\phi^\mu = -\frac{i}{2} \bar{\chi} \gamma^\mu \chi = s \xi^\mu - \xi^\nu \partial_\nu \xi^\mu\) is now given by

\[
\phi_m = -\chi_m \chi_0, \quad \phi_{\bar{m}} = -\chi_{\bar{m}n} \chi_n
\] (24)

When the parameter of vector supersymmetry vanishes, \(\chi_m = 0\), then the vector field \(\phi\) vanishes 4.

A consistent interpretation of the twisted supersymmetry only involves fermionic global charges. Thus, in what follows, \(\chi_m, \chi_{\bar{m}n}, \chi_0\) will be treated as constant ghosts. We will build a set of corresponding generators \(\delta_{\bar{m}}, \delta_{mn}, \delta\) that satisfy anticommutation relation that close

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4This condition means that \(\chi\) is a pure spinor, and it is not surprising that it entails great simplifications in the formalism, as in [13].
independently of the equations of motion (off-shell closure), but possibly modulo bosonic gauge
transformations. We will consider the operation

\[ Q = \chi_m \delta \bar{m} + \chi_{\bar{m}n} \delta_{mn} + \chi_0 \delta \]

(25)

For a vanishing gravitino field, \( Q \) is nilpotent, off-shell and modulo bosonic gauge transformations. It turns out that the global \( Q \)-invariance is a sufficiently strong condition to determine the supergravity action. In fact, it gives a Ward identity that is sufficient to control the quantum perturbative behavior of the theory generated by the \( Q \)-invariant action, once all its gauge invariances are gauge-fixed in a BRST invariant way. When the gravitino field is not zero, the closure algebra is more involved. We will see that it involves supersymmetry transformations with gravitino field dependent structure coefficients.

In this construction, the supergravity action is however fully determined by the global supersymmetry operation \( Q \). Local supersymmetry is warranted due to the systematic construction of the charges \( \delta \bar{m}, \delta_{mn}, \delta \) in a way that is compatible with the Bianchi identities of all field curvatures.

The four generators \( (\delta, \delta \bar{m}, \delta_{mn}) \) must act on all the twisted fields of the multiplet (1), with the following \( g \)-grading assignments.

| Field  | Grading | Field  | Grading |
|--------|---------|--------|---------|
| \( e_m \) | 0       | \( A \) | 0       |
| \( e_{\bar{m}} \) | 0       | \( B_2 \) | 0       |
| \( \Psi_m \) | 1       | \( \omega_{mn} \) | 0       |
| \( \Psi_0 \) | -1      | \( \omega_{m\bar{m}} \) | 0       |
| \( \Psi_{\bar{m}n} \) | -1      | \( \omega_{m\bar{m}} \) | 0       |

The commutation properties of the various fields are always obtained by computing the sum of the form degree and the grading \( g \) of fields (for instance \( e_m \) is an anticommuting object since the form degree is one and \( g = 0 \), \( \Psi_m \) is a commuting object since the form degree is one and \( g = 1 \), etc.). After having obtained a classical action that is invariant under the twisted nilpotent global supersymmetry \( Q \), one must in principle check that it remains invariant under local supersymmetry by giving a coordinate dependence to \( (\chi_0, \chi_m, \chi_{\bar{m}n}) \). This is in fact automatically realized, since all derivatives will appear as super-covariantized ones.

If we now generalize \( (\chi_0, \chi_m, \chi_{\bar{m}n}) \) into local commuting (twisted) Faddeev Popov ghosts,
one gets the operator
\[ \hat{s} = \chi_0(x)\delta + \chi_m(x)\delta_m + \chi_{\bar{m}\bar{n}}(x)\delta_{mn} \] (26)

Its action on the classical fields is the same as that of the standard BRST transformations in twisted form.

In the flat space \( N = 1 \) super-Yang–Mills theory [1], the three nilpotent symmetry generators \( \delta \) and \( \delta_p \) satisfy the off-shell closure anticommutation relations \( \delta^2 = 0, \{\delta_p, \delta_q\} = 0, \{\delta, \delta_p\} = \partial_p \).

The situation is more complicated in supergravity. In this case, one has indeed the property \( \hat{s}^2 = \mathcal{L}_\phi \), where the vector field \( \phi \) has been defined in (24). One has also the transformation law \( \hat{s}\chi \sim i\phi\Psi \) (see Appendix A), which remains true even when the supersymmetry ghosts are assumed to be constant. This implies the following supergravity generalization
\[ \delta^2 = 0, \{\delta_p, \delta_q\} = 0, \{\delta, \delta_p\} = \mathcal{L}_\phi - \sum_{a=0,m,\bar{m}} \Psi_{\bar{p},a}\delta_a \] (27)

These anticommutation relations hold modulo bosonic gauge transformations. The derivative \( \mathcal{L}_\phi \) is the Lie derivative along the vector field dual to the vielbein component \( e_{\bar{p}} \).

In fact, the supersymmetry generators that occur in the expression of \( \{\delta, \delta_p\} \) occur proportionally to the components \( \Psi_{\bar{p},a} \) of the gravitino field \( \Psi_a \equiv \Psi_{m,a}e_m - \Psi_{\bar{m},a}e_{\bar{m}} \).

One thus recognizes the expected feature of supergravity: the anticommutator \( \{\delta, \delta_p\} \) closes on supersymmetry generators with field-dependent coefficients, proportionally to gravitino-field components.

Therefore, it is expected that the anticommutator \( \{\delta, \delta_q\} \) involves the fourth symmetry generator \( \delta_{pq} \), whose existence can be checked afterwards in the twisted method.

Once \( \delta \) and \( \delta_p \) are determined, the \( \delta \) and \( \delta_p \) invariant action turns out to be automatically invariant under a \( \delta_{pq} \) symmetry. In the four-dimensional supergravity, the relations between \( \delta_{pq} \) and the other generators \( \delta \) and \( \delta_p \) are satisfied off-shell.

The fermionic scalar operator \( \delta \) can be extended as a globally well-defined object (provided there is a complex structure). We will mainly focus on the question of its direct construction. In fact, \( \delta_p \) and \( \delta_{pq} \) can only be given a geometrical interpretation on a coordinate patch.

5 The supergravity curvatures in the \( U(2) \subset SO(4) \) invariant formalism

In section 3, we have shown that both the Einstein and Rarita–Schwinger actions only depend on the selfdual or antiselfdual components of the spin-connection. In \( SU(2) \subset SO(4) \) notations, the selfduality condition of an antisymmetric Lorentz tensor, \( F_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd} \) reads
\[ F_{\bar{m}\bar{n}} = F_{mn} = 0 \quad F_{m\bar{n}} \equiv iF^{m\bar{n}}F_{m\bar{n}} = 0 \] (28)
while the antiselfduality condition \( F_{ab} = -\frac{1}{2} \epsilon_{abcd} F^{cd} \) reads
\[
F_{\bar{m}\bar{n}} - iJ_{\bar{m}\bar{n}} F_{\bar{p}\bar{p}} = 0
\]
(29)

Thus, the spin-connection \( \omega^{ab} = \omega^{+ab} + \omega^{-ab} \equiv (\omega_{mn}, \omega_{\bar{m}\bar{n}}, \omega_{m\bar{n}}) \) splits in selfdual and antiselfdual parts, respectively
\[
\omega^{+ab} \sim (0, 0, \omega_{m\bar{n}} - iJ_{m\bar{n}} \omega)
\]
\[
\omega^{-ab} \sim (\omega_{mn}, \omega_{\bar{m}\bar{n}}, iJ_{m\bar{n}} \omega)
\]
(30)

where \( \omega \equiv iJ_{m\bar{n}} \omega_{m\bar{n}} \).

The \( \text{SO}(4) \) Lie algebra is the product of two \( \text{SU}(2) \) corresponding to the selfdual and antiselfdual generators. Therefore, the antiselfdual part of the curvature 2-form \( R^- \sim (R_{mn}, R_{\bar{m}\bar{n}}, R) \) and its Bianchi identities only depend on the antiselfdual part of the connection \( \omega^{-ab} \):
\[
R = d\omega + 2\omega_{mn}\omega_{\bar{m}\bar{n}}
\]
\[
R_{mn} = d\omega_{mn} - \omega_{\bar{m}\bar{n}}
\]
\[
R_{\bar{m}\bar{n}} = d\omega_{m\bar{n}} + \omega_{m\bar{n}}
\]
\[
dR = 2R_{mn}\omega_{\bar{m}\bar{n}} - 2\omega_{mn}R_{\bar{m}\bar{n}}
\]
\[
dR_{mn} = R_{mn}\omega - R\omega_{mn}
\]
\[
dR_{\bar{m}\bar{n}} = R_{m\bar{n}}\omega + R\omega_{m\bar{n}}
\]
(31)

The \( \text{SO}(4) \) symmetry only acts as this \( \text{SU}(2) \) on \( \Psi_0 \) and \( \Psi_{\bar{m}\bar{n}} \), due to chirality properties. One can thus define the \( \text{SU}(2) \) covariant curvatures for \( \Psi_0 \) and \( \Psi_{\bar{m}\bar{n}} \)
\[
\rho_0 = d\Psi_0 - \left( \frac{1}{2} \omega - A \right) \Psi_0 + \omega_{mn} \Psi_{\bar{m}\bar{n}}
\]
\[
\rho_{\bar{m}\bar{n}} = d\Psi_{\bar{m}\bar{n}} + \left( \frac{1}{2} \omega + A \right) \Psi_{\bar{m}\bar{n}} - \omega_{m\bar{n}} \Psi_0
\]
(32)

Their Bianchi identities are
\[
D\rho_0 = \left( -\frac{1}{2} R + F \right) \Psi_0 + R_{mn} \Psi_{\bar{m}\bar{n}}
\]
\[
D\rho_{\bar{m}\bar{n}} = \left( \frac{1}{2} R + F \right) \Psi_{\bar{m}\bar{n}} - R_{\bar{m}\bar{n}} \Psi_0
\]
(33)

The curvature \( \rho_m \) of \( \Psi_m \) only involves the selfdual part of the spin-connection. We can skip its definition, since it is not needed in the supergravity action.

The torsion involves both selfdual and antiselfdual components of the spin-connection
\[
T_m = de_m + \omega_{mn} e_{\bar{n}} - \omega_{m\bar{n}} e_n + \Psi_m \Psi_0
\]
\[
T_{\bar{m}} = de_{\bar{m}} + \omega_{m\bar{n}} e_n - \omega_{\bar{m}\bar{n}} e_m + \Psi_{\bar{m}} \Psi_n
\]
(34)

\[
DT_m = R_{mn} e_{\bar{n}} - R_{m\bar{n}} e_n + \rho_m \Psi_0 - \Psi_m \rho_0
\]
\[
DT_{\bar{m}} = R_{mn} e_{\bar{n}} - R_{m\bar{n}} e_n + \rho_{\bar{m}} \Psi_n - \Psi_{\bar{m}} \rho_n
\]
(35)
We now use the $SU(2)$ notation to decompose the Einstein and Rarita–Schwinger Lagrangians as a sum of terms that are separately $SU(2)$ invariant, using the expressions (13) and (18)

\[ I_E = \int - \left( Re_m e_{\bar{m}} + R_{mn} e_{\bar{m}} e_n + R_{\bar{m}\bar{n}} e_{\bar{m}} e_n \right) - \left( \Psi_m \Psi_0 T_{\bar{m}} - \Psi_{\bar{m}} \Psi_n T_m \right) + T_m T_{\bar{m}} \]  

(36)

\[ I_{RS} = \int - \left( 2 \rho_{\bar{m}n} \Psi_n e_m - 2 \rho_0 \Psi_m e_{\bar{m}} \right) + \left( \Psi_m \Psi_0 T_{\bar{m}} - \Psi_{\bar{m}} \Psi_n T_m \right) \]  

(37)

Eqs. (36) and (37) are interesting. However, at first sight, they are not yet very suggestive about the existence of a twisted scalar supersymmetry.

In fact, to build the scalar supersymmetry, we depart from the method used in [6]. The so-called 1.5 order formalism, once adapted to the twisted fields of supergravity, will neatly separate the various terms of the invariant actions (36) and (37).

6 1.5 order formalism with $SU(2)$ covariant curvatures

The justification of the 1.5 order formalism for supergravity is detailed in [14]. One first builds a supersymmetry that acts on all fields but the spin-connection $\omega$. The later is taken not to transform under supersymmetry in a first step.

The second order formalism transformation law of $\omega$ is the one compatible with all Bianchi identities of the theory, including that of the Riemann curvature.

In the 1.5 order formalism, it is particularly simple to obtain the twisted scalar supersymmetry on all fields but the spin-connection, by imposing consistent constraints on the ghost-dependent curvatures.

The ghost-dependent curvatures are obtained by the substitutions

\[ d \rightarrow \hat{d} = d + \chi_0 \delta_{1.5} + i \phi, \quad \Psi \rightarrow \hat{\Psi} = \Psi + \chi \]  

(38)

We are only concerned with the scalar supersymmetry for the moment. Thus, we only retain a constant $\chi_0$ as the only non-vanishing component in $\chi$. Since $\chi_m = 0$, one has $\phi_m = \phi_{\bar{m}} = 0$ and $\hat{d} = d + \chi_0 \delta_{1.5}$. The property $\hat{d}^2 = 0$ implies $\delta^2_{1.5} = 0$ on all fields. The 1.5 order formalism constraints that are compatible with the Bianchi identities are

\[ \hat{R} = R, \quad \hat{R}_{mn} = R_{mn}, \quad \hat{R}_{\bar{m}\bar{n}} = R_{\bar{m}\bar{n}}, \quad \hat{F} = F \]

\[ \hat{\rho}_0 = \rho_0, \quad \hat{\rho}_{\bar{m}n} = \rho_{\bar{m}n}, \quad \hat{\rho}_m = \rho_m \]

\[ \hat{G}_3 = G_3, \quad \hat{T}_m = T_m, \quad \hat{T}_{\bar{m}} = T_{\bar{m}} \]  

(39)

where $G_3$ is the field strength of the 2-form $B_2$, defined in twisted form as

\[ \hat{G}_3 = \hat{d}B_2 + \hat{\Psi}_m \hat{\Psi}_0 e_{\bar{m}} - \hat{\Psi}_{\bar{m}} \hat{\Psi}_n e_m \]  

(40)
We now use Eq. (38) and pick up the term with ghost number one in Eq. (39). This gives the \( \delta_{1.5} \)-transformation laws for all fields:

\[
\begin{array}{|c|c|}
\hline
\text{field} & \delta_{1.5} \\
\hline
\epsilon_m & -\Psi_m \\
\epsilon_\bar{m} & 0 \\
\bar{\Psi}_m & 0 \\
\Psi_0 & \frac{1}{2} \omega - A \\
\bar{\Psi}_{\bar{m}\bar{n}} & \omega_{\bar{m}\bar{n}} \\
\omega_{mn} & 0 \\
\omega_{\bar{m}\bar{n}} & 0 \\
\omega & 0 \\
A & 0 \\
B_2 & -\Psi_m e_\bar{m} \\
\hline
\end{array}
\]  

(41)

The curvatures transform as

\[
\begin{align*}
\delta_{1.5} R &= 0 \\
\delta_{1.5} R_{mn} &= 0 \\
\delta_{1.5} R_{\bar{m}\bar{n}} &= 0 \\
\delta_{1.5} F &= 0 \\
\delta_{1.5} \rho_0 &= -\frac{1}{2} R + F \\
\delta_{1.5} \rho_{\bar{m}\bar{n}} &= -R_{\bar{m}\bar{n}} \\
\delta_{1.5} \rho_m &= 0
\end{align*}
\]  

(42)

We can therefore build three \( \delta_{1.5} \)-invariant Lagrangians that respectively contain the three independent \( SU(2) \)-invariant pieces \( R_{mn} \epsilon_m \epsilon_\bar{n} \), \( R_{m\bar{n}} \epsilon_m \epsilon_\bar{n} \) and \( R_{\bar{m}\bar{n}} \epsilon_m \epsilon_\bar{n} \) of the Einstein Lagrangian:

\[
\begin{align*}
R_{mn} \epsilon_m \epsilon_\bar{n} \\
R_{m\bar{n}} \epsilon_m \epsilon_\bar{n} + 2 \rho_{m\bar{n}} \Psi_n \epsilon_m \\
R \epsilon_m \epsilon_\bar{m} - 2 \rho_0 \Psi_m \epsilon_\bar{m} - 2 F B_2
\end{align*}
\]  

(43)

The action

\[
I = \int \alpha R_{mn} \epsilon_m \epsilon_\bar{n} + \beta (R_{m\bar{n}} \epsilon_m \epsilon_\bar{n} + 2 \rho_{m\bar{n}} \Psi_n \epsilon_m) + \gamma (R \epsilon_m \epsilon_\bar{m} - 2 \rho_0 \Psi_m \epsilon_\bar{m} - 2 F B_2)
\]  

(44)

is thus invariant under the transformations (41), for all possible values of the coefficients \( \alpha \), \( \beta \) and \( \gamma \). Lorentz symmetry is obtained when \( \alpha = \beta = \gamma \).

Alternatively, in a method that is closer to the one used in [6], one can directly check the invariance of the action (44) by computing the following quantities, using the Bianchi identities.
for the curvatures:

\[
\begin{align*}
\dot{D}(R_{mn}e_m e_n) &= 2\dot{R}_{mn}(\dot{T}_m - \dot{\Psi}_{\bar{m}\bar{p}}\dot{\Psi}_{\bar{p}})e_n \\
\dot{D}(\dot{R}_{\bar{m}n}e_m e_n) &= 2\dot{R}_{\bar{m}n}(\dot{T}_m - \dot{\Psi}_{m\bar{p}}\dot{\Psi}_{\bar{p}})e_n \\
\dot{D}(\dot{e}_m e_n) &= \dot{R}(\dot{T}_m - \dot{\Psi}_{m\bar{p}} \dot{\Psi}_{\bar{p}})e_m - \dot{\Psi}_{m\bar{p}}(\dot{T}_m - \dot{\Psi}_{m\bar{p}}\dot{\Psi}_{\bar{p}})
\end{align*}
\tag{45}
\]

Taking the part with ghost number 1 of these equations and retaining only \(\chi_0 \neq 0\), one obtains the \(\delta_{1,5}\) transformations of the various terms in the action:

\[
\begin{align*}
\delta_{1,5}(R_{mn}e_m e_n) &= 0 \\
\delta_{1,5}(R_{\bar{m}n}e_m e_n) &= -2R_{\bar{m}n}\Psi_{m\bar{p}} e_n \\
\delta_{1,5}(Re_m e_m) &= -R\Psi_{m\bar{p}} e_m \\
\delta_{1,5}(\rho_{\bar{m}n}\Psi_{m\bar{p}} e_m) &= -R_{\bar{m}n}\Psi_{m\bar{p}} e_m \\
\delta_{1,5}(\rho_0\Psi_{m\bar{p}} e_m) &= (-\frac{1}{2}R + F)\Psi_{m\bar{p}} e_m \\
\delta_{1,5}(FB_2) &= -F\Psi_{m\bar{p}} e_m
\end{align*}
\tag{46}
\]

which ensure that \(\delta_{1,5}(I) = 0\).

The formulas (45) are actually quite useful to directly compute the action of the vector supersymmetry \(\delta_p^{1,5}\), by generalizing to the case where \(\chi_p \neq 0\). Using a ghost expansion as for the scalar symmetry, one gets

\[
\begin{align*}
\delta_p^{1,5}(R_{mn}e_m e_n) &= 2R_{mn}\Psi_{m\bar{p}} e_n \\
\delta_p^{1,5}(R_{\bar{m}n}e_m e_n) &= 2R_{\bar{m}n}\Psi_{m\bar{p}} e_n \\
\delta_p^{1,5}(Re_m e_m) &= -R\Psi_{m\bar{p}} e_m + Re_m\Psi_{m\bar{p}} \\
\delta_p^{1,5}(\rho_{\bar{m}n}\Psi_{m\bar{p}} e_m) &= (-\frac{1}{2}R + F)\Psi_{m\bar{p}} e_m \\
\delta_p^{1,5}(\rho_0\Psi_{m\bar{p}} e_m) &= (-\frac{1}{2}R + F)\Psi_{m\bar{p}} e_m \\
\delta_p^{1,5}(FB_2) &= -F(\Psi_{m\bar{p}} e_m + \Psi_{m\bar{p}} e_m)
\end{align*}
\tag{47}
\]

One finds that \(\delta_p^{1,5}\) is another symmetry of the complete action, provided that \(\alpha = \beta = \gamma\), in which case the \(SU(2)\) symmetry is enlarged to \(SO(4)\).

However, one must be careful in the interpretation of this vector symmetry, since it cannot be obtained by twisting the supersymmetry generators \((Q^\alpha, \bar{Q}_\bar{\alpha})\). Indeed, \(\delta_{1,5}\) and \(\delta_p^{1,5}\) do not have the right anticommutation relations, since \(\{\delta_{1,5}, \delta_p^{1,5}\} \Psi = 0\), in contradiction with the twisted
supersymmetry algebra \((27)\). In fact the 1.5 order formalism, which is useful to determine the invariant action, does not properly define the supersymmetry generators. One must determine the \(\omega\) transformations consistent with the constraints, which appear as equations of motion in the 1.5 order formalism.

With the invariant action \((44)\), the equations of motion of the antiselfdual spin-connection give \(12 = 3 \times 4\) equations that can be solved algebraically to determine the 12 components of the three 1-forms \(\omega_{mn}, \omega_{\bar{m}\bar{n}}\) and \(\omega\), as functions of \(e\) and \(\Psi\). The precise values then depend on the parameters \(\alpha, \beta\) and \(\gamma\).

One can then compute the \(\delta_{1.5}\) transformations of these functions through the chain rule to obtain the transformations of \(\omega_{mn}, \omega_{\bar{m}\bar{n}}, \omega\). Since \(\delta_{1.5}\) is nilpotent on \(e\) and \(\Psi\), this procedure gives a nilpotent transformation in the second order formalism, where \(\omega_{mn}, \omega_{\bar{m}\bar{n}}\) and \(\omega\) are not independent fields.

The case of interest is for the rotationally invariant action \((44)\), which has \(\alpha = \beta = \gamma\). In this case, the spin-connection equations of motion give

\[
\frac{\delta}{\delta \omega} I(e, \Psi, B_2, \omega) = e_m T_m^{(\omega)} = 0 \\
\frac{\delta}{\delta \omega_{mn}} I(e, \Psi, B_2, \omega) = e_{[m} T_{n]}^{(\omega)} = 0 \\
\frac{\delta}{\delta \omega_{\bar{m}\bar{n}}} I(e, \Psi, B_2, \omega) = e_{[m} T_{\bar{n}]}^{(\omega)} = 0
\]

(48)

Here \(T^{(\omega)}\) is a function only of \(\omega\),

\[
T_m^{(\omega)} = de_m + \omega_{mn} e_{\bar{n}} + \Psi_m \Psi_0 \\
T_{\bar{m}}^{(\omega)} = de_{\bar{m}} - \omega_{\bar{m}\bar{n}} e_n + \Psi_{\bar{m}\bar{n}} \Psi_n
\]

These 12 equations fix the 12 components of the antiselfdual part of the spin-connection, \(\omega = \omega(e, \Psi), \omega_{mn} = \omega_{mn}(e, \Psi)\) and \(\omega_{\bar{m}\bar{n}} = \omega_{\bar{m}\bar{n}}(e, \Psi)\), as functions of the vielbein and the twisted gravitino. These components are the antiselfdual parts of the complete spin-connection which satisfy the constraint \(T_m = T_{\bar{m}} = 0\).

As a consequence of the chain rule, \(\omega(e, \Psi), \omega_{mn}(e, \Psi)\) and \(\omega_{\bar{m}\bar{n}}(e, \Psi)\) transform under supersymmetry, and the 1.5 formalism guarantees that

\[
I = - \int R_{mn} e_{\bar{m}} e_{\bar{n}} + (R_{\bar{m}\bar{n}} e_m e_n + 2\rho_{\bar{m}\bar{n}} \Psi_n e_m) + (R e_m e_{\bar{m}} - 2\rho_0 \Psi_m e_{\bar{m}} - 2 F B_2)
\]

(49)

is still supersymmetric.

To avoid the heavy calculations from the chain rule, one can use the formalism used in [6] and determine modified horizontality conditions for the field strengths \(\hat{R}\) and \(\hat{F}\) at ghost numbers 1 and 2, such that the Bianchi identities are satisfied and the constraints are invariant. The
invariance of the constraints is equivalent to the satisfaction of the chain rule. One defines
\begin{align}
\hat{R} &= R + R^{(1)} + R^{(2)} \\
\hat{F} &= F + F^{(1)} + F^{(2)}
\end{align}
while we keep
\begin{align}
\hat{T} &= T \\
\hat{\rho} &= \rho \\
\hat{G}_3 &= G_3
\end{align}

The ghost number two part of the Bianchi identity on the torsion $\hat{T}$ ensures that, when $\chi_m = 0$, $R^{(2)} = F^{(2)} = 0$. The condition $\hat{G}_3 = G_3$ implies $\delta B_2 = -\Psi_m e_{\bar{m}}$ and
\begin{align}
\hat{\rho}_0 &= (d + s)\hat{\Psi}_0 - \left(\frac{1}{2} \hat{\omega} - \hat{A}\right) \hat{\Psi}_0 + \hat{\omega}_{mn} \hat{\Psi}_{\bar{m}\bar{n}} = \rho_0 \\
\hat{\rho}_{\bar{m}\bar{n}} &= (d + s)\hat{\Psi}_{\bar{m}\bar{n}} + \left(\frac{1}{2} \hat{\omega} + \hat{A}\right) \hat{\Psi}_{\bar{m}\bar{n}} - \hat{\omega}_{\bar{m}\bar{n}} \hat{\Psi}_0 = \rho_{\bar{m}\bar{n}}
\end{align}
together with their respective Bianchi identities imply
\begin{align}
R^{(1)} &= 2F^{(1)} \\
R^{(1)}_{\bar{m}\bar{n}} &= 0
\end{align}
Finally, the part with ghost number 1 of the Bianchi identity on $\hat{T}$ (34) implies
\begin{align}
R^{(1)}_{mn} &= -\frac{1}{2} \left( \rho_{p[n,m]} e_p + \rho_{\bar{p}[\bar{n},m]} e_{\bar{p}} \right) \\
R^{(1)} &= (\rho_{p\bar{m},m} e_p + \rho_{\bar{p}\bar{m},m} e_{\bar{p}})
\end{align}
These values of $R^1$ and $F^1$ determine the transformation laws of $\omega$ and $A$, so that the second order scalar supersymmetry transformations that leave invariant the action (49) are

| \delta (\text{with } \delta^2 = 0) | e_m | -\Psi_m | e_{\bar{m}} | 0 | \Psi_m | 0 | \Psi_{\bar{m}\bar{n}} | \omega_{\bar{m}\bar{n}} | \omega_{mn} | -\frac{1}{2} \left( \rho_{p[n,m]} e_p + \rho_{\bar{p}[\bar{n},m]} e_{\bar{p}} \right) | \omega_{\bar{p}\bar{m},m} e_p + \rho_{\bar{p}\bar{m},m} e_{\bar{p}} | A | \frac{1}{2} (\rho_{p\bar{m},m} e_p + \rho_{p\bar{m},m} e_{\bar{p}}) | B_2 | -\Psi_m e_{\bar{m}} |

We used a notation where $\rho_{mn}$, $\rho_{\bar{m}\bar{n}}$ and $\rho_{\bar{m}n}$ are the components of the two-form $\rho$ on the vielbein basis, \textit{i.e.}, $\rho = \frac{1}{2} \left( \rho_{mn} e_{\bar{m}} e_{\bar{n}} + \rho_{\bar{m}\bar{n}} e_m e_n + \rho_{\bar{m}n} e_{\bar{m}} e_n \right)$. The indices on the right of the comma refer to the twisted spinor indices $0, m$ or $\bar{m}\bar{n}$. 15
7 Vector supersymmetry and non-vanishing torsion

There is no vector supersymmetry $\delta \bar{\rho}$ for the action (49) that can satisfy the off-shell closure relation $\{\delta, \delta \bar{\rho}\} = \mathcal{L}_{\bar{\rho}} - \Psi_{\bar{\rho},\alpha} \delta_{\alpha}$. Indeed, suppose that such a symmetry exists. The off-shell closure means $d^2 = (d + \chi_0 \delta + \chi_p \delta \bar{\rho} + i \phi)^2 = 0$, with $\phi_m = -\chi_m \chi_0 \neq 0$. Thus, the Bianchi identity,

$$\hat{d} G_3 = -\hat{\Psi}_m \bar{\rho} \hat{e}_m + \hat{\rho}_m \hat{\Psi}_0 \hat{e}_m + \hat{\Psi}_m \hat{\Psi}_0 \hat{T}_m - \hat{\rho}_{\bar{m}n} \hat{\Psi}_n \hat{e}_m + \hat{\Psi}_{\bar{m}n} \bar{\rho}_n \hat{e}_m - \hat{\Psi}_{\bar{m}n} \hat{\Psi}_n \hat{T}_m$$  \hspace{1cm} (57)

has a non-trivial ghost number 2 part, which is

$$i \phi G_3 = \chi_m \chi_0 T_m.$$  \hspace{1cm} (58)

Therefore, the torsion cannot be taken identically equal to zero, which implies that the Lagrangian found in the previous section must be modified by terms that have an off-shell relevance. To remain in the context of a Lorentz invariant action, we use the following constraints on the torsion, which generalize Eq. (58):

$$T_m = d e_m + \omega_{mn} e_n - \omega_{m\bar{n}} e_n + \Psi_m \Psi_0 = \frac{1}{2} (G_{m\bar{p}q} e_p e_q - G_{mpq} \bar{e}_p e_q)$$  \hspace{1cm} (59)

The value of the spin-connection is therefore changed and the distortion on the horizontality condition (50) becomes:

$$R_{mn}^{(1)} = -\frac{i}{2} (\rho_{p[n,m]} e_p + \rho_{[n,m]} e_p + G_{mn} \Psi_p)$$  \hspace{1cm} (60)

The scalar supersymmetry transformations are now:

| $\delta$ (with $\delta^2 = 0$) |
|-----------------------------|
| $e_m$ | $-\Psi_m$ |
| $e_{\bar{m}}$ | 0 |
| $\Psi_m$ | 0 |
| $\Psi_0$ | $\frac{1}{2} \omega - A$ |
| $\Psi_{\bar{m}n}$ | $\omega_{\bar{m}n}$ |
| $\omega_{mn}$ | $-\frac{1}{2} (\rho_{p[n,m]} e_p + \rho_{[n,m]} e_p + G_{mn} \Psi_p)$ |
| $\omega_{\bar{m}n}$ | 0 |
| $\omega$ | $\rho_{\bar{p}m,n} e_p + \rho_{pm,m} e_p + G_{m\bar{p}n} \Psi_p$ |
| $A$ | $\frac{1}{2} (\rho_{\bar{p}m,n} e_p + \rho_{pm,m} e_p + G_{m\bar{p}n} \Psi_p)$ |
| $B_2$ | $-\Psi_m e_{\bar{m}}$ |
With $T \neq 0$, the variation of the action found in the previous section involves new terms proportional to $T \delta \omega$, with must be compensated by the variation of new terms quadratic in $G$.

One has

$$
\delta G_{m\bar{p}q} = \rho_{\bar{q}m,\bar{p}} - G_{m\bar{p}q} \Psi_{\bar{q},r} - 2G_{m\bar{r}q} \Psi_{p,r},
$$

$$
\delta G_{m\bar{p}q} = \rho_{\bar{q}m,\bar{p}} - 2G_{m\bar{r}q} \Psi_{p,r},
$$

$$
\delta e = -\frac{1}{2} \epsilon_{\bar{s}mrs} e_{\bar{s}r} e_{\bar{s}m} 
$$

$$
\delta (\ast G_3 G_3) = -e G_{m\bar{p}q}(G_{\bar{r}\bar{q}m} \Psi_{r,m} + 2G_{m\bar{r}q} \Psi_{p,r})
$$

(62)

Here $\ast G_3$ denotes the Hodge dual of $G_3$ and $e$ is the volume form built from $(e_m, e_{\bar{m}})$.

From the relation between the torsion and the 3-form $G_3$, Eq. (58) one has:

$$
T_m T_{\bar{m}} + \ast G_3 G_3 = -\frac{1}{4}(G_{m\bar{p}q} G_{\bar{r}\bar{m}s} e_{\bar{p}} e_{\bar{q}} e_{r} e_{\bar{s}} + G_{m\bar{p}q} G_{\bar{r}\bar{m}s} e_{\bar{q}} e_{r} e_{\bar{s}})
$$

(63)

We thus add the term $T_m T_{\bar{m}} + \ast G_3 G_3$ to the action (49), which cancels the effect of the variations of the spin-connection given in (61) under the $\delta$ symmetry. The resulting invariant action is

$$
I_{\text{tot}} = -\int R_{mn} e_{\bar{m}} e_{\bar{n}} + (R_{\bar{m}\bar{n}} e_m e_n + 2\rho_{\bar{m}\bar{n}} \Psi_{n} e_m) + (R e_m e_{\bar{m}} - 2\rho_0 \Psi_m e_{\bar{m}} - 2FB_2) - T_m T_{\bar{m}} - \ast G_3 G_3
$$

(65)

Using Eqs. (36) and (37), this action can be written as

$$
I_{\text{tot}} = \int L_E + L_{RS} + 2FB_2 + \ast G_3 G_3
$$

(66)

This is nothing more that the complete supergravity action of Eq. (3).

This action is also invariant under $\delta_{\bar{p}}$ and $\delta_{pq}$, since it is equivalent to the one determined to be invariant under the complete untwisted BRST symmetry operator in [6]. The transformations under all twisted supersymmetry generators of the fields are:

| $\delta$ | $\delta_{\bar{p}}$ | $\delta_{pq}$ |
|---|---|---|
| $e_m$ | $-\Psi_m$ | $iJ_{mp} \Psi_0$ | $0$ |
| $e_{\bar{m}}$ | $0$ | $\Psi_{\bar{m}n}$ | $-2iJ_{\bar{m}[p} \Psi_q]$ |
| $\Psi_m$ | $0$ | $iJ_{\bar{m}m} (\frac{1}{2} \omega - A) + \omega_{\bar{m}m}$ | $0$ |
| $\Psi_0$ | $\frac{1}{2} \omega - A$ | $0$ | $-\omega_{pq}$ |
| $\omega_{\bar{m}m}$ | $\omega_{\bar{m}m}$ | $2J_{\bar{m}[p} | J_{\bar{n}[q]} (\frac{1}{2} \omega + A)$ | $0$ |
| $\omega_{mn}$ | $X_{[m,n]}$ | $-\frac{1}{2} J_{\bar{m}p} (\rho_{\bar{q}m,0} e_q + \rho_{\bar{q}m,0} e_q) - \frac{1}{2} G_{\bar{m}n} \Psi_0$ | $0$ |
| $\omega_{\bar{m}n}$ | $0$ | $-\frac{1}{2} (\rho_{\bar{n},m} e_q + \rho_{\bar{n},m} e_q + G_{\bar{m}n} \Psi_{\bar{q}})$ | $-iJ_{\bar{m}[p} X_{\bar{n}[q]}$ |
| $\omega_{\bar{m}n}$ | $2X_{\bar{m},\bar{n}}$ | $\frac{1}{2} J_{\bar{m}p} (\rho_{\bar{q}m,0} e_q + \rho_{\bar{q}m,0} e_q) + \frac{1}{2} (G_{\bar{m}n} \Psi_0 - G_{\bar{m}n} \Psi_{\bar{q}})$ | $0$ |
| $A$ | $X_{m,m}$ | $\frac{1}{2} (\rho_{pq,0} e_q + \rho_{pq,0} e_q - G_{\bar{m}n} \Psi_{0} + G_{\bar{m}n} \Psi_{\bar{q}})$ | $X_{p,q}$ |
| $B_2$ | $-\Psi_m e_{\bar{m}}$ | $-\Psi_0 e_{\bar{p}} - \Psi_{\bar{p}m} e_m | -2\Psi_{[p} e_q|$ |
with the twisted $X$ spinor in (9) defined as
\[ X_{m,n} = -\frac{1}{2}(\rho_{pm,me_p} + \rho_{pm,me_p} + G_{mnp}\Psi_p) \] (67)

Since these transformations are obtained directly from the Bianchi identities and the modified horizontality conditions for field strengths, the three anticommutation relations (27) hold true.

## 8 Matter and vector multiplets coupled to supergravity

In this section, we will compute both the scalar and vector symmetries acting on the matter fields, so we will retain $(\chi_0, \chi_p)$ $\neq 0$ when we expand the curvature equations in ghost number. The invariant actions for both multiplets can be expressed as $\delta$ exact terms, in a way that generalizes the flat space case [1].

### 8.1 The Wess–Zumino multiplet

The Wess–Zumino matter multiplet is $(P, \sigma, H)$ where $P$ is a complex scalar field, $\sigma$ a Majorana spinor (higgsino) and $H$ a complex auxiliary field, twisted into $(\phi, \bar{\phi}, \sigma_0, \sigma_m, \sigma_{mn}, B_{\bar{m}\bar{n}}, B_{mn})$. The various field strengths are

\[ \hat{P} = \hat{D}\phi + \hat{\Psi}_m\sigma_m \]
\[ \hat{P} = \hat{D}\bar{\phi} - \hat{\Psi}_{\bar{0}}\sigma_0 - \hat{\Psi}_{\bar{m}}\bar{\sigma}_{mn} \]
\[ \hat{\Sigma}_0 = \hat{D}\sigma_0 + B_{mn}\hat{\Psi}_{\bar{m}} \]
\[ \hat{\Sigma}_m = \hat{D}\sigma_m - B_{\bar{m}}\hat{\Psi}_{\bar{n}} \]
\[ \hat{\Sigma}_{mn} = \hat{D}\sigma_{mn} + B_{mn}\hat{\Psi}_0 \]
\[ \hat{H}_{mn} = \hat{D}B_{mn} \]
\[ \hat{H}_{\bar{m}\bar{n}} = \hat{D}B_{\bar{m}\bar{n}} \]

(68)

with the covariant derivative $D$ explicitly defined as

\[ \hat{D}\phi = \hat{d}\phi + w\hat{A}\phi \]
\[ \hat{D}\bar{\phi} = \hat{d}\bar{\phi} - w\hat{A}\bar{\phi} \]
\[ \hat{D}\sigma_0 = \hat{d}\sigma_0 + (\frac{1}{2}\hat{\omega} - w'\hat{A})\sigma_0 + \hat{\omega}_{\bar{m}}\bar{\sigma}_{mn} \]
\[ \hat{D}\sigma_m = \hat{d}\sigma_m + (\frac{1}{2}\hat{\omega} + w'\hat{A})\sigma_m - \hat{\omega}_{mn}\sigma_0 \]
\[ \hat{D}\sigma_{mn} = \hat{d}\sigma_{mn} - (\frac{1}{2}\hat{\omega} + w'\hat{A})\sigma_{mn} - \hat{\omega}_{mn}\sigma_0 \]
\[ \hat{D}B_{mn} = \hat{d}B_{mn} - w''\hat{A}B_{mn} \]
\[ \hat{D}B_{\bar{m}\bar{n}} = \hat{d}B_{\bar{m}\bar{n}} + w''\hat{A}B_{\bar{m}\bar{n}} \]

(69)

The explicit verification is non trivial, since it relies on the expression of the spin-connection, expressed as a solution of Eq. (59).
To have Bianchi identities, one must have \( w' = w + 1 \) and \( w'' = w + 2 \). One obtains:

\[
\begin{align*}
\hat{\dot{P}} &= \dot{w} \dot{F} \phi + \dot{\tilde{\rho}}_m \sigma_{\tilde{m}} - \dot{\tilde{\Psi}}_m \hat{\Sigma}_m \\
\hat{\dot{\Sigma}}_0 &= \left( \frac{1}{2} \hat{\dot{F}} - (w + 1) \hat{F} \right) \sigma_0 + \hat{\dot{R}}_{\tilde{m} \tilde{n}} \sigma_{\tilde{m}} + \hat{\dot{H}}_{\tilde{m} \tilde{n}} \dot{\tilde{\Psi}}_{\tilde{n}} - B_{\tilde{m} \tilde{n}} \dot{\tilde{\rho}}_{\tilde{n}} \\
\hat{\dot{\Sigma}}_{\tilde{m}} &= \left( \frac{1}{2} \hat{\dot{F}} + (w + 1) \hat{F} \right) \sigma_{\tilde{m}} - \hat{\dot{R}}_{\tilde{m} \tilde{n}} \sigma_{\tilde{n}} - \hat{\dot{H}}_{\tilde{m} \tilde{n}} \dot{\tilde{\Psi}}_{\tilde{n}} - B_{\tilde{m} \tilde{n}} \dot{\tilde{\rho}}_{\tilde{n}} \\
\hat{\dot{H}}_{\tilde{m} \tilde{n}} &= -(w + 2) \hat{F} B_{\tilde{m} \tilde{n}} \\
\hat{\dot{\Sigma}}_{\tilde{m} \tilde{n}} &= (w + 2) \hat{F} B_{\tilde{m} \tilde{n}}
\end{align*}
\]

The distorted horizontality conditions that are compatible with the Bianchi identities and warrant off-shell closure, are the following:

\[
\begin{align*}
\dot{\tilde{P}} &= \tilde{P} \\
\dot{\tilde{\Psi}}_0 &= \tilde{\Psi}_0 \theta \left( \tilde{P}_{\tilde{\beta}} - \frac{\tilde{w}}{2} \tilde{G}_{\tilde{m} \tilde{n} \tilde{p}} \theta \right) \\
\dot{\tilde{\Sigma}}_{\tilde{m}} &= \tilde{\Sigma}_{\tilde{m}} - \tilde{\Psi}_0 \left( \tilde{P}_{\tilde{m}} + \frac{\tilde{w}}{2} \tilde{G}_{\tilde{m} \tilde{n} \tilde{p}} \theta \right) \\
\dot{\tilde{\Sigma}}_{\tilde{m} \tilde{n}} &= \tilde{\Sigma}_{\tilde{m} \tilde{n}} + \tilde{\Psi}_0 \left( \tilde{P}_{\tilde{m}} + \frac{\tilde{w}}{2} \tilde{G}_{\tilde{m} \tilde{n} \tilde{p}} \theta \right) \\
\dot{\tilde{H}}_{\tilde{m} \tilde{n}} &= \tilde{H}_{\tilde{m} \tilde{n}} + \tilde{\Psi}_0 \left( \sigma_{\tilde{m} \tilde{n}} + \tilde{S}_{\tilde{m} \tilde{n}, \tilde{m} \tilde{n}} \right) - \tilde{R}_{\tilde{m} \tilde{n}} \sigma_{\tilde{n}} - \dot{\tilde{\Psi}}_{\tilde{n}} \left( \tilde{P}_{\tilde{n}} + \frac{\tilde{w}}{2} \tilde{G}_{\tilde{n} \tilde{m} \tilde{p}} \theta \right)
\end{align*}
\]

where

\[
\begin{align*}
\tilde{S}_{\tilde{m} \tilde{n}, \tilde{m} \tilde{n}} &= \left( \tilde{P}_{\tilde{m}} - \frac{\tilde{w}}{2} \tilde{G}_{\tilde{m} \tilde{n} \tilde{p}} \theta \right) \tilde{\Psi}_{\tilde{n}, \tilde{m}} + \left( \tilde{P}_{\tilde{n}} - \frac{\tilde{w}}{2} \tilde{G}_{\tilde{n} \tilde{m} \tilde{p}} \theta \right) \tilde{\Sigma}_{\tilde{n}, \tilde{m}} - \tilde{G}_{\tilde{m} \tilde{n} \tilde{p}} \sigma_{\tilde{p}} - \frac{\tilde{w}}{2} \rho_{\tilde{m} \tilde{n} \tilde{p}} \sigma_{\tilde{p}} - \frac{\tilde{w}}{2} \rho_{\tilde{q} \tilde{q} \tilde{m} \tilde{n}} \sigma_{\tilde{p}}
\end{align*}
\]

The ghost number 1 parts of these equations give the scalar and vector transformations of the fields:

\[
\begin{align*}
\delta \phi &= 0 \\
\delta \tilde{\phi} &= \sigma_0 \\
\delta \tilde{\Sigma}_0 &= 0 \\
\delta \tilde{\sigma}_{\tilde{m}} &= -\tilde{P}_{\tilde{m}} - \frac{\tilde{w}}{2} \tilde{G}_{\tilde{m} \tilde{n} \tilde{p}} \theta \\
\delta \tilde{\sigma}_{\tilde{n}} &= -\tilde{P}_{\tilde{n}} - \frac{\tilde{w}}{2} \tilde{G}_{\tilde{n} \tilde{m} \tilde{p}} \theta
\end{align*}
\]
\[ \delta \sigma_{mn} = -B_{mn} \]
\[ \delta \rho \sigma_{mn} = i(\hat{P}_{[m]n} - \frac{\gamma}{2} G_{qq[m]p}) J_{n} \]
\[ \delta B_{mn} = 0 \]
\[ \delta \rho B_{mn} = (\Sigma_{\rho, mn} + S_{\rho, mn}) - iJ_{\rho}[m](\Sigma_{n,0} + S_{n,0}) \]
\[ \delta B_{\bar{m}n} = -2(\Sigma_{[\bar{m}, n]} + S_{[\bar{m}, n]}) \]
\[ \delta \rho B_{\bar{m}n} = 0 \]

The anticommutation relations (27) can be explicitly verified on all fields, in a much easier way than for the supergravity multiplet (see the Appendix D).

### 8.2 The vector multiplet

The twisted vector multiplet is \((B, \xi_m, \xi_{\bar{m}n}, \xi_0, h)\), with \(B\) a \(U(1)\) gauge field, \((\xi_m, \xi_{\bar{m}n}, \xi_0)\) its twisted Majorana supersymmetric partner and \(h\) a real auxiliary field. The field strengths are

\[ \hat{F} = \hat{d}B - (\hat{\Psi}_0 \xi_m + \hat{\Psi}_m \xi_0) e_m - (\hat{\Psi}_p \xi_{\bar{m}n} + \hat{\Psi}_{\bar{m}n} \xi_p) e_m \]
\[ \hat{\Xi}_0 = \hat{D} \xi_0 - h \hat{\Psi}_0 \]
\[ \hat{\Xi}_m = \hat{D} \xi_m + h \hat{\Psi}_m \]
\[ \hat{\Xi}_{\bar{m}n} = \hat{D} \xi_{\bar{m}n} - h \hat{\Psi}_{\bar{m}n} \]
\[ \hat{\mathcal{H}} = \hat{d}h \]

where \(\hat{D}\) is given by \(\hat{D} \xi_0 = \hat{d} \xi_0 - \frac{1}{2} \hat{\omega} \xi_0 + \hat{A} \xi_0 + \hat{\omega}_{mn} \xi_{\bar{m}n}\), etc. The Bianchi identities for these field strengths are

\[ \hat{d} \hat{F} = \left( \hat{\Psi}_0 \hat{\Xi}_m + \hat{\Psi}_m \hat{\Xi}_0 - \hat{\rho}_0 \xi_m - \hat{\rho}_m \xi_0 \right) \hat{e}_m + \left( \hat{\Psi}_p \hat{\Xi}_{\bar{m}n} + \hat{\Psi}_{\bar{m}n} \hat{\Xi}_p - \hat{\rho}_p \xi_{\bar{m}n} - \hat{\rho}_{\bar{m}n} \xi_p \right) \hat{e}_m \]
\[ + (\hat{\Psi}_m \xi_0 + \hat{\Psi}_0 \xi_m) \hat{T}_m + (\hat{\Psi}_{\bar{m}n} \xi_p + \hat{\Psi}_p \xi_{\bar{m}n}) \hat{T}_m \]
\[ \hat{D} \hat{\Xi}_0 = -\left( \frac{1}{2} \hat{R} + \hat{\mathcal{F}} \right) \xi_0 + \hat{R}_{mn} \xi_{\bar{m}n} - \hat{\mathcal{H}} \hat{\Psi}_0 - h \hat{\rho}_0 \]
\[ \hat{D} \hat{\Xi}_m = -\left( \frac{1}{2} \hat{R} + \hat{\mathcal{F}} \right) \xi_m + \hat{R}_{pm} \xi_p + \hat{\mathcal{H}} \hat{\Psi}_m + h \hat{\rho}_m \]
\[ \hat{D} \hat{\Xi}_{\bar{m}n} = \left( \frac{1}{2} \hat{R} + \hat{\mathcal{F}} \right) \xi_{\bar{m}n} - \hat{R}_{m\bar{n}} \xi_0 - \hat{\mathcal{H}} \hat{\Psi}_{\bar{m}n} - h \hat{\rho}_{\bar{m}n} \]
\[ \hat{d} \hat{\mathcal{H}} = 0 \]

The supersymmetry is defined by the constraints

\[ \hat{F} = \mathcal{F} \]
\[ \hat{\Xi}_0 = \Xi_0 + \mathcal{F}_{mn} \hat{\Psi}_{\bar{m}n} \]
\[ \hat{\Xi}_m = \Xi_m - \mathcal{F}_{pm} \hat{\Psi}_{\bar{m}n} \]
\[ \hat{\Xi}_{\bar{m}n} = \Xi_{\bar{m}n} + \mathcal{F}_{mn} \hat{\Psi}_0 \]
\[ \hat{\mathcal{H}} = \mathcal{H} + \hat{\Psi}_p (\Xi_{p,0} + G_{mpn} \xi_{\bar{m}n}) \]
which give

\[ \begin{align*}
\delta B &= \xi_m \epsilon_m \\
\delta_\xi_0 &= h \\
\delta \xi_0 &= 0 \\
\delta \xi_m &= 0 \\
\delta \xi_{\bar{m} \bar{n}} &= F_{\bar{m} \bar{n}} \\
\delta h &= 0 \\
\delta_\xi_0 &= \Xi_{\bar{p},0} + G_{\bar{m} \bar{n}} \xi_{\bar{m} \bar{n}}
\end{align*} \]

(76)

The algebra closure relations (27) are satisfied on all fields (see the Appendix D).

9 Conclusion and outlook

We have shown that the supergravity action is essentially determined by its invariance under a single scalar supersymmetry generator. This scalar generator is nilpotent and formally similar to a BRST operator. It is singled out from the multiplet of supersymmetry generators by a twist and is therefore quite analogous to the one encountered in the twisted super-Yang–Mills theory in four dimensions. The supergravity action has parts which are independently invariant under this scalar generator and induce an interesting decomposition of both the Einstein and Rarita–Schwinger actions in twisted form.

In the twisted form, there is also a vector supersymmetry generator \( \delta_\bar{p} \). Its anticommutation with the scalar generator gives rise to translations, but with additional field dependent gauge transformations. These commutation relations are best related to the BRST transformations of the ghost fields, with a consistency derived from Bianchi identities. Nevertheless, when the gravitino field vanishes, the fourth symmetry \( \delta_{\bar{m} \bar{n}} \) can be safely ignored. This additional symmetry does not add any new constraint to the action.

There is an underlying localization around gravitational instantons that seems of interest in this construction. A twisted formulation of the Wess–Zumino and vector multiplets coupled to the supergravity multiplet has also been obtained. Generalizations to higher dimensional supergravities could be of interest and a analogous twist could be used to split the Poincaré symmetry of, for example, \( d = 10 \) supergravity into smaller and (hopefully) simpler sectors.

Acknowledgments

The work of V. R. is presently supported by the ERC Advanced Grant No. 246974, 'Supersymmetry: a window to non-perturbative physics'.
A The BSRT symmetry from horizontality conditions

The supergravity transformations can be expressed as BRST transformations, in a way that merely generalizes the Yang–Mills case (ghost unification, horizontality equations for the curvatures, etc.) [6]. Call $s$ the BRST operator of the supergravity transformation, and its ghost $\xi$. The other ghosts are those of local SUSY ($\chi$), Lorentz symmetry ($\Omega$), the chiral $U(1)$ symmetry ($c$) and the 2-form gauge symmetry ($B^1$). One gets the usual transformation laws of classical fields by changing the ghosts into local parameters, with the opposite statistics. Their off-shell closure property is equivalent to the nilpotency of the graded differential operator $s$. The difficult part of the supergravity BRST symmetry is its dependence on the supersymmetry ghost $\chi$. The reparametrization invariance can be absorbed, by redefining $\hat{s}$ as $\hat{s} = s - L_\xi$, with $s\xi^\mu = \xi^\nu \partial_\nu \xi^\mu + \frac{i}{2} \chi\gamma^\mu \chi$. With this property, the off-shell closure relation $s^2 = 0$ is equivalent to $\hat{s}^2 = L_\chi^\nu \chi$. Reparametrization invariance is decoupled by the operation $\exp(-i\xi)$, when classical and ghost fields are unified into graded sums, a property that was found for the study of gravitational anomalies but turns out to be very useful for the construction of supergravity BRST symmetries. For the $N = 1, d = 4$ supergravity in the new minimal scheme, the action of the operator $\hat{s}$ is as follows

$$
\hat{s} e^a = -\Omega^{ab} e_b - i\bar{\chi}\gamma^a \lambda \\
\hat{s} \lambda = -D\chi - \Omega^{ab} \gamma_{ab} \lambda - c\gamma^5 \lambda \\
\hat{s} B_2 = -dB_1^1 - i\bar{\chi}\gamma^a \lambda e_a \\
\hat{s} A = -dc - \frac{i}{2} \bar{\chi}\gamma^5 \gamma^a X_a \\
\hat{s} \omega^{ab} = -(D\Omega)^{ab} - i\bar{\chi}\gamma^{[a} X^{b]} 
$$

(A.1)

where the spinor $X_a$ is $X_a = \rho_{abc} e^b - \frac{i}{2} G_{abc} \gamma^{[bc} + \frac{1}{6} \epsilon_{abcd} G^{bcde} \gamma^5 \lambda$. $X_a$ vanishes when one uses the equations of motion of the gravitino and of the (propagating) auxiliary fields. The property $s^2 = 0$, equivalent to $\hat{s}^2 = L_\chi^\nu \chi$ is warranted by the ghost transformation laws [6]. At the root of these equations, there is a unification between classical fields and ghosts [6], which is analogous to the one that occurs when analyzing anomalies by descent equations. In fact, everything boils down to computing constraints on the curvatures, which satisfy the following Bianchi identities:

$$
\hat{T}^a \equiv \hat{d} e^a + (\omega + \Omega)^{ab} e_b + \frac{i}{2} (\bar{\lambda} + \bar{\chi}) \gamma^a (\lambda + \chi) = -\frac{1}{2} G_{abc} e^b e^c \\
\hat{\rho} \equiv \hat{d}(\lambda + \chi) + (\omega + \Omega + A + c)(\lambda + \chi) = \frac{1}{2} \rho_{abc} e^a e^b \\
\hat{G}_3 \equiv \hat{d}(B_2 + B_1^1 + B_0^2) + \frac{i}{2} (\bar{\lambda} + \bar{\chi}) \gamma^a (\lambda + \chi) e^a = \frac{1}{6} G_{abc} e^a e^b e^c \\
\hat{R}^{ab} \equiv \hat{d}(\omega + \Omega) + (\omega + \Omega)^2 = R^{ab} - i\bar{\chi}\gamma^{[a} X^{b]} - \frac{i}{4} \bar{\chi}\gamma^5 \chi G_{abc} \\
\hat{F} \equiv \hat{d}(A + c) = F - \frac{i}{2} \bar{\chi}\gamma^5 \gamma^a X_a - \frac{i}{24} \bar{\chi}\gamma^a \chi \epsilon_{abc} G^{bcd} 
$$

(A.2)
By expansion at ghost number one, one finds the transformation laws in Eq. (8) and at ghost number two, one finds those of the ghosts:

\[
\begin{align*}
\hat{s}\chi &= -i\phi\lambda -\Omega\chi - c\chi \\
\hat{s}c &= -i\phi A - \frac{i}{24}\chi\gamma^a\chi\epsilon_{abcd}G_{abcd} \\
\hat{s}B_1^1 &= -i\phi B - dB_0^2 - \frac{i}{2}\chi\gamma^a\chi c \\
\hat{s}B_2^2 &= -i\phi B_1^1 \\
\hat{s}\Omega^{ab} &= -i\phi\omega^{ab} - \frac{i}{2}\Omega^{ab} - \frac{i}{2}\chi\gamma^c\chi G^{ab}
\end{align*}
\]

(A.3)

B  Tensor and chirality conventions

The normalization of the completely antisymmetric four-index symbol with tangent space indices is

\[
\epsilon_{0123} = 1
\]

(B.1)

Once twisted, this is taken to be

\[
\epsilon_{1\bar{1}2\bar{2}} = 1
\]

(B.2)

The dual of an antisymmetric Lorentz tensor is

\[
\tilde{F}_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd}
\]

(B.3)

The selfdual and antiselfdual parts of \(F_{ab}\) are

\[
F_{\pm ab} = \frac{1}{2}(F_{ab} \pm \tilde{F}_{ab})
\]

(B.4)

We take \(\gamma_5\) such that \((\gamma_5)^2 = -1\) and define the chiral projections

\[
\lambda^\pm = \frac{1 \pm i\gamma_5}{2}\lambda
\]

\[
\bar{\lambda}^\pm = \frac{1 \pm i\gamma_5}{2}
\]

(B.5)

in order to have \(\lambda = \lambda^+ + \lambda^-\) and \(\bar{\lambda} = \bar{\lambda}^+ + \bar{\lambda}^-\). Then, we have the useful identity

\[
\bar{\lambda}^+\gamma^a\lambda^+ = \bar{\lambda}^+\gamma_5\gamma^a\gamma_5\lambda^+ = -i\bar{\lambda}^+\gamma^a(-i)\lambda^+ = -\bar{\lambda}^+\gamma^a\lambda^+ = 0
\]

(B.6)

and similarly \(\bar{\lambda}^-\gamma^a\lambda^- = 0\). Finally, once in twisted form, the chiral projections of spinor separate its various components according to

\[
\lambda^+ \sim (0, \Psi_p, 0)
\]
\[
\lambda^- \sim (\Psi_0, 0, \Psi_{\bar{m}\bar{n}})
\]
C The action of $\gamma$ matrices on twisted spinors

The action of a $\gamma$ matrix on a twisted spinor with components $(\Psi_0, \Psi_m, \bar{\Psi}_{\bar{m}})$ is defined as follows

\[
\begin{array}{c|ccc}
\gamma_m \Psi & 0 & \bar{p} & pq \\
\hline
\gamma_m \Psi & i\Psi_m & -J_{mp}\Psi_0 & 0 \\
\bar{\gamma}_{\bar{m}} \Psi & i\bar{\Psi}_{\bar{m}} & 2J_{\bar{m}\bar{p}}\Psi_{\bar{q}} \\
\end{array}
\] (C.1)

Similarly, the action of a $\gamma$ matrix on a twisted spinor with components $(\sigma_0, \sigma_{\bar{m}}, \sigma_{mn})$, as the one appearing in the Wess–Zumino multiplet, is

\[
\begin{array}{c|ccc}
\gamma_m \sigma & 0 & p & \bar{p}q \\
\hline
\gamma_m \sigma & 0 & i\sigma_{mp} & -J_{m[p}\sigma_{q]} \\
\bar{\gamma}_{\bar{m}} \sigma & i\sigma_{\bar{m}} & -J_{\bar{p}m}\sigma_0 & 0 \\
\end{array}
\] (C.2)

These conventions allow us to retrieve the Clifford algebra for the twisted $\gamma$ matrices

\[
\{\gamma_m, \gamma_n\} = 0 \\
\{\gamma_{\bar{m}}, \gamma_{\bar{n}}\} = 0 \\
\{\gamma_m, \gamma_{\bar{n}}\} = -iJ_{m\bar{n}} \equiv g_{m\bar{n}}
\]

We also define the $\gamma_{ab}$ matrices in twisted form as

\[
\begin{align*}
\gamma_{m\bar{n}} &= \gamma_m \gamma_{\bar{n}} - \gamma_{\bar{n}} \gamma_m \\
\gamma_{mn} &= \gamma_m \gamma_n - \gamma_n \gamma_m \\
\gamma_{\bar{m}\bar{n}} &= \gamma_{\bar{m}} \gamma_{\bar{n}} - \gamma_{\bar{n}} \gamma_{\bar{m}}
\end{align*}
\]

which act on the two kinds of twisted spinors according to the following tables

\[
\begin{array}{c|ccc}
\gamma_{mn} \Psi & 0 & p & \bar{p}q \\
\hline
\gamma_{mn} \Psi & 0 & 0 & -2J_{m[p}\bar{n]}\Psi_{\bar{q}]} \\
\gamma_{mn} \Psi & 2\Psi_{\bar{m}\bar{n}} & 0 \\
\gamma_{mn} \Psi & iJ_{m\bar{n}}\Psi_0 & 2iJ_{\bar{p}m}\Psi_m - iJ_{m\bar{n}}\Psi_p & -iJ_{m\bar{n}}\Psi_{\bar{p}q} \\
\end{array}
\] (C.3)

\[
\begin{array}{c|ccc}
\gamma_{mn} \sigma & 0 & p & pq \\
\hline
\gamma_{mn} \sigma & 2\sigma_{mn} & 0 & 0 \\
\gamma_{mn} \sigma & 0 & 0 \\
\gamma_{mn} \sigma & -iJ_{m\bar{n}}\sigma_0 & -\frac{3i}{2}J_{m\bar{p}}\sigma_{\bar{n}} + \frac{i}{2}J_{m\bar{n}}\sigma_{\bar{p}} & iJ_{m\bar{n}}\sigma_{p\bar{q}} \\
\end{array}
\] (C.4)
D Algebra closure on the fields of matter and vector multiplets

In this appendix, we give some examples of the anticommutation relations (27) on some matter fields of the Wess–Zumino and vector multiplets.

Starting with the $\phi$ and $\bar{\phi}$ fields of the Wess–Zumino multiplet, one needs their transformation laws under the pseudo-scalar symmetry in order to check (27). These are obtained in the same way as the scalar and vector symmetry transformation laws, i.e. by isolating the part of ghost number 1 in the horizontality conditions on $\hat{P} = P$ and $\hat{\bar{P}} = \bar{P}$ and keeping $\chi_{mn} \neq 0$. This yields

$$\delta_{mn} \phi = 0 \quad \text{and} \quad \delta_{mn} \bar{\phi} = \sigma_{mn} \quad (D.1)$$

The transformation laws in (72) allow us to compute straightforwardly

$$\delta^2 \phi = 0$$

$$\{\delta_p, \delta_q\} \phi = -(B_{pq} + B_{qp}) = 0 \quad (D.2)$$

$$\{\delta, \delta_p\} \phi = \left( P_p + \frac{w}{2} G_{m\bar{p}m} \phi \right) = 0$$

$$= \partial_p \phi + \left( w A_p + \frac{w}{2} G_{m\bar{p}m} \right) \phi + \Psi_{\bar{p},m} \chi_{m\bar{m}}$$

$$= \partial_p \phi + \delta_{\text{gauge}}(A, G) \phi - \sum_{a=0, m, \bar{m}} \Psi_{\bar{p},a} \delta_a \phi$$

where the last equality is a consequence of (72) and (D.1).

Similarly, on $\bar{\phi}$:

$$\delta^2 \bar{\phi} = \delta \sigma_0 = 0$$

$$\{\delta_{\bar{p}}, \delta_{\bar{q}}\} \bar{\phi} = 0 \quad (D.3)$$

$$\{\delta, \delta_{\bar{p}}\} \bar{\phi} = \left( \bar{P}_{\bar{p}} - \frac{w}{2} G_{\bar{m}\bar{p}m} \bar{\phi} \right)$$

$$= \partial_{\bar{p}} \bar{\phi} - \left( w A_{\bar{p}} + \frac{w}{2} G_{\bar{m}\bar{p}m} \right) \bar{\phi} - \left( \Psi_{\bar{p},0} \sigma_0 + \Psi_{\bar{p},\bar{m}n} \sigma_{mn} \right)$$

$$= \partial_{\bar{p}} \bar{\phi} + \delta_{\text{gauge}}(A, G) \bar{\phi} - \sum_{a=0, m, \bar{m}} \Psi_{\bar{p},a} \delta_a \bar{\phi}$$

using again (72) and (D.1) for the last equality.

Turning to the $B$ field of the vector multiplet, the horizontality condition on its field strength $\hat{F} = F$ allows us to compute

$$\delta_{mn} B = \xi_n e_m \quad (D.4)$$
and the transformation laws (76) of the vector multiplet fields yield:

\[
\delta^2 B = \delta(\xi_m e_m) = 0
\]

\[
\{\delta_p, \delta_q\} B = \xi_0 \Psi_{p\bar{q}} + \xi_{\bar{p}q} \Psi_0 + \xi_0 \Psi_{\bar{q}p} + \xi_{\bar{q}p} \Psi_0 = 0
\]

\[
\{\delta, \delta_{\bar{p}}\} B = h e_p + \mathcal{F}_{m\bar{p}} e_m - \xi_{m\bar{p}} \Psi_m + \mathcal{F}_{\bar{m}p} e_{\bar{m}} - i J_{m\bar{p}} h e_{\bar{m}} + \xi_m \Psi_{\bar{m}p}
\]

\[
\quad = \mathcal{F}_{\bar{m}p} e_{\bar{m}} - \mathcal{F}_{\bar{p}m} e_m - \xi_{m\bar{p}} \Psi_m + \xi_m \Psi_{\bar{m}p}
\]

\[
\quad = \partial_{\bar{p}} B - (\Psi_{\bar{p},0} \xi_m + \Psi_{\bar{p},m} \xi_0) e_m - (\Psi_{\bar{p},q} \xi_{\bar{m}q} + \Psi_{\bar{p},\bar{m}q} \xi_q) e_m
\]

\[
\quad = \partial_{\bar{p}} B - \sum_{a=0,m,\bar{m}} \Psi_{\bar{p},a} \delta_a B
\]

where, for the last equality, we’ve used the B transformations given by (76) and (D.4).
References

[1] L. Baulieu and G. Bossard, Reconstruction of N=1 supersymmetry from topological symmetry, Phys. Lett. B 632 (2006) 138 [hep-th/0507004];
L. Baulieu, SU(5)-invariant decomposition of ten-dimensional Yang-Mills supersymmetry, Phys. Lett. B 698 (2011) 63 [arXiv:1009.3893[hep-th]].

[2] L. Baulieu and I. M. Singer, The Topological Sigma Model, Commun. Math. Phys. 125 (1989) 227;
L. Baulieu and I. M. Singer, Conformally invariant gauge fixed actions for 2-D topological gravity, Commun. Math. Phys. 135 (1991) 253;
L. Baulieu and I. M. Singer, Topological Yang-mills Symmetry, Nucl. Phys. Proc. Suppl. 5B (1988) 12;
L. Baulieu, H. Kanno and I. M. Singer, Special quantum field theories in eight-dimensions and other dimensions, Commun. Math. Phys. 194 (1998) 149 [hep-th/9704167].

[3] V. P. Akulov, D. V. Volkov and V. A. Soroka, Generally Covariant Theories of Gauge Fields on Superspace, Theor. Math. Phys. 31 (1977) 285.

[4] A. S. Galperin, V. I. Ogievetsky and E. S. Sokatchev, Geometries Inherent To N=1 Supergravities, JINR-E2-81-854;
On Matter Couplings In N=1 Supergravities, Nucl. Phys. B 252 (1985) 435.

[5] M. S. Sohnius and P. C. West, An Alternative Minimal Off-Shell Version of N=1 Supergravity, Phys. Lett. B 105 (1981) 353;
The tensor calculus and matter coupling of the alternative minimal auxiliary field formulation of N = 1 supergravity, Nucl. Phys. B 198 (1982) 493–507.

[6] L. Baulieu and M. P. Bellon, p-forms and supergravity: gauge symmetries in curved space, Nucl. Phys. B 266 (1986) 75.

[7] A. Johansen, Twisting of N = 1 SUSY gauge theories and heterotic topological theories, Int. J. Mod. Phys. A 10 (1995) 4325 [hep-th/9403017].

[8] E. Witten, Supersymmetric Yang-Mills theory on a four manifold, J. Math. Phys. 35 (1994) 5101 [hep-th/9403195].

[9] A. D. Popov, Holomorphic analogs of topological gauge theories, Phys. Lett. B 473 (2000) 65 [hep-th/9909135];
T. A. Ivanova and A. D. Popov, Dressing symmetries of holomorphic BF theories, J. Math. Phys. 41 (2000) 2604 [hep-th/0002120].
[10] C. Hofman and J. -S. Park, *Cohomological Yang-Mills theories on Kahler 3 folds*, Nucl. Phys. B 600 (2001) 133 [hep-th/0010103];
J. -S. Park, *N=2 topological Yang-Mills theory on compact Kahler surfaces*, Commun. Math. Phys. 163 (1994) 113 [hep-th/9304060];
J. -S. Park, *Holomorphic Yang-Mills theory on compact Kahler manifolds*, Nucl. Phys. B 423 (1994) 559 [hep-th/9305095].

[11] L. Baulieu and A. Tanzini, *Topological symmetry of forms, N=1 supersymmetry and S-duality on special manifolds*, J. Geom. Phys. 56 (2006) 2379 [hep-th/0412014].

[12] H. Nicolai, *A Possible Constructive Approach To (Super $\Phi^3$) In Four-Dimensions. 1. Euclidean Formulation Of The Model*, Nucl. Phys. B 140 (1978) 294.

[13] N. Berkovits, *Perturbative Super-Yang–Mills from the Topological AdS(5)×S5 Sigma Model*, JHEP 0809 (2008) 088 [arXiv:0806.1960 [hep-th]].

[14] P. Van Nieuwenhuizen, *Supergravity*, Phys. Rept. 68 (1981) 189.