THIN POSITION FOR TANGLES

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Abstract. If a tangle, $K \subset \mathbb{B}^3$, has no planar, meridional, essential surfaces in its exterior then thin position for $K$ has no thin levels.

1. Introduction

In [1], D. Gabai introduced a complexity for embeddings of knots in $S^3$, called width. A knot that has been isotoped to have minimal possible width is said to be in thin position. In [7], A. Thompson showed that if a knot, with no planar, meridional essential surfaces in its exterior, is put in thin position then all of its maxima appear above all of its minima. This establishes a connection between width, and bridge number, a classical complexity introduced by H. Schubert [5].

This paper gives a similar result for tangles in $\mathbb{B}^3$. If a tangle in $\mathbb{B}^3$, with no planar, meridional, essential surfaces in its exterior, is put in thin position then all of its maxima (if there are any) appear above all of its minima. This is equivalent to saying that thin position for such a tangle has no thin levels (see the next section for the relevant definitions). This result has already been obtained when the tangle is the one-skeleton of a triangulation (see [3]).

The proof breaks into two cases. The first is handled by Thompson’s argument identically as in the situation of a knot. The second is reminiscent of Claim 4.5 of Thompson’s [4]. In this case the lightbulb trick produces a contradiction to thin position (as opposed to Thompson’s “fluorescent lightbulb trick”). This change in technique is unavoidable; in [3] Thompson begins with a thick level of a tangle, whereas we must start with a thin level.

It has come to our attention that the second case may also be handled with an argument of S. Matveev’s [2].

Date: November 21, 2018.
1To appear in the Journal of Knot Theory and its Ramifications.
2. Definitions

Suppose that \( K \subset \mathbb{B}^3 \) is a tangle, i.e. a properly embedded 1-manifold. Let \( h : \mathbb{B}^3 \to I \) be the standard height function, so that \( h^{-1}(t) \) is a 2-sphere for \( t \in (0, 1) \), \( h^{-1}(0) \) is a single point, and \( h^{-1}(1) = \partial \mathbb{B}^3 \).

Suppose that \( h|_K, h \) restricted to \( K \), is a Morse function. Let \( \{c_i\} \) denote the critical values of \( h|_K \), and let \( r_i \) be a regular value immediately above \( c_i \).

**Definition.** The width of \( K \) is the quantity
\[
w(K) = \sum_i |h^{-1}(r_i) \cap K|
\]

**Definition.** The tangle \( K \) is in thin position if \( w(K) \leq w(K') \) for any tangle \( K' \) properly isotopic to \( K \).

**Definition.** A level, \( L_i = h^{-1}(r_i) \), is thin if \( |K \cap L_i| < |K \cap L_{i-1}| \) and \( |K \cap L_i| < |K \cap L_{i+1}| \).

Let \( M_K \) be the exterior of \( K \), that is, \( \mathbb{B}^3 \) with a small neighborhood of \( K \) removed. A surface properly embedded in \( M_K \) is essential if it is incompressible and not boundary parallel. Finally, a properly embedded surface is meridional if all boundary components also bound small meridional disks for \( K \).

3. The Main Theorem

**Theorem 3.1.** Let \( K \) be a tangle in thin position in \( \mathbb{B}^3 \). Suppose that \( K \) has a thin level. Then the exterior of \( K \) contains a meridional, planar, essential surface.

**Proof.** Pick a thin level, \( S \). If \( S \) does not intersect \( K \) then \( K \) is split, and \( S \) is essential. If not, then note that \( S_K = S \cap M_K \) is not boundary parallel in \( M_K \). Compress \( S \) as much as possible in the complement of \( K \) to obtain a collection of spheres, \( P \), in \( \mathbb{B}^3 \). For a contradiction assume that each component of \( P_K = P \cap M_K \) is boundary parallel in \( M_K \).

Suppose some component of \( P_K \) is an annulus, parallel into the boundary of a neighborhood of \( K \). Pick an innermost such annulus, \( P'_K \). Following Thompson’s proof of the main result of [4], let \( D \) be a disk in \( \mathbb{B}^3 \) such that \( \partial D = \delta \cup \gamma \), where \( D \cap K = \delta \) and \( D \cap P = D \cap P' = \gamma \).

We now reverse the compressions used to obtain \( P \) from \( S \). Each time a compression is reversed, we attach a tube to some components of \( P \). These tubes may intersect \( D \), but only in its interior. In the end,
This situation is similar to Claim 4.5 from [6].

Recall that all boundary components of $P$ persist as a disk in $D$ via an isotopy fixing $K$ setwise. This situation is similar to Claim 4.5 from [6].

To fix notation, let $\{D_i\}_{i=1}^n$ be a sequence of disks, and $\{S_i\}_{i=0}^n$ the sequence of surfaces, so that $S_0 = S$, $S_n = P$, and $S_i$ is obtained from $S_{i-1}$ by compressing along $D_i$ in the complement of $K$. That is, remove a small neighborhood, $A_i$, of $\partial D_i$ from $S_{i-1}$. Construct $S_i$ by gluing two parallel copies of $D_i$ onto $\partial A_i$. Denote these by $B_i$ and $C_i$. So $A_n \cup B_n \cup C_n$ bounds a ball homeomorphic to $D_n \times I$. Finally, let $\alpha$ be the image of $\{pt\} \times I \subset D_n \times I$ by such a homeomorphism. The arc $\alpha$ is a co-core for the compression $D_n$.

Let $Q$ be the component of $S_{n-1}$ which meets $D_n$. Let $\{P_j\}_{j=0}^m$ denote the components of $P = S_n$, numbered consecutively, so that $P_0$ is innermost. Then there is a $j$ such that compressing $Q$ along $D_n$ yields $P_j$ and $P_{j+1}$. Let $N$ denote the submanifold of $B^3$ bounded by $P_j$ and $P_{j+1}$. Note $\alpha \subset N$.

Choose a homeomorphism $g : S^2 \times I \to N$ such that $K \cap N$ is a collection of straight arcs and $\pi(g^{-1}(B_n)) = \pi(g^{-1}(C_n))$. Here a straight arc is one of the form $g(\{pt\} \times I)$ and $\pi : S^2 \times I \to S^2$ is projection onto the first factor.

**Claim.** There is an isotopy, $\tilde{H} : B^3 \times I \to B^3$, fixing the complement of $N$ pointwise such that $\tilde{H}(\alpha, 1)$ is a straight arc.

This is the usual lightbulb trick. See [3], for example. Since width is measured with respect to the height function, $h$, we must use the reverse of $\tilde{H}$ to move $K$. Let $H(\cdot, t) = (\tilde{H}(\cdot, t))^{-1}$. The isotopy $H$ moves $K$ to $H(K, 1)$; this probably increases the width enormously. Another isotopy of $K$ is needed to bring the width back under control.

Let $R = Q \cap S$ ($= Q \setminus (\bigcup_{i=0}^{n-1} B_i \cup C_i)$) and set $R' = H(R, 1)$. Let $\{\beta_i\}_{i=1}^m$ denote the arcs of $K \cap N$.

**Claim.** There is an isotopy, $J : B^3 \times I \to B^3$, such that

- $J$ fixes the complement of $N$ pointwise,
- the arcs $J(\beta_i, 1)$ lie within a very small neighborhood of $R'$, and
- the arcs $H(J(\beta_i, 1), 1)$ have only one critical point with respect to the height function $h$.

**Proof.** Note that $\beta_i$ and $\tilde{H}(\alpha, 1)$ are straight. Hence, there is an isotopy, $J$, supported in $N$, which moves the arcs $\beta_i$ near $R'$ (see Figure [4]). The
image of $R'$ under $H(\cdot, 1)$ is the surface, $R$, which is part of the level, $S$. The foliation defined by the levels of $h$ is a product near $R$, so the arcs $H(J(\beta_i, 1), 1)$ lie in a product region. Adjust $J$ so that each arc has only one critical point with respect to $h$.

For each $i \in \{1, \ldots, m\}$, let $\beta_i' = H(J(\beta_i, 1), 1)$. For each $j \in \{1, \ldots, m\}$, let $K_j$ be the 1-manifold which consists of the subset of $K$ which lies outside of $N$, together with the arcs $\{\beta_i'\}_{i=1}^{j}$ and the arcs $\{\beta_i\}_{i=j+1}^{m}$. Finally, let $K_0 = K$. Hence, for each $i \in \{1, \ldots, m\}$, $K_i$ can be obtained from $K_{i-1}$ by isotoping the arc, $\beta_i$, to be near $S$ (refer to Figure 2). $K_i$ is not necessarily isotopic to $K_{i-1}$. However, $K_m$ is isotopic to $K_0$, since $K_m = H(J(K, 1), 1)$.

We now assume that the arcs $\beta_i$ lie above $S$ with respect to the height function $h$. (The other case is identical.)

**Claim.** For each $i \in \{1, \ldots, m\}$, $w(K_{i-1}) \geq w(K_i)$. Here equality implies that the first critical point of $h|K_{i-1}$ above $S$ is a maximum.

Proving this claim will produce the desired contradiction. For if $w(K_0) > w(K_m)$ then $K = K_0$ was not thin. On the other hand, if $w(K_0) = w(K_m)$ then $S$ was not a thin level.

**Proof.** Let $\{r_j\}$ denote regular values of $h|K_i$ which occur just above each critical value. Let $b'$ be the unique critical value of $\beta_i'$ and let $b$ be
the largest critical value of $\beta_i$. Suppose $r_l$ is just above $b'$ and $r_{l+k}$ is the highest regular value below $b$. Finally, let $s$ be a regular value just above $b$ (see schematic Figure 2).

\[ \sum_j |h^{-1}(r_j) \cap K_i| \]

Note that $|h^{-1}(r_j) \cap K_i|$ for all values of $j > l + k$ or $j < l$. Hence the difference $w(K_{i-1}) - w(K_i)$ is at least:

\[ \sum_{j=l+1}^{l+k} (|h^{-1}(r_j) \cap K_{i-1}| - |h^{-1}(r_j) \cap K_i|) \]

Also, for each $j \in \{l+1, ..., l+k\}$ we have $|h^{-1}(r_j) \cap K_i| \geq 2 + |h^{-1}(r_j) \cap K_i|$, so the above difference in width is at least

\[ \Delta = 2k + |h^{-1}(s) \cap K_{i-1}| - |h^{-1}(r_i) \cap K_{i-1}|. \]

Note that $\Delta$ is equal to zero only when all of the critical values of $h|K_{i-1}$ in the interval $[b', b]$ are maxima. Otherwise, $\Delta$ is strictly
greater than zero. In particular, if \( w(K_{i-1}) - w(K_i) = 0 \) then the first critical point of \( h|_{K_{i-1}} \) above \( S \) is a maximum.

This completes the proof of Theorem 3.1.

References

[1] D Gabai. Foliations and the topology of three-manifolds iii. J. Diff. Geom., 26:479–536, 1987.
[2] S. V. Matveev. Algorithms for the recognition of the three dimensional sphere (after A. Thompson). Math USSR-Sb, 186:695–710, 1995.
[3] Dale Rolfsen. Knots and links. Publish or Perish Inc., Berkeley, Calif., 1976. Mathematics Lecture Series, No. 7.
[4] S. Schleimer. Tightening almost normal surfaces. preprint, 2000.
[5] H. Schubert. Ueber eine numerische Knoteninvariante. Math. Z., 61:245–288, 1954.
[6] A. Thompson. Thin position and the recognition problem for the 3-sphere. Math. Research Letters, 1:613–630, 1994.
[7] A. Thompson. Thin position and bridge number for knots in the 3-sphere. Topology, 36:505–507, 1997.