EXTREMAL EXPONENTS OF RANDOM PRODUCTS OF CONSERVATIVE
DIFEOMORPHISMS

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Abstract. We show that for a $C^1$-open and $C^r$-dense subset of the set of ergodic iterated function systems of conservative diffeomorphisms of a finite-volume manifold of dimension $d \geq 2$, the extremal Lyapunov exponents do not vanish. In particular, the set of non-uniform hyperbolic systems contains a $C^1$-open and $C^r$-dense subset of ergodic random products of i.i.d. conservative surface diffeomorphisms.

1. Introduction

The notion of uniform hyperbolicity introduced by Smale in [Sma67] was early shown to be less generic that initially thought [AS70, New70]. In order to describe a large set of dynamical systems Pesin theory [Pes77] provides a weaker notion called non-uniform hyperbolicity. These systems are described in terms of non-zero Lyapunov exponents of the linear cocycle defined by the derivative transformation, the so-called differential cocycle. In contrast with the non-density of hyperbolicity, we had to wait some decades to construct the first examples of systems with robustly zero Lyapunov exponents [KN07, BBD16]. Even in the conservative setting there are open sets of smooth diffeomorphisms with invariant sets of positive measure where all the Lyapunov exponents vanish identically (see [CS89, Her90, Xia92]). However, recently in [LY17] it was showed that conservative diffeomorphisms without zero exponent in a set of positive volume are $C^1$-dense.

On the other hand, abundance of non-uniform hyperbolicity has been obtained in the general framework of linear cocycles when the base driving dynamics is fixed and the matrix group is perturbed in many different contexts [Fur63, Kni92, AC97, Via08, Avi11]. However, nothing is known for random product of i.d.d. non-linear dynamics. That is, for cocycles driving by a shift map endowed with a Bernoulli probability to value in the group of diffeomorphisms of a compact manifold. To perturbe the Lyapunov exponents of this cocycles one must change the non-linear dynamics similar as in the case of differential cocycles. Examples of iterated function systems (IFSs) of diffeomorphisms with robust zero extremal Lyapunov exponents with respect to some ergodic measure that not project on a Bernoulli measure were provided in [BBD14]. The authors in [BBD14] question about the possibility of construct examples of IFSs of conservative diffeomorphisms by taking as the ergodic measure the product measure of a Bernoulli measure on the base and the volume measure on the fiber. We give a negative answer of this question by showing that the non-uniform hyperbolic systems contain a $C^1$-open and $C^r$-dense subset of ergodic IFSs generated by conservative surface $C^r$-diffeomorphisms. In higher dimension we get the same result for the extremal Lyapunov exponents.
1.1. Random products of conservative diffeomorphisms. An iterated function system (IFS) can also be thought of as a finite collection of functions which can be applied successively in any order. We will focus in the study of IFSs generated by $C^r$-diffeomorphisms $f_1, \ldots, f_k$ of a finite-volume Riemannian manifold $M$ of dimension $d \geq 2$ which preserve the normalized Lebesgue measure $m$. We will denote

$$f_{\omega}^0 = \text{id}, \quad f_{\omega}^n = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0} \quad \text{for } \omega = (\omega_i)_{i \geq 0} \in \Omega_+ \overset{\text{def}}{=} \{1, \ldots, k\}^\mathbb{N} \text{ and } n > 0.$$

According to the random Oseledec’s multiplicative theorem [LQ06] there are real numbers $\lambda_-(x) \leq \lambda_+(x)$ called extremal Lyapunov exponents such that for $\mathbb{P}_+$-almost every $\omega \in \Omega_+$ and $m$-almost every $x \in M$,

$$\lambda_-(x) = \lim_{n \to +\infty} \frac{1}{n} \log \|Df_{\omega_0}^n(x)^{-1}\|^{-1} \quad \text{and} \quad \lambda_+(x) = \lim_{n \to +\infty} \frac{1}{n} \log \|Df_{\omega_0}^n(x)\|.
$$

Here $\mathbb{P}_+ = p^\mathbb{Z}$ is a Bernoulli measure on $\Omega_+$ where $p = p_1 \delta_1 + \cdots + p_k \delta_k$ with $p_1 > 0$ and $p_1 + \cdots + p_k = 1$. We will assume that $m$ is an ergodic measure for the group generated by $f_1, \ldots, f_k$. This means that any strictly $f_i$-invariant measurable subset of $M$ for all $i = 1, \ldots, k$ has either null or co-null measure. Hence, $\lambda_{\pm} = \lambda_{\pm}(x)$ are constant $m$-almost everywhere. Thus, $\lambda_{\pm}$ only depends on the conservative $C^r$-diffeomorphisms $f_1, \ldots, f_k$. Actually, these Lyapunov exponents also depend on the Bernoulli probability but we will consider the weights $p_i$ fixed and thus we will not explicit this dependance. Thereby we will denote $\lambda_{\pm} = \lambda_{\pm}(f_1, \ldots, f_k)$. Moreover, since the diffeomorphisms are conservative, we have that the sum of all Lyapunov exponents must be zero, so that in particular, $\lambda_- \leq 0 \leq \lambda_+$.

Our main result is the following:

**Theorem A.** Given $r \geq 1$ and $k \geq 2$, consider conservative $C^r$-diffeomorphisms $f_1, \ldots, f_{k-1}$ of $M$ such that the Lebesgue measure is ergodic for the group generated by these maps. Then there is a $C^1$-open and $C^r$-dense set $\mathcal{U}$ of conservative $C^r$-diffeomorphisms such that for any $f_k$ in $\mathcal{U}$,

$$\lambda_-(f_1, \ldots, f_k) < 0 < \lambda_+(f_1, \ldots, f_k).$$

After the conclusion of this work, Obata and Poletti sent us a preprint [OP18] where they get a similar result using different approach. They proved that the set of systems with positive integrated extremal Lyapunov exponent contains a $C^1$-open and $C^1$-dense subset of random products of i.i.d. conservative diffeomorphisms of a compact connected oriented surface.

The scheme of the proof of Theorem A is the following. Let us denote $\mathbb{P}(TM)$ the projective tangent space of $M$. Given a diffeomorphism $f$ the differential cocycle $(f, Df)$ naturally acts on $\mathbb{P}(TM)$. First, by using an invariance principle, we obtain that if $\lambda_-(f_1, \ldots, f_k) = \lambda_+(f_1, \ldots, f_k)$ then the cocycles $(f_i, Df_i)$ share a common invariant measure on $\mathbb{P}(TM)$ projecting on $m$. Then, we characterize the invariant measures by $(f_i, Df_i)$ projecting on $m$ for all $i = 1, \ldots, k$ as product measures. The ergodicity assumption will be used at this point. Finally, we prove that any conservative $C^r$-diffeomorphism $f$ can be perturbed so that $(f, Df)$ do not have any invariant measure given by the previous characterization concluding the result. The main step is the second one, that is, the classification of the invariant measure. We will actually classify the invariant measures of measurable cocycles in a more general setup.
1.2. Measurable random cocycles. Consider an invertible measure preserving transformation \( f \) of a standard Borel probability space \((X, \mu)\). Let \( G \) be a locally compact topological group whose operation is denoted by juxtaposition. Given a measurable function \( A : X \to G \) we define the \( G \)-valued cocycle over \( f \) by the dynamical defined products
\[
A^0(x) = \text{id}, \quad A^n(x) = A(f^{n-1}(x)) \cdots A(f(x))A(x) \quad \text{and} \quad A^{-n}(x) = A^n(f^{-n}(x))^{-1} \quad \text{if} \quad n > 0.
\]
As usual, we denote this cocycle by \((f, A)\). We say that \((f, A)\) is a linear cocycle if \( G \) is a subgroup of the group \( \text{GL}(d) \) of invertible \( d \times d \) matrices. We can always neglect sets of null measure, and we shall identify cocycles which coincide \( \mu \)-almost surely.

There is a natural group structure on the set of \( G \)-cocycles over invertible transformations on \((X, \mu)\). The product of two cocycles \((f, A)\) and \((g, B)\) is defined by
\[
(g, B) \cdot (f, A) = (g \circ f, (B \circ f)A).
\]
In particular, the powers of \((f, A)\) are given by \((f, A)^n = (f^n, A^n)\). Thus, the natural way to study the action of several cocycles is the notion of cocycle over a group. If \( T \) is a group of measure preserving transformations of \((X, \mu)\), the classical definition (see [Zim84]) of a \( G \)-valued cocycle over \( T \) is a Borel function \( \alpha : T \times X \to G \) such that
\[
\alpha(ts, x) = \alpha(t, s(x))\alpha(s, x) \quad \text{for all} \quad t, s \in T \quad \text{and} \quad \mu\text{-almost every} \quad x \in X.
\]
Then, given invertible \( \mu \)-preserving transformations \( f_i : X \to X \) and measurable functions \( A_i : X \to G \) for all \( i \in Y \), one can define a cocycle \( \alpha \) over the group \( T \) generated by these transformations such that \( \alpha(f_i(x)) = A_i(x) \). However, we will not use this formalism. With our terminology, by denoting \( A(t) = \alpha(t, x) \) for \( t \in T \), we identify \( \alpha \) with the group of cocycles \( \hat{T} = \{(t, A_t) : t \in T\} \). In order to study the action of several cocycles, instead of looking at a cocycle over a group, we will actually prefer to use the the formalism of random dynamical systems, by defining the notion of random cocycles as random walks on the group \( \hat{T} \). To do this, we will start by considering a particular class of cocycles.

1.2.1. Random cocycle. Denote by \( \Omega \) the product space \( Y^Z \) endowed with the product measure \( \mathbb{P} = p^Z \) where \((Y, p)\) is some probability space. Let \( \theta : \Omega \to \Omega \) be the shift map on \( \Omega \). Set \( \tilde{X} = \Omega \times X \) and \( \tilde{\mu} = \mathbb{P} \times \mu \). A random transformation is a measurable invertible skew-shift
\[
f : \tilde{X} \to \tilde{X}, \quad f(x) = (\theta \omega, f_\omega(x)) \quad \text{and} \quad \tilde{x} = (\omega, x) \in \tilde{X}
\]
where \( f_\omega = f_i \) depends only on the zeroth coordinate \( \omega_0 = i \) of \( \omega = (\omega_n)_{n \in \mathbb{Z}} \in \Omega \). We will assume that \( \tilde{\mu} \) is a \( f \)-invariant ergodic measure. Observe that the invariance of this measure implies that \( \mu \) is a \( f_i \)-invariant measure \( p \)-almost surely (cf. [LQ06]). Finally,

Definition 1.1. We say that \((f, A)\) is a random \( G \)-valued cocycle if \( A : \tilde{X} \to G \) defines a cocycle over a \( \tilde{\mu} \)-preserving ergodic random transformation \( f : \tilde{X} \to \tilde{X} \) such that
\[
A(\tilde{x}) = A_t(x) \quad \text{for} \quad \tilde{\mu} \text{-almost every} \quad \tilde{x} = (\omega, x) \in \tilde{X} = \Omega \times X \quad \text{with} \quad \omega = (\omega_n)_{n \in \mathbb{Z}} \quad \text{and} \quad \omega_0 = i.
\]
Obviously, a deterministic cocycle is a particular case of random cocycle by taking \( \Omega \) being a one-point space. Thus, we can also see the set of random cocycles as an extension of the deterministic case.
1.2.2. Cohomologous random cocycles. Two G-valued cocycles \((f, A)\) and \((f, B)\) over the same \(\mu\)-preserving invertible transformation \(f : X \to X\) are called cohomologous if there exists a measurable map \(P : X \to G\) such that

\[
B(x) = P(f(x))^{-1}A(x)P(x) \quad \text{for \(\mu\)-almost all } x \in X.
\]

In other words, \((f, B) = \phi_P^{-1} \circ (f, A) \circ \phi_P\) where \(\phi_P\) denotes the cocycle \((\text{id}, P)\). Thus the relation of cohomology defines an equivalence between measurable cocycles. We are interested to study the measurable cohomological reduction of random cocycles. First, observe that the cohomology class of a random cocycle could contain non-random cocycles. To keep it into study the measurable cohomological reduction of random cocycles. First, observe that the cohomology class of a random cocycle could contain non-random cocycles. To keep it into the class of random cocycles we need to ask that the conjugacy \(P : X \to G\) actually does not depend on \(\omega \in \Omega\). That is,

**Definition 1.2.** Two random G-valued cocycles \((f, A)\) and \((f, B)\) are cohomologous if and only if there is a measurable function \(P : X \to G\) such that

\[
B(\bar{x}) = P(f_\omega(x))^{-1}A(x)P(x) \quad \text{for } \bar{\mu}\text{-almost every } \bar{x} = (\omega, x) \in \bar{X} = \Omega \times X.
\]

Equivalently, if for \(\nu\)-almost every \(i \in Y\) it holds that

\[
B_i(x) = P(f_i(x))A_i(x)P(x) \quad \nu\text{-almost every } x \in X.
\]

1.2.3. Invariant measure of random cocycles. Let \((f, A)\) be a random G-valued cocycle. Since these cocycles are locally constant, i.e., only depends on the zeroth coordinate of the random sequence, for each \(i \in Y\) we have a G-valued cocycle \((f_i, A_i)\). Denote by \(Z\) a G-space. That is, a topological space where G acts by automorphisms. The cocycles \((f_i, A_i)\) naturally act on \(X \times Z\) by means of the skew-product maps

\[
F_i \equiv (f_i, A_i) : X \times Z \to X \times Z, \quad F_i(x, z) = (f_i(x), A_i(x)z).
\]

**Definition 1.3.** A measure \(\hat{\mu}\) on \(X \times Z\) is called a \((\bar{f}, A)\)-invariant if \(\hat{\mu}\) projects on \(\mu\) and it is \(F_i\)-invariant for \(\nu\)-almost every \(i \in Y\). Similarly, \(\hat{\mu}\) is called \((\bar{f}, A)\)-stationary if \(\hat{\mu}\) project on \(\mu\) and

\[
\hat{\mu} * p \overset{\text{def}}{=} \int_Y F_i \hat{\mu} dp(i) = \hat{\mu}.
\]

1.3. Characterization of invariant measures of random linear cocycles. We are interested in the projective invariant measures of random linear cocycles \((f, A)\). That is, we want to understand the \((\bar{f}, A)\)-invariant measure on \(X \times Z\) of a random G-valued cocycles where \(G = \text{GL}(d)\) and \(Z = \mathbb{P}(\mathbb{R}^d)\) is the projective space of \(\mathbb{R}^d\). This problem was previously studied by Arnold, Cong and Osceledets in [ACO97] using ingredients of the Zimmer amenable reduction theorem [Zim84]. They proved that up to cohomology all the projective invariant measure are deterministic measures:

**Theorem B ([ACO97, Thm. 3.8]).** Let \((\bar{f}, A)\) be a random GL(d)-valued cocycle. Then there is a random linear cocycle \((\bar{f}, B)\) cohomologous to \((\bar{f}, A)\) such that every \((\bar{f}, B)\)-invariant measure \(\hat{\mu}\) on \(X \times \mathbb{P}(\mathbb{R}^d)\) is a product measure, i.e., it is of the form \(\hat{\mu} = \mu \times \nu\) where \(\nu\) is a measure on \(\mathbb{P}(\mathbb{R}^d)\).
1.4. Invariant principle for random linear cocycles. Let \((f, A)\) be a random linear cocycle. When \(\log^+ \|A^{\pm 1}\|\) are both \(\tilde{\mu}\)-integrable functions, by Furstenberg-Kesten theorem (see [Via14]) there are real numbers \(\lambda_-(A) \leq \lambda_+(A)\), called extremal Lyapunov exponent of \((f, A)\), such that

\[
\lambda_-(A) = \lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x)\|^{-1} \quad \text{and} \quad \lambda_+(A) = \lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x)\|
\]

for \(\tilde{\mu}\)-almost every \(x \in \tilde{X}\). An invariant principle is an statement on the rigidity of the projective invariant measure under the assumption of the coincidence of extremal Lyapunov exponents. As a consequence of an invariant principle of Ledrappier [Led86] (see also [Cra90, Via14]) and the classification of invariant measure obtained in Theorem B, we will get the following:

**Theorem C.** Let \((f, A)\) be a random \(\text{GL}(d)\)-valued cocycle satisfying the integral conditions. If \(\lambda_-(A) = \lambda_+(A)\) then there is a random linear cocycle \((\tilde{f}, B)\) cohomologous to \((f, A)\) and a probability measure \(\nu\) on \(\mathcal{P}(\mathbb{R}^d)\) such that \(B_i(x) \nu = \nu\) for \(p\)-almost every \(i \in Y\) and \(\mu\)-almost every \(x \in X\).

**Remark 1.4.** In Theorem B and C the random \(\text{GL}(d)\)-valued cocycles \((f, A)\) and \((\tilde{f}, B)\) are cohomologous by means of a measurable function \(P : X \to \text{SL}^\pm(d)\). Here \(\text{SL}^\pm(d)\) denotes the group of \(d \times d\) matrices with determinant \(\pm 1\).

1.5. Organization of the paper. In the following sections we will prove Theorem A using Theorem B and assuming Theorem C. The proof of Theorem C will be provided in Section \(\S 3\). Finally, in Section \(\S 4\) we will give a direct (without using Zimmer amenable reduction theory) and alternative proof (different from [ACO97]) of Theorem B in the case \(d = 2\). We believe that this work can be helpful to understand better Theorem B and provides a self-contained proof of the main result (Theorem A) at least in dimension two.

2. Extremal exponents: Proof of Theorem A

We fix a finite-volume Riemannian manifold \(M\) of dimension \(d \geq 2\). At several places, we will need to look the matrix of a differential of a diffeomorphism of \(M\) in some basis. In view of this, we denote by \(\mathcal{B}\) the set of measurable maps \(P\) on \(M\) such that \(P(x)\) is a linear map from \(T_xM\) to \(\mathbb{R}^d\) for any \(x \in M\). We also denote by \(\mathcal{O}\) the subset of \(\mathcal{B}\) such that for any \(P \in \mathcal{O}\) we have that \(P(x)\) is an isometry (where \(\mathbb{R}^d\) is endowed with the Euclidean norm and \(T_xM\) with the Riemannian structure). Then, given \(P \in \mathcal{B}\) and a \(C^r\)-diffeomorphism \(f\) preserving the normalize Lebesgue measure \(m\), we define

\[
J_P f(x) = P(f(x)) \circ Df(x) \circ P(x)^{-1} \quad \text{for all} \ x \in M.
\]

Thus, \(J_P f(x)\) belongs to \(\text{GL}(d)\) and actually to \(\text{SL}^\pm(d)\) if \(P \in \mathcal{O}\). Similarly we define the class

\[
\mathcal{J} = \{ P \in \mathcal{B} : P = RQ \text{ with } R \in \text{SL}^\pm(d) \text{ and } Q \in \mathcal{O}\}.
\]

Notice that \(\mathcal{O} \subset \mathcal{J} \subset \mathcal{B}\) and for \(J_P f(x) \in \text{SL}^\pm(d)\) for all \(P \in \mathcal{J}\).
2.1. **Invariant principle.** We fix $k \geq 2$ and let $f_1, \ldots, f_k$ be $C^r$-diffeomorphisms of $M$ preserving $m$. Assume that the group generated by these maps is ergodic with respect to $m$. Consider a probability measure $p = p_1 \delta_1 + \cdots + p_k \delta_k$ on $Y = \{1, \ldots, k\}$ and set $\bar{\Omega} = \mathbb{P} \times \mathbb{Z}$. Let $\bar{f}$ be the skew-shift on $\bar{M} = \bar{\Omega} \times M$ given by $\bar{x} = (\omega, x) \mapsto (\theta \omega, f_\omega(x))$ where $f_\omega = f_i$ if the zeroth coordinate of $\omega = (\omega_n)_{n \in \mathbb{Z}}$ is $\omega_0 = i$. Observe that $\bar{f}$ preserves the measure $\bar{m} = \mathbb{P} \times m$. For a fixed $F \in \mathcal{B}$ we define the measurable map

$$J_F \bar{f} : \bar{M} \to \text{GL}^+(d), \quad J_F \bar{f}(\bar{x}) = J_F f_\omega(x) \text{ for } \bar{x} = (\omega, x) \in \bar{M}.$$ Then, $(\bar{f}, J_F \bar{f})$ is a random $\text{GL}(d)$-valued cocycle.

As we did in the introduction, we define the extremal Lyapunov exponents $\lambda_\pm(f_1, \ldots, f_k)$. Notice that if $\lambda_-(f_1, \ldots, f_k) = \lambda_+(f_1, \ldots, f_k)$ then $\lambda_-(J_F \bar{f}) = \lambda_+(J_F \bar{f})$ for all $F \in \mathcal{B}$. Thus, as a consequence of Theorem C (actually of Proposition 3.1), we have the following result.

**Proposition 2.1.** Assume that $\lambda_-(f_1, \ldots, f_k) = \lambda_+(f_1, \ldots, f_k)$. Then for every $F \in \mathcal{B}$, the random linear cocycle $(\bar{f}, J_F \bar{f})$ admits an invariant measure.

**Proof.** Let $Q \in \mathcal{B}$. Consider the random $\text{SL}^+(d)$-valued cocycle $(\bar{f}, J_Q \bar{f})$. From the assumption we obtain that $\lambda_-(J_Q \bar{f}) = \lambda_+(J_Q \bar{f})$. Hence, according to Theorem C, we find a random cocycle $(\bar{f}, B)$ cohomologous to $(\bar{f}, J_Q \bar{f})$ with a product invariant measure on $M \times \mathbb{P}(\mathbb{R}^d)$. In particular $(\bar{f}, J_Q \bar{f})$ has an invariant measure. This proves the proposition since for any $F \in \mathcal{B}$, $J_F f_i(x) = R^{-1}(\bar{f}_i(x))J_Q f_i(x)R(x)$ where $R(x) = Q(x)P(x)^{-1}$. □

2.2. **Breaking the invariance by perturbation.** Now, we are going to prove the following:

**Proposition 2.2.** For any $F \in \mathcal{B}$, the set $\mathcal{R}_F$ of the maps $f$ in $\text{Diff}^r_m(M)$ such that $(f, J_F f)$ has no invariant measure on $M \times \mathbb{P}(\mathbb{R}^d)$ of the form $m \times \nu$ is open and dense in $\text{Diff}^r_m(M)$.

The openness of $\mathcal{R}_F$ is obvious so we focus in proving the density. A small issue is the a priori absence of continuity points of $F$. We bypass the problem by defining points of regularity of $F$ in a weaker sense: by Lusin Theorem, there exists a increasing sequence $(E_j)_{j \in \mathbb{N}}$ of measurable subsets of $M$ whose Lebesgue measure goes to 1 such that $F$ restricted to $E_j$ is continuous. Moreover, up to replace $E_j$ by the subset of its density points, we can assume that every point of $E_j$ is a density point. Finally we set $E = \bigcup_j E_j$. The set $E$ has full measure and will be considered as the regularity points of $F$. The following lemma express the way we will use this regularity:

**Lemma 2.3.** Let $f$ be a conservative $C^r$-diffeomorphism of $M$ and consider $x_0 \in E \cap f^{-1}(E)$. If the linear cocycle $(f, J_F f)$ has an invariant measure of the form $m \times \nu$, then $J_F f(x_0)\nu = \nu$.

**Proof.** The assumption implies that $J_F f(x)\nu = \nu$ for $m$-almost every $x \in M$. In particular this equality holds for $m$-almost every $x$ in $E_j$. Take $j$ large enough so that $x_0$ is a density point of $E_j \cap f^{-1}(E_j)$. Since $J_F f$ is continuous on this set, the equality holds for $x = x_0$. □

Now, for $\alpha = (a, a', b, b')$ in $E^4$ with $a \neq b$ we consider the set

$$\mathcal{D}_\alpha = \{f \in \text{Diff}^r_m(M) : f(a) = a' \text{ and } f(b) = b'\}.$$
We defined also the application
\[ \Phi : \mathcal{D}_a \rightarrow \text{SL}^+(d) \times \text{SL}^+(d), \quad \Phi(f) = (Pf(a), Pf(b)). \]

We have the following properties.

**Lemma 2.4.** Consider \( f \in \mathcal{D}_a \) and let \( \mathcal{V} \) be a neighborhood of \( f \) in \( \text{Diff}_m^+(M) \). Then \( \Phi(\mathcal{V} \cap \mathcal{D}_a) \) has non-empty interior and \( \Phi(\mathcal{V} \cap \mathcal{D}_a \setminus \mathcal{R}_P) \) has empty interior.

**Proof.** We write \( g = f \circ u \) for the elements of \( \mathcal{V} \cap \mathcal{D}_a \). The map \( u \) runs over the set \( \mathcal{V}_0 \cap \mathcal{D}_0 \) where \( \mathcal{V}_0 \) is some neighborhood of \( \text{id} \) in \( \text{Diff}_m^+(M) \) and
\[ \mathcal{D}_0 = \{ u \in \text{Diff}_m^+(M) : u(a) = a \text{ and } u(b) = b \}. \]

A simple computation gives
\[ \Phi(g) = (Pf(a)P(a)Du(a)P(a)^{-1}, Pf(b)P(b)Du(b)P(b)^{-1}). \]

Now, since \( \text{SL}^+(d) \) is a normal subgroup of \( \text{GL}(d) \) and \( a \neq b \), the range of the map
\[ u \in \mathcal{B}_0 \cap \mathcal{D}_0 \mapsto (P(a)Du(a)P(a)^{-1}, P(b)Du(b)P(b)^{-1}) \]
contains an open ball in \( \text{SL}^+(d) \times \text{SL}^+(d) \). Hence, the range of \( \Phi \) restricted to \( \mathcal{V} \cap \mathcal{D}_a \) also contains an open set. This concludes the first item.

We will prove now that \( \Phi(\mathcal{V} \cap \mathcal{D}_a \setminus \mathcal{R}_P) \) has empty interior in \( \text{SL}^+(d) \times \text{SL}^+(d) \). Indeed, if \( f \) belongs to the complement of \( \mathcal{R}_P \) then \( (f, Pf) \) has an invariant measure of the form \( m \times \nu \). Moreover, if \( f \in \mathcal{D}_a \) then, by Lemma 2.3, we have that \( Pf(a)\nu = Pf(b)\nu = \nu \). In particular \( \Phi(\mathcal{V} \cap \mathcal{D}_a \setminus \mathcal{R}_P) \) is included in the set of pair of matrices \( (A, B) \) in \( \text{SL}^+(d) \times \text{SL}^+(d) \) such that there is a probability measure \( \nu \) on \( \mathcal{P}(\mathbb{R}^d) \) satisfying that \( A\nu = B\nu = \nu \). From [ASV13, Prop. 8.14 and Rem. 8.15] (see also [Via14, Sec. 7.4.2]) this set has empty interior in \( \text{SL}^+(d) \times \text{SL}^+(d) \) and consequently \( \Phi(\mathcal{V} \cap \mathcal{D}_a \setminus \mathcal{R}_P) \) also has it. This completes the proof. \( \square \)

Using the above lemma, we can prove the density of \( \mathcal{R}_P \):

**Proof of Proposition 2.2.** Let \( \mathcal{V} \) be any open ball of \( \text{Diff}_m^+(M) \). Consider \( f \in \mathcal{V} \) and take any pair of different points \( a, b \) in \( E \cap f^{-1}(E) \). Setting \( u = (a, f(a), b, f(b)) \) we have that \( \mathcal{V} \cap \mathcal{D}_a \) is not empty (it contains \( f \)). From Lemma 2.4, we obtain that \( \mathcal{V} \cap \mathcal{D}_a \) cannot be included in \( \mathcal{D}_a \setminus \mathcal{R}_P \). Hence \( \mathcal{V} \) must intersect \( \mathcal{R}_P \). Thus \( \mathcal{R}_P \) is dense. As we previously mentioned, \( \mathcal{R}_P \) is also open and then we conclude the proof of the proposition. \( \square \)

### 2.3. Proof of Theorem A

Let \( f_1, \ldots, f_{k-1} \) be \( C^r \)-diffeomorphisms of \( M \) preserving \( m \) and such that it group action is ergodic. Consider \( Q \in \mathcal{O} \) and take the random cocycle \((\hat{f}^r, J_0\hat{f}^r)\) defined by these \( k-1 \) maps. According to Theorem B, we find a random cocycle \((\tilde{f}^r, B)\) cohomologous to \((\hat{f}^r, J_0\hat{f}^r)\) such that every \((\tilde{f}^r, B)\)-invariant measure on \( M \times \mathcal{P}(\mathbb{R}^d) \) is a product measure. Moreover, according to Remark 1.4, there is \( R : M \rightarrow \text{SL}^+(d) \) such that \( B_i(x) = R(f_i(x))^{-1}J_0f_i(x)R(x) \) for \( m \)-almost every \( x \in M \) and every \( i = 1, \ldots, k-1 \). In particular, taking \( P(x) = R(x)^{-1}Q(x) \in \mathcal{F} \) we have that \( B_i(x) = Pf_i(x) \). Thus, \( B = Pf^r \).
By Proposition 2.2, one can find a dense open set $\mathcal{R}_P$ such that $(f, J_P f)$ has no invariant measure on $M \times \mathcal{P}(\mathbb{R}^d)$ of the form $m \times \nu$. In consequence, for every $f_k$ in $\mathcal{R}_P$ the maps $(f_i, J_P f_i)$ for $i = 1, \ldots, k$ have not a common invariant probability measure projecting on $m$ of the form $m \times \nu$. Moreover, the random cocycle $(\tilde{f}, J_P \tilde{f})$ defined by theses $k$ maps does not admit any invariant probability measure. Indeed, if $\tilde{\mu}$ is a $(\tilde{f}, J_P \tilde{f})$-invariant probability measure also it is $(\tilde{f}, J_P \tilde{f})$-invariant. Consequently $\tilde{\mu} = m \times \nu$ and thus $(f_k, J_P f_k)$ has a product invariant measure which is not possible. Hence, according to Proposition 2.1, $\lambda_+(f_1, \ldots, f_k) \neq \lambda_+(f_1, \ldots, f_k)$. This concludes Theorem A.

3. Invariant principle: Proof of Theorem C

Let $\tilde{(f}, A)$ be a random $\text{GL}(d)$-valued cocycle. Suppose that $\log^+ \|A^k\|$ are both $\tilde{\mu}$-integrable functions and recall the definition of the extremal Lyapunov exponents $\lambda_+(A)$ given in the introduction.

**Proposition 3.1.** Let $(\tilde{f}, A)$ be a random cocycle as above and assume that $\lambda_-(A) = \lambda_+(A)$. Then there exists a $(\tilde{f}, A)$-invariant probability measure $\tilde{\mu}$ on $X \times \mathcal{P}(\mathbb{R}^d)$.

**Proof.** Since $\mathcal{P}(\mathbb{R}^d)$ is a compact metric space, by Proposition A.2 in Appendix A we can take a $(\tilde{f}, A)$-stationary probability measure $\tilde{\mu}$ on $X \times \mathcal{P}(\mathbb{R}^d)$. We are going to prove that under the assumption $\lambda_-(A) = \lambda_+(A)$, the probability $\tilde{\mu}$ is in fact invariant. This measure has some disintegration $d\tilde{\mu} = \nu_x d\mu(x)$. We want to prove that this measure is invariant which is equivalent to say that $\nu_{f(x)} = A_i(x)\nu_x$ for $\mu$-almost every $x$ and $p$-almost every $i$ in $Y$.

Let us define the non-invertible skew-shift associated with the random dynamics. We set $\Omega_+ = Y^N$, $\mathcal{P}_+ = p^N$ and $\theta : \Omega_+ \to \Omega_+$ the shift map. Then we set $\bar{X}_+ = \Omega_+ \times X$, $\bar{\mu}_+ = \mathcal{P}_+ \times X$, and $f_\omega : \bar{X}_+ \to \bar{X}_+$ given by $\bar{x} = (\omega, x) \mapsto (\theta(\omega), f_\omega(x))$ where $\omega = (\omega_n)_{n \geq 0}$ and $\omega_0 = i$. The triplet $(\bar{X}_+, \bar{\mu}_+, f_\omega)$ is a non-invertible ergodic measure preserving dynamical system. Observe that since $A : \bar{X} \to \text{GL}(d)$ is locally constant then it also define a $\text{GL}(d)$-valued cocycle over this non-invertible system which we simply denote by $(\tilde{f}_+, A)$. Notice that, by the assumption, the extremal Lyapunov exponents of this new linear cocycle are also coincident.

Now we use the non-invertible version of the invariance principle of Ledrappier [Led86, Prop. 2 and Thm. 3] (see also [Via14, Thm. 7.2]). First, we consider the action of this cocycle $(\tilde{f}_+, A)$ on $\bar{X}_+ \times \mathcal{P}(\mathbb{R}^d)$. Then, since $\tilde{\mu}$ is an $(\tilde{f}, A)$-stationary measure then the measure $\eta_+ = \mathcal{P}_+ \times \tilde{\mu}$ is invariant for this action (see [LQ06]). Its projection on $\bar{X}_+$ is $\bar{\mu}_+$ and the corresponding disintegration is $d\eta_+ = \nu_x d\bar{\mu}(\bar{x})$ where $\nu_x = \nu_x$ only depends on the second coordinate of $\bar{x} = (\omega, x) \in \Omega_+ \times X = \bar{X}_+$. The invariance principle of Ledrappier says that the coincidence of the extremal Lyapunov exponents implies the equality $\nu_{f(x)} = A_i(x)\nu_x$ for $\bar{\mu}_+$-almost every $\bar{x} = (\omega, x) \in \bar{X}_+$. This is equivalent to say that $\nu_{f(x)} = A_i(x)\nu_x$ for $\mu$-almost every $x \in X$ and $p$-almost every $i \in Y$. This concludes the proof. □

As a corollary of the above proposition and Theorem B we get Theorem C:


Proof of Theorem C. If $(f, A)$ is a random $GL(d)$-valued cocycle satisfying the integrability conditions and $\lambda_{\ast}(A) = \lambda_{+}(A)$, according to Proposition 3.1 we get a $(\bar{f}, A)$-invariant probability measure $\hat{\mu}$ on $X \times \mathbb{P}(\mathbb{R}^d)$. From Theorem B, there is a random linear cocycle $(\bar{f}, B)$ cohomologous to $(f, A)$ such that every $(\bar{f}, B)$-invariant probability measure is a product measure. In particular, the $(\bar{f}, A)$-invariant measure $\hat{\mu}$ is transported to a $(\bar{f}, B)$-invariant measure $\hat{\mu}_1$. Then we get that $\hat{\mu}_1 = \mu \times \nu$ where $\nu$ is a probability measure on $\mathbb{P}(\mathbb{R}^d)$. Consequently $B_i(x)\nu = \nu$ for $p$-almost every $i \in Y$ and $\mu$-almost $x \in X$. This completes the proof.  \[\Box\]

4. Classification of invariant measures: Proof of Theorem B

In this section we are going to prove Theorem B (in dimension two). We did this by splitting the proof in two parts. First we will deal with essentially unbounded random linear cocycles (see definition below). After that, we will prove Theorem B for essentially bounded random cocycle as a consequence of a more general theorem on the classification of invariant measures of random cocycles with values to an arbitrary group $G$.

We say that a sequence $(u_n)_n$ of real numbers converges essentially to infinity if for every $K > 0$ the lower asymptotic density of $D_K = \{n \geq 0 : u_n \leq K\}$ is zero. That is, if

$$d(D_K) \overset{\text{def}}{=} \liminf_{n \to \infty} \frac{\#\{0, n] \cap D_K\}}{n} = 0.$$  

Let $(f, A)$ be a linear cocycle over an ergodic preserving invertible transformation $f$ of a standard Borel probability space $(X, \mu)$. Notice that the set of points $x \in X$ such that the sequence $(u_n)_n$ with $u_n = \|A^n(x)\|$ converges essentially to infinity is $f$-invariant. Thus, by the ergodicity of the measure $\mu$, this set is either $\mu$-null or $\mu$-conull. This implies the following dichotomy:

i) $\|A^n(x)\|$ converges essentially to infinity for $\mu$-almost every $x \in X$;

ii) $\|A^n(x)\|$ does not converge essentially to infinity for $\mu$-almost every $x \in X$.

This dichotomy allows us to classify the linear cocycles as follows:

Definition 4.1. A linear cocycle $(f, A)$ is said to be essentially unbounded if (i) holds. Otherwise, i.e., in the case (ii), or equivalently, if there are $K > 0$, $\delta > 0$ and a set $E \subset X$ of positive $\mu$-measure such that $d(\{n \geq 0 : \|A^n(x)\| \leq K\}) \geq \delta$ for all $x \in E$, the cocycle $(f, A)$ is called essentially bounded.

Notice that a random linear cocycle $(f, A)$ is a particular case of a linear cocycle over an ergodic $\bar{\mu}$-preserving invertible transformation $\bar{f}$. In particular, the notions of essentially bounded and essentially unbounded linear cocycle applies for random linear cocycles.

4.1. Essentially unbounded cocycles. We are going to characterize first the invariant measures of essentially unbounded deterministic linear cocycles $(f, A)$. To study these measures, we can normalize the cocycles by dividing $A$ by $|\det A|^{1/2}$, and hence assume that it is a random $SL^+(d)$-valued cocycle.
Theorem 4.2. Let \((f, A)\) be a essentially unbounded \(SL^+(2)\)-valued cocycle. Then, there are measurable families of one-dimensional linear subspaces \(E_1(x)\) and \(E_2(x)\) such that any \((f, A)\)-invariant measure \(\hat{\mu}\) on \(X \times \mathbb{P}(\mathbb{R}^2)\) is of the form
\[
d\hat{\mu} = (\lambda d\mu_{E_1(x)} + (1 - \lambda) d\mu_{E_2(x)})
\]
for some \(0 \leq \lambda \leq 1\).

Remark 4.3. If the linear cocycle satisfies the integrability conditions \(\log^+ ||A^x|| \in L^1(\mu)\) and has different extremal Lyapunov exponents one gets that \(E_1 = E^+\) and \(E_2 = E^-\) where \(\mathbb{R}^2 = E^+(x) \oplus E^-(x)\) is the Oseledets decomposition. See [Via14, Lemma 5.25].

Before proving the above theorem, we will get Theorem B in the case of essentially unbounded random \(GL(2)\)-valued cocycle.

Proposition 4.4. Let \((\tilde{f}, A)\) be an essentially unbounded random \(GL(2)\)-valued cocycle. Then there is a random cocycle \((f, B)\) cohomologous to \((\tilde{f}, A)\) such that every \((f, B)\)-invariant probability measure \(\hat{\mu}\) on \(X \times \mathbb{P}(\mathbb{R}^2)\) is of the form \(\hat{\mu}_0 = \mu \times \nu\) where \(\nu\) is a probability measure on \(\mathbb{P}(\mathbb{R}^2)\).

Proof. Let \((\tilde{f}, A)\) be a random \(GL(2)\)-valued cocycle. Since the action of \(A\) and \(A \cdot |\det A|^{-1/2}\) on \(\mathbb{P}(\mathbb{R}^2)\) coincides we can assume that \((f, A)\) is a random \(SL^+(2)\)-valued cocycle. Now, let \(\hat{\mu}\) be a \((f, A)\)-invariant measure on \(X \times \mathbb{P}(\mathbb{R}^2)\). That is, \(\hat{\mu}\) is a \((f, A)\)-invariant measure for \(p\)-almost every \(i \in Y\) of the form \(d\hat{\mu} = \nu_x d\mu(x)\) with \(\nu_x\) a measure on \(\mathbb{P}(\mathbb{R}^2)\). Consider the product measure \(\hat{\mu}_0 = \mathbb{P} \times \hat{\mu}\) on \(\Omega \times (X \times \mathbb{P}(\mathbb{R}^2))\). Observe that \(\hat{\mu}_0\) is an invariant measure for the skew-product \(F \equiv (f, A) : X \times \mathbb{P}(\mathbb{R}^2) \to X \times \mathbb{P}(\mathbb{R}^2)\). Moreover, since \(d\hat{\mu}_0 = \nu_x d\mu(x)d\mathbb{P}(\omega)\), \(\hat{\mu}_0\) projects down on \(\tilde{X} = \Omega \times X\) over \(\hat{\mu} = \mathbb{P} \times \mu\). According to Theorem 4.2, there are \(0 \leq \lambda \leq 1\) and measurable families of one-dimensional linear subspace \(E_1(\tilde{x})\) and \(E_2(\tilde{x})\) such that
\[
\nu_x = \lambda d\mu_{E_1(\tilde{x})} + (1 - \lambda) d\mu_{E_2(\tilde{x})}
\]
for some \(\tilde{x}\)-almost every \(\tilde{x} = (\omega, x) \in \tilde{X} = \Omega \times X\).

Since \(\nu_x\) does not depend on \(\omega\), then \(E_1\) and \(E_2\) does not depend either. Thus,
\[
\nu_x = \lambda d\mu_{E_1(x)} + (1 - \lambda) d\mu_{E_2(x)}
\]
for some \(\mu\)-almost every \(x \in X\).

Then \(E_1(x)\) and \(E_2(x)\) are \(A_i\)-invariant linear subspace for \(p\)-almost every \(i \in Y\), i.e.,
\[
A_i(x)[E_1(x), E_2(x)] = [E_1(f_i(x)), E_2(f_i(x))] \text{ for } \mu\text{-almost every } x \in X.
\]

Assume first that \(\mu\)-almost surely, \(\mathbb{R}^2 = E_1(x) \oplus E_2(x)\). Hence, we take in a measurable way \(P(x) = [U(x), V(x)]\) where \(U(x)\) and \(V(x)\) are vectors in \(E_1(x)\) and \(E_2(x)\) respectively. Then for \(p\)-almost every \(i \in Y\),
\[
[R_i(x)U(f_i(x)), S_i(x)V(f_i(x))] = [A_i(x)U(x), A_i(x)V(x)].
\]

Thus, we get that
\[
B_i(x) = P(f_i(x))^{-1}A_i(x)P(x) \text{ is of the form } \begin{pmatrix} R_i(x) & 0 \\ 0 & S_i(x) \end{pmatrix} \text{ or } \begin{pmatrix} 0 & R_i(x) \\ S_i(x) & 0 \end{pmatrix}.
\]

Otherwise, \(\mu\)-almost surely \(\mathbb{R}^2 \neq E_1(x) + E_2(x)\). Hence, we take in a measurable way the matrix \(P(x) = [U(x), V(x)]\) where \(U(x)\) is a vector in \(E_1(x) = E_2(x)\) and \(V(x)\) is other non-collinear (ortogonal) vector. Then for \(p\)-almost every \(i \in Y\), we get that
\[
B_i(x) = P(f_i(x))^{-1}A_i(x)P(x) \text{ is of the form } \begin{pmatrix} R_i(x) & T_i(x) \\ 0 & S_i(x) \end{pmatrix}.
\]
Consequently, any \((f, B)\)-invariant measure \(\hat{\mu}\) on \(X \times \mathbb{P}(\mathbb{R}^2)\) will be of the form
\[
d\hat{\mu} = \lambda d\delta_{F_1} + (1 - \lambda)d\delta_{F_2} \, d\mu(x)
\]
for some \(0 \leq \lambda \leq 1\)
where \(F_1\) and \(F_2\) are linear subspaces generated by canonical axis on \(\mathbb{R}^2\). This completes the proof of the proposition. \(\square\)

4.1.1. *Proof of Theorem 4.2.* Before proving the theorem, we will need some estimates. First, we identify the projective space \(\mathbb{P}(\mathbb{R}^2)\) with \(S^1\). Namely we identify the one-dimensional vector space \(E\) with a unitary vector \(h \in \mathbb{R}^2\) so that in polar form \(h = e^{i\theta}\) with \(\theta \in S^1 \equiv \mathbb{R} \bmod \pi\).

Then we can see the projective action of a \(SL^+(2)\)-matrix \(A\) as a map
\[
A : S^1 \to S^1 \text{ given by } AE = \text{span}(e^{iA(\theta)}) \text{ where } E = \text{span}(e^{i\theta}).
\]

**Lemma 4.5.** There is \(\theta_0 \in S^1\) such that for every \(\varepsilon > 0\)
\[
|A(\theta_1) - A(\theta_2)| \leq \frac{2\pi^3}{||A||^2\varepsilon^2} \quad \text{for all } \theta_1, \theta_2 \in S^1 \text{ with } |\theta_i - \theta_0| \geq \varepsilon \text{ for } i = 1, 2.
\]

*Proof.* Let \(h_1\) and \(h_2\) be two linearly independent unitary vectors in \(\mathbb{R}^2\). We denote by \(m\) the maximum between \(||Ah_1||\) and \(||Ah_2||\). Set \(h = ah_1 + bh_2\) with \(a, b \in \mathbb{R}\) and \(||h|| = 1\). By means of Cramer formula we get that \(|a|, \, |b| \leq 2/|\det(h_1, h_2)|\).

Hence, \(||Ah|| \leq (|a| + |b|)m \leq 4m/|\det(h_1, h_2)|\) and thus
\[
||A|| = \sup(||Ah|| : ||h|| = 1) \leq \frac{4m}{|\det(h_1, h_2)|}.
\]

This implies that
\[
||A|| \cdot |\det(h_1, h_2)| \leq 4 \max(||Ah_1||, ||Ah_2||) \quad \text{for all } h_1, h_2 \text{ in } \mathbb{R}^2 \text{ with } ||h_1|| = ||h_2|| = 1.
\]

Let \(h_0 = e^{i\theta_0}\) be such that \(||Ah_0|| = \min(||Ah|| : ||h|| = 1)\) with \(\theta_0 \in S^1\) Notice that for every \(h = e^{i\theta}\) with \(\theta \in S^1\) holds that \(|\det(h, h_0)| = \sin(|\theta - \theta_0|) \geq 2|\theta - \theta_0|/\pi\). Hence if \(|\theta - \theta_0| \geq \varepsilon\),
\[
\frac{2}{\pi}||A||\varepsilon \leq 4||Ah|| \quad \text{for every } h = e^{i\theta} \text{ with } \theta \in S^1.
\]

Finally, for any \(h_1 = e^{i\theta_1}\) and \(h_2 = e^{i\theta_2}\) so that \(|\theta_j - \theta_0| \geq \varepsilon\) for \(j = 1, 2\) it holds
\[
\frac{2}{\pi}|A(\theta_1) - A(\theta_2)| \leq \sin(|A(\theta_1) - A(\theta_2)|) = \frac{|\det(Ah_1, Ah_2)|}{||Ah_1|| ||Ah_2||} \leq \frac{4\pi^2}{||A||^2\varepsilon^2}.
\]

This completes the proof. \(\square\)

Now we are ready to provide the proof of Theorem 4.2.

**Proof of Theorem 4.2.** Let \(M(\mu)\) be the set of all \((f, A)\)-invariant Borel probability measures on \(X \times \mathbb{P}(\mathbb{R}^2)\). We want to prove that \(M(\mu)\) is actually a segment. Take \(\hat{\nu} \in M(\mu)\) and write \(d\hat{\nu} = v_x \, d\mu(x)\). Since \(f\) is invertible and \(\hat{\nu}\) is \((f, A)\)-invariant then \(v_x(\lambda) = A(\lambda)v_x\) for \(\mu\)-almost every \(x \in X\). Consider
\[
||v_x|| = \sup\{v_x(E) : E \in \mathbb{P}(\mathbb{R}^2)\}.
\]

**Claim 4.6.** \(|v_x|| \geq 1/2\) for \(\mu\)-almost every \(x \in X\).
Proof. We identify $\mathbb{P}(\mathbb{R}^2)$ with $S^1 \equiv \mathbb{R} \mod \pi$ and see $\nu_x$ as a measure on $S^1$. Let $\varepsilon > 0$ be small enough. Consider $C_\varepsilon = \varepsilon^{-2} > 0$. Set
\[
\varphi_\varepsilon(x) = \sup \{\nu_x(I) : |I| < \varepsilon\}, \quad B_\varepsilon = \{x \in X : \varphi_\varepsilon(x) < 1/2\} \quad \text{and} \quad B_\varepsilon(x) = \{n \geq 0 : f^n(x) \in B_\varepsilon\}.
\]
By Birkhoff ergodic theorem if $\mu(B_\varepsilon) > 0$ then $A_\varepsilon(x)$ has asymptotic positive density for $\mu$-almost every $x \in X$. In particular, $B_\varepsilon(x)$ has lower asymptotic positive density for $\mu$-almost every point in $A_\varepsilon$. On the other hand, if $x \in B_\varepsilon$ and $n \in B_\varepsilon(x)$ then $|A^n(x)| \leq C_\varepsilon$. Indeed, assume that on the contrary $|A^n(x)| > C_\varepsilon$. Hence, applying Lemma 4.5 there are arcs $I, J$ in $S^1$ with $|I|, |J| < \varepsilon$ such that $A^n(x)(S^1 \setminus I) \subset J$. Hence
\[
1 - \nu_x(I) = \nu_x(S^1 \setminus I) \leq \nu_x(A^n(x)^{-1}J) = \nu_{f^n(x)}(J)
\]
and thus
\[
1 \leq \nu_x(I) + \nu_{f^n(x)}(J) = \varphi_\varepsilon(x) + \varphi_\varepsilon(f^{-n}(x)) < 1/2 + 1/2 = 1
\]
which is impossible. Therefore, we get that if $\mu(B_\varepsilon) > 0$ then $\mu$-almost every $x \in B_\varepsilon$ there are a constant $C_\varepsilon > 0$ and a set $B_\varepsilon(x)$ of positive lower density such that $|A^n(x)| \leq C_\varepsilon$ for all $n \in B_\varepsilon(x)$. That is, $|A^n(x)|$ does not converges essentially to infinity for $\mu$-almost every $x \in X$. Consequently, since this is contrary to the assumption, $\mu(B_\varepsilon) = 0$ for all $\varepsilon > 0$ small enough.

This implies that $\varphi_\varepsilon(x) \geq 1/2$ for all small $\varepsilon > 0$ and $\mu$-almost every $x \in X$. From this,
\[
\|\nu_x\| = \lim_{\varepsilon \to 0} \varphi_\varepsilon(x) \geq \frac{1}{2} \quad \text{for} \quad \mu\text{-almost every} \quad x \in X.
\]

Assume now that $\hat{\nu}$ is ergodic. Since $\hat{\nu}$ is $(f, A)$-invariant then $\|\nu_{f^n(Y)}\| = \|A(x)\nu_x\| = \|\nu_x\|$ and thus $\varphi(x) = \|\nu_x\|$ is $f$-invariant. By the ergodicity, $\|\nu_x\| = C \geq 1/2$ for $\mu$-almost every $x \in X$. This implies that $\nu_x$ has either one or two atoms of maximal measure. Let $Y$ be the set of $x \in X$ such that $\nu_x$ has only one atom. Since $\hat{\nu}$ is $(f, A)$-invariant is not difficult to see that $Y$ must be $f$-invariant. Thus, again by the ergodicity of $\mu$, it follows that $Y$ has $\mu$-measure either zero or one. Namely,

i) if $\mu(Y) = 0$ then $\nu_x$ has two atoms $E_x$ and $F_x$ for $\mu$-almost $x \in X$. Moreover,
\[
\nu_x(E_x) = \nu_x(F_x) = \frac{1}{2} \quad \text{and thus} \quad \nu_x = \frac{\delta_{E_x} + \delta_{F_x}}{2}.
\]

ii) if $\mu(Y) = 1$ then $\nu_x$ has only one atom for $\mu$-almost $x \in X$. Moreover, $E_x = A(x)E_x$ and thus the measure $d\hat{\nu} = \delta_E d\mu(x)$ is $(f, A)$-invariant. Since $\nu_x d\mu(x) \geq C d_{E_x} d\mu(x)$ then $\hat{\nu}$ is absolutely continuous with respect to $\nu$. By standard arguments, since $\hat{\nu}$ is ergodic, it follows that $\hat{\nu} = \hat{\rho}$ and therefore $\nu_x = \delta_{E_x}$.

Now, we will prove that there are at most two $(f, A)$-invariant ergodic measures on $X \times \mathbb{P}(\mathbb{R}^2)$.

If we are in the above case (i), then we have a measure $d\hat{\nu} = \frac{1}{2}(\delta_{E_x} + \delta_{F_x}) d\mu(x)$. Suppose that we have another $(f, A)$-invariant measure of the form $d\hat{\nu} = \delta_{E'_x} d\mu(x)$ or $d\hat{\nu} = \frac{1}{2} (\delta_{E'_x} + \delta_{F'_x}) d\mu(x)$. If $E'_x = E_x$ for $\mu$-almost every $x \in X$ then $\delta_{E_x} d\mu(x) \leq 2 d\hat{\nu}$ or $\delta_{F'_x} d\mu(x) \leq 2 d\hat{\nu}$. Notice that either $\delta_{E_x} d\mu(x)$ or $\delta_{F'_x} d\mu(x)$ is a $(f, A)$-invariant measure. Then, similar as above, the ergodicity of $\hat{\nu}$ implies that either $2 d\hat{\nu} = \delta_{E_x} d\mu(x)$ or $2 d\hat{\nu} = \delta_{F'_x} d\mu(x)$ which, in both cases, is impossible. Therefore, $\hat{\nu}$ and $\hat{\rho}$ can not have projective atoms in common. Consequently, for any $0 < \varepsilon < 1$ the measure $\lambda_x d\mu(x) = (1 - \varepsilon) d\hat{\nu} + \varepsilon d\hat{\rho}$ is $(f, A)$-invariant and $\|\lambda_x\| \leq \max\{\frac{1}{2}(1 - \varepsilon), \varepsilon\}$. Taking
\( \varepsilon > 0 \) small enough we get that \( \|\lambda_x\| < 1/2 \) which contradicts Claim 4.6. Observe that in this case we have proved in fact that \( \mathcal{M}(\mu) = \{\hat{\nu}\} \).

If we are in the case (ii), we have \( \hat{\nu} = \delta_{E_i} \mu(\cdot) \). Suppose that we have another two different \( (f, A) \)-invariant ergodic measures \( \hat{\rho} \) and \( \hat{\lambda} \). By the above observation, necessarily \( \hat{\rho} = \delta_{E_i} \mu(\cdot) \) and \( \hat{\lambda} = \delta_{E_i} \mu(\cdot) \). Then, all the projective atoms must be different and consequently, the \( (f, A) \)-invariant measure \( \hat{\nu} = \rho_\kappa \mu(\cdot) \) has
\[
\|\rho_\kappa\| < \frac{1}{2} \quad \text{with} \quad \rho_\kappa = \frac{1}{3}(\delta_{E_1} + \delta_{E_2} + \delta_{E_2}).
\]
However, this is impossible according to Claim 4.6. Thus, we can only have at most two \( (f, A) \)-invariant ergodic measures.

This completes the proof of the theorem. \( \square \)

4.2. Essentially bounded cocycles. In this subsection we are going to classify the invariant measures of essentially bounded random linear cocycles. The following result proves Theorem B for essentially bounded cocycles.

**Proposition 4.7.** Let \((f, A)\) be an essentially bounded random \( \text{GL}(d) \)-valued cocycle. Then, there is a random linear cocycle \((\tilde{f}, B)\) with values on a compact subgroup of \( \text{GL}(d) \) and cohomologous to \((f, A)\) such that every \((\tilde{f}, B)\)-stationary probability measure \( \hat{\mu} \) on \( X \times \mathcal{P}(\mathbb{R}^d) \) is of the form \( \hat{\mu} = \mu \times \nu \), where \( \nu \) is a probability measure on \( \mathcal{P}(\mathbb{R}^d) \). In particular, every stationary measure is, in fact, invariant.

**Definition 4.8.** A random \( G \)-valued cocycle \((f, A)\) is called essentially bounded if there are a compact set \( K \) of \( G \), a constant \( \delta > 0 \) and a subset \( E \subset \hat{X} \) of positive \( \hat{\mu} \)-measure satisfying that
\[
d(\{n \geq 0 : A^n(x) \in K\}) \geq \delta \quad \text{for all} \quad x \in E.
\]

It is easy to check that that the new definition coincides with the older when \( G = \text{GL}(d) \).

**Theorem 4.9.** Let \((f, A)\) be an essentially bounded random \( G \)-valued cocycle. Then there is a random cocycle \((\tilde{f}, B)\) with values on a compact subgroup \( H \) of \( G \) such that

i) \((\tilde{f}, A)\) is cohomologous with \((f, B)\), and

ii) \( \mu \times m_H \) is the unique \((f, B)\)-stationary probability measure on \( X \times H \).

Moreover, any \((f, B)\)-stationary probability measure \( \hat{\pi} \) on \( X \times G \) is a product measure of the form
\[\hat{\pi} = \mu \times \nu\] with \( \nu = m_H \ast \omega \), where \( \omega \) is a measure on \( G \) and \( m_H \) is the Haar measure on \( H \). In particular, every stationary probability measure is, in fact, invariant.
The compactness of $H$ implies that must be essentially contained in the maximal compact subgroup of $G$. That is, we can find $Q \in G$ such that $Q^{-1}HQ$ is contained in the maximal compact subgroup of $G$. Thus, we get the following remark:

**Remark 4.10.** Up to conjugacy, if $G = GL(d)$ we have that $H \subset O(d)$ where $O(d)$ denotes the orthogonal group of $d \times d$ matrices with real coefficients.

Before proving this theorem, let us explain how it implies Proposition 4.7.

**Proof of Proposition 4.7.** By Theorem 4.9 and Remark 4.10 we have a random $O(d)$-valued cocycle $(f, B)$ cohomologous to $(f, A)$ such that any $(f, B)$-stationary probability on $X \times GL(d)$ is a product measure. Let $\hat{\mu}$ be a $(f, B)$-stationary measure on $X \times \mathbb{P}^d$. We write

$$d\hat{\mu} = \nu_x \, d\mu(x) \quad \text{where } \nu_x \text{ is a measure on } \mathbb{P}^d.$$

Let $m$ be the normalized Lebesgue measure on $\mathbb{P}^d$. We see $\mathbb{P}^d$ as the unit $(d - 1)$-sphere in $\mathbb{R}^d$ with antipodal points identified. One can further restrict to a hemisphere which remains a topological $(d - 1)$-sphere that we will denote by $S^{d-1}$. Thus, we see $\nu_x$ and $m$ as a probability measures on $\mathbb{R}^d$ supported on $S^{d-1}$. For each $x \in X$, define the probability measure

$$\pi_x = \nu_x \times m \times \dot{\ldots} \times m \quad \text{on } \mathbb{R}^d \times \ldots \times \mathbb{R}^d.$$

Now, identifying $\mathbb{R}^d \times \ldots \times \mathbb{R}^d$ with the set $M(d)$ of $d \times d$ matrices with real coefficients. Again, we can see $\pi_x$ as a measure on $M(d)$. 

**Claim 4.11.** The measure $\pi_x$ is supported on $GL(d)$.

**Proof.** We will prove that $\pi_x$-almost every $M$ in $M(d)$ is invertible. To do this, we will write the matrices $M$ as $M = [h_1, \ldots, h_d]$ and consider the sets

$$\mathcal{E} = \{M : h_i \in \mathbb{R}^d, \det M = 0\} \quad \text{and} \quad \mathcal{E}_0 = \{M : h_i \in S^{d-1}, \det M = 0\}.$$

Since $\pi_x = \nu_x \times m \times \dot{\ldots} \times m$ is supported on $S^{d-1} \times \ldots \times S^{d-1}$ we have that $\pi_x(\mathcal{E}) = \pi_x(\mathcal{E}_0)$. Assume that $h_1, \ldots, h_k$ are linearly independent vectors in $S^{d-1}$. Then, $h_{k+1} \in S^{d-1}$ is linearly combination of these vectors if and only if $h_{k+1}$ belongs to $\mathbb{P}^{\text{span}(h_1, \ldots, h_k)} \equiv S^{k-1}$. Hence,

$$\mathcal{E}_0 = \bigcup_{k=1}^{d-1} \mathcal{E}_k \quad \text{where } \mathcal{E}_k = \{M : h_i \in S^{d-1}, h_{k+1} \in \mathbb{P}^{\text{span}(h_1, \ldots, h_k)} \equiv S^{k-1}\}.$$

Thus

$$\pi_x(\mathcal{E}_k) = \int \mathcal{E} \big(\mathcal{E}(h_1, \ldots, h_k)\big) d(\nu_x \times m \times \dot{\ldots} \times m)(h_1, \ldots, h_{k+1}, \ldots, h_d)$$

where

$$\mathcal{E}(h_1, \ldots, h_k) = \{h \in S^{d-1} : h \in \text{span}(h_1, \ldots, h_k)\} = \mathbb{P}^{\text{span}(h_1, \ldots, h_k)} \equiv S^{k-1}.$$

Since $m$ is the Lebesgue measure on $S^{d-1}$ then $m(S^{k-1}) = 0$ for all $k < d$. Thus $\pi_x(\mathcal{E}_k) = 0$ for all $k = 1, \ldots, d - 1$. Therefore $\pi_x(\mathcal{E}_0) = 0$ and so $\pi_x$-almost every $M$ in $M(d)$ is invertible. □
Now, we consider the probability measure \( \tilde{\mu} \) defined as \( d\tilde{\mu} = \pi_x \, d\mu(x) \). We can write
\[
\tilde{\mu} = \tilde{\mu} \times m^{d-1} \quad \text{on} \quad (X \times \mathbb{R}^d) \times (\mathbb{R}^d)^{d-1} \quad \text{where} \quad m^{d-1} = m \times \cdots \times m.
\]
Moreover, according to the above claim \( \tilde{\mu} \) can be seen as a measure on \( X \times GL(d) \).

**Claim 4.12.** The probability \( \tilde{\mu} \) is a \((\tilde{f}, B)\)-stationary measure on \( X \times GL(d) \).

**Proof.** For each \( i \in Y \) we have that \( F_i = (f_i, B_i) \) acts on \( X \times GL(d) \) as \( F_i(x, M) = (f_i(x), B_i(x)M) \).

We write \( M = [h_1, \ldots, h_d] \) with \( h_i \in \mathbb{R}^d \). Then \( B_i(x)M = [B_i(x)h_1, \ldots, B_i(x)h_d] \) and
\[
B_i(x)\pi_x = B_i(x)\nu_x \cdot (B_i(x)m)^{d-1} \quad \text{for all} \quad i \in Y.
\]

Observe that since \( B_i(x) \in O(d) \) for \( \mu \)-almost \( x \in X \) then \( B_i(x)m = m \). Hence, using that \( \mu \) is \( f_i \)-invariant for \( p \)-almost every \( i \in Y \) we have that
\[
\int_Y F_i \tilde{\mu} \, dp(i) = \int_Y \int_X B_i(x)\nu_x \cdot (B_i(x)m)^{d-1} \, df_i \mu(x) \, dp(i)
\]
\[
= \int_Y \int_X B_i(x)\nu_x \cdot m^{d-1} \, d\mu(x) \, dp(i) = \int_Y F_i \tilde{\mu} \, dp(i) \times m^{d-1}
\]
\[
= \tilde{\mu} \times m^{d-1} = \tilde{\mu}.
\]
The last inequality holds because of \( \tilde{\mu} \) is a \((\tilde{f}, B)\)-stationary measure. \( \square \)

From this claim and according to Theorem 4.9, \( \tilde{\mu} \) is a product measure of the form \( \tilde{\mu} = \mu \times \nu \) with \( \nu \) a measure on \( GL(d) \). This implies that \( \pi_x = \nu \) for \( \mu \)-almost every \( x \in X \). Hence, since \( \pi_x = \nu_x \times m \times \cdots \times m \), then \( \nu_x \) does not depend on \( x \) and thus \( \tilde{\mu} \) is also a product measure.

This completes the proof of Proposition 4.7. \( \square \)

Now we will prove Theorem 4.9. We will split the proof in three steps.

4.2.1. **Step 1: Existence of stationary measure.** First we will study the existence of stationary measures for essentially bounded random cocycle.

**Proposition 4.13.** Let \((\tilde{f}, A)\) be an essentially bounded random \( G \)-valued cocycle. Then there exists a \((\tilde{f}, A)\)-stationary Borel probability regular measure \( \hat{\mu} \) on \( X \times G \).

**Proof.** Set \( \delta_e \) the probability measure on \( G \) supported on the identity element \( e \). Let us consider the sequence \((\hat{\nu}^n)_n\) of probability measures \( \hat{\nu}^n \) on \( X \times G \) defined by means of the disintegration \( d\hat{\nu}^n = \gamma^n \, d\mu(x) \) where \( \gamma^n : x \mapsto \gamma^n \) is given by
\[
\gamma^n = \frac{1}{n} \sum_{j=0}^{n-1} \mathcal{P}^j \delta_e \in L^\infty(X; \mathcal{P}(G))
\]
being \( \mathcal{P}^* \) the adjoint transfer operator on \( L^\infty(X; \mathcal{M}(G)) \) introduced in Appendix A. According to Lemma A.3, there exists a finite regular Borel \((\tilde{f}, A)\)-stationary measure \( \hat{\nu} \) on \( X \times Z \) which is an accumulation point in the weak’ topology of \((\hat{\nu}^n)_n\). Notice that the mass of the probability measures \( \hat{\nu}_n \) could be escaping to infinite and thus \( \hat{\nu} \) could be zero. However, this is not the case, since \((\tilde{f}, A)\) is essentially bounded. Indeed, there are compact set \( K \) of \( G \) and a
constant $\delta > 0$ satisfying that $D_K(\bar{f}) = \{m \geq 0 : A^m(\bar{f}) \in K\}$ has $\delta$ lower density on a set of positive $\hat{\mu}$-measure. In particular for any $n \geq 0$ large enough,

$$\hat{\nu}(X \times K) = \int \nu^x(K) d\mu(x) = \frac{1}{n} \sum_{j=0}^{n-1} \int \int A^{-i}(\omega, x)^{-1}\delta_e(K) d\IP(\omega) d\mu(x)$$

$$= \frac{1}{n} \sum_{j=0}^{n-1} A^i(\bar{f}^{-i}(\bar{x})) \delta_e(K) d\hat{\mu}(\bar{x}) = \frac{1}{n} \sum_{j=0}^{n-1} A^i(\bar{y}) \delta_e(K) d\hat{\mu}(\bar{y})$$

$$= \int \# \{0 \leq j \leq n-1 : A^i(\bar{y}) \in K\} d\hat{\mu}(\bar{y})$$

$$= \int \# \{(0, n-1] \cap D(\bar{y})\} \frac{1}{n} d\hat{\mu}(\bar{y}) \geq \delta.$$  

From this we have that $\hat{\nu}$ can not be equal to zero. Therefore, we get that the normalized measure is a probability measure.

Finally, we will prove that the normalized measure of $\hat{\nu}$ projects on $\mu$. To see this, first we observe that $\pi\hat{\nu}(B) \leq \mu(B)$ for all measurable set $B \subset X$ where $\pi$ is the projection onto $X$. Indeed, let $E \subset B \subset V$ be, respectively, a compact set and an open set of $X$ approximating the $\mu$-measure of $B$. Hence

$$\hat{\nu}(E \times G) \leq \hat{\nu}(V \times G) \leq \liminf_{n \to \infty} \hat{\nu}^n(V \times G) = \mu(V).$$

The second inequality holds from the weak* convergence (c.f. [EG92, Thm. 1 Sec. 1.9]). Thus, since both, $\hat{\nu}$ and $\mu$, are regular measures we get that

$$\pi\hat{\nu}(B) = \hat{\nu}(B \times G) = \sup \{\hat{\nu}(E \times K) : E \text{ and } K \text{ compact and } E \times K \subset B \times G\}$$

$$\leq \inf \{\mu(V) : V \text{ open and } B \subset V\} = \mu(B).$$

Therefore $\pi\hat{\nu}$ is absolute continuous with respect to $\mu$. Also $\pi\hat{\nu}$ is $(\theta, f)$-stationary since $\pi \circ F_i = f_i \circ \pi$ for all $i \in Y$ where recall that $F_i \equiv (f_i, A_i) : X \times G \to X \times G$. Moreover, $\pi\hat{\nu}$ is proportional to $\mu$. Indeed, according to Radon-Nikodym theorem (see [Gui06, thm. 13.18]) we can writing $d\pi\hat{\nu} = \phi(x) d\mu(x)$ with $\phi \in L^1(\mu)$. The ergodicity of $\mu$ implies that $\phi$ is constant $\mu$-almost everywhere. In fact, we obtain that $\phi(x) = \pi\hat{\nu}(X)$ for $\mu$-almost every $x \in X$. Therefore, we get that the normalized measure of $\hat{\nu}$ is a $(\bar{f}, A)$-stationary Borel probability regular measure and the proof of the proposition is completed. 

\section{4.2.2. Step 2: Reduction of the cocycle.}

Consider $h \in G$ and define

$$\Phi_h : X \times G \to X \times G \quad \Phi_h(x, g) = (x, gh).$$

\begin{proposition}
Let $\hat{\tau}$ be a $(\bar{f}, A)$-stationary ergodic Borel probability regular measure on $X \times G$. Then, $H(\hat{\tau}) = \{h \in G : \Phi_h \hat{\tau} = \hat{\tau}\}$ is a compact subgroup of $G$ and there exists a measurable function $P : X \to G$ such that

$$B(\omega, x) = P(f_\omega(x))^{-1} A(\omega, x) P(x) \in H(\hat{\tau}) \quad \text{for } (P \times \mu) \text{-almost every } (\omega, x) \in \Omega \times X.$$
\end{proposition}
Proof. First of all, notice that $H \equiv H(\hat{r})$ is group since $(\Phi_h)^{-1} = \Phi_{h^{-1}}$ and $\Phi_{hf} = \Phi_f \Phi_h$. Also, $H$ is closed because the map $h \mapsto \Phi_h \hat{r}$ is continuous. Then, to get that $H$ is compact we only need to prove that $H$ is bounded.

By contradiction, if $H$ is not bounded then for any pair of compact sets $K$ and $L$ of $G$ there is $h \in H$ such that $L \cap Kh = \emptyset$. Otherwise, $H \subset K^{-1}L$ which is a compact set. Since $\hat{r}$ is a Borel regular probability measure on $X \times G$ we can find a compact set $K$ of $G$ so that $\hat{r}(X \times K) > 0$. By induction one gets a sequence $\{h_n\}$ of elements of $H$ such that $Kh_n \cap (Kh_{n-1} \cup \cdots \cup Kh_1) = \emptyset$ for all $n \in \mathbb{N}$. Thus the sets $\Phi_{h_n}(X \times K)$ are pairwise disjoints for all $n$. Since $\Phi_{h_n}$ preserves $\hat{r}$ because of $h_n \in H$ then, all these sets have the same positive measure. This implies that $\cup_n \Phi_{h_n}(X \times K)$ has infinite measure which is a contradiction.

Now, we will construct the measurable function $P : X \to G$. Consider $E = \{(x, g) \in X \times G : \text{generic point for } \hat{r} \}$ and set $E_x = \{M : (x, g) \in E\}$.

As before, a generic point $(x, g)$ in $X \times G$ of the ergodic $(f, A)$-stationary measure $\hat{r}$ is understand in the sense that

$$\frac{1}{N} \sum_{n=0}^{N-1} \varphi(F^n_\omega(x, g)) \rightarrow \int \varphi \ d\hat{r} \ \text{P-almost surely}$$

for all continuous maps $\varphi : X \times G \to \mathbb{R}$ with compact support where

$$F^n_\omega(x, g) = (f^n_\omega(x), B^n(\bar{x})g), \quad \bar{x} = (\omega, x) \in \Omega \times X \text{ and } g \in G, \ n \geq 0.$$

Claim 4.15. If $g, \ell \in E_x$ then $h = g^{-1} \ell \in H$.

Proof. Since $\Phi_{h}F_i = F_i \Phi_{h}$ for all $i \in Y$ it holds that

$$\frac{1}{N} \sum_{n=0}^{N-1} \varphi(F^n_\omega(x, \ell)) = \frac{1}{N} \sum_{n=0}^{N-1} \varphi \circ F^n_\omega(\Phi_h(x, g)) = \frac{1}{N} \sum_{n=0}^{N-1} \varphi \circ \Phi_h(F^n_\omega(x, g)).$$

As both $(x, \ell)$ and $(x, g)$ are generic point for $\hat{r}$ then taking limit we get that

$$\int \varphi \ d\hat{r} = \int \varphi \circ \Phi_h \ d\hat{r} = \int \varphi \ d\Phi_h \hat{r}$$

for all continuous maps $\varphi : X \times G \to \mathbb{R}$ with compact support. This implies that $\Phi_h \hat{r} = \hat{r}$ and thus $h \in H$. \qed

For $\mu$-almost every $x \in X$, let $P(x) \in E_x$ chosen in a measurable way. This is follows from [Kec12, Corollary 18.7] since $d\hat{r} = \pi_x \ d\mu(x)$ and $\pi_x(E_x) = 1$ for $\mu$-almost every $x \in X$. Now we have that

$$(x, P(x)) \in E \quad \text{and} \quad (f_\omega(x), A(\omega, x)P(x)) = F_\omega(x, P(x)) \in E \quad \text{for } \text{P-almost every } \omega \in \Omega.$$
4.2.3. Step 3: Proof of Theorem 4.9. We start by considering the closed group $H_0 = G$ and the random cocycle $(\bar{f}, A)$ acting on $X \times H_0$ by means of 

$$(\bar{f}, A) : X \times H_0 \to X \times H_0, \quad (x, h) \mapsto (\bar{f}(x), A(x)h).$$

By Proposition 4.13, we find an ergodic $(\bar{f}, A)$-stationary probability measure $\hat{\pi}_0$ on $X \times H_0$. If $H(\hat{\pi}_0) \neq H_0$ we can cohomologically reduce the random $H_0$-valued cocycle $(\bar{f}, A)$ by means of Proposition 4.14 to a random $H_1$-valued cocycle $(\bar{f}, A_1)$ where $H_1 \triangleq H(\hat{\pi}_0)$. Arguing inductively we can get more. Pick again an ergodic $(\bar{f}, A_1)$-stationary probability measure $\hat{\pi}_1$ on $X \times H_1$. Reduce the random $H_1$-valued cocycle $(\bar{f}, A_1)$ to a random $H_2$-valued cocycle $(\bar{f}, A_2)$ in the case that $H_2 \triangleq H(\hat{\pi}_1) \leq H_1$. This induction defines a partial order and thus by Zorn’s lemma we can find a minimal random $H$-valued cocycle $(\bar{f}, B)$ which cannot be reduced. This implies that $H(\hat{\pi}) = H$ for every ergodic $(\bar{f}, B)$-stationary probability measure $\hat{\pi}$ on $X \times H$. Thus, for every $h \in H$, it holds that $\Phi_h \hat{\pi} = \hat{\pi}$. In particular, if $d \hat{\pi} = \pi_x \, d\mu(x)$ then $h\pi_x = \pi_x$ for $\mu$-almost every $x \in X$, i.e.,

$$\pi_x(E) = h\pi_x(E) = \pi_x(Eh^{-1}) \quad \text{for all Borel subsets } E \subset H \text{ and } h \in H.$$ Consequently $\pi_x$ is a right-translation-invariant probability measure on $H$. Since $H$ is a compact group then $\pi_x$ is the Haar measure of $H$ for $\mu$-almost every $x \in X$. Therefore, $\hat{\pi} = \mu \times m_H$ where $m_H$ is the Haar measure concluding the proof.

On the other hand, let $\hat{\pi}_1$ and $\hat{\pi}_2$ be two ergodic $(\bar{f}, B)$-stationary measures on $X \times G$. Since both measure project on the same measure $\mu$ we can take two generic points of $\hat{\pi}_1$ and $\hat{\pi}_2$ of the form $(x, g_1)$ and $(x, g_2)$ respectively. Recall that a generic point is understand in the sense that

$$\frac{1}{N} \sum_{n=0}^{N-1} \varphi(F^n_\omega(x, g_j)) \to \int \varphi \, d\hat{\pi}_j \quad \text{P-almost surely,} \quad j = 1, 2$$

for all continuous maps $\varphi : X \times G \to \mathbb{R}$ with compact support where

$$F^n_\omega(y, g) = F_{\omega^{-1}} \circ \cdots \circ F_{\omega}(F^n_\omega(y), B^n(\bar{g})g), \quad \bar{g} = (\omega, y) \in \Omega \times X \text{ and } g \in G \quad n \geq 0.$$ Take $h = g_1^{-1}g_2$. Since $\Phi_h \circ F_i = F_i \circ \Phi_h$ for all $i \in Y$ we get that

$$\frac{1}{N} \sum_{n=0}^{N-1} \varphi(F^n_\omega(x, g_2)) = \frac{1}{N} \sum_{n=0}^{N-1} \varphi \circ F^n_\omega(\Phi_h(x, g_1)) = \frac{1}{N} \sum_{n=0}^{N-1} \varphi \circ \Phi_h(F^n_\omega(x, g_2)).$$

Taking limit we have that $\Phi_h \hat{\pi}_1 = \hat{\pi}_2$. Therefore we relate any pair of ergodic stationary measure by the map $\Phi_h$ for some $h \in G$. Observe that the measure $\hat{\pi} = \mu \times m_H$ obtained above it also is a $(\bar{f}, B)$-stationary ergodic measure on $X \times G$. Since $\hat{\pi}$ is a product measure we get that also any other $(\bar{f}, B)$-stationary ergodic measure on $X \times G$ must be a product measure. Consequently, we get that any $(\bar{f}, B)$-stationary probability measure on $X \times G$ is a product measure of the form $\mu \times \nu$ with $\nu = m_H * \omega$ where $\omega$ is a measure on $G$. This concludes the proof of Theorem 4.9.
4.3. **Equivalent definitions of essentially bounded.** Proposition 4.14 shows that if a random G-valued cocycle \((\tilde{f}, A)\) admits a stationary measure on \(X \times G\) then it is cohomologous with a random cocycle \((f, B)\) with values on a compact group. Conversely we have the following:

**Theorem 4.16.** Let \((\tilde{f}, A)\) be a random G-valued cocycle. Then, it is equivalent:

i) \((\tilde{f}, A)\) is essentially bounded,

ii) there is a \((\tilde{f}, A)\)-stationary probability measure on \(X \times G\),

iii) \((\tilde{f}, A)\) is cohomologous to a random cocycle with values on a compact subgroup of G.

**Proof.** By Proposition 4.13 we have (i) implies (ii). Also (ii) implies (iii) it follows from Proposition 4.14. To complete the equivalence we will see (iii) implies (i).

Let \((f, B)\) be a random cocycle cohomologous to \((\tilde{f}, A)\) with values in a compact subgroup H of G. So there is \(P : X \to G\) measurable such that

\[
B^n(\bar{x}) = P(f^n(x))^{-1}A^n(\bar{x})P(x) \in H \quad \text{for } \bar{\mu}\text{-almost every } \bar{x} = (\omega, x) \in \bar{X} = \Omega \times X \text{ and } n > 0.
\]

By regularity of the measure \(P\mu\) on G, there exists a compact subset K of G such that the set \(E = \{\bar{x} = (\omega, x) \in \bar{X} : P(x) \in K\}\) has \(\bar{\mu}\)-positive measure. Then, for any integer \(n > 0\), if both, \(\bar{x} = (\omega, x)\) and \(f^n(\omega, x) = (\theta^n\omega, f^n_0(x))\) belong to E then \(A^n(\bar{x}) = P(f^n_0(x))B^n(\bar{x})P(x)^{-1}\) is in compact set \(L = \{abc^{-1} : a \in K, b \in H, c \in K\}\). Setting \(\delta = \bar{\mu}(E) > 0\), we have by Birkhoff Theorem that \(d(|n \in \mathbb{N} : f^n(x) \in E|) = \delta\) for \(\bar{\mu}\)-almost every \(\bar{x}\) in E and hence \(d(|n \in \mathbb{N} : A^n(\bar{x}) \in L|) \geq \delta\). Thus, the random cocycle \((\tilde{f}, A)\) is essentially bounded. \(\square\)

**Remark 4.17.** According to Theorem 4.9 every stationary measure of an essentially bounded random cocycle is in fact invariant. Then, we also have that \((\tilde{f}, A)\) is essentially bounded if and only if there is a \((\tilde{f}, A)\)-invariant probability measure \(\eta\) on \(\bar{X} \times G\) of the form \(\eta = P \times \hat{\mu}\) where \(\hat{\mu}\) is a probability measure on \(X \times G\).

Here we relate our results with other literature.

**Remark 4.18.** A G-valued cocycle \((f, A)\) is said to be bounded if for every \(\varepsilon > 0\) there is a compact set \(K \subset G\) such that \(\bar{\mu}(|x \in X : A^n(x) \in K|) > 1 - \varepsilon\) for all \(n > 0\). According to [Sch81], a cocycle \((f, A)\) is bounded if and only if it is cohomologous to a cocycle with values in a compact subgroup of G. Moreover, the equivalence between a bounded cocycle \((f, A)\) and the existence of a \((f, A)\)-invariant probability measure on \(X \times G\) it is follows from [OO96].

**Remark 4.19.** Theorem 4.16 does not follow from Remark 4.18. Indeed, notice that by definition a \((f, A)\)-invariant probability \(\eta\) on \(\bar{X} \times G\) is a measure that \(\eta\) projects on \(\bar{\mu} = P \times \mu\) and has a disintegration \(d\eta = \nu_{\bar{x}}d\bar{\mu}(\bar{x})\) such that \(A(\bar{x})\nu_{\bar{x}} = \nu_{f(\bar{x})}\) for \(\bar{\mu}\)-almost every \(\bar{x} \in \bar{X}\). However, \(\eta\) does not necessarily projects on \(X \times G\) over a \((f, A)\)-stationary measure. Similarly, a random cocycle \((\tilde{f}, A)\) could be cohomologous to a cocycle \((f, B)\) with values in a compact subgroup but not necessarily random. Thus, a priori, a bounded random cocycle is not necessarily an essentially bounded random cocycles.

**Remark 4.20.** Theorem 4.16 includes the results in the literature given in Remark 4.18 for deterministic cocycles. This follows from the fact that in this case (being \(\Omega\) a one-point set) the notion of stationary and invariant measures coincides (see also Remark 4.17). Thus, a posteriori, the notions of essentially bounded and bounded cocycle are also equivalent.
Appendix A. Stationary measure

Let us consider two standard Borel probability spaces \((X, \mu)\) and \((\Omega,\mathbb{P})\) and endow the product space \(\bar{X} = \Omega \times X\) with the product measure \(\bar{\mu} = \mathbb{P} \times \mu\). Consider a \(\bar{\mu}\)-preserving measurable skew-product map
\[
\tilde{f} : \bar{X} \to \bar{X}, \quad f(\omega, x) = (\theta \omega, f_\omega(x))
\]
where \(\theta\) is an ergodic \(\mathbb{P}\)-invariant invertible continuous transformation of \(\Omega\) and \(f_\omega : X \to X\) are continuous \(\mu\)-preserving maps for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\). Let \(Z\) be a locally compact Hausdorff topological space and consider the induce Borel \(\sigma\)-algebra. Again, consider a skew-product map
\[
F : \bar{X} \times Z \to \bar{X} \times Z, \quad F(\bar{x}, z) = (f(\bar{x}), A(\bar{x}) z)
\]
where \(A(\bar{x}) : Z \to Z\) are continuous maps \(\mu\)-almost surely. Sometimes we write \(F = f \circ A(\bar{x})\) or \(F = (f,A)\) when no confusion can arise to emphasize the base map and the fiber maps of \(F\). Observe that since \(\tilde{f} = \theta \circ f_\omega\) is also a skew-product map, we can also rewrite \(F = \theta \circ f_\omega\) where \(F_\omega = f_\omega \circ A(\omega, x)\). That is, \(\tilde{F} = \theta \circ f_\omega \circ A(\omega, x)\).

**Definition A.1.** A measure \(\hat{\nu}\) on \(X \times Z\) is \((f,A)\)-stationary if \(\hat{\nu}\) projects down on \(X\) over \(\mu\) and
\[
\hat{\nu} = \int F_\omega \hat{\nu} d\mathbb{P}(\omega).
\]

Notice that by definition a stationary measure is not necessarily a probability measure. Denote by \(\mathcal{M}(Z)\) be the Banach space of signed finite (not necessarily probability) Borel measures on \(Z\) with variation norm. This Banach space can be identified with the dual of \(C_0(Z)\), the space of bounded continuous real-valued functions on \(Z\) vanishing at infinity with supremum norm. Then we can endow \(\mathcal{M}(Z)\) with the weak* topology and consider the Borel \(\sigma\)-algebra induced by this topology. Let \(L^\infty(X;\mathcal{M}(Z))\) be the Banach space of (equivalence classes of) essentially bounded measurable mappings from \(X\) to \(\mathcal{M}(Z)\). Given \(\nu : x \mapsto \nu_x\) in \(L^\infty(X;\mathcal{M}(Z))\) define the measure \(\hat{\nu}\) on \(X \times Z\) by
\[
\hat{\nu}(E \times B) = \int_E \nu_x(B) d\mu(x)
\]
for \(E, B\) measurable sets on \(X\) and \(Z\) respectively and extending to the product \(\sigma\)-algebra. By definition \(\hat{\nu}\) has as marginal \(\mu\) and disintegration \(\nu : x \mapsto \nu_x\). That is, \(d\hat{\nu} = \nu_x d\mu(x)\).

Notice that \(L^\infty(X;\mathcal{M}(Z))\) can be identified with the dual of the Banach space
\[
L^1(X;C_0(Z)) \overset{\text{def}}{=} \{ h : X \to C_0(Z) : \| h_x \|_\infty \in L^1(X,\mu) \}.
\]
We introduce the transfer operator \(\mathcal{P}\) on \(L^1(X;C_0(Z))\) defined by
\[
\mathcal{P} \varphi \overset{\text{def}}{=} \int \varphi \circ F_\omega d\mathbb{P}(\omega) \in L^1(X;C_0(Z)) \text{ for } \varphi \in L^1(X;C_0(Z)).
\]
It is clear that \(\mathcal{P}\) is a bounded linear operator. The adjoint transition operator \(\mathcal{P}^*\) acts on \(L^\infty(X;\mathcal{M}(Z))\) by taking \(\nu : x \mapsto \nu_x\) in \(L^\infty(X;\mathcal{M}(Z))\) and defining \(\mathcal{P}^*\nu \in L^\infty(X;\mathcal{M}(Z))\) as
\[
\mathcal{P}^*\nu : x \mapsto (\mathcal{P}^*\nu)_x = \int A^{-1}(\omega, x)^{-1} \nu_z(f^{-1}_\omega(x)) d\mathbb{P}(\omega).
\]
Consequently $\mathcal{P}^*$ is also a bounded linear operator. Moreover, if $\mathcal{P}^*\nu = \nu$ then $\hat{\nu}$ is a $(\bar{f}, A)$-stationary measure on $X \times Z$. Indeed,

$$
\int F_x \hat{\nu}(E \times B) \, d\mathbb{P}(\omega) = \int \int A(\Theta^{-1}\omega, f^{-1}_x(\omega)) \nu \, d\mu(B) \, d\mathbb{P}(\omega)
$$

$$
= \int E (\mathcal{P}^*\nu)_x(B) \, d\mu(x) = \int E \nu_x(B) \, d\mu(x) = \hat{\nu}(E \times B)
$$

for all $E, B$ measurable sets on $X$ and $Z$ respectively. Hence $\hat{\nu}$ is a $(\bar{f}, A)$-stationary measure.

According to Banach-Alaoglu’s theorem the unit ball in $L^\infty(X; M(Z))$ is a compact set. Moreover, since $X$ is a standard probability space, its $\sigma$-algebra is countably generated. This implies that $L^1(X; C_0(Z))$ is separable and thus $L^\infty(X; M(Z))$ is metrizable [Cra86]. Consequently the unit ball in $L^\infty(X; M(Z))$ is also sequentially compact. Now, the existence of stationary probability measures for random cocycles follows from standard arguments when $Z$ is compact.

**Proposition A.2.** Let $(\bar{f}, A)$ be skew-product as (A.1) acting on $\bar{X} \times Z$. If $Z$ is a compact Hausdorff topological space then the set of $(\bar{f}, A)$-stationary probability measures on $X \times Z$ is nonvoid.

**Proof.** Denote by $\mathcal{P}(Z)$ the subset of $M(Z)$ of probability measure. Notice that

$$
L^\infty(X; \mathcal{P}(Z)) = \{ \nu \in L^\infty(X; \mathcal{P}(Z)) : \nu_x \in \mathcal{P}(Z) \text{ $\nu$-almost surly} \}
$$

is a convex subset of the unit ball in $L^\infty(X; M(Z))$. Moreover, $\mathcal{P}^*$ leaves $L^\infty(X; \mathcal{P}(Z))$ invariant. Since, by assumption $Z$ is compact, $L^\infty(X; \mathcal{P}(Z))$ is closed and hence compact in the weak* topology. Brouwer’s fixed-point theorem yields the existence of a $\mathcal{P}^*$-invariant element $\nu \in L^\infty(X; \mathcal{P}(Z))$. This yields a $(\bar{f}, A)$-stationary measure $\hat{\nu}$ on $X \times Z$ defined by $d\hat{\nu} = \nu_x \, d\mu(x)$ and completes the proof. $\square$

When $Z$ is not compact we can not guarantee in general that the set of $(\bar{f}, A)$-stationary probability measure is nonvoid. However the following lemma provides a powerful method to find stationary finite measure.

**Lemma A.3.** Let $\nu \in L^\infty(X; \mathcal{P}(Z))$. Then, the set of accumulation point $\eta \in L^\infty(X; M(Z))$ of the sequence $(\nu_n)_n$ given by

$$
\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} \mathcal{P}^j \nu \in L^\infty(X; \mathcal{P}(Z)) \quad \text{for } n \in \mathbb{N}
$$

is nonvoid. Moreover, any accumulation point $\eta$ of $(\nu_n)_n$ defines a $(\bar{f}, A)$-stationary finite measure $\hat{\eta}$ on $X \times Z$ whose disintegration is $\eta$. Consequently $\hat{\eta}$ is an accumulation point in the weak* topology of the sequence of probability measures $(\hat{\nu}_n)_n$ on $X \times Z$ defined by the disintegrations $(\nu_n)_n$.

**Proof.** Since the unit ball of $L^\infty(X; M(Z))$ is sequentially compact then we can extract a convergent subsequence from $(\nu_n)_n$ and thus the set of accumulation points is not empty. Moreover, any accumulation point belongs to this ball. Thus $(\nu_n)_n$ is a finite measure $\mu$-almost surly. On the other hand, by well know arguments the limit $\eta$ of any convergent sequences of $(\nu_n)_n$ is also $\mathcal{P}^*$-invariant and thus $\hat{\eta}$ is a $(\bar{f}, A)$-stationary measure on $X \times Z$. $\square$
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