Approximately Clean Quantum Probability Measures

Douglas Farenick and Remus Floricel

Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan S4S 0A2, Canada

Sarah Plosker

Department of Mathematics and Statistics, University of Guelph, Guelph, Ontario N1G 2W1, Canada

(Dated: May 8, 2013)

A quantum probability measure—or quantum measurement—is said to be clean if it cannot be irreversibly connected to any other quantum probability measure via a quantum channel. The notion of a clean quantum measure was introduced by Buscemi et al in 3 for finite-dimensional Hilbert space, and was studied subsequently by Kahn 15 and Pellonpää 18. The present paper provides new descriptions of clean quantum probability measures in the case of finite-dimensional Hilbert space. For Hilbert spaces of infinite dimension, we introduce the notion of “approximately clean quantum probability measures” and characterise this property for measures whose range determines a finite-dimensional operator system.

PACS numbers: 02.30.Cj, 02.50.Cw, 03.65.Aa, 03.67.-a

Introduction

In this paper we investigate the mathematical ramifications of the notion of cleanness for positive operator valued probability measures, which was originally introduced and studied by Buscemi et al 3 in connection with the addition of a preprocessing step to the measurement of a quantum system. A related work of Heinonen 12, which appeared at roughly the same time as 3, addressed the very general issue of optimality in quantum measurements. Subsequent studies of clean quantum measurements were undertaken by Kahn 15 and Pellonpää 18. Pellonpää in fact uses a slightly weaker definition than the original definition of clean quantum measurement put forward in 3. To explain this, we first recall some of the standard nomenclature in quantum mechanics.

A quantum system is represented by a Hilbert space $\mathcal{H}$ and a measurement of the system is represented by a positive operator valued probability measure $\nu : \mathcal{O}(X) \rightarrow \mathcal{B}(\mathcal{H})$, where $X$ represents a set of measurement outcomes for the system, $\mathcal{O}(X)$ is a $\sigma$-algebra of measurement events, and $\mathcal{B}(\mathcal{H})$ is the space of all bounded linear operators acting on $\mathcal{H}$. The corresponding measurement statistics, which capture the probability that event $E \in \mathcal{O}(X)$ is measured by the apparatus $\nu$ when the system $\mathcal{H}$ is in state $R$, are given by the real numbers $\text{Tr}_{\mathcal{H}}(R\nu(E))$.

Returning to the ideas introduced in 3, assume that $\nu$ is a measurement of a quantum system $\mathcal{H}$. Suppose that a preprocessing step is introduced through the use an irreversible quantum channel $\Phi$ that maps the states of $\mathcal{H}$ to states in some other system $\mathcal{H}'$ which is to be measured by $\nu'$. This preprocessing step is represented mathematically in the Heisenberg picture by $\nu = \Phi^* \circ \nu'$, where $\Phi^*$ is the dual of $\Phi$, and it transforms the measurement statistics according to the equation $\text{Tr}_{\mathcal{H}'}(R\nu'(E)) = \text{Tr}_{\mathcal{H}'}(\Phi(R)\nu'(E))$. The measurement apparatus $\nu'$ is thought to be cleaner than $\nu$ because of the quantum noise introduced by the quantum channel $\Phi$. Put differently, $\nu$ is obtained from a cleaner quantum probability measure $\nu'$ by irreversible preprocessing, a relation that is denoted by $\nu \ll_{\text{cl}} \nu'$. A clean quantum measurement $\nu$ is one in which there is no $\nu'$ from which $\nu$ is obtained irreversibly by preprocessing.

The authors of 3 were interested primarily in the quantum systems represented by Hilbert spaces of finite dimension; however, in the present paper we aim to develop mathematical techniques that are suited to the case of infinite-dimensional Hilbert space as well. Although there is precedent for this objective in the works of Heinonen 12, Kahn 15, and Pellonpää 18, we allow for an even weaker notion of quantum noise, and this leads us to a new concept: namely, that of an approximately clean quantum measurement. There is a corresponding order relation in which $\nu \ll_{\text{ap-cl}} \nu'$ denotes that $\nu'$ is approximately cleaner than $\nu$. Such a notion is imposed upon us by the mathematical realities of infinite-dimensional Hilbert spaces. There is an extensive literature on quantum theory in the context of infinite-dimensional spaces; the monographs of Davies 10 and Holevo 13, 14 treat the fundamentals of this theory. The Schrödinger–Heisenberg duality manifests itself in our work through the fact that $\mathcal{B}(\mathcal{H})$ is the Banach space $\mathcal{B}(\mathcal{H})$.

*Electronic address: douglas.farenick@uregina.ca
‡Electronic address: douglas.farenick@uregina.ca
†Electronic address: remus.floricel@uregina.ca
‡Electronic address: ploskers@brandonu.ca
Furthermore, if a completely positive linear map \( p \) which gives an analytic description of the order relation \( \nu \approx \nu' \) measures, as well as the order relation \( \nu \approx \nu' \) for quantum probability measures, and Theorem II.2 which determines the structure of approximately clean quantum probability measures. The paper concludes with Section IV which contains a number of remarks and observations about the main results. In particular we show that the relations \( \ll \approx \approx \) and \( \ll \approx \approx \) are distinct in infinite dimensions, and that there are approximately clean quantum measurements that are not clean.

Lastly, because we are using Pellonpää’s definition of clean quantum probability measure, which is more stringent than the definition used in [6] or [15], our main theorem (Theorem II.3) neither implies nor is implied by the results of [6] or [15].

I. OPERATOR THEORY BACKGROUND

All Hilbert spaces under consideration are assumed to be separable. If \( \mathcal{H} \) and \( \mathcal{K} \) are Hilbert spaces, then \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) is the Banach space of all bounded linear operators \( T : \mathcal{H} \to \mathcal{K} \). With \( \mathcal{K} = \mathcal{H} \), denote \( \mathcal{B}(\mathcal{H}, \mathcal{H}) \) by \( \mathcal{B}(\mathcal{H}) \), and \( \mathcal{T}(\mathcal{H}) \) denotes the ideal of trace-class operators, where the canonical trace on \( \mathcal{B}(\mathcal{H}) \) is denoted by \( \text{Tr}(\cdot) \) or by \( \text{Tr}_{\mathcal{H}}(\cdot) \) if more than one Hilbert space is being considered. The von Neumann algebra \( \mathcal{B}(\mathcal{H}) \) is the dual space of \( \mathcal{T}(\mathcal{H}) \) in the sense that every bounded linear functional \( \psi \) on \( \mathcal{T}(\mathcal{H}) \) has the form \( \psi(R) = \text{Tr}(AR) \), \( R \in \mathcal{T}(\mathcal{H}) \), for some uniquely determined operator \( A \in \mathcal{B}(\mathcal{H}) \). Thus, every bounded linear operator \( \Omega : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H}) \) admits an adjoint operator \( \Omega^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) such that, for every \( A \in \mathcal{B}(\mathcal{H}) \), the equation \( \text{Tr}(\Omega(R)A) = \text{Tr}(R\Omega^*(A)) \) holds for all \( R \in \mathcal{T}(\mathcal{H}) \).

Considered as the dual of \( \mathcal{T}(\mathcal{H}) \), \( \mathcal{B}(\mathcal{H}) \) carries a weak*-topology, which in terms of operator topologies coincides with the ultraweak topology (or, \( \sigma \)-weak topology, as it is often called). Namely, a \{ \{X_\alpha\}_\alpha \} \subset \mathcal{B}(\mathcal{H}) converges ultraweakly to \( X \in \mathcal{B}(\mathcal{H}) \) if and only if \( \text{Tr}(RX) = \lim_\alpha \text{Tr}(RX_\alpha) \) for every \( R \in \mathcal{T}(\mathcal{H}) \). A \{ \phi_\alpha \}_\alpha \) of bounded linear maps \( \phi_\alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) converges to a bounded linear map \( \phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) in the point-ultraweak topology if, for every \( X \in \mathcal{B}(\mathcal{H}) \), \( \phi_\alpha(X) \) is the ultraweak limit of the net \( \{ \phi_\alpha(X) \}_\alpha \) \subset \mathcal{B}(\mathcal{H}) \). Lastly, the set of all bounded linear maps \( \phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) of norm \( \| \phi \| \leq 1 \) is compact in the point-ultraweak topology.

The cone of all positive \( Q \in \mathcal{B}(\mathcal{H})^+ \) is denoted by \( \mathcal{B}(\mathcal{H})^+ \). A positive trace-class operator \( R \in \mathcal{T}(\mathcal{H}) \) of trace \( \text{Tr}(R) = 1 \) is called a density operator.

**Definition I.1.** If \( Q \in \mathcal{B}(\mathcal{H})^+ \), then \( \lambda_{\min}(Q) \) and \( \lambda_{\max}(Q) \) are the nonnegative real numbers

\[
\lambda_{\min}(Q) = \min\{\lambda : \lambda \in \text{Sp}(Q)\} \quad \text{and} \quad \lambda_{\max}(Q) = \max\{\lambda : \lambda \in \text{Sp}(Q)\},
\]

where \( \text{Sp}(Z) \) denotes the spectrum of an operator \( Z \in \mathcal{B}(\mathcal{H}) \).

**Definition I.2.** ([6], [17]) An operator system is a linear (not necessarily norm-closed) subspace \( \mathcal{S} \subset \mathcal{B}(\mathcal{H}) \) with the properties that \( 1 \in \mathcal{S} \) and \( \mathcal{S}^* \subset \mathcal{S} \) for every \( \mathcal{S} \in \mathcal{S} \).

The operator algebra \( \mathcal{B}(\mathcal{H}) \) is an operator system, as is every unital C*-subalgebra \( \mathcal{A} \subset \mathcal{B}(\mathcal{H}) \).

**Definition I.3.** Assume that \( \mathcal{S} \subset \mathcal{B}(\mathcal{H}) \) and \( \mathcal{T} \subset \mathcal{B}(\mathcal{K}) \) are operator systems acting on Hilbert space \( \mathcal{H} \) and \( \mathcal{K} \). A linear map \( \phi : \mathcal{S} \to \mathcal{T} \) is completely positive if, for every \( p \in \mathbb{N} \), the linear map \( \phi^{(p)} : \mathcal{M}_p(\mathcal{S}) \to \mathcal{M}_p(\mathcal{T}) \) in which

\[
\phi^{(p)}[S_{ij}]^p_{i,j=1} = [\phi(S_{ij})]^p_{i,j=1}
\]

has the property of mapping the positive cone \( \mathcal{M}_p(\mathcal{S})^+ \) of \( \mathcal{M}_p(\mathcal{S}) \) into the positive cone \( \mathcal{M}_p(\mathcal{T})^+ \) of \( \mathcal{M}_p(\mathcal{T}) \), where a \( p \times p \) matrix \( X \) of operators is positive if \( X \) is positive as an operator acting on the Hilbert space \( \mathcal{H}^{(p)} = \mathcal{H} \oplus \cdots \oplus \mathcal{H} \). Furthermore, if a completely positive linear map \( \phi : \mathcal{S} \to \mathcal{T} \) is such that \( \phi(1) = 1 \), then \( \phi \) is unital and \( \phi \) is called a ucp map.
Complete positive linear maps of $\mathcal{B}(\mathcal{H})$ or, more generally, of $C^*$-algebras admit a Stinespring decomposition [17, Chapter 4], which is an extremely important tool by which one studies complete positivity. In contrast, there is no Stinespring decomposition for completely positive maps on operator systems that are not $C^*$-algebras, which adds a degree of difficulty in working with such structures.

**Definition II.1.** A map $\phi : \mathcal{S} \to \mathcal{T}$ is a finite set (endowed the discrete topology), then $\{\phi\}$ is a map that is not normal.

X is a sample space of measurement outcomes and $\sigma$ is a $\sigma$-algebra of subsets of $X$. In the language of probability, $X$ is a sample space of measurement outcomes and $\mathcal{O}(X)$ is a $\sigma$-algebra of measurement events. If $X$ is a locally compact Hausdorff space, then $\mathcal{O}(X)$ is assumed to be the $\sigma$-algebra of Borel sets of $X$. In particular, if $X$ is a finite set (endowed the discrete topology), then $\mathcal{O}(X)$ is assumed to be the power set of $X$.

**Definition II.1.** A unital completely positive linear map $\phi : \mathcal{M} \to \mathcal{N}$ of von Neumann algebras is normal if, for every bounded increasing net $\{H_\alpha\}_\alpha \subset \mathcal{M}$ of hermitian operators, $\phi(\sup_\alpha H_\alpha) = \sup_\alpha \phi(H_\alpha)$ in $\mathcal{N}$.

The definition II.1. is a finite set (endowed the discrete topology), then $\{\phi\}$ is a map that is not normal. In Section IV we give an example of an approximately normal ucp $\phi : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ as follows. Consequently, $\Phi(S) = \sum_k A_k^* S A_k$, $\forall S \in \mathcal{B}(\mathcal{K})$, for some countable set $\{A_k\}_k \subset B(\mathcal{H}, \mathcal{K})$ for which $\sum_k A_k^* A_k = 1 \in B(\mathcal{H})$. (Convergence of sums is with respect to the ultraweak topology of $B(\mathcal{H})$.)

Normality is an important consideration for our study because a unital completely positive linear map $\phi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ is the adjoint $\phi^* = \Phi^*$ of some quantum channel $\Phi : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ if and only if $\phi$ is normal. (Of course, if $\dim \mathcal{K}$ is finite, every completely positive linear map $\mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ is normal.)

**Definition II.7.** A unital completely positive linear map $\phi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ is approximately normal if there exists a net $\{\phi_\alpha\}_\alpha$ of ucp maps $\phi_\alpha : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ such that each $\phi_\alpha$ is normal and $\phi_\alpha \to \phi$ in the point-ultraweak topology. In finite dimensions, ucp = normal ucp = approximately normal ucp, but in infinite dimensions there is a distinction between all these notions. It is rather well known that any ucp map $\phi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ for which $\phi(K) = 0$ for every compact operator $K \in \mathcal{B}(\mathcal{K})$ cannot be normal. In Section IV we give an example of an approximately normal ucp map that is not normal.

Variants of Definition II.7 may be used for other classes of maps (such as a positive rather than completely positive linear maps) or with other topologies. But because of the dual relationship between quantum channels and normal ucp maps, we focus only the class of maps and the topology that are natural from the perspective of duality.

**II. APPROXIMATELY CLEAN QUANTUM PROBABILITY MEASURES**

Throughout $X$ shall denote a nonempty set and $\mathcal{O}(X)$ will denote a $\sigma$-algebra of subsets of $X$. In the language of probability, $X$ is a sample space of measurement outcomes and $\mathcal{O}(X)$ is a $\sigma$-algebra of measurement events. If $X$ is a locally compact Hausdorff space, then $\mathcal{O}(X)$ is assumed to be the $\sigma$-algebra of Borel sets of $X$. In particular, if $X$ is a finite set (endowed the discrete topology), then $\mathcal{O}(X)$ is assumed to be the power set of $X$.

**Definition II.1.** A map $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ is a positive operator valued probability measure (POVM), or a quantum probability measure, if

(i) $\nu(E) \in \mathcal{B}(\mathcal{H})_+$ for every $E \in \mathcal{O}(X)$,

(ii) $\nu(X) = 1 \in \mathcal{B}(\mathcal{H})$, and
(iii) for every countable collection \( \{ E_k \}_{k \in \mathbb{N}} \subseteq \mathcal{O}(X) \) with \( E_j \cap E_k = \emptyset \) for \( j \neq k \) we have

\[
\nu \left( \bigcup_{k \in \mathbb{N}} E_k \right) = \sum_{k \in \mathbb{N}} \nu(E_k),
\]

where the convergence on the right side of the equation above is with respect to the ultraweak topology of \( \mathcal{B}(\mathcal{H}) \).

In addition:

(iv) if \( \nu(E \cap F) = \nu(E)\nu(F) \) for all \( E, F \in \mathcal{O}(X) \), then \( \nu \) is a projective quantum measure.

We are particularly interested in an ultraweakly closed vector space \( \mathcal{T}_\nu \) determined by the range \( \mathcal{R}_\nu \) of a quantum probability measure \( \nu \).

**Definition II.2.** If \( \nu \) is a quantum probability measure on \( (X, \mathcal{O}(X)) \), then the range of \( \nu \) is the set

\[
\mathcal{R}_\nu = \{ \nu(E) : E \in \mathcal{O}(X) \} \subset \mathcal{B}(\mathcal{H})_+,
\]

and the measurement space of \( \nu \) is the vector space

\[
\mathcal{T}_\nu = (\text{Span}_\mathbb{C} \mathcal{R}_\nu)^{\sigma-\text{wk}} \subset \mathcal{B}(\mathcal{H}),
\]

the ultraweak closure of all linear combinations of operators of the form \( \nu(E) \), for \( E \in \mathcal{O}(X) \).

By using the ultraweak closure of \( \text{Span}_\mathbb{C} \mathcal{R}_\nu \), the operator system \( \mathcal{T}_\nu \) becomes a dual operator system in the sense of [2]. In particular, the space \( \mathcal{T}_\nu \) can be identified with the dual space of its predual \( \mathcal{T}_{\nu^*} \), that is the Banach space of all weak*-continuous linear functionals on \( \mathcal{T}_\nu \).

**Definition II.3.** A measurement basis for a quantum probability measure \( \nu \) is a finite or countably infinite set \( \mathcal{B}_\nu \) of positive operators such that

(i) \( \mathcal{B}_\nu = \{ \nu(E) : E \in \mathcal{F}_\nu \} \) for some finite or countable family \( \mathcal{F}_\nu \subseteq \mathcal{O}(X) \) of pairwise disjoint sets,

(ii) for every \( Z \in \mathcal{T}_\nu \) there exists a unique sequence \( \{ \alpha_{A,Z} \}_{A \in \mathcal{B}_\nu} \) of complex numbers such that \( Z = \sum_{A \in \mathcal{B}_\nu} \alpha_{A,Z} A \) in the weak*-topology,

(iii) for every \( A \in \mathcal{B}_\nu \), the coefficient functional \( \varphi_A(Z) = \alpha_{A,Z}, Z \in \mathcal{T}_\nu \), is a normal positive linear functional.

If \( E_0 = X \setminus (\bigcup_{E \in \mathcal{F}_\nu} E) \), then the operator \( A_0 = \nu(E_0) \) is called the basis residual for \( \mathcal{B}_\nu \); if \( A_0 = 0 \), then \( \mathcal{B}_\nu \) is said to admit a trivial basis residual.

Note that \( 1 = A_0 + \sum_{A \in \mathcal{B}_\nu} A \), if \( \mathcal{B}_\nu \) is measurement basis for \( \nu \).

Our primary concern in this paper is with quantum probability measures for which the measurement space \( \mathcal{T}_\nu \) has finite dimension. The general situation requires a different approach, which will be addressed elsewhere.

**Proposition II.4.** If \( \omega \) is a projective measurement such that \( \mathcal{T}_\omega \) has finite dimension, then \( \mathcal{T}_\omega \) has a measurement basis and every measurement basis for \( \mathcal{T}_\omega \) has trivial residual.

**Proof.** Assume that \( \omega : \mathcal{O}(X) \to \mathcal{B}(\mathcal{K}) \) is a projective measurement and that \( \{ \omega(F_1), \ldots, \omega(F_m) \} \) is a linear basis for \( \mathcal{T}_\omega \). Set \( Q_j = \omega(F_j) \) for each \( j \); thus, \( Q_1, \ldots, Q_m \) are linearly independent (pairwise commuting) projections. Let \( E_1 = F_1 \) and define, iteratively,

\[
E_j = F_j \setminus \left( \bigcup_{i=1}^{j-1} F_i \right), \quad j = 2, \ldots, m.
\]

The sets \( E_1, \ldots, E_m \) are pairwise disjoint and nonempty, and therefore the projections \( P_1, \ldots, P_m \), where each \( P_j = \omega(E_j) \), are nonzero and pairwise orthogonal. Thus, \( \{ P_1, \ldots, P_m \} \) is a set of \( m \) linearly independent operators whose linear span is a subspace of the \( m \)-dimensional space \( \mathcal{T}_\omega \). Hence, \( \{ P_1, \ldots, P_m \} \) is a measurement basis for \( \mathcal{T}_\omega \). Note that \( P_0 = 1 - \sum_{j=1}^m P_j \) is orthogonal to each \( P_j \), and thus \( P_0 \) is either zero or is linearly independent of \( P_1, \ldots, P_m \). But \( m = \dim \mathcal{T}_\omega \) yields \( P_0 = 0 \), showing that the measurement basis \( \mathcal{B}_\omega = \{ P_1, \ldots, P_m \} \) for \( \omega \) has trivial residual. Indeed this latter argument shows that every measurement basis for \( \mathcal{T}_\omega \) has trivial residual. \( \square \)
The decomposition of $X$ into a finite disjoint-union of measurable sets $E_0, \ldots, E_m$ in the proof of Proposition II.4 is an idea that appears in other works on quantum measurement—for example, as in the decomposition of phase space into cells given in [3, (10.19)]. Pellonpää’s definition of clean quantum probability measure is stated below.

**Definition II.5.** (18) Assume that $\nu_1$ and $\nu_2$ are quantum probability measures on $(X, \mathcal{O}(X))$ with values in $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ respectively.

1. $\nu_1$ is cleaner than $\nu_2$, denoted by $\nu_2 \ll_{cl} \nu_1$, if $\nu_2 = \Phi^* \circ \nu_1$ for some quantum channel $\Phi : \mathcal{T}(\mathcal{H}_2) \to \mathcal{T}(\mathcal{H}_1)$.
2. $\nu_1$ and $\nu_2$ are cleanly equivalent, denoted by $\nu_2 \simeq_{cl} \nu_1$, if $\nu_1 \ll_{cl} \nu_2$ and $\nu_2 \ll_{cl} \nu_1$.
3. $\nu_1$ is clean if $\nu_2 \ll_{cl} \nu_1$ for every quantum probability measure $\nu_2$ satisfying $\nu_1 \ll_{cl} \nu_2$.

As noted in the Introduction, if $\nu_2 \ll_{cl} \nu_1$ via a quantum channel $\Phi$, then the measurement statistics satisfy

$$\text{Tr}_{\mathcal{H}_2}(R \nu_2(E)) = \text{Tr}_{\mathcal{H}_1}(\Phi(R) \nu_1(E)), \quad \forall R \in \mathcal{T}(\mathcal{H}_2), E \in \mathcal{O}(X).$$

As in classical probability, another of the most basic of partial orderings is the partial order that arises from absolute continuity.

**Definition II.6.** If $\nu_j : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_j)$ is a quantum probability measure on $(X, \mathcal{O}(X))$ for $j = 1, 2$, then $\nu_2$ is absolutely continuous with respect to $\nu_1$, denoted by $\nu_2 \ll_{ac} \nu_1$, if $\nu_2(E) = 0$ for all $E \in \mathcal{O}(X)$ for which $\nu_1(E) = 0$.

There is a Radon-Nikodým derivative that arises from the ordering $\nu_2 \ll_{ac} \nu_1$; see [11] for further information. Pellonpää’s result below shows that for projective measures the clean ordering is the same as the ordering by absolute continuity.

**Theorem II.7.** (Pellonpää [18 Theorem 3]) Every projective measurement is clean and the following statements are equivalent for quantum probability measures $\omega$ and $\nu$ on $(X, \mathcal{O}(X))$, where $\omega$ is projective:

1. $\nu \ll_{ac} \omega$;
2. $\nu \ll_{cl} \omega$.

We now introduce a new form of dominance for quantum probability measures.

**Definition II.8.** Assume that $\nu_1$ and $\nu_2$ are quantum probability measures on $(X, \mathcal{O}(X))$ with values in $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ respectively.

1. $\nu_1$ is approximately cleaner than $\nu_2$, denoted by $\nu_2 \ll_{ap-cl} \nu_1$, if $\nu_2 = \phi \circ \nu_1$ for some approximately normal ucp map $\phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$.
2. $\nu_1$ and $\nu_2$ are approximately cleanly equivalent, denoted by $\nu_2 \simeq_{ap-cl} \nu_1$, if $\nu_1 \ll_{ap-cl} \nu_2$ and $\nu_2 \ll_{ap-cl} \nu_1$.
3. $\nu_1$ is approximately clean if $\nu_2 \ll_{ap-cl} \nu_1$ for every quantum probability measure $\nu_2$ satisfying $\nu_1 \ll_{ap-cl} \nu_2$.

If $\nu_2 \ll_{ap-cl} \nu_1$ via an approximately normal ucp map $\phi$, then the corresponding measurement statistics are related by

$$\text{Tr}_{\mathcal{H}_2}(R \nu_2(E)) = \lim_\alpha \text{Tr}_{\mathcal{H}_1}(\Phi_\alpha(R) \nu_1(E)), \quad \forall R \in \mathcal{T}(\mathcal{H}_2), E \in \mathcal{O}(X),$$

where $\{\Phi_\alpha\}_\alpha$ is a net of quantum channels $\mathcal{T}(\mathcal{H}_2) \to \mathcal{T}(\mathcal{H}_1)$ for which $\Phi_\alpha^* \to \phi$ in the point-ultraweak topology.

**III. STRUCTURE OF APPROXIMATELY CLEAN QUANTUM PROBABILITY MEASURES WITH FINITE-DIMENSIONAL MEASUREMENT SPACES**

Our first objective is to characterise the order relation $\nu_2 \ll_{ap-cl} \nu_1$ in the case where the measurement space $\mathcal{T}_{\nu_1}$ has finite dimension.

**Theorem III.1.** If $\mathcal{B}_0 = \{\nu(E_1), \ldots, \nu(E_m)\}$ is a finite measurement basis for a quantum probability measure $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$, then the following statements are equivalent for a quantum probability measure $\nu' : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}')$:

1. $\nu' \ll_{ap-cl} \nu$;
2. \( \nu' = \psi \circ \nu \) for some ucp map \( \psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}') \);

3. for all \( L_0, \ldots, L_m \in \mathcal{M}_p(\mathbb{C}) \) and every \( p \in \mathbb{N} \),

\[
\left\| \sum_{j=0}^m \nu'(E_j) \otimes L_j \right\| \leq \left\| \sum_{j=0}^m \nu(E_j) \otimes L_j \right\|. \tag{1}
\]

Proof. \((1) \Rightarrow (2)\). If \( \nu' \ll_{\text{sp-cl}} \nu \), then \( \nu' = \phi \circ \nu \) for some approximately normal ucp map \( \phi \). Take \( \phi = \psi \).

\((2) \Rightarrow (3)\). Because \( \nu' = \psi \circ \nu \) for some ucp map \( \psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}') \), and because ucp maps are completely contractive, \( \left\| \sum_{j=0}^m \nu'(E_j) \otimes L_j \right\| \leq \left\| \sum_{j=0}^m \nu(E_j) \otimes L_j \right\| \) for all \( L_0, \ldots, L_m \in \mathcal{M}_p(\mathbb{C}) \) and every \( p \in \mathbb{N} \). That is, inequality \((1)\) holds.

\((3) \Rightarrow (4)\). Assume that for all \( L_0, \ldots, L_m \in \mathcal{M}_p(\mathbb{C}) \) and every \( p \in \mathbb{N} \), inequality \((1)\) holds. Let \( \mathcal{H}^{(\infty)} = \mathcal{H} \otimes \ell^2(\mathbb{N}) \) and \( \mathcal{H}'^{(\infty)} = \mathcal{H}' \otimes \ell^2(\mathbb{N}) \). Because \( \mathcal{H}^{(\infty)} \) and \( \mathcal{H}'^{(\infty)} \) are infinite-dimensional separable Hilbert spaces, they are isomorphic and we may identify them; that is, we assume \( \mathcal{H}^{(\infty)} = \mathcal{H}'^{(\infty)} \) without loss of generality. Let \( A_j = \nu(E_j) \) and \( B_j = \nu'(E_j) \), for \( j = 0, \ldots, m \), and let \( \pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}^{(\infty)}) \) be given by \( \pi(Z) = Z \otimes 1 \). Define \( \pi' : \mathcal{B}(\mathcal{H}') \to \mathcal{B}(\mathcal{H}'^{(\infty)}) \) analogously. Thus, \( \pi \) and \( \pi' \) a normal, unital homomorphisms. Set \( Z^{(\infty)} = \pi(Z) \) and \( Y^{(\infty)} = \pi'(Y) \), for every \( Z \in \mathcal{B}(\mathcal{H}) \) and \( Y \in \mathcal{B}(\mathcal{H}') \), and note that inequality \((1)\) implies

\[
\left\| \sum_{j=0}^m \mathcal{B}_j^{(\infty)} \otimes L_j \right\| \leq \left\| \sum_{j=0}^m A_j^{(\infty)} \otimes L_j \right\|, \tag{2}
\]

for all \( L_0, \ldots, L_m \in \mathcal{M}_p(\mathbb{C}) \) and every \( p \in \mathbb{N} \).

Consider now the operator systems \( \mathcal{X}_\nu = \text{Span} \{ A_1^{(\infty)}, \ldots, A_m^{(\infty)} \} \) and \( \mathcal{X}_{\nu'} = \text{Span} \{ B_1^{(\infty)}, \ldots, B_m^{(\infty)} \} \), and let \( \psi_0 : \mathcal{X}_\nu \to \mathcal{X}_{\nu'} \) denote the linear function defined by

\[
\psi_0 \left( \sum_{j=1}^m \alpha_j A_j^{(\infty)} \right) = \sum_{j=1}^m \alpha_j B_j^{(\infty)},
\]

for \( \alpha_1, \ldots, \alpha_m \in \mathbb{C} \). Inequality \((2)\) shows that the unital linear map \( \psi_0 \) is completely contractive. Hence, by Arveson’s extension theorem \([12]\), \( \psi_0 \) has a completely contractive extension to a ucp map \( \psi : \mathcal{B}(\mathcal{H}^{(\infty)}) \to \mathcal{B}(\mathcal{H}'^{(\infty)}) \), which maps the operator system \( \mathcal{X}_\nu \) into \( \mathcal{X}_{\nu'} \).

Let \( \mathcal{A} = C^*(\mathcal{X}_\nu) \), which is a separable unital C*-algebra, and let \( \theta = \psi|_\mathcal{A} \). Because the C*-algebra \( \mathcal{A} \) contains no nonzero compact operators, Voiculescu’s theorem \([6, \text{II.5}]\) implies that there is a sequence of isometries \( V_n \in \mathcal{B}(\mathcal{H}^{(\infty)}) \) such that \( \lim_n \|\theta(T) - V_n^*TV_n\| = 0 \) for every \( T \in \mathcal{A} \). If \( \{\omega_j\} \) and \( \{\epsilon_k\}_{k \in \mathbb{N}} \) are orthonormal bases of \( \mathcal{H} \) and \( \ell^2(\mathbb{N}) \), then the map \( V : \mathcal{H}' \to \mathcal{H}^{(\infty)} \) in which \( V\omega_j = \omega_j \otimes \epsilon_1 \), for all \( j \), is an isometry. Thus, for each \( n \), the map \( \phi_n : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) given by \( \phi_n(Z) = V^*V_n^*Z^{(\infty)}V_nV \) is a normal ucp map such that \( \lim_n \|B_j - \phi_n(A_j)\| = 0 \) for all \( j = 1, \ldots, m \). Since the space of all ucp maps \( \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}') \) is compact in the point-ultraweak topology, the sequence \( \{\phi_n\}_n \) admits a cluster point \( \phi \), and so there is a subsequence, which for notational simplicity we denote again by \( \{\phi_n\}_n \), such that \( \phi(Z) \) is the ultraweak limit of \( \{\phi_n(Z)\}_n \) for every \( Z \in \mathcal{B}(\mathcal{H}) \), and such that \( B_j = \phi(A_j) \) for each \( j = 1, \ldots, m \).

Our second objective is to characterise approximately clean quantum probability measures that have finite-dimensional measurement spaces. To do so, we begin with a useful preliminary result (Lemma \( \text{III.3} \)) related to the inequalities in statement \((3)\) of Theorem \( \text{III.1} \) above.

**Definition III.2.** If \( A_1, \ldots, A_m \) are pairwise commuting positive operators such that \( \sum_{j=1}^m A_j = 1 \), then the joint spectrum of the \( m \)-tuple \( (A_1, \ldots, A_m) \) is the subset \( \text{Sp}(A_1, \ldots, A_m) \subset \mathbb{R}^m \) of all \( \lambda \in \mathbb{R}^m \) for which there exists a unital homomorphism \( \rho : C^*(A_1, \ldots, A_m) \to \mathbb{C} \) such that \( \lambda_j = \rho(A_j) \) for all \( j = 1, \ldots, m \), where \( C^*(A_1, \ldots, A_m) \) is the unital abelian C*-algebra generated by \( A_1, \ldots, A_m \).

**Lemma III.3.** If \( A_1, \ldots, A_m \in \mathcal{B}(\mathcal{H}) \) are positive operators such that \( \sum_{j=1}^m A_j = 1 \), then for every Hilbert space \( \mathcal{K} \) and operators \( L_1, \ldots, L_m \in \mathcal{B}(\mathcal{K}) \) we have

\[
\left\| \sum_{j=1}^m A_j \otimes L_j \right\| \leq \max_{1 \leq j \leq m} \|L_j\|.
\]
If, moreover, $A_1, \ldots, A_m$ are pairwise commuting with joint spectrum $\Lambda \in \mathbb{R}^m$, then

$$\left\| \sum_{j=1}^{m} A_j \otimes L_j \right\| = \max_{\lambda \in \Lambda} \left\| \sum_{j=1}^{m} \lambda_j L_j \right\|.$$ 

In particular, if $A_1, \ldots, A_m$ are nonzero projections, then

$$\left\| \sum_{j=1}^{m} A_j \otimes L_j \right\| = \max_{1 \leq j \leq m} \| L_j \| .$$

**Proof.** Let $G_j = (A_j^{1/2} \otimes 1_\mathcal{K}) \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$; thus, $G_1, \ldots, G_m \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ are positive operators such that $\sum_j G_j^2 = 1 \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Pass to matrices of operators: let $L = (1_\mathcal{H} \otimes L_1) \oplus \cdots \oplus (1_\mathcal{H} \otimes L_m)$ and

$$G = \begin{bmatrix} G_1 & 0 & \cdots & 0 \\ G_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_m & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, $\sum_{j=1}^{m} A_j \otimes L_j = \sum_{j=1}^{m} G_j (1_\mathcal{H} \otimes L_j) G_j$ and

$$\left\| \sum_{j=1}^{m} G_j (1_\mathcal{H} \otimes L_j) G_j \right\| = \| G^* L G \| \leq \| G \|^2 \| L \| = \| G^* G \| \max_{1 \leq j \leq m} \| 1_\mathcal{H} \otimes L_j \| .$$

Because $\| G^* G \| = \left\| \sum_{j=1}^{m} G_j^2 \right\| = 1$ and $\| 1_\mathcal{H} \otimes L_j \| = \| L_j \|$, the desired general inequality follows.

Suppose now that $A_1, \ldots, A_m$ are pairwise commuting. Then the maximal ideal space $\mathfrak{M}$ of the abelian C*-algebra $C^* (A_1, \ldots, A_m)$ is homeomorphic to $\Lambda$ under the map $\rho \mapsto (\rho(A_1), \ldots, \rho(A_m))$. Hence, every operator $A_j$ is a complex-valued continuous function on $\mathfrak{M}$ whereby $\rho \mapsto \rho(A_j)$. Because the C*-algebra $C(\mathfrak{M}) \otimes \mathcal{B}(\mathcal{K})$ is isometrically isomorphic to the C*-algebra of continuous functions on $\mathfrak{M}$ with values in $\mathcal{B}(\mathcal{K})$, we deduce that

$$\left\| \sum_{j=1}^{m} A_j \otimes L_j \right\| = \max_{\rho \in \mathfrak{M}} \left\| \sum_{j=1}^{m} \rho(A_j) \otimes L_j \right\| = \max_{\lambda \in \Lambda} \left\| \sum_{j=1}^{m} \lambda_j L_j \right\| .$$

Lastly, if $A_1, \ldots, A_m$ are nonzero projections, then the equation $\sum_{j=1}^{m} A_j = 1$ implies that $A_1, \ldots, A_m$ are pairwise orthogonal and, hence, pairwise commuting. Fix $i$ and choose any $\rho$ in the maximal ideal space $\mathfrak{M}$ of $C^* (A_1, \ldots, A_m)$ for which $\rho(A_i) = 1$. (Such a $\rho$ exists because the spectrum of $A_i$ contains 1.) Then $1 = A_i + \sum_{j \neq i} A_j$ implies that $\rho(1) = \rho(A_i) + \sum_{j \neq i} \rho(A_j)$, and so $\sum_{j \neq i} \rho(A_j) = 0$. As each $\rho(A_j) \geq 0$, we obtain $\rho(A_j) = 0$ for every $j \neq i$. Hence, the joint spectrum $\Lambda$ of $(A_1, \ldots, A_m)$ contains only the canonical coordinate vectors $e_1, \ldots, e_m$ of $\mathbb{R}^m$ and, therefore,

$$\max_{\lambda \in \Lambda} \left\| \sum_{j=1}^{m} \lambda_j L_j \right\| = \max_{1 \leq j \leq m} \| L_j \| ,$$

which completes the proof.

The following theorem is the main result of the present paper.

**Theorem III.4.** The following statements are equivalent for a quantum probability measure $\nu$ such that the measurement space $\mathcal{T}_\nu$ has finite dimension:

1. $\nu$ is approximately clean;
2. there is a Hilbert space $\mathcal{K}$ and a projective quantum probability measure $\omega : \mathcal{O}(X) \to \mathcal{B}(\mathcal{K})$ such that $\mathcal{T}_\nu$ and $\mathcal{T}_\omega$ are unitally completely order isomorphic.
3. for all Hilbert spaces $K$, all operators $L_1, \ldots, L_m \in B(K)$, and every measurement basis $B_\nu = \{A_1, \ldots, A_m\}$ for $\nu$, the following two properties hold:

(a) $B_\nu$ has trivial residual, and

\[
\left\| \sum_{j=1}^m A_j \otimes L_j \right\| = \max_{1 \leq j \leq m} \| L_j \|;
\]

(b) for every measurement basis $B_\nu = \{A_1, \ldots, A_m\}$ for $\nu$, the following two properties hold:

(a) $B_\nu$ has trivial residual, and

(b) $\lambda_{\text{max}}(A_j) = 1$ and $\lambda_{\text{min}}(A_j) = 0$ for every $j = 1, \ldots, m$;

5. for each measurement basis $B_\nu = \{A_1, \ldots, A_m\}$ for $\nu$, the following two properties hold:

(a) $B_\nu$ has trivial residual, and

(b) there exist a (possibly nonseparable) Hilbert space $H_\nu$, a faithful unital representation $\phi : B(H) \to B(H_\nu)$, a subspace $H_0 \subset H_\nu$, positive operators $Y_1, \ldots, Y_m \in B(H_0)$, and pairwise-orthogonal projections $Q_1, \ldots, Q_m \in B(H_0^2)$ such that, for every $j = 1, \ldots, m$, $\| Y_j \| \leq 1$ and $\phi(A_j) = Q_j \oplus Y_j \in B(H_0^2 \oplus H_0)$.

If, in addition, $H$ has finite dimension, then the representation $\phi$ in (5b) is the identity and the Hilbert space $H_\nu$ is $H$ itself.

Proof. (1) $\Rightarrow$ (2). Suppose that $\{B_1, \ldots, B_m\}$ is a linear basis for $T_\nu$, where $B_j = \nu(F_j)$ for some $F_j \in O(X)$. By Naimark’s Dilatation Theorem there are a Hilbert space $K$, an isometry $V : H \to K$, and a projective measurement $\omega : O(X) \to B(K)$ such that $\nu(E) = V^* \omega(E) V$ for all $E \in O(X)$. If $\Phi : T(H) \to T(K)$ denotes the quantum channel $\Phi(R) = V R V^*$ and if $\phi = \Phi^*$, then Naimark’s Dilatation Theorem takes the form $\nu = \phi \circ \omega$. Because $\nu$ is approximately clean, $\omega = \psi \circ \nu$ for an approximately normal ucp map $\psi : B(H) \to B(K)$. Thus $T_\omega$ and $T_\nu$ necessarily have the same dimension, namely $m$.

Let $Q_j = \omega(F_j) \in T_\omega$ for each $j$. By Proposition (14) there are pairwise disjoint sets $E_1, \ldots, E_m \in O(X)$ such that, if $P_j = \omega(E_j)$ for each $j$, then $\{P_1, \ldots, P_m\}$ is a measurement basis for $T_\omega$ with trivial residual. For each $j$ let $A_j = \nu(E_j)$ and note that $P_j = \psi(A_j)$. Thus, $\{A_1, \ldots, A_m\}$ is a measurement basis for $T_\nu$ with trivial residual.

Moreover, by Lemma (13) for every $p \in \mathbb{N}$ and all $L_1, \ldots, L_m \in M_p(\mathbb{C})$, we have

\[
\max_{1 \leq j \leq m} \| L_j \| = \left\| \sum_{j=1}^m P_j \otimes L_j \right\| = \left\| \sum_{j=1}^m \psi(A_j) \otimes L_j \right\|
\leq \sum_{j=1}^m \left\| A_j \otimes L_j \right\|
\leq \max_{1 \leq j \leq m} \| L_j \|.
\]

Thus, $\psi|_{T_\nu}$ is a unital completely contractive map of $T_\nu$ onto $T_\omega$, and so $\psi|_{T_\nu}$ is a unital complete order isomorphism [17, Proposition 2.11].

(2) $\Rightarrow$ (3). Fix a measurement basis $B_\nu = \{A_1, \ldots, A_m\}$ for $\nu$. By assumption, there are a Hilbert space $K$ and a projective quantum probability measure $\omega : O(X) \to B(K)$ such that $T_\omega$ and $T_\nu$ are unital completely order isomorphic. Because every measurement basis for $T_\omega$ has trivial residual, the same is true of $T_\nu$. If $\psi : T_\nu \to T_\omega$ denotes this complete order isomorphism, then for any finite-dimensional Hilbert space $K$ and $L_1, \ldots, L_m \in B(K)$, we have, by Lemma (13) that

\[
\left\| \sum_{j=1}^m A_j \otimes L_j \right\| = \left\| \sum_{j=1}^m \psi(A_j) \otimes L_j \right\| = \max_{1 \leq j \leq m} \| L_j \|.
\]

Now if $T$ is a bounded linear operator on a Hilbert space $L$ of dimension at least 2, then for every $\varepsilon > 0$ there is a unit vector $\xi \in L$ such that $\| T \| < \| T \xi \| + \varepsilon$. If $L_\xi = \text{Span}\{\xi, T\xi\}$, and if $Q \in B(L)$ is the projection with range $L_\xi$, then $\| QTQ \| < \| T \xi \| + \varepsilon.$ That is, to know the norm of an operator $T$, it is sufficient to know the norms of all
compressions $QTQ$ of $T$ using projections of rank 2. Hence, if the norm equalities above hold for all finite-dimensional Hilbert spaces $K$, then they also hold for arbitrary Hilbert spaces $K$.

$(3) \Rightarrow (4)$. Let $B_v = \{A_1, \ldots, A_m\}$ be a measurement basis for $\nu$. Fix $i$ and let $L_i = 1$ and $L_j = 0$ for $j \neq i$. Then, by assumption, $\|A_i \otimes L_j\| = \|L_j\|$ and so $\|A_i\| = 1$. Since $\lambda_{\text{max}}(A_i) = \|A_i\|$, we deduce that $\lambda_{\text{max}}(A_i) = 1$. Because every spectral point of a hermitian operator is an approximate eigenvalue, the fact that 1 is in the spectrum of $A_i$ implies that there is a sequence of unit vectors $\xi_n \in H$ such that $\lim_n \|A_i \xi_n - \xi_n\| = 0$. Because, $A_i - 1 = -\sum_{j \neq i} A_j \xi_n = 0$. Hence, by the Cauchy–Schwarz inequality,

$$\lim_{n \to \infty} \left( \sum_{j \neq i} A_j \xi_n, \xi_n \right) = 0.$$  

That is,

$$0 = \lim_{n \to \infty} \sum_{j \neq i} \langle A_j^{1/2} \xi_n, A_j^{1/2} \xi_n \rangle = \sum_{j \neq i} \left( \lim_{n \to \infty} \|A_j^{1/2} \xi_n\|^2 \right).$$

and so $0 = \lim_{n \to \infty} \|A_j^{1/2} \xi_n\|$ for each $j \neq i$. Thus, 0 in the spectrum of $A_j$ for all $j \neq i$. That is, $\lambda_{\text{min}}(A_j) = 0$ for all $j \neq i$. Our choice of $i$ was arbitrary; thus, for every $j = 1, \ldots, m$ we necessarily have $\lambda_{\text{max}}(A_j) = 1$ and $\lambda_{\text{min}}(A_j) = 0$.

$(4) \Rightarrow (3)$. Fix a measurement basis $B_v = \{A_1, \ldots, A_m\}$ for $\nu$. Let $g : B(H) \to B(H_\nu)$ be the unital representation of $B(H)$ that is described in $(3)$, which has the property that $\lambda$ is an eigenvalue of $g(Y)$ for every spectral point $\lambda$ of a hermitian operator $Y \in B(H)$. (Hence, $H_\nu$ is necessarily non separable if $H$ has infinite dimension.) Therefore, by the hypothesis that $\lambda_{\text{max}}(A_j) = 1$ and $\lambda_{\text{min}}(A_j) = 0$ for each $j$, we deduce that 0 and 1 are eigenvalues of the positive operators $g(A_j)$ for all $j = 1, \ldots, m$.

For simplicity of notation, we shall drop the reference to the unital homomorphism $g$ and assume that 0 and 1 are eigenvalues of the positive operators $A_j$ for all $j = 1, \ldots, m$. (In the cases where $\dim H$ is finite, we may dispense with $g$ altogether because spectral points are necessarily eigenvalues in such cases.)

Decompose $H$ as $H = \ker(A_1 - 1) \oplus H_1$. Because both $A_j$ and $1 - A_j$ are positive, and because $\sum_{j=1}^m A_j = 1$, the decomposition $H = \ker(A_1 - 1) \oplus H_1$ implies that

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & G_1 \end{bmatrix} \text{ and } A_j = \begin{bmatrix} 0 & 0 \\ 0 & G_j \end{bmatrix} \text{ for all } j \geq 2,$$

for some positive contractions $G_1, \ldots, G_m \in B(H_1)$ satisfying $\sum_{j=1}^m G_j = 1$. Because 1 is an eigenvalue of $A_2$, the matrix representation of $A_2$ above shows that $\ker(A_2 - 1)$ must lie within the subspace $H_1$. Thus, we decompose the Hilbert space $H$ further as $H = \ker(A_1 - 1) \oplus \ker(A_2 - 1) \oplus H_2$ so that

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K_2 \end{bmatrix}, \quad \text{and } A_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_j \end{bmatrix}$$

for all $j \geq 3$ and for some positive contractions $K_1, \ldots, K_m \in B(H_2)$ satisfying $\sum_{j=1}^m K_j = 1$. As before, $A_3$ has an eigenvalue 1; the corresponding eigenspace $\ker(A_3 - 1)$ must lie in $H_3$, by the matrix representation of $A_3$ above. Therefore, an iteration of the argument used to this point produces, after $m$ steps in total, a decomposition of $H$ as

$$H = \left( \bigoplus_{j=1}^m \ker(A_j - 1) \right) \bigoplus H_m,$$

for a subspace $H_m \subset H$, and where, for every $j$, $A_j = Q_j \oplus Y_j$ for some positive contractions $Y_j \in B(H_m)$ such that $\sum_{j=1}^m Y_j = 1 \in B(H_m)$ and where each $Q_j$ is the projection with range $\ker(A_j - 1)$.

$(5) \Rightarrow (1)$. Suppose that $\nu'$ is a quantum probability measure and $\nu \ll_{\text{ap-cl}} \nu'$. The proof of Proposition $(1.3)$ shows that there exists pairwise-disjoint measurable sets $E_1, \ldots, E_m \in \mathcal{O}(X)$ such that, if $A_j = \nu(E_j)$ for each $j$, then $\{A_1, \ldots, A_m\}$ is a measurement basis for $\nu$. By assumption $(5)$, there exist a (possibly nonseparable) Hilbert space $H_{\nu}$, a faithful unital representation $g : B(H) \to B(H_\nu)$, a subspace $H_\nu \subset H$, positive operators $Y_1, \ldots, Y_m \in B(H_\nu)$, and pairwise-orthogonal projections $Q_1, \ldots, Q_m \in B(H_\nu^*)$ such that, for every $j = 1, \ldots, m$, $\|Y_j\| \leq 1$ and $g(A_j) = \sum_{\nu \in \mathcal{O}(X)} \sum_{j=1}^m \sum_{j=1}^m Q_j Y_j Q_j \nu \in \mathcal{O}(X)$. The proof of Proposition $(1.3)$ shows that there exists pairwise-disjoint measurable sets $E_1, \ldots, E_m \in \mathcal{O}(X)$ such that, if $A_j = \nu(E_j)$ for each $j$, then $\{A_1, \ldots, A_m\}$ is a measurement basis for $\nu$. By assumption $(5)$, there exist a (possibly nonseparable) Hilbert space $H_{\nu}$, a faithful unital representation $g : B(H) \to B(H_\nu)$, a subspace $H_\nu \subset H$, positive operators $Y_1, \ldots, Y_m \in B(H_\nu)$, and pairwise-orthogonal projections $Q_1, \ldots, Q_m \in B(H_\nu^*)$ such that, for every $j = 1, \ldots, m$, $\|Y_j\| \leq 1$ and $g(A_j) = \sum_{\nu \in \mathcal{O}(X)} \sum_{j=1}^m \sum_{j=1}^m Q_j Y_j Q_j \nu \in \mathcal{O}(X)$.
That is, for any finite-dimensional Hilbert space $K$ and operators $L_1, \ldots, L_m \in \mathcal{B}(K)$,

$$\left\| \sum_{j=1}^{m} A_j \otimes L_j \right\| = \left\| \sum_{j=1}^{m} q(A_j) \otimes L_j \right\| = \left\| \sum_{j=1}^{m} (Q_j \oplus Y_j) \otimes L_j \right\|
$$

$$= \max \left\{ \left\| \sum_{j=1}^{m} Q_j \otimes L_j \right\|, \left\| \sum_{j=1}^{m} Y_j \otimes L_j \right\| \right\}
$$

$$= \max_{1 \leq j \leq m} \| L_j \|,$$

since, by Lemma III.3,

$$\left\| \sum_{j=1}^{m} \nu_j \otimes L_j \right\| = \max_{1 \leq j \leq m} \| L_j \| \quad \text{and} \quad \left\| \sum_{j=1}^{m} \nu_j \otimes L_j \right\| \leq \max_{1 \leq j \leq m} \| L_j \| .$$

Therefore, by Lemma III.3 again,

$$\left\| \sum_{j=1}^{m} \nu'(E_j) \otimes L_j \right\| \leq \max_{1 \leq j \leq m} \| L_j \| = \left\| \sum_{j=1}^{m} \nu(E_j) \otimes L_j \right\| .$$

Hence, Theorem III.1 implies that $\nu' \ll_{ap-cl} \nu$. \hfill $\square$

IV. OBSERVATIONS AND APPLICATIONS

A. Approximately clean 1-0 measurements

A 1-0 measurement is one in which the sample space of outcomes is given by $X = \{0, 1\}$ and the space of events by $\mathcal{O}(X) = \{\emptyset, \{0\}, \{1\}, X\}$. A direct application of Theorem III.4 leads to the following equivalent statements for a quantum probability measures $\nu$ on $(X, \mathcal{O}(X))$:

1. $\nu$ is approximately clean;
2. $\lambda_{\min}(\nu(\{0\})) = 0$ and $\lambda_{\max}(\nu(\{0\})) = 1$.

This spectral condition above also appears in another notion of optimality for quantum measurements. By the notation $\nu_2 \ll_i \nu_1$, one means that $\nu_2$ is a fuzzy version of $\nu_1$, which is to say that there is a confidence mapping $\Lambda$ on $\mathcal{O}(X)$ such that $\text{Tr}(R \nu_2(E)) = \text{Tr}(R \nu_1(\Lambda(E)))$ for all measurable sets $E$ and density operators $R$ [12]. With respect to this partial order, $\nu$ is optimal if $\nu \ll_i \nu'$ implies $\nu' \ll_i \nu$. In the case where $X = \{0, 1\}$ and $\mathcal{O}(X)$ is the power set of $X$, a 1-0 quantum measurement $\nu$ is optimal with respect to $\ll_i$ if and only if $\nu(\{0\})$ satisfies the spectral property [2] above [12, Proposition 3]. Hence, we deduce that a 1-0 measurement is approximately clean if and only if it is optimal with respect to the fuzzy ordering $\ll_i$.

B. Approximately clean but not clean

Proposition IV.1. Assume that $\mathcal{H}$ is a separable Hilbert spaces of infinite dimension and that $X = \{0, 1\}$ is an outcome space for 1-0 measurements.

1. There exists an approximately normal ucp map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\phi$ is not normal.
2. There exist approximately clean 1-0 measurements $\omega, \nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ such that $\omega \ll_{ap-cl} \nu$, but $\omega \ll_{cl} \nu$.
3. There exist approximately clean 1-0 measurements $\omega, \nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ such that $\omega \ll_{cl} \omega$, but $\omega \not\ll_{cl} \nu$. 




Proof. To prove statement (1), assume without loss of generality that \( H = L^2([0,1]) \) and suppose that \( A \in \mathcal{B}(H) \) is the multiplication operator \( Af(t) = tf(t), f \in H \), and let \( P \in \mathcal{B}(H) \) be a projection such that \( P \) and \( 1 - P \neq 0 \) are of infinite rank. Thus, \( A \) and \( P \) are positive and \( \lambda_{\min}(A) = \lambda_{\min}(P) = 0 \) and \( \lambda_{\max}(A) = \lambda_{\max}(P) = 1 \). Therefore, by Theorem III.1 and its proof, there is an approximately normal ucp map \( \phi : \mathcal{B}(H) \to \mathcal{B}(H) \) for which \( P = \phi(A) \). Suppose, further, that \( \phi \) is normal. Thus, \( P = \sum_k W_k^* AW_k \) for some sequence \( \{W_k\}_k \subset \mathcal{B}(H) \) of contractions. Choose any nonzero \( \xi \in H \) such that \( P\xi = 0 \); thus,

\[
0 = \langle P\xi, \xi \rangle = \sum_k \langle AW_k\xi, W_k\xi \rangle = \sum_k \|A^{1/2}W_k\xi\|^2.
\]

Therefore, \( AW_k\xi = 0 \) for every \( k \). Because \( A \) has no eigenvalues, necessarily \( W_k\xi = 0 \) for every \( k \), and so also \( W_k^*W_k\xi = 0 \) for every \( k \). Thus, \( 0 \neq \xi = \sum_k W_k^*W_k\xi = 0 \), which is a contradiction. Thus, there cannot be a normal ucp map \( \phi \) for which \( P = \phi(A) \). Hence, \( \phi \) is approximately normal but not normal.

For statement (2), let \( \nu : \mathcal{O}(X) \to \mathcal{B}(H) \) denote the unique 1-0 quantum measurements for which \( \omega(\{0\}) = P \) and \( \nu(\{0\}) = A \). The proof of assertion (1) shows that there is an approximately normal \( \phi \) such that \( \omega = \phi \circ \nu \) but there are no normal ucp \( \psi \) satisfying \( \omega = \psi \circ \nu \). Thus, \( \omega \ll_{\text{ac}} \nu \), but \( \nu \not\ll_{\text{ac}} \omega \).

To prove statement (3), let us use \( \omega \) and \( \nu \) as above. Because (2) already asserts that \( \omega \ll_{\text{ac}} \nu \), we need only show that \( \nu \ll_{\text{cl}} \omega \). Theorem III.3 implies that \( \|A \otimes L + 1 \otimes K\| = \|P \otimes L + 1 \otimes K\| \) for all \( n \times n \) matrices \( L, K \) and all \( n \in \mathbb{N} \). Hence, by Lemma 2.4.2(iv), there is a separable Hilbert space \( K \), a homomorphism \( \pi : C^*(P) \to \mathcal{B}(K) \) (where \( C^*(P) \) is the unit C*-algebra generated by \( P \)) and an isometry \( V : H \to K \) such that \( A = V^*\pi(P)V \). Because \( A \) has infinite rank and \( 1 - A \) have infinite rank, so do \( \pi(P) \) and \( 1 - \pi(P) \). Thus, there is a unitary operator \( U : H \to K \) such that \( \pi(P)U = UP \). Define a normal ucp map \( \psi : \mathcal{B}(H) \to \mathcal{B}(H) \) by \( \psi(X) = (U^*V)^*X(U^*V) \) and observe that \( A = \psi(P) \). That is, \( \nu = \psi \circ \omega \) for a normal ucp map \( \psi \), and therefore \( \nu \ll_{\text{cl}} \omega \).

Corollary IV.2. There exists a 1-0 quantum measurement that is approximately clean but not clean.

Proof. Let \( \omega \) and \( \nu \) be the approximately clean 1-0 measurements discussed in the proof of Proposition IV.1. Because \( \nu \ll_{\text{ac}} \omega \) and \( \omega \not\ll_{\text{ac}} \nu \), the approximately clean quantum probability measure \( \nu \) fails to be clean.

C. Clean qubit measurements are projective

If \( H \) is finite-dimensional, then a quantum measure is clean if and only if it is approximately clean. Furthermore, if \( \dim H = 2 \), then Theorem III.3 implies that \( \nu \) is clean if and only if \( \nu \) is a projection-valued measure. In contrast, a qubit measurement \( \nu \) with sample space \( X = \{x_1, \ldots, x_n\} \) is clean in the original sense of if and only if \( \nu(\{x_j\}) \) has rank-1 for every \( j = 1, \ldots, n \) (Theorem 11.2, [15]). Thus, the stricter criteria for cleanness used herein leads to a smaller class of clean quantum measurements than was found in [15, 16].

D. Structure of quantum probability measures with finite-dimensional measurements spaces

Proposition IV.3. If \( \{A_1, \ldots, A_m\} \) is a measurement basis for a quantum probability measure \( \nu \), then there exist finite signed measures \( \nu_1, \ldots, \nu_m \) on \( (X, \mathcal{O}(X)) \) such that each \( \nu_j \ll_{\text{ac}} \nu \) (in the sense that \( \nu(E) = 0 \) implies \( \nu_j(E) = 0 \)) and

\[
\nu(E) = \sum_{j=1}^m \nu_j(E)A_j, \quad \text{for all } E \in \mathcal{O}(X).
\]

Proof. For every \( E \in \mathcal{O}(X) \) there exist unique \( \alpha_{E,1}^{E}, \ldots, \alpha_{E,m}^{E} \in \mathbb{R} \) such that \( \nu(E) = \sum_{j=1}^m \alpha_{E,j} A_j \). Define \( \nu_j : \mathcal{O}(X) \to \mathbb{R} \) by \( \nu_j(E) = \alpha_{E,j}^{E} \), \( j = 1, \ldots, m \). By the linear independence of \( A_1, \ldots, A_m \), \( \nu_j(\emptyset) = 0 \) for each \( j \), while \( \nu_j(X) = 1 \) for all \( j \). To show that \( \nu_j \) is countably additive, suppose that \( \{E_k\}_{k \in \mathbb{N}} \) is a countable collection of pairwise disjoint measurable sets. By the countable additivity of \( \nu \),

\[
\nu \left( \bigcup_{k \in \mathbb{N}} E_k \right) = \sum_{k \in \mathbb{N}} \nu(E_k) = \sum_{k \in \mathbb{N}} \sum_{j=1}^m \alpha_{E_k,j} A_j = \sum_{j=1}^m \left( \sum_{k \in \mathbb{N}} \alpha_{E_k,j} \right) A_j,
\]

which proves—again by the linear independence of \( A_1, \ldots, A_m \)—that

\[
\nu_j \left( \bigcup_{k \in \mathbb{N}} E_k \right) = \sum_{k \in \mathbb{N}} \nu_j(E_k),
\]
for every $j = 1, \ldots, m$. Lastly, if $\nu(E) = 0$, then the linear independence of $A_1, \ldots, A_m$ yields $v_j(E) = 0$ for every $j$, whence each $v_j \ll_{ac} \nu$.

In Proposition [IV.3] it is not generally true that the signed measures $v_j$ are in fact positive measures.

**Definition IV.4.** A measurement basis $B_\nu = \{A_1, \ldots, A_m\}$ for a quantum probability measure $\nu$ is perfect if there exist (positive) probability measures $\mu_1, \ldots, \mu_m$ on $(X, \mathcal{O}(X))$ such that

$$\nu(E) = \sum_{j=1}^{m} \mu_j(E) A_j, \text{ for all } E \in \mathcal{O}(X).$$

It is not difficult to verify that every measurement basis $B_\omega$ of a projective quantum probability measure $\omega$ is perfect.

In quantum information theory it is a standard practice to define a POVM associated with an $n$-outcome quantum measurement to be an $n$-tuples of positive operators $M_1, \ldots, M_n$ such that $\sum_{j=1}^{n} M_j = 1$. In this scenario, the underlying quantum probability measure $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ is the unique POVM for which $\nu(\{x_j\}) = M_j$ for each $j$, where $X = \{x_1, \ldots, x_n\}$. Observe that $\mathcal{O}(X)$ is the power set of $X$ and that

$$\mathcal{R}_\nu = \left\{ \sum_{j \in E} M_j : E \in \mathcal{O}(X) \right\} \subset \text{Span} \{M_1, \ldots, M_n\}.$$

Thus, $\text{Span}\{M_1, \ldots, M_n\} = T_\nu$. However, any measurement basis $\{A_1, \ldots, A_n\}$ for $T_\nu$ is drawn from the set $\mathcal{R}_\nu$ rather than from $\{M_1, \ldots, M_n\}$. This is an essential difference in our approach from other studies, where the POVM is analysed by studying the spanning set $\{M_1, \ldots, M_n\}$. By way of the observation above it is clear that $\dim T_\nu \leq |X|$, the cardinality of the sample space $X$.

**Proposition IV.5.** If $\{M_1, \ldots, M_n\}$ is a POVM associated with a quantum probability measure $\nu$ on a sample space $X$ of cardinality $n$, then the following statements are equivalent:

1. $\dim T_\nu = n$;
2. $\{M_1, \ldots, M_n\}$ is a perfect measurement basis for $\nu$.

**Proof.** If $\dim T_\nu = n$, then $\text{Span}\{M_1, \ldots, M_n\}$ is $n$-dimensional, which implies that $\{M_1, \ldots, M_n\}$ is a basis of $T_\nu$. Moreover, $\{M_1, \ldots, M_n\}$ is clearly a measurement basis and

$$\nu = \sum_{j=1}^{n} \delta_{\{x_j\}} M_j,$$

where $\delta_{\{x_j\}}$ is the Dirac probability measure with mass concentrated at the point set $\{x_j\}$. The converse is obvious. \qed

E. Quantity of information versus quality of information

It was noted by Buscemi et al in [5] that in passing to cleaner quantum measurements there is a trade off between the quantity of information and quality of information afforded by such measurements. As a specific example, we indicate below that quantum measurements that yield the greatest amount of information are never approximately clean.

Recall that a quantum probability measure $\nu$ on $(X, \mathcal{O}(X))$ with values in $\mathcal{B}(\mathcal{H})$ is *informationally complete* if, for any two states $R_1, R_2 \in T(\mathcal{H})$, the equation

$$\text{Tr} (R_1 \nu(E)) = \text{Tr} (R_2 \nu(E))$$

holds for every event $E \in \mathcal{O}(X)$ if and only if $R_1 = R_2$.

**Proposition IV.6.** No informationally complete quantum measurement with values in $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space of finite dimension $d \geq 2$, is clean.

**Proof.** It is well-known that if $\nu$ is informationally complete, then the only operators that commute with every effect $\nu(E), E \in \mathcal{O}(X)$, are the scalar operators. On the other hand, statement [6] of Theorem [III.3] asserts that if $\nu$ is clean and $\dim \mathcal{H} \geq 2$, then there is a non-trivial projection in the commutant of $T_\nu$. \qed
V. CONCLUSION

The notion of cleanness for a quantum probability measure is one way in which a positive operator valued measure can be viewed as being optimal. Our work herein is strongly motivated by the paper of Buscemi et al. [5] in the context of finite-dimensional spaces. In particular, we replace their condition that the underlying Hilbert space have finite dimension with the weaker condition that the measurement space associated with a quantum probability measure have finite dimension, which is of practical interest for the measurement of quantum systems represented by an infinite-dimensional Hilbert space in which only finite many outcomes of the measurement are expected. Even in this case, however, the use of infinite-dimensional spaces leads some further challenges, one of which being that the space of normal ucp maps is not closed in the point-ultraweak topology. For this reason, our use of Pellonpää’s refinement of the notion of cleanness [18] involves approximately normal rather than normal ucp maps. Although there is a true difference between clean and approximately clean in the contact of infinite-dimensional spaces, the latter notion allows for a complete characterisation (Theorem III.4) of such quantum probability measures when the measurement space has finite dimension. In the case of finite-dimensional Hilbert space, Theorem III.4 also represents a new result, as previous work in finite dimensions did not take Pellonpää’s definition of cleanness into account.

The notion of projective quantum probability measure dates back to von Neumann [19] and is associated with nondestructive measurements. Such quantum measures are clean, as Pellonpää [18] demonstrates. Herein we proved in Theorem III.4 that a non-projective quantum probability measure \( \nu \) with finite-dimensional measurement space \( \mathcal{T}_\nu \) is approximately clean if and only if there is a projective measure \( \nu^\rho \) such that \( \mathcal{T}_\nu \) and \( \mathcal{T}_{\nu^\rho} \) are completely order isomorphic. Thus, although the measurement outcomes of \( \nu \) are not registered as projections, the resulting measurement space is, from the perspective of linearity and matricial order, exactly the same as that space \( \mathcal{T}_{\nu^\rho} \) which does arise from measurement outcomes that are registered as projections. Therefore, in the category of finite-dimensional operator systems [17], approximately clean and projective measurements determine the same objects.

In practice, quantum measurements are rarely nondestructive, and so the introduction of approximately clean measurements allows for the opportunity to model (nearly) nondestructive measurements while at the same time accommodating within the mathematical model the very real presence of noise in the measuring apparatus. Indeed, in restricting to quantum systems modelled by finite-dimensional Hilbert spaces, Theorem III.4 demonstrates that a clean quantum measurement \( \nu \) has a component that is projective and an orthogonal component that is noisy, and that it is this projective component that is responsible for the cleanness of \( \nu \). Consequentially, all clean qubit measurements are projective. Some strategies for the preprocessing of noisy quantum measurements to achieve cleaner ones have been considered recently in [8].

Acknowledgements

This work is supported in by the Discovery Grant and Postgraduate Scholarship programs of the Natural Sciences and Engineering Research Council of Canada. We wish to thank the referee for a useful critique of the initial version of this paper and for drawing our attention to the references [8, 12].

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