Impedance of a circular coil of arbitrary orientation in a conducting tube

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Abstract
We have developed a procedure to determined analytical expressions for the electromagnetic field and impedance of a circular induction coil having an arbitrary orientation in a conductive tube. Initially we express the field of a circular current filament in free space in terms of the global coordinate system referenced to tube axis. This is done by representing the filament field as a single layer potential and forming an integral over the layer using the source coordinates of a static Green's kernel. A coordinate transform to local source coordinates referenced to the filament axis allows one to determine the coil source function from that of the filament via integration. The effects of induced current in the tube wall on the coil impedance can then be determined for an internal coil. By extension of the present results one can predict the arbitrary orientated coil impedance for an external coil and the impedance variations of a tilted coil due to flaws in tubes.

1. Introduction
The analysis of circular coil fields and their interaction with conductive plates, tubes, circular rods, and the corresponding layered systems, are studied to advance the theory of induced current and its applications to nondestructive evaluation. As a result, algorithms for evaluation of inductive probe fields can be embedded in integral equation codes to provide fast and accurate numerical predictions of flaw signals [1, 2] or flaw shape via inversion methods [3]. At an early stage in these developments, analytical results were obtained for the field and impedance of a circular coil whose axis is normal to the surface of a planar plate and a coil coaxial with a circularly cylindrical conductor, such as a heat exchanger tube[4]. Subsequently, expressions have been derived for the field of an arbitrary orientated coil above a plate [5]. Similarly, the field of a coil whose position and axial direction is arbitrary with respect to the axis of a tube or hole has been determined [6]. Here we give an alternative solution to the latter problem for an internal coil whose location and orientation is arbitrary, figure 1. The present approach is similar to one used previously for the particular case where the coil axis is perpendicular to that of the tube [7], but for coil of arbitrary orientation, the details are naturally more complicated. The results presented here are restricted to the case of an coil in a tube whose axis does not intersect the smallest circular cylinder enclosing the coil, figure 1. However, the procedure can be adapted to deal with this case and with an external coil.

In general, the analysis of the field of a circular coil near a cylindrical conductor requires two coordinate systems; one local and one global. The local coordinates are cylindrical coordinates referenced to the tube axis. The local coordinates are reference to the coil axis. A basic example of an eddy current problem using two coordinate systems is that of an internal coil whose axis parallel to that of the tube. In that case we simply have two cylindrical coordinate systems with parallel axes. For this case it is not difficult to carry out a transformation between coordinate systems to compute the offset coil field or the coil impedance variation with frequency due to the tube [8]. In another example, the axis of a circular induction coil is perpendicular to that of a uniform bore-hole or tube. The analytical expression for the field and impedance of a coil in this case was found by using
the Biot-Savart law to give a result in terms of an infinite integral [9]. This was done first for a circular filament and the result integrated over the coil cross section to get the coil field. Later, the derivation of the circular filament field with its axis perpendicular to that of a tube was formulated as a single layer potential problem [7]. Practical applications of this configuration include a circumferentially-sensitive pancake coil array [10].

An important motivation for treating the arbitrary orientation coil is its potential application to semi-automatic eddy current inspection systems. As these evolve toward full automation, they need to be not only self-guided but also self-correcting by using sensor data. Practical examples include the automatic inspection of aircraft engine discs, inspection during space flight and the potential need for inspection under liquid sodium in a fast reactor. The evolution of such systems places additional emphasis on the use of inductive probes as sensors of the surface geometry as well as functioning as flaw detectors. In these applications, the impedance variations of the coil provides the data that can be used to estimate the shape and location of the surface of the material via an inverse problem. In addition, one can potentially use the present solution as the foundation for including the effect a cylindrical core on the probe field [11, 12] an thereby extend the possibilities for surface sensing.

2. Field analysis

2.1. Scalar decomposition of the field

A quasi-static time-harmonic field varying as the real part of $e^{-i\omega t}$ can be represented by the transverse electric (TE) potential $\psi_1$ and transverse magnetic (TM) potential $\psi_2$ defined with respect to the unit vector $\hat{z}$, which is the direction of the global coordinate axis and also the tube axis, figure 1. In a source-free homogeneous region the magnetic field can be expressed as

$$
H = \nabla \times \nabla \times (\hat{z} \psi_i) + k^2 \nabla \times (\hat{z} \psi_2),
$$

where $k^2 = \omega \mu \sigma$, $\mu$ is the permeability of the tube material and $\sigma$ its electrical conductivity. The electric field intensity then takes the form

$$
E = \omega \mu [\nabla \times (\hat{z} \psi_1) + \nabla \times \nabla \times (\hat{z} \psi_2)].
$$

For a conductive region, both scalar potentials satisfy the Helmholtz equation,

$$
(\nabla^2 + k^2) \psi_i = 0, \quad i = 1, 2.
$$

For non-conductive regions, $k^2 = 0$, the transverse potentials satisfy the Laplace equation, and (1) reduces to

$$
H = \nabla \frac{\partial \psi_i}{\partial z}.
$$

Figure 1. A circular induction coil of arbitrary orientation inside a conductive tube.
Note that the magnetic field of the coil in a non-conductive region is defined solely in terms of the TE potential and this is also the case as we switch to local coordinates, denote by a zero subscript, figure 2. In local coordinates we write as \( \mathbf{y} = \nabla y_0 \) where transverse electric potential \( \psi_0 \) is defined with respect to the direction of the filament axis. In non-conducting regions, the potential satisfies the Laplace equation and

\[
\mathbf{y} = \nabla y_0 = \nabla \nabla \psi_0.
\]

By comparing (4) and (5), it is evident that recovering a solution in terms of \( \psi_1 \) is done simply by using

\[
\frac{\partial \psi_1}{\partial z} = -\frac{\partial \psi_0}{\partial x_0}.
\]

Equation (5) is similar to the case where the magnetic field is written in terms of the gradient of a magnetic scalar potential as \( \mathbf{H} = \nabla \Phi \). In both cases we have to deal with a potential conflict with Ampère’s law which requires that \( \nabla \times \mathbf{H} = 0 \) in a region where electric current flows. Even if an explicit representation of the current in the region is omitted, we still have to avoid a conflict with the circuital law due to the fact that a line integral of the gradient of a continuous scalar quantity with continuous derivatives is zero for a closed path and therefore not dependent on the current. The difficulty is traditionally overcome by introducing a discontinuity in the magnetic scalar potential \([13]\) at a surface bounded by the filament, usually chosen to be a circular disc. The discontinuity in \( \Phi \) is physically equivalent to the introduction of a magnetic dipole layer, in which case the magnetic scalar potential is by definition, a double layer potential. In using the TE potential instead, one chooses the appropriate discontinuity to be in the normal gradient of \( \psi_0 \) at the surface bounded by the current filament. This is effectively equivalent to the introduction of a magnetic monopole layer associated with a single layer potential \([14]\).

2.2. Field of a circular current filament

Consider a circular filament coaxial with the local \( x_0 \)-coordinate axis, figure 2, and centered at point Q. The origin of the local coordinates is at the point P located, in terms of global coordinates at \( (x, y, z) = (x_0, 0, 0) \). Starting on the global \( x \)-axis, the \( x_0 \)-axis can be located by rotating it through an azimuthal angle \( \phi \) about the point P and by tilting the \( z_0 \)-axis through a polar angle \( \theta \) relative to the line PV, figure 2 which is parallel to the global \( z \)-axis. The filament lies in the plane \( x_0 = x_0 \). Later we treat \( x_0 \) and the filament radius, \( r_0 \), as variables of integration to determine the coil field. To ensure the continuity of tangential electric field, at the bounding surface of the filament \( S_0 \), \( \psi_0 \) must be continuous this surface, taken here to be a circular disk.
restricted form of the circuital law, the line integral of the magnetic field cannot cross the filament disk at which we impose a discontinuity at \( S_0 \):
\[
\left[ \frac{\partial \psi_0}{\partial x_0} \right]_{S_0} = -I. \tag{7}
\]

The essence of the circuital law is thereby reproduced by integrating between adjacent points on opposite sides of the disk to get \( \pm I \) depending on the path direction.

The potential \( \psi_0 \) is a solution of the Laplace equation that can be expressed in cylindrical polar coordinates as an inverse Fourier transform:
\[
\psi_0(r) = \frac{I}{4\pi^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} D_m(v) \left( I_m(|v|\rho) \right) \left( K_m(|v|\rho) \right) e^{i m \phi} dv,
\]
where the source function, \( D_m(v) \), is to be determined in the present case for a filament centered at \( Q \), figure 2. The radial limit, \( s_1 \) in (8), is the distance from the \( z \)-axis to the closest point on the filament and is the upper limit of the solution containing \( I_m(|v|\rho) \). Similarly, the lower radial limit \( s_2 \) is the radial distance between the \( z \)-axis and the farthest point on the filament, figure 2. It is the lower limit of the solution in equation (8) containing \( K_m(|v|\rho) \). The solution in the region \( s_1 < \rho < s_2 \) is not needed for present purposes.

By using Green’s second theorem [15] for a single layer potential, \( \psi_0 \) can be determined from
\[
\psi_0(r) = \frac{I}{4\pi} \int_{S_0} G(r'|r) dS' = \frac{I}{4\pi} \int_{S_0} \frac{1}{R} dS',
\tag{9}
\]
A consequence of the dependence \( R = |r - r'| \) is that we can write
\[
\frac{\partial \psi_0}{\partial z} = -\frac{\partial \psi_0}{\partial x_0}, \tag{10}
\]
instead of (6) to determine \( \psi_1 \). Since \( R \) can be expressed as [16]
\[
\frac{1}{R} = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi' - \phi)} \times \int_{-\infty}^{\infty} I_m(|v|\rho) K_m(|v|\rho') e^{i m \phi'} dv,
\tag{11}
\]
where \( \rho_+ \) is the greater and \( \rho_- \) the lesser of \( \rho \) and \( \rho' \), one can substitute (11) into (9) and compare with (8) to get
\[
D_m(v) = \int_{S_0} e^{-im \phi'} \left( K_m(|v|\rho') \right) \left( I_m(|v|\rho') \right) e^{im \phi} dS',
\tag{12}
\]
which can be evaluated for a circular filament of arbitrary orientation following a co-ordinate transform.

### 3. Coordinate transformation

The filament source function, equation (12), can be expressed in a form that can be integrated to produce the coil source function. This can be done using Graf’s addition theorem [17] and a set of equations to transform between global and local coordinate systems.

We begin with Graf’s theorem which provides two results of relevance to the present task:
\[
K_\nu(\rho) e^{-i\nu x} = \sum_{n=-\infty}^{\infty} K_{\nu+n}(\alpha) I_n(\beta) e^{-i\nu \zeta}, \tag{13}
\]
and
\[
I_\nu(\rho) e^{-i\nu x} = \sum_{n=-\infty}^{\infty} (-1)^n I_{\nu+n}(\alpha) I_n(\beta) e^{-i\nu \zeta}. \tag{14}
\]
In both cases the arguments are related by the cosine rule, \( \alpha^2 = \alpha^2 + \beta^2 - 2\alpha\beta \cos \zeta \), for the triangle shown in figure 3. Equation (13) is used to transform the source integral (12) for the case where \( \rho < s_1 \) before evaluating the field integral, (9), containing \( I_n(|v|\rho) \). We seek instead the field of an internal coil since this is the more usual case in practical tube inspections. The integral for the internally sourced field contains \( K_{\nu}(|v|\rho) \) for the region where \( \rho > s_2 \) and the source integral is transformed in section 4 using (14).

The integration over the surface \( S_0 \) is divided into two parts at the line \( AB \) where the filament disk intersects the \( y = 0 \) plane, figure 4. This means that the addition theorem is applied twice with reference to \( \Delta \) RST and
where T is a point on the line AB, while U and S have the same $z_0$ coordinate. Also, we need to define the limits of the integration at AB and at the filament using local cartesian coordinates $x_0, y_0, z_0$. With the origin of the local coordinates on the global $x$-axis at $x = x_1$, which is point P in figure 4, global coordinates are related to the local coordinates as follows.

\[
\begin{align*}
    x &= x_0 \cos \phi \cos \theta - y_0 \sin \phi - z_0 \cos \phi \sin \theta + x_t \\
    y &= x_0 \cos \theta \sin \phi + y_0 \cos \phi - z_0 \sin \sin \phi \\
    z &= x_0 \sin \theta + z_0 \cos \theta
\end{align*}
\]

The surface $S_0$, bounded by the filament, is in the plane $x_0 = x_0'$, and coaxial with the $x_0$ axis. In order to define the integration over $S_0$ in local coordinates, one uses equations for the $x_0$ and $y_0$ coordinates of points on the line AB expressed as linear functions of $z_0$. From (16) with $y = 0$ and $x_0 = x_0'$, the perpendicular distance from the $y_0 = 0$ plane to a point on AB, figure 6, is given by

\[
Y_0(z_0) = \tan \phi (z_0 \sin \theta - x_0' \cos \theta).
\]

Similarly, the perpendicular distance from the $z_0 = 0$ plane to a point on AB is given by

\[
Z_0(y_0) = \frac{x_0'}{\tan \phi} + \frac{y_0}{\tan \phi \sin \theta}
\]

To define the perpendicular distance from a point on the $z$ axis to a point on AB, figure 4, we put $y_0 = Y_0(z_0)$ and use (18) in (15) together with $x_0 = x_0'$ for any point on $S_0$, to get

\[
X(z_0) = \frac{\cos \theta}{\cos \phi} x_0' - \frac{\sin \theta}{\cos \phi} z_0 + x_t.
\]

where $X(z_0)$ represents the distance RT, figure 4, measured in the $x$-direction from the $z$ axis and expressed as a function of the local coordinate variable $z_0$. 

ΔRUT, figure 5, where T is a point on the line AB, while U and S have the same $z_0$ coordinate. Also, we need to define the limits of the integration at AB and at the filament using local cartesian coordinates $x_0, y_0, z_0$. With the origin of the local coordinates on the global $x$-axis at $x = x_1$, which is point P in figure 4, global coordinates are related to the local coordinates as follows.

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    x &= x_0 \cos \phi \cos \theta - y_0 \sin \phi - z_0 \cos \phi \sin \theta + x_t \\
    y &= x_0 \cos \theta \sin \phi + y_0 \cos \phi - z_0 \sin \sin \phi \\
    z &= x_0 \sin \theta + z_0 \cos \theta
\end{align*}
\]

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X(z_0) = \frac{\cos \theta}{\cos \phi} x_0' - \frac{\sin \theta}{\cos \phi} z_0 + x_t.
\]

where $X(z_0)$ represents the distance RT, figure 4, measured in the $x$-direction from the $z$ axis and expressed as a function of the local coordinate variable $z_0$. 

Figure 3. Triangle used showing addition theorem variables related by the cosine rule.

Figure 4. Circular filament center at point Q. The filament surface $S_0$ intersects with $y = 0$ plane at the line AB.
4. Evaluation of the filament source function

4.1. Filament with a polar tilt and an azimuthal rotation
In applying the addition theorem in the form given in (14), with reference to the variables shown in figure 3, we let \( \kappa = |v|\rho' \), \( \alpha = |v|X(z_0) \) and \( \beta = |v|[y_0' - Y_0(z_0)] \). For the case where \( \rho > s_2 \) and \( \varphi' > 0 \),

\[
I_m(|v|\rho')e^{-im\varphi'} = \sum_{n=-\infty}^{\infty} (-1)^n I_{m+n}(|v|X) \times I_n(|v|[y_0' - Y_0])e^{-in(\varphi - \phi)}.
\]  

(21)

Figure 5. Plan view of the xy-plane intersected by the circular filament centered at Q.

Figure 6. Integration over the surface \( S_0 \) bounded by the circular filament is divided at the line AB which is at the intersection of the \( y = 0 \) plane with the filament disk. The linear function \( Y_0(z_0) \), shown in the diagram, defines the perpendicular distance of this line from the line at the intersection of the \( y_0 = 0 \) plane with the disk.
For \( \rho > s_2 \) and \( \varphi' < 0 \),

\[
I_m(|v|\rho')e^{-im\varphi'} = \sum_{n=-\infty}^{\infty} (-1)^n I_{m+n}(|v|X) \\
\times I_n(|v|y'_0 - Y_0) e^{i(n+\varphi')}.
\]  

(22)

The integration over the filament disk is with respect to the local source coordinates \( y'_0 \) and \( z'_0 \). The limits of the integration with respect to \( y'_0 \) are defined using the points \( y_A \) and \( y_B \) which are \( y_0 \)-coordinates of the points A and B, figure 6. Addition limits are defined by the filament boundary at which

\[
y'_0^2 + z'_0^2 = \rho_0^2
\]  

(23)

For the case where \( \varphi' > 0 \), the \( y'_0 \) integral is from \( y_B \) to \( y_0 \). For the case where \( \varphi' < 0 \) the integration is from \( -\rho_0 \) to \( y_A \), figure 6. To find expressions for \( y_A \) and \( y_B \) we put \( y_0 = Y_0(z_0) \) in (23) to find points on the line AB. Then use the resulting equation to eliminate \( z_0 \) from (18). The result is a quadratic equation for \( Y_0 \) whose solutions are

\[
y_A = -x'_0 \cos \phi \sin \phi \cos \theta + \frac{\xi}{\cos^2 \phi + \sin^2 \phi \sin^2 \theta}
\]

(24)

\[
y_B = -x'_0 \cos \phi \sin \phi \cos \theta - \frac{\xi}{\cos^2 \phi + \sin^2 \phi \sin^2 \theta}
\]

(25)

where

\[
\xi = \sin \phi \sin \theta \sqrt{\rho_0^2 (\cos^2 \phi + \sin^2 \phi \sin^2 \theta)} - x'_0^2 \sin^2 \phi \cos^2 \theta
\]

(26)

The limits of integration with respect to \( z'_0 \) are at the line AB and at the filament itself, figure 6. For \( \varphi' > 0 \), the lower limit of the integration with respect to \( z'_0 \) is \( -a(y'_0) \) where

\[
a(y'_0) = \sqrt{\rho_0^2 - y'_0^2},
\]

(27)

and the positive root is selected. The upper limit is \( z_a(y'_0) \) defined as a piece-wise function

\[
z_a(y'_0) = \begin{cases} 
Z_0(y'_0) & y_B \leq y'_0 < y_A \\
-a(y'_0) & y_A \leq y'_0 < \rho_0 
\end{cases}
\]

(28)

Similarly, for \( \varphi' < 0 \), the lower limit of integration is \( z_b(y'_0) \) defined as

\[
z_b(y'_0) = \begin{cases} 
-a(y'_0) & -\rho_0 \leq y'_0 < y_B \\
Z_0(y'_0) & y_B \leq y'_0 < y_A 
\end{cases}
\]

(29)

and the upper limit is \( a(y'_0) \).

By substituting (21), (22) and \( z' = x'_0 \sin \theta + z'_0 \cos \theta \) using the coordinate transform (17) into (12) and applying the limits discussed above, we can rewrite (12) as

\[
D_m(v) = \sum_{n=-\infty}^{\infty} e^{i(m+\varphi - n\varphi_0)}
\]

\[
\times \left\{ \int_{j_0}^{\rho_0} \int_{-a(y'_0)}^{z_b(y'_0)} h_{m,n}(y'_0, z'_0, v) dz_0' dy_0' + (-1)^n \int_{-\rho_0}^{y_A} \int_{a(y'_0)}^{Z_0(y'_0)} h_{m,n}(y'_0, z'_0, v) dz_0' dy_0' \right\}
\]

(30)

where

\[
h_{m,n}(y'_0, z'_0, v) = I_{m+n}(|v|X(z'_0)) \\
\times I_n(|v|y'_0 - Y_0(z'_0)) e^{-imz'_0 \cos \theta}
\]

(31)

The above derivation for a filament source function of arbitrary orientation has included a parameter representing the filament position \( x'_0 \) on the \( x_0 \) axis so that the coil source function can be determined by integration with respect to \( x'_0 \) and \( \rho_0 \). Special cases of both the filament and coil source functions are given in appendix A through C.
5. Free space coil field

Having determined the integrals for the TE potential $\psi_0$ of a filament with arbitrary orientation, we next determine $\psi_1$ for a circular coil in free space, figure 7, which is expressed in the form

$$\psi_1(r) = \frac{1}{4\pi^2} \sum_{m=-\infty}^{\infty} e^{im\phi} \int_{-\infty}^{\infty} C_m(v) I_m(|v|\rho) J_m(|v|\rho) e^{in\varphi} d\nu \quad \rho < \beta_1$$

where the radial limits, $\beta_1$ and $\beta_2$, are respectively the shortest and longest distances from $z$-axis to a point on the coil and we shall not need the solution in the region $\rho < \rho_1$ and $K_m(|v|\rho)$ is used for $\rho > \beta_2$. The coil source function, $C_m(v)$, is determined from the filament source function $D_m(v)$ by using the similar procedure to that given in [7].

By using (10) and integrating the filament field with respect to $x_0'$ and $\rho_0$ over the coil cross-section, figure 7, we obtain the source function, $C_m(v)$, for the coil field

$$C_m(v) = \frac{N}{l} \int_{r_1}^{r_2} [D_m(v, 1/2) - D_m(v, -1/2)] d\rho_0$$

where $\nu = N/l(r_2 - r_1)$ is the turns density of the coil, $N$ is number of turns and $l$ is the axil length. The integral limits, $r_1$ and $r_2$, are the inner and outer radii of the coil respectively.

6. Coil impedance variation and calculated results

6.1. Coil impedance variation formulation

The coil impedance variation, $\Delta Z$, due to the presence of tube is expressed in terms of transverse electric potentials derive from a reciprocity relationship [7],

$$P^2 \Delta Z = -\kappa \mu_0 a \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial \psi_1^{(r)}}{\partial z} \frac{\partial \psi_0^{(0)}}{\partial z} - \frac{\partial \psi_0^{(0)}}{\partial z} \frac{\partial \psi_1^{(r)}}{\partial z} \frac{\partial \psi_1^{(r)}}{\partial \rho} d\varphi d\zeta$$

where $\psi_0^{(0)}$ is the TE potential without the presence of tube and $\psi_1^{(r)}$ is the TE potential due to the reflection at the tube interface. By a transformation developed from Paseval’s relation

$$\Delta Z = \frac{\kappa \mu_0 a}{4\pi^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} v^2 C_m(-v) \Gamma_m(v) C_m(v) dv$$

where $\Gamma_m(v)$ is the reflection coefficient with the source inside of the tube, derived in [7].

6.2. Calculated results

Computation of the $C_m(v)$, impedance changes and induced eddy current could be facilitated by the use of numerical integration method. The finite integral can use Gauss-Legendre method. The infinite range integrals can be computed with existing automatic adaptive quadrature method. For example, (C.3), in double precision arithmetic, the CPU time for the computation of $C_m(v)$ on a typical laptop with CORE-i5 CPU is less than 1 min. The artificial PEC or PMC boundary conditions could be used to transform the infinite integration into summation to speed up the calculation. In addition, computation of induced eddy current density can use 2D Fast Fourier Transform.
The coil impedance variation, $\Delta Z$, due to presence of an inconel 600 tube ($\mu_r = 1$) of conductivity 0.84 MS/m, has been calculated for various coil orientations. The coil and tube parameters are given in Table 1. The coil center liftoff, $\lambda$, given in the table denotes the shortest distance between coil center point P, figure 7, and the tube inner surface. Impedance data is normalized by dividing by the isolated coil reactance, $X_0 = \omega L_0$.

In the first example, $\Delta Z$ for a circular bobbin coil inside and co-axial with the tube is calculated by using the present general formulation by setting $\theta = 90^\circ$, with $x_1 = 0$ and comparing the results with that calculated by using Dodd and Deed’s model [4]. The comparison, shown in figures 8 and 9 shows good agreement. For the case with $\theta = 0^\circ$ and $\phi = 0^\circ$, the general formulation gives results in agreement with the theoretical results and experimental results in [7].

Coil impedance variations due to azimuthal rotation and polar rotation are analyzed and presented separately. Figure 10 is the real part of $\Delta Z$ due to azimuthal rotation and figure 11 is the imaginary part. We can find $\phi = 0^\circ$ case has the strongest impedance changes and azimuthal tilt reduces the coil impedance changes. But the changes are not significant, which means the impedance variation is not sensitive to azimuthal rotation for the large liftoff case (compared with the dimension of the coil. Here $\lambda > r_2$). Figure 12 is the real part of $\Delta Z$ due to polar rotation and figure 13 is the imaginary part. We can find polar tilt increases the impedance changes and $\theta = 90^\circ$ has biggest reaction. The reason might be the coil axis is not perpendicular to the tube surface for polar rotation and the bigger polar tilt angle will make the coil closer to the tube surface and hence has stronger reaction.

This method is valid for analyzing an arbitrary tilted coil above a planar structure as well. By assigning a relatively large tube radius, this model should approximate the model shown in [5]. The impedance variation and induced eddy current results generated by the proposed method matches with this alternative method.

In order to observe the rotation effects visually, the magnitude of electrical field intensity distribution on the tube inner surface has been calculated shown in figure 14 to figure 15 in terms of distance in $z$ direction and angles ($\phi$) in the

| Table 1. Coil and inconel steam generator tube parameters. |
|----------------------------------------------------------|
| Coi n inner radius, $r_1$ | 1.529 mm |
| Coi n outer radius, $r_2$ | 3.918 mm |
| Coi n thickness | 1.044 mm |
| Number of turns | 305 |
| Isolated DC coil inductance, $L_0$ | 465 $\mu$H |
| Tube inner diameter | 16.64 mm |
| Tube outer diameter | 18.99 mm |
| Conductivity (MS/m) | 0.84 |
| Relative magnetic permeability, $\mu_r$ | 1 |
| Coi n center liftoff, $\lambda$ | 5.675 mm |

Figure 8. Comparison between coaxial bobbin coil and polar tilt coil with $\theta = 90^\circ$, $\phi = 0^\circ$ and $x_1 = 0$ of normalized resistance changes for a coil inside a tube of different frequency.
Figure 9. Comparison between coaxial bobbin coil and polar tilt coil with $\theta = 90^\circ$, $\phi = 0^\circ$ and $x_1 = 0$ of normalized reactance changes for a coil inside a tube of different frequency.

Figure 10. The effect of tilted angle $\phi$ on normalized resistance changes for a circular coil with different $\phi$ and $\theta = 0^\circ$ inside a tube.

Figure 11. The effect of tilted angle $\phi$ on normalized reactance changes for a circular coil with different $\phi$ and $\theta = 0^\circ$ inside a tube.
Figure 12. The effect of tilted angle $\theta$ on normalized resistance changes for a circular coil with different $\theta$ and $\phi = 0^\circ$ inside a tube.

Figure 13. The effect of tilted angle $\theta$ on normalized reactance changes for a circular coil with different $\theta$ and $\phi = 0^\circ$ inside a tube.

Figure 14. The magnitude of electric field, $|E|$, distribution on the tube inner surface with tilt angle $\theta = \phi = 0^\circ$ at 10 kHz. $z$ is the distance of a point on the surface away from $z = 0$ plane and angles $\Phi$ is angle in the circumferential direction referenced to global coordinates.
circumferential direction referenced to global coordinates at frequency 10 kHz with the dimensions in table 1. Once obtaining the electrical field distribution, the magnitude of current density distribution can be easily evaluated by multiplying the conductivity of tube material. In figure 14, the symmetry of field distribution is easily noticed and this matches the geometry symmetry of the coil to the tube. In addition, it has two spots where the field is stronger due to that the coil has two points which has the shortest distance to the tube surface (for current filament case, figure A1, the point D and E will be the closest point to the inner tube surface). Since the field is canceled along the axis of coil the field is negligibly weak at the center. The field distribution with \( \phi = \theta = 45^\circ \) is presented in figure 15, one can find the field is stronger in the area where is closer to the coil which also matches the geometrical relations of coil with respect to tube.

7. Conclusion

This article provides an efficient analytical method for evaluating the electromagnetic field and impedance of circular coil with arbitrary orientation inside a tube. The source functions used in the field expressions are obtained by solving a single layer potential problem via coordinate transform. Then the field in the tube and coil impedance variation due to the tube can be determined. In addition, the effects of azimuthal and polar rotations on impedance variations have been analyzed separately with numerical examples. This theory can be used in automated inspection system for determining the probe position and orientation with respect to a tube by monitoring coil impedance variations with position.

Compared with FEM method, this analytical method is more efficient. Noted that, \( C_m(\nu) \), is independent of frequency and tube. For a specific coil, It’s only required to calculate once and reuse it for all the other frequencies and tube the coil interacts with. The overall calculation time can be significantly reduced. However, the FEM solver needs to solve for all different frequency points.

Another advantage of this method is that it can easily be integrated with moment of method (MoM) or FEM to generate the incident field for solving more complicated application, for example, flaw detection. It can avoid recalculation of incident field for any different coil scan position relatively to object in non-destructive evaluation application.

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Appendix A. Azimuthally rotated filament

We next determine the source function, (30), for the case where the polar tilt angle \( \theta \) is zero and the filament has an arbitrary azimuthal rotation angle \( \phi \). To define the source integral (31), we need the \( X_0 \) and \( Y_0 \) coordinates, of the line at the intersection of the \( y = 0 \) plane with the filaments disk. This is a line normal to the plane of the diagram at A and in the absence of a polar tilt is independent of \( z_0 \). By substituting \( \theta = 0^\circ \) into (18) and (20) we
find, with (24), (25) and (26) that

\[ Y_0 = -x_0' \tan \phi = y_A = y_0 \]  
\[ X = \frac{x_0'}{\cos \phi} + x_0. \]  

In addition, we find that the limits of integration with respect to \( \zeta \) simplify to give

\[ z_\text{a} = -z_\text{b} = a(y_0') = \sqrt{\rho_0^2 - y_0'^2}. \]  

Substituting these results into (30) and carrying out the integral gives

\[ \int \frac{\rho_0^2 - \rho^2}{\rho_0^2 - \rho^2} e^{-\pi \zeta^2} d\zeta = \frac{2}{\pi} \sin (\sqrt{\rho_0^2 - y_0'^2}), \]  

and hence

\[ D_m(\nu) = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} e^{i(m+n)\phi} I_{m+n}(\nu|x|X \times [F_m'(Y_0, \rho_0) + (-1)^n F_m'(-Y_0, \rho_0)] \]  

where

\[ F_m'(r, \rho_0) = \int_{-\rho_0}^{\rho_0} I_m(|r| |r| - \rho_0) \sin (\nu \sqrt{\rho^2 - \tilde{\zeta}^2}) d\rho_0. \]

Note that if the tilt angle \( \phi \) is big enough or the circular coil is extensive in the axial direction, some filaments of the coil might be totally above or below the \( y = 0 \) plane. Then only one \( F_m(\rho, \rho_0) \) function in (A.5) will be included and the integration in (A.6) will be between the limits \(-\rho_0 \) to \( \rho_0 \).

For a coil with an azimuthal rotation, one substitutes (A.5) into (33), to get

\[ C_m(\nu) = \frac{2\nu}{\nu^2} \sum_{n=-\infty}^{\infty} e^{i(m+n)\phi} \left[ I_{m+n}(\nu |r| \left| \frac{l}{2 \cos \phi} + x_1 \right) \right. \]

\[ \left. - (-1)^n I_{m+n}(\nu |r| \left| -\frac{l}{2 \cos \phi} + x_1 \right) \right] \]

\times [F_0(\tau_1, r_2, Y_0) + (-1)^n F_0(-\tau_1, r_2, -Y_0)] \]  

where \( Y_0 \) is listed as (A.1).

\[ E_0(\chi_1, \chi_2, \eta) = \int_{\chi_1}^{\chi_2} \int_{\eta}^{\eta} I_\nu(|\xi - \eta|) \sin (\nu \sqrt{\rho_0^2 - \xi^2}) d\xi d\rho_0 \]  

Note that \( C_m(\nu) \) is an odd function of \( \nu \), which can simplify the numerical calculation.

**Appendix B. Polar tilted filament**

Consider the case where the filament has a zero azimuthal tilt angle and the an arbitrary polar rotation angle \( \theta \), figure B1. By substituting \( \phi = 0^\circ \) into (18), (20), (24) and (25), we get
Similarly, $z_a$ and $z_b$ can be reduced as follow
\begin{equation}
    z_a = -z_b = a(y_0') = \sqrt{\rho_0^2 - y_0'^2}
\end{equation}

By substituting these equations into (30), we get
\begin{equation}
    D_m(v) = 2 \sum_{n=-\infty}^{\infty} \cos \left( \frac{n\pi}{2} \right) e^{-n\pi y_0' \sin \theta} \int_0^{y_0'} I_n(|v| y_0') \, dy_0' \\
    \times \int_{-\sqrt{\rho_0^2 - y_0'^2}}^{\sqrt{\rho_0^2 - y_0'^2}} I_{m+n}(|v| X(z_0')) e^{-n\pi z_0' \cos \theta} \, dz_0' \, dy_0'
\end{equation}

After integrating with respect to $y_0'$ and $z_0'$ over the surface $S_0$ bounded by the filament, the source function, $D_m(v)$, for $\rho > s_2$ can be obtained.

For polar tilt, substituting (B.5) into (33), shows that $C_m(v)$ is given by
\begin{equation}
    C_m(v) = 2 \mu \sum_{n=-\infty}^{\infty} \cos \left( \frac{n\pi}{2} \right) \left[ e^{-n\pi z_0' \sin \theta} - e^{-n\pi y_0' \sin \theta} \right] E_n(v)
\end{equation}

where
\begin{equation}
    E_n(v) = \int_{-d}^{d} \int_{-\sqrt{\rho_0^2 - y_0'^2}}^{\sqrt{\rho_0^2 - y_0'^2}} I_n(|v| y_0') \, dy_0' \\
    \times e^{-n\pi z_0' \cos \theta} \, dz_0' \, dy_0'
\end{equation}

where $a = \sqrt{\rho_0^2 - y_0'^2}$. Note that, for all odd $n$, $C_m(v) = 0$ and $C_m(-v)$ conjugates with $C_m(v)$.

Appendix C. Filament axis perpendicular to that of the global coordinates

The rotary filament case can be obtained by setting $\theta = 0^\circ$ and $\phi = 0^\circ$. It also can be achieved by either setting $\theta = 0^\circ$ from polar rotation case or setting $\phi = 0^\circ$ for azimuthal rotation case. Here we substitute $\theta = 0^\circ$ into the expressions of the polar rotation case. Then (B.2) reduces to
\begin{equation}
    X = x_0' + x_1
\end{equation}

Substituting this into (B.5) and using (A.4), we find that
\begin{equation}
    D_m(v) = 4 \sum_{n=-\infty}^{\infty} \cos \left( \frac{n\pi}{2} \right) I_{m+n}(|v| X) \\
    \times \int_{-\sqrt{\rho_0^2 - y_0'^2}}^{\sqrt{\rho_0^2 - y_0'^2}} I_n(|v| y_0') \sin \left( \sqrt{\rho_0^2 - y_0'^2} \right) dy_0'
\end{equation}

which is the source function for a filament whose axis is perpendicular to the global coordinate $z$-axis.
To determine the coil source function in the case where its axis is perpendicular to that of the global z-axis, one substitutes (C.2) into (33), to get

\[
C_m(\nu) = \frac{4\mu}{v^2} \sum_{n=-\infty}^{\infty} \cos\left(\frac{n\pi}{2}\right) \times \left[ I_{m+\nu}\left(|v|; x_0 + \frac{l}{2}\right) - I_{m+\nu}\left(|v|; x_0 - \frac{l}{2}\right)\right] \times \int_0^{2\rho_0} \int_0^{\rho_0} I_n(|v| \rho_0') \sin\left(v\sqrt{\rho_0'^2 - y_0'^2}\right) d\rho_0' \quad (C.3)
\]

Note that \(C_m(\nu)\) is an odd function of \(\nu\) and that the double integral can be transformed into series that can be expressed in terms of Struve functions [7].

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