Cosmological string models from Milne spaces and \( SL(2, \mathbb{Z}) \) orbifold

Jorge G. Russo *

Departament ECM, Facultat de Física, Universitat de Barcelona, Diagonal 647, E-08028, Barcelona, Spain
Institució Catalana de Recerca i Estudis Avançats (ICREA)

Abstract

The \( n + 1 \)-dimensional Milne Universe with extra free directions is used to construct simple FRW cosmological string models in four dimensions, describing expansion in the presence of matter with \( p = \kappa \rho, \kappa = (4 - n)/3n \). We then consider the \( n = 2 \) case and make \( SL(2, \mathbb{Z}) \) orbifold identifications. The model is surprisingly related to the null orbifold with an extra reflection generator. The study of the string spectrum involves the theory of harmonic functions in the fundamental domain of \( SL(2, \mathbb{Z}) \). In particular, from this theory one can deduce a bound for the energy gap and the fact that there are an infinite number of excitations with a finite degeneracy. We discuss the structure of wave functions and give examples of physical winding states becoming light near the singularity.

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* e-mail address: jrusso@mail.cern.ch
1. Introduction

Deep questions in cosmology, such as the nature of the big bang singularity or initial boundary conditions, need to be addressed in the context of a quantum gravity theory. Understanding string theory in time-dependent backgrounds leads to a number of technical problems which have been considered in many recent works [1-16]. In particular, in [8] the null orbifold model [17] – a time dependent geometry consisting in flat space with some identifications – was studied in detail. The string model of this Lorentzian orbifold contains some instabilities which are reflected in IR divergences in scattering amplitudes. Their physical origin is a large back reaction in the geometry as soon as particles are introduced [3,11,12].

In the present work we will be interested in the general class of time-dependent locally flat spacetimes obtained from the $n + 1$ dimensional Milne Universe. They can be viewed as a particular case of hyperbolic compactification in string theory (see [18]), which have recently attracted some interest as they lead to interesting cosmologies [19]. We will consider in more detail the $n = 2$ case, which, with proper identifications, is related to the null orbifold. Here we make orbifold identifications on the upper half plane by using the (elliptic) modular group $SL(2, \mathbb{Z})$.

Milne spaces in the context of inflationary cosmology were studied in [20]. String models in 1+1 dimensional Milne space were discussed in [17,1,3,4]. Discussions on higher dimensional Milne spaces can be found in [17,18], and more recently in [16].

In section 2 we discuss the simplest string cosmology based on flat Milne space. We will see that already these simple backgrounds can be used to obtain interesting cosmologies in four dimensions. This section makes a connection with recent work on hyperbolic compactification [19], and it can be viewed as a physical motivation for string models based on Milne space. In section 3 we introduce the $SL(2, \mathbb{Z})$ orbifold model. The study of this model involves harmonic analysis on symmetric spaces and number theory, from which we will extract some results for the string spectrum. We also discuss the construction of wave functions and physical string states.

2. String cosmology from $n + 1$-dimensional Milne Universe

The $n + 1$-dimensional Milne space is described by the metric

$$ds^2 = -dt^2 + t^2 dH_n^2,$$ (2.1)
where $dH^2_n$ is the arc element of the hyperboloid or upper half $n$ plane. In Poincaré coordinates, it is

$$dH^2_n = \frac{1}{z^2}(dz^2 + dz_1^2 + ... + dz_{n-1}^2) .$$

(2.2)

The space is flat, as it is evident upon introducing Cartesian coordinates as follows:

$$U = \frac{t}{z} , \quad V = tz + \frac{t}{z} \sum_{i=1}^{n-1} z_i^2 , \quad X_i = \frac{tz_i}{z} .$$

(2.3)

This provides the embedding of the hyperboloid in $n + 1$ Minkowski space,

$$ds^2 = -dUdV + dX_i^2 ,$$

where the hyperboloid is described by

$$t^2 = UV - X_1^2 - ... - X_{n-1}^2 ,$$

(2.4)

which exhibits the $SO(1, n)$ isometry of $H_n$.

Alternatively, one can use angular or Milne coordinates

$$dH^2_n = d\rho^2 + \sinh^2 \rho \, d\Omega^2_{n-1} ,$$

(2.5)

and define

$$T = t \cosh \rho , \quad R = t \sinh \rho ,$$

(2.6)

in terms of which the metric (2.1) reads

$$ds^2 = -dT^2 + dR^2 + R^2 d\Omega^2_{n-1} .$$

(2.7)

Now let us consider type II string theory in the following background

$$ds^2_{10} = -dt^2 + t^2 dH^2_n + dx_1^2 + dx_2^2 + dx_3^2 + dy_1^2 + ... + dy_{6-n}^2 ,$$

(2.8)

where the $y$ coordinates describe compact internal dimensions. All other supergravity fields are trivial. This is an exact solution (to all alpha prime orders) of string theory, since the Riemann tensor identically vanishes (though it may receive quantum string-loop corrections). Similarly, one can write down the analogous eleven dimensional metric, which is a solution of M theory. The internal space described by the $y$ coordinates can be replaced by any Ricci flat space, giving a more general class of cosmological backgrounds which are solutions to leading order in $\alpha'$.
It is surprising that an interesting four dimensional FRW cosmology can be obtained from the model (2.8). First, we replace the hyperboloid $H_n$ by a finite volume space $H_n/\Gamma$, where $\Gamma$ is a discrete subgroup of $SO(1,n)$ such that the space has finite volume, as in [18,19]. Then we compactify to four dimensions. To obtain the four dimensional Einstein frame metric, we write the ten-dimensional metric in the form

$$ds^2 = e^{2a(t)}ds^2_{4E} + e^{2b(t)}dH_n^2 + dy_1^2 + ... + dy_{6-n}^2,$$

and the condition that $ds^2_{4E}$ is in the Einstein frame is $e^{2a}e^{nb} = 1$. Comparing to (2.8), one obtains

$$ds^2_{4E} = t^n(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2),$$

or

$$ds^2_{4E} = -d\tau^2 + \frac{2n}{3n} (dx_1^2 + dx_2^2 + dx_3^2).$$

This corresponds to 4d Einstein equations coupled to an energy momentum tensor of a perfect fluid with

$$p = \kappa \rho, \quad \kappa = \frac{4-n}{3n}.$$

Although we have started with vacuum Einstein equations in ten dimensions, the four dimensional Einstein metric describes a homogeneous and isotropic space in presence of matter. This matter is of course the scalar field associated with the modulus representing the volume of the hyperbolic space. Interestingly, the above metric is the asymptotic (large time) form of the models of [19]. For $n = 4$, it describes a universe filled with dust, and for $n > 4$ a universe filled with negative pressure matter.

Since the models are based on a flat ten dimensional geometry, the $n+1$ dimensional Milne Universes provide a simple setup for the study of interesting cosmological string models.

So let us consider strings propagating in this space. An important question is whether the string model is exactly solvable. To start with, consider the model (2.8) based on $H_n$ with no identification, i.e. $\Gamma = 1$. The string equations of motion in Cartesian coordinates (2.3) are solved explicitly in terms of right and left moving modes. However, from the relation (2.4) it follows that

$$UV - X_i^2 \geq 0.$$
If the physical space is restricted only to this Milne patch, say with $t > 0$, then the string coordinates are subject to the constraint that the string lives in the interior of the future directed light cone; the space is not geodesically complete and a full description requires boundary conditions at the light cone surface. In string theory, it is possible that consistency also requires the inclusion of the past light cone. In this case, the geometry would describe a universe contracting to a big crunch which makes a transition to an expanding big bang universe.

It is non-trivial to impose the condition (2.13) in string theory. On the other hand, if the full space $U, V, X_i$ is considered, closed timelike curves can arise in the exterior of light cone as a result of identifications.

We briefly comment on a possible approach with a boundary on the light cone surface. A string which at a given instant fully lies in the interior of the future light cone, will satisfy the condition $UV - X_i^2 > 0$ at all future times. Causal propagation implies that all closed strings in the physical space emerge from the light cone surface, so there is some interval of time where the condition $UV - X_i^2 > 0$ is satisfied only for a piece of the string (an example is given at the end of section 3). It is natural to describe the strings intersecting the light cone in terms of open strings with ends attached to the null light-cone surface, which can be thought of as a “null D brane”. In this picture, the set of open string states attached to the null surface determine the initial state, i.e. they would be described by some “in” state in the open string Hilbert space. The closed strings living in the interior of the light cone would originate by emission from the null D brane. Thus the time evolution of the cosmology would be determined by the initial state on the null D brane and the decay properties of this initial state.

Let us now comment on the supersymmetry of these models. The hyperbolic upper half $n$-plane $H_n$ has Killing spinors transforming in the spinorial representation of $SO(1, n-1)$. They were explicitly constructed in [21] (see also discussions in [22,18,23]). The simplest model (2.8) with $\Gamma = 1$ is supersymmetric. When $\Gamma$ is not trivial and the space $H_n/\Gamma$ has finite volume, it was shown in [18] that for $n =$even supersymmetries are always broken by the identifications. For $n =$odd and $n \geq 5$, the analysis of [18] does not exclude that an appropriate choice of $\Gamma$ could give a supersymmetric model with finite volume hyperbolic space. It would be interesting to find some example.
3. \(SL(2, \mathbb{Z})\) orbifold

Let us consider the case \(n = 2\) and recall its relation to the null orbifold [17]. The metric is
\[
ds^2 = -dt^2 + \frac{t^2}{y^2} (dx^2 + dy^2)
\]  
(3.1)
The constant time slices describe the hyperbolic upper half plane \(H_2\). Introducing \(U = t/y\), \(V = ty\), we obtain
\[
ds^2 = -dUdV + U^2 dx^2
\]  
(3.2)
This form exhibits an orbifold singularity at \(U = 0\) moving at the speed of light. Introducing new coordinates
\[
u = U , \quad v = V + U x^2 , \quad X = U x
\]  
(3.3)
we get
\[
ds^2 = -dudv + dX^2 , \quad u = X_0 - X_1 , \quad v = X_0 + X_1
\]  
(3.4)
The null (or “parabolic”) orbifold geometry of [17], studied by Liu, Moore and Seiberg [3], is obtained by orbifolding the isometry \(x = x + 1\). In the coordinates \((u, v, X)\) this leads to the identification
\[
(u, v, X) \equiv (u, v + un^2 + 2Xn, X + n u)
\]  
(3.5)

Now we consider another geometry: we replace \(H_2\) in (3.1) by \(F = SL(2, \mathbb{Z}) \setminus H_2\). \(SL(2, \mathbb{Z})\) transformations are generated by \(z \to z + 1\) and \(z \to -1/z\). The first transformation is precisely what leads to the null orbifold (3.5). Now we have the extra orbifold identification \(z \to -1/z\), with \(z = x + iy\) (and \(t \to t\)). This isometry is surprisingly simple in the coordinates \((u, v, X)\). It just corresponds to the spatial reflection:
\[
u \to u , \quad v \to -u , \quad X \to -X
\]  
(3.6)
where we have used the relation to the original coordinates \((t, x, y)\) given by
\[
u = \frac{t}{y} , \quad v = \frac{t}{y} (x^2 + y^2) , \quad X = \frac{tx}{y}
\]  
(3.7)
\(^1\) In the notation of [3] one has \(x^+ = u, \ x^- = v/2\).
\(^2\) A brief discussion of this geometry is in section 5 of [24].
It should be noted that it is not sufficient to add this orbifold identification (3.6) to the string model of [6] in order to have the fundamental domain $F$ as constant time slices. In addition, one has the restriction $uv - X^2 \geq 0$. So one can distinguish two related but different models: the “null orbifold with reflection” or just “$SL(2,\mathbb{Z})$ orbifold”, obtained simply by adding the reflection identification (3.6) to the null orbifold model of [7], and the “restricted $SL(2,\mathbb{Z})$ orbifold model” (3.1) having the finite volume space $F$ as constant time slices. Both models arise as a quotient by the $SL(2,\mathbb{Z})$ subgroup of the Poincaré isometry group $SL(2,\mathbb{R})$ and have $SL(2,\mathbb{Z})$ symmetry. The former has another patch $uv - X^2 < 0$ (i.e. $UV < 0$), where the metric (3.2) can be put into the form $ds^2 = dt^2 + \frac{t^2}{y^2}(dx^2 - dy^2) = dt^2 + \frac{t^2}{y^2}dz_+dz_-$, with $U = -t/y$, $V = ty$ and $z_\pm = x \pm y$. In this patch, the identification (3.7) corresponds to the isometry $z_+ \to 1/z_+$, $z_- \to 1/z_-$, $t \to t$. In the null orbifold model with reflection it seems more convenient to work in $(u,v,X)$ coordinates. This model has closed time-like curves arising in the region $uv - X^2 < 0$. To see this, one can consider the image of the point $u = v = 0$, $X = 1$ under an $SL(2,\mathbb{Z})$ transformation with parameters $\{a,b,c,d\}$, $ad-bc = 1$. For $bc > 0$, the images $(u',v',X') = (2ab,2cd,1+2bc)$ are time-like separated from the original point. This transformation is a combination of reflection (3.6) and the parabolic orbifold identification (3.3). The identification (3.6) alone, or the identification (3.3) alone, do not produce CTC.

Here we shall use properties of modular functions in $F$ so in the remainder we will consider the restricted $SL(2,\mathbb{Z})$ orbifold model. The two string models should be closely related.

In addition to the singularity at $U = t/y = 0$, the geometry contains the singularities of $F$ at $|z| = 1$, $x = \pm 1/2$ for all $t$. In the flat coordinates, it corresponds to the subspace

$$u = v = \pm 2X .$$

There are no surviving supersymmetries, they are broken by the $SL(2,\mathbb{Z})$ identifications.

Let us now investigate the string spectrum of this model and the structure of vertex operators. To leading order in $\alpha'$, physical modes of the string spectrum obey, in a suitable gauge, the Klein-Gordon equation. This applies to massless and massive scalar fluctuations, as well as to non-zero helicity components of a massive or massless mode. The only exception are winding modes, which cannot be described in terms of local fields of the low

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3 We thank Hong Liu for providing this example.
energy string effective action. We will construct examples of invariant winding states at the end.

First, consider a massless scalar field propagating in this geometry. To exhibit the \( SL(2, \mathbb{Z}) \) invariance, it is convenient to work in \((t, x, y)\) coordinate system. We set the transverse momentum components \( p_i = 0 \). The Laplace equation \( \Delta_3 \Psi = 0 \) is

\[
\Delta_3 \Psi = \partial_t (t^2 \partial_t \Psi) ,
\]

\[
\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} .
\]

We can solve it by separation of variables, \( \Psi = \psi_s(x, y) \varphi_s(t) \), with

\[
\Delta_2 \psi_s = s(s-1) \psi_s .
\]

This implies that the solution is given in terms of Maass waveforms. These are modular functions which are eigenfunctions of the Laplace operator \( \Delta_2 \) with at most polynomial growth at \( y \to \infty \) [25].

Let us first obtain the wave function by simply starting with the flat solution and summing over images. In the coordinates \((u, v, X)\) the solutions of the wave equation \( \Delta_3 \Psi = 0 \) are plane waves with momentum \((p_u, p_v, p_X)\), with \( 4p_u p_v = p_X^2 \). Then an invariant wave function would have the generic form

\[
\Psi = \sum_{\text{images}} \int dp_u dp_v dp_X \varphi(p_u, p_v, p_X) e^{ip_u u + ip_v v + ip_X X}
\]

\[
= \int dp_u dp_v \bar{\varphi}(p_u, p_v) \sum_{a, b, c, d} \left( \frac{\chi}{p_v} \right)^{|(a+c)z+b+d|} , \quad \chi = \sqrt{p_v/p_u} ,
\]

where \( \sum' \) stands for a sum over integers \( a, b, c, d \) obeying \( ad - bc = 1 \), and we have used the relation (3.7). Normalizability of the wave function (3.10) (with the invariant measure \( dx dy / y^2 \)) is a delicate issue. To illustrate this point, consider plane waves with \( (p_u, p_v) = q(m^2, n^2) \), with \( p_X \) again fixed by the mass shell condition \( 4p_u p_v = p_X^2 \). They are of the form

\[
\Psi = \int dq \ c(q) \sum_{(m, n) \neq (0, 0)} e^{iq \frac{z}{p} |m+nz|^2} .
\]

Now consider a \( c(q) \) of the form: \( c(q) = c_s q^{s-1} \theta(q) \), with \( s \) being a complex number and \( \theta(q) \) is the step function. The integral in \( q \) is a Mellin transform. Computing this integral, one obtains

\[
\Psi = 2c_s \Gamma(s) \zeta(2s) \ t^{-s} \ E_s(z) ,
\]
where $E_s$ is the non-holomorphic Eisenstein series

$$E_s = \frac{1}{2\zeta(2s)} \sum_{(m,n) \neq (0,0)} \frac{y^s}{|m + n|^{2s}}. \quad (3.13)$$

It obeys the functional relation $E_s = \text{const} \ E_{1-s}$, so with no loss of generality one can restrict to $\Re s \geq 1/2$. At $y = \infty$, $E_s$ behaves as $y^s$. Therefore none of these wave functions (3.12) are normalizable.

Normalizable wave functions can be written in terms of cusp forms, which are automorphic functions that decrease exponentially at infinity. They are bounded in $H_2$. Let us summarize some known facts, referring to [25] for details:

i) If $\Re s > 1/2$ and $s \neq 1$, then the vector space $V_s$ of Maass waveforms with eigenvalue $s(s-1)$ is generated by the non-holomorphic Eisenstein series $E_s$. For $s = 1$, the vector space $V_s$ is generated by the constant.

ii) If $\Re s = 1/2$, the vector space $V_s$ is generated by $E_s$ and the cusp forms $v_n$, $n \geq 1$.

iii) The vector space of cusp forms $v_n$ is different from $\{0\}$ only for eigenvalues $s(s-1)$ with $\Re s = 1/2$, for an infinite number of values $s$. The set of values is discrete. However, an explicit basis is not known, nor an explicit analytic example of a cusp form (they can be easily found numerically).

Thus cusp forms constitute the discrete part of the spectrum, while $E_s$ constitute a continuous part of the spectrum.

A general eigenfunction in $L^2(SL(2, \mathbb{Z}) \setminus H)$ is given in terms of the Roelcke-Selberg expansion, which is the analog of a Fourier expansion in $\mathcal{F}$. Define the inner product in the standard way as $(f,g) = \int_F \bar{g}(\bar{z})f(z)y^{-2}dxdy$. Then any $\psi$ in $L^2(SL(2, \mathbb{Z}) \setminus H)$ has the expansion

$$\psi(z) = \sum_{n \geq 0} (\psi, v_n)v_n + \frac{1}{4\pi i} \int_{\Re s = \frac{1}{4}} (\psi, E_s)E_s(z)ds, \quad (3.14)$$

where $v_0 = \sqrt{3/\pi}$.

Any cusp form is part of a vertex operator of the string model having non-zero quantum numbers in $\mathcal{F}$. A similar discussion applies for the wave function of a massive scalar field: we solve the equation

$$\left(\Delta_3 - M^2\right)\Psi = 0, \quad (3.15)$$

$\footnote{E_s has a pole at s = 1. E_s with s = 1/2 has a zero which is cancelled in (3.12) by the factor \zeta(1), leading to a y^{1/2} \log y behavior at y = \infty.}$

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by separation of variables, $\Psi = \psi_s(x, y) \phi_s(t)$. The function $\psi(x, y)$ is then an eigenfunction of the Laplace operator on $F$, so it is also given in terms of cusp forms. Using eq. (3.9), $\phi_s$ is then determined by

$$[\partial_t^2 \partial_t - s(s - 1) + M^2 t^2] \phi_s(t) = 0.$$  \hspace{1cm} (3.16)$$

Hence $\phi_s(t) = J_{\pm(s - \frac{1}{2})}(Mt)/\sqrt{t}$.

The cusp form can be expanded in terms of Bessel functions as follows:

$$\psi_s(z) = \sum_{n \neq 0} a_n y^{1/2} K_{s - \frac{1}{2}}(2\pi|n|y) \exp(2\pi inx), \quad s = \frac{1}{2} + ib, \quad b \in \mathbb{R},$$ \hspace{1cm} (3.17)

where $a_n$ are constrained by eq. (3.9). Alternatively, one can make calculations in the covering space and compute S-matrix elements by using plane-wave vertex operators of flat space and adding the images. However, we have seen above that naive vertices constructed by a sum over images as in (3.11) may not be normalizable (e.g. $c(q) = \sum_{k=0}^{\infty} c_k q^k$ does not give a normalizable wave function (3.11)). $SL(2, \mathbb{Z})$ invariance severely constrains the form of vertices and amplitudes. In what follows we shall deduce a few interesting facts about the spectrum using theorems about cusp forms.

The ten dimensional string theory based on the $SL(2, \mathbb{Z})$ orbifold has additional free directions ($x_1, ..., x_7$). For a massless scalar fluctuation obeying $\Delta_{10} \Psi = 0$, with no string oscillations in the $t, x, y$ directions, the vertex is of the form

$$\Psi = \int d^7 p \, a(p) \, \phi_s(t) \, \psi_s(x, y) \, \partial x. \partial x \, e^{ip_i x_i},$$ \hspace{1cm} (3.18)

where $\phi_s(t)$ is determined by (3.16) with $M^2 \to M^2 + p_i^2$ (in this case, $M^2 = 0$). In the frame $p_i = 0$, the equation is of the form $\partial_t^2 \partial_t \phi_e = e \phi_e$. The parameter $e$, being the eigenvalue of the time-derivative part of the Laplace operator, measures the energy scale of the Kaluza-Klein excitation (bearing in mind that in general energy is not conserved since the background is not static). The spectrum formula is just given by

$$e = s(s - 1),$$ \hspace{1cm} (3.19)

and $\phi_s(t) = a_s t^{-s} + b_s t^{s-1}$. For $s = 0$ we have $e = 0$, corresponding to a state with trivial quantum numbers in $F$, being the only normalizable state with $\text{Re } s \neq 1/2$. For other

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5 For massless excitations, the energy parameter $e$ is associated with the scaling $t \to \Lambda t$. 

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states, normalizability requires that \( s = 1/2 + ib \), so that \( e = -1/4 - b^2 \) is negative definite. Which is the value of \( e \) for the first non-trivial excitation? A theorem \[25\] establishes that \( s(s - 1) \leq -3\pi^2/2 \). This implies that the energy \( e \) of the first Kaluza-Klein state obeys the bound \( e \leq -3\pi^2/2 \). Precise numerical estimates of the first and the next eigenvalues are summarized in \[25\]. Because the spectrum contains both a discrete as well as a continuum part, one should not expect that at low energy the theory will look \( D - 2 \) dimensional.

Another important theorem establishes that the vector space of cusp forms with a given eigenvalue is finite dimensional. This means that there is a finite number of normalizable excitations at each energy level \( e \).

Finally, property iii) implies that the levels for cusp forms are discrete and that the total number of normalizable excitations is infinite. There is also an asymptotic formula from which one can estimate the total number \( N(e_0) \) of normalizable excitations above an energy level \( e_0 \) (or with \( |e| < |e_0| \)), counted with multiplicity. For large \( |e_0| \), one finds the remarkable formula \( N(e_0) \cong |e_0|/12 \). This is a consequence of the Selberg trace formula \[25\].

Similar results apply to quantum numbers of massive string modes.

Let us now construct physical winding states associated with the twisted sector of the orbifold identifications (3.5), (3.6). In the coordinates \((u,v,X)\), the solution to the string equations are expressed in terms of free right and left moving oscillators. We only need to impose boundary conditions and solve the constraints. We first consider the winding solution \((\sigma_\pm = \sigma \pm \tau)\)

\[
\begin{align*}
    u &= n \sigma_v \tau \\
    X &= \frac{n \sigma_v}{8\pi} (\sigma_+^2 - \sigma_-^2) = \frac{n \sigma_v}{2\pi} \sigma \tau \\
    X(\sigma + 2\pi) &= X(\sigma) + n \ u.
\end{align*}
\]  

(3.20)

Then \( v = v_+(\sigma_+) + v_-(\sigma_-) \) is determined by solving the constraint equations \( T_{++} = T_{--} = 0 \). This gives

\[
\begin{align*}
    v &= \frac{n^2 \sigma_v}{24\pi^2} (\sigma_+^3 - \sigma_-^3) = \frac{n^2 \sigma_v}{12\pi^2} (\tau^3 + 3\sigma^2 \tau) \\
    v(\sigma + 2\pi) &= v(\sigma) + n^2 u(\sigma) + 2nX(\sigma).
\end{align*}
\]  

(3.21)

In terms of \((t,x,y)\) coordinates, the solution takes the form

\[
\begin{align*}
    t &= \frac{n \sigma_v}{2\sqrt{3}\pi} \tau^2, \\
    x &= \frac{n \sigma}{2\pi}, \\
    y &= \frac{n}{2\sqrt{3}\pi} \tau.
\end{align*}
\]  

(3.22)
For this state to be \( SL(2, \mathbb{Z}) \) invariant in \( H_2 \), one adds all the images in the upper half plane.

Next, consider the following solution which is a winding state for the twisted sector of (3.6):

\[
X = 2L \sin(\sigma/2) \cos(\tau/2), \quad X_1 = 2L \cos(\sigma/2) \cos(\tau/2),
\]

\[
u = X_0 - X_1, \quad v = X_0 + X_1.
\]

From the constraints, one gets \( X_0 = L\tau \). This solution satisfies the boundary condition

\[
u(\sigma + 2\pi) = v(\sigma), \quad v(\sigma + 2\pi) = u(\sigma), \quad X(\sigma + 2\pi) = -X(\sigma).
\]

In the covering space, it describes a pulsating circular string loop, which contract to zero at \( \tau = \pi \) and has maximum radius at \( \tau = 0, 2\pi, \text{etc.} \)

In the \((t, x, y)\) coordinate system, the solution becomes

\[
t = L\sqrt{\tau^2 - 4 \cos^2(\tau/2)},
\]

\[
x = \frac{2 \sin(\sigma/2) \cos(\tau/2)}{\tau - 2 \cos(\sigma/2) \cos(\tau/2)}, \quad y = \frac{\sqrt{\tau^2 - 4 \cos^2(\tau/2)}}{\tau - 2 \cos(\sigma/2) \cos(\tau/2)},
\]

with the addition of the images. Note that in these coordinates the solution exists after a \( \tau_0 \approx 1.48 \) such that \( \tau^2 - 4 \cos^2(\tau/2) > 0 \). For \( \tau < \tau_0 \), the condition \( uv - X^2 > 0 \) is not satisfied for all points of the string, i.e. part of the string is in the patch \( uv - X^2 < 0 \) which is not covered by the \((t, x, y)\) coordinates. Different points of the string pass through the singularity \( u = 0 \) at different times \( X_0 = L\tau \). As \( \tau \) gradually increases and goes over \( \tau_0 \), the string fully enters into the region \( uv - X^2 > 0 \). At this point, the string has already left the singularity \( u = 0 \). In the upper half plane, when \( \tau \) is slightly above \( \tau_0 \), the string describes a long loop. When \( \tau = \pi \), it contracts to a point at \( x = 0, y = 1 \). Then it keeps pulsating around this point, with a maximum radius that decreases with time.

In the Hilbert space, these classical string configurations (3.22), (3.24) are described by physical string states with large quantum numbers, so that the classical description applies. This requires large \( p_v \) in the case of (3.24), and large \( L \) – which implies large occupation number of the lowest frequency string oscillator – in the case of (3.23), (3.24).

The state (3.22) which winds around \( x \) becomes “massless” near \( U = 0 \) (the light-cone energy \( p_u \) is proportional to \( U^2 n^2/p_v \), see (3.2), (3.3)). This is a reflection of the singularity of the geometry at \( U = 0 \).
The instabilities pointed out in [12] may be absent in the $SL(2, \mathbb{Z})$ orbifold model. Indeed, this instability originated from the gravitational interaction of plane waves and their images. Although here the spectrum has also a continuum part which may lead to wave interactions, this part is severely restricted by $SL(2, \mathbb{Z})$ symmetry, so the argument of [12] does not seem to directly apply to this model. The states in the discrete part of the spectrum have finite motion. The corresponding wave functions are regular (bounded) on $\mathcal{F}$, though they exhibit a singular behavior at $t = 0$. It would be interesting to investigate the partition function and higher point scattering amplitudes to establish if the string model is regular or singular. In the $SL(2, \mathbb{Z})$ orbifold model, the extra reflection symmetry (3.6) adds new terms to the partition function computed in [6].

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