Distinction of the Steinberg representation

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With an appendix by F. Courtès
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Abstract

We prove Dipendra Prasad’s conjecture on distinction of the Steinberg representation [Pr] for symmetric spaces of the form $G(E)/G(F)$, when $G$ is a split reductive group defined over $F$, and $E/F$ an unramified quadratic extension of non-archimedean local fields.

Introduction

Let $G$ be a connected reductive group defined over a non-archimedean local field $F$, and let $E/F$ be a quadratic galois extension of $F$. If $\pi$ is a smooth representation of $G(E)$ and $\chi$ a smooth character of $G(F)$, one says that $\pi$ is $\chi$-distinguished if the intertwining space

$$\text{Hom}_{G(F)}(\pi, \chi)$$

is non-trivial.

Let $\text{St}_E$ denote the Steinberg representation of $G(E)$. In [Pr], Dipendra Prasad defines an explicit quadratic abelian character $\chi_F$ of $G(F)$ and makes the following conjecture.

Conjecture. ([Pr], Conjecture 3, page 77). Assume that the derived subgroup of $G$ is quasi-split. Then:

(a) The Steinberg representation of $G(E)$ is $\chi_F$-distinguished.

(b) For any other smooth character $\chi$ of $G(F)$, different from $\chi_F$, the Steinberg representation of $G(E)$ is not $\chi$-distinguished.

This conjecture is proved for $\text{GL}(n)$ (by Prasad [Pr2] when $n = 2$, and by Anandavardhan and Rajan [AR], Theorem 1.5, for any $n$ and without restriction on the quadratic field extension $E/F$).
In this article, we first prove the following result.

**Theorem 1.** Assume that

(i) $E/F$ is unramified,

(ii) the residue field $k_F$ of $F$ is large enough.

(iii) the algebraic group $G$ is split over $F$, and to make our proof less technical:

(iv) The root system of $G$ relative to any maximal split torus is irreducible.

Then there exists an explicit quadratic character $\epsilon_F$ of $G(F)$, such that $\text{St}_E$ is $\epsilon_F$-distinguished.

We think that conditions (ii) and (iv) are not necessary. On the other hand, conditions (i) and (iii) are crucial for our proof.

Hence, in a particular case, we obtain a proof of part (a) of Prasad’s conjecture modulo the fact that $\epsilon_F = \chi_F$. This equality is true for $GL(n)$ and when $G$ is simply connected (in this case $\chi_F = \epsilon_F = 1$). We expect it to be always true.

The idea of the proof is to use the model of the Steinberg representation as a space of harmonic functions on the chambers of $X_E$, the building of $G(E)$. The building $X_F$ of $G(F)$ embeds in $X_E$ as a sub-simplicial complex of same dimension. The $G(F)$-equivariant linear form is then simply the ”period” obtained by summing a function over the sub-building. The Iwahori-spherical vector is a test vector of this linear form. The difficult point is to prove that the restriction of a harmonic function to $X_F$ is $L^1$.

We then prove the following.

**Theorem 2.** Assume that assumptions (i), (iii) and (iv) of Theorem 1 hold. Let $G^{\text{der}}$ be the derived group of $G$. We have the multiplicity 1 result:

$$\dim_{\mathbb{C}} \text{Hom}_{G^{\text{der}}(F)}(\text{St}_E, \mathbb{C}) \leq 1.$$ 

As a consequence, under the assumptions of Theorem 1, points (a) and (b) of Prasad’s conjecture hold.

Theorem 2 is a consequence of a transitivity property of the action of $G(F)$ on the chambers of $X_E$. The proof of this property is provided by F. Courtès in an appendix to this article.
Since the Steinberg representation factors through $\mathbb{G}(E)/Z_E$, where $Z_E$ is the center of $\mathbb{G}(E)$, we will assume that the group $\mathbb{G}$ is semi-simple.

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1 Notation: groups and buildings

We fix a locally compact non-archimedean and non-discrete field $F$. We do not assume that the (residue) characteristic of $F$ is not 2. We let $E/F$ be an unramified quadratic extension of $F$.

If $K$ is any locally compact non-archimedean and non-discrete field, we denote by

- $\mathfrak{o}_K$ the ring of integers of $K$,
- $\mathfrak{p}_K$ the maximal ideal of $\mathfrak{o}_K$,
- $k_K = \mathfrak{o}_K/\mathfrak{p}_K$ the residue field,
- $q_K = |k_K|$ the cardinal of $k_K$.

We in particular have $q_E = q_F^2$.

We fix a connected semisimple group $\mathbb{G}$ split and defined over $F$. We denote by $d$ its rank and by $G = G_F$ its group of $F$-rational points. For simplicity, we shall assume that the root system of $\mathbb{G}$ is irreducible.

We fix a maximal split torus $T$ of $G$ and we denote by $N$ the normalizer of $T(F)$ in $G$. Let $T^0$ be the subgroup of $T$ generated by the $\xi(u)$, where $\xi$ runs over the rational cocharacters of $T$ and $u$ over $\mathfrak{o}_F^*$. Then $T^0$ is the maximal compact subgroup of $T$.

Let $X = X_F$ be the semi-simple Bruhat-Tits building of $G$. This is a locally compact topological space on which $G$ acts continuously. It has dimension $d$. The space $X$ is naturally the geometric realization of a simplicial complex and the group $G$ acts by preserving the simplicial structure.

Let us fix a chamber $C_0$ in the apartment $A$ of $X$ attached to $T$ and write $I$ for the Iwahori subgroup of $G$ attached to $C_0$. This is the pointwise stabilizer of $C$ in $G$.

By [IM] (also see [I]), the affine Weyl group $W = N(T)/T^0$ of $T$ may be written as a semidirect product $W = \Omega \ltimes W_0$ of a coxeter group $W_0$ by a finite abelian group $\Omega$, in such a way that:

(a) $\Omega$ normalizes $I$, 

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(b) if \( N^0 \) is the inverse image of \( W_0 \) in \( N \), \((I, N_0)\) is a Tits system (or \( BN \)-pair),

(c) the set
\[
G_0 = IW_0I = \bigcup_{w \in W_0} IwI
\]
is a normal subgroup of \( G \) and \( G/G_0 \simeq \Omega \).

The pair \((I, N)\) is a **generalized Tits system**. When \( G \) is simply connected, we have \( \Omega = \{1\} \).

As a simplicial complex \( X_F \) is the building of the \( BN \)-pair \((I, N_0)\) \([BT]\). In particular \( X_F \) is labellable (in the sense of [Br], Appendix C, page 29) and \( G_0 \) acts on \( X_F \) by preserving the labelling of simplices.

The group \( \Omega \) acts on \( A_0 \) and stabilizes the chamber \( C_0 \). For \( \omega \in \Omega \), we denote by \( \epsilon(\omega) \) the signature of the permutation induced by the action of \( \omega \) on the vertex set of \( C_0 \). We define a quadratic character \( \epsilon_{G_F} \) of \( G_F \) by
\[
\epsilon_{G_F} = \epsilon \circ p_0
\]
where \( p_0 : G \to G/G_0 \) denotes the canonical projection.

We fix an unramified quadratic extension \( E/F \). By [T] there is a canonical embedding
\[
j : X_F \to X_E
\]
of \( X_F \) in the semisimple building \( X_E \) of \( G \) over \( E \). The Galois group \( \text{Gal}(E/F) \) acts on \( X \) and \( j \) is \( \text{Gal}(E/F) \times G_E \)-equivariant. Moreover since \( G \) is split and \( E/F \) is unramified, we have that :
- \( j(X_F) \) is the set of \( \text{Gal}(E/F) \)-fixed points in \( X_E \),
- \( j \) is simplicial.
- \( X_F \) and \( X_E \) share the same dimension \( d \), and \( j \) maps chambers to chambers.

We shall identify \( X_F \) as a subsimplicial complex of \( X_E \) by viewing \( j \) as an inclusion.

Let \( \Delta_d \) be the standard abstract simplex of dimension \( d \). We view its set of simplices as the power set of \( \{0, 1, \ldots, d\} \). Since \( X_E \) is labellable, there exists a simplicial map
\[
\lambda_E : X_E \to \Delta_d
\]
which preserves the dimension of simplices. If $\sigma$ is a simplex of $X_E$, we call $\lambda_E(\sigma)$ its label or type. The restriction $\lambda_F = (\lambda_E)|_{X_F}$ is a labelling of $X_F$ preserved by the action of $G_0$.

Let $C$ be a chamber of $C$ and $g \in G_E$. Let $(s_0, \ldots, s_d)$ (resp. $(t_0, \ldots, t_d)$) be an ordering of the vertices of $C$ (resp. of $gC$) such that $\lambda_E(s_i) = \{i\}$, $i = 0, \ldots, d$ (resp. $\lambda_E(t_i) = i$, $i = 0, \ldots, d$). We denote by $\epsilon(g, C)$ the signature of the permutation:

$$
\begin{pmatrix}
  g.s_0 & g.s_1 & \ldots & g.s_d \\
  t_0 & t_1 & \ldots & t_d
\end{pmatrix}
$$

Lemma 1.1 With the previous notation, we have:

(i) the signature $\epsilon(g, C)$ does not depend on $C$.
(ii) The map $g \mapsto \epsilon(g) = \epsilon(g, C_0)$ is a character of $G_E$.
(iv) The character $\epsilon$ satisfies $\epsilon|_{G_F} = \epsilon|_{G_E}$.

Proof. It is easy and based on the fact that the subgroup $G^*_E$ of $G_E$, formed of those elements preserving the labelling $\lambda_E$, acts transitively on chambers of $X_E$. Details are left to the reader. □

2 The Steinberg representation

There are several equivalent definitions of the Steinberg representation $\text{St}_E$ of $G_E$. That we shall use comes from the following beautiful theorem due to Borel and Serre.

Theorem 2.1 (BS) The representation of $G_E$ in $H^d_c(X_E, \mathbb{C})$, the $d$-th cohomology space with compact support, with coefficient in $\mathbb{C}$, where $d$ is the $E$-rank of $G$, is equivalent to the Steinberg representation.

Let $\text{Ch}_E$ denote the set of chambers of $X_E$ and $\mathbb{C}[\text{Ch}_E]$ the $\mathbb{C}$-vector space of complex valued functions on $\text{Ch}_E$ of arbitrary support. A function $f \in \mathbb{C}[\text{Ch}_E]$ is called a harmonic cocycle if for all codimension 1 simplex $D$ of $X_E$, we have

$$
\sum_{C \supseteq D} f(C) = 0
$$

where the sum is over the chambers of $X_E$ that contain $D$ as a subsimplex. We denote by $\mathcal{H}(X_E)$ the $\mathbb{C}$-vector space of harmonic cocycles on $X_E$.

We define a linear representation $(\pi_E, \mathcal{H}(X_E))$ of $G_E$ in $\mathcal{H}(X_E)$ by the formula:

$$
[\pi_E(g).f](C) = \epsilon(g)f(g^{-1}C) , \quad g \in G, \ C \in \text{Ch}_E.
$$

This representation is not smooth in general and we denote by $(\pi_E, \mathcal{H}(X_E)^\infty)$ its smooth part.
Proposition 2.2 The representation $(\pi_E, \mathcal{H}(X_E)^\infty)$ is equivalent as a $G_E$-representation to the contragredient of $\text{St}_E$.

Proof. For $k = d - 1, d$, let $C^k_c(X_E)^\text{alt}$ be the $\mathbb{C}$-vector space of alterned $k$-cochains on $X_E$ with coefficients in $\mathbb{C}$, the field of complex numbers. Denote by $\text{Ch}_E^*$ the set of pairs $(C, \sigma)$ formed of a chamber $C$ of $X_E$ together with a bijection $\sigma$ from the vertex set of $C$ to $\{0, 1, ..., d\}$. We let $G_E$ act on $\text{Ch}_E^*$ by $g.(C, \sigma) = (g.C, \sigma \circ g^*)$, where $g^*$ is the bijection from the vertex set of $C$ to the vertex set of $g.C$ induced by $g$. Then $C^d_c(X_E)^\text{alt}$ is the set of maps $f : \text{Ch}_E^* \rightarrow \mathbb{C}$ satisfying:

- $f$ has finite support,
- for all $(C, \sigma) \in \text{Ch}_E^*$ and for all permutation $\tau$ of $\{0, ..., d - 1\}$, we have $f(C, \tau \circ \sigma) = \epsilon(\tau)f(C, \sigma)$

where $\epsilon(\sigma)$ denotes the signature of $\sigma$.

The group $G_E$ naturally acts on $C^d_c(X_E)^\text{alt}$. Similarly we define the $G_E$-module $C^{d-1}_c(X_E)^\text{alt}$. The coboundary map

$$d : C^{d-1}_c(X_E)^\text{alt} \rightarrow C^d_c(X_E)^\text{alt}$$

is given by

$$dh(C, \sigma) = \sum_{D \subset C} h(D, \sigma|_D), \ (C, \sigma) \in \text{Ch}_E^*$$

where $\sigma|_D$ denotes the restriction of $\sigma$ to the vertex set of $D$.

For $k = d - 1, d$, let $C^k_c(X_E)$ be the $\mathbb{C}$-vector space of usual $k$-cochains with finite support. By orienting the simplices of $X_E$ thanks to the labelling $\lambda$, we obtain a coboundary map $d : C^{d-1}_c(X_E) \rightarrow C^d_c(X_E)$ given by

$$dh(C) = \sum_{D \subset C} (-1)^{\lambda(C\setminus D)} h(D).$$

For $k = d - 1, d$, we have an isomorphism of $\mathbb{C}$-vector spaces

$$C^k_c(X_E)^\text{alt} \rightarrow C^k_c(X_E)$$

given by

$$f \mapsto \{C \mapsto f(C, \lambda|_C)\}$$

where $\lambda|_C$ denotes the restriction of the labelling $\lambda$ to the vertex set of the simplex $C$. These isomorphisms are $G_E$-equivariant if one lets $G_E$ act on $C^k_c(X_E)$ via

$$(g.f)(D) = \epsilon_{G_E}(g) f(g^{-1}.D), \ D \text{ k-simplex of } X_E.$$
Moreover the isomorphisms are compatible with the coboundary maps.

The space $H^d_c(X_E)$ is known to be isomorphic to $C^d_c(X_E)^{alt}/dC^{d-1}_c(X_E)^{alt}$ as a $G_E$-module. So it is isomorphic to $C^d_c(X_E)/dC^{d-1}_c(X_E)$ as a $G_E$-module.

By letting $V^*$ denote the algebraic dual of a $C$-vector space $V$, we have

$$(H^d_c(X_E))^* = \{ \omega \in C^d_c(X_E)^* ; f|_{dC^{d-1}_c(X_E)} = 0 \}.$$ 

We may identify $C^d_c(X_E)^*$ with $C[Ch_E]$ by using the pairing

$$\langle \omega, f \rangle = \sum_{C \in Ch_E} \omega(C) f(C), \ \omega \in C[Ch_E], \ f \in C^d_c(X_E).$$

Then for $\omega \in C[Ch_E]$, the condition $\omega|_{dC^{d-1}_c(X_E)} = 0$ writes $\langle \omega, dh \rangle = 0$, for all $h \in C^{d-1}_c(X_E)$. This may be rewritten

$$\langle d^*\omega, h \rangle = 0, h \in C^{d-1}_c(X_E)$$

that is $d^*\omega = 0$

where $d^*: C^d_c(X_E)^* \longrightarrow C^{d-1}_c(X_E)^*$ is the adjoint of $d$. But a simple computation shows that

$$d^*\omega(D) = \sum_{C \supset D} \omega(C), \ D \ (d-1)\text{-simplex}$$

so that $d^*\omega = 0$ is the harmonicity condition. □

Note that the Steinberg representation of $G_E$ is self-dual.

3 Some geometric lemmas

We denote by $d_g$ the combinatorial distance on $X_E$ defined as follows. For $C, D \in Ch_E$, $d_g(C, D)$ is the length $k$ of a minimal gallery $(D_0, D_1, ..., D_k)$ satisfying $D_0 = C$ and $D_k = D$. The following result, due to F. Bruhat, will be very useful.

**Lemma 3.1** (Lemma 4.1 of [Bo]) Let $U$ be a compact open subgroup of $G_E$.

There exists an integers $k_0 = k_0(U)$ satisfying the following property. For all chamber $C$ such that $d_g(C_0, C) \geq k_0$, there exists a chamber $D$ adjacent to $C$ such that:

(i) $d_g(C_0, D) = d(C_0, C) - 1$;

(ii) the group $U$ acts transitively on the set of chambers $C'$ such that $C' \neq D$ and $C' \cap D = C \cap D$. 

Lemma 3.2 Let $D$ be a codimension 1 simplex in $X_F$ (resp. in $X_E$). Then $D$ is contains in $q_F + 1$ chambers of $X_F$ (resp. in $q_E + 1$ chambers of $X_E$).

Proof. We give a proof for $X_F$. Let $P_D$ be the parahoric subgroup of $G_F$ attached to $D$ and $P_D^1$ its pro-unipotent radical. Then $P_D/P_D^1 = G_D(k_F)$, where $G_D$ is a reductive group defined over $k_F$ and of $k_F$-rank 1. The chambers $C$ of $X_F$ containing $D$ are in bijection with the Borel subgroup of $P_D/P_D^1$ by

$$C \mapsto P_C \mod P_D^1$$

where $P_C$ denotes the Iwahori subgroup attached to $C$. But $G_D$ being of $k_F$-rank 1, $G_D(k_F)$ possesses $q_F + 1$ Borel subgroups. $\Box$

For any non negative integer $k$, we denote by $\Sigma_F(k)$ the set of chambers of $X_F$ at distance $k$ from $C_0$ and set $N_k(k) = |\Sigma_F(k)|$.

Lemma 3.3 We have

$$N_F(k) \leq (d + 1) d^{k-1} q_F^k , \ k \geq 1 .$$

Proof. Any chamber of $X_F$ has $d + 1$ codimension 1 faces. A chamber in $\Sigma_F(1)$ contains one of the $d + 1$ codimension 1 faces of $C_0$. By Lemma (3.2), such a face is contained in $q_F$ chambers different from $C_0$, so that

$$N_F(1) = (d + 1) q_F .$$

Moreover, for $k \geq 1$, any chamber in $\Sigma_F(k)$ is adjacent to at most $dq_K$ chambers at distance from $C_0$ greater than $k$. The formula follows by induction on $k$. $\Box$

Lemma 3.4 Let $f \in \mathcal{H}(X_E)^\infty$. There exist an integer $k_f$ and a positive real number $K_f$ such that the following holds. For all $C \in \text{Ch}_E$ such that $d_g(C_0, C) \geq k_f$, we have

$$|f(C)| \leq K_f \cdot q_E^{-d_g(C_0, C)} .$$

Proof. Since $f$ is smooth under the action of $G$, it is fixed by an open compact subgroup $U$ small enough. Set $k_f = k_0(U)$. For $k \geq 0$, set

$$M_k = \text{Max} \{ |f(C)| ; \ C \in \Sigma_E(k) \} .$$

We are going to prove that for $k \geq k_f$ we have $M_{k+1} \leq q_E^{-1} M_k$; the lemma will follow.
Let $C \in \Sigma_E(k+1)$. By applying Lemma (3.1), there exists $D \in \Sigma_E(k)$ such that $U$ acts transitively on

$$[C, D] := \{G \in \text{Ch}_E \ ; \ G \neq D \text{ and } G \cap D = C \cap D\}.$$

It follows that $f$ is constant on $[C, D]$. By applying the harmonicity condition at the codimension 1 face $C \cap D$, we get

$$q_E f(C) + f(D) = 0,$$

since $[C, D]$ has $q_E$ elements. So $\lvert f(C) \rvert = q_E^{-1} \lvert f(D) \rvert$, and our assertion follows. □

**Lemma 3.5** Assume that $q_F > d$. Let $f \in \mathcal{H}(X_E)^\infty$. Then we have

$$f_{|\text{Ch}_F} \in L^1(\text{Ch}_F)$$

where $L^1(\text{Ch}_F)$ denotes the set of complex functions $g$ on $\text{Ch}_F$ such that

$$\sum_{C \in \text{Ch}_F} \lvert g(C) \rvert < +\infty.$$

**Proof.** We may write

$$\sum_{C \in \text{Ch}_F} \lvert f(C) \rvert = \sum_{k \geq 0} \sum_{C \in \Sigma_F(k)} \lvert f(C) \rvert.$$

By the previous lemmas, for $k$ large enough and for some constant $K > 0$, we have:

$$\sum_{C \in \Sigma_F(k)} \lvert f(C) \rvert \leq K \left(\frac{d q_F}{q_E}\right)^k.$$

with $q_E = q_F^2$. The result follows. □

**Remark.** If $G$ is of rank 1, then the condition $q_F > d$ is automatically satisfied.

4 **Constructing $G_F$-equivariant linear forms**

In this section, we assume, as in Lemma (3.5), that we have $q_F > d$.

Thanks to lemma (3.5), the linear map $\lambda$ on $\mathcal{H}(G_E)^\infty$ given by

$$\lambda(f) = \sum_{C \in \text{Ch}_F} f(C)$$
is well defined. For \( g \in G_F \) and \( f \in \mathcal{H}(G_E)^\infty \), we have
\[
\lambda(\pi_E(g).f) = \sum_{C \in \text{Ch}_F} \epsilon_{G_F}(g)f(g^{-1}C) = \epsilon_{G_F}(g)\lambda(f).
\]
Hence we have \( \lambda \in \text{Hom}_{G_E}(\text{St}_E, \epsilon_{G_F}) \).

**Theorem 4.1** The Steinberg representation of \( G_E \) is \( \epsilon_{G_F} \)-distinguished. More precisely, a non-zero Iwahori-spherical vector is a test vector for \( \lambda \).

**Proof.** It suffices to prove that \( \lambda \) is not trivial. Let \( f \) be the Iwahori-spherical vector in \( \mathcal{H}(G_E)^\infty \) normalized in such a way that \( f(C_0) = 1 \). In Lemma (3.1), if \( U = I \), we may take \( k_0 = 0 \). It follows from the proof of Lemma (3.4) that, for all \( k \geq 0 \), \( f \) has constant value \( (\frac{-1}{q_E})^{-k} \) on \( \Sigma_F(k) \). As a consequence
\[
\lambda(f) = \sum_{k \geq 0} \left( \sum_{C \in \Sigma_F(k)} f(C) \right)
\]
is an alternating series. In particular we have
\[
\sum_{C \in \Sigma_F(0)} f(C) > \lambda(f) > \sum_{C \in \Sigma_F(0)} f(C) + \sum_{C \in \Sigma_F(1)} f(C)
\]
that is
\[
1 > \lambda(f) > 1 - \frac{d + 1}{q_F} \geq 0.
\]
and our Theorem follows. \( \square \)

Note that if the \( F \)-rank of \( G \) is 1 the value \( \lambda(f) \) may be explicitly computed. Indeed in that case, \( X_F \) is a regular tree of valence \( q_F + 1 \), and we have \( N_F(k) = 2q_F^k \), \( k \geq 1 \). Hence
\[
\lambda(f) = 1 + \sum_{k \geq 1} 2q_F^k \left( \frac{-1}{q_E} \right)^k = 1 - \frac{2}{q_F + 1}.
\]

### 5 Multiplicity 1

In this section we release the condition \( q_F > d \) and prove Theorem 2 without restriction on the size of \( k_F \).

Set \( \mathbb{H} = \mathbb{G}^{\text{der}} \) and \( H = \mathbb{H}(F) \). Note that \( \mathbb{H} \) and \( \mathbb{G} \) share the same (semisimple) Bruhat-Tits building over \( F \) (resp. over \( E \)). This essentially
comes from the fact that the inclusion $H \to G$ is $B-N$-adapté in the sense of [BT] (1.2.13), page 18 (cf. [BT] §2.7., page 49). By Proposition (2.2), we have

$$\text{Hom}_H(\text{St}_E, \mathbb{C}) = \mathcal{H}(X_E)^H,$$

the space of harmonic cochains on $X_E$ fixed by $H$.

Our proof relies on the following fundamental result whose proof is given in the appendix.

**Theorem 5.1** Let $C$ be a chamber of $X_E$ at combinatorial distance $\delta \geq 0$ from $X_F$. Then $G_F$ acts transitively on the set $\text{Ch}[C, \delta + 1]$ of chambers $D$ of $X_E$ satisfying:

- $D$ and $C$ are adjacent,
- $d(D, X_F) = \delta + 1$.

Note that since $H$ is contained in $G_0$ ($G/G_0$ is abelian), it acts on $\mathcal{H}(X_E)$ via the formula

$$h.\omega(C) = \omega(h^{-1}C), \ h \in H, \ \omega \in \mathcal{H}(X_E).$$

Let $\omega \in \mathcal{H}(X_E)^H$. Since $H$ acts transitively on $X_F$, the value $\omega(C)$ does not depend on the chamber $C$ of $X_F$. Let us denote it by $\varphi(\omega)$. Theorem 2 is a consequence of the following:

**Lemma 5.2** The linear map

$$\varphi : \mathcal{H}(X_E)^H \to \mathbb{C}, \ \omega \mapsto \varphi(\omega)$$

is injective.

**Proof.** For all integers $\delta \geq 0$, let $\text{Ch}(X_F, \delta)$ denote the set of chambers in $X_E$ at combinatorial distance $\delta$ from $X_F$ (in particular $\text{Ch}(X_F, 0)$ is the set of chambers of $X_F$). Let $\omega \in \mathcal{H}(X_E)^H$ and $\delta \geq 0$ be an integer. We prove that the restriction $\omega|_{\text{Ch}(X_F, \delta + 1)}$ is entirely determined by the restriction $\omega|_{\text{Ch}(X_F, \delta)}$. The lemma will obviously follow.

Let $D \in \text{Ch}(X_F, \delta + 1)$. Fix a chamber $C \in \text{Ch}(X_F, \delta)$ adjacent to $D$ and set $M = C \cap D$. The harmonicity condition at the codimension 1 face $M$ writes

$$\sum_{\Delta \in C_M} \omega(\Delta) = 0,$$
where $C_M$ is the set of chambers of $X_E$ containing $M$. We may split the set $C_M$ into two subsets: $C_M^{δ+1} := \text{Ch}[C, δ + 1]$ and its complement $C_M^δ$, contained in $\text{Ch}(X_F, δ)$. By theorem (5.1) and the $H$-invariance of $ω$, we have
\[
\sum_{Δ ∈ C_M^{δ+1}} ω(Δ) = |C_M^{δ+1}| × ω(C).
\]
Hence the harmonicity condition gives
\[
ω(C) = -\frac{1}{|C_M^{δ+1}|} × \sum_{Δ ∈ C_M^δ} ω(Δ)
\]
This proves that the value $ω(C)$ depends only on the restriction $ω|_{\text{Ch}(X_F, δ)}$, and we are done. □.

A A transitivity result

For every facet $A ⊂ X_E$, we shall denote by $K_{A,E}$ (resp. $K_{A,F}$) the connected fixator of $A$ in $G_E$ (resp. the intersection with $G_F$ of that connected fixator). More generally, for every subset $S$ of $X_E$, we shall denote by $K_{S,E}$ (resp. $K_{S,F}$) the intersection of the $K_{A,E}$ (resp. $K_{A,F}$), where $A$ runs over the set of facets of $X_E$ whose intersection with $S$ is nonempty. Let $\overline{S}$ be the closure of $S$; we have of course $K_{S,E} = K_{S,E}$ and $K_{S,F} = K_{S,F}$.

Proposition A.1 Let $d$ be a nonnegative integer, and let $C$ be any chamber of $X_E$ such that the combinatorial distance between $C$ and $X_F$ is $d$. Let $C'$ be a chamber of $X_E$ neighbouring $C$ and whose combinatorial distance from $X_F$ is $d + 1$, let $A$ be the unique facet of codimension 1 of $X_E$ contained in both $C$ and $C'$, and let $Δ$ be the set of chambers of $X_E$ containing $A$ in their closure and whose combinatorial distance from $X_F$ is $d + 1$. Then the group $K_{C,F}$ acts transitively on $Δ$.

Proof. Assume first $d = 0$, that is $C$ is contained in $X_F$. Let $A$ be an apartment of $X_F$ containing $C$, let $T$ be the corresponding $F$-split maximal torus of $G_E$, let $Φ$ be the root system of $G_E$ relatively to $T_E$ and let $±α$ be the elements of $Φ$ corresponding to the hyperplane $H$ of $A$ containing $A$. For every $β ∈ Φ$, let $U_β = U_β,E$ be the corresponding root subgroup of $G_E$, and let $v$ be a normalized valuation (that is a valuation such that for every $β$, $v(U_β) = \mathbb{Z} ∪ \{∞\}$; such a valuation exists because $G$ is split over an unramified extension of $E$) on the root datum $(G, T, (U_β))$ such that, with the subgroups $U_β,1$ of $U_β$ being defined according to that valuation, we have $U_β,1 \cap K_A = U_β,0$; we shall assume $α$ is the one such that $U_α,1 \cap K_C = U_α,1$. 


Let \( \phi \) (resp. \( \phi' \)) be a \( F \)-isomorphism between \( E \) and \( U_{\alpha,E} \) (resp. \( U_{-\alpha,E} \)) preserving the valuation, that is such that for every integer \( i \), \( U_{\alpha,i} \) (resp. \( U_{-\alpha,i} \)) is the image by \( \phi \) (resp. \( \phi' \)) of the elements of \( E \) of valuation \( \geq i \); the elements of \( \Delta \) are the chambers of the form \( \phi(x)C \), where \( x \) is an element of \( \mathfrak{o}_E \) which belongs neither to \( F \) nor to \( \mathfrak{p}_E \); moreover, \( \phi(x)C \) depends only of the class of \( x \) modulo \( \mathfrak{p}_E \). we can thus label the elements of \( \Delta \) as \( C_x = \phi(x)C \), where \( x \) is an element of \( k_E - k_F \).

Let now \( \Phi^\gamma \) be the system of coroots of \( T_E \) associated to \( \Phi \), and let \( \alpha^\gamma \) be the 1-parameter subgroup of \( T_E \) corresponding to \( \alpha \) in \( \Phi^\gamma \); for every \( y \in \mathfrak{o}_E^* \), we have \( \alpha^\gamma(y)C = C \), and if \( y \) is an element of \( \mathfrak{o}_E^* + \mathfrak{p}_E \), \( \alpha^\gamma(y) \) permutes the elements of \( \Delta \). Moreover, \( \alpha^\gamma(y) \) depends only of the class of \( y \modulo 1 + \mathfrak{p}_E \), hence we can view \( y \) as an element of \( k_E^* \).

Let \( x \) be an element of \( k_E - k_F \) (arbitrarily fixed for the moment). For every \( a \in k_F^* \) and every \( b \in k_F \), we have:

\[
\phi(b)\alpha^\gamma(a)C_x = \phi(b)\left(\text{Ad}(\alpha^\gamma(a))\phi(x)\right)C = \phi(b)\phi(a^2x)C = C_{a^2x+b}.
\]

Hence for every element of \( k_E - k_F \) of the form \( y = a^2x + b \), \( C_y \) is in the \( G_F \)-orbit of \( C_x \). If \( \text{char}(k_E) = 2 \), every element of \( k_F \) is a square, and since \( (1,x) \) is a basis of the \( k_F \)-vector space \( k_E \), every element of \( k_E - k_F \) is of that form, which proves the proposition in that case.

Now assume \( p \neq 2 \); there exists then \( x \in k_E - k_F \) such that \( \frac{1}{c} = x^2 \in k_F \); \( c \) is then not a square in \( k_F \). Let \( D \) be the chamber of \( A \) such that \( D \cap C = \overline{A} \); we have \( D = nC \), where \( n \) is any representative in the normalizer of \( T_F \) in \( K_A \) of the element \( s_\alpha \) in the Weyl group of \( \Phi \). Moreover, according to [BT 6.1.3 a) and b]1), we can assume that every such element is of the form:

\[
n = \phi'(y)\phi(-y^{-1})\phi'(y),
\]

with \( y \in \mathfrak{o}_E^* \). We then have:

\[
\phi'(y)D = \phi'(y)\phi'(-y)\phi(y^{-1})\phi'(-y)C = C_{y^{-1}},
\]

since \( \phi'(y)C = C \). Hence \( C_x = \phi(x^{-1})D = \phi(xc)D \). By the same reasoning as above, for every \( a \in k_F^* \) and every \( b \in k_F \), \( \phi'(a^2xc + b)D = C_{\frac{1}{a^2xc+b}} \). On the other hand, we have:

\[
\frac{1}{a^2xc+b} = \frac{a^2xc-b}{(a^2xc+b)(a^2xc-b)} = \frac{a^2xc-b}{a^4c-b^2} = \frac{x - \frac{b}{a^2c}}{a^2 - \frac{b^2}{a^2c}}.
\]

On the other hand, it is well-known and easy to check that there exists \( a, b \) such that \( a^2 - \frac{b^2}{a^2c} \) is not a square; we thus obtain that there exists \( a', b' \), such
that \( a' \) is not a square and \( C'_{a'x+b'} \) is in the \( K_{C,F} \)-orbit of \( C_x \). By the same reasoning as above once again, we obtain that it is true for every \( C'_{a''x+b''} \), \( a \in k_F^* \), \( b \in k_F \). Since \( (k_F^*)^2 \) is of index 2 in \( k_F \), we finally obtain that all of the \( C_x \), \( x \in k_E - k_F \), are in the same \( K_{C,F} \)-orbit, which completes the proof of the proposition when \( C \subset X_F \).

Now assume \( d > 0 \). Set \( \Gamma = \text{Gal}(E/F) \), and let \( \gamma \) be the unique nontrivial element of \( \Gamma \). First we prove the following lemma:

**Lemma A.2** There exists a \( \Gamma \)-stable apartment of \( X_E \) containing both \( C \) and \( \gamma(C) \).

**Proof.** Let \( \mathcal{A} \) be any apartment of \( X_E \) containing both \( C \) and \( \gamma(C) \); such an apartment exists by [BT, proposition 2.3.1]. Obviously, \( \gamma(\mathcal{A}) \) satisfies the same property; there exists then \( g \in G_E \) such that \( g\mathcal{A} = \gamma(\mathcal{A}) \), and we can assume \( g \in K_{C,E} \cap K_{\gamma(C),E} \). The element \( \gamma(g) \) then also belongs to \( K_{C,E} \cap K_{\gamma(C),E} \), and we have \( \gamma(g)\gamma(\mathcal{A}) = \mathcal{A} \). Hence \( \gamma(g) \) fixes \( \mathcal{A} \) pointwise, which means that it belongs to the unique parahoric subgroup \( K_T \) of the \( E \)-split maximal torus \( T \) of \( G_E \) associated to \( \mathcal{A} \).

Let now \( F_{nr} \) be the maximal unramified extension of \( F \), let \( G_{F_{nr}} \) be the group of \( F_{nr} \)-points of \( G \), and let \( K_{C,F_{nr}} \) be the connected fixator of \( C \) viewed as a chamber of the Bruhat-Tits building \( X_{F_{nr}} \) of \( G_{F_{nr}} \). By [Con, lemma 5.1], there exists an element \( h \in K_{F_{nr}} \) such that \( g = F(h)^{-1}h \), with \( F \) being the Frobenius element of \( \text{Gal}(F_{nr}/F) \). Moreover, the restriction of \( F \) to \( E \) is \( \gamma \), and we have:

\[
\gamma(g)g = F^2(h)^{-1}h \in K_T,
\]

Let \( T_{nr} \) be the maximal torus of \( G_{F_{nr}} \) associated to \( \mathcal{A} \), and let \( K_{T_{nr}} \) be its unique parahoric subgroup; we have \( K_T = K_{T_{nr}} \cap G_E \). Moreover, the Frobenius element of \( \text{Gal}(F_{nr}/E) \) is \( F^2 \); by [Con, lemma 5.1] again, there exists then \( t \in K_{T_{nr}} \) such that \( \gamma(g)g = F^2(t)^{-1} \). Hence \( ht = F^2(ht) \), which simply means that \( ht \in G_E \). We finally obtain:

\[
ht\mathcal{A} = h\mathcal{A} = \gamma(h)\gamma(\mathcal{A}) = \gamma(h\mathcal{A}) = \gamma(ht\mathcal{A}),
\]

hence \( ht\mathcal{A} \) is a \( \Gamma \)-stable apartment of \( X_E \) containing both \( C \) and \( \gamma(C) \) and the lemma is proved. \( \square \).

Now we designate by \( \mathcal{A} \) the apartment given by the above lemma, and by \( T \) the corresponding \( E \)-split maximal torus of \( G_E \); \( T \) is defined over \( F \), but not \( F \)-split. Let \( \Phi \) be the root system of \( G_E \) relatively to \( T \), and let \( \alpha \in \Phi \) be defined as in the case \( d = 0 \). Since \( T \) is defined over \( F \), \( \Gamma \) acts on \( \Phi \).

Let \( D \) be the unique chamber of \( \mathcal{A} \) such that \( \overline{C} \cap \overline{D} = \overline{A} \). Since \( \Delta \) is nonempty, the combinatorial distance between \( D \) and \( B_F \) must be either \( d \) or \( d + 1 \).
Lemma A.3 Assume \( H = \gamma(H) \). Then the combinatorial distance between \( D \) and \( F \) is \( d \).

Proof. Let \( s_H \) be the orthogonal reflection on \( A \) whose kernel is \( H \). Since \( H = \gamma(H) \), \( \gamma \) and \( s_H \) commute, hence there exists \( g_H \in G_F \) such that \( g_H \) acts on \( A \) via \( s_H \). Let \( C = C_0, \ldots, C_d \) be a minimal gallery of length \( d \) between \( C \) and some chamber \( C_d \) of \( X_F \). Then \( D = g_H C_0, \ldots, g_H C_d \) is also a minimal gallery and \( g_H C_d \subseteq X_F \), hence the combinatorial distance between \( D \) and \( X_F \) is at most \( d \). The other inequality follows from the above remarks. □

Note that the fact that \( H = \gamma(H) \) implies in particular that \( \gamma(\alpha) = \pm \alpha \). Conversely, we have:

Lemma A.4 Assume \( \gamma(\alpha) = \alpha \). Then \( H = \gamma(H) \).

Proof. Let \( \overline{\alpha} \) be the affine root of \( T \) corresponding to \( H \); it is an affine linear form on the affine space \( A \), and the corresponding linear form on the vector space \( (X_s(T)/X_s(Z)) \times \mathbb{R} \), where \( Z \) is the center of \( G \), is \( \alpha \). Hence \( \gamma(\overline{\alpha}) \) is of the form \( \alpha + c \), with \( c \) being some constant. We then have \( \gamma^2(\overline{\alpha}) = \alpha + 2c \); since \( \gamma^2 \) is trivial, it implies \( c = 0 \), hence \( H = \gamma(H) \). □

Note that it is not true when \( \gamma(\alpha) = -\alpha \).

Now we prove the proposition when \( H = \gamma(H) \). Consider the rank 1 subgroup \( G_\alpha \) of \( G_E \) generated by \( T \), \( U_\alpha \) and \( U_{-\alpha} \); it is defined over \( F \), and the fact that \( H = \gamma(H) \) implies that \( G_\alpha \cap K_A = G_\alpha \cap K_{\gamma(A)} \), hence \( K_A \) is \( \Gamma \)-stable. The elements of \( \Delta \) are then of the form \( uC \), where \( u \) is an element of \( U_\alpha \) not belonging to \( G_F \), and we can finish the proof the same way as in the case \( d = 0 \).

Assume now \( H \neq \gamma(H) \). Let \( C \) be the connected component of \( A - (H \cup \gamma(H)) \) containing \( C \). Assume \( C \) contains \( \gamma(C) \) as well. Consider an apartment of \( X_E \) of the form \( \phi(x)A \), where \( \phi \) is defined as in the case \( d = 0 \) for a given normalized valuation \( v \) on \( (G, T, (U_\beta)) \), and \( x \) is an element of \( E \) of valuation \( i \), where \( i \) is such that \( U_\alpha \cap K_{C,E} = U_{\alpha,i} \). Then \( \phi(x)A \) contains at the same time a chamber \( C'' \) distinct from \( C \) whose closure contains \( A \) and the half-apartment of \( A \) delimited by \( H \) and containing \( C \), which itself contains the closure of \( C \cup \gamma(C) \cup \gamma(D) \). We deduce from this that we have \( \gamma(\phi(x))\phi(x)\gamma(D) = \gamma(C'') \), hence \( \phi(x)\gamma(\phi(x))D = C'' \).

Moreover, \( \phi(x)\gamma(\phi(x)) \) is contained in \( K_{C,E} \), which is a pro-solvable group, hence if \( \gamma(\alpha) = -\alpha \), the commutator \( [\phi(x)^{-1}, \gamma(\phi(x)^{-1})] \) is an element of the subgroup \( K' \) of \( K_{C,E} \) generated by \( K_T, U_{\alpha,i+1} \) and \( \gamma(U_{\alpha,i+1}) \), which is itself contained in \( K_{D,\gamma}(D)_E \). If now \( \gamma(\alpha) \neq \pm \alpha \) (remember that by the previous lemma we cannot have \( \gamma(\alpha) = \alpha \)), then \( [\phi(x)^{-1}, \gamma(\phi'(x)^{-1})] \) is an element of the intersection with \( K_{C,E} \) of the subgroup of \( G \) generated by the \( U_{\lambda\alpha+\mu\beta} \).
where $\lambda$ and $\mu$ are positive integers such that $\lambda \alpha + \mu \beta$ is a root. We’ll also denote by $K'$ this last subgroup; it is also contained in $K_{D \cap \gamma(D),E}$.

In both cases, we can apply [Cou, lemma 5.1] to see that there exists $k \in K'$ such that $[\phi(x)^{-1}, \gamma(\phi'(x)^{-1})] = \gamma(k)k^{-1}$, hence $\phi(x)\gamma(\phi(x))k = \gamma(\phi(x))\phi(x)\gamma(k)$. We thus have proved that $\phi(x)\gamma(\phi(x))k$ is an element of $K_{C,F}$ sending $D$ to $C''$; since this is true for any $C''$ and in particular for $C'$, $\Delta$ must contain all of them and $K_{C,F}$ acts transitively on them, which proves the proposition in this case.

Assume now that $C$ does not contain $\gamma(C)$, or in other words that $C$ and $\gamma(C)$ are separated by at least one of $H$ and $\gamma(H)$. Then they are separated by both of them, which means that $D$ and $\gamma(D)$ are in the same connected component. We can then apply the same reasoning as above with $C$ and $D$ switched, and we obtain that for every chamber $C''$ of $X_E$ containing $A$ in its closure and distinct from $D$, there exists an element $g$ of $G_F$ such that $gC = C''$, which implies in particular that the combinatorial distance between $C''$ and $X_F$ must be $d$. Since by our hypothesis this is not true for $C'$, we must have $C' = D$ and even $\Delta = \{D\}$, and the result of the proposition is then trivial. $\square$

Remark. Actually, this very last case turns out to be impossible. To see that, we can for example observe that the combinatorial distance between $C$ and $X_F$ is equal to the combinatorial distance between $C$ and some facet of $X_F \cap A$ of maximal dimension plus the dimension of the $F$-anisotropic component of $T$, and that there exists a minimal gallery between $C$ and some chamber of $X_F$ whose closure contains the barycenter $b$ of $C \cup \gamma(C)$ (which is itself an element of $X_F$); with the hypotheses of the last case, it is easy to check that the closure of $C \cup \{b\}$ must contain $D$, hence a contradiction.

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