Article

On Hermite Hadamard-type Inequalities for Strongly log-convex Functions

Mehmet Zeki SARIKAYA, Hatice YALDIZ

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY
E-mail: sarikayamz@gmail.com, yaldizhatice@gmail.com

Article history: Received 28 November 2012, Accepted 25 February 2013, Published 1 March 2013.

Abstract: In this paper, the notation of strongly log-convex functions with respect to $c > 0$ is introduced and versions of Hermite Hadamard-type inequalities for strongly logarithmic convex functions are established.

Keywords: Hermite-Hadamard's inequalities, log-convex functions, strongly convex with modulus $c > 0$.

Mathematics Subject Classification: 26D07, 26D10, 26D15

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important as described in the literature (see, e.g. [8], [3]). These inequalities state that if $f : I \rightarrow \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

\[
 f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \tag{1.1}
\]

The inequality (1.1) has evoked the interest of many mathematicians. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained, to mention a few, see ([1]-[12]) and the references cited therein.

Definition 1.1: The function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$, is said to be convex if the following inequality holds
\[ f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \]

for all \( x, y \in [a, b] \) and \( \lambda \in [0,1] \). We say that \( f \) is concave if \( (-f) \) is convex.

In [6], Pearce et. al. generalized this inequality to \( r \)-convex positive function \( f \) which defined on an interval \([a,b]\), for all \( x, y \in [a,b] \) and \( t \in [0,1] \)

\[
f(tx + (1-t)y) \leq \left\{ \begin{array}{ll}
 t[f(x)]^r + (1-t)[f(y)]^r & \text{if } r \neq 0 \\
 [f(x)][f(y)]^{t-r} & \text{if } r = 0.
\end{array} \right.
\]

We have that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions.

Recently, the generalizations of the Hermite-Hadamard's inequality to the integral power mean of a positive convex function on an interval \([a,b]\), and to that of a positive \( r \)-convex function on an interval \([a,b]\) are obtained by Pearce and Pecaric, and others (see [6]-[12]).

A function \( f : I \to [0,\infty) \) is said to be log-convex or multiplicatively convex if \( \log t \) is convex, or, equivalently, if for all \( x, y \in I \) and \( t \in [0,1] \) one has the inequality:

\[
f(tx + (1-t)y) \leq [f(x)][f(y)]^{t-r}.
\]

We note that if \( f \) and \( g \) are convex and \( g \) is increasing, then \( g \circ f \) is convex; moreover, since \( f = \exp(\log f) \), it follows that a log-convex function is convex, but the converse may not necessarily be true [6]. This follows directly from (1.2) because, by the arithmetic-geometric mean inequality, we have

\[
[f(x)][f(y)]^{t-r} \leq tf(x) + (1-t)f(y)
\]

for all \( x, y \in I \) and \( t \in [0,1] \).

For some results related to this classical results, (see [3], [4], [9], [10]) and the references therein. Dragomir and Mond [3] proved the following Hermite-Hadamard type inequalities for the log -convex functions:
\[ f\left(\frac{a + b}{2}\right) \leq \exp\left[\frac{1}{b - a} \int_a^b \ln[f(x)] \, dx\right] \]
\[ \leq \frac{1}{b - a} \int_a^b G(f(x), f(a + b - x)) \, dx \]
\[ \leq \frac{1}{b - a} \int_a^b f(x) \, dx \]
\[ \leq L(f(a), f(b)) \]
\[ \leq \frac{f(a) + f(b)}{2}, \quad (1.3) \]

where \( G(p, q) = \sqrt{pq} \) is the geometric mean and \( L(p, q) = \frac{p^q - q^p}{\ln p - \ln q} \) (\( p \neq q \)) is the logarithmic mean of the positive real numbers \( p, q \) (for \( p = q \), we put \( L(p, q) = p \)).

Recall also that a function \( f : I \to R \) is called strongly convex with modulus \( c > 0 \), if
\[ f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2 \]
for all \( x, y \in I \) and \( t \in (0,1) \). Strongly convex functions have been introduced by Polyak in [13] and they play an important role in optimization theory and mathematical economics. Various properties and applications of them can be found in the literature see ([13]-[16]) and the references cited therein.

In this paper we introduce the notation of strongly logarithmic convex with respect to \( c > 0 \) and versions of Hermite-Hadamard-type inequalities for strongly logarithmic convex with respect to \( c > 0 \) are presented. This result generalizes the Hermite-Hadamard-type inequalities obtained in [3] for log-convex functions with \( c = 0 \).

2. Main Results

We will say that a positive function \( f : I \to (0, \infty) \) is strongly log-convex with respect to \( c > 0 \) if
\[ f(\lambda x + (1-\lambda)y) \leq [f(x)]^{\lambda}[f(y)]^{1-\lambda} - c\lambda(1-\lambda)(x-y)^2 \]
for all \( x, y \in I \) and \( \lambda \in (0,1) \). In particular, from the above definition, by the arithmetic-geometric mean inequality, we have
\[ f(\lambda x + (1-\lambda)y) \leq [f(x)]^{\lambda}[f(y)]^{1-\lambda} - c\lambda(1-\lambda)(x-y)^2 \]
\[ \leq \lambda f(x) + (1-\lambda) f(y) - c\lambda(1-\lambda)(x-y)^2 \]
\[ \leq \max\{f(x), f(y)\} - c\lambda(1-\lambda)(x-y)^2 \quad (2.1) \]
Theorem 2.1 If a function $f : I \to (0, \infty)$ be a strongly log-convex with respect to $c > 0$ and Lebesgue integrable on $I$, we have

$$f\left(\frac{a+b}{2}\right) + \frac{c(b-a)^2}{12} \leq \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))dx$$

$$\leq \frac{1}{b-a}\int_a^b f(x)dx$$

$$\leq L(f(a), f(b)) - \frac{c(b-a)^2}{6}$$

$$\leq \frac{f(a) + f(b)}{2} - \frac{c(b-a)^2}{6} \quad (2.2)$$

for all $a, b \in I$ with $a < b$.

**Proof.** From (2.1), we have

$$f(\lambda x + (1-\lambda)y) \leq [f(x)]^\lambda[f(y)]^{1-\lambda} - cf(1-\lambda)(x-y)^2$$

$$\leq \lambda f(x) + (1-\lambda) f(y) - c\lambda(1-\lambda)(x-y)^2. \quad (2.3)$$

Since $f$ is a strongly log-convex function on $I$, we have for $x, y \in I$ with $\lambda = \frac{1}{2}$

$$f\left(\frac{x+y}{2}\right) \leq \sqrt{f(x)f(y)} - \frac{c(x-y)^2}{4}$$

$$\leq \frac{f(x) + f(y)}{2} - \frac{c(x-y)^2}{4} \quad (2.4)$$

i.e., with $x = ta + (1-t)b$, $y = (1-t)a + tb$,

$$f\left(\frac{a+b}{2}\right)$$

$$\leq \sqrt{f(ta + (1-t)b)f((1-t)a + tb)} - \frac{c(b-a)^2(1-2t)^2}{4}$$

$$\leq f(ta + (1-t)b) + f((1-t)a + tb) - \frac{c(b-a)^2(1-2t)^2}{4}. \quad (2.5)$$

Integrating the inequality (2.5) with respect to $t$ over $(0,1)$, we obtain
\[
\int f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x) f(a+b-x) dx - \frac{c(b-a)^2}{12} \\
\leq \frac{1}{b-a} \int_a^b A(f(x), f(a+b-x)) dx - \frac{c(b-a)^2}{12},
\]

and so for \( \int_a^b f(x) dx = \int_a^b f(a+b-x) dx, \)

\[
f(\frac{a+b}{2}) + \frac{c(b-a)^2}{12} \leq \frac{1}{b-a} \int_a^b G(f(x), f(a+b-x)) dx \leq \frac{1}{b-a} \int_a^b f(x) dx. \tag{2.6}
\]

Since \( f \) is a strongly log-convex function on \( I \), for \( x = a \) and \( y = b \), we write

\[
f(ta+(1-t)b) \leq [f(a)] [f(b)] \cdot t - c(1-y)(a-b)^2 \\
\leq tf(a) + (1-t)f(b) - ct(1-t)(a-b)^2. \tag{2.7}
\]

Integrating the inequality (2.7) with respect to \( t \) over \( (0,1) \), we obtain,

\[
\frac{1}{b-a} \int_a^b f(x) dx \leq \int_0^1 \left[ \frac{f(a)}{f(b)} \right]^t dt - c(b-a)^2 \int_0^1 t(1-t) dt \\
\leq f(a) \int_0^1 t dt + f(b) \int_0^1 (1-t) dt - c(b-a)^2 \int_0^1 t(1-t) dt,
\]

and so

\[
\frac{1}{b-a} \int_a^b f(x) dx \leq L(f(a), f(b)) - \frac{c(b-a)^2}{6} \leq \frac{f(a) + f(b)}{2} - \frac{c(b-a)^2}{6}. \tag{2.8}
\]

Thus, from (2.6) and (2.8), we obtain the inequality of (2.2). This completes the proof.

**Theorem 2.2** Let a function \( f : I \to [0, \infty) \) be a strongly log-convex with respect to \( c > 0 \) and Lebesgue integrable on \( I \), then the following inequality holds:

\[
\frac{1}{b-a} \int_a^b f(x) f(a+b-x) dx \leq f(a) f(b) + \frac{c^2(b-a)^4}{30} \\
- \frac{4c(b-a)^2}{[\ln(f(b)-f(a))]^2} [A(f(a), f(b)) + L(f(a), f(b))]. \tag{2.9}
\]
for all $a, b \in I$ with $a < b$.

**Proof.** Since $f$ is strongly log-convex with respect to $c > 0$, we have that for all $t \in (0, 1)$

$$f(ta + (1-t)b) \leq [f(a)]^t [f(b)]^{1-t} - ct(1-t)(b-a)^2$$

$$\leq tf(a) + (1-t)f(b) - ct(1-t)(b-a)^2$$

(2.10)

and

$$f((1-t)a + tb) \leq [f(a)]^{-r} [f(b)]^r - ct(1-t)(b-a)^2$$

$$\leq (1-t)f(a) + tf(b) - ct(1-t)(b-a)^2.$$  

(2.11)

Multiplying both sides of (2.10) by (2.11), it follows that

$$f(ta + (1-t)b)f((1-t)a + tb) \leq f(a)f(b) + c^2(b-a)^4 t^2(1-t)^2$$

$$-c(b-a)^2 t(1-t) \left[ f(b) \left( \frac{f(a)}{f(b)} \right)^{t} + f(a) \left( \frac{f(b)}{f(a)} \right)^r \right].$$

(2.12)

Integrating the inequality (2.12) with respect to $t$ over $(0, 1)$, we obtain

$$\int_a^b f(ta + (1-t)b)f((1-t)a + tb) \, dt \leq \int_0^1 f(a)f(b) \, dt + c^2(b-a)^4 \int_0^1 t^2(1-t)^2 \, dt$$

$$-c(b-a)^2 \int_0^1 t(1-t) \left[ f(b) \left( \frac{f(a)}{f(b)} \right)^t + f(a) \left( \frac{f(b)}{f(a)} \right)^r \right] \, dt$$

$$= \int_0^1 f(a)f(b) \, dt + c^2(b-a)^4 \int_0^1 t^2(1-t)^2 \, dt - c(b-a)^2 f(b)I_1 - c(b-a)^2 f(a)I_2.$$  

(2.13)

Integrating by parts for $I_1$ and $I_2$ integrals, we obtain
\[ I_1 = \int_{0}^{1} t(1-t) \left[ \frac{f(a)}{f(b)} \right]' \, dt \]

\[ = (1-t) \frac{1}{\ln \left[ \frac{f(a)}{f(b)} \right]} \left[ \frac{f(a)}{f(b)} \right]' \bigg|_{0}^{1} - \frac{1}{\ln \left[ \frac{f(a)}{f(b)} \right]} \int_{0}^{1} (1-2t) \left[ \frac{f(a)}{f(b)} \right]' \, dt \]

\[ = - \frac{1}{\ln \left[ \frac{f(a)}{f(b)} \right]} \left[ (1-2t) \left[ \frac{f(a)}{f(b)} \right]' \bigg|_{0}^{1} + \frac{2}{\ln \left[ \frac{f(a)}{f(b)} \right]} \int_{0}^{1} \left[ \frac{f(a)}{f(b)} \right]' \, dt \right] \]

\[ = \frac{1}{f(b)} \frac{f(a) + f(b)}{[\ln(f(a) - f(b))]^2} + \frac{2f(b) - 2f(a)}{[\ln(f(a) - f(b))]^2}, \quad (2.14) \]

and similarly we get,

\[ I_2 = \int_{0}^{1} t(1-t) \left[ \frac{f(b)}{f(a)} \right]' \, dt \]

\[ = \frac{1}{f(a)} \frac{f(a) + f(b)}{[\ln(f(b) - f(a))]^2} + \frac{2f(b) - 2f(a)}{[\ln(f(b) - f(a))]^2}. \quad (2.15) \]

Putting (2.14) and (2.15) in (2.13), and if we change the variable \( x := ta + (1-t)b \), \( t \in (0,1) \), we get the required inequality in (2.9). This proves the theorem.

References

[1] M. Alomari and M. Darus, On the Hadamard's inequality for log-convex functions on the coordinates, *Journal of Inequalities and Applications*, vol. 2009, Article ID 283147, 13 pages, 2009.

[2] M. K. Bakula and J. Pečarić, Note on some Hadamard-type inequalities, *Journal of Inequalities in Pure and Applied Mathematics*, 5(3)(2004): 74.

[3] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[4] S. S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math.Lett.*, 11(5)(1998): 91-95.
[5] S. S. Dragomir and B. Mond, Integral inequalities of Hadamard type for Hadamard type for
log \( \log \) convex functions, *Demonstratio*, 31(1998): 354-364

[6] C.E.M. Pearce, J. Pecaric, V. Šimic, Stolarsky means and Hadamard's inequality, *J. Math. Anal. Appl.* 220(1998): 99-109.

[7] P. M. Gill, C. E. M. Pearce, and J. Pečarić, Hadamard's inequality for \( r \)-convex functions, *Journal of Mathematical Analysis and Applications*, vol. 215, no. 2, pp. 461-470, 1997.

[8] J.E. Pečarić, F. Proschan and Y.L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.

[9] E. Set, M. E. Özdemir, and S. S. Dragomir, On the Hermite-Hadamard inequality and other integral inequalities involving two functions, *Journal of Inequalities and Applications*, Article ID 148102, 9 pages, 2010.

[10] E. Set, M. E. Özdemir, and S. S. Dragomir, On Hadamard-Type inequalities involving several kinds of convexity, *Journal of Inequalities and Applications*, Article ID 286845, 12 pages, 2010.

[11] G.-S. Yang, D.-Y. Hwang, Refinements of Hadamard's inequality for \( r \)-convex functions, *Indian J. Pure Appl. Math.* 32 (2001): 1571-1579.

[12] N.P.G. Ngoc, N.V. Vinh and P.T.T. Hien, Integral inequalities of Hadamard-type for \( r \)-convex functions, *International Mathematical Forum*, 4 (2009): 1723-1728.

[13] B.T. Polyak, Existence theorems and convergence of minimizing sequences in extremum problems with restrictions, *Soviet Math. Dokl.*, 7(1966): 72-75.

[14] N. Merentes and K. Nikodem, Remarks on strongly convex functions, *Aequationes Math.* 80(1-2) (2010): 193-199.

[15] K. Nikodem and Zs. Páles, Characterizations of inner product spaces be strongly convex functions, *Banach J. Math. Anal.* 5(1) (2011): 83-87.

[16] H. Angulo, J. Gimenez, A. M. Moros and K. Nikodem, On strongly \( h \)-convex functions, *Ann. Funct. Anal.*, 2(2) (2011): 85-91.