Muirhead inequality for convex orders and a problem of I. Raşa on Bernstein polynomials

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Abstract

We present a new, very short proof of a conjecture by I. Raşa, which is an inequality involving basic Bernstein polynomials and convex functions. It was affirmed positively very recently by J. Mrowiec, T. Rajba and S. Wasowicz (2017) by the use of stochastic convex orderings, as well as by Abel (2017) who simplified their proof. We give a useful sufficient condition for the verification of some stochastic convex ordering relations, which in the case of binomial distributions are equivalent to the I. Raşa inequality. We give also the corresponding inequalities for other distributions. Our methods allow us to give some extended versions of stochastic convex orderings as well as the I. Raşa type inequalities. In particular, we prove the Muirhead type inequality for convex orderings for convolution polynomials of probability distributions.

Keywords: Bernstein polynomials, stochastic ordering, stochastic convex ordering, convex functions, functional inequalities related to convexity, Muirhead inequality

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1. Introduction

For \( n \in \mathbb{N} \) and \( i = 0, 1, \ldots, n \), let

\[
b_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad x \in [0, 1].
\]

denote the Bernstein basic polynomials. For \( i > n \), we define \( b_{n,i}(x) = 0 \). The Bernstein operator \( B_n : C([0, 1]) \to C([0, 1]) \) associates to each continuous function \( \varphi : [0, 1] \to \mathbb{R} \) the function \( B_n(\varphi) \) given by

\[
B_n(\varphi)(x) = \sum_{i=0}^{n} b_{n,i}(x) \varphi\left(\frac{i}{n}\right) \quad \text{for } x \in [0, 1].
\]
Recently, J. Mrowiec, T. Rajba and S. Wąsowicz [9] proved the following theorem on inequality for Bernstein operators.

**Theorem 1.1.** Let \( n \in \mathbb{N} \) and \( x, y \in [0, 1] \). Then

\[
\sum_{i=0}^{n} \sum_{j=0}^{n} \left[ b_{n,i}(x)b_{n,j}(x) + b_{n,i}(y)b_{n,j}(y) - 2b_{n,i}(x)b_{n,j}(y) \right] \varphi \left( \frac{i+j}{n} \right) \geq 0
\]

for all convex functions \( \varphi \in C([0, 1]) \).

This inequality involving Bernstein basic polynomials and convex functions was conjectured as an open problem 25 years ago by I. Raşa. J. Mrowiec, T. Rajba and S. Wąsowicz [9] showed that the conjecture is true. Their proof makes heavy use of probability theory. As a tool they applied a concept of stochastic convex orderings (which they proved for binomial distributions) as well as the so-called concentration inequality. Later U. Abel [1] gave an elementary proof of the above theorem, which was much shorter than that given in [9]. In this paper, we present a new, very short proof of the above theorem, which is significantly simpler and shorter than that given by U. Abel [1] (cf. Theorem 2.3 and Theorem 2.7). As a tool we use both stochastic convex orders as well as the usual stochastic order.

In [1], U. Abel studied Mirakyan-Szász operators \( S_n : D_S \to C([0, \infty)) \) (where \( D_S \subset C([0, \infty)) \)) consists of functions of at most exponential growth) given by

\[
S_n(\varphi)(x) = \sum_{i=0}^{\infty} s_i(nx)\varphi \left( \frac{x}{n} \right) \quad \text{for } x \in [0, \infty),
\]

where

\[
s_i(x) = e^{-x} \frac{x^i}{i!}, \quad x \in [0, \infty),
\]

are the corresponding basic functions, and he proved the following inequality (corresponding to the inequality (1.1)) for these operators and convex \( \varphi \in D_S 
\]

\[
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( s_i(x)s_j(x) + s_i(y)s_j(y) - 2s_i(x)s_j(y) \right) \varphi(i + j) \geq 0 \quad \text{for } x, y \in [0, \infty).
\]

U. Abel [1] considered also Baskakov operators \( V_n : D_{V_n} \to C([0, \infty)) \) (where \( D_{V_n} \subset C([0, \infty)) \)) given by

\[
V_n(\varphi)(x) = \sum_{i=0}^{\infty} v_n,i(x)\varphi \left( \frac{x}{n} \right) \quad \text{for } x \in [0, \infty),
\]

where

\[
v_n,i(x) = \frac{(n + i - 1)}{i} \frac{x^i}{(1 + x)^{n+i}}, \quad x \in [0, \infty),
\]

are the Baskakov basic functions, and proved the following inequality

\[
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( v_n,i(x)v_n,j(x) + v_n,i(y)v_n,j(y) - 2v_n,i(x)v_n,j(y) \right) \varphi(i + j) \geq 0 \quad \text{for } x, y \in [0, \infty).
\]

Note, that the basic functions, which appear in the formulas for the operators \( B_n, S_n \) and \( V_n \), correspond to the probabilities of binomial, Poisson, and negative binomial distributions, respectively. In this paper, we study some other families of probability distributions that can be used as basic functions for operators and inequalities associated with these operators.

In Section 2, we prove Theorem 2.3 on convex ordering, which is a useful tool for proving inequalities (1.1), (1.3), (1.5) and many similar inequalities (we call them I. Raşa type inequalities). In particular, using this theorem for binomial distributions, Poisson distributions and negative binomial distributions, we obtain new, very short proofs of inequalities (1.1), (1.3) and (1.5).
In the paper we consider several operators: $B_n$ (Bernstein-Schnabl operators), $S_n$ (Mirakyan-Szász operators), $V_n$ (Baskakov operators), $N_B$, $\Gamma$, $V_\alpha$, $B_t$ and $N_\sigma^2$ (see Section 2). Arguments and values of these operators are continuous functions defined on $[0, 1]$, $[0, \infty)$, $\mathbb{R}$, $[0, \infty) \times [0, 1)$ and similar spaces. If $K$ is a topological space, then $C(K)$ denotes the space of continuous, real-valued functions defined on $K$. We do not specify the domains of the investigated operators explicitly. In each case the domain is the set of continuous functions for which the definition of an operator makes sense and the resulting function is continuous. Operators $B_n$, $S_n$, $V_n$, $B_t$ and $N_\sigma^2$ are Markov operators and they share the following property: if $T$ is any of these operators and function $\phi$ is a affine, then $T(\phi) = \phi$. This nice property has been investigated by many authors (see [2, 3, 4]). In particular, in [2, 3, 4] the authors discussed inequalities concerning these operators ([3] Remark 3.4.4 and Examples 3.4.5-3.4.11, [2] Corollary 3.5 and Remark 3.6), which are parallel to those satisfied for Bernstein operators $B_n$ (see [9]). In this paper we introduce new examples of such operators (see Remark 2.24).

In Section 3, we give a strong generalization of Theorem 2.3 and use it to obtain subsequent generalizations of the inequality (1.1). The main result of Section 3 (Theorem 3.10) is the Muirhead type inequality for convex orderings for convolution polynomials of probability distributions.

2. The I. Raşa type inequalities

In the sequel we make use of the theory of stochastic orders. Let us recall some basic notations and results (see [12]) on stochastic ordering. As usual, $F_X(x) = P(X < x)$ $(x \in \mathbb{R})$ stands for the probability distribution function of a real-valued random variable $X$, while $\mu_X$ is the distribution corresponding to $X$.

If $X$ and $Y$ are two random variables such that

$$F_X(x) \geq F_Y(x) \quad \text{for all} \quad x \in \mathbb{R},$$

then $X$ is said to be smaller than $Y$ in the usual stochastic order (denoted by $X \leq_{st} Y$ or $F_X \leq_{st} F_Y$ or $\mu_X \leq_{st} \mu_Y$).

An important characterization of the usual stochastic order is the following theorem (here $=_{st}$ denotes the equality in law).

**Theorem 2.1** ([12], p. 5). Two random variables $X$ and $Y$ satisfy $X \leq_{st} Y$ if, and only if, there exist two random variables $\hat{X}$ and $\hat{Y}$ defined on the same probability space, such that

$$\hat{X} =_{st} X, \quad \hat{Y} =_{st} Y \quad \text{and} \quad P(\hat{X} \leq \hat{Y}) = 1.$$ 

If a random variable $X$ has the binomial distribution with parameters $n \in \mathbb{N}$ and $p \in [0, 1]$ (denoted by $X \sim B(n, p)$), then

$$P(X = i) = b_{n, p}(i) = \binom{n}{i} p^i (1 - p)^{n-i} \quad \text{for} \quad i = 0, 1, \ldots, n.$$ 

**Lemma 2.2** ([12], p. 14). Let $X \sim B(n, p_1)$ and $Y \sim B(n, p_2)$ with $n \in \mathbb{N}$ and $p_1, p_2 \in [0, 1]$. Then

$$X \leq_{st} Y \iff p_1 \leq p_2.$$ 

If $X$ and $Y$ are two random variables such that
\[
\mathbb{E}[\varphi(X)] \leq \mathbb{E}[\varphi(Y)] \quad \text{for all convex functions } \varphi : \mathbb{R} \to \mathbb{R},
\]
provided the expectations exist, then $X$ is said to be smaller than $Y$ in the convex stochastic order (denoted as $X \lesssim_{cx} Y$ or $F_X \leq_{cx} F_Y$ or $\mu_X \lesssim_{cx} \mu_Y$).

In the following theorem, we give a very useful sufficient condition that will be used for a verification of some convex stochastic orderings.

**Theorem 2.3.** Let $X$ and $Y$ be two independent random variables with finite means, such that $X \leq_{st} Y$ or $Y \leq_{st} X$. Let $(X_1, X_2)$ and $(Y_1, Y_2)$ be two pairs of independent random variables such that $X, X_1, X_2$ are identically distributed and $Y, Y_1, Y_2$ are identically distributed. Then
\[
F_{X+Y} \leq_{cx} \frac{1}{2} (F_{X_1+X_2} + F_{Y_1+Y_2}).
\]

To prove Theorem 2.3, we will need the following special case of the Hardy-Littlewood-Pólya inequality ([6], Theorem 108):

**Remark 2.4.** Let $E \subset \mathbb{R}$ be a convex subset of the real line and let $\varphi : E \to \mathbb{R}$ be a convex function. If $a \leq b, c \leq d$ are in $E$ and $a + d = b + c$, then
\[
\varphi(b) + \varphi(c) \leq \varphi(a) + \varphi(d).
\]

Indeed, if $a = d$, then (2.1) is obvious. Otherwise, to prove (2.1), it is enough to take the sum of the following two Jensen inequalities:
\[
\varphi(b) \leq \frac{d - b}{d - a} \cdot \varphi(a) + \frac{b - a}{d - a} \cdot \varphi(d),
\]
\[
\varphi(c) \leq \frac{d - c}{d - a} \cdot \varphi(a) + \frac{c - a}{d - a} \cdot \varphi(d).
\]

**Proof (Theorem 2.3).** Assume that $X$ and $Y$ are two independent random variables with finite means, such that $X \leq_{st} Y$, i.e. $F_X(x) \geq F_Y(x)$ for all $x \in \mathbb{R}$. Then, by Theorem 2.1, there exist two independent random vectors $(X_1, Y_1)$ and $(X_2, Y_2)$ such that
\[
X_1 =_{\mu} X_2 =_{\mu} X \quad \text{and} \quad Y_1 =_{\mu} Y_2 =_{\mu} Y.
\]
\[
P(X_1 \leq Y_1) = 1 \quad \text{and} \quad P(X_2 \leq Y_2) = 1. \tag{2.2}
\]

Then we have $F_{X+Y} = F_{X_1+Y_2} = F_{Y_1+X_2}$, which implies
\[
F_{X+Y} = \frac{1}{2} (F_{X_1+X_2} + F_{Y_1+Y_2}). \tag{2.3}
\]
By (2.2)
\[
P(X_1 + X_2 \leq X_1 + Y_2 \leq Y_1 + Y_2) = 1 \quad \text{and} \quad P(X_1 + X_2 \leq X_2 + Y_1 \leq Y_1 + Y_2) = 1,
\]
and obviously
\[
P((X_1 + Y_2) + (X_2 + Y_1) = (X_1 + X_2) + (Y_1 + Y_2)) = 1.
\]
Thus, by Remark 2.4, we conclude that
\[
P(\varphi(X_1 + Y_2) + \varphi(X_2 + Y_1) \leq \varphi(X_1 + X_2) + \varphi(Y_1 + Y_2)) = 1
\]
for all convex functions \( \varphi : \mathbb{R} \to \mathbb{R} \), which implies
\[
\frac{1}{2} \mathbb{E} \left( (\varphi(X_1 + Y_2) + \varphi(X_2 + Y_1)) \right) \leq \frac{1}{2} \mathbb{E} \left( \varphi(X_1 + X_2) + \varphi(Y_1 + Y_2) \right). 
\]
(2.4)

Writing (2.4) in terms of convex ordering, we have
\[
\frac{1}{2} \left( F_{X_1 + Y_2} + F_{X_2 + Y_1} \right) \preceq_{\text{st}} \frac{1}{2} \left( F_{X_1 + X_2} + F_{Y_1 + Y_2} \right).
\]
Taking into account (2.3), we obtain \( F_{X+Y} \preceq_{\text{st}} \frac{1}{2} \left( F_{X_1 + X_2} + F_{Y_1 + Y_2} \right) \). The theorem is proved.

**Remark 2.5.** In the proof of Theorem 2.3 we considered a convex function \( \varphi : \mathbb{R} \to \mathbb{R} \). Instead, we could consider any convex function \( \varphi \) for all convex functions \( \varphi \) we could consider any convex function \( \varphi \) in the proof of Theorem 2.3 we considered a convex function \( \varphi \).

Of course, neither the condition \( X \preceq_{\text{st}} Y \) nor \( Y \preceq_{\text{st}} X \) is satisfied. Let \((X_1, X_2)\) and \((Y_1, Y_2)\) be two pairs of independent random variables such that \(X_1, X_2\) are identically distributed and \(Y_1, Y_2\) are identically distributed. Then

\[
\begin{align*}
\mu_{X+Y} &= \frac{3}{4} \delta_{-3} + \frac{1}{4} \delta_1, \\
\mu_{X_1+X_2} &= \frac{1}{4} \delta_{-6} + \frac{1}{4} \delta_{2}, \\
\mu_{Y_1+Y_2} &= \frac{9}{16} \delta_0 + \frac{3}{8} \delta_4 + \frac{1}{16} \delta_8
\end{align*}
\]

and \( \frac{1}{2} \left( F_{X_1 + X_2} + F_{Y_1 + Y_2} \right) \) is the distribution function corresponding to the probability measure
\[
\frac{1}{2} \left( \frac{3}{4} \delta_{-6} + \frac{1}{4} \delta_{-2} + \frac{9}{16} \delta_0 + \frac{3}{8} \delta_4 + \frac{1}{16} \delta_8 \right).
\]
To prove that
\[
F_{X+Y} \preceq_{\text{st}} \frac{1}{2} \left( F_{X_1 + X_2} + F_{Y_1 + Y_2} \right), 
\]
we take a convex function \( \varphi : \mathbb{R} \to \mathbb{R} \). Then, by the Jensen inequality (cf. 7), we obtain
\[
\frac{1}{2} \left( \mathbb{E} \left[ \varphi(X_1 + X_2) + \varphi(Y_1 + Y_2) \right] \right) - \mathbb{E} \left[ \varphi(X + Y) \right] =
\]
\[
\frac{1}{2} \left( \frac{1}{4} \varphi(-6) + \frac{1}{4} \varphi(-2) + \frac{1}{16} \varphi(0) + \frac{1}{4} \varphi(2) + \frac{1}{4} \varphi(4) + \frac{1}{4} \varphi(8) \right) - \left( \frac{1}{4} \varphi(-3) + \frac{1}{4} \varphi(1) + \frac{1}{4} \varphi(5) \right) =
\]
\[
\frac{1}{2} \left( \frac{1}{4} \varphi(-6) + \frac{1}{4} \varphi(-2) + \frac{1}{16} \varphi(0) - \varphi(-3) \right) + \frac{1}{2} \left( \frac{1}{4} \varphi(-2) + \frac{1}{16} \varphi(0) + \frac{5}{4} \varphi(2) + \frac{7}{4} \varphi(4) - \varphi(1) \right) + \frac{1}{4} \left( \frac{1}{4} \varphi(4) + \frac{1}{4} \varphi(8) - \varphi(5) \right) \geq 0,
\]
which implies (2.5)

5
Using Theorem 2.7, we can give a new proof of the result of J. Mrowiec, T. Rajba and S. Wąsowicz [9] (Theorem 1.1).

**Theorem 2.7.** Let $n \in \mathbb{N}$ and $x, y \in [0, 1]$. Then
\[
\sum_{i=0}^{n} \sum_{j=0}^{n} \left( b_{n,i}(x)b_{n,j}(x) + b_{n,i}(y)b_{n,j}(y) - 2b_{n,i}(x)b_{n,j}(y) \right) \varphi \left( \frac{xy}{n^2} \right) \geq 0
\] (2.6)
for all convex functions $\varphi \in C([0, 1])$.

**Proof (new proof).** Let $x, y \in [0, 1]$. Let $X$ and $Y$ be two independent random variables such that $X \sim B(n, x)$ and $Y \sim B(n, y)$. By Lemma 2.8, $X \leq_{st} Y$ or $Y \leq_{st} X$. Let $(X_1, X_2)$ and $(Y_1, Y_2)$ be two pairs of independent random variables such that $X, X_1, X_2$ are identically distributed and $Y, Y_1, Y_2$ are identically distributed. This implies that the assumptions of Theorem 2.8 are satisfied. Consequently, we obtain the inequality
\[
F_{X+Y} \leq_{st} \frac{1}{2} (F_{X_1} + F_{Y_1}).
\] (2.7)
Since (2.7) is equivalent to the inequality
\[
E \varphi \left( \frac{X+Y}{n^2} \right) \leq \frac{1}{2} \left[ E \varphi \left( \frac{X_1+Y_1}{n^2} \right) + E \varphi \left( \frac{X_2+Y_2}{n^2} \right) \right]
\]
for all convex functions $\varphi \in C([0, 1])$, which can be rewritten in the form (2.6), the theorem is proved.

**Remark 2.8.** Note, that the I. Raşa inequality (2.6) is equivalent to the inequality (2.7) with binomially distributed random variables $X$ and $Y$. We can consider the inequality (2.7) for random variables $X$ and $Y$ with distributions from some other families of probability distributions. As a result we obtain several new inequalities, which are analogues of (1.1).

If a random variable $X$ has the Poisson distribution with the parameter $\lambda > 0$ (denoted by $X \sim \text{Poiss}(\lambda)$), then
\[
P(X = i) = s_i(\lambda) = e^{-\lambda} \cdot \frac{\lambda^i}{i!} \quad \text{for } i = 0, 1, \ldots.
\] (2.8)
By convention, we say that $X \sim \text{Poiss}(0)$, if $\mu_X = 0$. Then $s_0(0) = 1$ and $s_i(0) = 0$ for $i = 1, 2, \ldots$.

**Lemma 2.9.** Let $X \sim \text{Poiss}(\lambda_1)$ and $Y \sim \text{Poiss}(\lambda_2)$ with $\lambda_1, \lambda_2 \geq 0$. Then
\[
X \leq_{st} Y \iff \lambda_1 \leq \lambda_2.
\]

**Proof.** It is enough to show $\lambda_1 \leq \lambda_2 \Rightarrow X \leq_{st} Y$. Assume that $0 \leq \lambda_1 \leq \lambda_2$. Let $\hat{X} \sim \text{Poiss}(\lambda_1)$ and $\hat{Z} \sim \text{Poiss}(\lambda_2-\lambda_1)$ be independent random variables. Then $\hat{Y} := \hat{X} + \hat{Z}$ satisfies $\hat{Y} \sim \text{Poiss}(\lambda_2)$ and $P(\hat{X} \leq \hat{Y}) = P(\hat{Z} \geq 0) = 1$. Hence Theorem 2.7 yields $X \leq_{st} Y$.

Using the probabilities $s_i(\lambda)$ (given by (2.8)) as basic functions for an operator, we obtain the Mirakyan-Szász operator $S_\lambda : D_\lambda \to C([0, \infty))$ (where $D_\lambda \subset C([0, \infty))$) consists of functions of at most exponential growth, see (1.2)
\[
S_\lambda(\varphi)(x) = \sum_{i=0}^{\infty} s_i(\lambda) \varphi \left( \frac{x}{\lambda} \right) \quad \text{for } x \in [0, \infty).
\]
Note that 

\[ S_n(\varphi)(x) = \mathbb{E}\left[ \varphi\left( \frac{X}{n} \right) \right], \]

where \( X \sim \text{Poisss}(nx) \).

By Lemma 2.9 as an immediate consequence of Theorem 2.3, we obtain the following Raşa type inequality:

**Theorem 2.10.** If \( x, y \in [0, \infty) \), then

\[ 2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} s_i(x) \cdot s_j(y) \cdot \varphi(i + j) \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( s_i(x) \cdot s_j(x) + s_i(y) \cdot s_j(y) \right) \cdot \varphi(i + j) \quad (2.9) \]

for all convex functions \( \varphi: [0, \infty) \to \mathbb{R} \).

**Proof.** Let \( x, y \in [0, \infty) \). Without loss of generality we may assume that \( x \leq y \). By Lemma 2.9 and Theorem 2.3

\[ F_{X+Y} \leq_{\text{cx}} \frac{1}{2} \left( F_{X_1+X_2} + F_{Y_1+Y_2} \right), \quad (2.10) \]

where \((X, Y), (X_1, X_2)\) and \((Y_1, Y_2)\) are the pairs of independent random variables such that \( X, X_1, X_2 \sim \text{Poisss}(x) \) and \( Y, Y_1, Y_2 \sim \text{Poisss}(y) \). Since the inequality (2.10) is equivalent to the inequality (2.9), the theorem is proved.

**Remark 2.11.** The inequality (2.9) is also proved in [1]. U. Abel [1] gave there an elementary proof of the inequality (2.9), but the new proof given in this paper is significantly simpler and shorter than that given in [1].

Denote by \( F_{\text{Poisss}(\lambda)} \) the distribution function corresponding to the random variable \( X \sim \text{Poisss}(\lambda) \). If \( X \) and \( Y \) are two independent random variables such that \( X \sim \text{Poisss}(\lambda_1) \) and \( Y \sim \text{Poisss}(\lambda_2) \), then \( X + Y \sim \text{Poisss}(\lambda_1 + \lambda_2) \). Consequently, the inequality (2.10) is equivalent to the following inequality:

**Corollary 2.12.** Let \( x, y \in [0, \infty) \), then

\[ F_{\text{Poisss}(\lambda)} \leq_{\text{cx}} \frac{1}{2} \left( F_{\text{Poisss}(\lambda_1)} + F_{\text{Poisss}(\lambda_2)} \right), \]

which is equivalent to

\[ 2 \sum_{i=0}^{\infty} s_i\left( \frac{\lambda}{\lambda'} \right) \varphi(i) \leq \sum_{i=0}^{\infty} \left[ s_i(x) + s_i(y) \right] \varphi(i) \]

for all convex functions \( \varphi: [0, \infty) \to \mathbb{R} \) and, consequently,

\[ S_n(\varphi)\left( \frac{\lambda}{\lambda'} \right) \leq \frac{1}{2} \left[ S_n(\varphi)(x) + S_n(\varphi)(y) \right] \quad (2.11) \]

for all convex functions \( \varphi \in D_{\delta} \).

If a random variable \( X \) has the negative binomial distribution with parameters \( r > 0 \) and \( 0 \leq p < 1 \) (denoted by \( X \sim \text{NB}(r, p) \)), then

\[ P(X = k) = \text{nb}_d(r, p) = \binom{k + r - 1}{k} p^r (1 - p)^k = \frac{\Gamma(k + r)}{\Gamma(r) \cdot k!} p^r (1 - p)^k \quad \text{for} \quad k = 0, 1, \ldots. \]

By convention, we say that if \( 0 \leq p < 1 \), then \( \text{NB}(0, p) = \delta_0 \), i.e., \( \text{nb}_d(0, p) = 1 \) and \( \text{nb}_d(0, p) = 0 \) for \( k > 0 \). The geometric distribution is a special case of the negative binomial distribution, namely \( \text{Geom}(p) = \text{NB}(1, 1 - p) \).
Lemma 2.13. Let $X \sim NB(r_1, p_1)$ and $Y \sim NB(r_2, p_2)$ with $r_1, r_2 \geq 0$ and $p_1, p_2 \in [0,1)$. Then

$$(r_1 \leq r_2 \text{ and } p_1 \leq p_2) \implies X \leq_{st} Y.$$ 

Proof. We shall use the following observation: Let $r > 0$ and $p \in [0,1)$. If $(N_i)_{i \geq 0}$ is the Poisson process with intensity $\lambda = 1$ and $T \sim \Gamma(r,1)$ is independent of $(N_i)_{i \geq 0}$, then $N_{T \sim \Gamma(r,1)} \sim NB(r,p)$. Indeed, for $k = 0,1, \ldots$ we have

$$P(N_T = k) = \int_0^\infty \frac{t^{k-1}e^{-t}}{\Gamma(r)} \cdot \left(\frac{e^{-t}t^r}{k!}\right)^k dt = \frac{\Gamma(k+r)}{\Gamma(r) \cdot k!} \cdot (1-p)^r \cdot \frac{\Gamma(k+r+1)}{\Gamma(k+r)} = \left(\frac{k+r-1}{k}\right)p^r(1-p)^r \cdot 1.$$ 

Assume that $0 < r_1 \leq r_2$ and $0 \leq p_1 \leq p_2 < 1$, thus $0 \leq \frac{p_1}{1-p_1} \leq \frac{p_2}{1-p_2}$.

Let $(N_i)_{i \geq 0}$ (the Poisson process with intensity $\lambda = 1$), $T \sim \Gamma(r_1,1)$ and $\hat{Z} \sim NB(r_2-r_1, p_2)$ be independent. We set

$$\hat{X} := N_{\frac{r_1 T}{p_1}}, \quad \hat{Y} := N_{\frac{r_2 T}{p_2}} + \hat{Z}.$$ 

Then $\hat{X} \sim NB(r_1, p_1)$, $\hat{Y} \sim NB(r_2, p_2)$ and

$$P(\hat{X} \leq \hat{Y}) = P\left(\frac{r_1 T}{p_1} \leq \frac{r_2 T}{p_2} + \hat{Z} \leq 0\right) = 1.$$ 

Hence Theorem 2.11 yields $X \leq_{st} Y$.

We consider the following operator $NB : D_{NB} \to C([0,\infty) \times [0,1))$ (where $D_{NB} \subset C([0,\infty))$). The basic functions of $NB$ are $nb_2(r,p)$ and it is defined by $NB(\varphi)(r,p) = E[\varphi(X)]$, where $X \sim NB(r,p)$. We have

$$NB(\varphi)(r,p) = \sum_{k=0}^{\infty} nb_2(r,p) \cdot \varphi(k) = \sum_{k=0}^{\infty} \left(\frac{k+r-1}{k}\right)p^r(1-p)^r \varphi(k).$$

As an immediate consequence of Lemma 2.13 and Theorem 2.2 we obtain the following Ràşa type inequality, which is associated with the operator $NB$:

Theorem 2.14. Let $r_1, r_2 > 0$ and $p_1, p_2 \in [0,1)$. If $(r_1 - r_2)(p_1 - p_2) \geq 0$, then

$$2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} nb_i(r_1, p_1) \cdot nb_j(r_2, p_2) \cdot \varphi(i+j) \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[ nb_i(r_1, p_1) \cdot nb_j(r_1, p_1) + nb_i(r_2, p_2) \cdot nb_j(r_2, p_2) \right] \cdot \varphi(i+j) \quad (2.12)$$

for all convex functions $\varphi : [0,\infty) \to \mathbb{R}$. 

8
If \( r_1 = r_2 \) or \( p_1 = p_2 \), then the assumption \((r_1 - r_2)(p_1 - p_2) \geq 0\) is satisfied. Note that \( r_1 = r_2 = 1 \) corresponds to the case of the geometric probability distributions.

We recall that, if \( X \sim NB(r_1, p) \) and \( Y \sim NB(r_2, p) \) are two independent random variables, then \( X + Y \sim NB(r_1 + r_2, p) \). Denote by \( F_{NB(r, p)} \) the distribution function corresponding to the random variable \( X \sim NB(r, p) \). Taking into account Lemma 2.13 and Theorem 2.3 (applied with \( X \sim NB(\frac{r}{2}, p) \) and \( Y \sim NB(\frac{r}{2}, p) \)) we obtain the following corollary:

**Corollary 2.15.** Let \( p \in [0, 1) \) and \( r_1, r_2 \geq 0 \). Then

\[
F_{NB(\frac{r_1 + r_2}{2}, p)} \leq \frac{1}{2} (F_{NB(r_1, p)} + F_{NB(r_2, p)}),
\]

which is equivalent to

\[
2 \sum_{i=0}^{\infty} nb_i(\frac{r_1 + r_2}{2}, p) \cdot \varphi(i) \leq \sum_{i=0}^{\infty} [nb(r_1, p) + nb(r_2, p)] \cdot \varphi(i)
\]

for all convex functions \( \varphi: [0, \infty) \to \mathbb{R} \) and, consequently,

\[
NB(\varphi)\left(\frac{r_1 + r_2}{2}, p\right) \leq \frac{1}{2} [NB(\varphi)(r_1, p) + NB(\varphi)(r_2, p)]
\]

for all convex functions \( \varphi \in D_{NB} \).

**Remark 2.16.** Operators closely related to the negative binomial probability distribution are the Baskakov operators (see (1.4)). They are given by \( V_r(\varphi)(x) = NB(\frac{2x}{r}, p) \) for \( \varphi: [0, \infty) \to \mathbb{R} \) and \( x \geq 0 \), where \( \hat{\varphi}(t) = \varphi \left( \frac{1}{t} \right) \) and \( r \) is a positive parameter. Note that \( x \mapsto \frac{1}{x} \) is a continuous bijection from \([0, \infty)\) onto \([0, 1)\). We have \( V_r(\varphi)(x) = \sum_{i=0}^{\infty} v_r(x) \cdot \varphi \left( \frac{i}{r} \right) \), where

\[
v_r(x) = \binom{i + r - 1}{k} \frac{x^i}{(1 + x)^{r+1}} \quad \text{for } x \geq 0.
\]

Applying (2.12) with \( r_1 = r_2 = r, p_1 = \frac{1}{r+1} \) and \( p_2 = \frac{1}{r_1} \) (with \( x, y \geq 0 \)) results in the following I. Rasta type inequality, which is associated with the Baskakov operator \( V_r \): 

\[
2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} v_r(x) \cdot v_r(y) \cdot \varphi(i + j) \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} [v_r(x) \cdot v_r(y) + v_r(y) \cdot v_r(y)] \cdot \varphi(i + j)
\]

for all convex functions \( \varphi: [0, \infty) \to \mathbb{R} \). The special case of (2.13) (when \( r \) is a natural number) was proved by U. Abel in [1].

Now we are going to study continuous distributions (more precisely absolutely continuous with respect to the Lebesgue measure). A random variable \( X \) that is gamma distributed with shape \( a > 0 \) and rate \( \beta > 0 \) is denoted \( X \sim \Gamma(a, \beta) \). The corresponding probability density function in the shape-rate parametrization is

\[
\gamma_{a, \beta}(x) = \frac{\beta^a x^{a-1} e^{-\beta x}}{\Gamma(a)} \quad \text{for } x > 0,
\]

and \( \gamma_{a, \beta}(x) = 0 \) for \( x \leq 0 \). By convention, we define \( \Gamma(0, \beta) = \delta_0 \) for every \( \beta > 0 \).

**Lemma 2.17.** If \( X \sim \Gamma(\alpha_1, \beta_1) \) and \( Y \sim \Gamma(\alpha_2, \beta_2) \) with \( \alpha_1, \alpha_2 \geq 0 \) and \( \beta_1, \beta_2 > 0 \), then

\[
(\alpha_1 \leq \alpha_2 \text{ and } \beta_1 \geq \beta_2) \implies X \leq_{st} Y.
\]
γ functions are X-type inequality, which is associated with the operator Γ. By Lemma 2.17 and Theorem 2.3, we obtain the following I. Raşa type inequality, which is equivalent to

\[ \Gamma(\varphi)(0, \beta) = \int_0^\infty \varphi(u) \cdot \gamma_{\alpha, \beta_1}(u) \cdot \gamma_{\alpha, \beta_2}(v) \cdot \varphi(u + v) \, du \]

By Lemma 2.17 and Theorem 2.3 we obtain the following I. Raşa type inequality, which is associated with the operator Γ:

**Theorem 2.18.** Let \( \alpha_1, \alpha_2 > 0 \) and \( \beta_1, \beta_2 > 0 \) satisfy \( (\alpha_1 - \alpha_2)/(\beta_1 - \beta_2) \leq 0 \). Then

\[ 2 \int_0^\infty \int_0^\infty \gamma_{\alpha_1, \beta_1}(u) \cdot \gamma_{\alpha_2, \beta_2}(v) \cdot \varphi(u + v) \, du \, dv \leq \int_0^\infty \int_0^\infty \left[ \gamma_{\alpha_1, \beta_1}(u) \cdot \gamma_{\alpha_2, \beta_2}(v) + \gamma_{\alpha_2, \beta_1}(u) \cdot \gamma_{\alpha_1, \beta_2}(v) \right] \cdot \varphi(u + v) \, du \, dv \]  

(2.14)

for all convex functions \( \varphi : [0, \infty) \to \mathbb{R} \).

If \( \alpha_1 = \alpha_2 \) or \( \beta_1 = \beta_2 \) then the assumption \( (\alpha_1 - \alpha_2)/(\beta_1 - \beta_2) \leq 0 \) is satisfied. Note that \( \alpha_1 = \alpha_2 = 1 \) corresponds to the case of the exponential probability distribution.

We recall that if \( X \sim \Gamma(\alpha_1, \beta) \) and \( Y \sim \Gamma(\alpha_2, \beta) \) are two independent random variables, then \( X + Y \sim \Gamma(\alpha_1 + \alpha_2, \beta) \). Denote by \( F_{\Gamma(\alpha_1, \beta)} \) the distribution function corresponding to the random variable \( X \sim \Gamma(\alpha_1, \beta) \). By Lemma 2.17 and Theorem 2.3 (applied with \( X \sim \Gamma(\alpha_1, \beta) \) and \( Y \sim \Gamma(\alpha_2, \beta) \)) we obtain the following corollary:

**Corollary 2.19.** If \( \beta > 0 \) and \( \alpha_1, \alpha_2 > 0 \) then

\[ F_{\Gamma(\alpha_1, \beta)} \leq \frac{1}{2} \left( F_{\Gamma(\alpha_1, \beta)} + F_{\Gamma(\alpha_2, \beta)} \right), \]

which is equivalent to

\[ 2 \int_0^\infty \gamma_{\alpha_1, \beta}(u) \cdot \varphi(u) \, du \leq \int_0^\infty \left[ \gamma_{\alpha_1, \beta}(u) + \gamma_{\alpha_2, \beta}(u) \right] \cdot \varphi(u) \, du. \]

for all convex functions \( \varphi : [0, \infty) \to \mathbb{R} \) and, consequently,

\[ \Gamma(\varphi)(\alpha_1, \beta) \leq \frac{1}{2} \left[ \Gamma(\varphi)(\alpha_1, \beta) + \Gamma(\varphi)(\alpha_2, \beta) \right] \]

for all convex functions \( \varphi \in D_{\Gamma} \).

**Remark 2.20.** The following operators are continuous counterparts of the Baskakov operators. Let \( \alpha > 0 \) and \( \varphi \in D_{\Gamma} \subset C([0, \infty)) \). We define \( \overline{V}_\alpha(\varphi)(x) = \Gamma(\varphi)(\alpha, \frac{x}{\alpha}) \) for \( x > 0 \) and \( \overline{V}_\alpha(\varphi)(0) = \varphi(0) \). Applying (2.14) with \( \alpha_1 = \alpha_2 = \alpha, \beta_1 = \frac{\alpha}{\alpha_1}, \) and \( \beta_2 = \frac{\alpha}{\alpha_2} \) (with \( x, y > 0 \)) results in an I. Raşa type inequality, which is associated with the operator \( \overline{V}_\alpha \).
If a random variable $X$ has the beta distribution with parameters $\alpha, \beta > 0$, then we write $X \sim B(\alpha, \beta)$. The corresponding probability density function is

$$
\overline{b}_{\alpha, \beta}(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad \text{for } x \in (0, 1),
$$

and $\overline{b}_{\alpha, \beta}(x) = 0$ for $x \notin (0, 1)$. By convention, we define $B(0, \beta) = \delta_0$ and $B(\alpha, 0) = \delta_1$ for every $\alpha, \beta > 0$.

**Lemma 2.21.** Let $X \sim B(\alpha_1, \beta_1)$ and $Y \sim B(\alpha_2, \beta_2)$ with $\alpha_1, \beta_1, \alpha_2, \beta_2 \geq 0$ satisfying $\alpha_1 + \beta_1 > 0$ and $\alpha_2 + \beta_2 > 0$. We have

$$(\alpha_1 \leq \alpha_2 \text{ and } \beta_1 \geq \beta_2) \implies X \leq_{st} Y.
$$

**Proof.** Let $U \sim \Gamma(\alpha_1, 1)$, $V \sim \Gamma(\alpha_2 - \alpha_1, 1)$, $W \sim \Gamma(\beta_2, 1)$ and $Z \sim \Gamma(\beta_1 - \beta_2, 1)$ be independent random variables. We set $\hat{X} := \frac{U}{U+V}$ and $\hat{Y} := \frac{U+V}{U+V+W}$. Then $\hat{X} \sim B(\alpha_1, \beta_1)$, $\hat{Y} \sim B(\alpha_2, \beta_2)$ and $P(\hat{X} \leq \hat{Y}) = 1$. Theorem 2.23 yields $X \leq_{st} Y$.

By Lemma 2.21 and Theorem 2.23 we obtain one more I. Raşa type inequality:

**Theorem 2.22.** Let $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$ satisfy $(\alpha_1 - \alpha_2)(\beta_1 - \beta_2) \leq 0$. Then

$$
2 \int_0^1 \int_0^1 \overline{b}_{\alpha_1, \beta_1}(u) \cdot \overline{b}_{\alpha_2, \beta_2}(v) \cdot \varphi(u+v) \, du \, dv \leq \int_0^1 \int_0^1 \left[ \overline{b}_{\alpha_1, \beta_1}(u) \cdot \overline{b}_{\alpha_2, \beta_2}(v) + \overline{b}_{\alpha_2, \beta_2}(u) \cdot \overline{b}_{\alpha_1, \beta_1}(v) \right] \cdot \varphi(u+v) \, du \, dv
$$

for all convex functions $\varphi : [0, \infty) \to \mathbb{R}$.

Let $t > 0$. We define an operator $\overline{B}_t : C([0, 1]) \to C([0, 1])$ with

$$
\beta_{t,x}(u) := \overline{b}_{t,1-x}(u) = \frac{t}{1-x} x^{t-1}(1-u)^{1-x-1} \quad x, u \in (0, 1)
$$

as basic functions, by $\overline{B}_t[\varphi](x) = \mathbb{E}[\varphi(X)]$, where $X \sim B(x, (1-x)t)$. Clearly, $\overline{B}_t[\varphi](0) = \varphi(0)$ and $\overline{B}_t[\varphi](1) = \varphi(1)$. If $x \in (0, 1)$, then

$$
\overline{B}_t[\varphi](x) = \int_0^x \varphi(u) \cdot \beta_{t,x}(u) \, du = \int_0^1 \varphi(u) \cdot \frac{t}{1-x} u^{t-1}(1-u)^{1-x-1} \, du.
$$

If $t > 0$, $x, y \in [0, 1]$, $X \sim B(x, (1-x)t)$ and $Y \sim B(y, (1-y)t)$, then by Lemma 2.21 we have that $X \leq_{st} Y$ or $X \geq_{st} Y$. Therefore, by Theorem 2.23 we obtain the following I. Raşa type inequality, which is associated with the operator $\overline{B}_t$:

**Theorem 2.23.** Let $t > 0$ and $x, y \in (0, 1)$. Then

$$
2 \int_0^1 \int_0^1 \beta_{t,x}(u) \cdot \beta_{t,y}(v) \cdot \varphi(u+v) \, du \, dv \leq \int_0^1 \int_0^1 \left[ \beta_{t,x}(u) \cdot \beta_{t,y}(v) + \beta_{t,y}(u) \cdot \beta_{t,x}(v) \right] \cdot \varphi(u+v) \, du \, dv
$$

for all convex functions $\varphi : [0, 1] \to \mathbb{R}$.
Remark 2.24. Operators $\overline{B}_i$ share several nice properties with the Bernstein (Bernstein-Schnabl) operators $B_n$. They are endomorphisms of $C([0,1])$ and $[0,1]$ is compact, they are positive and satisfy $\overline{B}_i(\phi) = \phi$ for every affine $\phi \in C([0,1])$. Indeed, if $t > 0$, $x \in [0,1]$, $X = B(x t, (1 - x)t)$ and $\phi$ is affine, then $\overline{B}_i(\phi)(x) = E[\phi(X)] = \phi(EX) = \phi(x)$. Consequently, the operators $\overline{B}_i$ fit into the theory studied in [2, 3] and they form a new example of operators (see [2, 3]). Examples 1.1, p. 5) satisfying some conditions discussed in [2, 3], conditions (c1) and (c2), p. 15.

It is easy to construct many other operators with these properties. Let $(Y_i)_{i \in \mathbb{N}}$ be any weakly continuous stochastic process, with positive increments $(P(Y_i \leq Y_i) = 1$ and $P(Y_i < Y_i) > 0$ whenever $s < t$ and such that $\lim_{t \to +\infty} Y_i = -\infty$ (weakly) and $\lim_{t \to +\infty} Y_i = +\infty$ (weakly). As an example one may consider the process given by

$$
X_t = \int_0^t f(X_s) ds,
$$

where $f$ is a Lipschitz continuous stochastic process, with positive increments ($N$-variables satisfying some conditions discussed in [2, 4] (see [2], conditions (c1) and (c2), p. 15).

Let $X_0 = 0$, $X_1 = 1$ and $X_u = f(Y_{\text{int}(u)})$ for $u \in (0,1)$. The process $(X_u)_{u \in (0,1)}$ is weakly continuous, and it has positive increments (in particular $X_u \leq u X_1$ for every $0 \leq u \leq v \leq 1$) and $E X_u = u$ for every $u \in [0,1]$. Let $\mu_u = \mu_{X_u}$ be the distribution corresponding to $X_u$. We define the operator $T : C([0,1]) \to C([0,1])$ by $T(\phi)(u) = E \phi(X_u) = \int \phi(x) \mu_u(dx)$. Then $T$ is a Markov operator satisfying $T(\phi) = \phi$ for affine functions. Indeed, $T(\phi)(u) = E \phi(X_u) = E \phi(u)$ for $u \in [0,1]$. Moreover, by Theorem 2.26, the operator $T$ satisfies the following I. Raşa type inequality:

$$
2 \int_0^1 \int_0^1 \phi(x + y) \mu_u(dx) \mu_v(dy) \leq \int_0^1 \int_0^1 \phi(x) \mu_u(dx) \mu_v(dy) + \int_0^1 \int_0^1 \phi(x + y) \mu_v(dx) \mu_u(dy)
$$

for every convex function $\phi \in C([0,1])$ and $u, v \in [0,1]$.

If a random variable $X$ has the Gaussian distribution with the mean $m \in \mathbb{R}$ and the variance $\sigma^2 > 0$, we write $X \sim N(m, \sigma^2)$. The corresponding probability density function is

$$
f_{m, \sigma^2}(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x - m)^2}{2\sigma^2}} \quad (x \in \mathbb{R}).
$$

Lemma 2.25 ([12], p. 14). Let $X \sim N(m_1, \sigma_1^2)$ and $Y \sim N(m_2, \sigma_2^2)$ with $m_1, m_2 \in \mathbb{R}$ and $\sigma_1^2, \sigma_2^2 > 0$. Then

$$
X \leq_{st} Y \quad \iff \quad m_1 \leq m_2 \quad \text{and} \quad \sigma_1^2 \leq \sigma_2^2.
$$

Let $\sigma^2 > 0$. We define the operator $N_{\sigma^2} : D_{N_{\sigma^2}} \to C(\mathbb{R})$ by $N_{\sigma^2}(\phi)(m) = E[\phi(X)]$, where $X \sim N(m, \sigma^2)$. We have

$$
N_{\sigma^2}(\phi)(m) = \int \phi(u) \cdot f_{m, \sigma^2}(u) du = \int \phi(u) \cdot \frac{1}{\sqrt{2\pi} \sigma^2} e^{-\frac{(u - m)^2}{2\sigma^2}} du.
$$

By Lemma 2.25 and Theorem 2.26, we obtain the following I. Raşa type inequality, which is associated with the operators $N_{\sigma^2}$.

Theorem 2.26. If $\sigma^2 > 0$ and $m_1, m_2 \in \mathbb{R}$, then

$$
2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{m_1, \sigma^2}(u) \cdot f_{m_2, \sigma^2}(v) \cdot \phi(u + v) du dv \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ f_{m_1, \sigma^2}(u) \cdot f_{m_1, \sigma^2}(v) + f_{m_2, \sigma^2}(u) \cdot f_{m_2, \sigma^2}(v) \right] \cdot \phi(u + v) du dv \quad (2.15)
$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$.
If $X \sim N(m_1, \sigma^2)$ and $Y \sim N(m_2, \sigma^2)$ are two independent random variables, then $X + Y \sim N(m_1 + m_2, 2\sigma^2)$. Denote by $F_{N(m, \sigma^2)}$ the distribution function corresponding to the random variable $X \sim N(m, \sigma^2)$. By Lemma [2.23] and Theorem [2.3] (applied with $X \sim N(\frac{m_1}{n}, \frac{\sigma}{n})$ and $Y \sim N(\frac{m_2}{n}, \frac{\sigma}{n})$), we obtain the following corollary:

**Corollary 2.27.** Let $\sigma^2 > 0$ and $m_1, m_2 \in \mathbb{R}$. Then

$$F_{N\left(\frac{m_1+m_2}{n}, \sigma^2\right)} \leq \frac{1}{2} \left(F_{N(m_1, \sigma^2)} + F_{N(m_2, \sigma^2)}\right),$$

which is equivalent to

$$2 \int_{-\infty}^{\infty} f_{N\left(\frac{m_1+m_2}{n}, \sigma^2\right)}(u) \cdot \varphi(u) du \leq \int_{-\infty}^{\infty} \left[f_{N(m_1, \sigma^2)}(u) + f_{N(m_2, \sigma^2)}(u)\right] \cdot \varphi(u) du$$

for all convex functions $\varphi: \mathbb{R} \to \mathbb{R}$ and, consequently,

$$N_{\varphi^2}(\varphi) \left(\frac{m_1+m_2}{n}\right) \leq \frac{1}{2} \left[N_{\varphi^2}(\varphi)(m_1) + N_{\varphi^2}(\varphi)(m_2)\right]$$

for all convex functions $\varphi \in D_{N_{\varphi^2}}$.

### 3. The Muirhead type inequality for convex orders

Considering $x_1, \ldots, x_m \in [0, 1]$ ($m \geq 2$) instead of just two numbers $x, y \in [0, 1]$, the following generalization of the I. Raşa inequality (1.1) was proved in [9].

**Theorem 3.1.** Let $m, n \in \mathbb{N}$, $m \geq 2$. Then

$$\sum_{i_1, \ldots, i_m=0}^{n} \left(b_{i_1, i_2} x_1 \cdots b_{i_m, i_1} x_m + \cdots + b_{i_1, i_m} x_1 \cdots + b_{i_m, i_1} x_m\right) - mb_{i_1, i_2} x_1 \cdots b_{i_m, i_1} x_m \varphi \left(\frac{x_{i_1} + \cdots + x_{i_m}}{mn}\right) \geq 0$$

(3.1)

for each convex function $\varphi \in C([0, 1])$ and for all $x_1, \ldots, x_m \in [0, 1]$.

The inequality (3.1) is equivalent to the inequality

$$E\varphi\left(\frac{X^{(1)} + \cdots + X^{(m)}}{mn}\right) \leq \frac{1}{m} \left[E\varphi\left(\frac{X_1^{(1)} + \cdots + X_1^{(i)}}{mn}\right) + \cdots + E\varphi\left(\frac{X_m^{(1)} + \cdots + X_m^{(i)}}{mn}\right)\right],$$

(3.2)

where $X^{(1)}, \ldots, X^{(m)}$ are independent random variables and for all $i \in [1, \ldots, m]$ the random variables $X^{(1)}_i, \ldots, X^{(i)}_i$ are independent with $X^{(i)}_1, \ldots, X^{(i)}_m \sim B(n, x_i)$.

Furthermore, (3.2) is equivalent to the inequality

$$E\varphi\left(X^{(1)} + \cdots + X^{(m)}\right) \leq \frac{1}{m} \left[E\varphi\left(X^{(1)}_1 + \cdots + X^{(1)}_i\right) + \cdots + E\varphi\left(X^{(1)}_m + \cdots + X^{(1)}_m\right)\right]$$

(3.3)

for all convex functions $\varphi: \mathbb{R} \to \mathbb{R}$.

Note that the inequality (3.3) can be written in terms of convex ordering

$$F_{X^{(1)}_1 + \cdots + X^{(1)}_m} \leq \frac{1}{m} \left[F_{X^{(1)}_1 + \cdots + X^{(1)}_m} + \cdots + F_{X^{(1)}_m + \cdots + X^{(1)}_m}\right].$$

(3.4)

In the next theorem we give a sufficient condition for random variables $X^{(1)}, \ldots, X^{(m)}$ to satisfy the inequality (3.4). Then, applying this theorem, we give a new short proof of inequality (3.1).
Theorem 3.2. Let $X^{(1)}, \ldots, X^{(m)}$ be independent random variables with finite means, such that for all $i, j \in \{1, \ldots, m\}$ ($X^{(i)} \leq_{st} X^{(j)}$ or $X^{(j)} \leq_{st} X^{(i)}$). Let $X^{(1)}_{1}, \ldots, X^{(m)}_{1} (i \in \{1, \ldots, m\})$ be independent random variables such that $X^{(i)}_{1}, X^{(i)}_{2}, \ldots, X^{(i)}_{m}$ are identically distributed. Then

$$F_{X^{(1)}_{1} + \cdots + X^{(m)}_{1}} \leq_{st} \frac{1}{m} \left[ F_{X^{(1)}_{1}} + \cdots + F_{X^{(m)}_{1}} \right]. \quad (3.5)$$

Proof. Theorem 3.2 is the special case of Theorem 3.10 (presented later) for $k = n = m$, $(p) = (1, \ldots, 1)$ and $(q) = (m, 0, \ldots, 0)$.

Remark 3.3. The inequality (3.1), which is a generalization of the I. Raşa inequality (1.1), is an immediate consequence of Theorem 3.2 and Lemma 2.2.

Remark 3.4. Taking Poisson, negative binomial, gamma, exponential, beta or Gaussian distributions, as an immediate consequence of Theorem 3.2 and Lemmas 2.9, 2.13, 2.17, 2.25 we can obtain generalizations of the I. Raşa type inequalities (2.9), (2.12), (2.14), (2.15), respectively.

In particular, by (3.5), we obtain the following property of Mirakyan-Szász operators $S_{n}$, which is a generalization of the inequality (2.11).

Corollary 3.5. Let $n, m \in \mathbb{N}$ ($m > 1$) and $x_{1}, \ldots, x_{m} \in [0, \infty)$. Then

$$S_{n}(\varphi) \left( \frac{x_{1} + \cdots + x_{n}}{m} \right) \leq \frac{1}{n} \left[ S_{n}(\varphi)(x_{1}) + \cdots + S_{n}(\varphi)(x_{m}) \right]$$

for all convex functions $\varphi \in D_{c}$.

Note that Theorems 2.3 and 3.2 can be rewritten in terms of convolutions of probability measures as follows.

Theorem 3.6. Let $\mu$ and $\nu$ be two probability distributions with finite first moments, such that $\mu \leq_{st} \nu$ or $\nu \leq_{st} \mu$. Then

$$\mu \ast \nu \leq_{st} \frac{1}{\pi} (\mu \ast \mu + \nu \ast \nu).$$

Theorem 3.7. Let $\mu_{1}, \ldots, \mu_{m}$ be probability distributions with finite first moments, such that for all $i, j \in \{1, \ldots, m\}$ $\mu_{i} \leq_{st} \mu_{j}$ or $\mu_{j} \leq_{st} \mu_{i}$. Then

$$\mu_{1} \ast \cdots \ast \mu_{m} \leq_{st} \frac{1}{m} \left[ (\mu_{1})^{\ast m} + \cdots + (\mu_{m})^{\ast m} \right].$$

Now we are going to give an inequality, which is a generalization of Theorem 3.7. Before we state the theorem, we need to present two definitions.

Definition 3.8. Let $k \in \mathbb{N}$ and let $\Pi$ be the set of all permutations of the set $\{1, \ldots, k\}$. We consider the $k$-tuple $(p) = (p_{1}, \ldots, p_{k})$ of non-negative integers satisfying $p_{1} \geq \ldots \geq p_{k}$. For the given probability distributions $\mu_{1}, \ldots, \mu_{k}$ and $\pi \in \Pi$, we define $\mu_{\pi}^{(p)}$ as the following convolution of probability distributions:

$$\mu_{\pi}^{(p)} := (\mu_{\pi(1)})^{p_{1}} \ast (\mu_{\pi(2)})^{p_{2}} \ast \cdots \ast (\mu_{\pi(k)})^{p_{k}}$$

We also define

$$\mu^{(p)} := \frac{1}{k} \sum_{\pi \in \Pi} \mu_{\pi}^{(p)}.$$
Observe, that if we replace \((\mu_1, \ldots, \mu_k)\) by any permutation \((\mu_{\sigma(1)}, \ldots, \mu_{\sigma(k)})\), then \(\mu^{(p)}\) remains unaltered. In the set of all the \(k\)-tuples \((p)\) introduced in Definition 3.8, we consider the following order.

**Definition 3.9.** We say that \((p) \prec (q)\) if

\[
\begin{align*}
\sum_{i=1}^{k} p_i & = \sum_{i=1}^{k} q_i, \\
\sum_{i=1}^{m} p_i & \leq \sum_{i=1}^{m} q_i \text{ for } m = 1, \ldots, k.
\end{align*}
\]

The above order is a special case of majorization, which has been studied in [6] (before Theorem 45), [8], and many other sources. Our goal is to prove the following theorem.

**Theorem 3.10.** Let \(k \in \mathbb{N}\). Let \(\mu_1, \ldots, \mu_k\) be probability distributions with finite first moments \((\int |x|\mu(dx) < \infty\) for \(l = 1, \ldots, k)\). If \(\mu_1, \ldots, \mu_k\) are pairwise comparable in the usual stochastic order (for each \(1 \leq i, j \leq k\) we have \(\mu_i \leq_{st} \mu_j\) or \(\mu_i \geq_{st} \mu_j\)), then

\[(p) \prec (q) \implies \mu^{(p)} \leq_{cx} \mu^{(q)}.
\]

**Remark 3.11.** Theorem 3.10 is an analogue of Muirhead Inequality (see [6]. Theorem 45 or [8], Section 3G) with positive numbers replaced by probability distributions, multiplication replaced by convolution, and \(\leq\) replaced by \(\leq_{cx}\). Moreover, if \(x_1, \ldots, x_k > 0\), then applying Theorem 3.10 with \(\mu_l = \delta_{l, x_l}\) (for \(l = 1, \ldots, k)\) and the convex function \(\varphi(z) = e^z\), we obtain the classical Muirhead Inequality with integer exponents.

**Example 3.12.** If we apply Theorem 3.10

- for \(k = 2\), \((p) = (1, 1)\) and \((q) = (2, 0)\), then we obtain \(\mu \ast v \leq_{cx} \frac{1}{2}(\mu \ast \mu + v \ast v)\) (Theorem 3.3).
- for \(k = m\), \((p) = (1, \ldots, 1)\) and \((q) = (m, 0, \ldots, 0)\), we get Theorem 3.2
- for \(k = 3\), \((p) = (1, 1, 1)\), \((q) = (2, 1, 0)\), then we get

\[
\mu \ast v \ast k \leq_{cx} \frac{1}{6}(\mu \ast \mu \ast v + \mu \ast \mu \ast \mu + v \ast v \ast \mu + v \ast v \ast k + v \ast v \ast k \ast \mu + k \ast k \ast k \ast v).
\]

(3.6)

In the proof of Theorem 3.10 the following condition \((\ast)\) plays an important role.

**Definition 3.13.** We say that a pair \((p) \prec (q)\) satisfies condition \((\ast)\), if

there exist \(1 \leq l_1 < l_2 \leq k\) such that

\[
\begin{align*}
q_{l_1} & = p_{l_1} + 1, \\
q_{l_2} & = p_{l_2} - 1, \\
q_l & = p_l \text{ for } l \notin \{l_1, l_2\}.
\end{align*}
\]

**Proof (Theorem 3.10).** Theorem 3.10 is an immediate consequence of two subsequent lemmas:

**Lemma 3.14.** If \((p) \prec (q)\) then there exist \((p) = (p^0) \prec (p^1) \prec \cdots \prec (p^i) = (q)\) such that \((p^{i-1}) \prec (p^i)\) satisfies \((\ast)\) for \(i = 1, \ldots, I\).
Proof. Aiming for a contradiction, suppose that the conclusion of the lemma does not hold for some pair \((p, q) < (q)\). Among all such pairs we consider a pair minimizing the value of \(\sum_{m=1}^{k} \sum_{l=1}^{m} (q_l - p_l)\) (note that \((p, q) < (q)\) implies that \(\sum_{m=1}^{k} \sum_{l=1}^{m} (q_l - p_l)\) is a non-negative integer).

We have \((p) \neq (q)\), hence at least one of the non-negative numbers \(\sum_{m=1}^{k} (q_l - p_l)\) (with \(m = 1, \ldots, k\)) is strictly positive. Let \(1 \leq l_1 < l_2 \leq k\) be such that \(\sum_{m=1}^{l_1} (q_l - p_l) > 0\) for \(l_1 \leq m < l_2\) and \(\sum_{m=1}^{l_1} (q_l - p_l) = 0\). We define \((r)\) as follows:

\[
    r_l = \begin{cases} 
        p_l + 1 & \text{for } l = l_1, \\
        p_l - 1 & \text{for } l = l_2, \\
        p_l & \text{for } l \not\in \{l_1, l_2\}.
    \end{cases}
\]

We will show that \(r_1 > r_2 \geq \ldots \geq r_k\). In view of \(p_1 \geq p_2 \geq \ldots \geq p_k\), it is enough to show \(p_{l_1} < r_1\), \(p_{l_2} > p_{l_2+1}\) (if \(l_2 < k\)).

If \(l_1 > 1\), then \(\sum_{i=1}^{l_1-1} (q_i - p_i) \geq 0\), \(\sum_{i=1}^{l_1} (q_i - p_i) = 0\) and \(\sum_{m=1}^{l_1} (q_l - p_l) > 0\) imply \(q_{l_1-1} < p_{l_1-1}\) and \(q_{l_1} > p_{l_1}\). Taking into account \(q_{l_1} \geq q_{l_1-1}\), we obtain \(p_{l_1-1} > p_{l_1}\).

Similarly, if \(l_2 < k\), then \(\sum_{m=1}^{l_2} (q_l - p_l) \geq 0\), \(\sum_{m=1}^{l_2} (q_l - p_l) = 0\) and \(\sum_{m=1}^{l_2-1} (q_l - p_l) > 0\) imply \(q_{l_2+1} \geq p_{l_2+1}\) and \(q_{l_2} < p_{l_2}\). Taking into account \(q_{l_2} \geq q_{l_2+1}\), we get \(p_{l_2} > p_{l_2+1}\).

Now, we will show that \((p) < (r) < (q)\). If \(l_1 \leq m < l_2\), then \(\sum_{m=1}^{l_1} (r_l - p_l) = 1\) and \(\sum_{m=1}^{l_1} (q_l - r_l) = \sum_{m=1}^{l_1} (q_l - p_l) - 1 \geq 0\). If \(m < l_1\) or \(m \geq l_2\), then \(\sum_{m=1}^{l_1} (r_l - p_l) = 0\) and \(\sum_{m=1}^{l_1} (q_l - r_l) = \sum_{m=1}^{l_1} (q_l - p_l) \geq 0\). It follows that \((p) < (r) < (q)\). By the definition of \((r)\) we obtain that \((\star)\) holds for \((p) < (r)\).

We have

\[
    \sum_{m=1}^{k} \sum_{l=1}^{m} (q_l - r_l) = \sum_{m=1}^{k} \sum_{l=1}^{m} (q_l - p_l) - (l_2 - l_1) < \sum_{m=1}^{k} \sum_{l=1}^{m} (q_l - p_l) - (l_2 - l_1),
\]

By the minimality of the pair \((p) < (q)\), we infer that the pair \((r) < (q)\) satisfies the conclusion of the lemma, hence there exist \((r) = (p^{(1)} < p^{(2)} < \cdots < p^{(k)} = (q)\) such that \((p^{(i-1)} < p^{(i)}\) satisfies \((\star)\) for \(i = 2, \ldots, I\). Then \((p) = (p^{(1)}) < (p^{(2)}) < \cdots < (p^{(I)}) = (q)\) and \((p^{(i-1)}) < (p^{(i)})\) satisfies \((\star)\) for \(i = 1, \ldots, I\), which is a contradiction. It follows that the conclusion of the lemma holds for each pair \((p) < (q)\).

**Lemma 3.15.** Let \(k \in \mathbb{N}\). Assume that the probability distributions \(\mu_1, \ldots, \mu_k\) have finite first moments and they are pairwise comparable in the usual stochastic order. If \((p) < (q)\) satisfies condition \((\star)\), then \(\mu^{(p)} \preceq_{st} \mu^{(q)}\).

**Proof.** Reordering if necessary, we may assume without loss of generality, that \(\mu_1 \leq_{st} \mu_2 \leq_{st} \ldots \leq_{st} \mu_k\). Let \(\varphi : \mathbb{R} \to \mathbb{R}\) be a convex function. We need to show

\[
    \frac{1}{k} \cdot \sum_{\pi \in \Pi} \mathbb{E} \varphi(X^{(p)}_\pi) \leq \frac{1}{k} \cdot \sum_{\pi \in \Pi} \mathbb{E} \varphi(X^{(q)}_\pi),
\]

where \(X^{(p)}_\pi\) and \(X^{(q)}_\pi\) are any random variables satisfying \(X^{(p)}_\pi \sim \mu^{(p)}_\pi\) and \(X^{(q)}_\pi \sim \mu^{(q)}_\pi\).

We fix \(1 \leq l_1 < l_2 \leq k\) from condition \((\star)\) for \((p) < (q)\). We have \(q_{l_1} = p_{l_1} + 1 > p_{l_1} \geq p_{l_2} > p_{l_1} = 0\) and \(q_{l_2} = p_{l_2}\) for \(l \not\in \{l_1, l_2\}\). We fix any \(1 \leq u < v \leq k\) and we consider any permutation \(\pi \in \Pi\) satisfying \(\pi(l_1) = u\) and \(\pi(l_2) = v\). Let \(\pi' \in \Pi\) be given by \(\pi'(l_1) = \pi(l_2) = v\) and \(\pi'(l_2) = \pi(l_1) = u\) for \(l \not\in \{l_1, l_2\}\). We define the random variables \(Z^{(l)}_j\) (with \(l \in \{1, \ldots, k\} \setminus \{l_1, l_2\}\) and \(1 \leq s \leq p_l\)) such that

- all random variables \(Z^{(l)}_j\) and random vectors \((U_j, V_j)\) are independent,
Finally, we put

\[ Z_i = \sum_{b \in \{0, 1\}} \sum_{s=1}^{p_i} Z_i^s, \quad A := Z + \sum_{i=1}^{p_1} U_i + \sum_{i=p_1+2}^{p_1+p_2} V_i, \quad D := Z + \sum_{i=1}^{p_1-1} U_i + \sum_{i=p_1+1}^{p_1+p_2} V_i \]

\[ B := Z + \sum_{i=1}^{p_1} U_i + \sum_{i=p_1+1}^{p_1+p_2} V_i, \quad C := Z + \sum_{i=1}^{p_1-1} U_i + \sum_{i=p_1+1}^{p_1+p_2} V_i + U_{p_1+1} + \sum_{i=p_1+2}^{p_1+p_2} V_i. \]

We have: \( B \sim \mu^{(p)}_\ast, C \sim \mu^{(p)}_\ast, A \sim \mu^{(q)}_\ast, D \sim \mu^{(q)}_\ast \) and \( A + D = B + C \). Moreover, \( P(A \leq B, C \leq D) = 1 \). By Remark 2.3 we obtain \( P(\varphi(B) + \varphi(C) \leq \varphi(A) + \varphi(D)) = 1 \), hence

\[ \mathbb{E}(\varphi(X^{(p)}_\ast) + \varphi(X^{(q)}_\ast)) = \mathbb{E}(\varphi(B) + \varphi(C)) \leq \mathbb{E}(\varphi(A) + \varphi(D)) = \mathbb{E}(\varphi(X^{(q)}_\ast) + \varphi(X^{(q)}_\ast)). \quad (3.8) \]

Note that the left side or both sides of (3.8) can be equal to \( +\infty \). However, our assumption about finite first moments of \( \mu_1, \ldots, \mu_k \) ensures that neither left nor right side of (3.8) can be equal to \( -\infty \). Therefore, we may take the sum of the inequalities (3.8) over all \( \pi \in \Pi \) satisfying \( \pi(l_1) = u, \pi(l_2) = v \) and over all \( u < v \). We obtain

\[ \sum_{\pi \in \Pi} \mathbb{E}(\varphi(X^{(p)}_\ast) + \varphi(X^{(q)}_\ast)) \leq \sum_{1 \leq a < b \leq k} \sum_{\pi \in \Pi} \mathbb{E}(\varphi(X^{(p)}_\ast) + \varphi(X^{(q)}_\ast)) \]

\[ = \sum_{1 \leq a < b \leq k} \sum_{\pi \in \Pi} \mathbb{E}(\varphi(X^{(q)}_\ast) + \varphi(X^{(q)}_\ast)), \]

which is equivalent to (3.7). The proof is finished.

**Corollary 3.16.** Taking binomial, Poisson, negative binomial, gamma, exponential, beta or Gaussian distributions, as an immediate consequence of Theorem 3.10 and Lemmas 2.2, 2.9, 2.13, 2.14, 2.21, 2.23, we can obtain several generalizations of the I. Rasa type inequalities. For example, for \( \mu_1 = B(n, x), \mu_2 = B(n, y), \mu_3 = B(n, z) \) (with \( x, y, z \in [0, 1] \) and \( n \in \mathbb{N} \), \( (p) = (1, 1, 1) \) and \( (q) = (2, 1, 0) \) we obtain

\[ \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{l=0}^{n} b_{n,i}(x)b_{n,j}(y)b_{n,l}(z)\varphi \left( \frac{in+jl}{mn} \right) \leq \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{l=0}^{n} b_{n,i}(x)b_{n,j}(x)b_{n,l}(y) + b_{n,i}(x)b_{n,j}(x)b_{n,l}(z) + b_{n,i}(y)b_{n,j}(y)b_{n,l}(y) + b_{n,i}(y)b_{n,j}(z)b_{n,l}(z) + b_{n,i}(z)b_{n,j}(z)b_{n,l}(x) + b_{n,i}(z)b_{n,j}(z)b_{n,l}(y) \varphi \left( \frac{in+jl}{mn} \right) \]

for all convex functions \( \varphi \in C([0, 1]) \) (cf. 3.6 and Theorem 1.7).

In particular, by (3.6), we obtain the following property of Mirakyan-Szász \( S_n(\varphi)(x) \) operators.
Corollary 3.17. Let $n \in \mathbb{N}$ and $x, y, z \in (0, \infty)$. Then

$$S_n(\varphi)(x + y + z) \leq \frac{1}{6}[S_n(\varphi)(2x + y) + S_n(\varphi)(2x + z) + S_n(\varphi)(2y + z) + S_n(\varphi)(2y + x) + S_n(\varphi)(2z + x) + S_n(\varphi)(2z + y)]$$

for all convex functions $\varphi \in D_S$.

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