INTEGRAL TATE MODULES AND SPLITTING OF PRIMES IN TORSION FIELDS OF ELLIPTIC CURVES

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Abstract. Let $E$ be an elliptic curve over a finite field $k$, and $\ell$ a prime number different from the characteristic of $k$. In this paper we consider the problem of finding the structure of the Tate module $T_\ell(E)$ as an integral Galois representations of $k$. We show that the characteristic polynomial of the arithmetic Frobenius and the $j$-invariant of $E$ suffice to this purpose in almost all cases. Hilbert Class Polynomials of imaginary quadratic orders play an important role. We give an application to the splitting of primes in torsion fields arising from elliptic curves over number fields.

1. Introduction

Let $E$ be an elliptic curve over a finite field $k$, and let $\ell$ be a prime $\neq p$. Denote by $G_k$ the absolute Galois group of $k$, with respect to an algebraic closure $\bar{k}$ of $k$, and by $\text{Frob}_k$ the arithmetic Frobenius in $G_k$. The rational $\ell$-adic Tate module $V_\ell(E) = T_\ell(E) \otimes \mathbb{Q}_\ell$ is an isogeny invariant of $E$ which, by a result of Tate, defines a semi-simple $\mathbb{Q}_\ell$-linear representation of $G_k$. Thus, up to isomorphism, $V_\ell(E)$ is determined by the characteristic polynomial $f_E(x) = x^2 - a_E x + |k| - |E(k)|$ of the action of $\text{Frob}_k$, where $a_E$ is the “error term” $|k| + 1 - |E(k)|$.

On the other hand, strictly speaking, the integral Galois representation $T_\ell(E)$ is not an isogeny invariant of $E$, and the sole knowledge of $a_E$ does not suffice in general to determine its isomorphism class.

The main result of this paper (cf. Theorem 6.1) gives a recipe for finding, in almost all cases, a two-by-two matrix describing the action of an arithmetic Frobenius of $G_k$ on $T_\ell(E)$ starting from the polynomial $f_E(x)$ and the $j$-invariant $j_E$ of $E$. We will proceed in two steps: first we observe that the key invariant that allows one to identify the $G_k$-structure of $T_\ell(E) \subset V_\ell(E)$, among those of all $G_k$-stable lattices of $V_\ell(E)$, is the $\ell$-part of the index $b_E = [\text{End}_k(E) : \mathbb{Z}[\pi_E]]$ (cf. Theorem 4.1), where $\mathbb{Z}[\pi_E]$ is the subring of the $k$-endomorphisms of $E$ generated by the Frobenius isogeny $\pi_E : E \to E$ relative to $k$.

Next, the main theorem gives a procedure to recover the index $b_E$, whenever possible, from $f_E(x)$ and $j_E$. Hilbert Class Polynomials associated to imaginary quadratic orders play here a crucial role. In the ordinary case the method we use is known, in the supersingular case it is more delicate and requires some observations in the unstable case where $\text{End}_k(E)$ is “just” an order in an imaginary quadratic field (cf. 4).

Finally, our result has an application to the study of prime splitting in certain field extensions arising from elliptic curves over number fields. Theorem 7.1 gives a reciprocity law for the $N$-torsion field extension $K(E[N])/K$ arising from an elliptic curve $E$ over a number field $K$.

1This has essentially already been observed in [2].
In §2 we study reduction mod \( p \) of Hilbert Class Polynomials and recast Deuring’s Lifting Lemma in this perspective. In §3 we recall all basic facts on elliptic curves over finite fields needed in the paper. In §4 and §6 we study the Galois structure of Tate modules of elliptic curves over finite. In §5 we analyze in detail the so called supersingular unstable case. Finally, in §7 the above mentioned application of our result is explained.

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2. Hilbert Class Polynomials and their mod \( p \) reduction

Thanks to Deuring’s Lifting Lemma, and to the properties of good reduction of characteristic zero CM elliptic curves, roots of Hilbert Class Polynomials in characteristic \( p \) are known to be \( j \)-invariants of elliptic curves with prescribed Complex Multiplication. Expanding on [4] §1, we recall several facts on mod \( p \) reduction of characteristic zero singular \( j \)-invariants.

**Definition 2.1.** Let \( \kappa \) be a field, and \( \bar{\kappa} \) an algebraic closure of it. An elliptic curve \( A \) over \( \kappa \) has Complex Multiplication by an order \( O \) of an imaginary quadratic field (abbreviated CM by \( O \)) if there exists an embedding \( \iota : O \to \text{End}_\kappa(A \otimes_\kappa \bar{\kappa}) \) that is maximal, in the sense that all the endomorphisms of \( A \otimes_\kappa \bar{\kappa} \) lying in \( \iota(O) \otimes \mathbb{Q} \) belong to \( \iota(O) \).

Let \( D \in \mathbb{Z} \) be a negative discriminant, by which we mean an integer \( D < 0 \) such that \( D \equiv 0 \) or \( 1 \mod 4 \), and let \( O_D \) be the imaginary quadratic order of discriminant \( D \), viewed inside a fixed algebraic closure \( \mathbb{Q} \) of \( \mathbb{Q} \). The class group \( \text{Cl}_D \) of \( O_D \) is the group of isomorphism classes of rank one, projective \( O_D \)-modules, the product structure being induced by \( \otimes_{O_D} \). Its cardinality, the class number of \( O_D \), is denoted by \( h_D \).

Once an embedding of \( O_D \) in the field \( \mathbb{C} \) of complex numbers is chosen, elements of \( \text{Cl}_D \) can be thought of as homothety classes of lattices \( \Lambda \subset \mathbb{C} \) such that \( \text{End}_\mathbb{C}(\Lambda) = O_D \). Equivalently, \( \text{Cl}_D \) parametrizes isomorphism classes of complex elliptic curves with CM by \( O_D \). The theory of Complex Multiplication says that the \( j \)-invariants of elements of \( \text{Cl}_D \) are algebraic integers describing a single \( G_{\mathbb{Q}} \)-orbit in \( \mathbb{Q} \subset \mathbb{C} \) and generating an abelian extension of \( \mathbb{Q}(\sqrt{D}) \) known as Hilbert Class Ring attached to \( O_D \) (cf. [3], §9, §11).

The Hilbert Class Polynomial \( P_D(x) \) associated to \( D \) is defined as

\[
P_D(x) = \prod_{a \in \text{Cl}_D} (x - j_{\mathbb{C}/a}),
\]

it has integer coefficients, its degree is equal to \( h_D \), and it is irreducible in \( \mathbb{Q}[x] \).

The choice of the other embedding of \( O_D \) into the complex numbers has the effect of replacing \( j_{\mathbb{C}/a} \) by its complex conjugate, and does not affect the definition of \( P_D(x) \).

Let now \( p \) be a prime number. If \( O \) is any abstract imaginary quadratic order, denote by \( O^{(p)} \) the order of \( O \otimes \mathbb{Q} \) which is maximal at \( p \) and coincides with \( O \).
locally at any prime $\ell \neq p$. Fix a prime $p$ of $\mathbb{Q}$ of residual characteristic $p$, and denote by $\overline{F}_p$ the algebraic closure of $F_p$ given by the residue field of $p$. Since for any $a \in C_p$ the complex number $j_{C/a}$ is an algebraic integer, there exist a number field $K \subset C$ and a $K$-model $E_a$ of $C/a$ which has good reduction $\overline{E}_a$ at $p$ (cf. [6], §2). Up to enlarging $K$, we can assume the CM of $C/a$ be attained by $E_a$.

For any prime $\ell \neq p$, there is a natural identification of $\ell$-adic Tate modules $T_{\ell}(E_a) = T_{\ell}(\overline{E}_a)$ (loc. cit., Lem. 2) using which one can show that the torsion of the cokernel of the reduction map

$$r : \text{End}_K(E_a) \rightarrow \text{End}_{k_p}(\overline{E}_a),$$

is a $p$-group, where $k_p$ denotes the residue field of $K$ at $p$ (cf. [5], 13 §3 Lem. 1). More precisely, it can be shown that (cf. [10], Thm. 4.2):

$$(\text{Im}(r) \otimes \mathbb{Q}) \cap \text{End}_{k_p}(\overline{E}_a) = \text{Im}(r)^{(p)},$$

where the intersection on the left hand side of the equality takes place in $\text{End}_{k_p}(\overline{E}_a) \otimes \mathbb{Q}$. Since the injection induced by extension of scalars

$$\text{End}_{k_p}(\overline{E}_a) \rightarrow \text{End}_{\overline{F}_p}(\overline{E}_a \otimes k_p \overline{F}_p),$$

has torsion free cokernel (cf. [6], §4), we conclude that:

**Proposition 2.2.** Any root of the reduction modulo $p$ of $P_D(x)$ is the $j$-invariant of an elliptic curve $\overline{E}$ over $\overline{F}_p$ with CM by $O_D^{(p)}$.

From the Lifting Lemma of Deuring (cf. [5], 15 §5 Thm. 14), and using the observations preceding Proposition 2.2, we deduce the following converse statement:

**Proposition 2.3.** Let $\overline{E}$ be an elliptic curve over $\overline{F}_p$ which admits CM by $O_D$. Then $j_{\overline{E}}$ is a root of the mod $p$ reduction of $P_D(x)$.

The propositions above are useful to extract information on the size of the endomorphism ring of an elliptic curve in characteristic $p$ from its $j$-invariant. They will be used crucially in §3 and §6.

### 3. Elliptic Curves over Finite Fields

We recall a few basic facts on elliptic curves over finite fields, focusing on their endomorphism rings. For the proofs the reader will be referred to [8], [9], and [10]. The notation established in this section will be enforced in §§4 and §6.

Let $p$ be a prime number, $k$ a finite field of characteristic $p$ and size $p^r$, and $E$ an elliptic curve over $k$. Denote by $R_E$ the ring of $k$-endomorphisms of $E$, and by $\pi_E$ its distinguished element given by the Frobenius isogeny of $E$ relative to $k$. The “error term” $p^r + 1 - |E(k)|$ is denoted by $a_E$, while $b_E$ denotes the index $[R_E : \mathbb{Z}[$\pi_E$]]$, finite or infinite, of the subring generated by $\pi_E$ in $R_E$. If $\tilde{a}$ is the dual isogeny of a given nonzero $a \in R_E$, then the degree of $a$ is the positive integer given by the product $a \tilde{a}$ (cf. [8], III Thm. 6.1). The degree $\pi_E \pi_E$ of the purely inseparable $\pi_E$ is $p^r$ (loc. cit., II Prop. 2.11), and that of the separable $1 - \pi_E$ (loc. cit., III Cor. 5.5) is $(1 - \pi_E)(1 - \pi_E) = |E(k)|$. It follows that the trace $\pi_E + \pi_E$ is equal to the error term $a_E$, and $\pi_E$ satisfies in $R_E$ the polynomial

$$f_E(x) = (x - \pi_E)(x - \pi_E) = x^2 - a_E x + p^r \in \mathbb{Z}[x].$$

The integer $a_E$ is divisible by $p$ if and only if $E$ is supersingular (loc. cit., V §4), the discriminant $\Delta_E = a_E^2 - 4p^r$ of $f_E(x)$ is a non-positive integer (loc. cit., V Thm.
This is to say that \( \pi_E \) is a Weil \( k \)-number lying in the subfield \( \mathbb{Q}(\pi_E) \) of the division algebra \( R_E \otimes \mathbb{Q} \).

Honda-Tate theory of abelian varieties over finite fields, when applied to the above setting, says that the polynomial \( f_E(x) \) determines the \( k \)-isogeny class of \( E \) and gives a description of \( R_E \otimes \mathbb{Q} \) (cf. \([9]\)). As it is well known, we have

\[
R_E \otimes \mathbb{Q} \simeq \begin{cases} 
\mathbb{Q}(\sqrt{\Delta_E}) & \text{if } \Delta_E < 0, \\
\mathbb{Q}_{p,\infty} & \text{if } \Delta_E = 0;
\end{cases}
\]

where \( \mathbb{Q}_{p,\infty} \) denotes the unique \( \mathbb{Q} \)-quaternion ramified at \( p \) and infinity. The ring \( R_E \) is an order of \( R_E \otimes \mathbb{Q} \) containing \( \pi_E \otimes 1 \) that is maximal locally at \( p \) (cf. \([10]\), Thm. 4.2). The index \( b_E \) is finite if and only if \( \Delta_E < 0 \), in which case \( b_E^2 \) divides \( \Delta_E \) and \( R_E \) has discriminant \( \delta_{R_E} = \Delta_E/b_E^2 \). If \( E \) is ordinary, then \( \Delta_E < 0 \) and \( p \) splits completely in the imaginary quadratic field \( R_E \otimes \mathbb{Q} \). One way to see this is by remarking that if \( \Delta_E < 0 \) and the imaginary quadratic field \( R_E \otimes \mathbb{Q} \) has only one prime \( p \) lying above \( p \), then the equation \( \pi_E \tilde{\pi}_E = p^r \) implies \( \pi_E \equiv \tilde{\pi}_E \equiv 0 \mod p \), therefore \( a_E \equiv 0 \mod p \) and \( E \) is supersingular. If \( \Delta_E = 0 \), then \( E \) is supersingular, \( r = 2m \) is even, \( \pi_E = a_E/2 = \pm p^m \), and \( R_E \) is a maximal order of the definite quaternion \( R_E \otimes \mathbb{Q} \) (loc. cit., Thm. 4.2). Conversely, if \( E \) is supersingular then \( \Delta_E \) need not be zero. However, there exists a finite extension \( k'/k \) such that the polynomial \( f_E(x) \) attached to \( E' = E \otimes_k k' \) has discriminant zero. If \( E \) is supersingular and \( \Delta_E < 0 \) then we shall say that \( E \) belongs to the supersingular unstable case, which will be investigated in \([9]\). This terminology is justified from the fact that the ring of rational endomorphisms of \( E \) increases after a suitable base extension. Lastly, since for any finite extension \( k'/k \) the cokernel of the natural inclusion \( R_E \subset R_{E'} \) is a free \( \mathbb{Z}_{l'} \)-module of rank one (cf. \([6]\), §4), we observe that \( E \) has CM by \( R_E \) (cf. Definition \([2,1]\)). Moreover, for the same reason, \( R_{E'} \) is larger than \( R_E \) for some \( k' \), if and only if \( E \) falls in the supersingular unstable case.

4. The action of \( \text{Frob}_k \) on \( T_\ell(E) \) (I)

We keep the notation and assumptions of the previous section, and denote by \( G_k \) the absolute Galois group of \( k \), with respect to an algebraic closure \( \bar{k} \) of \( k \) that will be fixed throughout. If \( \ell \) is a prime number different from \( p \), then the \( \ell \)-adic Tate module \( T_\ell(E) \) of \( E \) is a free \( \mathbb{Z}_{\ell} \)-module of rank two on which \( G_k \) acts continuously and \( \mathbb{Z}_{\ell} \)-linearly. In this section we construct a matrix \( \sigma_E \in \mathbb{M}_2(\mathbb{Z}) \) describing the action of the arithmetic Frobenius \( \text{Frob}_k \in G_k \) on \( T_\ell(E) \), for any \( \ell \neq p \). While the polynomial \( f_E(x) \) suffices to determine the isomorphism class of the \( \mathbb{Q}[G_k] \)-module \( V_\ell(E) = T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \), finding the integral representation \( T_\ell(E) \), which is not in general an isogeny invariant, requires the extra data of the \( \ell \)-part of \( b_E \) (cf. Thm. \([4,1]\) and Rem. \([1,2]\)). Accordingly, the recipe for constructing \( \sigma_E \) involves the extra data of the index \( b_E \). Duke and Tóth have essentially already obtained this result (cf. \([2]\), Thm. 2.1), the proof given here is different from theirs and mainly relies on the fact that if \( \Delta_E < 0 \) then \( T_\ell(E) \) is a free \( R_E \otimes \mathbb{Z}_{\ell} \)-module of rank one (cf. \([6]\), §4 Remark).

The action of \( \text{Frob}_k \) on \( T_\ell(E) \) is the same as that induced from the isogeny \( \pi_E \) via functoriality of \( T_\ell \), thus our problem amounts to make explicit the structure of \( T_\ell(E) \) as a \( \mathbb{Z}_l[T_\ell(\pi_E)] \)-module. To do this, we begin by recalling that the \( \mathbb{Z}_{l} \)-linear extension of the natural map

\[ R_E \supset \psi \longrightarrow T_\ell(\psi) \in \text{End}_{\mathbb{Z}_{l}}(T_\ell(E)) \]
The first statement of the theorem follows. The isomorphism class of Theorem 4.1., what concerns the integral structure determined by ∆, the two-by-two matrix defined as $\pi Z f x$, which is not given by multiplication by any scalar in $Z f$, is an isomorphism of $R V \sqrt{a}$, since $R V E \pi E$, then $E R E Z f Z TATE MODULES AND TORSION FIELDS 5$. Then for any prime $\ell \neq p$ there exists a $Z f$-basis $B$ of $T \ell(E)$ such that the action of Frobenius in the coordinates induced by $B$ is given by $\sigma_E$. If $\Delta_E < 0$, then $T \ell(E)$ is free of rank one over $R E \otimes Z \ell$. Proof. If $\Delta_E = 0$, then $f_E(x) = (x - a_E/2)^2$, therefore the semi-simple operator $V_x(\pi_E)$ is multiplication by $a_E/2$, and so is $T \ell(\pi_E)$. The theorem follows easily in this case. Assume then $\Delta_E < 0$ or, equivalently, assume $b_E$ finite. We begin by showing the last statement of the theorem. The ring $R E \otimes Z \ell$ is a free $Z \ell$-module of rank two, it admits a $Z \ell$-basis of the form $(1, \pi')$, for some $\pi' \in R E \otimes Z \ell$. Observe that the mod $\ell$ reduction of $r_{\ell}(\pi') \in End_{Z \ell[G_\ell]}(T \ell(E))$ is an endomorphism of $T \ell(E)/\ell T \ell(E)$ which is not given by multiplication by any scalar in $Z/\ell Z$. For otherwise $r_{\ell}(\pi') - s$ would be divisible by $\ell$ in $End_{Z \ell[G_\ell]}(T \ell(E))$, for some $s \in Z \ell$, and so would be $\pi' - s$ in $R E \otimes Z \ell$, since $r_{\ell}$ is an isomorphism. This would contradict the fact that $(1, \pi')$ is a $Z \ell$-basis of $R E \otimes Z \ell$. Since $r_{\ell}(\pi')$ mod $\ell$ is not a scalar, there exists $t \in T \ell(E) - \ell T \ell(E)$ such that $r_{\ell}(\pi') \cdot t \notin Z \ell \cdot t + \ell T \ell(E)$. Therefore the pair $(t, r_{\ell}(\pi') \cdot t)$ is a $Z \ell$-basis $T \ell(E)$, since the mod $\ell$ reductions of its components generate $T \ell(E)/\ell T \ell(E)$. It is now clear that the map

$$R E \otimes Z \ell \ni a \mapsto r_{\ell}(a) \cdot t \in T \ell(E)$$

is an isomorphism of $R E \otimes Z \ell$-modules, thus $T \ell(E)$ is free of rank one over $R E \otimes Z \ell$.

Now, since $R E \otimes Z \ell$ is the unique order of $R E \otimes Q \ell = Q \ell[x]/(f_E(x))$ whose $Z \ell$-discriminant is the ideal $(\Delta_E/b_E^2)$, from the isomorphism (5) we deduce that $f_E(x)$ and the $\ell$-part of $b_E$ suffice to determine the isomorphism class of $T \ell(E)$. The first statement of the theorem follows.

Finally, to complete the proof, consider the pair

$$B Z = (1, (\Delta_E + b_E \sqrt{\Delta_E})/2b_E^2),$$

where $\sqrt{\Delta_E} \in R E$ is the square root of $\Delta_E$ given by $2\pi_E - a_E$. A discriminant computation shows that $B Z$ is a $Z$-basis of $R E$, furthermore multiplication by $\pi_E$
on $R_E$ is given in the coordinates induced by $\mathcal{B}_Z$ by the matrix $\sigma_E$. The same matrix a fortiori describes multiplication by $\pi_E \otimes 1$ on $R_E \otimes \mathbb{Z}_\ell$, with respect to the $\mathbb{Z}_\ell$-basis of $R_E \otimes \mathbb{Z}_\ell$ deduced from $\mathcal{B}_Z$. Since $T_\ell(E) \simeq R_E \otimes \mathbb{Z}_\ell$ as $R_E \otimes \mathbb{Z}_\ell$-modules, we conclude that $\sigma_E$ describes the multiplication action of $\pi_E$ on $T_\ell(E)$ as well, in the $\mathbb{Z}_\ell$-coordinates of a suitable basis $\mathcal{B}$. The theorem is proved. \hfill \Box

Remark 4.2. We observe that if $\Delta_E < 0$ and $\ell$ is a prime $\neq p$ which does not divide $b_E$ (for example $\ell \nmid \Delta_E$), then the ring $R_E \otimes \mathbb{Z}_\ell$ is isomorphic to $\mathbb{Z}_\ell[\pi_E]$ and from the theorem it follows that the action of $\text{Frob}_k$ on $T_\ell(E)$ is simply given by the matrix

$$
\begin{pmatrix}
0 & -p^r \\
1 & a_E
\end{pmatrix},
$$

which represents multiplication by $\pi_E$ on $\mathbb{Z}_\ell[\pi_E]$.

Remark 4.3. If $\mathcal{C}$ is a $k$-isogeny class of ordinary elliptic curves with associated polynomial $f_\mathcal{C}(x)$, then a special case of a result of Deligne (cf. [1]) says that there is an equivalence between $\mathcal{C}$ and the category of pairs $(T, F)$ where $T$ is a free $\mathbb{Z}$-module of rank two, and $F : T \to T$ is an endomorphism of $T$ which satisfies the polynomial $f_\mathcal{C}(x)$. It should be pointed out that the matrix $\sigma_E$ constructed above is unrelated to Deligne’s description, even if it defines an endomorphism of $\mathbb{Z}^2$ satisfying $f_\mathcal{C}(x)$. In fact $\sigma_E$, which depends only on the rings $\mathbb{Z}[\pi_E] \subset R_E$ and on the size of $k$, is meaningful only when used to describe the action of $\text{Frob}_k$ on $T_\ell(E)$ or on $E[N]$.

A corollary of the theorem is (cf. [2], Thm. 2.1):

**Corollary 4.4.** Let $N$ be a positive integer not divisible by $p$. Then $\sigma_E$ modulo $N$ describes the action of $\text{Frob}_k$ on the $k$-valued points of the $N$-torsion $E[N]$ of $E$.

**Proof.** This follows readily from the theorem, since $E[N]$ decomposes into the product of its $\ell$-parts, and since there is a natural identification $E[N] = T_\ell(E)/(\ell^n)$ of $G_\ell$-modules for all $n \geq 0$. \hfill \Box

5. **THE SUPERSINGULAR UNSTABLE CASE**

We keep the notation of [3] and investigate in some detail $k$-isogeny classes of supersingular unstable elliptic curves. Our analysis is targeted to the study of the index $b_E = [R_E : \mathbb{Z}[\pi_E]]$. We will see that, possibly up to a factor of two, $b_E$ is a power of $p$ which can be determined from the Weil polynomial $f_E(x)$ of the supersingular unstable curve $E$. This is due to a feature of the supersingular unstable case for which the order $\mathbb{Z}[\pi_E]$ is maximal locally at every prime $\ell \nmid 2p$. We also include a lemma that is useful to decide (in “most cases”) whether or not $2$ divides $b_E$ using the extra data of the $j$-invariant of $E$.

Recall that an elliptic curve $E$ over $k$ is supersingular unstable if it is supersingular and the discriminant $\Delta_E$ of $f_E(x) = x^2 - a_1x + p^r$ is negative. In this case $R_E$ is an imaginary quadratic order, its discriminant is $\Delta_E/b_E^2$. The curve $E$ acquire its quaternionic multiplication only over a non-trivial extension of $k$. Since $R_E \otimes \mathbb{Q}$ embeds in $\text{End}_k(E \otimes_k k) \otimes \mathbb{Q}$, the quaternion over $\mathbb{Q}$ ramified at $p$ and infinity, we have that $R_E \otimes \mathbb{Q}_p$ is a division ring. In particular, supersingular unstable polynomials $f_E(x)$ are subject to the constraint that $p$ not be complete split in $\mathbb{Q}(\sqrt{\Delta_E})$. This requirement, together with $p|a_E$ and $\Delta_E < 0$, characterizes them (cf. [3], Théorème 1). The Weil $k$-numbers showing up as their roots become
real when raised to an appropriate power. They are imaginary quadratic integers of the form $\zeta p^{r/2}$, where $\zeta$ is some root of unity and $p^{r/2}$ a square root of $p$. The non-splitting condition above forces the following few possibilities for $f_E(x)$ (cf. [10], Thm. 4.1):

| $f_E(x)$ | $p$ | $r$ | $\Delta_E$ | $b_E$ |
|----------|-----|-----|-------------|-------|
| $x^2 + p^{2m+1}$ | -   | $2m + 1$ | $-4p^{2m+1}$ | $p^m$ or $2p^m$ |
| $x^2 + p^{2m}$ | $\not\equiv 1 \, \text{mod} \, 4$ | $2m$ | $-4p^{2m}$ | $p^m$ |
| $x^2 \pm p^m x + p^{2m}$ | $\not\equiv 1 \, \text{mod} \, 3$ | $2m$ | $-3p^{2m}$ | $p^m$ |
| $x^2 \pm p^{m+1} x + p^{2m+1}$ | $2$ or $3$ | $2m + 1$ | $-(4 - p)p^{2m+1}$ | $p^m$ |

Table 1. Supersingular unstable Weil polynomials for $k$

The values of the index $b_E$ appearing in the last column of the table are readily computed from the corresponding $\Delta_E$, taking into account that $R_E$ is maximal locally at $p$ (loc. cit., Thm. 4.2). The point is that, except for the case where $p \equiv 3 \, \text{mod} \, 4$, $r = 2m + 1$ is odd, and $f_E(x) = x^2 + p^{2m+1}$, the order $\mathcal{O}[\pi_E]$ is maximal at every prime $\ell \neq p$, and thus $R_E$ is the maximal order and the equality $b_E = p^m$ follows. On the other hand, if $p \equiv 3 \, \text{mod} \, 4$ and $f_E(x) = x^2 + p^{2m+1}$ then $b_E$ is either $2p^m$ or $p^m$, and both cases do arise for suitable $E$ (loc. cit., Thm. 4.2). Therefore $b_E$ is not constant on this isogeny class and cannot be determined from $f_E(x)$ alone. If, however, the $j$-invariant $j_E$ of $E$ is not $1728$ then there is the following lemma:

**Lemma 5.1.** Let $p \equiv 3 \, \text{mod} \, 4$, assume that $r = 2m + 1$ is odd, and let $E$ be an elliptic curve over $k$, with $f_E(x) = x^2 + p^{2m+1}$. If $j_E \neq 1728$, then $E$ has CM by $O_{-p}$ or $O_{-4p}$ but not by both these rings. In the first case $b_E = 2p^m$, in the second one $b_E = p^m$.

**Proof.** Choose a square root $\sqrt{-p}$ of $-p$ inside the imaginary quadratic field $O_{-p} \otimes \mathbb{Q}$. By the maximality of $R_E$ at $p$ already mentioned, the inclusion $\mathbb{Z}[\pi_E] \subset R_E$ induces an embedding $\tau : O_{-4p} \to R_E$, sending $\sqrt{-p}^{2m+1}$ to $\pi_E$. If $\tau$ extends to an embedding of the maximal order $O_{-p}$ then $b_E = 2p^m$ and $E$ has CM by $O_{-p}$, otherwise $b_E = p^m$ and $E$ has CM by $O_{-4p}$.

To prove the lemma, we need to show that if $j_E \neq 1728$ then it is not possible to find two embeddings $\iota_1 : O_{-p} \to \text{End}_k(E \otimes_k \bar{k})$ and $\iota_2 : O_{-4p} \to \text{End}_k(E \otimes_k \bar{k})$ which are both maximal, in the sense of Definition [24]. Since 1728 is the only supersingular invariant when $p = 3$, we may and will continue the proof assuming $p > 3$. We will argue in two steps: first we show that the existence of a maximal embedding $\iota : O \to \text{End}_k(E \otimes_k \bar{k})$, where $O = O_{-p}$ or $O_{-4p}$, ensures the existence of a $k$-form $E_\theta$ of $E$ which is $k$-isogenous to $E$ and for which $R_{E_\theta} \simeq O$. Next, using that $j_E \neq 1728$, we show that the ring of $k$-endomorphisms of any $k$-form of $E$ is isomorphic to $R_E$.

Before carrying out our plan, we make a digression on the study $k$-forms of $E$ (cf. [3] for more details). The absolute Galois group $G_k$ acts in a natural way on the left of the group $\text{Aut}_k(E \otimes_k \bar{k})$. A 1-cocycle $\theta$ of this action defines an elliptic curve $E_\theta$ over $k$ and an isomorphism $\varphi_\theta : E_\theta \otimes_k \bar{k} \to E \otimes_k \bar{k}$ such that, if $\sigma \in G_k$ is the arithmetic Frobenius of $k$, the isogeny $\pi_{E_\theta}$ corresponds to $\theta(\sigma)\pi_E$ under the identification $\text{End}_k(E_\theta \otimes_k \bar{k}) = \text{End}_k(E \otimes_k \bar{k})$ induced by $\varphi_\theta$. In
particular, extension of scalars identifies \(\text{End}_k(E_\theta)\) with the subring of \(\text{End}_k(E \otimes_k \bar{k})\) given by the centralizer of \(\theta(\sigma)\pi_E\). This construction induces a bijection between 
\(H^1(G_k, \text{Aut}_k(E \otimes_k \bar{k}))\) and the set of \(k\)-forms of \(E\), considered up to \(k\)-isomorphism.

Since \(\pi_E^2 = -p^{2m+1}\), the curve \(E\) acquires all of its geometric endomorphisms over the degree 2 extension of \(k\) inside \(\bar{k}\), therefore the Galois action of \(G_k\) on \(\text{End}_k(E \otimes_k \bar{k})\), which is non-trivial, becomes trivial when restricted to its index 2 subgroup. Moreover, the \(G_k\)-invariant subring of this action is the ring \(R_E\) viewed inside \(\text{End}_k(E \otimes_k \bar{k})\) via extension of scalars.

Since any automorphism of \(\text{End}_k(E \otimes_k \bar{k}) \otimes \mathbb{Q}\) is an inner one, we deduce that \(\sigma\) acts on \(\text{End}_k(E \otimes_k \bar{k})\) via the involution \(\sigma(\varphi) = \pi_E \varphi \pi_E^{-1}\). Since \(p > 3\), the unit group of \(\text{Aut}_k(E \otimes_k \bar{k})\) is cyclic of order 2, 4, or 6, and intersects \(R_E\) only in \(\pm 1\). Therefore the action of \(\sigma \in G_k\) on \(\text{Aut}_k(E \otimes_k \bar{k})\) is given by inversion, and evaluation of cocycles at \(\sigma\) induces an isomorphism

\[
H^1(G_k, \text{Aut}_k(E \otimes_k \bar{k})) \simeq \text{Aut}_k(E \otimes_k \bar{k})/\text{Aut}_k(E \otimes_k \bar{k})^2,
\]
so that 
\(H^1(G_k, \text{Aut}_k(E \otimes_k \bar{k}))\) has order two.

Let now \(O\) be either \(O_{-p}\) or \(O_{-4p}\), and let \(i : O \to \text{End}_k(E \otimes_k \bar{k})\) be a maximal embedding. The unique ideal \(I_p\) of \(\text{End}_k(E \otimes_k \bar{k})\) of reduced norm \(p\) is principal and generated by \(i(\sqrt{-p})\). Since the reduced norm of \(\pi_E\) is \(p^{2m+1}\), there exists a unit \(u \in \text{End}_k(E \otimes_k \bar{k})\) such that

\[
i(\sqrt{-p}^{2m+1}) = u\pi_E.
\]
If \(\theta\) is the 1-cocycle of \(G_k\) valued in \(\text{Aut}_k(E \otimes_k \bar{k})\) such that \(\theta(\sigma) = u\), the construction described above leads to a \(k\)-form \(E_\theta\) of \(E\) such that, \(\pi_{E_\theta}\) corresponds to \(u\pi_E\) and the ring \(R_{E_\theta}\) corresponds to \(i(O)\), the centralizer of \(u\pi_E\) in \(\text{End}_k(E \otimes_k \bar{k})\). Therefore \(R_{E_\theta} \simeq O\), and the first step of our program is complete.

To prove the second step, observe that the assumption \(j_E \neq 1728\) ensures that \(-1 \in \text{Aut}_k(E \otimes_k \bar{k})\) is not a square. Therefore the 1-cocycle sending \(\sigma\) to \(-1\) on the one hand describes the only non-trivial \(k\)-form of \(E\), on the other hand, it defines an elliptic curve over \(k\) whose ring of \(k\)-endomorphisms is isomorphic to that of \(E\), since the centralizer of \(-\pi_E\) in \(\text{End}_k(E \otimes_k \bar{k})\) is the same as that of \(\pi_E\). We conclude that if \(j_E \neq 1728\) the two non-isomorphic \(k\)-forms of \(E\) have isomorphic \(k\)-endomorphism rings. This completes the proof of the lemma.

Notice that in the supersingular unstable case the value of \(b_E\) is an invariant associated with the \(k\)-form of \(E \otimes_k \bar{k}\) given by \(E\). Under its assumptions, the lemma basically says that \(b_E\) can in fact be read off from the ring \(\text{End}_k(E \otimes_k \bar{k})\), which is only a geometric invariant of \(E\). Using Propositions \(\ref{prop_2.2}\) and \(\ref{prop_2.3}\) we deduce a criterion to decide whether 2 divides \(b_E\) or not:

**Corollary 5.2.** Let \(p \equiv 3 \mod 4\), \(r = 2m + 1\) be odd, and \(f_E(x) = x^2 + p^{2m+1}\). If \(j_E \neq 1728\), then \(j_E\) is a root of \(P_{-4p}(x)\) mod \(p\) or of \(P_{-p}(x)\) mod \(p\) according to whether \(b_E\) is equal to \(p^m\) or to \(2p^m\), respectively.

We make the following definition:

**Definition 5.3.** Let \(p\) be a prime \(\equiv 3 \mod 4\), and \(r = 2m + 1\) be odd. We will say that \(E\) over \(k\) is **special** if \(f_E(x) = x^2 + p^{2m+1}\) and \(j_E = 1728\).

**Remark 5.4.** It is easy to see that special elliptic curves do exist. For example if \(p \equiv 3 \mod 4\) and \(p \neq 3\), then any elliptic curve \(E\) over the prime field \(\mathbb{F}_p\), with
j_E = 1728 is special. For E is supersingular, since Q(\sqrt{-1}), in which p is inert, embeds inside End_{F_p}(E \otimes_{F_p} F_p), and the only possibility for \( f_E(x) \) is \( x^2 + p \) (cf. Table 1). Similarly, if \( p = 3 \) then one can show that an elliptic curve E over F_3 with \( j_E = 1728 \) has an F_3-form which is special. Furthermore, every special elliptic curve over an (odd-degree) extension of F_p arises from base change from F_p. Lastly, we remark that Lemma 5.1 fails for special elliptic curves: using a computation on Table 1). Similarly, if E is the Hilbert Class Polynomial associated to \( O \) that the former has over the latter is explaining how to recover the index \( b_E \) of \( P \) containing \( O \) and \( O - 4p \) (cf. [2], §). Remark 5.5. If E is any elliptic curve over k with \( f_E(x) = x^2 + p^{2m+1} \), special or not, then in order to find \( b_E \) when \( p \equiv 3 \) mod 4 one may appeal to the two-torsion subgroup \( E[2] \) of E. In fact, \( b_E = 2p^m \) if and only if \( \pi_E - 1 \) is divisible by 2 in \( R_E \), which is to say if and only if \( E[2] \) is all defined over k. This criterion is practical for computations.

6. The action of Frob_k on T_{\ell}(E) (II)

This section completes the study of \( T_{\ell}(E) \) started in §4 and contains a solution to the problem motivating the paper that was exposed in the introduction. More precisely, Theorem 6.1 describes a recipe for making the action of Frob_k on \( T_{\ell}(E) \) explicit in terms of \( a_E, |k|, \) and \( j_E \), where \( \ell \) is a prime \( \neq p \) which is assumed odd if E is special.

Theorem 6.1 is an extension of the previous Theorem 4.1. The extra feature that the former has over the latter is explaining how to recover the index \( b_E \) from the above triple of invariants, at least when E is not special. The method for doing this requires verifying whether the mod p reductions of certain Hilbert Class Polynomials \( P_D(x) \) vanish or not on \( j_E \). In the ordinary case this procedure is known (cf. for example [2], §5). The observation that it remains valid when E is supersingular, and that a uniform statement including both cases can be given, is the main point of the paper, and might contain some novelty.

For a negative discriminant D set

\[
P_D(x) = \prod_{O_D \subset O_D'} P_{D'}(x),
\]

where \( O_D' \) ranges through all the orders of \( O_D \otimes \mathbb{Q} \) containing \( O_D \), and \( P_{D'}(x) \) is the Hilbert Class Polynomial associated to \( O_D' \) (cf. [2]). Notice that an integer \( D' \equiv 0 \) or mod 4 is the discriminant of an order O containing \( O_D \) if and only if \( D'h^2 = D \), for some integer \( h > 0 \), in which case \( h = [O : O_D] \). Extend the above definition of \( P_D(x) \) to all \( D \leq 0 \) by setting \( P_0(x) = 0 \), and \( P_D(x) = 1 \) when \( D < 0 \) and \( D \equiv 2 \) or mod 4. If \( P(x) \in \mathbb{Z}[x] \) is any polynomial, denote its reduction modulo p by \( \bar{P}(x) \in \mathbb{F}_p[x] \).

**Theorem 6.1.** Let E be an elliptic curve over k, define

\[
b = \sup_{h > 0} \{h^2 |_{\Delta_E} \text{ and } \bar{P}_{\Delta_E/h^2}(j_E) = 0\}.
\]

Set moreover
Then for any $\ell \neq p$ there exists a $\mathbb{Z}_l$-basis $B$ of $T_\ell(E)$ such that the action of $\text{Frob}_k$ in the coordinates of $B$ is given by $\tau_E$ provided that $\ell$ is odd if $E$ is special. If $E$ is not special, then $b_E = b$.

**Proof.** We first show that $b_E = b$ provided that $E$ not be special. We then deduce the theorem from Theorem 4.1, since $\tau_E = \sigma_E$. We complete the proof treating the special case separately.

If $\Delta_E = 0$, or equivalently if $b_E$ is infinite, then the equality $b_E = b$ trivially holds, since $\Delta_E = 0$ and $P_0(x) = 0$ by definition. Therefore we continue assuming $\Delta_E < 0$, so that $R_E$ is isomorphic to the imaginary quadratic order $O_{\Delta_E/b_E^2}$. Besides having CM by its endomorphism ring $R_E$ (cf. [3]), the curve $E$ has Complex Multiplication by $O_{\Delta_E/b^2}$, as it follows from the definition of $b$ and from Deuring's Lifting Lemma (cf. Proposition 2.3). Therefore there are inclusions

$$\begin{equation}
O_{\Delta_E/b_E^2} \cong R_E \hookrightarrow \text{End}_k(E \otimes \bar{k}) \hookrightarrow O_{\Delta_E/b^2},
\end{equation}$$

both having torsion free cokernel. In the ordinary case $\text{End}_k(E \otimes \bar{k})$ is an imaginary quadratic order, hence the two inclusions above both are isomorphisms. Thus $\Delta_E/b_E^2 = \Delta_E/b^2$, and $b_E = b$ readily follows. In the remaining case where $E$ is supersonic unstable, the argument just given breaks down, since $\text{End}_k(E \otimes \bar{k})$ is an order in the definite $\mathbb{Q}$-quaternion ramified at $p$, and has rank four over $\mathbb{Z}$.

To show that $b_E = b$ still holds if $E$ is not special we appeal to the maximality properties of $R_E$ investigated in [5].

From Deuring’s Lifting Lemma it follows that, among the orders of $O_{\Delta_E} \otimes \mathbb{Q}$ by which $E$ has CM, the ring $O_{\Delta_E/b^2}$ is the one with minimum discriminant. Therefore if $R_E$ is the maximal order then we must have $O_{\Delta_E/b^2} \cong R_E$, and $b_E = b$ follows. If $R_E$ is not maximal, then $p \equiv 3 \mod 4$, $r = 2m + 1$, $f_E(x) = x^2 + p^{2m+1}$, $b_E = p^m$, and $R_E$ sits in the maximal order of $R_E \otimes \mathbb{Q}$ with index 2 (cf. [4]). The minimality of its discriminant implies that $O_{\Delta_E/b^2}$ is isomorphic to either $R_E$ or to the maximal order. In the first case $b_E = p^m$, in the second one $b_E = 2p^m$. Assuming $E$ not special, Corollary [5] says that $\text{End}_k(j_E) = \text{End}_k(j_E)$ is nonzero, since $b_E = p^m$. Therefore $b \neq 2p^m$, and $b_E = b$ follows.

Assume now that $E$ is special, so that, in particular, $a_E = 0$, $\Delta_E = -4p^{2m+1}$, where $p^{2m+1}$ is the size of $k$, and $b_E \in \{p^m; 2p^m\}$ (cf. [4]). The action of $\text{Frob}_k$ on $T_\ell(E)$, for any $\ell \neq p$, is described by

$$\sigma' = \begin{pmatrix} 2p^{m+1} & -p^{m+1}(1 + 4p) \\ p^m & -2p^m \end{pmatrix} \quad \text{or} \quad \sigma'' = \begin{pmatrix} p^{m+1} & -p^{m+1}(1 + p)/2 \\ 2p^m & -p^{m+1} \end{pmatrix}$$

according to whether $b_E = p^m$ or $2p^m$, respectively (cf. Thm. 1.1). Notice that for any $\ell \neq 2p$, the matrices $\sigma'$ and $\sigma''$ define the same $\text{GL}_2(\mathbb{Z}_\ell)$-conjugacy class. To see this, set

$$\alpha = \begin{pmatrix} 1 & -p \\ 0 & 2 \end{pmatrix} \in \text{GL}_2(\mathbb{Z}[1/2]),$$
and observe that $\alpha \sigma' \alpha^{-1} = \sigma''$. Reasoning as above, we see that $b$ is either $p^m$ or $2p^m$ and $\tau_E$ gives $\sigma'$ or $\sigma''$, respectively. Since for any $\ell \nmid 2p$ these matrices lie in the same $\text{GL}_2(\mathbb{Z}_\ell)$-coinjugacy class, the theorem follows. \hfill \Box

**Corollary 6.2.** Let $N$ be a positive integer not divisible by $p$. Then $\tau_E \mod N$ describes the action of Frobenius on the $k$-valued points of $E[N]$, provided that $N$ is odd if $E$ is special.

The next corollary gives a criterion for the $N$-torsion of $E$ to be entirely defined over $k$, where $N$ is a positive integer not divisible by $p$.

**Corollary 6.3.** Let $N$ be a positive integer not divisible by $p$. Assume that $N > 2$ if $E$ is special. Then Frobenius acts on $E[N]\{k\}$ as multiplication by a scalar in $\mathbb{Z}/N\mathbb{Z}$ if and only if
\begin{equation}
N^2 | \Delta_E \quad \text{and} \quad \overline{\mathcal{P}}_{\Delta_E/N^2}(j_E) = 0.
\end{equation}
Moreover, $E[N]\{k\}$ consists entirely of $k$-rational points if and only if in addition to (7) we have
\begin{equation}
a_E \equiv 2 + \frac{\Delta_E}{N} \mod N^*,
\end{equation}
where $N^* = N$ if $N$ is odd and $N^* = 2N$ if $N$ is even.

**Proof.** If $\Delta_E = 0$, or equivalently if $b_E$ is infinite, then by Theorem 6.1 the action of Frobenius on $E[N]\{k\}$ is given by multiplication by $a_E/2 \mod N$. Since (7) is satisfied and $a_E/2 \equiv 1 \mod N$ is equivalent to $a_E \equiv 2 \mod N^*$, the corollary is proved in this case, and in what follows we assume $\Delta_E < 0$.

By Theorem 6.1 the action of Frobenius on $E[N]\{k\}$ is described by $a_E \mod N$, in a suitable $\mathbb{Z}/N\mathbb{Z}$-basis. Thus Frobenius operates on $E[N]\{k\}$ as scalar multiplication if and only if
\begin{equation}
b_E \quad \text{and} \quad \frac{\Delta_E (b_E^2 - \Delta_E)}{4b_E^2} \quad \text{are both divisible by } N.
\end{equation}

Since $\Delta_E = \delta_{R_E} b_E^2$, where $\delta_{R_E}$ is the discriminant of $R_E$ (cf. 3), we have that (9) is actually equivalent to $N | b_E$. If $E$ is not special, then $b_E = b$ by Theorem 6.1, thus $N | b_E$ is equivalent to (7), and the first part of the corollary follows. If $E$ is special, then $b_E$ is equal to $p^m$ or to $2p^m$, where $|k| = p^{2m+1}$, and $\Delta_E = -4p^{2m+1}$. The first part of the corollary follows, since for any prime-to-$p$ integer $N > 2$ we have that $b_E$ is not divisible by $N$ and that $\Delta_E$ is not divisible by $N^2$.

To prove the second part of the corollary, observe that $E[N]\{k\}$ is entirely defined over $k$ if and only if $\sigma_E \equiv \text{id}_2 \mod N$, where $\text{id}_2$ is the two-by-two identity matrix. By the first part of the corollary, this is equivalent to (7) and to the congruences
\begin{equation}
\frac{a_E b_E - \Delta_E}{2b_E} \equiv \frac{a_E b_E + \Delta_E}{2b_E} \equiv 1 \mod N.
\end{equation}
Since $b_E^2$ divides $\Delta_E$, we see that if (7) is satisfied then
\begin{equation}
\frac{a_E b_E - \Delta_E}{2b_E} \equiv \frac{a_E b_E + \Delta_E}{2b_E} \mod N;
\end{equation}

---

2Elkies’ computation in (14), §2 shows that $b = 2p^m$. We do not need this fact.
therefore \( \sigma_E \equiv \text{id}_2 \mod N \) if and only if (7) holds and
\[
a_E b_E - \Delta_E \equiv 1 \mod N.
\]
Since this last congruence is equivalent to
\[
a_E - \frac{\Delta_E}{b_E} \equiv 2 \mod N^*,
\]
the corollary follows once we remark that if (7) holds, then \( \Delta_E/b_E \equiv \Delta_E/N \mod N^* \), as can be easily verified. \( \square \)

7. An application to elliptic curves over number fields

Let \( K \) be a number field, \( \bar{K} \) an algebraic closure of it, and \( G_K \) the absolute Galois group \( \text{Gal}(\bar{K}/K) \) of \( K \). If \( p \) is a finite prime of \( K \), denote by \( k_p \) its residue field, by \( K_p \) the corresponding completion of \( K \), and by \( G_{K_p} \) the decomposition group of \( G_K \) at \( p \) with respect to the choice of a prime \( \bar{p} \) of \( \bar{K} \) extending \( p \). Denote moreover by \( \bar{k}_p \) the algebraic closure of \( k_p \) given by the residue field of \( \bar{p} \), and by \( G_{k_p} \) the Galois group \( \text{Gal}(\bar{k}_p/k_p) \).

Let \( E \) be an elliptic curve over \( K \). If \( p \) is a prime of \( K \) at which \( E \) has good reduction \( E_p \), then denote by \( a_p \) the error term \( |k_p| + 1 - |E_p(k_p)| \), and by \( \Delta_p \) the discriminant \( a_p^2 - 4|k_p| \). If \( N \) is an integer \( \geq 1 \), the \( N \)-torsion subgroup \( E[N] \) is a finite group scheme over \( K \) of rank \( N^2 \) whose group of \( L \)-valued points, for any \( K \)-algebra \( L \), is given by \( E[N](L) = \text{Hom}(\mathbb{Z}/N\mathbb{Z}, E(L)) \). We will identify \( E[N] \) with \( E[N](\bar{K}) \), an abelian group isomorphic to \( \langle \mathbb{Z}/N\mathbb{Z} \rangle^2 \) equipped with a continuous action of \( G_K \). By Galois Theory, the representation
\[
\rho_{E[N]} : G_K \longrightarrow \text{Aut}(E[N]) \simeq \text{GL}_2(\mathbb{Z}/N\mathbb{Z})
\]
defines a finite Galois extension \( K(E[N])/K \) whose Galois group is \( \text{Im}(\rho_{E[N]}) \). As it is well known, \( \rho_{E[N]} \) is unramified at every finite prime \( \mathfrak{p} \) of \( K \) not dividing \( N \) and at which \( E \) has good reduction. More precisely, the reduction map induces an identification
\[
E[N](\bar{K}) = E_p[N](\bar{k}_p)
\]
which is equivariant with respect to the Galois actions of \( G_K \) and \( G_{k_p} \). One can see this observing that there is a natural identification \( E[N](\bar{K}) = E[N](\bar{k}_p) \), where \( \bar{k}_p \) is the completion of \( k_p \) at \( \bar{p} \), and using Lemma 2 of [3] to get \( E[N](\bar{k}_p) = E_p[N](\bar{k}_p) \).

The results of [3] can therefore be applied to describe the conjugacy class of \( \rho_{E[N]}(\text{Frob}_p) \) in \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) in terms of \( a_p \), of the size of \( k_p \), and of the \( j \)-invariant \( j_{E_p} \), provided that \( N \) satisfies the usual constraint of being odd if \( p \) is one of the finitely many primes of \( K \) such that \( E_p \) is special. We will confine ourselves to the following application of Corollary 7.3 and of the Galois equivariant identity (10), which gives a reciprocity law for the extension \( K(E[N])/K \):

**Theorem 7.1.** Let \( N \) be a positive integer, and \( p \) a finite prime of \( K \) not dividing \( N \) and at which \( E \) has good reduction. If \( N = 2 \), assume furthermore that \( E_p \) is not special. Then \( p \) splits completely in \( K(E[N])/K \) if and only if the following conditions hold:

1. \( N^2 | \Delta_p \) and \( \mathcal{P}_{\Delta_p/N^2}(j_E) \equiv 0 \mod p \);
2. \( a_p \equiv 2 + \frac{\Delta_p}{N} \mod N^* \);

where \( N^* = N \) if \( N \) is odd and \( N^* = 2N \) if \( N \) is even.
Under the assumptions of the theorem, from Corollary 6.3 it also follows that \( \text{Frob}_p \) acts as scalar multiplication on \( E[N] \) if and only if condition i) in the theorem holds. Notice that in the special case where \( N = \ell \) is prime this gives a criterion for deciding whether or not \( \text{Frob}_p \) acts on \( E[\ell] \) in a semi semi-simple fashion in the critical case when \( \ell | \Delta_p \), i.e., when such an action has only one eigenvalue. This problem has been emphasized in [7].

References

1. P. Deligne, Variétés abéliennes ordinaires sur un corps fini, Invent. math. 8, 238-243 (1969).
2. W. Duke, Á. Tóth, The splitting of primes in division fields of elliptic curves, Experiment. Math. 11 (2002), no. 4, 555-565 (2003).
3. D. Cox, Primes of the form \( x^2 + ny^2 \), John Wiley & Sons, 1989 New York.
4. N. Elkies, The existence of infinitely many supersingular primes for every elliptic curve over \( \mathbb{Q} \), Invent. math. 89, 561-567 (1987).
5. S. Lang, Elliptic functions, 1987 Springer-Verlag, second edition.
6. J.-P. Serre, J. Tate, Good reduction of abelian varieties, Annals of Mathematics, 88, No. 3, 492-517 (1968).
7. G. Shimura, A reciprocity law in non–solvable extensions, J. Reine Angew. Math 221 (1966), 209–220.
8. J. H. Silverman, The arithmetic of elliptic curves, Graduate texts in Mathematics 106, 1986 Springer-Verlag, New York.
9. J. Tate, Classes d’isogénie des variétés abéliennes sur un corps fini, Sém. Bourbaki 21e année, 1968/69, no 352.
10. W.C. Waterhouse, Abelian varieties over finite fields, Ann. scient. Éc. Norm. Sup., 4e série, t. 2, 1969, 521–560.
11. W.C. Waterhouse, J.S. Milne, Abelian varieties over finite fields, 1969 Number Theory Institute (Proc. Sympos. Pure Math., Vol. XX, State Univ. New York, Stony Brook, N.Y., 1969), pp. 53–64. Amer. Math. Soc., Providence, R.I., 1971