Homogeneous geodesics of left invariant Finsler metrics

Dariush Latifi

Department of Mathematics, Mohaghegh Ardabili University, P.O. Box. 56199-11367, Ardabil, Iran
dlatifi@gmail.com

Abstract

In this paper, we study the set of homogeneous geodesics of a left-invariant Finsler metric on Lie groups. We first give a simple criterion that characterizes geodesic vectors. As an application, we study some geometric properties of bi-invariant Finsler metrics on Lie groups. In particular a necessary and sufficient condition that left-invariant Randers metrics are of Berwald type is given. Finally a correspondence of homogeneous geodesics to critical points of restricted Finsler metrics is given. Then results concerning the existence homogeneous geodesics are obtained.

Keywords: Invariant Finsler metrics, Homogeneous geodesics, Geodesic vectors, Randers spaces.
PACS numbers: 02.40.Ky, 02.40.Sf, 4520J
Mathematics Subject Classifications: 53C60; 53C35; 53C30; 53C22

1 Introduction

A classical problem of differential geometry is to study geodesics of Riemannian manifolds \((M, g)\). Of particular interest are geodesics with some special properties, for example homogeneous geodesics. A geodesic of a Riemannian manifold \((M, g)\) is called homogeneous if it is an orbit of a one-parameter group of isometries of \(M\). For results on homogeneous geodesics in homogeneous Riemannian manifolds we refer to [8], [14], [12], [11]. Homogeneous geodesics have important applications to mechanics. For example, the equation of motion of many systems of classical mechanics reduces to the geodesic equation in an appropriate Riemannian manifold \(M\). Geodesics of left-invariant Riemannian metrics on Lie groups were studied by V. I. Arnold extending Euler’s theory of rigid-body motion [1]. A major part
of V. I. Arnold’s paper is devoted to the study of homogeneous geodesics. Homogeneous geodesics are called by V. I. Arnold “relative equilibria.” The description of such relative equilibria is important for qualitative description of the behavior of the corresponding mechanical system with symmetries. There is a big literature in mechanics devoted to the investigation of relative equilibria. Homogeneous geodesics are interesting also in pseudo-Riemannian geometry and light-like homogeneous geodesics are of particular interest. For results on homogeneous geodesics in homogeneous pseudo-Riemannian manifolds we refer for example to [18], [19], [21], [7], [3], [6]. In [18], [21] and [7], the authors study plane-wave limits (Penrose limits) of homogeneous spacetimes along light-like homogeneous geodesics.

About the existence of homogeneous geodesics in a general homogeneous Riemannian manifold, we have, at first, a result due to V. V. Kajzer who proved that a Lie group endowed with a left-invariant Riemannian metric admits at least one homogeneous geodesic [10]. More recently O. Kowalski and J. Szenthe extended this result to all homogeneous Riemannian manifolds [13]. An extension of result of [13] to reductive homogeneous pseudo-Riemannian manifolds has been also obtained [21], [6]. Homogeneous geodesics of left-invariant Lagrangians on Lie groups were studied by J. Szenthe [24]. In this paper, we study the set of homogeneous geodesics of a left-invariant Finsler metric on Lie groups.

2 Preliminaries

2.1 Finsler spaces

In this section, we recall briefly some known facts about Finsler spaces. For details, see [2], [23], [4].

Let \( M \) be a \( n \)-dimensional \( C^\infty \) manifold and \( TM = \bigcup_{x \in M} T_x M \) the tangent bundle. If the continuous function \( F : TM \to R_+ \) satisfies the conditions that it is \( C^\infty \) on \( TM \setminus \{0\} \); \( F(tu) = tF(u) \) for all \( t \geq 0 \) and \( u \in TM \), i.e, \( F \) is positively homogeneous of degree one; and for any tangent vector \( y \in T_x M \setminus \{0\} \), the following bilinear symmetric form \( g_y : T_x M \times T_x M \to R \) is positive definite:

\[
g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t}[F^2(x, y + su + tv)]|_{s=t=0},
\]

then we say that \( (M, F) \) is a Finsler manifold.

Let

\[
g_{ij}(x, y) = \left( \frac{1}{2} F^2 \right)_{y^i y^j}(x, y).
\]

By the homogeneity of \( F \), we have

\[
g_y(u, v) = g_{ij}(x, y)u^i v^j, \quad F(x, y) = \sqrt{g_{ij}(x, y)y^i y^j}.
\]
Let $\gamma : [0, r] \rightarrow M$ be a piecewise $C^\infty$ curve. Its integral length is defined as

$$L(\gamma) = \int_0^r F(\gamma(t), \dot{\gamma}(t)) dt.$$ 

For $x_0, x_1 \in M$ denote by $\Gamma(x_0, x_1)$ the set of all piecewise $C^\infty$ curve $\gamma : [0, r] \rightarrow M$ such that $\gamma(0) = x_0$ and $\gamma(r) = x_1$. Define a map $d_F : M \times M \rightarrow [0, \infty)$ by

$$d_F(x_0, x_1) = \inf_{\gamma \in \Gamma(x_0, x_1)} L(\gamma).$$

Of course, we have $d_F(x_0, x_1) \geq 0$, where the equality holds if and only if $x_0 = x_1$; $d_F(x_0, x_1) \leq d_F(x_0, x_1) + d_F(x_1, x_2)$. In general, since $F$ is only a positive homogeneous function, $d_F(x_0, x_1) \neq d_F(x_1, x_0)$, therefore $(M, d_F)$ is only a non-reversible metric space.

Let $\pi^*TM$ be the pull-back of the tangent bundle $TM$ by $\pi : TM \setminus \{0\} \rightarrow M$. Unlike the Levi-Civita connection in Riemannian geometry, there is no unique natural connection in the Finsler case. Among these connections on $\pi^*TM$, we choose the Chern connection whose coefficients are denoted by $\Gamma^i_{jk}$ (see [2,p.38]). This connection is almost $g$-compatible and has no torsion. Here $g(x, y) = g_{ij}(x, y)dx^i \otimes dx^j = (\frac{1}{2}F^2)_{y'j}y^i dx^i \otimes dx^j$ is the Riemannian metric on the pulled-back bundle $\pi^*TM$.

The Chern connection defines the covariant derivative $D_T U$ of a vector field $U \in \chi(M)$ in the direction $V \in T_pM$. Since, in general, the Chern connection coefficients $\Gamma^i_{jk}$ in natural coordinates have a directional dependence, we must say explicitly that $D_T U$ is defined with a fixed reference vector. In particular, let $\sigma : [0, r] \rightarrow M$ be a smooth curve with velocity field $T = T(t) = \dot{\sigma}(t)$. Suppose that $U$ and $W$ are vector fields defined along $\sigma$. We define $D_T U$ with reference vector $W$ as

$$D_T U = \left[ \frac{dU^i}{dt} + U^j T^k (\Gamma^i_{jk})(\sigma, W) \right] \frac{\partial}{\partial x^i} \big|_{\sigma(t)}.$$ 

A curve $\sigma : [0, r] \rightarrow M$, with velocity $T = \dot{\sigma}$ is a Finslerian geodesic if

$$D_T \left[ \frac{T}{F(T)} \right] = 0,$$

with reference vector $T$.

We assume that all our geodesics $\sigma(t)$ have been parameterized to have constant Finslerian speed. That is, the length $F(T)$ is constant. These geodesics are characterized by the equation

$$D_T T = 0,$$

with reference vector $T$.

Since $T = \frac{d\sigma^i}{dt} \frac{\partial}{\partial x^i}$, this equation says that

$$\frac{d^2 \sigma^i}{dt^2} + \frac{d\sigma^j}{dt} \frac{d\sigma^k}{dt} (\Gamma^i_{jk})(\sigma, T) = 0.$$
If $U, V$ and $W$ are vector fields along a curve $\sigma$, which has velocity $T = \dot{\sigma}$, we have the derivative rule

$$\frac{d}{dt} g_w(U, V) = g_w(D_T U, V) + g_w(U, D_T V)$$

whenever $D_T U$ and $D_T V$ are with reference vector $W$ and one of the following conditions holds:

i) $U$ or $V$ is proportional to $W$, or

ii) $W = T$ and $\sigma$ is a geodesic.

2.2 Left-invariant Finsler metrics on Lie groups

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g} = T_eG$. We may identify the tangent bundle $TG$ with $G \times \mathfrak{g}$ by means of the diffeomorphism that sends $(g, X) \to (L_g)^* X \in T_gG$.

**Definition 2.1** A Finsler function $F : TG \to \mathbb{R}^+$ will be called $G$-invariant if $F$ is constant on all $G$-orbits in $TG = G \times \mathfrak{g}$; that is, $F(g, X) = F(e, X)$ for all $g \in G$ and $X \in \mathfrak{g}$.

The $G$-invariant Finsler functions on $TG$ may be identified with the Minkowski norms on $\mathfrak{g}$. If $F : TG \to R_+$ is an $G$-invariant Finsler function, then we may define $\tilde{F} : \mathfrak{g} \to R_+$ by $\tilde{F}(X) = F(e, X)$, where $e$ denotes the identity in $G$. Conversely, if we are given a Minkowski norm $\tilde{F} : \mathfrak{g} \to R_+$, then $\tilde{F}$ arises from an $G$-invariant Finsler function $F : TG \to R_+$ given by $F(g, X) = \tilde{F}(X)$ for all $(g, X) \in G \times \mathfrak{g}$.

Let $G$ be a connected Lie group, $L : G \times G \to G$ the action being defined by the left-translations $L_g : G \to G$, $g \in G$ and $T_L : G \times TG \to TG$ the action given by the tangent linear maps $T_L g : TG \to TG$, $g \in G$ of the left-translations. A smooth vector field $X : TG - \{0\} \to TTG$ is said to be left-invariant if

$$TTL_g \circ X \circ TL_g^{-1} = X \quad \forall g \in G.$$

By a classical argument of calculus of variation we have the following proposition.

**Proposition 2.2** If $F : TG \to R_+$ is a left-invariant Finsler metric then its geodesic spray $X$ is left-invariant as well.

3 Homogeneous geodesics of left invariant Finsler metrics

**Definition 3.1** Let $G$ be a connected Lie group, $\mathfrak{g} = T_eG$ its Lie algebra identified with the tangent space at the identity element, $\tilde{F} : \mathfrak{g} \to R_+$ a Minkowski
norm and $F$ the left-invariant Finsler metric induced by $\tilde{F}$ on $G$. A geodesic \( \gamma : R_+ \rightarrow G \) is said to be homogeneous if there is a $Z \in \mathfrak{g}$ such that \( \gamma(t) = \exp(tZ)\gamma(0) \), $t \in R_+$ holds. A tangent vector $X \in T_eG - \{0\}$ is said to be a geodesic vector if the 1-parameter subgroup $t \rightarrow \exp(tX)$, $t \in R_+$, is a geodesic of $F$.

The geodesic defined by a geodesic vector is obviously a homogeneous one. Conversely, let $\gamma$ be a geodesic with $\gamma(0) = g$ which is homogeneous with respect to a 1-parameter group of left-translations, namely

\[
\gamma(t) = \exp(tY)g, \quad t \in R_+, 
\]

then a homogeneous geodesic $\tilde{\gamma}$ is given by

\[
\tilde{\gamma}(t) = L_{g^{-1}} \circ \gamma(t) = L_{g^{-1}} \circ R_g \circ \exp(tY) \\
= \exp(Ad(g^{-1})tY).e = \exp(Ad(g^{-1})tY)\tilde{\gamma}(0),
\]

which means that $X = Ad(g^{-1})Y$ is a geodesic vector.

For results on homogeneous geodesics in homogeneous Finsler manifolds we refer to [16]. The basic formula characterizing geodesic vector in the Finslerian case was derived in [16], Theorem 3.1. In the following theorem we present a new elementary proof of this theorem for left invariant Finsler metrics on Lie groups.

**Theorem 3.2** Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$, and let $F$ be a left-invariant Finsler metric on $G$. Then $X \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if

\[
g_X([X,Z]) = 0
\]

holds for every $Z \in \mathfrak{g}$.

Proof: Following the conventions of [9] a left-invariant vector field associated to an element $X$ in $T_eG$ is denoted by $\tilde{X} : G \rightarrow TG$; that is $\tilde{X}_x = L_{xe^*}X$. For any left invariant vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$ on $G$, we have

\[
\tilde{Y}g_{\tilde{X}}(\tilde{Z},\tilde{X}) = g_{\tilde{X}}(D_{\tilde{Y}}\tilde{Z},\tilde{X}) + g_{\tilde{X}}(\tilde{Z},D_{\tilde{Y}}\tilde{X}) \quad \text{with reference } \tilde{X} \quad (1)
\]

Similarly,

\[
\tilde{Z}g_{\tilde{X}}(\tilde{Y},\tilde{X}) = g_{\tilde{X}}(D_{\tilde{Z}}\tilde{Y},\tilde{X}) + g_{\tilde{X}}(\tilde{Y},D_{\tilde{Z}}\tilde{X}) \quad (2)
\]

\[
\tilde{X}g_{\tilde{X}}(\tilde{Z},\tilde{X}) = g_{\tilde{X}}(D_{\tilde{X}}\tilde{Z},\tilde{X}) + g_{\tilde{X}}(\tilde{Z},D_{\tilde{X}}\tilde{X}) \quad (3)
\]

All covariant derivatives have $\tilde{X}$ as reference vector.

Subtracting (2) from the summation of (1) and (3) we get

\[
g_{\tilde{X}}(\tilde{Z},D_{\tilde{X}+\tilde{Y}}\tilde{X}) + g_{\tilde{X}}(\tilde{X} - \tilde{Y},D_{\tilde{Z}}\tilde{X}) = \tilde{Y}g_{\tilde{X}}(\tilde{Z},\tilde{X}) - \tilde{Z}g_{\tilde{X}}(\tilde{Y},\tilde{X}) + \tilde{X}g_{\tilde{X}}(\tilde{Z},\tilde{X}) \\
- g_{\tilde{X}}(\tilde{Y},\tilde{Z},\tilde{X}) - g_{\tilde{X}}(\tilde{X},\tilde{Z},\tilde{X}),
\]
where we have used the symmetry of the connection, i.e., $DZ\tilde{X} - D\tilde{X}Z = [\tilde{Z}, \tilde{X}]$.

Set $\tilde{Y} = \tilde{X} - \tilde{Z}$ in the above equation, we obtain

$$2g_X(\tilde{Z}, D\tilde{X}\tilde{X}) = 2\tilde{X}g_X(\tilde{Z}, \tilde{X}) - \tilde{Z}g_X(\tilde{X}, \tilde{X}) - 2g_X([\tilde{X}, \tilde{Z}], \tilde{X}).$$

(4)

Since $F$ is left-invariant, $dL_x$ is a linear isometry between the spaces $T_eG = \mathfrak{g}$ and $T_xG$, $\forall x \in G$. Therefore for any left-invariant vector field $\tilde{X}, \tilde{Z}$ on $G$, we have

$$g_X(\tilde{Z}, \tilde{X}) = g_X(Z, X)$$

i.e., the functions $g_X(\tilde{Z}, \tilde{X})$, $g_X(\tilde{X}, \tilde{X})$ are constant. Therefore from (4) the following is obtained

$$g_X(\tilde{Z}, D\tilde{X}\tilde{X}) |_e = -g_X([\tilde{X}, \tilde{Z}], \tilde{X}) |_e = -g_X([X, Z], X).$$

Consequently the assertion of the theorem follows. □

The following Proposition is well known for left-invariant Riemannian metrics.

**Proposition 3.3** Let $G$ be a connected Lie group furnished with a left-invariant Finsler metric $F$. Then the following are equivalent,

1. $F$ is right-invariant, hence bi-invariant.
2. $F$ is $\text{Ad}(G)$ invariant.
3. $g_Y([X, U], V) + g_Y(U, [X, V]) + 2C_Y([X, Y], U, V) = 0$, $\forall Y \in \mathfrak{g} - \{0\}, X, U, V \in \mathfrak{g}$, where $C_Y$ is the Cartan tensor of $F$ at $Y$.

If the Finsler structure $F$ is absolutely homogeneous, then one also has.

4. The inversion map $g \rightarrow g^{-1}$ is an isometry of $G$.

Proof: The equivalence of the first two assertion is routine, and we omit the details. The equivalence between (1) and (3) is a result of S. Deng and Z. Hou [5]. If $F$ is absolutely homogeneous, one can check quite easily that (4) is equivalent to (1). □

**Corollary 3.4** If $G$ is a Lie group endowed with a bi-invariant Finsler metric, then the geodesics through the identity of $G$ are exactly one-parameter subgroups.

Proof: Since $F$ is bi-invariant, we have

$$g_Y([X, U], V) + g_Y(U, [X, V]) + 2C_Y([X, Y], U, V) = 0$$

$\forall Y \in \mathfrak{g} - \{0\}, X, U, V \in \mathfrak{g}$. It follows from the homogeneity of $F$ that $C_Y(Y, V, W) = 0$. So we have

$$g_Y([X, Y], Y) = 0.$$

The result now follows from the Theorem 3.2. □

A connected Finsler space $(M, F)$ is said to be symmetric [15] if to each $p \in M$ there is associated an isometry $s_p : M \rightarrow M$ which is
(i) involutive ($s_p^2$ is the identity).

(ii) has $p$ as an isolated fixed point, that is, there is a neighborhood $U$ of $p$ in which $p$ is the only fixed point of $s_p$.

$s_p$ is called the symmetry of the point $p$.

**Theorem 3.5** Suppose $G$ is a Lie group with a bi-invariant absolutely homogeneous Finsler metric, then $G$ is a symmetric Finsler space.

Proof: Consider the smooth mapping $f \colon G \to G$, $f : x \mapsto x^{-1}$. Then $f_* : T_xG \to T_xG$ maps a vector $\xi \in T_xG$ to $-\xi$; in particular, $df$ is an isometry of $g = T_xG$. Clearly, $f = R_{g^{-1}}f_{L_{g^{-1}}}$. Therefore, $df : T_xG \to T_{g^{-1}}G$ is an isometry for any $g \in G$.

Let $s_g(x) = gx^{-1}g$, $g, x \in G$. The mapping $s_g$ is an isometry, because $s_g = R_gf_{L_{g^{-1}}}$. Thus, $s_g$ is an isometry of $G$, obviously fixing the point $g$. Furthermore, $s_g^2(x) = g(x^{-1})g^{-1}g = x$. To show that $s_g$ is the symmetry used in the definition of a symmetric Finsler space, it suffices to show that $(s_g)_s\xi = -\xi$ whenever $\xi \in T_xG$.

Let us start with the case $g = e$. Let $\xi = \frac{d}{dt}\gamma(t)|_{t=0} \in T_eG$ where $\gamma(t)$ is a one-parameter subgroup of $G$. Then $\gamma(t)^{-1} = \gamma(-t)$, and $(s_g)_s\xi = \frac{d}{dt}\gamma(-t) = -\xi$. Now, if $\xi$ is in $T_xG$ for an arbitrary $g \in G$, then $ds_g = dR_gdf_{L_{g^{-1}}}$, so $ds_g(\xi) = dR_g(df(dR_{g^{-1}}(\xi))) = dR_g(-dR_{g^{-1}}(\xi)) = -\xi$. \hfill $\square$

Let $M$ be a smooth n-dimensional manifold, a Randers metric on $M$ consists of a Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ on $M$ and a 1-form $b = b_i dx^i$. [2], [22]. Here $\tilde{a}$ and $b$ define a function $F$ on $TM$ by

$$F(x, y) = \alpha(x, y) + \beta(x, y) \quad \quad x \in M, y \in T_xM$$

where $\alpha(x, y) = \sqrt{\tilde{a}_{ij}y^i y^j}$, $\beta(x, y) = b_i(x)y^i$. $F$ is Finsler structure if $\|b\| = \sqrt{b^i b^j} < 1$ where $b^i = \tilde{a}^{ij}b_j$, and $(\tilde{a}^{ij})$ is the inverse of $(\tilde{a}_{ij})$. The Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ induces the musical bijections between 1-forms and vector fields on $M$, namely $b : T_xM \to T^*_xM$ given by $y \mapsto \tilde{a}_x(y, \cdot)$ and its inverse $\tilde{z} : T^*_xM \to T_xM$. In the local coordinates we have

$$(y^b)_i = \tilde{a}_{ij}y^j \quad y \in T_xM$$

$$(\tilde{z}^\theta) = \tilde{a}^{ij}\theta_j \quad \theta \in T^*_xM$$

Now the corresponding vector field to the 1-form $b$ will be denoted by $b^\theta$, obviously we have

$\|b\| = \|b^\theta\|$ and $\beta(x, y) = (b^\theta)_i(y) = \tilde{a}_x(b^\theta, y)$. Thus a Randers metric $F$ with Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ and 1-form $b$ can be showed by

$$F(x, y) = \sqrt{\tilde{a}_x(y, y)} + \tilde{a}_x(b^\theta, y) \quad x \in M, y \in T_xM$$

where $\tilde{a}_x(b^\theta, b^\theta) < 1 \quad \forall x \in M$. 

7
Let $F(x, y) = \sqrt{a_x(y, y)} + \tilde{a}_x(X, y)$ be a left invariant Randers metric. It is easy to check that the underlying Riemannian metric $\tilde{a}$ and the vector field $X$ are also left invariant.

**Theorem 3.6** Let $G$ be a Lie group with a left-invariant Randers metric $F$ defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ and the vector field $X$. Then the Randers metric $F$ is of Berwald type if and only if $\text{ad}_X$ is skew-adjoint with respect to $\tilde{a}$ and $\bar{a}(X, [g, g]) = 0$.

Proof: For all $Y, Z \in \mathfrak{g}$,

$$2\bar{a}(Y, \nabla_Z X) = \tilde{a}(Z, [Y, X]) + \tilde{a}(X, [Y, Z]) - \tilde{a}(Y, [X, Z]).$$

(5)

where $\nabla$ is the Levi-Civita connection of $(M, \tilde{a})$.

If $\text{ad}_X$ is skew-adjoint, then first and last terms of (5) sum to 0. If additionally $\bar{a}(X, [g, g]) = 0$, then the middle term is also 0. So $\nabla_Z X = 0$ for all $Z \in \mathfrak{g}$, which means that $X$ is parallel. By theorem 11.5.1. of [2] the Randers metric is of Berwald type if and only if $X$ is parallel with respect to $\bar{a}$.

Conversely, assume that the Randers metric is of Berwald type, so the left side of (5) equals 0 for all $Y, Z \in \mathfrak{g}$. When $Y = Z$, this yields $2\bar{a}(Y, [Y, X]) = 0$ for all $Y \in \mathfrak{g}$, which implies that $\text{ad}_X$ is skew-adjoint. This property makes the first and third terms of (5) sum to zero, so $\bar{a}(X, [Y, Z]) = 0$ for all $Y, Z \in \mathfrak{g}$. In other words $\bar{a}(X, [g, g]) = 0$. □

By a simple modification of the previous procedure, we can easily obtain the following.

**Theorem 3.7** Let $(M = \mathbb{R}/\mathbb{Z}, F)$ be a homogeneous Randers space with $F$ defined by the Riemannian metric $\tilde{a}$ and the vector field $X$. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the inner product induced on $\mathfrak{g}$ by $\bar{a}$. Then the Randers metric $F$ is of Berwald type if and only if $(\text{ad}_X)_{\mathfrak{m}}$ is skew-adjoint and $\bar{a}(X, [\mathfrak{m}, \mathfrak{m}]) = 0$, where $(\text{ad}_X)_{\mathfrak{m}}$ denotes $(\text{ad}_X)_{\mathfrak{m}} : \mathfrak{m} \rightarrow \mathfrak{m}$, $(\text{ad}_X)_{\mathfrak{m}}(y) = [X, y]_{\mathfrak{m}}$.

**Theorem 3.8** Let $G$ be a Lie group with a bi-invariant Randers metric $F$ defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ and the vector field $X$. Then the Randers metric $F$ is of Berwald type.

Proof: Let $F(p, y) = \sqrt{\bar{a}_p(y, y)} + \tilde{a}_p(X, y)$.

Now for $s, t \in \mathbb{R}$

$$F^2(y + su + tv) = \bar{a}(y + su + tv, y + su + tv) + \tilde{a}^2(X, y + su + tv)$$

$$+ 2\sqrt{\bar{a}(y + su + tv, y + su + tv)}\tilde{a}(X, y + su + tv)$$

By definition

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial r \partial s} F^2(y + ru + sv) |_{r = s = 0}.$$ 

So by a direct computation we get
\[ g_y(u, v) = \tilde{a}(u, v) + \tilde{a}(X, u)\tilde{a}(X, v) \]
\[ + \frac{\tilde{a}(u, v)\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}} - \frac{\tilde{a}(v, y)\tilde{a}(u, y)\tilde{a}(X, y)}{\tilde{a}(y, y)} \]
\[ + \frac{\tilde{a}(X, u)\tilde{a}(v, y)\sqrt{\tilde{a}(y, y)}}{\tilde{a}(y, y)} + \frac{\tilde{a}(X, v)\tilde{a}(u, y)\sqrt{\tilde{a}(y, y)}}{\tilde{a}(y, y)}. \]

So for all \( y, z \in g \) we have
\[ g_y(y, [y, z]) = \tilde{a}(y, [y, z]) + \tilde{a}(X, y)\tilde{a}(X, [y, z]) \]
\[ + \frac{\tilde{a}(y, [y, z])\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}} + \frac{\tilde{a}(X, [y, z])\sqrt{\tilde{a}(y, y)}}{\tilde{a}(y, y)} \]
\[ = \tilde{a}(y, [y, z]) \left( 1 + \frac{\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}} \right) \]
\[ + \tilde{a}(X, [y, z]) \left( \tilde{a}(X, y) + \sqrt{\tilde{a}(y, y)} \right). \]

So we have
\[ g_y(y, [y, z]) = \tilde{a}(y, [y, z]) \left( \frac{F(y)}{\sqrt{\tilde{a}(y, y)}} \right) + \tilde{a}(X, [y, z])F(y) \quad (6) \]

Since \( \tilde{a} \) is bi-invariant, \( \tilde{a}(y, [y, z]) = 0 \) and \( \text{ad}(x) \) is skew-adjoint for every \( x \in g \). Since \( F \) is bi-invariant, \( g_y(y, [y, z]) = 0 \). So from (6) we get \( \tilde{a}(X, [y, z]) = 0 \) for all \( y, z \in g \). Therefore, by Theorem 3.6, we see that \((G, F)\) is of Berwald type. □

**Corollary 3.9** Let \( G \) be a Lie group with a left-invariant Randers metric \( F \) defined by the Riemannian metric \( \tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j \) and the vector field \( X \). If the Randers metric \( F \) is of Berwald type then \( X \) is a geodesic vector.

The following lemma can be found in [20, p.301].

**Lemma 3.10** (Milnor) Let \( G \) be a Lie group endowed with a left-invariant Riemannian metric \( \tilde{a} \). If \( x \in g \) is \( \tilde{a} \)-orthogonal to the commutator ideal \( [g, g] \), then Ricci\((x) \leq 0 \), with equality if and only if \( \text{ad}_x \) is skew-adjoint with respect to \( \tilde{a} \).

**Corollary 3.11** Let \( G \) be a Lie group with a left-invariant Randers metric \( F \) defined by the Riemannian metric \( \tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j \) and the vector field \( X \). If the Randers metric \( F \) is of Berwald type then the Ricci curvature of \( \tilde{a} \) in the direction \( u = \frac{X}{\sqrt{\tilde{a}(X,X)}} \) is zero.

**Proof:** The corollary is a direct consequence of Theorem 3.6 and Lemma 3.10. □
4 Homogeneous geodesics and the critical points of the restricted Finsler function

Let $G$ be a connected Lie group, $\mathfrak{g} = T_eG$ its Lie algebra, $Ad : G \times \mathfrak{g} \to \mathfrak{g}$ the adjoint action, $G(X) = \{ Ad(g)X \mid g \in G \} \subset \mathfrak{g}$ the orbit of an element $X \in \mathfrak{g}$ and $G_X < G$ the isometry subgroup at $X$. The set $\frac{G}{G_X}$ of left-cosets of $G_X$ endowed with its canonical smooth manifold structure admits the canonical left-action

$$\Lambda : G \times \frac{G}{G_X} \to \frac{G}{G_X}$$

$$\quad (g, aG_X) \mapsto gaG_X,$$

which is also smooth. Moreover, a smooth bijection $\rho : \frac{G}{G_X} \to G(X)$ is defined by $\rho(aG_X) = Ad(a)X$ which thus yields an injective immersion into $\mathfrak{g}$ which is equivariant with respect to the actions $\Lambda$ and $Ad$.

Now consider a Minkowski norm $\tilde{F} : \mathfrak{g} \to \mathbb{R}$, then $F$ defines a left-invariant Finsler metric on $G$ by

$$F(x, U) = \tilde{F}(dL_x U), \quad U \in T_xG,$$

where $L_x : G \to G$ is the left translation by $x \in G$. Let $Q(Z) = \tilde{F}^2(Z)$, $Z \in \mathfrak{g}$. Using the formula $\tilde{F}(Z) = \sqrt{g_Z(Z, Z)}$, we have $Q(Z) = g_Z(Z, Z)$.

The smooth function $q = Q \circ \rho : \frac{G}{G_X} \to \mathbb{R}$ will be called the restricted Minkowski norm on $\frac{G}{G_X}$.

In the following, we give an extension of results of [25] to left-invariant Finsler metrics. We use some ideas from [25], [26] in our proofs.

**Theorem 4.1** Let $G$ be a connected Lie group and $\tilde{F}$ a Minkowski norm on its Lie algebra $\mathfrak{g}$. For $X \in \mathfrak{g} - \{0\}$ let $U \in \mathfrak{g}$ be such that $X \in G(U)$ for the corresponding adjoint orbit and let $gGU \in \frac{G}{G_U}$ be the unique coset with $\rho(gGU) = X$. Then $X$ is a geodesic vector if and only if $gGU$ is a critical point of $q = Q \circ \rho$ the restricted Minkowski norm on $\frac{G}{G_U}$.

Proof: The coset $gGU$ is a critical point of $q$ if and only if $vq = 0$ for $v \in T_{gGU}(\frac{G}{G_U})$. But as $\frac{G}{G_U}$ is homogeneous, for each $v$ there is a $Z \in \mathfrak{g}$ such that $v = \tilde{Z}(gGU)$ where $\tilde{Z} : \frac{G}{G_U} \to T_{gGU}(\frac{G}{G_U})$ is the infinitesimal generator of the action $\Lambda$ corresponding to $Z$. Consider also the infinitesimal generator $\tilde{Z} : \mathfrak{g} \to Tg$ of the adjoint action corresponding to $Z$. Since the injective immersion $\rho$ is equivariant with respect to the action $\Lambda$ and $Ad$ the following holds: $\tilde{Z} \circ \rho = \ldots$
$T \rho \circ \tilde{Z}$. But then the following is valid:

\[
v(q) = (T \rho \tilde{Z}) |_{gG U} = \tilde{Z}(Q \circ p) |_{gG U}
\]

\[
= \left( \frac{d}{dt} \bigg|_{t=0} (Ad\text{exp}(Z))X \right) Q
\]

\[
= \frac{d}{dt} |_{t=0} Q(Ad\text{exp}(Z)X)
\]

\[
= \frac{d}{dt} |_{t=0} g_{Ad(\text{exp}Z)X} (Ad\text{exp}(Z)X, Ad\text{exp}(Z)X)
\]

\[
= g_X([Z, X], X) + g_X(X, [Z, X]) + 2C_X([Z, X], X, X)
\]

\[
= 2g_X([Z, X], X),
\]

where $C_X$ is the Cartan tensor of $F$ at $X$. It follows from the homogeneity of $F$ that $C_X([Z, X], X, X) = 0$. Since the map $\alpha : g \to T_{gG U} (\tilde{Z}, Z \to \tilde{Z}(gG U))$ is an epimorphism, the assertion of the theorem follows. □

Corollary 4.2 Let $G$ be a compact connected semi-simple Lie group and $\tilde{F}$ a Minkowski norm on its Lie algebra $g$. Then each orbit of the adjoint action $Ad : G \times g \to g$ contains at least two geodesic vectors.

Proof: Consider an orbit $G(X)$ of the adjoint action, the corresponding coset manifold $\frac{G}{G X}$ and the injective immersion $\rho : \frac{G}{G X} \to g$. Since $G$ is compact and semi-simple then the manifold $\frac{G}{G X}$ becomes compact, and the restricted Minkowski norm $q = Q \circ \rho : \frac{G}{G X} \to R$ has at least two critical points. □

The following corollary is a consequence of the preceding corollary. Two geodesics are considered different if their images are different.

Corollary 4.3 Let $G$ be compact connected semi-simple Lie group of rank $\geq 2$ and $\tilde{F}$ a Minkowski norm on its Lie algebra. Then the left-invariant Finsler metric $F$ induced by $\tilde{F}$ on $G$ has infinitely many homogeneous geodesic issuing from the identity element.

Proof: The proof is similar to the Riemannian case, so we omit it [25]. □

5 Some examples

Example 5.1
Let $G$ be a three-dimensional connected Lie group endowed with a left-invariant Riemannian metric $\tilde{a}$.

1. Let $G$ be an unimodular Lie group. According to a result due to J. Milnor (see [20, Theorem 4.3, p.305], [17]) there exist an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$ such that

\[ [e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2. \]

Let $F$ be a left invariant Randers metric on $G$ defined by the Riemannian metric $\tilde{a}$ and the vector field $X = \epsilon e_1$, $0 < \epsilon < 1$, i.e.

\[ F(p, y) = \sqrt{\tilde{a}_p(y, y)} + \tilde{a}_p(X, y). \]

We note, by using Theorem 3.6, that $(G, F)$ is not of the Berwald type.

We want to describe all geodesic vectors of $(G, F)$.

For $s, t \in \mathbb{R}$

\[ F^2(y + su + tv) = \tilde{a}(y + su + tv, y + su + tv) + \tilde{a}^2(X, y + su + tv) + 2\sqrt{\tilde{a}(y + su + tv, y + su + tv)}\tilde{a}(X, y + su + tv) \]

By definition

\[ g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial r \partial s} F^2(y + ru + sv) \bigg|_{r=s=0}. \]

So by a direct computation we get

\[ g_y(u, v) = \tilde{a}(u, v) + \tilde{a}(X, u)\tilde{a}(X, v) \]

\[ + \frac{\tilde{a}(u, v)\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}} + \frac{\tilde{a}(v, y)\tilde{a}(u, y)\tilde{a}(X, y)}{\tilde{a}(y, y)\sqrt{\tilde{a}(y, y)}} \]

\[ + \frac{\tilde{a}(X, v)\tilde{a}(u, y)}{\sqrt{\tilde{a}(y, y)}} + \frac{\tilde{a}(X, u)\tilde{a}(v, y)}{\sqrt{\tilde{a}(y, y)}}. \]

So for all $z \in \mathfrak{g}$ we have

\[ g_y(y, [y, z]) = \tilde{a}\left( X + \frac{y}{\sqrt{\tilde{a}(y, y)}}, [y, z]\right) F(y) \tag{7} \]

Using Theorem 3.2 and (7) we can check easily that $e_1$ is a geodesic vector.

By using Theorem 3.2 and (7) a vector $y = y_1 e_1 + y_2 e_2 + y_3 e_3$ of $\mathfrak{g}$ is a geodesic vector if and only if

\[ \tilde{a}\left( ee_1 + \frac{y_1 e_1 + y_2 e_2 + y_3 e_3}{\sqrt{y_1^2 + y_2^2 + y_3^2}}, [y_1 e_1 + y_2 e_2 + y_3 e_3, e_j]\right) = 0 \]
for each $j = 1, 2, 3$.
So we get:

$$(\lambda_2 - \lambda_3)y_2y_3 = 0,$$

$$-\epsilon y_3\lambda_1 - \frac{1}{\sqrt{y_1^2 + y_2^2 + y_3^2}} y_1y_3\lambda_1 + \frac{1}{\sqrt{y_1^2 + y_2^2 + y_3^2}} y_1y_3\lambda_3 = 0,$$

$$-\epsilon y_2\lambda_1 + \frac{1}{\sqrt{y_1^2 + y_2^2 + y_3^2}} y_1y_2\lambda_1 - \frac{1}{\sqrt{y_1^2 + y_2^2 + y_3^2}} y_1y_2\lambda_2 = 0.$$

As a special case, if $\lambda_1 = \lambda_2 = \lambda_3 \neq 0$ we conclude that all geodesic vectors $y$ are those from the set $\text{Span}\{e_1\}$. Consequently, there is only one homogeneous geodesic.

2. Let $G$ be a non-unimodular Lie group. According to a result due to J. Milnor (see[20, Lemma 4.10, p.309],[17]) there exists an orthogonal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$ such that

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = \gamma e_2 + \delta e_3,$$

where $\alpha, \beta, \gamma, \delta$ are real numbers such that the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

has trace $\alpha + \delta = 2$ and $\alpha \gamma + \beta \delta = 0$. Let $F$ be a left invariant Randers metric on $G$ defined by the Riemannian metric $\bar{a}$ and the vector field $X = \epsilon e_1, \quad 0 < \epsilon < 1$.

By using Theorem 3.2 and (7), a vector $y = y_1e_1 + y_2e_2 + y_3e_3$ of $\mathfrak{g}$ is a geodesic vector if and only if

$$\bar{a}\left(\epsilon e_1 + \frac{y_1e_1 + y_2e_2 + y_3e_3}{\sqrt{y_1^2 + y_2^2 + y_3^2}}, [y_1e_1 + y_2e_2 + y_3e_3, e_j]\right) = 0$$

for each $j = 1, 2, 3$.

This condition leads to the system of equations

$$y_2(-\alpha y_2 - \gamma y_3) + y_3(-\alpha y_3 - \gamma y_3) = 0,$$

$$y_1y_2\alpha + y_1y_3\beta = 0,$$

$$y_1y_2\gamma + y_1y_3\delta = 0.$$

Putting $\alpha = 2, \delta = 0, \gamma = 0$ the above equations take the form

$$2y_2\left(y_2 + \frac{\beta}{2}y_3\right) = 0,$$

$$2y_1\left(y_2 + \frac{\beta}{2}y_3\right) = 0.$$

So a vector $y$ of $\mathfrak{g}$ is a geodesic vector if and only if:

- $y \in \text{Span}\{e_1, e_3\}$ for $\beta = 0$.
- $y \in \text{Span}\{e_1\} \cup \text{Span}\{e_3\} \cup \text{Span}\left(\frac{\beta}{2}e_2 - e_3\right)$ for $\beta \neq 0$. 

13
References

[1] V. I. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l’hydrodynamique des fluides parfaits, Ann. Inst. Fourier (Grenoble), 16 (1960), 319-361.

[2] D. Bao, S. S. Chern and Shen, An Introduction to Riemann-Finsler geometry, Springer-Verlag, New-York, 2000.

[3] G. Calvaruso and R. A. Marinosci, Homogeneous geodesics of three-dimensional unimodular Lorentzian Lie groups, Mediterr. J. Math. 3 (2006), 467-481.

[4] S. Deng and Z. Hou, The group of isometries of a Finsler space, Pacific J. Math 207 (1) (2002), 149-155.

[5] S. Deng and Z. Hou, Invariant Finsler metrics on homogeneous manifolds, J. Phys. A: Math. Gen. 37 (2004) 8245-8253.

[6] Z. Dušek, O. Kowalski, Light-like homogeneous geodesics and the geodesic lemma for any signature, (to appear in Publ. Math. Debrecen).

[7] J. Figueroa-O’Farrill, P. Meessen and S. Philip, Homogeneity and plane-wave limits, J. High. Energy Physics 5 (2005), 050.

[8] C. Gordon, Homogeneous Riemannian manifolds whose geodesics are orbits, Prog. Nonlinear Differential Equations Appl. 20 (1996) 155-174. S.

[9] Helgason, Differential Geometry, Lie groups and Symmetric Spaces, Academic Press, New York, 1978.

[10] V. V. Kajzer, Conjugate points of left-invariant metrics on Lie groups, Soviet Math. 34 (1990), 32-44.

[11] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry II, Interscience Publishers, New York, 1969.

[12] O. Kowalski, S. Nikčević and Z. Vlášek, Homogeneous geodesics in homogeneous Riemannian manifolds-Examples, Geometry and Topology of Submanifolds (Beijing/Berlin 1999)(2000), World Sci. Publishing Co., River Edge, NJ, 104-112.

[13] O. Kowalski and J. Szenthe, On the existence of homogeneous geodesics in homogeneous Riemannian manifolds, Geom. Dedicata, 81 (2000), 209-214. Erratum: Geom. Dedicata, 84 (2001), 331-332.

[14] O. Kowalski and L. Vanhecke, Riemannian manifolds with homogeneous geodesics, Boll. Un. Mat. Ital., 5 (1991), 189-246.

[15] D. Latifi and A. Razavi, On homogeneous Finsler spaces, Rep. Math. Phys, 57 (2006), 357-366.
[16] D. Latifi, Homogeneous geodesics in homogeneous Finsler spaces, J. Geom. Phys. 57 (2007) 1421-1433.

[17] R. A. Marinosci, Homogeneous geodesics in a three-dimensional Lie group, Comm. Math. Univ. Carolinae, 43 (2002), 261-270.

[18] P. Meessen, Homogeneous Lorentzian spaces admitting a homogeneous structure of type $\tau_1 \oplus \tau_3$, J. Geom. Phys. 56 (2006), 754-761.

[19] P. Meessen, Homogeneous Lorentzian spaces whose null-geodesics are canonically homogeneous, Lett. Math. Phys. 75 (2006), 209-212.

[20] J. Milnor, curvature of left-invariant metrics on Lie groups, Advances in Math. 21 (1976), 293-329.

[21] S. Philip, Penrose limits of homogeneous spaces, J. Geom. Phys. 56 (2006), 1516-1533.

[22] G. Randers, On an asymmetrical metric in the four-space of general relativity, Phys. Rev. 59 (1941)195-199.

[23] Z. Shen, Differential Geometry of Sprays and Finsler Space, Kluwer Academic Publishers, 2001.

[24] J. Szenthe, Existence of stationary geodesics of left-invariant Lagrangians, J. Phys. A: Math. Gen., 34 (2001), 165-175.

[25] J. Szenthe, homogeneous geodesics of left-invariant metrics, Univ. Iagel. Acta Math. 38 (2000), 99-103.

[26] J. Szenthe, On the set of homogeneous geodesics of a left-invariant metric, Univ. Iagel. Acta Math. 40 (2002), 171-181.