Abstract

In 1985, Vladimir Scheffer discussed partial regularity results for what he called solutions to the “Navier-Stokes inequality”. These maps essentially satisfy the incompressibility condition as well as the local and global energy inequalities and the pressure equation which may be derived formally from the Navier-Stokes system of equations, but they are not required to satisfy the Navier-Stokes system itself.

In a previous work, the author extended this notion to a system considered by Fang-Hua Lin and Chun Liu in the mid 1990s related to models of the flow of nematic liquid crystals, which include the Navier-Stokes system when the ‘director field’ \(d\) is taken to be zero. In addition to an extended Navier-Stokes system, the Lin-Liu model includes a further parabolic system which implies an a priori maximum principle for \(d\) which they use to establish the same type of partial regularity result, in terms of the parabolic Hausdorff dimension of sets, as that which was known in the Navier-Stokes setting. For the analogous ‘inequality’ one loses this maximum principle and the author previously explored the consequences for such Hausdorff-dimensional partial regularity.

The current work explores similar consequences for partial regularity, but with respect to the ‘parabolic fractal dimension’ \(\dim_{pf}\) (also called the upper box-counting, capacity or Minkowski dimension). In 2018, relying (as did Lin and Liu) on the boundedness of \(d\) coming from the maximum principle, Qiao Liu proved that solutions to the Lin-Liu model satisfy \(\dim_{pf}(\Sigma_\cap K) \leq \frac{25}{63}\) for any compact set \(K\), where \(\Sigma_\) is the set of ‘forward-singular’ space-time points near which the solution blows up forwards in time. For solutions to the corresponding ‘inequality’, the author proves here that, without any compensation for the lack of maximum principle, one has \(\dim_{pf}(\Sigma_\cap K) \leq \frac{55}{13}\). A range of criteria is also established, including as just one example the boundedness of \(d\), any one of which is shown to furthermore imply that \(\dim_{pf}(\Sigma_\cap K) \leq \frac{25}{63}\) for solutions to the inequality, just as Q. Liu proved for solutions to the full Lin-Liu system.

1 Introduction

In [LL95] and [LL96], Fang-Hua Lin and Chun Liu prove existence and partial regularity (similar to results in [CKNS2] for Navier-Stokes) of certain solutions to the following system, which reduces to the classical Navier-Stokes system in the case \(d \equiv 0\) (here we have set various parameters equal to one for simplicity):

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + \nabla T \cdot (u \otimes u + \nabla d \odot \nabla d) + \nabla p &= 0 \\
\nabla \cdot u &= 0 \\
\frac{\partial d}{\partial t} - \Delta d + (u \cdot \nabla) d + f(d) &= 0
\end{align*}
\]

with \(f = \nabla F\) for the scalar field \(F\) given by \(F(x) := (|x|^2 - 1)^2\) so that \(f(x) = 4(|x|^2 - 1)x\) (and in particular \(f(0) = 0\)). In (1.1), for vector fields \(v\) and \(w\), the matrix fields \(v \otimes w\) and \(\nabla v \odot \nabla w\) are...
defined to be the ones with entries

\[(v \otimes w)_{ij} = v_i w_j \quad \text{and} \quad (\nabla v \otimes \nabla w)_{ij} = v_i \cdot w_j := \frac{\partial v_k}{\partial x_i} \frac{\partial w_k}{\partial x_j}\]

(summing over the repeated index \(k\) as per the Einstein convention), and for a matrix field \(J = (J_{ij})\), we define the vector field \(\nabla^T \cdot J\) by

\[(\nabla^T \cdot J)_{i} := J_{ij,j} := \frac{\partial J_{ij}}{\partial x_j} \quad \text{(summing again over } j)\],

where we think formally of \(\nabla\) (as well as any vector field) as a column vector and \(\nabla^T\) as a row vector.

We take the spatial dimension to be three, so that for some \(\Omega \subseteq \mathbb{R}^3\) and \(T > 0\), we are seeking maps of the form

\[u, d : \Omega \times (0,T) \to \mathbb{R}^3 \quad \text{and} \quad p : \Omega \times (0,T) \to \mathbb{R}\]
satisfying (1.1), where \(F: \mathbb{R}^3 \to \mathbb{R}\) and \(f: \mathbb{R}^3 \to \mathbb{R}^3\) are fixed as above. As usual, \(u\) represents the velocity vector field of a fluid, \(p\) is the scalar pressure in the fluid, and, as in nematic liquid crystals models, \(d\) corresponds roughly to the ‘director field’ representing the local orientation of rod-like molecules, with \(u\) also giving the velocities of the centers of mass of those anisotropic molecules.

We note that by formally taking the divergence \(\nabla \cdot\) of the first line in (1.1) we obtain the usual ‘pressure equation’

\[-\Delta p = \nabla \cdot (\nabla^T \cdot [u \otimes u + \nabla d \otimes \nabla d]) . \quad (1.2)\]

As in the Navier-Stokes \((d \equiv 0)\) setting, one may formally deduce (see e.g. [Koc21] for more details) from (1.1) the following global and local energy inequalities which one may expect ‘sufficiently nice’ solutions of (1.1) to satisfy:

\[
\frac{d}{dt} \int_{\Omega} \left[ \frac{|u|^2}{2} + \frac{|
abla d|^2}{2} + F(d) \right] \, dx + \int_{\Omega} \left[ |
abla u|^2 + |\Delta d - f(d)|^2 \right] \, dx \leq 0 \quad (1.3)
\]

for each \(t \in (0,T)\), as well as a localized version

\[
\frac{d}{dt} \int_{\Omega} \left[ \left( \frac{|u|^2}{2} + \frac{|
abla d|^2}{2} \right) \phi \right] \, dx + \int_{\Omega} \left( |
abla u|^2 + |\nabla^2 d|^2 \right) \phi \, dx \\
\leq \int_{\Omega} \left[ \left( \frac{|u|^2}{2} + \frac{|
abla d|^2}{2} \right) (\phi_t + \Delta \phi) + \left( \frac{|u|^2}{2} + \frac{|
abla d|^2}{2} + p \right) u \cdot \nabla \phi \right]
\]

\[+ \ u \otimes \nabla \phi : \nabla d \otimes \nabla d - \phi \nabla^T [f(d)] : \nabla^T d \] \quad \text{dx} \quad (1.4)

for \(t \in (0,T)\) and each smooth, compactly supported in \(\Omega\) and non-negative scalar field \(\phi \geq 0\). (For Navier-Stokes, i.e. when \(d \equiv 0\), one may omit all terms involving \(d\), even though \(0 \neq F(0) \notin L^1(\mathbb{R}^3)\).) As explained in [Koc21], a Grönwall argument shows that this also formally implies a local energy inequality of the form (1.9) below for some \(\bar{C} > 0\) depending only on \(T\).

In the Navier-Stokes setting, it was asserted by Vladimir Scheffer in [Sch85] that in fact the proof of the partial regularity result (see below) of Caffarelli-Kohn-Nirenberg [CKNS82] does not require the full set of equations in (1.1). He mentions that the key ingredients are membership of the global energy spaces, the local energy inequality (1.3), the divergence-free condition \(\nabla \cdot u = 0\) and the pressure equation (1.2) (with \(d \equiv 0\) throughout). Scheffer called pairs \((u,p)\) satisfying these four requirements
solutions to the “Navier-Stokes inequality”, equivalent to solutions to the Navier-Stokes equations with a forcing $g$ which satisfies $g \cdot u \leq 0$ everywhere. (The Navier-Stokes solutions considered in \cite{CKNS82}, called “suitable weak solutions”, are roughly speaking solutions to Scheffer’s ‘inequality’ which moreover satisfy (1.1) itself – with, of course, $d = 0$.) In contrast, the results in \cite{LL96} (for a similar class of ‘suitable weak solutions’ of (1.1) for more general $d$) do very strongly use the third equation in (1.1) in that it implies a maximum principle for $d$, making it natural to assume $d \in L^\infty(\Omega \times (0, T))$.

In this paper, as in the author’s previous paper \cite{Koc21}, we continue to explore what happens if one considers the analog of Scheffer’s “Navier-Stokes inequality” for the system (1.1) when $d \neq 0$. That is, we consider triples $(u, d, p)$ with global regularities implied (at least when $\Omega$ is bounded and under suitable assumptions on the initial data) by (1.3) which satisfy (1.2) and $\nabla \cdot u = 0$ weakly as well as (a formal consequence of) (1.4), but are not necessarily weak solutions of the first and third equations (i.e., the two vector equations) in (1.1). In particular, we will not assume that $d \in L^\infty(\Omega \times (0, T))$, which would have been reasonable in view of the third equation in (1.1). Specifically, we will address the following scenario:

Fixing an open set $\Omega \subseteq \mathbb{R}^3$ and $T, \bar{C} \in (0, \infty)$ and setting $\Omega_T := \Omega \times (0, T)$, we consider $u, d : \Omega_T \to \mathbb{R}^3$ and $p : \Omega_T \to \mathbb{R}$ which satisfy the following four assumptions:\footnote{See \cite{Koc21} for more explanation as to why these assumptions are natural from the viewpoint of (1.1).}

1. $u, d$ and $p$ belong to the following spaces:\footnote{For a vector field $f$ or matrix field $J$ and scalar function space $X$, by $f \in X$ or $J \in X$ we mean that all components or entries of $f$ or $J$ belong to $X$; by $\nabla^2 f \in X$ we mean all second partial derivatives of all components of $f$ belong to $X$; etc.}

$$d \in L^\infty(0, T; L^4(\Omega)), \quad u, \nabla d \in L^\infty(0, T; L^2(\Omega)),$$

and

$$p \in L^\infty(\Omega_T); \quad (1.5)$$

2. $u$ is weakly divergence-free:

$$\nabla \cdot u = 0 \quad \text{in} \quad \mathcal{D}'(\Omega_T); \quad (1.6)$$

3. The following pressure equation holds weakly:

$$- \Delta p = \nabla \cdot [\nabla^T \cdot (u \otimes u + \nabla d \otimes \nabla d)] \quad \text{in} \quad \mathcal{D}'(\Omega_T); \quad (1.7)$$

4. The following local energy inequality holds:\footnote{Locally integrable functions, e.g., $u \in L^1_{\text{loc}}(\Omega_T)$ in (1.8) and $u \otimes u + \nabla d \otimes \nabla d \in L^1_{\text{loc}}(\Omega_T)$ in (1.9), will always be associated to the standard distribution whose action is integration against a suitable test function so that, e.g., $[\nabla \cdot u](\psi) := -\int u \cdot \nabla \psi$ for $\psi \in \mathcal{D}(\Omega_T)$.}

\[
\begin{align*}
\int_{\Omega \times \{1\}} \left( |u|^2 + |\nabla d|^2 \right) \phi dx + \int_0^T \int_{\Omega} \left( |\nabla u|^2 + |\nabla^2 d|^2 \right) \phi dx dt \\
\leq \bar{C} \int_0^T \int_{\Omega \times \{t\}} \left( |u|^2 + |\nabla d|^2 \right) |\phi_t + \Delta \phi| + |d|^2 |\nabla d|^2 \phi \right) dx \\
+ \int_{\Omega \times \{t\}} \left( |u|^2 + |\nabla d|^2 \right) |u \cdot \nabla \phi + u \otimes \nabla \phi : \nabla d \otimes \nabla d \right) dx \right) \right) dx dt \quad (1.9)
\end{align*}
\]

\[
\text{for a.e. } t \in (0, T) \quad \text{and} \quad \forall \phi \in C^\infty_0(\Omega \times (0, \infty)) \text{ s.t. } \phi \geq 0. \]

Remark 1. In the case $\Omega = \mathbb{R}^3$, the condition (1.0) on the pressure follows from (1.5) and (1.8) if $p$ is taken to be the potential-theoretic solution to (1.8), since (1.2) implies that $u, \nabla d \in L^4(\Omega_T)$.

\[\int_{\omega \times \{t\}} g dx := \int_\omega g(x, t) dx.\]
by interpolation and Sobolev embeddings, and then \(1.8\) gives \(p \in L^{\frac{7}{4}}(\Omega_T)\) by Calderon-Zygmund estimates. For a more general \(\Omega\), the existence of such a \(p\) can be derived from the motivating equation \(1.1\) (e.g. by estimates for the Stokes operator), see [LL96] and the references therein. Here, however, we will not refer to \(1.1\) at all and simply assume \(p\) satisfies \(1.6\) and address the partial regularity of such a hypothetical set of functions satisfying \(1.5\) - \(1.9\).

The partial regularity results of [CKN82] and [LL96] are given in terms of the (outer) parabolic Hausdorff measure \(P^k\) defined as follows (see [CKN82, pp.783-784]), where \(B_r(x)\) is the standard Euclidean ball in \(\mathbb{R}^3\) of radius \(r > 0\) centered at \(x \in \mathbb{R}^3\):

**Definition 1** (Parabolic Hausdorff measure and dimension). For any set \(A \subset \mathbb{R}^3 \times \mathbb{R}\) and \(k \geq 0\), its \(k\)-dimensional (outer) parabolic Hausdorff measure \(P^k(A)\) is defined as

\[
P^k(A) := \lim_{\delta \to 0} P^k_\delta(A),
\]

where

\[
P^k_\delta(A) := \inf \left\{ \sum_{j=1}^{\infty} r_j^k \mid A \subset \bigcup_{j=1}^{\infty} Q_{r_j}, r_j < \delta \forall j \in \mathbb{N} \right\}
\]

and \(Q_r\) is any parabolic cylinder of ‘radius’ \(r > 0\), i.e.

\[
Q_r = Q_r(x, t) := B_r(x) \times (t-r^2, t) \subset \mathbb{R}^3 \times \mathbb{R}
\]

for some \(x \in \mathbb{R}^3\) and \(t \in \mathbb{R}\). The parabolic Hausdorff dimension \(\dim_{P}(A)\) of \(A\) is then defined as

\[
\dim_{P}(A) := \inf\{ k \geq 0 \mid P^k(A) = 0 \}.
\]

The results in [CKN82] and [LL96] state roughly that for suitable weak solutions to \(1.1\) (including the Navier-Stokes setting \(d \equiv 0\) of [CKN82]), the singular set \(\Sigma\) of space-time points \((x, t)\) about each of which \(u\) and \(\nabla d\) are unbounded in some space-time neighborhood satisfies \(P^1(\Sigma) = 0\) (hence \(\dim_{P}(\Sigma) \leq 1\)). In these settings the vector-field \(d\) is bounded due to \(1.1\), while in [Koc21], for solutions to the corresponding ‘inequality’, the author proved that (without any compensation for the lack of boundedness of \(d\)) one has \(P^{\frac{d}{2}+\delta}(\Sigma) = 0\) for any \(\delta > 0\) (hence \(\dim_{P}(\Sigma) \leq \frac{d}{2}\)); the author moreover provided a range of criteria, including as just one example the boundedness of \(d\), any one of which was shown to furthermore imply (as in [CKN82, LL96]) that \(P^1(\Sigma) = 0\) even for solutions to the inequality.

In [Liu18], on the other hand, Qiao Liu proves (along the lines of other papers such as, e.g., [KP12] and the references therein), for the same type of solutions to \(1.1\) considered by [LL96], a partial regularity result in terms of the parabolic fractal dimension (also called the upper box-counting or capacity dimension, see [KP12] and the references therein, as well as the upper Minkowski dimension as it is referred to in [Liu18]) defined as follows:

**Definition 2** (Parabolic fractal dimension, see e.g. [KP12]). For any bounded set \(A \subset \mathbb{R}^3 \times \mathbb{R}\), its parabolic fractal dimension \(\dim_{Pf}(A)\) is defined as

\[
\dim_{Pf}(A) := \limsup_{r \to 0} \frac{\log N(A, r)}{\log(\frac{1}{r})}
\]

where, for any \(r > 0\),

\[
N(A, r) := \min \left\{ N \in \mathbb{N} \mid A \subset \bigcup_{j=1}^{N} Q^*_r(z_j) \text{ for some } \{z_j\}_{j=1}^{N} \subset \mathbb{R}^3 \times \mathbb{R} \right\}
\]

is the minimum number of (centered) parabolic cylinders of the form \(Q^*_r(z, t) := B_r(x) \times (t-r^2, t+r^2) \quad (x \in \mathbb{R}^3, t \in \mathbb{R}, r > 0)\) (1.11) with the fixed ‘radius’ \(r\) required to cover \(A\).

\(^5\)It is pointed out in [Liu18] that, in general, \(\dim_{P}(A) \leq \dim_{Pf}(A)\), where strict inequality is known to occur in some instances.

\(^6\)Note that in view of (1.10) and (1.11) one always has \(Q_r(z) \subset Q^*_r(z)\) for any \(z = (x, t) \in \mathbb{R}^3 \times \mathbb{R}\) and \(r > 0\).
Roughly speaking, for suitable weak solutions to (1.1), Q. Liu proves (see Theorem 1 below) that $\dim_{pf}(\Sigma_\gamma \cap K) \leq \frac{55}{63}$ for any compact space-time set $K$, where $\Sigma_\gamma$ is the set of ‘forward-singular’ space-time points, near each of which the solution blows up forwards in time. As we will describe below, the proof of Theorem 1 given in [Liu18] relies again on the fact that $d$ is bounded, a consequence of the third equation in (1.1). In contrast, for solutions to the corresponding ‘inequality’, we will prove in Theorem 3 below that, without any compensation for the lack of maximum principle and hence when $d$ is potentially unbounded, one can at least prove $\dim_{pf}(\Sigma_\gamma \cap K) \leq \frac{55}{63}$. It is reasonable that Theorem 3 which may be viewed as an adaptation of Q. Liu’s Theorem 1 yields a weaker conclusion (the bound $\frac{55}{63}$ rather than the smaller $\frac{29}{40}$), as one has removed the boundedness of $d$ from the set of essential properties used in the proof. As in [Koc21], we will also establish (in Theorem 2 below) a range of criteria (namely (2.2)) for solutions to the inequality, including as just one example the boundedness of $d$, any one of which would furthermore imply (without any reference to (1.1) itself) that $\dim_{pf}(\Sigma_\gamma \cap K) \leq \frac{55}{63}$: Theorem 2 may therefore be viewed as a generalization of Q. Liu’s Theorem 1.

Our key observation which allows us to work without any maximum principle is that, in view of the global energy (1.3) and the particular forms of $F$ and $f$, it is reasonable (see [Koc21]) to assume (1.4) which implies (at least for bounded $\Omega$ and $T < \infty$) that $d \in L^\infty(0,T; L^6(\Omega)) \cap L^{10}(\Omega \times (0,T))$, which is sufficient for our purposes. Indeed, the assumptions in (1.4) imply (locally in space, or from the left, i.e. moving forward in time, for $d$), so that $\frac{29}{40}$ below) a range of criteria (namely (2.2)) for solutions to the inequality, including as just one example $\gamma$ and $\Omega$ in the same Navier-Stokes-type energy spaces as does $u$. (Taking $\alpha : = \frac{3}{5}$ gives the usual fact that $u, \nabla d \in L^\infty(\Omega \times (0,T))$.) Taking $\alpha : = \frac{1}{5}$, we see that (for bounded $\Omega$ and $T < \infty$) $d \in L^{10}(0,T; W^{1,\frac{10}{3}}(\Omega)) \hookrightarrow L^{10}(\Omega \times (0,T))$ by Sobolev embedding.

2 Main results and context

In [Liu18], Q. Liu proves (for the spatial setting $\Omega = \mathbb{R}^3$) a theorem (Theorem 1) which may be stated essentially as follows:

Theorem 1 ([Liu18], Theorem 1). Fix an open set $\Omega \subseteq \mathbb{R}^3$ and $T, \bar{C} \in (0, \infty)$, set $\Omega_T : = \Omega \times (0,T)$, suppose $u, d : \Omega_T \to \mathbb{R}^3$ and $p : \Omega_T \to \mathbb{R}$ satisfy the assumptions (1.9), (1.10), (1.11) and (1.12) and suppose additionally that $d \in L^\infty(\Omega_T)$ (and that (1.1) holds weakly). Define the forward(-in-time)-singular set $\Sigma_\gamma \subseteq \Omega_T$ by

$$
\Sigma_\gamma := \{ \, z_0 \in \Omega_T \mid \| u \|_{L^\infty(Q_r(z_0) \cap \Omega_T)} = +\infty \ \forall r > 0 \, \}.
$$

Then for any compact set $K \subset \Omega_T$,

$$
\dim_{pf}(\Sigma_\gamma \cap K) \leq \frac{2}{5} - \gamma \quad \text{for any } \gamma \in (0, \frac{10}{43}),
$$

so that

$$
\dim_{pf}(\Sigma_\gamma \cap K) \leq \frac{2}{5} - \frac{10}{43} = \frac{6}{39}.
$$

Here, we will prove two related theorems. The first one (Theorem 2) shows that one does not need to assume that (1.1) itself holds as long as one assumes that $d$ is either bounded (as in Theorem 1) or satisfies an alternative assumption (see (2.2)) which plays the same role in the proof:

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7 As mentioned above, in [Liu18] it is actually assumed that $\Omega = \mathbb{R}^3$, but it is clear from the proof that one may establish Theorem 1 as stated here; this is in fact carried out in a more general setting in Theorem 2 below.

8 In fact, (1.1) is not needed, see Theorem 2 below.

9 Recall the definition (1.10) used to define the (standard parabolic) cylinder $Q_r(z_0)$ which appears in the definition of $\Sigma_\gamma$. In the notation $\Sigma_\gamma$, the “$\gamma$” can therefore be viewed as indicating blowup for times approaching $t_0 \in (0,T)$ from the left, i.e. moving forward in time, for $z_0 = (x_0, t_0) \in \Sigma_\gamma$.

10 During the preparation of this paper, the author discovered that the bound in (2.1) appears to have been recently improved slightly in [Liu21] from $\dim_{pf}(\Sigma_\gamma \cap K) \leq \frac{29}{40} \approx 1.51$ to $\dim_{pf}(\Sigma_\gamma \cap K) \leq \frac{6}{39} \approx 1.36$. 

---
Theorem 2. Fix an open set $\Omega \subseteq \mathbb{R}^3$ and $T, \tilde{C} \in (0, \infty)$, set $\Omega_T := \Omega \times (0, T)$ and suppose $u, d : \Omega_T \to \mathbb{R}^3$ and $p : \Omega_T \to \mathbb{R}$ satisfy the assumptions (\ref{1.5}), (\ref{1.6}), (\ref{1.7}), (\ref{1.8}) and (\ref{1.9}). Suppose as well that

$$g_\sigma := \sup_{\{r, z_0 \mid Q_r(z_0) \subseteq \Omega_T\}} \left( \frac{1}{r^{2+\frac{\sigma}{2}}} \int_{Q_r(z_0)} |d|^{\sigma} |\nabla d|^{3(1-\frac{\sigma}{2})} \, dz \right) < \infty \quad \text{for some } \sigma \in (5, 6) . \quad (2.2)$$

Define the forward-singular set $\Sigma_- \subseteq \Omega_T$ by

$$\Sigma_- := \{ z_0 \in \Omega_T \mid \|u\| + |\nabla d| \|L^\infty(Q_r(z_0) \cap \Omega_T)\| = +\infty \ \forall r > 0 \} .$$

Then for any compact set $K \subset \Omega_T$,

$$\dim_{pf}(\Sigma_- \cap K) \leq \frac{5}{4} - \gamma \quad \text{for any } \gamma \in (0, \frac{10}{63}) ,$$

so that

$$\dim_{pf}(\Sigma_- \cap K) \leq \frac{55}{63} .$$

Note that if $d \in L^\infty(\Omega_T)$ (as in Theorem 1), then\footnote{In fact, the two assumptions are equivalent for $\sigma = 6$ due to Lebesgue’s differentiation theorem.} holds with $\sigma = 6$ and hence Theorem 2 in particular implies Theorem 1. The proofs of Theorem 1 and Theorem 2 are, however, very similar, and the endpoint $\sigma = 6$ is included in (2.2) mainly to point out that the proof of Theorem 1 given by Q. Liu in \cite{Liu18} does not in fact require the full set of assumptions (as will be essentially clear from the proof of Theorem 2 given below).

The second theorem (Theorem 3) addresses the scenario where one removes the assumption (2.2) altogether from Theorem 2 (or, indeed, from Theorem 1 when $\sigma = 6$):

Theorem 3. Fix an open set $\Omega \subseteq \mathbb{R}^3$ and $T, \tilde{C} \in (0, \infty)$, set $\Omega_T := \Omega \times (0, T)$ and suppose $u, d : \Omega_T \to \mathbb{R}^3$ and $p : \Omega_T \to \mathbb{R}$ satisfy the assumptions (\ref{1.5}), (\ref{1.6}), (\ref{1.7}), (\ref{1.8}) and (\ref{1.9}). Define the forward-singular set $\Sigma_- \subseteq \Omega_T$ by

$$\Sigma_- := \{ z_0 \in \Omega_T \mid \|u\| + |\nabla d| \|L^\infty(Q_r(z_0) \cap \Omega_T)\| = +\infty \ \forall r > 0 \} .$$

Then for any compact set $K \subset \Omega_T$,

$$\dim_{pf}(\Sigma_- \cap K) \leq 5 - \delta \quad \text{for any } \delta \in (\frac{1}{4}, \frac{10}{11}) ,$$

so that

$$\dim_{pf}(\Sigma_- \cap K) \leq 5 - \frac{10}{11} = \frac{55}{11} .$$

The main mechanism used to establish results such as Theorems 1, 2 and 3 is the following general proposition which was already available in the literature prior to \cite{Liu18} (see, e.g., \cite{KP12} and \cite{Liu18} and the references therein):

Proposition 1. Fix any open set $\Omega \subseteq \mathbb{R}^3$ and $\lambda, \bar{r}, c_0, T \in (0, \infty)$ and set $\Omega_T := \Omega \times (0, T)$. Suppose $S \subseteq \Omega_T$ and that $H : \Omega_T \to [0, \infty]$ is a non-negative Lebesgue-measurable function such that

$$0 \leq H \in L^1(\Omega_T) , \quad (2.3)$$

and suppose that the following property holds in general (recall (\ref{1.11})):

$$z_0 \in S, \quad 0 < r \leq \bar{r}, \quad Q_r(z_0) \subseteq \Omega_T \quad \implies \quad \frac{1}{r^n} \int_{Q_r(z_0)} H(z) \, dz \geq c_0 . \quad (2.4)$$

Then for any compact set $K \subset \Omega_T$,

$$\dim_{pf}(S \cap K) \leq \lambda .$$
The novel and essential elements of Theorems 1, 2, and 3 are therefore the following more specific ‘epsilon-regularity’ type lemmas, each of which fairly immediately implies the corresponding theorem in view of Proposition 1. The first, proved in [Liu18], is the essence of [Liu18, Theorem 1] (i.e., Theorem 1 above) and may be stated essentially as follows:

**Lemma 1** ([Liu18], Lemma 6). Fix any \( \gamma \in (0, \frac{10}{3}) \) and \( \bar{C}, D \in (0, \infty) \). There exist numbers \( \epsilon^* = \epsilon^*(\gamma, \bar{C}, D) \in (0, 1) \) and \( r^* = r^*(\gamma) \in (0, 1) \) so small that the following holds for any fixed open set \( \Omega \subseteq \mathbb{R}^3 \) and \( T \in (0, \infty) \):

Set \( \Omega_T := \Omega \times (0, T) \), suppose \( u, d : \Omega_T \to \mathbb{R}^3 \) and \( p : \Omega_T \to \mathbb{R} \) satisfy the assumptions \((1.5), (1.0), (1.7), (1.8)\) and \((1.9)\) and suppose additionally that

\[
d \in L^\infty(\Omega_T) \quad \text{with} \quad \|d\|_{L^\infty(\Omega_T)} \leq D
\]

(and that \((1.1)\) hold weakly). For any \( z_0 \in \Omega_T \) and \( r_1 \in (0, r^*] \) such that \( Q_{r_1}(z_0) \subseteq \Omega_T \), if

\[
\frac{1}{r_1^{5-\gamma}} \int_{Q_{r_1}(z_0)} \left( |\nabla u|^2 + |\nabla^2 d|^2 + |u|^{\frac{10}{3}} + |\nabla d|^{\frac{10}{3}} + |p|^{\frac{10}{3}} \right) \, dz \leq \epsilon^*,
\]

then

\[
u, \nabla d \in L^\infty(Q_{r_1}(z_0))
\]

for some small \( r_0 \in (0, r_1) \).

Similarly, the essence of Theorems 2 and 3 are the following Lemmas 2 and 3 which we will prove in Section 3 below:

**Lemma 2.** Fix any \( \gamma \in (0, \frac{16}{3}) \), \( \sigma \in (5, 6] \) and \( \bar{C}, D \in (0, \infty) \). There exist numbers \( \epsilon^* = \epsilon^*(\gamma, \sigma, \bar{C}, D) \in (0, 1) \) and \( r^* = r^*(\gamma) \in (0, 1) \) so small that the following holds for any fixed open set \( \Omega \subseteq \mathbb{R}^3 \) and \( T \in (0, \infty) \):

Set \( \Omega_T := \Omega \times (0, T) \), suppose \( u, d : \Omega_T \to \mathbb{R}^3 \) and \( p : \Omega_T \to \mathbb{R} \) satisfy the assumptions \((1.5), (1.0), (1.7), (1.8)\) and \((1.9)\) and suppose additionally that

\[
g_\sigma < \infty \quad \text{with} \quad g_\sigma \leq D,
\]

with \( g_\sigma \) defined as in \((2.2)\). For any \( z_0 \in \Omega_T \) and \( r_1 \in (0, r^*] \) such that \( Q_{r_1}(z_0) \subseteq \Omega_T \), if

\[
\frac{1}{r_1^{5-\gamma}} \int_{Q_{r_1}(z_0)} \left( |\nabla u|^2 + |\nabla^2 d|^2 + |u|^{\frac{10}{3}} + |\nabla d|^{\frac{10}{3}} + |p|^{\frac{10}{3}} + |d|^{10} \right) \, dz \leq \epsilon^*,
\]

then

\[
u, \nabla d \in L^\infty(Q_{r_1}(z_0))
\]

for some small \( r_0 \in (0, r_1) \).

Note that, in the particular case when \( d \in L^\infty(\Omega_T) \), one can take \( \sigma := 6 \) and \( D := \|Q_1\|_{L^\infty(\Omega_T)} \) in Lemma 2 to recover the type of result in Lemma 1.

---

12 As noted above, Q. Liu [Liu18] actually assumes \( \Omega = \mathbb{R}^3 \) so that the Calderon-Zygmund estimates apply directly to \((1.8)\); however this is unnecessary (see Lemma 2 below) in the present context in view of the assumption \((1.5)\) on the pressure.

13 In fact, \((1.1)\) is not needed, as in Lemma 2 below.

14 Note, in contrast to Lemma 1 the appearance of the additional term \(|d|^{10}\) in the smallness assumption in both Lemma 2 and Lemma 3. This is the key new idea which allows us to remove the requirement that \( d \in L^\infty \).
Remark 2. For $\gamma \approx \frac{10}{27}$, setting

$$M_\gamma := \frac{154}{27(1 - \frac{6N - 7}{108r})} \quad \text{and} \quad N \in \mathbb{N} \cap \{M_\gamma, M_\gamma + 1\},$$

it will be clear from the proof of Lemma 3 that its conclusion is true, for example, with

$$r^* := \left(\frac{4}{5}\right)^{6M_\gamma + 11} \quad \text{and} \quad r_0 := 2\left(\frac{4}{5}\right)^{\frac{6N - 7}{108r} + \frac{64}{108}} \geq 2\left(\frac{4}{5}\right)^{\frac{1}{2} + \frac{64}{108}}$$

so that, in particular,

$$u, \nabla d \in L^\infty(Q_{\frac{4}{5}}(z_0)).$$

Lemma 3. Fix any $\delta \in \left(\frac{1}{5}, \frac{10}{27}\right)$ and $\bar{C} \in (0, \infty)$. There exist numbers $\epsilon^* = \epsilon^*(\delta, \bar{C}) \in (0, 1)$ and $r^* = r^*(\delta) \in (0, 1)$ so small that the following holds for any fixed open set $\Omega \subseteq \mathbb{R}^3$ and $T \in (0, \infty)$:

Set $\Omega_T := \Omega \times (0, T)$ and suppose $u, d : \Omega_T \to \mathbb{R}^3$ and $p : \Omega_T \to \mathbb{R}$ satisfy the assumptions \([155], [150], [179], [184]\) and \([199]\).

For any $z_0 \in \Omega_T$ and $r_1 \in (0, r^*)$ such that $Q_{r_1}(z_0) \subseteq \Omega_T$, if

$$\frac{1}{r_1^\gamma} \int_{Q_{r_1}(z_0)} \left(|\nabla u|^2 + |\nabla^2 d|^2 + |u|_{L^4}^{14} + |\nabla d|_{L^4}^{14} + |p|_{L^5}^{6} + |d|_{L^5}^{10}\right) dz \leq \epsilon^*,$$

then

$$u, \nabla d \in L^\infty(Q_{\frac{4}{5}}(z_0))$$

for some small $r_0 \in (0, r_1)$.

Remark 3. It will be clear from the proof of Lemma 3 that its conclusion is true, for example, with

$$r^* := \left(\frac{4}{5}\right)^{6M_\gamma + 2} \quad \text{and} \quad r_0 := 2\left(\frac{4}{5}\right)^{\frac{6N - 7}{108r} + \frac{64}{108}} \geq 2\left(\frac{4}{5}\right)^{\frac{1}{2} + \frac{64}{108}}$$

so that, in particular,

$$u, \nabla d \in L^\infty(Q_{\frac{4}{5}}(z_0)).$$

Let us now briefly outline how each theorem (for completeness, we include Theorem 1) follows easily from the corresponding lemma along with Proposition 1.

Proofs of Theorems 1, 2 and 3. Let us set

$$H := |\nabla u|^2 + |\nabla^2 d|^2 + |u|_{L^4}^{14} + |\nabla d|_{L^4}^{14} + |p|_{L^5}^{6} + |d|_{L^5}^{10}.$$

Under the assumptions of Theorem 1, 2 or 3 it follows (see, e.g., [Koc21]) that $H$ satisfies (2.3) (at least for any bounded $\Omega$ such that $K \subset \Omega_T \subseteq \Omega_T$).

Using Lemma 1 for Theorem 1, Lemma 2 for Theorem 2 and Lemma 3 for Theorem 3 in each setting there exists some $\lambda > 0$ ($\lambda = \frac{5}{2} - \gamma$ for some $\gamma \in \left(0, \frac{10}{27}\right)$ in Lemmas 1 and 2, $\lambda = 5 - \delta$ for some $\delta \in \left(\frac{1}{5}, \frac{10}{27}\right)$ in Lemma 3) as well as some $\epsilon^* > 0$ and $r^* > 0$ such that if \[16\]

$$\left(\frac{1}{r_1^\gamma} \int_{Q_{r_1}(z_0)} H(z) \, dz \leq \epsilon^* \right), \left(\frac{1}{r_1^\gamma} \int_{Q_{r_1}(z_0)} H(z) \, dz \leq \epsilon^* \right).$$

\[15\] For the proof of Theorem 1 one can in fact remove the term $|d|^{10}$ from $H$, but doing so does not change the conclusion reached by this method of proof. Moreover, the observation that (in view of 1.5) $H$ satisfies (2.3) even if $|d|^{10}$ is included is crucial to the proofs of Theorems 2 and 3.

\[16\] Recall that $Q_r(z) \subset Q_r^*(z)$ in general.
for some \( z_0 \in \Omega_T \) and \( r_1 \in (0, r^*] \) such that \( Q^*_{r_1}(z_0) \subseteq \Omega_T \), then \( u, \nabla d \in L^\infty(Q^*_{r_1}(z_0)) \) for some small \( r_0 \in (0, r_1) \), and hence \( z_0 \notin \Sigma_- \). This shows that for any \( z_0 \in \Sigma_- \), we must in particular have

\[
\frac{1}{r_1^2} \int_{Q^*_{r_1}(z_0)} H(z) \, dz \geq \epsilon^*
\]

for any such small \( r_1 > 0 \), and we may conclude by Proposition 1 with \( S := \Sigma_- \), \( \bar{r} := r^* \) and \( c_0 := \epsilon^* \).

To put these lemmas (and hence the theorems) into perspective, one should recall the following more classical type of ‘epsilon-regularity’ criteria, due primarily (when \((1.1)\) also holds) to [CKNS2] (or even [Sch77]) when \( d \equiv 0 \) and then extended to general \( d \in L^\infty \) in [LL96] and recently by the author in [Koc21] (see Section 4 below) to more general \( d \) when \((1.1)\) itself need not hold:

**Lemma 4.** Fix any \( C, D \in (0, \infty) \). There exists \( \bar{\epsilon} = \bar{\epsilon}(C, D) \in (0, 1) \) and, for each \( q \in (5, 6] \), there exists \( \bar{\epsilon}_q = \bar{\epsilon}_q(C) \in (0, 1) \) so small that the following holds for any fixed open set \( \Omega \subseteq \mathbb{R}^5 \) and \( T \in (0, \infty) \):

Set \( \Omega_T := \Omega \times (0, T) \) and suppose \( u, d : \Omega_T \to \mathbb{R}^3 \) and \( p : \Omega_T \to \mathbb{R} \) satisfy the assumptions \((1.5), (1.6), (1.7), (1.8)\) and \((1.9)\).

For any \( z_0 \in \Omega_T \) and \( r_0 \in (0, 1] \) such that \( Q_{r_0}(z_0) \subseteq \Omega_T \), if either

\[
\frac{1}{r_0^2} \int_{Q_{r_0}(z_0)} \left( |u|^3 + |\nabla d|^3 + |p|^\frac{3}{2} \right) \, dz + \frac{1}{r_0^2} \int_{Q_{r_0}(z_0)} |d|^q |\nabla d|^{3(1-\frac{q}{2})} \, dz \leq \bar{\epsilon}_q \quad \text{for some } q \in (5, 6]
\]

or

\[
d \in L^\infty(\Omega_T) \quad \text{with } ||d||_{L^\infty(\Omega_T)} \leq D \quad \text{and} \quad \frac{1}{r_0^2} \int_{Q_{r_0}(z_0)} \left( |u|^3 + |\nabla d|^3 + |p|^\frac{3}{2} \right) \, dz \leq \bar{\epsilon},
\]

then

\[
u, \nabla d \in L^\infty(Q^*_{r_1}(z_0)) \right). \tag{2.8}
\]

Lemmas 1, 2 and 3 all rely, in a fundamental way, on some version of Lemma 4. For Lemma 4 Q. Liu ([Lim18]) used the part for \( d \in L^\infty \) in the form given in [LL96] for solutions to \((1.1)\), whereas here we rely on \((2.6)\) of the more general Lemma 4 to prove Lemmas 2 and 3.

As Lemma 4 is of a similar form to Lemmas 1, 2 and 3, one may wonder how close one may come to Theorems 1, 2 and 3 using only Lemma 4 and Proposition 1 and the method of proof outlined above. When \( d \) is bounded as in Theorem 1, for example, if one uses the epsilon regularity criterion \((2.7)\) directly and applies Proposition 1 as in the proof above, one would only reach the conclusion that \( \dim_{\text{pf}}(\Sigma_\cap K) \leq 2 \), as \( r_0 \) appears in \((2.7)\) raised to the power \(2\). (One should compare this to the result \( \dim_{\text{pf}}(\Sigma) \leq 2 \) for \( d \equiv 0 \) proved by Scheffer in [Sch77].) One can, however, in fact reach \( \dim_{\text{pf}}(\Sigma_\cap K) \leq \frac{5}{2} \) (which should be compared to the improved\(^{17}\) result \( \dim_{\text{pf}}(\Sigma) \leq \frac{5}{2} \) for \( d \equiv 0 \) obtained by Scheffer in [Sch80], corresponding to the lower bound \( \gamma = 0 \) appearing in Lemma 1 and Theorem 1 more or less immediately from \((2.7)\) along with Hölder’s inequality. Indeed, one sees easily that Lemma 4 has the following immediate corollary, which more closely resembles Lemmas 1, 2 and 3.

\(^{17}\)We will see in Section 4 below how Lemma 4 in this form follows easily from the results given in [Koc21].

\(^{18}\)Scheffer’s result in [Sch80] was subsequently improved in [CKNS2] from \( \dim_{\text{pf}}(\Sigma) \leq \frac{5}{2} \) to \( \dim_{\text{pf}}(\Sigma) \leq 1 \), but this required a highly non-trivial application of (some version of) Lemma 4.
Corollary 1. Fix any $\bar{C}, D \in (0, \infty)$. There exists $\bar{c}^* = \bar{c}^*(\bar{C}, D) \in (0,1)$ and, for each $q \in (5,6]$, there exists $\bar{c}^*_q = \bar{c}^*_q(\bar{C}) \in (0,1)$ so small that the following holds for any fixed open set $\Omega \subseteq \mathbb{R}^3$ and $T \in (0,\infty)$:

Set $\Omega_T := \Omega \times (0,T)$ and suppose $u, d : \Omega_T \to \mathbb{R}^3$ and $p : \Omega_T \to \mathbb{R}$ satisfy the assumptions \[ (1.6), (1.7), (1.8) \text{ and } (1.9). \]

For any $z_0 \in \Omega_T$ and $r_0 \in (0,1]$ such that $Q_{r_0}(z_0) \subseteq \Omega_T$, if either

\[
\frac{1}{r_0^3} \int_{Q_{r_0}(z_0)} \left( |u|^\frac{10}{3} + |\nabla d|^{\frac{10}{3}} + |p|^{\frac{5}{3}} \right) \, dz + \frac{1}{r_0^3} \int_{Q_{r_0}(z_0)} |d|^{10} \, dz \leq \bar{c}^*_q \quad \text{for some } q \in (5,6] \tag{2.9}
\]

or

\[
d \in L^\infty(\Omega_T) \text{ with } \|d\|_{L^\infty(\Omega_T)} \leq D \quad \text{and} \quad \frac{1}{r_0^3} \int_{Q_{r_0}(z_0)} \left( |u|^\frac{10}{3} + |\nabla d|^{\frac{10}{3}} + |p|^{\frac{5}{3}} \right) \, dz \leq \bar{c}^*, \tag{2.10}
\]

then

\[
u, \nabla d \in L^\infty(Q_{2r_0}(z_0)).
\]

Proof of Corollary 1. Corollary 1 is a simple consequence of Lemma 1 along with Hölder’s inequality, in view of the convexity estimate $|d|^q |\nabla d|^{(1-\frac{q}{6})} \leq \frac{q}{6} |d|^q + (1-\frac{q}{6}) |\nabla d|^q$. \[ \square \]

For $d \in L^\infty(\Omega_T)$, one can now see clearly from the appearance of the power $\frac{q}{6}$ on $r_0$ in (2.10) how one would use Corollary 1 along with Proposition 1 to obtain the bound $\text{dim}_{\text{pf}}(\Sigma_+ \cap K) \leq \frac{25}{19}$. Therefore the novelty of Q. Liu’s Lemma 1 was that it reached $\frac{3}{7}$ (i.e., $\gamma = 0$) which appears in (2.10); more precisely, Lemma 1 extended this improvement all the way to $\gamma \approx \frac{10}{63}$, so in some sense $\frac{10}{63}$ quantifies the improvement\footnote{However, since the smaller cylinders of radius $\frac{4}{5}$ in Lemma 3 and Corollary 1 have radii depending linearly on the larger cylinders of radius $r_0$, those results would in fact yield (see e.g. \cite{CKNS2} \cite{Koc21} for details) estimates on the dimension of the full singular set $\Sigma$, e.g. $\text{dim}_{\text{pf}}(\Sigma \cap K) \leq \frac{2}{5}$, which is defined similar to $\Sigma_+$, but with $Q_\tau(z_0)$ replaced by the larger, and centered, $Q_\tau^*(z_0)$. On the other hand, as pointed out in Remark 2, the smaller radii $\frac{4}{5}$ in Lemma 1 and Lemma 2 depend non-linearly on $r_1 (\approx (\frac{49}{25})^1)$ for some $\eta > 0$ and it therefore seems difficult to conclude from them an estimate on the potentially larger set $\Sigma$ (which would also include points where, roughly speaking, blow-up occurs backwards in time). Remark 3 shows the situation to be similar in the context of Lemma 3.} of Lemma 1 over Corollary 1 (and hence over Lemma 1 in this context. Lemma 2 similarly obtains this improvement of $\frac{19}{39}$, even though one may relax slightly the assumption that $d$ is bounded by assuming (2.7) instead.

In the case (as in Theorem 3) where (potentially) $d \notin L^\infty(\Omega_T)$, if one instead uses the epsilons of Theorem 3 of Corollary 1 and applies Proposition 1, then due to the appearance of the larger power $5$ on $r_0$ in (2.10), one would only obtain the bound $\text{dim}_{\text{pf}}(\Sigma_+ \cap K) \leq 5$ which would correspond to $\delta = 0$ in the language of Lemma 3 and Theorem 3. In that setting, however, the epsilons of Corollary 3 (with $q \approx 5$) of Lemma 4 itself yields (due to the larger power $2 + \frac{5}{3}$ on $r_0$ in (2.10) the slightly better bound of $\text{dim}_{\text{pf}}(\Sigma_+ \cap K) \leq 2 + \frac{5}{3} = \frac{11}{3}$ corresponding to the lower bound $\delta = \frac{1}{2}$ in Lemma 3 and Theorem 3. (Note that the criterion (2.6) was established in \cite{Koc21} and used there to obtain the similar estimate $\text{dim}_{\text{pf}}(\Sigma) \leq \frac{9}{2}$, see \cite{Koc21}, Theorem 1.) On the other hand, applying Proposition 1 to Lemma 3 allows one to reach the bound $\text{dim}_{\text{pf}}(\Sigma_+ \cap K) \leq 5 - \delta$ even for $\delta \approx \frac{10}{63}$ ($\approx \frac{3}{7}$), and hence Lemma 3 (which in fact remains true for any $\delta \in (0,\frac{10}{63})$) still provides an improvement (in this context), roughly speaking, of $\frac{19}{39} - \frac{1}{2} = \frac{7}{39}$ over Lemma 3.

3 Proof of Proposition 1

In order to prove Proposition 1 (whose proof will certainly already be known to experts – we include a proof here for clarity and the benefit of the reader), we first note that the parabolic Vitali covering lemma recalled, for example, in \cite{CKNS2} Lemma 6.1], is based on the following fact, which is geometrically clear:
**Proposition 2.** Fix any \( \eta \in (0, 1) \) and \( \beta > 0 \), and for any \( r > 0 \) and \( z := (x, t) \in \mathbb{R}^3 \times \mathbb{R} \) define the parabolic cylinder

\[
Q^{\beta, \eta}_r(z) := B_r(x) \times I^{\beta, \eta}_r(t) \quad \text{with} \quad I^{\beta, \eta}_r(t) := (t - (1 - \eta)\beta r^2, t + \eta \beta r^2).
\]

Suppose \( A, B > 0 \) with

\[
A \geq \max \left\{ 2B + 1, \sqrt[3]{\frac{B^2}{\min\{\eta, 1 - \eta\}} + 1} \right\}.
\]

Then for any \( z_0, z_1 \in \mathbb{R}^3 \times \mathbb{R} \) and \( r_0, r_1 > 0 \), if

\[
r_1 \leq Br_0 \quad \text{and} \quad Q^{\beta, \eta}_{r_0}(z_0) \cap Q^{\beta, \eta}_{r_1}(z_1) \neq \emptyset
\]

then

\[
Q^{\beta, \eta}_{r_1}(z_1) \subseteq Q^{\beta, \eta}_{Ar_0}(z_0).
\]

The proof is simple, as it is geometrically clear that if even \( B_{r_0}(x_0) \cap B_{r_1}(x_1) \neq \emptyset \) (with the extreme being that the intersection is a single point) and \( r_0 + 2r_1 \leq Ar_0 \) then \( B_{r_1}(x_1) \subseteq B_{Ar_0}(x_0) \), and if even \( I^{\beta, \eta}_{r_0}(t_0) \cap I^{\beta, \eta}_{r_1}(t_1) \neq \emptyset \) (with the extreme being that the intersection is a single point) and both \( \eta \beta r_0^2 + \beta r_1^2 \leq \eta \beta (Ar_0)^2 \) as well as \((1 - \eta)\beta r_0^2 + \beta r_1^2 \leq (1 - \eta)\beta (Ar_0)^2\), then \( I^{\beta, \eta}_{r_1}(t_1) \subseteq I^{\beta, \eta}_{Ar_0}(t_0) \). (We omit the remaining details.)

In the proof of \[\text{[CKN82] Lemma 6.1}\], as they define the cylinders \( Q^* := Q^{1, 2} \), Proposition 2 is used with \( B := \frac{1}{2} \) (in fact, any \( B > 1 \) would work, with \( A \) adjusted appropriately) and with \( A := 5 \), which is suitable as

\[
\max\{2 \cdot \frac{3}{2} + 1, \sqrt{8 \cdot \left(\frac{3}{2}\right)^2 + 1}\} = \sqrt{19} \leq 5.
\]

However, the proof would similarly go through if we took \( Q^* := Q^{2, 2, \frac{1}{2}} \), which are centered parabolic cylinders of the form \( (1.11) \), in which case one may even take \( A = 4 \) (so taking \( A = 5 \) would certainly work as well)

\[
\max\{2 \cdot \frac{3}{2} + 1, \sqrt{2 \cdot \left(\frac{3}{2}\right)^2 + 1}\} = 4.
\]

We may therefore use \[\text{[CKN82] Lemma 6.1}\] as stated, but with the parabolic cylinders \( Q^*_r(z) \) defined as in \( (1.11) \) rather than as \( Q^1 r^2(z) \); let us call such a result \[\text{[CKN82] Lemma 6.1} (\text{1.11})\].

**Proof of Proposition 1.** As \( K \) is compact and \( \Omega_T \) is open, there exists some \( \bar{r} \in (0, \tilde{r}] \) sufficiently small that \( Q^r_{\bar{r}}(z_0) \subseteq \Omega_T \) for any \( z_0 \in K \). Indeed, if not then for any \( r \in (0, \bar{r}] \), there exists \( z(r) \in K \) and \( w(r) \notin \Omega_T \) such that \( w(r) \in Q^r_{\bar{r}}(z(r)) \). As \( K \) is (sequentially) compact, there exists a sequence \( 0 < r_k \to 0 \) as \( k \to \infty \) and \( \bar{z} \in K \) such that \( z(r_k) \to \bar{z} \) as \( k \to \infty \). By the triangle inequality, this implies as well that \( w(r_k) \to \bar{z} \) as \( k \to \infty \), which (as \( \bar{z} \in K \subseteq \Omega_T \)) contradicts the fact that \( \Omega_T \) is open, thus proving the assertion.

Fix \( \bar{r} \) as above. For any \( r \in (0, 5\bar{r}] \), let us consider the cover of the closure \((\overline{S \cap K})\) of \( S \cap K \) by the family of parabolic cylinders centered in \( S \cap K \) with radius \( \frac{r}{5} \):

\[
\overline{S \cap K} \subseteq \bigcup_{z \in \overline{S \cap K}} Q^r_{\bar{r}}(z)
\]

(as \( r > 0 \) is fixed for all \( z \), this indeed provides a cover of the closure). Since \( \overline{S \cap K} \) is also compact, there exists a finite sub-cover, i.e. there is some \( J \in \mathbb{N} \) and a set of distinct points \( \{z_j\}_{j=1}^J \subseteq \overline{S \cap K} \) such that

\[
\overline{S \cap K} \subseteq \bigcup_{j=1}^J Q^r_{\bar{r}}(z_j) \quad (\subseteq \Omega_T \text{ as } \frac{r}{5} < \bar{r}).
\]

By \[\text{[CKN82] Lemma 6.1} (1.11)\], there exists a further sub-set of this finite cover whose members (if there is more than one) are pair-wise disjoint, i.e. there exists \( K \in \mathbb{N} \) with \( K \leq M \) and a set of strictly
increasing sub-indices \( \{ j_k \}_{k=1}^K \subseteq \{ j \}_{j=1}^M \) such that 
\[ Q^*_T \left( z_{j_k} \right) \cap Q^*_T \left( z_{j_k} \right) = \emptyset \text{ if } 1 \leq k < \ell \leq K, \]
with the additional property that
\[
\left( S \cap K \subseteq \right) \bigcup_{j=1}^M Q^*_T \left( z_j \right) \subseteq \bigcup_{k=1}^K Q^*_T \left( z_{j_k} \right);
\]
in particular, the union on the right of \( K \) cylinders with radii \( (r) \) five times those \( (z) \) of the original cover constitutes an additional finite open cover. According to Definition 2, this implies that
\[
N(S \cap K, r) \leq K. \tag{3.1}
\]
On the other hand, as \( \frac{r}{5} \leq \tilde{r} \leq r \) and \( z_{j_k} \in S \) for \( 1 \leq k \leq K \), we deduce from (2.4) (along with the definition of \( \tilde{r} \)) that
\[
\frac{1}{(\frac{r}{5})^\lambda} \int_{Q^*_T \left( z_{j_k} \right)} H(z) \, dz \geq c_0 \text{ for } 1 \leq k \leq K.
\]
As the cylinders \( Q^*_T \left( z_{j_k} \right) \subseteq \Omega_T \) are disjoint, we then have
\[
K c_0 \left( \frac{r}{5} \right)^\lambda = \sum_{k=1}^K c_0 \left( \frac{r}{5} \right)^\lambda \leq \sum_{k=1}^K \int_{Q^*_T \left( z_{j_k} \right)} H(z) \, dz = \int_{\bigcup_{k=1}^K Q^*_T \left( z_{j_k} \right)} H(z) \, dz \leq \int_{\Omega_T} H(z) \, dz
\]
so that
\[
K \leq c_{\lambda,H} \left( \frac{1}{r} \right)^\lambda \tag{3.2}
\]
with
\[
c_{\lambda,H} := \frac{5^\lambda}{c_0} \int_{\Omega_T} H(z) \, dz < \infty.
\]
According to (3.1) and (3.2) and in view of Definition 2 we deduce that
\[
\dim_{pr}(S \cap K) = \limsup_{r \searrow 0} \frac{\log N(S \cap K, r)}{\log (\frac{1}{r})} \leq \limsup_{r \searrow 0} \frac{\log c_{\lambda,H} + \lambda \log (\frac{1}{r})}{\log (\frac{1}{r})} = \lambda
\]
which completes the proof. \( \square \)

### 4 Known epsilon-regularity criteria

In this section we show how Lemma 4 follows easily from certain results established in [Koc21]. In the following, for a given \( z_0 = (x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R} \) and \( r > 0 \), we will adopt the following the notation for the components of a standard parabolic cylinder \( Q_r(z_0) \):
\[
Q_r(z_0) := B_r(x_0) \times I_r(t_0), \text{ where } I_r(t_0) := (t_0 - r^2, t_0). \tag{4.1}
\]
Later on, we will also use the shorthand notations (for any set \( O \))
\[
\|U\|_{q,O} := \|U\|_{L^q(O)} \quad \text{and} \quad \|V\|_{q,F,Q_r(z_0)} := \|V\|_{L^r(I_r(t_0) ; L^q(B_r(x_0)))}
\]  \( \tag{4.2} \)

The following lemma was proved\(^{20}\) in [Koc21]:

---

\(^{20}\)Strictly speaking, in [Koc21], the first assumption on \( d \) in (1.3) appears as \( d \in L^\infty(I_1(t); L^2(B_1(x))) \) rather than \( d \in L^\infty(\Omega_1(t); L^2(B_1(x))) \); as described in [Koc21], the \( L^1 \) assumption is however the most natural one (in view of (1.3), as \( F \) is essentially quartic) and of course implies the \( L^2 \) assumption as the underlying set is bounded.
Proof of Lemma 4.

Fix any $\tilde{\epsilon}, D \in (0, \infty)$. There exists $\epsilon = \tilde{\epsilon}(\tilde{\epsilon}, D) \in (0, 1)$ and, for each $q \in (5, 6]$, there exists $\bar{\epsilon}_q = \tilde{\epsilon}_q(\tilde{\epsilon}) \in (0, 1)$ so small that the following holds for any fixed $\varepsilon = (\tilde{x}, \tilde{t}) \in \mathbb{R}^3 \times \mathbb{R}$ and $\rho \in (0, 1)$:

Suppose $u, d : Q_1(\varepsilon) \to \mathbb{R}^3$ and $p : Q_1(\varepsilon) \to \mathbb{R}$ satisfy (see (4.1))

$$d \in L^\infty(I_1(\tilde{t}); L^4(B_1(\tilde{x}))), \ u, \nabla d \in L^\infty(I_1(\tilde{t}); L^2(B_1(\tilde{x}))), \ \nabla u, \nabla^2 d \in L^2(Q_1(\varepsilon))$$

and $p \in L^{\mathfrak{Q}}(Q_1(\varepsilon))$,

$$\nabla \cdot u = 0 \quad \text{in} \ D'(Q_1(\varepsilon)) \quad \text{and}$$

$$\Delta p = \nabla \cdot \left(\nabla^T \cdot (u \otimes u + \nabla d \otimes \nabla d)\right) \quad \text{in} \ D'(Q_1(\varepsilon)),$$

along with the following local energy inequality

$$\int_{B_1(\tilde{x}) \times \{t\}} \left(\|u\|^2 + |\nabla d|^2\right) \phi \, dx + \int_{t-1}^{t} \int_{B_1(\tilde{x})} \left(\|u\|^2 + |\nabla^2 d|^2\right) \phi \, dx \
\leq \tilde{C} \int_{t-1}^{t} \left\{ \int_{B_1(\tilde{x}) \times \{t\}} \left[ \left(\|u\|^2 + |\nabla d|^2\right) \phi + \Delta \phi + (|u|^3 + |\nabla d|^3) |\nabla \phi| + \rho |d|^3 |\nabla d|^3 \phi \right] \, dx \
+ \left| \int_{B_1(\tilde{x}) \times \{t\}} p u \cdot \nabla \phi \, dx \right| \right\} \, dt$$

for a.e. $t \in I_1(\tilde{t})$ and $\forall \phi \in C_0^\infty(B_1(\tilde{x}) \times (t-1, \tilde{t}))$ s.t. $\phi \geq 0$.

Set

$$E_{3,q} := \int_{Q_1(\varepsilon)} \left(\|u\|^3 + |\nabla d|^3 + |p|^\frac{q}{2} + |d|^q |\nabla d|^{3(1-\frac{q}{2})}\right) \, dz$$

and

$$E_3 := \int_{Q_1(\varepsilon)} \left(\|u\|^3 + |\nabla d|^3 + |p|^\frac{q}{2}\right) \, dz.$$

The following then holds:

1. If $E_{3,q} \leq \bar{\epsilon}_q$ for some $q \in (5, 6]$, then $u, \nabla d \in L^\infty(Q_\frac{1}{2}(\varepsilon))$ with

$$\|u\|_{L^\infty(Q_\frac{1}{2}(\varepsilon))}, \|\nabla d\|_{L^\infty(Q_\frac{1}{2}(\varepsilon))} \leq \bar{\epsilon}_q^{2/9}.$$

2. (cf. [LL96, Theorem 2.6]) If $d \in L^\infty(Q_1(\varepsilon))$ with $\|d\|_{L^\infty(Q_1(\varepsilon))} \leq D$ and if $E_3 \leq \tilde{\epsilon}$, then similarly $u, \nabla d \in L^\infty(Q_\frac{1}{2}(\varepsilon))$ with

$$\|u\|_{L^\infty(Q_\frac{1}{2}(\varepsilon))}, \|\nabla d\|_{L^\infty(Q_\frac{1}{2}(\varepsilon))} \leq \tilde{\epsilon}_q^{2/9}.$$

For completeness, let us now use Lemma 5 to outline the simple proof (which can essentially be found in [Koc21]) of Lemma 4.

Proof of Lemma 4

If $z_0 = (x_0, t_0)$ with $x_0 \in \Omega$ and $t_0 \in (0, T)$, setting

$$u_{z_0,r_0}(x, t) := r_0 u(x_0 + r_0 x, t_0 + r_0^2 t), \quad p_{z_0,r_0}(x, t) := r_0^3 p(x_0 + r_0 x, t_0 + r_0^2 t)$$

and

$$d_{z_0,r_0}(x, t) := d(x_0 + r_0 x, t_0 + r_0^2 t),$$

21 See Footnote 11
22 Note that $E_{3,q} < \infty$ by (4.3) and standard embeddings.
23 An outline of the proof of this second part was given in the introduction to [Koc21]; see also [LL96] when (4.1) holds as well.
it follows (see [Koc21]) from the assumptions of Lemma 3 that the re-scaled triple \((u_{z_0,r_0}, d_{z_0,r_0}, p_{z_0,r_0})\) satisfies the conditions of Lemma 5 with \(\bar{z} := (0, 0)\) and \(\bar{\rho} := r_0^2\), for some \(\bar{C} = \bar{C}(\bar{C})\) depending only on the constant \(\bar{C}\) in \([149]\). If we set

\[ \tilde{e}(\bar{C}, D) := \tilde{e}(\bar{C}(\bar{C}), D) \quad \text{and} \quad \tilde{e}_q(\bar{C}) := \tilde{e}_q(\bar{C}(\bar{C})) , \]

then the smallness assumptions in \((2.6)\) or \((2.7)\) of Lemma 4 moreover imply that

\[ \frac{1}{r_0^2} \int_{Q_{r_0}(z_0)} (|u|^3 + |\nabla d|^3 + |p|^\frac{3}{2}) \, dz + \frac{1}{r_0^{2+\frac{q}{2}}} \int_{Q_{r_0}(z_0)} \mu \cdot |d|^q |\nabla d|^{3(1-\frac{q}{2})} \, dz = \]

\[ = \int_{Q_{(0,0)}} (|u_{z_0,r_0}|^3 + |\nabla d_{z_0,r_0}|^3 + |p_{z_0,r_0}|^\frac{3}{2} + \mu \cdot |d_{z_0,r_0}|^q |\nabla d_{z_0,r_0}|^{3(1-\frac{q}{2})}) \, dz \leq \epsilon_\mu \]

with

\[ \epsilon_\mu := (1 - \mu) \tilde{e}(\bar{C}, D) + \mu \tilde{e}_q(\bar{C}) = (1 - \mu) \tilde{e}(\bar{C}(\bar{C}), D) + \mu \tilde{e}_q(\bar{C}) \]

for some \(\mu \in (0, 1)\). Lemma 5 then implies that \(|u_{z_0,r_0}|, |\nabla d_{z_0,r_0}| \leq \frac{\epsilon_\mu}{r_0}\) on \(Q_2(0, 0)\), from which

\[ |u(y,s), |\nabla d(y,s)| \leq \frac{\epsilon_\mu}{r_0} \quad \text{for a.e.} \ (y,s) \in Q_{\frac{r_0}{2}}(z_0) \]

(and hence the conclusion of Lemma 3) follows in view of \((4.8)\). \(\square\)

5 Proofs of supporting lemmas

In what follows, let us set (recalling \((4.1)\))

\[
\begin{align*}
A_{z_0}(r) &:= \frac{1}{r} \text{ess sup}_{t \in I_r(z_0)} \int_{B_r(z_0)} (|u(t)|^2 + |\nabla d(t)|^2) \, dx, \\
E_{z_0}(r) &:= \frac{1}{r} \int \int_{Q_r(z_0)} (|\nabla u|^2 + |\nabla^2 d|^2) \, dz, \\
C_{z_0}(r) &:= \frac{1}{r^2} \int \int_{Q_{r}(z_0)} (|u|^3 + |\nabla d|^3) \, dz, \\
D_{z_0}(r) &:= \frac{1}{r^2} \int \int_{Q_{r}(z_0)} |p|^{3/2} \, dz, \\
\text{and} \quad G_{q,z_0}(r) &:= \frac{1}{r^{2+\frac{q}{2}}} \int \int_{Q_{r}(z_0)} |d|^q |\nabla d|^{3(1-\frac{q}{2})} \, dz.
\end{align*}
\]

(5.1)

We will make use of the following interpolation-type estimate (see [Koc21]) for the range of the quantities \(G_{q,z_0}\) (including \(G_{0,z_0} \leq C_{z_0}\)), a simple consequence of Hölder's inequality:

\[ 0 \leq q \leq \sigma \leq 6 \quad \implies \quad G_{q,z_0}(r) \leq G_{\sigma,z_0}(r) C_{z_0}^{1-\frac{q}{\sigma}}(r) \quad \forall \ r > 0. \]

(5.2)

Let us also set

\[ \mathcal{E}_{z_0}(r) := \int_{Q_{r}(z_0)} \left( |u|^{\frac{10}{3}} + |\nabla d|^{\frac{10}{3}} + |p|^{\frac{5}{2}} + |d|^{10} \right) \, dz \]

and

\[ \mathcal{F}_{z_0}(r) := r E_{z_0}(r) = \int_{Q_{r}(z_0)} (|\nabla u|^2 + |\nabla^2 d|^2) \, dz. \]

Note, in particular, that if we were to assume

\[ \mathcal{F}_{z_0}(R) = \int_{Q_{R}(z_0)} (|\nabla u|^2 + |\nabla^2 d|^2) \, dz \leq \epsilon_\gamma R^{\frac{5}{2} + \mu - \gamma} \]

(5.3)
Fix any Lemma 6. Lemma 2 and Lemma 3 will be consequences of the following technical lemma:

\[
\text{for some } \varepsilon \text{ and, in view of (5.2), with }
\]

\[
\text{and so that }
\]

\[
\text{(i.e., } \rho = 2r^\alpha \text{ and } \theta = r^\beta \text{) one has the estimate}^{24}
\]

\[
D_{z_0}(2r, N\beta + \alpha) \leq c_N \epsilon_\star^\frac{N}{\alpha} (r^\alpha A + r^B + r^{\bar{c}' C}) ,
\]

\[
G_{q, z_0}(2r, N\beta + \alpha) \leq c_{\epsilon_\star}^\frac{\mu}{N} r^{\bar{A}} ,
\]

\[
C_{z_0}(2r, N\beta + \alpha) \leq c_{\epsilon_\star}^\frac{\mu}{N} (r^B + r^{\bar{C}}) ,
\]

with

\[
A := \frac{3}{2} N\beta + \frac{3}{2} (\mu - \gamma) - [\alpha - 1] , \quad \bar{A} := \frac{3}{2} (\frac{\mu}{N} + \mu - \gamma) - (N\beta + \alpha) ,
\]

\[
B := (\frac{N}{\alpha} - 1)\beta + \frac{9}{10} \mu - \frac{9}{20} \gamma + [\alpha - 1] , \quad \bar{B} := \frac{9}{10} \mu - \frac{9}{20} \gamma + (N\beta + [\alpha - 1]) \quad \text{and}
\]

\[
C := -(N + 2)\beta + \frac{1}{\alpha} + \frac{5}{2} \mu - \frac{10}{20} \gamma - [\alpha - 1] , \quad \bar{C} := \frac{1}{\alpha} + \frac{5}{2} \mu - \frac{10}{20} \gamma - (N\beta + [\alpha - 1]) ,
\]

so that \( \bar{B} = B + (\frac{3}{2} N + 1)\beta > B \) and \( \bar{C} = C + 2\beta > C \). In particular (as \( r < 1 \)),

\[
C_{z_0}(2r, N\beta + \alpha) + D_{z_0}(2r, N\beta + \alpha) \overset{5.5, 5.7}{\leq} (c + c_N) \epsilon_\star^\frac{\mu}{N} (r^A + r^B + r^{\bar{C}}) \quad (5.8)
\]

and, in view of (5.5, 5.7),

\[
G_{q, z_0}(2r, N\beta + \alpha) \overset{5.5, 5.7}{\leq} g_\sigma^\frac{\mu}{N} (r^B + r^{\bar{C}})^{1 - \frac{\mu}{N}} \quad \text{as long as } \quad 0 \leq q \leq \sigma \leq 6 \quad (5.9)
\]

(with \( g_\sigma \in [0, \infty] \) defined as in (2.2)) and (as \( \epsilon_\star \leq 1 \))

\[
G_{q, z_0}(2r, N\beta + \alpha) \overset{5.5, 5.7}{\leq} c_{\epsilon_\star}^\frac{\mu}{N} (r^B + r^{\bar{C}})^{1 - \frac{\mu}{N}} \quad \text{as long as } \quad 0 \leq q \leq 6 . \quad (5.10)
\]

We will use (5.8) and (5.9) to prove Lemma 2 while we will use (5.8) and (5.10) to prove Lemma 3. Postponing momentarily the proof of Lemma 6, let us use it to prove Lemma 2 and Lemma 3.

---

24 Note that \( 2r^{N\beta + \alpha} \leq \theta^N \rho \leq \left( \frac{1}{2} \right)^N \cdot 2r < 2r \), so that \( Q_{2r, N\beta + \alpha}(z_0) \subset Q_{2r}(z_0) \subset \Omega_T \).
Proof of Lemma 2. Under the assumptions of Lemma 6, conclusions (5.8) and (5.9) imply that

\[ C_{z_0}(r_0) + D_{z_0}(r_0) + G_{q,z_0}(r_0) \leq (c + c_N)\epsilon_*^\beta (r^{3\gamma_2} + r^{3\theta C} + r^{2\theta C}) \leq \frac{\gamma}{\gamma_*} [\epsilon_*^\beta (r^{3\theta C} + r^{2\theta C})]^{1-\frac{\gamma}{\gamma_*}} \]

with \( r_0 := 2r^{N\beta + \alpha} < 2r \), as long as \( 0 \leq q \leq \sigma \leq 6 \). If we knew that

\[ A, B, C \geq 0, \quad (5.11) \]

this along with assumption (2.8) for some \( \sigma \in [5, 6] \) and \( D < \infty \) would imply (as \( r, \epsilon_* < 1 < c \)) that

\[ C_{z_0}(r_0) + D_{z_0}(r_0) + G_{q,z_0}(r_0) \leq \tilde{c}_{N,\sigma,C,D} \epsilon_*^{\beta \gamma} \leq \epsilon_{q_\sigma} \]

for any fixed \( q_\sigma \in (5, \sigma) \), for example \( q_\sigma := \frac{\gamma + 3}{\gamma} \), with \( \tilde{c}_{N,\sigma,C,D} := (3 + 2D) c(C) + 3c_N(C) \), provided that

\[ \epsilon_* \leq \left( \frac{\epsilon_{q_\sigma}}{\tilde{c}_{N,\sigma,C,D}} \right)^{\frac{10}{9(1 - \frac{\gamma}{\gamma_*})}} =: \epsilon^* = e^*(N, \sigma, C, D). \quad (5.12) \]

The main conclusion of Lemma 2 would then follow from Lemma 4 if, for example, \( \epsilon_* = \epsilon^* \). (To prove Lemma 2, we will need to take a certain \( N = N(\gamma) \), so that \( \epsilon^* = \epsilon^*(\gamma, \sigma, C, D) \).) As we’ll soon see, (5.11) can only hold if \( \gamma < \tilde{\gamma} \), and our goal will be to show that for any \( \tilde{\gamma} \in (0, 1) \), there exists some \( N, \beta, \alpha \) such that (5.11) holds with \( \mu = 0 \) and \( \gamma = \frac{10}{63} \tilde{\gamma} \).

Setting \( \mu := 0 \) and \( \gamma := \frac{16}{63} \tilde{\gamma} \) for some \( \tilde{\gamma} > 0 \), note first that (5.11) would imply that

\[ \tilde{\gamma} \leq 6(N - 1) \beta \leq \frac{2(N - 1)}{2N + 9} (3 - 2\tilde{\gamma}) \quad (5.13) \]

which (ignoring the intermediate inequality involving \( \beta \)) implies

\[ \tilde{\gamma} \leq 6(N - 1) \left[ \frac{1}{2N + 9} \right] \leq \frac{2(N - 1)}{2N + 9} (3 - 2\tilde{\gamma}). \quad (5.14) \]

(The two inequalities in (5.14) are equivalent, and clearly imply that \( \tilde{\gamma} < 1 \) as long as \( N \geq 2 \).) If (5.14) holds for some \( N \in N \cap [2, \infty) \) and \( \tilde{\gamma} \in (0, 1) \), we see easily that (5.13) will hold if we take

\[ \beta := \frac{1}{6N + 5} \quad ( > 0) \quad \iff \quad N\beta = \frac{1 - 5\beta}{6}. \]

For such a choice of \( \beta \) (and \( \mu \) and \( \gamma \)), we see that (5.11) says that

\[ \frac{1}{14} \tilde{\gamma} + \frac{23}{18} \beta - \frac{1}{18} \leq \alpha - 1 \leq \frac{1}{63} - \frac{7}{9} \beta + \min \left\{ \frac{11}{18} \beta + \frac{1}{21} (1 - \tilde{\gamma}), \frac{1}{2}, \frac{19}{63} (1 - \tilde{\gamma}) \right\}. \quad (5.15) \]

Note that

\[ \min \left\{ \frac{11}{18} \beta + \frac{1}{21} (1 - \tilde{\gamma}), \frac{1}{2}, \frac{19}{63} (1 - \tilde{\gamma}) \right\} > 0 \quad \text{for any } \beta > 0 \text{ and } \tilde{\gamma} < 1, \]

that

\[ \frac{1}{14} \tilde{\gamma} + \frac{23}{18} \beta - \frac{1}{18} \leq \frac{1}{63} - \frac{7}{9} \beta \quad \iff \quad \beta \leq \frac{9(1 - \tilde{\gamma})}{7 \cdot 44} \]

and that

\[ \frac{1}{63} - \frac{7}{9} \beta \geq 0 \quad \iff \quad \beta \leq \frac{2}{7 \cdot 21}; \]

hence for such \( \beta \) (equivalently, \( N \)), (5.15) will hold with (for example)

\[ \alpha := 1 + \left[ \frac{1}{63} - \frac{7}{9} \beta \right] \geq 1. \]

Fix now any \( \tilde{\gamma} \in (0, 1) \), set

\[ M_{\tilde{\gamma}} := \frac{1}{6} \max \left\{ \frac{7 \cdot 44}{9(1 - \tilde{\gamma})}, \frac{7 \cdot 21}{2} \right\} \quad ( > 2) \quad (5.16) \]
Note that if \( r < r^*(\bar{\gamma}) \), then the main conclusion of Lemma 3 would follow from Lemma 4 if, for example, \( q = q(\bar{\gamma}) \). We will see that one may take a certain \( \delta < \mathcal{N}(\bar{\gamma}) \). For such fixed \( N = N(\bar{\gamma}) \), set also
\[
\beta_N := \frac{1}{6N + 5} \quad \text{and} \quad \alpha_N := \frac{64}{63} - \frac{7}{5} \beta_N ,
\]
and note that
\[
\frac{1}{6(M_\beta + 1) + 5} < \beta_N \leq \frac{1}{6M_\beta + 5} \leq \min \left\{ \frac{9(1 - \bar{\gamma})}{7 \cdot 44}, \frac{2}{7 \cdot 21} \right\}
\]
which in turn implies that
\[
\alpha_N \geq \frac{64}{63} - \frac{7}{6} \cdot \frac{2}{7 \cdot 21} = 1
\]
and hence that (5.16) holds with \( \beta := \beta_N \) and \( \alpha := \alpha_N \). For any \( r > 0 \), let us now set
\[
\theta_{N,r} := r^{\beta_N} \quad \text{and} \quad \rho_{N,r} := 2r^{\alpha_N} .
\]
Note that if \( r \in (0, r^*) \subset (0, 1) \), then
\[
0 < \theta_{N,r} \leq \left( \frac{4}{5} \right)^{6(M_\beta + 11)\beta_N} \leq \left( \frac{4}{5} \right)^{6M_\beta + 11} = \frac{2}{5} \quad \text{and} \quad 0 < \rho_{N,r} \leq 2r ;
\]
in particular,
\[
r_0 := 2r^{N\beta_N + \alpha_N} = (\theta_{N,r})^N \rho_{N,r} \leq \left( \frac{4}{5} \right)^N \cdot 2r < 2r .
\]
Taking \( \epsilon^* := \epsilon^*(N(\bar{\gamma}), \sigma, \bar{C}, D) \) and \( r^* := r^*(\bar{\gamma}) \) as in (5.12), (5.10) and (5.17) for any \( \gamma \in (0, \frac{10}{63}) \), \( \sigma \in (5, 6) \) and \( \bar{C}, D \in (0, \infty) \), Lemma 4 therefore follows from Lemma 3 (with \( r := \bar{r}_0 = (0, \frac{1}{2}^\ast) \subset (0, 1) \), \( \epsilon^* = \epsilon^* \), \( \mu := 0 \), \( \rho := \rho_{N,\bar{r}_0} \), \( \bar{C} \) and \( \theta := \theta_{N,\bar{r}_0} \)), and taking \( q := \frac{5\cdot\bar{\gamma}}{6} \) in (5.9) and Lemma 3. \( \square \)

**Proof of Lemma 3.** Under the assumptions of Lemma 3, conclusions (5.8) and (5.10) imply that
\[
C_{z_0}(r_0) + D_{z_0}(r_0) + G_{q,z_0}(r_0) \leq (c + \epsilon_N)\epsilon^\frac{4}{3}\bar{A}^2 + r^{3\bar{A}} + r^{2\bar{C}} + 2c\epsilon^\frac{8}{3}(r^{3\bar{A}} + r^{2\bar{B}})
\]
with
\[
r_0 := 2r^{N\beta + \alpha} , \quad \bar{A}' := \eta_q\bar{A} + (1 - \eta_q)\bar{B}, \quad \text{and} \quad \bar{B}' := \eta_q\cdot2\bar{A} + (1 - \eta_q)\bar{C} , \quad \text{where} \quad \eta_q := \frac{q}{6} .
\]
If we knew that
\[
A, B, C, A'_q, B'_q \succeq 0
\]
for some \( q \in (5, 6) \), this would imply (as \( r, \epsilon^* < 1 \)) that
\[
C_{z_0}(r_0) + D_{z_0}(r_0) + G_{q,z_0}(r_0) \leq \tilde{c}_{N,\bar{C}}\epsilon^\frac{4}{3} \leq \epsilon_q
\]
with \( \tilde{c}_{N,\bar{C}} := 7c(\bar{C}) + 3c_N(\bar{C}) \), provided that
\[
\epsilon_\ast \leq \left( \frac{\epsilon_q}{\tilde{c}_{N,\bar{C}}} \right) \frac{4}{3} := \epsilon^* = \epsilon^*(N, \bar{C}, q) .
\]

The main conclusion of Lemma 3 would then follow from Lemma 4 if, for example, \( \epsilon_\ast = \epsilon^* \). (To prove Lemma 3, we will see that one may take a certain \( q = q(\delta) \) and \( N = 3 \), so that \( \epsilon^* = \epsilon^*(\delta, \bar{C}) \).)

We now claim that (5.18) holds for some \( q = q(\delta) \in (5, 6) \) provided that \( \gamma = 0, \alpha = 1 \) and \( \mu = \frac{10}{3} - \delta \) for some \( \delta < \frac{10}{13} \), i.e. \( \mu > \frac{100}{99} \), for some \( \beta > 0 \) and \( N \geq 3 \) such that
\[
\beta \in \left( 0, \frac{5}{2} \right) \quad \text{and} \quad N\beta \in \left( 0, \frac{44}{10} \right) .
\]
It is clear that $A, B \geq 0$ if $\gamma = 0$, $\alpha = 1$, $\beta \geq 0$, $\mu \geq 0$ and $N \geq 3$. Under the same assumptions, $C \geq 0$ provided that 
$$(N + 2)\beta \leq \frac{1}{3} \cdot \frac{4}{7} \mu,$$
and hence $C \geq 0$ if $\mu > 0$ and $\beta$ and $N$ satisfy (for example) (5.20), as then 
$$(N + 2)\beta < \frac{\mu}{10} + \frac{\mu}{2} = \frac{3}{5} \mu < \frac{1}{3} \cdot \frac{4}{7} \mu.$$ Now, when $\gamma = 0$ and $\alpha = 1$ we have 
$$\tilde{A} = -\frac{2}{\mu} + \frac{5}{7} \mu - N \beta, \quad \tilde{B} = \frac{3}{10} \mu + N \beta \quad \text{and} \quad \tilde{C} = \frac{1}{3} + \frac{4}{7} \mu - N \beta$$
so that 
$$A'_q = \left[\frac{1}{3} \eta_q + \frac{5}{7} (1 - \eta_q) + (5 \eta_q - 1) N \beta - \frac{2}{7} \eta_q \right] \geq 0$$
$$\iff \mu \geq 10 \left( 2 \eta_q - 1 \right) N \beta + \frac{20 \eta_q}{3 (3 - \eta_q)} = 120 \left( \frac{q - 3}{18 - q} \right) N \beta + \frac{20 q}{54 - 3 q} \approx \frac{40}{13} N \beta + \frac{100}{39}$$
for $q \approx 5$. Hence for any $\mu > \frac{100}{39}$ and $N \geq 3$, we can choose $q > 5$ sufficiently close to $5$ and then $\beta \in (0, \frac{100}{39})$ sufficiently small (depending on $q$ and $N$) to ensure that $A'_q \geq 0$ as well. For a fixed $\mu > \frac{100}{39}$, one can for example take $q = 5 + \frac{39 \mu - 100}{720 + (39 \mu - 100)}$ and $\beta = \beta_{\mu} := \frac{39 \mu - 100}{1200}$. (5.21) Similarly, we have 
$$B'_q = \frac{1}{3} - \frac{5}{7} \eta_q + \left( \frac{4}{5} - \frac{2}{7} \eta_q \right) \mu - (\eta_q + 1) N \beta \geq 0$$
$$\iff \mu \geq \frac{10}{3} \cdot \frac{5 \eta_q - 1}{4 - 2 \eta_q} + \frac{5}{4} \eta_q + \frac{1}{4} - 2 \eta_q N \beta = \frac{10}{3} \cdot \frac{5 q - 6}{24 - 2 q} + 5 \cdot \frac{q + 6}{24 - 2 q} N \beta \approx \frac{54}{14} N \beta$$
for $q \approx 5$. As $\frac{95}{42} \leq \frac{100}{39}$, this should not, in principle, impose additional constraints if $\mu > \frac{100}{39}$ and one can in fact check $\mu > \frac{100}{39}$ as well for $N$, $q$ and $\beta$ as in (5.21).

\footnote{Indeed, setting $\kappa := \frac{39 \mu - 100}{2}$ so that $\mu = \frac{100 + \kappa}{39}$, we have 
$$2 \eta_q \leq \frac{20 q}{54 - 3 q} \leq \frac{100 + \kappa}{39} \iff q \leq 5 + \frac{13 \kappa}{360 + \kappa},$$
which holds for example with $q := 5 + \frac{\kappa}{360 + \kappa}$ as in (5.21). Next, to make the sum less than or equal to $\mu = \frac{100 + \kappa}{39}$, we require (taking $N := 3$) 
$$2 \eta_q \leq \frac{5}{3} \cdot \frac{5 \eta_q - 1}{4 - 2 \eta_q} + \frac{5}{4} \eta_q + \frac{1}{4} - 2 \eta_q N \beta = \frac{10}{3} \cdot \frac{5 q - 6}{24 - 2 q} + 5 \cdot \frac{q + 6}{24 - 2 q} N \beta \approx \frac{54}{14} N \beta$$
for $q \approx 5$. As $\frac{95}{42} \leq \frac{100}{39}$, this should not, in principle, impose additional constraints if $\mu > \frac{100}{39}$ and one can in fact check $\mu > \frac{100}{39}$ as well for $N$, $q$ and $\beta$ as in (5.21).}
Setting $\mu(\delta) := \frac{10}{3} - \delta$ and taking $\epsilon^* := \epsilon^*(3, \bar{C}, q\mu(\delta)) = \epsilon^*(3, \bar{C}, 5 + \frac{10 - 13\delta}{200 - 13\delta})$ and $\gamma^* := \left(\frac{1}{2}\right)^{7\mu(\delta)} = \left(\frac{1}{2}\right)^{\frac{400}{10 - 13\delta}}$ as in (5.19) and (5.21) for any $\delta \in \left(\frac{1}{2}, \frac{10}{11}\right)$ (or even smaller) and $\bar{C} \in (0, \infty)$, Lemma 3 now follows from Lemma 6 (with $N := 3$, $r := \frac{5}{2}$, $\epsilon_\delta := \epsilon^*$, $\mu := \frac{10}{3} - \delta$, $\gamma := 0$, $\rho := r_1 = 2r$ and $\theta := \left(\frac{5}{2}\right)^{\frac{400}{10 - 13\delta}}$) so that $\alpha = 1$ and $\beta = \frac{400 - 13\delta}{200 - 13\delta}$, and taking $q := 5 + \frac{10 - 13\delta}{200 - 13\delta}$ in (5.10) and Lemma 4.

To prove Lemma 3 we will rely crucially on the following proposition (which is only slightly different from the corresponding result in Liu18) which is a consequence of the local energy inequality. (The reader should recall the notation 4.2 for Lebesgue norms which we will use in all of what follows.)

**Proposition 3.** There exists $C > 0$ such that the following holds:

Fix an open set $\Omega \subseteq \mathbb{R}^3$ and $\bar{C}, T \in (0, \infty)$, set $\Omega_T := \Omega \times (0, T)$ and suppose $u, d : \Omega_T \to \mathbb{R}^3$ and $p : \Omega_T \to \mathbb{R}$ satisfy the assumptions (1.5), (1.6) and (1.9).

Suppose

$$E_{z_0}(2r) = \int_{Q_{2r}(z_0)} \left( |u|^2 + |\nabla d|^2 + |\nabla \phi|^2 + |d|^4 \right) \, dz \leq \epsilon_\delta(2r)^{\frac{3}{2} + \mu - \gamma} \tag{5.22}$$

for some $z_0 \in \Omega_T$ and $r \in (0, 1]$ such that $Q_{2r}(z_0) \subseteq \Omega_T$ and some $\mu, \gamma \geq 0$ (one may assume $\mu \cdot \gamma = 0$) and $\epsilon^* \in (0, 1]$. Then (see 5.27)

$$A_{z_0}(r) = r^{-1} \left\| |u|^2 + |\nabla d|^2 \right\|_{1, \infty; Q_r(z_0)} \leq CC\epsilon^*_r r^{\frac{3}{2} + \mu - \gamma}. \tag{5.23}$$

**Proof of Proposition 3.** Using the backwards heat kernel and a suitable cut-off function, it is not hard to see (see, for example, Koc21, Liu18 for more details) that for any $z_0 \in \Omega_T$ and $0 < r \leq \frac{5}{2} \leq 1$, one can construct a test function $0 \leq \phi \in C_0^\infty(Q_r(z_0))$ with the following properties:

$$\frac{1}{r^3} \lesssim \phi \quad \text{on} \quad Q_r(z_0) \quad \tag{5.24}$$

and

$$\phi \lesssim \frac{1}{r^3}, \quad |\nabla \phi| \lesssim \left(\frac{p}{r} + 1\right) \frac{1}{r^4} \quad \text{and} \quad |\phi_t + \Delta \phi| \lesssim \frac{1}{r^p} \quad \text{on} \quad Q_r(z_0). \quad \tag{5.25}$$

Applying (1.9) for such a $\phi$, we have (with the constant in the second inequality being $\bar{C}$)

$$A_{z_0}(r) = r^{-1} \left\| |u|^2 + |\nabla d|^2 \right\|_{1, \infty; Q_r(z_0)} \lesssim r^2 \left\| |u|^2 + |\nabla d|^2 \right\|_{1, \infty; Q_r(z_0)}$$

$$\lesssim r^2 \left\| |\nabla d|^2 \right\|_{1, Q_r(z_0)} + \left(\frac{\rho}{r} + 1\right) \frac{1}{r^2} \left\| |u|^2 + |\nabla d|^2 \right\|_{1, Q_r(z_0)} + \frac{1}{r} |||d|^2|\nabla d|^2\phi||_{1, Q_r(z_0)}$$

$$\lesssim \left(\frac{p^2}{p^3} + \frac{1}{r^3}\right) \left\| |u|^2 + |\nabla d|^2 \right\|_{1, Q_r(z_0)} + \left(\frac{p}{r} + 1\right) \frac{1}{r^2} \left\| |u|^2 + |\nabla d|^2 \right\|_{1, Q_r(z_0)} + \frac{1}{r} \left\| |d|^2|\nabla d|^2\phi||_{1, Q_r(z_0)}$$

$$\lesssim \left(\frac{p^2}{p^3} + \frac{1}{r^3}\right) \left\| |u|^2 + |\nabla d|^2 \right\|_{1, Q_r(z_0)} + \left(\frac{p}{r} + 1\right) \frac{1}{r^2} \left\| |u|^2 + |\nabla d|^2 \right\|_{1, Q_r(z_0)} + \frac{1}{r} \left\| |d|^2|\nabla d|^2\phi||_{1, Q_r(z_0)}$$

$$\lesssim \left(\frac{p^2}{p^3} + \frac{1}{r^3}\right)^\frac{3}{2} \left\| |u|^2 + |\nabla d|^2 \right\|_{2, Q_r(z_0)} + \left(\frac{p}{r} + 1\right) \frac{1}{r^2} \left\| |u|^2 + |\nabla d|^2 \right\|_{1, Q_r(z_0)} + \frac{1}{r} \left\| |d|^2|\nabla d|^2\phi||_{1, Q_r(z_0)}.$$
Therefore, taking \( \rho := 2r \) so that, in particular,
\[
\frac{\rho}{r}, r, \epsilon_* \leq 2,
\]
assumption \([9,22]\) implies that
\[
A_{z_0}(r) \lesssim [\epsilon_* r^{\mu - \gamma}]^\frac{3}{8} + [\epsilon_* r^{\mu - \gamma}]^\frac{1}{4} + [\epsilon_* r^{\mu - \gamma}]^\frac{1}{8} = \left( r^{\frac{\mu + \gamma}{2}} + \epsilon_* r^{\frac{\mu + \gamma}{4}} + \epsilon_* r^{\frac{\mu + \gamma}{8}} \right) \lesssim \epsilon_* r^{\frac{\mu + \gamma}{8}}
\]
(as \( \mu, \gamma \geq 0 \)). If we keep track of the constants involved, the conclusion of the proposition follows. \( \square \)

As in \[Liu18\], we will also need the following two propositions, the first of which is a consequence of the Sobolev embeddings and Poincaré inequality while the second is a consequence of the elliptic theory; both can be found in the literature (for example, as indicated):

**Proposition 4** (Interpolation inequality; see, e.g., \[CY15\], Lemma 2). There exists a constant \( C > 0 \) such that, for any \( 0 < r' \leq r < \infty \) and any measurable function \( U : Q_r(z_0) \to \mathbb{R}^3 \), the estimate
\[
\|U\|_{3, Q_r(z_0)} \leq C \left( r'^{\frac{3}{2}} \|U\|_{2, \infty; Q_r(z_0)} \|\nabla U\|_{2, Q_r(z_0)} + r^{\frac{3}{2}} \|U\|_{2, \infty; Q_r(z_0)} \right)
\]
holds provided the right-hand side is well-defined. In particular, for any \( r > 0 \) and \( \eta \in (0, 1) \), one has an estimate of the form (see \([9,1]\))
\[
C_{z_0}(\eta r) \lesssim \eta \frac{3}{2} A_{z_0}(r) + \eta^{-\frac{3}{2}} A_{z_0}(r) E_{z_0}(r) \quad \text{for any } r > 0, \ \eta \in (0, 1], \ z_0 \in \mathbb{R}^{3+1}
\]
(5.26) provided that the right-hand side is well-defined.

**Proposition 5** (Interior elliptic estimate; see \[9,24\], e.g., \[GKT07\], Lemma 3.4). For any \( q \in (1, \infty) \) and \( n \in \mathbb{N} \), there exists a constant \( C_{q,n} > 0 \) such that if \( U \) is a weak solution to \(-\Delta U = \nabla \cdot (\nabla U \cdot F)\) in \( B_R(x_0) \subset \mathbb{R}^n \) for some \( R > 0 \) and \( x_0 \in \mathbb{R}^n \), then
\[
\|U\|_{q, B_{\theta R}(x_0)} \leq C_{q,n} \left( \|F\|_{q, B_{\theta R}(x_0)} + \theta^{\frac{n}{2}} \|U\|_{q, B_{\frac{\theta}{2} R}(x_0)} \right), \quad \text{for any } \theta \in (0, \frac{1}{2}],
\]
(5.27)
provided \( p \) satisfies the pressure equation (1.8) and \( Q_R(z_0) \subseteq \Omega_T \).

As in \[Liu18\], let us now use these propositions to prove Lemma 6.

**Proof of Lemma 6**. Under the assumptions of Lemma 6 taking \( \eta := \theta^{\frac{j}{2}} \) in (5.26) for any \( j \in \mathbb{N} \), Proposition 3 and Proposition 4 imply that
\[
C_{z_0}(\theta^j \rho) \lesssim \left( \theta^j \frac{3}{2} A_{z_0}(r) + \theta^j \frac{3}{2} A_{z_0}(r) E_{z_0}(r) \right)
\]
\[
\lesssim C \left( \theta^j \frac{3}{2} \left[ \epsilon_* r^{\frac{\mu + \gamma}{8}} \right]^\frac{3}{4} + \theta^j \frac{3}{4} \left[ \epsilon_* r^{\frac{\mu + \gamma}{4}} \right]^\frac{3}{2} + \theta^j \frac{3}{8} \left[ \epsilon_* r^{\frac{\mu + \gamma}{8}} \right]^\frac{1}{2} \right)
\]
\[
\lesssim \epsilon_* \left( \theta^j \rho \right)^3 \left[ \epsilon_* r^{\frac{\mu + \gamma}{8}} \right]^\frac{3}{2} r^{3+3+3} + \theta^j \rho \left[ \epsilon_* r^{\frac{\mu + \gamma}{4}} \right]^\frac{3}{2} \left[ \epsilon_* r^{\frac{\mu + \gamma}{8}} \right]^\frac{1}{2} \right)
\]
(5.28)
which, when \( j = N \), implies (5.7) for some \( c = c(\bar{C}) > 1 \). Moreover, noting that
\[
D_{z_0}(\rho) \lesssim \rho^{-\frac{n}{2}} \left\| p \right\|_{1, Q_2(z_0)}^{\frac{n}{2}} \leq \rho^{-\frac{n}{2}} \left\| p \right\|_{1, Q_2(z_0)}^{\frac{n}{2}} \leq \rho^{-\frac{n}{2}} \left[ \epsilon_{z_0}(2r) \right]^{\frac{n}{2}} \leq \rho^{-\frac{n}{2}} \left[ \epsilon_{z_0}(2r) \right]^{\frac{n}{2}},
\]
(5.29)
\[GKT07\] states the result for any \( \theta \in (0, \frac{1}{2}] \), but the author believes this to be a typographical error.
repeated applications of Proposition 3 (with $R := \theta^{k-1} \rho$ for $k \leq N$) along with \[5.28\] imply that
\[
D_{z_0}(\theta^N \rho) \lesssim_N \theta^N D_{z_0}(\rho) + \theta^{N-3} \sum_{j=0}^{N-1} \theta^{-j} C_{z_0}(\theta^j \rho) \lesssim_C \epsilon_*^{\frac{\theta}{2}} \left( \theta^N \rho - \frac{\theta}{2} \left[ \epsilon_*(2r)^{\frac{N}{2}} \right]^{\frac{3}{4}} + \theta^{N-3} \sum_{j=0}^{N-1} \left[ \theta^2 j \rho^3 \frac{\rho^2}{r^2 \gamma^3 + \theta^{-\frac{N}{2}} \theta^{-2} \rho^2 + \theta^{-N} \frac{1}{2} \rho^2} \right] \right)
\]
\[\left( \theta < 1 \right) \leq \epsilon_*^{\frac{\theta}{2}} \left( \theta^N \rho - \frac{\theta}{2} \left[ \epsilon_*(2r)^{\frac{N}{2}} \right]^{\frac{3}{4}} + \theta^{N-3} \left[ \rho^3 \frac{\rho^2}{r^2 \gamma^3 + \theta^{-\frac{N}{2}} \theta^{-2} \rho^2 + \theta^{-N} \frac{1}{2} \rho^2} \right] \right)
\]
which implies \[5.6\] for some $c_N = c_N(C) > 1$. Finally, setting $r_0 := \theta^N \rho \leq \left( \frac{1}{4} \right)^N (2r) < 2r$, we have
\[
G_{6,z_0}(r_0) \lesssim r_0^{-3} \left\| d^{10} \right\|_{1,Q_{r_0}(z_0)} \leq r_0^{-3} \left\| d^{10} \right\|_{1,Q_{2r}(z_0)} \leq r_0^{-3} \left[ \epsilon_{z_0}(2r) \right]^{\frac{1}{2}} \leq r_0^{-3} \left[ \epsilon_*(2r)^{\frac{N}{2}} \right]^{\frac{1}{2}}
\]
which implies \[5.6\] for some $c > 1$ (which in fact does not depend on $C$). \hfill \Box

References

[CKN82] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 35(6):771–831, 1982.

[CY15] Hi Jun Choe and Min Suk Yang. Hausdorff measure of the singular set in the incompressible magnetohydrodynamic equations. *Comm. Math. Phys.*, 336(1):171–198, 2015.

[GKT07] Stephen Gustafson, Kyungkeun Kang, and Tai-Peng Tsai. Interior regularity criteria for suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 273(1):161–176, 2007.

[Koc21] Gabriel S. Koch. Partial regularity for Navier-Stokes and liquid crystals inequalities without maximum principle. [arXiv:2001.04098](http://arxiv.org/abs/2001.04098) (to appear in *Analysis & PDE*), 2021.

[KP12] Igor Kukavica and Yuan Pei. An estimate on the parabolic fractal dimension of the singular set for solutions of the Navier-Stokes system. *Nonlinearity*, 25(9):2775–2783, 2012.

[Liu18] Qiao Liu. Dimension of singularities to the 3d simplified nematic liquid crystal flows. *Nonlinear Anal. Real World Appl.*, 44:246–259, 2018.

[Liu21] Qiao Liu. Partial regularity and the Minkowski dimension of singular points for suitable weak solutions to the 3D simplified Ericksen-Leslie system. *Discrete Contin. Dyn. Syst.*, 41(9):4397–4419, 2021.

[LL95] Fang-Hua Lin and Chun Liu. Nonparabolic dissipative systems modeling the flow of liquid crystals. *Comm. Pure Appl. Math.*, 48(5):501–537, 1995.

[LL96] Fang-Hua Lin and Chun Liu. Partial regularity of the dynamic system modeling the flow of liquid crystals. *Discrete Contin. Dyn. Systems*, 2(1):1–22, 1996.

[Sch77] Vladimir Scheffer. Hausdorff measure and the Navier-Stokes equations. *Comm. Math. Phys.*, 55(2):97–112, 1977.

[Sch80] Vladimir Scheffer. The Navier-Stokes equations on a bounded domain. *Comm. Math. Phys.*, 73(1):1–42, 1980.

[Sch85] Vladimir Scheffer. A solution to the Navier-Stokes inequality with an internal singularity. *Comm. Math. Phys.*, 101(1):47–85, 1985.

\[30\] Here, as in [Liu18], we conclude very crudely using $L^\infty \hookrightarrow L^1_{\text{loc}}$; that is, we do not make use of any convergent geometric series.