The width of the color flux tube at 2-Loop order

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Abstract: The color flux tube connecting a static quark-anti-quark pair in Yang-Mills theory supports massless transverse fluctuations, which are the Goldstone bosons of spontaneously broken translation invariance. Just as in chiral perturbation theory, the dynamics of these Goldstone bosons is described by a systematic low-energy effective field theory. We use the effective theory to calculate the width of the fluctuating string at the 2-loop level, using both cylindrical and toroidal boundary conditions. At zero temperature, the string width diverges logarithmically with the quark-anti-quark distance $r$. On the other hand, at low but non-zero temperature $T = 1/\beta$, for $r \gg \beta$ the string width diverges linearly.

Keywords: Nonperturbative Effects, Lattice Gauge Field Theories, Lattice Quantum Field Theory, Bosonic Strings
1. Introduction

Non-perturbative phenomena emerging in strongly coupled systems are quite common in physics. For example, in condensed matter physics, strongly coupled electrons are responsible for quantum antiferromagnetism and high-temperature superconductivity, whose origin remains mysterious even after decades of intensive research. In particle physics, the confinement of quarks is one of the most fundamental features of the strong interactions, and our understanding of the non-perturbative dynamics underlying quark confinement is still far from being complete. In general, it is very challenging to understand the dynamical mechanisms that are responsible for non-perturbative effects, also because collective modes play a crucial role.

However, even if the origin of some non-perturbative effect remains unclear, one may be able to quantitatively understand some of its consequences. In particular, one may analyze the symmetries — internal or related to space-time — of the ground state of a non-perturbative system, and then work out the consequences of spontaneous symmetry breaking, even if the dynamics leading to this phenomenon may remain unclear. In this framework, the typical energy scale of the non-perturbative phenomenon represents an upper bound for the energy of the phenomena that one wants to investigate. When a continuous symmetry breaks spontaneously, the resulting Goldstone bosons are the natural degrees of freedom to be taken into account in the low-energy regime of the theory. The Lagrangian
of the corresponding systematic low-energy effective theory consists of all terms respecting
the internal and space-time symmetries of the system. The terms in the Lagrangian are
multiplied by low-energy parameters, whose explicit values can be derived only from the
underlying quantum theory. They represent the high-energy non-perturbative input for
the low-energy dynamics.

The effective field theory approach is a very powerful tool for investigating strongly cou-
pled systems. In particular, it provides a way to perform analytic calculations in physical
systems that would otherwise be unmanageable. For example, chiral Lagrangians describe
pion and pion-nucleon systems in a very accurate manner. Similarly, a systematic low-
energy theory for magnons provides an excellent description of the low-energy dynamics of
quantum antiferromagnets.

In 1980, Lüscher, Symanzik, and Weisz [1] proposed a low-energy effective string de-
scription for the color flux tube connecting a static quark-anti-quark pair in the confined
phase of $d$-dimensional Yang-Mills theory. During its time-evolution, the flux string sweeps
out a 2-dimensional world-sheet, thereby spontaneously breaking the translation invariance
in the transverse directions. The $(d-2)$ Goldstone bosons resulting from that breaking
are the relevant degrees of freedom in the low-energy effective description of the string
dynamics. The leading term of the effective action describes the dynamics of a thin string
in the free-field approximation. Two non-trivial results follow from this observation. First,
the leading correction to the linear term in the static potential [2] is $-\pi(d-2)/(24r)$, where
$r$ is the distance between the two static sources. Second, the flux tube width is not fixed
but increases like $(d-2)\log(r/r_0)/(2\pi\sigma)$ [3], with the string tension $\sigma$ and a length scale
$r_0$ entering the effective theory as low-energy parameters.

Many numerical lattice simulations have successfully demonstrated the validity of the
predictions of the effective theory [4–19]. However, the very high numerical accuracy of the
Monte Carlo simulations calls for improving the analytic results of the low-energy effective
description. The sub-leading corrections to the free-field action of the $(d-2)$ Goldstone
bosons depend on a number of parameters that, in general, can be fixed only by matching
to the underlying Yang-Mills theory. Remarkably, Lüscher and Weisz have pointed out that
the low-energy effective string theory has a symmetry that had been overlooked before [20].
This new symmetry implies some relations between the values of the low-energy parameters
of the effective theory. Interestingly, the Nambu-Goto action satisfies the constraints that
one finds at next-to-leading order. In the special case of $d = 3$, the values of the low-
energy parameters are completely fixed up to the next-to-next-to-leading order and they
turn out to be the same as those of the Nambu-Goto action [21]. In this paper we report
on the computation of the correction to the increase of the flux tube width resulting from
the sub-leading correction to the free-field action in $d$ dimensions. The final expressions
involve the Dedekind $\eta$ function and the Eisenstein series $E_2$ and $E_4$. Due to the modular
inversion property of those functions, we have the next-to-leading correction to the string
width both at zero and at finite (but low) temperature.

The paper is organized as follows. Section 2 contains the description of the systematic
low-energy effective string theory for the dynamics of the confining string. In section 3
toroidal boundary conditions, which describe a closed string wrapping around a compact
spatial dimension, are investigated. Similarly, section 4 discusses cylindrical boundary conditions, which correspond to the propagation of an open string that ends in the static quark-anti-quark charges. For both boundary conditions the width of the string is analytically calculated at the 2-loop level. We present our conclusions in section 5. Technical details are described in four appendices.

2. Effective string theory

In this section we set up the framework for our computation. The Goldstone bosons resulting from the breaking of translation invariance are represented by a \((d-2)\)-component real-valued scalar field \(\vec{h}(x, t)\) living in a 2-dimensional rectangular base-space \((x, t)\) of size \(r \times \beta\). The field \(\vec{h}\) describes the displacement of the vibrating string from the minimal length arrangement in the transverse \((d-2)\) dimensions. The leading-order free-string approximation is given by the following action containing two derivatives

\[
S_2[\vec{h}] = \sigma \int_0^{\beta} dt \int_0^r dx \left( \partial_\mu \vec{h} \cdot \partial_\mu \vec{h}, \quad \mu \in \{x, t\}, \right) \tag{2.1}
\]

where \(\sigma\) is the string tension. In the effective theory the string tension is a low-energy parameter whose explicit value can be obtained only from the underlying Yang-Mills theory.

In our computation we consider toroidal boundary conditions, i.e.

\[
\vec{h}(x, t) = \vec{h}(x, t + \beta); \quad \vec{h}(x, t) = \vec{h}(x + r, t), \tag{2.2}
\]

as well as cylindrical boundary conditions, i.e.

\[
\vec{h}(x, t) = \vec{h}(x, t + \beta); \quad \vec{h}(0, t) = \vec{h}(r, t) = \vec{0}, \tag{2.3}
\]

for the string. When one interprets the \(t\)-direction as Euclidean time, cylindrical boundary conditions describe the propagation of an open string at finite temperature. On the other hand, when one interprets the \(x\)-direction as Euclidean time, the same boundary conditions correspond to the propagation of a closed string that is created at “time” \(x = 0\) and annihilated at \(x = r\). This dual interpretation gives rise to an open-closed string duality of the string theory.

The first bulk correction to the free-string action contains four derivatives and is given by

\[
S_4[\vec{h}] = \sigma \int_0^{\beta} dt \int_0^r dx \left[ c_2(\partial_\mu \vec{h} \cdot \partial_\mu \vec{h})^2 + c_3(\partial_\mu \vec{h} \cdot \partial_\nu \vec{h})^2 \right]. \tag{2.4}
\]

The open-closed string duality [20] constrains the value of the two low-energy parameters \(c_2\) and \(c_3\) by

\[
(d - 2) c_2 + c_3 = \frac{d - 4}{8}. \tag{2.5}
\]

Note that, for \(d = 3\), i.e. for a 1-component field \(h(x, t)\), the two terms in eq. (2.4) are identical and hence the first correction is completely fixed. As shown in [21], for general
A generalization of Lüscher and Weisz’s argument provides a further constraint on the next-to-leading order coefficients

$$c_2 + c_3 = -\frac{1}{8}. \quad (2.6)$$

Thus, for any value of $d$, the two independent constraints eq. (2.5) and eq. (2.6) completely fix the effective action at this perturbative order with $c_2 = \frac{1}{8}$ and $c_3 = -\frac{1}{4}$. Interestingly, the expansion of the Nambu-Goto action

$$S_{NG}[\vec{h}] = \sigma \int_{\beta}^{\beta} dt \int_{0}^{r} dx \sqrt{1 + \partial_x \vec{h} \cdot \partial_x \vec{h} + \partial_t \vec{h} \cdot \partial_t \vec{h} + (\partial_x \vec{h} \times \partial_t \vec{h})^2} \quad (2.7)$$

satisfies these two constraints.

Since cylindrical boundary conditions explicitly break translation invariance in the $x$-direction, one would expect surface terms (located at the boundaries at $x = 0$ and $x = r$) to appear in the effective action. Remarkably, as was shown by Lüscher and Weisz, due to open-closed string duality such terms are absent at leading order (i.e. there are no boundary terms with two derivatives). Boundary terms with an odd number of derivatives are excluded by parity symmetry. However, boundary terms with four derivatives do indeed exist. Fortunately, such terms contribute at one order higher than the four-derivative terms in the bulk that we discussed before. As a result, in our study boundary terms need not be taken into account.

The squared width of the string is defined as the second moment of the field $\vec{h}$, i.e.

$$w^2(x, t) = \langle (\vec{h}(x, t) - \vec{h}_0)^2 \rangle = \frac{\int D\vec{h} (\vec{h}(x, t) - \vec{h}_0)^2 \exp(-S[\vec{h}])}{\int D\vec{h} \exp(-S[\vec{h}]).} \quad (2.8)$$

Here $S[\vec{h}]$ is the effective string action and

$$\vec{h}_0 = \frac{1}{\beta r} \int_{0}^{\beta} dt \int_{0}^{r} dx \, \vec{h}(x, t) \quad (2.9)$$

is the equilibrium position of the string. For cylindrical boundary conditions, we have $\vec{h}_0 = 0$. At next-to-leading order, the string action is given by $S[\vec{h}] = S_2[\vec{h}] + S_4[\vec{h}]$. Similarly, at next-to-leading order, the field is replaced by $\vec{h}(x, t) \rightarrow \vec{h}(x, t) + \alpha \partial_{\mu} \partial_{\mu} \vec{h}(x, t)$, where $\alpha$ is a low-energy parameter. Expanding around the free-string action, the squared width of the string is given by

$$w^2(x, t) = w_{lo}^2(x, t) - \langle (\vec{h}(x, t) - \vec{h}_0^2) \rangle \left( S_4 \right)_0 + 2\alpha \langle (\partial_{\mu} \vec{h}(x, t))^2 \rangle_0 + \alpha^2 \langle (\partial_{\mu} \partial_{\mu} \vec{h}(x, t))^2 \rangle_0 - \frac{2\alpha}{\beta r} \int dx \langle \vec{h}_0 \cdot \partial_{\mu} \vec{h}(x, t) \rangle_0$$

$$- \frac{\alpha^2}{(\beta r)^2} \int dt \, dx \, dt' \, dx' \, \langle \partial_{\mu} \partial_{\mu} \vec{h}(x, t) \cdot \partial_{\mu} \partial_{\mu} \vec{h}(x', t') \rangle_0. \quad (2.10)$$

Here $\langle \ldots \rangle_0$ represents the vacuum expectation value with respect to the free-string action and

$$w_{lo}^2(x) = \langle \vec{h}(x, t)^2 \rangle_0 - \langle \vec{h}_0^2 \rangle_0 \quad (2.11)$$
Appendix B, the single-component free field propagator can be written as
\[ (\tilde{h}(x,t)^2 S_4)_0 = 4(d-2) \{ [(d-2)c_2 + c_3] T_1 + [2c_2 + (d-1)c_3] T_2 \} . \] (2.12)

Using the two constraints eq. (2.3) and eq. (2.6) this implies
\[ (\tilde{h}(x,t)^2 S_4)_0 = \frac{(d-2)^2}{2}(T_1 - 2T_2) - (d-2)T_1. \] (2.13)

The two terms \( T_1 \) and \( T_2 \) are given by
\[ T_1 = \lim_{\epsilon, \epsilon' \to 0} \int_0^\beta dt' \int_0^r dx' \partial_\mu' G(x,t;x',t') \partial_{\nu'} G(x'',t'',x,t) \partial_\mu \partial_{\nu'} G(x',t';x'',t''), \] (2.14)
as well as
\[ T_2 = \lim_{\epsilon, \epsilon' \to 0} \int_0^\beta dt' \int_0^r dx' \partial_\mu' G(x,t;x',t') \partial_{\nu'} G(x'',t'',x,t) \partial_\mu \partial_{\nu'} G(x',t';x'',t''), \] (2.15)
where \( x'' = x' + \epsilon \) and \( t'' = t' + \epsilon' \). Since they are ultraviolet divergent for \( (x',t') = (x'',t'') \), the integrals defined above have been regularized using the point-splitting method. Finally, we have
\[ \langle \tilde{h}(x,t) \partial_\mu' \partial_{\nu'} \tilde{h}(x',t') \rangle_0 = (d-2) \partial_\mu \partial_{\nu'} G(x,t;x',t'), \]
\[ \langle \partial_\mu \partial_{\nu'} \tilde{h}(x,t) \partial_\mu' \partial_{\nu'} \tilde{h}(x',t') \rangle_0 = (d-2) \partial_\mu \partial_\mu' \partial_{\nu'} \partial_{\nu'} G(x,t;x',t'). \] (2.16)

3. Toroidal boundary conditions

In this section we present the computation of the string width with toroidal boundary conditions at next-to-leading order in the low-energy effective theory. As we show in Appendix [3], the single-component free field propagator can be written as
\[ G(x,t) = \frac{t(t-\beta)}{2\sigma \beta r} + \frac{1}{2\pi \sigma} \sum_{n=1}^{\infty} \cos \left( \frac{2\pi n x}{r} \right) \frac{e^{-2\pi n t/r} + q^n e^{2\pi n t/r}}{n(1-q^n)} + K, \] (3.1)
where
\[ u = \frac{\beta}{r}, \quad q = e^{-2\pi u}, \quad K = \frac{\beta}{12\sigma r} + \frac{1}{\pi \sigma} \log \eta(iu). \] (3.2)

It should be noted that \( t \in [0, \beta] \). In eq. (3.1) we have used translation invariance, i.e. \( \langle \tilde{h}(x,t) h(x',t') \rangle = G(x-x',t-t') \). Translation invariance also implies that the string width \( w(x) = w \) does not depend on the position \( x \). At leading order, the squared width, \( w_{lo}^2 \), is ultraviolet divergent and we regularize it using the point-splitting method
\[ w_{lo}^2 = \lim_{\epsilon, \epsilon' \to 0} \langle \tilde{h}(x,t) \tilde{h}(x',t') \rangle_0 - \langle \tilde{h}_0 \rangle_0 = (d-2) \left\{ G(\epsilon, \epsilon') - \int_0^\beta \frac{dt}{\beta} \int_0^r \frac{dx}{r} G(x,t) \right\} . \] (3.3)
Using eq. (B.2) one immediately obtains
\[ G(\epsilon, \epsilon') = \frac{1}{2\pi\sigma} \log \frac{r}{r_0}, \quad r_0 = 2\pi\sqrt{\epsilon^2 + \epsilon'^2} \] (3.4)
as well as
\[ \int_0^r \frac{dt}{\beta} \int_0^r \frac{dx}{r} G(x, t) = -\frac{\beta}{12\sigma r} + K, \] (3.5)
such that
\[ w_{lo}^2 = \frac{d - 2}{2\pi\sigma} \log \frac{r}{r_0} - \frac{d - 2}{\pi\sigma} \log \eta(iu). \] (3.6)
The quantity \( r_0 \) is a low-energy parameter of dimension [length].

Let us now consider the corrections to this behavior resulting from the next-to-leading term, \( S_4 \), of the effective string action. By explicit calculation, it turns out that \( \langle h_0^2 S_4 \rangle_0 = 0 \). Hence, it remains to evaluate eq. (2.13) with \( T_1 \) and \( T_2 \) given by eq. (2.14) and eq. (2.15). We find
\[ T_1 = -\frac{1}{2\pi \sigma^2 \beta r} \left[ \log \frac{r}{r_0} - 2 \log \eta(iu) \right], \]
\[ T_2 = \frac{T_1}{2} + \frac{\pi u E_2(iu)}{72 \sigma^2 r^2} \left( \frac{E_2(iu)}{12 \sigma^2 r^2} + \frac{1}{8\pi \sigma^2 \beta r} \right). \] (3.7)
Furthermore, using the two identities
\[ \partial_x \partial_x G(x - x', t - t') = -\partial_t \partial_t G(x - x', t - t') + \frac{1}{\sigma \beta r}, \]
\[ \partial_{x'} \partial_{x'} \partial_t \partial_t G(x - x', t - t') = \partial_{t'} \partial_{t'} \partial_t \partial_t G(x - x', t - t'), \] (3.8)
we find that the terms proportional to \( \alpha \) cancel and those proportional to \( \alpha^2 \) vanish. Hence, at next-to leading order the squared width of the string is given by
\[ w^2 = \left( 1 - \frac{1}{\sigma \beta r} \right) w_{lo}^2 + \frac{(d - 2)^2}{4 \sigma^2 \beta r} \left( \frac{\pi}{18} \left[ u E_2(iu) \right]^2 - \frac{u E_2(iu)}{3} + \frac{1}{2\pi} \right). \] (3.9)
It should be noted that, order by order, this expression is modular invariant. In particular, it is invariant under the interchange of \( r \) and \( \beta \).

4. Cylindrical boundary conditions

Together with Appendix D, this section contains the calculation of the string width at next-to-leading order for cylindrical boundary conditions. In Appendix C, we show that the single-component free field propagator can be written as
\[ G(x, t; x', t') = \frac{1}{\pi\sigma} \sum_{n=1}^{\infty} \sin \left( \frac{n\pi x}{r} \right) \sin \left( \frac{n\pi x'}{r} \right) \frac{e^{-n\pi(t-t')/r} + q^n}{n(1 - q^n)} e^{n\pi(t-t')/r}, \] (4.1)
with
\[ u = \frac{\beta}{2r}, \quad q = e^{-2\pi u}. \] (4.2)
In eq. (4.1) we have used that \((t - t') \in [0, \beta]\). We now calculate the string width \(w(r/2)\) at the midpoint \(x = r/2\). It should be noted that, due to translation invariance in the \(t\)-direction, the string width does not depend on \(t\). At leading order, the squared width \(w_{lo}^2(r/2)\) is ultraviolet divergent and is again regularized using the point-splitting method. It turns out that

\[
w_{lo}^2(r/2) = \frac{d - 2}{2\pi \sigma} \log \frac{r}{r_0} + \frac{d - 2}{\pi \sigma} \log \frac{\eta(2iu)}{\eta^2(iu)},
\]

where the low-energy parameter \(r_0\) is now given by

\[
r_0 = \frac{\pi}{2} \sqrt{\epsilon^2 + \epsilon'^2}.
\]

For \(\beta \gg r\), the second term on the right-hand side of eq. (4.3) gives only exponentially small corrections to the leading logarithmic increase of the string width. The regime \(r \gg \beta\) [16] can be obtained using the inversion transformation rule given by eq. (A.13). Then we have

\[
w_{lo}^2(r/2) = \frac{d - 2}{2\pi \sigma} \log \frac{\beta}{4r_0} + \frac{d - 2}{4\beta \sigma} r + \mathcal{O}(e^{-2\pi r/\beta}).
\]

Interestingly, this equation shows that at finite but low temperature, the squared string width increases linearly with the distance. Similar to the case of toroidal boundary conditions, we have to evaluate eq. (2.13). Since eq. (4.1) is well-defined only for \((t - t') \in [0, \beta]\), it is convenient to split the integral over \(t\), i.e. \(\int_0^\beta dt = \int_0^{t'} dt + \int_{t'}^\beta dt\). We then obtain

\[
T_1 = \frac{\pi}{\sigma^2 r^4} \sum_{n=1}^\infty \sum_{m=1}^\infty (-1)^{m+n} \sum_{k=1}^\infty \frac{k}{1 - q^k} \left( e^{-\pi ke'}/r + q^k e^{\pi ke'}/r \right) \times \int_0^r dx \cos \frac{\pi(2kx + xt)}{r} \left[ \cos \frac{\pi(2(m-n)x + (2m-1)\epsilon)}{r} \right.
\]

\[
\times \frac{1}{(1 - q^{2n-1})(1 - q^{2m-1})} \int_{t'}^\beta dt \left( e^{-2\pi(n+m-1)t/r} + e^{-2\pi(n+m-1)(\beta - t)/r} \right)
\]

\[
+ \cos \frac{\pi(2(m-n)x + (2m-1)\epsilon)}{r} \left. \right] \left[ \cos \frac{\pi(2(m-n)x + (2m-1)\epsilon)}{r} \right] \times \frac{1}{(1 - q^{2n-1})(1 - q^{2m-1})} \int_{t'}^\beta dt \left( e^{-2\pi(n-m)t/r} q^{2m-1} + e^{-2\pi(n-m)t/r} q^{2n-1} \right),
\]

as well as

\[
T_2 = \frac{T_1}{2} - \sum_{n=1}^\infty \sum_{m=1}^\infty (-1)^{m+n} \frac{E_2(iu)}{24r^4 \sigma^2} \int_0^r dx \cos \frac{2\pi(m-n)x}{r} \times \frac{1}{(1 - q^{2n-1})(1 - q^{2m-1})} \int_{t'}^\beta dt \left( e^{-2\pi(n-m)t/r} q^{2m-1} + e^{-2\pi(n-m)t/r} q^{2n-1} \right)
\]

\[
- \frac{E_2(iu)}{96\sigma^2 r^2}.
\]

The last term is the contribution of the integration \(\int_0^{t'} dt\) after the limit \(\epsilon' \to 0\) has been taken. The explicit evaluation of \(T_1\) and \(T_2\) is rather involved, and is thus relegated to Appendix D.
The two identities
\[
\partial_x \partial_x G(x, t; x', t') = -\partial_t \partial_t G(x, t; x', t'), \\
\partial_{x'} \partial_{x'} \partial_x \partial_x G(x, t; x', t') = \partial_{t'} \partial_{t'} \partial_t \partial_t G(x, t; x', t'),
\]
(4.8)
imply that \( \langle (\partial_{\mu} \vec{h}(x, t))^2 \rangle_0 = 0 \) and \( \langle (\partial_{\mu} \partial_{\mu} \vec{h}(x, t))^2 \rangle_0 = 0 \). Hence the contribution to the squared width coming from the terms proportional to \( \alpha \) and \( \alpha^2 \) vanishes at next-to-leading order. Finally, inserting eq. (D.12) and eq. (D.2) in eq. (2.13) we obtain
\[
w^2(r/2) = w^2_{lo}(r/2) + \frac{\pi}{12\sigma r^2} [E_2(iu) - 4E_2(2iu)] \left( w^2_{lo}(r/2) - \frac{d-2}{4\pi\sigma} \right) \\
+ \frac{(d-2)\pi}{12\sigma^2 r^2} \left\{ u \left( q \frac{d}{dq} - \frac{d-2}{12} E_2(iu) \right) [E_2(2iu) - E_2(iu)] - \frac{d-2}{8\pi} E_2(iu) \right\}.
\]
(4.9)

5. Conclusions

By now, a lot of numerical evidence supporting the validity of the low-energy effective string description of the long-distance static quark-anti-quark potential has been accumulated in lattice Yang-Mills theory. In several cases, the numerical data are so accurate that higher-order corrections to the leading free string approximation must be taken into account. Remarkably, open-closed string duality completely determines the terms in the effective action at next-to-leading order, without any additional low-energy parameters. In this paper we have presented the details of an analytic computation of the width of the color flux tube in the low-energy effective theory at next-to-leading order. Our result has been crucial for accurately describing the width of the color flux tube obtained in numerical simulations of Yang-Mills theory [18]. The results are expressed in closed form in terms of the Dedekind \( \eta \) function and of the Eisenstein series \( E_2 \) and \( E_4 \). The modular inversion transformation property of those functions yields the next-to-leading order correction to the width both at zero and at finite (but low) temperatures. The calculation has been performed using both toroidal and cylindrical boundary conditions.

The effective theory that we used describes string fluctuations in the continuum. In order to apply the results of our calculation to lattice field theories, one must be sufficiently close to the continuum limit. Before one reaches the continuum limit, the confining string in a lattice Yang-Mills theory is also affected by lattice artifacts. First of all, at very strong coupling the world-sheet swept out by the lattice string is rigid, i.e. it follows the discrete lattice steps and does not even have massless excitations. Only at weaker coupling, after crossing the roughening transition, the string world-sheet supports massless excitations and thus becomes rough. Consequently, the effective theory is applicable only in the rough phase. Since the lattice theory is invariant only under discrete rotations and not under the full Poincaré group, before one reaches the continuum limit additional terms proportional to \( \sum_{\mu=1,2} (\partial_{\mu} \partial_{\mu} h)^2 \) and \( \sum_{\mu=1,2} (\partial_{\mu} h)^4 \) enter the effective action in the bulk. Since these terms contain four derivatives, they are of sub-leading order. Hence, they have no effect on the Lüscher term or on the leading logarithmic behavior of the string width. As a result, the Lüscher term is completely universal. Provided its world-sheet is rough, even a
lattice string supports exactly massless modes which contribute $-\pi/24r$ to the static quark potential. In order to incorporate lattice artifacts in the effective theory in a systematic manner, one must investigate whether additional boundary terms arise in the effective action. One must also reinvestigate the consequences of open-closed string duality, which were derived in [20] assuming full Poincaré invariance. These are interesting problems for future studies, which may eventually be important for the correct description of numerical simulation data when the lattice spacing is not sufficiently small.

Acknowledgments

M. P. and U.-J. W. gratefully acknowledge helpful discussions with P. Hasenfratz, F. Niedermayer, R. Sommer, and P. Weisz. We like to thank the anonymous referee for very insightful remarks and useful suggestions. This work is supported in part by funds provided by the Schweizerischer Nationalfonds (SNF). The “Albert Einstein Center for Fundamental Physics” at Bern University is supported by the “Innovations- und Kooperationsprojekt C-13” of the Schweizerische Universitätskonferenz (SUK/CRUS).

A. Infinite sums and products

In this appendix we list the infinite sums and products which appear in our calculation. Some of them can be expressed in terms of the Dedekind $\eta$ function and the Eisenstein series. They are respectively defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{i2\pi \tau} \tag{A.1}$$

and

$$E_{2k}(\tau) = 1 + (-1)^k \frac{4k}{B_k} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1 - q^n}, \tag{A.2}$$

where $B_k$ are the Bernoulli numbers, defined through the expansion

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} - \sum_{k=1}^{\infty} (-1)^k \frac{B_k}{(2k)!} z^{2k}. \tag{A.3}$$

They are also related to the Riemann $\zeta$ function of positive even integers and negative odd integers

$$\zeta(2k) = \frac{B_k (2\pi)^{2k}}{(2k)!}, \quad \zeta(1 - 2k) = (-1)^k \frac{B_k}{2k}. \tag{A.4}$$

In the one-loop calculation one encounters the sum

$$\sum_{n=1}^{\infty} \frac{n^{-1} q^n}{1 - q^n} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{kn}}{n} = - \sum_{k=1}^{\infty} \log \left(1 - q^k\right) = - \log \varphi(\tau), \tag{A.5}$$

where $\varphi(\tau)$ is the Euler function, related to $\eta$ by

$$\eta(\tau) = q^{1/24} \varphi(\tau). \tag{A.6}$$
In two-loop calculations one finds sums of the type

$$
\sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n} \equiv \sum_{k=1}^{\infty} \frac{q^k}{(1 - q^k)^2} = \frac{1}{24} [1 - E_2(\tau)] ,
$$

(A.7)

where the first identity is simply obtained by writing $q^n/(1 - q^n) = \sum_{k=1}^{\infty} q^{nk}$ (like in eq. (A.5)) and then inverting the order of the two sums. The second identity follows from the definition of $E_2(\tau)$ or from eq. (A.5) using the relation

$$
\frac{\eta'(\tau)}{\eta(\tau)} = \frac{i \pi}{12} E_2(\tau) .
$$

(A.8)

The identity (A.7) also implies the following three equations:

$$
\sum_{m=1}^{\infty} \frac{q^{2m-1}}{(1 - q^{2m-1})^2} = \frac{1}{24} [E_2(2\tau) - E_2(\tau)] ,
$$

(A.9)

as well as

$$
\sum_{j=1}^{\infty} \frac{q^j}{(1 + q^j)^2} = -\frac{7}{8} + \frac{E_2(2\tau)}{6} - \frac{E_2(\tau)}{24} ,
$$

(A.10)

and

$$
\sum_{k=1}^{\infty} \frac{(2m - 1)q^{2m-1}(1 + q^{2m-1})}{(1 - q^{2m-1})^3} = \frac{q}{24} \frac{d}{dq} [E_2(2\tau) - E_2(\tau)] .
$$

(A.11)

The derivative of $E_2(\tau)$ can be expressed in terms of other Eisenstein series using

$$
q \frac{d}{dq} E_2(\tau) = \frac{1}{12} [E_2(\tau)^2 - E_4(\tau)] .
$$

(A.12)

The Dedekind $\eta$ function and the Eisenstein series $E_{2k}$ obey the following transformation rules under the inversion $\tau \rightarrow -1/\tau$

$$
\eta(\tau) = \frac{1}{\sqrt{-i\tau}} \eta(-\frac{1}{\tau}) , \quad E_2(\tau) = \frac{1}{\tau^2} E_2(-\frac{1}{\tau}) - \frac{6}{i \pi \tau} .
$$

(A.13)

as well as

$$
E_{2k}(\tau) = \frac{1}{\tau^{2k}} E_{2k}(-\frac{1}{\tau}) , \quad k > 1 .
$$

(A.14)

These relations ensure fast convergence of the expressions in both the regimes $\beta \gg r \gg 1/\sqrt{\sigma}$ (the zero-temperature limit) and $r \gg \beta \gg 1/\sqrt{\sigma}$ (the finite (but low) temperature case).

B. Propagator on the torus

The free field two-point function on a torus of size $r \times \beta$ satisfies

$$
-\Delta G(x, t; x', t') = \frac{1}{\sigma} \delta(x - x') \delta(t - t') - \frac{1}{\sigma \beta r} .
$$

(B.1)
Here the term $1/\sigma \beta r$ accounts for the zero-mode subtraction, which makes the Laplace operator $\Delta$ invertible in the orthogonal subspace. The solution of this equation can be expressed in closed form in various ways. We begin with the explicit expression given in Appendix A of [5]

$$G(z) = -\frac{1}{2\pi \sigma} \Re \left[ \log \frac{2\pi z}{r} - \sum_{k=1}^{\infty} \frac{G_k(2\pi)^{2k}}{2k} \left( \frac{z}{r} \right)^{2k} \right] + \frac{(t-t')^2}{2\sigma \beta r}$$  \hspace{1cm} (B.2)

with

$$z = x - x' + i(t - t'), \quad G_k = 2\frac{\zeta(2k)}{r^{2k}} E_{2k}(iu), \quad u = \frac{\beta}{r},$$  \hspace{1cm} (B.3)

where the Eisenstein series $E_{2k}(\tau)$ are defined in our appendix A. Inserting this definition in eq. (B.2) we get

$$2\pi \sigma G(z) = \frac{\pi(3mz)^2}{\beta r} - \Re \left[ \log \frac{2\pi z}{r} - \sum_{k=1}^{\infty} \frac{B_k(2\pi)^{2k}}{(2k)!} \frac{1}{2k} \left( \frac{z}{r} \right)^{2k} \right] + 2\Re \left[ \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \left( \frac{2\pi z}{r} \right)^{2k} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1 - q^n} \right].$$  \hspace{1cm} (B.4)

The terms enclosed in the first square brackets can be re-summed using eq. (A.3)

$$\sum_{k=1}^{\infty} \frac{(-1)^k B_k w^{2k}}{(2k)!} \frac{1}{2k} = \sum_{n=1}^{\infty} e^{-nw} - \frac{w}{n} - \log w = -\log(1 - e^{-w}) - \frac{w}{2} + \log w,$$  \hspace{1cm} (B.5)

with $w = -2\pi iz/r$. Interchanging the order of the two sums in the last bracket of eq. (B.4), we recognize the Taylor expansion of the cosine and arrive at the simpler expression

$$2\pi \sigma G(z) = \frac{\pi(t-t')(t-t' - \beta)}{\beta r} \Re \left[ -\log(1 - e^{2\pi iz/r}) + 2 \sum_{n=1}^{\infty} \frac{n^{-1} q^n \cos \frac{2\pi nz}{r}}{1 - q^n} + 2\log \varphi(iu) \right],$$  \hspace{1cm} (B.6)

where the Euler function $\varphi$ has already been defined in Appendix A. Using the identity (A.3) and expanding the first logarithm in powers of $e^{2\pi iz/r}$, we obtain the Gaussian correlator on the torus in the compact form of eq. (B.1), where we have put $G(x,t) = G(z)$. It should be noted that this expression converges in the range $q < 1$ and $0 \leq t - t' \leq \beta$.

C. Propagator on the cylinder

The Gaussian correlator $G(x,t';x',t') \equiv \langle h(x,t) h(x',t') \rangle_0$ on a cylinder of size $r \times \beta$ with fixed boundary conditions at $x = 0$ and $x = r$ and periodic boundary conditions in $t$ with period $\beta$ can be conveniently written in terms of correlators $G(z - z')$ and $G(z + z'\sigma)$ on a torus of size $2r \times \beta$ (with $z' = x' + it'$ and $z'' = x' - it'$)

$$G(x,t; x', t') = G(z - z') - G(z + z'\sigma) = \frac{1}{\pi \sigma} \sum_{n=1}^{\infty} \frac{\pi nx}{r} \sin \frac{\pi nx'}{r} e^{-\pi n(t-t')/r} + e^{-\pi n(\beta-t+t')/r}/n(1 - q^n).$$  \hspace{1cm} (C.1)

This is the expression used in section 4.
D. Evaluation of $T_1$ and $T_2$ on the cylinder

In this appendix we evaluate explicitly the terms $T_1$ and $T_2$ given in eq. (4.4) and eq. (4.7). Performing the corresponding integrations one obtains

$$T_2 - \frac{1}{2} T_1 = \frac{\pi}{\sigma^2 r^2} \frac{\beta}{r} \sum_{m=1}^{\infty} \left[ -\frac{1}{12} E_2(iu) \frac{q^{2m-1}}{(1-q^{2m-1})^2} \right] - \frac{1}{\sigma^2 r^2} \frac{E_2(iu)}{96}. \quad (D.1)$$

Using eq. (A.9), the sum may be written in closed form, which then yields

$$T_2 - \frac{1}{2} T_1 = -\frac{\pi}{\sigma^2 r^2} u E_2(iu) \left[ E_2(2iu) - E_2(iu) \right] - \frac{1}{\sigma^2 r^2} \frac{E_2(iu)}{96}. \quad (D.2)$$

Performing the integrations in $T_1$ and putting $s = e^{-\pi \epsilon/r}$, we obtain

$$T_1 = \frac{1}{\sigma^2 r^2} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^k \left[ \frac{k}{k+2n-1} \cos \frac{\pi(k+2n-1)\epsilon}{R} \frac{s^k + q^k}{1-q^k} \right.$$

$$\times \left( \frac{2}{s^{2k+2n-1} + s^{2n-1}} + \frac{q^{2k+2n-1}}{1-q^{2k+2n-1}} + \frac{q^{2n-1}}{1-q^{2n-1}} \right)$$

$$- \frac{k+2n-1}{k} \cos \frac{\pi k \epsilon}{r} \frac{s^{k+2n-1} + q^{k+2n-1}}{1-q^{k+2n-1}} \left( \frac{q^{2k+2n-1}}{1-q^{2k+2n-1}} - \frac{q^{2n-1}}{1-q^{2n-1}} \right) \right] + \frac{\pi}{\sigma^2 r^2} \frac{\beta}{r} \sum_{m=1}^{\infty} \left[ \frac{(2m-1)q^{2m-1}(1+q^{2m-1})}{(1-q^{2m-1})^3} \right]. \quad (D.3)$$

The last sum may be written in terms of Eisenstein series using eq. (A.11) or eq. (A.12).

Introducing the two functions

$$A(a,b) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^k \frac{k}{k+2n-1} a^k b^{k+2n-1}, \quad (D.4)$$

as well as

$$B(a,b) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^k \frac{k+2n-1}{k} a^k b^{k+2n-1}, \quad (D.5)$$

and putting $t = e^{i\pi(\epsilon+i\epsilon')/r}$, we can rewrite eq. (D.3) in the form

$$T_1 = \frac{1}{\sigma^2 r^2} \Re \left\{ \frac{A(1,t) + A(s^2,t)}{2} \right. \left.$$

$$+ \sum_{j=1}^{\infty} \left[ A(q^j,q^j) + A(q^{-j},q^j) + 2 A(q^j,t) - B(q^j,s q^j) + B(q^{-j},s q^j) + 2 A(1,q^j) \right.$$\n
$$+ 2 \sum_{k=1}^{\infty} \left[ A(q^{j+k},q^j) - B(q^j,q^{j+k}) + B(q^{-j},q^{j+k}) \right] + 2 \sum_{k=1,k\neq j}^{\infty} A(q^{k-j},q^j) \right\}$$

$$\left. + \frac{\pi u}{12\sigma^2 r^2} q \frac{d}{dq} \left[ E_2(2iu) - E_2(iu) \right]. \quad (D.6)$$
The two functions $A$ and $B$ can be written in terms of elementary functions

$$A(a, b) = -\frac{a^2 b}{(1 - a^2)(1 + ab)} - \frac{a(1 + a^2)}{(1 - a^2)^2} \log(1 + ab) + \frac{a \log(1 - b)}{2(1 + a)^2} + \frac{a \log(1 + b)}{2(1 - a)^2},$$

$$B(a, b) = -\frac{a b^2}{(1 - b^2)(1 + ab)} - \frac{b(1 + b^2)}{(1 - b^2)^2} \log(1 + ab).$$

The divergences of $A(a, b)$ for $a \to 1$ or $b \to 1$ are only apparent. In particular, the useful ultraviolet limits to apply in eq. (D.6) are

$$A(1, q) = \frac{q}{4(1 + q)^2} + \frac{1}{8} \log \frac{1 - q}{1 + q},$$

$$A(q, t) = \frac{q(1 - q) + (1 + q^2) \log \frac{2}{1 + q}}{(1 - q^2)^2} - \frac{q}{2(1 + q)^2} \log \frac{2r}{\pi |\epsilon + i\epsilon'|} + \mathcal{O}(\epsilon + i\epsilon'),$$

$$\Re \left[ \frac{A(1, t) + A(s, t)}{2} \right] = \frac{1}{16} - \frac{1}{8} \log \frac{2r}{\pi |\epsilon + i\epsilon'|} + \mathcal{O}(\epsilon + i\epsilon').$$

We now rewrite this quantity in terms of Eisenstein series. We begin by explicitly writing the logarithmic terms of the double sum in eq. (D.6), namely those arising from

$$\sum_{j=1}^{\infty} \left\{ 2 \sum_{k=1}^{\infty} \left[ A(q^{-j}, q^j) - B(q^j, q^{j+k}) + B(q^{-j}, q^{j+k}) \right] + 2 \sum_{k=1, k \neq j}^{\infty} A(q^{k-j}, q^j) \right\},$$

which are given by

$$\sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\infty} \left[ \frac{q^{j+k}}{(1 + q^{j+k})^2} \log(1 - q^k) + \frac{q^{j+k}}{(1 - q^{j+k})^2} \log(1 + q^k) \right. \right.$$

$$+ \left. \frac{q^{j-k}}{(1 + q^{j-k})^2} \log(1 - q^k) - 2 \frac{q^{j+k}(1 + q^{2(j+k)})}{(1 - q^{2(j+k)})^2} \log(1 + q^k) \right]\right.$$  

$$+ \sum_{k=1, k \neq j}^{\infty} \left[ \frac{q^k \log(1 + q^k)}{q^k(1 - q^k)^2} - 2 \frac{q^k(1 + q^{2(k-j)})}{q^k(1 - q^{2(k-j)})^2} \log(1 + q^k) \right]\right.$$  

$$= \sum_{k=1}^{\infty} \frac{1 - q^k}{1 + q^k} \sum_{j=1}^{\infty} \frac{q^{j+k}}{(1 + q^{j+k})^2} + \sum_{j=1, j \neq k}^{\infty} \frac{q^{j-k}}{(1 + q^{j-k})^2}$$

$$= 2 \sum_{m=1}^{\infty} \frac{q^m}{(1 + q^m)^2} \sum_{k=1}^{\infty} \frac{1 - q^k}{1 + q^k} - \sum_{j=1}^{\infty} \frac{q^j}{(1 + q^j)^2} \log \frac{1 - q^j}{1 + q^j}. \quad (D.10)$$

The last sum cancels exactly against the logarithmic terms resulting from the single sums (i.e. the second line) in eq. (D.6). Apart from the first term in eq. (D.10), the only remaining terms are those associated with eq. (D.8). Putting all these terms together we obtain

$$T_1 = \frac{1}{\sigma^2 r^2} \left( \sum_{j=1}^{\infty} \frac{q^j}{(1 + q^j)^2} + \frac{1}{8} \right) \left[ \frac{1}{2} - \log \left( \frac{2r}{\pi |\epsilon + i\epsilon'|} \prod_{k=1}^{\infty} (1 + q^k) \right) \right]$$

$$+ \frac{\pi u}{12 \sigma^2 r^2} \frac{d}{dq} [E_2(2iu) - E_2(iu)] \quad (D.11)$$
By comparison with eq. (4.3) in the second factor of the first term we recognize the contribution to the squared width of the flux tube at leading order. Applying the identity (A.10) and the definition (A.1) of the Dedekind \( \eta \) function, we finally obtain

\[
T_1 = \frac{1}{24\sigma^2 r^2} \left\{ \left[ E_2(iu) - 4E_2(2iu) \right] \left( \log \frac{r \eta^2(2iu)}{r_0 \eta^4(iu)} - \frac{1}{2} \right) + 2\pi u q \frac{d}{dq} \left[ E_2(2iu) - E_2(iu) \right] \right\},
\]

where \( r_0 \) has been defined in eq. (4.4). Inserting eq. (D.12) in eq. (D.2) one immediately obtains an explicit expression for \( T_2 \) in terms of Eisenstein series.

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