Seiberg-Witten theory and duality in integrable systems

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Abstract
These lectures are devoted to the low energy limit of $\mathcal{N} = 2$ SUSY gauge theories, which is described in terms of integrable systems. A special emphasis is on a duality that naturally acts on these integrable systems. The duality turns out to be an effective tool in constructing the double elliptic integrable system which describes the six-dimensional Seiberg-Witten theory. At the same time, it implies a series of relations between other Seiberg-Witten systems.

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Part I

Seiberg-Witten theory: gauge versus integrable theories

The main target of these lectures is to present an integrable framework for one of the most intriguing discoveries of recent years in quantum field theory – a possibility of exact, non-perturbative treating the low energy behaviour of supersymmetric gauge theories proposed by N. Seiberg and E. Witten \[1, 2\]. The results obtained turned out to be most effectively described by integrable systems \[3\]. Since then, many new features of Seiberg-Witten theories have been discovered, all of them this or that way related to integrable systems (an intensive discussion of integrability in Seiberg-Witten theory and related structures (Whitham hierarchies and WDVV equations) can be found in the book \[4\] that contains recent comprehensive reviews). Here we are going to concentrate on the most recent and puzzling structure, a duality present in integrable systems of the Toda/Calogero/Ruijsenaars family \[5\].

Therefore, technically the lectures are divided into two parts. The first part is devoted to the Seiberg-Witten theories and their treatment within the integrable approach. The second part deals with the duality and its role in the Seiberg-Witten theories.

1 Seiberg-Witten solution

Being one of the most fascinating breakthroughs in field theory, the Seiberg-Witten approach opened new possibilities to deal with nonperturbative physics at strong coupling that amounted from the exact derivation of the low energy effective action for the $\mathcal{N} = 2$ SYM theory \[1, 2\]. In principle, to get the exact answer one needs to perform the summation of infinite instanton series. Therefore, the result by Seiberg and Witten was the first example of the exact derivation of the total instanton sum. However, the authors did not calculate instantonic series, using instead indirect and quite sophisticated methods. The calculation of the exact mass spectrum of stable particles was their second essentially new result. What is important, this theory is non-trivial in one loop and belongs to the class of asymptotically free theories. In fact, the situation is a little bit unusual, since, despite the fact that the direct instanton summation has not been performed yet, there are no doubts in the results derived which survives under different tests.

In order to get the effective action unambiguously, the authors of \[1, 2\] used the three ideas: holomorphy \[6\], duality \[7\] and their compatibility with the renormalization group flows. Holomorphy implies that the low energy effective action can be only of the form $\int \text{Im} \mathcal{F}(\Psi)$ depending on a single holomorphic function $\mathcal{F}$ called prepotential. The duality principle which provides the relation between the strong and weak coupling regimes is natural in the UV finite \[8\], say $\mathcal{N} = 4$ SUSY \[8\], theory, where the evident modular parameter built from the coupling constant and the $\theta$-term

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi},$$

(1.1)

is not renormalized and coincides with its asymptotic value. In the theory with non-trivial renormalizations, the duality transformation does no longer interchange the weak and strong coupling regimes of the same theory. Instead, it is much similar to the usual electric-magnetic duality and interchanges the low energy regimes of different theories. To construct this transformation explicitly, one needs an artificial trick, to introduce into the game an additional object, a higher genus Riemann surface whose period matrix coincides with the matrix of couplings depending on the values of condensate in a given vacuum.

In the $\mathcal{N} = 2$ SYM theory this scheme is roughly realized as follows \[1, 2\]. First of all, in the Lagrangian there is a potential term for the scalar fields of the form

$$V(\phi) = \text{Tr}[\phi, \phi^+]^2,$$

(1.2)

where the trace is taken over adjoint representation of the gauge group (say, $SU(N_c)$). This potential gives rise to the valleys in the theory, when $[\phi, \phi^+] = 0$. The vacuum energy vanishes along the valleys,
hence, the supersymmetry remains unbroken. One may always choose the v.e.v.'s of the scalar field to lie in the Cartan subalgebra $\phi = \text{diag}(a_1, ..., a_n)$. These parameters $a_i$ can not serve, however, good order parameters, since there is still a residual Weyl symmetry which changes $a_i$ but leave the same vacuum state. Hence, one should consider the set of the gauge invariant order parameters $u_k = \langle \text{Tr}\phi^k \rangle$ that fix the vacuum state unambiguously. Thus, we obtain a moduli space parametrized by the vacuum expectation values of the scalars, which is known as the Coulomb branch of the whole moduli space of the theory. The choice of the point on the Coulomb branch is equivalent to the choice of the vacuum state and simultaneously yields the scale which the coupling is frozen on. At the generic point of the moduli space, the theory becomes effectively abelian after the condensation of the scalar.

As soon as the scalar field acquires the vacuum expectation value, the standard Higgs mechanism works and there emerge heavy gauge bosons at large values of the vacuum condensate. To derive the effective low energy action one has to sum up the loop corrections as well as the multiinstanton contributions.

The explicit summation over instantons has not been done yet, however, additional arguments allow one to define the effective action indirectly.

The initial action of $\mathcal{N} = 2$ theory written in $\mathcal{N} = 2$ superfields has the simple structure $S(\Psi) = \text{Im}\int \text{Tr}\Psi^2$ with $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{\tau}$. Now the $\mathcal{N} = 2$ supersymmetry implies that the low energy effective action gets renormalized only by holomorphic contributions so that it is ultimately given by a single function known as prepotential $S_{\text{eff}}(\Psi) = \text{Im} \int F(\Psi)$. The prepotential is a holomorphic function of moduli $u_k$ (or v.e.v.'s) except for possible singular points at the values of moduli where additional massless states can appear disturbing the low energy behaviour.

Thus, the problem effectively reduces to the determination of one holomorphic function. If one manage to fix its behaviour nearby singularities, the function can be unambiguously restored. One of the singularities, corresponding to large values of v.e.v.'s, i.e. to the perturbative limit is under control (since the theory is asymptotically free). All other singularities are treated with the use of duality and of the non-renormalization theorems for the central charges of the SUSY algebra. A combination of these two ideas allows one to predict the spectrum of the stable BPS states which become massless in the deep non-perturbative region and are in charge of all other singularities.

The duality transformation can be easily defined in the finite $\mathcal{N} = 4$ SUSY theory just as the modular transformations generated by $\tau \rightarrow \tau^{-1}$ and $\tau \rightarrow \tau + 1$. This makes a strong hint that the duality can be related with a modular space of some Riemann surfaces, where the modular group acts. In fact, the naive application of duality meets some serious difficulties. The reason is that, in the asymptotically free theory, one has to match the duality with the renormalization group\footnote{In fact, there are more problems with duality even in $\mathcal{N} = 2$ SUSY theories. In particular, it unifies into one duality multiplet the monopole and gauge boson supermultiplets, while they have different spins, see, e.g.,\footnote{We define the symbols $\oint$ and res with additional factors $(2\pi i)^{-1}$ so that $\text{res}_\infty \frac{d\xi}{\xi} = - \text{res}_0 \frac{d\xi}{\xi} = \oint \frac{d\xi}{\xi} = 1$.}}. This is non-trivial, since now $\tau$ depends on the scale which is supposedly involved into the duality transformation. The solution of this problem is that one is still able to connect the duality and modular transformations if considering different low energy theories connected by the duality. Then, the duality acts on the moduli space of vacua and this moduli space is associated with the moduli space of an auxiliary Riemann surface, where the modular transformations act.

At the next step, one has to find out proper variables whose modular properties fit the field theory interpretation. These variables are the integrals\footnote{In fact, there are more problems with duality even in $\mathcal{N} = 2$ SUSY theories. In particular, it unifies into one duality multiplet the monopole and gauge boson supermultiplets, while they have different spins, see, e.g.,\footnote{We define the symbols $\oint$ and res with additional factors $(2\pi i)^{-1}$ so that $\text{res}_\infty \frac{d\xi}{\xi} = - \text{res}_0 \frac{d\xi}{\xi} = \oint \frac{d\xi}{\xi} = 1$.}} of a meromorphic 1-form $dS$ over the cycles on the Riemann surface, $a_i$ and $a_D^i$

\begin{align}
\begin{aligned}
a_i &= \oint_{A_i} dS, \\
a_D &= \oint_{B_i} dS,
\end{aligned}
\end{align}

(where $i, j = 1, ..., N_c - 1$ for the gauge group $SU(N_c)$).

These integrals play the two-fold role in the Seiberg-Witten approach. First of all, one may calculate the prepotential $F$ and, therefore, the low energy effective action through the identification of $a_D$ and
\[ \frac{\partial F}{\partial a} \text{ with } a \text{ defined as a function of moduli (values of condensate) by formula (1.3). Then, using the property of the differential } dS \text{ that its variations w.r.t. moduli are holomorphic one may also calculate the matrix of coupling constants} \]

\[ T_{ij}(u) = \frac{\partial^2 F}{\partial a_i \partial a_j}, \quad (1.4) \]

The second role of formula (1.3) is that, as was shown these integrals define the spectrum of the stable states in the theory which saturate the Bogomolny-Prasad-Somm erfeld (BPS) limit. For instance, the formula for the BPS spectrum in the \( SU(2) \) theory reads as

\[ M_{n,m} = |na(u) + ma_D(u)|, \quad (1.5) \]

where the quantum numbers \( n, m \) correspond to the “electric” and “magnetic” states. The reader should not be confused with the spectrum derived from the low energy behaviour which fixes arbitrarily heavy BPS states. The point is that the BPS spectrum is related to the central charge of the extended SUSY algebra \( Z_{\{m\}, \{n\}} \) and, therefore, has an anomaly origin. On the other hand, anomalies are not renormalized by the quantum corrections and can be evaluated in either of the UV and IR regions.

Note that the column \((a_i, a_D^i)\) transforms under the action of the modular group \( SL(2, \mathbb{Z}) \) as a section of the linear bundle. Its global behaviour, in particular, the structure of the singularities is uniquely determined by the monodromy data. As we discussed earlier, the duality transformation connects different singular points. In particular, it interchanges “electric”, \( a_i \) and “magnetic”, \( a^D_i \) variables which describe the perturbative degrees of freedom at the strong and weak coupling regimes respectively. Manifest calculations with the Riemann surface allow one to analyze the monodromy properties of dual variables when moving in the space of the order parameters. For instance, in the simplest \( SU(2) \) case, on the \( u \)-plane of the single order parameter there are three singular points, and the magnetic and electric variables mix when encircling these points. Physically, in the theory with non-vanishing \( \theta \)-term (not pure imaginary \( \tau \)), the monopole acquires the electric charge, while the polarization of the instanton medium yields the induced dyons.

## 2 Seiberg-Witten theory and theory of prepotential

Although the both auxiliary objects, the Riemann surface and the differential \( dS \) have come artificially, an attempt to recognize them in SUSY gauge theories results into discovery of the integrable structures responsible for the Seiberg-Witten solutions.

Now let us briefly formulate the structures underlying Seiberg-Witten theory. Later on, we often refer to Seiberg-Witten theory as to the following set of data:

- Riemann surface \( \mathcal{C} \)
- moduli space \( \mathcal{M} \) (of the curves \( \mathcal{C} \)), the moduli space of vacua of the gauge theory
- meromorphic 1-form \( dS \) on \( \mathcal{C} \)

How it was pointed out in 3, this input can be naturally described in the framework of some underlying integrable system.

To this end, first, we introduce bare spectral curve \( E \) that is torus \( y^2 = x^3 + g_2 x^2 + g_3 \) for the UV-finite gauge theories with the associated holomorphic 1-form \( d\omega = dx/y \). This bare spectral curve degenerates into the double-punctured sphere (annulus) for the asymptotically free theories (where dimensional transmutation occurs): \( x \to w + 1/w, \ y \to w - 1/w, \ d\omega = dw/w \). On this bare curve, there are given a matrix-valued Lax operator \( L(x, y) \). The corresponding dressed spectral curve \( \mathcal{C} \) is defined from the formula \( \det(L - \lambda) = 0 \).

This spectral curve is a ramified covering of \( E \) given by the equation

\[ \mathcal{P}(\lambda; x, y) = 0 \quad (2.6) \]

In the case of the gauge group \( G = SU(N_c) \), the function \( \mathcal{P} \) is a polynomial of degree \( N_c \) in \( \lambda \).
Thus, we have the spectral curve $C$, the moduli space $\mathcal{M}$ of the spectral curve being given just by coefficients of $\mathcal{P}$. The third important ingredient of the construction is the generating 1-form $dS \cong \lambda \omega$ meromorphic on $C$ ("$\cong$" denotes the equality modulo total derivatives). From the point of view of the integrable system, it is just the shortened action "$pdq$" along the non-contractible contours on the Hamiltonian tori. This means that the variables $a_i$ in (1.3) are nothing but the action variables in the integrable system. The defining property of $dS$ is that its derivatives with respect to the moduli (ramification points) are holomorphic differentials on the spectral curve. This, in particular, means that
\[
\frac{\partial dS}{\partial a_i} = d\omega_i
\]
where $d\omega_i$ are the canonical holomorphic differentials of $C$. Integrating this formula over $B$-cycles and using that $a_D = \partial F/\partial a$, one immediately obtains (1.4).

So far we reckoned without matter hypermultiplets. In order to include them, one just needs to consider the surface $C$ with punctures. Then, the masses are proportional to the residues of $dS$ at the punctures, and the moduli space has to be extended to include these mass moduli. All other formulas remain in essence the same (see [11] for more details).

The prepotential $F$ and other "physical" quantities are defined in terms of the cohomology class of $dS$, formula (1.3). Note that formula (1.4) allows one to identify the prepotential with logarithm of the $\tau$-function of the Whitham hierarchy [12]: $F = \log \tau$.

### 3 Seiberg-Witten theory and integrable systems

Thus, we have fixed the set of data, that is, a Riemann surface, the corresponding moduli space and the differential $dS$ that gives the physical quantities, in particular, prepotential, and explained how it can be associated with an integrable system. In the case of the $SU(2)$ pure gauge theory, which has been worked out in detail in [1], this correspondence can be explicitly checked [3]. In particular, one may manifestly check formulas (1.3), since all necessary ingredients are obtained in [1]. In this way, one can actually prove that this Seiberg-Witten theory corresponds to the integrable system (periodic Toda chain with two particles, see below). However, to repeat the whole Seiberg-Witten procedure for theories with more vacuum moduli is technically quite tedious if possible at all. Meanwhile, only in this way one could unquestionably prove the correspondence in other cases.

In real situation, the identification between the Seiberg-Witten theory and the integrable system comes via comparing the three characteristics

- number of the vacuum moduli and external parameters
- perturbative prepotentials
- deformations of the two theories

The number of vacuum moduli (i.e. the number of scalar fields that may have non-zero v.e.v.’s) on the physical side should be compared with dimension of the moduli space of the spectral curves in integrable systems, while the external parameters in gauge theories (bare coupling constant, hypermultiplet masses) should be also some external parameters in integrable models (coupling constants, values of Casimir functions for spin chains etc).

As for second item, we already pointed out the distinguished role of the prepotential in Seiberg-Witten theory (which celebrates a lot of essential properties, see e.g. [3, 4]). In the prepotential, the contributions of particles and solitons/monopoles (dyons) sharing the same mass scale, are still distinguishable, because of different dependencies on the bare coupling constant, i.e. on the modulus $\tau$ of the bare coordinate elliptic curve (in the UV-finite case) or on the $\Lambda_{QCD}$ parameter (emerging after dimensional transmutation in UV-infinite cases). In the limit $\tau \to i\infty$ ($\Lambda_{QCD} \to 0$), the solitons/monopoles do not contribute and the prepotential reduces to the "perturbative" one, describing the spectrum of non-interacting particles. It is immediately given by the SUSY Coleman-Weinberg formula [11]:

\[4\text{i.e. satisfying the conditions}
\int_{A_i} d\omega_j = \delta_{ij}, \quad \int_{B_i} d\omega_j = T_{ij}\]
Seiberg-Witten theory (actually, the identification of appropriate integrable system) can be used to construct the non-perturbative prepotential, describing the mass spectrum of all the “light” (non-stringy) excitations (including solitons/monopoles). Switching on Whitham times [14] presumably allows one to extract some correlation functions in the “light” sector.

The perturbative prepotential (3.8) has to be, certainly, added with the classical part,

\[ F_{\text{pert}}(a) \sim \sum_{\text{reps } R, i} (\gamma)^{R} \text{Tr}_R(a + M_i)^2 \log(a + M_i) \]  (3.8)

\[ F_{\text{class}} = -\tau a^2 \]  (3.9)

since the bare action is \(S(\Psi) = \text{Im} \int \text{Tr} \Psi^2\). Here \(\tau\) is the value of the coupling constant in the deeply UV region. It plays the role of modular parameter of the bare torus in the UV finite theories. On the other hand, in asymptotically free theories, where the perturbative correction behaves like \(a^2 \log(a/\Lambda_{QCD})\), \(\tau\) is just contained in \(\Lambda_{QCD}\), since it can be invariantly defined neither in integrable, nor in gauge theories.

In fact, the problem of calculation of the prepotential in physical theory is simple only at the perturbative level, where it is given just by the leading contribution, since the \(\beta\)-function in \(N = 2\) theories is non-trivial only in one loop. However, the calculation of all higher (instantonic) corrections in the gauge theory can be hardly done at the moment. Therefore, the standard way of doing is to make an identification of the Seiberg-Witten theory and an integrable system and then to rely on integrable calculations. This is why establishing the “gauge theories ↔ integrable theories” correspondence is of clear practical (apart from theoretical) importance.

Now let us come to the third item in the above correspondence, namely, how one can extend the original Seiberg-Witten theory. There are basically three different ways to extend the original Seiberg-Witten theory.

First of all, one may consider other gauge groups, from other simple classical groups to those being a product of several simple factors. The other possibility is to add some matter hypermultiplets in different representations. The two main cases here are the matter in fundamental or adjoint representations. At last, the third possible direction to deform Seiberg-Witten theory is to consider 5- or 6-dimensional theories, compactified respectively onto the circle of radius \(R_5\) or torus with modulus \(R_5/R_6\) (if the number of dimensions exceeds 6, the gravity becomes obligatory coupled to the gauge theory).

Now we present a table of known relations between gauge theories and integrable systems, see Fig.1.

Let us briefly describe different cells of the table.

The original Seiberg-Witten model, which is the 4d pure gauge \(SU(N)\) theory (in fact, in their papers [1,2], the authors considered the \(SU(2)\) case only, but the generalization made in [10] is quite immediate), is the upper left square of the table. The remaining part of the table contains possible deformations. Here only two of the three possible ways to deform the original Seiberg-Witten model are shown. Otherwise, the table would be three-dimensional. In fact, the third direction related to changing the gauge group, although being of an interest is slightly out of the main line.

One direction in the table corresponds to matter hypermultiplets added. The most interesting is to add matter in adjoint or fundamental representations, although attempts to add antisymmetric and symmetric matter hypermultiplets were also done (see [16] for the construction of the curves and [17] for the corresponding integrable systems). Adding matter in other representations in the basic \(SU(N)\) case leads to non-asymptotically free theories.

**Columns: Matter in adjoint vs. fundamental representations of the gauge group.** Matter in adjoint representation can be described in terms of a larger pure SYM model, either with higher SUSY or in higher dimensional space-time. Thus models with adjoint matter form a hierarchy, naturally associated with the hierarchy of integrable models \(\text{Toda chain} \leftrightarrow \text{Calogero} \leftrightarrow \text{Ruijsenaars} \leftrightarrow \text{Dell} \leftrightarrow \text{XYZ} \leftrightarrow \text{XXX} \leftrightarrow \text{XXZ} \leftrightarrow \text{XYZ} \leftrightarrow \text{Toda chain} \). Similarly, the models with fundamental matter are related to the hierarchy of spin chains originated from the Toda chain: \(\text{Toda chain} \leftrightarrow \text{XXX} \leftrightarrow \text{XXZ} \leftrightarrow \text{XYZ} \leftrightarrow \text{Toda chain}\).

Note that, while coordinates in integrable systems describing pure gauge theories and those with fundamental matter, live on the cylinder (i.e. the dependence on coordinates is trigonometric), the

\[ \int_{\tau} = \int_{\tau} = \int_{\tau} \]
| SUSY physical theory | Pure gauge SYM theory, gauge group $G$ | SYM theory with adj. matter | SYM theory with fund. matter |
|----------------------|-------------------------------|--------------------------|--------------------------|
| 4d                   | inhomogeneous periodic Toda chain for the dual affine $\hat{G}^\vee$ | elliptic Calogero model (trigonometric Calogero model) | inhomogeneous periodic XXX spin chain (non-periodic chain) |
| 5d                   | periodic relativistic Toda chain (non-periodic chain) | elliptic Ruijsenaars model (trigonometric Ruijsenaars) | periodic XXZ spin chain (non-periodic chain) |
| 6d                   | periodic “Elliptic” Toda chain (non-periodic chain) | Dell system (dual to elliptic Ruijsenaars, elliptic-trig.) | periodic XYZ (elliptic) spin chain (non-periodic chain) |

Figure 1: SUSY gauge theories $\Longleftrightarrow$ integrable systems correspondence. The perturbative limit is marked by the italic font (in parenthesis).

coordinates in the Calogero system (adjoint matter added) live on a torus\(^6\). However, when one takes the perturbative limit, the coordinate dependence becomes trigonometric.

**Lines: Gauge theories in different dimensions.** Integrable systems relevant for the description of vacua of $d=4$ and $d=5$ models are respectively the Calogero and Ruijsenaars ones (which possess the ordinary Toda chain and “relativistic Toda chain” as Inozemtsev’s limits \(^25\)), while $d=6$ theories are described by the double elliptic (Dell) systems. When we go from 4d (Toda, XXX, Calogero) theories to (compactified onto circle) 5d (relativistic Toda, XXZ, Ruijsenaars) theories the momentum-dependence of the Hamiltonians becomes trigonometric (the momenta live on the compactification circle) instead of rational. Similarly, the (compactified onto torus) 6d theories give rise to an elliptic momentum-dependence of the Hamiltonians, with momenta living on the compactification torus. Since adding the adjoint hypermultiplet ellipitizes the coordinate dependence, the integrable system corresponding to 6d theory with adjoint matter celebrates both the coordinate- and momentum-dependencies elliptic. A candidate for “the elliptic Toda chain” was proposed in \(^{26}\).

Further in this part we show how the Riemann surfaces and the meromorphic differentials arise when considering a concrete integrable many-body system. We are going to explain in more details the simplest top left corner of the table and make some comments on the adjoint matter case in different dimensions. Then, after discussing duality, we shall come to the most general adjoint matter system, the right bottom cell in the table which is our ultimate target in these lectures. It can be, however, done only with the use of the coordinate-momentum duality\(^8\).

### 4 4d pure gauge theory: Toda chain

Thus, we start with the simplest case of the 4d pure gauge theory, which was studied in details in the original paper \(^1\) for the $SU(2)$ gauge group and straightforwardly generalized to the $SU(N_c)$ case in \(^{10}\).

This system is described by the periodical Toda chain with period $N_c$ \(^3\) whose equations of motion read

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\(^6\)Since these theories are UV finite, they depend on an additional (UV-regularizing) parameter, which is exactly the modulus $\tau$ of the torus.

\(^7\)Let us point out that by adding the adjoint matter we always mean soft breaking of higher supersymmetries.

\(^8\)This duality has nothing to do with the duality of \(^1\) which we discussed earlier.
are momenta. The symplectic structure is
\[ \frac{\partial q_i}{\partial t} = p_i, \quad \frac{\partial p_i}{\partial t} = e^{q_{i+1} - q_i} - e^{q_i - q_{i-1}} \]  (4.10)
with periodic boundary conditions \( p_{i+N_c} = p_i, q_{i+N_c} = q_i \) imposed. In physical system, there are \( N_c \)
moduli and there are no external parameters (\( \Lambda_{QCD} \) can be easily removed by rescaling). In the Toda
system, there are exactly \( N_c \) conservation laws. These conservation laws can be constructed from the
Lax operator defined for any integrable system. The eigenvalues of the Lax operator do not evolve, thus,
any function of the eigenvalues is an integral of motion.

There are two different Lax representation describing the periodic Toda chain. In the first one, the
Lax operator is represented by the \( N_c \times N_c \) matrix depending on dynamical variables
\[
\mathcal{L}^{TC}(w) = \begin{pmatrix}
 p_1 & e^{\frac{i}{2}(q_2 - q_1)} & 0 & \ldots & 0 \\
 e^{\frac{i}{2}(q_2 - q_1)} & p_2 & e^{\frac{i}{2}(q_3 - q_2)} & \ldots & 0 \\
 0 & e^{\frac{i}{2}(q_3 - q_2)} & -p_3 & 0 & \ldots \\
 \frac{1}{w} & e^{\frac{i}{2}(q_1 - q_{N_c})} & 0 & 0 & p_{N_c}
\end{pmatrix}
\]  (4.11)
The characteristic equation for the Lax matrix
\[
\mathcal{P}(\lambda, w) = \det_{N_c \times N_c} \left( \mathcal{L}^{TC}(w) - \lambda \right) = 0
\]  (4.12)
generates the conservation laws and determines the spectral curve
\[
w + \frac{1}{w} = 2P_{N_c}(\lambda)
\]  (4.13)
where \( P_{N_c}(\lambda) \) is a polynomial of degree \( N_c \) whose coefficients are integrals of motion.

The spectral curve (4.13) is exactly the Riemann surface introduced in the context of \( SU(N_c) \) gauge
theory. The integrals of motion parametrize the moduli space of the complex structures of the hyperelliptic
surfaces of genus \( N_c - 1 \), which is the moduli space of vacua in physical theory.

If one considers the case of two particles (\( SU(2) \) gauge theory), \( P(\lambda) = \lambda^2 - u, u = p^2 - \cosh q \),
\( p = p_1 = -p_2, q = q_1 - q_2 \). Thus we see that the order parameter of the SUSY theory plays the
role of Hamiltonian in integrable system. In the perturbative regime of the gauge theory, one of the
exponentials in \( \cosh q \) vanishes, and one obtains the non-periodic Toda chain. In the equation (4.13) the
perturbative limit implies vanishing the second term in the l.h.s., i.e. the spectral curve becomes the
sphere \( w = 2P_{N_c}(\lambda) \).

After having constructed the Riemann surface and the moduli space describing the 4d pure gauge
theory, we turn to the third crucial ingredient of Seiberg-Witten theory that comes from integrable
systems, the generating differential \( dS \). The general construction was explained in the Introduction, this
differential is in essence the “shorten” action \( pdq \). Indeed, in order to construct action variables, \( a_i \)
oneeds to integrate the differential \( d\tilde{S} = \sum_i p_i dq_i \) over \( N_c - 1 \)-non-contractable cycles in the Liouville
torus which is nothing but the level submanifold of the phase space, i.e. the submanifold defined by
values of all \( N_c - 1 \) integrals of motion fixed. On this submanifold, the momenta \( p_i \) are functions of the
coordinates, \( q_i \). The Liouville torus in Seiberg-Witten theory is just the Jacobian corresponding to the
spectral curve (4.13) (or its factor over a subgroup).

In the general case of a \( g \)-parameter family of complex curves (Riemann surfaces) of genus \( g \), the
Seiberg-Witten differential \( dS \) is characterised by the property \( d\delta S = \sum_{i=1}^{g} \delta u_i dv_i \), where \( dv_i(z) \) are \( g \)
holomorphic 1-differentials on the curves (on the fibers), while \( \delta u_i \) are variations of \( g \) moduli (along the
base). In the associated integrable system, \( u_i \) are integrals of motion and \( \pi_{i} \), some \( g \) points on the curve
are momenta. The symplectic structure is
\[
\sum_{i=1}^{g} da_i \wedge dp_i^{fac} = \sum_{i,k=1}^{g} du_i \wedge dv_i(\pi_k)
\]  (4.14)
The vector of the angle variables,
\[ p_i^{Jac} = \sum_{k=1}^{g} \int_{\pi_k} d\omega_i \]  

(4.15)

is a point of the Jacobian, and the Jacobi map identifies this with the \( g \)-th power of the curve, \( Jac \cong C^g \).

Here \( d\omega_i \) are \textit{canonical} holomorphic differentials, \( dv_i = \sum_{j=1}^{g} d\omega_j f_{ij} dv_i \). Some details on the symplectic form on the finite-gap solutions can be found in [23].

Technical calculation is, however, quite tedious. It is simple only in the 2-particle (SU(2)) case, when the Jacobian coincides with the curve itself. In this case, the spectral curve is

\[ w + \frac{1}{w} = 2(\lambda^2 - u) \]  

(4.16)

while \( u = p^2 - \cosh q \). Therefore, one can write for the action variable

\[ a = \int pdq = \oint \sqrt{u - \cosh q} dq = \oint \lambda \frac{dw}{w}, \quad ds = \lambda \frac{dw}{w} \]  

(4.17)

where we made the change of variable \( w = e^q \) and used equation (4.16).

Now one can naturally assume that this expression for the differential \( ds \) is suitable for generic \( N_c \). A long calculation (which can be borrowed, say, from the book by M.Toda [30]) shows that this is really the case. One can easily check that the derivatives of \( ds \) w.r.t. to moduli are holomorphic, up to total derivatives. Say, if one parametrizes

\[ P_{N_c}(\lambda) = -\lambda^{N_c} + s_{N_c-2} \lambda^{N_c-2} + ... \]  

and note that

\[ dS = \lambda dw/w = \lambda dP_{N_c}(\lambda)/Y = P_{N_c}(\lambda) d\lambda/Y + \text{total derivatives} \]

then

\[ \frac{\partial dS}{\partial s_k} = \frac{\lambda^k d\lambda}{Y} \]  

(4.18)

and these differentials are holomorphic if \( k \leq N_c - 2 \). Thus, \( N_c - 1 \) moduli gives rise to \( N_c - 1 \) holomorphic differentials which perfectly fits the genus of the curve (we use here that there is no modulus \( s_{N_c-1} \)).

## 5 Perturbative prepotentials

In the previous section, we established in the concrete example the connection between the Seiberg-Witten integrable theory and the gauge theory. However, so far we studied only a kind of general landscape in the theory, since the real, numerical results enter the game only with the prepotential calculated. Moreover, the correspondence between gauge and integrable theories is supported by comparing the perturbative prepotentials. We discuss it now.

The technical tool that allows one to proceed with effective perturbative expansion of the prepotential is “the residue formula”, the variation of the period matrix, i.e. the third derivatives of \( F(a) \) (see, e.g. [11]):

\[ \frac{\partial^3 F}{\partial a_i \partial a_j \partial a_k} = \frac{\partial T_{ij}}{\partial a_k} = \text{res}_{d\xi=0} \frac{d\omega_i d\omega_j d\omega_k}{\delta dS}, \]  

(5.1)

where \( \delta dS \equiv d \left( \frac{d\xi}{d\xi} \right) d\xi \), or, explicitly \( \delta dS = d\lambda d\xi \). Here the differential \( d\xi \) given on the bare torus degenerates into \( dw/w \) when torus degenerates into the punctured sphere (Toda chain). We remark that although \( d\xi \) does not have zeroes on the \textit{bare} spectral curve when it is a torus or doubly punctured sphere, it does in general however possess them on the covering \( \hat{C} \).

In order to construct the perturbative prepotentials, one merely can note that, at the leading order, the Riemann surface (spectral curve of the integrable system) becomes rational. Therefore, the residue formula allows one to obtain immediately the third derivatives of the prepotential as simple residues on sphere. In this way, one can check that the prepotential has actually the form \[ F(a) \]  

Later we discuss these calculations, but right now we shall see what the perturbative prepotentials are in in gauge theory.

---

\(^9\)The absence of the term \( \lambda^{N_c-1} \) is due to the SU(\( N_c \)) group and corresponds to the total momentum equal to zero, i.e. to the centre mass frame. This is important for the counting of genus.

\(^{10}\)We reserve notation \( \xi \) for the variable living on the bare spectral curve (torus), while shall denote later through \( \zeta \) the variable living on the compactification torus in 6 dimensions.
As a concrete example, let us consider the $SU(N_c)$ gauge group. Then, say, the perturbative prepotential for the pure gauge theory acquires the form

$$F^V_{\text{pert}} = \frac{1}{8\pi^2} \sum_{ij} (a_i - a_j)^2 \log (a_i - a_j)$$  \hspace{1cm} (5.2)

This formula establishes that when v.e.v.’s of the scalar fields in the gauge supermultiplet are non-vanishing (perturbatively $a_r$ are eigenvalues of the vacuum expectation matrix $\langle \phi \rangle$), the fields in the gauge multiplet acquire masses $m_{+r'} = a_r - a_{r'}$ (the pair of indices $(r, r')$ label a field in the adjoint representation of $SU(N_c)$). The eigenvalues are subject to the condition $\sum_i a_i = 0$. Analogous formula for the adjoint matter contribution to the prepotential is

$$F^A_{\text{pert}} = -\frac{1}{8\pi^2} \sum_{ij} (a_i - a_j + M)^2 \log (a_i - a_j + M)$$  \hspace{1cm} (5.3)

These formulas can be almost immediately extended to the “relativistic” $5d$ $\mathcal{N} = 2$ SUSY gauge models with one compactified dimension. One can understand the reason for this “relativization” in the following way. Considering four plus one compact dimensional theory one should take into account the contribution of all Kaluza-Klein modes to each 4-dimensional field. Roughly speaking it leads to the 1-loop contributions to the effective charge of the form $T_{ij} \sim$$ \sum_{m} \log \sum_{M} \log \left( \frac{a_i + \frac{m}{R_5}}{R_5} \right) \sim \log \prod_{M} \left( R_5 a_{ij} + M \right) \sim \log \sinh R_5 a_{ij}$  \hspace{1cm} (5.4)

i.e. coming from $4d$ to $5d$ one should make a substitution $a_{ij} \to \sinh R_5 a_{ij}$, at least, in the formulas for perturbative prepotentials. Adding the adjoint matter in this case would give the effective charge (or, equivalently, the period matrix of the spectral curve of the Ruijsenaars-Schneider system $[20]$, see table 1):

$$T_{ij} \sim \log \left( \frac{\sinh (R_5 a_{ij} + \epsilon) \sinh (R_5 a_{ij} - \epsilon)}{\sinh R_5 a_{ij}} \right)$$  \hspace{1cm} (5.5)

The same general argument can be equally applied to the $6d$ case, or to the theory with two extra compactified dimensions, of radii $R_5$ and $R_6$. Indeed, the account of the Kaluza-Klein modes allows one to predict the perturbative form of charges in the $6d$ case as well. Namely, one should expect them to have the form\footnote{We denote through $\theta_+$ the $\theta$-function with odd characteristics $[3]$.}$

$$T_{ij} \sim \sum_{m, n} \log \sum_{M} \log \left( \frac{a_i + \frac{m}{R_5} + \frac{n}{R_6}}{R_5} \right) \sim$$

$$\sim \log \prod_{M} \left( R_5 a_{ij} + M + \frac{R_5}{R_6} \right) \sim \log \theta_+ \left( \frac{R_5 a_{ij}}{i \frac{R_5}{R_6}} \right)$$  \hspace{1cm} (5.6)

i.e. coming from $4d$ ($5d$) to $6d$ one should replace the rational (trigonometric) expressions by the elliptic ones, at least, in the formulas for perturbative prepotentials. In this case adding the adjoint matter would lead to the effective charge

$$T_{ij} \sim \log \left( \frac{\theta_+ \left( R_5 a_{ij} + \epsilon \right) \left( \frac{R_5 a_{ij}}{i \frac{R_5}{R_6}} \right)}{\theta_+ \left( R_5 a_{ij} - \epsilon \right) \left( \frac{R_5 a_{ij}}{i \frac{R_5}{R_6}} \right)} \right)$$  \hspace{1cm} (5.7)

6 4d theory: a simple example of calculation

Here we describe the method of calculating prepotentials in the simplest example of the Toda chain. In the perturbative limit ($\frac{\text{perc.}}{\text{pho.}} \to 0$) the second term in (4.13) vanishes, the curve acquires the form
\[ W = 2P_{N_c}(\lambda), \quad P_{N_c}(\lambda) = \prod (\lambda - \lambda_i) \]  \hspace{1cm} (6.8)

(a rational curve with punctures \footnote{These punctures emerge as a degeneration of the handles of the hyperelliptic surface so that the A-cycles encircle the punctures.}) and the generating differential (4.17) turns into

\[ dS^{(4)}_{\text{pert}} = \lambda d\log P_{N_c}(\lambda) \]  \hspace{1cm} (6.9)

Now the set of the \(N_c - 1\) independent canonical holomorphic differentials is

\[ d\omega_i = \left( \frac{1}{\lambda - \lambda_i} - \frac{1}{\lambda - \lambda_{N_c}} \right) d\lambda = \frac{\lambda_{iN_c} d\lambda}{(\lambda - \lambda_i)(\lambda - \lambda_{N_c})}, \quad i = 1, \ldots, N_c - 1, \quad \lambda_{ij} \equiv \lambda_i - \lambda_j \]  \hspace{1cm} (6.10)

and the \(A\)-periods \(a_i = \lambda_i\) (the independent ones are, say, with \(i = 1, \ldots, N_c - 1\)) coincide with the roots of polynomial \(P_{N_c}(\lambda), \lambda_i\). The prepotential can be now computed via the residue formula

\[ F_{\text{pert},ijk} = \text{res}_{d\log P_{N_c}=0} \frac{d\omega_i d\omega_j d\omega_k}{d\log P_{N_c}(\lambda) d\lambda} \]  \hspace{1cm} (6.11)

which can be explicitly done. The only technical trick is that it is easier to compute the residues at the poles of \(d\omega_i\)’s instead of the zeroes of \(d\log P\). This can be done immediately since there are no contributions from the infinity \(\lambda = \infty\). The final results have the following form

\[
2\pi i F_{\text{pert},iii} = \sum_{k \neq i} \frac{1}{\lambda_{ik}} + \frac{6}{\lambda_{iN_c}} + \sum_{k \neq N_c} \frac{1}{\lambda_{kN_c}}, \\
2\pi i F_{\text{pert},ijj} = \frac{3}{\lambda_{iN_c}} + \frac{2}{\lambda_{jN_c}} + \sum_{k \neq i, j, N_c} \frac{1}{\lambda_{kN_c}} - \frac{\lambda_{jN_c}}{\lambda_{iN_c} \lambda_{ij}}, \quad i \neq j, \\
2\pi i F_{\text{pert},ijk} = 2 \sum_{l \neq N_c} \frac{1}{\lambda_{iN_c}} - \sum_{l \neq i, j, k, N_c} \frac{1}{\lambda_{N_c}}, \quad i \neq j \neq k;
\]  \hspace{1cm} (6.12)

giving rise to the prepotential formula

\[ F_{\text{pert}} = \frac{1}{8\pi i} \sum_{ij} f^{(4)}(\lambda_{ij}), \quad f^{(4)}(x) = x^2 \log x^2 \]  \hspace{1cm} (6.13)

This formula perfectly matches the perturbative prepotential expected in the gauge theory (5.2).

**Part II**

**Dualities in the many-body systems and gauge theories**

In this part, we are going to discuss a property of some integrable systems which is called duality. This property will allow us to fill in the right bottom corner of table 1, i.e. to construct a double elliptic system.

In fact, as in string theory there are several dualities (S-, T- and U-dualities), as in integrable systems there are several kinds of duality \footnote{These punctures emerge as a degeneration of the handles of the hyperelliptic surface so that the A-cycles encircle the punctures.}. Here we concentrate on the most interesting, important and still remaining mysterious one, the coordinate-momentum duality. This is a duality between pairs of dynamical systems (which is self-duality for some systems) \footnote{These punctures emerge as a degeneration of the handles of the hyperelliptic surface so that the A-cycles encircle the punctures.}. Note that this duality at the moment is attributed only with the Hitchin type systems, i.e. the Seiberg-Witten theories with adjoint matter, while spin chains are still not embedded into this treatment.

Let us consider a system with the phase space being a hyperkahler four dimensional manifold. Let this manifold involve at most two tori, i.e. be the elliptically fibered K3 manifold. One torus is that where...
the momenta live and the other one is that where the coordinates live. The duality at hands actually
interchanges momentum and coordinate tori and, therefore, can be called $pq$-duality. Note that typically
one deals with degenerations of these tori.

One can also consider a theory which is self-dual. This integrable system is called Dell (double elliptic)
system\footnote{In fact, this is an example of the Dell system with 2 degrees of freedom. One may equally consider the Dell system with arbitrary many degrees of freedom, see below.} and describes the Seiberg-Witten theory for the six-dimensional theory with adjoint matter.

The other cases correspond to some degenerations of Dell system. Say, the degeneration of the
momentum torus to $C/Z_2$ leads to the five-dimensional theory, while the degeneration to $R^2$ leads to the
four-dimensional theory. Since the modulus of the coordinate torus has meaning of the complexified bare
coupling constant in field theory, interpretation of the degenerations of the coordinate torus is different.
In particular, the degeneration to the cylinder corresponds to switching off the instanton effects, i.e. to
the perturbative regime.

A generic description of the $pq$-duality expresses a relationship between two completely integrable
systems $S_1, S_2$ on a fixed symplectic manifold with given symplectic structure $(M, \omega)$ and goes back to
\footnote{\[1\]}. We say the Hamiltonian systems are dual when the conserved quantities of $S_1$ and $S_2$ together form
a coordinate system for $M$. Consider for example free particles, $H^{(1)}_k = \sum_i p_i^k/k$. For this system the
free particles momenta are identical to the conserved quantities or action variables. Now consider the
Hamiltonian $H^{(2)}_k = \sum_i q_i^k/k$ with conserved quantities $q_i$. Together $\{p_i, q_i\}$ form a coordinate system
for phase space, and so the two sets of Hamiltonians are dual. Duality then in this simplest example
is a transformation which interchanges momenta and coordinates. For more complicated interacting
integrable systems finding dual Hamiltonians is a non-trivial exercise. Note that this whole construction
manifestly depends on the particular choice of conserved quantities. A clever choice may result in the
dual system arising by simply interchanging the momentum and coordinate dependence, as in the free
system.

Some years ago Ruijsenaars \footnote{\[2\, 35\]} observed such dualities between various members of the Calogero-
Moser and Ruijsenaars families: the rational Calogero and trigonometric Ruijsenaars models were dual to
themselves while trigonometric Calogero model was dual with the rational Ruijsenaars system (see
\footnote{\[23\, 35\]} for more examples). These dualities were shown by starting with a Lax pair $L = L(p, q)$ and an
auxiliary diagonal matrix $A = A(q)$. When $L$ was diagonalized the matrix $A$ became the Lax matrix for
the dual Hamiltonian, while $L$ was a function of the coordinates of the dual system. Dual systems for a
model possessing a Lax representation are then related to the eigenvalue motion of the Lax matrix. We
discuss this approach in more details below.

Another approach to finding a dual system is to make a canonical transformation which substitutes
the original set of Poisson-commuting coordinates $q_i$, $\{q_i, q_j\} = 0$, by another obvious set of the Poisson-
commuting variables: the Hamiltonians $h_i(\hat{p}, \hat{q})$ or, better, the action variables $a_i(\hat{h}) = a_i(\hat{p}, \hat{q})$. It will
be clear below that in practice really interesting transformations are a little more sophisticated: $h_i$
are identified with certain functions of the new coordinates (these functions determine the Ruijsenaars
matrix $A(q)$), which – in the most interesting cases – are just the same Hamiltonians with the interactions
switched-off. Such free Hamiltonians are functions of momenta alone, and the dual coordinates substitute
these momenta, just as one had for the system of free particles.

The most interesting question for our purposes is: what are the duals of the elliptic Calogero and
Ruijsenaars systems ? Since the elliptic Calogero (Ruijsenaars) is rational (trigonometric) in momenta
and elliptic in the coordinates, the dual will be elliptic in momenta and rational (trigonometric) in coordinates.
Having found such a model the final elliptization of the coordinate dependence is straightforward,
providing us with the wanted double-elliptic systems.

7 $pq$-duality: two-body system

The way of constructing dual systems via a canonical transformation is technically quite tedious for
many-particle systems, but perfectly fits the case of $SU(2)$ which, in the center-of-mass frame, has
only one coordinate and one momentum. In this case the duality transformation can be described
explicitly since the equations of motion can be integrated in a straightforward way. Technically, given
two Hamiltonian systems, one with the momentum $p$, coordinate $q$ and Hamiltonian $h(p,q)$ and another
with the momentum $P$, coordinate $Q$ and Hamiltonian $H(P,Q)$ we may describe duality by the relation
\[ h(p, q) = f(Q), \]
\[ H(P, Q) = F(q). \]  
(7.14)

Here the functions \( f(Q) \) and \( F(q) \) are such that

\[ dp \wedge dq = -dp \wedge dq, \]  
(7.15)

which expresses the fact we have an (anti-)canonical transformation. This relation entails that

\[ F'(q) \frac{\partial h(p, q)}{\partial p} = f'(Q) \frac{\partial H(P, Q)}{\partial P}. \]  
(7.16)

At this stage the functions \( f(Q) \) and \( F(q) \) are arbitrary. However, when the Hamiltonians depend on a coupling constant \( \nu^2 \) and are such that their “free” part can be separated and depends only on the momenta, the free Hamiltonians provide a natural choice for these functions: \( F(q) = h_0(q) \) and \( f(Q) = H_0(Q) \) where

\[
\begin{align*}
  h(p, q)_{|_{\nu^2=0}} &= h_0(p), \\
  H(P, Q)_{|_{\nu^2=0}} &= H_0(P).
\end{align*}
\]  
(7.18)

With such choice the duality equations become

\[ h_0(Q) = h(p, q), \]
\[ H_0(q) = H(P, Q), \]
\[ \frac{\partial h(p, q)}{\partial p} H'_0(q) = h'_0(Q) \frac{\partial H(P, Q)}{\partial P}. \]  
(7.19)

Free rational, trigonometric and elliptic Hamiltonians are \( h_0(p) = \frac{p^2}{2}, \) \( h_0(p) = \cosh p \) and \( h_0(p) = \text{cn}(p|k) \) respectively.

Note that from the main duality relation,

\[ H_0(q) = H(P, Q) \]  
(7.20)

it follows that

\[
\begin{align*}
  \frac{\partial q}{\partial P} |_{Q} &= \frac{1}{H'_0(q)} \frac{\partial H(P, Q)}{\partial P}.
\end{align*}
\]  
(7.21)

which together with (7.19) implies:

\[
\begin{align*}
  \frac{\partial q}{\partial P} |_{Q} &= \frac{1}{h'_0(Q)} \frac{\partial h(p, q)}{\partial p}.
\end{align*}
\]  
(7.22)

When compared with the Hamiltonian equation for the original system,

\[ \frac{\partial q}{\partial t} = \frac{\partial h(p, q)}{\partial p}, \]  
(7.23)

\footnote{Note that this kind of duality relates the weak coupling regime for \( h(p, q) \) to the weak coupling regime for \( H(P, Q) \). For example, in the rational Calogero case

\[
\begin{align*}
  h(p, q) &= \frac{p^2}{2} + \frac{\nu^2}{q^2} - \frac{Q^2}{2} \\
  H(P, Q) &= \frac{P^2}{2} + \frac{\nu^2}{Q^2} = \frac{q^2}{2}.
\end{align*}
\]  
(7.17)

We recall that, on the field theory side, the coupling constant \( \nu \) is related to the mass of adjoint hypermultiplet and thus remains unchanged under duality transformations.}
we see that $P = h_0'(Q)t$ is proportional to the ordinary time-variable $t$, while $h_0(Q) = h(p, q) = E$ expresses $Q$ as a function of the energy $E$. This is a usual feature of classical integrable systems, exploited in Seiberg-Witten theory \[3\]: in the SU(2) case the spectral curve $q(t)$ can be described by

$$h\left(p \left(\frac{\partial q}{\partial t}, q\right), q\right) = E.$$ 

(7.24)

where $p$ is expressed through $\partial q/\partial t$ and $q$ from the Hamiltonian equation $\partial q/\partial t = \partial H/\partial p$. In other words, the spectral curve is essentially the solution of the equation of motion of integrable system, where the time $t$ plays the role of the spectral parameter and the energy $E$ that of the modulus.

Let us consider several simple examples. The simplest one is the rational Calogero. In this case, the duality transformation connects two identical Hamiltonians, (7.17). Therefore, this system is self-dual. Somewhat less trivial example is the trigonometric Calogero-Sutherland model \[36, 37\]. It leads to the following dual pair

$$h(p, q) = p^2 + \frac{\nu^2}{\sin^2 q} = \frac{Q^2}{2},$$ 

$$H(P, Q) = \sqrt{1 - \frac{2\nu^2}{Q^2}} \cos P = \cos q.$$ 

(7.25)

One may easily recognize in the second Hamiltonian the rational Ruijsenaars Hamiltonian \[38, 39\]. Thus, the Hamiltonian rational in coordinates and momenta maps onto itself under the duality, while that with rational momentum dependence and trigonometric coordinate dependence maps onto the Hamiltonian with inverse, rational coordinate and trigonometric momentum dependencies. Therefore, we can really see that the pq-duality exchanges types of the coordinate and momentum dependence. This implies that the Hamiltonian trigonometrically dependent both on coordinates and momenta is to be self-dual. Indeed, one can check that the trigonometric Ruijsenaars-Schneider system is self-dual:

$$h(p, q) = \sqrt{1 - \frac{2\nu^2}{Q^2}} \cos p = \cos Q,$$

$$H(P, Q) = \sqrt{1 - \frac{2\nu^2}{Q^2}} \cos P = \cos q.$$ 

(7.26)

Now we make the next step and introduce into the game elliptic dependencies.

### 8 Two-body system dual to Calogero and Ruijsenaars

We begin with the elliptic Calogero Hamiltonian\[37, 40\]

$$h(p, q) = \frac{p^2}{2} + \frac{\nu^2}{\sin^2(q|k)},$$ 

(8.27)

and seek a dual Hamiltonian elliptic in the momentum. Thus $h_0(p) = \frac{p^2}{2}$ and we seek $H(P, Q) = H_0(q)$ such that $H_0(q) = \text{cn}(q|k)$. Eqs.\[7.20\] become

$$\frac{Q^2}{2} = \frac{p^2}{2} + \frac{\nu^2}{\sin^2(q|k)}, \quad \text{cn}(q|k) = H(P, Q), \quad p \cdot \text{cn}'(q|k) = Q \frac{\partial H(P, Q)}{\partial P}.$$ 

(8.28)

Upon substituting

$$\text{cn}'(q|k) = -\text{sn}(q|k)\text{dn}(q|k) = -\sqrt{(1 - H^2)(k^2 + k^2H^2)},$$ 

(8.29)

(this is because $\text{sn}^2 q = 1 - \text{cn}^2 q$, $\text{dn}^2 q = k^2 + k^2\text{cn}^2 q$, $k^2 + k^2 = 1$ and $\text{cn}q = H$) we get for (8.28):

$$\left(\frac{\partial H}{\partial P}\right)^2 = \frac{p^2}{Q^2(1 - H^2)(k^2 + k^2H^2)}.$$ 

(8.30)

\[15\] The standard definitions of the Jacobi functions we use below can be found in [31].
Now from the first eqn. (8.28) \( p^2 \) can be expressed through \( Q \) and \( \text{sn}^2(q|k) = 1 - \text{cn}^2(q|k) = 1 - H^2 \) as

\[
\frac{p^2}{Q^2} = 1 - \frac{2\nu^2}{Q^2(1 - H^2)}, \tag{8.31}
\]

so that

\[
\left( \frac{\partial H}{\partial P} \right)^2 = \left( 1 - \frac{2\nu^2}{Q^2} - H^2 \right) (k'^2 + k^2 H^2). \tag{8.32}
\]

Therefore \( H \) is an elliptic function of \( P \), namely \( [37, 21] \)

\[
H(P, Q) = \text{cn}(q|k) = \alpha(Q) \cdot \text{cn} \left( P, \sqrt{k^2 + k^2 \alpha^2(Q)} \right) \tag{8.33}
\]

with

\[
\alpha^2(Q) = \alpha^2_{\text{rat}}(Q) = 1 - \frac{2\nu^2}{Q^2}. \tag{8.34}
\]

In the limit \( \nu^2 = 0 \), when the interaction is switched off, \( \alpha(q) = 1 \) and \( H(P, Q) \) reduces to \( H_0(P) = \text{cn}(P|k) \), as assumed in (8.28).

We have therefore obtained a dual formulation of the elliptic Calogero model (in the simplest \( SU(2) \) case). At first glance our dual Hamiltonian looks somewhat unusual. In particular, the relevant elliptic curve is “dressed”: it is described by an effective modulus

\[
k_{\text{eff}} = \frac{\alpha(Q)}{\sqrt{k^2 + k^2 \alpha^2(Q)}} = \frac{k\alpha(Q)}{\sqrt{1 - k^2(1 - \alpha^2(Q))}}, \tag{8.35}
\]

which differs from the “bare” one \( k \) in a \( Q \)-dependent way. In fact \( k_{\text{eff}} \) is nothing but the modulus of the “reduced” Calogero spectral curve \([19]\).

Now one can rewrite (8.33) in many different forms, in particular, in terms of \( \theta \)-functions, in hyper-elliptic parametrization etc. This latter is rather essential and we refer the reader for the details to the paper \([21]\). In particular, this is the hyperelliptic parametrization, where one can conveniently check that the symplectic structure \( dP \wedge dQ \) is actually equal to \( d\theta_{\text{Jac}} \wedge da \), i.e. to the action-angle variables in the Calogero model. Here \( p_{\text{Jac}}, (4.15) \) is the angle along the cycle in the Jacobian and \( a \) is the action variable given by \( (1.3) \).

All of the above formulae are straightforwardly generalised from the Calogero (rational-elliptic) system to the Ruijsenaars (trigonometric-elliptic) system. The only difference ensuing is that the \( q \)-dependence of the dual (elliptic-trigonometric) Hamiltonian is now trigonometric rather than rational \([21]\).

\[
\alpha^2(q) = \alpha^2_{\text{trig}}(q) = 1 - \frac{2\nu^2}{\sinh^2 q}. \tag{8.36}
\]

Rather than giving further details we will proceed directly to a consideration of the double-elliptic model.

9 Double-elliptic two-body system

In order to get a double-elliptic system one needs to exchange the rational \( Q \)-dependence in (8.28) for an elliptic one, and so we substitute \( \alpha^2_{\text{rat}}(Q) \) by the obvious elliptic analogue \( \alpha^2_{\text{eff}}(Q) = 1 - \frac{2\nu^2}{\text{sn}^2(Q|k)} \).

Moreover, now the elliptic curves for \( q \) and \( Q \) need not in general be the same, i.e. \( \tilde{k} \neq k \).

Instead of (8.28) the duality equations now become

\[
\begin{align*}
\text{cn}(q|k) &= H(P, Q|k, \tilde{k}), \\
\text{cn}(Q|\tilde{k}) &= H(p, q|\tilde{k}, k), \\
\text{cn}'(Q|\tilde{k}) \frac{\partial H(P, Q|k, \tilde{k})}{\partial P} &= \text{cn}'(q|k) \frac{\partial H(p, q|\tilde{k}, k)}{\partial p}, \tag{9.37}
\end{align*}
\]
and the natural ansatz for the Hamiltonian (suggested by \(8.33\)) is

\[
H(p, q|\vec{k}, k) = \alpha(q|\vec{k}, k) \cdot \text{cn} \left( p \beta(q|\vec{k}, k) \mid \gamma(q|\vec{k}, k) \right) = \alpha_{\text{cn}}(p\beta\gamma),
\]

\[
H(P, Q|\vec{k}, \tilde{k}) = \alpha(Q|\vec{k}, \tilde{k}) \cdot \text{cn} \left( P \beta(Q|\vec{k}, \tilde{k}) \mid \gamma(Q|\vec{k}, \tilde{k}) \right) = \tilde{\alpha}_{\text{cn}}(P\tilde{\beta}\tilde{\gamma}).
\]

(9.38)

Substituting these ansatz into \(9.37\) and making use of \(8.29\), one can arrive, after some calculations \(^1\), at \(^2\)

\[
\alpha^2(q|\vec{k}, k) = \alpha^2(q|k) = 1 - \frac{2\nu^2}{\text{sn}^2(q|k)}, \quad \beta^2(q|\vec{k}, k) = \vec{k}^2 + k^2\alpha^2(q|k), \quad \gamma^2(q|\vec{k}, k) = \frac{k^2\alpha^2(q|k)}{\vec{k}^2 + k^2\alpha^2(q|k)},
\]

(9.39)

with some constant \(\nu\) and finally the double-elliptic duality becomes \(^3\)

\[
H(P, Q|\vec{k}, \tilde{k}) = \text{cn}(q|k)\text{cn} \left( P\sqrt{k^2 + \vec{k}^2\alpha^2(q|\vec{k})} \mid \frac{k\alpha(Q|\vec{k})}{\sqrt{k^2 + \vec{k}^2\alpha^2(Q|\vec{k})}} \right),
\]

(9.40)

\[
H(p, q|\vec{k}, \tilde{k}) = \text{cn}(Q|k)\text{cn} \left( p\sqrt{k^2 + \vec{k}^2\alpha^2(q|\vec{k})} \mid \frac{\vec{k}\alpha(q|k)}{\sqrt{k^2 + \vec{k}^2\alpha^2(q|k)}} \right).
\]

(9.41)

We shall now consider various limiting cases arising from these and show that the double-elliptic Hamiltonian \(9.41\) contains the entire Ruijsenaars-Calogero and Toda family as its limiting cases, as desired. (Of course we have restricted ourselves to the \(SU(2)\) members of this family.)

In order to convert the elliptic dependence of the momentum \(p\) into the trigonometric one, the corresponding “bare” modulus \(\vec{k}\) should vanish: \(\vec{k} \to 0\), \(k^2 = 1 - \vec{k}^2 \to 1\) (while \(k\) can be kept finite). Then, since \(\text{cn}(x|k = 0) = \cosh x\),

\[
H^{\text{dell}}(p, q) \rightarrow \alpha(q) \cosh p = H^{\text{Ru}}(p, q)
\]

(9.42)

with the same

\[
\alpha^2(q|k) = 1 - \frac{2\nu^2}{\text{sn}^2(q|k)}.
\]

(9.43)

Thus we obtain the \(SU(2)\) elliptic Ruijsenaars Hamiltonian. The trigonometric and rational Ruijsenaars as well as all of the Calogero and Toda systems are obtained through further limiting procedures in the standard way.

The other limit \(k \to 0\) (with \(\vec{k}\) finite) gives \(\alpha(q|k) \to \alpha_{\text{trig}}(q) = 1 - \frac{2\nu^2}{\cosh q}\), and

\[
H^{\text{dell}}(p, q) \rightarrow \alpha_{\text{trig}}(q) \cdot \text{cn} \left( p\sqrt{k^2 + \vec{k}^2\alpha^2_{\text{trig}}(q)} \mid \frac{\vec{k}\alpha_{\text{trig}}(q)}{\sqrt{k^2 + \vec{k}^2\alpha^2_{\text{trig}}(q)}} \right) = \tilde{H}_{\text{Ru}}(p, q).
\]

(9.44)

This is the elliptic-trigonometric model, dual to the conventional elliptic Ruijsenaars (i.e. the trigonometric-elliptic) system. In the further limit of small \(q\) this degenerates into the elliptic-rational model with \(\alpha_{\text{trig}}(q) \to \alpha_{\text{rat}}(q) = 1 - \frac{2\nu^2}{q}\), which is dual to the conventional elliptic Calogero (i.e. the rational-elliptic) system, analysed above.

Our approach has been based on choosing appropriate functions \(f(q)\) and \(F(Q)\) and implementing duality. Other choices of functions associated with alternative free Hamiltonians may be possible. Instead of the duality relations \(8.33\), one could consider those based on \(h_0(p) = \text{sn}(p|\vec{k})\) instead of \(\text{cn}(p|\vec{k})\). With this choice one gets somewhat simpler expressions for \(\beta_s\) and \(\gamma_s\):\(^4\)

\(^1\)For ease of expression hereafter we suppress the dependence of \(\alpha, \beta, \gamma\) on \(k\) and \(\vec{k}\) in what follows using \(\alpha(q)\) for \(\alpha(q|k, k)\) and \(\tilde{\alpha}(Q)\) for \(\alpha(Q|k, \tilde{k})\) etc.
\[ \beta_s = 1, \quad \gamma_s(q|k, k) = \kappa \alpha_s(q|k), \quad \alpha_s(q|k) = 1 - \frac{2v^2}{c n^2(q|k)} \quad (9.45) \]

and the final Hamiltonian is now
\[ H_s(p, q|k, k) = \alpha_s(q|k) \cdot \sin(p) \kappa \alpha_s(q|k)). \quad (9.46) \]

Although this Hamiltonian is somewhat simpler than our earlier choice, the limits involved in obtaining the Ruijsenaars-Calogero-Toda reductions are somewhat more involved, and that is why we chose to present the Hamiltonian \((9.41)\) first.

One might further try other elliptic functions for \( h_0(p) \). Every solution we have obtained by making a different ansatz has been related to our solution \((9.41)\) via modular transformations of the four moduli \( k, k, k_{eff} = \gamma \) and \( k_{eff} = \gamma \).

10 \ pq-duality, \( \tau \)-functions and Hamiltonian reduction

Now one should extend the results of the two-particle case to the generic number of particles. Unfortunately, it is still unknown how to get very manifest formulas. Here we describe a method that ultimately allows one to construct commuting Hamiltonians of the \( N \)-particle Dell system, but very explicit expressions will be obtained only for first several terms in a perturbation theory.

However, in order to explain the idea, we need some preliminary work. In fact, we start from the old observation \([11, 12]\) that, if one study the elliptic (trigonometric, rational) solution to the KP hierarchy, the dynamics (dependence on higher times of the hierarchy) of zeroes of the corresponding \( \tau \)-function w.r.t. the first time is governed by the elliptic (trigonometric, rational) Calogero system. Moreover, if one starts with the two-dimensional Toda lattice hierarchy and studies the zeroes of the \( \tau \)-function w.r.t. the zero (discrete) time, the corresponding system governing zeroes is the Ruijsenaars system \([43]\).

Let us illustrate this with the simplest example of the trigonometric solution of the KP hierarchy. This is, in fact, the \( N \)-solitonic solution, the corresponding \( \tau \)-function being
\[
\tau(\{t_k\} | \{\nu_i, \mu_i, X_i\}) = \det_{N \times N} \left( \delta_{ij} - L_{ij} e^{\sum_k t_k (\nu_i^k - \nu_j^k)} \right), \quad L_{ij} \equiv \frac{X_i}{\nu_i - \mu_j} \quad (10.47)
\]

Here \( \nu_i, \mu_i \) and \( X_i \) are the soliton parameters. Let us impose the constraint \( \nu_i = \mu_i + \epsilon \). Then, the standard Hamiltonian structure of the hierarchy on the soliton solution gives rise to the Poisson brackets (in \([14, 15]\) similar calculation was done for the sine-Gordon case)
\[
\{X_i, X_j\} = -\frac{4X_i X_j e^{2}}{(\mu_j - \mu_j)(\mu_i - \mu_j - \epsilon)(\mu_i - \mu_j + \epsilon)}, \quad \{\nu_i, \nu_j\} = 0, \quad \{X_i, \mu_j\} = X_i \delta_{ij} \quad (10.48)
\]

Note that, upon identification \( \mu_i = q_i \), \( X_i \equiv e^{\nu_i} \prod_{l \neq i} \left( \frac{1}{(q_i - q_l)^{1/2}} - \frac{1}{\epsilon} \right) \), the matrix \( L_{ij} \) in \((10.47)\) becomes the rational Ruijsenaars Lax operator, with the proper symplectic structure \( \{p_i, q_j\} = \delta_{ij} \).

At the same time, \((10.47)\) can be rewritten in the form
\[
\tau(\{t_k\} | \{\nu_i, \mu_i, X_i\}) \sim \det_{N \times N} \left(e^{xz} \delta_{ij} - L_{ij} e^{\sum_k t_k (\nu_i^k - \nu_j^k)} \right) \quad (10.49)
\]

where \( x = t_1 \). This determinant is the generating function of the rational Ruijsenaars Hamiltonians. On the other hand, the \( N \) zeroes of the determinant as the function of \( x \) are just logarithms of the eigenvalues of the Lax operator. These zeroes are governed, as we know from \([14, 12]\) by the trigonometric Calogero system, and are nothing but the \( N \) particle coordinates in the Calogero system:
\[
\tau(\{t_k\} | \{\nu_i, \mu_i, X_i\}) \sim \sum_k e^{kx} H_k^R \sim \prod_i \sin(\epsilon x - q_i) \quad (10.50)
\]

Therefore, the eigenvalues of the (rational Ruijsenaars) Lax operator are the exponentials \( e^{\nu} \) of coordinates in the (trigonometric Calogero) dual system, while the \( \tau \)-function simultaneously is the generating
function of the Hamiltonians in one integrable systems and the function of coordinates in the dual system. This is exactly the form of duality, as it was first realized by S.Ruijsenaars \cite{5}. Its relation with \( \tau \)-functions was first observed by D.Bernard and O.Babelon \cite{46} (see also \cite{45}).

Similarly, one can consider the \( N \)-solitonic solution in the two-dimensional Toda lattice hierarchy. Then,

\[
\tau_n\{\{t_k\}|\{\nu_i, \mu_i, X_i\}\} = \text{det}_{N \times N} \left( \delta_{ij} - L_{ij} \left( \frac{\mu_i}{\nu_i} \right)^n e^{\sum_k t_k(\mu_i^k - \nu_i^k)} + \tau_k \left( \mu_i^{-k} - \nu_i^{-k} \right) \right), \quad L_{ij} = \frac{X_j}{\nu_i - \mu_j}
\] (10.51)

This time one should make a reduction \( \nu_i = e^t \mu_i \). Then, \( L_{ij} \) becomes the Lax operator of the trigonometric Ruijsenaars, and the zeroes w.r.t. to \( \nu_i \) are governed by the same, trigonometric Ruijsenaars system. This is another check that this system is really self-dual.

Thus constructed duality can be also interpreted in terms of the Hamiltonian reduction \cite{33, 34, 35}. In doing so, one starts with the moment map which typically is a constraint for two matrices or two matrix-valued functions. In order to solve it, one diagonalizes one of the matrices, its diagonal elements ultimately being functions (exponentials) of the coordinates, while the other matrix gives the Lax operator (its traces are the Hamiltonians). Now what one needs in order to construct the dual system is just to diagonalize the second matrix in order to get coordinates in the dual system, while the first matrix will provide one with the new Lax operator (its traces are the Hamiltonians in the dual system).

As an illustrative example, let us consider the trigonometric Calogero-Sutherland system \cite{47}. In order to get it, one can start from the free Hamiltonian system \( H = \text{tr} A^2 \) given on the phase space \((A, B)\) of two \( N \times N \) matrices, with the symplectic form \( \text{tr}(\delta A \wedge B^{-1} \delta B) \) (i.e. \( A \) lies in the \( gl(N) \) algebra, while \( B \) lies in the \( GL(N) \) group). Then, one can impose the constraint

\[
A - BAB^{-1} = \nu(I - P)
\] (10.52)

where \( \nu \) is a constant, \( I \) is the unit matrix and \( P \) is a matrix of rank 1. Now one has to make the Hamiltonian reduction with this constraint. First of all, one solves the constraint diagonalizing matrix \( B \) and using the gauge freedom of the system to transform \( P \) into the matrix with all entries unit. It gives

\[
A_{ij} = p_i \delta_{ij} + (1 - \delta_{ij}) \frac{a_{ib_j}}{\nu_i - \nu_j},
\]

where \( b_i \) are the diagonal elements of \( B \). This is nothing but the Lax operator of the trigonometric Calogero-Sutherland model. Traces of powers of this Lax operators give the Calogero Hamiltonians.

One can also diagonalize the other matrix, \( A \). Then, (10.52) can be rewritten in the form

\[
AB - BA = \nu(B - \overline{P})
\] (10.53)

where \( \overline{P} \) is another rank 1 matrix which is can be gauged out to the matrix with all entries depending only on the number of row (let denote them \( X_i/\nu \)). Solving then this constraint, one obtains \( B = \frac{X_i}{a_i - \nu} \), where \( a_i \) are the diagonal elements of the matrix \( A \). This is nothing but the Lax operator of the rational Ruijsenaars-Schneider system, traces of its powers give a set of the dual Hamiltonians.

Certainly, this picture nicely explaining the Ruijsenaars observation of duality is just the \( \tau \)-function approach told in different words.

11 Dual Hamiltonians for many-body systems

Unfortunately, the scheme described in the previous section does not work for elliptic models with arbitrary number of particles. The reason is that even having the dual Lax operator calculated, one is typically not able in elliptic cases to construct the dual Hamiltonians, i.e. invariant combinations involving the Lax operator.

Instead, in order to obtain dual Hamiltonians, here we use the other approach discussed above, that dealing with zeroes of the \( \tau \)-functions \cite{21, 22}. This approach gives no very explicit form of the dual Hamiltonians. Therefore, we construct later a kind of perturbative procedure for the Hamiltonians which allows one to get them absolutely explicitly term by term \cite{22}.

\textsuperscript{17}Since (10.52) does not fix the diagonal elements of \( A \), they are parametrized by arbitrary numbers \( p_i \).
Consider $SU(N)$ ($N = g + 1$) system. The whole construction is based on the fact that the spectral curve of the original integrable system (Toda chain, Calogero, Ruijsenaars or the most interesting double elliptic system) has a period matrix $T_{ij}(\vec{a})$, $i, j = 1, \ldots, N$ with the special property:

$$\sum_{j=1}^{N} T_{ij}(\vec{a}) = \tau, \quad \forall i$$

(11.54)

where $\tau$ does not depend on $a$. As a corollary, the genus-$N$ theta-function is naturally decomposed into a linear combination of genus-$g$ theta-functions:

$$\Theta^{(N)}(p_i|T_{ij}) = \sum_{n_i \in \mathbb{Z}} \exp \left( i\pi \sum_{i,j=1}^{N} T_{ij} n_i n_j + 2\pi i \sum_{i=1}^{N} n_i p_i \right) = \sum_{k=0}^{N-1} \exp \left( i\pi \sum_{i,j=1}^{N} T_{ij} n_i^2 + 2\pi i \sum_{i=1}^{N} n_i \bar{p}_i \right)$$

(11.55)

where

$$\Theta_k \equiv \tilde{\Theta}_k(p_i|T_{ij}) = \sum_{n_i \in \mathbb{Z}} \exp \left( -i\pi \sum_{i,j=1}^{N} \tilde{T}_{ij} n_i^2 + 2\pi i \sum_{i=1}^{N} n_i \bar{p}_i \right)$$

(11.56)

and $p_i = \zeta + \bar{p}_i, \sum_{i=1}^{N} \bar{p}_i = 0; T_{ij} = \tilde{T}_{ij} + \pi T, \sum_{i=1}^{N} \tilde{T}_{ij} = 0, \forall i$.

Now we again use the argument that zeroes of the KP (Toda) $\tau$-function (i.e. essentially the Riemannian theta-function), associated with the spectral curve (13.73) are nothing but the coordinates $q_i$ of the original (Calogero, Ruijsenaars) integrable system. In more detail, due to the property (11.54), $\Theta^{(N)}(p|T)$ as a function of $\zeta = \frac{1}{\pi} \sum_{i=1}^{N} \bar{p}_i$ is an elliptic function on the torus $(1, \tau)$ and, therefore, can be decomposed into an $N$-fold product of the genus-one theta-functions. Remarkably, their arguments are just $\zeta - q_i$:

$$\Theta^{(N)}(p|T) = c(p, T, \tau) \prod_{i=1}^{N} \theta(\zeta - q_i(p, T)|\tau)$$

(11.57)

(In the case of the Toda chain when $\tau \to i\infty$ this “sum rule” is implied by the standard expression for the individual $e^{p_i}$ through the KP $\tau$-function.) Since one can prove that $q_i$ form a Poisson-commuting set of variables with respect to the symplectic structure (11.58) [11], this observation indirectly justifies the claim ([21], [22]) that all the ratios $\Theta_k/\Theta_l$ (in order to cancel the non-elliptic factor $c(p, T, \tau)$) are Poisson-commuting with respect to the Seiberg-Witten symplectic structure

$$\sum_{i=1}^{N} dp_i^{Jac} \wedge da_i$$

(11.58)

(where $a_N \equiv -a_1 - \ldots - a_{N-1}$, i.e. $\sum_{i=1}^{N} a_i = 0$ [9]

$$\left\{ \frac{\Theta_k}{\Theta_l}, \frac{\Theta_m}{\Theta_n} \right\} = 0 \quad \forall k, l, m, n$$

(11.59)

The Hamiltonians of the dual integrable system can be chosen in the form $H_k = \Theta_k/\Theta_l$, $k = 1, \ldots, g$. However, these dual Hamiltonians are not quite manifest, since they depend on the period matrix of the original system expressed in terms of its action variables. Nevertheless, what is important, one can work with the $\theta$-functions as with series and, using the instantonic expansion for the period matrix $T_{ij}$, construct Hamiltonians term by term in the series.

\[ \text{It can be equivalently written as} \]

$$\Theta_i\{\Theta_j, \Theta_k\} + \Theta_j\{\Theta_i, \Theta_k\} + \Theta_k\{\Theta_i, \Theta_j\} = 0 \quad \forall i, j, k$$

or

$$\{\log\Theta_i, \log\Theta_j\} = \left\{ \frac{\Theta_k}{\Theta_j}, \frac{\Theta_l}{\Theta_i} \right\} \quad \forall i, j, k$$

\[ \text{18} \]
12 Dual Hamiltonians perturbatively

Let us see how it really works. We start with the simplest case of the Seiberg-Witten family, the periodic Toda chain.

12.1 Perturbative approximation

First we consider the perturbative approximation which is nothing but the open Toda chain, see (6.8) and (6.9). This is the case of the perturbative 4\(d\) pure \(N = 2\) SYM theory with the prepotential

\[ F_{\text{pert}}(a) = \frac{1}{2i\pi} \sum_{i<j}^N a_{ij}^2 \log a_{ij} \]  

(12.60)

Then, the period matrix is singular and only finite number of terms survives in the series for the theta-function:

\[ \Theta^{(N)}(p|T) = \sum_{k=0}^{N-1} e^{2\pi i k \zeta} H_k^{(0)}(p, a), \]

(12.61)

\[ H_k^{(0)}(p, a) = \sum_{I, |I|=k} \prod_{i \in I} \prod_{j \not\in I} Z_{ij}^{(0)}(a) \]

(12.62)

Here

\[ Z_{ij}^{(0)}(a) = e^{-i\pi T_{ij}^{(0)}} = \frac{\Lambda_{QCD}}{a_{ij}} \]

(12.63)

and \(I\) are all possible partitions of \(N\) indices into the sets of \(k = |I|\) and \(N - k = |\bar{I}|\) elements. Parameter \(\Lambda_{QCD}\) becomes significant only when the system is deformed: either non-perturbatively or to more complex systems of the Calogero–Ruijsenaars–double-elliptic family. The corresponding \(\tau\)-function \(\Theta^{(N)}(p|T), eq.(12.61)\), describes an \(N\)-soliton solution to the KP hierarchy and is equal to the determinant (10.47). The Hamiltonians \(H_k^{(0)}\), eq.(12.62) dual to the open Toda chain, are those of the degenerated rational Ruijsenaars system, and they are well-known to Poisson-commute with respect to the relevant Seiberg-Witten symplectic structure (11.58). This is not surprising since the open Toda chain is obtained by a degeneration from the trigonometric Calogero system which is dual to the rational Ruijsenaars.

The same construction for other perturbative Seiberg-Witten systems ends up with the Hamiltonians of more sophisticated systems.

For the trigonometric Calogero system (perturbative 4\(d\) \(\mathcal{N} = 4\) SYM with SUSY softly broken down to \(\mathcal{N} = 2\) by the adjoint mass \(M\)) the Poisson-commuting (with respect to the same (11.58)), whose perturbative prepotential is given by the sum of (5.2) and (5.3), Hamiltonians \(H_k^{(0)}\) are given by (12.62) with

\[ Z_{ij}^{(0)}(a) = e^{-i\pi T_{ij}^{(0)}} = \frac{\sqrt{a_{ij}^2 - M^2}}{a_{ij}} \]

(12.64)

i.e. are the Hamiltonians of the rational Ruijsenaars system which are really dual to the trigonometric Calogero model.

For the period matrix (effective charges) (5.4) of the trigonometric Ruijsenaars system (perturbative 5\(d\) \(\mathcal{N} = 2\) SYM compactified on a circle with an \(\epsilon\) twist as the boundary conditions) the Hamiltonians are given by (12.62) with

\[ Z_{ij}^{(0)}(a) = \frac{\sqrt{\sinh(a_{ij} + \epsilon) \sinh(a_{ij} - \epsilon)}}{\sinh a_{ij}} \]

(12.65)

i.e. are the Hamiltonians of the trigonometric Ruijsenaars system, which is, indeed, self-dual.

Finally, for the perturbative limit of the most interesting self-dual double-elliptic system (the explicit form of its spectral curves, whose period matrix is supposed to be (5.7), is yet unknown but in the case of two particles, see the next section) the relevant Hamiltonians are those of the elliptic Ruijsenaars system, given by the same (12.62) with (cf. with (5.7))
\[ Z_{ij}^{(0)}(a) = \sqrt{1 - \frac{2g^2}{\text{sn}^2(\tau)}} \sim \frac{\sqrt{\theta'(a_{ij} + \varepsilon |\tau|)\theta'(a_{ij} - \varepsilon |\tau|)}}{\theta'(a_{ij})} \]  \hspace{1cm} (12.66)

where \( \tilde{\tau} \) is the modulus of the second torus associated with the double elliptic system and \( \tilde{a}_{ij} \) are \( a_{ij} \) rescaled, [33].

In all these examples \( H_0^{(0)} = 1 \), the theta-functions \( \Theta^{(N)} \) are singular and given by determinant (solitonic) formulas with finite number of items (only terms with \( n_i = 0,1 \) survive in the series expansion of the theta-function), and Poisson-commutativity of arising Hamiltonians is analytically checked within the theory of Ruijsenaars integrable systems.

### 12.2 Beyond the perturbative limit

Beyond the perturbative limit, the analytical evaluation of \( \Theta^{(N)} \) becomes less straightforward.

The non-perturbative deformation (11.13) of the curve (12.6), is associated with somewhat sophisticated prepotential of the periodic Toda chain,

\[ F(a) = \frac{1}{4\pi i} \sum_{i<j}^N a_{ij}^2 \log \frac{a_{ij}}{\Lambda} + \sum_{k=1}^N \Lambda^2 F^{(k)}(a) \]  \hspace{1cm} (12.67)

The period matrix is

\[ \pi T_{ij} = \frac{\partial^2 F}{\partial a_i \partial a_j} = -2\pi i \delta_{ij} + \log \frac{a_{ij}}{\Lambda} + \sum_{k=1}^N \Lambda^2 \frac{\partial^2 F^{(k)}(a)}{\partial a_i \partial a_j}, \ i \neq j, \]  \hspace{1cm} (12.68)

and \( \tau \) in this case can be removed by the rescaling of \( \Lambda \). Then,

\[ \Theta^{(N)}(p|T) = \sum_{k=0}^{N-1} e^{2\pi i k \zeta} \Theta_k(p,a) = \sum_{k=0}^{N-1} e^{2\pi i k \zeta} \sum_{n_i, n_j, n_k = 0} e^{-i\pi \sum_i T_{ij} n_i^2} e^{2\pi i \sum_i n_i p_i} =  \\
= \left( 1 + \sum_{i \neq j}^N e^{2\pi i (p_i - p_j)} Z_{ij}^4 \prod_{k \neq i,j} Z_{ik} Z_{jk} + \right) +  \\
+ \sum_{i \neq j \neq k \neq l}^N e^{2\pi i (p_i - p_j - p_k + p_l)} Z_{ik}^4 Z_{jl}^4 \prod_{m \neq i,j,k,l} Z_{im} Z_{jm} Z_{km} Z_{lm} + \ldots \right) +  \\
+e^{2\pi i \zeta} \left( \sum_{i=1}^N e^{2\pi i p_i} \prod_{j \neq i} Z_{ij} + \sum_{i \neq j \neq k} e^{2\pi i (p_i + p_j - p_k)} Z_{ik}^4 Z_{jk}^4 \prod_{l \neq i, j, k} Z_{il} Z_{jl} Z_{kl} + \right) +  \\
+ \sum_{i \neq j}^N e^{2\pi i (2p_i - p_j)} Z_{ij}^9 \prod_{k \neq i, j} Z_{ik}^4 Z_{jk} + \ldots \right) +  \\
+e^{4\pi i \zeta} \left( \sum_{i \neq j}^N e^{2\pi i (p_i + p_j)} \prod_{k \neq i, j} Z_{ik} Z_{jk} + \sum_{i} e^{4\pi i p_i} \prod_{k \neq i} Z_{ik}^4 + \ldots \right) +  \\
+ \ldots \]

with

\[ Z_{ij} = e^{-i\pi T_{ij}} = Z_{ij}^{(0)} \left( 1 - i\pi \frac{\partial^2 Z^{(1)}(a)}{\partial a_i \partial a_j} + \ldots \right), \ i \neq j \]  \hspace{1cm} (12.70)
The first few corrections $Z^{(k)}$ to the prepotential are explicitly known in the Toda-chain case \[18\], for example,

$$Z^{(1)} = -\frac{1}{2\pi} \sum_{i=1}^{N} \prod_{k \neq i} \left( Z_{ik}^{(0)} \right)^{2} \tag{12.71}$$

The coefficients $\Theta_k$ in (12.69) are expanded into powers of $(\Lambda/a)^{2N}$ and the leading (zeroth-order) terms are exactly the perturbative expressions (12.63). Thus, the degenerated Ruijsenaars Hamiltonians (12.62) are just the perturbative approximations to the $H_k = \Theta_k/\Theta_0$ – the full Hamiltonians of the integrable system dual to the Toda chain.

The same expansion can be constructed for the other, elliptic systems. Note that the period matrix in these cases is expanded into powers of $q = e^{2\pi i r}$:

$$\pi iT_{ij} = \frac{\partial^2 F}{\partial a_i \partial a_j} = -2\pi i r \delta_{ij} + \log \frac{a_{ij}}{\Lambda} + \mu^2 \sum_{k=1}^{\infty} q^k \frac{\partial^2 F^{(k)}(a)}{\partial a_i \partial a_j}, \ i \neq j \tag{12.72}$$

and the dimensional constant $\mu$ depends on the system. E.g., in the Calogero model $\mu = i M$ etc. The limit to the Toda system corresponds to $q \to 0$, $q\mu^{2N}$-fixed.

Note that the first (instanton-gas) correction to the prepotential is indeed known to be of the same universal form (12.71) not only for the Toda chain, but also for the Calogero system [49]. For the Ruijsenaars and double-elliptic systems eq.(12.71) is not yet available in the literature.

In [28] the perturbative expansions for the Hamiltonians were tested for Poisson commutativity, up to few first terms and small number of particles $N$. In particular, it was checked that (12.73) is true for both the Ruijsenaars and Dell systems.

The main lesson we could get from this consideration is that if the universal expressions like (12.71) in terms of perturbative $Z_{ij}^{(0)}$, the same for all the systems, will be found for higher corrections to the prepotentials,\footnote{To avoid possible confusion, the recurrent relation [28] for the Toda-chain prepotential,}

$$\frac{\partial^2 F}{\partial \log \Lambda^2} \sim \frac{\partial^2 F}{\partial \log \Lambda \partial a_i} \frac{\partial^2 F}{\partial \log \Lambda \partial a_j} \frac{\partial^2}{\partial p_i \partial p_j} \log \Theta_0 \left|_{p=0} \right.$$ 

\footnote{does not immediately provide such universal expressions. Already for $Z^{(1)}$ this relation gives}

$$Z^{(1)} \sim \sum_{i,j}^{N} \left( Z^{(0)}_{ij} \right)^{2} \prod_{k \neq i,j}^{N} Z_{ik}^{(0)} Z_{jk}^{(0)} \tag{12.73}$$

which coincides with (12.71) for the Toda-chain $Z^{(0)}_{ij}$, eq.(12.63), but is not true (in variance with (12.71)) for Calogero $Z^{(0)}_{ij}$, eq. (12.64). Meanwhile, the recurrent relations of ref.[21] for the Calogero system provide more promising expansion.

13 Dell systems: 6d adjoint matter

Thus, we have constructed the Dell system and a set of dualities that acts on the Toda-Calogero-Ruijsenaars-Dell family. They are all put together in Fig.2.

\[19\] To avoid possible confusion, the recurrent relation [28] for the Toda-chain prepotential,
Since the theories of this family describes low energy limits of different SUSY gauge theories, this table describes the dualities between different gauge theories too. Say, perturbative limit of the 4d theory (trigonometric Calogero) is dual to a special degeneration of the perturbative limit of the 5d theory (rational Ruijsenaars), while full, non-perturbative 5d theory (elliptic Ruijsenaars) is dual to the perturbative limit of the 6d theory (Dell system).

To conclude our discussion of dualities and Dell systems, we calculate the perturbative prepotential of the 2-particle Dell system and show that it coincides with what is to be expected for the 6d theory with adjoint matter, supporting the identification of the Dell system with the 6d gauge theory.

Let us start from the full Dell model for 2-particles. Then, it is given by the Hamiltonian (9.41) parametrized by two independent (momentum and coordinate) elliptic curves with elliptic moduli \( k \) and \( \tilde{k} \). As usual, for two particles we can construct the full spectral curve from the Hamiltonian (9.41),

\[
H(\zeta,\xi|k,\tilde{k}) = u \quad (= \text{cn}(Q|k))
\]

It is characterized by the effective elliptic moduli

\[
k_{\text{eff}} = \frac{k\alpha(q|\tilde{k})}{\beta(q|k)}, \quad \tilde{k}_{\text{eff}} = \frac{\tilde{k}\alpha(q|k)}{\beta(q|k)}
\]

where the functions \( \alpha \) and \( \beta \) are manifestly given in (9.39). Coordinate-momentum duality interchanges \( k \leftrightarrow \tilde{k} \), \( k_{\text{eff}} \leftrightarrow \tilde{k}_{\text{eff}} \) (and \( q,p \leftrightarrow Q,P \)), while in general \( SU(N) \) case, the model describes an interplay between the four tori: the two bare elliptic curves and two effective Jacobians of complex dimension \( g = N - 1 \).

The generating differential in 6d theories is of the form \( dS = \zeta d\xi \), with \( \xi \) living on the coordinate torus and \( \zeta \) – on the momentum torus.

Now we calculate the leading order of the prepotential expansion in powers of \( \tilde{k} \) when the bare spectral torus degenerates into sphere. In the forthcoming calculation we closely follow the line of [20].

When \( \tilde{k} \to 0 \), \( \text{sn}(q|\tilde{k}) \) degenerates into the ordinary sine. For further convenience, we shall parametrize the coupling constant \( 2\nu^2 \equiv \text{sn}^2(\epsilon|k) \). Now the spectral curve (13.73) acquires the form

\[
\alpha(\xi) \equiv \sqrt{1 - \frac{\text{sn}^2(\epsilon|k)}{\sin^2 \xi}} = \frac{u}{\text{cn}(\zeta\beta|k_{\text{eff}})}
\]

Here the variable \( \xi \) lives in the cylinder produced after degenerating the bare coordinate torus. So does the variable \( x = 1/\sin^2 \xi \). Note that the \( A \)-period of the dressed torus shrinks on the sphere to a contour around \( x = 0 \). Similarly, \( B \)-period can be taken as a contour passing from \( x = 0 \) to \( x = 1 \) and back.

The next step is to calculate variation of the generating differential \( dS = \eta d\xi \) w.r.t. the modulus \( u \) in order to obtain a holomorphic differential:
\[ dv = \left( -\text{sn}(\epsilon|k)\sqrt{k'^2 + k^2u^2} \right)^{-1} \frac{dx}{x\sqrt{(x-1)(U^2 - x)}} \]  

(13.76)

where \( U^2 \equiv \frac{1-u^2}{\text{sn}^2(\epsilon|k)} \). Since

\[ \frac{\partial \alpha}{\partial u} = \oint_{x=0} \frac{\partial \tilde{dS}}{\partial u} = \oint_{x=0} dv = \frac{1}{\sqrt{(1-u^2)(k'^2 + k^2u^2)}} \]  

(13.77)

we deduce that \( u = \text{cn}(a|k) \) and \( U = \frac{\text{sn}(a|k)}{\text{sn}(\epsilon|k)} \). The ratio of the B- and A-periods of \( dv \) gives the period matrix

\[ T = \frac{U}{\pi} \int_0^1 \frac{dx}{x\sqrt{(x-1)(U^2 - x)}} = -\frac{1}{i\pi} \lim_{\kappa \to 0} \left( \log \frac{\kappa}{4} \right) + \frac{1}{i\pi} \log \frac{U^2}{1-U^2} \]  

(13.78)

where \( \kappa \) is a small-\( x \) cut-off. The \( U \) dependent part of this integral is finite and can be considered as the “true” perturbative correction, while the divergent part just renormalizes the bare “classical” coupling constant \( \tau \), i.e. classical part of the prepotential (see [20] for further details). Therefore, the perturbative period matrix is finally

\[ T_{\text{finite}} = \frac{i}{\pi} \log \frac{\text{sn}^2(\epsilon|k) - \text{sn}^2(a|k)}{\text{sn}^2(a|k)} + \text{const} \rightarrow \frac{i}{\pi} \log \frac{\theta_1(a + \epsilon)\theta_1(a - \epsilon)}{\theta_1^2(a)} \]  

(13.79)

and the perturbative prepotential is the elliptic tri-logarithm. Remarkably, it lives on the bare momentum torus, while the modulus of the perturbative curve \( (13.75) \) is the dressed one.

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