Computation of the State Bias and Initial States for Stochastic State Space Systems in the General 2-D Roesser Model Form

José A. Ramos  
Nova Southeastern University  
College of Engineering and Computing  
Department of Engineering and Technology  
3301 College Avenue  
Fort Lauderdale, FL 33314  
Email: jr1284@nova.edu  
and  
Guillaume Mercère  
Université de Poitiers  
Laboratoire d’Informatique et d’Automatique pour les Systèmes  
2 rue Pierre Brouse, bâtiment B25, TSA 41105  
86073 Poitiers cedex 9, France  
Email: guillaume.mercere@univ-poitiers.fr

August 12, 2018
Abstract

Recently [Ramos and Mercère (2017a)] presented a subspace system identification algorithm for 2-D purely stochastic state space models in the general Roesser form. However, since the exact problem requires an oblique projection of $Y^h_f$ projected onto $W^h_p$ along $\hat{X}^{vh}_f$, where $W^h_p = \begin{bmatrix} \hat{X}^{vh}_p \\ Y^h_p \end{bmatrix}$, this presents a problem since $\{\hat{X}^{vh}_p, \hat{X}^{vh}_f\}$ are unknown. In the above mentioned paper, the authors found that by doing an orthogonal projection $Y^h_f/Y^h_p$, one can identify the future horizontal state matrix $\hat{X}^h_f$ with a small bias due to the initial conditions that depend on $\{\hat{X}^{vh}_p, \hat{X}^{vh}_f\}$. Nevertheless, the results on modeling 2-D images were very good despite lack of knowledge of $\{\hat{X}^{vh}_p, \hat{X}^{vh}_f\}$. In this note we delve into the bias term and prove that it is insignificant, provided $i$ is chosen large enough and the vertical and horizontal states are uncorrelated. That is, the cross covariance of the state estimates $x^h_{r,s}$ and $x^v_{r,s}$ is zero, or $P_{hv} = 0_{n_x \times n_x}$ and $P_{vh} = 0_{n_x \times n_x}$. Our simulations use $i = 30$. We also present a second iteration to improve the state estimates by including the vertical states computed from a vertical data processing step, i.e., by doing an orthogonal projection $Y^v_f/Y^v_p$. In this revised algorithm we include a step to compute the initial states. This new portion, in addition to the algorithm presented in [Ramos and Mercère (2017a)], forms a complete 2-D stochastic subspace system identification algorithm.
1 Problem Formulation

The general 2-D stochastic Roesser model has the state-space form

\[
\begin{align*}
\dot{x}_{r,s}^h &= A_1 x_{r,s}^h + A_2 x_{r,s}^v + w_r^h, \quad (1a) \\
\dot{x}_{r,s}^v &= A_3 x_{r,s}^h + A_4 x_{r,s}^v + w_r^v, \quad (1b) \\
y_{r,s} &= C_1 x_{r,s}^h + C_2 x_{r,s}^v + v_{r,s}, \quad (1c)
\end{align*}
\]

where \(x_{r,s}^h \in \mathbb{R}^{n_h}, x_{r,s}^v \in \mathbb{R}^{n_v}, \) and \(y_{r,s} \in \mathbb{R}^{n_y}\) denote, respectively, the local horizontal state, local vertical state, and output vectors at the \((r,s)\)th location of a finite domain \(\mathbb{D} = \{(r,s) | 0 \leq r \leq N \text{ and } 0 \leq s \leq M\}\). The system matrices \(\{A,C\}\), given by

\[
A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad (2a)
\]

have partitioned dimensions \(A_1 \in \mathbb{R}^{n_h \times n_h}, A_2 \in \mathbb{R}^{n_h \times n_v}, A_3 \in \mathbb{R}^{n_v \times n_h}, A_4 \in \mathbb{R}^{n_v \times n_v}, C_1 \in \mathbb{R}^{n_y \times n_h}, \) and \(C_2 \in \mathbb{R}^{n_y \times n_v}\). The noise vectors \(w_{r,s}^h \in \mathbb{R}^{n_h}, w_{r,s}^v \in \mathbb{R}^{n_v}, \) and \(v_{r,s} \in \mathbb{R}^{n_y}\) are assumed to be white Gaussian noise processes with mean and joint covariance matrix given, respectively, by

\[
\mathbb{E}\left\{\begin{bmatrix} w_{r,s}^h \\
w_{r,s}^v \\
v_{r,s} \end{bmatrix}\right\} = \begin{bmatrix} 0_{n_h \times 1} \\ 0_{n_v \times 1} \\ 0_{n_y \times 1} \end{bmatrix}, \quad (3a)
\]

\[
\mathbb{E}\left\{\begin{bmatrix} \begin{bmatrix} w_{r,s}^h \\ w_{r,s}^v \end{bmatrix} & (w_{r',s'}^h)^\top & (w_{r',s'}^v)^\top & v_{r',s'}^\top \end{bmatrix}\right\} = \begin{bmatrix} Q_{hh} & Q_{hv} & S_h \\ Q_{vh} & Q_{vv} & S_v \end{bmatrix} \cdot \delta_{r-r'} \cdot \delta_{s-s'}, \quad (3b)
\]

where \(Q_{hh} \in \mathbb{R}^{n_h \times n_h}, Q_{hv} \in \mathbb{R}^{n_h \times n_v}, Q_{vh} \in \mathbb{R}^{n_v \times n_h}, Q_{vv} \in \mathbb{R}^{n_v \times n_v}, S_h \in \mathbb{R}^{n_h \times n_y}, S_v \in \mathbb{R}^{n_v \times n_y}, \) and \(R \in \mathbb{R}^{n_y \times n_y}\), \(n_x = n_h + n_v\) is the dimension of the combined system, \(\mathbb{E}\) is the expectation operator, \(M^\top\) denotes the transpose of \(M\), \(\delta_{k-k'}\) is the Kronecker delta function, \(0_{m \times n}\) denotes an \((m \times n)\) matrix with all its elements equal to zero, and \(\{Q,R,S\}\) are the covariance and cross-covariance matrices of the noise terms.

The noise and state vectors are uncorrelated with each other, i.e.,

\[
\mathbb{E}\left\{x_{r,s}^h \left[\begin{bmatrix} (w_{r',s'}^h)^\top \\ (w_{r',s'}^v)^\top \end{bmatrix} & v_{r',s'}^\top \end{bmatrix}\right]\right\} = 0_{n_h \times (n_x+n_y)}, \quad \forall r' \geq r \text{ and } s' \geq s, \quad (4a)
\]

\[
\mathbb{E}\left\{x_{r,s}^v \left[\begin{bmatrix} (w_{r',s'}^h)^\top \\ (w_{r',s'}^v)^\top \end{bmatrix} & v_{r',s'}^\top \end{bmatrix}\right]\right\} = 0_{n_v \times (n_x+n_y)}, \quad \forall r' \geq r \text{ and } s' \geq s. \quad (4b)
\]

Furthermore, the states \(x_{r,s}^h\) and \(x_{r,s}^v\) evolve with the following statistical properties: zero mean

\[
\mathbb{E}\left\{\begin{bmatrix} x_{r,s}^h \\ x_{r,s}^v \end{bmatrix}\right\} = \begin{bmatrix} 0_{n_h \times 1} \\ 0_{n_v \times 1} \end{bmatrix}, \quad r = 0, 1, \ldots, N \text{ and } s = 0, 1, \ldots, M \quad (5)
\]
and positive definite state covariance matrix

\[
\Pi = \mathbb{E}\left\{ \begin{bmatrix} x_{r,s}^h \\ x_{r,s}^v \end{bmatrix} \right\} = \begin{bmatrix} \Pi_h & 0_{n_h \times n_v} \\ 0_{n_v \times n_h} & \Pi_v \end{bmatrix}. \tag{6}
\]

Let us now define the covariance of the state update as

\[
\Pi' = \mathbb{E}\left\{ \begin{bmatrix} x_{r+1,s}^h \\ x_{r+1,s}^v \end{bmatrix} \right\} = \begin{bmatrix} \Pi_h' & 0_{n_h \times n_v} \\ 0_{n_v \times n_h} & \Pi_v \end{bmatrix}, \tag{7}
\]

where \(\Pi_{hv} = A_1 \Pi_h A_3^T + A_2 \Pi_v A_4^T + Q_{hv},\) \(\Pi_{vh} = \Pi_{hv}^T,\) and the dimensions are \(\Pi_h \in \mathbb{R}^{n_h \times n_h},\)
\(\Pi_{hv} \in \mathbb{R}^{n_h \times n_v},\) \(\Pi_{vh} \in \mathbb{R}^{n_v \times n_h},\) and \(\Pi_v \in \mathbb{R}^{n_v \times n_v}.\) The state covariance update equation becomes

\[
\Pi' = \Pi \Pi^T + Q, \tag{8}
\]

where \(\Pi = \Pi^T\) and \(\Pi' = (\Pi')^T.\) Note that \(\Pi\) is not a matrix Lyapunov state covariance equation since \(\Pi' \neq \Pi.\) However, by partitioning \(\Pi,\) one can decompose it into a pair of coupled horizontal and vertical matrix Lyapunov type equations (Ramos & Mercére, 2016b).

Nevertheless, one can enforce the constraint \(\Pi_{hv} = 0_{n_h \times n_v},\) which results in the joint matrix Lyapunov equation

\[
\begin{bmatrix} \Pi_h & 0_{n_h \times n_v} \\ 0_{n_v \times n_h} & \Pi_v \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \Pi_h & 0_{n_h \times n_v} \\ 0_{n_v \times n_h} & \Pi_v \end{bmatrix} + \begin{bmatrix} Q_{hh} & Q_{hv} \\ Q_{vh} & Q_{vv} \end{bmatrix},
\]

or, more compactly,

\[
\Pi = \Pi \Pi^T + Q. \tag{9}
\]

Throughout the rest of this note we will use the symbol \(> 0\) \((\geq 0)\) to indicate that a matrix is positive definite (positive semi-definite). Model (1a) – (1c) then satisfies the following constraints, also known as the positive real conditions:

\[
\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0, \quad Q \geq 0, \quad R > 0, \quad \Pi > 0. \tag{10}
\]

The 2-D output autocovariance sequence \(\Lambda_{k,m} \in \mathbb{R}^{n_y \times n_y}\) is given in terms of the Markov parameters of the system as

\[
\Lambda_{k,m} = \mathbb{E}\left\{ y_{r+k,s+m} y_{r,s}^T \right\} = \begin{cases} C_1 \Pi_h C_1^T + C_2 \Pi_v C_2^T + R, & \text{if } k = 0, \ m = 0 \\ C_1 A_1^{k-1} G_1, & \text{if } k \geq 1, \ m = 0 \\ C_2 A_4^{m-1} G_2, & \text{if } k = 0, \ m \geq 1 \\ C A^{k-1,m} G^{1,0} + C A^{k,m-1} G^{0,1}, & \text{if } k \geq 1, \ m \geq 1, \end{cases} \tag{11}
\]

where \(G_1\) and \(G_2\) are defined, respectively, as the horizontal and vertical partitions of the matrix \(G \in \mathbb{R}^{n_x \times n_y},\) obtained from

\[
G = \mathbb{E}\left\{ \begin{bmatrix} x_{r+1,s}^h \\ x_{r+1,s}^v \end{bmatrix} \right\} y_{r,s}^T = \Pi \Pi^T + S. \tag{12a}
\]
with \( G_1 \in \mathbb{R}^{n_h \times n_y} \) and \( G_2 \in \mathbb{R}^{n_v \times n_y} \) given as

\[
G_1 = A_1 \Pi_h C_1^T + A_2 \Pi_v C_2^T + S_h \\
G_2 = A_3 \Pi_h C_1^T + A_4 \Pi_v C_2^T + S_v,
\] (12b)

and

\[
G^{1,0} = \begin{bmatrix} G_1 \\ 0_{n_v \times n_y} \end{bmatrix}, \quad G^{0,1} = \begin{bmatrix} 0_{n_h \times n_y} \\ G_2 \end{bmatrix}
\]

\[
A^{0,0} = I_n, \quad A^{1,0} = \begin{bmatrix} A_1 \\ 0_{n_v \times n_h} \\ A_2 \end{bmatrix}, \quad A^{0,1} = \begin{bmatrix} 0_{n_h \times n_h} \\ A_3 \\ A_4 \end{bmatrix}
\]

\[
A = A^{1,0} + A^{0,1}, \quad A^{k,m} = A^{1,0} A^{k-1,m} + A^{0,1} A^{k,m-1}, \quad \text{for } (k, m) > (0, 0)
\]

\[
A^{-k,m} = A^{k,m} = 0_{n_x \times n_x}, \quad \text{for } k \geq 1, m \geq 1.
\]

The problem can now be stated as follows:

**Definition 1.** Given a data matrix \( Y \in \mathbb{R}^{n_y(N+1) \times (M+1)} \) corresponding to the output sequence \( y_{r,s} \in \mathbb{R}^{n_y} \), for \( r = 0, 1, \ldots, N \) and \( s = 0, 1, \ldots, M \), find: (i) the system orders \( n_h \) and \( n_v \) such that \( n_x = n_h + n_v \), (ii) parameter matrices \( \{A, C, G\} \) up to a similarity transformation, (iii) covariance matrices \( \{\Pi, Q, R, S\} \), and (iv) the initial conditions \( \{x_{0,s}^h\}_{s=0}^M \) and \( \{x_{r,0}^v\}_{r=0}^N \), subject to the constraints (10), so that the 2nd-order statistics of the output of the system match those of the given output data.

In order to simplify the analysis, we also formulate the problem in the innovations form as

\[
\hat{x}_{r+1,s}^h = A_1 \hat{x}_{r,s}^h + A_2 \hat{x}_{r,s}^v + K_1 e_{r,s} \\
\hat{x}_{r,s+1}^v = A_3 \hat{x}_{r,s}^h + A_4 \hat{x}_{r,s}^v + K_2 e_{r,s} \\
y_{r,s} = C_1 \hat{x}_{r,s}^h + C_2 \hat{x}_{r,s}^v + e_{r,s},
\] (13a)

(13b)

(13c)

where \( \hat{x}_{r,s}^h \in \mathbb{R}^{n_h} \) and \( \hat{x}_{r,s}^v \in \mathbb{R}^{n_v} \) are, respectively, the horizontal and vertical state estimates, with state estimate covariance matrices \( P_h = \mathbb{E}\{\hat{x}_{r,s}^h (\hat{x}_{r,s}^h)^\top\} \in \mathbb{R}^{n_h \times n_h} \) and \( P_v = \mathbb{E}\{\hat{x}_{r,s}^v (\hat{x}_{r,s}^v)^\top\} \in \mathbb{R}^{n_v \times n_v} \). Furthermore, we assume that \( P_{hv} = \mathbb{E}\{\hat{x}_{r,s}^h (\hat{x}_{r,s}^v)^\top\} = 0_{n_h \times n_v} \). These state estimate covariance matrices satisfy the joint Riccati equation

\[
P = APA^\top + (G - APC^\top)(\Lambda_{0,0} - CPC^\top)^{-1}(G - APC^\top)^\top,
\] (14)

where

\[
P = \begin{bmatrix} P_h & 0_{n_h \times n_v} \\ 0_{n_v \times n_h} & P_v \end{bmatrix}
\] (15)

is a positive definite matrix. We further define the innovations covariance matrix \( R_e = \mathbb{E}\{e_{r,s} e_{r,s}^\top\} \) as

\[
R_e = \Lambda_{0,0} - C_1 P_h C_1^\top - C_2 P_v C_2^\top
\] (16)
and state estimate errors and state estimate error covariance matrices, respectively, as

\[
\begin{align*}
\tilde{x}_{r,s}^h &= \tilde{x}_{r,s}^h - \tilde{x}_{r,s}^h \in \mathbb{R}^{n_h} \\
\tilde{x}_{r,s}^v &= \tilde{x}_{r,s}^v - \tilde{x}_{r,s}^v \in \mathbb{R}^{n_v} \\
\Sigma_h &= \mathbb{E}\left\{ \tilde{x}_{r,s}^h (\tilde{x}_{r,s}^h)^\top \right\} = \Pi_h - P_h \in \mathbb{R}^{n_h \times n_h} \\
\Sigma_v &= \mathbb{E}\left\{ \tilde{x}_{r,s}^v (\tilde{x}_{r,s}^v)^\top \right\} = \Pi_v - P_v \in \mathbb{R}^{n_v \times n_v}.
\end{align*}
\]

Then \(\Sigma_h\) and \(\Sigma_v\) satisfy the joint Riccati equation

\[
\Sigma = A\Sigma A^\top + Q + (A\Sigma C^\top + S)(C\Sigma C^\top + R)^{-1}(A\Sigma C^\top + S)^\top,
\]

where

\[
\Sigma = \begin{bmatrix} \Sigma_h & 0_{n_h \times n_v} \\
0_{n_v \times n_h} & \Sigma_v \end{bmatrix} = \begin{bmatrix} \Pi_h & 0_{n_h \times n_v} \\
0_{n_v \times n_h} & \Pi_v \end{bmatrix} - \begin{bmatrix} P_h & 0_{n_h \times n_v} \\
0_{n_v \times n_h} & P_v \end{bmatrix}.
\]

Finally, the Kalman gain matrix is given by either of the following two expressions

\[
K = (G - APC^\top)(A_{0,0} - CPC^\top)^{-1} \in \mathbb{R}^{n_x \times n_y}
\]

(18a)

\[
K = (A\Sigma C^\top + S)(C\Sigma C^\top + R)^{-1} \in \mathbb{R}^{n_x \times n_y},
\]

(18b)

where

\[
K = \begin{bmatrix} K_1 \\
K_2 \end{bmatrix},
\]

with dimensions \(K_1 \in \mathbb{R}^{n_h \times n_y}\) and \(K_2 \in \mathbb{R}^{n_v \times n_y}\).

## 2 Horizontal Data Processing

Let the horizontal and vertical past and future state matrices for \(k = 0, 1, \ldots, M\) and \(N = 2i + j - 2\) be defined as

\[
\tilde{X}_p(k) = \begin{bmatrix} \tilde{x}_{0,k}^h & \tilde{x}_{1,k}^h & \cdots & \tilde{x}_{j-1,k}^h \end{bmatrix} \in \mathbb{R}^{n_h \times j}
\]

(19)

\[
\tilde{X}_f(k) = \begin{bmatrix} \tilde{x}_{i+1,k}^h & \tilde{x}_{i+2,k}^h & \cdots & \tilde{x}_{i+j-1,k}^h \end{bmatrix} \in \mathbb{R}^{n_h \times j}
\]

(20)

\[
\tilde{X}_p(k) = \begin{bmatrix} \tilde{x}_{0,k}^v & \tilde{x}_{1,k}^v & \cdots & \tilde{x}_{j-1,k}^v \\
\tilde{x}_{i+1,k}^v & \tilde{x}_{i+2,k}^v & \cdots & \tilde{x}_{i+j-1,k}^v \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{x}_{2i-1,k}^v & \tilde{x}_{2i,k}^v & \cdots & \tilde{x}_{2i+j-2,k}^v \end{bmatrix} \in \mathbb{R}^{n_v \times j}
\]

(21)

\[
\tilde{X}_f(k) = \begin{bmatrix} \tilde{x}_{i+1,k}^v & \tilde{x}_{i+2,k}^v & \cdots & \tilde{x}_{i+j-1,k}^v \\
\tilde{x}_{i+1,k}^v & \tilde{x}_{i+2,k}^v & \cdots & \tilde{x}_{i+j-1,k}^v \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{x}_{2i-1,k}^v & \tilde{x}_{2i,k}^v & \cdots & \tilde{x}_{2i+j-2,k}^v \end{bmatrix} \in \mathbb{R}^{n_v \times j}
\]

(22)
where throughout the sequel, subscripts \( p \) and \( f \) denote past and future, respectively, superscripts \( h \) and \( v \) denote horizontal and vertical, respectively, \( vh \) denotes vertical from horizontal data processing, and \( i \) and \( j \) are fixed integer constants such that \( j \gg i \) and \( n_{y} i \gg \max\{n_{h}, n_{v}\} \).

Likewise, we define the horizontal past and future innovations and output data matrices for \( k = 0, 1, \ldots, M \) and \( N = 2i + j - 2 \) as follows:

\[
E_p^h(k) \triangleq \begin{bmatrix}
e_{0,k} & e_{1,k} & e_{2,k} & \cdots & e_{j-1,k} \\
e_{1,k} & e_{2,k} & e_{3,k} & \cdots & e_{j,k} \\
e_{2,k} & e_{3,k} & e_{4,k} & \cdots & e_{j+1,k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
e_{i-1,k} & e_{i,k} & e_{i+1,k} & \cdots & e_{i+j-2,k}
\end{bmatrix} \in \mathbb{R}^{n_y \times i \times j} \quad (23)
\]

\[
E_f^h(k) \triangleq \begin{bmatrix}
e_{i,k} & e_{i+1,k} & e_{i+2,k} & \cdots & e_{i+j-1,k} \\
e_{i+1,k} & e_{i+2,k} & e_{i+3,k} & \cdots & e_{i+j,k} \\
e_{i+2,k} & e_{i+3,k} & e_{i+4,k} & \cdots & e_{i+j+1,k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
e_{2i-1,k} & e_{2i,k} & e_{2i+1,k} & \cdots & e_{2i+j-2,k}
\end{bmatrix} \in \mathbb{R}^{n_y \times i \times j} \quad (24)
\]

\[
Y_p^h(k) \triangleq \begin{bmatrix}
y_{0,k} & y_{1,k} & y_{2,k} & \cdots & y_{j-1,k} \\
y_{1,k} & y_{2,k} & y_{3,k} & \cdots & y_{j,k} \\
y_{2,k} & y_{3,k} & y_{4,k} & \cdots & y_{j+1,k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
y_{i-1,k} & y_{i,k} & y_{i+1,k} & \cdots & y_{i+j-2,k}
\end{bmatrix} \in \mathbb{R}^{n_y \times i \times j} \quad (25)
\]

\[
Y_f^h(k) \triangleq \begin{bmatrix}
y_{i,k} & y_{i+1,k} & y_{i+2,k} & \cdots & y_{i+j-1,k} \\
y_{i+1,k} & y_{i+2,k} & y_{i+3,k} & \cdots & y_{i+j,k} \\
y_{i+2,k} & y_{i+3,k} & y_{i+4,k} & \cdots & y_{i+j+1,k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
y_{2i-1,k} & y_{2i,k} & y_{2i+1,k} & \cdots & y_{2i+j-2,k}
\end{bmatrix} \in \mathbb{R}^{n_y \times i \times j} \quad (26)
\]

One can easily show that the following equations are satisfied for \( k = 0, 1, \ldots, M \)

\[
Y_p^h(k) = \Gamma_i^h \hat{X}_p^h(k) + \Gamma_i^{vh} \hat{X}_p^{vh}(k) + K_i^h E_p^h(k) \quad (27)
\]

\[
Y_f^h(k) = \Gamma_i^h \hat{X}_f^h(k) + \Gamma_i^{vh} \hat{X}_f^{vh}(k) + K_i^h E_f^h(k) \quad (28)
\]

\[
\hat{X}_f^h(k) = A_i^h \hat{X}_p^h(k) + \Phi_i^{vh} \hat{X}_p^{vh}(k) + L_i^h E_p^h(k), \quad (29)
\]

where \( \{\Gamma_i^h, \Phi_i^{vh}, L_i^h\} \) and other related matrices are defined as follows:

\[
\Gamma_i^h \triangleq \begin{bmatrix}
C_1 \\
C_1 A_1 \\
\vdots \\
C_1 A_1^{i-1}
\end{bmatrix} \in \mathbb{R}^{n_y \times i \times n_h} \quad (30)
\]

\[
L_i^h \triangleq [A_1^{i-1} K_1 | A_1^{i-2} K_1 | \cdots | K_1] \in \mathbb{R}^{n_h \times n_y i} \quad (31)
\]

\[
\Phi_i^h \triangleq [A_1^{i-1} | A_1^{i-2} | \cdots | I_{n_h}] \in \mathbb{R}^{n_h \times n_y i} \quad (32)
\]
\[
\Phi_{i}^{vh} = \begin{bmatrix}
A_{i}^{1}A_{2} & A_{i}^{2}A_{2} & \cdots & A_{2}
\end{bmatrix} \in \mathbb{R}^{n_{h} \times n_{v}} = \Phi_{i}^{h} \cdot (I_{i} \otimes A_{2})
\]
\[\Theta_{i}^{h} \triangleq \begin{bmatrix}
I_{n_{h}} \\
A_{1} \\
\vdots \\
A_{i}^{-1}
\end{bmatrix} \in \mathbb{R}^{n_{h} \times n_{h}}, \]

and \(I_{k}\) denotes a \((k \times k)\) identity matrix. Finally, we define the lower triangular block Toeplitz matrices \(\{G_{A_{1}}^{h}, \Gamma_{i}^{vh}, K_{i}^{h}\}\) as

\[
G_{A_{1}}^{h} \triangleq \begin{bmatrix}
0_{n_{y} \times n_{v}} \\
I_{n_{h}}_{n_{y} \times n_{v}} \\
\vdots \\
A_{1}^{-2}A_{1}^{-3} & \cdots & 0_{n_{y} \times n_{v}}
\end{bmatrix} \in \mathbb{R}^{n_{h} \times n_{h} n_{y} \times n_{v}}
\]
\[
\Gamma_{i}^{vh} \triangleq \begin{bmatrix}
C_{2} \\
C_{1}A_{2} & C_{2} \\
\vdots \\
C_{1}A_{i}^{-2}A_{2} & C_{1}A_{i}^{-3}A_{2} & \cdots & C_{2}
\end{bmatrix} \in \mathbb{R}^{n_{h} \times n_{h} n_{y} \times n_{v}}
\]
\[
K_{i}^{h} \triangleq \begin{bmatrix}
I_{n_{y}} \\
C_{1}K_{1} & I_{n_{y}} \\
\vdots \\
C_{1}A_{i}^{-2}K_{1} & C_{1}A_{i}^{-3}K_{1} & \cdots & I_{n_{y}}
\end{bmatrix} \in \mathbb{R}^{n_{y} \times n_{y} n_{h} \times n_{h} n_{y} \times n_{v}}
\]

For the purpose of horizontal data processing we will work with the equivalent horizontal subsystem

\[
\hat{x}_{r+1,s}^{h} = A_{1}\hat{x}_{r,s}^{h} + A_{2}\hat{x}_{r,s}^{v} + K_{1}e_{r,s}
\]
\[
y_{r,s} = C_{1}\hat{x}_{r,s}^{h} + C_{2}\hat{x}_{r,s}^{v} + e_{r,s},
\]

for \(r = 0, 1, \ldots, N\) and \(s = 0, 1, \ldots, M\). However, at this point we need to make the following notational simplification \(\tilde{n}_{h} \triangleq n_{h}(M + 1), \tilde{n}_{y} \triangleq n_{y}(M + 1),\) and \(\tilde{j} \triangleq j(M + 1)\). Then, by defining

\[
Y_{p}^{h} \triangleq \begin{bmatrix}
Y_{p}^{h}(0) & Y_{p}^{h}(1) & \cdots & Y_{p}^{h}(M)
\end{bmatrix} \in \mathbb{R}^{n_{y} \times \tilde{j}}
\]
\[
Y_{f}^{h} \triangleq \begin{bmatrix}
Y_{f}^{h}(0) & Y_{f}^{h}(1) & \cdots & Y_{f}^{h}(M)
\end{bmatrix} \in \mathbb{R}^{n_{y} \times \tilde{j}}
\]
\[
\hat{X}_{p}^{h} \triangleq \begin{bmatrix}
\hat{X}_{p}^{h}(0) & \hat{X}_{p}^{h}(1) & \cdots & \hat{X}_{p}^{h}(M)
\end{bmatrix} \in \mathbb{R}^{n_{h} \times \tilde{j}}
\]
\[
\hat{X}_{f}^{h} \triangleq \begin{bmatrix}
\hat{X}_{f}^{h}(0) & \hat{X}_{f}^{h}(1) & \cdots & \hat{X}_{f}^{h}(M)
\end{bmatrix} \in \mathbb{R}^{n_{h} \times \tilde{j}}
\]
\[
\hat{X}_{p}^{vh} \triangleq \begin{bmatrix}
\hat{X}_{p}^{vh}(0) & \hat{X}_{p}^{vh}(1) & \cdots & \hat{X}_{p}^{vh}(M)
\end{bmatrix} \in \mathbb{R}^{n_{v} \times \tilde{j}}
\]
we get the horizontal subspace equations

$$\begin{align*}
\hat{X}_f^{vh} & \triangleq \left[ \begin{array}{c}
\hat{X}_f^{vh}(0) \\
\hat{X}_f^{vh}(1) \\
\vdots \\
\hat{X}_f^{vh}(M)
\end{array} \right] \in \mathbb{R}^{n_v \times j} \\
E_p^h & \triangleq \left[ \begin{array}{c}
E_p^h(0) \\
E_p^h(1) \\
\vdots \\
E_p^h(M)
\end{array} \right] \in \mathbb{R}^{n_p \times j} \\
E_f^h & \triangleq \left[ \begin{array}{c}
E_f^h(0) \\
E_f^h(1) \\
\vdots \\
E_f^h(M)
\end{array} \right] \in \mathbb{R}^{n_p \times j},
\end{align*}$$

(45)  
(46)  
(47)

2.1 Propagating the Vertical Hankel State Matrices

We will now propagate the state equation (14) backward until we reach the initial vertical states. By assuming zero initial vertical states, then the remaining vertical states are a function of the innovations and horizontal states only. Since (21) and (22) are Hankel matrices, we need to convert (14) into a pair of past and future Hankel type matrix equations. This is rather straightforward since (21) and (22) have partial horizontal dynamics (i.e., only through $X_{v}^{h}(k)$ and $\hat{X}_{f}^{vh}(k)$). Thus, by substituting $\hat{x}_{r,s}^{h}$, $\hat{x}_{r,s}^{v}$, and $e_{r,s}$ in (14) by their matrix equivalents, \{$\hat{X}_{p}^{h}(k), \hat{X}_{p}^{vh}(k), E_{p}^{h}(k)$\} and \{$\hat{X}_{f}^{h}(k), \hat{X}_{f}^{vh}(k), E_{f}^{h}(k)$\}, we obtain, respectively, the past and future vertical state equations given by

$$\begin{align*}
\hat{X}_p^{vh}(k+1) & = \Theta_i^{vh} \hat{X}_p^{h}(k) + A_i^{vh} \hat{X}_p^{vh}(k) + K_i^{vh} E_p^{h}(k) \\
\hat{X}_f^{vh}(k+1) & = \Theta_i^{vh} \hat{X}_f^{h}(k) + A_i^{vh} \hat{X}_f^{vh}(k) + K_i^{vh} E_f^{h}(k),
\end{align*}$$

(51)  
(52)

where

$$\Theta_i^{vh} \triangleq \left[ \begin{array}{ccc}
A_3 & & \\
A_3 A_1 & & \\
\vdots & & \\
A_3 A_1^{i-1} & & 
\end{array} \right] = (I_i \otimes A_3) \cdot \Theta_i^{h} \in \mathbb{R}^{n_v \times n_h}$$

(53)

$$\begin{align*}
A_i^{vh} & \triangleq \left[ \begin{array}{cccc}
A_4 & & & \\
A_3 A_2 & & & A_4 \\
\vdots & & & \vdots \\
A_3 A_1^{i-2} A_2 & A_3 A_1^{i-3} A_2 & \cdots & A_4
\end{array} \right] \in \mathbb{R}^{n_v \times n_v}
\end{align*}$$

(54)

$$\begin{align*}
K_i^{vh} & \triangleq \left[ \begin{array}{cccc}
K_2 & & & \\
A_3 K_1 & & & K_2 \\
\vdots & & & \vdots \\
A_3 A_1^{i-2} K_1 & A_3 A_1^{i-3} K_1 & \cdots & K_2
\end{array} \right] \in \mathbb{R}^{n_v \times n_y}
\end{align*}$$

(55)
Let us now solve (51) and (52) recursively for \( k = 0, 1, \ldots, M \) as follows:

\[
\begin{align*}
\hat{X}_p^v(0) & = \hat{X}_p^v(0) \\
\hat{X}_f^v(0) & = \hat{X}_f^v(0) \\
\hat{X}_p^v(1) & = \Theta_i^v \hat{X}_p^h(0) + A_i^v \hat{X}_p^v(0) + K_i^v E_p^h(0) \\
\hat{X}_f^v(1) & = \Theta_i^v \hat{X}_f^h(0) + A_i^v \hat{X}_f^v(0) + K_i^v E_f^h(0) \\
\hat{X}_p^v(2) & = \Theta_i^v \hat{X}_p^h(1) + A_i^v \Theta_i^v \hat{X}_p^v(0) + (A_i^v)^2 \hat{X}_p^v(0) + K_i^v E_p^h(1) + A_i^v K_i^v E_p^h(0) \\
\hat{X}_f^v(2) & = \Theta_i^v \hat{X}_f^h(1) + A_i^v \Theta_i^v \hat{X}_f^v(0) + (A_i^v)^2 \hat{X}_f^v(0) + K_i^v E_f^h(1) + A_i^v K_i^v E_f^h(0) \\
& \hspace{1cm} \vdots \\
\hat{X}_p^v(M) & = \Theta_i^v \hat{X}_p^h(M) + \sum_{k=0}^{M-1} (A_i^v)^{M-k-1} \Theta_i^v \hat{X}_p^h(k) + \sum_{k=0}^{M-1} (A_i^v)^{M-k-1} K_i^v E_p^h(k) \\
\hat{X}_f^v(M) & = \Theta_i^v \hat{X}_f^h(M) + \sum_{k=0}^{M-1} (A_i^v)^{M-k-1} \Theta_i^v \hat{X}_f^h(k) + \sum_{k=0}^{M-1} (A_i^v)^{M-k-1} K_i^v E_f^h(k).
\end{align*}
\]

Now we use \( \{ E_p^h(k), E_f^h(k), \hat{X}_p^h(k), \hat{X}_f^h(k) \} \) for \( k = 0, 1, \ldots, M \) to construct upper triangular block Toeplitz matrices such as

\[
\begin{align*}
E_p^* & \triangleq \begin{bmatrix} 
E_p^h(0) & E_p^h(1) & \cdots & E_p^h(M) \\
E_p^h(0) & & \ddots & \vdots \\
& & \ddots & E_p^h(0) \\
& & \cdots & E_p^h(0)
\end{bmatrix} \in \mathbb{R}^{n_y \times j} \\
E_f^* & \triangleq \begin{bmatrix} 
E_f^h(0) & E_f^h(1) & \cdots & E_f^h(M) \\
E_f^h(0) & & \ddots & \vdots \\
& & \ddots & E_f^h(0) \\
& & \cdots & E_f^h(0)
\end{bmatrix} \in \mathbb{R}^{n_y \times j} \\
\hat{X}_p^* & \triangleq \begin{bmatrix} 
\hat{X}_p^h(0) & \hat{X}_p^h(1) & \cdots & \hat{X}_p^h(M) \\
\hat{X}_p^h(0) & & \ddots & \vdots \\
& & \ddots & \hat{X}_p^h(0) \\
& & \cdots & \hat{X}_p^h(0)
\end{bmatrix} \in \mathbb{R}^{\tilde{n}_h \times j} \\
\hat{X}_f^* & \triangleq \begin{bmatrix} 
\hat{X}_f^h(0) & \hat{X}_f^h(1) & \cdots & \hat{X}_f^h(M) \\
\hat{X}_f^h(0) & & \ddots & \vdots \\
& & \ddots & \hat{X}_f^h(0) \\
& & \cdots & \hat{X}_f^h(0)
\end{bmatrix} \in \mathbb{R}^{\tilde{n}_h \times j}.
\end{align*}
\]

Notice that \((58) - (59)\) contain block Hankel entries, thus are block Toeplitz with Hankel blocks (BTHB). Finally, we define the controllability-like matrices

\[
\begin{align*}
A_M^v & \triangleq \begin{bmatrix} 
\Theta_i^v & A_i^v \Theta_i^v & \cdots & (A_i^v)^{M-1} \Theta_i^v \\
K_i^v & A_i^v K_i^v & \cdots & (A_i^v)^{M-1} K_i^v
\end{bmatrix} \in \mathbb{R}^{n_y \times n_h M} \\
K_M^v & \triangleq \begin{bmatrix} 
K_i^v & A_i^v K_i^v & \cdots & (A_i^v)^{M-1} K_i^v \\
K_i^v & A_i^v K_i^v & \cdots & (A_i^v)^{M-1} K_i^v
\end{bmatrix} \in \mathbb{R}^{n_y \times n_y i M}
\end{align*}
\]
\[
\begin{align*}
\Delta \hat{X}_p^{vh}(0) & \triangleq \left[ \begin{array}{c}
\hat{X}_p^{vh}(0) \\
A_i^v \hat{X}_p^{vh}(0) \\
\vdots \\
(A_i^v)^M \hat{X}_p^{vh}(0)
\end{array} \right] \in \mathbb{R}^{n_x \times j} \\
\Delta \hat{X}_f^{vh}(0) & \triangleq \left[ \begin{array}{c}
\hat{X}_f^{vh}(0) \\
A_i^v \hat{X}_f^{vh}(0) \\
\vdots \\
(A_i^v)^M \hat{X}_f^{vh}(0)
\end{array} \right] \in \mathbb{R}^{n_x \times j}.
\end{align*}
\]

It can now be easily shown that the vertical states satisfy a pair of Hankel matrix equations such as
\[
\begin{align*}
\hat{X}_p^{vh} &= \Delta \hat{X}_p^{vh}(0) + \left[ 0_{n_v \times n_h} \right] \hat{X}_p^{v} + \left[ K_M^{vh} \right] E_p^* \\
\hat{X}_f^{vh} &= \Delta \hat{X}_f^{vh}(0) + \left[ 0_{n_v \times n_h} \right] \hat{X}_f^{v} + \left[ K_M^{vh} \right] E_f^*.
\end{align*}
\]

If we now assume that \( \hat{X}_p^{vh}(0) = \hat{X}_f^{vh}(0) = 0_{n_v \times j} \), then \( \Delta \hat{X}_p^{vh}(0) = 0_{n_v \times j} \) and \( \Delta \hat{X}_f^{vh}(0) = 0_{n_v \times j} \). We then obtain the final expressions for \( \hat{X}_p^{vh} \) and \( \hat{X}_f^{vh} \) as
\[
\begin{align*}
\hat{X}_p^{vh} &= \left[ 0_{n_v \times n_h} \right] \hat{X}_p^{v} + \left[ K_M^{vh} \right] E_p^* \\
\hat{X}_f^{vh} &= \left[ 0_{n_v \times n_h} \right] \hat{X}_f^{v} + \left[ K_M^{vh} \right] E_f^*.
\end{align*}
\]

2.2 Computing the Orthogonal Projection \( Y_f^{h}/Y_p^{h} \)

Since the vertical state matrices are now functions of horizontal states and innovations, we can substitute these in the horizontal state equation \((50)\), to get
\[
\hat{X}_f^{*} = A_M^h \hat{X}_p^{*} + K_M^h E_p^*.
\]

where
\[
A_M^h \triangleq \left[ \begin{array}{cccc}
A_1^i \\
\Phi_i^{vh} \Theta_i^{vh} \\
\Phi_i^{vh} A_i^v \Theta_i^{vh} \\
\vdots \\
\Phi_i^{vh} (A_i^v)^{M-1} \Theta_i^{vh} \\
A_1^i \\
\Phi_i^{vh} (A_i^v)^2 \Theta_i^{vh} \\
\vdots \\
\Phi_i^{vh} (A_i^v)^{M-2} \Theta_i^{vh} \\
A_1^i \\
\Phi_i^{vh} (A_i^v)^3 \Theta_i^{vh} \\
\vdots \\
A_1^i \\
\Phi_i^{vh} (A_i^v)^{M-3} \Theta_i^{vh}
\end{array} \right] \in \mathbb{R}^{n_h \times n_h}.
\]

\[
K_M^h \triangleq \left[ \begin{array}{cccc}
\mathcal{L}_i^h \\
\Phi_i^{vh} K_i^{vh} \\
\Phi_i^{vh} A_i^v K_i^{vh} \\
\vdots \\
\Phi_i^{vh} (A_i^v)^{M-1} K_i^{vh} \\
\mathcal{L}_i^h \\
\Phi_i^{vh} K_i^{vh} \\
\Phi_i^{vh} A_i^v K_i^{vh} \\
\vdots \\
\Phi_i^{vh} (A_i^v)^{M-2} K_i^{vh} \\
\mathcal{L}_i^h \\
\Phi_i^{vh} K_i^{vh} \\
\vdots \\
\mathcal{L}_i^h \\
\Phi_i^{vh} (A_i^v)^{M-3} K_i^{vh}
\end{array} \right] \in \mathbb{R}^{n_h \times n_y i}.
\]

Let us now re-visit \((68) - (69)\) and further substitute \((70)\) in \((69)\), i.e.,
\[
\begin{align*}
\hat{X}_p^{vh} &= Q_1 \hat{X}_p^{*} + Q_2 E_p^* \\
\hat{X}_f^{vh} &= Q_1 \hat{X}_f^{*} + Q_2 E_f^* \\
&= Q_1 (A_M^h \hat{X}_p^{*} + K_M^h E_p^*) + Q_2 E_f^* \\
&= P_1 \hat{X}_p^{*} + P_2 E_p^* + Q_2 E_f^*.
\end{align*}
\]
where

\[
Q_1 = \begin{bmatrix} 0_{n_v \times n_h} & A_{M}^{v_h} \end{bmatrix} \in \mathbb{R}^{n_v \times n_h M}
\]

(75)

\[
Q_2 = \begin{bmatrix} 0_{n_v \times n_g i} & K_{M}^{v_h} \end{bmatrix} \in \mathbb{R}^{n_v \times n_g i M}
\]

(76)

\[
P_1 = \begin{bmatrix} 0_{n_v \times n_h} & A_{M}^{v_h} \end{bmatrix} \cdot A_{M}^{h} \in \mathbb{R}^{n_v \times n_h}
\]

(77)

\[
P_2 = \begin{bmatrix} 0_{n_v \times n_h} & A_{M}^{v_h} \end{bmatrix} \cdot K_{M}^{h} \in \mathbb{R}^{n_v \times n_g i}
\]

(78)

We now compute the orthogonal projection \( Y^h_f / Y^h_p \) as

\[
Y^h_f / Y^h_p = \Gamma_i X_f^h / Y^h_p + \Gamma_i v^{vh} X_f^h / Y^h_p + K_i E^h_f / Y^h_p
\]

\[
= \frac{1}{j} \left( \Gamma_i X_f^h (Y^h_p)^{\top} + \Gamma_i v^{vh} X_f^h (Y^h_p)^{\top} + K_i E^h_f (Y^h_p)^{\top} \right) (R_{pp}^h)^{-1} Y^h_p,
\]

(79)

where \( R_{pp}^h = \frac{1}{j} Y^h_p (Y^h_p)^{\top} \). Furthermore, we now substitute \( (Y^h_p)^{\top} \) to get

\[
Y^h_f / Y^h_p = \frac{1}{j} \Gamma_i X_f^h \left[ \left( \Gamma_i^h \right)^{\top} + \left( \Gamma_i^{vh} \right)^{\top} \right] (R_{pp}^h)^{-1} Y^h_p
\]

\[
= \frac{1}{j} \Gamma_i X_f^h \left[ \left( \Gamma_i^h \right)^{\top} + \left( \Gamma_i^{vh} \right)^{\top} \right] + \frac{1}{j} K_i E^h_f \left[ \left( \Gamma_i^h \right)^{\top} + \left( \Gamma_i^{vh} \right)^{\top} \right] (R_{pp}^h)^{-1} Y^h_p
\]

Let us now look at each term individually. We start with

\[
\frac{1}{j} \Gamma_i X_f^h \left( \Gamma_i^h \right)^{\top} = \frac{1}{j} \Gamma_i \left( A_i X_p^h + \Phi_i v^{vh} X_p^h \right) \left( X_p^h \right)^{\top} \left( \Gamma_i^h \right)^{\top}
\]

\[
= \frac{1}{j} \Gamma_i A_i X_p^h \left( X_p^h \right)^{\top} \left( \Gamma_i^h \right)^{\top} + \frac{1}{j} \Gamma_i \Phi_i v^{vh} \left( Q_1 \chi_p^h + Q_2 \chi_p^h \right) \left( X_p^h \right)^{\top} \left( \Gamma_i^h \right)^{\top}
\]

\[
+ \frac{1}{j} \Gamma_i \left( \chi_p^h \right)^{\top} \left( \Gamma_i^h \right)^{\top}
\]

\[
= \Gamma_i A_i P_h \left( \Gamma_i^h \right)^{\top} + \Gamma_i \Phi_i v^{vh} \left[ 0_{n_v \times n_h} \right] A_{M}^{v_h} \left[ \begin{array}{c} P_h \\ 0_{n_h \times n_h} \\ \vdots \\ 0_{n_h \times n_h} \end{array} \right] \left( \Gamma_i^h \right)^{\top} + 0_{n_g i \times n_g i}
\]

\[
= \Gamma_i A_i P_h \left( \Gamma_i^h \right)^{\top}.
\]
We continue with \( \frac{1}{j \Gamma^h_i} \hat{X}^h_f \left( \hat{X}^v_h \right)^T \left( \Gamma^{vh}_i \right)^T \),

\[
\frac{1}{j \Gamma^h_i} \hat{X}^h_f \left( \hat{X}^v_h \right)^T \left( \Gamma^{vh}_i \right)^T \Gamma^{vh}_i
= \frac{1}{j \Gamma^h_i} \left( A^h_i \hat{X}^h_p + \Phi^{vh}_i \hat{X}^v_h + \mathcal{L}^h E^h_p \right) \left( \hat{X}^v_h \right)^T \left( \Gamma^{vh}_i \right)^T
= \frac{1}{j \Gamma^h_i} A^h_i \hat{X}^h_p \left( \left( \hat{X}^v_h \right)^T Q^T_1 + \left( E^*_p \right)^T Q^T_2 \right) \left( \Gamma^{vh}_i \right)^T
+ \frac{1}{j \Gamma^h_i} \Phi^{vh}_i \hat{X}^v_h \left( \hat{X}^v_h \right)^T \left( \Gamma^{vh}_i \right)^T
+ \frac{1}{j \Gamma^h_i} \mathcal{L}^h E^h_p \left( \left( \hat{X}^v_h \right)^T Q^T_1 + \left( E^*_p \right)^T Q^T_2 \right) \left( \Gamma^{vh}_i \right)^T.
\]

Thus we get,

\[
\frac{1}{j \Gamma^h_i} \hat{X}^h_f \left( \hat{X}^v_h \right)^T \left( \Gamma^{vh}_i \right)^T = \Gamma^h_i A^i \left[ \begin{array}{c} P_h \mid 0_{n_k \times n_h} \cdots \mid 0_{n_k \times n_h} \end{array} \right] \left[ \begin{array}{c} 0_{n_h \times n_o i} \end{array} \right] \left( A^{vh} \right)^T \Gamma^{vh}_i
+ \Gamma^h_i \Phi^{vh}_i \left( I_i \otimes P_v \right) \left( \Gamma^{vh}_i \right)^T + 0_{n_i \times n_y_i}
+ \Gamma^h_i \left[ \begin{array}{c} R_e \mid 0_{n_y \times n_y} \cdots \mid 0_{n_y \times n_y} \end{array} \right] \left[ \begin{array}{c} 0_{n_y \times n_i} \end{array} \right] \left( K^{vh} \right)^T
= \Gamma^h_i \Phi^{vh}_i \left( I_i \otimes P_v \right) \left( \Gamma^{vh}_i \right)^T.
\]

Let us continue with the next term \( \frac{1}{j \Gamma^h_i} \hat{X}^h_f \left( E^h_p \right)^T \left( K^h_i \right)^T \)

\[
\frac{1}{j \Gamma^h_i} \hat{X}^h_f \left( E^h_p \right)^T \left( K^h_i \right)^T
= \frac{1}{j \Gamma^h_i} \left( A^h_i \hat{X}^h_p + \Phi^{vh}_i \hat{X}^v_h + \mathcal{L}^h E^h_p \right) \left( E^h_p \right)^T \left( K^h_i \right)^T
= \frac{1}{j \Gamma^h_i} A^h_i \hat{X}^h_p \left( E^h_p \right)^T \left( K^h_i \right)^T
+ \frac{1}{j \Gamma^h_i} \Phi^{vh}_i \left( Q_1 \hat{X}^v_p + Q_2 E^*_p \right) \left( E^h_p \right)^T \left( K^h_i \right)^T
+ \frac{1}{j \Gamma^h_i} \mathcal{L}^h E^h_p \left( E^h_p \right)^T \left( K^h_i \right)^T
= 0_{n_i \times n_y} + 0_{n_i \times n_y} + \Gamma^h_i \Phi^{vh}_i \left[ \begin{array}{c} 0_{n_i \times n_y} \mid K^{vh} \end{array} \right] \left[ \begin{array}{c} R_e \mid 0_{n_y \times n_y} \mid 0_{n_y \times n_y} \end{array} \right] \left( K^h_i \right)^T
= \Gamma^h_i \left( I_i \otimes R_e \right) \left( K^h_i \right)^T.
\]

11
We then continue with the term \( \frac{1}{j} \Gamma_{i}^{v_{h}} \hat{X}_{f}^{v_{h}} (Y_{p}^{h})^{\top} \), i.e.,

\[
\frac{1}{j} \Gamma_{i}^{v_{h}} \hat{X}_{f}^{v_{h}} (Y_{p}^{h})^{\top} = \frac{1}{j} \Gamma_{i}^{v_{h}} \hat{X}_{f}^{v_{h}} \left( \left( \hat{X}_{p}^{h} \right)^{\top} (\Gamma_{i}^{h})^{\top} + \left( \hat{X}_{p}^{h} \right)^{\top} \left( \Gamma_{i}^{h} \right)^{\top} + (E_{p}^{h})^{\top} (K_{i}^{h})^{\top} \right)
\]

\[
= \frac{1}{j} \Gamma_{i}^{v_{h}} \hat{X}_{f}^{v_{h}} \left( \hat{X}_{p}^{h} \right)^{\top} (\Gamma_{i}^{h})^{\top} + \frac{1}{j} \Gamma_{i}^{v_{h}} \hat{X}_{f}^{v_{h}} \left( \hat{X}_{p}^{h} \right)^{\top} (\Gamma_{i}^{h})^{\top} + \frac{1}{j} \Gamma_{i}^{v_{h}} \hat{X}_{f}^{v_{h}} (E_{p}^{h})^{\top} (K_{i}^{h})^{\top}
\]

\[
= \frac{1}{j} \Gamma_{i}^{v_{h}} \left( P_{1} \hat{X}_{f}^{h} + P_{2} E_{p}^{h} + Q_{2} E_{f}^{h} \right) \left( \hat{X}_{p}^{h} \right)^{\top} \left( \Gamma_{i}^{h} \right)^{\top} + \frac{1}{j} \Gamma_{i}^{v_{h}} \hat{X}_{f}^{v_{h}} \left( \hat{X}_{p}^{h} \right)^{\top} (\Gamma_{i}^{h})^{\top}
\]

\[
+ \frac{1}{j} \Gamma_{i}^{v_{h}} \left( P_{1} \hat{X}_{f}^{h} + P_{2} E_{p}^{h} + Q_{2} E_{f}^{h} \right) \left( E_{p}^{h} \right)^{\top} (K_{i}^{h})^{\top}
\]

\[
= \Gamma_{i}^{v_{h}} \left[ \begin{array}{c}
0_{n_{i} \times n_{h}} \mid A_{M}^{v_{h}} \end{array} \right] \cdot A_{M}^{h} \left[ \begin{array}{c}
P_{h} \\
\vdots \\
0_{n_{h} \times n_{h}} \end{array} \right] (\Gamma_{i}^{h})^{\top} + 0_{n_{i} \times n_{y_{i}}} + 0_{n_{i} \times n_{y_{i}}}
\]

\[
+ \frac{1}{j} \Gamma_{i}^{v_{h}} \hat{X}_{f}^{v_{h}} \left( \hat{X}_{p}^{h} \right)^{\top} (\Gamma_{i}^{h})^{\top} + 0_{n_{i} \times n_{y_{i}}}
\]

\[
+ \Gamma_{i}^{v_{h}} \left[ \begin{array}{c}
0_{n_{i} \times n_{h}} \mid A_{M}^{h} \end{array} \right] \cdot K_{M}^{h} \left[ \begin{array}{c}
R_{e} \\
\vdots \\
0_{n_{y} \times n_{y}} \end{array} \right] (K_{i}^{h})^{\top} + 0_{n_{i} \times n_{y_{i}}}
\]

\[
= \frac{1}{j} \Gamma_{i}^{v_{h}} \hat{X}_{f}^{v_{h}} \left( \hat{X}_{p}^{h} \right)^{\top} (\Gamma_{i}^{h})^{\top}.
\]

Finally, the last term \( \frac{1}{j} K_{i}^{h} E_{p}^{h} (Y_{p}^{h})^{\top} \) is zero since the future innovations are uncorrelated with the past data.

Now collecting all terms, we obtain

\[
Y_{f}^{h} / Y_{p}^{h} = \Gamma_{i}^{h} \left( A_{i}^{h} P_{h} (\Gamma_{i}^{h})^{\top} + \Phi_{i}^{v_{h}} (I_{i} \otimes P_{v}) (\Gamma_{i}^{h})^{\top} + L_{i}^{h} (I_{i} \otimes R_{e}) (K_{i}^{h})^{\top} \right) (R_{pp}^{h})^{-1} Y_{p}^{h}
\]

\[
+ \frac{1}{j} \Gamma_{i}^{v_{h}} \hat{X}_{f}^{v_{h}} \left( \hat{X}_{p}^{h} \right)^{\top} (\Gamma_{i}^{h})^{\top} (R_{pp}^{h})^{-1} Y_{p}^{h}.
\]

One can show that

\[
\Delta_{i}^{h} = A_{i}^{h} P_{h} (\Gamma_{i}^{h})^{\top} + \Phi_{i}^{v_{h}} (I_{i} \otimes P_{v}) (\Gamma_{i}^{h})^{\top} + L_{i}^{h} (I_{i} \otimes R_{e}) (K_{i}^{h})^{\top}
\]

\[
= \left[ \begin{array}{c}
A_{i}^{h} G_{1} \\
A_{i}^{h} G_{1} \\
\vdots \\
G_{1}
\end{array} \right].
\]

Therefore, we have

\[
Y_{f}^{h} / Y_{p}^{h} = \Gamma_{i}^{h} \cdot \Delta_{i}^{h} (R_{pp}^{h})^{-1} Y_{p}^{h} + \frac{1}{j} \Gamma_{i}^{v_{h}} \hat{X}_{f}^{v_{h}} \left( \hat{X}_{p}^{h} \right)^{\top} (\Gamma_{i}^{h})^{\top} (R_{pp}^{h})^{-1} Y_{p}^{h}.
\]

We define the bias term as

\[
bias = \frac{1}{j} \Gamma_{i}^{v_{h}} \hat{X}_{f}^{v_{h}} \left( \hat{X}_{p}^{h} \right)^{\top} (\Gamma_{i}^{h})^{\top} (R_{pp}^{h})^{-1} Y_{p}^{h}.
\]
We now need to find a closed form expression for $\frac{1}{J} \hat{X}_f^{vh} \left( \hat{X}_p^{vh} \right)^T$. For this, we will use equations (29) and (51) – (52), along the following state estimate covariance equations

\[
\begin{align*}
P_h &= A_1 P_h A_1^T + A_2 P_v A_2^T + K_1 R_e K_1^T \\
P_{hv} &= A_1 P_h A_3^T + A_2 P_v A_4^T + K_1 R_e K_2^T \\
P_{vh} &= A_3 P_h A_1^T + A_2 P_v A_2^T + K_2 R_v K_1^T \\
P_v &= A_3 P_h A_3^T + A_4 P_v A_4^T + K_2 R_v K_2^T.
\end{align*}
\]

One can easily prove the following results.

\[
\begin{align*}
P_0 &= \Theta_i^{vh} \cdot \left[ A_1^T P_h \left( \Theta_i^{vh} \right)^T + \Phi^{vh} \left( I_i \otimes P_v \right) \left( A_i^{vh} \right)^T + \mathcal{L}_i^h \left( I_i \otimes R_e \right) \left( K_i^{vh} \right)^T \right] \\
&= \Theta_i^{vh} \Phi_i^{vh} \left( I_i \otimes P_{hv} \right) \\
\mathcal{Q} &= \Theta_i^{vh} P_h \left( \Theta_i^{vh} \right)^T + A_i^{vh} \left( I_i \otimes P_v \right) \left( A_i^{vh} \right)^T + K_i^{vh} \left( I_i \otimes R_v \right) \left( K_i^{vh} \right)^T \\
&= (I_i \otimes P_v) + (I_i \otimes A_3) G_{A_3} \left( I_i \otimes P_{hv} \right) + (I_i \otimes P_{vh}) G_{A_3}^T \left( I_i \otimes A_3^T \right).
\end{align*}
\]

Now, since $\frac{1}{J} \hat{X}_f^{vh} \left( \hat{X}_p^{vh} \right)^T$ can be represented as

\[
\frac{1}{J} \hat{X}_f^{vh} \left( \hat{X}_p^{vh} \right)^T = \frac{1}{J} \sum_{k=0}^{M} \hat{X}_f^{vh}(k) \left( \hat{X}_p^{vh}(k) \right)^T
\]

Let us consider each product term for $k = 0, 1, \ldots, M$. Starting with $k = 0$, we have that

\[
\frac{1}{J} \hat{X}_f^{vh}(0) \left( \hat{X}_p^{vh}(0) \right)^T = 0_{n_v \times n_v}.
\]

Continuing with $k = 1$, we have

\[
\begin{align*}
\hat{X}_p^{vh}(1) &= \Theta_i^{vh} \hat{X}_p^{h}(0) + A_i^{vh} \hat{X}_p^{vh}(0) + K_i^{vh} E_p^{h}(0) \\
\hat{X}_f^{vh}(1) &= \Theta_i^{vh} \hat{X}_f^{h}(0) + A_i^{vh} \hat{X}_f^{vh}(0) + K_i^{vh} E_f^{h}(0) \\
&= \Theta_i^{vh} A_i^{h} \hat{X}_p^{h}(0) + \Theta_i^{vh} \Phi_i^{vh} \hat{X}_p^{vh}(0) + \Theta_i^{vh} \mathcal{L}_i^h E_p^{h}(0) + A_i^{vh} \hat{X}_f^{vh}(0) + K_i^{vh} E_f^{h}(0).
\end{align*}
\]

Now computing the covariance $\frac{1}{J} \hat{X}_f^{vh}(1) \left( \hat{X}_p^{vh}(1) \right)^T$, we get

\[
\frac{1}{J} \hat{X}_f^{vh}(1) \left( \hat{X}_p^{vh}(1) \right)^T = \frac{1}{J} \left( \Theta_i^{vh} \hat{X}_f^{h}(0) + A_i^{vh} \hat{X}_f^{vh}(0) + K_i^{vh} E_f^{h}(0) \right) \left( \hat{X}_p^{h}(0) \right)^T \left( \Theta_i^{vh} \right)^T + \left( \hat{X}_p^{vh}(0) \right)^T \left( A_i^{vh} \right)^T + \left( E_p^{h}(0) \right)^T \left( K_i^{vh} \right)^T
\]

\[
= \Theta_i^{vh} \left[ A_i^T P_h \left( \Theta_i^{vh} \right)^T + \Phi_i^{vh} \left( I_i \otimes P_v \right) \left( A_i^{vh} \right)^T + \mathcal{L}_i^h \left( I_i \otimes R_e \right) \left( K_i^{vh} \right)^T \right]
\]

\[
= \Theta_i^{vh} \Phi_i^{vh} \left( I_i \otimes P_{hv} \right)
\]

\[
P_0.
\]
Likewise, for \( k = 2 \), we have

\[
\begin{align*}
\hat{X}_p^{vh}(2) &= \Theta_i^{vh} \hat{X}_p^h(1) + A_i^{vh} \Theta_i^{vh} \hat{X}_p^h(0) + (A_i^{vh})^2 \hat{X}_p^{vh}(0) + K_i^{vh} E_p^h(1) + A_i^{vh} K_i^{vh} E_p^h(0) \\
\hat{X}_f^{vh}(2) &= \Theta_i^{vh} \hat{X}_f^h(1) + A_i^{vh} \Theta_i^{vh} \hat{X}_f^h(0) + (A_i^{vh})^2 \hat{X}_f^{vh}(0) + K_i^{vh} E_f^h(1) + A_i^{vh} K_i^{vh} E_f^h(0)
\end{align*}
\]

Continuing further, for \( k \geq 3 \), we obtain the general expression for \( \frac{1}{j} \Gamma_i^{vh} \hat{X}_f^{vh} \left( \hat{X}_p^{vh} \right)^\top \) as

\[
\begin{align*}
\frac{1}{j} \Gamma_i^{vh} \hat{X}_f^{vh} \left( \hat{X}_p^{vh} \right)^\top \left( \Gamma_i^{vh} \right)^\top &= \sum_{k=0}^{M-\ell} \sum_{\ell=1}^M (M - \ell - k + 1) \Gamma_i^{vh} \left( A_i^{vh} \right)^k \mathcal{P}_0 \left( (A_i^{vh})^k \right)^\top \left( \Gamma_i^{vh} \right)^\top \\
&\quad + \sum_{k=0}^{M-\ell-1} \sum_{\ell=1}^{M-1} (M - \ell - k) \Gamma_i^{vh} \left( A_i^{vh} \right)^\ell - 1 \Theta_i^{vh} \Phi_i^{vh} \left( A_i^{vh} \right)^k \\
&\quad \times \mathcal{Q}_0 \left( \left( (A_i^{vh})^{\ell+k} \right)^\top \left( \Gamma_i^{vh} \right)^\top \right). \quad (80)
\end{align*}
\]

Analyzing (80) one can see that each term in the first sum is a function of \( \mathcal{P}_0 \) and each term in the second sum is a function of \( \mathcal{Q}_0 \), both of which are functions of \( P_{hv} \) and/or \( P_{vh} \), which by (15) are zero matrices. Thus, we conclude that the bias term is zero. Thus,

\[
\text{bias} = \frac{1}{j} \Gamma_i^{vh} \hat{X}_f^{vh} \left( \hat{X}_p^{vh} \right)^\top \left( \Gamma_i^{vh} \right)^\top \left( R_{pp}^h \right)^{-1} Y_p^h = 0_{n_{yi} \times j_h}
\]

and

\[
Y_f^h / Y_p^h = \Gamma_i^h \cdot \Delta_i^h \left( R_{pp}^h \right)^{-1} Y_p^h \\
= \Gamma_i^h \cdot \hat{X}_f^h.
\]

14
2.3 Improving The State Estimates

Since the orthogonal projection is not exact, there is a small bias introduced that may affect the identification of the system parameters. Despite the fact that the bias is rather small, one can iterate the procedure in order to improve the state estimates and eliminate the bias. We now propose an oblique projection approach to improve the state estimates. Along the way we also propose a procedure for computing the initial states. We start by assuming that the vertical states are available from an orthogonal projection in the vertical direction, i.e., $Y_f^{v}/Y_p^{v} \simeq \Gamma_{i}^{v} \cdot \hat{X}_f$ (see Ramos and Mercere (2017a) for details), where $\hat{X}_f \in \mathbb{R}^{n_v \times j}$. Then we assemble the vertical from horizontal data processing Hankel state matrix $\hat{X}_f^{vh} \in \mathbb{R}^{n_v \times i \times j}$.

The second stage of the algorithm starts by defining $W_p^{v}$ as

$$W_p^{v} = \begin{bmatrix} \hat{X}_p^{vh} \\ Y_p^{h} \end{bmatrix} \in \mathbb{R}^{(n_v+n_y) i \times j}. \tag{81}$$

We then compute the RQ decomposition of the past/future data as follows:

$$\begin{bmatrix} \hat{X}_f^{vh} \\ W_p^{h} \\ Y_f^{h} \end{bmatrix} = \begin{bmatrix} R_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R_{33} \end{bmatrix} \begin{bmatrix} Q_{1}^{\top} \\ Q_{2}^{\top} \\ Q_{3}^{\top} \end{bmatrix}, \tag{82}$$

where $R_{11} \in \mathbb{R}^{n_v \times n_v}$, $R_{21} \in \mathbb{R}^{(n_v+n_y) \times (n_v+n_y)}$, $R_{22} \in \mathbb{R}^{(n_v+n_y) \times (n_v+n_y)}$, $R_{31} \in \mathbb{R}^{n_y \times n_v}$, $R_{32} \in \mathbb{R}^{n_y \times (n_v+n_y)}$, $R_{33} \in \mathbb{R}^{n_y \times n_y}$, $Q_{1} \in \mathbb{R}^{i \times n_v}$, $Q_{2} \in \mathbb{R}^{j \times (n_v+n_y)}$, and $Q_{3} \in \mathbb{R}^{j \times n_y}$.

From (82) one can find an expression for $Y_f^{h}$ using the R and Q parameters, along with using $Q_{1}^{\top} = R_{11}^{-1} \hat{X}_f^{vh}$ and $Q_{2}^{\top} = R_{22}^{-1} (W_p^{h} - R_{21} Q_{1}^{\top})$. That is,

$$Y_f^{h} = R_{31} Q_{1}^{\top} + R_{32} Q_{2}^{\top} + R_{33} Q_{3}^{\top} = R_{31} Q_{1}^{\top} + R_{32} R_{22}^{-1} (W_p^{h} - R_{21} Q_{1}^{\top}) + R_{33} Q_{3}^{\top} = R_{32} R_{22}^{-1} W_p^{h} + (R_{31} - R_{32} R_{22}^{-1} R_{21}) R_{11}^{-1} \hat{X}_f^{vh} + R_{33} Q_{3}^{\top} = \Gamma_{i}^{h} \cdot \hat{X}_f^{h} + \Gamma_{i}^{vh} \cdot \hat{X}_f^{vh} + K_{i}^{h} \cdot E_f^{h}.$$

It is now clearly evident that

$$\begin{align*}
\Gamma_{i}^{h} \cdot \hat{X}_f^{h} &= R_{32} R_{22}^{-1} W_p^{h} \\
\Gamma_{i}^{vh} \cdot \hat{X}_f^{vh} &= (R_{31} - R_{32} R_{22}^{-1} R_{21}) R_{11}^{-1} \hat{X}_f^{vh} \\
K_{i}^{h} \cdot E_f^{h} &= R_{33} Q_{3}^{\top}.
\end{align*}$$

Without computing the system parameters, our aim here is to compute $\Gamma_{i}^{v}$, then $\Gamma_{i}^{vh}$ and $K_{i}^{h}$ with the right lower triangular Toeplitz structure. Computing $\Gamma_{i}^{h}$ is straight forward, thus we assume it is already known. We now concentrate on computing $\Gamma_{i}^{vh}$ and $K_{i}^{h}$.

\footnote{Here we assume that one can compute the entire vertical state sequence $x_{r,s}^{v}$, for $r = 0, 1, \ldots, N$ and $s = 0, 1, \ldots, M$.}
Using the Lower Triangular Toeplitz System Solver (LTSS) procedure in [Ramos and Mercère (2016b)], we compute \( \Gamma^{vh}_i \) by solving the linear system of equations

\[
I_{n_y} \cdot \Gamma^{vh}_i \cdot R_{11} = (R_{31} - R_{32}R_{22}^{-1}R_{21}) ,
\]

subject to \( \Gamma^{vh}_i \) being lower triangular Toeplitz. The solution is

\[
\Gamma^{vh}_i = \text{LTSS}\{I_{n_y}, R_{11}, (R_{31} - R_{32}R_{22}^{-1}R_{21}) , n_y, n_v, i\}.
\]

We now define \( E_f \) and \( E_{f_1} \) as

\[
E_f = R_{33}Q_3^\top = [ E_f(0) \mid E_f(1) \mid \cdots \mid E_f(M) ] \in \mathbb{R}^{n_y \times j},
\]

\[
E_{f_1} = [ E_{f_1}(0) \mid E_{f_1}(1) \mid \cdots \mid E_{f_1}(M) ] \in \mathbb{R}^{n_y \times (j-i+1)(M+1)},
\]

where

\[
E_f(k) = \begin{bmatrix}
e_{k,0} & e_{k,1} & e_{k,2} & \cdots & e_{k,j-i} & e_{k,j-i+1} & \cdots & e_{k,j-1} \\
e_{k,1} & e_{k,1} & e_{k,2} & \cdots & e_{k,j-i} & e_{k,j-i+1} & \cdots & e_{k,j-1} \\
e_{k,2} & e_{k,2} & e_{k,2} & \cdots & e_{k,j-i} & e_{k,j-i+1} & \cdots & e_{k,j-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
e_{k,i-1,0} & e_{k,i-1,1} & e_{k,i-1,2} & \cdots & e_{k,i-1,j-i} & e_{k,i-1,j-i+1} & \cdots & e_{k,i-1,j-1} 
\end{bmatrix}
\]

and \( \times \) denotes a matrix that is not relevant to the discussion. Furthermore, since the main diagonal blocks of \( K^h_i \) are all equal to \( I_{n_y} \) and all elements above the main diagonal blocks are \( 0_{n_y \times n_y} \), we observe that the first \( n_y \) rows of \( E_f \) contains a sequence of innovations, from which \( K^h_i \) can be computed. That is,

\[
\begin{bmatrix}
I_{n_y} & 0_{n_y \times n_y} & \cdots & 0_{n_y \times n_y}
\end{bmatrix} E_f = \begin{bmatrix}
e_0(0) & e_0(1) & \cdots & e_0(M)
\end{bmatrix} \in \mathbb{R}^{n_y \times j},
\]

where

\[
e_0(k) = \begin{bmatrix}
e_{0,0}^k & e_{0,1}^k & \cdots & e_{0,j-1}^k
\end{bmatrix} \cong \begin{bmatrix}
e_{i,k} & e_{i+1,k} & \cdots & e_{i+j-1,k}
\end{bmatrix} \in \mathbb{R}^{n_y \times j}.
\]

Let us now form the array of Hankel matrices using \( e_0(k) \), for \( k = 0, 1, \ldots, M \), i.e.,

\[
E_{f_2} = [ E_{f_2}(0) \mid E_{f_2}(1) \mid \cdots \mid E_{f_2}(M) ] \in \mathbb{R}^{n_y \times (j-i+1)(M+1)},
\]

where

\[
E_{f_2}(k) = \begin{bmatrix}
e_{0,0}^k & e_{0,1}^k & e_{0,2}^k & \cdots & e_{0,j-i}^k \\
e_{0,1}^k & e_{0,2}^k & e_{0,3}^k & \cdots & e_{0,j-i+1}^k \\
e_{0,2}^k & e_{0,3}^k & e_{0,4}^k & \cdots & e_{0,j-i+2}^k \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
e_{0,j-i}^k & e_{0,i}^k & e_{0,i+1}^k & \cdots & e_{0,j-1}^k
\end{bmatrix} \in \mathbb{R}^{n_y \times (j-i+1)}.
\]
Notice that if we knew $E_f^h$, then $E_f^h(k)$ would be the first $j - i + 1$ columns of $E_f^h(k)$, for $k = 0, 1, \ldots, M$. That is,

$$E_f^h(k) =
\begin{bmatrix}
e_i,k & e_{i+1,k} & e_{i+2,k} & \cdots & e_{j,k} & e_{j+1,k} & \cdots & e_{i+j-1,k} \\
e_{i+1,k} & e_{i+2,k} & e_{i+3,k} & \cdots & e_{j+1,k} & e_{j+2,k} & \cdots & e_{i+j,k} \\
e_{i+2,k} & e_{i+3,k} & e_{i+4,k} & \cdots & e_{j+2,k} & e_{j+3,k} & \cdots & e_{i+j+1,k} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\
e_{2i-1,k} & e_{2i,k} & e_{2i+1,k} & \cdots & e_{i+j-1,k} & e_{i+j,k} & \cdots & e_{2i+j-2,k}
\end{bmatrix}.
$$

Let us now define the covariance matrices $V_1 \in \mathbb{R}^{n_y \times n_y}$ and $V_2 \in \mathbb{R}^{n_y \times n_y}$ as

$$V_1 = \frac{1}{(j - i + 1)(M + 1)} E_f^h (E_f^h)^\top,
$$

$$V_2 = \frac{1}{(j - i + 1)(M + 1)} E_f^h (E_f^h)^\top.
$$

We can now find a relationship between $V_1$ and $V_2$ as follows:

$$V_1 = I_{n_y} \cdot K_i^h \cdot V_2,$$

and upon applying the vec operator on both sides, we obtain

$$\text{vec}\{V_1\} = (V_2^T \otimes I_{n_y}) \cdot \text{vec}\{K_i^h\}. \quad (83)$$

Now, since $K_i^h$ is a lower triangular Toeplitz matrix, $\text{vec}\{K_i^h\}$ will contain repeated elements. To remove these redundancies, we apply the identity

$$\text{vec}\{K_i^h\} = \mathcal{F}_{K_i^h} \cdot k_i^h,$$

where $\mathcal{F}_{K_i^h} \in \mathbb{R}^{n_y \times n_y^2}$ is a permutation matrix with elements equal to 0 and 1 and $k_i^h$ contains all the elements of $K_i^h$, i.e.,

$$k_i^h =
\begin{bmatrix}
\text{vec}\{k_0\} \\
\text{vec}\{k_1\} \\
\vdots \\
\text{vec}\{k_{i-1}\}
\end{bmatrix},$$

with

$$k_s = \begin{cases}I_{n_y}, & \text{if } s = 0 \\
C_1 A_1^{s-1} K_1, & \text{if } s \geq 1.\end{cases}$$

The (LTTSS) procedure performs this operation and assembles the full Toeplitz matrix. Thus, we obtain $K_i^h$ as

$$V = \text{LTTSS}\{I_{n_y}, V_2, V_1, n_y, n_y, i\},
$$

$$K_i^h = V (I_i \otimes K_0^{-1}).$$

17
where $K_0 \in \mathbb{R}^{n_y \times n_y}$ is the first $(n_y \times n_y)$ block of $V$. Knowing $K^h_i$, we can now compute $E^h_i$ from $R_{33}Q_Q^T$. For $k = 0, 1, \ldots, M$, we need to solve for $E_f(k)$ using the Hankel System Solver (HSS) procedure outlined in Ramos and Mercère (2016a). That is, we solve the following linear system of equations

$$E_f(k) = K^h_i \cdot E^h_i(k) \cdot I_j,$$

subject to $E_f(k)$ being a Hankel matrix. Upon applying the vec operator on both sides, we get

$$\text{vec}\{E_f(k)\} = (I_j \otimes K^h_i) \cdot \text{vec}\{E^h_i(k)\}.$$  \hspace{1cm} (84)

However, since $E^h_i(k)$ is a Hankel matrix, $\text{vec}\{E^h_i(k)\}$ will be inefficient for solving (84) because of the repeated elements. In order to compute the minimum number of elements from $\text{vec}\{E^h_i(k)\}$, we use the property

$$\text{vec}\{E^h_i(k)\} = \mathcal{F}_{E^h_i(k)} \cdot e^h_i(k),$$

where $\mathcal{F}_{E^h_i(k)} \in \mathbb{R}^{n_y i \times n_y (i+j-1)}$ is a permutation matrix with elements equal to 0 and 1 and

$$e^h_i(k) = \begin{bmatrix} \text{vec}\{e_{i,k}\} \\ \text{vec}\{e_{i+1,k}\} \\ \vdots \\ \text{vec}\{e_{2i+j-2,k}\} \end{bmatrix}.$$  \hspace{1cm} (85)

The HSS procedure handles the removal of repeated elements and assembles the full Hankel matrix $E^h_i(k)$. Thus, we get

$$E^h_i(k) = \text{HSS}\{K^h_i, I_j, E_f(k), n_y, 1, i, j\},$$

for $k = 0, 1, \ldots, M$.

Once we have all $M + 1$ solutions, we can then assemble the full matrix $E^h_f$.

We now need to find $\hat{X}^{vh}_f$ using $\{Y^h_f, \hat{X}^h_f, E^h_f, \Gamma^h_i, \Gamma^{vh}_i, K^h_i\}$. That is, let us define

$$\hat{Z}^{vh}_f = Y^h_f - \Gamma^h_i \hat{X}^h_f - K^h_i E^h_f = \Gamma^{vh}_i \hat{X}^h_f = \begin{bmatrix} \hat{Z}^{vh}_f(0) \\ \hat{Z}^{vh}_f(1) \\ \vdots \\ \hat{Z}^{vh}_f(M) \end{bmatrix} \in \mathbb{R}^{n_y i \times j}.$$  \hspace{1cm} (84)

Once again, we have $M + 1$ systems of equations of the form

$$\hat{Z}^{vh}_f(k) = \Gamma^{vh}_i \cdot \hat{X}^{vh}_f(k) \cdot I_j,$$  \hspace{1cm} for $k = 0, 1, \ldots, M$.

If we now apply the vec operator on both sides, we get

$$\text{vec}\{\hat{Z}^{vh}_f(k)\} = (I_j \otimes \Gamma^{vh}_i) \cdot \mathcal{F}_{\hat{X}^{vh}_f(k)} \cdot \hat{x}^{vh}_f(k),$$  \hspace{1cm} (85)

where $\mathcal{F}_{\hat{X}^{vh}_f(k)} \in \mathbb{R}^{n_y iij \times n_y (i+j-1)}$ is a permutation matrix with elements equal to 0 and 1 and

$$\hat{x}^{vh}_f(k) = \begin{bmatrix} \text{vec}\{\hat{x}^{v}_{i,k}\} \\ \text{vec}\{\hat{x}^{v}_{i+1,k}\} \\ \vdots \\ \text{vec}\{\hat{x}^{v}_{2i+j-2,k}\} \end{bmatrix}.$$
By applying the (HSS) procedure, we obtain
\[ \tilde{X}_f^{vh}(k) = \text{HSS}\{\Gamma_i^{vh}, I_j, \tilde{Z}_f^{vh}(k), n_v, 1, i, j\}, \text{ for } k = 0, 1, \ldots, M. \]

In horizontal data processing, as it relates to future data, we have two instances where we need to use the HSS procedure. This operation could be computationally expensive since it has to be done \((M+1)\) times. Nevertheless, the solution will give us the right structure for \(E_f^{vh}\) and \(\tilde{X}_f^{vh}\). Now that we have \(\tilde{X}_f^{vh}(k)\) for \(k = 0, 1, \ldots, M\), we can assemble the full matrix \(\tilde{X}_f^{vh}\).

Let us now define the following matrices:
\[
\begin{align*}
T_2^h &= \begin{bmatrix} - (\Gamma_i^h)^\dagger \Gamma_i^{vh} & - (\Gamma_i^h)^\dagger K_i^h \end{bmatrix} \in \mathbb{R}^{n_h \times (n_v+2n_y)i} \\
H_f^h &= \begin{bmatrix} \tilde{X}_f^{vh} \end{bmatrix} \\
E_f^{vh} &= \begin{bmatrix} Y_f^{vh} 
\end{bmatrix} \in \mathbb{R}^{(n_v+2n_y)i \times j},
\end{align*}
\]

where \((\Gamma_i^h)^\dagger\) is the pseudo-inverse of \(\Gamma_i^h\). Notice that the future horizontal state matrix \(\tilde{X}_f^{vh}\) is related to \(T_2^h\) and \(H_f^h\) via
\[ \tilde{X}_f^{vh} = T_2^h H_f^h. \]

The next step is to recover \(\tilde{X}_p^{vh}\). Let us re-visit the QR decomposition of the data, i.e.,
\[
\begin{bmatrix}
\tilde{X}_f^{vh} \\
\tilde{X}_p^{vh} \\
Y_p^{vh} \\
Y_f^{vh}
\end{bmatrix} =
\begin{bmatrix}
R_{11} & R_{12}^1 & R_{12}^2 & R_{12}^3 & Q_1^T \\
R_{21}^1 & R_{22}^1 & R_{22}^2 & R_{22}^3 & Q_2^T \\
R_{31} & R_{32}^1 & R_{32}^2 & R_{32}^3 & Q_3^T
\end{bmatrix},
\]

where \(R_{12}^1 \in \mathbb{R}^{n_v \times n_v i}, R_{12}^2 \in \mathbb{R}^{n_y \times n_v i}, R_{12}^3 \in \mathbb{R}^{n_y \times n_v i}, R_{22}^1 \in \mathbb{R}^{n_y \times n_v i}, R_{22}^2 \in \mathbb{R}^{n_y \times n_y i}, R_{32}^1 \in \mathbb{R}^{n_x \times n_v i}, R_{32}^2 \in \mathbb{R}^{n_y \times n_y i}, Q_{21} \in \mathbb{R}^{j \times n_v i}, \) and \(Q_{22} \in \mathbb{R}^{j \times n_y i}. \) Then \(Q_{21}^T\) and \(Y_p^{vh}\) can be expressed as
\[
Q_{21}^T = (R_{22}^1)^{-1} \tilde{X}_p^{vh} - (R_{22}^1)^{-1} R_{21}^1 Q_{11}^T \\
Y_p^{vh} = R_{21}^1 Q_{11}^T + R_{22}^1 Q_{22}^T + R_{22}^2 Q_{22}^T \\
= R_{21}^1 Q_{11}^T + R_{22}^1 \left( (R_{22}^1)^{-1} \tilde{X}_p^{vh} - (R_{22}^1)^{-1} R_{21}^1 Q_{11}^T \right) + R_{22}^1 Q_{22}^T \\
= \left( R_{21}^1 - R_{22}^1 (R_{22}^1)^{-1} R_{21}^1 \right) Q_{11}^T + R_{22}^1 (R_{22}^1)^{-1} \tilde{X}_p^{vh} + R_{22}^2 Q_{22}^T.
\]

We can now isolate \(R_{22}^1 (R_{22}^1)^{-1} \tilde{X}_p^{vh}\) from the rest. That is, we define
\[
\tilde{Z}_p^{vh} = Y_p^{vh} - \left( R_{21}^1 - R_{22}^1 (R_{22}^1)^{-1} R_{21}^1 \right) Q_{11}^T - R_{22}^1 Q_{22}^T = \underbrace{R_{22}^1 (R_{22}^1)^{-1}}_{\Gamma_i^{vh}} \tilde{X}_p^{vh}
\]
\[= \left[ \tilde{Z}_p^{vh}(0) \ | \ \tilde{Z}_p^{vh}(1) \ | \ \cdots \ | \ \tilde{Z}_p^{vh}(M) \right] = \Gamma_i^{vh} \cdot \left[ \tilde{X}_p^{vh}(0) \ | \ \tilde{X}_p^{vh}(1) \ | \ \cdots \ | \ \tilde{X}_p^{vh}(M) \right].\]
where $\hat{Z}^{vh}_p \in \mathbb{R}^{n_yi \times j}$ and $\hat{Z}^{vh}_p(k) \in \mathbb{R}^{n_yi \times j}$. It is clear that we have $M + 1$ equations of the form

$$\hat{Z}^{vh}_p(k) = \Gamma^{vh}_i \cdot \hat{X}^{vh}_p(k) \cdot I_j, \text{ for } k = 0, 1, \ldots, M. \quad (87)$$

If we apply the vec operator on both sides of (87), we get

$$\text{vec}\{\hat{Z}^{vh}_p(k)\} = (I_j \otimes \Gamma^{vh}_i) \cdot \mathcal{F}_{\hat{X}^{vh}_p(k)} \cdot \hat{x}^{vh}_p, \quad (88)$$

where $\mathcal{F}_{\hat{X}^{vh}_p(k)} \in \mathbb{R}^{n_yi \times n_y(i+j-1)}$ is a permutation matrix with elements equal to 0 and 1 and

$$\hat{x}^{vh}_p(k) = \begin{bmatrix}
\hat{x}^{vh}_0, k \\
\hat{x}^{vh}_1, k \\
\vdots \\
\hat{x}^{vh}_{i+j-2}, k
\end{bmatrix}. $$

Now applying the HSS procedure, which solves (88) and reconstructs the Hankel matrix, we can find the individual solutions from

$$\hat{X}^{vh}_p(k) = \text{HSS}\{\Gamma^{vh}_i, I_j, \hat{Z}^{vh}_p(k), n_y, 1, i, j\}, \text{ for } k = 0, 1, \ldots, M.$$ Knowing all $\hat{X}^{vh}_p(k), \text{ for } k = 0, 1, \ldots, M,$ we can now assemble the full $\hat{X}^{vh}_p$ matrix.

We now let

$$Y_p = \left(\Gamma^h_i\right)^\perp \left(Y_p^{vh} - \Gamma^{vh}_i \cdot \hat{X}^{vh}_p\right) = \left(\Gamma^h_i\right)^\perp \Gamma^h_i \hat{X}^{vh}_p + \left(\Gamma^h_i\right)^\perp K^h_i E^h_p$$

$$= \left[ Y_p(0) \mid Y_p(1) \mid \ldots \mid Y_p(M) \right] = \left(\Gamma^h_i\right)^\perp K^h_i \left[ E^h_p(0) \mid E^h_p(1) \mid \ldots \mid E^h_p(M) \right],$$

where $Y_p(k) \in \mathbb{R}^{(n_yi-n_yh) \times j}$ and $\left(\Gamma^h_i\right)^\perp \in \mathbb{R}^{(n_yi-n_yh) \times n_yi}$ is the orthogonal complement of $\Gamma^h_i$.

We now have $M + 1$ equations of the form

$$Y_p(k) = \left(\Gamma^h_i\right)^\perp K^h_i \cdot E^h_p(k) \cdot I_j, \text{ for } k = 0, 1, \ldots, M. \quad (89)$$

Let us now apply the vec operator on both sides of (89), to get

$$\text{vec}\{Y_p(k)\} = \left(I_j \otimes \left(\Gamma^h_i\right)^\perp K^h_i\right) \cdot \mathcal{F}_{E^h_p(k)} \cdot e^h_p(k), \text{ for } k = 0, 1, \ldots, M,$$

where $\mathcal{F}_{E^h_p(k)} \in \mathbb{R}^{n_yi \times n_y(i+j-1)}$ is a permutation matrix with elements equal to 0 and 1 and

$$e^h_p(k) = \begin{bmatrix}
\text{vec}\{e_{0,k}\} \\
\text{vec}\{e_{1,k}\} \\
\vdots \\
\text{vec}\{e_{i+j-2,k}\}
\end{bmatrix}. $$

Once again, the solution can be found by applying the HSS procedure,

$$E^h_p(k) = \text{HSS}\{\left(\Gamma^h_i\right)^\perp K^h_i, I_j, Y_p(k), n_y, 1, i, j\}, \text{ for } k = 0, 1, \ldots, M,$$
from which one can assemble the full $E_p^h$ matrix.

Knowing $X_p^{ih}$ and $E_p^h$, one can now compute $X_p^h$ from

$$X_p^h = (\Gamma_i^h)^\dagger \left( Y_p^h - \Gamma_i^{vh} X_p^{vh} - K_i^h E_p^h \right)$$

$$= \begin{bmatrix} - (\Gamma_i^h)^\dagger \Gamma_i^{vh} - (\Gamma_i^h)^\dagger K_i^h \left( \Gamma_i^h \right)^\dagger \end{bmatrix} \begin{bmatrix} X_p^{vh} \\ E_p^h \\ Y_p^h \end{bmatrix}$$

$$= T_2^h H_p^h,$$

where

$$H_p^h = \begin{bmatrix} X_p^{vh} \\ E_p^h \\ Y_p^h \end{bmatrix} \in \mathbb{R}^{(n_v+2n_y)i \times j}.$$ 

Let us now define $J \in \mathbb{R}^{n_h \times (n_v+(n_v+n_y)i)}$ and $H \in \mathbb{R}^{(n_h+(n_v+n_y)i) \times j}$ as

$$J = \begin{bmatrix} A_i^1 \\ \Phi_i^{vh} \\ L_i^h \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} X_p^h \\ X_p^{vh} \\ E_p^h \end{bmatrix}.$$ 

Then since $X_f^h = J H$, solving for $J$ would require solving a large system of equations. In order to avoid this, we compute the covariances $Z_1$ and $Z_2$ as

$$Z_1 = \frac{1}{J} H H^\top \in \mathbb{R}^{(n_h+(n_v+n_y)i) \times (n_h+(n_v+n_y)i)}$$

$$Z_2 = \frac{1}{J} X_f^h H^\top \in \mathbb{R}^{n_h \times (n_h+(n_v+n_y)i)}.$$ 

Then, the solution for $J$ becomes

$$J = Z_2 Z_1^{-1},$$

from which $A_i^1$, $\Phi_i^{vh}$, and $L_i^h$ can be computed, i.e.,

$$A_i^1 = J(:, 1 : n_h)$$

$$\Phi_i^{vh} = J(:, n_h + 1 : n_h + n_v i)$$

$$L_i^h = J(:, n_h + n_v i + 1 : n_h + (n_v + n_y) i).$$
Now substituting $\hat{X}_p^h$ into $\hat{X}_f^h$ we get
\[
\hat{X}_f^h = A_1^i \hat{X}_p^h + \Phi_{i}^{vh} \hat{X}_p^h + \mathcal{L}_i^h E_p^h
\]
\[
= A_1^i \left[ - (\Gamma_i^h)^\dagger \Gamma_i^{vh} \right] - (\Gamma_i^h)^\dagger K_i^h \left[ A_1^i (\Gamma_i^h)^\dagger \right] \begin{bmatrix} \hat{X}_p^{vh} \\ E_p^h \\ Y_p^h \end{bmatrix} + \Phi_{i}^{vh} \hat{X}_p^h + \mathcal{L}_i^h E_p^h
\]
\[
= \left[ \Phi_{i}^{vh} - A_1^i (\Gamma_i^h)^\dagger \Gamma_i^{vh} \right] \mathcal{L}_i^h - A_1^i (\Gamma_i^h)^\dagger K_i^h \left[ A_1^i (\Gamma_i^h)^\dagger \right] \begin{bmatrix} \hat{X}_p^{vh} \\ E_p^h \\ Y_p^h \end{bmatrix}
\]
\[
= T_1^h H_p^h,
\]
where $T_1^h \in \mathbb{R}^{n_h \times (n_v + 2n_y)^i}$ is defined as
\[
T_1^h = \left[ \Phi_{i}^{vh} - A_1^i (\Gamma_i^h)^\dagger \Gamma_i^{vh} \right] \mathcal{L}_i^h - A_1^i (\Gamma_i^h)^\dagger K_i^h \left[ A_1^i (\Gamma_i^h)^\dagger \right].
\]
The final task involves computing $\hat{X}_{f_+}^h$ from
\[
\hat{X}_{f_+}^h = T_1^h H_f^h = \left[ \hat{X}_{f_+}^h(0) \left| \hat{X}_{f_+}^h(1) \right| \cdots \left| \hat{X}_{f_+}^h(M) \right] \right] \in \mathbb{R}^{n_h \times j},
\]
where
\[
\hat{X}_{f_+}^h(k) = \left[ \hat{x}_{2i,k} \left| \hat{x}_{2i+1,k} \right| \cdots \left| \hat{x}_{2i+j-1,k} \right] \right] \in \mathbb{R}^{n_h \times j}.
\]
Now, between $\hat{X}_p^h(k)$ and $\hat{X}_{f_+}^h(k)$ there is an overlap of $\{\hat{x}_{2i,k}, \hat{x}_{2i+1,k}, \ldots, \hat{x}_{2i+j-1,k}\}$. That is,
\[
\begin{bmatrix}
\hat{x}_{0,k} & \hat{x}_{1,k} & \hat{x}_{2,k} & \cdots & \hat{x}_{2i-1,k} & \hat{x}_{2i,k} & \hat{x}_{2i+1,k} & \cdots & \hat{x}_{2i+j-1,k} \\
\end{bmatrix} = \hat{X}_p^h(k)
\]
and
\[
\begin{bmatrix}
\hat{x}_{0,k} & \hat{x}_{1,k} & \hat{x}_{2,k} & \cdots & \hat{x}_{2i-1,k} & \hat{x}_{2i,k} & \hat{x}_{2i+1,k} & \cdots & \hat{x}_{2i+j-1,k} \\
\end{bmatrix} = \hat{X}_{f_+}^h(k).
\]
Therefore, the entire horizontal state sequence can be recovered from
\[
\left[ \hat{X}_p^h(:, 0 : 2i - 1)(k) \left| \hat{X}_{f_+}^h(k) \right] \right], \text{ for } k = 0, 1, \ldots, M.
\]
Now form the $(n_h(N + 1) \times (M + 1))$ matrix of horizontal states $\hat{X}_p^h$ by vectorizing the individual terms in $\hat{X}_p^h(k)$ and $\hat{X}_{f_+}^h(k)$, for $k = 0, 1, \ldots, M$, then recalling that $N = 2i + j - 2$, i.e.,
\[
\begin{align*}
\hat{X}_p^h & \triangleq \begin{bmatrix}
\text{vec}\{\hat{X}_p^h(0)\} & \text{vec}\{\hat{X}_p^h(1)\} & \cdots & \text{vec}\{\hat{X}_p^h(M)\}
\end{bmatrix} \in \mathbb{R}^{n_hj \times (M+1)} \quad (90) \\
\hat{X}_{f_+}^h & \triangleq \begin{bmatrix}
\text{vec}\{\hat{X}_{f_+}^h(0)\} & \text{vec}\{\hat{X}_{f_+}^h(1)\} & \cdots & \text{vec}\{\hat{X}_{f_+}^h(M)\}
\end{bmatrix} \in \mathbb{R}^{n_hj \times (M+1)}, \quad (91)
\end{align*}
\]
and finally compute $\hat{X}^h$ from

$$
\hat{X}^h \triangleq \begin{bmatrix}
\hat{X}^h_p(0 : 2i - 1, :)
\hat{X}^h_f(0 : j - 2, :)
\end{bmatrix}
= \begin{bmatrix}
\hat{x}^h_{0,0} & \hat{x}^h_{0,1} & \cdots & \hat{x}^h_{0,M} \\
\hat{x}^h_{1,0} & \hat{x}^h_{1,1} & \cdots & \hat{x}^h_{1,M} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{x}^h_{2i-1,0} & \hat{x}^h_{2i-1,1} & \cdots & \hat{x}^h_{2i-1,M} \\
\hat{x}^h_{2i,0} & \hat{x}^h_{2i,1} & \cdots & \hat{x}^h_{2i,M} \\
\hat{x}^h_{2i+1,0} & \hat{x}^h_{2i+1,1} & \cdots & \hat{x}^h_{2i+1,M} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{x}^h_{N,0} & \hat{x}^h_{N,1} & \cdots & \hat{x}^h_{N,M}
\end{bmatrix} \in \mathbb{R}^{n_h(N+1) \times (M+1)}.
$$

(92)

This completes the computation of the horizontal states. The same procedure must be applied in the vertical direction to get the vertical state estimates.

References

Ramos, J. A., & Mercère, G. (2016a, October). Image modeling based on a 2-D stochastic subspace system identification algorithm. *Multidimensional Systems and Signal Processing, 28*, 1133–1165.

Ramos, J. A., & Mercère, G. (2016b). Subspace algorithms for identifying separable-in-denominator 2d systems with deterministic-stochastic inputs. *International Journal of Control, 89*(12), 2584–2610. Retrieved from [http://dx.doi.org/10.1080/00207179.2016.1172258](http://dx.doi.org/10.1080/00207179.2016.1172258)

Ramos, J. A., & Mercère, G. (2017a). A stochastic subspace system identification algorithm for state space systems in the general 2-d roesser model form. *International Journal of Control*(submitted for publication).