Assisted Common Information with Applications to Secure Two-Party Computation

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Abstract—Secure multi-party computation is a central problem in modern cryptography. An important sub-class of this are problems of the following form: Alice and Bob desire to produce sample(s) of a pair of jointly distributed random variables. Each party must learn nothing more about the other party’s output than what its own output reveals. To aid in this, they have available a set up — correlated random variables whose distribution is different from the desired distribution — as well as unlimited noiseless communication. In this paper we present an upperbound on how efficiently a given set up can be used to produce samples from a desired distribution.

The key tool we develop is a generalization of the concept of common information of two dependent random variables [Gács-Körner, 1973]. Our generalization — a three-dimensional region — remedies some of the limitations of the original definition which captured only a limited form of dependence. It also includes as a special case Wyner’s common information [Wyner, 1975]. To derive the cryptographic bounds, we rely on a monotonicity property of this region: the region of the “views” of Alice and Bob engaged in any protocol can only monotonically expand and not shrink. Thus, by comparing the regions for the target random variables and the given random variables, we obtain our upperbound.

I. INTRODUCTION

Finding a meaningful definition for the “common information” of a pair of dependent random variables $X$ and $Y$ has received much attention starting from the 1970s [6], [16], [19], [1]. We propose a new measure — a three-dimensional region — which brings out a detailed picture of the extent of common information of a pair. This gives us an expressive region — which brings out a detailed picture of the extent of common information in (a single instance of) $(X,Y)$. However, in the same paper [6], Gács and Körner showed (a result later strengthened by Witsenhausen [16]) that this rate is still just the largest $H(Q)$ for such that $X$ and $Y$ can be written as $(X',Q)$ and $(Y',Q)$ respectively. In other words, this definition captures only an explicit form of common information in (a single instance of) $(X,Y)$.

One limitation of the common information defined by Gács and Körner is that it ignores information which is almost common. Our approach could be viewed as a strict generalization of theirs which uncovers extra layers of “almost common information.” Technically, we introduce an omniscient genie who has access to both the observations $X$ and $Y$ and can send separate messages to the two observers over rate-limited noiseless links. See Fig. 1(b). The objective is for the observers to agree on a “common” random variable as before, but now with the genie’s assistance. This leads to a trade-off region trading-off the rates of the noiseless links and the resulting common information (or the resulting residual dependency). We characterize these trade-off regions and show that, in general, they exhibit non-trivial behavior, but reduce to the trivial behaviour discussed above when the rates of the noiseless links are zero.

Our new measure has an immediate application to cryptography (Section III). Distributed random variables with non-trivial correlations form an important resource in the cryptographic task of secure multi-party computation. A fundamental problem here is for two parties to “securely generate” a certain pair of random variables, given another pair of random variables, by means of a protocol. We show that the region of residual dependency of the views of two parties engaged in such a protocol can only monotonically expand and not shrink. Thus, by comparing the regions for the target random variables and the given random variables, we obtain improved upperbounds on the efficiency with which one pair can be used to securely generate another pair.

1Hence, after removing the maximal such $Q$, the contribution to the common information from $X'$ and $Y'$ is zero, even if they are highly correlated. Other approaches which do not necessarily suffer from this drawback have been suggested, notably [19], [1], [21].

2We use the term common information primarily to maintain continuity with [6].
II. ASSISTED COMMON INFORMATION REGION

A. Characterization

We say that a rate pair \((R_1, R_2)\) enables a common information rate \(R_{CI}\) if for every \(\epsilon > 0\), there is a large enough integer \(n\) and (deterministic) functions \(f_k : X^n \times Y^n \rightarrow \{1, \ldots, 2^{n(R_1 + \epsilon)}\}\), \((k = 1, 2)\), \(g_1 : X^n \times \{1, \ldots, 2^{n(R_1 + \epsilon)}\} \rightarrow Z\), and \(g_2 : Y^n \times \{1, \ldots, 2^{n(R_2 + \epsilon)}\} \rightarrow Z\) (where \(Z\) is the set of integers) such that

\[
\Pr (g_1(X^n, f_1(X^n, Y^n)) \neq g_2(Y^n, f_2(X^n, Y^n))) \leq \epsilon, \quad (1)
\]

\[
\frac{1}{n} \sum_{x,y} I(X^n, Y^n; g_1(X^n, f_1(X^n, Y^n))) \geq R_{CI} - \epsilon. \quad (2)
\]

We denote the closure of the set of all rate pairs which enable a common information rate \(R_{CI}\) by \(\mathcal{R}_{CI}(R_{CI})\). We call this the rate-region for enabling a common information rate of \(R_{CI}\). Note that the largest value of \(R_{CI}\) we need consider is \(H(X, Y)\). For larger values of \(R_{CI}\), \(\mathcal{R}_{CI}(R_{CI})\) is clearly empty.

Similarly, we define the rate-region \(\mathcal{R}_{RD}(R_{RD})\) for enabling a residual dependency rate of \(R_{RD}\) as the closure of the set of all rate pairs which enable a residual dependency rate \(R_{RD}\), where the definition of what it means for a rate pair to enable a residual dependency rate \(R_{RD}\) is exactly as above except (2) is replaced by

\[
\frac{1}{n} \sum_{x,y} I(X^n; Y^n|g_1(X^n, f_1(X^n, Y^n))) \leq R_{RD} + \epsilon.
\]

We also define the following “single-letter” regions

\[
\mathcal{R}_{CI}(R_{CI}) = \{(I(Y; Q|X), I(X; Q|Y)) : I(X,Y;Q) \geq R_{CI}\},
\]

\[
\mathcal{R}_{RD}(R_{RD}) = \{(I(Y; Q|X), I(X; Q|Y)) : I(X,Y|Q) \leq R_{RD}\}.
\]

Here \(Q\) is any random variable dependent on \((X,Y)\).

The main result of this section is a characterization of the rate-regions defined above (proof is sketched in section II-F):

**Theorem 2.1:**

\[ R_{CI} = R_{\ast CI}, \]

\[ R_{RD} = R_{\ast RD}. \]

Further, the cardinality of the alphabet \(Q\) of \((3)-(4)\) can be restricted to \(|X||Y| + 2\).

B. Behavior at \(R_1 = R_2 = 0\) and Connection to Gács-Körner [6]

As discussed in the introduction, Gács-Körner showed that when there is no genie, the common information rate is zero unless \(X = (X', Q), Y = (Y', Q')\), and \(H(Q) > 0\). Since the absence of links from the genie is a more restrictive condition than zero-rate links from the genie, we can ask whether introducing an omniscient genie, but with zero-rate links to the observers, changes the conclusion of Gács-Körner. The corollary below answers this question in the negative. Also note that the result of Gács-Körner can be obtained as a simple consequence of this corollary.

Let \(R_{CI,0} = \sup \{R_{CI} : (0,0) \in \mathcal{R}_{CI}(R_{CI})\}\), and \(R_{RD,0} = \inf \{R_{RD} : (0,0) \in \mathcal{R}_{RD}(R_{RD})\}\).

**Corollary 2.2:** \(R_{CI,0} > 0\) (or, \(R_{RD,0} < I(X; Y)\)) only if there are \(X', Y', Q'\) such that \(X = (X', Q'), Y = (Y', Q')\), \(R_{CI,0} = H(Q')\), and \(R_{RD,0} = I(X; Y|Q')\).

**Proof sketch:** We first observe that the only \(Q\)'s allowed in (3) and (4) if the rate pair \((0,0)\) is a member are such that \(I(Q; Y|X) = I(Q; X|Y) = 0\). Thus, the joint p.m.f. of \(X, Y, Q\) has the form

\[
p(x, y, q) = p(x, y)p(q|x) = p(x, y)p(q|y).
\]

Hence, for all \((x, y)\) such that \(p(x, y) > 0\), we must have \(p(q|x) = p(q|y)\), \(\forall q\). This implies that, if we consider the bipartite graph with vertices in \(X \cup Y\) and an edge between \(x \in X\) and \(y \in Y\) if and only if \(p(x, y) > 0\), for all vertices in the same connected component, \(p(q|v\text{ertex})\) is the same. Using this, and defining \(Q'\) to be the connected component to which \(X\) (or, equivalently \(Y\)) belongs, we can show that

\[
I(X; Y; Q) = I(Q'; Q) \leq H(Q'),
\]

\[
I(X; Y|Q) = H(Q'|Q) + I(X; Y|Q') \geq I(X; Y|Q').
\]

If there is only one connected component, this implies that \(R_{CI,0} = 0\) and \(R_{RD,0} = I(X; Y)\). Hence, if \(R_{CI,0} > 0\) (or, \(R_{RD,0} < I(X; Y)\)) more than one connected component must exist; moreover \(R_{CI,0} = H(Q')\) and \(R_{RD,0} = I(X; Y|Q')\).

Thus, at zero rates, common information exhibits trivial behavior. However, for positive rates, the behavior is, in general, non-trivial. Presently, we will demonstrate this through a few examples. But before that, we will show that Wyner’s common information can also be obtained as a special case of our characterization.
be the optimal jointly Gaussian choice. The optimal $R_{\text{CI}}$ is at least as much as shown and the optimal $R_{\text{RD}}$ is at most what is shown. Note that $R_{\text{CI}}$ is strictly positive for all $R > 0$.

![Fig. 2](image1.png)

**Fig. 2:** An achievable trade-off between $R_1 = R_2 = R$ and $R_{\text{CI}}$ (also $R_{\text{RD}}$) for jointly Gaussian $X, Y$ of unit variance and correlation $\rho = 0.95$. The trade-off is obtained by choosing $Q$ in (3) and (4) to be the optimal jointly Gaussian choice. The optimal $R_{\text{CI}}$ is at least as much as shown and the optimal $R_{\text{RD}}$ is at most what is shown. Note that $R_{\text{CI}}$ is strictly positive for all $R > 0$.

![Fig. 3](image2.png)

**Fig. 3:** $U, V$ are binary random variables with joint p.m.f. $p(0, 0) = p(1, 1) = p, p(0, 1) = 1 - 2p$, and $p(0, 1) = 0$. Boundary of $R_{\text{RD}}(0)$ for $p = 1/3$ is shown. The marked point is the minimum sum-rate point.

**C. Connection to Wyner’s Common Information [19]**

Wyner offered an alternative definition for common information in [19]. Briefly, Wyner’s common information is the “minimum binary rate of the common input to two independent processors that generate an approximation to $X, Y$.” From [19], Wyner’s common information is

$$C_{\text{Wyner}} = \inf I(X; Y; U),$$

where the infimum is taken over $U$ such that $X - U - Y$ is a Markov chain. It is easy to show that $C_{\text{Wyner}} \geq I(X; Y)$. Wyner’s common information can be obtained as a special case of our characterization: (proof omitted due to space constraints)

**Corollary 2.3:**

$$C_{\text{Wyner}} - I(X; Y) = \min_{(R_1, R_2) \in R_{\text{RD}}(0)} R_1 + R_2.$$

**D. Non-Trivial Behavior at Non-Zero Rates**

**Example 2.1: Jointly Gaussian random variables.** We consider jointly Gaussian random variables $X, Y$ each of unit variance and correlation coefficient $\rho$. Let the rates of the links from the genie to the two observers be the same, $R_1 = R_2 = R$.

3While the discussion has been for discrete random variables, it extends directly to continuous random variables.

$X, Y$ are dependent random variables whose joint p.m.f is shown. The solid lines each carry a probability mass of $\frac{1}{8}$ and the lighter ones $\frac{1}{8}$. In the plot, all points on the dotted lines belong to $R_{\text{RD}}(0)$.

**Figure 3** plots an achievable $R_{\text{CI}}$ and $R_{\text{RD}}$ by choosing $Q$ in (3) and (4) to be the optimal jointly Gaussian choice (jointly Gaussian with $X, Y$); i.e., the optimal $R_{\text{CI}}$ is at least as much as shown and the optimal $R_{\text{RD}}$ is at most what is shown. Note that $R_{\text{CI}} = 0$ when $R = 0$ consistent with Corollary 2.2, but $R_{\text{CI}}$ is strictly positive for all $R > 0$.

**Example 2.2: A binary example.** Figure 3 shows the joint p.m.f. of a pair of dependent binary random variables $U, V$. The boundary of the rate region $R_{\text{RD}}(0)$ is plotted in Figure 3. This is the optimal trade-off of rates at which the genie can communicate with the observers so that they may produce a common random variable which can render their observations practically conditionally independent.

**Example 2.3:** Figure 4 shows the joint p.m.f. of a pair of dependent random variables $X, Y$. When $\delta = 0$, they have the simple dependency structure of $X = (X', Q), Y = (Y', Q)$ where $X', Y', Q$ are independent. This is the trivial case in the introduction, and the observers can each produce, without any assistance from the genie, $Q$ which renders their observations conditionally independent. Thus, $R_{\text{RD}}(0)$ is the entire positive quadrant. For small values of $\delta$ we intuitively expect the random variables to be “close” to this case. A measure such as the common information of Gács and Körner fails to bring this out (common information is discontinuous in $\delta$ jumping from $H(Q) = 1$ at $\delta = 0$ to $0$ for $\delta > 0$). However, the intuition is borne out by our trade-off regions. For instance, for $\delta = 0.05$, Figure 4 shows that $R_{\text{RD}}(0)$ is nearly all of the positive quadrant.

In Section III, we will use the characterization developed in this section to compare the pairs of random variables in the last two examples in a cryptographic context. See Example 3.1.

**E. Relationship between $R_{\text{CI}}$ and $R_{\text{RD}}$**

The residual dependency rate-region can be written in terms of the common information rate-region: (proof is omitted due to space constraints)

**Corollary 2.4:**

$$R_{\text{RD}}(R_{\text{RD}}) = \{(R_1, R_2) : \exists (r_1, r_2) \in R_{\text{CI}}(r_{\text{RD}}) \text{ s.t. } r_{\text{CI}} \geq I(X; Y) - R_{\text{RD}} + r_1 + r_2, R_1 \geq r_1, \text{ and } R_2 \geq r_2\},$$
where
\[ \delta \mathcal{R}_C(R_{C_2}) = \{(R_1, R_2) \in \mathcal{R}_C(R_{C_2}) : \beta(r_1, r_2) \in \mathcal{R}_C(R_{C_2}) \text{ s.t. } r_1 \leq r_1', r_2 \leq r_2', \text{ and } (r_1, r_2) \neq (r_1', r_2') \} \].

**F. Sketch of Proof of Theorem 2.1**

Proof of achievability (\( \mathcal{R}_a \subseteq \mathcal{R} \)), which is based on Wyner-Ziv's source coding with side-information [20], is omitted in the interest of space. The cardinality bound can be shown using Carathéodory’s theorem.

To prove the converse, let \( \epsilon > 0 \) and \( n, f_1, f_2, g_1, g_2 \) be such that (1) and (2) hold. Let \( C_k = f_k(X^n, Y^n) \), for \( k = 1, 2, \) and \( W_1 = g_1(X^n, C_1) \) and \( W_2 = g_2(Y^n, C_2) \). Then,
\[
R_1 + \epsilon \geq \frac{1}{n} I(C_1; X^n) \geq \frac{1}{n} I(C_1; X^n) \geq \frac{1}{n} I(W_1; X^n)
\]
\[
\geq \frac{1}{n} I(Y^n_1; W_1^n, X^n)
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} H(Y_i|X_i) - H(Y_i|Y_{i-1}^i, X_i^n, W_i)
\]
\[
\geq \frac{1}{n} \sum_{i=1}^{n} H(Y_i|X_i) - H(Y_i|X_i, W_i, Y_{i-1}^{i-1}, X_{i-1})
\]
\[
= \sum_{i=1}^{n} I(Y_i; Q_i|X_i) \quad \text{(a)}
\]
\[
\geq I(Y; Q|X, J) \quad \text{(c)}
\]
where (a) follows from the independence of \((X_i, Y_i)\) pairs across \( i \). In (b), we define \( J \) to be a random variable uniformly distributed over \( \{1, \ldots, n\} \) and independent of \((X^n, Y^n)\). And (c) follows from the independence of \( J \) and \((X^n, Y^n)\). Similarly,
\[
R_2 + \epsilon \geq \frac{1}{n} I(C_2; Y^n) \geq \frac{1}{n} I(W_2; Y^n)
\]
\[
= \frac{1}{n} I(W_2; Y^n) = \frac{1}{n} I(W_2; W_i)
\]
\[
\geq H(W_i|X^n) - H(W_i|W_i) \quad \text{(a)}
\]
\[
\geq \frac{1}{n} I(X^n; W_i|Y^n) - \kappa \epsilon \quad \text{(b)}
\]
\[
\geq I(X; W_i|Q_i) - \kappa \epsilon
\]
where (a) (with \( \kappa := 1 + \log |X||Y| \)) follows from Fano’s inequality and the fact that the range of \( g_1 \) can be restricted without loss of generality to a set of cardinality \(|X^n||Y^n|\). And (b) can be shown along the same lines as the chain of inequalities which gave a lower bound for \( R_1 \) above. Moreover,
\[
\frac{1}{n} I(X^n, Y^n; W_1) = \frac{1}{n} \sum_{i=1}^{n} H(X_i, Y_i) - H(X_i, Y_i|W_1, X_{i-1}^{i-1}, Y_{i-1}^{i-1})
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} I(X_i, Y_i|W_1, X_{i-1}^{i-1}, Y_{i-1}^{i-1})
\]
\[
= I(X_j, Y_j; Q).\]

Since \( X_j, Y_j \) has the same joint distribution as \( X, Y \), the converse (\( \mathcal{R}_{C_2} \subseteq \mathcal{R}_{C_2} \)) for common information follows. Similarly, the converse (\( \mathcal{R}_{RD} \subseteq \mathcal{R}_{RD} \)) for residual dependency can be shown using
\[
\frac{1}{n} I(X^n; Y^n|W_1) = \frac{1}{n} \sum_{i=1}^{n} I(X_i; Y_i|W_1, X_{i-1}^{i-1}, Y_{i-1}^{i-1})
\]
\[
\geq \frac{1}{n} \sum_{i=1}^{n} I(X_i; Y_i|W_1, X_{i-1}^{i-1}, Y_{i-1})
\]
\[
= I(X_j; Y_j|Q).
\]

### III. Cryptographic Application

#### A. Background

Secure multi-party computation is a central problem in modern cryptography. Roughly, the goal of secure multi-party computation is to carry out computations on inputs distributed among two (or more) parties, so as to provide each of them with no more information than what their respective inputs and outputs reveal to them. Our focus in this section is on an important sub-class of such problems — which we shall call secure 2-party sampling — in which the computation has no inputs, but the outputs to the parties are required to be from a given joint distribution (and each party should not learn anything more than just its own output). Also we shall restrict ourselves to the case of honest-but-curious adversaries. It is well-known (see for instance [18] and references therein) that very few distributions can be sampled from in this way, unless the computation is aided by a set up — some correlated random variables that are given to the parties at the beginning of the protocol. The set up itself will be from some distribution \((X, Y)\) (Alice gets \( X \) and Bob gets \( Y \)) which is different from the desired distribution \((U, V)\) (Alice getting \( U \) and Bob getting \( V \)). The fundamental question then is, which set ups \((X, Y)\) can be used to securely sample which distributions \((U, V)\), and how efficiently.

While the feasibility question can be answered using combinatorial analysis (as, for instance, was done in [12]), information theoretic tools have been put to good use to show bounds on efficiency of protocols (e.g. [2], [5], [15], [10], [17], [7], [4], [14]). Our work continues on this vein of using information theory to formulate and answer efficiency questions in cryptography. Specifically, the quantities explored in the previous section lead to effective tools in providing new and improved upper-bounds on the rate at which samples from a distribution \((U, V)\) can be securely generated, per sample drawn from a set up distribution \((X, Y)\). Below we sketch the outline of this application, which is further developed in [13].

**a) Secure Protocols:** A two-party protocol II is specified by a pair of (possibly randomized) functions \( \pi_{Alice} \) and \( \pi_{Bob} \), that are used by each party to operate on its current state \( W \) to produce a message \( m \) (that is sent to the other party) and a new state \( W' \) for itself. The initial state of the parties may consist of correlated random variables \((X, Y)\), with Alice’s state being \( X \) and Bob’s state being \( Y \); such a pair is called a set up for the protocol. The protocol proceeds by the parties taking turns to apply their respective functions to their state, and sending the resulting message to the other
party; this message is added to the state of the other party. \(\pi_{\text{Alice}}\) and \(\pi_{\text{Bob}}\) also specify when the protocol terminates and produces output (instead of producing the next message in the protocol). A protocol is considered valid only if both parties terminate in a finite number of rounds (with probability 1). The view of a party in an execution of the protocol is a random variable which is defined as the collection of its states so far in the protocol execution. For a valid protocol \(\Pi = (\pi_{\text{Alice}}, \pi_{\text{Bob}})\), we shall denote the final views of the two parties as \((\Pi_{\text{Alice}}^\text{view}(X, Y), \Pi_{\text{Bob}}^\text{view}(X, Y))\). Also, we shall denote the outputs as \((\Pi_{\text{Alice}}^\text{out}(X, Y), \Pi_{\text{Bob}}^\text{out}(X, Y))\).

For a protocol \(\Pi\) to be a secure realization of \((U, V)\) given a set up \((X, Y)\), firstly, the outputs \((\Pi_{\text{Alice}}^\text{out}(X, Y), \Pi_{\text{Bob}}^\text{out}(X, Y))\) must be identically distributed as \((U, V)\). Secondly, if either Alice or Bob is “curious” (or “passively corrupt”), the protocol should give that party no more information about the other party’s output than what their own output provides. This is formalized using a simulatability requirement. In case of information theoretic security (as opposed to computational security) these can be stated in terms of independence of the view, given one’s own output. Formally these three requirements can be stated as follows:

\[
(\Pi_{\text{Alice}}^\text{out}(X, Y), \Pi_{\text{Bob}}^\text{out}(X, Y)) = (U, V)
\]

\[
\Pi^\text{view}_{\text{Alice}}(X, Y) \leftrightarrow \Pi^\text{out}_{\text{Alice}}(X, Y) \leftrightarrow \Pi^\text{out}_{\text{Bob}}(X, Y)
\]

\[
\Pi^\text{out}_{\text{Alice}}(X, Y) \leftrightarrow \Pi^\text{out}_{\text{Bob}}(X, Y) \leftrightarrow \Pi^\text{view}_{\text{Bob}}(X, Y)
\]

B. Towards Measuring Cryptographic Content

In [17] three information theoretic quantities were used to quantify the cryptographic content of a pair of correlated random variables \(X\) and \(Y\), which we shall rephrase as below:

\[
H(Y \upharpoonright X|X) = \min_{Q: H(Q|Y)=I(X;Y|Q)=0} H(Q|X)
\]

\[
H(X \upharpoonright Y|Y) = \min_{Q: H(Q|X)=I(X;Y|Q)=0} H(Q|Y)
\]

\[
I(X;Y|X \land Y) = \min_{Q: H(Q|X)=H(Q|Y)=0} I(X;Y|Q)
\]

As shown in [17], these quantities are “monotones” that can only decrease in a protocol, and if the protocol securely realizes a pair of correlated random variables \((U, V)\) using a set up \((X, Y)\), then each of these quantities should be at least as large for \((X, Y)\) as for \((U, V)\). While these quantities do capture several interesting cryptographic properties, they paint a partial picture. For instance, two pairs of correlated random variables \((X, Y)\) and \((X', Y')\) may have vastly different values for these quantities, even if they are statistically close to each other, and hence have similar “cryptographic content.”

Instead, we shall consider the triplet \(K[X;Y|Q]\) defined as

\[
K[X;Y|Q] := (I(Q;Y|X), I(Q;X|Y), I(X;Y|Q)).
\]

\[\text{for an arbitrary random variable } Q. \text{ By considering all random variables } Q \text{ we define the region}^3
\]

\[
\mathbb{K}(X;Y) := \{(x, y, z) : \exists Q \text{ s.t. } K[X;Y|Q] \leq (x, y, z)\}.
\]

This generalizes the three quantities considered in [17], as (using arguments similar to that used for Corollary 2.2) it can be shown that the region \(\mathbb{K}(X;Y) \subseteq \mathbb{R}^{d+3}\) intersects the coordinate axes at the points \((H(Y \upharpoonright X|X), 0, 0), (0, H(X \upharpoonright Y|Y)), (0, 0, I(X;Y|X \land Y))\). In the following sections we point out that \(\mathbb{K}\) also satisfies a monotonicity property: the region can only expand in a protocol, and if the protocol securely realizes a pair of correlated random variables \((U, V)\) using a set up \((X, Y)\), then \(\mathbb{K}(X;Y)\) should be smaller than \(\mathbb{K}(U;V)\). As we shall see, since the region \(\mathbb{K}(X;Y)\) has a non-trivial shape (see for instance, Example 2.2), \(\mathbb{K}\) can yield much better bounds on the rate than just considering the axis intercepts; in particular \(\mathbb{K}\) can differentiate between pairs of correlated random variables that have the same axis intercepts. Further \(\mathbb{K}(X;Y)\) is continuous as a function of \((X, Y)\), and as such one can derive bounds on rate that are applicable to statistical security as well as perfect security.

C. Monotone Regions for 2-Party Secure Protocols

Given a pair of random variables \((X, Y)\) denoting the views of the two parties in a 2-party protocol we are interested in capturing the “cryptographic content” of this pair. We shall do so by defining a region in multi-dimensional real space, that intuitively, consists of witnesses of “weakness” in the cryptographic nature of the random variables \((X, Y)\); thus smaller this region, the more cryptographically useful the variables are. The region has a monotonicity property: a secure protocol that involves only communication (over noiseless links) and local computations (i.e., without using trusted third parties) can only enlarge the region.

Our definition of a monotone region from [13] given below, strictly generalizes that suggested by [17]. The monotone in [17], which is a single real number \(m\), can be interpreted as a one-dimensional region \([m, \infty)\) to fit our definition. (Note that a decrease in the value of \(m\) corresponds to the region \([m, \infty)\) enlarging.)

**Definition 3.1:** We will call a function \(M\) that maps a pair of random variables \((X, Y)\), to an upward closed subset\(^6\) of \(\mathbb{R}^{d}\) (points in the \(d\)-dimensional real space with non-negative co-ordinates) a monotone region if it satisfies the following properties:

1. **(Local computation cannot shrink it.)** For all random variables \((X, Y)\) with \(X \leftrightarrow Y \leftrightarrow Z\), we have \(M(XYZ) \supseteq M(Y;Z) \supseteq M(X;YZ) \supseteq M(X;Y)\).

2. **(Communication cannot shrink it.)** For all random variables \((X, Y)\) and functions \(f\) (over the support of

\[\text{Here} \leq \text{stands for coordinate-wise comparison. Note that } K(X;Y) \text{ is equivalent to } \{(R_{\Pi^0}(R_{\Pi^0}), R_{\Pi^0}) : R_{\Pi^0} \in [0, I(X;Y))]\}. \text{ We use this notation to make the dependence on } X \text{ and } Y \text{ explicit.}\)

\[\text{A subset } M \text{ of } \mathbb{R}^{d} \text{ is called upward closed if } \alpha \in M \text{ and } \alpha' \geq \alpha \text{ (i.e., each co-ordinate of } \alpha' \text{ is no less than that of } \alpha \text{ implies that } \alpha' \in M.}\)
X or Y), we have $\mathcal{M}(X;Yf(X)) \supseteq \mathcal{M}(X;Y)$ and $\mathcal{M}(Xf(Y);Y) \supseteq \mathcal{M}(X;Y)$.

3) (Securely derived outputs do not have smaller regions.) For all random variables $(X, U, V, Y)$ with $X \leftrightarrow U \leftrightarrow V$ and $U \leftrightarrow V \leftrightarrow Y$, we have $\mathcal{M}(U;V) \supseteq \mathcal{M}(X;Y)$.

4) (Cryptographic content in independent pairs add up.) For independent pairs of random variables $(X_0, Y_0)$ and $(X_1, Y_1)$, we have $\mathcal{M}(X_0;Y_0) + \mathcal{M}(X_1;Y_1) = \mathcal{M}(X_0;Y_0) + \mathcal{M}(X_1;Y_1)$, where the $+$ sign denotes Minkowski sum. That is, $\mathcal{M}(X_0;Y_0) \cup \mathcal{M}(X_1;Y_1) = \{a_0 + a_1 \mid a_0 \in \mathcal{M}(X_0;Y_0) \text{ and } a_1 \in \mathcal{M}(X_1;Y_1)\}$. (Here addition denotes coordinate-wise addition.)

Note that since $\mathcal{M}(X_0;Y_0)$ and $\mathcal{M}(X_1;Y_1)$ have non-negative co-ordinates and are upward closed, $\mathcal{M}(X_0;Y_0) + \mathcal{M}(X_1;Y_1)$ is smaller than both of them. This is consistent with the intuition that more cryptographic content (as would be the case with having more independent copies of the random variables) corresponds to a smaller region.

D. $\mathbb{K}$ as a Monotone Region.

In [13] we prove the theorem below, and obtain the following corollary.

**Theorem 3.1:** $\mathbb{K}$ is a monotone region as defined in Definition 3.1.

**Corollary 3.2:** If $n_1$ independent copies of a pair of correlated random variables $(U, V)$ can be securely realized from $n_2$ independent copies of a pair of correlated random variables $(X, Y)$, then $n_1 \mathbb{K}(X;Y) \subseteq n_2 \mathbb{K}(U;V)$. (Here multiplication by an integer $n$ refers to $n$-times repeated Minkowski sum.) Intuitively, $\mathbb{K}(X;Y)$ captures the cryptographic content of the correlated random variables $(X, Y)$: the farther it is from the origin, the more cryptographic content it has. In particular, if $\mathbb{K}(X;Y)$ contains the origin, then $(X, Y)$ is cryptographically “trivial,” in the sense that $(X, Y)$ can be securely realized with no set ups. This triviality property can be inferred from the three quantities considered by [17] as well, since those quantities correspond to the axis intercepts of our monotone region. However, what makes the monotone region more interesting is when the pair of correlated random variables is non-trivial, as illustrated in the following example.

**Example 3.1:** Consider the question of securely realizing $n_1$ independent pairs of random variables distributed according to $(U, V)$ in Example 2.2 from $n_2$ independent pairs of $(X, Y)$ in Example 2.3. While the monotones in [17] will give a lower-bound of $0.5182$ on $n_2/n_1$, we show that $n_2/n_1 \geq 1.8161$. (For this we use the intersection of $\mathbb{K}(U;V)$ with the plane $z = 0$ (Figure 3) and one point in the region $\mathbb{K}(X;Y)$ (marked in Figure 4), and apply Corollary 3.2.)

Hence, the axis intercepts of this monotone region (one of which is the common information of Gács and Körner) do not by themselves capture subtle characteristics of correlation that are reflected in the shape of the monotone region. As discussed in [13], $\mathbb{K}(X;Y)$ is a convex region, and for a fixed set of axis intercepts, the cryptographic quality of a pair of random variables is reflected in how little it bulges towards the origin.

We leave as an open question whether our bound is indeed tight.

**REFERENCES**

[1] R. Ahlswede and J. Körner, “On common information and related characteristics of correlated information sources,” in Proc. of the 7th Prague Conference on Information Theory, 1974.

[2] D. Beaver, “Correlated pseudorandomness and the complexity of private computations,” in Proc. 28th STOC, pp. 479–488. ACM, 1996.

[3] D. Beaver, “Precomputing oblivious transfer,” in Don Coppersmith, editor, CRYPTO, vol. 963 of Lecture Notes in Computer Science, pp. 97–109. Springer, 1995.

[4] L. Csiszár and R. Ahlswede, “On oblivious transfer capacity,” in Proc. International Symposium on Information Theory (ISIT), pp. 2061–2064, 2007.

[5] Y. Dodis and S. Micali, “Lower bounds for oblivious transfer reductions,” in Jacques Stern, editor, EUROCRYPT, vol. 1592 of Lecture Notes in Computer Science, pp. 42–55. Springer, 1999.

[6] P. Gács and J. Körner, “Common information is far less than mutual information,” Problems of Control and Information Theory, 2(2):119–162, 1973.

[7] H. Imai, K. Morozov, and A. C. A. Nascimento, “On the oblivious transfer capacity of the erasure channel,” in Proc. International Symposium on Information Theory (ISIT), pp. 1428–1431, 2006.

[8] Hideki Imai, Kirill Morozov, and Anderson C. A. Nascimento. Efficient oblivious transfer protocols achieving a non-zero rate from any non-trivial noisy correlation. In International Conference on Information Theoretic Security (ICITS), 2007.

[9] Hideki Imai, Kirill Morozov, Anderson C. A. Nascimento, and Andreas Winter. Efficient protocols achieving the commitment capacity of noisy correlations. In International Symposium on Information Theory (ISIT), pages 1432–1436, 2006.

[10] Hideki Imai, Jörn Müller-Quade, Anderson C. A. Nascimento, and Andreas Winter. Rates for bit commitment and coin tossing from noisy correlation. In International Symposium on Information Theory (ISIT), pages 45–, 2004.

[11] J. Kilian, “Founding cryptography on oblivious transfer,” in Proc. STOC, pp. 20–31. ACM, 1988.

[12] J. Kilian, “More general completeness theorems for secure two-party computation,” in Proc. 32nd STOC, pp. 316–324. ACM, 2000.

[13] H. Maji, M. Prabhakaran, V. Prabhakaran, and M. Rosulek, “On cryptographic capacity,” work in progress.

[14] S. Winkler and J. Wullschleger. “Statistical impossibility results for oblivious transfer reductions.” Cryptology ePrint Archive, Report 2009/508, 2009. http://eprint.iacr.org/.

[15] A. Winter, A. C. A. Nascimento, and H. Imai. “Commitment capacity of discrete memoryless channels.” In Kenneth G. Paterson, editor, IMA Int. Conf., vol. 2898 of Lecture Notes in Computer Science, pp. 35–51. Springer, 2003.

[16] H. S. Witsenhausen, “On sequences of pairs of dependent random variables,” SIAM Journal of Applied Mathematics, 28:100–113, 1975.

[17] S. Wolf and J. Wullschleger. “New monotonones and lower bounds in unconditional two-party computation,” IEEE Transactions on Information Theory, 54(6):2792–2797, 2008.

[18] J. Wullschleger. Oblivious-Transfer Amplification. Ph.D. thesis, Swiss Federal Institute of Technology, Zürich. http://arxiv.org/abs/cs.CR/0608076.

[19] A. D. Wyner, “The common information of two dependent random variables,” IEEE Transactions on Information Theory, 21(2):163–179, 1975.

[20] A. D. Wyner and J. Ziv, “Rate-distortion function for source coding with side information at the decoder,” IEEE Transactions on Information Theory, 22(1):1–11, 1976.

[21] H. Yamamoto, “Coding theorems for Shannon’s cipher system with correlated source outputs, and common information,” IEEE Transactions on Information Theory, 40(1):85–95, 1994.