Spinors on a curved noncommutative space: coupling to torsion and the Gross–Neveu model

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Abstract
We analyse the Dirac action on the truncated Heisenberg algebra and in particular, the nonminimal couplings to the background gravitational field. By projection to the Heisenberg algebra we obtain a renormalisable model: the noncommutative extension of the Gross–Neveu model. This result indicates that, as on the commutative curved backgrounds, nonminimal couplings with torsion and curvature are necessary (and sufficient) for renormalisability of scalar and spinor theories on the curved noncommutative spaces.

Keywords: noncommutative geometry, noncommutative gravity, renormalisation

1. Introduction
The Grosse–Wulkenhaar (GW) model is a noncommutative model which has attracted much attention and initiated a large amount of work in the past decade [1]. It describes a real scalar field \( \phi \) on the noncommutative Moyal space evolving in the external oscillator potential, in two and four Euclidean dimensions\(^4\),

\[
S_{GW} = \int \left( \frac{1}{2} \left( \partial_\mu \phi \right) \left( \partial^\mu \phi \right) + \frac{1}{2} \Omega^2 \xi_\mu \xi^\mu \phi^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4, \right.
\]

where \( \xi_\mu \) is related to \( x^\mu \) through (1.6). The model has exceptionally good properties in quantisation which have been established and analysed in many papers since 2003, and include perturbative renormalisability to all orders and vanishing of the \( \beta \)-function at the self-duality point [2]; it is likely to be perturbatively solvable. There is considerable progress in

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\(^4\) As we do not specify the representation we do not use the \( \ast \)-product notation: the product here is the operator product.
the nonperturbative treatment of the GW model as well; for recent results and developments, see [3]. Although initially treated in the matrix base, the Grosse–Wulkenhaar model was subsequently analysed by multiscale analysis in the coordinate base [4], and that analysis revealed many interesting mathematical properties and enabled generalisations. Similar though in many aspects different models in the external magnetic field were proposed even before [5] as exactly solvable quantum field theories; one of the most important properties which these models possess is the Langmann–Szabo (LS) duality, a new kind of symmetry which is also present in the GW model.

Many attempts have been made to understand the physical reasons underlying renormalisability of the Grosse–Wulkenhaar model, and to generalise it to other physical fields. One method of generalisation is straightforward: by constructing Lagrangians which have the Mehler kernel as propagator. For spinors this was done successfully in [6]; the proposed spinor action was

$$S_{\text{Dir}} = \int \bar{\psi} D^\psi \psi = \int \bar{\psi} \left( i \Gamma^\mu \partial_\mu + \Omega \Gamma^\mu \xi_\mu \right) \psi. \quad (1.2)$$

The action is defined on the space of spinors $\psi(x^\mu), \mu = 1,...,4$ which carry a double-dimensional spinor representation, $\{ \Gamma^A, \Gamma^B \} = 2 \delta^{AB}$. The square of the Dirac operator $D^\psi$ gives, up to a constant coordinate-independent matrix $\Sigma$, exactly the Hamiltonian of the massless GW model: in consequence, action (1.2) is renormalisable. The other possibility of taking ‘the square root of the harmonic potential’ was proposed by Vignes-Tourneret in [7, 8]

$$S_{\text{VT}} = \int \bar{\psi} D_{\text{VT}} \psi = \int \bar{\psi} \left( -i \gamma^\mu \partial_\mu + \Omega \gamma^\mu \xi_\mu + \bar{m} + \kappa \gamma_5 \right) \psi, \quad (1.3)$$

and it is a noncommutative extension of the Gross–Neveu model [9]. Remarkably, (1.3) is also renormalisable although $D_{\text{VT}}$ is not exactly equal to the kinetic term of (1.1); the parity breaking $\gamma_5$-term appears as a counterterm when spinors are massive.

Generalisation of the GW model to the gauge fields has been more difficult, and indeed a construction of a renormalisable gauge model is still an open problem. At first sight the problem is easy to understand. In order to have an oscillator-type external potential and correspondingly the Mehler propagator, one has to include coordinate dependent terms to the action: but these terms break the gauge symmetry. However, in noncommutative geometry this can be solved surprisingly simply. Namely, the momentum operators $p_\mu$ which define the partial derivatives by

$$\partial_\mu \phi = \left[ p_\mu, \phi \right] \quad (1.4)$$

can and often do belong to the algebra of coordinates $A$. For example on the space with constant nondegenerate noncommutativity

$$[x^\mu, x^\nu] = i k \delta^{\mu\nu} = \text{const}, \quad (1.5)$$

of which the Moyal space is a representation, the momenta are given by

$$p_\mu = (i k J_\mu)^{-1} x^\mu = -\frac{i}{2} \tilde{x}_\mu. \quad (1.6)$$

Therefore covariant momenta $\tilde{X}_\mu = p_\mu + A_\mu$, operators which define covariant derivatives, $D_\mu \phi = [\tilde{X}_\mu, \phi]$, also belong to the spacetime $A$. Covariant momenta transform covariantly, in the adjoint representation of the gauge group. Using $\tilde{X}_\mu$ or covariant coordinates,

$^5$ $A_\mu$ denotes the gauge potential: it is in applications usually the potential of the noncommutative $U(1)$ or the $U(N)$ gauge group.
\( X^\mu = x^\mu + i\tilde{\epsilon} J^\mu \), one can find gauge invariant and coordinate dependent actions \([10, 11]\), which in some cases do not have commutative analogues. Still, additional physical tools to construct acceptable generalisations of the Yang–Mills action are needed. In \([12]\), the oscillator potential was introduced through the ghost sector. A promising action was obtained in \([13]\) as the effective action for the \(U(1)\) gauge field coupled to the GW scalar, after integration of the scalar modes. Though the above-mentioned models can be written using the covariant coordinates and possess the LS duality, they have difficulties related mainly to the vacuum structure and none has proved to be renormalisable, \([14]\). For comprehensive recent reviews of the gauge models see for example \([15]\).

Another logic of generalisation of the Grosse–Wulkenhaar model was proposed in \([16]\): it is based on the observation that the harmonic potential can be seen as the scalar curvature of an appropriately defined noncommutative space. This geometric interpretation gives a straightforward way to obtain the action for other fields: it is simply the action on the curved spacetime. There are however additional details. Since two-dimensional space \([1.5]\) can be considered as a contraction of three-dimensional algebra \([2.7]\) which has finite-dimensional matrix representations, the oscillator term in the GW action can be understood as a geometric regularisation, or discretisation. We use a Kaluza–Klein reduction followed by rescaling or renormalisation of the physical fields. Along with geometric interpretation of the GW model \([16]\), the geometric approach gave an interesting gauge model \([17]\) with an improved vacuum structure and the explicit Becchi–Rouet–Stora–Tyutin (BRST) invariance. The gauge model is however relatively complicated as it contains both gauge and scalar fields.

Our attempts to obtain the spinor action in the geometric framework were in the beginning not successful in the sense that coordinate dependent terms were absent. The reason was, as we show here, simple: we treated only fermions minimally coupled to gravity. But even for commutative manifolds in two dimensions, the action for the minimally coupled spinors has no explicit dependence on the connection, \([18]\): one has to couple spinors to torsion nonminimally. A construction of such an action is the main content of this paper, and as we shall see it gives a known Lagrangian: the Vignes-Tourneret (VT) model, \([7]\). In fact in the gravitational framework renormalisability is a natural consequence of the property well established in four dimensions: renormalisable theories on the curved backgrounds necessarily contain (specific) nonminimal interactions with the torsion and the curvature \([19]\).

The plan of the paper is the following: in section 2 we briefly review the properties of the noncommutative space we work with and calculate its torsion. In section 3 we introduce the action for the massive Dirac fermions and the nonminimal couplings to torsion, and reduce these actions to two dimensions. In section 4 we discuss our results.

2. The truncated Heisenberg space

We will briefly introduce the main geometric objects which are of relevance. The truncated Heisenberg algebra is defined by commutation relations

\[
\begin{align*}
[\mu x, \mu y] &= i\epsilon (1 - \mu z), \\
[\mu x, \mu z] &= i\epsilon (\mu y \mu z + \mu z \mu y), \\
[\mu y, \mu z] &= -i\epsilon (\mu x \mu z + \mu z \mu x).
\end{align*}
\]  

(2.7)

The \(\mu\) is a constant of dimension of the inverse length; \(\epsilon\) is a dimensionless parameter which indicates the strength of noncommutativity and we denote \(k = \epsilon \mu^2\). For \(\epsilon = 1\) algebra (2.7) has finite representations by \(n \times n\) matrices for every integer \(n\), \([16]\); \(\epsilon = 0\) is the commutative ‘limit’. We usually assume that parameters \(\mu\) and \(\epsilon\) are independent; \(\mu\) is related
to some relevant length or mass scale (like for example the cosmological constant), while $\epsilon$ is related to noncommutativity that is quantisation of spacetime. One can assume instead that the combination $\kappa$ gives the Planck length, $\kappa = l_P$.

Contraction $\mu \to 0$ of (2.7) gives the Heisenberg algebra

$$[x^\mu, x^\nu] = i\kappa\epsilon^{\mu\nu}, \quad \mu, \nu = 1, 2,$$

(2.8)

which has only infinite-dimensional representations. The relation between the Heisenberg and the truncated Heisenberg algebras can be seen in the Fock-space representation of the former as truncation of infinite matrices to the finite ones; the limit $n \to \infty$ is a weak limit. Although the limit is weak, it is in many aspects consistent to treat this limit as a projection to the subspace $z = 0$ of the initial space [20]. Symmetries of the algebra (2.7) are rotations in the $xy$-plane: their generator is $M = \mu^2x^2 + \mu^2y^2 + \mu z$. Parity, on the other hand, is not a symmetry and we shall see that after dimensional reduction parity breaking becomes manifest in the Dirac Lagrangian.

Besides coordinates we have on $\mathcal{A}$ vector fields and $p$-forms. In the approach which we use [11], the space of 1-forms is spanned by frame $\{\theta^a\}$ characterised by

$$[\phi, \theta^a] = 0$$

(2.9)

for any function $\phi(x)$. Dual to frame 1-forms $\theta^a$ are derivations $e_a$, $\theta^a(e_b) = \delta^a_b$. Differential $d$ of a function $f$ is defined through the $e_a$,

$$d\phi = (e_a, \phi) \theta^a = \left[p_a, \phi \right] \theta^a.$$  

(2.10)

The $e_a$ are always inner derivations in finite matrix spaces; their generators $p_a \in \mathcal{A}$ are called the momenta. We will assume that $e_a$ and $d$ are always of the form (2.10), and by convention, that $\theta^a$ are antihermitian. Condition (2.9) is sufficient to introduce consistently the metric which in the frame basis has constant components. In our particular geometry this metric is Euclidean,

$$g_{\alpha\beta} = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3.$$  

(2.11)

As seen from (2.10) the choice of momenta is equivalent to the choice of $d$, which means that the differential calculus is neither uniquely fixed, nor do we have a canonical one like the de Rham calculus in commutative geometry. We choose for the truncated Heisenberg space, [16],

$$e_1 = i\mu^2y, \quad e_2 = -i\mu^2x, \quad e_3 = i\mu(\mu z - \frac{1}{2}).$$

(2.12)

so that on $z = 0$ the differential projects to the usual differential on the Heisenberg algebra.

The momentum algebra can be used to define the exterior product of 1-forms and to extend this product to 2-forms, 3-forms and so on [11]. On space (2.7) we find

$$\begin{align*}
(\theta^1)^2 &= 0, \quad (\theta^2)^2 = 0, \quad (\theta^3)^2 = 0, \quad \{\theta^1, \theta^2\} = 0, \\
\{\theta^1, \theta^3\} &= i\epsilon (\theta^2\theta^3 - \theta^3\theta^2), \quad \{\theta^2, \theta^3\} = i\epsilon (\theta^3\theta^1 - \theta^1\theta^3).
\end{align*}$$

(2.13)

From (2.13) and associativity of the exterior product follow the rules of multiplication of three 1-forms:

On $z = 0$ this generator reduces to $M = \mu^2x^2 + \mu^2y^2 = i\epsilon (xp_2 - yp_1)$. 

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\[ \theta^1 \theta^2 \theta^1 = \theta^2 \theta^3 \theta^2, \quad \theta^2 \theta^3 \theta^1 = 0, \quad \theta^1 \theta^2 \theta^3 = 0, \]
\[ \theta^2 \theta^3 = - \theta^2 \theta^3 = \theta^3 \theta^2 \theta^1 = - \theta^3 \theta^2 \theta^1 = i \frac{\epsilon^2 - 1}{2\epsilon} \theta^2 \theta^3 \theta^2, \]
\[ \theta^1 \theta^3 \theta^2 = - \theta^3 \theta^1 \theta^1 = i \frac{\epsilon^2 + 1}{2\epsilon} \theta^2 \theta^3 \theta^2. \] (2.14)

From these relations we can see that there is only one (linearly independent) 3-form, which means that the volume element is well defined, that is, unique. We denote it by \( \Theta \) and choose
\[ \Theta = - \frac{i}{2\epsilon} \theta^2 \theta^3 \theta^2 \] (2.15)
in order that \( \Theta \) reduce to \( \theta^2 \theta^3 \theta^3 \) in the commutative limit: this is, in principle, important to properly identify the Lagrangian. We need in addition the Hodge-\( * \) operation. One possibility to define it is
\[ \star \left[ \theta^1, \theta^2 \right] = 2\theta^3, \quad \star \left[ \theta^2, \theta^3 \right] = 2\theta^1, \quad \star \left[ \theta^3, \theta^1 \right] = 2\theta^2; \] (2.16)
a short analysis of this definition and its properties is given in the appendix.

Other differential-geometric quantities like the affine connection, the torsion and the curvature can be defined as well. The connection 1-form used in [16] to define the parallel transport on the truncated Heisenberg space is
\[ \omega_{12} = - \omega_{21} = \left( - \frac{\mu}{2} + 2i\epsilon \mu \right) \theta^3 = \mu \left( \frac{1}{2} - 2\mu \right) \theta^3, \]
\[ \omega_{13} = - \omega_{31} = \frac{\mu}{2} \theta^2 + 2i\epsilon \mu \theta^3 = \frac{\mu}{2} \theta^2 + 2\mu^2 x \theta^3, \]
\[ \omega_{23} = - \omega_{32} = - \frac{\mu}{2} \theta^1 - 2i\epsilon \mu \theta^3 = - \frac{\mu}{2} \theta^1 + 2\mu^2 y \theta^3. \] (2.17)

This parallel transport preserves lengths, that is (2.17) is compatible with the metric (2.11).

Having the connection, the torsion and the curvature tensors are defined as usual:
\[ T^a = d\theta^a + \omega^a \beta \theta^\beta, \quad \Omega^a \beta = d\omega^a \beta + \omega^a \gamma \omega^\gamma \beta. \] (2.18)

Denoting
\[ \Omega^a \beta = \frac{1}{2} R^a \beta \gamma \theta^\gamma \theta^\beta, \] (2.19)
by contractions we obtain the scalar curvature of the truncated Heisenberg space,
\[ R = \eta^{\beta \gamma} R^a \beta \alpha \theta^\alpha \theta^\beta = \frac{\mu^2}{2} \left( 15 - 8\mu \gamma - 16\mu^2 \left( x^2 + y^2 \right) \right). \] (2.20)

For the torsion 2-form we find
\[ T^1 = - i \frac{\mu}{2} \left( 1 - 2\mu z \right) \left[ \theta^1, \theta^3 \right], \quad T^2 = - i \frac{\mu}{2} \left( 1 - 2\mu z \right) \left[ \theta^2, \theta^3 \right], \]
\[ T^3 = - i \epsilon \mu \left[ \theta^2, \theta^3 \right] + i \epsilon \mu \left[ \theta^1, \theta^3 \right]. \] (2.21)
and therefore we have
\[ \star T^1 = i \epsilon \mu \left( 1 - 2\mu z \right) \theta^2, \quad \star T^2 = - i \epsilon \mu \left( 1 - 2\mu z \right) \theta^1, \]
\[ \star T^3 = - 2i \epsilon \mu \left( 1 - 2\mu z \right) \theta^3. \] (2.22)
and
\[( \ast T_\alpha ) T^\alpha = 2 \mu^2 e^3 \left( 2 e^2 (1 - \mu z) - (1 - 2\mu z)^2 \right) \Theta. \]

(2.23)

3. Spinors on the truncated Heisenberg space

Let us briefly review expressions for the Dirac Lagrangian on a curved commutative space, mainly to fix the notation. We assume that the space is Euclidean,
\[\{ \gamma^\alpha, \gamma^\beta \} = 2 \delta^\alpha_\beta, \]

(3.24)

\(\alpha, \beta = 1, 2, 3\), so the \(\gamma\)-matrices are Hermitian. Dirac spinor \(\psi(x)\) transforms, under the local frame rotations, in the spinor representation: for an infinitesimal rotation \(L = \lambda_{\alpha \beta} g^\alpha g^\beta\)
the representation is given by \(S(L) = 1 + \frac{1}{2} \lambda_{\alpha \beta} \gamma^\alpha \gamma^\beta\). The covariant derivative is
\[D\psi = d\psi + \frac{1}{4} \omega^\beta \gamma^\alpha \gamma^\beta \psi = (D_\alpha \psi) \theta^\alpha, \]

(3.25)

that is,
\[D_\alpha \psi = e_\alpha \psi + \Gamma_\alpha \psi, \quad D_\alpha \psi = e_\alpha \psi - \bar{\psi} \Gamma_\alpha, \quad \Gamma_\alpha = \frac{1}{4} \omega^\beta \gamma^\alpha \gamma^\beta. \]

(3.26)

Since the group generators are Hermitian, \(\bar{\psi} = \psi^\dagger\). The Dirac operator, \(\mathcal{D} = \gamma^\alpha D_\alpha\) defines the action
\[S = \int \sqrt{g} \bar{\psi} (i \mathcal{D} - m) \psi. \]

(3.27)

It can be seen easily by partial integration that (3.27) is real only if the torsion vanishes, more precisely if
\[\omega^\alpha_{\alpha \gamma} = \frac{1}{\sqrt{g}} \partial_\mu \left( e^\mu_{\gamma} \sqrt{g} \right); \]

(3.28)

if not, the spinor action is defined by symmetrisation,
\[S = \frac{1}{2} (S + S^\ast). \]

(3.29)

Action (3.27) can be rewritten in the language of forms [21]. If we introduce a matrix-valued 1-form \(V = \theta^\alpha \gamma_\alpha\), in \(d\) dimensions we have
\[\int \text{Tr} (D\psi) \bar{\psi} V ... V \gamma_3 = -i (d - 1)! \int \Theta \bar{\psi} \gamma^\beta (D_\alpha \psi) \]

(3.30)

where \(\text{Tr}\) is the trace in \(\gamma\)-matrices; the product \(VV ... V\) contains \((d - 1)\) factors. On commutative spaces 1-forms anticommute so the Hermitian part of (3.30) is
\[S_{\text{kin}} = \frac{1}{2} \int \text{Tr} \left( (D\psi) \bar{\psi} - \psi (D\bar{\psi}) \right) VV ... V \gamma_3. \]

(3.31)
Similarly, the mass term of the Dirac action can be written using
\[ m \int \Theta \psi \phi V V \ldots V \gamma_5 = -id! \int \Theta m \phi \psi, \] (3.32)
where now the product of 1-forms \( V V \ldots V \) has \( d \) factors.

Since we wish to construct a Dirac spinor on a three-dimensional space and then reduce it to two dimensions we need both of the spinor representations. In two and three spacetime dimensions the irreducible spinor representations are two-dimensional. A natural choice of the \( \gamma \)-matrices in 2D are Pauli matrices \( \gamma_1 = \sigma_1, \gamma_2 = \sigma_2 \). From them we obtain the ‘\( \gamma_5 \) -matrix’, \( \gamma_5 = -i\gamma_1\gamma_2 = \sigma_3 \); it is the chirality operator. This representation is, up to unitary equivalence, unique. In three dimensions, Pauli matrices \( \gamma_\alpha = \sigma_\alpha, \alpha = 1, 2, 3 \), also give a representation. For the \( \gamma_5 \) we have, \( \gamma_4 = -i\gamma_1\gamma_2\gamma_3 = 1 \). The other, inequivalent irreducible representation is \( \gamma_1 = \sigma_1, \gamma_5 = \sigma_2, \gamma_3 = -\sigma_3, \) and \( \gamma_4 = -1 \).

Therefore on a commutative three-dimensional manifold the Dirac action is the sum of terms
\[ S_{\text{kin}} = \frac{1}{4} \int \Theta (D\phi)^2 - \psi (D\phi) V V \ldots V, \quad S_{\text{mass}} = \frac{i}{6} \int \Theta \phi \psi V V V V. \] (3.33)
Let us by analogy construct the Dirac action on noncommutative space (2.7); for simplicity we calculate the kinetic and mass terms separately. We have
\[ S_{\text{kin}}^* = -\frac{1}{2} \int \Theta \psi (D\phi)^2 V V = \frac{1}{2} \int \Theta \Xi_0 \gamma_5 \theta^a \theta^b, \] (3.34)
where we introduced
\[ \Xi_0 = \psi (D_0 \phi) = \psi ( (e_0 \phi) - \phi \Gamma_0), \]
\[ \Xi_1 = \psi \left( e_1 \phi - \frac{i\mu}{4} \psi \gamma_1 \right), \]
\[ \Xi_2 = \psi \left( e_2 \phi + \frac{i\mu}{4} \psi \gamma_2 \right), \]
\[ \Xi_3 = \psi \left( e_3 \phi - \frac{i\mu}{4} \psi \gamma_3 + i\mu^2 \psi (x\gamma_2 - y\gamma_1 + z\gamma_3) \right). \] (3.35)
Using (3.35) and the algebra of forms (2.14) we obtain
\[ S_{\text{kin}}^* = \frac{1}{2} \int \Theta \Theta \left( i \Xi_1 \gamma_1 + i \Xi_2 \gamma_2 + i (1 - \epsilon^2) \Xi_3 \gamma_3 - \epsilon \Xi_1 \gamma_2 + \epsilon \Xi_2 \gamma_1 \right) = \frac{1}{2} \int \Theta \left( i(e_1 \phi) \gamma_1 \psi + i(e_2 \phi) \gamma_2 \psi + i(1 - \epsilon^2)(e_3 \phi) \gamma_3 \psi \right) \]
\[ - \frac{\mu}{4} (1 + \epsilon^2) \psi \overline{\psi} + \frac{\mu\epsilon}{2} \psi \beta \psi - \mu^2 (1 - \epsilon^2) \psi \overline{\psi} \]
\[ - \epsilon (e_1 \phi) \gamma_2 \psi + \epsilon (e_2 \phi) \gamma_1 \psi - i\mu^2 (1 - \epsilon^2) \psi (x\gamma_1 + y\gamma_2) \psi. \] (3.36)
The terms in the last line are purely imaginary. Taking the Hermitian part and projecting to the subspace \( z = 0, \ e_3 \phi = 0 \), from (3.36) we obtain
\[ S_{\text{kin}} = \frac{1}{2} \int \Theta \left( i\psi \gamma^\alpha (e_\alpha \phi) - i(e_\alpha \phi) \gamma^\alpha \psi + \frac{1}{2} \mu \left( 1 + \epsilon^2 \right) \psi \overline{\psi} - \mu \epsilon \psi \beta \psi \right). \] (3.37)
where the summation is now in \( \alpha = 1, 2 \). We observe that as in the scalar case, a part of the connection components after dimensional reduction manifest as mass. Similarly, from (3.32)
we obtain
\[
S_{\text{mass}} = \frac{m}{6} \int \text{Tr} \, \psi \bar{\psi} \, VVV = -m \int \Theta \left( \left( 1 - \frac{\epsilon^2}{3} \right) \bar{\psi} \psi - \frac{2 \epsilon}{3} \bar{\psi} \gamma_5 \psi \right).
\] (3.38)

This expression is apparently the same before and after dimensional reduction, which is due to our somewhat imprecise notation. To be completely precise we should have denoted the volume element before the reduction by \( \Theta^{(3)} \) and after it by \( \Theta^{(2)} \). Similarly, we should have distinguished spinors in two and three dimensions notationally, writing for example \( \psi^{(2)} = \sqrt{Z} \psi^{(3)} \) where \( Z \) is the volume of the projected-out third dimension; in the interacting case \( Z \) renormalises the coupling constants too. However such details are quite straightforward and thus they are omitted.

In the absence of noncommutativity \( \epsilon = 0 \), mass term (3.38) reduces to the usual one. But noncommutative algebra (2.7) is not invariant under the space inversion: this is reflected in the \( \bar{\psi} \gamma_5 \psi \) terms in (3.37) and (3.38). Therefore the spinors of different chirality have different masses: for \( \epsilon = 1 \) for example, \( \bar{\psi} \gamma_5 \) in (3.38) is massless while \( m_L = 4 m/3 \). Parity breaking is present in the kinetic term also, so even in the massless case the reduction gives \( m_{L,R} = \mu (1 \pm \epsilon^2)/4 \).

As mentioned before, the minimally coupled Lagrangian expressed through (3.37) and (3.38) does not contain coordinate dependent terms. Therefore, if we wish to introduce the harmonic potential or some term similar in a consistent geometric way we have to include the nonminimal interaction with torsion. Various interaction terms are possible, [19]; from dimensional analysis and invariance arguments it follows that they have to be linear in torsion and bilinear in spinors. The interactions in 3D are
\[
S'_{\text{tor}} = \int \text{Tr} \, \psi \bar{\psi} \, T_\alpha \gamma^\alpha V, \quad S''_{\text{tor}} = \int \text{Tr} \, \psi \bar{\psi} \big( * \, T_\alpha \big) \gamma^\alpha VV,
\] (3.39)
but in fact these two expressions are proportional, \( S''_{\text{tor}} = 2iS'_{\text{tor}} \). We find in our case
\[
S'_{\text{tor}} = 2 \epsilon \int \Theta \, \bar{\psi} \left( (\epsilon - \gamma_5) \left( \left( \mu - 2 \mu^2 \pm \gamma_3 \right) \left( \mu^2 x^\gamma_5 - \mu^2 y^\gamma_5 \right) \right) \bar{\psi} \right.
\]
\[- \left. 2 i \epsilon^2 \int \Theta \, \bar{\psi} \left( \mu^2 x^\gamma_1 + \mu^2 y^\gamma_3 \right) \bar{\psi} \right).
\] (3.40)

There are two independent coupling terms, the real and the imaginary parts of \( S'_{\text{tor}} \). After projection to \( z = 0 \) we obtain
\[
S_{\text{tor}} = \frac{\eta}{2} \int \Theta \mu \left( \epsilon \, \bar{\psi} \psi - \bar{\psi} \gamma_5 \psi \right) + \frac{1}{2} \int \Theta \, \bar{\psi} \left( \eta \epsilon_{\alpha\beta} + \varepsilon \gamma_{\alpha\beta} \right) \mu^2 x^\alpha \gamma^\beta \psi,
\] (3.41)
for the interaction part of the action, where \( \eta \) and \( \varepsilon \) are arbitrary real coefficients and the summation is in \( \alpha = 1, 2 \).

In conclusion: the Lagrangian which describes the Dirac spinors on the truncated Heisenberg space after reduction to the Heisenberg algebra is given by
\[
\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{mass}} + \mathcal{L}_{\text{tor}}
\]
\[
= \frac{1}{2} \left( i \bar{\psi} \gamma^\alpha (e^\alpha \psi) - i (e^\alpha \bar{\psi} \gamma^\alpha \psi) \right) + \frac{1}{2} \bar{\psi} \left( \eta \epsilon_{\alpha\beta} + \varepsilon \gamma_{\alpha\beta} \right) \mu^2 x^\alpha \gamma^\beta \psi
\]
\[- \frac{m}{3} \left( 3 - \epsilon \right) \bar{\psi} \psi - 2 \epsilon \bar{\psi} \gamma_5 \psi + \frac{\mu}{4} \left( \left( 1 + 2 \eta \epsilon + \epsilon^2 \right) \bar{\psi} \psi - 2 (1 + \eta) \bar{\psi} \gamma_5 \psi \right) \right).
\] (3.42)
Writing it in the form $\mathcal{L} = \bar{\psi} \slashed{\mathcal{D}} \psi$ we find the corresponding Dirac operator

$$\slashed{\mathcal{D}} = i \gamma^a e_a - A - B \gamma_3 + \frac{1}{2} \left( \eta \gamma^{\alpha\beta} + \omega \delta^{\alpha\beta} \right) \mu^2 x^\alpha \gamma^\beta,$$

(3.43)

with

$$A = \frac{m}{3} \left( 3 - \epsilon^2 \right) - \frac{\mu}{4} \left( 1 + 2 \eta \epsilon + \epsilon^2 \right), \quad B = - \frac{2 m \epsilon}{3} + \frac{\mu}{2} (\eta + \epsilon),$$

(3.44)

and

$$\slashed{\mathcal{D}}^2 = - e_a \gamma^a - 2 A \gamma^a i e_a + \frac{1}{4} \left( \eta^2 + \omega^2 \right) \mu^2 x_\alpha x^\alpha$$

$$+ \left( A^2 + B^2 \right) + \left( 2 A B - \mu^2 \eta - \frac{1}{4} \mu^2 \epsilon \left( \eta^2 + \omega^2 \right) \right) \gamma_3$$

$$- A \left( \eta \gamma^{\alpha\beta} + \omega \delta^{\alpha\beta} \right) \mu^2 x_\alpha \gamma_\beta + \frac{1}{2} \left( \eta \gamma^{\alpha\beta} + \omega \delta^{\alpha\beta} \right) \{ i e_a, \mu^2 x_\beta \}.$$ 

(3.45)

The obtained $\slashed{\mathcal{D}}^2$ is a generalisation of the usual Lichnerowicz spinor Laplacian: it contains an additional dependence on the connection coming from the interaction with the torsion; there are also terms induced by dimensional reduction.

4. Discussion

The purpose and the results of this paper are twofold. We wished on the one hand to investigate the scope of the noncommutative-frames definition of noncommutative geometry and its capacity for physical description of fields and interactions. For this investigation the truncated Heisenberg algebra proved to be an interesting and nontrivial example, because it is a curved space with torsion with both discrete and continuous representations. On the other hand, as often pointed out in the literature, if a lower bound to measurement of distances is related to gravity, gravity is expected to regularise quantum field theories [22]. This was our second motivation, to investigate possible relations of renormalisable noncommutative models to noncommutative gravity. A specific relation of this kind was established before for the Grosse–Wulkenhaar model of the interacting scalar field [16]; in this paper the analysis is extended to spinors. We find that indeed the action for the massive Dirac spinors coupled to torsion can be straightforwardly related to the second renormalisable model, the noncommutative extension of the Gross–Neveu model, or the Vignes-Tourneret model [7]. Moreover, renormalisability of both models is in complete accordance with renormalisation properties of fields propagating on four-dimensional curved background spaces [19].

The equivalence of the Dirac action (3.42) with the Vignes-Tourneret model (1.3) is established easily. Comparing notations we identify $\theta = - k$; then the parameters of the spinor actions (1.3) and (3.42) are associated as

$$\hat{m} = A, \quad \kappa = B, \quad \Omega = \frac{\eta \epsilon}{4}, \quad \omega = 0.$$ 

(4.46)

In fact having calculated (2.23), we observe that the harmonic term of the GW model, instead as an interaction with the curvature scalar,

$$S_{\psi, \text{cur}} = \frac{\xi}{2} \int R \phi^2 = \frac{\xi \mu^2}{4} \int \left( 15 - 16 \mu^2 \left( x^2 + y^2 \right) \right) \phi^2,$$

(4.47)
can be interpreted as an interaction with the torsion. The corresponding interaction term projected to two dimensions is

\[ S_{\phi,\text{tor}} = \frac{\zeta}{2} \int T^\alpha (s T_i) \phi^2 = \zeta \mu^2 \epsilon^2 \int \left( 2 \epsilon^2 - 1 - 2 \mu^2 (x^2 + y^2) \right) \phi^2. \]

(4.48)

The physical effects of both interactions are similar: they introduce the harmonic potential and modify the scalar-field mass.

Thereby we have identified, in this particular geometry, three nonminimal interactions with the background gravity which are necessary and also sufficient for renormalisability of the scalar and spinor theories. Exactly three essential nonminimal couplings have been found in [19] in four commutative dimensions\(^7\): the corresponding coupling constants are related as \( \xi \mapsto \xi_0, \zeta \mapsto \zeta_0, \eta \mapsto \eta_0 \). Term ‘essential’ means that, in renormalisation, coupling constants \( \xi_0, \zeta_0 \) and \( \eta_0 \) run and cannot be set to zero: the corresponding interaction terms appear always as counterterms. Essentially the same result we have obtained in three noncommutative dimensions. We can thus conclude that the physical reason for appearance of coordinate dependent terms in the renormalisable Grosse–Wulkenhaar and Vignes-Tourneret models is the fact that they are defined on a curved background space. Along with that, we confirm that geometric attributes introduced through the noncommutative frame formalism are not only natural, but physically very meaningful.

There are other effects contained in our model which deserve further investigation: the creation of mass and the parity breaking. The fact that the gravitational field (seen as curvature, or torsion) manifests itself as inertia is intuitively clear; as a possible additional source of the particle mass here we have the dimensional reduction. The parity breaking is also not hard to understand: since we start from a three-dimensional space which is not invariant under the space inversion, the spinor Lagrangian does not have this symmetry either; the property remains after projection to two dimensions. Parity breaking is manifested as a difference between masses of the right and the left components of the Dirac field:

\[ m_{R,L} = A \pm B = \frac{m}{3} (1 \pm \epsilon) (3 \pm \epsilon) - \frac{\mu}{4} (1 \pm \epsilon) (1 \pm \epsilon \mp 2 \eta). \]

(4.49)

It is interesting to note that the effect of parity breaking can be produced solely by a coupling to torsion which can be seen by putting \( \epsilon = 0, \eta = 0 \). Both of the mentioned effects give interesting possibilities for model building in particle physics: they might provide for example new versions of the see-saw mechanism.

There are also points in the presented result which require better understanding, and this before all concerns the role of the dimensional reduction procedure. It would be interesting to perform quantisation of the original model in three dimensions and then compare to the results and interpretation obtained in two dimensions. From a technical point of view however, this would most probably be equivalent to a comparison of the results obtained in the matrix base and in the coordinate base.

\(^7\) In four dimensions the scalar and the spinor interact with the torsion pseudotrace, \( S_{\phi} = \epsilon_{\mu \nu \rho} T^{\mu \nu \rho} \); the essential interaction terms are of the form \( \xi \phi \psi \) and \( \eta \tilde{\gamma} \gamma_i \gamma_j \phi \psi \). These terms do not look the same in three dimensions because of the differences in the reduction of \( T^{\mu \nu} \) to invariant parts.
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Appendix: Hodge-* and the volume form

A part of the algebra of 3-forms (2.14), rewritten as

\[
\begin{align*}
\theta^1 \theta^2 = \theta^3 [\theta^1, \theta^2] &= 2 \left(1 - \epsilon^2\right) \Theta, \\
\theta^2 \theta^3 = \theta^1 [\theta^2, \theta^3] &= 2 \Theta, \\
\theta^3 \theta^1 = \theta^2 [\theta^3, \theta^1] &= 2 \Theta,
\end{align*}
\]

suggests definition (2.16) the *-operation. This definition is further in accordance with the usual convention for the double action of the Hodge-* in three Euclidean dimensions is \(\ast (\ast \omega) = \omega\) for all \(p\)-forms \(\omega\). It looks however as if (2.16) changes the usual rules related to the volume element as for example

\[
\left(\ast \left(\frac{1}{2} [\theta^1, \theta^2]\right)\right) \frac{1}{2} [\theta^1, \theta^2] = (1 - \epsilon^2) \Theta, \\
\left(\ast \left(\frac{1}{2} [\theta^2, \theta^3]\right)\right) \frac{1}{2} [\theta^2, \theta^3] = \Theta.
\]

At this point we should recall that, in the noncommutative case, the commutators of 1-forms are not natural as a basis in the space of 2-forms: we should rather use the twisted commutators, \(\tilde{\theta}^{\alpha \beta} = P^{\alpha \beta}_{\gamma \delta} \theta^\gamma \theta^\delta\), as twisted commutators enclose the properties of the noncommutative product. We have, [17]

\[
\begin{align*}
\theta^1 \theta^2 &= \tilde{\theta}^{12} = P^{12}_{\gamma \delta} \theta^\gamma \theta^\delta = \frac{1}{2} \left[ \theta^1, \theta^2 \right], \\
\theta^1 \theta^3 &= \tilde{\theta}^{13} = P^{13}_{\gamma \delta} \theta^\gamma \theta^\delta = \frac{1}{2} \left[ \theta^1, \theta^3 \right] + \frac{i \epsilon}{2} \left[ \theta^2, \theta^3 \right], \\
\theta^2 \theta^3 &= \tilde{\theta}^{23} = P^{23}_{\gamma \delta} \theta^\gamma \theta^\delta = \frac{1}{2} \left[ \theta^2, \theta^3 \right] - \frac{i \epsilon}{2} \left[ \theta^1, \theta^3 \right].
\end{align*}
\]

The main drawback of the basis \(\{\tilde{\theta}^{\alpha \beta}\}\) is that is not Hermitian. Applied to this basis elements the Hodge-* gives

\[
\left(\ast \tilde{\theta}^{12}\right) \theta^{12} = \left(\ast \tilde{\theta}^{13}\right) \theta^{13} = \left(\ast \tilde{\theta}^{23}\right) \theta^{23} = \theta^i \theta^j \theta^k
\]

as one would expect; also, the order of factors does not matter. From (4.53) we see that in fact the volume 3-form should have been identified as

\[
\tilde{\Theta} = \theta^i \theta^j \theta^k = (1 - \epsilon^2) \Theta.
\]

However because of non-hermiticity, we have for example \((\ast \tilde{\theta}^{13}) \tilde{\theta}^{12} = 0\) but rather

\[
\left(\ast \tilde{\theta}^{13}\right) \tilde{\theta}^{12} + \tilde{\theta}^{12} \left(\ast \tilde{\theta}^{13}\right) = 0.
\]

Analogous relations hold for other components. We have not in our calculation redefined the volume 3-form \(\Theta\) to \(\tilde{\Theta}\), as this redefinition changes only the overall factor in the action. But formula (4.54) shows that the limit to the matrix case, \(\epsilon = 1\), is not smooth; the space of \(p\)-forms becomes a kind of fragmented. In a similar way the commutative limit, \(\epsilon = 0\), is singular because in this limit the momenta \(p_m\), (1.6) diverge. Perhaps a more detailed analysis of the tangent and cotangent spaces for \(\epsilon = 1\) could reveal some interesting or characteristic
properties of the matrix geometries or the quantum groups, of which the truncated Heisenberg algebra is one, somewhat exotic, example.

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