Weak-Pseudo-Hermiticity of Non-Hermitian Hamiltonians with Position-Dependent Mass

S.-A. Yahiaoui, M. Bentaiba
LPTHIRM, Département de Physique, Faculté des Sciences, Université Saad DAHLAB de Blida, Algeria.

Abstract

We extend the definition of \( \eta \)-weak-pseudo-Hermiticity to the class of potentials endowed with position-dependent mass. The construction of non-Hermitian Hamiltonians through some generating function are obtained. Special cases of potentials are thus deduced.

Keywords: \( \eta \)-weak-pseudo-Hermiticity; Non-Hermitian Hamiltonians; \( \mathcal{PT} \)–symmetry; Effective mass.

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1 Introduction

The Hamiltonians are called \( \mathcal{PT} \)-invariant if they are invariant under a joint transformation of parity \( \mathcal{P} \) and time-reversal \( \mathcal{T} \) [1-8]. A conjecture due to Bender and Boettcher [1] has relaxed \( \mathcal{PT} \)-symmetry as a necessary condition for the reality of the spectrum. Here, the Hermiticity assumption \( \mathcal{H} = \mathcal{H}^\dagger \) is replaced by the \( \mathcal{PT} \)-symmetric one; i.e. \( [\mathcal{PT}, \mathcal{H}] = 0 \), where \( \mathcal{P} \) denotes the parity operator (space reflection) and has as effects: \( x \rightarrow -x \), \( p \rightarrow -p \) and \( \mathcal{T} \) mimics the time-reversal and has as effects: \( x \rightarrow x \), \( p \rightarrow -p \), and \( i \rightarrow -i \). Note that \( \mathcal{T} \) changes the sign of \( i \) because it preserves the fundamental commutation relation of the quantum mechanics known as the Heisenberg algebra, i.e. \( [x, p] = i\hbar \) [1-3].

\[ \text{Corresponding author:} \]
E-mail address: bentaiba@hotmail.com
bentaiba1@caramail.com
According to Mostafazadeh [9-12], the basic mathematical structure underlying the properties of $\mathcal{PT}$-symmetry is explored and can now be found to be connected to the concept of a pseudo-Hermiticity. The pseudo-Hermiticity has been found to be a more general concept than those of Hermiticity and $\mathcal{PT}$-symmetry. As a consequence of this, the reality of the bound-state eigenvalues can be associated with it.

In terms of these settings, a Hamiltonian $\mathcal{H}$ is called pseudo-Hermitian if it obeys to [9,11]

$$\mathcal{H}^\dagger = \eta \mathcal{H} \eta^{-1},$$  

where $\eta$ is a Hermitian invertible linear operator and a dagger ($^\dagger$) stands for the adjoint of the corresponding operator. A non-Hermitian Hamiltonian has a real spectrum if and only if it is pseudo-Hermitian with respect to a linear Hermitian automorphism [10], and may be factored as

$$\eta = \mathcal{D}^\dagger \mathcal{D},$$  

where $\mathcal{D} : \mathfrak{H} \to \mathfrak{H}$ is a linear automorphism ($\mathfrak{H}$ is the Hilbert space). Note that choosing $\eta = 1$ reduces the assumption (1) to the Hermiticity of the Hamiltonian.

On the other hand, Bagchi and Quesne [13] have established that the twin concepts of pseudo-Hermiticity and weak-pseudo-Hermiticity are complementary to one another. In the pseudo-Hermiticity case, $\eta$ can be written as a first-order differential operator and may be anti-Hermitian, while in the weak-pseudo-Hermitian case, $\eta$ is a second-order differential operator and must be necessarily Hermitian.

The quantum mechanical systems with position-dependent mass have attracted, in recent years, much attention on behalf of physicists [15-20]. The effective mass Schrödinger equation was first introduced by BenDaniel and Duke in order to explain the behaviors of electrons in semi-conductors [15]. It also have many applications in the fields of materials science and condensed matter physics [20,21].

In the present paper, a class of non-Hermitian Hamiltonians, known in the literature, as well as their accompanying ground-state wavefunctions are generated as a by-product of the generalized $\eta$–weak-pseudo-Hermiticity endowed with position-dependent mass. Here our primary
concern is to point out that, being different from the realization of Ref.[13] considering therein $A ( x )$ as a pure imaginary function, there is no inconsistency if a shift on the momentum $p$ of the type $p \rightarrow p - \frac{A ( x )}{U ( x )}$ is used, where $A ( x )$ and $U ( x ) ( \neq 0 )$ are, respectively, complex- and real-valued functions. It opens a way towards the construction of non-Hermitian Hamiltonians (not necessarily $\mathcal{PT}$-symmetric). On these settings, Eq.(2) becomes $\eta \rightarrow \tilde{\eta} = \tilde{D} \tilde{D}^\dagger$. Such operator, i.e. $\tilde{D}$, may be looked upon as a gauge-transformed version of $\mathcal{D}$, depending essentially on the function $A ( x )$. Consequently, it is found that the wavefunction is also subjected to a gauge transformation of the type $\psi ( x ) \rightarrow \xi ( x ) = \Lambda ( x ) \psi ( x )$ where $\Lambda ( x ) = \exp \left[ i \int^x dy \frac{A ( y )}{U ( y )} \right]$.

2 Generalized pseudo-Hermitian Hamiltonians

The general form of the Hamiltonian introduced by von Roos [16] for the spatially varying mass $M ( x ) = m_0 m ( x )$ reads

$$\mathcal{H} = \frac{1}{4} \left[ m^\alpha ( x ) p m^\beta ( x ) p m^\gamma ( x ) + m^\gamma ( x ) p m^\beta ( x ) p m^\alpha ( x ) \right] + V ( x ), \quad (3)$$

where the constraint $\alpha + \beta + \gamma = -1$ holds and $V ( x ) = V_{\text{Re}} ( x ) + i V_{\text{Im}} ( x )$ is a complex-valued potential. Here, $p \left( = -i \frac{d}{dx} \right)$ is a momentum with $\hbar = m_0 = 1$, and $m ( x )$ is dimensionless real-valued mass function.

Using the restricted Hamiltonian from the $\alpha = \gamma = 0$ and $\beta = -1$ constraints, the Hamiltonian (3) becomes

$$\mathcal{H} = p U^2 ( x ) p + V ( x ), \quad (4)$$

with $U^2 ( x ) = \frac{1}{2m ( x )}$. The shift on the momentum $p$ in the manner

$$p \rightarrow p - \frac{A ( x )}{U ( x )}, \quad (5)$$
where \( A : \mathbb{R} \to \mathbb{C} \) is a complex-valued function, allows to bring the Hamiltonian of Eq.(4) in the form

\[
\mathcal{H} \to \mathcal{H}' = \left[ p - \frac{A(x)}{U(x)} \right] U^2(x) \left[ p - \frac{A(x)}{U(x)} \right] + V(x).
\]  

(6)

In Ref.[11], it was showed that for every anti-pseudo-Hermitian Hamiltonian \( \mathcal{H} \), there is an antilinear operator \( \tau \) fulfilling the condition

\[
\mathcal{H}^\dagger = \tau \mathcal{H} \tau^{-1}.
\]  

(7)

Let us extend the proof of Ref.[12] to our Hamiltonian (6). To this end, \( \tau \) should be constructed suitably. According to Mostafazadeh [12], \( \tau = \mathcal{T} e^{i\alpha(x)} \) is the product of linear and antilinear operators, and \( \alpha : \mathbb{R} \to \mathbb{C} \) is a complex-valued function. Therefore, the Hermiticity of \( \tau \) is established straightforwardly

\[
\tau^\dagger = e^{-i\alpha^*(x)} \mathcal{T}^\dagger = e^{-i\alpha^*(x)} \mathcal{T} = \mathcal{T} e^{i\alpha(x)} = \tau,
\]  

(8)

where the identities \( \mathcal{T}^\dagger = \mathcal{T} \) and \( \mathcal{T} f(x) \mathcal{T} = f^*(x) \) are used and \( f : \mathbb{R} \to \mathbb{C} \).

According to Mostafazadeh in Ref.[12], the function \( \alpha(x) \) can be gen-
eralized to \( \alpha(x) = -2 \int x \, dy \frac{A(y)}{U(y)} \), therefore

\[
\tau \mathcal{H} \tau^{-1} = T e^{i \alpha(x)} \left[ p - \frac{A(x)}{U(x)} \right] U^2(x) \left[ p - \frac{A(x)}{U(x)} \right] e^{-i \alpha(x)} T
\]

\[
+ T e^{i \alpha(x)} V(x) e^{-i \alpha(x)} T
\]

\[
= T \left[ p - \frac{A(x)}{U(x)} - \partial_x \alpha \right] e^{i \alpha(x)} U^2(x) e^{-i \alpha(x)} \left[ p - \frac{A(x)}{U(x)} - \partial_x \alpha \right] T
\]

\[
+ V^*(x)
\]

\[
= T \left[ p - \frac{A(x)}{U(x)} - \partial_x \alpha \right] U^2(x) \left[ p - \frac{A(x)}{U(x)} - \partial_x \alpha \right] T + V^*(x)
\]

\[
= T \left[ p + \frac{A^*(x)}{U(x)} \right] U^2(x) \left[ -p + \frac{A^*(x)}{U(x)} \right] + V^*(x)
\]

\[
= \left[ p - \frac{A^*(x)}{U(x)} \right] U^2(x) \left[ p - \frac{A^*(x)}{U(x)} \right] + V^*(x)
\]

\[
= \mathcal{H}^t,
\]

where for every differential function \( \alpha(x) \), the following identity holds

\[
e^{-i \alpha(x)} p e^{i \alpha(x)} = p + \partial_x \alpha(x)
\]

while the position \( x \) commutes with \( e^{i \alpha(x)} \) and remains unaffected under a last transformation; i.e. \( e^{-i \alpha(x)} x e^{i \alpha(x)} = x \).

Here we note that for every function \( f : \mathbb{R} \to \mathbb{C} \), the identity \( T f(x,p) \mathcal{T} = f^*(x,-p) \) is used.

In the other hand, and according to Ref.[11], it was checked that \( \mathcal{PT} \)–symmetry \((\mathcal{PT}, \mathcal{H}) = 0\) and anti-pseudo-Hermiticity operator \( \tau \) imply pseudo-Hermiticity of \( \mathcal{H} \) with the respect of a linear Hermitian automorphism \( \eta : \mathcal{H} \to \mathcal{H} \) according to

\[
\eta = \tau \mathcal{PT},
\]

and it turns out that the choice of \( \eta \) is not unique. As was made for \( \tau \), let us generalize \( \eta \) according to

\[
\eta = \exp \left[ 2i \int x \, dy \frac{A^*(y)}{U(y)} \right] \mathcal{P},
\]
then the Hermiticity of $\eta$ is established straightforwardly

$$
\eta^\dagger = \mathcal{P} \exp \left[ -2i \int^x dy \frac{A(y)}{U(y)} \right] = \exp \left[ -2i \int^{-x} dy \frac{A(y)}{U(y)} \right] \mathcal{P}
$$

$$
= \exp \left[ 2 \int^x dy i \text{Re} A(-y) - \text{Im} A(-y) \right] \mathcal{P}
$$

$$
= \exp \left[ 2 \int^x dy \frac{i \text{Re} A(y) + \text{Im} A(y)}{U(y)} \right] \mathcal{P}
$$

$$
= \exp \left[ 2i \int^x dy \frac{A^*(y)}{U(y)} \right] \mathcal{P}
$$

$$
= \eta,
$$

(12)

where we use $\mathcal{P}^\dagger = \mathcal{P}$ and, for every function $f : \mathbb{R} \rightarrow \mathbb{C}$, the following identity holds $\mathcal{P} f(x) \mathcal{P} = f(-x)$. In Eq.(12), the real and imaginary parts of $A(x)$ are, respectively, even and odd functions; i.e. $\text{Re} A(-x) = \text{Re} A(x)$, $\text{Im} A(-x) = -\text{Im} A(x)$ and $U(x)$ must be an even function, i.e. $U(x) = U(-x)$.

In summary, the $\mathcal{P}\mathcal{T}$-symmetry and anti-pseudo-Hermiticity with respect to $\tau$ imply pseudo-Hermiticity with respect to $\tau\mathcal{P}\mathcal{T}$ and which coincides with the $\eta$ operator [11]. Therefore, it is obvious that the (weak-) pseudo-Hermiticity as defined in Eq.(10) adapts very well to the problems relating with position-dependent effective mass.
3 The generalized weak-pseudo-Hermiticity generators

As \( \eta \) is weak-pseudo-Hermitian, then the operators \( D \) and \( D^\dagger \) are connected to the first-order differential operator through [14]

\[
D = U(x) \partial_x + \phi(x),
\]

\[
\eta = \text{weak-pseudo-Hermitian}
\]

\[
D^\dagger = -\partial_x U(x) + \phi^*(x),
\]

\[
\eta = \text{weak-pseudo-Hermitian}
\]

where we have used the abbreviation \( \partial_x = \frac{d}{dx} \). Here \( \phi : \mathbb{R} \to \mathbb{C} \) is a complex-valued function. It is obvious that the operator \( D \) becomes, under transformation (5),

\[
\tilde{D} = iU(x) \left[ p - \frac{A(x)}{U(x)} \right] + \phi(x),
\]

\[
\tilde{D}^\dagger = -i\partial_x U(x) + \phi^*(x),
\]

\[
\tilde{\eta} = \text{weak-pseudo-Hermitian}
\]

Therefore, the operator \( \tilde{D} \) may be looked upon as a gauge-transformed version of \( D \), depending on \( A(x) \) such that \( \tilde{D} = D - iA(x) \). In terms of these, \( \tilde{\eta} \) becomes

\[
\tilde{\eta} = \tilde{D}^\dagger \tilde{D} = \left[ D^\dagger + iA^*(x) \right] \left[ D - iA(x) \right] = D^\dagger D - iD^\dagger A(x) + iA^*(x)D + A^*(x)A(x),
\]

and taking into account that \( \phi(x) = f(x) + ig(x) \) and \( A(x) = a(x) + ib(x) \), (15) can be recast as

\[
\tilde{\eta} = D^\dagger D + 2iuU(x) a(x) \partial_x + i [U(x) A(x)]' - i\phi^*(x) A(x) + i\phi(x) A^*(x) + |A(x)|^2,
\]

where prime denotes derivative with respect to \( x \). At this point, let us now
evaluate $\eta$ appearing in Eq.(16) using Eq.(13), we obtain

$$D^\dagger D = \left[ -\partial_x U (x) + \phi (x) \right] \left[ U (x) \partial_x + \phi (x) \right]$$
$$= -U^2 (x) \partial_x^2 - 2 U (x) \left[ U' (x) + i g (x) \right] \partial_x + |\phi (x)|^2$$
$$- [U (x) \phi (x)]', \quad (17)$$

Combining Eq.(17) with Eq.(16), we obtain a second-order differential operator of

$$\tilde{\eta} = -U^2 (x) \partial_x^2 - 2 K (x) \partial_x + \mathcal{L} (x), \quad (18)$$

where $K (x)$ and $\mathcal{L} (x)$ are defined as

$$K (x) = U (x) U' (x) + i U (x) g (x) - i U (x) a (x), \quad (19.a)$$
$$\mathcal{L} (x) = |\phi (x)|^2 + |A (x)|^2 - [U (x) \phi (x)]' + i [U (x) A (x)]'$$
$$- i \phi^* (x) A (x) + i \phi (x) A^* (x). \quad (19.b)$$

One can easily check that $\tilde{\eta}$ given in Eq.(18) is, indeed, Hermitian since it is written in the form $\tilde{\eta} = \tilde{D}^\dagger \tilde{D}$. On the other hand, taking into account $p = -i \partial_x$, the Hamiltonian of Eq.(6) may be expressed as

$$\mathcal{H}' = -U^2 (x) \partial_x^2 - 2 M_1 (x) \partial_x + N_1 (x) + V (x), \quad (20)$$

where, by definition

$$M_1 (x) = U (x) U' (x) - i U (x) A (x), \quad (21.a)$$
$$N_1 (x) = i [U (x) A (x)]' + A^2 (x). \quad (21.b)$$

The adjoint of the Hamiltonian (20) reads as

$$\mathcal{H}'^\dagger = -U^2 (x) \partial_x^2 - 2 M_2 (x) \partial_x + N_2 (x) + V^* (x), \quad (22)$$

with

$$M_2 (x) = U (x) U' (x) - i U (x) A^* (x), \quad (23.a)$$
$$N_2 (x) = i [U (x) A^* (x)]' + A^{*2} (x). \quad (23.b)$$
It should be noted that $D$ and $D^\dagger$ are two intertwining operators, therefore, the defining condition (1) may be expressed as $\eta H = H^\dagger \eta$. Thereupon, a generalization beyond the pair $\tilde{\eta}$ and $H'$ is straightforward, given

$$\tilde{\eta} H' = H'^\dagger \tilde{\eta}. \quad (24)$$

Letting both sides of (24) act on every function, e.g. on a wavefunction. Using Eqs.(18), (20), (22) and comparing between their varying differential coefficients, we can easily recognize from the coefficients corresponding to the third derivative that $A(x)$ must be real function, i.e. $b(x) = 0$.

By comparing both coefficients corresponding to the second derivative, one deduces the expression connecting the potential to its conjugate through

$$V(x) = V^*(x) - 4iU(x)g'(x). \quad (25)$$

On the other hand, the coefficients corresponding to the first derivative give the shape of the potential

$$V''(x) = 2f(x)f'(x) - 2g(x)g'(x) - [U(x)f(x)]'' + 2i[U(x)g'(x)]', \quad (26)$$

and by integrating Eq.(26) taking into account its conjugate, we get

$$V(x) \equiv V_{\Re}(x) + iV_{\Im}(x) = f^2(x) - g^2(x) - [U(x)f(x)]' - 2iU(x)g'(x) + \delta, \quad (27)$$

with $\delta$ is a constant of integration. It is obvious that both imaginary parts of Eqs.(25) and (27) coincide.

The last remaining coefficients correspond to the null derivative and give the following pure-imaginary expression

$$-4U(x)f(x)f'(x)g'(x) - 4U(x)f^2(x)g'(x) + 4U^2(x)f'(x)g'(x)$$

$$+ 4U(x)U'(x)f'(x)g(x) + 4U(x)U'(x)f(x)g'(x) + 2U^2(x)f''(x)g(x)$$

$$+ 3U^2(x)U''(x)g''(x) + 2U(x)U''(x)f(x)g(x) - U^2(x)U''(x)g'(x)$$

$$- 2U(x)U'(x)U''(x)g(x) + U^3(x)g'''(x) - U^2(x)U'''(x)g(x) = 0. \quad (28)$$
Using Eq.(24) together with the eigenvalues of the Schrödinger equation for the Hamiltonian and its adjoint, namely $\mathcal{H}'|\xi_i\rangle = \mathcal{E}'_i|\xi_i\rangle$ and $\langle \xi_j | \mathcal{H}'^\dagger = \langle \xi_j | \mathcal{E}'_j^*$, where $|\xi_q\rangle \in \mathcal{H}$ ($q = i, j$), and then multiplying them by $\tilde{\eta}$ on the left- and right-hand sides, respectively, we can easily obtain due to Eq.(24), on subtracting, that any two eigenvectors $|\xi_i\rangle$ and $|\xi_j\rangle$ satisfy

\[
\langle \xi_j | (\mathcal{H}'^\dagger \tilde{\eta} - \tilde{\eta} \mathcal{H}') | \xi_i \rangle = \langle \xi_j | (\mathcal{E}'_j^* \tilde{\eta} - \mathcal{E}'_i \tilde{\eta}) | \xi_i \rangle = (\mathcal{E}'_j^* - \mathcal{E}'_i) \langle \xi_j | \tilde{\eta} | \xi_i \rangle = (\mathcal{E}'_j^* - \mathcal{E}'_i) \langle \xi_j \parallel \xi_i \rangle_{\tilde{\eta}} \equiv 0,
\]

(29)

where $\langle \xi_j \parallel \xi_i \rangle_{\tilde{\eta}} \equiv \langle \xi_j | \tilde{\eta} | \xi_i \rangle$ is the Hermitian indefinite inner product of the Hilbert space $\mathcal{H}$ defined by $\tilde{\eta} [9,11]$. According to the proposition 2 in Ref.[9], a direct implication of Eq.(29) has the following properties

(i) The eigenvectors with non-real eigenvalues have a vanishing $\eta$–norm, i.e. $\mathcal{E}'_i \notin \mathbb{R}$ implies that $\| \langle \xi_i \parallel \xi_i \rangle_{\tilde{\eta}} = \langle \xi_i \parallel \xi_i \rangle_{\tilde{\eta}} = 0$.

(ii) Any two eigenvectors are $\eta$–orthogonal unless their eigenvalues are complex conjugates, i.e. $\mathcal{E}'_i \neq \mathcal{E}'_i^*$ implies that $\langle \xi_i \parallel \xi_j \rangle_{\tilde{\eta}} = 0$.

The inner product $\langle \cdot \parallel \cdot \rangle_{\tilde{\eta}}$ is generally positive-definite, i.e. $\langle \cdot \parallel \cdot \rangle_{\tilde{\eta}} > 0$. Thus, the Hilbert space equipped with this inner product may be identified as the physical Hilbert space $\mathcal{H}_{\text{phys}} [1-3]$. Therefore, according to Eq.(29), it is obvious that $\mathcal{E}' = \mathcal{E}'^*$. Hence, the eigenvalue $\mathcal{E}'$ is real, i.e. $\mathcal{E}'_{\text{Im}} = 0$. In terms of these, $\eta$–orthogonality suggests that the eigenvector (wavefunction), here $\xi(x)$, is related to $\mathcal{H}'$ through the identity $\tilde{\eta} \xi(x) = 0 [14]$, i.e.

\[
\tilde{D} \xi(x) = 0,
\]

(30)

and keeping in mind Eq.(14), and after integration, we obtain the ground-
state wavefunction (not necessarily normalizable)

\[ \xi(x) = \Lambda(x) \psi(x) = \exp \left[ i \int^x dy \frac{A(y)}{U(y)} \right] \psi(x) \propto \exp \left[ - \int^x dy \frac{f(y)}{U(y)} - i \int^x dy \frac{g(y) - a(y)}{U(y)} \right], \tag{31} \]

where \( \psi(x) \) is the ground-state wavefunction when the restriction \( A(x) = 0 \) holds. Then \( \xi(x) \), as for \( \tilde{D} \), is also subjected to a gauge transformation in the manner of \( \psi(x) \rightarrow \xi(x) = \Lambda(x) \psi(x) \).

In these settings, letting \( \tilde{D} \) acts on both sides of (31), we obtain

\[ \tilde{D}\xi(x) \equiv [U(x) \partial_x - iA(x) + \phi(x)] \Lambda(x) \psi(x) = U(x) \Lambda'(x) \psi(x) + U(x) \Lambda(x) \psi'(x) - iA(x) \Lambda(x) \psi(x) + \phi(x) \Lambda(x) \psi(x) = \Lambda(x) [U(x) \partial_x + \phi(x)] \psi(x) \implies D\psi(x) = 0, \tag{32} \]

where \( \Lambda'(x) = i \frac{A(x)}{U(x)} \Lambda(x) \). That means that the wavefunctions thus obtained can be deduced either by \( \tilde{D}\xi(x) = 0 \) or by \( D\psi(x) = 0 \).

In the remainder of the article, we write \( \mathcal{E} \) instead of \( \mathcal{E}' \). Now, using the Schrödinger equation \( \mathcal{H}'\xi(x) = \mathcal{E}\xi(x) \), with \( \mathcal{H}' \) given in Eq.(20), \( \xi(x) \) in Eq.(31) and \( \mathcal{E} = \mathcal{E}_{\text{Re}} + i\mathcal{E}_{\text{Im}} \), we end up by relating \( f(x) \) to \( g(x) \) and \( U(x) \) through

\[ f(x) = \frac{U'(x) g(x) - U(x) g'(x)}{2g(x)}, \tag{33} \]

where for the sake of simplicity we considere \( \delta \equiv \mathcal{E}_{\text{Re}} \). Hence, it becomes clear that \( g(x) \) is our generating function leading to identify the function \( f(x) \), and then the potential \( V(x) \).

This in turn leads to the following question. Is (33) the equation connecting \( f(x) \) to the generating function \( g(x) \)? The answer to this question amounts to check for the satisfaction of Eq.(28). It is then straightforward,
after a long calculation, to be convinced that \( f(x) \), as defined in (33), is a farfetched function (solution).

In order to deal with position-dependent mass, we introduce the auxiliary function defined by the mapping \( \mu(x) \equiv \int_x^y \frac{dy}{U(y)} \), where \( \mu(x) \) is a dimensionless mass integral which will appear frequently in subsequent developments. The function \( f(x) \) can be written as

\[
f(x) = -\frac{g'(x)}{2\mu'(x)g(x)} - \frac{\mu''(x)}{2\mu'^2(x)},
\]

and the potential \( V(x) \) acquires the form

\[
V_{\text{eff}}(x) - E_{\text{Re}} = -g^2(x) - \frac{g'^2(x)}{4g^2(x)\mu'^2(x)} + \frac{g''(x)}{2g(x)\mu'^2(x)} - \frac{g'(x)\mu''(x)}{2g(x)\mu'^3(x)}
- 2i\frac{g'(x)}{\mu'(x)},
\]

where \( V_{\text{eff}}(x) \) is called the effective potential and is related to \( V(x) \) by

\[
V(x) = V_{\text{eff}}(x) - V_\mu(x),
\]

with

\[
V_\mu(x) = \frac{\mu'''(x)}{\mu'^3(x)} - \frac{5\mu''^2(x)}{4\mu'^4(x)}.
\]

4 Effective potentials and corresponding wavefunctions

The strategy to determine both effective potentials and ground-state wavefunctions is as follows. As \( g(x) \) is a generating function, all expressions depend on it. We may choose various generating functions \( g(x) \) and obtain all others expressions such as \( f(x) \), \( V_{\text{eff}}(x) \) and \( \tilde{\eta} \). Knowing \( f(x) \) and \( g(x) \), the proper ground-state wavefunctions can be found from Eq.(32), i.e. without the gauge-term. Without giving the details of our calculation which are straightforward, we present the results of various expressions in standard form.
4.1 3D–Harmonic oscillator potential

\[ g(x) = \alpha \mu(x), \quad (38.a) \]

\[ f(x) = -\frac{1}{2\mu(x)} - \frac{\mu''(x)}{2\mu'^2(x)}, \quad (38.b) \]

\[ V_{\text{HO}}(x) = -\alpha^2\mu^2(x) - \frac{1}{4\mu^2(x)} - 2i\alpha, \quad (38.c) \]

\[ \psi^{(0)}_{\text{HO}}(x) \propto \frac{\sqrt{\mu(x)}}{U(x)} \exp \left[ -\frac{i\alpha}{2} \mu^2(x) \right]. \quad (38.d) \]

4.2 Morse potential

\[ g(x) = \exp [-\alpha \mu(x)], \quad (39.a) \]

\[ f(x) = \frac{\alpha}{2} - \frac{\mu''(x)}{2\mu'^2(x)}, \quad (39.b) \]

\[ V_{\text{M}}(x) = -\exp [-2\alpha \mu(x)] + 2i\alpha \exp [-\alpha \mu(x)] + \frac{\alpha^2}{4}, \quad (39.c) \]

\[ \psi^{(0)}_{\text{M}}(x) \propto \frac{1}{U(x)} e^{-\frac{1}{2} \mu(x)} \exp \left[ \frac{2i}{\alpha} e^{-\alpha \mu(x)}(x) \right]. \quad (39.d) \]

4.3 Scarf II potential

\[ g(x) = \text{sech} [\alpha \mu(x)], \quad (40.a) \]

\[ f(x) = \frac{\alpha}{2} \tanh [\alpha \mu(x)] - \frac{\mu''(x)}{2\mu'^2(x)}, \quad (40.b) \]

\[ V_{\text{Sc}}(x) = -\left( 1 + \frac{3\alpha^2}{4} \right) \text{sech}^2 [\alpha \mu(x)] \]

\[ + 2i\alpha \text{sech} [\alpha \mu(x)] \tanh [\alpha \mu(x)] + \frac{\alpha^2}{4}, \quad (40.c) \]

\[ \psi^{(0)}_{\text{Sc}}(x) \propto \frac{1}{U(x) \sqrt{\cosh [\alpha \mu(x)]}} \exp \left[ -\frac{i}{\alpha} \arctan \tanh \frac{\alpha}{2} \mu(x) \right]. \quad (40.d) \]
4.4 Generalized Pöschl-Teller potential

\[ g(x) = \text{cosech} [\alpha \mu(x)], \quad (41.a) \]
\[ f(x) = \frac{\alpha}{2} \text{coth} [\alpha \mu(x)] - \frac{\mu''(x)}{\mu'(x)}, \quad (41.b) \]
\[ V_{\text{GPT}}(x) = - \left( 1 - \frac{3\alpha^2}{4} \right) \text{cosech}^2 [\alpha \mu(x)] + 2i \alpha \text{cosech} [\alpha \mu(x)] \text{coth} [\alpha \mu(x)] + \frac{\alpha^2}{4}, \quad (41.c) \]
\[ \psi^{(0)}_{\text{GPT}}(x) \propto \frac{1}{U(x) \sqrt{\sinh [\alpha \mu(x)]}} \tanh^{-\frac{2\alpha}{\alpha}} \left[ \frac{\alpha \mu(x)}{2} \right]. \quad (41.d) \]

4.5 Pöschl-Teller potential

\[ g(x) = \text{sech} [\alpha \mu(x)] \text{cosech} [\alpha \mu(x)], \quad (42.a) \]
\[ f(x) = \alpha \text{coth} [2\alpha \mu(x)] - \frac{\mu''(x)}{2\mu'(x)}, \quad (42.b) \]
\[ V_{\text{PT}}(x) = \left( \frac{3\alpha^2}{4} - 1 + 2i\alpha \right) \text{cosech}^2 [\alpha \mu(x)] - \left( \frac{3\alpha^2}{4} - 1 - 2i\alpha \right) \text{sech}^2 [\alpha \mu(x)] + \alpha^2, \quad (42.c) \]
\[ \psi^{(0)}_{\text{PT}}(x) \propto \frac{1}{U(x) \sqrt{\sinh [2\alpha \mu(x)]}} \tanh^{-\frac{2\alpha}{\alpha}} \left[ \alpha \mu(x) \right]. \quad (42.d) \]

The above models are displayed in their usual forms and give quite well-known exact solvable non-Hermitian effective potentials as well as their accompanying ground-state wavefunctions. The first one represents a generalized \( \eta \)–weak-pseudo-Hermitian \( 3D \)–harmonic oscillator. The second model corresponds to the non-\( \mathcal{PT} \)–symmetric Morse potential and is already obtained by [22,23], where the \( \gamma = b_R \) constraint is considered therein, using \( \mathfrak{sl}(2, \mathbb{C}) \) potential algebra as a complex Lie algebra by a simple complexification of the coordinates in a group theoretical point of view.
and also in [24], labelled LIII according to Lévai [25], once a substitution $b \rightarrow ib$ is made therein. The remainder models belong to so called PI class [25] which contains five individual potentials. The third model represents a generalized $\eta-$weak-pseudo-Hermitian $\mathcal{PT}-$symmetric Scarf II Potential, labelled PI$_1$, which is established in [22,23,24] with the same constraints quoted above. Finally, the two last models represent, respectively, a generalized $\eta-$weak-pseudo-Hermitian generalized Pöschl-Teller (PI$_2$) and a generalized $\eta-$weak-pseudo-Hermitian Pöschl-Teller (PI$_5$) potentials and are already established, respectively, in [22,23,24] and [24].

5 Conclusion

A well-known class of non-Hermitian Hamiltonians endowed with position-dependent mass are generated as a by-product of a generalized $\eta-$weak-pseudo-Hermiticity thanks to a shift on the momentum $p$ of the type $p \rightarrow p - A(x) U(x)$, and which allows to avoid the Hermitian invertible linear operator $\eta$ for the benefit of $\tilde{\eta}$. We show that, being different from the realization of Ref.[13], there is no inconsistency to generate a well-known class of non-Hermitian Hamiltonians if the last shift is used, leading then to consider that $\tilde{\mathcal{D}}$ may be looked upon as a gauge-transformed version of $\mathcal{D}$ and depending essentially on the function $A(x)$, i.e. $\delta\mathcal{D} \equiv \tilde{\mathcal{D}}-\mathcal{D} = -iA(x)$. As a consequence of this, the wavefunction $\xi(x)$ is also subjected to a gauge transformation in the manner $\psi(x) \rightarrow \xi(x) = \Lambda(x) \psi(x)$, with $\Lambda(x) = \exp \left[ i \int^x dy \frac{A(y)}{U(y)} \right]$ and where $\psi(x)$ is the ground-state wavefunction when the $A(x) = 0$ constraint holds.

References

[1] C. M. Bender, S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243.
[2] C. M. Bender, S. Boettcher, P. N. Meisenger, J. Math. Phys. 40 (1999) 2201.
[3] C. M. Bender, G. V. Dunne, P. N. Meisenger, Phys. Lett. A 252 (1999) 272.

[4] P. Dorey, C. Dunning, R. Tateo, J. Phys. A : Math. Gen. 34 (2001) 5679.

[5] B. Bagchi, F. Cannata, C. Quesne, Phys. Lett. A 269 (2000) 79.

[6] S.-A. Yahiaoui, O. Cherroud, M. Bentaiba, J. Math. Phys. 48 (2007) 113503.

[7] M. Bentaiba, S.-A. Yahiaoui, L. Chetouani, Phys. Lett. A 331 (2004) 175.

[8] M. Bentaiba, L. Chetouani, A. Mazouz, Phys. Lett. A 295 (2002) 173.

[9] A. Mostafazadeh, J. Math. Phys. 43 (2002) 205.

[10] A. Mostafazadeh, J. Math. Phys. 43 (2002) 2814.

[11] A. Mostafazadeh, J. Math. Phys. 43 (2002) 3944.

[12] A. Mostafazadeh, Mod. Phys. Lett. A 17 (2002) 1973.

[13] B. Bagchi, C. Quesne, Phys. Lett. A 301 (2002) 173.

[14] O. Mustafa, S. Habib Mazharimousavi, Phys. Lett. A 357 (2006) 295;
O. Mustafa, S. Habib Mazharimousavi, Czech. J. Phys. 56 (2006) 967.

[15] D. J. BenDaniel, C. B. Duke, Phys. Rev. 152 (1966) 683.

[16] O. von Roos, Phys. Rev. B 27 (1983) 7547.

[17] L. Dekar, L. Chetouani, F. T. Hammann, J. Phys. A : Math. Gen. 39 (1998) 2551;
L. Dekar, L. Chetouani, F. T. Hammann, Phys. Rev. A 59 (1999) 107.

[18] A. D. Alhaidari, Phys. Rev. A 65 (2002) 042109;
A. D. Alhaidari, Int. J. Theo. Phys. 42 (2003) 2999.

[19] B. Roy, P. Roy, J. Phys. A : Math. Gen. 36 (2003) 8105;
B. Roy, P. Roy, Phys. Lett. A 340 (2005) 70.

[20] G. Bastard, ”Wave Mechanics Applied to Heterostructures”, les Ulis,
les éditions de physique, 1989.
[21] C. Weisbach, B. Vinter, "Quantum Semiconductor Heterostructures", Academic Press, New York, 1993.

[22] B. Bagchi, C. Quesne, Phys. Lett. A 273 (2000) 285.

[23] B. Bagchi, S. Mallik, C. Quesne, Int. J. Mod. Phys. A 16 (2001) 2859.

[24] G. Lévai, J. Phys. A : Math. Gen. 27 (1994) 3809.

[25] G. Lévai, J. Phys. A : Math. Gen. 22 (1989) 689.