A COMBINATORIAL PROOF OF THE POINTWISE ERGODIC
THEOREM FOR ACTIONS OF AMENABLE GROUPS ALONG
TEMPelman FøLNER SEQUENCES

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Abstract. We give a short and combinatorial proof of the pointwise ergodic theorem for
pmp actions of amenable groups along increasing Tempelman Følner sequences. We do
this by tiling the space arbitrarily well with contradictory Følner sets, thus generalizing A.
Tserunyan’s proof of the pointwise ergodic theorem for \( \mathbb{Z} \).

1. Introduction

For a group \( \Gamma \) acting on a probability space \( X \) and a sequence \( (F_n) \) of finite subsets of \( \Gamma \),
the pointwise ergodic property for \( \Gamma \) along \( (F_n) \) says that the action of \( \Gamma \) is ergodic if and
only if for every \( L^1 \) function \( f \) on \( X \), the integral (the global average) of \( f \) over \( X \) is equal
to the limit of the averages of \( f \) over \( F_n \cdot x \) (the pointwise average) for almost every \( x \in X \).
The classical ergodic theorem, due to G. D. Birkhoff in 1931 [Bir31], says that probability
measure preserving (pmp) actions of \( \mathbb{Z} \) along the sequence \([0, n)\) have the pointwise ergodic
property. In 2001, E. Lindenstrauss proved that actions of amenable groups along tempered
Følner sequences have the pointwise ergodic property [Lin01].

A. Tserunyan in [Tse17] gives a short, combinatorial proof of the classical pointwise ergodic
theorem (for \( \mathbb{Z} \)) by reducing it to showing that the following tiling property holds for \( \Gamma = \mathbb{Z} \)
along the intervals \( F_n = [0, n) \):

Definition 1 (Tiling Property). We say that a countable group \( \Gamma \) has the tiling property
along a sequence \( (F_n) \) of its finite subsets if for any pmp action of \( \Gamma \) on a standard probability
space \( (X, \mu) \), a pointwise increasing sequence of functions \( l_n : X \to \mathbb{N} \), \( n \in \mathbb{N} \), and \( \epsilon > 0 \),
there are arbitrarily large finite subsets \( T \subseteq \Gamma \) such that for a set of points \( x \) of measure at
least \((1 - \epsilon)\), \(Tx\) can be covered up to \( \epsilon \) fraction by disjoint sets of the form \( F_{l_n(y)} \cdot y \).

Another proof of the ergodic theorem for \( \mathbb{Z} \) revolving around the same idea was given in
[KP06]. In this paper, we prove that the tiling property holds for amenable groups along
increasing Tempelman Følner sequences \( (F_n) \) by finding Vitali covers with Følner tiles on
multiple scales. As a consequence, we prove the corresponding pointwise ergodic theorem:

Theorem 2 (Pointwise ergodic). Fix a pmp action of an amenable group \( \Gamma \) on a standard
probability space \( (X, \mu) \) and an increasing Tempelman Følner sequence \( (F_n) \). Then the action
of \( \Gamma \) on \( X \) is ergodic if and only if for every \( f \in L^1(X, \mu) \),

\[
\lim_n A_f[F_n x] = \int_X f(x) d\mu(x) \quad \text{a.e.}
\]

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where \( A_f[F_n x] := \frac{1}{|F_n|} \sum_{\gamma \in F_n} f(\gamma \cdot x) \).

Although this is less general than Lindenstrauss’s theorem, our proof is shorter (yet self-contained) and offers the advantage that the methods used are more elementary.

Many people have shown this result for increasing Tempelman Følner sequences. The shortest proof of Theorem 2 that the authors are aware of is given in [OW83], which uses a Vitali covering lemma along with basic functional analysis: a function \( f \in L^1(X, \mu) \) is approximated by functions for which the ergodic theorem holds trivially, and the error is controlled by applying the Vitali covering lemma. Other proofs include [Eme74] and [Tem67], which also use a Vitali covering lemma along with analysis. However, none of these proofs yield the tiling property, unlike our proof.

As in [Tse17], the reduction of Theorem 2 to the tiling property goes as follows: we argue by contradiction, so we assume that the pointwise limsup of finite averages of the given function \( f \) is larger than the mean of \( f \). We want to tile the space \( X \) with Følner tiles, chosen such that the average of the function \( f \) on each tile is larger than the mean of \( f \). If we successfully tile most of \( X \) by proving the tiling property, we will be able to say that the mean is actually just an integral of averages over these tiles, which is a contradiction.

However, compared to [Tse17], the tiling property is much harder to establish for general increasing Tempelman Følner sequences. For example, tiling \( \mathbb{Z}^d \) with boxes of different given sizes for each center is harder than tiling \( \mathbb{Z} \) with intervals. The key idea in mitigating this difficulty is to iterate the Vitali covering lemma to find covers on multiple scales. We essentially zoom very far out, cover some constant fraction (coming from the Tempelman condition) of the space with large sets, and then zoom in on the spots we miss, and fill those in as best we can with smaller sets, and so on and so forth. Since we cover a constant fraction on each scale, if we zoom out far enough at the beginning, once we zoom all the way back in, we will have covered nearly the whole space.

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## 2. Definitions and notation

Let \( (X, \mu) \) be a standard probability space, and a function \( f : X \to \mathbb{R} \). For a finite set \( A \subseteq X \), define the average of \( f \) over \( A \), \( A_f[A] := \frac{1}{|A|} \sum_{x \in A} f(x) \). For a finite equivalence relation \( F \) on \( X \), define \( A_f[F](x) := A_f[[x]_F] \). Given a group \( \Gamma \) and a finite set \( R \), define the \( R \)-boundary of a set \( S \), denoted \( \partial_R S \), to be the set of points \( s \) for which \( Rs \cap S \neq \emptyset \) and \( Rs \cap S^c \neq \emptyset \).

A sequence \((F_n)_{n \in \mathbb{N}}\) of finite subsets of \( \Gamma \) is a Følner sequence if \( \Gamma = \bigcup_n F_n \) and \( \lim_n \frac{|\partial_R F_n|}{|F_n|} = 0 \) for all finite sets \( R \). A group \( \Gamma \) is called amenable if it admits a Følner sequence. For this paper, we will assume that our Følner sequences are increasing.

Given a Følner sequence \((F_n)\), we say \((F_n)\) is tempered if there is some natural number \( C \) such that for all \( n \),

\[
\left| \bigcup_{k<n} F_{k}^{-1} F_n \right| \leq C|F_n|
\]
and Tempelman if there is $C$ such that for all $n$,

$$\left| \bigcup_{k \leq n} F_k^{-1}F_n \right| \leq C|F_n|$$

in the latter case, we’ll call the smallest such $C$ the Tempelman constant of $(F_n)$. Note that any Tempelman Følner sequence is tempered.

Every amenable group has a tempered Følner sequence (in fact, every Følner sequence has a tempered subsequence). In [Lin01, Example 4.2], an example is given of an amenable group without a Tempelman Følner sequence. However, [Hoc07, Theorem 3.4] gives a sufficient condition for the existence of a Tempelman Følner sequence:

**Theorem 3** (Hochman 2007). *If for a countable, abelian, amenable group $G$, we have*

$$r(G) = \sup\{n \in \mathbb{N} : G \text{ contains a subgroup isomorphic to } \mathbb{Z}^n\} < \infty$$

*then $G$ possesses at least one Tempelman Følner sequence.*

### 3. Tiling with different sizes of tiles

In this section, we prove the following:

**Lemma 4.** *The tiling property holds for amenable groups along increasing Tempelman Følner sequences.*

This will almost immediately imply Theorem 2. In order to prove this lemma, we need a Vitali covering lemma. For the rest of this section, fix an amenable group $\Gamma$ and Tempelman Følner sequence $F_i$ with Tempelman constant $C$, standard probability space $(X, \mu)$ on which $\Gamma$ acts in a pmp way, and $\epsilon > 0$.

**Lemma 5** (Vitali covering). *Given a function $l : X \to N$, a finite subset $S \subseteq X$, there exists a set $K$, which is a disjoint union of sets of the form $F_i(x)x$, $x \in S$, such that $|K| \geq \frac{1}{C}|S \cup K|$.*

*Proof. Put $= S = \{x_1, ..., x_n\}$, $D_0 = K_0 = \emptyset$. We will add to $K = \bigcup K_i$ and $D = \bigcup D_i$ until $S \setminus D = \emptyset$. Assume $S \setminus D_i \neq \emptyset$. Let $t = \max_{x \in S \setminus D_i} l(x)$, and let $j$ be least such that $x_j \in S \setminus D_i$ and $l(x_j) = t$. Put $K_{i+1} := K_i \cup F_i x_j$ and $D_{i+1} := D_i \cup F_i^{-1} F_i x_j$. Iterate this (up to $n$ times) until $S \setminus D = \emptyset$.

We claim that the selected $F_i x_j$ are actually pairwise disjoint. If not, suppose the at the $(i+1)^{th}$ step there is some $y \in F_i x_j \cap K_i$. Then $x_j \in F_i^{-1} y$. But since $y \in K_i$, there is some $t' \geq t$ and $j'$ such that $y \in F_{t'} x_{j'}$. Hence $x_j \in F_{t'}^{-1} F_{t'} x_{j'} \subseteq F_{t'}^{-1} F_{t'} x_{j'} \subseteq D_i$, contradicting our choice of $x_j$.

So at each step, we add exactly $|F_i|$ elements to $K$ and at most $C|F_i|$ elements to $D$. Hence, $|K| \geq \frac{1}{C}|D| \geq \frac{1}{C}|S \cup K|$ since $S \cup K \subseteq D$. \qed

Now we may begin proving Lemma 4. The idea is to fix many different sized Følner shapes for a large fraction of points, and tile our space with large tiles. To cover a large tile, we’ll start by covering as much as we can with the largest sized Følner shapes, and then proceed with smaller sized shapes.

*Proof of Lemma 4.* First, fix $r \in \mathbb{N}$ large enough so that $(\frac{C-1}{C})^r < \frac{\epsilon}{2}$. We will ultimately pick $r$ many “good” sizes of tiles for a large fraction of the points in $X$. Fix $r$ many functions $G_i : [0,1] \to \mathbb{R}$ such that $G_1(x) \geq \frac{2}{\epsilon} (x)$ and for $i > 1$, $G_i(x) \geq \frac{C-1}{C} G_{i-1}(x) + (i+1)(x)$
where each $G_i$ is continuous and $G_i(0) = 0$. For example, $G_i(x) := \left(\frac{C-1}{e}\right)^i x + \sum_{k=1}^{i+1} kx$ is such a collection of functions. Fix $\alpha$ small enough so that $\beta \leq \alpha \implies G_r(\beta) < \frac{\epsilon}{2}$. Put $\eta := \min(\alpha, \epsilon)$.

For each $p \in \mathbb{N}$, let $A^{(p)}_n := \{x \in X : (\exists i \in \mathbb{N}) \ l_i(x) \in [p, n]\}$. Since the $l_i$ are strictly increasing in $x$ for all $x$, $\bigcup_n A^{(p)}_n = X$. Hence, there is some large enough $p^*$ such that $\mu(A^{(p^*)}_n) > 1 - \frac{\eta^2}{r}$. This means that we can fix $r$ many values $p_j$ such that $\mu(\{x \in X : (\forall j < r) \ (\exists i \in \mathbb{N} \ l_i(x) \in [p_j, p^*_j])\}) > 1 - \eta^2$. We will think of the $[p_j, p^*_j]$ as different sizes for our tiles.

We define a sequence of natural numbers of length $r$ as follows. Let $L_0$ be large enough so that $\frac{|\partial_{F_{r-1}} F_n|}{|F_n|} < \eta$ for all $n > L_0$. For $i < r$, define $R_i := L_i^*$, and $L_i+1 > R_i$ large enough so that

$$\frac{|\partial_{F_{r-1}} F_n|}{|F_n|} < \frac{\eta}{|F_{r-1}|}$$

for all $n \geq L_{i+1}$. Put $\delta := \frac{\eta}{|F_{r-1}|}$. Let $T \subseteq \Gamma$ satisfy $\frac{|\partial_{F_{r-1}} T|}{|T|} < \delta$.

Define partial functions $p_i(x) := l_j(x)$ where $j$ is smallest such that $l_j(x) \in [L_i, R_i]$ if such a $j$ exists. Let $P := \{x \in X : (\forall i < r) \ x \in \text{dom}(p_i)\}$. Hence, $\mu(P) > 1 - \eta^2$, so $\mu(\{x \in X : A_{x, p}[Tx] < 1 - \eta\}) < \eta$, because otherwise,

$$1 - \eta^2 < \mu(P) = \int_X \mathbb{1}_P(x) d\mu(x)$$

[by the invariance of $\mu$]

$$= \int_X \left(\frac{1}{|T|} \sum_{\gamma \in T} \mathbb{1}_P(\gamma \cdot x)\right) d\mu(x)$$

$$= \int_X A_{x, p}[Tx] d\mu(x)$$

$$< 1 - \eta + \eta(1 - \eta) = 1 - \eta^2.$$

Hence, at least $1 - \eta$ fraction of points in $X$ have all $p_i$ defined on at least $1 - \eta$ fraction of $Tx$. It now suffices to show that for a point $x$ such that at least $1 - \eta$ fraction of $Tx$ is contained in $P$, we can tile $Tx$ up to $\epsilon$ fraction with tiles of the form $F_{l(i)} x$. We claim that in $k$ steps, $k \leq r$, we can tile $Tx$ up to $\left(\frac{C-1}{e}\right)^k + G_k(\eta)$ fraction. As discussed earlier, we will start by tiling with our largest Følner shapes, i.e. $l_i(x) \in [L_r, R_r]$, and in each step we will move down a size.

In step $k$, apply Lemma 5 with $l = p_{r-k}$ and

$$S := \{x \in Tx \cap \text{dom}(p_{r-k}) : F_{r-k} x \text{ is contained in the set of uncovered points in } T\}$$

so that the constructed set $K$ is contained in the set of uncovered points in $T$. $S$ is almost all of the uncovered points in $T$ except possibly:

1. A small strip along the boundary of $T$, of size $|\partial_{F_{r-k}} T|\cdot |F_{r-k}| < \eta|T|$ since $\frac{|\partial_{F_{r-k}} T|}{|T|} < \frac{\eta}{|F_{r-k}|}$.

2. The set of points on which $p_{r-k}$ is not defined, which has fewer than $\eta|T|$ points.

3. A small strip along the boundary of the covered points from each of the previous $k - 1$ steps. Fix $j < k$, and let $A$ be the set of covered points in the $j^{th}$ step. We might miss a strip of size $|\partial_{F_{r-k}} A|\cdot |F_{r-k}|$. Note that since the boundary of $A$ consists of Følner
shapes of size \([L_{r-j}, R_{r-j}]\), we have \(\frac{|\partial F_{R_{r-k}} A|}{|A|} \leq \frac{|\partial F_{R_{r-j}} F_n|}{|F_n|}\) for some \(F_{L_{r-j}} \leq n \leq F_{R_{r-j}}\), so
\[
|\partial F_{R_{r-k}} A||F_{R_{r-k}}| \leq \frac{|\partial F_{R_{r-j}} F_n|}{|F_n|}|A||F_{R_{r-k}}| \\
\leq \frac{\eta}{|F_{R_{r-j-1}}|}|A||F_{R_{r-k}}| \\
\leq \eta|T|,
\]
where the penultimate inequality comes from our choice of \(L_{r-j}\) to be large enough that \(n \geq L_{r-j}\) implies \(\frac{|\partial F_{R_{r-k}} F_n|}{|F_n|} < \frac{\eta}{|F_{R_{r-j-1}}|}\), and the final inequality comes from \(A \subseteq T\) and the fact that \(r - k \leq r - j - 1\) for any \(j < k\).

In total, \(S\) is missing at most \((k + 1)\eta|T|\) uncovered points from \(T\). If \(k = 1\), we have that \(K\) covers at least \(\frac{1}{C}\) fraction of \(S \cup K\), and \(|S \cup K| \geq (1 - 2\eta)|T|\). So \(K\) covers at least \(\frac{1}{C}(1 - 2\eta)|T|\), and we are left with \(\frac{C - 1 + 2\eta}{C}|T|\), so we cover all but \(\frac{C - 1}{C} + G_1(\eta)\) fraction of \(|T|\).

If \(k \geq 2\), assume we have covered all but \((\frac{C - 1}{C})^{k-1} + G_{k-1}(\eta)\) fraction of \(|T|\) after the \((k - 1)^{th}\) step. Notice that
\[
|S \cup K| \leq \left( \left( \frac{C - 1}{C} \right)^{k-1} + G_{k-1}(\eta) \right)|T|,
\]
since both \(S\) and \(K\) are contained in the set of uncovered points of \(T\). Since \(K\) covers at least \(\frac{1}{C}\) fraction of \(|S \cup K|\), at most \(\frac{C - 1}{C}\) fraction of \(|S \cup K|\) is left uncovered. So \(K\) covers all of \(T\) but
\[
(k + 1)\eta|T| + \frac{C - 1}{C}|S \cup K| \leq \left( (k + 1)\eta + \left( \frac{C - 1}{C} \right)^k + \frac{C - 1}{C}G_{k-1}(\eta) \right)|T| \\
\leq \left( \left( \frac{C - 1}{C} \right)^k + G_k(\eta) \right)|T|
\]
many points. This concludes the proof of our claim. Iterate this algorithm \(r\) times so that we have covered all but \((\frac{C - 1}{C})^r + G_r(\eta)\) fraction of \(|T|\). Since both \((\frac{C - 1}{C})^r, G_r(\eta) < \frac{\eta}{2}\), this concludes the proof. \(\square\)

4. Deriving the Ergodic Theorem

Here we follow [Tse17] to derive Theorem 2 from Lemma 4.

**Theorem 2** (Pointwise ergodic). Fix a pmp action of an amenable group \(\Gamma\) on a standard probability space \((X, \mu)\) and a Tempelman Følner sequence \(F_n\). Then the action of \(\Gamma\) on \(X\) is ergodic if and only if for every \(L^1\) function \(f : X \to \mathbb{R}\),
\[
\lim_n A_n f[F_n x] = \int_X f(x) d\mu(x) \ a.e.
\]

**Remark.** The proof below, after a minor modification, would also give the more general theorem for not necessarily ergodic actions, with the mean of \(f\) replaced by its conditional expectation with respect to the \(\sigma\)-algebra of invariant sets.
We will show that \( \overline{\text{f}} := \limsup_n A_f(T_n x) \leq 0 \) a.e., and an analogous argument shows that \( \liminf_n A_f(T_n x) \geq 0 \) a.e., implying the desired result.

Since \( \overline{f} \) is \( \Gamma \)-invariant, ergodicity implies that it is some constant \( c \) almost everywhere. Suppose by way of contradiction that \( c > 0 \). Let \( \delta > 0 \) be small enough so that if \( \mu(A) < \delta \) then \( \int_A (f - \Delta) d\mu < \frac{\Delta}{3} \). Let \( M \) be large enough so that \( \mu(f^{-1}((\infty, -M])) < \delta \). Put \( \epsilon := \min\{\frac{\Delta}{3M}, \delta\} \). Apply Lemma 4 to \( X \) with amenable group \( \Gamma \), Tempelman Følner sequence \( F_n \), functions \( l_i \), and \( \epsilon \) as defined above. Set \( B := f^{-1}((\infty, -M)) \) and \( C := \{x \in X : Tx \text{ cannot be tiled up to } \epsilon \text{ fraction}\} \). Then

\[
\int_{X \setminus B} (f - \Delta)(x) d\mu(x) = \frac{1}{|T|} \sum_{\gamma \in T} \int_X (f - \Delta)(\mathbb{1}_X \setminus B)(\gamma \cdot x) d\mu(x)
\]

[by the invariance of \( \mu \)]

\[
= \frac{1}{|T|} \sum_{\gamma \in T} \int_X (f - \Delta)(\mathbb{1}_X \setminus B)(\gamma \cdot x) d\mu(x)
\]

\[
\geq -\epsilon M - \frac{\Delta}{3} = -\frac{2\Delta}{3}.
\]

Hence, \( \int_X f(x) d\mu(x) = \int_X (f - \Delta)(x) d\mu(x) + \Delta \geq -\frac{2\Delta}{3} + \int_B (f - \Delta)(x) d\mu(x) + \Delta > -\frac{2\Delta}{3} - \frac{\Delta}{3} + \Delta = 0. \)

This contradiction implies that \( \limsup_n A_f(T_n x) \leq 0 \) a.e., as desired. \( \square \)

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