Converting two-atom singlet state into three-atom singlet state via quantum Zeno dynamics

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\textbf{Abstract.} An approach is presented for converting a two-atom singlet state into a three-atom singlet state based on the quantum Zeno dynamics induced by continuous coupling. The scheme can be achieved within one step through appropriately regulating the Rabi frequencies of the classical fields. The effects of decoherence such as atomic spontaneous emission and the loss of cavity are also considered in virtue of the master equation. The numerical simulation result shows that this proposal is especially robust against the cavity decay, since no cavity-photon population is involved during the whole process because of the quantum Zeno dynamics. Furthermore, if a multilevel atom and a multi-mode cavity are applicable, the $N$-atom singlet state could be derived directly from the $(N - 1)$-atom singlet state with the same principle, which provides a scalable way for the preparation of $|S_N\rangle$ in theory.

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1. Introduction

Owing to the quantum Zeno effect, the dissipation of an unstable quantum system is suppressed and the transition between quantum states is frozen if measurements are performed frequently. Since the first experiment of the quantum Zeno effect on the oscillating system was carried out by Cook [1], many different physical systems have been applied to test the quantum Zeno effect, such as photon polarization [2], cavity QED [3, 4], SQUID [5, 6], Bose–Einstein condensates [7], etc. The quantum Zeno dynamics [8] found by Facchi et al. can be considered as a generalization of the usual quantum Zeno effect [9] in two aspects. One is that the frequent measurements do not necessarily hinder the evolution of the quantum system, but the system can evolve away from the initial state via the measurements [10]. The other is that the quantum Zeno effect can be obtained by a continuous coupling between the system of interest and the apparatus performing the observation without making use of von Neumann’s projections and non-unitary dynamics [11]. So far, many protocols have been put forward to implement quantum logic gates and create entanglement states via the quantum Zeno effect and quantum Zeno dynamics [12]–[23].

In 2002, Cabello proposed a special type of entangled state, the so-called $N$-particle $N$-level singlet states, which can be expressed as [24]

$$|S_N\rangle = \frac{1}{\sqrt{N!}} \sum_{|n_1\rangle} \epsilon_{n_1,\ldots,n_N}|\alpha_{n_1}, \ldots,\alpha_{n_N}\rangle,$$

(1)

where $\epsilon_{n_1,\ldots,n_N}$ is the generalized Levi-Civita symbol, and the state $|\alpha_{n_i}\rangle$ denotes one base of the qudit. These entanglement states are not only in connection with violations of Bell’s inequalities [25] and can be used to construct decoherence-free subspaces, which are robust against collective decoherence [26], but also they are the key resource for the solutions of ‘$N$-strangers’, ‘secret sharing’ and ‘liar detection’ problems, which have no classical solutions [24]. Nevertheless, the generation of these states for $N \geq 3$ is still a formidable physical challenge both in theory and experiment, and only two schemes have been proposed to realize the three-atom singlet state

$$|S_3\rangle = \frac{1}{\sqrt{6}}(|012\rangle - |102\rangle - |210\rangle + |120\rangle + |201\rangle - |021\rangle)$$

(2)
in the context of cavity quantum electrodynamics (QED), i.e. the scheme of generating a supersinglet of three three-level atoms in microwave cavity QED, based on the resonant
atom–cavity interaction offered by Jin et al [27], and the scheme for preparation of a singlet state with three atoms via Raman transitions raised by Lin et al [28]. The first scheme is more sensitive to the loss of cavity, since three atoms are sequentially sent through three different cavities, and cavity fields act as memories. Although the system in the second scheme does not involve cavity-photon population during the operation, it becomes probabilistic once the cavity decay and spontaneous emission of the atom are considered [29]. Different from the above schemes for generating the state $|S_3\rangle$ beginning with a direct product state of three atoms, we find that if a two-atom singlet state $|S_2\rangle$ is ready-made, it can directly be converted into $|S_3\rangle$ via the quantum Zeno dynamics.

We start off by giving an elementary introduction to the quantum Zeno dynamics, induced by continuous coupling [10, 30]. Suppose that the dynamical evolution of the whole system is governed by the following Hamiltonian:

$$H_K = H + KH_c,$$

(3)

where $H$ is the Hamiltonian of the subsystem to be investigated, $H_c$ is an additional interaction Hamiltonian that performs the measurement, and $K$ is the corresponding coupling constant. For a strong coupling limit $K \to \infty$, the subsystem of interest is dominated by the following evolution operator:

$$U(t) = \lim_{K \to \infty} \exp(iKH_c t)U_K(t),$$

(4)

which can be shown to have the form [10]

$$U(t) = \exp(-iH_Z t),$$

(5)

where

$$H_Z = \sum_n P_n HP_n$$

(6)

is termed the Zeno Hamiltonian and $P_n$ is the eigenprojection of the $H_c$ belonging to the eigenvalue $\eta_n$

$$H_c = \sum_n \eta_n P_n.$$  

(7)

Thus the whole system is governed by the limiting evolution operator

$$U_K(t) \sim \exp(-iKH_c t)U(t) = \exp\left(-i \sum_n K\eta_n P_n t + P_n HP_n t\right),$$

(8)

with the effective Hamiltonian as

$$H_{\text{eff}} = \sum_n K\eta_n P_n + P_n HP_n.$$  

(9)

Equation (9) shows that in the presence of a continuous coupling with a large coupling constant $K$, the Hamiltonian $H$ can govern the evolution of states in a certain subspace corresponding to the same eigenvalue of $H_c$, while leaving other states unchanged. Interestingly, if the system is initialized in the dark state with respect to $H_c$, the effective Hamiltonian will be reduced to $H_Z$. 

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Figure 1. The level configuration of the $i$th atomic four-level tripod system. The transition between the levels $|L\rangle_i \leftrightarrow |1\rangle_i$ ($|2\rangle_i$) is coupled to the cavity mode $a$ ($b$) resonantly with the coupling constant $g^a_i$ ($g^b_i$), and the transition $|L\rangle_i \leftrightarrow |0\rangle_i$ is driven by the classical pulse with the coupling strength $\Omega_i$.

2. Converting two-atom singlet state into three-atom singlet state

Now let us illustrate our scheme in detail. We consider three four-level atoms with tripod configuration simultaneously interacting with a bimodal vacuum cavity field. The transition between the levels $|L\rangle_i \leftrightarrow |1\rangle_i$ ($|2\rangle_i$) is coupled to the cavity mode $a$ ($b$) resonantly with the coupling constant $g^a_i$ ($g^b_i$), and the transition $|L\rangle_i \leftrightarrow |0\rangle_i$ is driven by the classical pulse with the coupling strength $\Omega_i$, as shown in figure 1. The Hamiltonian of the system, in the interaction picture with the rotating wave approximation (RWA), can be written as ($\hbar = 1$)

$$H_{\text{total}} = H_I + H_{\text{laser}},$$

$$H_I = \sum_{i=1}^{3} g^a_i (a^\dagger |1\rangle_i \langle L| + |L\rangle_i \langle 1| a) + g^b_i (b^\dagger |2\rangle_i \langle L| + |L\rangle_i \langle 2| b),$$  

(10)

$$H_{\text{laser}} = \sum_{i=1}^{3} \Omega_i (|L\rangle_i \langle 0| + |0\rangle_i \langle L|),$$

where $a^\dagger$ ($b^\dagger$) and $a$ ($b$) are the creation and annihilation operators for the cavity mode $a$ ($b$), respectively. In the present paper, we assume $g_i^{a,b} = g$, $\Omega_2 = \Omega_3 = \Omega$, and $\Omega_1$ to be real for the sake of convenience. Thus, in the following closed subspace:

$$|012\rangle|0_a0_b\rangle, \quad |L12\rangle|0_a0_b\rangle, \quad |112\rangle|1_a0_b\rangle, \quad |212\rangle|0_a1_b\rangle, \quad |121\rangle|1_a0_b\rangle, \quad |210\rangle|0_a0_b\rangle, \quad |120\rangle|0_a1_b\rangle, \quad |201\rangle|0_a0_b\rangle, \quad |221\rangle|0_a1_b\rangle, \quad |L21\rangle|0_a0_b\rangle, \quad |021\rangle|0_a0_b\rangle,$$

(11)
the Hamiltonian of the system can be expressed as

$$H_{\text{sub}} = H_i + H'_\text{laser},$$

$$H'_i = g(|L12⟩|0_a 0_b⟩(|12⟩|1_a 0_b⟩ + |L12⟩|0_a 0_b⟩|212⟩|0_a 1_b⟩ + |112⟩|1_a 0_b⟩|1L2⟩|0_a 0_b⟩ + 212⟩|0_a 1_b⟩|21L⟩|0_a 0_b⟩ + 122⟩|0_a 1_b⟩ + 21L⟩|0_a 0_b⟩|211⟩|1_a 0_b⟩ + 122⟩|0_a 1_b⟩|12L⟩|0_a 0_b⟩|211⟩|1_a 0_b⟩ + 21L⟩|0_a 0_b⟩|221⟩|0_a 1_b⟩ + |121⟩|1_a 0_b⟩|L21⟩|0_a 0_b⟩ + 221⟩|0_a 1_b⟩|L21⟩|0_a 0_b⟩ + \text{H.c.}),$$

$$H'_\text{laser} = (Ω_1|L12⟩|0_a 0_b⟩|012⟩|0_a 0_b⟩ + Ω|L12⟩|0_a 0_b⟩|102⟩|0_a 0_b⟩ + Ω|21L⟩|0_a 0_b⟩|210⟩|0_a 0_b⟩ + Ω|12L⟩|0_a 0_b⟩|120⟩|0_a 0_b⟩ + Ω|21L⟩|0_a 0_b⟩|201⟩|0_a 0_b⟩ + Ω_1|L21⟩|0_a 0_b⟩|021⟩|0_a 0_b⟩ + \text{H.c.}),$$

where we have divided the Hamiltonian of this subsystem into two parts, so that they make an analogy to equation (3). Under the condition $Ω_i ≪ g$, this Hilbert subspace is split into seven invariant Zeno subspaces [30]

$$\mathcal{H}_{P_1} = \{|012⟩|0_a 0_b⟩, |021⟩|0_a 0_b⟩, |102⟩|0_a 0_b⟩, |120⟩|0_a 0_b⟩,\rangle \}
\mathcal{H}_{P_2} = \{|ψ_3⟩\}, \mathcal{H}_{P_3} = \{|ψ_4⟩, |ψ_5⟩\}, \mathcal{H}_{P_4} = \{|ψ_6⟩, |ψ_7⟩\}, \mathcal{H}_{P_5} = \{|ψ_9⟩, |ψ_{10}⟩\}, \mathcal{H}_{P_6} = \{|ψ_{11}⟩, |ψ_{12}⟩\},$$

corresponding to the projections

$$P_i^\alpha = |α⟩⟨α|, \quad (|α⟩ \in \mathcal{H}_{P_i}),$$

with eigenvalues $η_1 = 0$, $η_2 = -2g$, $η_3 = -g$, $η_4 = g$, $η_5 = 2g$, $η_6 = -\sqrt{3}g$ and $η_7 = \sqrt{3}g$. Here,

$$|ψ_1⟩ = \frac{1}{√6}(|L12⟩|0_a 0_b⟩ - |1L2⟩|0_a 0_b⟩ - |21L⟩|0_a 0_b⟩ + |12L⟩|0_a 0_b⟩ + 21L⟩|0_a 0_b⟩ - |L21⟩|0_a 0_b⟩),$$

$$|ψ_2⟩ = \frac{1}{√6}(|112⟩|1_a 0_b⟩ - |212⟩|0_a 1_b⟩ - |122⟩|0_a 1_b⟩ + |211⟩|1_a 0_b⟩ + 121⟩|1_a 0_b⟩ - |221⟩|0_a 1_b⟩),$$

$$|ψ_3⟩ = \frac{1}{2√3}(|L12⟩|0_a 0_b⟩ - |112⟩|1_a 0_b⟩ - |212⟩|0_a 1_b⟩ - |112⟩|1_a 0_b⟩ + |1L2⟩|0_a 0_b⟩ + |21L⟩|0_a 0_b⟩ - |222⟩|0_a 1_b⟩ - |211⟩|1_a 0_b⟩ + |12L⟩|0_a 0_b⟩ + |2L1⟩|0_a 0_b⟩ - |121⟩|1_a 0_b⟩ - |221⟩|0_a 1_b⟩ + |L21⟩|0_a 0_b⟩),$$
\[ |\psi_4\rangle = \frac{1}{2\sqrt{2}} (|L12\rangle|0_a0_b\rangle - |212\rangle|0_a1_b\rangle - |1L2\rangle|0_a0_b\rangle + |212\rangle|0_a1_b\rangle + |211\rangle|1_a0_b\rangle + |2L1\rangle|0_a0_b\rangle - |2L1\rangle|0_a0_b\rangle + |121\rangle|1_a0_b\rangle + |L21\rangle|0_a0_b\rangle), \]

\[ |\psi_5\rangle = \frac{-1}{2\sqrt{6}} (|L12\rangle|0_a0_b\rangle - 2|112\rangle|1_a0_b\rangle + |212\rangle|0_a1_b\rangle + |1L2\rangle|0_a0_b\rangle + 2|21L\rangle|0_a0_b\rangle + |122\rangle|0_a1_b\rangle + |211\rangle|1_a0_b\rangle - 2|12L\rangle|0_a0_b\rangle + |2L1\rangle|0_a0_b\rangle + |121\rangle|1_a0_b\rangle - 2|221\rangle|0_a1_b\rangle + |L21\rangle|0_a0_b\rangle), \]

\[ |\psi_6\rangle = \frac{1}{2\sqrt{2}} (|L12\rangle|0_a0_b\rangle + |212\rangle|0_a1_b\rangle - |1L2\rangle|0_a0_b\rangle - |122\rangle|0_a1_b\rangle - |211\rangle|1_a0_b\rangle - |2L1\rangle|0_a0_b\rangle - |211\rangle|1_a0_b\rangle + |L21\rangle|0_a0_b\rangle), \]

\[ |\psi_7\rangle = \frac{1}{2\sqrt{6}} (|L12\rangle|0_a0_b\rangle + 2|112\rangle|1_a0_b\rangle - |212\rangle|0_a1_b\rangle + |1L2\rangle|0_a0_b\rangle - 2|21L\rangle|0_a0_b\rangle + |122\rangle|0_a1_b\rangle - |211\rangle|1_a0_b\rangle - 2|12L\rangle|0_a0_b\rangle + |2L1\rangle|0_a0_b\rangle + |121\rangle|1_a0_b\rangle + 2|221\rangle|0_a1_b\rangle + |L21\rangle|0_a0_b\rangle), \]

\[ |\psi_8\rangle = \frac{1}{2\sqrt{3}} (|L12\rangle|0_a0_b\rangle + |112\rangle|1_a0_b\rangle + |212\rangle|0_a1_b\rangle + |1L2\rangle|0_a0_b\rangle + |21L\rangle|0_a0_b\rangle + |122\rangle|0_a1_b\rangle + |211\rangle|1_a0_b\rangle + |12L\rangle|0_a0_b\rangle + |2L1\rangle|0_a0_b\rangle + |121\rangle|1_a0_b\rangle + |221\rangle|0_a1_b\rangle + |L21\rangle|0_a0_b\rangle), \]

\[ |\psi_9\rangle = \frac{-1}{2\sqrt{6}} (|L12\rangle|0_a0_b\rangle - |212\rangle|0_a1_b\rangle - |L12\rangle|0_a0_b\rangle + 2|21L\rangle|0_a0_b\rangle + |122\rangle|0_a1_b\rangle - |1L2\rangle|0_a0_b\rangle + |122\rangle|0_a1_b\rangle - |211\rangle|1_a0_b\rangle - 2|12L\rangle|0_a0_b\rangle + |2L1\rangle|0_a0_b\rangle + |121\rangle|1_a0_b\rangle - |L21\rangle|0_a0_b\rangle), \]

\[ |\psi_{10}\rangle = \frac{1}{2\sqrt{6}} (\sqrt{3}|L12\rangle|0_a0_b\rangle - 2|112\rangle|1_a0_b\rangle + |1L2\rangle|0_a0_b\rangle + \sqrt{3}|L2\rangle|0_a0_b\rangle - |122\rangle|0_a1_b\rangle + |211\rangle|1_a0_b\rangle - |211\rangle|1_a0_b\rangle + |122\rangle|0_a1_b\rangle - |211\rangle|1_a0_b\rangle + |122\rangle|0_a1_b\rangle - |211\rangle|1_a0_b\rangle + |122\rangle|0_a1_b\rangle - |211\rangle|1_a0_b\rangle + |L21\rangle|0_a0_b\rangle), \]

\[ |\psi_{11}\rangle = \frac{-1}{2\sqrt{6}} (|L12\rangle|0_a0_b\rangle + |212\rangle|0_a1_b\rangle - |1L2\rangle|0_a0_b\rangle + 2|21L\rangle|0_a0_b\rangle - \sqrt{3}|212\rangle|0_a1_b\rangle + |211\rangle|1_a0_b\rangle - 2|12L\rangle|0_a0_b\rangle + |2L1\rangle|0_a0_b\rangle - \sqrt{3}|212\rangle|0_a1_b\rangle + |211\rangle|1_a0_b\rangle - 2|12L\rangle|0_a0_b\rangle + |2L1\rangle|0_a0_b\rangle - \sqrt{3}|212\rangle|0_a1_b\rangle + |211\rangle|1_a0_b\rangle - 2|12L\rangle|0_a0_b\rangle + |2L1\rangle|0_a0_b\rangle). \]
\[ |\psi_{12} \rangle = \frac{-1}{2\sqrt{6}} (\sqrt{3}|L12\rangle|0_a,0_b\rangle + 2|112\rangle|1_a,0_b\rangle + |212\rangle|0_a,1_b\rangle + \sqrt{3}|1L2\rangle|0_a,0_b\rangle + |122\rangle|0_a,1_b\rangle - |211\rangle|1_a,0_b\rangle - \sqrt{3}|2L1\rangle|0_a,0_b\rangle - |121\rangle|1_a,0_b\rangle - 2|221\rangle|0_a,1_b\rangle - \sqrt{3}|L21\rangle|0_a,0_b\rangle) \]  

(28)

According to equation (9), the Hamiltonian of this subsystem approximately equals

\[ H_{\text{sub}} \simeq \sum_{i,a,\beta} \eta_i P_i^a + P_i^a H_{\text{laser}} P_i^b \]

\[ = -2g|\psi_3\rangle\langle\psi_3| - g(|\psi_4\rangle\langle\psi_4| + |\psi_5\rangle\langle\psi_5|) + g(|\psi_6\rangle\langle\psi_6| + |\psi_7\rangle\langle\psi_7|) \]

\[ + 2g|\psi_8\rangle\langle\psi_8| - \sqrt{3}g(|\psi_9\rangle\langle\psi_9| + |\psi_{10}\rangle\langle\psi_{10}|) + \sqrt{3}g(|\psi_{11}\rangle\langle\psi_{11}| + |\psi_{12}\rangle\langle\psi_{12}|) \]

\[ + \frac{1}{\sqrt{6}} \left[ \Omega_1(|012\rangle|0_a,0_b\rangle - |021\rangle|0_a,0_b\rangle)\langle\psi_1| + \Omega|\psi_1\rangle(|120\rangle|0_a,0_b\rangle \right. \]

\[ - |102\rangle|0_a,0_b\rangle + |201\rangle|0_a,0_b\rangle - |210\rangle|0_a,0_b\rangle + \text{H.c.} \], \quad (|\alpha\rangle, |\beta\rangle \in \mathcal{H}_R). \]

(29)

If the initial state is

\[ |\varphi\rangle = \frac{1}{\sqrt{2}} |0\rangle (|12\rangle - |21\rangle)|0_a,0_b\rangle, \]

(30)

where the state \((1/\sqrt{2})(|12\rangle - |21\rangle)\) can be considered as a two-atom singlet state (for the difference between this state and the genuine two-atom singlet state \(|S_2\rangle\) is just the labels of the energy level, and we will term this state as \(|S_2'\rangle\) so as to avoid confusion), \(|0\rangle\) is the state of the ancillary atom, and the cavity \(a \ (b)\) is initially in the vacuum state; the Hamiltonian \(H_{\text{sub}}\) reduces to

\[ H_{\text{eff}} = \frac{1}{\sqrt{6}} \left[ \Omega_1(|012\rangle|0_a,0_b\rangle - |021\rangle|0_a,0_b\rangle)\langle\psi_1| + \Omega|\psi_1\rangle(|120\rangle|0_a,0_b\rangle \right. \]

\[ - |102\rangle|0_a,0_b\rangle + |201\rangle|0_a,0_b\rangle - |210\rangle|0_a,0_b\rangle + \text{H.c.} \], \quad (31)

and the general evolution form of equation (30) in time \(t\) is

\[ |\Psi_t\rangle = \frac{2\Omega^2 + \Omega_1^2 \cos \left( \frac{\sqrt{2}\Omega^2 + \Omega_1^2}{2\sqrt{3}} t \right)}{2\Omega^2 + \Omega_1^2} \frac{1}{\sqrt{2}} (|012\rangle|0_a,0_b\rangle - |021\rangle|0_a,0_b\rangle) \]

\[ - i\Omega_1 \sin \left( \frac{\sqrt{2}\Omega^2 + \Omega_1^2}{2\sqrt{3}} t \right) |\psi_1\rangle + \sqrt{2}\Omega_1 \left[ -1 + \cos \left( \frac{\sqrt{2}\Omega^2 + \Omega_1^2}{2\sqrt{3}} t \right) \right] \]

\[ \times \frac{1}{2} (|120\rangle|0_a,0_b\rangle - |102\rangle|0_a,0_b\rangle + |201\rangle|0_a,0_b\rangle - |210\rangle|0_a,0_b\rangle). \]

(32)

Therefore, the three-atom singlet state \(|S_3\rangle\) is achieved by setting \(\Omega_1 = -(\sqrt{3} - 1)\Omega\), and selecting the interaction time \(t = \sqrt{3}\pi / \sqrt{2\Omega^2 + \Omega_1^2}\).
3. Analysis of the fidelity and the effect of decoherence

In the above derivation, the quantum Zeno dynamics requires that the Rabi frequencies of the classical fields \( \Omega_i \) should be set much smaller than the coupling strengths between the atoms and the cavity \( g \), i.e. \( \Omega_i \ll g \). Thus it is essential to quantify the range of \( \Omega_i/g \), in order to generate the three-atom singlet state \( |S_3\rangle \) effectively. In figure 2 we plot the fidelity of the prepared state \( |S_3\rangle \) versus different ratios between \( \Omega_i \) and \( g \) through the relation \( F = \langle S_3 | \rho(t) | S_3 \rangle \).

\[
F = \langle S_3 | \rho(t) | S_3 \rangle , \tag{33}
\]

where \( \rho(t) \) is the density matrix of the final state obtained in our scheme. The result reveals that the fidelity decreases in an oscillating form with the increase of the \( \Omega_i/g \) ratio. Even for a relatively large ratio of \( \Omega_i \) to \( g \), i.e. \( \Omega_i = 0.1g \), the fidelity remains above 99.00\%, which guarantees that the effective Hamiltonian in equation (31) is valid. One may also find that a better Zeno requirement can be satisfied when \( \Omega_i/g \leq 0.01 \), corresponding to a near 100\% fidelity in the ideal situation. Nevertheless, a smaller ratio \( \Omega_i/g \) means a longer operation time, which may cause our scheme to be more susceptible to the decoherence induced by atomic spontaneous emission. For this reason, we will adopt the value \( \Omega_i/g = 0.1 \) in the following calculations. As we will see later, this choice can make our scheme suit the current cavity QED experiments. In the idealized situation, we have supposed that all atoms have the same coupling to the optical field. But this assumption is hard to achieve in practice; hence, we need to investigate the variations in the couplings induced by the experimental imperfection. Suppose that there are some deviations in the selected experimental parameters, i.e. \( g_1^a = g_1^b = g, \ g_2^a = g_2^b = g - \delta_1 \) and \( g_3^a = g_3^b = g - \delta_2 \); we plot the fidelity versus \( \delta_1/g \) and \( \delta_2/g \) in figure 3. This result signifies that under the condition \( \delta_1, \delta_2 \in [0, 0.3g] \), the fidelity of prepared state \( |S_3\rangle \) is always higher than 92\%; in other words, our scheme can be considered as immune to the variations of the couplings between atoms and the cavity.
Next, we will study the influence of atomic spontaneous decay (at the rate $\gamma$) and photon leakage out of the cavities (at the rate $\kappa_a, \kappa_b$) on the generation of the three-atom singlet state $|S_3\rangle$. The master equation of the whole system is expressed as

$$
\dot{\rho} = -i[H_{\text{total}}, \rho] - \frac{\kappa_a}{2} (a \rho a^{\dagger} - 2a^{\dagger} \rho a + \rho a^{\dagger} a) - \frac{\kappa_b}{2} (b \rho b^{\dagger} - 2b^{\dagger} \rho b + \rho b^{\dagger} b) - \sum_{j=0.1,2} \sum_{m=1,2,3} \gamma_{mL_j} (\sigma_{jL_j} \rho - 2 \sigma_{jL_j}^{\dagger} \rho \sigma_{jL_j}^{\dagger} + \rho \sigma_{jL_j}),
$$

(34)

where $\gamma_{mL_j}^{L_0}$ represents the branching ratio of the atom decay from $|L\rangle_m$ to $|j\rangle_m$ ($m = 1, 2, 3$), and we assume $\gamma_{mL_0} = \gamma_{mL_1} = \gamma_{mL_2} = \gamma/3$, $\kappa_a = \kappa_b = \kappa$ for simplicity. By solving this master equation numerically in the subspace spanned by equation (11) and $|112\rangle|0_a0_b\rangle$, $|122\rangle|0_a0_b\rangle$, $|211\rangle|0_a0_b\rangle$, $|212\rangle|0_a0_b\rangle$, $|121\rangle|0_a0_b\rangle$ and $|221\rangle|0_a0_b\rangle$ with the quantum optics toolbox [32], we obtain the fidelity versus $\gamma/g$ and $\kappa/g$ in figure 4. From this, we can see that the prepared three-atom singlet state $|S_3\rangle$ is especially immune to the cavity decay, since the fidelity is still approaching 99.00% even for a large decay rate $\kappa/g = 0.1$. The physical principle behind this phenomenon is that once the Zeno requirements are satisfied, the cavity-photon population is excluded during the operation; hence the process of converting the two-atom singlet state into the three-atom singlet state is always in the decoherence-free subspace with respect to cavity decay. Because one state in the Zeno subspace $|\psi_1\rangle$ involves the excited level $|L\rangle$, the fidelity of $|S_3\rangle$ is more sensitive to the effect of decoherence induced by atomic spontaneous emission; it drops to 96.12% from 99.36% when $\gamma/g$ increases from 0 to 0.01 for an ideal bimodal cavity.

4. Discussion and generalization in theory

Now we give a brief discussion on the experimental realization. The atomic configuration may be realized in cesium atoms for our proposal. The states $|0\rangle$ correspond to $F = 4, m = 3$.
Figure 4. The evolution of the fidelity for the prepared $|S_3\rangle$ state versus $\kappa/g$ and $\gamma/g$ with $\Omega = 0.1g$.

The hyperfine state of $6^2S_{1/2}$ electronic ground state, $|1\rangle$ corresponds to $F = 3$, $m = 2$ hyperfine state of $6^2S_{1/2}$ electronic ground state, $|2\rangle$ corresponds to $F = 3$, $m = 4$ hyperfine state of $6^2S_{1/2}$ electronic ground state, and the excited state $|L\rangle$ corresponds to $F = 4$, $m = 3$ hyperfine states of $6^2P_{1/2}$ electronic excited states, respectively. In the current cavity QED system [33], the parameter conditions $g/2\pi = 750$ MHz, $\gamma/2\pi = 2.62$ MHz and $\kappa/2\pi = 3.5$ MHz have been achievable in a toroidal microcavity system with the cavity mode wavelength of about 852 nm. By substituting these typical parameters into equation (34), we obtain the fidelity of the three-atom singlet state $|S_3\rangle$ is about 98%. It is interesting to note that based on the same physical principle (quantum Zeno dynamics), an $N$-atom singlet state $|S_N\rangle$ can be obtained immediately from an $(N-1)$-singlet state $|S'_{N-1}\rangle$. Suppose that there are $N$ identical atoms with the energy level diagram shown in figure 5. The level transition between $|L_i\rangle \leftrightarrow |M_j\rangle$ ($M = 1, 2, \ldots, N - 1$) is coupled to the cavity mode $k$ ($k = 1, 2, \ldots, N - 1$), with the coupling strength $g_k$, and the transition between $|L_i\rangle \leftrightarrow |0\rangle$ is driven by the classical field with the Rabi frequency $\Omega_1$. For convenience, we set $g_k^i = g$, $\Omega_2 = \Omega_3 = \ldots = \Omega_N = \Omega$, and $\Omega_1$ to be real. If the Zeno requirements are achieved, i.e. $\Omega_1, \Omega \ll g$, the effective Hamiltonian for governing the evolution of the system reads

$$H_E = \frac{\Omega_1}{\sqrt{N}} |0\rangle \langle S'_{N-1}| \langle \chi | + \frac{\sqrt{N!} - (N - 1)! \Omega}{\sqrt{N!}} |\chi\rangle \langle \lambda_N | + \text{H.c.}, \quad (35)$$

where

$$|\chi\rangle = \frac{1}{\sqrt{N!}} \sum_{L,1,2,\ldots,N-1} \epsilon_{L,1,2,\ldots,N-1} |L, 1, 2, \ldots, N-1\rangle, \quad (36)$$
Figure 5. A potential atomic energy-level diagram to be used for converting $|S'_{N-1}\rangle$ into $|S_N\rangle$.

and

$$|\lambda_N\rangle = \frac{1}{\sqrt{N! - (N-1)!}} \left[ (-1) \sum \epsilon_{0,2,3,...,N-1} |0, 2, 3, \ldots, N - 1\rangle 
+ |2\rangle \sum \epsilon_{0,1,3,...,N-1} |0, 1, 3, \ldots, N - 1\rangle - |3\rangle \sum \epsilon_{0,1,2,...,N-1} |0, 1, 2, \ldots, N - 1\rangle 
+ \cdots + (-1)^{N-1} |N - 1\rangle \sum \epsilon_{0,1,2,...,N-2} |0, 1, 2, \ldots, N - 2\rangle \right].$$  \hspace{1cm} (37)

provided that the initial state is in the $(N - 1)$-atom singlet state

$$|0\rangle |S'_{N-1}\rangle = \frac{1}{\sqrt{(N-1)!}} |0\rangle \sum \epsilon_{1,2,...,N-1} |1, 2, \ldots, N - 1\rangle.$$  \hspace{1cm} (38)

In the above equations, we have omitted the state of the cavity as all modes of the cavity are always in the vacuum state during the whole operation due to the quantum Zeno dynamics. If the corresponding parameters satisfy the conditions

$$\sqrt{\alpha^2 + \beta^2} t = \pi,$$

$$\frac{\beta^2 - \alpha^2}{\sqrt{(N-1)!}} = \frac{-2\alpha\beta}{\sqrt{N! - (N-1)!}},$$  \hspace{1cm} (39)

where $\alpha = \Omega_1/\sqrt{N}$, $\beta = \sqrt{N! - (N-1)!} \Omega/\sqrt{N!}$, then the $N$-atom singlet state

$$|S_N\rangle = \frac{1}{\sqrt{N!}} \sum \epsilon_{0,1,2,...,N-1} |0, 1, 2, \ldots, N - 1\rangle$$  \hspace{1cm} (40)

can be generated. Without loss of generality, we take the case of generating a four-atom singlet state $|S_4\rangle$ as an example. In figure 6, we plot the evolution for the populations of states $|S_3\rangle'$ and $|\lambda_4\rangle$ with the effective Hamiltonian equation (35) (solid line) for $N = 4$ compared with the ones obtained from the full Hamiltonian (dashed line). It is clear that the agreement between the
Figure 6. The time evolution of the populations of $|S_3\rangle$ and $|\lambda_4\rangle$ for the preparation of $|S_4\rangle$, where $\Omega = 0.1g$.

exact and effective models is excellent under the given parameters. Once the time is selected, the final state is obtained as

$$\frac{1}{2}|0\rangle|S_3\rangle + \frac{\sqrt{3}}{2}|\lambda_4\rangle = \frac{1}{\sqrt{6}} \left( |123\rangle - |132\rangle - |213\rangle + |312\rangle + |231\rangle - |321\rangle \right)$$

$$+ \frac{\sqrt{3}}{4} \times \frac{1}{\sqrt{18}} \left( -|1023\rangle + |1032\rangle + |1203\rangle - |1302\rangle - |1230\rangle + |1320\rangle + |2013\rangle - |2031\rangle - |2103\rangle + |2301\rangle + |2130\rangle - |2310\rangle - |3012\rangle + |3021\rangle - |3201\rangle + |3102\rangle - |3120\rangle \right)$$

$$= |S_4\rangle.$$  

Although the two-atom singlet state of equation (30) is the simplest two-qubit entanglement state, and it has been created in the laboratory, it is necessary to illustrate how to prepare this state in our system. We find that it can also be carried out via the quantum Zeno dynamics. Suppose two four-level tripod atoms simultaneously interacting with a single-mode cavity, whose frequency is only resonant with the transition between $|L_n\rangle$ and $|1_n\rangle$, as shown in figure 7. If we assume $g_1^\alpha = g_2^\alpha = g$ and $\Omega_n \ll g$, i.e. the Zeno requirements are satisfied, then the effective Hamiltonian can be written as

$$H_{\text{eff}} = \frac{1}{\sqrt{2}}(\Omega_2|12\rangle|0_a\rangle \langle D| - \Omega_1|D\rangle\langle 21|)(|0_a\rangle + \text{H.c.}),$$

where

$$|D\rangle = \frac{1}{\sqrt{2}}(|1L\rangle - |L1\rangle)|0_a\rangle,$$

then the state $(|12\rangle - |21\rangle)/\sqrt{2}$ is actualized within one step by choosing $\Omega_1 = -(\sqrt{2} + 1)\Omega_2$ after time $t = \sqrt{2} \pi / \sqrt{\Omega_1^2 + \Omega_2^2}$, provided that the atoms are initialized in the state $|12\rangle|0_a\rangle$. 

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Two atoms are exploited for generating the two-atom singlet state $(|12⟩ - |21⟩)/\sqrt{2}$. The atoms interact with a single-mode cavity, whose frequency is only resonant with the transition between $|L_n⟩$ and $|1_n⟩$. Two classical fields are applied to couple the transitions $|L_n⟩ \leftrightarrow |2_n⟩$.

Based on the above analysis, we can safely say that the $N$-atom singlet state $|S_N⟩$ can be prepared in a scalable way from the initial state $|0, 1, 2, \ldots, N - 1⟩$.

5. Summary

In summary, we have proposed an approach for generating the three-atom singlet state from a two-atom singlet state by means of quantum Zeno dynamics. The result shows that a relatively high fidelity can be obtained when the effects of decoherence are taken into account. We further generalize this scheme to implement an arbitrary $N$-atom singlet state, on the condition that the $(N - 1)$-atom singlet state is ready-made. However, there has been no report about the strong interaction between the multimode field and a multilevel atom simultaneously inside one cavity experimentally yet; besides, it is hard to get rid of the many other states in the multiplets of a multi-level atom. Thus the generalization only originates from a theoretical viewpoint, but even so it can pave the way for the creation of multi-atom singlet state in the future as there are no other reports about the generation of multi-atom singlet state even in theory. Finally, the method of preparing the initial two-atom singlet state through the same physical principle is also provided. We hope that our work may be useful for the experimental realization of quantum information in the near future.

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