Interferometric geometry from symmetry-broken Uhlman gauge group and applications to topological phase transitions

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(Dated: October 15, 2020)

We provide a natural generalization of a Riemannian structure, i.e., a metric, recently introduced by Sjöqvist [1] for the space of non degenerate density matrices, to the degenerate case, i.e., in which the eigenspaces have dimension greater or equal to one. We present a physical interpretation of the metric in terms of an interferometric measurement. We study this metric, physically interpreted as an interferometric susceptibility, to the study of topological phase transitions at finite temperatures for band insulators. We compare the behaviors of this susceptibility and the one coming from the well-known Bures metric, showing them to be dramatically different. While both infer zero temperature phase transitions, only the former predicts finite temperature phase transitions as well. The difference in behaviours can be traced back to a symmetry breaking mechanism, akin to Landau-Ginzburg theory, by which the Uhlmann gauge group is broken down to a subgroup determined by the type of system’s density matrix (i.e., the ranks of its spectral projectors).

I. INTRODUCTION

Recent advances in information geometry have provided new methods for studying quantum matter and describing macroscopic critical phenomena based on quantum effects. Topological phases of matter are described in terms of global topological invariants that are robust against continuous perturbations of the system. An example of these invariants is the Thouless-Kohmoto-Nightingale-den Nijs (TKNN) invariant, mathematically a Chern number associated to the vector bundle of occupied Bloch states over the Brillouin zone. This invariant captures topological phases of matter that could not be understood previously, such as the case of the anomalous Hall insulator [2], which falls into the class of Chern insulators. The classification of topological phases of gapped free fermions is encoded in the so-called periodic table of topological insulators and superconductors [3]. However, by now we know that these phases of matter were just the tip of an iceberg see [4–7]. The theory underlying topological phases constitutes a change of paradigm with respect to the Landau theory of phase transitions [8]. The latter is described by means of a local order parameter, within the framework of the symmetry-breaking mechanism.

One can study phases of matter and the associated phase transitions (in particular topological ones) through a Riemannian metric on the space of quantum states. One such commonly used structure is based on the notion of fidelity, which is an information theoretical quantity that measures the distinguishability between quantum states. It has been widely used in the study of phase transitions [9–19], since its non-analytic behaviour signals phase transitions.

Within the context of dynamical phase transitions, occurring when one performs a quench on a system, the
information geometric methods based on state distinguishability were applied [26]. In particular, for finite temperature studies, besides the standard notion of fidelity induced Loschmidt echo, a notion of interferometric Loschmidt echo based on the interferometric phase introduced by Sjöqvist et al. in [27], was also considered. With regards to the associated infinitesimal counterparts, i.e., Riemannian metrics, their behaviour is significantly different.

For two-band Chern insulators, the fidelity susceptibility, one of the components of the Bures metric, was considered in detail in Ref. [23]. In particular, it was rigorously proven that the thermodynamic and zero temperature limits do not commute – the Bures metric is regular in the thermodynamic limit as one approaches the zero temperature limit.

In this paper, we provide, through what is called the Ehresmann connection, a natural generalization of a Riemannian structure over the space of non degenerate density matrices, introduced by Sjöqvist in Ref. [1], to the degenerate case. Our natural construction reveals a symmetry breaking mechanism by reducing the gauge group of the Uhlmann principal bundle [28], to a smaller subgroup preserving the type, i.e., the ranks of its spectral projectors, of the density matrix (see Sec. II for details). This symmetry breaking mechanism explains the natural enhanced distinguishability provided by the interferometric Riemannian metric. Introducing the notion of a generalized purification, we naturally generalize Sjöqvist’s result to the case of degenerate density matrices, see Sec. III.

In Sec. IV, we discuss an interferometric measurement probing the Riemannian metric derived. In Sec. V, we apply the derived metric to study finite temperature phase transitions in the context of band insulators. We present results for this metric in the case of the massive Dirac model, a Chern insulator, in two spatial dimensions, and compare them with those obtained using Bures metric. Our analysis of equilibrium phase transitions showed to be consistent with the previous study of dynamical phase transitions – the interferometric metric is more sensitive to the change of the parameters than the Bures one. Finally, we present conclusions in Sec. VI.

II. THE GEOMETRY OF THE SJÖQVIST METRIC AND NATURAL GENERALISATIONS TO DEGENERATE CASES

Consider a quantum system with the corresponding $n$-dimensional Hilbert space $\mathcal{H}$. Its general mixed state (density matrix) $\rho$ can be, using the spectral decomposition, written as

$$\rho = \sum_{i=0}^{k} p_i P_i,$$

where the real eigenvalues satisfy $p_0 = 0$ and $(i \neq j \Rightarrow p_i \neq p_j)$, while the orthogonal projectors satisfy $(i > 0 \Rightarrow \text{Tr} P_i \equiv r_i > 0)$, and $\sum_{i=1}^{k} r_i = r$. We call $r \in \{1, \ldots, n\}$ the rank of the state. Note that we do not require for the kernel of $\rho$ to be nontrivial (i.e., $r_0 \equiv \text{Tr} P_0 \geq 0$), while all other eigenspaces, $\mathcal{H}_i$, are at least one-dimensional (such that $\mathcal{H} = \bigoplus_{i=1}^{k} \mathcal{H}_i$). We call the $k$-tuple $\tau \equiv (r_1, r_2, \ldots, r_k) \in \mathcal{T}$, with $k \in \{1, \ldots, n\}$ and $(1 \leq r_1 \leq r_2 \leq \cdots \leq r_k)$, the type of the state $\rho$, where $\mathcal{T}$ is the set of all possible types. Note that as a consequence of the normalization of density matrices we have the additional constraint

$$\sum_{i=1}^{k} r_i p_i = 1.$$  \hfill (2)

Consider the set of all density operators of type $\tau$, denoted by $B_\tau$. The union, over the types $\tau \in \mathcal{T}$, of all sets $B_\tau$ forms the set of all possible states of a given system,

$$B = \bigcup_{\tau \in \mathcal{T}} B_\tau,$$

$$= \{ \rho \in \mathcal{H} \otimes \mathcal{H}^* : \rho^\dagger = \rho \text{ and } \rho \geq 0 \text{ and } \text{Tr} \rho = 1 \}. $$  \hfill (3)

We would like to analyse the geometry of the $B_\tau$’s, and see whether it is possible to induce a Riemannian metric on them along the lines of the metric introduced by Sjöqvist [1], for the case of type $\tau = (1,1,\ldots,1)$, for some $r = k$. We will do so by introducing gauge invariant Riemannian metrics and associated Ehresmann connections in suitably chosen principal bundles $P_\tau$ with corresponding base spaces $B_\tau$. Observe that every state $\rho$ is completely specified in terms of its “classical part”, the vector of (rescaled by the rank) probabilities $\sqrt{\mathbf{P}} = (\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_k})$ and its “quantum part”, the mutually orthogonal projectors $P_1, P_2, \ldots, P_k$ (note that $P_0$ is then determined unambiguously, $P_0 = I - \sum_{i=1}^{k} P_i$), which we compactly denote by $\mathbf{P} = (P_1, P_2, \ldots, P_k)$. We will explore a particular gauge degree of freedom in describing the quantum part in our construction. Namely, each eigenspace projector $P_i$ is uniquely specified by an orthonormal basis $\beta_i = \{ |e_{i,j} \rangle : j = 1, \ldots, r_i \}$. However, the basis $\beta_i$ itself is not uniquely determined by $P_i$. Indeed, every basis $U \beta_i = \{ |e_{i,j} \rangle : j = 1, \ldots, r_i \}$ with $U$ a unitary that acts non-trivially only on the image of $P_i$, the subspace $\mathcal{H}_i$, defines the same projector $P_i$.

We then define (the total space of) a principal bundle $P_\tau$ as the set of all $k$-tuples of pairs $p_i = (p_i, \beta_i)_{i=1}^{k}$, such that $(\sqrt{\mathbf{P}}, \mathbf{P})$ give rise to well-defined type $\tau$ density operators (observe that $p_i \neq p_j$ for all $i \neq j$). This space comes equipped with an obvious projection to the base space $B_\tau$ is given by

$$\pi_\tau(p_\tau) \equiv \sum_{i=1}^{k} p_i P_i = \rho,$$  \hfill (4)

with the fibers being isomorphic to the product of the
corresponding unitary groups in the type $\tau$,

$$G_\tau \equiv \prod_{i=1}^{k} U(r_i).$$  \hfill (5)

The group $G_\tau$ acts on the right in the obvious way, for $U_i \in U(r_i)$, we write $U_i = [(U_i)^j_j]_{1 \leq j \leq r_i} \in U(r_i)$ and then $\beta_i \cdot U_i$ is given by

$$|e_{i,j}^j \mapsto \sum_{j' = 1}^{r_i} |e_{i,j'}^j)(U_i)^j_j', \quad j = 1, \ldots, r_i.$$  \hfill (6)

By introducing *generalized amplitudes* $w_i \in \mathbb{C}^{n \times r_i}$ as matrices whose columns are vectors $|e_{i,j}^j⟩ \in \mathbb{C}^n$, $j = 1, \ldots, r_i$, i.e., $w_i \equiv (|e_{i,1}⟩, \ldots, |e_{i,r_i}⟩)$, $i = 1, \ldots, k$, we can see $P_\tau$ as

$$P_\tau = \{(p_i, w_i) : \big(\sum_{i=1}^{k} p_i w_i w_i^\dagger \in B_\tau \big) \quad \text{and} \quad w_i \in \mathbb{C}^{n \times r_i}, \forall i = 1, \ldots, k, \text{such that } w_i \neq w_j \text{, for all } i \neq j\},$$  \hfill (7)

and the right action of the gauge group is given by $w_i \mapsto w_i \cdot U_i$, with $U_i \in U(r_i)$. With this notation, we finally introduce a suitable “Hermitian form” (note that it is not a scalar product, as $P_\tau$ is not a linear space), that will define Horizontal subspaces, by the formula

$$\langle p_r, p_r' \rangle_\tau \equiv \sum_{i=1}^{k} \sqrt{p_i p_i'} \text{Tr}(w_i^\dagger w_i')$$  \hfill (8)

Observe that it is clear that this pairing arises from the restriction of the usual Hermitian inner product in $\bigoplus_{i=1}^{k} \mathbb{C}^{n \times r_i} \cong \mathbb{C}^{n \times r}$. Additionally, this allows for a convenient comparison with the Uhlmann principal bundle

$$P_\tau^{\text{Uhlmann}} = \{w \in \mathbb{C}^{n \times r} : \pi(w) = pw = \rho \in B, \quad \text{with } \text{rank}(\rho) = r\},$$  \hfill (9)

where the typical fibre is $U(r) \subset \mathbb{C}^{r \times r}$, whose elements act from the right ($w \mapsto w \cdot U$), and the Hermitian form, induced by the Hilbert-Schmidt scalar product on the space of linear operators from $\mathbb{C}^{r \times r}$, is

$$\langle w, w' \rangle = \text{Tr}(w^\dagger w').$$  \hfill (10)

Note that the base space for the Uhlmann bundle is the set of density matrices with rank $r$, which is the union of all $B_\tau$ sharing the same rank. Observe that for one such $\tau$, $P_\tau$ can be identified as a subset of $P_\tau^{\text{Uhlmann}}$. This follows from the map

$$P_\tau \ni (p_i, w_i)_{i=1}^{k} \mapsto (\sqrt{p_1} w_1, \ldots, \sqrt{p_k} w_k) \in \bigoplus_{i=1}^{k} \mathbb{C}^{n \times r_i},$$  \hfill (11)

being an embedding of $P_\tau$. Moreover, once we identify $\bigoplus_{i=1}^{k} \mathbb{C}^{n \times r_i} \cong \mathbb{C}^{n \times r}$, the image sits precisely in $P_\tau^{\text{Uhlmann}}$. In other words $P_\tau \subset P_\tau^{\text{Uhlmann}}$ and also $\pi_\tau$ equals the restriction of the projection of the Uhlmann bundle to $P_\tau$ ($p_i \neq p_j$, for all $i \neq j$, guarantees this), the image being precisely $B_\tau$. We remark that the gauge group of the Uhlmann bundle is far larger than the one for the principal bundle $P_\tau \rightarrow B_\tau$. By passing to a preferred type, we performed a symmetry breaking operation from $U(r)$ to $G_\tau = \prod_{i=1}^{k} U(r_i) \subset U(r)$. This is another way to see why interferometric-like quantities, like the interferometric Loschmidt echo, in certain applications develop non-analyticities, while the ones based on the fidelity don’t (see for example [29] and the references therein): the former have smaller space to “go through”, while the latter can, following the “broader” Uhlmann connection, instead of the interferometric ones, avoid possible sources of non-analyticities.

### III. DISTANCE MEASURES AND RIEMANNIAN METRICS

Consider now two points, $p_\tau = (p_i, w_i)_{i=1}^{k}$ and $q_\tau = (q_i, v_i)_{i=1}^{k} \in P_\tau$. By making use of Eq. (8) one can define a distance between elements $p_\tau$ and $q_\tau$ in the total space of the principal bundle given by

$$d_\tau^2(p_\tau, q_\tau) = 2 \left(1 - \text{Re}(\langle p_\tau, q_\tau \rangle_\tau)\right) = 2 \left(1 - \sum_{i=1}^{k} \sqrt{p_i q_i} \text{ Re}(\text{Tr}(w_i^\dagger v_i))\right).$$  \hfill (12)

The fact that $d_\tau$ is a distance follows from the fact that it is the restriction of the usual distance in $\bigoplus_{i=1}^{k} \mathbb{C}^{n \times r_i}$, where we see $P_\tau$ as a subset of this space through the map of Eq. (11). One can use this distance to define a distance on $B_\tau$, through the formula:

$$d_\tau^2(\rho, \sigma) = \inf \{d_\tau^2(p_\tau, q_\tau) : \pi(p_\tau) = \rho, \pi(q_\tau) = \sigma, \text{ for } p_\tau, q_\tau \in P_\tau\}.$$  \hfill (13)

The associated infinitesimal counterparts of the distances defined above are Riemannian metrics on $P_\tau$ and $B_\tau$, respectively. The Riemannian metric on $P_\tau$, which is gauge invariant, allows for the definition of what is called an Ehresmann connection over $P_\tau$ and this, in turn, defines a metric downstairs over the base space $B_\tau$.

Another way to see that $d_\tau^2(p_\tau, q_\tau)$ is indeed a metric is through what we call “generalised purifications”. Let us introduce “ancilla” amplitudes $w_i \in \mathbb{C}^{k_i \times 1}$, with $i = 1, 2, \ldots, k$, such that $w_i w_i^\dagger = P_i \in \mathbb{C}^{n \times n}$ are fixed orthogonal projectors of rank $1$ (i.e., $P_i$ do not depend on the choice of the state), satisfying $P_i P_j = \delta_{ij} I_k$ and $\sum_{i=1}^{k} P_i = I_k$. Define a generalised purification of state...
$\rho$, associated to the corresponding $p_r$, as

$$|p_r\rangle = \sum_{i=1}^{k} \sqrt{p_i} w_i \otimes w_i. \quad (14)$$

Then, we have that the scalar product between $|p_r\rangle$ and $|q_r\rangle$, induced by the Hilbert-Schmidt scalar product in the corresponding factor spaces, is

$$\langle p_r, q_r \rangle = \sum_{i,j=1}^{k} \sqrt{p_i q_j} \langle w_i, v_j \rangle \langle w_i, w_j \rangle = \sum_{i=1}^{k} \sqrt{p_i q_i} \langle w_i, v_i \rangle$$

$$= \sum_{i=1}^{k} \sqrt{p_i q_i} \sum_{j=1}^{k} \sqrt{p_j q_j} \langle w_i, w_j \rangle = \sum_{i=1}^{k} \sqrt{p_i q_i} \text{Tr}(w_i^\dagger v_i)$$

$$= \langle p_r, q_r \rangle_r, \quad \text{where the second equality is because } w_i \text{ and } w_j \text{ are orthogonal for } i \neq j. \quad (15)$$

Thus, the distance $d_r(p_r, q_r)$ is nothing but the standard Hilbert-Schmidt distance between the generalised purifications $|p_r\rangle$ and $|q_r\rangle$.

As in Eq. (7), if we take the $w_i$’s as (row) vectors $|w_i\rangle = \left[|e_{i,1}\rangle |e_{i,2}\rangle \ldots |e_{i,r_i}\rangle\right]$ whose entries are (column) vectors $|e_{i,j}\rangle$, one can by analogy generalise the quantum part of the metric for the non-degenerate case, the so-called “interferometric metric”, as

$$g^Q_i = \sum_{i=1}^{k} p_i \langle dw_i |(I_n - w_i w_i^\dagger)|dw_i\rangle = \sum_{i=1}^{k} p_i \langle de_{i,1} |(I_n - |e_{i,1}\rangle\langle e_{i,1}|)|de_{i,1}\rangle, \quad (16)$$

which is the “infinitesimal” distance between $\rho$ and $\rho' = \rho + \delta \rho$, where $\delta \rho = \dot{\rho} \delta t$ (see Appendix B for a detailed proof). Note that in the case of the Hadamard matrix, given by $|\ell\rangle \rightarrow (|0\rangle + (-1)^{\ell}|1\rangle)/\sqrt{2}$, with $\ell \in \{0,1\}$, the roles of arms 0 and 1 are exchanged.

IV. INTERFEROMETRIC MEASUREMENT INTERPRETATION

Consider the following experiment depicted in FIG 1. A particle is entering the Mach-Zehnder interferometer from the input arm 0, given by the state $|0\rangle$, with its internal degree of freedom in a mixed state $\rho$. Both the input and the output beam-splitters are balanced, described by the same unitary matrix, say, the one given by $|0\rangle \rightarrow (|0\rangle + i|1\rangle)/\sqrt{2}$. In arm 0 a unitary $V = \sum_{i=0}^{k} P_i V P_i$ is applied to the internal degree of freedom, i.e., $V$ is the most general unitary that commutes with $\rho$. In arm 1 a unitary $U = U(\delta t) \in U(n)$ is applied for a time period $\delta t$, changing the state of the internal degree of freedom to $\rho' = U \rho U^\dagger$. The particle is detected at detectors D0 and D1, with the corresponding probabilities $p_{r_0}$ and $p_{r_1}$. In our case of the Hadamard beam-splitters, we have that $p_{r_0} \leq p_{r_1}$, and for $U = V$ we have full constructive interference at the output arm 0, giving $p_{r_0} = 1$. In general, we have that

$$p_{r_1}^\text{max} = \max_{\{V_i\}} (p_{r_1}) = 1 - \frac{1}{4} g^Q_i(\rho, \rho + \delta \rho), \quad (18)$$

where $g^Q_i(\rho, \rho + \delta \rho) \approx g_1(\dot{\rho}, \delta \rho) \delta t^2$ is the “infinitesimal” distance between $\rho$ and $\rho' = \rho + \delta \rho$, where $\delta \rho = \dot{\rho} \delta t$ (see Appendix B for a detailed proof). Note that in the case of the Hadamard matrix, given by $|\ell\rangle \rightarrow (|0\rangle + (-1)^{\ell}|1\rangle)/\sqrt{2}$, with $\ell \in \{0,1\}$, the roles of arms 0 and 1 are exchanged.

![FIG. 1: Interferometric measurement to probe the generalised metric $g_i$.](image-url)
V. INTERFEROMETRIC METRIC IN THE CONTEXT OF BAND INSULATORS

Suppose we have a family of band insulators with two bands described by the Hamiltonian
\[ H(M) = \int_{BZ^d} \frac{d^d k}{(2\pi)^d} \psi^\dagger_k d^\mu k \sigma^\mu \psi_k, \]
parametrized by \( M \) (\( M \) can be some intrinsic parameter, such as the hopping), where \( \sigma_\mu, \mu = 1, 2, 3, \) are the Pauli matrices, \( k \) is the crystalline momentum in a \( d \)-dimensional Brillouin zone \( BZ^d \), with \( d = 1, 2, 3 \), and \( \Psi_k \) is an array of 2 creation operators for fermions at momentum \( k \). We assume that the system is gapped for generic values of \( M \), meaning that the vector \( d = (d^1, d^2, d^3) \) is non-vanishing as a function of \( k \). For a certain value of \( M \), we assume that the vector has isolated zeroes. This assumption is generically correct for the \( d = 1, 2 \) momenta coordinates plus the mass \( M \), as one needs to tune three parameters for an Hermitian matrix to have two eigenvalues cross.

The pullback of the interferometric metric that we have described in Sec. III,
\[ g = \frac{1}{4} \sum_i r_i \frac{d \rho_i^2}{\rho_i} + \sum_i \rho_i \text{Tr} (P_i d P_i d P_i), \]
with \( \rho = \sum_i p_i P_i \) and \( \text{Tr} P_i = r_i \), by the map induced by the Gibbs state
\[ M \mapsto \rho(M) = Z^{-1} \exp(-\beta H(M)), \]
where \( Z \) is the partition function, is given by
\[ ds^2 = \frac{1}{4} \int_{BZ^d} \frac{d^d k}{(2\pi)^d} \left[ \frac{1}{\cosh(\beta E) + 1} \right. \]
\[ \left. \times \left( \beta^2 \left( \frac{\partial E}{\partial M} \right)^2 + \cosh(\beta E) \delta_{\mu \nu} \frac{\partial n^\mu}{\partial M} \frac{\partial n^\nu}{\partial M} \right) \right] d^2 M, \]
where we omitted the obvious dependence on \( k \) and \( M \) of the quantities \( E \) and \( n^\mu \). We provide a technical derivation of this result in Appendix C. This result should be compared to the pullback of the Bures metric for \( d = 2 \), which yields (see Ref. [23])
\[ g_{\text{Bures}} = \frac{1}{4} \int_{BZ^d} \frac{d^d k}{(2\pi)^d} \left[ \frac{1}{\cosh(\beta E) + 1} \right. \]
\[ \left. \times \left( \beta^2 \left( \frac{\partial E}{\partial M} \right)^2 + \cosh(\beta E) \delta_{\mu \nu} \frac{\partial n^\mu}{\partial M} \frac{\partial n^\nu}{\partial M} \right) \right] d^2 M. \]

The two expressions have dramatically different behaviours, when it comes to taking the zero temperature limit.

Naively, one would say that both yield the pullback of the Fubini-Study metric, which is the pure-state metric,
\[ g_0 = \frac{1}{4} \int_{BZ^d} \frac{d^d k}{(2\pi)^d} \delta_{\mu \nu} \frac{\partial n^\mu}{\partial M} \frac{\partial n^\nu}{\partial M} d^2 M. \]

Note that for gapless points the vector \( n \) is not defined and the expression for \( g_0 \) becomes (potentially) singular. However, due to the gapless points, the integrands must be carefully analysed in the neighbourhoods of these points, as the singularities can be avoided in some cases. In fact, it was shown that if the gapless points are isolated in momentum space, then an expansion near these points of the integrand function yields a regular result [23]. Namely, because of the inequality
\[ \frac{1}{2} \frac{1}{\cosh(x)} < \frac{1}{\cosh(x) + 1}, \quad \text{for all } x \in \mathbb{R}, \]
we can write,
\[ \frac{1}{\cosh(\beta E) + 1} \beta^2 \left( \frac{\partial E}{\partial M} \right)^2 + \cosh(\beta E) - 1 \delta_{\mu \nu} \frac{\partial n^\mu}{\partial M} \frac{\partial n^\nu}{\partial M} \]
\[ < \frac{1}{\cosh(\beta E)} \left[ \beta^2 \left( \frac{\partial E}{\partial M} \right)^2 + (\cosh(\beta E) - 1) \delta_{\mu \nu} \frac{\partial n^\mu}{\partial M} \frac{\partial n^\nu}{\partial M} \right]. \]

Expansion for small \( \beta E \) yields up to \( O \left( (\beta E)^4 \right) \) the integrand is upper bounded by
\[ \frac{\beta^2}{\cosh(\beta E)} \delta_{\mu \nu} \frac{\partial n^\mu}{\partial M} \frac{\partial n^\nu}{\partial M}, \]
which is regular in the limit \( \beta \to \infty \). Hence, the potential singularities arising from the gapless region are regularized by the Bures prescription. However, in the case of the interferometric metric, considering the integrand
\[ \frac{1}{\cosh(\beta E) + 1} \left[ \beta^2 \left( \frac{\partial E}{\partial M} \right)^2 + \cosh(\beta E) \delta_{\mu \nu} \frac{\partial n^\mu}{\partial M} \frac{\partial n^\nu}{\partial M} \right] d^2 M, \]
\[ \text{near } E = 0 \text{ gives us} \]
\[ \frac{1}{\cosh(\beta E) + 1} \left[ \beta^2 \left( \frac{\partial E}{\partial M} \right)^2 + (1 + \frac{1}{2} \beta^2 E^2) \delta_{\mu \nu} \frac{\partial n^\mu}{\partial M} \frac{\partial n^\nu}{\partial M} \right] + O \left( (\beta E)^4 \right). \]

In this case, we cannot get rid of the singular factor
\[ \delta_{\mu \nu} \frac{\partial n^\mu}{\partial M} \frac{\partial n^\nu}{\partial M}. \]
which appears once in the second term without the regularizing coefficient \( \beta^2 E^2 \) which above allowed for the identification of the regular quantity
\[ \beta^2 \left( \frac{\partial E}{\partial M} \right)^2 + \beta^2 E^2 \delta_{\mu \nu} \frac{\partial n^\mu}{\partial M} \frac{\partial n^\nu}{\partial M} = \beta^2 \delta_{\mu \nu} \frac{\partial d^\mu}{\partial M} \frac{\partial d^\nu}{\partial M}. \]

This implies that the limit \( \beta \to \infty \) yields singular behaviour for \( g \), provided the same happens with \( g_0 \). But not the other way around, i.e., singular behaviour on
the finite temperature metric does not imply zero temperature singular behaviour. In other words, while in the case of the Bures metric the thermodynamic and the zero temperature limits did not commute, in the interferometric case they do, because the singular behaviour of the gapless points is recovered, as one considers a small neighbourhood of these points and takes the zero temperature limit. In the following, we will consider the massive Dirac model to illustrate the different behaviours of the two metrics.

A. Massive Dirac model

We consider the massive Dirac model, a band insulator in two spatial dimensions, described by Eq. (19), with

\[ d(k; M) = (\sin(k_x), \sin(k_y), M - \cos(k_x) - \cos(k_y)) , \]

(32)

where \( k = (k_x, k_y) \) is the quasi-momentum in the two-dimensional Brillouin zone \( BZ^2 \) and \( M \) is a real parameter. The model exhibits topological phase transitions [30]. We will focus at the one occurring at \( M = 0 \), where the Chern number goes from +1, for \( M \to 0^- \), to \(-1\), for \( M \to 0^+ \). The following two figures describe the interferometric metric (Fig. 2a) and the Bures metric (Fig. 2b) in the thermodynamic limit.

As argued above, the Bures metric is regular if one considers the thermodynamic limit and then the zero temperature limit. The same does not hold for the interferometric metric. In fact, we can see that the interferometric metric knows about the quantum phase transition taking place at \( T = 0 \) even at finite temperatures. The reason is that in passing from one metric to the other the symmetry was broken, namely \( U(r) \to \prod_{i=1}^{r_i} U(r_i) \), and, therefore, there is enhanced distinguishability. Indeed, in the interferometric case, whenever the gap closes, we expect a phase transition, even at finite temperatures, because then there are states which according to a Boltzmann-Gibbs distribution become degenerate in probability, hence the gap closing changes the type of the density matrix involved. Whether such singular behaviour of the interferometric metric is indeed observable for macroscopic many-body systems is an open question.

While the straightforward implementation of the interferometric experiment described in Sec. IV seems to be, at least technologically, infeasible, as it would require maintaining Schrödinger cat-like macroscopic states, possible variations are argued to be able to reveal the singular behaviour of the interferometric metric at finite temperatures (see Sec. V of Ref. [31]).

VI. CONCLUSIONS

In this work, we have generalized Sjöqvist’s interferometric metric introduced in [1], to the degenerate case.

For this purpose, we have introduced generalized amplitudes and purifications. We have analysed an interpretation of the metric in terms of a suitably generalized interferometric measurement, accommodating for the non-Abelian character of our gauge group, as opposed to the Abelian gauge group used in the non degenerate case. We have applied the induced Riemannian structure, physically interpreted as a susceptibility, to the study of topological phase transitions at finite temperatures for band insulators. To the best of our knowledge, this is the first study of finite-temperature equilibrium phase transitions using interferometric geometry. The inferred critical behaviour is very different from that of the Bures metric. The interferometric metric is more sensitive to the change of parameters than the Bures one, and unlike the latter, in addition to zero temperature phase transitions, infers finite temperature phase transitions as well. This sensitivity can be traced back to a symmetry breaking mechanism, much in the same spirit of Landau-Ginzburg theory. In our case, by fixing the type of the density matrix considered, a gauge group is broken down to a subgroup.

It would be very interesting to analyse the interferometric curvature, an analogue of the usual Berry curvature, generalized to this mixed setting, associated with
the Ehresmann connection presented in this manuscript. Since the curvature is intrinsically related to topological phenomena, this analysis might very well unravel new symmetry protected topological phases in the mixed state case and potentially help refining the classification of topological matter. It would be also interesting to compare the critical behaviour of different many-body systems in terms of interferometric metrics corresponding to different types of density matrices. Recent study of the fidelity susceptibility indicated that its singular behaviour around regions of criticality has preferred directions on the parameter space [32]. Performing a similar analysis for the interferometric critical geometry is another possible line of future research. Finally, probing experimentally the introduced interferometric metrics is a relevant topic of future investigation.

**ACKNOWLEDGEMENTS**

B.M. and N.P. are thankful for the support from SQIG – Security and Quantum Information Group, the Instituto de Telecomunicações (IT) Research Unit, Ref. UIDB/50008/2020, funded by Fundação para a Ciência e a Tecnologia (FCT), European funds, namely, H2020 project SPARTA, as well as projects QuantMining POCI-01-0145-FEDER-031826 and PREDICT PTDC/CCI-CIF/29877/2017. N.P. acknowledges FCT Estímulo ao Emprego Científico grant no. CEECIND/04594/2017/CP1393/CT0006.

**Appendix A: Induced Riemannian metrics**

Let us look again at the principal bundle $P_r$, for a fixed type $\tau = (r_1, ..., r_k)$. In this case, a point in $P_r$ is given by $p_r = (\{p_i, w_i\})_{i=1}^k$ and can be equivalently represented as $p_r = (\{p_i\}_{i=1}^{k_1},\{w_i\}_{i=1}^{k_2})$. With this identification, we can separate $p_r$ into its “classical” and “quantum” parts:

(i) A classical probability amplitude vector $\sqrt{p} = (\sqrt{p_1}, ..., \sqrt{p_k})$, with $\sum_{i=1}^{k} p_i = 1$ and, for each $i \in \{1, ..., k\}$, $p_i > 0$. Note that the set of all classical probability amplitudes is in fact contained in the $k-1$-dimensional sphere and the associated classical Fisher metric is, up to a factor of $1/4$, the usual round metric in the sphere $S^{k-1}$.

(ii) A quantum part which is a $k$-tuple, i.e., a sequence of matrices $\{w_1, ..., w_k\}$, each of them identifying a $r_i$-unitary frame in $\mathbb{C}^n$, i.e., $w_i \in V_{r_i}(\mathbb{C}^n)$, where

$$V_{r_i}(\mathbb{C}^n) = \{w_i \in \mathbb{C}^{n \times r_i} : w_i^\dagger w_i = I_k\} \subset \mathbb{C}^{n \times r_i},$$

commonly known as the Stiefel manifold of $r_i$-unitary frames in $\mathbb{C}^n$.

Our aim is to compute the Riemannian metric in the base space $B_r$ for a given type $\tau = (r_1, ..., r_k)$. For this purpose, we will first look at the tangent space at a point $p_r$, which is isomorphic to the direct sum

$$T_{p_r} P_r \cong T_{\sqrt{p}} S^{k-1} \oplus \left( \bigoplus_{i=1}^{k} T_{w_i} V_{r_i}(\mathbb{C}^n) \right). \quad (A2)$$

This isomorphism follows from the factorization into classical and quantum parts: for every curve in the total space $P_r$, there will be a tangent vector for each of the curves induced by projection in the different factors of $P_r$.

The classical components have no gauge ambiguity. The quantum components, however, have a $U(r_i)$ gauge degree of freedom for each matrix $w_i$, $i = 1, ..., k$. This gauge ambiguity corresponds to variations along the fibres, as we will mention later on. From a physical standpoint, the exact point in the fibre has no significance, since the matrices $w_i$ will be projected onto the base space, where the projectors $P_i$ are gauge invariant: namely, $w_i$ and $w_i U$, for $U \in U(r_i)$, give rise to the same projector $P_i = w_i w_i^\dagger$, for all $i = 1, ..., k$. Hence, we need to define the horizontal subspaces of the tangent spaces to $P_r$, in order to uniquely represent the tangent spaces to the base space upstairs, i.e., in the tangent spaces to $P_r$. Mathematically, this notion is referred to as an Ehresmann connection, see, for example, Sec. 6.3 of Ref. [33].

Before we proceed, let us focus on one of the Stiefel manifolds, say for a fixed $i \in \{1, ..., k\}$, $V_{r_i}(\mathbb{C}^n)$. For convenience, we define the projection onto the space of projectors of rank $r_i$, identified with the Grassmannian of $r_i$-planes in $\mathbb{C}^n$, i.e., the manifold of linear subspaces of dimension $r_i$ in $\mathbb{C}^n$,

$$\pi_i : V_{r_i}(\mathbb{C}^n) \to Gr_{r_i}(\mathbb{C}^n) \quad (A3)$$

Consider a curve in the Stiefel manifold

$$\gamma_{w_i} : [0, 1] \ni t \mapsto \gamma_{w_i}(t) \in V_{r_i}(\mathbb{C}^n) \quad (A4)$$

subject to the initial conditions $\gamma_{w_i}(0) = w_i$ and

$$\frac{d\gamma_{w_i}}{dt} \bigg|_{t=0} = \dot{w}_i \equiv \ddot{v}. \quad (A5)$$

The vertical space at $w_i \in V_{r_i}(\mathbb{C}^n)$ is the set of tangent vectors in $T_{w_i} V_{r_i}(\mathbb{C}^n)$, such that its infinitesimal projection onto the base space is zero, that is

$$\frac{d}{dt} (\pi_i (\gamma_{w_i}(t))) \bigg|_{t=0} = 0 \quad \Leftrightarrow \frac{d}{dt} (\gamma_{w_i}(t) \gamma_{w_i}^\dagger(t)) \bigg|_{t=0} = w_i w_i^\dagger + w_i^\dagger w_i = 0. \quad (A6)$$
The vertical space is then given by

$$V_{w_i} = \{ \dot{w}_i \in T_{w_i} V_{r_i}(\mathbb{C}^n) : \dot{w}_i w_i^\dagger + w_i \dot{w}_i^\dagger = 0 \}. \quad (A7)$$

The projection $\pi_r$ has derivative, $d\pi_r = w_i dw_i^\dagger + dw_i w_i^\dagger$, and the vertical tangent vectors are in the kernel of this linear map. Given a fiber of $\pi_r$ and a choice of a $w_i$ in this fiber, then we can diffeomorphically identify the fiber with $U(r_i)$ by right multiplication. Suppose we take $X \in u(r_i)$, identified as an anti-Hermitian matrix in the usual way, and choose a curve $t \mapsto w_i(t) = w_i \cdot e^{tX}$. Clearly, the projection onto the base is invariant under this transformation

$$w_i(t) w_i^\dagger(t) = w_i e^{tX} (w_i e^{tX})^\dagger = w_i e^{tX} e^{-tX} w_i^\dagger = w_i w_i^\dagger. \quad (A8)$$

The tangent vector to the fiber can now be written as

$$\frac{dw_i}{dt}\big|_{t=0} = \dot{w}_i = w_i \cdot X,$$ which satisfies the condition for vertical matrices

$$w_i w_i^\dagger + \dot{w}_i \dot{w}_i^\dagger = w_i X w_i^\dagger + w_i X^\dagger w_i = w_i X w_i^\dagger - w_i X w_i^\dagger = 0. \quad (A9)$$

Hence, by dimensionality, our vertical space can also be seen as

$$V_{w_i} = \{ \dot{w}_i \in T_{w_i} V_{r_i}(\mathbb{C}^n) : \dot{w}_i = w_i \cdot X, \text{where } X^\dagger = -X \}. \quad (A10)$$

We are now in condition to define the horizontal subspaces, which will simply be the collection of tangent vectors $\dot{w}_i$ that are orthogonal to $V_{w_i}$

$$H_{w_i} = (V_{w_i})^\perp = \{ \dot{w} \in T_{w_i} V_{r_i}(\mathbb{C}^n) : \langle \dot{w}, \dot{w}_i^\dagger \rangle = 0, \text{where } \dot{w}_i^\dagger \in V_{w_i} \}. \quad (A11)$$

Note that the operation $\langle \cdot, \cdot \rangle$ is not the Hermitian form defined in Eq. (8). It is instead the standard inner product in the space of complex matrices seen as a real vector space $\langle A, B \rangle \equiv \text{Re Tr}(A^\dagger B)$. The condition in (A11) is then given by

$$\text{Re Tr} \left( \dot{w}_i w_i^\dagger \cdot X \right) = 0, \text{for every } X \in u(r_i) \implies \dot{w}_i w_i^\dagger - w_i^\dagger \dot{w}_i = 0, \quad (A12)$$

where the implication stems from the fact that $X$ is anti-Hermitian, so that $\dot{w} \dot{w}$ can only be Hermitian. [34]

We can go further by making use of the condition in Eq. (A5), yielding $\dot{w}_i w_i = -w_i \dot{w}_i$, and substituting this in Eq. (A12) we get

$$\dot{w}_i w_i - w_i^\dagger \dot{w}_i = -2w_i^\dagger \dot{w}_i = 0 \implies w_i^\dagger \dot{w}_i = 0. \quad (A13)$$

Finally, now that we have a notion of horizontal subspaces of the tangent spaces to $V_{r_i}(\mathbb{C}^n)$, we have unique isomorphisms of $H_{w_i} \cong T_{w_i} \text{Gr}_{r_i}(\mathbb{C}^n)$ provided by the projection $\pi_r$. This means that for each $v \in T_{w_i} \text{Gr}_{r_i}(\mathbb{C}^n)$ there exists a unique $\bar{v}^H \in H_{w_i} \subset T_{w_i} V_{r_i}(\mathbb{C}^n)$, such that its projection is $v$, i.e., $\pi_r(\bar{v}^H) = \bar{v}^H w_i^\dagger + w_i \bar{v}^H w_i^\dagger = v$, and the converse is also true. This lift is called the “horizontal lift” for obvious reasons. Any other lift of $v$ to $T_{w_i} V_{r_i}(\mathbb{C}^n)$, i.e., any tangent vector projecting to $v$, would differ from the horizontal by an element of the kernel of the derivative of the projection, i.e., a vertical vector. As a consequence of this isomorphism, the Riemannian metric in the base space is $g_r(v_1, v_2) := \langle \bar{v}_1^H, \bar{v}_2^H \rangle = \text{Re Tr} \left( \bar{v}_1^H \bar{v}_2^H \right)$, where $\bar{v}^H$, are horizontal lifts of tangent vectors $v_1, v_2 \in T_{w_i} \text{Gr}_{r_i}(\mathbb{C}^n)$. Moreover, the expression $g_r(v_1, v_2)$ does not depend on the point of the fiber over $P_i$, because the horizontal subspaces are $U(r_i)$-equivariant and the metric is $U(r_i)$-invariant. Indeed, if $\bar{v}^H \in H_{w_i}$ is an horizontal lift of $v \in T_{w_i} \text{Gr}_{r_i}(\mathbb{C}^n)$, then $\bar{v}^H \cdot U$ is a horizontal lift belonging to $H_{w_i \cdot U}$, for every $U \in U(r_i)$: $w_i^\dagger \bar{v}^H = 0 \implies (w_i^\dagger U)^\dagger \bar{v}^H U = U^\dagger w_i^\dagger \bar{v}^H U = 0$. Note that, in $\bar{v}^H \cdot U$, right multiplication should be understood as the tangent map of right multiplication at $w_i$. Finally, $\text{Re Tr} \left[ \bar{v}_1^H \bar{v}_2^H \right] = \text{Re Tr} \left[ (\bar{v}^H \cdot U)^\dagger \bar{v}_2^H \cdot U \right]$, by the cyclic property of the trace, which shows that this expression defines a metric in the base space.

Now every tangent vector $\bar{v} \in T_{w_i} V_{r_i}(\mathbb{C}^n)$ is uniquely projected to a horizontal vector $\bar{v}^H \in H_{w_i}$, which is mapped to a base space tangent vector $v \in T_{w_i} \text{Gr}_{r_i}(\mathbb{C}^n)$. Given the decomposition $T_{w_i} V_{r_i}(\mathbb{C}^n) = V_{w_i} \oplus H_{w_i}$, we can always find unique projection operators onto the vertical and horizontal subspaces, that perform the splitting

$$\bar{v} = \bar{v}^V + \bar{v}^H, \text{ where } \bar{v}^V \in V_{w_i}, \bar{v}^H \in H_{w_i}. \quad (A14)$$

We have the identity

$$g(v_1, v_2) = \langle \bar{v}_1^H, \bar{v}_2^H \rangle. \quad (A15)$$

Additionally, due to the splitting of subspaces, we can write

$$\bar{v} = \bar{v}^H - \bar{v}^V, \quad (A16)$$

In the following, we determine the form of the projection onto the vertical subspaces, in order to obtain a more compact form for the metric on the base space.

We claim that the vertical projection of a general tangent vector $\bar{v}$ is given by

$$\bar{v}^V = P_i \bar{v} = w_i \bar{v}_i^\dagger \bar{v}. \quad (A17)$$

Let us see why this is true. For this tangent vector to be vertical it must comply with Eq (A7), i.e.,

$$(P_i \bar{v}) w_i^\dagger + w_i (P_i \bar{v})^\dagger = w_i w_i^\dagger \bar{v} w_i^\dagger + w_i \bar{v}^\dagger w_i w_i^\dagger = 0. \quad (A18)$$

However, we know that $\bar{v}^\dagger$ is a tangent vector, that is, we know that $\bar{v}^\dagger w_i = -w_i^\dagger \bar{v}$. Replacing this in the expression above we have

$$w_i w_i^\dagger \bar{v} w_i^\dagger - w_i^\dagger \bar{v} w_i = 0. \quad (A19)$$
Hence, we have verified that $P_{i} \tilde{\nu}$ is a vertical tangent vector and the map $\tilde{\nu} \rightarrow w_i w_i^{\dagger} \tilde{\nu}$ is a projection onto the vertical space. The horizontal projection is then given by

$$\tilde{\nu}^H = \tilde{\nu} - (w_i w_i^{\dagger}) \tilde{\nu}. \quad (A20)$$

Meanwhile, the metric in $Gr_r(\mathbb{C}^n)$ is, using the horizontal projections, given by the following compact formula

$$g_i = \text{Re} \text{Tr} \left[ (dw_i^{\dagger} - dw_i^{\dagger} w_i w_i^{\dagger}) \left( dw_i - w_i w_i^{\dagger} dw_i \right) \right]$$

$$= \text{Re} \text{Tr} \left[ dw_i^{\dagger} dw_i - dw_i^{\dagger} w_i w_i^{\dagger} dw_i 
- dw_i^{\dagger} w_i w_i^{\dagger} dw_i + dw_i^{\dagger} (w_i w_i^{\dagger})^2 dw_i \right]. \quad (A21)$$

We know that $w_i (w_i^{\dagger} w_i) w_i^{\dagger} = w_i w_i^{\dagger}$, since $w_i^{\dagger} w_i = I_k$, so the last two terms cancel each other, giving

$$g_i = \text{Re} \text{Tr} \left[ dw_i^{\dagger} dw_i \right]$$

$$= \text{Re} \text{Tr} \left[ dw_i^{\dagger} (1 - w_i w_i^{\dagger}) dw_i \right]. \quad (A22)$$

Now, this expression is written in terms of the elements defined in the principal bundle so we want to write it in terms of the elements in the base space — the projectors $P_i$. For this purpose, notice that $w_i = (w_i w_i^{\dagger}) w_i = P_i w_i$ which, by derivation gives $dw_i = dP_i w_i + P_i dw_i$. The same can be done for the hermitian $w_i^{\dagger} = w_i^{\dagger} (w_i w_i^{\dagger})^{\dagger} w_i = P_i^{\dagger}$ that gives us $dw_i^{\dagger} = dw_i^{\dagger} P_i + w_i^{\dagger} dP_i$. Replacing these in Eq. (A22), we get

$$g_i = \text{Re} \text{Tr} \left[ dw_i^{\dagger} (1 - w_i w_i^{\dagger}) dw_i \right]$$

$$= \text{Re} \text{Tr} \left[ (dw_i^{\dagger} P_i + w_i^{\dagger} dP_i) (1 - P_i) (dP_i w_i + P_i dw_i) \right]$$

$$= \text{Re} \text{Tr} \left[ (dw_i^{\dagger} P_i + w_i^{\dagger} dP_i - dw_i^{\dagger} P_i - w_i^{\dagger} dP_i P_i) \right. \right.$$\n
$$\left. \left( dP_i w_i + P_i dw_i \right) \right]$$

$$= \text{Re} \text{Tr} \left[ \left( dw_i^{\dagger} P_i dP_i w_i + w_i^{\dagger} dP_i P_i w_i 
- dw_i^{\dagger} P_i dP_i w_i - w_i^{\dagger} dP_i P_i w_i 
+ dw_i^{\dagger} P_i dP_i w_i + w_i^{\dagger} dP_i P_i w_i \right. \right.$$\n
$$\left. \left. - dw_i^{\dagger} P_i dP_i w_i - w_i^{\dagger} dP_i P_i w_i \right) \right]$$

$$= \text{Re} \text{Tr} \left[ \left( w_i^{\dagger} dP_i P_i w_i - w_i^{\dagger} dP_i P_i w_i \right) \right. \right.$$\n
$$\left. \left. - \left( dP_i P_i + P_i dP_i P_i \right) \right) \right]$$

$$= \text{Re} \text{Tr} \left[ \left( P_i dP_i P_i P_i - \text{Tr} \left( P_i dP_i P_i P_i \right) \right) \right. \right.$$\n
$$\left. \left. - \text{Tr} \left( P_i dP_i P_i P_i \right) \right) \right]. \quad (A23)$$

Moreover, since $P_i^2 = P_i$, we have that $dP_i = d(P_i^2) = P_i dP_i + dP_i P_i$. Multiplying this expression by $P_i$ on both sides we get $P_i dP_i P_i = 2P_i dP_i P_i$ and we can conclude that $P_i dP_i P_i = 0$. The last term on the last expression is then zero and we see that the metric is given by

$$g_i = \text{Re} \text{Tr} \left( P_i dP_i P_i P_i \right) = \text{Re} \text{Tr} \left( P_i dP_i P_i P_i \right)$$

$$= \text{Tr} \left( P_i dP_i P_i P_i \right). \quad (A24)$$

Now we wish to determine the metric on the total space of the principal bundle, i.e., the metric that encompasses both the classical and quantum parts. For this purpose, consider a curve in the principal bundle space given by $t \rightarrow p_r(t) = \left( \sqrt{p_i(t)}, w(t) \right)$ and compute the distance between two infinitesimally close points $t$ and $t + \delta t$. For the first case, we consider a static $w(t) = w$ and compute the distance

$$d_t^2 (p_r(t), p_r(t + \delta t)) = 2 \left( 1 - \sum_{i=1}^{k} \sqrt{p_i(t) p_i(t + \delta t)} \text{Re} \text{Tr} (w_i^{\dagger} w_i) \right). \quad (A25)$$

We have $\text{Tr} (w_i^{\dagger} w_i) = \text{Tr} P_i = r_i$, hence

$$d_t^2 (p_r(t), p_r(t + \delta t)) = 2 \left( 1 - \sum_{i=1}^{k} r_i \sqrt{p_i(t) p_i(t + \delta t)} \right). \quad (A26)$$

Let us look more closely at the expression $\sqrt{p_i(t) p_i(t + \delta t)}$. We can Taylor expand $p_i(t + \delta t)$ to second order in $\delta t$ to obtain

$$\sqrt{p_i(t) p_i(t + \delta t)} = \sqrt{p_i(t)} \left( p_i(t) + \dot{p}_i \delta t + \frac{1}{2} \frac{\ddot{p}_i}{p_i} \delta t^2 \right)$$

$$= p_i(t) \left( 1 + \frac{\dot{p}_i}{p_i} \delta t + \frac{1}{2} \frac{\ddot{p}_i}{p_i} \delta t^2 \right). \quad (A27)$$

We can then approximate the quantity inside the square root by $\sqrt{1 + x} \approx 1 + \frac{1}{2} x - \frac{1}{8} x^2$, which, ignoring higher order terms, yields

$$\sqrt{p_i(t) p_i(t + \delta t)} \approx p_i \left[ 1 + \frac{1}{2} \frac{\dot{p}_i}{p_i} \delta t + \frac{1}{2} \frac{\ddot{p}_i}{p_i} \delta t^2 \right.$$\n
$$- \frac{1}{8} \left( \frac{\dot{p}_i}{p_i} + \frac{1}{2} \frac{\ddot{p}_i}{p_i} \delta t^2 \right)^2 \right]$$

$$= p_i \left[ 1 + \frac{1}{2} \frac{\dot{p}_i}{p_i} \delta t + \frac{1}{2} \frac{\ddot{p}_i}{p_i} \delta t^2 - \frac{1}{8} \left( \frac{\dot{p}_i}{p_i} \right)^2 \delta t^2 \right]$$

$$= p_i + \frac{1}{2} \frac{\dot{p}_i}{p_i} \delta t + \frac{1}{2} \frac{\ddot{p}_i}{p_i} \delta t^2 - \frac{1}{8} \frac{\dot{p}_i^2}{p_i} \delta t^2. \quad (A28)$$

Replacing this in Eq. (A26), we get

$$d_t^2 (p_r(t), p_r(t + \delta t)) = 2 \left[ 1 - \sum_{i=1}^{k} r_i \left( p_i + \frac{1}{2} \frac{\dot{p}_i}{p_i} \delta t + \frac{1}{2} \frac{\ddot{p}_i}{p_i} \delta t^2 - \frac{1}{8} \frac{\dot{p}_i^2}{p_i} \delta t^2 \right) \right]. \quad (A29)$$

Using the condition $\sum_{i=1}^{k} r_i \dot{p}_i = 0$ and $\sum_{i=1}^{k} r_i \ddot{p}_i = 0$. Applying these results in the expression above, we finally arrive at the
Fisher-Rao metric

\[(ds_{BA}^Q)^2 = \frac{1}{4} \sum_{i=1}^{k} r_i \frac{dp_i^2}{p_i} \delta t^2 = \frac{1}{4} \sum_{i=1}^{k} \frac{dp_i^2}{p_i} \]  \hspace{1cm} (A30)

in terms of the probability distribution “coordinates” \(\sqrt{p}\).

Next, consider the case of a static classical part \(p(t) = p\). The distance is then

\[d_s^2 (p_r(t), p_r(t + \delta t)) = 2 \left(1 - \sum_{i=1}^{k} p_i \Re \Tr(w_i(t)^\dagger w_i(t + \delta t))\right). \hspace{1cm} (A31)\]

Expanding \(w_i(t + \delta t)\) to second order \(w_i(t + \delta t) \approx w_i(t) + \dot{w}_i(t)\delta t + \frac{1}{2} \ddot{w}_i(t)\delta t^2\) we have

\[\Re \Tr(w_i(t)^\dagger w_i(t + \delta t)) = \Re \Tr(w_i^\dagger w_i) + \frac{1}{2} \Re \Tr(w_i^\dagger \ddot{w}_i)\delta t^2 + r_i + \frac{1}{2} \Tr(w_i^\dagger \dot{w}_i + w_i^\dagger \dot{w}_i)\delta t + \frac{1}{4} \Tr(w_i^\dagger \dot{w}_i + w_i^\dagger \dot{w}_i)\delta t^2. \hspace{1cm} (A32)\]

From condition (A5) for tangent vectors, the first order term is zero. From this same condition one can infer that \(\ddot{w}_i w_i + \dot{w}_i \dot{w}_i = -2\dot{w}_i^\dagger \dot{w}_i\) and Eq. (A32) becomes

\[\Re \Tr(w_i(t)^\dagger w_i(t + \delta t)) = r_i - \frac{1}{2} \Tr(\dot{w}_i^\dagger \dot{w}_i)\delta t^2. \hspace{1cm} (A33)\]

Using this expression in Eq. (A31) we get

\[d_s^2 (p_r(t), p_r(t + \delta t)) = 2 \left(1 - \sum_{i=1}^{k} r_i p_i + \frac{1}{2} \sum_{i=1}^{k} p_i \Tr(\dot{w}_i^\dagger \dot{w}_i)\delta t^2\right). \hspace{1cm} (A34)\]

Since \(\sum_{i=1}^{k} r_i p_i = 1\), we have

\[(ds_{BA}^Q)^2 = \frac{1}{4} \sum_{i=1}^{k} p_i \Tr(\dot{w}_i^\dagger \dot{w}_i)\delta t^2 = \frac{1}{4} \sum_{i=1}^{k} p_i \Tr(dw_i^\dagger dw_i). \hspace{1cm} (A35)\]

From the derivation of Eq. (A24), it becomes clear that, restricting to the horizontal subspaces, one obtains the induced quantum part of the metric in the base space

\[(ds_{BA}^Q)^2 = \sum_{i=1}^{k} p_i \Tr(P_i dP_i dP_i). \hspace{1cm} (A36)\]

So, the quantum part of the metric in the base space is the sum for \(i \in \{1, ..., k\}\) of the metric on the Grassmannian given by Eq. (A24) weighed by the relative proportions of the distribution \(p_i\).

Finally, we are left with the task of taking a general variation, where both \(\sqrt{p}(t)\) and \(w(t)\) are non-constant, to make sure that we do not get cross terms. We have,

\[d_s^2 (p_r(t), p_r(t + \delta t)) = 2 \left(1 - \sum_{i=1}^{k} \sqrt{p_i(t)p_i(t + \delta t)} \Re \Tr(w_i(t)^\dagger w_i(t + \delta t))\right). \hspace{1cm} (A37)\]

We can Taylor expand, as before, to obtain

\[d_s^2 (p_r(t), p_r(t + \delta t)) = 2 \left[1 - \sum_{i=1}^{k} \left(p_i + \frac{1}{2} p_i \dot{p}_i + \frac{1}{8} \ddot{p}_i p_i \delta t^2\right)\right]. \hspace{1cm} (A38)\]

Collecting the terms up to second order we get

\[d_s^2 (p_r(t), p_r(t + \delta t)) = 2 \left[1 - \sum_{i=1}^{k} \left(p_i \Tr(\dot{w}_i^\dagger \dot{w}_i)\delta t^2 + \frac{1}{2} r_i \ddot{p}_i p_i \delta t^2\right)\right], \hspace{1cm} (A39)\]

which, using the same arguments as before, reduces to

\[ds_{p_r}^2 = \sum_{i=1}^{k} \left(\frac{1}{4} r_i \frac{dp_i^2}{p_i} + p_i \Tr(\dot{w}_i^\dagger \dot{w}_i)\delta t^2\right)\]

\[= \sum_{i=1}^{k} \left(\frac{1}{4} r_i \frac{dp_i^2}{p_i} + p_i \Tr(dw_i^\dagger dw_i)\right). \hspace{1cm} (A40)\]

Hence, the metric in the principal bundle is just the sum of the respective classical and quantum metrics. We want to arrive at the metric for the base space: the classical probability distributions \(\sqrt{p}\) have no gauge freedom so they have no vertical or horizontal components and their projection is trivial; meanwhile, the horizontal projection in the quantum part described by the amplitudes \(w_i\) proceeds as in the Stiefel manifold case, for each \(i = 1, ..., k\), so that our final interferometric metric \(g_i\) is

\[g_i = ds_{BA}^2 = (ds_{BA}^C)^2 + (ds_{BA}^Q)^2 = \frac{1}{4} \sum_{i=1}^{k} r_i \frac{dp_i^2}{p_i} + \sum_{i=1}^{k} p_i \Tr(P_i dP_i dP_i). \hspace{1cm} (A41)\]

Appendix B: The proof of the maximal output probability in the interferometric experiment

The input state is \(|0\rangle\langle 0| \otimes \rho\). The first beam splitter \(BS_1 \otimes I\) acts on this state giving
\[ \frac{1}{2} (|0\rangle\langle 0| \otimes V \rho V^\dagger - i |0\rangle\langle 1| \otimes V \rho U^\dagger \\
+ i |1\rangle\langle 0| \otimes U \rho V^\dagger + |1\rangle\langle 1| \otimes U \rho U^\dagger) \]  

(B1)

Upon passing through a second beam splitter and measuring the \( |1\rangle \) state yields

\[ \frac{1}{4} |1\rangle\langle 1| \otimes \left[ V \rho V^\dagger + V \rho U^\dagger + U \rho V^\dagger + U \rho U^\dagger \right]. \]  

(B2)

Tracing out this quantity gives

\[ \frac{1}{4} \left[ \text{Tr} U \rho U^\dagger + \text{Tr} V \rho V^\dagger + 2 \text{Re} \text{Tr} U \rho V^\dagger \right]. \]  

(B3)

We know that \( \text{Tr} U \rho U^\dagger = \text{Tr} V \rho V^\dagger = 1 \), hence

\[ \frac{1}{2} \left[ 1 + \text{Re} \text{Tr} U \rho V^\dagger \right]. \]  

(B4)

Recall that \( V = \sum_{i=0}^k P_i V P_i \), and that since we can write, in terms of a choice of amplitudes \( w_i \), \( i = 1, ..., k \),

\[ P_i = w_i w_i^\dagger, \quad i = 1, ..., k, \]  

(B5)

then,

\[ V = P_0 V P_0 + \sum_{i=1}^k w_i V_i w_i^\dagger, \]  

(B6)

where \( V_i = w_i^\dagger V w_i \) is an \( r_i \times r_i \) unitary matrix, for \( i = 1, ..., k \). Observe that

\[ \text{Tr} [V^\dagger U \rho] = \sum_{i,j=0}^k p_i \text{Tr} [P_j V^\dagger P_i U P_j] \]

\[ = \sum_{i} p_i \text{Tr} [V_i^\dagger P_i U P_1], \]  

(B7)

where in the last step we used the cyclic property of the trace and \( P_i P_j = \delta_{ij} P_i \), \( i, j = 0, ..., k \). Finally, introducing the expression for \( V \) of Eq. (B6) we can write, using \( w_i^\dagger w_i = I_{r_i}, \) \( i = 1, ..., r_1 \), and \( p_0 = 0 \),

\[ \sum_{i=1}^k p_i \text{Tr} [V_i^\dagger P_i U P_1] \]

\[ = \sum_{i=1}^k p_i \text{Tr} [(V_i^\dagger w_i^\dagger U) w_i] \]

\[ = \sum_{i=1}^k p_i \text{Tr} [(U_i^\dagger w_i V_i) w_i] \]  

(B8)

observe that if we write

\[ p_r = ((p_i, w_i))_{i=1}^k \quad \text{and} \quad q_r = ((p_i, U_i^\dagger w_i V_i))_{i=1}^k, \]  

(B9)

then,

\[ \sum_{i=1}^k p_i \text{Tr} [V_i^\dagger w_i^\dagger U w_i] = \langle q_r, p_r \rangle, \]  

(B10)

where \( \langle q_r, p_r \rangle \) is the Hermitian form defined in Eq. (8). Hence,

\[ \text{pr}_1 = \frac{1}{2} \left( 1 + \text{Re} \text{Tr} U \rho V^\dagger \right) \]

\[ = 1 - \frac{1}{2} \left( 1 - \sum_{i=1}^k p_i \text{Re} \text{Tr} [P_i V^\dagger P_i U P_1] \right) \]

\[ = 1 - \frac{1}{2} \left( 1 - \sum_{i=1}^k p_i \text{Re} \langle q_r, p_r \rangle \right) \]

\[ = 1 - \frac{1}{4} d_\tau^2 (q_r, p_r), \]  

(B11)

where \( d_\tau \) is the distance over the total space of the principal bundle \( B_r \rightarrow B \). Maximizing over the the gauge degree of freedom given by the collection of unitary \( r_i \times r_i \) matrices, \( V_i, \) \( i = 1, ..., k \) (note that \( P_0 V P_0 \) is irrelevant), one gets the distance \( d_\tau (\rho, U^\dagger \rho U) \) over the base space \( B_r \) of as explored in the main text.

**Appendix C: Pullback of interferometric metric to parameter space**

We wish to find the metric obtained by pulling back the interferometric metric

\[ g = \frac{1}{4} \sum_i r_i \frac{dp_i^2}{p_i} + \sum_i p_i \text{Tr} (P_i dP_i dP_i), \]

with \( \rho = \sum_i p_i P_i \) and \( \text{Tr} P_i = r_i, \)

(C1)

by the map induced by the Gibbs state

\[ M \mapsto \rho(M) = Z^{-1} \exp(-\beta H(M)), \]  

(C2)

with \( H(M) \) given by Eq. (19) and where \( Z \) is the partition function. The first thing to notice is that if \( \rho = \rho_1 \otimes \rho_2 \), with \( \rho_\alpha = \sum_{i_\alpha} p_{i_\alpha} P_{i_\alpha}, \alpha = 1, 2 \) we have the decomposition

\[ \rho = \sum_I p_I P_I = \sum_{i_1, i_2} p_{i_1} p_{i_2} P_{i_1} \otimes P_{i_2}, \]  

(C3)

where \( I = (i_1, i_2) \) is a multi-index describing the joint system labels. Note that,

\[ \sum_{i_1, i_2} r_{i_1} \frac{dp_{i_1}^2}{p_{i_1}} \]

\[ = \sum_{i_1, i_2} r_{i_1} r_{i_2} \left( p_{i_2}^2 d_{p_{i_1}} d_{p_{i_2}} + 2 p_{i_1} d_{p_{i_1}} d_{p_{i_2}} + p_{i_1}^2 d_{p_{i_2}}^2 \right) \]

\[ = \sum_{i_1} r_{i_1} \frac{dp_{i_1}^2}{p_{i_1}} + \sum_{i_2} r_{i_2} \frac{dp_{i_2}^2}{p_{i_2}}, \]  

(C4)
and

\[ \sum_i p_i \text{Tr} (P_i dP_i dP_i) \]
\[ = \sum_{i_1,i_2} p_{i_1} \text{Tr} [P_{i_1} \otimes P_{i_2} d(P_{i_1} \otimes P_{i_2}) d(P_{i_1} \otimes P_{i_2})] \]
\[ = \sum_{i_1} p_{i_1} \text{Tr} (P_{i_1} dP_{i_1} dP_{i_1}) + \sum_{i_2} p_{i_2} \text{Tr} (P_{i_2} dP_{i_2} dP_{i_2}), \]
(C5)

where we used \( PdPP = 0 \) for any projector \( P \). As a consequence, the interferometric metric, much like the Bures metric, converts tensor product states into orthogonal sum metrics.

Because the Hamiltonian is diagonal in momentum space, the density matrix factors over the momenta – it follows that the metric becomes an integral over the momentum space of individual contributions of each momentum sector. The pullback of the classical term, which also appears in the Bures metric,

\[ \frac{1}{4} \sum_i \frac{dp_i^2}{p_i} \]
(C6)

was computed in the Appendix of Ref. [23] and it yields

\[ \frac{\beta^2}{4} \int_{BZ^d} \frac{d^d k}{(2\pi)^d} \frac{1}{\cosh(\beta E(k; M)) + 1} \left( \frac{\partial E(k; M)}{\partial M} \right)^2 dM^2, \]
(C7)

where \( E(k; M) = |d(k; M)| \) is the magnitude of \( d(k, M) \). With regards to the second term, one can use the mathematical fact that the embedding of the space of \( k \)-dimensional subspaces of \( \mathbb{C}^N \), \( Gr_k(\mathbb{C}^N) \) on the space of 1-dimensional subspaces of the Fock space \( \mathcal{F} \), is isometric. In the previous equation \( c_i \) stand for creation operators for \( |i\rangle \), i.e., at the single particle level, \( c_i |0\rangle = |i\rangle \), \( i = 1, ..., k \). The embedding being isometric means, in this context, that if we write the rank \( k \) single-particle projector

\[ P = \sum_{i=1}^k |i\rangle\langle i| \]
(C9)

and the rank 1 many-body projector

\[ P = c_1^\dagger ... c_k^\dagger |0\rangle\langle 0| c_1 ... c_k, \]
(C10)

we have

\[ \text{Tr} (\tilde{P}dPdP) = \text{Tr} (PdPdP). \]
(C11)

In particular, this means that in the gapped case for each \( k \in BZ^d \) we will have four classes of orthogonal eigenstates,

\[ |0\rangle, c_{1,k}^\dagger |0\rangle, c_{2,k}^\dagger |0\rangle, c_{1,k}^\dagger c_{2,k}^\dagger |0\rangle, \]
(C12)

where \( c_{i,k}^\dagger \), \( i = 1, 2 \), are the Bogoliubov quasiparticle creation operators of \( \mathcal{H} \) with energies \( E(k; M) \) and \( -E(k; M) \), respectively. The energies of the classes of eigenstates are, respectively, 0, \( E(k; M) \), \( -E(k; M) \) and 0. The associated single-particle \( 2 \times 2 \) projectors are, respectively, the 0 projector, \( P_1(k; M) = c_{1,k}^\dagger |0\rangle\langle 0| c_{1,k} \), \( P_2(k; M) = c_{2,k}^\dagger |0\rangle\langle 0| c_{2,k} \) and the \( 2 \times 2 \) identity matrix \( I_2 \). Only \( P_1(k) \) and \( P_2(k) \) are non-trivial and moreover, if we introduce the unit vector \( n = d/|d| \), we can write,

\[ P_1(k; M) = \frac{1}{2} (I_2 + n^\mu(k; M) \sigma_\mu) \]
\[ P_2(k; M) = I_2 - P_1(k; M), \]
(C13)

As a consequence, using the identity \( \text{Tr} (PdPdP) = (1/2) \text{Tr} (dPdP) \) and using the fact that the Pauli matrices are traceless, we get,

\[ \text{Tr} (P_1 dP_1 dP_1) = \text{Tr} (P_2 dP_2 dP_2) \]
\[ = \frac{1}{4} \delta_{\mu\nu} \frac{\partial h^\mu(k; M)}{\partial M} \frac{\partial h^\nu(k; M)}{\partial M} dM^2. \]
(C14)

Finally, taking into account the partition function factor \( Z_k = (2 + 2 \cosh(\beta E(k; M))) \), we get that the quantum contribution is

\[ \frac{1}{4} \int_{BZ^d} \frac{d^d k}{(2\pi)^d} \frac{1}{\cosh(\beta E(k; M)) + 1} \left( \cosh(\beta E(k; M)) \right) \]
\[ \times \delta_{\mu\nu} \frac{\partial h^\mu(k; M)}{\partial M} \frac{\partial h^\nu(k; M)}{\partial M} dM^2. \]

Finally, we obtain,

\[ g = \frac{1}{4} \int_{BZ^d} \frac{d^d k}{(2\pi)^d} \left[ \frac{1}{\cosh(\beta E) + 1} \right. \]
\[ \times \left( \beta^2 \left( \frac{\partial E}{\partial M} \right)^2 + \cosh(\beta E) \delta_{\mu\nu} \frac{\partial h^\mu}{\partial M} \frac{\partial h^\nu}{\partial M} \right) \]
\[ \left. \right] dM^2, \]
(C15)

where we omitted the obvious dependence on \( k \) and \( M \) of the quantities \( E \) and \( n^\mu \).
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[34] To see this, observe that a complex matrix can be split into its Hermitian and anti-Hermitian components: $Z = Z^H + Z^{AH}$, where $Z^H = \frac{1}{2}(Z + Z^\dagger)$ and $Z^{AH} = \frac{i}{2}(Z - Z^\dagger)$. This real-linear decomposition divides the full matrix into two orthog-
onal components. Indeed, \[ \text{Re} \text{Tr} \left( (Z_1^{AH})^\dagger Z_2^H \right) = \frac{1}{2} \left\{ \text{Tr} \left[ (Z_1^{AH})^\dagger Z_2^H \right] + \text{Tr} \left[ (Z_2^H)^\dagger Z_1^{AH} \right] \right\} = \frac{1}{2} \left( -\text{Tr} \left[ (Z_1^{AH})^\dagger Z_2^H \right] + \text{Tr} \left[ (Z_2^H)^\dagger Z_1^{AH} \right] \right) = 0. \] Moreover, since the real vector space of Hermitian matrices and anti-Hermitian matrices both have dimension $k \times k$, we conclude that if a complex matrix is (real-)orthogonal to an anti-Hermitian matrix, then it must be Hermitian.