Multivariable Invariants of Colored Links
Generalizing the Alexander Polynomial *

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Abstract

We discuss multivariable invariants of colored links associated with the $N$-
dimensional root of unity representation of the quantum group. The invariants
for $N > 2$ are generalizations of the multi-variable Alexander polynomial. The
invariants vanish for disconnected links. We review the definition of the in-
variants through $(1,1)$-tangles. When $(N,3) = 1$ and $N$ is odd, the invariant
does not vanish for the parallel link (cable) of the knot $3_1$, while the Alexander
polynomial vanishes for the cable link.

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1 Introduction

The multi-variable Alexander polynomial is an isotopy invariant of colored links. A variable is assigned on each component of colored links. A hierarchy of multi-variable invariants of colored links which generalize the multi-variable Alexander polynomial was proposed through representations of the colored braid group. \[1\] In order to define link invariants, however, we need a proper method for regularization of the Markov trace. \[2\] The purpose of this paper is to review the definition of the multi-variable invariants of colored links from the viewpoint of the crossing symmetry, and to show a systematic calculation of the invariants. We see that the (multi-variable) invariants are stronger than the Alexander polynomial.

The new invariants are related to the quantum group $U_q(sl(2))$. It was shown that the representation of the (colored) braid group for the invariant \[1\] is equivalent to the R-matrix of an N-dimensional root of unity representation of the quantum group. \[16\] ($N = 2, 3, \ldots$) Hereafter we call the root of unity representation nilpotent representation.

The main problem we encountered was the following. The state sum for the nilpotent root of unity representation vanishes for any link diagrams. The standard method for the Markov trace gives a trivial Markov trace which vanishes for any braid, and we have a trivial invariant that vanishes for any knot. For the case of the Alexander polynomial, however, we can define a non-trivial invariant although the standard state sum vanishes.

The state models for the Alexander polynomial have also this vanishing property. \[8, 13, 23, 24\] For the Alexander polynomial there are methods for regularizations by using the Hecke algebra or the skein relation. It seems, however, that these methods do not hold for the cases of the invariants associated with the nilpotent representations.

This paper consists of the following. In §2 we derive the R-matrix for the nilpotent representation. In §3 we review the regularization given in \[2\]. In §4 we give the Clebsch-Gordan coefficients (CGC) of the nilpotent representation, and then we discuss the crossing symmetry, which is quite different from the crossing symmetry of the standard CGC of $U_q(su(2))$. Using the crossing symmetry of the nilpotent CGC we prove a proposition important for the definition of the multi-variable invariants. In §5 we calculate the multi-variable invariants for the parallel links of 2-braid knots.

2 R matrix for nilpotent rep.

We introduce the defining relations of $U_q(sl(2))$ for the generators $e, f,$ and $k$. \[20, 22\]

\[kek^{-1} = qe, \quad kf k^{-1} = q^{-1}f, \quad [e, f] = \frac{k^2 - k^{-2}}{q - q^{-1}}. \quad (2.1)\]
The following formulas for the commultiplication $\Delta$, the antipode $S$, and the counit $\epsilon$ on the generators define the structure of the Hopf algebra

$$\begin{align*}
\Delta(k^{\pm 1}) &= k^{\pm 1} \otimes k^{\pm 1}, \\
\Delta(e) &= e \otimes k + k^{-1} \otimes e, \\
\Delta(f) &= f \otimes k + k^{-1} \otimes f, \\
S(k^{\pm 1}) &= k^{\mp 1}, \\
S(e) &= -qe, \\
S(f) &= -q^{-1}f, \\
\epsilon(k^{\pm 1}) &= 1, \\
\epsilon(e) &= \epsilon(f) = 0.
\end{align*}$$

Hereafter we shall sometimes use the notation $k = q^{H/2}$. We shall use the following $q$-analog notations:

$$\begin{align*}
[n]_q &= \frac{q^n - q^{-n}}{q - q^{-1}}, \\
[n]_q! &= \prod_{k=1}^{n} [k]_q, \\
[p; n]_q! &= \prod_{k=0}^{n-1} [p - k]_q.
\end{align*}$$

where $n$ is a positive integer and $p$ is a complex parameter. In particular, we assume that $[0]_q! = [p; 0]_q! = 1$. The universal $R$ matrix of $U_q(sl(2))$ (q generic case) is given by

$$R = q^{H \otimes H/2} \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{[n]_q!} (q - q^{-1})^{H/2} e \otimes f q^{-H/2}.$$  

Let us consider the nilpotent root of unity representation. Hereafter we assume that $q^{2N} = 1$ and $q^2$ is a primitive $N$-th root of unity. When $N$ is even, $q$ is $2N$-th primitive root of unity ($q^N = -1$, while when $N$ is odd, $q$ is $2N$-th ($q^N = -1$) or $N$-th primitive root of unity ($q^N = 1$).

We introduce $x = e^N, y = f^N, \text{ and } k^{2N},$ which are central elements of $U_q(sl(2))$ at $q$ roots of unity. For the nilpotent root of unity representation they are given by $x = y = 0$, and $z = q^{2Np},$ where $p$ is a complex parameter. We note that for the cyclic representations $x, y, z$ are given by generic complex parameters.

We denote by $V(p)$ the left $U_q(sl(2))$ module for the nilpotent representation. The dimension of $V(p)$ is given by $N$. The nilpotent representation has the following fusion rule:

$$V(p_1) \otimes V(p_2) = \sum_{p_3} N_{p_1 p_2}^{p_3} V(p_3),$$

where

$$N_{p_1 p_2}^{p_3} = \begin{cases} 1 & \text{if } p_3 = p_1 + p_2 - n, (n = 0, 1, \cdots N - 1), \\
0 & \text{otherwise.}
\end{cases}$$

The fusion rule is quite different from that for the finite dimensional representations of $U_q(sl(2))$. We note that the fusion rule for the semi-periodic representation is given in [3], where $x = 0, y \neq 0$ and $z \neq 1$ or $x \neq 0, y = 0$ and $z \neq 1$.  

We now consider a matrix representation $\pi_p$ induced from the module $V(p)$. Let $e_0, \ldots, e_{N-1}$ be the basis of the module $V(p)$. For $a, b = 0, 1, \cdots N - 1$, we assume the following matrix representations (2.7)

$$
\begin{align*}
\pi_p(e)_{ab} &= ([2p - a][a + 1])^{1/2} \delta_{a+1,b} \\
\pi_p(f)_{ab} &= ([2p - a + 1][a])^{1/2} \delta_{a-1,b} \\
\pi_p(k)_{ab} &= (q^{H/2})_{ab} = q^{p-a} \cdot \delta_{a,b}.
\end{align*}
$$

(2.7)

Putting the representations of the generators (2.7) into the universal $R$-matrix (2.4), we have the matrix representation of the $R$-matrix

$$
R_{p1}^{p2}(+)_{b1b2} = ((\pi_{p1} \otimes \pi_{p2})R)_{b1b2}^{a1a2}
$$

$$
= \delta_{a1b1-n} \delta_{a2b2+n} q^{2(p1-b1)(p2-b2)-n(p1-b1)+n(p2-b2)-(n^2-n)/2} \times
$$

$$
\frac{(q - q^{-1})^n}{[n]!} \sqrt{[b1;n]![a2;n]![2p1-a1;n]![2p2-b2;n]}.
$$

(2.8)

It is shown [10] that the representation of the colored braid group [1] is equivalent to the $R$-matrix (2.8) for the nilpotent representation through some gauge transformations.

3 Regularization

We now discuss the construction of the colored link invariants [2]. For oriented tangles we follow the notation of [37] and [2].

It is straightforward to construct invariants of oriented colored tangles from the representations [1] of the colored braid group. We can define the tangle invariants by checking the Reidemeister moves [37, 38] for the tangle diagrams (see also [2]).

We denote by $\phi(T, \alpha)$ the invariant for a colored oriented tangle $(T, \alpha)$, where $\alpha = (p_1, \cdots, p_n)$ and $p_j$ denotes the color of the component $j$ of the oriented tangle $T$.

We first note the following property:

**Proposition 3.1** For any $(0,0)$-tangle $(T, \alpha)$, the value $\phi(T, \alpha)$ vanishes.

In order to have a nontrivial (multi-variable) invariant, we introduce a regularization procedure. The regularization consists of the following 2 steps. [2]

1. **1-st step**
   Let $T$ be a $(1,1)$-tangle. We denote by $\hat{T}$ the link obtained by closing the open string of $T$. It is not difficult to show the following proposition [2].
Proposition 3.2  Let $T_1$ and $T_2$ denote two $(1,1)$-tangles. If $\hat{T}_1$ is isotopic to $\hat{T}_2$ as a tangle in $S^3$ by an isotopy which carries the closing component of $\hat{T}_1$ to that of $\hat{T}_2$, then $T_1$ is isotopic to $T_2$ as a $(1,1)$-tangle.

We give a comment on Prop. 3.2, which is useful for the regularization. Let $T_1, T_2$ be the two $(1,1)$-tangles defined as in Prop. 3.2. Thanks to Prop. 3.2, we have $\phi(T_1, \alpha) = \phi(T_2, \alpha)$. We recall that $\phi$ is invariant under the Reidemeister moves for tangles. Thus for a link $L$ we can define an invariant by the value of the tangle invariant $\phi(T, \alpha)$, where $\hat{T}$ is equivalent to the link $L$.

Let us consider the value of $\phi$ precisely. Let $s$ denote the closing component (or edge) of $\hat{T}$. We denote by $\phi(T, \alpha)^a_b$ the value $\phi$ for the colored tangle $(T, \alpha)$ with variables $a$ and $b$ on the closing component (or, on the two edges of the closing component $s$). Then we can show [2] that the value of $\phi(T, \alpha)^a_b$ does not depend on $a$ or $b$:

$$\phi(T, \alpha)^a_b = \lambda \delta_{ab}.$$  \hfill (3.1)

The property (3.1) is derived from the irreducibility of the nilpotent representation.

We now define an invariant $\Phi$. Let $(L, \alpha)$ be a colored link and $T$ be a $(1,1)$-tangle such that $\hat{T} = L$. Let $s$ denote the closing component of $T$. Then we define $\Phi$ by $\Phi(L, s, \alpha) = \phi(T, \alpha)^0_0$. Recall that $\Phi$ is well-defined, i.e. $\Phi(L, s, \alpha)$ does not depend on a choice of $T$.

(2) 2-nd step

We note that the value $\Phi(L, s, \alpha)$ depends on the cutting component $s$. The invariant $\Phi(L, s, \alpha)$ is an isotopy invariant of colored link $L$ with a particular choice of the component $s$ of the link $L$. We now construct an isotopy invariant which does not depend on $s$. In order to give a proper regularization, we use the following proposition.

Proposition 3.3 ([2])  For a link $L$ and its color $\alpha = (p_1, \cdots, p_n)$, we have

$$\Phi(L, s, \alpha)((2p_s; N-1)!)^{-1} = \Phi(L, r, \alpha)((2p_r; N-1)!)^{-1}.$$  \hfill (3.2)

The proof of this proposition has been given in Appendix C of [2]. We shall show Prop. 3.3 by using the crossing symmetry of CGC in §4.

Thus we arrive at new invariants $\hat{\Phi}(L, \alpha)$ of colored oriented links $(L, \alpha)$.

Definition 3.4 ([2])  For a colored oriented link $(L, \alpha)$, we define an isotopy invariant $\hat{\Phi}$ of $(L, \alpha)$ by

$$\hat{\Phi}(L, \alpha) = \Phi(L, s, \alpha)((2p_s; N-1)!)^{-1}.$$  \hfill (3.3)

The new colored link invariants are related to the multi-variable Alexander polynomial. It was shown by J. Murakami [29] that a colored link invariant which corresponds to $\hat{\Phi}(L, \alpha)$ for the $N=2$ case a version of the multivariable Alexander polynomial (the Conway potential function). Therefore the new
colored link invariants $\hat{\Phi}(L, \alpha)$ for $N = 3, 4, \cdots$, are generalizations of the multivariable Alexander polynomial.

We make a comment on the invariants for non-colored links. We can consider invariants $\Phi(L, s, \alpha)$ also for non-colored links $(L, \alpha)$ for which all the colors of the components are equal: $p_1 = \cdots = p_n = p$ ($\alpha = (p, \cdots, p)$). We emphasize the following fact. It is not trivial to show that when $(L, \alpha)$ is a non-colored link with $n \geq 2$ ($\alpha = (p, \cdots, p)$) the invariant $\Phi(L, s, \alpha)$ is independent of $s$, i.e., independent of the choice of the cutting component. Thanks to Prop. 3.3, however, we can show that $\Phi(L, s, \alpha)$ is independent of $s$ when $(L, \alpha)$ is non-colored ($\alpha = (p, \cdots, p)$). Thus we can define a link invariant $\Phi(L)$ by

$$\Phi(L) = \Phi(L, 1, \alpha),$$

(3.4)

where $L$ is a link, and $\alpha = (p_1, \cdots, p_n)$ with $p_1 = \cdots = p_n = p$.

Let us consider the one component case $n = 1$. We consider a knot $K$. We can define a knot invariant $\Phi$ by making use of Prop. 3.2 and the property (3.1). We define the invariant $\Phi(K)$ by

$$\Phi(K) = \Phi(K, 1, \alpha).$$

(3.5)

## 4 CGC and crossing symmetry

Let us discuss the Clebsch-Gordan coefficients of the nilpotent representations of $U_q(sl(2))$ [17]. We note that the CGC of finite dimensional representations of $U_q(su(2))$ are given in [26].

Let us consider the left $U_q(sl(2))$-module $V(p)$ in $\S 2$. Let the symbol $|p, z >$ ($z = 0, 1, \cdots, N - 1$) be the basis vector of $V(p)$. [34, 17] The actions of the generators on the basis of the module are given by

$$e|p, z > = \sqrt{[2p - z + 1][z]}|p, z - 1 >, \quad f|p, z > = \sqrt{[2p - z][z]}|p, z + 1 >,$$

$$k|p, z > = q^{p-\bar{z}}|p, z > .$$

(4.1)

We define an inner product among the vectors in the module $V(p)$ by

$$\langle |p, z_1 >, |p, z_2 > \rangle = \delta_{z_1, z_2}.$$  

(4.2)

We assume that two vectors belonging to modules with different values of $p$ are orthogonal:

$$\langle |p_1, z_1 >, |p_2, z_2 > \rangle = 0, \text{ if } p_1 \neq p_2.$$  

(4.3)

For the real forms of $U_q(sl(2))$ the inner products were discussed in [27].

We consider the fusion rule (2.6) in terms of the modules. Let $V(p_3)$ be one of the modules obtained from the decomposition of the tensor product $V(p_1) \otimes V(p_2)$. We note that $p_3$ is given by $p_3 = p_1 + p_2 - n$, where $n$ is an integer with the condition $0 \leq n \leq N - 1$. Let $|p_1, p_2; p_3, z_3 >$ be the basis vector of the module $V(p_3)$ with the value $z$ in (4.1) is given by $z_3$. Let us
introduce some symbols. Let $m, n, z, w$ be integers with $0 \leq m, n, z, w \leq N - 1$. We recall that $z_1, z_2, z_3$ are integers satisfying $0 \leq z_1, z_2, z_3 \leq N - 1$. Let the symbol $p(n)$ denote $p(n) = p_1 + p_2 - n$. We define the Clebsch-Gordan coefficients for the nilpotent representations by the relation

$$|p_1, p_2; p_3, z_3> = \sum_{z_1, z_2} C(p_1, p_2, p_3; z_1, z_2, z_3)|p_1, z_1 > \otimes |p_2, z_2 >,$$  \quad (4.4)

and the normalization condition

$$\sum_{z_1=0}^{\min(n+z, n+w)} C(p_1, p_2, p(n); z_1, n + z - z_1, z) \times$$

$$\times C(p_1, p_2, p(m); z_1, n + w - z_1, w + n - m) = \delta_{nm} \delta_{zw}.$$  \quad (4.5)

An explicit expression \cite{17} of the nilpotent CGC is given by

$$C(p_1, p_2, p_3; z_1, z_2, z_3) = \delta(z_3, z_1 + z_2 - n) \times$$

$$\times \sqrt{[2p_1 + 2p_2 - 2n + 1]} [n!![z_1]!![z_2]!![z_3]!!] \times$$

$$\times \sum_{\nu = \max(0, n - z_2)}^{\min(n, z_1)} \frac{(-1)^{\nu} q^{-\nu(p_1 + p_2 + p_3 + 1)} \times q^{(n - n^2)/2 + (n - z_2)p_1 + (n + z_1)p_2}}{[\nu]!![n - \nu]!![z_1 - \nu]!![z_2 - n + \nu]!! \times}$$

$$\times \sqrt{[2p_1 - n; z_1 - \nu]!!! [2p_1 - z_1; n - \nu]!![2p_2 - n; z_2 + \nu - n]!![2p_2 - z_2; \nu]!! [2p_1 + 2p_2 - n + 1; z_1 + z_2 + 1]!!}.$$  \quad (4.6)

We note that the sum over the integer $\nu$ in (4.6) is taken under the condition: max $\{0, n - z_2\} \leq \nu \leq \min \{n, z_1\}$. A derivation of the expression (4.6) is shown in \cite{17}. However, we shall give a simple derivation of the nilpotent CGC in Appendix A.

Let us consider dual representations. We introduce some symbols for them. Let $(\pi, V)$ be a set of a representation and its module. Let $V^* = Hom(V, C)$ be the dual vector space of $V$. We denote $\nu^*(w)$ by $< \nu^*, w >$, where $\nu^* \in V^*$ and $w \in V$. The transposed representation $\pi^t$ of $\pi$ is defined by $< \pi^t \nu^*, w >= <\nu^*, \pi w >$. We now consider the nilpotent representation $\pi_p$ given in (2.7) and the module $V(p)$. We define the dual representation $\pi_p^*$ of $\pi_p$ by

$$\pi_p^*(a) = \pi_p^t o S(a) = \pi_p^t(S(a)), \quad \text{for all } a \in U_q(sl(2)).$$  \quad (4.7)

We now show that the dual representation is equivalent to a nilpotent representation. Let us define $\bar{p}$ and $\bar{z}$ by

$$\bar{p} = N - 1 - p, \quad \bar{z} = N - 1 - z.$$  \quad (4.8)

We call $\bar{p}$ and $\bar{z}$ conjugates of $p$ and $z$, respectively. Let us introduce an operator $w \in Hom(V(p), V(p)^*)$. We define $w$ by the following matrix elements on the basis vectors of $V(p)$ and $V(p)^*$.

$$(w)_{ab} = (-q)^{-b} q^{-\bar{p}(N-1)} \delta_{ab}, \quad \text{for } a, b = 0, \ldots, N - 1.$$  \quad (4.9)
Then we see that $\pi^*_p$ corresponds to $\pi_{\bar{p}}$ through the following relation

$$\pi^*_p(a) = w \pi_{\bar{p}}(a) w^{-1}, \quad \text{for all } a \in U_1(sl(2)). \quad (4.10)$$

We can show (4.10) by checking for the relations for the generators $\pi^1_p(e) = \pi_p(f)$, $\pi^1_p(f) = \pi_p(e)$, $\pi^1_p(k) = \pi_p(k)$ and

$$\pi_p(e)_{ab} = \pi_p(f)_{ab}, \quad \pi_p(f)_{ab} = \pi_p(e)_{ab}, \quad \pi_p(k)_{ab} = \pi_p(k^{-1})_{ab}. \quad (4.11)$$

We note that in the case of the finite dim. rep. of $U_1(su(2))$ [26] the dual representation of a finite dim. rep. $j$ is equivalent to the rep. $j$ itself.

We shall consider the crossing symmetry of the nilpotent CGC. Hereafter we employ the following expression of CGC. (We fix the phase factor related to the square root so that the CGCs satisfy the simplest form of the crossing symmetry.)

$$\pi^*_p(e)_{ab} = \pi_p(f)_{ab}, \quad \pi_p(f)_{ab} = \pi_p(e)_{ab}, \quad \pi_p(k)_{ab} = \pi_p(k^{-1})_{ab}. \quad (4.11)$$

We note that the sum over $\nu$ is taken under the condition: $\max \{0, n - z_2\} \leq \nu \leq \min \{n, z_1\}$. Now we show the crossing symmetry of the nilpotent CGCs (4.12) which is consistent with the correspondence (4.10).

**Proposition 4.1 Crossing symmetry**

$$C(p_3, \bar{p}_2, p_1; z_3, \bar{z}_2, z_1) = (-1)^{z_3-z_1} q^{(p_2-z_2)-Np_2} \sqrt{\frac{[2p_3; N-1]!}{[2p_1; N-1]!}} \times C(p_1, p_2, p_3; z_1, \bar{z}_2, z_3), \quad (4.13)$$

where the symbols $\bar{p}_2$ and $\bar{z}_2$ denote the conjugates of $p_2$ and $z_2$, respectively, and are given by

$$\bar{p}_2 = N - 1 - p_2, \quad \bar{z}_2 = N - 1 - z_2. \quad (4.14)$$

(Proof) We shall give a proof of (4.13) through induction on $z_1$ in Appendix B.

The crossing symmetry of the nilpotent CGC is different from that of the CGC of the finite dim. representations [26]. We note that the crossing symmetry of the nilpotent CGC is consistent with the crossing symmetry of the colored braid matrix (the nilpotent R matrix) given in §6 of [2]. We also note that the latter can be derived from the relation $(S \otimes 1) R = R^{-1} \otimes 1$ by using the correspondence (4.10).

Making use of the crossing symmetry of the nilpotent CGC we prove the next proposition.
Proposition 4.2 \[13\]

\[
\begin{align*}
N-1 \sum_{z_2=0}^{\infty} (C(p_1, p_2, p_3; z_1, z_2, z_1 + z_2 - n))^2 q^{2\rho(p_2, z_2)} &= \frac{[2p_3 + 1][2p_1; N - 1]!}{[2p_1 + 2p_2 + 1; N]!}, \\
N-1 \sum_{z_1=0}^{\infty} (C(p_1, p_2, p_3; z_1, z_2, z_1 + z_2 - n))^2 q^{-2\rho(p_1, z_1)} &= \frac{[2p_3 + 1][2p_2; N - 1]!}{[2p_1 + 2p_2 + 1; N]!},
\end{align*}
\]

where \( p_3 = p_1 + p + 2 - n \) and \( \rho(p, z) \) is given by

\[
\rho(p, z) = (p - z) - Np. \quad (4.15)
\]

(Proof) Using the crossing symmetry \((4.13)\) and the orthonormal condition \((4.5)\) of the nilpotent CGC we have the following.

\[
\begin{align*}
\sum_{z_2} (C(p_1, p_2, p_3; z_1, z_2, z_1 + z_2 - n))^2 q^{2\rho(p_2, z_2)} &= \sum_{z_2} C(p_3, \bar{p}_2, p_1; z_1 - \bar{z}_2 + \bar{n}, \bar{z}_2, z_1) \\
&\times (-1)^{(z_3 - z_1)} q^{-(p_2 - z_2) + Np} \sqrt{\frac{[2p_1; N - 1]!}{[2p_3; N - 1]!}} \left[ 2p_3 + 1 \right] \left[ 2p_1 + 2p_2 + 1; N \right]!.
\end{align*}
\]

Let us discuss Prop. 3.3 in \$3\$. It is straightforward to show Prop. 3.3 from Prop. 4.2. Thus from the crossing symmetry we can derive Prop. 3.3 that is important in the definition of the multi-variable invariants.

5 Evaluation for cables of 2-braid knots

We now discuss calculation of the invariant \( \Phi(L) \) for the parallel link of a 2-braid knot \( b_1^k \). It is known that the Alexander polynomial vanishes for the parallel link (the cable) of any knot. Let us consider the parallel link \( L \) of the trefoil knot. We note that the trefoil knot is equivalent to the closed braid of \( b_1^3 \). The cable link \( L \) gives an example that it is not a split link but it has split Seifert surfaces, and therefore the Alexander polynomial vanishes for it. \[2\]

Let us introduce a technique for evaluation. We consider spectral decomposition of the R matrix in terms of the nilpotent CGC. We denote \( C(p_1, p_2, p_1 + 2) \). Prof. H.R. Morton suggested that the author should calculate the cable of the trefoil knot. \[28\]
\[ p_2 - n; a_1, a_2, a_1 + a_2 - n \] by \( C_{12}(n; a_1, a_2) \). We recall that \( p_1, p_2 \in \mathbb{C} \) and \( n = 0, \ldots, N - 1 \). We define \( \lambda_n(p_1, p_2) \) by

\[
\lambda_n(p_1, p_2) = (-1)^n q^{2p_1 p_2 - 2n(p_1 + p_2) + n(n-1)}. \tag{5.1}
\]

Then we have the following

\[
R^{p_1 p_2}(\pm) = \sum_{n=0}^{N-1} (\lambda_n(p_1, p_2))^{\pm 1} C_{12}(n; b_1, b_2) C_{21}(n; a_2, a_1). \tag{5.2}
\]

We give a comment on (5.2). The \( \lambda_n(p_1, p_2) \) corresponds to the eigenvalue of the nilpotent R-matrix. For the link polynomial associated with the finite dimensional representation of \( U_q(sl(2)) \) [3, 26, 31] the eigenvalues of the R-matrix give the higher order skein relation. From the decomposition (5.2) we can also derive "higher order" skein relation for the multi-variable invariant. For evaluation of the multi-variable invariants, however, we shall use the decomposition (5.2) rather than the skein relation.

Let us discuss the normalization of the R-matrix. We note the following relation [2] which corresponds to the Reidemeister move I (or the Markov trace property)

\[
\sum_{b=0}^{N-1} R^{p p}(\pm)_{ab} q^{2(p-b) - 2Np} = q^{\pm 2pp}. \tag{5.3}
\]

Let \( f(p_1, p_2; q) \) be an arbitrary function but satisfy the next condition:

\[
f(p_1, p_1; q) = q^{2p_1 \bar{p}_1} = q^{2p_1(N-1-p_1)}. \tag{5.4}
\]

For example, we can consider \( f(p_1, p_2; q) = q^{p_1 \bar{p}_2 + p_1 p_2} \). We define \( \tilde{R}(\pm) \) by

\[
\tilde{R}^{p_1 p_2}(\pm) = R^{p_1 p_2}(\pm)(f(p_1, p_2; q))^{\pm 1}. \tag{5.5}
\]

We use \( \tilde{R} \) for calculation of the (multi-variable) invariants. From (5.3) we see that the normalization is consistent with the Reidemeister move I (or the Markov trace property). We note that the property (5.3) was shown in [3] (see also [14]). We shall show another proof of (5.3) in Appendix C.

Using the decomposition (5.2) and Prop. 4.2 we now calculate \( \Phi(K) \) for 2 braid knots \( K \). We assume that the knot \( K \) is equivalent to the closed braid of \( b_k^1 \). (We also assume that \( k \) is odd.) Let the symbol \( p_n \) denote \( p_n = 2p - n \). Then we have

\[
\Phi(K) = \sum_{n=0}^{N-1} \frac{[2p; N-1]!}{[2p_n; N-1]!} (\lambda_n(p, p))^k q^{-k(2p^2 - 2(N-1)p)}
\]

\[\text{[3]}\text{We note that we can derive (5.2) from the spectral decomposition of the solvable vertex and IRF models associated with the nilpotent representation [17], which we called colored vertex models and colored IRF models. The colored vertex and colored IRF models can be constructed by applying Jimbo’s method in [22, 23].} \]
We recall that
\[ p_\alpha = \left(\frac{q^{2Np} - q^{-2Np}}{(q^{4Np} - q^{-4Np})(q^{2p+1} - q^{-2p-1})} \times \sum_{n=0}^{N-1} (-1)^n q^{k(2(N-1)p-4np+n(n-1)) (q^{4p-2n+1} - q^{-4p+2n-1})}. \] (5.6)

From the expression (5.6) we see that for \( N \geq 2 \) we have \( \Phi(b_1^k) \neq \Phi(b_1^{-k}) \), in general. We give, for example, the result for the trefoil knot (\( k = 3 \)). We use the notation that \( \omega = q^{-2}, Z = q^{-4p} \). Then we have
\[ \Phi(b_1^3) = Z^{-2} + \omega Z^{-1} + \omega^2 - \omega + Z + \omega Z^2, \]
\[ \Phi(b_1^{-3}) = Z^2 + \omega^2 Z + \omega - \omega^2 + Z^{-1} + \omega^2 Z^{-2}. \] (5.7)

For an illustration of the normalization, we show a calculation of \( \hat{\Phi}(L, \alpha) \), where \( L \) is equivalent to the closed (colored) braid of \( b_1^k \) with \( k \) even, and \( \alpha = (p_1, p_2) \). We cut the first component with color \( p_1 \) and consider \( \Phi(L, 1, \alpha) \). We recall that \( p(n) = p_1 + p_2 - n \). Then we have
\[ \Phi(L, 1, \alpha) = \frac{[2p_1; N - 1]!}{[2p_1 + 2p_2 + 1; N]!} (f(p_1, p_2; q))^k \sum_{n=0}^{N-1} [2p(n) + 1] (\lambda_n(p_1p_2; q))^k. \] (5.8)

We now divide \( \Phi(L, 1, \alpha) \) by \( [2p_1; N - 1]! \), and obtain \( \hat{\Phi}(L, \alpha) \):
\[ \hat{\Phi}(L, \alpha) = \frac{(f(p_1, p_2; q))^k}{[2p_1 + 2p_2 + 1; N]!} \sum_{n=0}^{N-1} [2p(n) + 1] (\lambda_n(p_1p_2; q))^k. \] (5.9)

Let \( Z_{12} \) denote \( Z_{12} = q^{2(p_1+p_2)} \). If we choose the normalization as \( f(p_1, p_2; q) = q^{p_1p_2+p_1p_2} \), then we have
\[ \hat{\Phi}(L, \alpha) = \frac{(q - q^{-1})^N Z_{12}^{(N-1)k/2}}{(Z_{12}^N - Z_{12}^{-N}) q^{N(N+1)/2}} \times \sum_{n=0}^{N-1} \left( Z_{12}^{-kn+1} q^{kn(n-1)-2n+1} - Z_{12}^{-kn-1} q^{kn(n-1)+2n-1} \right). \] (5.10)

Let us consider evaluation for the parallel links. We assume that the link \( L(k) \) corresponds to the closed braid of \( (b_2 b_1 b_2 b_2) b_3^{-2k} \) with \( k \) odd, which is the parallel link of the 2 braid knot \( b_1^k \). We again make use of Prop. 4.2 and the decomposition (5.4). We have the following.
\[ \Phi(L) = \frac{[2p; N - 1]!}{[8p + 1; N]!(q - q^{-1})} \times \sum_{m=0}^{N-1} (-1)^m q^{-4k(2m-N+1)} p + k(m^2 - m) \times \left( q^{8p+1-2m} A_{N,m,k} - q^{-8p+1+2m} B_{N,m,k} \right), \]
where
\[ A_{N,m,k} = \begin{cases} N, & \text{if } (2m + 1)k - 2 = 0 \pmod{N}, \\ 0, & \text{otherwise}, \end{cases} \]
\[ B_{N,m,k} = \begin{cases} N, & \text{if } (2m + 1)k + 2 = 0 \pmod{N}, \\ 0, & \text{otherwise}. \end{cases} \]

From the evaluation (5.11) we obtain the following result.

**Proposition 5.1** The invariant for the parallel link of 2-braid knot \( b_1^k \) is non-zero if and only if \( (k, N) = 1 \) and \( N \) is odd.

For an illustration we show the invariants for the parallel links in some non-vanishing cases.

\[ \Phi(L(k)) = \begin{cases} \mp 5q^{-1}[2p; 4]!(q - q^{-1})^4, & \text{if } (k, N) = (3, 5) \text{ and } q^5 = \pm 1, \\ \mp 7q^{-1}[2p; 6]!(q - q^{-1})^6, & \text{if } (k, N) = (3, 7) \text{ and } q^7 = \pm 1, \\ \pm 3q[2p; 2]!(q - q^{-1})^2(q^{24p} + q^{-24p}), & \text{if } (k, N) = (5, 3) \text{ and } q^3 = \pm 1. \end{cases} \]

From the results of the parallel links and the 2 braid knots we can conclude that the (multi-variable) invariants for the \( N \) dim. nilpotent representations give generalizations of the (multi-variable) Alexander polynomial.

### 6 Discussion

In the paper we have calculated the multi-variable invariants using the Clebsch-Gordan coefficients of the nilpotent representation. It is an interesting problem to evaluate the multi-variable invariants for closed 3 braids and their parallel links. We can use the expressions of the Clebsch-Gordan coefficients and the Racah coefficients of the nilpotent representations obtained in [17]. The results of the calculation might suggest some unknown connections between the knot theory [4, 8] and the invariants of the quantum group.

Let us consider the nilpotent representation from the viewpoint of cyclic representation of \( U_q(sl(2)) \) where \( x, y, \) and \( z \) are (generic) complex parameters. The nilpotent representation could be viewed as a special case of the cyclic representation with parameters \( x, y, z \). However, the two representations give quite different R-matrices [7, 15] and different link invariants. There are many works associated with the cyclic, semi-periodic, and nilpotent representations of \( U_q(sl(n)) \). [4, 10, 36] Precise connections among the R-matrices and link invariants for the cyclic and the nilpotent representations should be discussed elsewhere.
We now discuss the multi-variable invariant from the viewpoint of the Hopf algebras \cite{21}, and consider the universal invariant related to the multi-variable invariant of the knot case. \cite{10} When \( q \) is a root of unity, the center of the universal enveloping algebra is larger than that for the \( q \) generic case, and we can consider quotient algebras. Reshetikhin and Turaev obtained a universal R-matrix for the standard root of unity representations of \( U_q(sl(2)) \), where the generators \( e \) and \( f \) are nilpotent and \( k^{2N} = id \). \cite{32} In the case of the nilpotent representation, the generators satisfy relations \( e^N = f^N = 0 \) and \( k^{2N} = q^{2Np}, p \in \mathbb{C} \). We may consider the defining relations of the quotient algebra related to the nilpotent representation as a generalization of those associated with the standard root of unity representation. Rosso obtained an expression of R matrices for the nilpotent root of unity representations of \( U_q(g) \). \cite{35} However, the R-matrices depend on the representations explicitly, and are not operator-valued. A \emph{colored ribbon Hopf algebra} is defined in order to employ the relation \( k^N = q^{2Np} \) as one of the defining relations of the quotient algebra, and then a universal R-matrix for the (colored) quotient algebra and a universal invariant, whose value is given by an element of the colored ribbon Hopf algebra, are obtained. \cite{30}

Finally we give comments on the crossing symmetry of the nilpotent CGC. The crossing symmetry of the nilpotent CGC could be considered as one of the most characteristic properties of the multi-variable invariants associated with the nilpotent representations. There are several reasons. (1) We can consider Prop 3.3 as a consequence of the crossing symmetry of the nilpotent CGC. We recall that Prop. 3.3 is important for the definition of the multi-variable invariants. (2) There are many different points in the crossing symmetry of the finite dimensional representations of \( U_q(sl(2)) \) and that of the nilpotent representations. (3) From the crossing symmetry of the nilpotent CGC we can derive the crossing symmetries for the colored braid matrix (the nilpotent R matrix) \cite{2} and for the lattice models \cite{15, 17}. The multi-variable invariants of colored framed graphs \cite{17} also have the crossing symmetry.

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A Appendix A

In [17] an explicit proof for the expression (4.12) is given. Here, however, we give a simple derivation of CGC (4.12) using an operator notation. We give the highest weight vector in the module $V(p_3)$ in an operator notation, which does not depend on an explicit matrix representation. In Appendix A it is not necessary to assume that $q$ is the root of unity, and the discussion holds also for the $q$ generic case. [17]

Let the $q$-combinatorial for a complex number $p$ and a non-negative integer $z$ be

$$\left[ \frac{p}{z} \right] = \frac{[p,z]!}{[z]!}. \quad (A.1)$$

Let $n$ be a non-negative integer. We introduce a vector $|12; n, 0>$ by

$$|12; n, 0> = \sum_{z_1=0}^{n} (-1)^{z_1} q^{-z_1(p_3+1)} \left[ \frac{2p_1 - z_1}{n - z_1} \right] \times$$

$$\times \left[ \frac{2p_2 - (n - z_1)}{z_1} \right] f^{z_1} \otimes f^{n-z_1} |p_1, 0> \otimes |p_2, 0>. \quad (A.2)$$

We shall show that $|p_1, p_2; p_3, 0>$ is given by $N(p_1, p_2, p_3)|12; n, 0>$ with a normalization factor $N(p_1, p_2, p_3)$. We show the following.

Lemma A.1 If we assume that $|p_i, 0>$ are highest : $e|p_i, 0> = 0$ for $i = 1$ and 2, then the vector $|12; n, 0>$ is also highest:

$$\Delta(e)|n, 0> = 0. \quad (A.3)$$
(Proof) Let us define $z_2$ by $z_2 = n - z_1$. We first calculate $[\Delta(e), f^{z_1} \otimes f^{z_2}]$.

\[
[\Delta(e), f^{z_1} \otimes f^{z_2}] = [e \otimes k, f^{z_1} \otimes f^{z_2}] + [k^{-1} \otimes e, f^{z_1} \otimes f^{z_2}]
\]

\[
= [e, f^{z_1}] \otimes k f^{z_2} + (1 - q^{z_2}) f^{z_1} e \otimes k f^{z_2} + k^{-1} f^{z_1} \otimes [e, f^{z_2}] + (1 - q^{-z_1}) k^{-1} f^{z_1} \otimes f^{z_2} e. \quad (A.4)
\]

Then we have

\[
[\Delta(e), f^{z_1} \otimes f^{z_2}]|p_1, 0 > \otimes |p_2, 0 > = \left( \frac{[z_1]}{q - q^{-1}} q^{-z_2} f^{z_1 - 1}(q^{1-z_1} k^2 - q^{z_1 - 1} k^{-2}) \otimes f^{z_2} k \right. \\
+ \left. \frac{[z_2]}{q - q^{-1}} q^z_1 f^{z_1 - 1} \otimes f^{z_2 - 1}(q^{1-z_2} k^2 - q^{z_2 - 1} k^{-2}) \right)|p_1, 0 > \otimes |p_2, 0 > \\
= \left( [z_1] q^{p_2 - 1} [2 p_1 - z_1 + 1] f^{z_1 - 1} \otimes f^{z_2} + \\
+ [z_2] q^{p_1} [2 p_2 - z_2 + 1] f^{z_1} \otimes f^{z_2 - 1} \right)|p_1, 0 > \otimes |p_2, 0 >. \quad (A.5)
\]

Substituting (A.3) and (A.2) into $\Delta(e)|12; n, 0 >$ we have the proposition.

We now assume the properties (4.2) and (4.3) of the inner product for $V(p_1)$ and $V(p_2)$. Let us assume the actions (4.1) of the generators on the base of the modules $V(p_1)$ and $V(p_2)$, then we have

\[
|p_j, z_j >= f^z_j |p_j, 0 > / \sqrt{[2 p_j; z_j]!![z_j]!}, \text{ for } j = 1, 2. \quad (A.6)
\]

We recall that $p(n) = p_1 + p_2 - n$, and that $z$ denotes a positive integer. We now define vector $|12; n, z >$ by

\[
|12; n, z >= (\Delta(f))^2 |12; n, 0 > / \sqrt{[2 p(n); z]!![z]!}. \quad (A.7)
\]

We can show that $|12; n, 0 >$ is orthogonal to $|12; m, z >$ if $m \neq n$. In particular, we can show the following by using (A.6) (see [17])

\[
(|12; n, 0>, |12; m, n - m >) = 0, \text{ if } 0 < m \leq n. \quad (A.8)
\]

If we take the normalization factor $N(p_1, p_2, p_3)$ as

\[
N(p_1, p_2, p_3) = q^{(n-n^2)/2+n p_2} \sqrt{\frac{[n]!}{[2 p_3 + n + 1; n]!![2 p_1; n]!![2 p_2; n]!}}, \quad (A.9)
\]

then we can show that the vector $N(p_1, p_2, p_3)|12; n, z >$ satisfies the normalization condition of (1.5). Thus we have shown that the vector $|12; n, 0 >$ with the normalization factor $N(p_1, p_2, p_3)$ gives the normalized highest weight vector in the module $V(p_3)$ with $p_3 = p(n)$.

We define $|p_1, p_2, p_3, 0 >$ by $|p_1, p_2, p_3, 0 > = N(p_1, p_2, p_3)|12; n, 0 >$. By applying $\Delta(f)$ to $|p_1, p_2, p_3, 0 > = N(p_1, p_2, p_3)|12; n, 0 >$ and by using (A.7) we obtain the Clebsch-Gordan coefficients of the nilpotent representation.
Appendix B

We prove the crossing symmetry of the CGC of the nilpotent representations by using induction on \( z_1 \).

(1) For the case \( z_1 = 0 \), we can show the relation (4.13) by explicit calculation using the expression (4.12). We use the following properties of \( q \)-integers.

\[
\begin{align*}
[N - p] &= \lfloor p \rfloor, \quad [p + 1; N]! = (-1)^{[p; N]}! \quad \text{if } q^N = -1, \\
[N - p] &= (-1)^{[p]} \quad \text{if } q^{-p} = 1.
\end{align*}
\]  

(2) We assume that the relation (4.13) of the crossing symmetry holds for the case \( z_1 = z_1(0) \). We consider the recurrence relations among CGCs. We derive the following two recursion relations by considering the actions of \( \Delta(e) \) and \( \Delta(f) \) on the vectors of \( V(p_3) \) for the first and the second relations, respectively.

\[
\begin{align*}
\sqrt{z_3 + 1}[2p_3 - z_3]C(p_1, p_2, p_3; z_1, z_2 + 1, z_3 + 1) &= \\
\sqrt{[p_1 + 1 - z_1]}q^{p_2 - z_2 - 1}C(p_1, p_2, p_3; z_1, z_2 - 1, z_3) \\
&+ \sqrt{[z_2 + 1]}[2p_2 - z_2]q^{-p_1 + z_1}C(p_1, p_2, p_3; z_1, z_2, z_3), \\
\sqrt{[z_3]}[2p_3 + 1 - z_3]C(p_1, p_2, p_3; z_1, z_2 + 1, z_3 - 1) &= \\
\sqrt{[z_1 + 1]}[2p_1 - z_1]q^{p_2 - z_2 + 1}C(p_1, p_2, p_3; z_1 + 1, z_2, z_3) \\
&+ \sqrt{[z_2]}[2p_2 + 1 - z_2]q^{-p_1 + z_1}C(p_1, p_2, p_3; z_1, z_2, z_3).
\end{align*}
\]  

In the relation (B.2) we replace \( p_1, p_2, p_3, z_1, z_2, \) and \( z_3 \) by \( p_3, p_2, p_1, z_3 + 1, z_2 - 1, \) and \( z_1(0) \), respectively, and we have (B.2)*. We apply (B.2)* to the left hand side of the relation (4.13) for \( z_1 = z_1(0) \). To the derived expression we apply the second relation (B.3) with \( z_2 \) and \( z_3 \) replaced by \( z_2 + 1 \) and \( z_3 + 1 \), respectively. We then obtain the crossing symmetry (4.13) for the case \( z_1 = z_1(0) + 1 \).

Appendix C

We discuss the Markov trace property (the Reidemeister move I) for the nilpotent R-matrix. Using the decomposition (5.2) of the R-matrix and Prop. 4.2 we show the following.
Lemma C.1

\[ \sum_{b=0}^{N-1} R_{ab}^{bp}(\pm) q^{2(p-b) - 2Np} = q^{|2p|}. \quad (C.1) \]

(Proof) We consider the case of + sign.

\[
(LHS) = \frac{[2p; N-1]!}{[4p+1; N]!} \sum [4p+1 - 2n] \lambda_n(p, p; q) = \frac{[2p; N-1]!}{[4p+1; N]!} q^{-2p}|SUM| \frac{1}{q - q^{-1}},
\]

where the \((SUM)\) is given by

\[
(SUM) = \sum (-1)^n q^{2(N-1-2n)p+n(n-1)} (q^{4p+1-2n} - q^{-4p-1+2n})
= \sum_{n=1}^{N-2} - \sum_{n=1}^{N} (-1)^{n-1} q^{2(N-1-2n)p+n(n-1)-1}
= q^{2(N+1)p+1} - q^{2(N-1)p-1} - (-1)^N q^{-2(N-1)p+N(N-1)+1} +
+ (-1)^N q^{-2(N+1)p+N(N-1)-1}. \quad (C.2)
\]

We now note the following.

\[ [p; N)! = q^{(N-1)N/2} (q^{Np} - q^{-Np}) (q - q^{-1})^{-N}. \quad (C.3) \]

Then we have

\[
(LHS) = \frac{q^{2Np} - q^{-2Np}}{(q^{4Np+N} - q^{-4Np-N}) (q^{2p-N+1} - q^{-2p+N-1}) (SUM) q^{2p^2-2(N-1)p}}
= q^{2p^2-2(N-1)p}. \quad (C.4)
\]

The last equation can be shown by an explicit calculation using \(q^N = \pm 1\).