PRYM–BRILL–NOETHER LOCI OF SPECIAL CURVES

STEVEN CREECH, YOAV LEN, CAELAN RITTER, AND DEREK WU

Abstract. We use Young tableaux to compute the dimension of $V^r$, the Prym–Brill–Noether locus of a folded chain of loops of any gonality $k$. This tropical result yields a new upper bound on the dimensions of algebraic Prym–Brill–Noether loci. Moreover, we prove that $V^r$ is pure-dimensional and connected in codimension 1 when $\dim V^r \geq 1$. We then compute the genus of this locus for even gonality when the dimension is exactly 1, and compute the cardinality when the locus is finite and the edge lengths are generic.

Contents

1. Introduction 1
2. Preliminaries 3
3. Dimensions of Prym–Brill–Noether loci 4
4. Tropological properties 9
5. Discrete properties 17
References 22

1. Introduction

Let $f : \tilde{X} \to X$ be an unramified double cover of either tropical or algebraic curves, and let $f_* \overline{\cdot}$ be the induced map on divisor classes. The corresponding Prym–Brill–Noether locus is

$$V^r(X, f) = \{ [D] \in \text{Jac}(\tilde{X}) \mid f_*(D) = K_X, r(D) \geq r, r(D) \equiv r \pmod{2} \},$$

where $K_X$ is the canonical divisor of $X$. It is a variation of the classical Brill–Noether locus $W^d_d(\tilde{X})$ that also takes symmetries of $\tilde{X}$ into account. The Prym–Brill–Noether locus naturally lives inside the Prym variety $\text{Prym}(X, f)$, namely a connected component of the fiber of $K_X$ in the Jacobian of $\tilde{X}$ (see Section 2 for more details).

The dimension and topological properties of the usual Brill–Noether locus have been studied extensively in classical algebraic geometry [GH80, Gie82, FL81] and in tropical geometry [CDPR12, JR17, Len14]. More recently, dimensions of non-maximal components of Brill–Noether loci were computed using both tropical and non-tropical techniques [CPJ19, Lar19].

On the other hand, much less is known for Prym varieties. Bertram and Welters computed the dimension of the Prym–Brill–Noether locus for curves that are general in the moduli [Ber87, Wel85], and Welters has also shown that the locus is generically smooth. The tropical study of Prym varieties was initially introduced in joint work of the second author with Jensen [JL18], and further studied in joint work with Ulirsch [LU19] (see [LUZ19] for higher degree covers). As they show, tropical Prymes are abelian of the expected dimension and behave well with respect to tropicalization, leading to a new bound on the dimension of algebraic Prym–Brill–Noether loci of general even-gonal curves.

Our first result is an extension of these techniques to curves of any gonality.
Theorem A. Let $\varphi: \overline{\Gamma} \to \Gamma$ be a $k$-gonal uniform folded chain of loops, and denote $l = \lceil \frac{k}{2} \rceil$. Then
\[
\text{codim } V^r(\Gamma, \varphi) = \begin{cases} 
\frac{l(l+1)}{2} + l(r-l) & \text{if } l \leq r-1, \\
\frac{l(l+1)}{2} & \text{if } l > r-1.
\end{cases}
\]

By uniform we mean that each of its loops has the same torsion, see Section 2 for more details. As it turns out, the odd gonality case is far trickier than the even gonality one, which necessitates a variety of new combinatorial tools.

Denote the quantity expressed in Eq. (1.1) by $n(r,k)$. Note that we adopt the convention that a set whose dimension is negative is empty, hence, $V^r(\Gamma, \varphi)$ is empty if $\text{codim } V^r(\Gamma, \varphi) > g-1$. As a consequence of the theorem, we obtain an upper bound on the dimensions of Prym–Brill–Noether loci for algebraic curves that are general in moduli of $k$-gonal curves.

Corollary B. Let $r \geq -1$ and $k \geq 2$. Then there is a nonempty open subset of the $k$-gonal locus of $R_g$ such that for every unramified double cover $f: \overline{C} \to C$ in this open subset we have
\[
\dim V^r(C, f) \leq g - 1 - n(r,k).
\]

We then turn our attention to more subtle tropical (namely, topological properties of tropical varieties) properties of Prym–Brill–Noether loci of folded chains of loops.

Theorem C. $V^r(\Gamma, \varphi)$ is pure-dimensional for any gonality $k$. If $\dim V^r(\Gamma, \varphi) \geq 1$ then it is also connected in codimension 1.

By connected in codimension 1 we mean that any two maximal components are connected by a sequence of components whose codimension is at most 1. The different properties mentioned in the theorem are proven as part of Proposition 4.8 and Proposition 4.7. The pure dimensionality of the locus is quite surprising since Brill–Noether loci of general $k$-gonal curves may very well have maximal components of different dimension (see, for instance [CPJ19, Example 2.4]). We don’t know at this point whether this phenomena is special to tropical Prym curves, or carries on to algebraic ones as well.

When $r$ and $k$ are chosen so that $\dim V^r(\Gamma, \varphi) = 0$, the Prym–Brill–Noether locus is a finite collection of points. If the gonality is also assumed to be even, we compute its cardinality by constructing a bijection between its points and certain lattice paths (Proposition 5.1). If the dimension is 1, the tropical Prym–Brill–Noether locus is a graph within the Prym variety, whose first homology we compute in the generic case.

Theorem D. Let $\varphi: \overline{\Gamma} \to \Gamma$ be a generic chain of loops such that $\dim V^r(\Gamma, \varphi) = 1$. Then the genus of $V^r(\Gamma, \varphi)$ is
\[
\frac{r \cdot f^{\lambda} \cdot \left(\frac{r+1}{2}\right) + 1}{2},
\]
where $f^{\lambda}$ is the number of ways to fill a staircase tableau of length $r$ with distinct symbols.

Moreover, all of our results rely on the correspondence between certain Young tableaux and divisors on tropical curves (cf. [CDPRI2, Pfl17b]). The key tool that we develop to enumerate such tableaux is the notion of a non-repeating strip, a special subset that determines the rest of the tableau (see Section 4.1 for more detail). We hope that this and other techniques presented in this paper will lead to additional results concerning dimensions and Euler characteristics of tropical and algebraic Brill–Noether loci.

Acknowledgements. We thank Dave Jensen for insightful remarks on a previous version of this manuscript. This research was conducted at the Georgia Institute of Technology with the support of RTG grant GR10004614 and REU grant GR10004803.
2. Preliminaries

We assume throughout that the reader is familiar with the theory of divisors on tropical curves and with harmonic morphisms of graphs. A beautiful introduction to the topic may be found in the survey paper [BJ16]. A morphism $\varphi: \tilde{\Gamma} \to \Gamma$ of metric graphs is called a double cover if it is harmonic of degree 2 in the sense of [ABBR15b]. The morphism is called unramified if, in addition, it pulls back the canonical divisor of $\Gamma$ to the canonical divisor of $\tilde{\Gamma}$.

Fix a divisor class $[D]$ on $\Gamma$. The fiber $\varphi^{-1}([D])$ consists of either one or two connected components in the Picard group of $\Gamma$ [JL18, Proposition 6.1]. Each of them is referred to as a Prym variety, and their elements are called Prym divisors. Prym varieties are principally polarized tropical abelian varieties [LU19, Theorem 2.3.7]. Unless stated otherwise, the divisor $[D]$ will always be the canonical divisor $K_{\Gamma}$. Fixing an integer $r$, the Prym–Brill–Noether locus $V^r(\Gamma, \varphi)$ consists of the Prym divisor whose rank is at least $r$ and has the same parity as $r$.

We will mostly be interested in a particular double cover known as the folded chain of loops. In this case, the target of the map is the chain of loops that recently appeared in various celebrated papers (e.g. [JP16, Pfl17a, JR17]). It consists of $g$ loops, denoted $\gamma_1, \ldots, \gamma_g$ and connected by bridges. The source graph is a chain of $2g - 1$ loops, as exemplified in Figure 2.1. Each pair of loops $\tilde{\gamma}_a$ and $\tilde{\gamma}_{2g-a}$ (for $a < g$) maps down to $\gamma_a$, while each edge of $\tilde{\gamma}_g$ maps isometrically onto the loop $\gamma_g$. See [LU19, Section 5.2] for a detailed explanation.

![Figure 2.1. A Prym divisor on the 4-gonal folded chain of 7 loops and its image under $\varphi_*$ in the 4-gonal chain of 7 loops.](image)

The torsion of a loop $\gamma_a$ is the least positive integer $k$ such that $\ell_a + m_a$ divides $k \cdot m_a$, where $m_a$ and $\ell_a$ are the lengths of the lower and upper arcs of $\gamma_a$ respectively. The chain of loops is uniform $k$-gonal if each loop has torsion $k$. Note that a uniform $k$-gonal chain of loops is indeed a $k$-gonal metric graph in the sense of [ABBR15a, Section 1.3.2]. A double cover as above is said to be uniform $k$-gonal if $\Gamma$ is. Note that $\tilde{\Gamma}$ is not in itself uniform $k$-gonal, since the loop $\tilde{\gamma}_g$ has torsion 2.

2.1. Prym tableaux. We study divisors only indirectly, making use of a correspondence between sets of divisors on chains of loops and Young tableaux as introduced in [Pfl17a, LU19]; here we shall recall only the essential definitions and introduce some helpful notation.

For our purposes, a tableau on a subset $\lambda \subseteq \mathbb{N}^2$ is a map $t: \lambda \to \{1, 2, \ldots, n\}$ satisfying the tableau condition: $t(x, y) < t(x + 1, y)$ and $t(x, y) < t(x, y + 1)$ whenever these values are defined. We call an element $(x, y) \in \lambda$ a box of $t$, and its image $t(x, y) \in \mathbb{N}$ the symbol contained in the box $(x, y)$. We say that a box $(x, y)$ is below $(x', y')$ if $x \leq x'$, $y \leq y'$, and $(x, y) \neq (x', y')$. The tableau condition implies that $t(x, y) < t(x', y')$ whenever $(x, y)$ is below $(x', y')$. When $\lambda$ is a partition of $n$ and $t$ is injective, then $t$ is a standard Young tableau.

The tableau $t$ is called a $k$-uniform displacement tableau if

$$t(x, y) = t(x', y') \text{ only if } x - y \equiv x' - y' \mod k.$$
We refer to 0-uniform displacement tableau as generic (for reasons that will become clear later). Note that such tableau are exactly the standard Young tableau. Notice that this displacement condition partitions \( \lambda \) into \( k \) regions, which we shall call diagonals modulo \( k \). To be precise, we define the \( i \)-th diagonal modulo \( k \) to be

\[
D_i = \{ (x, y) \in \lambda \mid x - y \equiv i \pmod{k} \};
\]

then \( \lambda \) is the disjoint union of \( D_i \) for \( i \in \{0, 1, \ldots, k-1\} \), and the fiber of each element in the codomain of \( t \) is contained within some \( D_i \).

The \( n \)-th anti-diagonal \( A_n \) is the set of all boxes \((x, y)\) such that \( x + y = n + 1 \). Define the lower triangle of size \( n \) to be \( T_n = \bigcup_{i=1}^{n} A_i \); we shall refer to \( A_n \) as the main anti-diagonal in this context. For example, Figure 2.2 shows a lower-triangular tableau of size 6. \( D_1 \) is colored blue, \( A_6 \) is red, and their intersection is purple. In the French notation, the bottom-left box is \((1, 1)\), with the first coordinate increasing to the right and the second coordinate increasing upwards. Every box here not colored red or purple is below \( A_6 \).

![Figure 2.2](image)

**Figure 2.2.** A typical example of a lower-triangular tableau of size 6 with torsion 3.

As explained in [Pfl17a], \( k \)-uniform displacement tableaux on the rectangle \([g - d + r] \times [r + 1]\) with image contained in \([g]\) give rise to divisors of rank at least \( r \) on the uniform \( k \)-gonal chain of \( g \) loops. These definitions extend naturally to the folded chain, as it is itself a chain of loops; the only difference is that while the fiber of a symbol \( a \neq g \) is contained within a single diagonal modulo \( k \), the fiber of \( g \) must occur within one of the two diagonals modulo 2. (Equivalently, each box containing \( g \) must be even cab distance from each other such box.) By abuse of terminology, we shall refer to such tableau as \( k \)-uniform.

We note that for a \( k \)-gonal chain of \( g \) loops \( \Gamma \), the genus of the folded chain \( \overline{\Gamma} \) is \( 2g - 1 \). Moreover, Prym divisors map down to \( K_\Gamma \) and must therefore have degree \( 2g - 2 \). We call a \([r + 1] \times [r + 1]\) square tableau \( t \) on \([2g - 1]\) symbols Prym of type \((g, r, k)\) if \( t \) is \( k \)-uniform and satisfies the following Prym condition: \( t(x, y) = 2g - t(x', y') \) only if \((x, y)\) and \((x', y')\) both lie in the same diagonal modulo \( k \).

Such tableaux give rise to a set \( P(t) \) of Prym divisors of rank at least \( r \) on the \( k \)-uniform folded chain of \( 2g - 1 \) loops. In fact, every Prym divisor of rank at least \( r \) is obtained this way [LU19, Corollary 5.3.10].

### 3. Dimensions of Prym–Brill–Noether loci

Our primary focus in this section is to prove Theorem A, by constructing Prym tableaux that minimize the number of symbols used. Under the correspondence between tableaux and divisors, each symbol in the tableau determines the position of a chip on the corresponding loop. In the case of Prym divisors, the position of a chip on the \( a \)-th loop determines the position on the \( 2g - a \)-th loop and vice versa. In particular, if a symbol \( a \) appears in \( t \), then the placement of chips on the loops \( a \) and \( 2g - a \) are determined in \( P(t) \); the symbol \( 2g - a \) may then appear in the tableau “for free”, in the sense that it does not affect the codimension of the set of divisors (provided that the Prym condition is satisfied).
Similarly, the Prym condition stipulates that the chip on the $g$-th loop is at one of two coordinates, so an appearance of the symbol $g$ in the tableau does not increase the codimension.

It is therefore reasonable to expect that the codimension is minimized precisely for those Prym tableaux in which symbols $a$ and $2g - a$ appear in pairs. This motivates the notion of reflective Prym tableau.

### 3.1. Reflective tableaux.

Given a Prym tableau $t$ with domain $\lambda = [r + 1] \times [r + 1]$, consider the map $\rho: \lambda \to \lambda$ defined by $\rho(x, y) = (r + 2 - y, r + 2 - x)$; in other words, $\rho$ picks out the box which is the reflection of $(x, y)$ across the main anti-diagonal. We say that a box $(x, y)$ is reflective if $t(x, y) = 2g - t(\rho(x, y))$ (i.e., if the symbol in the box is the dual of the symbol in its reflection).

**Definition 3.1.** A tableau $t$ is said to be reflective if every box of $t$ is reflective.

Note that a displacement tableau is reflective only if it is Prym.

Given two Prym tableaux $t$ and $s$ of type $(g, r, k)$, we shall say that $s$ dominates $t$ if $P(s) \supset P(t)$. If $s$ and $t$ each dominate the other, we shall call them equivalent. By a slight abuse of notation, we define $\text{codim}(t)$ to be the codimension of the corresponding set of Prym divisors $P(t)$ regarded as a subset of the Prym($\Gamma, \varphi$). By our earlier remarks, $\text{codim}(t)$ counts the pairs of symbols $\{a, 2g - a\}$ for which $a \neq g$ and either $a$ or $2g - a$ appears in $t$.

**Remark 3.2.** A tableau $s$ dominates $t$ precisely when for each $(x, y) \in \lambda$, there exists $(x', y') \in \lambda$ in the same diagonal modulo $k$ such that either $s(x, y) = t(x', y')$ or $s(x, y) = 2g - t(x', y')$.

If $s$ dominates $t$, then $\text{codim}(s) \leq \text{codim}(t)$. Therefore, for the purpose of computing the dimension of $V^r(\Gamma, \varphi)$, we may restrict our attention to the tableaux that are maximal with respect to the partial order given by dominance. The main result of this section is the following.

**Proposition 3.3.** Let $t$ be a Prym tableau. Then there exists a reflective tableau $s$ that dominates $t$.

The following definition from [Pfl17b] will be used repeatedly during the proof. Given a partition $\lambda$ and subset $S \subset \mathbb{Z}/k\mathbb{Z}$, the upward displacement of $\lambda$ by $S$, denoted $\text{disp}^+(\lambda, S)$, is equal to $\lambda \cup L$, where $L$ consists precisely of those boxes $(x, y) \notin \lambda$ such that:

- $(x - 1, y) \in \lambda$ or $x = 1$,
- $(x, y - 1) \in \lambda$ or $y = 1$, and
- $x - y \equiv i \pmod{k}$ for some $i \in S$.

The boxes in $L$ are known as the loose boxes of $\lambda$ with respect to $S$. When $S = \mathbb{Z}/k\mathbb{Z}$, we use the shorthand $\text{disp}^+(\lambda)$ and note the following: if $\lambda$ is a partition, then so is $\text{disp}^+(\lambda)$; $L$ is nonempty; every box in $\lambda$ is below some box in $L$; and every box in $\mathbb{N}^2 \setminus \text{disp}^+(\lambda)$ is above some box in $L$. The usefulness of this operation on partitions is made evident in the following example, which outlines the subsequent proof of Proposition 3.3.

**Example 3.4.** Consider the initial Prym tableau of type $(g, r, k) = (11, 4, 3)$ in the sequence illustrated in Fig. 3.1. This tableau is far from being reflective, but at each step we make small changes so that the resulting tableau is closer to being reflective and dominates the preceding one.

At each step, the boxes previously dealt with are depicted in blue; we look at the symbols in the loose boxes with respect to the lower-left blue partition and choose the minimum $a$; we look at the symbols contained in the reflection of the loose boxes and choose the maximum $b$; finally, denote $c$ the minimum of $a$ and $2g - b$. If $c$ was obtained at the box $(x, y)$, replace the value at each loose box in the same diagonal modulo $k$ (depicted in red) with $c$, and the value at their reflection (depicted in red as well) with $2g - c$. The final tableau is reflective and dominates the initial tableau.
The basic operation of the algorithm is to repeatedly reflect symbols, i.e., given a box $\omega$, to insert the dual symbol, $2g - t(\omega)$, into the reflection, $\rho(\omega)$. The following lemma ensures that the result is still a Prym tableau, granted that the tableau condition holds; then the proof of Proposition 3.3 will make the rest of the algorithm precise.

**Lemma 3.5.** Given a Prym tableau $t$ such that the box $(x, y)$ is not reflective, the tableau $s$ obtained by defining

$$s(\omega) = \begin{cases} 
2g - t(x, y) & \text{for } \omega = \rho(x, y) \\
t(\omega) & \text{otherwise}
\end{cases}$$

(3.1)

satisfies the Prym and displacement conditions.

**Proof.** The only box at which either of the conditions might fail is at $\rho(x, y)$. However, taking the difference of the coordinates of $\rho(x, y) = (r + 2 - y, r + 2 - x)$, we find that $\rho(x, y) \in D_{x - y}$. The Prym condition is immediately satisfied, and it is not hard to see that, since any other box containing the symbol $2g - t(x, y)$ would need to be in $D_{x - y}$, the displacement condition is also satisfied. $\square$

**Proof of Proposition 3.3.** Denote $s_0 = t$. We describe an algorithm which at each step, given a Prym tableau $s_i$, will produce a Prym tableau $s_{i+1}$ that dominates $s_i$. After a finite number of steps, the algorithm will produce Prym tableau $s_f$ which is reflective away from the main anti-diagonal and which dominates $t$ by transitivity. In the final step, the symbols along the main anti-diagonal of $s_f$ are replaced with $g$ to obtain $s$.

Suppose that after the $i$-th step we have a Prym tableau $s_i$ that dominates $s_{i-1}$. Define $\kappa_i$ to be the subpartition of boxes below the main anti-diagonal whose symbols in $s_i$ are at most $n_i$. (Note that the blue-colored boxes in Example 3.4 are precisely $\kappa_i \cup \rho(\kappa_i)$.) Suppose that $\kappa_i$ is reflective and that $s_i(\omega) = t(\omega)$ for each box $\omega$ not in $\kappa_i \cup \rho(\kappa_i)$. If $\kappa_i = T_{\ell_r}$, we are ready to perform the final step. Otherwise, let $L_i$ be the set of loose boxes of $\kappa_i$ that lie below the main anti-diagonal, and note that $L_i$ is nonempty.
Consider the minimal positive integer \( n_{i+1} \) among the set of symbols \( s_i(L_i) \cup (2g - s_i(\rho(L_i))) \). We claim that \( n_{i+1} \) exists and is at most \( g - 1 \). Indeed, given any \( \omega \in L_i \), if \( s_i(\omega) \leq g - 1 \), the claim is true. Otherwise, \( s_i(\omega) \geq g \); since \( \omega \) lies below its reflection \( \rho(\omega) \), it follows that \( s_i(\rho(\omega)) \geq g + 1 \); moreover, it must be the case that \( s_i(\rho(\omega)) \leq 2g - 1 \); from both of these inequalities we find that \( 1 \leq 2g - s_i(\rho(\omega)) \leq g - 1 \), as desired.

Note also that \( n_{i+1} \geq n_i + 1 \); otherwise, any boxes containing \( n_{i+1} \) would be in \( \kappa_i \) and any boxes containing its dual would be in \( \rho(\kappa_i) \) (and so they would not appear in either \( L_i \) or \( \rho(L_i) \), respectively). Then every box in \( T_r \setminus \text{disp}^r(\kappa_i) \) must be above some box of \( L_i \), so the fact that \( n_{i+1} \) is (in particular) minimal among the symbols in \( L_i \) implies that \( n_{i+1} \) cannot appear in any box of \( T_r \setminus \text{disp}^r(\kappa_i) \); such a case would violate the tableau condition. Analogously, \( 2g - n_{i+1} \) is at least \( g + 1 \) and at most \( 2g - (n_i + 1) \) and does not appear in the reflected set \( \rho(T_r \setminus \text{disp}^r(\kappa_i)) \).

From these considerations, we find that the next tableau, \( s_{i+1} \), constructed by defining

\[
s_{i+1}(\omega) = \begin{cases} 
2g - n_{i+1} & \text{for } \omega \in \rho(s_i^{-1}(n_{i+1}) \cap T_r) \\
\rho(\omega) & \text{for } \omega \in s_i^{-1}(2g - n_{i+1}) \cap \rho(T_r), \\
s_i(\omega) & \text{otherwise}
\end{cases}
\]

(3.2)

satisfies the tableau condition. In terms of Example 3.4, this construction amounts to the following: color every box below \( A_{r+1} \) which contains \( n_{i+1} \) and every box above \( A_{r+1} \) which contains \( 2g - n_{i+1} \) red; then color each box which is the reflection of a red box red; now replace the symbols in each of these latter boxes with either \( 2g - n_{i+1} \) or \( n_{i+1} \) as appropriate. Then it becomes clear that the result still satisfies the tableau condition, since \( n_{i+1} \) is larger than every lower-left blue box and smaller than every uncolored box and \( 2g - n_{i+1} \) is larger than any uncolored box and smaller than any upper-right blue box.

It follows from repeated application of Lemma 3.5 that the displacement and Prym conditions are preserved in \( s_{i+1} \). Given this, it is straightforward to see that \( s_{i+1} \) dominates \( s_i \) and—by transitivity—\( t \). Note also that adding a selection of loose boxes to a partition forms another partition, so in particular, the next set \( \kappa_{i+1} \) will be a partition. It will also be equal to the set \( s_{i+1}^{-1}([n_{i+1}]) \cap T_r \). Therefore, all the inductive hypotheses are satisfied.

Since \( \kappa_{i+1} \) strictly contains \( \kappa_i \), it follows that in a finite number of steps, \( T_r \) will be reflective. After the final step, we replace all boxes on the main anti-diagonal \( A_{r+1} \) with symbol \( g \), and the resulting tableau is reflective.

A reflective tableau is determined by its restriction to \( T_r \), so we may as well only consider this subset.

**Definition 3.6.** A **staircase Prym tableau of type** \((g, r, k)\) is a lower-triangular \( k \)-uniform displacement tableau of length \( r \) with image in \([g - 1]\).

We extend all definitions regarding Prym tableaux to staircase Prym tableaux in the natural way; for instance, if \( s \) is a reflective square tableau which extends a staircase Prym tableau \( t \), then \( P(t) \) equals, by definition, \( P(s) \).

**3.2. Proof of Theorem A.** Throughout this section, \( \varphi: \overline{\Gamma} \to \Gamma \) will represent a folded chain of loops of genus \( g \), where the edge lengths of \( \Gamma \) are either generic or the torsion of each loop is \( k \). For the sake of brevity, we will refer to the folded chain of loops and its corresponding Prym tableaux in the former case as **generic** and in the latter as **k-gonal**.

The dimension of \( V^r(\Gamma, \varphi) \) is known in the generic case and when \( k \) is even; see [LU19, Theorem 6.1.4, Corollary 6.2.2]. When \( k \) is odd, [LU19, Remark 6.2.3] provides an upper and a lower bound for the dimension. In this section we show that the dimension of \( V^r(\Gamma, \varphi) \) in fact coincides with the lower bound. As a consequence, we obtain an upper bound on the dimension of \( V^r \) for generic \( k \)-gonal algebraic curves. We restate the precise result here.
Theorem A. Let $\varphi \colon \Gamma \to \Gamma$ be a $k$-gonal uniform folded chain of loops, and denote $l = \lfloor \frac{k}{2} \rfloor$. Then
\[
\text{codim} V'(\Gamma, \varphi) = \begin{cases} 
\frac{l+1}{2} + l(r-l) & \text{if } l \leq r-1, \\
\frac{l+1}{2} & \text{if } l > r-1.
\end{cases}
\tag{1.1}
\]

We have phrased this in terms of codimension rather than dimension because of the close relationship between codimension of $P(t)$ and the number of pairs of symbols in $t$. In fact, because we are concerned with computing the minimal codimension of $P(t)$ over all Prym tableaux $t$ of type $(g, r, k)$, Proposition 3.3 implies that it suffices to consider staircase Prym tableaux: given any Prym tableau, we apply the reflection algorithm to obtain a dominating Prym tableau, which, per the remarks at the end of Section 3.1, may be regarded as the staircase Prym tableaux which constitutes its restriction to the lower triangle. Then, for $t$ a staircase Prym tableaux, we have the convenient formula $\text{codim}(t) = \lim t$.

The second case in Eq. (1.1) corresponds to generic edge lengths. Note that $\binom{r+1}{2}$ counts the number of boxes in $T_r$. The cab distance between any two boxes is at most $2r - 2 \leq 2l - 2 < k$, so each must contain a unique symbol; it follows that the number of symbols in any such tableau is precisely $\binom{r+1}{2}$.

The same reasoning explains the presence of the $\binom{l+1}{2}$ term in the first case: it counts the number of symbols in $T_l$, which are all necessarily unique. Any repeats occur above $T_l$. In fact, we claim that a tableau of minimal codimension contains precisely $l$ new symbols on each subsequent anti-diagonal, of which there are $r-l$; this accounts for the $l(r-l)$ term. Precisely, we say that a set of symbols $S \subset t(A_n)$ is new if $S \cap t(T_{n-1})$ is empty.

Proposition 3.7. Given a staircase Prym tableau $t$ of type $(g, r, k)$, there exist at least $l$ new symbols in $A_n$ for each $n \geq l + 1$.

The following lemma establishes a restriction on symbols which will go most of the way toward proving Proposition 3.7.

Lemma 3.8. Let $t$ be a staircase Prym tableau of type $(g, r, k)$, and fix $n \leq r$. For any boxes $(x, y) \in D_i$ and $(x', y') \in D_{i+1}$ that lie below $A_n$, there exists a box $\omega = (\omega_1, \omega_2) \in A_n \cap (D_i \cup D_{i+1})$ such that $t(\omega)$ is greater than both $t(x, y)$ and $t(x', y')$.

Proof. Denote $a = t(x, y)$ and $b = t(x', y')$. Since $a$ and $b$ lie in different diagonals modulo $k$, we know that $a \neq b$. We will assume that $a < b$; the proof will follow the same way when the converse inequality holds. We want to show that there is a box $\omega$ in $A_n \cap (D_i \cup D_{i+1})$ which lies above $(x', y')$, since this would force $t(\omega) > b$.

Indeed, define $\delta = n + 1 - x' - y'$. We know that $x' + y' \leq n$ because $(x', y')$ sits below $A_n$, so $\delta \geq 1$. If $\delta$ is even, then we define
\[
\omega := \left(x' + \frac{\delta}{2}, y' + \frac{\delta}{2}\right).
\tag{3.3}
\]
Note that $\omega_1$ and $\omega_2$ are both positive integers, $\omega_1 + \omega_2 = n + 1$, and $\omega_1 - \omega_2 = x' - y' \equiv i + 1 \pmod{k}$; moreover, $\omega$ sits above $(x', y')$, as desired.

Suppose instead that $\delta$ is odd, and define
\[
\omega := \left(x' + \frac{\delta - 1}{2}, y' + \frac{\delta + 1}{2}\right).
\tag{3.4}
\]
Then the desired properties once again hold (although in this case, $\omega \in D_i$).

Proof of Proposition 3.7. Given $n$ such that $l+1 \leq n \leq r$, we note first that $T_{n-1} \cap D_i$ is nonempty. Indeed, we may write $i \in \{-l + 1, \ldots, l - 1\}$. If $i \geq 0$, we have that $(1+i, 1) \in T_{n-1} \cap D_i$; if $i < 0$, then $(1-i, 1) \in T_{n-1} \cap D_i$.

For each $i$, choose $\omega_i \in T_{n-1} \cap D_i$ such that $t(\omega_i)$ is maximal among $t(T_{n-1} \cap D_i)$. Then apply Lemma 3.8 to each pair $\omega_i, \omega_{i+1}$ to obtain a box $\eta_i \in A_n \cap (D_i \cup D_{i+1})$ such that $t(\eta_i) > t(\omega_i)$ and $t(\eta_i) > t(\omega_{i+1})$. Hence, $t(\eta_i) > t(\omega)$ for every box $\omega \in T_{n-1} \cap (D_i \cup D_{i+1})$ and so is new in $A_n$. 

\[\square\]
Therefore, for each pair \( \{i, i+1\} \subset \mathbb{Z}/k\mathbb{Z} \), the set \((D_i \cup D_{i+1}) \cap A_n\) contains at least one new symbol, which we shall denote \(b_i\). Note that if \(\{i, i+1\}\) and \(\{j, j+1\}\) are disjoint, then their respective symbols \(b_i\) and \(b_j\) must lie in different diagonals modulo \(k\), and so must be distinct. Thus, the minimum number of new symbols in \(A_n\) coincides with the minimum number of elements we can choose from \(\mathbb{Z}/k\mathbb{Z}\) such that we have at least one element in each pair \(\{i, i+1\}\). Suppose for the sake of contradiction that we could achieve this with \(l-1\) elements. Each is a member of two pairs, so we cover at most \(2(l-1) < k\) pairs. This is insufficient, as there are \(k\) pairs, so the minimum size of such a set is \(l\).

Proof of Theorem A. We have already proved the case where \(l > r\), so assume otherwise. From Proposition 3.7 and our earlier remarks, we get that \(T_r\) contains at least \(\binom{\frac{l+1}{2}}{2} + l(r-l)\) distinct symbols. Hence, \(\text{codim} V_\nu(\Gamma, \varphi)\) is bounded below by this quantity. Meanwhile, [LU19, Corollary 6.2.2, Remark 6.2.3] implies that it is also an upper bound, so we are done.

3.3. Relation to algebraic geometry. We are now in a position to prove Corollary B, restated below.

**Corollary B.** Let \(r \geq -1\) and \(k \geq 2\). Then there is a nonempty open subset of the \(k\)-gonal locus of \(R_g\) such that for every unramified double cover \(f: \tilde{C} \to C\) in this open subset we have

\[
\dim V_\nu(C, f) \leq g - 1 - n(r, k).
\]

(1.2)

**Proof.** Let \(r \geq -1\) and let \(k\) any integer. The proof will be complete once we produce at least one unramified double cover \(f: \tilde{X} \to X\) of genus \(g\) in the \(k\)-gonal locus of \(R_g\) whose Prym–Brill–Noether locus has dimension bounded by \(g - 1 - n(r, k)\).

Let \(\varphi: \tilde{\Gamma} \to \Gamma\) be a uniform \(k\)-gonal folded chain of loops, and let \(f: \tilde{X} \to X\) be a smoothing over a non-Archimedean field \(K\) [LU19, Lemma 7.0.1]. By Theorem A, the dimension of \(V_\nu(\Gamma, \varphi)\) equals \(g - 1 - n(r, k)\), and from Baker’s specialization inequality [Bak08, Corollary 2.11] we obtain

\[
\text{Trop}(V_\nu(X, f)) \subseteq V_\nu(\Gamma, \varphi).
\]

If \(g - 1 < n(r, k)\), then the tropical Prym–Brill–Noether locus \(V_\nu(\Gamma, \varphi)\) is empty and so the algebraic Prym–Brill–Noether locus \(V_\nu(X, f)\) is empty as well.

Otherwise, since both \(\Gamma\) and \(\tilde{\Gamma}\) are trivalent and without vertex-weights, both of their Jacobians and Prym varieties are maximally degenerate. Therefore we may apply Gubler’s Bieri–Groves Theorem for maximally degenerate abelian varieties [Gub07, Theorem 6.9] to conclude that

\[
\dim V_\nu(X, f) = \dim \text{Trop}(V_\nu(X, f)) \leq \dim V_\nu(\Gamma, \varphi) = g - 1 - n(r, k).
\]

Note that a general curve of genus \(g \leq 2k-2\) is \(k\)-gonal, so by [Wel85], the codimension of the Prym–Brill–Noether locus of a general curve is \(\binom{r+1}{2}\). However, we believe that in all other cases, the bound we found is tight.

**Conjecture 3.9.** Suppose that \(g > 2k - 2\), and let \(f: \tilde{C} \to C\) be a generic Prym curve. Then

\[
\dim V_\nu(C, f) = g - 1 - n(r, k).
\]

4. Tropological properties

As before, fix a folded chain of loops \(\varphi: \tilde{\Gamma} \to \Gamma\) of genus \(g\) and gonality \(k\). In this section, we prove that the Prym–Brill–Noether locus \(V_\nu(\Gamma, \varphi)\) is pure-dimensional and path-connected when the dimension is positive (in fact, we show that it is connected in codimension 1). To accomplish this, we develop the notions of strips and non-repeating tableaux, which will also be necessary for our genus computations in Section 5.
4.1. Strips and non-repeating tableaux. We focus our attention on Prym tableaux of minimal codimension. Since Proposition 3.3 implies that any such tableau is equivalent to a staircase Prym tableau, it suffices to consider this restricted type. To simplify our terminology, we shall say that a tableau is minimal if it is a staircase Prym tableau of minimal codimension.

In the generic case (which, by a slight abuse of terminology, we take to include both the case of generic edge lengths and the non-generic case where \( r \leq l \)), minimal tableaux are relatively easy to classify, since they are precisely the standard Young tableaux on \( T_r \). By contrast, the cases of even and odd torsion both elude such a concise description. However, the job is not impossible; as we will presently make precise, there are subsets of \( T_r \) that we call strips on which minimal tableaux are determined up to equivalence.

**Definition 4.1.** A subset \( \mu \subset T_r \) is a *strip* if \( T_l \subset \mu \) and there exists a box in \( \mu \cap A_n \) for each \( n \in \{ l, l + 1, \ldots, r \} \) called the \( n \)-th leftmost box that satisfies the following properties:

- \((1, l)\) is the \( l \)-th leftmost box,
- if \((x, y)\) is the \( n \)-th leftmost box, then the \((n + 1)\)-th leftmost box is \((x, y + 1)\) or \((x + 1, y)\), and
- if \((x, y)\) is the \( n \)-th leftmost box, then the boxes of \( \mu \cap A_n \) are precisely those of the form \((x + i, y - i)\) for each \( i \in \{0, 1, \ldots, l - 1\} \).

If \((x, y)\) is the \( n \)-th leftmost box, then we call \((x + l - 1, y - l + 1)\) the \( n \)-th rightmost box.

Note that \( \mu \cap A_n \) contains precisely \( \min \{ n, l \} \) boxes, any two of which are separated by cab distance at most \(2l - 2\). This implies that any \( k \)-uniform tableau on \( T_r \) must be injective on \( \mu \cap A_n \) for all \( n \) (though not necessarily on \( T_r \)). Moreover, since we designate \((1, l)\) as the \( l \)-th leftmost box and choose each subsequent leftmost box out of two possibilities, it follows that \( \mu \) may take on any of \( 2^{r-l} \) distinct shapes.

\( T_r \setminus \mu \) consists of two (possibly empty) contiguous components, which we shall call the left and right, respectively. In particular, the left component of \( T_r \setminus \mu \) (if it exists) is the one that contains \((1, r)\). We refer to the strip whose right component is empty as the horizontal strip and denote it by \( \mu_0 \).

We now introduce a subclass of maps \( T_r \to [g-1] \) that will play a key role for the rest of the paper. As we shall see, these maps are in fact minimal tableaux. The even and odd cases differ; in what follows, take \( \epsilon = k \pmod{2} \).

**Definition 4.2.** Given a strip \( \mu \) and a map \( t : T_r \to [g-1] \) such that \( t|_{\mu} \) satisfies the tableau and displacement conditions, we say that \( t \) is *non-repeating in \( \mu \)* if

(a) \( t(x, y) = t(x + l - \epsilon, y - l) \) for each \((x, y)\) in the left component of \( T_r \setminus \mu \),
(b) \( t(x, y) = t(x - l, y + l - \epsilon) \) for each \((x, y)\) in the right component of \( T_r \setminus \mu \), and
(c) for all \( n \in \{ l, l + 1, \ldots, r - 1 \} \), writing the \( n \)-th leftmost box as \((x, y)\), then \((x, y + 1)\) is the \((n+1)\)-th leftmost box if and only if \( t(x, y) > t(x + l - \epsilon, y - l + 1) \).

We refer to condition (c) as the *gluing condition.*

See Fig. 4.1 for an example. The usefulness of non-repeating tableaux is elucidated by the following result.
Proposition 4.3. Given a strip \( \mu \), if \( t \) is non-repeating in \( \mu \), then \( t \) is a minimal tableau.

Proof. We first show that \( t \) is staircase Prym. The displacement condition follows from conditions (a) and (b) in Definition 4.2, since every symbol in \( T_n \setminus \mu \) is copied from a box that is distance \( k \) away.

By definition, \( t|_\mu \) satisfies the tableau condition, so it remains to check that \( t|_{T_n \setminus \mu} \) does as well. Consider the case where \( k \) is odd. Suppose the tableau condition holds for the symbols in \( T_n \), and let \((x,y) \in A_{n+1}\) be a box in the left component; then \((x-1,y)\) is also in the left component. Their symbols are copied from \((x+l-1,y-l)\) and \((x+l-2,y-l)\), respectively. The former lies above the latter, and both are in \( T_n \), so \( t(x+l-1,y-l) > t(x+l-2,y-l) \). Then \( t(x,y) > t(x-1,y) \), as we had hoped. A similar conclusion follows if \((x,y-1)\) is in the left component. However, it is possible that \((x,y-1)\) lies in \( \mu \). If it does, then it must be the \( n \)-th leftmost box; then the \((n+1)\)-th leftmost box must be \((x+1,y-1)\), which by the gluing condition implies that \( t(x,y-1) < t(x+l-1,y-l) = t(x,y) \), as desired.

By transposing the first and second coordinates, we see that the argument above also works for boxes in the right component. Moreover, a very similar argument proves it in the case that \( k \) is even. Then in both cases, \( t \) satisfies the tableau condition everywhere by induction, so \( t \) is staircase Prym.

To see that \( t \) is minimal, observe that \( t \) has precisely \( l \) new symbols on each anti-diagonal, since every symbol not in \( \mu \) is repeated from within \( \mu \). Then it is not hard to see that \( \text{codim}(t) = n(r,k) \), as desired.

Lemma 4.4. Fix \( \mu \) a strip and \( D_i \) a diagonal modulo \( k \). Then for any map \( t \) non-repeating in \( \mu \) and any boxes \( \omega \in \mu \cap A_m \cap D_i \) and \( \omega' \in \mu \cap A_n \cap D_i \) with \( m < n \), we have that \( t(\omega) < t(\omega') \).

Proof. The statement is true for \( n \leq l \) by the tableau condition since \( D_i \cap T_l \) is contained in a single diagonal. We proceed by induction on \( n \). Suppose that the statement is true in \( T_n \), and let \( \omega' \) be a box in \( \mu \cap A_{n+1} \cap D_i \). Then it suffices to show that \( t(\omega) < t(\omega') \) for \( \omega \in \mu \cap A_m \cap D_i \) where \( m \leq n \) is the maximum index such that \( \mu \cap A_m \cap D_i \) is nonempty.

When \( k \) is odd, let \((x,y)\) be the \( n \)-th leftmost box, and assume without loss of generality (by transposing the coordinates if necessary) that \((x+1,y)\) is the \((n+1)\)-th leftmost box. If \( \omega' \) is the \((n+1)\)-th rightmost box \((x+l,y-1)\), then \( m = n \) and \( \omega = (x,y) \). The desired result then obtains by the gluing condition. If \( \omega' \) is any other cell \((x',y')\) in \( \mu \cap A_{n+1} \cap D_i \), it is not hard to see that \( \mu \cap A_m \cap D_i \) is empty and \( \mu \cap A_{n+1} \cap D_i \) contains precisely one box, namely, \((x'-1,y'-1)\). Then the tableau condition implies the desired result.

The case where \( k \) is even follows in a similar way; we omit the details here. □
Proposition 4.5. For \( k \) odd, given two tableaux \( t \) and \( s \) that are non-repeating in \( \mu \) and \( \nu \) respectively, \( t \) and \( s \) are equivalent if and only if \( \mu = \nu \) and \( t|\mu = s|\nu \). For \( k \) even, the statement above holds only if we enforce the condition that \( \mu = \nu \).

Proof of Proposition 4.5. Let \( k \) be odd. Suppose that \( t \) and \( s \) are equivalent, and assume for the time being that \( \mu = \nu \). Because \( t(D_i) = s(D_i) \) for each \( i \), the total ordering on the boxes of \( \mu \cap D_i \) given by Lemma 4.4 forces \( t|_\mu = s|_\mu \). Now suppose for the sake of contradiction that \( \mu \neq \nu \). Let \( n \) be the smallest index such that \( \mu \cap A_{n+1} \neq \nu \cap A_{n+1} \); note that \( n \geq l \). Applying the argument above to the restricted domain \( T_{n+l} \), we have that \( t|_{\mu \cap T_n} = s|_{\nu \cap T_n} \). Then the gluing condition on \( \mu \) and \( \nu \) forces the \((n+1)\)-th leftmost box of each to be the same, so \( \mu \cap A_{n+1} = \nu \cap A_{n+1} \), a contradiction. The converse trivially follows from Definition 4.2.

If \( k \) is even, we observe that every minimal tableau that is non-repeating on some strip is non-repeating on every strip simultaneously. The necessity of the additional condition follows.

Proof of Proposition 4.6. Given a staircase Prym tableau \( t \), there exist a strip \( \mu \) and a tableau \( s \) that is non-repeating in \( \mu \) such that \( s \) dominates \( t \). Moreover, in the even case, this strip may be chosen to be horizontal.

Proof. First suppose that \( k \) is odd. We begin by defining a tableau \( s_l = t|_{T_l} \) and a strip \( \mu_l = T_l \), and proceed by induction: suppose that we have defined a tableau \( s_n \) on \( T_n \) that is non-repeating on a strip \( \mu_n \subset T_n \), and suppose that \( s_n|_{\mu_n} = t|_{\mu_n} \). Let \( (x,y) \) be the \( n \)-th leftmost box; then \((x+l-1,y-l+1)\) is the \( n \)-th rightmost box. Recall that \( t(x,y) \neq t(x+l-1,y-l+1) \) because \( t|_{\mu \cap A_n} \) must be injective. Define the strip \( \mu_{n+1} \) so that \( \mu_{n+1}|_{T_n} = \mu_n \) and the \((n+1)\)-leftmost box is \((x,y+1)\) just if \( t(x,y) > t(x+l-1,y-l+1) \). Then take \( s_{n+1} \) to be the map so that \( s_{n+1}|_{\mu_{n+1}} = t|_{\mu_{n+1}} \) and \( s|_{A_{n+1} \setminus \mu} \) is defined according to conditions (a) and (b) of Definition 4.2. Clearly, \( s_{n+1} \) is non-repeating in \( \mu_{n+1} \) and dominates \( t|_{T_{n+1}} \), so take \( s = s_r \). Having defined \( s \) on all of \( T_r \), observe that \( s \) is non-repeating in \( \mu \), so \( s \) is staircase Prym by Proposition 4.3, and hence has minimal codimension. Moreover, \( s|_\mu = t|_\mu \) and \( s \) contains no other symbols besides those in \( \mu \), so \( s \) dominates \( t \).

If \( k \) is even, we may find a dominating non-repeating strip similarly to the odd case with minor adjustments. However, as we now show, we may choose this strip to be the horizontal strip \( \mu_0 \). In other words, we need to construct a minimal dominating tableau \( s \) such that \( s(x,l) < s(x+l,1) \) for every \( x \).

First, define \( s|_{T_l} = t|_{T_l} \). These symbols are all unique in \( T_l \), and \( s|_{T_l} \) obey the tableau condition. We now proceed to define \( s \) by induction on the anti-diagonals \( A_n \). Let \( n \geq l \), and suppose that \( s|_{T_n} \) is defined and obeys the tableau and displacement conditions. For each \( (x,y) \in A_{n+1} \cap D_i \), define \( s(x,y) \) to be the maximal value of \( t \) on \( A_{n+1} \cap D_i \). We claim that the tableau condition is still satisfied. In particular, given \( (x,y) \in A_{n+1} \cap D_i \), we claim that \( s(x,y) \) is greater than both \( s(x-1,y) \) and \( s(x,y-1) \) (whenever these values are defined). Indeed, \( s(x-1,y) = t(x',y') \) for \( (x',y') \in A_n \cap D_{i-1} \). Therefore, \( t(x',y') < t(x+1,y') \). But \( (x'+1,y') \in A_{n+1} \cap D_i \), so by construction \( s(x,y) = t(x+1,y') \). The other inequality holds by a similar argument.

Note that \( s \) dominates \( t \) by construction. Moreover, \( s \) has exactly \( l \) new symbols on every anti-diagonal \( A_n \) for \( n \geq l \), and is therefore minimal. Moreover, by construction along with the tableau condition we have \( s(x,l) < s(x,l+1) = s(x+l,1) \), so the gluing condition is satisfied as well.

The fact that the Prym–Brill–Noether locus is pure dimensional readily follows from the results of this section.

Proposition 4.7. \( V'(\Gamma, \varphi) \) is pure-dimensional for any gonality \( k \).

Proof. Given a Prym tableau \( t \), we want to find a Prym tableau \( s \) that dominates \( t \) and attains \( \text{codim}(s) = n(r,k) \). Apply the reflection algorithm of Proposition 3.3 to \( t \); the resulting tableau \( v \) dominates \( t \). It is sufficient to consider the staircase Prym tableau \( u = v|_{T_r} \). In the generic case, every symbol is \( u \) is necessarily unique, so we are done. Otherwise, apply Proposition 4.6 to obtain a minimal tableau \( s \) which dominates \( u \).

\( \square \)
4.2. Path-connectedness. In this section, we shall occupy ourselves with the following result.

**Proposition 4.8.** If \( \dim V^r(\Gamma, \varphi) \geq 1 \), then \( V^r(\Gamma, \varphi) \) is connected in codimension 1.

We may write \( V^r(\Gamma, \varphi) = \bigcup_{\alpha \in I} P(t_\alpha) \) for a finite indexing set \( I \) and some collection of staircase Prym tableau \( \{ t_\alpha \}_{\alpha \in I} \). Without loss of generality, we choose this collection to be minimal in the sense that \( t_\alpha \) dominates \( t_\beta \) only if \( \alpha = \beta \). By Proposition 4.7, we know that each \( t_\alpha \) is minimal.

Each subspace \( P(t_\alpha) \) is a torus. To prove that \( V^r(\Gamma, \varphi) \) is connected in codimension 1, it suffices to show that for any \( \beta \) and \( \gamma \) in \( I \), there is a sequence \( (\alpha_p)_{p=0}^k \) in \( I \) with \( \alpha_0 = \beta \) and \( \alpha_k = \gamma \), such that \( P(t_{\alpha_p}) \) and \( P(t_{\alpha_{p+1}}) \) intersect at a torus of codimension 1 for each \( p \). Note that the last condition is equivalent to the property that for any indices \( i \neq j \in \{0, 1, \ldots, k-1\} \), the sets of symbols \( t_{\alpha_i}(D_i) \) and \( t_{\alpha_{j+1}}(D_j) \) have empty intersection; indeed, a symbol \( h \) appearing in both sets would impose contradicting conditions on the placement of chips on the \( h \)-th loop. Call any two such staircase Prym tableaux adjacent. An observation that will be useful in proving Proposition 4.8 in the odd case is that two staircase Prym tableaux are adjacent if there exists a third that is dominated by each of them.

We now define several terms that will be useful for reliably generating paths of adjacent tableaux. Let \( t \) be a minimal tableau. Since \( \dim V^r(\Gamma, \varphi) \geq 1 \), there is some symbol \( a \) not appearing in \( t \). Choose some box \( (x, y) \), and define \( s(x, y) = a \) and \( s(\omega) = t(\omega) \) for all \( \omega \neq (x, y) \). We call this procedure swapping \( a \) into \( (x, y) \). In general, \( s \) will not satisfy the tableau condition; we need to check that \( t(x-1, y) < a \), \( t(x, y-1) < a \), \( t(x+1, y) > a \), and \( t(x, y+1) \). (We may also check that \( t(x, y) > a \) or \( t(x, y) < a \) to verify the last two or last two inequalities, respectively.) Given that the tableau condition is satisfied, it is not hard to see that \( s \) is a staircase Prym tableau that is adjacent to \( t \). However, \( s \) is minimal if and only if the symbol \( t(x, y) \) does not appear anywhere else in \( t \).

Thus, we introduce the related notion of swapping \( a \) in for \( b \). Given \( t \) and \( a \) as above, pick a symbol \( b \). If \( b \) does not appear in the tableau, define \( s = t \) (i.e., do nothing). Otherwise, for each box \( \omega \in t^{-1}(b) \), define \( s(\omega) = a \), and for each \( \omega' \in t^{-1}(b) \), define \( s(\omega') = t(\omega') \). The tableau condition must again be checked, this time at each box \( \omega \). Supposing that it holds, \( s \) satisfies the displacement condition because \( t \) does, \( t \) and \( s \) are adjacent, and \( s \) is minimal.

It is straightforward to check that we may always swap \( a \) in for \( a+1 \) and \( a \) in for \( a-1 \). Hence, if \( t \) and \( a \) are as above, and there is a symbol \( b > a \) that we want to pull out of the tableau, we iterate the following procedure: at the \( i \)-th step, \( a+i \) is not in the tableau, swap \( a+i \) in for \( a+i+1 \). After \( b-a \) steps, each symbol \( a+i \) in \( t \) for \( i \in [b-a] \) has been decremented by 1. In particular, \( b \) no longer appears in the tableau. An analogous procedure may be used in the case that \( b < a \); in either case, we call this cycling out \( b \).

**Remark 4.9.** Both swapping and cycling use at most a single additional symbol than is already in the tableau. As a consequence, any path of tableaux produced via these operations is connected in codimension 1.

Given any subset \( \lambda \) of \( \mathbb{N}^2 \), we establish a total order on its boxes as follows: given \( (x, y) \in \lambda \cap A_m \) and \( (x', y') \in \lambda \cap A_n \), say that \( (x, y) < (x', y') \) if \( m < n \), or if both \( m = n \) and \( x < x' \). Let \( Q_\lambda(\omega) \) be the number of boxes \( \omega' \in \lambda \) for which \( \omega' \leq \omega \). Then we define an \( \mathbb{N} \)-valued function \( V_\lambda \) on the set of tableaux that are injective on \( \lambda \) by

\[
V_\lambda(t) := |\lambda| - \max \{ Q_\lambda(\omega) \mid t(\omega') = Q_\lambda(\omega') \text{ for all } \omega' \leq \omega \}.
\]

We denote by \( \bar{t} \) the unique tableau for which \( V_\lambda(\bar{t}) = 0 \); call it the standard increasing tableau. For example, if \( \lambda = T_4 \), then \( \bar{t} \) is the final tableau in Fig. 4.2. Intuitively, \( V_\lambda \) measures how far a given tableau is from being identical to \( \bar{t} \).

We are now equipped to prove Proposition 4.8 in the case of generic edge length.

**Proof in the generic case.** We will show by induction on \( V_{T_r} \) that any injective tableau \( t \) defined on \( T_r \) has a path to the standard increasing tableau \( \bar{t} \). If \( V_{T_r}(t) = 0 \), then the statement is trivially true since it
must be the case that \( t = \tilde{t} \). Otherwise, suppose that any tableau \( s \) with \( V_T(s) < V_T(t) \) is connected by a path to \( \tilde{t} \). Then it suffices to show that \( t \) has a path to some such \( s \).

Let \((x, y)\) be the smallest box such that \( t(x, y) = Q_T(x, y) \). Denote \( a = Q_T(x, y) \). Let \( S \) be the set of boxes that are strictly smaller than \((x, y)\); then each of these boxes \( \omega \) contains the corresponding symbol \( Q_T(\omega) \), which is less than \( a \). We aim to produce a tableau that has a path to \( t \), agrees with \( t \) (and hence with \( \tilde{t} \)) on \( S \), and has \( a \) in the box \((x, y)\). Indeed, cycle out \( a \) and call the resulting tableau \( u \); since \( t(x, y) \) and \( a \) are both greater than \( a - 1 \), no box of \( S \) is affected. Then swap \( a \) into \((x, y)\) and call the resulting tableau \( s \). Again, this does not alter symbols in \( S \). Moreover, \( s \) satisfies the tableau condition since \( a < u(x, y) \) and \((x - 1, y) \) and \((x, y - 1) \) both are in \( S \) and hence contain symbols that are smaller than \( a \). Finally, \( s \) has \( a \) in the correct box, so \( V_T(s) \leq V_T(t) - 1 \), completing the proof. \(\square\)

**Example 4.10.** In Fig. 4.2, we exhibit a sequence of tableaux beginning at the given injective tableau on \( T_4 \) and terminating at the standard increasing tableau. The shaded boxes at each step represent the set \( S \), i.e., the set of boxes whose symbols are in the correct position. Note that \( V_T(\cdot) \) maps the first tableau to 8. We have that \( g \geq 12 \) since \( \dim V^4(\Gamma, \varphi) \geq 1 \). Hence, 11 is a free symbol (and in general, the only free symbol). The first step cycles out 3; notice that each symbol \( a \geq 3 \) is replaced with \( a + 1 \) in the process. The second step swaps 3 into \((2, 1)\), thereby removing 6 from the tableau. We continue cycling and swapping as appropriate until every symbol is in the correct position according to the order.

![Figure 4.2](image-url)

**Figure 4.2.** Example application of the algorithm from the proof of Proposition 4.8 in the generic case.

To prove the even and odd cases, we make use of the theory of non-repeating tableaux developed in Section 4.1. We shall define our ordering on the horizontal strip \( \mu \); the fact that the proof of Proposition 4.8 in the generic case was more or less agnostic as to the particular shape of the tableau will allow us to skip many of the details in the subsequent proofs.

**Proof of Proposition 4.8 in the even case.** Let \( \tilde{t} \) be the unique tableau non-repeating in the horizontal strip \( \mu_0 \) that extends the standard increasing tableau on \( \mu_0 \). Given any \( t \) non-repeating in \( \mu_0 \), we can repeat the procedure from the proof in the generic case that inducts on \( V_{\mu_0}(t) \): as before, we cycle out \( a \) to produce \( u \), but instead of swapping \( a \) just into \((x, y)\), we swap it in for the symbol \( u(x, y) \). In particular, this operation swaps \( a \) into the boxes \( \omega_i = (x - il, y + il) \) for each \( i \in \mathbb{Z}_{\geq 0} \) such that \( \omega_i \in T_r \). Again, call the resulting tableau \( s \). It suffices to check that \( s \) satisfies the tableau condition at each \( \omega_i \); as long as this holds, the fact that \( V_{\mu_0}(s) \leq V_{\mu_0}(t) - 1 \) finishes the proof.

Suppose first that \( 2 \leq y \leq l - 1 \). Then the same argument that we used in the proof of the generic case demonstrates that \( s \) satisfies the tableau condition at \( \omega_0 \), and the fact that the boxes distance 1 from \( \omega_i \) are all copied from the respective boxes that are distance 1 from \( \omega_0 \) implies that the tableau condition is satisfied everywhere. If \( y = l \), the same argument works once we note that \( u(x, y + 1) = u(x + l, 1) \)
and \((x + l, 1)\) is not in \(S\). Finally, in the case that \(y = 1\), the tableau condition holds at \(\omega_0\), but since there is no box \((x, y - 1)\), we may worry that \(a < s(x - l, l)\), thereby violating the tableau condition at \(\omega_1 = (x - l, l + 1)\). These worries are not warranted: \((x - l, l)\) is in \(S\) and hence contains a symbol less than \(a\). Then the tableau condition is satisfied at \(\omega_i\) for \(i > 1\) for much the same reason as before. \(\square\)

The odd case is more difficult than the even case because we cannot only consider the horizontal strip: by Proposition 4.6, each of the \(2^{r-1}\) strips determine a unique set of maximal cells of \(V^r(\Gamma, \varphi)\). We must therefore show not only that there is a path between any two tableau non-repeating in the horizontal strip, but also that there is a path between any tableau non-repeating in some strip and a tableau non-repeating in the horizontal strip. It is the latter that we will primarily be occupied with proving.

We construct a height function as follows: given a tableau \(t\) of odd torsion which is non-repeating in \(\mu\), define \(H(t)\) to be the second coordinate of the \(r\)-th leftmost box of \(\mu\). Note that \(H\) is well-defined by Proposition 4.6. Moreover, \(H(t) = l\) if and only if \(\mu\) is the horizontal strip. To simplify the notation in the following proof, we introduce the unit vectors \(\hat{x}\) and \(\hat{y}\) to describe boxes relative to other boxes. For example, if \(\omega = (x, y)\), then \(\omega + \hat{x} = (x + 1, y)\) and \(\omega - \hat{y} = (x, y - 1)\).

**Proof of Proposition 4.8 in the odd case.** Let \(\tilde{t}\) be the odd tableau non-repeating in the horizontal strip \(\mu_0\) such that \(\tilde{t}\langle_{\mu_0}\) is the standard increasing tableau on \(\mu_0\). Suppose that \(t\) is another tableau non-repeating in \(\mu_0\); then an argument analogous to the one given in the even case yields a path between \(t\) and \(\tilde{t}\). Therefore, the statement holds for any tableau \(t\) for which \(H(t) = l\).

Now take \(\tilde{t}\) as before, but let \(\mu\) be any strip and \(t\) any tableau non-repeating in \(\mu\). To prove that there is a path from \(t\) to \(\tilde{t}\), we induct on \(H(t)\). We just argued that the base case, \(H(t) = l\), holds. Suppose then that every \(s\) for which \(H(s) < H(t)\) has a path to \(\tilde{t}\). Denote by \((x, y)\) the unique box in \(\mu\) for which \(y = H(t) \) and \((x - 1, y) \notin \mu\). Denote its anti-diagonal by \(A_{q, r}\) and let \(n = r - q\). Then define \(\psi_i := (x + i, y)\) for each \(i \in \{0, 1, \ldots, n\}\). Since \(H(t) = y\), \(\psi_i\) is the \((q + i)\)-th leftmost box for all \(i\), and in particular, \(\psi_n\) is the \(r\)-th leftmost box.

Our goal is to connect \(t\) by a path to a tableau \(s\) non-repeating in \(\nu\), where \(\nu\) is the strip that agrees with \(\mu\) up to \(A_{q-1}\) but has every subsequent leftmost box one step in the \(\hat{x}\) direction. (In particular, the \(q\)-th leftmost box of \(\nu\) is \((x + 1, y - 1)\), not \((x, y)\).) Since \(H(s) = H(t) - 1\), if we can construct such an \(s\), then we are done.

Observe that \(\psi_i\) is in the left component of \(T_r \setminus \nu\). Hence, by Definition 4.2, we need \(s(\psi_i) = s(\omega_i, 0)\), where \(\omega_i, 0 := (x + l - 1 + i, y - l)\). Note that \(\omega_0, 0\) is in \(\mu\), while for each \(i \geq 1\), \(\omega_i, 0\) is in the right component of \(T_r \setminus \mu\). Then for each \(i \geq 1\), we have that \(t(\omega_i, 0) = t(\psi_i - \hat{x} - \hat{y})\). More generally, for each \(j \geq 0\) we define \(\omega_{i,j} := (x + l - 1 + i + j, y - l - j(l - 1))\); then for all \(i\) and \(j \geq 1\), we have \(t(\omega_{i,j}) = t(\omega_{i,0})\) and \(s(\omega_{i,j}) = s(\omega_{i,0})\); this again follows by Definition 4.2, since \(\omega_{i,j}\) is in the right component of both \(T_r \setminus \mu\) and \(T_r \setminus \nu\) for \(j \geq 1\). See Fig. 4.3 for a schematic diagram of our notations.

To go from \(t\) to \(s\), we could try to replace the symbol in \(\omega_{i,j}\) with the symbol in \(\psi_i\) for each \(i\) and \(j\), and leave all other symbols unchanged. This operation does not change the diagonal modulo \(k\) in which any symbol lives; this, combined with the fact that \(t\) is minimal, would imply that \(t\) and \(s\) are equivalent. This should be an immediate cause for worry, since, by Proposition 4.5, two tableaux that are non-repeating on different strips cannot be equivalent. The tableau condition must fail somewhere. In fact, it fails precisely at \(\omega_{n,0}\), which lies on \(A_{r-1}\). Indeed, we have \(t(\psi_n) > t(\psi_n - \hat{y}) = t(\omega_{n,0} + \hat{x})\); it is also possible (though not necessary) that \(t(\psi_n) > t(\omega_{n,0} + \hat{y})\). This cause a failure of the tableau condition when we attempt to copy the symbol \(t(\psi_n)\) into \(\omega_{n,0}\). We shall modify \(t\) so that these issues are avoided; in particular, we will put the two largest symbols, \(g - 2\) and \(g - 1\), into \(\omega_{n,0} + \hat{y}\) and \(\omega_{n,0} + \hat{x}\), respectively. Afterwards, we shall verify that the tableau condition does not fail anywhere else.

Cycle out \(g - 2\) and swap it into \(\omega_{n,0} + \hat{y}\). Then cycle out \(g - 1\) and call the resulting tableau \(v\). We need to verify that the swap preserves the tableau condition. (This is sufficient because cycling always
preserves the tableau condition.) Indeed, note that \( \omega_{n,0} + \hat{y} \) is the \( r \)-th rightmost box of \( \mu \); neither \( \omega_{n,0} \) nor \( \omega_{n,0} - \hat{x} + \hat{y} \) can contain the symbol \( g - 1 \) after the first cycling operation (since neither box is on \( A_r \)), and every other symbol is smaller than \( g - 2 \), so the tableau condition is satisfied. The same is true in \( v \). Moreover, \( v \) is non-repeating in \( \mu \) and has a path to \( t \).

Now we swap \( g - 1 \) into \( \omega_{n,0} + \hat{x} \) to produce a tableau \( u \). Much as before, the tableau condition is satisfied. However, \( \omega_{n,0} + \hat{x} \) is in the right component of \( T_r \), \( \mu \) symbol, and in particular, \( v(\omega_{n,0} + \hat{x}) = v(\psi_n - \hat{y}) \); hence, we have added a symbol (namely, \( g - 1 \)) to the tableau without removing every instance of another, so the codimension of \( P(u) \) relative to \( V'(\Gamma, \varphi) \) is 1. The consequence is that we cannot perform any more swaps or cycles: in general, \( \dim V'(\Gamma, \varphi) \) is no more than 1, so there are not necessarily any symbols in \([g - 1]\) that do not appear in \( u \) and with which we may perform those operations. Our only recourse is to show that \( u \), which has a path to \( t \) and is dominated by \( v \), is also dominated by some \( s \) non-repeating on \( \nu \).

To do this, we construct \( s \) from \( u \) in the same way that we attempted to construct \( s \) from \( t \). Precisely, for each \( i \) and \( j \), we define \( s(\omega_{i,j}) = u(\psi_j) \) and let \( s \) coincide with \( u \) everywhere else. Then it is not difficult to see that \( s \) satisfies the displacement condition and (strictly) dominates \( u \). Moreover, \( s \) is non-repeating in \( \nu \) provided that the tableau condition holds everywhere.

We need to check that the tableau condition is preserved at each \( \omega_{i,j} \). In fact, it suffices to check the case where \( j = 0 \) because the symbols in the boxes distance 1 from \( \omega_{i,j} \) are copied from the respective boxes distance 1 from \( \omega_{i,0} \). To that end, consider first \( \omega_{i,0} \) for \( 1 \leq i \leq n - 1 \). Then satisfaction of the tableau condition at \( \omega_{i,0} \) follows from the following computations:

\[
\begin{align*}
    s(\omega_{i,0}) &= u(\psi_i) > u(\psi_{i-1}) = s(\omega_{i-1,0}) \quad (4.1) \\
    s(\omega_{i,0}) &= u(\psi_i) > u(\psi_i - \hat{x} - 2\hat{y}) = u(\omega_{i,0} - \hat{y}) = s(\omega_{i,0} - \hat{y}) \quad (4.2) \\
    s(\omega_{i,0}) &= u(\psi_i) < u(\psi_{i+1}) = s(\omega_{i+1,0}) \quad (4.3) \\
    s(\omega_{i,0}) &= u(\psi_i) < u(\psi_i + \hat{y}) = u(\omega_{i,0} + \hat{y}) = s(\omega_{i,0} + \hat{y}) \quad (4.4)
\end{align*}
\]

Eqs. (4.3) and (4.4) apply also to the case where \( i = 0 \), while the other two are replaced by

\[
\begin{align*}
    s(\omega_{0,0}) &= u(\psi_0) > u(\psi_0 - \hat{y}) = u(\omega_{1,0}) > u(\omega_{0,0} - \hat{x}) = s(\omega_{0,0} - \hat{x}) \quad (4.5) \\
    s(\omega_{0,0}) &= u(\psi_0) > u(\psi_0 - \hat{y}) = u(\omega_{1,0}) > u(\omega_{0,0} - \hat{y}) = s(\omega_{0,0} - \hat{y}) \quad (4.6)
\end{align*}
\]

If \( i = n \), Eqs. (4.1) and (4.2) apply, and we have

\[
\begin{align*}
    s(\omega_{n,0}) &< g - 1 = u(\omega_{n,0} + \hat{x}) = s(\omega_{n,0} + \hat{x}) \quad (4.7) \\
    s(\omega_{n,0}) &< g - 2 = u(\omega_{n,0} + \hat{y}) = s(\omega_{n,0} + \hat{y}) \quad (4.8)
\end{align*}
\]
It is straightforward that the tableau condition is satisfied at $\omega_{0,0}$ in the special case that $n = 0$. \hfill \Box

**Example 4.11.** Consider the tableau in Fig. 4.4, where $g = 22$, $r = 8$, and $k = 5$. We shade the strip $\mu$ in blue for each tableau. In our example, we note that $H(\mu) = 5$, so $(x, y) = (3, 5)$ as $(3, 5) \in \mu$, but $(2, 5) \notin \mu$; we note that $t(x, y) = 17$. Furthermore, we have that $n = 1$ as $(x + n, y) = (4, 5)$ is the $r$-th leftmost box; this is the box containing 19. Thus, $(x + n + l - 1, y - l + 1) = (6, 3)$ is the $r$-th rightmost box; this is the box containing 21. The first step of the algorithm is to cycle out $g - 2 = 21$, and then swap it into $(x', y')$. For the first tableau in Fig. 4.4, these two operations do not change the tableau. The next step is to cycle out $g - 1 = 22$ and swap it into $(x' + 1, y' - 1)$. We note that 22 does not appear in the first tableau, so we do not need to cycle it out. Thus, this operation is captured going from the first tableau to the second by swapping 22 into $(x' + 1, y' - 1)$ this is the tableau called $u$ in the proof. Since this tableau is not minimal, we do not highlight a strip. Now the boxes $\omega_{i,j}$ are $\omega_{0,0} = (5, 2)$ and $\omega_{1,0} = (6, 2)$; these are all the values where $\omega_{i,j}$ are defined. The last step, we do the replacement of the boxes $\omega_{i,j}$ we just copy these from the corresponding symbols on the top row. We again shade the strip $\mu$, and we remark that the height has decreased. Another iteration of this process would give a tableau non-repeating in the horizontal strip.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4_4.png}
\caption{Figure 4.4}
\end{figure}

5. Discrete properties

Now that we have established a few general facts about Prym–Brill–Noether loci, we can begin to look at some of their enumerative properties. We start with counting the number of divisors in 0-dimensional loci before looking at 1-dimensional loci.

5.1. **Cardinality of finite Prym–Brill–Noether loci.** In this section, we fix $g - 1 = n(r, k)$. The Prym–Brill–Noether locus is then finite, and its points correspond to staircase Prym tableaux where every
symbol in \([g - 1]\) is used. Denote by \(C(r, k)\) the number of divisor classes in \(V'^r(\Gamma, \varphi)\). This number has been computed \([LU19, \text{Corollary 6.1.5}]\) for generic edge length or \(k > 2r - 2\) using the hook-length formula.

For even gonality \(k \leq 2r - 2\), we now use the observations from Section 4 to obtain a bijection between tableaux and certain lattice paths, giving rise to the following formula.

**Proposition 5.1.** For even \(k \leq 2r - 2\), the number of divisors in a 0-dimensional locus is

\[
C(r, k) = n! \sum \frac{1}{(r + \alpha_1 k)!} \frac{1}{(r - 2 + \alpha_2 k)!} \cdots \frac{1}{(r - k + 2 + \alpha_k k)!} \frac{1}{(r + 1 + \alpha_1 k)!} \frac{1}{(r - 1 + \alpha_2 k)!} \cdots \frac{1}{(r - l + 1 + \alpha_l k)!} \frac{1}{(r + l - 3 + \alpha_2 k)!} \cdots \frac{1}{(r - l + 1 + \alpha_l k)!}
\]

where \(n = n(r, k) = g - 1\) is the codimension and the sum is taken over all \(l\)-tuples \((\alpha_i)_{i=1}^l\) for which \(\alpha_i \in \mathbb{Z}\) and \(\sum_{i=1}^l \alpha_i = 0\).

**Proof.** The set of divisors that we want to enumerate is in bijection with the set of tableaux that are non-repeating in the horizontal strip \(\mu_0\). We describe the bijection in the following way: given \(\mu\), we create a lattice path in \(\mathbb{Z}^l\) that starts at the point \((l, l - 1, \ldots, 1)\), where each step is a standard unit vector. If the symbol \(a\) appears in box \((x, y)\) of \(\mu\), then the \(a\)-th step of the lattice path is a unit vector in the \(y\)-th coordinate.

By the tableau condition, the \(a\)-th step of the path is the \(x\)-th step in the \(y\)-coordinate. Moreover, since the symbols below \((x, y)\) in the same column are all smaller than \(a\), the number of symbols taken in indices smaller than \(y\) are at least \(x\). Therefore, considering that the indices of the starting point of the lattice path already satisfied \(z_1 > z_2 > \cdots > z_l\), this inequality remains true throughout the entire path.

We also must consider the gluing condition, which says \(t(x + l, 1) > t(x, l)\); on the lattice path, this means that the \(x + l\)-th step in the first index must come after the \(x\)-th step in the \(l\)-th index. At the starting point, the first index is already \(l - 1\) greater than the \(l\)-th index, and the gluing condition allows this gap to grow to at most \(k - 1\), giving us the final inequality \(z_1 > z_1 - k\). Counting the number of boxes per row to find the end point, this implies a lattice path from \((l, l - 1, \ldots, 1)\) to \((r + l, r + l - 2, \ldots, r - l + 2)\) that lies within the hyperplanes given by \(z_1 > z_2 > \cdots > z_l > z_1 - k\).

Conversely, given a lattice path from \((l, l - 1, \ldots, 1)\) to \((r + l, r + l - 2, \ldots, r - l + 2)\) within the regions \(z_1 > z_2 > \cdots > z_l > z_1 - k\), we may reverse the construction to get a non-repeating strip \(\mu\). If the \(a\)-th step in the lattice path is the \(x\)-th step in the \(y\)-th index, then the symbol \(a\) goes into box \((x, y)\); the first \(l - 1\) inequalities on the indices verify the tableau condition, and the last inequality verifies the gluing condition. From [Bón15, Theorem 10.18.6], Eq. (5.1) is exactly the number of lattice paths that lie within that region, which coincides with the number of divisors in the locus. \[\square\]

See Fig. 5.1 for various values of \(C(r, k)\).

| \(r\) | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 2 | 4 | 8 | 16 | 32 |
| 6 | 1 | 2 | 16 | 128 | 1024 | 8178 |
| 8 | 1 | 2 | 16 | 768 | 35840 | 1671168 |
| * | 1 | 2 | 16 | 768 | 292864 | 1100742656 |

**Figure 5.1.** \(C(r, k)\) for several values of \(r\) and \(k\). The * indicates the generic case.
Example 5.2. For low values of $k$, we may exhibit all the horizontal strips (and therefore all the divisor classes) directly. For instance, when $k = 2$, we have $C(r, k) = 1$ for every $r$. Indeed, the Prym tableau with minimal codimension are uniquely determined by the bottom row, and the only way to fill out the tableau is to use the symbols 1 through $g - 1$ in increasing order.

When $k = 4$, we use induction to show that $C(r, k) = 2^{r - 1}$. By Theorem A, the assumption that the locus is finite implies that $g = 2r$, so the tableau contains all the symbols in $[g - 1]$. When $r = 1$, there is a unique way of filling the tableau. Now, assume that the formula holds for $r$ at most $m$, and let $r = m + 1$. The tableau is uniquely determined by the horizontal strip, which consists of $r$ boxes in the bottom row, and $r - 1$ boxes in the row above it.

We note that it is not possible for both $2r - 1$ and $2r - 2$ to appear in the second row: the largest possible symbol that could appear at the end of the first row is $2r - 3$, violating the gluing condition. Thus, $2r - 1$ and $2r - 2$ appear in the boxes $(r, 1)$ and $(r - 1, 2)$. Once those symbols are placed (in any order), the remaining boxes produce a tableau of size $r - 1$. The inductive hypothesis implies that there are $2^{r - 2}$ such tableaux, so we are done.

When the Prym–Brill–Noether locus has positive dimension, its top dimensional cells are still uniquely determined by fillings of a horizontal strip with $n(r, k)$ symbols, leading to the following formula.

Proposition 5.3. Let $k \leq 2r - 2$ be an even integer. The number of top-dimensional components of $V^r(\Gamma, \varphi)$ equals $\binom{g - 1}{n(r, k)} \cdot C(r, k)$.

5.2. Genus of 1-dimensional loci. We now turn our attention to loci of dimension 1, by choosing $r$ and $k$ so that $g - 1 = n(r, k) + 1$. The Prym–Brill–Noether locus is then a graph embedded in the Prym variety. Since each tableau corresponds to a circle, $V^r(\Gamma, \varphi)$ is a 4-regular graph. This section is devoted to calculating its genus in the generic and $k = 2, 4$ cases, as well as other combinatorial properties. We begin with a simple observation.

Lemma 5.4. The genus of $V^r(\Gamma, \varphi)$ equals the number of vertices plus 1.

Proof. Since the graph is 4-valent, the number of edges $e$ equals twice the number of vertices $v$. The genus is therefore $e - v + 1 = 2v - v + 1 = v + 1$. \hfill \Box

Each circle in the graph corresponds to a staircase tableau with exactly one missing symbol $m_i$, which we refer to as the free symbol. The reason for this terminology is that in the corresponding divisors, the position of the chip on the $m_i$th and $2g - m$ loop is not determined. Accordingly, we refer to these loops as free as well. Two tableaux $t$ and $t'$ with missing symbols $m$ and $m'$ respectively give rise to intersecting circles precisely when we can swap $m$ with a symbol appearing in $t$ to create $t'$.

Proof of Theorem D. For any skew shape $\lambda$, denote $f^\lambda$ the number of ways of filling $\lambda$ with distinct symbols. When $\lambda$ is a staircase tableau of length $r$, $f^\lambda$ is just $C(r, 0)$. Since we assume that the edge lengths are generic, the number of symbols required for a length $r$ staircase tableau is $\binom{r + 1}{2}$. Since the Prym–Brill–Noether locus is 1-dimensional, the total number of symbols is $\binom{r + 1}{2} + 1$. Every choice of symbols gives $C(r, 0)$ different tableaux, so $V^r(\Gamma, \varphi)$ consists of $C(r, 0) \cdot \left(\binom{r + 1}{2} + 1\right)$ circles. The claim will be proven once we show that the average number of vertices for each circle is $r$ (keeping in mind that every vertex is double counted this way).

From [CLMPTiB18, Theorem 2.9], it follows that the average number of vertices per circle is

$$E = 2 \left( r + \sum_{i=1}^{r} \frac{r - i}{n + 1} \cdot \frac{f^\lambda}{f^\lambda} - \sum_{i=1}^{r + 1} \frac{r + 1 - i}{n + 1} \cdot \frac{f^\lambda}{f^\lambda} \right),$$
where the terms \( i^\lambda \) and \( \lambda^i \) describe the tableaux obtained by adding a box to the left or the right respectively in the \( i \)-th row. We claim that in our case, this entire expression equals \( r \).

First, let us consider the term

\[
\sum_{i=1}^{r} \frac{r-i}{n+1} \cdot \frac{f^{i\lambda}}{f^\lambda}.
\]

Observe that for \( i \neq r \), the resulting shape of \( i^\lambda \) is not a skew tableau, so \( f^{i\lambda} = 0 \), the number of fillings of this shape is 0. When \( i = r \), while it is a skew tableau, we have \( r-i = 0 \) in the numerator. Thus, this term in the expression is 0, and does not need to be considered.

Next, we look at

\[
\sum_{i=1}^{r} \frac{r+1-i}{n+1} \cdot \frac{f^{\lambda_i}}{f^\lambda} = \sum_{i=1}^{r} (r+1-i) \cdot \frac{f^{\lambda_i}}{(n+1)f^\lambda}.
\]

In this sum we need to enumerate the tableaux obtained by adding a box to the end of each row of the staircase tableau. Each of these number can be computed using the hook length formula. We note that in the staircase tableau, the boxes on the largest anti-diagonal have hook length 1, the boxes on the next largest have hook length 3, and so on until the hook length of the bottom left square is \( 2r-1 \).

\[
\begin{array}{ccc}
1 & 1 & 1 \\
3 &  &  \\
5 & 3 & 1 \\
7 & 5 & 3 & 1 \\
9 & 7 & 5 & 3 & 1 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & 1 \\
3 &  &  \\
6 & 4 & 2 & 1 \\
7 & 5 & 3 & 2 \\
9 & 7 & 5 & 4 & 1 \\
\end{array}
\]

\textbf{Figure 5.2.} Hook lengths of each box, before and after adding the box for \( r = 5, i = 3 \).

When a box is added, the hook length of every box in its row and column increases by 1, while all other hook lengths remain the same. Thus, the fraction \( \frac{f^{\lambda_i}}{(n+1)f^\lambda} \) simplifies down to the ratio of the differing hook lengths:

\[
\frac{f^{\lambda_i}}{(n+1)f^\lambda} = \frac{(n+1)! \prod h_{\lambda}(i,j)}{(n+1)n! \prod h_{\lambda'}(i,j)} = \frac{(2(r-i)+1)!!(2i-3)!!}{(2(r-i+1))!!(2i-2)!!},
\]

where \((-1)!!\) is defined as 1. We observe that

\[
\frac{(2i-3)!!}{(2i-2)!!} = \frac{2i-3}{2i-2} = \frac{(2i-2)!}{2^{i-1}(i-1)!},
\]

\[
\frac{(2r-i+1)!!}{(2r-i+1)!!} = \frac{2^{r-i+1}}{2^{r-i+1}}.
\]

A similar calculation gives us

\[
\frac{(2i-2)!!}{2^{r-i+1}} = \frac{2^{r-i+1}}{2^{r-i+1}}.
\]

Thus, we have

\[
\frac{f^{\lambda_i}}{(n+1)f^\lambda} = \frac{(2i-2)!!}{2^{r-i+1}} \cdot \frac{(2r-i+1)!!}{2^{r-i+1}}.
\]

We can reindex the sum by setting \( j = r-i+1 \), thus becoming

\[
\sum_{i=1}^{r} (r-i+1) \cdot \frac{(2i-2)!!}{2^{r-i+1}} \cdot \frac{(2r-i+1)!!}{2^{r-i+1}} = \sum_{j=1}^{r} j \cdot \frac{(2j)!!}{2^{2r+j}} \cdot \frac{(2r-j)!!}{2^{2r+j}}.
\]
We observe for \( j \) and \( r - j \), we have
\[
\frac{2j}{j} \cdot \frac{\binom{2r}{j}}{(r-j)} + (r-j) \cdot \frac{\binom{2(r-j)}{r-j}}{2^{2r}} = r \cdot \frac{\binom{2(r-j)}{r-j}}{2^{2r}}.
\]
Thus, grouping \( j \) and \( r - j \) together, this becomes
\[
\sum_{j=1}^{r} \frac{2j}{j} \cdot \frac{\binom{2r}{r-j}}{2^{2r}} = \frac{r}{2} \sum_{j=1}^{r} \frac{\binom{2(r-j)}{r-j}}{2^{2r}}.
\]
Finally, by [Sve84], the sum is equal to 1, and the term is equal to \( \frac{r}{2} \). Plugging this value back in the formula for \( E \), we conclude that the average number of vertices at each circle is \( r \).

We finish by computing the genus of the Prym–Brill–Noether curve for low even gonality.

**Proposition 5.5.** Suppose that \( k = 2 \) and that the Prym–Brill–Noether locus is 1-dimensional. Then it contains \( r + 1 = g - 1 \) circles, and has genus \( r + 1 \).

**Proof.** In this case, the tableau contains \( g - 2 \) symbols and is determined by the bottom \( 1 \times r \) rectangle; the positions of the symbols in the strip are determined after choosing which symbol to leave out. When 1 or \( g - 1 \) is the free symbol, it may only swap into the first or last cell in the strip, respectively, so the corresponding circle only has a single vertex. If any other symbol \( m \) is left out, it can swap with either the symbol \( m - 1 \) or \( m + 1 \), so the corresponding circle has two vertices. Thus, the locus is a chain of \( r + 1 = g - 1 \) circles wedged together, which has a genus of \( r + 1 \).

In the \( k = 4 \) case, we compute the genus, and find the number vertices each circle has.

**Proposition 5.6.** Suppose that \( k = 4 \) and that the Prym–Brill–Noether locus is 1-dimensional. Then it has the following structure.

(i) The circles corresponding to the free symbol 1 have a single vertex.
(ii) The circles corresponding to any odd free symbol have two vertices.
(iii) The circles corresponding to the free symbol 2 have three vertices.
(iv) The circles corresponding to the free symbol 2r have two vertices.
(v) The circles corresponding to any other even free symbol have four vertices.

The graph has \( 2^{r-1} \cdot 2r \) circles and genus \( 2^{r-1}(3r - 2) + 1 \).

**Proof.** Since \( k = 4 \), the genus and rank are related by \( g = 2r + 1 \). From the gluing condition, it follows that the pair of symbols in each of the boxes \((m+1,1)\) and \((m,2)\) is strictly bigger than the pair of symbols in \((m,1)\) and \((m-1,2)\) (see Fig. 5.3). In total, for any missing symbol there are \( 2^{r-1} \) ways of filling the tableau, giving rise to \( 2^{r-1} \cdot (2r) \) circles.

Next, we calculate the number of vertices in the graph, by finding the number of ways of swapping in a free symbol. If the free symbol is 1, it may only be swapped with 2, which must be in the bottom left corner. Therefore, any circle corresponding to a tableau with missing symbol 1 has exactly one vertex. Similarly, a missing 2 may only be swapped for the first three boxes, and a missing 2r may only be swapped for the two rightmost boxes.

Suppose that the strip is missing an even symbol \( 2 < 2m < 2r \). Then the symbols in the boxes \((m+1,1)\) and \((m,2)\) are \( 2m - 2 \) and \( 2m - 1 \), and the symbols in the boxes to to right are \( 2m + 1 \) and \( 2m + 2 \). The symbol \( 2m \) may be swapped in for any of them. If, on the other hand, the strip is missing the odd symbol \( 2 < 2m + 1 < 2r \), then the boxes \((m+1,1)\) and \((m,2)\) are \( 2m \) and \( 2m + 2 \), and the symbols to the right are \( 2m + 3 \) and \( 2m + 4 \). Our symbol \( 2m + 1 \) may only be swapped in for of \( 2m \) or \( 2m + 2 \) without violating either the tableau or gluing condition.
Altogether, we see that there are
\[
\frac{(4(r - 2) + 2(r - 1) + 1 + 3 + 2) \cdot 2^{r-1}}{2} = 2^{r-1}(3r - 2)
\]
vertices, so the genus is \(2^r \cdot (3r - 2) + 1\) by Lemma 5.4.

\[
\begin{array}{|c|c|}
\hline
3 & \cdots \\
\hline
1 & 2 \\
\hline
\end{array} \quad \begin{array}{|c|c|}
\hline
2m-3 & 2m-2m+1 \\
\hline
2m-4 & 2m-12m+2 \\
\hline
\end{array}
\]

**Figure 5.3.** The bottom 2 rows of a tableau. The symbol \(2m\) is missing, and may be swapped in four different boxes.

**Example 5.7.** Let \(g = 7, k = 4, \) and \(r = 3\). The Prym–Brill–Noether locus is depicted in Fig. 5.4a. In this case, \(n(r, k) = 5\), and \(C(r, k) = 2^{3-1} = 4\). Proposition 5.3 shows that the locus consists of \(4 \cdot \binom{6}{5} = 24\) circles, and Proposition 5.6 implies that the genus is \(4(3(3) - 2) + 1 = 29\).

\[
\begin{array}{|c|c|c|}
\hline
5 & 1 & 2 \\
\hline
4 & 6 & \\
\hline
\end{array}
\]

(b) The tableau corresponding to the highlighted circle in the locus.

(a) \(V^r\) for \((g, k, r) = (7, 4, 3)\).

The four circles with 4 vertices each correspond to tableaux with free symbol 4. The two circles with only a single vertex correspond to the free symbol 1, and the circles they intersect with correspond to 2 being the free symbol. The highlighted circle in red is the circle corresponding to the tableaux on the right, which has free symbol 3. The highlighted point of intersection corresponds to swapping the symbols 4 and 3.

**References**

[ABBR15a] O. Amini, M. Baker, E. Brugallé, and J. Rabinoff. Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta. *Res. Math. Sci.*, 2:Art. 7, 67, 2015.

[ABBR15b] O. Amini, M. Baker, E. Brugallé, and J. Rabinoff. Lifting harmonic morphisms II: Tropical curves and metrized complexes. *Algebra Number Theory*, 9(2):267–315, 2015.

[Bak08] M. Baker. Specialization of linear systems from curves to graphs. *Algebra Number Theory*, 2(6):613–653, 2008.

[Ber87] A. Bertram. An existence theorem for prym special divisors. *Invent. Math.*, 90(3):669–671, 1987.

[BJ16] M. Baker and D. Jensen. *Degeneration of Linear Series from the Tropical Point of View and Applications*, pages 365–433. Springer International Publishing, Cham, 2016.

[Bón15] M. Bóna. *Handbook of enumerative combinatorics*, volume 87. CRC Press, 2015.
[CDPR12] F. Cools, J. Draisma, S. Payne, and E. Robeva. A tropical proof of the Brill-Noether theorem. *Adv. Math.*, 230(2):759–776, 2012.

[CLMPTBiB18] M. Chan, A. López Martín, N. Pflueger, and M. Teixidor i Bigas. Genera of bril-noether curves and staircase paths in young tableaux. *Transactions of the American Mathematical Society*, 370(5):3405–3439, 2018.

[CPJ19] K. Cook-Powell and D. Jensen. Components of brill-noether loci for curves with fixed gonality. *arXiv preprint:1907.08366*, 2019.

[FL81] W. Fulton and R. Lazarsfeld. On the connectedness of degeneracy loci and special divisors. *Acta Mathematica*, 146(3-4):271, 1981.

[GH80] P. Griffiths and J. Harris. On the variety of special linear systems on a general algebraic curve. *Duke Math. J.*, 47(1):233–272, 1980.

[Gie82] D. Gieseker. Stable curves and special divisors: Petri’s conjecture. *Inventiones Mathematicae*, 66(2):251, 1982.

[Gub07] W. Gubler. Tropical varieties for non-Archimedean analytic spaces. *Invent. Math.*, 169(2):321–376, 2007.

[JL18] D. Jensen and Y. Len. Tropicalization of theta characteristics, double covers, and Prym varieties. *Selecta Math.*, 24(2):1391–1410, 2018.

[JP16] D. Jensen and S. Payne. Tropical independence ii: The maximal rank conjecture for quadrics. *Algebra Number Theory*, 10(8):1601–1640, 2016.

[JR17] D. Jensen and D. Ranganathan. Brill-noether theory for curves of a fixed gonality. *Preprint arXiv:1701.06579*, 2017.

[Lar19] H. Larson. A refined brill-noether theory over hurwitz spaces. *arXiv preprint:1907.08597*, 2019.

[Len14] Y. Len. The Brill-Noether rank of a tropical curve. *J. Algebraic Combin.*, 40(3):841–860, 2014.

[LU19] Y. Len and M. Ulirsch. Skeletons of Prym varieties and Brill–Noether theory. *arXiv preprint:1902.09410*, 2019.

[LUZ19] Y. Len, M. Ulirsch, and D. Zakharov. Abelian tropical covers. *arXiv:1906.04215*, 2019.

[Pfl17a] N. Pflueger. Brill-Noether varieties of k-gonal curves. *Adv. Math.*, 312:46–63, 2017.

[Pfl17b] N. Pflueger. Special divisors on marked chains of cycles. *J. Combin. Theory Ser. A*, 150:182–207, 2017.

[Sve84] M. Sved. Counting and recounting: The aftermath. *Math. Intelligencer*, 6:44–46, 1984.

[Wel85] G.E. Welters. A theorem of gieseker-petri type for prym varieties. *Ann. Sci. École Norm. Sup.*, 18(4):671–683, 1985.