ON THE DIFFERENTIAL SMOOTHNESS OF 3-DIMENSIONAL SKEW POLYNOMIAL ALGEBRAS AND DIFFUSION ALGEBRAS

ARMANDO REYES AND CRISTIAN SARMIENTO

Dedicated to professor Oswaldo Lezama for his brilliant academic career at the Universidad Nacional de Colombia - Sede Bogotá

Abstract. In this paper, we study the differential smoothness of 3-dimensional skew polynomial algebras and diffusion algebras.

1. Introduction

Brzeziński [6] investigated the construction of an algebraic differential structure of an affine algebra \( A \) by defining a notion of smoothness over the algebra \( A \). He considered the existence of two complexes of \( A \)-modules related through an isomorphism which is analogous to the star isomorphism of Hodge's theory, as it was obtained for \( M_n(\mathbb{C}) \) (the set of matrices of size \( n \) times \( n \) with entries in the set of complex numbers \( \mathbb{C} \)) in the work of Dubois et al., [11] (this isomorphism motivates the features of the definition of integrable differential calculus), where the dimensionality of these structures is conditioned by the Gel’fand-Kirillov dimension of \( A \).

In this way, Brzeziński et al., [7] and [8], studied the notion of twisted multiderivation \((\sigma, \partial)\), where \( \sigma : A \rightarrow M_n(A) \) is an automorphism of algebras and \( \partial : A \rightarrow A^\sigma \) is a \( \mathbb{K} \)-linear function (\( \mathbb{K} \) a field) such that \( \partial(ab) = \partial(a)\sigma(b) + a\partial(b) \), for all \( a, b \in A \), with the purpose of introduce a first-order differential calculus \((\Omega^1(A), d)\) ([7], Definition 4.1), accompanied by a divergence function, also called a hom-connection \( \nabla \), that is, a \( \mathbb{K} \)-linear function \( \nabla : \text{Hom}_A(\Omega^1(A), A) \rightarrow A \) that satisfies \( \nabla(fa) = \nabla(f)a + f(d(a)) \), for all \( f \in \text{Hom}_A(\Omega^1(A), A) \) and \( a \in A \). Using a set of skew derivations relative to automorphisms of \( A \), Brzeziński et al., [8] built a twisted multiderivation, where \( \sigma \in M_n(\text{End}(A)) \) is a diagonal matrix ([8], p. 292). This idea allowed to Brzeziński [7] to find examples of these first-order differential calculus and hom-connections for generalized Weyl algebras of degree one (introduced by Bavula [4]), and skew polynomial rings (defined by Ore [27]) on \( \mathbb{C}[x] \), see Brzeziński [6]. It is important to say that using a hom-connection \( \nabla \), we obtain a notion of first-order integral calculus (Brzeziński [7], Definition 4.6), and a resolution of \( A \)-modules called the complex of integral forms \((\mathcal{I}A, \nabla)\), extending \( \nabla \) (Brzeziński [8]). The existence of a complex isomorphism between \((\mathcal{I}A, \nabla)\) and \((\Omega(A), d)\) determines the differential smoothness of \( A \) (Brzeziński [10]). Therefore,

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it is interesting to study the conditions of the twisted derivations in order to construct the Brzeziński’s differential calculus and then study the property of being differentially smooth, at least for affine algebras in a field.

With all the above facts in mind and following the ideas developed by Brzeziński, our purpose in this article is to study the differential smoothness of noncommutative algebras of polynomial type known as 3-dimensional skew polynomial algebras defined by Bell and Smith \cite{5}, and diffusion algebras introduced by Isaev et al., \cite{17} (cf. Pyatov and Twarock \cite{28}).

The article is organized as follows. Section 2 contains the study of the differential smoothness of 3-dimensional skew polynomial algebras. Next, Section 3 presents several results about the differential smoothness of diffusion algebras. Finally, we present some ideas for a possible future work.

Throughout the paper, \(\mathbb{N}\) denotes the set of natural numbers including zero, and \(K\) and \(\mathbb{K}\) denote a commutative ring with identity and a field, respectively. If \(S\) is a set, \(|S|\) denotes its number of elements. The word ring means an associative ring with identity not necessarily commutative.

2. Differential smoothness of 3-dimensional skew polynomial algebras

We start establishing some preliminary definitions and results about differential calculi.

Following Gianchetta et al., \cite{13}, Section 1.6, a graded algebra \(\Omega^*\) over \(K\) is defined as a direct sum \(\Omega^* = \bigoplus_{k \in \mathbb{N}} \Omega^k\) of \(K\)-modules, provided with an associative multiplication law \(a \wedge b, a, b \in \Omega^*\), such that \(a \wedge b \in \Omega^{|a|+|b|}\), where \(|a|\) denotes the degree of an element \(a \in \Omega^{|a|}\). Note that \(\Omega^0\) is a (noncommutative) \(K\)-algebra \(A\), while \(\Omega^{k>0}\) are \(A\)-bimodules and \(\Omega^*\) is an \((A - A)\)-algebra. A graded algebra \(\Omega^*\) is said to be graded commutative if \(a \wedge b = (-1)^{|a||b|} b \wedge a\), for elements \(a, b \in \Omega^*\).

A graded algebra \(\Omega^* = \bigoplus_{k \in \mathbb{N}} \Omega^k\), where \(\Omega^0 \cong A\), is said to be a differential calculus over \(A\) if it is a cochain complex of \(K\)-modules (also known as de Rham complex) of the differential graded algebra \((\Omega^*, d)\)

\[
0 \rightarrow \mathbb{K} \rightarrow A \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots
\]

with respect to a coboundary operator \(d\) which obeys the graded Leibniz rule \(d(ab) = d(a) \wedge b + (-1)^{|a||b|} a \wedge d(b)\), for all pair of homogeneous elements \(a, b \in \Omega\).

In particular, \(d : A \rightarrow \Omega^1\) is a \(\Omega^1\)-valued derivation of a \(K\)-algebra \(A\).

The following definition gives the general features of differential calculi that we study in this paper.

\textbf{Definition 2.1.} \cite[p. 416]{10} We say that a differential calculus \((\Omega A, d)\), where \(\Omega A = \bigoplus_{a \in \mathbb{N}} \Omega^a\), with \(\Omega A = \bigwedge_{j=1}^n \Omega^j A\) and \(\Omega A = A\), is of dimension \(n\), if \(\Omega^a A \neq 0\), and \(\Omega^m A = 0\), for all \(n < m\). We say that \(\Omega^n A\) admits a volume form if \(\Omega^n A\) is isomorphic to \(A\) as a right and left module (not necessary as a bimodule). If this is the case, and \(\omega\) is a right generator of \(\Omega^n A\), we say that \(\omega \in \Omega^n A\) is a volume form. We call a differential calculus \((\Omega, d)\) connected if \(\text{Ker}(d|_{\Omega^n}) = \mathbb{K}\).

For a differential calculus that admits a volume form, we define for each right generator \(\omega \in \Omega^n A\), the algebra automorphism \(\nu_\omega : A \rightarrow A\) such that for each \(a \in A\), \(a \omega = \omega \nu_\omega(a)\), and the \(A\)-module isomorphism \(\pi_\omega : \Omega^n (A) \rightarrow A\), such that for all \(a \in A\), we have \(\pi_\omega(\omega a) = a\).
The following definition of an integrable differential calculus seeks to portray a version of Hodge star isomorphisms between the complex of differential forms of a differentiable manifold and a complex of dual modules of it (Brzeziński [6], p. 112).

**Definition 2.2.** ([10], Definition 2.1) An $n$-dimensional differential calculus $\Omega A$ is said to be integrable if $\Omega A$ admits a complex of integral forms $(\mathcal{I}, \nabla)$ ([8], Section 2), where $\mathcal{I}_k A = \text{Hom}_A(\Omega^k(A), A)$, for $0 \leq i \leq n$, and for which there exist an algebra automorphism $\nu$ of $A$ and $A$-bimodule isomorphism $\Theta_k : \Omega^k A \to \mathcal{I}_{-k} A$, $k = 0, \ldots, n$, such that for each $k$, we have the commutative diagram given by

$$
\begin{array}{ccc}
\Omega^k A & \xrightarrow{d} & \Omega^{k+1} A \\
\downarrow{\Theta_k} & & \downarrow{\Theta_{k+1}} \\
\mathcal{I}_{-k} A & \xrightarrow{\nabla_k} & \mathcal{I}_{-(k+1)} A
\end{array}
$$

The following result allows us to guarantee the integrability of a differential calculus $\Omega(A)$ by considering the existence of finitely generator elements that allow to determine left and right components of any homogeneous element of $\Omega(A)$, using the volume isomorphisms $\nu_\omega$ and $\pi_\omega$.

**Lemma 2.3.** ([10], Lemma 2.7) Let $\Omega(A)$ be an $n$-dimensional calculus over $A$ admitting a volume form $\omega$. Assume that for all $k = 1, \ldots, n - 1$, there exists a finite number of forms $\omega^k, \pi^k \in \Omega^k(A)$ such that for all $\omega' \in \Omega^k(A)$,

$$
(2.1) \quad \omega' = \sum_i \omega^k_i \pi^k_i (\pi^k_i^{-1} \wedge \omega') = \sum_i \nu_\omega^{-1}(\pi_\omega(\omega' \wedge \omega^k_i))\omega^{k_i}_i.
$$

Then $\omega$ is an integral form and all the $\Omega^k(A)$ are finitely generated and projective as left and right $A$-modules.

Now, we recall the key notion for the paper.

**Definition 2.4.** ([10], p. 421) An affine algebra $R$ of integer Gelfand-Kirillov dimension $m$ is said to be differentially smooth if there exists a connected, $m$-dimensional, integrable differential calculus on $R$.

About the differential smoothness, from Brzeziński [6] and [8] we know that this property holds for some tensor products and skew polynomial rings (introduced by Ore [27]) of automorphism type $R[y; \sigma, \delta]$ over the polynomial ring $R = \mathbb{C}[x]$ with a non-trivial derivation $\delta$ over $R$. Precisely, related to these objects, we recall below the first family of noncommutative rings that interest us in this work.

**Definition 2.5** ([5]; [35], Definition C4.3). A 3-dimensional algebra $A$ is a $\mathbb{K}$-algebra generated by the indeterminates $x, y, z$ subject to the relations $yz - \alpha xy = \lambda$, $zx - \beta xz = \mu$, and $xy - \gamma yx = \nu$, where $\lambda, \mu, \nu \in \mathbb{K}x + \mathbb{K}y + \mathbb{K}z + \mathbb{K}$, and $\alpha, \beta, \gamma \in \mathbb{K}^*$. $A$ is called a 3-dimensional skew polynomial $\mathbb{K}$-algebra if the set $\{x^iy^jz^k \mid i, j, k \geq 0\}$ forms a $\mathbb{K}$-basis of the algebra.

Different authors have studied ring-theoretical and computational properties of 3-dimensional skew polynomial algebras (see [12], [14], [18], [19], [20], [22], [36], [29], [30], and [32]). Next, we recall the classification of these algebras.

**Proposition 2.6** ([35], Theorem C4.3.1). Up to isomorphism, a 3-dimensional skew polynomial $\mathbb{K}$-algebra $A$ is given by the following relations:
(1) If \( |\{\alpha, \beta, \gamma\}| = 3 \), then \( \mathcal{A} \) is defined by the relations \( yz - \alpha yz = 0 \), \( zx - \beta xz = 0 \), and \( xy - \gamma yx = 0 \).

(2) If \( |\{\alpha, \beta, \gamma\}| = 2 \) and \( \beta \neq \alpha = \gamma = 1 \), then \( \mathcal{A} \) is defined by one of the following rules:

(a) \( yz - zy = z, \) \( zx - \beta xz = y, \) and \( xy - yx = x \).
(b) \( yz - zy = z, \) \( zx - \beta xz = b, \) and \( xy - yx = x \).
(c) \( yz - zy = 0, \) \( zx - \beta xz = y, \) and \( xy - yx = 0 \).
(d) \( yz - zy = 0, \) \( zx - \beta xz = b, \) and \( xy - yx = 0 \).
(e) \( yz - zy = az, \) \( zx - \beta xz = 0, \) and \( xy - yx = x \).
(f) \( yz - zy = z, \) \( zx - \beta xz = 0, \) and \( xy - yx = 0 \).

Here \( a, b \in \mathbb{K} \) are arbitrary; all nonzero values of \( b \) yield isomorphic algebras.

(3) If \( \alpha = \beta = \gamma \neq 1 \), and if \( \beta \neq \alpha = \gamma \neq 1 \), then \( \mathcal{A} \) is one of the following algebras:

(a) \( yz - azy = 0, \) \( zx - \beta xz = y + b, \) and \( xy - \alpha yx = 0 \).
(b) \( yz - azy = 0, \) \( zx - \beta xz = b, \) and \( xy - \alpha yx = 0 \).

Here \( a, b \in \mathbb{K} \) is arbitrary; all nonzero values of \( b \) yield isomorphic algebras.

(4) If \( \alpha = \beta = \gamma \neq 1 \), then \( \mathcal{A} \) is determined by the relations \( yz - \alpha yz = a_1 x + b_1, \)
\( zx - \alpha xz = a_2 y + b_2, \) and \( xy - \alpha yx = a_3 z + b_3 \).

If \( a_1 = 0 \), then all nonzero values of \( b_i \) yield isomorphic algebras.

(5) If \( \alpha = \beta = \gamma = 1 \), then \( \mathcal{A} \) is isomorphic to one of the following algebras:

(a) \( yz - zy = x, \) \( zx - xz = y, \) and \( xy - yx = z \).
(b) \( yz - zy = 0, \) \( zx - xz = 0, \) and \( xy - yx = z \).
(c) \( yz - zy = 0, \) \( zx - xz = 0, \) and \( xy - yx = b \).
(d) \( yz - zy = -y, \) \( zx - xz = x + y, \) and \( xy - yx = 0 \).
(e) \( yz - zy = az, \) \( zx - xz = x, \) and \( xy - yx = 0 \).

Here \( a, b \in \mathbb{K} \) are arbitrary; all nonzero values of \( b \) yield isomorphic algebras.

Now, we are going to formulate the first result of the article, Theorem 1. This allows us to study the differential smoothness of some 3-dimensional skew polynomial algebras.

**Theorem 2.7.** Let \( \mathcal{A} \) be the \( \mathbb{K} \)-algebra generated by the indeterminates \( x, y \) and \( z \) subject to the relations given by
\[
yz - z(\alpha y + a) = 0, \quad zx - \beta xz = b, \quad xy - (\gamma y + d)x = 0,
\]
where \( \gamma, \alpha, \beta \in \mathbb{K} \setminus \{0\}, \) \( b \in \mathbb{K} \), such that (i) \( d = a = 0 \) or (ii) \( \alpha = \gamma = 1 \). If \( \text{GKdim}(\mathcal{A}) = 3 \), then \( \mathcal{A} \) is differentially smooth.

**Proof.** In both cases (i) and (ii), one can check that the automorphisms of \( \mathcal{A} \) given by
\[
\nu_x(x) = \beta^{-1} x, \quad \nu_x(y) = \gamma^{-1} y - d, \quad \nu_x(z) = \beta z,
\nu_y(x) = \gamma x, \quad \nu_y(y) = y, \quad \nu_y(z) = \alpha^{-1} z,
\nu_z(x) = \beta^{-1} x, \quad \nu_z(y) = \alpha y + a, \quad \nu_z(z) = \beta z,
\]
commute with each other. Having in mind the left structure defined with these automorphisms \( \nu_a \), \( w = x, y, z \), over the right free \( \mathcal{A} \)-module \( \Omega^1(\mathcal{A}) := d(x)A \oplus d(y)A \oplus d(z)A \), with left action defined, for all \( a \in \mathcal{A} \), by \( ad(w) = d(w)\nu_a(w) \), and since the extension of the assignments \( w \mapsto d(w) \) to a derivative \( d : \mathcal{A} \to \Omega^1(\mathcal{A}) \) is well-defined by the definition of \( \nu_a \), if \( \nu_a(w) := \theta_w w \), then for all \( i \in \mathbb{N} \setminus \{0\} \), one can assert that \( d(w^i) = d(w)(\sum_{j=0}^{i-1} \theta_j^i)w^{i-1} \). Note that if \( \theta_w = 1 \), then
\[ d(w^i) = d(w)d(w^i). \] In this way, we have the well-defined \( K \)-maps \( \partial_w : A \rightarrow A \), such that \( d(a) = \sum_{w \in \{x, y, z\}} d(w)\partial_w(a) \), where for all \( x^iy^jz^k \in A \),

\[
d(x^iy^jz^k) = d(x)\left( \sum_{t=0}^{i-1} \theta_t^x \right)x^{i-1}y^jz^k + d(y)\left( \sum_{t=0}^{j-1} \theta_t^y \right)\nu_y(x^i)y^jz^{k-1} + d(z)\left( \sum_{t=0}^{k-1} \theta_t^z \right)\nu_z(x^i)\nu_z(y^j)z^{k-1}.
\]

Note that \( a \in \text{Ker}(d) \) if and only if \( a \in \text{Ker}(\partial_x) \cap \text{Ker}(\partial_y) \cap \text{Ker}(\partial_z) = K \), which shows that \( (\Omega(A), d) \) is connected, where \( \Omega(A) = \bigoplus_{i=0}^{3} \Omega^i(A) \). In \( \Omega^2(A) \) we get that, in both cases (i) and (ii), the following relations hold:

\[
d(y) \wedge d(x) = -\gamma^{-1}d(x) \wedge d(y),
\]

\[
d(z) \wedge d(x) = -\beta d(x) \wedge d(z),
\]

\[
d(z) \wedge d(y) = -\alpha^{-1}d(y) \wedge d(z),
\]

Since the automorphisms \( \nu_w, w \in \{x, y, z\} \), commute with each other, there are no additional relationships to the previous ones, so that \( \Omega^2(A) = d(x) \wedge d(y)A \bigoplus d(z)A \bigoplus d(y) \wedge d(z)A \). Thus, \( \Omega^3 = \omega A \cong A \) as a right and left \( A \)-module, with \( \omega := d(x) \wedge d(y) \wedge d(z) \), where \( \nu_w = \nu_x \circ \nu_y \circ \nu_z \), i.e., \( \omega \) is a volume form of \( A \). From Lemma 2.3, we get that \( \omega \) is an integral form by setting

\[
\omega_1^1 = \omega_1 = d(x), \quad \omega_2^1 = \omega_2 = d(y), \quad \omega_3^1 = \omega_3 = d(z)
\]

\[
\omega_1^2 = d(y) \wedge d(z), \quad \omega_2^2 = -\gamma d(x) \wedge d(z), \quad \omega_3^2 = \alpha \beta^{-1} d(x) \wedge d(y),
\]

\[
\omega_1^3 = \gamma \beta^{-1} d(y) \wedge d(z), \quad \omega_2^3 = -\alpha d(x) \wedge d(z), \quad \omega_3^3 = d(x) \wedge d(y).
\]

Of course, this shows that \( A \) is differentially smooth. \( \square \)

**Remark 2.8.** Suppose that for some generators \( x_i, x_j \) and \( x_k \) of an algebra \( A \) generated by the indeterminates \( x_1, \ldots, x_n \) one has the relationships given by \( x_ix_j - ax_ix_j = bx_i + cx_j + f x_k + e \). If we have a first order differential calculus \( (\Omega^1, d) \) and \( d \) is a well defined derivation \( d : A \rightarrow \Omega^1 \), we get that

\[
d(x_i)x_j + x_i d(x_j) - ad(x_j)x_i - ax_j d(x_i) = bd(x_i) + cd(x_j) + fd(x_k) + e,
\]

whence \( d(x_k) \) is generated in the \( A \)-bimodule by the elements \( d(x_i) \) and \( d(x_j) \). Since \( \Omega^1 \) is generated as \( A \)-bimodule by \( d(A) \), then \( \Omega^1 \) is generated by \( n-1 \) elements, and hence \( \Omega^n = \bigwedge_{i=1}^n \Omega^1 = 0 \). Besides, if \( \text{GKdim}(A) = n \), we get that \( A \) cannot be differentially smooth because there is no a differential graded calculus of dimension \( n \).

From Theorem 2.7, we can guarantee the differential smoothness of 3-dimensional skew polynomial \( K \)-algebras appearing in (1), (2)(b), (2)(d), (2)(e), (2)(f), (3)(b) and (5)(c), and by other hand, Remark 2.8 shows us that the algebras (2)(a), (2)(c), (3)(a), (4) with \( |a_1| + |a_2| + |a_3| \geq 0 \), (5)(a), (5)(b) and (5)(d), are no differentially smooth. For the algebra (5)(c), though this algebra does not hold the conditions of Theorem 2.7, we have that the automorphisms \( \nu_w := \text{id}_A \) guarantee a structure in the sense of Theorem 2.7. However, for the algebra (5)(c), we cannot conclude its
differential smoothness using this reasoning because applying $d$ to $zx - x(z + 1) = 0$
we obtain the following facts:

$$
d(z)x + zd(x) - d(x)(z + 1) - xd(z) = 0
$$

$$
d(z)x + d(x)\nu_x(z) - d(x)(z + 1) - d(z)\nu_x(x) = 0
$$

$$
d(z)[x - \nu_x(x)] + d(x)[\nu_x(z) - (z + 1)] = 0.
$$

Due to the right free structure of $\Omega^1$, necessarily $\nu_x(x) = x$ and $\nu_x(z) = z + 1$. In
the same way, since $xy - yx = 0$ then $\nu_x(y) = y$ and $\nu_y(x) = x$. Thus, $\nu_x$ respect
$zy - (y + a)z = 0$ if and only if $a = 0$. In other words, if $a \neq 0$, we cannot define a left
structure on $\Omega^1(\mathcal{A})$ such that there exists a well-defined derivation $d : \mathcal{A} \to \Omega^1(\mathcal{A})$,
which is our main trouble about the differential smoothness of algebra (5)(e).

2.1. **Special $n$-dimensional skew polynomial algebras.** In the proof of Theorem 2.7 we saw that we need automorphisms $\nu_x$, of $\mathcal{A}$, for each generator $x_i$, such
that the following conditions hold:

- the automorphisms allow to define a differential $d : \mathcal{A} \to \Omega^1(\mathcal{A})$ under the
  left action $ad(x_i) = d(x_i)\nu_{x_i}(a)$, for $a \in \mathcal{A}$, and $1 \leq i \leq n$;
- for any pair of generators $x_i$ and $x_j$, $[\nu_{x_i}, \nu_{x_j}] = 0$.

In a general way, if $\mathcal{A}$ is an algebra generated by a set of indeterminates $\{x_1, \ldots, x_n\}$
satisfying the relations

\begin{equation}
  \begin{align*}
  x_i x_j - a_{ij} x_j x_i &= b_{ij} x_i + c_{ij} x_j + e_{ij}, \quad \text{for all } 1 \leq i < j \leq n,
  \\
  \end{align*}
\end{equation}

where $a_{ij}, b_{ij}, c_{ij}, e_{ij} \in \mathbb{K}$, $a_{ij} \neq 0$, if we want to define $d$ we need that $\nu_{x_i}(x_j) = a_{ij}^{-1} x_j - a_{ij}^{-1} b_{ij}$ and $\nu_{x_j}(x_i) = a_{ij} x_i + c_{ij}$, for $i < j$. Thus, for any $x_k$, if $j > k$ then
we have that $\nu_{x_j}(x_k) = a_{jk} x_k + c_{jk}$, and if $j < k$ then $\nu_{x_j}(x_k) = a_{jk}^{-1} x_k - a_{jk}^{-1} b_{jk}$.

In order to establish if the automorphisms $\nu_x$ are well defined, we proceed in the
following way:

1. If $j < k$, we have $\nu_{x_k}(x_j) = a_{jk} x_j + c_{jk}$, and since $x_j x_k - a_{jk} x_k x_j = b_{jk} x_j + c_{jk} x_j + e_{jk}$, when we apply $\nu_{x_k}$ we obtain
   \begin{equation}
   \begin{align*}
   \nu_{x_k}(x_j) &= a_{jk} x_j - a_{jk}^{-1} b_{jk}.
   \\
   \end{align*}
   \end{equation}
   If $\nu_{x_k}(x_j) = a_{jk} x_j - b_{kk}$, we get the expression $0 x_k + (a_{jk} b_{kk} - b_{jk}) c_{jk} = 0$. In this way,
\begin{equation}
  \begin{align*}
  b_{kk} (a_{jk} - 1) + b_{jk} (a_{kk} - 1) &= 0, \\
  (a_{kk} a_{jk} - 1) e_{jk} + (a_{jk} b_{kk} - b_{jk}) c_{jk} &= 0.
  \\
  \end{align*}
\end{equation}

2. If $k < j$, we have $\nu_{x_k}(x_j) = a_{kj}^{-1} x_j - a_{kj}^{-1} b_{kj}$, and since $x_k x_j - a_{kj} x_j x_k = b_{kj} x_j + c_{kj} x_j + e_{kj}$, when we apply $\nu_{x_k}$ we get $\nu_{x_k}(x_j) = a_{kj}^{-1} x_j - a_{kj}^{-1} b_{kj}$
   \begin{equation}
   \begin{align*}
   \nu_{x_k}(x_j) &= a_{kj}^{-1} x_j - a_{kj}^{-1} b_{kj}.
   \\
   \end{align*}
   \end{equation}
   If $\nu_{x_k}(x_j) = a_{kk} x_j - b_{kk}$, then
\begin{equation}
  \begin{align*}
  (a_{kk} a_{kj} - 1) c_{kj} + b_{kk} (a_{kk} - 1) &= 0, \\
  c_{kj} (a_{kk} - a_{kj}) + (b_{kk} + c_{kj}) b_{kj} &= 0.
  \\
  \end{align*}
\end{equation}
Now, for a relation that do not involve $x_k$, we mean a relation between $x_j$ and $x_t$ (without lost of generality with $j < t$), i.e., $x_j x_t - a_{jt} x_t x_j = b_{jt} x_j + c_{jt} x_t + e_{jt}$, if we apply $\nu_{x_k}$ we get the following three cases:

1. If $j < t < k$, we have that $\nu_{x_k} (x_j) = a_{jk} x_j - c_{jk}$ and $\nu_{x_k} (x_t) = a_{tk} x_t - c_{tk}$, whence

   $$(a_{jt} c_{tk} a_{jk} - a_{jk} c_{tk} - b_{jt} a_{jk} + a_{jk} a_{tk} b_{jt}) x_j$$

   $$+ (a_{jt} a_{tk} c_{jk} - c_{jk} a_{tk} - c_{jt} a_{tk} + a_{jk} a_{tk} c_{jt}) x_t$$

   $$+ a_{jk} a_{tk} e_{jt} + c_{jk} e_{tk} - a_{jt} c_{tk} c_{jk} + b_{jt} c_{jk} + c_{jt} e_{tk} - e_{jt} = 0,$$

   which is equivalent to the following equalities:

   $$(a_{jt} - 1) e_{tk} + (a_{tk} - 1) b_{jt} = 0,$$

   $$(a_{jt} - 1) c_{jk} + (a_{jk} - 1) c_{jt} = 0,$$

   $$(a_{jk} a_{tk} - 1) c_{jt} + b_{jt} c_{jk} + c_{jt} e_{tk} + (1 - a_{jt}) c_{tk} c_{jk} = 0.$$

(2.5)

2. If $j < k < t$, we have that $\nu_{x_k} (x_j) = a_{jk} x_j - c_{jk}$ and $\nu_{x_k} (x_t) = a_{kt}^{-1} x_t - a_{kt}^{-1} b_{kt}$, so

   $$a_{jt}^{-1} c_{jk} - c_{jt} a_{kt}^{-1} - c_{jt} a_{kt}^{-1} + a_{jk} a_{kt}^{-1} c_{jt} x_t$$

   $$+ (a_{jt} a_{kt}^{-1} b_{kt} a_{jk} - b_{jt} a_{jk} - a_{jk} a_{kt}^{-1} b_{kt} + a_{jk} a_{kt}^{-1} b_{jt}) x_j$$

   $$- e_{jt} + b_{jt} c_{jk} + c_{jt} a_{kt}^{-1} b_{kt} + c_{jk} a_{kt}^{-1} b_{kt} - a_{jt} a_{kt}^{-1} b_{kt} c_{jk} + a_{jk} a_{kt}^{-1} e_{jt} = 0,$$

   that is,

   $$(a_{jt} - 1) c_{jk} + (a_{jk} - 1) c_{jt} = 0,$$

   $$(a_{jt} - 1) b_{kt} + b_{jt} (1 - a_{kt}) = 0,$$

   $$(a_{jk} - a_{kt}) e_{jt} + (c_{jt} + c_{jk}) b_{kt} + (b_{jt} a_{kt} - a_{jt} b_{kt}) c_{jk} = 0.$$

(2.6)

3. If $k < j < t$, we have that $\nu_{x_k} (x_j) = a_{kj}^{-1} x_j - a_{kj}^{-1} b_{kj}$ and $\nu_{x_k} (x_t) = a_{kt}^{-1} x_t - a_{kt}^{-1} b_{kt}$, which implies that

   $$b_{jt} - a_{jt} b_{kt} - a_{jt} b_{kt} x_j$$

   $$+ (a_{jt} b_{kj} - b_{kj} - c_{jt} a_{kj} + c_{jt}) x_t$$

   $$+ e_{jt} + b_{jt} b_{kj} a_{kt} b_{kj} + a_{kj} b_{jt} b_{kj} + a_{kj} c_{jt} b_{kt} - a_{kt} a_{kj} e_{jt} = 0,$$

   or what is the same,

   $$(a_{jt} - 1) b_{kt} + (1 - a_{kt}) b_{jt} = 0,$$

   $$(a_{jt} - 1) c_{jt} + (1 - a_{kj}) c_{jt} = 0,$$

   $$b_{kj} (a_{jt} - 1) + (1 - a_{kj}) c_{jt} = 0,$$

   $$(1 - a_{kj} a_{kt}) e_{jt} + b_{kt} (b_{kj} + a_{kj} c_{jt}) + b_{kj} (a_{kt} b_{jt} - a_{jt} b_{kt}) = 0.$$

(2.7)

Additionally, if we want that the automorphisms $\nu$’s commute with each other, since for any $x_k$, if $j > k$, we have that $\nu_{x_j} (x_k) = a_{kj} x_k - c_{kj}$, and if $j < k$, $\nu_{x_j} (x_k) = a_{jk}^{-1} x_k - a_{jk}^{-1} b_{jk}$, and $\nu_{x_k} (x_k) = a_{kk} x_k - b_{kk}$, we need the commutativity
equations given by the following:

1. If \( j, t > k \), then \( c_{kj}(a_{kt} - 1) = c_{kt}(a_{kj} - 1) \).
2. If \( j > k > t \), then \( c_{kj}(1 - a_{kt}) = b_{kk}(a_{kj} - 1) \).

\[(2.8) \]

3. If \( j, t < k \), then \( b_{jk}(1 - a_{kt}) = b_{kk}(1 - a_{jk}) \).
4. If \( k < j \), then \( c_{kj}(a_{kk} - 1) = b_{kk}(a_{kj} - 1) \).
5. If \( j < k \), then \( b_{kk}(1 - a_{jk}) = b_{jk}(a_{kk} - 1) \).

**Remark 2.9.** Now, we have to guarantee that the differential calculus \((\Omega(A), d)\), where \( \Omega^1(A) = \bigoplus_{j=1}^{n} dx_{j}A \) and \( \Omega^j(A) = \bigwedge_{j=1}^{n} \Omega^j(A) \), is a differential connected calculus. Since \( \nu_{x_k}(x_k) = b_{kk}x_k - b_{kk} \), the following equalities hold:

\[
d(x^j_k) = \sum_{j=1}^{i} x^{i-1}_k d(x^j_k)x^{j-1}_k = \sum_{j=1}^{i} d(x_k)\nu_{x_k}(x^{i-1}_k)x^{j-1}_k,
\]

\[
= d(x_k)\left(\sum_{j=1}^{i} (a_{kk}x_k - b_{kk})^{j-1}x^{i-1}_k\right)
\]

\[
= d(x_k)\left(\sum_{j=1}^{i} \sum_{t=0}^{j-1} \binom{j-1}{t} a_{kk}^t x^{i+t-1}_k (-b_{kk})^{j-1-t}\right).
\]

With these facts, we have that \( d(a) = \sum_{i=1}^{n} d(x_i)\nu_{x_i}(a) \), where

\[
\overline{\mathcal{D}}(x^{l_1}_{1} \cdots x^{l_n}_{n}) = \left(\prod_{j=1}^{l_1} \nu_{x_j}(x^{l_1}_{j})\right) \left(\prod_{j=1}^{l_2} \sum_{j=1}^{l_1} \sum_{t=0}^{j-1} \binom{j-1}{t} a_{11}^t x^{l_1+t-1}_1 (-b_{11})^{j-1-t}\right) x^{l_2}_{2} \cdots x^{l_n}_{n}.
\]

If \( i = 1 \), then for \( \sum_{r \in \Gamma} \alpha_r x^{l_1}_{1} \cdots x^{l_n}_{n} \in A \),

\[
\overline{\mathcal{D}}(\sum_{r} \alpha_r x^{l_1}_{1} \cdots x^{l_n}_{n}) = \sum_{r} \alpha_r \left(\sum_{j=1}^{l_1} \sum_{t=0}^{j-1} \binom{j-1}{t} a_{11}^t x^{l_1+t-1}_1 (-b_{11})^{j-1-t}\right) x^{l_2}_{2} \cdots x^{l_n}_{n} = 0,
\]

which is equivalent to

\[
\sum_{r \in \Gamma_r} \alpha_r \left(\sum_{j=1}^{l_1} \sum_{t=0}^{j-1} \binom{j-1}{t} a_{11}^t x^{l_1+1+t-1}_1 (-b_{11})^{j-1-t}\right) = 0,
\]

where \( r' \in \Gamma_r \) if and only if \( x^{l_2}_{2} \cdots x^{l_n}_{n} = x^{l_2'}_{2} \cdots x^{l_n'}_{n} \), i.e., we can rewrite this expression as \( \overline{\mathcal{D}}(\sum_{r} \alpha_r x^{l_1}_{1} \cdots x^{l_n}_{n}) = 0 \), and hence \( \overline{\mathcal{D}}(\sum_{r} \alpha_r x^{l_1}_{1} \cdots x^{l_n}_{n}) = 0 \), for all \( r \). If \( p_r(x_1) = \sum_{m=0}^{l_{1r}} q_m x_1^m \), with \( q_m \in \mathbb{K} \), then

\[
0 = \overline{\mathcal{D}}(p_r(x_1)x^{l_2}_{2} \cdots x^{l_n}_{n})
\]

\[
= \sum_{m=0}^{l_{1r}} q_m \left(\sum_{j=1}^{m} (a_{11} x_1 - b_{11})^{j-1} x_1^{m-j}\right) x^{l_2}_{2} \cdots x^{l_n}_{n}
\]

\[
= \sum_{m=0}^{l_{1r}} q_m \left(\sum_{j=1}^{m} \sum_{t=0}^{j-1} \binom{j-1}{t} a_{11}^t (-b_{11})^{j-1-t} x_1^{m+t-1}\right) x^{l_2}_{2} \cdots x^{l_n}_{n}.
\]

Since in the last expression the unique way to obtain the highest exponent of \( x_1 \) is considering \( m = l_{1r}, j = 1 \) and \( t = 0 \), we have that \( q_{l_{1r}} (1^{-1}) a_{11}^0 (-b_{11})^{-1} = q_{l_{1r}} = \).
with the aim of obtaining an integrable differential calculus with volume form \( \omega = d(x_1) \wedge \cdots \wedge d(x_n) \) and automorphism \( \nu_\omega = \nu_{x_1} \circ \cdots \circ \nu_{x_n} \), for each injective increasing maps \( \varphi_i : \{1, \ldots, k\} \to \{1, \ldots, n\} \) and \( \varphi_i \): \{1, \ldots, (n-k)\} \to \{1, \ldots, n\} \setminus \text{Im}(\varphi_i) \), that determine an \( n \)-permutations \( \varphi_i \) such that \( \varphi_i \varphi_i^{-1} = \varphi_i \) if \( p \leq k \), and \( \varphi_i \varphi_i^{-1} = \varphi_i^{-1} \) if \( k < p \leq n \), and analogously \( \varphi_i \varphi_i^{-1} = \varphi_i^{-1} \), we define the elements \( \omega_k^i, \varphi_i, \in \Omega^k(\mathcal{A}) \) as,

\[
\begin{align*}
\omega^i_k &= A_{ik} d(x_{\varphi_i(1)}) \wedge \cdots \wedge d(x_{\varphi_i(k)}), \\
\varphi_i^{-1} &= A_{i(n-k)} d(x_{\varphi_i(1)}) \wedge \cdots \wedge d(x_{\varphi_i(n-k)}),
\end{align*}
\]

where \( A_{ik}, A_{i(n-k)} \in \mathbb{K} \) are defined (to obtain \( \varphi_i^{-1} \wedge \omega_k^i = \omega \)) as follows: if \( i < j \), then \( x_j d(x_i) = d(x_i)^{-1}(x_j) = d(x_i)(a_{ij}^{-1} x_j - a_{ij}^{-1} b_{ij}) \), and therefore applying \( d \), \( d(x_i) \wedge d(x_j) = -a_{ij} d(x_j) \wedge d(x_i) \). In this way,

- If \( \varphi_i(1) = 1 \), then
  \[
  A_{ik} = \prod_{s=1}^{n-k-1} \prod_{t=1}^{n-k} (-1)a_{\varphi_i(s)}(t), \quad A_{i(n-k)} = \prod_{s=1}^{n-k} \prod_{t=s+1}^{n-k} (-1)a_{\varphi_i(t)}(s),
  \]

- If \( \varphi_i(1) = 1 \), then
  \[
  A_{ik} = \prod_{s=1}^{k-1} \prod_{t=s+1}^{k} (-1)a_{\varphi_i(s)}(t), \quad A_{i(n-k)} = \prod_{s=1}^{k-1} \prod_{t=\varphi_i(s)+1}^{n-k} (-1)a_{\varphi_i(t)}(s),
  \]

Since \( \varphi_i(p) = \varphi_i^{-1} \), for \( 1 \leq p \leq k \), and the set \( \{d(x_{i_1}) \wedge \cdots \wedge d(x_{i_k}) \mid 1 \leq i_1 < \cdots < i_k \leq n\} \) forms a right (and left) base of \( \Omega^k(\mathcal{A}) \), for an arbitrary \( \omega' = \sum_{1 \leq i_1 < \cdots < i_k \leq n} d(x_{i_1}) \wedge \cdots \wedge d(x_{i_k}) a_{i_1 \cdots i_k} \in \Omega^k(\mathcal{A}) \), where \( a_{i_1 \cdots i_k} \in \mathcal{A} \), we have the equalities

\[
\begin{align*}
\omega^i_k \pi_\omega (\varphi_i^{-1} \wedge \omega') &= \omega^i_k \pi_\omega (\varphi_i^{-1} \wedge d(x_{i_1}) \wedge \cdots \wedge d(x_{i_k}) a_{i_1 \cdots i_k}) \]
&= \omega^i_k \pi_\omega (d(x_{i_1}) \wedge \cdots \wedge d(x_{i_k}) a_{i_1 \cdots i_k}),
\end{align*}
\]

\[
\nu^{-1}_\omega (\pi_\omega (\omega' \wedge \omega^{-1}_{i} \wedge \omega^{-1}_{i} \wedge \cdots \wedge \omega^{-1}_{i})) = \nu^{-1}_\omega (\pi_\omega (d(x_{i_1}) \wedge \cdots \wedge d(x_{i_k}) a_{i_1 \cdots i_k} \wedge \omega^{-1}_{i} \wedge \cdots \wedge \omega^{-1}_{i}))
\]

where \( b_{i_1 \cdots i_k} := \nu_\varphi \circ \cdots \circ \nu_\varphi^{-1} \). With this, we get that equations (2.1) hold, and by Lemma 2.3, \( \Omega(\mathcal{A}) \) is an integrable differential calculus.

From [12], Theorem 7.4.1, the algebra \( \mathcal{A} \) defined by relations (2.2) has Gelfand-Kirillov dimension \( n \), that is, \( \text{GKdim}(\mathcal{A}) = n \), so the following result follows.
Theorem 2.11. Let $\mathcal{A}$ be the $\mathbb{K}$-algebra generated by $x_1, \ldots, x_n$ subject to the relations given by
\[ x_ix_j - a_{ij}x_jx_i = b_{ij}x_i + c_{ij}x_j + e_{ij}, \]
where $a_{ij}, b_{ij}, c_{ij}, e_{ij} \in \mathbb{K}$, $a_{ij} \neq 0$,
for all $1 \leq i < j \leq n$. If relations given in (2.3), (2.4), (2.5), (2.6), (2.7) and commutativity equations (2.8) hold, then $\mathcal{A}$ is differentially smooth.

From Theorem 2.11 we get the differential smoothness of algebras such as the algebra of linear partial differential operators, the algebra of linear partial $\phi$-dilation operators, the additive analogue of Weyl algebra and the multiplicative analogue of the Weyl algebra (see [12], Chapter 2, for a detailed definition of every algebra).

From the above results, we obtain the following corollary.

Corollary 2.12. Consider the Ore extension $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n][y; \sigma, \delta]$ with $\sigma(x_i) = b_ix_i + a_i$ and $\delta(x_i) = c_ix_i$, satisfying one of the following conditions:
\- $a_i \neq 0$ and $c_i = 0$, for all $i = 1, \ldots, n$; or
\- $a_i = 0$, for all $i = 1, \ldots, n$, and $c_i(b_k - 1) + c_k(b_i - 1) = 0$, for all pair of elements $i \neq k$.

If $\text{GKdim}(\mathcal{A}) = n$, then $\mathcal{A}$ is a differentially smooth algebra.

Remark 2.13. There is a natural question: Why do we choose for automorphism $\nu_{x_k}$ the assignment $\nu_{x_k}(x_k) \in \mathbb{K}x_k + \mathbb{K}$? We make this choice because we want that the automorphisms $\nu$’s commute between them; of course, there are other possibilities. For instance, in [25] we find that the automorphisms of $\mathbb{K}[x, y]$ are defined considering the assignments $x \mapsto x$, $y \mapsto y + h(x)$, or $x \mapsto a_{11}x + a_{12}y + a_{13}$, $y \mapsto a_{21}x + a_{22}y + a_{23}$, with $a_{11}a_{22} - a_{12}a_{21} \neq 0$. If $\nu_y(x) = a_{11}x + a_{13}$, i.e., $a_{12} = 0$, and $\nu_y(y) \notin \mathbb{K}y + \mathbb{K}$, we have the following possibilities:
\begin{enumerate}
\item $\nu_y(x) = x$ and $\nu_y(y) = y + h(x)$ with $0 \neq h(x) \in \mathbb{K}[x]$. In this case, $\nu_x(x) = a'_{11}x + a'_{12}y + a'_{13}$ and $\nu_y(y) = a'_{21}y + a'_{23}$. If $\nu_y(y) = 0$, then $\nu'_y(h(x)) + a'_{21}y + a'_{23} = h(a'_{11}x + a'_{12}y + a'_{13})$, which implies that
\[ a'_{21}h(x) = h(a'_{11}x + a'_{12}y + a'_{13}), \]
whence $a'_{21} = 1$, and $h(x) \in \mathbb{K}x$ or $\nu_y(x) = x$.
\item $\nu_y(x) = a_{11}x + a_{13}$ and $\nu_y(y) = a_{21}x + a_{22}y + a_{23}$ with $a_{21} \neq 0$. In this way, for $\nu_x(x) = a_{11}x + a_{12}y + a_{13}$ and $\nu_y(y) = a'_{21}y + a'_{23}$, if $\nu_y(y) = 0$, we get that
\[ a'_{21}a_{21}x + a'_{12}a_{22}y + a'_{11}a_{13} + a'_{13} = a_{11}a'_{12}y + a_{11}a'_{13} + a_{13}. \]
This equality implies that $a'_{12}a_{21} = 0$, $a'_{13}a_{22} = a_{11}a'_{12}$ and $a_{12}a_{23} + a'_{11}a_{13} + a'_{13} = a_{11}a'_{13} + a_{13}$. Since $a_{21} \neq 0$, we obtain $a'_{12} = 0$ and $(a'_{11} - 1)a_{13} = (a_{11} - 1)a_{13}$. With respect to $y$,
\[ a_{21}(a'_{11}x + a'_{13}) + a_{22}(a'_{21}y + a'_{23}) + a_{23} = a'_{21}(a_{21}x + a_{22}y + a_{23}) + a'_{23} \]
\[ a_{21}a'_{11}x + a_{21}a'_{13} + a_{22}a'_{23} + a_{23} = a'_{21}a_{21}x + a'_{21}a_{23} + a'_{23}, \]
whence $a_{21}a'_{11} = a'_{21}a_{21}$ and $a_{21}a'_{13} + (a_{22} - 1)a'_{23} = (a'_{21} - 1)a_{23}$. Since $a_{21} \neq 0$, we obtain $a'_{11} = a'_{21}$.
\end{enumerate}
In the case of automorphisms that commute with each other, it may be that $\nu_y(y) \notin \mathbb{K}y + \mathbb{K}$. However, if we use polynomial expressions of degree greater than one, such as in Case 1, then the commutativity between the automorphisms may fail.
3. Differential smoothness of diffusion algebras

Diffusion algebras were introduced formally by Isaev et al., [17] as quadratic algebras that appear as algebras of operators that model the stochastic flow of motion of particles in a one dimensional discrete lattice. However, its origin can be found in Krebs and Sandow [21].

Let us start by recalling its definition.

Definition 3.1. ([17], p. 5817) The diffusion algebras type 1 are affine algebras \( D \) that are generated by \( n \) indeterminates \( D_1, \ldots, D_n \) over a field \( K \) that admit a linear PBW basis of ordered monomials of the form \( D_{a_1}^{k_1} D_{a_2}^{k_2} \cdots D_{a_n}^{k_n} \) with \( k_j \in \mathbb{N} \) and \( \alpha_1 > \alpha_2 > \cdots > \alpha_n \), and there exist elements \( x_1, \ldots, x_n \in K \) such that for all \( 1 \leq i < j \leq n \), there exist \( \lambda_{ij} \in K^* \) such that

\[
\lambda_{ij} D_i D_j - \lambda_{ji} D_j D_i = x_j D_i - x_i D_j. \tag{3.1}
\]

Fajardo et al., [12] studied ring-theoretical properties of a graded version of these algebras; its definition is the following:

Definition 3.2. ([12], Section 2.4) The diffusion algebras type 2 are affine algebras \( D \) generated by \( 2n \) variables \( \{D_1, \ldots, D_n, x_1, \ldots, x_n\} \) over a field \( K \) that admit a linear PBW basis of ordered monomials of the form \( B_{a_1}^{k_1} B_{a_2}^{k_2} \cdots B_{a_n}^{k_n} \) with \( B_{a_i} \in \{D_1, \ldots, D_n, x_1, \ldots, x_n\} \), for all \( i \leq 2n \), \( k_j \in \mathbb{N} \) and \( \alpha_1 > \alpha_2 > \cdots > \alpha_n \), such that, for all \( 1 \leq i < j \leq n \), there exist elements \( \lambda_{ij} \in K^* \) such that

\[
\lambda_{ij} D_i D_j - \lambda_{ji} D_j D_i = x_j D_i - x_i D_j. \tag{3.2}
\]

Different physical applications of algebras type 1 and 2 have been studied in the literature. From the point of view of ring-theoretical, homological and computational properties, several papers have been published (see [12], [15], [16], [23], [24], [26], [31], [33], [34], and [37]).

Remark 3.3. About the above definitions of diffusion algebras, we make the following comments:

- Isaev et al., [17] and Pyatov and Twarok [28] defined diffusion algebras type 1 by taking \( K = \mathbb{C} \). Nevertheless, for the results obtained in this paper we can take any field not necessarily \( \mathbb{C} \).
- Hinchcliffe [16], Definition 2.1.1, considered the following definition of diffusion algebras. Let \( R \) be the algebra generated by \( n \) indeterminates \( x_1, x_2, \ldots, x_n \) over \( \mathbb{C} \) subject to relations \( a_{ij} x_i x_j - b_{ij} x_j x_i = r_j x_i - r_i x_j \), whenever \( i < j \), for some parameters \( a_{ij} \in \mathbb{C} \setminus \{0\} \), for all \( i < j \) and \( b_{ij}, r_i \in \mathbb{C} \), for all \( i < j \). He defined the standard monomials to be those of the form \( x_1^{a_{12}} x_2^{a_{23}} \cdots x_{n-1}^{a_{n-2}} x_1^{a_{12}} \). \( R \) is called a diffusion algebra if it admits a PBW basis of these standard monomials. In other words, \( R \) is a diffusion algebra if these standard monomials are a \( \mathbb{C} \)-vector space basis for \( R \). In what follows, we will consider the notation presented in Definitions 3.1 and 3.2, i.e., over any field \( K \).
- Following Krebs and Sandow [21], the relations (3.1) are consequence of subtracting (quadratic) operator relations of the type

\[
\Gamma_{\gamma \delta}^{\alpha \beta} D_\alpha D_\beta = D_\gamma X_\delta - X_\gamma D_\delta, \quad \text{for all } \gamma, \delta = 0, 1, \ldots, n - 1,
\]
where $\Gamma_{a,b}^{\gamma,\delta} \in K$, and $D_i$'s and $X_j$'s are operators of a particular vector space, such that not necessarily $[D_i, X_j] = 0$ holds (see [21], p. 3168, for more details).

Next, we present some combinatorial properties appearing in diffusion algebras (Lemma 3.5 and Proposition 3.6). It is very possible that these are found in the literature; however, we could not find them explicitly somewhere. Before, we consider the following definition.

**Definition 3.4.** If $D_i$ and $D_j$ are generators of a diffusion algebra $\mathcal{D}$ (as in Definitions 3.1 or 3.2), for all pair of elements $k, n \in \mathbb{N}$ such that $k \leq n$, the numbers $P^k_n, Q^n_k \in K$ are defined as

\[
P^k_n = \sum_{t=1}^{k} \binom{n - k + t - 1}{n - k} \lambda_{ji}^{t-1} \lambda_{ij}^{k-1}, \quad Q^n_k = \binom{n}{k-1} \lambda_{ji}^{k-1},
\]

where $\binom{n-k+t-1}{n-k}$ are the entries of the $k$-diagonal of the triangle formed with the first $n+1$ levels of the Pascal’s triangle.

**Lemma 3.5.** If we consider elements $k, n \in \mathbb{N}$ such that $k \leq n$, then:

1. $P^{n+1}_k = P^n_{k-1} \lambda_{ij} + Q^n_k$ and $P^{n+1}_{n+1} = P^n_n \lambda_{ij} + \lambda^n_{ji}$.
2. $Q^{n+1}_k = Q^n_{k-1} \lambda_{ji} + Q^n_k$ and $Q^{n+1}_{n+1} = Q^n_n \lambda_{ji} + \lambda^n_{ji}$.
3. $P^n_1 = Q^n_1 = 1$.

**Proof.** (1) As is well known, $\binom{a+b}{c} = \binom{a}{b}$, for all $a, b \in \mathbb{N}$, which implies the equalities

\[
P^n_{k-1} \lambda_{ij} + Q^n_k = \sum_{t=1}^{k-1} \binom{n - k + 1 + t - 1}{n - k + 1} \lambda_{ji}^{t-1} \lambda_{ij}^{k-1} + \binom{n}{k-1} \lambda_{ji}^{k-1}
= \sum_{t=1}^{k-1} \binom{n + 1 - k + t - 1}{n + 1 - k} \lambda_{ji}^{t-1} \lambda_{ij}^{k-t}
+ \binom{n + 1 - k + k - 1}{k - 1} \lambda_{ij}^{k-1}
= \sum_{t=1}^{k} \binom{n + 1 - k + t - 1}{n + 1 - k} \lambda_{ji}^{t-1} \lambda_{ij}^{k-t} = P^{n+1}_k,
\]

and

\[
P^n_n \lambda_{ij} + \lambda^n_{ji} = \sum_{t=1}^{n} \lambda_{ji}^{t-1} \lambda_{ij}^{n-t+1} + \lambda^n_{ji} = P^{n+1}_{n+1}.
\]

(2) Since $\binom{n}{k-2} + \binom{n}{k-1} = \binom{n+1}{k-1}$, for all $n, k \in \mathbb{N}$, we can assert that

\[
Q^n_{k-1} \lambda_{ji} + Q^n_k = \binom{n}{k-2} \lambda_{ji}^{k-2} \lambda_{ij} + \binom{n}{k-1} \lambda_{ji}^{k-1}
= \left( \binom{n}{k-2} + \binom{n}{k-1} \right) \lambda_{ji}^{k-1} = \binom{n+1}{k-1} \lambda_{ji}^{k-1} = Q^{n+1}_k,
\]

and also,

\[
Q^n_k \lambda_{ji} + \lambda^n_{ji} = \left( \binom{n}{n-1} + \binom{n}{n} \right) \lambda^n_{ji} = \binom{n+1}{n} \lambda^n_{ji} = Q^{n+1}_{n+1}.
\]
(3) It follows from a short computation.

\[ \lambda_{ij}^{n+1} D_i D_j = \lambda_{ij}^n D_i D_j + \sum_{k=1}^{n} (-1)^{k+n} P_k^n x_i^{n-k} x_j D_k + (-1)^{n+k-1} Q_k^n x_i^{n-k+1} D_j D_i^{k-1}. \]

**Proof.** For \( n = 1 \), it is clear that (3.3) coincides with (3.1). We suppose that the assertion holds for a fixed \( n \in \mathbb{N} \). Then,

\[
\begin{align*}
\lambda_{ij}^{n+1} D_i D_j &= \lambda_{ij}^n D_i D_j + \sum_{k=1}^{n} (-1)^{k+n} P_k^n x_i^{n-k} x_j D_k + (-1)^{n+k-1} Q_k^n x_i^{n-k+1} D_j D_i^{k-1} \\
&= \lambda_{ij}^n (\lambda_{ji} D_i + x_j D_i - x_i D_j) D_i^n + \sum_{k=1}^{n} (-1)^{k+n} P_k^n x_i^{n-k} x_j \lambda_{ji} D_i^{k+1} \\
&\quad + (-1)^{n+k-1} Q_k^n x_i^{n-k+1} (\lambda_{ji} D_i + x_j D_i - x_i D_j) D_i^{k-1} \\
&= \lambda_{ij}^n D_i D_j^{n+1} + \lambda_{ji} D_j D_i^n + \sum_{k=1}^{n} (-1)^{k+n} P_k^n x_i^{n-k} x_j \lambda_{ji} D_i^{k+1} \\
&\quad + (-1)^{n+k-1} Q_k^n x_i^{n-k+1} x_j D_i^{k} + (-1)^{n+k} Q_k^n x_i^{n-k+2} D_j D_i^{k-1}
\end{align*}
\]

By the associative law of the sum, and separating the term \( k = n \) of the first sum and the term \( k = 1 \) of the second, we obtain the equalities

\[
\begin{align*}
\lambda_{ij}^{n+1} D_i D_j &= \lambda_{ij}^n D_i D_j^{n+1} + \lambda_{ji} x_j D_i^{n+1} + \lambda_{ji} x_i D_j D_i^n \\
&\quad + P_k^n x_i \lambda_{ji} D_i^{n+1} - Q_k^n x_i \lambda_{ji} D_j D_i^n \\
&\quad + \sum_{k=2}^{n-1} (-1)^{k+n} P_k^n x_i^{n-k} x_j \lambda_{ji} D_j^{k+1} + (-1)^{n+k-1} Q_k^n x_i^{n-k+1} \lambda_{ji} D_j D_i^{k-1} \\
&\quad + \sum_{k=2}^{n} (-1)^{n+k-1} Q_k^n x_i^{n-k+1} x_j D_i^{k} + (-1)^{n+k} Q_k^n x_i^{n-k+2} D_j D_i^{k-1} \\
&\quad + (-1)^n Q_k^n x_i^{n+1} x_j D_i + (-1)^{n+1} Q_k^n x_i^{n+1} D_j
\end{align*}
\]

or equivalently,
\[
= \lambda_{ji}^n D_j D_i^{n+1} + (\lambda_{ji}^n + P_{\lambda_{ij}}^n) x_j D_i^{n+1} - \lambda_{ji}^n x_i D_j D_i^n
+ \sum_{k=2}^n (-1)^{k-1+n} P_k x_i^{n-k+1} x_j \lambda_{ji} D_i^k
+ (-1)^{n+k-2} Q_k x_i^{n-k-2} \lambda_{ji} D_j D_i^{k-1}
\]
\[
+ \sum_{k=2}^n \left[ (-1)^{k-1+n} Q_k x_i^{n-k+1} x_j D_i^k (+ (-1)^{n+k} Q_k x_i^{n-k+2} D_j D_i^{k-1}) \right]
\]
\[
+ (-1)^n P_1 x_i^n D_i + (-1)^{n+1} Q_1 x_i^{n+1} D_j
= \lambda_{ji}^n D_j D_i^{n+1} + P_{\lambda_{ij}}^n x_j D_i^{n+1} - Q_1 x_i D_j D_i^n
+ \sum_{k=2}^n \left[ (-1)^{k-1+n} P_k x_i^{n-k+1} x_j D_i^k (+ (-1)^{n+k} Q_k x_i^{n-k+2} D_j D_i^{k-1}) \right]
+ (-1)^n P_1 x_i^n D_i + (-1)^{n+1} Q_1 x_i^{n+1} D_j,
\]
where the second equality is due to a substitution on the index of the first sum and Lemma 3.5 in the last term, and both last equations are due to the distributivity and Lemma 3.5, parts 1 and 2. This proves that the assumption is true for \(n + 1\), which concludes the proof. \(\square\)

**Remark 3.7.** Since in diffusion algebras type 2 the generators \(x_i\)'s are central elements, for these algebras Proposition 3.6 holds.

Using a similar reasoning to the proof presented in Proposition 3.6, we can prove the following result.

**Proposition 3.8.** If \(D_i\) and \(D_j\) are generators of the diffusion algebra \(D\), then for all \(n \geq 1\),

\[
\lambda_{ji}^n D_i D_j^n = \lambda_{ji}^n D_j D_i^n + \sum_{k=1}^n Q_k x_j^{n-k+1} D_j D_i^{k-1} - P_k x_j^{n-k} x_i D_i^k.
\]

**Remark 3.9.** Propositions 3.6 and 3.8 allow to show the following facts. For the inner derivations, \(\partial_i, \partial_j : D \rightarrow D\) defined by \(\partial_k(a) = D_k a - a D_k\) with \(k = i, j\) and \(i < j\), we have that

\[
\lambda_{ji}^n \partial_j (D_i^n) = \lambda_{ji}^n D_j D_i^n - \lambda_{ji}^n D_j D_i^n
= (\lambda_{ji}^n - \lambda_{ij}^n) D_j D_i + \sum_{k=1}^n (-1)^{k+n} P_k x_j^{n-k} x_j D_i^k + (-1)^{n+k-1} Q_k x_j^{n-k-1} D_j D_i^{k-1}.
\]
which means that for the basic element $D_j^m D_i^n$,

$$\lambda^m_{ij} \partial_i(D_j^m D_i^n) = D_j^m \lambda^m_{ij} \partial_i(D_i^n)$$

$$= D_j^m [(\lambda^m_{ij} - \lambda^n_{ij}) D_i^n D_i]$$

$$+ \sum_{k=1}^{n} (-1)^{k+1} P_k^m x^{m-k} x_j D_i^k + (-1)^{n+k-1} Q_k^m x^{m-k+1} D_i x_i D_i^{k-1}]$$

$$= (\lambda^m_{ij} - \lambda^n_{ij}) D_j^{m+n} D_i$$

$$+ \sum_{k=1}^{n} (-1)^{k+1} P_k^m x^{m-k} x_j D_i^m D_i^k + (-1)^{n+k-1} Q_k^m x^{m-k+1} D_j^{m+1} D_i^{k-1}.$$  

On the other hand,

$$\lambda^m_{ij} \partial_t(D_j^m D_i^n) = \lambda^m_{ij} D_j^m D_t - \lambda^n_{ij} D_t D_j^m$$

$$= (\lambda^m_{ij} - \lambda^n_{ij}) D_j^m D_t - \sum_{k=1}^{m} \left[ Q_k^m x^{m-k+1} D_j D_i^k - P_k^m x^{m-k} x_j D_i^k \right].$$

which implies that

$$\lambda^m_{ij} \partial_t(D_j^m D_i^n) = \lambda^m_{ij} \partial_t(D_j^m D_i^n)$$

$$= (\lambda^m_{ij} - \lambda^n_{ij}) D_j^m D_t^{n+1}$$

$$- \sum_{k=1}^{m} \left[ Q_k^m x^{m-k+1} D_j D_i^{n+k-1} - P_k^m x^{m-k} x_j D_i^{n+k} \right].$$

Diffusion algebras of $n$ generators are constructed in such a way that the subalgebras of three generators are also. As we can see in [17] and [28], diffusion algebras type 1 of three generators can be classified into 4 families, $A, B, C,$ and $D,$ and these in turn are divided into classes as shown below.

**Theorem 3.10.** ([28], p. 3270) If $\mathcal{D}$ is a diffusion algebra type 1 generated by $D_1, D_2$ and $D_k$ with $i < j < k$, then $\mathcal{D}$ belongs to some of the following classes of diffusion algebras type $I$:

1. **Class A**: $\lambda_{ij} = \lambda_{ij} = \lambda_{ik} = \lambda_{kj} \neq 0, x_i, x_j, x_k \in \mathbb{K} \setminus \{0\}.$
2. **Class A1L**: $\lambda_{ij} = g_{ij} - g_{ji}$ for all $\alpha, \beta \in \{i, j, k\}$ with $\alpha < \beta$, where $g_i = \lambda_{ij} + \lambda_{jk} - \lambda_{ik}, g_j = \lambda_{jk} - \lambda_{ik}$ and $g_k = -\lambda_{ik}; \lambda_{ij} \neq 0$ if $\alpha < \beta,$ and $\lambda_{ij} = 0$ if $\alpha > \beta, x_i, x_j, x_k \in \mathbb{K} \setminus \{0\}.$
3. **Class B**: $\lambda_{ij} = \lambda_{jk} \neq 0, \lambda_{ij} = \lambda_{kj}, \lambda_{ij} - \lambda_{ij} = \lambda, \lambda_{ij} - \lambda_{ij} = \lambda, \lambda_{ij} - \lambda_{ij} = \lambda, \lambda_{ij} - \lambda_{ij} = \lambda, x_k = 0.$
4. **Class B1L**: $\lambda_{ij} = \lambda_{ij} = 0, \lambda_{ij}, x_i, x_k \in \mathbb{K} \setminus \{0\}.$
5. **Class B1**: $\lambda_{ij} = \lambda_{ij} = 0, \lambda_{ij} = \lambda_{ij} = 0, \lambda_{ij} - \lambda_{ij} = \lambda, \lambda_{ij} - \lambda_{ij} = \lambda, x_k = 0.$
6. **Class B1V**: $\lambda_{ij} = \lambda_{ij} = 0, x_i = 0, \lambda_{ij} = \lambda_{ij} = \lambda, \lambda_{ij} = \lambda, \lambda_{ij} = \lambda, x_k = 0.$
7. **Class C**: $\lambda_{ij} = \lambda_{ij} = \lambda_{ij}, \lambda_{ij}, x_i, x_k = 0; x_j = x_k = 0.$
8. **Class C1L**: $\lambda_{ij} = \lambda_{ij} = \lambda_{ij} = 0, \lambda_{ij} = \lambda_{ij} = 0, \lambda_{ij} = \lambda_{ij} = 0, x_k = 0.$
9. **Class D**: $\lambda_{ij} = \lambda_{ij} = \lambda_{ij} = 0, \lambda_{ij} = \lambda_{ij} = 0, \lambda_{ij} = \lambda_{ij} = 0, \lambda_{ij} = \lambda_{ij} = 0, x_k = 0.$

**Remark 3.11.** From the results above, we conclude that diffusion algebras of type 1 for which $\lambda_{ij} \neq 0$, with $i < j$, are skew polynomial algebras. In this way, the algebras presented in Theorem 3.10 must be also distributed in the classification...
of skew polynomial algebras of Theorem 2.6. More precisely, diffusion algebras of class $C_I$ and $D$ are skew polynomial algebras of type $2(e)$ and $I$, respectively, but the cases of diffusion algebras of classes $A_I$ and $B_I$ not belong to any of the list of Theorem 2.6. This mean that to consider the classes $A_I$ and $B_I$ as skew polynomial algebras, we need first make an identification (by establishing an automorphism of skew polynomial algebras of Theorem 16 ARMANDO REYES AND CRISTIAN SARMIENTO) that $v_\alpha$ is the vector of coefficients of degree one of $\sigma$. The rest of 3-degree diffusion algebras of type 1 are not skew polynomial algebras, at least not if we consider the same generators, by the fact that $\lambda_{ji}$'s cannot be equals to zero.

We finish this section with an analysis of the impossibility of orthogonal pairs of derivations in diffusion algebras type 2. Let $\text{Aut}_L(\mathcal{D})$ be the set of automorphisms of $K$-algebras of $\mathcal{D}$, where for an element $\sigma \in \text{Aut}_L(\mathcal{D})$ the relation $\sigma(\{D_1, D_2, x_1, x_2\}) \subseteq K D_1 + K D_2 + K x_1 + K x_2 + K$ holds. Consider $\sigma \in \text{Aut}_L(\mathcal{D})$ and $A_\alpha, B_\alpha, S_\alpha, H_\alpha \in K$, for $\alpha \in \{D_1, D_2, x_1, x_2, k\}$. The coefficients of $\sigma(D_1), \sigma(D_2), \sigma(x_1)$ and $\sigma(x_2)$ are given by

$$
\begin{align*}
\sigma(D_1) &= A D_1 D_1 + A D_2 D_2 + A x_1 x_1 + A x_2 x_2 + A_k, \\
\sigma(D_2) &= B D_1 D_1 + B D_2 D_2 + B x_1 x_1 + B x_2 x_2 + B_k, \\
\sigma(x_1) &= S D_1 D_1 + S D_2 D_2 + S x_1 x_1 + S x_2 x_2 + S_k, \\
\sigma(x_2) &= H D_1 D_1 + H D_2 D_2 + H x_1 x_1 + H x_2 x_2 + H_k,
\end{align*}
$$

which motivates the following result.

**Proposition 3.12.** If $\sigma : \mathcal{D} \to \mathcal{D}$ is an automorphism defined as in (3.4), then $\det(A) \neq 0$, where

$$
A = \begin{pmatrix}
A_{D_1} & B_{D_1} & S_{D_1} & H_{D_1} \\
A_{D_2} & B_{D_2} & S_{D_2} & H_{D_2} \\
A_{x_1} & B_{x_1} & S_{x_1} & H_{x_1} \\
A_{x_2} & B_{x_2} & S_{x_2} & H_{x_2}
\end{pmatrix}.
$$

**Proof.** Suppose that $\det(A) = 0$. Then, the set $\{v_1, v_2, v_3, v_4\}$ of columns of $A$\footnote{If we say that $D_1$ is the first generator $a_1$, $D_2$ is $a_2$, $x_1$ is $a_3$ and $x_2$ is $a_4$, we mean that $A_i$ is the vector of coefficients of degree one of $\sigma(a_i)$.} is linear dependent, i.e., there exist elements $\alpha_i \in K$ with $i \in \{1, 2, 3, 4\}$, such that $v_{\beta(1)} = \alpha_2 v_{\beta(2)} + \alpha_3 v_{\beta(3)} + \alpha_4 v_{\beta(4)}$, where $\beta$ is a permutation of $\{1, 2, 3, 4\}$. Then, $\sigma(v_{\beta(1)} + \epsilon) = \alpha_2 \sigma(v_{\beta(2)}) + \alpha_3 \sigma(v_{\beta(3)}) + \alpha_4 \sigma(v_{\beta(4)})$, where $\epsilon = \alpha_2 k_{\beta(2)} + \alpha_3 k_{\beta(3)} + \alpha_4 k_{\beta(4)} - k_{\beta(1)}$, with $k_1 = A_k, k_2 = B_k, k_3 = S_k$ and $k_4 = H_k$. Since $v_{\beta(1)} + \epsilon \neq \alpha_2 v_{\beta(2)} + \alpha_3 v_{\beta(3)} + \alpha_4 v_{\beta(4)}$, we obtain that $\sigma$ is not injective, which is a contradiction. Thus, $\det(A)$ is different from zero. \hfill $\Box$

\[\text{1}\]
For the next proposition, consider the following terms that can be obtained using the expression (3.4):

\[
\lambda_{12}\sigma(D_1)\sigma(D_2) = \lambda_{12}A_1D_1B_1 + A_2D_2B_2 + A_3x_1 + A_4x_2 + A_5)(B_1D_1A_1 + B_2D_2B_1 + B_3x_1 + B_4x_2 + B_5)
\]

\[
\lambda_{21}\sigma(D_2)\sigma(D_1) = \lambda_{21}B_1D_1A_1B_2 + A_2D_2B_2 + A_3x_1 + A_4x_2 + A_5)(A_1D_1B_1 + A_2D_2B_1 + A_3x_1 + A_4x_2 + A_5)
\]

\[
\lambda_{12}\sigma(D_1)\sigma(D_2) = \lambda_{12}A_1D_1B_1 + A_2D_2B_2 + A_3x_1 + A_4x_2 + A_5)(B_1D_1A_1 + B_2D_2B_1 + B_3x_1 + B_4x_2 + B_5)
\]

\[
\lambda_{21}\sigma(D_2)\sigma(D_1) = \lambda_{21}B_1D_1A_1B_2 + A_2D_2B_2 + A_3x_1 + A_4x_2 + A_5)(A_1D_1B_1 + A_2D_2B_1 + A_3x_1 + A_4x_2 + A_5)
\]

\[
\lambda_{12}\sigma(D_1)\sigma(D_2) + \lambda_{21}\sigma(D_2)\sigma(D_1) = \lambda_{12}A_1D_1B_1 + A_2D_2B_2 + A_3x_1 + A_4x_2 + A_5)(B_1D_1A_1 + B_2D_2B_1 + B_3x_1 + B_4x_2 + B_5)
\]

\[
\lambda_{21}\sigma(D_2)\sigma(D_1) + \lambda_{12}\sigma(D_1)\sigma(D_2) = \lambda_{21}B_1D_1A_1B_2 + A_2D_2B_2 + A_3x_1 + A_4x_2 + A_5)(A_1D_1B_1 + A_2D_2B_1 + A_3x_1 + A_4x_2 + A_5)
\]

\[
\lambda_{12}\sigma(D_1)\sigma(D_2) + \lambda_{21}\sigma(D_2)\sigma(D_1) = \lambda_{12}A_1D_1B_1 + A_2D_2B_2 + A_3x_1 + A_4x_2 + A_5)(B_1D_1A_1 + B_2D_2B_1 + B_3x_1 + B_4x_2 + B_5)
\]

Proposition 3.13. If \(\sigma : D \rightarrow D\) is an automorphism defined as in (3.4), then \(A_k = B_k = S_k = H_k = 0\).

Proof. As we must guarantee that (3.2) is respected by \(\sigma\), we need to verify that the following relation holds:

\[
\sigma(x_2)\sigma(D_1) = \lambda_{12}\sigma(D_1)D_1\sigma(D_2) - \lambda_{21}\sigma(D_2)\sigma(D_1) - \sigma(x_2)\sigma(D_1) + \sigma(x_1)\sigma(D_2) = 0.
\]

Therefore, we should have that the coefficients of the terms of degree one \(D_1, D_2, x_1\) and \(x_2\) must be zero (this due to the fact that \(D\) is a quadratic algebra). With this in mind, we vanish the coefficients of the elements \(D_1, D_2, x_1\) and \(x_2\) in expression (3.6), whence we obtain the following equations:

\[
D_1 = \lambda_{12}A_1D_1B_1 + \lambda_{12}A_2D_2B_1 + \lambda_{21}B_1D_1A_1 - \lambda_{21}B_1D_2 + A_1x_1 - A_2x_2 + A_3
\]

\[
D_2 = \lambda_{12}A_1D_1B_1 + \lambda_{12}A_2D_2B_1 + \lambda_{21}B_1D_1A_1 - \lambda_{21}B_1D_2 + A_1x_1 - A_2x_2 + A_3
\]

\[
x_1 = \lambda_{12}A_1B_1 + \lambda_{12}A_2B_2 - \lambda_{21}B_1D_1A_1 - \lambda_{21}B_1D_2 + A_1x_1 - A_2x_2 + A_3
\]

\[
x_2 = \lambda_{12}A_1B_1 + \lambda_{12}A_2B_2 - \lambda_{21}B_1D_1A_1 - \lambda_{21}B_1D_2 + A_1x_1 - A_2x_2 + A_3
\]

\[
k = \lambda_{12}A_1B_1 + \lambda_{12}A_2B_2 - \lambda_{21}B_1D_1A_1 - \lambda_{21}B_1D_2 + A_1x_1 - A_2x_2 + A_3
\]
From the algebraic properties in \( K \), we obtain the following equalities:

\[
\begin{align*}
D_1 : & \quad (\lambda - \lambda_1)B_{D_1} - H_{D_1}A_k + (\lambda - \lambda_2)A_{D_1} + S_{D_1}B_k + B_{D_1}S_k - A_{D_1}H_k = 0, \\
D_2 : & \quad (\lambda - \lambda_1)B_{D_2} - H_{D_2}A_k + (\lambda - \lambda_2)A_{D_2} + S_{D_2}B_k + B_{D_2}S_k - A_{D_2}H_k = 0, \\
x_1 : & \quad (\lambda - \lambda_1)B_{x_1} - H_{x_1}A_k + (\lambda - \lambda_2)A_{x_1} + S_{x_1}B_k + B_{x_1}S_k - A_{x_1}H_k = 0, \\
x_2 : & \quad (\lambda - \lambda_1)B_{x_2} - H_{x_2}A_k + (\lambda - \lambda_2)A_{x_2} + S_{x_2}B_k + B_{x_2}S_k - A_{x_2}H_k = 0, \\
k : & \quad (\lambda - \lambda_1)A_kB_k + S_kB_k - H_kA_k = 0,
\end{align*}
\]

Equivalently, the equations obtained in the generators \( D_1, D_2, x_1 \) and \( x_2 \) can be expressed as the linear system \( \Gamma \mathbf{x} = 0 \), where

\[
\Gamma = \begin{pmatrix}
(\lambda - \lambda_1)B_{D_1} - H_{D_1} & (\lambda - \lambda_2)A_{D_1} + S_{D_1} & B_{D_1} & -A_{D_1} \\
(\lambda - \lambda_1)B_{D_2} - H_{D_2} & (\lambda - \lambda_2)A_{D_2} + S_{D_2} & B_{D_2} & -A_{D_2} \\
(\lambda - \lambda_1)B_{x_1} - H_{x_1} & (\lambda - \lambda_2)A_{x_1} + S_{x_1} & B_{x_1} & -A_{x_1} \\
(\lambda - \lambda_1)B_{x_2} - H_{x_2} & (\lambda - \lambda_2)A_{x_2} + S_{x_2} & B_{x_2} & -A_{x_2}
\end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} A_k \\ B_k \\ S_k \\ H_k \end{pmatrix}.
\]

By the properties of the determinant function, we get \( \det(\Gamma) = \det(A) \), see (3.5). In this way, Proposition 3.12 implies that \( \det(A) \neq 0 \), whence the system \( \Gamma \mathbf{x} = 0 \) has the unique solution \( \mathbf{x} = 0 \), that is, \( A_k = B_k = S_k = H_k = 0 \).

Now, for \( \sigma : \mathcal{D} \rightarrow \mathcal{D} \) defined by (3.4), and for a \( \sigma \)-derivation \( \partial : \mathcal{D} \rightarrow \mathcal{D} \) defined on basic elements as follows

\[
\partial(D_1) = a_{D_1}D_1 + a_{D_2}D_2 + a_{x_1}x_1 + a_{x_2}x_2 + a_k,
\]
\[
\partial(D_2) = b_{D_1}D_1 + b_{D_2}D_2 + b_{x_1}x_1 + b_{x_2}x_2 + b_k,
\]
\[
\partial(x_1) = c_{D_1}D_1 + c_{D_2}D_2 + c_{x_1}x_1 + c_{x_2}x_2 + c_k,
\]
\[
\partial(x_2) = d_{D_1}D_1 + d_{D_2}D_2 + d_{x_1}x_1 + d_{x_2}x_2 + d_k,
\]

we can check that

\[
\begin{align*}
\lambda_{12}\partial(D_1)\sigma(D_2) &= \lambda_{12}(a_{D_1}D_1 + a_{D_2}D_2 + a_{x_1}x_1 + a_{x_2}x_2 + a_k)(B_{D_1}D_1 + B_{D_2}D_2 + B_{x_1}x_1 + B_{x_2}x_2 + B_k) \\
&= \lambda_{12}a_{D_1}B_{D_1}D_1^2 + a_{D_1}B_{D_2}(\lambda_{12}B_{D_1}D_1 + x_1D_1 - x_1D_2) + \lambda_{12}a_{D_2}B_{D_2}(\lambda_{12}B_{D_1}D_1 + x_1D_1 - x_1D_2) + \lambda_{12}a_{D_1}B_{D_1}(\lambda_{12}B_{D_2}D_2 + x_1D_1 - x_1D_2) + \lambda_{12}a_{D_2}B_{D_2}(\lambda_{12}B_{D_1}D_1 + x_1D_1 - x_1D_2) \\
&+ \lambda_{12}a_{D_1}B_{x_1}D_1 + \lambda_{12}a_{D_2}B_{D_1}D_1 - \lambda_{12}a_{D_2}B_{D_2}D_1^2 + \lambda_{12}a_{D_2}B_{D_1}D_1 + \lambda_{12}a_{D_2}B_{D_2}D_1^2 + \lambda_{12}a_{D_2}B_{D_1}D_1 + \lambda_{12}a_{D_2}B_{D_2}D_1^2 + \lambda_{12}a_{D_2}B_{D_1}D_1 + \lambda_{12}a_{D_2}B_{D_2}D_1^2 + \lambda_{12}a_{D_2}B_{D_1}D_1 + \lambda_{12}a_{D_2}B_{D_2}D_1^2 \\
&+ \lambda_{12}a_{D_1}B_{x_1}D_1 + \lambda_{12}a_{D_2}B_{D_1}D_1 - \lambda_{12}a_{D_2}B_{D_2}D_1^2 + \lambda_{12}a_{D_2}B_{D_1}D_1 + \lambda_{12}a_{D_2}B_{D_2}D_1^2 + \lambda_{12}a_{D_2}B_{D_1}D_1 + \lambda_{12}a_{D_2}B_{D_2}D_1^2 + \lambda_{12}a_{D_2}B_{D_1}D_1 + \lambda_{12}a_{D_2}B_{D_2}D_1^2 \\
&+ \lambda_{12}a_{D_1}B_{x_1}D_1 + \lambda_{12}a_{D_2}B_{D_1}D_1 - \lambda_{12}a_{D_2}B_{D_2}D_1^2 + \lambda_{12}a_{D_2}B_{D_1}D_1 + \lambda_{12}a_{D_2}B_{D_2}D_1^2 + \lambda_{12}a_{D_2}B_{D_1}D_1 + \lambda_{12}a_{D_2}B_{D_2}D_1^2 + \lambda_{12}a_{D_2}B_{D_1}D_1 + \lambda_{12}a_{D_2}B_{D_2}D_1^2
\end{align*}
\]
\[ \lambda_2 \partial(D_2) \sigma(D_1) = \lambda_2 (b_{D_1} D_1 + b_{D_2} D_2 + b_{x_1} x_1 + b_{x_2} x_2 + b_k)(A_{D_1} D_1 + A_{D_2} D_2 + A_{x_1} x_1 + A_{x_2} x_2 + A_k) \]
\[ = \lambda_2 b_{D_1} A_{D_1} D_1^2 + \lambda_2 b_{D_2} A_{D_2} D_1^2 + \lambda_2 b_{x_1} A_{x_1} x_1 D_1 + \lambda_2 b_{x_2} A_{x_2} x_2 D_1 + \lambda_2 b_k A_k D_1 \]
\[ + \lambda_2 b_{D_1} A_{D_1} D_2 + \lambda_2 b_{D_2} A_{D_2} D_2 + \lambda_2 b_{x_1} A_{x_1} x_1 D_2 + \lambda_2 b_{x_2} A_{x_2} x_2 D_2 + \lambda_2 b_k A_k D_2 \]
\[ + \lambda_2 b_{D_1} b_{D_2} A_{D_1} A_{D_2} D_1 + \lambda_2 b_{x_1} b_{x_2} A_{x_1} x_1 x_2 + \lambda_2 b_k b_k A_k \]
\[ = \lambda_2 b_{D_1} b_{D_2} (a_{D_1} D_1 + a_{D_2} D_2 + a_{x_1} x_1 + a_{x_2} x_2 + a_k) \]
\[ = \lambda_2 b_{D_1} b_{D_2} D_1 + \lambda_2 a_{D_1} D_2 + \lambda_2 a_{D_2} D_1 + \lambda_2 a_{x_1} D_2 x_1 + \lambda_2 a_{x_2} D_2 x_2 + \lambda_2 a_k D_2. \]

\[ \partial(x_2)\sigma(D_1) = (d_{D_1} D_1 + d_{D_2} D_2 + d_{x_1} x_1 + d_{x_2} x_2 + d_k)(A_{D_1} D_1 + A_{D_2} D_2 + A_{x_1} x_1 + A_{x_2} x_2 + A_k) \]
\[ = d_{D_1} A_{D_1} D_1^2 + d_{D_1} A_{D_2} D_1^2 + d_{D_1} A_{x_1} x_1 D_1 + d_{D_1} A_{x_2} x_2 D_1 + d_{D_1} A_k D_1 \]
\[ + d_{D_2} A_{D_1} D_2 + d_{D_2} A_{D_2} D_2 + d_{D_2} A_{x_1} x_1 D_2 + d_{D_2} A_{x_2} x_2 D_2 + d_{D_2} A_k D_2 \]
\[ + d_{x_1} A_{D_1} x_1 D_1 + d_{x_1} A_{D_2} x_1 D_2 + d_{x_1} A_{x_1} x_1 D_1 + d_{x_1} A_{x_2} x_2 D_1 + d_{x_1} A_k D_1 \]
\[ + d_{x_2} A_{D_1} x_2 D_1 + d_{x_2} A_{D_2} x_2 D_2 + d_{x_2} A_{x_1} x_1 D_2 + d_{x_2} A_{x_2} x_2 D_2 + d_{x_2} A_k D_2 \]
\[ + d_k A_{D_1} D_1 + d_k A_{D_2} D_2 + d_k A_{x_1} x_1 D_2 + d_k A_{x_2} x_2 D_2 + d_k A_k. \]

\[ x_2 \sigma(D_1) = x_2 (b_{D_1} D_1 + b_{D_2} D_2 + b_{x_1} x_1 + b_{x_2} x_2 + b_k) \]
\[ = b_{D_1} x_2 D_1 + b_{D_2} x_2 D_2 + b_{x_1} x_1 x_1 + b_{x_2} x_2 x_2 + b_k x_2. \]

\[ \partial(x_1)\sigma(D_2) = (c_{D_1} D_1 + c_{D_2} D_2 + c_{x_1} x_1 + c_{x_2} x_2 + c_k)(B_{D_1} D_1 + B_{D_2} D_2 + B_{x_1} x_1 + B_{x_2} x_2 + B_k) \]
\[ = c_{D_1} B_{D_1} D_1^2 + c_{D_1} B_{D_2} D_1^2 + c_{D_1} B_{x_1} x_1 D_1 + c_{D_1} B_{x_2} x_2 D_1 + c_{D_1} B_k D_1 \]
\[ + c_{D_2} B_{D_1} D_2 + c_{D_2} B_{D_2} D_2 + c_{D_2} B_{x_1} x_1 D_2 + c_{D_2} B_{x_2} x_2 D_2 + c_{D_2} B_k D_2 \]
\[ + c_{x_1} B_{D_1} x_1 D_1 + c_{x_1} B_{D_2} x_1 D_2 + c_{x_1} B_{x_1} x_1 x_1 + c_{x_1} B_{x_2} x_2 x_1 + c_{x_1} B_k x_1 \]
\[ + c_{x_2} B_{D_1} x_2 D_1 + c_{x_2} B_{D_2} x_2 D_2 + c_{x_2} B_{x_1} x_1 x_2 + c_{x_2} B_{x_2} x_2 x_2 + c_{x_2} B_k x_2 \]
\[ + c_k B_{D_1} D_1 + c_k B_{D_2} D_2 + c_k B_{x_1} x_1 + c_k B_{x_2} x_2 + c_k B_k. \]

\[ x_1 \sigma(D_2) = x_1 (b_{D_1} D_1 + b_{D_2} D_2 + b_{x_1} x_1 + b_{x_2} x_2 + b_k) \]
\[ = b_{D_1} x_1 D_1 + b_{D_2} x_1 D_2 + b_{x_1} x_1 x_1 + b_{x_2} x_1 x_2 + b_k x_1. \]

With these previous terms, we obtain the following proposition.

**Proposition 3.14.** If \( \sigma: D \to D \) is an automorphism defined as in (3.4) such that \( \text{span}_k(S, H) = \text{span}_k(L_1, L_2) \), and \( \partial: D \to D \) is a \( \sigma \)-derivation as in (3.7), then the elements \( \partial(D_1), \partial(D_2), \partial(x_1) \) and \( \partial(x_2) \) have no zero degree terms, where

\[
L := \begin{pmatrix}
0 & \lambda_{12} & A_{D_1} & B_{D_1} \\
-\lambda_{21} & 0 & A_{D_2} & B_{D_2} \\
0 & 1 & A_{x_1} & B_{x_1} \\
1 & 0 & A_{x_2} & B_{x_2}
\end{pmatrix}.
\]
Proof. From the one degree terms of $\partial$ applied to the relation (3.2), we obtain the equations:

\[
D_1 : \lambda_{12} a_1 B_{D1} + \lambda_{21} b_1 B_{D1} + \lambda_{12} b_2 A_{D1} - \lambda_{21} b_2 A_{D1} - d_{D1} A_k - d_k A_{D1} + c_{D1} B_k + c_k B_{D1} = 0
\]
\[
D_2 : \lambda_{12} a_2 B_{D2} + \lambda_{21} b_2 B_{D2} - \lambda_{21} b_1 A_{D2} - \lambda_{12} b_2 A_{D2} - d_{D2} A_k - d_k A_{D2} + c_{D2} B_k + c_k B_{D2} = 0
\]
\[
x_1 : \lambda_{12} a_1 B_{x1} + \lambda_{21} b_1 A_{x1} - \lambda_{21} b_1 A_{x1} - d_{x1} A_k - d_k A_{x1} + c_{x1} B_k + c_k B_{x1} + b_k = 0
\]
\[
x_2 : \lambda_{12} a_2 B_{x2} + \lambda_{21} b_2 A_{x2} - \lambda_{21} b_1 A_{x2} - d_{x2} A_k - d_k A_{x2} + c_{x2} B_k + c_k B_{x2} + a_k = 0
\]

Now, since $\det(A) \neq 0$, then Proposition 3.13 implies that $A_k = B_k = 0$. In this way, the previous equations can be expressed as

\[
D_1 : \lambda_{12} a_1 B_{D1} - \lambda_{21} b_1 A_{D1} - d_{D1} A_k + c_k B_{D1} = -\lambda_{12} b_k, \\
D_2 : \lambda_{12} a_2 B_{D2} - \lambda_{21} b_2 A_{D2} - d_{D2} A_k + c_k B_{D2} = \lambda_{21} a_k, \\
x_1 : \lambda_{12} a_1 B_{x1} - \lambda_{21} b_1 A_{x1} - d_{x1} A_k + c_k B_{x1} = -b_k, \\
x_2 : \lambda_{12} a_2 B_{x2} - \lambda_{21} b_2 A_{x2} - d_{x2} A_k + c_k B_{x2} = -a_k,
\]

which is a linear system of equations in the variables $a_k, b_k, c_k$ and $d_k$, i.e., it is a system $\Theta \overline{y} = 0$, where

\[
\Theta = \begin{pmatrix}
\lambda_{12} B_{D1} & \lambda_{12} A_{D1} & -A_{D1} & B_{D1} \\
\lambda_{12} B_{D2} & -\lambda_{21} A_{D2} & -A_{D2} & B_{D2} \\
\lambda_{12} A_{x1} & 1 - \lambda_{21} A_{x1} & -A_{x1} & B_{x1} \\
\lambda_{12} A_{x2} + 1 & -\lambda_{21} A_{x2} & -A_{x2} & B_{x2}
\end{pmatrix}, \quad \overline{y} = \begin{pmatrix} a_k \\ b_k \\ d_k \\ c_k \end{pmatrix}.
\]

It is clear that

\[
\det(\Theta) = \det\left(\begin{pmatrix}
\lambda_{12} B_{D1} & \lambda_{12} A_{D1} & -A_{D1} & B_{D1} \\
\lambda_{12} B_{D2} & -\lambda_{21} A_{D2} & -A_{D2} & B_{D2} \\
\lambda_{12} A_{x1} & 1 - \lambda_{21} A_{x1} & -A_{x1} & B_{x1} \\
\lambda_{12} A_{x2} & -\lambda_{21} A_{x2} & -A_{x2} & B_{x2}
\end{pmatrix}\right) \\
+ \det\left(\begin{pmatrix}
0 & \lambda_{12} - \lambda_{21} A_{D1} & -A_{D1} & B_{D1} \\
0 & -\lambda_{21} A_{D2} & -A_{D2} & B_{D2} \\
1 & -\lambda_{21} A_{x1} & -A_{x1} & B_{x1} \\
1 & -\lambda_{21} A_{x2} & -A_{x2} & B_{x2}
\end{pmatrix}\right) \\
= \det\left(\begin{pmatrix}
0 & \lambda_{12} - \lambda_{21} A_{D1} & -A_{D1} & B_{D1} \\
0 & -\lambda_{21} A_{D2} & -A_{D2} & B_{D2} \\
0 & 1 - \lambda_{21} A_{x1} & -A_{x1} & B_{x1} \\
0 & 1 - \lambda_{21} A_{x2} & -A_{x2} & B_{x2}
\end{pmatrix}\right) \neq 0.
\]
Since det $\Theta$ is the determinant of the homogeneous linear system $\Theta$, we have that the matrices $A^T$ and $L^T$ are row equivalent, and therefore $\det(L) \neq 0$, because $\det(A) \neq 0$. Since $\det(\Theta) = -\det(L)$, $\det(\Theta) \neq 0$ which implies that the unique solution to the homogeneous linear system $\Theta \overline{y} = 0$ is the trivial solution $\overline{y} = 0$, i.e., $a_k = b_k = c_k = d_k = 0$.

From results above, it follows the next assertion.

**Corollary 3.15.** If $\sigma : D \rightarrow D$ is an automorphism defined as in (3.4) such that the linear components satisfy $\text{span}_K(S, H) = \text{span}_K(L_1, L_2)$ as in Proposition 3.14, and if $\partial : D \rightarrow D$ is a $\sigma$-derivation, then $\partial(D) \cap K = \emptyset$.

By Corollary 3.15, we cannot guarantee the density conditions ([1], p. 6), which, as we have seen, we want for the construction of the differential calculus used in Theorem 2.7 and Theorem 2.11. Therefore, when we work in diffusion algebras type 2 with 2 generators, we prefer to choose skew derivations of graded automorphism such that $\text{span}_K(S, H) \neq \text{span}_K(L_1, L_2)$.

4. Conclusions and future work

In this article, we have studied the differential smoothness of several algebras, and we have realized that the GK dimension is a very important notion to guarantee the non-existence of the differential calculus on the algebras. It is a pending task to characterize the differential smoothness of 3-dimensional skew polynomial algebras type 5(e).

Now, since the 3-dimensional skew polynomial and diffusion algebras are part of more general families of PBW noncommutative structures (see for example [12], [31], and [34]) for which its GK dimension has been computed explicitly in [12], a possible future work is to study differential smoothness in this general context. For example, in the setting of skew polynomial rings considered by Artamonov et al., [3], which are examples of the PBW algebras studied in [12], we can see that the coefficients do not commute with the variables, as they do with the algebras studied here, so new calculations will have to be developed. Surely, the derivations of these structures following Artamonov’s ideas [2], and the automorphisms of the objects characterized by Venegas [38] will have to considered.

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