Random Scaling of Gumbel Risks

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Abstract. In this paper we consider the product of two positive independent risks $Y_1$ and $Y_2$. If $Y_1$ is bounded and $Y_2$ has distribution in the Gumbel max-domain of attraction with some auxiliary function which is regularly varying at infinity, then we show that $Y_1 Y_2$ has also distribution in the Gumbel max-domain of attraction. Additionally, if both $Y_1, Y_2$ have log-Weibullian or Weibullian tail behavior, we show that $Y_1 Y_2$ has log-Weibullian or Weibullian asymptotic tail behavior, respectively.

Keywords and phrases: Gumbel max-domain of attraction; product of random variables; log-Weibullian tail behavior; Weibullian tail behavior; supremum of Gaussian process.

1. Introduction

Consider $Y_1$ and $Y_2$, two positive independent random variables (rvs). If $Y_2$ is bounded, say $Y_2 \leq 1$ almost surely, then $X = Y_1 Y_2$ is referred to as a random contraction, see e.g., Pakes and Navarro (2007). In such a contraction model we expect that the asymptotic tail behavior of $X$ is essentially determined by that of $Y_1$. This intuition is confirmed in Theorem 1.1 below for the case $Y_1$ has a distribution with unbounded support, being further in the Gumbel max-domain of attraction, i.e.,

$$\lim_{u \to \infty} \frac{\Pr \{ Y_1 > u + a(u) t \}}{\Pr \{ Y_1 > u \}} = \exp(-t), \quad \forall t \geq 0$$

for some positive scaling function $a(\cdot)$, which is regularly varying at infinity with index $-\tau$ for $\tau \geq -1$, i.e.,

$$\lim_{u \to \infty} \frac{a(u x)}{a(u)} = x^{-\tau}, x > 0.$$ We abbreviate (1.1) as $Y_1 \in GMDA(a)$ and refer to, e.g., Resnick (1987), for details on the Gumbel max-domain of attraction and regular variation.

Theorem 1.1. If condition (1.1) holds with $a(\cdot)$ being regularly varying at infinity with index $-\tau$ for $\tau \geq -1$ and $Y_2$ has distribution with right endpoint equal to 1, then $X = Y_1 Y_2 \in GMDA(a)$.

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In view of Lemma 3.2 in Arendarczyk and Dębicki (2011) a direct implication of Theorem 1.1 is
\[ X = \sup_{t \in [0,S]} B_H(t) \in GMDA(a), \quad \text{where } a(x) = 1/x, \]
provided that \( S > 0 \) is a bounded risk being independent of a standard fractional Brownian motion \( \{B_H(t), t \in \mathbb{R}\} \) (with mean zero, variance function \( t^{2H} \)) with Hurst index \( H \in (0,1) \).

A canonical example for \( Y_1 \in GMDA(a) \) is when
\[ \mathbb{P}\{ Y_1 > u \} \sim C_1 u^{\alpha_1} \exp(-L_1 u^{p_1}), \quad u \to \infty, \quad (1.2) \]
where \( C_1, L_1, p_1 \) are positive constants and \( \alpha_1 \in \mathbb{R} \); note that \( f_1(u) \sim f_2(u) \) means \( \lim_{u \to \infty} f_1(u)/f_2(u) = 1 \).

Clearly, if \( (1.2) \) holds, then \( Y_1 \in GMDA(a) \), where \( a(u) = u^{1-p_1}/L_1 \). Consequently, the assumption of Theorem 1.1 on \( a(\cdot) \) holds with \( \tau = p_1 - 1 \).

If \( Y_1 \) and \( Y_2 \) can simultaneously take large values with non-zero probability, then the asymptotic tail behavior of \( X \) is known in few cases. In particular, if also \( Y_2 \) satisfies \( (1.2) \) with some constants \( \alpha_2 \in \mathbb{R}, C_2 > 0, L_2 > 0, p_2 > 0 \), then in light of Arendarczyk and Dębicki (2011)
\[ \mathbb{P}\{ X > u \} \sim \left( \frac{2\pi p_2 L_2}{p_1 + p_2} \right)^{1/2} C_1 C_2 A^{p_2/2 + \alpha_2 - \alpha_1} u^{\frac{2\alpha_1 + 2\alpha_2 + p_2}{2(p_1 + p_2)}} \exp \left( -Bu^{p_1+p_2} \right), \quad (1.3) \]
holds as \( u \to \infty \), where
\[ A = [(p_1 L_1)/(p_2 L_2)]^{1/(p_1 + p_2)} \quad \text{and} \quad B = L_1 A^{-p_1} + L_2 A^{p_2}. \quad (1.4) \]

Our second result shows that the asymptotic tail behavior of \( X \) can also be derived for a more general case when the power term in the tail expansion of \( Y_1 \) and \( Y_2 \) is substituted by some regularly varying function, see Theorem 2.1 in Section 2. We refer to, e.g., Berman (1992), Cline and Samorodnitsky (1994), Maulik and Resnick (2004), Nadarajah (2005), Dębicki and van Uitert (2006), Jessen and Mikołajczyk (2006), D’Auria and Resnick (2006, 2008), Resnick (2007), Charpentier and Segers (2007, 2009), Schlüter and Fischer (2012), Dębicki et al. (2013), Farkas and Hashorva (2013), Hashorva and Weng (2013), Tan and Hashorva (2013), Yang and Wang (2013) for related results and numerous motivations of investigation of tail behavior of the distribution of products of rvs.

As an illustration of Theorems 1.1 and 2.1 we analyze:

- limiting behavior of the maximum of randomly scaled Gaussian processes,
- exact asymptotic tail behavior of the supremum of Gaussian processes with stationary increments over a random interval with length which has Weibullian tail behavior.

We organize this paper as follows: Section 2 derives the tail asymptotics of the product of two independent Weibullian-type rvs. Our applications are presented in Section 3. Proofs of all results are relegated to Section 4, which concludes this article.
2. Log-Weibullian and Weibullian Risks

We say that $Y_i, i = 1, 2$ has a log-Weibullian tail behavior (or alternatively $Y_i$ is a log-Weibullian rv), if

$$\lim_{u \to \infty} \frac{\log(\mathbb{P}\{Y_i > u\})}{u^{p_i}} = -L_i \quad (2.1)$$

for some positive constants $p_i, L_i$. The main result in this section is Theorem 2.1 statement (a) therein shows that if (2.1) holds, then $X = Y_1 Y_2$ has also a log-Weibullian tail behavior.

The definition of Weibullian tail behavior is formulated (motivated by (1.2)) in terms of the following condition:

$$\mathbb{P}\{Y_i > u\} \sim g_i(u) \exp(-L_i u^{p_i}), \quad u \to \infty \quad (2.2)$$

holds for $i = 1, 2$, where $g_1, g_2$ are two given regularly varying at infinity functions and $L_i, p_i, i = 1, 2$ are positive constants. We say alternatively that $Y_1$ and $Y_2$ are Weibullian-type rvs. We note that if a rv is of Weibullian-type, then it is log-Weibullian.

For $g_1, g_2$ being regularly varying and ultimately monotone Hashorva and Weng (2013) shows that a similar result to (1.3) holds. In statement (b) of Theorem 2.1 we remove the assumption that $g_1$ and $g_2$ are ultimately monotone.

**Theorem 2.1.** Let $Y_1, Y_2$ be two independent positive rvs, and let $L_i, p_i, i = 1, 2$ be positive constants.

(a) If $Y_i, i = 1, 2$ satisfy (2.1) with $p_i, L_i, i = 1, 2$, then with $B$ given in (1.4)

$$\lim_{u \to \infty} \frac{\log(\mathbb{P}\{Y_1 Y_2 > u\})}{u^{p_1 + p_2}} = -B. \quad (2.3)$$

(b) Assume that for $i = 1, 2$ condition (2.2) holds with $L_i, p_i, i = 1, 2$ and $g_1, g_2$ two regularly varying functions at infinity. If $A, B$ are two constants as in (1.4), then we have

$$\mathbb{P}\{Y_1 Y_2 > u\} \sim Du^{p_1 p_2 / (p_1 + p_2)} g_1(u/c_u) g_2(c_u) \exp(-Bu^{p_1 p_2 / (p_1 + p_2)}) \sim Du^{p_1 p_2 / (p_1 + p_2)} \mathbb{P}\{Y_1 > u/c_u\} \mathbb{P}\{Y_2 > c_u\} \quad (2.4)$$

as $u \to \infty$, where

$$c_u = Au^{p_1 / (p_1 + p_2)}, \quad \text{and} \quad D = \left(\frac{2\pi (p_1 L_1)^{p_1 / (p_1 + p_2)} (p_2 L_2)^{p_2 / (p_1 + p_2)}}{p_1 + p_2} \right)^{1/2}.$$

**Remark 2.1.** Theorem 2.1 straightforwardly extends to the case of the product of $n$ rvs. Namely, if $Y_i, i \leq n$ are positive independent rvs with tail asymptotics given by (2.2), then $X = \prod_{i=1}^n Y_i$ also satisfies the condition (2.2) with some $g^*, L^*$ and $p^* = (\sum_{i=1}^n 1/p_i)^{-1}$.

Hereafter by $h^R(u) := \inf\{x : h(x) \geq u\}$ we denote the generalized inverse of the function $h$. 
Remark 2.2. Let \((Y_{n1}, Y_{n2}), n \geq 1\) be independent copies of \((Y_1, Y_2)\) and let \(F^+\) and \(H^+\) be the generalized inverse of the distributions of \(Y_1\) and \(Y_2\), respectively. Define next

\[ b_n := F^+(1 - 1/n), \quad \tilde{b}_n := H^+(1 - 1/n), \quad n > 1. \]

Under the assumptions of Theorem 1.1 for \(Y_1\) and \(Y_2\), we have that \(a(b_n) \sim a(\tilde{b}_n)\) and \(\tilde{b}_n \sim b_n\) and further

\[ \lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| P \left\{ \max_{1 \leq k \leq n} Y_{k2} \leq a(b_n)x + b_n \right\} - \exp(-\exp(-x)) \right| = 0. \]

In view of Theorem 1.1 it follows that \(Y_1Y_2\) is in Gumbel max-domain of attraction. Furthermore, since \(H^+(1 - 1/x)\) is a slowly varying function at infinity, we obtain

\[ \lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| P \left\{ \max_{1 \leq k \leq n} Y_{k1}Y_{k2} \leq a(b_n)x + \tilde{b}_n \right\} - \exp(-\exp(-x)) \right| = 0. \]

3. Applications

In this section we present two applications of Theorem 1.1 and Theorem 2.1. The first one focuses on the maximum of randomly scaled Gaussian processes. The second one, which combines Theorem 2.1 with an interesting finding of Arendarczyk and Dębicki (2011), derives the asymptotic behavior of the tail distribution of supremum of Gaussian processes with stationary increments over Weibullian and log-Weibullian random intervals.

3.1. Limit law of the maximum of deflated Gaussian processes. This application is motivated by a key finding of Kabluchko (2011). Instead of Gaussian processes treated therein, we consider here deflated Gaussian processes. Let therefore \(\Gamma(\cdot, \cdot)\) be a negative definite kernel in \(\mathbb{R}^2\) and define a Brown-Resnick stochastic process with Gaussian points as

\[ \eta_{BR}(t) = \max_{i \geq 1} \left( Y_i + Z_i(t) - \sigma^2(t)/2 \right), \quad t \in \mathbb{R}, \] (3.1)

where \(\{Z_i(t), t \in \mathbb{R}\}, i \geq 1\) are mutually independent centered Gaussian processes with incremental variance function \(Var(Z_i(t_1) - Z_i(t_2)) = \Gamma(t_1, t_2), i \geq 1\) and variance function \(\sigma^2(\cdot) > 0\), being further independent of the Poisson point process \(\Xi = \sum_{i=1}^{\infty} \xi_{\tau_i}\), with intensity measure \(\exp(-x)dx\), see for more details Kabluchko (2011).

In the following, for scaling the Gaussian process, we shall use a generic positive rv \(S\), which has either a distribution with right endpoint 1, or it has a Weibullian tail behavior satisfying (2.2) with some \(p, L\) and \(g\) being regularly varying at infinity. Our next result generalizes Theorem 5.1 in Hashorva (2013).

Theorem 3.1. Let \(\{X_{ni}(t), t \in \mathbb{R}\}, 1 \leq i \leq n, n \geq 1\) be independent Gaussian processes with mean-zero, unit variance function and correlation function \(\rho_n(s, t), s, t \in \mathbb{R}\). Let \(S_{ni}, i, n \geq 1\) be independent copies of \(S\), and let \(H^+\) be the generalized inverse of the distribution \(H\) of \(SX_{11}(1)\). Assume that \(S, S_{ni}, X_{ni}(t), t \in \mathbb{R}\) are

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mutually independent for any \(i = 1, \ldots, n\). For \(d_n = H^+(1 - 1/n)\) set \(c_n = 1/d_n\) if \(S\) is bounded, and set \(c_n = d_n(2 + p)/(2p \log n)\) otherwise. If further
\[
\lim_{n \to \infty} \frac{2d_n}{c_n} (1 - \rho_n(t_1, t_2)) = \Gamma(t_1, t_2) \in (0, \infty), \quad t_1 \neq t_2 \in \mathbb{R},
\]
(3.2)
then, as \(n \to \infty\)
\[
c_n \left( \max_{1 \leq i \leq n} S_n X_n(t) - d_n \right) \implies \eta_{BR}(t), \quad t \in \mathbb{R},
\]
(3.3)
where \(\implies\) means the weak convergence of the finite-dimensional distributions. Furthermore, \(d_n = (1 + o(1)) \sqrt{2 \log n}\) if \(S\) is bounded and \(d_n = (1 + o(1))((\log n)/B)^{(2+p)/(2p)}\) otherwise with \(B\) as in (1.4).

3.2. Supremum over random intervals for Gaussian processes with stationary increments. The main result of Arendarczyk and Dębicki (2011) derives the exact asymptotics (as \(u \to \infty\)) of
\[
P \left\{ \sup_{t \in [0, T]} X(t) > u \right\},
\]
where \(\{X(t), t \geq 0\}\) with \(X(0) = 0\) a.s. is a mean-zero Gaussian process with stationary increments and a.s. continuous sample paths being independent of \(T > 0\), which has tail asymptotics given by (1.2). The following result extends Theorem 3.1 in the aforementioned paper.

**Theorem 3.2.** Let \(T\) be a nonnegative log-Weibullian rv that satisfies (2.7) with some \(L, p > 0\) and let \(\{X(t), t \geq 0\}\) be, an independent of \(T\), centered Gaussian process with stationary increments and continuously differentiable variance function \(\sigma^2(t) = \text{Var}(X(t))\). Suppose that \(\sigma^2(\cdot)\) is convex, regularly varying at infinity with index \(\alpha \in (1, 2]\). If further \(\sigma^2(t) \leq Kt^\alpha\) holds for any \(t > 0\) and some positive constant \(K > 0\), then we have
\[
P \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim P \{\sigma(T)\mathcal{N} > u\}, \quad u \to \infty,
\]
(3.4)
where \(\mathcal{N}\) is an \(N(0, 1)\) rv independent of \(T\).

A combination of Theorem 2.1 with Theorem 3.2 leads to the following corollary.

**Corollary 3.1.** Under the setup of Theorem 3.2 suppose further that \(\sigma(t) \sim Ct^{\alpha/2}\) as \(t \to \infty\) with \(\alpha \in (1, 2]\) and some constant \(C > 0\).

(a) Then \(\sigma(T)\) satisfies (2.7) with \(\bar{p} = \frac{2p}{\alpha}\) and \(\bar{L} = \frac{L}{\alpha}\) and
\[
\lim_{u \to \infty} \frac{\log \left( P \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \right)}{u^{2p/(\alpha+2)}} = -\bar{L}(\bar{L} \bar{p})^{-\frac{\bar{p}}{\alpha+2}} - \frac{1}{2} (\bar{L} \bar{p})^{-\frac{2}{2\alpha+2}} =: -\bar{B}.
\]
(3.5)
(b) If \(\sigma(T)\) satisfies (2.7) with \(\bar{p}, \bar{L}\) and some regularly varying at infinity function \(\bar{g}\), then
\[
P \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim (\bar{p} + 2)^{-\frac{p}{\alpha}} g \left( (\bar{L} \bar{p})^{-\frac{1}{\alpha+2}} u^{-\frac{2}{\alpha+2}} \right) \exp \left( -\bar{B} u^{\frac{2}{\alpha+2}} \right), \quad u \to \infty.
\]
(3.6)
Remark 3.1. Clearly, if we specify in the assumptions of Theorem 3.2 that \( \sigma^2(x) = Cx^\alpha \) (i.e., \( X \) is a fractional Brownian motion with Hurst index \( \alpha/2 \)) and \( T \) is Weibullian, then both \( \sigma(T) \) and \( N \) are Weibullian-type rvs, and thus assumptions of Corollary 3.1 (b) are satisfied. Hence Corollary 3.1 is an extension of Theorem 4.1 in Arendarczyk and Dębicki (2011).

4. Proofs

It is well-known that for some rv \( U \) which has distribution with right endpoint equal to infinity the assumption \( U \in GMDA(a) \) implies that the tail of \( U \) is rapidly varying at infinity, i.e.,

\[
\lim_{u \to \infty} \frac{\mathbb{P}\{U > \lambda u\}}{\mathbb{P}\{U > u\}} = 0
\]

holds for any \( \lambda > 1 \). First we present a result on the random scaling of rvs with rapidly varying tails which is of some interest on its own.

Lemma 4.1. Let \( S, Y, Y^* \) be three independent rvs. Suppose that \( S \geq 0 \) has distribution \( G \) with right endpoint equal to 1. If further \( Y \) has a rapidly varying tail and \( \mathbb{P}\{Y > u\} \sim L(u)\mathbb{P}\{Y^* > u\} \) as \( u \to \infty \) for some slowly varying function \( L(\cdot) \), then

\[
\mathbb{P}\{SY > u\} \sim \mathbb{P}\{SY > u, S > w\} \sim L(u)\mathbb{P}\{SY^* > u\}
\] (4.1)

holds for any \( w \in (0, 1) \).

Proof of Lemma 4.1 By the independence of \( S \) and \( Y \) for any \( u > 0 \) and \( w \in (0, 1) \), we have

\[
\mathbb{P}\{SY > u\} = \mathbb{P}\{SY > u, S \leq w\} + \mathbb{P}\{SY > u, S > w\} \leq \mathbb{P}\{Y > u/w\} + \mathbb{P}\{SY > u, S > w\}.
\]

The assumption that \( Y \) has a rapidly varying tail implies for any \( t \in (w, 1) \)

\[
\frac{\mathbb{P}\{Y > u/w\}}{\mathbb{P}\{SY > u, Y > w\}} \leq \frac{\mathbb{P}\{Y > u/w\}}{\mathbb{P}\{SY > u, S > t\}} \leq \frac{\mathbb{P}\{Y > u/w\}}{\mathbb{P}\{Y > u/t\} \mathbb{P}\{S > t\}} \to 0, \quad u \to \infty
\]

hence for any \( w \in (0, 1) \)

\[
\mathbb{P}\{SY > u\} \sim \int_w^1 \mathbb{P}\{Y > u/s\} \, dG(s), \quad u \to \infty.
\]

By the uniform convergence theorem for regularly varying functions (e.g., Embrechts et al. (1997))

\[
\int_w^1 \mathbb{P}\{Y > u/s\} \, dG(s) \sim L(u)\int_w^1 \mathbb{P}\{Y^* > u/s\} \, dG(s), \quad u \to \infty.
\]

The assumption \( \mathbb{P}\{Y > u\} \sim L(u)\mathbb{P}\{Y^* > u\} \) as \( u \to \infty \) yields that \( Y^* \) has also a rapidly varying tail at infinity. Hence in view of the above arguments and the fact that \( S \) and \( Y^* \) are independent, we have that

\[
\int_w^1 \mathbb{P}\{Y^* > u/s\} \, dG(s) \sim \mathbb{P}\{SY^* > u\}, \quad u \to \infty
\]
Next, we have

\[ \mathbb{P} \{ Y_1 > u + za(u) \} = \mathbb{P} \{ Y_1 > u \} \rightarrow \exp(-x), \quad u \rightarrow \infty \]  \quad (4.2)

holds uniformly for \( x \) on compact sets of \( \mathbb{R} \). Since \( Y_1 \) has a rapidly varying tail at infinity, then by Lemma \[4.1\] for any fixed \( z \geq 0 \) and \( w \in (0, 1) \)

\[ \mathbb{P} \{ Y_1 Y_2 > u + a(u)z \} \sim \int_{w}^{1} \mathbb{P} \{ Y_1 > (u + za(u))/s \} dG(s), \quad u \rightarrow \infty \]

holds with \( G \) the distribution of \( Y_2 \). By the uniform convergence theorem for regularly varying functions

\[ \lim_{u \rightarrow \infty} \frac{a(u z)}{a(u)} = x^{-\tau} \]

holds uniformly for \( x \in [w, 1] \), with \( w \in (0, 1) \) some arbitrary constant. Hence

\[ z_{u,s} := \frac{z}{s \ a(u/s)} \rightarrow \frac{z}{s^{1+\tau}}, \quad u \rightarrow \infty \]

uniformly for \( s \in [w, 1] \), and thus

\[ \frac{\mathbb{P} \{ Y_1 > u/s + za(u)/s \}}{\mathbb{P} \{ Y_1 > u/s \}} = \frac{\mathbb{P} \{ Y_1 > u/s + a(u/s)z_{u,s} \}}{\mathbb{P} \{ Y_1 > u/s \}} \rightarrow \exp(-z/s^{1+\tau}), \quad u \rightarrow \infty \]

uniformly for \( s \in [w, 1] \). For any \( \varepsilon > 0 \) we can find \( w \in (0, 1) \) such that for all \( s \in [w, 1] \)

\[ (1 - \varepsilon) \exp(-z) \leq \exp(-z/s^{1+\tau}) < (1 + \varepsilon) \exp(-z) \]

implying that as \( u \rightarrow \infty \)

\[ \mathbb{P} \{ Y_1 Y_2 > u + a(u)z \} \sim \int_{w}^{1} \mathbb{P} \{ Y_1 > u/s + a(u/s)z_{u,s} \} dG(s) \sim \exp(-z)\mathbb{P} \{ Y_1 Y_2 > u \} . \]

Hence \( Y_1 Y_2 \in GMDA(a) \), and thus the proof is complete. \(\square\)

**Proof of Theorem \[2.1\]**  Ad. (a). Since for any \( u > 0 \)

\[ \mathbb{P} \{ Y_1 Y_2 > u \} \geq \mathbb{P} \left\{ Y_1 > \left( \frac{p_2 L_2}{p_1 L_1} \right)^{1/(p_1 + p_2)} u^{p_2/(p_1 + p_2)} \right\} \mathbb{P} \left\{ Y_2 > \left( \frac{p_1 L_1}{p_2 L_2} \right)^{1/(p_1 + p_2)} u^{p_1/(p_1 + p_2)} \right\} , \]

then we immediately get

\[ \lim_{u \rightarrow \infty} \frac{\log(\mathbb{P} \{ Y_1 Y_2 > u \})}{u^{p_1/p_2 + p_2/(p_1 + p_2)}} \geq - \left( L_1 \left( \frac{p_2 L_2}{p_1 L_1} \right)^{p_1/(p_1 + p_2)} + L_2 \left( \frac{p_1 L_1}{p_2 L_2} \right)^{p_2/(p_1 + p_2)} \right) =: -B. \]

Next, we have

\[ \mathbb{P} \{ Y_1 Y_2 > u \} \leq \sum_{k = [u^{p_2/(2(p_1 + p_2))}]}^{[u^{(p_2 + p_1)/(p_1 + p_2))]} \mathbb{P} \{ Y_1 \in [k, k + 1), Y_1 Y_2 > u \} \]

\[ + \mathbb{P} \left\{ Y_1 < \left[ u^{p_2/(2(p_1 + p_2))} \right], Y_1 Y_2 > u \right\} + \mathbb{P} \left\{ Y_1 > \left[ u^{(p_2 + p_1)/(p_1 + p_2)} \right], Y_1 Y_2 > u \right\} \]

\[ = \Sigma + P_1 + P_2. \]
Now observe that, as $u \to \infty$

$$\log(P_1) \leq \log \left( \mathbb{P} \left\{ Y_2 > u^{1-p_2/(2(p_1+p_2))} \right\} \right) \sim -L_2 u^{(p_1+p_2/2)p_2/(p_1+p_2)} \quad (4.3)$$

and

$$\log(P_2) \leq \log \left( \mathbb{P} \left\{ Y_1 > [u^{(p_2+p_1/2)/(p_1+p_2)}] \right\} \right) \sim -L_1 u^{(p_2+p_1/2)p_1/(p_1+p_2)} . \quad (4.4)$$

Moreover, for each $\varepsilon > 0$, sufficiently large $u$ and $k \in \left[ [u^{p_2/(2(p_1+p_2))}], [u^{(p_2+p_1/2)/(p_1+p_2)}] \right]$

$$\log \left( \mathbb{P} \left\{ Y_1 \in [k, k+1], Y_1 Y_2 > u \right\} \right) \leq \log \left( \mathbb{P} \left\{ Y_1 \geq k, Y_2 > u/(k+1) \right\} \right)$$

$$\leq -(1 - \varepsilon)(L_1 k^{p_1} + L_2(u/k)^{p_2})$$

$$\leq -(1 - \varepsilon)B u^{p_1 p_2/(p_1+p_2)}, \quad (4.5)$$

where (4.5) follows from the fact that $f(x) = L_1 x^{p_1} + L_2 \left( \frac{x}{2} \right)^{p_2}$ attains its minimum $f(x_u) = B u^{p_1 p_2/(p_1+p_2)}$ at $x_u = \left( \frac{2L_1}{L_2} \right) \frac{1}{p_2/(p_1+p_2)} u^{p_2/(p_1+p_2)}$ and for any $\delta \in (0, 1)$ and all $u$ large $k/(k+1) > 1 - \delta$. Thus, using the fact that $\Sigma$ consists of a polynomial (with respect to $u$) number of elements, we have that

$$\limsup_{u \to \infty} \frac{\log(\Sigma)}{u^{p_1 p_2/(p_1+p_2)}} \leq -B . \quad (4.6)$$

The combination of (4.3), (4.4) with (4.6) completes the proof of statement (a).

Ad. (b). Suppose without loss of generality that $L_1 = L_2 = 1$. With the same arguments as in the proof of Lemma 3.2 in Hashorva and Ji (2013), if $Y_1^*$ and $Y_2^*$ are two positive independent rvs tail equivalent to $Y_1$ and $Y_2$, respectively, then

$$\mathbb{P} \left\{ Y_1 Y_2 > u \right\} \sim \mathbb{P} \left\{ Y_1^* Y_2^* > u \right\}, \quad u \to \infty.$$

We define next $Y_i^* = S_i Z_i$ where $S_i$ has distribution $G_i$, $i = 1, 2$ with right endpoint equal to 1, and $Z_1, Z_2$ are independent of $S_1, S_2$. Let $\alpha_i^*$ and $\alpha_2^*$ be the index of the regular variation of $g_1$ and $g_2$, respectively. Let $\alpha_i > \alpha_i^*, i = 1, 2$ be two arbitrary constants. The functions $\tilde{g}_i(x) = g_i(x) x^{-\alpha_i}$ are regularly varying at infinity with index $\alpha_i^* - \alpha_i < 0$. Hence, we can assume without loss of generality, that

$$\mathbb{P} \left\{ S_i > 1 - a_i(u)/u \right\} = \frac{1}{\Gamma(\alpha_i - \alpha_i^* + 1)} \tilde{g}_i(u), \quad u \to \infty,$$

where $a_i(u) = u^{1-p_i}, i = 1, 2, u > 0$. In view of Example 1 in Hashorva (2012) (see also Theorem 3.1 in Hashorva et al. (2010)) for $i = 1, 2$ we obtain

$$\mathbb{P} \left\{ S_i Z_i > u \right\} \sim \mathbb{P} \left\{ Y_i > u \right\}, \quad u \to \infty,$$

where $S_i, Z_i, i = 1, 2$ are independent and positive rvs, and

$$\mathbb{P} \left\{ Z_i > u \right\} \sim u^{\alpha_i} \exp(-u^{p_i}), \quad u \to \infty.$$
Consequently, as \( u \to \infty \)

\[
\mathbb{P} \{ Y_1 Y_2 > u \} \sim \mathbb{P} \{ S_1 Z_1 Z_2 > u \} \sim \mathbb{P} \{ U W > u \},
\]

where \( U = S_1 S_2 \) and \( W = Z_1 Z_2 \). The tail asymptotics of \( U \) follows by a direct application of Theorem 2.1 in Farkas and Hashorva (2013) whereas the tail asymptotics of \( W \) follows from [1.3]. Hence, the tail asymptotics of \( U W \) follows by applying again the result of the aforementioned example, and thus the proof is complete. \( \square \)

**Proof of Theorem 3.1** The proof follows by the same arguments as the proof of Theorem 5.1 in Hashorva (2013). When \( S \) is a bounded rv, then in view of Remark 2.2, we have that \( d_n = (1 + o(1)) \sqrt{2 \log n} \) and since the scaling function \( a(x) = 1/x \), then \( c_n = 1/d_n \) follows. For the case \( S \) has a Weibullian tail behavior, the relation between \( c_n \) and \( d_n \) can be established using the same idea as in the proof of the aforementioned theorem. \( \square \)

**Proof of Theorem 3.2** For chosen constants \( \gamma_1 = 2/(\alpha + 2p), \gamma_2 = 4/(2\alpha + p) \) and \( \delta = \delta(u) = 2\sigma_u^2/(\sigma_u u^2 log^2(u)) \) we have

\[
P \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \leq \int_0^{u^{-\gamma_1}} \mathbb{P} \left\{ \sup_{t \in [0, s]} X(t) > u \right\} dF_T(s) + \int_{u^{-\gamma_1}}^{u^{-\gamma_2}} \mathbb{P} \left\{ \sup_{t \in [0, s-\delta]} X(t) > u \right\} dF_T(s) + \int_{u^{-\gamma_2}}^{\infty} \mathbb{P} \left\{ \sup_{t \in [0, s]} X(t) > u \right\} dF_T(s)
\]

\[
= I_1 + I_2 + I_3 + I_4.
\]

As in the proof of Theorem 3.1 in Arendarczyk and Dębicki (2011), we conclude that

\[ I_1 + I_2 = o(\mathbb{P} \{ X(T) > u \}) \]

as \( u \to \infty \) and for each \( \varepsilon > 0 \) and \( u \) large enough

\[ I_3 \leq (1 + \varepsilon)\mathbb{P} \{ X(T) > u \} = (1 + \varepsilon)\mathbb{P} \{ \sigma(T)N > u \}, \]

where \( N \) is an \( N(0,1) \) rv independent of \( T \). Thus it suffices to show that

\[ I_4 = o(\mathbb{P} \{ X(T) > u \}) \] \hspace{1cm} (4.7)

as \( u \to \infty \). Indeed, since for all large \( u \) we have

\[ I_4 \leq \mathbb{P} \{ T > u^{-\gamma_2} \}, \]

then

\[ \limsup_{u \to \infty} \frac{\log(I_4)}{u^{4p/(2\alpha + p)}} \leq -L. \]

On the other hand, for each \( \varepsilon \in (0, \alpha/2) \) and sufficiently large \( u \), the assumption that \( \sigma(\cdot) \) is regularly varying at \( \infty \) with index \( \alpha/2 \) implies

\[ \mathbb{P} \{ \sigma(T) > u \} \geq \mathbb{P} \left\{ T^{\alpha/2-\varepsilon} > u \right\}. \]

Hence, for some \( K > 0 \) by statement (a) of Theorem 2.1

\[ \liminf_{u \to \infty} \frac{\log(\mathbb{P} \{ X(T) > u \})}{u^{2p/(\alpha-2\varepsilon)}} \geq \liminf_{u \to \infty} \frac{\log(\mathbb{P} \{ T^{\alpha/2-\varepsilon}N > u \})}{u^{2p/(\alpha-2\varepsilon)}} \geq -K. \]
Consequently, since for sufficiently small $\varepsilon > 0$, we have $2p/(p + \alpha - 2\varepsilon) < 4p/(p + 2\alpha)$, then (4.7) holds. □

**Proof of Corollary 3.1** The proof boils down to checking, that for both cases (a) and (b) the conditions imposed on $\sigma(\cdot)$ imply that $T$ satisfies the assumptions of Theorem 3.2, therefore we omit the details. □

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