Actions for Vacuum Einstein’s Equation with a Killing Symmetry

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Abstract

In a space-time $M$ with a Killing vector field $\xi^a$ which is either everywhere timelike or everywhere spacelike, the collection of all trajectories of $\xi^a$ gives a 3-dimensional space $S$. Besides the symmetry-reduced action from that of Einstein-Hilbert, an alternative action of the fields on $S$ is also proposed which gives the same fields equations as those reduced from the vacuum Einstein equation on $M$.

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1 Introduction

Dimensional reduction is a crucial step in any high dimensional theory of physics such as, Kaluza-Klein theory [1] and string theory [2], in order to make contact with the 4-dimensional sensational world. It is also a useful approach to study 4-dimensional space-time with symmetries in general relativity.

In the appendix of [3], Geroch introduced a Killing reduction formalism of 4-dimensional space-time. Let $(M,g_{ab})$ be a space-time with a Killing vector field $\xi^a$, and $\xi^a$ is either everywhere timelike or everywhere spacelike. The collection of all trajectories of $\xi^a$ gives a 3-dimensional space $S$. Geroch’s discussion shows that there is a one-to-one correspondence between tensor fields and tensor operations on $S$ and certain tensor fields and tensor operations on $M$. Thus, the differential geometry of $S$ will be mirrored in $M$. Geroch gives the curvature tensor and geometric properties of $S$ and finally the field equations on $S$, which is equivalent to the vacuum Einstein equation on $M$. If the Killing vector field $\xi^a$ is hypersurface orthogonal, there is a well-known conclusion that 3+1 gravity is equivalent to 2+1 gravity coupled to a massless scalar field.

Geroch’s work concerned only the reduction of the equation of motion. In this paper, we will study the Killing reduction from the viewpoint of variation principle. It turns out that, if one starts with the symmetry-reduced Hilbert

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action, the 4-metric components rather than the variables of Geroch have to be regarded as the arguments in order to get the correct reduced fields equations. An alternative action in the arguments of Geroch’s 3-dimensional fields is also obtained, which gives the same field equations as Geroch’s on $S$. If we confined the configuration to static space-time, both actions are reduced to that of 3-dimensional Euclidean gravity coupled to a massless Klein-Gordon field, which is consistent with the reduced action in [4].

2 Killing reduction of space-time

Let $(M, g_{ab})$ be a space-time with Killing vector field $\xi^a$, which is either everywhere timelike or everywhere spacelike. Let $S$ denote the collection of all trajectories of $\xi^a$. An element of $S$ is an inextendible curve in $M$ which is everywhere tangent to $\xi^a$.

A mapping $\psi$ from $M$ to $S$ can be defined as follows: For each point $p$ of $M$, $\psi(p)$ is the trajectory of $\xi^a$ passing through $p$. Assume $S$ is given the structure of a differentiable 3-manifold such that $\psi$ is a smooth mapping.

If the Killing field $\xi^a$ were hypersurface orthogonal, it would be possible to represent $S$ as one of the hypersurfaces in $M$ which is everywhere orthogonal to $\xi^a$. Each trajectory of $\xi^a$ would intersect this hypersurface in exactly one point.

In the non-hypersurface-orthogonal case, it is most natural to regard $S$ as a quotient space of $M$ rather than a subspace. Geroch shows that there is a one-to-one correspondence between tensor fields $T^{b_{\cdots d}}_{a_{\cdots c}}$ on $S$ and tensor fields $T^{b_{\cdots d}}_{a_{\cdots c}}$ on $M$ which satisfy

$$\xi^a T^{b_{\cdots d}}_{a_{\cdots c}} = 0, \quad \cdots, \quad \xi^d T^{b_{\cdots d}}_{a_{\cdots c}} = 0,$$

$$\mathcal{L}_\xi T^{b_{\cdots c}}_{a_{\cdots d}} = 0.$$ (1)

The entire tensor algebra on $S$ is completely and uniquely mirrored by tensors on $M$ subject to (1). So the primes will be dropped hereafter: Speaking of tensor fields being on $S$ is merely as a shorthand way of saying that the field on $M$ satisfies (1).

The metric and inverse metric on $S$ are defined as

$$h_{ab} = g_{ab} - \lambda^{-1} \xi_a \xi_b,$$ (2)

$$h^{ab} = g^{ab} - \lambda^{-1} \xi^a \xi^b,$$ (3)

$$h^a_b = \delta^a_b - \lambda^{-1} \xi_a \xi^b,$$ (4)

where $\lambda$ is the norm of Killing field $\xi^a$, i.e.,

$$\lambda = \xi^m \xi_m.$$ (5)

Note that the indices of any tensor on $S$ can be raised or lowered with either $h$ or $g$ with the same result.
The covariant derivative operator on $S$ is defined as
\[ D_{c}T_{a...c}^{b...d} = h_{a}^{m}...h_{c}^{n}h_{r}^{b}...h_{s}^{d}h_{p}^{v}\nabla_{p}T_{m...n}^{r...s}, \tag{6} \]
which satisfies all the conditions of derivative operator and
\[ D_{c}h_{ab} = 0. \tag{7} \]

The twist of $\xi^{a}$ is defined by
\[ \omega_{a} = \epsilon_{abcd}\xi^{b}\nabla^{c}\xi^{d}. \tag{8} \]

The Ricci tensor $R_{ab}$ of $h_{ab}$ on $S$ is related to that of $g_{ab}$ on $M$ by
\[ R_{ab} = \frac{1}{2}\lambda^{-2}(\omega_{a}\omega_{b} - h_{ab}\omega^{c}\omega_{c}) + \frac{1}{2}\lambda^{-1}D_{a}D_{b}\lambda - \frac{1}{4}\lambda^{-2}(D_{a}\lambda)(D_{b}\lambda) + h_{a}^{c}h_{b}^{d}R_{cd}. \tag{9} \]

Contracting Eq. (9) we get the relation between the scalar curvatures $R$ of $h_{ab}$ and $R$ of $g_{ab}$ as
\[ R = -\frac{1}{2}\lambda^{-2}\omega_{a}\omega_{b}h^{ab} + \lambda^{-1}h^{ab}D_{a}D_{b}\lambda - \frac{1}{4}\lambda^{-2}h^{ab}(D_{a}\lambda)(D_{b}\lambda) + R. \tag{10} \]

As shown in [3], in the source-free case (the 4-dimensional Ricci tensor $R_{ab} = 0$), $\omega_{a}$ is a gradient:
\[ \omega_{a} = D_{a}\omega. \tag{11} \]

The Ricci tensor $\mathcal{R}_{ab}$ on $S$ then takes the form
\[ \mathcal{R}_{ab} = \frac{1}{2}\lambda^{-2}[D_{a}\omega](D_{b}\omega) - h_{ab}(D^{m}\omega)(D_{m}\omega)] + \frac{1}{2}\lambda^{-1}D_{a}D_{b}\lambda - \frac{1}{4}\lambda^{-2}(D_{a}\lambda)(D_{b}\lambda). \tag{12} \]

and the scalar fields on $S$ satisfy
\[ D^{2}\lambda = \frac{1}{2}\lambda^{-1}(D^{m}\lambda)(D_{m}\lambda) - \lambda^{-1}(D^{m}\omega)(D_{m}\omega), \tag{13} \]
\[ D^{2}\omega = \frac{3}{2}\lambda^{-1}(D^{m}\lambda)(D_{m}\omega). \tag{14} \]

Eqs. (12), (13) and (14) are equivalent to the vacuum Einstein’s equation of $g_{ab}$.
3 Alternative actions on $S$

3.1 The symmetry-reduced Hilbert action

The Einstein-Hilbert action on $M$ is defined by

$$S_{EH}[g^{ab}] = \int_M \mathcal{L}_G = \int_M \sqrt{-g} R,$$  \hspace{1cm} (15)

where $g$ is the determinant of 4-metric components $g_{\mu\nu}$ in some basis on $M$.

Suppose the Killing vector field is everywhere timelike. For practical calculations, it is convenient to take a coordinate system adapted to the congruence:

$$t^a = \left(\frac{\partial}{\partial t}\right)^a = \xi^a. \hspace{1cm} (16)$$

Then the line element of $g_{ab}$ on $M$ can be written as

$$ds^2 = h_{\mu\nu} dx^\mu dx^\nu + \lambda (dt + B_\mu dx^\mu)^2. \hspace{1cm} (17)$$

The components of metric are

$$g_{\mu\nu} = h_{\mu\nu} + \lambda^{-1} \xi_\mu \xi_\nu, \quad g_{\mu 4} = \xi_\mu = \lambda B_\mu, \quad g_{44} = \xi_4 = \lambda. \hspace{1cm} (18)$$

We also have

$$2B_{[\mu,\nu]} = |\lambda|^{-3/2} \epsilon_{\mu\rho\sigma} \omega^\rho, \quad \epsilon_{\mu\rho\sigma} = |\lambda|^{-1/2} \epsilon_{\mu\nu\rho\sigma} \xi_\sigma. \hspace{1cm} (19)$$

Let $h$ denote the determinant of $h_{\mu\nu}$ on $S$. Straightforward calculations give

$$g = \lambda h. \hspace{1cm} (20)$$

This formula is still valid in the case where $\xi^a$ is everywhere spacelike.

Since the principle of symmetric criticality is valid in the one Killing vector model of general relativity [6], one expects that the 3-dimensional fields Eqs. (12), (13) and (14) could be obtained by the variations of the action from symmetric reduction of (15), which can be read from Eq. (10) as

$$S_{EH} = \int_S |\lambda|^{1/2} \sqrt{|h|} (\mathcal{R} + \frac{1}{2} \lambda^{-2} \omega_a \omega_b h^{ab}), \hspace{1cm} (21)$$

where a 3-dimensional total divergence term has been neglected. This reduced action cannot lead to the correct reduced fields equations by variations with respect to the Geroch’s variables $(h^{ab}, \lambda, \omega_a)$. For instance, the variation of (21) with respect to $\omega_a$ leads to $\omega_a = 0$, which is not the case in general. This is not a very surprising result since the relation of $\omega_a$ and the 4-metric components is rather nontrivial, as one can see from Eq. (19). Although the variation of Hilbert action with respect to the 4-metric gives certainly the Einstein equation, there is no guarantee to get the same equation if one varies it with respect to
other variables arbitrarily concocted such as $\omega_a$. Eq. (19) suggests a one-form $A_a$ at least locally on $S$ such that

$$(dA)_{ab} = |\lambda|^{-3/2} \varepsilon_{bac} \omega_c.$$  

(22)

Substituting $A_a$ for $\omega_a$ in Eq. (21), we get the familiar form of the symmetry-reduced action

$$S_{EH}[h^{ab}, A_c, \lambda] = \int_S |\lambda|^{1/2} \sqrt{|h|} [\mathcal{R} - \frac{1}{4} \lambda h^{ab} h^{cd} (dA)_{ac}(dA)_{bd}].$$  

(23)

The variations of this action with respect to $(h^{ab}, A_c, \lambda)$ will give the correct reduced fields equations on $S$, that can be checked directly by substituting $\omega_a$ back for $A_a$ after variations.

### 3.2 Hypersurface orthogonal case

We now consider a special case where the Killing field $\xi^a$ is hypersurface orthogonal. It is then possible to represent $S$ as one of the hypersurfaces in $M$ which is everywhere orthogonal to $\xi^a$. The symmetry-reduced Einstein-Hilbert action on $S$ reads

$$S_{EH}[h^{ab}] = \int_S \mathcal{L}_G = \int_S |\lambda|^{1/2} \sqrt{|h|} \mathcal{R}. \quad (24)$$

To obtain regular form of action on $S$, we now conformally transform $h_{ab}$ as $\tilde{h}_{ab}$:

$$\tilde{h}_{ab} = \Omega^{-2} h_{ab}. \quad (25)$$

Let $\tilde{D}_a$ be the covariant derivative operator determined by metric $\tilde{h}_{ab}$, i.e.,

$$\tilde{D}_a \tilde{h}_{bc} = 0. \quad (26)$$

$\tilde{D}_a$ is related to $D_a$ by

$$D_a v_b = \tilde{D}_a v_b - C^c_{ab} v_c, \quad (27)$$

where

$$C^c_{ab} = \frac{1}{2} h^{cd} (\tilde{D}_a h_{bd} + \tilde{D}_b h_{ad} - \tilde{D}_d h_{ab}) = \tilde{h}^c_{\alpha} \tilde{D}_a \ln \Omega + \tilde{h}^c_{\beta} \tilde{D}_b \ln \Omega - \tilde{h}_{ab} \tilde{h}^{cd} \tilde{D}_d \ln \Omega. \quad (28)$$

After the conformal transform, the form of action (24) becomes

$$S_{EH}[\tilde{h}^{ab}] = \int_S |\lambda|^{1/2} \sqrt{\Omega^6 |\tilde{h}|} \Omega^{-2} \left[ \tilde{\mathcal{R}} - 4 \tilde{h}^{ab} \tilde{D}_a \tilde{D}_b \ln \Omega - 2 \tilde{h}^{ab} (\tilde{D}_a \ln \Omega)(\tilde{D}_b \ln \Omega) \right]. \quad (29)$$

Let

$$\Omega = |\lambda|^{-1/2}, \quad (30)$$

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then

\[ S_{EH}[\tilde{h}^{ab}] = \int_S \sqrt{\tilde{h}} \left[ \tilde{\mathcal{R}} - 4\tilde{h}^{ab}\tilde{D}_a\tilde{D}_b \ln \Omega - 2\tilde{h}^{ab}(\tilde{D}_a \ln \Omega)(\tilde{D}_b \ln \Omega) \right]. \]  

Note that the second term in (31) is a total divergence term that can be ignored for variations. Let

\[ \Lambda \equiv \sqrt{2 \ln \Omega}, \]

straightforward calculations of variations show that the action (31) gives the same equations of motion as those of 3-gravity \( \tilde{h}^{ab} \) coupled to a massless Klein-Gordon field \( \Lambda \), which is defined by the coupled action:

\[ S_E + S_{KG} = \int_S \sqrt{\tilde{h}} \left[ \tilde{\mathcal{R}} - \tilde{h}^{ab}(\partial_a \Lambda)(\partial_b \Lambda) \right]. \]

Therefore, a 4-dimensional space-time with a hypersurface orthogonal Killing vector field which is either everywhere timelike or everywhere spacelike, is “conformally” equivalent to 3-dimensional gravity coupled to a massless scalar field. Static space-time is of course a typical case [3].

### 3.3 An alternative action

In Geroch’s reduction, the 4-dimensional Einstein vacuum \( (M, g^{ab}) \) reduces in general to a 3-dimensional gravity coupled to two scalar fields on \( S \) satisfying Eqs. \( (12), (13) \) and \( (14) \). One may ask if there is an action in the arguments of Geroch’s variables \( \omega \) and \( \lambda \) on \( S \), whose variations give the above Geroch’s equations. Note that in hypersurface orthogonal case, besides \( \tilde{h}^{ab} \) only one scalar field \( \lambda \) exists on \( S \). Up to a boundary term the action (31) can be written as

\[ S_{EH}[\tilde{h}^{ab}] = \int_S \sqrt{\tilde{h}} \left[ \tilde{\mathcal{R}} - \frac{1}{2} \lambda^{-2}\tilde{h}^{ab}(\tilde{D}_a \lambda)(\tilde{D}_b \lambda) \right]. \]

While in general case, another scalar field \( \omega \) exists on \( S \). By carefully observing Geroch’s equations and the special action (34), we propose an action for the reduced fields equations on \( S \) as

\[ S[\tilde{h}^{ab}] = \int_S \sqrt{|\tilde{h}|} \left[ \tilde{\mathcal{R}} - \frac{1}{2} \lambda^{-2}\tilde{h}^{ab}(\tilde{D}_a \lambda)(\tilde{D}_b \lambda) - \frac{1}{2} \lambda^{-2}\tilde{h}^{ab}(\tilde{D}_a \omega)(\tilde{D}_b \omega) \right]. \]

We now show that the variations of (35) give exactly the same field equations as those of Geroch’s work.

The variation of (35) with respect to \( \tilde{h}^{ab} \) gives

\[ \delta S = \int_S \sqrt{|\tilde{h}|} \left[ \tilde{\mathcal{R}}_{ab} - \frac{1}{2} \tilde{\mathcal{R}} \tilde{h}_{ab} + \frac{1}{4} \lambda^{-2}\tilde{h}_{ab}\tilde{h}^{cd}(\tilde{D}_c \lambda)(\tilde{D}_d \lambda) - \frac{1}{2} \lambda^{-2}(\tilde{D}_a \lambda)(\tilde{D}_b \lambda) 
+ \frac{1}{4} \lambda^{-2}\tilde{h}_{ab}\tilde{h}^{cd}(\tilde{D}_c \omega)(\tilde{D}_d \omega) \right] \delta \tilde{h}^{ab}. \]
The vanishing of (36) gives the Einstein field equation of gravity with sources on $S$ as:

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{R}h_{ab} = \frac{1}{2}\lambda^{-2}(\tilde{D}_a\lambda)(\tilde{D}_b\lambda) - \frac{1}{4}\lambda^{-2}\tilde{h}_{ab}\tilde{h}^{cd}(\tilde{D}_c\lambda)(\tilde{D}_d\lambda) + \frac{1}{2}\lambda^{-2}(\tilde{D}_a\omega)(\tilde{D}_b\omega) - \frac{1}{4}\lambda^{-2}\tilde{h}_{ab}\tilde{h}^{cd}(\tilde{D}_c\omega)(\tilde{D}_d\omega),$$

(37)

or

$$\tilde{R}_{ab} = \frac{1}{2}\lambda^{-2}(\tilde{D}_a\lambda)(\tilde{D}_b\lambda) + \frac{1}{2}\lambda^{-2}(\tilde{D}_a\omega)(\tilde{D}_b\omega).$$

(38)

Likewise the variation principle gives the equations of motion of the matter fields $\lambda$ and $\omega$ determined by action (35) respectively as:

$$\tilde{h}_{ab}\tilde{D}_a\tilde{D}_b\lambda = \lambda^{-1}\tilde{h}_{ab}(\tilde{D}_a\lambda)(\tilde{D}_b\lambda) - \lambda^{-1}\tilde{h}_{ab}(\tilde{D}_a\omega)(\tilde{D}_b\omega),$$

(39)

and

$$\tilde{h}_{ab}\tilde{D}_a\tilde{D}_b\omega = 2\lambda^{-1}\tilde{h}_{ab}(\tilde{D}_a\lambda)(\tilde{D}_b\omega).$$

(40)

To compare with Geroch’s work [3], we make the conformal transformation (25) inversely. From (27), (28) and (30), we have

$$\tilde{h}_{ab}\tilde{D}_a\tilde{D}_b\lambda = \Omega^2 [h_{ab}D_aD_b\lambda - h_{ab}(D_a\lambda)(D_b\ln\Omega)]$$

$$= \Omega^2 \left[ D^2\lambda + \frac{1}{2}\lambda^{-1}h_{ab}(D_a\lambda)(D_b\lambda) \right],$$

(41)

and

$$\tilde{h}_{ab}\tilde{D}_a\tilde{D}_b\omega = \Omega^2 [h_{ab}D_aD_b\omega - h_{ab}(D_a\omega)(D_b\ln\Omega)]$$

$$= \Omega^2 \left[ D^2\omega + \frac{1}{2}\lambda^{-1}h_{ab}(D_a\omega)(D_b\lambda) \right].$$

(42)

Substituting (41) into (39), one gets equation (13). Substituting (42) into (40), one gets equation (14).

The relation between $R_{ab}$ and $\tilde{R}_{ab}$ reads (see the appendix D of [7])

$$R_{ab} = \tilde{R}_{ab} - D_aD_b\ln\Omega - h_{ab}h^{cd}D_cD_d\ln\Omega$$

$$-(D_a\ln\Omega)(D_b\ln\Omega) + h_{ab}h^{cd}(D_c\ln\Omega)(D_d\ln\Omega).$$

(43)

Substituting (43) into (38), we obtain

$$R_{ab} = -\frac{1}{4}\lambda^{-2}(D_a\lambda)(D_b\lambda) + \frac{1}{2}\lambda^{-2}(D_a\omega)(D_b\omega)$$

$$+ \frac{1}{2}\lambda^{-1}D_aD_b\lambda - \frac{1}{4}\lambda^{-2}h_{ab}(D^m\lambda)(D_m\lambda) + \frac{1}{2}\lambda^{-1}h_{ab}D^2\lambda.$$  

(44)

Using (12), one gets equation (12).

In conclusion, the variations of the symmetry-reduced Hilbert action [21] with respect to Geroch’s variables on $S$ cannot lead to the correct reduced
field equations. While, the variations of action $35$ defined on $S$ give the field equations for 3-dimensional gravity coupled to two scalar fields, which are “conformally” equivalent to Geroch’s equations reduced from the vacuum Einstein equation on $M$. In this sense, we may regard the action $35$ as an alternative action for vacuum Einstein’s equation with a Killing symmetry. Geroch’s equations can thus be obtained from the variation principle.

The above discussion could be generalized to higher dimensional case $8$, and hence might lead some new sight on the mechanism of dimensional reduction.

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