Formulas for the Walsh coefficients of smooth functions and their application to bounds on the Walsh coefficients

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Abstract

We establish formulas for the $b$-adic Walsh coefficients of functions in $C^\alpha[0,1]$ for an integer $\alpha \geq 1$ and give upper bounds on the Walsh coefficients of these functions. We also study the Walsh coefficients of functions in periodic and non-periodic reproducing kernel Hilbert spaces.

Keywords: Walsh series, Walsh coefficient, Sobolev space, smooth function

1. Introduction

The Walsh coefficients are the generalized Fourier coefficients for the Walsh system. The Walsh analysis has been used for numerical integration, see the comprehensive book [8] and references therein. In particular, the decay of the Walsh coefficients of smooth functions was considered and used to give explicit constructions of quasi-Monte Carlo rules which achieve the optimal rate of convergence in [5, 6]. In this paper, we also focus on the decay of the Walsh coefficients of smooth functions.

Throughout the paper we use the following notation: Assume that $b \geq 2$ is a positive integer. We assume that $k$ is a nonnegative integer whose $b$-adic expansion is $k = \kappa_1 b^{\alpha_1} + \cdots + \kappa_v b^{\alpha_v}$ where $\kappa_i$ and $\alpha_i$ are integers with $0 < \kappa_i \leq b - 1$, $\alpha_1 > \cdots > \alpha_v \geq 1$. For $k = 0$ we assume that $v = 0$ and $\alpha_0 = 0$. We denote by $\mathbb{N}_0$ the set of nonnegative integers. Let $\omega_b := \exp(2\pi \sqrt{-1}/b)$.

The Walsh functions were first introduced by Walsh [13], see also [3, 4]. For $k \in \mathbb{N}_0$, the $b$-adic $k$-th Walsh function $\text{wal}_k(x)$ is defined as

$$\text{wal}_k(x) := \omega^{\sum_{i=1}^v \kappa_i \xi_{a_i}}_b,$$

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for $x \in [0, 1)$ whose $b$-adic expansion is given by $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$, which is unique in the sense that infinitely many of the digits $\xi_i$ are different from $b-1$. We also consider $s$-dimensional Walsh functions. For $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $x = (x_1, \ldots, x_s) \in [0, 1)^s$, the $b$-adic $k$-th Walsh function $\text{wal}_k(x)$ is defined as

$$\text{wal}_k(x) := \prod_{j=1}^s \text{wal}_{k_j}(x_j).$$

For $k \in \mathbb{N}_0^s$ and $f : [0, 1)^s \to \mathbb{C}$, we define the $k$-th Walsh coefficient of $f$ as

$$\hat{f}(k) := \int_{[0,1)^s} f(x) \text{wal}_k(x) \, dx.$$

It is well-known that the Walsh system $\{\text{wal}_k : k \in \mathbb{N}_0^s\}$ is a complete orthonormal system in $L^2[0,1)^s$ for any positive integer $s$ (for a proof, see e.g., [8, Theorem A.11]). Hence we have a Walsh series expansion

$$f(x) \sim \sum_{k \in \mathbb{N}_0^s} \hat{f}(k) \text{wal}_k(x)$$

for any $f \in L^2[0,1)^s$. Let $s = 1$ at this moment. It is known that if $f \in C^\infty[0,1]$ has bounded variation then $f$ is equal to its Walsh series expansion, that is, for all $x \in [0,1)$ we have $f(x) = \sum_{k \in \mathbb{N}_0} \hat{f}(k) \text{wal}_k(x)$, see [13]. More information of the Walsh analysis can be found in the books [11, 10].

There are several studies for the decay of the Walsh coefficients. Fine considered the Walsh coefficients of functions which satisfy a Hölder condition in [2]. Dick proved the decay of the Walsh coefficients of functions of smoothness $\alpha \geq 1$ in [3, 6] and studied it in more detail in [7]. It was proved that if a function $f$ has $\alpha - 1$ derivatives for which $f^{(\alpha-1)}$ satisfies a Lipschitz condition, then $|\hat{f}(k)| \in O(b^{-a_1 - \cdots - a_{\min(\alpha,v)}})$ [6]. Dick also proved that this order is the best possible. That is, for $f$ of smoothness $\alpha$, if there exists $1 \leq r \leq \alpha$ such that $\hat{f}(k)$ decays faster than $b^{-a_1 - \cdots - a_r}$ for all $k \in \mathbb{N}_0$ and $v \geq r$, then $f$ is a polynomial of degree at most $r - 1$ [7, Theorem 20].

Recently, Yoshiki gave a method to analyze the dyadic (i.e., 2-adic) Walsh coefficients in [14]. He introduced dyadic differences of (maybe discontinuous) functions and gave a formula in which the dyadic Walsh coefficients are given by dyadic differences multiplied by constants. Dyadic differences of a smooth function are expressed in terms of derivatives of the function. This enabled him to establish a formula for the dyadic Walsh coefficients of smooth functions expressed in terms of those derivatives. From this formula, he obtained a bound on the dyadic Walsh coefficients for $\alpha$ times continuously differentiable functions for $\alpha \geq 1$.

In this paper, we establish a formula in which the $b$-adic Walsh coefficients of smooth functions are expressed in terms of those derivatives as

$$\hat{f}(k) = (-1)^v \int_0^1 f^{(v)}(x) W(k)(x) \, dx,$$
where the function $W(k)(\cdot) : [0, 1) \to \mathbb{C}$ is given by the iterated integral of Walsh functions as in Definition 2.1. This formula is a generalization of the formula for the dyadic Walsh coefficients of smooth functions in [14], however our method is different from that in [14]. Our main idea is first separating the interval $[0,1)$ to appropriate intervals on which particular Walsh functions take constant values, and then applying integration by parts iteratively. We also establish another formula for the Walsh coefficients to use all of the smoothness of functions.

Furthermore, we give bounds on the $b$-adic Walsh coefficients for $\alpha$ times continuously differentiable functions. Our assumption is somewhat stronger than that of [7]. Instead, we obtain bounds asymptotically better with respect to $\alpha$ than results in [7]. Our bounds for the dyadic case recover results for smooth functions in [14]. Moreover, we obtain a class of infinitely smooth functions whose Walsh coefficients decay as $|\hat{f}(k)| \in O(b^{-\alpha_1-\cdots-\alpha_v})$. We also obtain improved bounds on the Walsh coefficients for functions in periodic and non-periodic reproducing kernel Hilbert spaces which are considered in [7].

The rest of the paper is organized as follows. We give two formulas for the Walsh coefficients of smooth functions in Sections 2 and 4. Bounds on the Walsh coefficients of smooth functions and Bernoulli polynomials are given in Sections 3 and 5, respectively. In Section 6 (resp. Section 7), we give a bound on the Walsh coefficients of functions in non-periodic (resp. periodic) reproducing kernel Hilbert spaces.

2. Integral formula for the Walsh coefficients of smooth functions

We introduce further notation which is used throughout the paper. For $k > 0$, let $k' = k - \kappa_v b^{a_v-1}$. Let $v(k) := v$ be the number of non-zero digits of $k$.

In this section, we define the function $W(k)(\cdot)$ and establish a formula in which the Walsh coefficients of smooth functions are expressed in terms of $W(k)(\cdot)$ and derivatives of the functions.

**Definition 2.1.** For $k \in \mathbb{N}_0$, we define functions $W(k)(\cdot) : [0, 1] \to \mathbb{C}$ recursively as

$$W(0)(x) := 1,$$

$$W(k)(x) := \int_0^x \text{wal}_{\kappa_v, b^{a_v-1}}(y) W(k')(y) \, dy,$$

and the integral value of $W(k)(\cdot)$ as

$$I(k) := \int_0^1 W(k)(x) \, dx.$$

By definition, $W(k)(x)$ is continuous for all $k \in \mathbb{N}_0$. Note that we have

$$W(k)(x) = \int_0^x W(k')(y) \, dy \quad \text{for } x \in [0, b^{-a_v}]$$
Lemma 2.2. Let \( k \in \mathbb{N}_0 \). Let \( x \in [0, 1) \) and \( x = cb^{-a_v} + x' \), where \( 0 \leq c < b^{a_v} \) is an integer and \( 0 \leq x' < b^{-a_v} \) is a real number. Then we have

\[
W(k)(x) = \frac{1 - \omega_b^{\kappa_{a_v}}}{1 - \omega_b^{\kappa_{a_v}}} W(k)(b^{-a_v}) + \omega_b^{\kappa_{a_v}} W(k)(x').
\]

In particular, \( W(k)(\cdot) \) is a periodic function with period \( b^{-a_v+1} \) if \( v > 0 \).

Proof. We prove the lemma by induction on \( x \in \mathbb{N}_0 \). We have \( \omega_b^{\kappa_{a_v}} = 1 \) for all \( y \in [0, b^{-a_v}) \). We show the periodicity of \( W(k)(\cdot) \) in the next lemma.

Theorem 2.3. Let \( k \in \mathbb{N}_0 \). Assume that \( f \in C^\alpha[0, 1] \) for a positive integer \( \alpha \). Then for an integer \( 0 \leq n \leq \min(\alpha, v) \) we have

\[
\hat{f}(k) = (-1)^n \int_0^1 f^{(n)}(x) \text{wal}_{b^{a_v}}(x) W(k_{a_v}^n)(x) \, dx.
\]

Proof. We prove the formula by induction on \( n \). For \( n = 0 \), the result holds by the definition of the Walsh coefficients. Hence assume now that \( n > 0 \) and that the result holds for \( n - 1 \). We have \( \text{wal}_{k_{a_v}^{n-1}}(x) = \text{wal}_{k_{a_v}^n}(x) \text{wal}_{\kappa_{a_v}^{n-1}}(x) \) for all \( x \in [0, 1) \) and

\[
\text{wal}_{k_{a_v}^n}(x) = \text{wal}_{k_{a_v}^n}(ib^{-a_v+1}) \quad \text{for} \quad x \in [ib^{-a_v+1}, (i+1)b^{-a_v+1})
\]
for each integer \(0 \leq i < b^{-n+1}\). Hence we have

\[
\hat{f}(k) = (-1)^{n-1} \int_0^1 f^{(n-1)}(x) \text{wal}_{k-n-1}^{-1}(x) W(k_{\leq}^{n-1})(x) \, dx
\]

\[
= (-1)^{n-1} \sum_{i=0}^{b^{-n+1}-1} \text{wal}_{k-n-1}^{-1}(i b^{-n+1}) \times \\
\int_{ib^{-n+1}}^{(i+1)b^{-n+1}} f^{(n-1)}(x) \text{wal}_{k-n-1}(x) W(k_{\leq}^{n-1})(x) \, dx
\]

\[
= (-1)^{n-1} \sum_{i=0}^{b^{-n+1}-1} \text{wal}_{k-n-1}^{-1}(i b^{-n+1}) \left( \int_{ib^{-n+1}}^{(i+1)b^{-n+1}} f^{(n)}(x) W(k_{\leq}^{n})(x) \, dx \right)
\]

\[
= (-1)^n \sum_{i=0}^{b^{-n+1}-1} \text{wal}_{k-n-1}^{-1}(i b^{-n+1}) \int_{ib^{-n+1}}^{(i+1)b^{-n+1}} f^{(n)}(x) W(k_{\leq}^{n})(x) \, dx
\]

where we use the induction assumption for \(n - 1\) for the first equality and \(W(k_{\leq}^{n})(i b^{-n+1}) = W(k_{\leq}^{n})(i+1)b^{-n+1}) = 0\) by Lemma 2.2 for the fourth equality, respectively. This proves the result for \(n\).

Now we consider the \(s\)-variate case. For a function \(f: [0,1]^s \to \mathbb{R}\), let \(f^{(n_1,\ldots,n_s)} := (\partial/\partial x_1)^{n_1} \cdots (\partial/\partial x_s)^{n_s} f\) be the \((n_1,\ldots,n_s)\)-th derivative of \(f\). Considering coordinate-wise integration, we have the following.

**Theorem 2.4.** Let \(k = (k_1,\ldots,k_s) \in \mathbb{N}_0^s\). Assume that \(f: [0,1]^s \to \mathbb{R}\) has continuous mixed partial derivatives up to order \(\alpha_j\) in each variable \(x_j\). Let \(n_j\) be integers with \(0 \leq n_j \leq \min(\alpha_j,v(k_j))\) for \(1 \leq j \leq s\). Then we have

\[
\hat{f}(k) = (-1)^{n_1+\cdots+n_s} \int_{[0,1]^s} f^{(n_1,\ldots,n_s)}(x) \prod_{j=1}^s \text{wal}_{k_j-n_j}(x_j) W(k_{\leq}^{n_j})(x_j) \, dx.
\]

### 3. The Walsh coefficients of smooth functions

Let \(f \in C^\alpha[0,1]\) and \(p,q \in [1,\infty]\) with \(1/p + 1/q = 1\). By Theorem 2.3 for \(n = \min(\alpha,v)\) and Hölder’s inequality, we have

\[
|\hat{f}(k)| \leq \int_0^1 \left| f^{(\min(\alpha,v))}(x) \text{wal}_{k_{\leq}^{\min(\alpha,v)}}(x) W(k_{\leq}^{\min(\alpha,v)})(x) \right| \, dx
\]

\[
\leq \|f^{(\min(\alpha,v))}\|_{L^p} \|W(k_{\leq}^{\min(\alpha,v)})(\cdot)\|_{L^q}.
\]
Thus, it suffices to bound $\|W(k_x^v)(\cdot)\|_{L^q}$ to bound $|\hat{f}(k)|$. We give bounds on $\|W(k_x^v)(\cdot)\|_{L^\infty}$ for the non-dyadic case, $\|W(k_x^v)(\cdot)\|_{L^q}$ for the dyadic case and $|\hat{f}(k)|$ in Sections 3.1, 3.2 and 3.3, respectively.

We introduce a function $\mu$ as follows. For $k \in \mathbb{N}_0$, we define

$$
\mu(k) := \begin{cases} 
0 & \text{for } k = 0, \\
\alpha_1 + \cdots + \alpha_v & \text{for } k \neq 0.
\end{cases}
$$

For $k = (k_1, \cdots, k_s) \in \mathbb{N}_0^s$, we define $\mu(k) := \sum_{j=1}^s \mu(k_j)$.

For subsequent analysis, we give the exact values of $I(k)$ and $W(k)(b^{-a_v})$ in the next lemma.

**Lemma 3.1.** For $k \in \mathbb{N}_0$, we have the following.

(i) $I(k) = \prod_{i=1}^v \frac{b^{-\mu(k)}}{1 - \omega_b^{\alpha_i}}$,

(ii) $W(k)(b^{-a_v}) = \prod_{i=1}^{v-1} \frac{b^{-\mu(k)}}{1 - \omega_b^{\alpha_i}}$.

(iii) Let $x \in [0, 1)$ and $x = cb^{-a_v} + x'$ where $0 \leq c < b^{a_v}$ is an integer and $0 \leq x' < b^{-a_v}$ is a real number. Then we have

$$
W(k)(x) = (1 - \omega_b^{\alpha_v})I(k) + \omega_b^{\alpha_v}W(k)(x').
$$

Here, the empty products $\prod_{i=1}^0$ and $\prod_{i=1}^{v-1}$ are defined to be 1.

**Proof.** By Lemma 2.2, we have

\[
I(k) = \sum_{i=0}^{b^{a_v}-1} \int_{cb^{-a_v}}^{(i+1)b^{-a_v}} W(k)(x) \, dx
= \sum_{i=0}^{b^{a_v}-1} \int_0^{b^{-a_v}} \left( \frac{1 - \omega_b^{\alpha_v}}{1 - \omega_b^{\alpha_v}} W(k)(b^{-a_v}) + \omega_b^{\alpha_v}W(k)(x) \right) \, dx
= W(k)(b^{-a_v}) \frac{b^{a_v}-1}{1 - \omega_b^{\alpha_v}} \sum_{i=0}^{b^{a_v}-1} (1 - \omega_b^{\alpha_v}) + \sum_{i=0}^{b^{a_v}-1} \omega_b^{\alpha_v} \int_0^{b^{-a_v}} W(k)(x) \, dx
= W(k)(b^{-a_v}) \frac{b^{a_v}-1}{1 - \omega_b^{\alpha_v}},
\]

Furthermore, $W(k)(b^{-a_v})$ is computed as

\[
W(k)(b^{-a_v}) = \int_0^{b^{-a_v}} W(k')(x) \, dx
= b^{-a_v}I(k'),
\]

where we use the fact that $W(k')(\cdot)$ is periodic with period $b^{-a_v}$, which follows from Lemma 2.2 in the last equality. Using equations (3) and (4) iteratively, we have (i) and (ii). Combining (4) and Lemma 2.2 we have (iii).
In the following, we consider two cases in order to bound $\|W(k)(\cdot)\|_{L^\infty}$: the non-dyadic case and the dyadic case. We define two positive constants $m_b$ and $M_b$ as

$$m_b := \min_{c=1,2,...,b-1} |1 - \omega^c_b| = 2\sin(\pi/b),$$

$$M_b := \max_{c=1,2,...,b-1} |1 - \omega^c_b| = \begin{cases} 2 & \text{if } b \text{ is even}, \\ 2\sin((b + 1)\pi/2b) & \text{if } b \text{ is odd}. \end{cases}$$

### 3.1. Non-dyadic case

The following lemmas are needed to bound $\sup_{x' \in [0, b-a v]} |W(k)(x')|$. 

**Lemma 3.2.** Let $A, B$ be complex numbers and $r$ be a positive real number. Then we have

$$\sup_{x \in [0, r]} |Ax + B| = \max(|B|, |rA + B|).$$

**Proof.** We have

$$\sup_{x \in [0, r]} |Ax + B| = \sqrt{\sup_{x \in [0, r]} |Ax + B|^2} = \sup_{x \in [0, r]} (|A|^2 x^2 + 2\Re(AB)x + |B|^2).$$

Since $|A|^2 x^2 + 2\Re(AB)x + |B|^2$ is a convex function on $[0, r]$, its maximum value occurs at its endpoints. □

**Lemma 3.3.** Let $a$ and $1 \leq \kappa \leq b - 1$ be positive integers. Then we have

$$\sup_{c' = 0, 1, ..., ab} \left| \sum_{i=0}^{c'-1} (1 - \omega^i_b) \right| \leq ab.$$}

**Proof.** Since $\sum_{c=0}^{ab-1} \omega^{c\kappa}_b = 0$, we have

$$\sup_{c' = 0, 1, ..., ab} \left| \sum_{i=0}^{c'-1} (1 - \omega^i_b) \right| = \sup_{c' = 0, 1, ..., ab} \left| c' + \sum_{i=c'}^{ab-1} \omega^i_b \right| \leq \sup_{c' = 0, 1, ..., ab} \left( c' + \sum_{i=c'}^{ab-1} |\omega^i_b| \right) = ab.$$ □

We now have an upper bound on $\sup_{x' \in [0, b-a v]} |W(k)(x')|$. 

**Lemma 3.4.** Let $k$ be a positive integer. If $b > 2$, then we have

$$\sup_{x' \in [0, b-a v]} |W(k)(x')| \leq \frac{b(2)^k - 1}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right).$$
Proof. We prove the lemma by induction on $v$. If $v = 1$, we have
\[
\sup_{x' \in [0, b^{-a_1}]} |W(k)(x')| = \sup_{x' \in [0, b^{-a_1}]} \left| \int_0^{x'} W(0)(y) \, dy \right| = \sup_{x' \in [0, b^{-a_1}]} |x'| = b^{-a_1} = b^{-\mu(k)}.
\]
Hence the lemma holds for $v = 1$.

Thus assume now that $v > 1$ and that the result holds for $v - 1$. Let $x' \in [0, b^{-a_v}]$ be a real number and $x' = c'b^{-a_{v-1}} + x''$ where $0 \leq c' < b^{-a_v+a_{v-1}}$ is an integer and $0 \leq x'' < b^{-a_{v-1}}$ is a real number. Then by Lemma 3.1 (iii) we have
\[
|W(k)(x')| = \left| \int_0^{x'} W(k')(y) \, dy \right| = \sum_{i=0}^{c'-1} \int_0^{b^{-a_{v-1}}} \left( (1 - \omega_b^{\iota_{k-1}})I(k') + \omega_b^{\iota_{k-1}} W(k')(y) \right) \, dy + \int_0^{x''} \left( (1 - \omega_b^{\iota_{k-1}})I(k') + \omega_b^{\iota_{k-1}} W(k')(y) \right) \, dy \\
\leq b^{-a_{v-1}} \sum_{i=0}^{c'-1} \left( (1 - \omega_b^{\iota_{k-1}})I(k') + x''(1 - \omega_b^{\iota_{k-1}})I(k') \right) + \sum_{i=0}^{c'-1} \omega_b^{\iota_{k-1}} \int_0^{b^{-a_{v-1}}} W(k')(y) \, dy + \omega_b^{\iota_{k-1}} \int_0^{x''} W(k')(y) \, dy.
\]

We evaluate the supremum of the first term of (5). Note that the first term of (5) is equal to $|b^{-a_{v-1}} \sum_{i=0}^{c'-1} (1 - \omega_b^{\iota_{k-1}})I(k')|$ if $x'' = b^{-a_{v-1}}$. By Lemmas 3.2 [33] and 3.3 [33], we have
\[
\sup_{0 \leq c' < b^{-a_v+a_{v-1}}} \sup_{x'' \in [0, b^{-a_{v-1}}]} \left| b^{-a_{v-1}} \sum_{i=0}^{c'-1} (1 - \omega_b^{\iota_{k-1}})I(k') + x''(1 - \omega_b^{\iota_{k-1}})I(k') \right| \\
= \sup_{0 \leq c' < b^{-a_v+a_{v-1}}} \left| b^{-a_{v-1}} \sum_{i=0}^{c'-1} (1 - \omega_b^{\iota_{k-1}})I(k') \right| \\
\leq b^{-a_{v-1}} \frac{b^{-\mu(k)}}{m_b^{a_{v-1}}} b^{-a_v+a_{v-1}} \\
= \frac{b^{-\mu(k)}}{m_b^{a_{v-1}}}.
\]

8
We move on to the evaluation of the second term of (5). We have

\[
\sup_{c', x''} \left| \sum_{i=0}^{c'-1} \frac{\omega_{b_{i+1} - a_{v-1}}}{\omega_b} \int_0^{b_{i+1} - a_{v-1}} W(k')(y) \, dy + \frac{\omega_{b_{c'} - a_{v-1}}}{\omega_b} \int_0^{b_{c'} - a_{v-1}} W(k')(y) \, dy \right|
\]

\[
= \sup_{c', x''} \left| \sum_{i=0}^{c'-1} \frac{\omega_{b_{i+1} - a_{v-1}}}{\omega_b} \int_0^{b_{i+1} - a_{v-1}} W(k')(y) \, dy + \frac{\omega_{b_{c'+1} - a_{v-1}}}{\omega_b} \int_0^{b_{c'+1} - a_{v-1}} W(k')(y) \, dy \right|
\]

\[
\leq \sup_{x''} \frac{M_b}{m_b} \left| b^{-a_{v-1}} - x'' \right| + \frac{M_b}{m_b} x'' \cdot \sup_{y \in [0, b^{-a_{v-1}}]} |W(k')(y)|
\]

\[
\leq \frac{M_b}{m_b} b^{-a_{v-1}} \cdot \frac{b^{-\mu(k')}}{m_b^{v-1}} \cdot \frac{b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^{v-1} \right)
\]

\[
\leq \frac{b^{-\mu(k)}}{m_b^{v-1}} \frac{M_b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right),
\]

where we use the induction assumption for \( v - 1 \) in the forth inequality and \( b \cdot b^{-a_{v-1}} \leq b^{-a_v} \) in the last inequality.

By summing up the bounds obtained on each term of (5), we have

\[
\sup_{x' \in [0, b^{-a_v}]} |W(k)(x')| \leq \frac{b^{-\mu(k)}}{m_b^{v-1}} \frac{M_b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right)
\]

\[
= \frac{b^{-\mu(k)}}{m_b^{v-1}} \frac{b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right).
\]

Using the above lemma, we obtain an upper bound on \( \|W(k)\|_{L^\infty} \).

**Proposition 3.5.** Let \( k \in \mathbb{N}_0 \). If \( b > 2 \), we have

\[
\|W(k)(\cdot)\|_{L^\infty} \leq \frac{b^{-\mu(k)}}{m_b^{v-1}} \left( M_b + \frac{b m_b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right) \right)^{\min(1, v)}.
\]

**Proof.** The case \( k = 0 \) is obvious. We assume that \( k > 0 \). Let \( x \in [0, 1] \) and \( x = cb^{-a_v} + x' \), where \( 0 \leq c < b^{a_v} \) is an integer and \( 0 \leq x' < b^{-a_v} \) is a real number. By Lemmas 3.1 and 3.3, we have

\[
|W(k)(x)| = \left| (1 - \alpha_{c=k}^m) I(k) + \alpha_{c=k}^m W(k)(x') \right|
\]

\[
\leq M_b |I(k)| + \sup_{x' \in [0, b^{-a_v}]} |W(k)(x')|
\]

\[
\leq \frac{b^{-\mu(k)}}{m_b^{v-1}} \left( M_b + \frac{b m_b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right) \right)^{\min(1, v)},
\]

which proves the proposition.

\( \square \)
3.2. Dyadic case

In this subsection, we assume that \( b = 2 \). In the dyadic case, we can obtain the exact values of \( \|W(k)(\cdot)\|_{L^1} \) and \( \|W(k)(\cdot)\|_{L^\infty} \). First we show properties of \( W(k)(x) \) for the dyadic case.

**Lemma 3.6.** Let \( k \in \mathbb{N}_0 \). Assume that \( b = 2 \) and \( x_1, x_2 \in [0,1) \). Then we have the following.

(i) Assume that \( x_1 + x_2 \) is a multiple of \( 2^{-a_v+1} \). Then we have \( W(k)(x_1) = W(k)(x_2) \).

(ii) Assume that \( x_1 + x_2 \) is a multiple of \( 2^{-a_v} \) and not a multiple of \( 2^{-a_v+1} \). If \( k \neq 0 \), then we have \( W(k)(x_1) + W(k)(x_2) = W(k)(2^{-a_v}) \).

(iii) The function \( W(k)(x) \) is nonnegative.

**Proof.** We prove the lemma by induction on \( v \). The results hold for \( v = 0 \) since \( W(0)(x) = 1 \) for all \( x \in [0,1) \). Hence assume now that \( v > 0 \) and that the results hold for \( v - 1 \).

First we assume that \( x_1 + x_2 \) is the multiple of \( 2^{-a_v+1} \). Since \( W(k)(\cdot) \) has a period \( 2^{-a_v+1} \) by Lemma 2.2, we can assume that \( x_1, x_2 \in [0,2^{-a_v+1}) \). Then we can assume that \( x_1 \in [0,2^{-a_v}] \) and that \( x_2 = 2^{-a_v+1} - x_1 \). Now we prove that \( W(k)(x_1) = W(k)(x_2) \). By the induction assumption of (i) for \( v - 1 \), we have

\[
W(k)(x) = W(k)(2^{-a_v+1}) - \int_{x_2}^{2^{-a_v+1}} \text{Wal}_{2^{a_v-1}}(y) W(k')(y) \, dy
\]

\[
= 0 - \int_{x_2}^{2^{-a_v+1}} (-1) W(k')(2^{-a_v+1} - y) \, dy
\]

\[
= \int_{0}^{x_1} W(k')(y) \, dy
\]

\[
= W(k)(x_1),
\]

which proves (i) for \( v \).

Second we assume that \( x_1 + x_2 \) is a multiple of \( 2^{-a_v} \) and not a multiple of \( 2^{-a_v+1} \). Similar to the first case, we can assume that \( x_1, x_2 \in [0,2^{-a_v}] \) and that \( x_2 = 2^{-a_v} - x_1 \). By the induction assumption of (ii) for \( v - 1 \), we have
\[ W(k')(y) = W(k')(2^{-a_v} - y) \text{ for all } y \in [0, 2^{-a_v}]. \] Hence we have
\[
W(k)(x_1) + W(k)(x_2) = \int_0^{x_1} W(k')(y) \, dy + \int_0^{x_2} W(k')(y) \, dy
= \int_0^{x_1} W(k')(y) \, dy + \int_0^{x_2} W(k')(2^{-a_v} - y) \, dy
= \int_0^{x_1} W(k')(y) \, dy + \int_{2^{-a_v} - x_2}^{2^{-a_v}} W(k')(y) \, dy
= \int_0^{2^{-a_v}} W(k')(y) \, dy
= W(k)(2^{-a_v}),
\]
which proves (iii) for \( v \).

Finally we prove that \( W(k)(x) \) is nonnegative. By the induction assumption of (iii) for \( v - 1 \), \( W(k')(x) \) is nonnegative. For \( x \in [0, 2^{-a_v}] \), we have \( W(k)(x) = \int_0^x W(k')(y) \, dy \), and thus \( W(k)(x) \) is nonnegative for \( x \in [0, 2^{-a_v}] \). Hence by (i) for \( v \) and Lemma 2.2, \( W(k)(x) \) is nonnegative for \( x \in [0, 1) \).

Now we are ready to consider \( \| W(k)(\cdot) \|_{L^q} \) for \( 1 \leq q \leq \infty \).

First we consider \( \| W(k)(\cdot) \|_{L^1} \). By Lemmas 3.1 (i) and 3.6 (iii), we have
\[
\| W(k)(\cdot) \|_{L^1} = \int_0^1 |W(k)(x)| \, dx = \int_0^1 W(k)(x) \, dx = 2^{-\mu(k)-v}.
\]

Second we consider \( \| W(k)(\cdot) \|_{L^\infty} \). If \( k = 0 \), we have \( \| W(k)(\cdot) \|_{L^\infty} = 1 \). We assume that \( k > 0 \). Considering the symmetry and the non-negativity of \( W(k)(x) \) given by Lemma 3.6 we have
\[
\| W(k)(\cdot) \|_{L^\infty} = \sup_{x \in [0, 2^{-a_v}]} |W(k)(x)| \, dx
= \sup_{x \in [0, 2^{-a_v}]} \left| \int_0^x W(k')(y) \, dy \right|
= \int_0^{2^{-a_v}} W(k')(y) \, dy
= W(k)(2^{-a_v}) = 2^{-\mu(k)-v+1}.
\]

Thus we have \( \| W(k)(\cdot) \|_{L^\infty} \leq 2^{-\mu(k)-v+\min(1,v)} \) for all \( k \in \mathbb{N}_0 \).

Finally we consider \( \| W(k)(\cdot) \|_{L^q} \). By Hölder’s inequality, we have
\[
\| W(k)(\cdot) \|_{L^q} = \left( \int_{[0,1]} |W(k)(x)| \cdot |W(k)(x)|^{q-1} \, dx \right)^{1/q}
\leq \left( \| W(k)(\cdot) \|_{L^1} \| W(k)(\cdot) \|_{L^{q-1}} \right)^{1/q}
\leq 2^{-\mu(k)-v+(1-1/q)\min(1,v)}.
\]

We have shown the following proposition.
Proposition 3.7. Let $b = 2$. For $k \in \mathbb{N}_0$ and $1 \leq q \leq \infty$, we have
\[
\|W(k)(\cdot)\|_{L^q} \leq 2^{-\mu(k) - v(1)\min(1,v)},
\]
and the equality holds if $q = 1$ or $q = \infty$.

3.3. Bounds on the Walsh coefficients of smooth functions

For a positive integer $\alpha$ and $k \in \mathbb{N}_0$, we define
\[
\mu_\alpha(k) := \mu(k) = \begin{cases} 
0 & \text{for } k = 0, \\
a_1 + \cdots + a_v & \text{for } 1 \leq v \leq \alpha, \\
a_1 + \cdots + a_\alpha & \text{for } v \geq \alpha, 
\end{cases}
\]
as in [7]. By (1), Proposition 3.5 and Proposition 3.7, we obtain the following bound on the Walsh coefficients of smooth functions.

Theorem 3.8. Let $f \in C^\alpha[0,1]$ and $k \in \mathbb{N}_0$. If $b > 2$, we have
\[
|\hat{f}(k)| \leq \|f^{(\min(\alpha,v))}\|_{L^1} \frac{b^{-\mu_\alpha(k)}}{m_b^{\min(1,v)}} \times
\left( M_b + \frac{bm_b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^{\min(1,v)} \right) \right)^{\min(1,v)}.
\]
If $b = 2$, for $1 \leq p \leq \infty$ we have
\[
|\hat{f}(k)| \leq \|f^{(\min(\alpha,v))}\|_{L^p} \cdot 2^{-\mu_\alpha(k) - \min(\alpha,v) + \min(1,v)/p}.
\]

The $s$-variate case follows in the same way as the univariate case.

Theorem 3.9. Let $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$. Assume that $f : [0,1]^s \to \mathbb{R}$ has continuous mixed partial derivatives up to order $\alpha_j$ in each variable $x_j$. Let $n_j := \min(\alpha_j, v(k_j))$ for $1 \leq j \leq s$. Then, if $b > 2$, we have
\[
|\hat{f}(k)| \leq \|f^{(n_1,\ldots,n_s)}\|_{L^1} \prod_{j=1}^s \frac{b^{-\mu_{\alpha_j}(k_j)}}{m_b^{n_j}} \times
\left( M_b + \frac{bm_b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^{n_j} \right) \right)^{\min(1,v(k_j))}.
\]
If $b = 2$, for $1 \leq p \leq \infty$ we have
\[
|\hat{f}(k)| \leq \|f^{(n_1,\ldots,n_s)}\|_{L^p} \prod_{j=1}^s 2^{-\mu_{\alpha_j}(k_j) - n_j + \min(1,v(k_j))/p}.
\]

As a corollary, we give a sufficient condition for a infinitely smooth function that its Walsh coefficients decay with order $O(b^{-\mu(k)})$. 

12
Corollary 3.10. Let \( f \in C^\infty[0, 1]^s \) and \( r_j > 0 \) be positive real numbers for \( 1 \leq j \leq s \). Assume that there exists a positive real number \( D \) such that

\[
\|f^{(n_1, \ldots, n_s)}\|_{L^1} \leq D \prod_{j=1}^{s} r_j^{n_j}
\]

holds for all \( n_1, \ldots, n_s \in \mathbb{N}_0 \). Then for all \( k \in \mathbb{N}_0^s \) we have

\[
|\hat{f}(k)| \leq Db^{-\mu(k)} \prod_{j=1}^{s} (r_j m_b^{-1})^{v(k_j)} C_b^{\min(1,v(k_j))},
\]

where \( C_b \) is a constant defined as

\[
C_b = \begin{cases} 
2 & \text{for } b = 2, \\
M_b + \frac{bm_b}{b-M_b} & \text{for } b \neq 2.
\end{cases}
\]

In particular, if \( r_j = m_b \) holds for all \( 1 \leq j \leq s \), then \( |\hat{f}(k)| \in O(b^{-\mu(k)}) \) holds.

4. Another formula for the Walsh coefficients

In this section, we give another formula for the Walsh coefficients. For this purpose, we introduce functions \( W_j(k)(\cdot) \) and their integration values \( I_j(k) \) for \( j, k \in \mathbb{N}_0 \).

Definition 4.1. For \( j, k \in \mathbb{N}_0 \), we define functions \( W_j(k)(\cdot) : [0, 1] \to \mathbb{C} \) and complex numbers \( I_j(k) \) recursively as

\[
W_0(k)(x) := W(k)(x),
\]

\[
I_j(k) := \int_{0}^{1} W_j(k)(x) dx,
\]

\[
W_{j+1}(k)(x) := \int_{0}^{x} (W_j(k)(x) - I_j(k)) dy.
\]

We note that \( W_j(k)(0) = W_j(k)(1) = 0 \) for all \( j, k \in \mathbb{N}_0 \) with \( (j, k) \neq (0, 0) \).

We now establish another formula for the Walsh coefficients of smooth functions.

Theorem 4.2. Let \( k, r \in \mathbb{N}_0 \) and \( f \in C^{v+r}[0, 1] \). Then we have

\[
\hat{f}(k) = \sum_{i=0}^{r} (-1)^{v+i} I_i(k) \int_{0}^{1} f^{(v+i)}(x) dx + (-1)^{v+r} \int_{0}^{1} f^{(v+r)}(x)(W_r(k)(x) - I_r(k)) dx.
\]
Proof. We prove the theorem by induction on \( r \). We have already proved the case \( r = 0 \) in Theorem 2.3. Thus assume now that \( r \geq 1 \) and that the result holds for \( r - 1 \). By the induction assumption for \( v - 1 \), we have

\[
\hat{f}(k) = \sum_{i=0}^{r-1} (-1)^{v+i} I_i(k) \int_0^1 f^{(v+i)}(x) \, dx \\
+ (-1)^{v+r-1} \int_0^1 f^{(v+r-1)}(x)(W_{r-1}(k)(x) - I_{r-1}(k)) \, dx \\
= \sum_{i=0}^{r-1} (-1)^{v+i} I_i(k) \int_0^1 f^{(v+i)}(x) \, dx \\
+ (-1)^{v+r-1} \left( \left[ f^{(v+r-1)}(x)W_r(k)(x) \right]_0^1 - \int_0^1 f^{(v+r)}(x)W_r(k)(x) \, dx \right) \\
= \sum_{i=0}^{r} (-1)^{v+i} I_i(k) \int_0^1 f^{(v+i)}(x) \, dx \\
+ (-1)^{v+r} \int_0^1 f^{(v+r)}(x)(W_r(k)(x) - I_r(k)) \, dx,
\]

where we use \( W_r(k)(0) = W_r(k)(1) = 0 \) in the third equality. This proves the result for \( r \).

5. The Walsh coefficients of Bernoulli polynomials

In this section, we analyze the decay of the Walsh coefficients of Bernoulli polynomials.

For \( r \geq 0 \), we denote \( B_r(x) \) the Bernoulli polynomial of degree \( r \) and \( b_r(x) = B_r(x)/r! \). For example, we have \( B_0(x) = 1 \), \( B_1(x) = x - 1/2 \), \( B_2(x) = x^2 - x + 1/6 \) and so on. Those polynomials have the following properties: For all \( r \geq 1 \) we have

\[
b_r'(x) = b_{r-1}(x) \quad \text{and} \quad \int_0^1 b_r(x) \, dx = 0, \tag{7}
\]

and for all \( r \in \mathbb{N}_0 \) we have

\[
b_r(1 - x) = (-1)^r b_r(x), \tag{8}
\]

see [1, Chapter 23]. We clearly have \( b_0'(x) = 0 \) and \( \int_0^1 b_0(x) = 1 \).

The Walsh coefficients of Bernoulli polynomials are given as follows. If \( r < v \), then by Theorem 2.3 and (7) we have

\[
\hat{b}_r(k) = (-1)^v \int_0^1 b_r^{(v)}(x)W(k)(x) \, dx = 0.
\]

14
If $r \geq v$, then by Theorem 4.2 and 7 we have
\[
\hat{b}_{r}(k) = \sum_{i=0}^{r-v} (-1)^{v+i} I_{i}(k) \int_{0}^{1} b_{r}^{(v+i)}(x) \, dx \\
+ (-1)^{r} \int_{0}^{1} b_{r}^{(r)}(x)(W_{r-v}(k)(x) - I_{r-v}(k)) \, dx \\
= (-1)^{r} I_{r-v}(k).
\]
Now we proved:

**Lemma 5.1.** For positive integers $k$ and $r$, we have
\[
\hat{b}_{r}(k) = \begin{cases} 
0 & \text{if } r < v, \\
(-1)^{r} I_{r-v}(k) & \text{if } r \geq v.
\end{cases}
\]

In the following, we give upper bounds on $\|W_{j}(k)(\cdot) - I_{j}(k)\|_{L^{\infty}}$, $|I_{j}(k)|$ and $\|W_{j}(k)(\cdot)\|_{L^{\infty}}$, which give bounds on the Walsh coefficients of Bernoulli polynomials and smooth functions. First we compute $W_{j}(k)(\cdot)$ and $I_{j}(k).

**Lemma 5.2.** Let $k, j \in \mathbb{N}_{0}$. Let $x \in [0, 1)$ and $x = c b^{-a_{v}} + x'$ with $c \in \mathbb{N}_{0}$ and $x' \in [0, b^{-a_{v}})$. Then we have
\[
(i) \ W_{j}(k)(x) = \frac{1 - \omega_{b}^{k_{v}}}{1 - \omega_{b}^{k_{v}}} W_{j}(k)(b^{-a_{v}}) + \omega_{b}^{k_{v}} W_{j}(k)(x'),
\]
\[
(ii) \ I_{j}(k) = \frac{W_{j}(k)(b^{-a_{v}})}{1 - \omega_{b}^{k_{v}}}.
\]

**Proof.** We prove the lemma by induction on $j$. We have already proved the case $j = 0$ in Lemmas 2.2 and 3.1. Thus assume now that $j \geq 1$ and that the result holds for $j - 1$. Then we have
\[
\begin{align*}
W_{j}(k)(x) &= \int_{0}^{x} (W_{j-1}(k)(y) - I_{j-1}(k)) \, dy \\
&= \sum_{i=0}^{c-1} \int_{0}^{b^{-a_{v}}} \left( \frac{-\omega_{b}^{k_{v}}}{1 - \omega_{b}^{k_{v}}} W_{j-1}(k)(b^{-a_{v}}) + \omega_{b}^{k_{v}} W_{j-1}(k)(y) \right) \, dy \\
&\quad + \int_{0}^{x'} \left( \frac{-\omega_{b}^{k_{v}}}{1 - \omega_{b}^{k_{v}}} W_{j-1}(k)(b^{-a_{v}}) + \omega_{b}^{k_{v}} W_{j-1}(k)(y) \right) \, dy \\
&= \sum_{i=0}^{c-1} \omega_{b}^{k_{v}} W_{j}(k)(b^{-a_{v}}) + \omega_{b}^{k_{v}} W_{j}(k)(x') \\
&= \frac{1 - \omega_{b}^{k_{v}}}{1 - \omega_{b}^{k_{v}}} W_{j}(k)(b^{-a_{v}}) + \omega_{b}^{k_{v}} W_{j}(k)(x'),
\end{align*}
\]
where we use the induction assumption for $j - 1$ in the second and third equalities and the definition of $W_{j}(k)(\cdot)$ in the third equality. This proves (i) for $j$.

Now we compute $I_{j}(k)$. Replacing $W(k)(x)$ to $W_{j}(k)(x)$ in (3), we have $I_{j}(k) = W_{j}(k)(b^{-a_{v}})/(1 - \omega_{b}^{k_{v}})$, which proves (ii) for $j$. \(\square\)
Lemma 5.3. Let \( j \in \mathbb{N}_0 \). If \( b \neq 2 \), for any positive integer \( k \) we have
\[
\|W_j(k)\cdot - I_j(k)\|_{L^\infty} \leq \frac{b^{\mu(k) - j \alpha_v}}{m_b^{v + j}} \left( 1 + \frac{b m_b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right) \right).
\]

Proof. Let \( x \in [0, 1) \) and \( x = cb^{-a_v} + x' \) with \( c \in \mathbb{N}_0 \) and \( x' \in [0, b^{-a_v}) \). First assume that \( j = 0 \). Then it follows from Lemmas 3.1 and 3.4 that
\[
|W_0(k)(x) - I_0(k)| = |\overline{w}_b^{\kappa_v} I(k) + \overline{w}_b^{\kappa_v} W(k)(x')|
\leq |I(k)| + \sup_{x' \in [0, b^{-a_v})} |W(k)(x')|
\leq \frac{b^{\mu(k)}}{m_b^v} \left( 1 + \frac{b m_b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right) \right),
\]
which proves the case \( j = 0 \).

Now we assume that \( j > 0 \). Then it follows from Lemma 5.2 that
\[
|W_j(k)(x) - I_j(k)| = \left| \frac{1}{1 - \overline{w}_b^{\kappa_v}} W_j(k)(b^{-a_v}) + \overline{w}_b^{\kappa_v} W_j(k)(x') - \frac{W_j(k)(b^{-a_v})}{1 - \overline{w}_b^{\kappa_v}} \right|
\leq \frac{1}{m_b} \left| \int_{x'}^{b^{-a_v}} (W_{j-1}(k)(y) - I_{j-1}(k)) \, dy \right|
\leq \frac{1}{m_b} (b^{-a_v} - x') \sup_{y \in [0, b^{-a_v}]} |W_{j-1}(k)(y) - I_{j-1}(k)|
\leq \frac{b^{-a_v}}{m_b} \|W_{j-1}(k)(\cdot) - I_{j-1}(k)\|_{L^\infty}.
\]

Using the case \( j = 0 \) and this evaluation inductively, we have the case \( j > 0 \). □

Lemma 5.4. Let \( j \) and \( k \) be positive integers. If \( b > 2 \), then we have
\[
|I_j(k)| \leq \frac{b^{\mu(k) - j \alpha_v}}{m_b^{v + j}} \left( 1 + \frac{b m_b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right) \right).
\]
Proof. By Lemmas 5.2 and 5.3 we have

\[ |I_j(k)| = |W_j(k)(b^{-a_v}/(1 - \omega_b^{a_v}))| \]
\[ \leq \frac{1}{m_b} \int_0^{b^{-a_v}} |W_{j-1}(k)(y) - I_{j-1}(k)| \, dy \]
\[ \leq \frac{b^{-a_v}}{m_b} \|W_{j-1}(k)(y) - I_{j-1}(k)\|_{L^\infty} \]
\[ \leq \frac{b^{-\mu(k) - ja_v}}{m_b^{j+1}} \left( 1 + \frac{bm_b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right) \right). \]

\[ \square \]

Lemma 5.5. Let \( j \) and \( k \) be positive integers. If \( b > 2 \), then we have

\[ \|W_j(k)(\cdot)\|_{L^\infty} \leq \frac{b^{-\mu(k) - ja_v}}{m_b^{j+1}} M_b \left( 1 + \frac{bm_b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right) \right). \]

Proof. Let \( x \in [0, 1) \) and \( x = cb^{-a_v} + x' \), where \( 0 \leq c < b^{a_v} \) is an integer and \( 0 \leq x' < b^{-a_v} \) is a real number. Then we have

\[ W_j(k)(x) = \frac{1 - \omega_b^{a_v}}{1 - \omega_b^{a_v}} W_j(k)(b^{-a_v}) + \omega_b^{a_v} W_j(k)(x') \]
\[ = \frac{1 - \omega_b^{a_v}}{1 - \omega_b^{a_v}} (W_j(k)(b^{-a_v}) - W_j(k)(x')) + \frac{1 - \omega_b^{(c+1)a_v}}{1 - \omega_b^{a_v}} W_j(k)(x') \]
\[ = \frac{1 - \omega_b^{a_v}}{1 - \omega_b^{a_v}} \int_{x'}^{b^{-a_v}} (W_{j-1}(k)(y) - I_{j-1}(k)) \, dy \]
\[ + \frac{1 - \omega_b^{(c+1)a_v}}{1 - \omega_b^{a_v}} \int_0^{x'} (W_{j-1}(k)(y) - I_{j-1}(k)) \, dy. \]

Thus we have

\[ |W_j(k)(x)| \leq \left| \frac{1 - \omega_b^{a_v}}{1 - \omega_b^{a_v}} \int_{x'}^{b^{-a_v}} (W_{j-1}(k)(y) - I_{j-1}(k)) \, dy \right| \]
\[ + \left| \frac{1 - \omega_b^{(c+1)a_v}}{1 - \omega_b^{a_v}} \int_0^{x'} (W_{j-1}(k)(y) - I_{j-1}(k)) \, dy \right| \]
\[ = \frac{M_b}{m_b} b^{-a_v} \|W_j(k)(\cdot) - I_{j-1}(k)\|_{L^\infty} \]
\[ \leq \frac{b^{-\mu(k) - ja_v}}{m_b^{j+1}} M_b \left( 1 + \frac{bm_b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right) \right). \]

\[ \square \]

We also consider the dyadic case.

Lemma 5.6. Let \( k \) be a positive integer and \( j \in \mathbb{N}_0 \). If \( b = 2 \), then we have the following.

(i) \( \|W_j(k)(x) - I_j(k)\|_{L^\infty} \leq 2^{-j(a_v+1) - \mu(k) - v} \),
Proof. Lemma 5.3 and Proposition 5.4 imply (ii) and (iii) for \( j = 0 \).

Since \( W_0(k)(x) \) and \( I_0(k) \) are nonnegative, we have

\[
\|W_0(k)(x) - I_0(k)\|_{L^\infty} \leq \max \left( \|W_0(k)(\cdot)\|_{L^\infty} - \|I_0(k)\|, \|I_0(k)\| - \|W_0(k)(\cdot)\|_{L^\infty} \right) \leq 2^{-\mu(k) - v},
\]

and thus (i) for \( j = 0 \) holds.

For the proof for the case \( j > 0 \), we note that parts of proofs of Lemmas 5.3, 5.4 and 5.5 are valid even in the dyadic case. For \( b = 2 \) we have

\[
|W_j(k)(x) - I_j(k)| \leq \frac{b^{-a_v}}{m_b} \|W_{j-1}(k)(\cdot) - I_{j-1}(k)\|_{L^\infty},
\]

\[
|I_j(k)| \leq \frac{b^{-a_v}}{m_b} \|W_{j-1}(k)(y) - I_{j-1}(k)\|_{L^\infty},
\]

\[
|W_j(k)(x)| \leq \frac{M_k b^{-a_v}}{m_b} \|W_{j-1}(k)(\cdot) - I_{j-1}(k)\|_{L^\infty}.
\]

Combining these inequalities and the case \( j = 0 \), we have (i), (ii) and (iii) for \( j > 0 \).

Now we assume that \( j \) is odd and prove \( I_j(k) = 0 \). By Lemma 5.1 we have

\[
\hat{b}_{v+j}(k) = (-1)^{v+j} I_j(k).
\]

Hence it suffices to show \( \hat{b}_{v+j}(k) = 0 \). Since \( j \) is odd, by (8) we have \( b_{v+j}(x) = (-1)^{v+1} b_{v+j}(1 - x) \). Furthermore, \( \text{wal}_k(x) = (-1)^v \text{wal}_k(1 - x) \) holds for all but finitely many \( x \in [0, 1) \), since we have \( \text{wal}_{2^e-1}(x) = -\text{wal}_{2^e-1}(1 - x) \) for \( x \in [0, 1) \), \( \{l/2^a \mid 0 \leq l < 2^a \} \) and \( \text{wal}_k(x) = \prod_{i=1}^a \text{wal}_{2^i-1}(x) \). Hence we have

\[
\hat{b}_{v+j}(k) = \int_0^{1/2} b_{v+j}(x) \text{wal}_k(x) \, dx + \int_{1/2}^1 b_{v+j}(x) \text{wal}_k(x) \, dx
\]

\[
= \int_0^{1/2} b_{v+j}(x) \text{wal}_k(x) \, dx + \frac{1}{2} b_{v+j}(1 - x) \text{wal}_k(1 - x) \, dx
\]

\[
= \int_0^{1/2} b_{v+j}(x) \text{wal}_k(x) \, dx - \frac{1}{2} b_{v+j}(x) \text{wal}_k(x) \, dx
\]

\[
= 0. \quad \blacksquare
\]

Now we are ready to analyze the decay of the Walsh coefficients of Bernoulli polynomials. For a positive integer \( \alpha \) and \( k \in \mathbb{N}_0 \), we define

\[
\mu_{\alpha,\text{per}}(k) = \begin{cases} 
0 & \text{for } k = 0, \\
\alpha v + v + (\alpha - v) a_v & \text{for } 1 \leq v \leq \alpha, \\
a_1 + \cdots + a_{\alpha} & \text{for } v \geq \alpha,
\end{cases}
\] (9)
as in [7]. By Lemmas 5.1, 5.4 and 5.6 we have the following bound on the Walsh coefficients of Bernoulli polynomials.

**Theorem 5.7.** For positive integers \( k \) and \( r \), we have

\[
\left| \hat{b}_r(k) \right| = \begin{cases} 
0 & \text{if } r < v, \\
0 & \text{if } r \geq v, \ r - v \text{ is odd and } b = 2, \\
\leq 2^{-\mu_{c,p,v}(k) - r} & \text{if } r \geq v, \ r - v \text{ is even and } b = 2, \\
\leq \frac{b^{-\mu_{c,p,v}(k)}}{m_b} c_{b,v} & \text{if } r \geq v \text{ and } b \neq 2,
\end{cases}
\]

where \( c_{b,v} := 1 + \frac{b M_b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right) \).

6. The Walsh coefficients of functions in Sobolev spaces

In this section, we consider functions in the Sobolev space \( \mathcal{H}_\alpha := \{ f : [0,1] \to \mathbb{R} | f^{(i)}: \text{abs. conti. for } i = 0, \ldots, \alpha - 1, f^{(\alpha)} \in L^2[0,1] \} \) for which \( \alpha \geq 1 \) as in [7]. The inner product is given by

\[
\langle f, g \rangle_\alpha = \sum_{i=0}^{\alpha-1} \int_0^1 f^{(i)}(x) \, dx \int_0^1 g^{(i)}(x) \, dx + \int_0^1 f^{(\alpha)}(x) g^{(\alpha)}(x) \, dx.
\]

and the corresponding norm in \( \mathcal{H}_\alpha \) is given by \( \| f \|_{\text{Sob},\alpha} := \sqrt{\langle f, f \rangle_\alpha} \). The space \( \mathcal{H}_\alpha \) becomes a reproducing kernel Hilbert space (see [2] for general information of reproducing kernel Hilbert space). The reproducing kernel for this space is given by

\[
K(x, y) = \sum_{i=0}^{\alpha} b_i(x) b_i(y) - (-1)^\alpha \bar{b}_{2\alpha}(x - y),
\]

where

\[
\bar{b}_\alpha(x - y) := \begin{cases} 
b_\alpha(|x - y|), & \alpha: \text{even}, \\
(-1)^{1_{x<y} \bar{b}_\alpha(|x - y|)}, & \alpha: \text{odd},
\end{cases}
\]

where we define \( 1_{x<y} \) is 1 for \( x < y \) and 0 otherwise, see [4, Lemma 2.1]. We have

\[
f(y) = \langle f, K(\cdot, y) \rangle_\alpha = \sum_{i=0}^{\alpha} \int_0^1 f^{(i)}(x) \, dx b_i(y) - (-1)^\alpha \int_0^1 f^{(\alpha)}(x) \bar{b}_\alpha(x - y) \, dx,
\]

which implies that

\[
\hat{f}(k) = \sum_{i=0}^{\alpha} \int_0^1 f^{(i)}(x) \, dx \hat{b}_i(k) - (-1)^\alpha \int_0^1 f^{(\alpha)}(x) \int_0^1 \bar{b}_\alpha(x - y) w_{alk}(y) \, dy \, dx.
\]
However, we have already proved two formulas for the Walsh coefficients: For $f \in C^\alpha[0,1]$, in the case $\alpha \geq v$ we have Theorem 4.2 for $r = \alpha - v$, which is written as

$$\hat{f}(k) = \sum_{i=v}^\alpha (-1)^i I_{\alpha-v}(k) \int_0^1 f^{(i)}(x) \, dx$$

$$+ (-1)^n \int_0^1 f^{(n)}(x)(W_{\alpha-v}(k)(x) - I_{\alpha-v}(k)) \, dx,$$

(12)

and in the case $\alpha < v$ we have Theorem 2.3 for $n = \alpha$, which is written as

$$\hat{f}(k) = (-1)^\alpha \int_0^1 f^{(\alpha)}(x)\text{walk}_x(\alpha)(x) \, dx.$$  

(13)

In this section, we show that Formulas (12) and (13) are also valid for $f \in H_\alpha$ and give an upper bound for the Walsh coefficients of functions in $H_\alpha$.

6.1. Formula for the Walsh coefficients of functions in Sobolev spaces

First we consider the case $\alpha \geq v$. The following lemma is needed to show that (12) is also valid for $f \in H_\alpha$.

**Lemma 6.1.** Assume $\alpha \geq v$. Define functions $h_1, h_2: [0,1] \to \mathbb{C}$ as

$$h_1(x) := - \int_0^1 b_\alpha(x-y)\text{walk}(y) \, dy,$$

$$h_2(x) := W_{\alpha-v}(k)(x) - I_{\alpha-v}(k).$$

Then $h_1(x) = h_2(x)$ holds for all $x \in [0,1]$.

**Proof.** For $f \in C^\alpha[0,1]$ both formulas (11) and (12) hold. Furthermore, by Lemma 5.1 the first term of each formula is equal. Hence we have

$$\int_0^1 f^{(\alpha)}(x)h_1(x) \, dx = \int_0^1 f^{(\alpha)}(x)h_2(x) \, dx$$

for all $f \in C^\alpha[0,1]$. It is well known that if $h: [0,1] \to \mathbb{C}$ is continuous and $\int_0^1 g(x)h(x) = 0$ holds for all continuous functions $g \in C^0[0,1]$, then $h(x) = 0$ holds. Thus it suffices to show that $h_1$ and $h_2$ are continuous.

By definition, $h_2$ is continuous. Now we prove that $h_1$ is continuous. Fix $\epsilon > 0$. Since $b_\alpha(z)$ is uniformly continuous on $z \in [0,1]$, there exists $\delta_1$ such that $|b_\alpha(z) - b_\alpha(z')| < \epsilon/2$ for all $z, z' \in [0,1]$ with $|z - z'| < \delta_1$. Let $\delta_2 = \min \left(4^{-1}\epsilon \max_{z \in [0,1]} |b_\alpha(z)|^{-1}, \delta_1 \right)$. We fix $x \in [0,1]$ and prove $|h_1(x) - h_1(x')| \leq \epsilon$ for all $x' \in [0,1]$ with $|x - x'| < \delta_2$. Without loss of generality, we
can assume that \( x < x' \). Then we have
\[
\left| \int_0^1 \tilde{b}_\alpha(x-y) \text{wal}_k(y) \, dy - \int_0^1 \tilde{b}_\alpha(x'-y) \text{wal}_k(y) \, dy \right|
\leq \left( \int_x^x + \int_x^{x'} + \int_x^1 \right) \left| \tilde{b}_\alpha(x-y) - \tilde{b}_\alpha(x'-y) \right| \, dy
\leq x \max_{y \in [0,x]} |b_\alpha(x-y) - b_\alpha(x'-y)| + (x' - x) \max_{y \in [x,x']} (|b_\alpha(y - x)| + |b_\alpha(x' - y)|)
+ (1 - x') \max_{y \in [x',1]} |b_\alpha(y - x) - b_\alpha(y - x')|
\leq x \epsilon/2 + 2 \delta_2 \max_{z \in [0,1]} |\tilde{b}_\alpha(z)| + (1 - x') \epsilon/2
\leq \epsilon,
\]
which implies the continuity of \( h_1 \).

The following result follows now from the above lemma, Lemma 5.1 and (11).

**Proposition 6.2.** Assume \( \alpha \geq v \). Then for \( f \in H_\alpha \) we have
\[
\hat{f}(k) = \sum_{i=v}^{\alpha} (-1)^i I_{i-v}(k) \int_0^1 f^{(i)}(x) \, dx 
+ (-1)^\alpha \int_0^1 f^{(\alpha)}(x)(W_{\alpha-v}(k)(x) - I_{\alpha-v}(k)) \, dx.
\]

Now we treat the case \( \alpha < v \). Note that \( \text{wal}_{k_\alpha}(x)W(k_\alpha^2)(x) \) is continuous since \( W(k_\alpha^2)(x) \) equals 0 on the set where \( \text{wal}_{k_\alpha}(x) \) is not continuous. In the same way as the case \( \alpha \geq v \), we have the following.

**Proposition 6.3.** Assume \( \alpha < v \). Then we have
\[
- \int_0^1 \tilde{b}_\alpha(x-y) \text{wal}_k(y) \, dy = \text{wal}_{k_\alpha}(x)W(k_\alpha^2)(x).
\]

In particular, for \( f \in H_\alpha \) we have
\[
\hat{f}(k) = (-1)^\alpha \int_0^1 f^{(\alpha)}(x)W_{\alpha-v}(k)(x) \, dx.
\]

6.2. Upper bound on the Walsh coefficients of functions in Sobolev spaces

In this subsection, we give a bound on the Walsh coefficients of functions in \( H_\alpha \).

By Propositions 6.2 and 6.3 for \( f \in H_\alpha \) we have
\[
|\hat{f}(k)| \leq \sum_{i=v}^{\alpha} |I_{i-v}(k)| \int_0^1 |f^{(i)}(x)| \, dx + N_\alpha \int_0^1 |f^{(\alpha)}(x)| \, dx,
\]

21
where \( N_\alpha = \|W_{\alpha-v}(k)(\cdot) - I_{\alpha-v}(k)\|_{L^\infty} \) if \( \alpha \geq v \) and \( N_\alpha = \|W(k^\circ_\alpha)(\cdot)\|_{L^\infty} \) otherwise. Thus, by Propositions 3.5 and 3.7 and Lemmas 5.3 and 5.6, we have the following.

**Theorem 6.4.** Let \( \alpha \) and \( k \) be positive integers. Assume \( f \in \mathcal{H}_\alpha \). If \( b > 2 \), we have

\[
|\tilde{f}(k)| \leq \sum_{i=v}^{\alpha} \left| \int_0^1 f^{(i)}(x) \, dx \right| \frac{b^{-\mu_{i,\text{per}}(k)}}{m_b^i} \left( 1 + \frac{bm_b}{b-M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right) \right)
+ \int_0^1 |f^{(\alpha)}(x)| \, dx \frac{b^{-\mu_{\alpha,\text{per}}(k)}}{m_b^\alpha} \left( M_b + \frac{bm_b}{b-M_b} \left( 1 - \left( \frac{M_b}{b} \right)^v \right) \right),
\]

and if \( b = 2 \), we have

\[
|\tilde{f}(k)| \leq \sum_{i=v \text{ mod } 2}^{\alpha} \left| \int_0^1 f^{(i)}(x) \, dx \right| \frac{b^{-\mu_{i,\text{per}}(k)}}{2^i} + \int_0^1 |f^{(\alpha)}(x)| \, dx \frac{b^{-\mu_{\alpha,\text{per}}(k)}}{2^{\alpha-1}},
\]

where for \( v > \alpha \) the empty sum \( \sum_{i=v}^{\alpha} \) is defined to be 0.

For an integer \( i \) with \( v \leq i \leq \alpha \), \( \mu_{i,\text{per}}(k) \geq \mu_{\alpha}(k) \) holds for all \( k \in \mathbb{N}_0 \) by the definitions of \( \mu_{i,\text{per}}(k) \) and \( \mu_{\alpha}(k) \). Thus, applying Hölder’s inequality to Theorem 6.4, we obtain the following corollary.

**Corollary 6.5.** Let \( \alpha \) and \( k \) be positive integers. Then, for all \( f \in \mathcal{H}_\alpha \), we have

\[
|\tilde{f}(k)| \leq b^{-\mu_{\alpha}(k)} C_{b,\alpha,q} \|f\|_{p,\alpha},
\]

where \( \|f\|_{p,\alpha} := \left( \int_0^1 f^{(i)}(x) \, dx \right)^p + \int_0^1 |f^{(\alpha)}(x)|^p \, dx \right)^{1/p} \), where \( 1/p + 1/q = 1 \), and

\[
C_{b,\alpha,q} := \left( \sum_{i=1}^{\alpha} \frac{1}{m_b^q} \left( 1 + \frac{bm_b}{b-M_b} \right)^q + \frac{1}{m_b^q} (M_b + \frac{bm_b}{b-M_b})^q \right)^{1/q}
\]

for \( b > 2 \) and \( C_{2,\alpha,q} := (\sum_{i=1}^{\alpha} 2^{-iq} + 2^{-(\alpha-1)q})^{1/q} \) for \( b = 2 \).

**Remark 6.6.** This corollary can be generalized to tensor product spaces, for which the reproducing kernel is just the product of the one-dimensional kernel, as [3, Section 14.6].

7. The Walsh coefficients of smooth periodic functions

As [7], we consider a subset of the previous reproducing kernel Hilbert space, namely, let \( \mathcal{H}_{\alpha,\text{per}} \) be the space of all functions \( f \in \mathcal{H}_\alpha \) which satisfy the condition \( \int_0^1 f^{(i)}(x) \, dx = 0 \) for \( 0 \leq i < \alpha \). This space also has a reproducing kernel, which is given by

\[
K_{\alpha,\text{per}}(x,y) = b_\alpha(x)b_\alpha(y) + (-1)^{\alpha+1}b_{2\alpha}(x-y),
\]
The inner product is given by
\[ \langle f, g \rangle_{\alpha, \text{per}} = \int_0^1 f^{(\alpha)}(x) g^{(\alpha)}(x) \, dx, \]
see [12, (10.2.4)]. We also have the representation
\[ f(y) = \langle f, K_{\alpha, \text{per}}(\cdot, y) \rangle_{\alpha, \text{per}} \]
\[ = \int_0^1 f^{(\alpha)}(x) \, dx \, b_\alpha(y) + (-1)^{\alpha+1} \int_0^1 f^{(\alpha)}(x) \bar{b}_\alpha(x - y) \, dx \]
and
\[ \hat{f}(k) = \int_0^1 f^{(\alpha)}(x) \, dx \, \hat{b}_\alpha(k) + (-1)^{\alpha+1} \int_0^1 f^{(\alpha)}(x) \int_0^1 \bar{b}_\alpha(x - y) \text{wal}_k(y) \, dx \, dy. \]

By the condition \( \int_0^1 f^{(i)}(x) \, dx = 0 \) for \( 0 \leq i < \alpha \) and Propositions 6.2 and 6.3, we have the following.

**Lemma 7.1.** Let \( \alpha \) and \( k \) be positive integers. Assume \( f \in H_{\alpha, \text{per}} \). If \( \alpha \geq \nu \), then we have
\[ \hat{f}(k) = (-1)^{\alpha} \int_0^1 f^{(\alpha)}(x) W_{\alpha-\nu}(k)(x) \, dx. \]
If \( \alpha < \nu \), then we have
\[ \hat{f}(k) = (-1)^{\alpha} \int_0^1 f^{(\alpha)}(x) \text{wal}_k(\cdot) W(k^{\alpha}_\nu)(x) \, dx. \]

This lemma, Propositions 5.5 and 5.7 and Lemmas 5.4 and 5.6 imply the following bound.

**Theorem 7.2.** Let \( \alpha \) and \( k \) be positive integers. Assume \( f \in H_{\alpha, \text{per}} \). If \( b > 2 \), then we have
\[ |\hat{f}(k)| \leq \int_0^1 |f^{(\alpha)}(x)| \, dx \frac{b^{-\mu_{\alpha, \text{per}}(k)}}{m_b^\alpha} M_b \left( 1 + \frac{b m_b}{b - M_b} \left( 1 - \left( \frac{M_b}{b} \right)^\nu \right) \right). \]
If \( b = 2 \), then we have
\[ |\hat{f}(k)| \leq \int_0^1 |f^{(\alpha)}(x)| \, dx \frac{b^{-\mu_{\alpha, \text{per}}(k)}}{2^{\alpha-1}}. \]

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