k*-METRIZABLE SPACES AND THEIR APPLICATIONS

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Abstract. In this paper we introduce and study so-called k*-metrizable spaces forming a new class of generalized metric spaces, and display various applications of such spaces in topological algebra, functional analysis, and measure theory. By definition, a Hausdorff topological space X is k*-metrizable if X is the image of a metrizable space M under a continuous map f : M → X having a section s : X → M that preserves precompact sets in the sense that the image s(K) of any compact set K ⊂ X has compact closure in X.

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Introduction

In this paper we introduce and study so-called $k^*$-metrizable spaces forming a new class of generalized metric spaces, and display various applications of such spaces in topological algebra, functional analysis, and measure theory.

By definition, a Hausdorff topological space $X$ is $k^*$-metrizable if $X$ is the image of a metrizable space $M$ under a subproper map $\pi : M \to X$. A map $\pi : M \to X$ is called subproper if it admits a section $s : X \to M$ that preserves precompact sets in the sense that the image $s(K)$ of any compact set $K \subset X$ has compact closure in $M$. Since each subproper map is compact-covering, all compact subsets of a $k^*$-metrizable space are metrizable. Conversely, if all compact subsets of a space $X$ are metrizable, then $X$ is $k^*$-metrizable if and only if $X$ is cs*-metrizable in the sense that $X$ is the image of a metrizable space $M$ under a continuous map $\pi : M \to X$ having a section $s : X \to M$ that is cs*-continuous in the sense that for any convergent sequence $(x_n)$ in $X$ the sequence $(s(x_n))$ has a convergent subsequence. Thus the $k^*$-metrizability decomposes into two weaker properties: the cs*-metrizability and the metrizability of all compact subsets.

The class of $k^*$-metrizable spaces is closed under many countable (and some uncountable) topological operations. Regular $k^*$-metrizable spaces can be characterized as spaces with $\sigma$-compact-finite $k$-network, see Theorem 6.4. This characterization shows that the class of $k^*$-metrizable spaces is sufficiently wide and contains all Lašnev spaces (closed images of metrizable spaces), all $\aleph_0$-spaces (images of metrizable separable spaces under compact-covering maps) and all $\aleph_0$-spaces (regular spaces possessing a $\sigma$-locally finite $k$-network). Many results known for latter classes of spaces (like metrizability criteria) still hold in a more general framework of $k^*$-metrizable spaces. The choice of the term “$k^*$-metrizable space” was motivated by a characterization of regular $k^*$-metrizable spaces as spaces $X$ possessing a metric $\rho$ such that (i) each $\rho$-convergent sequence converges in $X$; (ii) a $\rho$-Cauchy sequence converges in $X$ if and only if it contains a subsequence convergent in $X$ and (iii) each compact subset $K \subset X$ is totally bounded with respect to the metric $\rho$. Replacing the last condition with (iii) ‘each convergent sequence in $X$ contains a $\rho$-Cauchy subsequence’, we obtain a metric characterization of cs*-metrizable spaces.

Luckily, our motivation for a thorough study of $k^*$-metrizable spaces was outside the theory of generalized metric spaces and came from probability theory. According to a celebrated result of A.V. Skorohod [71], for every sequence of Borel probability measures $\mu_n$ on a complete separable metric space $X$ that is weakly convergent to a Borel probability measure $\mu_0$, one can find Borel functions $\xi_n : [0,1] \to X$, $n = 0, 1, \ldots$, such that $\lim_{n \to \infty} \xi_n(t) = \xi_0(t)$ for almost all $t \in [0,1]$ and the image of Lebesgue measure $\lambda$ under $\xi_n$ is $\mu_n$ for every $n \geq 0$. Various extensions of this result have been found since then (see, e.g., [14], [18], [23], [29], [35], [41], [68], and the references therein). The most important for us is the extension discovered independently by Blackwell and Dubbins [14] and Fernique [35], according to which all Borel probability measures on $X$ can be parametrized simultaneously by mappings from $[0,1]$ with the preservation of the above correspondence. It was shown in [18] that this result can be derived from its simple 1-dimensional case and certain deep topological selection theorems.

More precisely, it was shown in [18] that to every Borel probability Radon measure $\mu$ on a metrizable $X$ one can associate a Borel function $\xi_\mu : [0,1] \to X$ such that $\mu$ is the image of Lebesgue measure $\lambda$ under $\xi_\mu$ and if measures $\mu_n$ on $X$ converge weakly to $\mu$, then $\lim_{n \to \infty} \xi_{\mu_n}(t) = \xi_\mu(t)$ for almost all $t \in [0,1]$. This property of the space $X$ was called the strong Skorohod property in [18]. Thus, each metrizable space has the strong Skorohod property. The situation changes beyond the class of metrizable spaces, see [8], [9], [10]. It was observed
in [18, 4.1] that the inductive limit $\mathbb{R}^\infty = \lim_{n\to\infty} \mathbb{R}^n$ of finite-dimensional Euclidean spaces (in some sense, the simplest non-metrizable locally convex space) fails to have the strong Skorohod property.

On the other hand, it was noticed in [18, 4.4] that the space $\mathbb{R}^\infty$ possesses a weakened version of the strong Skorohod property which was called the weak Skorohod property. Namely, a topological space $X$ is defined to have the weak Skorohod property if to each probability Radon measure $\mu$ on $X$ one can assign a Borel function $\xi_\mu : [0, 1] \to X$ such that $\mu$ is the image of the Lebesgue measure $\lambda$ under $\xi_\mu$ and for every uniformly tight sequence $(\mu_n)$ of probability Radon measures on $X$ the function sequence $(\xi_{\mu_n})$ contains a subsequence that converges almost surely.

In Lemma 4.2(ii) of [18] it was shown that proper maps preserve the weak Skorohod property. In Theorem 10.1 we make one step further and show that this property is preserved by subproper maps, thus generalizing the aforementioned lemma from [18]. Since metrizable spaces have the weak Skorohod property we conclude that any $k^*$-metrizable space also has that property.

In light of this result it was natural to study subproper maps more deeply. This will be done in Section 1. In Section 2 we show that subproper maps are preserved by many topological constructions. In Section 3 we introduce $k^*$-metrizable spaces and present their metric characterization in Theorem 3.4. In this section we also show that $k^*$-metrizability is preserved by many topological operations.

In Section 4 we decompose the $k^*$-metrizability into the $cs^*$-metrizability plus the sequential compactness of all compact subsets, and study $cs^*$-metrizable spaces in more details. Section 5 is devoted to cardinal characteristics of $k^*$-metrizable spaces. The main result here is Theorem 5.3 asserting that many cardinal characteristics of $k^*$-metrizable $k$-spaces (intermediate between extent and $k$-network weight) coincide.

In Section 6 we present a characterization of $cs^*$-metrizable spaces in terms of $\sigma$-$cs$-finite $cl_1$-osed $cs^*$-networks and derive from this characterization a characterization of $k^*$-metrizable spaces as topological spaces having a $\sigma$-compact-finite $cl_1$-osed $k$-network. In Section 7 we apply the characterizations of $k^*$-metrizable spaces to study the interplay between $k^*$-metrizable spaces and other generalized metric spaces such as metrizable, stratifiable, semi-stratifiable, monotonically normal, Lašnev, $\aleph_0$- or $\aleph$-spaces. In particular, we show that the class of $\aleph_0$-spaces coincides with the class of regular $k^*$-metrizable spaces having countable network.

In Section 8 we apply the $k$-network characterization of $k^*$-metrizable spaces to detect such spaces among function spaces $C_K(X, Y)$. In Sections 9 and 10 we pay tribute to our initial motivation and apply subproper maps and $k^*$-metrizable spaces to spaces of measures. In particular, we show that under some mild restrictions the functor of probability Radon measures preserves subproper maps as well as $k^*$-metrizable spaces. In Theorem 10.1 we notice that the weak Skorohod property is preserved by subproper maps and derive from this that submetrizable $k^*$-metrizable spaces have the weak Skorohod property, thus giving many natural examples of non-metrizable spaces with that property.

In Section 11 we observe that $k^*$-metrizable spaces naturally appear in the theory of locally convex spaces as results of application of some operations to metrizable locally convex spaces. In particular we show that often operator spaces lead to $k^*$-metrizable spaces. Section 12 is devoted to very special operator spaces, namely Banach spaces endowed with the weak topology. We observe that such a space $(X, \text{weak})$ is $k^*$-metrizable if either the dual Banach space $X^*$ is separable or else $X$ has the Shur property (= weakly convergent sequences are norm convergent). These two opposite cases are near to exhaust all $k^*$-metrizable spaces of the form $(X, \text{weak})$ where $X$ is a separable Banach space: if $(X, \text{weak})$ is $k^*$-metrizable,
then either the dual $X^*$ is separable or else $X$ contains a closed infinite-dimensional subspace with the Shur property, see Theorem 12.3. This result allows us to construct an open linear operator $T : X \to Y$ between separable Banach spaces such that $(X, \text{weak})$ is $k^*$-metrizable while $(Y, \text{weak})$ is not. This example shows that the $k^*$-metrizability is not preserved by open maps.

In the final Section 13 we penetrate into the structure of sequential $k^*$-metrizable groups. In Theorem 6.3 we show that any such a group $G$ either is metrizable or else contains an open $k_\omega$-subgroup. Applying this classification to locally convex spaces, we show in Theorem 13.7 that there are only two topological types of non-metrizable sequential $k^*$-metrizable locally convex spaces: $\mathbb{R}^\infty = \lim \mathbb{R}^n$ and $[0,1]^\omega \times \mathbb{R}^\infty$. A similar characterization holds also for non-metrizable zero-dimensional sequential $k^*$-metrizable groups, see Theorem 13.6.

1. Subproper maps

In this section we define subproper maps and study their relationship with other known classes of maps.

Throughout, the term “map” always means a continuous map unlike the term “function”. We consider only Hausdorff topological spaces. As usual, $\bar{A}$ or $\overline{\text{cl}}(A)$ stands for the closure of a subset $A$ of a topological space $X$.

A subset $A$ of a topological space $X$ is defined to be precompact if it has compact closure in $X$. Observe that a map $f : X \to Y$ between topological spaces is proper if and only if the preimage $f^{-1}(K)$ of any precompact set $K \subset Y$ is a precompact subset of $X$.

Definition 1.1. A map $f : X \to Y$ is defined to be subproper if there exists a subset $Z \subset X$ such that $f(Z) = Y$ and for any precompact subset $K \subset Y$ the set $Z \cap f^{-1}(K)$ is precompact in $X$.

There is a simple characterization of subproper maps in terms of sections preserving precompact sets.

A (possibly discontinuous) function $s : Y \to X$ is called a section of a map $f : X \to Y$ if $f \circ s(y) = y$ for each point $y \in Y$. We say that a function $f : X \to Y$ between topological spaces preserves precompact sets (or else is precompact-preserving) if the image $s(K)$ of any precompact set $K \subset Y$ is precompact in $X$.

A map $f : X \to Y$ is called compact-covering if each compact subset $K \subset Y$ is the image of some compact set $C \subset X$ under $f$. Equivalently, $f$ is compact-covering if it induces a surjective function $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$, $\mathcal{K}(f) : K \mapsto f(K)$, between the families of compact subsets of $X$ and $Y$.

Theorem 1.2. For a surjective map $f : X \to Y$ between Hausdorff topological spaces the following conditions are equivalent:

1) $f$ is subproper;
2) $f$ has a section preserving precompact sets;
3) the induced function $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ admits a section $s : \mathcal{K}(Y) \to \mathcal{K}(X)$ which is monotone in the sense that $s(K) \subset s(K')$ for any compact subsets $K \subset K'$ of $Y$;
4) the induced function $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ admits a section $s : \mathcal{K}(Y) \to \mathcal{K}(X)$ which is additive in the sense that $s(K \cup K') = s(K) \cup s(K')$ for any compact subsets $K, K' \subset Y$.

Proof. We shall prove the implications (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (4) Assuming that $f : X \to Y$ is subproper, find a subset $Z \subset X$ such that $f(Z) = Y$ and for any compact subset $K \subset Y$ the set $s(K) = \overline{\text{cl}}_X(Z \cap f^{-1}(K))$ is compact. The continuity of $f$ implies that $\mathcal{K}(f) \circ s(K) = K$ and hence $s : \mathcal{K}(Y) \to \mathcal{K}(X)$ is a section of
the induced function $K(f) : K(X) \to K(Y)$. It is clear that for any compact subsets $A, B \subset Y$ we get

\[ s(A \cup B) = \overline{cl}(Z \cap f^{-1}(A) \cup f^{-1}(B)) = \overline{cl}(Z \cap (f^{-1}(A) \cup f^{-1}(B))) = \overline{cl}(Z \cap (f^{-1}(A) \cup f^{-1}(B))) = s(A) \cup s(B) \]

which means that $s : K(Y) \to K(X)$ is an additive section of $K(f)$.

The implication (4) $\Rightarrow$ (3) is trivial since for an additive section $s : K(Y) \to K(X)$ of $K(f)$ and two compacta $A \subset B$ in $Y$ we get $s(B) = s(A) \cup s(B)$ which implies $s(A) \subset s(B)$.

(3) $\Rightarrow$ (2) Assume that $l : K(Y) \to K(X)$ is a monotone section of $K(f)$. Given a point $y \in Y$ pick any point $s(y)$ in the compact set $l(\{y\}) \subset X$. Since $f(l(\{y\})) = \{y\}$, the so-defined function $s : Y \to X$ is a section of $f$. Next, assume that $K \subset Y$ is a precompact subset of $Y$. Then $s(K) \subset \bigcup_{y \in K} l(\{y\}) \subset l(K)$ lies in the compact subset $l(K)$ of $X$ and thus $s(K)$ is a precompact subset of $X$.

(2) $\Rightarrow$ (1) Assume that $f$ admits a precompact-preserving section $s : Y \to X$. Let $Z = s(Y)$. Then for any precompact subset $K \subset Y$ the intersection $s(K) = Z \cap f^{-1}(K)$ is a precompact subset in $X$ and thus $f$ is a subproper map. □

Precompact-preserving functions are tightly connected with cs*-continuous functions. A function $s : Y \to X$ between topological spaces is called

- sequentially continuous (or else cs-continuous) if for any convergent sequence $(y_n)$ in $Y$ the sequence $(s(y_n))$ converges in $X$;
- cs*-continuous if for any convergent sequence $(y_n)$ in $Y$ the sequence $(s(y_n))$ has an accumulation point $x_\infty$ in $X$ (the latter means that each neighborhood of $x_\infty$ contains infinitely many points $s(y_n)$).

It is clear that each precompact-preserving function $f : X \to Y$ is cs*-continuous. The converse is true if compact subsets of $Y$ are sequentially compact and $X$ is $\mu$-complete.

We recall that a topological space $X$ is called sequentially compact if each sequence $(x_n)$ in $X$ contains a convergent subsequence.

A topological space $X$ is called $\mu$-complete if each bounded subset of $X$ has compact closure, where a subset $B \subset X$ is bounded if for any locally finite collection $U$ of open sets in $X$ only finitely many sets $U \in U$ meet $B$, see [13]. It is easily seen (and well-known) that a subset $B$ of a Tychonoff space is bounded if and only if for any continuous real-valued function $f : X \to \mathbb{R}$ the image $f(B)$ is bounded in $\mathbb{R}$. According to [31, 8.5.13], each Dieudonné-complete space (in particular, each paracompact space) is $\mu$-complete. On the other hand, the ordinal segment $[0, \omega_1)$ endowed with the natural interval topology is not $\mu$-complete.

**Proposition 1.3.** Let $X, Y$ be topological spaces such that $X$ is $\mu$-complete and each compact subset of $Y$ is sequentially compact. A function $s : Y \to X$ is precompact-preserving if and only if $s$ is cs*-continuous.

**Proof.** The “only if” part is trivial and holds without any assumptions on $X$ and $Y$. To prove the “if” part, assume that $s : Y \to X$ is a cs*-continuous function from a space $Y$ whose all compact subsets are sequentially compact into a $\mu$-complete space $X$. To show that $s$ is precompact-preserving, take any compact subset $K \subset Y$. We claim that the image $s(K)$ is precompact. Assuming that it is not so, we get that $s(K)$ is not bounded by the $\mu$-completeness of $X$. Consequently there is an infinite locally finite family $\mathcal{U} = \{U_n : n \in \omega\}$.
of open subsets of $X$ such that for every $n \in \omega$ the intersection $U_n \cap s(K)$ contains a point $x_n$. Pick any point $y_n \in K$ with $s(y_n) = x_n$ and use the sequentially compactness of $K$ to find a convergent subsequence $(y_{n_k})$ of the sequence $(y_n)$. Since $s$ is $cs^*$-continuous, the sequence $(s(y_{n_k}))$ has an accumulation point $x_\infty$. Then each neighborhood $W$ of $x_\infty$ contains infinitely many points $s(y_{n_k}) = x_{n_k}$ and thus meets infinitely many sets $U_{n_k} \ni x_{n_k}$ which contradicts the local finity of $U$. □

This proposition implies another characterization of subproper maps.

**Theorem 1.4.** A map $f : X \to Y$ from a $\mu$-complete space $X$ into a space $Y$ whose all compact subsets are sequentially compact is subproper if and only if it has a $cs^*$-continuous section $s : Y \to X$.

It turns out that each closed map has a $cs^*$-continuous section.

**Proposition 1.5.** Any section $s : Y \to X$ of a closed map $f : X \to Y$ is $cs^*$-continuous. Moreover if the space $X$ is $\mu$-complete, then $s$ preserves precompact sets.

**Proof.** Assume that $f : X \to Y$ is closed and take any section $s : Y \to X$ of $f$. We claim that $s$ is $cs^*$-continuous. Given a convergent sequence $(y_n)_{n \in \omega}$ in $Y$ we should find an accumulation point for its image $(s(y_n))$ in $X$. Without loss of generality, $y_n \neq y_\infty = \lim_{n \to \infty} y_n$ for all $n$. Assuming that the sequence $(s(y_n))$ has no accumulation point in $X$, we will get that $\{s(y_n) : n \in \omega\}$ is a closed discrete subset in $X$. Since $f$ is closed, the set $\{y_n : n \in \omega\} = f(\{s(y_n) : n \in \omega\})$ is closed in $Y$ which is not possible because this set has an accumulation point $y_\infty \notin \{y_n : n \in \omega\}$.

Next, assume additionally that the space $X$ is $\mu$-complete. We shall show that the the image $s(K)$ of any compact set $K \subset Y$ is precompact in $X$. Assuming the converse, use the $\mu$-completeness of $X$ to find a locally finite countable collection $\mathcal{U}$ of open subsets of $X$ meeting the image $s(K)$. For each $U \in \mathcal{U}$ pick a point $x_U \in U \cap s(K)$. Then the set $\{x_U : U \in \mathcal{U}\} \subset Y$ is closed and discrete in $X$. Since the map $f$ is closed, $\{f(x_U) : U \in \mathcal{U}\}$ is a closed countable subset of the compactum $\bar{K}$. Consequently, this set has a non-isolated point and there is an infinite subcollection $\mathcal{W} \subset \mathcal{U}$ such that the set $\{f(x_U) : U \in \mathcal{W}\}$ is not closed in $Y$, which contradicts the closedness of the set $\{x_U : U \in \mathcal{U}\}$ in $X$ and the closedness of the map $f$. □

This proposition can be partly reversed. First we recall some definitions.

A map $f : X \to Y$ is defined to be *inductively closed* (resp. *inductively perfect*) if there is a closed subset $Z \subset X$ such that $f(Z) = Y$ and the restriction $f|Z : Z \to Y$ of $f$ is a closed (resp. perfect) map. (A map $f : X \to Y$ is *perfect* if it is closed and the preimage $f^{-1}(y)$ of each point $y \in Y$ is compact, see [31 §3.7]). It is well-known that each perfect map is proper [31 3.7.2] and each proper map into a $k$-space is perfect [31 3.7.18].

A topological space $X$ is called a *Fréchet–Urysohn space* if for each subset $A \subset X$ and each point $a \in A$ there is a sequence $(a_n) \subset A$ convergent to $a$. It is clear that each metrizable space is Fréchet–Urysohn. A standard example of a non-metrizable Fréchet–Urysohn space of arbitrarily large character is the *sequential fan* $S_\kappa$ where $\kappa$ is an infinite cardinal $\kappa$. By definition, $S_\kappa = \{2^{-n} : n \leq \infty\} \times \kappa / \{0\} \times \kappa$ is the quotient space of the discrete sum of $\kappa$ many convergent sequences with their limit point glued together.

**Theorem 1.6.** For a surjective map $f : X \to Y$ from a $\mu$-complete space $X$ into a Fréchet-Urysohn space $Y$ the following conditions are equivalent:

1) $f$ is subproper;
2) $f$ has a section preserving precompact sets;
3) \( f \) has a cs\(^n\)-continuous section;  
4) \( f \) is inductively closed.
Moreover, if \( Y \) contains no closed subspace homeomorphic to the sequential fan \( S_\omega \), then (1)–(4) are equivalent to  
5) \( f \) is inductively perfect.

**Proof.** The equivalences (1) \( \iff \) (2) \( \iff \) (3) follow from Theorem 1.3 and the fact that each Fréchet-Urysohn compact space is sequentially compact. The implication (4) \( \Rightarrow \) (3) has been proved in Proposition 1.3 and (5) \( \Rightarrow \) (4) is trivial.

(3) \( \Rightarrow \) (4) Assume that \( f \) has a cs\(^n\)-continuous section \( s : Y \to X \) and let \( Z = \text{cl}_X(s(Y)) \). We claim that the restriction \( f|Z : Z \to Y \) is a closed map. Assume that this is not true. Then we can find a closed subset \( A \subset Z \) whose image \( f(A) \) is not closed in \( Y \). Since \( Y \) is a Fréchet-Urysohn space, there is a sequence \( (a_n) \subset A \) such that the sequence \( (f(a_n)) \) converges to a point \( y_0 \in Y \setminus f(A) \). It can be easily shown that \( B = \{a_n : n \in \mathbb{N}\} \) is a closed discrete subset of \( A \). Since \( B \) is not compact and the space \( X \) is \( \mu \)-complete, the set \( B \) is not bounded in \( X \). Consequently, there is a locally finite countable collection \( \mathcal{U} = \{U_n : n \in \omega\} \) of open sets of \( X \) meeting the set \( B \). For every \( i \in \mathbb{N} \) pick a point \( b_i \in B \cap U_i \). Since \( \mathcal{U} \) is an infinite locally finite family, the set \( \{b_i : i \in \mathbb{N}\} \subset B \) is infinite (and discrete in \( X \)). Then the sequence \( (f(b_i)) \) converges to \( y_0 \), being a subsequence of \( (a_n) \).

For each \( i \in \mathbb{N} \) fix an open neighborhood \( O(f(b_i)) \) of \( f(b_i) \) in \( Y \) whose closure does not contain the point \( y_0 \) and let \( W_i = U_i \cap f^{-1}(O(f(b_i))) \). Then \( \{W_i : i \in \mathbb{N}\} \) is a locally finite family of open subsets of \( X \) and for every \( i \in \mathbb{N} \) the closure of \( f(W_i) \) in \( Y \) does not contain the point \( y_0 \). Since \( b_i \in \text{cl}_X(W_i \cap s(Y)) \), by the continuity of \( f \), we get that \( f(b_i) \in \text{cl}_Y(f(W_i \cap s(Y))) \). Then \( y_0 \) lies in the closure of the union \( \bigcup_{i \in \mathbb{N}} f(W_i \cap s(Y)) \) in \( Y \). Since the space \( Y \) is Fréchet-Urysohn, there is a sequence \( y_n = \bigcup_{i \in \mathbb{N}} f(W_i \cap s(Y)) \) convergent to \( y_0 \). For each \( n \in \mathbb{N} \) find \( i(n) \in \mathbb{N} \) such that \( y_n \in f(W_{i(n)} \cap s(Y)) \). Passing to a subsequence, we can assume that \( i(n) \neq i(n') \) for any \( n \neq n' \). Then \( s(y_n) \in W_{i(n)} \) and hence \( \{s(y_n) : n \in \mathbb{N}\} \) is an infinite closed discrete subset of \( X \) which has no accumulation point in \( X \), a contradiction with the cs\(^n\)-continuity of \( s \) and the convergence of \( (y_n) \).

(4) \( \Rightarrow \) (5) Assume \( f \) is inductively closed and \( Y \) contains no closed subspace homeomorphic to the sequential fan \( S_\omega \). Let \( Z \) be a closed subset of \( X \), such that \( f(Z) = Y \) and the restriction \( f|Z : Z \to Y \) is a closed map. We shall show that the map \( f|Z \) is perfect. Take any section \( s : Y \to Z \) of the map \( f \). We loose no generality assuming that \( s(Y) \) is dense in \( Z \). Fix any point \( y_0 \in Y \) and assume that the pre-image \( f^{-1}(y_0) \cap Z \) is not compact. Since the space \( X \) is \( \mu \)-complete, the set \( f^{-1}(y_0) \cap Z \) is not bounded and thus there is an infinite locally finite family \( \mathcal{U} = \{U_n : n \in \omega\} \) of open sets in \( X \) meeting \( f^{-1}(y_0) \cap Z \). Without loss of generality, we may assume that \( s(y_0) \notin \cup \mathcal{U} \). For every \( n \in \omega \) pick a point \( z_n \in U_n \cap f^{-1}(y_0) \cap Z \). Since \( z_n \in \text{cl}_X(U_n \cap s(Y)) \), the continuity of \( f \) ensures that \( f(z_n) = y_0 \in \text{cl}_Y(f(U_n \cap s(Y))) \). Since the space \( Y \) is Fréchet-Urysohn, there is a sequence \( y_0 = \{y_{n,i}\}_{i=1}^{\infty} \subset f(U_n \cap s(Y)) \) convergent to \( y_0 \). This sequence is not trivial because \( s(y_0) \notin \cup U_n \cap s(Y) \) and hence \( y_0 \notin f(U \cap s(Y)) \). Passing to a subsequence we may assume that \( y_{n,i} \neq y_{n,j} \) for all \( i \neq j \).

We claim that there is an increasing number sequence \( (n_k) \) such that the sets \( T_{n_k} = \{y_{n_k,i}\}_{i=1}^{\infty} \subset f(U_n \cap s(Y)) \) converge to \( y_0 \). This completes the inductive construction.
We claim that $T = \{y_0, y_{n_k,i} : k, i \in \mathbb{N}\}$ is a closed subset of $Y$ homeomorphic to the sequential fan $S_\omega = S_1 \times \omega/\{0\} \times \omega$, where $S_1 = \{2^{-i} : i \in \omega\}$ is a convergent sequence. For this consider the map $h: S_\omega \to T$ assigning to the non-isolated point of $S_\omega$ the point $y_0$ and to each point $(2^{-i}, k) \in S_\omega$ the point $y_{n_k,i}$.

First we check that $T$ is closed in $Y$. Since $Y$ Fréchet-Urysohn, it suffices to verify that for each compact subset $K \subset Y$ the intersection $T \cap K$ is compact. Since the section $s: Y \to Z$ preserves precompact sets, the image $s(K)$ of $K$ is precompact in $X$ and hence meets only finitely many sets $U_{n_k} \supseteq s(T_{n_k})$. Consequently, $K$ meets only finitely many sets $T_{n_k}$ with $k \leq l$ for some $l$ and $K \cap T = \bigcup_{k \leq l} K \cap \text{cl}(T_{n_k})$ is compact.

The above argument shows also that each convergent sequence in $T$ lies in the union of finitely many sets $\text{cl}(T_{n_k})$, which implies the continuity of the map $h: T \to S_\omega$. The analogous property of the sequential fan ensures the continuity of the inverse map $h^{-1}: S_\omega \to T$.

**Remark 1.7.** The implication $(4) \Rightarrow (5)$ is due to A.Arkhangeski [2]. If $X$ and $Y$ are separable metrizable and $X$ is Polish or $Y$ is $\sigma$-compact, then the five equivalent conditions of Theorem 1.6 hold if and only if the map $f$ is compact-covering, see [21], [60], [63], [64], [42], [56]. Under certain additional set-theoretic assumptions (namely, the determinacy of all analytic games) each compact-covering map $f: X \to Y$ between coanalytic spaces is inductively perfect, see [25]. On the other hand, there is a model of ZFC (namely, Gödel’s Constructive Universe) in which there exists a compact-covering map $f: X \to \mathbb{N}^\omega$ from an $F_\sigma$-subset $X$ of the Baire space $\mathbb{N}^\omega$, which is not inductively perfect and thus is not subproper, see [26], [27]. This shows that the interplay between compact-covering and subproper maps is highly non-trivial and subtle even in the realm of separable metrizable spaces.

### 2. Operations over subproper maps

In this section we show that subproper maps are preserved by many topological constructions. We start from three simple observations.

**Proposition 2.1.** If $f: X \to Y$ is a subproper map, then for every subspace $Z \subset Y$ the map $f|f^{-1}(Z): f^{-1}(Z) \to Z$ is subproper.

**Proposition 2.2.** The composition $f \circ g : X \to Z$ of two subproper maps $g: X \to Y$ and $f: Y \to Z$ is subproper.

**Proposition 2.3.** For a family $\{f_i : X_i \to Y_i\}_{i \in I}$ of subproper maps the induced map $\prod_{i \in I} f_i : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ between Tychonoff products is subproper.

Subproper maps are also preserved by the construction of hyperspace. We recall that the hyperspace $\exp(X)$ of a topological space $X$ is the space of all non-empty compact subsets of $X$ endowed with the Vietoris topology generated by the base consisting of the sets

\[\{U_1, \ldots, U_n\} = \{K \in \exp(X) : K \subseteq \bigcup_{i=1}^n U_i, K \cap U_i \neq \emptyset \text{ for all } i \leq n\},\]

where $U_1, \ldots, U_n$ run over all open subsets of $X$. Observe that $K(X) = \exp(X) \cup \{\emptyset\}$. It is well-known [31, 3.12.26] that for any compact Hausdorff space $X$ its hyperspace $\exp(X)$ is compact as well. If the topology of a space $X$ is generated by a metric $d$, then the Vietoris topology on $\exp(X)$ is generated by the Hausdorff metric

\[d_H(A, B) = \max\{\max_{a \in A} d(a, B), \max_{b \in B} d(A, b)\}, \quad A, B \in \exp(X)\].

Given a map $f: X \to Y$ between topological spaces let $\exp(f): \exp(X) \to \exp(Y)$ be the map between their hyperspaces acting as $\exp(f)(K) = f(K)$ for $K \in \exp(X)$. 
Proposition 2.4. A map \( f : X \to Y \) is subproper if and only if so is the map \( \exp(f) : \exp(X) \to \exp(Y) \).

Proof. Assuming that the map \( f : X \to Y \) is subproper, let \( s : Y \to X \) be a section of \( f \) preserving precompact sets. Next, define a section \( l : \exp(Y) \to \exp(X) \) of \( \exp(f) \) letting \( l(K) = \text{cl}_X(s(K)) \) for \( K \in \exp(Y) \). We claim that the section \( l \) preserves precompact sets. Let \( K \) be any compact subset of \( \exp(Y) \).

First we show that the union \( \bigcup K \) is a compact subset of \( Y \). Let \( U \) be a cover of \( \bigcup K \) by open subsets of \( Y \). Assuming that no finite subfamily \( F \subset U \) covers \( \bigcup K \), find a compact set \( K_F \subseteq K \) with \( K_F \not\subseteq \bigcup F \). We can consider the family \( \{U\}^{<\omega} \) of finite subsets of \( U \) as a partially ordered set ordered by the inclusion relation. The compactness of \( K \) ensures the existence of an accumulation point \( K_\infty \in K \) of the net \( (K_F)_{F \in \{U\}^{<\omega}} \). The latter means that for any open neighborhood \( O(K_\infty) \subset \exp(Y) \) and any finite subset \( F \subset \{U\}^{<\omega} \) there is another finite subset \( E \subset \{U\}^{<\omega} \) containing \( F \) and such that \( K_F \in O(K_\infty) \). Using the compactness of \( K_\infty \in \bigcup K \subset \bigcup U \) find a finite family \( F \subset \{U\} \) with \( K_\infty \subset \bigcup F \). Then \( (\bigcup F) = \{K \in \exp(Y) : K \subset \bigcup F\} \) is an open neighborhood of \( K_\infty \) in \( \exp(Y) \). The accumulation property of \( K_\infty \) ensures the existence of a finite set \( E \subset \bigcup U \) containing \( F \) and such that \( K_E \in (\bigcup F) \). Then \( K_E \subset \bigcup F \subset \bigcup E \). On the other hand, \( K_E \not\subseteq \bigcup E \) by the choice of \( K_E \). This contradiction completes the proof of the compactness of \( \bigcup K \).

Since the section \( s \) is precompact-preserving, the set \( C = \text{cl}_X(s(\bigcup K)) \) is compact in \( X \) and \( \exp(C) \) is a compact subset of \( \exp(X) \). Observing that

\[
 l(K) \subset \text{cl}_X(s(K)) \subset \text{cl}_X(s(\bigcup K)) = C
\]

for every \( K \in K \), we see that \( l(K) \subset \exp(C) \) and thus \( l(K) \) is a precompact subset of \( \exp(X) \).

Next, assume conversely that the map \( \exp(f) : \exp(X) \to \exp(Y) \) is subproper. Then there is a section \( l : \exp(Y) \to \exp(X) \) of \( \exp(f) \) preserving precompact sets. For each \( y \in Y \) let \( s(y) \) be any point of the compact subset \( l(y) \) of \( X \). Since \( \exp(f)(l(y)) = f(l(y)) = \{y\} \), we get that the so-defined function \( s : Y \to X \) is a section of the map \( f \). To verify that \( s \) preserves precompact subsets, fix any compact subset \( K \) of \( Y \). It follows that \( K = \{y \in K \} \) is a compact subset of \( \exp(Y) \) and hence \( l(K) = \{l(y) : y \in K \} \) is a precompact subset of \( \exp(X) \) contained in some compact subset \( C \) of \( \exp(X) \). Then \( \bigcup C \) is a compact subset of \( X \) containing all sets \( l(y) \) for \( y \in K \). Hence \( s(K) \subset \bigcup C \) is a precompact subset of \( X \).

The constructions of the hyperspace \( \exp \) is an example of a functorial construction on the category \( \text{Haus} \) of Hausdorff spaces and their continuous maps. The above results suggest the following general

**Problem 2.5.** Which functorial topological constructions do preserve subproper maps?

We shall answer this question for functors with compact continuous support. For basic concepts of categorial topology we refer the reader to [74]. Let \( F : \text{Haus} \to \text{Haus} \) be a functor on the category \( \text{Haus} \) of Hausdorff spaces and let \( X \) be a Hausdorff space. We say that a point \( a \in F(X) \) is supported by a subset \( K \subset X \) if \( a \in F(e_K)(F(K)) \), where \( e_K : K \to X \) stands for the natural inclusion. We say that a functor \( F : \text{Haus} \to \text{Haus} \) has compact continuous support if there is a natural transformation \( \text{supp} : F \to \text{exp} \) of the functor \( F \) into the hyperspace functor \( \text{exp} \) such that for every Hausdorff space \( X \) the component \( \text{supp}_X : F(X) \to \text{exp}(X) \) is a continuous map such that each element \( a \in F(X) \) is supported by the compact set \( \text{supp}_X(a) \subset X \). We say that a functor \( F : \text{Haus} \to \text{Haus} \) preserves surjective maps between compact spaces if for every surjective map \( f : X \to Y \) between compact Hausdorff spaces the spaces \( F(X) \) and \( F(Y) \) are compact and the map \( F(f) : F(X) \to F(Y) \) is surjective.
Proposition 2.6. Suppose $F : \text{Haus} \to \text{Haus}$ is a functor with compact continuous support that preserves surjective maps between compact spaces. Then for every subproper map $f : X \to Y$ the map $F(f) : F(X) \to F(Y)$ is subproper.

Proof. Let $f : X \to Y$ be a subproper map and let $\text{supp} : F \to \exp$ be the natural transformation such that for each Hausdorff space $Z$ the component $\text{supp}_Z : F(Z) \to \exp(Z)$ is a continuous map and each $a \in F(Z)$ is supported by the compact set $\text{supp}_Z(a)$. Let $s : Y \to X$ be a section preserving precompact sets. Define a section $l : F(Y) \to F(X)$ of the map $F(f)$ as follows. Given a point $a \in F(Y)$ let $C = \text{supp}_Y(a)$ and $K = \text{cl}_X(s(C))$. Find a point $c \in F(C)$ such that $F(e_C)(c) = a$ where $e_C : C \to Y$ is the identity inclusion. Since the functor $F$ preserves surjective maps between compacta, there is a point $b \in F(K)$ such that $F(f|K)(b) = c$, where $f|K : K \to C$ is the restriction of $f$. Let finally $l(a) = F(e_K)(b) \in F(X)$ where $e_K : K \to X$ is the identity embedding.

It follows that

$$F(f)(l(a)) = F(f) \circ F(e_K)(b) = F(f \circ e_K)(b) = F(e_C \circ f|K)(b) = F(e_C) \circ F(f|K)(b) = F(e_C)(c) = a$$

and thus the so-defined function $l : F(Y) \to F(X)$ is a section of the map $F(f)$.

To show that this section preserves precompact sets, fix any compact subset $K \subset F(Y)$. By the continuity of the map $\text{supp}_Y$, the image $\text{supp}_Y(K)$ of $K$ is a compact subset of the hyperspace $\exp(Y)$. Then the union $A = \cup \text{supp}_Y(K)$ is a compact subset of $Y$. Let $B = \text{cl}_X(s(A))$. Since $F$ preserves surjective maps between compacta, the space $F(B)$ is compact and so is its image $F(e_B)(F(B))$ in $F(X)$ where $e_B : B \to X$ stands for the identity inclusion. It follows from the construction that $l(a) \in F(e_B)(F(B))$ for each $a \in K$ which yields that $l(K)$ is a precompact subset of $F(X)$. □

Remark 2.7. There are many examples of functors satisfying the requirements of Proposition 2.6; see [34, VII.§1] or [74] §2.7. Among them there are the functor $(\cdot)^n$ of finite power, the functor $SP^n$ of symmetric power, the subfunctors $\exp_n$ of the hyperspace functor, etc.

We finish this section by establishing an “upper semicontinuity” property of $c^*$-continuous functions. Let us recall that the upper limit $\overline{\lim}_{n \to \infty} F_n$ of a sequence $(A_n)_{n \in \omega}$ of subsets of a topological space $X$ is the set of all points $x \in X$ such that any neighborhood $U_x$ of $x$ meets infinitely many sets $A_n$. We shall write that $(A_n) \nearrow \overline{\lim}_{n \to \infty} A_n$ if each neighborhood of $\overline{\lim}_{n \to \infty} A_n$ contains all but finitely many sets $A_n$.

Proposition 2.8. Let $s : Y \to X$ be a $c^*$-continuous function and $(A_n)_{n \in \omega}$ be a sequence of subsets of $Y$ such that $K = \overline{\lim}_{n \to \infty} A_n$ is metrizable compact and $(A_n) \nearrow K$. Then the upper limit $B = \overline{\lim}_{n \to \infty} s(A_n)$ is closed and bounded in $X$ and $(s(A_n)) \nearrow B$. If the space $X$ is $\mu$-complete, then $B$ is compact.

Proof. It is clear that $B = \overline{\lim}_{n \to \infty} s(A_n)$ is closed. Assuming that this set is unbounded, find an infinite locally finite family $\mathcal{U} = \{U_k : k \in \omega\}$ of open subsets of $X$ that intersect $B$. By induction construct an increasing number sequence $(n_k)_{k \in \omega}$ and a sequence of points $y_k \in A_{n_k}$ such that $s(y_k) \in U_k$. It follows from the compactness and metrizability of $K$ and the convergence $(A_n) \nearrow K$ that the set $L = K \cup \{y_k : k \in \omega\}$ is compact and metrizable. Consequently, the sequence $(y_k)$ has a convergent subsequence $(y_{k_i})$. By the $c^*$-continuity of $s$ the sequence $(s(y_{k_i}))$ has an accumulation point $x_\infty$ in $X$. Any neighborhood of this point meets infinitely many sets $U_{k_i}$ which contradicts the local finity of $\mathcal{U}$.

Therefore $B$ is closed and bounded. If $X$ is $\mu$-complete, then $B$ is compact by the definition of $\mu$-completeness.
Next, we show that $(s(A_n)) \not\subset B$. Assuming the converse, we can find a neighborhood $W$ of $B$ in $X$ such that the set $J = \{n \in \omega : s(A_n) \not\subset W\}$ is infinite. Then we can construct a sequence $(y_n)_{n \in J}$ such that each $y_n \in A_n$ and $s(y_n) \not\in W$. It follows from $(A_n) \not\subset K$ that the set $K \cup \{y_n : n \in J\}$ is compact and metrizable and the sequence $(y_n)_{n \in J}$ has a convergent subsequence $(y_n)$. By the cs$\ast$-continuity of the section $s$, the sequence $\{s(y_n)\}$ has an accumulating point $x_{\infty}$ in $X$. By definition, $x_{\infty} \in \lim_{n \to \infty}s(A_n) = B$. On the other hand, $x_{\infty} \in \text{cl}\{s(y_n) : n \in J\} \subset X \setminus W \subset X \setminus B$, which is a contradiction. \hfill \Box

3. $k^\ast$-Metrizable Spaces

It is well-known that the image of a metrizable space under a perfect map is metrizable \cite{31, 4.4.15}. The images of metrizable spaces under (sub)proper need not be metrizable, which allows us to introduce two new classes of generalized metric spaces.

**Definition 3.1.** A topological space $X$ is defined to be $k$-metrizable (resp. $k^\ast$-metrizable) if $X$ is the image of a metrizable space $M$ under a proper (resp. subproper) map $\pi : M \to X$.

In fact, the $k$-metrizability of a space $X$ is equivalent to the metrizability of the $k$-coreflexion of $X$. By the $k$-coreflexion $kX$ of a topological space $X$ we understand the set $X$ endowed with the strongest topology inducing the original topology on each compact subset $K \subset X$. It is clear that the identity map $i : kX \to X$ is continuous. Moreover, for any map $f : Z \to X$ from a $k$-space $Z$ the composition $i^{-1} \circ f : Z \to kX$ still is continuous. Thus $kX$ carries the weakest $k$-space topology that is stronger than the original topology of $X$. We recall that a topological space $X$ is a $k$-space if a subset $F \subset X$ is closed if and only if for any compact set $K \subset X$ the intersection $K \cap F$ is closed in $K$.

$k$-Metrizable spaces admit a simple characterization.

**Proposition 3.2.** A topological space $X$ is $k$-metrizable if and only if its $k$-coreflexion $kX$ is metrizable.

*Proof.* Since the identity map $i : kX \to X$ is proper, the space $X$ is $k$-metrizable if $kX$ is metrizable. Now assume conversely that $X$ is $k$-metrizable and find a proper map $\pi : M \to X$ from a metrizable space $M$. It follows that the composition $i^{-1} \circ \pi : M \to kX$ is proper and hence perfect because $kX$ is a $k$-space, see \cite{31} 3.7.17. Then the space $kX$ is metrizable, being a perfect image of a metrizable space $M$, see \cite{31} 4.4.15. \hfill \Box

The $k^\ast$-metrizability also can be reduced to considering the $k$-coreflexion.

**Proposition 3.3.** A space $X$ is $k^\ast$-metrizable if and only if its $k$-coreflexion $kX$ is $k^\ast$-metrizable.

*Proof.* Let $i : kX \to X$ denote the identity map. Assuming that $X$ is $k^\ast$-metrizable, find a subproper map $\pi : M \to X$ for a metrizable space $M$. Since $M$ is a $k$-space, the composition $i^{-1} \circ \pi : M \to kX$ is continuous and subproper, which implies that $kX$ is $k^\ast$-metrizable.

Assuming conversely that $kX$ is $k^\ast$-metrizable, find a subproper map $\pi : M \to kX$ from a metrizable space and consider the composition $i \circ \pi : M \to X$ which is a subproper map because the function $i^{-1} : X \to kX$ preserves compacta. \hfill \Box

It turns out that sequentially regular $k^\ast$-metrizable spaces carry a nice metric. We define a topological space to be *sequentially regular* if for each point $x \in X$ and a neighborhood $U \subset X$ of $x$ there is another neighborhood $V \subset X$ of $x$ such that $\text{cl}_1(V) \subset U$, where $\text{cl}_1(V)$ is the set of the limit points of sequences $\{x_n\}_{n \in \omega} \subset V$, convergent in $X$. It is clear that each regular space is sequentially regular.
Theorem 3.4. A (sequentially regular) space \( X \) is \( k^* \)-metrizable (if and) only if \( X \) has a metric \( \rho \) such that

(a) each compact subset \( K \subset X \) is totally bounded with respect to the metric \( \rho \);
(b) each \( \rho \)-convergent sequence converges in \( X \);
(c) a \( \rho \)-Cauchy sequence converges in \( X \) if and only if it has a convergent subsequence in \( X \).

Proof. To prove the “only if” part, assume that \( X \) is the image of a metric space \((M, d)\) under a subproper map \( \pi : M \to X \). Find a section \( s : X \to M \) of \( \pi \) that preserves precompact sets and consider the metric \( \rho(x, x') = d(s(x), s(x')) \) on the space \( X \), induced by the metric \( d \) of \( M \). It is easy to check that the so-defined metric \( \rho \) on \( X \) has the three properties indicated in the theorem.

Now assume conversely that a sequentially regular space \( X \) admits metric \( \rho \) with properties (a)–(c). Consider the metric space \( Z = (X, \rho) \) and its completion \( \tilde{Z} \). The condition (b) implies that the identity map \( \pi : Z \to X \) is continuous. Let \( M \) be the set of all points \( z \in \tilde{Z} \) such that for any sequence \((z_n) \subset Z\) convergent to \( z \) the sequence \((\pi(z_n)) \) converges to some point \( x \). Letting \( \pi(z) = x \) one defines a map \( \tilde{\pi} : M \to X \) extending the identity map \( \pi : Z \to X \). The sequential regularity of \( X \) can be used to show that the map \( \tilde{\pi} : M \to X \) is continuous. We claim that \( \tilde{\pi} \) is subproper. Fix any compact subset \( K \subset X \). This set is totally bounded in the metric space \((Z, \rho)\) and thus has compact closure \( \overline{K} \subset \tilde{Z} \). It remains to check that \( \overline{K} \subset M \). Assuming that this is not so, find a point \( z \in \overline{K} \setminus M \). The compact space \( K \), being the continuous image of a totally bounded (and hence separable) metric space, has countable network and thus is metrizable. Let \((x_{2n})_{n \in \omega} \subset K \) be a sequence convergent to the point \( z \) in the metric space \( \tilde{Z} \). The compactness of \( K \) implies that this sequence contains a subsequence convergent to some point \( x \) in \( K \). The condition \( z \notin M \) implies the existence of a sequence \((x_{2n+1})_{n \in \omega} \subset Z \) convergent to \( z \) in \( \tilde{Z} \) whose image \((\pi(x_{2n+1}))_{n \in \omega} \) diverges in \( X \). Then the sequence \((x_n)_{n \in \omega} \) is \( \rho \)-Cauchy and contains a subsequence convergent to \( x \) in \( X \). The property (c) of the metric \( \rho \) implies that the sequence \((x_n)_{n \in \omega} \) converges to \( x \) in the space \( X \) which is not possible because \((x_{2n+1})_{n \in \omega} \) does not converge to \( x \). This contradiction shows that the set \( \pi^{-1}(K) = Z \cap \pi^{-1}(K) \subset \overline{K} \subset M \) is precompact in \( M \), which means that \( X \) is \( k^* \)-metrizable, being the image of the metrizable space \( M \) under the subproper map \( \tilde{\pi} : M \to X \).

Let us remark some obvious properties of \( k^* \)-metrizable spaces.

Theorem 3.5. 1. A subspace of a \( k^* \)-metrizable space is \( k^* \)-metrizable.
2. A countable product of \( k^* \)-metrizable spaces is \( k^* \)-metrizable.
3. The topological sum of arbitrary family of \( k^* \)-metrizable spaces is \( k^* \)-metrizable.
4. The image of a \( k^* \)-metrizable space under a subproper map is \( k^* \)-metrizable.
5. Each sequentially compact subset of a \( k^* \)-metrizable space is metrizable.
6. A \( k \)-metrizable space is sequential if and only if it is a \( k \)-space.

Therefore the \( k^* \)-metrizability is preserved by subproper maps. In the class of \( k \)-spaces the same is true for closed maps.

Proposition 3.6. If \( f : X \to Y \) is a closed map from a \( k^* \)-metrizable \( k \)-space \( X \), then \( Y \) is \( k^* \)-metrizable as well.

Proof. Let \( \pi : M \to X \) be a subproper map of a metrizable space \( M \) onto the \( k^* \)-metrizable \( k \)-space \( X \). The proposition will be proven as soon as we check that the composition \( f \circ \pi : M \to Y \) is subproper. Let \( s : Y \to X \) be any section of the map \( \pi \) and \( \sigma : X \to M \) be a precompact-preserving section of \( \pi \). We claim that \( \sigma \circ s : Y \to M \) also preserves precompact
Proposition 3.7. A $k^*$-metrizable space $X$ is compact and metrizable iff the $k$-coreflexion $kX$ of $X$ is countably compact.

Proof. Let $\pi : M \to X$ be a subproper map from a metrizable space $M$ onto $X$ and $s : X \to M$ is a section of $\pi$ preserving precompact sets. Suppose that $kX$ is countably compact. We claim that the closure $Z = s(X)$ of $s(X)$ in $M$ is compact. Assuming that this is not true, we can find an infinite subset $D \subset s(X)$ that is closed and discrete in $X$. Then the image $\pi(D)$ has finite intersection with each compact subset of $X$ and hence closed and discrete in the $k$-coreflexion $kX$, which is not possible as $kX$ is countably compact. Therefore $Z$ is compact metrizable space and $X$, being the continuous image of $Z$, is compact and metrizable too. □

Remark 3.8. Surprisingly, but a countably compact $k^*$-metrizable space need not be metrizable. It was shown by Frolik \[37\] (see also \[39, 3.23\]) that $\beta N$ contains a countably compact subspace $X \subseteq N$ of size $2^c$ and (network) weight $\mathfrak{c}$ such that all compact subsets of $X$ and $\beta N \setminus X$ are finite. It follows that the $k$-coreflexion of $X$ is discrete and hence both $kX$ and $X$ are $k^*$-metrizable. On the other hand $X$ is a non-metrizable separable, countably compact space.

Next, we show that a space $X$ is $k^*$-metrizable if $X$ admits a special cover by $k^*$-metrizable spaces.

We define a subset $A$ of a topological space $X$ to be $k$-closed if for each compact subset $K \subset X$ the intersection $A \cap K$ is closed in $K$. This is equivalent to saying that $A$ is closed in the $k$-coreflexion $kX$ of $X$.

A collection $\mathcal{F}$ of subsets of a space $X$ is defined to be compact-finite if for every compact subset $K \subset X$ the set $\{F \in \mathcal{F} : F \cap K \neq \emptyset\}$ is finite. It is clear that each locally finite collection is compact-finite and each compact-finite collection is point-finite. In Fréchet-Urysohn spaces each compact-finite family is locally-finite.

Theorem 3.9. Let $X$ be a topological space. Then

1) $X$ is $k^*$-metrizable if and only if there is a compact finite cover $\mathcal{C}$ of $X$ such that each $C \in \mathcal{C}$ lies in a $k$-closed $k^*$-metrizable subspace of $X$;

2) $X$ is $k^*$-metrizable if and only if there is a countable collection $(X_n)_{n=1}^\infty$ of $k$-closed $k^*$-metrizable subspaces of $X$ such that every convergent sequence $S \subset X$ is contained in some $X_n$.

Proof. 1. Let $\mathcal{C}$ be a compact-finite cover of $X$ such that for each set $C \in \mathcal{C}$ there is a $k$-closed subspace $\tilde{C} \supset C$ of $X$ such that $\tilde{C}$ is $k^*$-metrizable and hence $\tilde{C}$ is the image of a metrizable space $M_{\tilde{C}}$ under a continuous map $\pi_{\tilde{C}} : M_{\tilde{C}} \to \tilde{C}$ admitting a section $s_{\tilde{C}} : \tilde{C} \to M_{\tilde{C}}$ that preserves precompact sets. It is clear that the topological sum $M = \bigoplus_{C \in \mathcal{C}} M_{C}$ is a metrizable space and the map $\pi = \bigcup_{C \in \mathcal{C}} \pi_{C} : M \to \bigcup_{C \in \mathcal{C}} \tilde{C} = X$ is continuous. Given a point $x \in X$, find any subset $C(x) \in \mathcal{C}$ containing $x$ and let $s(x) = s_{C(x)}(x) \in M_{C(x)} \subset M$. We claim that the section $s : X \to M$ of $\pi$ preserves precompact sets. Given a compact subset $K \subset X$ observe...
that the family $C' = \{ C \in C : C \cap K \neq \emptyset \}$ is finite and thus $s(K)$ lies in the precompact subset $\bigcup_{C \in C'} s_C'(K \cap C)$ of $M$.

2. Suppose that $\{X_n\}_{n \in \mathbb{N}}$ is a countable collection of $k$-closed $k^*$-metrizable subspaces of $X$ such that each convergent sequence $K \subset X$ lies in some $X_n$. For every $n \in \mathbb{N}$ let $Y_n = X_n \setminus \bigcup_{i \leq n} X_i$. The $k^*$-metrizability of $X$ will follow from the preceding item as soon as we check that the family $\{Y_n\}_{n \in \mathbb{N}}$ is compact-finite. Take any compact subset $K \subset X$. Assuming that $K \not\subset X_n$ for all $K$, construct a sequence $(x_n)$ in $K$ such that $x_n \not\in X_n$ for all $n$. We claim that the sequence $(x_n)$ has a convergent subsequence. Since each set $X_n$ is $k$-closed, the intersection $K \cap X_n$, being a compact subset of the $k^*$-space $X_n$ is metrizable. Then $K = \bigcup_{n} K \cap X_n$, being the countable union of metrizable compacta, has countable network of the topology and hence is metrizable and sequentially compact. So we can find a convergent subsequence $(x_{n_k})$ of $(x_n)$ lying in no subset $X_n$, which contradicts the choice of the sets $X_n$.

We saw in Theorem 3.5(2) that the countable product of $k^*$-metrizable spaces is $k^*$-metrizable. The same is true for the box products, that is Cartesian products $\prod_{i \in I} X_i$ endowed with the box-product topology generated by sets of the form $\bigcap_{i \in J} X_i$ for subsets $J \subset I$.

**Theorem 3.10.** Let $\{X_i : i \in I\}$ be a family of $k^*$-metrizable spaces. Then the box-product $\Box_{i \in I} X_i$ is $k^*$-metrizable.

**Proof.** Elements of the box-product $\Box_{i \in I} X_i$ can be thought as functions $x : I \to \bigcup_{i \in I} X_i$ such that $x(i) \in X_i$ for $i \in I$. For two functions $x, y \in \Box_{i \in I} X_i$ let

$$\{ x \neq y \} = \{ i \in I : x(i) \neq y(i) \} \text{ and } \sigma(x) = \{ y \in \Box_{i \in I} X_i : |\{ y \neq x \}| < \omega \}.$$

Let us show that each set $\sigma(x)$ is closed in $\Box_{i \in I} X_i$. Indeed, any point $y \in \Box_{i \in I} X_i \setminus \sigma(x)$ can be separated from $\sigma(x)$ by the box-neighborhood $\prod_{i \in I} U_i$ of $y$, where $U_i = X_i \setminus \{ x(i) \}$ if $i \in \{ x \neq y \}$ and $U_i = X_i$ otherwise. Observe also that for two points $x, x' \in \Box_{i \in I} X_i$, the sets $\sigma(x)$ and $\sigma(x')$ either coincide or are disjoint. Then $\Sigma = \{ \sigma(x) : x \in \Box_{i \in I} X_i \}$ is a disjoint closed cover of the box-product $\Box_{i \in I} X_i$. We claim that $\Sigma$ is a compact-finite cover of $X$ consisting of closed $k^*$-metrizable subspaces of $\Box_{i \in I} X_i$.

First we check that the cover $\Sigma$ is compact-finite. Take any compact set $K$ and assume that it intersects infinitely many sets $\sigma(x_n)$, where $x_n \not\in \sigma(x_m)$ for all $n \neq m$. By the compactness of $K$ the sequence $(x_n)$ has an accumulation point $x_\infty$ in $K$. This point $x_\infty$ can belong to at most one set $\sigma(x_n)$. Without loss of generality, $x_\infty \not\in \sigma(x_n)$ for all $n$, which means that the sets $\{ x_n \neq x_\infty \}$ are infinite. This allows us to construct an injective sequence $(i_n)$ in $I$ such that $x_\infty(i_n) \neq x_n(i_n)$ for all $n$. For every index $i \in I$ let

$$U_i = \begin{cases} X_i \setminus \{ x_n(i_n) \} & \text{if } i = i_n \text{ for some } n \\ X_i & \text{otherwise} \end{cases}$$

Then $\prod_{i \in I} U_i$ is an open neighborhood of $x_\infty$ in $\Box_{i \in I} X_i$ that contains no point of the sequence $(x_n)$, which is impossible as $x_\infty$ is an accumulation point of $(x_n)$. Therefore $\Sigma$ is a compact-finite cover of $\Box_{i \in I} X_i$.

Next, we check that each set $\sigma(x) \in \Sigma$ is $k^*$-metrizable. Given a finite subset $F \subset I$ let

$$\sigma_F(x) = \{ y \in \sigma(x) : \{ x \neq y \} = F \} \text{ and } \Pi_F(x) = \{ y \in \sigma(x) : \{ x \neq y \} \subset F \}.$$

It is clear that $\sigma_F(x) \subset \Pi_F(x)$ and $\Pi_F(x)$ is a closed subspace of $\sigma(x)$ homeomorphic to the finite product $\prod_{i \in F} X_i$, which $k^*$-metrizable according to Theorem 3.5(2). Repeating the preceding argument, we may show for each compact subset $K \subset \sigma(x)$ the set $F = \bigcup_{y \in K} \{ y \not\in \}$
\(x\) is finite and hence \(K \subset \prod_{F}(x)\). This implies that the cover \(\{\sigma_{F}(x) : F \subset I, |F| < \infty\}\) of \(\sigma(x)\) is compact-finite. Since each \(\sigma_{F}(x)\) lies in the closed \(k^{*}\)-metrizable subspace \(\prod_{F}(x)\) of \(\sigma(x)\), we can apply Theorem 3.10 to conclude that the space \(\sigma(x)\) is \(k^{*}\)-metrizable.

By Theorem 3.10 the box-product \(\square_{i \in I}X_{i}\) is \(k^{*}\)-metrizable being the union of a compact-finite cover consisting of closed \(k^{*}\)-metrizable subspaces. \(\square\)

**Remark 3.11.** The \(\sigma\)-product \(\sigma(x) \subset \prod_{i \in I}X_{i}\) of metrizable spaces \(X_{i}\) endowed with the Tychonoff product-topology is \(k^{*}\)-metrizable if and only if it is metrizable. This follows from the fact that any nonmetrizable space \(\sigma(x)\) contains a compact subspace homeomorphic to the one-point compactification of an uncountable discrete space. Then \(\sigma(x)\) cannot be the image of a metrizable space under a compact-covering map.

The \(k^{*}\)-metrizability is also preserved by the construction of hyperspace. This follows from Proposition 2.4.

**Proposition 3.12.** The hyperspace \(\exp(X)\) of any \(k^{*}\)-metrizable space is \(k^{*}\)-metrizable.

Applying Theorem 9.6, we get a more general

**Proposition 3.13.** Let \(F : \text{Haus} \rightarrow \text{Haus}\) be a functor with compact continuous support that preserves metrizable spaces and surjective maps between compact spaces. Then for every \(k^{*}\)-metrizable space the space \(F(X)\) is \(k^{*}\)-metrizable.

4. \(cs^{*}\)-Metrizable Spaces

In fact, the \(k^{*}\)-metrizability decomposes into two weaker properties: the \(cs^{*}\)-metrizability and the sequential compactness of all compact subsets.

**Definition 4.1.** A topological space \(X\) is defined to be \(cs^{*}\)-metrizable (resp. \(cs\)-metrizable) if \(X\) is the image of a metrizable space \(M\) under a map \(\pi : M \rightarrow X\) having a \(cs^{*}\)-continuous (resp. sequentially continuous) section.

The \(k\)-metrizability (resp. \(k^{*}\)-metrizability) can be characterized via \(cs^{*}\)-metrizability (resp. \(cs^{*}\)-metrizability) as follows.

**Proposition 4.2.** A topological space \(X\) is \(k^{*}\)-metrizable (resp. \(k\)-metrizable) if and only if \(X\) is \(cs^{*}\)-metrizable (resp. \(cs\)-metrizable) and all compact subsets of \(X\) are sequentially compact.

**Proof.** Assuming that a space \(X\) is \(k^{*}\)-metrizable, find a subproper map \(\pi : M \rightarrow X\) from a metrizable space. The space \(M\), being metrizable is \(\mu\)-complete. Then Proposition 1.3 implies that \(\pi\) has a \(cs^{*}\)-continuous section, which means that \(\pi\) is \(cs^{*}\)-metrizable. Since \(\pi\) is compact-covering, all compact subsets of \(X\) are metrizable and hence sequentially compact.

Now assume conversely that \(X\) is \(cs^{*}\)-metrizable and all compact subsets of \(X\) are sequentially compact. Find a map \(\pi : M \rightarrow X\) possessing a \(cs^{*}\)-continuous section. By Proposition 1.3 this map is subproper and thus \(X\) is \(k^{*}\)-metrizable.

The equivalence of the \(k\)-metrizability to the \(cs\)-metrizability plus the sequential compactness of all compact subsets can be proved by analogy.

By analogy with the \(k\)-metrizability (which is equivalent to the metrizability of the \(k\)-coreflexion), the \(cs\)-metrizability of a space \(X\) is equivalent to the metrizability of the sequential coreflexion \(sX\) of \(X\).

We recall that a topological space \(X\) is sequential if each sequentially closed subset of \(X\) is closed. A subset \(F \subset X\) is sequentially closed in \(X\) if it contains the limits of all sequences \((x_{n}) \subset F\) convergent in \(X\).
By the sequential coreflexion $sX$ of a space $X$ we understand $X$ endowed with the topology consisting of sequentially open subsets of $X$. A subset $U \subset X$ is called sequentially open if its complement $X \setminus U$ is sequentially closed in $X$. It is clear that the identity maps $sX \to kX \to X$ are continuous and all these maps are homeomorphisms if the space $X$ is sequential. In fact, the identity map $sX \to X$ always is a sequential homeomorphism. A bijective function $h: X \to Y$ between topological spaces is called a sequential homeomorphism if both $h$ and $h^{-1}$ are sequentially continuous. Topological spaces $X, Y$ are sequentially homeomorphic if there is a sequential homeomorphism $h: X \to Y$.

The following theorem is a counterpart of Propositions 3.2, 3.3 and Theorem 3.4 and can be proved by analogy.

**Theorem 4.3.**

1. A topological space $X$ is cs-metrizable iff its sequential coreflexion $sX$ is metrizable.
2. A topological space $X$ is cs*-metrizable iff its sequential coreflexion $sX$ is a cs*-metrizable iff $sX$ is $k^*$-metrizable.
3. A regular space $X$ is cs*-metrizable if and only if $X$ has a metric $\rho$ such that
   (a) each $\rho$-convergent sequence converges in $X$;
   (b) each convergent sequence in $X$ contains a $\rho$-Cauchy subsequence;
   (c) a $\rho$-Cauchy sequence converges in $X$ if and only if it has a convergent subsequence in $X$.

Properties of cs*-metrizable spaces are analogous to those of $k^*$-metrizable spaces. We define a family $\mathcal{F}$ of subsets cs-finite if each convergent sequence meets only finitely many sets $F \in \mathcal{F}$.

**Theorem 4.4.**

1. A subspace of a cs*-metrizable space is cs*-metrizable.
2. The topological sum of arbitrary family of cs*-metrizable spaces is cs*-metrizable.
3. A space $X$ is cs*-metrizable if $X$ is the union of a cs-finite cover $C$ such that each $C \in C$ lies in a sequentially closed cs*-metrizable subspace of $X$.
4. A space $X$ is cs*-metrizable if $X$ is the union $X = \bigcup_{n \in \omega} X_n$ of a sequence of sequentially closed cs*-metrizable subspaces such that each convergent sequence lies in some $X_n$.
5. The countable product of cs*-metrizable spaces is cs*-metrizable.
6. The box-product $\Box_{i \in I} X_i$ of arbitrary family of cs*-metrizable spaces is cs*-metrizable.
7. Each sequentially compact subset of a cs*-metrizable space is metrizable.
8. The image of a sequential cs*-metrizable space under a closed or subproper map is cs*-metrizable.
9. A space $X$ is countably compact.

Thus the $k^*$-, $k^*$-, cs-, and cs*-metrizability properties relate as follows:

- metrizable $\longrightarrow$ k-metrizable $\longrightarrow$ k*-metrizable $\downarrow$
- cs-metrizable $\longrightarrow$ cs*-metrizable

**Remark 4.5.** None arrow in this diagram can be reversed:

(a) the Banach space $(l^1, \text{weak})$ endowed with the weak topology is an example of a $k^*$-metrizable space that is not metrizable;
(b) the inductive limit $\mathbb{R}^\infty = \lim_{\longrightarrow} \mathbb{R}^n$ is $k^*$-metrizable but not $k$-metrizable, see Theorem 11.1.
(c) the Stone-Čech compactification $\beta \omega$ of $\omega$ contains no non-trivial convergent sequences and hence is cs-metrizable but fails to be $k^*$-metrizable.
(d) The product $\beta \omega \times \mathbb{R}^\infty$ is cs*-metrizable but is neither cs-metrizable nor $k^*$-metrizable.

5. Cardinal invariants of $k^*$-metrizable spaces

In this section we calculate some cardinal invariants of $k^*$-metrizable spaces. We recall that for a topological space $X$

- $w(X)$, the weight of $X$, is the smallest size of a base of the topology of $X$;
- $nw(X)$, the network weight of $X$, is the smallest size of a network $\mathcal{N}$ for $X$ ($\mathcal{N}$ is a network if for any open set $U \subset X$ and a point $x \in U$ there is $N \in \mathcal{N}$ with $x \in N \subset U$);
- $knw(X)$, the $k$-network weight of $X$, is the smallest size of a $k$-network $\mathcal{N}$ ($\mathcal{N}$ is a network if for any open set $U \subset X$ and a compact set $K \subset U$ there is a finite subfamily $\mathcal{F} \subset \mathcal{N}$ with $K \subset \cup \mathcal{F} \subset U$);
- $l(X)$, the Lindelöf number, is the smallest cardinal $\kappa$ such that each open cover $\mathcal{U}$ of $X$ has a subcover $\mathcal{V} \subset \mathcal{U}$ of size $|\mathcal{V}| \leq \kappa$;
- $ml(X)$, the meta-Lindelöf number, is the smallest cardinal $\kappa$ such that each open cover of $X$ has an open refinement $\mathcal{U}$ such that $|\{U \in \mathcal{U} : x \in U\}| \leq \kappa$ for every point $x \in X$;
- $s(X) = \sup\{|D| : D$ is a discrete subspace of $X\}$ is the spread of $X$;
- $ext(X) = \sup\{|D| : D$ is a closed discrete subspace of $X\}$ is the extent of $X$;
- $d(X)$, the density of $X$, is the smallest size of a dense set in $X$.

For any metrizable space $X$ the cardinals $w(X), knw(X), nw(X), l(X), s(X), ext(X), d(X)$ coincide (see [31, 4.1.15]) while for any topological space $X$ we have the inequalities:

$$ext(X) \leq s(X), l(X) \leq nw(X) \leq knw(X) \leq w(X)$$

and

$$d(X) \leq nw(X) \text{ and } ml(X) \leq l(X) \leq ml(X) \cdot d(X).$$

In addition to the density $d(X)$ we consider its sequential versions $d_\alpha(X)$ dependent on the ordinal parameter $\alpha$. By the sequential closure of a set $A$ in a topological space $X$ we understand the set $cl_1(A)$ consisting of the limits of all sequences $(a_n) \subset A$ that converge in $X$. Next, for any ordinal $\alpha$ define the $\alpha$-th sequential closure $cl_\alpha(A)$ of $A$ by transfinite induction: $cl_\alpha(A) = cl_1(cl_\beta(A))$ if $\alpha = \beta + 1$ is a successor ordinal and $cl_\alpha(A) = \bigcup_{\beta < \alpha} cl_\beta(A)$ if $\alpha$ is a limit ordinal. It is well-known (and easy to see) that $cl_\alpha(A) = cl_{\omega_1}(A)$ for all ordinals $\alpha \geq \omega_1$ and $cl_{\omega_1}(A)$ coincides with the closure of $A$ in the sequential coreflexion $sX$ of $X$.

The sequential order $so(X)$ of a topological space $X$ is the smallest ordinal $\alpha$ such that $cl_\alpha(A) = cl_{\omega_1}(A)$ for any subset $A \subset X$. Observe that a sequential space $X$ is Fréchet-Urysohn if and only if $so(X) \leq 1$.

By the $\alpha$-th sequential density $d_\alpha(X)$ of a space $X$ we understand the smallest size $|D|$ of a subset $D \subset X$ whose $\alpha$-th sequential closure $cl_\alpha(D)$ of $X$. It is clear that $d_\alpha(X) \geq d_\beta(X)$ for any ordinals $\alpha \leq \beta$ and $d_{so(X)}(X) = d_{\omega_1}(X)$ equals the density $d(sX)$ of the sequential coreflexion of $X$.

Let also $d_{\omega_1}(X) = \min_{\alpha < \omega_1} d_\alpha(X)$. Note that the cardinal $d_{\omega_1}(X)$ can be strictly larger than $d_{\omega_1}(X)$. A suitable counterexample yields the space $B(\mathbb{R}) \subset \mathbb{R}^{\mathbb{R}}$ of all Borel-measurable functions. By the classical Lebesgue-Hausdorff Theorem (see [31, 11.6]), $B(\mathbb{R}) = cl_{\omega_1}(C(\mathbb{R}))$ where $C(\mathbb{R})$ stands for the set of all continuous functions and hence $d_{\omega_1}(B(\mathbb{R})) = \aleph_0$. On the other hand, it can be shown that $d_{\omega_1}(B(X)) > d_{\omega_1}(B(X))$.

Since a space $X$ and its sequential coreflexion $sX$ have the same convergent sequences, the following equality holds.

**Proposition 5.1.** For any space $X$ and any ordinal $\alpha$ we get $d_\alpha(X) = d_\alpha(kX) = d_\alpha(sX)$. 
It turns out that cs*-continuous sections of continuous maps do not increase the cardinal invariant $d_\alpha$ for $\alpha < \omega_1$.

**Proposition 5.2.** Let $\pi : M \to X$ be a map from a metrizable space $M$ and $s : X \to M$ be a cs*-continuous section of $\pi$. Then for any subset $A \subset X$ and any ordinal $\alpha$ we get

1. $d(s(\text{cl}_1(A))) \leq d(s(A))$;
2. $d(s(\text{cl}_\alpha(A))) \leq \max\{|\alpha|, |A|\}$;
3. $d(s(X)) \leq \min\{d_{<\omega_1}(X), \text{so}(X) \cdot d_{\omega_1}(X)\}$.

**Proof.** 1. Assuming that $d(s(\text{cl}_1(A))) > d(s(A))$, find a subset $D \subset s(\text{cl}_1(A))$ of size $|D| > d(s(A))$ that is closed and discrete in $M$.

For every point $y \in D$ use the inclusion $s^{-1}(y) \in \text{cl}_1(A)$ to find a sequence $(a_n^y) \subset A$ convergent to the point $x_y = s^{-1}(y) = \pi(y)$. By the cs*-continuity of $s$ the sequence $(s(a_n^y))$ has a limit point $z_y \in \text{cl}(s(A))$ in $M$. By the continuity of $\pi$ the point $z_y$ projects into $x_y$.

The set $Z = \{z_y : y \in D\}$ has size $|Z| = |D| > d(s(A)) = d(s(A))$ and hence contains a convergent sequence $S \subset Z$ whose image $\pi(S)$ is a convergent sequence in $\pi(Z) = \pi(D)$. Then the cs*-continuity of $s$ implies that the sequence $s \circ \pi(S) \subset s \circ \pi(D) = D$ has a limit point in $M$ which is not possible as $D$ is closed and discrete.

2. The second item by transfinite induction. For $\alpha = 1$ it follows from the first item. Assume that for some ordinal $\beta \leq \omega_1$ the inequality $d(s(\text{cl}_\alpha(A))) \leq \max\{|\alpha|, |A|\}$ has been proved for all ordinals $\alpha < \beta$. If $\beta = \alpha + 1$ is a successor ordinal, then $\text{cl}_\beta(A) = \text{cl}_1(\text{cl}_\alpha(A))$ and by the first item $d(s(\text{cl}_1(A))) \leq d(s(\text{cl}_\alpha(A))) \leq \max\{|\alpha|, |A|\} \leq \max\{|\beta|, |A|\}$ by the inductive hypothesis. If $\beta$ is a limit ordinal, then $\text{cl}_\beta(A) = \bigcup_{\alpha < \beta} \text{cl}_\alpha(A)$ and

$$d(s(\text{cl}_\beta(A))) = d\left(\bigcup_{\alpha < \beta} s(\text{cl}_\alpha(A))\right) \leq \sum_{\alpha < \beta} d(s(\text{cl}_\alpha(A))) \leq \sum_{\alpha < \beta} \max\{|\alpha|, |A|\} \leq \max\{|\beta|, |A|\}.$$

3. The inequality $d(s(X)) \leq d_{<\omega_1}(X)$ follows from the preceding item. To show that $d(s(X)) \leq \text{so}(X) \cdot d_{\omega_1}(X)$, consider two cases. If $\text{so}(X) < \omega_1$, then $d_{\omega_1}(X) = d_{<\omega_1}(X)$ and hence $d(s(X)) \leq d_{<\omega_1}(X) \leq \text{so}(X) \cdot d_{\omega_1}(X)$. If $\text{so}(X) = \omega_1$, then $d(s(X)) \leq \max\{\aleph_1, d_{\omega_1}(X)\} \leq \text{so}(X) \cdot d_{\omega_1}(X)$ by item (2).

Cardinal characteristics of $k^*$-metrizable $k$-spaces are evaluated in the following

**Theorem 5.3.** If $X$ is a $k^*$-metrizable $k$-space, then

$$d(X) \leq knw(X) = nw(X) = s(X) = \text{eext}(X) = l(X) = ml(X) \cdot d(X) = d_1(X) \leq \text{so}(X) \cdot d(X).$$

**Proof.** Assuming that $X$ is a $k^*$-metrizable $k$-space, find a subproper map $\pi : M \to X$ of a metrizable space $M$ onto $X$ and take a section $\sigma : X \to M$ of $\pi$ that preserves precompact sets. Without loss of generality, $\sigma(X)$ is dense in $M$ and hence

$$w(M) = d(M) = d(\sigma(X)) \leq \min\{d_{<\omega_1}(X), \text{so}(X) \cdot d_{\omega_1}(X)\}$$

by Proposition 5.2. The continuity of the map $\pi$ and the metrizability of $M$ imply the inequalities

$$d(X) = d_{\omega_1}(X) \leq d_{<\omega_1}(X) \leq d_1(X) \leq d_1(M) = d(M) = w(M).$$

Therefore,

$$w(M) = d_{<\omega_1}(X) = d_1(X) \leq \text{so}(X) \cdot d_{\omega_1}(X) = \text{so}(X) \cdot d(X).$$

The latter equality holds because $X$, being a $k^*$-metrizable $k$-space, is sequential, which yields $d(X) = d_{\omega_1}(X)$. 
Next, we show that \( w(M) \leq \text{ext}(X) \). Otherwise we may take any \( \sigma \)-discrete base \( \mathcal{B} \) of the topology of \( M \) and find a discrete family \( \mathcal{U} \subset \mathcal{B} \) of size \( |\mathcal{U}| > \text{ext}(X) \). Using the density of \( \sigma(X) \) in \( M \), for every \( U \in \mathcal{U} \) pick a point \( x_U \in X \) with \( \sigma(x_U) \in U \). Since \( \sigma \) preserves precompact sets, the set \( D = \{ x_U : U \in \mathcal{U} \} \) has finite intersection with compact subsets of \( X \) and thus is closed and discrete in the \( k \)-space \( X \). Consequently \( \text{ext}(X) \geq |D| > \text{ext}(X) \), which is a contradiction. This proves the inequality \( w(M) \leq \text{ext}(X) \).

Then for any base \( \mathcal{B} \) of the topology of \( M \) with size \( |\mathcal{B}| = w(M) \) the family \( \mathcal{N} = \{ f(B) : B \in \mathcal{B} \} \) is a \( k \)-network for \( X \) and hence \( \text{knw}(X) \leq |\mathcal{N}| \leq |\mathcal{B}| = w(M) \leq \text{ext}(X) \). This inequality combined with the trivial inequalities \( \text{ext}(X) \leq \text{knw}(X) \) and \( \text{ml}(X) \leq \text{l}(X) \leq \text{ml}(X) \cdot d(X) \leq \text{nw}(X) \) implies the equalities \( \text{ext}(X) = s(X) = \text{nw}(X) = \text{knw}(X) = \text{l}(X) = \text{ml}(X) \cdot d(X) \).

In light of Theorem 5.3 the following problem is natural.

**Problem 5.4.** Is \( d(X) = \text{nw}(X) \) for any regular \( k^* \)-metrizable \( k \)-space?

We shall give an affirmative answer to this problem assuming (CH), the Continuum Hypothesis.

We shall need \( + \)-modifications of the cardinal characteristics \( \text{ext}(X) \), \( l(X) \), and \( \text{ml}(X) \). For a topological space \( X \) let

- \( \text{ext}^+(X) \) be the smallest cardinal \( \kappa \) such that no closed discrete subset of \( X \) has cardinality \( \kappa \);
- \( l^+(X) \) be the smallest cardinal \( \kappa \) such that each open cover of \( X \) has a subcover having size \( < \kappa \);
- \( \text{ml}^+(X) \) be the smallest cardinal \( \kappa \) such that each open cover \( U \) of \( X \) has an open refinement \( V \) such that \( |\{ V \in V : x \in V \}| < \kappa \) for every point \( x \in X \).

It is easy to see that the cardinals \( \text{ext}^+(X) \), \( l(X) \), and \( \text{ml}^+(X) \) completely determine the values of the related cardinal characteristics \( \text{ext}(X) \), \( l(X) \), and \( \text{ml}(X) \):

\[
\text{ext}(X) = \sup \{ \kappa : \kappa < \text{ext}^+(X) \},
\]

\[
l(X) = \sup \{ \kappa : \kappa < l^+(X) \},
\]

\[
\text{ml}(X) = \sup \{ \kappa : \kappa < \text{ml}^+(X) \}.
\]

It turns out that the meta-Lindelöf number of a regular \( k^* \)-metrizable \( k \)-space is bounded by a local version of the extent called the extent-character of a topological space.

For a point \( x \) of a topological space \( X \) by the extent-character \( \text{ext}_X(x; X) \) (resp. extent-character \( \text{ext}^+_X(x; X) \)) of \( X \) at \( x \) we shall understand the largest cardinal \( \kappa \) such that for any family \( \mathcal{U} \) of neighborhoods of \( x \) having size \( |\mathcal{U}| \leq \kappa \) (resp. \( |\mathcal{U}| < \kappa \)) there is an injective map \( f : \mathcal{U} \to X \) such that the image \( f(\mathcal{U}) \) is a closed discrete subset of \( X \) and \( f(U) \in U \) for every set \( U \in \mathcal{U} \).

The value of the cardinal \( \text{ext}_X(x; X) \) is fully determined by the value of \( \text{ext}^+_X(x; X) \):

\[
\text{ext}_X(x; X) = \sup \{ \kappa : \kappa < \text{ext}^+_X(x; X) \}.
\]

Let also

\[
\text{ext}_X(x) = \sup_{x \in X} \text{ext}_X(x; X), \quad \text{ext}^+_X(x) = \sup_{x \in X} \text{ext}^+_X(x; X)
\]

and observe that \( \text{ext}_X(X) \leq \text{ext}(X) \) and \( \text{ext}^+_X(X) \leq \text{ext}^+(X) \).

The extent-character \( \text{ext}^+_X(x; X) \) is also bounded by the usual character \( \chi(x; X) \) equal to the smallest size of a neighborhood base at \( x \).
Proposition 5.5. Let \( x \) be a point of a topological space. Then
\[
\text{ext}_\chi(x; X) \leq \text{ext}_\chi^+(x; X) \leq \chi(x; X) \text{ and } \text{ext}_\chi(X) \leq \text{ext}_\chi^+(X) \leq \chi(X).
\]
Applications of the extent-character to \( k^* \)-metrizable spaces will rely on the following

Proposition 5.6. Let \( \pi: M \to X \) be a map possessing a \( cs^* \)-continuous section \( s: X \to M \) and let \( \mathcal{U} \) be a point-finite open cover of \( M \). If the space \( X \) is sequential, then each point \( x \in X \) has a neighborhood \( O_x \) whose image \( s(O_x) \) meets less than \( \text{ext}_\chi^+(x; X) \) sets \( U \in \mathcal{U} \).

Proof. Assume the converse; the image \( s(O_x) \) of any neighborhood \( O_x \) of some point \( x \) meets at least \( \text{ext}_\chi^+(x; X) \) sets \( U \in \mathcal{U} \). The definition of the cardinal \( \kappa = \text{ext}_\chi^+(x; X) \) yields a family \( \mathcal{B} \) of open neighborhoods of \( x \) such that \( |\mathcal{B}| = \kappa \) and for each injective map \( f: \mathcal{B} \to X \) with \( f(B) \in B \) for \( B \in \mathcal{B} \) the image \( f(\mathcal{B}) \) is not closed and discrete in \( X \).

Enumerate the family \( \mathcal{B} \) as \( \mathcal{B} = \{B_\alpha: \alpha < \kappa\} \). By transfinite induction we shall construct a transfinite sequence of points \( \{x_\alpha: \alpha < \kappa\} \subset X \) such that \( s_\alpha \in B_\alpha \) and
\[
s(x_\alpha) \notin \bigcup_{\beta < \alpha} \{U \in \mathcal{U} : s(x_\beta) \in U\}
\]
for every \( \alpha < \kappa \). Let \( x_0 = x \). Assuming that for some ordinal \( \alpha < \kappa \) the points \( x_\beta, \beta < \alpha \), have been constructed, we shall find a point \( x_\alpha \). Use the point-finiteness of the cover \( \mathcal{U} \) to conclude that the family \( \mathcal{U}_{<\alpha} = \bigcup_{\beta < \alpha} \{U \in \mathcal{U} : s(x_\beta) \in U\} \) has size \( < \kappa \). By our hypothesis, the image \( s(B_\alpha) \) meets at least \( \kappa \) sets \( U \in \mathcal{U} \). Consequently, there is a point \( x_\alpha \in B_\alpha \) with \( s(x_\alpha) \notin \bigcup \mathcal{U}_{<\alpha} \). This completes the inductive construction.

The choice of the family \( \{B_\alpha\}_{\alpha < \kappa} \) implies that the set \( D = \{x_\alpha: \alpha < \kappa\} \) is not closed and discrete in the sequential space \( X \). The \( cs^* \)-continuity of the section \( s: X \to M \) implies that the image \( s(D) \) is not closed and discrete in \( M \). Consequently, there is a point \( z \in M \) whose any neighborhood contains infinitely many points \( s(x_\alpha) \). In particular, any neighborhood \( U \in \mathcal{U} \) of \( z \) contains two points \( s(x_\alpha), s(x_\beta) \) with \( \alpha < \beta \), which is not possible because \( s(x_\beta) \notin \{U \in \mathcal{U} : s(x_\alpha) \in U\} \).

The proof of the following theorem essentially is due to S. Lin [46].

Theorem 5.7. If \( X \) is a \( k^* \)-metrizable \( k \)-space, then \( ml(X) \leq \text{ext}_\chi(X) \). If the cardinal \( \text{ext}_\chi(X) \) is regular, then also \( ml^+(X) \leq \text{ext}_\chi^+(X) \).

Proof. Let \( \kappa = \text{ext}_\chi(X) \). Let \( \pi: M \to X \) be a subproper map from a metric space \((M,d)\) and \( s: X \to M \) be a precompact-preserving section of \( \pi \) having dense image \( s(X) \) in \( M \). Let \( \mathcal{U}_{-1} = \{X\} \) and by induction select a sequence \((\mathcal{U}_k)_{k \in \omega}\) of locally finite open covers of \( M \) such that each cover \( \mathcal{U}_k \) refines \( \mathcal{U}_{k-1} \) and consists of sets with diameter \( < 2^{-k} \). By Proposition 5.6 each point of \( X \) has a neighborhood meeting at most \( \kappa \) sets \( s^{-1}(U), U \in \mathcal{U}_k \).

For a family \( \mathcal{F} \) of subsets of \( X \) let \( \text{ord}(\mathcal{F}) \) (resp. \( \text{Ord}(\mathcal{F}) \)) be the smallest cardinal \( \lambda \) such that each point of \( X \) belongs to (resp. has a neighborhood meeting) at most \( \lambda \) sets \( F \in \mathcal{F} \).

Claim 5.8. For each family \( \mathcal{F} \) of subsets of \( X \) with \( \text{Ord}(\mathcal{F}) \leq \kappa \) there is a family \( O(\mathcal{F}) = \{O(F): F \in \mathcal{F}\} \) of open neighborhoods \( O(F) \) of the sets \( F \in \mathcal{F} \) in \( X \) with \( \text{ord}O(\mathcal{F}) \leq \kappa \).

Let \( \omega^{<\omega} \) denote the set of all finite sequences of non-negative integer numbers. For each sequence \( \sigma \in \omega^{<\omega} \) we define a family \( O_\sigma(\mathcal{F}) = \{O_\sigma(F): F \in \mathcal{F}\} \) with \( \text{Ord}(O_\sigma(\mathcal{F})) \leq \kappa \), by induction. For the empty sequence \( \emptyset \) we put \( O_\emptyset(F) = F \) for all \( F \in \mathcal{F} \). Assume that for some finite sequence \( \sigma = (n_0, \ldots, n_k) \) the sets \( O_\sigma(F), F \in \mathcal{F} \), have been defined so that the family \( O_\sigma(\mathcal{F}) = \{O_\sigma(F): F \in \mathcal{F}\} \) has \( \text{Ord}(O_\sigma(\mathcal{F})) \leq \kappa \). Given a number \( i \in \omega \) consider the
sequence $\sigma i = (n_0, \ldots, n_k, i)$ and let

$$N(\sigma i) = \{N \in s^{-1}(U_i) : |\{F \in \mathcal{F} : N \cap O_{\sigma}(F) \neq \emptyset\}| \leq \kappa\}$$

$$O_{\sigma i}(F) = \{N \in N(\sigma i) : N \cap O_{\sigma}(F) \neq \emptyset\}$$

To show that $\text{Ord}(O_{\sigma i}(F)) \leq \kappa$, take any point $x \in X$ and find a neighborhood $W_x$ meeting at most $\kappa$-sets of the family $O_{\sigma}(\mathcal{F})$. By Proposition 5.6, the neighborhood $W_x$ can be chosen so small that it meets at most $\kappa$ sets of the family $s^{-1}(U_i) = \{s^{-1}(U) : U \in U_i\}$. Then the definition of the family $O_{\sigma i}(\mathcal{F})$ implies that $W_x$ meets at most $\kappa$ sets from this family.

After completing the inductive construction, put $O(F) = \bigcup_{\sigma \in \omega^{<\omega}} O_{\sigma}(F)$ for $F \in \mathcal{F}$ and consider the family $O(\mathcal{F}) = \{O(F) : F \in \mathcal{F}\}$. It follows from $\text{Ord}(O_{\sigma}(\mathcal{F})) \leq \kappa$ that $\text{ord}(O(\mathcal{F})) \leq \kappa$.

It remains to show that each set $O(F), F \in \mathcal{F}$ is open in $X$. Since $X$ is a sequential space, it suffices to check that $O(F)$ is sequentially open. Assuming the converse, we could find a sequence $\{x_n\}_{n \in \omega}$ in $X \setminus O(F)$, convergent to a point $x_\omega \in O(F)$. Using the $\sigma^*$-continuity of the section $s$, we may assume that the sequence $s(x_n)$ converges in $M$ to some point $z_\omega$. The continuity of the map $\pi$ implies that $\pi(z_\omega) = x_\omega$. Since $x_\omega \in O(F)$, there is a finite number sequence $\sigma \in \omega^{<\omega}$ with $x_\omega \in O_{\sigma}(F)$. Since $\text{Ord}(O_{\sigma}(\mathcal{F})) \leq \kappa$, the point $x_\omega$ has a neighborhood $W \subset X$ meeting at most $\kappa$ sets of the family $O_{\sigma}(\mathcal{F})$. Then we can find $i \in \omega$ so large that each set $U \in U_i$ containing $z_\omega$ lies in $\pi^{-1}(W)$. Then the set $s^{-1}(U)$ lies in $W$ and hence meets at most $\kappa$ sets of the family $O_{\sigma}(\mathcal{F})$. The definition of the set $O_{\sigma i}(F)$ ensures that $s^{-1}(U) \subset O_{\sigma i}(F) \subset O(F)$. Since $U$ contains almost all element of the sequence $(s(x_n))$, the set $s^{-1}(U)$ contains almost all elements of the sequence $(x_n)$, a contradiction with $\{x_n\} \subset X \setminus O(F)$. This contradiction completes the proof of the claim.

Now the proof of the theorem is quite easy. Given an open cover $\mathcal{W}$ of $X$, for every $n \in \omega$ consider the subcollection $\mathcal{F}_n = \{F \in s^{-1}(U_n) : F \subset W(F) \text{ for some } W(F) \in \mathcal{W}\}$ having $\text{Ord}(\mathcal{F}_n) \leq \kappa$ by Proposition 5.6. Since $\bigcup_{n \in \omega}s^{-1}(U_n)$ is a network for $X$, the family $\bigcup_{n \in \omega}\mathcal{F}_n$ covers $X$.

The preceding claim yields us a family $O(\mathcal{F}_n) = \{O(F) : F \in \mathcal{F}_n\}$ of open neighborhoods of the sets $F \in \mathcal{F}_n$ with $\text{ord}(O(\mathcal{F}_n)) \leq \kappa$. Replacing each $O(F)$ by the intersection $O(F) \cap W(F)$, if necessary, we may assume that $O(F) \subset W(F)$ for all $F \in \mathcal{F}_n$. Then $\mathcal{V} = \bigcup_{n \in \omega}O(\mathcal{F}_n)$ is an open refinement of $\mathcal{W}$ with $\text{ord}(\mathcal{V}) \leq \kappa$, which completes the proof of the inequality $\text{ml}(X) \leq \kappa = \text{ext}^+(X)$.

A minor modification of this proof yields also the inequality $\text{ml}^+(X) \leq \text{ext}^+(X)$ in case of a regular cardinal $\text{ext}^+(X)$.

The preceding theorem will help us to evaluate the $k$-network weight $\text{knw}(X)$ of a $k^*$-metrizable $k$-space via the extent-characters and the density.

**Theorem 5.9.** If $X$ is a $k^*$-metrizable $k$-space, then

1. $\text{knw}(X) = \text{ml}(X) \cdot d(X) = \text{ext}^+(X) \cdot d(X)$;
2. $\text{knw}(X) < 2^d(X)$ if $X$ is a regular space.

**Proof.** 1. The inequality $\text{ext}^+(X) \cdot d(X) \leq \text{knw}(X)$ follows immediately from the trivial inequalities $\text{ext}^+(X) \leq \text{ext}(X) \leq \text{nw}(X) \leq \text{knw}(X)$ and $d(X) \leq \text{nw}(X) \leq \text{knw}(X)$. On the other hand, Theorems 5.3 and 5.7 yield $\text{knw}(X) = \text{ml}(X) \cdot d(X) \leq \text{ext}^+(X) \cdot d(X)$.

2. By Theorem 5.3, $d(X) \leq \text{knw}(X) \leq \text{nw}(X) \leq \text{knw}(X)$ leads to a contradiction. Observe that the latter equality implies that $d(X) = \text{nw}(X) = 2^d(X)$. So it remains to show that the assumption $\text{knw}(X) = 2^d(X)$ leads to a contradiction. By Theorem 1.5.6, the space $X$, being regular, has weight $w(X) \leq 2^d(X)$. Consequently, $\text{ext}^+(X) \leq \chi(X) \leq w(X) \leq 2^8 = 8_1$ and hence
ext\(\chi(X)\) \leq \aleph_0. Now the first item implies \(\knw(X) = ext\chi(X) \cdot d(X) \leq \aleph_0\), which contradicts \(\knw(X) = \aleph_1\).

Combining Theorem 5.3 with the second item of Theorem 5.9 we get

**Corollary 5.10.** Under (CH), every regular \(k^*\)-metrizable \(k\)-space satisfies the equality \(\knw(X) = d(X)\).

This corollary is specific for \(k\)-spaces and does not hold in general.

**Remark 5.11.** The Frolik’s space \(X\) from Remark 3.5 has cardinal characteristics

\[\aleph_0 = d(X) < \nw(X) = \knw(X) = w(X) = 2^{\aleph_0} < 2^{2^{\aleph_0}} = |kX| = \knw(kX)\]

which show that even for a \(k\)-metrizable space \(X\) the gap between \(d(X)\) and \(\knw(X)\) can be very large.

Nonetheless we can estimate the cardinal characteristics of \(k^*\)-metrizable spaces via the cardinal characteristics of their \(k\)-coreflexions.

**Corollary 5.12.** If \(X\) is a \(k^*\)-metrizable space, then

\[d(X) \leq \nw(X) \leq \knw(X) \leq d_{\omega_1}(X) = d_{\omega_1}(kX) = \snw(X) \cdot d(kX).\]

The preceding discussion displays the importance of the local cardinal invariant \(ext\chi(X)\).

It turns out that this cardinal characteristic can be bounded from below by another local cardinal invariant, which is a quantitative version of the Arkhangel’ski’s property (\(\alpha_4\)).

Following [3] we say that a topological space \(X\) has the property (\(\alpha_4\)) at a point \(x \in X\) if for any countable family \(\mathcal{S}\) consisting of non-trivial sequences convergent to \(x\) there is a sequence \(T \subset X\) convergent to \(x\) and intersecting infinitely many sequences \(S \in \mathcal{S}\). (Besides the usual meaning, by a non-trivial convergent sequence in a space \(X\) we shall understand a countable infinite subset \(S \subset X\) whose closure in \(X\) is compact that has a unique non-isolated point \(x = \lim S\) called the limit of \(S\)).

A quantification of the property (\(\alpha_4\)) yields two local cardinal characteristics.

For a point \(x\) of a topological space \(X\) let \(\alpha_4(x; X)\) be the smallest cardinal \(\tau\) such that for any cardinal \(\kappa < \tau\) there is a family \(\mathcal{S}\) of size \(|\mathcal{S}| = \kappa\) that consist of non-trivial sequences convergent to \(x\) and is such that each sequence \(T \subset X\) convergent to \(x\) meets only finitely many sequences \(S \in \mathcal{S}\). If such a cardinal \(\tau\) does not exist (which happen if \(x\) is an isolated point of the sequential coreflexion \(sX\)), then we put \(\alpha_4(x; X) = 0\).

Another way of quantifying the property (\(\alpha_4\)) leads to a cardinal invariant \(\alpha_4^+(x; X)\) equal to the smallest cardinal \(\kappa \geq 1\) such that such that for any family \(\mathcal{S}\) of size \(|\mathcal{S}| = \kappa\) that consist of non-trivial sequences convergent to \(x\) there is a sequence \(T \subset X\) that converges to \(x\) and meets infinitely many sequences \(S \in \mathcal{S}\). We put \(\alpha_4^+(x; X) = 1\) if \(x\) is a non-isolated point of \(sX\).

Observe that \(\alpha_4(x; X) = \sup\{\kappa : \kappa < \alpha_4^+(x; X)\}\) and hence the cardinal \(\alpha_4^+(x; X)\) completely determined the value of \(\alpha_4(x; X)\).

We put \(\alpha_4(X) = \sup_{x \in X} \alpha_4(x; X)\) and \(\alpha_4^+(X) = \sup_{x \in X} \alpha_4^+(x; X)\).

Observe that \(\alpha_4^+(X) = \aleph_0\) if and only if \(x\) is a non-isolated point in \(sX\) and \(X\) has property (\(\alpha_4\)) at \(x\). On the other hand, for any infinite cardinal \(\kappa\) the sequential fan \(S_\kappa\) with \(\kappa\) nodes has \(\alpha_4(S_\kappa) = \kappa\) and \(\alpha_4^+(S_\kappa) = \kappa^+\).

The cardinal \(\alpha_4^+(X)\) will be used to detect the presence of a sequential copies of the sequential fan \(S_\kappa\) in the space. For a point \(x\) of a topological space \(X\) we write \((S_\kappa, *) \subset \text{cl} (X, x)\) if there is a closed subset \(F \subset X\) containing point \(x\) and a homeomorphism \(h : S_\kappa \to F\) mapping the unique non-isolated point of the fan \(S_\kappa\) onto \(x\).
Theorem 5.13. Let $\pi : M \to X$ be a map from a metrizable space and $s : X \to M$ be a cs*-continuous section of $\pi$ such that $s(X)$ is dense in $M$. Then for any point $x \in X$ that is not isolated in the sequential coreflexion $sX$ we get

1. $\alpha_4(x;X) = \sup\{\kappa : (S_\kappa, *) \subset_\text{cl} (sX, x)\} = \max\{N_0, \text{ext}(\pi^{-1}(x))\} \leq \text{ext}_\chi(x; sX)$;
2. $\alpha_4^+(x;X) = \min\{\kappa : (S_\kappa, *) \not\subset_\text{cl} (sX, x)\} = \max\{N_0, \text{ext}^+(\pi^{-1}(x))\} \leq \text{ext}_\chi^+(x; sX)$.

Proof. Since the values of the cardinals in the second item determine the values of the corresponding cardinals in the first item, it suffices to prove item (2) only.

Let

$$\kappa_1 = \alpha_4^+(x; X),$$
$$\kappa_2 = \max\{N_0, \text{ext}^+(\pi^{-1}(x))\},$$
$$\kappa_3 = \min\{\kappa : (S_\kappa, *) \not\subset_\text{cl} (sX, x)\}.$$

We shall prove the inequalities $\kappa_1 \leq \kappa_2 \leq \kappa_3 \leq \kappa_1$ and $\kappa_3 \leq \text{ext}_\chi^+(x; sX)$.

($\kappa_1 \leq \kappa_2$). Assume conversely that $\alpha_4^+(x; X) = \kappa_1 > \kappa_2$. The definition of $\alpha_4^+(x; X)$ yields an infinite family $S$ consisting of $\kappa_2$ many sequences convergent to $x$ such that any other sequence $T \subset X$ with $\lim T = x$ meets only finitely many sequences $S \in S$. Since the section $s : X \to M$ is cs*-continuous, we may replace each sequence $S \in S$ with a suitable subsequence and assume that the sequence $s(S)$ converges to some point $z_S \in M$. The continuity of $\pi$ implies that $z_S \in \pi^{-1}(x)$. We claim that the set $\{z_S : S \in S\}$ is a closed discrete subset in $\pi^{-1}(x)$. Observe that any convergent sequence $T \subset M$ meets only finitely many sequences $s(S), S \in S$ (because its image $\pi(T)$ meets only finitely many sequences $S \in S$). Then the indexed set $\{z_S : S \in S\}$ cannot have accumulating points in $M$ and hence $D = \{z_S : S \in S\}$ is a closed discrete subset of $\pi^{-1}(x)$ with size $|S| = \kappa_2 \geq \text{ext}^+(\pi^{-1}(x))$. But this contradicts the definition of the cardinal $\text{ext}^+(\pi^{-1}(x))$. This proves the inequality $\kappa_1 \leq \kappa_2$.

($\kappa_2 \leq \kappa_3$) also will be proved by contradiction. Assume conversely that $\text{ext}^+(\pi^{-1}(x)) = \kappa_2 > \kappa_3$. The definition of $\text{ext}^+(\pi^{-1}(x))$ provides a closed discrete subset $D \subset \pi^{-1}(x)$ having size $|D| = \kappa_3$. We may additionally assume that $s(x) \notin D$. The space $M$, being metrizable, is collectively normal. Hence for each point $z \in D$ we can find a neighborhood $O_z$ such that the family $\{O_z : z \in D\}$ is discrete. The density of $s(X)$ in $M$ implies that for any point $z \in D$ we can find a sequence $S(z) \subset O_z \cap s(X)$ convergent to $z$. Using the cs*-continuity of the section $s$ it can be shown that the set $F = \{x\} \cup \bigcup_{z \in D} \pi(S(z))$ with distinguished point $x$ is a closed copy of the fan $(S_{\kappa_1}, *)$ in the sequential coreflexion $sX$ of $X$. But this contradicts the definition of the cardinal $\kappa_3$.

To prove the inequality $\kappa_3 \leq \kappa_1$, assume conversely that $\kappa_3 > \kappa_1 = \alpha_4^+(x; X)$. Then the definition of the cardinal $\kappa_3$ yields us a topological copy of the fan $(S_{\kappa_1}, *)$ in $(sX, x)$. Since each sequence convergent to $x$ meets only finitely many nodes of the fan $S_{\kappa_1}$, the space $X$ has $\alpha_4^+(X; x) > \kappa_1$, a contradiction with the definition of the cardinal $\kappa_1$.

The final inequality $\kappa_3 \leq \text{ext}_\chi(x; sX)$ follows from the observation that $\text{ext}_\chi^+(S_\kappa) = \kappa^+$ for each infinite cardinal $\kappa$. $\square$

Theorem 5.13 gives an upper bound $\text{ext}_\chi(x; X)$ for $\alpha_4(x; X)$. A lower bound is given by another local cardinal invariant called the cs-character of $X$ at $x$.

A family $\mathcal{N}$ of subsets of a space $X$ is called a cs-network (resp. cs*-network) at $x$ if for any neighborhood $U \subset X$ of $x$ and any sequence $\{x_n\}_{n \in \omega} \subset U$ convergent to $x$ there is a set $N \subset U$ in $\mathcal{N}$ such that $N$ contains all but finitely many (resp. infinitely many) points $x_n$.

The cs-character (resp. cs*-character) of a space $X$ at a point $x \in X$ is the smallest size $|\mathcal{N}|$ of a cs-network (resp. cs*-network) at $x$. The cardinal $\text{cs}_\chi(X) = \sup_{x \in X} \text{cs}_\chi(x, X)$ (resp. $\text{cs}_\chi(X) = \sup_{x \in X} \text{cs}_\chi(x, X)$) is called the cs-character (resp. cs*-character) of $X$. 

Proposition 5.14. If $X$ is a $cs^*$-metrizable space, then $cs^*_M(s; X) \leq \alpha_4(s; X) \leq \alpha_4(x; X) \leq \text{ext}_M(s; X)$. If $cs^*_M(s; X) \leq \aleph_0$, then $\alpha_4(x; X) = cs^*_M(s; X) = cs^*_M(x; X) \in \{1, \aleph_0\}$.

Proof. Since $X$ is a $cs^*$-metrizable, there is a map $\pi : M \to X$ from a metrizable space and a $cs^*$-continuous section $s : X \to M$ such that $s(X)$ is dense in $M$. By Theorem 5.13, $\text{ext}(\pi^{-1}(x)) \leq \alpha_4(x; X)$. Let $B$ be a $\sigma$-discrete base of the topology of the metrizable space $M$. It follows that the subfamily $C = \{U \in B : U \cap \pi^{-1}(y)\}$ has size $|C| \leq d(\pi^{-1}(x)) = \text{ext}(\pi^{-1}(x)) \leq \alpha_4(x; X)$.

Then the family $C_\cup = \{\cup F : F \in C, |C| < \infty\}$ also has size $\leq \alpha_4(x; X)$. We claim that $N = \{\pi(C) : C \in C_\cup\}$ is a cs-network at $x$. Indeed, take any neighborhood $O_x \subset X$ of $x$ and a sequence $S = \{x_n\} \subset U$ convergent to $x$. The $cs^*$-continuity of the section $s$ implies that the set $s(S)$ has compact closure $K$ in $M$. Let $F \subset B$ be a finite subcover of the set $K \cap \pi^{-1}(x)$ such that $\cup F \subset \pi^{-1}(O_x)$. The continuity of $\pi$ implies that all points of $K \setminus \pi^{-1}(x)$ are isolated and hence the set $K \setminus \cup F$ is finite. Then $\pi(\cup F) \subset O_x$ is an element of the family $N$, containing almost all elements of the sequence $(x_n)$. This shows that $N$ is a cs-network at $x$ and hence $cs^*_M(x; X) \leq |N| \leq \alpha_4(x; X)$. The inequality $\alpha_4(x; X) \leq \text{ext}_M(x; X)$ was proved in Theorem 5.13.

Now assume that $cs^*_M(x; sX) \leq \aleph_0$. If $x$ an isolated point in $sX$, then $\alpha_4(x; X) = cs^*_M(x; X) = cs^*_M(x; X) = 1$. So, assume that $x$ is not isolated in $sX$, which means that there is a non-trivial convergent sequence in $X$. In light of the preceding paragraph, it suffices to check that $\alpha_4(x; X) \leq \aleph_0$. Assuming the converse and applying Theorem 5.13 find a closed embedding $(S_{\omega_1}, *)$ into $(sX, x)$. Then $cs^*_M(S_{\omega_1}) \leq \alpha_4(x; sX) \leq \aleph_0$ and hence there is a countable $cs^*$-network $N$ for $S_{\omega_1}$. Write $S_{\omega_1} = \{\star\} \cup \{x_{\alpha}^\alpha : n \in \omega, \alpha < \omega_1\}$ where each sequence $T_\alpha = \{x_{\alpha}^\alpha\}_{n<\omega}$ converges to the non-isolated point $\star \notin T_\alpha$ of $S_{\omega_1}$, the sequences $T_\alpha$, $\alpha < \omega_1$, are pairwise disjoint and determine the topology of $S_{\omega_1}$ in the sense that a subset $F \subset T_\omega_1$ is closed in $S_{\omega_1}$ if for any $x < \omega_1$ the intersection $F \cap T_\alpha$ is compact. For a set $A \subset S_{\omega_1}$, let $\text{supp}(A) = \{\alpha \in \omega_1 : A \cap T_\alpha \neq \emptyset\}$. Let $N_\infty$ be the subfamily of the $cs^*$-network $N$, consisting of all sets $N \in N$ with infinite support $\text{supp}(N)$. Since $N_\infty$ is at most countable, we can inductively construct an at most countable subset $D \subset S_{\omega_1}$ such that $D \cap N \neq \emptyset$ for any $N \in N_\infty$ and $|D \cap T_\alpha| \leq 1$ for any $\alpha \in \omega_1$. By definition of the topology of $S_{\omega_1}$, the set $D$ is closed and discrete in $S_{\omega_1}$. Then $U = S_{\omega_1} \setminus D$ is an open neighborhood of the non-isolated point $\star$ of $S_{\omega_1}$. Pick any countable ordinal $\alpha$ that does not belong to the countable set $\text{supp}(D) \cup \bigcup_{N \in N \setminus N_\infty} \text{supp}(N)$. Then $T_\alpha$ is a convergent sequence in $U$ and hence there is a set $N_0 \subset U$ in $N$ having infinite intersection with $T_\alpha$. This set $N_0$ cannot belong to $N_\infty$ because no set $N \in N_\infty$ lies in $U$. Then $N_0 \in N \setminus N_\infty$, which also is not possible because $\alpha \notin \text{supp}(N_0)$. The obtained contradiction completes the proof of the inequality $\alpha_4(x; X) \leq \aleph_0$.

Question 5.15. Does $cs^*_M(x; X) \leq \aleph_0$ imply $\alpha_4(x; X) = cs^*_M(x; X)$? (The answer is affirmative if $X$ is a sequential space).

Question 5.16. Is $\alpha_4(x; X) = \text{ext}_M(x; X)$ for a point $x$ of a regular $k^*$-metrizable $k$-space?

6. Characterizing $k^*$-metrizable spaces in terms of $k$-networks

In this section for each space $X$ we describe a canonical construction of a map $\text{lim} : \text{Nw}(X) \to X$ from the so-called Network Hyperspace over $X$ such that $X$ is $k^*$-metrizable (resp. $cs^*$-metrizable) if and only if the map $\text{lim}$ has a precompact-preserving (resp. $cs^*$-continuous) section.
Given a topological space $X$ by $\mathcal{P}(X)$ we denote the power-set of $X$ endowed with the discrete topology. For each point $x \in X$ consider the subset $\text{Nw}(x)$ of $\mathcal{P}(X)^\omega$ consisting of all sequences $(A_n)_{n \in \omega}$ of non-empty subsets of $X$ such that

1. $A_{n+1} \subset A_n$ for all $n$;
2. for any neighborhood $U$ of $x$ there is a number $n \in \omega$ with $\text{cl}_1(A_n) \subset U$.

Therefore $\{\{x\} \cup A_n\}_{n \in \omega}$ is a decreasing network at $x$. We recall that $\text{cl}_1(A)$ stands for the 1-st sequential closure of a set $A$ in a topological space $X$. By definition, $\text{cl}_1(A)$ consists of limit points of sequences $(a_n) \subset A$ that converge in $X$.

The subspace $\text{Nw}(X) = \bigcup_{x \in X} \text{Nw}(x)$ of the Tychonoff product $\mathcal{P}(X)^\omega$ is called the Network Hyperspace over $X$. Since $X$ is Hausdorff, $\text{Nw}(x) \neq \text{Nw}(y)$ for distinct points $x, y$ of $X$, which allows us to define a map $\lim : \text{Nw}(X) \to X$ letting $\lim^{-1}(x) = \text{Nw}(x)$. It is easy to see that the so-defined map $\lim$ is continuous.

The construction of the Network Hyperspace is functorial in the sense that each continuous map $f : X \to Y$ between Hausdorff spaces induces a continuous map $f^\omega : \text{Nw}(X) \to \text{Nw}(Y)$, $f^\omega : (A_n) \mapsto (f(A_n))$ making the diagram

\[
\begin{array}{ccc}
\text{Nw}(X) & \xrightarrow{\lim} & X \\
\downarrow f^\omega & & \downarrow f \\
\text{Nw}(Y) & \xrightarrow{\lim} & Y
\end{array}
\]

commutative.

The Network Hyperspace will be our principal tool in characterizing $k^*$-metrizable spaces in terms of $k$-networks.

A family $\mathcal{N}$ of subsets of a space $X$ is called a $k$-network (resp. $cl_1$-osed $k$-network) for $X$ if for any open set $U \subset X$ and a compact set $K \subset U$ there is a finite subfamily $F \subset \mathcal{N}$ such that $K \subset \bigcup F \subset U$ (resp. $K \subset \bigcup F \subset \text{cl}_1(\bigcup F) \subset U$).

A family $\mathcal{N}$ is a cs$^*$-network (resp. $cl_1$-osed cs$^*$-network) for $X$ if for any open set $U \subset X$ and a sequence $(x_n)_{n \in \omega} \subset U$, convergent to a point $x_\infty \in U$ there is a set $N \in \mathcal{N}$ containing infinitely many points $x_n$ and such that $N \subset U$ (resp. $\text{cl}_1(N) \subset U$).

More detail information on $k$- and cs$^*$-networks can be found in [45], [53], [54], [72]. It is clear that a family $\mathcal{N}$ is a $k$-network (resp. cs$^*$-network) for a sequentially regular space $X$ if and only if $\mathcal{N}$ is a $cl_1$-osed $k$-network (resp. $cl_1$-osed cs$^*$-network) for $X$. We recall that a topological space $X$ to be sequentially regular if for each point $x \in X$ and a neighborhood $U \subset X$ of $x$ there is another neighborhood $V \subset X$ of $x$ with $\text{cl}_1(V) \subset U$.

A family $\mathcal{A}$ of subsets of a space $X$ is called compact-finite (resp. cs-finite) if for any compact subset (resp. convergent sequence) $K \subset X$ the family $\mathcal{F} = \{A \in \mathcal{A} : A \cap K \neq \emptyset\}$ is finite. A family $\mathcal{A}$ is $\sigma$-compact-finite (resp. $\sigma$-cs-finite) if it can be written as the countable union $\mathcal{A} = \bigcup_{n \in \omega} \mathcal{A}_n$ of compact-finite (resp. cs-finite) subfamilies.

**Proposition 6.1.** If $\mathcal{A}$ is a $\sigma$-compact-finite (resp. $\sigma$-cs-finite) family of subsets of a space $X$, then the family $\mathcal{A}_\infty = \{\cap \mathcal{F} : \mathcal{F} \subset \mathcal{A}, |\mathcal{F}| < \infty\}$ also is $\sigma$-compact-finite (resp. $\sigma$-cs-finite).

**Proof.** Write $\mathcal{A}$ as the countable union $\mathcal{A} = \bigcup_{k \in \omega} \mathcal{A}_k$ of compact-finite families. For any finite subset $F \subset \omega$ consider the compact-finite family $\mathcal{A}_F = \{\bigcap_{k \in F} A_k : A_k \in \mathcal{A}_k, k \in F\}$ and observe that $\mathcal{A}_\infty = \bigcup_F \mathcal{A}_F$ is $\sigma$-compact-finite, being the countable union of the compact-finite families $\mathcal{A}_F$.

By analogy we can prove the cs-finite case. \[ \square \]

It is clear that each $\sigma$-compact-finite $k$-network for $X$ is a $\sigma$-cs-finite cs$^*$-network. If all compact subsets of $X$ are countably compact, then the converse is also true.
Proposition 6.2. Let $X$ be a topological space whose all compact subsets are sequentially compact. Then each $\sigma$-cs-finite (cl$_1$-osed) cs$^*$-network $\mathcal{N}$ for $X$ is a $\sigma$-compact-finite (cl$_1$-osed) k-network for $X$.

Proof. First we check that each $\sigma$-finite family $\mathcal{F}$ of subsets of $X$ is compact-finite. Assuming the converse, find a sequence $(F_k)_{k\in\omega}$ of pairwise distinct sets from $\mathcal{F}$ meeting some compact set $K$. For every $k \in \omega$ pick a point $x_k \in K \cap F_k$ and use the sequential compactness of $K$ to find a convergent subsequence $(x_{k_i})_{i \in \omega}$ of $(x_k)$. Since the set $\{x_k : i \in \omega\}$ intersects infinitely many sets from $\mathcal{F}$, the family $\mathcal{F}$ cannot be cs-finite, which is a contradiction.

Therefore, each $\sigma$-cs-finite cs$^*$-network $\mathcal{N}$ for $X$ is $\sigma$-compact-finite. Next, we show that $\mathcal{N}$ is a k-network for $X$. Fix any open set $U \subset X$ and a compact subset $K \subset U$. Since $\mathcal{N}$ is $\sigma$-compact-finite, the subfamily $\mathcal{N}' = \{ N \in \mathcal{N} : N \subset U, N \cap K \neq \emptyset \}$ is at most countable and hence can be enumerated as $\mathcal{N}' = \{ N_k : k \in \omega \}$. We claim that $K \subset N_0 \cup \cdots \cup N_m$ for some finite $m$. Assuming the converse, we would construct a sequence $(x_n)_{n \in \omega}$ with $x_n \in K \setminus (N_0 \cup \cdots \cup N_n)$ for all $n \in \omega$. By the sequential compactness of $K$ this sequence has a convergent subsequence $(x_{n_i})$. The family $\mathcal{N}'$, being a cs$^*$-network, contains a set $N \in \mathcal{N}'$ such that $N \subset U$ and $N$ meets infinitely many points of the sequence $(x_{n_i})$. Since $N$ meets $K$ and lies in $U$, it belongs to the subfamily $\mathcal{N}'$ and hence equals $N_m$ for some $m$. Now the choice of the points $x_k$ implies that $x_{n_i} \notin N_m = N$ for all $i$ with $n_i > m$, which is a contradiction.

Repeating the above prove for the family $\mathcal{N}'_1 = \{ N \in \mathcal{N}' : \pi_1(N) \subset U, N \cap K \neq \emptyset \}$, we can prove that each $\sigma$-cs-finite cl$_1$-osed cs$^*$-network for $X$ is a $\sigma$-compact-finite cl$_1$-osed k-network for $X$. \qed

The following theorem gives an inner characterization of cs$^*$-metrizable spaces in terms of cs$^*$-networks.

Theorem 6.3. For a space $X$ the following conditions are equivalent:

1. $X$ is cs$^*$-metrizable;
2. the map $\lim : \text{Nw}(X) \to X$ has a cs$^*$-continuous section $s : X \to \text{Nw}(X)$;
3. $X$ has a $\sigma$-cs-finite cl$_1$-osed cs$^*$-network.

Proof. The implication (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (3) Assuming that $X$ is cs$^*$-metrizable, find a map $\pi : M \to X$ from a metrizable space $M$ having a cs$^*$-continuous section $s : X \to M$. Let $Z = s(X)$. The space $M$, being metrizable, admits a $\sigma$-locally-finite base $\mathcal{B}$ of the topology. For each $B \in \mathcal{B}$ let $N_B = s^{-1}(B)$. We claim that $\mathcal{N} = \{ N_B : B \in \mathcal{B} \}$ is a $\sigma$-cs-finite cl$_1$-osed cs$^*$-network for $X$. First we show that $\mathcal{N}$ is a cl$_1$-osed cs$^*$-network for $X$. Take an open set $U \subset X$ and a sequence $(x_n)_{n \in \omega} \subset U$, convergent to a point $x_\infty \in U$. It follows from the cs$^*$-continuity of $\pi$ that the sequence $(s(x_n))$ has an accumulating point $z_\infty \in M$ which projects onto $x_\infty$ by the continuity of $\pi$. Find a basic set $B \in \mathcal{B}$ such that $z_\infty \in B \subset \overline{B} \subset \pi^{-1}(U)$. Then $B$ contains infinitely many points $s(x_n)$ and hence $N_B = s^{-1}(B)$ contains infinitely many points $x_n$. The cs$^*$-continuity of $s$ implies that $\pi_1(N_B) \subset \pi(B) \subset U$. This proves that $\mathcal{N}$ is a cl$_1$-osed cs$^*$-network for $X$.

To show that the cs$^*$-network $\mathcal{N}$ is $\sigma$-cs-finite, write the base $\mathcal{B}$ as the countable union $\mathcal{B} = \bigcup_{k \in \omega} \mathcal{B}_k$ of locally finite families in $M$. Then $\mathcal{N} = \bigcup_{k \in \omega} \mathcal{N}_k$ where $\mathcal{N}_k = \{ N_B : B \in \mathcal{B}_k \}$. It remains to check that each family $\mathcal{N}_k$ is cs-finite in $X$. Fix any convergent sequence $K \subset X$. It follows from the cs$^*$-continuity of $s$ that the image $s(K)$ is precompact in $M$ and thus meets only finitely many sets of the locally finite family $\mathcal{B}$. Then its image $K = \pi(s(K))$ meets only finitely many elements of the family $\mathcal{N}_k$, $B \in \mathcal{B}_k$, which proves the cs-finity of $\mathcal{N}_k$. Therefore each cs$^*$-metrizable space admits a $\sigma$-cs-finite cl$_1$-osed cs$^*$-network.

(3) $\Rightarrow$ (2) Now assume that $X$ has a $\sigma$-cs-finite cl$_1$-osed cs$^*$-network $\mathcal{N}$. According to Proposition 6.1 we may assume that $\mathcal{N}$ is closed under finite intersections.
The network $\mathcal{N}$, being $\sigma$-cs-finite, can be written as the countable union $\mathcal{N} = \bigcup_{k \in \omega} \mathcal{N}_k$ of cs-finite families so that for any $m \in \omega$ the family $\bigcup_{k \geq m} \mathcal{N}_k$ still is a $c_{1\omega}$-osed $cs^*$-network for $X$. Define an increasing sequence of ordinals $(\beta_k)_{k \in \omega}$ letting $\beta_0 = 0$ and $\beta_{k+1} = \beta_k + |\mathcal{N}_k|$. Let also $\beta_\omega = \sup_{k \in \omega} \beta_k$. Then the family $\mathcal{N}$ can be enumerated as $\mathcal{N} = \{N_\alpha : \alpha < \beta_\omega\}$ so that $\mathcal{N}_k = \{N_\alpha : \beta_k \leq \alpha < \beta_{k+1}\}$ for all $k \in \omega$. The cs-finity of the families $\mathcal{N}_k$ implies that for every point $x \in X$ the set $\Lambda(x) = \{\alpha < \beta_\omega : x \in N_\alpha\}$ has finite intersection with each interval $[\beta_k, \beta_{k+1})$ and consequently, $\Lambda(x)$ has order type of $\omega$ (here we also use the fact that for every $k \in \omega$ the family $\{N_\alpha : \beta_k \leq \alpha < \beta_\omega\}$ is a network for $X$). Consequently, there is an increasing bijective map $\alpha_x : \omega \to \Lambda(x)$.

Finally, let $s(x) = (A_k)_{k \in \omega}$, where $A_k = \bigcap_{i \leq k} N_{\alpha(i)}$. It is easy to see that $s(x) \in Nw(x)$ and $\lim_{n} s(x) = x$, which means that $s$ is a section of the map $\lim : Nw(X) \to X$.

It remains to check that this section is $cs^*$-continuous. Take any sequence $(x_n)$ in $X$ convergent to a point $x_\omega \in X$. Let $K = \{x_n : n \leq \infty\}$ and consider the family $\Lambda(K) = \{\alpha < \beta_\omega : K \cap N_\alpha \neq \emptyset\}$. The cs-finity of the families $\mathcal{N}_k$ implies that for every $k \in \omega$ the intersection $\Lambda(K) \cap [0, \beta_{k+1})$ is finite and hence $\Lambda(K)$ is a countable set. Using the fact that for every $k \in \omega$ the family $\{N_\alpha : \beta_k \leq \alpha < \beta_{\omega}\}$ is a $cs^*$-network for $X$, construct an increasing number sequence $(n_k)_{k \in \omega}$ such that $K \subseteq \bigcup_{\beta_k \leq \alpha < \beta_{k+1}} N_\alpha$.

For every $n \in \omega$ consider the set $\Lambda(x_n) = \{\alpha < \beta_\omega : x_n \in N_\alpha\}$ having order type $\omega$ and let $\alpha_n = \alpha_{x_n} : \omega \to \Lambda(x_n)$ be the increasing bijective map. It follows that for every $i \in \omega$ we get $\alpha_n(i) < \beta_{k_{i+1}}$ and thus $\alpha_n(i) \in \Lambda_i(K)$ where $\Lambda_i(K) = \Lambda(K) \cap [0, \beta_{k_{i+1}})$ for $i \in \omega$. Endow each finite set $\Lambda_i(K)$ with discrete topology and consider the countable product $\Pi = \prod_{i \in \omega} \Lambda_i(K)$, which is a metrizable compact space. For every $n$ the function $\alpha_n : \omega \to \Lambda_i(K)$ can be considered as a point of the metrizable compact space $\Pi$. Then the sequence $(\alpha_n)_{n \in \omega}$ contains a subsequence $(\alpha_{n})_{n \in J}$ that converges to some point $\alpha_\infty \in \Pi$ (here $J$ is an infinite subset of $\omega$). Moreover, we can assume that $\alpha_n(i) = \alpha_\infty(i)$ for all $n \in J$ with $n \geq i$.

Consequently, $x_n \in N_{\alpha_\infty(i)}$ for all $n \in J$ with $n \geq i$ and hence $x_\infty = \lim_{n} x_n \in cl_1(N_{\alpha_\infty(i)})$ for all $i \in \omega$. Consider the sequence $(N_{\alpha_\infty(i)})_{i \in \omega}$. It follows from the convergence of the sequence $(\alpha_n)_{n \in J}$ to $\alpha_\infty$ that the sequence $(s(x_n))_{n \in J}$ converges to the sequence $\tilde{n} = (\cap_{j \leq i} N_{\alpha_\infty(i)})_{i \in \omega}$ in $\mathcal{P}(X)$. It remains to check that $\tilde{n}$ belongs to the subset $Nw(X) \subset \mathcal{P}(X)^\omega$. This will follows as soon as we prove that each neighborhood $U$ of $x_\infty$ contains some set $cl_1(N_{\alpha_\infty(i)})$. Since $\mathcal{N}$ is a $c_{1\omega}$-osed $cs^*$-network, there is a set $N_\gamma \in \mathcal{N}$ with $\gamma < \beta_\omega$ and $cl_1(N_\gamma) \subset U$ containing infinitely many points of the sequence $(x_n)_{n \in J}$. So the set $J_1 = \{n \in J : x_n \in N_\gamma\}$ is infinite. For every $n \in J_1$, the ordinal $\gamma$ belongs to the set $\Lambda(x_n)$ and hence $\gamma = \alpha_{x_n}(k_n)$ for some number $k_n \leq |\Lambda(K) \cap [0, \gamma]|$. Since the set $\Lambda(K) \cap [0, \gamma]$ is finite, for some number $m$ the set $J_2 = \{n \in J_1 : k_n = m\}$ is infinite. The convergence of the sequence $(\alpha_n)_{n \in J}$ to $\alpha_\infty$ implies that $\alpha_\infty(m) = \alpha_{x_n}(m) = \gamma$ for all sufficiently large numbers $n \in J$. Consequently, $N_{\alpha_\infty(m)} = N_\gamma \subset cl_1(N_\gamma) \subset U$, which implies that $\tilde{n} = (N_{\alpha_\infty(i)})_{i \in \omega} \in M$ by the definition of $M$.

Theorem 6.3 combined with Proposition 6.2 implies a characterization of $k^*$-metrizable spaces in terms of $c_{1\omega}$-osed $k$-networks.

**Theorem 6.4.** For a topological space $X$ the following conditions are equivalent:

1. $X$ is $k^*$-metrizable;
2. $X$ is $cs^*$-metrizable and all compact subsets of $X$ are sequentially compact;
3. the map $\lim : Nw(X) \to X$ has a section that preserves precompact sets;
4. all compact subsets of $X$ are sequentially compact and $X$ has a $\sigma$-cs-finite $c_{1\omega}$-osed $cs^*$-network;
5. $X$ has a $\sigma$-compact-finite $c_{1\omega}$-osed $k$-network.
If the space $X$ is sequentially regular, then the conditions (1)–(5) are equivalent to
(6) $X$ has a $\sigma$-compact-finite $k$-network.

Proof. The equivalences (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) follow from Theorem 6.3 and Propositions 4.2 and 1.3. The implication (4) $\Rightarrow$ (5) follows from Proposition 6.2. To prove that (5) $\Rightarrow$ (4) it suffices to check that for a space $X$ possessing a $\sigma$-compact-finite $\text{cl}_1$-osed $k$-network $\mathcal{N}$ each compact subset $K \subset X$ is sequentially compact. For this observe that the family
$$\mathcal{A} = \{N \cap K : N \in \mathcal{N}, \ N \cap K \neq \emptyset\}$$
is a countable $k$-network in $K$, which implies the metrizability (and hence sequential compactness) of $K$.

The equivalence (6) $\iff$ (5) for a sequentially regular space $X$ is obvious. $\square$

Remark 6.5. The $\sigma$-compact-finite $k$-networks in the characterization of $k^*$-metrizable spaces cannot be replaced by compact-countable $k$-networks: the ordinal space $[0, \omega_1)$ has a compact-countable base of the topology and hence has a compact-countable $k$-network. Yet, it is not $k^*$-metrizable, being sequentially compact but not compact.

Each continuous binary operation $\circ : X \times X \to X$ on a regular space $X$ induces a continuous binary operation
$$\bullet : \text{Nw}(X) \times \text{Nw}(X) \to \text{Nw}(X), \ (A_n) \bullet (B_n) \mapsto (A_n \circ B_n)$$on $\text{Nw}(X)$, where $A \circ B = \{a \circ b : a \in A, \ b \in B\}$. If $\circ$ is an associative (commutative) operation on $X$, then $\bullet$ is an associative (commutative) operation on $\text{Nw}(X)$. Moreover the limit operator $\lim : \text{Nw}(X) \to X$ preserves the operation in the sense that $\lim((A_n) \bullet (B_n)) = \lim(A_n) \circ \lim(B_n)$.

Combining this observation with Theorem 6.3 we obtain an algebraic characterization of $k^*$-metrizable topological semigroups.

Corollary 6.6. A regular (commutative) topological semigroup $X$ is $k^*$-metrizable if and only if there is a continuous homomorphism $h : M \to X$ from a (commutative) metrizable topological semigroup $M$ admitting a section $s : X \to M$ that preserves precompact sets.

We recall that a topological semigroup is a pair $(S, \circ)$ consisting of a topological space $S$ and a continuous associative operation $\circ : S \times S \to S$.

7. INTERPLAY BETWEEN $k^*$-METRIZABLE SPACES AND OTHER GENERALIZED METRIC SPACES

In this section we study the interplay between $k^*$-metrizable spaces and other classes of generalized metric space such as Lašnev or $\mathbb{R}$-spaces. We remind that a Hausdorff space $X$ is called a Lašnev space if it is the image of a metrizable space under a closed continuous map.

We need to recall the definition of the Arens’ fan, which is the subset
$$S_2 = \{(0,0), (\frac{1}{n},0), (\frac{1}{n}, \frac{1}{m}) : n, m \in \mathbb{N}\}$$of the plane, endowed with the strongest topology inducing the original topology on each compact subset $K_n = \{(0,0), (\frac{1}{k},0), (\frac{1}{j}, \frac{1}{j}) : k, j \in \mathbb{N}, \ i \leq n\}, \ n \in \mathbb{N}$, see [1].

The following theorem is close by spirit to the results of [17], [51], [59], [73].

Theorem 7.1. Let $X$ be a Hausdorff space. Then
(1) $X$ is a Lašnev space iff $X$ is a Fréchet-Urysohn $k^*$-metrizable space iff $X$ is a Fréchet-Urysohn $cs^*$-metrizable space iff $X$ is a sequential $cs^*$-metrizable space not containing a topological copy of the Arens’ fan $S_2$;
(2) $X$ is cs-metrizable if and only if $X$ is a cs*-metrizable space containing no (sequentially closed) subspace sequentially homeomorphic to the fans $S_\omega$ or $S_2$.

(3) $X$ is k-metrizable if and only if $X$ is a k*-metrizable space containing no (sequentially closed) subspace sequentially homeomorphic to the fans $S_\omega$ or $S_2$.

(4) $X$ is metrizable iff $X$ is a sequential cs-metrizable space iff $X$ is a k-metrizable k-space.

Proof. 1. If $X$ is a Ćapić space, then there is a closed map $\pi : M \to X$ of a metrizable space onto $X$ and $X$ is a Fréchet-Urysohn space by [31, 2.4.C]. By Proposition 1.5 each section $s : M \to X$ is cs*-continuous. Consequently, $X$ is cs*-metrizable. Being Fréchet-Urysohn, $X$ is a k*-metrizable space. Since the Arens’ fan $S_2$ is not Fréchet-Urysohn, $X$ cannot contain a topological copy of $S_2$.

Now assume conversely that $X$ is a Fréchet-Urysohn cs*-metrizable space and find a metrizable space $M$ and a map $\pi : M \to X$ having a cs*-continuous section. By Theorem 1.6 the map $\pi$ is inductively closed and hence $X$ is a Ćapić space. If $X$ is a sequential space containing no copy of the Arens’ fan, then $X$ is Fréchet-Urysohn by [70].

2. Assume that $X$ is cs-metrizable, which is equivalent to the metrizability of the sequential coreflexion $sX$ of $X$. We need to show that $X$ contains no subspace $W$ sequentially homeomorphic to the Fréchet-Urysohn or Arens fan. Actually, we can prove more: $X$ contains no subspace sequentially homeomorphic to a non-metrizable sequential space $Z$. Suppose that this is not true. Then we can find a sequential homeomorphism $h : W \to Z$ of a subspace $W \subset X$ onto $Z$. Since the identity map $i : sX \to X$ is a sequential homeomorphism, the composition $h \circ i^{-1}(W) : i^{-1}(W) \to Z$ is a sequential homeomorphism. Since both spaces $i^{-1}(W)$ and $Z$ are sequential, the function $h \circ i^{-1}(W)$ is a homeomorphism. Then the space $Z$, being homeomorphic to the metrizable space $i^{-1}(W)$, is metrizable, which is a contradiction.

Now assume conversely that $X$ is a cs*-metrizable but not cs-metrizable space. We will find a sequentially closed subspace in $X$ sequentially homeomorphic to a Fréchet-Urysohn or Arens’ fan. By Theorem 4.3(1) the sequential coreflexion $sX$ of $X$ is cs*-metrizable but not metrizable. Since all compact subsets of $sX$ are sequentially compact (by the sequentiality of $sX$), the space $sX$ is k*-metrizable and hence $sX$ is the image of a metrizable space $M$ under a subproper map $\pi : M \to sX$. Let $\sigma : sX \to M$ be a section of $\pi$ that preserves precompact sets. Without loss of generality, $\sigma(sX)$ is dense in $M$.

The space $sX$ is not k-metrizable, being a non-metrizable k-space. Consequently, the map $\pi$ fails to be proper and for some compact set $K \subset sX$ the preimage $\pi^{-1}(K)$ is not compact and hence contains a infinite closed discrete subset $\{z_n : n \in \mathbb{N}\}$. The sequentiality of $X$ implies the sequential compactness of the compactum $K$. Passing to a suitable subsequence we can assume that the sequence $(\pi(z_n))_{n=1}^\infty \subset K$ converges to a point $x_\infty$ of the sequentially compact space $K$. Again passing to a subsequence we can assume that either $\pi(z_n) = x_\infty$ for all $n$ or $\pi(z_n) \neq \pi(z_m) \neq x_\infty$ for all $n \neq m$.

Since the set $\{z_n : n \in \mathbb{N}\}$ is closed and discrete in $Z$ while $\sigma(K)$ is precompact, only finitely many points $z_n$ belong to $\sigma(K)$. Without loss of generality, we can assume that $z_n \notin \sigma(K)$ for all $n$ and thus $\{z_n : n \in \mathbb{N}\} \subset Z \setminus \sigma(K)$. Pick pairwise disjoint neighborhoods $O(z_n)$ of the points $z_n$ in $Z$ such that the collection $\{O(z_n) : n \in \mathbb{N}\}$ is discrete in $Z$. Since $\sigma(sX)$ is dense in $Z$ we can find for every $n \in \mathbb{N}$ a sequence $(z_{n,m})_{m=1}^\infty \subset \sigma(sX) \cap O(z_n)$ convergent to $z_n$. Then the sequence $(\pi(z_{n,m}))_{m=1}^\infty$ converges to $\pi(z_n)$. It can be shown that the set $W = \{x_\infty, \pi(z_n), \pi(z_{m,n}) : n, m \in \mathbb{N}\}$ is closed in $sX$. We claim that $W$ is sequentially homeomorphic either to the Fréchet–Urysohn fan or to the Arens fan.

Indeed, if $\pi(z_n) = x_\infty$ for all $n$, we define a homeomorphism $h : S_\omega \to W$ from the Fréchet–Urysohn fan letting $h(*) = x_\infty$ and $h(2^{-m},n) = \pi(z_{n,m})$ for $(2^{-m},n) \in S_\omega \setminus \{\ast\}$. Here $\ast = \{0\} \times \omega$ is the non-isolated point of the fan $S_\omega = \{2^{-m} : m \leq \omega\} \times \omega / \{0\} \times \omega$. If
\[ \pi(z_n) \neq \pi(z_m) \neq x_\infty \] for all \( n \neq m \), we define a homeomorphism \( h : S_2 \to W \) from the Arens fan letting \( h(0,0) = x_\infty \), \( h \left( \frac{1}{n}, 0 \right) = \pi(z_n) \), and \( h \left( \frac{1}{n}, \frac{1}{m} \right) = \pi(z_{n,m}) \) for all \( n, m \in \mathbb{N} \).

Since the identity map \( i : sX \to X \) is a sequential homeomorphism, the image \( i(W) \) is a sequentially closed subset of \( X \), sequentially homeomorphic to the Fréchet-Urysohn or Arens’ fans.

3. If \( X \) is \( k \)-metrizable, then it is \( k^* \)-metrizable and cs-metrizable. By the preceding item, \( X \) contains no subspace sequentially homeomorphic to the Fréchet-Urysohn or Arens’ fan. Now assume conversely that \( X \) is a \( k^* \)-space containing no sequentially closed subspace homeomorphic to the Fréchet-Urysohn or Arens’ fan. By the preceding item, \( X \) is cs-metrizable. Being a \( k^* \)-space, \( X \) is the image of a metrizable space under a subproper map. Consequently all compact subsets of \( X \) are metrizable and hence sequentially compact. Now Proposition 4.2 implies that \( X \) is \( k \)-metrizable.

4. The last item trivially follows from the fact that the (sequential) \( k \)-coreflexion of \( X \) coincides with \( X \) if \( X \) is a (sequential) \( k \)-space.

Next, we establish the interplay between \( k^* \)- and cs*-metrizable spaces and generalized metric spaces defined with help of \( k \)- or cs*-networks (like \( \mathcal{N}_0 \)- or \( \mathcal{N} \)-spaces). \( \mathcal{N}_0 \)-Spaces were introduced by E. Michael [55] as regular spaces with countable \( k \)-network. He also proved that a regular space \( X \) is an \( \mathcal{N}_0 \)-space if and only if \( X \) is the image of a metrizable separable space under a compact-covering map. We are going to show that a bit more is true: each regular \( \mathcal{N}_0 \)-space is the image of a metrizable separable space under a subproper map. A version of the following theorem was proved by Chuan Liu and Y. Tanaka in [54].

**Theorem 7.2.** For a regular topological space \( X \) the following conditions are equivalent:

1. \( X \) is an \( \mathcal{N}_0 \)-space;
2. \( X \) is the image of a metrizable separable space \( M \) under a subproper map \( \pi : M \to X \);
3. \( X \) is a \( k^* \)-metrizable space with countable network;
4. \( X \) is a cs*-metrizable space with countable network;
5. the \( k \)-coreflexion \( kX \) of \( X \) is a \( k^* \)-metrizable space with countable extent;
6. the sequential coreflexion \( sX \) of \( X \) is a cs*-metrizable space with countable extent.
7. \( kX \) is the image of a metrizable separable space \( M \) under a quotient map \( f : M \to kX \).
If \( sX \) is regular and \( (CH) \) holds, then the conditions (1)–(6) are equivalent to:

8. \( sX \) is separable and cs*-metrizable.

**Proof.** We shall prove the implications (1) \( \Rightarrow \) (3) \( \Rightarrow \) (4) \( \Rightarrow \) (5) \( \Rightarrow \) (6) \( \Rightarrow \) (2) \( \Rightarrow \) (7) \( \Rightarrow \) (1) \( \Leftrightarrow \) (8).

The implication (1) \( \Rightarrow \) (3) follows from Theorem 6.3 while (3) \( \Rightarrow \) (4) is trivial.

(4) \( \Rightarrow \) (5) If \( X \) is cs*-metrizable space with countable network, then each compact subset of \( X \) has countable network and hence is metrizable. By Proposition 4.2, the space \( X \) is \( k^* \)-metrizable and so is the \( k \)-coreflexion \( kX \) of \( X \), see Proposition 3.3. The space \( X \) has countable network, and hence is the image of a separable metrizable space \( Z \) under a continuous map \( g : Z \to X \), see [39, 4.9]. Observe that the map \( g : Z \to kX \) is continuous. Consequently, the space \( kX \) has countable network and cannot contain uncountable closed discrete subset.

(5) \( \Rightarrow \) (6) Assume that the \( k \)-coreflexion \( kX \) of \( X \) is a \( k^* \)-metrizable space with countable extent. Then each compact subset of \( kX \) is metrizable and hence the space \( kX \) is sequential and coincides with the sequential coreflexion \( sX \) of \( X \). Then \( sX = kX \) is a cs*-metrizable space with countable extent.

(6) \( \Rightarrow \) (2) Assume that \( sX \) is a cs*-metrizable space with countable extent. By Theorem 4.3, the space \( X \) is cs*-metrizable. So we can take a map \( \pi : M \to X \) from a metrizable space
that has a $\pi^*$-continuous section $\sigma : X \to M$ with dense image $s(X)$ in $M$. We may assume that $\sigma(X)$ is dense in $M$. We claim that the space $M$ is separable. Otherwise, we could find an uncountable subset $D \subset \sigma(X)$ that is closed and discrete in $X$. Then the set $\pi(D)$ has finite intersection with each convergent sequence in $sX$ and consequently, $\pi(D)$ is an uncountable closed discrete subset in $sX$, which is not possible as $sX$ has countable extent. Therefore, the space $M$ is separable and consequently, each compact subset of $X$ has countable network and is metrizable. By Proposition 1.3, the map $\pi : M \to X$ is subproper.

(2) $\Rightarrow$ (7) If $\pi : M \to X$ is a subproper map from a metrizable separable space, then $\pi$, considered as a map from $M$ to $kX$ is compact-covering and thus quotient.

(7) $\Rightarrow$ (1) This is due to E.Michael [53]. Let $f : M \to kX$ be a quotient map of a metrizable separable space $M$ and let $\mathcal{B}$ be a countable base of the topology of $M$. We claim that $f(\mathcal{B}) = \{f(B) : B \in \mathcal{B}\}$ is a $k$-network for $X$. Take any open set $U \subset X$ and a compact subset $K \subset U$. Let $\{C_i : i \in \omega\}$ be an enumeration of the family $\mathcal{C} = \{f(B) : B \in \mathcal{B}, f(B) \subset U\}$. We claim that $K \subset C_1 \cup \cdots \cup C_n$ for some $n \in \omega$. Assuming the converse, we can find a sequence $(x_n)$ in $K$ such that $x_n \notin C_1 \cup \cdots \cup C_n$ for all $n \in \omega$. The compactum $K$, being the image of a separable metrizable space, has countable network and hence is metrizable. Then, passing to a subsequence, we may assume that $(x_n)$ converges to some point $x \in K$ such that $x \neq x_n$ for all $n \in \omega$. Since $f : M \to kX$ is quotient and $A = \{x_n : n \in \omega\}$ is not closed in $kX$, the set $\pi^{-1}(A)$ is not closed in $M$. Let $z \in \pi^{-1}(A) \setminus \pi^{-1}(A)$. Then $z \in \pi^{-1}(K) \subset \pi^{-1}(U)$. Choose $B \in \mathcal{B}$ with $z \in B \subset \pi^{-1}(U)$. Then $f(B) \in \mathcal{C}$ and $f(B)$ contains infinitely many $x_n$'s. This contradicts the way the $x_n$'s were chosen.

The equivalence $(1) \Leftrightarrow (8)$ follows from Corollary 5.10. \hfill $\square$

**Remark 7.3.** The countably compact space $X \subset \beta\mathbb{N}$ from Remark 3.8 is a separable $k$-metrizable space without countable network (and thus $X$ is not an $\aleph_0$-space). Another example of this sort is the Lindelöfication $\{\infty\} \cup D$ of an uncountable discrete space $D$ which is Lindelöf and $k$-metrizable but has no countable network. These examples show that the network weight $nw(X)$ in Theorem 7.2(3) cannot be replaced by the density $d(X)$ or Lindelöf number $l(X)$. We do not know if this can be done for the hereditary density or hereditary Lindelöf number.

**Question 7.4.** Has a regular $k^*$-metrizable space $X$ countable network if it is hereditarily separable and hereditarily Lindelöf?

$\aleph_0$-Spaces form a subclass in a more general class of $\aleph$-spaces introduced by O’Meara [62]. We recall that a regular space $X$ is defined to be an $\aleph$-space if it has a $\sigma$-locally-finite $k$-network, see [45]. Since each locally-finite family is compact-finite, we may apply Theorem 6.4 to conclude that the class of $k^*$-metrizable spaces contains all $\aleph$-spaces.

**Corollary 7.5.** Each $\aleph$-space is $k^*$-metrizable.

It is interesting to mention that $\aleph$- and $k$-spaces can be characterized as regular $k$-spaces possessing a $\sigma$-compact-finite closed $k$-network. Such a characterization holds because each compact-finite family of closed subsets of a $k$-space is locally-finite. In spite of the similarity in characterizations of $k^*$-metrizable and $\aleph$-spaces, there are $k^*$-metrizable spaces which are not $\aleph$-spaces. The simplest example is the uncountable sequential fan $S_{\omega}$, which is a regular $k^*$-metrizable $k$-space that fails to be an $\aleph$-space. The following theorem helps to detect $\aleph$-spaces among $k^*$-metrizable spaces with small character.

**Proposition 7.6.** A regular paracompact $k^*$-metrizable $k$-space $X$ with $\text{ext}_\chi(X) \leq \aleph_0$ is an $\aleph$-space.
Proof. Fix a subproper map $\pi : M \to X$ of a metric space $(M, d)$ possessing a precompact-preserving section $s : X \to M$ with dense image $s(X)$ in $M$. For every $n \in \omega$ fix a locally finite open cover $U_n$ of the metric space $M$ by sets of diameter < $2^{-n}$. By Proposition 5.7, each point $x \in X$ has a neighborhood $O_n(x)$ meeting at most countably many sets $s^{-1}(U)$, $U \in U_n$. The paracompactness of $X$ yields a $\sigma$-discrete open cover $V_n$ of $X$ refining the cover $\{O_n(x) : x \in X\}$. Write $V_n = \bigcup_{m \in \omega} V_{n,m}$, where each collection $V_{n,m}$ is discrete in $M$. For every $V \in V_{n,m}$ the image $s(V)$ lies in the union of a countable subcollection $U_V \subset U_n$. Enumerate this subcollection as $U_V = \{U_{V,k} : k < |U_V|\}$ and put $U_{V,k} = \emptyset$ for $k \geq |U_V|$. Finally let
\[ N_{n,m,k} = \{s^{-1}(U_{V,k}) : V \in V_{n,m}\} \]
and observe that the family $N_{n,m,k}$ is discrete in $X$. On the other hand, it can be easily shown that the union $N = \bigcup_{n,m,k \in \omega} N_{n,m,k}$ is a $k$-network for $X$. Thus $X$ admits a $\sigma$-discrete $k$-network and hence $X$ is an $\aleph$-space.

Problem 7.7. Characterize $\aleph$-spaces within the class of $k^*$-metrizable spaces. In particular, is a regular $k^*$-metrizable $k$-space $X$ an $\aleph$-space if $\alpha_4(X) \leq \aleph_0$? (The answer is affirmative if $X$ carries a topological group structure, see Theorem 13.2).

Next we investigate the interplay between $k^*$-metrizable and semi-stratifiable spaces. We recall that a regular space $X$ is semi-stratifiable if there is a function $G$ which assigns to each $n \in \omega$ and a closed subset $F \subset X$, an open set $G(n, F)$ containing $F$ such that
\begin{enumerate}
  \item[(a)] $F = \bigcap_n G(n, F)$;
  \item[(b)] $F \subset K \Rightarrow G(n, F) \subset G(n, K)$.
\end{enumerate}

If also
\begin{enumerate}
  \item[(a')] $F = \bigcap_n \overline{G(n, F)}$,
\end{enumerate}
then $X$ is called stratifiable, see [39, §5].

By Theorem [39, 5.9], each regular space with countable network is semi-stratifiable. In particular, each $\aleph_0$-space is semi-stratifiable. However, an $\aleph_1$-space need not be stratifiable. The simplest example is the Banach space $(l^1, \text{weak})$ endowed with the weak topology, which is not stratifiable according to [38]. Another example is the space $C_k(\mathbb{Q})$ of continuous functions on rationals, endowed with the compact-open topology. By a recent result of P.Nyikos [61], $C_k(\mathbb{Q})$ fails to be a stratifiable space but is an $\aleph_0$-space according to a classical result of E.Michael [55]. Also $k^*$-metrizable spaces need not be semi-stratifiable.

Example 7.8. The Frolik’s space $X$ from Example 7.3, being separable and not Lindelöf, fails to be subparacompact and hence is not semi-stratifiable, see [39, 5.11]. Nonetheless, $X$ is a $k$-metrizable space.

In the meantime, $cs^*$-metrizable spaces are semi-stratifiable in a weaker sequential sense. We shall say that a subset $U$ of a space $X$ is a sequential barrier for a set $F \subset X$ if for any sequence $(x_n) \subset X$ convergent to a point $x_{\infty} \in F$ there is $m \in \omega$ such that $x_n \in U$ for all $n \geq m$. It is clear that each open set $U$ is a sequential barrier for any set $F \subset U$. The converse is true in Fréchet-Urysohn spaces.

Theorem 7.9. For each $cs^*$-metrizable space $X$ there is a function $G$ which assigns to each $n \in \omega$ and a closed subset $F \subset X$, a sequential barrier $G(n, F)$ for $F$ such that
\begin{itemize}
  \item $F = \bigcap_n G(n, F)$;
  \item $F \subset K \Rightarrow G(n, F) \subset G(n, K)$.
\end{itemize}
Proof. Given a cs*-metrizable space $X$, find a map $\pi : M \to X$ of a metrizable space $M$ having a cs*-continuous section $s : X \to M$. We may additionally assume that the set $Z = s(X)$ is dense in $M$.

The space $M$, being metrizable, is stratifiable and hence has a function $G_M$ assigning to each $n \in \omega$ and a closed subset $F \subset X$, an open set $G_M(n,F)$ containing $F$ and such that

1. $F = \bigcap_n G_M(n,F)$;
2. $F \subset K \Rightarrow G_M(n,F) \subset G_M(n,K)$.

Using the stratifying function $G_M$ define a function $G$ letting

$$G(n,F) = s^{-1}(G_M(n,\pi^{-1}(F))),$$

where $n \in \omega$ and $F$ is a closed subset of $X$. We claim that $G(n,F)$ is a sequential barrier at $F$. Indeed, assume that there is a sequence $(x_n) \in X \setminus G(n,F)$ convergent to a point $x_\infty \in F$. The cs*-continuity of the function $s$ implies the existence of a limit point $z_\infty$ for the sequence $(s(x_n))$. Then $\pi(z_\infty) = x_\infty$ by the continuity of $\pi$. Since $G_M(n,\pi^{-1}(F))$ is an open neighborhood of $z_\infty$, infinitely many points of the sequence $(s(x_n))$ belong to $G_M(n,\pi^{-1}(F))$ and then infinitely many points of the sequence $(x_n)$ belong to the set $G(n,F) = s^{-1}(G_M(n,\pi^{-1}(F)))$ which contradicts the choice of the sequence $(x_n)$.

In obvious way the two properties of the function $G_M$ imply the corresponding two properties of the function $G$ indicated in Theorem 7.9.

Also cs*-spaces have a property close to the monotonical normality. We recall that a topological space $X$ is called **monotonically normal** if to each pair $(H, K)$ of disjoint closed subsets of $X$ one can assign an open set $D(H, K) \subset X$ such that

1. $H \subset D(H, K) \subset \overline{D(H, K)} \subset X \setminus K$;
2. if $H \subset H'$ and $K \supset K'$, then $D(H, K) \subset D(H', K')$.

The function $D$ is called a **monotone normality operator** for $X$, see [39, 5.15] or [22]. Observe that for such an operator $D$ we have

$$\text{cl}(D(H, K)) \cap K = \emptyset \quad \text{and} \quad \text{cl}(X \setminus D(H, K)) \cap H = \emptyset$$

for any disjoint closed subsets $H, K$ of $X$.

According to a characterization in [39, 5.22], a space $X$ is stratifiable if and only if the product $X \times S_1$ of $X$ and a convergent sequence $S_1 = \{2^{-n} : n \leq \infty\}$ is monotonically normal.

We define a topological space $X$ to be **monotonically sequentially normal** if to each pair $(H, K)$ of disjoint closed subsets of $X$ one can assign a subset $D(H, K) \subset X$ such that

1. $\text{cl}_1(D(H, K)) \cap K = \emptyset$ and $\text{cl}_1(X \setminus D(H, K)) \cap H = \emptyset$;
2. if $H \subset H'$ and $K \supset K'$, then $D(H, K) \subset D(H', K')$.

Here $\text{cl}_1(A)$ stands for the 1-st sequential closure of a subset $A \subset X$. Since $\text{cl}_1(A) = \text{cl}(A)$ for every subset $A$ of a Fréchet–Urysohn space, we see that each monotonically sequentially normal Fréchet–Urysohn space is monotonically normal.

**Proposition 7.10.** Each cs*-metrizable space is monotonically sequentially normal.

Proof. Given a cs*-metrizable space let $\pi : M \to X$ be a continuous map from a metrizable space $M$ and let $s : X \to M$ be a cs*-continuous section of $\pi$. Without loss of generality, the image $Z = s(X)$ is dense in $M$. The metrizable space $M$ is monotonically normal, hence admits a monotone normality operator $D_M$.

Given a pair $(H, K)$ of disjoint closed subsets of $X$, let

$$D(H, K) = s^{-1}(D_M(\pi^{-1}(H), \pi^{-1}(K))).$$
If \((H', K')\) are disjoint closed subsets of \(X\) with \(H \subset H'\) and \(K \supset K'\), then
\[D_M(\pi^{-1}(H), \pi^{-1}(K)) \subset D_M(\pi^{-1}(H'), \pi^{-1}(K'))\]
and hence \(D(H, K) \subset D(H', K')\).

It can be shown that
\[
\text{cl}_1(D(H, K)) \subset \pi(Z \cap D_M(\pi^{-1}(H), \pi^{-1}(K))) = \pi(D_M(\pi^{-1}(H), \pi^{-1}(K))).
\]
Since \(\overline{D_M(\pi^{-1}(H), \pi^{-1}(K))} \cap \pi^{-1}(K) = \emptyset\), we get \(\text{cl}_1(D(H, K)) \cap K = \emptyset\).

Similarly, we obtain
\[
\text{cl}_1(X \setminus D(H, K)) = \text{cl}_1(\pi(Z \setminus D_M(\pi^{-1}(H), \pi^{-1}(K)))) \subset \pi(Z \setminus D_M(\pi^{-1}(H), \pi^{-1}(K))).
\]
Since \(Z \setminus D_M(\pi^{-1}(H), \pi^{-1}(K)) \subset M \setminus D_M(\pi^{-1}(H), \pi^{-1}(K))\) and the latter set misses \(\pi^{-1}(H)\), we get that \(\text{cl}_1(X \setminus D(H, K)) \cap H = \emptyset\).

It is interesting to remark that there exist \(\sigma\)-compact \(\aleph_0\)-spaces which are monotonically sequentially normal but not monotonically normal.

**Example 7.11.** For every infinite-dimensional Banach space \(X\) whose dual \(X^*\) is separable, the space \((X, \text{weak})\) is an \(\aleph_0\)-space (see Proposition [12.1] and thus is monotonically sequentially normal, but \((X, \text{weak})\) is not monotonically normal, see [38].

8. \(k^*\)-Metrizable Function Spaces

Given topological spaces \(X\) and \(Y\) let \(C(X, Y)\) be the set of all continuous functions from \(X\) to \(Y\). There is a general way of topologizing \(C(X, Y)\) using a family \(\mathcal{K}\) of compact subsets of \(X\). Namely we endow the set \(C(X, Y)\) with the topology generated by the sub-base consisting of the sets
\[\{F, U\} = \{f \in C(X, Y) : f(F) \subset U\}\]
where \(F\) is a closed subset of a compact set \(K \in \mathcal{K}\) and \(U\) runs over open subsets of \(Y\), and denote the obtained topological space by \(C_{\mathcal{K}}(X, Y)\). It is easy to see that the space \(C_{\mathcal{K}}(X, Y)\) is Hausdorff if \(Y\) is Hausdorff and \(\cup \mathcal{K}\) is dense in \(X\). If, in addition, the space \(Y\) is (completely) regular, then so is the function space \(C_{\mathcal{K}}(X, Y)\), see Theorems 3.4.13 and 3.4.15 of [31].

If the union \(\cup \mathcal{K} \subset X\) is separable and all compact subsets of a space \(Y\) are metrizable, then all compact subsets of the function space \(C_{\mathcal{K}}(X, Y)\) are metrizable too. Indeed, take any countable dense subset \(D \subset \mathcal{K}\) and consider the restriction operator \(R : C_{\mathcal{K}}(X, Y) \to Y^D\) which is continuous and injective. Then each compact subset \(K \subset C_{\mathcal{K}}(X, Y)\) is homeomorphic to the compact subset \(R(K)\) of the countable product \(Y^D\) and hence is metrizable.

Two particular choices of the family \(\mathcal{K}\) are especially important. If \(\mathcal{K}\) is the family of all finite (resp. compact) subsets of \(X\), then we write \(C_p(X, Y)\) (resp. \(C_k(X, Y)\)) instead of \(C_{\mathcal{K}}(X, Y)\) and say about the topology of pointwise convergence (the compact-open topology) on the set \(C(X, Y)\).

By a classical result of E.Michael [53] the function space \(C_k(X, Y)\) is an \(\aleph_0\)-space for any \(\aleph_0\)-spaces \(X\) and \(Y\). This result was generalized to \(\aleph\)-spaces by O’Meara [62].

**Theorem 8.1** (O’Meara). The space \(C_k(X, Y)\) of continuous functions from an \(\aleph_0\)-space to an \(\aleph\)-space is an \(\aleph\)-space, and hence \(C_k(X, Y)\) is \(k^*\)-metrizable.

We shall derive from this theorem a bit more general result treating subspaces of \(C_k(X, Y)\).

**Theorem 8.2.** Let \(X\) be an \(\aleph_0\)-space, \(Y\) be an \(\aleph\)-space and \(\mathcal{K}\) be a family of compact subsets of a space \(X\) with dense union \(\cup \mathcal{K}\) in \(X\). A subspace \(\mathcal{F} \subset C_k(X, Y)\) is \(k^*\)-metrizable if the evaluation operator \(e : \mathcal{F} \times X \to Y\), \(e : (f, x) \mapsto f(x)\) is sequentially continuous.
**Proof.** Let \((\mathcal{F}, \tau_k)\) (resp. \((\mathcal{F}, \tau_K)\)) denote the set \(\mathcal{F}\) endowed with the topology inherited from the function space \(C_k(X,Y)\) (resp. \(C_K(X,Y)\)). It is clear that the identity map \(id : (\mathcal{F}, \tau_k) \to (\mathcal{F}, \tau_K)\) is continuous. The sequential continuity of the evaluation operator \(e : \mathcal{F} \times X \to Y\) and the metrizability of compact subsets of \(X\) imply that the inverse function \(id^{-1} : (\mathcal{F}, \tau_K) \to (\mathcal{F}, \tau_k)\) is sequentially continuous. Moreover, this function is continuous on each compact subset of \((\mathcal{F}, \tau_K)\) because compact subsets of the function space \(C_K(X,Y) \supset \mathcal{F}\) are metrizable (this follows from the separability of \(\mathcal{K}\) and metrizability of compact subsets of \(Y\)). Therefore the identity map \(id : (\mathcal{F}, \tau_k) \to (\mathcal{F}, \tau_K)\) is proper.

By Theorem 8.3 the space \(C_k(X,Y)\) is \(k^*\)-metrizable and so is its subspace \((\mathcal{F}, \tau_k)\). Since the \(k^*\)-metrizability is preserved by proper maps, the space \(\mathcal{F} = (\mathcal{F}, \tau_k)\) is \(k^*\)-metrizable. \(\square\)

Unfortunately we do not know the answer to the following intriguing

**Question 8.3.** Is \(C_k(X,Y)\) \(k^*\)-metrizable for any \(\aleph_0\)-space \(X\) and any regular \(k^*\)-metrizable space \(Y\)?

As an application of Theorem 8.2 we prove the \(k^*\)-metrizability of spaces of continuous homomorphisms between topological groups. For topological groups \(G, H\) and a family \(\mathcal{K}\) of compact subsets of \(G\) let \(\text{Hom}_\mathcal{K}(G, H)\) be the subspace of the function space \(C_K(G, H)\), consisting of all continuous group homomorphisms.

**Corollary 8.4.** Let \(G, H\) be topological groups such that \(G\) is a Baire \(\aleph_0\)-space and \(H\) is an \(\aleph\)-space. Then for any family \(\mathcal{K}\) of compact subsets of \(G\) with \(\cup \mathcal{K} = G\) the space \(\text{Hom}_\mathcal{K}(G, H)\) is \(k^*\)-metrizable.

**Proof.** This corollary will follow from Theorem 8.2 as soon as we check that the evaluation operator \(e : \text{Hom}_\mathcal{K}(G, H) \times G \to H\) is sequentially continuous. Take any two compact sets \(A \subset \text{Hom}_\mathcal{K}(G, H)\) and \(B \subset G\). Since \(\cup \mathcal{K} = G\), the set \(A\) is compact in \(C_p(G, H)\). Applying Troallic’s Theorem 4.4 [76], we conclude that \(A\) is compact in \(C_k(G, H)\) which implies that the topology of pointwise convergence on \(A\) coincides with the compact-open topology. This implies that the restriction of the evaluation operator \(e\) to \(A \times B\) is continuous and hence \(e\) is sequentially continuous. \(\square\)

9. **Subproper maps between spaces of measures**

By a *measure* on a topological space \(X\) we understand a countably additive function \(\mu : \mathcal{B}(X) \to [0, \infty]\) defined on the \(\sigma\)-algebra of Borel subsets of \(X\). A measure \(\mu\) on \(X\) is called a *probability measure* if \(\mu(X) = 1\). A measure \(\mu\) on \(X\) is *Radon* if for every \(\varepsilon > 0\) and every Borel subset \(B \subset X\) there is a compact subset \(K \subset B\) with \(\mu(B \setminus K) < \varepsilon\).

For a Hausdorff topological space \(X\) by \(P_r(X)\) we denote the space of all probability Borel Radon measures on \(X\) endowed with the topology generated by the sub-basis consisting of the sets \(\{\mu \in P_r(X) : \mu(U) > a\}\) where \(a \in \mathbb{R}\) and \(U\) runs over open subsets of \(X\).

It is well-known that for a Tychonoff space \(X\) the topology of \(P_r(X)\) is generated by the sub-basis consisting of the sets \(\{\mu \in P_r(X) : \int_X f \, du > 0\}\) where \(f\) runs over all bounded real-valued continuous functions on \(X\). For a compact Hausdorff space \(X\) the space \(P_r(X)\) is known to be compact and Hausdorff [16, 8.5.1], [17, Ch. 8], [34, VII.3.5] or [32]; for a metrizable space \(X\) the space \(P_r(X)\) is metrizable [16, 8.5.1], [17, Ch. 8], or [4].

For a Borel function \(f : X \to Y\) between topological spaces let \(\text{Pr}(f) : P_r(X) \to P_r(Y)\) be the function assigning to each measure \(\mu \in P_r(X)\) the measure \(\eta = \text{Pr}(f)(\mu)\) such that \(\eta(B) = \mu(f^{-1}(B))\) for any Borel subset \(B \subset Y\). It is well-known that for every continuous map \(f : X \to Y\) between Hausdorff spaces the function \(\text{Pr}(f) : P_r(X) \to P_r(Y)\) is continuous; for an injective map \(f\) between Tychonoff spaces the map \(\text{Pr}(f)\) is injective; for a surjective map...
f between compact Hausdorff spaces the map \( P_r(f) \) is surjective; for a topological embedding \( f: X \to Y \) of Tychonoff spaces the map \( P_r(f) \) is a topological embedding (see [17, Ch. 8.9]).

A family \( \mathcal{M} \subset P_r(X) \) of measures on \( X \) is called uniformly tight if for every \( \varepsilon > 0 \) there is a compact subset \( K_{\varepsilon} \subset X \) such that \( \mu(X \setminus K_{\varepsilon}) < \varepsilon \) for all \( \mu \in \mathcal{M} \). Every uniformly tight collection \( \mathcal{M} \subset P_r(X) \) of measures on a Tychonoff space is precompact in \( P_r(X) \). The converse is not always true [65]. A Tychonoff space \( X \) is called Prohorov if each compact subset of \( P_r(X) \) is uniformly tight. If each convergent sequence in \( P_r(X) \) is uniformly tight, then the space \( X \) is called sequentially Prohorov. The class of Prohorov spaces includes all Čech-complete spaces (hence all locally compact spaces and all complete metric spaces) and all \( k_\text{w} \)-spaces, see [16, 8.3.5], [17, Ch. 8]. However, the space \( \mathbb{Q} \) of rational numbers is not Prohorov, see [65]. On the other hand, each Tychonoff space of countable type (in particular, each metrizable space) is sequentially Prohorov [16, 8.3.15]. We recall that a topological space \( X \) has countable type if each point \( x \in X \) lies in a compact subset of \( X \) possessing a countable fundamental system of neighborhoods, see [31, 3.1.E].

**Theorem 9.1.** Suppose a map \( f: X \to Y \) between Tychonoff spaces admits a section \( s: Y \to X \) that preserves precompact sets. Then the map \( P_r(f): P_r(Y) \to P_r(X) \) admits a section \( l: P_r(Y) \to P_r(X) \) such that \( l(\mu)(s(K)) = \mu(K) \) for every compact subset \( K \subset Y \) and every measure \( \mu \in P_r(Y) \).

**Proof.** Let \( \beta f: \beta X \to \beta Y \) be the continuous extension of the map \( f \) onto the Stone-Čech compactifications of \( X \) and \( Y \). Since the functor \( P_r \) preserves embeddings [4, 2.4], we can identify the space \( P_r(X) \) with a subspace of \( P_r(\beta X) \). Given a probability Radon measure \( \mu \) on \( Y \) and a compactum \( K \subset \beta(Y) \) consider the closed subset

\[
M_K = \{ \eta \in P_r(\beta X): P_r(\beta f)(\eta) = \mu \text{ and } \eta(s(K)) \geq \mu(K) \}
\]

of the compact space \( P_r(\beta X) \). We claim that the intersection \( \bigcap_{K \in K(Y)} M_K \) is not empty. Since the sets \( M_K \) are closed in the compact space \( P_r(\beta X) \), it suffices to verify that \( \bigcap_{K \in \mathcal{F}} M_K \neq \emptyset \) for every finite collection \( \mathcal{F} \subset K(Y) \). Let \( \mathcal{A} \) be the smallest algebra of subsets of \( X \) containing the family \( \mathcal{F} \) and let \( \{ A_1, \ldots, A_m \} \) denote the set of atoms (that is the minimal non-empty subsets) in \( \mathcal{A} \). It is clear that the sets \( A_i \) are pairwise disjoint Borel subsets of \( X \).

Since the measure \( \mu \) is Radon, for every \( i \leq m \) we can find a countable collection \( K_i \) of pairwise disjoint compact subsets of \( A_i \) such that \( \mu(A_i) = \sum_{K \in K_i} \mu(K) \). Next, for every \( K \in K_i \) find a measure \( \eta_K \) on the compactum \( s(K) \) whose image under the map \( f \) is the restriction \( \mu|K \) of the measure \( \mu \) onto the compactum \( K \). Finally, let \( \eta = \sum_{i=1}^m \sum_{K \in K_i} \eta_K \). It is easy to see that \( P_r(\beta f)(\eta) = \mu \) and \( \eta(s(K)) \geq \mu(K) \) for every \( K \in \mathcal{F} \). Thus \( \eta \in \bigcap_{K \in \mathcal{F}} M_K \neq \emptyset \) and the intersection \( \bigcap_{K \in K(Y)} M_K \) contains some measure \( l(\mu) \).

Thus we define a function \( l: P_r(Y) \to P_r(\beta X) \). It is clear that \( P_r(\beta f)(l(\mu)) = \mu \) and \( l(\mu)(f^{-1}(K)) \geq l(\mu)(s(K)) \geq \mu(K) = l(\mu)(f^{-1}(K)) \) for each \( K \in K(Y) \). Let us show that the measure \( l(\mu) \) can be identified with a Radon measure on \( X \). Since the measure \( \mu \) is Radon, for every \( \epsilon > 0 \) there is a compact subset \( K \subset Y \) with \( \mu(K) > 1 - \epsilon \). Then \( l(\mu)(s(K)) \geq \mu(K) \geq 1 - \epsilon \) and thus \( l(\mu) \) is a Radon measure on \( X \).

**Corollary 9.2.** Let \( f: X \to Y \) be a subproper map between Tychonoff spaces. Then the map \( P_r(f): P_r(X) \to P_r(Y) \) admits a section that preserves uniformly tight families of measures. Consequently, if the space \( Y \) is Prohorov, then the map \( P_r(f) \) is subproper.

In spite of the fact that a metric space does not need to be Prohorov, we shall show that the functor \( P_r \) preserves subproper maps between metrizable spaces.
Theorem 9.3. Suppose \( f : X \to Y \) is a map from a Tychonoff space \( X \) onto a sequentially Prohorov space \( Y \). If \( f \) has a precompact-preserving section, then the map \( P_r(f) : P_r(X) \to P_r(Y) \) has a cs*-continuous section.

Proof. By Corollary 9.2 the map \( P_r(f) : P_r(X) \to P_r(Y) \) admits a section \( s : P_r(Y) \to P_r(X) \) preserving uniformly tight families of measures. Since \( Y \) is sequentially Prohorov, each convergent sequence \( (\mu_n)_{n \in \omega} \) in \( P_r(Y) \) is uniformly tight and hence its image \( \{s(\mu_n)\}_{n \in \omega} \), being uniformly tight, is precompact in \( P_r(X) \) and thus has an accumulation point in \( P_r(X) \), which means that \( s \) is cs*-continuous. \( \square \)

Combining Theorem 9.3 with Theorem 1.4 we get

Corollary 9.4. Let \( f : X \to Y \) be a subproper map between Tychonoff spaces. Then the map \( P_r(f) : P_r(X) \to P_r(Y) \) is subproper provided the space \( P_R(X) \) is \( \mu \)-complete, \( Y \) is sequentially Prohorov, and each compact subset of \( P_R(Y) \) is sequentially compact.

Remark 9.5. It should be mentioned that for a \( \mu \)-complete Tychonoff space \( X \) the space \( P_r(X) \) does not need to be \( \mu \)-complete: according to [33] the space \( P_r(\mathbb{R}^{\omega_1}) \) contains a closed topological copy of the ordinal segment \([0, \omega_1)\) and hence is not \( \mu \)-complete.

A topological space \( X \) is called submetrizable if it admits a continuous bijective map \( f : X \to M \) onto a metrizable space \( M \).

Theorem 9.6. If \( f : X \to Y \) is a subproper map from a metrizable space \( X \) onto a submetrizable sequentially Prohorov Tychonoff space \( Y \), then the map \( P_r(f) : P_r(X) \to P_r(Y) \) is subproper.

Proof. Since the space \( P_r(X) \) is metrizable, it is \( \mu \)-complete. The space \( Y \), being submetrizable, admits an injective continuous map \( g : Y \to M \) into a metrizable space \( M \). By [4, 2.1], the map \( P_r(g) : P_r(Y) \to P_r(M) \) is injective. Then for every compact subset \( K \subset P_r(Y) \) the restriction \( P_r(g)|K \) is an embedding of \( K \) into the metrizable space \( P_r(M) \) which implies that the compactum \( K \) is metrizable and thus sequentially compact. Now we can apply Corollary 9.4 to conclude that the map \( P_r(f) : P_r(X) \to P_r(Y) \) is subproper. \( \square \)

The following simple assertion shows that the requirement on \( Y \) to be sequentially Prohorov is essential in the last theorem.

Proposition 9.7. Let \( f : X \to Y \) be a map between Tychonoff spaces. If the space \( X \) is Prohorov and the map \( P_r(f) : P_r(X) \to P_r(Y) \) is subproper, then the space \( Y \) is Prohorov.

According to [36] the Banach space \( l^1 \) endowed with the weak topology is not sequentially Prohorov. Since each weakly convergent sequence in \( l^1 \) converges in norm, the identity map \( \text{id} : l^1 \to (l^1, \text{weak}) \) from the norm into the weak topology is proper. Yet, the map \( P_r(\text{id}) : P_r(l^1) \to P_r(l^1, \text{weak}) \) is not subproper (because the space \( l^1 \) is Prohorov while \( (l^1, \text{weak}) \) is not). Thus we get the following example complementing Theorem 9.6.

Example 9.8. There is a bijective proper map \( f : X \to Y \) from a Polish space \( X \) onto a Tychonoff submetrizable space \( Y \) such that the map \( P_r(f) : P_r(X) \to P_r(Y) \) is not subproper.

The results on preservation of subproper maps by the functors of Radon measures will help us to detect \( k^* \)-metrizable spaces of probability measures.

Theorem 9.9. Let \( X \) be a Tychonoff sequentially Prohorov space and \( P_r(X) \) be the space of probability Radon measures on \( X \).

(1) If \( P_r(X) \) is \( k^* \)-metrizable or cs*-metrizable, then so is the space \( X \);
(2) If $X$ is $k^*$-metrizable, then $P_r(X)$ is $cs^*$-metrizable;
(3) If $X$ is submetrizable and $k^*$-metrizable, then $P_r(X)$ is $k^*$-metrizable;
(4) $X$ is an $\aleph_0$-space if and only if $P_r(X)$ is an $\aleph_0$-space.

10. $k^*$-metrizable spaces and Skorohod properties

Following [18] and [8] we say that a Tychonoff space $X$ has the strong Skorohod property if to each measure $\mu \in P_r(X)$ one can assign a Borel function $\xi_{\mu} : [0, 1] \to X$ such that $\mu$ is the image of Lebesgue measure under the function $\xi_{\mu}$ and for every sequence $(\mu_n) \subset P_r(X)$ convergent to a measure $\mu_0 \in P_r(X)$ the function sequence $(\xi_{\mu_n})$ converges to $\xi_{\mu_0}$ almost surely on $[0, 1]$. The uniformly tight strong Skorohod property is defined by requiring the latter only for uniformly tight weakly convergent sequences $(\mu_n)$.

A topological space $X$ has the weak Skorohod property if to each measure $\mu \in P_r(X)$ one can assign a Borel function $\xi_{\mu} : [0, 1] \to X$ such that $\mu$ is the image of Lebesgue measure under the function $\xi_{\mu}$ and for every uniformly tight sequence $(\mu_n) \subset P_r(X)$ the function sequence $(\xi_{\mu_n})$ contains a subsequence that converges almost surely in $X$ with respect to Lebesgue measure $\lambda$ on $[0, 1]$.

It was shown in [18, 3.11] and [8, 2.4] that each metrizable space has the strong Skorohod property and each $k$-metrizable space has the uniformly tight strong Skorohod property.

It turns out that the weak Skorohod property is preserved by subproper maps.

**Theorem 10.1.** A Tychonoff space $Y$ has the weak Skorohod property if and only if there is a subproper map $f : X \to Y$ from a space $X$ possessing the weak Skorohod property.

**Proof.** The “only if” part is trivial. To prove the “if” part, let $f : X \to Y$ be a subproper map. Assume that the space $X$ has the weak Skorohod property, i.e., to each probability Radon measure $\eta$ on $X$ we can assign a Borel map $\xi_{\eta} : [0, 1] \to X$ so that $\eta$ is the image of Lebesgue measure $\lambda$ under the map $\xi_{\eta}$ and for every uniformly tight sequence $(\eta_n)$ of probability Radon measures on $X$ the function sequence $(\xi_{\eta_n})$ contains a subsequence that converges almost surely.

By Corollary 9.2 the map $P_r(f) : P_r(X) \to P_r(Y)$ admits a section $s : P_r(Y) \to P_r(X)$ preserving uniformly tight families of measures. Let us assign to each measure $\mu \in P_r(Y)$ the Borel function $\zeta_{\mu} = f \circ \xi_{s(\mu)} : [0, 1] \to Y$. It is clear that

$$P_r(\zeta_{\mu})(\lambda) = P_r(f \circ \xi_{s(\mu)})(\lambda) = P_r(f) \circ P_r(\xi_{s(\mu)})(\lambda) = P_r(f)(s(\mu)) = \mu.$$ 

Thus $\mu$ is the image of Lebesgue measure $\lambda$ under the Borel function $\zeta_{\mu}$.

Now assume that $(\mu_n) \subset P_r(Y)$ is a uniformly tight sequence of probability Radon measures on $Y$. Since the section $s$ preserves uniformly tight families of measures, the sequence $(s(\mu_n))$ of measures on $X$ is uniformly tight. By our choice of the maps $\xi_{\mu}$, the function sequence $(\xi_{s(\mu_n)})$ contains a subsequence $(\xi_{s(\mu_{nk})})$ that converges almost surely. Then by the continuity of $f$, the subsequence $(\zeta_{\mu_{nk}})$ of the sequence $(\xi_{\mu_n})$ converges almost surely in $Y$ which completes the proof of the weak Skorohod property of the space $Y$. \qed

Since each $k$-metrizable space has the uniformly tight strong Skorohod property [8, 2.4] and each metrizable space has the weak Skorohod property [18, §4], Theorems 9.9 and 10.1 imply the following result.

**Theorem 10.2.** Let $X$ be a Tychonoff space.

(1) If $X$ is metrizable, then $X$ has the strong Skorohod property;
(2) If $X$ is $k$-metrizable, then $X$ has the uniformly tight strong Skorohod property.
(3) If $X$ is $k^*$-metrizable, then $X$ has the weak Skorohod property.
Remark 10.3. It was shown in [8, 3.4] that the ordinal segment $[0, \omega_1]$ endowed with the interval topology has the strong and weak Skorohod properties. Since $[0, \omega_1]$ is a non-metrizable sequentially compact space, it cannot be $cs^*$-metrizable. This shows that the class of Tychonoff spaces with the weak Skorohod property is strictly wider than the class of $k^*$-metrizable spaces.

11. $k^*$-METRIZABLE LOCALLY CONVEX SPACES

In this section we consider locally convex spaces that are $k^*$-metrizable. We recall that a locally convex space $X$ is the strict inductive limit of a sequence $X_1 \subset X_2 \subset \ldots$ of its subspaces (denoted by $\text{ind} X_n$) if $X = \bigcup_{n=1}^{\infty} X_n$ and $X$ carries the strongest locally convex topology inducing the original topology on each space $X_n$. It is well known that each bounded subset $B$ of the strict inductive limit $X = \text{ind} X_n$ lies in some $X_n$, see [67, II.6.5]. Also we will use the well-known fact that the topology of the locally convex sum $\bigoplus_{i \in I} X_i$ of a collection of locally convex spaces coincides with the corresponding box-product topology, see Exercises II.12 and I.1 in [67]. Theorems 3.5, 3.9, and 3.10 imply that the class of $k^*$-metrizable locally convex spaces has the following permanence properties.

**Theorem 11.1.** 1. Every subspace of a $k^*$-metrizable locally convex space $k^*$-metrizable.

2. The countable product of $k^*$-metrizable locally convex is $k^*$-metrizable.

3. The locally convex sum $\bigoplus_{i \in I} X_i$ of arbitrary collection of $k^*$-metrizable locally convex spaces is $k^*$-metrizable.

4. The projective limit of a sequence of $k^*$-metrizable locally convex spaces is $k^*$-metrizable.

5. The strict inductive limit $\text{ind} X_n$ of an increasing sequence of $k$-closed $k^*$-metrizable spaces is $k^*$-metrizable.

6. A locally convex space $X$ is $k^*$-metrizable if there is a sequence $\{X_n\}_{n=1}^{\infty}$ of compactly closed $k^*$-metrizable subspaces of $X$ such that every convergent sequence of $X$ lies in some $X_n$.

Next, we look for $k^*$-metrizable spaces among operator spaces. Given two linear topological spaces $X$ and $Y$ let $\mathcal{L}(X, Y)$ denote the linear space of all linear continuous operators from $X$ to $Y$. For a collection $\mathcal{S}$ of subsets of $X$, the topology of uniform convergence on elements of $\mathcal{S}$ (or briefly, the $\mathcal{S}$-topology) is the locally convex topology whose base at the origin consists of the sets

$$\{ L \in \mathcal{L}(X, Y) : L(S_1 \cup \ldots \cup S_n) \subset U \},$$

where $S_1, \ldots, S_n \in \mathcal{S}$ and $U$ is an open neighborhood of the origin in $Y$, see [67, III.3]. The space $\mathcal{L}(X, Y)$ endowed with the $\mathcal{S}$-topology will be denoted by $\mathcal{L}_\mathcal{S}(X, Y)$. The following important theorem follows immediately from Theorem 8.2.

**Theorem 11.2.** Let $X$ be a linear topological $\aleph_0$-space and $\mathcal{S}$ be a family of compact subsets of $X$ with dense union $\cup \mathcal{S}$ in $X$. For any linear topological $\aleph$-space $Y$ the operator space $\mathcal{L}_\mathcal{S}(X, Y)$ endowed with the $\mathcal{S}$-topology is $k^*$-metrizable provided the evaluation operator

$$e : \mathcal{L}_\mathcal{S}(X, Y) \times X \to Y, \ e : (T, x) \mapsto T(x)$$

is sequentially continuous.

A subset $\mathcal{F} \subset \mathcal{L}(X, Y)$ is called equicontinuous if for each neighborhood $U \subset Y$ of zero there is a neighborhood $V \subset X$ of zero such that $f(V) \subset U$ for each $f \in \mathcal{F}$. It is easy to see that for any equicontinuous family $\mathcal{F} \subset \mathcal{L}(X, Y)$ the evaluation operator $\mathcal{F} \times X \to Y$ is continuous and hence preserves precompact sets.

A barrel in a locally convex space $X$ is a closed convex symmetric subset of $X$. A locally convex space $X$ is barrelled if each barrel in $X$ is a neighborhood of zero, see [67, II.7]. It is
well known that each Baire locally convex space is barrelled. On the other hand, there are separable normed barrelled spaces of the first Baire category, see [44, 5.3.6] or [6].

**Corollary 11.3.** Let $X$ be a linear topological $\aleph_0$-space and $S$ be a family of compact subsets of $X$ with $\cup S = X$. Assume that either $X$ is a Baire space or else $X$ is a barrelled locally convex space. Then for any linear topological $\aleph$-space $Y$ the space $L_S(X, Y)$ is $k^*$-metrizable.

**Proof.** It suffices to check that the evaluation operator $e : L_S(X, Y) \times X \to Y$ is sequentially continuous. Take two compact sets $K \subset L$ of $X$. Then for any linear topological $\aleph$-space $Y$ the space $L_S(X, Y)$ is $k^*$-metrizable. 

**Another corollary of Theorem 11.2 concerns the compact-open topology on the operator spaces.**

**Corollary 11.4.** Let $X$ be a linear topological $\aleph_0$-space and $S$ be a family of compact subsets of $X$ containing all convergent sequences in $X$. For any linear topological $\aleph$-space $Y$ the operator space $L_S(X, Y)$ is $k^*$-metrizable.

Observing that the dual space $X'$ to a locally convex space $X$ coincides with the operator space $L(X, \mathbb{R})$ and taking into account that regular $k^*$-metrizable spaces with countable network are $\aleph_0$-spaces, we get

**Corollary 11.5.** Let $X$ be a linear topological $\aleph_0$-space and $S$ be a family of compact subsets of $X$ such that $\cup S$ is dense in $X$. The dual space $X'$ endowed with the $S$-topology is an $\aleph_0$-space if the duality map $X' \times X \to \mathbb{R}$, $(f, x) \mapsto f(x)$, is sequentially continuous.

**Corollary 11.6.** Let $X$ be a linear topological $\aleph_0$-space and either $X$ is Baire or $X$ is locally convex and barrelled. Then the dual space $X'$ endowed with the topology of pointwise convergence is an $\aleph_0$-space.

**Remark 11.7.** It follows from Corollary 11.6 and Proposition 7.10 that for any Baire (or barrelled locally convex) linear topological $\aleph_0$-space $X$ the dual space $X'$ endowed with the topology of pointwise convergence is monotonically sequentially normal. On the other hand, by [38] for any infinite-dimensional locally convex space $X$ the dual space $X'$ endowed with the topology of pointwise is not monotonically normal.

Next, we consider the compact-open topology on dual spaces.

**Corollary 11.8.** For any linear topological $\aleph_0$-space $X$ and any collection $S$ containing all convergent sequences in $X$ the dual space $X'$ endowed with the topology of $S$-convergence is an $\aleph_0$-space.

For metrizable separable spaces $X$ the compact-open topology of the dual space $X'$ can be described in more details.

We recall that a topological space $X$ is a $k_\omega$-space if there is a countable collection $C$ of compact subsets of $X$ such that $X$ carries the strongest topology inducing the original topology on each compactum $C \subset C$. It is clear that each compact subset of a metrizable space is metrizable. The converse statement is true for $k_\omega$-spaces: if each compact subset of a $k_\omega$-space $X$ is metrizable, then the space $X$ is metrizable. It is well-known that each metrizable $k_\omega$-space is a stratifiable $\aleph_0$-space.

**Theorem 11.9.** Let $X$ be a separable metrizable locally convex space and let $S$ be a collection of totally bounded subsets of $X$ containing all sequences convergent to zero in $X$. Then the dual space $X'$ endowed with the $S$-topology is a metrizable $k_\omega$-space.
Proof. Let \( \{U_n\}_{n \in \mathbb{N}} \) be a neighborhood base at the origin of \( X \) such that \( U_{n+1} \subset U_n \) for all \( n \). Let \( U_n^\circ = \{ f \in X' : |f(x)| \leq 1 \text{ for each } x \in U_n \} \), \( n \in \mathbb{N} \). It is clear that \( X' = \bigcup_{n \in \mathbb{N}} U_n^\circ \) and each set \( U_n^\circ \) is equicontinuous. By the Alaoglu–Bourbaki theorem [67, III.4.3] and [67, Ch. III, §4, Theorem 4.7] the sets \( U_n^\circ \), \( n \in \mathbb{N} \), are metrizable compacta in the topology of simple convergence on \( X' \). By the Banach–Dieudonné theorem [67, IV.6.3], the \( S \)-topology coincides with the strongest topology inducing the topology of simple convergence on equicontinuous subsets of \( X' \). It follows from the above remarks that the dual space \( X' \) endowed with the \( S \)-topology is a submetrizable \( k_\omega \)-space.

Corollary 11.10. Let \( \{X_n : n \in \mathbb{N}\} \) be a countable cover of a locally convex space \( X \) by separable metrizable linear subspaces and let \( S \) be a collection of totally bounded subsets of \( X \) such that \( S \) includes all sequences convergent to zero in the spaces \( X_n \) and each set \( S \in S \) lies in some \( X_n \). Then the dual space \( X' \) endowed with the \( S \)-topology is a stratifiable \( \aleph_0 \)-space.

Proof. For every \( n \in \mathbb{N} \) let \( S_n = \{ S \in S : S \subset X_n \} \). It follows that the dual space \( X \) endowed with the \( S \)-topology is a subspace of the product \( \prod_{n \in \mathbb{N}} X_n' \), where each dual space \( X_n' \) is endowed with the \( S_n \)-topology. By Theorem [11.9] each dual space \( X_n' \) is a submetrizable \( k_\omega \)-space and thus a stratifiable \( \aleph_0 \)-space. Since the class of stratifiable \( \aleph_0 \)-spaces is closed under countable products and passage to subspaces, we conclude that \( X' \) is a stratifiable \( \aleph_0 \)-space.

Corollary 11.11. Let \( X \) be a locally convex space that is a countable union \( X = \bigcup_{n=1}^\infty X_n \) of separable metrizable linear subspaces such that each compact subset of \( X \) lies in some \( X_n \). Then the dual space \( X' \) endowed with the compact-open topology is a stratifiable \( \aleph_0 \)-space.

We recall that the strong dual topology on the dual space \( X' \) to a locally convex space \( X \) is the \( S \)-topology where \( S \) is the collection of all bounded subsets of \( X \). A locally convex space \( X \) is called boundedly-compact if each closed bounded subset of \( X \) is compact, see [31, 8.4.7]. It is clear that for such a space \( X \) the strong dual topology on the dual space \( X' \) coincides with the topology of precompact convergence.

Corollary 11.12. Let \( X \) be a locally convex space that is a countable union \( X = \bigcup_{n=1}^\infty X_n \) of separable metrizable linear subspaces such that each compact subset of \( X \) lies in some \( X_n \). If the space \( X \) is boundedly-compact, then the dual space \( X' \) endowed with the strong dual topology is a stratifiable \( \aleph_0 \)-space.

We recall that a locally convex space \( X \) is called an (LF)-space if \( X \) is the strict inductive limit of a sequence of complete metrizable linear subspaces of \( X \).

Corollary 11.13. If \( X \) is a separable (LF)-space, then the dual space \( X' \) endowed with the topology of simple convergence and the dual space \( X' \) endowed with the compact-open topology are \( \aleph_0 \)-spaces. If the space \( X \) is boundedly-compact, then \( X' \) endowed with the strong dual topology is a stratifiable \( \aleph_0 \)-space.

A classical example of a boundedly-compact (LF)-space is the space \( D \) of test functions. Its dual \( D' \) endowed with the strong dual topology is the space of distributions.

Corollary 11.14. The space \( D \) of test functions and its dual \( D' \) endowed with the topology of simple convergence are \( \aleph_0 \)-spaces. The dual \( D' \) to \( D \) endowed with the strong dual topology is a stratifiable \( \aleph_0 \)-space.

It was shown in [7] that the space of distributions \( D' \) endowed with the strong dual topology is homeomorphic to the countable power \( (\mathbb{R}^\infty)^\omega \) of the strict inductive limit \( \mathbb{R}^\infty \) of finite-dimensional Euclidean spaces.
Remark 11.15. It follows from Corollary 11.11 and [38] that for an infinite-dimensional (LF)-space $X$ the topology of compact convergence on the dual $X'$ is monotonically normal, but the topology of simple convergence is not. Yet, both topologies are monotonically sequentially normal.

12. The $k^*$-metrizability of the weak topology on Banach spaces

In this section we detect Banach spaces whose weak topology is $k^*$-metrizable. For a Banach space $X$ by $(X, \text{weak})$ we denote $X$ endowed with the weak topology. Since $(X, \text{weak})$ is a subspace of the second dual space $X^{**}$ endowed with the topology of simple convergence, we can apply Corollary 11.6 to get

Proposition 12.1. If $X$ is a normed space with a separable dual, then $(X, \text{weak})$ is an $\aleph_0$-space.

There also exists a Banach space $X$ with non-separable dual such that $(X, \text{weak})$ is an $\aleph_0$-space. We recall that a normed space $X$ has the Shur property if each weakly convergent sequence of $X$ converges in norm. A standard example of a Banach space with the Shur property is $l^1$, the Banach space of all absolutely convergent series. It is known that each infinite-dimensional Banach space with the Shur property contains a subspace isomorphic to $l^1$, see [28, p. 212]. It follows from the definition that for a Banach space $X$ with the Shur property, the sequential coreflexion of $(X, \text{weak})$ coincides with $X$, which means that $(X, \text{weak})$ is cs-metrizable. It turns out that the converse assertion also is true.

Proposition 12.2. For a normed space $X$ the following conditions are equivalent:

1. $(X, \text{weak})$ is k-metrizable;
2. $(X, \text{weak})$ is cs-metrizable;
3. $X$ has the Shur property.

Proof. By Eberlein-Smulian Theorem [10, p.52] compact subsets of $(X, \text{weak})$ are sequentially compact. Now we see that the equivalence (1) $\iff$ (2) follows from Proposition 11.2.

$(2) \implies (3)$. Assume that the space $(X, \text{weak})$ is cs-metrizable but $X$ fails to have the Shur property. Then we can find a sequence $(x_n)_{n \in \omega}$ on the unit sphere of $X$ that weakly converges to zero. Observe that for every $m \in \mathbb{N}$ the sequence $(m \cdot x_n)_{n \in \omega}$ also weakly converges to zero. The cs-metrizability of $(X, \text{weak})$ implies that the space $(X, \text{weak})$ has Arkhangel’ski’s property $(\alpha_4)$, which allows us to select two increasing number sequences $(m_k)$ and $(n_k)$ such that the sequence $(m_k \cdot x_{n_k})_{k \in \omega}$ weakly converges to zero, which is impossible because this sequence is unbounded.

$(3) \implies (2)$. If $X$ has the Shur property, then the identity map $\text{id} : X \to (X, \text{weak})$ has sequentially continuous inverse witnessing the cs-metrizability of the space $(X, \text{weak})$. \hfill \Box

Thus the Shur property is responsible of the $k^*$-metrizability of the weak topology of a Banach space. The problem of characterization of Banach spaces with $k^*$-metrizable weak topology is more delicate.

Theorem 12.3. Let $X$ be a normed space and $B$ be the closed unit ball of $X$. Then the following conditions are equivalent:

1. $(X, \text{weak})$ is $k^*$-metrizable;
2. $(X, \text{weak})$ is cs$^*$-metrizable;
3. $(B, \text{weak})$ is $k^*$-metrizable;
4. $(B, \text{weak})$ is cs$^*$-metrizable.
Moreover, if the completion of $X$ contains no isomorphic copy of $l^1$, then the conditions (1)–(3) are equivalent to

1. $(B, \text{weak})$ is cs-metrizable;
2. $(B, \text{weak})$ is $k$-metrizable;
3. $(B, \text{weak})$ is metrizable;
4. $X^*$ is separable.

**Proof.** The equivalences (1) ⇔ (2), (3) ⇔ (4), (5) ⇔ (6) follow from Proposition 4.2 combined with the Eberlein-Smulian Theorem [40, p.52]. The equivalence (2) ⇔ (4) can be easily derived from Theorem 14(1,3). The equivalence (7) ⇔ (8) is well-known while (7) ⇒ (5) ⇒ (4) are trivial.

So, it remains to prove that (3) ⇒ (7) under the assumption that $l^1$ does not embed into $X$. In this case we may apply the famous Rosenthal $l^1$-Theorem to conclude that bounded subsets of $X$ and $X^2$ are Fréchet–Urysohn with respect to the weak topology, see [28, pp. 215, 216]. In particular, the unit ball $B$ of $X$ is Fréchet–Urysohn in the weak topology. Assuming that the space $(X, \text{weak})$ is $k^*$-metrizable but not metrizable, we would get that $(B, \text{weak})$ is a non-metrizable Fréchet–Urysohn $k^*$-metrizable space which by [57, Corollary 1.8] or Theorem 7.1 is a Lašnev space containing a subspace homeomorphic to the Fréchet–Urysohn fan. Then the square $(B^2, \text{weak})$ is a Fréchet–Urysohn space containing the square of the Fréchet–Urysohn fan. This implies that the square of the Fréchet–Urysohn fan is a Fréchet–Urysohn space which is not true, see [57, 1.6].

It follows from Theorem 1.c.9 of [49] that a Banach space $X$ with an unconditional basis contains an isomorphic copy of $l^1$ if and only if the dual $X^*$ is not separable.

**Question 12.4.** Let $X$ be a Banach space with an unconditional basis. Is $(X, \text{weak})$ $k^*$-metrizable? Does $(X, \text{weak})$ have the weak Skorohod property?

A standard example of a separable Banach space having a non-separable dual and containing no isomorphic copy of $l^1$ is the James tree space $JT$, see [40, p.215], [48]. By Theorem 12.3 $(JT, \text{weak})$ fails to be $k^*$-metrizable. Since each separable Banach space (in particular, $JT$) is a quotient of $l^1$, we arrive at the following result.

**Proposition 12.5.** There is a Banach space $X$ possessing a closed linear subspace $Y \subset X$ such that $(X, \text{weak})$ is $k$-metrizable, but $(X/Y, \text{weak})$ is not even $cs^*$-metrizable. Thus the $k^*$-, $k^*$-, $cs^*$-, and $cs^*$-metrizability is not preserves by open maps.

In the following proposition we describe a situation where such a pathology cannot appear.

**Proposition 12.6.** Let $Y$ be a reflexive subspace of a Banach space $X$. If $(X, \text{weak})$ is $k^*$-metrizable, then so is the space $(X/Y, \text{weak})$.

**Proof.** Let $Q: X \to X/Y$ be the quotient operator. By [12, Proposition 1.19] there is a continuous section $s: X/Y \to X$ of $Q$ such that $s$ is positively homogeneous, i.e., $s(\lambda x) = \lambda s(x)$ for each $x \in X/Y$ and $\lambda > 0$. It follows from the continuity and positive homogeneity of $s$ that $s$ maps bounded subsets of $X/Y$ onto bounded subsets of $X$.

To finish the proof it remains to apply Theorem 3.3(4) and show that for each weakly compact subset $K \subset X/Y$ the weak closure of $s(K)$ in $X$ is weakly compact. By the Davis–Figiel–Johnson–Peczynski factorization theorem [24] there is an injective linear continuous operator $T: R \to X/Y$ from a reflexive Banach space $R$ such that the image $T(B)$ of the unit ball $B \subset R$ contains the set $K$. In the product $R \times X$ consider the closed linear subspace $Z = \{(y, x) \in R \times X : T(y) = Q(x)\}$. Observe that the projection $pr_X: Z \to X$ is injective while the kernel of the projection $pr_R: Z \to R$ is isomorphic to the reflexive space.
Y. Since the reflexivity is a tree-space property \cite[4.1]{20}, we conclude that the Banach space $Z$ is reflexive. Since $s$ preserves bounded sets, the image $s(K)$ is bounded in $X$. Then the preimage $\text{pr}_X^{-1}(s(K)) \subset Z$ is contained in the bounded set $(B \times s(K)) \cap Z$ of $Z$. Take any closed bounded convex set $D \subset Z$ with $D \supset \text{pr}_Z^{-1}(s(K))$. By the reflexivity of $Z$ the set $D$ is weakly compact and so is its image $\text{pr}_X(D)$ in $X$. Since $s(K) \subset \text{pr}_X(D)$, we conclude that the weak closure of $s(K)$ in $X$ is weakly compact. Thus $Q$: $(X, \text{weak}) \to (X/Y, \text{weak})$ is a subproper map and we can apply Theorem \ref{thm:3.5}(4) to finish the proof. \hfill $\square$

In light of Propositions \ref{thm:12.5} and \ref{thm:12.6} the following question appears naturally.

**Question 12.7.** Let $X$ be a Banach space and let $Y$ be a closed linear subspace of $X$. Suppose that $(Y, \text{weak})$ and $(X/Y, \text{weak})$ are $k^*$-metrizable. Is $(X, \text{weak})$ $k^*$-metrizable?

In the following proposition we describe situations when an answer to this question is affirmative. We recall that $c_0$ is the Banach space of all real sequences convergent to zero.

**Proposition 12.8.** Let $X$ be a normed space and let $Y$ be a closed linear subspace of $X$ such that both $(Y, \text{weak})$ and $(X/Y, \text{weak})$ are $k^*$-metrizable. Then $(X, \text{weak})$ is $k^*$-metrizable if one of the following conditions holds:

1. $Y$ is complemented in $X$;
2. there is a linear continuous operator $T$: $X \to Z$ into a normed space $Z$ such that $T|Y$ is an isomorphic embedding and $(Z, \text{weak})$ is $k^*$-metrizable;
3. $X$ is separable and $Y$ is isomorphic to a subspace of $c_0$;
4. $X$ is complete and $X/Y$ has the Shur property.

**Proof.**

1. The first statement follows immediately from the second.

2. Suppose that $T$: $X \to Z$ is a continuous linear operator into a normed space such that $T|Y$ is an isomorphic embedding and the space $(Z, \text{weak})$ is $k^*$-metrizable. By Theorem \ref{thm:3.5} the product $(X/Y, \text{weak}) \times (Z, \text{weak})$ is $k^*$-metrizable. Denote by $Q$: $X \to X/Y$ the quotient operator. It can be shown that the operator $(Q,T)$: $X \to (X/Y) \times Z$ is an isomorphic embedding. Then $(X, \text{weak})$, being a subspace of $(X/Y, \text{weak}) \times (Z, \text{weak})$, is $k^*$-metrizable.

3. Next, assume that the space $X$ is separable and there is an isomorphism $h$: $Y \to Z$ of $Y$ onto a subspace $Z$ of $c_0$. By the Sobczyk theorem \cite[Theorem 94]{40}, there is a continuous linear operator $T$: $X \to c_0$ such that $T|Y = h$. Since the space $c_0$ has the separable dual $c_0^* = l^1$, the space $(c_0, \text{weak})$ is an $\mathbb{N}_0$-space according to Proposition \ref{thm:12.1}. Now the previous statement yields that $(X, \text{weak})$ is $k^*$-metrizable.

4. Finally assume that $X$ is a Banach space and the quotient $X/Y$ has the Shur property. It follows from the Michael selection theorem that there is a continuous section $s$: $X/Y \to X$ of the quotient operator $Q$: $X \to X/Y$. By Theorem \ref{thm:3.5} the product $(Y, \text{weak}) \times (X/Y)$ is $k^*$-metrizable. Consider the bijective continuous map $f$: $(Y, \text{weak}) \times (X/Y) \to X$ defined by $f(y, x) = y + s(x)$. To show that $(X, \text{weak})$ is $k^*$-metrizable, it suffices to verify that the map $f$ is proper. Let $K$ be a weakly compact subset of $X$. Then $Q(K)$ is a weakly compact subset of $X/Y$ and by Eberlein-Šmulyan Theorem \cite[p.52]{40}, $Q(K)$ is sequentially compact in the weak topology of $X/Y$. Since the space $X/Y$ has the Shur property, the set $Q(K)$ is norm compact. Consequently, $s(Q(K))$ is a norm compact subset of $X$ and $C = (K - s(Q(K))) \cap Y$ is a weakly compact subset of $Y$. Observing that $f^{-1}(K) \subset C \times Q(K)$ we see that $f$ is proper. \hfill $\square$

Finally, we show that the equivalence of the conditions $(1) \iff (8)$ in Theorem \ref{thm:12.3} cannot be proved without any restrictions on $X$. A suitable counterexample can be constructed using the operation of the $l_1$-sum $\left( \sum_{i \in \omega} X_i \right)_{l_1}$ of a sequence of Banach spaces $X_i$ with norms $\| \cdot \|_i$. 

We recall that

\[(\sum_{i\in\omega} X_i)_{l_1} = \{(x_i)_{i\in\omega} \in \prod_{i=1}^{\infty} X_i : \sum_{i\in\omega} \|x_i\| < \infty\}\]

has norm \(\|(x_i)\| = \sum_{i\in\omega} \|x_i\| < \infty\).

**Proposition 12.9.** For any sequence \(\{(X_i, \|\cdot\|_i)\}\) of normed spaces whose weak topology is \(k^*\)-metrizable the weak topology of the \(l_1\)-sum \(X = (\sum_{i\in\omega} X_i)_{l_1}\) is \(k^*\)-metrizable. If infinitely many spaces \(X_i\) fail to have the Shur property, then the weak topology of the unit ball of \(X\) is not \(\sigma\)-metrizable.

**Proof.** For every \(i \in \mathbb{N}\) fix a \(\sigma\)-compact-finite \(k\)-network \(N_i\) for the \(k^*\)-metrizable space \((X_i, \text{weak})\). Let \(B\) denote the closed unit ball of the normed space \(X = (\sum_{i\in\omega} X_i)_{l_1}\). Given a number \(\varepsilon \in \{2^{-m} : m \in \omega\}\), a finite subset \(F \subset \omega\), and a sequence of sets \((N_i)_{i\in F} \in \prod_{i\in F} N_i\), consider the set

\[N[(N_i)_{i\in F}, \varepsilon] = \{(x_i) \in X : \forall i \in F \ x_i \in N_i \text{ and } \sum_{i\in\omega\setminus F} \|(x_i)\| \leq \varepsilon\}\].

It is easy to check that the family \(N\) of all such sets \(N[(N_i)_{i\in F}, \varepsilon]\) is \(\sigma\)-compact-finite in \((X, \text{weak})\). It remains to check that \(N\) is a \(k\)-network for \((X, \text{weak})\). According to Proposition 6.2 this will follow as soon as we show that all compact subsets of \((X, \text{weak})\) are sequentially compact and \(N\) is a \(cs^*\)-network for \((X, \text{weak})\).

First we check that each weakly compact subset \(K\) of \(X\) is metrizable. Given a number \(i \in \omega\) consider the projection \(K_i\) of \(K\) onto the factors \(X_i\). Then \(K_i\), being a compact subset of the \(k^*\)-metrizable space \((X_i, \text{weak})\), is metrizable and separable. Consequently, the linear hull \(L_i\) of \(K_i\) is separable in \((X_i, \text{weak})\) and also in \(X_i\) because weak and norm separability is equivalent for convex sets. Then \(K\), being a subset of the separable normed space \((\sum_{i\in\omega} L_i)_{l_1}\), has countable network both in norm and weak topologies. Consequently, the weak topology of \(K\) is metrizable because it has countable network.

Next, we check that \(N\) is a \(cs^*\)-network for \((X, \text{weak})\). Fix an open set \(W\) in \((X, \text{weak})\) and a sequence \(\{x^n\}_{n\in\omega} \subset W\) weakly convergent to a point \(x^\infty \in W\). It will be more convenient to think of elements of \(X\) as functions \(x : \omega \rightarrow \bigcup_{i\in\omega} X_i\) with \(x(i) \in X_i\) for all \(i \in \omega\).

The compactness of the set \(S = \{x^n : n \leq \infty\}\) in \(W\) implies the existence of an open neighborhood \(W_0\) of the origin in \((X, \text{weak})\) such that \(S + W_0 + W_0 \subset W\). Next, find \(\varepsilon \in \{2^{-m} : m \in \omega\}\) such that \(W_0\) contains the closed \(2\varepsilon\)-ball centered at the origin.

We claim that for any \(\varepsilon > 0\) there is \(m \in \omega\) such that \(\sum_{i=m}^{\infty} \|x^n(i)\|_i \leq \varepsilon\) for all but finitely many numbers \(n\). Assume the converse: there is \(\varepsilon > 0\) such that for any \(m \in \omega\) there are infinitely many numbers \(n \in \omega\) such that \(\sum_{i=m}^{\infty} \|x^n(i)\|_i > \varepsilon\). Inductively we can construct two increasing number sequences \((m_k)\) and \((n_k)\) such that for any \(k \in \omega\)

\[\sum_{i=m_{k+1}+1}^{m_{k+1}} \|x^{n_k}(i)\|_i > \varepsilon.\]

By the Hahn-Banach Theorem for every \(k \in \omega\) we can find a linear functional \(f_k\) on \((\sum_{i=m_{k+1}+1}^{m_{k+1}} X_i)_{l_1}\) with norm \(\|f_k\| = 1\) such that \(f_k(x^{n_k}(m_k, m_{k+1})) > \varepsilon\). These functionals form a continuous operator

\[f : X \rightarrow l_1, \quad f : x \mapsto (f_k(x)|_{m_k, m_{k+1}})_{k\in\omega}\]

on \(X\). The weak convergence of \((x^{n_k})\) to \(x^\infty\) implies the weak convergence of \((f(x^{n_k}))\) to \(f(x^\infty)\). Since \(l_1\) has the Shur property, this sequence converges in norm, which yields a
number \(i_0 \in \omega\) with
\[
\sum_{i \geq i_0} |f_i(x^n||m_i, m_{i+1})| < \varepsilon \quad \text{and hence} \quad |f_k(x^n||m_k, m_{k+1})| < \varepsilon
\]
for all \(k \geq i_0\). But this contradicts the choice of the functional \(f_k\).

So we may take a number \(m \in \omega\) such that \(\sum_{i=m}^{\infty} |x^n(i)|_i \leq \varepsilon\) for all numbers \(n \leq \infty\). Let \(\text{pr}_m : X \to (\sum_{i=m}^{\infty} X_i)_{l_1} = \{(z_i) \in X : z_i = 0 \text{ for all } i \geq m\}\). The sequence \((\text{pr}_m(x^n))_{n \in \omega}\) weakly converges to \(\text{pr}_m(x^\infty)\). Since the norm distance between \(x^n\) and its projection \(\text{pr}_m(x^n)\) does not exceed \(\varepsilon\), we get \(\{\text{pr}_m(x^n) : n \in \omega\} \subset S+W_0\). Since \(N_i\) are \(k\)-networks in \((X_i, \text{weak})\), we can select sets \(N_i \in N_i\) for \(i < m\) whose product \(\prod_{i=m}^{\infty} X_i\) lies in \((S+W_0) \cap (\sum_{i=m}^{\infty} X_i)_{l_1}\) and contains infinitely many points of the sequence \((\text{pr}_m(x^n))\). Then \(N[(N_i)_{i<m}, \varepsilon] \in N\) lies in \(S+W_0+\varepsilon B \subset W\) and contains infinitely many points of the sequence \((x^n)\), witnessing that \(N\) is a \(cs^*\)-network for \((X, \text{weak})\).

By Proposition 6.2, the \(\sigma\)-compact finite \(cs^*\)-network \(N\) is a \(k\)-network for \((X, \text{weak})\) and by Theorem 6.4, the space \((X, \text{weak})\) is \(k^*\)-metrizable.

Now assuming that infinitely many spaces \(X_i\) fail to have the Shur property, we shall show that the weak unit ball \((B, \text{weak})\) of \(X\) is not \(cs\)-metrizable. Without loss of generality, all the spaces \(X_k\) fail the Shur property, which allows us in each space \(X_k\) to find a sequence \((x_n^k)_{n \in \omega}\) on the unit sphere that weakly converges to zero. Identify each space \(X_k\) with the subspace \(\{(z_i)_{i \in \omega} \in X : z_i = 0 \text{ for all } i \neq k\}\) of \(X\) and consider the set \(V = \{0, x_n^k : k, n \in \omega\}\) in \((B, \text{weak})\). It can be shown that the space \(V\) fails to have the Arkhangel’ski’s property \((\alpha_4)\), which implies that \((B, \text{weak})\) cannot be \(cs\)-metrizable. \(\square\)

Using the above Proposition one can show that the weak unit ball of the space \(L_1[0, 1]\) is not \(k^*\)-metrizable.

**Problem 12.10.** Is the space \(L_1[0, 1]\) \(k^*\)-metrizable in its weak topology?

More generally, we can pose a

**Problem 12.11.** Characterize Banach spaces \(X\) whose weak unit ball is \(k\)-metrizable (resp. \(k^*\)-metrizable).

Let us note that in the realm of nonseparable Banach spaces the \(k^*\)-metrizability of the weak topology is "orthogonal" to the WCG-property. We recall that a Banach space \(X\) is weakly compact generated (briefly, WCG) if there is a weakly compact subset \(K\) whose linear hull is dense in \(X\).

**Proposition 12.12.** If a WCG Banach space \(X\) has \(k^*\)-metrizable weak topology, then \(X\) is separable.

The proof follows from the fact that compact subsets of \(k^*\)-metrizable spaces are metrizable and hence separable.

**Remark 12.13.** Propositions 12.1, 12.2, and 12.8, 12.9 give examples of infinite-dimensional Banach spaces \(X\) which being endowed with the weak topology are \(k^*\)-metrizable spaces and hence have the weak Skorohod property. In contrast, the space \((X, \text{weak})\) has the \emph{strong} Skorohod property if and only if \(X\) is finite-dimensional, see [10].
13. The Structure of Sequential $k^*$-Metrizable Groups

Results of the preceding section give us many examples of non-metrizable $k^*$-metrizable topological groups. However such topological groups rarely are sequential in view of the following result [52, 2.7].

**Theorem 13.1** (Liu-Sakai-Tanaka). A sequential topological group $G$ with point-countable cosmic $k^*$-network contains a cosmic open subgroup.

A $k$-network $N$ will be called cosmic if each space $N \in N$ is cosmic in the sense that $N$ is the continuous image of a separable metrizable space. This theorem of C.Liu, M.Sakai, Y.Tanaka can be completed by the following result of T.Banakh and L.Zdomskyy [11].

**Theorem 13.2** (Banakh-Zdomskyy). A sequential topological group $G$ with countable cs*-character is a stratifiable $\aleph$-space. More precisely, $G$ is either metrizable or contains an open $k_{\omega}$-subgroup.

A topological group $G$ is called a $k_{\omega}$-group if its underlying topological space is a $k_{\omega}$-space.

We shall use these two theorems to prove a structure theorem for sequential $k^*$-metrizable topological groups.

**Theorem 13.3.** Each sequential $k^*$-metrizable group $G$ has countable cs*-character and hence is a stratifiable $\aleph$-space. More precisely, $G$ is either metrizable or contains an open $k_{\omega}$-subgroup.

**Proof.** This theorem will follow from Theorem 13.2 as soon as we check that $G$ has countable cs*-character. Assuming the converse and applying Theorem 5.13 we will get that $\alpha_4(G) > \aleph_0$ and hence $\alpha_4^+(G) > \aleph_1$, which means that $G$ contains a closed copy of the uncountable sequential fan $S_{\omega_1}$. In particular, $G$ contains a closed topological copy of $S_{\omega_1}$. By Lemma 4 from [11], a sequential topological group cannot simultaneously contain closed copies of the sequential fan $S_{\omega_1}$ and its metrizable counterpart, the sequential hedgehog

$$H_\omega = \{0, 2^{-m}e_n : n, m \in \omega\} \subset l^2,$$

where $(e_n)$ is the standard orthonormal base of the separable Hilbert space $l^2$. The space $H_\omega$ is not locally compact at its unique non-isolated point 0, and is the smallest non-locally compact space among first countable non-locally compact spaces: each first countable non-locally compact space contains a closed copy of $H_\omega$.

Therefore, the group $G$ contains no closed copy of $H_\omega$. We shall use this fact to show that $G$ has a point-countable cosmic $k^*$-network.

Fix a subproper map $\pi : M \to X$ of a metrizable space $M$ onto $X$ and let $s : X \to M$ be a section of $\pi$ that preserves precompact sets.

**Claim 13.4.** Each point $x \in M$ has a neighborhood $U \subset M$ whose image $\pi(U)$ is precompact in $G$.

Assuming the converse, fix a countable neighborhood base $(U_n)$ at $x$. For every $U_n$ the image $\pi(U_n)$ is not precompact, which means that the closure $\overline{\pi(U_n)}$ is not compact. The sequentiality of the $k^*$-metrizable space $G$ implies that each countably compact space is compact, see Proposition 5.7. Hence $\overline{\pi(U_n)}$ is not countably compact and contains a countable infinite closed discrete subset $D_n$ not containing the point $y = \pi(x)$. By induction we can construct an increasing number sequence $(n_k)$ such that $\overline{\pi(U_{n_k})} \cap \bigcup_{i \leq n_k} D_i = \emptyset$ for all $k$.

The “convergence” of the sets $U_n$ to $x$ implies that $(D_{n_k})$ converges to $y = \pi(x)$ in the sense that each neighborhood of $y$ contains all but finitely many sets $D_n$. This fact can be used to
show that the space $D = \{y\} \cup \bigcup_k D_{n_k}$ is a closed copy of the hedgehog $H_\omega$ in $G$, which is not possible. □

So we can find a $\sigma$-locally-finite base $\mathcal{B}$ for $M$ consisting of set $U \subset M$ with precompact images $\pi(U)$ in $G$. Since each compact subset of $G$ is metrizable, the sets $s^{-1}(U) \subset \pi(U)$, $U \in \mathcal{B}$, are metrizable and separable. The precompact preserving property of $s$ implies that $\mathcal{N} = \{s^{-1}(U) : U \in \mathcal{B}\}$ is a $\sigma$-compact-finite $k$-network consisting of metrizable separable subsets of $G$. In particular, $\mathcal{N}$ is a point-countable cosmic $k$-network for $G$. By Theorem 13.1 the group $G$ contains an open cosmic subgroup $H$. This subgroup has countable network and, being a $k^*$-metrizable space, has countable $k$-network by Theorem 7.2. By Proposition 5.14 $cs^*(G) = cs^*_c(H) \leq \alpha_4(H) \leq ext(H) \leq knw(H) \leq \aleph_0$. So $G$ has countable $cs^*$-character. Application of Theorem 13.2 completes the proof. □

Theorem 13.3 will be applied to show that many cardinal characteristics of a sequential $k^*$-metrizable group coincide with a specific group-topological invariant $ib(G)$ called the index of boundedness of $G$ and equal to the smallest cardinal $\kappa$ such that for any open neighborhood $U \subset G$ of the neutral element $e \in G$ there is a subset $F \subset G$ of size $|F| \leq \kappa$ with $G = F \cdot U = \{x \cdot y : x \in F, y \in U\}$. It is known that $ib(G) \leq \min\{ext(G), d(G)\}$ for any topological group.

**Theorem 13.5.** If $G$ is an infinite sequential $k^*$-metrizable group, then $ib(G) = d(G) = ext(G) = s(G) = l(G) = nw(G) = knw(G)$.

**Proof.** The assertion is trivial if $G$ is metrizable. So, assume that $G$ is not metrizable. By Theorem 13.3 the group $G$ contains an open $k_\omega$-subgroup $H$. Since all compact subsets in $G$ are metrizable, we get $knw(H) = \aleph_0$. By the definition of $ib(G)$, there is a subset $F \subset G$ of size $|F| \leq ib(G)$ with $G = F \cdot H$. Then for any countable $k$-network $\mathcal{N}$ in $H$ the family $F \cdot \mathcal{N} = \{f \cdot N : f \in F, N \in \mathcal{N}\}$ is a $k$-network for $G$ having size $|F \cdot \mathcal{N}| \leq |F| \cdot |\mathcal{N}| = ib(G)$, which yields $knw(G) \leq ib(G)$. Combining this inequality with $ib(G) \leq \min\{ext(G), d(G)\} \leq nw(G) \leq knw(G)$, we get the desired equalities. □

Theorem 13.3 will be also applied to the problem of topological classification of non-metrizable sequential $k^*$-metrizable groups. This can be done for two classes of groups: punctiform groups and locally convex spaces.

A topological space $X$ is called punctiform if each compact subset of $X$ is zero-dimensional. A topological classification of punctiform $k^*$-metrizable group involves the notion of the compact scatteredness rank.

Given a topological space $X$ let $X^{(1)} \subset X$ denote the set of all non-isolated points of $X$. For each ordinal $\alpha$ define the $\alpha$-th derived set $X^{(\alpha)}$ of $X$ by transfinite induction: $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$. By the scatteredness height $sch(X)$ of $X$ we understand the smallest ordinal $\alpha$ such that $X^{(\alpha+1)} = X^{(\alpha)}$. A topological space $X$ is scattered if $X^{(\alpha)} = \emptyset$ for some ordinal $\alpha$. By the compact scatteredness rank of a topological space $X$ we understand the ordinal $scr(X) = \sup\{sch(K) : K$ is a scattered compact subspace of $X\}$. In particular, $scr(X) = \omega_1$ for any uncountable compact metrizable space $X$ (such a space $X$ contains a copy of the Cantor set and hence a copy of each countable compactum).

**Theorem 13.6.** Two non-metrizable sequential punctiform $k^*$-metrizable groups $G, H$ are homeomorphic if and only if $d(G) = d(H)$ and $scr(G) = scr(H)$.

This theorem was proved in [11] for sequential groups with countable $cs^*$-character. By Theorem 13.3 each sequential $k^*$-metrizable group has countable $cs^*$-character, so the theorem follows.
Also we can classify non-metrizable sequential $k^*$-metrizable locally convex spaces. In this cases all of them are homeomorphic either to $\mathbb{R}^\infty = \lim_{\rightarrow} \mathbb{R}^n$, the strict inductive limit of Euclidean spaces, or to the product $\mathbb{R}^\infty \times Q$ of $\mathbb{R}^\infty$ and the Hilbert cube $Q = [0, 1]^{\omega}$.

**Theorem 13.7.** Each non-metrizable sequential $k^*$-metrizable locally convex space is homeomorphic to $\mathbb{R}^\infty$ or $\mathbb{R}^\infty \times Q$.

This theorem follows from the corresponding classification of sequential locally convex spaces with countable cs$^*$-character proved in [11].

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