AN AXIOMATIC CHARACTERIZATION OF STEENROD’S CUP-i PRODUCTS

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Abstract. We show that any construction of cup-i products on the normalized chains of simplicial sets is isomorphic— not just homotopy— to Steenrod’s original construction if it is natural, minimal, non-degenerate, irreducible and free. We use this result to prove that all cup-i constructions in the literature represent the same isomorphism class.

1. Introduction

In [Ste47], Steenrod introduced by means of formulas the cup-i products on the cochains of spaces. These bilinear maps were used to define cohomology operations

$$\text{Sq}^k : H^\vee(X; \mathbb{F}_2) \rightarrow H^\vee(X; \mathbb{F}_2),$$

the Steenrod squares, laying at the heart of stable homotopy theory.

Steenrod’s formulas for the cup-i products extend the Alexander-Whitney product on cochains. This non-commutative product $\sim_0$ induces the commutative algebra structure on cohomology, and we can interpret the higher cup-i products

$$\sim_i : N^\vee(X; \mathbb{F}_2) \otimes N^\vee(X; \mathbb{F}_2) \rightarrow N^\vee(X; \mathbb{F}_2)$$

as coherent homotopies witnessing the derived commutativity of $\sim_0$ at the cochain level.

In later work by Steenrod [SE62] and others, an indirect argument based on the acyclic carrier theorem is used to establish the existence of cup-i products and consequently of Steenrod squares. This approach became the standard since any set of choices for the cup-i products homotopic to Steenrod’s original gives rise to the same cohomology operations which, by then, had been axiomatically characterized. As a consequence, the need to interact with a specific set of choices for the cup-i products largely declined.

Interest in actual cochain representatives and operations, like the cup-i products, has recently been rekindled by their use in the study of topological phases of matter [GK16; KT17; Bar+21] (see also [BM16; BM18]) and topological data analysis [ZC05; LMT18]. Additionally, the combinatorics of Steenrod’s cup-i products has been related to higher category theory by deducing the nerve construction from them [Str87; Med20b], while their geometry has been explored using fibrations of convex polytopes [LM22].
In the present work we introduce an axiomatic characterization of Steenrod’s original cup-$i$ construction up to isomorphism – not just homotopy. Loosely expressed, our axioms demand that a cup-$i$ construction be natural with respect to simplicial maps, parameterized by the smallest possible complex, not identically 0 or reducible to subsimplices, and as free as possible with respect to transpositions.

The relevance of this result is twofold. On the one hand, as we overview below, it clarifies that all available cup-$i$ constructions represent the same isomorphism class. For two of these, no previous proof of the equivalence with Steenrod’s existed in print. On the other hand, it highlights the fundamental nature of Steenrod’s cup-$i$ construction illuminating its unexpected connections with convex geometry and higher category theory.

The first alternative construction of cup-$i$ products is due to Real [Rea96] and it is based on the Eilenberg–Zilber contraction. It was further developed by González–Díaz–Real [GR99] and used as the basis for an algorithm, implemented by Palmieri on SAGE, for the computation of Steenrod squares. We will review this construction in §7.2 and use our axioms to show it is isomorphic – in fact equal – to Steenrod’s original.

In [Med21c], new formulas defining a cup-$i$ construction were introduced and used to present a faster algorithm for the computation of Steenrod squares for finite simplicial complexes. This algorithm was implemented in the package steenroder computing persistence Steenrod barcodes for topological data analysis. The question of comparing the resulting cup-$i$ construction with either Steenrod’s or Real’s was not addressed. We use our axiomatic characterization to show in §7.1 that these agree.

A generalization of the notion of cup-$i$ product structure is that of $E_{\infty}$-algebra. Using work by [Ben98, §4.5] generalizing Steenrod’s formulas, McClure–Smith [MS03] and Berger–Fresse [BF04] constructed natural $E_{\infty}$-algebra structures on the normalized cochains of simplicial sets. As we describe in §7.3, these structures induce a bijection between the set of bases of the arity 2 part of their operads and the set of cup-$i$ constructions satisfying our axioms.

These operadic constructions are obtained by dualizing $E_{\infty}$-coalgebra structures that extend the Alexander–Whitney diagonal coproduct. In [Med20a], the normalized chains of standard simplicial sets where equipped with a natural $\mathcal{M}$-bialgebra structure generated by the Alexander–Whitney coproduct, the augmentation map, and an algebraic version of the join of simplices. This structure generalizes that of McClure–Smith and Berger–Fresse. In particular, Steenrod’s cup-$i$ coproducts are given by explicit compositions of the Alexander–Whitney and join (co)operations.

A (non-degenerate) $E_{\infty}$-coalgebra structure on the normalized chains of simplicial sets defines operations on their mod $p$ cohomology [Ste53; May70]. When $p = 2$ these agree with Steenrod squares. Analogues of Steenrod’s cup-$i$ coproducts effectively defining these operations were introduced in [KM21] and implemented in CosCH. These cup-$(p, i)$ coproducts are expressible in terms of the Alexander–Whitney and join (co)products, and we expect to study their moduli in future work.

Outline. Section 2 reviews the required preliminaries from the theory of simplicial sets. Our axiomatic characterization is presented in Section 3. The reader is
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encouraged to start the paper in this section referring back if needed. We discuss Steenrod squares in Section 4. This section, serving to provide context for the subject, is logically independent of the rest of the paper. Before presenting any proofs, we recast in Section 5 the main definitions and theorems of Section 3 in terms of a functor naturally isomorphic to that of normalized chains. The proof of our main result occupies Section 6. We finish this work by showing in Section 7 that all other cup-i constructions in the literature are isomorphic to Steenrod’s original.

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2. Preliminaries

In this section we review the basic theory of simplicial sets including the construction of their normalized chains.

2.1. Simplicial sets. We denote the set of non-negative integers by \( \mathbb{N} \). The simplicial category is defined to have an object \( [n] = \{0, \ldots, n\} \) for every \( n \in \mathbb{N} \) and a morphism in \( \Delta([m],[n]) \) for each order-preserving function from \( [m] \) to \( [n] \). As is commonly done we denote the identity \( [n] \rightarrow [n] \) simply as \( [n] \). The morphisms \( \delta_i: [n-1] \rightarrow [n] \) and \( \sigma_i: [n+1] \rightarrow [n] \) defined for \( 0 \leq i \leq n \) by

\[
\begin{align*}
\delta_i(k) &= \begin{cases} 
  k & k < i, \\
  k+1 & i \leq k,
\end{cases} \quad \text{and} \\
\sigma_i(k) &= \begin{cases} 
  k & k \leq i, \\
  k-1 & i < k,
\end{cases}
\end{align*}
\]

generate all morphisms in the simplex category. They satisfy the cosimplicial identities:

\[
\begin{align*}
\delta_j \delta_i &= \delta_i \delta_{j-1}, & i < j, \\
\sigma_j \sigma_i &= \sigma_i \sigma_{j+1}, & i \leq j, \\
\sigma_j \delta_i &= \begin{cases} 
  \delta_i \sigma_{j-1}, & i < j, \\
  \text{id}, & i = j, j+1, \\
  \delta_{i-1} \sigma_j, & i > j+1,
\end{cases}
\end{align*}
\]

which can be used to uniquely express any morphism in the form

\[\delta_{u_p} \cdots \delta_{u_1} \sigma_{v_1} \cdots \sigma_{v_q}\]

for some sets of non-negative integers \( U = \{u_1 < \cdots < u_p\} \) and \( V = \{v_1 < \cdots < v_q\} \). We simplify notation and write the canonical factorization (2) simply as \( \delta_U \sigma_V \).

A simplicial set \( X \) is a contravariant functor from the simplex category to the category of sets and a simplicial map is a natural transformation between two simplicial sets. We denote this category by \( s\text{Set} \). As is customary we use the notation:

\[
X([n]) = X_n, \quad X(\delta_i) = d_i, \quad X(\sigma_i) = s_i.
\]

Elements in \( X_n \) are referred to as \( n \)-simplices with the integer \( n \) called its dimension and omitted when unspecified. Simplices in the image of any \( s_i \) are termed degenerate. We extend the notation of canonical factorizations writing
\( X(\delta_U \sigma_V s_V) = s_V d_U \) with \( d_U \) and \( s_V \) representing the identity. We say that a simplex \( y \) is a (proper) face of a simplex \( x \) if \( y = d_U(x) \) for some (non-empty) \( U \).

For each \( n \in \mathbb{N} \), the simplicial set \( \Delta^n \), referred to as the \( n \)th standard simplicial set, is defined by

\[
\Delta^n_m = \Delta([m],[n]), \quad d_i(\delta_U \sigma_V) = \delta_U \sigma_V \delta_i, \quad s_i(\delta_U \sigma_V) = \delta_U \sigma_V \sigma_i.
\]

Natural constructions on simplicial sets are controlled by their action on standard simplicial sets since for any simplicial set \( X \) we have

\[
X_n \cong \operatorname{colim} \Delta^n.
\]

2.2. Normalized chains. We will work with homologically graded chain complexes of \( \mathbb{F}_2 \)-modules, regarding, in the usual way, the set of linear maps between them and their tensor product as chain complexes:

\[
\operatorname{Hom}(C,C')_m = \{ f \mid \forall k \in \mathbb{Z}, f(C_k) \subseteq C'_{k+m} \}, \quad \partial f = \partial \circ f + f \circ \partial,
\]

\[
(C \otimes C')_m = \bigoplus_{p+q=m} C_p \otimes C'_q, \quad \partial(c \otimes c') = \partial c \otimes c' + c \otimes \partial c'.
\]

The functor of normalized chains (with \( \mathbb{F}_2 \)-coefficients) \( N : \mathbf{sSet} \to \mathbf{Ch} \) is defined on objects as follows:

\[
N(X)_n = \frac{\mathbb{F}_2 \{ X_n \}}{\mathbb{F}_2 \{ s(X_{n-1}) \}}
\]

where \( s(X_{n-1}) = \bigcup_{i=0}^{n-1} s_i(X_{n-1}) \), and \( \partial_n : N(X)_n \to N(X)_{n-1} \) is given by

\[
\partial_n = \sum_{i=0}^{n} d_i.
\]

The tensor product functor \( N \otimes N \) will play an important role as well. The functor of normalized cochains \( N^\vee \) is defined by composing \( N \) with the linear duality functor \( \operatorname{Hom}(\cdot, \mathbb{F}_2) \). Notice that in this definition \( N^\vee(X) \) is concentrated in non-positive degrees. If \( x \) is a \( n \)-simplex we abuse notation and use the same symbol \( x \) to denote the associated basis element in \( N(X)_n \) and its dual basis element in \( N^\vee(X)_{-n} \).

3. Main statement

In this section we introduce the notion of cup-\( i \) construction and the properties that characterize Steenrod’s. We state our main result and review a specific presentation of a cup-\( i \) construction satisfying these properties.

3.1. Cup-\( i \) constructions. Let \( S_2 \) be the group with one non-identity element \( T \) and let

\[
W = \left( \mathbb{F}_p[S_2]\{e_0\} \xleftrightarrow{1+T} \mathbb{F}_p[S_2]\{e_1\} \xleftrightarrow{1+T} \cdots \right)
\]

be the minimal projective resolution of \( \mathbb{F}_2 \) by \( \mathbb{F}_2[S_2] \)-modules.
Definition 1. A cup-$i$ product structure on a chain complex $A$ is a chain map
\[ W \otimes_{\mathbb{F}_2[S_2]} A^{\otimes 2} \to A \]
where $T$ acts by transposition on $A^{\otimes 2}$. We denote the image of $[e_i \otimes \alpha \otimes \beta]$ by $\alpha \cup_i \beta$.

Remark 2. Consider a cup-$i$ product structure on $A$. The induced map
\[ F_2 \otimes_{\mathbb{F}_2[S_2]} H^{\otimes 2} \to H \]
on the homology $H$ of $A$ defines a commutative product on it. We can therefore think of the structure on $A$ as a product $\cup_0 : A \otimes A \to A$ that is commutative up to coherent homotopies, which are given by the maps $\cup_i : A \otimes A \to A$ for $i > 0$.

Definition 3. An isomorphism of cup-$i$ product structures on $A$ is an automorphism $\phi$ of $W$ making the diagram
\[ W \otimes_{\mathbb{F}_2[S_2]} A \xrightarrow{\phi \otimes \text{id}} W \otimes_{\mathbb{F}_2[S_2]} A \]
commute.

Definition 4. A cup-$i$ construction (more accurately termed cup-$i$ product construction for simplicial sets) is a cup-$i$ product structure on $N^\vee(X)$ for every simplicial set $X$ that is natural with respect to simplicial maps. An isomorphism of cup-$i$ constructions is an isomorphism of cup-$i$ structures on $N^\vee(X)$ for every simplicial set $X$ that is natural with respect to simplicial maps.

3.2. Uniqueness. The first axiom alluded to in the introduction, naturality, has been explicitly included in our definition of cup-$i$ construction; whereas the second, minimality, is manifested in the use of $W$ instead of an arbitrary (non-minimal) projective resolution of $F_2$.

Definition 5. A cup-$i$ construction is non-degenerate if for any simplex $x$
\[ x \cup_0 x \neq 0 \]
whenever $|x| = 0$. It is irreducible if for any proper face $y$ of $x$
\[ \left( y^{(1)} \cup_i y^{(2)} \right)(x) = 0 \]
for any two faces $y^{(1)}$ and $y^{(2)}$ of $y$. It is free if for any two simplices $x$ and $y$
\[ x \cup_i y = y \cup_i x \implies x \cup_i y = 0 \]
whenever $|x| \neq i$ or $|y| \neq i$.

We can now state our main result.

Theorem 6. There is up to isomorphism only one non-degenerate, irreducible and free cup-$i$ construction.

We will use this result to prove in Section 7 that all cup-$i$ constructions available in the literature are isomorphic to Steenrod’s original [Ste47].
3.3. Existence. To prove our main result we will use formulas defining one such cup-\(i\) construction introduced in [Med21c].

**Notation.** For any \(n\)-simplex \(x\) and set \(U = \{u_1 < \cdots < u_r\} \subseteq \{0, \ldots, n\}\) we write \(d_U(x)\) for \(d_{u_1} \cdots d_{u_r}(x)\), with \(d_{\emptyset}(x) = x\).

**Definition 7 ([Med21c]).** Let \(X\) be a simplicial set, \(x \in X_n\) and \(\alpha, \beta \in N^\lor(X)\).

\[
(\alpha \bowtie_i \beta)(x) = \sum_{U \subseteq \{0, \ldots, n\}} (\alpha \otimes \beta) d_U(x) \otimes d_{U^c}(x)
\]

if \(i \in \{0, \ldots, n\}\),

\[0\]

otherwise,

where the sum is taken over all \(U = \{u_1 < \cdots < u_{n-i}\} \subseteq \{0, \ldots, n\}\) and

\[U^0 = \{u_j \in U \mid u_j \equiv j \text{ mod } 2\}, \quad U^1 = \{u_j \in U \mid u_j \not\equiv j \text{ mod } 2\}.
\]

**Example 8.** For \(i = 0\), Equation (3) gives

\[
(\alpha \bowtie_0 \beta)(x) = \sum_{j=0}^n \alpha(d_{j+1} \cdots d_n(x)) \cdot \beta(d_0 \cdots d_{j-1}(x)),
\]

the so-called Alexander–Whitney product.

The verification that our formulas in Equation (3) define a cup-\(i\) construction is given in [Med21c]. We will refer to it as the **canonical cup-\(i\) construction.** Furthermore, it is non-degenerate, irreducible, and free (Theorem 19) and agrees with Steenrod’s original cup-\(i\) construction (Theorem 27).

4. Steenrod squares

This section is included to provide context to our results and is logically independent of the rest of this paper.

By an acyclic carrier argument [EM53] all non-degenerate cup-\(i\) constructions are homotopic to each other. Any of them gives rise to the following celebrated cohomology operations introduced by Steenrod in [Ste47] through an explicit cup-\(i\) construction reviewed in §7.1.

**Definition.** For any non-degenerate cup-\(i\) construction the \(k\)th **Steenrod square** is defined on cocycle representatives by

\[
\text{Sq}^k : H^{-n} \longrightarrow H^{-n-k}
\]

\[\left[\alpha\right] \mapsto \left[\alpha \bowtie_{n-k} \alpha\right].
\]

These operations have been axiomatically characterized [SE62], and one can interpret Theorem 6 as a continuation of this result capturing the isomorphism type, and not just the homotopy type, of Steenrod’s original construction.

**Remark (Discrimination tools).** We can illustrate the additional discriminatory power offered by these operations considering the following isomorphisms:

1. As graded vector spaces, \(H^\lor(\mathbb{R}P^2; \mathbb{F}_2) \cong H^\lor(S^1 \wedge S^2; \mathbb{F}_2)\) but they are distinguished by \(\text{Sq}^1\).
2. As graded modules, \(H^\lor(\mathbb{C}P^2; \mathbb{Z}) \cong H^\lor(S^2 \wedge S^4; \mathbb{Z})\) but they are distinguished by \(\text{Sq}^2\).
(3) As graded rings, \( H^\vee(\Sigma C P^2; \mathbb{Z}) \cong H^\vee(\Sigma(S^2 \wedge S^4); \mathbb{Z}) \) but they are distinguished by \( Sq^2 \).

**Remark (Persistent homology).** Persistent homology is a method used on highly intensive data analysis tasks [Car+08; CCR13; Lee+17] and for which various software projects exist [Bau21; The22; Tau+21]. Based on [LMT18], the project **steenroder** incorporates into the persistent homology pipeline the additional information Steenrod squares provide.

**Remark (Relations).** Cup-\( i \) products provide an effective construction of coboundaries coherently witnessing the commutativity relation of the cup product in cohomology at the cochain level. There are two notable relations satisfied by Steenrod squares; the first one, known as the Cartan relation, expresses the interaction between these operations and the cup product:

\[
Sq^k([\alpha][\beta]) = \sum_{i+j=k} Sq^i([\alpha]) Sq^j([\beta]),
\]

whereas the second, the Adem relation [Ade52], expresses dependencies appearing through iteration:

\[
Sq^i Sq^j = \sum_{k=0}^{[i/2]} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k.
\]

Explicit cochains witnessing them were recently defined in [Med20c] and [BMM21] respectively.

**Remark (Transverse intersections).** For any space, Thom showed that every mod 2 homology class is represented by the push-forward of the fundamental class of a closed manifold \( W \) along some map to the space. Furthermore, if the target \( M \) is a closed manifold, and therefore satisfies Poincaré duality

\[
PD: H^k(M; \mathbb{F}_2) \to H_{|M|-k}(M; \mathbb{F}_2),
\]

the cohomology class dual to the homology class represented by the intersection of two transverse maps \( V \to M \) and \( W \to M \) is the cohomology class \([\alpha][\beta]\) where \([\alpha]\) and \([\beta]\) are respectively dual to the homology classes represented by \( V \to M \) and \( W \to M \). By taking \([\alpha] = [\beta]\) of degree \(-k\) we have that \( Sq^k([\alpha]) \) is represented by the transverse self-intersection of \( W \to M \). A comparison lifting this one between intersection and the cup-0 product of cochains was given in [FMS21]. A generalization of this result to cup-\( i \) products is the focus of current research [FMS22].

5. **Reformulation**

In this section we give a more category theoretic description of the notion of cup-\( i \) construction using the functor of normalized chains. We also introduce a functor naturally isomorphic to it – interesting on its own right – that simplifies the presentation of the canonical cup-\( i \) construction and facilitates the proof of our main theorem.
5.1. Natural linear transformations. Recall the functors $N$ and $N \otimes N$ from $sSet$ to $Ch$, and the notion of natural linear transformation $N \to N \otimes N$ between them, i.e., a linear map $N(X) \to N(X) \otimes N(X)$ for every simplicial set $X$ that is natural with respect to simplicial maps.

**Lemma 9.** A cup-$i$ construction is canonically equivalent to a collection of natural linear transformations $\Delta_i : N \to N \otimes N$ for $i \in \mathbb{N}$ satisfying
\[
\partial \circ \Delta_i + \Delta_i \circ \partial = (1 + T)\Delta_{i-1}
\]
for all $i \geq 0$ with the convention $\Delta_{-1} = 0$.

**Proof.** Let $C^n = \text{Hom}(C, \mathbb{F}_2)$ with $C$ a finite dimensional chain complex. Using the hom-tensor adjunction and the finite dimensionality of $C$ we have
\[
\text{Hom}(W \otimes \mathbb{F}_2[2], (C^n) \otimes 2, C^n) \cong \text{Hom}_\mathbb{F}_2[2](W, \text{Hom}((C^n) \otimes 2, C^n)) \\
\cong \text{Hom}_\mathbb{F}_2[2](W, \text{Hom}(C, C^{\otimes 2}))
\]
as chain complexes of $\mathbb{F}_2$-modules. In other words, the linear duality functor induces a bijection between cup-$i$ product structures on $C^n$ and $\mathbb{F}_2[2]$-linear chain maps $\Delta : W \to \text{Hom}(C, C^{\otimes 2})$. The latter are canonically equivalent to linear maps $\Delta_i = \Delta(e_i)$ satisfying
\[
\partial \circ \Delta_i + \Delta_i \circ \partial = (1 + T)\Delta_{i-1}
\]
for all $i \geq 0$ with $\Delta_{-1} = 0$, since
\[
\partial \Delta(e_i) + \Delta \partial(e_i) = \partial \Delta(e_i) + \Delta(1 + T)(e_{i-1}) \\
= \partial \circ \Delta_i + \Delta_i \circ \partial + (1 + T)\Delta_{i-1}.
\]

By naturality, a cup-$i$ structure is determined by its restriction to representable simplicial sets $\Delta^n$. Since $N(\Delta^n)$ is finite dimensional, the previous argument involving only canonical equivalences proves the claim. \qed

Motivated by this formulation of cup-$i$ structure we present the following.

**Lemma 10.** A natural linear transformation $f : N \to N \otimes N$ is canonically equivalent to a collection of elements $f[n] \in N(\Delta^n) \otimes 2$ for $n \in \mathbb{N}$ whose image under $N(\sigma_j)^{\otimes 2}$ is 0 for each codegeneracy map $\sigma_j : [n] \to [n - 1]$.

**Proof.** By naturality $f$ is determined by its restriction to $N(\Delta^n)$ for $n \in \mathbb{N}$. Furthermore, for any non-degenerate simplex $(x : [m] \to [n]) \in \Delta^n_m$ we have
\[
f(x) = f(x \circ [m]) = (f \circ N(x))[m] = (N(x) \otimes N(x))f[m],
\]
so the elements $f[m]$ with $m \in \mathbb{N}$ determine $f$. Here $[m]$ denotes the identity of the object $[m]$ and we are using $x$ to also denote the simplicial map $\Delta^m \to \Delta^n$ defined by $y \mapsto x \circ y$.

The simplex associated to a codegeneracy map $\sigma_j : [n] \to [n - 1]$ is degenerate in $\Delta^{n-1}$ so it is 0 in $N(\Delta^{n-1})$. Therefore,
\[
0 = f(0) = f(\sigma_j) = f(\sigma_j \circ [n]) = (f \circ N(\sigma_j))[n] = (N(\sigma_j) \otimes N(\sigma_j))f[n]
\]
as claimed. \qed
5.2. The functor $\mathcal{P}$. It will be convenient to use an alternative to the functor of normalized chains which is naturally isomorphic to it. This functor is constructed using the fact that every non-degenerate simplex of $\Delta^n$ is a face of the identity $[n]$.

**Definition 11.** Let $\mathcal{P}^{n-m}_n$ be the set of subsets of $\{0, \ldots, n\}$ whose cardinality is $n - m$. Define the degree $m$ part of a chain complex $\mathcal{P}(\Delta^n)$ by

\[
\mathcal{P}(\Delta^n)_m = \begin{cases} \mathbb{F}_2 \mathcal{P}^{n-m}_n, & \text{if } 0 \leq m \leq n, \\ 0, & \text{otherwise,} \end{cases}
\]

and its differential by

\[
\partial U = \sum_{\bar{a} \in \{0, \ldots, n\} \setminus U} \{\bar{a}\} \cup U.
\]

**Lemma 12.** The linear map

\[
\Psi_n : \mathcal{P}(\Delta^n) \to N(\Delta^n)
\]

\[
\begin{array}{c}
U \\
\mapsto d_U[n]
\end{array}
\]

is a chain isomorphism for every $n \in \mathbb{N}$.

**Proof.** It can be easily seen that $\Psi_n$ induces a degree preserving bijection of basis elements. We will verify that this assignment induces a chain map. Let $U = \{u_1 < \cdots < u_{n-m}\} \in \mathcal{P}_{n-m}^n$. Using the relation $d_j d_u = d_u d_{j+1}$ if $u \leq j$ we have

\[
\partial \Psi_n(U) = \partial d_U[n] = \sum_{j \in \{0, \ldots, m\}} d_j d_{u_1} \cdots d_{u_{n-m}}[n]
\]

\[
= \sum_{\bar{a} \in \{0, \ldots, n\} \setminus U} d_{\bar{a}} \cdots d_{u_{n-m}}[n]
\]

\[
= \sum_{\bar{a} \in \{0, \ldots, n\} \setminus U} d_{\bar{a}}[n] = \Psi_n(\partial U),
\]

as claimed. \qed

**Definition 13.** Let $\mathcal{P} : s\text{Set} \to \text{Ch}$ be the functor defined on standard simplicial sets as in Definition 11 and such that for $U \in \mathcal{P}(\Delta^n)_m$ and coface $\delta_j : [n] \to [n+1]$

\[
\mathcal{P}(\delta_j)(U) = \{u_1 < \cdots < u_{k-1} < j < u_k + 1 < \cdots < u_{n-m} + 1\}
\]

where $k$ is determined by the inequalities, and for any codegeneracy $\sigma_j : [n] \to [n-1]$

\[
(5) \quad \mathcal{P}(\sigma_j)(U) = \begin{cases} U \setminus \{j+1\}, & j+1 \in U, \\ U \setminus \{j\}, & j+1 \notin U \text{ and } j \in U, \\ 0, & j+1 \notin U \text{ and } j \notin U. \end{cases}
\]

**Lemma 14.** The chain isomorphisms $\{\Psi_n\}_{n \in \mathbb{N}}$ define a natural equivalence $\Psi$ between the functors $\mathcal{P}$ and $N$.

**Proof.** This follows from two straightforward computations using the cosimplicial identities (1). \qed

A direct consequence of Equation (5) is the following.
Lemma 15. For any codegeneracy $\sigma_j : [n] \to [n - 1]$ a basis element $V \otimes W \in P(\Delta^n)\otimes^2$ is in $\ker (P(\sigma_j)\otimes^2)$ if and only if both $j$ and $j + 1$ are missing from either $V$ or $W$.

5.3. Cup-$i$ constructions. We can use Lemma 9 and the natural isomorphism $\Psi$ of Lemma 14 to interpret cup-$i$ constructions in terms of natural linear transformations $\Delta_i : P \to P \otimes P$ satisfying

$$\partial \circ \Delta_i + \Delta_i \circ \partial = (1 + T) \Delta_{i-1}$$

for all $i \in \mathbb{N}$ with the convention $\Delta_{-1} = 0$. Additionally, by Lemma 10, any $\Delta_i$ is determined by a collection of elements $\Delta_i[n] \in P(\Delta^n)_{i+n}^2$ indexed by $n \in \mathbb{N}$ with

$$\Delta_i[n] = \sum_{\lambda \in \Lambda(i,n)} V_{\lambda} \otimes W_{\lambda}$$

in the kernel of $P(\sigma_j)\otimes^2$ for every codegeneracy $\sigma_j : [n] \to [n - 1]$, where $\Lambda(i,n)$ is a finite (possibly empty) indexing set, and $V_{\lambda} \otimes W_{\lambda}$ is a basis element for each $\lambda \in \Lambda(i,n)$.

We will identify a cup-$i$ construction and its associated set $\{\Delta_i[n]\}_{i,n \in \mathbb{N}}$.

5.4. Canonical cup-$i$ construction.

Definition 16. For any $U \in P_{n-i}^n$ the index function of $U$ is given by

$$\text{ind}_U : U \to \mathbb{F}_2$$

$$u_j \mapsto u_j + j \mod 2.$$

Notation. For any $U \in P_{n-i}^n$ and $\varepsilon \in \mathbb{F}_2 \cong \{0, 1\}$ we write $U^{\varepsilon}$ instead of $\text{ind}_{U}^{-1}(\varepsilon)$.

By inspecting Definition 7 we have the following.

Lemma 17. The canonical cup-$i$ construction is given by the collection of elements $\Delta_i[n] \in P(\Delta^n)_{i+n}^2$ for $i, n \in \mathbb{N}$ with

$$\Delta_i[n] = \sum_{U \in P_{n-i}^n} U^0 \otimes U^1$$

if $i \leq n$ and $\Delta_i[n] = 0$ otherwise.

6. Proof

In this section we present the proof of our main theorem: any cup-$i$ construction that is non-degenerate, irreducible and free is isomorphic to the canonical one. We divide this proof into several parts. In §6.1 we revisit the properties defining our axiomatic characterization in light of the previous section. A useful consequence of freeness is recorded in §6.2. The base case of an induction argument is given in §6.3; whereas in §6.4 we recall a fact about our presentation of the canonical cup-$i$ construction. We devote §6.5 to prepare an induction step and §6.6 to present it. Our theorem is finally proven in §6.7 using the foreshadowed induction argument.
6.1. Properties.

**Lemma 18.** Consider for each \(i, n \in \mathbb{N}\) an element

\[
\Delta_i[n] = \sum_{\lambda \in \Lambda(i, n)} V_\lambda \otimes W_\lambda
\]

with each \(V_\lambda \otimes W_\lambda\) in the basis of \(\mathcal{P}(\Delta^n)^{\otimes 2}_{i+n}\). If \(\{\Delta_i[n]\}_{i,n \in \mathbb{N}}\) defines a cup-i construction, this satisfies the following equivalences:

1. It is non-degenerate iff \(\triangle_0[0] \neq 0\).
2. It is irreducible iff \(\forall i, n \in \mathbb{N}, \forall \lambda \in \Lambda(i, n), V_\lambda \cap W_\lambda = \emptyset\).
3. It is free iff \(\forall i, n \in \mathbb{N}, \forall \lambda_1, \lambda_2 \in \Lambda(i, n), V_{\lambda_1} \otimes W_{\lambda_1} \neq W_{\lambda_2} \otimes V_{\lambda_2}\) if \(i \neq n\).

**Proof.** By naturality, it suffices to verify these equivalences for the cup-i product structure defined on \(\Lambda_i(n)\) for every \(n \in \mathbb{N}\). In this case, the claims follow from directly comparing with Definition 5. □

The following is deduced from Lemma 18 by inspecting Lemma 17.

**Theorem 19.** The canonical cup-i construction is non-degenerate, irreducible, and free.

6.2. A consequence of freeness.

**Lemma 20.** Let \(\{\Delta_i[n]\}_{i,n \in \mathbb{N}}\) be a free cup-i construction. If

\[
(1 + T)\Delta_i[n] = (1 + T) \sum_{\lambda \in \Lambda(i, n)} V_\lambda \otimes W_\lambda
\]

for \(i, n \in \mathbb{N}\) with \(i \neq n\) and each \(V_\lambda \otimes W_\lambda\) is a basis element of \(\mathcal{P}(\Delta^n)^{\otimes 2}_{i+n}\), then there is a partition of \(\Lambda(i, n) = \Lambda_1 \sqcup \Lambda_2\) with

\[
\Delta_i[n] = \sum_{\lambda_1 \in \Lambda_1} V_{\lambda_1} \otimes W_{\lambda_1} + \sum_{\lambda_2 \in \Lambda_2} W_{\lambda_2} \otimes V_{\lambda_2}.
\]

**Proof.** We directly have that (6) holds up to an element \(\kappa\) in the kernel of \((1 + T)\). This kernel is generated by elements of the form \(U \otimes U\) and \(V \otimes W + W \otimes V\), so Item 3 in Lemma 18 implies that \(\kappa = 0\). □

6.3. Special cases.

**Lemma 21.** Let \(\{\Delta_i[n]\}_{i,n \in \mathbb{N}}\) be free and non-degenerate cup-i construction.

1. \(\forall i, n \in \mathbb{N}, \Delta_i[n] = 0\) if \(i > n\).
2. \(\forall n \in \mathbb{N}, \Delta_n[n] = 0 \otimes \emptyset\).

**Proof.** The chain complex \(\mathcal{P}(\Delta^n)^{\otimes 2}\) is 0 in degrees greater than \(2n\) and it is generated by \(\emptyset \otimes \emptyset\) in degree \(2n\).

The claim in Item 1 is immediate since \(\Delta_i[n]\) is in degree \(n+i > 2n\) if \(i > n\).

If the conclusion of the claim in Item 2 does not hold, there exists \(n \in \mathbb{N}\) smallest such that \(\Delta_n[n] = 0\). If \(n > 0\) then

\[
(1 + T)\Delta_{n-1}[n] = \partial \Delta_n[n] + \Delta_n \partial [n] = 0,
\]

and Lemma 20 implies \(\Delta_{n-1}[n] = 0\). From this and the assumption

\[
\Delta_{n-1}[n-1] = \emptyset \otimes \emptyset
\]
we obtain
\[(1 + T)\Delta_{n-2}[n] = \partial \Delta_{n-1}[n] + \Delta_{n-1} \partial[n] = \sum_{u=0}^{n} \{u\} \otimes \{u\},\]

which is a contradiction since \(\sum_{u} \{u\} \otimes \{u\}\) is not in the image of \((1 + T)\).

The previous argument shows that \(\Delta_{n}[n] = 0\) for every \(n \in \mathbb{N}\). This serves as the base case of an induction argument over \(n - i\) proving that \(\Delta_{i}[n] = 0\) for every \(i, n \in \mathbb{N}\); a contradiction to the non-degeneracy of the cup-\(i\) construction. For the induction step, consider
\[(1 + T)\Delta_{i-1}[n] = \partial \Delta_{i}[n] + \Delta_{i} \partial[n] = 0,\]

which, by Lemma 20, implies \(\Delta_{i-1}[n] = 0\). \(\square\)

**Lemma 22.** Let \(\{\Delta_{i}[n]\}_{i,n \in \mathbb{N}}\) be free and non-degenerate cup-\(i\) construction. For all integer \(n \geq 1\) either \(\Delta_{n-1}[n]\) or \(T\Delta_{n-1}[n]\) is equal to
\[\Delta_{n-1}[n] = \sum_{u \in \{0, \ldots , n\}} \{u\} \otimes \emptyset + \sum_{u \in \{0, \ldots , n\}} \emptyset \otimes \{u\}.\]

**Proof.** By Lemma 21 we have \(\Delta_{n}[n] = \emptyset \otimes \emptyset\) and \(\Delta_{n} \partial[n] = 0\) for all \(n \in \mathbb{N}\). Therefore,
\[(1 + T)\Delta_{n-1}[n] = (\partial \otimes \text{id} + \text{id} \otimes \partial)(\emptyset \otimes \emptyset)\]
\[= (1 + T) \sum_{u=0}^{n} \{u\} \otimes \emptyset\]

and we need to show that the partition of the indexing set \(\{0, \ldots , n\} = \Lambda_{0} \sqcup \Lambda_{1}\) provided by Lemma 20 is determined by the parity of the associated integer. Let us argue by contradiction assuming some \(j\) and \(j + 1\) belong to the same \(\Lambda_{\pm}\). With no loss of generality we have
\[\Delta_{n-1}[n] = (\{j\} + \{j + 1\}) \otimes \emptyset + O(j, j + 1)\]
where \(O(j, j + 1)\) is a sum of basis elements missing \(\{j\}\) and \(\{j + 1\}\) from both of its tensor factors. Since \(\Delta_{n-1} \partial[n] = \sum_{u=0}^{n} \{u\} \otimes \{u\}\),
\[(1 + T)\Delta_{n-2}[n] = (1 + T)(\{j\} \otimes \{j + 1\}) + P(j, j + 1)\]
where \(P(j, j + 1)\) is a sum of basis elements with \(j\) and \(j + 1\) missing from at least one of its tensor factors.

By Lemma 15 every basis element in \(P(j, j + 1)\) is in the kernel of \(P(\sigma_{j}) \otimes P(\sigma_{j})\). Using Lemma 20 in the above equation implies that \(\Delta_{n-2}[n]\), an element in \(\ker (P(\sigma_{j}) \otimes P(\sigma_{j}))\), is equal to either \(\{j\} \otimes \{j + 1\}\) or \(\{j + 1\} \otimes \{j\}\) plus an element in this kernel. This is a contradiction since neither of these two basis elements is in \(\ker (P(\sigma_{j}) \otimes P(\sigma_{j}))\) by Lemma 15. \(\square\)

6.4. A fact about our formulas.

**Notation.** For \(U \in P_{n}^{n-i}\) we write \(u \notin U\) if \(u \notin \{0, \ldots , n\} \setminus U\). We simplify notation writing \(u.U\) instead of \(\{u\} \cup U\) if \(u \notin U\) and \(U \setminus u\) instead of \(U \setminus \{u\}\) if \(u \in U\).
The following is proven as Lemma 21 in [Med21c].

**Proposition 23.** For $i, n \in \mathbb{N}$ with $i < n$ we have:

$$\Delta_i \partial [n] = \sum_{U \in \mathcal{P}^n_{n-i}} \left( \sum_{u \in U^1} u.U^0 \otimes U^1 + \sum_{u \in U^0} U^0 \otimes u.U^1 \right).$$

6.5. **Preparing the induction step.**

**Notation.** Given a function $\xi : \mathcal{P}^n_{n-i} \to \mathbb{F}_2$ we denote by $\bar{\xi} : \mathcal{P}^n_{n-i} \to \mathbb{F}_2$ the function defined by the condition $\bar{\xi}(U) \neq \xi(U)$ for all $U \in \mathcal{P}^n_{n-i}$. We will simplify notation writing $U^\xi$ and $\bar{U}^\xi$ instead of $U^{\xi(U)}$ and $U^{\bar{\xi}(U)}$.

**Lemma 24.** Let $\{\Delta_i[n]\}_{i,n \in \mathbb{N}}$ be a free and non-degenerate cup-$i$ construction and $i, n \in \mathbb{N}$ with $i \leq n - 2$. If there is a function $\xi : \mathcal{P}^n_{n-i} \to \mathbb{F}_2$ with

$$\Delta_i[n] = \sum_{U \in \mathcal{P}^n_{n-i}} U^\xi \otimes U^{\bar{\xi}}$$

and either

$$\Delta_i[n - 1] = \Delta_i[n - 1] \quad \text{or} \quad \Delta_i[n - 1] = T \Delta_i[n - 1],$$

then $\xi$ is constant, i.e.,

$$\Delta_i[n] = \Delta_i[n] \quad \text{or} \quad \Delta_i[n] = T \Delta_i[n].$$

**Proof.** Let us assume $\Delta_i[n - 1] = \Delta_i[n - 1]$. The other case is proven analogously. Applying the boundary of $\mathcal{P}(\Delta^\otimes^2)$ – Equation (4) – to Equation (8) we have

$$\partial \Delta_i[n] = \sum_{U \in \mathcal{P}^n_{n-i}} \left( \sum_{u \in U^1} u.U^\xi \otimes U^{\bar{\xi}} + \sum_{u \in U^0} U^\xi \otimes u.U^{\bar{\xi}} \right)$$

$$+ \sum_{U \in \mathcal{P}^n_{n-i}} \left( \sum_{u \in U^1} u.U^\xi \otimes U^{\bar{\xi}} + \sum_{u \in U^0} U^\xi \otimes u.U^{\bar{\xi}} \right).$$

By assumption and Proposition 23 we have

$$\Delta_i \partial [n] = \Delta_i \partial [n] = \sum_{U \in \mathcal{P}^n_{n-i}} \left( \sum_{u \in U^1} u.U^0 \otimes U^1 + \sum_{u \in U^0} U^0 \otimes u.U^1 \right).$$

Adding this last two identities together we have

$$\Delta_i \partial [n] = \Delta_i \partial [n] = \sum_{U \in \mathcal{P}^n_{n-i}} \left( \sum_{u \in U^1} u.U^0 \otimes U^1 + \sum_{u \in U^0} U^0 \otimes u.U^1 \right) \quad \text{(9)}$$

$$= \sum_{U \in \mathcal{P}^n_{n-i}} \left( \sum_{u \in U^1} u.U^\xi \otimes U^{\bar{\xi}} + \sum_{u \in U^0} U^\xi \otimes u.U^{\bar{\xi}} \right) \quad \text{(10)}$$

$$+ (1 + T) \sum_{U \in \mathcal{P}^n_{n-i}} \left( \sum_{u \in U^1} u.U^0 \otimes U^1 + \sum_{u \in U^0} U^0 \otimes u.U^1 \right) \quad \text{(11)}$$

We will use Lemma 15 to show that summand (10) is in the kernel $K^{(j)}$ of $\mathcal{P}(\sigma_j) \otimes \mathcal{P}(\sigma_j)$ for every codegeneracy $\sigma_j : [n] \to [n - 1]$. Consider $U \in \mathcal{P}^n_{n-i}$. If $\{j, j +
The claim follows from the existence of the bijections \( \Gamma \) and \( \xi \) that relate two elements \( U \sim \phi \) in the analysis. This applies to any \( j \) and \( f \), and since \( \Delta \) is free, \( \phi \) is defined on \( \sum_{j} j \). Let \( \Lambda = \Lambda \). Let \( \Gamma \), \( \Gamma \), and \( \Gamma \) be the subset of \( \Lambda = \Lambda \). Let \( \Lambda \) be the subset of \( \Lambda = \Lambda \) consisting of elements \( U \in \Lambda \). We define \( \Lambda \), \( \Lambda \), and \( \Lambda \) analogously. The set \( \Lambda \) is defined by the conditions \( \xi(U) \neq 0 \) and \( j, j + 1 \in U \).

Observe that the sum

\[
\sum_{\Lambda, \Lambda, \Lambda} \left( \sum_{u \in U} u \cdot U \otimes U + \sum_{u \in U} U \otimes u \cdot U \right)
\]

is in \( K^{(j)} \) since, given that \( \text{ind}_U(j) = \text{ind}_U(j + 1) \), the only non-zero summands associated to \( (U, j) \) and \( (U, j + 1) \) cancel each other. Therefore, applying \( \mathcal{P}(\sigma_j) \otimes \mathcal{P}(\sigma_j) \) to

\[
\sum_{\Gamma, \Gamma, \Gamma} \left( \sum_{u \in U} u \cdot U \otimes U + \sum_{u \in U} U \otimes u \cdot U \right)
\]

yields

\[
\sum_{\Lambda, \Lambda, \Lambda} U \otimes \{j\} \otimes U + \sum_{\Lambda, \Lambda, \Lambda} U \otimes U \{j\}
\]

which must be in the kernel of \( (1 + T) \). This implies the existence of an involution \( \phi^{(j)} \) of \( \Lambda = \Lambda \cup \Lambda \cup \Lambda \) defined by a choice of canceling pairs. By the freeness of \( \Delta \) and since \( i \leq n - 2 \), this involution has no fixed points. It follows that two elements \( U \) and \( U' \) with \( U \cap \{j, j + 1\} = U' \cap \{j, j + 1\} \) cannot be related by \( \phi^{(j)} \) since then \( U = U' \). Therefore, \( \phi^{(j)}(U) = U \) if and only if \( U' = j.(U \setminus \{j + 1\}) \) or \( U' = (j + 1).(U \setminus \{j + 1\}) \). Recall that by definition \( \xi(U) = \xi(U') \neq 0 \). This analysis applies to any \( j \in \{0, \ldots, n\} \) and we introduce a relation in \( \mathcal{P}_{n-i} \), writing \( U \sim U' \) if \( U' = j.(U \setminus \{j + 1\}) \) or \( U' = (j + 1).(U \setminus \{j + 1\}) \) for some \( j \). By...
the previous analysis, if $U \sim U'$ then $\xi(U) = \xi(U')$. The lemma now follows from observing that any two elements $V$ and $W$ in $P^{n}_{n-i}$ are related by a sequence

$$V \sim \cdots \sim W,$$

so $\xi: P^{n}_{n-i} \to \mathbb{F}_2$ must be constant. \hfill \qed

**Lemma 25.** Let $\{\triangle_i[n]\}_{i,n \in \mathbb{N}}$ be a non-degenerate, irreducible, and free cup-$i$ construction and $i, n \in \mathbb{N}$ with $i \leq n - 2$. If $\triangle_i[n] = \triangle_i[n]$ or $\triangle_i[n] = T\triangle_i[n]$ then the following implications hold:

$$\triangle_i[n - 1] = \triangle_i[n - 1] \implies \triangle_i[n] = \triangle_i[n],$$

$$\triangle_i[n - 1] = T\triangle_i[n - 1] \implies \triangle_i[n] = T\triangle_i[n].$$

**Proof.** We use the convention $\triangle_{-1}[n] = \triangle_{-1}[n] = 0$ for all $n \in \mathbb{N}$. We will establish the first of the implications above using a proof by contradiction for which we assume $\triangle_i[n - 1] = \triangle_i[n - 1]$ and $\triangle_i[n] = T\triangle_i[n]$. The second implication is proven analogously. We have

$$(1 + T)\triangle_{i-1}[n] = \partial \triangle_i[n] + \triangle_i \partial[n]$$

$$= \partial T\triangle_i[n] + \triangle_i \partial[n]$$

$$= T \partial \triangle_i[n] + \partial \triangle_i[n] + \partial \triangle_i[n] + \partial \triangle_i[n]$$

$$= (1 + T) \partial \triangle_i[n] + (1 + T) \triangle_{i-1}[n]$$

$$= (1 + T)\triangle_i \partial[n] + (1 + T)\triangle_{i-1}[n].$$

Using Proposition 23 and Lemma 17, i.e.

$$\triangle_i \partial[n] = \sum_{U \in P^{n}_{n-i}} \left( \sum_{u \in U^1} u. U^0 \otimes U^1 + \sum_{u \in U^0} u. U^1 \right),$$

$$\triangle_{i-1}[n] = \sum_{V \in P^{n}_{n-i+1}} V^0 \otimes V^1,$$

we have

$$(1 + T)\triangle_{i-1}[n] = (1 + T) \sum_{U \in P^{n}_{n-i}} \left( \sum_{u \in U^1} u. U^0 \otimes U^1 + \sum_{u \in U^0} u. U^1 \right)$$

$$+ (1 + T) \sum_{V \in P^{n}_{n-i+1}} V^0 \otimes V^1.$$

By Lemma 20 there are functions $\eta: P^{n}_{n-i} \to \mathbb{F}_2$ and $\zeta: P^{n}_{n-i+1} \to \mathbb{F}_2$ such that

$$\triangle_{i-1}[n] = \sum_{U \in P^{n}_{n-i}} \left( \sum_{u \in U^\eta} u. U^\eta \otimes U^\eta + \sum_{u \in U^\eta} u. U^\eta \right)$$

$$+ \sum_{V \in P^{n}_{n-i+1}} V^\zeta \otimes V^\zeta.$$ 

This contradicts the irreducibility of $\{\triangle_i[n]\}_{i,n \in \mathbb{N}}$ as expressed in Lemma 18. \hfill \qed
6.6. Induction step.

Lemma 26. Let \( \{ \Delta_i[n] \}_{i,n \in \mathbb{N}} \) be a free non-degenerate and irreducible cup-i construction. Let \( \{ p(i,n) \}_{i,n \in \mathbb{N}} \) and \( \{ q(i,n) \}_{i,n \in \mathbb{N}} \) each be one of the following two families of propositions:

\[
\{ \Delta_i[n] = \Delta_i[n] \}_{i,n \in \mathbb{N}} \quad \text{or} \quad \{ \Delta_i[n] = T \Delta_i[n] \}_{i,n \in \mathbb{N}}.
\]

For all \( i, n \in \mathbb{N} \) with \( i \leq n - 2 \) the following implication holds:

\[
p(i+1,n) \land p(i+1,n-1) \land q(i,n-1) \implies q(i,n)
\]

Proof. Let both \( \{ p(i,n) \}_{i,n \in \mathbb{N}} \) and \( \{ q(i,n) \}_{i,n \in \mathbb{N}} \) be the family \( \{ \Delta_i[n] = \Delta_i[n] \}_{i,n \in \mathbb{N}} \). The other three combinations are treated analogously.

From \( p(i+1,n) \) and \( p(i+1,n-1) \) we have

\[
\partial \Delta_{i+1}[n] + \Delta_{i+1} \partial [n] = \partial \Delta_{i+1}[n] + \Delta_{i+1} \partial [n]
\]

or, equivalently,

\[
(1 + T) \Delta_i[n] = (1 + T) \Delta_i[n] \overset{\text{def}}{=} (1 + T) \sum_{U \in P_{n-i}} U^0 \otimes U^1
\]

Lemma 20 implies the existence of a function \( \xi : P^i_{n-i} \to \mathbb{F}_2 \) such that

\[
\Delta_i[n] = \sum_{U \in P_{n-i}} U^\xi \otimes U^\xi.
\]

Lemma 24 implies, using \( q(i,n-1) \), that

\[
\Delta_i[n] = \Delta_i[n] \quad \text{or} \quad \Delta_i[n] = T \Delta_i[n].
\]

Finally, Lemma 25 implies \( \Delta_i[n] = \Delta_i[n] \), i.e., \( q(i,n) \). \( \square \)
6.7. Complete proof. Let \( \{ \Delta_i[n] \}_{i,n \in \mathbb{N}} \) be a non-degenerate, irreducible, and free cup-\(i\) construction. We will use an induction argument over \( k = n - i \) to show that for each \( i \in \mathbb{N} \) either \( \Delta_i[n] = \Delta_i[n] \) or \( \Delta_i[n] = T \Delta_i[n] \) for all \( n \in \mathbb{N} \). By Lemma 21 \( \Delta_i[n] = \Delta_i[n] \) and \( \Delta_i[n] = T \Delta_i[n] \) for \( r \leq 0 \). By Lemma 22 \( \Delta_i[n] = \Delta_i[n] \) or \( \Delta_i[n] = T \Delta_i[n] \) for \( r = 1 \). This serves as the base case of the induction and Lemma 26 as the induction step. We have now proven Theorem 6.

7. Other constructions

In this section we show in §7.1 that the canonical cup-\(i\) construction is equal to Steenrod’s original. We then review Real’s approach in §7.2 and show that the resulting cup-\(i\) construction is also equal to the canonical one. The agreement of these three constructions appear in print for the first time here. We devote §7.3 to review more general structures on the normalized chains of simplicial chains due to McClure–Smith, Berger–Fresse, and the author.

7.1. Original construction. We now review Steenrod’s cup-\(i\) construction [Ste47, p.293]. Following Lemma 9, we will describe it as a set of elements \( \Delta^S_i[n] \in N(\Delta^n)^{\otimes 2} \) with \( i, n \in \mathbb{N} \). If \( i > n \) then \( \Delta^S_i[n] = 0 \), otherwise it is given by the sum over all ordered sequences of integers

\[
0 \leq p_1 < \cdots < p_i+1 \leq n
\]

of the basis element

\[
[0, \ldots , p_1] \ast [p_2, \ldots , p_3] \ast \cdots \ast [p_i, \ldots , p_{i+1}]
\]

(12)

\[
\otimes [p_1, \ldots , p_2] \ast \cdots \ast [p_i, \ldots , p_{i+1}]
\]

if \( i \) is odd, and of

\[
[0, \ldots , p_1] \ast [p_2, \ldots , p_3] \ast \cdots \ast [p_i, \ldots , p_{i+1}]
\]

(13)

\[
\otimes [p_1, \ldots , p_2] \ast [p_3, \ldots , p_4] \ast \cdots \ast [p_{i+1}, \ldots , n]
\]

if \( i \) is even, where \( \ast \) denotes the join of simplices:

\[
[p_{k-1}, \ldots , p_k] \ast [p_{k+1}, \ldots , p_{k+2}] = [p_{k-1}, \ldots , p_k, p_{k+1}, \ldots , p_{k+2}].
\]

Theorem 27. Steenrod’s cup-\(i\) construction agrees with the canonical one.

Proof. We will use Theorem 6 to prove that they are isomorphic. We can then conclude their equality by inspecting that \( \Delta^S_i[i+1] \neq T \Delta_i[i+1] \).

Steenrod’s cup-\(i\) construction is non-degenerate since \( \Delta_0([0]) = [0] \otimes [0] \neq 0 \). It is irreducible since for each basis element in \( \Delta^S_i[n] \) with \( i \leq n \), all integers \( \{0, \ldots , n\} \) appear in at least one of the tensor factors. To prove it is free let us assume \( i \) is odd with \( i < n \). The case where \( i \) is even is done analogously. If it is not free, then there exist two distinct sequences

\[
0 = p_0 \leq p_1 < \cdots < p_i+1 \leq p_{i+2} = n
\]

\[
0 = q_0 \leq q_1 < \cdots < q_i+1 \leq q_{i+2} = n
\]

such that

\[
[p_0, \ldots , p_1] \ast [p_2, \ldots , p_3] \ast \cdots \ast [p_i, \ldots , p_{i+1}]
\]

\[
[q_1, \ldots , q_2] \ast [q_3, \ldots , q_4] \ast \cdots \ast [q_{i+1}, \ldots , q_{i+2}]
\]
and
\[
[q_0, \ldots, q_1] * [q_2, \ldots, q_3] * \cdots * [q_i, \ldots, q_{i+1}] = \\
[p_1, \ldots, p_2] * [p_3, \ldots, p_4] * \cdots * [p_{i+1}, \ldots, p_{i+2}].
\]

We will prove that \( p_{r+1} = q_{r+1} = r \) for \( 0 \leq r \leq i \), in particular, this will imply the contradiction \( i = n \). We have the base case of an induction argument since
\[
p_0 = q_1 = p_0 = q_1 = 0.
\]

The induction step follows from the identities
\[
[p_r] * [p_{r+1}] = [q_r, q_r + 1],
\]
\[
[q_r] * [q_{r+1}] = [p_r, p_r + 1].
\]

Theorem 6 proves that \( \Delta_i^R = \Delta_i \) or \( \Delta_i^S = T\Delta_i \) for any \( i \in \mathbb{N} \). Consider the element \( U = \{0\} \in \mathbb{P}_{i+1}^1 \) giving rise to the summand \( U^1 \otimes U^0 = \{0\} \otimes \emptyset \) in \( T\Delta_i[i+1] \). Applying the isomorphism \( \Psi^{\otimes 2} : \mathbb{P}(\Delta^n)^{\otimes 2} \rightarrow \mathbb{N}(\Delta^n)^{\otimes 2} \) we obtain the basis element \([1, \ldots, i+1] \otimes [0, \ldots, i+1]\) which is not a summand of \( \Delta_i^R[i+1] \). This concludes the proof.

\]

7.2. AW–EZ contraction. In work by Real [Rea96], further developed by González-Díaz–Real [GR99; GR+03; GR05], an alternative cup-i construction was introduced based on the Alexander–Whitney and Eilenberg–Zilber linear natural transformations
\[
\text{AW: } \mathbb{N}(\Delta^n) \xrightarrow{\text{AW}} \mathbb{N}(\Delta^n) \otimes \mathbb{N}(\Delta^n) : \text{EZ},
\]
and an explicit natural chain homotopy SHI between their non-trivial composition EZ AW and the identity. The natural linear transformations defining Real’s cup-i construction are
\[
\Delta_i^R = \text{AW}(T \text{SHI})^i.
\]

In [GR99, Corollary 3.2] these authors unravelled the above definition in terms of face maps. We use Lemmas 10 and 14 to describe these formulas as a set of elements \( \Delta_i^R[n] \in \mathbb{P}(\Delta^n)^{\otimes 2} \) with \( i, n \in \mathbb{N} \). If \( i > n \) then \( \Delta_i^R[n] = 0 \), otherwise it is given by
\[
\Delta_i^R[n] = \sum_{j_i = S(i)}^{n} \sum_{j_{i-1} = S(i-1)}^{j_i-1} \cdots \sum_{j_1 = S(1)}^{j_2-1} \{j_0 + 1 < \cdots < j_{i-1} - 1\} \cup \{j_2 + 1 < \cdots < j_3 - 1\} \cup \cdots \cup \{j_{i-1} + 1 < \cdots < j_n\} \otimes \{0 < \cdots < j_0 - 1\} \cup \{j_1 + 1 < \cdots < j_{i-1} - 1\} \cup \cdots \cup \{j_{i-1} + 1 < \cdots < j_{i-1} - 1\}
\]
if \( n \) is even and by
\[
\Delta_i^R[n] = \sum_{j_i = S(i)}^{n} \sum_{j_{i-1} = S(i-1)}^{j_i-1} \cdots \sum_{j_1 = S(1)}^{j_2-1} \{j_0 + 1 < \cdots < j_{i-1} - 1\} \cup \{j_1 + 1 < \cdots < j_{i-1} - 1\} \cup \cdots \cup \{j_{i-1} + 1 < \cdots < j_{i-1} - 1\} \otimes \{0 < \cdots < j_0 - 1\} \cup \{j_1 + 1 < \cdots < j_{i-1} - 1\} \cup \cdots \cup \{j_{i-1} + 1 < \cdots < j_{i-1} - 1\}
\]
if \( n \) is odd, where
\[
S(k) = j_{k+1} - j_{k+2} + \cdots + (-1)^{k+i-1}j_i + (-1)^{k+i} \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor.
\]
Remark 28. To compare the above formulas to those appearing in [GR99, Corollary 3.2] we mention that \( i \) and \( n \) here are respectively equal to \( j - i = n \) and \( i + j = m \) there.

A small variation of the proof given for Theorem 27 establishes the following.

**Theorem 29.** Real’s cup-\( i \) construction agrees with the canonical one.

7.3. **Operads.** In this section, independent of the main results of this work, we assume familiarity with the theory of operads over the category of chain complexes (of \( F_2 \)-modules). We discuss the relationship between Steenrod’s cup-\( i \) construction and the \( E_\infty \)-structures of McClure–Smith [MS03], Berger–Fresse [BF04], and the author [Med20a; Med21b].

7.3.1. An \( \mathcal{M} \)-bialgebra is a chain complex \( B \) with three operations

\[
\Delta : B \to B \otimes B, \quad \varepsilon : B \to F_2, \quad * : B \otimes B \to B,
\]

such that the first two define the structure of a counital coalgebra on \( B \), and the third one satisfies

\[
\varepsilon \circ * = 0,
\]

\[
\partial \circ * + * \circ (\partial \otimes \text{id}) + * \circ (\text{id} \otimes \partial) = \varepsilon \otimes \text{id} + \text{id} \otimes \varepsilon.
\]

As proven in [Med20a], any \( \mathcal{M} \)-bialgebra \( B \) is an \( E_\infty \)-coalgebra by discarding compositions of the generators defining maps of the form \( B^{\otimes s} \to B^{\otimes r} \) for \( s \neq 1 \). The \( E_\infty \)-operad controlling this structure is denoted \( U(\mathcal{M}) \).

The complex of normalized chains of a standard simplicial set is naturally an \( \mathcal{M} \)-bialgebra with the Alexander–Whitney coproduct \( \Delta \), the augmentation map \( \varepsilon \), and the join product \( * : N(\Delta^n) \to N(\Delta^n) \) defined by

\[
*([v_0, \ldots, v_p] \otimes [v_{p+1}, \ldots, v_q]) = \begin{cases} [v_{\pi(0)}, \ldots, v_{\pi(q)}] & \text{if } v_i \neq v_j \text{ for } i \neq j, \\ 0 & \text{if not,} \end{cases}
\]

where \( \pi \) is the permutation that orders the vertices. The natural \( U(\mathcal{M}) \)-coalgebra structure on \( N(\Delta^n) \) extends to \( N(X) \) for every simplicial set \( X \). Using this \( E_\infty \)-coalgebra structure, a cup-\( i \) construction is recursively defined by

\[
\Delta_0 = \Delta, \quad \Delta_i = (\ast \otimes \text{id}) \circ (\text{id} \otimes T \Delta_{i-1}) \circ \Delta.
\]

We can directly compare Equation (14) to Equations (12) and (13) to conclude that this cup-\( i \) construction agrees with Steenrod’s original.

7.3.2. The Alexander–Whitney coproduct and the join product are coassociative and associative respectively and we write

\[
\Delta^1 = \Delta, \quad *^1 = *, \quad \Delta^{m+1} = (\Delta^m \otimes \text{id}) \circ \Delta, \quad *^{m+1} = \ast \circ (\ast^m \otimes \text{id}).
\]

Consider a surjection \( s : \{1, \ldots, \ell\} \to \{1, \ldots, r\} \) which we represent by its order image \( (s(1), \ldots, s(\ell)) \). The natural linear transformation associated to \( s \) is

\[
(s^{s^{-1}(1)} \otimes \cdots \otimes s^{s^{-1}(r)}) \circ \pi_s \circ \Delta^{\ell-1}
\]
where $s^0 = \text{id}$ and $\pi_s$ is the shuffle permutation defined by
\[
(\pi_s(1), \ldots, \pi_s(\ell)) = (1, \ldots, 1, r, \ldots, r).
\]
This assignment defines the **surjection operad** $\mathcal{X}$ of McClure–Smith [MS03] as a suboperad of the endomorphism operad of $N$. Furthermore, it is clear that for any simplicial set $X$ the $X$-coalgebra on $N(X)$ is controlled by its $U(M)$-coalgebra structure i.e., by the Alexander–Whitney coproduct and the join product. Additionally, we can inspect Equation (14) to conclude that the surjections \{(1, 2), (1, 2, 1), (1, 2, 1, 2), \ldots\} define Steenrod’s original cup-i construction. In fact, $\mathcal{X}(2)$ is isomorphic to $W$ so its set of bases is in bijection with the set of a cup-i constructions satisfying our axioms.

7.3.3. Let $G$ be a finite group and consider the simplicial set
\[
EG_m = \{(g_0, \ldots, g_m) \mid g_i \in G\},
\]
\[
d_j(g_0, \ldots, g_m) = (g_0, \ldots, \hat{g}_j, \ldots, g_m),
\]
\[
s_j(g_0, \ldots, g_m) = (g_0, \ldots, g_j, g_j, \ldots, g_m).
\]
The partial composition of permutations
\[
\circ_j : \mathbb{S}_p \times \mathbb{S}_q \to \mathbb{S}_{p+q-1}
\]
induces an operad structure on the collection of normalized chains $E(r) = N(E\mathbb{S}_r)$. This $E_\infty$-operad, introduced by Berger–Fresse [BF04], is referred to as the Barratt–Eccles operad and is here denoted by $\mathcal{E}$. These authors also define a quasi-isomorphism of operads $\text{TR}: \mathcal{E} \to \mathcal{X}$ and use it to associate to any simplex in $E\mathbb{S}_r$ a natural linear transformation $N \to N \otimes p^r$. In particular, for any simplicial set $X$ the $\mathcal{E}$-coalgebra structure on $N(X)$ is controlled by its $U(M)$-coalgebra structure, i.e., by the Alexander–Whitney coproduct and the join product. Since for the isomorphism TR: $\mathcal{E}(2) \to \mathcal{X}(2)$ we have
\[
\text{TR}(\text{id}, T, \text{id}, \ldots, T^i) = \begin{cases} 
(1, 2, \ldots, 2, 1) & \text{if } m \text{ is even}, \\
(1, 2, \ldots, 2, 1) & \text{if } m \text{ is odd},
\end{cases}
\]
the elements \{(id, T, id, \ldots, T^i)\}$_{i \in \mathbb{N}}$ define a cup-i construction that agrees with Steenrod’s, and the set of bases of $\mathcal{E}(2)$ is in bijection with the set of a cup-i constructions satisfying our axioms.

7.4. Steenrod squares are parameterized by classes on the mod 2 homology of $S_2$. Steenrod used this group homology viewpoint to non-constructively define for any prime $p$ operations on the mod $p$ cohomology of spaces [Ste52; Ste53]. To define these constructively for odd primes, May’s operadic viewpoint [May70] was used in [KM21] to provide explicit cup-$(p, i)$ constructions, which were implemented in the computer algebra system **ComCH** [Med21a]. A cup-$(p, i)$ construction is an equivariant chain map
\[
W(p) \to \text{Hom}(N, N^\otimes p)
\]
where
\[
W(p) = \left( F_p[S_p]\{e_0\} \xrightarrow{T^{-1}} F_p[S_p]\{e_1\} \xleftarrow{N} \cdots \right)
\]
is the minimal resolution of $F_p$ as an $F_p[S_p]$-module. Studying the moduli of cup-
$(p, i)$ constructions as done here for the $p = 2$ case is left to future work.

References

[Ade52] José Adem. “The iteration of the Steenrod squares in algebraic topology”. Proc. Nat. Acad. Sci. U.S.A. 38 (1952) (cit. on p. 7).
[Bar+21] Maissam Barkeshli et al. “Classification of (2+1)D invertible fermionic topological phases with symmetry” (2021) (cit. on p. 1).
[Bau21] Ulrich Bauer. “Ripser: efficient computation of Vietoris-Rips persistence barcodes”. J. Appl. Comput. Topol. 5.3 (2021) (cit. on p. 7).
[Ben98] D. J. Benson. Representations and cohomology. II. Second. Vol. 31. Cambridge Studies in Advanced Mathematics. Cohomology of groups and modules. Cambridge University Press, Cambridge, 1998 (cit. on p. 2).
[BF04] Clemens Berger and Benoit Fresse. “Combinatorial operad actions on cochains”. Math. Proc. Cambridge Philos. Soc. 137.1 (2004) (cit. on pp. 2, 19, 20).
[BM16] Greg Brumfiel and John Morgan. “The Pontrjagin Dual of 3-Dimensional Spin Bordism”. arXiv e-prints (2016) (cit. on p. 1).
[BM18] Greg Brumfiel and John Morgan. “The Pontrjagin Dual of 4-Dimensional Spin Bordism”. arXiv e-prints (2018) (cit. on p. 1).
[BMM21] Greg Brumfiel, Anibal Medina-Mardones, and John Morgan. “A cochain level proof of Adem relations in the mod 2 Steenrod algebra”. J. Homotopy Relat. Struct. (2021) (cit. on p. 7).
[Car+08] Gunnar Carlsson et al. “On the local behavior of spaces of natural images”. Int. J. Comput. Vis. 76.1 (2008) (cit. on p. 7).
[CCR13] Joseph Minhow Chan, Gunnar Carlsson, and Raul Rabadan. “Topology of viral evolution”. Proceedings of the National Academy of Sciences 110.46 (2013) (cit. on p. 7).
[EM53] Samuel Eilenberg and Saunders MacLane. “Acyclic models”. Amer. J. Math. 75 (1953) (cit. on p. 6).
[FMS21] Greg Friedman, Anibal M. Medina-Mardones, and Dev Sinha. “Flowing from intersection product to cup product”. arXiv e-prints (2021). Submitted (cit. on p. 7).
[FMS22] Greg Friedman, Anibal M. Medina-Mardones, and Dev Sinha. “Co-orientations, pull-back products, and the foundations of geometric cohomology” (2022). In preparation (cit. on p. 7).
[GK16] Davide Gaiotto and Anton Kapustin. “Spin TQFTs and fermionic phases of matter”. International Journal of Modern Physics A 31.28n29 (2016) (cit. on p. 1).
[GR+03] Rocio Gonzalez-Diaz, Pedro Real, et al. “Computation of cohomology operations of finite simplicial complexes”. Homology, Homotopy and Applications 5.2 (2003) (cit. on p. 18).
[GR05] Rocio Gonzalez-Diaz and Pedro Real. “HPT and cocyclic operations”. Homology Homotopy Appl. 7.2 (2005) (cit. on p. 18).
[GR99] Rocío González-Díaz and Pedro Real. “A combinatorial method for computing Steenrod squares”. Vol. 139. 1-3. Effective methods in algebraic geometry (Saint-Malo, 1998). 1999 (cit. on pp. 2, 18, 19).

[KM21] Ralph M. Kaufmann and Aníbal M. Medina-Mardones. “Cochain level May–Steenrod operations”. Forum Math. (2021) (cit. on pp. 2, 20).

[KT17] Anton Kapustin and Ryan Thorngren. “Fermionic SPT phases in higher dimensions and bosonization”. J. High Energy Phys. 10 (2017) (cit. on p. 1).

[Lee+17] Yongjin Lee et al. “Quantifying similarity of pore-geometry in nanoporous materials”. Nature communications 8.1 (2017) (cit. on p. 7).

[LM22] Guillaume Laplante-Anfossi and Aníbal M. Medina-Mardones. “Fiber polytopes and the Steenrod construction”. In preparation. 2022 (cit. on p. 1).

[LMT18] Umberto Lupo, Aníbal M. Medina-Mardones, and Guillaume Tauszin. “Persistence Steenrod modules”. arXiv e-prints (2018). Submitted (cit. on pp. 1, 7).

[May70] J. Peter May. “A general algebraic approach to Steenrod operations”. *The Steenrod Algebra and its Applications*. Lecture Notes in Mathematics, Vol. 168. Springer, Berlin, 1970 (cit. on pp. 2, 20).

[Med20a] Aníbal M. Medina-Mardones. “A finitely presented $E_{\infty}$-prop I: algebraic context”. High. Struct. 4.2 (2020) (cit. on pp. 2, 19).

[Med20b] Aníbal M. Medina-Mardones. “An algebraic representation of globular sets”. Homology Homotopy Appl. 22.2 (2020) (cit. on p. 1).

[Med20c] Aníbal M. Medina-Mardones. “An effective proof of the Cartan formula: the even prime”. J. Pure Appl. Algebra 224.12 (2020) (cit. on p. 7).

[Med21a] Aníbal M. Medina-Mardones. “A computer algebra system for the study of commutativity up to coherent homotopies”. arXiv e-prints (2021). To appear in Tbilisi Math. J. (cit. on p. 20).

[Med21b] Aníbal M. Medina-Mardones. “A finitely presented $E_{\infty}$-prop II: cellular context”. High. Struct. 5.1 (2021) (cit. on p. 19).

[Med21c] Aníbal M. Medina-Mardones. “New formulas for cup-i products and fast computation of Steenrod squares”. arXiv e-prints (2021). Submitted (cit. on pp. 2, 6, 13).

[MS03] James E. McClure and Jeffrey H. Smith. “Multivariable cochain operations and little n-cubes”. J. Amer. Math. Soc. 16.3 (2003) (cit. on pp. 2, 19, 20).

[Rea96] Pedro Real. “On the computability of the Steenrod squares”. Ann. Univ. Ferrara Sez. VII (N.S.) 42 (1996) (cit. on pp. 2, 18).

[SE62] N. E. Steenrod and D. B. A. Epstein. *Cohomology Operations: Lectures by N. E. Steenrod*. Princeton University Press, 1962 (cit. on pp. 1, 6).

[Ste47] N. E. Steenrod. “Products of cocycles and extensions of mappings”. Ann. of Math. (2) 48 (1947) (cit. on pp. 1, 5, 6, 17).

[Ste52] N. E. Steenrod. “Reduced powers of cohomology classes”. Ann. of Math. (2) 56 (1952) (cit. on p. 20).

[Ste53] N. E. Steenrod. “Cyclic reduced powers of cohomology classes”. Proc. Nat. Acad. Sci. U.S.A. 39 (1953) (cit. on pp. 2, 20).
REFERENCES

[Str87] Ross Street. “The algebra of oriented simplexes”. J. Pure Appl. Algebra 49.3 (1987) (cit. on p. 1).

[Tau+21] Guillaume Tauzin et al. “giotto-tda: A Topological Data Analysis Toolkit for Machine Learning and Data Exploration”. Journal of Machine Learning Research 22.39 (2021) (cit. on p. 7).

[The22] The GUDHI Project. GUDHI User and Reference Manual. 3.5.0. GUDHI Editorial Board, 2022 (cit. on p. 7).

[ZC05] Afra Zomorodian and Gunnar Carlsson. “Computing persistent homology”. Discrete & Computational Geometry 33.2 (2005) (cit. on p. 1).

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