WEIGHTED NORM INEQUALITIES FOR POTENTIALS WITH APPLICATIONS TO SCHRÖDINGER OPERATORS, FOURIER TRANSFORMS, AND CARLESON MEASURES

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I. Introduction. A new characterization of the trace inequality for potential operators is given and used to sharpen recent results of C. L. Fefferman and D. H. Phong on the distribution of eigenvalues of Schrödinger operators. It is also used to study the domain and essential spectrum of Schrödinger operators, to obtain weighted norm inequalities for Fourier transforms, and to determine the Carleson measures for Dirichlet-type spaces.

THEOREM 1. Suppose \(K\) is a nonnegative, locally integrable, radial function on \(\mathbb{R}^n\), which is decreasing as a function of \(|x|\). For \(f\) in the class \(P(\mathbb{R}^n)\) of nonnegative, measurable functions on \(\mathbb{R}^n\) and \(x \in \mathbb{R}^n\), set

\[
(Tf)(x) = (K * f)(x) = \int_{\mathbb{R}^n} K(x - y)f(y) \, dy,
\]

provided this integral exists for almost all \(x \in \mathbb{R}^n\). Then given \(1 < p < \infty\) and \(v \in P(\mathbb{R}^n)\), there exists \(C > 0\) so that the trace inequality

\[
\int_{\mathbb{R}^n} (Tf)(x)^p v(x) \, dx \leq C \int_{\mathbb{R}^n} f(x)^p \, dx, \quad f \in P(\mathbb{R}^n),
\]

holds if and only if \(C' > 0\) exists with

\[
\int_Q T(\chi_Q v)(x)^p' \, dx \leq C' \int_Q v(x) \, dx < \infty \quad \text{for all dyadic cubes } Q,
\]

where, as usual, \(p' = \frac{p}{p - 1}\).

Alternative characterizations of the trace inequality in terms of \(L^p\) capacities have been obtained in [1 and 4].

The trace inequality (1), for \(p = 2\), and the potential kernel \(K^\alpha(x)\), with \(K^\alpha(\zeta) = (\alpha + |\zeta|^2)^{-1/2}\), arises in estimating the eigenvalues\(^3\) of a Schrödinger operator \(H\). Let

\[
(I_2 f)(x) = \int_{\mathbb{R}^n} |x - y|^{2-n} f(y) \, dy
\]

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\(^3\)By eigenvalues we mean the numbers \(\lambda_1 < \cdots < \lambda_N < \cdots\), where \(\lambda_N\) is the maximum over all \(N - 1\) tuples \(\phi_1, \ldots, \phi_{N-1}\) of the quantity \(\inf(Hu, u) / (u, u)\), the infimum being over all \(u \in Q(H)\), \(u \perp \phi_j\), \(j = 1, \ldots, N - 1\). Here \(Q(H)\) denotes the form domain of \(H\). See [10].
denote the Newtonian potential of \( f \). The following result refines the estimates of the least eigenvalue of \( H \) given in Theorem 5 of [3].

**Theorem 2.** Let \( H = -\Delta - v \), where \( v \in P(R^n), n \geq 3 \). Denote the \( v \) measure of \( Q \), \( \int_Q v(x) \, dx \), by \( |Q|_v \). There are positive constants \( C \) and \( c \), depending only on the dimension \( n \), such that the least eigenvalue, \( \lambda_1 \), of \( H \) satisfies

\[
E_{sm} \leq -\lambda_1 \leq E_{big},
\]

where

\[
E_{sm} = \sup \left\{ |Q|^{-2/n} : |Q|_v^{-1} \int_Q I_2(x_Qv)v \geq C \right\},
\]

\[
E_{big} = \sup \left\{ |Q|^{-2/n} : |Q|_v^{-1} \int_Q I_2(x_Qv)v \geq c \right\}.
\]

A similar refinement of Theorems 6 and 6' in [3] is given in

**Theorem 3.** Let \( H = -\Delta - v \), where \( v \in P(R^n), n \geq 3 \). There are positive constants \( C \) and \( c \), depending only on the dimension \( n \), such that

(A) \( H \) has at least \( N \) eigenvalues \( \leq -\lambda \), \( \lambda > 0 \), provided there exists a collection of \( N \) cubes \( Q_1, \ldots, Q_N \) of side length at most \( \lambda^{-1/2} \), whose doubles are pairwise disjoint, with \( |Q_j|_v^{-1} \int_Q I_2(x_Qv)v \geq C \), \( 1 \leq j \leq N \).

Conversely,

(B) \( H \) having at least \( CN \) eigenvalues \( \leq -\lambda \) implies there is a collection of \( N \) pairwise disjoint dyadic cubes \( Q_1, \ldots, Q_N \), of side length at most \( \lambda^{-1/2} \), that satisfy

\[
|Q_j|_v^{-1} \int_Q I_2(x_Qv)v \geq c, \quad 1 \leq j \leq N.
\]

**Remarks.**

1. Roughly speaking, Theorem 3 says that the negative eigenvalues of \( H \) are approximately given by \( -|Q|^{-2/n} \) as \( Q \) ranges over all the minimal dyadic cubes satisfying \( |Q|_v^{-1} \int_Q I_2(x_Qv)v \geq C \).

2. As an illustration of Theorem 2, consider Example V in [3]: a particle in a rectangular box \( B = B_1 \times B_2 \times \cdots \times B_n \) with side lengths \( \delta_1 \leq \delta_2 \leq \cdots \leq \delta_n \). Let \( v = \chi_B \) and \( x_B \) denote the centre of \( B \). Since

\[
\sup_Q |Q|_v^{-1} \int_Q I_2(x_Qv)v \cong I_2 v(x_B) \cong \delta_1^2 + \delta_1 \delta_2 (1 + \log \delta_3/\delta_2)
\]

\[
\cong \delta_1 \delta_2 \log(1 + \delta_3/\delta_2),
\]

Theorem 2 yields the correct order of magnitude for the energy, \( E_{critical} \), needed to trap a particle in \( B \), namely

\[
E_{critical} = \sup \{ E \geq 0: -\Delta - Ev \geq 0 \} \cong (\delta_1 \delta_2 \log(1 + \delta_3/\delta_2))^{-1}.
\]

3. The quantity \( |Q|_v^{-1} \int_Q I_2(x_Qv)v \) is, in a sense, intermediate between the simpler ones used in [3] for the results corresponding to (A) and (B). Indeed, it is possible to show that for \( p > 1 \),

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\[ |Q|^{2/n-1} \int_Q v \leq C|Q|^{-1} \int_Q I_2(\chi_Q v) v \leq C_p \sup_{Q' \subset Q} |Q'|^{2/n-1/p} \left( \int_{Q'} v^p \right)^{1/p}. \]

The trace inequality also arises in questions concerning the domain and essential spectrum of Schrödinger operators. For example, conditions like (2) determine when the operator \( T \) in (1) is compact. This leads to conditions sufficient for \( H \) to have the same essential spectrum as \(-\Delta\), that is, \([0, \infty)\).

Another application of Theorem 1 is to weighted inequalities for Fourier transforms on \( R \).

**Theorem 4.** Suppose \( u(x) \) is an even, locally integrable function on \( R \) which is convex and decreases to 0 on \((0, \infty)\). Then for arbitrary \( v(x) \geq 0 \),

\[
\int_{-\infty}^{\infty} |f(x)u(x)|^2 \, dx \leq C \int_{-\infty}^{\infty} |f(x)v(x)|^2 \, dx \quad \text{for all } f \in L^1(R)
\]

if and only if

\[
\int_I \tilde{M}(\chi_I v^{-2})(x)^2 \, dx \leq C' \int_I v(x)^{-2} \, dx \quad \text{for all intervals } I,
\]

where

\[
(\tilde{M}f)(x) = \sup_{x \in I} \left[ \int_0^{||I||^{-1}} u(y) \, dy \right] \int_I |f(y)| \, dy.
\]

For earlier conditions guaranteeing (3) see [5, 6, and 7].

Our final application is to Carleson measures for the Dirichlet-type spaces \( h^p_K \) introduced in [9]. The space \( h^p_K \) consists of the Poisson integrals, \( u \), of potentials \( K \ast f \), \( f \in L^p(R^n) \). A positive measure \( \mu \) on \( R^{n+1}_+ \) is said to be a Carleson measure for \( h^p_K \) if \( \|u\|_{L^p(\mu)} \leq C\|f\|_p \) for all \( f \in L^p(R^n) \).

**Theorem 5.** Suppose \( K(x) \) is nonnegative and radial on \( R^n \) and is decreasing as a function of \(|x|\). Then for \( 1 < p < \infty \), a positive Borel measure \( \mu \) on \( R^{n+1}_+ \) is a Carleson measure for \( h^p_K \) if and only if

\[
\int_Q \tilde{M}(\chi_T(Q)\mu)(x)^p' \, dx \leq C \int_{T(Q)} d\mu < \infty \quad \text{for all cubes } Q.
\]

Here, \( Q \) is a cube in \( R^n \) and \( T(Q) \) denotes the cube in \( R^{n+1}_+ \) having \( Q \) as a face. The Carleson maximal function, \( \tilde{M} \nu \), is given at \( x \in R^n \) by

\[
\tilde{M} \nu(x) = \sup_{x \in Q} \left[ |Q|^{-1} \int_{|y| \leq |Q|^{1/n}} K(y) \, dy \right] \int_{T(Q)} d\nu.
\]

A characterization of Carleson measures in terms of \( L^p \) capacities can be found in [9 and 12].

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II. Sketch of proofs.

 Proof of Theorem 1 By duality, (1) is equivalent to
\[ \int T(gv)^p' \leq C' \int g^p' v, \quad g \in P(R^n). \]

Extensions of theorems in [8] show that (4) amounts to the same inequality with \( T \) replaced by the dyadic maximal operator
\[ (Mf)(x) = \sup_{x \in Q} \left( \frac{|Q|}{|x|} \right)^{-1} \int_{|y| \leq |Q|^{1/n}} K(y) \, dy \int_{Q} |f|, \quad Q \text{ dyadic.} \]

The methods of [11] now yield (2), with \( M \) instead of \( T \), as necessary and sufficient for the latter inequality. Finally, as above, \( M \) and \( T \) are interchangeable, so the proof is complete.

 Proof of Theorem 2. We have
\[ \frac{-\lambda_1}{2} \equiv \sup_{u \in Q(H)} \frac{\langle Hu, u \rangle}{\langle u, u \rangle} = \inf\{ \alpha > 0 : C_\alpha \leq 1 \}, \]
where \( C_\alpha \) is the least constant such that \( \int_{R^n} (I_1 f)^2 v \leq C_\alpha \int f^2 \) for all \( f \in P(R^n) \), and \( I_1^\alpha \) has kernel \( K_1^\alpha \) with \( \hat{K}_1^\alpha(\zeta) = (\alpha + |\zeta|^2)^{-k/2} \). This is so since
\[
\sup_{u \in Q(H)} \frac{\langle Hu, u \rangle}{\langle u, u \rangle} = \inf \left\{ \alpha > 0 : \int |u|^2 v \leq \int (\alpha |u|^2 + |\nabla u|^2) = \int (\alpha + |\zeta|^2)|\hat{u}(\zeta)|^2 d\zeta \right\} = \inf \{ \alpha > 0 : C_\alpha \leq 1 \}.
\]

Theorem 1 now yields \( C_\alpha \approx \sup_{Q} |Q|^{-1} \int I_1^\alpha(\chi_Q v)^2 \). Standard estimates on Bessel kernels show it suffices to take this supremum over cubes of side length at most \( \alpha^{-1/2} \), so Theorem 2 follows readily, since
\[ \int I_1^\alpha(\chi_Q v)^2 = \int_Q I_2^\alpha(\chi_Q v) v \approx \int_Q I_2(\chi_Q v) v \]
for such cubes.

 Proof of Theorem 3. (A) As in [3], it suffices to construct an \( N \)-dimensional subspace \( S \) of \( Q(H) \) such that \( \int (|\nabla u|^2 + \lambda |u|^2) \leq \int |u|^2 v, \: u \in S \). With some computation, one verifies this inequality for
\[ S = \text{Span}\{\theta_j I_2^\alpha(\chi_Q v)\}_{j=1}^N, \]
the \( \theta_j \) being dilates and translates of a fixed \( C^\infty \) function \( \theta \) with \( \theta_j \equiv 1 \) on \( 3^j Q_j, \: \text{supp} \theta_j \subset 2Q_j, \: j = 1, \ldots, N \).

(B) We sketch the case \( \lambda = 0 \), following the line of proof in [3]. Thus, we prove (B) by showing that if \( Q_1, \ldots, Q_N \) are all the minimal dyadic cubes satisfying \( |Q|^{-1} \int Q I_2(\chi_Q v) v \geq c \), then \( H \) has at most \( CN \) negative eigenvalues. This is done by constructing a subspace \( S \) of codimension \( CN \) in \( L^2 \) such that \( \int |u|^2 v \leq \int |\nabla u|^2, \: u \in S \cap Q(H) \). We define additional cubes \( Q_{N+1}, \ldots, Q_M, \: M \leq CN \), and sets \( E_j, \: 0 \leq j \leq M \), in analogy with those in [3]. A modification of arguments in [3] shows that if
\[ v_j = \chi_{E_j} v, \] then \[ |Q|^{-1} \int_Q I_2(\chi_{Q}v_j) v_j \leq c \] for all dyadic cubes \( Q \) and, thus, \[ \int f f^2 v_j \leq \int f^2, \quad f \in P(R^n), \quad 0 \leq j \leq M, \] by Theorem 1.

It is possible to find cubes \( Q_j \) (not necessarily dyadic or pairwise disjoint) such that \( \bigcup_j Q_j = E_j \) for \( 0 \leq j \leq M \) and such that the total number of \( Q_j \) does not exceed \( C_n M \). Let \( S = \{ u \in L^2 : \int_{Q_j} u = 0 \text{ for all } i, j \} \). Lemma 1.4 of \cite{2} shows

\[ |u(x)| \leq CI_1(\chi_{Q_j} |\nabla u|)(x) \leq CI_1(\chi_{E_j} |\nabla u|)(x) \]

for \( x \in Q_j, u \in S \), and so

\[
\int |u|^2 v = \sum_{j=0}^{M} \int |u|^2 v_j \leq C \sum_{j=0}^{M} \int \left[ I_1(\chi_{E_j} |\nabla u|) \right]^2 v_j \leq \sum_{j=0}^{M} \int_{E_j} |\nabla u|^2 = \int |\nabla u|^2
\]

for \( u \in S \cap Q(H) \), as required.

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