COUNTING ISOTROPIC TANGENT LINES OF HYPERSURFACES

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Abstract. Consider the standard symplectic $(\mathbb{R}^{2n}, \omega_0)$, a point $p \in \mathbb{R}^{2n}$ and an immersed closed orientable hypersurface $\Sigma \subset \mathbb{R}^{2n} \setminus \{p\}$, all in general position. We study the following passage/tangency question: how many lines in $\mathbb{R}^{2n}$ pass through $p$ and tangent to $\Sigma$ parallel to the 1-dimensional characteristic distribution $\ker (\omega_0|_{T\Sigma}) \subset T\Sigma$ of $\omega_0$. We count each such line with a certain sign, and present an explicit formula for their algebraic number. This number is invariant under regular homotopies in the class of a general position of the pair $(p, \Sigma)$, but jumps (in a well-controlled way) when during a homotopy we pass a certain singular discriminant. It provides a low bound to the actual number of these isotropic lines.

1. INTRODUCTION

1.1. Immersions and their invariants. The study of the topology of the space of immersions $\text{Imm}(X, Y)$ of a smooth manifold $X$ into a smooth manifold $Y$ is a famous classical problem. Recall that a smooth map $f : X \to Y$ is called an immersion if its differential $df$ is everywhere injective. The space $\text{Imm}(X, Y)$ equipped with the $C^\infty$-topology is an open subset of the Fréchet manifold $C^\infty(X, Y)$.

The study of $\text{Imm}(X, Y)$ becomes especially interesting if $X$ and $Y$ are orientable and oriented manifolds. For example, one can study the path-connectedness of $\text{Imm}(X, Y)$. Two immersions are regularly homotopic if they can be connected by a continuous path of immersions, i.e. they belong to the same path-component of $\text{Imm}(X, Y)$. Already in the simplest cases one gets non-trivial results. In particular, Whitney [5] classified immersions of the oriented unit circle $\mathbb{S}^1$ into the oriented Euclidean plane $\mathbb{R}^2$, i.e. oriented plane immersed curves $\Gamma : \mathbb{S}^1 \to \mathbb{R}^2$ up to regular homotopy: path-components of $\text{Imm}(\mathbb{S}^1, \mathbb{R}^2)$ are in a...
natural bijection with integers $\mathbb{Z}$. The bijection is given by the Whitney index $\text{ind}(\Gamma)$ - the degree of the tangential Gauss map $G_T : S^1 \to S^1$ given by $G_T(s) = \frac{\Gamma'(s)}{\|\Gamma'(s)\|}$. Note that the Whitney index $\text{ind} : \text{Imm}(S^1, \mathbb{R}^2) \to \mathbb{Z}$ is a locally constant function, i.e. it is an invariant of immersions up to regular homotopies. The Whitney index $\text{ind}(\Gamma)$ can also be expressed in terms of the normal Gauss map as follows. Fix the standard orientations: $o_{S^1}$ on $S^1$ and $o_{\mathbb{R}^2}$ on $\mathbb{R}^2$. Let us coorient $\Gamma(S^1)$ by choosing a normal vector field $N$ on it satisfying $N \times o_{\Gamma(S^1)} = o_{\mathbb{R}^2}$. Then $\text{ind}(\Gamma)$ equals to the degree of the normal Gauss map $G_N : S^1 \to S^1$ given by $G_N(s) = N(\Gamma(s))$.

The following Morse theoretical interpretation of $\text{ind}(\Gamma)$ provides an interesting connection between the Morse theory and the degree theory. Let $h : \Gamma(S^1) \to \mathbb{R}$ be a Morse function, such that its set of critical points $\text{Crit}(h)$ does not contain singular points of the curve $\Gamma$. For example, one can take a generic height function $h(p) := \langle p - p_0, v \rangle$, where $p_0 \in \mathbb{R}^2$ and $v \in S^1$, see Figure 1.

![Figure 1. Whitney index via Morse function.](image)

Then $\text{Crit}(h)$ splits into $\text{Crit}^+(h) \sqcup \text{Crit}^-(h)$, where

$$\text{Crit}^+(h) := \{ p \in \text{Crit}(h) | N(p) = +v \}$$

and

$$\text{Crit}^-(h) := \{ p \in \text{Crit}(h) | N(p) = -v \}.$$ 

In particular, we can use "half" of critical points of $h$ to compute the Whitney index. Namely, if we denote by $\mu_h(p)$ the Morse index of a
critical point $p$ of the function $h$, then

$$\text{ind}(\Gamma) = \sum_{p \in \text{Crit}^+(h)} (-1)^{\dim(S^1)-\mu_h(p)} = \sum_{p \in \text{Crit}^-(h)} (-1)^{\mu_h(p)}.$$ 

Indeed, $\text{ind}(\Gamma) = \deg(G_N)$ and since $\pm v$ are regular values of $G_N$ we have

$$\deg(G_N) = \sum_{G_N(p)=+v} \deg_p G_N = \sum_{G_N(p)=-v} \deg_p G_N.$$ 

Finally, we observe that local degrees satisfy $\deg_p G_N = (-1)^{\dim(S^1)-\mu_h(p)}$, if $p \in \text{Crit}^+(h)$ and $\deg_p G_N = (-1)^{\mu_h(p)}$, if $p \in \text{Crit}^-(h)$. It is worth pointing out that

$$\sum_{p \in \text{Crit}^+(h)} (-1)^{\dim(S^1)-\mu_h(p)} = - \sum_{p \in \text{Crit}^+(h)} (-1)^{\mu_h(p)}$$

and thus,

$$\chi(S^1) = \sum_{p \in \text{Crit}} (-1)^{\mu_h(p)} = \sum_{p \in \text{Crit}^+(h)} (-1)^{\mu_h(p)} + \sum_{p \in \text{Crit}^-(h)} (-1)^{\mu_h(p)} = 0.$$ 

Note also that while the Whitney index $\text{ind}(\Gamma)$ was defined as a discrete sum of signs (signed points), it can as well be expressed as a continuous sum. Namely, by Hopf’s Umlaufsatz [3], we have the integral formula

$$\int_{S^1} G_N^* \mu = \text{ind}(\Gamma),$$

where $\mu \in \Omega^1(S^1; \mathbb{R})$ is the volume form normalized by $\int_{S^1} \mu = 1$.

The above description immediately generalizes to higher dimensions. Indeed, consider the space $\text{Imm}(S, \mathbb{R}^m)$, where $S$ is a smooth closed orientable manifold of dimension $m-1$. Fix orientations: the standard $\sigma_\mathbb{R}^m$ on $\mathbb{R}^m$ and $\sigma_S$ on $S$. Let $\iota: S \to \mathbb{R}^m$ be an immersion. Then $\Sigma := \iota(S) \subset \mathbb{R}^m$ is an immersed orientable hypersurface oriented by $\sigma_\Sigma := \iota_*(\sigma_S)$. Let us coorient $\Sigma$ by choosing a normal vector field $N$ on it satisfying $N \times \sigma_\Sigma = \sigma_\mathbb{R}^m$. Define the Whitney index $\text{ind}(\Sigma)$ of $\Sigma$ to be the degree of the normal Gauss map $G_N: S \to S^{m-1}$, $G_N(s) = N(\iota(s))$. By choosing a Morse height function $h$ on $\Sigma$ as before, we get that

$$\text{ind}(\Sigma) = \sum_{p \in \text{Crit}^+(h)} (-1)^{\dim(S)-\mu_h(p)} = \sum_{p \in \text{Crit}^-(h)} (-1)^{\mu_h(p)}.$$
Now, we observe the principal difference between even and odd dimensional cases. Indeed, if \( \dim(S) \) is even, then

\[
\sum_{p \in \text{Crit}^\uparrow(h)} (-1)^{\dim(S) - \mu_h(p)} = \sum_{p \in \text{Crit}^\uparrow(h)} (-1)^{\mu_h(p)}
\]

and so

\[
2 \text{ind}(\Sigma) = \sum_{p \in \text{Crit}(h)} (-1)^{\mu_h(p)} = \chi(S),
\]

where \( \chi(S) \) is the Euler characteristics of \( S \). It means that \( \text{ind} \) is a constant function \( \frac{1}{2} \chi(S) \) on the space \( \text{Imm}(S, \mathbb{R}^m) \). In the odd dimensional case, \( \text{ind} \) is already a locally constant function, which depends on topological types of immersions. Note also that in the latter case we have \( \chi(S) = 0 \). So we have a constant function \( \chi(S) \) on the space \( \text{Imm}(S, \mathbb{R}^m) \), which, in the case of \( \dim(S) \) is odd, naturally splits into an invariant of immersions up to regular homotopies.

The above description of \( \text{ind}(\Sigma) \) has the following enumerative meaning: \( 2 \text{ind}(\Sigma) \) it is an algebraic (with the above Morse signs) number of affine tangent spaces to \( \Sigma \) parallel to a given one. In turn, this can be seen as an algebraic count of affine tangent spaces to \( \Sigma \) that pass through \((m - 2)\)-dimensional space at ”infinity”. So one can generalize this very special case to a (algebraic) count of affine tangent spaces to \( \Sigma \) that pass through a generic \((m - 2)\)-dimensional affine space \( P \subset \mathbb{R}^m \setminus \Sigma \). In Morse theoretical terms it means that we consider now a Morse function \( h \), which is a projection of \( \Sigma \) from \( P \) onto a generic directed line, see Figure 2. The plane case of \( m = 2 \) was considered in [4]. Namely, let \( \mathcal{L} := \mathcal{L}(\rho, \Gamma) \) be a set of lines in \( \mathbb{R}^2 \) passing
through a fixed point \( p \) and tangent to a (generic) oriented immersed plane closed curve \( \Gamma \). Each such line \( l \in \mathcal{L} \) was counted with a certain sign \( \varepsilon_l \), so that the total algebraic number \( \mathcal{N}(p, \Gamma) = \sum_{l \in \mathcal{L}} \varepsilon_l \) of lines does not change under homotopy of \( \Gamma \) in \( \mathbb{R}^2 \setminus p \). One can guess such a sign rule as follows. Under a deformation shown in Figure 3a, two new lines appear, so their contributions to \( \mathcal{N}(p, \Gamma) \) should cancel out. Thus, their signs should be opposite and one gets the sign rule shown in Figure 3b. It follows (see [4]) that

\[
\mathcal{N}(p, \Gamma) = 2 \text{ind}(\Gamma) - 2 \text{ind}_p(\Gamma).
\]

Here, \( \text{ind}_p(\Gamma) \) is the index of \( p \) w.r.t. \( \Gamma \), i.e. the number of turns made by the vector connecting \( p \) to a point \( q \in \Gamma \), as \( q \) passes once along \( \Gamma \) following the orientation. It may be computed as the intersection number \( I([p, \infty], \Gamma; \mathbb{R}^2) \) of a 1-chain \([p, \infty]\) (i.e. an interval connecting \( p \) with a point near infinity of \( \mathbb{R}^2 \)) with an oriented 1-cycle \( \Gamma \) in \( \mathbb{R}^2 \). The appearance of \( \text{ind}(\Gamma) \) and \( \text{ind}_p(\Gamma) \) in the above formula comes as no surprise: in fact, these are the only invariants of the curve \( \Gamma \) under its homotopy in the class of immersions in \( \mathbb{R} \setminus p \).

In contrast to the plane case, in higher dimensions we do not have a natural choice of a tangent direction on an immersed hypersurface \( \Sigma := \iota(S) \subset \mathbb{R}^m \), \( \iota \in \text{Imm}(S, \mathbb{R}^m) \), such that the degree of the corresponding tangent Gauss map is a locally constant function on \( \text{Imm}(S, \mathbb{R}^m) \). Nevertheless, such a tangent direction can be chosen in the presence of additional structures on the target space \( \mathbb{R}^m \). Indeed, let us consider the following equivalent formulation of the plane case. Equip \( \mathbb{R}^2 \) with the standard symplectic structure \( \omega_0 \) and ask how many lines in \( \mathbb{R}^2 \) pass through \( p \) and tangent to \( \Gamma \) parallel to 1-dimensional characteristic distribution \( \ker(\omega_0|_{T\Gamma}) \) of \( \omega_0 \). Indeed, in the 2-dimensional case \( \Gamma \) is a Lagrangian immersed submanifold, i.e. \( \ker(\omega_0|_{T\Gamma}) = T\Gamma \), and the new formulation is equivalent to the previous one. So in the present paper we study the following question. Consider the standard symplectic \((\mathbb{R}^{2n}, \omega_0)\), \( n \in \mathbb{N} \) with the standard orientation, a point \( p \in \mathbb{R}^{2n} \) and an immersed closed oriented hypersurface \( \Sigma \subset \mathbb{R}^{2n} \setminus \{p\} \), all in general
position. How many lines in \( \mathbb{R}^{2n} \) pass through \( p \) and tangent to \( \Sigma \) parallel to the 1-dimensional characteristic distribution \( \ker (\omega_0|_{T\Sigma}) \subset T\Sigma \) of \( \omega_0 \)? Such lines will be called isotropic lines of \( \Sigma \) passing through \( p \).

1.2. Main results and the structure of the paper. Let \( p \in \mathbb{R}^{2n} \) and \( \Sigma \subset \mathbb{R}^{2n} \setminus \{p\} \) be as before. We use the natural almost contact structure and the shape operator of \( \Sigma \) in order to equip each isotropic line with a certain sign of tangency. We count each such line with this sign and present an explicit formula for their algebraic number. This number is invariant under regular homotopies in the class of a general position of the pair \((p, \Sigma)\), but jumps (in a well-controlled way) when during a homotopy we pass a certain singular discriminant. It provides a low bound to the actual number of these isotropic lines.

The paper is organized in the following way. In Section 2 we introduce objects of our study, define signs of tangency, list the requirements of a general position, and formulate the main theorem. Section 3 is dedicated to the proofs. We interpret the desired number of lines as a certain intersection number; the main claim follows from different ways of its calculation. We also obtain an integral formula for that number of lines.

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2. Statement of the main results.

2.1. Setting. Consider the standard symplectic \((\mathbb{R}^{2n}, \omega_0)\) with the fixed standard orientation \(o_{\mathbb{R}^{2n}}\). Let \( p \in (\mathbb{R}^{2n}, \omega_0) \) be a fixed point, let \( J_0 = \text{Id} \) be the standard almost complex structure on \((\mathbb{R}^{2n}, \omega_0) = (\mathbb{C}^n, \omega_0)\), i.e. the multiplication by \( \sqrt{-1} \). Suppose that \( S \) is a smooth closed orientable manifold of dimension \( 2n - 1 \) with a fixed orientation \( o_S \) and \( \iota : S \looparrowright \mathbb{R}^{2n} \setminus \{p\} \) is an immersion. Then \( \Sigma := \iota(S) \subset \mathbb{R}^{2n} \setminus \{p\} \) is an immersed orientable hypersurface oriented by \( o_{\Sigma} := \iota^*(o_S) \). Let us coorient \( \Sigma \) by choosing a normal vector field \( N \) on it satisfying \( N \times o_{\Sigma} = o_{\mathbb{R}^{2n}} \). The tangent vector field \( J_0 N \) on \( \Sigma \) spans the kernel of \( \omega_0|_{T\Sigma} \), i.e. spans isotropic lines of \( \omega_0|_{\Sigma} \). Let \((P, -J_0 N, \eta)\) be the natural almost contact structure on \( \Sigma \) induced by \( J_0 \). Namely, for \( X \in \Gamma(T\Sigma) \) the \((1, 1)\)-tensor \( P \in \text{End}(\Gamma(T\Sigma)) \) and the 1-form \( \eta \in \Omega^1(\Sigma) \) are given by \( P(X) = J_0(X) - \langle J_0(X), N \rangle N \) and \( \eta(X) = \langle J_0(X), N \rangle \), where
\langle \cdot, \cdot \rangle is the standard Euclidean inner product on \( \mathbb{R}^{2n} \) – see [2] for more details.

Finally, denote by \( \mathcal{L} := \mathcal{L}(p, \Sigma, \omega_0) \) the set of lines in \( \mathbb{R}^{2n} \) passing through a fixed point \( p \) and tangent to \( \Sigma \) parallel to the 1-dimensional characteristic distribution \( \ker(\omega_0|_{T\Sigma}) \subset T\Sigma \) of \( \omega_0 \).

2.2. General position for the pair \((p, \Sigma)\) and signs of lines. We shall assume that the following (generic) conditions hold:

1. The hypersurface \( \Sigma \) is generically immersed in \( \mathbb{R}^{2n} \setminus \{p\} \), i.e. all its self-intersections are transversal.
2. Every \( \ell \in \mathcal{L} \) is tangent to \( \Sigma \) at only one non-singular point.
3. If a line \( \ell \in \mathcal{L} \) is tangent to the hypersurface \( \Sigma \) at a point \( \iota(s) \), then \( \det(A_s \pm \lambda_s P_s) \neq 0 \), where \( A_s \) is the shape operator of the immersion \( \iota : S \mapsto \mathbb{R}^{2n} \setminus \{p\} \) at \( s \), \( \lambda_s := \|\iota(s) - p\|^{-1} \) and \( P_s \) is a short notation for \( (d\iota(s)^{-1} \circ P_{\iota(s)} \circ d\iota(s) \). Recall, that the shape operator is equal to the differential \( d\iota G \) of the Gauss map \( G : S \to S^{2n-1} \), \( G(s) = N(\iota(s)) \) at the point \( s \).

Now, to each \( \ell \in \mathcal{L} \) we assign a sign \( \varepsilon_\ell \in \{\pm 1\} \) as follows.

**Definition 2.1.** Suppose that a line \( \ell \) is tangent to \( \Sigma \) at a point \( \iota(s) \), such that its direction vector \( \xi_{\iota(s)} := \frac{\iota(s) - p}{\|\iota(s) - p\|} \) equals to \( \pm J_0 N(\iota(s)) \), then define

\begin{equation}
\varepsilon_\ell := \text{sgn}(\det(A_s \pm \lambda_s P_s)).
\end{equation}

Note that the sign \( \varepsilon_\ell \) is invariant under a homothety of \( \mathbb{R}^{2n} \setminus \{p\} \) with a positive ratio and the center at \( p \). Indeed, under such a homothety with a positive ratio \( c \), both operators \( A_s \) and \( \lambda_s P_s \) are multiplied by \( c^{-1} \).

In addition, for \( n = 1 \), i.e. the case of the toy model from Section [1], we have that \( P = 0 \). In particular, the sign \( \varepsilon_\ell \) equals to the sign of the curvature of a curve at the point of tangency. As a consequence, up to an orientation of the curve it coincides with Polyak's sign. In the higher dimensional case \((n > 1)\), when \( \lambda_s << 1 \), i.e. the point \( p \) is far away from \( \Sigma \), we have

\[ \varepsilon_\ell = \text{sgn}(\det(A_s \pm \lambda_s P_s)) = \text{sgn}(\det(A_s)) = \text{sgn}(K_s), \]

where \( K_s \) is the Gauss curvature of the immersion at \( s \). On the other hand, when we are passing zeros of the polynomial \( K_s(t) = \det(A_s + tP_s) \), the sign \( \text{sgn}(K_s) \) may change. As it will follow from the computations below - see Section [3] we may choose an orientating
frame on \( T_sS \), such that the operator \( P_s \) is represented by the matrix
\[
0 \oplus \pm \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \oplus \ldots \oplus \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
\]

Let \((a_{ij})\) be a symmetric \((2n - 1) \times (2n - 1)\) matrix representing the shape operator \( A_s \) in the above frame. Then, for example, in the case \( n = 2 \) we have that
\[
\det(A_s + tP_s) = \det \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & a_{22} & a_{23} - t \\
a_{13} & a_{23} + t & a_{33}
\end{pmatrix} = a_{11}t^2 + K_s.
\]

2.3. The statement of the main result. Let \( N := N(p, \Sigma, \omega_0) \) be the algebraic number
\[
N := \sum_{\ell \in \mathcal{L}} \varepsilon_\ell
\]
of lines in \( \mathbb{R}^{2n} \) passing through \( p \) and tangent to \( \Sigma \) parallel to the 1-dimensional characteristic distribution \( \ker (\omega_0|_{T\Sigma}) \subset T\Sigma \) of \( \omega_0 \). Denote \( \text{ind}(\Sigma) := \deg(G) \) – the degree of the Gauss map (higher dimensional Whitney index). Denote also \( \text{ind}_p(\Sigma) := \deg(\xi) \) – the degree of the map \( \xi : S \to S^{2n-1} \) given by \( \xi(s) = \xi_{t(s)} := \frac{t(s) - p}{\|t(s) - p\|} \) (higher dimensional index of \( p \) w.r.t. \( \Sigma \)). It can be interpreted as the linking number of a 0-chain \( \{\infty\} - \{p\} \) with \( \Sigma \) in \( \mathbb{R}^{2n} \), where \( \{\infty\} \) is a generic point "near infinity" of \( \mathbb{R}^{2n} \). Note that the linking number is the intersection number of a 1-chain \([p, \infty]\) with \( \Sigma \) in \( \mathbb{R}^{2n} \). The main result of this work is the following

**Theorem 2.2.** Let \((p, \Sigma)\) be in general position as in Section 2.2. Then
\[
N = 2 \text{ind}(\Sigma) - 2 \text{ind}_p(\Sigma)
\]

In particular, the number \( N \) is invariant under local regular homotopies of the pair \((p, \Sigma)\) in the class of general position.

Note that the number \( N \) provides a low bound to the actual number of counted isotropic lines.

3. The proof of the main results.

Consider a smooth manifold \( M := S^{2n-1} \times S^{2n-1} \). Let \( X_\pm := \phi_\pm(S) \) be immersed orientable submanifolds of \( M \) of dimension \( 2n - 1 \), where
immersions $\phi_{\pm} : S \to M$ are given by
\[
\phi_{\pm}(s) := (\xi, \pm J_0 \circ G)(s) = \left( \xi_q := \frac{q - p}{\|q - p\|}, \pm J_0(N(q)) \right), \; q = \iota(s).
\]
Denote $X := X_+ \cup X_-$ and note that $X_+ \cap X_- = \emptyset$. In addition, let $\mathcal{D} := \Delta(S^{2n-1})$, where $\Delta : S^{2n-1} \to M$ is the diagonal embedding $\Delta(q) = (q, q)$. Every point $x \in X \cap \mathcal{D}$ corresponds bijectively to some line $\ell(x) \in \mathcal{L}$. Since the pair $(p, \Sigma)$ is in general position, we have that $X$ and $\mathcal{D}$ intersect transversally in finitely many points. Consider the intersection number $I(\mathcal{D}, X; M)$ of $\mathcal{D}$ with $X$ in $M$
\[
I(\mathcal{D}, X; M) = \sum_{x \in X \cap \mathcal{D}} I_x(\mathcal{D}, X; M),
\]
where $I_x(\mathcal{D}, X; M)$ is the local intersection number.

**Proposition 3.1.** For every $x \in X \cap \mathcal{D}$, we have $I_x(\mathcal{D}, X; M) = \varepsilon_{\ell(x)}$.

**Proof.** Suppose that $x = (\xi_q, \pm J_0 N(q)) \in X \cap \mathcal{D}$, in particular, $\xi_q = \pm J_0 N(q)$. Let us find the local intersection number $I_x(\mathcal{D}, X; M)$ of $\mathcal{D}$ with $X$ in $M$ at the point $x$. We fix an orientation $o_M := o_{S^{2n-1}} \times o_{S^{2n-1}}$ on $M$, where $o_{S^{2n-1}}$ is the orientation on $S^{2n-1}$ defined by the outer normal vector field $\nu$, so that $\nu \times o_{S^{2n-1}} = o_{\mathbb{R}^{2n}}$. Note that $T_q(\mathbb{R}^{2n} \setminus \{p\}) = \mathbb{R}\xi_q \oplus (\mathbb{R}\xi_q)\perp$, and $(\mathbb{R}\xi_q)\perp = T_{\xi_q}S^{2n-1}$. So one can choose mutually orthogonal unit vectors $e_1, \ldots, e_{n-1} \in T_{\xi_q}S^{2n-1}$, such that $T_q(\mathbb{R}^{2n} \setminus \{p\}) = \text{Span}_{\mathbb{R}}(\xi_q, J_0\xi_q, e_1, J_0e_1, \ldots, e_{n-1}, J_0e_{n-1})$ and the orientation of the ordered basis $(\xi_q, J_0\xi_q, e_1, J_0e_1, \ldots, e_{n-1}, J_0e_{n-1})$ equals $o_{\mathbb{R}^{2n}}$. Note that in this basis we also have that $T_{\xi_q}S^{2n-1} = \text{Span}_{\mathbb{R}}(J_0\xi_q, e_1, J_0e_1, \ldots, e_{n-1}, J_0e_{n-1})$ and the orientation of the ordered basis $(J_0\xi_q, e_1, J_0e_1, \ldots, e_{n-1}, J_0e_{n-1})$ equals $o_{S^{2n-1}}$. Next, since $T_q\Sigma = (\mathbb{R}N(q))\perp = (\mathbb{R}(-J_0\xi_q))\perp$ we have that $T_q\Sigma = \text{Span}_{\mathbb{R}}(\pm\xi_q, e_1, J_0e_1, \ldots, e_{n-1}, J_0e_{n-1})$ and the orientation of the ordered basis $(\pm\xi_q, e_1, J_0e_1, \ldots, e_{n-1}, J_0e_{n-1})$ equals $o_{\Sigma}$. Now, if $s = \iota^{-1}(q) \in S$, the local intersection number $I_x(\mathcal{D}, X; M)$ equals to the sign of $\text{det}(d_\xiq \Delta + ds\phi_{\pm})$ (depending on $x \in X_+$ or $x \in X_-$) in the following frames:
\[
\text{Span}_{\mathbb{R}} \left( \left. J_0\xi_q; (e_i, J_0e_i)_{i=1}^{n-1} \right|_{i=1}^{n-1} \right) \oplus (ds)^{-1} \left( \text{Span}_{\mathbb{R}} \left( \pm\xi_q, (e_i, J_0e_i)_{i=1}^{n-1} \right) \right)
\]
for $T_{\xi_q}S^{2n-1} \times T_s S$ and
\[
\text{Span}_{\mathbb{R}} \left( \left. J_0\xi_q; (e_i, J_0e_i)_{i=1}^{n-1} \right|_{i=1}^{n-1} \right) \oplus \text{Span}_{\mathbb{R}} \left( J_0\xi_q, (e_i, J_0e_i)_{i=1}^{n-1} \right).
\]
for $T_{(\xi,\xi_\ell)}M$. In these frames the matrix $A_\pm$ of $d_{\xi_\ell}^*\Delta + d_\xi^*\phi_\pm$ equals to
\begin{equation}
A_\pm = \begin{pmatrix}
1_{2n-1} & \lambda_\pm \text{Diag}(0, 1, \ldots, 1) \\
1_{2n-1} & B_\pm
\end{pmatrix},
\end{equation}
where $\lambda_\pm = \|q-p\|^{-1}$ and $B_\pm$ is the matrix of the differential $d_s (\pm J_0 \circ G) = (\pm J_0|_{\Sigma}) \circ d_s G$.

It follows that $\det(A_\pm) = \det(B_\pm - \lambda_\pm \text{Diag}(0, 1, \ldots, 1))$. In the chosen frames the matrix of $\pm J_0|_{\Sigma}$ equals
\begin{equation}
C_\pm := 1 \oplus \pm \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)_{n-1}
\end{equation}
and the matrix of $\pm \lambda_\pm P_\Sigma$ equals $\lambda_\pm C_\pm \text{Diag}(0, 1, \ldots, 1)$. Moreover, $C_+ C_- = C_- C_+ = 1_{2n-1}$ and $\det(C_\pm) = 1$. As a consequence,
\begin{equation}
\det(A_\pm) = \det(d_s G \pm \lambda_\pm P_\Sigma) = \det(A_s \pm \lambda_\pm P_\Sigma)
\end{equation}
and hence $I_{\Sigma}(\mathcal{D}, X; M) = \varepsilon_{\ell(\Sigma)}$. \hfill \Box

**Corollary 3.2.** We have $N = \sum_{\ell \in \mathcal{L}(p, \Sigma)} \varepsilon_{\ell} = I(\mathcal{D}, X; M)$.

The next proposition finishes the proof of the main theorem

**Proposition 3.3.** We have $I(\mathcal{D}, X; M) = 2 \text{ind}(\Sigma) - 2 \text{ind}_p(\Sigma)$.

**Proof.** We shall use the homological interpretation of the intersection number. Namely,
\begin{equation}
I(\mathcal{D}, X; M) = \Delta_*[S^{2n-1}] \bullet (\phi_\pm)_*\Sigma + \Delta_*[S^{2n-1}] \bullet (\phi_-)_*\Sigma,
\end{equation}
where $\bullet : H_{2n-1}(M; \mathbb{Z}) \times H_{2n-1}(M; \mathbb{Z}) \to H_0(M; \mathbb{Z}) \cong \mathbb{Z}$ is the homological intersection product, i.e.
\begin{equation}
\alpha \bullet \beta = (PD(\alpha) \cup PD(\beta)) \cap [M] = PD(\beta) \cap \alpha = -PD(\alpha) \cap \beta.
\end{equation}

Note that for any $\theta_1, \theta_2 \in S^{2n-1}$, the class $\Delta_*[S^{2n-1}]$ splits as
\begin{equation}
\Delta_*[S^{2n-1}] = (\Delta_{\theta_1})_*[S^{2n-1}] + (\Delta_{\theta_2})_*[S^{2n-1}],
\end{equation}
where the embeddings $\Delta_{\theta_1}, \Delta_{\theta_2} : S^{2n-1} \to M$ are given by
\begin{equation}
\Delta_{\theta_1}(\theta) = (\theta_1, \theta) \text{ and } \Delta_{\theta_2}(\theta) = (\theta, \theta_2).
\end{equation}

Next we fix $\theta_1, \theta_2 \in S^{2n-1}$, such that $\theta_1$ is a regular value of the first coordinate $\xi : S \to S^{2n-1}$ of the map $\phi_\pm$, $\theta_2$ is a regular value of the
second coordinate $\pm J_0 \circ G : S \to \mathbb{S}^{2n-1}$ of the map $\phi_{\pm}$ and a smooth submanifold $D_{\theta_i} := \Delta_{\theta_i} (\mathbb{S}^{2n-1})$, $i = 1, 2$ intersects transversally with $X$ in nonsingular points. Since the manifold $D_{\theta_i}$ represents the class $(\Delta_{\theta_i})_* [\mathbb{S}^{2n-1}]$ for $i = 1, 2$, it follows

$$I(D, X; M) = I(D_{\theta_1}, X; M) + I(D_{\theta_2}, X; M).$$

It remains to prove the following

**Lemma 3.4.** We have $I(D_{\theta_1}, X; M) = -2 \text{ind}_p(\Sigma)$ and $I(D_{\theta_2}, X; M) = 2 \text{ind}(\Sigma)$.

**Proof of Lemma 3.4.** Firstly, we treat

$$I(D_{\theta_1}, X; M) = I(D_{\theta_1}, X_+; M) + I(D_{\theta_1}, X_-; M).$$

We have

$$I(D_{\theta_1}, X_{\pm}; M) = \sum_{x_{\pm} \in D_{\theta_1} \cap X_{\pm}} I_{x_{\pm}}(D_{\theta_1}, X_{\pm}; M)$$

and in the chosen frames

$$I_{x_{\pm}}(D_{\theta_1}, X_{\pm}; M) = \text{sgn det} \begin{pmatrix} 0_{2n-1} & d_s(x_{\pm}) \xi \\ 1_{2n-1} & d_s(x_{\pm}) (\pm J_0 \circ G) \end{pmatrix},$$

where $s(x_{\pm}) = \phi_{\pm}^{-1}(x_{\pm})$. It follows that

$$I(D_{\theta_1}, X_{\pm}; M) = \sum_{x_{\pm} \in D_{\theta_1} \cap X_{\pm}} -\text{sgn det} \left( d_s(x_{\pm}) \xi \right) = -\deg(\xi)$$

and hence, $I(D_{\theta_1}, X; M) = -2 \text{ind}_p(\Sigma)$.

In the same way we get that

$$I(D_{\theta_2}, X_{\pm}; M) = \sum_{x_{\pm} \in D_{\theta_2} \cap X_{\pm}} I_{x_{\pm}}(D_{\theta_2}, X_{\pm}; M)$$

and in the chosen frames

$$I_{x_{\pm}}(D_{\theta_2}, X_{\pm}; M) = \text{sgn det} \begin{pmatrix} 1_{2n-1} & d_s(x_{\pm}) \xi \\ 0_{2n-1} & d_s(x_{\pm}) (\pm J_0 \circ G) \end{pmatrix},$$

where $s(x_{\pm}) = \phi_{\pm}^{-1}(x_{\pm})$. Recall that $\det(\pm J_0 |_{T_{x_{\pm}}(\Sigma)}) = 1$. It follows that

$$I(D_{\theta_2}, X_{\pm}; M) = \sum_{x_{\pm} \in D_{\theta_2} \cap X_{\pm}} \text{sgn det} \left( d_s(x_{\pm}) (\pm J_0 \circ G) \right) = \deg(\Sigma)$$

and hence, $I(D_{\theta_2}, X; M) = 2 \text{ind}(\Sigma)$. \qed
Using the Poincaré duality, we immediately get an integral formula for $\mathcal{N}$. Indeed, let $\delta \in H^{2n-1}_d(M; \mathbb{R})$ be the Poincaré dual class of $\Delta_* [\mathbb{S}^{2n-1}]$ in the de Rham cohomology. Explicitly, the class $\delta$ is given by $[pr_1^* \mu] + [pr_2^* \mu]$, where $\mu \in \Omega^{2n-1}(\mathbb{S}^{2n-1}; \mathbb{R})$ is the volume form normalized by $\int_{\mathbb{S}^{2n-1}} \mu = 1$ and $pr_1, pr_2 : \mathbb{S}^{2n-1} \times \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$ are the natural projections.

**Corollary 3.5.** We have 

$$\mathcal{N} = - \int_S (\phi_+^*(\delta) + \phi_-^*(\delta)).$$

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