One-loop renormalization group equations of
the general framework with two Higgs doublets

G. Cvetič
Inst. für Physik, Universität Dortmund, 44221 Dortmund, Germany

S.S. Hwang and C.S. Kim
Department of Physics, Yonsei University, Seoul 120-749, Korea

(March 26, 2022)

Abstract

We derive one-loop renormalization group equations (RGE’s) for Yukawa coupling parameters of quarks and for the vacuum expectation values of the Higgs doublets in a general framework of the Standard Model with two Higgs doublets (2HDM “type III”). In the model, the neutral-Higgs-mediated flavor-changing neutral currents are allowed but are assumed to be reasonably suppressed at low energies. The popular “type II” and “type I” models are just special cases of this framework. We also present a numerical example for the RGE flow of Yukawa coupling parameters and masses of quarks. Detailed investigation, through RGE’s, of connection between the low energy and the high energy structure of these 2HDM’s would give us some additional insights into the viability of such frameworks, i.e., it would tell us for which choices of phenomenologically acceptable values of low energy parameters such frameworks can be regarded as reasonably natural (– without excessive changes in the parameter structure as energy of probes increases over a wide range). Furthermore, such analyses could provide us with possible signals of new physics at high energies.

PACS: 11.10.Hi; 12.15.Ff; 12.15.Mm; 12.38.Bx; 12.60.Fr

1 Introduction

Low energy experiments show that flavor-changing neutral currents (FCNC’s) are very suppressed in nature. For example,

\[ Br(K_L^0 \to \mu^+ \mu^-) \simeq (7.4 \pm 0.4) \cdot 10^{-9} , \quad |m_{B_L^0} - m_{B_L^0}| \simeq (3.36 \pm 0.40) \cdot 10^{-10} \text{MeV} , \]

\[ |m_{K_L} - m_{K_S}| \simeq (3.510 \pm 0.018) \cdot 10^{-12} \text{MeV} , \quad |m_{D_L^0} - m_{D_L^0}| < 1.32 \cdot 10^{-10} \text{MeV} , \]

\[ Br(b \to s\gamma) = (2.32 \pm 0.67) \cdot 10^{-4} \text{ etc.} \]
The various alternative models of electroweak interactions – extensions of the minimal Standard Model (MSM) – must take into account this FCNC suppression. The most conservative extensions of the MSM are apparently the models with two Higgs doublets (2HDM’s). The conditions for the one-loop FCNC suppression of contributions coming from gauge boson loops, \( i.e., \) the allowed representations of fermions, have been investigated some time ago [1]. In addition to the MSM (one Higgs doublet model), Glashow and Weinberg [1] proposed for the Higgs sector the “type I” and “type II” 2HDM’s. They proposed them as models which, in a “natural” way, have the zero value for the flavor-changing renormalized (\( i.e., \) low energy) Yukawa couplings in the neutral sector (called from now on: FCNC renormalized Yukawa couplings). These two types of the 2HDM’s have been widely discussed in the literature.

Later on, extensions with more than one Higgs doublet, other than the 2HDM(I) and (II), have been proposed and investigated – the general “type III” 2HDM’s [2]-[20]. This is the framework in which the renormalized (\( i.e., \) low energy) FCNC Yukawa coupling parameters are in general nonzero, but must be sufficiently suppressed. By “sufficiently” we mean that the measured low energy flavor-changing processes, including the \( \Delta F = 2 \) processes, \( K^0 - \bar{K}^0 \), \( B^0 - \bar{B}^0 \) and \( D^0 - \bar{D}^0 \), do not get enhanced by tree level neutral Higgs exchange diagrams beyond the bounds given by low energy experiments. However, it may well be that the suppression of FCNC’s is not relevant for the top quark, \( i.e., \) we may have appreciable renormalized coupling of \( t \) and \( c \) to neutral scalars. This is so because, up to date, no stringent experimental bounds for these direct FCNC couplings of the heavy top quarks exist.

The general 2HDM(III) framework has been first mentioned already in 1973 by T.D. Lee [2], for the hadronic sector and with the emphasis on the CP-violating phenomena originating from the nonzero relative phase of the two vacuum expectation values (VEV’s). Later on, the model has been investigated by several authors [2]-[11], [16], who mainly investigated the bounds on the low energy (\( i.e., \) renormalized) parameters in the scalar mass and in the Yukawa parameter sector – the bounds resulting from available phenomenological data on CP-violating phenomena, such as \( \varepsilon \) and \( \varepsilon’ \) parameters of the kaon physics, and the neutron electric dipole moment.

In 2HDM(III), specific ansätze for the FCNC Yukawa coupling parameters have been proposed by Cheng, Sher and Yuan (CSY) [1], [8], and by Antaramian, Hall and Rašin (AHR) [11]. These two groups of authors also investigated bounds on the masses of scalars and on FCNC Yukawa coupling parameters arising from available phenomenological data of FCNC phenomena, such as the \( K - \bar{K} \) and \( B - \bar{B} \) mass differences and rare \( B \) decays. The basic messages of their works are the following:

- their ansätze are reasonably natural (or, more cautiously: not “unnatural”) from the aspect of actual hierarchy of fermionic masses, since they are motivated to a large degree by this mass hierarchy.

- their ansätze allow the masses of neutral scalars to be as low as \( \sim 10^2 \) GeV while still not violating available (low energy) data of FCNC phenomena.

CSY ansatz is given explicitly in Section 2. Later on, several authors further investigated implications of the CSY, AHR and/or related ansätze, and of specific assumed ranges of

\footnote{A more precise expression would be “neutral flavor-changing scalar (Yukawa) coupling,” since these couplings have no four-vector current structure involving \( \gamma^\mu \).}

\footnote{Implicitly, all these coupling parameters are the renormalized (or: nearly renormalized) parameters, \( i.e., \) they are parameters at the low evolution energies since the authors investigated their tree level contributions to (low energy) FCNC phenomena for which phenomenological data are available.}
masses of scalars, for the FCNC phenomena measured presently or to be measured in various possible future colliders, such as: (tree level effects in) the decays $B \rightarrow \mu^+\mu^-$ \footnote{Ref. \cite{19} contains, in addition, an analysis of constraints from the electroweak $\rho$-parameter and from experimental data of various low energy processes such as $\Delta F = 2$ processes, $Z \rightarrow b\bar{b}$, rare B decays, $e^+e^- \rightarrow b\bar{b}$.}; one-loop processes $t \rightarrow c\gamma$ and $t \rightarrow cZ$ \footnote{Near the Landau pole of the top quark mass the theory starts behaving generally in a nonperturbative manner and the perturbative (one-loop) RGE’s start losing predictive power.}; two-loop effects in $\mu \rightarrow e\gamma$ \footnote{– additional to the low energy arguments of CSY and AHR.}; $H^0 \rightarrow tc$ \footnote{Hall and Weinberg \cite{12} pointed out that, while such 2HDM(III) models generically (naturally) suppress FCNC reaction rates to acceptable levels (e.g., by possessing approximate global flavor $U(1)$ symmetries leading to AHR-type of ansätze), they may in general give too much CP violation in the neutral kaon mass matrix. Therefore, they argued that these models must also possess CP as a good approximate symmetry.}; (tree level) process $\mu^+\mu^- \rightarrow H^0 \rightarrow tc$ \footnote{\cite{14}; gauge boson ($WW$ and $ZZ$) fusion processes $e^+e^- \rightarrow t\bar{c}\nu_\ell\ell$, $t\bar{c}e^+e^-$ \footnote{5}; rare decays $t \rightarrow cW^+W^-$, $cZZ$ \footnote{\cite{20}}.}; one-loop process $e^+e^- \rightarrow \gamma\gamma$, $Z^* \rightarrow tc$ \footnote{In the present work, we construct RGE’s for the Yukawa coupling parameters of quarks (and for quark masses) in the discussed 2HDM(III) and thus obtain a means for investigations of the high energy behavior of such theories. The main motivation for the latter investigations can be summarized in the following question: For which (if any) phenomenologically acceptable ansätze of the FCNC Yukawa coupling parameters at low energies do we have a reasonable behavior of these parameters at higher energies of evolution? Under the “reasonable” behavior we understand a rather tame evolution of these parameters as the energy of probes increases by several orders of magnitude. Stated otherwise, this is the requirement that these parameters do not increase by order(s) of magnitude in the region of the evolution energy which is not very close (on the logarithmic scale) to the Landau pole of the top quark mass. Therefore, the theory would not change qualitatively in the FCNC Yukawa sector up until the energies of probes where the framework starts behaving nonperturbatively due to the large “mass” Yukawa coupling of the top quark. Such a reasonable behavior would then provide us with additional arguments that the discussed 2HDM(III) frameworks, at least for some of the specific phenomenologically acceptable low energy choices of Yukawa parameters, are not unnatural.}

$$L_{\text{Y(III)}}^{(E)} = - \sum_{i,j=1}^{3} \left\{ \tilde{D}_{ij}^{(1)}(\tilde{q}_L^{(i)}\Phi^{(1)})\tilde{u}_R^{(j)} + \tilde{D}_{ij}^{(2)}(\tilde{q}_L^{(i)}\Phi^{(2)})\tilde{u}_R^{(j)} \right\} + \left\{ \tilde{U}_{ij}^{(1)}(\tilde{q}_L^{(i)}\Phi^{(1)})\tilde{d}_R^{(j)} + \tilde{U}_{ij}^{(2)}(\tilde{q}_L^{(i)}\Phi^{(2)})\tilde{d}_R^{(j)} + h.c. \right\} + \{ \Phi\ell-terms \} .$$ (1)

The tilde above the Yukawa coupling parameters and above the quark fields means that these quantities are in an arbitrary SU(2)$_L$-basis (not in the mass basis). The superscript $(E)$ for the Lagrangian density means that the theory has a finite effective energy cutoff $E$, and
the reference to this evolution energy $E$ was omitted at the fields and at the Yukawa coupling parameters in order to have simpler notation ($E \sim 10^2$ GeV for renormalized quantities). The following notations are used:

\[
\Phi^{(k)} = \left(\phi^{(k)+}_i, \phi^{(k)}_i\right) = \frac{1}{\sqrt{2}} \left(\phi^{(k)}_1 + i\phi^{(k)}_4, \phi^{(k)}_2 + i\phi^{(k)}_3\right), \quad \bar{\Phi}^{(k)} = i\tau_2 \Phi^{(k)*} = \frac{1}{\sqrt{2}} \left(\phi^{(k)}_1 - i\phi^{(k)}_4, -\phi^{(k)}_2 + i\phi^{(k)}_3\right),
\]

(2)

\[
\bar{q}^{(i)} = \left(\bar{u}^{(i)}, \bar{d}^{(i)}\right), \quad \bar{q}^{(1)} = \left(\bar{u}, \bar{d}\right), \quad \bar{q}^{(2)} = \left(\bar{c}, \bar{s}\right), \quad \bar{q}^{(3)} = \left(\bar{t}, \bar{b}\right),
\]

(3)

\[
\langle \Phi^{(1)} \rangle_0 = \frac{e^{i\eta_1}}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \quad \langle \Phi^{(2)} \rangle_0 = \frac{e^{i\eta_2}}{\sqrt{2}} \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \quad v_1^2 + v_2^2 = v^2.
\]

(4)

In (1), $v \equiv v(E)$ is the usual VEV needed for the electroweak symmetry breaking, i.e., $v(E_{\text{ew}}) \approx 246$ GeV. The phase difference $\eta \equiv \eta_2 - \eta_1$ between the two VEV’s in (1) may be nonzero; it represents CP violation originating from the purely scalar 2HD sector (cf. [22]). The leptonic sector has been omitted in (1).

We note that the popular “type I” and “type II” models are special cases (subsets) of this framework, with part of FCNC Yukawa coupling parameters being exactly zero

\[
2\text{HDM(I)}: \quad U^{(1)} = D^{(1)} = 0 \quad 2\text{HDM(II)}: \quad U^{(1)} = D^{(2)} = 0.
\]

(5)

In the 2HDM(II), the family symmetry in $\mathcal{L}_Y$ enforcing this complete FCNC Yukawa parameter suppression at all evolution energies $E$ is of U(1)-type: $d_R^{(j)} \to e^{i\alpha} d_R^{(j)}$, $\Phi^{(1)} \to e^{-i\alpha} \Phi^{(1)}$ $(j=1,2,3)$, the other fields remaining unchanged. This symmetry ensures that, in the course of renormalization, no loop-induced (ln $\Lambda$ cutoff-dependent) Yukawa couplings other than those of the form of the 2HDM(II) can appear. In the 2HDM(I), the family symmetry is similar: $d_R^{(j)} \to e^{i\alpha} d_R^{(j)}$, $u_R^{(j)} \to e^{-i\alpha} u_R^{(j)}$, $\Phi^{(1)} \to e^{-i\alpha} \Phi^{(1)}$. In contrast to type I and type II, in type III 2HD’s there is no exact (family) symmetry enforcing the complete suppression of the FCNC Yukawa coupling parameters. Stated otherwise, while FCNC Yukawa parameters in this general framework, at least those not involving the top quark, must be made quite small at low energies of probes $E \sim E_{\text{ew}}$, they in general may increase when the energy of probes $E$ increases. If they increase by order(s) of magnitude in the energy region well below the Landau pole, then such models should be regarded as rather unnatural – their behavior in the FCNC Yukawa sector is then drastically different at higher energies of probes (not just very near the Landau pole) from the behavior at low (electroweak) energies.

In Section 2, we discuss relations between various notations for the scalar isodoublets and between various bases of the quark fields, and the ensuing changes in the representation of the Yukawa coupling parameters – in order to highlight the suppression conditions imposed at low evolution energies on the FCNC Yukawa coupling parameters of the 2HDM(III). In Section 3, we derive one-loop RGE’s of the Yukawa coupling parameters and of the scalar fields (and hence of their VEV’s) in the described 2HDM(III) model, with the purpose of investigating the behavior of the framework at evolving energies of probes $E$. In Section 4, we show and discuss one typical numerical example of the evolution of Yukawa coupling parameters in this general framework, in particular the evolution of the FCNC Yukawa parameters. The values of these parameters at low energies ($E \sim E_{\text{ew}}$) were chosen according to the CSY ansatz, and therefore they fulfill the FCNC suppression restrictions discussed in this Introduction and in Section 2. Section 5 is a summary of the results and conclusions.
2 Conditions of FCNC suppression at low evolution energies

The Lagrangian density (1) can be written in a form more convenient for consideration of FCNC Yukawa coupling parameters by redefining the scalar isodoublets in the following way:

\[
\Phi'^{(1)} = (\cos \beta) \Phi^{(1)} + (\sin \beta) e^{-i \eta} \Phi^{(2)}, \\
\Phi'^{(2)} = -(\sin \beta) \Phi^{(1)} + (\cos \beta) e^{-i \eta} \Phi^{(2)},
\]

(6)

where

\[
\eta = \eta_2 - \eta_1
\]

(7)

and

\[
\tan \beta = \frac{v_2}{v_1} \Rightarrow \cos \beta = \frac{v_1}{v}, \quad \sin \beta = \frac{v_2}{v}.
\]

(8)

Therefore, the VEV’s of the redefined scalar isodoublets are

\[
e^{-i \eta} \langle \Phi'^{(1)} \rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \langle \Phi'^{(2)} \rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

(9)

The isodoublet \( \Phi'^{(1)} \) is therefore responsible for the masses of the quarks, and \( \Phi'^{(2)} \) with its couplings to the quarks is responsible for the FCNC couplings, as will be seen below. The original Yukawa Lagrangian density (1) of 2HDM(III) can then be rewritten in terms of these redefined scalar fields as

\[
L^{(E)}_{Y(III)} = - \sum_{i,j=1}^{3} \left\{ \tilde{G}^{(D)}_{ij} (\bar{q}^{(i)}_L \Phi'^{(1)}) \bar{d}^{(j)}_R + \tilde{G}^{(U)}_{ij} (\bar{q}^{(i)}_L \Phi'^{(1)}) \bar{u}^{(j)}_R + \text{h.c.} \right\}
- \sum_{i,j=1}^{3} \left\{ \tilde{D}^{(D)}_{ij} (\bar{q}^{(i)}_L \Phi'^{(2)}) \bar{d}^{(j)}_R + \tilde{U}^{(U)}_{ij} (\bar{q}^{(i)}_L \Phi'^{(2)}) \bar{u}^{(j)}_R + \text{h.c.} \right\},
\]

(10)

where the Yukawa matrices \( \tilde{G}^{(U)} \) and \( \tilde{G}^{(D)} \) are rescaled mass matrices, and \( \tilde{U} \) and \( \tilde{D} \) the corresponding “complementary” Yukawa matrices, in an (arbitrary) \( SU(2)_L \)-basis

\[
\tilde{G}^{(U)} = \frac{\sqrt{2}}{v} \tilde{M}^{(U)} = (\cos \beta) \tilde{U}^{(1)} + (\sin \beta) e^{-i \eta} \tilde{U}^{(2)}, \\
\tilde{G}^{(D)} = \frac{\sqrt{2}}{v} \tilde{M}^{(D)} = (\cos \beta) \tilde{D}^{(1)} + (\sin \beta) e^{+i \eta} \tilde{D}^{(2)};
\]

(11)

\[
\tilde{U} = -(\sin \beta) \tilde{U}^{(1)} + (\cos \beta) e^{-i \eta} \tilde{U}^{(2)}, \\
\tilde{D} = -(\sin \beta) \tilde{D}^{(1)} + (\cos \beta) e^{+i \eta} \tilde{D}^{(2)}.
\]

(12)

By a biunitary transformation involving unitary matrices \( V^U_L, V^U_R, V^D_L \) and \( V^D_R \), the Yukawa parameters can be expressed in the mass basis of the quarks, where the (rescaled) mass

\[
\text{energy } E \text{ at the quark fields, at the scalar fields and their VEV’s and at the Yukawa coupling parameters.}
\]

---

\[5\] Throughout this Section we omit, for simpler notation, reference to the arbitrary evolution (cutoff) energy \( E \) at the quark fields, at the scalar fields and their VEV’s and at the Yukawa coupling parameters.
matrices \( G^{(U)} \) and \( G^{(D)} \) are diagonal and real

\[
G^{(U)} = \frac{\sqrt{v}}{v} M^{(U)} = V_L^U \bar{G}^{(U)} V_R^{U\dagger}, \quad M^{(U)}_{ij} = \delta_{ij} m_i^{(u)}; \\
U = V_L^U \bar{U} V_R^{U\dagger}; \\
G^{(D)} = \frac{\sqrt{2}}{v} M^{(D)} = V_L^D \bar{G}^{(D)} V_R^{D\dagger}, \quad M^{(D)}_{ij} = \delta_{ij} m_i^{(d)}; \\
D = V_L^D \bar{D} V_R^{D\dagger};
\]

(13)

(14)

\[
u_L = V_L^U \bar{u}_L, \quad u_R = V_R^U \bar{u}_R, \quad d_L = V_L^D \bar{d}_L, \quad d_R = V_R^D \bar{d}_R.
\]

(15)

The lack of tildes above the Yukawa coupling parameters and above the quark fields means that these quantities are in the quark mass basis (at a given evolution energy \( E \)). The Lagrangian density \( \mathcal{L}_{\text{Y(III)}} \) can be written now in the quark mass basis. The neutral current part of the Lagrangian density in the quark mass basis is

\[
\mathcal{L}_{\text{Y(III) neutral}}^{\text{(E)}} = -\frac{1}{\sqrt{2}} \sum_{i,j=1}^{3} \left\{ G^{(D)}_{ij} \bar{d}_L^{(i)} d_R^{(j)} (\phi_3^{(1)} + i \phi_4^{(1)}) + \\
+ G^{(U)}_{ij} \bar{u}_L^{(i)} u_R^{(j)} (\phi_3^{(1)} - i \phi_4^{(1)}) + \text{h.c.} \right\} \\
- \frac{1}{\sqrt{2}} \sum_{i,j=1}^{3} \left\{ D_{ij} \bar{\phi}_L^{(i)} \phi_R^{(j)} (\phi_3^{(2)} + i \phi_4^{(2)}) + \\
+ U_{ij} \bar{\phi}_L^{(i)} \phi_R^{(j)} (\phi_3^{(2)} - i \phi_4^{(2)}) + \text{h.c.} \right\}.
\]

(16)

On the other hand, the charged current part of the Lagrangian density in the quark mass basis is

\[
\mathcal{L}_{\text{Y(III) charged}}^{\text{(E)}} = -\frac{1}{\sqrt{2}} \sum_{i,j=1}^{3} \left\{ (V^\dagger G^{(D)})_{ij} \bar{d}_L^{(i)} d_R^{(j)} (\phi_1^{(1)} + i \phi_2^{(1)}) - \\
-(V^\dagger G^{(U)})_{ij} \bar{u}_L^{(i)} u_R^{(j)} (\phi_1^{(1)} - i \phi_2^{(1)}) + \text{h.c.} \right\} \\
- \frac{1}{\sqrt{2}} \sum_{i,j=1}^{3} \left\{ (V D)_{ij} \bar{d}_L^{(i)} d_R^{(j)} (\phi_1^{(2)} + i \phi_2^{(2)}) - \\
-(V^\dagger U)_{ij} \bar{u}_L^{(i)} u_R^{(j)} (\phi_1^{(2)} - i \phi_2^{(2)}) + \text{h.c.} \right\}.
\]

(17)

Here, we denoted by \( V \) the Cabibbo-Kobayashi-Maskawa (CKM) matrix

\[
V \equiv V_{\text{CKM}} = V_L^U V_L^{D\dagger}.
\]

(18)

We see from \( \text{(13)} \) that the \( U \) and \( D \) matrices, as defined by \( \text{(12)} \) and \( \text{(13)-(14)} \) through the original Yukawa matrices \( \bar{U}^{(j)} \) and \( \bar{D}^{(j)} \) of the 2HDM(III) Lagrangian density \( \text{(1)} \), allow the model to possess in general scalar-mediated FCNC’s. Namely, in the quark mass basis only the (rescaled) quark mass matrices \( G^{(U)} \) and \( G^{(D)} \) of \( \text{(13)-(14)} \) [cf. also \( \text{(14)} \)] are diagonal, but the matrices \( U \) and \( D \) in this general framework are in general not diagonal

\[
\mathcal{L}_{\text{Y(III) FCNC}}^{\text{(E)}} = -\frac{1}{2\sqrt{2}} \sum_{i,j=1}^{3} \left[ (D + D^\dagger)_{ij} \left( \bar{d}_L^{(i)} d_R^{(j)} \right) \phi_3^{(2)} + i \left( D - D^\dagger \right)_{ij} \left( \bar{d}_L^{(i)} d_R^{(j)} \right) \phi_4^{(2)} \right].
\]
\[ + (D + D^\dagger)_{ij} \left( \bar{d}^{(i)} i\gamma_5 d^{(j)} \right) \phi_4^{(2)} - i \left( D - D^\dagger \right)_{ij} \left( \bar{d}^{(i)} i\gamma_5 d^{(j)} \right) \phi_3^{(2)} \]
\[- \frac{1}{2\sqrt{2}} \sum_{i,j=1}^{3} \left[ \left( U + U^\dagger \right)_{ij} \left( \bar{u}^{(i)} u^{(j)} \right) \phi_3^{(2)} - i \left( U - U^\dagger \right)_{ij} \left( \bar{u}^{(i)} u^{(j)} \right) \phi_4^{(2)} \right] \]
\[- \left( U + U^\dagger \right)_{ij} \left( \bar{u}^{(i)} i\gamma_5 u^{(j)} \right) \phi_4^{(2)} - i \left( U - U^\dagger \right)_{ij} \left( \bar{u}^{(i)} i\gamma_5 u^{(j)} \right) \phi_3^{(2)} \]. \quad (19)

It should be noted that the original four Yukawa matrices \( \tilde{U}^{(j)} \) and \( \tilde{D}^{(j)} \) \((j = 1, 2)\) in an \( SU(2)_L \)-basis are already somewhat constrained by the requirement that (at low energy) the squares of the linear combinations\(^7\) \( M^{(U)} \) and \( M^{(D)} \) are diagonalized by unitary transformations involving such unitary matrices \( V_L^U \) and \( V_L^D \), respectively, which are related to each other by \( V_L^U V_L^{D\dagger} = V \). Here, \( V \) is the CKM matrix which is, for any specific chosen phase convention, more or less known at low energies.

In order to have at low evolution energies \( E \sim E_{ew} \) a phenomenologically viable suppression of the scalar-mediated FCNC’s, the authors Cheng, Sher and Yuan (CSY) \(^8\) basically argued that the elements of the \( U \) and \( D \) matrices (in the quark mass basis and at low evolution energies \( E \)) should have the form:

\[ U_{ij}(E) = \xi_{ij}^{(u)} \frac{\sqrt{2}}{v} \sqrt{m_i^{(u)} m_j^{(u)}} \], \quad D_{ij}(E) = \xi_{ij}^{(d)} \frac{\sqrt{2}}{v} \sqrt{m_i^{(d)} m_j^{(d)}} \], \quad (20)

where
\[ \xi_{ij}^{(u)}, \xi_{ij}^{(d)} \sim 1 \] for \( E \sim E_{ew} \). \quad (21)

This form is in general phenomenologically acceptable and is motivated basically only by the actual mass hierarchies of quarks and the requirement that there is [at a given low energy of evolution \( \sim E_{ew} \)] no fine-tuning in which large Yukawa terms \( U_{jk}^{(i)} \) (and: \( D_{jk}^{(i)} \)) add together via \((12)\) to make small terms \( U_{jk} \) (\( D_{jk} \)).\(^9\) Therefore, this (CSY) form is considered to be reasonably natural. Similar (but not identical) ansätze have been proposed by the authors of \([11]\) (AHR), motivated by their requirement that the Yukawa interactions have approximate \( U(1) \) flavor symmetries.

From the CSY ansatz \((20)-(21)\) we see that the scalar-mediated FCNC vertices involving the heavy top quark are the only ones that are not strongly suppressed (at low evolution energies), since, as mentioned in the Introduction, FCNC processes involving the top quark vertices (not loops) are not constrained by present experiments. Later in Section 4 we will use low energy conditions \((20)-(24)\) for a numerical example of RGE flow of FCNC Yukawa coupling parameters.

### 3 Renormalization Group Equations (RGE’s) in the general 2HDM

\(^7\)Strictly speaking, the following “squares”: \( M^{(U)} M^{(U)} \) and \( M^{(D)} M^{(D)} \).

\(^8\)To visualize this point, Eqs. \((11)\) and \((12)\) should be inspected, but this time in the quark mass basis \((i.e., \text{no tildes over the matrices})\). Then \((11)\) would suggest that \( U_{jk}^{(i)} (D_{jk}^{(i)}) \) is in general non-diagonal and of the order of \( \sqrt{m_j^{(u)} m_k^{(u)}} / v \left( \sqrt{m_j^{(d)} m_k^{(d)}} / v \right) \). As a result, \((12)\) would suggest that \( U_{jk} (D_{jk}) \) is also of that order of magnitude, unless there is some peculiar fine-tuning on the right-hand side (RHS) of \((12)\). For the complete suppression of FCNC Yukawa couplings \( U_{jk} = 0 = D_{jk} \) for \( j \neq k \) we would need fine-tuning on the RHS of \((12)\).
3.1 RGE’s for the scalar fields

Here we present a derivation of the one-loop RGE’s for the scalar fields (2). The one-loop RGE’s for the Yukawa coupling matrices $\tilde{D}^{(k)}$, $\tilde{U}^{(k)}$ ($k = 1, 2$) will be presented in the next Subsection. In both derivations we will follow the finite-cutoff interpretation of RGE’s as discussed, for example, by Lepage [23].

In order to calculate evolution of the scalar fields $\phi_i^{(k)}(E)$ with “cutoff” energy $E$, we need to calculate first the truncated (one-loop) two-point Green functions $-i \Sigma_{ij}^{(k,\ell)}(p^2; E^2)$ represented diagrammatically in Fig. 1. More specifically, we calculate their cutoff-dependent parts $\propto p^2 \ln E^2$ which will be ultimately responsible for the effective kinetic-energy-type terms $\sim \partial_\nu \phi_i^{(k)}(E) \partial^\nu \phi_j^{(\ell)}(E)$. In the course of the calculations, we ignore all the masses $m \sim E_{ew}$ of the relevant particles in the diagram. This would be consistent with the picture of a framework with a finite but large ultraviolet energy cutoff $E \gg E_{ew}$. For this reason, we don’t have to work in the mass basis of the relevant particles – these particles are regarded as effectively massless in the approximation, the transformations between the original bases of the relevant fields and their mass bases are unitary, and therefore the (mass-independent parts of the) calculated Green functions are the same in both bases.

Calculation of the mentioned two-point Green functions $-i \Sigma_{ij}^{(k,\ell)}(p^2; E^2)$, whose external (scalar) legs $\phi_i^{(k)}$ and $\phi_j^{(\ell)}$ are truncated, is in the mentioned framework rather straightforward. The relevant (massless) integrals over internal quark-loop momenta $q$ can be carried out, for example, in Euclidean metric $[\bar{q} = (-iq^0, -q^i), \bar{p} = (-ip^0, -p^i)]$, where the upper bound in the loop integral is: $\bar{q}^2 \leq E^2$. After rotating back into Minkowski metric ($\bar{p}^2 \mapsto -p^2$), we end up with the following results:

1. Green functions with the external legs $\phi_i^{(k)}$ and $\phi_j^{(\ell)}$ having the same scalar indices $(i = j)$:

$$-i \Sigma_{ij}^{(k,\ell)}(p^2; E^2) = i\frac{\kappa}{2} p^2 \ln \left( \frac{E^2}{m^2} \right) \times \text{Tr} \left[ \tilde{U}^{(k)} \tilde{U}^{(\ell)\dagger} + \tilde{\bar{D}}^{(k)} \tilde{\bar{D}}^{(\ell)\dagger} \right](E), (22)$$

\footnote{For clearer notation, we denote in this Section the evolving (UV cutoff) energy $E$ at the fields not as a superscript, but rather as an argument.}

\footnote{Throughout this Section, we implicitly ignore the cutoff-independent parts of the Green functions, since these parts are irrelevant for the mass-independent ($\overline{\text{MS}}$-type of) RGE framework considered here.}
where \( j = 1, 2, 3, 4 \) (no running over the repeated indices \( j \)); \( k, \ell = 1, 2; \) \( m \) is an arbitrary but fixed mass of the order of electroweak scale \( (m \sim E_{ew}) \); and \( \kappa \) stands for

\[
\kappa \equiv \frac{N_c}{16\pi^2} .
\]

2. Green functions with the external legs \( \phi_{i}^{(k)} \) and \( \phi_{j}^{(\ell)} \) having different scalar indices \( (i \neq j) \):

\[
-i \Sigma_{1,2}^{(1,2)}(p^2; E^2) = -i \Sigma_{3,4}^{(1,2)}(p^2; E^2)
\]

\[
= -\frac{\kappa}{2}p^2 \ln \left( \frac{E^2}{m^2} \right) \text{Tr} \left[ \tilde{U}^{(1)}\tilde{U}^{(2)\dagger} - \tilde{U}^{(2)}\tilde{U}^{(1)\dagger} - \tilde{D}^{(1)}\tilde{D}^{(2)\dagger} + \tilde{D}^{(2)}\tilde{D}^{(1)\dagger} \right] (E) .
\]  

(23)

The above Green functions are antisymmetric under the exchange of the Higgs generation indices \( k \neq \ell \), and also under the exchange of the scalar indices \( i \neq j \)

\[
-i \Sigma_{1,2}^{(1,2)} = +i \Sigma_{1,2}^{(2,1)} = -i \Sigma_{2,1}^{(2,1)} = +i \Sigma_{2,1}^{(1,2)}
\]

\[
= -i \Sigma_{3,4}^{(1,2)} = +i \Sigma_{3,4}^{(2,1)} = -i \Sigma_{4,3}^{(2,1)} = +i \Sigma_{4,3}^{(1,2)} .
\]

(24)

3. The other Green functions are zero

\[
-i \Sigma_{i,j}^{(k,\ell)} = 0
\]

(25)

for \( i = 1, 2 \) and \( j = 3, 4; \) for \( i = 3, 4 \) and \( j = 1, 2; \) for \( i \neq j \) and \( k = \ell \).

All these Green functions can be induced at the tree level by kinetic energy terms. For example, in theory with the UV cutoff \( E \), the kinetic energy term \( \partial_{\nu}\phi_{i}^{(k)}(E)\partial_{\nu}\phi_{j}^{(\ell)}(E) \) induces (at the tree level) the two-point Green function value \( -i \Sigma_{i,j}^{(k,\ell)}(p^2; E^2) = ip^2 \) if \( \phi_{i}^{(k)}(E) \neq \phi_{j}^{(\ell)}(E) \), and the value \( 2ip^2 \) if \( \phi_{i}^{(k)}(E) \equiv \phi_{j}^{(\ell)}(E) \). Now, following the finite-cutoff interpretation of RGE’s as described, for example, by Lepage [23], we compare the kinetic energy terms in the theory with the UV cutoff \( E \) and the equivalent theory with the slightly different cutoff \( (E + dE) \). The two-point Green functions in these two equivalent theories must be identical. When imposing this requirement in the tree + one-loop approximation, this leads to the following relation:

\[
\frac{1}{2} \sum_{j=1}^{4} \partial_{\nu} \left[ \phi_{j}^{(1)}, \phi_{j}^{(2)} \right] (E) \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \partial_{\nu} \left[ \phi_{j}^{(1)}, \phi_{j}^{(2)} \right] (E) =
\]

\[
\frac{1}{2} \sum_{j=1}^{4} \partial_{\nu} \left[ \phi_{j}^{(1)}, \phi_{j}^{(2)} \right] (E + dE) \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \partial_{\nu} \left[ \phi_{j}^{(1)}, \phi_{j}^{(2)} \right] (E + dE)
\]

\[
+ \frac{\kappa}{2} (d\ln E^2) \sum_{j=1}^{4} \partial_{\nu} \left[ \phi_{j}^{(1)}, \phi_{j}^{(2)} \right] (E) \left[ A_{11}(E) \ A_{12}(E) \ A_{21}(E) \ A_{22}(E) \right] \partial_{\nu} \left[ \phi_{j}^{(1)}, \phi_{j}^{(2)} \right] (E)
\]

\[
+ \frac{\kappa}{2} (d\ln E^2) \sum_{(i,j)} (-1)^i \partial_{\nu} \left[ \phi_{i}^{(1)}, \phi_{j}^{(2)} \right] (E) \left[ \begin{array}{cc} 0 & B_{12}(E) \\ B_{21}(E) & 0 \end{array} \right] \partial_{\nu} \left[ \phi_{i}^{(1)}, \phi_{j}^{(2)} \right] (E) ,
\]

(26)

where: the summation in the last sum runs over \( (i,j) = (1,2), (2,1), (3,4), (4,3); \) \( d\ln E^2 \equiv \ln(E + dE)^2 - \ln E^2 = 2dE/E; \) and the real matrix elements \( A_{kl} \) and \( B_{kl} \) are related to the
one-loop two-point Green function expressions (22) and (23)-(25), respectively:

\[ A_{kl}(E) = \frac{1}{2} \text{Tr} \left[ \tilde{U}^{(k)} \tilde{U}^{(l)} + \tilde{U}^{(l)} \tilde{U}^{(k)} + \tilde{D}^{(k)} \tilde{D}^{(l)} + \tilde{D}^{(l)} \tilde{D}^{(k)} \right](E) \],

\[ B_{12}(E) = \frac{i}{2} \text{Tr} \left[ \tilde{U}^{(1)} \tilde{U}^{(2)} - \tilde{U}^{(2)} \tilde{U}^{(1)} - \tilde{D}^{(1)} \tilde{D}^{(2)} + \tilde{D}^{(2)} \tilde{D}^{(1)} \right](E) = B_{21}(E) \].

Equation (22) is described in the following way: the sum on the left-hand side (LHS) and the first sum on the RHS represent the full kinetic energy terms of the scalars in the formulation with the UV cutoff \( E \) and \( (E + dE) \), respectively. The one-loop contributions of Fig. 1 with the loop momentum \( |\vec{q}| \) in the energy interval \( E \leq |\vec{q}| \leq \Lambda \) are already contained in the kinetic energy terms of the LHS effectively at the tree level (\( \Lambda \) is a large cutoff where the theory is presumed to break down). On the other hand, the kinetic energy terms of the \( (E + dE) \) cutoff formulation [the first sum on the RHS of (26)] effectively contain, at the tree level, the one-loop effects of Fig. 1 for the slightly smaller energy interval: \( (E + dE) \). This is illustrated in Fig. 2.

Figure 2: Diagrammatic illustration of the RGE relation (21) leading to the evolution of the scalar fields. \( \phi^{(k)}_i \) stands for \( \phi_i^{(k)}(E) \), and \( d\phi \) stands for \( \phi(E + dE) - \phi(E) \) (\( \phi \) is a generic notation for \( \phi^{(i)} \)'s). The cross represents the contribution of the change of the kinetic energy terms originating from the changes \( d\phi \) of scalar fields.

In order to find RGE’s for the scalar fields \( \phi_i^{(k)}(E) \), we make the following ansatz for a solution of Eq. (26):

\[ \tilde{\phi}_i(E + dE) \equiv \left[ \phi_i^{(1)} \phi_i^{(2)} \right](E + dE) = \tilde{\phi}_i(E) + d\alpha(i; E)\tilde{\phi}_i(E) + d\beta(i; E)\tilde{\phi}'(E) \],

where \( d\alpha(i; E) \) and \( d\beta(i; E) \) are infinitesimally small \( 2 \times 2 \) matrices

\[ d\alpha(i; E) \equiv \begin{bmatrix} d\alpha_{11} & d\alpha_{12} \\ d\alpha_{21} & d\alpha_{22} \end{bmatrix} \quad d\beta(i; E) \equiv \begin{bmatrix} 0 & d\beta_{12} \\ d\beta_{21} & 0 \end{bmatrix} \],

and the index \( i' \) in (29) is complementary to index \( i \):

\[ i = 1 \mapsto i' = 2, \quad i = 2 \mapsto i' = 1; \quad i = 3 \mapsto i' = 4, \quad i = 4 \mapsto i' = 3. \]

Inserting ansatz (23) into the RGE relation (26), we end up with the following set of relations:

\[ d\alpha_{kl}(i; E) + d\alpha_{lk}(i; E) = -\kappa \left( d\ln E^2 \right) A_{kl}(E), \]

11 More precisely: the corresponding effective kinetic energy terms.
In principle, these relations alone do not define the elements \( d\alpha_{kl}(i; E) \) and \( d\beta_{kl}(i; E) \). However, there is another requirement that should be imposed on these transformation coefficients: the resulting RGE evolution of the isodoublet fields \( \Phi^{(1)}(E) \) and \( \Phi^{(2)}(E) \) should be invariant under the exchange of Higgs generation indices 1 \( \leftrightarrow \) 2, because these two Higgs doublets appear in the original Lagrangian density \( [\Pi] \) in a completely 1 \( \leftrightarrow \) 2 symmetric manner. We will see in retrospect that this discrete symmetry is respected once we impose the conditions

\[
d\alpha_{kl}(i; E) = d\alpha_{lk}(i; E),
\]

\[
d\beta_{12}(1; E) = d\beta_{21}(2; E), \quad d\beta_{12}(2; E) = d\beta_{21}(1; E),
\]

\[
d\beta_{12}(3; E) = d\beta_{21}(4; E), \quad d\beta_{12}(4; E) = d\beta_{21}(3; E).
\]

Solutions (31)-(32) of the RGE condition (26), together with the symmetry conditions (33)-(34), lead to specific expressions for the evolution coefficients \( d\alpha_{kl}(i; E) \) and \( d\beta_{kl}(i; E) \). When inserting these coefficients back into the scalar field evolution ansatz (29), we end up with the following RGE’s for the evolution of the scalar fields:

\[
\frac{16\pi^2}{N_c} \frac{d}{d\ln E} \phi_j^{(1)}(E) = -\text{Tr} \left[ \tilde{U}^{(1)} \tilde{U}^{(1)\dagger} + \tilde{D}^{(1)} \tilde{D}^{(1)\dagger} \right] \phi_j^{(1)}
\]

\[-\frac{1}{2} \text{Tr} \left[ \tilde{U}^{(1)} \tilde{U}^{(2)\dagger} + \tilde{U}^{(2)} \tilde{U}^{(1)\dagger} + \tilde{D}^{(1)} \tilde{D}^{(2)\dagger} + \tilde{D}^{(2)} \tilde{D}^{(1)\dagger} \right] \phi_j^{(2)}
\]

\[+i(-1)^{j_1} \frac{1}{2} \text{Tr} \left[ \tilde{U}^{(1)} \tilde{U}^{(2)\dagger} - \tilde{U}^{(2)} \tilde{U}^{(1)\dagger} - \tilde{D}^{(1)} \tilde{D}^{(2)\dagger} + \tilde{D}^{(2)} \tilde{D}^{(1)\dagger} \right] \phi_{j_2}^{(2)},
\]

\[
\frac{16\pi^2}{N_c} \frac{d}{d\ln E} \phi_j^{(2)}(E) = -\text{Tr} \left[ \tilde{U}^{(2)} \tilde{U}^{(2)\dagger} + \tilde{D}^{(2)} \tilde{D}^{(2)\dagger} \right] \phi_j^{(2)}
\]

\[-\frac{1}{2} \text{Tr} \left[ \tilde{U}^{(1)} \tilde{U}^{(2)\dagger} + \tilde{U}^{(2)} \tilde{U}^{(1)\dagger} + \tilde{D}^{(1)} \tilde{D}^{(2)\dagger} + \tilde{D}^{(2)} \tilde{D}^{(1)\dagger} \right] \phi_j^{(1)}
\]

\[+i(-1)^{j_1} \frac{1}{2} \text{Tr} \left[ \tilde{U}^{(1)} \tilde{U}^{(2)\dagger} - \tilde{U}^{(2)} \tilde{U}^{(1)\dagger} - \tilde{D}^{(1)} \tilde{D}^{(2)\dagger} + \tilde{D}^{(2)} \tilde{D}^{(1)\dagger} \right] \phi_{j_2}^{(1)},
\]

where again \( j’ \) is the scalar index complementary to index \( j \): \( (j, j’) = (1, 2), (2, 1), (3, 4), (4, 3) \). These RGE’s lead to RGE’s for scalar isodoublets \( \Phi^{(k)} \):

\[
\frac{16\pi^2}{N_c} \frac{d}{d\ln E} \Phi^{(k)}(E) = -\sum_{\ell=1}^2 \text{Tr} \left[ \tilde{U}^{(k)} \tilde{U}^{(\ell)\dagger} + \tilde{D}^{(k)} \tilde{D}^{(\ell)\dagger} \right] \Phi^{(\ell)}.
\]

We see indeed that this set of one-loop RGE’s is invariant under the exchange 1 \( \leftrightarrow \) 2, as required by the form of the Yukawa Lagrangian density \( [\Pi] \) of the 2HDM(III).

In addition to quark loops, there are also loops of the electroweak gauge bosons contributing to one-loop two-point Green functions of the scalars. However, since these gauge bosons couple to the Higgs isodoublets identically as in the minimal Standard Model (MSM), their contributions to the RHS of RGE’s (33)-(34) and (37) are the same as in the MSM. Consequently, the full one-loop RGE’s for the evolution of the scalar isodoublets in 2HDM(III)

\[\text{[12] For these contributions of EW gauge bosons in the MSM, see for example Arason et al. [24], App. A. However, note that they use for the } U(1)_{Y'} \text{ gauge coupling } g_1 \text{ a different, GUT-motivated, convention: } (g_1^2)_{\text{Arason et al.}} = (5/3)(g_1^2)_{\text{here}}.\]
\[ 16\pi^2 \frac{d}{d\ln E} \Phi^{(k)}(E) = -N_c \sum_{\ell=1}^{2} \text{Tr} \left[ \tilde{U}^{(k)} \tilde{U}^{(\ell)\dagger} + \tilde{D}^{(\ell)} \tilde{D}^{(k)\dagger} \right] \Phi^{(\ell)} \\
+ \left[ \frac{3}{4} g_1^2(E) + \frac{9}{4} g_2^2(E) \right] \Phi^{(k)}(E). \] 

These RGE’s are simultaneously also RGE’s for the corresponding VEV’s \(\Phi^{(\ell)}\) \((\ref{VEV})\):

\[ 16\pi^2 \frac{d}{d\ln E} \left( e^{i\eta v_k} \right) = -N_c \sum_{\ell=1}^{2} \text{Tr} \left[ \tilde{U}^{(k)} \tilde{U}^{(\ell)\dagger} + \tilde{D}^{(\ell)} \tilde{D}^{(k)\dagger} \right] \left( e^{i\eta v_{\ell}} \right) \\
+ \left[ \frac{3}{4} g_1^2(E) + \frac{9}{4} g_2^2(E) \right] \left( e^{i\eta v_k} \right). \] 

In this paper we don’t discuss the question of quadratic cutoff terms \(\Lambda^2\) which appear in the radiative corrections to VEV’s in any SM framework. In the MSM, their consideration – under the assumption of the top quark dominance of the radiative corrections in the scalar sector – leads to severe upper bounds on the ultraviolet cutoff \(\Lambda\) for a substantial subset of values of the bare doublet mass and of the bare scalar self-interaction parameters \(M^2(\Lambda)\) and \(\lambda(\Lambda)\) – cf. Ref. \([25]\).

### 3.2 RGE’s for the Yukawa matrices

In order to derive one-loop RGE’s for the Yukawa matrices \(\tilde{U}^{(k)}\) and \(\tilde{D}^{(k)}\), we will need the results of the previous Subsection concerning evolution of the scalar fields. In addition, we will need evolution of the quark fields \(\tilde{u}^{(j)}_{L,R}\) and \(\tilde{d}^{(j)}_{L,R}\). The latter can be derived in close analogy with the derivation of the evolution of scalar fields of the previous Subsection. Now, the diagrams (Green functions) of Figs. 1 and 2 are replaced by those of Figs. 3 and 4, and the scalar field kinetic energy terms in \((\ref{G1})\) are replaced by those of the quark fields. The one-loop two-point Green function of Fig. 3, with the incoming \(\tilde{u}^{(i)}\) and outgoing \(\tilde{u}^{(j)}\) of momentum \(p\), in the framework with UV cutoff \(E\), is

\[ -i \Sigma(p; E; \tilde{u}^{(i)}, \tilde{u}^{(j)}) = \frac{i}{64\pi^2} \ln \left( \frac{E^2}{m^2} \right) \delta^2 \left\{ (1 - \gamma_5) \sum_{\ell=1}^{2} \left[ \tilde{U}^{(\ell)} \tilde{U}^{(\ell)\dagger} + \tilde{D}^{(\ell)} \tilde{D}^{(\ell)\dagger} \right]_{ji} \\
+ 2 (1 + \gamma_5) \sum_{\ell=1}^{2} \left[ \tilde{U}^{(\ell)} \tilde{U}^{(\ell)\dagger} \right]_{ji} \right\}. \]
The Green function with the incoming \( \tilde{d}^{(i)} \) and outgoing \( \tilde{d}^{(j)} \) of momentum \( p \) is obtained from the above expression by simply exchanging \( \tilde{U}^{(i)} \leftrightarrow \tilde{D}^{(i)} \) and \( \tilde{U}^{(i)\dagger} \leftrightarrow \tilde{D}^{(i)\dagger} \). We now make the ansatz for the running of the quark fields

\[
d\tilde{u}^{(k)}(E)_{L,R}\left\{\equiv \tilde{u}^{(k)}(E + dE)_{L,R} - \tilde{u}^{(k)}(E)_{L,R}\right\} = df_u(E)^{(L,R)}_{k\ell} \tilde{u}^{(\ell)}(E)_{L,R}, \tag{41}
\]

\[
d\tilde{d}^{(k)}(E)_{L,R}\left\{\equiv \tilde{d}^{(k)}(E + dE)_{L,R} - \tilde{d}^{(k)}(E)_{L,R}\right\} = df_d(E)^{(L,R)}_{k\ell} \tilde{d}^{(\ell)}(E)_{L,R}, \tag{42}
\]

where subscripts \( L, R \) denote handedness of the quark fields: \( \tilde{q}_L \equiv (1 - \gamma_5)\tilde{q}/2, \tilde{q}_R \equiv (1 + \gamma_5)\tilde{q}/2 \) (\( \tilde{q} = \tilde{u}^{(k)}, \tilde{d}^{(k)} \)). In complete analogy with the previous Subsection, we obtain from these ansätze and from the RGE relation illustrated in Fig. 4\textsuperscript{13} the following relations

\[
\sum_{ijL,R} \left[ \int_{E - |\tilde{q}| < E + dE} \frac{4}{d\tilde{q}} (2\pi)^4 \left(\begin{array}{c} \tilde{u}^{(i)} \\
\tilde{u}^{(j)} \end{array}\right) \left(\begin{array}{c} \tilde{q} \\
\tilde{p} - \tilde{q} \\
\tilde{p} \end{array}\right) + \left(\begin{array}{c} \tilde{u}^{(i)} \\
\tilde{u}^{(j)} \end{array}\right) \left(\begin{array}{c} \tilde{p} \tilde{d} \tilde{u} \\
\tilde{p} \end{array}\right) \right] = 0
\]

Figure 4: Diagrammatic illustration of the RGE relation leading to the evolution of quark fields. Physically, this relation means that the two-point Green functions with truncated external (quark) legs, at one-loop level, are the same in the theory with \( E \) cutoff and in the theory with \( E + dE \) cutoff. Conventions are the same as in previous figures.

for the quark field evolution matrices \( df_u \):

\[
df_u(E)^{(L)}_{ij} + df_u(E)^{(L)}_{ji} = -\frac{1}{32\pi^2} (d\ln E^2) \sum_{k=1}^{2} \left[ \tilde{U}^{(k)\dagger} \tilde{U}^{(k)} + \tilde{D}^{(k)\dagger} \tilde{D}^{(k)} \right]_{ji} (E), \tag{43}
\]

\[
df_u(E)^{(R)}_{ij} + df_u(E)^{(R)}_{ji} = -\frac{2}{32\pi^2} (d\ln E^2) \sum_{k=1}^{2} \left[ \tilde{U}^{(k)\dagger} \tilde{U}^{(k)} \right]_{ji} (E). \tag{44}
\]

The relations for the \( df_d \) evolution matrices of the down-type sector are obtained from the above by simple exchanges \( \tilde{U}^{(i)} \leftrightarrow \tilde{D}^{(i)} \) and \( \tilde{U}^{(i)\dagger} \leftrightarrow \tilde{D}^{(i)\dagger} \). A solution to all these relations for the quark field evolution matrices is

\[
df_u(E)^{(L)}_{ij} = -\frac{1}{64\pi^2} (d\ln E^2) \sum_{k=1}^{2} \left[ \tilde{U}^{(k)\dagger} \tilde{U}^{(k)} + \tilde{D}^{(k)\dagger} \tilde{D}^{(k)} \right]_{ij} (E) = df_d(E)^{(L)}_{ij}, \tag{45}
\]

\[
df_u(E)^{(R)}_{ij} = -\frac{2}{64\pi^2} (d\ln E^2) \sum_{k=1}^{2} \left[ \tilde{U}^{(k)\dagger} \tilde{U}^{(k)} \right]_{ij} (E), \tag{46}
\]

\[
df_d(E)^{(R)}_{ij} = -\frac{2}{64\pi^2} (d\ln E^2) \sum_{k=1}^{2} \left[ \tilde{D}^{(k)\dagger} \tilde{D}^{(k)} \right]_{ij} (E). \tag{47}
\]

Another Green function needed for the derivation of the one-loop RGE’s of Yukawa matrices is the one represented by the diagram of Fig. 5. When the external legs there are \( \tilde{u}^{(i)} \) (incoming, with momentum \( k \)), \( \tilde{u}^{(j)} \) (outgoing, with momentum \( p + k \)), and \( \phi^{(i)}_3 \) [or \( \phi^{(j)}_4 \)],

\textsuperscript{13} The RGE relation represented by Fig. 4 is analogous to relation (26) represented diagrammatically in Fig. 2, but this time the kinetic energy terms are those of the quark fields.
it turns out that only the diagram with the charged scalar exchange contributes, and the resulting truncated three-point Green function, in the framework with the UV cutoff $E$, is

$$ G^{(3)}(k, p; E; \tilde{u}^{(i)}, \tilde{u}^{(j)}; \phi^{(r)}_3) = -\frac{i}{32\pi^2\sqrt{2}} \ln\left(\frac{E^2}{m^2}\right) \times \sum_{r=1}^{2} \left\{ (1 + \gamma_5) \left[ \tilde{D}^{(r)} \tilde{D}^{(r)\dagger} \tilde{U}^{(r)} \right]_{ji} + (1 - \gamma_3) \left[ \tilde{U}^{(r)\dagger} \tilde{D}^{(r)} \tilde{D}^{(r)\dagger} \right]_{ji} \right\}. $$

The corresponding Green function with the down-type quark external legs is obtained from the above expression by the simple exchanges $\tilde{U}^{(s)} \leftrightarrow \tilde{D}^{(s)}$ and $\tilde{U}^{(s)\dagger} \leftrightarrow \tilde{D}^{(s)\dagger}$.

Now, the one-loop RGE’s for the Yukawa matrices are obtained in analogy with the reasoning leading, in the case of two-point scalar Green functions, to the RGE relation (26) in the previous Subsection [cf. Fig. 2]. It is straightforward to check that the contribution of the quark loops in the scalar external leg cancel the contributions coming from the renormalizations of the scalar fields in the kinetic energy terms of the scalars – this is illustrated in Fig. 6. Furthermore, it can be checked that the contributions of the scalar exchanges on the external quark legs cancel the contributions coming from the renormalizations of the quark fields in the kinetic energy terms of the quarks – this is illustrated in Fig. 7. All in all, the 1PR one-loop contributions are canceled by the contributions of field renormalizations in the kinetic energy terms. Therefore, the only one-loop terms contributing to the evolution of the $\tilde{U}^{(k)}$ Yukawa matrices are those depicted in Fig. 8. The three diagrams with crosses there correspond to contributions of the following changes in the Yukawa coupling terms:

- Yukawa matrix change (renormalization) $d\tilde{U}^{(k)} \equiv \tilde{U}^{(k)}(E + dE) - \tilde{U}^{(k)}(E)$ – Fig. 8(b);
\[
\left[ \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \left( \begin{array}{c}
\mathbf{p} \\
\mathbf{q}
\end{array} \right) + \left( \begin{array}{c}
\mathbf{p} \\
\mathbf{q}
\end{array} \right) \right] = 0
\]

\[E < q < E + dE\]

Figure 7: Cancellation of contributions from the scalar exchange on the quark legs (1PR) with those of the quark field renormalizations in the kinetic energy term of the quarks, for the energy cutoff interval \((E, E + dE)\).

\[\begin{array}{c}
\begin{array}{c}
\tilde{U}(E) \\
(a)
\end{array} + \begin{array}{c}
d\tilde{U} \\
(b)
\end{array} + \begin{array}{c}
d\phi \\
(c)
\end{array} + \begin{array}{c}
d\tilde{u} \\
(d)
\end{array}
\end{array}
\plus \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \left( \begin{array}{c}
\mathbf{p} \\
\mathbf{q}
\end{array} \right) \right] = \tilde{U}(E)
\]

\[E < q < E + dE\]

Figure 8: Diagrammatic representation of the RGE for the up-type Yukawa matrix \(\tilde{U}\). Only the 1PI scalar exchange [(e)] and the effects of the renormalizations of the Yukawa matrix, of the scalar fields and the quark fields in the Yukawa couplings [(b), (c), (d), respectively] contribute when the cutoff is changed from \(E\) (RHS) to \(E + dE\) (LHS). Note that \(d\tilde{U}\) stands for \(\tilde{U}(E + dE) - \tilde{U}(E)\), etc. The contributions of the gauge boson exchanges were not considered in the figure.

- the scalar field renormalization \(d\tilde{\phi}_s^{(k)}\) \([\equiv \tilde{\phi}_s^{(k)}(E + dE) - \tilde{\phi}_s^{(k)}(E)]\) – Fig. 8(c);
- the quark field renormalization \(d\tilde{u}^{(i)}\) and \(d\tilde{u}^{(j)}\) – Fig. 8(d).

Figure 8 is a diagrammatic representation of the physical requirement that the three-point (quark-antiquark-scalar) Green function, at one-loop level, be in the theory with the cutoff \(E + dE\) [left-hand side of Fig. 8: (a) + … + (e)] the same as it is in the theory with the slightly lower cutoff \(E\) (right-hand side).

Using the results of this and the previous Subsection, we can then write down the one-loop RGE for \(\tilde{U}^{(k)}\) corresponding to Fig. 8, at the right-handed component \([\propto (1 + \gamma_5)]\) of the three-point Green function

\[
\tilde{U}^{(k)}_{ji} + d\tilde{U}^{(k)}_{ji} + \frac{1}{32\pi^2} \left( d \ln E^2 \right) \left\{ - N_c \sum_{\ell=1}^2 \text{Tr} \left[ \tilde{U}^{(k)} \tilde{U}^{(\ell)} + \tilde{D}^{(k)} \tilde{D}^{(\ell)} \right] \tilde{U}^{(i)} \right. \\
- \frac{1}{2} \sum_{\ell=1}^2 \left[ \left( \tilde{U}^{(k)} \tilde{U}^{(\ell)} + \tilde{D}^{(k)} \tilde{D}^{(\ell)} \right) \tilde{U}^{(k)} + 2 \tilde{U}^{(k)} \tilde{U}^{(\ell)} \right] \tilde{U}^{(i)} \\
\left. + 2 \sum_{\ell=1}^2 \left[ \tilde{D}^{(k)} \tilde{D}^{(\ell)} \tilde{U}^{(\ell)} \right] \right\} = \tilde{U}^{(k)}_{ji}.
\]

The first sum on the LHS \((\propto N_c)\) corresponds to Fig. 8(c) [cf. Eq. (37)], the second sum to Fig. 8(d) [cf. Eqs. (45), (46)], and the third sum to Fig. 8(e) [cf. Eq. (48)]. The left-handed
part of the Green function yields just the Hermitian conjugate of the above matrix relation. The analogous consideration of the three-point Green functions with the down-type external quark legs \( \tilde{d}^{(i)} \) and \( \tilde{d}^{(j)} \) gives relations which can be obtained from the above relation again by the simple exchanges \( \tilde{U}^{(s)} \leftrightarrow \tilde{D}^{(s)} \) and \( \tilde{U}^{(s)\dagger} \leftrightarrow \tilde{D}^{(s)\dagger} \). These relations can be rewritten in a more conventional form

\[
16\pi^2 \frac{d}{d\ln E} \tilde{U}^{(k)}(E) = \left\{ N_e \sum_{\ell=1}^{2} \text{Tr} \left[ \tilde{U}^{(k)} \tilde{U}^{(\ell)\dagger} + \tilde{D}^{(k)} \tilde{D}^{(\ell)\dagger} \right] \tilde{U}^{(\ell)} \right. \\
+ \frac{1}{2} \sum_{\ell=1}^{2} \left[ \tilde{U}^{(\ell)} \tilde{U}^{(\ell)\dagger} + \tilde{D}^{(\ell)} \tilde{D}^{(\ell)\dagger} \right] \tilde{U}^{(k)} + \tilde{U}^{(k)} \sum_{\ell=1}^{2} \tilde{U}^{(\ell)\dagger} \tilde{U}^{(\ell)} \\
- 2 \sum_{\ell=1}^{2} \left[ \tilde{D}^{(\ell)} \tilde{D}^{(k)\dagger} \tilde{U}^{(\ell)} \right] - A_U \tilde{U}^{(k)} \right\} ,
\]

(50)

and an analogous RGE for \( \tilde{D}^{(k)} \). These RGE’s still don’t contain one-loop effects of exchanges of gauge bosons. However, since the couplings of quarks and the Higgs doublets to the gauge bosons are identical to those in the usual MSM, 2HDM(I) and 2HDM(II), their contributions on the RHS of the above RGE’s are identical to those in these theories. Therefore, the final form of the one-loop RGE’s for the Yukawa matrices in the general 2HDM(III) now reads

\[
16\pi^2 \frac{d}{d\ln E} \tilde{U}^{(k)}(E) = \left\{ N_e \sum_{\ell=1}^{2} \text{Tr} \left[ \tilde{U}^{(k)} \tilde{U}^{(\ell)\dagger} + \tilde{D}^{(k)} \tilde{D}^{(\ell)\dagger} \right] \tilde{U}^{(\ell)} \right. \\
+ \frac{1}{2} \sum_{\ell=1}^{2} \left[ \tilde{U}^{(\ell)} \tilde{U}^{(\ell)\dagger} + \tilde{D}^{(\ell)} \tilde{D}^{(\ell)\dagger} \right] \tilde{U}^{(k)} + \tilde{U}^{(k)} \sum_{\ell=1}^{2} \tilde{U}^{(\ell)\dagger} \tilde{U}^{(\ell)} \\
- 2 \sum_{\ell=1}^{2} \left[ \tilde{D}^{(\ell)} \tilde{D}^{(k)\dagger} \tilde{U}^{(\ell)} \right] - A_U \tilde{U}^{(k)} \right\} ,
\]

(51)

\[
16\pi^2 \frac{d}{d\ln E} \tilde{D}^{(k)}(E) = \left\{ N_e \sum_{\ell=1}^{2} \text{Tr} \left[ \tilde{D}^{(k)} \tilde{D}^{(\ell)\dagger} + \tilde{U}^{(k)} \tilde{U}^{(\ell)\dagger} \right] \tilde{D}^{(\ell)} \right. \\
+ \frac{1}{2} \sum_{\ell=1}^{2} \left[ \tilde{U}^{(\ell)} \tilde{U}^{(\ell)\dagger} + \tilde{D}^{(\ell)} \tilde{D}^{(\ell)\dagger} \right] \tilde{D}^{(k)} + \tilde{D}^{(k)} \sum_{\ell=1}^{2} \tilde{D}^{(\ell)\dagger} \tilde{D}^{(\ell)} \\
- 2 \sum_{\ell=1}^{2} \left[ \tilde{D}^{(\ell)} \tilde{U}^{(k)\dagger} \tilde{D}^{(\ell)} \right] - A_D \tilde{D}^{(k)} \right\} ,
\]

(52)

where the functions \( A_U \) and \( A_D \), characterizing the contributions of the gauge boson exchanges, are the same as in the MSM, 2HDM(I) and 2HDM(II)

\[
A_U = 3 \left( \frac{N_c^2 - 1}{N_c} \right) g_3^2 + \frac{9}{4} g_2^2 + \frac{17}{12} g_1^2 ,
\]

\[
A_D = A_U - g_1^2 ,
\]

(53)

and the gauge coupling parameters \( g_j \) satisfy the one-loop RGE’s

\[
16\pi^2 \frac{d}{d\ln E} g_j = -C_j g_j^3 ,
\]

(54)
with the coefficients $C_j$ being those for the 2HDM’s $(N_H = 2)$

$$C_3 = \frac{1}{3}(11N_c - 2n_q), \quad C_2 = 7 - \frac{2}{3}n_q, \quad C_1 = -\frac{1}{3} - \frac{10}{9}n_q. \quad (55)$$

Here, $n_q$ is the number of effective quark flavors – e.g., for $E > m_t$ we have $n_q \approx 6$; for $m_b < E < m_t$ we have $n_q \approx 5$, etc.

The obtained relevant set of RGE’s for the VEV’s $e^{ik}\nu_k$ (39) and the Yukawa matrices $\tilde{U}(k)$ and $\tilde{D}(k)$ (11)–(52) determines in principle also the running of the quark masses. Instead, we can rewrite all these RGE’s in a representation involving the VEV parameters $v \equiv \sqrt{v_1^2 + v_2^2}$, $\tan \beta \equiv v_2/v_1$ and $\eta \equiv \eta_2 - \eta_1$ [cf. (3)], and matrices $\tilde{G}^{(U)}$, $\tilde{G}^{(D)}$, $\tilde{U}$ and $\tilde{D}$ [cf. (11), (12)] – this representation is more convenient for discerning the running of the quark masses and of the FCNC couplings. Applying lengthy, but straightforward, algebra to the hitherto obtained RGE’s results in the RGE’s of the latter set of parameters

$$16\pi^2 \frac{d}{d\ln E} \left( v^2 \right) = -2N_c \text{Tr} \left[ \tilde{G}^{(U)}\tilde{G}^{(U)t} + \tilde{G}^{(D)}\tilde{G}^{(D)t} \right] v^2 + \left[ \frac{3}{2}g_1^2 + \frac{9}{2}g_2^2 \right] v^2, \quad (56)$$

$$16\pi^2 \frac{d}{d\ln E} (\tan \beta) = -\frac{N_c}{2\cos^2 \beta} \left\{ \text{Tr} \left[ \tilde{U}\tilde{G}^{(U)t} + \tilde{G}^{(D)}\tilde{D}^t \right] + \text{Tr} \left[ \tilde{G}^{(U)}\tilde{U}^t + \tilde{D}\tilde{G}^{(D)t} \right] \right\}, \quad (57)$$

$$16\pi^2 \frac{d}{d\ln E} (\eta) = \frac{N_c}{i \sin(2\beta)} \left\{ \text{Tr} \left[ \tilde{G}^{(U)}\tilde{U}^t - \tilde{U}\tilde{G}^{(U)t} \right] - \text{Tr} \left[ \tilde{G}^{(D)}\tilde{D}^t - \tilde{D}\tilde{G}^{(D)t} \right] \right\}, \quad (58)$$

$$16\pi^2 \frac{d}{d\ln E} (\bar{\tilde{U}}) = N_c \left\{ 2\text{Tr} \left[ \tilde{U}\tilde{G}^{(U)t} + \tilde{G}^{(D)}\tilde{D}^t \right] \tilde{G}^{(U)} + \text{Tr} \left[ \tilde{U}\tilde{U}^t + \tilde{D}\tilde{D}^t \right] \tilde{U} 
- \frac{1}{2} (\cot \beta) \text{Tr} \left[ \tilde{G}^{(U)}\tilde{U}^t - \tilde{U}\tilde{G}^{(U)t} \right] \tilde{U} + \frac{1}{2} (\cot \beta) \text{Tr} \left[ \tilde{G}^{(D)}\tilde{D}^t - \tilde{D}\tilde{G}^{(D)t} \right] \tilde{U} 
+ \left\{ \frac{1}{2} \left[ \tilde{U}\tilde{U}^t + \tilde{D}\tilde{D}^t + \tilde{G}^{(U)}\tilde{G}^{(U)t} + \tilde{G}^{(D)}\tilde{G}^{(D)t} \right] \tilde{U} + \tilde{U} \right\} + \left[ \tilde{U}\tilde{U}^t + \tilde{G}^{(U)}\tilde{G}^{(U)t} \right] 
- 2\tilde{D}\tilde{D}^t \tilde{U} - 2\tilde{G}^{(D)}\tilde{D}^t \tilde{G}^{(U)} - A_U \tilde{U} \right\}, \quad (59)$$

$$16\pi^2 \frac{d}{d\ln E} (\bar{\tilde{D}}) = N_c \left\{ 2\text{Tr} \left[ \tilde{D}\tilde{G}^{(D)t} + \tilde{G}^{(U)}\tilde{U}^t \right] \tilde{G}^{(D)} + \text{Tr} \left[ \tilde{U}\tilde{U}^t + \tilde{D}\tilde{D}^t \right] \tilde{D} 
- \frac{1}{2} (\cot \beta) \text{Tr} \left[ \tilde{G}^{(D)}\tilde{D}^t - \tilde{D}\tilde{G}^{(D)t} \right] \tilde{D} + \frac{1}{2} (\cot \beta) \text{Tr} \left[ \tilde{G}^{(U)}\tilde{U}^t - \tilde{U}\tilde{G}^{(U)t} \right] \tilde{D} 
+ \left\{ \frac{1}{2} \left[ \tilde{U}\tilde{U}^t + \tilde{D}\tilde{D}^t + \tilde{G}^{(U)}\tilde{G}^{(U)t} + \tilde{G}^{(D)}\tilde{G}^{(D)t} \right] \tilde{D} + \tilde{D} \right\} + \left[ \tilde{U}\tilde{U}^t + \tilde{G}^{(U)}\tilde{G}^{(U)t} \right] 
- 2\tilde{U}\tilde{U}^t \tilde{D} - 2\tilde{G}^{(U)}\tilde{U}^t \tilde{G}^{(D)} - A_D \tilde{D} \right\}, \quad (60)$$
The system of RGE’s was solved numerically, using Runge-Kutta subroutines with adaptive stepsize control (given in [26]).

The latter values of quark masses correspond to:

\[ m_c = 0.77 \text{ GeV}, \quad m_s = 0.11 \text{ GeV}, \quad m_b = 3.2 \text{ GeV}, \quad \text{and} \quad m_t = 171.5 \text{ GeV}. \]

The latter values of quark masses correspond to: \( m_c(m_c) \approx 1.3 \text{ GeV}, \quad m_s(1\text{GeV}) \approx 0.2 \text{ GeV}, \quad m_b(m_b) \approx 4.3 \text{ GeV}, \) and \( m_t^{\text{phys}} \approx 174 \text{ GeV} \) \( m_t(m_t) \approx 166 \text{ GeV}. \) For \( \alpha_3(E) \) we used two-loop evolution formulas, with threshold effect at \( E \approx m_t^{\text{phys}} \) taken into account; for \( \alpha_j(E) \) \( (j = 1, 2) \) we used one-loop evolution formulas.

The described simplified framework resulted in 18 coupled RGE’s [for 18 real parameters: \( v^2, \tan \beta, \tilde{U}_{ij}, \tilde{D}_{ij}, \tilde{G}^{(U)}_{ij}, \tilde{G}^{(D)}_{ij} \)], with the mentioned boundary conditions at \( E = M_Z \). The system of RGE’s was solved numerically, using Runge-Kutta subroutines with adaptive stepsize control (given in [26]). The numerical results were cross-checked in several ways.
including the following: FORTRAN programs for the RGE evolution and for the biunitary transformations were constructed independently by two of the authors (S.S.H. and G.C.), and they yielded identical numerical results presented in this Section.

The results for the FCNC Yukawa parameter ratios $U_{ij}(E)/U_{ij}(M_Z)$ and $D_{ij}(E)/D_{ij}(M_Z)$ ($i \neq j$) are depicted in Fig. 9. From this Figure we immediately notice that the FCNC coupling parameters of the down-type ($b$-$c$) sector are remarkably stable as the evolution energy increases. Even the up-type FCNC ratios, although involving the heavy top quark, remain rather stable. Only very close to the top-quark-dominated Landau pole ($E_{\text{pole}} \approx 0.84 \cdot 10^{13}$ GeV)\footnote{The value of $E_{\text{pole}}$ is strongly dependent on the given value of parameter $\xi$, as shown later in Fig. 13.} the coupling parameters start to increase substantially. For example, in the down-type FCNC sector ($b$-$c$) the corresponding ratio $D_{21}(E)/D_{21}(M_Z)$ acquires its double initial value (i.e., value 2) at $E \approx 0.7E_{\text{pole}}$, which is very near the (Landau) pole.\footnote{Approximately the same is true also for $U_{21}(E)/U_{21}(M_Z)$.}

The value of $E_{\text{pole}}$ is strongly dependent on the given value of parameter $\xi$, as shown later in Fig. 13.

![Figure 9: Evolution of the FCNC Yukawa parameter ratios $U_{ij}(E)/U_{ij}(M_Z)$, $D_{ij}(E)/D_{ij}(M_Z)$ ($i \neq j$) in 2HDM(III). These parameters are in the quark mass basis. The choice of parameters of the model at the starting low energy $E = M_Z$ is specified in Sec. 4. Yukawa couplings of the first generation were neglected; $i = 1, 2$ correspond to the second and third quark generation, respectively.](image)

For the ratio $D_{12}(E)/D_{12}(M_Z)$ the corresponding energy is even closer to $E_{\text{pole}}$. One may ask whether the mentioned stability features in the evolution of all Yukawa coupling parameters in the framework. This doesn’t seem to be the case. For example, in Figs. 10 and 11 we depicted, with the described case of initial conditions, evolution of the Yukawa coupling ratios connected with no FCNC’s: $U_{jj}(E)/U_{jj}(M_Z)$, $D_{jj}(E)/D_{jj}(M_Z)$, $G^{(U)}_{jj}(E)/G^{(U)}_{jj}(M_Z)$ and $G^{(D)}_{jj}(E)/G^{(D)}_{jj}(M_Z)$ ($j = 1, 2$). We see that evolution behavior of many of these ratios, in stark contrast to the FCNC parameter case of Fig. 9, is far from stable. For example, the ratio $D_{11}(E)/D_{11}(M_Z)$, connected with $s$ quark, acquires double its ini-
Figure 10: Same as in Fig. 9, but for the neutral current Yukawa coupling parameters $U_{jj}$ and $D_{jj}$ ($j = 1, 2$) which don’t change flavor.

In Fig. 12 we depicted evolution of the quark masses for the discussed case. Note that they are not simply proportional to the “mass” Yukawa parameters $G_{jj}^{(u)}(E)$ and $G_{jj}^{(d)}(E)$, because the VEV $v(E)$ also evolves with energy [cf. Eq. (58)]. Logarithmic scale was chosen for the masses in order to include $m_t(E)$ in the figure. From this figure we see that we have a rather strong variation of $m_b(E)$ [and also of $m_t(E)$] continuously when the evolution energy increases. For example, $m_b(E)$ reaches half its initial value [$m_b(E) = m_b(M_Z)/2$] at $E \approx 10^{-6}E_{\text{pole}}$, which is very far away from $E_{\text{pole}}$ – this again contrasts with the behavior of FCNC Yukawa parameters of Fig. 9.

The discussed numerical example of the 2HDM(III) framework tells us that there definitely exist choices of reasonably suppressed (i.e., phenomenologically acceptable) FCNC Yukawa coupling parameters at low energies such that these parameters remain largely unchanged (suppressed) up to energy regions very close to the Landau pole. On the other hand, this behavior doesn’t feature in the entire sector of the Yukawa coupling parameters.

It should be stressed that these results are independent of the chosen value of the VEV ratio $\tan \beta$ at $E = M_Z$. This is connected with our choice of the CSY boundary conditions (20)-(21) at $E = M_Z$ for the Yukawa matrices in the quark mass basis ($\xi_{ij}^{(u)} = \xi_{ij}^{(d)} = 1$) and the reality of the chosen CKM matrix at $E = M_Z$. These boundary conditions result
in real and \( \beta \)-independent Yukawa matrices \( \tilde{U}, \tilde{D}, \tilde{G}^{(U)}, \tilde{G}^{(D)} \) in a weak \([SU(2)_L] \) basis\(^{16}\) at \( E = M_Z \). The RGE’s (59)-(62) then imply that these matrices remain real and independent of \( \beta \) at any evolution energy \( E \), and that also their counterparts \( U, D, G^{(U)} \) and \( G^{(D)} \) in the quark mass basis, as well as the CKM matrix \( V \), remain real and independent of \( \beta \) at any energy \( E \). Stated otherwise, if there is \( \beta \)-independence and no CP violation (neither in

original Yukawa matrices nor in the scalar sector) at a low energy (\( E = M_Z \)), then these properties persist at all higher energies of evolution.\(^{17}\)

This feature is in stark contrast with the situation in the 2HDM(II) where the Yukawa matrices strongly depend on \( \beta \) already at low energies – e.g., \( g_t(M_Z) = m_t(M_Z)\sqrt{2}/v_u = m_t(M_Z)\sqrt{2}/[v\sin(\beta(M_Z))] \). Also the location of the Landau pole in the 2HDM(II) then crucially depends on \( \beta(M_Z) \) – smaller \( \beta(M_Z) \) implies larger \( g_t(M_Z) \) and hence a drastically lower Landau pole.

On the other hand, the 2HDM(III) framework treats the up-type and the down-type sectors of quarks (the two VEV’s \( v_1 \) and \( v_2 \) ) non-discriminatorily. Therefore, it should be

---

\(^{16}\) We chose at \( E = M_Z \) the following weak basis: \( \tilde{U} = U, \tilde{G}^{(U)} = G^{(U)}, \tilde{D} = VD, \tilde{G}^{(D)} = VG^{(D)} \), where \( V \) is the CKM matrix (at \( E = M_Z \)). According to relations (11)-(12), the reality of the Yukawa matrices \( \tilde{U}, \tilde{D}, \tilde{G}^{(U)} \) and \( \tilde{G}^{(D)} \) at the low energy \( E = M_Z \) would follow, for example, from: the requirement of no CP violation in the Yukawa sector (i.e., the original Yukawa matrices \( \tilde{U}^{(j)} \) and \( \tilde{D}^{(j)} \) are all real) together with the requirement of no CP violation in the scalar sector (i.e., the VEV phase difference \( \eta = 0 \)) at that low energy.

\(^{17}\) CP conservation in the pure scalar sector at a low energy \( E = M_Z \) (i.e., \( \eta = 0 \)) also persists then at all higher energies of evolution, since \( d\eta/d\ln E = 0 \) by the reality of the Yukawa matrices, according to RGE (58).
Figure 12: Evolution of the quark masses

\begin{align*}
m_j^{(u)}(E) &= G^{(U)}_{jj}(E)v(E)/\sqrt{2} \\
m_j^{(d)}(E) &= G^{(D)}_{jj}(E)v(E)/\sqrt{2}
\end{align*}

\(m_1^{(d)} = m_s, m_2^{(d)} = m_b; m_1^{(u)} = m_c, m_2^{(u)} = m_t\), for the discussed numerical example.

expected that any reasonable boundary conditions for Yukawa coupling parameters at low energies should also be independent of \(\beta\) in such frameworks, and this independence then persists to a large degree also at higher energies. Also the locations of the Landau poles (i.e., of the approximate scales of the onset of new physics) should then be expected to be largely \(\beta\)-independent. In this sense, 2HDM(III) has more similarity to the minimal SM (MSM) than to the 2HDM(II). The persistence of complete \(\beta\)-independence of the Yukawa coupling parameters at high energies and of the Landau poles, however, can then be “perturbed” by CP violation – because RGE’s (59)-(62) are somewhat \(\beta\)-dependent when the Yukawa matrices \(\bar{U}\), etc., are not real. Also the VEV phase difference \(\eta\) is then not a constant when the energy increases [cf. (58)].

In addition to the connection between (low energy) CP violation and \(\beta\)-dependence of high energy results, there is yet another feature that distinguishes the 2HDM(III) framework from the MSM – the Landau pole of a 2HDM(III) framework is in general much lower than that of the MSM. We can see that in the following way: let us consider that only the Yukawa parameters connected with the top quark degree of freedom are substantial, i.e.,

\[G^{(U)}_{22} = g_t \sim 1\] and \(U_{22} = g_t' \sim 1\). We have: \(g_t(E) = m_t(E)\sqrt{2}/v(E)\), as in the MSM, and \(g_t'(E)\) is an additional large Yukawa parameter – both crucially influence location of the Landau pole. Inspecting RGE’s (59) and (61) for this special approximation of two variables \(g_t\) and \(g_t'\), we see that RGE for \(g_t\) is similar to that in the MSM, but with an additional large positive term on the right: \((3/2)(g_t')^2g_t\). RGE for \(g_t'\) has a similar structure as RGE for \(g_t\), but with substantially larger coefficients at the positive terms on the right. As a result, \(g_t'(E)\) is in general larger than \(g_t(E)\). Our specific numerical example [cf. Figs. 10 and 11 for \(U_{22}\) and \(G^{(U)}_{22}\)] shows that \(g_t'(E)\) is on average (average over the whole evolution energy
range) almost twice as large as $g_t(E)$. If we then simply replace in the mentioned additional term $(3/2)(g'_t)^2 g_t$ the parameter $(g'_t)^2$ by $3.5g_t^2$, we obtain from the resulting “modified” MSM RGE for $g_t$ a value for the Landau pole in the region of $10^{12} - 10^{13}$ GeV, which is roughly in agreement with the actual value of the Landau pole of our numerical example $E_{\text{pole}} \approx 0.84 \cdot 10^{13}$ GeV. And this value is much lower than $E_{\text{pole}}$ in the MSM which is above the Planck scale. Of course, when we allow the $\xi_{ij}^{(u)}$ parameters of the CSY ansatz (20)-(21) at $E = M_Z$ to deviate from 1, we obtain larger $\log(E_{\text{pole}})$ for smaller $\xi_{ij}^{(u)}$, and smaller $\log(E_{\text{pole}})$ for larger $\xi_{ij}^{(u)}$. In Fig. 13 we depicted this variation of the Landau pole energy when the CSY low energy parameters are varied.

\section{Summary and Conclusions}

We have derived the one-loop RGE’s for the quark Yukawa coupling matrices and the VEV’s in the Standard Model framework with the most general two-Higgs-doublet Yukawa sector [2HDM(III)]. A simple – and at low energies phenomenologically acceptable – numerical example for the resulting evolution of these Yukawa parameters suggests that the framework cannot be dismissed as “unnatural” from the FCNC-RGE point-of-view. Stated otherwise, the numerical example shows remarkable stability of the suppressed FCNC Yukawa coupling parameters when the energy of probes increases continuously all the way to the vicinity of the top quark Landau pole. The Landau pole is in general well below the Planck scale in this framework. We believe that further numerical investigations are warranted, in order to see

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure13.png}
\caption{Variation of the Landau pole energy when the low energy ($E = M_Z$) parameters $\xi_{ij}^{(u)} = \xi_{ij}^{(d)} \equiv \xi$ of the CSY ansatz (20)-(21) are varied. For $\xi = 2.5$, the onset scale of new physics is already quite low: $E_{\text{pole}} \approx 2$ TeV.}
\end{figure}
whether and/or to what degree this behavior survives when we scan over certain reasonable (phenomenologically acceptable) ranges the values of low energy parameters of the model – i.e., the Yukawa coupling parameters and tan $\beta$ at $E \sim E_{\text{ew}}$.

6 Abbreviations used in the article:

AHR – Antaramian, Hall and Rašin;
CKM – Cabibbo, Kobayashi and Maskawa;
CSY – Cheng, Sher and Yuan;
FCNC – flavor-changing neutral current;
LHS – left-hand side;
MSM – minimal Standard Model;
RHS – right-hand side;
RGE – renormalization group equation;
SM – Standard Model;
VEV – vacuum expectation value;
1PI – one-particle-irreducible;
1PR – one-particle-reducible;
2HDM – two-Higgs-doublet (Standard) Model;
2HDM(III) – general two-Higgs-doublet (Standard) Model – “type III”.

7 Acknowledgments

The work of GC was supported in part by the Bundesministerium fuer Bildung, Wissenschaft, Forschung und Technologie, Project. No. 057DO93P(7). The work of CSK and SSH was supported in part by the CTP of SNU, in part by Yonsei University Faculty Research Fund of 1997, in part by the BSRI Program, Ministry of Education, Project No. BSRI-97-2425, and in part by the KOSEF-DFG large collaboration project, Project No. 96-0702-01-01-2.
References

[1] E.A. Paschos, Phys. Rev. D 15 (1977) 1966 (1977);
    S.L. Glashow and S. Weinberg, Phys. Rev. D 15 (1977) 1958;
    K. Kang and J.E. Kim, Phys. Lett. B 64 (1976) 93.
[2] T. D. Lee, Phys. Rev. D 8 (1973) 1226.
[3] P. Sikivie, Phys. Lett. B 65 (1976) 141.
[4] A.B. Lahanas and C.E. Vayonakis, Phys. Rev. D 19 (1979) 2158.
[5] G.C. Branco, A.J. Buras and J.-M. Gérard, Nucl. Phys. B 259 (1985) 306.
[6] J. Liu and L. Wolfenstein, Nucl. Phys. B 289 (1987) 1.
[7] T.P. Cheng and M. Sher, Phys. Rev. D 35 (1987) 3484.
[8] M. Sher and Y. Yuan, Phys. Rev. D 44 (1991) 1461.
[9] M.J. Savage, Phys. Lett. B 266 (1991) 135.
[10] M. Luke and M.J. Savage, Phys. Lett. B 307 (1993) 387.
[11] A. Antaramian, L.J. Hall and A. Rašin, Phys. Rev. Lett. 69 (1992) 1971.
[12] W.S. Hou, Phys. Lett. B 296 (1992) 179.
[13] D. Chang, W.S. Hou and W.Y. Keung, Phys. Rev. D 48 (1993) 217.
[14] W. S. Hou and G. L. Lin, Phys. Lett. B 379 (1996) 261.
[15] L.J. Hall and S. Weinberg, Phys. Rev. D 48 (1993) 979.
[16] Y.-L. Wu, in Intersections between Particle and Nuclear Physics, Proceedings of the 5th Conference on the Intersections of Particle and Nuclear Physics, ed. S. J. Seestrom (AIP, 1995) (also: hep-ph/9406306);
    Y.-L. Wu and L. Wolfenstein, Phys. Rev. Lett. 73 (1994) 1762.
[17] D. Atwood, L. Reina and A. Soni, Phys. Rev. Lett. 75 (1995) 3800.
[18] D. Atwood, L. Reina and A. Soni, Phys. Rev. D 53 (1996) 1199.
[19] D. Atwood, L. Reina and A. Soni, Phenomenology of two Higgs doublet models with flavor changing neutral currents, hep-ph/9609279.
[20] S. Bar-Shalom, G. Eilam, A. Soni and J. Wudka, Probing the flavor-changing $tc$ vertex via tree level processes: $e^+e^- \rightarrow t\bar{c}\nu_e\bar{\nu}_e$, $t\bar{c}e^+e^-$ and $t \rightarrow cW^+W^-$, Univ. of California (Riverside) preprint UCRHEP-T185/97, hep-ph/9703224.
[21] G. Cvetič, in Spontaneous Symmetry Breaking, Proceedings of the Budapest Workshop, Budapest, July 1994, pp. 67-83, eds. F. Csikor and G. Pócsik (World Scientific, Singapore, 1995) (also: hep-ph/9411436).
[22] J.F. Gunion, H.E. Haber, G. Kane and S. Dawson, The Higgs Hunter’s Guide (Addison-Wesley, 1990).

[23] P.G. Lepage, What is renormalization, Cornell University preprint CLNS-89-870, June 1989, 26 pp., invited lectures given at TASI’89 Summer School, Boulder, CO, USA, June 1989, published in Boulder ASI 1989, pp. 483-508.

[24] H. Arason et al., Phys. Rev. D 46 (1992) 3945.

[25] J. P. Fatelo, J.-M. Gérard, T. Hambye and J. Weyers, Phys. Rev. Lett. 74 (1995) 492; T. Hambye, Phys. Lett. B 371 (1996) 87; G. Cvetič, Int. J. Mod. Phys. A 11(1996) 5405.

[26] W.H. Press et al., Numerical recipes in FORTRAN – the art of scientific computing, (Cambridge University Press, 1986).