The inverse cyclotomic Discrete Fourier Transform algorithm

Sergei V. Fedorenko

Abstract—The proof of the theorem concerning to the inverse cyclotomic Discrete Fourier Transform algorithm over finite field is provided.

I. INTRODUCTION

The discrete Fourier transform (DFT) can be applied in error correcting codes and code-based cryptography. The cyclotomic DFT method [1], [2] is the best one for computing the DFT. Correcting codes and code-based cryptography. The cyclotomic DFT [2, (6)] in the example of paper [2] had not been proved. We have corrected this mistake and introduced the proof.

II. BASIC NOTIONS AND DEFINITIONS

The DFT of length $n = 2^m - 1$ of a vector $f = (f_i)$, $i \in [0, n - 1]$, $f_i \in GF(2^m)$, is the vector $F = (F_j)$

$$F_j = \sum_{i=0}^{n-1} f_i \alpha^{ij}, \quad j \in [0, n - 1],$$

where $\alpha$ is an element of order $n$ in $GF(2^m)$. Let us write the DFT in matrix form

$$F = W f,$$  \hspace{1cm} (1)

where $W = (\alpha^{ij})$, $i, j \in [0, n - 1]$, is a Vandermonde matrix. We assume that the length of the $n$-point Fourier transform over $GF(2^m)$ is $n = 2^m - 1$.

Let us consider cyclotomic cosets modulo $n = 2^m - 1$ over $GF(2)$

$$\{c_0\} = \{0\},$$
$$\{c_1, c_1^2, c_1^{2^2}, \ldots, c_1^{2^{m-1}}\},$$
$$\ldots,$$

$$\{c_l, c_l^2, c_l^{2^2}, \ldots, c_l^{2^{m-1}}\},$$

where $c_k \equiv c_l^{2^m} \mod n$, $l + 1$ is the number of cyclotomic cosets modulo $n$ over $GF(2)$.

Let us introduce the set of indices modulo $n$

$$Z = \{Z_i\} = \{c_0, c_1, c_1^2, c_1^{2^2}, \ldots, c_1^{2^{m-1}}, \ldots, c_l, c_l^2, c_l^{2^2}, \ldots, c_l^{2^{m-1}}\}, \quad i \in [0, n - 1].$$

Then, we define a permutation matrix $\Pi = (\Pi_{i,j})$, $i, j \in [0, n - 1]$,

$$\Pi_{i,j} = \begin{cases} 1, & \text{if } j = Z_i, \quad i \in [0, n - 1] \\ 0, & \text{otherwise}. \end{cases}$$

Let us denote a basis $\beta_k = (\beta_k, 0, \ldots, \beta_k, m_k - 1)$ of the subfield $GF(2^{m_k}) \subset GF(2^m)$.

Then we can write the cyclotomic DFT [1], [2]

$$F_j = \sum_{k=0}^{m_k-1} \sum_{s=0}^{\beta_k^2 s} f_{k, j, s} \beta_{k, s}^{2^s},$$

where $a_{k,j,s} \in GF(2)$. This equation can be represented in matrix form as

$$F = AL(\Pi f),$$  \hspace{1cm} (2)

where $A$ is a matrix with elements $a_{k,j,s} \in GF(2)$ and $L$ is a block diagonal matrix with elements $\beta_{k, s}^{2^s}$. If one chooses the normal basis $\beta_k$, then all the blocks of the matrix $L$ are circulant matrices.

The inverse DFT in the field $GF(2^m)$ is

$$f = W^{-1} F.$$

It is easily shown that

$$W^{-1} = EW,$$  \hspace{1cm} (3)

where $E$ is a matrix

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

III. THE INVERSE CYCLOTOMIC DFT

Since both matrices $A$ and $L$ are invertible, from (2) the following representation of the inverse DFT can be derived

$$\Pi f = L^{-1} A^{-1} F.$$

Lemma 1 (23): Suppose $\beta_k$ are the normal bases, then it is possible to show that blocks of $L^{-1}$ consist of elements of bases $\beta_k^l$ which are dual to $\beta_k$, that is, the blocks of $L^{-1}$ are also circulant matrices.

Theorem 1: The inverse cyclotomic DFT for $GF(2^m)$ is

$$(\Pi E)F = L^{-1} A^{-1} f.$$

Proof: From (1) and (2) we have

$$W = AL\Pi.$$
and
\[ W^{-1} = \Pi^{-1} L^{-1} A^{-1}. \]

From the last formula and (3) we obtain
\[ W^{-1} = EW = \Pi^{-1} L^{-1} A^{-1}. \]

and
\[ W = E^{-1} \Pi^{-1} L^{-1} A^{-1}. \]

Hence,
\[ F = W f = (E^{-1} \Pi^{-1}) L^{-1} A^{-1} f. \]

and
\[ (\Pi E) F = L^{-1} A^{-1} f. \]

IV. EXAMPLES

A. DFT of length \( n = 7 \)

Let \((\gamma, \gamma^2, \gamma^4, \gamma^8)\) be a normal basis of \(GF(2^3)\), where \(\gamma = \alpha^3\) and \(\alpha\) is a root of the primitive polynomial \(x^3 + x + 1\). Then the cyclotomic DFT can be represented as

\[
\begin{pmatrix}
F_0 \\
F_1 \\
F_2 \\
F_3 \\
F_4 \\
F_5 \\
F_6
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma^{-1} & \gamma^2 & \gamma^4 & 0 & 0 & 0 & 0 \\
0 & \gamma^4 & \gamma^1 & 0 & 0 & 0 & 0 \\
0 & \gamma^4 & \gamma^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma^1 & \gamma^2 & \gamma^4 & 0 \\
0 & 0 & 0 & 0 & \gamma^4 & \gamma^1 & \gamma^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
f_0 \\
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Finally note that the last formula coincides with formula \[2\] (6).

This algorithm requires 6 multiplications and 24 additions and appears to be the best known 7-point DFT for \(GF(2^3)\).

B. DFT of length \( n = 15 \)

Let \((\alpha^3, \alpha^6, \alpha^{12}, \alpha^9)\) and \((\alpha^{11}, \alpha^7, \alpha^{14}, \alpha^{13})\) be the normal bases of \(GF(2^5)\), where \(\alpha\) is a root of the primitive polynomial \(x^5 + x + 1\). Then the cyclotomic DFT can be represented as formula (4) or \(F = AL(\Pi f)\). The inverse cyclotomic DFT can be written as formula (5) or \((\Pi E) F = L^{-1} A^{-1} f\).

ACKNOWLEDGMENT

The author would like to thank Alexey Maevskiy for pointing out a methodological mistake in the paper \[2\], and Peter Trifonov for helpful discussions.

REFERENCES

[1] P. V. Trifonov and S. V. Fedorenko. A method for fast computation of the Fourier transform over a finite field. Problems of Information Transmission, vol. 39, no. 3, pp. 231–238, 2003. Translation of Problemy Peredachi Informatsii.

[2] E. Costa, S. V. Fedorenko, and P. V. Trifonov. On computing the syndrome polynomial in Reed–Solomon decoder. European Transactions on Telecommunications, vol. 15, no. 3, pp. 337–342, 2004.

[3] J. Hong and M. Vetterli. Computing in DFT’s over GF(q) with one DFT over GF(qm). IEEE Transactions on Information Theory, vol. 39, no. 1, pp. 271–274, January 1993.
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha^7 & \alpha^{14} & \alpha^{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha^{13} & \alpha^{13} & \alpha^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha^{11} & \alpha^{11} & \alpha^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha^{11} & \alpha^7 & \alpha^{14} & \alpha^{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha^{14} & \alpha^{13} & \alpha^{13} & \alpha^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha^{14} & \alpha^{13} & \alpha^{11} & \alpha^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha^{13} & \alpha^{11} & \alpha^7 & \alpha^{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha^5 & \alpha^{10} & \alpha^5 & \alpha^5 & \alpha^5 & \alpha^5 & \alpha^5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\times
\begin{pmatrix}
111111111111111 \\
10011110101100 \\
10010001110101 \\
11011100100011 \\
10110010011110 \\
10111101111011 \\
11011110111101 \\
11110111101110 \\
11101111011110 \\
11011011011101 \\
110111011000100 \\
111000100110101 \\
10111001001101 \\
10011010111000 \\
101101101101101 \\
110110110110110 \\
\end{pmatrix}
\begin{pmatrix}
F_0 \\
F_1 \\
F_2 \\
F_3 \\
F_4 \\
F_5 \\
F_6 \\
F_7 \\
F_8 \\
F_9 \\
F_{10} \\
F_{11} \\
F_{12} \\
F_{13} \\
F_{14} \\
\end{pmatrix}
\]
\[(5)\]