A MULTINOMIAL ASYMPTOTIC REPRESENTATION OF ZENGA'S DISCRETE INDEX, ITS INFLUENCE FUNCTION AND DATA-DRIVEN APPLICATIONS

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Abstract. In this paper, we consider the Zenga index, one of the most recent inequality index. We keep the finite-valued original form and address the asymptotic theory. The asymptotic normality is established through a multinomial representation. The Influence function is also given. The results are simulated and applied to Senegalese data.

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keywords and phrases. Inequality measures; Asymptotic behaviour; Asymptotic representations; functional empirical process.

AMS 2010 Mathematics Subject Classification : 62G05; 62G20; 62G07; 91B82; 62P20.

1. Introduction

Over the years, a number of measures of inequality have been developed. Examples include the generalized entropy, the Atkinson, the Gini, the quintile share ratio and the Zenga measures (see e.g. Zenga (1984) and Zenga (1990)), Cowell and Flachaire (2007); Cowell et al. (2009); Hulliger and Schoch (2009). Recently, Mergane and Lo (2013) gathered a significant number of inequality measures under the name of Theil-like family.
Such inequality measures are very important in capturing inequality in income distributions. They also have applications in many other branches of Science, e.g. in ecology (see e.g. Magurran (1991)), sociology (see e.g. Allison (1978)), demography (see e.g. White (1986)) and information science (see e.g. Rousseau (1993)).

The inequality measure of Zenga (2006) is one of the most recent one. It is receiving a considerable attention from researchers for its novelty indeed, but for its interesting properties. Papers dealing with that measure cover theoretical aspects including asymptotic theory and statistical inference (Greselin et al. (2010b), Eldin and Marilou (1999)) and applied works to income data (Greselin et al. (2010a)), etc.

In this paper, we focus on the discrete form as introduced by Zenga (2006). We justify the asymptotic study of the discrete and finite form by a number of reasons. In some situations, only aggregated data exists. Although this is hardly conceivable today, it is still possible and it is highly probable that the researcher does not have access to the original data and has in hand only data in form of frequency tables. Some other times, frequency tables may be available while the full data is destroyed or lost. Right now, in Gambia, health data collected from the health centers are stored in daily books and the national health direction extract frequency tables from those books and this type of data is the only one available in their computerized system. So one of the main reason to work on the finite discrete data is the lack of accessibility to the full data for one reason or another. The second main is that an asymptotic theory on such king of data will give the structure of the limit results with also no severe conditions. By replacing the discrete finite probability law of the aggregated data by a general probability law, we get the precise general asymptotic case. From that simplified study, we see what might be expected in general theory before we proceed it.

Here, we suppose that the full data has been summarized into a frequencies table of the form

Each class \((c_{i-1}, c_i)\) in Table 1 is represented by a single point \(x_i^*\), usually taken as the middle of class \(x_i^* = (c_{i-1} + c_i)/2\) (other possible choices are the mean of the median of observation falling in the class). So we may adopt approximatively reconstitute the \(n \geq 1\) data as follows
In the sequel, we suppose that the data itself is discrete and takes a predetermined number of \( m \) value. First, we will give an asymptotic theory which will be given in the form of representation in multinomial laws, in opposite to representation in Brownian Bridges in the general case. Next, the influence function will be derived by direct computations and this usually allows to find again the asymptotic variance and some times as in our case, to find a different but equivalent expression of that variance.

The works presented here will be applied to incomes available in an aggregated form. At the same time, they serve as a paving way to a more general approach.

Let us suppose that the income variable \( X \) is discrete and takes the \( m \) \((m > 1)\) ordered values \( 0 = -\infty < x_1 < \ldots < x_m < x_{m+1} = +\infty \) with the probabilities \( p_j > 0 \), \( j \in \{1,...,m\} \) with \( p_1 + p_2 + \ldots + p_m = 1 \). If the income continuously observed, we have a sequence of random replications \( X_1, X_2, \ldots \) defined on the same probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \). For each \( n \geq 1 \), the empirical distribution of \( X \) on the sample is characterized by the empirical frequencies

\[
n_0 = 0, \quad n_j = \#\{h \in \{1,...,n\}, \ X_h = x_j\}, \ j \in \{1,...,m\},
\]

and their normalized and cumulative forms respectively

\[
f_0 = 0, \quad f_j = \frac{n_j}{n}, \ j \in \{1,...,m\}
\]

| classes \((c_{i-1}, c_i)\) | Represents \(x_i^*\) | frequencies \(n_i\) |
|--------------------------|----------------|------------------|
| \((c_0, c_1)\) | \(x_1^*\) | frequencies \(n_1\) |
| \((c_1, c_2)\) | \(x_2^*\) | frequencies \(n_2\) |
| \(\vdots\) | \(\vdots\) | \(\vdots\) |
| \((c_{m-1}, c_m)\) | \(x_m^*\) | frequencies \(n_m\) |
| Total | \(x_i^*\) | \(n\) |

Table 1. Frequencies Tables
and
\[ n_0^* = f_0^* = 0, n_j^* = \sum_{h=1}^{j} n_h, \quad f_j^* = \sum_{h=1}^{j} f_h, \quad j \in \{1, \ldots, m\}, \]

with
\[ \sum_{j=1}^{m} n_j = n, \quad \sum_{j=1}^{m} f_j = 1, \quad n_m^* = n, \quad f_m^* = 1. \]

We also define
\[ p_0^* = 0, \quad p_j^* = \sum_{h=1}^{j} p_h, \quad p_m^* = 1. \]

The empirical and discrete Zenga (2006)’s index is given by
\[ Z_{d,n} = 1 - \sum_{j=1}^{m-1} f_j (n_j^*/n)^{-1} \sum_{1 \leq h \leq j} n_h x_h \left[ \frac{1}{(1 - (n_j^*/n))^{-1} \sum_{j+1 \leq h \leq m} n_h x_h} \right], \]

which is obtained by summing Formula (3.1) in Zenga (2006) over \( j \in \{1, \ldots, m\} \) and presented as a synthetic measure of inequality. The empirical cumulative distribution function (cdf) based on the sample of size \( n \geq 1 \) is
\[ F_n(x) = \frac{1}{n} \sum_{h=1}^{m} n_h 1_{[x_h, x_{h+1}[}(x), \quad x \in \mathbb{R} \]

and is the non-parametric estimator of the true (cdf)
\[ F_n(x) = \sum_{h=1}^{m} p_j 1_{[x_h, x_{h+1}[}(x), \quad x \in \mathbb{R} \]

We also have the empirical probability generated by the sample is given by
\[ P_{X,n}(A) = \frac{1}{n} \sum_{j=1}^{m} 1_A(x_j) \]

We may express \( Z_{n,d} \) in terms of the empirical probability measure by
\[ Z_{d,n} = 1 - \sum_{j=1}^{m-1} \mathbb{P}_{X,n}(x_j) \frac{\left( \int_{0}^{1} 1_{[0,x_j]}(t) d\mathbb{P}_{X,n}(t) \right)^{-1} \left( \int_{1}^{\infty} 1_{[x_j,\infty]}(t) d\mathbb{P}_{X,n}(t) \right)^{-1} \left( \int_{1}^{\infty} 1_{[x_j]}(t) d\mathbb{P}_{X,n}(t) \right)^{-1}}{\left( \int_{1}^{\infty} d\mathbb{P}_{X,n}(t) \right)^{-1} \left( \int_{1}^{\infty} 1_{[x_j]}(t) d\mathbb{P}_{X,n}(t) \right)^{-1} \left( \int_{1}^{\infty} 1_{[x_j,\infty]}(t) d\mathbb{P}_{X,n}(t) \right)^{-1}}. \]

Finally by considering the discrete measure \( \nu = \sum_{1 \leq j \leq n} \delta_{x_j} \), where \( \delta_{x_j} \) is the Dirac measure concentrated at \( x_j \) with mass one, we may also write

\[ Z_{d,n} = 1 - \int \frac{\left( \int_{0}^{1} 1_{[0,s]}(t) d\mathbb{P}_{X,n}(t) \right)^{-1} \left( \int_{1}^{\infty} 1_{[s,\infty]}(t) d\mathbb{P}_{X,n}(t) \right)^{-1} \left( \int_{1}^{\infty} 1_{[s]}(t) d\mathbb{P}_{X,n}(t) \right)^{-1}}{\left( \int_{1}^{\infty} d\mathbb{P}_{X,n}(t) \right)^{-1} \left( \int_{1}^{\infty} 1_{[s]}(t) d\mathbb{P}_{X,n}(t) \right)^{-1} \left( \int_{1}^{\infty} 1_{[s,\infty]}(t) d\mathbb{P}_{X,n}(t) \right)^{-1}} \mathbb{P}_{X,n}(s) d\nu(s). \]

It is clear, by the convergence in law of the sequence of probability measures \( \mathbb{P}_{X,n} \) to the \( \mathbb{P}_{X} = \mathbb{P}_{X}^{-1} \) (the probability law of \( X \)), we see that \( Z_{n,d} \) converges to

\[ Z_d = 1 - \int \frac{\left( \int_{0}^{1} 1_{[0,x]}(t) d\mathbb{P}_{X}(t) \right)^{-1} \left( \int_{1}^{\infty} 1_{[x,\infty]}(t) d\mathbb{P}_{X}(t) \right)^{-1} \left( \int_{1}^{\infty} 1_{[x]}(t) d\mathbb{P}_{X}(t) \right)^{-1}}{\left( \int_{1}^{\infty} d\mathbb{P}_{X}(t) \right)^{-1} \left( \int_{1}^{\infty} 1_{[x]}(t) d\mathbb{P}_{X}(t) \right)^{-1} \left( \int_{1}^{\infty} 1_{[x,\infty]}(t) d\mathbb{P}_{X}(t) \right)^{-1}} \mathbb{P}_{X}(s) d\nu(s). \]

In this simple setting, the convergence are easily justified because of the finiteness of the summations and of the functions. In terms of cdf and on mathematical expectation, we have

\[ Z_d = 1 - \int_{x_1}^{x_m} \frac{1}{F(s)} \int_{0}^{s} t d\mathbb{P}_{X}(t) \mathbb{P}_{X}(s) d\nu(s), (X) \]

The integral in the last expression should be read as

\[ \int_{x_1}^{x_m} \frac{1}{F(s)} \int_{0}^{s} t d\mathbb{P}_{X}(t) \mathbb{P}_{X}(s) d\nu(s) = \int_{1}^{\infty} \frac{1}{1-F(s)} \int_{s}^{\infty} t d\mathbb{P}_{X}(t) \mathbb{P}_{X}(s) d\nu(s), \]

so that neither \( 1 - F(s) \) nor \( F(s) \) never vanishes on the integration domain.

On one side, we are going to draw an asymptotic normality theory of \( Z_{n,d} \) using the \( m \)-multivariate binomial laws. On an other side, the sensitivity of a statistic \( T(F) \) and the impact of extreme observations on it are also two recurrent questions in the research in the field (see Cowell and Flachaire (2007))
In that context, the asymptotic variance of the plug-in estimator $T(F_n)$ of statistic $T(F)$ is of the form $\sigma^2 = \int L(x, T(F))^2 dF(x)$. From this, we may say that the influence function behaves in nonparametric estimation as the score function does in the parametric setting (See Wasserman (2006), page 19). To define the notion of IF, Let us consider the contaminated probability law $P_X^{(\varepsilon)}$ of $P_X$ at $x$ with mass $\varepsilon > 0$ by

\begin{equation}
P_X^{(\varepsilon)} = (1 - \varepsilon)P_X + \varepsilon\delta_x.
\end{equation}

and a functional of $P_X$, namely $T(P_X)$. The influence function of the functional $T$ at $x$, if it exists, is given by

\begin{equation}
IF(T, x) = \lim_{\varepsilon \to 0} \frac{T(P_X^{(\varepsilon)}) - T(P_X)}{\varepsilon}.
\end{equation}

The previous remarks motivate us to derive the IF function of $Z_d(P_X)$ and to compare it with the asymptotic variance the plug-in Zenga’s estimator.

Before we proceed to our a task, we point out that asymptotic normality results for Zenga’s index are available in the literature, among them those of Greselin et al. (2010b) and Eldin and Marilou (1999). We will come back to these results in the coming paper where we deal with other version of asymptotic versions in the general case.

Here is how is organized the paper, we give our asymptotic results as described above in Section 2 in Theorems 1 and 2. Section 3 is devoted to simulation studies and data-driven application to Senegalese Data. A conclusion and perspectives section ends the paper.

2. Asymptotic Theory for the discrete Zenga measure

(A) - Asymptotic normality.

Let begin by the following reminder. For each $n \geq 1$, the random vector $(n_1, ..., n_m)$ follows a $m$-dimensional multimonial law of parameters $n \geq 1$ and $p = (p_1, ..., p_m)^t$. In such a case a classical result of weak convergence (See Lo et al. (2016), for example., as $n \to +\infty$, is the following
\[
\left( \frac{n_1 - np_1}{\sqrt{np_1}}, \ldots, \frac{n_m - np_m}{\sqrt{np_m}} \right)^t \\
\equiv (N_{1,n}, \ldots, N_{m,n})^t \\
\sim Z = (Z_1, \ldots, Z_m)^t \sim N_m(0, \Sigma),
\]

the variance-covariance matrix \( \Sigma = \sigma_{h,k} \) of \( Z \) is defined, for \((h, k) \in \{1, \ldots, m\}^2, h \neq k, \) by

\[
\sigma_{hh} = \mathbb{E}(Z_h^2) = 1 - p_h \quad \text{and} \quad \sigma_{hk} = \mathbb{E}(Z_h Z_k) = -\sqrt{p_h p_k}.
\]

We invoke the Skorohod-Wichura Theorem (See Wichura (1970)) to suppose that \( Z \) is defined on the same probability space and that

\[
(N_{1,n}, \ldots, N_{m,n})^t \rightarrow_P Z, \quad \text{as} \quad n \rightarrow +\infty.
\]

Let us give some notation. Define vectors \( C = (c_1, \ldots, c_m)^t \) such that

\[
c_j = \sqrt{p_j} \left( \frac{1/p_j^*}{(1/(1 - p_j^*)) \mu(j)} \right) 1_{(j \neq m)}, j \in \{1, \ldots, m\},
\]

for \( j \in \{1, \ldots, m - 1\} \), \( i \in \{1, 2\} \), \( D_{j,i} = (d_{j,i,1}, \ldots, d_{j,i,m})^t \) such that

\[
d_{j,1,h} = (x_h \sqrt{p_h}) 1_{(h \leq j)}, \quad d_{j,2,h} = - (x_h \sqrt{p_h}) 1_{(h > j + 1)}
\]

\[
\gamma_{j,1} = p_j \left( \frac{1/p_j^*}{(1/(1 - p_j^*)) \mu(j)} \right), \quad \gamma_{j,2} = p_j \left( \frac{1/p_j^*}{(1/(1 - p_j^*)) \mu(j)} \right)^2
\]

and let \( E_j = (e_{j,1}, \ldots, e_{j,m})^t \) be the vector defined by its components as follows

\[
e_{j,h} = - (\sqrt{p_h}) 1_{(h \leq j)}.
\]

Finally, let us defined

\[-H = C + \sum_{j=1}^{m-1} \left( \gamma_{j,1} D_{j,1} + \gamma_{j,2} D_{j,2} + (p_j)^{-2} E_j \right).\]

**Theorem 1.** Under the notation given above, we have, as \( n \rightarrow +\infty, \)

\[
\sqrt{n}(Z_{d,n} - Z_d) \sim N_m \left( 0, H^t \Sigma H \right). \quad \diamond
\]
Proof of Theorem 1. Let us fix $n \geq 1$. We have
\[
Z_{n,d} = 1 - \sum_{j=1}^{m-1} \frac{n_j}{n} \left( \frac{n}{n_j^*} - 1 \right) \frac{\sum_{1 \leq h \leq j} n_h x_h}{\sum_{j+1 \leq h \leq m} n_h x_h}.
\]

We define
\[
Z^*_{d,n} = \sum_{j=1}^{m-1} \frac{n_j}{n} \left( \frac{n}{n_j^*} - 1 \right) \frac{\sum_{1 \leq h \leq j} n_h x_h}{\sum_{j+1 \leq h \leq m} n_h x_h}
\]
and for $1 \leq j \leq m-1$,
\[
\mu(j) = \sum_{h=1}^{j} p_h x_h \quad \text{and} \quad \mu^{(j)} = \sum_{h=j+1}^{m} p_h x_h.
\]

We have
\[
\frac{\sum_{1 \leq h \leq j} n_h x_h}{\sum_{j+1 \leq h \leq m} n_h x_h} - \frac{\mu(j)}{\mu^{(j)}}
= \frac{\sum_{1 \leq h \leq j} n_h x_h}{\sum_{j+1 \leq h \leq m} n_h x_h} - \frac{n \mu(j)}{\sum_{j+1 \leq h \leq m} n_h x_h}
+ \frac{n \mu(j)}{\sum_{j+1 \leq h \leq m} n_h x_h} - \frac{\mu^{(j)}}{\mu^{(j)}}
= \frac{\sum_{h=1}^{j} x_h N_{h,n} \sqrt{p_h}}{\sqrt{n} \sum_{j+1 \leq h \leq m} n_h x_h / n} - \frac{\mu(j)}{\mu^{(j)}} \frac{\sum_{h=j+1}^{m} x_h N_{h,n} \sqrt{p_h}}{\sqrt{n} \mu^{(j)} \left( \sum_{j+1 \leq h \leq m} n_h x_h / n \right)}.
\]

Then
\[
Z^*_{d,n} = \sum_{j=1}^{m-1} \frac{n_j}{n} \left( \frac{n}{n_j^*} - 1 \right) \frac{\mu(j)}{\mu^{(j)}}
+ \frac{1}{\sqrt{n}} \sum_{j=1}^{m-1} \frac{n_j}{n} \left( \frac{n}{n_j^*} - 1 \right) \left( \frac{\sum_{h=1}^{j} x_h N_{h,n} \sqrt{p_h}}{\sum_{j+1 \leq h \leq m} n_h x_h / n} - \frac{\mu(j)}{\mu^{(j)}} \frac{\sum_{h=j+1}^{m} x_h N_{h,n} \sqrt{p_h}}{\sqrt{n} \mu^{(j)} \left( \sum_{j+1 \leq h \leq m} n_h x_h / n \right)} \right)
= : Z^*_{d,n}(1) + R_n(1,1)
\]

We also have
\[
\frac{n}{n^*_j} - 1 - \frac{1}{p^*_j} - 1 = \left( \frac{n}{n^*_j} - 1 \right) - \frac{n}{\sum_{h=1}^{j} np_h} - 1
\]
\[
= - \frac{\sum_{h=1}^{j} n_h - \sum_{h=1}^{j} p_h}{\left( \sum_{h=1}^{j} p_h \right) \left( \sum_{h=1}^{j} n_h \right)}
\]
\[
= - \frac{1}{\sqrt{n}} \frac{\sum_{h=1}^{j} \sqrt{p_h N_{h,n}}}{\left( \sum_{h=1}^{j} n_h / n \right)}.
\]

This leads to

\[
Z^{*}_{d,n}(1) = \sum_{j=1}^{m-1} n_j \left( \frac{1}{p^*_j} - 1 \right) \frac{\mu(j)}{\mu(j)} - \sum_{j=1}^{m-1} \frac{n \sqrt{n} \sum_{h=1}^{j} \sqrt{p_h N_{h,n}}}{\left( \sum_{h=1}^{j} n_h \right) \left( \sum_{h=1}^{j} np_h \right)} \frac{\mu(j)}{\mu(j)}
\]
\[
= : Z^{*}_{d,n}(2) + R_n(1, 2)
\]

Finally, we have

\[
Z^{*}_{d,n}(2) = \sum_{j=1}^{m-1} p_j \left( \frac{1}{p^*_j} - 1 \right) \frac{\mu(j)}{\mu(j)} + \frac{1}{\sqrt{n}} \sum_{j=1}^{m-1} \sqrt{p_j N_{j,n}} \left( \frac{1}{p^*_j} - 1 \right) \frac{\mu(j)}{\mu(j)}
\]
\[
= \sum_{j=1}^{m-1} p_j \left( \frac{1}{p^*_j} - 1 \right) \frac{\mu(j)}{\mu(j)} + \frac{1}{\sqrt{n}} \sum_{j=1}^{m-1} \sqrt{p_j N_{j,n}} \left( \frac{1}{p^*_j} - 1 \right) \frac{\mu(j)}{\mu(j)} \tag{L2}
\]
\[
= \sum_{j=1}^{m-1} \left( \frac{1}{p^*_j} \right) \frac{\mu(j)}{\mu(j)} + \frac{1}{\sqrt{n}} \sum_{j=1}^{m-1} \sqrt{p_j N_{j,n}} \left( \frac{1}{p^*_j} - 1 \right) \frac{\mu(j)}{\mu(j)}
\]
\[
= : Z^{*}_d + R_n(3).
\]

It is clear that

\[
Z_d = 1 - Z^{*}_d.
\]

We finally get

\[
\sqrt{n}(Z^{*}_{d,n} - Z^{*}_d) = \sqrt{n} R_n(1) + \sqrt{n} R_n(2) + \sqrt{n} R_n(3).
\]
By using the convergence (strong and weak) on binomial probabilities, we get

\[
\sqrt{n}R_n(1, 1) = \sum_{j=1}^{m-1} \frac{n_j}{n} \left( \frac{n}{n_j} - 1 \right) \left( \frac{\sum_{h=1}^{j} (x_h \sqrt{p_h}) N_{h,n}}{\sum_{j+1 \leq h \leq m} n_h x_h / n} \right) - \frac{\mu(j)}{\mu^{(j)}} \left( \frac{\sum_{h=j+1}^{m} (x_h \sqrt{p_h}) N_{h,n}}{\sum_{j+1 \leq h \leq m} n_h x_h / n} \right)
\]

\[
\rightarrow_p \sum_{j=1}^{m-1} \frac{(1/p_j^*)}{(1/(1 - p_j^*))} \left( \frac{\mu^{(j)}}{\mu^{(j)}} \right) \frac{\sum_{h=1}^{j} (x_h \sqrt{p_h}) Z_h}{\sum_{h=j+1}^{m} (x_h \sqrt{p_h}) Z_h}, \quad (A1)
\]

Next

\[
\sqrt{n}R_n(1, 2) = -\frac{\sum_{h=1}^{j} \sqrt{p_h} N_{h,n}}{\left( \sum_{h=1}^{j} p_h \right) \left( \sum_{h=1}^{j} n_h / n \right)}
\]

\[
\rightarrow_p -\frac{\sum_{h=1}^{j} \sqrt{p_h} Z_h}{(p_j^*)^2} \quad (A2)
\]

and finally

\[
\sqrt{n}R_n(3) = \sum_{j=1}^{m-1} \sqrt{p_j} \left( \frac{1}{p_j^*} - 1 \right) \frac{\mu(j)}{\mu^{(j)}} N_{j,n}
\]

\[
\rightarrow_p \sum_{j=1}^{m-1} \sqrt{p_j} \frac{(1/p_j^*)\mu(j)}{(1/(1 - p_j^*))\mu^{(j)}} Z_j, \quad (A3)
\]

By combining Developments (A1), (A2) and (A3), we get
\[
\sqrt{n}(Z^*_{d,n} - Z^*_d) \\
\rightarrow \sum_{j=1}^{m-1} p_j \frac{(1/p^*_j)}{(1/(1 - p^*_j))} \left( \frac{\sum_{h=1}^j (x_h \sqrt{P_h}) Z_h}{\mu(j)} - \frac{\mu(j) \sum_{h=j+1}^m (x_h \sqrt{P_h}) Z_h}{(\mu(j))^2} \right) \\
- \sum_{h=1}^j \sqrt{P_h} Z_h \\
+ \sum_{j=1}^{m-1} \sqrt{p_j} \frac{(1/p^*_j) \mu(j)}{(1/(1 - p^*_j)) \mu(j)} Z_j \\
= \left( \sum_{j=1}^{m-1} \langle \gamma_{j,1} D_{j,1}, Z \rangle + \langle \gamma_{j,2} D_{j,2}, Z \rangle + \langle (p^*_j)^{-2} E_j, Z \rangle \right) + \langle C, Z \rangle.
\]

We conclude that
\[
\sqrt{n}(Z^*_{d,n} - Z^*_d) \rightarrow_{P} H^t Z. \quad \square
\]
Theorem 2. Under the notations given below, the Influence function of $Z_d$ is given, for $x_1 \leq x \leq x_m$, by

$$IF(Z_d, x) = \int \mathbb{P}_X(s) \left( \frac{R_1(s)}{R_2(s)^2 (1 - F(s))} 1_{[s, +\infty]}(x) - \frac{1}{R_2(s) F(s)} 1_{[0, s]}(x) \right) x d\nu$$

$$+ \int \mathbb{P}_X(s) \left( \frac{R_1(s)}{R_2(s) F(s)} 1_{[0, s]}(x) - \frac{R_1(s)}{R_2(s) (1 - F(s))} 1_{[s, +\infty]}(x) \right) d\nu$$

$$- \int \delta_x(s) \frac{R_1(s)}{R_2(s)} d\nu + \int \mathbb{P}_X(s) \frac{R_1(s)}{R_2(s)} d\nu.$$

Proof of Theorem 2. Let us write, for $s \in \mathcal{R}$,

$$R_1(s) = R_1(s, \mathbb{P}_X) = \frac{\int t 1_{[0, s]}(t) d\mathbb{P}_X(t)}{\int 1_{[0, s]}(t) d\mathbb{P}_X(t)},$$

and

$$R_2(s) = R_2(s, \mathbb{P}_X) = \frac{\int t 1_{[s, +\infty]}(t) d\mathbb{P}_X(t)}{\int 1_{[s, +\infty]}(t) d\mathbb{P}_X(t)}.$$

We have

$$Z_d(\mathbb{P}_X) = Z_d = 1 - \int \frac{R_1(s)}{R_2(s)} \frac{d\mathbb{P}_X(s)}{d\mathbb{P}_X} d\nu(s).$$

By using Formula (1.1), we have

$$\frac{d(\mathbb{P}_X^{(\varepsilon)} - \mathbb{P}_X)}{\varepsilon} = -d\mathbb{P}_X + d\delta_x$$

For short, we write

$$R_i(s, \mathbb{P}_X) = R_i(s) \text{ and } R_i(s, \mathbb{P}_X^{(\varepsilon)}) = R_i(s, \varepsilon), i \in \{1, 2\}.$$
\[
Z_d(P^\varepsilon_X) - Z_d(P_X) = -(1 - \varepsilon) \int P_X(s) \frac{R_1(s, \varepsilon)}{R_2(s, \varepsilon)} d\nu - \varepsilon \int \delta_x(s) \frac{R_1(s, \varepsilon)}{R_2(s, \varepsilon)} d\nu
+ \int P_X(s) \frac{R_1(s)}{R_2(s)} d\nu
= - \int P_X(s) \left( \frac{R_1(s, \varepsilon)}{R_2(s, \varepsilon)} - \frac{R_1(s)}{R_2(s)} \right) d\nu
+ \varepsilon \int P_X(s) \frac{R_1(s, \varepsilon)}{R_2(s, \varepsilon)} d\nu - \varepsilon \int \delta_x(s) \frac{R_1(s, \varepsilon)}{R_2(s, \varepsilon)} d\nu.
\]

Let us apply the definition of the \(\text{IF}\) as in Formula (1.2). Since \(P^\varepsilon_X \to P_X\) as \(\varepsilon \to 0\) (The convergence being meant as a convergence in law), we have no problem to see that

\[
\lim_{\varepsilon \to 0} \frac{Z_d(P^\varepsilon_X) - Z_d(P_X)}{\varepsilon} = \int P_X(s) \frac{R_1(s)}{R_2(s)} d\nu - \int \delta_x(s) \frac{R_1(s)}{R_2(s)} d\nu
- \int P_X(s) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \frac{R_1(s, \varepsilon)}{R_2(s, \varepsilon)} - \frac{R_1(s)}{R_2(s)} \right) d\nu.
\]

(2.1)

So we have to find the influence function of \(R_1(s)/R_2(s)\). By formally representing the differentiation of a functional \(T(P_X)\) by

\[
\frac{\partial T(P_X)}{\partial \lambda}
\]

we have that the influence function of \(R_1(s)/R_2(s)\) is given by

\[
IF(R_1(s)/R_2(s), x) = \frac{R_2(s) \frac{\partial R_1(s)}{\partial \lambda} - R_1(s) \frac{\partial R_2(s)}{\partial \lambda}}{R_2(s)^2}.
\]

But
\[ R_1(s, \varepsilon) - R_1(s) = \frac{\int t1_{[0,\varepsilon]}(t) d\mathbb{P}_X(t)}{\int 1_{[0,\varepsilon]}(t) d\mathbb{P}_X(t)} - \frac{\varepsilon \int t1_{[0,\varepsilon]}(t) d\mathbb{P}_X(t)}{\int 1_{[0,\varepsilon]}(t) d\mathbb{P}_X(t)} + \frac{\varepsilon \int t1_{[0,\varepsilon]}(t) d\Delta X(t)}{\int 1_{[0,\varepsilon]}(t) d\mathbb{P}_X(t)} - \frac{\varepsilon \int t1_{[0,\varepsilon]}(t) d\Pi^{(e)}_X(t)}{\int 1_{[0,\varepsilon]}(t) d\mathbb{P}_X(t)} - \frac{\varepsilon \int t1_{[0,\varepsilon]}(t) d\Pi^{(e)}_X(t)}{\int 1_{[0,\varepsilon]}(t) d\mathbb{P}_X(t)} \int t1_{[0,\varepsilon]}(t) d\mathbb{P}_X(t). \]

We get

\[ \lim_{\varepsilon \to 0} \frac{R_1(s, \varepsilon) - R_1(s)}{\varepsilon} = \frac{\int t1_{[0,\varepsilon]}(t) d(-\mathbb{P}_X(t) + \delta_x)}{\int 1_{[0,\varepsilon]}(t) d\mathbb{P}_X(t)} - \frac{\int 1_{[0,\varepsilon]}(t) d(-\mathbb{P}_X(t) + \delta_x)}{\left(\int 1_{[0,\varepsilon]}(t) d\mathbb{P}_X(t)\right)^2} \int t1_{[0,\varepsilon]}(t) d\mathbb{P}_X(t) = -\frac{\int t1_{[0,\varepsilon]}(t) d\mathbb{P}_X(t) + 1_{[0,\varepsilon]}(x)}{\left(\int 1_{[0,\varepsilon]}(t) d\mathbb{P}_X(t)\right)^2} \int t1_{[0,\varepsilon]}(t) d\mathbb{P}_X(t) \]

We get

\[ \frac{\partial R_1(s)}{\partial \lambda} = -R_1(s) + \frac{x1_{[0,\varepsilon]}(x)}{F(s)} + R_1(s) - \frac{R_1(s)}{F(s)} 1_{[0,\varepsilon]}(x). \]

By treating \( R_2(s) \) in the same manner we have (We should not forget that we differentiate in the probability)
\[
\frac{\partial R_1(s)}{\partial \lambda} = \frac{x_{1|0,s]}(x)}{F(s)} - \frac{R_1(s)}{F(s)} 1_{[0,s]}(x) \\
\frac{\partial R_2(s)}{\partial \lambda} = \frac{x_{1|s,+\infty}(x)}{1 - F(s)} - \frac{R_2(s)}{1 - F(s)} 1_{[s,+\infty]}(x)
\]

Thus

\[
\lim_{\varepsilon \to 0} \frac{R_1(s, \varepsilon) - R_1(s)}{\varepsilon} = \left( \frac{1_{[0,s]}(x)}{R_2(s) F(s)} - \frac{R_1(s) 1_{[s,+\infty]}(x)}{R_2(s)(1 - F(s))} \right) x \\
+ \left( \frac{R_1(s)}{R_2(s)(1 - F(s))} 1_{[s,+\infty]}(x) - \frac{R_1(s)}{R_2(s) F(s)} 1_{[0,s]}(x) \right);
\]

By replacing this limit with its expression in the equation (2.1) we get:

\[
\lim_{\varepsilon \to 0} \frac{Z_d(P^{(s)}_X) - Z_d(P_X)}{\varepsilon} = \int \mathbb{P}(X = x) R_1(s) \frac{R_1(s)}{R_2(s)} d\nu - \int \delta_x(s) R_1(s) \frac{R_1(s)}{R_2(s)} d\nu \\
+ \int \mathbb{P}(X = x) \left( \frac{R_1(s)}{R_2(s) F(s)} 1_{[0,s]}(x) - \frac{R_1(s)}{R_2(s)(1 - F(s))} 1_{[s,+\infty]}(x) \right) x d\nu \\
+ \int \mathbb{P}(X = x) \left( \frac{R_1(s)}{R_2(s)(1 - F(s))} 1_{[s,+\infty]}(x) - \frac{R_1(s)}{R_2(s) F(s)} 1_{[0,s]}(x) \right) d\nu.
\]

From this, the proof is directed concluded. ■

3. Data-driven Applications

Simulation Study.

Quality of the convergence. We choose a Probability distribution of yearly income supported by \( m = 10 \) points with lower endpoint \( x_1 = 4.515.000 \) XOF (9.030 nearly) and upper endpoint \( x_m = 9.000.000 \) XOF(170.490 nearly), characterized as in Table 2.

| values | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) | \( x_5 \) | ... |
|--------|--------|--------|--------|--------|--------|-----|
| \( \mathbb{P}(X = x_i) \) | 0.05   | 0.05   | 0.05   | 0.05   | 0.1    | ... |

Table 2. Underlying Probability Law (to be continued)
Table 2 shows the good performance of estimation the Zenga’s discrete for size samples from \( n = 100 \) to \( n = 1500 \). Such sizes are comparable with those of sample survey from population counted in dozen of millions.

| Size  | 100   | 200   | 500   | 750   | 1000  | 750   |
|-------|-------|-------|-------|-------|-------|-------|
| ERM   | 3.6 \(10^{-3}\) | -5.36 \(10^{-3}\) | 10\(^{-3}\) | -8.41 \(10^{-4}\) | 4.56 \(10^{-5}\) | -1.441 \(10^{-5}\) |
| MSE   | 6.4 \(10^{-2}\) | 3.35 \(10^{-2}\) | 2.49 \(10^{-2}\) | 2.16 \(10^{-2}\) | 1.9 \(10^{-2}\) | 1.64 \(10^{-2}\) |

Table 4. Mean errors (ERM), Mean Square Errors (MSE)

Figure 2 shows the pretty good asymptotic normality approximation of the centered and normalized empirical Zenga’s estimator.

**(B) Data-driven Applications.**

We use the income Data in Senegal (2001-2002) from the database related to ANSD : Senegalese Survey from Households (2001-2202) . The incomes are given by households. We should use an adult-equivalence scale to consider to be able to compare households. The notion of adult-equivalence has already been described in Lo (2016) and implemented on different sets of data, among them the data just described above. The data are available for the whole country (Senegal) and for the 10 areas given in the following order :

**(OA)**: Dakar, Diourbel, Fatik, Kaolack, Louga, Saint-Louis, Tamba, Thies, Ziguinchor, Kolda.

Dakar in the most urbanized area of Senegal and includes the capital of the country, named also after Dakar. It concentrated almost 23.1% of the population.

The Zenga and the Gini index have been computed for the 11 areas from the aggregate data, and are display in Table 5 (continued in Table 6).
Figure 1. Histograms, Parzen Estimators and QQ-plots for sample sizes 500, 1000 and 1500 from left to right.

Table 5. Zenga and Gini index measures for Senegal’s administrative areas (2000), to be continued.

| Index | Senegal | Dakar  | Diourbel | Fatick  | Kaolack | Louga  |
|-------|---------|--------|----------|---------|---------|--------|
| Zenga | 80.65   | 93.33  | 81.34    | 92.54   | 81.11   | 84.00  |
| Gini  | 75.00   | 80.90  | 75.26    | 80.39   | 75.16   | 16.25  |
Table 6. Continuation of Table 5

| Index   | Saint-Louis | Tamba | Thies | Ziguinchor | Kolda |
|---------|-------------|-------|-------|------------|-------|
| Zenga   | 87.69       | 86.64 | 82.61 | 82.11      | 80.24 |
| Gini    | 78.83       | 77.26 | 75.72 | 75.52      | 47.86 |

Table 6. Continuation of Table 5

Through the values in these tables, the 11 areas are ordered from the least inequality index to the greatest as follows:

**Ordering by Zenga’s index**: Kolda (1), Senegal (2), Kaolack (3), Diourbel (4), Ziguinchor (5), Thies (6), Louga (7), Tamba (8), Saint-Louis (9), Fatick (10), Dakar (11).

**Ordering by Gini’s index**: Louga (1), Kolda (2), Senegal (3), Kaolack (4), Diourbel (5), Ziguinchor (6), Thies (7), Tamba (8), Saint-Louis (9), Fatick (10), Dakar (11).

These orderings are illustrated in Figure 2.

The most striking fact is that the two index do not order the areas in an exact similar way. The most unfair areas (with the greatest values of the inequality index) are the same with the same ordering, form areas 8 to 11.
From areas 1 to 7, the ordering is slightly changed but the case of Louga is remarkable. It is ranked first by Gini and seventh by Zenga.

One may think that the inequality should be greater in urban areas than in rural zone. Indeed we see that with the areas of Thies, Saint-Louis, Dakar. But Factik and Tamba are so urbanized areas. Investigating why the inequality indices (Both Zenga and Gini) are high should be investigated in accordance with local realities.

In this simple study, we are concerned with a large scale comparison studies between Zenga’s and Gini’s either but simulation studies or by theoretical investigations. This would be certainly in coming papers.

4. Conclusion and perspectives

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