Bäcklund transformation and solitonic solutions for a parametric coupled Korteweg-de Vries system

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Abstract. We analyze a parametric coupled KdV system and we find a Bäcklund transformation. For a positive value of the parameter the system reduces to two KdV decoupled equations. For negative value of the parameter the system has non trivial coupling and presents multisolitonic solutions generated by the Bäcklund transformation. We compare the results with the already known in the literature.

1. Introduction

Historically, Bäcklund transformations arose as a useful tool to be used in differential geometry and differential equations [1, 2], for example, in the study of surfaces in classical differential geometry [3], and can be understood in a natural way as a generalization of contact transformations [4, 5].

More recently, Bäcklund transformations were used to find new solutions from known ones in the context of non linear partial differential equations [6, 7, 8]. It seems also that there exists a direct relation between such transformations and the integrability properties of the associate problems, in particular those obtained from the inverse scattering method for solving them.

Coupled Korteweg-de Vries (KdV) systems have generated great interest by virtue of its outstandings properties and wide range of applications [9, 10, 11, 12, 13].

In this work we find a Bäcklund transformation for a parametric coupled KdV system with relevant physical applications, following the approach developed by Whalquist and Estabrook [7]. We then generate multisolitonic solutions and we compare our results with those reported in the literature.

2. The Bäcklund transformation for a coupled Korteweg-de Vries system

We consider a particular coupled Korteweg-de Vries system, describing a two-layer liquid model studied in references [11, 14, 15, 16]. The system is formulated in terms of two independent functions $u(x,t)$ and $v(x,t)$ and given by the following partial differential equations:

$$u_t + uu_x + u_{xxx} - vv_x = 0$$  \hspace{1cm} (1)

$$v_t + u_xv + v_{xx}u + v_{xxx} = 0.$$  \hspace{1cm} (2)

We notice that by redefining $u$ and $v$ as $-6u$ and $6v$ respectively we obtain the system considered in [11]. Both systems (1),(2) and the one in [11] are completely equivalent.
We will follow the approach of Whalquist and Estabrook [7] in order to obtain a Bäcklund transformation for the system given by (1) and (2).

Moreover, we will consider a generalization of the system given by (1) and (2) in terms of an arbitrary parameter \( \lambda \), more explicitly we will analyze the system

\[
\begin{align*}
  u_t + uu_x + u_{xxx} + \lambda v_x &= 0 \\
  v_t + u_x v + v_x u + v_{xxx} &= 0.
\end{align*}
\]

By a redefinition of \( v \) we may reduce the value of \( \lambda \) to be +1 or −1. Both systems are not equivalent. The case \( \lambda = -1 \) corresponds the the system given by (1) and (2).

We notice that the system given by (3) and (4) is invariant under a Galileo transformation of the coordinates \( x, t \). In fact, if

\[
\begin{align*}
  x &\rightarrow x + ct \\
  t &\rightarrow t
\end{align*}
\]

and

\[
\begin{align*}
  u &\rightarrow u - c \\
  v &\rightarrow v
\end{align*}
\]

then (3) and (4) remain invariant for any value of \( \lambda \).

The system (1),(2) may be derived from a lagrangian

\[
\mathcal{L} = -\frac{1}{2} w_x w_t - \frac{1}{6} (w_x)^3 - \frac{1}{2} w_{xxx} w_x + \frac{1}{2} w_x (y_x)^2 + \frac{1}{2} y_x y_t + \frac{1}{2} y_{xxx} y_x
\]

expressed in terms of the fields \( w(x,t) \) and \( y(x,t) \) related to the original fields \( u(x,t), v(x,t) \) by the following relations:

\[
\begin{align*}
  u &= w_x, \\
  v &= y_x.
\end{align*}
\]

The hamiltonian and associated Poisson structure may be derived from the above lagrangian. The field equations obtained from the lagrangian (7) by taking variations on \( w \) and \( y \) are the following, once an integration on \( x \) is performed,

\[
\begin{align*}
  w_t + \frac{1}{2} (w_x)^2 + w_{xxx} + \frac{1}{2} \lambda (y_x)^2 + C(t) &= 0 \\
  y_t + y_{xxx} + w_x y_x + \tilde{C}(t) &= 0
\end{align*}
\]

for \( \lambda = -1 \), \( C(t) \) and \( \tilde{C}(t) \) are integration constants which depend only on the \( t \) variable. Hence they may be eliminated by a redefinition of \( w \) and \( y \):

\[
\begin{align*}
  w(x,t) &\rightarrow w(x,t) + \int_0^t C(\tau) d\tau \\
  y(x,t) &\rightarrow y(x,t) + \int_0^t \tilde{C}(\tau) d\tau.
\end{align*}
\]

We end up with the following system in terms of \( w(x,t), y(x,t) \), equivalent to the original system (1),(2) for \( \lambda = -1 \),

\[
\begin{align*}
  w_t + \frac{1}{2} (w_x)^2 + w_{xxx} + \frac{1}{2} (y_x)^2 &= 0 \\
  y_t + y_{xxx} + w_x y_x &= 0.
\end{align*}
\]
We propose the following Bäcklund transformation, whose integrability conditions yield (10), (11) for any value of $\lambda$:

\[ w_x + w'_x = 2\eta - \frac{1}{12}(w - w')^2 - \frac{\lambda}{12}(y - y')^2, \]  

(12)

\[ w_t + w'_t = \frac{1}{6}(w - w') (w_{xx} - w'_{xx}) + \frac{\lambda}{6} (y - y') (y_{xx} - y'_{xx}) - \frac{1}{3} w_x^2 - \frac{1}{3} w_x w'_x - \frac{\lambda}{3} y_x^2 - \frac{\lambda}{3} y'_x, \]  

(13)

\[ y_x + y'_x = 2\mu - \frac{1}{6}(w - w') (y - y'), \]  

(14)

\[ y_t + y'_t = \frac{1}{6}(w - w') (y - y')_{xx} + \frac{1}{3}(w - w')_{xx} (y - y') - (\frac{2}{3} w_x y_x + \frac{2}{3} w'_x y'_x + \frac{1}{3} w_x y'_x + \frac{1}{3} w'_x y_x). \]  

(15)

We define

\[ Q_1(w, y) \equiv w_t + \frac{1}{6} w_x^2 + w_{xxx} + \frac{1}{2} \lambda y_x^2 \]  

(16)

\[ Q_2(w, y) \equiv y_t + w_x y_x + y_{xxx}. \]  

(17)

It then follows that

\[ (12)_{xx} + (13) = 0 \]

implies

\[ Q_1(w, y) + Q_1(w', y') = 0, \]  

(18)

\[ -(12)_t + (13)_x = 0 \]

implies

\[ [Q_1(w, y) - Q_1(w', y')] (w - w') + \lambda [Q_2(w, y) - Q_2(w', y')] (y - y') = 0, \]  

(19)

\[ (14)_{xx} + (15)_x = 0 \]

implies

\[ Q_2(w, y) + Q_2(w', y') = 0, \]  

(20)

and

\[ -(14)_t + (15)_x = 0 \]

implies

\[ [Q_1(w, y) - Q_1(w', y')] (y - y') + [Q_2(w, y) - Q_2(w', y')] (w - w') = 0. \]  

(21)

Consequently, if

\[ (w - w')^2 - \lambda (y - y')^2 \neq 0, \]  

(22)

then a solution of (12), (13), (14), (15) implies

\[ Q_1(w, y) = 0, Q_2(w, y) = 0 \]

\[ Q_1(w', y') = 0, Q_2(w', y') = 0 \]

that is, a solution of the system (10), (11).
Moreover, if \( w', y' \) is a solution of the system (10),(11), then a solution \( w, y \) of (12),(13),(14),(15) is a solution of (10),(11). That is, given a (may be a trivial) solution of (10),(11) by resolving (12),(13),(14),(15) we obtain a new solution of the original system (10),(11).

We notice that condition (22) when \( \lambda \) is negative, in particular \( \lambda = -1 \), implies \( (w, y) \neq (w', y') \) and vice versa. That is, if \( (w, y) \neq (w', y') \) are solutions of (12),(13),(14),(15) then \( (w, y) \) and \( (w', y') \) are different solutions of (10),(11).

When \( \lambda \) is positive, by redefining \( y \) we may reduce to consider \( \lambda = 1 \). In that case equations (10),(11) reduce to two decoupled equations in terms of \( (w + y) \) and \( (w - y) \). The equations are exactly the KdV equation for each variable.

Using the above Bäcklund transformation and starting from trivial solution of the coupled system one can generate solitonic solutions. These solutions agree with the already reported multisolitonic solutions in [11].

3. Conclusions
We obtained, following the approach given in [7], a Bäcklund transformation for a parametric coupled system, including as a special case the system studied by [11]. We can then generate multisolitonic solutions, they agree with the already reported in the literature.

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