Orbits of Lie Group Actions are Weakly Embedded

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Abstract. In this note we prove that whenever a Lie group $G$ acts on a manifold $X$, then the orbit $Gx$ through any point $x$ of $X$ is a weakly embedded submanifold of $X$. The investigation of this problem was inspired by an application to Catastrophe Theory.

1 Introduction

When a Lie group $G$ acts on a manifold $X$, the orbit $Gx$ through a point $x$ of $X$ is a special immersed submanifold of $X$, namely it is weakly embedded. According to [P], an immersed submanifold $Y$ of $X$ is said to be weakly embedded, if every smooth map $g : S \to X$ of a manifold $S$ into $X$ with image in $Y$ induces a smooth map $g : S \to Y$. In particular, a smooth curve $c$ in $X$, whose image lies in $Y$, is actually a smooth curve in $Y$. Immersed submanifolds, which are not regular submanifolds, i.e. whose topology is strictly finer than the induced one, need not possess this property. A simple illustration is the following Figure Eight.

The curve, whose image is composed of parts 1 and 2 as well as the point $x$, lies in the Figure Eight but is not continuous as a map into the Figure Eight. Consequently, this Figure Eight is not weakly embedded.

The main step in proving that orbits of Lie group actions are weakly embedded consists in showing that any $C^1$ curve whose image lies in an orbit locally

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coincides with the image of a curve in the group under the orbital map (Lifting
Theorem). The idea for studying this problem arose from analyzing its relevance
to one of the major theorems in Catastrophe Theory, namely John Mather’s
necessary condition for determining when a smooth function of several variables
coincides with its own Taylor polynomial up to a local diffeomorphism (see
\[Ca/Ha\,\text{Chap.}\,4\]).

2 The Linearization Lemma

In this note a manifold \(X\) means a smooth manifold, i.e. a \(C^\infty\) manifold.
\(X\) need not be Hausdorff nor have a countable base. An immersed submanifold
\(Y\) of \(X\) is a manifold which is a subset of \(X\) such that the inclusion \(i : Y \to X\)
is an immersion. Clearly, the topology of \(Y\) is finer than the relative topology of
\(X\). When these topologies coincide, \(Y\) is called a regular submanifold.
Weakly embedded submanifolds are intermediary between regular and immersed
submanifolds (see \[Mo\], \[S\]). Note that in the definition of weakly embedded, the
map \(g : S \to X\) is smooth if it is continuous \[W, 1.13\]. More generally, for every
\(k\) it follows as in \[W\] that every \(C^k\) map \(g : S \to X\) with image in \(Y\) induces a
\(C^k\) map \(g : S \to Y\) if \(g : S \to Y\) is only continuous. Furthermore, to notify that
an immersed submanifold \(Y\) of \(X\) is weakly embedded, i.e. possesses the lifting
property of smooth maps given above, it suffices to consider just the lifting of
\(C^1\) curves:

**Proposition** Let \(Y\) be an immersed submanifold of a manifold \(X\) such
that every \(C^1\) curve in \(X\) with image in \(Y\) induces a continuous curve in \(Y\).
Then \(Y\) is weakly embedded in \(X\).

**Proof:** Otherwise, for a manifold \(S\) and a smooth map \(g : S \to X\) with
g(\(S\)) \(\subset Y\), there would be a sequence \((s_n)\) in \(S\) converging to a point \(s_0\) in
\(S\) whereas its image \((g(s_n))\) would not converge to \(g(s_0)\) in \(Y\). In charts,
let \(s_0\) be the origin. Without restriction, \(t_n := |s_n| < 1\) is strictly monotonically
decreasing. By taking chords between successive points \(s_n\) and
\(s_{n+1}\) and smoothing the corners, one obtains a smooth curve \(c : [0, 1[\to S\)
with \(c(t_n) = s_n\) which is continuously extendable to the origin by setting
\(c(0) := 0\). A further refinement involving a reparametrisation using the
third power of the arc length results in a smooth curve \(c : [0, 1[\to S\)
with \(c(t_n) = s_n\) which is \(C^1\) extendable to \([-1, 1]\) by defining \(c(t) = 0, t \leq 0\).
Thus, \(g \circ c : [-1, 1[\to X\) is \(C^1\) with image in \(Y\) but is not continuous as
a map into \(Y\), contradicting the hypothesis of the proposition.

Leaves of a foliation are classical examples of weakly embedded submanifolds
(see \[Mc, S\]). The objective here is to prove that if a Lie group \(G\) acts on a
manifold \(X\), the orbits \(Gx\) through points \(x\) in \(X\) are weakly embedded. This
will be proved by showing that the property of $C^1$ curves given in the above proposition is satisfied. The proof of this requires a Linearization Lemma and a Lifting Theorem. The latter will be proved by solving a differential equation. Tangent vector will always refer to vectors tangent to $C^1$ curves.

A Lie group $G$ is assumed to be separable and, consequently, $\sigma$–compact. Let $(\xi, x) \mapsto \xi x$ be a smooth action of $G$ on $X$. Recall that for $x \in X$ the canonical map $G/G_x \to X$, $\xi G_x \mapsto \xi x$, is an injective immersion and its image $Gx$ is an immersed submanifold of $X$. Denote by $\rho : G \to X$ the orbital map $\xi \mapsto \xi x$. The proof of this requires a Linearization Lemma and a Tangent vector will always refer to vectors tangent to $C^1$ curves.

**Proof:** Let $d$ be the dimension of $G$ and $N$ that of $X$ around $x \in X$. By (I), the Rank Theorem can be applied to the orbital map $\rho$ so that in charts $u$ for $G$ at $e$ with $u(e) = 0$ and $v$ for $X$ at $x$ with $v(x) = 0$, $\rho$ becomes $p \mapsto (pr_r(p), 0)$ for $0 \in \mathbb{R}^{N-r}$, where $pr_r$ is the projection onto the first $r$ coordinates. Let $c$ be defined on $I = ]-1, 1[$.

For any $\epsilon \in ]0, 1[$ there exist an $s \in I$ with $|s| \leq \epsilon$ and convergent sequences $(t_n)$ in $I \setminus \{s\}$ and $(\xi_n)$ in $G$ such that $t_n \to s$ and $c(t_n) = \rho(\xi_n)$ hold. To see this, for every $t$ in $I_\epsilon := [-\epsilon, \epsilon]$ choose any $\xi_t$ in $G$ with $c(t) = \rho(\xi_t)$. The $\sigma$–compactness of $G$ insures the existence of infinitely many $t \in I_\epsilon$ such that all $\xi_t$ lie in some compact subset of $G$. Choose a sequence $(t_i)$ of such $t$'s for which $(\xi_{t_i})$ converges. Then any convergent subsequence of $(t_i)$ proves the claim.

Assume first that $s = 0$. Let $\xi := \lim_n \xi_n$ and put $\xi'_n := \xi_n \xi^{-1}$. Then $\xi'_n \to e$ and $c(t_n) = \rho(\xi'_n)$ holds. Therefore $v \circ c(t_n) \in \mathbb{R}^r \times \{0\}$ and $(v \circ c)'(0) = (k, 0)$ for some $k \in \mathbb{R}^r$ and $0 \in \mathbb{R}^{N-r}$ due to $v \circ c(0) = 0$. Now $C(t) := u^{-1}(tk, 0)$, $0 \in \mathbb{R}^{d-r}$, defines a curve $\rho \circ C$ in the orbit $Gx$ satisfying $v \circ \rho \circ C(t) = (tk, 0)$. Since it also has $(k, 0)$ as its tangent at $x$, we are finished.

Next consider $s \neq 0$. Let $\rho_t : G \to X$ denote the orbital map $\xi \mapsto \xi c(t)$ of $G$ into $X$ defined by $c(t)$ for $t$ in $I$. Obviously, $\rho_0 = \rho$, and by hypothesis $Gc(t) = Gx$. Since $c(t_n) = \rho_n(\xi'_n)$, similar to the argument for $s = 0$ we obtain $\frac{dc}{dt}c \in T_{c(t)}(Gx)$.  

An appropriate continuity argument will now show that (II) holds for $s = 0$. For every $t$ in $I$ the dimension of the tangent space $T_t := \mathbb{R}^{N-r}$.
\( T_{c(t)}(Gx) = d_e \rho_t(T_eG) \) is \( r \) by \([\text{1}].\) Choose elements \( \Gamma_1, \ldots, \Gamma_r \) in \( T_eG \) so that \( d_e \rho(\Gamma_j) \), \( 1 \leq j \leq r \), form a basis of \( T_0 \). Since the maps
\[
I \to TX, \ t \mapsto \Gamma_j(t) := d_e \rho_t(\Gamma_j)
\]
are continuous, the \( \Gamma_j(t), \ 1 \leq j \leq r \), constitute a basis of \( T_t \) for small \( t \). The following calculations are best done in charts. Using an orthogonal basis obtained from \( \Gamma_j(t) \), \( 1 \leq j \leq r \), an expression for the length \( l(t) \) of the component of \( d_t c \) in \( \mathbb{R}^N \) orthogonal to \( T_t \) is deduced which depends continuously on \( t \). For the special parameters \( s, (\text{3}) \), is equivalent to \( l(s) = 0 \), and we obtain \( l(0) = 0 \), implying \( d_0 c \in T_0 \).

\[ \square \]

A special case of this Linearization Lemma is treated in [Ca/Ha, 4.37].

3 The Lifting Theorem

Even more than is claimed in the Linearization Lemma holds. Around the origin, \( c \) actually coincides with \( \rho \circ C \) where \( C \) is a curve in \( G \).

**Lifting Theorem** Let \( G \) be a Lie group acting on a manifold \( X \). Then any \( C^k \) curve \( c \) in \( X \) based at a point \( x \) in \( X \), whose image lies in the orbit \( Gx \) through \( x \), can be locally lifted to a \( C^k \) curve \( C \) in \( G \) based at the identity, i.e., \( C(t)x = c(t) \) for small \( t \), for \( k \geq 1 \).

**Proof:** The curve \( C \) will be the solution of a differential equation. To set up this equation choose a right inverse \( r(t) : T_t \to T_eG \) of \( d_e \rho_t \), i.e. \( (d_e \rho_t)r(t)\gamma = \gamma \) for all \( \gamma \in T_t \), which depends smoothly on \( t \) for small \( t \). Let \( R_{\xi} : G \to G, \eta \mapsto \xi \eta \), denote right multiplication on \( G \). Because the Linearization Lemma implies \( c'(t) := d_t c \in T_t \), the following differential equation for a curve \( C \) in \( G \) based at \( e \) is well-defined:
\[
C'(t) = d_e R_{C(t)} r(t) c'(t).
\]  
(3)

It remains to check that \( C(t)C(0) = c(t) \) holds or equivalently \( (C(t)^{-1}c(t))' = 0 \) for small \( t \). For \( \xi \in G \) let \( \alpha_\xi \) denote the diffeomorphism \( X \to X, \ y \mapsto \xi y \). Then
\[
(C(t)^{-1}c(t))' = d_{C(t)^{-1}} \rho_t (C(t)^{-1})' + d_{c(t)} \alpha_{C(t)^{-1}} c'(t).
\]  
(4)

The second summand is the negative of the first. To see this, observe that \([\text{3}] \), implies
\[
c'(t) = d_e \rho_t(d_e R_{C(t)})^{-1} C'(t).
\]
Differentiating \( e = C(t)^{-1}C(t) \) yields \( 0 = (d_e R_{C(t)})^{-1} C'(t) + d_{C(t)^{-1}} L_{C(t)} (C(t)^{-1})' \), where \( L_{\xi} \) denotes left multiplication on \( G \) and
where \( d_{C(t)} R_{C(t)}^{-1} = (d_{e} R_{C(t)})^{-1} \) has been used. The second summand in (4) now becomes

\[-d_{e(t)} \alpha_{C(t)}^{-1} d_{e} \rho_{t} d_{C(t)}^{-1} L_{C(t)}(C(t)^{-1})'.\]

However, since \( \rho_{t} = \alpha_{C(t)}^{-1} \circ \rho_{t} \circ L_{C(t)} \) is true, this term is the negative of the first summand in (4).

\[\blacksquare\]

A special case of this theorem is in [Ca/Ha, Exercise 4.7]. Because the orbital map \( G \to Gx, \xi \mapsto \xi x \), is smooth, we can conclude with

**Corollary 1** Every \( C^k \) curve in \( X \) based at a point \( x \) in \( X \), whose image lies in the orbit \( Gx \) through \( x \), is a \( C^k \) curve in that orbit, for \( k \geq 1 \).

**Corollary 2** Every \( C^k \) curve in an immersed submanifold \( Z \) of \( X \), which is contained in the orbit \( Gx \), defines a \( C^k \) curve in that orbit, for \( k \geq 1 \).

**Corollary 3** The orbits of \( G \) in \( X \) are weakly embedded.

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