Local geometry from entanglement entropy

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Abstract

Constructing the corresponding geometries from given entanglement entropies of a boundary QFT is a big challenge and leads to the grand project \textit{it from Qubit}. Based on the observation that the AdS metric in the Riemann Normal Coordinates (RNC) can be summed into a closed form, we find that the AdS\textsubscript{3} metric in RNC can be straightforwardly read off from the entanglement entropy of CFT\textsubscript{2}. We use the finite length or finite temperature CFT\textsubscript{2} as examples to demonstrate the identification.
Quantum entanglement is one of the most distinct features of quantum systems. When a quantum system is divided into two parts, the natural way to measure the correlation between these two subsystems is to calculate the Entanglement Entropy (EE). We divide the system under consideration into two regions: $A$ and $B$. The total Hilbert space is therefore decomposed into $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Then, one traces out the degrees of freedom of region $B$ to get the reduced density matrix of region $A$: $\rho_A = \text{Tr}_{\mathcal{H}_B} \rho$. The entanglement entropy of the region $A$ is evaluated by the von Neumann entropy, $S_A = -\text{Tr}(\rho_A \ln \rho_A)$. It is widely believed that the entanglement entropy is of great help to realize AdS/CFT correspondence [1], the correspondence between the quantum gravity in the bulk of AdS and gauge theory on the AdS conformal boundary. Though no proof for this correspondence exists so far, many evidences have been proposed to support this conjecture. Based on AdS/CFT, a holographic way to calculate the entanglement entropy from the dual gravity theory was proposed by Ryu and Takayanagi (RT) [2] and has been verified extensively on AdS$_3$/CFT$_2$, referring to a recent review [3] and references therein. To approach the entanglement entropy from the dual gravity side, RT generalized the well-known Bekenstein-Hawking entropy $S_{BH}$ of black hole [4, 5]:

$$S_{BH} = \frac{\text{Area (horizon)}}{4G_N^{(D)}} , \quad (1)$$

which turns out to be the same as the entanglement entropy $S_A$ of region $A$ on the boundary where the conformal theory lives. To be specific, considering an AdS$_{d+1}$, on the boundary $\mathcal{B}$, taking a spatial slice $\Sigma_B \subset \mathcal{B}$, for a subset $A \subset \Sigma_B$, whose boundary is $\partial A$, a minimal surface $\gamma_A$ in the bulk of AdS$_{d+1}$ can be identified. The minimal surface $\gamma_A$ shares the same boundary of $A$, namely $\partial A = \partial \gamma_A$. Then, comparing with the Bekenstein-Hawking entropy, one can treat the area of the minimal surface $\gamma_A$ as a “horizon” of $A$. Thus we can define the entanglement entropy of $A$ as

$$S_A = \frac{\text{Area (}\gamma_A\text{)}}{4G_N^{(D)}} . \quad (2)$$

Since the entanglement entropies calculated from the holographic CFT and dual gravity are in agreement as expected from AdS/CFT, it is natural to ask if we can construct the dual geometry from given entanglement entropies. This idea leads to the grand project it from Qubit. However, when we calculate the area of minimal surface in the bulk, the geometric structure, specifically the metric, is lost. It is then difficult to realize the relation between the spacetime geometry and quantum entanglement. In [6, 7, 8], the authors made efforts to build connections between the spacetime geometry and the quantum entanglement in QFT. This idea was further developed by Maldacena and Susskind [9]. They conjectured an equivalence between Einstein-Rosen bridge (ER) and Einstein-Podolsky-Rosen (EPR) paradox. In a recent work [10], the authors made a nice progress on this direction by introducing the concept of kinematic space, defined on the oriented geodesics of AdS$_3$. After identifying the Crofton form with the second derivatives of the given entanglement entropy of CFT$_2$, they read the metric of dS$_2$ which is the kinematic space of AdS$_3$. 

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Based on our previous work [11], in this paper, without any auxiliary fields, we give a transparent construction of the local geometry from a given entanglement entropy, specifically, AdS$_3$ metric as an instance. To start with, let us recall the strategy of RT proposal in AdS$_3$/CFT$_2$. To calculate the holographic entanglement entropy, on the geometry side, one needs to calculate the area of the corresponding minimal surface. In AdS$_3$ static configuration, the minimal surface is simplified as geodesics. Therefore, geodesics play a central role in this formalism. On the other hand, it is inspiring to notice that geodesics are the basis of the Riemann Normal Coordinates (RNC). We are thus led to conjecture RNC may be of use to construct the geometry from given entanglement entropy. Moreover, the holographic entanglement entropy is usually determined by two length scales, $l$ of the visible subsystem $A$ and $L$ of the whole system under consideration. While in RNC, the metric for a point $P$ is determined by its distance from the origin $O$ and the scale of the whole bulk geometry.

Remarkably, it turns out indeed we can read the RNC metric straightforwardly from the entanglement entropy. This is possible by the observation that in RNC, the perturbatively expanded metric can be summed into a closed form for maximally symmetric spaces. Moreover, the sum for AdS$_{d+1}$ matches precisely the string worldsheet topology sum in flat spacetime. We are thus led to conclude in [11] that AdS genus zero worldsheet corresponds to all genus expansion of string theory in flat spacetime. This correspondence connects geometry and topology.

Let us demonstrate how this works in RNC. On a manifold $M$, taking a point $O \in M$, any vector in the tangent space $V \in T_O M$ defines a unique geodesic $\gamma(s)$ passing through $O$ by the exponential map $\exp: T_O M \to M$. The tangent vector of the geodesic at $O$ is $V, \gamma(0) = O, \frac{d\gamma}{ds}|_{s=0} = V, \exp(V) = \gamma(1)$.

Then set the coordinates $x^\mu(s) = sV^\mu$ to build the RNC. From the construction, the RNC takes geodesics as coordinates and we have $g_{ij}(O) = \delta_{ij}, \partial g_{ij}(O) = 0$. It is known that RNC fails around singularities where geodesics cannot reach (geodesically incomplete), or the neighborhood of some point where different geodesics emanating from the origin $O$ cross. So at least for manifolds we are concerned like AdS, the RNC is well defined for finite regions in the bulk. Without loss of generality, we set the coordinate of the origin $O$ as 0 and expand the metric in the defining region, which is not necessarily local. The Taylor expansion is greatly simplified in RNC

$$g_{ij}^{RNC}(X) = \delta_{ij} + \frac{1}{3} R_{iklj} X^k X^l + \frac{1}{6} D_k R_{ilmj} X^k X^l X^m + \frac{1}{20} \left( D_k D_l R_{ilmj} + \frac{8}{9} R_{iklp} R^p_{mnj} \right) X^k X^l X^m X^n + \ldots \quad (3)$$

We now set the background as a $d + 1$-dimensional AdS spacetime which is a maximally symmetric space with $D_m R_{ikjl} = 0$ and $R_{ikjl} = -\frac{1}{R_{AdS}} (g_{ij} g_{kl} - g_{ik} g_{jl})$. We choose Euclidean signature because what we are going to compare with is the static case of the CFT$_2$ entanglement entropy. It is remarkable that, referring to the derivations in the Appendix of [11], the expansion can be summed over as a closed form.

$$g_{ij}^{RNC}(X) = \delta_{ij} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{2n+2}}{(2n + 2)!} \delta_{il} \ell_{a_1} \ell_{a_2} \ldots \ell_{a_{n-1}}$$
\[
\begin{align*}
\left( \ell^2 \right)^a_{\ b} & \equiv \delta^a_b X^2 + X^a X_b. \\
\left( \ell^2 \right)^a_{\ b} & \equiv -\delta^a_b X^2 + X^a X_b.
\end{align*}
\]

Or we can write the metric in a more explicit but less compact form

\[
\begin{align*}
g_{\text{RNC}}^{ij}(X) &= \left( \delta_{ij} - \frac{X_i X_j}{X^2} \right) \frac{\sin^2 \left( \frac{\ell X_i}{R_{\text{AdS}}} \right)}{X^2 / R_{\text{AdS}}^2} + \frac{X_i X_j}{X^2},
\end{align*}
\]

where the indices are raised and lowered by \( \delta_{ij} \) and \( |X| = \sqrt{X^2} \). It is very interesting to note that the distance \( |X| \) is not measured by the AdS geometry, but by the flat metric, which again motivates us to identify it with the length scale of the subsystem on the boundary in the following discussions.

We now look at the entanglement entropies calculated from CFT. It is not easy to compute the von Neumann entropy directly in CFT. Instead, the replica trick is employed to calculate Renyi entropy \( S^{(n)}_A \) \cite{14,15} and the entanglement entropy is obtained as follows

\[
S_{EE} \equiv \lim_{n \to 1} S^{(n)}_A = \lim_{n \to 1} \left[ \frac{1}{1-n} \ln \left( \text{tr} \rho^n_A \right) \right].
\]

In this work, we are focused on the static cases. The vacuum entanglement entropy of CFT in a finite region, corresponding to pure AdS\(_3\), is given by

\[
S^{\text{vac}}_{EE} = \frac{c}{6} \ln \left[ \frac{\sin^2 \left( \frac{\pi}{L} \right)}{\left( \frac{\pi}{L} \right)^2} \right] + \text{divergent terms},
\]

where \( l \) is the length of the subsystem under consideration and \( L \) is the length of the total system. It is immediate to see that one can read the RNC metric (4) directly from this entanglement entropy under the identifications

\[
L \leftrightarrow \pi R_{\text{AdS}}, \quad l^2 \leftrightarrow \ell^2.
\]

These two identifications are very physical. The identification \( L \leftrightarrow \pi R_{\text{AdS}} \) is inspiring since, in a rough sense, \( L \) represents the circumference of radius \( R_{\text{AdS}} \) geometry. One may be puzzled by the identification of \( l^2 \) and \( (\ell^2)^a_{\ b} \) since \( l \) is a number while the latter is a matrix. This can be easily understood by looking at eqn. (5) and (6). Basically, \( l \) corresponds to the length \( |X| \) of a set of vectors \( X^i \)'s, which implies that for a specific \( l \), we can determine the metric for all the points with the same distance \( l \) from the origin by using (5). This actually provides an explanation to the overdetermined puzzle, namely, in an asymptotically AdS\(_{d+1}\) spacetime, the metric usually is specified by \((d+1)(d+2)/2\) functions, in contrast to that the bipartition of a Cauchy slice is specified by two functions.

In eqn. (8), the divergent term takes a form of \( \frac{c}{a} \ln \frac{a}{\ell^2} \), where \( a \) is a cut-off. This terms accounts for the failure of the RNC near the AdS boundary. Therefore, we even get more information than naively expected, i.e. the existence of the asymptotic region.
We turn to another closely related configuration. The entanglement entropy with finite temperature is given by

\[ S_{EE}^{\text{fin}} = \frac{c}{6} \ln \left[ \frac{\sinh^2 \left( \frac{\pi l}{\beta} \right)}{\left( \frac{\pi l}{\beta} \right)^2} \right], \tag{10} \]

where \( \beta = \frac{1}{T} \) is an inverse of temperature and we dropped off the divergent term for reasons explained. It is straightforward to get the corresponding RNC metric by identifying

\[ i\beta \leftrightarrow \pi R_{\text{AdS}}, \quad \ell^2 \leftrightarrow \ell^2. \tag{11} \]

The corresponding global geometry can be easily found

\[ ds^2 = R_{\text{AdS}} \left( - \cosh^2 (\rho) \, dt_E^2 + d\rho^2 + \sinh^2 (\rho) \, d\theta^2 \right), \tag{12} \]

Note in order to compare with the static configuration consistently, we are using Euclidean AdS from the very beginning, so there is a minus sign for the \( dt_E^2 \) term, which is different from the usual conventions in literature. Therefore, this global geometry is nothing but the Euclidean BTZ black hole. More explicitly, based on the equivalence between the Euclidean BTZ black hole at temperature \( T \) and Euclidean AdS\(_3\) at temperature \( 1/T^4 \), utilizing the transformation

\[ r = r_+ \cosh \rho, \quad \tau = \frac{R_{\text{AdS}} t}{r_+}, \quad \varphi = -i \frac{R_{\text{AdS}} t_E}{r_+}, \tag{13} \]

we get the three dimensional Euclidean BTZ black hole

\[ ds^2 = (r^2 - r_+^2) \, d\tau^2 + \frac{R_{\text{AdS}}}{r^2 - r_+^2} \, dr^2 + r^2 \, d\varphi^2, \tag{14} \]

where \( 0 < \varphi < 2\pi \) and \( \tau \) is compactified as \( \tau \sim \tau + \frac{2\pi R}{r_+} \).

The eqn. (9) and (11) show transparent identifications of the local RNC metric from the dual CFT entanglement entropy. This result is quite surprising. How to interpret it? Let us think about the geometry side. As a local theory, basically, all the information about classical gravity is contained in the metric (with Levi-Civita connection). For most asymptotic AdS geometries we are concerned, the RNC can be defined in finite region - named as quasi-local. We already see that the RNC metric at point \( P \) depends on the location of the origin \( O \), in some sense, these two points are “entangled”. On the other hand, since the metric at \( O \) is flat, in other words, the information about the geometry at \( O \) is not reachable by solely referring to itself, but is stored in points like \( P \). Therefore, in a loose sense, RNC is the natural language for the gravity version of entanglement entropy.

We take the finite region or finite temperature CFT\(_2\) as examples in this work. Since the dual geometry is very simple, we can easily figure out the global geometries, AdS\(_3\) or BTZ black hole respectively. It is very tempting to ask: does this identification between RNC metric and entanglement entropy also work for other more non-trivial asymptotic AdS geometries? When the holographic entanglement entropy does not take a closed form, the identification of local metric seems still possible. But for the global metric, it may not be
It is inspiring to notice that in the RNC metric expansion for any geometry, what really control the expansion are the connections. Therefore, it looks possible to construct the global geometry via the connections. The system with a finite size at finite temperature for a free Dirac fermion in two dimensions is a good arena to test this conjecture.

Moreover, based on the fact that the RNC metric must respect the Einstein equation, it would not be very hard to construct the dynamical equation for the entanglement entropy through the identification.

The Renyi entropy, as a trick to calculate the entanglement entropy, has an obscured physical picture. It is calculated from the two point functions of twist fields. We know that the two point functions of CFT2 is completely determined by the distance and the conformal dimension of the operators. From our derivations, the information about the distance on the CFT2 is completely included in the RNC metric. This may provide some clues to build a correspondence between the CFT two point function and some gravity quantities.

In a previous paper, we demonstrated a correspondence between genus zero string worldsheet in AdS and the sum of all genus string worldsheet in flat geometry. Comparing with the observation in this paper, we are led to conjecture there might be another method to calculate the entanglement entropy in CFT, specifically, alike to the Gopakumar-Vafa (GV) method, lifting the theory to the equivalent M-theory and calculate the entanglement entropy in a physically transparent way. From what we learned from GV, we tend to believe it would represent a sum of the genus expansion of string worldsheets.

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