Super Yang-Mills Theory on Noncommutative Torus from Open Strings Interactions

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Abstract

Considering the scattering of massless open strings attached to a D2-brane living in the $B$ field background, we show that corresponding scattering up to the order of $\alpha'^2$ is exactly given by the gauge theory on noncommutative background, which is characterized by the Moyal bracket.

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Introduction

After the novel work of Witten [1], it was realized that the low energy dynamics of open strings attached to a D-brane is governed by the Super Yang-Mills (SYM) theory defined on the D-brane world volume. The gauge group of the corresponding SYM theory is $U(N)$, $N$ being the number of coincident D-branes. Elaborating on the point that such open strings describe the dynamics of D-branes [2], led to the M(atrix)-model [3]. M(atrix)-model as a conjecture for discrete light cone quantization (DLCQ) of M-theory should also contain the background three-form field of 11 dimensional supergravity, such a background field was studied in [4]. There Connes, Douglas and Schwarz (CDS) conjectured that the SYM theory on a noncommutative torus properly describes the three-form field background.

Since the eleven dimensional three-form is related to the NSNS two from field of string theory, the CDS conjecture means that the dynamics of D0-branes, or more generally any other D-brane, in a $B$ filed background is described by a gauge theory living on a noncommutative torus, $T^d_\theta$. It has been argued that the $\theta$ parameter of the torus should be identified with the constant $B$ field background [5]. Thereafter the D-brane dynamics or the M(atrix)-model on a noncommutative space has been vastly studied [6,7,8,9,10, 11,12,13,14,15,16,17]. It was argued in [5] that, we can realize the characteristics of the SYM on noncommutative torus, the Moyal bracket, in string theory. In [9,13] extending the ideas of [5], they studied the D0-brane dynamics in both constant and non-constant B field background and showed that this will modify the ordinary multiplication of field with the Moyal product.

In this paper using the usual string theory methods, we study the interaction of open strings attached to a D2-brane living in the B field background. We show that the same scattering amplitudes upto second order of $\alpha'$, can be obtained from the SYM theory on noncommutative torus. We show that $O(\alpha')$ corrections to SYM are cancelled at super open string tree level. By going to a dual picture of above, i.e. a D-string in B field background considering the open string scattering, we will give an intuitive way of finding the Fourier transformed of the interaction terms of the effective action.

Open string scattering

In order to calculate the open strings scattering amplitude, first we discuss their mode expansions. Although the following argument holds for the most general case, for simplicity let us consider a D2-brane winding around an orthogonal torus defined by:

$$\tau = \frac{iR_2}{R_1}, \quad \rho = iR_1R_2 + B.$$  \hspace{1cm} (1)

The open string attached to such a D2-brane satisfy the boundary conditions [18]
\[ \begin{align*} 
\partial_\sigma X^0 &= 0 \\
\partial_\sigma X^i + B^i_j \partial_\tau X^j &= 0 \quad i, j = 1, 2 \\
\partial_\tau X^a &= 0 \quad , \quad a = 3, ..., 9.
\end{align*} \]

Hence the mode expansion of these open strings are [15]

\[ \begin{align*} 
X^0 &= x^0 + p^0 \tau + \sum_{n \neq 0} a_n^0 \frac{\epsilon^{-in\tau}}{n} \cos n\sigma \\
X^i &= x^i + (p^i - B^i_j p^j) + \sum_{n \neq 0} \frac{\epsilon^{-in\tau}}{n} (i a^i_n \cos n\sigma + B^i_j a^j_n \sin n\sigma) \quad i, j = 1, 2 \\
X^a &= x^a + \sum_{n \neq 0} a^a_n \frac{\epsilon^{-in\tau}}{n} \sin n\sigma,
\end{align*} \]

where \( x^i \) show an arbitrary point on D2-brane and \( p^i \) are vectors on the dual torus:

\[ p^i = \frac{n_i}{R_i}, \quad n_i \in \mathbb{Z}. \]

We observe that in the case of non-zero B field the coefficient of \( \sigma \) in the mode expansion is proportional to \( p^i \). As it is shown in [15], this is the root of having noncommutative coordinates:

\[ [x^i, p^j] = i\eta^{ij}, \quad [x^i, x^j] = -2iB^{ij}. \]

This is a clue showing the field theory living on a D2-brane in B field background, which is described by these open strings, is not a conventional local field theory.

To find the effective low energy theory governing these open strings, we consider their scattering processes.

**Three super open string scattering**

Let us consider three massless vector vertex [19]:

\[ S_{int3} = \langle V_1 V_2 V_3 \rangle, \]

where

\[ V_a = \int d\sigma \xi_a \Pi \exp(ik_a X(\sigma, \tau)), \]

with \( \xi_a, k_a \) are the polarization and momentum of the vector state: \( \xi_a k_a = k_a^2 = 0 \). Here we only consider the nontrivial cases, i.e. \( \xi_a, k_a \) are vectors on the torus. \( \Pi \) is the conjugate momentum of \( X(\sigma, \tau) \), which is proportional to \( p^i \) [20], minus a fermionic part which is
$-\frac{1}{4} \epsilon_{i j} S^a S^b k^j \delta_{i j}$ [19] and $X$ is the mode expansion given by eq.(3).

Calculating $S_{int3}$ with string theory methods [19], we find

$$S_{int3} = g_\xi \xi_2 \xi_3 \xi_3 \sum_{a<b} i B_{ij} k_a^i k_b^j \delta \left( \sum_a k_a^i \right),$$

where

$$t_{\mu \nu \rho} = k_2 \eta_{\mu \nu} + k_3 \eta_{\mu \rho} + k_1 \eta_{\nu \rho}.$$ (9)

The term $\exp(\sum_{a<b} i B_{ij} k_a^i k_b^j)$, which shows the B dependence of the interaction, considering the momentum conservation, can be written as

$$\exp(i B_{ij} k_a^i k_b^j) \quad a < b.$$ (10)

So $S_{int3}$ corresponds to the cubic interaction derived from the $tr F^2$ in SYM on noncommutative torus. More explicitly, $S_3$ shows the Fourier transformed from

$$\int d^3 x \partial \mu A_\nu(x) \exp(i B_{\mu \nu \rho} \partial \mu \partial \nu \partial \rho) A_\mu(x, x^1, x^2) A_\nu(x, x^1, x^2) \big| x^\mu = x^\nu, x^\rho = x^\nu.$$ (10)

Again by the usual string theory calculations:

$$S_{int4} = \langle V_1 V_2 V_3 V_4 \rangle.$$ (11)

Four open string scattering

To obtain the full effective action, we should also consider the $M$ open string scattering, but it is easy to show that all the higher $M \ (M > 4)$ are reduced to the three and four open string scattering. Hence we consider the four string massless vector vertex:

$$S_{int4} = \langle V_1 V_2 V_3 V_4 \rangle.$$ (11)

Again by the usual string theory calculations:

$$S_{int4} = -\frac{1}{2} g^2 \xi_1^\mu \xi_2^\nu \xi_3^\alpha \xi_4^\beta K_{\mu \nu \alpha \beta} \exp(\sum_{a<b} i B_{ij} k_a^i k_b^j) \delta(\sum_a k_a^i),$$ (12)

where the kinematic factor $K$, is symmetric under the exchange of 1,2 with 3,4, and anti-symmetric under exchange of 1 with 2 and 3 with 4, and contains two type of terms: The term proportional to $-\frac{1}{4} \eta_{\mu \nu} \eta_{\alpha \beta}$ and its permutations, and the terms of the form $\frac{1}{2} k_a^a k_a^b \eta_{\nu \beta}$. Since

$$\exp(\sum_{a<b} i B_{ij} k_a^i k_b^j) = \exp(i B_{ij} k_1^i k_2^j) \exp(i B_{ij} k_3^i k_4^j)$$

2 All other permutations in result is also possible.
Taking only the massless poles in $K$, the first type gives rise to the terms

$$[A_{\mu}, A_{\nu}]_{M.B.}[A^{\mu}, A^{\nu}]_{M.B.},$$

in the effective action, and the second type, is described by

$$\partial_{\mu} A_\nu \ [A_{\mu}, A_\alpha]_{M.B.} \partial_\alpha A_\nu.$$  

(14)

As we see in the tree level of the super open string scattering there is no $O(\alpha')$ corrections to the effective action of SYM on noncommutative torus.

Putting the results of three and four super open string scattering together, the low energy effective action reads as:

$$S = \int tr F^2 d^3x,$$

with

$$F_{\mu\nu} = \partial_{\mu} A_\nu - \partial_\nu A_\mu + [A_{\mu}, A_\nu]_{M.B.}.$$  

(16)

and the $tr$ is performed on the noncommutative torus.

**Intuitive calculations**

The calculations given above can also be understood in an intuitive way. To do so, we consider a D-string wound around the cycle $R_1$, with a B field background. We will momentarily show that generalizing the ideas of [5], one can obtain the full SYM on $T^2_\theta$. As we will see in the D-string version, we should perform a Fourier transformation to obtain the Moyal bracket structure, i.e. the T-duality in string theory is like the Fourier expansion in related field theory.

In three open string scattering, two open strings can interact when the end of the first coincides with the beginning of the second. It is worth noting that the open strings attached to D-string are oriented. Here we deal with three open strings attached to such a D-string. Mode expansion of these open strings are [15]

$$\begin{cases}
X^i = x^i_0 + p^i \tau + L^i \sigma + Oscil. \ , \ i = 1, 2 \\
X^0 = x^0_0 + p^0 \tau + Oscil. \\
X^a = x^a_0 + Oscil. \ , \ a = 3, ..., 9
\end{cases}$$

(17)

where

$$p^1 = \frac{r}{R_1}, \quad p^2 = 0$$

$$L^1 = \frac{qB}{R_1}, \quad L^2 = qR_2 \quad r, q \in Z.$$  

(18)
Hence the beginning and the end of these open strings up to the torus identifications are

\[
\text{beginning: } \\
\begin{array}{c}
(1) \\
(2) \\
(3)
\end{array}
\begin{array}{c}
(x^1, 0) \\
(x^2, 0) \\
(x^3, 0)
\end{array}
\begin{array}{c}
(x^1 + l^1, 0) \\
(x^2 + l^2, 0) \\
(x^3 - l^3, 0)
\end{array}
\]

(19)

with \( l^i = \frac{a^i B}{R_1} \). The interaction condition reads as:

\[
x^1 + l^1 = x^2, \quad x^2 + l^2 = x^3 - l^3, \quad x^3 = x^1.
\]

(20)

Resulting in:

\[
l^1 + l^2 + l^3 = 0
\]

(21)

Whenever the field associated with the open string, aside from its tensorial structure, is denoted by \( \Phi(x, l) \), the interaction of three open string is given by:

\[
S_{int3} = \sum_{q^i} \int dx^1 dx^2 dx^3 \Phi_1(x^1, l^1) \Phi_2(x^2, l^2) \Phi_3(x^3, l^3) \delta(x^3 - x^1) \delta(x^2 - x^1 - l^1) \delta(q^1 + q^2 + q^3)
\]

\[
= \sum_{q^1, q^2} \int dx^1 \Phi_1(x^1, q^1) \Phi_2(x^1 + l^1, q^2) \Phi_3(x^1, -q^1 - q^2).
\]

(22)

After Fourier transformation and putting \( \sigma_i = \frac{R_i x^i}{l} \) [5]

\[
S_{int3} = \int d\sigma^1 d\sigma^2 \Phi_1(\sigma^1, \sigma^2) \Phi_2(\sigma^1, \sigma^2) \Phi_3(\sigma^1, \sigma^2).
\]

(23)

We can also have the four open string scattering. Let us consider two open strings as:

\[
\text{beginning: } \\
\begin{array}{c}
(1) \\
(2) \\
(3)
\end{array}
\begin{array}{c}
(x^1, 0) \\
(x^2, 0)
\end{array}
\begin{array}{c}
(x^1 + l^1, 0) \\
(x^2 + l^2, 0)
\end{array}
\]

(24)

These open strings can split at some mid-point and join again, so that after scattering we have two open strings starting at \( (x^1, 0) \) and \( (x^2, 0) \) and ending at \( (x^2 + l^2, 0) \) and \( (x^1 + l^1, 0) \) respectively:

\[
\text{beginning: } \\
\begin{array}{c}
(4) \\
(3)
\end{array}
\begin{array}{c}
(x^1, 0) \\
(x^2, 0)
\end{array}
\begin{array}{c}
(x^2 + l^2, 0) \\
(x^1 + l^1, 0)
\end{array}
\]

(25)

Imposing the condition that these open strings should end on the D-string forces

\[
x^1 + l^1 - x^2 = l,
\]

(26)
where \( l \) is of the form of \( \frac{nB}{l_i} \) for arbitrary integer \( n \). Hence the effective interaction can be written as

\[
S_{\text{int}} = \sum_{q_1,q_2,n} \int dx^1 dx^2 \Phi_1(x^1, l^1) \Phi_2(x^2, l^2) \Phi_3(x^1, l^1 - l) \Phi_4(x^2, l).
\] (27)

Then one can Fourier transform eq.(27) to obtain:

\[
S_{\text{int}} = \int d\sigma^1 d\sigma^2 \exp(iB \frac{\partial}{\partial \eta^1} \frac{\partial}{\partial \eta^2}) \Phi_1(\sigma^1, \eta^2) \Phi_2(\eta^1, \sigma^2)|_{\eta^1=\sigma^1} \exp(iB \frac{\partial}{\partial \xi^1} \frac{\partial}{\partial \xi^2}) \Phi_3(\sigma^1, \xi^2) \Phi_4(\xi^1, \sigma^2)|_{\xi^1=\sigma^1}
\] (28)

So the interaction part of the action is:

\[
S_{\text{int}} = S_{\text{int}3} + S_{\text{int}4} = \int d\sigma^1 d\sigma^2 (\Phi_1 [\Phi_2, \Phi_3]_{M.B} + [\Phi_1, \Phi_2]_{M.B} [\Phi_3, \Phi_4]_{M.B}).
\] (29)

In our case we know that the \( \Phi \) fields describing the light states of open strings are vector states, so eq.(29) gives the interaction part of the SYM on \( T^2_\theta \), with \( \theta \) identified with \( B \).

**Discussion**

In this paper considering three and four open strings scattering, we explicitly showed that up to tree open string level at low energies, the SYM on \( T^2_\theta \) effectively describes the massless vector open strings.

Appearance of the phase factor in eq.(8) and eq.(12) is a result of using mixed boundary conditions, the eq.(2). This phase factor first pointed out in [22]. The background field considered there ([22]) was the electric, which as discussed in [15,20] will not lead to any noncommutativity in space-time.

We should note that considering the super strings removes the \( \alpha' \) corrections to the effective action at tree level and hence the first correction to the SYM is of the order of \( \alpha'^2 \). These correction terms as calculated in [21], are

\[
L = \frac{-1}{4} tr F^2 + \frac{1}{8} (2\pi \alpha')^2 (F^4 - \frac{1}{4} (F^2)^2) + O(\alpha'^3),
\] (30)

but in our case, i.e. when we have a non-zero background B field, \( F_{\mu\nu} = \partial_{[\mu} A_{\nu]}(x) + [A_{\mu}, A_{\nu}]_{M.B.} \).

This correction is in perfect agreement with the leading terms appearing in the expansion of DBI action on \( T^2_\theta \). So it seems that one can regain the full DBI action by considering all order corrections to the SYM theory.

In this paper, we studied scattering amplitudes of open strings attached to a D-brane with the B field turned on and verified that they are consistent with the SYM theory on
noncommutative torus. It would be interesting to generalize the calculations of [23], the N coincident D-branes, to the case with non-vanishing B field background. In this case we expect to see both the Moyal and the gauge group commutators in the effective action.

Obtaining the SYM on noncommutative torus from the open strings scattering, tells us that this theory, although being a non-local field theory, may be renormalizable. Moreover we can show that this gauge theory can be obtained from a dual gauge theory on a commutative background [24]. In this way, one can hope to handle the question of renomalizability of SYM on noncommutative torus.

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