SU(2) slave-boson formulation of spin nematic states in $S = \frac{1}{2}$ frustrated ferromagnets

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An SU(2) slave boson formulation of bond-type spin nematic orders is developed in frustrated ferromagnets, where the spin nematic states are described as the resonating spin-triplet valence bond (RVB) states. The $d$-vectors of spin-triplet pairing ansatzes play the role of the directors in the bond-type spin quadrupolar states. The low-energy excitations around such spin-triplet RVB ansatzes generally comprise the (potentially massless) gauge bosons, massless Goldstone bosons, and spinon individual excitations. Extending the projective symmetry group argument to the spin-triplet ansatzes, we show how to identify the number of massless gauge bosons efficiently. Applying this formulation, we next (i) enumerate possible mean field solutions for the spin one half frustrated ferromagnetic $J_1$-$J_2$ Heisenberg model on the square lattice, with ferromagnetic nearest neighbor $J_1$ and competing antiferromagnetic next-nearest neighbor $J_2$, and (ii) argue their stability against small gauge fluctuations. As a result, two stable spin-triplet RVB ansatzes are found in the intermediate coupling regime around $J_1 : J_2 \simeq 1 : 0.4$. One is the $Z_2$ Balian-Werhamer (BW) state stabilized by the Higgs mechanism and the other is the SU(2) chiral $p$-wave (Anderson-Brinkman-Morel) state stabilized by the Chern-Simon mechanism. The former $Z_2$ BW state in fact shows the same bond-type spin quadrupolar order as found in the previous exact diagonalization study [N. Shannon et al., Phys. Rev. Lett. 96, 027213 (2006)].

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I. INTRODUCTION

Recent theoretical progress has revealed that a certain class of frustrated magnets shows spin nematic states, as their magnetic ground states, where the spin quadratic tensor, $K_{j\mu,\nu} = \langle S_{j\mu} S_{j\nu} \rangle - \frac{1}{2} \rho_{\mu,\nu} \langle S_j \cdot S_j \rangle$ with $(\mu,\nu = 1, 2, 3)$, exhibits a long-range order, while the spin moment $\langle S_{j\mu} \rangle$ remains disordered. Such spin nematic states can be classified into the chiral type (p-nematic) and non-chiral type (n-nematic) states, according to the parity of the spin quadratic tensor. Namely, the antisymmetric quadratic tensor $P_{j\mu,\lambda} = \epsilon_{\mu,\nu,\lambda} K_{j\mu,\nu}$ is nothing but the vector chirality, while the symmetric part — non-chiral one — plays the role of the spin quadrupolar moment, $Q_{j\mu,\nu} \equiv \frac{1}{3} \rho_{\mu,\nu} (K_{j\mu,\nu} + K_{j\nu,\mu})$. The latter ordered state is a spin analogue of the nematic state well known in liquid crystals, where the order parameter is characterized by the so-called ‘director vector’ $d(r)$ in the form

$$Q_{\mu,\nu}(r) = d_\mu(r) d_\nu(r) - \frac{1}{3} \delta_{\mu,\nu} |d(r)|^2.$$  \hspace{1cm} (1)

From this analogy, the spin quadrupolar states are often dubbed simply as the ‘spin-nematic’ states.

Depending on how the spin quadrupolar moments are microscopically organized, spin nematic states have two distinct classes; (i) site-type nematic states, and (ii) bond-type nematic states. The former types of nematic orders are realized in the spin 1 bilinear-biquadratic model, $\mathcal{H}_{S=1} = \sum_{(ij)} [S_i \cdot S_j + K (S_i \cdot S_j)^2]$, where the quadrupolar moments constituted at respective sites exhibit the long-range order due to the strong biquadratic coupling. Ground state wavefunctions of these site-type nematic states can be essentially factorized into decoupled ‘vacuums’, which are defined on respective sites. Thus, their spin-wave theories, including low-energy effective theories, were well-established. Namely, the elementary excitation around such a site-factorized vacuum is also given by a linear combination of bosons introduced on respective sites.

The simplest localized spin models which allow the second class of spin nematic states — bond-type nematic states — are the spin one half frustrated ferromagnets, which could be realized in a certain family of layered cuprates, and vanadates and also in solid 3He films. For example, in (CuX)LaNb$_2$O$_7$ (X=Cl,Br), Cu$^{2+}$ ions, having a localized spin $\frac{1}{2}$, compose a square lattice, while the anion X$^-$ locates at the center of the square instead of the bond center. As a result, the nearest neighbor (NN) exchange interaction $J_1$ between the localized spins becomes ferromagnetic because of the Goodenough-Kanamori rule, while the next nearest neighbor (NNN) interaction $J_2$ becomes antiferromagnetic; the model-Hamiltonian is given by

$$\mathcal{H} = -J_1 \sum_{(ij)} S_i \cdot S_j + J_2 \sum_{((ij))} S_i \cdot S_j,$$ \hspace{1cm} (2)

with $J_1, J_2 > 0$. The preceding exact diagonalization (ED) studies for this spin one half square lattice $J_1$-$J_2$ model indicated that the $d$-wave bond-type spin nematic order develops in the intermediate parameter region, $J_1 \simeq 2J_2$. Namely, strong ferromagnetic exchange interactions favor the spin-triplet valence bond formations between two neighboring spin one halves, while, simultaneously, these two spin one halves try to change their partners quantum-mechanically by way of the NNN antiferromagnetic exchange interactions. This leads to a
kind of resonating spin-triplet valence bond state, where the quadrupolar moment organized at each neighbor bond exhibits the following antiferro-type configuration with the uniform amplitude:

$$Q_{j,j+\hat{x}},22 - Q_{j,j+\hat{x}},11 = Q_{j,j+\hat{y}},11 - Q_{j,j+\hat{y}},22 > 0$$

(3)

Similar bond nematic order phases were also found in other frustrated ferromagnets, such as a zigzag spin chain\textsuperscript{2,6,7} containing ferromagnetic $J_1$ and a triangular lattice multiple-spin exchange model\textsuperscript{2,9,30}. In contrast to the site-type nematic states, however, when attempting to construct a mean-field description of these bond nematic states [as well as their spin wave theories], one could immediately reach a more fundamental question: how their ground state wavefunctions are no longer described by any kind of magnetic bonds [in the square lattice case], their ground equally mental question; how their ground state wavefunctions of these bond nematic states [as well as their spin wave excitations] is crucial to the instability of the starting mean-field ansatzes\textsuperscript{30,31,33,34,35}. Thus, we will next argue the spin-triplet extension of the projector symmetry group (PSG) arguments. Without resorting to any microscopic calculations, this extension enables us to identify the number of the massless gauge bosons for any given mixed ansatz having both spin-triplet and spin-singlet link variables.

Armed with these general formulations, we study in Section III the ferromagnetic $J_1$-$J_2$ Heisenberg square lattice model defined in Eq. (2), thereby finding two stable spin-triplet RVB ansatzes in the intermediate coupling region, $J_1 \simeq 2J_2$. One is the Balian-Werthamer (BW) type triplet pairing state\textsuperscript{29} having the coplanar configurations of the $d$-vector, $\hat{d}(k) \propto \hat{x}k_x + \hat{y}k_y$, while the other is the chiral $p$-wave state\textsuperscript{30} having its $d$-vector all pointing in the same direction $\hat{d}(k) \propto \hat{z}(k_x + ik_y)$ [see Fig. 2(b)]. The PSG arguments indicate that, in general, all the non-magnetic (gauge) excitations in the BW state have finite Higgs mass. Thus, this ansatz — $Z_2$ BW state — is stable against any type of small gauge fluctuations. On the other hand, the chiral $p$-wave state does not break any of the $SU(2)$ gauge symmetry. Instead, it breaks the time-reversal symmetry and all the mirror symmetries. As a result, non-magnetic (gauge) bosons are endowed by the Chern-Simon term with the topologically-induced mass. Thus, this $SU(2)$ chiral $p$-wave state is also stable against any small gauge fluctuation. Though both the BW and chiral $p$-wave states exhibit spin quadrupolar orders, the BW state especially shows the same configuration of quadrupolar moments as the bond-type spin-nematic order found in Ref.\textsuperscript{3}. Hence, we further discuss possible experimental features of this BW state, mainly focusing on its magnetic excitations.

Section IV is devoted to the summary and open issues. The relation between our $Z_2$ BW state and the time-reversal topological insulator recently discussed in the various literatures\textsuperscript{41,42,43,44,45} is briefly mentioned. We also propose those combinations of the triplet and singlet ansatzes which describe the vector chiral order having no finite director vector $\hat{x}$ i.e. $P_{j,l,\mu} \neq 0$ and $Q_{j,l,\mu} = 0$. Those readers who want to make the $SU(2)$ slave-boson study in frustrated ferromagnets to be a controlled analysis might as well consult the appendix A, where we describe the large-$N$ generalization of frustrated ferromagnetic spin models.
II. $SU(2)$-FORMULATION OF SPIN-TRIPLET RVB STATE

A. Matrix representation

The slave-boson formulation begins with describing the spin operator by the bilinear of fermion fields; $2S_{j\mu} \equiv f_{j\alpha}^\dagger [\sigma_\mu]_{\alpha\beta} f_{j\beta}$. The enlarged (fermion’s) Hilbert space reduces to the physical (spin’s) Hilbert space, provided that the following local constraints are strictly observed at each site:

$$f_{j\alpha}^\dagger [\sigma_\mu]_{\alpha\beta} f_{j\beta} = 1, \quad f_{j1}^\dagger f_{j1} = f_{j1} f_{j1} = 0.$$ 

In the partition function, these local constraints are implemented as the coupling between the fermion (spinon) fields and the temporal $SU(2)$ gauge fields $a^\nu_{j,\tau}$ ($\nu = 1, 2, 3$).

$$Z \equiv \int d\bar{\Psi} \, d\Psi \exp \left[ -\int_0^\beta d\tau \mathcal{L} \right],$$ 

$$\mathcal{L} = \frac{1}{2} \sum_j \text{Tr} \left[ \Psi_j^\dagger (\partial_\tau \sigma_0 + \sum_{\nu=1}^3 i a^\nu_{j,\tau} \sigma_{\nu}) \Psi_j \right] + \mathcal{H},$$

where $\Psi_j$ and $\Psi_j^\dagger$ stand for the $2 \times 2$ matrices

$$\Psi_j = \begin{bmatrix} f_{j1} & f_{j1}^\dagger \\ f_{j1}^\dagger & -f_{j1} \end{bmatrix}, \quad \Psi_j^\dagger = \begin{bmatrix} f_{j1}^\dagger & f_{j1} \\ f_{j1} & -f_{j1}^\dagger \end{bmatrix}. \quad (10)$$

The spin Hamiltonian part $\mathcal{H}$ becomes quartic in the fermion field ($\Psi$-field). Depending on the sign of the exchange interaction, we decompose this quartic term into the Stratonovich-Hubbard variables in two alternative ways:

$$Z = \int dU_{\text{sin}} \, dU_{\text{tri}} \, d\bar{\alpha} \, d\bar{\Psi} \, d\Psi \exp \left[ -\int_0^\beta d\tau \mathcal{L} \right],$$ 

$$\mathcal{L} = \frac{1}{2} \sum_j \text{Tr} \left[ \Psi_j^\dagger (\partial_\tau \sigma_0 + \sum_{\nu=1}^3 i a^\nu_{j,\tau} \sigma_{\nu}) \Psi_j \right]$$

$$- \frac{J_1}{4} \sum_{\langle jl \rangle} \left\{ -E_{jl}^2 - D_{jl}^2 + \text{Tr} \left[ \Psi_j U_{\text{tri}}^{\dagger} \Psi_j \sigma_{\mu} \sigma_{\mu}^T \right] \right\},$$

$$- \frac{J_2}{4} \sum_{\langle jl \rangle} \left\{ -\chi_{jl}^2 - \eta_{jl}^2 + \text{Tr} \left[ \Psi_j U_{\text{sin}}^{\dagger} \Psi_j \right] \right\}. \quad (12)$$

Namely, the triplet and singlet link-variables, $U_{\text{sin}}^{\dagger}_{ij}$ and $U_{\text{tri}}^{\dagger}_{ij,\mu}$, are introduced as the auxiliary fields for the ferro- and antiferromagnetic bonds, respectively. This is simply because the sign of the ferromagnetic exchange interaction generally allows us to perform the gaussian-integration only over the $d$-vectors in the excitonic/Cooper channel. In fact, this integration precisely reproduces the ferromagnetic exchange interaction,

$$-4S_j \cdot S_l = -\sum_{\mu=1}^3 \left( f_{j\alpha}^\dagger \sigma_{\mu} [\sigma_\mu]_{\alpha\beta} f_{j\beta} \right) f_{l\gamma}^\dagger \sigma_{\gamma} f_{l\delta},$$

while that over the singlet variable leads to the antiferromagnetic exchange interaction.

Thus, the slave-boson formulation of mixed Heisenberg magnets generally requires us to use the spin-triplet link-variable $U_{\text{tri}}^{\dagger}_{ij,\mu}$ for every ferromagnetic bond and the spin-singlet link variable $U_{\text{sin}}^{\dagger}_{ij}$ for every antiferromagnetic bond.

The saddle point solutions of Eq. (12) lead to the coupled gap equations for these link-variables, i.e. Eqs. (14), (15), and (5), whose right hand sides are self-consistently given by these mean-fields themselves. In terms of $U_{\text{tri}}^{\dagger}_{ij,\mu}$ and $U_{\text{sin}}^{\dagger}_{ij}$ this determined, the spin quadrupolar moment and vector chirality are given by

$$-2Q_{j,\mu\nu} = \text{Tr} \left[ U_{\text{tri}}^{\dagger}_{ij} U_{\text{tri}}^{\dagger}_{ij,\nu} \right] - \frac{\delta_{\mu\nu}}{3} \sum_\lambda \text{Tr} \left[ U_{\text{tri}}^{\dagger}_{ij,\lambda} U_{\text{tri}}^{\dagger}_{ij,\lambda} \right], \quad \text{(14)}$$

$$-2iP_{j,\lambda} = \text{Tr} \left[ U_{\text{sin}}^{\dagger}_{ij,\lambda} U_{\text{tri}}^{\dagger}_{ij,\lambda} \right]. \quad \text{(15)}$$

Comparing Eq. (12) with Eq. (16), notice that the present $J_1$-$J_2$ model can have spin quadrupolar order on ferromagnetic bonds, but cannot have vector chirality on any links, since $U_{\text{sin}}^{\dagger}_{ij,\lambda} = 0$. Within our formalism, a naive mean-field description of vector chiral orders becomes possible only in those spin models having either symmetric anisotropic exchange interactions or antisymmetric anisotropic one. In the next section, without making any distinction between the $n$-mematic states and $p$-mematic ones, we will widely call those mean-field ansatzes having both finite triplet ansatz and singlet ansatz as spin-triplet RVB states.

B. Low-energy excitations around spin-triplet RVB states

To see the low-energy excitations around the spin-triplet RVB ansatzes, let us first express the spin operator in terms of the $2 \times 2$ matrix representation $S_{j\mu} \equiv \frac{1}{2} \text{Tr} \left[ \Psi_j^\dagger \sigma_{\mu} \sigma_{\mu}^T \right]$. Namely, a spin rotation is described by an $SU(2)$ matrix, say $h_j$, applied from the left (right) hand side of $\Psi_j^\dagger (\Psi_j)$,

$$\Psi_j \rightarrow \Psi_j h_j^T, \quad \Psi_j^\dagger \rightarrow h_j^* \Psi_j^\dagger,$$
while physical quantities are invariant under any local $SU(2)$ gauge transformation applied from the right (left) hand side of $\Psi^\dagger_j (\Psi_j)$:

$$
\Psi_j \rightarrow g_j \Psi_j, \quad \Psi^\dagger_j \rightarrow \Psi^\dagger_j g^\dagger_j,
$$

$$\{U_{jl,\mu}^{tri}, U^\text{sin}_{jl}\} \rightarrow g_j \{U_{jl,\mu}^{tri}, U^\text{sin}_{jl}\} g^\dagger_j.
$$

For example, both parts of the spin quadratic tensor, Eqs. (12) and (15), are invariant under this local $SU(2)$ gauge transformation. In regard to these two symmetries, any spin-triplet mean-field ansatz is generally accompanied by two types of low-energy excitations: the magnetic ones (Goldstone bosons) and the non-magnetic ones (gauge bosons). Specifically, one can expand the effective action in terms of two types of low-energy excitations: the magnetic ones (Goldstone bosons) and the non-magnetic ones (gauge bosons). The former excitations are semiclassically described by the deformations of the $d$-vectors around its mean-field configuration,

$$U_{jl,\mu}^{tri} \equiv \sum_{\nu=1}^{3} U_{jl,\mu}^{tri} R_{\nu\mu} \left(\frac{j + 1}{2}, \tau\right) \quad (16)$$

for $\mu = 1, 2, 3$ with the $3 \times 3$ rotational matrix $R(x, \tau)$. Such deformations cost infinitesimally small energy in spin models with spin continuous symmetry, provided that the variation of the rotation is sufficiently slow in space and time. This type of deformations describe the Goldstone modes accompanying the spontaneous symmetry breaking.

In addition to this conventional excitation, a certain non-magnetic (gauge) excitation also become massless, when our starting mean-field ansatz is invariant under a continuous gauge symmetry. For example, assume that the invariant gauge group (IGG) contains the $U(1)$ gauge symmetry $\{e^{i \theta_s} | \theta \in [0, 2\pi)\}$. Namely, our mean-field ansatz is invariant under any rotation around the 3-axis in the gauge space,

$$e^{i \theta_s} \{U_{jl,\mu}^{tri}, U^\text{sin}_{jl}\} e^{-i \theta_s} = \{\tilde{U}_{jl,\mu}^{tri}, \tilde{U}^\text{sin}_{jl}\} \quad (17)$$

for $\mu = 1, 2, 3$ and $\tilde{a}_{jl,\tau} = \delta_{\nu3} a_{jl,\tau}^\nu$. Then, we can argue that the following non-magnetic deformation also comprises the gapless excitation:

$$\{U_{jl,\mu}^{tri}, U^\text{sin}_{jl}\} \equiv \{\tilde{U}_{jl,\mu}^{tri}, \tilde{U}^\text{sin}_{jl}\} e^{i a_{jl,\tau}^3},$$

$$a_{jl,\tau} = \tilde{a}_{jl,\tau} + a_0(j, \tau),$$

where $a_{jl}$ relates to the spatial components of “gauge fluctuations” $a_\alpha(j, \tau) (\alpha = 1, \cdots, d)$ in the form

$$a_{jl}(\tau) = (j-l)a_\alpha(j, \tau).$$

Specifically, one can expand the effective action in terms of these variations $a_\alpha(j, \tau) (\alpha = 0, 1, \cdots, d)$, assuming these fluctuations to be much smaller than their units, $a_\alpha(j, \tau) \ll 2\pi$. Up to their quadratic order, the effective action generally reads as follows:

$$F_{\text{gauge}} = \sum_{\alpha, \beta=0}^{d} M_{\alpha\beta}(Q) a_\alpha(Q) a_\beta(-Q) + \cdots, \quad (21)$$

$$a_\alpha(Q) = \frac{1}{\sqrt{N}} \sum_{\omega_n} e^{i \omega_n \epsilon} a_\alpha(j, \tau) \quad (22)$$

with $Q = (q, \omega_m)$. Then, taking into account the $U(1)$ gauge symmetry of the mean-field ansatz, one can specify the form of the $(d + 1) \times (d + 1)$ matrix $M(Q)$, such that the quadratic part in the following expression reduces to the $U(1)$ gauge invariant form as in Eq. (23).

To see this, introduce the following local $U(1)$ gauge transformation in Eq. (12):

$$\Psi^\dagger_j(\tau) \rightarrow \Psi^\dagger_j(\tau) e^{i \theta_j(\tau) \sigma_3},$$

$$\Psi_j(\tau) \rightarrow e^{-i \theta_j(\tau) \sigma_3} \Psi_j(\tau),$$

where $\theta_j(\tau)$ varies slowly in space and time. Under this transformation, all changes in the link variables [13] are put into the transformation, $a_{jl} \rightarrow a_{jl} + \theta_l - \theta_j$ and $a_0 \rightarrow a_0 + \partial_\tau \theta$, due to the $U(1)$ symmetry in IGG. Thus the effective action around $Q \approx 0$ is literally transformed as

$$F_{\text{gauge}} \rightarrow \sum_{\alpha, \beta=0}^{d} M_{\alpha\beta}(Q) (a_\alpha + \partial_\tau \theta)(a_\beta + \partial_\tau \theta)(-Q).$$

However, the free energy should have been invariant under any gauge transformation, since gauge degrees of freedom can be absorbed into the integral variables, $\Psi$-fields. This requires that $M(Q)$ must precisely reduce to zero at $Q = 0$, so that the quadratic part of the action takes the $U(1)$ gauge invariant form, e.g.

$$F_{\text{gauge}}^{U(1)} = \frac{1}{8\pi} \sum_{Q=0}^{d} \sum_{\alpha=0}^{d} \frac{1}{g_\alpha^2} f_\alpha(Q) f_\alpha(-Q) + \cdots, \quad (23)$$

where $f_\alpha \equiv \epsilon_{\alpha\beta\gamma} \partial_\gamma a_\beta \bar{a}_\beta$ stands for the field strength [2] [34]. It is well-known that this Maxwell form does not suppress the gauge fluctuation efficiently. Especially, when the mean-field ansatz have its fermionic excitations fully gapped and when $d = 2$, these massless gauge fluctuations destroy the mean-field ansatz itself [34, 35, 38] apart from some exceptional cases [28, 50, 51, 52, 53, 54, 55]. Following the literature [34], we call in this paper such spin-triplet mean-field ansatz as the gapped $U(1)$ [or $SU(2)$] state.

On the other hand, if the starting mean-field ansatz has no continuous invariant gauge group (IGG) like in Eq. (17), the local minimum condition imposed on mean-field ansatzes generally requires all the eigenvalues of $M(Q)$ to be positive. Therefore, all the gauge fields have finite Higgs mass around any $Q$:

$$F_{\text{gauge}}^{Z_2} = \sum_{Q=0}^{d} \sum_{\alpha=0}^{d} M_\alpha(Q) \bar{a}_\alpha(Q) \bar{a}_\alpha(-Q) + \cdots \quad (24)$$
with $\tilde{M}_\nu(Q) > 0$. In contrast to the maxwell form discussed above, this finite Higgs mass suppresses any small gauge fluctuation completely, so that the starting mean-field ansatz is always guaranteed to be (at least locally) stable. Such ansatzes are usually dubbed as the $Z_2$ state.

The efficient way to confirm the absence of the continuous IGG was introduced by Wen, \textsuperscript{34,48} where he pointed out the sufficient condition for its absence. We can extend his argument to the spin-triplet RVB states also. To see this, let us begin with the calculation of the SU(2) flux defined on a plaquette by multiplying link-variables along the closed loop in a regular sequence, where either $U_{ij}^{\text{tri}}$ or one of $U_{ij}^{\text{tri}}$ should be chosen on each link. For example, when the loop is given by a triangular path $i \rightarrow j \rightarrow k \rightarrow i$, one can have an SU(2) flux by $U_{ij}U_{jk}U_{ki}$, which always transforms in a gauge-covariant way;

$$U_{ij}U_{jk}U_{ki} \rightarrow g_i^1 \cdot U_{ij}U_{jk}U_{ki} \cdot g_i,$$

under $\Psi_j \rightarrow g_j \Psi_j$. As such, the relative angle subtended by two distinct SU(2) fluxes derived from the same base-site, such as $U_{ij}U_{jk}U_{ki}$ and $U_{ij}U_{ji}U_{ii}$, contains non-trivial gauge-independent information, provided that the two triangular paths, $(ijk(i))$ and $(ijl(i))$, are different with each other. Note that, even out of the same triangular loop, we can have two distinct fluxes, when one of its three links has two different types of spin-triplet ansatzes, $U_{ij}^{\text{tri}} \neq U_{ij}^{\text{tri}}$. In this case, we should regard $U_{ij}^{\text{tri}} \cdot U_{jk}^{\text{tri}}$ and $U_{ij}^{\text{tri}} \cdot U_{jk}^{\text{tri}} \cdot U_{ki}^{\text{tri}}$ are two distinct fluxes obtained from the same base-site $i$.

Having all SU(2) fluxes thus obtained in hand, one can readily see that, (i) if two distinct SU(2) fluxes obtained from the same base-site are not collinear with each other, there is no continuous IGG in that mean-field ansatz. (ii) If all the distinct fluxes obtained from the same base-site are pointed along one direction in the gauge space, say along the 3-axis, the ansatz could have a certain $U(1)$ gauge symmetry around this 3-axis, just like in Eq. (17). One can also confirm that, (iii) the ansatz can be invariant under a certain SU(2) gauge symmetry [so-called SU(2) state], if all the SU(2) fluxes are proportional to the unit matrix.

This ‘non-collinearity’ argument of the SU(2) fluxes concludes the (local) stability of each ansatz against gauge fluctuations very efficiently, without resorting to any microscopic calculation. Thus, it substantially helps us to find a better spin-triplet mean-field ansatz as in the case of spin-single RVB ansatzes.\textsuperscript{34,48}

III. $J_1$-$J_2$ FRUSTRATED FERROMAGNETIC SQUARE LATTICE HEISENBERG MODEL

In this section, we will apply the spin-triplet slave-boson mean-field formulation onto the spin-$1/2$ $J_1$-$J_2$ mixed Heisenberg model \textsuperscript{6} on the square lattice with ferromagnetic nearest neighbor (NN) $J_1$ and antiferromagnetic next nearest neighbor (NNN) $J_2$. As was described in the previous section, we always decompose the ferromagnetic NN bond into the spin-triplet ansatz and the antiferromagnetic NNN bond into the spin-singlet ansatz.

A. Mean-field solutions

To be specific, we have numerically studied the various local ‘stable’ minima of the mean-field free energy given in Eq. (12), assuming that the magnetic unit cells (MUC) are either (i) original square-lattice unit cell or (ii) $2 \times 2$ of the original unit cell. The dimension of the (real-valued) parameter space in each case becomes (i) 32(+3) and (ii) 128(+12). Starting from a randomly chosen initial point in these multiple dimensional parameter spaces, we perform the Newton-Raphson method, only to reach a certain local minimum of the mean-field free energy $E_{\text{mf}}$ (per the magnetic unit cell);

$$E_{\text{mf}} \equiv \frac{J_1}{4} \sum_{(j\ell) \in \text{MUC}} \left( |E_{j\ell}|^2 + |D_{j\ell}|^2 \right) + \frac{J_2}{4} \sum_{(j\ell) \in \text{MUC}} \left( |\chi_{j\ell}|^2 + |\eta_{j\ell}|^2 \right) - \frac{1}{16\pi^2} \sum_{\alpha=1}^\nu \int_{\text{MBZ}} dk_x dk_y |\lambda_\alpha|,$$  \hspace{1cm} (25)

with (i) $\nu = 4$ or (ii) $\nu = 16$. Here, the summation over $j\ell$ is taken within each magnetic unit cell and $\lambda_\alpha$ denotes the spinon energy band. We have repeated this procedure from 50 times to 300 times for each parameter point, i.e. $(J_1, J_2) = (\sin \theta, \cos \theta)$ with $0 < \theta \leq \frac{\pi}{2}$. In this way, we enumerated various spin-triplet RVB ansatzes.

Throughout this extensive search, we found basically three distinct RVB ansatzes having both spin-triplet link-variable on each NN bond and spin-singlet link-variable on each NNN bond. All of these three do not break any translational symmetries of the original unit cell, i.e. $T_x$ and $T_y$.

1. Z$_2$ Balian-Werthamer state

The first one is a sort of the Balian-Werthamer (BW) state\textsuperscript{39} where the $d$-vector on the NN $x$-link is perpendicular to that on the $y$-link,

$$U_{(j\ell, j+\hat{x})}^{\text{tri}} = i\delta_{\mu 1} D_{\sigma 2}, \quad U_{(j\ell, j+\hat{y})}^{\text{tri}} = i\delta_{\mu 2} D_{\sigma 2},$$

$$U_{(j\ell, j+\hat{x})}^{\text{sing}} = \chi_{\sigma 3} \pm \eta_{\sigma 1}, \quad i\alpha_\nu = 0.$$  \hspace{1cm} (26)

‘$D$', ‘$\chi$' and ‘$\eta$' above correspond to the real parts of Eqs. (5) and (8), respectively. This RVB state exhibits the same antiferro-type configuration of quadrupolar moments as the bond nematic state found in Ref.\textsuperscript{5}. Namely, the nematic order parameters on NN bonds show

$$Q_{j\ell, 11} = -\frac{2}{3} D_2^2, \quad Q_{j\ell, 22} = Q_{j\ell, 33} = \frac{1}{3} D_2^2$$  \hspace{1cm} (27)
for the $x$-direction and

$$Q_{j,22} = \frac{-2}{3} D^2, \quad Q_{j,11} = Q_{j,33} = \frac{1}{3} D^2$$  \hspace{1cm} (28)$$

for the $y$-direction, where $Q_{j,\mu\nu} = 0$ for $\mu \neq \nu$ (see Fig. 1). While this mean-field ansatz breaks the mirror symmetry $P_{xy}$ which interchanges $x$-link and $y$-link, it is invariant under the following combined symmetry and gauge transformations: $G_x T_x, G_y T_y, G_{P_x} P_x, G_{P_y} P_y, G_{P_{xy}} P_{xy}$ and $G_\pi T$. The respective gauge transformations read

$$G_x = G_y = \sigma_0, \quad G_{P_x} = i \sigma_1(-1)^{j_x}, \quad G_{P_y} = i \sigma_1(-1)^{j_y},$$  \hspace{1cm} (29)$$

$$G_{P_{xy}} = i \sigma_2(-1)^{j_z}, \quad G_\pi = (-1)^i_{x+y}. $$

Here $T$ refers to the time-reversal symmetry, while $P_{xy}$ stands for the mirror symmetry $P_{xy}$ accompanied by an appropriate spin-rotation about the $3$-axis by $\pi/2$.

FIG. 1: (color online) (a) 2$z^2-x^2-y^2$ type quadrupole moment formed by two $S = 1/2$ spins on each bond. (b) $J_1$-$J_2$ model and the configuration of the quadrupole moments on bonds in the $Z_2$ BW state [see Eqs. (27) and (28)].

Provided $\eta \neq 0$, the ansatz supports two non-collinear $SU(2)$ gauge fluxes,

$$U_{j,1}^{\text{tri}}(j,\tilde{\omega}), U_{j,1}^{\text{tri}}(j+\tilde{\omega}, j+\tilde{\omega}+\tilde{\mu}), U_{j,1}^{\text{sin}}(j+\tilde{\omega}+\tilde{\mu}, j) \propto \chi \sigma_3 + \eta \sigma_1, $$  \hspace{1cm} (30)$$

$$U_{j,1}^{\text{tri}}(j,\tilde{\omega}), U_{j,1}^{\text{tri}}(j+\tilde{\omega}, j+\tilde{\omega}+\tilde{\mu}), U_{j,1}^{\text{sin}}(j+\tilde{\omega}+\tilde{\mu}, j) \propto \chi \sigma_3 - \eta \sigma_1. $$  \hspace{1cm} (31)$$

Hence it is protected from any small gauge fluctuation by finite Higgs mass. We call this ansatz as the $Z_2$ BW state. The spinon’s band dispersion $\lambda_0$ of this $Z_2$ state is comprised of two doubly degenerate bands, both of which are always separated by a finite energy gap in the entire Brillouin zone, $[-\pi, \pi] \times [-\pi, \pi]$:

$$\lambda_{1,2} = -\lambda_{3,4} \equiv \{ A^2(s_1^2 + s_2^2) + B^2 s_1^2 s_2^2 + C^2 s_1^2 s_2^2 \}^{1/2} \hspace{1cm} (32)$$

with $(s_\mu, c_\mu) \equiv (\sin k_\mu, \cos k_\mu)$ and $(2A, B, C) \equiv (J_1 D, J_2 \chi, J_2 \eta)$.

2. $SU(2)$ chiral $p$-wave state

The second ansatz we found is the chiral $p$-wave [Anderson-Brinkman-Morel (ABM)] state in which all the $\mathbf{d}$-vectors on the NN-bonds are collinear, while the $\mathbf{d}$-vector on the $x$-link acquires extra phase $i$ in relative to that on the $y$-link,

$$U_{j,1}^{\text{tri}}(j+\tilde{x}, j+\tilde{x}) - i \delta \pi D \sigma_2, \quad U_{j,1}^{\text{tri}}(j+\tilde{y}, j) - i \delta \pi D \sigma_1, \quad U_{j,1}^{\text{sin}}(j+\tilde{y}, j+\tilde{y}) = \chi \sigma_3 + i \alpha \nu = 0. $$  \hspace{1cm} (33)$$

Namely, two ‘$D$’ appearing in the first line stand for the real and imaginary part of the $\mathbf{d}$-vector respectively. Because of this relative phase factor, this ansatz has its fermionic band-dispersion fully gapped in the whole momentum space:

$$\lambda_{1,2} = -\lambda_{3,4} = \lambda = \{ A^2(s_1^2 + s_2^2) + B^2 s_1^2 s_2^2 + C^2 s_1^2 s_2^2 \}^{1/2}. $$  \hspace{1cm} (34)$$

In this state, all NN bonds have the same ferro-nematic order $Q_{j,33} = -\frac{1}{2} D^2$, $Q_{j,11} = Q_{j,22} = \frac{1}{4} D^2$.

The IGG of this chiral $p$-wave state contains the following three continuous gauge symmetries:

$$\{ e^{i(-1)^{j_2+y} \sigma_1}, e^{i(-1)^{j_2+y} \sigma_2}, e^{i(-1)^{j_2+y} \sigma_2} | \theta \in [0, 2\pi) \}. $$  \hspace{1cm} (35)$$

Correspondingly, the low-energy effective theory in the gauge (non-magnetic) part consists of three maxwell forms around $q = (\pi, \pi), (\pi, 0)$ and $(0, \pi)$ respectively. Namely, above continuous gauge symmetries require that the following three types of non-magnetic deformations constitute the $U(1)$ gauge invariant effective actions:

$$\{ U_{j,1}^{\text{tri}}, U_{j,1}^{\text{sin}} \} = \{ U_{j,1}^{\text{tri}}, U_{j,1}^{\text{sin}} \} e^{i(j_{l-\omega}-(l, \tau)) a_{n_{l,\tau}}, \sigma_3 3} \hspace{1cm} (36)$$

$$\{ U_{j,1}^{\text{tri}}, U_{j,1}^{\text{sin}} \} = \{ U_{j,1}^{\text{tri}}, U_{j,1}^{\text{sin}} \} e^{i(j_{-\omega}-(l, \tau)) a_{n_{l,\tau}}, \sigma_1 1} \hspace{1cm} (37)$$

$$\{ U_{j,1}^{\text{tri}}, U_{j,1}^{\text{sin}} \} = \{ U_{j,1}^{\text{tri}}, U_{j,1}^{\text{sin}} \} e^{i(j_{\omega}-(l, \tau)) a_{n_{l,\tau}}, \sigma_2 2} \hspace{1cm} (38)$$

Though these three types of gauge fluctuations are not suppressed by finite Higgs mass, the ansatz itself is still protected by the so-called Chern-Simon mechanism $\omega \subseteq 34, 35, 36, 30, 31, 32, 53$.

To see this, notice that the ansatz breaks all the mirror symmetries $P_x, P_y, P_{xy}$ and the time-reversal symmetry $T$. Instead, it is invariant only under these mirror symmetries accompanied by the time-reversal symmetry $G_{PT} \cdot PT$ or under the spatial inversion symmetry $G_{R_x} R_\pi$. The respective gauge transformations are given by

$$G_{P_{xy}} T = \sigma_0, \quad G_{R_x} = G_{P_{xy}} T = (-1)^{j_2+j_y}, \quad G_{P_{xy}} T = i(-\sigma_3)_{j_2+j_y}. $$  \hspace{1cm} (39)$$

This magnetic point group clearly allows the spontaneous Hall conductance of the ‘spinon’, like in the chiral spin
state.\textsuperscript{51,52,53} In fact, corresponding to the three continuous gauge symmetries given in Eq. (35), we have five conserved ‘charges’, all of which are accompanied by finite quantized transverse conductance \( \sigma_{xy} = \frac{e}{2c} \). As a result, the effective actions around \( q = (\pi, \pi) \), \((0, \pi)\) and \((\pi, 0)\) acquire the Chern-Simon term in addition to the maxwell form.\textsuperscript{34,35,51,52,53}

\[
F_{\text{gauge}} \equiv \int dx^2 d\tau \frac{\sigma_{xy}}{2} \partial_x a_\mu \partial_\tau a_\lambda \epsilon_{\mu\nu\lambda} + (\text{maxwell form}).
\]

This Chern-Simon term endows the apparently massless gauge boson with a finite energy gap.\textsuperscript{52}

3. \( Z_2 \) collinear state

The third stable ansatz we found is the ‘collinear’ state, where all \( d \)-vectors are pointing to the same direction,

\[
U^{\text{tri}}_{(j,j+\pm\hat{x})}\mu = U^{\text{tri}}_{(j,j+\pm\hat{y})}\mu = i\delta_{\mu3}D\sigma_2, \\
U^{\text{sin}}_{(j,j+\pm\hat{x}\pm\hat{y})} = \chi \sigma_3 \pm \eta \sigma_1, \\
i\alpha_{j,x}^{\pm} \neq 0,
\]

showing ferro-nematic order \( Q_{j,33} = -\frac{2}{3} D^2 \), \( Q_{j,11} = Q_{j,22} = \frac{1}{3} D^2 \). Although having the same spin-quadrupolar moment as the previous one, this collinear ansatz is a distinct quantum order state from the \( SU(2) \) chiral \( p \)-wave state. It preserves mirror symmetries as well as the time-reversal symmetry. In fact, one can see that all the discrete symmetries of the original square lattice are recovered, when combined with the following gauge transformations:

\[
G_x = G_y = \sigma_0, \\
G_{P_\pm} = i\sigma_1(-1)^{\pm}, \\
G_{P_y} = i\sigma_1(-1)^{\mp}, \\
G_T = (-1)^{\pm+jy}.
\]

Having the non-collinear \( SU(2) \) gauge fluxes as in Eqs. (39) and (41), all the gauge fluctuations around this ansatz are suppressed by finite Higgs mass. We hence call this state as \( Z_2 \) collinear state.

B. Phase diagram

The mean-field energy for these three ansatze are plotted in Fig. 2(a) with \( (J_1, J_2) = J(\sin \theta, \cos \theta) \). Let us begin with the lowest energy mean-field solution in the well-studied limit, \( J_2 \gg J_1 \). In the strong \( J_2 \) limit, our model reduces to the two decoupled antiferromagnetic square lattice, so that the knowledges of the saddle-point solutions in this limit have been well-established.\textsuperscript{30,31,32,34,35,55,56,57,58} Namely, the \( \pi \)-flux state defined on each square lattice,

\[
U^{\text{sin}}_{(j,j+\hat{x}\pm\hat{y})} = \chi \sigma_3 \pm \eta \sigma_1, \\
U^{\text{tri}}_{(j,j+\hat{\nu}),\mu} = i\alpha_{j,x}^{\nu} \neq 0
\]

with \( \chi = \eta \), becomes global minimum, when the magnetic unit cell (MUC) is restricted to the original square lattice unit cell. On the other hand, when the MUC is enlarged up to the \( 2 \times 2 \), the global minimum state becomes the staggered dimer state introduced on each decoupled square lattice, e.g.

\[
U^{\text{sin}}_{(j,j+\hat{x}\pm\hat{y})} = U^{\text{sin}}_{(j,j+\hat{\nu}),\mu} = i\alpha_{j,x}^{\nu} = 0
\]

However, using the variational Monte Carlo (VMC) calculations, Gros and his co-workers\textsuperscript{58} have demonstrated that, when projected onto the original (spin) Hilbert space, the \( \pi \)-flux state eventually wins over this isolated dimer state. In fact, it is well-established\textsuperscript{58} that the projected \( \pi \)-flux state gives the second best variational energy in the strong \( J_2 \) limit (the best variational estimate is obtained from the Neel order state).\textsuperscript{22}

When increasing the NN ferromagnetic interaction \( J_1 \), a finite spin-triplet ansatz continuously develops on the top of this \( \pi \)-flux state, while simultaneously the parameters \( \eta \) start to deviate from \( \chi \), i.e., \( \eta \neq \chi \). This leads to either \( Z_2 \) BW state or \( Z_2 \) collinear state for \( \theta_{c1} \equiv 0.66 < \theta \). Thus, the transitions from the \( \pi \)-flux state to these two \( Z_2 \) states are both the second order at the mean-field level. Energetically speaking, the \( Z_2 \) BW state gives a slightly lower mean-field energy than that of the \( Z_2 \) collinear state.

Notice also that these two \( Z_2 \) states are clearly preempted by the staggered dimer state, Eq. (43), at the mean-field level [see Fig. 2(a)]. Observing the situation in the strong \( J_2 \) limit, however, one can naturally expect that, when projected onto the physical (spin) Hilbert space, both \( Z_2 \) states would win over this isolated dimer state in the case of a finite \( J_1 \). Namely, since our \( Z_2 \) states are constructed based on the decoupled \( \pi \)-flux states [compare Eqs. (26,40) with Eq. (42)], they would certainly acquire substantial resonance energies in the same way as the \( \pi \)-flux state does. On the other hand, being factorisable, any isolated dimer state cannot gain such resonance energies, irrespective of finite ferromagnetic exchange interactions. Moreover, Fig. 2(a) indicates that the \( Z_2 \) BW ansatz is quite energetically tunable in the presence of the ferromagnetic exchange interaction. Thus, we presume that the \( Z_2 \) BW state finally dominates in this intermediate coupling region, \( \theta_{c2} \equiv 0.66 < \theta \).

When \( \theta_{c2} \equiv 0.76 < \theta \), this \( Z_2 \) BW state reduces to the \( U(1) \) state having no finite \( \eta \). Namely, with \( \eta = 0 \), two \( SU(2) \) gauge fluxes given in Eqs. (39) and (41) become collinear with each other. Simultaneously, this \( U(1) \) BW state becomes energetically degenerate with the \( SU(2) \) chiral \( p \)-wave state. Namely, both of them have precisely the same mean-field band dispersions \( \pm \lambda_k \) [compare Eq. (34) with Eq. (42) having \( \eta = 0 \)].

This \( U(1) \) BW state is destroyed by the infinitesimally small gauge fluctuation. Namely, in the absence of finite
FIG. 2: (color online) (a) Mean-field energies (per site) of the various ansatzes in the $S = 1/2$ square lattice ferromagnetic $J_1$-$J_2$ model. Note that $(J_1, J_2) \equiv |J| (\sin \theta, \cos \theta)$, where the energy unit is taken to be $|J|$. The Blue line (labeled as $B$) is for the Balian-Werthamer (BW) state, which is the $Z_2$ state for $\theta_{c1} < \theta < \theta_{c2}$ and which reduces to the $U(1)$ state for $\theta_{c2} < \theta$. The green line (labeled as $C$) is for the $Z_2$ collinear state, while the red line (labeled as $D$) stands for the $SU(2)$ chiral p-wave state. The red dotted line (labeled as $A$) is the doubled $\pi$-flux state, where both the $A$-sublattice and the $B$-sublattice support flat-fermion states respectively [see Eq. (12)]. The Blue dotted line (labeled as $E$) is for a set of ‘flat-band’ states ($E_{flat}^{\text{dimer}} = -\frac{1}{2}|J| \cos \theta$), all of which give the same best mean-field energy for $\theta_{c1} < \theta$. The green dotted line (labeled as $F$) is for the staggered dimer state ($E_{\text{dimer}}^{\text{staggered}} = -\frac{1}{2}|J| \cos \theta$), where both the $A$-sublattice and the $B$-sublattice support staggered dimer states respectively [see Eq. (38) for its example]. These isolated dimer states are known to be overcome energetically by the doubled $\pi$-flux state $E_{\text{flat}}^{\pi}$ when they are projected onto the physical Hilbert space. Since the $Z_2$ BW state is composed on the top of the $\pi$-flux state, this staggered dimer state is also expected to be overcome by the projected $Z_2$ BW state. (b) Expected mean-field phase diagram in the intermediate coupling region. The transition at $\theta_{c3}$ is the 2nd order, since the magnetic space group of the $Z_2$ BW state belongs to that of the $\pi$-flux state. On the other hand, the transition at $\theta_{c3}$ is the 1st order at the mean-field level, which one can see directly from the Figure (a).

\[ \eta \], the non-magnetic deformations defined in Eq. (19) constitute the following Maxwell form around $q = (\pi, \pi)$,

\[ F_{\text{gauge}} = \int_0^\beta d\tau \int d^2x \{ u e^2 + \frac{1}{2} K b^2 \} + \cdots \]

where $e_\alpha$ ($\alpha = 1, 2$) and $b$ are defined, from Eqs. (19)-(20), as $e_\alpha(j, \tau) \equiv (-1)^{j - j_y} (\partial_\tau a_\alpha - \partial_\tau a_\alpha)$, and $b(j, \tau) \equiv (-1)^{j + j_y} (\partial_\tau a_1 - \partial_\tau a_2)$. Since the fermionic excitations are fully gapped even without $\eta$ [see Eq. (22)], this Maxwell form is free from any dissipation effect. e.g.

\[ u = \int \int_{[-\pi, \pi]^2} d^2k A^4 s_z^2 s_y^2 + A^2 B^2 s_z^2 (s_x^2 + s_y^2) > 0. \]

Having the time-reversal symmetry [see Eq. (29)], the massless gauge fluctuation is not suppressed by the Chern-Simon term either. Consequently, infinitesimally small fluctuations of this type of gauge fields lead the $U(1)$ BW state into a confining phase having no gapped free spinon in its excitation. More specifically, those space-time instantons (monopoles) which are allowed by the corresponding compact QED action, $\int d\tau \int d^2x \{ u e^2 - K \cos (\epsilon_{\alpha \beta} \Delta_\alpha a_\beta) \}$, proliferate in the $2 + 1$ dimensional space, lowering a certain magnetic symmetries enumerated in Eq. (29). To capture the resulting magnetic space group of the confining phase, one generally need to identify the quantum number carried by this monopole creation field, $\phi_{2D}$.

For $\theta_{c3} \equiv 0.775 < \theta$, these two degenerate ansatzes, – $SU(2)$ chiral p-wave state and $U(1)$ BW state –, are further overcome (energetically) by another ansatz, which we dubbed as the ‘flat-band’ states,

\[ U_{jl}^{\sin} = 0, \quad U_{jl}^{\text{tri}} \neq 0, \quad \lambda_{jl, \tau} 
eq 0. \]

(44)

These ‘flat-band’ states do not have any finite singlet ansatzes anymore and keep on giving the lowest mean-field energy ($E_{\text{flat}} = -\frac{1}{2} J_1$) for the remaining ferromagnetic side, $\theta_{c3} < \theta < \frac{\pi}{2}$. However, these ‘flat-band’ states do not necessarily refer to a specific configuration of the spin-triplet ansatzes. Instead, they refer to a group of the states all of which give precisely the same mean-field energy. For example, these ‘flat-band’ states include the following parameterization of the spin-triplet ansatz:

\[
\begin{align*}
E''_{(j, j+y)} &= 0, \\
E''_{(j, j+y)} &= \alpha n_1 \cdot m_1^T, \\
E''_{(j, j+y)} &= 0, \\
E''_{(j, j+y)} &= \beta n_1 \cdot m_2^T, \\
E''_{(j, j+y)} &= \alpha n_2, \\
E''_{(j, j+y)} &= \beta n_3, \\
\end{align*}
\]

where $\alpha^2 + \beta^2 = \frac{1}{4}$, and $n_j$ and $m_j$ can be arbitrary unit vectors that observe Eq. (19). Here $E''_{jl, \mu}$ and $D''_{jl, \mu}$ stand for the real part of $E_{jl, \mu}$ and $D_{jl, \mu}$, respectively, while $E''_{jl, \mu}$ and $D''_{jl, \mu}$ are their respective imaginary parts. Thus, only the first one is parity even $E''_{jl, \mu} = E''_{jl, \mu}$, while the others are odd, $E''_{jl, \mu} = -E''_{jl, \mu}$ and $D''_{jl, \mu} = -D''_{jl, \mu}$. Bearing these in mind, one can easily see that this mean-field ansatz always gives the two doubly degenerate spinon bands, which are totally flat in the entire Brillouin zone,

\[ \lambda_{1,2} = -\lambda_{3,4} = \frac{J_1}{4}. \]

Because of this feature, all the spin-triplet ansatizes parameterized by Eq. (19) give the same mean-field energy (per site) $E_{\text{flat}} = -\frac{1}{8} J_1$. 
The emergence of these ‘huge’ numbers of ‘flat-band’ states in the strong $J_1$ limit reflects the fact that the ground-state order parameter of any Heisenberg ferromagnet (total spin moment) and the corresponding spin Hamiltonian are simultaneously diagonalizable. When projected onto the physical (spin) Hilbert space, we expect that these flat-band states reduce to a fully polarized state (ferromagnetic state).

Observing Fig. 2 please notice that our $Z_2$ BW phase appears in larger $J_2$ region in comparison with the previous ED studies. Namely, Fig. 2 indicates that its phase boundaries are given by $J_1 : J_2 = 1 : 1.29$ at $\theta = \theta_{c1}$ and $J_1 : J_2 = 1 : 1.05$ at $\theta = \theta_{c2}$, while d-wave bond nematic order phase was found in $0.4 \lesssim J_2/J_1 \lesssim 0.6$ in the previous finite-size studies. This discrepancy simply stems from the so-called ‘factor 3’ difference, often encountered between the Hartree-Fock (HF) spin-singlet ansatz and the HF spin-triplet ansatz. If one employed a more numerics-oriented formulation, $\frac{J_1}{J_2}$ appearing in Eq. (12) is replaced by $\frac{J_0}{J_2}$, while $\frac{J_0}{J_2}$ is replaced by $\frac{J_0}{J_2}$.

Consequently, we have $J_1 : J_2 = 1 : 0.43$ ($\theta = \theta_{c1}$) and $J_1 : J_2 = 1 : 0.36$ ($\theta = \theta_{c2}$), which would be relatively comparable with the previous ED result. More quantitative comparison, however, requires the variational Monte Carlo studies based on these spin-triplet ansatzes.

In summary, we have argued that three spin-triplet RVB ansatzes — $Z_2$ and $U(1)$ BW states and $SU(2)$ chiral $p$-wave state — become the lowest mean-field states in the intermediate coupling region, $J_1 \approx 2J_2$ [see Fig. 2(b)]. Among them, both the $Z_2$ BW state and the $SU(2)$ chiral $p$-wave state are stable against any (infinitesimally) small gauge fluctuation, while in the $U(1)$ BW state the effect of gauge fluctuation is crucial, making spinons confined. Using Eq. (3), one can easily see that the BW states show the $d$-wave bond-type spin quadrupolar order precisely as in Eq. (3).

C. Magnetic excitations in the BW states

Here we briefly discuss magnetic excitations in the $Z_2$ BW state. The ‘low-energy’ excitation around the $Z_2$ BW state is composed of three parts; (i) gapped non-magnetic excitations (gauge bosons), (ii) gapless magnetic excitations (Goldstone bosons), and (iii) gapped fermionic ($\Psi$-field) individual excitations. The gapped gauge boson plays only a subdominant role in the spin-structure factor, while the latter two contribute significantly to magnetic excitations. Up to the Hartree-Fock level, one can easily see that the gapped fermionic excitation constitutes the continuum spectrum above $\omega > \max(J_1|D|, J_2|\chi|)$. When one further takes into account the random-phase approximation terms, the gapless bosonic dispersions emerge below this spinon continuum, whose low-energy limit can be described by the matrix-form non-linear $\sigma$ model,

$$F_{\text{magnetic}} = \sum_{\mu=\sigma,\pi,\tau} \text{Tr}[\hat{A}_\mu \partial_\mu \hat{R}^{-1} \partial_\mu \hat{R}]. \quad (46)$$

Namely, the $3 \times 3$ matrix $\hat{R}$ is nothing but the spatio-temporally varying rotational matrix of the director vector used in Eq. (13). The symmetry argument dictates that the diagonal matrices $\hat{A}_\mu$ generally take the following form:

$$\{\hat{A}_\tau, \hat{A}_x, \hat{A}_y\} \equiv \begin{pmatrix} c_0 & c_1 & c_1 \\ c_2 & c_3 & c_4 \\ c_1 & c_4 & c_3 \end{pmatrix}, \quad (47)$$

where the director coplanar plane was taken to be the $2$-$3$ plane. In terms of the semiclassical (gradient) expansion, one can directly calculate their respective coupling constants:

$$c_0 \equiv 0, \quad c_1 \equiv \int_{[-\pi,\pi]^2} d^2k \frac{A^2(s_2^2 + s_3^2)}{64\pi^2\lambda_k^3}, \quad (48)$$

$$c_2 \equiv \int_{[-\pi,\pi]^2} d^2k \frac{A^2(s_2^2 + s_3^2)}{64\pi^2\lambda_k^3}, \quad (49)$$

$$c_3 \equiv \int_{[-\pi,\pi]^2} d^2k \left\{ \frac{J_1A^2s_2^2}{8\pi^2} - \frac{A^4s_2^2s_3^2}{64\pi^2\lambda_k^3} \right\}, \quad (50)$$

$$c_4 \equiv \int_{[-\pi,\pi]^2} d^2k \left\{ \frac{J_1A^2s_2^2}{8\pi^2} - \frac{A^4s_2^2s_3^2}{64\pi^2\lambda_k^3} \right\}, \quad (51)$$

with $J_1 \equiv 4^{-1}\lambda_k^{-1}\{(\partial_{k_x}n^T)(\partial_{k_x}n) + 2^{-1}\lambda_k^{-1}\partial_{k_x}^2\lambda_k\}$, and $n \equiv 2^{\frac{1}{2}}\lambda_k^{-1}(2As_x, 2As_y, Bc_xc_y, Cc_xc_y)$. In addition to these massless excitations, we could also have several gapped (‘optical’) magnetic modes, provided that they are not damped by the spinon individual excitations. One might also expect a certain characteristic behavior of the spectral weight themselves. In fact, Tsunetsugu et al. and Lauchli et al. demonstrated that the spin-structure factor in the site-nematic ordered state exhibits the vanishing spectral intensities of the Goldstone modes around the $\Gamma$-point.

IV. SUMMARY AND OPEN ISSUES

In this paper, we have introduced the spin-triplet slave-boson formulation as a mean-field theory for the bond-type spin nematic state, which was described as the spin-triplet RVB state. Namely, the $d$-vectors of the spin-triplet RVB ansatz constitute the quadrupolar order, while the combination of the spin-triplet and singlet link variables on the same link leads to the vector chiral order.

When applied to the $S = \frac{1}{2}$ square-lattice frustrated ferromagnetic Heisenberg model, our spin-triplet slave-boson analysis gives two non-trivial stable spin-triplet
RVB ansatzes in the intermediate coupling region around $J_1 : J_2 \simeq 1 : 0.4$. One is the $Z_2$ BW state stabilized by the Anderson-Higgs mechanism, while the other is the $SU(2)$ chiral $p$-wave state protected by the Chern-Simon mechanism. Our slave-boson analysis also found an unstable $U(1)$ BW state as a mean-field solution, which possibly gives a route to the realization of spinon confined quadrupolar ordered states with a certain symmetry reduction. The projective symmetry group of the $Z_2$ BW state as well as the $U(1)$ BW state is consistent with the magnetic space group of the $d$-wave bond-type spin nematic state discussed in Ref. [2]. Both of them exhibit the antiferro-type configuration of the bond quadrupolar moment shown in Fig. 1.

Contrary to a naive expectation, our BW state is classified into a ‘weak topological (ordinary) insulator’ instead of the ‘strong topological insulator’ defined in the recent literatures.\textsuperscript{11,12,13,14,15} Physically speaking, such a ‘weak topological insulator (WTI)’ is accompanied either by no spinon edge states at all or by even numbers of the helical edge states. To see that it is indeed a ‘WTI’, one can first deform this $Z_2$ ansatz into the $U(1)$ ansatz ($\eta \to 0$). Since the fermionic dispersion remains gapped, the $Z_2$ topological index associated with the filled spinon band\textsuperscript{11-13} is also unchanged. After reaching the simpler $U(1)$ ansatz, let us then utilize the Fermi surface argument recently introduced by Sato.\textsuperscript{16-18} His argument relates the $Z_2$ topological index in the superconducting state ($D \neq 0$) with the Fermi surface topology in the corresponding ‘normal’ state ($D = 0$). That is, if a Fermi surface in the normal state surrounds odd/even numbers of the time-reversal invariant momentum (TRIM) points, the BW state constructed on top of this normal state is accompanied by non-trivial/trivial $Z_2$ topological index. Since our normal state is composed of two decoupled $u$-RVB states at $\eta = 0$, the resulting Fermi surface clearly surrounds two time-reversal symmetric $k$-points, i.e. $(0, 0)$ and $(\pi, \pi)$. Thus, our $Z_2$ BW state should be classified into the ‘WTI’ ($Z_2$ even class $-\nu$).

In the followings, we will enumerate several open issues and possible extensions of the current work. The most immediate open issue is to identify the magnetic space group of the confining phase proximate to the $U(1)$ BW state based on the monopole field studies.\textsuperscript{63,64} Namely, such an analysis gives several complementary informations to the direct ED studies of the original spin model.\textsuperscript{2}

The fate of the $SU(2)$ chiral $p$-wave state observed at $\theta_{c3} < \theta < \theta_{c3}$ is not so clear either, although we have argued its stability against any (infinitesimally) small gauge fluctuation. Namely, previous exact diagonalization studies of the $SU(2)$ spin model did not find any $T$-symmetry breaking ferro-nematic states between the $d$-wave bond-nematic state and ferromagnetic state. In fact, it is also possible that, when projected onto the real (spin) Hilbert space, the strong gauge fluctuation could wipe out this time-reversal breaking ansatz.

Though we have mainly discussed the quadrupolar order in this paper, our formulation can also describe vector chiral order having no quadrupolar moment\textsuperscript{2} i.e. $P_{\mu,\nu} \neq 0$ and $Q_{\mu,\nu} = 0$. In fact, such vector chiral order state was observed in the spin one half frustrated Heisenberg model having the ring exchange coupling\textsuperscript{2}. When applying the current spin-triplet slave-boson formulation onto these quantum spin systems, one could use the following mean-field parameterization:

$$[E_{j\ell} J_{j\ell} J_{j\ell}'] = [n_1 n_2 n_3]$$

$$[\chi_{j\ell} - \eta_{j\ell} \eta_{j\ell}'] = [\gamma_1 \gamma_2 \gamma_3]$$

where $\{n_1, n_2, n_3\}$ are the normalized unit vectors orthogonal to one another. Namely, such an ansatz gives a finite vector chirality, $P_{\mu,\nu} = 2i \sum_{\alpha=1}^3 \gamma_\alpha n_\alpha$, without any quadrupolar moments. We generally have three alternative ways to parameterize this vector chiral order,

$$[E_{j\ell} J_{j\ell} J_{j\ell}'] = [n_1 n_2 n_3]$$

$$[\chi_{j\ell} - \eta_{j\ell} \eta_{j\ell}'] = [\gamma_1 \gamma_2 \gamma_3]$$

or

$$[E_{j\ell}'' J_{j\ell} J_{j\ell}'] = [\bar{n}_1 \bar{n}_2 \bar{n}_3]$$

$$[\chi_{j\ell} - \eta_{j\ell} \eta_{j\ell}'] = [\gamma_1 \gamma_2 \gamma_3]$$

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The mean-field analysis described in this paper becomes exact in the large $N$ limit of the following action:

$$Z = \int dU^{\text{spin}} dU^{\text{tri}} d\sigma d\Psi^a d\Psi^a \exp \left[ - \int_0^\beta d\tau \mathcal{L} \right]. \quad (A1)$$

$$\mathcal{L} = \frac{1}{2} \sum_j \text{Tr} \left[ \Psi_j^a \left( \partial_\tau \sigma_0 + \sum_{\mu=1}^3 i a^\mu_j \sigma_\mu \right) \Psi_j^a \right] - \frac{J_1}{4} \sum_{(jl)} N \left( |E_{jl}|^2 - |D_{jl}|^2 \right) + \text{Tr} \left[ \Psi_j^a U^{\text{tri}}_j, \Psi_j^a \sigma^T \right] \right\} \right)} \right), \quad (A2)

where the summations with respect to the fermion’s species index $a$ ($= 1, \cdots, N$) were made implicit. The integration over the auxiliary fields leads the following large-$N$ spin Hamiltonian for frustrated ferromagnets:

$$\mathcal{H}_N \equiv - \frac{J_1}{N} \sum_{(jl)} \{ S_j^{ab} \cdot S_l^{ba} + \psi_j^{ab} \psi_l^{ba} \} + \frac{J_2}{N} \sum_{(jl)} S_j^{ab} \cdot S_l^{ba}. \quad (A3)$$

Note that, in addition to the usual $SP(2N)$ spin operators, we have the density operator which is asymmetric in the fermion’s species index:

$$\psi_j^{ab} \equiv \frac{i}{2} \left( f_{a \dagger}^{(l)} f_{b}^{(l)} - f_{b \dagger}^{(l)} f_{a}^{(l)} \right), \quad S_j^{ab} \equiv \frac{1}{2} \left( f_{a \dagger}^{(l)} f_{b}^{(l)} + f_{b \dagger}^{(l)} f_{a}^{(l)} \right),$$

$$S_j^{ab+} \equiv \frac{1}{2} \left( f_{a \dagger}^{(l)} f_{b}^{(l)} + f_{b \dagger}^{(l)} f_{a}^{(l)} \right), \quad S_j^{ab-} \equiv \{ S_j^{ab+} \}^3. \quad (A4)$$

The Hilbert space of this generalized spin Hamiltonian is defined as the $SU(2)$-gauge invariant subspace of the fermionic Hilbert space. That is, any fermionic wavefunction which respects the following local constraints is an element of our Hilbert space:

$$\left\{ \sum_{a=1}^N f_{j \alpha}^a [\hat{\sigma}_\mu]_{\alpha \beta} f_{j \beta}^a \right\}_{\text{phy}} = 0, \quad \forall j, \mu = 1, 2, 3.$$
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