On the two-parameter Erdős-Falconer distance problem over finite fields

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Abstract

Given \( E \subseteq \mathbb{F}_q^d \times \mathbb{F}_q^d \), with the finite field \( \mathbb{F}_q \) of order \( q \) and the integer \( d \geq 2 \), we define the two-parameter distance set as \( \Delta_{d,d}(E) = \{ \|x_1 - y_1\|, \|x_2 - y_2\| : (x_1, x_2), (y_1, y_2) \in E \} \). Birklbauer and Iosevich (2017) proved that if \( |E| \gg q^{3d+1} \), then \( |\Delta_{d,d}(E)| = q^2 \). For the case of \( d = 2 \), they showed that if \( |E| \gg q^{10} \), then \( |\Delta_{2,2}(E)| \gg q^2 \). In this paper, we present extensions and improvements of these results.

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1 Introduction

The general Erdős distance problem asks to determine the number of distinct distances spanned by a finite set of points. In the Euclidean space, it is conjectured that for any finite set \( E \subset \mathbb{R}^d \), \( d \geq 2 \), we have \( |\Delta(E)| \gtrapprox |E|^\frac{d}{2} \), where \( \Delta(E) = \{ \|x - y\| : x, y \in E \} \). Here and throughout, \( X \ll Y \) means that there exists \( C > 0 \) such that \( X \leq CY \), and \( X \gtrapprox Y \) with the parameter \( N \) means that for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that \( X \leq C_\varepsilon N^\varepsilon Y \).

The finite field analogue of the distance problem was first studied by Bourgrain, Katz, and Tao [2] over prime fields. In this setting, the Euclidean distance among any two points \( x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{F}_q^d \), the \( d \)-dimensional vector space over the finite field of order \( q \), is defined as \( \|x - y\| = \sum_{i=1}^d (x_i - y_i)^2 \in \mathbb{F}_q \). For prime fields \( \mathbb{F}_p \) with \( p \equiv -1 \mod 4 \), they showed

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that if \( E \subset \mathbb{F}_p^2 \) with \(|E| = p^\delta\) for some \(0 < \delta < 2\), then the distance set satisfies \(|\Delta(E)| \gg |E|^\frac{1}{2} + \epsilon\), for some \(\epsilon > 0\) depending only on \(\delta\).

This bound does not hold in general for arbitrary finite fields \(\mathbb{F}_q\) as shown by Iosevich and Rudnev [6]. In this general setting, they considered the Erdos-Falconer distance problem to determine how large \(E \subset \mathbb{F}_q^d\) needs to be so that \(\Delta(E)\) spans all possible distances or at least a positive proportion of them. More precisely, they proved that \(\Delta(E) = \mathbb{F}_q\) if \(|E| > 2q^{\frac{d+1}{2}}\), where the exponent is sharp for odd \(d\). It is conjectured that in even dimensions, the optimal exponent will be \(d^2\). As a relaxed fractional variant for \(d = 2\), it was shown in [3] that if \(E \subset \mathbb{F}_q^2\) satisfies \(|E| \gg q^{\frac{4}{3}}\), then \(|\Delta(E)| \gg q\).

A recent series of other improvements and generalizations on the Erdos-Falconer distance problem can be found in [5, 8, 10, 11, 12].

Using Fourier analytic techniques, a two-parameter variant of the Erdos-Falconer distance problem for the Euclidean distance was studied by Birkbauer and Iosevich in [1]. More precisely, given \(E \subset \mathbb{F}_q^d \times \mathbb{F}_q^d\), where \(d \geq 2\), define the two-parameter distance set as

\[
\Delta_{d,d}(E) = \{((\|x_1 - y_1\|, \|x_2 - y_2\|) : (x_1, x_2), (y_1, y_2) \in E) \subseteq \mathbb{F}_q \times \mathbb{F}_q. \]

They proved the following results.

**Theorem 1.1.** Let \(E\) be a subset in \(\mathbb{F}_q^d \times \mathbb{F}_q^d\). If \(|E| \gg q^{\frac{d}{2} + \frac{1}{2}}\), then \(|\Delta_{d,d}(E)| = q^2\).

**Theorem 1.2.** Let \(E\) be a subset in \(\mathbb{F}_q^2 \times \mathbb{F}_q^2\). If \(|E| \gg q^{\frac{3}{4}}\), then \(|\Delta_{2,2}(E)| \gg q^2\).

In this short note, we provide an extension and an improvement of these results. Compared to the method in [1], our results are much elementary.

For \(x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{F}_q^d\) and for an integer \(s \geq 2\), we introduce

\[
\|x - y\|_s := \sum_{i=1}^d a_i(x_i - y_i)^s,
\]

where \(a_i \in \mathbb{F}_q\) with \(a_i \neq 0\) for \(i = 1, \ldots, d\). For any set \(E \subset \mathbb{F}_q^d \times \mathbb{F}_q^d\), define

\[
\Delta_{d,d}^s(E) = \{((\|x_1 - y_1\|_s, \|x_2 - y_2\|_s) : (x_1, x_2), (y_1, y_2) \in E) \subseteq \mathbb{F}_q \times \mathbb{F}_q. \}

Our first result reads as follows.

**Theorem 1.3.** Let \(E\) be a subset in \(\mathbb{F}_q^d \times \mathbb{F}_q^d\). If \(|E| \gg q^{\frac{2s+1}{s}}\), then \(|\Delta_{d,d}^s(E)| \gg q^2\).

It is worth mentioning that our method also works for the multi-parameter distance set defined for \(E \subset \mathbb{F}_q^{d_1 + \cdots + d_k}\), but we do not discuss such extensions herein. For the case of \(d = 2\), we get an
improved version of Theorem 1.2 for the usual distance function over prime fields.

**Theorem 1.4.** Let \( E \subseteq \mathbb{F}_p^2 \times \mathbb{F}_p^2 \). If \(|E| \gg p^{d + \frac{2}{d - 1}}\), then \( |\Delta_2(E)| \gg p^2\).

We note that the continuous version of Theorems 1.3 and 1.4 have been studied in \([4, 7]\). However, the authors do not know whether the method in this paper can be extended to that setting. Moreover, it follows from our approach that the conjecture exponent \( \frac{d}{2} \) of the (one-parameter) distance problem would imply the sharp exponent for two-parameter analogue, namely, \( \frac{3d}{2} \) for even dimensions. We refer the reader to \([1]\) for constructions and more discussions.

## 2 Proof of Theorem 1.3

The following lemma plays a key role in our proof for Theorem 1.3.

**Lemma 2.1** (Theorem 2.3, \([13]\)). Let \( X, Y \subseteq \mathbb{F}_q^d \). Define \( \Delta^s(X, Y) = \{ \|x - y\|_s : x \in X, y \in Y \} \).

If \(|X||Y| \gg q^{d + 1}\), then \(|\Delta^s(X, Y)| \gg q\).

**Proof of Theorem 1.3.** By assumption, we have \(|E| \geq Cq^{d + \frac{d + 1}{2}}\) for some constant \( C > 0 \). For \( y \in \mathbb{F}_q^d \), let \( E_y := \{ x \in \mathbb{F}_q^d : (x, y) \in E \} \), and define

\[
Y := \left\{ y \in \mathbb{F}_q^d : |E_y| > \frac{C}{2}q^{d + \frac{d + 1}{2}} \right\}.
\]

We first show that \(|Y| \geq \frac{C}{2}q^{d + \frac{d + 1}{2}}\). Note that

\[
|E| = \sum_{y \in Y} |E_y| + \sum_{y \in \mathbb{F}_q^d \backslash Y} |E_y| \leq q^d|Y| + \sum_{y \in \mathbb{F}_q^d \backslash Y} |E_y|,
\]

where the last inequality holds since \(|E_y| \leq q^d\) for \( y \in \mathbb{F}_q^d \). Combining it with the assumption on \(|E|\) gives the lower bound \( \sum_{y \in \mathbb{F}_q^d \backslash Y} |E_y| \geq Cq^{d + \frac{d + 1}{2}} - q^d|Y| \). On the other hand, by definition, we have \(|E_y| \leq \frac{C}{2}q^{d + \frac{d + 1}{2}}\) for \( y \in \mathbb{F}_q^d \backslash Y \) yielding the upper bound \( \sum_{y \in \mathbb{F}_q^d \backslash Y} |E_y| \leq \frac{C}{2}q^{d + \frac{d + 1}{2}} \). Thus, these two bounds altogether give \( Cq^{d + \frac{d + 1}{2}} - q^d|Y| \leq \frac{C}{2}q^{d + \frac{d + 1}{2}} \), proving the claimed bound \(|Y| \geq \frac{C}{2}q^{d + \frac{d + 1}{2}}\).

In particular, Lemma 2.1 implies \(|\Delta^s(Y, Y)| \gg q\), as \(|Y||Y| \gg q^{d + 1}\). On the other hand, for each \( u \in \Delta^s(Y, Y) \), there are \( z, t \in Y \) such that \( \|z - t\| = u \). One has \(|E_z|, |E_t| \gg q^{d + \frac{d + 1}{2}}\), therefore, again by Lemma 2.1 \(|\Delta^s(E_z, E_t)| \gg q\). Furthermore, for \( v \in \Delta^s(E_z, E_t) \), there are \( x \in E_z \) and \( y \in E_t \) satisfying \( \|x - y\|_s = v \). Note that \( x \in E_z \) and \( y \in E_t \) mean that \( (x, y) \in E \). Thus, \((v, u) = (\|x - y\|_s, \|z - t\|_s) \in \Delta^s_{d,d}(E)\). From this, we conclude that \(|\Delta^s_{d,d}(E)| \gg q|\Delta^s(Y, Y)| \gg q^2\), which completes the proof. \( \square \)
3 Proof of Theorem 1.4

To improve the exponent over prime fields $\mathbb{F}_p$, we strengthen Lemma 2.1 as follows. Following the proof of Theorem 1.3 with Lemma 3.1 below proves Theorem 1.4 then.

Lemma 3.1. Let $X, Y \subseteq \mathbb{F}_p^2$. If $|X|, |Y| \gg p^{\frac{5}{4}}$, then $|\Delta(X, Y)| \gg p$.

Proof. It is clear that if $X' \subseteq X$ and $Y' \subseteq Y$, then $\Delta(X', Y') \subseteq \Delta(X, Y)$. Thus, without loss of generality, we may assume that $|X| = |Y| = N$ with $N \gg p^{\frac{5}{4}}$. Let $Q$ be the number of quadruples $(x, y, x', y') \in X \times Y \times X \times Y$ such that $\|x - y\| = \|x' - y'\|$. It follows easily from the Cauchy-Schwarz inequality that

$$|\Delta(X, Y)| \gg \frac{|X|^2|Y|^2}{Q}.$$ 

Let $T$ be the number of triples $(x, y, y') \in X \times Y \times Y$ such that $\|x - y\| = \|x - y'\|$. By the Cauchy-Schwarz inequality again, one gets $Q \ll |X| \cdot T$. Next, we need to bound $T$. For this, denote $Z = X \cup Y$, then $N \leq |Z| \leq 2N$. Let $T'$ be the number of triples $(a, b, c) \in Z \times Z \times Z$ such that $\|a - b\| = \|a - c\|$. Obviously, one gets $T \leq T'$. On the other hand, it was recently proved (see [9, Theorem 4]) that

$$T' \ll \frac{|Z|^3}{p} + p^{2/3}|Z|^{5/3} + p^{1/4}|Z|^2,$$

which gives

$$T \ll \frac{N^3}{p} + p^{2/3}N^{5/3} + p^{1/4}N^2,$$

and then $T \ll \frac{N^3}{p}$ (since $N \gg p^{\frac{5}{4}}$). Putting all bounds together we obtain

$$\frac{N^3}{|\Delta(X, Y)|} = \frac{|X||Y|^2}{|\Delta(X, Y)|} \ll \frac{Q}{|X|} \ll T \ll \frac{N^3}{p},$$

or equivalently, $|\Delta(X, Y)| \gg p$, as required. \qed

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