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To cite this article: S. Gamse (2018) Dynamic modelling of displacements on an embankment dam using the Kalman filter, Journal of Spatial Science, 63:1, 3-21, DOI: 10.1080/14498596.2017.1330711

To link to this article: https://doi.org/10.1080/14498596.2017.1330711
Dynamic modelling of displacements on an embankment dam using the Kalman filter

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ABSTRACT

For embankment dams, the modelling of influences in the dam response becomes more difficult if the empirical model cannot be used due to changing external influences. In this work, an attempt at modeling the measured deformations as a dynamic stochastic process is presented. A discrete Wiener process acceleration model (DWPA), which is based on Kalman filtering, is implemented on geodetically measured displacements of the point on a rock-fill embankment dam. The acceleration is modelled as a zero-mean white sequence. The verification of a filter design and choosing an appropriate value of the process noise intensity scalar is controlled primarily with the compliance of the statistical tests in the domain of measurements and in the system state domain. In the case study, it was shown that the DWPA can detect statistically significant changes in measured displacements according to the previous behaviour and can be used for the identification of potential anomalies.

KEYWORDS

Dynamic process; Kalman filter; rock-fill embankment dam; stochastic model

1. Introduction

Modeling of physical systems and processes can be undertaken as either forward modeling or inverse modeling (Gibbs 2011). In both cases, the estimated functional model can be used for the prediction of future behaviour. In the case of inverse modelling, we try to infer the model that fits best to the measured values, but which still remains robust to outliers or large estimated anomalies. A well-known method of inverse modeling is, for example, regression analysis. In forward modelling, known or estimated parameters and measured inputs are used to predict the observed outputs of a system.

In engineering sciences, we often deal with the mathematical modelling of physical systems and processes, in which the values of parameters to be estimated, measurements, additional inputs, disturbances and expected outputs are uncertain and can vary in time. In the modeling of dynamic and kinematic processes, we must deal with the sequential evaluation of input–output time series in a near real-time manner, and standard techniques of the noise reduction of time series, including filtering and smoothing, should be used.

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This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.
The Kalman filter (KF) is an estimation method for time-varying non-stationary stochastic systems using a state-space model. It enables the allowance of process noise, its estimation and weighting. The KF with a stochastic process model can be used in different applications to account for the effects that are difficult to model or are subject to random disturbances (Gibbs 2011). The algorithm is described in several books, such as Bar-Shalom et al. (2001), Grewal et al. (2001), Simon (2006), Gibbs (2011), Catlin (1989), and implemented in many non-engineering and, especially, engineering tasks.

Modern sensors, which are installed on or in engineering structures, enable high precision measurements. The monitoring systems are usually configured as automated monitoring and alarming network compound systems of different complementary measurement sensors, which enable a high frequency of measurements. Moreover, the monitoring systems on large dams consist of several sensors, where the development dictates a full automation of measurements and their assessment. It is of the utmost importance that reliable evaluation models are implemented in such systems to detect significant deviations in a real-time manner and also to detect changes in a long-term trend of the structural behaviour.

In Gamse et al. (2016), the inverse modeling of geodetically measured displacements on a rock-fill embankment dam is presented using multiple linear regression of the hydrostatic-season-time model. The estimated empirical model, which models the influence of the water level in an impounding reservoir and the influence of water and air temperature on the dam, can be used for future predictions and as a real-time alarm system, under the presumption that the future external influences are in the same max-min range as for the modeling period. Due to the water regulation in impounding reservoirs, especially when the dams are constructed for electricity production, this is not always the case. Also, in the case of emptying the impounding reservoir, the estimated empirical model cannot be used for future predictions. For concrete arch dams, the well-established deterministic models, described in Arbeitsgruppe num. Methoden (2015), can also be used instead of empirical models. In the case of rock- or earth-fill embankment dams it is more difficult to model influences on the dam, if the empirical model cannot be used. In this work, an attempt at modeling the measured irreversible and reversible deformations of the control point on the dam crest as a dynamic stochastic process is presented. For one-step forward modeling of measured displacements, a kinematic model of the Kalman filtering (KF) is implemented. In KF, different stochastic process models can be implemented to account for effects that are difficult to model or are subject to random disturbances. The most commonly used is a stochastic model with a random walk, which is the discrete version of Brownian motion, also called the Wiener process (Gibbs 2011). In our case, a discrete Wiener process acceleration model is implemented, where the acceleration is modeled as a zero-mean white sequence.

The paper is organised as follows. After the introduction, the mathematical background of the Kalman filter functional and stochastic model is described in Section 2. In Section 3, the proposed model is implemented on geodetically measured displacements and the numerical results are interpreted. The compliance of the statistically significant anomalies in the domain of measurements and in the system state domain is used as the main criterion of the KF performance. The contribution of the work and suggestions for future work are discussed in Section 4.
2. Kalman filter

In an evaluation model, the redundant measurements are usually used to check the system and to reduce measurement and model errors and uncertainties. The redundant observations give the possibility to increase the precision of the computed unknowns, to estimate the standard deviation of the observations and the unknowns, to test the mathematical and stochastic model, to find gross-errors in the observations and to compute the reliability of the system (Bakker et al. 1995). In the case of kinematic and dynamic processes, which are observed with only one measurement system, KF incorporates the redundant measurements numerically.

KF is a data processing algorithm that estimates the system state from noisy measurements using a least-squares method. It gives the optimal system state estimate together with a measure of how certain it is that the system state estimate is the true state. It performs an optimal solution for a linear process with uncorrelated, white, zero-mean Gaussian process and measurement disturbances.

2.1. Functional model

The dynamic process can be modeled with two main equations (Bar-Shalom et al. 2001, Gibbs 2011):

$$\mathbf{x}_k = \mathbf{F}_{k-1} \cdot \mathbf{x}_{k-1} + \mathbf{G}_{k-1} \cdot \mathbf{u}_{k-1} + \mathbf{w}_{k-1}, \quad \text{dynamic plant equation,} \quad (1)$$

$$\mathbf{z}_k = \mathbf{H}_k \cdot \mathbf{x}_k + \mathbf{v}_k, \quad \text{measurement equation.} \quad (2)$$

Here, $\mathbf{x}_k \in \mathbb{R}^{n \times 1}$ and $\mathbf{z}_k \in \mathbb{R}^{m \times 1}$ is a system state vector and a vector of measurements respectively. Vector $\mathbf{w}_{k-1} \in \mathbb{R}^{n \times 1}$ and $\mathbf{v}_k \in \mathbb{R}^{m \times 1}$ are process and measurement noise, with zero-mean normal distribution, $\mathbf{w}_{k-1} \sim N(\mathbf{0}, \mathbf{Q}_{k-1})$ and $\mathbf{v}_k \sim N(\mathbf{0}, \mathbf{R}_k)$ respectively. Matrices $\mathbf{Q}_{k-1} \in \mathbb{R}^{n \times n}$ and $\mathbf{R}_k \in \mathbb{R}^{m \times m}$ are process and measurement covariance matrices respectively. Matrices $\mathbf{F}_{k-1} \in \mathbb{R}^{n \times n}$ and $\mathbf{H}_k \in \mathbb{R}^{m \times n}$ are the state transition matrix and the observation matrix, respectively. Matrix $\mathbf{G}_{k-1} \in \mathbb{R}^{n \times n_u}$ is a matrix that relates to the optional control input. Matrices $\mathbf{F}_{k-1}$, $\mathbf{H}_k$ and $\mathbf{G}_{k-1}$ can be functions of time, but for most processes they are constant.

To perform Kalman filtering an initial estimate of the system state, $\mathbf{x}_0^-$, and the associated a priori error covariance matrix, $\mathbf{P}_0^-$, at the time step $t_k$ must be known. The measurement-update equations then incorporate new measurements $\mathbf{z}_k$ in the a priori estimate to obtain an improved a posteriori estimate, $\mathbf{x}_k^+$, and an a posteriori error covariance matrix, $\mathbf{P}_k^+$. The measurement-update equations are composed of:

$$K_k = \mathbf{P}_k^- \cdot \mathbf{H}_k^T \cdot (\mathbf{H}_k \cdot \mathbf{P}_k^- \cdot \mathbf{H}_k^T + \mathbf{R}_k)^{-1} \quad (3)$$

$$\mathbf{x}_k^+ = \mathbf{x}_k^- + K_k \cdot (\mathbf{z}_k - \mathbf{H}_k \cdot \mathbf{x}_k^-) \quad (4)$$

$$\mathbf{P}_k^+ = (I - K_k \cdot \mathbf{H}_k) \cdot \mathbf{P}_k^- \cdot. \quad (5)$$
Matrix $I$ is an identity matrix of dimension $\Re^{n \times n}$. Matrix $K_k \in \Re^{n \times m}$ in Equation (4) is the Kalman gain matrix that weights the measurement residuals.

The dynamic plant equations project the system state, $x_k^+$, and the error covariance a posteriori estimate, $P_k^+$, forward in time to get an a priori estimate for the next epoch:

\[
x_{k+1}^- = F_k \cdot x_k^+ + G_k \cdot u_k
\]

\[
P_{k+1}^- = F_k \cdot P_k^+ \cdot F_k^T + Q_k.
\]

### 2.2. Stochastic model

After formalising a functional model as a mathematical relationship between sequential system state components and between measurements and system state vector, a stochastic model is needed for an assessment of propagation of random errors in the estimated parameters. The stochastic model of KF includes hypothesis testing and identification of significant inconsistencies in the domain of measurements and in the system state domain.

#### 2.2.1. Test statistics in the domain of measurements

The statistical testing in the domain of measurements represents a global test of KF estimation. Actual measurements, $z_k$, are compared with their best available prediction based on the system model and previous measurement, $H_k \cdot x_k^-$. The measurement residual, called innovation, $d_k$ (Bar-Shalom et al. 2001, Lippitsch et al. 2006):

\[
d_k = z_k - H_k \cdot x_k^-
\]

follows the $\chi^2$-distribution with the probability relationship $P\{\Omega_{d_k}^2 \leq \chi^2_{m,1-\alpha}\} = 1 - \alpha$. The number of observations at epoch $t_k$, $m$, corresponds to the degrees of freedom. $\alpha$ is a significance level. The null-hypothesis supposes that the sample observations result purely from chance, $H_0 : E(d_k) = 0$, and can be tested against an alternative hypothesis, $H_a : E(d_k) \neq 0$. The test statistics $\Omega_{d_k}^2$, called normalised innovation squared (NIS), is given as Bar-Shalom et al. (2001):

\[
\Omega_{d_k}^2 = d_k^T \cdot D_k^{-1} \cdot d_k,
\]

where $D_k$ is the covariance matrix of innovation (Lippitsch et al. 2006):

\[
D_k = R_k + H_k \cdot P_k^- \cdot H_k^T.
\]

The test statistics is scaled to some reference variance $\sigma_{0(NIS)}^2$. The innovation sequence is zero-mean, white (uncorrelated), and has a Gaussian distribution, $d_k \sim \mathcal{N}(0, D_k)$.

#### 2.2.2. Test statistics in the system state domain

In the system state domain, we test the estimated system state for the epoch $k$, $x_k^+$, with its predicted value, $x_k^-$, based on any previous knowledge we may have about the system. The system state correction or residual $v_{x,k}$ can be written as Bar-Shalom et al. (2001):

\[
v_{x,k} = x_k^+ - x_k^- = K_k \cdot d_k
\]
and is based on the weighting of measurement residuals with the Kalman gain matrix, $K_k$. The system state residual represents the difference between the system state estimate before and after a measurement update. The corresponding covariance matrix, $P_{v_{x,k}}$, is given by Lippitsch et al. (2006):

$$P_{v_{x,k}} = K_k \cdot D_k \cdot K_k^T. \quad (12)$$

The system state corrections have a normal distribution $v_{x,k} \sim N(0, P_{v_{x,k}})$. The test statistics for the system state corrections, called normalised state estimation error squared (NEES), is Bar-Shalom et al. (2001):

$$\Omega_{v_{x,k}}^2 = v_{x,k}^T \cdot P_{v_{x,k}}^{-1} \cdot v_{x,k} \sim \chi^2_{n,1-\alpha}, \quad (13)$$

with $n$ unknowns in epoch $t_k$. The test statistics is scaled to some reference variance $\sigma^2_{0(NEES)}$.

The following tests and indicators can also be used for an assessment of the KF performance (Gamse et al. 2014):

- autocorrelation of the system state components corrections and innovations,
- properties of the a posteriori system state covariance matrix, $P_k^+$: convergence and condition number,
- indicators of inner confidence, such as controllability and observability.

### 2.2.3. Convergence of the $P_k^+$ trace

The convergence of the trace of the a posteriori system state covariance matrix $P_k^+$ can be used to obtain preliminary characteristics of the reliability of the KF model. If the sum of diagonal elements, which represent variances (squares of standard deviations) of the system state components, converges towards some value, then we reason that single elements also converge towards some solution.

### 2.2.4. Condition number of $P_k^+$

In cases in which some elements of the system state vector $x$ are estimated to a much greater precision than other elements of $x$, matrix $P_k^+$ can become indefinite or non-symmetric (Simon 2006). The discrepancy can be identified by analysing eigenvalues of the matrix $P_k^+$ with the condition number of matrix $P_k^+$, defined in Bar-Shalom et al. (2001) as the common logarithm of the ratio of its largest to smallest eigenvalue:

$$\kappa(P_k^+) = \log_{10} \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}}. \quad (14)$$

The value of the condition number depends strongly on the value of the process noise intensity scalar $\sigma_w$, described in Subsection 3.2. The condition number can therefore reflect the ratio between process and measurement noise.

### 2.2.5. Controllability

A discrete-time system is controllable if, given an arbitrary point in the state space, there is an input sequence that will bring the system from any initial state to this point by a finite number of steps (Bar-Shalom et al. 2001). For the $n$-state discrete linear time-invariant system, the controllability condition is that the pair $F, G$ is controllable; that is, the controllability matrix:
\[ C = \begin{bmatrix} G & F & G^2 & G & \ldots & F^{(n-1)} & . & G \end{bmatrix} \] 

(15)

has a full rank \( n \), Bar-Shalom et al. (2001).

2.2.6. Observability

A deterministic system is observable if its initial state can be fully and uniquely recovered from a finite number of observations of its output estimates with the given knowledge of associated inputs. For the \( n \)-state discrete linear time-invariant system, the observability condition is that the pair \( F, H \) is observable; that is, the observability matrix:

\[ O = \begin{bmatrix} H & H & F & H & F^2 & \ldots & H & F^{(n-1)} \end{bmatrix}^T \] 

(16)

has a full rank \( n \) (Bar-Shalom et al. 2001).

2.2.7. Autocorrelation of system state components corrections and innovations

The KF is based on the assumption of white measurement and process noise. A commonly used tool for checking randomness – i.e. there is no time dependence in a data set and correlations between adjacent observations – is an autocorrelation function. To characterise the noise content and time correlation in analysed time series, the autocorrelation function can be used. In the case of randomness, autocorrelations should be near zero for any time lag-separation (Montgomery et al. 2012).

3. Case study: dynamic modelling of measured displacement on a rock-fill embankment dam

3.1. Measurements

The KF model was implemented on geodetically measured relative displacements of the geodetic point in a permanent geodetic network. The direct measurements of angles and distances were performed with the high-precision total stations for some time spans. The geodetic point was located in the middle of the upstream side on the crest of a rock-fill embankment dam. The result of the geodetic network adjustment were coordinates of points in the geodetic network in three perpendicular directions of a predefined local coordinate system. In the presented work, totalling 21-years of observations (January 1992 – December 2012, \( N = 252 \)) of relative displacements in the \( x \)-direction of the predefined local coordinate system were processed. Namely, the reversible and irreversible deformations are of the utmost significance in this direction. The coordinate system is defined as Gamse et al. (2016):

- \( x \)-direction is defined as the longitudinal direction along the water reservoir; a negative direction is defined in the direction of the upstream face;
- \( y \)-direction is along the crest, perpendicular to the \( x \)-axis;
- \( z \)-direction defines differences in height and is perpendicular to the \( x \)- and \( y \)-axes; a negative sign exposes a settlement.

A sketch of the local coordinate system is given in Figure 1. The data processed are relative displacements as monthly median values of daily measured coordinates reduced to zero value for the initial time step. The relative displacements are given in [m], Figure 8. The
available data are also the monthly median values of the water level in the impounding reservoir, given in \([m]\), Figure 8. Water levels are not directly modeled in the KF model; they are used for the interpretation of detected significant anomalies in the domain of measurements and in the system state domain.

### 3.2. Discrete Wiener process acceleration model

The main parameters of interest to be estimated, using measurements described in the previous subsection are relative displacements. Namely, the main intention of the contribution is to detect significant anomalies in relative displacements in a one-step back, one-step forward manner. The monthly median values of relative displacements can be considered as time-varying parameters and their dynamics can be described with simple equations of motion. The system state involves displacement, \(x_k\), velocity, \(v_k\), and acceleration, \(a_k\), in one direction of a predefined local coordinate system, \(x_k = \begin{bmatrix} x & v & a \end{bmatrix}^T\). Under the presumption that the changes in relative displacements are slow, which should be the case for any dam, they can be modelled by disturbances in acceleration. We assume that the movement of the point can be described as a movement with approximately constant acceleration \(a\) during each sampling period \(\Delta t\), \(\ddot{x} = \ddot{a} = 0\). We suppose that the acceleration increments during the \(k\)th sampling period are small and random and can be modelled as a zero-mean white sequence; i.e. the acceleration is a discrete-time Wiener process (Bar-Shalom et al. 2001, Gibbs 2011). The piecewise constant Wiener acceleration model, or discrete Wiener process acceleration model (DWPAM) is given with the system state equation:

\[
x_k = F_{k-1} \cdot x_{k-1} + \Gamma_{k-1} \cdot \tilde{w}_{k-1},
\]

(17)

where the discrete-process noise \(w_{k-1}\) in Equation (1) enters in the dynamic equation through a noise gain \(\Gamma_{k-1} \in \mathbb{R}^{n \times 1}\) and zero-mean white noise scalar \(\tilde{w}_{k-1}\), \(E[\tilde{w}_k] = 0\) and \(E[\tilde{w}_k \cdot \tilde{w}_j] = \sigma_w^2 \cdot \delta_{kj}\). A scalar \(\sigma_w^2\) is a process noise variance or process noise intensity scalar. A function \(\delta_{kj}\) is a Kronecker delta. The state transition matrix \(F_k\) has the form:

\[
F_k = \begin{bmatrix}
1 & \frac{(\Delta t)^2}{2} & \frac{(\Delta t)^3}{6}
0 & 1 & \Delta t
0 & 0 & 1
\end{bmatrix}.
\]

(18)

We are interested in the integrated effect of the random walk on the velocity and displacement as the second and first system state component respectively. The noise gain has
the form $\Gamma_k = \left[ \frac{\Delta t^2}{2} \Delta t \ 1 \right]^T$. For defining the process noise matrix $Q_k$, the process noise variance $\sigma^2_w$ is multiplied by the noise gain (Bar-Shalom et al. 2001):

$$Q_k = E[\Gamma_k \cdot \hat{w}_k \cdot \hat{w}_k^T \cdot \Gamma_k^T] = \Gamma_k \cdot \sigma^2_w \cdot \Gamma_k^T = \sigma^2_w \cdot \begin{bmatrix} \frac{\Delta t^4}{4} & \frac{\Delta t^3}{2} & \frac{\Delta t^2}{2} \\ \frac{\Delta t^3}{2} & \Delta t^2 & \Delta t \\ \frac{\Delta t^2}{2} & \Delta t & 1 \end{bmatrix}. \tag{19}$$

The sampling interval $\Delta t$ was one month in our case. The observation matrix $H_k$ in the measurement equation, Equation (2), is equal to $H_k = \begin{bmatrix} 1 \ 0 \ 0 \end{bmatrix}$, since the relative displacement is the system state component to be estimated and measured at the same time.

In our model we have no deterministic inputs, therefore the term $G_{k-1} \cdot u_{k-1}$ in Equation (1) is not included.

### 3.2.1. Initialisation

To perform the Kalman filtering, some a priori information has to be known. To start the KF loop, we need the initial values of the system state components, $x_{0-}$ (position, velocity, acceleration), and initial error covariance matrix, $P_{0-}$. They are defined from the first few measurements as well as from a few repetitions of the KF.

The covariance matrix of observations $R_k$ is usually defined externally using a known standard uncertainty of a used measurement system or earlier observations. In our case, the standard deviations of monthly median values of measured relative displacements, which present the KF measurements, were not available. We made the presumption that the monthly median values of relative displacements were defined with the same precision, which is lower than the precision of individual directly measured displacements. Therefore we could assume that the replacement of the high precision total stations during a 21-year period did not influence the precision of monthly median values. Namely, the spread of directly measured relative displacement values for the time span of one month was influenced mostly by the water level in the impounding reservoir and irreversible deformations. In Figure 2 the histogram of differences between adjacent monthly median values of relative displacements was plotted and can indicate the dispersion of daily relative displacements for one month. Further, the Allan variance (AVAR) for monthly median values of relative displacements was computed and amounts for averaging time $\tau_0 = 1$, $\sigma_{AVAR}(\tau_0 = 1) = 0.006$ m. Taking the AVAR value as a precision metric of monthly median values, several tests with different values of noise intensity scalar were performed but did not result in the consistency of two statistical tests. Most of the innovations were detected as anomalies. By considering the information from the distribution graph of differences between adjacent monthly median values in Figure 2, the value of $2 \ast$ AVAR was finally taken for the covariance matrix of observations $R_k$, i.e. $\sigma_R = 1.2$ cm. One can refer to Land et al. (2007) or Czerwinski et al. (2009) for more details on the Allan variance. The computations of the AVAR were partially based on the Matlab-code `allanv1_71.m`, by Fabian Czerwinski (Czerwinski et al. 2009).

Further, we needed the process noise covariance matrix $Q_k$, Equation (19). It depends strongly on the process noise variance $\sigma^2_w$, which is defined iteratively, by optimising statistical output parameters and tests.
3.3. Results

The evaluation of the KF model – discrete Wiener process acceleration model, proposed and described in subsection 3.2, is given for the longitudinal $x$-direction of a pre-defined local coordinate system. In this direction, the influence of the water level in the impounding reservoir is at its most significant. We dealt with a time-invariant system, using fixed time intervals, for which the matrices $F_k$, $H_k$, $R_k$, $Q_k$ and $\Gamma_k$ were constant throughout the process. A main challenge was to define the process noise covariance matrix $Q_k$ and corresponding process noise intensity scalar $\sigma^2_w$, which defines the weight between the process and measurement noise in an estimation problem. With tuning the process noise intensity scalar, the parameters of the KF were adjusted to achieve the compliance of the statistical tests and filtering verification parameters. A nearly constant acceleration model was obtained choosing a small process noise intensity scalar, $\sigma^2_w$, which meant high confidence in the model. Larger values of process noise intensity scalar, $\sigma^2_w$, meant a higher confidence in the measurements.

3.3.1. Test statistics in the domain of measurements and in the system state domain

Since, in practice, we cannot measure the performance of the model with respect to the error measures of the system state components directly, a statistical evaluation should be performed. The stochastic model includes a statistical test in the domain of measurements and in the system state domain.

The statistical test in the domain of measurements (NIS test) is based on the statistical testing of innovations, $d_k$, and is the differences between predicted and true measurements, Equation (8). The KF innovations were tested together with their covariances, $D_k$, using the test statistics, as given in Equation (9). For the corresponding probability relationship, a significance level value of $\alpha = 0.05$ was used. The number of degrees of freedom is equal to $r = m = 1$. At the 95% confidence level, the confidence region of the upper one-sided test was $\chi^2_{1,1-0.05} = 3.84$. The NIS test of innovations with significance level is given in
Figure 3. Normalised innovation squared, NIS, with the confidence region $\chi^2_{1,1-0.05} = 3.84$.

Figure 3. It resulted in 22 points out of $N = 252$ (8.7%) being estimated as statistically significant anomalies, with a 95% confidence level.

A second formal test for consistency of the KF DWPAM was to examine the normalised system state error. In the system state domain we tested if the filtered system state, $\mathbf{x}_k^\dagger$, was comparable to the previous knowledge about the system, $\mathbf{x}_{k-}$. The confidence region for a three-degree-of-freedom system state vector ($r = n = 3$) was $\chi^2_{3,1-0.05} = 7.815$ at a 95% confidence level. The NEES test is presented graphically in Figure 4. KF estimation resulted in 21 points out of $N = 252$ (8.3%) being statistically significant as anomalies with a 95% confidence level.

Both test statistics – in the domain of measurements and in the system state domain – were compared with an a priori variance, which was computed by preliminary iterations for the presented numerical case.

The compliance of NIS and NEES test of the proposed KF DWPAM on the data analysed was achieved for the process noise intensity scalar $\sigma_w^2 = 0.6 \text{ m}^2/\text{month}^4$, for the initial value of precision of monthly median values of relative displacements, $\sigma_R = 1.2 \text{ cm}$. For these values it can be summarised that there was no significant discrepancy between the system and measurement model and the statistically significant anomalies were detected at the same measurement epochs in both tests. The reasons of incompatibility or anomalies then had to be analysed. By choosing an inappropriate value of $\sigma_w^2$ a strong non-compliance of both tests occurs. Namely, a high confidence in the model by adopting a small value of $\sigma_w^2$ leads to an unrealistically high number of anomalies in the domain of measurements. On the other hand, a high confidence in the measurements by adopting a large value of $\sigma_w^2$ leads to an unrealistic high number of anomalies in the system state domain.

For the chosen optimal value of the process noise intensity scalar, $\sigma_w^2$, the estimated displacements (blue stars and blue solid line) with standard deviation of measurements (blue dashed line) in the $x$-direction are graphically presented in Figures 5 and 6. In these figures, the measured (red dots) and predicted (green stars) values of displacements are plotted. The updated estimate lies always between the prediction and the measured value and presents a weighted sum of both values. The system state vector also includes
Figure 4. Normalised estimation error squares, NEES, with the confidence region $\chi^2_{3.1-0.05} = 7.815$.

Figure 5. Displacements in the $x$-direction: estimated (dashed line with dots), measured (circles), predicted (solid line with stars) and $\sigma_R$-bound (dashed line).

velocity and acceleration, whose estimates are given in Figure 7. The estimated values for acceleration lie between $\pm 2.5$ cm/month$^2$. The estimated velocity lies in an interval $[-1.5, +3.0]$ cm/month, with some deviations, which are the strongest in a negative direction (upstream face) in the years 2011 and 2012.

The innovations with $\sigma_R$ and $2 \times \sigma_R$ bounds and statistically significant measurement epochs are plotted in Figure 10. In Figure 8 the monthly median values of measured relative displacements and water level with statistically significant measurement epochs are plotted. In Figure 9 details are plotted for significant anomalies in the period 1995–1996 (left) and 2011–2012 (right). The significant anomalies were detected if a much larger or smaller difference between previous two adjacent monthly median values of relative
displacements occurred in comparison to the current difference. The significant anomalies can be interpreted by one or several adjacent larger differences in the water level.

The estimated system state components – relative displacement, velocity and acceleration – follow the periodicities of the water level with two significant frequencies: with annual, 0.0833 cycle/year, and semi-annual periodicity, 0.1667 cycle/year, for monthly data. The underlying periodicites were computed using the fast Fourier transform and are given in Figure 11.

Beside NIS and NEES tests, further parameters and tests of the KF DWPAM model were performed. In KF analysis two presumptions are made:
Figure 8. Monthly median values of water level, in figure given with respect to a fixed constant, and monthly median values of relative displacements in the $x$-direction of a local coordinate system, [m].

Figure 9. Monthly median values of water level and monthly median values of relative displacements between 1995–1996 (left) and between 2011–2012 (right).

- it is assumed that measurement residuals – innovations – are normal Gaussian distributed variables with zero mean and covariance, given in $\mathbf{D}_k$;
- the acceleration increments during the sampling periods are small and random and can be modelled as a zero-mean white sequence.

The histograms with a normal density function and normal probability plots for system state corrections and innovations expose approximately linear patterns on the whole data; Figure 12. Deviations from the straight line indicate deviations from normality. The tail-end deviations indicate significant deviations from the assumed normal distribution and can indicate anomalies of measurement innovations and system state corrections. In Table 1,
Figure 10. Innovations with $\sigma_R$ and $2*\sigma_R$ bound, with statistically significant measurement epochs (stars) detected with NIS and NEES test.

Figure 11. Fast Fourier transform for water level and system state components.

Table 1. Parameters of normal distribution for system state corrections and innovations.

|                        | $\mu$            | Confidence interval | $\sigma$ | Confidence interval |
|------------------------|-------------------|---------------------|----------|---------------------|
| Corrections $x$ [m]    | $-5.6 \cdot 10^{-5}$ | $[-0.001, 0.001]$ | 0.010    | [0.009, 0.011]      |
| Corrections $v$ [m/month] | $-3.6 \cdot 10^{-5}$ | $[-0.002, 0.002]$ | 0.014    | [0.013, 0.015]      |
| Corrections $\sigma$ [m/months$^2$] | $-3.0 \cdot 10^{-5}$ | $[-0.001, 0.001]$ | 0.010    | [0.010, 0.011]      |
| Innovations [m]        | $-6.2 \cdot 10^{-5}$ | $[-0.001, 0.001]$ | 0.010    | [0.009, 0.011]      |

the estimates of mean $\mu$ and the estimate of the standard deviation $\sigma$ of the normal distribution for system state corrections and innovations are given at the 95% confidence interval.
Figure 12. Probability distributions for system state components corrections and innovations: histograms with a normal density function (left) and normal probability plots (right).
3.3.2. Autocorrelation of system state components corrections and innovations

Furthermore, we plotted autocorrelation functions to check the randomness and to detect possible time correlations in system state corrections and innovations, Figure 13. For a chosen optimal value of the process noise intensity scalar, $\sigma_w^2 = 0.6 \text{ m}^2/\text{month}^4$, only a few lags lay slightly outside the 95% confidence limits. Confidence bounds enclosed the lags for which the null hypothesis, which proposes the random distribution of the data, is assumed to hold. By choosing an inappropriate ratio of process to measurement noise, $\sigma_w^2/\sigma_v^2$, the system state corrections and innovation sequence could become strongly correlated and autocorrelation functions expose underlying periodicities.

The ratio of mean absolute deviation (MAD) to standard deviation (SD) as a further test of normality can be used. Its value should be near $\omega_{\text{MAD}/\text{SD}} = \sqrt{2/\pi} = 0.798$ (Geary 1935). In our case, for a chosen $\sigma_w^2$, the ratio is $\omega_{\text{MAD}/\text{SD}} = 0.772$.

3.3.3. Convergence of the trace and the condition number of $P_k^+$

The first important condition of the reliability of the KF model is the convergence of the trace of the a posteriori system state covariance matrix, $P_k^+$ and, accordingly, the convergence of the standard deviations of the system state components, which is satisfied for the proposed DWPAM. The convergence of the $P_k^+$-trace is given in Figure 14. If the initial values are not chosen correctly, the problem of filter divergence can occur, a consequence of which is the divergence of the trace of the a posteriori system state covariance matrix.

In Figure 14 the condition number of $P_k^+$ is plotted for the chosen initial values and $\sigma_w^2 = 0.6 \text{ m}^2/\text{month}^4$ value. It converges toward 2.94. A large condition number, $\kappa(P_k^+) > 6$, can indicate near-singularity (Bar-Shalom et al. 2001).

3.3.4. Controllability and observability

The controllability matrix $C \in \mathbb{R}^{3\times3}$ of the KF DWPAM was computed according to Equation (15), where instead of the matrix $G$, the vector $\Gamma_k$ is taken, because the system dynamics are independent of any control signal, i.e. $G = 0$, and the process noise enters in the system dynamics through the vector $\Gamma_k$. The rank of computed matrix $C$ is then $\text{rank}(C) = n = 3$,
equal to the number of system state components, \( n = 3 \). We concluded that the KF DWPA model is controllable for all three system state components.

The observability matrix \( \mathbf{O} \in \mathbb{R}^{3 \times 3} \) of the KF DWPM model was computed according to equation (16). Our model satisfies the observability condition, where the rank of the observability matrix is equal to the dimension, \( n \), of the system state vector \( \mathbf{x}_k \): \( \text{rank}(\mathbf{O}) = n = 3 \).

4. Conclusions

In the current work, an attempt to estimate geodetically measured displacements on a rockfill embankment dam with dynamic modelling using KF is presented. In dynamic processing by adopting the KF we get the redundant measurements computationally, which enable the statistical evaluation of the functional model. As such, the KF overcomes the problem of uniform defined unknown quantities.

The main objective of the proposed KF DWPAM is to detect, according to the previous behaviour, significant anomalies in monthly median values of relative displacements measured at the control point on the crest of the embankment dam. The model is a one-step back, one-step forward iterative algorithm. The monthly median values of relative displacements present the system state component and measurement in the dynamic plant and measurement equation respectively. The second and third system state components are velocity and acceleration, where the acceleration is modelled as a zero-mean white sequence. The advantage of these kinds of models is that the process noise intensity can be well related to physical characteristics of the motion (acceleration) (Bar-Shalom et al. 2001, Gibbs 2011).

To start the KF iterative loop, initial values have to be defined. Although a good initialisation of the system state components is desirable for a linear Kalman filter it is not essential; it only conditions the time of the estimator to stabilise. The initial system state components can be defined very precisely from the first few measurements. The most
important and influential value is the process noise intensity scalar, $\sigma_w^2$, which defines the weighting between the confidence in the model and measurements and is constant throughout the process. Its value is defined iteratively. The verification of a filter design and choosing an appropriate value of the process noise intensity scalar is controlled primarily with the compliance of the statistical tests in the domain of measurements and in the system state domain. The consistency was achieved for the process noise intensity scalar value $\sigma_w^2 = 0.6 \text{ m}^2/\text{month}^4$. At this value, the normality test also confirms the normal distribution of innovations and system state components corrections at the 5% significance level.

The main contributions of the work can be summarised as follows:

- to process monthly median values of geodetically measured relative displacements with dynamic modelling using the KF DWPAM;
- to detect statistically significant anomalies, when the new observation enters in the model;
- to define an optimal value of the process noise intensity scalar, $\sigma_w^2$, by achieving the compliance of the statistical tests in the domain of measurements and in the system state domain.

The KF DWPAM can detect statistically significant or unexpected changes in monthly median values of relative displacements according to the previous behaviour, but it cannot express any information about the underlying long-term trend or to detrend and estimate the influences such as water level and temperatures, on the relative displacements. The method could be implemented in a real-time manner, which is also the main purpose of any KF model, but further research is required in which the model will be tested on more dense time series with daily measurements. In this case it is expected that the estimated acceleration would follow the normal distribution even better due to a shorter time step – one hour instead of one month as in the case study presented in the paper.

**Note**

1. For a time-invariant system, using fixed time intervals, the matrices $F$, $G$ and $H$ are independent of the discrete time (Grewal et al. 2001).

**Acknowledgements**

The author wishes to acknowledge the hydropower company that provided the data for evaluation in this paper and the University of Innsbruck for financing the Open Access Publishing.

**Disclosure statement**

No potential conflict of interest was reported by the author.

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