COMPUTATIONAL HOLONY DECOMPOSITION OF
TRANSFORMATION SEMIGROUPS

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Abstract. We present an understandable, efficient, and streamlined proof of the Holonomy Decomposition for finite transformation semigroups and automata. This constructive proof closely follows the existing computational implementation. Its novelty lies in the strict separation of several different ideas appearing in the holonomy method. The steps of the proof and the constructions are illustrated with computed examples.

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1. INTRODUCTION

One of the fundamental concepts of science and computation is the notion of \textit{change}: a system goes from a state to another state due to external manipulations or due to internal processes at various time-scales. If the set of states is a continuum
then we study continuous functions and thus we do analysis. If we have a set of
discrete states then we do algebraic automata theory. A transformation semigroup
\((X, S)\) captures the concept of change in a rigorous and discrete way. It consists
of a set of states \(X\) (analogous to phase space), and a set \(S\) of transformations of
the state set, \(s : X \to X\) acting by \(x \mapsto x \cdot s\), that is closed under the associative
operation of function composition. Writing \(s_1 s_2 \in S\) for the composite function
\(s_1 \in S\) followed by \(s_2 \in S\), we have \(x \cdot (s_1 s_2) = (x \cdot s_1) \cdot s_2\), giving a (right) action of
\(S\) on \(X\). A fixed generating set for a transformation semigroup can be considered as
a set of input symbols, therefore automata (without specifying initial and accepting
states) and transformation semigroups are essentially the same concepts.

Another fundamental technique of the scientific method is decomposition. The
holonomy decomposition is a method for finding the building blocks of a transfor-
mation semigroup and compose them in a hierarchical structure. This composite
semigroup has a structure that promotes understanding and it is capable of emu-
lating the original transformation semigroup. Therefore, we say that the holonomy
decomposition is a way of understanding transformation semigroups.

Our aim here is to provide the simplest and most accessible proof for the holo-
nomy decomposition theorem by giving a construction which is ‘isomorphic’ to its
computational implementation [7, 5]. The novelty of this proof is the strict sepa-
ration of the several different ideas that appear in the holonomy decomposition.
Both separating them from each other and from the technical details.

1.1. General Ideas. There are four fundamental concepts used in the holonomy
decomposition. First we state them in their generality to aid intuition, then give a
short summary how they actually appear in the method.

- **Approximation:** gives less information about a system in a way that the
  partial description does not contradict the full description.
- **Emulation:** is a capability of one system producing the same dynamics as
  another one, not necessarily containing an exact copy.
- **Compression:** for repeated patterns stores the pattern once and record its
  occurrences.
- **Hierarchy:** is any system where the control information flows in one direction
  only and abstractions are natural operations.

In the holonomy decomposition, we study the action on chains of increasingly
smaller subsets of the state set, recovering the original transformations at the level
of singleton subsets (approximation). Whenever the semigroup acts the same way
on different subsets, we consider those subsets equivalent and only store the action
on the equivalence class representatives (compression). These representative local
actions are the building blocks and they are aligned according to a partial order
(hierarchy). The chain semigroup and its encoded form, the cascade product can
calculate everything the original transformation semigroup can (emulation).

1.2. Mathematical Preliminaries, Notation. A semigroup is a set \(S\) together
with an associative binary operation \(S \times S \to S\). A semigroup is a monoid if it
contains the identity element. Let \(S^1\) denote the monoid we get by adjoining an
identity to \(S\) in case \(S\) is not a monoid. A transformation semigroup \((X, S)\) is a
finite nonempty set \(X\) (the state set) together with a set \(S\) of total transformations
of \(X\) closed under function composition. The states are often denoted by a set of
integers \(n = \{1, \ldots, n\}\), and the transformations by the list of images \([j_1, j_2, \ldots, j_n]\),
where \( i \mapsto j \) for \( i, j \in n \). The action \( x \mapsto x \cdot s \) on the points (states) \( x \in X \) by transformations \( s \in S \) naturally extends to set of points: \( P \cdot s := \{ p \cdot s \mid p \in P \} \), \( P \subseteq X \), \( s \in S \), and we have \((P \cdot s_1) \cdot s_2 = P \cdot (s_1 s_2)\), for \( s_1, s_2 \in S \). Similarly, the action can also be extended to sets of sets of points or to tuples or sequences of points or sets of points.

The wreath product \((X, S) \wr (Y, T)\) of transformation semigroups is the transformation semigroup \((X \times Y, W)\) where

\[
W = \{(s, f) \mid s \in S, f \in T^X\},
\]

whose elements map \( X \times Y \) to itself as follows

\[
(x, y) \cdot (s, f) = (x \cdot s, y \cdot f(x))
\]

for \( x \in X, y \in Y \). Here \( T^X \) is the semigroup of all functions \( f \) from \( X \) to \( T \) (under pointwise multiplication). Note we have written \( y \cdot f(x) \) for the element \( f(x) \in T \) applied to \( y \in Y \). The wreath product construction is associative on the class of transformation semigroups (up to isomorphism) and can be iterated for any number of components.

The size of the iterated wreath product grows rapidly by increasing the number of components or by increasing their sizes. Explicit computation with wreath products is impractical. This motivates the definition of cascade products: efficient constructions of substructures of wreath products, induced by explicit dependency functions [6]. Essentially, cascade products are transformation semigroups glued together by functions in a hierarchical tree. More precisely, let \(((X_1, S_1), \ldots, (X_n, S_n))\) be a fixed list of transformation semigroups (here \( S_i \) are semigroups and \( X_i \) the sets on which they act), and define dependency functions to be functions of the form

\[
d_i : X_1 \times \ldots \times X_{i-1} \to S_i, \quad \text{for } i \in \{2 \ldots n\}.
\]

A transformation cascade is then defined to be an \( n \)-tuple of dependency functions \( d = (d_1, \ldots, d_n) \), where \( d_i \) is a dependency function of level \( i \). On the top level, \( d_1 \) is simply an element of the semigroup \( S_1 \). The transformation cascade \( d \) applied to \((x_1, \ldots, x_n)\) is defined coordinatewise by \( x_i \cdot d_i(x_1, \ldots, x_{i-1})\), applying the results of the evaluated dependency functions, so that the cascade product can be regarded as a special transformation representation on the set \( X_1 \times \ldots \times X_n \). The hierarchical structure allows us to conveniently distribute computation among the components, and perform abstractions and approximations of the system modelled as a cascade product. In the permutation group case it is basically the Schreier-Sims algorithm [14] put into product form [6].

1.3. Computational Tools. The constructive proof for the holonomy decomposition described here is implemented in the SgpDec [7, 5] software package for the GAP computer algebra system [10]. For the verification of the correctness of the software package we use a selection of transformation semigroups with interesting features and corner cases. We also have a shadow implementation of the algorithms based on partitioned binary relations in the kigen system [4].

1.4. Historical Notes. In Krohn-Rhodes theory, the holonomy method for cascade decomposition was originally developed by H. Paul Zeiger [21, 22], and subsequently improved by S. Eilenberg [8], and later by several others [3, 11, 13].
Variants [2, 18], and generalizations of the theorem to the infinite case [9, 12] and to categories [20] were also studied.

The term ‘holonomy’ is borrowed from differential geometry, since a roundtrip of composed bijective maps producing permutations is analogous to moving a vector via parallel transport along a smooth closed curve yielding change of the angle of the vector.

The current proof is a prime example of the observation on the development of mathematics, that proofs turn into definitions (see the introduction of [19]), as the way we define the chain semigroup is the key argument of the previous proofs.

2. Approximation

For a transformation semigroup \((X, S)\) we describe ways to approximate the states \(x \in X\) by subsets of \(X\), and to approximate the transformations in \(S\), the ‘behaviour’ of the semigroup.

2.1. Approximating states. What is the current state of the system? We can answer this question precisely by giving a single element, or we can give partial information by specifying a set of states with the condition that the current state is contained in the set. This way, any subset \(P \subseteq X\) such that \(x \in P\) can be considered as an approximation of the state \(x\).

For a particular transformation semigroup we do not need to consider all such elements of the power set \(\mathcal{P}(X)\), we can restrict to those that are generated by the semigroup action.

**Definition 2.1.** The set \(\mathcal{I}_S(X) = \{X \cdot s \mid s \in S\}\) is the image set of the transformation semigroup \((X, S)\).

Note that in general \(X\) itself and the singleton state sets are not necessarily included, so we may need to add them to the image set.

**Definition 2.2.** The extended image set of the state set under the action of the semigroup is \(\mathcal{I}'_S(X) = \mathcal{I}_S(X) \cup \{X\} \cup \{\{x\} \mid x \in X\}\).

When approximating, we may be interested in doing it step-by-step. Since approximations are subsets, we can build successive approximations by nested subset chains.

**Definition 2.3.** A chain \(C\) is a subset of \(\mathcal{P}(X)\) such that \(P \subseteq Q\) or \(Q \subseteq P\). A chain \(C\) is maximal if it is not properly contained in any other chain. We say that two chains \(C\) and \(D\) agree down to \(P\) if \(P \in C \cap D\) and for all subsets \(Q\) with \(P \subseteq Q \subseteq X\) we have \(Q \in C \Leftrightarrow Q \in D\).

**Observation.** Notice that \(S\) acts on subset chains in \(X\): Since \(P \subseteq Q\) implies \(P \cdot s \subseteq Q \cdot s\), necessarily \(C \cdot s\) is a chain if \(C\) is. Moreover, \((C \cdot s_1) \cdot s_2 = C \cdot s_1s_2\). However, the length of chains can become shorter under this action.

As mentioned before, for the holonomy decomposition we do not need the full power set. However, we need the extended image set if we want to describe all necessary stages of approximating a state by maximal chains.

**Definition 2.4.** Let \(\mathcal{C} = \mathcal{C}(X, S)\) denote the set of all maximal chains in \(\mathcal{I}'_S(X)\).

There is a surjective function \(\eta : \mathcal{C} \to X\) mapping each maximal chain \(C\) to the element of its unique singleton \(\{x\} \in C\). We say \(C\) is a lift of \(x \in X\) if \(\eta(C) = x\).
2.2. **Approximating Transformations.** A state \( x \) is lifted as a maximal subset chain starting from \( \{x\} \). Consequently, for lifting transformations we need to construct transformations mapping \( C \) to itself. However, simply acting on maximal chains, \( C \mapsto C \cdot s \) is not a well-defined action on \( C \), since \( C \cdot s \) may not be maximal.

**Definition 2.5.** A dominating chain of a chain \( C \) is a maximal chain \( D \) such that \( C \subseteq D \).

There can be more than one dominating chain. For instance, acting by a constant \( C \subseteq s \) may not be maximal.

For any fixed \( s \in S \) we can define a (non-unique) mapping \( \hat{s} : C \to C \) by \( \hat{s}(C) = D \), where \( D \) is any fixed maximal chain containing \( C \cdot s \). We can think of such an \( \hat{s} \) as mapping the nested approximations \( C \) of \( x = \eta(C) \) to nested approximations \( \hat{s}(C) \) of \( x \cdot s = \eta(\hat{s}(C)) \). We say \( \hat{s} \) is consistent with chain structure if \( C \) and \( C' \) agree down to \( P \) then \( \hat{s}(C) \) and \( \hat{s}(C') \) agree down to \( P \cdot s \).

One way to ensure this condition is to totally order \( \mathcal{I}'_S(X) \), and for example choose its least member that can be included when building a dominating chain. We observe there is always at least one way to choose \( \hat{s} \) so that it is consistent with chain structure.

**Lemma 2.6.** If \( \hat{s}_1 \) and \( \hat{s}_2 \) mapping \( C \) to itself are consistent with chain structure, then so is the composite mapping \( \hat{s}_1 \hat{s}_2 \).

**Proof.** If maximal chains \( C \) and \( C' \) agree down to \( P \in C \cap C' \) then, since \( \hat{s}_1 \) is consistent with chain structure, \( \hat{s}_1(C) \) and \( \hat{s}_1(C') \) agree down to \( P \cdot s_1 \). Since \( \hat{s}_2 \) is consistent too, we have that \( \hat{s}_2(\hat{s}_1(C)) \) and \( \hat{s}_2(\hat{s}_1(C')) \) agree down to \( (P \cdot s_1) \cdot s_2 = P \cdot (s_1 s_2) \). \( \square \)

**Definition 2.7** (Chain semigroup). Given a generating set \( A \), for \( s \), for each \( a \in A \) we choose a consistent \( \hat{a} \) and take \( \hat{S} = \langle \hat{a} \mid a \in A \rangle \). Then we call the transformation semigroup \( (C, \hat{S}) \) a chain semigroup.

We say \( \hat{a}_1 \cdots \hat{a}_k \) is a lift of \( s \in S \) if \( s = a_1 \cdots a_k \) for generators \( a_1, \ldots, a_k \in A \).

By Lemma 2.6 it follows that any \( \hat{s} = \hat{a}_1 \cdots \hat{a}_k \) is consistent, i.e.,

**Proposition 2.8.** All mappings in a chain semigroup \( (C, \hat{S}) \) are consistent with chain structure.

**Remark 2.9.**

1. We generally take just one lift \( \hat{a} \) for each generator \( a \) of \( S \) to generate a chain semigroup, since one would often like \( \hat{S} \) to be as small as possible. Different choices of lifts for the generators can result in different sized \( \hat{S} \).
2. Generally, there can be many different lifts \( \hat{s} \in \hat{S} \) for fixed \( s \) in \( S \), since \( s = a_1 \cdots a_k = a'_1 \cdots a'_\ell \) does not imply \( \hat{a}_1 \cdots \hat{a}_k \) and \( \hat{a}'_1 \cdots \hat{a}'_\ell \), although both are lifts of \( s \).
3. There is a unique maximal chain semigroup obtained by taking all possible consistent \( \hat{s} \) for \( s \in S \), and letting \( \hat{S} \) be the semigroup they generate.

In a sense chain semigroup contains approximations of \( (X, S) \). The rest of the holonomy decomposition is about putting an efficient notation (by embedding it into a wreath product) on this expanded semigroup.
3. Emulation

We need to show that a chain semigroup emulates the original semigroup.

**Lemma 3.1.** There is a surjective morphism of transformation semigroups

\[(X, S) \twoheadrightarrow (C, \hat{S}).\]

**Proof.** There is a semigroup homomorphism from \(\hat{S}\) to \(S\) determined by \(\hat{a} \mapsto a\), where we recall that \(a\) is a generator of \(S\). It is not hard to see this is well-defined.

(And it follows, e.g., from Proposition 1.10 in [3]). Since \(\eta(\hat{a}(C)) = x \cdot a\) for \(x = \eta(C)\), the action is respected. \(\Box\)

In the final form of the holonomy decomposition we will use the following notion of emulation.

3.1. Division. One transformation semigroup divides another, \((X, S) \mid (Y, T)\), if \((X, S)\) is a homomorphic image of a substructure of \((Y, T)\): precisely, there exists a subset \(Z \subseteq Y\) and a subsemigroup \(U \leq T\), with \(z \cdot u \in Z\) for all \(z \in Z, u \in U\), and a surjective function \(\theta_1 : Z \twoheadrightarrow Y\) and surjective homomorphism \(\theta_2 : U \rightarrow S\) such that \(\theta_1(z \cdot u) = \theta_1(z) \cdot \theta_2(u)\) for all \(z \in Z\) and \(u \in U\).

4. Compression

4.1. Equivalence of Subsets. On \(I'_{S}(X)\) we define an equivalence relation by

\[P \equiv_{S} Q \iff \exists s, t \in S\] such that \(P = Q \cdot s\) and \(Q = P \cdot t\).

This is the equivalence relation of ‘mutual reachability’ under the action of \(S\), and the equivalence classes are the strongly connected components of \((X, S)\) acting on \(I'_S(X)\).

It is immediate that \(P \equiv_{S} Q \implies |P| = |Q|\). As we will see \(S\) acts the same way on equivalent elements (see permutator and holonomy groups defined below), thus the equivalence classes provide the way to compress information in the decomposition. For each equivalence class there will be only one component in the hierarchical decomposition.

4.2. Group Actions. For a subset \(P \subseteq X\) we have the stabilizer semigroup \(S_P = \{s \in S \mid P \cdot s = P\}\). If we restrict the action of the stabilizer to \(P\) we get the permutator group \(G_P\). These groups are also called generalized Schützenberger groups [17].

In the holonomy decomposition we need the most coarse-grained approximation possible so we have to take another homomorphic image of \(G_P\). Considering the inclusion relation \((I'_S(X), \subseteq)\), we call a (lower) cover \(P_i\) of a non-singleton subset \(P \in I(X)\) a tile denoted by \(P_i \prec P\). The set of all tiles of \(P\) is denoted by \(T(P)\). These are the maximal subsets of \(P\) in \(I'_S(X)\). Obvious properties of tiles are:

\[P = \bigcup_{i=1}^{k} P_i, \quad P_i \subseteq P_j \implies P_i = P_j\]

where \(P_i \in T(P)\) and \(k = |T(P)|\). Important to note that tiles of a set may overlap, so one should think of roof tiles as the analogy.

The holonomy group \(H_P\) is the permutation group \((T(P), G_P)\) made faithful.
4.3. Constructing Holonomy Groups. If $P \equiv_S Q$, then there exist mappings $m_{P\to Q}, m_{Q\to P} \in S$ mapping $P$ to $Q$ bijectively ($Q$ to $P$ respectively), such that $m_{P\to Q}m_{Q\to P}$ is the identity map restricted to $P$ and $m_{Q\to P}m_{P\to Q}$ is the identity restricted to $Q$ (see e.g. [17]).

It can be shown that if $P \equiv_S Q$ then $G_P \cong G_Q$. Since there is a bijection between $\mathcal{T}(P)$ and $\mathcal{T}(Q)$, it follows that $H_P \cong H_Q$. Moreover, ‘roundtrips’ of mappings in the equivalence class induce permutations on elements of the equivalence class (see schematic drawing on Figure 1). We can get the generators of $G_R$ by contracting roundtrips of the form

$$m_{R\to P} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
It is easy to see that $\subseteq_S$ is a preorder: it is reflexive, since $P \subseteq P \cdot 1$, and it is transitive, since if $P \subseteq Q \cdot s_1$ and $Q \subseteq R \cdot s_2$ then $P \subseteq R \cdot s_2 s_1$, thus $P \subseteq_S R$.

Using a common technique for preorders, we define the $\equiv_S$ equivalence relation on $\mathcal{I}'_S(X)$ by taking subduction in both directions: $P \equiv_S Q \iff P \subseteq_S Q$ and $Q \subseteq_S P$.

### 5.2. Positioning the components: Height and Depth of Sets.

The *height* of a set $Q \in \mathcal{I}'_S(X)$ is given by the function $h : \mathcal{I}'_S(X) \to \mathbb{N}$, which is defined by $h_S(Q) = 0$ if $Q$ is a singleton, and for $|Q| > 1$, $h_S(Q)$ is defined by the length of the longest strict subduction chain(s) in the skeleton starting from a non-singleton set and ending in $Q$:

$$h_S(Q) = \max_{i}(Q_1 \subseteq_S \cdots \subseteq_S Q_i = Q),$$

where $|Q_1| > 1$. The height of $(X, S)$ is $h = h_S(X)$.

It is also useful to speak of *depth* values, which are derived from the height values:

$$d(Q) = h_S(X) - h_S(Q) + 1.$$

The top level is depth 1.

Calculating the height values establishes the hierarchical levels in the decomposition, i.e. the number of coordinate positions in the holonomy decomposition is $h_S(X)$.

**Fact 5.1 (Depth never decreases).** Let $P \in \mathcal{I}'$. Then $d(P \cdot s) \geq d(P)$. If $d(P \cdot s) = d(P)$ then $P \equiv_S P \cdot s$.

### 5.3. Positioned Chain Semigroup.

By using the depth function, we can know align the members of those maximal chains on which the chain semigroup acts.

**Definition 5.2 (Positioned chain).** For a maximal chain $C$

$$X = P_1 \supset P_2 \supset P_3 \supset \ldots \supset P_k = \{x\},$$

we take the associated *positioned chain* $C^{\text{pos}}$. This is a vector of length $h_S(X)$ where the slots are empty (denoted by *) except that $P_{i+1}$ is in position $d(P_i)$ for $1 \leq i < k$. For a positioned chain $C^{\text{pos}}$ the content at level $i$ is $C^{\text{pos}}[i]$.

This puts the members of chains into coordinate slots. By the maximality of the chain we have $C^{\text{pos}}[i] \prec P_i$. Note that a positioned chain omits $X$, since it is not a tile of anything.

We can identify the action of the chain semigroup with an action on positioned chains denoted by $C^{\text{pos}}(X, S)$:

**Fact 5.3.** $(C, S) \cong (C^{\text{pos}}, \hat{S})$.

**Proof.** The positioned chains are in one-to-one correspondence with the maximal chains of $C$ by the maps $C \leftrightarrow C^{\text{pos}}$, since the only missing element of the chain in the positioned chain is $X$ itself, so it can be added without any ambiguity when recovering the maximal chain.
At each level of depth we need to know how far the approximation proceeded so far, i.e., we need to know what subset of the state set are we acting on at the given depth. The value at the position is a tile, and tiles can belong to more than one set, so we need to look back to the first concrete value above.

**Definition 5.4** (state of approximation).

\[
\alpha_i(C^{\text{pos}}) = \begin{cases} 
X & \text{if } i = 1 \\
C^{\text{pos}}[j] & \text{otherwise, where } j = \max_j \{ C^{\text{pos}}[j] \neq * \text{ and } j < i \},
\end{cases}
\]

\[1 \leq i \leq h_S(X).\]

Since \( \alpha_i \) only depends on \( C^{\text{pos}}[j] \) where \( j < i \), \( \alpha_i \) is well-defined on prefixes of \( C \) of length at least \( i - 1 \). Moreover, we define \( \alpha(C^{\text{pos}}) = (\alpha_1(C^{\text{pos}}), \ldots, \alpha_{h_S(X)}(C^{\text{pos}})) \).

**Lemma 5.5.** For all maximal chains \( C \) and \( 1 \leq i \leq h_S(X) \),

\[ C^{\text{pos}}[i] \in \mathcal{T}(\alpha_i(C^{\text{pos}})) \]

when \( C^{\text{pos}}[i] \neq * \).

**Proof.** This is immediate from the definition of positioned chains (Def. 5.2). \( \square \)

**Lemma 5.6.** For all maximal chains \( C \) it always holds the \( d(\alpha_i(C^{\text{pos}})) \geq i \).

**Proof.** If \( i = 1 \) then \( \alpha_i(C^{\text{pos}}) = X \) and \( d(X) = 1 \) so the statement holds. If \( i > 1 \) then assume the statement holds by induction hypothesis for \( i \) so we show that it follows for \( i + 1 \):

- **Case 1:** \( d(\alpha_i(C^{\text{pos}})) = i \) then by the definition of positioned chains \( C^{\text{pos}}[i] \) is a tile of \( \alpha_i(C^{\text{pos}}) \), so at level \( i + 1 \) the value of \( \alpha_{i+1} \) will be this tile, which is of depth at least \( i + 1 \).

- **Case 2:** \( d(\alpha_i(C^{\text{pos}})) > i \) then \( \alpha_{i+1}(C^{\text{pos}}) = \alpha_i(C^{\text{pos}}) \) but still \( d(\alpha_{i+1}(C^{\text{pos}})) \geq i + 1 \).

\( \square \)

When lifting a transformation \( s \), we only need to act when we are on the right level, i.e., \( d(\alpha_i(s(C^{\text{pos}}))) = i \). The next lemma shows that the action of a lifted transformation respects approximation.

**Lemma 5.7.** For a transformation \( s \in S \) and a maximal chain \( C \), we have for all coordinate levels \( i \)

\[ (\alpha_i(C^{\text{pos}})) \cdot s \subseteq \alpha_i((\hat{s}(C))^{\text{pos}}) \cdot s.\]

**Proof.** Let \( P_i = \alpha_i(C^{\text{pos}}) \) and \( Q_i = \alpha_i((\hat{s}(C))^{\text{pos}}) \). \( P_1 = Q_1 = X \), so the statement is true for \( i = 1 \).

By induction hypothesis, the statement holds for levels down to and including \( i \). Trivially, \( P_{i+1} \subseteq P_i \) and \( Q_{i+1} \subseteq Q_i \). We show that \( P_{i+1} \cdot s \subseteq Q_{i+1} \).

If \( Q_{i+1} = Q_i \) then the statement holds since \( P_{i+1} \cdot s \subseteq P_i \cdot s \subseteq Q_i \). Otherwise, \( Q_{i+1} \subseteq Q_i \) and by the maximality of the chain \( Q_{i+1} \in \mathcal{T}(Q_i) \).

- **Case 1:** If \( P_i \cdot s = Q_i \), then \( d(P_i) \leq d(Q_i) \) as \( P_i \) cannot be deeper than \( Q_i \) and \( d(Q_i) = i \) since we are on the right level and \( d(P_i) \geq i \) always holds. Thus we have \( P_i \equiv_S Q_i \) and \( d(P_i) = d(Q_i) = i \). Also, \( d(P_{i+1}) \geq i + 1 \), therefore \( P_i \neq P_{i+1} \). Finally, \( P_{i+1} \cdot s \in \hat{s}(C^{\text{pos}}) \Rightarrow P_{i+1} \cdot s \subseteq Q_{i+1} \).

- **Case 2:** If \( P_i \cdot s \subset Q_i \) since \( \hat{s}(C) \) is a maximal chain containing \( P_i \cdot s \) and \( Q_{i+1} \) is a tile of \( Q_i \), so \( P_i \cdot s \subseteq Q_{i+1} \), whence \( P_{i+1} \cdot s \subseteq Q_{i+1} \).

\( \square \)
5.4. Holonomy Cascade Semigroup. We build a cascade product of the holonomy groups of \((X, S)\). First the components. Let \(R_1, \ldots, R_k\) be the representative sets of depth \(i\). Then the \(i\)th component of the cascade product is defined as the transformation semigroup

\[
\mathcal{H}_i = (\tau_i, \mathcal{H}_i) = (T(R_1) \cup \cdots \cup T(R_k) \cup \{\ast\}, \mathcal{H}_{R_1} \cup \cdots \cup \mathcal{H}_{R_k}), 1 \leq i \leq h_S(X).
\]

The set of states are the set of tiles of the representative sets of depth \(i\). These tile sets may overlap, thus we need to take the disjoint union. This causes no confusion since for each positioned chain we know the current state of approximation, hence we know which set of tiles we need to choose from.

The transformations come from the holonomy groups of the representatives of depth \(i\). How does \(H_i\) act on \(T_i\)? If \(P\) lies in the \(j\)th set \(T(R_j)\) of the disjoint union then \((h_1, \ldots, h_k) \in H_i\) acts on \(P\) by applying \(h_j\) and it acts on \(\ast\) trivially. Recall that \(H_i\) augments the group \(H_i\) with all constant maps on \(T_i\). Since \((\tau_i, \mathcal{H}_i)\) is a well defined transformation semigroup for \(1 \leq i \leq h_S(X)\), we can form their wreath product.

**Definition 5.8.** We call \(\mathcal{H}_1 \ast \cdots \ast \mathcal{H}_d = \mathcal{H}(X, S)\) the holonomy wreath product semigroup of \((X, S)\).

In practice, we only want a substructure of this potentially huge wreath product, so we need to construct a cascade product by giving explicit dependency functions in the transformation cascades induced by the generators of \(S\). The maps are \(s \mapsto \hat{s} \mapsto \text{enc}(\hat{s})\), where the final encoding describes \(\hat{s}\) in terms of the corresponding representative set.

5.4.1. Encoding and decoding. We encode the elements of a positioned chain, that are tiles of the current state of approximation, as tiles of the representative set of the corresponding height. If \(C_{\text{pos}}[j] \neq \ast\) then

\[
\text{enc}(C_{\text{pos}})[j] = C_{\text{pos}}[j] \cdot m_{\mathcal{T} 
abla P} \text{ where } P = \alpha_j(C_{\text{pos}}),
\]

otherwise the encoded value is \(\ast\). Since \(\alpha\) is not recursive, encoding can also be done independently for any level.

Decoding does the opposite, however we need to calculate the current unencoded state of approximation, therefore it is a recursive calculation. Let \(V = \text{enc}(C_{\text{pos}})\), the tuple of coordinate values. If \(V_{\text{pos}}[j] \neq \ast\) then

\[
\text{dec}(V_{\text{pos}})[j] = V_{\text{pos}}[j] \cdot m_{\mathcal{T} \nabla P} \text{ where } P = \alpha_j(\text{dec}(V_{\text{pos}})),
\]

otherwise the encoded value is \(\ast\). These are bijective maps, thus \(\text{dec}(\text{enc}(C_{\text{pos}})) = C_{\text{pos}}\) and \(\text{enc}(\text{dec}(V)) = V\).

5.4.2. Dependency functions. For \(\hat{s}\) in a chain semigroup \(\hat{S}\), let’s define \(\text{enc}(\hat{s})\) to be the transformation cascade given by the dependency functions

\[
(\text{enc}(\hat{s})): T_1 \times \cdots \times T_{i-1} \rightarrow \mathcal{H}_i.
\]

Let’s fix a positioned chain \(C_{\text{pos}}\), and thus \(P = \alpha_i(C_{\text{pos}}), Q = \alpha_i(\hat{s}(C_{\text{pos}}))\) and \(V = \text{enc}(C_{\text{pos}})\). By Lemma 5.7, these state approximations satisfy \(P \cdot \hat{s} \subseteq Q\), in which case we have a constant map (reset) to a tile or a permutation.

Precisely, if \(i \neq d(Q)\), then let \(\text{enc}(\hat{s})(V_1, \ldots, V_{i-1}) = \text{constant } \ast \in \mathcal{H}_i\).

There are two possibilities when \(i = d(Q)\):
**Theorem 5.9.** \((C, \hat{S}) \cong (C^{\text{pos}}, \hat{S}) \hookrightarrow H_1 \cdots \hookrightarrow H_d = \mathcal{H}(X, S), \) where \(d = h_S(X).\)

The image of such an embedding is called a holonomy (decomposition) cascade product.

**Proof.** The isomorphism was shown in Fact 5.3. We show \(\text{enc}\) is an embedding of transformation semigroups from \((C^{\text{pos}}, \hat{S})\) to the wreath product. For the states, \(\text{enc}(C^{\text{pos}}) \subseteq T_1 \times \cdots \times T_d\) holds trivially. We need to show that if \(V = \text{enc}(C^{\text{pos}}) \in \text{enc}(C^{\text{pos}})\) then \(\text{enc}(\hat{s})(V) = \text{enc}(\hat{s}(\text{dec}(V))).\)

By looking at the \(i\)th position for each \(1 \leq i \leq d,\) if \(i \neq d(\hat{s}(\alpha_i(C^{\text{pos}})))\) then \(\hat{s}(C^{\text{pos}})\) cannot have a tile in position \(i,\) it follows that \(\ast = \text{enc}(\hat{s}(C^{\text{pos}}))\), which is equal to

\[ V_i \cdot \text{enc}(\hat{s})_i(V_1, \ldots, V_{i-1}) = V_i \cdot \text{constant}, \]

as required.

Otherwise, \(i = d(\hat{s}(\alpha_i(C^{\text{pos}})))\), and we have two cases. If \(\text{enc}(\hat{s})_i(V_1, \ldots, V_{i-1})\) is a constant map to a tile, then the definition of \(\text{enc}(\hat{s})\) yields

\[ V_i \cdot \text{enc}(\hat{s})_i(V_1, \ldots, V_{i-1}) = V_i \cdot \text{constant} \cdot (\text{enc}(\hat{s}(C^{\text{pos}}))[i]) = (\text{enc}(\hat{s}(C^{\text{pos}}))[i]), \]

as required. Otherwise, the component action is a permutation, and then

\[ \text{enc}(\hat{s})(V)[i] = V_i \cdot \text{enc}(\hat{s})_i(V_1, \ldots, V_{i-1}) = \text{enc}(C^{\text{pos}})[i] \cdot m_{\mathcal{P} \rightarrow \mathcal{P}} s m_{\mathcal{Q} \rightarrow \mathcal{Q}} \]

\[ = C^{\text{pos}}[i] \cdot m_{\mathcal{P} \rightarrow \mathcal{P}} m_{\mathcal{P} \rightarrow \mathcal{P}} s m_{\mathcal{Q} \rightarrow \mathcal{Q}} \]

\[ = C^{\text{pos}}[i] \cdot s m_{\mathcal{Q} \rightarrow \mathcal{Q}} \]

\[ = \text{enc}(\hat{s}(C^{\text{pos}}))[i] = \text{enc}(\hat{s}(\text{dec}(V)))[i], \]

by the property that \(m_{\mathcal{P} \rightarrow \mathcal{P}} m_{\mathcal{P} \rightarrow \mathcal{P}} = 1_{\mathcal{P}}, \) the identity map on \(\mathcal{P},\) hence on its set of tiles, where \(P = \alpha_i(C^{\text{pos}})\) and \(Q = \alpha_i(\hat{s}(C^{\text{pos}})).\)
Since this holds for all \(1 \leq i \leq d\), we have
\[
\text{enc}(\tilde{s})(\text{enc}(C_{\text{pos}})) = \text{enc}(\tilde{s}(C_{\text{pos}})).
\]
It follows that
\[
(\text{enc}(\tilde{s}_2) \circ \text{enc}(\tilde{s}_1))(\text{enc}(C_{\text{pos}})) = \text{enc}(\tilde{s}_2)(\text{enc}(\tilde{s}_1)(\text{enc}(C_{\text{pos}})))
\]
\[
= \text{enc}(\tilde{s}_2)(\text{enc}(\tilde{s}_1(C_{\text{pos}})))
\]
\[
= \text{enc}(\tilde{s}_2(\tilde{s}_1(C_{\text{pos}})))
\]
\[
= \text{enc}(\tilde{s}_2 \circ \tilde{s}_1)(\text{enc}(C_{\text{pos}})).
\]
Thus, \(\text{enc}\) is clearly an (injective) semigroup homomorphism. Whence, \((\text{enc}(C_{\text{pos}}), \text{enc}(\tilde{S}))\) is isomorphic to \((C_{\text{pos}}, \tilde{S})\).

By Lemma 3.1 and Theorem 5.9, we have

**Corollary 5.10 (Holonomy Decomposition Theorem).** A finite transformation semigroup \((X, S)\) divides its holonomy wreath product
\[
(X, S) \mid H_1 \wreath \cdots \wreath H_d = \mathcal{H}(X, S),
\]
where \(d = h_S(X)\).

### 6. Computational Complexity

The holonomy decomposition algorithm given here enumerates the image set \(I_S(X)\) of the state set \(X\). The worst case is enumerating the powerset with \(2^{|X|}\) elements. It is easy to conclude that the algorithm given has time complexity at least exponential in the number of states (cf. Maler [18]). Moreover, by the Krohn-Rhodes prime decomposition theorem [15, 16], every simple group divisor of a finite semigroup must occur as a divisor of any cascade decomposition. Therefore it follows that a finite automata has no nontrivial subgroups (i.e., is aperiodic) if and only if all its holonomy groups are trivial. The results of Cho and Huynh [1] show that aperiodicity is \(PSPACE\)-complete, so it follows immediately that computing the holonomy decomposition is \(PSPACE\)-hard.

In practice we can calculate with huge semigroups (of size hundreds of thousands of elements). The size of the state set and the size of generator set or of the semigroup do not necessarily give a good guide to computational complexity in practice. It would be interesting to find the appropriate features and parameters and do parametrized complexity analysis for holonomy decompositions.

### 7. Computed Examples

**Example 7.1.** As a minimalistic but non-trivial example, let \((3, S)\) be the transformation semigroup generated by \(s_1 = [2, 1, 3]\) and \(s_2 = [1, 2, 2]\). From Figure 2 we can read off the maximal chains: \([\{1, 2, 3\}, \{1, 2\}, \{1\}],[\{1, 2, 3\}, \{1, 2\}, \{2\}],[\{1, 2, 3\}, \{2\}]\). Let’s see how from \(t = s_2s_1 = [211]\) we construct \(\tilde{t}\) acting on the chain representing state 1, i.e. doing the action on the members of the chain, removing duplicates then finding a dominating chain.

| Chain \(C\) | \(C \cdot t\) | \(C \cdot \tilde{t}\) |
|-----------------|-----------------|-----------------|
| \{1, 2, 3\}    | \{1, 2\}    | \{1, 2, 3\}    |
| \{1, 2\}       | \{1, 2\}       | \{1, 2\}       |
| \{1\}          | \{2\}          | \{2\}          |
Figure 2. The skeleton of a semigroup acting on 3 points (Example 7.1). The nodes are the elements of $T'_S(X)$. The boxes are the equivalence classes, the rectangular nodes the chosen representatives of a class. Shaded equivalence classes have nontrivial holonomy groups. The arrows point to the tiles of a representative set, the labels denote sequences of generators taking the set to its tile. Dotted arrows indicate tiles that are not images. On the side depth values are indicated.

In this very small example we have only a single dominating chain.

Example 7.2. Let $(6, S)$ be the transformation semigroup generated by transformations $s_1, \ldots, s_6$:

- $s_1 = [1, 2, 3, 1, 1]$ creates the image $\{1, 2, 3\}$,
- $s_2 = [4, 4, 5, 4, 6]$ is the transposition $(4, 5)$ and gives the image $\{4, 5, 6\}$,
- $s_3 = [4, 4, 5, 6, 4]$ is a cycle on $\{4, 5, 6\}$,
- $s_4 = [4, 4, 4, 5, 5]$ creates the image $\{4\}$,
- $s_5 = [4, 4, 1, 2, 3]$ maps $\{4, 5, 6\}$ to $\{1, 2, 3\}$, and $\{1, 2, 3\}$ to $\{4\}$,
- $s_6 = [2, 3, 1, 4, 4]$ is a cycle on $\{1, 2, 3\}$,

and its basic properties are $|S| = 138$, and $T'_S(X) = 19$. The ‘skeleton’ of its holonomy decomposition is depicted on Figure 3.

For $C = \{\{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 4\}, \{2, 4\}\}$ we have

| depth | $C^{pos}$ | $\text{enc}(C^{pos})$ | $\alpha(C^{pos})$ |
|-------|-----------|---------------------|------------------|
| 1     | $\{1, 2, 3\}$ | $\{1, 2, 3, 4\}$ | $\{1, 2, 3, 4, 5, 6\}$ |
| 2     | $\{2, 4\}$ | $\{2, 4\}$ | $\{1, 2, 3, 4\}$ |
| 3     | $\{2\}$ | $\{1\}$ | $\{2, 4\}$ |
| 4     | * | * | $\{2\}$ |
| 5     | * | * | $\{2\}$ |

demonstrating that an encoded positioned chain is not necessarily a chain.

Example 7.3. The full transformation semigroup $T_3$ has a canonical generating set consisting of two permutations (transposition and cycle) and an elementary collapsing. Figure 4 shows how these generators act on the set of maximal chains. The generators $[2, 1, 3]$ and $[2, 3, 1]$ are permutations of $\{1, 2, 3\}$ that map maximal chains to maximal chains. The lifts of these transformations to the chain semigroup are thus exactly as shown the Figure 4 and hence unique. The transformation $[1, 2, 1]$ gives subsets chains $\{\{1, 2\}, \{1\}\}$ and $\{\{1, 2\}, \{2\}\}$ that miss the full state set
Figure 3. The skeleton of a semigroup acting on 6 points (Example 7.2). Interesting feature is that \( d(\{1, 4\}) < d(\{1, 2, 3\}) \).

itself. Thus, \([1, 2, 1]\) does not map maximal chains to maximal chains, but it maps \(\{1, 2, 3\}\) to \(\{1, 2\}\). For any lift of \([1, 2, 1]\): All maximal chains trivially agree down to \(\{1, 2, 3\}\) so it must map to a chains agreeing down to \(\{1, 2\} = \{1, 2, 3\} \cdot [1, 2, 1]\), thus the lift the of \([1, 2, 1]\) maps each maximal chain \(C\) to \(\{1, 2\} \supset \{1, 2\} \supset \eta(C) \cdot [1, 2, 1]\), and so is uniquely determined.

However, having a unique dominating chain or unique lift is not a general property. Constant map \(c = [3, 3, 3]\) produces the chain \(\{3\}\) for which any maximal chain containing \(3\) is a dominating chain. Since any two maximal chains \(C_1\) and \(C_2\) both start with the top set \(X = \{1, 2, 3\}\), they agree at \(X\) and so, by consistency \(\hat{c}(C_1)\) and \(\hat{c}(C_2)\) must agree down to \(X \cdot c = \{3\}\). That is, \(\hat{c}(C_1) = \hat{c}(C_2)\), and \(\hat{c}\) is itself a constant map. Here there are two choices, \(\{1, 2, 3\} \supset \{1, 3\} \supset \{3\}\) or \(\{1, 2, 3\} \supset \{2, 3\} \supset \{3\}\), for the constant value of \(\hat{c}\).

The same argument applies to lifting any constant map in this holonomy method: the lift of a constant to the chain semigroup yields a (non-unique) constant.

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Figure 4. Action of the canonical generators (transposition, cycle, elementary collapsing) on the maximal chains.

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Appendix A. Notation

\( \subseteq S \)  
subduction relation

\( \triangleleft \)  
tile of relation

\( \alpha_i(C^{\text{pos}}) \)  
current state of approximation at depth \( i \) for a positioned chain

\( \mathbf{C}, \mathbf{D} \)  
chains

\( \mathbf{C}^{\text{pos}}, \mathbf{C}^{\text{pos}}[i] \)  
positioned chains, content of position \( i \)

\( \mathbf{C}_P, \mathbf{C}_x, \mathbf{C} \)  
maximal chains from \( P, \{x\} \) to \( X \), all maximal chains.

\( (\mathcal{C}, \hat{S}) \)  
chain semigroup

\( \mathcal{C}^{\text{pos}} \)  
all positioned tile chains

\( G_P \)  
the permutator (generalized Schützenberger) group of \( P \)

\( H_P \)  
holonomy permutation group of \( P \), \( (T(P), G_P) \) made faithful

\( \overline{H}_P \)  
holonomy permutation-reset transformation semigroup of \( P \)

\( h_S(P), d(P) \)  
height, depth of a set

\( m_{\mathbf{P} \rightarrow \mathbf{P'}, m_{\mathbf{P} \rightarrow \mathbf{P'}}} \)  
mapping from and to a representative

\( P, Q \in I_S(X), I_S'(X) \)  
images, image set, extended image set

\( P \equiv_S Q, \overline{P} \)  
equivalence, representative element

\( S_P \)  
setwise stabilizer semigroup of \( P \)

\( T(Q) \)  
the tiles of \( Q \)

\( V \)  
encoded coordinate values (tiles of representatives)

\( (X, S), (Y, T) \)  
transformation semigroups

\( x, y, z \in X \)  
states, stat set

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