A Simple Set of Separable States in a Commutative Simplex.

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In this note a very crude but simple approximation to the set of separable states in an arbitrary simplex of commutative states is given using the fact that on the lines connecting the maximally mixed state and an arbitrary pure state the positivity of the partial transpose is both necessary and sufficient condition for separability of states. The lower limit to the volume of separable states in a simplex is slightly improved.

0. The positive partial transpose (PPT)[1] is an extremely useful toll for detecting separability. In this note it will be used on the lines connecting pure states and the maximally mixed state where PPT is both sufficient and necessary condition for separability. Most of the procedures which will be used are slight modifications of some more general results and well known procedures. The result, a convex set of separable states is a slightly larger then the previous similar sets.

1. To start with, a commuting simplex over an orthogonal ray resolution of the identity in $C^n \otimes C^n$, is

$$S\{\{P_m\}\} = \text{conv}\{\{P_m\}_{m=1}^{n^2} \mid \sum_m \alpha_m P_m \mid \sum \alpha_m = 1, \alpha_m \geq 0\}$$

where

$$\sum_{m=1}^{n^2} P_m = I \quad , \quad P_m P_n = \delta_{mn} P_m \quad , \quad tr(P_m) = 1$$

With respect to separability it is a well known fact that the states on the lines containing the most mixed state and a pure state are easier to handle, and among the first examples inspected were mixtures

$$\rho(\alpha) = \alpha P + (1 - \alpha) \rho_o$$

where $\rho_o = I/n^2$ and $P$ projects onto the vector

$$|\Psi\rangle = N^{-1/2} \sum |i\rangle \otimes |i\rangle.$$ 

These states are separable for $\alpha \leq 1/(n + 1)$.

A generalization to an arbitrary pure state is given in [2]. A short description is the following one. Assume that the Schmidt form of the vector onto which the projector $P = |\Psi\rangle\langle \Psi|$ is projecting is $|\Psi\rangle = \Sigma \lambda_k |k\rangle \otimes |k\rangle$. Then, in the basis $\{\langle i| \otimes |j\rangle\}$ assuming $\sum \lambda_k^2 = 1$ the state

$$\rho(\alpha) = \alpha P + (1 - \alpha) \rho_o$$

has a non-negative PT when $(1 - \alpha)/n^2 \geq \alpha \lambda_k \lambda_r$ for all $k, r$. Denoting $\max(\lambda_k \lambda_r) = M$ we obtain the value of $\alpha_M = 1/(1 + n^2 M)$.

An explicit separable construction (cf [2]) may be the following one: e.g. take an (unnormalized) separable vector $|\psi\rangle \otimes |\phi\rangle$ both factors with components $(\sqrt{\lambda_1}, \sqrt{\lambda_2}... \sqrt{\lambda_n})$ in the Schmidt basis of the
entangled state. The projector on this state has matrix elements $P_{ik,mn} \sim \sqrt{\lambda_i \lambda_k \lambda_m \lambda_n}$. One can "twirl" [3] this state and obtain the mixture

$$\rho_p = \frac{1}{N} \sum_{k=1}^{N} ((U_1)^k \otimes (U_2)^k) P((U_1^\dagger)^k \otimes (U_2^\dagger)^k)$$

where $U_1 = U_2^*$ are diagonal unitary operators, complex conjugates of each other. The eigen-values of e.g. $U_1$ should look something like

$$(1, e^{i\phi}, e^{i(k+d)\phi}, e^{i(k+3d)\phi}, e^{i(k+7d)\phi}, ...)$$

where $\phi = 2\pi/N$ and $N$ is a large enough prime. This ensures that all differences in phases are different and the resulting state has the non-zero off-diagonal elements proportional to off-diagonal elements of the entangled state $P$. Then

$$\rho(\alpha_M) = \frac{(\sum \lambda_k)^2}{1 + n^2 M} \rho_p + \sum_{k \neq r} \frac{(M - (1 - \delta_{kr}) \lambda_k \lambda_r)}{(1 + n^2 M)} |k\rangle \langle k| \otimes |r\rangle \langle r| =$$

$$= \frac{1}{1 + n^2 M} P + \frac{n^2 M}{1 + n^2 M} \rho_o$$

is the first PPT state on the segment $\{P, \rho_o\}$ expressed as a separable state. One should also notice that $\rho_p$ in general, need not belong to the initial simplex and the addition of diagonal elements brings $\rho(\alpha_M)$ back into the initial simplex. On the other hand $\rho_p$ itself gives a decomposition of an entangled state into a sum of separable projectors.

Extending the segment $\{P, \rho_o\}$ beyond $\rho_o$ one gets the last point which is still a state, $\rho_{re} = (I - P)/(n^2 - 1)$, the baricenter of the face complementary to $P$, which is also separable. This state has PPT [4] and an explicit construction of its separability may be the following one:

first note, (cf. [5]), that the matrix (note the reversed order of $r$ and $k$)

2
\[
A_{kr} = \frac{1}{(\lambda_k + \lambda_r)^2} \left( \lambda_k^2 |kk\rangle\langle kk| - \lambda_k \lambda_r (|kr\rangle\langle rk| + |rk\rangle\langle kr|) + \right.
\]
\[
+ \lambda_k \lambda_r (|kr\rangle\langle rk| + |rk\rangle\langle kr|) + \lambda_r^2 |rr\rangle\langle rr| \right)
\]
is a separable state in the subspace defined by
\[
(|k\rangle\langle k| + |r\rangle\langle r|) \otimes (|k\rangle\langle k| + |r\rangle\langle r|).
\]
Now, the desired state is
\[
\rho_{pc} = \sum_k (\lambda_k + \lambda_r)^2 A_{kr} + \sum_{k \neq r} \frac{1 - (\lambda_k \lambda_r)}{n^2 - 1} |kr\rangle\langle kr|.
\]
Again, the first term may be outside of the simplex.

Therefore, in an arbitrary commuting simplex (eq. (1)), the convex hull
\[
\text{conv}\{ \{(\alpha_m P_m + (1 - \alpha_m)\rho_o, (I - P_m)/(n^2 - 1))\}_{m=1} \}
\]
is a crude but simple approximation to the set of separable states.

In both constructions one has to "pad" the matrix with diagonal elements, so before the "padding", the states are "farther" away from \(\rho_o\) and this simple approximation could be improved if some of these states belong to the initial simplex. The next improvement may come from the PT of the said simplex, again assuming the PT states belong to the original simplex.

3. One can also obtain a slight improvement to the lower limit of the volume of separable states in an arbitrary simplex. The set (2) is made out of one central simplex, centered at \(\rho_o\) and \(n^2\) smaller simplices, each having an \((n^2 - 1)\) order face of the central simplex as a base and the baricenter of the \((n^2 - 1)\) base as a vertex. To keep things simple one can make the central simplex a regular one by assuming that the initial simplex is scaled by factor \(\alpha_{\text{min}}\). The volume of this contraption gives a crude lower limit to the volume of separable states in a simplex as
\[
V_{\text{sep}} > \frac{\alpha_{\text{min}}^{n^2 - 2}}{(n^2 - 1)!} \times \left( \alpha_{\text{min}} \sqrt{\frac{n^2}{n^2 - 1}} + n^2 \frac{(1 - \alpha_{\text{min}})}{n^2(1 - 1)} \right)
\]
where \(\alpha_{\text{min}}\) is the minimal \(\alpha_k\) from eq. (2), first factor is the base of the central simplex divided by the dimension, the first term in the brackets is the height of the central simplex and the second, the sum of heights of \(n^2\) outer simplices.

To conclude, the procedures used to obtain set (2) can be applied to an arbitrary nonorthogonal resolution of the identity, or, as a matter of fact, to an arbitrary set of pure states.
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