THE FUNCTION \((b^x - a^x)/x\): LOGARITHMIC CONVEXITY
AND APPLICATIONS TO EXTENDED MEAN VALUES

FENG QI AND BAI-NI GUO

Abstract. In the present paper, we first prove the logarithmic convexity of
the elementary function \(b^x - a^x/x\), where \(x \neq 0\) and \(b > a > 0\). Basing on this,
we then provide a simple proof for Schur-convex properties of the extended
mean values, and, finally, discover some convexity related to the extended
mean values.

1. Introduction

For given numbers \(b > a > 0\), let
\[
g_{a,b}(t) = \begin{cases} 
\frac{b^t - a^t}{t}, & t \neq 0; \\
 b-a, & t = 0.
\end{cases}
\]

This elementary and special function was first dedicated to be investigated in [44, 45]. Subsequently, it was utilized to construct Steffensen pairs in [6, 26, 27, 28, 31] and its reciprocal was also used to generalize Bernoulli numbers and polynomials in [9, 14, 15, 16, 30]. It has something to do with the classical Euler gamma function \(\Gamma\) and the remainder of Binet’s first formula for the logarithm of \(\Gamma\) (see, for example, [7, 10, 13, 32, 33, 35, 41, 42, 52] and closely-related references therein).

More importantly, it was employed not only to provide alternative proofs for the
monotonicity of the extended mean values \(E(r, s; x, y)\) in [40, 46] but also to create
the logarithmic convexity and Schur-convex properties of \(E(r, s; x, y)\) in [3, 4, 19, 22, 21, 23, 25], where the extended mean values \(E(r, s; x, y)\) were defined in [12, 51] for \(x, y > 0\) and \(r, s \in \mathbb{R}\) as
\[
E(r, s; x, y) = \left(\frac{r}{s} \cdot \frac{y^r - x^s}{y^r - x^s}\right)^{1/(s-r)},
\]
\[
E(r, 0; x, y) = \left(\frac{1}{r} \cdot \frac{y^r - x^r}{\ln y - \ln x}\right)^{1/r},
\]
\[
E(r, r; x, y) = \left(\frac{x^r - y^r}{y^r - x^r}\right)^{1/(x^r - y^r)},
\]
\[
E(0, 0; x, y) = \sqrt{xy},
\]
\[
E(r, s; x, x) = x,
\]

There has been a lot of literature on the extended mean values \(E(r, s; x, y)\). For
more information, please refer to [2, 24] and related references therein.

In this paper, we first present the logarithmic convexity of the function \(g_{a,b}(t)\).
Basing on this, we then provide a concise proof for Schur-convex properties of the extended mean values \(E(r, s; x, y)\), and, finally, discover some monotonicity and

2000 Mathematics Subject Classification. Primary 26A51, 33B10; Secondary 26D07.

Key words and phrases. Schur-convex property, extended mean values, Lazarević’s inequality.
The first author was supported partially by the China Scholarship Council.
This paper was typeset using \(\text{AMS-LaTeX} \).
2. LOGARITHMIC CONVEXITY OF $g_{a,b}(t)$

For the sake of proceeding smoothly, we need the following definition which can be found in [38, 39] and related references therein.

**Definition 1.** A $k$-times differentiable function $f(t) > 0$ is said to be $k$-log-convex on an interval $I$ if

$$[\ln f(t)]^{(k)} \geq 0, \quad k \in \mathbb{N}$$

on $I$: If the inequality (2) reverses then $f$ is said to be $k$-log-concave on $I$.

Now we are in a position to state and prove the logarithmic convexity of the function $g_{a,b}(t)$ on $(-\infty, \infty)$.

**Theorem 1.** Let $b > a > 0$. Then the function $g_{a,b}(t)$ is logarithmic convex on $(-\infty, \infty)$, 3-log-convex on $(-\infty, 0)$, and 3-log-concave on $(0, \infty)$. Consequently, the function

$$h_{a,b}(t) = \begin{cases} \frac{b't \ln b - a't \ln a - 1}{b' - a'} - \frac{1}{t}, & t \neq 0 \\ \ln \sqrt{ab}, & t = 0 \end{cases}$$

is increasing on $(-\infty, \infty)$ and satisfies

$$\lim_{t \to -\infty} h_{a,b}(t) = \ln a \quad \text{and} \quad \lim_{t \to \infty} h_{a,b}(t) = \ln b.$$  \hfill (4)

**Proof.** For $t \neq 0$, taking the logarithm of $g_{a,b}(t)$ and differentiating yields

$$\ln g_{a,b}(t) = \ln |t|,$$

$$[\ln g_{a,b}(t)]' = \frac{b't \ln b - a't \ln a - 1}{b' - a'} - \frac{1}{t},$$

$$[\ln g_{a,b}(t)]'' = \frac{1}{t^2} - \frac{a'b'(\ln a - \ln b)^2}{(a' - b')^2}$$

and

$$[\ln g_{a,b}(t)]''' = \frac{a'b'(a' + b')(\ln a - \ln b)^3}{(a' - b')^3} - \frac{2}{t^3}$$

$$\quad = \frac{2(\ln a - \ln b)^3}{(a' - b')^3} \left\{ \frac{(a/b)^{t/2} + (b/a)^{t/2}}{2} - \frac{(a/b)^{t/2} - (b/a)^{t/2}}{(\ln a - \ln b)t} \right\}$$

$$\triangleq \frac{2(\ln a - \ln b)^3}{(a' - b')^3} Q_{a,b}(t),$$

where, by using Lazarević’s inequality in [1, p. 131] and [11, p. 300],

$$Q_{a,b} \left( \frac{2t}{\ln a - \ln b} \right) = \frac{e^{-t} + e^t}{2} - \left( \frac{e^{t} - e^{-t}}{2t} \right)^3 = \cosh t - \left( \frac{\sinh t}{t} \right)^3 < 0.$$  \hfill (3)

Consequently,

$$[\ln g_{a,b}(t)]''' \begin{cases} > 0, & t \in (-\infty, 0) \\ < 0, & t \in (0, \infty) \end{cases}$$

which implies that the function $[\ln g_{a,b}(t)]''$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Since

$$\lim_{t \to -\infty} \frac{a'b'}{(a' - b')^2} = \lim_{t \to -\infty} \frac{(a/b)^t}{[(a/b)^t - 1]^2} = \lim_{t \to -\infty} \frac{(b/a)^t}{[(b/a)^t - 1]^2} = 0.$$
and the function $[\ln g_{a,b}(t)]''$ is even on $\mathbb{R}$, then $[\ln g_{a,b}(t)]'' > 0$, and so the function $[\ln g_{a,b}(t)]' = h_{a,b}(t)$ is increasing on $\mathbb{R}$. Since 
\[
\frac{b^t \ln b - a^t \ln a}{b^t - a^t} = \frac{(b/a)^t \ln b - \ln a}{(b/a)^t - 1} = \frac{\ln b - (a/b)^t \ln a}{1 - (a/b)^t},
\]
then it follows easily that 
\[
\lim_{t \to -\infty} [\ln g_{a,b}(t)]' = \ln a \quad \text{and} \quad \lim_{t \to \infty} [\ln g_{a,b}(t)]' = \ln b.
\]

The L'Hôpital's rule reveals that 
\[
\lim_{t \to 0} [\ln g_{a,b}(t)]' = \lim_{t \to 0} \frac{t(b^t \ln b - a^t \ln a) - (b^t - a^t)}{t(b^t - a^t)} = \lim_{t \to 0} \frac{y^t(\ln b)^2 - x^t(\ln a)^2}{(b^t - a^t)/t + (b^t \ln b - a^t \ln a)} = \frac{\ln b + \ln a}{2}.
\]

The proof of Theorem 1 is thus completed. \qed

**Remark 1.** In the preprint [34], Theorem 1 was also verified by using the celebrated Hermite-Hadamard’s integral inequality [43, 47, 48, 49] instead of Lazarević’s inequality.

**Remark 2.** Theorem 1 provides important supplements to the work in [44, 45].

## 3. A Concise Proof of Schur-Convexity of $E(r, s; x, y)$

Let us recall [17, pp. 75–76] the definition of Schur-convex functions.

**Definition 2.** A function $f$ with $n$ arguments defined on $I^n$ is called Schur-convex if $f(x) \leq f(y)$ holds for each two $n$-tuples $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ on $I^n$ such that $x \prec y$, where $I$ is an interval with nonempty interior and the relationship of majorization $x \prec y$ means that

\[
\sum_{i=1}^k x[i] \leq \sum_{i=1}^k y[i] \quad \text{and} \quad \sum_{i=1}^n x[i] = \sum_{i=1}^n y[i] \quad \text{for} \quad 1 \leq k \leq n-1,
\]

where $x[i]$ denotes the $i$-th largest component in $x$.

A function $f$ is Schur-concave if and only if $-f$ is Schur-convex.

Based on intricate conclusions in [21, 22] and basic properties of $E(r, s; x, y)$, the following Schur-convex properties of the extended mean values $E(r, s; x, y)$ with respect to $(r, s)$ was first obtained in [19, 23].

**Theorem 2.** With respect to the $2$-tuple $(r, s)$, the extended mean values $E(r, s; x, y)$ are Schur-concave on $[0, \infty) \times [0, \infty)$ and Schur-convex on $(-\infty, 0) \times (-\infty, 0]$.

The aim of this section is to demonstrate a concise proof of Theorem 2 with the help of Theorem 1.

**Proof.** When $y > x > 0$, the extended mean values $E(r, s; x, y)$ may be represented in terms of $g_{x,y}(t)$ as

\[
E(r, s; x, y) = \begin{cases} 
\left[ \frac{g_{x,y}(s)}{g_{x,y}(r)} \right]^{1/(s-r)}, & (r - s)(x - y) \neq 0; \\
\exp \left[ \frac{g'_{x,y}(r)}{g_{x,y}(r)} \right], & r = s, \quad x - y \neq 0
\end{cases}
\]
and
\[
\ln E(r, s; x, y) = \begin{cases} 
\frac{1}{s - r} \int_r^x \frac{g'_x(t)}{g_{x,y}(t)} \, dt, & (r - s)(x - y) \neq 0; \\
\frac{g'_x(r)}{g_{x,y}(r)}, & r = s, x - y \neq 0.
\end{cases}
\] (6)

In virtue of Theorem 1, it follows that
\[
[\ln g_{x,y}(t)]^{(3)} = \begin{cases} 
\frac{g'_x(t)}{g_{x,y}(t)} \left( t - \frac{x}{s} \right), & t \in (0, \infty), \\
\frac{g'_x(t)}{g_{x,y}(t)} \left( t - \frac{x}{s} \right), & t \in (-\infty, 0).
\end{cases}
\] (7)

In [5], it was obtained that the integral arithmetic mean
\[
\phi(r, s) = \begin{cases} 
\frac{1}{s - r} \int_r^s f(t) \, dt, & r \neq s \\
f(r), & r = s
\end{cases}
\] (8)
of a continuous function \( f \) on \( I \) is Schur-convex (or Schur-concave, respectively) on \( I^2 \) if and only if \( f \) is convex (or concave, respectively) on \( I \). Consequently, by virtue of the formula (6) and Definition 2, it is not difficult to see that, in order that the extended mean values \( E(r, s; x, y) \) are Schur-convex (or Schur-concave, respectively) with respect to \( (r, s) \), it is sufficient to show the validity of (7), which may be deduced from Theorem 1 straightforwardly. Theorem 2 is thus proved. \( \square \)

**Remark 3.** In [50], an alternative proof of Theorem 2 was given, among other things.

### 4. Some logarithmic convexity related to \( E(r, s; x, y) \)

In Remark 6 of [36, 37], it was pointed out that the reciprocal of the exponential mean
\[
I_{s,t}(x) = \frac{1}{s - t} \int_t^x e^{(x + s)(t + s)}^{1/(s-t)}
\] (9)
for \( s \neq t \) is logarithmically completely monotonic on \((\min\{s, t\}, \infty)\) and that the exponential mean \( I_{s,t}(x) \) for \( s \neq t \) is also a completely monotonic function of first order on \((\min\{s, t\}, \infty)\).

In [18], it was remarked that the logarithmic mean
\[
L_{s,t}(x) = L(x + s, x + t)
\] (10)
is increasing and concave on \((\min\{s, t\}, \infty)\) for \( s \neq t \). In [20, 29], the logarithmic mean \( L_{s,t}(x) \) for \( s \neq t \) is further proved to be a completely monotonic function of first order on \((\min\{s, t\}, \infty)\).

For \( x, y > 0 \) and \( r, s \in \mathbb{R} \), let
\[
F_{r,s,x,y}(w) = E(r + w, s + w; x, y), \quad w \in \mathbb{R},
\] (11)
\[
G_{r,s,x,y}(w) = E(r, s; x + w, y + w), \quad w > -\min\{x, y\}
\] (12)
and
\[
H_{r,s,x,y}(w) = E(r + w, s + w; x + w, y + w), \quad w > -\min\{x, y\}.
\] (13)
By virtue of the monotonicity of the extended mean values \( E(r, s; x, y) \), it is easy to see that the functions \( F_{r,s,x,y}(w) \), \( G_{r,s,x,y}(w) \) and \( H_{r,s,x,y}(w) \) are increasing with respect to \( w \). Furthermore, since
\[
I_{s,t}(x) = E(1, 1; x + s, y + t) = G_{1,1,x,y}(w)
\]
and
\[
L_{s,t}(x) = E(0, 1; x + s, y + t) = G_{0,1,x,y}(w),
\]
the following problem was posed in [20, 29]: What about the logarithmic convexity of the functions \( F_{r,s,x,y}(w) \), \( G_{r,s,x,y}(w) \) and \( H_{r,s,x,y}(w) \) with respect to \( w \)?
The aim of this section is to supply a solution to the above problem about the function $F_{r,s;x,y}(w)$. Our main results are the following theorems.

**Theorem 3.** The function $F_{r,s;x,y}(w)$ is logarithmically convex on $(-\infty, -\frac{4}{3})$ and logarithmically concave on $(-\frac{4}{3}, \infty)$.

**Theorem 4.** The product $F_{r,s;x,y}(w) = F_{r,s;x,y}(w)F_{r,s;x,y}(-w)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

**Theorem 5.** If $s + r > 0$, the function $w\ln F_{r,s;x,y}(w)$ is convex on $(-\frac{4}{3}, 0)$; If $s + r < 0$, it is also convex on $(0, -\frac{4}{3})$.

**Proof of Theorem 3.** In the first place, we claim that if $f(t)$ is even on $(-\infty, \infty)$ and increasing on $(-\infty, 0)$, then the function

$$p(t) = f(t + \alpha) - f(t), \quad \alpha > 0$$

is positive on $(-\infty, -\frac{\alpha}{2})$ and negative on $(-\frac{\alpha}{2}, \infty)$. This can be verified as follows:

1. If $t + \alpha > t > 0$, since $f(t)$ is decreasing on $(0, \infty)$, then $F(t) < 0$;
2. If $t < t + \alpha < 0$, since $f(t)$ is increasing on $(-\infty, 0)$, then $F(t) > 0$;
3. If $t + \alpha > t > 0$,
   a. when $t + \alpha > -t > 0$, i.e., $t > -\frac{\alpha}{2}$, using the even and monotonic properties of $f(t)$ shows that $F(t) = f(t + \alpha) - f(-t)$ and it is negative;
   b. similarly, when $-t > t + \alpha > 0$, i.e., $t < -\frac{\alpha}{2}$, the function $F(t)$ is positive.

The claim is thus proved.

From (6), it follows that if $y > x > 0$ then

$$\frac{d^2 \ln F_{r,s;x,y}(w)}{dw^2} = \left\{ \begin{array}{ll} \frac{1}{s - r} \int_r^s \frac{d^2}{dw^2} \left[ \frac{g'_{x,y}(w + t)}{g_{x,y}(w + t)} \right] dt, & (r - s)(x - y) \neq 0; \\
\frac{d^2}{dw^2} \left[ \frac{g'_{x,y}(w + r)}{g_{x,y}(w + r)} \right], & r = s, x - y \neq 0. \end{array} \right.$$ \hspace{1cm} (15)

As shown in the proof of Theorem 1, the function $[\ln g_{x,y}(t)]''$ for $y > x > 0$ is even on $\mathbb{R}$ and increasing on $(-\infty, 0)$. Substituting $f(t)$ and $\alpha$ by $[\ln g_{x,y}(t)]''$ and $s - r > 0$ in (14) respectively and utilizing (15) demonstrates that

$$\frac{[\ln g_{x,y}(t + s - r)]'' - [\ln g_{x,y}(t)]''}{s - r} = \frac{d^2 \ln F_{r,s;x,y}(w)}{dw^2} > 0$$

for $t < -\frac{4}{3}$ and that $\frac{d^2 \ln F_{r,s;x,y}(w)}{dw^2} < 0$ for $t > -\frac{4}{3}$. As a result,

$$\frac{d^2 \ln F_{r,s;x,y}(w)}{dw^2} = \frac{[\ln g_{x,y}(w + s)]'' - [\ln g_{x,y}(w + r)]''}{s - r} \left\{ \begin{array}{ll} > 0, & w < -\frac{s + r}{2}; \\
< 0, & w > -\frac{s + r}{2}. \end{array} \right.$$ \hspace{1cm} (16)

Because $F_{r,s;x,y}(w) = F_{s,r;x,y}(w) = F_{s,r;x,y}(w)$, the equation (16) holds for all $r, s \in \mathbb{N}$ and $x, y > 0$ with $x \neq y$. Theorem 3 is proved. \hfill \Box

**Proof of Theorem 4.** It is easy to see that

$$[\ln F_{r,s;x,y}(w)]' = \frac{F'_{r,s;x,y}(w)}{F_{r,s;x,y}(w)} - \frac{F'_{r,s;x,y}(-w)}{F_{r,s;x,y}(-w)}. $$

Careful computation reveals that

$$\frac{F'_{r,s;x,y}(w)}{F_{r,s;x,y}(w)} = \frac{F'_{r,s;x,y}(-w - (s + r))}{F_{r,s;x,y}(-w - (s + r))}$$
for \( w \in (-\infty, \infty) \). Theorem 3 implies that the function
\[
q(w) = \frac{F'_{r,s;x,y}(w - (s + r)/2)}{F_{r,s;x,y}(w - (s + r)/2)}
\]
is increasing on \((-\infty, 0)\) and decreasing on \((0, \infty)\). It is also apparent that the function \( q(w) \) is even, that is, \( q(w) = q(-w) \) for \( w \in (-\infty, \infty) \). By virtue of the claim verified in the proof of Theorem 3, it is easy to see that the difference \( q(w + (s + r)) - q(w) \) is positive on \((-\infty, -\frac{s + r}{2})\) and negative on \((-\frac{s + r}{2}, \infty)\), equivalently, the function
\[
q\left(w + \frac{s + r}{2}\right) - q\left(w - \frac{s + r}{2}\right) = \frac{F'_{r,s;x,y}(w)}{F_{r,s;x,y}(w)} - \frac{F'_{r,s;x,y}(w - (s + r))}{F_{r,s;x,y}(w - (s + r))}
\]
is positive on \((-\infty, 0)\) and negative on \((0, \infty)\). On the other hand, since
\[
F_{r,s;x,y}(w) = \frac{xyF'_{r,s;x,y}(w)}{F_{r,s;x,y}(w - (s + r))},
\]
then the function (17) equals \([\ln F_{r,s;x,y}(w)]''\). Thus, Theorem 4 is proved.

**Proof of Theorem 5.** Direct calculation yields
\[
[w \ln F_{r,s;x,y}(w)]'' = 2[\ln F_{r,s;x,y}(w)]' + w[\ln F_{r,s;x,y}(w)]''.
\]
By Theorem 3, it follows that \([\ln F_{r,s;x,y}(w)]' > 0\) on \((-\infty, \infty)\), \([\ln F_{r,s;x,y}(w)]'' > 0\) on \((-\infty, -\frac{s + r}{2})\), and \([\ln F_{r,s;x,y}(w)]'' < 0\) on \((-\frac{s + r}{2}, \infty)\). Therefore,

1. if \( s + r < 0 \), then \([w \ln F_{r,s;x,y}(w)]'' > 0\), and so \(w \ln F_{r,s;x,y}(w)\) is convex on \((0, -\frac{s + r}{2})\);
2. if \( s + r > 0 \), then \([w \ln F_{r,s;x,y}(w)]'' > 0\), and so \(w \ln F_{r,s;x,y}(w)\) is convex on \((-\frac{s + r}{2}, 0)\).

The proof of Theorem 5 is complete.

**Remark 4.** Theorem 3 generalizes [4, Theorem 1 and Theorem 3] and [25, Theorem 1]. Theorem 4 generalizes [4, Theorem 2 and Theorem 3] and [25, Theorem 2]. Theorem 5 generalizes [4, Theorem 5]. Notice that the paper [3] is a preprint and a complete version of [4].

**Remark 5.** By the same method as in [4, Theorem 4], the function
\[
(w + s - r)[F_{r,s;x,y}(w)]^{s-r}, \quad s > r
\]
can be proved to be increasingly convex on \((-\infty, \infty)\) and logarithmically concave on \((-\frac{s + r}{2}, \infty)\).

**References**

[1] P. S. Bullen, *A Dictionary of Inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathematics 97, Addison Wesley Longman Limited, 1998.

[2] P. S. Bullen, *Handbook of Means and Their Inequalities*, Mathematics and its Applications, Volume 560, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.

[3] W.-S. Cheung and F. Qi, *Logarithmic convexity of the one-parameter mean values*, RGMIA Res. Rep. Coll. 7 (2004), no. 2, Art. 15, 331–342; Available online at http://www.staff.vu.edu.au/rgmia/v7n2.asp.

[4] W.-S. Cheung and F. Qi, *Logarithmic convexity of the one-parameter mean values*, Taiwanese J. Math. 11 (2007), no. 1, 231–237.

[5] N. Elezović and J. Pečarić, *A note on Schur-convex functions*, Rocky Mountain J. Math. 30 (2000), no. 3, 853–856.

[6] H. Gauchman, *Steffensen pairs and associated inequalities*, J. Inequal. Appl. 5 (2000), no. 1, 53–61.
Complete monotonicity of the logarithmic mean

F. Qi and Sh.-X. Chen, Generalisation of Bernoulli polynomials, Internat. J. Math. Ed. Sci. Tech. 33 (2002), no. 3, 428–431.

A class of completely monotonic functions related to the remainder of Binet’s formula with applications, Tamkang J. Math. Sci. 25 (2009), no. 1, 9–14.

J.-Ch. Kuang, Chángyóng Budúa (Applied Inequalities), 3rd ed., Shandong Science and Technology Press, Jinan City, Shandong Province, China, 2004. (Chinese)

E. B. Leach and M. C. Sholander, Extended mean values, Amer. Math. Monthly 85 (1978), no. 2, 84–90.

A.-Q. Liu, G.-F. Li, B.-N. Guo and F. Qi, Monotonicity and logarithmic concavity of two functions involving exponential function, Internat. J. Math. Ed. Sci. Tech. 39 (2008), no. 5, 686–691.

Q.-M. Luo, B.-N. Guo and F. Qi, Generalizations of Bernoulli’s numbers and polynomials, RGMIA Res. Rep. Coll. 5 (2002), no. 2, Art. 12, 353–359; Available online at http://www.staff. vu.edu.au/rgmia/v5n2.asp.

Q.-M. Luo, B.-N. Guo, F. Qi, and L. Debnath, Generalizations of Bernoulli numbers and polynomials, Internat. J. Math. Math. Sci. 2003 (2003), no. 59, 3769–3776.

Q.-M. Luo and F. Qi, Relationships between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials, Adv. Stud. Contemp. Math. (Kyungshang) (2003), no. 1, 11–18.

J. Pečarić, F. Proschan, and Y. L. Tong, Convex Functions, Partial Orderings, and Statistical Applications, Mathematics in Science and Engineering 187, Academic Press, 1992.

F. Qi, A new lower bound in the second Kershaw’s double inequality, J. Comput. Appl. Math. 214 (2008), no. 2, 610–616; Available online at http://dx.doi.org/10.1016/j.cam.2007.03.016.

F. Qi, A note on Schur-convexity of extended mean values, Rocky Mountain J. Math. 35 (2005), no. 5, 1787–1793.

F. Qi, Complete monotonicity of logarithmic mean, RGMIA Res. Rep. Coll. 10 (2007), no. 1, Art. 18; Available online at http://www.staff. vu.edu.au/rgmia/v10n1.asp.

F. Qi, Logarithmic convexity of extended mean values, Proc. Amer. Math. Soc. 130 (2002), no. 6, 1787–1796.

F. Qi, Logarithmic convexities of the extended mean values, RGMIA Res. Rep. Coll. 2 (1999), no. 5, Art. 5, 643–652; Available online at http://www.staff. vu.edu.au/rgmia/v2n5.asp.

F. Qi, Schur-convexity of the extended mean values, RGMIA Res. Rep. Coll. 4 (2001), no. 4, Art. 4, 529–533; Available online at http://www.staff. vu.edu.au/rgmia/v4n4.asp.

F. Qi, The extended mean values: Definition, properties, monotonicities, comparison, convexities, generalizations, and applications, Cubo Mat. Educ. 5 (2003), no. 3, 63–90.

F. Qi, P. Cerone, S. S. Dragomir and H. M. Srivastava, Alternative proofs for monotonic and logarithmically convex properties of one-parameter mean values, Appl. Math. Comput. 208 (2009), no. 1, 129–133; Available online at http://dx.doi.org/10.1016/j.amc.2008.11.023.

F. Qi and J.-X. Cheng, New Steffensen pairs, RGMIA Res. Rep. Coll. 3 (2000), no. 3, Art. 11, 431–436; Available online at http://www.staff. vu.edu.au/rgmia/v3n3.asp.

F. Qi and J.-X. Cheng, Some new Steffensen pairs, Anal. Math. 29 (2003), no. 3, 219–226.

F. Qi, J.-X. Cheng and G. Wang, New Steffensen pairs, Inequality Theory and Applications, Volume 1, Ed. Yeol Je Cho et al., 273–279, Nova Science Publishers, Huntington, NY, 2001.

F. Qi and Sh.-X. Chen, Complete monotonicity of the logarithmic mean, Math. Inequal. Appl. 10 (2007), no. 4, 799–804.

F. Qi and B.-N. Guo, Generalisation of Bernoulli polynomials, RGMIA Res. Rep. Coll. 4 (2001), no. 4, Art. 10, 691–695; Available online at http://www.staff. vu.edu.au/rgmia/v4n4.asp.

F. Qi and B.-N. Guo, On Steffensen pairs, J. Math. Anal. Appl. 271 (2002), no. 2, 534–541.

F. Qi and B.-N. Guo, Some properties of extended remainder of Binet’s first formula for logarithm of gamma function, Available online at http://arxiv.org/abs/0904.1118.

F. Qi and B.-N. Guo, Some properties of extended remainder of Binet’s first formula for logarithm of gamma function, Math. Slovaca (2010), in press.

F. Qi and B.-N. Guo, The function $(b^x - a^x)/x$: Logarithmic convexity, RGMIA Res. Rep. Coll. 11 (2008), no. 1, Art. 5; Available online at http://www.staff. vu.edu.au/rgmia/v11n1.asp.
[35] F. Qi, B.-N. Guo and Ch.-P. Chen, The best bounds in Gautschi-Kershaw inequalities, Math. Inequal. Appl. 9 (2006), no. 3, 427–436.

[36] F. Qi and S. Guo, New upper bounds in the second Kershaw’s double inequality and its generalizations, RGMIA Res. Rep. Coll. 10 (2007), no. 2, Art. 1; Available online at http://www.staff.vu.edu.au/rgmia/v10n2.asp.

[37] F. Qi, S. Guo and Sh.-X. Chen, A new upper bound in the second Kershaw’s double inequality and its generalizations, J. Comput. Appl. Math. 220 (2008), no. 1–2, 111–118; Available online at http://dx.doi.org/10.1016/j.cam.2007.07.037.

[38] F. Qi, S. Guo and B.-N. Guo, A class of k-log-convex functions and their applications to some special functions, RGMIA Res. Rep. Coll. 10 (2007), no. 2, Art. 1; Available online at http://www.staff.vu.edu.au/rgmia/v10n2.asp.

[39] F. Qi, S. Guo and B.-N. Guo, A class of k-log-convex functions and their applications to some special functions, Integral Transforms Spec. Funct. 19 (2008), no. 3, 195–200.

[40] F. Qi and Q.-M. Luo, A simple proof of monotonicity for extended mean values, J. Math. Anal. Appl. 224 (1998), no. 2, 356–359.

[41] F. Qi, D.-W. Niu and B.-N. Guo, Monotonic properties of differences for remainders of psi function, Internat. J. Pure Appl. Math. Sci. 4 (2007), no. 1, 3–5.

[42] F. Qi, D.-W. Niu and B.-N. Guo, Monotonic properties of differences for remainders of psi function, RGMIA Res. Rep. Coll. 8 (2005), no. 4, Art. 16, 683–690; Available online at http://www.staff.vu.edu.au/rgmia/v8n4.asp.

[43] F. Qi, Z.-L. Wei and Q. Yang, Generalizations and refinements of Hermite-Hadamard’s inequality, Rocky Mountain J. Math. 35 (2005), no. 1, 235–251.

[44] F. Qi and S.-L. Xu, Refinements and extensions of an inequality, II, J. Math. Anal. Appl. 211 (1997), no. 2, 616–620.

[45] F. Qi and S.-L. Xu, The function $(b^x - a^x)/x$: Inequalities and properties, Proc. Amer. Math. Soc. 126 (1998), no. 11, 3355–3359.

[46] F. Qi, S.-L. Xu and L. Debnath, A new proof of monotonicity for extended mean values, Internat. J. Math. Math. Sci. 22 (1999), no. 2, 415–420.

[47] F. Qi and M.-L. Yang, Comparisons of two integral inequalities with Hermite-Hadamard-Jensen’s integral inequality, Internat. J. Math. Math. Sci. 3 (2006), no. 1, 83–88.

[48] F. Qi and M.-L. Yang, Comparisons of two integral inequalities with Hermite-Hadamard-Jensen’s integral inequality, Octagon Math. Mag. 14 (2006), no. 1, 53–58.

[49] F. Qi and M.-L. Yang, Comparisons of two integral inequalities with Hermite-Hadamard-Jensen’s integral inequality, RGMIA Res. Rep. Coll. 8 (2005), no. 3, Art. 18, 535–540; Available online at http://www.staff.vu.edu.au/rgmia/v8n3.asp.

[50] J. Sándor, The Schur-convexity of Stolarsky and Gini means, Banach J. Math. Anal. 1 (2007), no. 2, 212–215.

[51] K. B. Stolarsky, Generalizations of the logarithmic mean, Mag. Math. 48 (1975), 87–92.

[52] Sh.-Q. Zhang, B.-N. Guo and F. Qi, A concise proof for properties of three functions involving the exponential function, Appl. Math. E-Notes 9 (2009), 177–183.

(F. Qi) Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300160, China
E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
URL: http://qifeng618.spaces.live.com

(B.-N. Guo) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China
E-mail address: bai.ni.guo@gmail.com, bai.ni.guo@hotmail.com
URL: http://guobaini.spaces.live.com