Loss-based risk statistics with scenario analysis

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Abstract Since the investors and regulators pay more attention to losses rather than gains, we will study a new class of risk statistics, named loss-based risk statistics in this paper. This new class of risk statistics can be considered as a kind of risk extension of risk statistics introduced by Kou, Peng and Heyde (2013), and also data-based versions of loss-based risk measures introduced by Cont et al. (2013) and Sun et al. (2018).

Keywords loss-based · risk statistics · data-based

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1 Introduction

In their seminal paper, Artzner et al. (1997)(1999) firstly introduced the class of coherent risk measures, by proposing four basic properties to be satisfied by every sound financial risk measure. Further, Föllmer and Schied (2002) and, independently, Frittelli and Rosazza Gianin (2002) introduced the broader class, named convex risk measure, by dropping one of the coherency axioms.

As pointed out by Cont et al. (2013), these axioms fail to take into account some key features encountered in the practice of risk management. In fact, sometimes, when measuring the risk of a portfolio, it is only relevant to consider the losses of this portfolio, not the gains. For this reason, we will study the risk based on losses, not gains.

On the other hand, from the statistical point of view by Kou, Peng and Heyde (2013), the behavior of a random variable can be characterized by its samples. At the same time, one can also incorporate scenario analysis into this framework. Therefore, a natural question is how about the discuss of loss-based risk with scenario analysis.

In the present paper, we will study convex and coherent loss-based risk statistics with scenario analysis, and dual representation results for them. Finally, the relationship between loss-based risk statistics and the convex risk statistics introduced by Tian and Suo (2012) will also be given to illustrate the loss-based risk statistics.

It is worth mentioning that the issue of risk measures with scenario analysis have already been studied by Delbaen (2002). It have also been extensively studied in the last decade. For example, see Kou, Peng and Heyde (2013), Ahmed, Filipovic, and Svindland (2008), Assa and Morales (2010), Tian and Suo (2012), Tian and Jiang (2015), and the references therein. From this point of view, the present paper can also be considered as a kind of risk extension of risk statistics.

The rest of the paper is organized as follows. In Section 2, we will briefly introduce some preliminaries. The main results will be stated in Section 3, and their proofs will be postponed to Section 4. In Section 5, we will provide the relationship between loss-based risk statistics and the convex risk statistics introduced by Tian and Suo (2012).
2 Preliminaries

In this section, we will briefly introduce some preliminaries. From now on, let \( N \geq 1 \) be a fixed positive integer. Let \( \mathcal{X} \) be a set of random losses, and \( \mathcal{X}^N \) the product space \( \mathcal{X}_1 \times \cdots \times \mathcal{X}_N \), where \( \mathcal{X}_i = \mathcal{X} \) for \( 1 \leq i \leq N \). Any element of \( \mathcal{X}^N \) is called a portfolio of random losses. In practice, the behavior of the \( N \)-dimensional random vector \( M = (X_1, \cdots, X_N) \) under different scenarios is represented by different sets of data observed or generated under those scenarios because specifying accurate models for \( M \) is usually very difficult. Some detailed notations can be found in Kou, Peng and Heyde (2013). Here, we suppose there always exist \( m \) scenarios. Specifically, suppose the behavior of \( M \) is represented by a collection of data \( M = (X_1, \cdots, X_N) \in \mathbb{R}^N \) which can be a data set based on historical observations, hypothetical samples simulated according to a model, or a mixture of observations and simulated samples.

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For any \( M_1 = (X_1^1, \cdots, X_N^1) \), \( M_2 = (X_1^2, \cdots, X_N^2) \in \mathbb{R}^N \), \( M_1 \leq M_2 \) means \( X_i^1 \leq X_i^2 \) for any \( i = 1, 2, \cdots, N \). And for any \( M = (X_1, \cdots, X_N) \in \mathbb{R}^N \), let \( M \wedge 0 := (\min\{X_1, 0\}, \cdots, \min\{X_N, 0\}) \). Given \( a \in \mathbb{R} \), denote \( a1 := (a, \cdots, a) \).

**Definition 21** \( \rho : \mathbb{R}^N \to [0, +\infty) \) is called a convex loss-based risk statistic if

(A.1) Normalization for cash losses: for any \( a \geq 0 \), \( \rho(-a1) = a \);

(A.2) Monotonicity: for any \( M_1, M_2 \in \mathbb{R}^N \), \( M_1 \leq M_2 \) implies \( \rho(M_1) \geq \rho(M_2) \);

(A.3) Loss-dependence: for any \( M \in \mathbb{R}^N \), \( \rho(M) = \rho(M \wedge 0) \);

(A.4) Convexity: for any \( M_1, M_2 \in \mathbb{R}^N \) and \( 0 < \alpha < 1 \),

\[
\rho(\alpha M_1 + (1 - \alpha) M_2) \leq \alpha \rho(M_1) + (1 - \alpha) \rho(M_2).
\]

A loss-based risk statistic \( \rho \) is called a coherent loss-based risk statistic if it still satisfies

(A.5) Positive homogeneity: for any \( \alpha \geq 0 \) and \( M \in \mathbb{R}^N \), \( \rho(\alpha M) = \alpha \rho(M) \).

3 Main results

**Theorem 31** \( \rho : \mathbb{R}^N \to [0, +\infty) \) is a convex loss-based risk statistic if and only if there exists a convex function \( \alpha : \mathbb{R}^N \to [0, +\infty] \), which is satisfied

\[
\min_{Q \in \mathbb{R}^N, \min Q_i \geq 1 - \epsilon} \alpha(Q) = 0 \quad \text{for any } \epsilon \in (0, 1)
\]

such that

\[
\rho(M) = \max_{Q \in \mathbb{R}^N} \{- \sum_{i=1}^N Q_i(X_i \wedge 0) - \alpha(Q)\}.
\]

The function \( \alpha \) for which \( [3.3] \) holds can be choose as \( \alpha_{\min}(Q) := \sup_{M \in \mathbb{R}^N} \{- \sum_{i=1}^N Q_i(X_i) - \rho(M)\} \) for any \( Q \in \mathbb{R}^N \). Moreover, \( \alpha_{\min} \) is the minimal penalty function in the sense that for any penalty function \( \alpha \) representing \( \rho \) satisfies \( \alpha(Q_1, \cdots, Q_N) \geq \alpha_{\min}(Q_1, \cdots, Q_N) \) for all \( (Q_1, \cdots, Q_N) \in \mathbb{R}^N \).

**Theorem 32** \( \rho : \mathbb{R}^N \to [0, +\infty) \) is a coherent loss-based risk statistic if and only if for any \( M \in \mathbb{R}^N \),

\[
\rho(M) = \max_{Q \in \mathbb{R}^N} \{- \sum_{i=1}^N Q_i(X_i \wedge 0)\}.
\]

**Remark 31** Compared to the representation result in Tian and Suo (2012), we have the cash-loss property. Moreover, the dual representation result in Theorem [37] depends only on the negative part of \( M \) due to the loss-dependence property. In Theorem [32] let \( N = 1 \), then representation result is reduced to the one-dimensional case which coincides with the representation results of Cont et al. (2013).
4 Proofs of Main results

In this section, we will provide proofs of main results in Section 3.

Proof of Theorem 31. Let \( f(X) = \rho(-X) \), then \( f \) is an increasing convex function. According to [Cheridito and Li, 2009, Th4.2], we have

\[
f(M) = \max_{M^* \in \mathbb{R}^N} \{M^*(M) - f^*(M^*)\}
\]

where

\[
f^*(M^*) = \sup_{M \in \mathbb{R}^N} \{M^*(-M) - \rho(M)\}.
\]

Hence

\[
\rho(M) = f(-M) = \max_{M^* \in \mathbb{R}^N} \{M^*(-M) - f^*(M^*)\}.
\]

Hence

\[
\rho(M) = \max_{Q \in \mathbb{R}^N} \left\{-\sum_{i=1}^{N} Q_i(X_i) - f^*(Q)\right\},
\]

where

\[
f^*(Q) = \sup_{Q \in \mathbb{R}^N} \left\{-\sum_{i=1}^{N} Q_i(X_i) - \rho(M)\right\}.
\]

Define \( \alpha_{\min} : \mathbb{R}^N \rightarrow [0, +\infty] \) by

\[
\alpha_{\min}(Q) := \sup_{Q \in \mathbb{R}^N} \left\{-\sum_{i=1}^{N} Q_i(X_i) - \rho(M)\right\},
\]

and by loss-dependence property of \( \rho \), we have

\[
\rho(M) = \max_{Q \in \mathbb{R}^N} \left\{-\sum_{i=1}^{N} Q_i(X_i \wedge 0) - \alpha_{\min}(Q)\right\}.
\]

Now, let \( \alpha \) be any penalty function for \( \rho \). Then, for any \( (Q_1, \cdots, Q_N) \in \mathbb{R}^N \) and \( M = (X_1, \cdots, X_N) \),

\[
\rho(M) \geq -\sum_{i=1}^{N} Q_i(X_i) - \alpha(Q_1, \cdots, Q_N).
\]

Hence,

\[
\alpha(Q_1, \cdots, Q_N) \geq -\sum_{i=1}^{N} Q_i(X_i) - \rho(M).
\]

Taking supremum over \( \mathbb{R}^N \) for \( M = (X_1, \cdots, X_N) \) in give rise to

\[
\alpha(Q) \geq \sup_{(X_1, \cdots, X_N) \in \mathbb{R}^N} \left\{-\sum_{i=1}^{N} Q_i(X_i) - \rho(M)\right\} = \alpha_{\min}(Q)
\]

Next, we check that \( \rho \) represented in (3.2) is a convex loss-based risk statistic. Obviously, \( \rho \) is a convex function and satisfies (A3). Hence, we only need to show \( \rho \) satisfies (A1) and (A2). To this end, for any \( a \geq 0 \)
and $1 < \epsilon < 1$,
\[
a = \rho (-a1)
\]
\[
= \max_{Q \in \mathbb{R}^N} \{ a \sum_{i=1}^{N} Q_i - \alpha_{\min}(Q) \}
\]
\[
\leq \max_{Q \in \mathbb{R}^N} \{ \max_{1 \leq i \leq N} Q_i < 1 - \epsilon, \min_{1 \leq i \leq N} \{ a \sum_{i=1}^{N} Q_i - \alpha_{\min}(Q) \} \}
\]
\[
\leq \max_{Q \in \mathbb{R}^N, \min_{1 \leq i \leq N} Q_i < 1 - \epsilon, \max_{1 \leq i \leq N} \min_{1 \leq i \leq N} Q_i \geq 1 - \epsilon} \{ a \sum_{i=1}^{N} Q_i - \alpha_{\min}(Q) \}
\]
\[
\leq \max_{Q \in \mathbb{R}^N, \min_{1 \leq i \leq N} Q_i < 1 - \epsilon, \max_{1 \leq i \leq N} \min_{1 \leq i \leq N} Q_i \geq 1 - \epsilon} \{ a \sum_{i=1}^{N} Q_i - \alpha_{\min}(Q) \}
\]
which implies $\alpha_{\min}$ satisfies (3.1). Now, let $M_1 := (X^1_1, \cdots, X^1_N), M_2 := (X^2_1, \cdots, X^2_N)$. Then, the relation $M_1 \leq M_2$ implies $X^1_i \wedge 0 \leq X^2_i \wedge 0$ for any $1 \leq i \leq N$. Hence for any $Q := (Q_1, \cdots, Q_N) \in \mathbb{R}^N$, we have
\[
\sum_{i=1}^{N} Q_i (X^1_i \wedge 0) \leq \sum_{i=1}^{N} Q_i (X^2_i \wedge 0),
\]
which implies $\rho(M_1) \leq \rho(M_2)$. The proof of Theorem 3.1 is completed.

**Proof of Theorem 3.2** If $\rho$ is a coherent loss-based risk statistic, then by the proof of Theorem 3.1 and the positive homogeneity of $\rho$, for any $Q \in \mathbb{R}^N$ and $\lambda > 0$, we have
\[
\alpha_{\min}(Q) = \sup_{M \in \mathbb{R}^N} \{ -\sum_{i=1}^{N} Q_i (-X_i) - \rho(M) \}
\]
\[
= \sup_{M \in \mathbb{R}^N} \{ -\sum_{i=1}^{N} Q_i (-\lambda X_i) - \rho(\lambda M) \}
\]
\[
= \lambda \sup_{M \in \mathbb{R}^N} \{ -\sum_{i=1}^{N} Q_i (-X_i) - \rho(M) \}
\]
\[
= \lambda \alpha_{\min}(Q)
\]
Hence, $\alpha_{\min}$ can take only the values 0 and $+\infty$. The proof of Theorem 3.2 is completed.

5 Loss-based version of convex risk statistics

For any convex risk statistic $\bar{\rho}$ on $\mathbb{R}^N$ defined in Tian and Suo (2012), we can define a new risk statistic $\rho$ by $\rho(M) := \bar{\rho}(M \wedge 0)$ for any $M \in \mathbb{R}^N$. Obviously, $\rho$ is a convex loss-based risk statistic. We call $\rho$ the loss-based version of $\bar{\rho}$.
We can prove that a convex loss-based risk statistic \( \rho \) is the loss-based version of some convex risk statistic if and only if it satisfies

(CLA) Cash-loss additivity: for any \( M \in \mathbb{R}^N \) and \( a \in \mathbb{R} \) with \( M \leq 0, a \geq 0, \)

\[
\rho(M - a1) = \rho(M) + a.
\]

On the one hand, if \( \rho(M) = \bar{\rho}(M \wedge 0) \) for certain convex risk statistic \( \bar{\rho} \) on \( \mathbb{R}^N \), then for any \( M \in \mathbb{R}^N, M \leq 0 \) and \( a \geq 0, \)

\[
\rho(M - a1) = \bar{\rho}(M - a1) = \rho(M) + a = \rho(M) + a.
\]

where the second equality is due to the cash-additivity property of \( \bar{\rho} \).

On the other hand, suppose a convex loss-based risk statistic \( \rho \) satisfies the cash-loss additivity property.

Define

\[
\tilde{\rho}(M) = \rho(M - a_M 1) - a_M
\]

for any \( M := (X_1, \cdots, X_N) \in \mathbb{R}^N \) where \( a_M \) is any upper-bound of each \( X_i \). By the cash-loss additivity property for \( \rho \), we know that \( \tilde{\rho} \) is well-defined. Next, we need to claim that \( \tilde{\rho} \) is a convex risk statistic with \( \rho(M) = \tilde{\rho}(M \wedge 0) \). To this end, for any \( M := (X_1, \cdots, X_N) \in \mathbb{R}^N \) and \( a \in \mathbb{R}, \)

\[
\tilde{\rho}(M - a1) = \rho(M - a1 - (a_M 1 - a1)) - (a_M - a)
\]

\[
= \rho(M - a_M 1) - a_M + a
\]

\[
= \tilde{\rho}(M) + a.
\]

Next, let \( M_1 := (X^1_1, \cdots, X^1_N), M_2 := (X^2_1, \cdots, X^2_N) \in \mathbb{R}^N \) with \( M_1 \leq M_2 \). Taking \( a_{M_1}, a_{M_2} \) to be the upper-bound of each \( X^1_i \) and \( X^2_i \). Then,

\[
\tilde{\rho}(M_1) = \rho(M_1 - a_{M_1} 1) - a_{M_1}
\]

\[
\geq \rho(M_2 - a_{M_2} 1) - a_{M_2}
\]

\[
= \tilde{\rho}(M_2),
\]

which yields \( \tilde{\rho} \) is monotonous. Finally, for any \( M_1, M_2 \in \mathbb{R}^N \) and \( 0 \leq t \leq 1, \)

\[
\tilde{\rho}(tM_1 + (1 - t)M_2) = \rho(t(M_1 + (1 - t)M_2) - ta_{M_1} 1 - (1 - t)a_{M_2} 1)
\]

\[
= \rho(t(M_1 - a_{M_1} 1) + (1 - t)(M_2 - a_{M_2} 1)) - ta_{M_1} - (1 - t)a_{M_2}
\]

\[
= t\rho(M_1 - a_{M_1} 1) + (1 - t)\rho(M_2 - a_{M_2} 1) - ta_{M_1} - (1 - t)a_{M_2}
\]

\[
= t\tilde{\rho}(M_1) + (1 - t)\tilde{\rho}(M_2),
\]

which implies \( \tilde{\rho} \) is convex.

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