Factoriality of $q$-Gaussian von Neumann algebras

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Abstract

We prove that the von Neumann algebras generated by $n$ $q$-Gaussian elements, are factors for $n \geq 2$.

1 Introduction

In the early 70’s, Frish and Bourret considered operators satisfying the $q$-canonical commutation relations, for $-1 < q < 1$:

$$l(e)^* l^*(f) - ql^*(f)l(e) = (e,f)Id.$$ 

Nevertheless their existence was proved only 20 years later by Bożejko and Speicher in [2]. Since, many people studied the von Neumann algebra $\Gamma_q(\mathcal{H}_\mathbb{R})$, generated by $q$-Gaussian random variables $\{l(e) + l^*(e); e \in \mathcal{H}_\mathbb{R}\}$, and some of their generalizations. It is well known that $\Gamma_q(\mathcal{H}_\mathbb{R})$ is type $II_1$. One of the interesting point is that these algebras realize a kind of interpolating scale between $\Gamma_1(\mathcal{H})$ which is commutative and $\Gamma_{-1}(H)$ the hyperfinite $II_1$ factor. For $q = 0$, we recover the algebra generated by Voiculescu’s semicircular elements, which is a central object in the free probability theory.

Among the known results, Bożejko and Speicher showed that $\Gamma_q(\mathcal{H}_\mathbb{R})$ is non injective under some condition on the dimension of $\mathcal{H}$, which was removed by Nou [4]. Recently, Shlyakhtenko [5] proved that they are solid for some values of $q$. The question of the factoriality of $\Gamma_q(\mathcal{H}_\mathbb{R})$ was studied by Bożejko, Kümmner and Speicher [1], they showed that if $\mathcal{H}$ is infinite dimensional then $\Gamma_q(\mathcal{H}_\mathbb{R})$ is a factor. This condition was partially released by Śniady [6], who showed that this is still true if the dimension of $\mathcal{H}$ is greater than a function of $q$.

2 Preliminaries

In this paper, $-1 < q < 1$ is a fixed real number, we will use standard notation and refer to the papers [3, 1, 4] for general background.

Let $\mathcal{H}$ be the complexification of some real Hilbert space $\mathcal{H}_\mathbb{R}$. By $\mathcal{H}^\otimes_n$ ($n \geq 1$), we denote the hilbertian $n$-tensor product of $\mathcal{H}$ with itself, this space is equipped with a scalar product that we write $(\ldots)$. Let $P_n : \mathcal{H}^\otimes_n \to \mathcal{H}^\otimes_n$ be given by

$$P_n(e_1 \otimes \ldots \otimes e_n) = \sum_{\sigma \in S_n} q^{||\sigma||} e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(n)} = \sum_{\sigma \in S_n} q^{||\sigma||}\phi(\sigma)(e_1 \otimes \ldots \otimes e_n),$$

where $S_n$ denotes the symmetric group of $n$ elements.
where $S_n$ is the symmetric group on $n$ elements, $|\sigma|$ is the number of inversion of $\sigma$, and $\phi$ is the natural action of $S_n$ on $\mathcal{H}^{\otimes 2n}$. It was shown in [3], that this operator is bounded and strictly positive, therefore we denote by $\mathcal{H}^{\otimes n}$, the hilbert space $\mathcal{H}^{\otimes 2n}$ equip with the new scalar product $\langle ., . \rangle$ given by

$$\forall x, y \in \mathcal{H}^{\otimes n} \quad \langle x, y \rangle = (x, P_n(y)).$$

From now on, if $x \in \mathcal{H}^{\otimes n}$, $\|x\|$ is the norm of $x$ with respect to this new scalar product. For instance, if $e \in \mathcal{H}$ and $\|e\| = 1$, then

$$\|e^{\otimes n}\|^2 = [n]_q!,$$

where $[k]_q = \frac{1-q^k}{1-q}$ and $[n]_q! = [1]_q \cdots [n]_q$.

**Remark 1** We will use as a key point that the sequence $([n]_q!)$ behave like a geometric sequence.

Moreover, it is known that the following algebraic relation holds :

$$P_n = R_{n,k}(P_{n-k} \otimes P_k) \quad \text{with} \quad R_{n,k} = \sum_{\sigma \in S_n/S_{n-k} \times S_k} q^{|\sigma|} \phi(\sigma^{-1}),$$

and the sum runs over the representatives of the right cosets of $S_{n-k} \times S_k$ in $S_n$ with minimal number of inversions. As a consequence, since $\|R_{n,k}\|_{B(\mathcal{H}^{\otimes 2n})} \leq C_q = \prod_{i \geq 1}(1 - |q|^i)^{-1}$, we get that the formal identity map

$$Id : \mathcal{H}^{\otimes n-k} \otimes_2 \mathcal{H}^{\otimes k} \to \mathcal{H}^{\otimes n}$$

has norm bounded by $\sqrt{C_q}$.

**Remark 2** As an application, we get that, if $e_1, ... e_n$ and $e$ are norm 1 vectors in $\mathcal{H}$, then

$$\|e_1 \otimes ... \otimes e_n \otimes e^{\otimes m}\|_{\mathcal{H}^{\otimes n+m}} \leq C_q^{m/2} \sqrt{[m]_q!}.$$

The $q$-deformed Fock space is the Hilbert space defined by

$$\mathcal{F}_q(\mathcal{H}_R) = \mathbb{C}\Omega \oplus \oplus_{n \geq 1} \mathcal{H}^{\otimes n},$$

where $\Omega$ is a unital vector, considered as the vacuum. Vectors in $\mathcal{H}$ will be called letters and an elementary tensor of letters in $\mathcal{H}^{\otimes n}$ will be called a word of length $n$.

For $e \in \mathcal{H}_R$, we consider left and right creation operators on $\mathcal{F}_q(\mathcal{H}_R)$, given by :

$$l(e)(e_1 \otimes ... \otimes e_n) = e \otimes e_1 \otimes ... \otimes e_n$$

$$l_r(e)(e_1 \otimes ... \otimes e_n) = e_1 \otimes ... \otimes e_n \otimes e$$

They are bounded endomorphisms of $\mathcal{F}_q(\mathcal{H}_R)$, more precisely if $\|e\| = 1$ then

$$\|l_r(e)\| = \|l(e)\| = \begin{cases} 1 & \text{if } q \leq 0 \\ \frac{1}{\sqrt{1-q}} & \text{if } q > 0 \end{cases}$$
Their adjoints in $B(\mathcal{F}_q(\mathcal{H}_\mathbb{R}))$ are the annihilation operators:

$$l^*(e)(e_1 \otimes \ldots \otimes e_n) = \sum_{1 \leq i \leq n} q^{l-1}(e, e_i) \otimes e_1 \otimes \ldots \otimes \hat{e}_i \otimes \ldots \otimes e_n$$

$$l^*_r(e)(e_1 \otimes \ldots \otimes e_n) = \sum_{1 \leq i \leq n} q^{n-l}(e, e_i) \otimes e_1 \otimes \ldots \otimes \hat{e}_i \otimes \ldots \otimes e_n$$

where $\hat{e}_i$ denotes a removed letter, if $n = 0$, we put $l^*(e) \Omega = l^*_r(e) \Omega = 0$.

The operators $l(e)$ satisfy the $q$-commutation relations:

$$l(e)l^*(f) - ql^*(f)l(e) = (e, f)Id$$

For $e \in \mathcal{H}_\mathbb{R}$, let

$$W(e) = l(e) + l^*(e) \quad \text{and} \quad W_r(e) = l_r(e) + l^*_r(e).$$

So for $e \in \mathcal{H}_\mathbb{R}$, $W(e)$ is self-adjoint.

$\Gamma_q(\mathcal{H}_\mathbb{R})$ stands for the von Neumann algebra generated by $(W(e))_{e \in \mathcal{H}_r}$

$$\Gamma_q(\mathcal{H}_\mathbb{R}) = \{ W(e) ; e \in \mathcal{H}_\mathbb{R} \}''.$$

And, $\Gamma_{q,r}(\mathcal{H}_\mathbb{R})$ stands for the von Neumann algebra generated by $(W_r(e))_{e \in \mathcal{H}_\mathbb{R}}$

$$\Gamma_{q,r}(\mathcal{H}_\mathbb{R}) = \{ W_r(e) ; e \in \mathcal{H}_\mathbb{R} \}''.$$

We recall some classical results on those algebras,

- The commutant of $\Gamma_q(\mathcal{H}_\mathbb{R})$ is $\Gamma_q(\mathcal{H}_\mathbb{R})' = \Gamma_{q,r}(\mathcal{H}_\mathbb{R})$.

- The vacuum vector $\Omega$ is separating and cyclic for both $\Gamma_q(\mathcal{H}_\mathbb{R})$ and $\Gamma_{q,r}(\mathcal{H}_\mathbb{R})$.

- The vector state $\tau(x) = \langle x \Omega, \Omega \rangle$ is a trace for both $\Gamma_q(\mathcal{H}_\mathbb{R})$ and $\Gamma_{q,r}(\mathcal{H}_\mathbb{R})$.

According to the second point, any $x \in \Gamma_q(\mathcal{H}_\mathbb{R})$ is uniquely determined by $\xi = x.\Omega \in \mathcal{F}_q(\mathcal{H}_\mathbb{R})$, so we will call it $x = W(\xi)$ (and similarly for $\Gamma_{q,r}(\mathcal{H}_\mathbb{R})$, $x = W_r(\xi)$). This notation is consistent with the definition of $W(e) = l(e) + l^*(e)$. The subspace $\Gamma_q(\mathcal{H}_\mathbb{R}).\Omega \subset \mathcal{F}_q(\mathcal{H}_\mathbb{R})$ of all such $\xi$ contains all tensors of finite rank, so it contains all words. If $e_1 \otimes \ldots \otimes e_n$ is a word in $\mathcal{F}_q(\mathcal{H}_\mathbb{R})$, there is a nice description of $W(e_1 \otimes \ldots \otimes e_n)$ in terms of $l(e_i)$ called the Wick formula:

$$W(e_1 \otimes \ldots \otimes e_n) = \sum_{m=0}^{n} \sum_{\sigma \in S_{n-m} \times S_m} q^{\sigma !}(l(e_{\sigma(1)})\ldots l(e_{\sigma(n-m)})l^*(e_{\sigma(n-m+1)})\ldots l^*(e_{\sigma(n)}),$$

where $\sigma$ is the representative of the right coset of $S_{n-m} \times S_m$ in $S_n$ with minimal number of inversions. There is a similar formula for $W_r$.

Actually, the algebras $\Gamma_q(\mathcal{H}_\mathbb{R})$ and $\Gamma_{q,r}(\mathcal{H}_\mathbb{R})$ are in standard form in $B(\mathcal{F}_q(\mathcal{H}_\mathbb{R}))$, but we won’t use it. If we denote by $S$, the anti-symmetry that inverses the order of words in $\mathcal{H}_\mathbb{R}$, then for any $\xi \in \Gamma_q(\mathcal{H}_\mathbb{R}).\Omega$:

$$W(\xi)^* = W(S\xi) \quad \text{and} \quad S\cdot W(\xi)\cdot S = W_r(S\xi).$$
3 \ THE MAIN RESULT

In particular $\Gamma_q(\mathcal{H}_\mathbb{R}).\Omega = \Gamma_{q,r}(\mathcal{H}_\mathbb{R}).\Omega$.

**Remark 3** For $\xi, \eta \in \Gamma_q(\mathcal{H}_\mathbb{R}).\Omega$, we will frequently use

$$W(\xi)\eta = W(\xi)W_r(\eta)\Omega = W_r(\eta)W(\xi)\Omega = W_r(\eta)\xi.$$ 

Let $T : \mathcal{H}_\mathbb{R} \to \mathcal{H}_\mathbb{R}$, be a $\mathbb{R}$-linear contraction, then there is a canonical $\mathbb{C}$-linear contraction, $\mathcal{F}_q(T)$, on $\mathcal{F}_q(\mathcal{H}_\mathbb{R})$ extending $T$, called the first quantization; formally

$$\mathcal{F}_q(T) = 1d_{\mathbb{C}\Omega} \oplus \oplus_{n \geq 1} \tilde{T}^\otimes n$$

with $\tilde{T}$, the complexification of $T$ on $\mathcal{H}$.

The second quantization of $T$, is the unique unital completely positive map $\Gamma_q(T)$ on $\Gamma_q(\mathcal{H}_\mathbb{R})$ satisfying, for $\xi \in \Gamma_q(\mathcal{H}_\mathbb{R}).\Omega$

$$\Gamma_q(T)(W(\xi)) = W(\mathcal{F}_q(T)\xi).$$

For instance, if $\mathcal{K}_\mathbb{R} \subset \mathcal{H}_\mathbb{R}$, the second quantization associated to the orthogonal projection $P_{\mathcal{K}_\mathbb{R}}$ on $\mathcal{K}_\mathbb{R}$ is a conditional expectation

$$\Gamma_q(P_{\mathcal{K}_\mathbb{R}}) : \Gamma_q(\mathcal{H}_\mathbb{R}) \to \Gamma_q(\mathcal{K}_\mathbb{R}) = \{W(e) ; e \in \mathcal{K}_\mathbb{R}\}''.$$

3 \ The main result

Let $e \in \mathcal{H}_\mathbb{R}$ of norm one and denote by $E_e$ the closed subspace of $\mathcal{F}_q(\mathcal{H}_\mathbb{R})$ spanned by the elements $\{e^{\otimes n} ; n \geq 0\}$, that is $E_e = \mathcal{F}_q(\mathbb{R}e)$. It is easy to check that for any $x = W(\xi) \in W(e)''$, we have $\xi \in E_e$. Conversely, assume $x = W(\xi)$ and that $\xi \in E_e$, then $x \in W(e)''$; by the second quantization, we have a conditional expectation $\Gamma_q(P_{\mathbb{R}e}) : \Gamma_q(\mathcal{H}_\mathbb{R}) \to W(e)''$, but then

$$\Gamma_q(P_{\mathbb{R}e})(x)\Omega = \mathcal{F}_q(P_{\mathbb{R}e})\xi = P_{E_e}\xi = \xi = x\Omega,$$

as $\Omega$ is separating, $x = \Gamma_q(P_{\mathbb{R}e})(x) \in W(e)''.

**Theorem 1** Assume that $\dim \mathcal{H} \geq 2$ and let $e \in \mathcal{H}_\mathbb{R}$, $\|e\| = 1$, then $W(e)''$ is a maximal abelian subalgebra in $\Gamma_q(\mathcal{H}_\mathbb{R})$.

**Corollary 2** $\Gamma_q(\mathcal{H}_\mathbb{R})$ is a factor as soon as $\dim \mathcal{H} \geq 2$.

**Proof :** Let $x \in \Gamma_q(\mathcal{H}_\mathbb{R}) \cap \Gamma_q(\mathcal{H}_\mathbb{R})'$, then there is $\xi \in \mathcal{F}_q(\mathcal{H}_\mathbb{R})$ such that $x = W(\xi)$.

By the theorem, we must have $x \in W(e)''$ for every $e \in \mathcal{H}_\mathbb{R}$, but then $\xi \in E_e$, so necessarily $x \in \mathbb{C}\Omega$.

**Proof :** Fix $(e_i)_{i \geq 0}$ an orthonormal basis in $\mathcal{H}_\mathbb{R}$, with $e_0 = e$.

Let $x = W(\xi) \in \Gamma_q(\mathcal{H}_\mathbb{R}) \cap W(e)'$, we have to show that $\xi \in E_e$. For any $y = W(\eta)$ with $\eta \in E_e$, we have

$$xy - yx = 0$$

$$(W(\xi)W(\eta) - W(\eta)W(\xi))\Omega = 0$$

$$(W_r(\eta) - W(\eta))\xi = 0$$
So $\xi \in \cap_{y=W(\eta)\in W(e)''} \ker (W_r(\eta) - W(\eta))$. By duality, we have to prove that

$$\text{span}\{\text{ran} (W_r(\eta) - W(\eta)) ; y = W(\eta) \in W(e)''\} \supset E^\perp_e.$$ 

$E^\perp_e$ is the closed linear span of the set of elementary tensors

$$F = \{e_{i_1} \otimes ... \otimes e_{i_n} ; n \geq 1, \text{ and } (i_1, ..., i_n) \in \mathbb{N}^n \backslash \{(0, ..., 0)\}\}$$ 

Let $z = e_{i_1} \otimes ... \otimes e_{i_n}$ be a word in $F$, it suffices to prove that $z$ is a weak-limit of elements in span{ran $(W_r(\eta) - W(\eta)) ; y = W(\eta) \in W(e)''$}.

The von Neumann algebra $W(e)''$ is commutative and diffuse and separably generated (see [1]), so we can assume that $W(e)'' = L_\infty([0, 1], dm)$, where $dm$ is the Lebesgue measure. With this identification, the Rademacher functions $r_i$ belong to $W(e)''$, so we have $r_i = W(\eta_i)$ for some $\eta_i \in E_e$. Obviously $W(\eta_i)$ is a self-adjoint symmetry and $W(\eta_i)^2 = 1$. Moreover, the sequence $(\eta_i)_{i \geq 1}$ converges to 0 for the weak topology on $\mathcal{F}_q(\mathcal{H}_R)$, since $r_i$ is an orthonormal basis in $L_2([0, 1], dm)$.

Consider

$$z_i = (W(\eta_i) - W_r(\eta_i))(W(\eta_i)(z)),$$

obviously $z_i \in \text{span}\{\text{ran} (W_r(\eta) - W(\eta)) ; y = W(\eta) \in W(e)''\}$ and a simple calculation gives

$$z_i = W(\eta_i)^2(z) - W_r(\eta_i)W(\eta_i)(z) = z - W_r(\eta_i)W(\eta_i)(z).$$

We will show that $y_i = W_r(\eta_i).W(\eta_i)(z)$ tends weakly to 0 in $\mathcal{F}_q(\mathcal{H}_R)$. As $\|y_i\| \leq \|z\|$, it suffices to prove that for any word $t = e_{j_1} \otimes ... \otimes e_{j_p}$, $\langle y_i, t \rangle \to 0$. We have,

$$\langle y_i, t \rangle = \langle W_r(\eta_i).W(\eta_i)(z), t \rangle$$
$$= \langle W_r(z)(\eta_i), W(t)(\eta_i) \rangle$$

This is the point where we use the Wick formula:

$$W(e_{j_1} \otimes ... \otimes e_{j_n}) = \sum_{m=0}^{n} \sum_{\sigma \in S_m/S_{m-n} \times S_m} q^{|\sigma|}(l(e_{j_{\sigma(1)}})...l(e_{j_{\sigma(m-n)}})l^*(e_{j_{\sigma(n-m+1)}})...l^*(e_{j_{\sigma(n)}}))$$

and similarly for $z$. Since the number of terms appearing after developing the sums is finite (it depends only on $n$ and $p$), we only need to show that

$$I_i = \langle l_r(e_{i_1})...l_r(e_{i_m})l^*_r(e_{i_{m+1}})...l^*_r(e_{i_n}) (\eta_i), l(e_{j_1})...l(e_{j_r})l^*(e_{j_{r+1}})...l^*(e_{j_p}) (\eta_i) \rangle \to 0,$$

as soon as at least one of the $i_k$'s is non zero. Let $v$ be the first $k$ such that $i_k \neq 0$. Since the letters in $\eta_i$ are only $e$, we can suppose that $v \leq m$, otherwise $l_r(e_{i_1})...l_r(e_{i_m})l^*_r(e_{i_{m+1}})...l^*_r(e_{i_n})(\eta_i) = 0$ (we have to cancel some $e_{i_n}$ in $\eta_i$ !). More generally, we can assume that $e_{i_{m+1}} = ... = e_{i_n} = e_{j_{r+1}} = ... = e_{j_p} = e$.

Recall that

$$l(e)^*e^{\otimes n} = [n]_q e^{\otimes n-1}.$$
Now, we write \( \eta = \sum_{k \geq 0} a_k^i e^{\otimes k} \), interchanging the sums and making simplifications gives that (with \( a_{-n} = 0 \) if \( n > 0 \)). The \( a_n \) are reals since \( r_i \) is self adjoint,

\[
I_i = \langle l_r(e_{i_1}) ... l_r(e_{i_m}) l_r^+(e_{i_{m+1}}) ... l_r^+(e_{i_n}) (\eta_i), l(e_{j_1}) ... l(e_{j_{r'}}) l_r^+(e_{j_{r'+1}}) ... l_r^+(e_{j_{r''}}) (\eta) \rangle
\]

\[
= \sum_{k \geq r, m} a_{k+n-2m}^i a_{k+p-2r}^i [k + n - 2m] q^1 /[k - m] q^1 [k + p - 2r] q^1 /[k - r] q^1
\]

\[
. (l_r(e_{i_1}) ... l_r(e_{i_m}) e^{\otimes k-m}, l(e_{j_1}) ... l(e_{j_{r'}}) e^{\otimes r-r})
\]

\[
= \sum_{k \geq r, m} a_{k+n-2m}^i a_{k+p-2r}^i [k + n - 2m] q^1 /[k - m] q^1 [k + p - 2r] q^1 /[k - r] q^1
\]

\[
. (l_r(e_{i_{v+1}}) ... l_r(e_{i_m}) e^{\otimes k-m}, l_r^+(e_{i_1}) ... l_r^+(e_{i_{v-1}}) (e_{j_1} \otimes ... \otimes e_{j_r} \otimes e^{\otimes k-r}))
\]

Assume that \( k \) is big (say \( k > N > 2(n + p) \)), by the definition of \( v \), we have that \( i_1 = ... = i_{v-1} = e \), so

\[
l_r^+(e_{i_{v+1}}) ... l_r^+(e_{i_{m}}) (e_{j_1} \otimes ... \otimes e_{j_r} \otimes e^{\otimes k-r})
\]

is obtained by cancelling \((v - 1)\) times the letter \( e \) in the word \( e_{j_1} \otimes ... \otimes e_{j_r} \otimes e^{\otimes k-r} \)

using some geometric weight \( q^n \),

\[
\sum_{1 \leq h_{v+1} \leq k - v - 2} ... \sum_{1 \leq h_2 \leq k - 1} \sum_{1 \leq h_1 \leq k} \delta_{h_1,...,q^{(\sum h_i)-v+1}} (e_{j_1} \otimes ... \otimes e_{j_r} \otimes e^{\otimes k-r})(h_1,...,h_{v+1})
\]

where \((e_{j_1} \otimes ... \otimes e_{j_r} \otimes e^{\otimes k-r})(h_1,...,h_{v+1})\) is obtained from \( e_{j_1} \otimes ... \otimes e_{j_r} \otimes e^{\otimes k-r} \) by removing the letter on the \( h_1 \)-th position from the right, then the letter at \( h_2 \)-th position in the remaining word and so on and where \( \delta_{h_1,...} \) is one if all the removed letters are \( e \) and 0 otherwise. To have a non zero term in

\[
l_r^+(e_{i_{v}})(e_{j_1} \otimes ... \otimes e_{j_r} \otimes e^{\otimes k-r})(h_1,...,h_{v+1})
\]

we have to cancel a letter that is not an \( e \), so it can happen only for the terms coming from \( e_{j_1} \otimes ... \otimes e_{j_r} \) (if there are some left !), as this word of length \( k - v + 1 \) ends with at least \( (k - r - v + 1) e \), we end up with a sum of at most \( r \) words in front of which there is a factor less than \( |q|^{k - r - v + 1} \). Moreover, by the remark \( \| \) the norm of such a word is less than \( C_q^r q^{v^2} \sqrt{|k - r - v + 1|} q^1 \).

If we sum up everything, we get that

\[
\| l_r^+(e_{i_{v-1}}) ... l_r^+(e_{i_{m}}) (e_{j_1} \otimes ... \otimes e_{j_r} \otimes e^{\otimes k-r}) \| \leq C(n, m, v, q) |q|^{k} \sqrt{|k|} q^1
\]

where \( C(n, m, v) \) does not depend on \( k \) (because \( |k| q \leq C_q \)).

Now we can estimate \( I_i \), by cutting the sum into two parts \( A_i + B_i = \sum_{k \leq N} |.| + \sum_{k \geq N} |.| \).

Since \( \eta_i \to 0 \) weakly, each \( a_j^i \) tends to 0, then

\[
A_i \leq \sum_{N > k \geq r, m} |a_{k+n-2m}^i| |a_{k+p-2r}^i| C(k, n, p) \to 0
\]
and as $\|\eta_i\| \leq 1$, we have $|a_k^i| \leq 1/\sqrt{|k|q!}$, so

$$B_i \leq \sum_{k \geq N} \frac{\sqrt{|k+n-2m|q!|k+p-2r|q!}}{|k-m|q!|k-r|q!} \|l_r^*(e_{i_{\nu}}) \ldots l_r^*(e_{i_1}) (e_{j_1} \otimes \ldots \otimes e_{j_q} \otimes e^{\otimes k-q})\|.$$

$$\leq \sum_{k \geq N} \frac{\sqrt{|k+n-2m|q!|k+p-2r|q!}}{|k-m|q!|k-r|q!} C|q|^k \sqrt{|k|q!} C(q)^m \sqrt{|k-m|q!}.$$

$$\leq \sum_{k \geq N} C|q|^k \leq C|q|^N.$$

Consequently, we get that $\limsup |I_i| \leq C|q|^N$ for every $N$, so $I_i \to 0$. 

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