DETERMINANT OF LAPLACIANS ON HEISENBERG MANIFOLDS

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Abstract. We give an integral representation of the zeta-regularized determinant of Laplacians on three dimensional Heisenberg manifolds, and study a behavior of the values when we deform the uniform discrete subgroups. Heisenberg manifolds are the total space of a fiber bundle with a torus as the base space and a circle as a typical fiber, then the deformation of the uniform discrete subgroups means that the “radius” of the fiber goes to zero. We explain the lines of the calculations precisely for three dimensional cases and state the corresponding results for five dimensional Heisenberg manifolds. We see that the values themselves are of the product form with a factor which is that of the flat torus. So in the last half of this paper we derive general formulas of the zeta-regularized determinant for product type manifolds of two Riemannian manifolds, discuss the formulas for flat tori and explain a relation of the formula for the two dimensional flat torus and the Kroneker’s second limit formula.

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Introduction

In the conformal field theory, especially in the string theory an infinite dimensional analogue of the determinant for elliptic operators appears and it is considered in the framework of the analytic continuation method of the spectral zeta-function of elliptic operators. In the string theory it seems to be most interesting to calculate

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the explicit values for compact Riemannian surfaces and also from the mathematical point of view the quantity relates with various aspects of special functions already in such two dimensional cases. As for the values for the spheres there are relations with the values of the Riemann $\zeta$-functions at the negative integers, for the two dimensional flat torus it is expressed by using the famous formula, so called “Kroneker’s second limit formula”, and for the cases of compact surfaces with constant negative curvature it was calculated by using Selberg’s trace formula and deep properties of modified Bessel functions ([10], [15], [5], [4], [7]). Thus for all two dimensional cases we already know the values of the zeta-regularized determinant of the Laplacians or relations with other quantities. A purpose of this note is to give an integral representation of the zeta-regularized determinant for three dimensional Heisenberg manifolds and state the corresponding results for five dimensional Heisenberg manifolds. It will be possible to give similar expressions for higher dimensional cases, but we restrict ourselves to these two cases and also we restrict ourselves to deal with a certain kind of uniform discrete subgroups of the Heisenberg group, since even in these cases they contain all necessary features for determining the values for any cases and make us the calculations to be simple. Then in the last half of this paper we derive a general formula of the zeta-regularized determinant for manifolds of the product form of two Riemannian manifolds and the formulas for flat tori of two, three and four dimensions.

In §1 we gather up the basic data of the spectrum of the three dimensional Heisenberg group and Heisenberg manifolds. In §2 first, we explain the zeta-regularized determinant of the Laplacian and give a calculation for the three dimensional Heisenberg manifolds based on an integral representation of the spectral zeta-function (= the Mellin transformation of the trace of heat kernel divided by a Gamma function). In §3 as an application of an integral representation of the spectral zeta-function given in §2 we give expressions of the all coefficients of the asymptotic expansion of the heat kernel for the three dimensional Heisenberg manifolds. In §4 we state the corresponding results in §2 and §3 for five dimensional Heisenberg manifolds. In §5 we give a general expression of the zeta-regularized determinant of the Laplacian on the product of two Riemannian manifolds. In §6 we give a precise form of the formula derived in §5 for a product type manifold with $S^1$. Finally in §7 we give such formulas for two, three and four dimensional flat tori and explain a relation of the formula for two dimensional flat torus and the Kroneker’s second limit formula.

1. Spectrum of three dimensional Heisenberg manifolds

Let $H_3$ be the three dimensional Heisenberg group:

\[
H_3 = \left\{ g = g(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.
\]
The Lie algebra is

\[ (1.2) \quad h_3 = \left\{ X = X(x, y, z) = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}. \]

It is decomposed into a direct sum in the form of

\[ (1.3) \quad h_3 = g_+ \oplus g_- \oplus \mathfrak{z}, \]

where

\[ g_+ = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x \in \mathbb{R} \right\}, \]

\[ g_- = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid y \in \mathbb{R} \right\} \]

and

\[ \mathfrak{z} = \text{the center} = \left\{ \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid z \in \mathbb{R} \right\}. \]

We identify \( h_3 \) and \( H_3 \) through the exponential map. Then the group multiplication of two elements \( X = X(x, y, z) \in h_3 \) and \( \tilde{X} = \tilde{X}(\tilde{x}, \tilde{y}, \tilde{z}) \in h_3 \) is given by

\[ (1.4) \quad Y = X \ast \tilde{X} = X + \tilde{X} + \frac{1}{2}[X, \tilde{X}], \text{ that is, } \exp(Y) = \exp X \cdot \exp \tilde{X}. \]

Left invariant Riemannian metrics are determined by its restriction to the tangent space at the identity element (\( \cong \) its Lie algebra), and among the left invariant Riemannian metrics we only consider such a metric that

\[ X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

and \( Z_0 = z_0 \cdot Z = z_0 \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) are being an orthonormal basis. Also we only consider uniform discrete subgroups \( \Gamma_\ell \) of the following form

\[ (1.5) \quad \Gamma_\ell = \left\{ \begin{pmatrix} 1 & m & k/2\ell \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \mid m, n, k \in \mathbb{Z} \right\}, \]

where \( \ell \) is a positive integer. These choices do not lose the essential features in treating with the spectrum of the Laplacian on the quotient space, \( H_3/\Gamma_\ell \), so called, Heisenberg manifolds, since the spectrum is given more or less in a similar form ([6], we state them later).
Then the inverse image of \( \Gamma_\ell \) by the exponential map is
\[
\exp^{-1} (\Gamma_\ell) = \left\{ \begin{pmatrix} 0 & m & k/2\ell \\ 0 & 0 & n \\ 0 & 0 & 0 \end{pmatrix} \mid m, n, k \in \mathbb{Z} \right\},
\]
which is a direct sum of two uniform lattices \( \Gamma_B \) in \( \mathfrak{g}_+ \oplus \mathfrak{g}_- \) and \( \Gamma_V (\ell) \) in \( \mathfrak{z} \) such that
\[
\Gamma_B = \left\{ \begin{pmatrix} 0 & m & 0 \\ 0 & 0 & n \\ 0 & 0 & 0 \end{pmatrix} \mid m, n \in \mathbb{Z} \right\}
\]
and
\[
\Gamma_V (\ell) = \left\{ \begin{pmatrix} 0 & 0 & k/2\ell \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid k \in \mathbb{Z} \right\}.
\]

The kernel \( K_t (g, \tilde{g}) \) of \( e^{-t\Delta} \), the heat kernel on the Heisenberg group \( H_3 \), is given by
\[
K_t (g, \tilde{g}) = K_t (x, y, z; \tilde{x}, \tilde{y}, \tilde{z})
\]
\[
= (2\pi)^{-2} \int_{-\infty}^{+\infty} e^{\sqrt{-1} \eta \cdot \{ \tilde{z} - z + \frac{1}{2}(\tilde{x}y - xy) \}} \cdot e^{-t|\eta|^2} \times
\]
\[
\times \frac{|\eta|}{2 \sinh t |\eta|} e^{-\frac{|\eta| \cosh t |\eta|}{4 \sinh t |\eta|} \cdot ((x-\tilde{x})^2 + (y-\tilde{y})^2)} d\eta,
\]
where we regard \( \eta \in \mathfrak{z}^* = \mathbb{R}Z^* \cong \mathbb{R} \) and \( g = xX_1 + yY_1 + zZ, \tilde{g} = \tilde{x}X_1 + \tilde{y}Y_1 + \tilde{z}Z \in \mathfrak{h}_3 \cong H_3 \) through the identification by the exponential map \( \exp : \mathfrak{h}_3 \rightarrow H_3 \).

Then the heat kernel \( k_{H_3/\Gamma_\ell} (t; [g], [\tilde{g}]) \) on the Heisenberg manifold \( H_3/\Gamma_\ell \) is expressed by making use of the heat kernel \( K_t (g, \tilde{g}) \) on the whole group, because of the invariance \( K_t (\gamma \cdot g, \gamma \cdot \tilde{g}) = K_t (g, \tilde{g}), \gamma \in H_3 \):
\[
k_{H_3/\Gamma_\ell} (t; [g], [\tilde{g}]) = \sum_{\gamma \in \Gamma_\ell} K_t (\gamma \cdot g, \tilde{g}).
\]

Its trace is calculated in the following form:

**Theorem 1.1.** ([9])
\[
\int_{H_3/\Gamma_\ell} k_{H_3/\Gamma_\ell} (t; [g], [\tilde{g}]) dg = \sum_{\gamma \in \Gamma_\ell} \int_{F_\ell} K_t (\gamma \cdot g, g) dg
\]
\[
= \sum_{\mu \in \Gamma_V (\ell)^* \setminus \{0\}} \sum_{m=0}^{\infty} Vol (\mathfrak{g}_+ \oplus \mathfrak{g}_- / \Gamma_B) \cdot |\mu| \cdot e^{-t \left\{ 4\pi^2 \|\mu\|^2 + 2\pi(2m+1)\|\mu\| \right\}}
\]
\[
+ \sum_{\nu \in \Gamma_B^*} e^{-4\pi^2 t \|\nu\|^2}.
\]
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Here $F_\ell$ denotes a fundamental domain of the uniform discrete subgroup $\Gamma_\ell$ and $\Gamma^*_B$ and $\Gamma^*_V(\ell)^*$ are dual lattices, i.e.,

$$\Gamma^*_B = \{ \nu \in (g_+ \oplus g_-)^* \mid \nu(\gamma) \in \mathbb{Z}, \text{ for any } \gamma \in \Gamma_B \}$$

$$\cong \{ \nu = \nu_1 X_1^* + \nu_2 Y_1^* \mid \nu, \{ X_1^*, Y_1^* \} \text{ are dual basis of } g_+ \oplus g_- \},$$

$$\Gamma^*_V(\ell)^* = \{ \mu \in (g_+ \oplus g_-/\Gamma_B) \mid \mu(\gamma) \in \mathbb{Z}, \text{ for any } \gamma \in \Gamma_V(\ell) \}$$

$$\cong \{ \mu = 2\ell \cdot k Z_1^* \mid k \in \mathbb{Z} \}, \| \mu \| = 2\ell |k|.$$
Then the Mellin transformation of the trace of the heat kernel,

\[ \frac{1}{\Gamma(s)} \int_0^\infty \left( \int_M k_M(t; x, x)dx - 1 \right) t^{s-1}dt = \sum_{\lambda \neq 0} \frac{m_\lambda}{\lambda^s}, \]

\( m_\lambda = \text{multiplicity of the eigenvalue } \lambda, \)

is meromorphically continued to the whole complex plane with only poles of order one (at most) at \( s = n/2, n/2 - 1, \ldots \), especially at \( s = 0 \) it is holomorphic. We put this function as \( \zeta_M(s) \) and call the spectral zeta-function of the Riemannian manifold \( M \). Then we can regard the value \( e^{-\zeta_M'(0)} \)

as a "determinant" (= product of non-zero eigenvalues) of the Laplacian \( \Delta_M \) acting on the space of functions orthogonal to the space of constant functions\( (\text{[14]}, \text{[13]}, \text{[7]} \) and call it as zeta-regularized determinant of the Laplacian. We denote it by \( \text{Det} \Delta_M \). Of course it can be defined in a same way for more general elliptic operators.

In our case of the three dimensional Heisenberg manifold \( M(\ell) = H_3/\Gamma_\ell \) we put

\[ \sum_{\mu \in \Gamma_V(\ell)^*, \mu \neq 0} \sum_{m=0}^\infty \left( \text{Vol}(g_+ \oplus g_- / \Gamma_B) \cdot |\mu| \right) \cdot e^{-t(4\pi^2 ||\mu||^2 + 2\pi(2m+1)||\mu||)} \]

\[ + \sum_{\nu \in \Gamma_B^*} e^{-4\pi^2 t||\nu||^2} \]

\( Z_{M(\ell)}(t) = Z_V(t) + Z_{T^2}(t), \)

then the second term is the trace of the heat kernel of the flat torus \( T^2 \cong (g_+ \oplus g_-) / \Gamma_B, \) and so

\[ \zeta_M(\ell)(s) = \frac{1}{\Gamma(s)} \int_0^\infty (Z_{M(\ell)}(t) - 1)t^{s-1}dt \]

\[ = \frac{1}{\Gamma(s)} \int_0^\infty Z_V(t)t^{s-1}dt + \frac{1}{\Gamma(s)} \int_0^\infty (Z_{T^2}(t) - 1)t^{s-1}dt = \zeta_V(s) + \zeta_{T^2}(s). \]

Hence we have

**Proposition 2.1.**

\[ e^{-\zeta_M(\ell)'(0)} = e^{-\zeta_V'(0)} \cdot e^{-\zeta_{T^2}'(0)}. \]

The value \( \zeta_{T^2}'(0) \) is given by the formula called "Kroneker’s second limit formula":

**Proposition 2.2.** (\( \text{[3]}, \text{[7]}, \text{[13]} \)) \( \text{Det} \Delta_{T^2} = e^{-\zeta_{T^2}'(0)} = e^{-\frac{\pi}{2}} \prod_{k=-\infty}^{\infty} \left( 1 - e^{-2\pi |k|} \right)^2. \)

We give an elementary proof of this formula and expressions of the zeta-reguralized determinant for the three and four dimensional flat tori in \( \text{§7}. \)
So in this section we only consider the value \( \zeta_V'(0) \). The Mellin transform of the function \( Z_V(t) \) is

\[
\frac{1}{\Gamma(s)} \int_0^\infty Z_V(t) t^{s-1} dt = 4\ell \sum_{n=1}^\infty \sum_{m=0}^\infty \frac{(4\pi \ell)^{2s}(n^2 + \frac{n}{4\pi \ell}(2m+1))^s}{\Gamma(s)\Gamma(s-1) \cdot (4\pi \ell)^{2s}} \int_0^\infty \int_0^\infty \frac{x+y}{e^{x+y} - 1} \frac{x}{e^x - e^{-\frac{x}{4\pi \ell}}} (xy)^{s-2} dxdy.
\]

Put

\[
f(x, y) = \frac{x+y}{e^{x+y} - 1} \frac{x}{e^x - e^{-\frac{x}{4\pi \ell}}},
\]

then by the transformation

\[
\begin{cases}
  (x, y) \mapsto (u, v), & u = x \text{ and } v = \frac{y}{x} \text{ on the domain } x > y \\
  (x, y) \mapsto (v, u), & u = y \text{ and } v = \frac{x}{y} \text{ on the domain } x < y
\end{cases}
\]

we have

\[
\int_0^\infty \int_0^\infty f(x, y) \frac{(xy)^s - 2}{x + y} dxdy = \int_0^1 \left( \int_0^\infty f(u, uv) u^{2s-4} du \right) \frac{v^{s-2}}{1+v} dv + \int_0^1 \left( \int_0^\infty f(uv, u) u^{2s-4} du \right) \frac{v^{s-2}}{1+v} dv
\]

\[
= \int_0^1 \left( \int_0^\infty G(u, v) u^{2s-4} du \right) \frac{v^{s-2}}{1+v} dv,
\]

where we put

\[
G(u, v) = f(u, uv) + f(uv, u).
\]

Let \( g(x) = \frac{x}{e^x - 1} \) and \( h(x) = \frac{2x}{e^x - e^{-x}} = \frac{x}{\sinh x} \),

then

\[
G(u, v) = 2\pi \ell \cdot g(u (1 + v)) \left( h\left( \frac{u}{4\pi \ell} \right) + h\left( \frac{uv}{4\pi \ell} \right) \right).
\]

Since the functions \( g \) and \( h \) are rapidly decreasing on the positive real axis, we have,

**Proposition 2.3.** For any integers \( k \) and \( l \)

\[
\lim_{u \to \infty} \frac{\partial^{k+l} G}{\partial u^k \partial v^l}(u, v) = 0
\]

uniformly for \( v \in [0, 1] \).
Next we consider the behavior of the functions \( \frac{\partial^{k+l} G}{\partial u^k \partial v^l} \), when \( u \downarrow 0 \).

Let

\[
g(x) = \frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \alpha_k x^k, \quad |x| < 2\pi
\]

and

\[
h(x) = \frac{2x}{e^x - e^{-x}} = \frac{x}{\sinh x} = \frac{\sqrt{1-x}}{x} = \sum_{k=0}^{\infty} (-1)^k \beta_{2k} x^{2k}, \quad |x| < \pi.
\]

Note that \( \alpha_0 = 1, \alpha_1 = -1/2, \alpha_2 = 1/12, \alpha_{2i+1} = 0 \) for \( i = 1, 2, 3, \cdots \) and \( \beta_0 = 1, \beta_2 = 1/6, \beta_4 = 7/360 \). The coefficients are expressed in the following forms with Bernoulli numbers \( B_{2k} = \frac{2(2k)!}{(2\pi)^{2k}} \zeta(2k) \):

\[
\beta_{2k} = \frac{(2^{2k} - 2) B_{2k}}{(2k)!},
\]

\[
\alpha_{2k} = \frac{B_{2k}}{(2k)!}.
\]

For \( |u| < \pi \) and \( v \in [0,1] \), the function \( G(u, v) \) is expanded as follows:

\[
G(u, v) = 2\pi \ell \sum_{n=0}^{\infty} \left( \sum_{i+2j=n} (-1)^j \frac{\alpha_i \beta_{2j}}{(4\pi \ell)^{2j}} (1 + v)^i (1 + v^{2j}) \right) \cdot u^n = 2\pi \ell \sum_{n=0}^{\infty} P_n(v) u^n,
\]

where we denote the polynomial \( P_n(v) \)

\[
P_n(v) = \sum_{i+2j=n} (-1)^j \frac{\alpha_i \beta_{2j}}{(4\pi \ell)^{2j}} (1 + v)^i (1 + v^{2j}).
\]

**Proposition 2.4.** For \( v \in [0,1] \)

\[
\lim_{u \downarrow 0} \frac{\partial^{k+l} G}{\partial u^k \partial v^l}(u, v) = 2\pi \ell \cdot k! \cdot \frac{d^l P_k(v)}{dv^l}.
\]

Let \( g(v, s) \) be a sufficiently many times differentiable function defined on a domain in \( \mathbb{R} \times \mathbb{C} \) including \([0,1] \times D\), where \( D = \{ s \in \mathbb{C} \mid \Re(s) > -\epsilon \} \) (\( \epsilon > 0 \) and fixed) and \( g \) is holomorphic on the domain \( D \) for each fixed \( t \in [0,1] \).

**Proposition 2.5.** The function defined by the integral

\[
\int_0^1 g(v, s) v^{s-2} dv
\]

has the Laurent expansion at \( s = 0 \) as

\[
\int_0^1 g(v, s) v^{s-2} dv = \frac{R_{-1}}{s} + R_0 + O(s),
\]

where \( R_{-1} \) and \( R_0 \) are given by the formulas:
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\[ R_{-1} = \frac{\partial g}{\partial v}(0, 0) \]

and

\[ R_0 = -\int_0^1 \frac{\partial^2 g}{\partial v^2}(v, 0) \log vd + \int_0^1 \frac{\partial^2 g}{\partial v}(v, 0) \log vd + \int_0^1 \frac{g(v, 0)}{1 + v} \]

By applying this to the function of the form \( \frac{g(v, s)}{1 + v} \) we have the Laurent expansion of the function

\[ \int_0^1 g(v, s) \frac{v^{s-2}}{1 + v} dv \]

as

**Corollary 2.6.**

\[ \int_0^1 g(v, s) \frac{v^{s-2}}{1 + v} dv = \left( \frac{\partial g}{\partial v}(0, 0) - g(0, 0) \right) \frac{1}{s} \]

\[ - \int_0^1 \frac{\partial^2 g}{\partial v^2}(v, 0) \log vd + \int_0^1 \frac{\partial g}{\partial v}(v, 0) \log vd + \int_0^1 \frac{g(v, 0)}{1 + v} \]

\[ + \frac{\partial^2 g}{\partial s \partial v}(0, 0) - \frac{\partial g}{\partial s}(0, 0) + \frac{\partial g}{\partial v}(0, 0) - g(1, 0) + O(1) \]

When we restrict the variable \( s \) in the domain \( \text{Re}(s) > 3/2 \), we have

\[ (2.6) \quad \int_0^\infty G(u, v) u^{2s-4} du = \frac{1}{(2s-3)(2s-2)(2s-1)2s} \int_0^\infty \frac{\partial^4 G(u, v)}{\partial u^4} u^{2s} du. \]

Now by Corollary (2.6) when we put

\[ \int_0^1 \left( \int_0^\infty G(u, v) u^{2s-4} du \right) \frac{v^{s-2}}{1 + v} dv \]

\[ = \frac{1}{(2s-3)(2s-2)(2s-1)2s} \int_0^1 \left( \int_0^\infty \frac{\partial^4 G(u, v)}{\partial u^4} u^{2s} du \right) \frac{v^{s-2}}{1 + v} dv \]

\[ = \frac{1}{(2s-3)(2s-2)(2s-1)2s} \left\{ \frac{R_{-1}}{s} + R_0 + O(1) \right\}, \]

then

**Proposition 2.7.**

\[ R_{-1} = 0, \]

\[ R_0 = 2 \int_0^\infty \frac{\partial G(u, 0)}{\partial v} \log u du - 2 \int_0^\infty \frac{\partial^4 G(u, 0)}{\partial u^4} \log u du \]

\[ = -4\pi \ell \int_0^\infty \frac{d^4}{du^4} \left( h \left( \frac{u}{2} \right)^2 \left( h \left( \frac{u}{4\pi \ell} \right) + 1 \right) \right) \log u du. \]
Proof. First we show
\[
R_{-1} = \int_{0}^{\infty} \frac{\partial^{5} G(u, 0)}{\partial v \partial u^{4}} \, du - \int_{0}^{\infty} \frac{\partial^{4} G(u, 0)}{\partial u^{4}} \, du
\]
\[
= \frac{\partial^{4} G(u, 0)}{\partial v \partial u^{3}} \big|_{0}^{\infty} - \frac{\partial^{3} G(u, 0)}{\partial u^{3}} \big|_{0}^{\infty}
\]
\[
= \frac{2\pi \ell}{(4\pi \ell)^{2}} \cdot g'(0) h''(0) - \frac{2\pi \ell}{(4\pi \ell)^{2}} \cdot (1 + v) \cdot g'(0) h''(0) \big|_{v=0} = 0.
\]
Next we calculate \( R_{0} \):
\[
R_{0} = -\int_{1}^{0} \left( \int_{0}^{\infty} \frac{\partial^{6} G(u, v)}{\partial v^{2} \partial u^{4}} \, du \right) \log v \, dv + \int_{1}^{0} \left( \int_{0}^{\infty} \frac{\partial^{5} G(u, v)}{\partial v \partial u^{4}} \, du \right) \log v \, dv
\]
\[
+ \frac{1}{1 + v} \int_{0}^{\infty} \frac{\partial^{3} G(u, 0)}{\partial v \partial u^{4}} \, du - \int_{0}^{\infty} \frac{\partial^{4} G(u, 0)}{\partial u^{4}} \, du
\]
\[
= \int_{0}^{1} \frac{\partial^{5} G(0, v)}{\partial v^{2} \partial u^{3}} \log v \, dv - \int_{0}^{1} \frac{\partial^{4} G(0, v)}{\partial v^{2} \partial u^{3}} \log v \, dv - \int_{0}^{1} \frac{\partial^{3} G(0, v)}{\partial u^{3}} \frac{1}{1 + v} \, dv
\]
\[
+ 2 \int_{0}^{\infty} \frac{\partial^{5} G(u, 0)}{\partial v \partial u^{4}} \log u \, du - 2 \int_{0}^{\infty} \frac{\partial^{4} G(u, 0)}{\partial u^{4}} \log u \, du
\]
\[
= 2\pi \ell \cdot 3! \cdot \int_{0}^{1} \left( \frac{d^{2} P_{3}(v)}{dv^{2}} - \frac{dP_{3}(v)}{dv} \right) \log v - \frac{P_{3}(v)}{1 + v} \, dv
\]
\[
+ 2\pi \ell \cdot 3! \cdot \left\{ -\frac{dP_{3}(0)}{dv} + P_{3}(1) \right\}
\]
\[
+ 2 \int_{0}^{\infty} \frac{\partial^{5} G(u, 0)}{\partial v \partial u^{4}} \log u \, du - 2 \int_{0}^{\infty} \frac{\partial^{4} G(u, 0)}{\partial u^{4}} \log u \, du
\]
\[
= 2\pi \ell \cdot 3! \cdot (-3) \cdot P_{3}(0) + 2\pi \ell \cdot 3! \cdot 3 \cdot P_{3}(0)
\]
\[
+ 2 \int_{0}^{\infty} \frac{\partial^{5} G(u, 0)}{\partial v \partial u^{4}} \log u \, du - 2 \int_{0}^{\infty} \frac{\partial^{4} G(u, 0)}{\partial u^{4}} \log u \, du
\]
\[
= 2 \int_{0}^{\infty} \frac{\partial^{5} G(u, 0)}{\partial v \partial u^{4}} \log u \, du - 2 \int_{0}^{\infty} \frac{\partial^{4} G(u, 0)}{\partial u^{4}} \log u \, du
\]
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\[-4\pi\ell \int_0^\infty \frac{d^4 u}{du^4} \left( h \left( \frac{u}{2} \right)^2 \left( h \left( \frac{u}{4\pi\ell} \right) + 1 \right) \right) \log u du.\]

□

Summing up we have

\[\frac{1}{\Gamma(s)} \cdot \int_0^\infty Z_V(t) t^{s-1} dt = 4 \cdot \ell \cdot (s-1) \cdot \frac{s^2}{(4\pi\ell)^{2s}} \cdot \frac{1}{(2s-3)(2s-2)(2s-1)2s} \left\{ R_0 + O(s) \right\},\]

and the zeta-regularized determinant of the Laplacian on the Heisenberg manifold \(H_3/\Gamma_\ell\) is given by the formula:

**Theorem 2.8.**

\[\text{Det} \Delta_{H_3/\Gamma_\ell} = \text{Det} \Delta_{T^2} \cdot e^{-\ell R_0/3} = e^{-\frac{\pi}{3}} \cdot \prod_{k=-\infty}^{\infty} (1 - e^{-2\pi|k|})^2 \cdot e^{\frac{4\pi^2}{3} \int_0^\infty \frac{d^4 u}{du^4} \left( \frac{u}{2\sinh \frac{u}{2}} \right)^2 \left( \frac{u}{4\pi\ell} \right)^2 \log u du} \]

(2.7)

**Corollary 2.9.** When \(\ell \to \infty\), \(\text{Det} \Delta_{H_3/\Gamma_\ell} \to 0\).

**Proof.** Since

\[\lim_{\ell \to \infty} \int_0^\infty \frac{d^4 u}{du^4} \left\{ h \left( \frac{u}{2} \right)^2 \left( h \left( \frac{u}{4\pi\ell} \right) + 1 \right) \right\} \log u du \]

(2.8)

we only determine the sign of this integral. For this purpose we decompose the integral (2.8) in the form

\[I = \int_0^\infty \frac{d^4 u}{du^4} \left\{ h \left( \frac{u}{2} \right)^2 \right\} \log u du \]

\[= \int_r^\infty \frac{d^4 u}{du^4} \left\{ h \left( \frac{u}{2} \right)^2 - T_0 - T_1 u - T_2 u^2 - T_3 u^3 \right\} \log u du \]

\[+ \int_r^\infty \frac{d^4 u}{du^4} \left\{ h \left( \frac{u}{2} \right)^2 \right\} \log u du,\]

where

\[T_0 = h(0)^2 = 1,\]

\[T_1 = \frac{d}{du} h \left( \frac{u}{2} \right) \bigg|_{u=0} = 0\]

\[T_2 = \frac{1}{2!} \frac{d^2}{du^2} \left\{ h \left( \frac{u}{2} \right)^2 \right\} \bigg|_{u=0} = -\frac{1}{12}\]
Then

\[ I = \frac{2}{r^3} - \frac{1}{2r} - 3! \left\{ \int_0^r \left( h \left( \frac{u}{2} \right)^2 - 1 + \frac{1}{12} u^2 \right) u^{-4} du + \int_r^\infty h \left( \frac{u}{2} \right)^2 u^{-4} du \right\}. \]

Next we prove that \( h(u/2)^2 - 1 + \frac{1}{12} u^2 \) takes positive values on the interval \([0,2]\). For this purpose, recall that \( h(u/2) = \frac{u}{\sinh u} \) is expanded for \( u \in (-\pi, \pi) \) as

\[ h(u) = \sum_{k=0}^\infty (-1)^k \beta_{2k} u^{2k}, \]

with the coefficients

\[ \beta_{2k} = \frac{2(2^{2k} - 2)}{(2\pi)^{2k}} \zeta(2k). \]

Then

\[ \frac{\beta_{2k+2}}{\beta_{2k}} < \frac{4 - \frac{2}{2\pi}}{(2\pi)^2(1 - \frac{2}{2\pi})} \frac{\zeta(2k + 2)}{\zeta(2k)} < \frac{3}{(2\pi)^2} < \frac{1}{12}. \]

and

\[ h(x)^2 = \sum_{k=1}^\infty (-1)^k \gamma_{2k} x^{2k}, \]

with positive coefficients \( \gamma_{2k} \) such that

\[ \gamma_{2k} = \sum_{l=0}^k \beta_{2k-2l} \beta_{2l}. \]

Hence

\[ \gamma_{2k} - \gamma_{2k+2} = \beta_0 (\beta_{2n} - 2\beta_{2n+2}) + \sum_{l=1}^k \beta_{2l} (\beta_{2k-2l} - \beta_{2k+2-2l}) \]

so \( \gamma_{2k} > \gamma_{2k+2} \) always.

From these estimates we see that the function \( h(u/2)^2 - 1 + \frac{1}{12} u^2 \) takes positive values on the interval \([0,2]\), since \( h(u/2)^2 - 1 + \frac{1}{12} u^2 = \sum_{k=2}^\infty (-1)^k \gamma_{2k} \left( \frac{u}{2} \right)^{2k} \) is the form of the alternative sum consisting of decreasing positive sequences when \( u \in [0,2] \). Now we take \( r = 2 \), then we see that \( I < 0 \), hence we have the desired result. \(\square\)
3. Heat Kernel Asymptotics

As an application of the formula (2.1) we calculate the heat kernel asymptotics for the three dimensional Heisenberg manifolds, which are given in terms of Bernoulli numbers.

We know that the heat kernel \( k_M(t; x, x) \) has the asymptotic expansion:

\[
k_M(t; x, x) \sim \frac{1}{(4\pi t)^{n/2}} \left\{ c_0(x) + c_1(x)t + c_2(x)t^2 + \cdots \right\}, \quad t \downarrow 0,
\]

\( n = \text{dimension of the manifold } M \).

Of course the coefficients are given in terms of quantities coming from metric tensors, but here we calculate the values \( c_k = \int_M c_k(x)dx \) explicitly by the method of analytic continuation by making use of the formula (2.1), since we know that the Mellin transform of the heat kernel has poles of order one (at most) at the points \( n/2 - k, \ k = 0, 1, 2, \cdots \), and the residue at the pole \( n/2 - k \) is given by the integral \( \int_M c_k(x)dx \).

In our cases the trace of the heat kernel \( Z_{H_3/\Gamma} (t) = \int k_{H_3/\Gamma}(t; [g], [g])dg \) is expanded as

\[
\int_{H_3/\Gamma} k_{H_3/\Gamma}(t; [g], [g])dg = Z_V(t) + Z_{T^2}(t) \sim \frac{1}{(4\pi t)^{3/2}} (c_0 + c_1 t + c_2 t^2 + \cdots),
\]

and the second term \( Z_{T^2}(t) \) is that corresponding to the two dimensional flat torus. So that it is enough to consider the Mellin transform of the first term \( Z_V(t) \):

\[
\frac{1}{\Gamma(s)} \int_0^\infty Z_V(t)t^{s-1}dt = 4\ell \sum_{n=1}^\infty \sum_{m=0}^\infty \frac{(4\pi \ell)^{2s(n^2 + n/4\ell(2m + 1))}}{(4\pi)^{2s}}
\]

\[
= \frac{4\ell}{\Gamma(s)\Gamma(s-1)} \cdot \frac{1}{(4\pi \ell)^{2s}} \cdot \int_0^1 \left( \int_0^\infty G(u, v)u^{2s-4}du \right) \frac{v^{s-2}}{1+v}dv
\]

\[
= \frac{4\ell}{\Gamma(s)\Gamma(s-1)} \cdot \frac{1}{(4\pi \ell)^{2s}} \times
\]

\[
\times \left( \int_0^1 \left( \int_0^1 G(u, v)u^{2s-4}du \right) \frac{v^{s-2}}{1+v}dv + \int_0^1 \left( \int_1^\infty G(u, v)u^{2s-4}du \right) \frac{v^{s-2}}{1+v}dv \right).
\]

(3.1)

Here,

\[
G(u, v) = \frac{u(v+1)}{e^{u(v+1)} - 1} \cdot \left\{ \frac{u}{e^{u/4\pi} - e^{-u/4\pi}} + \frac{uv}{e^{u/4\pi} - e^{-u/4\pi}} \right\}.
\]
Since the function defined by the integral
\[ \int_{1}^{\infty} G(u, v) u^{2s-4} du \]
is holomorphic for any \( s \in \mathbb{C} \) and so the second term in the above expression has poles of order at most one at the points \( s = 1, 0, -1, -2, \ldots \). Hence residues at these points must vanish. Consequently it is enough to consider the first term,

\[
\frac{4\ell}{\Gamma(s)\Gamma(s - 1)} \cdot \frac{1}{(4\pi \ell)^{2s}} \int_{0}^{1} \left( \int_{1}^{\infty} G(u, v) u^{2s-4} du \right) \frac{v^{s-2}}{1 + v} dv,
\]

for calculating the residues at the poles \( s = 3/2 - k, k = 0, 1, 2, 3 \cdots \).

Then we have
\[
\int_{0}^{1} \left( \int_{0}^{1} G(u, v) u^{2s-4} du \right) \frac{v^{s-2}}{1 + v} dv = 4\pi \ell \cdot \int_{0}^{1} \left( \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \alpha_i \beta_{2j}(u(1 + v))^i \left( \left( \frac{u}{4\pi \ell} \right)^{2j} + \left( \frac{uv}{4\pi \ell} \right)^{2j} \right) u^{2s-4} du \right) \frac{v^{s-2}}{1 + v} dv = 4\pi \ell \cdot \sum_{m=0}^{\infty} \int_{0}^{1} \frac{1}{2s - 3 + m} \sum_{i+2j=m} (-1)^j \alpha_i \beta_{2j} \frac{1}{(4\pi \ell)^{2j}} (1 + v)^i (1 + v^{2j}) \frac{v^{s-2}}{1 + v} dv.
\]

To calculate the residues at the poles \( s = 3/2 - k, k = 0, 1, 2, \ldots \), again it is enough to consider the terms corresponding to \( m = \text{even} = 2k \) in the sum.

Let \( W_{i,j}(s) = \int_{0}^{1} (1 + v)^{2i}(1 + v^{2j}) \frac{v^{s-2}}{1 + v} dv \), then of course \( W_{i,j}(s) \) is meromorphically continued to the whole complex plane and we have

**Lemma 3.1.** \( W_{i,j}(s) \) is holomorphic at \( s = 3/2 - (i + j) \) and for \( i > 0 \), \( W_{i,j}(3/2 - (i + j)) = 0 \).

**Proof.** Let \( i > 0 \), then
\[
\int_{0}^{1} (1 + v)^{2i}(1 + v^{2j}) \frac{v^{s-2}}{1 + v} dv = \sum_{r=0}^{2i-1} C_r \left( \frac{1}{r + s - 1} + \frac{1}{r + 2j + s - 1} \right).
\]
So at the point \( s = 3/2 - (i + j) \) it takes the form
\[
\sum_{r=0}^{2i-1} C_r \frac{1}{r + 1/2 - (i + j)}.
\]
\[ + \sum_{r=0}^{2i-1} 2i-1C_{2i-1-r} \frac{1}{2i - 1 - r + 2j + 3/2 - (i + j) - 1}. \]

Hence we have \( W_{i,j}(3/2 - (i + j)) = 0 \)

Now by Lemma (3.1), to calculate the residues of the function (2.6) at the points \( s = 3/2 - k \), it is enough to consider the function

\[ 4\pi \ell \sum_{k=0}^{\infty} \alpha_0 \beta_{2k} \frac{1}{(4\pi \ell)^{2k}} \frac{1}{2s - 3 + 2k} \int_0^1 \frac{1 + v^{2k}}{1 + v} v^{s-2} dv. \]

Let \( W_k(s) = \int_0^1 \frac{1 + v^{2k}}{1 + v} v^{s-2} dv \), then the value at \( s = 3/2 - k \),

\[ W_k(3/2 - k) = W_k \]

is given one by one by the following lemma

**Lemma 3.2.**

\[ W_k(3/2 - k) = \int_0^1 \frac{1 + v^{2k}}{1 + v} v^{s-2} dv|_{s=3/2-k} \]

\[ = \int_0^1 \frac{1 + v^{2k}}{1 + v} v^{s-2} dv|_{s=3/2-k} + \int_0^1 \frac{v^{k-1/2}}{1 + v} dv \]

\[ = \sum_{r=0}^{k-1} \frac{(-1)^r}{r - k + 1/2} + (-1)^{k} \int_0^1 \frac{1}{(1 + v)^{1/2}} dv \]

\[ + \int_0^1 \frac{v^{k}}{(1 + v)^{1/2}} dv \]

\[ = \sum_{r=0}^{k-1} \frac{(-1)^r}{r - k + 1/2} + (-1)^{k} \frac{\pi}{2} + 2 \sum_{r=0}^{k-1} kC_r \left( \frac{k - r}{k(2r + 1)} - J_r \right), \]

where we put

\[ J_r = \int_0^1 \frac{\theta^{2r}}{1 + \theta^2} d\theta. \]

\( J_r \) is determined by the formula

\[ J_r = \sum_{i=0}^{r-1} rC_i \left( \frac{r - i}{r(2i + 1)} - J_i \right), J_0 = \pi/4. \]

Also this value \( W_k = W_k(3/2 - k) \) is given by the formula:

\[ \sum_{r=0}^{k-1} \frac{(-1)^r}{r - k + 1/2} + (-1)^{k} \frac{\pi}{2} + \sum_{j=0}^{\infty} \frac{(-1)^{j}}{j + k + 1/2}. \]

Finally we have
Proposition 3.3. The residue $c_k = \int_{H_3/\Gamma_3} c_k(g) dg$ of the spectral zeta-function $\zeta_{H_3/\Gamma_3}(s)$ at the point $s = 3/2 - k$, $k = 0, 1, 2, \ldots$, is equal to

$$\frac{4\pi}{\Gamma(3/2 - k)\Gamma(1/2 - k)} \frac{\beta_{2k}}{2 \cdot (4\pi \ell)^2} \cdot W_k(3/2 - k).$$

Note that

$$\beta_{2k} = \frac{2^{2k} - 2}{(2k)!} B_{2k},$$

where $B_{2k} = \frac{2(2k)!}{(2\pi)^{2k}} \zeta(2k)$ is the Bernoulli number.

4. Five dimensional Heisenberg manifolds

So far we only considered three dimensional cases and illustrated the procedure to calculate the determinants and heat kernel asymptotics somehow precisely. These indicate that our method for calculating the determinants and heat kernel asymptotics for higher dimensional Heisenberg manifolds would also be valid. Even so the calculations and the results are so complicated, here we state the results for the cases of 5-dimensional Heisenberg manifolds.

Let $\mathfrak{h}_5$ be the 5-dimensional Heisenberg Lie algebra:

$$\mathfrak{h}_5 = \mathfrak{g}_+ \oplus \mathfrak{g}_- \oplus \mathfrak{z} = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & z \\ 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid x_i, y_i, z \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

where $\mathfrak{z}$ is the center and $[\mathfrak{g}_+, \mathfrak{g}_-] = \mathfrak{z}$.

The corresponding Lie group $H_5$ is realized as

$$H_5 = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x_i, y_i, z \in \mathbb{R} \right\}.$$

(4.1)

As in the three dimensional cases we only consider a left invariant Riemannian metric defined from such an inner product on the Lie algebra $\mathfrak{h}_5$ that $X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $Y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $Y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and
$Z_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

are orthonormal basis of $\mathfrak{h}_5$.

Also as in the three dimensional cases let us take uniform discrete subgroups $\Gamma_\ell$ ($\ell \in \mathbb{N}$) of the form

$$\Gamma_\ell = \begin{cases} \begin{pmatrix} 1 & m_1 & m_2 & \frac{k}{2\ell} \\ 0 & 1 & 0 & n_1 \\ 0 & 0 & 1 & n_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} & | n_i, m_i, k \in \mathbb{Z} \end{cases}.$$

We identify $H_5$ and $\mathfrak{h}_5$ by the exponential map

$$\exp : \mathfrak{h}_5 \to H_5$$

$$\exp : g = \begin{pmatrix} 0 & x_1 & x_2 & z \\ 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies \begin{pmatrix} 1 & x_1 & x_2 & z + 1/2 \sum x_i y_i \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then the uniform discrete subgroup $\Gamma_\ell \subset H_5$ is identified with a direct sum of two lattices $\Gamma_B \subset \mathfrak{g}_+ \oplus \mathfrak{g}_-$ and $\Gamma_V(\ell) \subset \mathfrak{z}$:

$$\Gamma_B = \begin{cases} \begin{pmatrix} 0 & m_1 & m_2 & 0 \\ 0 & 0 & 0 & n_1 \\ 0 & 0 & 0 & n_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} & | m_i, n_i \in \mathbb{Z} \end{cases},$$

$$\Gamma_V(\ell) = \begin{cases} \begin{pmatrix} 0 & 0 & 0 & k/\ell \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & | k \in \mathbb{Z} \end{cases},$$

$$\exp (\Gamma_V(\ell) + \Gamma_B) = \Gamma_\ell$$

The heat kernel on the whole group $H_5$ is given as the following form:

$$K_t(g, \tilde{g}) = K_t(x, y, z; \tilde{x}, \tilde{y}, \tilde{z})$$

$$= (2\pi)^{-3} \int_{-\infty}^{+\infty} e^{\sqrt{-1} \eta \left\{ \tilde{z} - z + \frac{1}{4} ([\tilde{x}, y] - [x, \tilde{y}]) \right\} } \cdot e^{-\ell |\eta|^2}$$
\[
\times \left( \frac{\lvert \eta \rvert}{2 \sinh t \lvert \eta \rvert} \right)^2 \prod_{i=1}^{2} e^{-\frac{\cosh \eta}{\sinh t \eta}} \left\{ (x_i - \tilde{x}_i)^2 + (y_i - \tilde{y}_i)^2 \right\} \, d\eta.
\]

where we regarded \( \eta \in \mathbb{H} = \mathbb{R}^* \cong \mathbb{R} \) and \( g = (x, y, z) = x_1X_1 + x_2X_2 + y_1Y_1 + y_2Y_2 + zZ_1 \), and similar to \( \tilde{g} \) through the exponential map.

Then the heat kernel \( k_{H_5/\Gamma_\ell}(t; [g], [\tilde{g}]) \) on the Heisenberg manifold \( M_\ell = H_5/\Gamma_\ell \), is expressed as :

\[
(4.4) \quad k_{H_5/\Gamma_\ell}(t; [g], [\tilde{g}]) = \sum_{\gamma \in \Gamma_\ell} K_t(\gamma \cdot g, \tilde{g}).
\]

and its trace is calculated in the following form:

**Theorem 4.1.** \((9)\)

\[
Z_\ell(t)
= \int_{H_5/\Gamma_\ell} k_{H_5/\Gamma_\ell}(t; [g], [\tilde{g}]) \, dg = \sum_{\gamma \in \Gamma_\ell} \int_{F_\ell} K_t(\gamma \cdot g, \tilde{g}) \, dg
= \sum_{\mu \in \Gamma_V(\ell)^*} \sum_{\mu \neq 0} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \text{Vol}(\mathfrak{g}_+ \oplus \mathfrak{g}_- / \Gamma_B) \|\mu\|^2 \cdot e^{-t \left\{ 4\pi^2 \|\mu\|^2 + 2\pi \sum_{i=1}^{2} (2m_i+1)\|\mu\| \right\}}
+ \sum_{\nu \in \Gamma_B^*} e^{-4\pi^2 t \|\nu\|^2}
= Z_V(t) + Z_{T^4}(t).
\]

Here \( F_\ell \) denotes a fundamental domain of the uniform discrete subgroup \( \Gamma_\ell \) and \( \Gamma_B^* \) and \( \Gamma_V(\ell)^* \) are dual lattices, as before.

The second term \( Z_{T^4}(t) \) is that corresponding to 4-dimensional flat torus \( (\mathfrak{g}_+ \oplus \mathfrak{g}_-) / \Gamma_B \).

Let \( \zeta_{\mathbb{M}_\ell}(s) \) be the function

\[
\zeta_{\mathbb{M}_\ell}(s) = \frac{1}{\Gamma(s)} \int_0^\infty (Z(t) - 1) t^{s-1} \, dt
= \frac{1}{\Gamma(s)} \int_0^\infty Z_V(t) t^{s-1} \, dt + \frac{1}{\Gamma(s)} \int_0^\infty (Z_{T^4}(t) - 1) t^{s-1} \, dt = \zeta_{V,\ell}(s) + \zeta_{T^4}(s),
\]

then

**Theorem 4.2.**

\[
\text{Det} \Delta_{H_5/\Gamma_\ell} = \text{Det} \Delta_{T^4} \cdot e^{-\zeta_{V,\ell}^{'(0)}},
\]

where \( \zeta_{V,\ell}^{'(0)} \) is given in Proposition \((4.4)\) below.
Corresponding to the formula (2.1) we have an integral representation of the function \( \zeta_{V,\ell}(s) \):

(4.5)

\[
\zeta_{V,\ell}(s) = \frac{1}{\Gamma(s)} \int_0^\infty Z_V(t) t^{s-1} dt
\]

\[
= 8\ell^2 \sum_{n=1}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(4\pi\ell)^{2s}(n^2 + \frac{n}{4\pi\ell}(2m_1 + 1 + 2m_2 + 1))^s}{\Gamma(s) \Gamma(s-2)} \cdot \frac{1}{(4\pi\ell)^{2s}} \cdot \int_0^\infty \int_0^\infty \frac{x+y}{e^{x+y} - 1} \left( \frac{x}{e^{x/\ell} - e^{-x/\ell}} \right)^2 \frac{(xy)^{s-3}}{x+y} dxdy
\]

\[
= \frac{8\ell^2}{\Gamma(s+1)} \cdot (4\pi\ell)^{2s} \cdot \frac{1}{(2s-5)(2s-4)(2s-3)(2s-2)} \cdot \int_0^1 \int_0^\infty \frac{d^6}{du^6} \left\{ g(u(1+v)) \left( h \left( \frac{u}{4\pi\ell} \right)^2 + h \left( \frac{uv}{4\pi\ell} \right)^2 \right) \right\} u^{2s} \frac{(uv)^{s-3}}{1+v} dudv.
\]

Proposition 4.3.

\[
\int_0^1 \int_0^\infty \frac{d^6}{du^6} \left\{ g(u(1+v)) \left( h \left( \frac{u}{4\pi\ell} \right)^2 + h \left( \frac{uv}{4\pi\ell} \right)^2 \right) \right\} u^{2s} \frac{(uv)^{s-3}}{1+v} dudv
\]

\[
= 2 \int_0^\infty \frac{d^6}{du^6} \left\{ h \left( \frac{u}{2} \right)^3 \left( \cosh \frac{u}{2} \right) \left( h \left( \frac{u}{4\pi\ell} \right)^2 + 1 \right) \right\} \log u du
\]

\[
= 2 \int_0^\infty \frac{d^6}{du^6} \left\{ \left( \frac{u}{2} \right)^3 \left( \cosh \frac{u}{2} \right) \left( \left( \frac{u}{2} \right)^2 + 1 \right) \right\} \log u du + O(s).
\]

Proposition 4.4.

\[
\zeta'_{V,\ell}(0) = -\frac{8}{15} \pi^2 \ell^4 \times
\]

\[
\int_0^\infty \frac{d^6}{du^6} \left\{ \left( \frac{u}{\sinh \frac{u}{2}} \right)^3 \cosh u/2 \left( \left( \frac{u}{2} \right)^2 + 1 \right) \right\} \log u du.
\]

Corollary 4.5.

\[
\lim_{\ell \to \infty} \det \Delta_{H_5/\ell} = \lim_{\ell \to \infty} \det \Delta_{T_4} \times e^{-\zeta'_{V,\ell}(0)} = 0.
\]
Proof. Put \( I(\ell) \)

\[
I(\ell) = \int_0^\infty \frac{d^6}{du^6} \left\{ \left( \frac{u/2}{\sinh u/2} \right)^3 \cosh u/2 \left( \frac{u/(4\pi \ell)}{\sinh u/(4\pi \ell)} \right)^2 + 1 \right\} \log u du
\]

\[- \frac{1}{3} \left( \frac{u}{4\pi \ell} \right)^2 e^{-u/2} \frac{u/2}{\sinh u/2} \log u du.
\]

Since

\[
\lim_{\ell \to \infty} I(\ell) = 2 \int_0^\infty \frac{d^6}{du^6} \left\{ \left( \frac{u/2}{\sinh u/2} \right)^3 \cosh u/2 \right\} \log u du,
\]

it is enough to see the sign of this integral as we did before in Corollary (2.9) to determine the behavior of the determinant \( \text{Det} \Delta_{H_5/\Gamma_\ell} \) when \( \ell \to \infty \). Then, by a similar calculation in Corollary 2.9 we have an expression of the integral

\[
\int_0^\infty \frac{d^6}{du^6} \left\{ \left( \frac{u/2}{\sinh u/2} \right)^3 \cosh u/2 \right\} \log u du
\]

\[
= \frac{4!}{r^5} - \frac{1}{2r} - 5! \int_0^r \left( \frac{u/2}{\sinh u/2} \right)^3 \cosh u/2 - 1 + \frac{1}{2^4 \cdot 3 \cdot 5} u^4 \right) u^{-6} du
\]

\[- 5! \int_r^\infty \left( \frac{u/2}{\sinh u/2} \right)^3 \cosh u/2 \right) u^{-6} du.
\]

So putting \( r = 2 \cdot \sqrt[4]{3} \), we have

\[
\int_0^\infty \frac{d^6}{du^6} \left\{ \left( \frac{u/2}{\sinh u/2} \right)^3 \cosh u/2 \right\} \log u du
\]

\[
= -5! \int_0^{2 \sqrt[4]{3}} \left( \frac{u/2}{\sinh u/2} \right)^3 \cosh u/2 - 1 + \frac{1}{2^4 \cdot 3 \cdot 5} u^4 \right) u^{-6} du
\]

\[- 5! \int_{2 \sqrt[4]{3}}^\infty \left( \frac{u/2}{\sinh u/2} \right)^3 \cosh u/2 \right) u^{-6} du.
\]

It is clear that the second integrand takes always positive values on the positive real axis. We can also prove that the integrand in the first integral takes positive values on the positive real axis. Since the coefficients of the Taylor expansion of the function

\[
\cosh x - \left( 1 - \frac{1}{15} x^4 \right) \left( \frac{\sinh x}{x} \right)^3 = \sum_{n=3}^\infty a_n x^{2n}
\]

are given as

\[
a_n = \frac{1}{(2n)!} + \frac{3^{2n-2} - 1}{20 \cdot (2n - 1)!} - \frac{3 \cdot 3^{2n+2} - 1}{4 \cdot (2n + 3)!}
\]
and all take positive values, which we can see from the expression of $a_n$ for $n \geq 4$,

$$a_n = \frac{1}{20 \cdot (2n+3)!} \left\{ \left(3^{2n-2} - 1\right) \left( (2n+3)(2n+2)(2n+1)(2n) - 15 \cdot 3^4 \right) \\
+ 20 \cdot (2n+3)(2n+2)(2n+1) - 15 \cdot (3^4 - 1) \right\},$$

$$a_4 = \frac{1}{20 \cdot (11)!} \left\{ \left(3^6 - 1\right) \left( (11 \cdot 10 \cdot 9 \cdot 8 - 15 \cdot 3^4 \right) \\
+ 20 \cdot 11 \cdot 10 \cdot 9 - 15 \cdot (3^4 - 1) \right\},$$

and for $n = 3$, $a_3 = \frac{4}{189}$. From these facts we can prove the desired result. \qed

Finally we list the heat asymptotics for five dimensional Heisenberg manifolds $M_\ell$. Let $c_k$ be the coefficients of the asymptotic expansion of the heat kernel $k_{H_5/\Gamma_\ell}(t; x, y)$ of the five dimensional Heisenberg manifolds $M_\ell = H_5/\Gamma_\ell$:

$$\int_{M_\ell} k_{M_\ell}(t; x, x)dx \sim \frac{1}{(4\pi t)^{5/2}} \left\{ c_0 + c_1 t + \cdots + c_k t^k + \cdots \right\}.$$

Let us denote the Taylor expansion of the function $h(x)^2$ as

$$(4.9) \quad h(x)^2 = \left( \frac{x}{\sinh x} \right)^2 = \sum_{k=0}^{\infty} (-1)^k \delta_k x^{2k},$$

then $\delta_k$ is given by

$$(4.10) \quad \delta_k = \sum_{j=0}^{k} \beta_{2j}\beta_{2(k-j)} = \sum_{j=0}^{k} \frac{4(2^{2(k-j)} - 2)(2^{2j} - 2)}{(2\pi)^k} \zeta(2(k-j))\zeta(2j).$$

**Proposition 4.6.**

$$(4.11) \quad c_k = \frac{(2k-5)!!(2k-1)!!}{2^{2k+4}} \frac{(-1)^k}{\pi^3 \ell} \delta_k W_k,$$

where $W_k$ is given in (3.8).

### 5. A FORMULA FOR PRODUCT MANIFOLDS

Let $(M, g)$ and $(N, h)$ be closed Riemannian manifolds, then the Laplacian $\Delta_{M \times N}$ on the product Riemannian manifold $M \times N$ is of the form $\Delta_M \otimes Id + Id \otimes \Delta_N$ and the spectrum $\text{Spec}(\Delta_{M \times N})$ is given by

$$\text{Spec}(\Delta_{M \times N}) = \{ \lambda_m + \mu_n | 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \in \text{Spec}(\Delta_M), \\
\text{and } 0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \in \text{Spec}(\Delta_N) \}.$$

In this section we give a formula of the zeta-regularized determinant of the Laplacian on the product Riemannian manifold $M \times N$ in terms of the each value of the zeta-regularized determinant and heat invariants.
The spectral zeta-function \( \zeta_{M \times N}(s) \) for the product Riemannian manifold \( M \times N \) is given by

\[
\zeta_{M \times N}(s) = \sum_{m,n=0,(m,n)\neq 0}^{\infty} \frac{1}{(\lambda_m + \mu_n)^s}.
\]

We express this as

\[
\sum_{m,n=0,(m,n)\neq 0}^{\infty} \frac{1}{(\lambda_m + \mu_n)^s} = \frac{1}{\Gamma(s)} \sum_{m=1}^{\infty} \frac{\lambda_m^s}{\lambda_m} \sum_{n=0}^{\infty} \int_0^{\infty} e^{-t(\lambda_m + \mu_n)s} \frac{dt}{t^{s-1}} + \zeta_N(s)
\]

(5.1)

\[
= \frac{1}{\Gamma(s)} \sum_{m=1}^{\infty} \lambda_m^s \int_0^{\infty} \left\{ \sum_{n=0}^{\infty} e^{-t\mu_n} \right\} \left( \frac{\lambda_m}{4\pi t} \right)^{N/2 \left[ (N+M)/2 \right]} \sum_{i=0}^{\infty} b_i \cdot \left( \frac{t}{\lambda_m} \right)^i - \left( \frac{\lambda_m}{4\pi t} \right)^{N/2 \left[ (N+M)/2 \right]} \sum_{i=0}^{\infty} b_i \cdot \left( \frac{t}{\lambda_m} \right)^i e^{-t^{s-1}dt}
\]

\[
+ \frac{1}{\Gamma(s)} \sum_{m=1}^{\infty} \lambda_m^s \int_0^{\infty} \left( \frac{\lambda_m}{4\pi t} \right)^{N/2 \left[ (N+M)/2 \right]} \sum_{i=0}^{\infty} b_i \cdot \left( \frac{t}{\lambda_m} \right)^i e^{-t^{s-1}dt} + \zeta_N(s)
\]

(5.2)

\[
= Q_0(s) + \sum_{i=0}^{\left[ (N+M)/2 \right]} b_i \cdot \frac{\Gamma(s+i-N/2)}{(4\pi)^{N/2} \cdot \Gamma(s)} \cdot \zeta_M(s+i-N/2) + \zeta_N(s),
\]

where \( N = \dim N, \ M = \dim M \) and \( \{ b_i \}_{i=0}^{\infty} \) are the coefficients of asymptotic expansion of the trace of the heat kernel \( k_N(t; x, y) \) on \( N \):

\[
k_N(t) = \int_N k_N(t; x, x) dx
\]

(5.3)

\[
= \sum_{n=0}^{\infty} e^{-t\mu_n} \sim \left( \frac{1}{4\pi t} \right)^{N/2} \left\{ b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \cdots \right\}.
\]

We also denote by \( \{ a_i \}_{i=0}^{\infty} \) the coefficients of the asymptotic expansion of trace of the heat kernel \( k_M(t; x, y) \) on the manifold \( M \):

\[
k_M(t) = \int_M k_M(t; x, x) dx
\]

(5.4)

\[
= \sum_{m=0}^{\infty} e^{-t\lambda_m} \sim \left( \frac{1}{4\pi t} \right)^{M/2} \left\{ a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \right\}.
\]

Since

\[
\left| k_N \left( \frac{t}{\lambda_m} \right) - \left( \frac{\lambda_m}{4\pi t} \right)^{N/2 \left[ (N+M)/2 \right]} \sum_{i=0}^{\infty} b_i \cdot \left( \frac{t}{\lambda_m} \right)^i \right| = O \left( \left( \frac{t}{\lambda_m} \right)^{\left[ (N+M)/2 \right]+1-N/2} \right),
\]

(5.5)
the first term $Q_0(s)$ is holomorphic on the domain $\{ s \in \mathbb{C} \mid \Re(s) > -1/2 \}$, at least under this expression for any case of the dimensions of the two manifolds. So we can put $s = 0$ in (5.1) and we have

$$Q_0(0) = 0$$

(5.5) \[ \sum_{m=1}^{\infty} \int_0^\infty \left\{ k_N \left( \frac{t}{\lambda_m} \right) - \left( \frac{\lambda_m}{4\pi t} \right)^{(N+M)/2} \sum_{i=0}^{[N+M]/2} b_i \cdot \left( \frac{t}{\lambda_m} \right)^i \right\} e^{-t^{-1}dt}. \]

Next we put the second term as

$$Q_1(s) = \frac{[N+M]/2}{(4\pi)^{N/2} \Gamma(s)} \cdot \zeta_M(s + i - N/2) = \sum_{i=0}^{[N+M]/2} q_i(s)$$

(5.6)

(5.7)

\[ \sum_{i=0}^{[N+M]/2} \frac{b_i}{(4\pi)^{N/2} \Gamma(s)} \int_0^\infty (k_M(t) - 1)t^{s+i-N/2-1}dt. \]

**Remark 5.1.** By the asymptotic expansion (5.4) it is well known that the Mellin transformation $\int_0^\infty (k_M(t) - 1)t^{s-1}dt$ is meromorphically continued to the whole complex plane and has possible poles of order one at points $M/2 - i, i = 0, 1, \ldots$ with the residue

$$\frac{a_i}{(4\pi)^M/2}$$

when $\dim M$ is odd, and when $\dim M$ is even then the residue at the pole $M/2 - i, i \neq M/2$ is

$$\frac{a_i}{(4\pi)^M/2}$$

and at $s = 0$ the residue is

$$\frac{a_{M/2}}{(4\pi)^M/2} - 1.$$

By the remark above we can describe the derivative at $s = 0$ of the each term

$$q_i(s) = \frac{b_i}{(4\pi)^{N/2} \Gamma(s)} \cdot \zeta_M(s + i - N/2)$$

in (5.6) in terms of the spectral zeta-function $\zeta_M(s)$. For this purpose we consider four cases separately.

**Proposition 5.2.** Let $\dim M = M$ be even and $\dim N = N$ odd. Since $\zeta_M(s)$ is holomorphic at each point $i - N/2$ we have

$$Q'_1(0) = \sum_{i=0}^{[N+M]/2} q'_i(0)$$
\[
= \sum_{i=0}^{[(N+M)/2]} \frac{b_i \cdot \Gamma(i - N/2)}{(4\pi)^{N/2}} \cdot \zeta_M(i - N/2)
\]

**Proposition 5.3.** Let \( \dim M = M \) be odd and \( \dim N = N \) even. Then

for \( 0 \leq i < N/2 \)

\[
q'_i(0) = \frac{(-1)^{N/2-i}b_i}{(4\pi)^{N/2}(N/2 - i)!} \cdot \zeta'_M(i - N/2).
\]

for \( i = N/2 \)

\[
q'_{N/2}(0) = \frac{b_{N/2}}{(4\pi)^{N/2}} \cdot \zeta'_M(0).
\]

for \( N/2 < i \leq [(N + M)/2] \)

\[
q'_i(0) = \frac{b_i(i - N/2 - 1)!}{(4\pi)^{N/2}} \cdot \zeta_M(i - N/2).
\]

Hence

\[
Q'_1(0) = \sum_{i=0}^{N/2-1} \frac{(-1)^{N/2-i}b_i}{(4\pi)^{N/2}(N/2 - i)!} \cdot \zeta'_M(0)
\]

\[
+ \frac{b_{N/2}}{(4\pi)^{N/2}} \cdot \zeta'_M(0)
\]

\[
+ \sum_{i=N/2+1}^{[(N+M)/2]} \frac{b_i}{(4\pi)^{N/2}(i - N/2 - 1)!} \cdot \zeta_M(i - N/2).
\]

**Proposition 5.4.** Let both of \( \dim M = M \) and \( \dim N = N \) be odd. Then,

\[
q'_i(0) = \frac{b_i}{(4\pi)^{N/2}} \left\{ -\Gamma'(1) \cdot \frac{a_i((N+M)/2-i)}{(4\pi)^{M/2}} \right\}
\]

\[
+ \left( \lim_{s \to 0} \Gamma(i - N/2) \cdot \zeta_M(s + i - N/2) - \frac{a_i((N+M)/2-i)}{(4\pi)^{M/2}} \cdot \frac{1}{s} \right)
\]

Hence

\[
Q'_1(0) = \sum_{i=0}^{(N+M)/2} \frac{b_i}{(4\pi)^{N/2}} \left\{ -\Gamma'(1) \cdot \frac{a_i((N+M)/2-i)}{(4\pi)^{M/2}} \right\}
\]

\[
+ \left( \lim_{s \to 0} \Gamma(i - N/2) \cdot \zeta_M(s + i - N/2) - \frac{a_i((N+M)/2-i)}{(4\pi)^{M/2}} \cdot \frac{1}{s} \right)
\]

**Proposition 5.5.** Let \( \dim M = M \) and \( \dim N = N \) be both even. Then,

for \( 0 \leq i < N/2 \)

\[
q'_i(0) = \frac{b_i}{(4\pi)^{N/2}} \left\{ -\Gamma'(1) \cdot \frac{a_i((N+M)/2-i)}{(4\pi)^{M/2}} \right\}
\]
for $i = N/2$

\[
q_N'(0) = \frac{b_{N/2}}{(4\pi)^{N/2}} \cdot \zeta_M(0) \quad \text{and}
\]

\[
q_i'(0) = \frac{b_i}{(4\pi)^{N/2}} \left\{ -\Gamma'(1) \cdot \frac{a_i((N+M)/2 - i)}{(4\pi)^{M/2}} \right. \\
+ \lim_{s \to 0} \Gamma(s + i - N/2) \cdot \zeta_M(s + i - N/2) - \frac{a_i((N+M)/2 - i)}{(4\pi)^{M/2}} \cdot \frac{1}{s} \left. \right\}.
\]

Hence,

\[
Q_i'(0) = \sum_{i=0}^{N/2-1} \frac{b_i}{(4\pi)^{N/2}} \left\{ -\Gamma'(1) \cdot \frac{a_i((N+M)/2 - i)}{(4\pi)^{M/2}} \right. \\
+ \lim_{s \to 0} \Gamma(s + i - N/2) \cdot \zeta_M(s + i - N/2) - \frac{a_i((N+M)/2 - i)}{(4\pi)^{M/2}} \cdot \frac{1}{s} \left. \right\} \\
+ \frac{b_{N/2}}{(4\pi)^{N/2}} \cdot \zeta_M(0) \\
+ \sum_{i=N/2+1}^{(N+M)/2} \frac{b_i}{(4\pi)^{N/2}} \left\{ -\Gamma'(1) \cdot \frac{a_i((N+M)/2 - i)}{(4\pi)^{M/2}} \right. \\
+ \lim_{s \to 0} \Gamma(s + i - N/2) \cdot \zeta_M(s + i - N/2) - \frac{a_i((N+M)/2 - i)}{(4\pi)^{M/2}} \cdot \frac{1}{s} \left. \right\}.
\]

\[
\text{Remark 5.6.} \quad -\Gamma'(1) = -\int_0^\infty e^{-t} \log t \, dt = C_e = 0.57721 \cdots = \text{Euler’s constant}.
\]

We can now write down an expression of the value $\text{Det} \, \Delta_{M \times N}$ corresponding to the each case \[\text{5.2, 5.3, 5.4 and 5.5} \] above. Here we only state for the case that both of $\dim M$ and $\dim N$ are even. In the next section we state a special case of $N = S^1$.

**Theorem 5.7.** Let $\dim N$ and $\dim M$ be both even, then

\[
\text{Det} \, \Delta_{M \times N} = e^{-Q_M'(0)} \cdot e^{-Q_N'(0)} \cdot \text{Det} \, \Delta_N
\]

\[
= \text{Det} \, \Delta_N \cdot \prod_{m=1}^\infty e^{-\int_0^\infty \left( \frac{b_m}{(4\pi)^{M/2}} \right) \sum_{i=0}^{N/2-1} \frac{1}{(4\pi)^{N/2}} \cdot \frac{1}{s} \left. \right\} e^{-t} \, dt \\
\times \prod_{i=0}^{N/2-1} e^{-\frac{b_i}{(4\pi)^{(N+M)/2}}} \cdot \left\{ \lim_{s \to 0} \frac{C_e \cdot b_i \cdot a_i((N+M)/2 - i)}{(4\pi)^{(N+M)/2}} \left[ \Gamma(s + i - N/2) \cdot \zeta_M(s + i - N/2) - \frac{a_i((N+M)/2 - i)}{(4\pi)^{(N+M)/2}} \cdot \frac{1}{s} \right] \right\} \\
\times (\text{Det} \, \Delta_M)^{\frac{b_{N/2}}{(4\pi)^{(N+M)/2}}} \cdot \prod_{i=N/2+1}^{(N+M)/2} e^{-\frac{b_i}{(4\pi)^{(N+M)/2}}} \times
\]
Theorem 6.1.

By interchanging $N$ and $M$ we have another expression of $\text{Det} \, \Delta_{N \times M}$:

$$\text{Det} \, \Delta_{N \times M} = \text{Det} \, \Delta_M \cdot \prod_{n=1}^\infty e^{- \int_0^\infty \left\{ \sum_{i=0}^{M/2-1} e^{-\frac{a_i}{(4\pi)^{M/2}}} \left\{ \lim_{s \to 0} \left( \frac{\Gamma(s+i-M/2)}{\Gamma(s+i-M/2)} \zeta_N(s+i-M/2) - \frac{b_i}{(4\pi)^{(M+1)/2}} \right) \right\} \right\} e^{-t^2} dt} \times$$

$$\times \prod_{i=0}^{M/2-1} e^{\frac{a_M}{(4\pi)^{M/2}}} \prod_{i=0}^{M/2-1} e^{\frac{a_i}{(4\pi)^{M/2}}} \left\{ \lim_{s \to 0} \left( \frac{\Gamma(s+i-M/2)}{\Gamma(s+i-M/2)} \zeta_N(s+i-M/2) - \frac{b_i}{(4\pi)^{(M+1)/2}} \right) \right\} \times$$

$$\times (\text{Det} \, \Delta_N) \frac{a_M}{(4\pi)^{M/2}} \prod_{i=M/2+1}^{(N+M)/2} e^{-\frac{a_i}{(4\pi)^{M/2}}} \left\{ \lim_{s \to 0} \left( \frac{\Gamma(s+i-M/2)}{\Gamma(s+i-M/2)} \zeta_N(s+i-M/2) - \frac{b_i}{(4\pi)^{(M+1)/2}} \right) \right\} \times$$

$$\times \prod_{i=M/2+1}^{(N+M)/2} e^{\frac{a_i}{(4\pi)^{M/2}}} \left\{ \lim_{s \to 0} \left( \frac{\Gamma(s+i-M/2)}{\Gamma(s+i-M/2)} \zeta_N(s+i-M/2) - \frac{b_i}{(4\pi)^{(M+1)/2}} \right) \right\} .$$

6. Product manifold $M \times S^1$

The formula we gave in the last section says that even in the product manifold case the zeta-regularized determinant is not expressed in a simple way in terms of each zeta-regularized determinant. In the paper [7] a formula of the zeta-regularized determinant for a manifold of the type $M \times S^1$ is given, in fact a formula is derived for a higher order operator of the form with the variable in $S^1$ being separated, by reducing the problem to a boundary value problem on the manifold $M \times [0, 1]$. In this section we give a direct proof of this formula by restricting ourselves to the case of Laplacians on $M \times S^1$ by following the same line as we did in the last section. Here in this special case the Poisson summation formula allows us to make an integration of the term (5.5) and we arrive at a precise formula.

Now in this case the Laplacian $\Delta_{M \times S^1}$ is given by $\Delta_{M \times S^1} = \Delta_M - \frac{d^2}{dx^2}$, where we regard $S^1 \cong \mathbb{R}/(2\pi \ell \cdot \mathbb{Z})$, $x \in \mathbb{R}$, $\ell > 0$ and the spectrum are

$$\left\{ \lambda_n + \left( \frac{k}{\ell} \right)^2 : 0 = \lambda_0 < \lambda_1 \leq \lambda_2, \cdots, \text{ are the spectrum of } M, \text{ and } k \in \mathbb{Z} \right\} .$$

Theorem 6.1.

(6.1) $$\text{Det} \, \Delta_{M \times S^1} = 4\pi^2 \ell^2 C_M \cdot \prod_{n=1}^\infty \left| 1 - e^{-2\pi \ell \sqrt{\lambda_n}} \right|^2 ,$$

where the constant $C_M$ is given by

(a) when $\dim M = \text{even} = 2m$, then

$$C_M = e^{2\pi \ell \zeta_M(-1/2)}$$
(b) when \( \dim \mathbf{M} = \text{odd} = 2m + 1 \), then

\[
\log \mathbf{C}_M = -\sqrt{\pi \ell} \left\{ \lim_{s \to 0} \left( 2 \sqrt{\pi} \cdot \zeta_M(s - 1/2) + \frac{a_{(M+1)/2}}{(4\pi)^{(M+1)/2}} \right) + \frac{\Gamma'(1)}{(4\pi)^{M/2}} a_{(M+1)/2} \right\}.
\]

Here \( a_{(M+1)/2} \) denotes \((M+1)/2\)-th coefficient of the asymptotic expansion of the heat kernel \( Z_M(t) \) on \( \mathbf{M} \).

**Proof.** We express the spectral zeta-function \( \zeta_{M \times S^1}(s) \) as

\[
\zeta_{M \times S^1}(s) = \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} \frac{1}{\lambda_n + (\frac{k}{2})^2} s + 2\ell^{2s} \cdot \zeta(2s)
\]

\[
= \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} \int_0^\infty \sum_{k \in \mathbb{Z}} \sqrt{\frac{\pi \lambda_n \ell^2}{x}} e^{-\frac{2\lambda_n k^2 \ell^2}{x}} e^{-x^{s-1}dx + 2\ell^{2s} \cdot \zeta(2s)}
\]

Then by using the Poisson’s summation formula this equals to

\[
= \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} \int_0^\infty \sum_{k \in \mathbb{Z}, k \neq 0} \sqrt{\frac{\pi \lambda_n \ell^2}{x}} e^{-\frac{2\lambda_n k^2 \ell^2}{x}} e^{-x^{s-1}dx}
\]

\[
+ \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} \int_0^\infty \sqrt{\frac{\pi \lambda_n \ell^2}{x}} e^{-x^{s-1}dx + 2\ell^{2s} \cdot \zeta(2s)}
\]

\[
= \mathcal{P}_0(s) + \frac{\sqrt{\pi \ell} \cdot \Gamma(s - 1/2)}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{s-1/2}} + 2\ell^{2s} \cdot \zeta(2s),
\]

where we put

\[
\mathcal{P}_0(s) = \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} \int_0^\infty \sum_{k \in \mathbb{Z}, k \neq 0} \sqrt{\frac{\pi \lambda_n \ell^2}{x}} e^{-\frac{2\lambda_n k^2 \ell^2}{x}} e^{-x^{s-1}dx}
\]

\[
= 2\sqrt{\pi \ell} \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{s-1/2}} \int_0^\infty \sum_{k=1}^{\infty} e^{-\frac{2\lambda_n k^2 \ell^2}{x}} e^{-x^{s-3/2}dx}.
\]

We denote the second term \( \frac{\sqrt{\pi \ell} \cdot \Gamma(s - 1/2)}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{s-1/2}} = \frac{\sqrt{\pi \ell} \cdot \Gamma(s - 1/2)}{\Gamma(s)} \cdot \zeta_M(s - 1/2) \) by \( \mathcal{P}_1(s) \).

Since the sum \( \sum_{k=1}^{\infty} e^{-\frac{2\lambda_n k^2 \ell^2}{x}} \) in the integrand satisfies the asymptotics

\[
\sum_{k=1}^{\infty} e^{-\frac{2\lambda_n k^2 \ell^2}{x}} = O \left( \left( \frac{x}{\lambda_n} \right)^N \right)
\]
for any \( N \in \mathbb{N} \), or the coefficients \( b_i \) are all zero except \( b_0 = 2\pi \ell \), the function \( \mathcal{P}_0(s) \) is holomorphic on the whole complex plane and we have

\[
\mathcal{P}_0(0) = 0,
\]

\[
\mathcal{P}'_0(0) = 2\sqrt{\pi \ell} \sum_{n=1}^{\infty} \sqrt{\lambda_n} \int_0^{\infty} \sum_{k=1}^{\infty} e^{-\frac{\pi^2 k^2 \ell^2}{x^2}} e^{-x} x^{-3/2} dx.
\]

To calculate the value \( \mathcal{P}'_0(0) \) recall a formula of a modified Bessel function \( K_{1/2}(z) = K_{-1/2}(z) \) (see [1]):

\[
\int_0^{\infty} e^{-\left(t + \frac{z^2}{4t}\right)} \frac{1}{\sqrt{t}} dt = \sqrt{\pi} e^{-z} = \sqrt{2z} K_{1/2}(z).
\]

Now we have

\[
\mathcal{P}'_0(0) = -2 \sum_{n=1}^{\infty} \log \left(1 - e^{-2\pi \ell \sqrt{\lambda_n}}\right).
\]

Hence from Propositions 5.2 and 5.4 the determinant \( \det \Delta_{\mathcal{M} \times S^1} \) is of the form

\[
\det \Delta_{\mathcal{M} \times S^1} = 4\pi^2 \ell^2 C_{\mathcal{M}} \prod_{n=1}^{\infty} \left(1 - e^{-2\pi \ell \sqrt{\lambda_n}}\right)^2,
\]

where the constant \( C_{\mathcal{M}} \) is given by

(a) when \( \dim \mathcal{M} \) is even, then

\[
C_{\mathcal{M}} = e^{2\pi \ell \zeta_\mathcal{M}(-1/2)}
\]

(b) when \( \dim \mathcal{M} = M \) is odd, then

\[
\log C_{\mathcal{M}} = -\sqrt{\pi \ell} \left\{ \lim_{s \to 0} \left(2\sqrt{\pi} \cdot \zeta_\mathcal{M}(s - 1/2) + \frac{a_{(M+1)/2}}{(4\pi)^{(M+1)/2}} \frac{1}{s}\right) + \frac{\Gamma'(1)}{(4\pi)^{M/2}} a_{(M+1)/2} \right\}.
\]

Example 6.2. As an application of our formula \((6.1)\) we give an expression of \( \det \Delta_{S^2 \times S^1} \).

For the standard 2-dimensional sphere \( S^2 \) the spectral zeta-function is

\[
\zeta_{S^2}(s) = \sum_{k=1}^{\infty} \frac{2k + 1}{k^s(k + 1)^s},
\]

which converges for \( \Re(s) > 1 \). We rewrite this as

\[
\zeta_{S^2}(s) = \frac{1}{2^s} + \sum_{k=2}^{\infty} \frac{1}{k^{2s-1}} \left\{ \left(1 + \frac{1}{k}\right)^{-s} + \left(1 - \frac{1}{k}\right)^{-s}\right\}
\]

\[
= \frac{1}{2^s} + \sum_{k=2}^{\infty} \frac{1}{k^{2s-1}} \sum_{m=0}^{\infty} \frac{2d_{2m}(-s)}{k^{2m}} \left(\frac{1}{k}\right)^{2m}
\]
\[ (6.4) \quad = \frac{1}{2^s} + 2 \sum_{m=0}^{2m \leq n} d_{2m}(-s)(\zeta(2s-1+2m) - 1) \]
\[ + 2 \sum_{2m>n} \sum_{k=2}^{\infty} d_{2m}(-s) \frac{1}{k^{2m-n}} \cdot \frac{1}{k^{2s-1+n}}, \]

where we used the expansion \((1 + z)^{\alpha} = \sum d_m(\alpha) z^m\), for \(|z| < 1\). Note that for \(\alpha > 0\) this series converges for \(-1 \leq z \leq 1\). Then for \(\Re(s) > (2 - n)/2\), by the estimate
\[ \sum_{2m>n} \sum_{k=2}^{\infty} |d_{2m}(-s)| \frac{1}{k^{2m-n}} \cdot \frac{1}{k^{2s-1+n}} \]
and the functional relation for the Riemann \(\zeta\)-function the expression \((6.4)\) gives us the analytic continuation of \(\zeta_{S^2}(s)\) to the complex plane of \(\Re(s) > (2 - n)/2\) for each \(n > 0\). So we can put \(s = -1/2\) in \((6.4)\) and we have an expression of \(\zeta_{S^2}(-1/2)\):
\[ \zeta_{S^2}(-1/2) \]
\[ = \sqrt{2} + 2 \{\zeta(-2) - 1\} + 2d_2(1/2) \{\zeta(0) - 1\} \]
\[ + 2 \sum_{m=3}^{\infty} d_{2m}(1/2) (\zeta(2m - 1) - 1) \]
\[ = \sqrt{2} - 2 \sum_{m=0}^{\infty} d_{2m}(1/2) + 2 \sum_{m=0}^{\infty} d_{2m}(1/2) (\zeta(2m - 2)) \]
\[ = - \sum_{m=0}^{\infty} \frac{(4m)!}{2^{4m-1} (4m-1)((2m)!)^2} \zeta(2m - 2). \]

This is also expressed as
\[ (6.5) \quad \zeta_{S^2}(-1/2) = \frac{4}{9\pi} \int_0^\infty \int_0^\infty \frac{\partial^2}{\partial x^2} \left( \frac{\partial^2}{\partial y^2} \left( \frac{x + y}{e^{x+y} - 1} \right) e^{-x} \right) \cdot \frac{1}{\sqrt{xy}} dxdy. \]

So, finally we have
\[ (6.6) \quad \text{Det} \Delta_{S^2 \times S^1} = 4\pi^2 \ell^2 \prod_{m=1}^{\infty} e^{-\pi \ell \frac{(4m)!}{2^{4m-1} (4m-1)((2m)!)^2} \zeta(2m-2)} \cdot \prod_{n=1}^{\infty} \left| 1 - e^{-2\pi \ell \sqrt{k(k+1)}} \right|^{2k+1}. \]

Again by applying Propositions \ref{5.4} and \ref{5.3} we have an alternative representation of the determinant \(\text{Det} \Delta_{M \times S^1}\).

**Corollary 6.3.** When \(\dim M\) is odd, then
\[ (6.7) \quad \text{Det} \Delta_{M \times S^1} \]
\[ \begin{align*}
&= \prod_{k=1}^{\infty} e^{-2 \int_0^\infty \left( k_M \left( \frac{t^2}{\kappa^2} \right) - \left( \frac{t^2}{4\pi\kappa^2} \right) \sum_{j=0}^{M/2} a_j \left( \frac{t^2}{\kappa^2} \right)^j \right) e^{-t^2} dt \\
&\times \prod_{i=0}^{[M/2]} e^{-\frac{2}{(4\pi\kappa^2)^{M/2}}} \cdot a_i \cdot \ell^{M-i} \cdot \Gamma(2i-M) \times e^{-\frac{2}{(4\pi\kappa^2)^{M/2}}} \cdot a_i \cdot \ell^{M-i} \cdot \Gamma(2i-M) \times \\
&\times e^{-\frac{2}{(4\pi\kappa^2)^{M/2}}} \cdot \Gamma(2i-M) \times \\
&\times \det \Delta_M.
\end{align*} \]

Here we used the formula \( \zeta(s) = 1/(s-1) + C e^s + O(s) \).

When \( \dim M \) is even, then

\[ \det \Delta_{M \times S^1} \]

\[ = \prod_{k=1}^{\infty} e^{-2 \int_0^\infty \left( k_M \left( \frac{t^2}{\kappa^2} \right) - \left( \frac{t^2}{4\pi\kappa^2} \right) \sum_{j=0}^{M/2} a_j \left( \frac{t^2}{\kappa^2} \right)^j \right) e^{-t^2} dt \\
\times \prod_{i=0}^{M/2-1} e^{-\frac{2}{(4\pi\kappa^2)^{M/2}}} \cdot a_i \cdot \ell^{M-i} \cdot \Gamma(2i-M) \times \\
\times 2\pi \ell \cdot \det \Delta_M. \]

Note that \( \zeta(-2k) = 0 \) for \( k = 1, 2, \cdots \).

Remark 6.4. Our formula (6.1) is of course a special case of formulas given in the paper \[7\] for more general elliptic operators on the product manifolds \( M \times S^1 \). However here we gave an expression of the constant \( C_M \), although the formula itself is not a computable form, especially for \( M \) being odd dimensional. To obtain a further information we must specify the manifolds \( M \). So in the next section we give a more precise form of this factor \( C_M \) for some flat tori.

7. Flat Tori

In the last two sections we considered the zeta-regularized determinant for manifolds of a product form as a Riemannian manifold. In this section we deal with the case that the manifolds are two, three and four dimensional flat tori, which are not always of a product form of lower dimensional tori as Riemannian manifolds.

We know by a similar calculation as we showed in \$2\$ and \$4\$ that the zeta-regularized determinant of \((2n+1)\)-dimensional Heisenberg manifolds are always of the product form with a factor which is the zeta-regularized determinant of a \( 2n \)-dimensional torus. So of course it is required to determine the zeta-regularized determinant of flat tori to complete the calculation for Heisenberg manifolds. In this section we give an expression of it for two, three and four dimensional flat tori. Although our expressions are not of a computable form within a finite step, the
expression for two dimensional cases are given by the famous limit formula of Kroneker as we cited in §2, and higher cases correspond to a generalization of this limit formula. The structure of a generalization was already stated and discussed focusing in their functional relations in the papers \[2\] and \[3\] for more general Dirichlet series than Epstein zeta-functions which are of our cases. Here we treat with the typical Epstein zeta-functions of two, three and four variables. Since it is enough for our purpose to give an explicit analytic continuation of the functions from a left half region in the complex plane to a region including zero, we give them based on the Jacobi identity and the Mellin transformation in a quite elementary way. For this purpose we fix the flat tori in the following way.

Let \( e_1, e_2, e_3, e_4 \) be the standard orthonormal basis on \( \mathbb{R}^4 \) and we fix a basis \( \{u_1, u_2, u_3, u_4\} \) of the following form

\[
\begin{align*}
  u_1 &= e_1, \\
  u_2 &= a_{1,2}e_1 + a_{2,2}e_2, \text{ we put this } = Ae_1 + Be_2, \\
  u_3 &= a_{1,3}e_1 + a_{2,3}e_2 + a_{3,3}e_3, \\
  u_4 &= a_{1,4}e_1 + a_{2,4}e_2 + a_{3,4}e_3 + a_{4,4}e_4 (a_{2,2}, a_{3,3}, a_{4,4} > 0).
\end{align*}
\]

(a) \( T_L^2 \cong \mathbb{R}^2/L \), where

\[
L = L_2 = \{n u_1 + m u_2 = (n + ma_{1,2}, ma_{2,2}) : n, m \in \mathbb{Z}\}
\]

is a lattice in \( \mathbb{R}^2 \),

(b) \( T^3 \cong \mathbb{R}^3/L_3 \), \( L_3 = \{n u_1 + m u_2 + l u_3 : n, m, l \in \mathbb{Z}\} = \{u_1, u_2, u_3\} \),

(c) \( T^4 \cong \mathbb{R}^4/L_4 \), \( L_4 = \{n u_1 + m u_2 + l u_3 + k u_4 : n, m, l, k \in \mathbb{Z}\} = \{u_1, u_2, u_3, u_4\} \),

of two, three and four dimensions. All flat tori of such dimensions reduce to these cases.

I. Kroneker’s second limit formula and two dimensional tori

Since the dual lattice \( L^* \) of \( L \) is given by

\[
L^* = \left\{ \left( n, \frac{m - nA}{B} \right) \in \mathbb{R}^2 : n, m \in \mathbb{Z} \right\},
\]

non-zero eigenvalues of the Laplacian on \( T_L^2 \) are

\[
\left\{ 4\pi^2 \left( n^2 + \frac{(m - nA)^2}{B^2} \right) : n, m \in \mathbb{Z}, (n, m) \neq (0, 0) \right\}.
\]

The spectral zeta-function \( \zeta_{T_L^2}(s) \), is

\[
\frac{1}{\Gamma(s)} \int_0^\infty \left( Z_{T_L^2}(t) - 1 \right) t^{s-1} dt = \zeta_{T_L^2}(s)
\]

\[
= \frac{2}{(4\pi^2)^s} \sum_{n=1}^\infty \sum_{m \in \mathbb{Z}} \frac{1}{n^2 + \left( \frac{1}{B^2} (m - nA)^2 \right)^s} + \frac{2B^{2s}}{(4\pi^2)^s} \cdot \zeta(2s).
\]

(7.1)
By using the Poisson’s summation formula we rewrite the first term as follows:

\[
\frac{2}{(4\pi^2)^s} \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} \frac{1}{(n^2 + \left(\frac{1}{B^2}(m - nA)^2\right))^s} \\
= \frac{2}{(4\pi^2)^s \cdot \Gamma(s)} \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} \frac{1}{n^{2s}} \int_0^{\infty} e^{-\left(1 + \frac{1}{B^2}(m - nA)^2\right)x} x^{-s-1} dx \\
= \frac{2}{(4\pi^2)^s \cdot \Gamma(s)} \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} \sqrt{\frac{\pi B^2 n^2}{x}} e^{-\frac{(\pi B n m)^2}{x}} e^{-2\sqrt{-1}Anm} e^{-x} x^{-s-1} dx \\
= \frac{2}{(4\pi^2)^s \cdot \Gamma(s)} \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \int_0^{\infty} \sum_{m \in \mathbb{Z}, m \neq 0} \sqrt{\frac{\pi B^2 n^2}{x}} e^{-\frac{(\pi B n m)^2}{x}} e^{-2\sqrt{-1}Anm} e^{-x} x^{-s-1} dx \\
+ \frac{2\sqrt{\pi} B \cdot \Gamma(s - 1/2)}{(4\pi^2)^s \cdot \Gamma(s)} \cdot \zeta(2s - 1),
\]

so \(\zeta_{T^2}^1(s)\) is of the following form:

**Proposition 7.1.** ([11], [3])

\[
\zeta_{T^2}^1(s) = \mathcal{H}_0(s) + \frac{2\sqrt{\pi} B \cdot \Gamma(s - 1/2)}{(4\pi^2)^s \cdot \Gamma(s)} \cdot \zeta(2s - 1) + \frac{2B^{2s}}{(4\pi^2)^s} \cdot \zeta(2s),
\]

where we put

\[
\mathcal{H}_0(s) = \frac{2}{(4\pi^2)^s \cdot \Gamma(s)} \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \int_0^{\infty} \sum_{m \in \mathbb{Z}, m \neq 0} \sqrt{\frac{\pi B^2 n^2}{x}} e^{-\frac{(\pi B n m)^2}{x}} e^{-2\sqrt{-1}Anm} e^{-x} x^{-s-1} dx \\
= \frac{2\sqrt{\pi} B}{(4\pi^2)^s \cdot \Gamma(s)} \sum_{n=1}^{\infty} e^{-\frac{2\sqrt{-1}Anm}{n^{2s}}} \int_0^{\infty} \sum_{m \in \mathbb{Z}, m \neq 0} e^{-\frac{(\pi B n m)^2}{x}} e^{-x} x^{-s-3/2} dx
\]

We know the integrand of \(\mathcal{H}_0(s)\) satisfies the asymptotics:

**Lemma 7.2.** For any \(N \in \mathbb{N}\)

\[
(7.2) \quad \sum_{m \in \mathbb{Z}, m \neq 0} \sqrt{\frac{\pi B^2 n^2}{x}} e^{-\frac{(\pi B n m)^2}{x}} e^{-2\sqrt{-1}Anm} e^{-x} x^{-s-1} = O\left(\frac{x^{N - \Re(s) - 3/2}}{n^{2N-1}}\right).
\]

Hence the first term \(\mathcal{H}_0(s)\) is a holomorphic function of \(s\) on the whole complex plane, and

\[
\mathcal{H}_0(s) \\
= 2\sqrt{\pi} B \sum_{n=1}^{\infty} \int_0^{\infty} \sum_{m \in \mathbb{Z}, m \neq 0} e^{-\frac{(\pi B n m)^2}{x}} e^{-2\sqrt{-1}Anm} e^{-x} x^{-1/2-1} dx \cdot s + O(s^2)
\]
Then again by making use of (6.2)

\[ H_0'(0) = 2\sqrt{\pi} B \sum_{n=1}^{\infty} ne^{-2\pi \sqrt{\sqrt{1}Anm}} \int_0^{\infty} \sum_{m \in \mathbb{Z}, m \neq 0} \frac{e^{-\frac{(xBnm)^2}{x}}}{m} e^{-x} x^{-1/2} dx \]

\[ = 2 \frac{\sqrt{\pi}}{B} \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}, m \neq 0} \frac{e^{-2\pi \sqrt{\sqrt{1}Anm}}}{m} \int_0^{\infty} \frac{e^{-\frac{(xBnm)^2}{x}}}{m} e^{-x} x^{-1/2} dx \]

\[ = 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \left\{ e^{-2\pi nm(B-\sqrt{\sqrt{1}A})} + e^{-2\pi nm(B+\sqrt{\sqrt{1}A})} \right\} \]

\[ (7.3) \quad = -2 \sum_{n=1}^{\infty} \left\{ \log \left( 1 - e^{2\pi n(B-\sqrt{\sqrt{1}A})} \right) + \log \left( 1 - e^{2\pi n(B+\sqrt{\sqrt{1}A})} \right) \right\}. \]

From the facts

\[ \zeta(-1) = -\frac{1}{12} \]

\[ \zeta(s) = -\frac{1}{2} - \frac{s}{2} \log 2\pi + O(s^2) \]

and (6.2) we have

\[ \zeta_T^2(s) = -1 \]

\[ + \left\{ \frac{\pi B}{3} - 2 \log B - 2 \sum_{n=1}^{\infty} \log \left( 1 - e^{-2\pi n(B-\sqrt{\sqrt{1}A})} \right) + \log \left( 1 - e^{-2\pi n(B+\sqrt{\sqrt{1}A})} \right) \right\} \cdot s \]

\[ + O(s^2). \]

Then the zeta-regularized determinant \( Det \Delta_T^2 \) is given by the formula:

**Theorem 7.3.**

\[ (7.4) \quad Det \Delta_T^2 = B^2 e^{-\frac{2\pi B}{3}} \prod_{n=1}^{\infty} \left| 1 - e^{-2\pi n(B-\sqrt{\sqrt{1}A})} \right|^4. \]

**Corollary 7.4.** From the expression (7.4) we can see easily that \( Det \Delta_T^2 \) is periodic with respect to the parameter \( A \), and when \( A = 0 \) we have both of

\[ \lim_{B \to 0} Det \Delta_T^2 = 0, \]

\[ \lim_{B \to \infty} Det \Delta_T^2 = 0. \]

Now we explain the relation of (7.4) with the Kroneker’s second limit formula (see [7] where another explanation is given.). By using the integral representation of the modified Bessel function \( K_\alpha(z) \) ([1]):

\[ K_\alpha(z) = \frac{1}{2} \left( \frac{z}{2} \right)^\alpha \int_0^{\infty} e^{-t - \frac{z^2}{4t}} t^{-\alpha-1} dt, |\text{arg} z| < \frac{\pi}{4}, \]
the function $\mathcal{H}_0(s)$ is expressed as

$$
\mathcal{H}_0(s) = \frac{8B^{s+1/2}}{(4\pi)^s \cdot \Gamma(s)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \cos \left(2\pi \sqrt{-Anm}\right) \left(\frac{m}{n}\right)^{-1/2} K_{1/2-s}(2\pi B nm).
$$

Then from this we have a functional relation of the function $\mathcal{H}_0(s)$:

**Proposition 7.5.**

$$
H_0(1-s) = \frac{(4\pi)^{2s-1} \cdot \Gamma(s)}{\Gamma(1-s) \cdot B^{2s-1}} \mathcal{H}_0(s),
$$

especially

$$
\mathcal{H}_0(1) = \frac{B}{4\pi} \mathcal{H}'(0).
$$

This relation (7.5) together with the functional relation of Riemann $\zeta$-function gives us a very simple functional relation of the spectral zeta-function $\zeta_{T_L^2}(s)$ for two dimensional flat torus $T_L^2$.

**Corollary 7.6.**

$$
\Gamma(1-s) \cdot \zeta_{T_L^2}(1-s) = \left(\frac{4\pi}{B}\right)^{2s-1} \cdot \Gamma(s) \cdot \zeta_{T_L^2}(s).
$$

From the formula (7.1) we can easily see that the function $\zeta_{T_L^2}(s)$ has (only) a pole of order one at $s = 1$, which comes from that of the second term in (7.1) and the **Kroneker’s second limit formula** gives the constant term at this pole, that is, by the above relation (7.7) of the first term $\mathcal{H}_0(s)$ we have

**Proposition 7.7. (Kroneker’s second limit formula)**

$$
\lim_{s \to 1} \left\{ \zeta_{T_L^2}(s) - \frac{1}{2s-2} \right\} = \mathcal{H}_0(1) + \lim_{s \to 1} \frac{2\sqrt{\pi} B \cdot \Gamma(s - 1/2)}{(4\pi)^s \cdot \Gamma(s)} \left\{ \zeta(2s - 1) - \frac{1}{2s-2} \right\} + \frac{2B^2}{4\pi^2} \cdot \zeta(2)
$$

$$
= \frac{B}{4\pi} \mathcal{H}'(0) + \frac{B}{2} C_e + \frac{2B^2}{4\pi^2} \cdot \zeta(2).
$$

This gives

**Corollary 7.8.**

$$
\log \text{Det} \Delta_{T^2} = 4\pi \lim_{s \to 1} \left\{ \zeta_{T_L^2}(s) - \frac{1}{2s-2} \right\} = 2\log B - 2\pi C_e - B \left\{ \frac{\pi}{3} + \frac{2}{\pi} \right\}.
$$

II. Three dimensional flat torus.
Let \( \mathfrak{A} = \begin{pmatrix} 1 & a_{1,2} & a_{1,3} \\ 0 & a_{2,2} & a_{2,3} \\ 0 & 0 & a_{3,3} \end{pmatrix} \) and \( \mathfrak{S} = ^t\mathfrak{A}^{-1} = (g_{i,j}) \), then the dual lattice of \( L_3 \) is generated by the basis \( \{ \mathbf{u}_1^*, \mathbf{u}_2^*, \mathbf{u}_3^* \} \), where \( \mathbf{u}_j^* = \sum_i g_{i,j} \mathbf{e}_i^* \) and we know

\[
\text{Spec}(\Delta_{T_L^3}) = \left\{ 4\pi^2 \left((l g_{3,3} + m g_{3,2} + n g_{3,1})^2 + (m g_{2,2} + n g_{2,1})^2 + (n g_{1,1})^2 \right) \mid n, m, l \in \mathbb{Z} \right\}
\]

Then the spectral zeta-function \( \zeta_{T_L^3}(s) \) is

\[
\frac{1}{(4\pi)^s} \sum_{\substack{n, m, l \in \mathbb{Z} \\mid n^2 + m^2 + l^2 \neq 0}} \frac{1}{I(n, m)^s} \int_0^\infty e^{-\left(1 + \frac{(l g_{3,3} + m g_{3,2} + n g_{3,1})^2}{I(n, m)}\right)x} x^{s-1} dx
\]

Put \( (m g_{2,2} + n g_{2,1})^2 + (n g_{1,1})^2 = I(n, m) \) and as before we express \( \zeta_{T_L^3}(s) \) as

\[
\zeta_{T_L^3}(s) = \frac{1}{(4\pi)^s} \frac{1}{\Gamma(s)} \sum_{\substack{n, m, l \in \mathbb{Z} \\mid n^2 + m^2 + l^2 \neq 0}} \frac{1}{I(n, m)^s} \int_0^\infty e^{-\left(1 + \frac{(l g_{3,3} + m g_{3,2} + n g_{3,1})^2}{I(n, m)}\right)x} x^{s-1} dx
\]

\[+ \frac{2}{(2\pi g_{3,3})^{2s}} \cdot \zeta(2s)
\]

Then this equals to the expression:

\[
\frac{1}{(4\pi)^s} \frac{1}{\Gamma(s)} \sum_{\substack{n, m, l \in \mathbb{Z} \\mid n^2 + m^2 + l^2 \neq 0}} \frac{1}{I(n, m)^s} \int_0^\infty \sum_{l \in \mathbb{Z}} e^{-\left(l + m \frac{g_{3,3}^2 + n \frac{g_{3,1}^2}{g_{3,3}}}{g_{3,3}}\right) x} \frac{2 \pi}{g_{3,3}} \frac{1}{I(n, m)^x} e^{-x x^{s-1} dx}
\]

\[+ \frac{2}{(2\pi g_{3,3})^{2s}} \cdot \zeta(2s)
\]

\[= \frac{1}{(4\pi)^s} \frac{1}{\Gamma(s)} \sum_{\substack{n, m, l \in \mathbb{Z} \\mid n^2 + m^2 + l^2 \neq 0}} \frac{1}{I(n, m)^s} \int_0^\infty \sqrt{\frac{\pi I(n, m)}{g_{3,3}^2 x}} \times
\]

\[\times \sum_{l \in \mathbb{Z}} e^{-\frac{\pi I^2(n, m)}{g_{3,3}^2 x}} e^{2 \pi \sqrt{\pi \left(m \frac{g_{3,2}^2 + n \frac{g_{3,1}^2}{g_{3,3}}}{g_{3,3}}\right)}} e^{-x x^{s-1} dx} + \frac{2}{(2\pi g_{3,3})^{2s}} \cdot \zeta(2s)
\]

\[= \frac{\sqrt{\pi}}{g_{3,3} (4\pi)^s} \frac{1}{\Gamma(s)} \times
\]

\[\times \sum_{\substack{n, m, l \in \mathbb{Z} \\mid n^2 + m^2 + l^2 \neq 0}} \frac{1}{I(n, m)^s-1/2} \sum_{l \in \mathbb{Z}} e^{2 \pi \sqrt{\pi \left(\frac{g_{3,2}^2 + n \frac{g_{3,1}^2}{g_{3,3}}}{g_{3,3}}\right)}} \int_0^\infty e^{-\frac{\pi I^2(n, m)}{g_{3,3}^2 x}} e^{-x x^{s-3/2} dx}
\]
\[
+ \frac{\sqrt{\pi}}{g_{3,3}(4\pi^2)^s} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{\substack{n,m \in \mathbb{Z} \\ n^2 + m^2 \neq 0}} \frac{1}{I(n, m)^{s-1/2}} + \frac{2}{(2\pi g_{3,3})^{2s}} \cdot \zeta(2s).
\]

We put this as \(A_0(s) + A_1(s) + A_2(s)\), and calculate each \(A'_i(0)\).

(a)

\[
A_0(0) = 0
\]

\[
A'_0(s) = \sum_{\substack{n,m \in \mathbb{Z} \\ n^2 + m^2 \neq 0}} \log \left( 1 - e^{-2\pi \sqrt{I(n,m)}/g_{3,3} + 2\pi \sqrt{-1} (n \cdot g_{3,3} + n \cdot g_{3,3})} \right)
\]

\[
- \sum_{\substack{n,m \in \mathbb{Z} \\ n^2 + m^2 \neq 0}} \log \left( 1 - e^{-2\pi \sqrt{I(n,m)/g_{3,3} - 2\pi \sqrt{-1} (m \cdot g_{3,3} + n \cdot g_{3,3})}} \right)
\]

\[
= -2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \log \left( 1 - e^{-2\pi \sqrt{I(n,m)/g_{3,3} + 2\pi \sqrt{-1} (m \cdot g_{3,3} + n \cdot g_{3,3})}} \right)
\]

\[
- 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \log \left( 1 - e^{-2\pi \sqrt{I(n,m)/g_{3,3} - 2\pi \sqrt{-1} (m \cdot g_{3,3} + n \cdot g_{3,3})}} \right)
\]

(b)

\[
A_1(s) = \frac{\sqrt{\pi}}{g_{3,3}(4\pi^2)^s} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{\substack{n,m \in \mathbb{Z} \\ n^2 + m^2 \neq 0}} \frac{1}{I(n, m)^{s-1/2}}
\]

\[
= \frac{1}{2\sqrt{\pi} g_{1,1}^{2s-1} g_{3,3}} \frac{\Gamma(s + 1/2)}{\Gamma(s + 1)} \frac{1}{s - 1/2} \cdot \zeta_{T_L^2}^2(s - 1/2),
\]

where \(\zeta_{T_L^2}(s)\) is the spectral zeta-function of two dimensional torus \(T_L^2\) corresponding to the lattice \(L = L_{A,B}\), \(A = \frac{g_{2,1}}{g_{2,2}}, B = \frac{g_{1,1}}{g_{2,2}}\). Then

\[
A_1(0) = 0
\]

\[
A'_1(0) = -\frac{g_{1,1}}{g_{3,3}} \cdot \zeta_{T_L^2}(-1/2).
\]
Next we express the value $\zeta_{T_L^2}(-1/2)$ in terms of a modified Bessel function $K_\alpha(z)$ for $\alpha = 1$:

$$K_\alpha(z) = \frac{1}{2} \left(\frac{z}{2}\right)^\alpha \int_0^\infty e^{-t \cdot \frac{z^2}{4}} t^{-\alpha} \, dt, \quad |\arg z| < \frac{\pi}{4}.$$ 

So we return to the expression (7.1):

$$\zeta_{T_L^2}(s) = \mathcal{H}_0(s) + \frac{2\sqrt{\pi}B \cdot \Gamma(s - 1/2)}{(4\pi^2)^s \cdot \Gamma(s)} \cdot \zeta(2s - 1) + \frac{2B^{2s}}{(4\pi^2)^s} \cdot \zeta(2s).$$

Then

$$\mathcal{H}_0(-1/2)
= -2\pi B \sum_{n=1}^{\infty} n^2 \int_0^\infty \sum_{m \in \mathbb{Z}, m \neq 0} e^{-\frac{(\pi Bnm)^2}{x}} e^{-2\pi \sqrt{-TAnm}} e^{-x} \, dx
= -8 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{n}{m}\right) K_1(2\pi Bnm) \cos(2\pi Anm),$$

$$\frac{2\sqrt{\pi}B \cdot \Gamma(s - 1/2)}{(4\pi^2)^s \cdot \Gamma(s)} \cdot \zeta(2s - 1) |_{s=-1/2} = -\frac{2B}{\pi} \cdot \zeta(3),$$

$$\frac{2B^{2s}}{(4\pi^2)^s} \cdot \zeta(2s) |_{s=-1/2} = -\frac{\pi}{3B}.$$ 

Hence

$$A_1'(0) = \frac{g_{1,1}}{g_{3,3}} \left\{ 8 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n}{m} K_1(2\pi Bnm) \cos(2\pi Anm) + \frac{2B}{\pi} \cdot \zeta(3) + \frac{\pi}{3B} \right\}.$$

(c)

$$A_2(0) = -1$$

$$A_2'(0) = 2 \log g_{3,3}.$$

Finally, for the lattice $L_3 = \{u_i\}_{i=1}^3$ we have an expression of the zeta-regularized determinant

**Theorem 7.9.**

$$\text{Det} \ \Delta_{T_L^3}$$

$$= \prod_{n=1}^{\infty} \prod_{m=1}^{\infty} \left| 1 - e^{-2\pi \left(\sqrt{\frac{\pi B(n,m)}{g_{3,3}}} + \sqrt{-\frac{mg_{3,2} + mg_{3,1}}{g_{3,3}}} \right)} \right|^4 \times$$

$$\times \prod_{n=1}^{\infty} \prod_{m=1}^{\infty} \left| 1 - e^{-2\pi \left(\sqrt{\frac{\pi B(n,m)}{g_{3,3}}} + \sqrt{-\frac{mg_{3,2} + mg_{3,1}}{g_{3,3}}} \right)} \right|^4 \times$$
\[
\times \prod_{n=1}^{\infty} \left| 1 - e^{2\pi n \left( \frac{\sqrt{g_{1,1}^2+g_{1,1}^2}}{g_{3,3}} + \frac{\sqrt{g_{3,3}^2}}{g_{3,3}} \right)} \right|^4 
\times e^{-\frac{g_{1,1}}{g_{3,3}} \left( \sum_{n=1}^{\infty} \frac{n}{m} \right) K_1 \left( 2\pi \frac{g_{1,1}}{g_{2,2}} nm \right) \cos \left( 2\pi \frac{g_{2,1}}{g_{2,2}} nm \right)} \right) \times \frac{2}{2 \pi \frac{g_{1,1}}{g_{3,3}}} \zeta(3) \cdot e^{-\frac{2}{2 \pi \frac{g_{1,1}}{g_{3,3}}}} \times \left( \frac{1}{g_{3,3}} \right)^2 .
\]

**Remark 7.10.** As a special case of $T^3$, let assume $a_{1,3} = a_{2,3} = 0$. Then the formula of $\text{Det} \Delta_{T^3}$ for this case coincides with the formula (6.1) for $T^2 \times S^1$.

### III. Four dimensional flat torus

Here we only state a formula for a four dimensional torus and a special case of them.

Let $\mathfrak{A} = \begin{pmatrix} 1 & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & a_{2,2} & a_{2,3} & a_{2,4} \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & 0 & a_{4,4} \end{pmatrix}$ and $u_j = \sum_i a_{i,j} e_i$ as explained in the beginning of this section and $\mathfrak{G} = \mathfrak{A}^{-1} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4} \\ \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & \alpha_{4,4} \end{pmatrix} = (g_{i,j})$.

Let $L = L(\mathfrak{A})$ be the lattice generated by $\{u_1, u_2, u_3, u_4\}$, then the dual lattice of $L(\mathfrak{A})$ is generated by the basis $\{u_1^*, u_2^*, u_3^*, u_4^*\}$, where $u_j^* = \sum_i \alpha_{i,j} e_i^*$.

Put

\[
(lg_{3,3} + mg_{3,2} + ng_{3,1})^2 + (mg_{2,2} + ng_{2,1})^2 + (ng_{1,1})^2 = I(n, m, l),
\]

and

\[
\frac{ng_{4,1} + mg_{4,2} + lg_{4,3}}{g_{4,4}} = \alpha(n, m, l)
\]

then the spectral zeta-function $\zeta_{T^4_{L(\mathfrak{A})}}(s)$ for this case is written as

\[
\zeta_{T^4_{L(\mathfrak{A})}}(s) = \frac{1}{(4\pi^2)^s} \sum_{n,m,l \in \mathbb{Z}} \frac{1}{I(n, m, l) + (ng_{4,1} + mg_{4,2} + lg_{4,3} + kg_{4,4})^2} s
\]

\[
= \frac{1}{(4\pi^2)^s} \frac{1}{\Gamma(s)} \sum_{n,m,l \in \mathbb{Z}} \frac{1}{I(n, m, l)} \int_0^\infty \sum_{k \in \mathbb{Z}} e^{-k + \alpha(n, m, l)} \frac{g_{4,4}^2}{I(n, m, l)^{2}} x e^{-x} x^{-1} dx
\]

\[+ \frac{2}{(2\pi g_{4,4})^{2s}} \cdot \zeta(2s)
\]

\[
= \frac{\sqrt{\pi}}{g_{4,4} (4\pi^2)^s \Gamma(s)} \times
\]
\begin{align*}
\times & \sum_{n,m,l \in \mathbb{Z}} \frac{1}{I(n, m, l)^{s-1/2}} \int_0^\infty \sum_{k \in \mathbb{Z}, k \neq 0} \frac{-e^{2i\pi l \cdot \sqrt{-1 \alpha(n, m, l)}}}{\sqrt{\pi} g_{4, 4}^2} I(n, m, l) e^{2\pi \sqrt{-1 \alpha(n, m, l)} \cdot x} x^{s-3/2} dx \\
+ & \frac{\sqrt{\pi}}{g_{4, 4} (2\pi)^{2s}} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{n,m,l \in \mathbb{Z}} \frac{1}{I(n, m, l)^{s-1/2}} + \frac{2}{(2\pi g_{4, 4})^{2s}} \cdot \zeta(2s).
\end{align*}

We put this as \( B_0(s) + B_1(s) + B_2(s) \) corresponding to each term.

Note that
\[ B_1(s) = \frac{1}{2\sqrt{\pi} g_{4, 4}} \frac{\Gamma(s + 1/2)}{\Gamma(s + 1)} \cdot \frac{s}{s - 1/2} \cdot \zeta_T(s - 1/2), \]
and at \( s = -1/2 \), \( \zeta_T(s) \) is holomorphic.

**Theorem 7.11.**

\[ \text{Det } \Delta_{T^4_{L(\mathfrak{a})}} = \prod_{i=0}^2 e^{-B'_i(0)}, \]

where each \( B'_i(0) \) is given as follows:

\[ B_0'(0) = 0, \]

\[ B_1'(0) = -\frac{1}{g_{4, 4}} \cdot \zeta_T(-1/2) = -\frac{1}{g_{4, 4}} \{ A_0(-1/2) + A_1(-1/2) + A_2(-1/2) \} \]

\[ = \frac{1}{g_{4, 4}} \left\{ \sum_{n,m,l \in \mathbb{Z}} \sum_{n^2 + m^2 + l^2 \neq 0} 4 \frac{I(n, m)}{l} \cos \left( 2\pi \frac{mg_{3,2} + ng_{4,1}}{g_{3,3}} \right) \cdot K_1 \left( \frac{2\pi}{g_{3,3}} \sqrt{\frac{I(n, m)}{l}} \right) \right. \\
+ 8 \sqrt{\pi} \frac{g_{2,1}}{g_{3,3}} \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{n^2 m^2}{g_{2,1} g_{2,2}} \cos \left( 2\pi \frac{g_{2,1} n m}{g_{2,2}} \right) K_1 \left( \frac{2\pi g_{1,1}}{g_{2,2}} n m \right) \\
+ \frac{3g_{1,1}^2}{4\pi^3 g_{3,3}} \cdot \zeta(4) + \frac{2g_{2,2}^2}{\pi g_{3,3}} \cdot \zeta(3) + \frac{\pi}{3} \frac{g_{3,3}}{g_{3,3}} \right\}, \]

\[ B_2(0) = -1, \]

\[ B_2'(0) = 2 \log g_{4, 4}. \]
The formula in Theorem 5.7 and similar one give us several formulas of the zeta-regularized determinant for flat tori defined by matrices of the form \( \mathbf{A} = \begin{pmatrix} 1 & a_{1,2} & a_{1,3} & 0 \\ 0 & a_{2,2} & a_{2,3} & 0 \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & 0 & a_{4,4} \end{pmatrix} \) or \( \mathbf{A} = \begin{pmatrix} 1 & a_{1,2} & 0 & 0 \\ 0 & a_{2,2} & 0 & 0 \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & 0 & a_{4,4} \end{pmatrix} \).

As an application of the formula (5.7) we state a formula for a torus defined by the latter one, that is, the torus is a direct product of two 2-dimensional tori as Riemannian manifold.

Let \( \mathbf{A} = \begin{pmatrix} 1 & a_{1,2} & 0 & 0 \\ 0 & a_{2,2} & 0 & 0 \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & 0 & a_{4,4} \end{pmatrix} \) and \( L_\mathbf{A} \) the lattice in \( \mathbb{R}^4 \) generated by \( \{ \mathbf{u}_1 = e_1, \mathbf{u}_2 = a_{1,2}e_1 + a_{2,2}e_2, \mathbf{u}_3 = a_{3,3}e_3, \mathbf{u}_4 = a_{3,4}e_3 + a_{4,4}e_4 \} \). Then the torus \( \mathbb{R}^4/L(\mathbf{A}) \) is a direct product of two tori \( M \times N \), where \( M = \mathbb{R}^2/L_M \), \( L_M = \{ \{ \mathbf{u}_1, \mathbf{u}_2 \} \} \) and \( N = \mathbb{R}^2/L_N \), \( L_N = \{ \{ \mathbf{u}_3, \mathbf{u}_4 \} \} \). Since each heat kernel asymptotics for flat tori vanishes except the first one = \( b_0 = \text{volume of the torus} \), we can rewrite the formula (5.7) for this case as

**Corollary 7.12.**

\[
\det \Delta_{\mathbb{R}^4/M} = \det \Delta_M \cdot \prod_{\lambda \in L_M^*, \lambda \neq 0} e^{-\int_0^\infty \left( K_N \left( \frac{t}{\| \lambda \|^2} \right) - \frac{\| \lambda \|^2 b_0}{\pi t} \right) e^{-t^{-1}dt}} \cdot e^{-\frac{b_0}{\pi} \lim_{s \to 0} (\Gamma(s-1) \cdot \zeta_M(s-1))},
\]

where

\[
\det \Delta_M = B^2 e^{-\frac{B}{4} \sum_{n=1}^\infty \left( 1 - e^{-2\pi n(B - \sqrt{-TA})} \right)^4},
\]

\[
\prod_{\lambda \in L_M^*, \lambda \neq 0} e^{-\int_0^\infty \left( K_N \left( \frac{t}{\| \lambda \|^2} \right) - \frac{\| \lambda \|^2 b_0}{\pi t} \right) e^{-t^{-1}dt}} = \prod_{\lambda \in L_M^*, \lambda \neq 0} \prod_{\gamma \in L_N, \gamma \neq 0} e^{-\frac{\| \lambda \|^2 b_0}{\pi \| \gamma \|^2} K_1(\| \lambda \|, \| \gamma \|)},
\]

and

\[
\lim_{s \to 0} \Gamma(s-1) \cdot \zeta_M(s-1) = \lim_{s \to -1} \left\{ \frac{2}{(4\pi^2)^s} \sum_{n=1}^\infty \frac{1}{n^{2s}} \int_0^\infty \sqrt{\frac{\pi B^2 n^2}{x}} e^{\frac{(\pi B n)^2}{x}} e^{-2\pi \sqrt{TA} n m} e^{-x} x^{s-1} dx \right. \\
+ \frac{2\sqrt{\pi B}}{(4\pi^2)^s} \cdot \Gamma(s-1/2) \cdot \zeta(2s-1) \\
+ \Gamma(s) \cdot B^{2s} \cdot \frac{\Gamma(s) \cdot B^{2s}}{(4\pi^2)^s} \cdot \zeta(2s) \right\}
\]

\[
= \frac{32\pi}{\sqrt{B}} \sum_{n=1}^\infty \sum_{m=1}^\infty \left( \frac{n}{m} \right)^{3/2} \cos(2\pi Anm) K_{3/2}(2\pi Bnm) + \frac{8\pi}{B} \cdot \zeta(4) + \frac{4}{B^2} \cdot \zeta(3)
\]
\[
\left(\frac{4\pi}{B}\right)^3 \Gamma(2)\zeta_{T^2}(2) = 4\pi B \sum_{n,m\in\mathbb{Z}, n^2+m^2\neq 0} \frac{1}{(Bn)^2 + (m-nA)^2},
\]

here \(B = \frac{g_{1,1}}{g_{2,2}}, \ A = -\frac{g_{2,1}}{g_{2,2}}\), \(b_0 = \text{volume of } N = a_{3,3}a_{4,4}\).

Note that the second term is obtained by making use of the Jacobi identity:

\[
\sum_{\mu\in L_N} e^{-t\|\mu\|^2} = \frac{b_0}{4\pi t} \sum_{\gamma\in L_N} e^{-\frac{\|\gamma\|^2}{4t}}.
\]

Finally, we note that in the most special case, that is, let the matrix \(A\) be the identity matrix, then we have a formula for the spectral zeta-function \(\zeta_{T^4}(s)\)

\[
(4\pi^2)^{-s}\zeta_{T^4}(s) = 8(1 - 2^{2-2s})\zeta(s)\zeta(s-1)
\]

(and in each dimension we have similar formulas). So by this formula we have simply

**Corollary 7.13.** Let the torus \(T^4\) be defined by the lattice \(\{e_1,e_2,e_3,e_4\}\), then the zeta-regularized determinant \(\det \Delta_{T^4}\) is given explicitly in the form

\[
\log \det \Delta_{T^4} = 2^4 \left(\log 2\pi + 2(\log 2)^2\right) \zeta(0)\zeta(-1) - 2^3 \left(\zeta'(0)\zeta(-1) + \zeta(0)\zeta'(-1)\right)
\]

It is possible to simplify this formula by using several formulas of Riemann zeta-function \(\zeta(s)\) and to compare with our formula.

**Remark 7.14.** Similar to the case of \(\zeta_{T^2}(s)\), the function \(\zeta_{T^4_L}(s)\) (respectively \(\zeta_{T^4}(s)\)) has only a pole at \(s = 3/2\) (resp. \(s = 2\)) of order one coming from the second term \(A_1(s)\) (respectively \(B_1(s)\)) and the term \(A_0(s)\) (resp. \(B_0(s)\)) will correspond to the term \(H_0(s)\) in the two dimensional cases and there are similar functional relations like (7.3) also in these cases which are derived from the Jacobi identity.

**References**

[1] G. E. Andrews, R. Askey and R. Roy: *Special functions, Encyclopedia of mathematics and its applications*, Vol. 71 (1999), Cambridge Univ. Press

[2] B. C. Berndt: *Identities involving the coefficients of a class of Dirichlet series V*, Trans. Amer. Math. Soc., Vol. 160 (1971), 139-156

[3] __________: *Identities involving the coefficients of a class of Dirichlet series VI*, Trans. Amer. Math. Soc., Vol. 160 (1971), 157-167

[4] J. Bolte and F. Steiner: *Determinants of Laplace-like Operators on Riemann Surfaces*, Comm. Math. Phy., Vol. 130 (1990), 581-597.

[5] E. D’Hoker and D. H. Phong: *The geometry of string perturbation theory, Rev Mod Phys Vol. 60*(1988), 917-1065

[6] C. Gordon and E. Wilson: *The spectrum of the Laplacian on Riemannian Heisenberg manifolds, Michigan Math. J. Vol.33*(1986), 253-271
[7] R. Forman: Functional determinants and geometry, Invent. math., Vol. 88 (1987), 447-493
[8] K. Furutani, K. Sagami and N. Otsuki: The spectrum of the Laplacian on a certain nilpotent Lie group, Comm. Partial Differential Equations, 18, No. 3&4 (1993), 533-555
[9] K. Furutani: The heat kernel and the spectrum of a class of nilmanifolds, Comm. Partial Differential Equations, 21, No. 3&4 (1996), 423-438
[10] H. P. Mckean: Selberg’s trace formula as applied to compact Riemann surfaces, Comm. Pure Appl. Math., Vol. 25 (1972), 225-271.
[11] Y. Motohashi: A new proof of the limit formula of Kronecker, Proc. Japan Acad. Vol. 44 (1968), 614-616.
[12] C. Nash and D.J. O’Connor: Determinants of Laplacians, Ray-Singer torsion on lens spaces and the Riemann zeta function, J. Math. Phys., Vol. 36 No. 3 (1995), 1462-1505.
[13] J. R. Quine, S.H. Heydari and R.Y. Song: Zeta Regularized Products, Transaction of the American Math. Soc. Vol. 338 (1993), No. 1, 213-231
[14] D.B. Ray and I.M. Singer: R-Torsion and the Laplacian on Riemannian manifolds, Adv. Math., 7 (1971), 145-210
[15] I. Vardi: Determinants of Laplacians and multiple gamma functions SIAM J. Math. Anal., Vol. 19 (1988), 493-507.

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