Entanglement detection in permutation symmetric states based on violation of local sum uncertainty relation

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(Dated: March 30, 2021)

We show that violation of variance based local sum uncertainty relation (LSUR) for angular momentum operators of a bipartite system, proposed by Hofmann and Takeuchi [Phys. Rev. A 68, 032103 (2003)], is necessary and sufficient for entanglement in two-qubit permutation symmetric state. Moreover, we also establish its one-to-one connection with negativity of covariance matrix [Phys. Lett. A 364, 203 (2007)] of the two-qubit reduced system of a permutation symmetric N-qubit state. Consequently, it is seen that the violation of the angular momentum LSUR serves as a necessary condition for pairwise entanglement in N-qubit system, obeying exchange symmetry. We illustrate physical examples of entangled permutation symmetric N-qubit systems, where violation of the local sum uncertainty relation manifests itself as a signature of pairwise entanglement.

PACS numbers: 03.65.Ud, 03.67.-a

I. INTRODUCTION

Uncertainty relations place fundamental limits on the precision achievable in measuring non-commuting observables. The original idea of uncertainty relation was first introduced by Heisenberg [1] for position and momentum observables \( Q, P \). Subsequently, a mathematically precise version (with \( \hbar = 1 \))

\[
(\Delta Q)^2 (\Delta P)^2 \geq \frac{1}{4}
\]  

(1)

was formulated by Kennard [2]. Here \( (\Delta \Gamma)^2 = \langle \Gamma^2 \rangle - \langle \Gamma \rangle^2 \) denotes the variance of the observable \( \Gamma = Q \) or \( P \) and the bracket \( \langle \cdots \rangle = \text{Tr} [\rho \cdots] \) corresponds to the expectation value in a quantum state \( \rho \).

Weyl [3] and Robertson [4] extended the uncertainty relation [1] for any arbitrary pair of physical observables \( A_1, A_2 \):

\[
(\Delta A_1)^2 (\Delta A_2)^2 \geq \frac{1}{4} |(A_1, A_2)|^2
\]  

(2)

which is commonly referred to as the Heisenberg-Robertson uncertainty relation in the literature. Here \( [A_1, A_2] = A_1 A_2 - A_2 A_1 \) denotes the commutator of the observables \( A_1 \) and \( A_2 \). The uncertainty relation [2] imposes restrictions on the product of variances \( (\Delta A_1)^2 \) and \( (\Delta A_2)^2 \) – essentially limiting the capability towards precise prediction of the measurement results of non-commuting observables. In general the Heisenberg-Robertson approach of placing limits on the uncertainties of incompatible observables in the given quantum state \( \rho \) sets the conventional framework for deriving preparatory uncertainty relations [2, 4].

Apart from their fundamental interest, uncertainty relations play a significant role in the field of quantum information processing, with several applications such as entanglement detection [2, 4], quantum cryptography [10, 11], quantum metrology [12] and foundational tests of quantum theory [13, 14]. Motivated by their applicability, there has been an ongoing interest in reformulating uncertainty relations expressing trade-off of more than two incompatible observables, formalized in terms of variances [7, 8, 20–34], or via information entropies [35–44]. Recently several experimental tests have been carried out to verify different forms of uncertainty relations [45–48].

In this paper we investigate variance based LSUR for local angular momentum operators of a bipartite quantum system, proposed by Hofmann and Takeuchi [1], violation of which witnesses entanglement. One of the main advantages of employing LSUR is the fact that it is possible to detect entanglement without a complete knowledge of the quantum state and it suffices to determine experimental friendly variances of local angular momentum observables for this purpose. We show that the angular momentum LSUR, which places lower bound on the set of all bipartite separable states, gets violated if and only if the covariance matrix of the two-qubit reduced system of the N-qubit permutation symmetric state is negative. Since it has been shown [49] that the covariance matrix negativity serves as a necessary and sufficient for entanglement in a two-qubit symmetric system, our result establishes a one-to-one connection between entanglement and violation of the LSUR. Furthermore, we show that negative covariance matrix of the two-qubit reduced system of a N-qubit symmetric state leads to a variance based test of pairwise entanglement.

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II. SUM UNCERTAINTY RELATION

It may be noted that the term $\langle [A_1, A_2] \rangle = \text{Tr}(\rho [A_1, A_2])$ appearing in the right hand side of the Heisenberg-Robertson uncertainty relation (2) vanishes in some specific quantum states $\rho$. In such cases, one ends up with a trivial relation $(\Delta A_1)^2(\Delta A_2)^2 \geq 0$ for the product of variances of non-commuting observables $A_1, A_2$. Moreover, variance vanishes in the eigenstate of one of the observables. In such cases the Heisenberg-Robertson uncertainty relation (2) fails to capture the intrinsic indeterminacy of non-commuting observables. To overcome such issues it is preferable to employ uncertainty relations placing non-trivial bounds on the sum of variances $(\Delta A_1)^2 + (\Delta A_2)^2$. In fact, a lower bound for the sum of variances may be found by using the inequality $\sum_{\alpha=1}^m a_\alpha/m \geq (\prod_{\alpha} a_\alpha)^{1/m}$. Choosing $a_1 = (\Delta A_1)^2$, $a_2 = (\Delta A_2)^2$ and using (2) one obtains a variance based sum uncertainty relation

$$ (\Delta A_1)^2 + (\Delta A_2)^2 \geq \big| \langle [A_1, A_2] \rangle \big|. $$

(3)

However the sum uncertainty relation (3) is a byproduct of the Heisenberg-Robertson inequality (2) and thus it is non-informative for some quantum states $\rho$ in which one of the variances and/or $\langle [A_1, A_2] \rangle$ vanish.

A. Sum uncertainty relations for angular momentum operators

Hofmann and Takeuchi $[7]$ proposed that a non-trivial lower bound $U > 0$ must exist for the sum of variances of a set $\{A_\alpha\}$, $\alpha = 1, 2, \ldots, m$ of non-commuting observables, as they do not share any simultaneous eigenstate:

$$ \sum_{\alpha=1}^m (\Delta A_\alpha)^2 \geq U. $$

(4)

The lower bound $U$ in the sum uncertainty relation (4) corresponds to the absolute minimum value $[\sum_{\alpha} (\Delta A_\alpha)^2]_{\text{min}}$ for any quantum state $\rho$. While it is, in general, tough to determine $U$ for any arbitrary set $\{A_\alpha\}$ of non-commuting observables, there are some important physical examples in finite dimensional Hilbert spaces, where the limiting value $U$ can be readily identified. For example, in the case of a spin-$j$ quantum system the components of angular momentum operators $\{J_1, J_2, J_3\}$ satisfy the following conditions:

$$ \langle (J_1^2 + J_2^2 + J_3^2) \rangle = j(j+1), $$

$$ \langle J_1^2 \rangle^2 + \langle J_2^2 \rangle^2 + \langle J_3^2 \rangle^2 \leq j^2. $$

Thus, one obtains a variance based sum uncertainty relation for the components $(J_1, J_2, J_3)$ of angular momentum operator:

$$ (\Delta J_1)^2 + (\Delta J_2)^2 + (\Delta J_3)^2 \geq j, $$

(5)

imposing a limit on the measurement precision of more than one of the angular momentum components.

In the specific example of $j = 1/2$ (i.e., a qubit), we have $J_\alpha = \sigma_\alpha/2, \alpha = 1, 2, 3$, where $\sigma_\alpha$ denote the Pauli matrices. One then obtains the sum uncertainty relation for qubits $[8]$:

$$ (\Delta \sigma_1)^2 + (\Delta \sigma_2)^2 + (\Delta \sigma_3)^2 \geq 2. $$

(6)

Given an arbitrary single qubit density matrix $\rho$, expressed in the standard basis $\{|0\rangle, |1\rangle\}$,

$$ \rho = \frac{1}{2} \left( I + \sum_{\alpha=1}^3 s_\alpha \sigma_\alpha \right), $$

(7)

$$ s_\alpha = \text{Tr}[\rho \sigma_\alpha]; \quad \sum_{\alpha} s_\alpha^2 \leq 1, $$

one obtains $(\Delta \sigma_1)^2 + (\Delta \sigma_2)^2 + (\Delta \sigma_3)^2 = 3 - \sum_{\alpha} s_\alpha^2 \geq 2$, in accordance with the sum uncertainty relation (6). Equality sign holds when $\sum_{\alpha} s_\alpha^2 = 1$ i.e., for pure states of qubit.

B. Local sum uncertainty relations for bipartite systems

Let us consider angular momentum operators $J_{A\alpha}, J_{B\alpha}$ acting on the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ of dimensions $d_A = (2j_A + 1), d_B = (2j_B + 1)$ respectively. They satisfy the sum uncertainty relations

$$ \sum_{\alpha=1}^3 (\Delta J_{A\alpha})^2 \geq j_A, \quad \sum_{\alpha=1}^3 (\Delta J_{B\alpha})^2 \geq j_B. $$

(8)

Then, the LSUR $[8]$

$$ \sum_{\alpha=1}^3 |\Delta (J_{A\alpha} + J_{B\alpha})|^2 \geq j_A + j_B $$

(9)

is necessarily satisfied by the set of all bipartite separable states $\rho_{AB}^{\text{sep}} = \sum_k p_k (\rho_{Ak} \otimes \rho_{Bk})$ in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Violation of the LSUR (9) serves as a clear signature of entanglement. This is readily seen by considering a bipartite system with $j_A = j_B = j$ prepared in a spin singlet state

$$ |\Psi_{AB}^{\text{singlet}}\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} (-1)^{j-m} |j, m\rangle_A \otimes |j, -m\rangle_B $$

in which one obtains

$$ (J_{A\alpha} + J_{B\alpha}) |\Psi_{AB}^{\text{singlet}}\rangle \equiv 0, \quad \alpha = 1, 2, 3, $$

leading to

$$ \sum_{\alpha=1}^3 |\Delta (J_{A\alpha} + J_{B\alpha})|^2 = 0. $$
In other words, the LSUR \[ \text{III. VIOLATION OF LSUR BY PERMUTATION SYMMETRIC N-QUBIT STATES} \]

Permutation symmetric \( N \)-qubit states draw attention due to their experimental feasibility and for the mathematical simplicity offered by them. This class of states belong to the \( d = 2j + 1 = N + 1 \) dimensional subspace of the \( 2^N \) dimensional Hilbert space, which corresponds to the maximum value \( j = N/2 \) of angular momentum of \( N \)-qubit system. Invariance under exchange of particles labeled by \( \alpha, \beta \) in a multiparty state \( \rho^{\text{sym}} \) is exhibited by the property

\[
\Pi_{\alpha\beta} \rho^{\text{sym}} = \rho^{\text{sym}} \Pi_{\alpha\beta} = \rho^{\text{sym}}
\]  

(13) where \( \Pi_{\alpha\beta} \) denotes the permutation operator corresponding to the interchange of labels \( \alpha, \beta \).

We focus our attention on permutation symmetric \( N \)-qubit states \( \rho^{\text{sym}} \) with even \( N \), i.e., \( N = 2n \), \( n = \text{integer} \) (which corresponds to the value \( j = N/2 = n \) associated with the set \( \{ J^\alpha_{\text{total}} = J_\alpha \otimes I_2^{\otimes n} + I_2^{\otimes n} \otimes J_\alpha, \alpha = 1, 2, 3 \} \) of collective angular momentum operators of the \( N \)-qubit system) and explore the LSUR \[ \text{III. VIOLATION OF LSUR BY PERMUTATION SYMMETRIC N-QUBIT STATES} \]

\[
J_{\alpha A} = J_\alpha \otimes I_2^{\otimes n} = \frac{1}{2} \sum_{k=1}^n \sigma_{k\alpha} \otimes I_2^{\otimes n}
\]

\[
J_{\alpha B} = I_2^{\otimes n} \otimes J_\alpha = I_2^{\otimes n} \otimes \frac{1}{2} \sum_{k=1}^n \sigma_{k\alpha},
\]  

(14) where \( \sigma_{k\alpha} = I_2 \otimes I_2 \otimes \ldots \otimes I_2 \) denotes \( n \)-th order tensor product of \( 2 \times 2 \) identity matrix \( I_2 \) and

\[
\sigma_{k\alpha} = I_2 \otimes I_2 \otimes \ldots \otimes I_2, \quad \alpha = 1, 2, 3
\]  

(15) with \( \sigma_{k\alpha} \), \( \alpha = 1, 2, 3 \) appearing at the position \( k \leq n \) in the \( n \)-th order tensor product. Using angular momentum algebra it readily follows that

\[
\sum_{\alpha=1}^3 \langle J_{\alpha A}^2 \rangle_{\rho^{\text{sym}}} = j_A(j_A + 1) = \frac{n(n+2)}{4}
\]

(16) and

\[
\sum_{\alpha=1}^3 \langle J_{\alpha B}^2 \rangle_{\rho^{\text{sym}}} = j_B(j_B + 1) = \frac{n(n+2)}{4}
\]  

(17)

Furthermore, the expectation values \( \langle J_{\alpha A} \rangle_{\rho^{\text{sym}}}, \langle J_{\alpha B} \rangle_{\rho^{\text{sym}}}, \langle J_{\alpha A} J_{\alpha B} \rangle_{\rho^{\text{sym}}} \) evaluated in the \( N \)-qubit symmetric state \( \rho^{\text{sym}} \) can be expressed in terms of two-qubit averages as follows:

\[
\langle J_{\alpha A} \rangle_{\rho^{\text{sym}}} = \langle J_\alpha \otimes I_2^{\otimes n} \rangle_{\rho^{\text{sym}}}
\]

\[
= \frac{1}{2} \sum_{k=1}^n \langle \sigma_{k\alpha} \otimes I_2^{\otimes n} \rangle_{\rho^{\text{sym}}}
\]

\[
= \frac{n}{2} \langle \sigma_{k\alpha} \otimes I_2 \rangle_{\rho^{\text{sym}}}
\]  

(18) and

\[
\langle J_{\alpha B} \rangle_{\rho^{\text{sym}}} = \langle I_2^{\otimes n} \otimes J_\alpha \rangle_{\rho^{\text{sym}}}
\]

\[
= \frac{1}{2} \sum_{l=1}^n \langle I_2^{\otimes n} \otimes \sigma_{l\alpha} \rangle_{\rho^{\text{sym}}}
\]

\[
= \frac{n}{2} \langle I_2 \otimes \sigma_{l\alpha} \rangle_{\rho^{\text{sym}}}
\]  

(19)

\[
\langle J_{\alpha A} J_{\alpha B} \rangle_{\rho^{\text{sym}}} = \langle J_\alpha \otimes J_\beta \rangle_{\rho^{\text{sym}}}
\]

\[
= \frac{1}{4} \sum_{k,l=1}^n \langle \sigma_{k\alpha} \otimes \sigma_{l\alpha} \rangle_{\rho^{\text{sym}}}
\]

\[
= \frac{n^2}{4} \langle \sigma_{\alpha} \otimes \sigma_{\alpha} \rangle_{\rho^{\text{sym}}}
\]  

(20)
Here it may be noticed that

\[
\begin{align*}
\langle \sigma_{k\alpha} \otimes I_2^{\otimes n} \rangle_{\rho_{AB}^{\text{sym}}} &= \langle \sigma_{\alpha} \otimes I_2 \rangle_{\rho^{\text{sym}}} \\
\langle I_2^{\otimes n} \otimes \sigma_{k\alpha} \rangle_{\rho_{AB}^{\text{sym}}} &= \langle I_2 \otimes \sigma_{\alpha} \rangle_{\rho^{\text{sym}}} \\
\langle \sigma_{k\alpha} \otimes \sigma(\alpha) \rangle_{\rho_{AB}^{\text{sym}}} &= \langle \sigma_{\alpha} \otimes \sigma(\alpha) \rangle_{\rho^{\text{sym}}} 
\end{align*}
\]

irrespective of the qubit labels \(k, l = 1, 2, \ldots n\) (because the system is symmetric under interchange of constituent qubits) \([43, 51]\), with \(\rho^{\text{sym}}\) denoting the two-qubit reduced state of any random pair \((k, l)\) of qubits, drawn from the \(n\)-qubit partitions \(A\) and \(B\) of the \(N = 2n\)-qubit state \(\rho_{AB}^{\text{sym}}\).

Expressing the two-qubit symmetric density matrix

\[
\rho^{\text{sym}} = \frac{1}{4} \left( I_2 \otimes I_2 + \sum_{\alpha=1}^{3} (\sigma_{\alpha} \otimes I_2 + I_2 \otimes \sigma_{\alpha}) s_{\alpha} + \sum_{\alpha, \beta=1}^{3} (\sigma_{\alpha} \otimes \sigma_{\beta}) t_{\alpha\beta} \right)
\]

in terms of its 8 real state parameters \([43, 51]\),

\[
\begin{align*}
s_{\alpha} &= \langle \sigma_{\alpha} \otimes I_2 \rangle_{\rho^{\text{sym}}} = \langle I_2 \otimes \sigma_{\alpha} \rangle_{\rho^{\text{sym}}} \\
t_{\alpha\beta} &= \langle \sigma_{\alpha} \otimes \sigma_{\beta} \rangle_{\rho^{\text{sym}}} = \langle \sigma_{\beta} \otimes \sigma_{\alpha} \rangle_{\rho^{\text{sym}}}
\end{align*}
\]

leads to the following identifications for the expectation values \(\langle J_{A\alpha} \rangle_{\rho_{AB}^{\text{sym}}}, \langle J_{B\alpha} \rangle_{\rho_{AB}^{\text{sym}}}, \langle J_{A\alpha} J_{B\alpha} \rangle_{\rho_{AB}^{\text{sym}}}\) (see \([18, 19, 20]\)):

\[
\begin{align*}
\langle J_{A\alpha} \rangle_{\rho_{AB}^{\text{sym}}} &= \frac{n}{2} s_{\alpha} = \langle J_{B\alpha} \rangle_{\rho_{AB}^{\text{sym}}} \\
\langle J_{A\alpha} J_{B\alpha} \rangle_{\rho_{AB}^{\text{sym}}} &= \frac{n^2}{4} t_{\alpha\alpha}
\end{align*}
\]

It may be noted that under \textit{identical} local unitary transformation \(\rho^{\text{sym}} \rightarrow U^{\otimes 2} \rho^{\text{sym}} (U^{\dagger})^{\otimes 2}, U \in SU(2)\), the qubit orientation vector \(s = (s_1, s_2, s_3)^T\) (see \([21]\)) and the \(3 \times 3\) real symmetric two-qubit correlation matrix \(T = (t_{\alpha\beta})\), \(\alpha, \beta = 1, 2, 3\) (see \([25]\)) transform as \([50]\)

\[
s \rightarrow s' = R s, \quad T \rightarrow T' = RT R^T
\]

where \(R \in SO(3)\) denotes the \(3 \times 3\) rotation matrix.

Let us consider the angular momentum operators of the partitions \(A\) and \(B\) of the even \(N\)-qubit density matrix \(\rho_{AB}^{\text{sym}}\) (See \([14]\))

\[
J_{A\alpha} = J_{\alpha} \otimes I_2^{\otimes n} = \frac{1}{2} \sum_{k=1}^{n} \sigma_{k\alpha} \otimes I_2^{\otimes n}, \quad J_{B\alpha} = I_2^{\otimes n} \otimes \sigma_{k\alpha}
\]

\[
J_{B\alpha} = I_2^{\otimes n} \otimes \left\{ \left[ U^\dagger R \hat{\alpha}, \hat{\theta} \right] \right\} \rceil_{\rho_{AB}^{\text{sym}}} J_{\alpha} \left[ U \hat{\alpha}, \hat{\theta} \rceil_{\rho_{AB}^{\text{sym}}}
\]

\[
= I_2^{\otimes n} \otimes \left\{ \frac{1}{2} \sum_{k=1}^{n} \left( \left[ U^\dagger R \hat{\alpha}, \hat{\theta} \right] \right) \rceil_{\rho_{AB}^{\text{sym}}} \sigma_{k\alpha} \left[ U \hat{\alpha}, \hat{\theta} \rceil_{\rho_{AB}^{\text{sym}}}
\]

\[
= I_2^{\otimes n} \otimes \left\{ \frac{1}{2} \sum_{k=1}^{n} \left( \sigma_{k\alpha} \right) \rceil_{\rho_{AB}^{\text{sym}}} \left[ U \hat{\alpha}, \hat{\theta} \rceil_{\rho_{AB}^{\text{sym}}}
\]

\[
= \frac{3}{2} \sum_{\beta=1}^{n} R_{\alpha\beta} \hat{H} \rceil_{\rho_{AB}^{\text{sym}}}
\]

where \(U \hat{\alpha}, \hat{\theta} \rceil_{\rho_{AB}^{\text{sym}}} \rho_{AB}^{\text{sym}} \rceil_{\rho_{AB}^{\text{sym}}} U^{-1} \hat{\alpha}, \hat{\theta} \)

\[
\sum_{\alpha=1}^{3} | \Delta (J_{A\alpha} + J_{B\alpha}) |^2 \geq n.
\]

is satisfied by the set of all separable symmetric states. Violation of the \textit{LSUR} \([24]\), i.e., \(\sum_{\alpha=1}^{3} | \Delta (J_{A\alpha} + J_{B\alpha}) |^2 < n\) reveals that the bipartite state \(\rho_{AB}^{\text{sym}}\) is entangled.

We define

\[
\chi(\hat{\alpha}, \hat{\theta}) = \frac{1}{n^2} \left( \sum_{\alpha=1}^{3} | \Delta (J_{A\alpha} + J_{B\alpha}) |^2 - n \right)
\]

\[
= \frac{1}{n^2} \left( \sum_{\alpha=1}^{3} \left( \Delta \left[ J_{A\alpha} + \sum_{\beta=1}^{n} R_{\alpha\beta} \hat{H} \right] \rceil_{\rho_{AB}^{\text{sym}}}
\]

\[
\quad \right)^2 - n \right)
\]

and observe that \(\chi(\hat{\alpha}, \hat{\theta}) < 0 \Rightarrow \) the \textit{LSUR} \([24]\) is violated. This in turn implies that \(\rho_{AB}^{\text{sym}}\) is entangled.

We now proceed to prove the following Lemma.

\textbf{Lemma.} In an even \(N\)-qubit permutation symmetric state \(\rho_{AB}^{\text{sym}}\) it is seen that

\[
\chi(\hat{\alpha}, \hat{\theta}) = \frac{1}{2} \left( 1 - s_3^2 \right) + \text{Tr} [ R \hat{\alpha}, \hat{\theta} ] C ]
\]

where \(C = T - s s^T\) is the \(3 \times 3\) real symmetric covariance matrix \([54, 51]\) defined in terms of the two-qubit correlation matrix \(T = (t_{\alpha\beta})\), \(\alpha, \beta = 1, 2, 3\) (see \([25]\)) and the qubit orientation vector \(s = (s_1, s_2, s_3)^T\) (see \([25]\)). The squared magnitude of the qubit orientation vector is denoted by \(s_0^2 = s^T s = s_1^2 + s_2^2 + s_3^2\).
Proof. Let us express

$$[\Delta(J_{Aa} + J_{Bb})]^2 = [\Delta(J_{Aa})]^2 + [\Delta(J_{Bb})]^2 + 2(J_{Aa}J'_{Bb} - J'_{Aa}J_{Bb})$$

and simplify the left hand side of the LSUR (29) by substituting (13)–(22), (24), (25) to obtain

$$\sum_{a=1}^{3} [\Delta(J_{Aa} + J'_{Bb})]^2 = \frac{n(n+2)}{2} - \frac{n^2}{2} s_0^2 + \frac{n^2}{2} \text{Tr} [\hat{R}(\hat{a}, \hat{\theta})C].$$

(32)

It readily follows that $\chi(\hat{a}, \hat{\theta})$ defined in 31 reduces to the simple form given by 34. □

Our main result is presented in the form of the following theorem:

**Theorem.** Permutation symmetric even N-qubit state violates the angular momentum LSUR (29) if the associated two-qubit covariance matrix $C = T - ss^T$ is negative.

Proof. Violation of the LSUR (29) is ensured whenever $\chi(\hat{a}, \hat{\theta}) < 0$. For any values of the axis-angle parameters $a = (a_1, a_2, a_3)^T$, $a^T a = a_1^2 + a_2^2 + a_3^2 = 1$, $0 \leq \theta \leq 2\pi$. Substituting the explicit form (6) for the elements of the rotation matrix $R(\hat{a}, \hat{\theta})$ i.e.,

$$R_{\alpha\beta} = \cos \theta \delta_{\alpha\beta} + (1 - \cos \theta) a_\alpha a_\beta - \sin \theta \epsilon_{\alpha\beta\gamma} a_\gamma,$$

$\alpha, \beta, \gamma = 1, 2, 3$ (33)

in terms of the axis-angle parameters and using the property $c_{\alpha\beta} = c_{\beta\alpha}$ of the covariance matrix $C = T - ss^T$, we obtain

$$\text{Tr} [R(\hat{a}, \hat{\theta})C] = \cos \theta \text{Tr} [C] + (1 - \cos \theta) a^T C a = \cos \theta (1 - s_0^2) + (1 - \cos \theta) a^T C a.$$

where we have substituted

$$\text{Tr} [C] = (1 - s_0^2).$$

(34)

Furthermore, choosing the rotation axis $a = (a_1, a_2, a_3)^T$ to be the eigenvector of the two-qubit covariance matrix $C$, corresponding to its least eigenvalue $c_L$, and setting the rotation angle to be $\theta = \pi$ in (34) we get

$$\text{Tr} [R(\hat{a}, \hat{\theta} = \pi)C] = -(1 - s_0^2) + 2 c_L.$$ (35)

Substituting (35) in (34) leads to

$$\chi(\hat{a}, \hat{\theta} = \pi) = c_L.$$ (36)

It is thus evident that the LSUR (29) is violated, whenever $c_L < 0$, thus proving the Theorem. □

In Ref. [50] it has been established by some of us that a two-qubit symmetric state $\rho_{\text{sym}}$ is entangled (negative under partial transpose) if and only if its associated covariance matrix $C = T - ss^T$ is negative. Thus the above theorem draws attention to the fact that entanglement in the two-qubit reduced state $\rho_{AB}^\text{sym}$ of the whole symmetric even N-qubit system $\rho_{\text{sym}}$ reflects itself in the violation of the LSUR (29).

Substituting (35) in (29), the left hand side of LSUR, one obtains

$$\sum_{a=1}^{3} [\Delta(J_{Aa} + J'_{Bb})]^2 = n(1 + n c_L).$$

(37)

Thus

$$\sum_{a=1}^{3} [\Delta(J_{Aa} + J'_{Bb})]^2 \geq n \Rightarrow (1 + n c_L) \geq 1$$ (38)

where LSUR (29) is expressed in terms of the least eigenvalue of the two-qubit covariance matrix $C$. As long as $c_L \geq 0$ (which happens to be the case for separable symmetric states) the LSUR (35) is obeyed. In the next section we discuss two specific physical examples of N-qubit symmetric states which violate (38).

**IV. EXAMPLES OF N-QUBIT SYMMETRIC STATES VIOLATING THE ANGULAR MOMENTUM LSUR**

**A. Symmetric multiqubit state generated by one-axis twisting Hamiltonian**

Kitagawa and Ueda [62] had proposed a non-linear Hamiltonian

$$\hat{H} = \chi J_1^2,$$

referred to as the *one-axis twisting Hamiltonian* for generating multiqubit spin squeezed states, where $J_1$ denotes one of the components of the collective angular momentum operator of the N-qubit system.

Dynamical evolution of an initially *spin-down* state $|j, -j\rangle$ of the $N = 2j$ qubit system, governed by the one-axis twisting Hamiltonian, results in [62]:

$$|\Psi_{\text{KU}}\rangle = \exp(-i\hat{H}t)|j, -j\rangle, \quad j = \frac{N}{2}.\quad (39)$$

In Ref. [49, 54], authored by some of us, the mean spin vector $s = (s_1, s_2, s_3)^T$ and the correlation matrix $T$ of a random pair of qubits drawn from the state $|\Psi_{\text{KU}}\rangle$ were explicitly given: The mean spin vector is given by

$$s^T = \begin{pmatrix} 0, & 0, & -\cos(N-1)\chi t \end{pmatrix}$$

(40)

and the the non-zero elements of the $3 \times 3$ real symmetric matrix $T$ take the form,

$$t_{11} = t_{13} = t_{23} = 0, \quad t_{12} = \cos(N-2)(\chi t) \sin(\chi t)$$

$$t_{22} = \frac{1}{2} \left[ 1 - \cos(N-2)(2\chi t) \right], \quad t_{33} = 1 - t_{22}$$

(41)
We then construct the covariance matrix $C = T - s s^T$ and evaluate its eigenvalues as a function of the number $N$ of qubits and the dimensionless dynamical parameter $\chi t$. Fig. 1 illustrates the behaviour of the left hand side of the LSUR (38) with respect to $\chi t$ for different choices of $n = N/2$.

\[ \sum_{n=1}^{N} |\Delta(j_{A_n} + j_{B_n})|^2 = \rho(n(1 + n c_L)) \text{ in the N-qubit symmetric state } |\Psi_{KL}\rangle \text{ given in (49), as a function of the dimensionless dynamical parameter } \chi t \text{ for different choices of } n = N/2. \] 

Violation of LSUR (38) is clearly seen as $1 + n c_L < 1$. This in turn highlights the pairwise entanglement in the symmetric N-qubit state (42), where bipartite divisions are characterized by the collective angular momenta $j_A = j_B = n/2$.

**B. One parameter family of W-class N qubit states**

We consider the one-parameter $N$-qubit symmetric state of the W-class (43):

\[ \left|\Psi\right\rangle_W = a \left|0_1,0_2,\ldots,0_N\right\rangle + \sqrt{1-a^2} \left|W\right\rangle, \quad 0 < a < 1, \] 

where

\[ \left|W\right\rangle = \frac{1}{\sqrt{N}} \left(|1_1,0_2,\ldots,0_N\rangle + |0_1,1_2,0_3,\ldots,0_N\rangle + \ldots + |0_1,0_2,\ldots,1_N\rangle\right) \] 

denotes the symmetric $N$-qubit W state.

Reduced two-qubit density matrix $\varrho^{\text{sym}} = \text{Tr}_{N-2} \left[ |\Psi\rangle_W \left\langle \Psi\right| \right]$ obtained by tracing any of the $N-2$ qubits is found to be (43):

\[ \varrho^{\text{sym}} = \frac{1}{A + 2D} \begin{pmatrix} A & B & B & 0 \\ B & D & D & 0 \\ B & D & D & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

where

\[ A = \frac{N^2 a^2 + (N-2)(1-a^2)}{N^2 a^2 + N(1-a^2)}, \quad B = \frac{a \sqrt{1-a^2}}{1+a^2(N-1)}, \]

\[ D = \frac{1-a^2}{N^2 a^2 + N(1-a^2)}. \]

The covariance matrix elements $c_{\alpha \beta} = t_{\alpha \beta} - s_{\alpha} s_{\beta}$ are readily evaluated in the two-qubit symmetric state $\varrho^{\text{sym}} = \text{Tr}_{N-2} \left[ |\Psi\rangle_W \left\langle \Psi\right| \right]$ and the associated $3 \times 3$ matrix $C = T - s s^T$ takes the form

\[ C = \frac{1}{A + 2D} \begin{pmatrix} 2D - 4B^2 & 0 & 2B(1-D) \\ 0 & 2D & 0 \\ 2B(1-D) & 0 & A - 2D - A^2 \end{pmatrix} \]

We have plotted the left hand side of the LSUR (38) as a function of the parameter $a$, for different choices of the angular momenta $j_A = j_B = n/2$ in Fig. 2. Violation of the LSUR i.e., $(1 + n c_L) \leq 1$ is manifestly seen in Fig. 2, thus highlighting the pairwise entanglement in the bipartite division (characterized by collective angular momenta $j_A = j_B = n/2$) of the W-class $N$-qubit symmetric state (42).

**V. CONCLUSION**

In this paper, it is shown that violation of local sum uncertainty relations (LSUR) in angular momentum operators is a necessary and sufficient condition for entanglement in two-qubit permutation symmetric states. The angular momentum LSUR for permutation symmetric even $N$-qubit states are shown to be violated when the least eigenvalue of the covariance matrix of the two-qubit reduced state is negative. The entanglement in the two-qubit reduced system is thus shown to imply entanglement between bipartite divisions of the even $N$-qubit
symmetric state. Our illustration of the result in two important classes of multiqubit states helps in discerning the bipartite entanglement through the parameters of two-qubit marginal.

Acknowledgements

HSK acknowledges the support of NCN through grant SHENG (2018/30/Q/ST2/00625). Sudha and ARU are supported by the Department of Science and Technology, India (Project No. DST/ICPS/QUST/Theme-2/2019).

[1] W. Heisenberg, Z. Phys. 43, 172 (1927).
[2] E. H. Kennard, Z. Phys. 44, 326 (1927).
[3] H. Weyl, Gruppentheorie und Quantenmechanik (Hirzel, Leipzig, 1928).
[4] H. P. Robertson, Phys. Rev. 34, 163 (1929).
[5] P. Bush, P. Lahti, and R. F. Werner, Rev. Mod. Phys., 86, 1261 (2014).
[6] Preparation uncertainty relations place intrinsic bounds on the spread of incompatible observables, measured in different statistical trials, using identical preparations of a quantum state. On the other hand, measurement uncertainty relations impose bounds on the error occurring in the measurement scheme of one of the observables and the corresponding postmeasurement disturbance caused on the other observable in an apparatus.

[7] H. F. Hofmann, and S. Takeuchi, Phys. Rev. A 68, 032103 (2003).
[8] H. F. Hofmann, Phys. Rev. A 68, 034307 (2003).
[9] O. Guhne, Phys. Rev. Lett. 92, 117903 (2004)
[10] M. Koashi, J. Phys.: Conf. Ser. 36, 98 (2006).
[11] M. Berta, M. Christandl, R. Colbeck, J. M. Renes, and R. Renner, Nature Physics 6, 659 (2010).
[12] E. Hanneke, and R. Renner, arXiv:1009.1833 [quant-ph].
[13] M. Tomamichel, and R. Renner, Phys. Rev. Lett. 106, 110506 (2011).
[14] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V.Scarani, and H. M. Wiseman, Phys. Rev. A 85, 010301(R) (2012).
[15] H. S. Karthik, A. R. Usha Devi, and A. K. Rajagopal, Phys. Rev. A 91, 012115 (2015).
[16] P. J. Coles, M. Berta, M. Tomamichel, and S. Wehner, Rev. Mod. Phys. 89, 015002 (2017).
[17] Q. Y. He, Shi-Guo Peng, P. D. Drummond, and M. D. Reid, Phys. Rev. A 84, 022107 (2011).
[18] M. O. Scully, B.-G. Englert, and H. Walther, Nature, 351, 111 (1991).
[19] Q. Y. He, Shi-Guo Peng, P. D. Drummond, and M. D. Reid, Phys. Rev. A 84, 022107 (2011).
[20] A.K. Pati and P. K. Sahu, Phys. Lett. A 367, 177 (2007).
[21] A. Rivas and A. Luis, Phys. Rev. A 77, 022105 (2008).
[22] Y. Huang, Phys. Rev. A 86, 024101 (2012).
[23] L. Maccone and A. K. Pati, Phys. Rev. Lett. 113, 260401 (2014).
[24] B. Chen and S. M. Fei, Sci. Rep. 5, 14238 (2015).
[25] S. Shabbir and G. Bjork, Phys. Rev. A 93, 052101 (2016).
[26] Y. Xiao, N. Jing, X. Li-Jost, and S. M. Fei, Sci. Rep. 6, 23201 (2016).
[27] S. Bagchi and A. K. Pati, Phys. Rev. A 94, 042104 (2016).
[28] W. Ma, B. Chen, Y. Liu, M. Wang, X. Ye, F. Kong, F. Shi, S. M. Fei, and J. Du, Phys. Rev. Lett. 118, 180402 (2017).
[29] Q. C. Song, J. L. Li, G. X. Peng and C. F. Qiao, Sci. Rep. 7, 44764 (2017).
[30] H. de Guise, L. Maccone, B. C. Sanders and N. Shukla, Phys. Rev. A 98, 042121 (2018).
[31] P. Giorda, L. Maccone and A. Riccardi, Phys. Rev. A 99, 052121 (2019).
[32] P. Busch, and O. Reardon-Smith, [arXiv:1901.03695 [quant-ph]]
[33] X. Zheng, S. Ma, and G. Zhang, Quantum Inf Process 19, 116 (2020).
[34] K. Szymański, and K. Życzkowski, J. Phys. A: Math. Theor. 53, 015302 (2020).
[35] I. I. Hirschman, Am. J. Math. 79, 152 (1957).
[36] W. Beckner, Ann. Math., 102, 159 (1975).
[37] I. Białynicki-Birula, and J. Mycielski, Commun. Math. Phys. 44, 129 (1975).
[38] D. Deutsch, Phys. Rev. Lett. 50, 631 (1983).
[39] M. H. Partovi, Phys. Rev. Lett. 50, 1883 (1983).
[40] I. Białynicki-Birula, Phys. Lett. A 103, 253 (1984).
[41] K. Kraus, Phys. Rev. D 35, 3070 (1987).
[42] H. Maassen, and J. B. M. Uffink, Phys. Rev. Lett. 60, 1103 (1988).
[43] S. Wehner, and A. Winter, New J. Phys. 12, 025009 (2010).
[44] I. Białynicki-Birula, and L. Rudnicki, in Statistical Complexity, edited by K. Sen (Springer, Dordrecht, 2011) p.1.
[45] K. Wang, et al, Phys. Rev. A 93, 052108 (2016).
[46] W. Ma, B. Chen, Y. Liu, M. Wang, X. Ye, F. Kong, F. Shi, S.-M. Fei, and J. Du, Phys. Rev. Lett. 118, 180402 (2017).
[47] Z.-X. Chen, J.-L. Li, Q.-C. Song, H. Wang, S. M. Zangi, and C.-F. Qiao, Phys. Rev. A 96, 062123 (2017).
[48] Z.-X. Chen, H.-L. Wang, J.-L Li, Q.-C. Song, and C.-F. Qiao, Sci. Rep. 9, 5687 (2019).
[49] A. R. Usha Devi, M. S. Uma, R. Prabhu, Sudha, Int. J. Mod. Phys. 20, 1917 (2006).
[50] A. R. Usha Devi, M. S. Uma, R. Prabhu, and A. K. Rajagopal, Phys. Lett. A 364, 203 (2007).
[51] A. R. Usha Devi, R. Prabhu, and A. K. Rajagopal, Phys. Rev. Lett. 98, 060501 (2007).
[52] A. Sorensen, L.-M. Duan, J. I. Cirac, and P. Zoller, Nature (London) 409, 63 (2001).
[53] J. K. Korbcz, J. I. Cirac, and M. Lewenstein, Phys. Rev. Lett. 95, 120502 (2005); 95, 259901(E) (2005).
[54] A. R. Usha Devi, M. S. Uma, R. Prabhu, and Sudha, J. Opt.B. \textbf{7}, S740 (2005).
[55] J. K. Korbicz, O. Gühne, M. Lewenstein, H. Häffner, C. F. Roos, and R. Blatt, Phys. Rev. A \textbf{74}, 052319 (2006).
[56] J. Martin, O. Giraud, P. A. Braun, D. Braun, and T. Bastin, Phys. Rev. A \textbf{81}, 062347 (2010).
[57] D. Markham, Phys. Rev. A \textbf{83}, 042332 (2011).
[58] A. R. Usha Devi, Sudha, and A. K. Rajagopal, Quantum Inf. Proc. \textbf{11}, 685 (2012).
[59] L. Lamata, C. E. López, B. P. Lanyon, T. Bastin, J. C. Retamal, and E. Solano, Phys. Rev. A \textbf{87}, 032325 (2013).
[60] A. Neven, J. Martin, and T. Bastin, Phys. Rev. A \textbf{98}, 062335 (2018).
[61] K. N. Srinivasa Rao, \textit{Rotation and Lorentz Groups for Physicists}, (Wiley, 1988).
[62] M. Kitagawa and M. Ueda, \textit{Phys. Rev. A} \textbf{47}, 5138 (1993).
[63] K. S. Akhilesh, B. G. Divyamani, Sudha, A. R. Usha Devi and K. S. Mallesh, Quantum Inf. Proc., \textbf{18}, 144 (2019)