Distributed Optimal Power Dispatch of On/Off Devices via Newton-like Neural Network Dynamics

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Abstract

This paper aims to solve a distributed optimization problem which naturally occurs in optimal dispatch of generators and flexible load devices that are limited to binary operational states, e.g., on and off. This problem is formulated as a binary program with a cost function defined by the summation of agent costs plus a global mismatch/penalty term. We propose a modification of the Hopfield Neural Network (HNN) dynamics in order to solve this problem while incorporating a novel Newton-like weighting factor. This addition lends itself to fast avoidance of saddle points, which the gradient-like HNN is susceptible to. Turning to a multi-agent setting, we reformulate the problem and develop a distributed implementation of the Newton-like dynamics. We show that if a local solution to the distributed reformulation is obtained, it is also a local solution to the centralized problem. A main contribution of this work is to show that the probability of converging to a saddle point of an appropriately defined energy function in both the centralized and distributed settings is zero under light assumptions. Finally, we enlarge our algorithm with an annealing technique which gradually “learns” a feasible binary solution. Simulation results demonstrate that the proposed methods are competitive with centralized greedy and SDP relaxation approaches in terms of solution quality, while the main advantage of our approach is a significant improvement in runtime over the SDP relaxation method and the distributed quality of implementation.

Key words: second-order methods; dynamical systems; distributed optimization; neural networks; binary optimization.

1 Introduction

Motivation. There has been an explosion of literature surrounding the design of distributed algorithms for convex optimization problems and how these pertain to the operation of future power grids. A common assumption of these algorithms is the property of convexity, which lends itself to provably optimal solutions which are scalable and fast. However, devices available for providing load-side frequency regulation are often limited to discrete on/off operational modes, such as air handling units and ventilation systems, household appliances, and manufacturing systems. It is even preferable to charge populations of electric vehicles in a discrete on/off manner due to nonlinear battery chemistries. The available tools in optimization for this nonconvex setting are less mature, and when considering a distributed setting in which devices act as agents that collectively compute a solution over a sparse communication graph, the available tools are significantly less developed. With this in mind, we are motivated to develop a scalable, fast approach for these on/off settings which is amenable to a distributed implementation.

Literature Review. Quadratic programs with nonconvex binary constraints are known to be NP-hard in general, see [7, 18]. Greedy algorithms [9] have been proposed for binary programs, such as the well-known Traveling Salesman Problem (TSP), but it is well documented that these methods can greatly suffer in performance [14] except in cases where the cost function is submodular [22, 28]. A more modern approach to solving optimization problems with a binary feasibility set is to cast them as a semidefinite program (SDP) with a nonlinear rank constraint, see [4, 25, 30] for some classical references or [19, 31] for more recent work on the topic. By relaxing the rank constraint, a convex problem is obtained whose solution can be shown to be equal to the optimal dual value of the original problem, see e.g. [23]. However, it is necessary in these approaches to either impose a single centralized coordinator to compute the solution and broadcast it to the actuators or agents, or schedule computations, which suffers from scalability issues, privacy concerns, and does not enjoy the simpler and more robust implementation of a distributed architecture in a large network.

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Neuro-dynamic programming is a different paradigm for addressing nonconvex problems with computational tractability, see [3] for a broad reference. A neural-network based method for binary programs was first developed by Hopfield in [15], which was originally proposed in order to address TSPs. We refer to this method from here on as a Hopfield Neural Network (HNN). This method provided a completely different avenue for approaching binary optimizations, and followup works are found in [2, 16, 20, 29]. These works formalize and expand the framework in which the HNN method is applicable. However, these algorithms essentially implement a gradient-descent on an applicable nonconvex energy function, which is susceptible to being slowed down by convergence to saddle-points. There are avenues for Newton-like algorithms in nonconvex environments to address this issue, which incorporate some treatment of the negative Hessian eigenvalues in order to maintain a monotonic descent of the cost function, see e.g. [10, 11]. A recently developed method employs a Positive-definite Truncated inverse (PT-inverse) operation on the Hessian of a nonconvex energy or cost function in order to define a nonconvex Newton-descent direction [24], although the technique does not presently address binary settings. Perhaps more importantly, all variants of existing HNN methods and the aforementioned works for nonconvex Newton-like algorithms are framed for centralized environments in which each agent knows global information about the state of all other agents, which is not scalable.

**Statement of Contributions.** The contributions of this paper are threefold. We start by considering a binary programming problem formulated as a summation of local costs plus a squared global term. We recall the following lemma that is used in the sequel.

\[ L_{ij} = \begin{cases} -1, & j \in \mathcal{N}_i, \\ -\sum_{k \neq i} L_{ik}, & i = j, \\ 0, & \text{otherwise}. \end{cases} \]

An immediate property is that 0 is an eigenvalue of \( L \) associated with the eigenvector \( 1_n \). It is simple iff \( \mathcal{G} \) is connected.

### 2 Preliminaries

This section establishes notation and background concepts to be used throughout the paper.

#### 2.1 Graph Theory

We refer the reader to [6] as a supplement to this section. Consider an undirected graph \( \mathcal{G} \) of nodes and edges \((\mathcal{N}, \mathcal{E})\), where \( i \in \mathcal{N} \) corresponds to an agent in a network and \((i, j) \in \mathcal{E}\) indicates that agents \( i, j \) are neighbors. Graphs will be used to model node interactions or communications. We denote by \( \mathcal{N}_i \) and \( \mathcal{N}_i^2 \) the one-hop and two-hop neighbor sets of agent \( i \); i.e. \( j \in \mathcal{N}_i \) iff \((i, j) \in \mathcal{E}\) and \( j \in \mathcal{N}_i^2 \) iff \( \exists k \) s.t. \((i, k), (k, j) \in \mathcal{E}\). One can define a Laplacian matrix \( L \) associated with a graph \( \mathcal{G} \) as follows:

### 2.2 Schur Complement

We recall the following lemma that is used in the sequel.

**Lemma 1 ([32]) (Matrix Definiteness via Schur Complement).** Consider a symmetric matrix \( M \) of the form

\[ M = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \]

The set of real numbers, real positive numbers, real \( n \)-dimensional vectors, and real \( n \times n \) matrices are written as \( \mathbb{R}, \mathbb{R}_+, \mathbb{R}^n, \) and \( \mathbb{R}^{n \times n} \), respectively. We denote by \( x_i \) the \( i \)-th element of \( x \in \mathbb{R}^n \) and \( A_{ij} \) the element in the \( i \)-th row and \( j \)-th column of \( A \in \mathbb{R}^{n \times n} \). For a square matrix \( A \), we denote by \( A^\top \) the Moore-Penrose pseudoinverse of \( A \). We use the shorthand \( 1_n = (1, \ldots, 1)^\top \in \mathbb{R}^n \) and \( 0_n = (0, \ldots, 0)^\top \in \mathbb{R}^n \). Cartesian products of sets are denoted by a superscript, for example, \( \{0, 1\}^n = \{0, 1\} \times \cdots \times \{0, 1\} \). The gradient of a function \( f : \mathbb{R}^n \to \mathbb{R} \) with respect to \( x \in \mathbb{R}^n \) is denoted by \( \nabla_x f(x) \in \mathbb{R}^n \), and the Hessian matrix of \( f \) at \( x \) is written as \( \nabla_{xx} f(x) \in \mathbb{R}^{n \times n} \). The operator \( \circ \) on vector arguments \( x, y \in \mathbb{R}^n \) denotes elementwise multiplication, e.g. \( x \circ y = \text{diag}(x) y \), and a vector \( x \in \mathbb{R}^n \) with a superscript indicates the elementwise power operation, e.g. \( x^2 = \text{diag}(x) x \), where \( \text{diag}(x) \) indicates a diagonal matrix with entries given by the elements of \( x \). The division operator \( c/x \) for \( c \in \mathbb{R} \) and \( x \in \mathbb{R}^n \) indicates elementwise division \( c/x = (c/x_1, \ldots, c/x_n)^\top \); similarly, \( \log(x) = (\log(x_1), \ldots, \log(x_n))^\top \) and \( e^x = (e^{x_1}, \ldots, e^{x_n})^\top \) are used elementwise. The notation \( B(x, \eta) \) denotes the closed ball of radius \( \eta \) centered at \( x \).
If $C$ is invertible, then the following properties hold:
(1) $M > 0$ if and only if $C > 0$ and $A - BC^{-1}B^\top \succ 0$.
(2) If $C \succ 0$, then $M \geq 0$ if and only if $A - BC^{-1}B^\top \geq 0$.

### 2.3 Stable Manifold Theorem

Consider the solution to the initial value problem

$$\dot{x} = \varphi(x), \quad x \in \mathbb{R}^n, \quad x(0) = x_0,$$

and let its linearization around a point $\bar{x}$ be given by

$$\dot{\zeta} = D\varphi(\bar{x})\zeta,$$

where $D\varphi(\bar{x})$ indicates the Jacobian matrix of $\varphi$ at $\bar{x}$.

We recall the following definition and theorem.

**Definition 2 ([13]) (Local Stable and Unstable Manifolds).** Let $x^* \in \mathbb{R}^n$ be a fixed point of the dynamics $\dot{x} = \varphi(x)$, i.e. $\varphi(x^*) = 0$. The local stable and unstable manifolds of $x^*$ are given by

$$W_s(x^*) = \left\{ x_0 \in U | x(t) \to x^* \text{ as } t \to \infty, \right. \left. \text{ and } x(t) \in U, \forall t \geq 0 \right\},$$

$$W_u(x^*) = \left\{ x_0 \in U | x(t) \to x^* \text{ as } t \to -\infty, \right. \left. \text{ and } x(t) \in U, \forall t \geq 0 \right\},$$

where $U \subset \mathbb{R}^n$ is a neighborhood of the fixed point $x^*$.

**Theorem 3 ([13]) (Stable Manifold Theorem for a Fixed Point).** Suppose that $\dot{x} = \varphi(x)$ has a hyperbolic fixed point $x^*$, i.e. $\varphi(x^*) = 0$ and the real parts of eigenvalues of $D\varphi(x^*)$ are nonzero. Then there exist local stable and unstable manifolds $W_s(x^*)$, $W_u(x^*)$ of the same dimensions $n_s, n_u$ as those of the eigenspaces $E_s, E_u$ of $D\varphi(x^*)$ corresponding to eigenvalues with positive and negative real parts, respectively.

Note that the theorem as stated here does not specify the behavior of the trajectories along center manifolds corresponding to eigenspaces associated with (real-part) zero eigenvalues of $D\varphi(x^*)$; the behavior of the trajectories contained in these manifolds is not necessary to state for the main results of this paper.

### 2.4 Positive-definite Truncation

Here, we introduce the Positive-definite Truncated inverse (PT-inverse) and its relevance to nonconvex Newton methods.

**Definition 4 ([27]) (PT-inverse).** Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with an orthonormal basis of eigenvectors $Q \in \mathbb{R}^{n \times n}$ and diagonal matrix of eigenvalues $\Lambda \in \mathbb{R}^{n \times n}$. Consider a constant $m > 0$ and define $|\Lambda|_m \in \mathbb{R}^{n \times n}$ by:

$$|\Lambda|_m = \begin{cases} |\Lambda|_i, & |\Lambda|_i \geq m, \\ m, & \text{otherwise}. \end{cases}$$

The PT-inverse of $A$ with parameter $m$ is defined by

$$|A|^{-1}_m = Q^{-1}(|\Lambda|_m)^{-1}Q > 0.$$
In particular, consider the quadratic energy function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by:

$$F(x) = \frac{1}{2} x^\top W x - v^\top x,$$

where $W \in \mathbb{R}^{n \times n}$ may represent weights of the connections between nodes and is typically defined to be symmetric but possibly indefinite. In a physical network, these weights can be thought of as a model for the interconnection of electronic amplifiers or interactions between neurons in a neural network. The linear $v \in \mathbb{R}^n$ term represents an input bias weighting.

The HNN dynamics on the internal state $u$ are given by:

$$\dot{u} = W g(u) - \frac{u}{\tau} + v,$$

where $\tau > 0$ is a time constant. With this dynamical model, the two physical systems we described (electronic amplifier network and neural network) tend toward low energy states upon applying the transformation $x = g(u)$. Within the scope of this paper, low energy states can be interpreted as lower cost solutions of an optimization problem.

### 3 Problem Statement and Dual Problem

In this section, we formally state the nonconvex optimization problem we wish to solve and formulate its dual for the sake of deriving a lower bound to the optimal cost.

#### 3.1 Problem Statement

We aim to find an adequate solution to an economic dispatch problem where the optimization variables take the form of binary (on/off) decisions over a population of $n$ devices. Let each binary device $i \in \{1, \ldots, n\}$ be endowed with a cost of operation $c_i \in \mathbb{R}$, a value which indicates the incremental cost of operating in the “on” state versus the “off” state. Each of these devices possesses a local binary decision variable $x_i \in \{0, 1\}$, where $x_i = 0$ indicates off and vice versa, and a real-power value $p_i$, where we adopt the convention that device $i$ consumes power $p_i$ if $x_i = 1$ and consumes zero if $x_i = 0$.

We are afforded some flexibility in the model by the constant $c_i$, so we design cost functions $f_i : [0, 1] \rightarrow \mathbb{R}$ such that $f_i(1) - f_i(0) = c_i$, $\forall i$. With this in mind, we adopt the following:

**Assumption 1 (Quadratic Cost Functions).** The local cost functions $f_i$ take the form

$$f_i(x_i) = \frac{a_i}{2}(x_i - b_i)^2, \quad a_i, b_i \in \mathbb{R}.$$

Note that, for any value $c_i = f_i(1) - f_i(0)$, there exists a family of coefficients $a_i, b_i$ such that $(a_i/2)((1-b_i)^2 - (a_i/2)b_i^2) = c_i$. The freedom in the choice of $a_i, b_i$ will be addressed in Section 4.

The problem we aim to solve can now be formulated as:

$$\mathcal{P} 1 : \min_{x \in \{0, 1\}^n} f(x) = \sum_{i=1}^{n} f_i(x_i) + \frac{\gamma}{2} (p^\top x - P_r)^2.$$

Here, $P_r$ is a given real-power reference value to be matched by the total power output $p^\top x$ of the devices, with $p \in \mathbb{R}^n$ having entries $p_i$. This matching is enforced by means of a penalty term with coefficient $\gamma > 0$ in $\mathcal{P} 1$. The signal $P_r$ is determined by an Independent System Operator (ISO) and communicated to a Distributed Energy Resource Provider (DERP) that solves $\mathcal{P} 1$ to obtain a real-time dispatch solution, see [1] for additional context on this setting.

#### 3.2 Dual Problem and Lower Bound

Before presenting the method for solving $\mathcal{P} 1$, we first derive the dual problem for the sake of efficiently computing a lower bound.

Rewrite each $x_i \in \{0, 1\}$ as $x_i(x_i - 1) = 0$ associated with dual variables $\mu_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$, and let $\mu \in \mathbb{R}^n$ be a vector with entries $\mu_i$. The Lagrangian $\mathcal{L}$ of $\mathcal{P} 1$ is given by

$$\mathcal{L}(x, \mu) = \sum_{i=1}^{n} [f_i(x_i) + \mu_i x_i(x_i - 1)] + \frac{\gamma}{2} (p^\top x - P_r)^2$$

$$= x^\top \left( \text{diag}(a/2 \mu) + \frac{\gamma}{2} p p^\top \right) x$$

$$- (a \circ b + \mu + \gamma P_r) x + \sum_{i=1}^{n} a_i b_i^2 + \frac{\gamma}{2} P_r^2$$

$$= x^\top \frac{Q(\mu)}{2} x - \xi(\mu)^\top x + \zeta,$$

where we have defined $Q : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, $\xi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (affine functions of $\mu$), and $\zeta \in \mathbb{R}$ (constant) to simplify the notation. This is a quadratic expression in $x$, and it follows that a minimizer $\hat{x}(\mu) = \underset{x}{\text{argmin}} \mathcal{L}(x, \mu)$ satisfies

$$\hat{x}(\mu) = \begin{cases} Q(\mu) \xi(\mu), & Q(\mu) \succeq 0, \\ \xi(\mu) \in \text{image } Q(\mu), & \text{unbounded, o.w.} \end{cases}$$

where $Q(\mu)$ can be computed via a Singular Value Decomposition, see e.g. [5] § A.5.4. It follows from (2)
and (3) that the dual function $\beta(\mu)$ is given by
\[
\beta(\mu) = \begin{cases} 
\frac{1}{2} \xi(\mu)^\top Q(\mu) \xi(\mu) + \zeta, & Q(\mu) \succeq 0, \\
-\infty, & \xi(\mu) \notin \text{image } Q(\mu), \\
\end{cases} \text{w.r.t.}
\]
with temperature parameter $T > 0$.

From here, it is possible to cast the dual problem as an SDP. Introduce a variable $\Delta \leq \beta(\mu)$, and let the objective be to maximize $\Delta$. By applying Lemma 1, the dual problem $D1$ to $P1$ is given by the semidefinite program
\[
D1 : \max_{\mu, \Delta} \Delta, \quad \text{subject to } \begin{bmatrix} \frac{1}{2} Q(\mu) & \xi(\mu) \\ \xi(\mu)^\top & \zeta - \Delta \end{bmatrix} \succeq 0.
\]

This is a standard semidefinite program in $\mu$ and $\Delta$ and can be solved in polynomial time to determine a lower bound $\Delta$ to the optimal value of $P1$.

4 Centralized Newton-like Neural Network

In this section, we develop the Centralized Newton-like Neural Network, or NNN-c, which is well suited for solving $P1$ in a centralized setting.

4.1 Statement and Discussion of Dynamics

To draw analogy with the classic Hopfield Neural Network approach we will briefly introduce an auxiliary variable $u_i$, whose relation to $x_i$ is given by the logistic function $g$ for each $i$:
\[
x_i = g(u_i) = \frac{1}{1 + e^{-u_i/T}}, \quad u_i \in \mathbb{R},
\]
\[
u_i = g^{-1}(x_i) = -T \log \left( \frac{1}{x_i} - 1 \right), \quad x_i \in (0, 1),
\]
with temperature parameter $T > 0$ as in Section 2.5.

Let $x \in (0, 1)^n$, $u \in \mathbb{R}^n$ be vectors with entries given by $x_i, u_i$. To establish our algorithm, it is appropriate to first define an energy function related to $P1$. Consider
\[
E(x) = f(x) + \frac{1}{\tau} \sum_i \int_0^{x_i} g^{-1}(\nu) d\nu,
\]
where $\tau > 0$ is a time-constant and for $z \in [0, 1], 
\int_0^{z} g^{-1}(\nu) d\nu = \begin{cases} 
T (\log(1 - z) - z \log(\frac{1}{x} - 1)), & z \in (0, 1), \\
0, & z \in \{0, 1\}.
\end{cases}
\]

The classic HNN implements dynamics of the form $\dot{u} = -\nabla x E(x)$, where the equivalent dynamics in $x$ can be computed as $\dot{x} = -\nabla x E(x) dx/du$. Recall that the domain of $x$ is $(0, 1)^n$ and $\log$ and division in the elementwise sense for vectors. Expressions for $\nabla x E(x)$ and $dx/du$ are:
\[
\nabla x E(x) = -Wx - v - \frac{T}{\tau} \log \left( \frac{1}{x} - 1 \right),
\]
\[
dx/du = \frac{x - x^2}{T},
\]
where $W, v$ are defined via $f$ (as in Section 2.5):
\[
W = \text{diag}(a) - \gamma pp^\top, \quad \in \mathbb{R}^{n \times n},
\]
\[
v = a \circ b + \gamma P_t p, \quad \in \mathbb{R}^n.
\]

From this point forward, we work mostly in terms of $x$ for the sake of consistency. Consider modifying the classic HNN dynamics with a PT-inverse $(|H(x)|_m)^{-1} > 0$ as in Section 2.4, where $H(x) = \nabla x E(x)$. The NNN-c dynamics are then given by:
\[
\dot{x} = -(|H(x)|_m)^{-1} \text{diag}(dx/du) \nabla x E(x)
\]
\[
= \gamma (|H(x)|_m)^{-1} \text{diag}((x - x^2)/T)
\]
\[
(Wx + v + \frac{T}{\tau} \log \left( \frac{1}{x} - 1 \right)).
\]

These dynamics lend to the avoidance of saddle points of $E$. To see this, consider the eigendecomposition $H(x) = Q^\top AQ$ at some $\tilde{x}$ near a saddle point, i.e. $\nabla x E(\tilde{x}) \approx 0$. If many entries of $\Lambda$ are small in magnitude and remain small in the proximity of $\tilde{x}$, then the gradient is changing slowly along the “slow” manifolds associated with the eigenspace of the small eigenvalues. This is precisely what the PT-inverse is designed to combat: the weighting of the dynamics is increased along these manifolds by a factor that is inversely proportional to the magnitude of the eigenvalues. Additionally, negative eigenvalues of the Hessian are flipped in sign, which causes attractive manifolds around saddle points to become repellent.

4.2 Properties of Energy Function and Equilibria

It is desirable for $E$ to be concave on most of its domain so the trajectories are pushed towards the feasible points of $P1$; namely, the corners of the unit hypercube. To examine this, the Hessian of $E$ can be computed as
\[
H(x) = \frac{d^2 f}{dx^2} + \frac{1}{\tau} \text{diag}(\frac{dg^{-1}(x)}{dx})
\]
\[
= -W + \frac{T}{\tau} \text{diag}(\frac{1}{x - x^2}).
\]
Notice that the second term is positive definite on \( x \in (0, 1)^n \) and promotes the convexity of \( E \), particularly for elements \( x_i \) close to 0 or 1. For a fixed \( T, \tau \), choosing \( a_i < -\gamma ||p||^2 - 4T/\tau , \forall i \) guarantees \( E(x) < 0 \) at \( x = (0.5) \mathbf{1}_n \). Generally speaking, choosing \( a_i \) to be negative and large in magnitude lends itself to concavity of \( E \) over a larger subset of its domain and to trajectories converging closer to the set \( \{0,1\}^n \). However, this comes at the expense of not exploring a rich subset of the domain. At the end of this section, we develop a Deterministic Annealing (DA) approach inspired by [26] for the online adjustment of \( T, \tau \) to obtain an effective compromise between exploration of the state space and convergence to a feasible point of \( P1 \).

We now characterize the equilibria of (7) for \( x \in [0,1]^n \). It would appear that \( x \) with some components \( x_i \in \{0,1\} \) are candidate equilibria due to the \( x_i - x_i^2 \) factor vanishing. However, the dynamics are not well defined here due to the log term. Additionally, note that

\[
\lim_{x_i \to \delta} e_i^T H(x) e_i = \infty, \quad \delta \in \{0,1\}, \forall i,
\]

where \( e_i \) is the \( i \)th canonical basis vector. Due to the \( \frac{T}{\tau(x_i - x_i^2)} \) term dominating \( W \) in the expression for \( H \) when \( x_i \) values are close to \( 0,1 \), it follows that an eigenvalue of \( (|H(x)|_m)^{-1} \) approaches zero as \( x_i \to 0 \) or 1 with corresponding eigenvector approaching \( v_i = e_i \):

\[
\lim_{x_i \to \delta} = v_i^T (|H(x)|_m)^{-1} v_i = \frac{T}{\tau}(x_i - x_i^2) = 0, \quad \delta \in \{0,1\}, \forall i.
\]

Using this fact, and ignoring \( T, \tau > 0 \), we can compute the undetermined limits in the components of \( \dot{x} \) as \( x_i \to \delta \in \{0,1\} \) by repeated applications of L’Hospital’s rule:

\[
\lim_{x_i \to \delta} \log \left( \frac{1}{x_i} - 1 \right) (x_i - x_i^2)^2 = \lim_{x_i \to \delta} \log \left( \frac{1}{x_i} - 1 \right) = \lim_{x_i \to \delta} \frac{1}{1/(x_i - x_i^2)^2} = \lim_{x_i \to \delta} \frac{x_i - 1}{2 - 4x_i} = \lim_{x_i \to \delta} \frac{-x_i(2x_i - 1)(x_i - 1)}{2} = \begin{cases} 0, & \delta = 0^+, \\ 0, & \delta = 1^- \end{cases}
\]

Thus, components \( x_i \in \{0,1\} \) constitute candidate equilibria. We will, however, return to the first line of (9) in the proof of Lemma 6 to show that they are unstable. As for components of \( x \) in the interior of the hypercube, the expression \( \dot{x} = 0 \) can not be solved for in closed form. However, we provide the following Lemma which shows that the set of equilibria is finite.

**Lemma 5 (Finite Equilibria).** Let \( \mathcal{X} \) be the set of equilibria of (7) satisfying \( \dot{x} = 0 \) on \( x \in [0,1]^n \). The set \( \mathcal{X} \) is finite.

**Proof.** First consider only \( \mathcal{X} \cap (0,1)^n \). Note that \( (|H(x)|_m)^{-1} > 0 \) (by construction) and \( \text{diag}(x - x^2 / T) > 0 \) on \( x \in (0,1)^n \), so we focus on

\[
Wx + \frac{T}{\tau} \log \left( \frac{1}{x} - 1 \right) + v = 0_n \tag{10}
\]

Examining the above expression elementwise, it is non-constant, continuous, and its derivative changes sign only a finite number of times. Therefore, the total number of zeros on \( (0,1)^n \) must be finite.

Now consider the \( i \)th element of (10) for \( x_j \to 0 \) or 1 for all \( j \) in an arbitrary permutation of \( \{1, \ldots, n\} \setminus \{i\} \). Since the number of these permutations is finite, and each permutation still gives rise to a finite number of solutions to (10) in the \( i \)th component, it follows that \( \mathcal{X} \) is finite. \( \square \)

To demonstrate the qualitative behavior of equilibria in a simple case, consider a one-dimensional example with \( a > -\gamma p^2 - 4T/\tau \) and recall that, for \( x \in (0,1) \), the sign of \( -\nabla x E(x) \) is the same as \( \dot{x} \). In Figure 1, we observe that \( -\nabla x E(x) \) monotonically decreases in \( x \), and a globally stable equilibrium exists in the interior \( x \in (0,1) \) near \( x_1 = 0.5 \). On the other hand, \( a < -\gamma p^2 - 4T/\tau \) gives way to 3 isolated equilibria in the interior (one locally unstable near \( x = 0.5 \) and two locally stable near \( x_1 \in \{0,1\} \)). This behavior extends in some sense to the higher-dimensional case. Therefore, for a scheme in which \( T \) and \( \tau \) are held fixed, we prescribe \( a < -\gamma ||p||^2 - 4T/\tau \). We provide a Deterministic Annealing (DA) approach inspired by [26] for the online adjustment of \( T, \tau \) in the following subsection which compromises with this strict design of \( a \).

Finally, we establish a Lemma about the domain of the trajectories of (7).

**Lemma 6 (Forward Invariance of the Open Hypercube).** The open hypercube \( (0,1)^n \) is a forward-invariant set under the NNN-C dynamics (7).

**Proof.** Consider again the terms of \( \dot{x} \) elementwise. There are two cases to consider for evaluating \( x_i; x_i = \varepsilon \) and \( x_i = 1 - \varepsilon \) for some \( 0 < \varepsilon \ll 1 \) sufficiently small such that the terms of \( (|H(x)|_m)^{-1} \) are still dominated by \((1/x_i - x_i^2)\) and the \( Wx + v \) are still dominated by the log term. Then, consider the expression

\[
\log \left( \frac{1}{x_i} - 1 \right) (x_i - x_i^2)^2 \tag{11}
\]
4.3 Convergence Analysis

Fig. 1. Illustration of $-\nabla_x E(x)$ (top) and $\dot{x}$ (bottom) for three instances of $a$. Case 1: $a > -\gamma \|p\|^2 - 4T/\tau$, Case 2: $a = -\gamma \|p\|^2 - 4T/\tau$, Case 3: $a < -\gamma \|p\|^2 - 4T/\tau$.

For $x_i = \varepsilon \approx 0$, (11) evaluates to a small positive value, and for $x_i = 1 - \varepsilon \approx 1$, (11) evaluates to a small negative value. We have argued that these are the dominating terms regardless of values of the remaining components of $x$, and so we conclude that $x_i \in \{0, 1\}$ are componentwise anti-stable and that elements of $x$ will never approach 0 or 1. Thus, the open hypercube is forward invariant.

\section*{Definition 7 (Critical Point).}

(1) A point $x^*$ is a critical point of $E$ if the gradient $\nabla_x E(x^*) = 0_n$.  
(2) A critical point $x^*$ is isolated if there is a neighborhood $U$ around $x^*$ and $x^*$ is the only critical point in $U$.  
(3) A critical point is a local minimum if there is a neighborhood $U$ around $x^*$ such that $E(x^*) \leq E(x)$ for all $x \in U$ and a local maximum if $E(x^*) \geq E(x)$.  
(4) A critical point is a saddle point if for all neighborhoods $U$ around $x^*$, there are $x, y \in U$ such that $E(x) \leq E(x^*) \leq E(y)$.

\section*{Definition 8 (Globally Stable Set).} The Globally Stable Set $W_s(x^*)$ of a critical point $x^*$ is the set of initial conditions under which NNN-c converges to $x^*$, i.e.

$$W_s(x^*) \triangleq \{x(0) \mid \lim_{t \to \infty} x(t) = x^* \text{ under NNN-c}\}.$$  

\section*{Assumption 2 (Random Initial Condition).} The initial condition $x(0) \in (0, 1)^n$ is chosen randomly according to a distribution $\mathbb{P}$ that is nonzero on sets that have nonzero volume in $(0, 1)^n$.

An appropriately unbiased initial condition for our algorithm is $x(0) \approx (0.5) 1_n$, which is adequately far from the local minima located near corners of the unit cube. So, we suggest choosing a uniformly random $x(0) \in \mathcal{B}((0.5) 1_n, \epsilon)$, where $0 < \epsilon \ll 1$.

\section*{Assumption 3 (Choice of $T, \tau$).} The constants $T, \tau > 0$ are each chosen randomly according to a distribution $\mathbb{P}$ that is nonzero on sets that have nonzero volume on $\mathbb{R}_+$.  

Similarly to $x(0)$, we suggest choosing these constants uniformly randomly in a ball around some nominal $T_0, \tau_0$, i.e. $T \in \mathcal{B}(T_0, \epsilon), \tau \in \mathcal{B}(\tau_0, \epsilon), 0 < \epsilon \ll 1$.

Now we state the main convergence result of NNN-c.

\section*{Theorem 9 (Convergence of NNN-c).} An initial condition $x(0) \in (0, 1)^n$ converges asymptotically under NNN-c to a critical point $x^*$ of $E$. In addition, under Assumption 2, on the random choice of initial conditions, and Assumption 3, on the random choice of $T, \tau$, the probability that $x(0)$ is in the set $\bigcup W_s(x^*)$, where $\hat{x}$ is a saddle-point or local maximum of $E$, is zero. In other words, for a random choice of $T, \tau$, an initial condition chosen randomly from $(0, 1)^n$ converges asymptotically to a local minimizer $x^*$ with probability one.

\section*{Proof.} Let $\mathcal{X}$ be the set of all critical points of $E$. We first establish that $E$ decreases along the trajectories of NNN-c and that $x(t)$ converges asymptotically to $\mathcal{X}$. Differentiating $E$ in time, we obtain:

$$\begin{align*}
\frac{dE}{dt} &= \dot{x}^T \nabla_x E(x) = \dot{x}^T (-W x - v + g^{-1}(x)/\tau) \\
&= -\dot{x}^T \text{diag}(T/(x - x^2)) |H(x)|_m \dot{x} < 0,
\end{align*}$$

(12) for $\dot{x} \neq 0$, $x \in (0, 1)^n$.

Recall that $x(t) \in (0, 1)^n$ for all $t \geq 0$ due to Lemma 6. From (7) and the discussion that followed on equilibria, $\dot{x} = 0$ implies $\nabla_x E(x) = 0$ due to $(H(x)|_m)^{-1} > 0$ and $\text{diag}((x - x^2)/T) > 0$ on $x \in (0, 1)^n$. The domain of $E$ is the compact set $[0, 1]^n$ (per the definition of the integral terms), and $E$ is continuous and bounded from
below on this domain, so at least one critical point exists. Combining this basic fact with (12) shows that the NNN-c dynamics monotonically decrease $E$ until reaching a critical point. More formally, applying the LaSalle Invariance Principle [17] tells us that the trajectories converge to the largest invariant set contained in the set $dE/dt = 0$. This set is $\mathcal{X}$, which is finite per Lemma 5. In this case, the LaSalle Invariance Principle additionally establishes that we converge to a single $x^* \in \mathcal{X}$.

The proof of the second statement of the theorem relies on an application of the Stable Manifold Theorem as stated in Section 2.3 as well as Lemma 5. Let $\dot{x} = \varphi_{T, \tau}(x)$ for a particular $T, \tau$. We aim to show that $\mathbb{P}[\cup_{\hat{x}} \{W_h(\hat{x}) \mid \hat{x} \text{ is a saddle or local maximum}\}] = 0$ under Assumptions 2-3. It is sufficient to show that, for each critical point $x^*$ such that $\varphi_{T, \tau}(x^*) = 0$, and almost all $T, \tau$, $D\varphi_{T, \tau}(x^*)$ is full rank and its eigenvalues have non-zero real parts. The reason for this argument is the following: let $x^*$ be a critical point with $D\varphi_{T, \tau}(x^*)$ full rank and eigenvalues with non-zero real parts. If the eigenvalues do not all have positive real parts, then some have negative real parts, which indicates that $x^*$ is a saddle or local maximum of $E$. These negative real-part eigenvalues induce an unstable manifold of dimension $n_u \geq 1$. As such, the globally stable set $W_h(x^*)$ is a manifold with dimension $n - n_u < n$, and $\mathbb{P}[x(0) \in W_h(x^*)] = 0$ per Assumption 2.

To argue this case, define $h : (0, 1)^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$h(x, T, \tau) = \det D\varphi_{T, \tau}(x).$$

We now leverage Assumption 3 and [21] to claim first that $\mathbb{P}[h(x^*, T, \tau) = 0] = 0$ for each $x^* \in \mathcal{X}$, i.e. $D\varphi_{T, \tau}(x^*)$ is full rank for each $x^*$ with probability one w.r.t. $\mathbb{P}$. We first address the points $x$ for which the function $h$ is discontinuous. Define $\hat{\mathcal{X}}$ as the set of $x$ for which the truncation of the eigenvalues of $H(x)$ becomes active, i.e. the discontinuous points of $h$. Although we do not write it as such, note that $H$ is implicitly a function of $T, \tau$ and that the eigenvalues of $H$ can be expressed as nonconstant real-analytic functions of $T, \tau$. Considering this fact and an arbitrary $x$, the set of $T, \tau$ which give $x \in \hat{\mathcal{X}}$ has measure zero with respect to $\mathbb{R}^2$ [21]. Thus, for particular $T, \tau$, $h$ is $C^\infty$ almost everywhere. Applying once more the argument in [21] and Assumption 3 with the fact that $h$ is a nonconstant real analytic function of $T, \tau$, we have that

$$\mathbb{P}[\hat{T}(\hat{x}) \triangleq \{(T, \tau) \mid h(\hat{x}, T, \tau) = 0\}] = 0, \quad \forall \hat{x} \notin \hat{\mathcal{X}}.$$ 

Now consider the set of critical points as an explicit function of $T, \tau$ and write this set as $\mathcal{X}(T, \tau)$. Recalling Lemma 5, the set of $\hat{x}$ that we are interested in reduces to a finite set of critical points $x^* \in \mathcal{X}(T, \tau)$. Thus, we can conclude that

$$\mathbb{P}[\cup_{x^* \in \mathcal{X}(T, \tau)} T(x^*)] \leq \sum_{x^* \in \mathcal{X}(T, \tau)} \mathbb{P}(T(x^*)) = 0.$$

There is an additional case which must be considered, which is that $h(x^*, T, \tau) \neq 0$, but some eigenvalues of $D\varphi_{T, \tau}(x^*)$ are purely imaginary and induce stable center manifolds, which could accommodate the case of a globally stable set which is an $n$-dimensional manifold (i.e. the “degenerate saddle” case). We consider the function $h$ mostly out of convenience, but the argument can be extended to a function $h : (0, 1)^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}^n$ which is a map to the roots of the characteristic equation of $D\varphi_{T, \tau}(x)$. We are concerned that each element of $h(x, T, \tau)$ should have a nonzero real part almost everywhere. To extend the previous case to this, consider the identification $\mathbb{C} \equiv \mathbb{R}^2$ and compose $h$ with the nonconstant real analytic function $\zeta(w, z) = w$, for which the zero set is $w = 0$, corresponding to the imaginary axis in our identification. From this, we obtain a nonconstant real-analytic as before whose zero set is the imaginary axis. Applying the argument in [21] in a similar way as above, $h(x, T, \tau)$ has nonzero real parts for almost all $(T, \tau)$ for each $x$. Therefore, the probability of a particular saddle point or local maximum $x^*$ having a nonempty stable center manifold is zero for arbitrary $x(0)$ satisfying Assumption 2 and $T, \tau$ satisfying Assumption 3.

\[\square\]

We now define a Deterministic Annealing (DA) variant inspired by [26] to augment the NNN-c dynamics and provide a method for gradually learning a justifiably good feasible point of $\mathcal{P}$. In [26], the author justifies the deterministic online tuning of a temperature parameter in the context of data clustering and shows that this avoids poor local optima by more thoroughly exploring the state space. Similarly, we aim to learn a sufficiently good solution trajectory by allowing the dynamics to explore the interior of the unit hypercube in the early stages of the algorithm, and then to force the trajectory outward to a feasible binary solution by gradually adjusting $T$ or $\tau$ online.

Consider either reducing the temperature $T$ or increasing the time constant $\tau$ during the execution of NNN-c. This reduces the terms in $E$ which promote convexity, particularly near the boundaries of the unit hypercube. As $T, \tau$ are adjusted, for $a \prec -\gamma\|p\|^2$, the domain of $E$ becomes gradually more concave away from the corners of the unit hypercube. Thus, starting with $T/\tau$ sufficiently large, the early stages of the algorithm promote exploration of the interior of the state space. As $T/\tau$ is reduced, the equilibria of $E$ are pushed closer to (and eventually converge to) the feasible points of $\mathcal{P}$. The update policy we propose is described formally in Algorithm 1, and we further explore its performance in simulation.

**Corollary 10 (Convergence to Feasible Points).** Under Assumptions 2-3 and $a \prec -\gamma\|p\|^2$, the NNN-c
Algorithm 1 Deterministic Annealing

1: procedure DET-ANNEAL($\beta > 1, T_0, \tau_0, t_d$)
2: Initialize $x(0)$
3: $T \leftarrow T_0, \tau \leftarrow \tau_0$
4: while true do
5: Implement NNN-c for $t_q$ seconds
6: $\tau \leftarrow \beta \tau$ or $T \leftarrow (1/\beta)T$
7: end while
8: end procedure

dynamics augmented with Algorithm 1 converge asymptotically to feasible points of $\mathcal{P} 1$.

The result of the Corollary is quickly verified by inspecting the terms of $H(x)$ in (8). The function $E$ is smooth, strictly concave near $x = (0.5) \mathbf{1}_n$ for small $T/\tau$ due to the design of $a_n$, and becomes strictly convex (corresponding to isolated local minima of $E$) as the elements of $x$ approach 0 or 1 due to the $T/\tau$ term dominating (8). As the quantity $T/\tau$ is reduced under Algorithm 1, these local minima are shifted asymptotically closer to corners of the unit hypercube, i.e. feasible points of $\mathcal{P} 1$.

5 Distributed Hopfield Neural Network

With the framework of the previous section we formulate a problem $\mathcal{P} 2$ which is closely related to $\mathcal{P} 1$, but for which the global penalty term can be encoded by means of an auxiliary decision variable. This formulation leads to the Distributed Newton-like Neural Network, or NNN-d, which we rigorously analyze for its convergence properties.

5.1 Problem Reformulation

It is clear from the PT-inverse operation and $W$ being nonsparse that NNN-c is indeed centralized. In this section, we design a distributed algorithm in which each agent $i$ must only know $p_j, j \in \mathcal{N}_i$ and the value of an auxiliary variable $y_j, j \in \mathcal{N}_i \cup \mathcal{N}^2_i$, i.e. it must have communication with its two-hop neighbor set. If two-hop communications are not directly available, the algorithm can be implemented with two communication rounds per algorithm step. We provide comments on a one-hop algorithm in Remark 18.

Assumption 4 (Graph Properties and Connectivity). The graph $\mathcal{G} = (\mathcal{N}, E)$ is undirected and connected; that is, a path exists between any two pair of nodes and, equivalently, its associated Laplacian matrix $L = L^\top$ has rank $n - 1$.

Now consider the $n$ linear equations:

$$p \circ x + Ly = \frac{P}{n} \mathbf{1}_n.$$

Notice that, by multiplying from the left by $\mathbf{1}_n^\top$ and applying $\mathbf{1}_n^\top L = \mathbf{0}_n^\top$, we recover $p^\top x = P_i$. Thus, by augmenting the state with an additional variable $y \in \mathbb{R}^n$, we can impose a distributed penalty term. We now formally state the distributed reformulation of $\mathcal{P} 1$.

$$\mathcal{P} 2 : \min_{x \in (0,1)^n, y \in \mathbb{R}^n} \hat{f}(x,y) = \sum_{i} f_i(x_i) + \frac{\gamma}{2} \sigma^\top \sigma,$$

where the costs $f_i$ again satisfy $f_i(1) - f_i(0) = c_i$, and we have defined $\sigma = p \circ x + Ly - (P_i/n) \mathbf{1}_n$ for notational simplicity. Before proceeding, we provide some context on the relationship between $\mathcal{P} 1$ and $\mathcal{P} 2$ and a remark on the importance of graph connectivity.

Claim 11 (Equivalence of $\mathcal{P} 1$ and $\mathcal{P} 2$). Let Assumption 4, on graph connectivity, hold, and let $(x^*, y^*)$ be a solution to $\mathcal{P} 2$. Then, $x^*$ is a solution to $\mathcal{P} 1$ and $f(x^*) = \hat{f}(x^*, y^*)$.

Proof. The equivalence stems from the global term and the flexibility in the unconstrained $y$ variable. Notice

$$\frac{\gamma}{2} \sigma^\top \sigma = \frac{\gamma}{2} \sigma^\top (I_n - \mathbf{1}_n \mathbf{1}_n^\top /n)\sigma + \frac{\gamma}{2} \sigma^\top (\mathbf{1}_n \mathbf{1}_n^\top /n)\sigma = \frac{\gamma}{2} \sigma^\top (I_n - \mathbf{1}_n \mathbf{1}_n^\top /n)\sigma + \frac{\gamma}{2} (p^\top x - P_i)^2.$$

We have recovered the original global term of $\mathcal{P} 1$ in the bottom line, so now we deal with the remaining term. The matrix $I_n - \mathbf{1}_n \mathbf{1}_n^\top /n \succeq 0$ has image $I_n - \mathbf{1}_n \mathbf{1}_n^\top /n = \text{span}\{\mathbf{1}_n\}^\perp = \text{image } L$, given that $L$ is connected. Thus, because $y$ is unconstrained and does not enter the cost anywhere else, we can compute the set of possible minimizers of $\hat{f}$ in closed form with respect to any $x$ as

$$y^* \in \{-L^1(p \circ x - (P_i/n) \mathbf{1}_n) + \theta \mathbf{1}_n \mid \theta \in \mathbb{R}\} = \{-L^1(p \circ x) + \theta \mathbf{1}_n \mid \theta \in \mathbb{R}\}.$$

Moreover, substituting a $y^*$ gives $\sigma \in \text{span}\{\mathbf{1}_n\}$, and it follows that the problem $\mathcal{P} 2$ reduces precisely to $\mathcal{P} 1$.

Remark 12 (Role of Graph Connectivity). The result of Claim 11 that $\mathcal{P} 2$ is equivalent to $\mathcal{P} 1$ is clearly dependent on the matrices $L$ and $I_n - \mathbf{1}_n \mathbf{1}_n^\top /n$ sharing the same image, otherwise, the same image, otherwise, the term $(\gamma/2) \sigma^\top (I_n - \mathbf{1}_n \mathbf{1}_n^\top /n)\sigma$ cannot be “cancelled” by $y$. If one attempts to solve $\mathcal{P} 2$ via a graph with $m$ distinct connected components, the null space of $L$ will be spanned by $m$ vectors with ones in the entries corresponding to the nodes within a particular component, and zeros elsewhere [6].

The effect that this has on $\mathcal{P} 2$ and the dynamics to follow is that each distinct component $\mathcal{I} \subset \mathcal{N}$ of size $n_2 < n$ attempts to minimize the term $\left(\sum_{i \in \mathcal{I}} p_i x_i - \frac{P_i n_2}{n}\right)^2$.
5.2 Distributed Newton-like Neural Net Dynamics

To define NNN-d, we augment the centralized NNN-c with gradient-descent dynamics in \( y \) on a newly obtained energy function \( \tilde{E} \) of \( \mathcal{P} \). Define \( \tilde{E} \) as

\[
\tilde{E}(x, y) = \tilde{f}(x) + \frac{1}{\tau} \sum_i \int_0^{x_i} g^{-1}(\nu) d\nu.
\]  (13)

In Section 4, we obtained a matrix \( W \) which was nonsparse. Define \( \tilde{W}, \tilde{v} \) for \( \tilde{E} \) in accordance with Section 2.5:

\[
\tilde{W} = -\text{diag}(a + \gamma p^2), \quad \tilde{v} = \alpha \circ b + \gamma \text{diag}(p) \left( \frac{P}{n} 1_n - Ly \right).
\]

Compute the Hessian of \( \tilde{E} \) with respect to only \( x \) as:

\[
\tilde{H}(x) = \nabla_{xx} \tilde{E}(x, y) = -\tilde{W} + \frac{T}{\tau} \text{diag}(1/x - x^2).
\]

Since \( \tilde{H}(x) \) is diagonal, the \( ii \)th element of the PT-inverse of \( \tilde{H}(x) \) can be computed locally by each agent \( i \) as:

\[
(|\tilde{H}(x)|_i)^{-1} = \begin{cases} |\tilde{H}(x)|^{-1}, & |\tilde{H}(x)_{ii}| \geq m, \\ 1/m, & \text{o.w.} \end{cases}
\]

where \( \tilde{H}(x)_{ii} = a_i + \gamma p^2 + T/\tau (x_i - x_i^2)^{-1} \). The NNN-d dynamics, which are PT-Newton descent in \( x \) and gradient descent in \( y \) on \( \tilde{E} \), are then stated as:

\[
\dot{x} = (|\tilde{H}(x)|)_i^{-1} \text{diag}((x - 2x_i)/T) \left( \tilde{W}x + \frac{T}{\tau} \log \left( \frac{1}{x} - 1_n \right) + \tilde{v} \right),
\]  (14)

\[
\dot{y} = -\alpha \gamma L (\text{diag}(p)x + Ly),
\]

where \( \alpha > 0 \) is an arbitrary gain constant. Due to the new matrices \( \tilde{W}, \tilde{v} \) and the sparsity of \( L \), \( \dot{x} \) can be computed with one-hop information and \( \tilde{y} \) with two-hop information (note the \( L^2 \) term); thus, (14) defines a distributed algorithm. Additionally, recalling the discussion on parameter design, the problem data \( a \) and \( b \) can now be locally designed.

Before proceeding, we establish a property of the domain of \( y \) and some distributed extensions of Lemmas 5 and 6.

**Lemma 13 (Domain of Auxiliary Variable).**

Given an initial condition \( y(0) \) with \( 1_n y(0) = \kappa \), the trajectory \( y(t) \) is contained in the set

\[
\mathcal{Y} = \{ \omega + (\kappa/n) 1_n \mid 1_n^\top \omega = 0 \}. \quad (15)
\]

**Proof.** The proof is trivially seen by multiplying \( \dot{y} \) in (14) from the left by \( 1_n \) and applying the null space of \( L \).

**Lemma 14 (Closed Form Auxiliary Solution).**

For an arbitrary fixed \( x \in [0, 1]^n \), the unique minimizer \( y^* \) contained in \( \mathcal{Y} \) of both \( \dot{f} \) and \( \tilde{E} \) is given by

\[
y^* = -L^1 (p \circ \tilde{x}) + \frac{\kappa}{n} 1_n.
\]  (16)

This is also the unique equilibrium of (14) in \( \mathcal{Y} \).

**Proof.** The first term is computed by setting \( \nabla_y \dot{f}(x, y^*) = 0_n \) (resp. \( \nabla_y \tilde{E}(x, y^*) = 0_n \)) and solving for \( y^* \). There is a hyperplane of possible solutions due to the rank deficiency of \( L \), but we are looking for the unique solution in \( \mathcal{Y} \). The second term therefore follows from (15). The fact that this point is also the unique equilibrium in \( \mathcal{Y} \) follows from the fact that \( \dot{y} = -\alpha \nabla_y \tilde{E}(x, y^*) \).

**Lemma 15 (Finite Equilibria (Distributed)).** Let \( \tilde{X} \times \tilde{Y} \) be the set of all equilibria of (14) satisfying \( (\tilde{x}, \tilde{y}) = 0 \) on \( (x, y) \in [0, 1]^n \times \mathcal{Y} \). The set \( \tilde{X} \times \tilde{Y} \) is finite.

**Proof.** The proof follows closely to the proof of Lemma 5 with the variation that \( \dot{\tilde{y}} \) in the expression for \( \tilde{x} \) is now a function of \( y \). Given the result of Lemma 14, we may directly substitute the unique \( y^* \) (16) for any \( x \). Because \( y^* \) is simply a linear expression in \( x \), the same argument as in Lemma 5 that \( \tilde{X} \) is finite follows.

We now extend the results of Section 4.3 to the distributed case of solving \( \mathcal{P} \) via NNN-d. We have the following theorem on the trajectories of \( (x(t), y(t)) \) under (14).

**Theorem 16 (Convergence of NNN-d).** An initial condition \( (x(0), y(0)) \in (0, 1)^n \times \mathbb{R}^n \) converges asymptotically under NNN-d to a critical point \((x^*, y^*)\) of \( \tilde{E} \). Under Assumption 3 on the choice of \( T, \tau \), and Assumption 2 on the initial condition \( x(0) \), the set of initial conditions \( \bigcup \tilde{W}_n(\tilde{x}, \tilde{y}) \), where \((\tilde{x}, \tilde{y})\) is a saddle-point or local maximum of \( \tilde{E} \), has measure zero on \((0, 1)^n \times \mathbb{R}^n \). Thus, an initial condition chosen randomly from a suitably dense distribution in \((0, 1)^n \times \mathbb{R}^n \) converges asymptotically to a local minimizer \((x^*, y^*)\) with probability one. Lastly, all local minima \((x^*, y^*)\) of \( \tilde{E} \) are globally optimal w.r.t. \( y \).
for the particular $x^*$, meaning that $\bar{E}(x^*, y) \geq \bar{E}(x^*, y^*)$ for all local minima $(x^*, y^*)$.

**Proof.** The first part of the proof to establish convergence to a critical point follows from a similar argument to the proof of Theorem 9. Differentiating $\bar{E}$ with respect to time gives:

$$
\frac{d\bar{E}}{dt} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}^T \begin{bmatrix} \nabla_x \bar{E}(x, y) \\ \nabla_y \bar{E}(x, y) \end{bmatrix}
= \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}^T \begin{bmatrix} -W x - \tilde{v} + g^{-1}(x)/\tau \\ -\alpha^{-1} \dot{y} \end{bmatrix}
= -\dot{x}^T \text{diag}(T/(x - x^2))|H(x)|m_\dot{x} - \alpha^{-1} \dot{y}^T \dot{y} < 0,
\dot{x} \neq 0 \text{ or } \dot{y} \neq 0.
$$

(17)

Thus, $\bar{E}$ monotonically decreases along the trajectories of NNN-d. Given (17), we now re-establish the forward invariance property of the open hypercube for the distributed case:

**Lemma 17 (Forward Invariance of the Open Hypercube (Distributed)).** The set $(0, 1)^n \times \mathcal{Y}$ is a forward-invariant set under the NNN-d dynamics (14).

**Proof.** The forward invariance of $\mathcal{Y}$ is already established per its definition and Lemma 13, but we must establish that the trajectories $y(t)$ remain bounded in order to apply the argument in Lemma 6. Compute the Hessian of $\bar{E}$ with respect to $y$ as:

$$
\nabla_{yy} \bar{E} = \gamma L^2 \succeq 0.
$$

Due to the connectedness of $L$, the eigenspace associated with the $n - 1$ strictly positive eigenvalues of $\gamma L^2$ is parallel to $\mathcal{Y}$. Therefore, $\bar{E}$ is strictly convex in $y$ on this subspace, and it follows that $\bar{E}$ is bounded from below on $\mathcal{Y}$. Due to $d\bar{E}/dt \leq 0$ (17) and the continuity of $\bar{E}$ in $y$, it follows that $y(t)$ is bounded for all $t$. Given this, the argument from Lemma 6 applies to the trajectories $x(t)$, and the set $(0, 1)^n \times \mathcal{Y}$ is forward invariant under NNN-D (14).

Due to the deficiency induced by $L$, $\bar{E}$ is not radially unbounded in $y$ over all of $\mathbb{R}^n$, so we must be careful before applying the LaSalle Invariance Principle. Instead, define $\tilde{E}$ only on $[0, 1]^n \times \mathcal{Y}$ in consideration of Lemma 13. Radial unboundedness in $\tilde{E}$ is then obtained given any $y(0)$, and it follows that the trajectories converge to largest invariant set contained in $d\tilde{E}/dt = 0$ per the LaSalle Invariance Principle [17]. This is the finite set of critical points of $\tilde{E}$ per Lemma 15, and so it additionally follows that we converge to a single critical point $(x^*, y^*)$.

Because $\tilde{E}$ is convex in $y$, it follows that for any fixed $x$ there exist only local minima of $\tilde{E}$ with respect to $y$. In consideration of this, we need only apply the Stable Manifold Theorem to $x$. The argument for this develops similarly to the proof of Theorem 9, and we conclude that the trajectories of NNN-d converge to a local minimizer $(x^*, y^*)$ of $\tilde{E}$ with probability one.

The final part of the Theorem statement that $\bar{E}(x^*, y) \geq \bar{E}(x^*, y^*)$, $\forall y \in \mathbb{R}^n$ can also be seen from the convexity of $\bar{E}$ in $y$ and applying the first-order condition of convexity:

$$
\bar{E}(x^*, y) \geq \bar{E}(x^*, y^*) + (y - y^*)^T \nabla_y \bar{E}(x^*, y^*)
$$

along with $\nabla_y \bar{E}(x^*, y^*) = 0_n$.

**Remark 18 (One-Hop Distributed Algorithm).** The proposed distributed algorithm requires two-hop neighbor information, which may be intractable in some settings. The source of the two-hop term stems from the squared flavor of the $\gamma$ penalty term. In fact, it is possible to define a one-hop distributed algorithm via a Lagrangian-relaxation route.

Consider posing $\mathcal{P} 2$ with the $\gamma$ term instead as a linear constraint: $\sqrt{\gamma/2}(p \circ x + Ly) = \sqrt{\gamma/2}(P_l/n) 1_n$. Writing the Lagrangian for this problem introduces a Lagrange multiplier on the linear constraint terms. From there, it would be appropriate to define a saddle-point-like algorithm along the lines of [8], in which gradient-ascent in the dual variable is performed on the Lagrangian. This changes the nature of the penalty from squared to linear, and the linearly constrained formulation may generally be infeasible, but it follows that this approach could be implemented with one-hop information.

### 6 Simulations

Our simulation study is split into two parts; the first focuses on numerical comparisons related to runtime and solution quality, and the second is a 2D visualization of the trajectories of the Distributed Annealing (DA) variants for both the centralized and distributed NNN methods.

#### 6.1 Runtime and Solution Quality Comparison

In this section, we compare to a greedy method stated as Algorithm 2 and a semidefinite programming (SDP) relaxation method stated as Algorithm 3. In short, the greedy method initializes the state as $x = 0_n$, and iteratively sets the element $x_i$ to one which decreases the
cost function the most. This is repeated until no element remains for which the updated state has lower cost than the current state. For the SDP method, a convex SDP is obtained as the relaxation of \( P_1 \), see e.g. [30]. We use the shorthand SDP$_{Prx}(\bullet)$ to indicate this in the statement of Algorithm 3. This SDP is solved using CVX software in MATLAB [12] and a lowest-cost partition is computed to construct a feasible solution. For the sake of convenience in stating both algorithms, we have defined \( f': 2^n \rightarrow \mathbb{R} \) to be the set function equivalent of \( f \), i.e. the cost of \( P_1 \). That is, \( f'(S) = f(x) \), where \( i \in S \) indicates \( x_i = 1 \) and \( i \notin S \) indicates \( x_i = 0 \).

Algorithm 2 Greedy Method

1. procedure Greedy\((f')\)
2. \( S \leftarrow \emptyset \)
3. done \( \leftarrow \) false
4. while done = false do
5. \( i^* \leftarrow \text{argmin } f'(S \cup \{i\}) \) \( \forall i \notin S \)
6. if \( f'(S \cup \{i^*\}) < f'(S) \) then
7. \( S \leftarrow S \cup \{i^*\} \)
8. else
9. done \( \leftarrow \) true
10. end if
11. end while
12. \( x_i \leftarrow \begin{cases} 0, & i \notin S, \\ 1, & i \in S. \end{cases} \)
13. return \( x \)
14. end procedure

Algorithm 3 SDP Relaxation Method

1. procedure SDP\((f')\)
2. \( P_{SDP} \leftarrow \text{SDP}_{Prx}(\mathcal{P}_1) \)
3. \( x^* \leftarrow \text{argmin } P_{SDP} \)
4. \( S \leftarrow \emptyset \)
5. done \( \leftarrow \) false
6. while done = false do
7. \( i^* \leftarrow \text{argmax } x_i \) \( \forall i \notin S \)
8. if \( f'(S \cup \{i^*\}) < f'(S) \) then
9. \( S \leftarrow S \cup \{i^*\} \)
10. else
11. done \( \leftarrow \) true
12. end if
13. end while
14. \( x_i \leftarrow \begin{cases} 0, & i \notin S, \\ 1, & i \in S. \end{cases} \)
15. return \( x \)
16. end procedure

As for algorithm performance as it pertains to the cost of the obtained solution, we fix \( n = 50 \) and additionally include DA variants of both NNN-c and NNN-d. We also omit the brute force method due to intractability. For the sake of comparison, we compute a performance metric \( Q \) and provide it for each method in Table 1. The metric \( Q \) is computed as follows: for each trial, sort the methods by solution cost. Assign a value of 6 for the best method, 5 for the second-best, and so on, down to the seventh-best (worst) receiving zero. Add up these scores for all 100 trials, and then normalize by a factor of 600 (the maximum possible score) to obtain \( Q \). It should be unsurprising that the tried-and-true centralized greedy and SDP methods perform the best. However, we note that they were beaten by our methods in a significant number of trials, which can be seen by noting that a \( Q \) score for two methods which perform best or second-best in all trials would sum to 1100/600 = 1.83, while \( Q \) (greedy) + \( Q \) (SDP) = 1.75, or a cumulative pre-scaled score of 1050, indicating that our methods outperformed these methods in net 50 “placement spots” over the 100 trials. In general, we find that the DA version of the NNN algorithms obtains better solutions than the non-DA version, confirming the benefit of this approach. We also find that NNN-d generally outperforms NNN-c. It’s possible that an initially “selfish” trajectory in \( x \) is beneficial, which would neglect the global penalty until \( y \) adequately converges, although this is speculative. Lastly, we note that the HNN method never performs better than worst, which we attribute to the steepest-descent nature of gradient algorithms which do not use curvature information of the energy function. It might be possible that the stopping criterion forces HNN to terminate near saddle-points, although we do not suspect this since we observe the Hessian is positive-definite in the majority of termination instances.
Table 1
Comparison of performance metric $Q$ for 100 randomized trials with $n = 50$.

| Method        | $Q$  |
|---------------|------|
| NNN-c         | 0.2161 |
| NNN-c-DA      | 0.2891 |
| NNN-d         | 0.5443 |
| NNN-d-DA      | 0.7005 |
| HNN           | 0    |
| Greedy        | 0.8411 |
| SDP           | 0.9089 |

Table 2
Problem data and parameter choices (where relevant) for performance comparison. Problem data $p_i, c_i$ is generated randomly from given distributions for each of 100 trials.

| Data or parameter | Value |
|-------------------|-------|
| $n$               | 50    |
| $p_i$             | $U[1, 50]$ |
| $c_i$             | $p_i^e, e \sim U[2, 3]$ |
| $P_i$             | 1500  |
| $\gamma$         | 1    |
| $T_0$             | 1    |
| $\tau_0$         | 0.1  |
| $m$               | 0.1  |
| $\alpha$         | 1    |
| Learning steps    | 10   |
| $\beta$          | 1.4  |
| $n$               | 50   |

As for parameter selection, we find that choosing $m \ll 1$ is generally best, since $m \geq 1$ would always produce a PT-inverse Hessian with eigenvalues contained in $(0, 1]$. This effectively scales down $\dot{x}$ in the eigenspace associated with Hessian eigenvalue magnitudes greater than 1, but does not correspondingly scale up $\dot{x}$ in the complementary eigenspace associated with small eigenvalues. Additionally, choosing $T/\tau$ greater than 1 in the fixed case tended to be effective. This may be related to selecting $a_i < -\gamma||p||^2 - 4T/\tau$ to guarantee anti-stability from $(0.5) \begin{bmatrix} 1 \\ n \end{bmatrix}$, and would explain why an initially high $T/\tau$ that decreases in the DA learning variant performs so well. Finally, all $\alpha \approx 1$ seemed to behave roughly the same, with only $\alpha \ll 1$ and $\alpha \gg 1$ behaving poorly (the former leading to slow convergence in $y$ and “selfish” behavior in $x$, and the latter being destabilizing in the discretization of $\hat{y}$).

6.2 Learning Steps and 2-D Trajectories

Next, for the sake of understanding how the learning rate $T/\tau$ affects the trajectories of the solutions, we have provided Figure 3 which plots the 2-D trajectories of NNN-c and NNN-d with $T/\tau$ being gradually reduced over 15 learning steps. The contours of the energy function for the final step are also plotted. The problem data and choice for $a$ is:

$$c = (2, 1)^T, \quad p = (3, 1)^T, \quad P_i = 2.8, \quad \gamma = 4, \quad a := -(10, 10)^T.$$  

Note that, in each case, the trajectory approaches the optimal solution $x^* = (1, 0)^T$. However, it is worth noting that a steep saddle point occurs around $x = (0.75, 0.6)^T$. Intuitively, this corresponds to a high risk of the trajectory veering away from the optimal solution had the DA not been implemented. With the opportunity to gradually learn the curvature of the energy function, as shown by stabilization to successive equilibria marked by $\times$, each algorithm is given the opportunity to richly explore the state space before stabilizing to the optimal solution $(1, 0)^T$. Further studying the learning-rate $T/\tau$ and a more complete analysis of Algorithm 1 and the parameter $\beta$ are subjects of future work.

7 Conclusion

This paper posed an optimal generator dispatch problem for settings in which the agents are generators with binary controls. We first showed that the centralized problem is amenable to solution via a Centralized NEWTON-LIKE NEURAL NETWORK approach and proved convergence to a local minimizer with probability one under light assumptions. Next, we developed an approach to make the dynamics computable in a distributed setting in which agents exchange messages with
Fig. 3. Centralized NNN-c (a) and distributed NNN-d (b) trajectories in 2D with 15 learning steps. Stable equilibrium points between learning steps indicated by ×, contours of $E$ and $\tilde{E}$ in final step indicated by dashed lines.

their two-hop neighbors in a communication graph. The methods scale and perform well compared to standard greedy and SDP-relaxation approaches, and the latter method enjoys the qualities of a distributed algorithm, unlike previous approaches. Future research directions include application of the methods to a broader class of problems which may include additional cost terms or constraints and a deeper analysis of the Deterministic Annealing variant as it pertains to the online adjustment of the learning-rate $T/\tau$.

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