A CLUSTER EXPANSION APPROACH TO RENORMALIZATION GROUP TRANSFORMATIONS

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Abstract. The renormalization group (RG) approach is largely responsible for the considerable success which has been achieved in developing a quantitative theory of phase transitions. This work treats the rigorous definition of the RG map for classical Ising-type lattice systems in the infinite volume limit at high temperature. A cluster expansion is used to justify the existence of the partial derivatives of the renormalized interaction with respect to the original interaction. This expansion is derived from the formal expressions, but it is itself well-defined and convergent. Suppose in addition that the original interaction is finite-range and translation-invariant. We will show that the matrix of partial derivatives in this case displays an approximate band property. This in turn gives an upper bound for the RG linearization.

1. Introduction

We consider renormalization group (RG) transformations for Ising-type lattice spin systems on $\mathbb{Z}^d$. The spins in the original lattice $L$ are denoted by $\sigma$, whereas the block spins in the image lattice $L'$ are denoted by $\sigma'$, and assumed to be of Ising-type also. This assumption allows for treatment of many important RG transformations such as decimation and majority rule, but is not applicable for more general types such as block-average transformations. $L'$ indexes a partition of $L$ into blocks, all with the same cardinality $s$. Thus for each site $y$ in $L'$, there is a corresponding block $y^o$ that is a subset of $L$. Also, $L$ is endowed with a metric $d$, and this naturally induces a metric $d'$ on $L'$.

Formally, the RG maps a Hamiltonian $H(\sigma) = -\sum_X J(X)\sigma_X$ into a renormalized Hamiltonian $H'(\sigma') = -\sum_Y J'(Y)\sigma'_Y$:

$$e^{-H'(\sigma')} = \sum_{\sigma'} \prod_{y \in L'} T_y(\sigma, \sigma'_y)e^{-H(\sigma)},$$

(1)

where $\sum_{\sigma}$ and $\sum_{\sigma'}$ (normalized sums) denote the product probability measures on $\{+1, -1\}^L$ and $\{+1, -1\}^{L'}$, respectively, and $T_y(\sigma, \sigma'_y)$ denotes a specific RG probability kernel, which depends only on $\sigma$ through the block corresponding to $y$, and satisfies both a symmetry condition,

$$T_y(\sigma, \sigma'_y) = T_y(-\sigma, -\sigma'_y),$$

(2)

and a normalization condition,

$$\sum_{\sigma'} T_y(\sigma, \sigma'_y) = 1$$

(3)

for every $\sigma$ and every $y$. Notice that because of (2) and (3),

$$\sum_{\sigma} T_y(\sigma, +1) = \sum_{\sigma} T_y(\sigma, -1) = 1.$$
Equation (4) will be of fundamental importance for the purpose of this paper (cf. proof of Proposition 2.2), whereas assumptions (2) and (3), which lead to (4), are not essential. This, however, would rule out RG transformations where different block spins have unequal occurrence probabilities. (Some references for this literature may be found, for instance, in a series of lectures by Brydges [2].)

Our basic assumption is that the original interaction $J$ lies in a Banach space $B_r$, with norm

$$||J||_r = \sup_{x \in \mathbb{L}, x \in X} |J(X)||e^r|X|, \quad (5)$$

where the constant $r > 0$ and $|X|$ denotes the cardinality of the set $X$. The properties of the RG transformation have been studied extensively by mathematical physicists over a period of many years. The first existence results of renormalized interactions in trivial unique-phase regimes were obtained by Griffiths and Pearce [6] in the low density (or high magnetic field) regime by cluster expansion methods. They also presented plausible arguments showing that the RG transformation exhibits a rather peculiar behavior at low temperatures. Robert Israel [8] justified the existence of the renormalized interaction $J'$ using beautiful and ingenious techniques involving Banach algebras and conditional expectations. Kashapov [9] worked with cumulants (semi-invariants), with estimates that relied on combinatorial methods of Malyshev [13], and showed that the RG map can be formalized rigorously in terms of the Hamiltonian of a Gibbs field. Martinelli and Olivieri [14] [15] investigated the stability and instability of pathologies of RG transformations under decimation. For block-average RG transformations, Cammarota [3] proved that the block spin interaction tends in norm to a one-body quadratic potential in the infinite volume limit at high temperature, whereas van Enter [20] constructed a counterexample showing that the renormalized measure may not be Gibbsian even when the temperature is above the critical temperature.

These results are extended by the recent work [22] which analyzes the spectrum of the RG maps corresponding to decimation and majority rule at infinite temperature and discovers that it is of an unusual kind: dense point spectrum for which the adjoint operators have no point spectrum at all, but only residual spectrum. The present investigation is a follow-up to my previous work and explores various existence properties of the RG at high temperature. It employs a reasonably straightforward application of the cluster expansion machinery and justifies the existence of the partial derivatives of the renormalized interaction $J'$ with respect to the original interaction $J$ (Theorem 3.6). Under the additional assumption that the original interaction $J$ is finite-range and translation-invariant, it will be shown that the matrix of partial derivatives displays an approximate band property (Theorem 4.3). This in turn gives an upper bound for the RG linearization (Theorem 5.3).

The real interest of the RG is to define the transformation at intermediate temperature, in particular, the critical temperature. This is a considerably more difficult enterprise: one could worry about the many issues raised by van Enter, Fernández, and Sokal [21]. Fortunately, there is some hope for progress in this area due to the fact that the correlation length of the constrained system relevant to the definition of the RG transformation may well be finite, and may even sometimes be used as a small parameter. Pioneering efforts were made by Olivieri and his various collaborators [17] [18]. Another approach in a similar spirit was developed in the important work of Kennedy [10]. Additional references may be found in [11] [4] [5] [12]. Using similar techniques as in the present work [23], it may be shown that parallel results for RG linearization hold under the condition proposed by Haller and Kennedy [7].
Another possible generalization of this work is in the context of Potts models. Working with transmissivities rather than coupling constants [19], results on spectral properties of the RG map are expected in the high temperature regime through a more involved application of the cluster expansion machinery.

2. CLUSTER EXPANSION

This section gives cluster expansion expressions which are valid for finite lattices.

**Proposition 2.1.** For every subset $W$ of the original lattice and every subset $Z$ of the image lattice, the partial derivative $\frac{\partial J'(Z)}{\partial J(W)}$ of the RG transformation is given by the expression

$$\frac{\partial J'(Z)}{\partial J(W)} = \sum_{\sigma'} \sigma'_Z \frac{\partial}{\partial J(W)} \left( \sum_{\sigma} \prod_{y \in L'} T_y(\sigma, \sigma'_y) e^{\sum_X J(X)\sigma_X} \right).$$

(6)

**Proof.** By the use of the Fourier series on the group $\{+1, -1\}^L$, we see that the renormalized coupling constants $J'$ are given by

$$J'(Z) = \sum_{\sigma'} \sigma'_Z \log \left( \sum_{\sigma} \prod_{y \in L'} T_y(\sigma, \sigma'_y) e^{\sum_X J(X)\sigma_X} \right).$$

(7)

We then take the derivative of both sides of (7) with respect to $J(W)$. □

To understand the following Proposition II.2, we need to introduce some combinatorial concepts. A hypergraph is a set of sites together with a collection $\Gamma$ of nonempty subsets. Such a nonempty set is referred to as a hyper-edge or link. Two links are block-connected if they both intersect some block. The support of a hypergraph is the set $\cup \Gamma$ of sites that belong to some set in $\Gamma$. A hypergraph $\Gamma$ is block-connected if the support of $\Gamma$ is nonempty and cannot be partitioned into nonempty sets with no block-connected links. In our current setting, a subset $X$ of $L$ defines a subset $X'$ of $L'$, corresponding to the set of blocks that have non-empty intersection with $X$. Thus a hypergraph $\Gamma$ on $L$ defines a hypergraph $\Gamma'$ on $L'$. We use $\Gamma_c$ to indicate block connectivity of the hypergraph $\Gamma_c$, and write $\Gamma'_c = \cup \Gamma'_c$ for the support of $\Gamma'_c$ in the image lattice.

**Proposition 2.2.** Let $W(\sigma')$ be the frozen-block-spin partition function

$$W(\sigma') = \sum_{\sigma} \prod_{y \in L'} T_y(\sigma, \sigma'_y) e^{\sum_X J(X)\sigma_X}. \tag{8}$$

For fixed values of the renormalized spins $\sigma'$, $W(\sigma')$ has the cluster representation

$$W(\sigma') = \sum_{\Delta} \prod_{N \in \Delta} w_N, \tag{9}$$

where $\Delta$ is a set of disjoint subsets $N$'s of $L'$, and

$$w_N = \sum_{\Gamma'_c = N} \alpha(N, \Gamma_c, \{\sigma'\}_N) \tag{10}$$

and the sum here is over block-connected hypergraphs $\Gamma_c$ on $L$ whose images in $L'$ have support $N$. The contribution of each block-connected hypergraph is given by

$$\alpha(N, \Gamma_c, \{\sigma'\}_N) = \sum_{\sigma} \prod_{y \in N} T_y(\sigma, \sigma'_y) \prod_{X \in \Gamma_c} \left( e^{J(X)\sigma_X} - 1 \right). \tag{11}$$
Therefore, our claim thus follows.

where \( G \) is a set of subsets \( X \)'s of \( \mathcal{L} \).

We are going to organize the sum over hypergraphs in (12) in the following way. Each hypergraph \( \Gamma \) on \( \mathcal{L} \) has a support in \( \mathcal{L}' \), which breaks up into block-connected parts. Let \( \Delta \) be the parts, and for \( N \in \Delta \), let \( S(N) \) be the corresponding block-connected hypergraph on this part, i.e., \( S(N)^* = N \). Then summing over hypergraphs \( \Gamma \) is equivalent to summing over \( \Delta \) and functions \( S \) with the appropriate property. Furthermore, the product over \( N \) in \( \Delta \) and the links in \( S(N) \) is equivalent to the product over the corresponding \( \Gamma \). We have

\[
W(\sigma') = \sum_{\sigma} \prod_{y \in \mathcal{L}'} T_y(\sigma, \sigma_y') \sum_{\Delta} \sum_{S} \prod_{N \in \Delta} \prod_{X \in S(N)} \left( e^{J(X)\sigma_X} - 1 \right). 
\]  

By independence, the sum over \( \sigma \) can be factored over \( \Delta \), and this gives

\[
W(\sigma') = \sum_{\Delta} \prod_{y \in \cup \Delta} T_y(\sigma, \sigma_y') \sum_{S} \prod_{N \in \Delta} \sum_{\sigma} \prod_{y \in N} T_y(\sigma, \sigma_y') \prod_{X \in S(N)} \left( e^{J(X)\sigma_X} - 1 \right). 
\]  

Notice that because of (14), many of the \( T_y \) factors sum to 1, (13) can be simplified,

\[
W(\sigma') = \sum_{\Delta} \sum_{S} \prod_{N \in \Delta} \alpha(N, S(N), \{\sigma'\}_N). 
\]  

And by the distributive law,

\[
\sum_{S} \prod_{N \in \Delta} \alpha(N, S(N), \{\sigma'\}_N) = \prod_{N \in \Delta} \sum_{\Gamma_c^* = N} \alpha(N, \Gamma_c, \{\sigma'\}_N). 
\]  

Therefore

\[
W(\sigma') = \sum_{\Delta} \prod_{N \in \Delta} \sum_{\Gamma_c^* = N} \alpha(N, \Gamma_c, \{\sigma'\}_N). 
\]

Our claim thus follows. 

We rewrite (12) in the following way to apply standard results on cluster expansion,

\[
\sum_{\Delta} \prod_{N \in \Delta} w_N = \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{N_1, \ldots, N_p} \prod_{\{i,j\}} (1 - c(N_i, N_j)) w_{N_1} \cdots w_{N_p} 
= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{N_1, \ldots, N_p} \sum_{G} \prod_{\{i,j\} \in G} (-c(N_i, N_j)) w_{N_1} \cdots w_{N_p}, 
\]

where \( G \) is a graph with vertex set \( \{1, \ldots, p\} \) and

\[
c(N_i, N_j) = \begin{cases} 1 & \text{if } N_i \text{ and } N_j \text{ overlap;} \\ 0 & \text{otherwise.} \end{cases}
\]
Proposition 2.3. The frozen-block-spin free energy \(\log(W(\sigma'))\) is given by the cluster expansion
\[
\log(W(\sigma')) = \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{N_1,\ldots,N_p} C(N_1,\ldots,N_p) w_{N_1} \cdots w_{N_p},
\]
where
\[
C(N_1,\ldots,N_p) = \sum_{G_c} \prod_{\{i,j\} \in G_c} (-c(N_i,N_j)),
\]
and \(G_c\) is a connected graph with vertex set \(\{1,\ldots,p\}\).

Proof. The effect of taking the logarithm is that the sum over graphs is replaced by the sum over connected graphs:
\[
\log(W(\sigma')) = \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{N_1,\ldots,N_p} \sum_{G_c} \prod_{\{i,j\} \in G_c} (-c(N_i,N_j)) w_{N_1} \cdots w_{N_p}.
\]
Our claim thus follows. \(\square\)

3. Existence of the partial derivatives

From now on, we work in the infinite volume limit of the lattice system. Following standard interpretation of statistical mechanics, our results hold when the original interaction \(J\) is at high temperature.

Theorem 3.1 (Kotecký-Preiss). Recall (19) and (21). Take \(1 < M < e^r\). Suppose that
\[
\sum_{N'} c(N,N') |w_{N'}| M^{|N'|} \leq |N| \log(M).
\]
Then the avoidance probability for every \(Y \subset L'\) has a convergent power series expansion,
\[
\left| \sum_{\Delta'} \prod_{N \in \Delta'} w_N / \sum_{\Delta} \prod_{N \in \Delta} w_N \right| \leq \exp \left( -\sum_{p=1}^{\infty} \frac{1}{p!} \sum_{N_1,\ldots,N_p} C(N_1,\ldots,N_p) c(Y, \cup_{i=1}^p N_i) w_{N_1} \cdots w_{N_p} \right) \leq M^{|Y|},
\]
where \(\Delta'\) is a set of disjoint subsets of \(L' \setminus Y\), and \(\Delta\) is a set of disjoint subsets of \(L'\). Notice that here we are only counting contributions of block-connected \(N_i\)'s that are also block-connected to \(Y\).

Proposition 3.2. Take \(1 < M < e^r\). Consider the original coupling constants \(J\) with the Banach space norm \(||J||_r\). Suppose \(J\) is at high temperature (||\(J\)||_r small),
\[
||J||_r \leq \frac{\log(M) c^2}{2s \left(e + \log(M)\right)},
\]

(25)
where \( c = \frac{1}{\sqrt{e}} - 1 \) and \( \epsilon = Me^{-r} \). Then for each block spin site \( y \), we have

\[
\sum_{y \in N} |w_N|M^{|y|} \leq \log(M). \tag{26}
\]

**Remark.** The inequality (26) is a standard sufficient condition for (23). It will be applied in the following Theorem 3.6.

**Proof.** We notice that when \( ||J||_r \) is small (say \( ||J||_r \leq \frac{1}{2} \), \( e^{||J(X)||} - 1 \leq 2|J(X)| \) by the mean value theorem. Also, it easily follows from (5) that for all \( X \) with cardinality \( m \) and containing a fixed \( x \), \( \sum |J(X)| \leq ||J||_re^{-rm} \). More importantly, for \( \Gamma_c^* = N, |N| \leq \sum |X| \) with \( X \in \Gamma_c \). We have

\[
\sum_{y \in N} |w_N|M^{|y|} \leq \sum_{y \in N \cap \Gamma_c^*} \sum_{X \in \Gamma_c} 2|J(X)|M^{|X|} \leq \sum_{y \in N \cap \Gamma_c^*} \prod_{X \in \Gamma_c} 2|J(X)|M^{|X|} = \sum_{y \in \Gamma_c^*} \prod_{X \in \Gamma_c} 2|J(X)|M^{|X|}. \tag{27}
\]

We say that a hypergraph \( \Gamma_c \) is block-rooted at \( y \) if its support intersects a fixed block \( y^0 \). Let \( a_n(y) \) be the contribution of all block-connected hypergraphs with \( n \) links that are block-rooted at \( y \),

\[
a_n(y) = \sum_{y \in \Gamma_c^* \cap \Gamma_c^*} \prod_{X \in \Gamma_c} 2|J(X)|M^{|X|}. \tag{28}
\]

Then

\[
\sum_{y \in N} |w_N|M^{|y|} \leq \sum_{n=1}^{\infty} \sup_{y \in \mathcal{L}'} a_n(y). \tag{29}
\]

Let \( a_n \) be the supremum over \( y \) of the contribution of block-connected hypergraphs with \( n \) links that are block-rooted at \( y \), i.e., \( a_n = \sup_{y \in \mathcal{L}'} a_n(y) \). It seems that once we show that \( a_n \) is exponentially small, the geometric series above will converge, and our claim might follow. To estimate \( a_n \), we relate to some standard combinatorial facts [16]. The rest of the proof follows from a series of lemmas.

**Lemma 3.3.** Let \( a_n \) be the supremum over \( y \) of the contribution of block-connected hypergraphs with \( n \) links that are block-rooted at \( y \). Then \( a_n \) satisfies the recursive bound

\[
a_n \leq 2s||J||_r \sum_{m=1}^{\infty} \epsilon^m \sum_{k=0}^{m} \binom{m}{k} \sum_{a_{n_1}, \ldots, a_{n_k} \in \{0, 1\}^n} a_{n_1} \cdots a_{n_k} \tag{30}
\]

for \( n \geq 1 \), where \( \binom{m}{k} \) is the binomial coefficient.

**Proof.** We first linearly order the points \( x \) in \( \mathcal{L} \) and also linearly order the subsets \( X \) of \( \mathcal{L} \). This naturally induces a linear ordering of the points \( y \) in \( \mathcal{L}' \). For a fixed but arbitrarily chosen \( y \) in \( \mathcal{L}' \), we examine (28). Write \( \Gamma_c = \{X_1\} \cup \Gamma_c^1 \), where \( X_1 \) is the least \( X \) in \( \Gamma_c \) with \( y^0 \cap X_1 \neq \emptyset \). There must be such a set, since \( y \in \Gamma_c^* \). Moreover, there must be some \( x \in y^0 \) such that \( x \in X_1 \), of which there are \( s \) possibilities, as the block cardinality is \( s \). Then

\[
a_n(y) \leq s \sum_{m=1}^{\infty} \sum_{X_1 \cap X_1 \cap |X_1| = m} 2|J(X_1)|M^{|m|} \prod_{X \in \Gamma_c^1} 2|J(X)|M^{|X|}. \tag{31}
\]
As a consequence,

\[ a_n(y) \leq \sum_{m=1}^{\infty} 2s||J||_r \epsilon^m \sum_{k=0}^{m} \prod_{X \in \Gamma_c} 2|J(X)|M|X|. \tag{32} \]

The remaining hypergraph \( \Gamma_c^1 \) has \( n - 1 \) subsets and breaks into \( k : k \leq m \) block-connected components \( \Gamma_1, \ldots, \Gamma_k \) of sizes \( n_1, \ldots, n_k \), with \( n_1 + \cdots + n_k = n - 1 \). The set of such components may be empty, or it could just be the original block-connected set. For each component \( \Gamma_i \), there is a least block \( y_i^0 \) through which it is block-connected to \( X_1 \). The image \( \{ y_i \} \) of these blocks is a subset of \( X_1^r \) in \( L' \), thus has no more than \( k : k \leq m \) points, and the components are block-rooted at these image sites. Furthermore, different \( \Gamma_i \)'s correspond to disjoint \( \Gamma_i' \)'s, as \( y_i \in \Gamma_i' \), the map from the components to this image is injective. So we have

\[ a_n(y) \leq \sum_{m=1}^{\infty} 2s||J||_r \epsilon^m \sum_{k=0}^{m} \sum_{a_{n_1}, \ldots, a_{n_k}: n_1 + \cdots + n_k + 1 = n} a_{n_1} \cdots a_{n_k}. \tag{33} \]

Our inductive claim follows by taking the supremum over all \( y \) in \( L' \). Finally, we look at the base step: \( n = 1 \). In this simple case, as reasoned above, we have

\[ a_1 = \sup_{y \in L'} \sum_{y' \in \Gamma_c} \prod_{X \in \Gamma_c} 2|J(X)|M|X| \]

\[ \leq s \sum_{m=1}^{\infty} \sup_{x \in L} \sum_{X : x \in X, |X| = m} 2|J(X)|M^m \]

\[ = \sum_{m=1}^{\infty} 2s||J||_r \epsilon^m, \tag{34} \]

and this verifies our claim. \( \square \)

Clearly, \( \sum_{y \in N} |w_N| |M| |N| \) will be bounded above by \( \sum_{n=1}^{\infty} a_n \), if

\[ a_n = 2s||J||_r \epsilon^m \sum_{k=0}^{m} \epsilon^m \sum_{a_{n_1}, \ldots, a_{n_k}: n_1 + \cdots + n_k + 1 = n} a_{n_1} \cdots a_{n_k} \tag{35} \]

for \( n \geq 1 \), i.e., equality is obtained in the above lemma.

**Lemma 3.4.** Consider the coefficients \( a_n \) that bound the contributions of block-connected and block-rooted hypergraphs with \( n \) links. Let \( w = \sum_{n=1}^{\infty} a_n z^n \) be the generating function of these coefficients. Then the recursion relation (33) for the coefficients is equivalent to the formal power series generating function identity

\[ w = 2s||J||_r z \sum_{m=1}^{\infty} \epsilon^m (1 + w)^m = 2s||J||_r z \frac{\epsilon(1 + w)}{1 - \epsilon(1 + w)}. \tag{36} \]

**Proof.** Notice that \( (1 + w)^m = \sum_{k=0}^{m} \binom{m}{k} w^k \), thus

\[ w = 2s||J||_r z \sum_{m=1}^{\infty} \epsilon^m \sum_{k=0}^{m} \binom{m}{k} w^k. \tag{37} \]
Writing completely in terms of $z$, we have
\[ \sum_{n=1}^{\infty} \bar{a}_n z^n = 2s||J||_r \sum_{m=1}^{\infty} \epsilon^m \sum_{k=0}^{m} \binom{m}{k} \sum_{a_{n_1}, \ldots, a_{n_k}, n_1 + \cdots + n_k + 1 = n} \bar{a}_{n_1} \cdots \bar{a}_{n_k} z^n. \] (38)

Our claim follows from term-by-term comparison. \(\square\)

Lemma 3.5. If $w$ is given as a function of $z$ as a formal power series by the generating function identity (36), then this power series has a nonzero radius of convergence $|z| \leq \frac{1}{2s||J||_r} c^2$. For big enough $c$, this radius of convergence is arbitrarily large, and in particular, the series will converge for $z = 1$, i.e., the sum of the bounds on the contributions of block-connected and block-rooted hypergraphs converges.

Proof. Without loss of generality, assume $z \geq 0$. Set $z_1 = 2s||J||_r z$. Solving (36) for $z_1$ gives
\[ z_1 = \frac{w (1 - \epsilon (1 + w))}{\epsilon (1 + w)}. \] (39)

By elementary calculus, this increases as $w$ goes from 0 to $c$ to have values $z_1$ from 0 to $c^2$. It follows that as $z_1$ goes from 0 to $c^2$, the $w$ values range from 0 to $c$. \(\square\)

Proof of Proposition 3.2 continued. We notice that in the above lemma, $w = \sum_{n=1}^{\infty} \bar{a}_n z^n = c$ corresponds to $z_1 = 2s||J||_r z = c^2$, which implies that for each $n$,
\[ \bar{a}_n \leq c(2s||J||_r)^n c^{-2n}. \] (40)

Gathering all the information we have obtained so far,
\[ \sum_{y \in \mathcal{N}} |w_N| M_\mathcal{N} | \leq \sum_{n=1}^{\infty} c(2s||J||_r)^n c^{-2n} \]
\[ = c \frac{2s||J||_r}{1 - \frac{2s||J||_r}{c^2}} \leq \log(M) \] (41)
by (25).

We have shown in (9) that the denominator of (6) has a cluster representation. We now examine the effect of multiplying $\sigma_W$ to this cluster representation as in the numerator of (6). There will be two kinds of terms. In some of these, none of the block-connected components intersect $W$, so for these terms one gets a product of $\sigma_W$ with a product of independent $w_N$'s. For the other terms one decomposes $\Delta$ into one block-connected component that is connected to $W$ and remaining block-connected components that are not. The result is the representation $\sum_{R, \Delta'} \bar{w}_R \prod_{N \in \Delta'} w_N$, where $R = \emptyset$ or $R \cap W' \neq \emptyset$, and $\bar{w}_R$ is a sum over hypergraphs $\Delta_R$ with $\cup \Delta_R = R$ such that $W$, $\Delta_R$ is block-connected. Therefore
\[ \frac{\partial J'(Z)}{\partial J(W)} = \sum_{\sigma'} \sum_{R, \Delta'} \frac{\bar{w}_R \prod_{N \in \Delta'} w_N}{\sum_{\Delta} \prod_{N \in \Delta} w_N}. \] (42)

Theorem 3.6. Suppose the original interaction $J$ is at high temperature (cf. (25)). Then for every subset $W$ of the original lattice and every subset $Z$ of the image lattice, the partial derivative $\frac{\partial J'(Z)}{\partial J(W)}$ of the RG transformation (42) is well-defined.
Proof. The proof of this theorem is an application of the Kotecký-Preiss result [11]. Recall that $N \in \Delta'$ implies $N \cap (R \cup W') = \emptyset$. By (21),
\[
\left| \sum_{\Delta'} \prod_{N \in \Delta'} w_N / \sum_{\Delta} \prod_{N \in \Delta} w_N \right| \leq M^{[R \cup W']}.
\] (43)
To verify our claim, we need to estimate
\[
\left| \partial J'(Z) / \partial J(W) \right| \leq \sum_R |\bar{u}_R|M^{[R \cup W']} \leq \sum_{\Delta_R} M^{|W|} \prod_{Y \in \Delta_R} |w_Y|M^{|Y|}.
\] (44)
But this is easy, remove $W$, the remaining hypergraph breaks up into $k : 0 \leq k \leq |W|$ block-connected components. So this last quantity is bounded by
\[
M^{|W|} \sum_{k=0}^{|W|} \binom{|W|}{k} (\log(M))^k = M^{|W|} (1 + \log(M))^{|W|}.
\] (45)
□

4. Band structure

In this section, we concentrate our attention on finite-range and translation-invariant Hamiltonians. We will show that the matrix of partial derivatives in this case displays an approximate band property.

Remark. Let
\[
diam(X) = \sup \{d(x, y) : x \in X, y \in X\}
\] (46)
be the volume of a subset $X$ of the original lattice. For later purposes, we point out that the finite-range assumption on the Hamiltonian implies a weaker assumption, finite-body, i.e., there is a constant $S$ such that $J(X) = 0$ for $diam(X) > S$ implies there is a constant $D$ such that $J(X) = 0$ for $|X| > D$, where $D$ only depends on the maximum possible range $S$ and the number of dimensions $d$.

Proposition 4.1. Suppose the original interaction $J$ is at high temperature (cf. (25)). Then for each block spin site $y$, we have
\[
\sum_{y \in N : |N| > P} |w_N|M^{|N|} \leq \epsilon(P),
\] (47)
where
\[
\epsilon(P) = c \left( \frac{2s||J||_r}{c^d} \right)^P 1 - \frac{2s||J||_r}{c^d},
\] (48)
and for a fixed $||J||_r$ that satisfies (25), $\epsilon(P) \to 0$ as $P \to \infty$.

Proof. Due to the finite-body assumption on the Hamiltonian, any block-connected hypergraph that is block-rooted at $y$ and with cardinality greater than $P$ will have at least $P/D$ links. By (40), this implies
\[
\sum_{y \in N : |N| > P} |w_N|M^{|N|} \leq \sum_{n=P/D}^{\infty} c(2s||J||_r)^n c^{-2n} = \epsilon(P)
\] (49)
□
Proposition 4.2. Suppose the original interaction $J$ is at high temperature (cf. (25)). Then for every $Y \subseteq \Lambda'$, we have

$$\sum_{p=1}^{\infty} \frac{1}{p!} \sum_{N_1, \ldots, N_p \mid N_1 + \cdots + N_p > P} |C(N_1, \ldots, N_p) \cdot (c(Y, \cup_{1}^{P} N_i) w_{N_1} \cdots w_{N_p})| \leq |Y| \epsilon(P).$$

(50)

Proof. This follows from Proposition 4.1. Remove $Y$, the remaining hypergraph is still block-connected by (21). Moreover, there can be at most $|Y|$ choices for where it is pinned down. □

Theorem 4.3. Suppose the original interaction $J$ is at high temperature (cf. (25)). Suppose also that it is finite-range and translation-invariant. Then there is an approximate band property for the matrix of partial derivatives: For subset $W$ of the original lattice and subset $Z$ of the image lattice that are sufficiently far apart, the partial derivative $\frac{\partial J'(Z)}{\partial J(W)}$ of the RG transformation (42) is arbitrarily small.

Remark. Recall (18). Let

$$l(W, Z) = \inf \{ d'(w, z) : w \in W', z \in Z \}$$

(51)

be the distance between $W$ and $Z$ measured in the image lattice. If

$$l(W, Z) > S(|W| P + Q K),$$

then

$$|\frac{\partial J'(Z)}{\partial J(W)}| \leq M^D (1 + \log(M))^D \left( \frac{\epsilon(P)}{\log(M)} + (\epsilon(Q) + \epsilon(K)) (1 + P) D M^{(1 + P) D} \right).$$

(53)

Proof. Fix a $P$ that is large enough. We rewrite (42) as

$$\frac{\partial J'(Z)}{\partial J(W)} = \sum_{\sigma'} \frac{\sum_{|R| > |W| P} \prod_{N \in \Delta'} w_N}{\sum_{\Delta'} \prod_{N \in \Delta'} w_N} + \sum_{\sigma'} \frac{\sum_{|R| > |W| P} \prod_{N \in \Delta'} w_N}{\sum_{\Delta'} \prod_{N \in \Delta'} w_N}.$$ 

(54)

Following, we will verify the smallness of (54) by examining the two terms on the right-hand side separately.

Case 1: $|R| > |W| P$. Similarly as in the proof of Theorem 3.6 we estimate (44). Remove $W$, the remaining hypergraph (with cardinality greater than $|W| P$) breaks up into $k : 0 \leq k \leq |W| P$ block-connected components, so at least one of them has cardinality greater than $P$. By (18), the contribution of this hypergraph is bounded by

$$M^{|W|} \epsilon(P) \sum_{k=0}^{[W]} \binom{|W|}{k} (\log(M))^{k-1} = \frac{\epsilon(P)}{\log(M)} M^{|W|} (1 + \log(M))^{|W|}.$$ 

(55)

Case 2: $|R| \leq |W| P$. We need to do a more careful analysis for this case. By the Kotecký-Preiss theorem [11], [26] implies

$$\sum_{\Delta'} \prod_{N \in \Delta'} w_N / \sum_{\Delta} \prod_{N \in \Delta} w_N$$

(56)

$$= \exp \left( - \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{N_1, \ldots, N_p} C(N_1, \ldots, N_p) c(R \cup W', \cup_{1}^{P} N_i) w_{N_1} \cdots w_{N_p} \right).$$
For notational convenience, we will denote the right-hand side of (56) by \( F(\infty, \infty) \), where the first parameter of \( F \) indicates the cardinality restriction over the subsets \( N_i \)'s under consideration, whereas the second parameter of \( F \) indicates the maximum number of \( N_i \)'s allowed in the expansion. It is straightforward that for fixed \( Q \) and \( K \),

\[
F(\infty, \infty) = F(\infty, \infty) - F(Q, \infty) + F(Q, \infty) - F(Q, K) + F(Q, K).
\]

(57)

We first examine \( F(\infty, \infty) - F(Q, \infty) \). For every subset \( N \) of \( L' \), define

\[
u_N = \begin{cases} \omega_N & \text{if } |N| \leq Q; \\ 0 & \text{otherwise}. \end{cases}
\]

(58)

The difference in \( F \) can then be interpreted as induced by evaluating (56) using two sets of parameters \( \omega_N \) and \( \nu_N \). By (24), these two parameter sets both lie in the region of analyticity of (56), thus intuitively, the difference can be as small as desired when \( Q \) is large enough. In fact, it is bounded by \(|R \cup W'\)| \( M^{|R \cup W'|} \epsilon(Q) \) by the mean value theorem, applied to (24) and (50). Fix such a \( Q \). We next examine \( F(Q, \infty) - F(Q, K) \). This difference can be regarded as the tail of the convergent series (56), thus should also be small when \( K \) is large enough. We again refer to (24) and (50), and conclude that it is bounded by \(|R \cup W'\)| \( M^{|R \cup W'|} \epsilon(K) \). Fix such a \( K \). For these two situations, the only thing left to show now is that

\[
\sum_{|R|\leq|W'|} |\tilde{\omega}_R| \quad (59)
\]

is finite, but this naturally follows from (45).

Finally, we examine \( F(Q, K) \). By (10) and (11), \( w_N \) only depends on image sites in \( N \). As \( R \cup W' \) and \( \cup_i^p N_i \) is block-connected, \( F(Q, K) \) will only depend on image sites in a finite region (roughly a ball with radius \( S(|W|P + QK) \)). If \( Z \) is outside this region, then

\[
\sum_{|R|\leq|W'|} \tilde{\omega}_R F(Q, K) \quad (60)
\]

is a constant with respect to \( \sigma'_{Z} \), thus, when summing over all possible image configurations \( \sigma' \) as in (54), it vanishes.

**Proposition 4.4.** Suppose the original interaction \( J \) is at high temperature (cf. (25)). Suppose also that it is finite-range and translation-invariant. Then for subset \( W \) of the original lattice and subset \( Z \) of the image lattice, as the distance \( l(W, Z) \) between \( W \) and \( Z \) gets large, the partial derivative \( \frac{\partial J'_{(Z)}}{\partial J(W)} \) decays sub-exponentially, a little slower than \( \exp(-l(W, Z)^{1/2}) \).

**Proof.** For notational convenience, we denote \( l(W, Z) \) simply by \( l \). Take

\[
P = \frac{1}{|W|} \left( \frac{l}{2S} \right)^\alpha,
\]

(61)

and

\[
Q = K = \left( \frac{l}{2S} \right)^\beta,
\]

(62)

where \( 0 < \alpha < \beta \leq 1/2 \). We examine (53) closely. The first factor,

\[
M^D \left( 1 + \log(M) \right)^D,
\]

is just a constant. The second factor is more complicated and thus merits more attention. The first term, \( \epsilon(P)/\log(M) \), decays like \( \exp(-l^\alpha) \), whereas the second term, \( (\epsilon(Q) + \epsilon(K)) (1 + P)DM^{(1+P)D} \),
decays like \( \exp\left(-l^\beta + l^\alpha\right) \sim \exp(-l^\beta) \). Piecing it all together, \( \left| \frac{\partial J'(Z)}{\partial J(W)} \right| \) decays sub-exponentially, like \( \exp(-l^\alpha) \).

\[ \]

\section{Upper bound for the RG linearization}

\textbf{Definition 5.1.} For every subset \( Z \) of the image lattice, the linearization \( L(J) \) of the RG transformation for \( J \) at high temperature is given by a linear function of \( K \) (which indicates variation from infinite temperature),

\[
    L(J)K(Z) = \sum_{W} \left| \frac{\partial J'(Z)}{\partial J(W)} \right| K(W),
\]

where \( W \) ranges over all finite subsets of the original lattice.

\textbf{Proposition 5.2.} Consider finite-range and translation-invariant Hamiltonians. Fix a subset \( Z \) of the image lattice. Let \( n(E) \) be the number of subsets \( W \) of the original lattice that are at most \( E \)-distance away from \( Z \),

\[
    n(E) = \# \{ W : l(W, Z) \leq E \}.
\]

Then \( n(E) \) grows polynomially in \( E \).

\textit{Proof.} Due to our finite-range and translation-invariant assumptions on the Hamiltonian,

\[
    n = \sup_{y \in L'} \# \{ W : y \in W' \} < \infty.
\]

Thus \( n(E) \) grows at the same rate as the volume of a \( d \)-dimensional ball with radius \( E \), i.e., polynomial growth \( E^d \). \( \square \)

\textbf{Theorem 5.3.} Suppose the original interaction \( J \) is at high temperature (cf. (23)). Suppose also that it is finite-range and translation-invariant. Then the linearization \( L(J) \) of the RG transformation (63) is well-defined and has an upper bound.

\textit{Proof.} This is mainly due to the fact that sub-exponential decay dominates polynomial growth. Take \( K \) with \( ||K||_r \) small. As \( ||K||_\infty \leq ||K||_r \), \( ||K||_\infty \) is small also. By Propositions 4.4 and 5.2

\[
    |L(J)K(Z)| \leq \sum_{n=0}^{\infty} \sum_{n \leq l(W, Z) < n+1} \left| \frac{\partial J'(Z)}{\partial J(W)} \right| |K(W)|
\]

\[ \]

\[
    \lesssim ||K||_\infty \sum_{n=0}^{\infty} \exp(-n^\alpha)(n + 1)^d.
\]

Our claim then follows from the integral test. \( \square \)

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A CLUSTER EXPANSION APPROACH TO RENORMALIZATION GROUP TRANSFORMATIONS

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