A Transport Equation for Quantum Fields with Continuous Mass Spectrum *

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Abstract

Within a relativistic real-time Green’s function formalism, a quantum transport equation for the phase-space distribution function is derived without a quasi-particle approximation. Dissipation is due to a nonzero spectral width, and can be separated into time-local and memory effects.

Dedicated to Prof. Hiroomi Umezawa on occasion of his 70th birthday.

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1 Introduction

Currently one of the main interests of nuclear physics is the investigation of hot, compressed nuclear matter. One of the most widely used theoretical tools for this investigation is the simulation of nuclear collisions by numerical solution of transport equations, like e.g. the Vlasov-Uehling-Uhlenbeck (VUU) equation [1]. This equation and its many relatives have their origin in the kinetic theory of molecular gases. Today however they are obtained from relativistic quantum field theory, i.e., by processing the Schwinger-Dyson equation for propagators of the interacting quantum fields with certain semi-classical approximations [2, 3, 4].

It is in principle possible to compare the solution of such transport equations to the solution of the full Schwinger-Dyson equation, but to common knowledge such an attempt has not been made but once [2]. The result was a significant difference in the time evolution, the full quantum system relaxing somewhat slower than the "molecular" transport solution. It has been made plausible that such an effect is due to "off-shell" propagation of particles [5, 6], which gives rise to memory effects in the hot and dense medium [7, 8]. A derivation of such memory effects within the framework of quantum field theory has not been presented so far.

It is the purpose of the present paper to bridge this gap between the full Schwinger-Dyson equation and the traditional "molecular" kinetic picture.

To this end, one has to go beyond the semi-classical (naive) particle picture when processing the integral equation for the propagator. This naive particle picture, traditionally labeled "quasi-particle" approximation, assumes the existence of infinitely long living elementary excitations even in statistical systems.

Experimentally, there are indications that this is not very well justified: Spectral functions of hadrons in the medium presumably determine the spectrum of emitted lepton pairs [9], and "momentum dependent forces" are thought to be important for the so-called equation of state in hadronic matter. In relativistic systems such a momentum dependence is equivalent to a nonzero imaginary part of the self energy, and hence again to a nontrivial hadronic spectral function. Hence one may adopt the viewpoint, that the extraction of quantitative information from nuclear collisions requires to go beyond the quasi-particle level.

Also theoretically the naive particle picture is not a good startpoint: Consider the breaking of Lorentz invariance in matter or temperature states [10]. It leads to the Narnhofer-Thirring theorem stating the impossibility of a perturbation theory with
quasi-particles at finite temperature \cite{11}. In other words, the entities to use for a perturbative description in statistical systems must have a continuous mass spectrum – they cannot be quasi-particles. How one may formulate a finite temperature perturbation theory for quantum fields with continuous mass spectrum has been shown not too long ago \cite{12}. Recently we have extended this picture to time dependent \cite{14, 15} as well as spatially inhomogeneous \cite{18} non-equilibrium situations.

Here, these recent developments will be extended to a general non-equilibrium system. The basic assumption for this work is, that nuclear phenomena can be described successfully using methods from relativistic quantum field theory. First, to ensure compatibility with existing literature, the Schwinger-Dyson equation for the full propagator of an interacting fermionic quantum field is processed according to standard procedures. Next, the equation for the spectral function is solved consistently. The final step of the present paper is a study of the equation resulting from insertion of the spectral function into the transport equation.

2 Matrix valued Schwinger-Dyson equation

As has been pointed out by various authors, the description of dynamical (time dependent) quantum phenomena in a statistical ensemble necessitates a formalism with a doubled Hilbert space \cite{2, 3}. For our purpose the relevant content of this formalism is, that its two-point Green functions are $2 \times 2$ matrix-valued. We leave it to the reader to choose either the conventional Schwinger-Keldysh, or Closed-Time Path (CTP) Green function formalism \cite{4, 13}, or the technically simpler method of Thermo Field Dynamics (TFD) \cite{15, 16}.

Although this choice does not affect the following considerations, it has to be stated that the principles of TFD are in accordance with a fundamental requirement: The existence of two mutually commuting representations of the field operator algebra. This requirement is not fulfilled by the more conventional CTP (or Schwinger-Keldysh) formalism, i.e., CTP may lead to mathematical pitfalls. The recently discussed factorization problem seems to point in this direction \cite{19}.

Within this matrix formulation, we consider the Schwinger-Dyson equation for the full fermion propagator of a given model

\[ S = S_0 + S_0 \odot \Sigma \odot S . \]  

(1)
Here $S_0$ is the free and $S$ the full two-point Green function of the fermion field, $\Sigma$ is the full self energy and the generalized product of these is to be understood as a matrix product (thermal and spinor indices) and an integration (each of the matrices is a function of two space coordinates):

$$
\Sigma_{xy}^{ij} \odot G_{yz}^{jk} = \sum_j \int d^4 y \Sigma_{xy}^{ij} G_{yz}^{jk} .
$$

Throughout this paper we use the convention to write space-time and momentum variables as lower indices, e.g. $\Sigma_{xy} \equiv \Sigma(x, y)$.

In the CTP formulation as well as in the $\alpha = 1$ parameterization of TFD [15], the matrix elements of $S$ obey

$$
S^{11} + S^{22} = S^{12} + S^{21},
$$

and a similar relation holds for the free propagator $S_0$. It follows that a linear relation exists also among the components of the self energy:

$$
\Sigma^{11} + \Sigma^{22} = -\Sigma^{12} - \Sigma^{21} .
$$

Thus, the four components of the Schwinger-Dyson equation are not independent, the matrix equation can be simplified by a linear transformation. This linear transformation, which one may conveniently express as a matrix transformation [3, 15], has a physical interpretation only in the TFD formalism: It is the Bogoliubov transformation defining stable statistical quasi-particle states [16, 17], not to be confused with the traditional quasi-particle approximation.

In view of this fact, the transformation matrices $B$ will be written in the form

$$
B(n) = \begin{pmatrix}
(1 - n) & -n \\
1 & 1
\end{pmatrix} ,
$$

depending on one parameter only. In appendix A some useful properties of such matrices are listed. Due to the linear relations (3) and (4) any pair of Bogoliubov matrices $B(n_1), B(n_2)$ gives

$$
B(n_1) \tau_3 S_0 \odot \Sigma \odot S (B(n_2))^{-1} = \begin{pmatrix}
S_0^R \odot \Sigma^R \odot S^R \\
S_0^A \odot \Sigma^A \odot S^A
\end{pmatrix} .
$$

Here, $\tau_3 = \text{diag}(1, -1)$, $\Sigma^{R,A}$ are the retarded and advanced full self energy function, and $S^{R,A}$ are the retarded and advanced full propagator (similarly for $S_0$)

$$
\Sigma^R = \Sigma^{11} + \Sigma^{12} , \quad \Sigma^A = \Sigma^{11} + \Sigma^{21}
$$

$$
S^R = S^{11} - S^{12} , \quad S^A = S^{11} - S^{21} .
$$
Together with eqn. (63) follows, that the diagonal elements of the transformed equation therefore are retarded and advanced Schwinger-Dyson equation. The off-diagonal element is a transport equation.

We proceed by applying Dirac differential operators to the diagonal parts of the simplified equation. They are acting on the free Green function $S_0$ of a fermion field with mass $M$ according to

$$\hat{S}_0^{-1} = i\hat{\theta}_x - M \implies \hat{S}_0^{-1} S_{0,xy}^{A,R} = \delta_{xy},$$

where $\hat{\theta}_x = \partial_x^\mu \gamma_\mu$ and $\delta_{xy} = \delta^4(x-y)$. The action of these differential operators on the off-diagonal elements $S_0^{12}$ and $S_0^{21}$ of the free propagator gives zero. For the retarded and advanced Schwinger-Dyson equation this yields

$$\hat{S}_0^{-1} S_{xy}^{R,A} = \delta_{xy} + \sum_{xz}^{R,A} \circ S_{zy}^{R,A}.$$  \hspace{1cm} (9)

To get rid of the dependence on coordinate differences present already in space-time homogeneous systems, one performs a Fourier transform of these equations into the mixed (or Wigner) representation, according to

$$\tilde{\Sigma}_{XP} = \int d^4(x-y) \exp(iP_\mu(x-y)^\mu) \Sigma_{xy}. \hspace{1cm} (10)$$

Here we have set $X = (x+y)/2$, and the "\sim" will be dropped from now on. The differential operators (inverse free propagators) occurring in the equations (9) then take the form

$$\hat{S}_0^{-1} = \hat{p} - M + \frac{i}{2} \hat{\theta}_X \implies \hat{S}_0^{-1} S_{0,XP}^{A,R} = 1.$$ \hspace{1cm} (11)

The Wigner transform of the convolution $\Sigma \circ G$ as defined in (2) is a nontrivial step: Formally it may be expressed as a gradient expansion

$$\int d^4(x-y) \exp(iP_\mu(x-y)^\mu) \Sigma_{xz} \circ G_{zy} = \exp(-i\diamond) \tilde{\Sigma}_{XP} \tilde{G}_{XP}. \hspace{1cm} (12)$$

$\diamond$ is a 2nd order differential operator acting on both functions appearing behind it,

$$\diamond = \frac{1}{2} \left( \partial_X^\Sigma \partial_P G - \partial_P^\Sigma \partial_X G \right).$$ \hspace{1cm} (13)

We have indexed the derivatives to avoid confusion on which factor they act. The index $\Sigma$ stands for real or imaginary part of the self energy function, and the index $G$ stands for action on real and imaginary part of the propagator. Explicitly, the first-order term is

$$\diamond \Sigma G = \frac{1}{2} \left( \frac{\partial \Sigma}{\partial X_\mu} \frac{\partial G}{\partial P^\mu} - \frac{\partial \Sigma}{\partial P_\mu} \frac{\partial G}{\partial X^\mu} \right). \hspace{1cm} (14)$$
Obviously, this is the Poisson bracket of the two quantities $G$ and $\Sigma$. The exponential of $\diamond$ is a differential operator of infinite order, which formally can be split into real and imaginary part as $\exp(-i\diamond) = \cos \diamond - i\sin \diamond$.

It is not completely clear how to justify the gradient expansion for arbitrary self energy functions, to infinite or to finite order. One may however adopt the following viewpoint on the gradient expansion: It is the expansion of the expectation value of a composite operator into a series of terms with increasing non-locality. Such an expansion is always associated with a length scale, and therefore we have to assume that the non-equilibrium system is studied only on length and time scales for which the gradient expansion is rapidly convergent.

Since each operator $\diamond$ needs a factor $\hbar$ to render it dimensionless, the gradient expansion is also an expansion in powers of $\hbar$. Truncating it therefore means, that quantum effects occurring on very short length or time scales must be absorbed into the propagator and the self energy function on larger scales. Indeed, to use a fashionable expression for this viewpoint: When taking the gradient expansion seriously, we are considering an effective field theory. This requires that the fields we are considering have a continuous mass spectrum [21].

The propagator and the self energy function in mixed representation are split into real and imaginary part according to

$$\begin{align*}
S_{XP}^{R,A} & = G_{XP} \mp i\pi A_{XP} \\
\Sigma_{XP}^{R,A} & = \Sigma_X^\delta + \text{Re}\Sigma_{XP} \mp i\pi \Gamma_{XP}.
\end{align*}$$

(15)

Each of these functions is assumed to be real, and they are spinor-valued. $\Sigma^\delta$ is the part of the self energy function which is local in space and time, which for a relativistic model implies that it is the Hartree part of $\Sigma$.

It is tempting to call $A$ the spectral function of the fermions. However, it cannot be guaranteed that the full fermion propagator in non-equilibrium states has a spectral decomposition [11]. Therefore, this interpretation has to be treated with caution for the moment.

The gradient expansion and the decomposition of the propagator and self energy function are inserted into eqns. (9), leading to

$$\begin{align*}
\left(\not{p} - M + \frac{i}{2} \not{\partial}_X - \exp(-i\diamond) \Sigma_X^\delta\right) A_{XP} & = \exp(-i\diamond) \left[\text{Re}\Sigma_{XP} A_{XP} + \Gamma_{XP} G_{XP}\right] \\
\left(\not{p} - M + \frac{i}{2} \not{\partial}_X - \exp(-i\diamond) \Sigma_X^\delta\right) G_{XP} & = 1 +
\end{align*}$$
The above equations contain a mixture of real and imaginary part, which can be separated easily. However, they are not independent in another sense – which can be seen when performing the separation and taking the trace over the Dirac indices. This gives two pairs of equations:

\[ \text{Tr} \left[ (\partial_X + 2 \sin \diamond \Sigma^X) A_{XP} \right] = -2 \sin \diamond \text{Tr} \left[ \text{Re} \Sigma_{XP} A_{XP} + \Gamma_{XP} G_{XP} \right] \]

\[ \text{Tr} \left[ (\partial_X + 2 \sin \diamond \Sigma^X) G_{XP} \right] = -2 \sin \diamond \text{Tr} \left[ \text{Re} \Sigma_{XP} G_{XP} - \pi^2 \Gamma_{XP} A_{XP} \right] \]

and

\[ \text{Tr} \left[ (\bar{\psi} - M - \cos \diamond \Sigma^X) A_{XP} \right] = \cos \diamond \text{Tr} \left[ \text{Re} \Sigma_{XP} A_{XP} + \Gamma_{XP} G_{XP} \right] \]

\[ \text{Tr} \left[ (\bar{\psi} - M - \cos \diamond \Sigma^X) G_{XP} \right] = \text{Tr} \left[ 1 \right] + \cos \diamond \text{Tr} \left[ \text{Re} \Sigma_{XP} G_{XP} - \pi^2 \Gamma_{XP} A_{XP} \right] \]

The first pair of these equations is obtained from the second pair by application of the differential operator \(2 \tan \diamond\). This can be traced back to the fact, that the inverse free propagator in eqn. (11) is a mixture of zeroth and first order in \(\diamond\), formally

\[ \left( \bar{\psi} - M + \frac{i}{2} \partial_X \right) G_{XP} = \left( 1 - i \tan \diamond \right) \left( \bar{\psi} - M \right) G_{XP} . \]

When the gradient expansion is truncated without considering this fact, this mixture may be the source of spurious terms in transport equations [20].

Obviously, eqn. (18) contains only even powers of the differential operator \(\diamond\). Solving it in zeroth order \(\diamond\) therefore gives a spectral function that is correct up to first order in \(\diamond\).

### 3 Determination of the spectral function

We now solve the diagonal components of the Schwinger-Dyson equation, by making an explicit ansatz for the Dirac spinor structure of the self energy function:

\[ \Sigma^X + \text{Re} \Sigma_{XP} = S(X, P) + V^\mu(X, P) \gamma_\mu + T^{\mu\nu}(X, P) \sigma_{\mu\nu} , \]

where \(S(X, P), V(X, P)\) and \(T(X, P)\) stand for (Lorentz) scalar, vector and tensor component of the real part of the self energy, \(\sigma_{\mu\nu} = i/2 [\gamma_\mu, \gamma_\nu]\) and \(T\) is antisymmetric. To avoid confusion with the symbol used for the fermion propagator, we
exploit that the scalar and vector piece of the real self energy function only appear in combination with $M$ and $P$. Hence in the following we make use of the abbreviations

$$M^* = M + S(X, P) \quad P^*_\nu = P_\nu - V_\nu(X, P)$$

(21)

for effective mass and momentum. Note, that in contrast to other derivations of transport equations, the above functions depend on both $X$ and $P$.

Usual derivations of transport equations assume that the self energy contains only Lorentz vector and scalar parts \[22, 24\]. It is however easy to see that for reasonable descriptions of nuclear dynamics a tensor part of the self energy should be taken into account. To this end we consider a system, where fermions are minimally coupled to bosons. In such a model, the Born diagrams for fermion-fermion scattering are obtained by cutting the diagrams in figure 1 along the dotted line. If we assume, that the Born terms are dominant contributions to the collision term in “molecular” transport equations, we thus have to take the diagrams of figure 1 into account in our self energy function.

The contribution of the left-hand diagram in the figure to the fermion self energy then contains products of up to three $\gamma$-matrices. The threefold product is equivalent to a pseudovector, which can be excluded in a parity eigenstate. The twofold product can be converted to a sum of unit matrix and tensor terms. Thus, if at least two independent Lorentz vectors can be specified for a momentum eigenstate, such
diagrams contribute tensor parts to the self energy functions also for homogeneous systems.

Clearly this is the case for a system at finite density, where the Lorentz invariance is broken: Independent Lorentz vectors are the (macroscopic) four-current $j^\mu$ of matter and the momentum of the single-particle state $P_\mu$ (or equivalently the effective momentum $P_\mu^*$). The tensor part of the self energy is then proportional to the commutator $[P_\nu^* \gamma^\nu, j_\mu \gamma^\mu]$. If one wants to consider phenomena like spin diffusion\[20\], it must not be disregarded.

By virtue of dispersion relations, a similar decomposition also holds for the imaginary part of the self energy,

$$\Gamma_{XP} = \Gamma_s + \Gamma_\nu^\mu \gamma_\mu + \Gamma_t^{\mu\nu} \sigma_{\mu\nu},$$

It follows from the Dyson equations in the previous section, that the same decomposition then holds for $G$ and the spectral function:

$$G_{XP} = G_s + G_\nu^\mu \gamma_\mu + G_t^{\mu\nu} \sigma_{\mu\nu},$$
$$A_{XP} = A_s + A_\nu^\mu \gamma_\mu + A_t^{\mu\nu} \sigma_{\mu\nu}.$$ \hspace{1cm} (23)

The ”off-shellness” of the four momentum $P_\nu$, and correspondingly the generalized width of the spectral distribution are given by

$$\mathcal{Y} = P_\nu^* P_{\nu'} - M^*^2 - 2T_{\mu\nu} T^{\mu\nu},$$
$$\mathcal{X} = \mathcal{Y} + \pi^2 \left( \Gamma_s^2 - \Gamma_{v,\mu}^\nu \Gamma_\nu^\mu + 2\Gamma_{t,\mu\nu} \Gamma_t^{\mu\nu} \right),$$
$$\mathcal{W} = 2 \left( P_\nu^* \Gamma_t^{\nu\mu} + M^* \Gamma_s^\mu + 2T_{\mu\nu} \Gamma_t^{\mu\nu} \right),$$ \hspace{1cm} (24)

and the propagator denominator is

$$\mathcal{D} = \mathcal{X}^2 + \pi^2 \mathcal{W}^2.$$ \hspace{1cm} (25)

As was stated after eqn.\[18\], we may take the solution for $A$ and $G$ as obtained in homogeneous systems\[22, 23\]: It is still correct in first order of the gradient expansion, when the local functions for effective mass and momentum are inserted.

For reasonably smooth systems, or rather on sufficiently large length scales, the equation for the spectral function therefore is always an algebraic equation.

Explicitly, the solutions are\[23\]

$$G_{XP} = \left[ \left( \bar{\psi}^* + M^* + T^{\mu\nu} \sigma_{\mu\nu} \right) \mathcal{X} - \pi^2 \left( \Gamma_s - \Gamma_\nu^\mu \gamma_\mu + \Gamma_t^{\mu\nu} \sigma_{\mu\nu} \right) \mathcal{W} \right] \mathcal{D}^{-1}$$ \hspace{1cm} (26)
$$A_{XP} = \left[ \left( \bar{\psi}^* + M^* + T^{\mu\nu} \sigma_{\mu\nu} \right) \mathcal{W} + \left( \Gamma_s - \Gamma_\nu^\mu \gamma_\mu + \Gamma_t^{\mu\nu} \sigma_{\mu\nu} \right) \mathcal{X} \right] \mathcal{D}^{-1}.$$ \hspace{1cm} (27)
Figure 2: Ghost poles in the perturbative propagator (schematically)

○ are the ghost poles of eqn. (27) in a vacuum state, ● at nonzero density. ⊘ are the particle poles present in certain approximations, like e.g. the one discussed in ref. [28].

These solutions may be obtained in several ways, a straightforward method is summarized in appendix B. The equations can be combined as

\[ G_{XP} \mp i\pi A_{XP} = \left( \not{P} - M - \text{Re}\Sigma_{XP} \pm i\pi \Gamma_{XP} \right)^{-1} \]

\[ \Leftrightarrow \quad S^{R,A}_{XP} = \left( \not{P} - M - \Sigma^{R,A}_{XP} \right)^{-1}. \]  

(28)

The inverse on the r.h.s. is to be understood as the proper inverse of a complex 4×4-matrix. Hence \( G \) and \( \pi A \) are real and imaginary part of the same complex function. While this may be used to determine the functional form of \( A \) in homogeneous states, the matrix inversion does not lead to our finding, that \( A \) in this form is correct also to first order in the gradients. The solution of the differential equations (16) to this order therefore is a necessary step of the derivation (see appendix B).

However, one may not use the above solutions directly. As is discussed in detail in ref. [28], a perturbative calculation of the self energy function \( \Sigma \) leads to the invalidation of the equation for \( G \). The reason is, that according to Weinberg’s theorem [29] the perturbative self energy has a certain asymptotic behavior, which leads to the appearance of complex poles in the propagator (28), see figure 2.
These Landau ghost poles are unphysical, because they appear in the wrong complex energy half plane for retarded and advanced propagator – and hence have to be removed. In ref. [28], three strategies were discussed for this removal: Direct subtraction of the poles, non-perturbative calculation of the self energy function or the definition of the full propagator as a dispersion integral

$$G_{XP} \mp i\pi \mathcal{A}_{XP} = \int_{-\infty}^{\infty} dE \ \mathcal{A}(X;E,\vec{p}) \frac{1}{p_0 - E \pm i\epsilon},$$

(29)

with $P = (p_0, \vec{p})$. It was also shown, that the three strategies are equivalent, and that furthermore the non-perturbative calculation of the self energy can be approximated by a form factor giving a better asymptotic behavior than the one implied by Weinberg’s theorem. For the purpose of the present work, eqn. (29) is considered the definition of $G$, in addition to (26). In coordinate space, the above dispersion relation implies

$$S_{xy}^{R,A} = \mp 2\pi i\Theta (\pm (x_0 - y_0)) \mathcal{A}_{xy}.$$  

(30)

For simplicity, consider the spectral function in the case of zero tensor part of the self energy. Clearly, this spectral function is not generally of the form

$$\mathcal{A}_{XP} = \left( \not{p}^* + M^* \right) \frac{\mathcal{W}}{(P^* \cdot M^*)^2 + \pi^2 \mathcal{W}^2},$$

(31)

with $\mathcal{W}$ linear in $\Gamma$. This equation for the fermionic spectral function is only approximately correct in case $\Gamma \ll \text{Re}\Sigma$. Such a functional form (which was obviously guessed from the non-relativistic result for the spectral function) has been proposed elsewhere also for arbitrary self energies [24], but it is not reproduced by the complete derivation carried out here.

The form (28) of the retarded and advanced propagator implies, how the limit of zero $\Gamma$ has to be taken, i.e. as

$$\lim_{\Gamma \rightarrow 0} \Sigma_{XP}^{R,A} = \mp i\epsilon \gamma^0,$$

(32)

to recover the vacuum boundary conditions for the Green functions [23].

This can be taken into account by adding an infinitely small part to the function $\Gamma$. Performing this modification in the above equations then gives for the case of zero imaginary part of the self energy:

$$G_{XP} = \not{p}^* + M^* + T^{\mu\nu}\sigma_{\mu\nu} \frac{\chi}{\mathcal{X}},$$

$$\mathcal{A}_{XP} = \left( \not{p}^* + M^* + T^{\mu\nu}\sigma_{\mu\nu} \right) \text{sign}(P_0^*) \delta (\mathcal{X}).$$

(33)
In the limit of zero tensor self energy, this turns into

\[ \mathcal{A}_{XP} = \left( \slashed{P}^* + M^* \right) \text{sign}(P_0^*) \delta \left( P^{*2} - M^{*2} \right) . \]  

(34)

Obviously the “particle”-like pole of the Green function is renormalized by the derivative of the self energy.

4 Transport equation

As was stated above, the off-diagonal component of the Schwinger-Dyson equation (in triangular form) is a transport equation. To study this more closely, we specify the Bogoliubov parameter of the transformation (3) as

\[ n_1 = n_2 = n_0 = \text{const.} \]  

(35)

Using eqn. (33) and (34), this off-diagonal component then has the form

\[ S^K = S_0^K + S_0^K \odot S^A \odot S^K + S_0^R \odot \Sigma^R \odot S^K - S_0^R \odot \Sigma^K \odot S^K , \]  

(36)

with retarded and advanced components as above and kinetic components

\[ S_0^K = (1 - n_0) S_0^{12} + n_0 S_0^{21} \]
\[ S^K = (1 - n_0) S^{12} + n_0 S^{21} \]
\[ \Sigma^K = (1 - n_0) \Sigma^{12} + n_0 \Sigma^{21} . \]  

(37)

As in the preceding section, Dirac differential operators are applied to the off-diagonal Schwinger-Dyson equation component. They act on the retarded and advanced propagator as specified in eqn. (8), and annihilate the kinetic component \( S_0^K \). The kinetic component of the matrix valued Schwinger-Dyson equation therefore is

\[ \widehat{S}_0^{-1} S_{xy} = \Sigma_{xz} \odot S_{yz} - \Sigma_{xz} \odot S_{zy} . \]  

(38)

Together with its adjoint, this equation can be expressed entirely in terms of the off-diagonal components of self energy function and propagator. These equations then are known as the Kadanoff-Baym equations, see ref. [30].

The next step is to transform to the mixed (Wigner) representation, and to insert the gradient expansion,

\[ \left( \slashed{P} - M + \frac{i}{2} \slashed{\partial}_X - \exp(-i\phi) \Sigma^X \right) S^K_{XP} \]
\[ = \exp(-i\phi) \left[ (\text{Re}\Sigma_{XP} - i\pi \Gamma_{XP}) S^K_{XP} + \Sigma^K_{XP} \left( G_{XP} + i\pi \mathcal{A}_{XP} \right) \right] \]  

(39)
The separation of real and imaginary part is a trivial step. However, for comparison with existing derivations we perform it explicitly and obtain the two equations

\[
\text{Tr} \left[ \left( \hat{j}_X + 2 \sin \diamond \left( \Sigma^\delta_X + \text{Re} \Sigma_{XP} \right) + \cos \diamond 2\pi \Gamma_{XP} \right) S^K_{XP} \right] = 2i \text{Tr} \left[ i \sin \diamond \Sigma^K_{XP} G_{XP} - \cos \diamond \Sigma^K_{XP} i\pi A_{XP} \right] \tag{40}
\]

\[
\text{Tr} \left[ \left( \hat{p} - M - \cos \diamond \left( \Sigma^\delta_X + \text{Re} \Sigma_{XP} \right) + \sin \diamond \pi \Gamma_{XP} \right) S^K_{XP} \right] = \text{Tr} \left[ \cos \diamond \Sigma^K_{XP} G_{XP} - i \sin \diamond \Sigma^K_{XP} i\pi A_{XP} \right] \tag{41}
\]

It is obvious, that these equations are not independent: As was already observed for the retarded and advanced equation, (40) follows from (11) by the application of a differential operator. Usually only eqn. (40) is processed further, leading to certain disadvantages that we will avoid below.

The final result of the processing will be an equation for the quantity \(N_{XP}\) defined by

\[
(1 - N_{XP}) S^{12}_{XP} + N_{XP} S^{21}_{XP} = 0 \tag{42}
\]

For the purpose of the present paper \(N_{XP}\) is taken as a scalar function, it has the physical meaning of a generalized phase-space distribution function [31]. The description of Phenomena like spin diffusion require to use a Dirac matrix valued \(N_{XP}\).

It follows according to eqns. (30) and (37), that

\[
S^K_{XP} = 2\pi i \left( N_{XP} - n_0 \right) A_{XP} \tag{43}
\]

Equation (42) has two aspects which are important for the understanding of transport phenomena. First, consider it in the equilibrium case: No dependence on \(X\) is present. The equation is then equivalent to the Kubo-Martin-Schwinger boundary condition for the propagator [32] – which implies the proper asymptotic property for the quantity \(N_{XP}\) we are searching.

The second aspect of the above equation only makes sense in the context of thermo field dynamics (TFD). There, it is the condition for a \textit{diagonalization} of the full propagator by (space-time dependent) Bogoliubov transformations, i.e., according to eqn. [33]

\[
B(\hat{N}_{XP}) \tau_3 S_{XP} \left( B(\hat{N}_{XP}) \right)^{-1} = \begin{pmatrix}
G_{XP} - i\pi A_{XP} \\
G_{XP} + i\pi A_{XP}
\end{pmatrix} \tag{44}
\]
Hence, the full propagator (in mixed representation) is diagonalized by a space-time dependent Bogoliubov transformation. One may insert this propagator into the Schwinger-Dyson equation – the action of the Dirac differential operator on the function $N_{XP}$ would have to be considered then. According to eqn. (54), this would only affect the off-diagonal component. The strategy of diagonalizing the full non-equilibrium propagator is therefore equivalent to the processing of the Schwinger-Dyson equation as done here.

For the following, we furthermore define a “pseudo-equilibrium” distribution function: The property (4) of the self energy function allows to diagonalize its $2 \times 2$ matrix structure by a Bogoliubov transformation (see eqn. (62)) with a parameter $N_{0XP}$ such that

$$\Sigma_{12}^{12} = 2\pi i N_{0XP}^0 \Gamma_{XP} \quad \Sigma_{21}^{21} = 2\pi i \left( N_{0XP}^0 - 1 \right) \Gamma_{XP} . \quad (45)$$

It is easy to show, that in equilibrium states $N^0$ is identical to $N$ as defined in (42) [15]. For non-equilibrium states however, $N^0$ may vary with time and be quite different from $N_{XP}$. Similar to eqn. (43) we obtain

$$\Sigma_{XP}^K = 2\pi i \left( N_{0XP}^0 - n_0 \right) \Gamma_{XP} . \quad (46)$$

We now return to the transport equation (39), perform a gradient expansion to first order in $\diamond$ and insert the above definitions:

$$\text{Tr} \left[ (1 - i \diamond) \left\{ \hat{P} - M - (\Sigma^\delta_X + \text{Re} \Sigma_{XP}) + i\pi \Gamma_{XP} \right\} \left\{ (N_{XP} - n_0) A_{XP} \right\} \right] =$$

$$\text{Tr} \left[ (1 - i \diamond) \left\{ (N_{0XP}^0 - n_0) \Gamma_{XP} \right\} \left\{ G_{XP} + i\pi A_{XP} \right\} \right] . \quad (47)$$

The curly brackets contain the quantities acted upon by the $\diamond$ operator according to (14).

The derivative of $A$ on the left side is eliminated using (17), and the non-derivative terms on the left side are switched over to the right side using (18). All terms involving $n_0$ drop out correctly, and the resulting full quantum transport equation is

$$\text{Tr} \left[ A_{XP} \left( -i \diamond \right) \left\{ \hat{P} - M - (\Sigma^\delta_X + \text{Re} \Sigma_{XP}) + i\pi \Gamma_{XP} \right\} \left\{ N_{XP} \right\} \right] =$$

$$\text{Tr} \left[ -N_{XP} (1 - i \diamond) \left\{ \Gamma_{XP} \right\} \left\{ G_{XP} + i\pi A_{XP} \right\} \right] + (1 - i \diamond) \left\{ N_{0XP}^0 \Gamma_{XP} \right\} \left\{ G_{XP} + i\pi A_{XP} \right\} . \quad (48)$$
5 Vlasov equation

As the next step, we rederive the "standard" Vlasov equation without collision term. To this end, the right side of eqn. (48) is dropped completely. Furthermore, the traditional quasi-particle approximation is made: The spectral function is chosen according to (34). Note, that according to eqn. (29) this is equivalent to an energy-independent real self energy function in Wigner representation. In coordinate space, such a picture is completely consistent only in an approximation, where all self energy functions are local in space and time. In a relativistic model this is the case only for the Hartree approximation, nonrelativistically one may also add the exchange (Fock) self energy [2].

In leaving the covariant notation, we set \( P = (p_0, \vec{p}) \) and \( X = (t, \vec{x}) \). Furthermore we assume, that the spectral \( \delta \)-function (34) has at least one pole along the real \( p_0 \)-axis, i.e. that one can rewrite it as

\[
\delta \left( P^*^2 - M^*^2 \right) = \sum_i \delta \left( p_0 - E_i(\vec{x}, \vec{P}, t) \right) \left| \frac{\partial (P^*^2 - M^*^2)}{\partial p_0} \right|^{-1},
\]

where \( E_i(\vec{x}, \vec{P}, t) \) is the (space-time and momentum dependent) generalized energy of the corresponding “particle”-like state.

Abbreviating the function \( N_{XP} \) at each of the poles as \( N_i(\vec{x}, \vec{P}, t) \), an integration over the energy variable then eliminates the \( \delta \)-function as well as the energy derivative, and one obtains from eqn. (48)

\[
\sum_i \left( \frac{\partial N_i(\vec{x}, \vec{P}, t)}{\partial t} + \frac{\partial E_i(\vec{x}, \vec{P}, t)}{\partial \vec{P}} \frac{\partial N_i(\vec{x}, \vec{P}, t)}{\partial \vec{x}} - \frac{\partial E_i(\vec{x}, \vec{P}, t)}{\partial \vec{x}} \frac{\partial N_i(\vec{x}, \vec{P}, t)}{\partial \vec{P}} \right) = 0.
\]

This is the Vlasov (or mean-field) quantum transport equation in standard notation: The system evolves in time free of dissipation.

It is quite instructive to rewrite this in a covariant notation. Using the spectral function defined in eqn. (34) and \( Y \) as defined in (24), one can transform equation (48) in a few lines into

\[
\delta(Y) \left\{ \frac{\partial Y}{\partial P^\mu} \frac{\partial N_{XP}}{\partial X^\mu} - \frac{\partial Y}{\partial X^\mu} \frac{\partial N_{XP}}{\partial P^\mu} \right\} = 0.
\]

This may be integrated over the variable \( Y \), hence the covariant Vlasov equation amounts to the vanishing of the expression in curly brackets on the hypersurface.
\( \mathcal{Y} = 0 \) \cite[pp.438]{1}. The characteristic curves of this partial differential equation are parameterized by \( s \),

\[
\frac{dX^\mu}{ds} = \frac{\partial Y}{\partial P_\mu}, \quad \frac{dP^\mu}{ds} = -\frac{\partial Y}{\partial X_\mu}.
\]

(52)

Hence eqn. (51) states, that the total derivative of \( N_{XP} \) with respect to \( s \) vanishes on the hypersurface defined by \( \mathcal{Y} = 0 \). The characteristic equations, together with the constraint, are Hamilton’s equations for an infinitesimal phase-space “fluid element” with proper time proportional to \( s \). In numerical simulations, these two equations determine the test-particle trajectories \cite{1}.

Note, that the sum over the poles appearing in (50) is also inherent to (51): To make the latter meaningful, one has to solve the constraint equation \( \mathcal{Y} = 0 \), and in general will find that the hypersurface it defines consists of two or more disjoint parts.

6 Transport equation with quantum effects

The second step in the study of the quantum transport equation (48) consists of neglecting only the gradient terms on the right hand side. Real and imaginary part separate easily,

\[
\text{Tr} \left[ A_{XP} \ (2\otimes) \ \left\{ \pi \Gamma_{XP} \right\} \ \left\{ N_{XP} \right\} \right] = -2 \left( N_{XP} - N_{0XP}^0 \right) \ \text{Tr} \left[ \Gamma_{XP} G_{XP} \right]
\]

(53)

\[
\text{Tr} \left[ A_{XP} \ (2\otimes) \ \left\{ \hat{P} - M - (\Sigma^X_X + \text{Re}\Sigma_{XP}) \right\} \ \left\{ N_{XP} \right\} \right] = 2\pi \left( N_{XP} - N_{0XP}^0 \right) \ \text{Tr} \left[ \Gamma_{XP} A_{XP} \right].
\]

(54)

Obviously, this is a transport equation in relaxation time approximation: The derivative of \( N_{XP} \) is proportional to the difference of \( N_{XP} \) and \( N_{0XP}^0 \).

Up to now the spectral function is completely general. However, to study the consequences of the off-shell effects in this equation, we have to relate the it as closely as possible to the Vlasov equation. This achieved by using the approximate spectral function \cite{1}, which amounts to an expansion of the complete solution \cite{27} up to lowest order in \( \Gamma \) and neglection of the tensor parts. It is inserted in eqn. (54), i.e., into the imaginary part of the full transport equation. One obtains, after
calculation of the trace over the Dirac matrices

\[
\left( \frac{\mathcal{W}}{\mathcal{Y}^2 + \pi^2 \mathcal{W}^2} \right) \left\{ \frac{\partial \mathcal{Y}}{\partial \mathcal{P}_\mu} \frac{\partial \mathcal{N}_{XP}}{\partial X^\mu} - \frac{\partial \mathcal{Y}}{\partial X^\mu} \frac{\partial \mathcal{N}_{XP}}{\partial \mathcal{P}_\mu} + 2\pi \mathcal{W} \left( \mathcal{N}_{XP} - \mathcal{N}_{XP}^0 \right) \right\} = 0 . \tag{55}
\]

While one may consider this equation for each value of \( p_0 \), it obtains a more specific meaning when we perform the integration over \( \mathcal{Y} \). The insertion of the spectral function (31) already implies a sufficiently small \( \mathcal{W} \), hence the overall factor outside the curly brackets can be approximated by a \( \delta \)-function in this case.

Then the physical interpretation of \( \mathcal{W} \) is that of a spectral width multiplied by twice the energy of an almost particle-like mode, and the energy of this mode is the positive solution of \( \mathcal{Y} = 0 \) (see (24)). For this mode we obtain

\[
\left( \frac{d\mathcal{N}_{XP}}{ds} \right)_{\mathcal{Y}=0} = \left( \frac{\partial \mathcal{Y}}{\partial \mathcal{P}_\mu} \frac{\partial \mathcal{N}_{XP}}{\partial X^\mu} - \frac{\partial \mathcal{Y}}{\partial X^\mu} \frac{\partial \mathcal{N}_{XP}}{\partial \mathcal{P}_\mu} \right)_{\mathcal{Y}=0} = -2\pi \mathcal{W} \left( \mathcal{N}_{XP} - \mathcal{N}_{XP}^0 \right)_{\mathcal{Y}=0} \tag{56}
\]

As argued above \( s \) is proportional to the proper time along the characteristic curves of the Vlasov equation, and these characteristic curves are the hamiltonian trajectories of an infinitesimal phase-space cell, see (52).

The standard Vlasov equation states, that the occupation probability of this phase space cell does not change with proper time. Conversely the equation (56) expresses, that such a change of the distribution function occurs in case the spectral function peak has a nonzero width. Hence, instead of the Vlasov equation, the result is a kinetic equation with dissipation.

An equation similar to (56) has been obtained by several authors in spatially homogeneous fermionic and bosonic systems [6, 7, 15, 24, 25, 27]. For the purpose of understanding relaxation phenomena it is of tremendous importance: It implies, that without an imaginary part of the self energy function and correspondingly a nonzero spectral width there is no relaxation in a quantum system.

The interpretation of trajectories in phase space is more complicated when proceeding to the full spectral function (27). Also, when using only eqn. (54) one obtains ugly expressions of the type \( \Gamma_s \partial_X M^* \ldots \). Hence one has to combine it with the real part of eqn. (18), which is written explicitly in (53). The latter then contains terms of the type \( M^* \partial_X \Gamma_s \ldots \).

The result of this combination is written down only for the sake of completeness: It demonstrates how the quantum transport equation including dissipation may be
written in covariant form. Using the abbreviations defined in (24), one obtains

\[
- \frac{X}{\pi W} \times \text{eqn. (53)} + \text{eqn. (54)} \Rightarrow \left( \frac{1}{X^2 + \pi^2 W^2} \right) \times \\
\left\{ W \left( \frac{\partial Y}{\partial P_\mu} \frac{\partial N_{XP}}{\partial X_\mu} - \frac{\partial Y}{\partial X_\mu} \frac{\partial N_{XP}}{\partial P_\mu} \right) - X \left( \frac{\partial W}{\partial P_\mu} \frac{\partial N_{XP}}{\partial X_\mu} - \frac{\partial W}{\partial X_\mu} \frac{\partial N_{XP}}{\partial P_\mu} \right) \right. \\
- \frac{X^2}{\pi^2 W} \left( \frac{\partial (X - Y)}{\partial P_\mu} \frac{\partial N_{XP}}{\partial X_\mu} - \frac{\partial (X - Y)}{\partial X_\mu} \frac{\partial N_{XP}}{\partial P_\mu} \right) \\
+ 2\pi \left( N_{XP} - N_{XP}^0 \right) \left( W^2 + \frac{X^2}{\pi^2} \right) + 2 \left( X - \frac{X^2}{W} \right) (X - Y) \right\} = 0. \tag{57}
\]

As before, this equation can be studied around a pronounced peak in the spectral function. To first order in the peak width, eqn. (53) is recovered.

Finally we put together all knowledge assembled so far, i.e., the quantum transport equation (48) is used without further approximation. The crucial problem is now to determine the role of the gradient terms neglected in the derivation of (57).

The right side of (48), which contains these terms, has the general structure

\[
C_{XP} = (1 - i \otide) \tilde{\Gamma}_{XP} (G_{XP} + i\pi A_{XP}) \approx \exp(-i \otide) \tilde{\Gamma}_{XP} (G_{XP} + i\pi A_{XP}) . \tag{58}
\]

\( \tilde{\Gamma} \) has an obvious meaning to be read off from (48), resembling either \( \Gamma_{XP} \) or \( N_{XP}^0 \Gamma_{XP} \).

In appendix C of the present paper it is shown, that if and only if the two terms on the right side are related through the dispersion relation (23), one may express \( C_{XP} \) up to first order in the gradient expansion as

\[
C_{XP} = \tilde{\Gamma}_{XP} (G_{XP} + i\pi A_{XP}) \\
+ \int d\tau dE \Theta(-\tau) \exp(-i\tau E) \otide \tilde{\Gamma}(t + \tau/2, \vec{x}; P) \ A(X; p_0 + E, \vec{p}) \tag{59}
\]

with \( X = (t, \vec{x}) \) and \( P = (p_0, \vec{p}) \). Hence the dissipative kernel of eqn. (58) can be split into the time-local piece we have considered already in eqn. (54), and an integral over the past of the system. While the original Schwinger-Dyson equation required a time integration reaching into the future of the system, the present equation is strictly causal.

Using the explicit meaning of \( \tilde{\Gamma} \) from eqn. (48), the final result of the present work is the equation

\[
\text{Tr} \left[ A_{XP} \left( 2 \otide \right) \left\{ \vec{P} - M - (\Sigma_X^\delta + \text{Re}\Sigma_{XP}) \right\} \left\{ N_{XP} \right\} \right] \]

18
\[\begin{aligned}
&= 2\pi \left( N_{XP} - N_{XP}^0 \right) \text{Tr} \left[ \Gamma_{XP} A_{XP} \right] \\
&+ N_{XP} \int d\tau dE \Theta(-\tau) \sin(\tau E) \text{Tr} \left[ (2\Delta) \left( \Gamma(t + \tau/2, \vec{x}; P) A(X; p_0 + E, \vec{p}) \right) \right] \\
&- \int d\tau dE \Theta(-\tau) \sin(\tau E) \text{Tr} \left[ (2\Delta) \left( N^0 \Gamma(t + \tau/2, \vec{x}; P) A(X; p_0 + E, \vec{p}) \right) \right].
\end{aligned}\]

The physical interpretation of the integral over the past history of the system is obvious: It leads to the memory effects observed in transport theory, and relates these to the spectral width of the "particles" propagating in the medium.

Note, that one may insert a diagrammatic expansion for the self energy function, and thereby for \(\Gamma_{XP}\) and \(N_{XP}^0 \Gamma_{XP}\). Consider e.g. the one depicted in figure 1: In quasi-particle approximation it leads to a standard collision integral on the right side of the above equation. However, beyond the quasi-particle approximation the oscillating factor \(\sin(\tau E)\) may lead to enhancement as well as suppression of relaxation.

## 7 Conclusions

In the present work, an equation of motion for the relativistic one-particle phase-space distribution function of a fermion system was derived within a real time Green function formalism. The only approximation made is an expansion to first order in the gradients present in a non-equilibrium system. In particular, no quasi-particle approximation was introduced: The full spectral information to the same order in the gradient expansion is retained in the dissipative equation (60). This places the present derivation in the gap between quantum field theory and traditional transport equations.

A result of this study is the observation, that dissipation occurs if and only if the spectral function deviates from a \(\delta\)-function, i.e., if the imaginary part of the self energy function is nonzero. For strongly peaked spectral functions, which one might associate with "particles", this implies that dissipation is equivalent to a finite lifetime of the excitation.

Such an interpretation is fully consistent with standard “molecular” transport theory: The Uehling-Uhlenbeck collision term in “molecular” transport equations is diagrammatically given by Born diagrams [1]. Their closure to a self energy function, as depicted in figure 1, gives the lowest order perturbative contribution to the width
of the quasi-particle pole in the propagator. This result makes the quasi-particle approach to the simulation of nuclear collisions quite dubious.

Close to equilibrium, the dissipative driving force is proportional to the difference between \( N_{XP} \) and \( N_{XP}^0 \), i.e., between the time local distribution function diagonalizing the propagator and the one diagonalizing the self energy function. The strategy of diagonalizing the full non-equilibrium propagator by thermal Bogoliubov transformations is therefore equivalent to the solution of the transport equation. One may put this into the form of a statement about the TFD formalism: The basic idea of thermo field dynamics, the Bogoliubov transformation, achieves the proper separation of spectral and statistical information even in a non-equilibrium system [13, 27].

To first order in the gradients the spectral function is the solution of an algebraic equation, eqn. (27). This may be cast together with the definition of the real part of the full propagator by dispersion integral, eqn. (29) and an ansatz to determine the self energy from the propagator. Such an ansatz was not discussed in the present work, it necessitates a perturbative expansion in terms of the full propagators that has been formulated elsewhere [23, 20, 27]. We then have at hand a closed system of equations for the Green function of the interacting fermion system, with controlled approximations.

The rewriting of the dissipative kernel in terms of the spectral function and the self energy was achieved by using a dispersion integral. This necessitates a proper understanding of the analytical structure of the retarded and advanced Green function of the system, before following the time evolution of occupation probabilities. Hence, the proper ghost-free definition of the real part of the propagator, as outlined before and discussed in ref. [28], is an essential ingredient of this step.

The previously made artificial distinction between the spectral width due to collisions and a “natural” spectral width, like e.g. for resonances [1], does not arise in the present work. However, a distinction is possible between time-local relaxation and memory effects present in the system. The influence exerted by the memory kernel of eqn. (21) depends on the details of spectral function and time evolution: One may not estimate its effect on the net relaxation time a priori.

In ref. [7] it was emphasized, that even in a simple model for relativistic heavy-ion collisions a slowdown as well as an enhancement may occur through memory effects. In [2] an average slow-down was found in a similar non-relativistic framework, by
comparing the solution of the Schwinger-Dyson equation with a “molecular” transport equation. Accordingly, the next logical step will be a numerical study of the derived transport equation (61), but this is beyond the scope of the present work.

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A Thermal Bogoliubov matrix

The thermal Bogoliubov transformations of thermo field dynamics exist in various parameterizations. As has been pointed out in ref. \[15, 16\], the special choice $\alpha = 1$ and $s = 1/2 \log(1 + \sigma n)$ makes them linear in a single parameter $n$, for fermionic fields the explicit form is given in eqn. (5). It follows e.g., that

$$
(B(n))^{-1} \frac{d}{dt} B(n) = (B(n))^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{dn}{dt} \end{pmatrix} B(n). \tag{61}
$$

Obviously, a time-dependence of the Bogoliubov parameter $n$ only affects the off-diagonal part of the Schwinger-Dyson equation.

Now let $\Sigma$ be any $2 \times 2$ matrix which fulfills the linear relation (4). Then one obtains

$$
B(n_1) \Sigma \tau_3 (B(n_2))^{-1} = \begin{pmatrix} \Sigma^R & \Sigma^{11}(n_2 - n_1) - (1 - n_2)\Sigma^{12} - n_1\Sigma^{21} \\ \Sigma^A & \Sigma^{12} + n_1\Sigma^{21} \end{pmatrix}. \tag{62}
$$

Quite similar relations exist for propagator matrices: Let $S$ be any $2 \times 2$ matrix which fulfills the linear relation (3). Then

$$
B(n_1) \tau_3 S (B(n_2))^{-1} = \begin{pmatrix} S^R & S^{11}(n_2 - n_1) + (1 - n_2)S^{12} + n_1S^{21} \\ S^A & S^{12} + n_1S^{21} \end{pmatrix}. \tag{63}
$$

For the purpose of deriving transport equations it is furthermore advantageous to exploit a similar relation for the product of three such matrices. Let $S_0$ also be a propagator matrix, whose matrix elements satisfy the relations (3). Then one obtains

$$
B(n_1) \tau_3 S_0 \Sigma S (B(n_2))^{-1} = \begin{pmatrix} S_0^R \Sigma^R X \\ S_0^A \Sigma^A S^A \end{pmatrix}, \tag{64}
$$

where we have used the definition of retarded and advanced components as found in eqn. (7), and

$$
X = S_0^K \Sigma^A S^A + S_0^R \Sigma^R S^K
+ S_0^R ( (n_2 - n_1)\Sigma^{11} + (n_2 - 1)\Sigma^{12} - n_1\Sigma^{21} ) S^A \tag{65}
$$

with the kinetic components defined in eqn. (37). Corresponding relations hold for bosonic Bogoliubov transformations, see ref. [15].
B Fermionic spectral function

In this section, the equations (16) are solved using (20) for the spinor structure of the self energy function. First, the trace is taken over the Dirac matrices:

\[
\cos \diamond (P^*_\mu G^\mu_v - M^* G_s - 2T_{\mu\nu}G^{\mu\nu}_t) = 1 - \pi^2 \cos \diamond (\Gamma_s A_s + \Gamma_{v,\mu} A^\mu_v + 2\Gamma_{t,\mu\nu} A^{\mu\nu}_t)
\]

\[
\cos \diamond (P^*_\mu A^\mu_v - M^* A_s - 2T_{\mu\nu}A^{\mu\nu}_t) = \cos \diamond (\Gamma_s G^\mu_v + \Gamma_{v,\mu} G^\mu_v + 2\Gamma_{t,\mu\nu} G^{\mu\nu}_t).
\]

(66)

Multiplication with \(\gamma^\nu\) before taking the trace gives the next equations. To eliminate the term \(\partial_\mu G^{\mu\nu}_t\) one uses (19),

\[
\cos \diamond (P^*_\nu G_s - M^* G^\nu_v) + 2\sin \diamond (P^*_\nu G^{\mu\nu}_t - T^{\mu\nu} G_{\nu,\mu})
\]

\[-\pi^2 \cos \diamond (\Gamma_s A^\nu_v + \Gamma^\nu_v A_s) - 2\pi^2 \sin \diamond (\Gamma^{\mu\nu}_t A_{\nu,\mu} + \Gamma_{v,\mu} A^{\mu\nu}_t)
\]

\[
\cos \diamond (P^*_\nu A_s - M^* A^\nu_v) + 2\sin \diamond (P^*_\nu A^{\mu\nu}_t - T^{\mu\nu} A_{\nu,\mu})
\]

\[= \cos \diamond (\Gamma_s G^\nu_v + \Gamma^\nu_v G_s) + 2\sin \diamond (\Gamma^{\nu\nu}_t G_{\nu,\mu} + \Gamma_{v,\mu} G^{\mu\nu}_t).
\]

(67)

The remaining spin components are obtained by multiplying with \(i\gamma^5\gamma^\alpha\) before taking the trace:

\[
e^{\alpha\nu\sigma\tau} \cos \diamond (P^*_\nu G_{t,\sigma\tau} - T_{\sigma\tau} G^\nu_v) = -\pi^2 e^{\alpha\nu\sigma\tau} \cos \diamond (\Gamma_{t,\sigma\tau} A^\nu_v + \Gamma_{v,\nu} A_{t,\sigma\tau})
\]

\[
e^{\alpha\nu\sigma\tau} \cos \diamond (P^*_\nu A_{t,\sigma\tau} - T_{\sigma\tau} A^\nu_v) = e^{\alpha\nu\sigma\tau} \cos \diamond (\Gamma_{t,\sigma\tau} G^\nu_v + \Gamma_{v,\nu} G_{t,\sigma\tau})
\]

where \(e^{\alpha\nu\sigma\tau}\) is the total antisymmetric tensor in four dimensions. Since \(X\) and \(P\) are independent variables, the \(\sin \diamond\)-terms from equation (67) cancel. Furthermore, in first order gradient expansion \(\cos \diamond = 1\). Reduced spinorial components are introduced as

\[
a_s = \frac{A_s}{M^*}, \quad a_v = -\frac{A^\mu_v P^*_\mu}{P^*_2}, \quad a_t = \frac{A^{\mu\nu}_t T^{\mu\nu}}{T^2},
\]

(68)

similar for \(g_s, v, t \leftrightarrow G\) and \(k_s, v, t \leftrightarrow \Gamma\). The scalar equations are

\[
P^{*2} g_v - M^{*2} g_s - 2T^2 g_t = 1 - \pi^2 (M^{*2} k_s a_s + \Gamma_{v,\mu} A^\mu_v + 2\Gamma_{t,\mu\nu} A^{\mu\nu}_t)
\]

\[
P^{*2} a_v - M^{*2} a_s - 2T^2 a_t = M^{*2} k_s g_s + \Gamma_{v,\mu} G^\mu_v + 2\Gamma_{t,\mu\nu} G^{\mu\nu}_t.
\]

(69)

The vector equations are contracted with \(P^*_\nu\) first:

\[
g_s - g_v = -\pi^2 (k_s a_v + k_v a_s)
\]

\[
a_s - a_v = k_s g_v + k_v g_s,
\]

(70)
and then with \( \Gamma_{v,\nu} \):

\[
P^{*2} k_v g_s - G^\nu_v \Gamma_{v,\nu} = -\pi^2 (k_s A^\nu_v \Gamma_{v,\nu} + a_s \Gamma^\nu_v \Gamma_{v,\nu})
\]

\[
P^{*2} k_v a_s - A^\nu_v \Gamma_{v,\nu} = k_s \Gamma^\nu_v \Gamma_{v,\nu} + g_s \Gamma^\nu_v \Gamma_{v,\nu}.
\]

(71)

The tensorial equations are contracted with \( P^{*\nu} T^{\sigma\tau} \), leading to

\[
g_t - g_v = -\pi^2 (k_t a_v + k_v a_t)
\]

\[
a_t - a_v = k_t g_v + k_v g_t,
\]

(72)

and with \( P^{*\nu} \Gamma^{\sigma\tau}_t \):

\[
G_{t,\sigma\tau} \Gamma^{\sigma\tau}_t - T^2 g_v k_t = -\pi^2 (a_v \Gamma_{t,\sigma\tau} \Gamma^{\sigma\tau}_t + k_v A_{t,\sigma\tau} \Gamma^{\sigma\tau}_t)
\]

\[
A_{t,\sigma\tau} \Gamma^{\sigma\tau}_t - T^2 a_v k_t = g_v \Gamma_{t,\sigma\tau} \Gamma^{\sigma\tau}_t + k_v G_{t,\sigma\tau} \Gamma^{\sigma\tau}_t,
\]

(73)

We can then use eqns. (72) and (73) to eliminate the products of \( A \) and \( G \) with \( \Gamma \). The abbreviations used in the following are defined in eqns. (24), (25), and the final solution eqn. (27) is obtained through

\[
\mathcal{D} a_s = k_s \mathcal{X} + \mathcal{W}, \quad \mathcal{D} a_v = -k_v \mathcal{X} + \mathcal{W}, \quad \mathcal{D} a_t = k_t \mathcal{X} + 2 \mathcal{W}
\]

\[
\mathcal{D} g_s = \mathcal{X} - k_s \pi^2 \mathcal{W}, \quad \mathcal{D} g_v = \mathcal{X} + k_v \pi^2 \mathcal{W}, \quad \mathcal{D} g_t = \mathcal{X} - k_t \pi^2 \mathcal{W}.
\]

(74)
C Dissipative kernel of the transport equation

In this section, the expression on the right side of the quantum transport equation (18) is evaluated to first order in the gradient expansion. First, we write it to infinite order in $\Gamma$

$$\mathcal{C}_{XP} = \exp(-i\Diamond) \tilde{\Gamma}_{XP} (G_{XP} + i\pi A_{XP}) .$$

(75)

The next step consists of using (30) in connection with (12) (See remark in the text after eqn. (58)). We remove the gradient expansion for this intermediate step,

$$\mathcal{C}_{XP} = 2\pi \int d^4z d^4Q (2\pi)^4 \Theta(-z_0) \exp \left( i(P - Q)z + \frac{i}{2} \frac{\partial^T}{\partial p_0} \right) \times \tilde{\Gamma}(X + z/2; P) iA(X; Q) .$$

(76)

The three-dimensional integrations can be carried out, with $X = (t, \vec{x})$ and $P = (p_0, \vec{p})$ one obtains

$$\mathcal{C}_{XP} = i \int d\tau dE \Theta(-\tau) \exp \left( -i\Diamond - i\tau E + \frac{i}{2} \frac{\partial^T}{\partial p_0} \right) \times \tilde{\Gamma}(t + \tau/2, \vec{x}; P) A(X; p_0 + E, \vec{p}) .$$

(77)

The solutions for $G$ and $A$ are known only to first order in the gradient expansion, hence this expression is also valid only in first order in $\Diamond$. Furthermore, the additional derivative term in the exponent is eliminated by shifting the argument of $\tilde{\Gamma}$:

$$\mathcal{C}_{XP} = i \int d\tau dE \Theta(-\tau) \exp(-i\tau E) \left( 1 + i\Diamond + \frac{i}{2} \frac{\partial^T}{\partial p_0} \right) \times \tilde{\Gamma}(t + \tau/2, \vec{x}; P) A(X; p_0 + E, \vec{p})$$

(78)

In the first part, the $t$-integration can be carried out,

$$\mathcal{C}_{XP} = \int dE \frac{1}{p_0 - E - i\epsilon} \tilde{\Gamma}(X; P) A(X; E, \vec{p})$$

$$+ \int d\tau dE \Theta(-\tau) \exp(-i\tau E) \Diamond \tilde{\Gamma}(t + \tau/2, \vec{x}; P) A(X; p_0 + E, \vec{p}) .$$

(79)

Together with (29), this is equivalent to eqn. (58).
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