ON CONJUGACY CLASSES IN A REDUCTIVE GROUP

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INTRODUCTION

0.1. Let $k$ be an algebraically closed field of characteristic exponent $p \geq 1$ and let $G$ be a connected reductive algebraic group over $k$. Let $W$ be the Weyl group of $G$. Let $cl(G)$ (resp. $cl(W)$) be the set of conjugacy classes of $G$ (resp. $W$). In this paper we prove the following result.

Theorem 0.2. There exist an equivalence relation $\sim$ on $cl(G)$ and an equivalence relation $\sim$ on $cl(W)$ such that the sets of equivalence classes $cl(G)/\sim$, $cl(W)/\sim$ are in canonical bijection and such that the following hold:

(a) The equivalence relation $\sim$ on $cl(W)$ depends only on $W$ (as a Coxeter group) and not on $p$ or the underlying root datum. Hence $cl(G)/\sim$ is indexed by a finite set which depends only on $W$ (as a Coxeter group) and not on $p$ or the underlying root datum.

(b) Any equivalence class for $\sim$ on $cl(G)$ is a union of conjugacy classes of the same dimension (at most one of which is unipotent) and this dimension depends only on $W$ (as a Coxeter group) and not on $p$ or the underlying root datum.

The proof is given in 1.8. From the theorem we see that $G$ itself is partitioned into finitely many strata (a stratum is the union of all conjugacy classes of $G$ in a fixed equivalence class for $\sim$ hence of constant dimension); one of the strata is the centre of $G$. We see also that if $n \geq 1$, then for any integer $k$, the following three conditions are equivalent:

- there exists a conjugacy class of dimension $k$ in $SO_{2n+1}(C)$;
- there exists a conjugacy class of dimension $k$ in $Sp_{2n}(C)$;
- there exists a conjugacy class of dimension $k$ in $Sp_{2n}(\overline{F}_2)$.

The proof shows that the following fourth condition is equivalent to the three conditions above: there exists a unipotent conjugacy class of dimension $k$ in $Sp_{2n}(\overline{F}_2)$.

We will give two approaches to the theorem. The first approach, see §1, is based on Springer’s correspondence (see [Spr] when $p = 1$ or $p \gg 0$ and [L2] for any $p$) connecting irreducible representations of Weyl groups with unipotent classes and on the results of [L5,L7] connecting $cl(W)$ with unipotent classes in}

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The second approach, see §2, is based on an extension of the ideas in [L5], and Springer’s correspondence does not appear in it.

In §3 we define an equivalence relation analogous to ~ in the case where $G$ is replaced by a loop group.

**0.3. Notation.** For an algebraic group $H$ over $k$ we denote by $H^0$ the identity component of $H$. For a subgroup $T$ of $H$ we denote by $N_HT$ the normalizer of $T$ in $H$. Let $G_{ad}$ be the adjoint group of $G$. Let $B$ be the variety of Borel subgroups of $G$. For $g \in G$ we denote by $Z_G(g)$ the centralizer of $g$ in $G$ and by $g_s$ (resp. $g_u$) the semisimple (resp. unipotent) part of $g$. Let $B_g = \{B \in B; g \in B\}$.

For any (finite) Weyl $\Gamma$ we denote by $\text{Irr}_\Gamma$ a set of representatives for the isomorphism classes of irreducible representations of $\Gamma$ over $\mathbb{Q}$. For any $\tau \in \text{Irr}W$ let $n_\tau$ be the smallest integer $i \geq 0$ such that $\tau$ appears in the $i$-th symmetric power of the reflection representation of $W$.

1. **Proof of Theorem 0.2**

1.1. We will view $W$ as an indexing set for the orbits of $G$ acting diagonally on $B \times B$; we denote by $O_w$ the orbit corresponding to $w \in W$. Note that $W$ is naturally a Coxeter group; its length function is denoted by $\ell: W \to \mathbb{N}$.

Let $g \in G$. Let $W_g$ be the Weyl group of the connected reductive group $H := Z_G(g_s)^0$. We can view $W_g$ as a subgroup of $W$ as follows. Let $\beta$ be a Borel subgroup of $H$ and let $T$ be a maximal torus of $\beta$. We define an isomorphism $b_{T,\beta}: N_HT/T \sim W_g$ by $n'T \mapsto H$-orbit of $(\beta, n'n'^{-1})$. Similarly for any $B \in B$ such that $T \subset B$ we define an isomorphism $a_{T,B}: N_GT/T \sim W$ by $n'T \mapsto G$-orbit of $(B, n'Bn'^{-1})$. Now assume that $B \in B$ is such that $B \cap H = \beta$. We define an imbedding $c_{T,\beta,B}: W_g \to W$ as the composition $W_g \xrightarrow{b_{T,\beta}^{-1}} N_HT/T \to N_GT/T \xrightarrow{a_{T,B}} W$ where the middle map is the obvious imbedding. If $B' \in B$ also satisfies $B' \cap H = \beta$ then we have $B' = nBn^{-1}$ for some $n \in N_GT$ and from the definitions we have $c_{T,\beta,B'}(w) = a_{T,B}(nT)c_{T,\beta,B}(w)a_BT(nT)^{-1}$ for any $w \in W_g$. Thus $c_{T,\beta,B}$ depends (up to composition with an inner automorphism of $W$) only on $T, \beta$ and we can denote it by $c_{T,\beta}$. Since the set of pairs $T, \beta$ as above form a homogeneous space for the connected group $H$ we see that $c_{T,\beta}$ is independent of $T, \beta$ (up to composition with an inner automorphism of $W$) hence it does not depend on any choice. We see that there is a well defined collection $\mathcal{C}$ of imbeddings $W_g \to W$ so that any two of them differ only by composition by an inner automorphism of $W$.

Now let $\rho$ be the irreducible representation of $W_g$ which under the Springer correspondence for $H$ corresponds to the $H$-conjugacy class of $g_u$ and the trivial local system on it. We choose $f \in \mathcal{C}$; then we can view $\rho$ as an irreducible representation of $f(W_g)$, a subgroup of $W$. Let $\tilde{\rho}$ be the irreducible representation of $W$ obtained from $\rho$ by $j$-induction [LS, 3.2] from $\rho$. (Note that the $j$-induction can be applied to $\rho$ since $\rho$ is good in the sense of [L4, 1.3], see [L4, 1.4].) Since $f$ is well defined up to composition by an inner automorphism of $W$, we see that $\tilde{\rho}$ is
independent of the choice of $f$. Thus we have a well defined map $\phi_G : G \to \text{Irr} W$, $g \mapsto \tilde{\rho}$ whose nonempty fibres are called the strata of $G$. The strata of $G$ are clearly unions of conjugacy classes of $G$; hence $\phi_G$ induces a map $\tilde{\phi}_G : \text{cl}(G) \to \text{Irr} W$.

For $\gamma, \gamma'$ in $\text{cl}(G)$ we write $\gamma \sim \gamma'$ if $\tilde{\phi}_G(\gamma) = \tilde{\phi}_G(\gamma')$. This is an equivalence relation on $\text{cl}(G)$. Let $\text{cl}(G)/\sim$ be the set of equivalence classes.

Let $\mathcal{R}_G(W)$ be the image of $\tilde{\phi}_G$ (or of $\tilde{\phi}_G$). Then $\tilde{\phi}_G$ induces a bijection $\text{cl}(G)/\sim \leftrightarrow \mathcal{R}_G(W)$.

1.2. Clearly, we have $\phi_G = \phi_{G_{ad}} \pi$, $\tilde{\phi}_G = \tilde{\phi}_{G_{ad}} \tilde{\pi}$, where $\pi : G \to G_{ad}$, $\tilde{\pi} : \text{cl}(G) \to \text{cl}(G_{ad})$ are the obvious (surjective) maps. Hence $\mathcal{R}_G(W) = \mathcal{R}_{G_{ad}}(W)$. If we assume that $G$ is adjoint, we have $G = \prod_{k \in K} G_k$, $W = \prod_{k \in K} W_k$, where $G_k$ is adjoint simple and $W_k$ is the Weyl group of $G_k$. From the definition we can identify $\mathcal{R}_G(W)$ with $\prod_{k \in K} \mathcal{R}_{G_k}(W_k)$ (via external tensor product).

1.3. Returning to the general case, we show:

(a) Let $\gamma \in \text{cl}(G)$ and let $n = n_{\tilde{\phi}(\gamma)}$, see 0.3; then $\dim \gamma = 2 \dim \mathcal{B} - 2n$. In particular, each stratum of $G$ is a union of conjugacy classes of the same dimension. Let $g \in \gamma$. Let $\rho$ (resp. $\tilde{\rho}$) be the irreducible representation of $W_g$ (resp. $W$) defined by $g_u$ as in 1.1. Let $n_\rho$ be the smallest integer $i \geq 0$ such that $\rho$ appears in the $i$-th symmetric power of the reflection representation of $W_g$. Let $n_{\tilde{\rho}}$ be as in 0.3. By the definition of $j$-induction we have $n_\rho = n_{\tilde{\rho}}$. By assumption we have $n_{\tilde{\rho}} = n$ hence $n_\rho = n$. By a known property of Springer’s representations, $n_\rho$ is equal to the dimension of the variety of Borel subgroups of $Z_G(g_s)^0$ that contain $g_u$; hence by a result of Steinberg (for $p = 1$) and Spaltenstein [Spa, 10.15] (for any $p$), $n_\rho$ is equal to

$$\frac{(\dim(Z_{G(G_s)^0}(g_u)^0 - \text{rk}(Z_G(g_s)^0))/2 = (\dim(Z_G(g)^0) - \text{rk}(G))/2.$$

It follows that $(\dim(Z_G(g)^0) - \text{rk}(G))/2 = n$ and (a) follows.

1.3. Let $r$ be either 1 or a prime number. Let $G_r$ be a connected reductive group of the same type as $G$ over an algebraically closed field of characteristic exponent $r$, whose Weyl group is identified with $W$. Let $\mathcal{U}_r$ be the set of unipotent classes of $G_r$. Let $\mathcal{X}^r(W)$ be the set of irreducible representations of $W$ associated by Springer correspondence to a unipotent class in $\mathcal{U}_r$ and the trivial local system on it. We have the following result.

Proposition 1.4. We have $\mathcal{R}_G(W) = \bigcup_{r \text{ prime}} \mathcal{X}^r(W)$. Moreover, $\mathcal{R}_G(W)$ depends only on $W$ (as a Coxeter group) and not on $p$ or the underlying root datum.

By the arguments in 1.2 we can assume that $G$ is adjoint simple. Now the sets $\mathcal{X}^r(W)$ are explicitly known for any reductive group; they are described in [L4, 1.4] in terms of $j$-induction of representations in $\mathcal{X}^1(W')$ for certain Weyl subgroups $W'$ of the Weyl group. Hence the set $\mathcal{R}_G(W)$ can be explicitly determined. It can be described as follows.

If $G$ is of type $A_n(n \geq 1)$ or $E_6$ we have $\mathcal{R}_G(W) = \mathcal{X}^1(W)$. 
If $G$ is of type $B_n(n \geq 2), C_n(n \geq 3), D_n(n \geq 4), F_4$ or $E_7$, we have $\mathcal{R}_G(W) = \mathcal{X}^2(W)$.

If $G$ is of type $G_2$ we have $\mathcal{R}_G(W) = \mathcal{X}^3(W)$.

If $G$ is of type $E_8$ we have $\mathcal{R}_G(W) = \mathcal{X}^2(W) \cup \mathcal{X}^3(W)$.

Now to get the first assertion of the proposition we use the following known results.

$\mathcal{X}^1(W) \subset \mathcal{X}^r(W)$ for any $r \geq 2$;

$\mathcal{X}^r(W) = \mathcal{X}^1(W)$ if $G$ is of type $A_n$ or $E_6$ ($r \geq 2$), or if $G$ is of type $B_n, C_n, D_n, F_4$ or $E_7$, ($r \geq 3$), or if $G$ is of type $G_2$, ($r = 2$ or $r \geq 5$) or if $G$ is of type $E_8$, $r \geq 5$.

In particular we see that $\mathcal{R}_G(W)$ does not depend on $p$. The last assertion of the proposition follows from the fact that the sets $\mathcal{X}^2(W)$ (resp. $\mathcal{X}^3(W)$) are compatible with the exceptional isogeny between groups of type $B_n, C_n$ with $p = 2$ and that between $F_4, F_4$ with $p = 2$ (resp. between $G_2, G_2$ with $p = 3$). This completes the proof.

1.5. In view of the proposition we can denote $\mathcal{R}_G(W)$ simply by $\mathcal{R}(W)$. In the case where $G$ is of type $E_8$ we have $|\mathcal{X}^1(W)| = 70, |\mathcal{X}^2(W)| = 74, |\mathcal{X}^3(W)| = 71, \mathcal{X}^2(W) \cap \mathcal{X}^3(W) = \mathcal{X}^1(W)$, hence $|\mathcal{R}(W)| = 74 + 71 - 70 = 75$.

1.6. In [L5] we have defined a surjective map $cl(W) \rightarrow \mathcal{U}_r$; we denote this map by $\Phi_r$. We can identify $\mathcal{U}_r = \mathcal{X}^r(W)$ (see 1.3) in an obvious way. Then $\Phi_r$ becomes a map $cl(W) \rightarrow \mathcal{X}^r(W)$. For $C \in cl(W)$ we define $\tilde{\Phi}(C) = \cup_{r \text{ prime}} \mathcal{X}^r(W)$ as follows. If $\Phi_r(C) \in \mathcal{X}^1(W)$ for all $r > 1$ (recall that $\mathcal{X}^1(W) \subset \mathcal{X}^r(W)$) then $\Phi_r(C)$ is independent of $r$ (see [L7, 0.4]) and we set $\tilde{\Phi}(C) = \Phi_r(C)$ for any $r > 1$. If $\Phi_r(C) \notin \mathcal{X}^1(W)$ for some $r > 1$ then $r$ is unique. (To prove this we can assume that $G$ is almost simple, simply connected; then the only case where there is an issue is in type $E_8$ in which case we use the tables in [L7, 2.6].) We then set $\tilde{\Phi}(C) = \Phi_r(C)$.

Thus we have defined a surjective map $\tilde{\Phi} : cl(W) \rightarrow \cup_{r \text{ prime}} \mathcal{X}^r(W)$ that is, $\tilde{\Phi} : cl(W) \rightarrow \mathcal{R}(W)$ (see 1.4, 1.5). For $C, C'$ in $cl(W)$ we write $C \sim C'$ if $\tilde{\Phi}(C) = \tilde{\Phi}(C')$; this is an equivalence relation on $cl(W)$. Let $cl(W)/ \sim$ be the set of equivalence classes. The equivalence classes are described explicitly in [L7]. Note that $\tilde{\Phi}$ induces a bijection $cl(W)/ \sim \leftrightarrow \mathcal{R}(W)$.

**Proposition 1.7.** The equivalence relation $\sim$ on $cl(W)$ and the bijection $cl(W)/ \sim \leftrightarrow \mathcal{R}(W)$ in 1.6 depend only on $W$ (as a Coxeter group) and not on $p$ or the underlying root datum.

We can assume that $G$ is almost simple. We then use the fact that the maps $\Phi_2$ (resp. $\Phi_3$) are compatible with the exceptional isogeny between groups of type $B_n, C_n$ with $p = 2$ and that between $F_4, F_4$ with $p = 2$ (resp. between $G_2, G_2$ with $p = 3$). This implies the result.

1.8. We prove Theorem 0.2. We define the bijection $cl(G)/ \sim \leftrightarrow cl(W)/ \sim$ as the composition of the bijection $cl(G)/ \sim \leftrightarrow \mathcal{R}(W)$ in 1.1 with the inverse of the
bijection \( cl(W) / \sim \leftrightarrow R(W) \) in 1.6. Then 0.2(a) follows from 1.7. Now 0.2(b) follows from 1.3(a) and 1.4. This completes the proof of Theorem 0.2.

1.9. Assume that \( G \) has type \( E_8 \). Using 1.5 we see that \( G \) has exactly 75 strata. If \( p \neq 2, 3 \) then exactly 70 strata contain unipotent elements. If \( p = 2 \) (resp. \( p = 3 \) then exactly 74 (resp. 71) strata contain unipotent elements. The unipotent class of dimension 58 is a stratum. If \( p \neq 2 \), there is a stratum which is a union of a semisimple class and a unipotent class (both of dimension 128); in particular this stratum is disconnected. The corresponding equivalence class in \( cl(W) \) consists of 5 conjugacy classes of involutions, one of which contains the longest element of \( W \).

1.10. One can show that any stratum of \( G \) is a union of pieces in the partition of \( G \) defined in [L2, 3.1]; in particular it is a constructible subset of \( G \).

1.11. Assume that \( G = GL(V) \) where \( V \) is a \( k \)-vector space of dimension \( n \geq 1 \). We choose a sufficiently large \( m \in \mathbb{N} \). Let \( g \in G \). For any \( x \in k^* \) let \( V_x \) be the generalized \( x \)-eigenspace of \( g : V \to V \) and let \( \lambda_1^x \geq \lambda_2^x \geq \cdots \geq \lambda_m^x \) be the sequence in \( \mathbb{N} \) whose terms are the sizes of the Jordan blocks of \( x^{-1}g : V_x \to V_x \). Let \( \lambda(g) \) be the sequence \( \lambda(g)_1 \geq \lambda(g)_2 \geq \cdots \geq \lambda(g)_m \) given by \( \lambda(g)_j = \sum_{x \in k^*} \lambda_j^x \). Now \( g \mapsto \lambda(g) \) defines a map from \( G \) onto the set of partitions of \( n \). From the definitions we see that the fibres of this map are exactly the strata of \( G \). If \( g \in G \) and \( \lambda(g) = (\lambda_1, \lambda_2, \ldots, \lambda_m) \) then \( \dim(Bg) = (m-1)n - \sum_{k \in \{1, m-1\}} (\lambda_1 + \lambda_2 + \cdots + \lambda_k) \) (with \( B_g \) as in 0.3).

1.12. Assume that \( G = Sp(V) \) where \( V \) is a \( k \)-vector space of dimension \( 2n \geq 2 \) with a fixed nondegenerate symplectic form. We choose a sufficiently large \( m \in \mathbb{N} \). We say that a bipartition \((\lambda_1 \geq \lambda_3 \geq \cdots \geq \lambda_{2m+1}), (\lambda_2 \geq \lambda_4 \geq \cdots \geq \lambda_{2m})\) (with entries in \( \mathbb{N} \)) is of type \((e, e')\) if \( \lambda_i \geq \lambda_{i+1} - e \) for \( i = 1, 2, \ldots, 2m-1 \) and \( \lambda_i \geq \lambda_i+1 - e' \) for \( i = 2, 4, \ldots, 2m \).

Let \( g \in G \). For any \( x \in k^* \) let \( V_x \) be the generalized \( x \)-eigenspace of \( g : V \to V \). For any \( x \in k^* \) such that \( x^2 \neq 1 \) let \( \lambda_1^x \geq \lambda_2^x \geq \cdots \geq \lambda_{2m+1}^x \) be the sequence in \( \mathbb{N} \) whose terms are the sizes of the Jordan blocks of \( x^{-1}g : V_x \to V_x \) (a partition of \( \dim V_x \)). For \( x \in k^* \) such that \( x^2 = 1 \) let \( \lambda_1^x, \lambda_2^x, \ldots, \lambda_{2m+1}^x \) be the sequence in \( \mathbb{N} \) such that \((\lambda_1^x \geq \lambda_3^x \geq \cdots \geq \lambda_{2m+1}^x), (\lambda_2^x \geq \lambda_4^x \geq \cdots \geq \lambda_{2m}^x)\) is the bipartition of \( \dim(V_x)/2 \) such that the corresponding irreducible representation of the Weyl group of type \( B_{\dim V_x/2} \) is the Springer representation attached to the unipotent element \( x^{-1}g \in Sp(V_x) \). (This bipartition is of type 1,1 if \( p \neq 2 \) and of type 2,2 if \( p = 2 \). Let \( \lambda(g) \) be the sequence \( \lambda(g)_1, \lambda(g)_2, \ldots, \lambda(g)_{2m+1} \) in \( \mathbb{N} \) given by \( \lambda(g)_j = \sum_x \lambda_j^x \) where \( x \) runs over a set of representatives for the orbits of the involution \( a \mapsto a^{-1} \) of \( k^* \). Note that \((\lambda(g)_1 \geq \lambda(g)_3 \geq \cdots \geq \lambda(g)_{2m+1}), (\lambda(g)_2 \geq \lambda(g)_4 \geq \cdots \geq \lambda(g)_{2m})\) is a bipartition of \( n \) of type 2,2. Now \( g \mapsto \lambda(g) \) defines a map from \( G \) onto the set of bipartitions of \( n \) of type 2,2. From the definitions we see that the fibres of this map are exactly the strata of \( G \).

If \( g \in G \) and \( \lambda(g) = (\lambda_1, \lambda_2, \ldots, \lambda_{2m+1}) \) then \( \dim(B_g) = 2mn - \sum_{k \in \{1, 2m\}} (\lambda_1 + \lambda_2 + \cdots + \lambda_k) \) (with \( B_g \) as in 0.3).
1.13. Assume that $G = SO(V)$ where $V$ is a $k$-vector space of dimension $2n+1 \geq 1$ with a fixed nondegenerate quadratic form. (When $p = 2$ this means that the associated bilinear form has 1-dimensional radical on which the quadratic form is nonzero.) We choose a sufficiently large $m \in \mathbb{N}$. Let $g \in G$. For any $x \in k^*$ let $V_x$ be the generalized $x$-eigenspace of $g : V \to V$. For any $x \in k^*$ such that $x^2 \neq 1$ let $\lambda_1^x \geq \lambda_2^x \geq \cdots \geq \lambda_{2m+1}^x$ be the sequence in $\mathbb{N}$ whose terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$ (a partition of dim $V_x$). For $x \in k^*$ such that $x^2 = 1$ let $\lambda_1^x, \lambda_2^x, \ldots, \lambda_{2m+1}^x$ be the sequence in $\mathbb{N}$ such that $((\lambda_1^x \geq \lambda_3^x \geq \cdots \geq \lambda_{2m+1}^x), (\lambda_2^x \geq \lambda_4^x \geq \cdots \geq \lambda_{2m}^x))$ is the bipartition of $(\dim(V_x) - 1)/2$ (if $x \neq -1$ or $p = 2$) or of $\dim(V_x)/2$ (if $x = -1$ and $p \neq 2$) such that the corresponding irreducible representation of the Weyl group of type $B_{(\dim(V_x) - 1)/2}$ (if $x \neq -1$ or $p = 2$) or of type $D_{\dim(V_x)/2}$ (if $x = -1$ and $p \neq 2$) is the Springer representation associated with the unipotent element $x^{-1}g \in SO(V_x)$.

Assume that $\lambda_1^x, \ldots, \lambda_{2m+1}^x$ runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $k^*$. Note that $((\lambda_1^x \geq \lambda_3^x \geq \cdots \geq \lambda_{2m+1}^x), (\lambda_2^x \geq \lambda_4^x \geq \cdots \geq \lambda_{2m}^x))$ is a bipartition of $n$ of type 2, 2.

Now $g \mapsto \lambda(g)$ defines a map from $G$ onto the set of bipartitions of $n$ of type 2, 2. From the definitions we see that the fibres of this map are exactly the strata of $G$.

If $g \in G$ and $\lambda(g) = (\lambda_1, \lambda_2, \ldots, \lambda_{2m+1})$ then $\dim(B_g) = 2mn - \sum_{k \in [1, 2m]} (\lambda_1 + \lambda_2 + \cdots + \lambda_k)$ (with $B_g$ as in 0.3).

1.14. Assume that $G = SO(V)$ where $V$ is a $k$-vector space of dimension $2n \geq 2$ with a fixed nondegenerate quadratic form. We choose a sufficiently large $m \in \mathbb{N}$. We say that a bipartition $((\lambda_1 \geq \lambda_3 \geq \cdots \geq \lambda_{2m-1}), (\lambda_2 \geq \lambda_4 \geq \cdots \geq \lambda_{2m}))$ (with entries in $\mathbb{N}$) is of type $(e, e')$ if $\lambda_i \geq \lambda_{i+1} - e$ for $i = 1, 3, \ldots, 2m - 1$ and $\lambda_i \geq \lambda_{i+1} - e'$ for $i = 2, 4, \ldots, 2m - 2$.

Let $g \in G$. For any $x \in k^*$ let $V_x$ be the generalized $x$-eigenspace of $g : V \to V$. For any $x \in k^*$ such that $x^2 \neq 1$ let $\lambda_1^x \geq \lambda_2^x \geq \cdots \geq \lambda_{2m}^x$ be the sequence in $\mathbb{N}$ whose terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$ (a partition of dim $V_x$). For $x \in k^*$ such that $x^2 = 1$ let $\lambda_1^x, \lambda_2^x, \ldots, \lambda_{2m}^x$ be the sequence in $\mathbb{N}$ such that $((\lambda_1^x \geq \lambda_3^x \geq \cdots \geq \lambda_{2m-1}^x), (\lambda_2^x \geq \lambda_4^x \geq \cdots \geq \lambda_{2m}^x))$ is the bipartition of $\dim(V_x)/2$ such that the corresponding irreducible representation of the Weyl group of type $D_{\dim(V_x)/2}$ is the Springer representation attached to the unipotent element $x^{-1}g \in SO(V_x)$. This bipartition is of type 0, 2 if $x = 1$, $p \neq 2$, of type 2, 0 if $x = -1$, $p \neq 2$ and of type 2, 2 if $x = 1$, $p = 2$. Let $\lambda(g)$ be the sequence $\lambda(g)_1, \lambda(g)_2, \ldots, \lambda(g)_{2m+1}$ in $\mathbb{N}$ given by $\lambda(g)_j = \sum x^j \lambda_j^x$ where $x$ runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $k^*$.

Now $g \mapsto \lambda(g)$ defines a map from $G$ onto the set of bipartitions of $n$ of type 0, 4, 0, 4. From the definitions we see that the fibres of this map are exactly the strata
of $G$ (except for the fibre over a bipartition $((\lambda_1 \geq \lambda_3 \geq \cdots \geq \lambda_{2m-1}), (\lambda_2 \geq \lambda_4 \geq \cdots \geq \lambda_{2m}))$ with $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4, \ldots, \lambda_{2m-1} = \lambda_{2m}$ in which case the fibre is a union of two strata).

If $g \in G$ and $\lambda(g) = (\lambda_1, \lambda_2, \ldots, \lambda_2m)$ then $\dim(B_g) = (2m-1)n - \sum_{k \in [1,2m-1]}(\lambda_1 + \lambda_2 + \cdots + \lambda_k)$ (with $B_g$ as in 0.3).

1.15. Assume that $G = SO_5(k)$. Then $cl(W) = \{C_4, C_4^2, C', C'', \{1\}\}$ where $C_4$ consists of the elements of order 4 and $C', C''$ are the two conjugacy classes of reflections. The obvious map $cl(W) \rightarrow cl(W)/\sim$ is a bijection. One stratum is the union of all conjugacy classes of dimension 8 (it corresponds to $C_4$); one stratum is the union of all conjugacy classes of dimension 6 (it corresponds to $C_4^2$); each of the two conjugacy classes of dimension 4 is a stratum (these two strata correspond to $C', C''$). The unit element is a stratum (it corresponds to $\{1\}$). If $p \neq 2$ then one stratum is a semisimple class, the other strata contain unipotent elements; if $p = 2$ all strata contain unipotent elements.

1.16. Repeating the definition of sheets in a semisimple Lie algebra over $C$ (see [Bo]) one can define the sheets of $G$ as the maximal irreducible subsets of $G$ which are unions of conjugacy classes of fixed dimension. One can show that if $G = GL_n(k)$, the sheets of $G$ are the same as the strata of $G$, as described in 1.11. (In this case, the sheets of $G$, or rather their Lie algebra analogue, are described in [Pe]. They are smooth varieties.) This is not true for a general $G$ (the sheets of $G$ do not usually form a partition of $G$; the strata of $G$ are not always irreducible).

1.17. Let $ce(W)$ be the set of two-sided cells of $W$. For two conjugacy classes $\gamma, \gamma'$ of $G$ we write $\gamma \approx \gamma'$ if $\tilde{\phi}(g), \tilde{\phi}(\gamma')$ belong to the same two-sided cell of $W$. This is an equivalence relation on $cl(G)$. Let $cl(G)/\approx$ be the set of equivalence classes. Note that $\tilde{\phi}$ induces a bijection $cl(G)/\approx \leftrightarrow ce(W)$. (We use that for any two-sided cell of $W$ there exists $\rho \in R(W)$ which belongs to that two-sided cell, namely the special representation attached to the cell.) Similarly, for two conjugacy classes $C, C'$ of $W$ we write $C \approx C'$ if $\tilde{\Phi}(C), \tilde{\Phi}(C')$ belong to the same two-sided cell of $W$. This is an equivalence relation on $cl(W)$. Let $cl(W)/\approx$ be the set of equivalence classes. Note that $\tilde{\Phi}$ induces a bijection $cl(W)/\approx \leftrightarrow ce(W)$.

We define the bijection

\[(a) \quad cl(G)/\approx \leftrightarrow cl(W)/\approx\]

as the composition of the bijection $cl(G)/\approx \leftrightarrow ce(W)$ above with the inverse of the bijection $cl(W)/\approx \leftrightarrow ce(W)$ above. We see that $G$ is partitioned into finitely many subsets said to be special strata: a special stratum is the union of all conjugacy classes of $G$ in a fixed equivalence class for $\approx$; it is also a union of strata of $G$. Note that the set of special strata of $G$ is canonically in bijection with $ce(W)$. The intersection of a special stratum with the unipotent variety is exactly one special piece (in the sense of [L3]) of that unipotent variety.
2. A second approach

2.1. In this section we sketch another approach to defining the strata of $G$ in which the theory of Springer representations does not appear.

For $w \in W$ let

$$G_w = \{ g \in G; (B, gBg^{-1}) \in O_w \text{ for some $B \in B$} \}.$$ 

For $C \in cl(W)$ let

$$C_{\min} = \{ w \in C; \downarrow: C \to N \text{ reaches minimum at $w$} \}$$

and let $G_C = G_w$ where $w \in C_{\min}$. As pointed out in [L5, 0.2], from [L5, 1.2(a)] and [GP, 8.2.6(b)] it follows that $G_C$ is independent of the choice of $w$ in $C_{\min}$.

From [L5] it is known that $G_C$ contains unipotent elements; in particular, $G_C \neq \emptyset$. Clearly, $G_C$ is a union of conjugacy classes. Let

$$\gamma \in cl(G); \gamma \subseteq G_C \dim \gamma \gamma_C = \bigcup_{\gamma \in cl(G); \gamma \subseteq G_C, \dim \gamma = \delta_C \gamma} \gamma,$$

Then $G_C$ is $\neq \emptyset$, a union of conjugacy classes of fixed dimension, $\delta_C$. We have the following result.

**Theorem 2.2.** Assume that $p$ is not a bad prime for $G$.

(a) A subset of $G$ is a stratum if and only if it is of the form $G_C$ for some $C \in cl(W)$. In particular, we have $\cup_{C \in cl(W)} G_C = G$; moreover, if $C, C' \in cl(W)$ then $G_C = G_{C'}$ if $C \sim C'$ and $G_C \cap G_{C'} = \emptyset$ if $C \not\sim C'$.

(b) Let $\gamma \in cl(G), C \in cl(W)$. The equivalence class of $\gamma$ under $\sim$ corresponds under the bijection 0.2 to the equivalence class of $C$ under $\sim$ if and only if $\gamma \subseteq G_C$.

We can assume that $G$ is almost simple and that $k$ is an algebraic closure of a finite field. The proof in the case of exceptional groups is reduced in 2.3 to a computer calculation. The proof for classical groups, which is based on combining the techniques of [L5], [L6] and [L8], will be given elsewhere. It is likely that the assumption on $p$ can be removed.

2.3. In this subsection we assume that $k$ is an algebraic closure of a finite field $F_q$ and that $G$ is simply connected, defined and split over $F_q$ with Frobenius map $F : G \to G$. Let $\gamma$ be an $F$-stable conjugacy class of $G$. Let $\gamma' = \{ g_s; g \in \gamma \}$, an $F$-stable semisimple conjugacy class in $G$. For every $s \in \gamma'$ let $\gamma(s) = \{ u \in Z_G(s); u \text{ unipotent, } us \in \gamma \}$, a unipotent conjugacy class of $Z_G(s)$. We fix $s_0 \in \gamma'^F$ and we set $H = Z_G(s_0)$, $\gamma_0 = \gamma(s_0)$. Let $W_H$ be the Weyl group of $H$. As in
1.1 we can regard $W_H$ as a subgroup of $W$ (the imbedding of $W_H$ into $W$ is canonical up to composition with an inner automorphism of $W$). By replacing if necessary $F$ by a power of $F$, we can assume that $H$ contains a maximal torus which is defined and split over $F_q$. For any $F$-stable maximal torus $T$ of $G$, $R^1_T$ is the virtual representation of $G^F$ defined as in [DL, 1.20] (with $\theta = 1$ and with $B$ omitted from notation). Replacing $T, G$ by $T', H$ where $T'$ is an $F$-stable maximal torus of $H$, we obtain a virtual representation $R^1_{T', H}$ of $H^F$. For any $z \in W$ we denote by $R^1_z$ the virtual representation $R^1_T$ of $G^F$ where $T$ is an $F$-stable maximal torus of $G$ of type given by the conjugacy class of $z$ in $W$. For any $z' \in W_H$ we denote by $R^1_{z', H}$ the virtual representation $R^1_{T', H}$ of $H^F$ where $T'$ is an $F$-stable maximal torus of $H$ of type given by the conjugacy class of $z'$ in $W_H$. For $E' \in \mathrm{Irr} W$ we set $R_{E'} = |W|^{-1} \sum_{y \in W} \mathrm{tr}(y, E') R^1_y$. Then for any $z \in W$ we have $R^1_z = \sum_{E' \in \mathrm{Irr} W} \mathrm{tr}(z, E') R_{E'}$.

Let $w \in W$. We show:

$$ |\{(g, B) \in \gamma^F \times B^F; (B, gBg^{-1}) \in O_w\}| = |G^F||H^F|^{-1} \sum_{E \in \mathrm{Irr} W, E' \in \mathrm{Irr} W, E'' \in \mathrm{Irr} W_H, y} \mathrm{tr}(T_w, E_q)(\rho_E, R_{E'}) $$

$$ \times (E'|_{W_H} : E'')|Z_{W_H}(y)|^{-1} \mathrm{tr}(y, E'') \sum_{u \in \gamma_0^F} \mathrm{tr}(u, R^1_{y, H}) $$

where $y$ runs over a set of representatives for the conjugacy classes in $W_H$ and $T_w, E_q, \rho_E$ are as in [L5, 1.2]. Let $N$ be the left hand side of (a). As in [L5, 1.2(c)] we see that

$$ N = \sum_{E \in \mathrm{Irr} W} \mathrm{tr}(T_w, E_q) A_E $$

with

$$ A_E = |G^F|^{-1} \sum_{g \in \gamma^F} \sum_{\mathcal{T}} |\mathcal{T}^F|(\rho_E, R^1_{\mathcal{T}}) \mathrm{tr}(g, R^1_{\mathcal{T}}) $$

where $\mathcal{T}$ runs over all maximal tori of $G$ defined over $F_q$. We have

$$ A_E = |G^F|^{-1} \sum_{s \in \gamma^F, u \in \gamma(s)^F} \sum_{\mathcal{T}} |\mathcal{T}^F|(\rho_E, R^1_{\mathcal{T}}) \mathrm{tr}(su, R^1_{\mathcal{T}}) $$

$$ = |H^F|^{-1} \sum_{u \in \gamma_0^F} \sum_{\mathcal{T}} |\mathcal{T}^F|(\rho_E, R^1_{\mathcal{T}}) \mathrm{tr}(su, R^1_{\mathcal{T}}). $$

By [DL, 4.2] we have

$$ \mathrm{tr}(s_0 u, R^1_{\mathcal{T}}) = |H^F|^{-1} \sum_{x \in G^F : x^{-1} \mathcal{T} x \subset H} \mathrm{tr}(u, R^1_{x^{-1} \mathcal{T} x, H}) $$
hence

\[ A_E = |H^F|^{-2} \sum_{u \in \gamma_0} \sum_{\mathcal{T}} |T^F| \left( \rho_E, R_{T^F}^1 \right) \sum_{x \in H^F : x^{-1} T x \subset H} \text{tr}(u, R_{x^{-1} T x, H}^1) \]

\[ = |G^F| |H^F|^{-2} \sum_{T' \subset H} |T'^F| \left( \rho_E, R_{T'^F}^1 \right) \sum_{u \in \gamma_0} \text{tr}(u, R_{T', H}^1) \]

where \( T' \) runs over the maximal tori of \( H \) defined over \( F_q \). Using the classification of maximal tori of \( H \) defined over \( F_q \) we obtain

\[ A_E = |G^F| |H^F|^{-1} |W_H|^{-1} \sum_{z \in W_H} \left( \rho_E, R_z^1 \right) \sum_{u \in \gamma_0} \text{tr}(u, R_{z, H}^1) \]

\[ = |G^F| |H^F|^{-1} |W_H|^{-1} \sum_{z \in W_H} \sum_{E' \in \text{Irr} W} \text{tr}(z, E')(\rho_E, R_{E'}) \sum_{u \in \gamma_0} \text{tr}(u, R_{z, H}^1). \]

This clearly implies (a).

Now assume that \( G \) is almost simple of exceptional type and that \( w \) has minimal length in its conjugacy class in \( W \). We can also assume that \( q - 1 \) is sufficiently divisible. Then the right hand side of (a) can be explicitly determined using a computer. Indeed it is an entry of the product of several large matrices whose entries are explicitly known. In particular the quantities \( \text{tr}(T_w, E_q) \) (known from the works of Geck and Geck-Michel, see [GP, 11.5.11]) are available through the CHEVIE package [CH]. The quantities \( (\rho_E, R_{E'}) \) are coefficients of the nonabelian Fourier transform in [L1, 4.15]. The quantities \( (E'|_{W_H} : E'') \) are available from the induction tables in the CHEVIE package. The quantities \( \text{tr}(y, E'') \) are available through the CHEVIE package. The quantities \( \text{tr}(u, R_{y, H}^1) \) are Green functions; I thank Frank Lübeck for providing to me tables of Green functions for groups of rank \( \leq 8 \) in GAP format. I also thank Gongqin Li for her help with programming in GAP to perform the actual computation using these data.

Thus the number \( \{|(g, B) \in \gamma^F \times B^F ; (B, gBg^{-1}) \in O_w \} | \) is explicitly computable. It turns out that it is a polynomial in \( q \). Note that \( \{|(g, B) \in \gamma \times B ; (B, gBg^{-1}) \in O_w \} \) is nonempty if and only if this polynomial is non zero. Thus the condition that \( \gamma \subset G_w \) can be tested. This can be used to check that Theorem 2.2 holds for exceptional groups.

2.4. If \( C \) is the conjugacy class containing the Coxeter elements of \( W \) then \( G_C = \bigcup_{\gamma \in \mathcal{C}} \mathcal{C}_\gamma \) is the union of all conjugacy classes of dimension \( \dim G - \text{rk}(G) \), see [St].

2.5. Let us replace \( G \) by a possibly disconnected reductive group \( G' \) over \( k \) with identity component \( G \). Let \( D \) be a connected component of \( G' \). The definitions in 2.1 can be extended to the present case as follows. Let \( cl(D) \) be the set of \( G \)-conjugacy classes in \( D \). Let \( cl_D W \) be the set of twisted conjugacy classes in \( W \) (as in [L8,0.1]; the twisting depends on \( D \)). For \( w \in W \) let

\[ D_w = \{ g \in D ; (B, gBg^{-1}) \in O_w \text{ for some } B \in \mathcal{B} \}. \]
For \( C \in \text{cl}_D(W) \) let
\[
C_{\text{min}} = \{ w \in C; \underline{\lambda} : C \to \mathbb{N} \text{ reaches minimum at } w \}.
\]
and let \( D_C = D_w \) where \( w \in C_{\text{min}} \). This is again independent of the choice of \( w \) in \( C_{\text{min}} \). One can show that \( D_C \neq \emptyset \). Clearly, \( D_C \) is a union of \( G \)-conjugacy classes. Let
\[
\delta_C = \min_{\gamma \in \text{cl}(D)} \dim \gamma,
\]
\[
\square D_C = \bigcup_{\gamma \in \text{cl}(D); \gamma \subseteq D_C, \dim \gamma = \delta_C} \gamma.
\]
Then \( \square D_C \neq \emptyset \), a union of \( G \)-conjugacy classes of fixed dimension, \( \delta_C \). The following variant of Theorem 2.2 will be proved elsewhere (some partial results in this direction are contained in [L8]).

(a) Assume that \( p \) is either 1 or a good prime for \( G \), not dividing \( |G'/G| \). We have \( \cup_{C \in \text{cl}_D(W)} \square D_C = D \). If \( C, C' \in \text{cl}_D(W) \), then \( \square D_C \cap \square D_{C'} \) are either equal or disjoint. If \( \gamma, \gamma' \in \text{cl}(D) \) we say that \( \gamma \sim \gamma' \) if for some \( C \in \text{cl}_D(W) \) we have \( \gamma \subseteq \square D_C \) and \( \gamma' \subseteq \square D_{C'} \); this is an equivalence relation on \( \text{cl}(D) \) (the set of equivalence classes is denoted by \( \text{cl}(D)/\sim \)). If \( C, C' \in \text{cl}_D(W) \) we say that \( C \sim C' \) if \( \square D_C = \square D_{C'} \); this is an equivalence relation on \( \text{cl}_D(W) \) (the set of equivalence classes is denoted by \( \text{cl}_D(W)/\sim \)). The correspondence \( \gamma \in \square D_C \) induces a bijection \( \text{cl}(D)/\sim \leftrightarrow \text{cl}_D(W)/\sim \).

Again, it is likely that the assumption on \( p \) can be removed.

3. Variants

3.1. In this section we fix a prime number \( l \) invertible in \( k \). For any algebraic variety \( X \) over \( k \) and \( i \in \mathbb{Z} \) we set \( H_i(X, \mathbb{Q}_l) = \text{Hom}(H^i_c(X, \mathbb{Q}_l), \mathbb{Q}_l) \) where \( H^i_c(X, \mathbb{Q}_l) \) is the \( l \)-adic cohomology with compact support of \( X \).

Let \( g \in G \). Let \( d = \dim B_g \). The imbedding \( h_g : B_g \to B \) induces a linear map \( h_{g*} : H_{2d}(B_g, \mathbb{Q}_l) \to H_{2d}(B, \mathbb{Q}_l) \). Now \( H^2_{c}(B_g, \mathbb{Q}_l), H^2_{c}(B, \mathbb{Q}_l) \) carry natural \( W \)-actions, see [L2], and this induces natural \( W \)-actions on \( H_{2d}(B_g, \mathbb{Q}_l), H_{2d}(B, \mathbb{Q}_l) \) which are compatible with \( h_* \). Hence \( W \) acts naturally on the subspace \( h_{g*}(H_{2d}(B_g, \mathbb{Q}_l)) \) of \( H_{2d}(B, \mathbb{Q}_l) \).

The following result gives an alternative description of the map \( g \mapsto \tilde{\rho} \) (in 1.1) from \( G \) to \( \text{Irr}W \).

**Proposition 3.2.** The \( W \)-module \( h_{g*}(H_{2d}(B_g, \mathbb{Q}_l)) \) of \( H_{2d}(B, \mathbb{Q}_l) \) (see 3.1) is irreducible and is isomorphic (after extension of scalars) to the \( W \)-module \( \tilde{\rho} \) associated to \( g \) in 1.1.

First we note that \( h_{g*}(H_{2d}(B_g, \mathbb{Q}_l)) \neq 0 \); indeed it is clear that for any irreducible component \( E \) of \( B_g \) (necessarily of dimension \( d \)) the image of the fundamental class of \( E \) under \( h_{g*} \) is nonzero (we ignore Tate twists). Let \( B' \) be the variety
of Borel subgroups of $Z_G(g_u)$. Let $B_{g_u}' = \{ \beta \in B'; g_u \in \beta \}$. Then $\dim B' = d$ and $W_{g}$ (see 1.1) acts naturally on $H_{2d}(B_{g_u}', Q_{l})$; from the definitions, the $W$-module $H_{2d}(B_{g}, Q_{l})$ is isomorphic to $\text{Ind}_{W_{g}}^{W} H_{2d}(B_{g_u}', Q_{l})$. Let $\tilde{\rho}'$ be the $W$-module obtained from $\tilde{\rho}$ by extension of scalars from $Q$ to $Q_{l}$. From the definitions, we have $n_{\tilde{\rho}'} = d$ and $\text{Ind}_{W_{g}}^{W} H_{2d}(B_{g_u}', Q_{l}) \cong \tilde{\rho}' \oplus \mathcal{T}$ where $\mathcal{T}$ is a $W$-module such that any irreducible constituent $\tau$ of $\mathcal{T}$ satisfies $n_{\tau} > d$. We can identify $\tilde{\rho}'$, $\mathcal{T}$ with complementary $W$-submodules of $H_{2d}(B_{g}, Q_{l})$. From the definition of $\mathcal{T}$ we see that $h_{g*}(\mathcal{T}) = 0$. Hence $h_{g*}(H_{2d}(B_{g}, Q_{l})) = h_{g*}(\tilde{\rho}') \neq 0$. Hence $h_{g*}(\tilde{\rho}')$ is an irreducible $W$-module. The proposition follows.

3.3. In this subsection we assume that $G$ is semisimple simply connected. Let $K$ be the field of formal power series $k((\epsilon))$ and let $\hat{G} = G(K)$. Let $\hat{B}$ be the set of Iwahori subgroups of $\hat{G}$ viewed as an increasing union of projective algebraic varieties over $k$. Let $\hat{W}$ be the affine Weyl group associated to $\hat{G}$ viewed as an infinite Coxeter group. Let $G(K)_{rsc}$ be the set of all $g \in G(K)$ which are compact (that is such that $\hat{B}_{g} = \{ B \in \hat{B}; g \in B \}$ is nonempty) and regular semisimple. If $g \in G(K)_{rsc}$ then $\hat{B}_{g}$ a union of projective algebraic varieties of fixed dimension $d = d_{g}$ (see [KL] for a closely related result) hence the homology space $H_{2d}(\hat{B}_{g}, Q_{l})$ is well defined and it carries a natural $\hat{W}$-action (see [L9]). Similarly the homology space $H_{2d}(\hat{B}, Q_{l})$ is well defined and it carries a natural $\hat{W}$-action. The imbedding $h_{g} : \hat{B}_{g} \to \hat{B}$ induces a linear map $h_{g*} : H_{2d}(\hat{B}_{g}, Q_{l}) \to H_{2d}(\hat{B}, Q_{l})$ which is compatible with the $\hat{W}$-actions. Hence $\hat{W}$ acts naturally on the (finite dimensional) subspace $E_{g} := h_{g*}(H_{2d}(\hat{B}_{g}, Q_{l}))$ of $H_{2d}(\hat{B}, Q_{l})$, but this action is not irreducible in general. Note that $E_{g}$ is the subspace of $H_{2d}(\hat{B}, Q_{l})$ spanned by the images of the fundamental classes of the irreducible components of $\hat{B}_{g}, Q_{l}$ (we ignore Tate twists) hence is $\neq 0$. For $g, g' \in G(K)_{rsc}$ we say that $g \sim g'$ if $d_{g} = d_{g'}$ and $E_{g} = E_{g'}$. This is an equivalence relation on $G(K)_{rsc}$. The equivalence classes for $\sim$ are called the strata of $G(K)_{rsc}$. Note that $G(K)_{rsc}$ is a union of countably many strata and each stratum is a union of conjugacy classes of $G(K)$ contained in $G(K)_{rsc}$.

3.4. In this subsection we assume that $k = C$. Let $\mathfrak{g}$ be the Lie algebra of $G$. We identify $B$ with the variety of Borel subalgebras of $\mathfrak{g}$. For any $\xi \in \mathfrak{g}$ let $B_{\xi} = \{ b \in B; \xi \in b \}$ and let $d = \dim B_{\xi}$. The subspace of $H_{2d}(B, Q_{l})$ spanned by the images of the fundamental classes of the irreducible components of $B_{\xi}$ is an irreducible $W$-module denoted by $\tau_{\xi}$. We also denote by $\tau_x$ the corresponding $W$-module over $Q$. Thus we have a well defined map $\mathfrak{g} \to \text{Irr} W$, $\xi \mapsto \tau_{\xi}$. The nonempty fibres of this map are called the strata of $\mathfrak{g}$. Each stratum of $\mathfrak{g}$ are unions of adjoint orbits of fixed dimension; exactly one of these orbits is nilpotent. The image of the map $\xi \mapsto \tau_{\xi}$ is exactly $\mathcal{X}^{1}(W)$ (see 1.3).

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