Entanglement Efficiencies in $\mathcal{PT}$-Symmetric Quantum Mechanics

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The degree of entanglement is determined for an arbitrary state of a broad class of $\mathcal{PT}$-symmetric bipartite composite systems. Subsequently we quantify the rate with which entangled states are generated and show that this rate can be characterized by a small set of parameters. These relations allow one in principle to improve the ability of these systems to entangle states. It is also noticed that many relations resemble corresponding ones in conventional quantum mechanics.

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I. INTRODUCTION

In conventional quantum mechanics, one demands that the Hamiltonian $H$ generating the time evolution has a real spectrum and that the corresponding time evolution operator $U$ is unitary. These conditions are fulfilled if the Hamiltonian is Hermitian, i.e. $H = H^\dagger$, which is usually considered an axiom of quantum mechanics. However, the condition of Hermiticity can be weakened. In the class of so-called $\mathcal{PT}$-symmetric Hamiltonians, one can ensure real eigenvalues and a unitary time evolution even for explicitly non-Hermitian Hamiltonians [2, 3]. A thorough review of the foundations of $\mathcal{PT}$-symmetric quantum mechanics can be found in [1].

In the following we will investigate entanglement phenomena in bipartite systems within this framework. An entangled state is a quantum state where two or more degrees of freedom are intertwined, so that they are not independent anymore. In this context a couple of historical discussions took place and gave deep insights into the nature of quantum mechanics, like the Einstein-Podolsky-Rosen paradox [8]. These states have a wide range of applications, for example in quantum information theory and quantum computing. The question of entanglement generation and entanglement efficiencies for conventional quantum mechanics was addressed earlier in [7].

Using $\mathcal{PT}$-symmetric quantum mechanics to describe entanglement phenomena was also done in [13]. Especially for the case of bipartite systems, relations for the degree of entanglement of given states and the entanglement capability of certain systems could be found. Although we confirmed many of these results, we have some different and new findings. For a particular initial state, as well as for general states, we give relations between the efficiency of the system to generate entangled states and the parameters of the Hamiltonian describing the dynamics of the system.

We will first introduce a measure for entanglement and a certain class of $\mathcal{PT}$-symmetric Hamiltonians. We then quantify the degree of entanglement of an arbitrary and of a generalized Einstein-Podolsky-Rosen state. Subsequently, we are dealing with the question of entanglement generation of a particular $\mathcal{PT}$-symmetric state and generalize for arbitrary states. The question we try to answer is how to characterize the rate of entanglement generation for this class of systems.

II. BIPARTITE SYSTEMS

A. A measure for entanglement

Let $\mathcal{H}_i = 1, \ldots, N$ denote a set of Hilbert spaces. We call a state of a composite system $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ entangled, if there is no decomposition of the form $|\Psi\rangle = |\chi_1\rangle \otimes \cdots \otimes |\chi_N\rangle$ with suitable $|\chi_i\rangle \in \mathcal{H}_i$. In the following we restrict ourselves to bipartite systems, i.e. $N = 2$.

A measure of entanglement are the entropies

$$E(\Psi) = -\text{tr}_1 (\rho_1 \log \rho_1) = -\text{tr}_2 (\rho_2 \log \rho_2),$$

where $\rho_1 = \rho_2 \rho$ and $\rho_2 = \text{tr}_1 \rho$ are the reduced density matrices, $\rho = |\Psi\rangle\langle\Psi|$ is the density matrix itself and $\text{tr}_i$ denotes the partial trace over the $i$th subsystem [12]. Here $E(\Psi) \in [0, 1]$ and $E(\Psi) = 0$ if and only if $|\Psi\rangle \in \mathcal{H}$ is not entangled.

B. General entanglement content

Consider the Hamiltonian

$$H = \begin{pmatrix} r e^{i\Theta} & s \\ s & r e^{-i\Theta} \end{pmatrix}, \quad r, s, \Theta \in \mathbb{R},$$

with $s^2 \geq r^2 \sin^2 \Theta$. Observe that $H$ is in general not represented by a Hermitian matrix, $H^\dagger \neq H$. But it
obeys $\mathcal{PT}$-symmetry with
\[ \mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \] (3)
and $\mathcal{T}$ complex conjugation [4] (see also erratum [5]), i.e. $[H, \mathcal{PT}] = 0$.

Define $\sin \varphi \equiv r/s \cdot \sin \Theta$ with $\varphi \in [-\pi/2, \pi/2]$. The simultaneous eigenstates of $H$ and $\mathcal{PT}$ are given by
\[
|\phi_+\rangle = \frac{1}{\sqrt{2 \cos \varphi}} \left( e^{i \varphi / 2} |\phi_1\rangle + e^{-i \varphi / 2} |\phi_2\rangle \right),
\]
\[
|\phi_-\rangle = \frac{1}{\sqrt{2 \cos \varphi}} \left( e^{-i \varphi / 2} |\phi_1\rangle - e^{i \varphi / 2} |\phi_2\rangle \right),
\] (4)
with eigenvalues $E_{\pm} = r \cos \Theta \pm s \cos \varphi$. These eigenstates are orthonormal with respect to the positive $\mathcal{CPT}$-inner product $(\langle \Phi | \Phi \rangle)_{\mathcal{CPT}}$ with $(\langle \Phi | \mathcal{CPT} | \Phi \rangle)_{\mathcal{T}}$, where $(\langle \Phi | \mathcal{CPT} | \Phi \rangle)_{\mathcal{T}}$ is of the form of (2) with $\sin \varphi_i \equiv r_i/s_i \cdot \sin \Theta_i$, $i = 1, 2$. The C operator reads $C_1 \otimes C_2$, where $C_i$ is of the form of (5) with corresponding parameters.

In the following we will consider the composite system $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ of Hilbert spaces $\mathcal{H}_i = \mathbb{C}^2$ with dynamics governed by the Hamiltonian $H = H_1 \otimes H_2$, where $H_i$ is of the form of (2) with $r_i$, $s_i$, $\Theta_i \in \mathbb{R}$, $s_i^2 \geq r_i^2 \sin^2 \Theta_i$, $\sin \varphi_i \equiv r_i/s_i \cdot \sin \Theta_i$, $i = 1, 2$. Our respective $\mathcal{C}$ operator reads $C_1 \otimes C_2$, where $C_i$ is the form of (5) with $[H, \mathcal{C}] = 0$.

Now consider the density matrix $\rho = \rho(t) = |\Psi(t)\rangle \langle \Psi(t)|_{\mathcal{CPT}}$ of a system in the state
\[
|\Psi(t)\rangle = \alpha(t)|\phi_+\rangle_1 \otimes |\phi_+\rangle_2 + \beta(t)|\phi_+\rangle_1 \otimes |\phi_-\rangle_2
\]
\[
+ \gamma(t)|\phi_-\rangle_1 \otimes |\phi_+\rangle_2 + \delta(t)|\phi_-\rangle_1 \otimes |\phi_-\rangle_2
\] (7)
with $|\Psi(t)\rangle \in \mathcal{H}$. The eigenvalues of $\rho_1(t) = \text{tr}_2 \rho(t)$ are given by $\lambda_{\pm}(t) = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \Xi(t)}$, where $\Xi(t) = 4|\alpha(t)\delta(t) - \beta(t)\gamma(t)|^2$, which is a simplification of the result in [13].

Hence the entanglement content is
\[
E(t) \equiv E(\Psi(t))
\]
\[
= -\lambda_+ (t) \log_2 \lambda_+ (t) - \lambda_- (t) \log_2 \lambda_- (t).
\] (8)
Note that $|\Psi(t)\rangle$ only separates if $\alpha(t_0)\delta(t_0) = \beta(t_0)\gamma(t_0)$ for $t = t_0$.

Our results are in disagreement with [13] mainly due to the author’s use of non-$\mathcal{CPT}$-normalized states.

We can consider the entanglement content
\[
E(\Psi^-) = -\lambda_+ \log_2 \lambda_+ - \lambda_- \log_2 \lambda_-.
\] (12)
as a function of $\varphi_1$ and $\varphi_2$, i.e. of the Hamiltonians $H_1$ and $H_2$. The result can be seen in figure 1. Note the cases $\varphi_1 = \varphi_2$ or $\varphi_1 + \varphi_2 = \pm \pi$, where (10) has the form of a $\mathcal{PT}$-symmetric Bell state.
III. ENTANGLEMENT GENERATION

A. Entanglement capability

We investigate the question how to increase the capability of a system to entangle states. More precisely we want to understand the dependencies of the entanglement capability $\Gamma(t) \equiv \frac{dE(t)}{dt}$ of the parameters of the system. In order to determine the time evolution operator of the Hamiltonian $H_1 \otimes H_2$ (see section II B) define

$$\vec{n}_i \equiv \frac{2}{\omega_i} (s_i, 0, ir_i \sin \Theta_i)^\dagger, \quad \omega_i \equiv 2s_i \cos \varphi_i.$$ (13)

Rewrite $H$ as

$$H = \left( r_1 \cos \Theta_1 \mathbb{1} + \frac{\omega_1}{2} \vec{n}_1 \cdot \vec{\sigma} \right) \otimes \left( r_2 \cos \Theta_2 \mathbb{1} + \frac{\omega_2}{2} \vec{n}_2 \cdot \vec{\sigma} \right),$$ (14)

with $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^\dagger$, $\sigma_i$ denoting the Pauli matrices and $\mathbb{1}$ the $2 \times 2$ identity matrix. Expanding the expression for $H$ yields four terms of which only one can generate entanglement. We restrict ourselves to this term and define $\hat{H} = \omega_1 \omega_2 / 4 \cdot (\vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma})$ resulting in a time evolution operator $U(t) = \exp(-i\hat{H}t/\hbar)$ given by

$$U(t) = \cos \left( \frac{\omega_1 \omega_2 t}{4\hbar} \right) (\mathbb{1} \otimes 1) - i \sin \left( \frac{\omega_1 \omega_2 t}{4\hbar} \right) (\vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma}).$$ (15)

Consider now a simple initial state, namely $|\Psi(t = 0)\rangle = |\uparrow\rangle \otimes |\uparrow\rangle$, and apply $U(t)$ to find

$$|\Psi(t)\rangle = \alpha(t) |\uparrow\rangle \otimes |\uparrow\rangle + \beta(t) |\uparrow\rangle \otimes |\downarrow\rangle$$
$$+ \gamma(t) |\downarrow\rangle \otimes |\uparrow\rangle + \delta(t) |\downarrow\rangle \otimes |\downarrow\rangle$$ (16)

with

$$\alpha(t) = \cos \left( \frac{\omega_1 \omega_2 t}{4\hbar} \right) + \frac{4ir_1 r_2 \sin \Theta_1 \sin \Theta_2}{\omega_1 \omega_2} \sin \left( \frac{\omega_1 \omega_2 t}{4\hbar} \right),$$

$$\beta(t) = \frac{4s_2 r_1 \sin \Theta_1}{\omega_1 \omega_2} \sin \left( \frac{\omega_1 \omega_2 t}{4\hbar} \right),$$

$$\gamma(t) = \frac{4s_1 r_2 \sin \Theta_2}{\omega_1 \omega_2} \sin \left( \frac{\omega_1 \omega_2 t}{4\hbar} \right),$$

$$\delta(t) = -\frac{4is_1 s_2}{\omega_1 \omega_2} \sin \left( \frac{\omega_1 \omega_2 t}{4\hbar} \right).$$ (17)

Up to here our results are in agreement with [13]. We now find the entanglement content to be given by

$$E(t) = -\lambda_+(t) \log_2 \lambda_+(t) - \lambda_-(t) \log_2 \lambda_-(t)$$ (18)

with

$$\lambda_{\pm}(t) = \frac{1}{2} \pm \frac{1}{2} \cos \left( \frac{\omega_1 \omega_2 t}{2\hbar} \right).$$ (19)

We find the entanglement rate to be

$$\Gamma(t) = \frac{dE(t)}{dt} = \frac{\omega_1 \omega_2}{\hbar} \sin \left( \frac{\omega_1 \omega_2 t}{2\hbar} \right) \log_{16} \cot^2 \left( \frac{\omega_1 \omega_2 t}{2\hbar} \right).$$ (20)

The maximal entanglement capability is $\Gamma_{\max} = \max_t \Gamma(t) = 0.4781 \omega_1 \omega_2 / \hbar$. Therefore by changing $\omega_i$ due to an adjustment of the parameters of the Hamiltonians $H_i$, we can control $\Gamma_{\max}$. The typical time dependency of the entanglement content and entanglement rate can be seen in figure 2.

B. Efficiency of general systems

If we have a given Hamiltonian one may ask how to maximize the entanglement rate of the system. We want to generalize some results of conventional quantum mechanics addressed in [7] to the $\mathcal{PT}$-symmetric case.

Consider an arbitrary $\mathcal{PT}$-symmetric two qubit system. Using the Schmidt-decomposition theorem we rewrite an arbitrary state $|\Psi(t)\rangle \in \mathcal{H}_1 \times \mathcal{H}_2$ as

$$|\Psi(t)\rangle = \sqrt{p(t)} |\varphi_\uparrow\rangle \otimes |\chi\rangle$$
$$+ e^{i\alpha} \sqrt{1-p(t)} |\varphi_\downarrow\rangle \otimes |\chi\rangle$$ (21)

with $p \in [0, 1]$ as Schmidt-coefficient and $\langle \varphi_\uparrow |\varphi_\uparrow\rangle_{\mathcal{CPT}} = \langle \chi |\chi\rangle_{\mathcal{CPT}} = 0$ as Schmidt-vectors. The entanglement content is given by

$$E(t) = -p(t) \log_2 p(t) - (1-p(t)) \log_2 (1-p(t))$$ (22)

and the entanglement rate factorizes in two terms, i.e. $\Gamma(t) = dE(t)/dp(t) \times dp(t)/dt$, where

$$\frac{dE(t)}{dp(t)} = \frac{2}{\log 2} \arctanh (1-2p(t)).$$ (23)

After choosing the phase $\alpha$ appropriately the evolution of the Schmidt-coefficient is determined by the differential equation
\[
\frac{dp(t)}{dt} = \frac{2}{\hbar} \sqrt{p(t)(1-p(t))} \times |\\langle \phi_t^r |\mathcal{CPT} \otimes \langle \chi_t^r |\mathcal{CPT} H (|\varphi_t^r \rangle \otimes |\chi_t^r \rangle) |.
\]

This relation is known from conventional quantum mechanics, but also holds for \(\mathcal{PT}\)-symmetric systems. Define
\[
\hbar \Omega = \max_{\|x\|=1} |\\langle \phi^r_t |\mathcal{CPT} \otimes \langle \chi_t^r |\mathcal{CPT} H (|\varphi_t^r \rangle \otimes |\chi_t^r \rangle) |.
\]

Then the time evolution of \(p(t)\) for an optimally prepared setup, i.e. a setup forcing the qubit states to be optimal at every instant of time
\[
|\\langle \phi_t^r |\mathcal{CPT} \otimes \langle \chi_t^r |\mathcal{CPT} H (|\varphi_t^r \rangle \otimes |\chi_t^r \rangle) | = \hbar \Omega
\]

for all \(t\), follows from the differential equation
\[
\frac{dp_{opt}(t)}{dt} = 2\Omega \sqrt{p_{opt}(t)(1-p_{opt}(t))}.
\]

We find \(p_{opt}(t) = \sin^2(\Omega t + \delta_0)\) with an integration constant \(\delta_0\), which is in agreement with results from conventional quantum mechanics in [7] (see also [13]). Hence we can characterize the entanglement rate of an optimally prepared setup completely in terms of \(\Omega\) via
\[
\Gamma_{opt}(t) = \frac{d\psi_{opt}(t)}{dt} \cdot \frac{dE(t)}{dp(t)}\bigg|_{p_{opt}(t)} = \frac{2\Omega}{\log 2} \arctanh (\cos (2\Omega t + 2\delta_0)) \sin (2\Omega t + 2\delta_0).
\]

and find for the maximal rate \(\Gamma_{max} = \max_t \Gamma_{opt}(t) = 1.9123\Omega\). The parameter \(\Omega\) is the critical value one needs to maximize to efficiently entangle states. We remark that this result also holds in conventional quantum mechanics, where \(\Omega\) is defined with conventional conjugates instead of \(\mathcal{CPT}\)-conjugates.

\[\text{IV. CONCLUSIONS}\]

For the case of a state from a bipartite system we determined the degree of entanglement and saw the emergence of symmetrical patterns (see figure 1) in the case of the Einstein-Podolsky-Rosen state. We quantified the capability of a given \(\mathcal{PT}\)-symmetric system to generate entangled states in terms of the parameters of the Hamiltonian. Their ability to entangle states can be described by the parameters \(\omega_{i=1,2}\) in (13) or in general by \(\Omega\) in (25). Many relations are similar to the corresponding ones in conventional quantum mechanics after replacing the usual inner product with the \(\mathcal{CPT}\)-inner product. However, these results are not obvious and need to be checked. For example, the recent discussion of the quantum brachistochrone problem showed that \(\mathcal{PT}\)-symmetric quantum mechanics can give some surprising results [6, 9–11].

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