Evolving convex curves by a generalized length-preserving flow

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Abstract This paper deals with a generalized length-preserving flow for convex curves in the plane. It is shown that the flow exists globally and deforms convex curves into circles as time tends to infinity.

Keywords convex curve, curvature flow, length-preserving, quasilinear parabolic equation

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1 Introduction

Let $X(\phi,t) : S^1 \times [0, \omega) \to \mathbb{R}^2$ be a family of smooth and closed curves in the plane. Denote by $\kappa(\phi, t)$ the (relative) curvature with respect to the Frenet frame $\{T(\phi, t), N(\phi, t)\}$ of the curve $X(\phi, t)$ at time $t$, where $T(\phi, t)$ and $N(\phi, t)$ are the unit tangent vector and inward unit normal vector, respectively. Suppose that $X_0(\phi)$ is a smooth and convex curve, where we call a curve convex if it is closed, embedded and has positive curvature everywhere. In this paper we consider a generalized length-preserving curvature flow for convex curves:

\[
\begin{align*}
\frac{\partial X}{\partial t}(\phi, t) &= \left[ F(\kappa(\phi, t)) - \frac{1}{2\pi} \int_0^L F(\kappa(s, t))\kappa(s, t)ds \right] N(\phi, t) \quad \text{in } S^1 \times (0, \omega) \\
X(\phi, 0) &= X_0(\phi) \quad \text{on } S^1,
\end{align*}
\]

(1.1)

where $s$ stands for the arc length parameter of the evolving curve $X(\phi, t)$, $L = L(t)$ its length and $F(\cdot)$ a smooth real function on $(0, +\infty)$.

In recent decades, curvature flows for plane curves have received a lot of attention. Besides the well known curve shortening flow (see [8], [9], [11] and [17]) and its generalizations

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(see [2, 3]), there are models of nonlocal flows as well, such as length-preserving flows by Pan-Yang [27] and Ma-Zhu [23] and area-preserving flows by Gage [10], Ma-Cheng [22] and Mao-Pan-Wang [25]. Lin and Tsai [20] summarised the previews length-preserving or area-preserving flows as the so called $\kappa^\alpha$-type and $\frac{1}{\kappa^\alpha}$-type nonlocal flows, where $\alpha > 0$ is a constant. Later, Wang and Tsai [28] proved that $\kappa^\alpha$-type length-preserving or area-preserving flows for convex curves exist globally and drive the evolving curve into circles as $t \to +\infty$. The $\alpha$-homogeneity of $F$ plays an essential role in Wang-Tsai’s research. The higher dimensional model for hypersurfaces in Euclidean space with $\alpha$-homogeneity assumption of $F$ has been treated by Andrews-McCoy [5]. Recently, Gao, Pan and Tsai [13, 14] studied $\frac{1}{\kappa^\alpha}$-type nonlocal flows for convex curves. A lot of blow-up phenomena of this kind of nonlocal flows have been found and all global flows are proved convergent as time tends to infinity.

The $\kappa^\alpha$-type length-preserving flow is a special case of the flow (1.1). So the research of this generalized model is the continuation of previous studies. The flow (1.1) with some concrete $F$ can be defined for general embedded curves, but it may blow up in a finite time if the function $F$ or the initial convex curve is not chosen properly. One can find some blow-up examples in [10, 24] (if $X_0$ is embedded but not convex) and in [13, 29] (if $X_0$ is convex). In order to obtain good results of the flow, including its global existence and convergence, some limitations on the function $F$ should be imposed. From now on, we assume $F$ is smooth and defined in the interval $(0, +\infty)$ and

(i) $F'(u) > 0$ and $\lim_{u \to 0^+} F'(u) \cdot u$ exists;
(ii) $F(u) > 0$ and $\lim_{u \to +\infty} F(u) = +\infty$ and
(iii) $F(u)$ and $F'(u)$ satisfy

$$\lim_{u \to +\infty} \frac{F'(u) \cdot u^2}{F(u)} = +\infty, \quad \lim_{u \to 0^+} \frac{F'(u) \cdot u^2}{F(u)} = 0.$$ 

Before the statement of our main result, there are two remarks. Comparing with abstract evolution models of hypersurfaces (see [1, 4, 16], etc.), there are no extra convexity condition on the function $F$. Apart from the case of $F = \kappa^\alpha (\alpha > 0)$ studied by Wang-Tsai [28], there are lots of other examples satisfying above conditions (i)-(iii), such as

$$F(\kappa) = \ln(1 + \kappa), \ e^\kappa, \ 2\kappa + \sin \kappa, \ \kappa^2 \ln \kappa + \kappa$$

and so on. Another remark is that the parabolic property $F' > 0$ and the positivity condition $F > 0$ imply the limit

$$\lim_{u \to 0^+} F'(u) \cdot u = 0$$

if this limit exists (see Lemma [3.1]). Therefore, the conditions (i)-(iii) can be summarized as the parabolic property, positivity and four limits of $F$. The main result of this paper is as follows.

**Theorem 1.1.** Let $X_0$ be a smooth convex curve in the plane. If a smooth function $F$ satisfies conditions (i)-(iii) above, then the flow (1.1) exists on time interval $[0, +\infty)$,
keeps the convexity of the evolving curve, preserves its length and deforms the curve into a finite circle as time tends to infinity.

The condition (i) guarantees the parabolicity of the evolution equation. Tsai in [29] revealed the blow up phenomenon of a convex curve evolving according to (1.1) with \( F(\kappa) = -1/\kappa \). The blow up phenomenon also occurs in some other nonlocal models with \( F < 0 \) [13, 14]. Therefore, this flow may blow up in a finite time without the condition (ii). The third condition can be used to bound the curvature uniformly from both above and below. The proofs rely on a subtle usage of the maximum principle which are inspired by Chou’s early work [6] and its developments (see [5, 15] and [28]). Without the uniform bounds of curvature, one can not expect the convergence of the flow.

Lin and Tsai in [20] suggested that one may consider the flow (1.1) under the assumption that \( F(u) \) is increasing if \( u > 0 \). This condition implies the parabolicity condition \( F' > 0 \) if \( F(u) > 0 \) holds for all positive \( u \). It is still open whether or not Lin-Tsai’s condition leads to the convergence of the flow (1.1).

The global existence of the flow (1.1) is proved in Section 2. The convergence of the curvature is proved in Section 3 and the convergence of the curve is given in Section 4.

## 2 Global existence

In this section, the long term existence of the flow (1.1) will be proved. It is shown that the flow exists in time interval \([0, +\infty)\) and the evolving curve keeps its convexity.

### 2.1 Short time existence and convexity

Denote by \( \theta \) the tangent angle of the evolving curve. By adding a proper tangent component to the original flow, one can obtain a new one:

\[
\frac{\partial \tilde{X}}{\partial t} = -\frac{1}{\kappa} \frac{\partial F}{\partial s} T + (F - \lambda) N \quad \text{in} \quad S^1 \times (0, \omega) \\
\tilde{X}(\phi, 0) = X_0(\phi) \quad \text{on} \quad S^1. \tag{2.1}
\]

If there is a family of convex curves evolving under the flow (1.1), then (2.1) also has a solution \( \tilde{X}(\cdot, t) \) which differs from \( X(\cdot, t) \) by a reparameterization and a translation. By straightforward computations (see for example, [12]), the tangent angle under the flow (2.1) satisfies that

\[
\frac{\partial \theta}{\partial t} = \left( \frac{-1}{\kappa} \frac{\partial F}{\partial s} \right) \cdot \kappa + \frac{\partial}{\partial s} (F - \lambda) \equiv 0.
\]

So \( \theta \) is independent of time. It can be used as the parameter of the evolving curve \( \tilde{X}(\cdot, t) \). Due to this reason, we consider the flow (2.1) from now on.
Since the curvature determines a curve up to a congruence, (2.1) can be reduced to the evolution equation of the curvature \( \kappa \) in a short time interval:

\[
\frac{\partial \kappa}{\partial t} = \kappa^2 \left[ F'(\kappa) \frac{\partial^2 \kappa}{\partial \theta^2} + F''(\kappa) \left( \frac{\partial \kappa}{\partial \theta} \right)^2 + F(\kappa) - \frac{1}{2\pi} \int_0^{2\pi} F(\kappa) d\theta \right],
\]

(2.2)

where the initial curvature \( \kappa_0(\theta) = \kappa(\theta, 0) \) satisfies the closing condition

\[
\int_0^{2\pi} \frac{e^{i\theta}}{\kappa_0(\theta)} d\theta = 0.
\]

(2.3)

The equation (2.2) is a quasilinear parabolic equation. Its linearization at the function \( u = \kappa_0 + tu_0 \) is as follows

\[
\frac{\partial \kappa}{\partial t} = u^2 F'(u) \frac{\partial^2 \kappa}{\partial \theta^2} + \text{lower order terms of } \kappa,
\]

where \( u_0 = u_0(\theta, t) \) is chosen to be smooth and bounded between two constants. Let \( t_* > 0 \) be small so that \( \kappa_0 + tu_0 \in (m_0/2, 2M_0) \), where \( m_0 = \min_\theta \kappa_0(\theta) > 0 \) and \( M_0 = \max_\theta \kappa_0(\theta) \).

Since \( F'(u(\theta, t)) \) on the domain \( S^1 \times [0, t_*) \) is bounded between two positive constants, the linearization of (2.2) is uniformly parabolic on the same domain. It follows from the linearization method \([7, 15]\) that the Cauchy problem (2.2) has a unique and smooth solution \( \kappa(\theta, t) \) which satisfies the closing condition (2.3). Thus the flow (2.1) exists in a short time interval:

**Lemma 2.1.** There is a unique family of smooth convex curves evolving under the flow (1.1) in a time interval \([0, \omega)\), where \( \omega \) is positive.

Since \( \kappa_0(\theta) \) is positive everywhere, one can show that \( \kappa(\theta, t) \) is always positive under the flow, i.e., (2.1) preserves the convexity of the evolving curve.

**Lemma 2.2.** Under the flow (2.1), the evolving curve is convex.

**Proof.** Suppose that the flow (2.1) exists on time interval \([0, \omega)\). By the continuity of the evolving curve, there is a small time interval such that \( X(\cdot, t) \) is convex on that time interval. Suppose the lemma does not hold and there exists \( t_0 \in (0, \omega) \) such that the evolving curve is strictly convex on \([0, t_0)\) but

\[
\min \{ \kappa(\theta, t_0) | \theta \in [0, 2\pi] \} = 0.
\]

By the assumption, there is a positive \( M \) such that

\[
\max \{ \kappa(\theta, t) | \theta \in [0, 2\pi], t \in [0, t_0] \} \leq M.
\]

If one defines a function

\[
f(t) := \frac{\frac{1}{2} \min_\theta \kappa_0(\theta)}{\frac{1}{2} F(M) \cdot \min_\theta \kappa_0(\theta) t + 1},
\]

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then
\[ f(0) = \frac{1}{2} \min_\theta \kappa_0(\theta) < \kappa_0(\theta) \quad \text{and} \quad f'(t) = -F(M)f^2(t) < 0. \]

For a positive number \( \varepsilon \), we define \( h(\theta, t) := \kappa(\theta, t) - f(t) + \varepsilon t \). Then on a small time interval \([0, t) \subset [0, t_0)\), we have \( h > 0 \). To draw a contradiction, it is suffice to show \( h > 0 \) on the time interval \([0, t_0]\) for any positive \( \varepsilon \).

It follows from the evolution equation of \( \kappa \) that
\[
\frac{\partial h}{\partial t} = \kappa^2 \left[ F'(\kappa) \frac{\partial^2 h}{\partial \theta^2} + F''(\kappa) \left( \frac{\partial h}{\partial \theta} \right)^2 \right] + \kappa^2 F(\kappa) - \kappa^2 \frac{1}{2\pi} \int_0^{2\pi} F(\kappa) d\theta
\]
\[ + F(M)f^2(t) + \varepsilon. \]

Since \( h > 0 \) on \([0, t) \subset [0, t_0)\), in this smaller time interval one can estimate
\[
k^2 F(\kappa) - \kappa^2 \frac{1}{2\pi} \int_0^{2\pi} F(\kappa) d\theta + F(M)f^2(t) \geq 0 - \kappa^2 F(M) + F(M)f^2(t)
\]
\[ = -F(M)(\kappa + f(t))(\kappa - f(t)) \geq -F(M)(\kappa + f(t))h \geq -F(M)(M + \min_\theta \kappa_0(\theta))h. \]

Substituting the above inequality into the evolution of \( h \), one obtains
\[
\frac{\partial h}{\partial t} \geq \kappa^2 \left[ F'(\kappa) \frac{\partial^2 h}{\partial \theta^2} + F''(\kappa) \left( \frac{\partial h}{\partial \theta} \right)^2 \right] - F(M)(M + \min_\theta \kappa_0(\theta))h + \varepsilon. \]

The maximum principle tells us that on the time interval \([0, t)\),
\[
h(\theta, t) \geq h_0(\theta)e^{-C_0t},
\]
where \( C_0 = F(M)(M + \min_\theta \kappa_0(\theta)) \) and \( h_0(\theta) = \kappa_0(\theta) - \frac{1}{2} \min_\theta \kappa_0(\theta) > 0 \). Therefore, by the continuity, \( h(\theta, t) \geq h_0(\theta)e^{-C_0t} > 0 \) for all \((\theta, t) \in [0, 2\pi] \times [0, t_0]\).

Now, on the time interval \([0, t_0]\), it follows from \( h(\theta, t) > 0 \) that
\[
\kappa(\theta, t) > f(t) - \varepsilon t.
\]

By the arbitrariness of \( \varepsilon \), one gets \( \kappa(\theta, t) \geq f(t) > 0 \) for all \( t \in [0, t_0] \). This contradicts to the existence of the time \( t_0 \). Thus, the continuity of \( \kappa \) implies that it is always positive for \( t \in [0, \omega) \).

As a lower bound of \( \kappa(\theta, t) \), \( f(t) \) is better than an exponential function with a negative power (c.f. \([18, 25]\)). Since the evolving curve \( \tilde{X}(\cdot, t) \) is always strictly convex, from now on, the curve can be parameterized by the tangent angle \( \theta \) whenever the flow exists.
2.2 Uniform bounds of the support function

**Lemma 2.3.** Under the flow \((L(t), A(t))\), the length \(L(t)\) of \(X(\cdot, t)\) is preserved and the area \(A(t)\) of the domain bounded by this curve is increasing.

**Proof.** The proof is a direct computation. Under the flow, the length and the area evolve according to

\[
\frac{dL}{dt} = -\int_0^{2\pi} (F(\kappa) - \lambda(t))d\theta = -\int_0^{2\pi} \left[ F(\kappa) - \frac{1}{2\pi} \int_0^{2\pi} F(\kappa)d\theta \right] d\theta = 0,
\]

\[
\frac{dA}{dt} = -\int_0^L (F(\kappa) - \lambda(t))ds = -\int_0^L F(\kappa)ds + \frac{L}{2\pi} \int_0^L F(\kappa)d\kappa.
\]

Since \(F\) is strictly increasing, Andrews’ inequality (Lemma I3.3 of [2]) implies that the area \(A(t)\) is increasing. \(\Box\)

Denote by \(I\) the isoperimetric ratio, i.e., \(I = \frac{L^2}{4\pi A}\). For a convex curve in the plane, the following Bonnesen’s inequality holds [26]:

\[rL - A - \pi r^2 \geq 0, \quad r \in [r_{in}, r_{out}],\]

where \(r_{in}\) and \(r_{out}\) are the inradius and outradius (circumradius) of the convex domain bounded by \(\tilde{X}\), respectively. So

\[
\frac{L - \sqrt{L^2 - 4\pi A}}{2\pi} \leq r_{in} \leq r_{out} \leq \frac{L + \sqrt{L^2 + 4\pi A}}{2\pi},
\]

which implies

\[
\frac{r_{out}}{r_{in}} \leq \frac{L + \sqrt{L^2 - 4\pi A}}{L - \sqrt{L^2 - 4\pi A}} = \left(\sqrt{I} + \sqrt{I - 1}\right)^2.
\]

Lemma 2.3 tells us that the function \(I(t)\) is decreasing under the flow \((L, A)\). In particular, one gets

\[
\frac{r_{out}(t)}{r_{in}(t)} \leq \left(\sqrt{I(0)} + \sqrt{I(0) - 1}\right)^2.
\]

Noticing that \(r_{out}(t) \geq \frac{L}{2\pi}\), one can obtain

\[
r_{in}(t) \geq \frac{L}{2\pi} \left(\sqrt{I(0)} + \sqrt{I(0) - 1}\right)^{-2} > 0. \tag{2.4}
\]

Let \(E_0\) be a circle with radius

\[r_0 := \frac{L}{2\pi} \left(\sqrt{I(0)} + \sqrt{I(0) - 1}\right)^{-2}\]
and centred at the center $O$ of a maximum inscribed circle of $X_0$. Shrink $E_0$ via the flow

$$\begin{cases} \frac{\partial Y}{\partial t} = F(\bar{\kappa})N_{in}, \\ Y(\cdot, 0) = E_0, \end{cases}$$

(2.5)

where $\bar{\kappa}(\cdot, t)$ is the curvature of the curve $Y(\cdot, t)$. By the maximum principle, $Y(\cdot, t)$ is a family of circles. Its radius $r = r(t)$ satisfies

$$\frac{dr}{dt} = -F\left(\frac{1}{r}\right).$$

As we know that if $r$ is positive then $F\left(\frac{1}{r}\right) > 0$. It follows from the evolution equation of $r$ that the function $r(t)$ is strictly decreasing. Define $T_1 := r_0/(2F(r_0/2))$. We shall show that if $t \in [0, T_1)$, then

$$r(t) > \frac{r_0}{2},$$

where $r_0 = r(0)$. Suppose there is a $\tilde{t} \in (0, T_1)$ such that $r(t) > \frac{r_0}{2}$ for $t \in (0, \tilde{t})$ and $r(\tilde{t}) = \frac{r_0}{2}$. Since $F$ is increasing, we have

$$F\left(\frac{1}{r}\right) < F\left(\frac{1}{r_0}\right) = F\left(\frac{2}{r_0}\right),$$

for $t \in (0, \tilde{t})$. So, in this time interval, one obtains $\frac{dr}{dt} > -F\left(\frac{2}{r_0}\right)$ which implies

$$r(t) \geq r_0 - F\left(\frac{2}{r_0}\right) t.$$  

(2.6)

By the continuity of $r(t)$, if $t$ tends to $\tilde{t}$ then

$$r(\tilde{t}) \geq r_0 - F\left(\frac{2}{r_0}\right) \tilde{t} > r_0 - F\left(\frac{2}{r_0}\right) T_1 = \frac{r_0}{2}.$$ 

This conflicts to the choice of $\tilde{t}$. Therefore, $r(t) \geq \frac{r_0}{2}$ if $t \in [0, T_1]$.

Suppose the flow (1.1) exists on time interval $[0, T_1)$. The support function of a convex curve is defined by $p := -\langle X, N \rangle$, where $N$ is the unit inward pointing normal vector. Since $X(\cdot, t)$ contains the circle $Y(\cdot, t)$ if $t \in [0, T_1)$, its support function satisfies

$$p(\theta, t) \geq r(t) \geq r(T_1) \geq \frac{r_0}{2}, \quad t \in [0, T_1].$$

Set $\delta = \frac{r_0}{4} > 0$. We have

$$2\delta \leq p(\theta, t) \leq \frac{L}{2}, \quad t \in [0, T_1].$$

(2.7)

The second inequality in (2.7) follows from the definition of the support function. Until now we do not know whether the flow exists on the time interval $[0, T_1)$ or not. The inequality (2.7) is a priori estimate.

If the flow (1.1) can be extended on time interval $[nT_1, (n+1)T_1)$, where $n$ is a natural number, then one can similarly prove that there exists a point $O_n$ such that the support function $p(\theta, t)$ with respect to $O_n$ still satisfies (2.7).
2.3 Extend the flow globally

Now we extend the flow on the time interval $[0, +\infty)$. Suppose the flow exists on a finite time interval $[0, \omega)$ and it blows up at $t = \omega > 0$. Let $k$ be a nature number such that $\omega \in (kT_1, (k + 1)T_1]$, where $T_1 = r_0/(2F\frac{2}{r_0})$ is a positive constant given in the above subsection. Next we shall show that the flow can be extended on the time interval $[0, \omega + \epsilon)$, where $\epsilon > 0$. So we draw a contradiction, which implies that the flow exists globally.

In every time interval $[0, T_1), [T_1, 2T_2), \ldots, [kT_1, \min\{\omega, (k + 1)T_1\})$, we consider the function

$$\varphi(\theta, t) := \frac{F}{p - \delta},$$

where we choose a proper origin so that the support function $p$ satisfies (2.7) in each time interval and the natural number $k$ may be 0. By some straightforward calculations, one obtains

$$\frac{\partial \varphi}{\partial \theta} = \frac{1}{p - \delta} \frac{\partial F}{\partial \theta} - \frac{F}{(p - \delta)^2} \frac{\partial p}{\partial \theta},$$

and

$$\frac{\partial^2 \varphi}{\partial \theta^2} = \frac{1}{p - \delta} \frac{\partial^2 F}{\partial \theta^2} - \frac{2}{(p - \delta)^2} \frac{\partial F}{\partial \theta} \frac{\partial p}{\partial \theta} - \frac{F}{(p - \delta)^2} \frac{\partial^2 p}{\partial \theta^2} + \frac{2F}{(p - \delta)^3} \left(\frac{\partial p}{\partial \theta}\right)^2.$$

By the evolution equation of $\kappa$, the function $F$ evolves according to

$$\frac{\partial F}{\partial t} = F'\kappa^2 \left[\frac{\partial^2 F}{\partial \theta^2} + F - \lambda(t)\right].$$

(2.8)

It follows from the definition of the support function $p$ and the flow equation (2.1),

$$\frac{\partial p}{\partial t} = \lambda(t) - F.$$ 

(2.9)

Now, combining (2.8) and (2.9), one gets the evolution equation of $\varphi(\theta, t)$:

$$\frac{\partial \varphi}{\partial t} = F'\kappa^2 \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{2F'\kappa^2}{p - \delta} \frac{\partial p}{\partial \theta} \frac{\partial \varphi}{\partial \theta} + \frac{F'\kappa^2}{p - \delta} \frac{\partial F}{\partial \theta} \frac{\partial^2 \varphi}{\partial \theta^2} - \frac{\delta F'\kappa^2}{p - \delta} - \frac{(p - \delta)}{p - \delta} \frac{\lambda F'\kappa^2}{p - \delta} - \lambda \frac{\varphi}{p - \delta} + \varphi^2.$$

In the above calculation, the identity $\frac{\partial^2 p}{\partial \theta^2} + p = \frac{1}{\kappa}$ is used.

**Lemma 2.4.** There is a positive constant $M$ independent of time such that

$$\kappa(\theta, t) \leq M$$

(2.10)

for all $(\theta, t) \in [0, 2\pi] \times [0, \omega)$.
Proof. In every time interval \([0, T_1), [T_1, 2T_2), \ldots, [kT_1, \omega)\), we can rewrite the evolution equation of \(\varphi\) as

\[
\frac{\partial \varphi}{\partial t} = F'\kappa^2 \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{2F'\kappa^2}{p - \delta} \frac{\partial p}{\partial \theta} \frac{\partial \varphi}{\partial \theta} - \frac{\lambda F'\kappa^2}{p - \delta} \frac{\lambda \varphi}{p - \delta} + \frac{F'\kappa}{p - \delta} \left(1 - \frac{\delta}{2}\right) \varphi - \left(\frac{\delta F'\kappa^2}{2F} - 1\right) \varphi^2, \tag{2.11}
\]

where a proper origin was chosen in the above every time interval so that the support function \(p\) satisfies (2.7). Since \(\lim_{u \to +\infty} \frac{F'(u)\cdot u^2}{F(u)} = +\infty\), there exists a \(U_0 > 0\) such that

\[
\frac{F'(u)\cdot u^2}{F(u)} > \frac{2}{\delta}
\]

for \(u > U_0\).

Now we have two cases. If the natural number \(k = 0\) then \(\min\{\omega, T_1\} = \omega\). Suppose that \(\varphi(\theta, t)\) attains its maximum at the point \((\theta_*, t)\) with respect to \(\theta\). If

\[
\varphi(\theta_*, t) > \frac{1}{\delta} F \left(\max \left\{\frac{2}{\delta}, U_0\right\}\right) \tag{2.12}
\]

then (2.7) tells us

\[
F(\kappa(\theta_*, t)) = (p - \delta)\varphi(\theta_*, t) > \frac{p - \delta}{\delta} F \left(\max \left\{\frac{2}{\delta}, U_0\right\}\right) > F \left(\max \left\{\frac{2}{\delta}, U_0\right\}\right).
\]

Noticing that \(F(u)\) is increasing, one obtains \(\kappa(\theta_*, t) > \max \{\frac{2}{\delta}, U_0\}\), and thus

\[
1 - \frac{\delta}{2}\kappa(\theta_*, t) < 0, \quad \frac{\delta F'(\kappa(\theta_*, t))\cdot \kappa(\theta_*, t)^2}{2F(\kappa(\theta_*, t))} - 1 > 0.
\]

Therefore, (2.12) implies

\[
\frac{\partial \varphi}{\partial t}(\theta_*, t) \leq 0.
\]

By the maximum principle, the function \(\varphi_{\max}(t) := \max\{\varphi(\theta, t) | \theta \in [0, 2\pi]\}\) is decreasing if (2.12) holds. Therefore, one gets

\[
\varphi(\theta, t) \leq \max \left\{\max \varphi(\theta, 0), \frac{1}{\delta} F \left(\max \left\{\frac{2}{\delta}, U_0\right\}\right)\right\}, \quad t \in [0, \omega). \tag{2.13}
\]

If the natural number \(k > 0\) then we consider the following time intervals one by one:

\([0, T_1), [T_1, 2T_1), \ldots, [kT_1, \omega)\).

Applying the above methods again, we can get the same estimate (2.13) in the time interval \([0, T_1)\). Similarly, for \(t \in [T_1, 2T_1)\), we have

\[
\varphi(\theta, t) \leq \max \left\{\max \varphi(\theta, T_1), \frac{1}{\delta} F \left(\max \left\{\frac{2}{\delta}, U_0\right\}\right)\right\}.
\]

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By the estimate (2.13) in the time interval [0, T_1), we know
\[ \varphi(\theta, T_1) \leq \max_\theta \varphi(\theta, 0), \quad \frac{1}{\delta} F \left( \max_\theta \left\{ \frac{2}{\delta}, U_0 \right\} \right). \]

Therefore, the estimate (2.13) holds in time interval [0, 2T_1]. By the mathematical induction, there is a positive constant C_0 (the R.H.S. of (2.13)) such that \( \varphi(\theta, t) \leq C_0 \) for \( t \in [0, T_1) \cup [T_1, 2T_1) \cup \ldots \cup [kT_1, \omega) = [0, \omega) \). So, it follows from (2.7) that
\[ F(\kappa) \leq \left( \frac{L}{2} - \delta \right) C_0, \quad t \in [0, \omega). \] (2.14)

Therefore, the condition (ii) of the function \( F \) implies that the curvature \( \kappa \) has an upper bound \( M \) independent of time, where \( t \in [0, \omega) \).

We have assumed that \( \omega \) is a finite number. By the uniform upper bound of \( \kappa \) and the positivity of \( \kappa \) (see Lemma 2.2), there is a constant
\[ m := f(\omega) = \frac{\frac{1}{T} \min_\theta \kappa_0(\theta)}{F(M) \cdot \min_\theta \kappa_0(\theta) + 1} > 0 \]
such that
\[ \kappa(\theta, t) \geq m, \quad (\theta, t) \in [0, 2\pi] \times [0, \omega). \] (2.15)

So the evolution equation of \( \kappa \) is uniformly parabolic. By the results in the next subsection, every higher derivative of the curvature also has uniform bounds.

Since \( \kappa \) has uniform bounds (see (2.10) and (2.15)) and \( F(\cdot) \) is smooth and increasing on \( (0, +\infty) \), the functions \( F(\kappa) \) and \( \lambda(t) \) can be uniformly bounded on the time interval \([0, \omega)\):
\[ F(m) \leq F(\kappa) \leq F(M), \quad F(m) \leq \lambda(t) \leq F(M). \]

So, integrating the flow equation (2.1), we obtain a convex curve
\[ X(\cdot, \omega) := \int_0^\omega \frac{\partial X}{\partial t}(\cdot, t) dt. \]

It follows from the Arzelà-Ascoli theorem that the curvature \( \kappa(\cdot, t) \) converges to a smooth function \( \kappa_\omega(\cdot) \) in the sense of \( C^\infty \) metric as \( t \to \omega \). So the curve \( X(\cdot, \omega) \) is smooth.

Let \( X(\cdot, \omega) \) be an initial curve evolving under the flow (2.1). Then we can extend this flow on a larger time interval \([0, \omega + \varepsilon)\) by the short time existence. This leads to a contradiction. Hence we can conclude:

**Theorem 2.5.** The flow (2.7) exists on the time interval \([0, +\infty)\).
2.4 Higher regularity of the flow

For the higher regularity of the flow, one needs to bound all derivatives of $\kappa$ on the finite time interval $[0, \omega)$ with $\omega > 0$. The proof relies on both integral estimates (see for example Gage-Hamiton [11], Jiang-Pan [18]) and a technique invented by Lin and Tsai [21]. If $t \in [0, \omega)$, the estimates (2.10), (2.14) and (2.15) imply that there are constants $C_i = C_i(\omega)$ such that

$$\left| \frac{d^i F}{du^i} \right| \leq C_i, \quad i = 1, 2, \cdots. \quad (2.16)$$

It follows from (2.15) that there is a constant $c_1 = c_1(\omega) > 0$ such that

$$F'(\kappa(\theta, t)) > c_1 \quad (2.17)$$

for all $(\theta, t) \in [0, 2\pi] \times [0, \omega)$.

**Lemma 2.6.** The integral $\int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^6 d\theta$ has at most polynomial growth.

**Proof.** Using the evolution equation of $F$, one can compute that

$$\frac{d}{dt} \frac{1}{6} \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^6 d\theta = \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^5 \frac{\partial^2 F}{\partial \theta \partial t} d\theta = -5 \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^4 \frac{\partial^2 F}{\partial \theta^2} \frac{\partial F}{\partial t} d\theta$$

$$= -5 \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^4 \frac{\partial^2 F}{\partial \theta^2} F' \kappa^2 \left[ \frac{\partial^2 F}{\partial \theta^2} + F - \lambda(t) \right] d\theta$$

$$= -5 \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^4 \left( \frac{\partial^2 F}{\partial \theta^2} \right)^2 d\theta$$

$$- 5 \int_0^{2\pi} F' \kappa^2 (F - \lambda(t)) \left( \frac{\partial F}{\partial \theta} \right)^4 \frac{\partial^2 F}{\partial \theta^2} d\theta.$$

Since

$$(F - \lambda(t)) \frac{\partial^2 F}{\partial \theta^2} \leq \frac{1}{2} \left[ \left( \frac{\partial^2 F}{\partial \theta^2} \right)^2 + (F - \lambda(t))^2 \right],$$

one gets

$$\frac{d}{dt} \frac{1}{6} \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^6 d\theta \leq - \frac{5}{2} \int_0^{2\pi} F' \kappa^2 \left( \frac{\partial F}{\partial \theta} \right)^4 \left( \frac{\partial^2 F}{\partial \theta^2} \right)^2 d\theta$$

$$+ \frac{5}{2} \int_0^{2\pi} F' \kappa^2 (F - \lambda(t))^2 \left( \frac{\partial F}{\partial \theta} \right)^4 d\theta,$$

which together with (2.15) and (2.16) implies

$$\frac{d}{dt} \frac{1}{6} \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^6 d\theta \leq \frac{5}{2} C_1 M^2 F(M)^2 \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^4 d\theta.$$
Lemma 2.7. The integral $\int_0^{2\pi} \left( \frac{\partial^2 F}{\partial \theta^2} \right)^2 d\theta$ has at most polynomial growth.

Proof. Compute that

$$\frac{d}{dt} \int_0^{2\pi} \left( \frac{\partial^2 F}{\partial \theta^2} \right)^2 d\theta = -\int_0^{2\pi} \frac{\partial F}{\partial \theta} \frac{\partial^3 F}{\partial \theta^3} d\theta - \int_0^{2\pi} F' \frac{\partial^2 F}{\partial \theta^2} \frac{\partial^3 F}{\partial \theta^3} d\theta - 2 \int_0^{2\pi} F' \frac{\partial^3 F}{\partial \theta^3} d\theta + \int_0^{2\pi} F' \frac{\partial^3 F}{\partial \theta^3} d\theta,$$

$$:= -\int_0^{2\pi} F' \kappa^2 \left( \frac{\partial^3 F}{\partial \theta^3} \right)^2 d\theta + I + II + III + IV + V.$$

Using the Cauchy-Schwarz inequality, one can estimate that

$$I = -\int_0^{2\pi} F' \frac{\partial^2 F}{\partial \theta^2} \frac{\partial^3 F}{\partial \theta^3} d\theta = -\int_0^{2\pi} F'' \frac{\partial^2 F}{\partial \theta^2} \frac{\partial^3 F}{\partial \theta^3} d\theta,$$

$$\leq \varepsilon \int_0^{2\pi} \left( \frac{\partial^3 F}{\partial \theta^3} \right)^2 d\theta + C_2 M^2 \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right) \left( \frac{\partial^2 F}{\partial \theta^2} \right)^2 d\theta,$$

$$II = -2 \int_0^{2\pi} F' \frac{\partial^3 F}{\partial \theta^2} \frac{\partial^2 F}{\partial \theta^3} d\theta,$$

$$\leq \varepsilon \int_0^{2\pi} \left( \frac{\partial^3 F}{\partial \theta^3} \right)^2 d\theta + M \varepsilon \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right) \left( \frac{\partial^2 F}{\partial \theta^2} \right)^2 d\theta,$$

$$III = -\int_0^{2\pi} F'' \frac{\partial^2 F}{\partial \theta^2} \frac{\partial^3 F}{\partial \theta^3} d\theta,$$

$$\leq \varepsilon \int_0^{2\pi} \left( \frac{\partial^3 F}{\partial \theta^3} \right)^2 d\theta + C_2 M^2 F(M) \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta,$$

$$IV = -2 \int_0^{2\pi} F' \frac{\partial^3 F}{\partial \theta^3} d\theta,$$

Integrating this can show that $\int_0^{2\pi} \left( \frac{\partial^2 F}{\partial \theta^2} \right)^2 d\theta$ is bounded by a cubic polynomial. \qed
\[
\begin{aligned}
\mathcal{L} &= -2 \int_0^{2\pi} \kappa \frac{\partial F}{\partial \theta} (F - \lambda(t)) \frac{\partial^3 F}{\partial \theta^3} \, d\theta \\
&\leq \varepsilon \int_0^{2\pi} \left( \frac{\partial^3 F}{\partial \theta^3} \right)^2 \, d\theta + \frac{MF(M)}{\varepsilon} \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 \, d\theta,
\end{aligned}
\]

and

\[
\mathcal{V} = - \int_0^{2\pi} \kappa^2 \frac{\partial F}{\partial \theta} \frac{\partial^3 F}{\partial \theta^3} \, d\theta \\
\leq \varepsilon \int_0^{2\pi} \left( \frac{\partial^3 F}{\partial \theta^3} \right)^2 \, d\theta + \frac{C_1 M^2}{4\varepsilon} \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 \, d\theta,
\]

where we have used \(-F(M) \leq -\lambda \leq F - \lambda \leq F(M)\) which implies \(|F - \lambda| \leq F(M)|. Therefore,

\[
\begin{aligned}
\frac{d}{dt} \int_0^{2\pi} \left( \frac{\partial^2 F}{\partial \theta^2} \right)^2 \, d\theta &\leq \int_0^{2\pi} \left( 5\varepsilon - F' \kappa^2 \right) \left( \frac{\partial^3 F}{\partial \theta^3} \right)^2 \, d\theta \\
&\quad + \left( \frac{C_2 M^2}{4\varepsilon c_1} + \frac{M}{\varepsilon} \right) \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 \left( \frac{\partial^2 F}{\partial \theta^2} \right)^2 \, d\theta \\
&\quad + \left( \frac{C_2 M^2 F(M)}{4\varepsilon c_1} + \frac{MF(M)}{\varepsilon} + \frac{C_1 M^2}{4\varepsilon} \right) \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 \, d\theta.
\end{aligned}
\]

Since

\[
\begin{aligned}
\int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 \left( \frac{\partial^2 F}{\partial \theta^2} \right)^2 \, d\theta &= \frac{1}{3} \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^3 \frac{\partial^3 F}{\partial \theta^3} \, d\theta \\
&\leq \frac{\varepsilon^2}{3} \int_0^{2\pi} \left( \frac{\partial^3 F}{\partial \theta^3} \right)^2 \, d\theta + \frac{1}{12\varepsilon^2} \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^6 \, d\theta,
\end{aligned}
\]

one can finally obtain

\[
\begin{aligned}
\frac{d}{dt} \int_0^{2\pi} \left( \frac{\partial^2 F}{\partial \theta^2} \right)^2 \, d\theta &\leq \int_0^{2\pi} \left[ 5\varepsilon - F' \kappa^2 + \frac{\varepsilon}{3} \left( \frac{C_2 M^2}{4c_1} + M \right) \right] \left( \frac{\partial^3 F}{\partial \theta^3} \right)^2 \, d\theta \\
&\quad + \frac{1}{12\varepsilon^2} \left( \frac{C_2 M^2}{4\varepsilon c_1} + \frac{M}{\varepsilon} \right) \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^6 \, d\theta \\
&\quad + \left( \frac{C_2 M^2 F(M)}{4\varepsilon c_1} + \frac{MF(M)}{\varepsilon} + \frac{C_1 M^2}{4\varepsilon} \right) \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 \, d\theta.
\end{aligned}
\]

Now one can choose \(\varepsilon\) small enough such that

\[
5\varepsilon - F' \kappa^2 + \frac{\varepsilon}{3} \left( \frac{C_2 M^2}{4c_1} + M \right) < 5\varepsilon - c_1 m^2 + \frac{\varepsilon}{3} \left( \frac{C_2 M^2}{4c_1} + M \right) < 0.
\]

By Lemma 2.6 both \(\int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^6 \, d\theta\) and \(\int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 \, d\theta\) have at most polynomial growth. So one can complete the proof by integrating the above differential inequality. \(\square\)
Sobolev’s inequality tells us that $|\frac{\partial F}{\partial \theta}|$ is bounded in time interval $[0, \omega)$ and so is $\frac{\partial \kappa}{\partial \theta} = \frac{\partial F}{\partial \theta}/F'$. Set $g = \frac{\partial^2 F}{\partial \theta^2} + \alpha \left( \frac{\partial F}{\partial \theta} \right)^2$ and compute that
\[
\frac{\partial g}{\partial t} = F' \kappa^2 \frac{\partial^2 g}{\partial \theta^2} + \left( 4\kappa \frac{\partial F}{\partial \theta} + F'' \kappa^2 \frac{\partial \kappa}{\partial \theta} - 2\alpha F' \kappa^2 \frac{\partial F}{\partial \theta} \right) \frac{\partial g}{\partial \theta} + \left( -2\alpha F' \kappa^2 + \frac{F''}{F'} \kappa^2 + 2\kappa \right) g^2 + \text{lower order terms of } g. \tag{2.18}
\]

Since $\kappa$ and $F'$ have lower bounds (see (2.15) and (2.17)), one can choose $\alpha > 0$ large enough so that the coefficient of $g^2$ in the above equation is negative. It follows from the maximum principle that the function $g_{\max}(t) = \max \{g(\theta, t) \mid \theta \in [0, 2\pi]\}$ is at most exponentially increasing. Therefore, $g$ has a uniformly upper bound on the finite time interval $[0, \omega)$. Similarly, one can choose $\alpha < 0$ with sufficiently large $|\alpha|$ so that the coefficient of $g^2$ is positive and $g$ has a uniformly lower bound on time interval $[0, \omega)$. Combining the above two cases, $\left| \frac{\partial^2 F}{\partial \theta^2} \right|$ is uniformly bounded in time interval $[0, \omega)$.

Since all higher derivatives $\frac{\partial^k F}{\partial \theta^k}(k = 3, 4, \ldots)$ satisfy a linear parabolic equation with uniformly bounded coefficients, $\left| \frac{\partial^k F}{\partial \theta^k} \right|$ is uniformly bounded on time interval $[0, \omega)$ and so are all derivatives of $\kappa$.

Until now we only prove that $\kappa$ is always positive. Whether the evolution equation of $\kappa$ is degenerate or not as $t \to +\infty$ is still unknown. In fact, the existence of uniformly lower positive bound of the curvature plays an essential role in the study of asymptotic behavior of the evolving curve. We leave this part to the next section.

### 3 Convergence of the curvature

The main task of this section is to prove that the curvature $\kappa(\theta, t)$ converges to a constant $\frac{2\pi}{L}$ as time tends to infinity.

**Lemma 3.1.** Suppose a $C^1$ and positive function $F$ is defined on the interval $(0, u_0)$ with some $u_0 > 0$ and $F'(u) > 0$ for all $u \in (0, u_0)$. If the limit $\lim_{u \to 0^+} F'(u) \cdot u$ exists then
\[
\lim_{u \to 0^+} F'(u) \cdot u = 0. \tag{3.1}
\]

**Proof.** Since $F'(u)u > 0$ for $u \in (0, u_0)$ and $\lim_{u \to 0^+} F'(u) \cdot u$ exists, this limit satisfies
\[
\lim_{u \to 0^+} F'(u) \cdot u \geq 0.
\]

Suppose that the limit is positive. By continuity, there exists a constant $C_0 > 0$ such that $F'(u)u > C_0$ for $u \in (0, u_0/2)$. So $F'(u) > C_0/u$ holds for $u \in (0, u_0/2)$. Integrating this inequality on the interval $(\varepsilon, u_0/2)$ gives us
\[
F(u_0/2) > F(\varepsilon) + C_0(\ln(u_0/2) - \ln \varepsilon) > C_0(\ln(u_0/2) - \ln \varepsilon),
\]
where $\varepsilon \in (0, u_0/2)$. Letting $\varepsilon$ tend to 0 from the right hand side, we find that $F(u_0/2)$ is unbounded, a contradiction to the definition of $F$. \qed
If we suppose $F(\cdot)$ is defined at $u = 0$ and has right derivative $F'_+(0)$ then the limit \((3.1)\) is a direct calculation:

$$
\lim_{u \to 0^+} F'(u) \cdot u = \lim_{u \to 0^+} \lim_{h \to 0^+} \frac{F(u + h) - F(u)}{h} \cdot u
= \lim_{h \to 0^+} \lim_{u \to 0^+} \frac{F(u + h) - F(u)}{h} \cdot u
= F'_+(0) \cdot 0 = 0.
$$

However, there indeed exist some concrete functions of $F$ without right derivative at $u = 0$ but still satisfying the conditions (i)-(iii). For example, one may choose the power function $F(u) = u^\alpha$, where $0 < \alpha < 1$. In order not to exclude these functions, we only assume that $F$ is defined and smooth on $(0, +\infty)$.

Using the evolution equation of $F$ (see \((2.8)\)), one can compute

$$
\frac{d}{dt} \frac{1}{2} \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta
= \int_0^{2\pi} \frac{\partial F}{\partial \theta} \frac{\partial^2 F}{\partial \theta \partial t} d\theta
= -\int_0^{2\pi} \frac{\partial F}{\partial t} \left( \frac{1}{F'(k)} \frac{\partial F}{\partial \theta} - F + \lambda \right) d\theta
= -\int_0^{2\pi} \frac{1}{F'(k)} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta + \frac{1}{2} \frac{d}{dt} \int_0^{2\pi} F^2 d\theta - \lambda \frac{d}{dt} \int_0^{2\pi} F d\theta.
$$

Since $\lambda = \frac{1}{2\pi} \int_0^{2\pi} F d\theta$, one gets

$$
\frac{d}{dt} \frac{1}{2} \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta \leq \frac{1}{2} \frac{d}{dt} \int_0^{2\pi} F^2 d\theta - \frac{1}{4\pi} \frac{d}{dt} \left( \int_0^{2\pi} F d\theta \right)^2,
$$

which gives us by integration

$$
\frac{1}{2} \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta \leq \frac{1}{2} \int_0^{2\pi} \left( \frac{\partial F_0}{\partial \theta} \right)^2 d\theta + \frac{1}{2} \int_0^{2\pi} F^2 d\theta - \frac{1}{2} \int_0^{2\pi} (F_0)^2 d\theta
- \frac{1}{4\pi} \left( \int_0^{2\pi} F d\theta \right)^2 + \frac{1}{4\pi} \left( \int_0^{2\pi} F_0 d\theta \right)^2
\leq \frac{1}{2} \int_0^{2\pi} \left( \frac{\partial F_0}{\partial \theta} \right)^2 d\theta + \pi F(M)^2 + \frac{1}{4\pi} \left( \int_0^{2\pi} F_0 d\theta \right)^2.
$$

Therefore, the function $F(k(\theta, t))$ is equicontinuous with respect to $\theta$. Since

$$
0 < F(k(\theta, t)) \leq F(M),
$$

the Arzelà-Ascoli Theorem tells us that there is a convergent subsequence $\{F(k(\theta, t_i))\}$, where $t_i$ tends to infinity.

**Lemma 3.2.** There is a positive constant $c_0$ independent of time such that

$$
\lambda(t) \geq c_0, \quad t \geq 0.
$$

(3.3)
Proof. We have assumed \( F(u) > 0 \) and \( F(u) \) is increasing for \( u > 0 \). There exists a limit 
\[
F(0) := \lim_{u \to 0^+} F(u).
\]
If \( F(0) > 0 \) then \( \lambda \) has a positive constant independent of time:
\[
\lambda(t) = \frac{1}{2\pi} \int_0^{2\pi} F(\kappa) d\theta \geq \frac{1}{2\pi} \int_0^{2\pi} F(0) d\theta = F(0) > 0,
\]
for all \( t \geq 0 \).

Suppose \( F(0) = 0 \). By the positivity of \( \kappa(\theta, t) \), \( F(\kappa(\theta, t)) \) is also positive and furthermore \( \lambda(t) > 0 \) if \( t \geq 0 \). Suppose there is a sequence \( t_i \) tending to \(+\infty\) such that \( \lambda(t_i) \to 0 \). Choose a subsequence \( \{\tilde{t}_i\} \) of \( \{t_i\} \) so that \( \{F(\kappa(\theta, \tilde{t}_i))\} \) converges uniformly. Let 
\[
F_{\infty}(\theta) := \lim_{\tilde{t}_i \to +\infty} F(\kappa(\theta, \tilde{t}_i)).
\]
The function \( F \) is positive for every \( t \), so the limit function \( F_{\infty}(\theta) \geq 0 \). Since \( F(\kappa(\theta, \tilde{t}_i)) \) converges to the continuous function \( F_{\infty}(\theta) \) uniformly and by the assumption of \( \lambda(\tilde{t}_i) \),
\[
0 = \lim_{\tilde{t}_i \to +\infty} \lambda(\tilde{t}_i) = \lim_{\tilde{t}_i \to +\infty} \frac{1}{2\pi} \int_0^{2\pi} F(\kappa(\theta, \tilde{t}_i)) d\theta = \frac{1}{2\pi} \int_0^{2\pi} F_{\infty}(\theta) d\theta,
\]
we have \( F_{\infty}(\theta) \equiv 0 \).

For fixed \( \theta \), by Lemma 2.2 and Lemma 2.4, \( \kappa(\theta, t) \) is bounded with respect to \( t \), so \( \{\kappa(\theta, \tilde{t}_i) | \tilde{t}_i \geq 0\} \) has a convergent subsequence. Denote by \( \kappa_{\infty}(\theta) \) the limit of such a convergent subsequence (still use the symbol \( \{\kappa(\theta, \tilde{t}_i)\} \)) as \( \tilde{t}_i \to +\infty \). Since \( F \) is continuous, one may compute the limit
\[
F(\kappa_{\infty}(\theta)) = \lim_{\tilde{t}_i \to +\infty} F(\kappa(\theta, \tilde{t}_i)) = F_{\infty}(\theta) = 0.
\]
Now, \( F(0) = 0 \) and \( F(u) > 0 \) (for all \( u > 0 \)) imply that \( \kappa_{\infty}(\theta) = 0 \). Its every convergent subsequence tends to the same limit \( 0 \), therefore for a fixed \( \theta \), \( \kappa(\theta, t) \) converges to \( 0 \) as \( t \to +\infty \). By the arbitrariness of \( \theta \), \( \lim_{t \to +\infty} \kappa(\theta, t) \equiv 0 \) in the sense of pointwise. Thus one has a limit
\[
L(t) = \int_0^{2\pi} \frac{1}{\kappa(\theta, t)} d\theta \to +\infty, \quad \text{as} \quad t \to +\infty.
\]
This is impossible, because the length of the evolving curve is fixed under the flow.

Combining the above two cases of \( F(0) > 0 \) and \( F(0) = 0 \), we complete the proof of this lemma.

Lemma 3.3. There is a positive constant \( m \) independent of time such that for all \( t \geq 0 \),
\[
\kappa(\theta, t) \geq m.
\]

Proof. Consider the time interval \([nT_1, (n+1)T_1] (n = 1, 2, \cdots)\), where \( T_1 \) (see Section 2) is a constant independent of time. Choose an original point \( O_n \) so that the support
function satisfies \( (2.1) \). Let \( \Delta \) be a positive constant such that \( \Delta \geq 2p \). For example, let \( \Delta = L \). Set \( \psi := \frac{F}{\Delta - p} \). Direct calculations can give us the evolution equation of \( \psi \):

\[
\frac{\partial \psi}{\partial t} = F' \kappa \frac{\partial^2 \psi}{\partial \theta^2} - \frac{2F' \kappa^2 \partial p \partial \psi}{\Delta - p} \frac{\partial \psi}{\partial \theta} - \frac{\Delta F' \kappa^2 \psi}{\Delta - p} + \frac{\lambda F' \kappa^2}{\Delta - p} \psi - \psi^2
\]

Since \( \lim_{u \to 0^+} \frac{F'(u) \cdot u^2}{F(u)} = 0 \) and \( \lambda \geq c_0 > 0 \), there exists a \( u_1 > 0 \) such that

\[
\frac{F'(u) \cdot u^2}{F(u)} < \frac{1}{2(\Delta - p)}, \quad u \in (0, u_1).
\]

From \( \lim_{u \to 0^+} F'(u) \cdot u = 0 \) it follows that there exists a \( u_2 > 0 \) such that

\[
F'(u) \cdot u < \frac{c_0}{4L} < \frac{\lambda}{4(\Delta - p)}, \quad u \in (0, u_2).
\]

If \( \psi < \frac{F(u_1)}{L} \), i.e. \( F(\kappa) < \frac{\Delta - p}{L} F(u_1) < F(u_1) \) then \( \kappa < u_1 \) and furthermore

\[
\frac{1}{2(\Delta - p)} - \frac{F'(\kappa) \cdot \kappa^2}{F(\kappa)} > 0.
\]

Suppose \( \psi < \min\{\frac{c_0}{4L}, \frac{F(u_1)}{L}\} \). One can get both

\[
\psi < \frac{c_0}{4L} < \frac{\lambda}{4(\Delta - p)}
\]

and

\[
F(\kappa) < \frac{\Delta - p}{L} F(u_2) < F(u_2).
\]

Since \( F \) is increasing, \( \kappa < u_2 \) and furthermore

\[
F'(\kappa) \cdot \kappa < \frac{c_0}{4L} < \frac{\lambda}{4(\Delta - p)},
\]

and we can obtain

\[
\frac{\lambda}{2(\Delta - p)} - \frac{F'(\kappa)}{\Delta - p} - \varphi > 0.
\]

Therefore, the function \( \psi_{\min}(t) := \min\{\psi(\theta, t) | \theta \in [0, 2\pi]\} \) is increasing if

\[
\psi(\theta, t) < \min \left\{ \frac{c_0}{4L}, \frac{F(u_1)}{L}, \frac{F(u_2)}{L} \right\},
\]

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where the constants $c_0, u_1$ and $u_2$ are independent of time. By the maximum principle,

$$
\psi_{\min}(t) \geq \min \left\{ \frac{c_0}{4L}, \frac{F(u_1)}{L}, \frac{F(u_2)}{L} \right\} := C, \quad t \geq 0.
$$

Thus, under the flow (2.1), one obtains that $F \geq (\Delta - \delta)C > 0$, where $\Delta = L, \delta$ and $C$ are positive constants independent of time.

Until now we have proved the function $F$ has a positive lower bound $(\Delta - \delta)C$ for all $(\theta, t) \in [0, 2\pi] \times [0, +\infty)$. If $\lim_{u \to 0^+} F(u) = 0$ then there is a positive constant $m$ such that $\kappa \geq m$ for all $(\theta, t) \in [0, 2\pi] \times [0, +\infty)$. In this case $\kappa$ can not be too small since under the flow $F$ has a uniformly positive lower bound.

On the other hand, if $F(0) := \lim_{u \to 0^+} F(u) > 0$ then we let $\tilde{F}(\cdot) = F(\cdot) - F(0)$. For example, if $F(\kappa) = e^\kappa$ then we denote $\tilde{F}(\kappa) = e^\kappa - 1$. We consider a new flow:

$$
\begin{aligned}
&\frac{\partial \tilde{X}}{\partial t} = -\frac{1}{\kappa} \frac{\partial \tilde{F}}{\partial s} \mathcal{T} + (\tilde{F} - \tilde{\lambda})N \quad \text{in } S^1 \times (0, \omega) \\
&\tilde{X}(\varphi, 0) = X_0(\varphi) \quad \text{on } S^1,
\end{aligned}
$$

where $\tilde{\lambda}(t) = \frac{1}{2\pi} \int_0^L \tilde{F}(\kappa(s, t)) \kappa(s, t) ds$. Mimicking the proof of Lemma 3.2, one can show that there is a positive constant $c_0$ independent of time such that

$$
\tilde{\lambda}(t) \geq c_0, \quad t \geq 0.
$$

Since $\tilde{F}$ also satisfies the equation (2.8), one can also use the same method in the above case to show that $\tilde{F} \geq (\Delta - \delta)C > 0$, where $\Delta = L, \delta$ and $C$ are positive constants independent of time. Now we have the limit $\lim_{u \to 0^+} \tilde{F}(u) = 0$, so the curvature $\kappa(\theta, t)$ also has a positive lower bound (3.4) independent of time under the flow.

This lemma tells us that the evolution equation of $\kappa$ does not degenerate as $t \to +\infty$. Then uniform bounds of $\kappa$ ($0 < m \leq \kappa \leq M$) imply that there exists a constant $C_n$ independent of time such that

$$
\left| \frac{d^n F}{du^n}(\kappa) \right| \leq C_n \quad \text{(3.5)}
$$

under the flow (2.1). So there is also a positive constant $c_1$ independent of time such that

$$
F'(\kappa(\theta, t)) \geq c_1 \quad \text{(3.6)}
$$

for all $(\theta, t) \in [0, 2\pi] \times [0, +\infty]$.

Noticing that

$$
\int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta = \int_0^{2\pi} \left( F' \right)^2 \left( \frac{\partial \kappa}{\partial \theta} \right)^2 d\theta \geq c_1^2 \int_0^{2\pi} \left( \frac{\partial \kappa}{\partial \theta} \right)^2 d\theta
$$
and using the upper bound of \( \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta \) (see (3.2)), one has a bound of the \( L^2 \)-norm of \( \frac{\partial \kappa}{\partial \theta} \):

\[
\int_0^{2\pi} \left( \frac{\partial \kappa}{\partial \theta} \right)^2 d\theta \leq \frac{c_1^2}{c_1} \left[ \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta + 2\pi F^2(M) + \frac{1}{2\pi} \left( \int_0^{2\pi} F_0 d\theta \right)^2 \right].
\] (3.7)

Since

\[
|\kappa(\theta_1, t) - \kappa(\theta_2, t)| = \left| \int_{\theta_1}^{\theta_2} \frac{\partial \kappa}{\partial \theta} d\theta \right| \leq \sqrt{|\theta_1 - \theta_2|} \sqrt{\int_0^{2\pi} \left( \frac{\partial \kappa}{\partial \theta} \right)^2 d\theta},
\]

\( \kappa(\theta, t) \) is equicontinuous with respect to \( \theta \).

The curvature has uniformly upper and lower bounds and it is equicontinuous, so there is a convergent subsequence \( \kappa(\theta, t_i) \) as \( t_i \to +\infty \). Furthermore, one can show that the curvature converges to a constant as \( t \to +\infty \).

**Lemma 3.4.** Under the flow (2.1), the curvature of the evolving curve has a limit:

\[
\lim_{t \to +\infty} \kappa(\theta, t) = \frac{2\pi}{L}.
\] (3.8)

**Proof.** By the evolution equations of the area \( A(t) \) and \( F(\kappa) \), one can compute that

\[
\frac{d^2 A}{dt^2} = \frac{d}{dt} \left[ -\int_0^{2\pi} \frac{F}{\kappa} d\theta + L \lambda \right] = -\int_0^{2\pi} F' \kappa' \left( \frac{\partial^2 F}{\partial \theta^2} + F - \lambda(t) \right) d\theta + \int_0^{2\pi} F \left( \frac{\partial^2 F}{\partial \theta^2} + F - \lambda(t) \right) d\theta + L \frac{2\pi}{\kappa^2} \int_0^{2\pi} F' \kappa d\theta + \lambda \int_0^{2\pi} F' \kappa d\theta
\]

\[
-\int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta + \int_0^{2\pi} F'^2 d\theta - \lambda \int_0^{2\pi} F d\theta - \frac{L}{2\pi} \int_0^{2\pi} \kappa^2 \frac{F''}{F'} \left( \frac{\partial F}{\partial \theta} \right)^2 \kappa d\theta - \frac{L}{\pi} \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 \kappa d\theta + \frac{L}{2\pi} \int_0^{2\pi} F' \kappa^2 F' d\theta - \frac{\lambda L}{2\pi} \int_0^{2\pi} F' \kappa^2 d\theta
\]

Since \( F, |F''|, F' \) and \( \kappa \) all have positive upper bounds independent of time and \( F' \) has a positive lower bound, \( \frac{d^2 A}{dt^2} \) also has an upper bound independent of time.
Noticing that \( \frac{dA}{dt} \) is nonnegative and
\[
\int_0^{+\infty} \frac{dA}{dt} \, dt = \lim_{t \to \infty} A(t) - A_0 \leq \frac{L^2}{4\pi},
\]
one can obtain
\[
\lim_{t \to \infty} \frac{dA}{dt} = 0. \quad (3.9)
\]
Substituting a convergent subsequence \( \kappa(\theta, t_i) \) into the evolution equation of \( A \) can yield
\[
\frac{dA}{dt}(t_i) = -\int_0^{2\pi} \frac{F(\kappa(\theta, t_i))}{\kappa(\theta, t_i)} \, d\theta + \frac{L}{2\pi} \int_0^{2\pi} F(\kappa(\theta, t_i)) \, d\theta.
\]
Denote by \( \kappa_\infty(\theta) \) the limit of the \( \kappa(\theta, t_i) \) as \( t_i \to +\infty \). Then letting \( t_i \to \infty \), one can get
\[
0 = -\int_0^{2\pi} \frac{F(\kappa_\infty(\theta))}{\kappa_\infty(\theta)} \, d\theta + \frac{L}{2\pi} \int_0^{2\pi} F(\kappa_\infty(\theta)) \, d\theta.
\]
It follows from the equality case in Andrews’ inequality [2] that \( \kappa_\infty(\theta) \) is a constant. Since the flow (2.11) preserves the length, we have
\[
\kappa_\infty(\theta) = \frac{2\pi}{L}.
\]
It is shown that all convergent subsequence of \( \kappa(\theta, t) \) tend to the same limit \( \frac{2\pi}{L} \). Therefore the curvature function also has the same limit. The limit (3.8) is proved. \( \square \)

It follows from (3.4) and (3.6) that the evolution equation of \( \kappa \) is uniformly parabolic with smooth coefficients. Set \( a^{11}(\kappa) := \kappa^2 F'(\kappa) \). By the uniform bounds of \( \kappa \) (see the equations (3.4) and (2.10)), there exist positive constants \( \beta_1, \beta_2 \) independent of time such that
\[
1 \leq \beta_2 a^{11}(x^2),
\]
for \( |x| \geq \beta_1 \). By Lieberman’s local gradient estimate (Theorem 11.18 in [19]) and the fact that \( \kappa(\theta, t) \) is periodic with respect to \( \theta \), there exists a constant \( M_1 \) independent of time such that
\[
\left| \frac{\partial \kappa}{\partial \theta}(\theta, t) \right| \leq M_1, \quad (\theta, t) \in [0, 2\pi] \times (0, +\infty). \quad (3.10)
\]
By the standard regularity theory of parabolic equations (see [19]) or a simple method by Lin-Tsai [21], there exist constants \( M_i \) independent of time such that
\[
\left| \frac{\partial^i \kappa}{\partial \theta^i}(\theta, t) \right| \leq M_i, \quad i = 2, 3, \ldots. \quad (3.11)
\]
So we also have constants \( \tilde{M}_i \) independent of time such that

\[
\left| \frac{\partial^i F}{\partial \theta^i}(\theta, t) \right| \leq \tilde{M}_i.
\] (3.12)

From (3.8) and (3.10)-(3.12), one can conclude

\[
\lim_{t \to +\infty} \frac{\partial \kappa}{\partial \theta}(\theta, t) = \lim_{t \to +\infty} \frac{\partial^i F}{\partial \theta^i}(\theta, t) = 0,
\] (3.13)

where \( i = 1, 2, \ldots \). One obtains the \( C^\infty \) convergence of the curvature.

**Remark 3.5.** We have shown that the curvature \( \kappa \) is uniformly bounded (see (2.10) and (3.4)). Another method to do the gradient estimate for \( \kappa \) is to use the evolution equation of \( F \) (see (2.8)). One may consider the function \( \varphi := \frac{\partial F}{\partial \theta} + \alpha F^2 \), where \( \alpha \) is a constant. By direct computations, one can obtain the evolution equation of \( \varphi \). Following a very useful method by Lin-Tsai [21], one can show that \( \varphi \) has uniform lower and upper bounds independent of time. So \( |\frac{\partial F}{\partial \theta}| \) is also uniformly bounded. By induction, the uniform estimate (3.12) can be obtained for all \( i \geq 2 \). Using (3.5) and (3.6), one can bound all derivatives of \( \kappa \), so (3.13) is also proved.

### 4 Convergence of the evolving curve

Until now, the evolving curve has been proved asymptotically circular. In order to prove that the flow (2.1) can efficiently evolve \( X_0 \) to a circle, one needs to show that the evolving curve indeed has a limit as time goes to infinity. In fact, if the speed of the flow (2.1) exponentially decays then the evolving curve \( X(\cdot, t) \) converges to a fixed limiting curve as time goes to infinity.

By the evolution equation of \( F \) (see (2.8)),

\[
\frac{d}{dt} \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta = -\int_0^{2\pi} \frac{\partial^2 F \partial F}{\partial \theta^2 \partial t} d\theta
\]

\[
= -\int_0^{2\pi} \frac{\partial^2 F}{\partial \theta^2} F' \kappa^2 \left[ \frac{\partial^2 F}{\partial \theta^2} + F - \lambda(t) \right] d\theta
\]

\[
= -\int_0^{2\pi} F'' \kappa^2 \left( \frac{\partial^2 F}{\partial \theta^2} \right)^2 d\theta + \int_0^{2\pi} F' \kappa^2 \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta + \int_0^{2\pi} \kappa (F - \lambda) \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta.
\]

By (3.8), one obtains

\[
\lim_{t \to +\infty} F'(\kappa) \cdot \kappa^2 = F'\left( \frac{2\pi}{L} \right) \cdot \left( \frac{2\pi}{L} \right)^2
\]
and
\[
\lim_{t \to +\infty} (F(\kappa) - \lambda(t)) = 0.
\]

For any \(\varepsilon > 0\), there exists a \(T_0 > 0\) such that \(|F - \lambda| < \varepsilon\) and
\[
F' \left( \frac{2\pi}{L} \right) \cdot \left( \frac{2\pi}{L} \right)^2 \leq \varepsilon \leq F'(\kappa) \cdot \kappa^2 \leq F' \left( \frac{2\pi}{L} \right) \cdot \left( \frac{2\pi}{L} \right)^2 + \varepsilon
\]
for \(t > T_0\). One has the following estimate for large \(t > T_0\):
\[
\frac{d}{dt} \left. \frac{1}{2} \right|_{0}^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 \leq - \left( \frac{F'}{2\pi} \left( \frac{2\pi}{L} \right) \cdot \left( \frac{2\pi}{L} \right)^2 \right) \int_{0}^{2\pi} \left( \frac{\partial F}{\partial \theta^2} \right)^2 d\theta + \left( \frac{F'}{2\pi} \left( \frac{2\pi}{L} \right) \cdot \left( \frac{2\pi}{L} \right)^2 + \varepsilon \right) \int_{0}^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta + \left( \frac{C_2}{c_1} M^2 \varepsilon + 2M \varepsilon \right) \int_{0}^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta.
\]

(4.1)

Since \(\frac{\partial F}{\partial \theta} = F'(\kappa) \left( \frac{1}{\rho} \frac{\partial \rho}{\partial \theta} \right)\) and \(\frac{\partial^2 F}{\partial \theta^2} = F''(\kappa) \left( \frac{1}{\rho^2} \frac{\partial \rho}{\partial \theta} \right)^2 - F'(\kappa) \frac{\partial^2 \rho}{\partial \theta^2} + F'(\kappa) \frac{\partial^2}{\partial \theta^2} \left( \frac{\partial \rho}{\partial \theta} \right)^2\), one can conclude from the Wirtinger type inequality which says
\[
\int_{0}^{2\pi} \left( \frac{\partial^2 \rho}{\partial \theta^2} \right)^2 d\theta \geq 4 \int_{0}^{2\pi} \left( \frac{\partial \rho}{\partial \theta} \right)^2 d\theta
\]
and the limits in (3.13) that if \(t\) is large enough then
\[
\int_{0}^{2\pi} \left( \frac{\partial F}{\partial \theta^2} \right)^2 d\theta \geq (4 - \varepsilon) \int_{0}^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta.
\]

(4.2)

Substituting (4.2) into (4.1) for large \(t\), one obtains
\[
\frac{d}{dt} \left. \frac{1}{2} \right|_{0}^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 \leq - \left( (3 - \varepsilon) F'(\kappa) \left( \frac{2\pi}{L} \right) \cdot \left( \frac{2\pi}{L} \right)^2 + 5\varepsilon \right) \int_{0}^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta + \left( \frac{C_2}{c_1} M^2 \varepsilon + 2M \varepsilon \right) \int_{0}^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta.
\]

Let \(\varepsilon \in (0, \frac{1}{2})\) be small enough such that \(5\varepsilon + \frac{C_2}{c_1} M^2 \varepsilon + 2M \varepsilon < 2F'(\kappa) \left( \frac{2\pi}{L} \right)^2\). Then one gets for large \(t\) that
\[
\frac{d}{dt} \int_{0}^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta \leq -F'(\kappa) \cdot \left( \frac{2\pi}{L} \right)^2 \int_{0}^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta
\]
and thus \(\int_{0}^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta\) decays exponentially.
Lemma 4.1. If $t$ is large enough then the speed of the flow (2.1) decays exponentially.

Proof. It follows from Sobolev’s inequality that

$$|F - \lambda| \leq \frac{1}{\sqrt{2\pi}} \left( \int_0^{2\pi} (F - \lambda)^2 d\theta \right)^{\frac{1}{2}} + \sqrt{2\pi} \left( \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta \right)^{\frac{1}{2}}.$$

Since $\int_0^{2\pi} (F - \lambda) d\theta = 0$, Wirtinger’s inequality implies that

$$\int_0^{2\pi} (F - \lambda)^2 d\theta \leq \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta. \quad (4.4)$$

By the inequality (4.4),

$$|F - \lambda| \leq \left( \frac{1}{\sqrt{2\pi}} + \sqrt{2\pi} \right) \left( \int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta \right)^{\frac{1}{2}}.$$

Since $\int_0^{2\pi} \left( \frac{\partial F}{\partial \theta} \right)^2 d\theta$ exponentially decays when $t > T_0$, we have done. \qed

Theorem 4.2. Under the flow (2.1), the evolving curve converges to a limiting circle as time goes to infinity.

Proof. Since the speed $|F - \lambda|$ decays exponentially if $t$ is large enough, one can obtain the result via integrating the equation (2.1). \qed

Combining Theorem 2.5, Lemma 3.4 and Theorem 4.2, one can prove Theorem 1.1.

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