A NON – SINGULAR COSMOLOGICAL MODEL WITH SHEAR AND ROTATION

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December 16, 2011

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Abstract

We have investigated a non-static and rotating model of the universe with an imperfect fluid distribution. It is found that the model is free from singularity and represents an ever expanding universe with shear and rotation vanishing for large value of time

AMS Mathematics Subject Classification: 83 F Key words and Phrases: Cosmology, non singular model
1 INTRODUCTION

The “Singularity theorems” of Hawking, Penrose and Geroch\(^1\) show that any general relativistic model universe (a) satisfying reasonable causality and generality condition (b) possessing a surface of the past (future) of which the light cones starts converging and (c) containing matter which satisfies the energy condition.

\[
\left( T_{ij} - \frac{1}{2} T g_{ij} \right) t^i t^j \geq 0 \tag{1}
\]

For any unit time like vector \( t^i \) must contain a time like curve with a past (future) end point at finite proper time (a singularity). In most known examples this end point is associated with an infinite curvature singularity. The actual universe is expected to satisfy generally all the above three requirements. Does this mean that the universe actually passed through a singularity in the past?

Some workers are resigned to the singularity and try to make it acceptable by conjecturing that it is of the mild variety\(^2\) or else by relegating the big bang singularity to the “infinite past” according to physically appealing time scale\(^3\). Other contends cosmological singularity is only a symptom of the incompleteness of the theory. They propose that it is absent in more fundamental theory in which Einstein’s Equations are appropriately modified by quantum effects\(^4\), phenomenological quadratic terms in the curvature\(^5\), or the effect of the torsion\(^6\).

It is known that once the energy condition (1) is given up, singularity free cosmologies become possible. The Examples have been provided by Hoyle-Narlikar C-field theory\(^7\),\(^12\),\(^13\), Murphy\(^8\), Fulling and Parker\(^9\), Bekenstein\(^10\) and Prigogine\(^11\). Recently Bianchi type II and III vacuum cosmologies have been a matter of interest to the cosmologists\(^14\)–\(^17\). In this paper we have investigated a non-singular and rotating model of the universe by considering the following Bianchi type III general metric

\[
ds^2 = e^{2\psi} \left[ -b(dx^1)^2 - e^{2x^1} a(dx^2)^2 - c(dx^3)^2 + 2e^{x^1} dx^4 dx^2 + (dx^4)^2 \right] \tag{2}
\]

Where \( a, b, c \) & \( \psi \) are functions of time \( x^4 \), as a generalization of Godel’s stationary model\(^18\) and subsequent rotating model given by Heckmann – Schucking\(^19\) and Reval – Vaidya\(^20\),\(^21\). The Energy momentum tensor has been taken as that of imperfect fluid in Lichnerowicz\(^22\).

\[
T^{ij} = (p + \rho) u^i u^j - pg^{ij} + (q - p) v^i v^j \tag{3}
\]

Where \( \rho \) is the energy density of the fluid, \( p \) is the pressure in \( x^1 \) or \( x^2 \) direction, \( q \) is the pressure in \( x^3 \) direction. The fluid flow vectors \( u^i \) is a normalized time – like vector, where as \( v^i \) is a space – like vector so that

\[
u^i u_i = 1
\]
\[ u^i = \left(0, 0, 0, \frac{1}{\sqrt{g_{44}}} \right), a \]

\[ v^i v_i = -1 \]

and

\[ v^i = \left(0, 0, \frac{1}{\sqrt{-g_{33}}} 0 \right) \]

in co-moving co-ordinates.

By solving Einstein’s field equations

\[ R^{ij} - \frac{1}{2} R g^{ij} = -T^{ij} + \Lambda g^{ij} \quad (4) \]

It is found that the model is free from singularity and represents an ever-expanding universe with shear and rotation vanishing for large value of time. The matter-density of the universe is positive throughout the span of the model but the pressure is found to be negative. Thus like other non-singular cosmological models, this model also violates energy-condition (1). In section (2), we have formed Einstein’s field equations (4), corresponding to metric (2) and Energy momentum tensor (3). In section (3), a particular solution of Einstein’s field equations have been obtained. We have also discussed various geometrical and physical properties of the model. The last section contains concluding remarks.

2. EINSTEIN’S FIELD EQUATION

The non-vanishing components of Ricci Tensor \( R_{ij} \) for the metric (2) are as follows.

\[ R_{11} = \left[ -\left( \frac{a b a_4}{2(a+1)} + \frac{a b a_4}{(a+1)} \right) \right] - \left( \frac{a b a_4}{2(a+1)} + \frac{a a_4}{(a+1)} \right) \left( 2 \psi_4 - \frac{a a_4}{2(\alpha + 1)} - \frac{b c}{2a} + \frac{c}{2c} \right) + \frac{a_4 (2a+1)}{2(a+1)} \]

\[ R_{22} = e^{2x^4} \left[ -\left( \frac{a b a_4}{2(a+1)} + \frac{a a_4}{(a+1)} \right) \right] - \left( \frac{a b a_4}{2(a+1)} + \frac{a a_4}{(a+1)} \right) \left( 2 \psi_4 - \frac{(a+2)a_4}{2a(a+1)} + \frac{b c}{2b} + \frac{c}{2c} \right) + \frac{a (2a+3)}{2b(a+1)} \]

\[ R_{33} = \left[ -\left( \frac{a c a_4}{2(a+1)} + \frac{a c a_4}{(a+1)} \right) \right] - \left( \frac{a c a_4}{2(a+1)} + \frac{a c a_4}{(a+1)} \right) \left( 2 \psi_4 - \frac{a_4}{2(\alpha + 1)} + \frac{b d}{2b} - \frac{c}{2c} \right) \]

\[ R_{42} = e^{2x^4} \left[ \left( \frac{a b a_4}{2(a+1)} + \frac{a a_4}{(a+1)} \right) \right] + \left( \frac{a b a_4}{2(a+1)} + \frac{a a_4}{(a+1)} \right) \left( 2 \psi_4 - \frac{a_4}{2(\alpha + 1)} + \frac{b d}{2b} + \frac{c}{2c} \right) - \frac{1}{2b(\alpha + 1)} \]

\[ R_{44} = \left( 4 \psi_4 + \frac{a b a_4}{2(a+1)} + \frac{b c}{2b} + \frac{c}{2c} \right) - \left( \frac{(a+2) a_4}{(a+1)} \right) - \left( 4 \psi_4 + \frac{a a_4}{2(\alpha + 1)} + \frac{b c}{2b} + \frac{c}{2c} \right) \left( \frac{(a+2) a_4}{(a+1)} \right) \]
\[ R_{41} = \left[ -\frac{\psi_4}{a+1} + \frac{(2a+3)a_4}{4(a+1)^2} - \frac{(2a+1)b_4}{4(a+1)b} - \frac{c_4}{4(a+1)c} \right] \]

\[ R_{21} = ae^{-x^4} \left[ \frac{\psi_4}{a+1} + \frac{(2+a)a_4}{4(a+1)^2a} - \frac{b_4}{4(a+1)b} - \frac{c_4}{4(a+1)c} \right] \]

The Einstein's field equation (4), corresponding to metric (2) and energy momentum tensor (3) yields

\[ R_{21} = 0 \tag{5} \]

\[ R_{41} = 0 \tag{6} \]

\[ R_{44} = e^{-x^4} R_{42} \tag{7} \]

\[ G_{11} = (p + \Lambda) g_{11} \tag{8} \]

\[ G_{22} = - \left[ (p + \rho) \frac{g_{22}^2}{g_{44}} - pg_{22} \right] + \Lambda g_{22} \tag{9} \]

\[ G_{33} = (q + \Lambda) g_{33} \tag{10} \]

Equation (5) and (6) give

\[ \frac{a_4}{a} = \frac{b_4}{b} (a + 1 \neq 0) \tag{11} \]

and

\[ 4\psi_4 = -\frac{a_4}{a(a+1)} - \frac{c_4}{c} \tag{12} \]

Equations (7) – (10) give by using (11) and (12)

\[ \frac{-a_4 c_4}{a+1} + \frac{a_4 c_4}{2ac} - \frac{c_4^2}{4c^2} + \frac{(2a^2 - 6a - 1) a_4^2}{4a^2 (a+1)^2} = 0 \tag{13} \]

\[ e^{2\psi} (p + \Lambda) \left[ -4\psi_4^4 - 2\psi_4^2 - \frac{2(a+2)a_4\psi_4}{a(a+1)} - \frac{2a_4\psi_4}{a} - \frac{c_4^4}{c} + \frac{c_4^2}{2c^2} + \frac{a_4^2}{2a(a+1)} - \frac{(a+2)a_4 c_4}{2ac(a+1)} + \frac{1}{2ab} \right] = 0 \tag{14} \]
\[ \frac{1}{2(a+1)} \left[ -4\psi_{14} - 2\psi_{14}^2 - \frac{2(a+2)a^4 \psi_4}{a(a+1)} - 2a^4 \psi_4 - \frac{a^4}{a} - \frac{a^4}{c} + \frac{a^4}{2c} + \frac{a^4}{2a(a+1)} - \frac{(a+2)a^6 c}{2a(a+1)} + \frac{4a+1}{2c^6} \right] = e^{2\psi} (\rho - \Lambda) \]  

(15)

\[ \frac{a}{(2a+1)} \left[ -4\psi_{14} - 2\psi_{14}^2 - \frac{2(a+2)a^4 \psi_4}{a(a+1)} - 2a^4 \psi_4 - \frac{a^4}{a} + \frac{a^4}{2a} + \frac{a^4}{2a(a+1)} - \frac{(a+2)a^2}{2a(a+1)} - \frac{4a+3}{2a^3} \right] = e^{2\psi} (q + \Lambda) \]  

(16)

3. A PARTICULAR SOLUTION

The functional form of \( \psi \) may be determined if we make a simplifying assumption that

\[ \frac{a_4}{a} = \frac{c_4}{c} \]  

(17)

This leads to this following solutions of equations (11) – (13)

\[ b = \beta a \]  

(18)

\[ c = \gamma a \]  

(19)

\[ e^{4\psi} = \kappa^4 \left( \frac{a + 1}{a^2} \right) \]  

(20)

and

\[ a_4 = A \frac{(a + 1)^{7/4}}{a} \]  

(21)

where \( \beta, \gamma, \kappa \) and \( A \) are arbitrary constants of integration. We may replace

\( \beta x^1 \rightarrow x^1 \) & \( \gamma x^3 \rightarrow x^3 \)

Hence we take, without loss of generality

\[ a = b = c \]

Changing time \( t \) to proper time \( T \) via transformation

\[ dT = e^\psi dt \]

and denoting new time derivative by \( a_T \), the equation (19) is transformed to

\[ a_T = A \frac{(a + 1)^{3/2}}{\kappa a^{1/2}} \]  

(22)
which yields on integration

\[ \ln \left( a^{1/2} + (a + 1)^{1/2} \right) - \frac{a^{1/2}}{(a + 1)^{1/2}} = \frac{A}{2\kappa T + L} \]  

(23)

where L is arbitrary constant of integration. The equation (23) shows that for proper time T to be real

\[ a \geq 0 \]

And T increases monotonically with “a”.

Thus the metric (2) takes the following form

\[ ds^2 = dT^2 + \frac{2(a + 1)^{1/4}}{a^{1/2}}e^{x^1}dTdx^2 - (a + 1)^{1/2} \left[ (dx^1)^2 + e^{2x^1}(dx^2)^2 + (dx^3)^2 \right] \]  

(24)

where we have taken arbitrary constant k = 1 for shake of simplicity.

4. Some Physical and geometrical properties of the model:

(i) The expansion scalar \( \theta \) is given by

\[ \theta = \frac{1}{3}u^k_{\; ; k} = \frac{1}{12}A \frac{(3a - 2)(a + 1)^{1/2}}{a^{3/2}} \]

This shows that when \( a < 2/3 \), \( \theta \) is negative, therefore the model shows contraction for \( a < 2/3 \). However this can be avoided if we choose the value of arbitrary constant L in equation (23) such that when \( a = 2/3 \), proper time \( T = 0 \).

This gives constant \( L = 0.111777 \).

Thus for proper time \( T \geq 0 \), the model will represent an ever expanding universe. For large value of ‘a’ expansion scalar \( \theta \) become stationary

\[ a \to \infty \Rightarrow \theta \to \frac{1}{4}A \]

(ii) Inserting value of \( a_4 \) and \( \psi \) in the field equations (14) – (16), we get –

\[ p + \Lambda = -\frac{3}{16}A^2 + \frac{1}{4(a + 1)^{3/2}} \]  

(25)

\[ q + \Lambda = -\frac{3}{16}A^2 + \frac{4a + 3}{4(a + 1)^{3/2}} \]  

(26)
\[ \rho - \Lambda = \frac{3}{16} A^2 - \frac{(4a + 1)}{4 (a + 1)^{3/2}} \] 

\[ \frac{3}{16} A^2 \geq \frac{4a + 1}{4 (a + 1)^{3/2}} \Rightarrow \rho - \Lambda \geq 0 \]

As \( \frac{4a + 1}{4(a+1)^{3/2}} \) is decreasing function of \( a \), its value is maximum at minimum value of \( a \) i.e. \( a=2/3 \).

This gives

\[ \frac{3}{16} A^2 \geq .426. \]

This choice of arbitrary constant \( A \) will make energy-density \( \rho \) always positive throughout the span of the model, but then pressures \( p \) and \( q \) will be negative barring the little span when

\[ \frac{3}{16} A^2 \leq \frac{1}{4 (a + 1)^{3/2}} \& \frac{3}{16} A^2 \leq \frac{4a + 3}{4 (a + 1)^{3/2}} \]

The negative value is very little nearly -.426.

(iii) Equation (1), in presence of \( \Lambda \) should be read as

\[ (T_{ij} - \frac{1}{2} g_{ij} + \Lambda g_{ij}) t^i t^j \geq 0 \]

If we take \( t^i = u^i \) as a special case, then the equation turns into

\[ \rho + 2p + q + 2\Lambda \geq 0 \]

in our case. Inserting values of \( \rho, p \) and \( q \) from equations (25-27) we get

\[ \rho + 2p + q + 2\Lambda = -\frac{3}{16} A^2 + \frac{1}{(a + 1)^{3/2}} \]

which is negative, although it is nearly -.213. As stated earlier this is the requirement of non gingular models which violates conditions laid by Hawking, Penrose and Geroch.\(^1\)

(iv) The non vanishing components of shear tensor

\[ \sigma_{ij} = u_{i;j} - (g_{ij} - u_i u_j) \theta \]

\[ \sigma_{11} = -A \frac{(a + 1)}{a^{3/2}} \]

\[ \sigma_{22} = A e^{2x^2} \frac{(a - 2) (a + 1)}{12a^{1/2}} \]
\[
\sigma_{33} = -A \frac{(a+1)}{6a^{3/2}}
\]

\[\therefore \text{shear– scalar}\]

\[\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = A^2 \frac{(3a^2 + 4a + 4)}{96 (a + 1) a^3}\]

\[\therefore a \to \infty \Rightarrow \sigma^2 \to 0\]

i.e. for large values of time, shear vanishes.

(v) The non-vanishing component of angular velocity vector

\[\omega^j = \frac{1}{2} \sqrt{(-g)} \varepsilon^{ijkl} (u_{j;k} - u_{k;j}) u_l\]

is given by

\[\omega^3 = \frac{1}{2} \frac{1}{(a + 1)}\]

\[\therefore a \to \infty \Rightarrow \omega \to 0\]

This shows that the model possesses a non zero finite rotation about \(x^3\) direction, but it goes on diminishing monotonically with the passage of time.

5. Conclusion

The main purpose of the paper is to investigate Cosmological model without singularity within framework of general relativity. We have proposed an ever expanding and nonsingular model of rotating universe. As stated in the introduction that the singularity which appears in almost all the standard Cosmological models of the universe is not accepted by workers in the field (see ref.[2-13]) this paper is a simple and elegant effort in this direction. The initial state of the universe was inhomogeneous and anisotropic, but slowly-slowly with the advent of time inhomogeneity and anisotropy ceased out and at present the universe is spatially homogeneous and isotropy. This behavior is very well depicted in our model. We started with universe filled with imperfect fluid showing shear and rotation but these elements are gradually decreasing with advent of time and ultimately the model will be turned up into a spatially homogeneous and isotropic one with constant values of density, pressure and expansion scalar.

It is noteworthy to state that our model is not a special case of rotating model given in the reference [19-21]. These models are valid only for a limited time interval whereas our model is ever lasting.
6. Acknowledgement

The author is highly grateful to Prof. V.B. Johri, emeritus Prof., Deptt. of Mathematics and Astronomy Lucknow University for his valuable suggestions and comments during the preparation of the paper.

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