EMBEDDINGS AND IMMERSIONS OF TROPICAL CURVES

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Abstract. We construct immersions of trivalent abstract tropical curves in the Euclidean plane and embeddings of all abstract tropical curves in higher dimensional Euclidean space. Since not all curves have an embedding in the plane, we define the tropical crossing number of an abstract tropical curve to be the minimum number of self-intersections, counted with multiplicity, over all its immersions in the plane. We show that the tropical crossing number is at most quadratic in the number of edges and this bound is sharp. For curves of genus up to two, we systematically compute the crossing number. Finally, we observe that our immersed tropical curves can be lifted to nodal algebraic curves using the results of Mikhalkin and Shustin.

1. Introduction

For many geometric structures, there are both abstract and embedded versions of the objects, and then it is natural to ask how the two compare. In this paper, we study realizations of abstract tropical curves in a real vector space. In particular, we prove the following:

Theorem 1.1. Let $\Gamma$ be an abstract tropical curve and suppose that $d$ is the maximum degree of a vertex of $\Gamma$. Then $\Gamma$ has a smooth embedding in $\mathbb{R}^n$ so long as $n$ is at least $\max\{3, d - 1\}$ and $\Gamma$ has an immersion in $\mathbb{R}^2$ if $d$ is at most 3.

For comparison, any algebraic curve can be immersed in $\mathbb{P}^2$ or embedded in $\mathbb{P}^3$. However, for tropical curves, the local model for a degree $d$ vertex only embeds in $\mathbb{R}^{d-1}$, so it is never possible to have an embedding or immersion in $\mathbb{R}^n$ when $n < d - 1$.

Any embedding or immersion of an abstract tropical curve is also an embedding or immersion respectively of the underlying graph. Not all graphs have an embedding in $\mathbb{R}^2$ and a quantitative measure for the failure of a graph to be planar is given by the minimum number of crossings among all planar immersions. We analogously define the tropical crossing number for abstract tropical curves to be the minimum number of crossings over all its planar immersions, counted with tropical multiplicities. These multiplicities are a special case of those used in the definition of stable intersection, as in [RGST05 Sec. 4]. Our proof of the existence of immersions gives the following bound on the crossing number:
Theorem 1.2. If $\Gamma$ is a trivalent graph with $e$ edges, then the tropical crossing number of $\Gamma$ is at most $O(e^2)$.

The quadratic bound on the crossing number from Theorem 1.2 is sharp up to a constant factor, even for planar graphs:

Theorem 1.3. There exists a family of trivalent tropical curves with $e$ edges, whose underlying graph is planar, but whose crossing number is $\Theta(e^2)$.

Thus, while the tropical crossing number is trivially greater than the graph-theoretic crossing number, the gap can be large. The trivalent tropical curves in Theorem 1.3 are the Brill-Noether general “chain of loops” tropical curves studied in [CDPR12], and indeed, the same lower bound on the crossing number would apply to any tropical curve with the divisorial gonality of a Brill-Noether general curve.

Theorems 1.1 and 1.3 are proved in Sections 2 and 3 respectively, after which we turn to applications of these theorems. In Section 4, we study the crossing numbers of abstract tropical curves of genus at most 2. Perhaps surprisingly, in Proposition 4.7 we give a case of an abstract tropical curve which has a planar embedding for generic values of the metric parameters, but not for specializations. As a consequence, the crossing number is neither lower nor upper semi-continuous in the metric parameters (see Remark 4.8).

A complementary analysis, going up to curves of genus 5, has been undertaken in [BJMS14]. They focus on planar curves, i.e. curves of crossing number 0, for which they describe the closure of the locus of such curves within the moduli space of all tropical curves in terms of inequalities on the metric parameters. Their techniques are computational and demonstrate that there are effective and practical algorithms for classifying planar tropical curves or, more generally, curves with specified crossing number.

Finally, in Section 5 we combine our results with Mikhalkin’s correspondence theorem to show that any abstract tropical curve can be lifted to a nodal plane curve whose tropicalization is also nodal. In forthcoming work of Cheung, Fantini, Park, and Ulirsch, this result will be generalized beyond the planar case [CFPU14]. Their paper will establish that any tropical curve is the skeleton of an algebraic curve with an embedding whose tropicalization is smooth.

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EMBEDDINGS AND IMMERSIONS OF TROPICAL CURVES

2. EMBEDDINGS AND IMMERSIONS OF TROPICAL CURVES

In this section, we present the construction of the smooth planar immersion and embedding of tropical curves in $\mathbb{R}^3$. We start by recalling the definitions of abstract and embedded tropical curves, as well as the relation between the two.

**Definition 2.1.** An abstract tropical curve is a finite connected graph possibly with loops or multiple edge together with either a positive real number or infinity attached to each edge, which will be known as the length of the edge. Any edge with infinite length must have a degree 1 vertex at one of its endpoints, which will be referred to as an infinite vertex.

A subdivision of an abstract tropical curve consists of replacing an edge with two consecutive edges whose lengths add up to the length of the original edge. If the original edge was infinite then the subdivided edge incident to the infinite vertex must also be infinite. Two tropical curves are equivalent if one can be transformed into the other by a series of subdivisions and reverse subdivisions.

Our definition of an abstract tropical curve is based on the one in [Mik05], but slightly more general because we don’t require all 1-valent vertices to be infinite. Because of this, our definition is equivalent to that in [ABBR13, Sec. 2.1].

**Remark 2.2.** The underlying graph of a tropical curve has a natural realization as a topological space and the lengths along the edges additionally give a metric on this realization, away from the infinite vertices. This metric realization gives an alternative characterization of abstract tropical curves up to equivalence as inner metric spaces which have a finite cover by open sets isometric to star shapes. See, for example, [ABBR13, Sec. 2.1], for a definition from this perspective.

We will also consider the coarser relation on tropical curves called tropical modification. An elementary tropical modification is formed by adding an infinite edge at a finite vertex. A tropical modification is any sequence of elementary tropical modifications, subdivisions, and reverse subdivisions.

**Example 2.3.** In the middle of Figure 1 is an abstract tropical curve, which is a tropical modification of the tropical curve on the right of that figure. This tropical modification is obtained by first subdividing each of the edges of the latter graph into 4 edges of equal length and then attaching infinite edges to 7 of the 9 newly created vertices. In Example 2.7, we will see that the leftmost diagram in Figure 1 gives an immersion of the modified graph.

We now turn to the embedded side and define smooth and nodal tropical curves in $\mathbb{R}^n$. For any integer $d$ in the range $2 \leq d \leq n + 1$, the standard smooth model of valence $d$ is the union of the $d$ rays generated by the coordinate vectors $e_1$ through $e_{d-1}$ and the vector $-e_1 - \cdots - e_{d-1}$. Up to changes of coordinates in $GL_n(\mathbb{Z})$ these are exactly the 1-dimensional
Figure 1. The immersed tropical curve on the left gives rise to the abstract tropical curve in the middle. The vertices and edges shown in gray are infinite and the other edges all have finite lengths, which happen to be equal. This abstract tropical curve is a tropical modification of the curve on the right, in which all edge lengths are equal to 4 times the finite edge lengths of its tropical modification. We will refer to the underlying graph of this last curve as a theta graph.

matroidal fans from [AK06] and such fans form the building blocks of tropical manifolds [MZ13, Def. 1.14].

**Definition 2.4.** A smooth tropical curve in $\mathbb{R}^n$ will be a union of finitely many segments and rays such that a neighborhood of any point is equal to a neighborhood of a standard model, after a translation and a change of coordinates taken from $GL_n(\mathbb{Z})$. □

For nodal curves, we have an additional local model. Recall that in algebraic geometry, a nodal curve singularity is one that is analytically isomorphic to a union of two distinct lines meeting at a point. For tropical curves, the local model for a node will analogously consist of two distinct (classical) lines with rational slopes, passing through the origin in $\mathbb{R}^2$. We will only ever consider nodal curves in the plane.

**Definition 2.5.** A nodal tropical curve in $\mathbb{R}^2$ is a union of finitely many segments and rays which is locally equal to either to a standard smooth local model or a nodal local model, again up to translation and the action of $GL_2(\mathbb{Z})$. □

The embedded and abstract tropical curves are related in that any embedded curve gives rise to an abstract one, as we now explain. By definition, each smooth tropical curve is a union of segments and rays, so if we add a vertex “at infinity” for each unbounded ray, we naturally get a finite graph. Thus, it only remains to assign lengths to the edges of this graph. In the standard local model, since we only allow changes of coordinates in $GL_n(\mathbb{Z})$, any segment is parallel to an integer vector. In other words, if $p_1$ and $p_2$ are the endpoints of a segment, then $p_1 - p_2 = \alpha v$, where $\alpha$ is a positive real number and $v$ is a primitive vector i.e., $v \in \mathbb{Z}^n$ is an integral vector. If we
further assume that the entries of $v$ are relatively prime, then $\alpha$ is uniquely determined and we use it as the length of the edge (cf. Mikhalkin [Mik05, Rmk. 2.4]).

For nodal curves in $\mathbb{R}^2$, we use the procedure as above for edges and smooth vertices. For the local model consisting of two lines passing through the origin, the abstract tropical curve has two vertices which map to the origin, one of which is an endpoint for the edges corresponding to one line and the other corresponding to the other line. This is illustrated in Figure 2.

**Definition 2.6.** Let $\Gamma$ be an abstract tropical curve. An embedding in $\mathbb{R}^n$ (resp. an immersion in $\mathbb{R}^2$) is a smooth (resp. nodal) tropical curve in $\mathbb{R}^n$ (resp. $\mathbb{R}^2$) which is isometric to a tropical modification of $\Gamma$. □

**Example 2.7.** The leftmost curve of Figure 1 is a nodal tropical curve with a single node. After resolving the node, we obtain the abstract tropical curve in the middle, so the leftmost curve is an immersion of the middle curve. Although the segments in the immersed curve have different lengths in the Euclidean metric, each finite edge of the realized graph has the same length.

Moreover, as we saw in Example 2.3, the middle abstract tropical curve is a tropical modification of the curve on the right, and thus the leftmost curve is also an immersion of the rightmost abstract tropical curve. □

We define the notion of multiplicities and then present a proof of Theorems 1.1 and 1.2. Recall from Definition 2.5 that a neighborhood of a nodal point of a tropical curve is the union of two lines. As in the construction of the edge lengths, we can assume that these lines are parallel to integral vectors $v = (v_1, v_2)$ and $u = (u_1, u_2)$ respectively, and that these integral vectors are primitive. Then the multiplicity of this node is

$$\left| \det \begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \end{pmatrix} \right|$$

This determinant is equivalent to the multiplicity of the stable intersection of the two lines, such as in [MS15, Def. 3.6.5]. We now prove our main theorems of this section.
Proof of Theorems 1.1 and 1.2. Fix $n \geq 2$ and let $\Gamma$ be a graph whose vertices have degree at most $n+1$. We will construct an embedding (if $n \geq 3$) or an immersion (if $n = 2$) of $\Gamma$, as follows.

If $\Gamma$ has any loops or parallel edges, we first subdivide them to obtain an equivalent graph which doesn’t have loops or parallel edges. We then label each vertex-edge incidence of $\Gamma$ with an integer from 0 to $n$ inclusive such that the labels are distinct at each vertex, and along each edge, the labels at the endpoints differ by $\pm 1$, modulo $n+1$. The first part of this condition can always be achieved since the degree of any vertex is at most $n+1$ and if the second part is violated at an edge, then we can resolve it by subdividing that edge sufficiently many times.

Fix a sufficiently small real number $L$ such that $nL$ is less than the length of any edge of $\Gamma$ and we will construct most of the embedding inside the $n$-dimensional cube $[0,L]^n$. We begin by defining $\iota: \Gamma \to \mathbb{R}^n$ on the finite vertices of $\Gamma$ by sending them to points inside this $n$-dimensional cube, and we will later assume that these points are generic.

To extend $\iota$ to a small neighborhood of each vertex, we let $e_1, \ldots, e_n$ denote the basis elements $(1,0,\ldots,0)$, $(0,1,\ldots,0)$, $\ldots$, $(0,0,\ldots,1)$ of $\mathbb{Z}^n$ and set $e_0 = (-1,\ldots,-1)$. We adopt the convention that the index $i$ in the expression $e_i$ is taken modulo $n+1$, so $e_{n+1} = e_0$, and so on. In a neighborhood of a vertex $v$, we send a small interval along each edge incident to $v$ to an interval in the direction $e_i$ if $i$ is the label on the edge-vertex pair.

In order to complete the local model, we add infinite edges at $v$ until its degree is $n+1$ and then send these infinite edges to rays in the direction $e_i$ for the indices $i$ which were not among the labels around $v$. We also send the infinite edges of $\Gamma$ to rays in the already determined directions.

We now extend the map $\iota$ to the finite edges of $\Gamma$ as follows. For each edge $E$ between vertices $u$ and $w$ in $\Gamma$, we already defined $\iota$ on neighborhoods of each endpoint, thus giving the initial tangent directions. We can assume that the tangent directions at $u$ and $w$ are $e_i$ and $e_{i+1}$ respectively, recalling that the indices are taken modulo $n+1$. Since $e_i, \ldots, e_{i+n-1}$ form a basis for $\mathbb{R}^n$, we can uniquely write:

$$\iota(w) - \iota(u) = \alpha_0 e_i + \alpha_1 e_{i+1} + \cdots + \alpha_{n-1} e_{i+n-1}$$

We set $m = |\alpha_0| + \cdots + |\alpha_{n-1}|$, and let $\ell$ be the length of the edge $E$. Recall that $\iota(w)$ and $\iota(u)$ are inside a box of side length $L$, so $m \leq nL < \ell$, by our choice of $L$.

We now let $\alpha'_0$, $\alpha''_0$, $\alpha'_1$, and $\alpha''_1$ be the unique numbers such that for $i = 0, 1$, we have $\alpha'_i > 0$, $\alpha''_i < 0$, $\alpha'_i + \alpha''_i = \alpha_i$, and

$$|\alpha'_i| + |\alpha''_i| - |\alpha_i| = (m - \ell)/2.$$

Then, we can map the edge $E$ to connect $\iota(u)$ and $\iota(w)$ in a piecewise linear fashion, with all segments parallel to one of $e_i, \ldots, e_{i+n-1}$. In particular, we
define $\iota(E)$ to linearly interpolate between the following points:

$$
\iota(u) + \alpha'_0 e_i + \alpha'_1 e_{i+1} + \alpha''_0 e_i + \alpha''_1 e_{i+1} + \cdots + \alpha'_2 e_{i+2} = \iota(w) - \alpha''_1 e_{i+1}
$$

The constants $\alpha'_i$ and $\alpha''_i$ are chosen such that this path has total length $\ell$, which is the desired edge length, and such that the tangent directions from $\iota(u)$ and $\iota(w)$ are $e_i$ and $e_{i+1}$, respectively. In order to complete the local model at each vertex other than the endpoints $\iota(u)$ and $\iota(v)$, we need to add an unbounded ray in the appropriate direction, corresponding to a subdivision and modification of $\Gamma$.

To show that $\iota$ defines an immersion for $n = 2$, we need to show that for generic choices of images for the vertices, the self intersections of $\iota$ will only occur at edges. Here, the key observation is that for any vertex in the image of $\iota$, both of its coordinates will be affected by at least one of its endpoints. Thus, by perturbing the endpoint slightly, we can move the vertex off of a point of multiple intersection. Furthermore, by choosing this perturbation to be sufficiently small, we won't introduce any additional self-intersections.

For $n \geq 3$, we want to show that for generic choices for an embedding of the vertices, $\iota$ is injective. Here, assume that we have an intersection between two different embedded edges. By our initial subdivisions, these edges share at most one endpoint in common. Moreover, by our choice of the directions for the edge near the endpoints, even if the edges share an endpoint, the intersection won't be in the first two segments next to that endpoint. Thus, at least two of the coordinates of this point of intersection will depend on the opposite endpoint, so there is at least a 2-dimensional space of perturbations that will move the segment. On the other hand, the perturbations parallel to the other segment will still have a self-intersection, but this will be at most a 1-dimensional subspace. Thus, we can find some perturbation which eliminates the intersection.

Finally, we need to prove the quadratic bound on the number of crossings in the case of immersions. The subdivisions to avoid loops or parallel edges can be done by replacing each edge by 2 or 3 edges respectively. Then, our immersion further subdivides each edge into 4 segments and introduces 3
infinite rays. Thus, each edge of $\Gamma$ results in a bounded number of segments in the immersion. Moreover, these segments are each parallel to one of the vectors $e_0$, $e_1$, and $e_2$, so the intersection multiplicity of any two is at most one. Thus, the total number of crossings of $\iota$ is $O(e^2)$, as desired.

\[ \square \]

Remark 2.8. We note that the embeddings and immersions constructed in the proof of Theorem 1.1 have the following additional properties, which will be relevant for the applications to realizability of curves in [CFPU14]. First, for every edge of $\Gamma$, the directions of the embeddings of the edges in the subdivision form a basis for $\mathbb{Z}^n$. Second, if all of the lengths of $\Gamma$ are rational, then all the vertices of the tropical curve can also be chosen to be rational. Third, vertices contained in three or more bounded edges are only adjacent to vertices contained in at most two bounded edges, because every edge of the original graph is subdivided.

\[ \square \]

Remark 2.9. In algebraic geometry and other fields, immersions and embeddings may be constructed by starting with an embedding in a high-dimensional space and projecting (for example, Prop. IV.3.5 and Thm. IV.3.10 in [Har77]). However, the key to such arguments is showing that generic projections are sufficient, but, in tropical geometry, projections which preserve smoothness, even at a single point are relatively rare and thus can not be considered to be the generic case.

\[ \square \]

3. Crossing numbers and gonality

By Theorem 1.1, any tropical curve has a planar immersion. We can define tropical crossing number of the curve to be the minimal number of nodes, counted with multiplicity, of any planar immersion. Thus, the tropical curve has a smooth planar embedding if and only if its crossing number is zero, and Theorem 1.2 establishes a quadratic upper bound on the crossing number. In this section, we establish a lower bound on the crossing number of an abstract tropical curve in terms of its genus and divisorial gonality. We use this to prove Theorem 1.3 in the form of Corollary 3.6.

A tool we will repeatedly use in this section and the next is the dual subdivision of a plane curve. We gather some basic facts about the dual subdivision which will come up in many of the proofs. Since any nodal tropical curve is balanced, [RGST05, Thm. 3.3] states that it is the non-differentiable locus of some concave piecewise-linear function. The dual subdivision $\Delta$ is the projection of the lower convex hull of the coefficients of the piecewise linear function. See [MS15, Sec. 1.3] for details. For us, the relevance will be the duality between the curve and the subdivision, in which the vertices and edges of the curve correspond to the polygons and edges of the subdivision respectively. Likewise, the bounded and unbounded regions of the complement of the curve correspond to the vertices in the interior and on the boundary of $\Delta$. An example of a tropical curve and its dual subdivision are shown in Figure 3.
Figure 3. A plane tropical curve on the left and the corresponding dual subdivision on the right. The triangles of the subdivision correspond to the trivalent vertices of the curve and the square represents the curve’s unique node.

Definition 3.1. The genus $g$ of an abstract tropical curve is the rank of the first homology of its underlying graph, i.e. $\dim_\mathbb{Q} H_1(\Gamma, \mathbb{Q})$. □

Proposition 3.2. If $\Gamma$ is a tropical curve of genus $g$ and $\iota(\Gamma)$ is an immersion with $n$ nodes, counted with multiplicities, then

$$i = g + n,$$

where $i$ is the number of integral points in the interior of the dual polygon.

We will use the following well-known result about the number of lattice points in an integral polygon.

Proposition 3.3 (Pick’s Theorem). Let $P$ be a polygon with integral vertices. If $P$ has area $A$ and $i$ lattice points, of which $b$ are on the boundary of the polygon, then we have the relation $A = i - b/2 - 1$.

Proof of Prop. 3.2. We first introduce some additional notation. We write $g'$ for the genus of $\iota(\Gamma)$, without resolving any nodes, and $n'$ for the number of such nodes, without any multiplicities. Since the procedure for resolving a node adds a vertex, but doesn’t change the number of edges, we have that $g = g' - n'$. Therefore, it will suffice to prove the relation

$$i = g' + n - n'.$$

The dual subdivision $\Delta$ will have a triangle corresponding to each trivalent vertex and a parallelogram corresponding to each node. We now count how the interior lattice points of $\Delta$ fall within the subdivision. Because of our local smooth model, the only lattice points in a triangle will be its vertices. Moreover, because all the edges of the curve have multiplicity 1, there are no lattice points on the interior of any edge of the subdivision. Thus, an internal lattice point of $\Delta$ is either a vertex of the subdivision or in the interior of a parallelogram. Each vertex of the former category defines a bounded region of the complement of $\iota(\Gamma)$ and thus there are $g'$ such vertices. Therefore, we will have verified (2) if we can show that a node of multiplicity $m$ is dual to a parallelogram containing $m - 1$ interior lattice points.
Let $P$ be such a parallelogram. A simple computation shows that the area of $P$ is equal to the multiplicity $m$ of the node as in \cite{1}. As noted above, the only lattice points on the boundary of $P$ are the 4 vertices. Therefore, Pick’s formula (Prop. 3.3) tells us that $P$ contains $m + 3$ lattice points and thus $m - 1$ interior lattice points. □

Our method for proving the asymptotic lower bound on the crossing number from Theorem 1.3 is the divisorial gonality of graph as in \cite{Bak08}. Specifically a graph $\Gamma$ is defined to have divisorial gonality $d$ if $d$ is the least integer such that there exists a divisor on $\Gamma$ of degree $d$ and rank 1. We refer to \cite[Sec. 1]{GK08} or \cite[Def. 7.1]{MZ08} for the definitions of the rank of a divisor. The divisor theory of an abstract tropical curve is related to its immersion by the following proposition.

**Proposition 3.4.** Let $\iota: \Gamma \to \mathbb{R}^2$ be an immersion of a tropical curve in the plane and suppose that $\iota(\Gamma)$ is not just a line. Then the stable intersection of $\Gamma$ with a straight line of rational slope defines a divisor with rank at least 1.

**Proof.** We first change coordinates so that the line $L$ is parallel to the y-axis, say defined by $x = a$. Let $D$ be the divisor formed by the stable intersection of $\Gamma$ with $L$. Then we need to show that for any point $p \in \Gamma$, there exists an effective divisor linearly equivalent to $D$ which contains $p$. We first let $L'$ be the vertical line containing $\iota(p)$, and say that $L'$ is defined by $x = a'$. Let $D'$ the stable intersection of $L'$ with $\Gamma$. We define $\phi: \mathbb{R}^2 \to \mathbb{R}$ to be the piecewise linear function

$$
\max\{x - a, 0\} - \max\{x - a', 0\}.
$$

Then, $D' = D + \text{div}(\phi \circ \iota)$, so $D'$ is linearly equivalent to $D$. Moreover, if $p$ is an isolated point of $L' \cap \iota(\Gamma)$, then $p$ is in $D'$ as required.

Otherwise, $p$ is contained in a positive-length interval of $L' \cap \iota(\Gamma)$. If this interval is bounded, then $D'$ contains both endpoints of this interval. These endpoints can be moved together the same distance so that one of them contains $p$. If the interval is unbounded, then by our assumption, it must still be bounded in one direction, and $D'$ will contain that endpoint. We can then move the point of $D'$ along the unbounded edge until it contains $p$. □

**Proposition 3.5.** If $\Gamma$ is an abstract tropical curve with divisorial gonality $d > 2$ and genus $g$, then the tropical crossing number of $\Gamma$ is at least

$$
\frac{3}{8}(d - 2)^2 - g + \frac{1}{2}.
$$

Our proof uses the same technique as in \cite[Thm. 3.3]{Smi14}, which bounds the gonality of a curve in a smooth toric variety.

**Proof.** Suppose that $\Gamma$ has divisorial gonality $d$ and we have a planar immersion. By \cite[Thm. 3.3]{RGST05}, $\Gamma$ is dual to a subdivision of a Newton polygon $\Delta$. If the lattice width of $\Delta$ is $w$, then Proposition 3.4 shows that the divisorial gonality of $\Gamma$ is at most $w$, i.e. $w \geq d$. We let $\Delta^{(1)}$ denote the convex hull of the interior points of $\Delta$ and let $w'$ denote the lattice width.
of $\Delta^{(1)}$. Then Theorem 4 from [CC12] shows that $w' = w - 2$ or $\Delta$ and $\Delta^{(1)}$ are unimodular simplices scaled by $w$ and $w - 3$ respectively. We first deal with the former case, for which we have $w' \geq d - 2$.

Then, Theorem 2 from [FTM74] implies that $A$, the area of $\Delta^{(1)}$, is at least $\frac{3(d-2)^2}{8}$. By our assumption that $d > 2$, we know that $w' > 0$, so $\Delta^{(1)}$ is not contained in a line. We thus apply Pick’s formula, Proposition 3.3, to get the relation $i = A + b/2 + 1$, where $i$ is the number of integral points in $\Delta^{(1)}$, and $b$ is the number of those integral points on the boundary of $\Delta^{(1)}$. Thus,

$$i \geq A + 1 \geq \frac{3(d-2)^2}{8} + 1.$$  

We now return to the case when $\Delta^{(1)}$ is a unimodular simplex scaled by $w - 3$. We can directly compute that in this case, with $i$ again the number of integral points in $\Delta^{(1)}$,

$$i = \frac{(w - 2)(w - 1)}{2} \geq \frac{(d - 2)(d - 1)}{2} = \frac{(d - 2)^2}{2} + \frac{d - 2}{2} \geq \frac{3(d-2)^2}{8} + \frac{1}{2},$$

because $1/2 > 3/8$ and $d > 2$. Thus, in either case, we can apply Proposition 3.3 to get that the number of nodes is $i - g$ and thus at least $\frac{3}{8}(d-2)^2 - g + \frac{1}{2}$, as claimed. $\square$

We prove Theorem 1.3 using the family of graphs from [CDPR12]. Their paper treats graphs constructed by linking a series of $g$ loops in a chain by edges, so, in particular, these graphs are planar. They assign generic edge lengths to these edges, for which it is sufficient that the vector of length assignments avoids a finite list of rational hyperplanes. We use the same family of graphs and genericity assumptions in the following, which shows this family of abstract tropical curves is far from having planar embeddings, for large $g$.

**Corollary 3.6.** If $\Gamma$ is the chain of $g \geq 3$ loops with generic edge lengths then the crossing number of $\Gamma$ is at least $\frac{3g^2}{32} - \frac{11g}{8} + \frac{7}{8}$. In particular, the crossing number of this family of graphs is quadratic in the number of edges.

**Proof.** By [CDPR12] Thm. 1.1, $\Gamma$ is Brill-Noether general, which implies that it has divisorial gonality $\lceil g/2 \rceil + 1$. In particular, when $g \geq 3$, the divisorial gonality is greater than 2. Substituting this into the expression in Proposition 3.3 and dropping the ceiling, we see that $\Gamma$ has crossing number at least $\frac{3g^2}{32} - \frac{11g}{8} + \frac{7}{8}$.

The last sentence follows from the fact that $\Gamma$ has $3g - 3$ edges and thus the crossing number is also quadratic in the number of edges. $\square$

4. Low genus curves

In this section, we examine the crossing numbers of tropical curves with genus at most 2. We begin with genus 0. The underlying graph of a genus 0 tropical curve is a tree and in Proposition 4.3, we characterize genus 0 tropical
curves with crossing number 0 in terms of its underlying graph. In genus 1, we only consider graphs consisting of infinite edges attached to the central loop, in which case Proposition 4.4 gives a sharp bound on the crossing number. For genus 2 curves, we restrict to stable tropical curves, by which we mean that every vertex has degree equal to 3. In particular, stable tropical curves have no infinite edges. Proposition 4.7 gives the crossing numbers of stable genus 2 tropical curves. In this case, the tropical crossing number depends on the edge length.

The following proposition is also proved as Proposition 8.3 in [BJMS14], where the curves satisfying the hypothesis are called sprawling.

**Proposition 4.1.** Let Γ be a trivalent abstract tropical curve and let v be a degree 3 vertex of Γ such that removing v disconnects Γ into three components A, B, and C. We consider each of these components to be a tropical curve by including v in each of them.

Suppose that A, B, and C each contain at least one trivalent vertex. Then Γ has a planar embedding if and only if A, B, and C each contain a single trivalent vertex, in which case Γ looks like the curve shown in Figure 4.

**Proof.** Consider a planar embedding of Γ. By making a change of coordinates, we may assume that the outgoing edges from v have directions (1, 0), (0, 1), and (−1, −1). Let $w_A$, $w_B$, and $w_C$ be the first trivalent vertices in these respective directions, which we’ve assumed to exist. We will show that v, $w_A$, $w_B$, and $w_C$ are the only trivalent vertices.

The local model for a smooth curve implies that the outgoing directions at $w_A$ are (−1, 0), (−a, −1), and (a + 1, 1) for some $a \in \mathbb{Z}$. Similarly, the outgoing directions at $w_B$ are (0, −1), (−1, −b), and (1, b + 1) for some $b \in \mathbb{Z}$ and at $w_C$ they are (1, 1), (c, c − 1), and (−c − 1, −c) for some $c \in \mathbb{Z}$.

Consider the component of the complement $\mathbb{R}^2 \setminus \Gamma$ which lies between $w_A$ and $w_B$. This component will be convex because the angle between any two rays of the standard smooth model is less than 180°, even after any change of coordinates. Furthermore, because the only paths between $w_A$ and $w_B$ pass through v, this component must be unbounded. On the other hand, we have the edge from $w_A$ in the direction (a + 1, 1), so in particular, with increasing y-coordinate, and we have an edge from $w_B$ with slope $b + 1$, so if
Figure 5. The lollipop abstract tropical curve, which does not have a planar embedding by Corollary 4.2.

Figure 6. Embedding of the caterpillar graph with 10 leaves. Caterpillar graphs with more leaves can also be embedded by adding more repetitions of the blocks in the dashed lines.

If $b + 1 \leq 0$, then these two directions would eventually meet, and thus they could not form edges of a convex, unbounded region. We conclude that $b \geq 0$. Applying the same analysis to the region between $w_C$ and $w_B$, we get that $b \leq 0$, so $b = 0$ and by symmetry $a = c = 0$ as well.

At this point, we know that, in a neighborhood of $v$, $w_A$, $w_B$, and $w_C$, the embedding of $\Gamma$ must look like in Figure 4. However, the region between $w_A$ and $w_B$ now has two edges parallel to $(1, 1)$ in its boundary, and as we’ve seen, this region must be unbounded, so these edges are also unbounded. By symmetry, each of the vertices $w_A$, $w_B$, and $w_C$ must have two unbounded edges from it. Thus, there are no further trivalent vertices, which is what we wanted to show. \qed

**Corollary 4.2.** The “lollipop curve” shown in Figure 5 has crossing number at least 1, for any lengths.

Corollary 4.2 is a strengthening of the last sentence of Proposition 2.3 from [BLMPR14]. In that paper, they studied smooth plane quartics and found examples of all combinatorial types of genus 3 graphs except for the lollipop graph. Proposition 4.1 further shows that the lollipop graph does not have a planar embedding, even if the curve is not required to be quartic.

**Proposition 4.3.** Let $\Gamma$ be a genus 0 abstract tropical curve. Then $\Gamma$ has crossing number 0 if and only if the underlying graph is a subdivision of the caterpillar graph or the windmill graph.

**Proof.** In the case of the caterpillar graph or the windmill graph, the embedding can be constructed as in Figures 6 and 4 respectively. Conversely, if the
Proposition 4.4. Let $\Gamma$ be a curve whose underlying graph is a sun: a cycle with $n$ infinite rays attached to the cycle. Then the crossing number of $\Gamma$ is at least $\lceil n/2 \rceil - 4$ if $n > 9$ and this bound is sharp for some edge lengths.

Proof. If we have an immersion of $\Gamma$ with $k$ nodes, then the dual triangulation will be a polygon with $k + 1$ interior lattice points. Since each ray of the sun graph will produce an unbounded edge, the polygon has at least $n$ edges. Since $n > 9$, then Scott’s inequality on polygons gives $n \leq 2(k + 1) + 6$ [Scor86], and so $k \geq n/2 - 4$. Since the crossing number can only be an integer, we get the desired inequality.

To show that this bound can be sharp, we give examples of immersions of tropical curves which achieve them. For $n \leq 9$, the embedding in Figure 7 justifies the requirement of $n > 9$ from the proposition statement. On the other hand, the pattern in Figure 8 with $k$ of the dotted blocks will give an
Lemma 4.6 says that the Newton polygon can be constrained to one of these two shaded regions. In either case, the interior points of the Newton polygon are the circled dots, which have coordinates $(0,0)$ and $(1,0)$.

immersion with $k$ crossings and $2k + 8$ infinite edges. For $n > 9$, by taking $k = \lceil n/2 \rceil - 4$, we have an example which has $n$ or $n + 1$ infinite edges and achieves the crossing number bound from the proposition statement. \hfill □

Lemma 4.5. Let $\Gamma$ be the theta graph as in the right of Figure 9 with all edge lengths equal. Then $\Gamma$ has crossing number 1.

The first step in the proof of Lemma 4.5 is the following Lemma 4.6, which constrains the possible shapes of the Newton polygon of an embedding of the theta graph, or of any other genus 2 graph. The possibilities in the conclusion of Lemma 4.6 are illustrated in Figure 9.

Lemma 4.6. Let $\Delta$ be the Newton polygon dual to a smooth embedding of an abstract tropical curve of genus two. Then we can choose an affine change of coordinates such that the interior points of $\Delta$ are $(0,0)$ and $(1,0)$ and such that either of the following inequalities hold:

1. The points of $\Delta$ are bounded by $-1 \leq y \leq 1$ and $x \leq 2$.
2. The points of $\Delta$ are bounded by $-1 \leq y \leq 1$ and $x \geq -1$.

Moreover, these two coordinates differ by a linear transformation that is the identity on the $y = 0$ line.

Proof. Since the embedding, by definition, has no nodes, Proposition 3.2 implies that $\Delta$ has exactly two integral points in its interior. After an affine change of coordinates which is common to both (1) and (2) from the statement, we can assume that these two interior points are $(0,0)$ and $(1,0)$. Now suppose that there is a vertex of the polygon $(a,b)$ with $b > 1$. We claim that the integral point $v = ([a/b], 1)$ would also be in the interior of $\Delta$, which would be a contradiction. To see this claim, let $r = \lceil a/b \rceil - a/b$, and note that $0 \leq r \leq 1 - 1/b$. We can then write:

$$v = \left( 1 - r - \frac{1}{b} \right) (0,0) + r(1,0) + \left( \frac{1}{b} \right) (a,b),$$

which is a convex linear combination by the previously noted inequalities. Since $(0,0)$, $(1,0)$, and $(a,b)$ are all in $\Delta$, so is $v$. Moreover, the former two points are in the interior of $\Delta$, and $v$ is not equal to $(a,b)$, so $v$ is in the interior of $\Delta$. Thus, we’ve proved the desired claim, and so there can be no
interior lattice point \((a, b)\) with \(b > 1\). By symmetry, there are no vertices \((a, b)\) in \(\Delta\) with \(b< -1\) either, and so we’ve proved the first half of either set of bounds.

We now turn to finding bounds on the horizontal extent of \(\Delta\). Suppose that \((a, 1)\) is the rightmost vertex of \(\Delta\) on the \(y = 1\) line. We now change coordinates by sending \((x, y)\) to \((x - (a - 2)y, y)\), after which \((2, 1)\) is the rightmost such point. We now claim that all points in \(\Delta\) are bounded by \(x \leq 2\). On the \(y = 1\) line, this is by our change of coordinates, and if there exists a vertex \((a, b)\) with \(a > 2\) and either \(b = 0\) or \(b = -1\), then the vertex \((2, 0)\) would be in the interior of \(\Delta\), which is a contradiction. Therefore, we’ve found coordinates satisfying the bounds in \((1)\).

To find the second set of coordinates, we start with the leftmost vertex on the \(y = 1\) and apply an analogous change of coordinates, which is the identity on the \(y = 0\) line and after which \(\Delta\) is bounded by \(x \geq -1\), as desired. □

Proof of Lemma 4.5. The immersion from Example 2.7 shows that the crossing number is at most 1, and so it remains to show that there is no planar embedding of \(\Gamma\).

We assume for the sake of contradiction that we have a planar embedding \(\iota\). We consider the dual triangulation of the Newton polygon \(\Delta\), for which we first assume we have coordinates for which \(\Delta\) is bounded as in Lemma 4.6(1). Since \(\Gamma\) is a theta graph, there must be an edge of the embedding separating the two bounded regions. Dually, the bounded regions correspond to the points \((0, 0)\) and \((1, 0)\), so there must be an edge of the triangulation between them, which then corresponds to a vertical edge in \(\iota(\Gamma)\). The triangles above and below the edge joining \((0, 0)\) and \((1, 0)\) correspond to the two trivalent vertices of \(\Gamma\). We label the edges of \(\Gamma\) as \(e_1, e_2, \) and \(e_3\), such that \(\iota(e_2)\) is the vertical edge, and the regions to the left and right of this edge are bounded by \(\iota(e_1 \cup e_2)\) and \(\iota(e_2 \cup e_3)\) respectively.

Now we consider the subset \(e_3'\) of \(e_3\) consisting of points \(p\) of \(e_3\) such that \(\iota(p) + (\epsilon, 0)\) is in an unbounded region for all sufficiently small \(\epsilon\). Equivalently, \(\iota(e_3')\) is the union of the segments of \(\iota(e_3)\) whose dual in the triangulation are edges connecting \((1, 0)\) and a point with first coordinate greater than 1. By the inequalities of Lemma 4.6(1), the only possibilities for the second point are \((2, 1), (2, 0),\) and \((2, -1)\), which correspond to edges in \(\iota(e_3')\) parallel to the vectors \((1, -1), (0, -1)\) and \((-1, -1)\), respectively. All of these vectors have \(-1\) in the second coordinate, so the total length of the segment \(e_3'\) equals the height of \(\iota(e_3')\). Since the length of \(e_3\) equals the length of \(e_2\), which equals the height of the vertical segment \(\iota(e_2)\), we know that \(e_3'\) must consist of all of \(e_3\). As a consequence, the only possible endpoints of edges of the triangulation containing \((1, 0)\) are \((0, 0), (2, 1), (2, 0),\) and \((2, -1)\). Thus, the triangles above and below the edge from \((0, 0)\) to \((1, 0)\) in the triangulation must contain the vertices \((2, 1)\) and \((2, -1)\) respectively. The midpoint of these two vertices will be \((2, 0)\) and note that this midpoint is preserved under linear changes of coordinates which also preserve the \(y = 0\) line.
Second, we consider coordinates such that $\Delta$ is bounded as in Lemma 4.6(2). By symmetry, the same argument applied to $e_1$ shows that in these coordinates, the triangles above and below the vertices $(0, 0)$ and $(1, 0)$ are $(-1, 1)$ and $(-1, -1)$ respectively. The midpoint of these two vertices is $(-1, 0)$, which would remain true when changing to the coordinates as in Lemma 4.6(1). Therefore, we have a contradiction with the previous paragraph, so there is no embedding of the graph $\Gamma$, so its crossing number is 1. 

\begin{figure}
\centering
\includegraphics[width=0.25\textwidth]{figure10.png}
\caption{Embedding of the barbell graph.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.25\textwidth]{figure11.png}
\caption{Embedding of the theta graph when $a < b \leq c$. In this figure, the middle edge has length $a$.}
\end{figure}

\begin{proposition}
Let $\Gamma$ be a stable tropical curve of genus two. Then $\Gamma$ has tropical crossing number 0 unless $\Gamma$ is the theta graph in Figure 1 with all edge lengths equal, in which case it has crossing number 1.
\end{proposition}

\begin{proof}
There are two combinatorial types of trivalent graphs of genus 2. In the case of the barbell graph, we can take the planar immersion shown in Figure 10 for all possible edge lengths. For the theta graph in Figure 1, there are three possibilities depending on the edge lengths $a$, $b$, and $c$. By symmetry, we can assume that $a \leq b \leq c$. If we have a strict inequality $a < b$ or if $a = b < c$, then we can use the embeddings in Figures 11 and 12 respectively. Otherwise, all of the edge lengths are equal, and the crossing number is 1 by Lemma 4.5.
\end{proof}
Remark 4.8. The example of the theta graph in Proposition 4.7 shows that the crossing number can increase for specializations of the metric parameters, i.e., the crossing number function is not lower semi-continuous. Moreover, it is not a semi-continuous because the crossing number can jump in specializations as well. For example, the chain of $g \geq 15$ loops with generic edge lengths has positive crossing number by Corollary 3.6, but if all edge lengths are equal then it is planar. Also, a consequence of the proof of Proposition 4.4 is that any embedding of a sun curve with 9 infinite edges is equivalent to that in Figure 7, up to change of coordinates in $GL_2(\mathbb{Z})$. However, such an embedding implies non-trivial conditions among the lengths of the edges of the cycle.

For curves of higher genus, we refer to [BJMS14, Sec. 5–8], where curves of crossing number 0 are characterized using computational techniques. In particular, they show that it is feasible to enumerate the possible Newton polygons of low genus curves and from these compute the defining inequalities of the tropical curves which can then arise.

5. Algebraic realizations of tropical curves

In this section, we consider applications of our results to realizations of tropical curves by algebraic curves. We recall that for any curve $C \subset \mathbb{G}_m^n$ over a field $K$ with valuation, the tropicalization of $C$ is a union of finitely many edges in $\mathbb{R}^2$. One characterization of the tropicalization $C \subset \mathbb{G}_m^n$ is as the projection of the Berkovich analytification $C_{\text{an}}$ using the valuations of the coordinate functions of $\mathbb{G}_m^n$.

Baker, Payne, and Rabinoff have shown that any algebraic curve $C$ has a faithful tropicalization with respect to any skeleton of $C_{\text{an}}$, meaning that there exists an embedding of a dense open set $C \supset C' \rightarrow \mathbb{G}_m^n$ such that the projection of $(C')_{\text{an}}$ to its tropicalization is an isometry on the specified skeleton [BPR11, Thm. 1.1]. We reverse this process by beginning with an abstract tropical curve and finding an algebraic curve $C \subset \mathbb{G}_m^2$ which satisfies a variant of the faithful tropicalization property in which we allow nodes in the algebraic and tropical curves. In particular, the map from the minimal skeleton of $C_{\text{an}}$ to its tropicalization is the immersion map as in Section 2.
Theorem 5.1. Let $\Gamma$ be a trivalent abstract tropical curve with rational edge lengths and let $K$ denote the field of convergent Puiseux series with complex coefficients. Then there exists a nodal $K$-curve $C \subset G_m^2$ whose skeleton contains $\Gamma$ and such that the tropicalization map from the minimal skeleton of $C_{\text{can}}$ is an isometry away from the finitely many points that lie above the nodes of the tropicalization.

Proof. By Theorem 1.1, $\Gamma$ admits a planar immersion $\iota : \Gamma \to \mathbb{R}^2$ and by Remark 2.8, we can assume that the vertices in this immersion have rational coordinates. Then, it will be sufficient to show that this nodal plane curve is the tropicalization of some algebraic curve, for which we use Mikhalkin’s correspondence theorem [Mik05], and more specifically the algebraic version due to Shustin [Shu05, Thm. 3]. While the cited theorem only asserts that a certain count of nodal algebraic curves equals a weighted count of the corresponding tropical curves, the proof works by constructing at least one algebraic curve for each nodal tropical curve. In particular, [Shu05, Sec. 3.7] shows that for any nodal tropical curve $\iota(\Gamma)$, it is possible to find certain auxiliary data, denoted by $S, F,$ and $R$ in that paper. Then, [Shu05, Lem. 3.12] states that from $\iota(\Gamma)$, together with this auxiliary data, we can find a nodal algebraic curve tropicalizing to $\iota(\Gamma)$. $\square$

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