APPOMIMATION OF HIILBERT-VALUED GAUSSIAN MEASURES ON DIRICHLET STRUCTURES

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\textbf{Abstract.} We introduce a framework to derive quantitative central limit theorems for the approximation of arbitrary non-degenerate Gaussian random variables taking values in a separable Hilbert space. In particular, our method provides an alternative to the usual (non-quantitative) finite dimensional distribution convergence and tightness argument for proving functional convergence of stochastic processes. We also derive four moments theorems for Hilbert-valued random variables with possibly infinite chaos expansion, which also include, as special cases, all finite-dimensional four moments theorems for Gaussian approximation in a diffusive context proved earlier by various authors. Our main ingredients are infinite-dimensional versions of Stein’s method as introduced by Shih, and the so-called Gamma calculus. As an application, we derive rates of convergence in the functional Breuer-Major theorem recently proved by Nourdin and Nualart.

1. Introduction

In this paper, we develop a framework which allows to derive quantitative limit theorems for the approximation of arbitrary non-degenerate Gaussian random variables taking values in a separable Hilbert space. More specifically, we develop a Dirichlet structure and the associated Gamma calculus in an infinite-dimensional context (which might be of independent interest), and combine it with the infinite-dimensional Stein method as introduced by Shih in [Shi11] in order to derive general carré du champ bounds on several probabilistic metrics (all of them at least metrizing weak convergence). In particular, in the case where the limiting random variable is a Gaussian process, our results yield a universal and quantitative method to prove functional central limit theorems, and can be regarded as an alternative to the (non-quantitative) strategy of proving convergence of finite-dimensional distributions and tightness of the approximating sequence. To illustrate this method, we quantify a functional version of the celebrated Breuer-Major theorem (recently proved in [NN18] (see also [CNN18, HN18] for related results), thus assessing the

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speed of convergence of stochastic processes \((U_t)_{t \geq 0}\) of the form
\[
U_t = \sum_{i=1}^{\lfloor nt \rfloor} \varphi(X_i)
\]
towards a scaled Brownian motion as \(n \to \infty\), where \(\varphi\) is a square integrable real-valued function, and \((X_i)_{i \in \mathbb{N}}\) is a square integrable stationary Gaussian sequence (see Section 4 for details and Theorem 4.1 for full details).

Another application of our results are four moments theorems. In this direction, we provide new results in both finite and infinite dimension. For example, let \(X = (X_t)_{t \geq 0}\) be a stochastic process taking values in a fixed Wiener chaos (or more general Markov chaos), \(Z = (Z_t)_{t \geq 0}\) be a non-degenerate Gaussian process, and assume that the trajectories of both processes are elements of some separable Hilbert space \(K\). Furthermore, for simplicity, assume that both processes have the same covariance operator \(S\). Then a consequence of our results is that
\[
d(X, Z) \leq C \sqrt{E[\|X\|_K^4] - E[\|X\|_K^2]^2 - 2 \|S\|_{HS}^2},
\]
where \(C\) is a positive constant, and \(d\) denotes an appropriate probabilistic metric implying weak convergence. This result contains, in a unified way, all previously established four moments theorems for Gaussian approximation in a diffusive, finite-dimensional context (see [Led12, ACP14, CNPP16, BCLT19, NP09, NPR10b, NN11, NOL08, NP05]). In particular, if \(K = \mathbb{R}\), we recover the one-dimensional fourth moment theorem by Nourdin and Peccati [NP09], which reads
\[
d(F, Z) \leq C \sqrt{E[F^4] - 3 E[F^2]^2},
\]
where \(C\) is again a positive constant, \(F\) is an element of a fixed Wiener chaos and \(Z\) is a real-valued Gaussian random variable. If \(K = \mathbb{R}^d, d \geq 2\), we provide a new and somewhat simpler bound for the vector-valued case [NN11]. We also prove such four moments bounds for Hilbert-valued random variables with infinite chaos expansion. Such bounds are new even when the Hilbert space has finite dimension.

The existing literature on quantitative functional limit theorems is rather scarce. Barbour extended Stein’s method to a functional setting in [Bar90] for diffusion approximations by a Brownian motion. This has recently been applied and extended by Kasprzak in [KDV17, Kas17a, Kas17b]. Shih later generalized Barbour’s approach to Gaussian measures on a Banach space in [Shi11]. Later on, Coutin and Decreusefond [CD13] combined Stein’s method with integration by parts techniques in a Hilbert space setting. While the spirit is similar to our approach, the results are very different: their bounds are stated in terms of partial traces and require explicit evaluations of isometries as all calculations are done in \(\ell^2(\mathbb{N})\); furthermore, no carré du champ, moment, or contraction bounds are provided.

Finally, let us point out that our results can also be used to obtain weak convergence in a Banach space setting. Indeed, as pointed out by Kuelbs in [Kue70], it is always possible to densely embed any separable Banach space \(B\) into a Hilbert space \(K\) such that the Borel sets of \(B\) are generated by the inner product of \(K\). Thus, weak convergence in this
Hilbert space $K$ implies weak convergence in $B$. More details on this fact are provided in Remark 3.7.

The rest of the paper is organized as follows. Section 2 contains the needed preliminaries on function spaces, probabilistic metrics, Dirichlet structures and Stein’s method on Banach spaces. The main results are contained in Section 3. We start by introducing the Dirichlet structure we are working with in Section 3.1, which also includes a generalized definition of Markov chaos. The carré du champ bounds can be found in Section 3.2, while the moment statements are stated and proved in Section 3.3. Section 3.4 is devoted to studying our approach in the Wiener space setting on which more structure is available. Here, bounds in terms of so called contractions are obtained which in many cases are easier to deal with than moments. Finally, in Section 4, we provide rates for the aforementioned functional Breuer-Major theorem (see [NN18] for the non-quantitative statement).

2. Preliminaries

2.1. Function spaces and notation. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. For a separable Banach space $B$ with norm $\|\cdot\|_B$, $L^p(\Omega; B)$ denotes the Banach space of all equivalence classes (under almost sure equality) of $B$-valued random variables with finite $p$-th moment, on which the norm is given by

$$\|\cdot\|_{L^p(\Omega; B)} = \left(\mathbb{E}[\|\cdot\|_B^p]\right)^{1/p}.$$ 

As $B$ is separable, the space $L^2(\Omega) \otimes B$ is separable as well and isomorphic to $L^2(\Omega; B)$, which implies that $L^2(\Omega; B)$ is separable.

Given a real separable Hilbert space $K$ with inner product $\langle \cdot, \cdot \rangle_K$ and associated norm $\|\cdot\|_K$, we denote by $S_1(K)$ the Banach space of all trace class operators on $K$. The norm $\|\cdot\|_{S_1(K)}$ on $S_1(K)$ is given, for any trace class operator $T$ on $K$, by

$$\|T\|_{S_1(K)} = \text{tr} |T|,$$

where $|T| = \sqrt{TT^*}$. The subspace of Hilbert-Schmidt operators will be denoted by $\text{HS}(K)$, its inner product and associated norm by $\langle \cdot, \cdot \rangle_{\text{HS}(K)}$ and $\|\cdot\|_{\text{HS}(K)}$, respectively. Recall that

$$\|\cdot\|_{\text{op}} \leq \|\cdot\|_{\text{HS}(K)} \leq \|\cdot\|_{S_1},$$

where $\|\cdot\|_{\text{op}}$ denotes the usual operator norm given by, for any operator $T$ on $K$,

$$\|T\|_{\text{op}} = \sup_{x \in K: \|x\|_K = 1} \|Tx\|_K.$$ 

When there is no ambiguity about what Hilbert space $K$ underlies $\langle \cdot, \cdot \rangle_K$, $\|\cdot\|_K$, $S_1(K)$ or $\text{HS}(K)$, we will drop the $K$ dependency and just write $\langle \cdot, \cdot \rangle$, $\|\cdot\|$, $S_1$ and $\text{HS}$, etc.

2.2. Banach space valued random variables. In this section, we briefly review Banach space valued random variables. For more details, see the monographs [BGL14, DPZ96].

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $B$ a separable Banach space and $\mathcal{B}(B)$ the Borel sets of $B$. A $B$-valued random variable $X$ is a measurable map from $(\Omega, \mathcal{F})$ to $(B, \mathcal{B}(B))$. Such random variables are characterized by the property that for any $\varphi \in B^*$, where, here and in the sequel, $B^*$ denotes the topological dual of $B$, the function $\varphi(X): \Omega \to \mathbb{R}$ is a (real valued) random variable. As usual, the distribution or law of a random variable $X$ is defined to be the push-forward probability measure $P \circ X^{-1}$ on
The set of all $B$-valued random variables is a vector space over the field of real numbers. If the Lebesgue integral $E(\|X\|_B) = \int_\Omega \|X\|_B \, dP$ exists and is finite, then the Bochner integral $\int_\Omega X \, dP$ exists in $B$ and is called the expectation of $X$. Slightly abusing notation, we denote this integral by $E(X)$ as well. It will always be clear from the context if $E(\cdot)$ denotes Lebesgue- or Bochner-integration with respect to $P$. For $p \geq 1$, we denote by $L^p(\Omega; B)$ the Banach space of all equivalence classes (under almost sure equality) of $B$-valued random variables $X$ with finite $p$-th moment, i.e., such that $\|X\|_{L^p(\Omega, B)} = (E(\|X\|_B^p))^{1/p} < \infty$.

Remark 2.1. Note that for all $X \in L^p(\Omega; B)$, the Bochner integral $E(X)$ exists.

In the sequel, images of $B$-valued random variables by densely defined, possibly unbounded linear operators will be encountered. In order to ensure that such images are still measurable (with respect to the Borel $\sigma$-algebra on $B$), we need some technical results.

Proposition 2.2 ([DPZ96], Proposition 1.6). Let $B_1, B_2$ be separable Banach spaces and $T: \text{dom}(T) \to B_2$ be a closed linear operator, where $\text{dom}(T)$ is a Borel subset of $B_1$. If $X: \Omega \to B_1$ is a random variable such that $X(\omega) \in \text{dom}(T)$ almost surely, then $TX$ is a $B_2$-valued random variable.

The domain of a densely defined closed linear operator between separable Banach spaces is always a Borel subset (see the following Lemma). By Proposition 2.2, such operators therefore preserve measurability.

Lemma 2.3. Let $T$ be defined as in Proposition 2.2. If $\text{dom}(T)$ is dense in $B_1$, then $\text{dom}(T)$ is a Borel subset of $B_1$.

Proof. $B_1 \times B_2$ is a separable Banach space under the norm $\|(u, v)\|_{B_1 \times B_2} = \|u\|_{B_1} + \|v\|_{B_2}$ and as $T$ is closed, its graph $\text{gr}(T) = \{(u, Tu): u \in \text{dom}(T)\}$ is a closed linear subspace of $B_1 \times B_2$ and therefore also a separable Banach space under $\|(\cdot)\|_{B_1 \times B_2}$. The projection map $\pi: \text{gr}(T) \to B_1$ defined by $\pi(u, Tu) = u$ is injective (as $T$ is linear) and continuous, with image $\text{dom}(T)$. A result by Lusin and Souslin (see for example [Kec95, Theorem 15.1]) now allows to conclude that $\text{dom}(T)$ is a Borel subset of $B_1$. 

Let now $X$ be a random variable taking values in some separable Hilbert space $K$. If $X \in L^2(\Omega; K)$, the covariance operator $S: K \to K$ of $X$ is defined by

$$SA = E(\langle \cdot, X \rangle \langle A, X \rangle), \quad A \in K.$$ 

Consequently, 

$$\langle SA, B \rangle = E(\langle A, X \rangle \langle B, X \rangle),$$

so that $S$ is positive and self-adjoint. Moreover, one can show that $S$ is a compact trace-class operator and that

$$\text{tr} S = E(\|X\|^2).$$
2.3. Gaussian measures and Stein’s method on abstract Wiener spaces. In this section, we first introduce Gaussian measures, the associated abstract Wiener spaces and then present Stein’s method on such spaces. Standard references for Gaussian measures and abstract Wiener spaces are the books [Bog98, Kuo75], Stein’s method on abstract Wiener space has been introduced by Shih in [Shi11].

2.3.1. Abstract Wiener spaces. Let $H$ be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and define a norm $\| \cdot \|$ on $H$ (not necessarily induced by another inner product) that is weaker than $\| \cdot \|_H$. Denote by $B$ the Banach space obtained as the completion of $H$ with respect to the norm $\| \cdot \|$ (note that if the $\| \cdot \|$ happens to be induced by an inner product, then $B$ is actually a Hilbert space), and define $i$ to be the canonical embedding of $H$ into $B$. Then, the triple $(i,H,B)$ is called an abstract Wiener space. We identify $B^*$ as a dense subspace of $H^*$ under the adjoint operator $i^*$ of $i$, so that we have the continuous embeddings $B^* \subset H \subset B$, where, as usual, $H$ is identified with its dual. The abstract Wiener measure $\mu$ on $B$ is characterized as the Borel measure on $B$ satisfying

$$\int_B e^{i\langle x,\eta \rangle_{B,B^*}} \mu(dx) = e^{-\frac{\|\eta\|_2^2}{2}},$$

for any $\eta \in B^*$, where $\langle \cdot, \cdot \rangle_{B,B^*}$ denotes the dual pairing in $B$.

2.3.2. Gaussian measures on Banach and Hilbert spaces. For a Banach space $B$, we denote by $B(B)$ its Borel sets.

Definition 2.4. Let $B$ be a real separable Banach space. A Gaussian measure $\nu$ is a probability measure on $(B,B(B))$, such that every linear functional $x \in B^*$, considered as random variable on $(B,B(B),\nu)$, has a Gaussian distribution (on $(\mathbb{R},B(\mathbb{R}))$). The Gaussian measure $\nu$ is called centered (or non-degenerate), if these properties hold for the distributions of every $x \in B^*$.

We see from the definition that every abstract Wiener measure is a Gaussian measure and, conversely, for any Gaussian measure $\nu$ on a separable Banach space $B$ there exists a Hilbert space $H$ such that the triple $(i,H,B)$ is an abstract Wiener space with associated abstract Wiener measure $\nu$ (see Lemma 2.1 in [Kue70]).

2.3.3. Stein characterization of abstract Wiener measures. Let $B$ be real separable Banach space with norm $\| \cdot \|$ and let $Z$ be a $B$-valued random variable on some probability space $(\Omega,\mathcal{F},P)$ such that the distribution $\mu_Z$ of $Z$ is a non-degenerate Gaussian measure on $B$ with zero mean. Let $(i,H,B)$ be the abstract Wiener space associated to the Wiener measure $\mu_Z$, as described in the previous subsection. Let $(P_t)_{t \geq 0}$ denote the Ornstein-Uhlenbeck semigroup associated with $\mu_Z$ and defined, for any $B(B)$-measurable function $f$ and $x \in B$, by

$$P_t f(x) = \int_B f(e^{-t}x + \sqrt{1-e^{-2t}}y) \mu_Z(dy), \quad t \geq 0,$$

provided such an integral exists. We have the following Stein lemma for abstract Wiener measures.

Theorem 2.5. Let $X$ be a $B$-valued random variable with distribution $\mu_X$. 

(i) If \( B \) is finite-dimensional, then \( \mu_X = \mu_Z \) if and only if
\[
E \left[ \langle X, \nabla f(X) \rangle_{B,B^*} - \Delta_G f(X) \right] = 0
\]
for any twice differentiable function \( f \) on \( B \) such that \( E \left[ \| \nabla^2 f(Z) \|_{\mathcal{S}_1(H)} \right] < \infty \).

(ii) If \( B \) is infinite-dimensional, then \( \mu_X = \mu_Z \) if and only if (2) holds for any twice \( H \)-differentiable function \( f \) on \( B \) such that \( \nabla f(x) \in B^* \) for any \( x \in B \),
\[
E \left[ \| \nabla^2 f(Z) \|_{\mathcal{S}_1(H)} \right] < \infty \text{ and } E \left[ \| \nabla f(Z) \|_{B^*}^2 \right] < \infty.
\]

The notion of \( H \)-derivative appearing in Theorem 3.4 was introduced by Gross in [Gro67] and is defined as follows. A function \( f: U \to W \) from an open set \( U \) of \( B \) into a Banach space \( W \) is said to be \( H \)-differentiable at a point \( x \in U \) if the map \( \phi(h) = f(x + h) \), \( h \in H \), regarded as a function defined in a neighborhood of the origin of \( H \) is Fréchet differentiable at 0. The Fréchet derivative \( \phi'(0) \) at 0 is called the \( H \)-derivative of \( f \) at \( x \in B \). The \( H \)-derivative of \( f \) at \( x \) in the direction \( h \in H \) is denoted by \( \langle \nabla f(x), h \rangle_H \). The \( k \)-th order \( H \)-derivatives of \( f \) at \( x \) can be defined inductively and are denoted by \( \nabla^k f(x) \) for \( k \geq 2 \), provided they exist. If \( f \) is scalar-valued, \( \nabla f(x) \in H^* \approx H \) and \( \nabla^2 f(x) \) is regarded as a bounded linear operator from \( H \) into \( H^* \) for any \( x \in U \), and the notation \( \langle \nabla^2 f(x)h, k \rangle_H \) stands for the action of the linear form \( \nabla^2 f(x)(h, \cdot) \), \( h \in H \), on \( k \in H \), denoted by \( \nabla^2 f(x)(h,k) \). Furthermore, if \( \nabla^2 f(x) \) is a trace-class operator on \( H \), we can define the so-called Gross Laplacian \( \Delta_G f(x) \) of \( f \) at \( x \) appearing in (2) by \( \Delta_G f(x) = \text{tr}_H(\nabla^2 f(x)) \).

Remark 2.6 (On the relation between Fréchet and \( H \)-derivatives). An \( H \)-derivative \( \nabla f(x) \) at \( x \in B \) determines an element in \( B^* \) if there is a constant \( C > 0 \) such that \( |\langle \nabla f(x), h \rangle_H| \leq C \| h \|_H \) for any \( h \in H \). Then, \( \nabla f(x) \) defines an element of \( B^* \) by continuity and we denote this element by \( \nabla f(x) \) as well. Now, if \( f \) is also twice Fréchet differentiable on \( B \), then \( \nabla f(x) \) coincides with the first-order Fréchet derivative \( f'(x) \) at \( x \in B \) and is automatically in \( B^* \). Furthermore, \( \nabla^2 f(x) \) coincides with the restriction of the second-order Fréchet derivative \( f''(x) \) to \( H \times H \) at \( x \in B \). In this framework, since for any \( x \in B \), \( f''(x) \) is a bounded linear operator from \( B \) into \( B^* \), Goodman’s theorem (see [Kuo75, Chapter I, Theorem 4.6]) implies that \( \nabla^2 f(x) \) is a trace-class operator on \( H \) and that, consequently, the Gross Laplacian \( \Delta_G f(x) \) is well-defined. Twice Fréchet differentiability hence constitutes a sufficient condition for the existence of the Gross Laplacian.

2.3.4. Stein’s equation and its solutions for abstract Wiener measures. In view of the above Stein lemma (Theorem 2.5), the associated Stein equation is given by
\[
\Delta_G f(x) - \langle x, \nabla f(x) \rangle_{B,B^*} = h(x) - E[h(Z)], \quad x \in B,
\]
where \( h \) is given in some class of test functionals. Shih showed in [Shi11] that
\[
f_h(x) = -\int_0^\infty (P_u h(x) - E[h(Z)]) \, du, \quad x \in B
\]
solves the Stein equation (3) whenever \( h \) is an element of \( \text{ULip-1}(B) \), the Banach space of scalar-valued uniformly 1-Lipschitz functions \( h \) on \( B \) with the norm \( \| h \| = \| h \|_{\text{ULip}} + |h(0)| \), where
\[
\| h \|_{\text{ULip}} = \sup_{x \neq y \in B} \frac{|h(x) - h(y)|}{\| x - y \|} < \infty.
\]
In what follows, we will consider test functions from the space $C^k_b(K)$ of real-valued, $k$-times Fréchet differentiable functions on a separable Hilbert space $K$ with bounded derivatives up to order $k$. A function $h$ thus belongs to $C^k_b(K)$ whenever

$$\|h\|_{C^k_b(K)} = \sup_{j=1, \ldots, k} \sup_{x \in K} \|\nabla^j h(x)\|_{K^j} < \infty.$$  

The following Lemma collects some properties of the Stein solution $f_h$ for a given function $h \in C^k_b(K)$.

**Lemma 2.7.** Let $K$ be a separable Hilbert space, $k \geq 1$ and $h \in C^k_b(K)$. Then the Stein solution $f_h$ defined in (4) also belongs to $C^k_b(K)$ and furthermore one has that

$$\sup_{u \in K} \|\nabla^j f_h(u)\|_{K^j} \leq \frac{1}{j} \|h\|_{C^k_b(K)}, \quad j \in \mathbb{N}, \quad j \leq k. \quad (5)$$

**Proof.** As for any $x \in K$, $f_h(x) = -\int_0^{\infty} (P_u h(x) - E[h(Z)]) \, du$, we have, for any $j = 1, \ldots, k$,

$$\nabla^j f_h(x) = -\int_0^{\infty} \nabla^j P_u h(x) \, du,$$

so that

$$\|f_h\|_{C^k_b(K)} = \sup_{j=1, \ldots, k} \sup_{x \in X} \left[ -\int_0^{\infty} \nabla^j P_u h(x) \, du \right]_{X^j} \leq \sup_{j=1, \ldots, k} \sup_{x \in X} \int_0^{\infty} \|\nabla^j P_u h(x)\|_{X^j} \, du.$$ 

Using the property of the semigroup $P$ that $\nabla^j P_u h(x) = e^{-ju} P_u \nabla^j h(x)$, and the fact that $P$ is contractive yields

$$\|f_h\|_{C^k_b(K)} \leq \sup_{j=1, \ldots, k} \sup_{x \in X} \int_0^{\infty} e^{-ju} \|P_u \nabla^j h(x)\|_{K^j} \, du$$

$$\leq \sup_{j=1, \ldots, k} \sup_{x \in X} \int_0^{\infty} e^{-ju} \|\nabla^j h(x)\|_{K^j} \, du$$

$$= \sup_{j=1, \ldots, k} \sup_{x \in K} \frac{1}{j} \|\nabla^j h(x)\|_{K^j}$$

$$\leq \|h\|_{C^k_b(K)} < \infty,$$

proving that $f_h \in C^k_b(K)$. The bound (5) can be derived similarly. \qed

### 2.4. Probabilistic metrics.

The proximity of two measures $\mu$ and $\nu$, both defined on a real separable Hilbert space $K$, will be quantified using probabilistic distances of the form

$$d_{\mathcal{U}}(\mu, \nu) = \sup_{h \in \mathcal{U}} \left| \int_K h(x) \mu(dx) - \int_K h(x) \nu(dx) \right|, \quad (6)$$

where $\mathcal{U} \subseteq L^1(K, \mu) \cap L^1(K, \nu)$ is a class of test functions $h: K \to \mathbb{R}$ which is separating, in the sense that if $\int_K h(x) \mu(dx) = \int_K h(x) \nu(dx)$ for all $h \in \mathcal{U}$, then $\mu = \nu$.

Equivalently, one can express the above from the point of view of random variables, by interpreting the measures $\mu$ and $\nu$ as the laws of two $K$-valued random variables, say $F$ and $G$. In this case, the difference of integrals on the right hand side of equation (6) can be
written as a difference of expectations, namely $\operatorname{E}(h(F)) - \operatorname{E}(h(G))$. The analogue of (6) thus reads

$$d(\mathcal{U}) = \max_{h \in \mathcal{U}} \left| \operatorname{E}(h(F)) - \operatorname{E}(h(G)) \right|,$$

where, with slight abuse of notation, $d(\mathcal{U}) = d(\mu, \nu)$. Both notations will be used interchangeably.

It is straightforward to check that $d(\mathcal{U})$ defines a metric on the set of all probability measures $\mu$ on $K$, such that $\mathcal{U} \subseteq L^1(K, \mu)$, and, of course, restricting or enlarging a given test class $\mathcal{U}$ of test functions weakens or strengthens the distance, respectively.

In the context of Gaussian approximation via Stein’s method, a natural class of test functions are the twice Fréchet differentiable functions with uniformly bounded first and second derivatives. We will denote this distance by $d_2$ and therefore have that

$$d_2(F, G) = \sup_{h \in C^2_b(K), \|h\|_{C^2_b(K)} \leq 1} \left| \operatorname{E}(h(F)) - \operatorname{E}(h(G)) \right|,$$

where $K$ is a separable Hilbert space and $C^2_b(K)$ is defined in Subsection 2.3.4.

As the next lemma shows, the class of test functions used to define $d_2$ is separating and, furthermore, $d_2$ metrizes convergence in law. Here, as usual, convergence in law (or weak convergence) of a sequence $(F_n)_{n \in \mathbb{N}_0}$ of $K$-valued random variables to a limiting random variable $F$ means that

$$\operatorname{E}(h(F_n)) \to \operatorname{E}(h(F))$$

as $n$ goes to infinity, for all $h \in C_b(K)$, the space of real-valued, bounded and continuous functions on $K$.

**Lemma 2.8.** On any separable Hilbert space $K$, the class of two times Fréchet differentiable functions with uniformly bounded first and second derivative is separating. In particular, if $(F_n)_{n \in \mathbb{N}_0}$ is a sequence of $K$-valued random variables such that $d_2(F_n, F_0) \to 0$, $\quad (n \to \infty)$, then the law of $F_n$ converges weakly to the law of $F_0$.

**Proof.** This lemma was proved in [CD13, Lemma 4.1] for the special case $K = \ell^2(\mathbb{N})$ but the proof continues to work without any modification for arbitrary separable Hilbert spaces.

As will become clear later, in order to strengthen the norm of the bounding carré du champ expression, we need to slightly weaken the distance $d_2$ by shrinking the class of test functions. Such a procedure (of a different type) was also necessary in [Bar90]. In our case, the restriction depends on a positive definite and self-adjoint trace-class operator on $K$. Such an operator, say $A$, introduces an inner product $\langle \cdot , \cdot \rangle_1$ on $K$ by

$$\langle x, y \rangle_1 = \langle \sqrt{A} x, \sqrt{A} y \rangle_K,$$

which is weaker than $\langle \cdot , \cdot \rangle_K$ as for the induced norms

$$\|x\|_1 \leq \|S\|_{\text{op}} \|y\|_K.$$

Let $K_1$ be the completion of $K$ with respect to $\|\cdot\|_1$. Then $K \subseteq K_1$ densely and as a Borel set.
Define
\[ \mathcal{U}_A = \left\{ h_{|K} : h \in C^2_b(K), \|h_{|K}\|_{C^2_b(K)} \leq 1 \right\} \]
and the corresponding distance \( d_A \) by
\[ d_A = \sup_{h \in \mathcal{U}_A} |E(h(F)) - E(h(G))|. \]
By (8), one has \( \mathcal{U}_A \subseteq \left\{ h \in C^2_b(K) : \|h\|_{C^2_b(K)} \leq 1 \right\} \) so that \( d_A \leq d_2 \). The following lemma shows that \( \mathcal{U}_A \) is separating and therefore \( d_A \) metrizes convergence in law.

**Lemma 2.9.** For a given separable Hilbert space \( K \) and a bijective, positive definite and self-adjoint trace class operator \( A \) on \( K \), the class \( \mathcal{U}_A \) defined by (9) is separating. In particular, if \( (F_n)_{n \in \mathbb{N}_0} \) is a sequence of \( K \)-valued random variables such that
\[ d_A(F_n, F_0) \to 0 \quad \text{as} \quad n \to \infty, \]
then, the law of \( F_n \) converges weakly to the law of \( F_0 \) as \( n \to \infty \).

**Proof.** Denote the laws of \( F_n \) by \( \mu_n, n \in \mathbb{N}_0 \). Then one can define laws \( \tilde{\mu}_n \) on \( (K_1, \mathcal{B}(K_1)) \), where \( \mathcal{B}(K_1) \) are the Borel sets of \( K_1 \) with respect to \( \|\cdot\|_1 \), by
\[ \tilde{\mu}_n(A) = \mu_n(A \cap K), \quad A \in \mathcal{B}(K_1). \]
Let \( g \in \mathcal{U}_S \). Then \( g = h_{|K} \) with \( h \in C^2_b(K_1) \). As \( g \) is measurable with respect to \( \mathcal{B}(K_1) \cap K \) and the topological support of \( \tilde{\mu}_n \) is \( K \), it follows that
\[ \int_K g(x) \mu_n(dx) = \int_K h_{|K}(x) \mu_n(dx) = \int_{K_1} h(x) \tilde{\mu}_n(dx) \]
for \( n \in \mathbb{N}_0 \), so that
\[ d_{\mathcal{U}_S}(\mu_n, \mu_0) = d_{C^2_b(K_1)}(\tilde{\mu}_n, \tilde{\mu}_0). \]
By Lemma 2.8, we get that \( \tilde{\mu}_n \to \tilde{\mu}_0 \) weakly on \( (K_1, \mathcal{B}(K_1)) \). Therefore, \( \mu_n \to \mu_0 \) weakly on \( (K, \mathcal{B}(K_1) \cap K) \). Now an application of Lemma 2.10 below gives that \( \mu_n \to \mu_0 \) weakly on \( (K, \mathcal{B}(K)) \).

**Lemma 2.10.** Let \( K_0 \) be a real separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle_0 \) and \( S \) be a bijective, positive definite and self-adjoint trace-class operator on \( K_0 \). Define an inner product \( \langle \cdot, \cdot \rangle_1 \) on \( K_0 \) by
\[ \langle x, y \rangle_1 = \left\langle \sqrt{S}x, \sqrt{S}y \right\rangle_0, \quad x, y \in K_0 \]
and denote the associated norm by \( \|\cdot\|_1 \). Let \( \mathcal{B}_0 \) and \( \mathcal{B}_1 \) be the Borel sets of \( K_0 \) with respect to the norms \( \|\cdot\|_0 \) and \( \|\cdot\|_1 \), respectively, so that \( \mathcal{B}_1 \subseteq \mathcal{B}_0 \). Then any two finite measures \( \mu \) and \( \nu \) on \( (K, \mathcal{B}_1) \) which agree on \( (K, \mathcal{B}_1) \) are equal.

**Proof.** Let \( K_1 \) be the completion of \( K_0 \) with respect to \( \|\cdot\|_1 \). Then \( K_0 \subseteq K_1 \) densely and as a Borel set and furthermore \( \mathcal{B}_1 = \mathcal{B}(K_1) \cap K_0 \).

Let \( \{k_i : i \in \mathbb{N}\} \) be an orthonormal basis of \( K_0 \) consisting of eigenfunctions of \( S \). Then, by construction, \( \{k_i : i \in \mathbb{N}\} \) is a complete orthogonal system in \( K_1 \).

For \( n \in \mathbb{N} \), let \( P_n \) be the orthogonal projector of \( K_1 \) onto
\[ V_n = \text{span} \{k_1, k_2, \cdots, k_n\} \]
and denote the restriction of $P_n$ to $K_0$ by $R_n$. Then $R_n$ is the orthogonal projector of $K_0$ onto $V_n$.

Let $B_r(y)$ be the closed ball in $K_0$ with respect to $\|\cdot\|_0$, centered at $y \in K_0$ and with radius $r > 0$. Then $B_n = R_n(B)$ is the closed ball in $V_n$ with respect to $\|\cdot\|_0$, centered at $R Ny$ and with radius $r$. Now define $A_{1,n} = P_n^{-1}(B_n)$ and $A_{0,n} = R_n^{-1}(B_n)$. Then $A_{1,n} \in \mathcal{B}(K_1)$ since $P_n$ is bounded (and hence continuous and measurable). Also, as $R_n$ is the restriction of $P_n$ to $K_0$, one has $A_{0,n} = A_{1,n} \cap K_0 \in \mathcal{B}(K_1) \cap K_0$. Therefore, by assumption, $\mu(A_{0,n}) = \nu(A_{0,n})$. As the sequence $(A_{0,n})_{n \in \mathbb{N}}$ is decreasing and its intersection is $B_r(y)$, it follows that $\mu(B_r(y)) = \nu(B_r(y))$. The assertion is now a consequence of the fact that finite Borel measures on a separable Banach space are determined by their values on balls (see [PT91]).

3. Main results

3.1. Dirichlet structures and Hilbert space valued Markov chaos. In this section, a Dirichlet structure for Hilbert-valued random variables is introduced, which then gives rise to a notion of Markov chaos (generalizing earlier definitions stated in [ACP14] and [CNPP16]). We start by recalling the well-known definition in the case of real-valued random variables (full details can for example be found in [BH91, FOT11, MR92]): given a probability space $(\Omega, \mathcal{F}, P)$, a Dirichlet structure $(\mathbb{D}, \mathcal{E})$ on $L^2(\Omega; \mathbb{R})$ with associated carré du champ operator $\Gamma$ consists of a Dirichlet domain $\mathbb{D}$, which is a dense subset of $L^2(\Omega; \mathbb{R})$ and a carré du champ operator $\Gamma : \mathbb{D} \times \mathbb{D} \to L^1(\Omega; \mathbb{R})$ characterized by the following properties.

- $\Gamma$ is bilinear, symmetric ($\Gamma(f, g) = \Gamma(g, f)$) and positive $\Gamma(f, f) \geq 0$,
- for all $m, n \in \mathbb{N}$, all Lipschitz and continuously differentiable functions $\varphi : \mathbb{R}^m \to \mathbb{R}$ and $\psi : \mathbb{R}^n \to \mathbb{R}$ and all $f = (f_1, \ldots, f_m) \in \mathbb{D}^m$, $g = (g_1, \ldots, g_n) \in \mathbb{D}^n$, it holds that

$$
\Gamma(\varphi(f), \psi(g)) = \sum_{i=1}^m \sum_{j=1}^n \partial_i \varphi(f) \partial_j \psi(g) \Gamma(f_i, g_j),
$$

(11)

- the induced positive linear form $f \mapsto \mathcal{E}(f, F)$, where

$$
\mathcal{E}(f, g) = \frac{1}{2} \mathbb{E}(\Gamma(f, g)),
$$

is closed in $L^2(\Omega; \mathbb{R})$, i.e., $\mathbb{D}$ is complete when equipped with the norm

$$
\|\cdot\|^2 = \|\cdot\|^2_{L^2(\Omega,F,P)} + \mathcal{E}(\cdot).
$$

Here and in the following, $\mathbb{E}(\cdot)$ denotes expectation on $(\Omega, \mathcal{F})$ with respect to $P$. The form $f \mapsto \mathcal{E}(f, f)$ is called a Dirichlet form, and, as is customary, we will write $\mathcal{E}(f)$ for $\mathcal{E}(f, f)$. Every Dirichlet form gives rise to a strongly continuous semigroup $(P_t)_{t \geq 0}$ on $L^2(\Omega; \mathbb{R})$ and an associated symmetric Markov generator $-L$, defined on a dense subset $\text{dom}(-L) \subseteq \mathbb{D}$. We will often switch between $-L$ or $L$, as these two operators only differ by sign. There are two important relations between $\Gamma$ and $L$: the first one is the integration by parts formula

$$
\mathbb{E}(\Gamma(f, g)) = - \mathbb{E}(fLg) = - \mathbb{E}(gLf),
$$

(12)
valid whenever \( f, g \in \mathbb{D} \), the second one is the relation
\[
\Gamma(f, g) = \frac{1}{2} (L(fg) - gLf - fLg),
\]
which holds for all \( f, g \in \text{dom}(L) \) such that \( fg \in \text{dom}(L) \).

If \(-L\) is diagonalizable with spectrum \( \{0 = \lambda_0 < \lambda_1 < \ldots\} \), a pseudoinverse \(-L^{-1}\) can be introduced via spectral calculus as follows: if \( f = \sum_{i=0}^{\infty} f_i \) with \( f_i \in \ker(L + \lambda_i \text{Id}) \), then
\[
-L^{-1} f = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} f_i.
\]

It follows that
\[
LL^{-1} f = f - E(f).
\]

Assume now such a Dirichlet structure on \( L^2(\Omega; \mathbb{R}) \) with diagonalizable generator as given and denote the Dirichlet domain, Dirichlet form, carré du champ operator, its associated infinitesimal generator and pseudo-inverse by \( \mathbb{D}, \mathcal{E}, L, \tilde{L} \) and \( L^{-1} \), respectively, in order to distinguish these objects from their extensions to the Hilbert-valued setting to be introduced below.

Given a separable Hilbert space \( K \), one has that \( L^2(\Omega; K) \) is isomorphic to \( L^2(\Omega; \mathbb{R}) \otimes K \). The Dirichlet structure on \( L^2(\Omega; \mathbb{R}) \) can therefore be extended to \( L^2(\Omega; K) \) via a tensorization procedure as follows.

Let \( \{0 = \lambda_0 < \lambda_1 < \ldots\} \) be the spectrum of \( -\tilde{L} \) and \( A \) the set of all functions \( F \) of the form
\[
F = \sum_{p \in I} f_p \otimes k_p,
\]
where \( I \subseteq \mathbb{N}_0 \) is finite, the \( f_p \) are eigenfunctions of \( -\tilde{L} \) with eigenvalue \( \lambda_p \geq 0 \) and \( k_p \in K \).

Then \( A \) is dense in \( L^2(\Omega; K) \) as any \( F \in L^2(\Omega; K) \) can be written in the form (13) with \( I = \mathbb{N}_0 \). For \( F \in A \) of the form (13) and analogously \( G = \sum_{p' \in I'} f_{p'} \times k_{p'} \in A \), define linear operators \( L, L^{-1} \) by
\[
LF = \sum_{p \in I} (\tilde{L}f_p) \otimes k_p = -\sum_{p \in I} \lambda_p f_p \otimes k_p,
\]
and a bilinear operator \( \Gamma \) by
\[
\Gamma(F, G) = \frac{1}{2} \sum_{p \in I} \sum_{p' \in I'} \tilde{\Gamma}(f_p, f_{p'}) \otimes (k_p \otimes k_{p'} + k_{p'} \otimes k_p),
\]
where we identify \( \Gamma(F, G) \in L^2(\Omega; \mathbb{R}) \otimes K \otimes K \simeq L^2(\Omega; \mathcal{L}(K, K)) \) with a random operator on \( K \). The action of \( \Gamma(F, G) \) is given by
\[
\Gamma(F, G)u = \frac{1}{2} \sum_{p \in I} \sum_{p' \in I'} \tilde{\Gamma}(f_p, f_{p'}) \left( (k_p, u) \otimes k_{p'} + (k_{p'}, u) \otimes k_p \right), \quad u \in K.
\]
For all \( F, G \in \mathcal{A} \), this defines a random trace class operator \( \Gamma(F, G) \), i.e., an element of \( L^1(\Omega; S_1) \). Recall from Subsection 2.2 that the norm on \( L^1(\Omega; S_1) \) is defined as \( \| \cdot \|_{L^1(\Omega; S_1)} = E(\text{tr} |\cdot|) \). As the next proposition shows, the operators \( L, L^{-1} \) and \( \Gamma \) are closable.

**Proposition 3.1.** The operators \( L: \mathcal{A} \to L^2(\Omega; K) \) and \( L^{-1}: \mathcal{A} \to L^2(\Omega; K) \) and \( \Gamma: \mathcal{A} \times \mathcal{A} \to L^1(\Omega; S_1) \) defined by (14) and (15), respectively, are closable in \( L^2(\Omega; K) \). The operator \( \text{dom}(\Gamma) = \mathcal{A} \times \mathcal{A} \) is closable in \( L^1(\Omega; S_1) \). Their maximal domains \( \text{dom}(L) \), \( \text{dom}(L^{-1}) \) and \( \text{dom}(\Gamma) = D \times D \) are given by

\[
\text{dom}(L) = \left\{ F \in L^2(\Omega; K); \sum_{p=1}^{\infty} \lambda_p^2 \tilde{\pi}_p \left( \| F \|_2^2 \right) < \infty \right\}
\]

\[
\text{dom}(L^{-1}) = L^2(\Omega; K)
\]

and

\[
D = \text{dom}(\varphi) = \left\{ F \in L^2(\Omega; K); \sum_{p=1}^{\infty} \lambda_p \tilde{\pi}_p \left( \| F \|_2^2 \right) < \infty \right\},
\]

respectively, where \( \tilde{\pi}_p \) denotes the orthogonal projection onto \( \ker(\tilde{L} + \lambda_p \text{Id}) \subseteq L^2(\Omega; \mathbb{R}) \). In particular, one has

\( \mathcal{A} \subseteq \text{dom}(L) \subseteq D \subseteq \text{dom}(L^{-1}) = L^2(\Omega; K) \),

where all inclusions are dense.

**Proof.** Closability and the form of the maximal domains for \( L \) and \( L^{-1} \) follow straightforwardly from the respective properties of \( \tilde{L} \) and \( \tilde{L}^{-1} \). Now let \( F \in \mathcal{A} \) be of the form (13) and \( u \in K \). Then, almost surely,

\[
\langle \Gamma(F, F)u, u \rangle_K = \sum_{p, p' \in I} \tilde{\Gamma}(f_p, f_{p'}) \langle k_p, u \rangle_K \langle k_{p'}, u \rangle_K \tilde{\Gamma}(f_p, f_{p'}) \geq 0,
\]

where the last inequality follows from the positivity of \( \tilde{\Gamma} \). Therefore \( \Gamma(F, F) \) is positive for all \( F \in K \). To prove closability and the form of the maximal domain \( D \times D \) of \( \Gamma \), it suffices to consider the operator \( F \mapsto \Gamma(F, F) \). The general case then follows by the polarization identity

\[
\Gamma(F, G) = \frac{1}{4} (\Gamma(F + G, F + G) - \Gamma(F - G, F - G)).
\]

Let \( (F^{(n)}) \subseteq \mathcal{A} \) such that \( \| F^{(n)} \|_{L^2(\Omega, K)} \to 0 \). By (13), each \( F^{(n)} \) is of the form

\[
F^{(n)} = \sum_{p \in I_n} f^{(n)}_p \otimes k_p,
\]
Theorem 3.2. For the Dirichlet structure \( L \) on \( \Gamma \) above, the following is true.

Throughout this article, the extensions of \( L, L^{-1} \) and \( \Gamma \) to their maximal domains will still be denoted by the same symbols. The operators just defined yield a Dirichlet domain \( \sim \Gamma(\Omega;K) \), consisting of a dense subspace \( \sim \mathbb{D} \) of \( L^2(\Omega;K) \) and a carré du champ operator \( \sim \Gamma: \sim \mathbb{D} \times \sim \mathbb{D} \to L^1(\Omega;S_1) \) as introduced above, the following is true.

(i) \( \sim \Gamma \) is bilinear, almost surely positive (i.e., \( \sim \Gamma(F,F) \geq 0 \) as an operator on \( K \) ), symmetric in its arguments and self-adjoint \( \langle \sim \Gamma(F,G)u,v \rangle \geq \langle u,\sim \Gamma(F,G)v \rangle \) for all \( u,v \in K \).

(ii) The Dirichlet domain \( \mathbb{D} \), endowed with the norm

\[
\|F\|_{\mathbb{D}}^2 = \|F\|_{L^2(\Omega;K)}^2 + \|\sim \Gamma(F,F)\|_{L^1(\Omega;S_1)}
\]

is complete, so that \( \sim \Gamma \) is closed.

(iii) For all Lipschitz and Fréchet differentiable operators \( \varphi, \psi \) on \( K \) and \( F,G \in \mathbb{D} \), one has that \( \varphi(F), \psi(G) \in \mathbb{D} \) and the diffusion identity

\[
\sim \Gamma(\varphi(F), \psi(G)) = \frac{1}{2} (\nabla \varphi(F)^* \sim \Gamma(F,G) \nabla \psi(G) + \nabla \psi(G)^* \sim \Gamma(F,G) \nabla \varphi(F))
\]

holds, where \( \nabla \varphi(F) \) and \( \nabla \psi(G) \) denote the Fréchet derivatives of \( \varphi \) and \( \psi \) at \( F \) and \( G \), respectively, and \( \nabla \varphi(F)^*, \nabla \psi(G)^* \) are their adjoints in \( K \).

(iv) The associated generator \(-\sim L\) acting on \( L^2(\Omega;K) \) is positive, symmetric, densely defined and has the same spectrum as \(-L\).
(v) There exists a compact pseudo-inverse $L^{-1}$ of $L$ such that
$$LL^{-1}F = F - E(F)$$
for all $F \in L^2(\Omega; K)$, where the expectation on the right is a Bochner integral (well defined in view of Remark 2.1 as $F \in L^2(\Omega; K)$).

(vi) The integration by parts formula
$$E(\text{tr} \Gamma(F,G)) = -E(\langle LF,G \rangle) = -E(\langle F, LG \rangle).$$
is satisfied for all $F, G \in \text{dom}(-L)$.

(vii) The carré du champ $\Gamma$ and the generators $L$ and $\tilde{L}$ are connected through the identity
$$\text{tr} \Gamma(F,G) = \frac{1}{2} \left( \tilde{L} \langle F,G \rangle - \langle LF,G \rangle - \langle F, LG \rangle \right),$$
valid for $F, G \in \text{dom}(L)$.

(viii) the fundamental identity
$$\langle \Gamma(F,G)u,v \rangle = \frac{1}{2} \left( \tilde{\Gamma} (\langle F,u \rangle, \langle G,v \rangle) + \tilde{\Gamma} (\langle G,u \rangle, \langle F,v \rangle) \right),$$
connecting $\Gamma$ and its one-dimensional counterpart $\tilde{\Gamma}$ is valid for all $F, G \in \mathbb{D}$ and all $u, v \in K$.

Proof. Parts (i)-(ii) and (iv)-(viii) can be verified without difficulty, using the definitions of $\Gamma$, $L$ and $L^{-1}$. To prove (iii), write
$$F = \sum_{p=0}^{\infty} \sum_{i=1}^{n} f_p \otimes k_i \quad \text{and} \quad G = \sum_{p=0}^{\infty} \sum_{i=1}^{n} g_p \otimes k_i,$$
where the $f_p$ and $g_p$ are eigenfunctions of $\tilde{L}$ with eigenvalue $-\lambda_p$, and $\{k_i : i \in \mathbb{N}\}$ is an orthonormal basis of $K$. Let $K_n = \text{span} \{k_i : 1 \leq i \leq n\}$ and $\rho_n$ be the orthogonal projection onto $L^2(\Omega; K_n)$, so that
$$\rho_n(F) = \sum_{p=0}^{\infty} \sum_{i=1}^{n} f_p \otimes k_i \quad \text{and} \quad \rho_n(G) = \sum_{p=0}^{\infty} \sum_{i=1}^{n} g_p \otimes k_i.$$
Denote by $i_n : K_n \to \mathbb{R}^n$ the canonical isometric isomorphism mapping $K_n$ to $\mathbb{R}^n$ so that $\xi_n = i_n \circ \rho_n(F) \in \mathbb{R}^n$ and $\nu_n = i_n \circ \rho_n(G) \in \mathbb{R}^n$.

Let $\tilde{\varphi}_n = \varphi \circ i_n^{-1}$ and $\tilde{\psi}_n = \psi \circ i_n^{-1}$. Then $\tilde{\varphi}_n : \mathbb{R}^n \to K$ is Lipschitz and Fréchet differentiable, with Fréchet derivative given by
$$\nabla \tilde{\varphi}_n(x)(y) = \nabla \varphi(i_n^{-1}(x))(i_n^{-1}(y))$$
for all $x, y \in \mathbb{R}^n$ and an analogous result is true for $\nabla \tilde{\psi}_n$. Therefore, via
$$\Gamma (\varphi(\rho_n(F)), \psi(\rho_n(G))) = \Gamma \left( \tilde{\varphi}_n(\xi_n), \tilde{\psi}_n(\nu_n) \right)$$
and identity (19), the assertion can be transformed into an equivalent assertion for $\tilde{\Gamma}$, which can then be verified by tedious but straightforward calculations, using the diffusion property (11) for $\tilde{\Gamma}$ and then letting $n \to \infty$. \qed
In analogy to the real-valued case, \( E(F, G) = E(\text{tr} \Gamma(F, G)) \) defines a bilinear, positive and symmetric form on \( D \times D \), and \( D \) is complete under the norm \( \|F\|_D = \|F\|_{L^2(\Omega; K)}^2 + E(F, F) \), so that the form \( F \mapsto E(F, F) \) is closed. Instead of directly constructing \( L \), it could also be obtained through \( E \) from the general theory of Dirichlet forms. Furthermore, note that \( L \) and \( \Gamma \) (as well as their domains) depend on \( K \) only through the dimension. Note that the Malliavin calculus on Hilbert spaces discussed in Subsection 3.4 is a particular case of a Dirichlet structure, where \( \Gamma(X, Y) = \langle DX, DY \rangle_{\Omega} \), \( D = D^{1,2} \), and \( L \) is the generator of the \( K \)-valued Ornstein-Uhlenbeck semigroup (see Subsection 3.4 for undefined notation).

Having established a Dirichlet structure on \( L^2(\Omega; K) \), the definition of (jointly) chaotic eigenfunctions can be extended as follows.

**Definition 3.3.** In the above setting, an eigenfunction \( F \in \ker (L + \lambda \text{Id}) \) is called chaotic, if
\[
\|F\|_K^2 \in \bigoplus_{\alpha \in \Lambda, \alpha \leq 2\lambda} \ker (L + \alpha \text{Id}) .
\]
Here, \( \Lambda \) denotes the spectrum of \( L \). Two eigenfunctions \( F \in \ker (L + \lambda \text{Id}) \) and \( G \in \ker (L + \eta \text{Id}) \) are called jointly chaotic, if
\[
\langle F, G \rangle_K \in \bigoplus_{\alpha \leq \lambda + \eta} \ker (L + \alpha \text{Id}) .
\]
A diagonalizable generator \(-L\) is called chaotic, if any two of its eigenfunctions are jointly chaotic.

### 3.2. General carré du champ bounds

In this section, we combine the Stein methodology presented in Subsection 2.3 with the Dirichlet structure introduced above in order to derive carré du champ bounds on probabilistic metrics for the approximation of Gaussian measures on the Hilbert space \( K \). From now on, we assume that we work with a Dirichlet structure on \( L^2(\Omega; K) \) as introduced in Subsection 3.1, where \( K \) is a separable Hilbert space. The generator, its spectrum and the carré du champ operator will be denoted by \( L \), \( \{\cdots < -\lambda_2 < -\lambda_1 < -\lambda_0 = 0\} \) and \( \Gamma \), respectively. Also, fix a positive-definite, self-adjoint trace-class operator \( A \) and recall the probabilistic distances \( d_2 \) and \( d_A \) introduced in Subsection 2.4.

#### 3.2.1. Abstract carré du champ bounds

The following general bound between the laws of a square integrable \( K \)-valued random variable in the Dirichlet domain \( D \) and an arbitrary Gaussian measure holds.

**Theorem 3.4.** Let \( Z \) be a non-degenerate Gaussian random variable on \( K \) with covariance operator \( S \). Then, for all \( F \in D \) one has
\[
d_2(F, Z) \leq \frac{1}{2} \left\| \Gamma(F, L^{-1}F) - S \right\|_{L^1(\Omega, S_1)} \tag{20}
\]
and
\[
d_A(F, Z) \leq C \left\| \Gamma(F, L^{-1}F) - S \right\|_{L^2(\Omega, HS)} , \tag{21}
\]
where
\[ C = \frac{\pi}{2} \| S \|_{\text{op}} \sqrt{\text{tr}(S)} \sqrt{\text{tr}(A)} \]
is a positive constant depending only on \( A \) and \( S \).

**Remark 3.5.** Note that in the case where \( K \) is finite-dimensional, the bound (21) becomes the main statement of [NPR10b, Theorem 3.5] with a different constant. In [NPR10b, Theorem 3.5], instead of \( C_Z \) the authors use matrix methods to obtain the constant
\[ C^* = \| S \|^{1/2} \| S^{-1} \|_{\text{op}}, \]
which explodes as the dimension grows to infinity (as the inverse of a compact operator is always unbounded in infinite dimension). It is to circumvent this problem that the enlarged Hilbert space \( K_1 \) was introduced when defining the probabilistic metric \( d_A \). Also recall that \( d_A \leq d_2 \) and \( \| \cdot \|_{L^2(\Omega;HS)} \leq \| \cdot \|_{L^1(\Omega;S_1)} \). Thus, comparing the bounds (20) and (21), we see that weakening the distance allows to strengthen the norm of the carré du champ expression.

Before proceeding with the proof of Theorem 3.4, let us remark that if \( Z \) is a \( K \)-valued Gaussian random variable with covariance operator \( S \), then \( (Z, -L^{-1}Z) = S \). This yields the following corollary.

**Corollary 3.6.** Let \( Z_1, Z_2 \) be two Gaussian random variables on \( K \) with covariance operators \( S_1, S_2 \), respectively. Then, it holds that
\[ d_2(Z_1, Z_2) \leq \frac{1}{2} \| S_1 - S_2 \|_{S_1} \]
and
\[ d_A(Z_1, Z_2) \leq C \| S_1 - S_2 \|_{HS}, \]
where \( C \) is a positive constant depending either only on \( A \) and \( S_2 \) (it could of course also be chosen to only depend on \( A \) and \( S_1 \)).

**Proof of Theorem 3.4.** Let \( h \in C_0^2(K) \). Identifying \( K^* \) with \( K \), using the integration by parts formula (18) and the diffusion property (17) for the carré du champ, we can write
\[
(F, \nabla f_h(F))_{K,K^*} = E(\langle \nabla f_h(F), F \rangle_K)
= E(\langle LL^{-1}F, \nabla f_h(F) \rangle_K)
= E(\text{tr}_K(\nabla f_h(F), -L^{-1}F))
= E(\text{tr}_K(\nabla^2 f_h(F) \Gamma(F, -L^{-1}F))).
\]
Now let \( H \) be the Hilbert space associated to \( Z \) as introduced in Section 2.3.2. As the covariance operator \( S \) of \( Z \) is compact and one-to-one (see [Kuo75] or [Bog98]), it holds that \( S = \sum_{i \in \mathbb{N}} \lambda_i \langle \cdot, e_{k_i} \rangle_K e_{k_i} \) for some \( \lambda_i > 0 \) and an orthonormal basis \((e_{k_i})_{i \in \mathbb{N}}\) of \( H \) consisting of eigenvectors. Then \((k_i)_{i \in \mathbb{N}}\), where \( k_i = \frac{1}{\sqrt{\lambda_i}}e_{k_i} \), is an orthonormal basis of \( K \), as \( H = \sqrt{S}(K) \). It thus follows that
\[
\text{tr}_H \nabla^2 f_h(F) = \sum_{i \in \mathbb{N}} \nabla^2 f_h(F)(e_i, e_i) = \sum_{i \in \mathbb{N}} \nabla^2 f_h(F) (S k_i, k_i) = \text{tr}_K (\nabla^2 f_h(F) S).
\]
Combining the last two calculations yields for any \( h \in C_0^2(K) \) that
\[
E \left( \langle F, \nabla f_h(F) \rangle_{K,K^*} - E \left( \text{tr}_H \nabla^2 f_h(F) \right) \right) = E \left( \text{tr}_K \left( \nabla^2 f_h(F) \left( \Gamma(F,-L^{-1}F) - S \right) \right) \right),
\]
and, taking absolute values,
\[
\left| E \left( \langle F, \nabla f_h(F) \rangle_{K,K^*} - E \left( \text{tr}_H \nabla^2 f_h(F) \right) \right) \right| = \left| E \left( \text{tr}_K \left( \nabla^2 f_h(F) \left( \Gamma(F,-L^{-1}F) - S \right) \right) \right) \right| \leq E \left( \text{tr}_K \left| \nabla^2 f_h(F) \left( \Gamma(F,-L^{-1}F) - S \right) \right| \right).
\]
(22)

To show the bound (20), note that
\[
E \left( \text{tr}_K \left| \nabla^2 f_h(F) \left( \Gamma(F,-L^{-1}F) - S \right) \right| \right) \leq E \left( \| \nabla^2 f_h(F) \|_{\text{op}(\Gamma)} \text{tr}_K \left( \Gamma(F,-L^{-1}F) - S \right) \right) \leq \left( \sup_{u \in K} \| \nabla^2 f_h(u) \|_{\text{op}(\Gamma)} \right) E \left( \text{tr}_K \left( \Gamma(F,-L^{-1}F) - S \right) \right) \leq \frac{1}{2} \| \Gamma(F,-L^{-1}F) - S \|_{L^1(\Omega;S_1(K))},
\]
where the last inequality is a consequence of (5).

To prove (21), one can again start from (22) and apply the Cauchy-Schwarz inequality for the Schatten norms. This gives
\[
E \left( \text{tr}_K \left| \nabla^2 f_h(F) \left( \Gamma(F,-L^{-1}F) - S \right) \right| \right) \leq E \left( \| \nabla^2 f_h(F) \|_{\text{HS}(\Gamma)} \| \Gamma(F,-L^{-1}F) - S \|_{\text{HS}(\Gamma)} \right) \leq \sqrt{E \left( \| \nabla^2 f_h(F) \|_{\text{HS}(\Gamma)}^2 \right) E \left( \| \Gamma(F,-L^{-1}F) - S \|_{\text{HS}(\Gamma)}^2 \right)}
\]
(23)

Now let \( h \in \mathcal{U}_A \), where \( \mathcal{U}_A \) is defined in (9). Then the bound (5.13) in [Shi11] can be applied (note that such \( h \) satisfies condition (5.2) in the aforementioned reference) and yields
\[
\| \nabla^2 f_h(F) \|_{L^2(\Omega;\text{HS}(\Gamma))}^2 \leq S \|_{\text{op}(\Gamma)} \sqrt{\text{tr}_K(S) \sup_{x \in K} \| \nabla^2 f_h(x) \|_{K_1,K_1^*}},
\]
where \( K_1 \) is the Hilbert space obtained by completing \( K \) with respect to the inner product \( \langle h,k \rangle_1 = \langle \sqrt{A} h, \sqrt{A} k \rangle_K \) (see Subsection 2.4 for definitions). From here, proceeding as in the proof of [Shi11, Theorem 4.9.ii], one can show that
\[
\| \nabla^2 f_h(x) \|_{K_1,K_1^*} \leq \frac{\pi}{2} \| h \|_{\text{Lip}(K_1)} \sqrt{\text{tr}_K(A)} \leq \frac{\pi}{2} \sqrt{\text{tr}_K(A)}
\]
so that in total
\[
\| \nabla^2 f_h(F) \|_{L^2(\Omega;\text{HS}(\Gamma))}^2 \leq \frac{\pi}{2} S \|_{\text{op}(\Gamma)} \sqrt{\text{tr}_K(S)} \sqrt{\text{tr}_K(A)} = C.
\]
Plugged back into (23) and taking the supremum over all \( h \in \mathcal{U}_A \), this gives

\[
d_A(F, Z) = \sup_{h \in \mathcal{U}_A} \left| E\left( h(F) \right) - E\left( h(Z) \right) \right|
\]

\[
= \sup_{h \in \mathcal{U}_A} \left| E\left( \langle F, \nabla f_h(F) \rangle_{K,K^*} - E\left( \text{tr}_H \nabla^2 f_h(F) \right) \right) \right|
\]

\[
\leq C \| \Gamma(F, -L^{-1}F) - S \|_{L^2(\Omega; \mathbb{HS}(K))}.
\]

\[
= \sup_{h \in \mathcal{U}_A} \left| E\left( \langle F, \nabla f_h(F) \rangle_{K,K^*} - E\left( \text{tr}_H \nabla^2 f_h(F) \right) \right) \right|
\]

\[
\leq C \| \Gamma(F, -L^{-1}F) - S \|_{L^2(\Omega; \mathbb{HS}(K))}.
\]

\[
\square
\]

**Remark 3.7.**

(1) Our bounds can also be used to prove weak convergence in a Banach space setting such as the Skorohod space or the space of continuous functions equipped with the supremum norm by the following procedure: Starting from a Gaussian random variable on a separable Banach space \( B \), it is always possible (see [Kue70, Lemma 2.1]) to densely embed \( B \) in a separable Hilbert space \( K \) such that the Borel sets of \( B \) are generated by the inner product of \( K \). Then, by applying our methods, one obtains weak convergence in \( K \), which in turn implies weak convergence in \( B \).

(2) Let us give an example of what Theorem 3.4 implies when we particularize the Hilbert space \( K \) to be finite-dimensional. In the case where \( K = \mathbb{R} \), we retrieve, as special cases, the quantitative bounds obtained in [Led12, ACP14] through the reformulation of Theorem 3.4 in this framework. More specifically, let \( Z \) be a real-valued, centered, Gaussian random variable with variance \( \sigma^2 \). Let \( F \in \mathbb{D} \) be centered. Then, Theorem 3.4 specialized to \( K = \mathbb{R} \) yields

\[
d_2(F, Z) \leq \frac{1}{2} \sqrt{E\left[ \Gamma(F, -L^{-1}F) - \sigma^2 \right]^2},
\]

which is the main result of [Led12]. In the case where \( K \) is taken to be \( \mathbb{R}^d \) for some \( d \geq 2 \), Theorem 3.4 yields a quantitative version of the main results of [CNPP16]. For more details on the finite-dimensional case, we refer to Subsections 3.3.2 and 3.4.4.

### 3.3. Fourth Moment bounds via chaos expansions.

In this section, we show how the carré du champ bounds obtained in Theorem 3.4 can be further estimated by the first four moments of the approximating random variable or sequence. The main ingredient is to decompose the approximating random variable as an orthogonal direct sum obtained by projecting it onto the eigenspaces of \( L \). This is known as a chaos expansion, and is a powerful tool for analysis of functionals of diffusive Markov operators. We start by introducing a covariance condition, which is technically not necessary for our results to hold, but allows to write our bounds in a basis-free bounds (see Remark 3.11 for more details on this point). We continue to assume as given a Dirichlet structure on \( L^2(\Omega; K) \), where \( K \) is a separable Hilbert space and a positive-definite, self-adjoint trace class operator \( A \) on \( K \).

**Definition 3.8.** A random variable \( F \in L^2(\Omega; K) \) is said to satisfy the covariance condition (C) if it holds that

\[
2 \text{ Cov} \left( \langle F, u \rangle, \langle F, v \rangle \right) \leq \text{ Cov} \left( \langle F, u \rangle^2, \langle F, v \rangle^2 \right)
\]

for any two orthogonal vectors \( u, v \in K \) with unit norm.
Remark 3.9. It will be proved later that such a condition is satisfied whenever $F$ is an eigenfunction of the Ornstein-Uhlenbeck generator.

3.3.1. Moment bounds.

**Theorem 3.10.** Let $Z$ be a Gaussian random variable on $K$ with covariance operator $S$ and let $F = \sum_{p=1}^{\infty} F_p$, where $L F_p = -\lambda_p F_p$. Assume that $L$ is chaotic and $F_p$ satisfies the covariance condition (24) for all $p \in \mathbb{N}$. Denote the covariance operators of $F_p$ by $S_p$. Furthermore, for any $N \in \mathbb{N}$, define $F_N = \sum_{p=1}^{N} F_p$, and denote by $T_N$ its covariance operator. Then

$$
\begin{align*}
 d_A(F, Z) & \leq C \left( \frac{1 + \sqrt{3}}{\sqrt{3}} \sum_{p=1}^{N} \sqrt{E(\|F_p\|^4)} \sqrt{E(\|F_p\|^4)} - \frac{E(\|F_p\|^2)}{\|S_p\|^1_{HS}} \right) \\
 & \quad + \sum_{1 \leq p, q \leq N \atop p \neq q} \frac{a_{p,q}}{\sqrt{3}} \sqrt{E(\|F_p\|^4)} \sqrt{E(\|F_q\|^4)} - \frac{E(\|F_p\|^2)}{\|S_p\|^1_{HS}} \right)^{1/2} \\
 & \quad + \frac{1}{2} \left( \|T_N - S\|^1_{HS} \right) + \sum_{p=N+1}^{\infty} E(\|F_p\|^2).
\end{align*}
$$

(25)

In particular, if $F = F_p$ for some eigenfunction $F_p \in \ker(L + \lambda_p \text{Id})$, then

$$
\begin{align*}
 d_A(F, Z) & \leq \left( \frac{1 + \sqrt{3}}{4\sqrt{3}} \sqrt{E(\|F\|^4)} \sqrt{E(\|F\|^4)} - \frac{E(\|F\|^2)}{\|S\|^1_{HS}} \right) \right)^{1/2} \\
 & \quad + \frac{1}{2} \|S_p - S\|^1_{HS}.
\end{align*}
$$

Proof of Theorem 3.10. Let $h \in C^2_2(K)$ and write $F = F_N + R_N$, where $F_N = \sum_{p=1}^{N} F_p$ and $R_N = F - F_N = \sum_{p=N+1}^{\infty} F_p$. Without loss of generality, assume that $F$ and $Z$ are defined on the same probability space. Then,

$$
 d_A(F, Z) \leq d_A(F, F_N) + d_A(F_N, Z).
$$
and, by Lipschitz continuity and the Cauchy-Schwarz inequality,
\[
\begin{align*}
    d_A(F, F_N) &\leq \sup_{h \in C_b^2(K)} |E(h(F) - h(F_N))| \\
    &\leq E \|F - F_N\| \\
    &\leq \sqrt{E \left( \|R_N\|^2 \right)} \\
    &= \sqrt{\sum_{p=N+1}^{\infty} E \left( \|F_p\|^2 \right)}.
\end{align*}
\]

It therefore remains to estimate \(d_A(F_N, Z)\). Applying Theorem 3.4, we get
\[
d_A(F_N, Z) \leq C \sqrt{E \left( \|\Gamma(F_N, -L^{-1}F_N) - S\|_{L^2(\Omega; HS)}^2 \right) + \|T_N - S\|_{HS}},
\]
where \(C\) is the constant appearing in the bounds of Theorem 3.4.

Before dealing with the first norm appearing in the above inequality, let \((e_i)_{i \in \mathbb{N}}\) be an orthonormal basis of \(K\), and denote, for any \(i \in \mathbb{N}\), \(F_i = \langle F, e_i \rangle\), as well as \(F_{p,i} = \langle F_p, e_i \rangle\).

Recalling that, by orthogonality, \(T_N = \sum_{p=1}^{N} S_p\), we have, for any \(i, j \in \mathbb{N}\),
\[
    T_N(e_i, e_j) = \sum_{p=1}^{N} S_p(e_i, e_j) = \sum_{p=1}^{N} E(F_{p,i}F_{p,j}) = \sum_{p,q=1}^{N} E(F_{p,i}F_{q,j}),
\]
where the last equality follows from the fact that \(E(\langle F_{p,i}, F_{q,j} \rangle) = 0\) whenever \(p \neq q\).

Hence,
\[
\begin{align*}
    E \left( \|\Gamma(F_N, -L^{-1}F_N) - T_N\|_{HS}^2 \right) &= \sum_{i,j=1}^{\infty} \sum_{p,q=1}^{N} E \left( \left( \Gamma(F_{p,i}, -L^{-1}F_{q,j}) - E(\Gamma(F_{p,i}, -L^{-1}F_{q,j})) \right)^2 \right) \\
    &= \sum_{p,q=1}^{N} \sum_{i,j=1}^{\infty} \text{Var} \left( \Gamma(F_{p,i}, -L^{-1}F_{q,j}) \right).
\end{align*}
\]

From a polarized version of [ACP14, Theorem 3.2], we get that
\[
\begin{align*}
    \text{Var} \left( \Gamma(F_{p,i}, -L^{-1}F_{q,j}) \right) &\leq a_{p,q} \left( E\left(\Gamma(F_{p,i}, -L^{-1}F_{q,j})\right) - E\left(F_{p,i}F_{q,j}\right)\right)^2,
\end{align*}
\]
where \(a_{p,q} = \frac{\lambda_p + \lambda_q}{2\lambda_q}\). The diffusion property and integration by parts yields
\[
\begin{align*}
    E\left(F_{p,i}F_{q,j} \Gamma(F_{p,i}, -L^{-1}F_{q,j})\right) &= \frac{1}{2} \left( E\left(\Gamma(F_{p,i}^2, -L^{-1}F_{q,j})\right) - E\left(F_{p,i}^2 \Gamma(F_{q,j}, -L^{-1}F_{q,j})\right)\right) \\
    &= \frac{1}{2} \left( E\left(F_{p,i}^2 F_{q,j}^2\right) - E\left(F_{p,i} \Gamma(F_{q,j}, -L^{-1}F_{q,j})\right)\right),
\end{align*}
\]
which, plugged into (26), gives

$$\text{Var} \left( \Gamma(F_{p,i}, -L^{-1}F_{q,j}) \right) \leq a_{p,q} \left( E \left( F^2_{p,i} F^2_{q,j} \right) - E \left( F^2_{p,i} \right) E \left( F^2_{q,j} \right) - 2 E \left( F_{p,i} F_{q,j} \right)^2 \right. \left. - E \left( F^2_{p,i} \left( \Gamma(F_{q,j}, -L^{-1}F_{q,j}) - E \left( F^2_{q,j} \right) \right) \right) \right).$$

Using the Parseval identity, we obtain from the above bound that

$$\sum_{i,j=1}^{\infty} \text{Var} \left( \Gamma(F_{p,i}, -L^{-1}F_{q,j}) \right) \leq a_{p,q} \left( E \left( \|F_p\|^2 \|F_q\|^2 \right) - E \left( \|F_p\|^2 \right) E \left( \|F_q\|^2 \right) - 2 \sum_{i,j=1}^{\infty} E \left( F_{p,i} F_{q,j} \right)^2 \right. \left. - \sum_{j=1}^{\infty} E \left( \|F_p\|^2 \left( \Gamma(F_{q,j}, -L^{-1}F_{q,j}) - E \left( F^2_{q,j} \right) \right) \right) \right) \leq a_{p,q} \left( E \left( \|F_p\|^2 \|F_q\|^2 \right) - E \left( \|F_p\|^2 \right) E \left( \|F_q\|^2 \right) - 2 \sum_{i,j=1}^{\infty} E \left( F_{p,i} F_{q,j} \right)^2 \right. \left. + \sqrt{E \left( \|F_p\|^4 \right) \sum_{j=1}^{\infty} \text{Var} \left( \Gamma(F_{q,j}, -L^{-1}F_{q,j}) \right)} \right) \leq a_{p,q} \left( E \left( \|F_p\|^2 \|F_q\|^2 \right) - E \left( \|F_p\|^2 \right) E \left( \|F_q\|^2 \right) - 2 \sum_{i,j=1}^{\infty} E \left( F_{p,i} F_{q,j} \right)^2 \right. \left. + \sqrt{E \left( \|F_p\|^4 \right) \sum_{j=1}^{\infty} \text{Var} \left( \Gamma(F_{q,j}, -L^{-1}F_{q,j}) \right)} \right) \leq a_{p,q} \left( E \left( \|F_p\|^4 \right) - E \left( \|F_p\|^2 \right)^2 - \|S_p\|^2 \right)$$

Now note that

$$E \left( F_{p,i} F_{q,j} \right) = \begin{cases} 0 & \text{if } p \neq q, \\ \|S_{F_p}\|_{\text{HS}} & \text{if } p = q \end{cases}$$

and also that, by a polarized version of [ACP14, Theorem 3.2] along with the implications of the covariance condition (24),

$$\sum_{j=1}^{\infty} \text{Var} \left( \Gamma(F_{q,j}, -L^{-1}F_{q,j}) \right) \leq \frac{1}{3} \sum_{j=1}^{\infty} \left( E \left( F^4_{q,j} \right) - 3 E \left( F^2_{q,j} \right)^2 \right)$$

$$= \frac{1}{3} \sum_{i,j=1}^{\infty} \left( E \left( F^2_{q,i} F^2_{q,j} \right) - E \left( F^2_{q,i} \right) E \left( F^2_{q,j} \right) - 2 E \left( F_{q,i} F_{q,j} \right)^2 \right)$$

$$\leq \frac{1}{3} \sum_{i,j=1}^{\infty} \left( E \left( F^2_{q,i} F^2_{q,j} \right) - E \left( F^2_{q,i} \right) E \left( F^2_{q,j} \right) - 2 E \left( F_{q,i} F_{q,j} \right)^2 \right)$$

$$= \frac{1}{3} \left( E \left( \|F_q\|^4 \right) - E \left( \|F_q\|^2 \right)^2 - \|S_q\|_{\text{HS}}^2 \right)$$
Plugged into (27), we get for $p = q$ that
\[
\sum_{i,j=1}^{\infty} \text{Var} \left( \Gamma(F_{p,i}, -L^{-1}F_{p,j}) \right) \leq E \left( \|F_p\|^4 \right) - E \left( \|F_p\|^2 \right) - 2 \|S_p\|_{\text{HS}}^2
\]
\[
+ \frac{1}{3} \frac{\sqrt{3}}{\sqrt{3}} E \left( \|F_p\|^4 \right) \sqrt{E \left( \|F_p\|^4 \right) - E \left( \|F_p\|^2 \right)^2 - 2 \|S_p\|_{\text{HS}}^2}
\]
\[
\leq 1 + \frac{\sqrt{3}}{\sqrt{3}} \frac{\sqrt{3}}{\sqrt{3}} \frac{\sqrt{E \left( \|F_p\|^4 \right) \sqrt{E \left( \|F_p\|^4 \right) - E \left( \|F_p\|^2 \right)^2 - 2 \|S_p\|_{\text{HS}}^2}}}{E \left( \|F_p\|^4 \right) - E \left( \|F_p\|^2 \right)^2 - 2 \|S_p\|_{\text{HS}}^2}. \tag{28}
\]
and for $p \neq q$ that
\[
\sum_{i,j=1}^{\infty} \text{Var} \left( \Gamma(F_{p,i}, -L^{-1}F_{q,j}) \right) \leq a_{p,q} \left( E \left( \|F_p\|^2 \|F_q\|^2 \right) - E \left( \|F_p\|^2 \right) E \left( \|F_q\|^2 \right) \right)
\]
\[
+ a_{p,q} \frac{\sqrt{3}}{\sqrt{3}} \sqrt{E \left( \|F_p\|^4 \right) E \left( \|F_q\|^4 \right) - E \left( \|F_q\|^2 \right)^2 - 2 \|S_q\|_{\text{HS}}^2}. \tag{28}
\]

The assertion now follows by summing over $p$ and $q$. \qed

**Remark 3.11.** The covariance assumption (24) is used in order to give a basis-free moment bound. If this assumption is not made, the conclusion of Theorem 3.2 is still valid, with the second sum in the bound (25) replaced by
\[
\sum_{1 \leq p,q \leq N} \sum_{p \neq q} \frac{a_{p,q}}{3} \sqrt{E \left( \|F_p\|^4 \right) E \left( \|F_q\|^4 \right) - 3 E \left( \|F_q\|^2 \right)^2} \sum_{i=1}^{\infty} \left( E \left( F_{q,i}^4 \right) - 3 E \left( F_{q,i}^2 \right)^2 \right),
\]
where $F_{q,i} = \langle F_q, e_i \rangle$ and $(e_i)_{i \in \mathbb{N}}$ is an orthonormal basis of $K$.

The next result is a Hilbert-valued counterpart to the fourth moment theorem derived in [ACP14]. It is a direct consequence of Theorem 3.10.

**Theorem 3.12 (Abstract fourth moment theorem).** Let $Z$ be a Gaussian random variable on $K$ with covariance operator $S$ and $(F_n)_{n \in \mathbb{N}}$ be a sequence of $K$-valued chaotic eigenfunctions with covariance operators $S_n$ such that $LF_n = -\lambda_n F_n$ and condition (24) is satisfied for all $n \in \mathbb{N}$. Assume that $\|S_n - S\|_{S_1} \to 0$ as $n \to \infty$. Consider the following two asymptotic relations, as $n \to \infty$:

(i) $F_n$ converges in distribution to $Z$;

(ii) $E \left( \|F_n\|^4 \right) \to E \left( \|Z\|^4 \right)$.

Then, (ii) implies (i), and the converse implication holds whenever the sequence \( \{\|F_n\|^4 : n \geq 1\} \) is uniformly integrable.
Proof. The fact that \((ii)\) implies \((i)\) is a direct consequence of Theorem 3.10, noting that \(\|S_n - S\|_{S_1} \to 0\) implies that \(E\left(\|F_n\|^2\right) \to E\left(\|Z\|^2\right)\). The converse implication follows immediately if the additional uniform integrability condition is assumed to hold. 

\(\square\)

Remark 3.13. Note that a sufficient condition to ensure the uniform integrability of the sequence \(\left(\|F_n\|^4\right)_{n \in \mathbb{N}}\) is given by \(\sup_{n \geq 1} E\left[\|F_n\|^4 + \varepsilon\right] < \infty\) for some \(\varepsilon > 0\).

For functionals with infinite chaos expansions, we can provide the following statement.

**Theorem 3.14.** Let \(Z\) be a Gaussian random variable on \(K\) with covariance operator \(S\) and let \((F_n)_{n \in \mathbb{N}}\) be a sequence of square integrable, \(K\)-valued random variables with chaos decomposition

\[F_n = \sum_{p=1}^{\infty} F_{p,n},\]

where, for each \(n, p \geq 1\), \(F_{p,n}\) is an eigenfunction associated to the eigenvalue \(-\lambda_p\). For \(n, p \in \mathbb{N}\), let \(S_n\) and \(S_{p,n}\) be the covariance operators of \(F_n\) and \(F_{p,n}\), respectively. Suppose that:

(i) For every \(p \in \mathbb{N}\), there exists a covariance operator \(S_p\) such that \(S_p = \sum_{p=1}^{\infty} S_p\), \(\|S_{p,n} - S_p\|_{HS} \to 0\) as \(n \to \infty\) and

\[(29) \quad \lim_{N \to \infty} \sup_{n \geq N} \sum_{p=1}^{\infty} \text{tr} S_{p,n} = 0\]

(ii) For all \(p, q \in \mathbb{N}\), it holds that

\[E\left(\|F_{p,n}\|^4\right) - E\left(\|F_{p,n}\|^2\right)^2 - 2 \|S_{p,n}\|^2_{HS} \to 0\]

and, if \(p \neq q\),

\[E\left(\|F_{p,n}\|^2 \|F_{q,n}\|^2\right) - E\left(\|F_{p,n}\|^2\right) E\left(\|F_{q,n}\|^2\right) \to 0\]

as \(n \to \infty\).

Then \(F_n\) converges weakly to \(Z\) as \(n \to \infty\).

Proof. We are going to show that \(d_A(F_n, Z)\) converges to zero as \(n \to \infty\). For \(N \in \mathbb{N}\), define \(F_{n,N} = \sum_{p=1}^{N} F_{p,n}\), \(R_{n,N} = \sum_{p=N+1}^{\infty} F_{p,n}\) and let \(Z_N\) be a Gaussian random variable on \(K\) with covariance operator \(\sum_{p=1}^{N} S_p\). Now, note that

\[d_A(F_n, Z) \leq d_A(F_n, F_{n,N}) + d_A(F_{n,N}, Z_N) + d_A(Z_N, Z).\]

Let \(\varphi \in \mathcal{U}_A\), \(x, y \in K\) and \(\varepsilon > 0\). By a Banach space version of Taylor’s theorem (see for example [Die69, 8.14.3]), there exists \(\delta > 0\) such that

\[|\varphi(x + y) - \varphi(x) - \nabla \varphi(x)y| \leq \varepsilon \|y\|\]
for all \( y \) satisfying \( \| y \| < \delta \). Therefore, for \( N \) large enough, we have

\[
|E(\varphi(F_n) - \varphi(F_{n,N}))| \leq E(|\varphi(F_n) - \varphi(F_{n,N})|) \leq \varepsilon E(\|R_{n,N}\|) + E(\|\nabla\varphi(F_{n,N})\| R_{n,N}) \\
\leq \varepsilon E(\|R_{n,N}\|) + E(\|\nabla\varphi(F_{n,N})\| \|R_{n,N}\|) \\
\leq \varepsilon E(\|R_{n,N}\|) + \left( \sup_{u \in K} \|\nabla\varphi(u)\| \right) E(\|R_{n,N}\|) \\
\leq (1 + \varepsilon) E(\|R_{n,N}\|) \\
\leq (1 + \varepsilon) \sqrt{E(\|R_{n,N}\|^2)} \\
= (1 + \varepsilon) \left( \sum_{p=N+1}^{\infty} E(\|F_{p,n}\|^2) \right) \\
= (1 + \varepsilon) \left( \sum_{p=N+1}^{\infty} \text{tr} S_{p,n} \right),
\]

so that \( d_A(F_n, F_{n,N}) \) converges to zero as \( N \to \infty \) due to the fact that (i) implies that the series \( \sum_{p=1}^{\infty} \text{tr} S_{p,n} = \sum_{p=1}^{\infty} E[\|F_{p,n}\|^2] \) is convergent, as for \( N \in \mathbb{N} \),

\[
\sum_{p=1}^{\infty} E[\|F_{p,n}\|^2] = \sum_{p=1}^{\infty} E[\langle F_{p,n}, F_{p,n} \rangle] \\
= \sum_{p=1}^{\infty} \sum_{i,j \geq 1} E[\langle F_{p,n}, e_i \rangle \langle F_{p,n}, e_j \rangle] \\
= \sum_{p=1}^{\infty} \|S_{p,n}\|_{\text{HS}} = \|S_n\|_{\text{HS}} \\
\leq \|S_n - S\|_{\text{HS}} + \|S\|_{\text{HS}} < \infty,
\]

so that the tail of this series, \( \sum_{p=N+1}^{\infty} \text{tr} S_{p,n} = \sum_{p=N+1}^{\infty} E[\|F_{p,n}\|^2] \) converges to zero as \( N \to \infty \).

On the other hand, note that

\[
\Gamma(Z_N, -L^{-1}Z_N) = \sum_{p=1}^{N} S_p,
\]

so that, by Theorem 3.4,

\[
d_A(Z_N, Z) \leq \frac{1}{2} \sum_{p=N+1}^{\infty} \text{tr} S_{p,n},
\]

which converges to zero as \( N \to \infty \) by the same observation as above. Finally, as (i) and (ii) hold, Theorem 3.10 ensures that \( d_A(F_{n,N}, Z_N) \) converges to zero as \( n \to \infty \), hence concluding the proof. \( \square \)
Remark 3.15. By using the above results, it is also possible to cover the convergence of $d$-dimensional vectors of processes instead of a unique process by taking the Hilbert space $K$ to be of the particular form $K = K_1 \otimes \cdots \otimes K_d$, where $K_1, \ldots, K_d$ are themselves real separable Hilbert spaces. This particular case is already included in the generality of our statements, and allow to immediately deduce Peccati-Tudor type (i.e., vector valued statements in the spirit of [PT05]) versions of all results stated here at no additional cost.

Let us illustrate this fact with $2$-dimensional vectors of processes for notational simplicity. Consider a $2$-dimensional process $F = (F_1, F_2)$. Another way to think about such a process is to consider $F_i$ to be a $K_i$-valued random variable, where $K_i$ is some real separable Hilbert space, $i = 1, 2$. Denoting by $K = K_1 \otimes K_2$, $F$ itself can be regarded as a $K$-valued random variable for which all of our results are applicable, where by construction, $K$ is again a real separable Hilbert space.

In this setting (where $F$ is a $K_1 \otimes K_2$-valued random variable), note that the quantity $\Gamma(F, -L^{-1} F)$ appearing in Theorem 3.4 can be regarded as an element of $L^2(\Omega; \mathcal{L}(K_1 \otimes K_2, K_1 \otimes K_2))$, so that $\Gamma(F, -L^{-1} F)$ can be identified with the random matrix

$$
\begin{pmatrix}
\Gamma(F_1, -L^{-1} F_1) & \Gamma(F_1, -L^{-1} F_2) \\
\Gamma(F_2, -L^{-1} F_1) & \Gamma(F_2, -L^{-1} F_2)
\end{pmatrix},
$$

where, for each $i, j = 1, 2$, $\Gamma(F_i, -L^{-1} F_j) \in L^2(\Omega; \mathcal{L}(K_i, K_j))$.

3.3.2. Finite-dimensional examples. In the case where $K$ is taken to be finite dimensional, the results of the above subsection yield the known moment bounds obtained in [Led12, ACP14] for $K = \mathbb{R}$, as well as the moment bounds obtained in [CNPP16] for $K = \mathbb{R}^d$ for some $d \geq 2$. To illustrate this fact, we state what Theorem 3.10 implies when setting $K = \mathbb{R}$.

**Theorem 3.16** (Corollary 7 in [Led12] and Theorem 3.1 in [NPR10a]). Let $Z$ be a real-valued, centered, standard Gaussian random variable. Let $F \in \mathbb{D}$ be an eigenfunction of some diffusive Markov generator $-L$ associated with the eigenvalue $\lambda$. Furthermore, assume that $E[F^2] = 1$. Then, it holds that

$$
d_A(F, Z) \leq \frac{(1 + \sqrt{3})}{6} \sqrt{E[F^4]} \sqrt{|E[F^4] - 3|}.
$$

In the case where $K = \mathbb{R}^d$, Theorem 3.10 yields the following statement, which is the main result of [CNPP16].

**Theorem 3.17** (Theorem 1.2 in [CNPP16] and Theorem 1.5 in [NN11]). Let $F = (F_1, \ldots, F_d)$ be an $\mathbb{R}^d$-valued random vector of eigenfunctions of some Markov generator $L$, and assume that the matrix $C = \{C_{ij}: 1 \leq i, j \leq d\}$ encodes the covariance of $F$. Let $Z$ denote a centered Gaussian vector with covariance matrix $C$. Then,

$$
d_A(F, Z) \\
\leq \frac{1}{2} \left(1 + \frac{\sqrt{3}}{\sqrt{3}} \sqrt{E(\|F\|_{\mathbb{R}^d}^4)} \sqrt{E(\|F\|_{\mathbb{R}^d}^4) - E(\|F\|_{\mathbb{R}^d}^2)^2} - 2 ||C||_{\text{HS}(\mathbb{R}^d)}^2 \right)^{1/2}.
$$
3.4. Hilbert-valued Wiener structures. This section is devoted to applying our results (obtained in the generality of functionals of a Hilbert-valued Markov generator) to the Hilbert-valued Wiener structure. We start by introducing elements of the Malliavin calculus on Hilbert spaces, the terminology specific to this particular framework, and show what some of the operators introduced in Section 3.1 become when assuming this additional structure. For further details on this topic, the reader is referred to the references [CT06, Nua06, PV14, Kru14].

3.4.1. The Malliavin derivation and divergence operators. Let \( \{W(h): h \in \mathcal{H}\} \) be an isonormal Gaussian process with underlying separable Hilbert space \( \mathcal{H} \), that is \( \{W(h): h \in \mathcal{H}\} \) is a centered family of Gaussian random variables and

\[
E[W(h_1)W(h_2)] = \langle h_1, h_2 \rangle_{\mathcal{H}}, \quad h_1, h_2 \in \mathcal{H}.
\]

Let \( K \) be another separable Hilbert space and denote by \( S \otimes K \) the class of smooth \( K \)-valued random variables \( F: \Omega \to K \) of the form \( F = f(W(h_1),\ldots,W(h_n)) \otimes v \), where \( f \in C^\infty_0(\mathbb{R}^n) \), \( h_1,\ldots,h_n \in \mathcal{H} \), \( v \in K \), and linear combinations thereof. \( S \otimes K \) is dense in \( L^2(\Omega; K) \) and for \( F \in S \otimes K \), define the Malliavin derivative \( DF \) of \( F \) as the \( \mathcal{H} \otimes K \)-valued random variable given by

\[
DF = \sum_{i=1}^n \partial_i f(W(h_1),\ldots,W(h_n)) h_i \otimes v.
\]

\( D \) is a closable operator from \( L^2(\Omega; K) \) into \( L^2(\Omega; \mathcal{H} \otimes K) \) and its domain is denoted by \( D \) again. The domain of \( D \), denoted by \( \mathbb{D}^{1,2}(K) \), is the closure of \( S \otimes K \) with respect to the Sobolev norm \( \|F\|_{\mathbb{D}^{1,2}(K)}^2 = \|F\|_{L^2(\Omega; K)}^2 + \|DF\|_{L^2(\Omega; \mathcal{H} \otimes K)}^2 \). Similarly, for \( k \geq 2 \), let \( \mathbb{D}^{k,2}(K) \) denote the closure of \( S \otimes K \) with respect to the Sobolev norm \( \|F\|_{\mathbb{D}^{k,2}(K)}^2 = \|F\|_{L^2(\Omega; K)}^2 + \sum_{i=1}^k \|D^i F\|_{L^2(\Omega; \mathcal{H} \otimes K)}^2 \). For any \( k \geq 2 \), the operator \( D^k \) can be interpreted as the iteration of the Malliavin derivative operator defined in (30).

Remark 3.18. In this context, the operator \( \Gamma \) defined in Subsection 3.1 takes the form \( \Gamma(F,G) = \langle DF, DG \rangle_{\mathcal{H}} \).

As \( D \) is a closed linear operator from \( \mathbb{D}^{1,2}(K) \) to \( L^2(\Omega; \mathcal{H} \otimes K) \), it has an adjoint operator, denoted by \( \delta \), which maps a subspace of \( L^2(\Omega; \mathcal{H} \otimes K) \) into \( L^2(\Omega; K) \) through the duality relation

\[
E[\text{tr}_K (\langle DF, \eta \rangle_{\mathcal{H}})] = E[\langle DF, \delta(\eta) \rangle_{\mathcal{H}}] = E[\langle F, \delta(\eta) \rangle_K],
\]

for any \( F \in \mathbb{D}^{1,2}(K) \) and \( \eta \in \text{dom}(\delta) \), where the domain of \( \delta \), denoted by \( \text{dom}(\delta) \), is the subset of random variables \( \eta \in L^2(\Omega; \mathcal{H} \otimes K) \) such that \( E[\langle DF, \eta \rangle_{\mathcal{H}}] \leq C_\eta \|F\|_{L^2(\Omega; K)} \), for all \( F \in \mathbb{D}^{1,2}(K) \), where \( C_\eta \) is a positive constant depending only on \( \eta \). Since \( D \) is a form of gradient, its adjoint \( \delta \) should be interpreted as a divergence, so that it is referred to as the divergence operator. Similarly, for any \( k \geq 2 \), we denote by \( \delta^k \) the adjoint of \( D^k \) as an operator from \( L^2(\Omega; \mathcal{H} \otimes K) \) to \( L^2(\Omega; K) \) with domain \( \text{dom}(\delta^k) \).
3.4.2. Hilbert-valued multiple integrals and chaos decomposition. Any $K$-valued random variable $F \in L^2(\Omega; K)$ can be decomposed as

$$F = \sum_{n=0}^{\infty} \delta^n(f_n),$$

where the kernel $f_n \in \mathcal{H}^\otimes n \otimes K$ are uniquely determined by $F$, where $\mathcal{H}^\otimes n$ denotes the $n$-fold symmetrized tensor product of $\mathcal{H}$. The representation (31) is called the chaos decomposition of $F$, and for each $n \geq 0$, $\delta^n(f_n)$ is an element of the closure of $\mathcal{H}_n \otimes K$ with respect to the norm on $L^2(\Omega; K)$, where the so-called $n$-th Wiener chaos $\mathcal{H}_n$ is defined to be closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(W(h)) : h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where $H_n$ is the $n$-th Hermite polynomial. For any $n \geq 0$, the $K$-valued random variable $\delta^n(f_n)$ is usually denoted by $I_n(f_n)$ and called the $(K$-valued) multiple Wiener integral of order $n$ of $f_n$. Denote by $J_n$ the linear operator on $L^2(\Omega)$ given by the orthogonal projection onto $\mathcal{H}_n$, and by $J_n^K$ the extension of $J_n \otimes \text{Id}_K$ to $L^2(\Omega; K)$. Then, it holds that $J_n^K F = I_p(f_n)$.

**Remark 3.19.** Note that, for any $p \geq 1$ and any $f \in \mathcal{H}^\otimes p \otimes K$, $I_p(f)$ is an eigenfunction of the operator $L$ defined in Proposition 3.1 associated to the eigenvalue $-p$.

**Remark 3.20.** Note that in the particular case where $K = \mathbb{R}$, the above-defined multiple Wiener integrals coincide with the usual ones as defined in [Nua06] for instance. Whenever the integrand $f$ appearing in a multiple Wiener integral of order $n$ of the form $I_n(f)$ is valued in $\mathbb{R}$, we can continue to use the notation $I_n(f)$ to denote the usual, $\mathbb{R}$-valued, multiple Wiener integral of order $n$.

Let $\{e_k : k \geq 0\}$ be an orthonormal basis of $\mathcal{H}$. Given $f \in \mathcal{H}^\otimes n$ and $g \in \mathcal{H}^\otimes m$, for every $r = 0, \ldots, n \wedge m$, the $r$-th contraction of $f$ and $g$ is the element of $\mathcal{H}^\otimes (n+m-2r)$ defined as

$$f \otimes_r g = \sum_{i_1, \ldots, i_r=0}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathcal{H}^\otimes r} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathcal{H}^\otimes r}.$$

Given an orthonormal basis $\{v_k : k \geq 0\}$ of $K$, the following multiplication formula is satisfied by $K$-valued multiple Wiener integrals: for two arbitrary basis elements $v_i, v_j$ of $K$ and for $f \in \mathcal{H}^\otimes n \otimes K$ and $g \in \mathcal{H}^\otimes m \otimes K$, denote by $f_i = \langle f, v_i \rangle_K$ and $g_j = \langle g, v_j \rangle_K$, then

$$I_n(f_i)I_m(g_j) = \sum_{r=0}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} I_{n+m-2r}(f_i \otimes_r g_j).$$

Finally, the action of the Malliavin derivative operator on a $K$-valued multiple Wiener integral of the form $I_n(f) \in L^2(\Omega; K)$, where $f \in \mathcal{H}^\otimes n \otimes K$ is given by $DI_n(f) = nI_{n-1}(f(\cdot)) \in L^2(\Omega; \mathcal{H}^\otimes K)$.

3.4.3. Results for Hilbert-valued Wiener structures. This subsection contains results specific to Wiener structures, obtained as consequences of our main results coupled to the particular Wiener structure. In the sequel, we always assume that the probability space we work on is equipped with the $\sigma$-algebra generated by the underlying Gaussian process, so that any element of $L^2(\Omega)$ has the desired measurability properties to admit a chaos decomposition as stated in the previous subsection. $K$ continues to denote a real separable Hilbert space.
Theorem 3.21 (Infinite-dimensional Fourth Moment Theorem). Let $Z$ be a Gaussian random variable on $K$ with covariance operator $S$, and let $(F_n = I_p(f_n))_{n \in \mathbb{N}}$ be a sequence of $K$-valued multiple integrals such that $\|S_n - S\|_{HS} \to 0$ as $n \to \infty$. Then, as $n \to \infty$, the following assertions are equivalent.

(i) $F_n$ converges in distribution to $Z$;
(ii) $E\left(\|F_n\|^4\right) \to E\left(\|Z\|^4\right)$;
(iii) $E\left(\|\langle DF_n, DF_n \rangle \rangle - pS_n\|_{HS}^2\right) \to 0$;
(iv) $\|f_n \otimes_r f_n \|_{\mathcal{F}^{(2p-2r)} \otimes K^2} \to 0$ for all $r = 1, \ldots, p - 1$;
(v) $\|f_n \otimes_r f_n \|_{\mathcal{F}^{(2p-2r)} \otimes K^2} \to 0$ for all $r = 1, \ldots, p - 1$.

Proof. As $\|S_n - S\|_{HS} \to 0$ as $n \to \infty$, the hypercontractivity of the Wiener chaos implies that for any $r \geq 2$, $\sup_n E[\|F_n\|^r] < \infty$, which yields that (i) implies (ii) by uniform integrability. (iii) follows from (ii) by (28) together with the hypothesis that $S_n \to S$ as $n \to \infty$. The fact that (iv) is implied by (iii) is a consequence of the product formula for multiple integrals (32) from which it follows, together with the Parceval identity, that

$$E\left(\|\langle DF_n, -DL^{-1}F_n \rangle \rangle - S_n\|_{HS}^2\right) = p^2 \sum_{r=1}^{p-1} c_{p,p}(r)^2 \|f_n \otimes_r f_n\|_{\mathcal{F}^{(2p-2r)} \otimes K^2}^2,$$

where $c_{p,p}(r)$ is the combinatorial constant appearing in Lemma 3.23. The fact that (v) follows from (iv) is a consequence of (33). Finally, (i) follows from (v) by Theorem 3.24 combined with the fact that $\|f_n \otimes_r f_n\|_{\mathcal{F}^{(2p-2r)} \otimes K^2} \leq \|f_n \otimes_r f_n\|_{\mathcal{F}^{(2p-2r)} \otimes K^2}$.

The following result states that the covariance condition one needs to assume in Subsection 3.3 in order to be able to state basis-free results always holds in a Wiener structure, hence proving that this additional condition appearing in the general case can be disregarded in the present context.

Lemma 3.22. Let $p \in \mathbb{N}$, $K$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and $f \in \mathcal{F}^{\otimes p} \otimes K$. Let $F = I_p(f)$ be a multiple integral with values in $K$. Then $F$ satisfies the covariance condition (24).

Proof. For better legibility, let us write $I_p(f_u) = I_p(\langle f, u \rangle)$ and $I_p(f_v) = I_p(\langle f, v \rangle)$. By the product formula for multiple integrals, we get that

$$E\left(I_p(f_u)^2 I_p(f_v)^2\right) = \sum_{r=0}^{p} a_{p,r}^2 (2p - 2r)! \|f_u \tilde{\otimes}_r f_v \|_{\mathcal{F}^{(2p-2r)}}^2 = (2p)! \|f_u \tilde{\otimes} f_v\|_{\mathcal{F}^{(2p-2r)}}^2 + (p!)^2 \langle f_u, f_v \rangle_{\mathcal{F}^{(p-r)}} + \sum_{r=1}^{p-1} a_{r}^2 (2p - 2r)! \|f_u \tilde{\otimes}_r f_v\|_{\mathcal{F}^{(2p-2r)}}^2;$$

where $a_{p,r} = r!(p-r)!$. Without loss of generality, let us assume that $\mathcal{F} = L^2(A, \mu)$. Let $\mathcal{S}_{2p}$ be the set of all permutations of $\{1, 2, \ldots, 2p\}$ and, for $0 \leq r \leq p$, denote by $\mathcal{S}_{2p,r} \subseteq \mathcal{S}_{2p}$ the set of those permutations $\sigma$ such that $\text{card}\{\sigma(1), \ldots, \sigma(p)\} \cap \{1, 2, \ldots, p\} = r$. Then
we have that $\mathcal{G}_{2p} = \bigcup_{r=0}^{p} \mathcal{G}_{2p,r}$, where the union is disjoint. Furthermore (see for example [NP12, p. 97]), card $\mathcal{G}_{2p,r} = p!2^p$. Using these notations and facts, we get that

\[
(2p)! \| f_u \otimes f_v \|_{\Lambda \otimes 2p}^2 = \frac{1}{(2p)!} \sum_{\sigma, \sigma' \in \mathcal{G}_{2p}} \int_{A^{2p}} f_u (t_{\sigma(1)}, \ldots, t_{\sigma(p)}) f_v (t_{\sigma(p+1)}, \ldots, t_{\sigma(2p)}) f_u (t_{\sigma'(1)}, \ldots, t_{\sigma'(p)}) f_v (t_{\sigma'(p+1)}, \ldots, t_{\sigma'(2p)}) \mu(dt_1) \ldots \mu(dt_{2p})
\]

\[
= \sum_{\sigma \in \mathcal{G}_{2p}} \int_{A^{2p}} f_u (t_1, \ldots, t_p) f_v (t_{p+1}, \ldots, t_{2p}) f_u (t_{\sigma(1)}, \ldots, t_{\sigma(p)}) f_v (t_{\sigma(p+1)}, \ldots, t_{\sigma(2p)}) \mu(dt_1) \ldots \mu(dt_{2p})
\]

\[
= \sum_{r=0}^{p} \sum_{\sigma \in \mathcal{G}_{2p,r}} \int_{A^{2p}} f_u (t_1, \ldots, t_p) f_v (t_{p+1}, \ldots, t_{2p}) f_u (t_{\sigma(1)}, \ldots, t_{\sigma(p)}) f_v (t_{\sigma(p+1)}, \ldots, t_{\sigma(2p)}) \mu(dt_1) \ldots \mu(dt_{2p})
\]

\[
= p! \sum_{r=0}^{p} \binom{p}{r}^2 \| f_u \otimes_r f_v \|_{\Lambda \otimes (2p-2r)}^2
\]

Plugged into (3), we get after rearranging terms that

\[
(33) \quad \mathbb{E} \left( I_p(f_u)^2 I_p(f_v)^2 \right) - (p!)^2 \| f_u \|_{\Lambda \otimes p}^2 \| f_v \|_{\Lambda \otimes p}^2 - 2(p!)^2 \langle f_u, f_v \rangle_{\Lambda \otimes 2p}
\]

\[
= \sum_{r=1}^{p-1} \binom{p}{r}^2 (2p-2r)! \| f_u \otimes_r f_v \|_{\Lambda \otimes (2p-2r)}^2 + p! \left( \binom{p}{r}^2 \| f_u \otimes_r f_v \|_{\Lambda \otimes (2p-2r)}^2 \right) > 0.
\]

The assertion follows by applying the Itô isometry in order to transform the norms on the left hand side into expectations.

The connection between the general bound appearing in Theorem 3.4 and norms of contractions is addressed below. This technical lemma will be helpful in the proof of our next result.

**Lemma 3.23.** For $K$-valued multiple integrals $F_p = I_p(f_p)$ and $F_q = I_q(f_q)$, it holds that

\[
\mathbb{E} \left( \| \Gamma(F_p, -L^{-1}F_q) - \delta_{p,q} S_p \|_{\text{HS}}^2 \right) \leq p^2 \left\{ \sum_{r=1}^{p \wedge q} c_{p,q}(r)^2 \| f_p \otimes_r f_q \|_{\Lambda \otimes (p+q-2r)}^2 \quad \text{if } p \neq q, \right.
\]

\[
\sum_{r=1}^{p-1} c_{p,q}(r)^2 \| f_p \otimes_r f_p \|_{\Lambda \otimes (2p-2r)}^2 \quad \text{if } p = q,
\]

\]
where $\delta_{p,q}$ denotes the Dirac function equal to 1 if $p = q$ and zero otherwise. Furthermore, the constants $c_{p,q}(r)$ are given by

\begin{equation}
    c_{p,q}(r) = (r-1)! \left( \frac{p-1}{r-1} \right) \left( \frac{q-1}{r-1} \right) (p+q-2r)!
\end{equation}

**Proof.** Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal basis of $K$ and abbreviate the inner products $\langle F_p, e_i \rangle$ by $F_{p,i}$. Then, by [NP12],

\[
    E \left( \Gamma(F_{p,i}, -L^{-1}F_{q,j}) - E(F_{p,i}F_{q,j})^2 \right)
    = p^2 \left\{ \begin{array}{ll}
        \sum_{r=1}^{p \wedge q} c_{p,q}(r) \| f_p \otimes_r f_q \|_{\mathcal{H}^\otimes (p+q-2r) \otimes K}^2 & \text{if } p \neq q, \\
        \sum_{r=1}^{p-1} c_{p,p}(r) \| f_p \otimes_r f_p \|_{\mathcal{H}^\otimes (2p-2r)}^2 & \text{if } p = q
    \end{array} \right.
\]

The assertion follows by the Parseval identity after summing over $i$ and $j$. \hfill \Box

**Theorem 3.24.** Let $Z$ be a Gaussian random variable on $K$ with covariance operator $S$ and $F$ be a square integrable $K$-valued random variable with covariance operator $T$ and chaos decomposition $F = \sum_{p=1}^{\infty} I_p(f_p)$, where, for each $p \geq 1$, $f_p \in \mathcal{H}^\otimes p \otimes K$. Then,

\[
    d_A(F, Z) \leq C \left( \left( \sum_{p=1}^{\infty} \sum_{r=1}^{p-1} c_{p,p}(r) \| f_p \otimes_r f_p \|_{\mathcal{H}^\otimes (2p-2r) \otimes K}^2 \right)^{1/2} \right.
\]

\[
    + \sum_{1 \leq p, q \leq \infty \atop p \neq q} \sum_{r=1}^{p \wedge q} p^2 c_{p,q}(r) \| f_p \otimes_r f_q \|_{\mathcal{H}^\otimes (p+q-2r) \otimes K}^2 \left. \right) + \| T - S \|_{HS}
\]

where $C$ is the constant defined in the statement of Theorem 3.4.

**Proof.** By linearity, we have that

\[
    \Gamma(F, -L^{-1}F) = \sum_{p,q=1}^{\infty} \Gamma(F_p, -L^{-1}F_q).
\]
Therefore, denoting the covariance operator of $F_p$ by $S_p$ (so that $T = \sum_{p=1}^{\infty} S_p$), it follows by Lemma 3.23 that

\[
\| \Gamma(F, -L^{-1}F') - T \|_{L^2(\Omega; HS)} = \left\| \sum_{p,q=1}^{\infty} \Gamma(F_p, -L^{-1}F_q) - \sum_{p=1}^{\infty} S_p \right\|_{L^2(\Omega; HS)}
\]

\[
\leq \sum_{p=1}^{\infty} \| \Gamma(F_p, -L^{-1}F_p) - S_p \|_{L^2(\Omega; HS)}
\]

\[
+ \sum_{1 \leq p,q \leq \infty, p \neq q} \| \Gamma(F_p, -L^{-1}F_q) \|_{L^2(\Omega; HS)}
\]

\[
\leq \left( \sum_{p=1}^{\infty} \sum_{r=1}^{p-1} c_{p,p}(r)^2 \| f_p \otimes_r f_p \|_{\mathcal{H}^2_{(2p-2r)\otimes K}^\otimes}^2
\]

\[
+ \sum_{1 \leq p,q \leq \infty, p \neq q} \sum_{r=1}^{p \wedge q} \| f_p \otimes_r f_q \|_{\mathcal{H}^2_{(p+q-2r)\otimes K}^\otimes}^2 \right)^{1/2}
\]

Using this bound, the conclusion is thus a consequence of Theorem 3.4.

\[\square\]

**Remark 3.25.** Note that

\[
\| f_p \otimes_r f_p \|_{\mathcal{H}^2_{(2p-2r)\otimes K}^\otimes}^2 = \left\| f_p \right\|_{\mathcal{H}^2_{(p-2r)\otimes K}^\otimes}^2.
\]

Let us now state a version of Theorem 3.14 in the context of Wiener structures.

**Theorem 3.26.** Let $Z$ be a Gaussian random variable on $K$ with covariance operator $S$ and let $(F_n)_{n \in \mathbb{N}}$ be a sequence of square integrable $K$-valued random variables with chaos decomposition

\[(35)\]

\[F_n = \sum_{p=1}^{\infty} I_p(f_{p,n}),\]

where, for each $n, p \geq 1$, $f_{p,n} \in \mathcal{H}^2_{\otimes p} \otimes K$. Suppose that:

(i) For every $p \in \mathbb{N}$, there exists a covariance operator $S_p$ such that $S = \sum_{p=1}^{\infty} S_p$, $\| S_{p,n} - S_p \|_{HS} \to 0$ as $n \to \infty$ and

\[(36)\]

\[\lim_{N \to \infty} \sup_{n \in \mathbb{N}} \sum_{p=N}^{\infty} \text{tr} S_{p,n} = \lim_{N \to \infty} \sup_{n \in \mathbb{N}} \sum_{p=N}^{\infty} p! \| f_{p,n} \|_{\mathcal{H}^2_{\otimes p} \otimes K}^2 = 0.
\]

(ii) For all $p \in \mathbb{N}$ and $r = 1, \ldots, p - 1$, it holds that

\[\| f_{p,n} \otimes_r f_{p,n} \|_{\mathcal{H}^2_{(p-r)\otimes K}^\otimes} \to 0
\]

as $n \to \infty$.

Then, the law of $F_n$ converges to the law of $Z$ as $n \to \infty$. 

Proof. Applying Theorem 3.14, we only have to show condition (ii) of this Theorem is satisfied. For this, let \((e_i)_{i \in \mathbb{N}}\) be an orthonormal basis of \(K\) and note that by the Parseval identity,

\[
\|f_{p,n} \otimes_r f_{p,n}\|_{\mathcal{L}_2(p-r) \otimes K \otimes 2}^2 = \sum_{i,j} \|f_{p,n,i} \otimes_r f_{p,n,j}\|_{\mathcal{L}_2(p-r) \otimes K \otimes 2}^2,
\]

where \(f_{p,n,i} = \langle f_{p,n}, e_i \rangle\). Therefore, by assumption

\[
\|f_{p,n,i} \otimes_r f_{p,n,j}\|_{\mathcal{L}_2(p-r) \otimes K \otimes 2}^2 \to 0
\]
as \(n \to \infty\) for all \(p,i,j \in \mathbb{N}\). Again using the Parseval identity and orthogonality of multiple integrals, we obtain

\[
E \left( \|F_{p,n}\|^2 \right) = (p!)^2 \sum_{i,j=1}^{\infty} \|f_{p,n,i}\|_{\mathcal{L}_p}^2 \|f_{p,n,j}\|_{\mathcal{L}_p}^2
\]
and, analogously,

\[
E \left( \|F_{p,n}\|^4 \right) = (p!)^2 \sum_{i,j=1}^{\infty} \langle f_{p,n,i}, f_{p,n,j} \rangle^2.
\]

By (33) and the well-known fact that

\[
\|f_{p,n,i} \otimes_r f_{p,n,j}\|_{\mathcal{L}_2(p-r)} \leq \|f_{p,n,i} \otimes_r f_{p,n,j}\|_{\mathcal{L}_2(p-r)},
\]
we thus get that

\[
E \left( \|F_{p,n}\|^4 \right) - E \left( \|F_{p,n}\|^2 \right)^2 - 2 \|S_{p,n}\|_{HS}^2 \to 0
\]
as \(n \to \infty\). Similar calculations also yield that if \(p \neq q\), then

\[
E \left( \|F_{p,n}\|^2 \|F_{q,n}\|^2 \right) - E \left( \|F_{p,n}\|^2 \right) E \left( \|F_{q,n}\|^2 \right) \to 0
\]
as \(n \to \infty\). Note that here one needs to apply a Cauchy-Schwarz argument to get that

\[
\|f_{p,n,i} \otimes_r f_{q,n,j}\|_{\mathcal{L}_{p+q-2r}} \leq \|f_{p,n,i} \otimes_r f_{p,n,j}\|_{\mathcal{L}_{p+q-2r}} \|f_{q,n,i} \otimes_r f_{q,n,j}\|_{\mathcal{L}_{p+q-2r}}.
\]

3.4.4. Finite dimensional examples. With the above Wiener space specific statements, we can once more retrieve all known quantitative and non-quantitative finite-dimensional statements available in the literature. For instance, whenever \(K = \mathbb{R}\), the one-dimensional result from [Led12] stated in Remark 3.7 yields the bound first established in [NP09], which can be stated as follows.

**Theorem 3.27** (Theorem 3.1 in [NP09]). Let \(Z\) be a real-valued, centered, Gaussian random variable with variance \(\sigma^2\). Let \(F \in \mathbb{D}^{1,2}\) be centered. Then, it holds that

\[
d_A(F,Z) \leq C \sqrt{E \left[ \left( \langle DF, -DL^{-1}F \rangle_{\mathcal{B}} - \sigma^2 \right)^2 \right]}.
\]
Quantitative bounds in terms of contractions can also be derived in the one-dimensional case. Applying Theorem 3.24 with $K = \mathbb{R}$ allows to also obtain such a contraction bound as shown by the result below.

**Theorem 3.28.** Let $Z$ be a real-valued, centered, standard Gaussian random variable. Let $F = I_p(f)$ be a multiple Wiener integral of order $p \geq 1$ such that $E[F^2] = 1$. Then, it holds that

$$d_A(F, Z) \leq C\left(\sum_{r=1}^{p-1} c_p(r)^2 \|f \otimes_r f\|^2_{\mathcal{H}^\otimes(2p-2r)}\right)^{1/2},$$

where $c_p(r) = p(r - 1)!\left(\frac{p - 1}{r - 1}\right)^2 (2p - 2r)!$.

In the case where $F$ has an infinite-chaos expansion as in (35), Theorem 3.24 with $K = \mathbb{R}$ yields the main result of [FT16], which we state below.

**Theorem 3.29 (Theorem 3.1 in [FT16]).** Let $Z$ be a real-valued, centered, standard Gaussian random variable. Let $F = \sum_{p=1}^{\infty} I_p(f_p) \in D^{1,4}$ such that $E[F^2] = 1$, where for all $p \geq 1$, $f_p \in \mathcal{H}^\otimes p$. Then,

$$d_A(F, Z) \leq C\left(\sum_{p=1}^{\infty} \sum_{r=1}^{p-1} p^2 c_{p,p}(r)^2 \|f_p \otimes_r f_p\|^2_{\mathcal{H}^\otimes(2p-2r)} + \sum_{1 \leq p \neq q \leq \infty} \sum_{r=1}^{p \wedge q} p^2 c_{p,q}(r)^2 \|f_p \otimes_r f_q\|^2_{\mathcal{H}^\otimes(p+q-2r)}\right)^{1/2},$$

where $c_{p,q}(r)$ is the constant defined in (34).

Whenever $K = \mathbb{R}^d$ for some $d \geq 2$, Theorem 3.4 specialized to $K = \mathbb{R}^d$ yields the main result of [NPR10b] and its consequences with a major difference that lies in the fact that our bound does not depend on the operator norm of the inverse of the covariance matrix of the limiting vector, which is not necessarily a problem in finite dimension, but was a major difficulty to overcome in our context as this norm always blows up when the dimension grows to infinity (when one considers the norm of the inverse of the covariance operator in our framework). Similarly, by taking $K = \mathbb{R}^d$ again in Theorem 3.10, we recover the moment bounds proposed in [NN11] (we omit the statements for the sake of brevity).

### 4. Quantifying the functional Breuer-Major Theorem

In this section, it is shown how the main results of this article can be applied to give rates of convergence for the functional Breuer-Major theorems recently proved in [CNN18] and [NN18].

To introduce the setting, let $X = \{X_t : t \geq 0\}$ be a centered, stationary Gaussian process and define $\rho(k) = E(X_0 X_k)$ such that $E(X_s X_t) = \rho(t - s) = \rho(s - t)$. Assume $\rho(0) = 1$, denote the standard Gaussian measure on $\mathbb{R}$ by $\gamma$ and let $\varphi \in L^2(\mathbb{R}, \gamma)$ be of Hermite rank $d \geq 1$, i.e. $\varphi$ can be expanded in the form

$$\varphi(x) = \sum_{i=d}^{\infty} c_i H_i(x), \quad c_d \neq 0,$$

where $H_i(x)$ are Hermite polynomials.
where \( H_i(x) = (-1)^i e^{x^2/2} \left( \frac{d}{dx} \right)^i e^{-x^2/2} \) is the \( i \)-th Hermite polynomial. The classical Breuer-Major Theorem proved in [BM83] states that under the condition

\[
\sum_{k \in \mathbb{Z}} \rho(k)^d < \infty,
\]

the finite-dimensional distributions of the stochastic process \((U_n(t))_{t \in [0,1]}\) given by

\[
U_n(t) = \frac{1}{\sqrt{n}} \sum_{i=0}^{\lfloor nt \rfloor} \varphi(X_i)
\]

converge in law to those of a scaled Brownian motion \( \sigma W \), where \( W = \{W_t : t \in [0,1]\} \) is standard Brownian motion and the scaling is given by

\[
\sigma^2 = \sum_{p=d}^{\infty} p! c_p^2 \sum_{k \in \mathbb{Z}} \rho(k)^p.
\]

A modern proof of this result using a combination of Malliavin calculus and Stein’s method can be found in [NP12, Chapter 7], and with straightforward adaptations one can also obtain convergence of the finite-dimensional distributions of

\[
V_n(t) = \frac{1}{\sqrt{n}} \int_0^{nt} \varphi(X_s) \, ds, \quad t \in [0,1],
\]

to those of \( \tilde{\sigma} W \), where

\[
\tilde{\sigma}^2 = \sum_{p=d}^{\infty} p! c_p^2 \int_{-\infty}^{\infty} \rho(k)^p
\]

and \( W \) is a standard Brownian motion as before. Recently, as a special case of [CNN18, Theorem 1.1], it was shown that if \( \varphi \in L^p([0,1], \gamma) \) for some \( p > 2 \), then the law of the whole process \( V_n \) converges to the one of \( \sigma W \) in \( C(0,1) \), endowed with the uniform norm. A crucial part of the proof has been a new technique to prove tightness, which has been introduced in [JN19]. The approach was then transferred to the discrete case in [NN18], yielding convergence of \( U_n \) in the Skorohod space (again assuming that \( \varphi \in L^p([0,1], \gamma) \) for some \( p > 2 \)).

In this section, it will be shown how our main results can be used to associate rates both of the aforementioned functional convergences in a natural and straightforward way, when considering \( U_n \) and \( V_n \) as random variables taking values in a suitable Hilbert space \( K \) containing \( D([0,1]) \) and \( C_0([0,1]) \), respectively. We will only treat \( U_n \) here and furthermore assume for simplicity that \( K = L^2([0,1]) \) and \( \rho(k) \sim k^{-\alpha} \) for some \( \alpha > 0 \). This latter assumption on \( \rho \) for example includes the case where \( X \) is the increment process of a fractional Brownian motion.

The corresponding results for \( V_n \) can of course be obtained, mutatis mutandis, in the same way and our main results also allow to treat more general covariance functions \( \rho \) and smaller Hilbert spaces \( K \) with finer topologies, such as the Besov-Liouville Hilbert spaces (see [SKM93] for definitions and [CD13] for proofs of related functional limit theorems in this space) or reproducing kernel Hilbert spaces. As the calculations are more involved and
also quite lengthy and technical, we decided to keep the present article within reasonable
bounds and provide further details on this topic in a dedicated followup work.

Let us briefly comment on the fact that the stochastic processes $U$, $V$ and $W$ can be interpreted as $L^2([0,1])$-valued random variables, thus inducing Borel-measures on $L^2([0,1])$: For the Brownian motion, note that Wiener measure on classical Wiener space $C([0,1])$ is concentrated on the closed subspace $C_0([0,1])$ (the space of continuous functions $h$ on $[0,1]$ such that $h(0) = 0$). Therefore, the restriction of this measure to $C_0([0,1])$ yields a non-degenerate Gaussian measure, which will be denoted by $\nu$. Let $K$ be the completion of $C_0([0,1])$ under the norm induced by the $L^2([0,1])$-inner product. Then it is straightforward to verify that $K = L^2([0,1])$. Indeed, one has $\|h\|_{L^2([0,1])} = \|h\|_\infty$ for all $h \in C_0([0,1])$ and the inclusions $C_0([0,1]) \subseteq C([0,1]) \subseteq L^2([0,1])$ are all dense with respect to the $L^2$-inner product. On $L^2([0,1])$, define a Gaussian measure $\gamma_W$ by

$$\gamma_W(A) = \nu(A \cap C_0[0,1]), \quad A \in \mathcal{B}(K).$$

Then $\gamma_W$ is the law of a standard Brownian motion $W$ on $L^2([0,1])$ and its covariance operator $S$ is a Hilbert-Schmidt integral operator on $L^2([0,1])$ with kernel $s \wedge t$, i.e.

$$Sf(t) = \int_0^1 (s \wedge t) f(s) \, ds, \quad f \in L^2([0,1]).$$

It is clear that the above procedure can also be carried out for a scaled Brownian motion $\sigma^2 W$, leading to a Gaussian measure $\gamma_{\sigma^2 W}$ on $L^2([0,1])$ with covariance operator $\sigma^2 S$.

The laws of $U$ and $V$ on $D([0,1])$ and $C([0,1])$, respectively, can be lifted to $L^2([0,1])$ in the same way (note that the Skorohod space $D([0,1])$ is densely and as a Borel set embedded in $L^2([0,1])$ as well).

The rates of convergence will depend on the so called chaotic gap introduced in [FT16]. For a function $\varphi \in L^2([0,1],\gamma)$ of Hermite rank $d$ (recall that $\gamma$ is the standard Gaussian measure on $\mathbb{R}$) and with Hermite expansion (37), this chaotic gap $l \in [1,\infty]$ is defined as

$$l = \sup \{ k \in \mathbb{N}_0 : c_q \neq 0 \Rightarrow c_{q+j} = 0 \text{ for all } q \geq d \text{ and } j = 1, \ldots, k-1 \}.$$

For example, the exponential function has Hermite rank zero and chaotic gap one, the absolute value has Hermite rank zero and chaotic gap two, the sine function has Hermite rank one and chaotic gap two and Hermite polynomials $H_p$ have Hermite rank $p$ and chaotic gap $\infty$.

Functional convergence of the laws of $U_n$ can now be quantified as follows.

**Theorem 4.1.** Let $\varphi \in L^2([0,1],\gamma)$ be of Hermite rank $d \geq 1$ and chaotic gap $l$, such that

$$\sum_{p=d}^{\infty} p c_p^2 (2 + \varepsilon)^{2p} < \infty$$

for some $\varepsilon > 0$, where the $c_p$ are the coefficients of the Hermite expansion (37) of $\varphi$. Let $X$ be a centered, stationary Gaussian process $X$ such that $\rho(k) = \mathbb{E}(X_0X_k) \sim k^{-\alpha}$ for large $k$ and some $\alpha > 0$. Let the $L^2([0,1])$-valued stochastic process $\{U_n(t) : t \in [0,1]\}$ be defined by (38). Finally, let $W$ be a standard Brownian motion on $L^2([0,1])$ and denote its covariance operator by $S$. Then, for any bijective, positive definite and self-adjoint trace class operator $A$ on $L^2([0,1])$ there exists a positive constant $C$, only depending on $A$, $\sigma$
and $S$, such that

$$d_A(U_n, \sigma W) \leq C r_{\alpha,d,l}(n)$$

where $\sigma$ is defined by (39) and the rate function $r_{\alpha,d,l}$ is given as follows: If $d = 2$ and $l = 1$, one has

$$r_{\alpha,2,1}(n) = \begin{cases} n^{-1/2} & \text{if } \alpha \in (1, 2), \\ n^{-\alpha/2} & \text{if } \alpha \in (\frac{2}{3}, 1), \\ n^{1-2\alpha} & \text{if } \alpha \in (\frac{1}{2}, \frac{2}{3}), \end{cases}$$

if $d = 2$ and $l \geq 2$, one has

$$r_{\alpha,2,l}(n) = \begin{cases} n^{-1/2} & \text{if } \alpha \in \left(\frac{3}{4}, 2\right), \\ n^{1-2\alpha} & \text{if } \alpha \in \left(\frac{1}{2}, \frac{3}{4}\right), \end{cases}$$

if $d \geq 3$ and $l = 1$, one has

$$r_{\alpha,d,1}(n) = \begin{cases} n^{-1/2} & \text{if } \alpha \in (1, 2), \\ n^{-\alpha/2} & \text{if } \alpha \in \left(\frac{2}{d+1}, 1\right), \\ n^{1-d\alpha} & \text{if } \alpha \in \left(\frac{1}{d}, \frac{2}{d+1}\right), \end{cases}$$

and if $d \geq 3$ and $l \geq 2$, one has

$$r_{\alpha,d,l}(n) = \begin{cases} n^{-1/2} & \text{if } \alpha \in \left(\frac{d}{2}, 2\right), \\ n^{-\alpha} & \text{if } \alpha \in \left(\frac{1}{d-1}, \frac{1}{d}\right), \\ n^{1-d\alpha} & \text{if } \alpha \in \left(\frac{1}{d}, \frac{2}{d-1}\right). \end{cases}$$

**Proof of Theorem 4.1.** Throughout this proof, $C$ denotes a positive constant which might change from line to line. Let $\mathcal{H}$ be the Hilbert space obtained by the closure of the set of all finite linear combinations of indicator functions $1_{[0,t]}$, $t \geq 0$ with respect to the inner product

$$\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{H}} = \rho(t - s),$$

and let $X$ be an isonormal Gaussian process on $\mathcal{H}$ (for details on this construction, see for example [NP12, Example 2.1.5]). Then

$$E(X(1_{[0,i]}), X(1_{[0,j]})) = \langle 1_{[0,i]}, 1_{[0,j]} \rangle_{\mathcal{H}} = \rho(j - i) = E(X_i, X_j),$$

where expectations are taken over the respective probability spaces of $X$ and $X$. Furthermore, $U_n$ has the same law as $\sum_{p=d}^{\infty} U_{p,n}$, where

$$U_{p,n}(t) = c_p I_p(f_{p,n,t})$$

and

$$f_{p,n,t}(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]-1} g_p(i, x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} 1_{[\frac{i}{n}, \frac{i+1}{n})}(t) g_p(i, x). \tag{43}$$

Here, $g_p(i, x) = \prod_{j=1}^{p} 1_{[0,1]}(x_j) \geq 0$. Let us denote by $T_{p,n}$ the covariance operator of $U_{p,n}$ so that the covariance operator $T_n$ of $U_n$ is given by $T_n = \sum_{p=d}^{\infty} T_{p,n}$. Also, define

$$\sigma_{p,n}^2 = p! c_p^2 \sum_{k \in \mathbb{Z}} \rho(k)^p \left(1 - \frac{|k|}{n}\right) 1_{\{|k| < n\}}$$
and
\begin{equation}
\sigma_n^2 = \sum_{p=d}^{\infty} \sigma_{p,n}
\end{equation}

By the triangle inequality, it holds that
\begin{equation}
d_A(U_n, \sigma W) \leq d_A(U_n, \sigma_n W) + d_A(\sigma_n W, \sigma W).
\end{equation}
Now, applying Corollary 3.6 alongside with (51), we obtain
\begin{equation}
d_A(\sigma_n W, \sigma W) \leq d_2(\sigma_n W, \sigma W) \leq \frac{1}{2} \| \sigma_n^2 S - \sigma^2 S \|_{L^1(\Omega; S_1(L^2([0,1])))}
\end{equation}
\begin{equation}
= \frac{1}{2} \| \sigma_n^2 - \sigma^2 \| \mathrm{tr} S = \frac{1}{2} |\sigma_n^2 - \sigma^2| \leq C n^{-1/2(1-\alpha d)},
\end{equation}
where the last inequality follows after a straightforward calculation, using formulas (39) and (44). Let $T_n$ be the covariance operator of $U_n$. Then, by Theorem 3.24,
\begin{equation}
d_A(U_n, \sigma_n W) \leq C \left( \sum_{p=1}^{\infty} \sum_{r=1}^{p-1} p^2 c_{p,p}(r)^2 \| f_{p,n} \circ f_{p,n} \|^2_{H^2((2p-2r) \circ L^2([0,1]))} \right)^{1/2}
\end{equation}
\begin{equation}
+ \left( \sum_{1 \leq p, q \leq \infty} \sum_{r=1}^{p \wedge q} p^2 c_{p,q}(r)^2 \| f_{p,n} \circ f_{q,n} \|^2_{H^2((p+q-2r) \circ L^2([0,1]))} \right)^{1/2}
\end{equation}
\begin{equation}
+ \| T_n - \sigma_n^2 S \|_{H^2([0,1])}.
\end{equation}
Lemma 4.2 yields that
\begin{equation}
\| T_n - \sigma_n^2 S \|_{H^2([0,1])} \leq C n^{-1/2(1-\alpha d)}.
\end{equation}
Now plugging (48) into (47), then together with (46) into (45) and noting that $n^{-1/2(1-\alpha d)}/r_{\alpha,d,l}(n) \to 0$ as $n \to \infty$, it remains to show that
\begin{equation}
\sum_{p=1}^{\infty} \sum_{r=1}^{p-1} p^2 c_{p,p}(r)^2 \| f_{p,n} \circ f_{p,n} \|^2_{H^2((2p-2r) \circ L^2([0,1]))} \leq C r_{\alpha,d,l}(n)^2
\end{equation}
\begin{equation}
+ \sum_{1 \leq p, q \leq \infty} \sum_{r=1}^{p \wedge q} p^2 c_{p,q}(r)^2 \| f_{p,n} \circ f_{q,n} \|^2_{H^2((p+q-2r) \circ L^2([0,1]))} \leq C r_{\alpha,d,l}(n)^2
\end{equation}
Now
\begin{equation}
\left( \frac{1}{n} \mathbb{1}_{[1/n,1]}(\cdot), \frac{1}{n} \mathbb{1}_{[1/n,1]}(\cdot) \right)_{L^2([0,1])} \leq 1,
\end{equation}
so that for $r = 1, \ldots, p \wedge q$,
\begin{equation}
\| f_{p,n,t} \circ f_{q,n,t} \|^2_{L^2(A \wedge \mathbb{A}(\cdot), f_{p,n,1} \circ f_{q,n,1} \circ L^2((p+q-2r) \circ L^2([0,1]))) \leq \| f_{p,n,1} \circ f_{q,n,1} \circ L^2(A \wedge \mathbb{A}(\cdot) \circ (p+q-2r)}
\end{equation}
In other words, the contraction norms of the kernels of the stochastic process \((U_n(t))_{t \in [0,1]}\) are bounded by those of the random variable \(U_n(1)\), so that (49) follows from the calculations in [FT16, proof of Theorem 4.1].

\[ \|T_n - \sigma_n^2 S\|_{\text{HS}(L^2([0,1])))} \leq C n^{-1\nu(1-d\alpha)} \]

**Lemma 4.2.** In the setting of Theorem 4.1, it holds that

\[ \|T_n - \sigma_n^2 S\|_{\text{HS}(L^2([0,1])))} \leq C n^{-1\nu(1-d\alpha)} \]

**Proof.** The operator \(K_n = T_n - \sigma_n^2 S\) is a Hilbert-Schmidt integral operator of the form

\[ K_n f(t) = \int_0^1 k_n(s,t) f(s) \, ds, \]

with kernel \(k_n\) given by

\[ k_n(s,t) = \mathbb{E}(U_n(s)U_n(t)) - (s \wedge t)\sigma_n^2. \]

Note that by orthogonality

\[ \mathbb{E}(U_n(s)U_n(t)) = \sum_{p=d}^{\infty} \mathbb{E}(U_{p,n}(s)U_{p,n}(t)) \]

\[ = \sum_{p=d}^{\infty} p! c_p^2 \langle f_{p,n,s}, f_{p,n,t} \rangle_{L^2(A,A,\mu)^{\otimes p}}. \]

where the kernels \(f_{p,n,\cdot}\) are given by (43). Now

\[ \langle f_{p,n,s}, f_{p,n,t} \rangle_{L^2(A,A,\mu)^{\otimes p}} = \frac{1}{n} \sum_{i,j=1}^{n} 1_{1\leq i\leq j\leq n}(s)1_{1\leq i\leq j\leq n}(t)\rho(|i-j|)^p \]

\[ = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n-i} 1_{1\leq i\leq j\leq n}(s)1_{1\leq i\leq j\leq n}(t)\rho(|j|)^p \]

\[ = \frac{1}{n} \sum_{i=1}^{n-1} \sum_{j=-(n-1)}^{n-i} 1_{1\leq i\leq j\leq n}(s)1_{1\leq i\leq j\leq n}(t)\rho(|j|)^p \]

\[ = \frac{1}{n} \sum_{i=1}^{n-1} \sum_{j=-(n-1)}^{n-i} 1_{1\leq i\leq j\leq n}(s)1_{1\leq i\leq j\leq n}(t)\rho(|j|)^p \]

\[ = A_n + B_n + C_n, \]

where the terms \(A_n, B_n\) and \(C_n\) are obtained after decomposing the sum over \(j\) according to

\[ \sum_{j=-(n-1)}^{n-1} = \sum_{j=-(n-1)}^{-1} + \sum_{j=0}^{0} + \sum_{j=1}^{n-1}, \]
so that
\[
A_n = \frac{1}{n} \sum_{j=-(n-1)}^{-(n-1)} \# \{1 \leq i \leq n: 1 - j \leq i, i \leq ns, i \leq nt - j\} \rho(|j|^p) \\
= \frac{1}{n} \sum_{j=-(n-1)}^{-(n-1)} ((ns \wedge (nt - j)] + j) \rho(|j|^p) \\
= \sum_{j=-(n-1)}^{-(n-1)} \rho(|j|^p) \times \left\{ \frac{\lfloor ns \rfloor}{n} + \frac{1}{n} \text{ if } t - s > \frac{1}{n}, \right. \left. \frac{\lfloor nt \rfloor}{n} \text{ if } t - s \leq \frac{1}{n} \right\},
\]
\[
B_n = \frac{1}{n} \# \{1 \leq i \leq n: i \leq ns, i \leq nt - j\} = \frac{\lfloor ns \wedge nt \rfloor}{n}
\]
and
\[
C_n = \frac{1}{n} \sum_{j=1}^{n-1} \# \{1 \leq i \leq n: i \leq n - j, i \leq ns, i \leq nt - j\} \rho(|j|^p) \\
= \frac{1}{n} \sum_{j=1}^{n-1} ((n-j) \wedge [ns \wedge (nt - j)]) \rho(|j|^p) \\
= \sum_{j=1}^{n-1} \rho(|j|^p) \times \left\{ \frac{\lceil ns \rceil}{n} - \frac{j}{n} \text{ if } t - s > \frac{j}{n}, \right. \left. \frac{\lceil nt \rceil}{n} - \frac{j}{n} \text{ if } t - s \leq \frac{j}{n} \right\}.
\]
Plugging (52) into (51) and using formula (44) for $\sigma_n$, this yields
\[
\mathbb{E} (U_n(s)U_n(t)) - (s \wedge t)\sigma_n \\
= \sum_{p=d}^{\infty} p!c_{p}^{2} \left( A_n + B_n + C_n - (s \wedge t) \sum_{j=-(n-1)}^{n-1} \rho(|j|^p) \left( 1 - \frac{|j|}{n} \right) \right)
\]
and after a tedious but straightforward calculation, one arrives at
\[
|k_n(s, t)| = |\mathbb{E} (U_n(s)U_n(t)) - (s \wedge t)\sigma_n| \lesssim \frac{1}{n} \left( 1 + \sum_{p=d}^{\infty} p!c_{p}^{2} \sum_{j=1}^{n-1} j \rho(|j|^p) \right) \\
\lesssim n\rho(n)^d + \frac{1}{n} \\
\lesssim n^{-\alpha d} + \frac{1}{n} \lesssim n^{-\alpha d + 1} \lesssim n^{-\alpha d + 1} \lesssim n^{-\alpha d + 1}
\]
where we have used that $\rho(n) \approx n^{-\alpha}$ to obtain the last estimate. Consequently,
\[
\|T_n - \sigma_n^2S\|_{\text{HS}(L^2([0,1])))} = \|k_n\|_{L^2([0,1]^2)} \leq \sup_{s,t \in [0,1]} |k_n(s, t)| \leq Cn^{-\alpha d + 1}
\]
as asserted.
References

[ACP14] Ehsan Azmoodeh, Simon Campese, and Guillaume Poly, Fourth Moment Theorems for Markov diffusion generators, J. Funct. Anal. 266 (2014), no. 4, 2341–2359. MR 3150163

[Bar90] A. D. Barbour, Stein’s method for diffusion approximations, Probab. Theory Related Fields 84 (1990), no. 3, 297–322. MR 1035659

[BCLT19] Soleyman Bourguin, Simon Campese, Nikolai Leonenko, and Murad S. Taqqu, Four moments theorems on Markov chaos, Ann. Probab. 47 (2019), no. 3, 1417–1446 (EN).

[BGL14] Dominique Bakry, Ivan Gentil, and Michel Ledoux, Analysis and geometry of Markov diffusion operators, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 348, Springer, Cham, 2014. MR 3155209

[Bar90] A. D. Barbour, Stein’s method for diffusion approximations, Probab. Theory Related Fields 84 (1990), no. 3, 297–322. MR 1035659

[BM83] Peter Breuer and Péter Major, Central limit theorems for nonlinear functionals of Gaussian fields, J. Multivariate Anal. 13 (1983), no. 3, 425–441. MR 716933

[Bog98] Vladimir I. Bogachev, Gaussian measures, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998. MR 1642391

[CD13] L. Coutin and L. Decreusefond, Stein’s method for Brownian approximations, Commun. Stoch. Anal. 7 (2013), no. 3, 349–372. MR 3167403

[CNN18] Simon Campese, Ivan Nourdin, and David Nualart, Continuous Breuer-Major theorem: tightness and non-stationarity, arXiv:1807.09740 [math] (2018), 1–34, arXiv: 1807.09740.

[CNPP16] Simon Campese, Ivan Nourdin, Giovanni Peccati, and Guillaume Poly, Multivariate Gaussian approximations on Markov chaoses, Electron. Commun. Probab. 21 (2016), 1–9 (EN).

[CT06] René A. Carmona and Michael R. Tehranchi, Interest rate models: an infinite dimensional stochastic analysis perspective, Springer Finance, Springer-Verlag, Berlin, 2006. MR 2235463

[Die67] J. Dieudonné, Foundations of modern analysis, Academic Press, New York-London, 1969. MR 0349288

[DPZ96] G. Da Prato and J. Zabczyk, Ergodicity for infinite-dimensional systems, London Mathematical Society Lecture Note Series, vol. 229, Cambridge University Press, Cambridge, 1996. MR 1417491

[FT16] Tobias Fissler and Christoph Thaele, A new quantitative central limit theorem on the Wiener space with applications to Gaussian processes, arXiv:1610.01456 [math, stat] (2016), 1–21, arXiv: 1610.01456.

[Gro67] Leonard Gross, Potential theory on Hilbert space, J. Functional Analysis 1 (1967), 123–181. MR 0227774

[HN18] Daniel Harnett and David Nualart, Central limit theorem for functionals of a generalized self-similar Gaussian process, Stochastic Process. Appl. 128 (2018), no. 2, 404–425. MR 3739502

[HN18] Daniel Harnett and David Nualart, Central limit theorem for functionals of a generalized self-similar Gaussian process, Stochastic Process. Appl. 128 (2018), no. 2, 404–425. MR 3739502

[JN19] Arturo Jaramillo and David Nualart, Functional limit theorem for the self-intersection local time of the fractional Brownian motion, Ann. Inst. Henri Poincaré Probab. Stat. 55 (2019), no. 1, 480–527. MR 3901653

[Kas17a] Mikolaj J. Kasprzak, Diffusion approximations via Stein’s method and time changes, arXiv:1701.07633 [math] (2017), 1–39, arXiv: 1701.07633.

[Kas17b] Mikolaj J. Kasprzak, Multivariate functional approximations with Stein’s method of exchangeable pairs, arXiv:1710.09262 [math] (2017), 1–46, arXiv: 1710.09263.

[KDV17] Mikolaj J. Kasprzak, Andrew B. Duncan, and Sebastian J. Vollmer, Note on A. Barbour’s paper on Stein’s method for diffusion approximations, Electron. Commun. Probab. 22 (2017), Paper No. 23, 8. MR 3645505

[Kec95] Alexander S. Kechris, Classical descriptive set theory, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597

[Kru14] Raphael Kruse, Strong and weak approximation of semilinear stochastic evolution equations, Lecture Notes in Mathematics, vol. 2093, Springer, Cham, 2014. MR 3154916
[Kue70] J. Kuelbs, *Gaussian measures on a Banach space*, J. Functional Analysis 5 (1970), 354–367. MR 0260010

[Kuo75] Hui Hsiung Kuo, *Gaussian measures in Banach spaces*, Lecture Notes in Mathematics, Vol. 463, Springer-Verlag, Berlin-New York, 1975. MR 0461643

[Led12] Michel Ledoux, *Chaos of a Markov operator and the fourth moment condition*, Ann. Probab. 40 (2012), no. 6, 2439–2459 (EN), Zentralblatt MATH identifier: 06114704.

[MR92] Zhi Ming Ma and Michael Röckner, *Introduction to the theory of (nonsymmetric) Dirichlet forms*, Universitext, Springer-Verlag, Berlin, 1992. MR 1214375

[NN11] Salim Noreddine and Ivan Nourdin, *On the Gaussian approximation of vector-valued multiple integrals*, J. Multivariate Anal. 102 (2011), no. 6, 1008–1017. MR 2793872

[NN18] Ivan Nourdin and David Nualart, *The functional Breuer-Major theorem*, arXiv:1808.02378 [math] (2018), 1–13, arXiv: 1808.02378.

[NOL08] David Nualart and Salvatore Ortiz-Latorre, *Central limit theorems for multiple stochastic integrals and Malliavin calculus*, Stochastic Process. Appl. 118 (2008), no. 4, 614–628. MR 2394845

[NP05] David Nualart and Giovanni Peccati, *Central limit theorems for sequences of multiple stochastic integrals*, Ann. Probab. 33 (2005), no. 1, 177–193. MR 2118863

[NP09] Ivan Nourdin and Giovanni Peccati, *Stein’s method on Wiener chaos*, Probab. Theory Related Fields 145 (2009), no. 1-2, 75–118. MR 2520122

[NP12] , *Normal approximations with Malliavin calculus*, Cambridge Tracts in Mathematics, vol. 192, Cambridge University Press, Cambridge, 2012, From Stein’s method to universality.

[NPR10a] Ivan Nourdin, Giovanni Peccati, and Gesine Reinert, *Invariance principles for homogeneous sums: universality of Gaussian Wiener chaos*, The Annals of Probability 38 (2010), no. 5, 1947–1985.

[NPR10b] Ivan Nourdin, Giovanni Peccati, and Anthony Réveillac, *Multivariate normal approximation using Stein’s method and Malliavin calculus*, Ann. Inst. Henri Poincaré Probab. Stat. 46 (2010), no. 1, 45–58. MR 2641769

[Nua06] David Nualart, *The Malliavin calculus and related topics*, second ed., Probability and its Applications (New York), Springer-Verlag, Berlin, 2006.

[PT91] D. Preiss and J. Tišer, *Measures in Banach spaces are determined by their values on balls*, Mathematika 38 (1991), no. 2, 391–397 (1992). MR 1147839

[PT05] Giovanni Peccati and Ciprian A. Tudor, *Gaussian limits for vector-valued multiple stochastic integrals*, Séminaire de Probabilités XXXVIII, Lecture Notes in Math., vol. 1857, Springer, Berlin, 2005, pp. 247–262.

[PV14] Matthijs Pronk and Mark Veraar, *Tools for Malliavin calculus in UMD Banach spaces*, Potential Anal. 40 (2014), no. 4, 307–344. MR 3201985

[Shi11] Hsin-Hung Shih, *On Stein’s method for infinite-dimensional Gaussian approximation in abstract Wiener spaces*, J. Funct. Anal. 261 (2011), no. 5, 1236–1283. MR 2807099

[SKM93] Stefan G. Samko, Anatoly A. Kilbas, and Oleg I. Marichev, *Fractional integrals and derivatives*, Gordon and Breach Science Publishers, Yverdon, 1993. MR 1347689