Recent results on permutations without short cycles

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Abstract. The density, denoted by $\kappa(n, r)$, of permutations having no cycles of length less than $r + 1$ in a symmetric group $S_n$ is explored. New asymptotic formulas for $\kappa(n, r)$ are obtained using the saddle-point method when $5 \leq r < n$ and $n \to \infty$.

Keywords: symmetric group, long cycles, Buchstab’s function, Dickman’s function, saddle-point method

The probability $\kappa(n, r)$ that a permutation sampled from the symmetric group $S_n$ uniformly at random has no cycles of length less than $r + 1$, where $1 \leq r < n$ and $n \to \infty$, is explored. New asymptotic formulas valid in specified regions are obtained using the saddle-point method. One of the results is applied to show that estimate of the total variation distance for permutations can be expressed only through the function $\nu(n, r)$ which is a probability that a permutation sampled from the $S_n$ uniformly at random has no cycles of length greater than $r$.

To address the problem, we need recollect the following functions. Buchstab’s function $\omega(v)$ is defined as a solution to difference-differential equation

$$(v\omega(v))' = w(v - 1)$$

for $v > 2$ with the initial condition $\omega(v) = 1/v$ if $1 \leq v \leq 2$. Dickman’s function $\varrho(v)$ is the unique continuous solution to the equation

$$v\varrho'(v) + \varrho(v - 1) = 0$$

for $v > 1$ with initial condition $\varrho(v) = 1$ if $0 \leq v \leq 1$.

The interest to the problem begins with the classical example of derangements

$$\kappa(n, 1) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} = e^{-1} + O \left( \frac{1}{n!} \right)$$

and the trivial case $\kappa(n, r) = 1/n$ if $n/2 \leq r < n$. There was a series of works concerning general asymptotic formulas of the probability $\kappa(n, r)$ the strongest of which are presented here as Proposition 1 and Proposition 2.
Proposition 1  For $1 \leq r < n$, we have

$$\kappa(n, r) = e^{-H_r + \gamma \omega(n/r)} + O\left(\frac{1}{r^2}\right).$$

See [3, Theorem 3].

Proposition 2  Let $u = n/r$. For $1 \leq r \leq n / \log n$,

$$\kappa(n, r) = e^{-H_r} + O\left(\frac{(u/e)^r - u}{r^2}\right).$$

If $r \geq 3$, we can replace $e$ by 1 in the error term.

See [13, Proposition 2]. Together these propositions provide stronger estimates of $\kappa(n, r)$ than those in [3], [8], [11]. New results are the following theorems:

Theorem 1  For $\sqrt{n \log n} \leq r < n$, we have

$$\kappa(n, r) = e^{-H_r + \gamma \omega(n/r)} + O\left(\frac{\varphi(n/r)}{r^2}\right).$$

Proof. The result is a corollary of Theorem 1 in [7]. It is obtained from the probability generating function using saddle-point method, the technique is elaborated in [11].

Theorem 2  For $(\log n)^4 \leq r < n$, we have

$$\kappa(n, r) = e^{-H_r} + O\left(\frac{\varphi(n/r)}{r}\right).$$

Proof. The saddle-point method is applied to the Cauchy’s integral representation of $\kappa(n, r)$, as in the proof of Theorem 1. However, there are some other technical difficulties one must to overcome.

Theorem 3  For $5 \leq r < n$, we have

$$\kappa(n, r) = e^{-H_r} + O\left(\frac{\nu(n, r)}{r}\right).$$

Proof. Quite the same technique to that used in the proof of Theorem 2 is employed, just a different approximation of the saddle point is taken and Corollary 5 of [8] is applied.

Theorem 1 and Theorem 2 (see also Corollary 2.3 in [5]) improve on Proposition 1 and Proposition 2. Theorem 3 is of separate interest; as we see, it can be useful in formulas where both probabilities $\kappa(n, r)$ and $\nu(n, r)$ are involved. Here is an example.
Let $k_j(\sigma)$ equal the number of cycles of length $j$ in a permutation $\sigma \in S_n$, $\mathbf{k}(\sigma) = (k_1(\sigma), k_2(\sigma), \ldots, k_n(\sigma))$, and $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$, where $Z_j$ are Poisson random variables such that $E Z_j = 1/j$, $j \in \mathbb{N}$. Thus, if $5 \leq r < n$, we have (see Lemma 3.1 on p. 69 of [1])

$$d_{TV}(n, r) = \sup_{V \subseteq \mathbb{Z}_r} \left| \frac{\# \{ \sigma : \mathbf{k}(\sigma) \in V \}}{n!} - \Pr(\mathbf{Z} \in V) \right|$$

$$= \frac{1}{2} \sum_{m=0}^{\infty} \nu(m, r) \left| \kappa(n - m, r) - e^{-H_r} \right|$$

$$= \frac{e^{-H_r}}{2} \sum_{m=n-r}^{\infty} \nu(m, r) + \frac{1}{2} \nu(n, r) + O \left( \frac{1}{r} \sum_{m=0}^{n-r-1} \nu(m, r) \nu(n - m, r) \right).$$

Consequently, only results on the probability $\nu(n, r)$ are needed attempting to improve on the order of notable estimate for $d_{TV}(n, r)$ in [3].

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