Lower order eigenvalues of the poly-Laplacian with any order on spherical domains

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Abstract. We consider the lower order eigenvalues of poly-Laplacian with any order on spherical domains. We obtain universal inequalities for them and show that our results are optimal.

Keywords: eigenvalue; poly-Laplacian.

Mathematics Subject Classification: Primary 35P15; Secondary 53C20.

1 Introduction

Let $\Omega$ be a connected bounded domain in an $n$-dimensional complete Riemannian manifold $M$. In this paper, we consider the Dirichlet eigenvalue problem of the poly-Laplacian with order $p$:

$$\begin{cases}
(-\Delta)^p u = \lambda u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{p-1} u}{\partial \nu^{p-1}} = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.1)

where $\Delta$ is the Laplacian in $M$ and $\nu$ denotes the outward unit normal vector field of $\partial \Omega$. Let $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty$ denote the successive eigenvalues for (1.1), where each eigenvalue is repeated according to its multiplicity.

When $p = 1$, the eigenvalue problem is called a fixed membrane problem. For $M = \mathbb{R}^2$ and $p = 1$, Payne-Pólya-Weinberger [10] proved

$$\lambda_2 + \lambda_3 \leq 6\lambda_1.$$

In 1993, for general dimensions $n \geq 2$, Ashbaugh and Benguria [2] proved

$$\sum_{i=1}^{n}(\lambda_{i+1} - \lambda_1) \leq 4\lambda_1.$$

Recently, the inequalities of eigenvalues of the fixed membrane problem have been generalized to some Riemannian manifolds. For the related research and improvement

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in this direction, see [3, 4, 9, 11] and the references therein. In particular, Sun-Cheng-Yang [11] proved that when $M$ is an $n$-dimensional unit sphere $S^n(1)$,

$$\sum_{i=1}^{n} (\lambda_{i+1} - \lambda_1) \leq 4\lambda_1 + n^2. \quad (1.2)$$

When $p = 2$, the eigenvalue problem (1.1) is called a clamped plate problem. For $M = \mathbb{R}^n$ and $p \geq 2$, Cheng-Ichikawa-Mametsuka proved in [8] that

$$\sum_{i=1}^{n} (\lambda_{i+1} - \lambda_1) \leq 4p(2p - 1)\lambda_1, \quad (1.3)$$

$$\sum_{i=1}^{n} (\lambda_{i+1}^p - \lambda_1^p)^{p-1} \leq (2p)^{p-1}\lambda_1^{p-1}. \quad (1.4)$$

Inequalities (1.3) and (1.4) include two universal inequalities of the clamped plate problem announced by Ashbauth in [1]. When $M$ is a general complete Riemannian manifold, for $p = 2$, Cheng-Huang-Wei [6] obtained

$$\sum_{i=1}^{n} (\lambda_{i+1}^p - \lambda_1^p)^{p-1} \leq (2p)^{p-1}\lambda_1^{p-1}, \quad (1.5)$$

where $H_0^2$ is a nonnegative constant which depends only on $M$ and $\Omega$. For $M = S^n(1)$, we have $H_0^2 = 1$ such that (1.5) becomes the following inequality:

$$\sum_{i=1}^{n} (\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} \leq (4\lambda_1^{\frac{1}{2}} + n^2H_0^2)^{\frac{1}{2}} \{(2n + 4)\lambda_1^{\frac{1}{2}} + n^2H_0^2\}^{\frac{1}{2}}, \quad (1.6)$$

We remark that when $\Omega = S^n(1)$, it holds that $\lambda_1 = 0$ and $\lambda_2 = \cdots = \lambda_{n+1} = n^2$. Therefore, the inequality (1.6) becomes equality. Hence, for $M = S^n(1)$, the inequality (1.6) is optimal.

In the present article, we consider the eigenvalue problem (1.1) with any $p$ when $M$ is a unit sphere $S^n(1)$. We obtain the following result:

**Theorem.** Let $\Omega$ be a bounded domain in an $n$-dimensional unit sphere $S^n(1)$. Let $\lambda_i$ be the $i$-th eigenvalue of the eigenvalue problem (1.1). Then we have

$$\sum_{i=1}^{n} (\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} \leq \left\{ \left( \frac{1}{\lambda_1^{\frac{1}{p}}} + n \right)^{p} - \lambda_1 + 4(2p - (p + 1))\lambda_1^{\frac{1}{p}} \left( \frac{1}{\lambda_1^{\frac{1}{p}}} + n \right)^{p-2} \right\}^{\frac{1}{2}} \times \left\{ 4\lambda_1^{\frac{1}{p}} + n^2 \right\}^{\frac{1}{2}}. \quad (1.7)$$

**Remark 1.** For $p = 2$, the inequality (1.7) becomes the optimal inequality (1.6).

**Remark 2.** For the unit sphere $S^n(1)$, by taking $\Omega = S^n(1)$, we know $\lambda_1 = 0$ and $\lambda_2 = \cdots = \lambda_{n+1} = n^p$. Hence, the inequality (1.7) becomes equality. Therefore, our result is optimal.

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2 Proof of Theorem

Let \( u_i \) be the orthonormal eigenfunction corresponding to eigenvalue \( \lambda_i \), that is,

\[
\begin{cases}
(-\Delta)^p u_i = \lambda_i u_i & \text{in } \Omega, \\
u_i = \frac{\partial u_i}{\partial \nu} = \cdots = \frac{\partial^{p-1} u_i}{\partial \nu^{p-1}} = 0 & \text{on } \partial \Omega, \\
\Omega \\
\int u_i u_j = \delta_{ij}.
\end{cases}
\]

Let \( x^1, x^2, \ldots, x^{n+1} \) be the standard Euclidean coordinate functions of \( \mathbb{R}^{n+1} \), then

\[
S^n(1) = \left\{ (x^1, x^2, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} ; \sum_{i=1}^{n+1} (x^i)^2 = 1 \right\}.
\]

It is well known that

\[
\Delta x^i = -nx^i, \quad i = 1, 2, \ldots, n+1.
\]

Assume that \( B \) is an \((n+1) \times (n+1)\)-matrix defined by \( B = (b_{ij}) \), where

\[
b_{ij} = \int_\Omega x^i u_1 u_{j+1}.
\]

Using the orthogonalization of Gram and Schmidt, we know that there exist an upper triangle matrix \( R = (r_{ij}) \) and an orthogonal matrix \( Q = (q_{ij}) \) such that \( R = QB \), that is,

\[
r_{ij} = \sum_{k=1}^{n+1} q_{ik} b_{kj} = \sum_{k=1}^{n+1} \int_\Omega q_{ik} x^k u_1 u_j = 0, \quad 2 \leq j \leq i \leq n+1.
\]

Defining \( h_i = \sum_{k=1}^{n+1} q_{ik} x^k \), one gets

\[
\int_\Omega h_i u_1 u_j = \sum_{k=1}^{n+1} \int_\Omega q_{ik} x^k u_1 u_j = 0, \quad 2 \leq j \leq i \leq n+1.
\]

Setting

\[
\varphi_i = h_i u_1 - u_1 \int_\Omega h_i u_1^2.
\]

Then

\[
\int_\Omega \varphi_i u_j = 0, \quad \text{for any } j \leq i.
\]

It follows from Rayleigh-Ritz inequality that

\[
\lambda_{i+1} \leq \frac{\int_\Omega \varphi_i (-\Delta)^p \varphi_i}{\|\varphi_i\|^2}, \quad (2.1)
\]
where \( \| f \|^2 = \int_{\Omega} |f|^2 \). By a direct calculation, we derive at

\[
\int_{\Omega} \varphi_i (-\Delta)^p \varphi_i = \int_{\Omega} \varphi_i (-\Delta)^p (h_i u_1) \\
= \int_{\Omega} \varphi_i \{ (-\Delta)^p (h_i u_1) - h_i (-\Delta)^p u_1 \} + \lambda_1 h_i u_1 \\
= \lambda_1 \| \varphi_i \|^2 + \int_{\Omega} \varphi_i (-\Delta)^p (h_i u_1) - h_i (-\Delta)^p u_1 \\
= \lambda_1 \| \varphi_i \|^2 + \int_{\Omega} h_i u_1 (-\Delta)^p (h_i u_1) - h_i (-\Delta)^p u_1 \\
- \int_{\Omega} h_i u_1^2 \int_{\Omega} u_1 (-\Delta)^p (h_i u_1) - h_i (-\Delta)^p u_1 \\
= \lambda_1 \| \varphi_i \|^2 + \int_{\Omega} h_i u_1 (-\Delta)^p (h_i u_1) - h_i (-\Delta)^p u_1.
\]

Defining

\[
\nabla^r = \begin{cases} \Delta^{r/2} & \text{when } r \text{ is even,} \\ \nabla (\Delta^{(r-1)/2}) & \text{when } r \text{ is odd.} \end{cases}
\]

Then (2.2) can be written as

\[
\int_{\Omega} \varphi_i (-\Delta)^p \varphi_i = \lambda_1 \| \varphi_i \|^2 + \| \nabla^p (h_i u_1) \|^2 - \lambda_1 \| h_i u_1 \|^2.
\]  

(2.3)

Putting (2.3) into (2.1) yields

\[
(\lambda_{i+1} - \lambda_1) \| \varphi_i \|^2 \leq \| \nabla^p (h_i u_1) \|^2 - \lambda_1 \| h_i u_1 \|^2.
\]  

(2.4)

One gets from integration by parts that

\[
\int_{\Omega} u_1 h_i^2 \langle \nabla h_i, \nabla u_1 \rangle = \frac{1}{4} \int_{\Omega} \langle \nabla (h_i^2), \nabla (u_1^2) \rangle = -\frac{1}{4} \int_{\Omega} u_1^2 \Delta (h_i^2) \\
= -\frac{1}{2} \int_{\Omega} u_1^2 h_i \Delta h_i - \frac{1}{2} \int_{\Omega} u_1^2 |\nabla h_i|^2.
\]
Hence,
\[
\int_\Omega \varphi_i \left( \langle \nabla h_i, \nabla u_1 \rangle + \frac{1}{2} u_1 \Delta h_i \right)
\]
\[= \int_\Omega u_1 h_i \langle \nabla h_i, \nabla u_1 \rangle + \frac{1}{2} \int_\Omega u_1^2 \Delta h_i
+ \frac{1}{2} \int_\Omega h_i u_1^2 \left( \int_\Omega \varphi_i \left( \langle \nabla h_i, \nabla u_1 \rangle + \frac{1}{2} u_1 \Delta h_i \right) \right)
\]
\[= \int_\Omega u_1 h_i \langle \nabla h_i, \nabla u_1 \rangle + \frac{1}{2} \int_\Omega u_1^2 \Delta h_i
\]
\[= - \frac{1}{2} \int_\Omega u_1^2 |\nabla h_i|^2
\]
\[= - \frac{1}{2} \| u_1 \nabla h_i \|^2.
\] (2.5)

By virtue of (2.4) and (2.5), it is easy to see
\[
(\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} \| u_1 \nabla h_i \|^2 = -2(\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} \int_\Omega \varphi_i \left( \langle \nabla h_i, \nabla u_1 \rangle + \frac{1}{2} u_1 \Delta h_i \right)
\]
\[\leq \delta (\lambda_{i+1} - \lambda_1) \| \varphi_i \|^2 + \frac{1}{\delta} \left\| \langle \nabla h_i, \nabla u_1 \rangle + \frac{1}{2} u_1 \Delta h_i \right\|^2
\]
\[\leq \delta \left\{ \| \nabla^p(h_i u_1) \|^2 - \lambda_1 \| h_i u_1 \|^2 \right\} + \frac{1}{\delta} \left\| \langle \nabla h_i, \nabla u_1 \rangle + \frac{1}{2} u_1 \Delta h_i \right\|^2,
\] (2.6)

where \( \delta \) is a positive constant. Summing over \( i \) from 1 to \( n+1 \) for (2.6), one finds that
\[
\sum_{i=1}^{n+1} (\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} \| u_1 \nabla h_i \|^2 \leq \delta \sum_{i=1}^{n+1} \left\{ \| \nabla^p(h_i u_1) \|^2 - \lambda_1 \| h_i u_1 \|^2 \right\}
\]
\[+ \frac{1}{\delta} \sum_{i=1}^{n+1} \left\| \langle \nabla h_i, \nabla u_1 \rangle + \frac{1}{2} u_1 \Delta h_i \right\|^2
\]
\[= \delta \sum_{i=1}^{n+1} \left\{ \| \nabla^p(x_i u_1) \|^2 - \lambda_1 \| x_i u_1 \|^2 \right\}
\]
\[+ \frac{1}{\delta} \sum_{i=1}^{n+1} \left\| \langle \nabla x_i, \nabla u_1 \rangle + \frac{1}{2} u_1 \Delta x_i \right\|^2
\] (2.7)

Making use of the same method as proof of Lemma 1 in [5], it is easy to prove
\[
\int_\Omega |\nabla u_1|^2 \leq \lambda_1^\frac{3}{2}.
\]
Thus,
\[
\sum_{i=1}^{n+1} \left\| (\nabla x_i, \nabla u_1) + \frac{1}{2} u_1 \Delta x_i \right\|^2 = \sum_{i=1}^{n+1} \int_{\Omega} \left( (\nabla x_i, \nabla u_1) + \frac{1}{2} u_1 \Delta x_i \right)^2 \\
= \sum_{i=1}^{n+1} \int_{\Omega} \left( \frac{1}{4} u_1^2 (\Delta x_i)^2 + (\nabla x_i, \nabla u_1)^2 + \frac{1}{2} \Delta x_i (\nabla x_i, \nabla (u_1^2)) \right) \\
= n^2 \frac{1}{4} + \int_{\Omega} |\nabla u_1|^2 \\
\leq n^2 \frac{1}{4} + \lambda_1^\frac{1}{p}. 
\] 
(2.8)

It has been shown in [7] (see Proposition 2.2 of [7]) that
\[
\sum_{i=1}^{n+1} \int_{\Omega} u_1 x_i \{(-\Delta)^p(u_1 x_i) - x_i(-\Delta)^p u_1\} \\
\leq (\lambda_1^\frac{1}{p} + n)^p - \lambda_1 + 4[2^p - (p + 1)]\lambda_1^\frac{1}{p} \left( \lambda_1^\frac{1}{p} + n \right)^{p-2}. 
\] 
(2.9)

Inserting (2.8) and (2.9) into (2.7), we infer
\[
\sum_{i=1}^{n+1} (\lambda_{i+1} - \lambda_1)^\frac{1}{2} |u_1 \nabla h_i|^2 \leq \delta \left\{ \left( \lambda_1^\frac{1}{p} + n \right)^p - \lambda_1 + 4[2^p - (p + 1)]\lambda_1^\frac{1}{p} \left( \lambda_1^\frac{1}{p} + n \right)^{p-2} \right\} \\
+ \frac{1}{\delta} \left\{ \lambda_1^\frac{1}{p} + \frac{n^2}{4} \right\}. 
\] 
(2.10)

Minimizing the right hand side of (2.10) as a function of \( \delta \) by choosing
\[
\delta = \left( \frac{\lambda_1^\frac{1}{p} + \frac{n^2}{4}}{(\lambda_1^\frac{1}{p} + n)^p - \lambda_1 + 4[2^p - (p + 1)]\lambda_1^\frac{1}{p} \left( \lambda_1^\frac{1}{p} + n \right)^{p-2}} \right)^\frac{1}{2},
\]
we obtain
\[
\sum_{i=1}^{n+1} (\lambda_{i+1} - \lambda_1)^\frac{1}{2} |u_1 \nabla h_i|^2 \leq \left\{ \left( \lambda_1^\frac{1}{p} + n \right)^p - \lambda_1 + 4[2^p - (p + 1)]\lambda_1^\frac{1}{p} \left( \lambda_1^\frac{1}{p} + n \right)^{p-2} \right\} \frac{1}{2} \\
\times \left\{ 4\lambda_1^\frac{1}{p} + n^2 \right\}^\frac{1}{2}. 
\] 
(2.11)

By a transformation of coordinates if necessary, for any point \( q \), one gets
\[
|\nabla h_i|^2 \leq 1 \quad \text{for any } i. 
\]
It follows that

\[ \sum_{i=1}^{n+1} (\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} |\nabla h_i|^2 = \sum_{i=1}^{n} (\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} |\nabla h_i|^2 + (\lambda_{n+1} - \lambda_1)^{\frac{1}{2}} |\nabla h_{n+1}|^2 \]

\[ = \sum_{i=1}^{n} (\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} |\nabla h_i|^2 + (\lambda_{n+1} - \lambda_1)^{\frac{1}{2}} \left( n - \sum_{i=1}^{n} |\nabla h_i|^2 \right) \]

\[ = \sum_{i=1}^{n} (\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} |\nabla h_i|^2 + (\lambda_{n+1} - \lambda_1)^{\frac{1}{2}} \sum_{i=1}^{n} (1 - |\nabla h_i|^2) \]

\[ \geq \sum_{i=1}^{n} (\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} |\nabla h_i|^2 + \sum_{i=1}^{n} (\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} (1 - |\nabla h_i|^2) \]

\[ = \sum_{i=1}^{n} (\lambda_{i+1} - \lambda_1)^{\frac{1}{2}}. \]

From (2.11) and (2.12), we obtain

\[ \sum_{i=1}^{n} (\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} \leq \left\{ \left( \lambda_1^{\frac{1}{2p}} + n \right)^{p} - \lambda_1 + 4[2^p - (p + 1)] \lambda_1^{\frac{1}{p}} \left( \lambda_1^{\frac{1}{p}} + n \right)^{p-2} \right\}^{\frac{1}{2}} \]

\[ \times \left\{ 4 \lambda_1^{\frac{1}{p}} + n^2 \right\}^{\frac{1}{2}}, \]

which concludes the proof of Theorem.

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