Dissipative search of an unstructured database

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The search of an unstructured database amounts to finding one element having a certain property out of $N$ elements. The classical search with an oracle checking one element at a time requires on average $N/2$ steps \cite{1}. One of the hallmarks of quantum computation is the Grover search algorithm which yields quadratic speedup of the search time – involves on average by some function (oracle) applied to each element at a time to be correctly identified. The asymptotic scaling of the search problem as a dissipative Markov process acting on an $N$-level system weakly coupled to a thermal bath. Assuming that the energy levels of the system represent the database elements, we show that, with a proper choice of the spectrum and physically admissible, long-range transition rates between the energy levels, the system relaxes to the ground state, corresponding to the sought element, in time $O(\ln N)$.

In a classical search of an unstructured set of $N$ elements, finding an element with a specific feature – verified by some function (oracle) applied to each element at a time – involves on average $N/2$ steps \cite{1}. One of the hallmarks of quantum computation is the Grover search algorithm which yields quadratic speedup of the search time $\tau = O(\sqrt{N})$ \cite{2,3}. The quantum search can be formulated as a Hamiltonian evolution of an analog quantum system \cite{3}. The $N$ elements of the set are associated with the orthonormal basis states $|w_k\rangle$. The Hamiltonian of the system is then

$$H_0 = \epsilon |w_\ell\rangle \langle w_\ell|,$$

where $\epsilon$ is a known constant, but $\ell$ is not known. To find $\ell$, and thereby $|w_\ell\rangle$, one prepares the system in the equally-weighted superposition state $|\psi(0)\rangle = |s\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^N |w_k\rangle$ and adds to $H_0$ the interaction Hamiltonian $V = \epsilon |s\rangle \langle s|$ which couples all the basis states,

$$\langle w_l|V|w_k\rangle = \frac{\epsilon}{N} \quad \forall l, k.$$

The system evolution is governed by the Schrödinger equation, and at time $\tau = \frac{\pi}{2\epsilon} \sqrt{N}$ the system attains the desired state $|\psi(\tau)\rangle = |w_\ell\rangle$. The time $\tau$ does not depend on $\ell$, and the evolution should be terminated immediately thereafter in order for the target state $|w_\ell\rangle$ to be correctly identified. The asymptotic scaling of the search time $\tau = O(\sqrt{N})$ is optimal for the coherent quantum evolution \cite{3}, and it can also be deduced from the time-energy uncertainty relation \cite{6} applied to the Hamiltonian $H_0 + V$ acting on $|\psi(0)\rangle$.

Here we demonstrate an exponential speedup of the quantum search by using a dissipative Markov dynamics, instead of the coherent Schrödinger dynamics. We begin again with the Hamiltonian $H_0$, assuming that $\epsilon < 0$ and therefore $|w_\ell\rangle$ is the ground state of $H_0$. We then add an auxiliary (known) Hamiltonian $H_1$ which lifts the degeneracy of all the energy levels $|w_k\rangle$, but still leaves $|w_\ell\rangle$ as a ground state of $H = H_0 + H_1$ for any $\ell$, with some energy gap $\Delta_\ell$. We next couple the system to a thermal bath at a temperature $T = 1/\beta$ and let it relax to the Gibbsian (equilibrium) state described by the density operator $\rho \propto e^{-\beta H}$. If $T$ is sufficiently low, such that $\Delta_\ell > 1$ for any $\ell = 1, 2, \ldots, N$, then $\rho \approx |w_\ell\rangle \langle w_\ell|$, and this approximation can be made arbitrary precise by increasing the energy gap $\Delta_\ell$ or decreasing the temperature $T$.

The working time of our dissipative analog device $\tau_{\text{res}}$ is proportional to the inverse relaxation rate $\alpha^{-1}$ of the system towards the equilibrium state $\rho$, and our main goal is to estimate this time, given the spectrum of the system and its coupling to the thermal reservoir, as detailed below. In contrast to the coherent quantum search, however, we need not demand that $\tau_{\text{res}}$ be independent on the unknown index $\ell$, because once the system reaches the equilibrium state $\rho \approx |w_\ell\rangle \langle w_\ell|$, it will remain in that state thereafter. We can then take the search time as $\tau_{\text{res}} = \max_{\ell} \{1/\alpha(\ell)\}$.

We now turn to a more quantitative description of the system. The Hamiltonian $H_1$ that shifts the energy levels $|w_k\rangle$ should leave $|w_\ell\rangle$ as a unique ground state of $H = H_0 + H_1$ for any $\ell$. Assuming for simplicity that $H_1$ commutes with $H_0$, we therefore require that spectrum$[H_1] = \{\eta_k\}_{k=1}^N$ satisfies the condition

$$\eta_N - \eta_1 < |\epsilon|,$$

where $\eta_N = \max \{\eta_k\}$ and $\eta_1 = \min \{\eta_k\}$. We thus have spectrum$[H] = \{\epsilon_k\}_{k=1}^N = \{\epsilon + \eta_k, \eta_k\}_{k\neq \ell}$ with the smallest energy gap between state $|w_\ell\rangle$ and all the other states $|w_k\rangle$ being $\Delta_N = \eta_1 - (\eta_N + \epsilon) > 0$.

The (incoherent) Markovian dynamics of the system weakly coupled to a thermal bath is governed by the rate equations for the populations $p_k$ of states $|w_k\rangle$ \cite{7}:

$$\dot{p}_k = \sum_{l=1}^N v_{kl} p_l - p_k \sum_{l=1}^N v_k,$$
where \( v_{kl} \geq 0 \) \((v_{kk} = 0)\) are the transition rates between the energy levels induced by the thermal bath at temperature \( T \). Hence, we have the detailed balance condition

\[
v_{kl} e^{-\beta \varepsilon_l} = v_{lk} e^{-\beta \varepsilon_k},
\]

which implies that there is a unique stationary state of the system determined by the Gibbs (equilibrium) probabilities \( p_k = Z^{-1} e^{-\beta \varepsilon_k} \) for \( k = 1, 2, \ldots, N \), with \( Z = \sum_{k=1}^{N} e^{-\beta \varepsilon_k} \). In view of the spectrum of the Hamiltonian \( H \), the probability of the ground state is

\[
p_1^{(eq)} = \left[ 1 - e^{\beta \varepsilon} \left( e^{\beta \eta} \sum_{k=1}^{N} e^{-\beta \eta_k} - 1 \right) \right]^{-1},
\]

and \( 1 - p_1^{(eq)} \) is then the error probability of our dissipative search. Obviously, the smallest ground state probability in the stationary state, and thus the largest error, would occur for \( \ell = N \) with the ground state having the largest possible energy \( \epsilon + \eta_N \), see Eq. (3). To ensure the ground state dominance, we therefore require that \( e^{\beta \varepsilon} \gg \left( e^{\beta \eta} \sum_{k=1}^{N} e^{-\beta \eta_k} - 1 \right) \).

We are interested in the dependence of the relaxation time to the equilibrium state on \( N \). We therefore demand that the total transition rate from any state \( |w_k\rangle \) be bounded,

\[
\gamma_k \equiv \sum_{l=1}^{N} v_{lk} \lesssim v \quad \forall k,
\]

where \( v \) is some constant independent on \( N \). This condition determines the physically acceptable transition rates \( v_{kl} \) that exclude parallel relaxation processes and hence prevent trivial acceleration of the dynamics, as discussed in §1 of [8]. Note that similar condition also holds for the Hamiltonian quantum search, where all the energy levels are coupled to each other, Eq. (2).

Condition (7) automatically holds for short-range interacting systems: \( v_{kl} \sim v \neq 0 \) only for \( |k-l| \leq r \) with some fixed interaction range \( r \) for any \( N \). But then, starting from any arbitrary state \( |w_k\rangle \), we will reach the desired ground state \( |w_1\rangle \), where the population accumulates, via diffusive transport, and the relaxation time will scale as \( \tau_{rlx} = O(N^2) \) [9, 10]. This is worse than using either ballistic transport or classical search with the \( \tau = O(N) \) scaling. We shall therefore consider long-range but bounded interactions, as per Eq. (7).

We can rewrite Eq. (4) for the vector \( p = [p_1, p_2, \ldots, p_N]^T \) in a matrix form,

\[
\dot{p} = Ap, \quad A_{kl} = v_{kl} - \delta_{kl} \gamma_k,
\]

where \( A_{\ell \neq l} \geq 0 \) and \( \sum_{k=1}^{N} A_{kl} = 0 \). If follows from Eq. (5) that \( B_{kl} = A_{kl} e^{-\beta (\varepsilon_l - \varepsilon_k)} = B_{lk} \) is a symmetric matrix that has the same eigenvalues as \( A_{kl} \). Hence the eigenvalues of \( A \) are real. Assuming that \( A \) is primitive, \( [A^m]_{k \neq l} > 0 \) for a sufficiently large integer \( m \), we can employ the Perron-Frobenius theorem [10, 11] to argue that the largest eigenvalue \( \alpha_1 \) of \( A \) is zero and non-degenerate, and hence all the other eigenvalues are negative, \( \alpha_1 = 0 > \alpha_2 \geq \alpha_3, \ldots \). The largest non-zero eigenvalue \( \alpha_2 \) of \( A \) defines the relaxation time \( \tau_{rlx} = 1/|\alpha_2| \) of the master equation (5), as shown in §2 of [8].

The detailed balance condition and the requirement that \( A \) be imprimitive still leaves some freedom in choosing the transition rates \( v_{kl} \). A well-known and widely-employed approach is to use the Glauber rates proposed phenomenologically [12] and deduced from the system-bath weak-coupling treatment [12, 14]. We thus assume that

\[
v_{kl} = \frac{v}{\max(n_{kl}, n_l)} \left( 1 + e^{-\beta \varepsilon_l + \beta \varepsilon_k} \right)^{-1},
\]

where \( v \) is the bare relaxation rate that depends on the strength of the system-bath coupling, and \( n_{kl} \) is the number of levels with energies not larger than \( \varepsilon_k \). The detailed balance condition [8] obviously holds. The physical meaning of Eq. (10) is that transitions from higher to lower energy levels are facilitated, while the reverse transitions are suppressed. The Glauber rates are usually written without the factor \( 1/\max(n_{kl}, n_l) \), but here we allow transitions between all energy levels, and this factor is needed to ensure that condition (7) is satisfied.

Consider first the trivial case of \( H_1 = 0 \), i.e. all states have the same energy equal to zero, and the ground state energy is \( \epsilon < 0 \). From Eqs. (4) and (9), we obtain for the ground state population

\[
\dot{p}_1 = \frac{1}{N(1 + e^{\beta \varepsilon})} - \frac{p_1}{\tau_{rlx}},
\]

\[
\frac{1}{\tau_{rlx}} = \frac{1}{N} \left( 1 + \frac{(N-1)e^{\beta \varepsilon}}{1 + e^{\beta \varepsilon}} \right),
\]

which leads to the equilibrium population \( p_1^{(eq)} = \left[ 1 + (N-1)e^{\beta \varepsilon} \right]^{-1} \) attained exponentially for times \( t \gg \tau_{rlx} \). Thus the ground state dominance, \( p_1^{(eq)} \approx 1 \) for \( (N-1)e^{\beta \varepsilon} \ll 1 \), leads to \( \tau_{rlx} = O(N) \), as seen from Eq. (11). This is the expected result for the classical search of an unstructured set.

To obtain more interesting results, we now consider the auxiliary Hamiltonian \( H_1 \) with a non-degenerate spectrum

\[
\eta_k = a \ln(k), \quad k = 1, \ldots, N,
\]

where \( a > 0 \). Hence, the energies \( \eta_k \) grow only slowly (logarithmically) with index \( k \), which justifies the possibility of long-range coupling between the energy levels. We then set \( \epsilon = -b \ln(N) \) with the parameter \( b > a \), to satisfy the condition (3). In §3 of [8] we present explicit expressions for the Glauber rates for the logarithmic spectrum [12] and show that condition (7) also holds. The ground state dominance, \( p_1^{(eq)} \approx 1 \), now requires that
\[ N^{b \beta} \gg (N^{a \beta} \sum_{k=1}^{N} k^{-a \beta} - 1), \text{ which for } N \gg 1 \text{ is satisfied if } b \beta > \max\{1, a \beta\}, \text{ see Eq. (9). Indeed, for } a \beta > 1 \text{ we have } N^{a \beta} \sum_{k=1}^{N} k^{-a \beta} = O(N^{a \beta}), \text{ while for } a \beta < 1 \text{ we have } N^{a \beta} \sum_{k=1}^{N} k^{-a \beta} = O(N). \] 

The error probability is

\[ 1 - p_1^{(eq)} \approx N^{-(b-a)\beta} \sum_{k=1}^{N} k^{-a \beta}, \quad (13) \]

which can be made arbitrary small, \( 1 - p_1^{(eq)} \approx N^{-(b-a)\beta} \), when \( a \beta > 1 \).

In Fig. 1 we show the dynamics of population of the ground state \( p_1 \), as obtained from the numerical solution of the rate equations (3), with the initial population equally distributed among all the energy levels, while the sought index is \( \ell = N \) corresponding to the smallest energy gap \( \Delta_N = (b-a) \ln N \). We observe that the time at which the ground state population exceeds some threshold value, e.g. \( p_1 \geq 0.95 \), grows very slowly with increasing the system size \( N \). In the Inset of Fig. 1 we show the total transition rates \( \gamma_k \) from levels \( k \), which remain bounded for any \( N \) as per condition (7).

In Fig. 2 we show the relaxation time \( \tau_{rlx} = 1/|\alpha_2| \) as a function of \( N \), obtained from diagonalization of matrix \( A \) in Eq. (8). We observe that, for \( a \beta > 1, b > a \) and large \( N \gg 1 \), the relaxation time growth logarithmically with \( N \)

\[ \tau_{rlx} = O(\ln N). \quad (14) \]

Note that \( \tau_{rlx} \) depends on the sought index \( \ell \), which determines the energy gap \( \Delta_{\ell} \) but also the spectrum of the excited states and its bandwidth, especially for small \( \ell \).

But once the ground state dominance condition is satisfied, the relaxation time does not change upon increasing \( (b-a) \).

The relaxation time \( \tau_{rlx} \) increases for smaller values of \( a \beta \) (or for smaller temperatures), since then the transition rates between excited levels tend to equalize, e.g. for \( \ell = N \) we have \( v_{kl}/v_{lk} = (l/k)^{a \beta} \) for \( l > k > 1 \); see §3 of 8. In other words, for smaller \( a \beta \), the system wanders longer among excited levels before relaxing to the ground state.

For \( a \beta \) sufficiently smaller than 1, the relaxation time follows a power-law

\[ \tau_{rlx} = O(N^{\kappa}), \quad (15) \]

with the exponent \( \kappa < 1 \) that depends on \( a \), see the Inset of Fig. 2. For \( a \to 0 \), we have \( \kappa \to 1 \), i.e. we are back to the classical search time. Now the error probability \( 1 - p_1^{(eq)} \approx N^{-(b-a)\beta} \) does not depend on \( a \). The transition from the logarithmic to the power-low dependence of \( \tau_{rlx} \) on \( N \) is gradual taking place in the vicinity of \( a \beta \simeq 1 \).

To summarize, we have shown that a dissipative Markov dynamics in a system with a weakly non-degenerate spectrum of \( N \gg 1 \) states can result in the relaxation of the system to the (unknown) ground state during time \( \tau_{rlx} = O(\ln N) \). The system can be viewed...
as an analog of an unstructured database of $N$ elements, for which the classical search time scales as $O(N)$ while the optimal quantum search time scales as $O(\sqrt{N})$. We can identify the relaxation time of the system with the search time that has exponentially better scaling with the system size $N$ than either the classical or the fully quantum search.

The necessary condition for achieving the short relaxation times of the system, apart from the (weakly) non-degenerate spectrum, is that the Markov process involves transitions between arbitrary energy levels. We should ensure, however, that this long-range interactions are bounded for any $N$, excluding thereby the parallel relaxation processes, since otherwise decreasing the search time with increasing $N$ would be trivial. Note that long-range coupling between energy levels is also present in the Hamiltonian analog of the quantum search \cite{4}.

Since the Markov dynamics of Eqs. (14) is described in terms of classical probabilities, it is natural to ask whether the considered dissipative search can be implemented on a classical computer using, e.g. Monte-Carlo simulations to reach the equilibrium state dominated by the ground state. It can of course be done, but will require large amount of calculations and computer memory. Indeed, the $N = 2^n$ energy levels may correspond to different configurations of $n$ bits or spins \{\sigma_i = \pm 1\}_{i=1}^n$. Recall that the energy levels in our dissipative analog device should be (weakly) non-degenerate, which means that we need to realize a weakly-interacting $n$-spin system. We will then have to calculate $\sim N$ different energies for $2^n$ different configurations and store them in the memory, in order to determine the transition probabilities between the different energy levels. And these transitions may involve up to $n$ simultaneous spin-flips ($\sigma_i \rightarrow -\sigma_i$). In contrast, in the usual Monte-Carlo simulations only one spin is flipped at a time, and the energy difference between the old and new configurations is easy to calculate at each time step.

In our study, we assumed weak system-bath coupling and employed the Glauber rates \cite{12, 14} for the transitions between the energy levels. But our results equally hold for other similar coupling schemes, e.g. Arrhenius rates often employed in chemical physics \cite{10, 13}. It would be interesting to consider the applicability of our results to complex quantum systems with weakly non-degenerate spectra and long-range transition rates. Finally, our results may have important and interesting implications for protein folding and similar problems, where macromolecules attain the target (minimal energy) conformations very fast, despite the available huge energy landscape \cite{16, 17}.

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SUPPLEMENTARY MATERIAL

Here we present a justification for bounded transition rates of the Markov process, formal solution of the master equation for the populations of energy levels of the system, and explicit expressions for the Glauber transition rates for the logarithmic energy spectrum.

§1. Transition rates of the Markov process

We can express the Markovian dynamics for populations of energy levels $k$ of Hamiltonian $H$ in a time-integrated form as

\[ p_k(t + \delta t) = \sum_{l=1}^{N} \Pi_{kl}(\delta t) p_l(t), \]

\[ \Pi_{kl}(\delta t) \geq 0, \quad \sum_{k=1}^{N} \Pi_{kl}(\delta t) = 1, \]

where $\delta t$ is the time-step, and $\Pi_{kl}(\delta t)$ is a Markov matrix that determines the conditional probability of the transition $l \rightarrow k$. Equations (2) ensure that $p_k(t + \delta t) \geq 0$ and $\sum_{k=1}^{N} p_k(t + \delta t) = 1$, provided $p_k(t)$ satisfy analogous conditions.

To obtain a differential form of the Markov dynamics, we assume small $\delta t$ and expand

\[ \Pi_{kl}(\delta t) = \delta_{kl} - \delta t \delta_{kl} \sum_{s=1}^{N} v_{sl} + \delta t v_{kl} + \mathcal{O}(\delta t^2), \]

where $\delta_{kl}$ is the Kronecker delta, and $v_{kl}$ are the $l \rightarrow k$ transition rates, while $v_{kk} = 0$.

In the main text, we are interested in the dependence of the relaxation time $\tau_{rlx}$ on the total number of energy levels $N$. We therefore should ensure that the time-step of the Markov dynamics does not depend on $N$. Consider $k = l$ in (3), which we write as

\[ 1 \gg \delta t \sum_{s=1}^{N} v_{sl}. \]

This condition should hold for any small $\delta t$ which does not depend on $N$. Hence, we should demand that

\[ \sum_{s=1}^{N} v_{sl} \sim v \quad \forall l \in [1,N], \]

with $v$ being independent on $N$. This the same condition (7) of the main text, and it plays an important role when deciding on physically acceptable transition rates $v_{sl}$. The physical meaning of Eq. (5) is as follows: if from a given level $l$ we have transitions to $\mathcal{O}(N)$ levels $s$, then each such transition should scale as $v_{sl} \propto v/N$. Hence (5) excludes parallel relaxation processes.

§2. Formal solution of the master equation

We write the master equation for the vector $|p\rangle$ of populations of states $\{|w_k\rangle\}_{k=1}^{N}$ using the Dirac notation,

\[ \partial_t |p\rangle = A |p\rangle, \]

where the matrix elements $A_{kl} = v_{kl} - \delta_{kl} \sum_{j=1}^{N} v_{jk}$, with $v_{kk} = 0$, satisfy $A_{k\neq l} \geq 0$ and $A_{kk} = -\gamma_k < 0$. The right eigenvector $|R_1\rangle$ of $A$ with eigenvalue $\alpha_1 = 0$ coincides with the stationary Gibbsian probability $\Pi_0$: $\sum_{l=1}^{N} A_{kl} e^{-\beta \epsilon_l} = 0$. The corresponding left eigenvector $\langle L_1 |$ has all its components equal to 1, as seen from $\sum_{k=1}^{N} A_{kl} = 0$. Writing the eigenresolution of $A$ as

\[ A = \sum_{k=1}^{N} \alpha_k |R_k\rangle \langle L_k |, \quad \langle L_k | R_l \rangle = \delta_{kl}, \]

\[ A |R_k\rangle = \alpha_k |R_k\rangle, \quad \langle L_k | A = \alpha_k \langle L_k |, \]

where $\{|R_k\rangle\}_{k=1}^{N}$ and $\{|L_k\rangle\}_{k=1}^{N}$ are the right and left eigenvectors, we can formally solve Eq. (4) via $e^{At} = \sum_{k=1}^{N} e^{\alpha_k t} |R_k\rangle \langle L_k |$ with $\alpha_1 = 0 > \alpha_2 \geq \alpha_3, \ldots$, leading to

\[ |p(t)\rangle = e^{At} |p(0)\rangle \sim |p^{(\alpha_1)}\rangle + e^{-|\alpha_1| t} \langle L_2 | p(0) \rangle |R_2 \rangle + \mathcal{O}(e^{-|\alpha_2| t}), \]

where $|p^{(\alpha_1)}\rangle = Z^{-1} \sum_{k=1}^{N} e^{-\beta \epsilon_k} |w_k\rangle \langle w_k |$, with $Z = \sum_{k=1}^{N} e^{-\beta \epsilon_k}$, is the stationary state. We can therefore define the relaxation time as $\tau_{rlx} = 1/|\alpha_2|$.

Spectral features of Markov matrices are needed in many applications and are extensively studied; see, e.g. [18].

§3. Glauber rates for the logarithmic spectrum

In the main text, we employ the Glauber rates

\[ v_{kl} = \frac{v}{\max(n_l, n_k)} \left( 1 + e^{-\beta \epsilon_l + \beta \epsilon_k} \right)^{-1}, \]

for the transitions between the states with energies $\epsilon_k$ and $\epsilon_l$, and here we present the corresponding explicit expressions for the logarithmic spectrum $\eta_k = a \ln(k)$ of the auxiliary Hamiltonian $H_1$. Consider first the case of the ground state of $H_0$ at $\ell = 1$ with energy $\epsilon = -b \ln(N)$. The energy levels of $H = H_0 + H_1$ are $(\epsilon_1, \epsilon_2, \ldots, \epsilon_N) = (-b \ln N, a \ln 2, \ldots, a \ln N)$, and the transition rates [10]
are
\[ v_{ll} = \frac{1}{l[1 + l^{-a\beta} N^{-b\beta}]}, \quad l > 1, \] (11a)
\[ v_{l1} = \frac{1}{l[1 + b\beta N^{-a\beta}]}, \quad l > 1, \] (11b)
\[ \frac{v_{ll}}{v_{l1}} = \frac{l a\beta N^{b\beta}}, \quad l > 1, \] (11c)
\[ v_{k<l} = \frac{1}{l[1 + (k/l)^{a\beta}]}, \quad l > 1, \ k > 1, \] (11d)
\[ v_{k>l} = \frac{1}{l[1 + (l/k)^{a\beta}]}, \quad l > 1, \ k > 1, \] (11e)
\[ \frac{v_{k<l}}{v_{k>l}} = \left(\frac{k}{l}\right)^{-a\beta}, \quad l > 1, \ k > 1 \] (11f)

Note that for high temperatures \( a\beta < 1 \) the transition rate \( v_{lk} \) from a state with lower energy \( \varepsilon_k \) to a state with higher energy \( \varepsilon_l \) is smaller, but comparable with the reverse transition rate \( v_{kl} \), which leads to longer relaxation times \( \tau_{kl} \) as is also confirmed numerically. The total transition rate from state \( |w_l\rangle \) is then
\[ \gamma_1 = \sum_{s=1}^{N} v_{s1} = \sum_{s=2}^{N} s^{-1} \left[ 1 + N^{b\beta s^{a\beta}} \right]^{-1}, \] (12a)
\[ \gamma_l = \sum_{s=1}^{N} v_{sl} = \frac{1}{l} \left[ 1 + N^{-b\beta l^{-a\beta}} \right]^{-1} + \frac{1}{l} \sum_{s=2}^{l-1} \left[ 1 + (s/l)^{a\beta} \right]^{-1} \]
\[ + \sum_{s=l+1}^{N} s^{-1} \left[ 1 + (s/l)^{a\beta} \right]^{-1}, \quad l > 1, \] (12b)

and \( \gamma_l \gtrsim 1 \) holds for any \( a\beta > 0 \), as required.

The same conclusions hold for other values of \( \ell \), e.g., for \( \ell = N \) the energy levels are \( \langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N \rangle = \{(a - b) \ln N, 0, a \ln 2, \ldots, a \ln(N - 1)\} \) and the transition rates are
\[ v_{ll} = \frac{1}{l[1 + (l-1)^{-a\beta} N^{-b\alpha}]}, \quad l > 1, \] (13a)
\[ v_{l1} = \frac{1}{l[1 + (l-1)^{-a\beta} N^{-b\alpha}]}, \quad l > 1, \] (13b)
\[ \frac{v_{ll}}{v_{l1}} = \frac{l a\beta N^{b\alpha}}, \quad l > 1, \] (13c)
\[ v_{k<l} = \frac{1}{l[1 + ((k-1)/(l-1))^{a\beta}]}, \quad l > 1, \ k > 1, \] (13d)
\[ v_{k>l} = \frac{1}{l[1 + ((l-1)/(k-1))^{a\beta}]}, \quad l > 1, \ k > 1, \] (13e)
\[ \frac{v_{k<l}}{v_{k>l}} = \left(\frac{k-1}{l-1}\right)^{-a\beta}, \quad l > 1, \ k > 1 \] (13f)

and for \( \varepsilon_k < \varepsilon_l \) we again have \( v_{lk} < v_{kl} \), provided \( b > a \) which is always assumed. The total transition rate from any state \( |w_l\rangle \) is
\[ \gamma_1 = \sum_{s=1}^{N} v_{s1} = \sum_{s=2}^{N} s^{-1} \left[ 1 + N^{(b-a)\beta (s-1)^{a\beta}} \right]^{-1}, \] (14a)
\[ \gamma_l = \sum_{s=1}^{N} v_{sl} = \frac{1}{l} \left[ 1 + N^{(a-b)\beta (l-1)^{-a\beta}} \right]^{-1} \]
\[ + \frac{1}{l} \sum_{s=2}^{l-1} \left[ 1 + \left(\frac{s-1}{l-1}\right)^{a\beta} \right]^{-1} \]
\[ + \sum_{s=l+1}^{N} s^{-1} \left[ 1 + \left(\frac{s-1}{l-1}\right)^{a\beta} \right]^{-1}. \] (14b)