Quasilinear elliptic equations in $\mathbb{R}^N$ via variational methods and Orlicz-Sobolev embeddings

A. Azzollini · P. d’Avenia · A. Pomponio

Abstract In this paper we prove the existence of a nontrivial non-negative radial solution for the quasilinear elliptic problem

$$\begin{cases}
- \nabla \cdot \left[ \phi'(|\nabla u|^2) \nabla u \right] + |u|^{q-2} u = |u|^{s-2} u, & x \in \mathbb{R}^N, \\
u(x) \to 0, & \text{as } |x| \to \infty,
\end{cases}$$

where $N \geq 2$, $\phi(t)$ behaves like $t^{q/2}$ for small $t$ and $t^{p/2}$ for large $t$, $1 < p < q < N$, $1 < \alpha \leq p^* q' / p'$ and $\max\{q, \alpha\} < s < p^*$, being $p^* = \frac{pN}{N-p}$ and $p'$ and $q'$ the conjugate exponents, respectively, of $p$ and $q$. Our aim is to approach the problem variationally by using the tools of critical points theory in an Orlicz-Sobolev space. A multiplicity result is also given.

Mathematics Subject Classification 35J62 · 46E30 · 46E35

1 Introduction

This paper deals with the following quasilinear elliptic equation

$$- \nabla \cdot \left[ \phi'(|\nabla u|^2) \nabla u \right] = f(u) \quad \text{in } \mathbb{R}^N, \ N \geq 2,$$

Communicated by P. Rabinowitz.

A. Azzollini
Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, Via dell’Ateneo Lucano 10, 85100 Potenza, Italy
e-mail: antonio.azzollini@unibas.it

P. d’Avenia · A. Pomponio
Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, Via E. Orabona 4, 70125 Bari, Italy
e-mail: a.pomponio@poliba.it

P. d’Avenia
e-mail: p.davenia@poliba.it
where \( \phi \in C^1(\mathbb{R}_+, \mathbb{R}_+) \) has a different growth near zero and infinity. Such a type of behaviour occurs, for example, when \( \phi(t) = 2(1 + t)^{\frac{1}{2}} - 1 \). In this case (1) becomes

\[
-\nabla \cdot \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = f(u),
\]

known as the prescribed mean curvature equation or the capillary surface equation.

Such a kind of problems has been deeply studied in the recent years: existence and non-existence results of solutions decaying to zero at infinity have been proved by [2,8,10–12,14,19,20,22], among others, under different assumptions on the nonlinearity \( f \) and on the function \( \phi \). Moreover, for bounded domains, we recall [7,13,17,18].

More precisely we are interested in the existence of solutions of the following quasilinear elliptic problem

\[
\begin{cases}
-\nabla \cdot \left[ \phi'(|\nabla u|^2) \nabla u \right] + |u|^{q-2}u = |u|^{s-2}u, & x \in \mathbb{R}^N, \\
u(x) \to 0, & \text{as } |x| \to +\infty,
\end{cases}
\]

(P)

where \( N \geq 2 \), \( \phi(t) \) behaves like \( t^{q/2} \) for small \( t \) and \( t^{p/2} \) for large \( t \), \( 1 < p < q < N \), \( 1 < \alpha \leq p^* q' / p' \) and \( \max\{q, \alpha\} < s < p^* = \frac{pN}{N-p} \), being \( p' \) and \( q' \) the conjugate exponents, respectively, of \( p \) and \( q \).

Our aim is to approach the problem variationally by using the tools of critical points theory. A non-trivial difficulty, which immediately appears, consists in identifying the right functional setting for the problem. Solutions of (P) are, at least formally, the critical points of the functional

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \phi(|\nabla u|^2) + \frac{1}{\alpha} \int_{\mathbb{R}^N} |u|^\alpha - \frac{1}{s} \int_{\mathbb{R}^N} |u|^s,
\]

where

\[
\phi(|\nabla u|^2) \simeq \begin{cases} 
|\nabla u|^p, & \text{if } |\nabla u| \gg 1, \\
|\nabla u|^q, & \text{if } |\nabla u| \ll 1.
\end{cases}
\]

This different growth at zero and at infinity of the principal part and the unboundedness of the domain advise us not to use classical Sobolev spaces and to introduce a new functional framework. So, we define a sort of Orlicz-Sobolev space with respect to which the functional is well defined and \( C^1 \). In this direction, a first step is to show, by suitable embedding theorems, that all the parts of the functional are finite and controlled by the norm of our space. In particular, since there are power-like nonlinearities, we need to study the embedding of our space into a Lebesgue ones. At this stage, we look at the results obtained in [3,4,9] on the sum of Lebesgue spaces to recover some useful known properties on our space and prove a fundamental continuous embedding theorem.

Afterwards we deal with the compactness properties of the functional. Actually, as in the situation of semilinear elliptic equations, the main difficulty to get compactness lies in the fact that in unbounded domains the group of translations constitutes an obstruction to compact embeddings. To overcome this difficulty, as in [5,25], we need to constrain the functional to a suitable space which is not invariant with respect to the translations. In view of this, we restrict the domain of the functional to the Orlicz-Sobolev space obtained by density starting from radially symmetric test functions. Proceeding in analogy with the well known result due to Strauss [25], we are able to get uniformly decaying estimates that we use to show that our space compactly embeds into certain Lebesgue spaces. As a consequence it is easy
to check that our functional satisfies the Palais-Smale condition, a first step in view of the
application of the Mountain Pass Theorem.

At this point, we test the geometrical assumptions of Mountain Pass Theorem in this
setting and get our goal.

In order to state more precisely our results, let $1 < p < q$ and $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be such that

1. $\phi(0) = 0$;
2. there exists a positive constant $c$ such that
   \[
   \begin{cases}
   ct^p \leq \phi(t), & \text{if } t \geq 1, \\
   ct^q \leq \phi(t), & \text{if } 0 \leq t \leq 1;
   \end{cases}
   \]
3. there exists a positive constant $C$ such that
   \[
   \begin{cases}
   \phi(t) \leq Ct^p, & \text{if } t \geq 1, \\
   \phi(t) \leq Ct^q, & \text{if } 0 \leq t \leq 1;
   \end{cases}
   \]
4. there exists $0 < \mu < 1$ such that
   \[
   \phi'(t)t \leq \frac{s\mu}{2}\phi(t), \quad \text{for all } t \geq 0;
   \]
5. the map $t \mapsto \phi(t^2)$ is strictly convex.

**Remark 1.1** Observe that by (Φ4) and (Φ5), we infer that

$$\phi(t) < 2\phi'(t)t \leq s\mu\phi(t), \quad \text{for all } t > 0,$$

and so $s\mu > 1$.

**Remark 1.2** As an example, a function that satisfies all the previous assumptions is

$$\phi(t) = \frac{2}{p} \left[ (1 + t^q)^\frac{p}{q} - 1 \right].$$

In this case the problem (P) becomes

\[
\begin{cases}
-\nabla \cdot \left( (1 + |\nabla u|^q)^\frac{p}{q} - 1 |\nabla u|^q - 2 \nabla u \right) + |u|^q - 2u = |u|^\alpha - 2u, & x \in \mathbb{R}^N, \\
u(x) \to 0, & \text{as } |x| \to +\infty.
\end{cases}
\]

Let us observe that even if Franchi, Lanconelli & Serrin in [11] treat a very general quasilinear equation, Theorem A does not apply to problem (2) since, with their notation, in this case we have $H(\infty) = \Omega(\infty) = F(\gamma) = \infty$ but $\lim_{u \to \infty} \frac{H^{-1}(F(u))}{u} = \infty$.

Now we state our main results.

**Theorem 1.3** Assuming that $1 < p < q < \min\{N, p^*\}$, $1 < \alpha \leq p^*q'/p'$, $\max\{q, \alpha\} < s < p^*$ and (Φ1 – Φ5), there exists a nontrivial non-negative radially symmetric solution of (P).

**Remark 1.4** By well known results by Pucci, Serrin & Zou [23,24], if $\alpha \geq q$, we infer that the solution found is positive; on the contrary it has compact support, if $\alpha < q$ and requiring in addition that there exists $1 < \nu \leq s\mu$ such that

$$\frac{\nu}{2}\phi(t) \leq \phi'(t)t,$$

for $t$ sufficiently small.
Theorem 1.5 Assuming that \( 1 < p < q < \min\{N, p^*\} \), \( 1 < \alpha \leq p^*q'/p' \), \( \max\{q, \alpha\} < s < p^* \) and \((\Phi1 - \Phi5)\), there exist infinitely many radially symmetric solutions of \((\mathcal{P})\).

Remark 1.6 Any couple \((p, q)\) in the interior of the coloured region in Figure 1 is admissible for our problem.

Remark 1.7 Theorem 1.3 holds also if in the right hand side of \((\mathcal{P})\), instead of a pure power nonlinearity, we consider a more general one which satisfies the Ambrosetti-Rabinowitz growth condition. More precisely, using slightly modified arguments, we can treat the following problem

\[
\begin{cases}
-\nabla \cdot \left( \phi'(|\nabla u|^2) \nabla u \right) + |u|^\alpha - 2u = f(u), & x \in \mathbb{R}^N, \\
u(x) \to 0, & \text{as } |x| \to \infty,
\end{cases}
\]

where \( f \in C(\mathbb{R}, \mathbb{R}) \) satisfies

(f1) \( f(t) = o(t^{\alpha-1}) \), as \( t \to 0^+ \),

(f2) \( f(t) = o(t^{p^*-1}) \), as \( t \to +\infty \),

(f3) if \( F(t) = \int_0^t f(z)dz \), there exists \( t > a \) such that

\[
0 < \theta F(t) \leq f(t)t, \quad \text{for all } t > 0,
\]

(f4) \( \lim_{t \to +\infty} \frac{F(t)}{t^q} = +\infty \), if \( \alpha < q \), and \( \phi \in C^1(\mathbb{R}_+, \mathbb{R}_+) \) satisfies \((\Phi1), (\Phi2), (\Phi3), (\Phi5)\) and \((\Phi4')\) there exists \( 0 < \mu < 1 \) such that

\[
2\phi'(t)t \leq \theta \mu \phi(t), \quad \text{for all } t \geq 0.
\]

Theorem 1.5 holds requiring also that

(f5) \( f \) is odd.

The paper is organized in the following way.

In Sect. 2 we introduce the functional framework and list some fundamental properties of the space. In particular in this part we study the relation between our space and the classical Lebesgue spaces and provide new continuous and compact embedding theorems.

In Sect. 3 we verify that the functional has a good geometry and compactness to apply both the classical Mountain Pass Theorem and its \( \mathbb{Z}_2 \)-symmetric version. We also show that, strengthening a little bit our assumptions, we are able to prove the existence of a ground state solution in the set of all the radially symmetric solutions.
2 The functional setting

This section is devoted to the construction of the functional setting.

As a first step, we will recall some well known facts on the sum of Lebesgue spaces.

**Definition 2.1** Let $1 < p < q$ and $\Omega \subset \mathbb{R}^N$. We denote with $L^p(\Omega) + L^q(\Omega)$ the completion of $C_c^\infty(\Omega, K)$ in the norm

$$
\|u\|_{L^p(\Omega) + L^q(\Omega)} = \inf \left\{ \|v\|_p + \|w\|_q \mid v \in L^p(\Omega), w \in L^q(\Omega), u = v + w \right\}. 
$$

(3)

We set $\|u\|_{p,q} = \|u\|_{L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)}$.

The spaces $L^p(\Omega) + L^q(\Omega)$ are extensively studied in [3, Sect. 2], where a slightly different definition is given, but it can be easily shown that the two definitions are equivalent. Moreover, in [3], it has been shown that $L^p(\Omega) + L^q(\Omega)$ are Orlicz spaces.

In the next proposition we give a list of properties that will be useful in the rest of the paper.

**Proposition 2.2** Let $\Omega \subset \mathbb{R}^N$, $u \in L^p(\Omega) + L^q(\Omega)$ and $\Lambda_u = \{x \in \Omega \mid |u(x)| > 1\}$. We have:

(i) if $\Omega' \subset \Omega$ is such that $|\Omega'| < +\infty$, then $u \in L^p(\Omega')$;

(ii) if $\Omega' \subset \Omega$ is such that $u \in L^\infty(\Omega')$, then $u \in L^q(\Omega')$;

(iii) $|\Lambda_u| < +\infty$;

(iv) $u \in L^p(\Lambda_u) \cap L^q(\Lambda_u^-)$;

(v) the infimum in (3) is attained;

(vii) $L^p(\Omega) + L^q(\Omega)$ is reflexive and $(L^p(\Omega) + L^q(\Omega))' = L^{p'}(\Omega) \cap L^{q'}(\Omega)$;

(viii) $\|u\|_{L^p(\Omega) + L^q(\Omega)} \leq \max\{\|u\|_{L^p(\Lambda_u)}$, $\|u\|_{L^q(\Lambda_u^-)}\}$.

Proof For the proof of properties (i)–(vii) we refer to [3, Sect. 2]. Here we give only the proof of (viii).

Let $u \in L^p(\Omega) + L^q(\Omega)$. Obviously $u|_B \in L^p(\Omega) + L^q(\Omega)$ and $u|_{\Omega \setminus B} \in L^p(\Omega \setminus B) + L^q(\Omega \setminus B)$. So, by (v), we can consider $v_1 \in L^p(B)$, $v_2 \in L^q(\Omega \setminus B)$, $w_1 \in L^q(B)$ and $w_2 \in L^q(\Omega \setminus B)$ such that

$$
\begin{align*}
u &= v_1 + w_1 \quad \text{on } B, \\
u &= v_2 + w_2 \quad \text{on } \Omega \setminus B,
\end{align*}
$$

Then, if

$$
v = \begin{cases} v_1 & \text{in } B \\ v_2 & \text{in } \Omega \setminus B \end{cases} \quad \text{and} \quad w = \begin{cases} w_1 & \text{in } B \\ w_2 & \text{in } \Omega \setminus B \end{cases},
$$

we have that $v \in L^p(\Omega)$, $w \in L^q(\Omega)$, $u = v + w$ and
\[
\|u\|_{L^p(\Omega) + L^q(\Omega)} \leq \|v\|_{L^p(\Omega)} + \|w\|_{L^q(\Omega)} \\
\leq \|v_1\|_{L^p(B)} + \|v_2\|_{L^p(\Omega \setminus B)} + \|w_1\|_{L^q(B)} + \|w_2\|_{L^q(\Omega \setminus B)} \\
= \|u\|_{L^p(B) + L^q(B)} + \|u\|_{L^p(\Omega \setminus B) + L^q(\Omega \setminus B)}.
\]

We can now define the Orlicz-Sobolev space where we will study our problem.

**Definition 2.3** Let and \( \alpha > 1 \). We denote with \( \mathcal{W} \) the completion of \( C_c^\infty(\mathbb{R}^N, \mathbb{R}) \) in the norm

\[
\|u\| = \|u\|_\alpha + \|\nabla u\|_{p,q}.
\]

Let us now study some properties of the space \( \mathcal{W} \).

**Proposition 2.4** \((\mathcal{W}, \| \cdot \|)\) is a Banach space.

**Proof** Let \( \{u_n\}_n \) be a Cauchy sequence in \( \mathcal{W} \). Then \( \{u_n\}_n \) is a Cauchy sequence in \( L^\alpha(\mathbb{R}^N) \) and \( \{\nabla u_n\}_n \) is a Cauchy sequence in \( L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N) \). Since \( L^\alpha(\mathbb{R}^N) \) is complete, there exists \( u \in L^\alpha(\mathbb{R}^N) \) such that \( \lim_n u_n = u \) in \( L^\alpha(\mathbb{R}^N) \). Since \( L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N) \) is complete, then there exists \( a \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N) \) such that \( \lim_n \nabla u_n = a \) in \( L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N) \). We want to prove that \( \nabla u = a \) in the distributions sense, i.e. that for every \( \varphi \in C_c^\infty(\mathbb{R}^N) \)

\[
\int_\mathbb{R}^N u \nabla \varphi = - \int_\mathbb{R}^N \varphi a.
\]

Obviously, for every \( \varphi \in C_c^\infty(\mathbb{R}^N) \) and for every \( n \in \mathbb{N} \)

\[
\int_\mathbb{R}^N u_n \nabla \varphi = - \int_\mathbb{R}^N \varphi \nabla u_n.
\]

So it is sufficient to prove that

\[
\lim_n \int_\mathbb{R}^N u_n \nabla \varphi = \int_\mathbb{R}^N \nabla \varphi \quad \text{and} \quad \lim_n \int_\mathbb{R}^N \varphi \nabla u_n = \int_\mathbb{R}^N \varphi a.
\]

Since \( \lim_n u_n = u \) in \( L^\alpha(\mathbb{R}^N) \), then

\[
\left| \int_\mathbb{R}^N (u_n - u) \nabla \varphi \right| \leq \|\nabla \varphi\|_{\alpha'} \|u_n - u\|_\alpha \to 0.
\]

Moreover, for every \( n \in \mathbb{N} \), from (v) of Proposition 2.2, we can consider \((v_n, w_n) \in L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N)\) such that

\[
\nabla u_n - a = v_n + w_n \quad \text{and} \quad \|\nabla u_n - a\|_{p,q} = \|v_n\|_p + \|w_n\|_q.
\]

Since \( \lim_n \|\nabla u_n - a\|_{p,q} = 0 \), then \( \lim_n \|v_n\|_p = \lim_n \|w_n\|_q = 0 \). Thus, if \( \varphi \in C_c^\infty(\mathbb{R}^N) \)

\[
\left| \int_\mathbb{R}^N \varphi (\nabla u_n - a) \right| = \left| \int_\mathbb{R}^N \varphi v_n + \int_\mathbb{R}^N \varphi w_n \right| \\
\leq \|\varphi\|_{p'} \|v_n\|_p + \|\varphi\|_{q'} \|w_n\|_q \to 0.
\]

\(\square\)
Theorem 2.6

If \( A \) as in [3], on \( W \) to \( \parallel \nabla \cdot \parallel ^* \) with \( \bar{\alpha} \) with \( \parallel \cdot \parallel \). Let

By (vi) of Proposition 2.2 we have

If we take \( t \) (\( \parallel \cdot \parallel \) is uniformly convex and then \( W \) such that \( \parallel \cdot \parallel ^* \) and the \( L^{\alpha} (\mathbb{R}^N) \) norm. By a well known general result, also the norm

is uniformly convex and then \( (W, \parallel \cdot \parallel ^*) \) is reflexive. But, since the norm \( \parallel \cdot \parallel ^* \) is equivalent to \( \parallel \cdot \parallel \), then, also \( (W, \parallel \cdot \parallel) \) is reflexive. \( \square \)

Adapting some classical arguments (see e.g. [6]) we prove the following embedding result.

Theorem 2.6 If \( 1 < p < \min(q, N) \) and \( 1 < \frac{p^* q'}{p} \) then, for every \( \alpha \in \left( 1, \frac{p^* q'}{p} \right) \), the space \( W \) is continuously embedded into \( L^{p^*} (\mathbb{R}^N) \).

Proof Let \( \varphi \in C_c^\infty (\mathbb{R}^N) \) and \( t \geq 1 \). It can be proved that (see [6, page 280])

By (vi) of Proposition 2.2 we have

If we take \( t \) such that \( \frac{tN}{N-1} = \frac{p}{p-1} (t-1) \), the inequality (4) can be written as

with \( \bar{\alpha} = \frac{Nq (p-1)}{(q-1)(N-p)} = \frac{p^* q'}{p} \).

We notice that, since \( p < q \), then \( \bar{\alpha} < p^* \).

Moreover, for every \( i = 1, \ldots, N \),

\[
\inf \left\{ \| v_i \|_p + \| w_i \|_q \mid v_i \in L^p (\mathbb{R}^N), w_i \in L^q (\mathbb{R}^N), \frac{\partial \varphi}{\partial x_i} = v_i + w_i \right\} \\
\leq \inf \left\{ \| v \|_p + \| w \|_q \mid v \in L^p (\mathbb{R}^N), w \in L^q (\mathbb{R}^N), \nabla \varphi = v + w \right\}.
\]

Then

\[
\prod_{i=1}^N \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{p,q}^{1/N} \leq \| \nabla \varphi \|_{p,q}.
\]
Proof Defining any $τ$ functions with compact support. We will follow some ideas of [15].

Let $W$ be a more precise description of the space $\sigma$. Applying (6) to $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$

$$\|\varphi\|_{p^\ast} \leq C \left(\|\varphi\|_{t-1}^{t-1} + \|\varphi\|_{p^\ast}^{t-1}\right)\|\nabla \varphi\|_{p,q}.$$ 

(6)

We claim that the previous inequality holds for any $u \in W$. Indeed, for every $u \in W$, we can consider a sequence $\{\varphi_n\}_n$ such that $\lim_n \varphi_n = u$ in $W$. Then

$$\lim_{n} \nabla \varphi_n = \nabla u \text{ in } L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N),$$

$$\lim_{n} \varphi_n = u \text{ in } L^\alpha(\mathbb{R}^N),$$

and

$$\lim_{n} \varphi_n = u \text{ a.e. in } \mathbb{R}^N.$$ 

(7)

Applying (6) to $\varphi = \varphi_n - \varphi_m$ for $n, m \geq 1$, we deduce that $\{\varphi_n\}_n$ is a Cauchy sequence in $L^{p^\ast}(\mathbb{R}^N)$ and, as a consequence, it converges in $L^{p^\ast}(\mathbb{R}^N)$ to a function $v$. On the other hand, by (7) and the uniqueness of the limit a.e., we deduce that $v = u$ and then

$$\lim_{n} \varphi_n = u \text{ in } L^{p^\ast}(\mathbb{R}^N).$$

So, applying (6) to $\{\varphi_n\}_n$ and passing to the limit we deduce our claim.

The continuous embedding $W \hookrightarrow L^{p^\ast}(\mathbb{R}^N)$ can be deduced reasoning as follows: if $\{u_n\}_n$ is a sequence in $W$ that converges to $u$ in $W$, we have that

$$\|u_n - u\|_{t^\ast} \leq C \left(\|u_n - u\|_{t-1}^{t-1} + \|u_n - u\|_{t^\ast}^{t-1}\right)\|\nabla u_n - \nabla u\|_{p,q}$$

and then $\lim_n u_n = u$ in $L^{p^\ast}(\mathbb{R}^N)$. ∎

Remark 2.7 By interpolation we have that $W$ is continuously embedded into $L^{r}(\mathbb{R}^N)$ for any $r \in [\alpha, \ p^\ast]$.

Requiring something more with respect to the assumptions of Theorem 2.6, we could have a more precise description of the space $W$.

Theorem 2.8 If $1 < p < \min\{q, N\}$, $1 < p^\ast q^\prime/p$ and $q < p^\ast$, then, for every $\alpha \in \left(1, p^\ast q^\prime/p\right]$, we have that

$$W = \{u \in L^\alpha(\mathbb{R}^N) \cap L^{p^\ast}(\mathbb{R}^N) \mid \nabla u \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)\}.$$ 

Proof Defining

$$\tilde{W} = \{u \in L^\alpha(\mathbb{R}^N) \cap L^{p^\ast}(\mathbb{R}^N) \mid \nabla u \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)\},$$

we have to show that $W = \tilde{W}$. By definition of $W$ and by Theorem 2.6, we have that $W \subset \tilde{W}$. Now, let $u \in \tilde{W}$, we have to prove that it can be approximated in the $W^\prime$-norm by smooth functions with compact support. We will follow some ideas of [15].

As a first step, we prove that $u$ can be approximated in the $W^\prime$-norm by compact support functions. Let $k : \mathbb{R}^N \to [0, 1]$ be a test function such that $k \equiv 1$ in $|x| \leq 1$ and $k \equiv 0$ in $|x| \geq 2$. For any $M > 0$, define $u_M = k_M u$, where $k_M(x) = k(x/M)$, and set $A_M = \{x \in \mathbb{R}^N \mid M \leq |x| \leq 2M\}$. Certainly $u_M$ has a compact support and it is in $L^\alpha(\mathbb{R}^N) \cap L^{p^\ast}(\mathbb{R}^N)$. Moreover, since $\nabla u_M = k_M \nabla u + u \nabla k_M$, we have that $\nabla u_M \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ if both the terms of the
sum are in $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$. Since $k_M \in L^\infty(\mathbb{R}^N)$ and $\nabla u \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$, of course $k_M \nabla u \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$. Since $\nabla k_M$ vanishes in $A_M^C$, $|A_M| < +\infty$, $\nabla k_M \in L^\infty(A_M)$ and $u \in L^{p^*}(\mathbb{R}^N)$, we deduce that also $u \nabla k_M \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$. We have easily that

$$\|u - v_M\|_q^q \leq \int_{B^c_M} |u(x)|^q \, dx = o_M(1),$$

where $o_M(1)$ denotes vanishing functions as $M \to +\infty$. Then we have to show that

$$\|\nabla u - \nabla v_M\|_{p,q} = o_M(1),$$
too. Using (iv) of Proposition 2.2, we deduce that $\nabla u \in L^p(A\nabla u) \cap L^q(A\nabla u^C)$. Let us observe that

$$\|\nabla u\|_{L^p(A\nabla u \cap B^c_M)} + \|\nabla u\|_{L^q(A\nabla u \cap B^c_M)} = o_M(1), \quad (8)$$

and

$$\|u\|_{L^{p^*}(A_M)} = o_M(1). \quad (9)$$

Since $u \in L^{p^*}(\mathbb{R}^N)$, by Hölder inequality, we get that $\nabla v_M \in L^p(A\nabla u \cap B^c_M) \cap L^q(A\nabla u^C \cap B^c_M)$ and by (8) and (viii) of Proposition 2.2, we have

$$\|\nabla u - \nabla v_M\|_{p,q} \leq \|\nabla u - \nabla v_M\|_{L^p(B^c_M) + L^q(B^c_M)} + \|\nabla u - \nabla v_M\|_{L^p(B^c_M) + L^q(B^c_M)}$$

$$\leq \|\nabla u - \nabla v_M\|_{L^p(A\nabla u \cap B^c_M) + L^q(A\nabla u \cap B^c_M)} + \|\nabla u - \nabla v_M\|_{L^p(A\nabla u^C \cap B^c_M) + L^q(A\nabla u^C \cap B^c_M)}$$

$$\leq \|\nabla v_M\|_{L^p(A\nabla u \cap B^c_M)} + \|\nabla v_M\|_{L^q(A\nabla u^C \cap B^c_M)} + o_M(1)$$

$$\leq \|u\nabla k_M\|_{L^p(A_M)} + \|u\nabla k_M\|_{L^q(A_M)} + o_M(1).$$

Since

$$\|u\nabla k_M\|_{L^p(A_M)} \leq \frac{C}{M} \|u\|_{L^{p^*}(A_M)} |A_M|^{\frac{1}{p^*}},$$

and $|A_M| = O(M^N)$, as $M \to +\infty$, by (9), we have that

$$\|u\nabla k_M\|_{L^p(A_M)} = o_M(1).$$

Analogously, if $p < q < p^*$,

$$\|u\nabla k_M\|_{L^q(A_M)} \leq \frac{C}{M} \|u\|_{L^{p^*}(A_M)} |A_M|^{\frac{1}{q^*} - \frac{1}{p^*}} = o_M(1).$$

Therefore, we can conclude that $v_M \to u$ in the $W$-norm, as $M \to +\infty$.

As a second step, let us show that $u \in W$ can be approximated by smooth functions.

Let $j : \mathbb{R}^N \to \mathbb{R}_+$ be in $C^\infty_c(\mathbb{R}^N)$ a function inducing a probability measure, $j_\varepsilon(x) = \varepsilon^{-N} j(\frac{x}{\varepsilon})$ and $u_\varepsilon = u * j_\varepsilon \in C^\infty_c(\mathbb{R}^N)$ the convolution product of $u$ with $j_\varepsilon$. Since $\{j_\varepsilon\}$ are approximations to the identity, certainly $u_\varepsilon \to u$ in $L^2(\mathbb{R}^N)$, as $\varepsilon \to 0$. Moreover if we write $\nabla u = a + b$, with $a \in L^p(\mathbb{R}^N)$ and $b \in L^q(\mathbb{R}^N)$, we have $\nabla u_\varepsilon = \nabla u * j_\varepsilon = a * j_\varepsilon + b * j_\varepsilon$, with of course $a * j_\varepsilon \in L^p(\mathbb{R}^N)$ and $b * j_\varepsilon \in L^q(\mathbb{R}^N)$. Therefore

$$\|\nabla u_\varepsilon - \nabla u\|_{p,q} \leq \|a * j_\varepsilon - a\|_p + \|b * j_\varepsilon - b\|_q \to 0.$$
The conclusion of the proof follows immediately observing that \( \{ v_{M \ast j_\varepsilon} \}_{M, \varepsilon} \) are in \( C^\infty_c(\mathbb{R}^N) \) and approximate \( u \) in the \( W \)-norm.

In order to prove some compactness results, we consider radially symmetric functions of \( W \).

**Definition 2.9** Let us denote with 
\[
(C^\infty_c(\mathbb{R}^N, \mathbb{R}))_{\text{rad}} = \{ u \in C^\infty_c(\mathbb{R}^N, \mathbb{R}) \mid u \text{ is radially symmetric} \},
\]
and let \( W_r \) be the completion of \( (C^\infty_c(\mathbb{R}^N, \mathbb{R}))_{\text{rad}} \) in the norm \( \| \cdot \| \), namely 
\[
W_r = (C^\infty_c(\mathbb{R}^N, \mathbb{R}))_{\text{rad}} \| \cdot \|.
\]

**Remark 2.10** In general it is not clear to see if \( W_r \) coincides with the set of radial functions of \( W \). While, if \( 1 < p < \min\{q, N\} \), \( 1 < \alpha < p^\ast \frac{q'}{p'} \) and \( q < p^\ast \), then, arguing as in the proof of Theorem 2.8, we can prove that the two sets are equal.

The following compact embedding result holds.

**Theorem 2.11** If \( 1 < p < q < N \) and \( 1 < p^\ast \frac{q'}{p'} \) then, for every \( \alpha \in \left( 1, p^\ast \frac{q'}{p'} \right] \), \( W_r \) is compactly embedded into \( L^\tau(\mathbb{R}^N) \) with \( \alpha < \tau < p^\ast \).

To show this result we apply [5, Theorem A.I], that we recall here.

**Theorem 2.12** Let \( P \) and \( Q : \mathbb{R} \to \mathbb{R} \) be two continuous functions satisfying 
\[
\lim_{|s| \to +\infty} \frac{P(s)}{Q(s)} = 0,
\]
\( \{ v_n \}_n \) be a sequence of measurable functions from \( \mathbb{R}^N \) to \( \mathbb{R} \) such that 
\[
\sup_n \int_{\mathbb{R}^N} |Q(v_n)| < +\infty,
\]
\[
P(v_n(x)) \to v(x) \text{ a.e. in } \mathbb{R}^N.
\]
Then \( \| P(v_n) - v \|_{L^1(B)} \to 0 \), for any bounded Borel set \( B \).

Moreover, if we have also 
\[
\lim_{s \to 0} \frac{P(s)}{Q(s)} = 0,
\]
\[
\lim_{|x| \to +\infty} \sup_n |v_n(x)| = 0,
\]
then \( \| P(v_n) - v \|_1 \to 0 \).

In order to use the previous result, we need a uniform decaying estimate on the functions of our space. The radial symmetry of the functions allows us to prove the following lemma which is the analogous of the well known result due to Strauss (see [5] or [25]).

**Lemma 2.13** If \( 1 < p < q < N \), there exists \( C > 0 \) such that for every \( u \in W_r \)
\[
|u(x)| \leq \frac{C}{|x|^{N-q}} \| \nabla u \|_{p,q}, \text{ for } |x| \geq 1.
\]

\( \square \) Springer
Proof Let \( u \in (C_c^\infty(\mathbb{R}^N))_{\text{rad}} \) and \( v \in L^p(\mathbb{R}^N) \) and \( w \in L^q(\mathbb{R}^N) \) such that \( \nabla u = v + w \). Denote by \( S^{N-1} \) the boundary of the \( N \) dimensional sphere. If \( r \geq 1 \),

\[
|u(r)| \leq \int_0^{+\infty} |u'(\rho)|d\rho
\]

\[
= \frac{1}{|S^{N-1}|} \int_{B_r^c} \frac{|\nabla u|}{|x|^{N-1}}
\]

\[
\leq \frac{1}{|S^{N-1}|} \left( \int_{B_r^c} \frac{|v|}{|x|^{N-1}} + \int_{B_r^c} \frac{|w|}{|x|^{N-1}} \right)
\]

\[
\leq \frac{1}{|S^{N-1}|} \left[ \|v\|_p \left( \int_{B_r^c} \frac{1}{(N-1)r^{p-1}} \right)^{\frac{p-1}{p}} + \|w\|_q \left( \int_{B_r^c} \frac{1}{(N-1)r^{q-1}} \right)^{\frac{q-1}{q}} \right]
\]

\[
\leq C(N, p, q) \frac{\|v\|_p}{r^{\frac{N-p}{p}}} + \frac{\|w\|_q}{r^{\frac{N-q}{q}}}
\]

\[
\leq \frac{C(N, p, q)}{r^{\frac{N-q}{q}}} (\|v\|_p + \|w\|_q).
\]

Passing to the infimum, we deduce that (10) holds for any \( u \in (C_c^\infty(\mathbb{R}^N))_{\text{rad}} \). By the density of \( (C_c^\infty(\mathbb{R}^N))_{\text{rad}} \) and the convergence a.e. in \( \mathbb{R}^N \), we have that (10) is true for every \( u \in \mathcal{W}_r. \)

Proof of Theorem 2.11 Let \( \{u_n\}_n \) be a bounded sequence in \( \mathcal{W}_r \). By Lemma 2.13 we have that \( \lim_{|x| \to +\infty} |u_n(x)| = 0 \) uniformly with respect to \( n \).

Up to a subsequence \( \{u_n\}_n \) converges weakly to a function \( u \in \mathcal{W} \).

We prove that \( u_n \to u \) a.e. in \( \mathbb{R}^N \).

For every \( n \in \mathbb{N} \) and for every \( K \subset \mathbb{R}^N \) bounded, by (i) Proposition 2.2, \( \nabla u_n \in L^p(K) \) and then \( u_n \in W^{1,\sigma}(K) \), with \( \sigma = \min(\alpha, p) \).

Moreover, if \( v_n \in L^p(\mathbb{R}^N) \) and \( w_n \in L^q(\mathbb{R}^N) \) such that \( \nabla u_n = v_n + w_n \), we have

\[
\|\nabla u_n\|_{L^p(K)} \leq \|v_n\|_{L^p(K)} + \|w_n\|_{L^p(K)}
\]

\[
\leq \|v_n\|_{L^p(K)} + C \|w_n\|_{L^q(K)} \leq C(\|v_n\|_p + \|w_n\|_q)
\]

and, passing to the infimum,

\[
\|\nabla u_n\|_{L^p(K)} \leq C\|\nabla u_n\|_{p,q}.
\]

If \( |K| \leq 1 \) the constant \( C \) does not depend on \( K \) and then \( \{u_n\}_n \) is bounded in \( W^{1,\sigma}(K) \). Thus we get that \( u_n \to u \) a.e. in \( K \). Covering \( \mathbb{R}^N \) with sets with measure less than 1, we deduce that \( u_n \to u \) a.e. in \( \mathbb{R}^N \) and \( u \in \mathcal{W}_r \).

Hence we apply Theorem 2.12 with \( P(t) = |t|^r \), \( Q(t) = |t|^q + |t|^p \), \( v_n = u_n - u \) and \( v = 0 \), and we get that \( \lim_n u_n = u \) (strongly) in \( L^r(\mathbb{R}^N) \). \( \square \)
3 Existence and multiplicity of solutions

In this section we prove our main existence and multiplicity results. From now on we suppose
that all the assumptions of Theorem 1.3 hold.

Let us define the functional $I : W \rightarrow \mathbb{R}$ as:

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \phi(|\nabla u|^2) + \frac{1}{\alpha} \int_{\mathbb{R}^N} |u|^{\alpha} - \frac{1}{s} \int_{\mathbb{R}^N} |u|^s.$$

Proposition 3.1 The functional $I$ is well defined and it is of class $C^1$.

Proof The conclusion follows easily from, for example, [9, Lemma 2.2].

We will find solutions of $(P)$ as critical points of the functional $I$.

In the following proposition, we show that the functional $I$ satisfies the geometrical
assumptions of the Mountain Pass Theorem. More precisely, we have:

Proposition 3.2 The functional $I$ verifies the following properties:

(i) $I(0) = 0$;

(ii) there exist $\rho, \bar{c} > 0$ such that $I(u) \geq \bar{c}$, for any $u \in W$ with $\|u\| = \rho$;

(iii) there exists $\bar{u} \in W$ such that $I(\bar{u}) < 0$.

Proof Trivially, $I(0) = 0$.

Let us check (ii).

If $\|u\|$ is sufficiently small, by (F2), (iv) of Proposition 2.2 and since $W \hookrightarrow L^s(\mathbb{R}^N)$, we have that

$$I(u) \geq c_1 \int_{\Lambda_{\nabla u}} |\nabla u|^q + c_2 \int_{\Lambda_{\nabla u}} |\nabla u|^p + \frac{1}{\alpha} \int_{\mathbb{R}^N} |u|^{\alpha} - \frac{1}{s} \int_{\mathbb{R}^N} |u|^s$$

$$\geq c \max\left(\int_{\Lambda_{\nabla u}} |\nabla u|^q, \int_{\Lambda_{\nabla u}} |\nabla u|^p\right) + \frac{1}{\alpha} \int_{\mathbb{R}^N} |u|^{\alpha} - \frac{1}{s} \int_{\mathbb{R}^N} |u|^s$$

$$\geq c \left[\|u\|_{p,q}^{q} + \|u\|_{\alpha}^{\alpha} - \|u\|_s^{s}\right].$$

Let us check (iii).

Let $u \in C_c^{\infty}(\mathbb{R}^N)$, then by (F3), for all $t > 0$, we get

$$I(tu) \leq C_1 \int_{\Lambda_{\nabla (tu)}} |\nabla (tu)|^q + C_2 \int_{\Lambda_{\nabla (tu)}} |\nabla (tu)|^p + \frac{1}{\alpha} \int_{\mathbb{R}^N} |tu|^{\alpha} - \frac{1}{s} \int_{\mathbb{R}^N} |tu|^s$$

$$\leq C \left[t^q \int_{\mathbb{R}^N} |\nabla u|^q + t^p \int_{\mathbb{R}^N} |\nabla u|^p + t^{\alpha} \int_{\mathbb{R}^N} |u|^{\alpha} - t^s \int_{\mathbb{R}^N} |u|^s\right].$$

Therefore, $I(tu) < 0$, for a $t$ sufficiently large.

By Remark 2.10, using the standard Palais’ result (see [21]), we infer that $W_t$ is a natural
constraint for the functional $I$. So we consider $I$ restricted to this space and we prove that
here the Palais-Smale condition holds.
Proposition 3.3 The functional $I|_{W_r}$ satisfies the Palais-Smale condition.

Proof Let $\{u_n\}_n \subset W_r$ be a PS-sequence for the $I$, namely for a suitable $\bar{c} \in \mathbb{R}$

$$I(u_n) \to \bar{c} \quad \text{and} \quad I'(u_n) \to 0 \text{ in } W'_r.$$ 

Let us show that $\{u_n\}_n$ is bounded. Indeed, by (Φ4), we have

$$\bar{c} + o_n(1)\|u_n\| = I(u_n) - \frac{1}{s} I'(u_n)[u_n]$$ 

$$= \int_{\mathbb{R}^N} \left[ \frac{1}{2} \phi(|\nabla u_n|^2) - \frac{1}{s} \phi'(|\nabla u_n|^2)|\nabla u_n|^2 \right] + \left( \frac{1}{\alpha} - \frac{1}{s} \right) \int_{\mathbb{R}^N} |u_n|^\alpha$$ 

$$\geq \frac{1 - \mu}{2} \int_{\mathbb{R}^N} \phi(|\nabla u_n|^2) + \left( \frac{1}{\alpha} - \frac{1}{s} \right) \int_{\mathbb{R}^N} |u_n|^\alpha$$ 

$$\geq c \left[ \min \left( \|\nabla u_n\|_{p,q}^q, \|\nabla u_n\|_{p,q}^p \right) + \|u_n\|_{\alpha}^{\alpha} \right].$$

Therefore, by Proposition 2.5 and Theorem 2.11, there exists $u_0 \in W_r$ such that

$$u_n \rightharpoonup u_0, \quad \text{weakly in } W_r,$$ 

$$u_n \to u_0, \quad \text{in } L^s(\mathbb{R}^N),$$ 

$$u_n \to u_0, \quad \text{a.e. in } \mathbb{R}^N.$$ 

Inspired by [16], we write $I(u) = A(u) - B(u)$, where $A(u) = A_1(u) + A_2(u)$ and

$$A_1(u) = \frac{1}{2} \int_{\mathbb{R}^N} \phi(|\nabla u|^2), \quad A_2(u) = \frac{1}{\alpha} \int_{\mathbb{R}^N} |u|^\alpha, \quad B(u) = \frac{1}{s} \int_{\mathbb{R}^N} |u|^s.$$

With these notations, we have

$$A(u_n) - B(u_n) \to \bar{c} \quad \text{and} \quad A'(u_n) - B'(u_n) \to 0 \text{ in } W'_r,$$

By (12), we infer that

$$B(u_n) \to B(u_0) \quad \text{and} \quad B'(u_n) \to B'(u_0) \text{ in } W'_r.$$

Therefore

$$A'(u_n) \to B'(u_0) \quad \text{in } W'_r. \tag{13}$$

Since $A$ is convex, we have

$$A(u_n) \leq A(u_0) + A'(u_0)[u_n - u_0],$$

and so, by (11) and (13), we get

$$\limsup_n A(u_n) \leq A(u_0).$$

Moreover, by the weak lower semicontinuity ($A$ is convex and continuous)

$$A(u_0) \leq \liminf_n A(u_n),$$

and therefore

$$A(u_n) \to A(u_0). \tag{14}$$
By (11) and arguing as in [15, page 208], we have
\begin{align}
\nabla u_n &\rightharpoonup \nabla u_0, \text{ weakly in } L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N), \\
u_n &\rightharpoonup u_0, \text{ weakly in } L^q(\mathbb{R}^N),
\end{align}
and so by the weak lower semicontinuity ($A_1$ and $A_2$ are convex and continuous)
\begin{align}
A_1(u_0) &\leq \liminf_n A_1(u_n), \\
A_2(u_0) &\leq \liminf_n A_2(u_n).
\end{align}
This, together with (14), implies that
\begin{align}
A_1(u_0) &= \lim_n A_1(u_n), \\
A_2(u_0) &= \lim_n A_2(u_n).
\end{align}
By (16) and (18), we infer that
\begin{align}
u_n &\to u_0, \text{ in } W^{r}.
\end{align}
By (15) and (17) and by [9, Lemma 2.3], we have that
\begin{align}
\nabla u_n &\rightharpoonup \nabla u_0, \text{ in } L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N).
\end{align}
Therefore
\begin{align}
u_n &\to u_0, \text{ in } W^{r}
\end{align}
and the proof is concluded.

\textbf{Proof of Theorem 1.3} The existence of a nontrivial solution follows immediately by Propositions 3.2 and 3.3.

To find a nontrivial and non-negative solution, we repeat all the previous arguments showing the existence of a nontrivial solution $\bar{u} \in W_{r}$ of the following problem
\begin{align}
\begin{cases}
-\nabla \cdot \left[ \phi'(|\nabla u|^2) \nabla u \right] + |u|^\alpha - 2u = g(u), & x \in \mathbb{R}^N, \\
\end{cases}
\end{align}
where $g : \mathbb{R} \to \mathbb{R}$ is so defined:
\begin{align}
g(t) = \begin{cases}
t^{t^{-1}}, & \text{if } t \geq 0, \\
0, & \text{if } t < 0.
\end{cases}
\end{align}
Since $\bar{u}$ solves (19) and since, by Theorem 2.8, $\bar{u}^- = \min\{\bar{u}, 0\} \in W_{r}$, multiplying the equation by $\bar{u}^-$, we get
\begin{align}
\int_{\Omega^-} \phi'(|\nabla \bar{u}|^2) |\nabla \bar{u}|^2 + |\bar{u}|^\alpha = 0,
\end{align}
where $\Omega^- = \{ x \in \mathbb{R}^N \mid \bar{u}(x) < 0 \}$. Since $\phi'(t) \geq 0$, for all $t \geq 0$, we argue that $\bar{u} \geq 0$ and so it is a nontrivial and non-negative solution of (P).

\textbf{Proof of Theorem 1.5} By the $\mathbb{Z}_2$-symmetric version of the Mountain Pass Theorem [1], we need only to prove that there exist $\{V_n\}_n$, a sequence of finite dimensional subspaces of $W_{r}$ with $\dim V_n = n$ and $V_n \subset V_{n+1}$, and $\{R_n\}_n$, a sequence of positive numbers, such that $I(u) \leq 0$ for all $u \in V_n \setminus B_{R_n}$.

\textcopyright Springer
Consider \( \{ \varphi_n \}_n \) a sequence of radially symmetric test functions such that, for any \( n \geq 1 \), the functions \( \varphi_1, \varphi_2, \ldots, \varphi_n \) are linearly independent. Denote by \( V_n = \text{span} \{ \varphi_1, \varphi_2, \ldots, \varphi_n \} \subset (C_0^\infty(\mathbb{R}^N, \mathbb{R}))_{\text{rad}} \subset \mathcal{W}_r \).

By (\( \Phi_3 \)) and since \( V_n \) is a finite dimensional space of test functions, we conclude observing that, if \( u \in V_n \setminus B_{R_n} \) and \( R_n \) is sufficiently large,

\[
I(u) \leq C \left[ \| \nabla u \|^q_q + \| u \|^\alpha_\alpha - \| u \|^s_s \right] \leq C \left[ R_n^q + R_n^\alpha - R_n^s \right] \leq 0.
\]

\[ \square \]

3.1 Ground state solution in \( \mathcal{W}_r \)

In this section, we will show how, requiring something slightly more on \( \varphi \), we can find a ground state solution in \( \mathcal{W}_r \).

Let us suppose that \( \varphi \) satisfies:

\( (\Phi_2') \) there exists a positive constant \( c \) such that

\[
\begin{cases}
ct \frac{p-1}{q-1} \leq \varphi'(t), & \text{if } t \geq 1, \\
ct \frac{q-1}{q-1} \leq \varphi'(t), & \text{if } 0 \leq t \leq 1.
\end{cases}
\]

Of course \( (\Phi_2') \) implies \( (\Phi_2) \).

Let us indicate with \( S \) the set of all nontrivial solutions of (\( P \)) in \( \mathcal{W}_r \), namely

\[ S = \{ u \in \mathcal{W}_r \setminus \{0\} \mid I'(u) = 0 \}. \]

By Theorem 1.3, we know that \( S \neq \emptyset \).

The following lemmas hold for the set \( S \).

**Lemma 3.4** There exists a positive constant \( \tilde{c} > 0 \) such that \( \| u \| \geq \tilde{c} \), for all \( u \in S \).

**Proof** By \( (\Phi_2') \) we have

\[
\| u \|^s_s = \int_{\mathbb{R}^N} \varphi'(|\nabla u|^2) |\nabla u|^2 + \int_{\mathbb{R}^N} |u|^\alpha \\
\geq c \max \left( \int_{\Lambda_{\nabla u}} |\nabla u|^q, \int_{\Lambda_{\nabla u}} |\nabla u|^p \right) + \int_{\mathbb{R}^N} |u|^\alpha \\
\geq c \left[ \| \nabla u \|^q_{p,q} + \| u \|^\alpha_\alpha \right] \geq c\| u \|_{\max\{\alpha,q\}}^{\max\{\alpha,q\}}.
\]

\[ \square \]

**Lemma 3.5** There exists a positive constant \( \tilde{c} > 0 \) such that \( I(u) \geq \tilde{c} \), for all \( u \in S \).

**Proof** Let \( u \in S \). Repeating the arguments of the proof of Proposition 3.3 and using by Lemma 3.4, we have

\[
I(u) = I(u) - \frac{1}{s} I'(u)[u] \geq c \left[ \min \left( \| \nabla u \|^q_{p,q}, \| \nabla u \|^p_{p,q} \right) + \| u \|^\alpha_\alpha \right] \geq \tilde{c}.
\]

\[ \square \]
Remark 3.6 Let us indicate with \( \mathcal{N} \) the Nehari manifold associated to the functional \( I \), namely

\[
\mathcal{N} = \{ u \in \mathcal{W}_r \setminus \{0\} | I'(u)[u] = 0 \}.
\]

Then Lemmas 3.4 and 3.5 hold also for \( \mathcal{N} \).

By Lemma 3.5, we infer that

\[
\sigma = \inf_{u \in S} I(u) > 0,
\]

and the next theorem shows that this infimum is achieved.

Theorem 3.7 Assuming that \( 1 < p < q < N \), \( \max\{q, \alpha\} < s < p^* \), \( 1 < \alpha \leq p^*q'/p' \) and \((\Phi_1, \Phi_2', \Phi_3 - \Phi_5)\), then in the space \( \mathcal{W}_r \), there exists a ground state solution for the problem \((P)\), namely there exists a nontrivial solution \( \bar{u} \in \mathcal{W}_r \), such that

\[
I(\bar{u}) = \min_{u \in S} I(u).
\]

Proof Let \( \{u_n\}_n \subset S \) be a minimizing sequence, namely

\[
I(u_n) \to \sigma \quad \text{and} \quad I'(u_n) = 0.
\]

Then \( \{u_n\}_n \) is a PS-sequence for \( I \) and we conclude by means of Proposition 3.3 \( \Box \)

Acknowledgments The authors are supported by M.I.U.R.-P.R.I.N. “Metodi variazionali e topologici nello studio di fenomeni non lineari” and by G.N.A.M.P.A. Project “Metodi variazionali e problemi ellittici non lineari”.

References

1. Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. J. Funct. Anal. 14, 349–381 (1973)
2. Badiale, M., Citti, G.: Concentration compactness principle and quasilinear elliptic equations in \( \mathbb{R}^n \). Commun. Partial Differ. Equ. 16, 1795–1818 (1991)
3. Badiale, M., Pisani, L., Rolando, S.: Sum of weighted Lebesgue spaces and nonlinear elliptic equations. NoDEA Nonlinear Differ. Equ. Appl. 18, 369–405 (2011)
4. Benci, V., Fortunato, D.: Towards a unified field theory for classical electrodynamics. Arch. Ration. Mech. Anal. 173, 379–414 (2004)
5. Berestycki, H., Lions, P.L.: Nonlinear scalar field equations. I. Existence of a ground state. Arch. Rational Mech. Anal. 82, 313–345 (1983)
6. Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York (2011)
7. Clément, Ph, García-Huidobro, M., Manásevich, R., Schmitt, K.: Mountain pass type solutions for quasilinear elliptic equations. Calc. Var. Partial Differ. Equ. 11, 33–62 (2000)
8. Conti, M., Gazzola, F.: Existence of ground states and free-boundary problems for the prescribed mean-curvature equation. Adv. Differ. Equ. 7, 667–694 (2002)
9. D’Aprile, T., Siciliano, G.: Magnetostatic solutions for a semilinear perturbation of the Maxwell equations. Adv. Differ. Equ. 16, 435–466 (2011)
10. del Pino, M., Guerra, I.: Ground states of a prescribed mean curvature equation. J. Differ. Equ. 241, 112–129 (2007)
11. Franchi, B., Lanconelli, E., Serrin, J.: Existence and uniqueness of nonnegative solutions of quasilinear equations in \( \mathbb{R}^n \). Adv. Math. 118, 177–243 (1996)
12. Fukagai, N., Ito, M., Narukawa, K.: Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev nonlinearity on \( \mathbb{R}^n \). Funkcial. Ekvac. 49, 235–267 (2006)
13. Fukagai, N., Narukawa, K.: On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problems. Ann. Mat. Pura Appl. (4) 186, 539–564 (2007)
14. Kusano, T., Swanson, C.A.: Radial entire solutions of a class of quasilinear elliptic equations. J. Differ. Equ. **83**, 379–399 (1990)
15. Lieb, E.H., Loss, M.: Analysis. American Mathematical Society, Providence, RI (2001)
16. Mihăilescu, M., Rădulescu, V.: Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: an Orlicz-Sobolev space setting. J. Math. Anal. Appl. **330**, 416–432 (2007)
17. Motreanu, D., Tanaka, M.: Existence of solutions for quasilinear elliptic equations with jumping nonlinearities under the Neumann boundary condition. Calc. Var. Partial Differ. Equ. **43**, 231–264 (2012)
18. Narukawa, K., Suzuki, T.: Nonlinear eigenvalue problem for a modified capillary surface equation. Funkcial. Ekvac. **37**, 81–100 (1994)
19. Ni, W.M., Serrin, J.: Existence and nonexistence theorems for ground states of quasilinear partial differential equations. The anomalous case. Accad. Naz. Lincei, Atti dei Convegni 77, 231–257 (1985)
20. Ni, W.M., Serrin, J.: Nonexistence theorems for quasilinear partial differential equations. Rend. Circ. Mat. Palermo (2) Suppl. **8**, 171–185 (1985)
21. Palais, R.S.: The principle of symmetric criticality. Commun. Math. Phys. **69**, 19–30 (1979)
22. Peletier, L.A., Serrin, J.: Ground states for the prescribed mean curvature equation. Proc. Am. Math. Soc. **100**, 694–700 (1987)
23. Pucci, P., Serrin, J., Zou, H.: A strong maximum principle and a compact support principle for singular elliptic inequalities. J. Math. Pures Appl. (9) **78**, 769–789 (1999)
24. Serrin, J., Zou, H.: Symmetry of ground states of quasilinear elliptic equations. Arch. Ration. Mech. Anal. **148**, 265–290 (1999)
25. Strauss, W.A.: Existence of solitary waves in higher dimensions. Commun. Math. Phys. **55**, 149–162 (1977)