Generalizations of Han’s Hook Length Identities

Laura L.M. Yang
Department of Mathematical and Statistical Sciences
University of Alberta, Edmonton, Alberta, Canada T6G 2G1
yanglm@hotmail.com

Abstract
Han recently discovered new hook length identities for binary trees. In this paper, we extend Han’s identities to binomial families of trees. Moreover, we present a bijective proof of one of the identities for the family of ordered trees.

1. Introduction

The hook length of a vertex $v$ of a rooted tree $T$ is the number $h_v$ of descendants of $v$ in $T$ (including $v$ itself). Several identities involving this parameter have been discovered, especially since the appearance of Postnikov’s identity in 2004 [10]; see, e.g., [1, 2, 3, 9, 11] and the references contained therein. Han [4, 5] recently found two more such identities, namely,

$$\sum_T \prod_{v \in T} \frac{1}{h_v 2^{h_v} - 1} = \frac{1}{n!}$$

and

$$\sum_T \prod_{v \in T} \frac{(z + h_v)^{h_v} - 1}{h_v (2z + h_v - 1)^{h_v} - 2} = \frac{2^n z}{n!} (z + n)^{n - 1},$$

where each sum is over the (incomplete) binary trees $T$ with $n$ vertices (in which each vertex has at most one left-child and at most one right-child). Our main object here is to extend Han’s identities to more general binomial families of trees. The definition of these families and our main results will be stated in Section 2. The proofs will be given in Section 3. Finally, in Section 4, we give a bijective proof of one of the identities for the family of ordered trees.

2. Definitions and Main Results

We recall that ordered trees are (finite) rooted trees with an ordering specified for the children of each vertex (see, e.g., Knuth [6, p. 306]). Let $s$ and $m$ be given constants such that $sm > 0$.
and $m$ is a positive integer if $s > 0$. And let $d_v$ denote the number of children of vertex $v$ in any given rooted tree $T$. If each ordered tree $T$ is assigned the weight

$$w(T) = \prod_{v \in T} \left( \frac{m}{d_v} \right)^{s^{d_v}},$$

then the resulting family of weighted ordered trees is called a binomial family or an $(s, m)$-family $F$. Let $F_n$ denote the subset of binomial trees that have $n$ vertices and let $y_n = \sum_{T \in F_n} w(T)$ denote the (weighted) number of trees in $F_n$. It follows readily from these definitions that the generating function $y = y(x) = \sum_{n=1}^\infty y_n x^n$ satisfies the relation

$$y = x(1 + sy)^m.$$

For additional remarks on these families, especially in the context of simply generated families may be sound in [9]. Notice, for example, that the binomial families include the incomplete $k$-ary and the ordered trees; but they do not include the complete binary trees in which every vertex has zero or two children. We now state our main results.

**Theorem.** Let $F_n$ denote the subset of the $(s, m)$-family of binomial trees that have $n$ vertices. Then

$$\sum_{T \in F_n} w(T) \prod_{v \in T} \frac{1}{h_v m^{h_v - 1}} = \frac{s^{n-1}}{n!}$$

and

$$\sum_{T \in F_n} w(T) \prod_{v \in T} \frac{(z + h_v)^{h_v - 1}}{h_v (mz + h_v - 1)^{h_v - 2}} = \frac{s^{n-1} m^n z^n}{n!} (z + n)^{n-1},$$

for $n = 1, 2, \ldots$.

### 3. Proof of Theorem

Let $p_n$ and $q_n$ denote the lefthand sides of identities (3) and (4) for $n = 1, 2, \ldots$. The proof will be by induction on $n$. It is easy to check that $p_1 = 1$ and $q_1 = mz$ so (3) and (4) hold when $n = 1$. Any non-trivial binomial tree $T$ with $n$ vertices in which the root has $d$ children may be constructed from an ordered collection of $d$ smaller binomial trees with $n - 1$ vertices altogether by attaching a new (root) vertex to the roots of the $d$ smaller trees and then introducing the appropriate weight factors. It follows readily from this observation and the definition of $p_n$, that if $n > 1$ then

$$p_n = \frac{1}{nm^{n-1}} \sum_{d \geq 1} \left( \frac{m}{d} \right)^{s^d} \sum_{j=0}^{d} \left( \frac{m}{d} \right)^{s^j} \cdot \frac{1}{j!} \cdot \sum_{j=0}^{d} \left( \frac{m}{d} \right)^{s^j},$$

where the inner sum is over all compositions $(j) = (j_1, \ldots, j_d)$ of $n - 1$ into $d$ positive integers. If we apply the induction hypothesis that $p_j = s_{j-1}/j!$ for $j < n$, simplify, and rewrite the righthand side of relation (5) in terms of generating functions, we find that

$$p_n = \frac{s^{n-1}}{nm^{n-1}} \sum_{d \geq 1} \left( \frac{m}{d} \right)^{s^d} \left[ x^{n-1} (e^x - 1)^d \right] = \frac{s^{n-1}}{n!}.$$
This suffices to prove identity (3).

Before proceeding to the proof of identity (4) we recall that if \( u = u(x) \) is a power series such that \( u = e^{xu} \), then it follows readily from Lagrange’s inversion formula that

\[
u^z = 1 + \sum_{n \geq 1} \frac{z(z + n)^{n-1}}{n!} x^n.
\]

(6)

for any \( z \).

We now consider identity (4) for the quantity \( q_n \). In this case the reasoning that led to relation (5) leads to the conclusion that if \( n > 1 \), then

\[q_n = \frac{(z + n)^{n-1}}{n(mz + n - 1)^{n-2}} \sum_{d \geq 1} \binom{m}{d} s^d \sum q_{j_1} \cdots q_{j_d}
\]

(7)

where the inner sum is over the same compositions \((j)\) as before. If we apply the induction hypothesis that \( q_j = s^{j-1}n^j \frac{z(z + j)^{j-1}}{j!} \) for \( j < n \), simplify, rewrite the right-hand side of relation (7) in terms of generating functions, and appeal to relation (6), we find that

\[q_n = \frac{(sm(z + n))^{n-1}}{n(mz + n - 1)^{n-2}} \sum_{d \geq 1} \binom{m}{d} [x^{n-1}] (u^z - 1)^d
\]

\[= \frac{(sm(z + n))^{n-1}}{n(mz + n - 1)^{n-2}} [x^{n-1}] (u^z - 1)^d
\]

(8)

This suffices to complete the proof of the theorem.

**Example.** The five ordered trees with \( n = 4 \) vertices are illustrated in Figure 1. If \( F \) is the

(1, \( k \))-family (of incomplete \( k \)-ary trees), then it follows from the Theorem that

\[\sum_T w(T) \prod_{v \in T} \frac{1}{h_v k^{h_v - 1}} = \frac{1}{n!}
\]

(9)

and

\[\sum_T w(T) \prod_{v \in T} \frac{(z + h_v)^{h_v - 1}}{h_v (kz + h_v - 1)^{h_v - 2}} = \frac{k^n z}{n!} (z + n)^{n-1},
\]

where the sums, here and elsewhere, are over the trees \( T \) in \( F_n \). In this case, the weights of \( T_1, T_2, T_3, T_4 \) and \( T_5 \) are \( \binom{k}{3}, \binom{k}{2} \binom{k}{1}, \binom{k}{2} \binom{k}{1} \binom{k}{1} \binom{k}{2} \) and \( \binom{k}{1}^3 \), respectively. Hence,

\[p_4 = \frac{\binom{k}{3}}{4 \cdot k^3} + \frac{\binom{k}{2} \binom{k}{1}}{4 \cdot k^3 \cdot 2 \cdot 1} + \frac{\binom{k}{2} \binom{k}{1}^2}{4 \cdot k^3 \cdot 3 \cdot k^2} + \frac{\binom{k}{1}^3}{4 \cdot k^3 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{4!}
\]
and

\[
q_4 = \frac{k^4(z + 4)^3}{4(kz + 3)^2(2kz)^{-2}} + \sum_{i=1}^{k} \frac{k^3(z + 4)^3(z + 2)}{4(kz + 3)^2 \cdot 3(kz + 2)(2kz)^{-2}} + \frac{(z + 4)^3(z + 2)}{4(kz + 3)^2 \cdot 2(kz)^{-2}} + \left(\frac{k}{1}\right) (z + 4)^3(z + 2)
\]

Notice that (8) and (9) reduce to Han’s identities when \(k = 2\).

If \(F\) is a \((-1, -k)\)-family, then

\[
\sum_T w(T) \prod_{v \in T} \frac{1}{h_v(-k)h_v^{-1}} = \frac{(-1)^{n-1}}{n!}
\]

and

\[
\sum_T w(T) \prod_{v \in T} \frac{(z + h_v)h_v^{-1}}{h_v(h_v - kz - 1)h_v^{-2}} = \frac{-k^n z}{n!} (z + n)^{n-1}.
\]

In particular, if \(F\) is the \((-1, -1)\)-family, i.e., the family of ordered trees, then

\[
\sum_T \prod_{v \in T} \frac{1}{h_v(-1)h_v^{-1}} = \frac{(-1)^{n-1}}{n!}
\]

and

\[
\sum_T \prod_{v \in T} \frac{(z + h_v)h_v^{-1}}{h_v(h_v - z - 1)h_v^{-2}} = \frac{-z}{n!} (z + n)^{n-1},
\]

where have omitted the weight factors here since they all equal one. In this case,

\[
p_4 = \frac{1}{4 \cdot (-1)^3} + \frac{1}{4 \cdot (-1)^3 \cdot 2 \cdot (-1)} + \frac{1}{4 \cdot (-1)^3 \cdot 2 \cdot (-1) \cdot 2 \cdot (-1)} = \frac{-1}{4!}
\]

and

\[
q_4 = \frac{(z + 4)^3}{4(3 - z)^2(2z)^{-3}} + \frac{(z + 4)^3(z + 2)}{4(3 - z)^2 \cdot 2(2z)^{-2}} + \frac{(z + 4)^3(z + 2)}{4(3 - z)^2 \cdot 2(z)^{-2}} + \frac{(z + 4)^3(z + 3)^2}{4(3 - z)^2 \cdot 3(2z)^{-2}} + \frac{(z + 4)^3(z + 3)^2(z + 2)}{4(3 - z)^2 \cdot 3(2z)^{-2} \cdot 2(z)^{-1}} = \frac{z(z + 4)^3}{4!}.
\]

If \(F\) is a \((1/m, m)\)-family, then

\[
\sum_T w(T) \prod_{v \in T} \frac{1}{h_v m h_v^{-1}} = \frac{1}{m^{n-1} n!}
\]

and

\[
\sum_T w(T) \prod_{v \in T} \frac{(z + h_v)h_v^{-1}}{h_v(mz + h_v - 1)h_v^{-2}} = \frac{mz}{n!} (z + n)^{n-1}.
\]
If we let \( z = 1/m \) in (15) and take the limit as \( m \) tends to infinity, we obtain the identity

\[
\sum_T \prod_{v \in T} \frac{1}{d_v!} = \frac{n^{n-1}}{n!},
\]

where the sum is over all ordered trees \( T \) with \( n \) vertices. This relation, which expresses the number \( n^{n-1} \) of rooted labelled trees with \( n \) vertices as a sum over the ordered trees with \( n \) vertices, with suitable weights taken into account, is equivalent to a relation given by Mohanty [8, p. 163].

4. An Involution on Increasing Ordered Trees

We conclude by giving a sign-reversing involution that establishes an alternate form of identity (12), namely,

\[
\sum_T n! \prod_{v \in T} \frac{1}{h_v(-1)^{h_v}} = -1,
\]

where the sum is over all ordered trees with \( n \) vertices.

It is well known that \( n!/\prod_{v \in T} h_v \) counts the number of ways to label the vertices of \( T \) with \( \{1, 2, \ldots, n\} \) such that the label of each vertex is less than that of its descendants [7, p.67, exer. 20]. Such a labelled tree is called increasing. We define the sign of a tree \( T \) to be \( \prod_{v \in T} (-1)^{h_v} \).

An increasing ordered tree \( T \) with \( n \) vertices is proper if the root of \( T \) has \( n-1 \) children and their labels are increasing from left to right. It is easy to check that the sign of any proper tree is \(-1\).

The involution is based on the non-proper increasing ordered trees. For any leaf \( v \) of a non-proper increasing ordered tree, suppose \( v \) is the \( i \)-th child of \( u \) and \( w \) is the \( i+1 \)-th child of \( u \) if it exists. We say that \( v \) is illegal if \( v \) is the rightmost child of \( u \) and the subtree rooted at \( u \) is proper or \( v \) is bigger than any vertex of the subtree rooted at \( w \) and the subtree rooted at \( w \) is proper.

Now the involution can be described as follows: Given any non-proper increasing ordered tree \( T \) let \( v \) be the first illegal leaf encountered when traversing the tree \( T \) in preorder. We now have two cases: (1) \( v \) is bigger than any vertex of the subtree rooted at \( w \) and the subtree rooted at \( w \) is proper; (2) \( v \) is the rightmost child of \( u \) and the subtree rooted at \( u \) is proper. In this case, \( u \) is not the root of \( T \).

For case (1), let \( u \) be the parent of \( v \). We cut off the edge between \( u \) and \( v \), and move \( v \) as the rightmost child of \( w \). Let \( T' \) be the resulting tree. Note that in the search process for \( T' \), the leaf \( v \) is still the first encountered illegal leaf.

For case (2), we may reverse the construction for case (1). Hence we obtain a sign-reversing involution. Figure 2 illuminates this involution on increasing ordered trees.
References

[1] W.Y.C. Chen and L.L.M. Yang, On Postnikov’s hook length formula for binary trees, European J. Combin. in press, 2008.

[2] R.R.X. Du and F. Liu, \((k, m)\)-Catalan numbers and hook length polynomials for plane trees, European J. Combin. 28 (2007) 1312–1321.

[3] I.M. Gessel and S. Seo, A refinement of Cayley’s formula for trees, Electron. J. Combin. 11(2) (2006) #R27.

[4] G.-N. Han, New hook length formulas for binary trees, arXiv:0804.3638v1.

[5] G.-N. Han, Yet another generalization of Postnikov’s hook length formula for binary trees, arXiv:0804.4268v1.

[6] D.E. Knuth, The Art of Computer Programming, Vol. 1, Addison-Wesley, Reading, 1973.

[7] D.E. Knuth, The Art of Computer Programming, Vol. 3, 2nd ed., Addison-Wesley, Reading, 1997.

[8] S.G. Mohanty, Lattice Path Counting and Applications, Academic Press, New York, 1979.

[9] J.W. Moon and L.L.M. Yang, Postnikov identities and Seo’s formulas, Bull. Inst. Combin. Appl., 49 (2007) 21–31.

[10] A. Postnikov, Permutahedra, associahedra, and beyond, Retrospective in Combinatorics: Honoring Richard Stanley’s 60th Birthday, MIT, Cambridge, Mass. June 22–24, 2004. http://www-math.mit.edu/~apost/talks/perm-slides.pdf. See also arXiv:math.CO/0507163.

[11] S. Seo, A combinatorial proof of Postnikov’s identity and a generalized enumeration of labelled trees, Electron. J. Combin. 11(2) (2005) #N3.