Research article

On $q$-analogue of meromorphic multivalent functions in lemniscate of Bernoulli domain

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Abstract: Utilizing the concepts from $q$-calculus in the field of geometric function theory, we introduce a subclass of $p$-valent meromorphic functions relating to the domain of lemniscate of Bernoulli. The well known problem of Fekete-Szego for this class is evaluated. Also some geometric results related to subordinations are evaluated for this class in connection with Janowski functions.

Keywords: meromorphic functions; $q$-calculus; lemniscate of Bernoulli; Janowski functions

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1. Introduction

1.1. Background of the research

The $q$-calculus (Quantum Calculus) is a branch of mathematics related to calculus in which the concept of limit is replaced by the parameter $q$. This field of study has motivated the researchers in the recent past with its numerous applications in applied sciences like Physics and Mathematics, e.g., optimal control problems, the field of ordinary fractional calculus, $q$-transform analysis, $q$-difference and $q$-integral equations. The applications of $q$-generalization in special functions and quantum physics are of high value which makes the study pertinent and interesting in these fields. While the $q$-difference operator has a vital importance in the theory of special functions and quantum theory,
number theory, statistical mechanics, etc. The \( q \)-generalization of the concepts of differentiation and integrations were introduced and studied by Jackson [1]. Similarly, Aral and Gupta [2, 3] used some what similar concept by introducing the \( q \)-analogue of operator of Baskakov Durmeyer by using \( q \)-beta function. Later, Aral and Anastassiu et. al. in [4, 5] generalized some complex operators, \( q \)-Gauss-Weierstrass singular integral and \( q \)-Picard operators. For more details on the topic one can see, for example [6–17]. Some of latest inovations in the field can be seen in the work of Arif et al. [18] in which they investigated the \( q \)-generalization of Harmonic starlike functions. While Srivastava with his co-authors in [19, 20] investigated some general families in \( q \)-analogue related to Janowski functions and obatained some interesting results. Later, Shafiq et al. [21] extended this idea of generalization to close to convex functions. Recently, more research seem to have diversifed this field with the introduction of operator theory. Some of the details of such work can be seen in the work of Shi and co-authors [22]. Also some new domains have been explored such as Sine domain in the recent work [23]. Motivated from the discussion above we utilize the concepts of \( q \)-calculus and introduce a subclass of \( p \)-valent meromorphic functions and investigate some of their nice geometric properties.

1.2. Preliminaries

Before going into our main results we give some basic concepts relating to our work. Let \( \mathcal{M}_p \) represents the class of meromorphic multivalent functions which are analytic in \( \mathcal{D}^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \) with the representation

\[
f(z) = \frac{1}{z^p} + \sum_{k=p+1}^{\infty} a_k z^k, \quad (z \in \mathcal{D}^*). \tag{1.1}
\]

Let \( f(z) \) and \( g(z) \) be analytic in \( \mathcal{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). Then the function \( f(z) \) is subordinated to \( g(z) \) in \( \mathcal{D} \), written as \( f(z) < g(z) \), if there exist a Schwarz function \( \omega(z) \) such that \( f(z) = g(\omega(z)) \), where \( \omega(z) \) is analytic in \( \mathcal{D} \), with \( w(0) = 0 \) and \( w(z) < 1, z \in \mathcal{D} \).

Let \( \mathcal{P} \) denote the class of analytic function \( l(z) \) normalized by

\[
l(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.2}
\]

such that \( Re(z) > 0 \).

We now consider a class of functions in the domain of lemniscate of Bernoulli. All functions \( l(z) \) will belong to such a class if it satisfy;

\[
h(z) < \sqrt{1 + z}. \tag{1.3}
\]

These functions lie in the right-half of the lemniscate of Bernoulli and with this geometrical representation is the reason behind this name.

With simple calculations the above can be written as

\[
|h(z)| < \sqrt{1 + z}.
\]

Similarly, \( SL^* \), in parallel comparison to starlike functions, for analytic functions is

\[
SL^* = \left\{ f(z) \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \sqrt{1 + z} \right\}, \tag{1.4}
\]
where $\mathcal{A}$ represents the class of analytic functions and $z \in \mathbb{D}$. Alternatively
\[
S.L' = \left\{ f(z) \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \right\},
\]
Sokol and Stankiewicz [24] introduced this along with some properties. Further study on this was made by different authors in [25–27]. Upper bounds for the coefficients of this class are evaluated in [28].

An important problem in the field of analytic functions is to study a functional
\[
\left| a_3 - v a_2^2 \right|
\]
called the Fekete-Szegö functional. Where $a_2$ and $a_3$ the coefficients of the original function with a parameter $v$ over which the extremal value of the functional is evaluated. The problem of obtaining the upper bound of this functional for subclasses of normalized functions is called the Fekete-Szegö problem or inequality. M. Fekete and G. Szegö [29], were the first to estimate this classical functional for the class $S$. While Pfluger [30] utilized Jenkin’s method to prove that this result holds for complex $\mu$ such that $Re\left\{ \mu \right\} \geq 0$. For other related material on the topic reader is referred to [31–33].

Similarly the class of Janowski functions is defined for the function
\[
J(z) = \frac{1 + A z}{1 + B z}
\]
equivalently the functions of this class satisfies
\[
\left| \frac{J(z) - 1}{A - BJ(z)} \right| < 1
\]
more details on Janowski functions can be seen in [34].

The $q$-derivative, also known as the $q$-difference operator, for a function is
\[
D_q f(z) = \frac{f(qz) - f(z)}{z(q-1)},
\]
with $z \neq 0$ and $0 < q < 1$. With simple calculations for $n \in \mathbb{N}$ and $z \in \mathbb{D}^*$, one can see that
\[
D_q \left( \sum_{n=1}^{\infty} a_n z^n \right) = \sum_{n=1}^{\infty} \left[ n \right]_q a_n z^{n-1},
\]
with
\[
\left[ n \right]_q = \frac{1 - q^n}{1 - q} = 1 + \sum_{i=1}^{n-1} q^i \quad \text{and} \quad [0]_q = 0.
\]
Now we define our new class and we discuss the problem of Fekete-Szegö for this class. Some geometric properties of this class related to subordinations are discussed in connection with Janowski functions.

1.3. The class $\textbf{MSL}^*_p,q$

We introduce $\textbf{MSL}^*_p,q$, a family of meromorphic multivalent functions associated with the domain of lemniscate of Bernoulli in $q$-analogue as:
If \( f(z) \in M_p \), then it will be in the class \( \mathcal{MSL}_p^* \) if the following holds

\[
- \frac{q^p z D_q f(z)}{[p]^q f(z)} < \sqrt{1 + z}, \tag{1.7}
\]

we note that \( \lim_{q \to 1^+} \mathcal{MSL}_p^* = \mathcal{MSL}_p^* \), where

\[
\mathcal{MSL}_p^* = \left\{ f(z) \in M_p : -\frac{zf'(z)}{pf(z)} < \sqrt{1 + z}, z \in \mathbb{D}^* \right\}.
\]

In this research article we investigate some properties of meromorphic multivalent functions in association with lemniscate of Bernoulli in \( q \)-analogue. The important inequality of Fekete-Szegő is evaluated in the beginning of main results. Then we evaluate some bounds of \( \xi \) which associate \( 1 + \xi \frac{zD_q f(z)}{[p]^q f(z)} \) with Janowski functions and \( z^n f(z) < \sqrt{1 + z} \).

Utilizing these theorems along with some conditions we prove that a function may be a member of \( \mathcal{MSL}_p^* \).

2. Sets of Lemma

The following Lemmas are important as they help in our main results.

2.1. Lemma

[35]. If \( l(z) \) is in \( \mathcal{P} \) given by (1.2), then

\[
\left| p_2 - \lambda p_1^2 \right| \leq 2 \max \{ 1, |2\lambda - 1| \}, \nu \in \mathbb{C}.
\]

2.2. Lemma

[35]. If \( l(z) \) is in \( \mathcal{P} \) given by (1.2), then

\[
\left| p_2 - \nu p_1^2 \right| \leq \begin{cases} -4\nu + 2 & (\nu \leq 0), \\ 2 & (0 \leq \nu \leq 1), \\ 4\nu - 2 & (\nu \geq 1). \end{cases}
\]

2.3. Lemma

[36]. (q-Jack’s lemma) For an analytic function \( \omega(z) \) in \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) with \( \omega(0) = 0 \). If \( |\omega(z)| \) attains its maximum value on the circle \( |z| = r \) at a point \( z_0 = re^{i\theta} \), for \( \theta \in [-\pi, \pi] \), we can write that for \( 0 < q < 1 \)

\[
z_0 D_q \omega(z_0) = m \omega(z_0),
\]

with \( m \) is real and \( m \geq 1 \).

3. Main results

In this section we start with Fekete-Szegő problem in the first two theorems. Then some important results relating to subordination are proved using q-Jack’s Lemma and with the help of these results the functions are shown to be in the class of \( \mathcal{MSL}_p^* \) in the form of some corollaries.
3.1. Theorem

Let \( f \in \mathcal{MSL}_{p,q} \) and are of the form (1.1), then

\[
|a_{p+2} - \lambda a_{p+1}^2| \leq \frac{[p]_q}{2([p+2]_q - [p]_q)} \max \{1, |\mu|\},
\]

where

\[
\mu = \frac{(q^p ([p+1]_q)^2 + 3q^p ([p]_q)^2 - 2\lambda ([p]_q)^2 + 2\lambda [p+2]_q [p]_q - 4q^p [p+1]_q [p]_q)^2}{4q^p ([p+2]_q - [p]_q)}. \]

Proof. Let \( f \in \mathcal{MSL}_{p,q} \), then we have

\[
-q^p z D_q f(z) = \frac{[p]_q}{f(z)} = \sqrt{1 + \omega(z)}, \tag{3.1}
\]

where \(|\omega(z)| \leq 1 \) and \( \omega(0) = 0 \). Let

\[
\Phi(\omega(z)) = \sqrt{1 + \omega(z)}. \]

Thus for

\[
l(z) = 1 + p_1 z + p_2 z^2 + \cdots = \frac{1 + \omega(z)}{1 - \omega(z)}, \tag{3.2}
\]

we have \( l(z) \) is in \( \mathcal{P} \) and

\[
\omega(z) = \frac{p_1 z + p_2 z^2 + p_3 z^3 + \cdots}{2 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots} = \frac{l(z) - 1}{l(z) + 1}.
\]

Now as

\[
\sqrt{\frac{2l(z)}{l(z) + 1}} = 1 + \frac{1}{4} p_1 z + \left(\frac{1}{4} p_2 - \frac{5}{32} p_1^2\right) z^2 + \cdots.
\]

So from (3.1), we get

\[
-q^p z D_q f(z) = [p]_q \sqrt{\frac{2l(z)}{l(z) + 1}} f(z),
\]

thus

\[
\frac{[p]_q}{z^p} - q^p \sum_{k=p+1}^{\infty} [k]_q a_k z^k = [p]_q \left(1 + \frac{1}{4} p_1 z + \left(\frac{1}{4} p_2 - \frac{5}{32} p_1^2\right) z^2 + \cdots\right)\left(\frac{[p]_q}{z^p} + \sum_{k=p+1}^{\infty} a_k z^k\right)
\]

By comparing of coefficients of \( z^{k+p} \), we get

\[
a_{p+1} = -\frac{[p]_q}{4q^p ([p+1]_q - [p]_q)} p_1, \tag{3.3}
\]
\[ a_{p+2} = -\frac{[p]_q}{q^{p}([p+2]_q - [p]_q)} \left( \frac{1}{4} p_2 - \frac{5 [p+1]_q - 7 [p]_q}{32 ([p+1]_q - [p]_q)^p_1} \right). \]  

Form (3.3) and (3.4)

\[ |a_{p+2} - \lambda a_{p+1}| = \frac{[p]_q}{4q^{p}([p+2]_q - [p]_q)} \left| \frac{p_2 - \frac{5q^p([p+1]_q)^2 - 12q^p[p+1]_q[p]_q + 7q^p([p]_q)^2 - 2\lambda([p]_q)^2 + 2[p+2]_q[p]_q}{8q^p([p+1]_q - [p]_q)^2} p_1}{p_1^2} \right|, \]

Using Lemma 2.1

\[ |a_{p+2} - \lambda a_{p+1}| \leq \frac{[p]_q}{2([p+2]_q - [p]_q)} \max\{1, |\mu|\}, \]

with \( \mu \) is defined as above. \( \square \)

3.2. Theorem

If \( f \in MSL_{p,q}^* \) and of the form (1.1), then

\[ |a_{p+2} - \lambda a_{p+1}| \leq \begin{cases} \frac{\gamma}{4q^p(\alpha - \gamma)} \frac{-q^p\beta^2 + 4q^p\beta\gamma - 3q^p\gamma^2}{2q^p(\beta - \gamma)^2} + \frac{\gamma}{4q^p(\alpha - \gamma)} \frac{2\alpha y - 2\gamma^2}{2q^p(\beta - \gamma)^2} \leq 0, \\ \frac{\gamma}{4q^p(\alpha - \gamma)} \frac{2q^p\beta^2 - 4q^p\beta\gamma + 3q^p\gamma^2}{2q^p(\beta - \gamma)^2} - \frac{\gamma}{4q^p(\alpha - \gamma)} \frac{2\alpha y - 2\gamma^2}{2q^p(\beta - \gamma)^2} \leq 0, \\ \frac{5q^p([p+1]_q)^2 - 12q^p[p+1]_q[p]_q + 7q^p([p]_q)^2 - 2\lambda([p]_q)^2 + 2[p+2]_q[p]_q}{8q^p([p+1]_q - [p]_q)^2} \leq 1, \end{cases} \]

where \( \lambda \in \mathbb{R}, \alpha = [p+2]_q, \beta = [p+1]_q \) and \( \gamma = [p]_q \).

Proof. From (3.3) and (3.4) it follows that

\[ a_{p+2} - \lambda a_{p+1} = \frac{[p]_q}{4q^{p}([p+2]_q - [p]_q)} (p_2 - \frac{5q^p([p+1]_q)^2 - 12q^p[p+1]_q[p]_q + 7q^p([p]_q)^2 - 2\lambda([p]_q)^2 + 2[p+2]_q[p]_q}{8q^p([p+1]_q - [p]_q)^2} p_1^2), \]

using above notations, we get

\[ a_{p+2} - \lambda a_{p+1} = \frac{\gamma}{4q^p(\alpha - \gamma)} \left( p_2 - \frac{5q^p\beta^2 - 12q^p\beta\gamma + 7q^p\gamma^2 - 2\lambda\gamma^2 + 2\alpha \gamma \lambda}{8q^p(\beta - \gamma)^2} \right) p_1^2. \]

Let \( v = \frac{5q^p\beta^2 - 12q^p\beta\gamma + 7q^p\gamma^2 - 2\lambda\gamma^2 + 2\alpha \gamma \lambda}{8q^p(\beta - \gamma)^2} \leq 0, \) using Lemma 2.2, we have

\[ |a_{p+2} - \lambda a_{p+1}| \leq \frac{\gamma}{4q^p(\alpha - \gamma)} \left| -4 \left( \frac{5q^p\beta^2 - 12q^p\beta\gamma + 7q^p\gamma^2 - 2\lambda\gamma^2 + 2\alpha \gamma \lambda}{8q^p(\beta - \gamma)^2} \right) + 2 \right| \]
Let \( v = \frac{5q\beta^2 - 12q^2\beta\gamma + 7q^3\gamma^2 - 24\gamma^2 + 2\alpha\gamma}{8q^2(\beta - \gamma)^2} \), where \( v \in [0, 1] \) using Lemma 2.2, we get the second inequality.

Now for \( v = \frac{5q\beta^2 - 12q^2\beta\gamma + 7q^3\gamma^2 - 24\gamma^2 + 2\alpha\gamma}{8q^2(\beta - \gamma)^2} \geq 1 \), using Lemma 2.2, we have

\[
\left| a_{p+2} - \Lambda a_{p+1}^2 \right| \leq \frac{\gamma}{4q^p(\alpha - \gamma)} \left[ 4 \left( \frac{5\beta^2 - 12\beta\gamma + 7\gamma^2 - 2\lambda\gamma}{8(\beta - \gamma)^2} \right) - 2 \right] - \frac{\gamma}{4q^p(\alpha - \gamma)} \frac{2\alpha\gamma - 2\gamma^2}{2q^p(\beta - \gamma)^2} \lambda,
\]

and hence the proof. \( \square \)

### 3.3. Theorem

If \( f(z) \in \mathcal{M}_p \), then for \(-1 \leq B < A \leq 1\) with

\[
|\xi| \geq \frac{2^\frac{3}{4}(A - B)[p]_q}{1 - |B| - 4p(1 + |B|)},
\]

and if

\[
1 + \xi \frac{z^p D_q f(z)}{[p]_q} < \frac{1 + Az}{1 + Bz},
\]

holds, then

\[
z^n f(z) < \sqrt{1 + z}.
\]

**Proof.** Suppose that

\[
J(z) = 1 + \xi \frac{z^{p+1} D_q f(z)}{[p]_q}
\]

and consider

\[
z^n f(z) = \sqrt{1 + \omega(z)}.
\]

Now to prove the required result it will be enough if we prove that \(|\omega(z)| < 1\).

Using (3.7) and (3.8)

\[
J(z) = 1 + \xi \frac{z D_q \omega(z)}{2 \sqrt{1 + \omega(z)}} - p \sqrt{1 + \omega(z)}
\]

and so

\[
\left| \frac{J(z) - 1}{A - BJ(z)} \right| = \left| \frac{\xi \frac{z D_q \omega(z)}{2 \sqrt{1 + \omega(z)}} - p \sqrt{1 + \omega(z)}}{A - B \left( 1 + \xi \frac{z D_q \omega(z)}{2 \sqrt{1 + \omega(z)}} - p \sqrt{1 + \omega(z)} \right)} \right|
\]

\[
= \frac{\xi z D_q \omega(z) - 2p \xi (1 + \omega(z))}{2[p]_q (A - B) \sqrt{1 + \omega(z)} - B \left( \xi z D_q \omega(z) - 2p \xi (1 + \omega(z)) \right)}
\]

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Now if \( \omega(z) \) attains its maximum value at some \( z = z_0 \), which is \( |\omega(z_0)| = 1 \). Then by Lemma 2.3, with \( m \geq 1 \) we have, \( \omega(z_0) = e^{i \theta} \) and \( z_0 D_q \omega(z_0) = m \omega(z_0) \), with \( \theta \in [-\pi, \pi] \) so

\[
\begin{align*}
\left| \frac{J(z_0) - 1}{A - BJ(z_0)} \right| & = \frac{\xi (m \omega(z_0) - 2p (1 + \omega(z_0)))}{2 (A - B) [p]_q \sqrt{1 + \omega(z_0) - B (\xi (m \omega(z_0) - 2p (1 + \omega(z_0))))}} \\
& \geq \frac{\xi (m - 2p (1 + e^{i \theta}))}{2 (A - B) [p]_q \sqrt{1 + e^{i \theta} + |B| (|\xi| (m - 2p (1 + e^{i \theta}))})} \\
& = \frac{|\xi| (m - 4p)}{|B| |\xi| (m + 4p) + 2^\frac{3}{2} (A - B) [p]_q}.
\end{align*}
\]

Consider

\[
\phi(m) = \frac{|\xi| (m - 4p)}{|B| |\xi| (m + 4p) + 2^\frac{3}{2} (A - B) [p]_q}
\]

\[
\Rightarrow \phi'(m) = \frac{8p |\xi|^2 |B| + 2^\frac{3}{2} |\xi| (A - B) [p]_q}{(|B| |\xi| (m + 4p) + 2^\frac{3}{2} (A - B) [p]_q)^2} > 0,
\]

showing the increasing behavior of \( \phi(m) \) so minimum of \( \phi(m) \) will be at \( m = 1 \) with

\[
\left| \frac{J(z_0) - 1}{A - BJ(z_0)} \right| \geq \frac{|\xi| (1 - 4p)}{2^\frac{3}{2} (A - B) [p]_q + |B| |\xi| (1 + 4p)},
\]

so from (3.5)

\[
\left| \frac{J(z_0) - 1}{A - BJ(z_0)} \right| \geq 1
\]

contradicting (3.6), thus \( |\omega(z)| < 1 \) and so we get the desired result. \( \square \)

3.4. Corollary

Let \(-1 \leq B < A \leq 1\) and \( f(z) \in M_p \). If

\[
|\xi| \geq \frac{2^\frac{3}{2} [p]_q (A - B)}{1 - |B| - 4 (1 + |B|) p},
\]

and

\[
1 - \left( 1 - p + \frac{zD_q^2 f(z)}{D_q f(z)} - \frac{zD_q f(z)}{f(z)} \right) \xi zD_q f(z) < \frac{1 + Az}{1 + Bz}, \tag{3.9}
\]

then \( f(z) \in MS_{p,q}^\infty \).
Proof. Suppose that
\[ k(z) = \frac{q^p z^{-p} D_q f(z)}{[p]_q f(z)}. \] (3.10)

From (3.10) it follows that
\[ z^{p+1} D_q k(z) = \left( 1 - p + \frac{z D_q^2 f(z)}{f(z)} - \frac{z D_q f(z)}{f(z)} \right) \frac{z D_q f(z)}{[p]^2 q f(z)}. \]

Using the condition (3.9), we have
\[ 1 - \xi z^{p+1} D_q k(z) < \frac{1 + Az}{1 + Bz}. \]

Now using Theorem 3.3, we get
\[ -z^p k(z) = -\frac{q^p z D_q f(z)}{[p]_q f(z)} < \sqrt{1 + z}, \]

thus \( f(z) \in MS\mathcal{L}_p^*. \) \( \square \)

3.5. Theorem

Let \(-1 \leq B < A \leq 1\) and \( f(z) \in \mathcal{M}_p \). If
\[ |\xi| \geq \frac{4 [p]_q (A - B)}{1 - |B| - 4 (1 + |B|) p} \] (3.11)

and
\[ 1 + \xi \frac{z D_q f(z)}{[p]_q f(z)} < \frac{1 + Az}{1 + Bz}, \] (3.12)

then
\[ z^p f(z) < \sqrt{1 + z}. \]

Proof. We define a function
\[ J(z) = 1 + \xi \frac{z D_q f(z)}{[p]_q f(z)}. \] (3.13)

Now as
\[ z^p f(z) = \sqrt{1 + \omega(z)}. \] (3.14)

Using Logarithmic differentiation on (3.14), from (3.13) we obtain that
\[ J(z) = 1 + \frac{\xi}{[p]_q} \left( \frac{z D_q \omega(z)}{2(1 + \omega(z))} - p \right) \]

and so
\[ \left| \frac{J(z) - 1}{A - BJ(z)} \right| = \left| \frac{\xi}{[p]_q} \left( \frac{z D_q \omega(z)}{2(1 + \omega(z))} - p \right) \right|. \]
3.6. Corollary

If at some \( z = z_0 \), \( \omega(z) \) attains its maximum value i.e. \( |\omega(z_0)| = 1 \). Then using Lemma 2.3, we have

\[
\frac{J(z_0) - 1}{A - BJ(z_0)} = \frac{\xi (m\omega(z_0) - 2p(1 + \omega(z_0)))}{2(A - B)[p]_q (1 + \omega(z_0)) - B(\xi m\omega(z_0) - 2p\xi(1 + \omega(z_0)))}.
\]

we have

\[
\frac{J(z_0) - 1}{A - BJ(z_0)} \geq \frac{(1 - 4p)|\xi|}{4(p|\xi|B + (A - B)[p]_q + |B||\xi|)}.
\]

hence by (3.11)

\[
\frac{J(z_0) - 1}{A - BJ(z_0)} \geq 1,
\]

which contradicts (3.12), therefore \( |\omega(z)| < 1 \) and so the desired result.

3.6. Corollary

Let \( -1 \leq B < A \leq 1 \) and \( f(z) \in M_p \). If

\[
|\xi| \geq \frac{4[p]_q(A - B)}{1 - |B| - 4(1 + |B||p|)}
\]

and

\[
1 - \left(1 - p + \frac{zD_q^2f(z)}{D_qf(z)} - \frac{zD_qf(z)}{f(z)}\right)\frac{\xi}{[p]_q} < \frac{1 + Az}{1 + Bz},
\]

then \( f(z) \in \mathcal{ML}^{\ast}_{p,q} \).
3.7. Theorem

Let \(-1 \leq B < A \leq 1\) and \(f(z) \in \mathcal{M}_p\). If

\[
|\xi| \geq \frac{2^{3} [p]_q (A - B)}{1 - |B| - 4 (1 + |B|) p}
\]

and

\[
1 + \xi \frac{z^{1-p} D_q f(z)}{[p]_q (f(z))^2} < \frac{1 + Az}{1 + Bz},
\]

then \(z^p f(z) < \sqrt{1 + z}\).

\textbf{Proof.} Here we define a function

\[
J(z) = 1 + \xi \frac{z^{1-p} D_q f(z)}{[p]_q f(z)^2}.
\]

So if

\[
z^p f(z) = \sqrt{1 + \omega(z)},
\]

using some simplification we obtain that

\[
J(z) = 1 + \xi \left( \frac{z D_q \omega(z)}{2 (1 + \omega(z))^2} - \frac{p}{\sqrt{1 + \omega(z)}} \right),
\]

and so

\[
\left| \frac{J(z) - 1}{A - BJ(z)} \right| = \left| \frac{\xi \left( \frac{z D_q \omega(z)}{2 (1 + \omega(z))^2} - \frac{p}{\sqrt{1 + \omega(z)}} \right)}{A - B \left( 1 + \xi \left( \frac{z D_q \omega(z)}{2 (1 + \omega(z))^2} - \frac{p}{\sqrt{1 + \omega(z)}} \right) \right)} \right|
\]

\[
= \left| \frac{\xi \left( z D_q \omega(z) - 2 p (1 + \omega(z)) \right)}{2 (A - B) [p]_q (1 + \omega(z))^2 + 2 p \xi B (1 + \omega(z)) - B \xi z D_q \omega(z)} \right|.
\]

Now if \(\omega(z)\) attains, at some \(z = z_0\), its maximum value which is \(|\omega(z_0)| = 1\). Then by Lemma 2.3, with \(m \geq 1\) we have, \(\omega(z_0) = e^{i\theta}\) and \(z_0 D_q \omega(z_0) = m \omega(z_0)\), with \(\theta \in [-\pi, \pi]\) so

\[
\left| \frac{J(z_0) - 1}{A - BJ(z_0)} \right| = \left| \frac{\xi (m \omega(z_0) - 2 p (1 + \omega(z_0)))}{2 (A - B) [p]_q (1 + \omega(z_0))^2 + 2 p \xi B (1 + \omega(z_0)) - B \xi m \omega(z_0)} \right|
\]

\[
\geq \frac{|\xi| m - 2 p |\xi| |1 + e^{i\theta}|}{2 (A - B) [p]_q |1 + e^{i\theta}|^2 + |B| |\xi| m + 2 p |\xi| |B| |1 + e^{i\theta}|}
\]

\[
\geq \frac{|\xi| m - 2 p |\xi| \sqrt{2 + 2 \cos \theta}}{2 (A - B) [p]_q (2 + 2 \cos \theta)^2 + |B| |\xi| m + 2 p |\xi| |B| \sqrt{2 + 2 \cos \theta}}
\]

\[
= \frac{(m - 4 p) |\xi|}{2^{3/2} (A - B) [p]_q + 4 p |\xi| |B| + |B| |\xi| m}.
\]
Now let

\[
\phi(m) = \frac{|\xi|(m - 4p)}{2^\frac{1}{2}(A - B)[p] + |B||\xi|(m + 4p)}
\]

\[\Rightarrow \phi'(m) = \frac{|\xi|(2^\frac{1}{2}(A - B) + 8p|\xi||B|)}{(2^\frac{1}{2}(A - B)[p] + 4p|\xi||B| + |B||\xi|m)^2} > 0,
\]

this shows \(\phi(m)\) an increasing function which implies that at \(m = 1\) it will have its minimum value and

\[
\left|\frac{J(z_0) - 1}{A - BJ(z_0)}\right| \geq \frac{(1 - 4p)|\xi|}{2^\frac{1}{2}(A - B)[p] + |B||\xi| + 4p|\xi||B|},
\]

now by (3.15) we have

\[
\left|\frac{J(z_0) - 1}{A - BJ(z_0)}\right| \geq 1,
\]

this is a contradiction as \(J(z) < \frac{1 + Az}{1 + Bz}\), thus \(|\omega(z)| < 1\) and so the result. \(\square\)

3.8. Corollary

Let \(-1 \leq B < A \leq 1\) and \(f(z) \in M_p\). If

\[
|\xi| \geq \frac{2^\frac{1}{2}[p](A - B)}{1 - |B| - 4(1 + |B|)p},
\]

and

\[
1 - \left(1 - p + \frac{zD^\frac{1}{2}f(z)}{D_qf(z)} \right) \frac{\xi f(z)}{zD_qf(z)} < \frac{1 + Az}{1 + Bz},
\]

then \(f(z) \in MSL_{r,q}^p\).

3.9. Theorem

Let \(-1 \leq B < A \leq 1\) and \(f(z) \in M_p\). If

\[
|\xi| \geq \frac{8[p](A - B)}{1 - |B| - 4(1 + |B|)p}
\]

(3.16)

and

\[
1 + |\xi|^{1 - 2p}D_qf(z) \left[\frac{1}{p}(f(z))^3\right] \leq \frac{1 + Az}{1 + Bz},
\]

(3.17)

then \(z^pf(z) < \sqrt{1 + z}\).

Proof. Suppose that

\[
J(z) = 1 + |\xi|^{1 - 2p}D_qf(z) \left[\frac{1}{p}(f(z))^3\right].
\]
Now if
\[ z^p f(z) = \sqrt{1 + \omega(z)}, \]
with simple calculations we can easily obtain
\[ J(z) = 1 + \frac{\xi}{[p]_q (1 + \omega(z))} \left( \frac{zD_q \omega(z)}{2[p]_q (1 + \omega(z))} - p \right), \]
and so
\[
\begin{aligned}
\left| \frac{J(z) - 1}{A - BJ(z)} \right| &= \left| \frac{\xi}{[p]_q (1 + \omega(z))} \left( \frac{zD_q \omega(z)}{2[p]_q (1 + \omega(z))} - p \right) \right|
&= \frac{\xi \left( zD_q \omega(z) - 2p (1 + \omega(z)) \right)}{2 (A - B) [p]_q (1 + \omega(z))^2 + 2p\xi B (1 + \omega(z)) - B\xi zD_q \omega(z)},
\end{aligned}
\]
if at some \( z = z_0, \omega(z) \) attains its maximum value i.e. \(|\omega(z_0)| = 1\). Then using Lemma 2.3,
\[
\begin{aligned}
\left| \frac{J(z_0) - 1}{A - BJ(z_0)} \right|
&= \left| \frac{\xi (m\omega(z_0) - 2p (1 + \omega(z_0)))}{2 (A - B) [p]_q (1 + \omega(z_0))^2 + 2p\xi B (1 + \omega(z_0)) + B\xi m\omega(z_0)} \right| |\xi| (m - 2p |1 + e^{i\theta}|)
&\geq \frac{2 (A - B) [p]_q |1 + e^{i\theta}|^2 + 2p |\xi| |B| |1 + e^{i\theta}| + |B| |\xi| m}{(m - 4p) |\xi|}
&\geq \frac{8 (A - B) [p]_q + |B| |\xi| (m + 4p)}{8 (A - B) [p]_q + |B| |\xi| (m + 4p)}.
\end{aligned}
\]
Now let
\[
\begin{aligned}
\phi(m) &= \frac{(m - 4p) |\xi|}{8 (A - B) [p]_q + |B| |\xi| (m + 4p)}
\Rightarrow \phi'(m) = \frac{8 |\xi| (A - B) [p]_q + 8p |\xi|^2 |B|}{(8 (A - B) [p]_q + |B| |\xi| m + 4p |\xi| |B|)^2} > 0
\end{aligned}
\]
which shows that the increasing nature of \( \phi(m) \) and so its minimum value will be at \( m = 1 \) thus
\[
\left| \frac{J(z_0) - 1}{A - BJ(z_0)} \right| \geq \frac{(1 - 4p) |\xi|}{8 (A - B) [p]_q + |B| |\xi| (1 + 8p)},
\]
and hence
\[
\left| \frac{J(z_0) - 1}{A - BJ(z_0)} \right| \geq 1,
\]
thus a contradiction by (3.17), so \(|\omega(z)| < 1\) and so we get the desired proof. \( \square \)
3.10. **Corollary**

Let \(-1 \leq B < A \leq 1\) and \(f(z) \in M_p\). If

\[
|\xi| \geq \frac{8(A - B)[p]_q}{1 - |B| - 4p(1 + |B|)}
\]

and

\[
1 - \xi[p]_q \left( 1 - p + \frac{zD_q^2 f(z)}{D_q f(z)} - \frac{zD_q f(z)}{f(z)} \right) \left( \frac{f(z)}{zD_q f(z)} \right)^2 < \frac{1 + Az}{1 + Bz}
\]

then \(f(z) \in \mathcal{MSL}^*_{p,q}\).

3.11. **Remark**

Letting \(q \to 1^+\) in our results we obtain results for the class \(\mathcal{MSL}^*_{p}\).

4. **Conclusion**

The main purpose of this article is to seek some applications of the \(q\)-calculus in Geometric Function theory, which is the recent attraction for many researchers these days. The methods and ideas of \(q\)-calculus are used in the introduction of a new subclass of \(p\)-valent meromorphic functions with the help of subordinations. The domain of lemniscate of Bernoulli is considered in defining this class. Working on the coefficients of these functions we obtained a very important result of Fekete-Szeg"o for this class. Furthermore the functionals \(1 + \xi[D_q^2 f(z)]_q, 1 + \xi[D_q f(z)]_q, 1 + \xi[z^{1-\epsilon}D_q f(z)]_q\) and \(1 + \xi[z^{1-\epsilon}D_q f(z)]_q\) are connected with Janowski functions with the help of some conditions on \(\xi\) which ensures that a function to be a member of the class \(\mathcal{MSL}^*_{p,q}\).

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**Conflict of interest**

The authors declare that they have no competing interests.

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