ONE HALF LOG DISCRIMINANT

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Abstract. We give a geometric proof that one may compute a particular generalized Mahler integral using equidistribution of preperiodic points of a dynamical system on the sphere. The dynamical system is associated to the multiplication by 2 map on an elliptic curve over a number field \( K \) with Weierstrass equation \( y^2 = P(x) \) (a Lattès dynamical system). At each finite place \( v \), we prove the local equidistribution formula

\[
v(\Delta) = \lim_{n \to \infty} \frac{1}{n^2} (D. H_n)_v,
\]

where \( H_n \) is the Zariski closure in \( \mathbb{P}^1_{\mathcal{O}_K} \) of the image in \( \mathbb{P}^1_K \) of the \( n \)-torsion minus the 2-torsion and \( \Delta \) is the discriminant of the polynomial \( P(x) \). One consequence of this result is the formula

\[
\frac{1}{2} \log |\text{Norm}_{K/\mathbb{Q}}(\Delta)| = \sum_{\sigma} \int_{\mathbb{P}^1(\mathbb{C})} \log |P(x)|_\sigma \frac{dx \wedge d\bar{x}}{\Im(\tau)_\sigma |P(x)|^2}.
\]

In [21], Szpiro, Ullmo, and Zhang proved that for any abelian variety \( A \) over \( \mathbb{Q} \), any continuous function \( g \) on \( A(\mathbb{C}) \), and any nonrepeating sequence of point \( \beta_n \in A(\overline{\mathbb{Q}}) \) with Néron-Tate height tending to zero, one has

\[
\frac{1}{|\text{Gal}(\beta_n)|} \sum_{\sigma \in \text{Gal}(\beta_n)} g(\sigma(\beta_n)) = \int_{A(\mathbb{C})} g \, d\mu,
\]

where \( d\mu \) is the normalized Haar measure on \( A \) and \( \text{Gal}(\beta_n) \) is the Galois group of the Galois closure of \( \mathbb{Q}(\beta_n) \) in \( \mathbb{C} \). This result says, in effect, that Galois orbits of points with small Néron-Tate height are equidistributed in \( A(\mathbb{C}) \). Ullmo [22] and Zhang [23] later used this fact to give proofs of the Bogomolov conjecture for abelian varieties.

When the abelian variety \( A \) is an elliptic curve, the multiplication by 2 map gives rise to a map on the projective line, called a Lattès map. Thus, in this case, the work of [21] can be viewed as an equidistribution result for a rational map on the projective line. Recently, a

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variety of authors have proven more general equidistribution results for arbitrary rational maps of degree greater than 1 on the projective line; see Autissier [2], Baker/Rumely [5], Bilu [6], Chambert-Loir [7], and Favre/Rivera-Letelier [12, 11], for example. Many of these results hold for measures at finite places as well as at archimedean places.

In [15], it is shown that for any nonconstant map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree greater than 1 over a number field $K$, the canonical height $h_{\varphi}(\alpha)$ of an algebraic point $\alpha$ with minimal polynomial $F$ can be calculated by integrating $\log |F|$ along the invariant measures for the map $\varphi$. This gives a generalization of the notion of a Mahler measure of a polynomial (see [14]). Everest, Ward, and Fhlathuin [10, 9] had previously extended the notion of Mahler measures to elliptic curves.

Additional difficulties arise, however, when one attempts to prove equidistribution results for the functions $\log |F|_v$. Indeed, the exact analog of the main result of [21] is not true when the continuous function $g$ is replaced by a function of the form $\log |F|$ (see [1] or [4]). On the other hand, it is possible to prove an equidistribution result for functions of the form $\log |F|_v$ provided that one averages over all points of period $n$ as $n$ goes to infinity rather than over Galois orbits of families of points of small height (see [19]). In the case of elliptic curves, Baker, Ih, and Rumely [4] were able to refine this to prove that for any algebraic number $\alpha$ and any Lattès map $\varphi$ one has

$$[K(\alpha) : \mathbb{Q}] h_{\varphi}(\alpha) = \sum_{\text{places } v \text{ of } K} \lim_{n \to \infty} \frac{1}{|\text{Gal}(\beta_n)|} \sum_{\sigma \in \text{Gal}(\beta_n)} \log |F(\beta_n^\sigma)|_v$$

for any nonrepeating sequence of algebraic points $\beta_n$ such that $h_{\varphi}(\beta_n) = 0$ for all $n$. Both [4] and [19] use results from diophantine approximation, specifically Roth’s theorem (16) and A. Baker’s work on linear forms in logarithms (3).

When one applies the results of [4] and [15] to the points of period 2 for a Lattès map corresponding to multiplication by 2 on the elliptic curve $E$ with Weierstrass equation $y^2 = P(x)$, one obtains the formula

$$\frac{1}{2} \log |\text{Norm}_{K/\mathbb{Q}}(\Delta)| = \sum_{\sigma : K \to \mathbb{C}} \lim_{n \to \infty} \frac{1}{n^2} \log \prod_{\beta \in \text{Supp } H_n} |P(\beta)|_\sigma$$

$$= \sum_{\sigma : K \to \mathbb{C}} \int_{\mathbb{P}^1(\mathbb{C})} \log |P(x)|_\sigma \frac{dx \wedge d\bar{x}}{3(\tau_\sigma)|P(x)|^2_\sigma},$$

where $\Delta$ is the discriminant of $F$ over $K$ and $\tau_\sigma$ denotes the element corresponding to the elliptic curve $E_\sigma$ in the fundamental domain for the action of $\text{SL}(2, \mathbb{Z})$ on the Poincare upper half space in $\mathbb{C}$. Using the product formula and the fact that $h_{\varphi}(\alpha) = 0$ for periodic points $\alpha$,
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this is equivalent to showing that at each nonarchimedean place $v$ of $K$, we have

$$\lim_{n \to \infty} \frac{1}{n^2} \log \prod_{\beta \in \text{Supp} \, H_n} |P(\beta)|_\sigma = \lim_{n \to \infty} \frac{1}{n^2} \log \text{Norm}_{H_n/O_K}(P_{H_n})$$

$$= \lim_{n \to \infty} \frac{1}{n^2} \sum_v (D,H_n)_v \log N(v),$$

where $N(v)$ is the cardinality of the residue field at $v$ and $H_n$ is the Zariski closure in $\mathbb{P}^1_{O_K}$ of the image in $\mathbb{P}^1_K$ of the $n$-torsion minus the 2-torsion.

We give here a local proof using blow-ups of closed points and intersection theory on $\mathbb{P}^1_V$. This proof uses resolution of singularities (in fact separation of branches) of one-dimensional schemes by blowing up. It also uses information about the special fiber of an elliptic curve with semistable reduction (see [17]). We do not use diophantine approximation. The proof we give is valid for equicharacteristic $V$ (local geometric case) as well as for unequal characteristic (local arithmetic case). Note that the case of positive characteristic does not follow from the results of [11] and [19], since the relevant approximation theorems are not valid in characteristic $p$. Relations between the discriminant of an elliptic curve and its $n$-torsion have been studied in [13] and in [20]. The main theorem of this paper is the following.

**Theorem.** Let $V$ be a discrete valuation ring with fraction field $K$. Let $y^2 = P(x)$ be the minimal Weierstrass equation with coefficients in $V$ of an elliptic curve $E$. Suppose that $E$ has semi-stable reduction over $V$. Let $D$ be the scheme of zeroes of $P(x)$ in $\mathbb{P}^1_V$ and let $H_n$ be the Zariski closure in $\mathbb{P}^1_V$ of the image in $\mathbb{P}^1_K$ of the kernel of the multiplication by $n$ in $E_K$ minus the 2-torsion. Then

$$\lim_{n \to \infty} \frac{1}{n^2} (D,H_n)_v = \frac{1}{2} v(\Delta),$$

where $v$ is the normalized valuation of $V$, $\Delta$ is the discriminant of $E$ over $V$, and $(.-.)_v$ is the geometric intersection pairing on the surface $\mathbb{P}^1_V$.

For simplicity we will assume that the roots of $P(x)$ are rational over $K$. Also for simplicity we assume that the residual characteristic of $V$ is not 2. These two conditions are not essential for the theorem but they make the proof easier. The valuation of the discriminant is then even; we write $2k = v(\Delta)$. One knows (see for example [17] or [8]) that the closed fiber of the minimal model $E$ for $E$ over $V$ is a cycle of $2k$ projective lines of self intersection $(-2)$; it is obtained by
blowing up the plane model for the elliptic curve $k$ times. Recall that the Néron model in this case is

$$E \setminus \{\text{singular points of the special fiber}\}$$

(see figure 3).

The strategy of the proof is to compute intersections in the Néron model for $E$ after a suitable base change. The multiplicative structure of the special fiber is simply $\mathbb{G}_m$ crossed with the group of components. One can easily see how the $n$-torsion distributes itself among the components, and that allows one to calculate intersections without difficulty.

We will use the fact that the Néron model for $E$ has $2k$ components in its special fiber (see [17, page 365]). It is naturally a $2 : 1$ cover of a model for $\mathbb{P}^1$ with $k + 1$ components. The hyperelliptic map induces a map on components that sends inverse component and its inverse to a single component; there are two components that are their own inverse (the identity and the component of order 2), which gives a total of $(k - 1)/2 + 2 = k + 1$ components on a model of $\mathbb{P}^1$.

We begin with the plane model $E$ for $E_K$ coming from the equation $y^2 = P(x)$.

Figure 1
Definition 1. Let $D_0$ denote the divisor $D$. We define the divisor $D_i$ recursively (for $i \leq k$) as the proper transform of $D_{i-1}$ for the blow-up $\sigma_i : X_i \to X_{i-1}$ centered at the point $P_{i-1}$ of multiplicity 2 on $D_{i-1}$.

Note that this is a horizontal divisor of degree 3 intersects the special fiber $F_0$ of $\mathbb{P}^1_r = X_0$ in 2 points: one $P_0$ of multiplicity 2 on $D_0$, the other one of multiplicity 1 on $D_0$. We now define the divisors in our models $X_i$ coming from $H_n$.

Definition 2. The horizontal divisor $C_0$ is defined to be $H_n$ for some fixed odd $n$. The divisor $C_i$ is the proper transform of $C_{i-1}$ in $X_i$.

The degree of $C_0$ is $(n^2 - 1)/2$ when $n$ is odd and $(n^2/2) - 3$ when $n$ is even. This follows from the fact that the hyperelliptic map sends each point and its inverse to the same point in $\mathbb{P}^1$.

Definition 3. Define $\wp_K : E_K \to \mathbb{P}^1$ to be the projection onto “the x axis” (i.e., $\wp$ is the Weierstrass $\wp$ function). We will, by abuse of language, note $\wp : E_i \to X_i$ to be the the extension of $\wp_K$ to model $E_i$ for $E_K$ over $V$.

The figure 2 illustrates the situation.

Figure 2
Lemma. Assume that \( n \) is odd or that the residual characteristic is not 2, then after \( k \) successive blow-ups of the points \( P_i \) of multiplicity 2 on \( D_i \), the proper transform \( D_k \) is étale and the proper transforms \( D_k \) and \( C_k \) do not meet.

Proof. (Of Lemma.) If \( \varphi^*(C_k) \) and \( \varphi^*(D_k) \) are both in the Néron model (i.e., if \( n \) and \( (2k) \) have a common factor \( m \)), then \( H_n \) and 2-torsion are distinct; hence, when the characteristic is not 2, they do not meet in the Néron model. If \( n \) is prime to \( 2k \) and \( \varphi^*(H_k) \) is not inside the Néron model, then \( \varphi^*(H_k) \cap \varphi^*(D_k) = \emptyset \), since \( \varphi^*(D_k) \) is in the Néron model (see figure 3). \( \square \)

Figure 3

\[ \begin{array}{c}
\text{D} \quad \text{Divisor of 2-torsion points} \\
\text{E} \quad \text{Divisor of } n\text{-torsion points} \\
\text{2k} \quad \text{components} \\
\text{(n, }2k\text{) = 1} \\
\end{array} \]

We are now ready to prove the main theorem.
Proof. We treat first the case when \( n \) is prime to \( 2k \). The exceptional divisor of \( \sigma_i \) will be denoted as \( F_i \). By abuse of language the proper transform of \( F_i \) will still be called \( F_i \) after \( \sigma_{i+1}, \ldots, \sigma_k \). We will let \( Q_i \) denote the point of intersection of \( F_i \) with \( F_{i-1} \) in \( X_i \) (see figure 2).

We will denote the usual pull-back map for divisors with \(*\). We denote the composed map \( \sigma_i \cdot \sigma_{i-1} \cdots \sigma_1 \) as \( \rho_i \). After \( i \) blow-ups, one has integers \( m_{j,i} \) such that

\[
\rho_i^*D = D_i + \sum_{j \leq i} m_{j,i}F_j
\]

and

\[
\pi_{i+1}^*D = \sigma_{i+1}^*D_i + \sum_{j \leq i} m_{j,i}\sigma_{i+1}^*F_j.
\]

Figure 2 (continued)  \hspace{1cm} \cdots \cdots \ \text{C}_{i-1} \hspace{0.5cm} \text{and} \hspace{0.5cm} \text{C}_i

\[\begin{array}{c}
D_1 \\
\vdots \\
D_i \\
F_i \\
X_i \\
\end{array}\hspace{1cm}\begin{array}{c}
D_1 \\
\vdots \\
D_i \\
F_i \\
X_{i-1} \\
\end{array}\]
As long as $i$ is less than $(k - 1)$, one has
\[ \sigma_{i+1}^* D_i = D_{i+1} + 2F_{i+1}, \]
since the multiplicity of $D_i$ at $P_i$ is still 2. Since $\sigma_{i+1}^* F_i = F_i + F_{i+1}$ we have $m_{j,i} = m_{j,j}$ for any $i \geq j$, so we have
\[ m_{j,i} = m_{j,j} = m_{j-1,j-1} + 2 \]
for all $i \geq j$. Thus, by induction, we have $m_{j,j} = 2j$ for each $j$, which means that $m_{j,i} = 2j$ for all $i \geq j$.

The intersection multiplicity we are looking for can be computed as follows
\[ ([*])(H_n, D) = ([C_k, D_k + \sum_{j \leq k} m_{j,k} F_j]) = 2 \sum_{j \leq k} j(C_k, F_j). \]
One is left with computing each $(C_k, F_j)$. We will achieve this by looking at the special fiber of various models of $E_K$ over $V$. By the projection formula for $\varphi$ we can compute intersections on the minimal model $E'_K$ or on the $k$-th blow-up $X_k$ of $\mathbb{P}^1$. In fact we will use the projection formula to compute intersections on the minimal model $E'_K$ for $E$ after the base change $\text{Spec } V[X]/(X^n - \pi) \to \text{Spec } V$ where $\pi$ is a uniformizing parameter of $V$. A description of the resolution of singularities of the base change can be found in [LS, Exposé 1, Propositio 2.2].

On the minimal model $E'_K$, the special fiber has $2kn$ components. Let $Z_0$ denote the component of the origin of the elliptic curve, and let us denote the other components as $Z_1, \ldots, Z_{2kn-1}$ in such a way that $Z_i$ meets $Z_{i+1}$ for $0 \leq i \leq (2kn - 1)$ and $Z_{2k-1}$ meets $Z_0$ (figure 4).

The divisor of $n$-torsion points meets only the components $Z_i$ for which $i$ is a multiple of $2k$; the multiplicity of each intersection is $n$. The components $Z_j$ for which $j$ is a multiple of $n$ are the only ones not contracted by the morphism to the plane model $E$. The contribution at $Q_j$ in the intersection number $(C_k, F_j)$ for $j \neq 0, k$ will be
\[ n \cdot |\{ m \text{ such that } (j - 1)n \leq 2km \leq jn \}| \]
Write $n = 2kq + r$ with $0 \leq r < 2k$. We have
\[ |\{ m \text{ such that } (j - 1)n \leq 2km \leq jn \}| - q \leq 1. \]
Thus, we have
\[ (***) \left| (C_k, F_j) - 2n \frac{n-r}{2k} \right| \leq 2n. \]
Since $(C_k, F_k) = n \frac{n-r}{k}$ we obtain
\[ (H_n, D) \simeq 2 \sum_{j \leq (k-1)} j 2n \frac{n-r}{2k} + 2kn \frac{n-r}{k} \]
with an error at most \(2 \sum_{j \leq (k-1)} j(2n) = \frac{2(k-1)}{2} 2n\). Hence, we have

\[|c = n-r nk| \leq k(k-1)2n,\]

so

\[
\lim_{n \to \infty} \frac{1}{n^2} (H_n.D) = k.
\]

**Figure 4**

Special fibers of minimal models over \(V'\) and \(V\)

\(V' = V[x]/(x^n - \pi)\)

\(E' = E\)

This finishes the proof in the case when \(n\) and \(2k\) are relatively prime. For the case where \(n\) and \(2k\) have a gcd \(m\) greater than 1 the formula (*) is still valid. The \(n\)-torsion distribute themselves in packets of \(m\) in components of the special fiber (see figure 5). Thus, the estimate (**) for \((C_k,F_i)\) has now an error term of at most \(m\).
$(2k, n) = m$

Adding as before, we now obtain

$$|\langle H_n.D \rangle - (n - r)nk| \leq \sum_{j \leq (k-1)} 2jm = mk(k - 1) \leq k(k - 1)(2k).$$

Letting $n$ go to $\infty$ we see again that

$$\lim_{n \to \infty} \frac{1}{n^2} \langle H_n.D \rangle = k.$$

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