ON MCKAY QUIVER AND COVERING SPACES

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ABSTRACT. In this paper, we study the relationship between the McKay quivers of a finite subgroup $G$ of special linear groups general linear groups, via some natural extension and embedding. We show that the McKay quiver of certain extension of a finite subgroup $G$ of $\text{SL}(m, \mathbb{C})$ in $\text{GL}(m, \mathbb{C})$ is a regular covering of the McKay quiver of $G$, and when embedding $G$ in a canonical way into $\text{GL}(m - 1, \mathbb{C})$, the new McKay quiver is obtained by adding an arrow from the Nakayama translation of $i$ back to $i$ for each $i$. We also show that certain interesting examples of McKay quivers are obtained in these two ways.

In 1980, John McKay introduced McKay quiver for a finite subgroup of the general linear group [1]. McKay observes that when $G$ is a subgroup of $\text{SL}(2, \mathbb{C})$, then its McKay quiver $Q = Q_G$ is a double quiver of the affine Dynkin diagram of type $A, D, E$ [1]. McKay observed that the McKay quivers describe the relationship between these groups and the Kleinian singularities.

McKay quiver has bridged many mathematical fields such as algebraic geometry, mathematics physics and representation theory (See, for example, [2]). In representation theory of algebra, for example, it appears in the study of the Auslander-Reiten quiver of Cohen-Macaulay modules [3,4], preprojective algebras of tame hereditary algebras [5,6,7,8] and quiver varieties [9], etc. We find that it can also play a critical role in classification of selfinjective Koszul algebras of complexity 2 [10].

Covering space is an important tool of the algebraic topology, it is introduced in the study of the representation theory of algebra by Bongartz and Gabriel in [11] and plays an important role here [12,13,14]. In this paper, we show that the covering maps also appear naturally in the McKay quivers. We show that when one extend a finite subgroup of $\text{SL}(m, \mathbb{C})$ via some nice cyclic group to a finite
subgroup of $\text{GL}(m, \mathbb{C})$, one gets natural covering for their McKay quivers. There are also covering maps between such extensions. So one can even construct the 'universal cover' of an McKay quiver.

Such coverings are not only geometric for the McKay quivers. They also induce coverings in the categories of the projective-injective modules of the corresponding skew group algebras over the exterior algebra of the given vector spaces. So their representation theory are related. Using the universal covering, one get an "universal algebra" which has simply connected quiver. This can be used to explain our early results for the case of $m = 2$ [15].

Though McKay quivers are well known for $\text{SL}(2, \mathbb{C})$. It is difficult to determine the McKay quiver in general. We describe the McKay quiver for certain finite subgroup of general linear group. Let $G$ be a finite subgroup of the general linear group, then we have that $N = G \cap \text{SL}(m, \mathbb{C})$ is a normal subgroup of $G$ and $\hat{G} = G/N$ is a finite cyclic group. We observe that under certain condition on $G$, the McKay quiver of $G$ is a covering quiver of the McKay quiver of $N$ with the group $\hat{G}$. We also show by example, how the "physics" quivers appearing in the study of D-branes [16] can be explained using our results.

1. Finite Subgroups in $\text{GL}(m, \mathbb{C})$ and $\text{SL}(m, \mathbb{C})$ 

Let $V$ be an $m$-dimensional vector space over $\mathbb{C}$ and let $G$ be a finite subgroup of $\text{GL}(m, \mathbb{C}) = \text{GL}(V)$. Let $Q_G = (Q_{G,0}, Q_{G,1})$ be the McKay of $G$. $V$ is naturally a faithful representation of $G$, and there are $a_{i,j}$ arrows from the vertex $i$ to the vertex $j$.

The need the following lemma.

**Lemma 1.1.** Let $G$ be a finite subgroup of $\text{GL}(m, \mathbb{C})$ and let $N = G \cap \text{SL}(m, \mathbb{C})$. Then $G/N$ is a cyclic group.

**Proof.** Consider the map $\text{det} : G \to \mathbb{C}^*$, which send each matrix $g \in G$ to its determinant. Clearly $\text{det}$ is a homomorphism with the kernel $N$. So $G/N$ is a subgroup of $\mathbb{C}^*$, and hence is a finite abelian group. By the Fundament Theorem of
the Structure of Finite Abelian Groups, $G/N$ is a direct sum of cyclic groups. Let $\bar{g}, \bar{h}$ be two elements in $G/N$ such that $|\bar{g}||\bar{h}|$. Then there is an $|\bar{h}|$th root $\xi$ of the unit, such that $\det(\bar{h}) = \xi$ and $\det(\bar{g}) = \xi^{|\bar{h}|}$. So we have that $\det(\bar{gh}^{|\bar{h}|}) = 1$ and $gh^{-|\bar{h}|} \in N$. This implies that $\bar{g} = \bar{h}^{|\bar{h}|}$ and one see easily that $G/N$ is cyclic. \qed

Let $Q$ be a quiver and $i \in Q_0$ be a vertex. Denote by $i^+$ the set of arrows ending at $i$ and $i^+$ the set of arrows starting at $i$.

Let $G$ be a group, a map $\pi : Q \to Q'$ of quivers is called a regular covering map with the group $G$ provided that for any vertex $j \in Q_0$ the group $G$ acts transitively and freely on $\pi^{-1}(j)$; and for each vertex $i \in Q_0$ the induced maps $\pi^+_i : i^+ \to \pi(i)^+$ and $\pi^-_i : i^- \to \pi(i^-)$ are both bijective. Clearly, this is exactly the regular covering map of oriented linear graphs with automorphism group $G$. (see Chapter 5 and 6 of [17]). When $Q = Q'$ and $G = \{1\}$, a covering map is just an automorphism of the quiver, that is, an automorphism is a regular covering map with the trivial group.

**Theorem 1.2.** Let $G$ be a finite subgroup of $GL(m, \mathbb{C})$ and let $N = G \cap SL(m, \mathbb{C})$. If every irreducible character of $N$ is extendible, then the McKay quiver of $G$ is a regular covering of the McKay quiver of $N$ with the automorphism group $G/N$.

**Proof.** Since every irreducible character of $N$ is extendible, If $\chi_1, \ldots, \chi_n$ are the irreducible characters of $N$, there exist irreducible characters $\chi'_1, \ldots, \chi'_n$ of $G$, such that $\chi'_i|_N = \chi_i$, for $i = 1, \ldots, n$. Assume the characters $\chi'_1, \ldots, \chi'_n$ are afforded by the irreducible representations $S_1, \ldots, S_n$ of $G$. Since the irreducible characters of $N$ are extendable, $S_1, \ldots, S_n$ are also irreducible representations of $N$. And they afford the irreducible characters $\chi_1, \ldots, \chi_n$ of $N$, respectively. If $\beta_1, \ldots, \beta_r$ are the irreducible characters of $G/N$ which are afforded by the irreducible representations $T_1, \ldots, T_r$ of $G/N$, they are naturally regarded as the representations of $G$. Since $G/N$ is cyclic by Lemma [14], $|G/N| = r$ and $T_1, \ldots, T_r$ are all 1-dimensional. By Corollary 6.17 of [15], $\{\chi'_i|_N = \chi_i, 1 \leq i \leq n, 1 \leq j \leq r\}$ is a set of irreducible characters of $G$ and $\chi'_i|_N = \chi'_j|_N$ if and only if $i = s$ and $j = t$. Clearly, $\chi'_i|_N = \chi'_j|_N$ is afforded by the representations $S_i \otimes T_j$ of $G$, so $S_i \otimes T_j$ is irreducible for $1 \leq i \leq n, 1 \leq j \leq r$. Since $|G| = |N| \cdot |G/N|$, by comparing the dimensions, we find that $\{S_i \otimes T_j | 1 \leq i \leq n, 1 \leq j \leq r\}$ is a complete set of the irreducible representations of $G$.

If $V = \bigoplus_{i=1}^n b_i S_i$ over $N$, then there is a $j_1, \ldots, j_n \in \{1, \ldots, n\}$ such that $V = \bigoplus_{i=1}^n \bigoplus_{j=1}^r b_{i,j} S_i \otimes T_j$. Then $b_i = \sum_{j=1}^r b_{i,j}$ Let $\bar{g}$ be a generator of $G/N$, there is a $r$th root $\xi$ of the unit such that $\bar{g}x = \xi^jx$ for any $x \in T_j$. Reindex the irreducible representations if necessary, we may assume that the index are taken from the set of residue classes modulo $r$. Then $T_i \otimes T_j \simeq T_{i+j}$. So we have that for each $i, k \in \{1, \ldots, n\}$, if $S_i \otimes S_k \simeq \bigoplus_{k=1}^r b_{i,k} c_{i,k} S_j$, and $S_i \otimes V \simeq \bigoplus_{j=1}^n a_{i,j} S_j$
as representation of $N$, then $a_{i,j} = \sum_{k=1}^{n} b_{i,k}c_{i,k,j}$. As representation of $G$, we have that for $i = 1, \ldots, n$ and $s = 1, \ldots, r$,

$$
(S_i \otimes T_s) \otimes V \cong \bigoplus_{k=1}^{n} S_i \otimes S_k \otimes T_s \otimes T_{jk} \cong \bigoplus_{k=1}^{n} \bigoplus_{l=1}^{n} c_{i,k,l}S_i \otimes T_{s+jk}.
$$

So we get that if $Q_{N,0} = \{(i)|i = 1, 2, \ldots, n\}$, then $\pi(i, t) = i$, whose fibre consists of single orbit of $G/N$ with $G/N$ acts freely. It also induces a bijection $(i, s)^+$ and $i^+$ in the McKay quiver and so it induces a regular covering map from $Q_G$ to $Q_N$ with the group $G/N$. \hfill \Box

**Remark.**

- Clearly, for any finite subgroup $N \subset \text{SL}(m, \mathbb{C})$ and any positive integer $r$, let $\xi_{mr}$ be an $mr$th root of the unit. Let $H_r$ be the subgroup generated by

$$
\left( \begin{array}{cccc}
\xi_{mr} & 0 & \cdots & 0 \\
0 & \xi_{mr} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \xi_{mr}
\end{array} \right)
$$

the diagonal matrix $H_r$ is a cyclic subgroup of $\text{GL}(m, \mathbb{C})$ of order $r$. The product $G = N \times H_r$ satisfy the conditions of Theorem 1.2. So the McKay quiver of $G$ is a regular covering space of the McKay quiver of $N$ with the group $H_r$. So given a finite subgroup $G$ of $\text{SL}(m, \mathbb{C})$ and finite cyclic group $H$, we can always find a subgroup of $\text{GL}(m, \mathbb{C})$ whose McKay quiver is a regular covering of that of $G$ with the group $H$.

- In the case $m = 2$ it follows from [14, 19] that the McKay quiver of a finite subgroup of $\text{GL}(2, \mathbb{C})$ should be a regular covering graph of the McKay quiver of some finite subgroup of $\text{SL}(2, \mathbb{C})$ (a double quiver of some affine Dynkin diagram). They are in fact a translation quiver with a slice of affine Dynkin quiver. But we don’t know the exact relationship between them.

In fact, given a finite subgroup $G \subset \text{GL}(m, \mathbb{C})$ such that every irreducible character of $N = G \cap \text{SL}(m, \mathbb{C})$ is extendible, the covering maps exist not only for the McKay quivers of $G$ and the subgroup $N$, but also for the McKay quivers of any subgroup $N \subset L \subset G$. It is obvious that such $L$ is a normal subgroup and every irreducible character of $L$ is extendible. In fact, our proof of Theorem 1.2 tell us the condition in the following Theorem already guarantees the existence of a covering map.
Theorem 1.3. Let $G$ be a finite subgroup of $\text{GL}(m, \mathbb{C})$ and $L$ be a normal subgroup of $G$ such that $G/L$ is cyclic. If every irreducible character of $L$ is extendible. Then the McKay quiver of $G$ is a regular covering space of the McKay quiver of $L$ with the group $G/L$.

Given a finite subgroup $G_0$ of $\text{GL}(m, \mathbb{C})$. Let $G_m(G_0)$ be the set of finite subgroups $G$ of $\text{GL}(m, \mathbb{C})$ containing $G_0$ as a normal subgroup such that $G/G_0$ is cyclic and all the irreducible character of $G_0$ is extendible to $G$. Then if $G \in G_m(G_0)$, we denote the regular covering map for their McKay quivers as $\pi_{G,G_0}$: $Q_G \to Q_{G_0}$.

The following theorem is obvious.

Theorem 1.4. Let $L$ be a finite subgroup of $\text{GL}(m, \mathbb{C})$. $N \in G_m(L)$ and $G \in G_m(N)$. Then $G \in G_m(L)$ and $\pi_{N,L} \circ \pi_{G,N} = \pi_{G,L}$.

Thus the regular covering map $\pi_{G,N}$ is in fact a homomorphism from the covering map $\pi_{G,L}$ to the regular covering map for $\pi_{N,L}$. For any finite subgroup $L \subset \text{SL}(m, \mathbb{C})$, the McKay quivers for the groups in $G_m(L)$ together with the regular covering maps between them is a category.

2. Skew Group Construction and Morphisms of Graphs with Relations

In [20], we relate a finite subgroup of $\text{GL}(m, \mathbb{C}) = \text{GL}(V)$ with a finite dimensional selfinjective Koszul algebra, its skew group algebra over the exterior algebra $\wedge V$ of $V$, which has the McKay quiver as its quiver. With the exterior construction, it seems that we have a finer description of the regular covering of the McKay quivers finer, by using the morphism of quiver with relations introduced by Green [21].

Let $V$ be an $m$–dimensional vector space over $\mathbb{C}$ and let $G$ be a finite subgroup of $\text{GL}(V)$. Let $\wedge V$ be the exterior algebra of $V$, construct the skew group algebra $\wedge V * G$ over the exterior algebra $\wedge V$ using the natural action of $G$ on $V$, we know that the quiver of $\wedge V * G$ is exactly the McKay quiver of $G$ and the map of determinant introduces Nakayama translation on the vertices of $Q_G$ [20].

Let $\Lambda$ be a finite dimensional selfinjective graded algebra over an algebraically closed field $k$, and let $Q$ be its quiver. The Nakayama translation is defined as the permutation $\sigma$ on the vertex set $Q_0$ of $Q$ such that for any $i \in Q_0$, $\sigma i$ is the vertex of the simple which is the socle of the projective cover of the simple associated to $i$ [22]. $\sigma$ defines a permutation on $Q_0$, called the Nakayama translation.

Let $\Lambda(G)$ be the basic algebra of $\wedge V * G$, we call it the basic algebra of $G$. Then there is an idempotent element $e \in \wedge V * G$, such that $\Lambda(G) = e \wedge V * G e$. In fact,
e = e_1 + \cdots + e_n for a set \{e_1, \ldots, e_n\} of orthogonal primitive idempotents such that \{\wedge V \ast Ge_i| i = 1, \ldots, n\} is a complete set of representatives of the isomorphism classes of the indecomposable projective modules. \Lambda(G) is a finite dimensional selfinjective Koszul algebra with gradation

\[
\Lambda(G) = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_m.
\]

J = \Lambda_1 + \cdots + \Lambda_m is its radical and \(J^i/J^{i+1} \simeq \Lambda_t\) as \(\Lambda_0\)-module.

By [20], we have that \(\Lambda(G) \simeq \mathbb{C}Q_G/I\) for some admissible ideal \(I = (\rho_G)\) of the path algebra \(\mathbb{C}Q_G\) which is induced by the relations defining \(\wedge V\). In this case, we have \(1 = (e = ) \sum_{i\in Q_{G,0}} e_i\) is a decomposition of \(1\) of \(\Lambda(G)\) as orthogonal primitive idempotent elements. By definition, the number of arrows from \(i\) to \(j\) is \(\dim_{\mathbb{C}} e_j V \ast Ge_i\). If \(\{v_1, \ldots, v_m\}\) is a bases of \(V\), they generates \(V \ast G\) as an \(kG\)-\(kG\)-bimodule. We have that \(\Lambda_0 = eV \ast Ge, \Lambda_1 = eV \ast Ge\) and \(e_j V \ast Ge_i = e_j \Lambda_1 e_i\). Choose a basis \(v_1, \ldots, v_m\) of \(V\), they defines the arrow of \(Q_G\), in fact, we have an arrow of type \(t\) from \(i\) to \(j\) provided that \(e_j v_i e_i \neq 0\).

Let \(G\) be a finite subgroup of \(\text{GL}(V)\) and let \(T_kV\) be the tensor algebra of \(V\), the skew group algebra \(T_k \ast G\) of \(G\) over \(T_kV\) has as its quiver the McKay quiver \(Q_G\) of \(G\). We have the following lemma.

**Lemma 2.1.** If \(\alpha\) and \(\beta\) are two arrows of different types starting at the same vertex in the quiver \(Q_G\) of \(\Lambda(G)\), then there are arrows \(\alpha'\) of the same type as \(\alpha\) and \(\beta'\) of the same type as \(\beta\), such that \(\alpha\beta'\) and \(\beta\alpha'\) will ending at the same vertex.

**Proof.** Let \(T_kV\) be the tensor algebra of \(V\), it is well known that the exterior algebra \(\wedge V \simeq T_kV/(\rho_0)\), where \((\rho_0)\) is the ideal generated by \(v_i \otimes v_i\) and \(v_i \otimes v_j + v_j \otimes v_i\) for all \(i, j\). If \(G\) is a finite subgroup of \(\text{GL}(V)\). In the skew group algebra \(T_kV \ast G\), we have that, if \(e_l, e_h\) are idempotents in \(kG \subset T_kV\) and if \(e_h = \sum_{g\in G} a_g g, a_g \in k\), then

\[
e_h v_i \otimes v_j e_i = \sum_{g\in G} g^{-1}(v_i) \otimes g^{-1}(v_i) a_g g e_i,
\]

and

\[
e_h v_j \otimes v_i e_i = \sum_{g\in G} g^{-1}(v_j) \otimes g^{-1}(v_i) a_g g e_i.
\]

So we see that \(e_h v_i \otimes v_j e_i \neq 0\) if and only if \(e_h v_j \otimes v_i e_i \neq 0\) in \(T_kV\). This proves the lemma.

\(\square\)

Now regard \(\Lambda(G)\) as the quotient algebra \(\Lambda_Q \simeq kQ_G/(\rho_G)\), the relation \(\rho_G\) comes from those of the exterior algebra. That is, for the basis \(v_1, \ldots, v_m\) of \(V\), for all \(s, t\)
The Nakayama translation is extended uniquely to an automorphism 
Lemma 2.2.

Proof. Since the socle of an indecomposable projective \(\Lambda(G)\)-module is simple, we have that \(\dim_k e_{\alpha} \Lambda_m e_i = 1\). Now for any arrow \(\alpha : i \to j\) of type \(t\), \(\alpha = e_j v_t e_i\). We have that in \(\wedge V v_i \ldots v_i = 0\) if there are \(1 \leq h < h' \leq s\) such that \(i_h = i_{h'}\). This implies that \(0 \neq \alpha_m \ldots \alpha_2 \alpha \in e_{\sigma} \Lambda_m e_i\) if and only if for \(h = 2, \ldots, m\), \(\alpha_h = e_j v_t e_i \neq 0\), \(i_1 = i_1 = j_1 = j\) such that \(i_{h+1} = j_h\), and \(t_1, \ldots, t_m\) are pairwise different. This shows that \(\beta \alpha_m \ldots \alpha_2 \neq 0\) implies \(\beta = e_j v_t e_{\sigma}\). On the other hand, since \(0 \neq \alpha_m \ldots \alpha_2 \in \Lambda_m e_i\) is not in \(\text{soc} \Lambda_{m-1} e_i\) and \(0 \neq V \alpha_m \ldots \alpha_2 \in \text{soc} \Lambda_{m-1} e_i\), so such \(\beta\) exists and we have that \(l = \sigma i_2\). Define \(\sigma \alpha = \beta\), this is an extension of \(\sigma\), and for any vertex \(i\) of \(Q_G\), \(\sigma\) induces an bijection from \(i^+\) to \((\sigma i)^+\).

So \(\sigma\) is extended to an automorphism of the quiver \(Q_G\).

A relation in a quiver \(Q\) is a subset of elements in \((kQ^2)\), the ideal of the path algebra \(kQ\) generated by paths of length 2. According to the definition of \([21]\), if \((Q, \rho)\) and \((Q', \rho')\) are quivers with relations, a regular covering map \(\pi : Q \to Q'\) is called a morphism of quivers with relations if \(\pi\) satisfies the following conditions:

1. \(\rho = \{L(x)|LQ' \to Q\ is a lifting and \ x \in \rho'\}\)
2. If \(x \in \rho\) and \(i', j' \in Q_0\), there exists \(i, j \in Q_0\) such that \(\pi(i) = i', \pi(j) = j'\) and \(\bar{\pi}(e_j x e_i) = e_{\rho'} \bar{\pi}(x) e_{\rho'}\), here \(\bar{\pi}\) is the homomorphism from \(kQ\) to \(kQ'\) induced by \(\pi\), that is, if \(x = \sum_t d_t p_t\) for \(d_t \in k\) and \(p_t\) paths in \(Q\), then \(\bar{\pi}(x) = \sum_t d_t \pi(p_t)\).

Let \(N\) be a normal subgroup of \(G\) such that the conditions of Theorem 1.2 are satisfied. Let \(Q_G\) and \(Q_N\) be the McKay quivers of \(G\) and \(N\), respectively and let \(\pi : Q_G \to Q_N\) be the covering map. Let \(\Lambda(G)\) and \(\Lambda(N)\) be their basic algebras, respectively, then \(\Lambda(G) \simeq kQ_G/(\rho_G)\) and \(\Lambda(N) \simeq kQ_N/(\rho_N)\). Now consider the relationship between \(\rho_N\) and \(\rho_G\). We have the following theorem

**Theorem 2.3.** Let \(G\) and \(N\) as in Theorem 1.2. The covering map \(\pi : Q_G \to Q_N\) defined in the proof of Theorem 1.2 is a morphism of quiver with relations.

Proof. By Theorem 1.2, we need only to prove the above conditions (1) and (2).
Use the notations of the proof of Theorem 1.2 if \( V = \bigoplus_{i=1}^{r} V_{i} \), where for \( b_{1} + \cdots + b_{t-1} < i \leq b_{1} + \cdots + b_{r} \) we have \( V_i \simeq S_t \) as irreducible representation of \( N \) and for \( b_{1} + \cdots + b_{t-1} + b_{t,1} + \cdots + b_{t,s} < i \leq b_{1} + \cdots + b_{t,1} + \cdots + b_{r,s} \), \( V_i \simeq S_t \otimes T_{j,s} \) as irreducible representation of \( G \). We may assume that our basis \( \{v_1, \ldots, v_m\} \) is a union of the bases of \( V_1, \ldots, V_n' \).

Now there is an arrow \( \alpha \) from \( i \) to \( j \) in the quiver \( Q_N \) of \( \Lambda(N) \) provided that we have a basic element \( v_r \) in \( V_r' \) such that \( S_j \) is a summand of \( V_r' \otimes S_i \); in this case \( \alpha \) is of the form \( e_j v_r e_i \). As we have done in the proof of Theorem 1.2 we may index the vertices (idempotents of \( kG \)) of \( Q_G \) as the quiver of \( \Lambda(G) \) by the pair \( \{(i, j) | i \in Q_{N,0}, j \in \mathbb{Z}/|G/N|\mathbb{Z}\} \). If \( i \) is lifted to \( (i, l) \) in \( Q_G \), then the arrow \( \alpha \) is lifted to an arrow \( \hat{\alpha} : (i, l) \to (j, l + j') \), for some lift \( (j, l + j') \) of \( j \) which is determined by the same element \( v_r \) of \( V_r' \) regarding as representation of \( G \). Now both relations \( \rho_N \) and \( \rho_G \) are quadratic and are induced by the relations \( \rho = \{v_i v_s + v_s v_i | 1 \leq t, s \leq m\} \). It follows from Lemma 2.1 that both conditions (1) and (2) hold. \( \square \)

Obviously, the covering maps in Theorem 1.4 can be regarded as morphisms of quiver with relations. Denote \( \mathcal{M}_m(N) \) the category of McKay quivers for the groups in \( G_m(L) \) with the relations as above together with morphisms of quiver with relations between them. It follows from Theorem 2.6 of [21], there is an unique universal cover \( (Q, \rho) \) for \( \mathcal{M}_m(N) \), if an (locally finite) infinite quiver and hence is not the an object in \( \mathcal{M}_m(N) \).

In the setting of Theorem 2.3 regard \( \Lambda(G) \) and \( \Lambda(N) \) as locally bounded categories with finitely many objects \([11]\), the morphism \( \pi \) of quiver with relation induces a covering functor from \( \Lambda(G) \) and \( \Lambda(N) \).

### 3. Finite Subgroups in \( \text{GL}(m, \mathbb{C}) \) and \( \text{SL}(m + 1, \mathbb{C}) \)

Let \( V \) be an \( m + 1 \)-dimensional vector space over \( \mathbb{C} \) and let \( G \) be a finite subgroup of \( \text{GL}(m + 1, \mathbb{C}) = \text{GL}(V) \). Let \( \det \) denote the one-dimensional representation \( \mathbb{C} \) defined by \( g \cdot x = \det(g)x \), here \( \det(g) \) denote the determinant of \( g \). Then \( S \to S \otimes \det \) define a bijection on the set of irreducible representations, which induces an automorphism on the McKay quiver. In the case of \( m = 2 \), this coincide with the translation defined in [19].

Let \( V \) be an \( m + 1 \)-dimensional vector space over \( \mathbb{C} \) and let \( V' \) be an \( m \)-dimensional subspace of \( V \). Take a basis of \( V' \) and extend it to a basis of \( V \). In this way we embed \( \text{GL}(m, \mathbb{C}) = \text{GL}(V') \) into \( \text{SL}(V) = \text{SL}(m + 1, \mathbb{C}) \) as follow. For each \( g \in \text{GL}(V') \)

\[
f : g \to \begin{pmatrix} g & 0 \\ 0 & \det(g)^{-1} \end{pmatrix}.
\]
The follow Theorem tells us how their McKay quivers are related.

**Theorem 3.1.** Let $G'$ be a finite subgroup of $GL(V')$ and let $G = f(G') \subset SL(V)$ be its image under the map defined above. Then the McKay quiver of $G'$ is obtained from that of $G'$ by adding an arrow from $\sigma_i$ to $i$ for each vertex $i$ in $Q_{G',0}$.

**Proof.** Let $v_1, \ldots, v_m$ be a basis of $V'$ and $v_1, \ldots, v_{m+1}$ be a basis of $V$. Consider the exterior algebras $\wedge V'$ and $\wedge V$. Then $\wedge V' \simeq \wedge V/(v_{m+1})$ naturally. Now consider the skew group algebra $\wedge V G$. Since the subspace $V'$ and subspace $\langle v_{m+1} \rangle$ spanned by $v_{m+1}$ of $V$ are both invariant under $G$. Consider now the ideal $(v_{m+1})$ generated by $v_{m+1}$, so we have that

$$\wedge V/(v_{m+1}) * G = \wedge V/(v_{m+1}) * G \simeq \wedge V' * G' \subset \wedge V/(v_{m+1}) * G.$$ 

Since $G \simeq G'$, their McKay quiver has the same number of vertices. In $\wedge V' * G'$, we have that the image of $v_{m+1}$ is zero and the longest paths in it are formed by $m$ arrows of different type going from each vertex to its Nakayama translation, while in $\wedge V * G$ the longest paths in it are formed by $m + 1$ arrows of different type going from each vertex to itself by [20], these paths are formed by exactly the $m$ arrows of different in $\wedge V' * G'$ adding one arrows of new type $v_{m+1}$. So we see that for each vertex $i$, the new arrow in $Q_G$ is of type $v_{m+1}$ going from the Nakayama translation $\sigma_i$ of the vertex back to $i$ itself. This proves our Theorem. \(\square\)

In this case, we say that the McKay quiver of $G$ is obtained from that of $G'$ by replacing Nakayama translation with arrows.

**Remark.** Adding an new arrow in certain quiver to get a new one in a similar way appears in the research on higher dimensional Auslander algebra and the cluster algebras recently, see [23, 24].

We have characterized the McKay quiver for finite abelian subgroup $G$ of $SL(m, \mathbb{C})$ in [25]. Since the irreducible representations of an abelian group are all one-dimensional, we can assume that all the elements in the group are diagonal. So we can regard it as an image of a subgroup of $GL(m - 1, \mathbb{C})$, this shows the following Proposition.

**Proposition 3.2.** Let $G$ be an abelian subgroup $G$ of $SL(m, \mathbb{C})$, then there is a subgroup $G'$ of $GL(m - 1, \mathbb{C})$, such that the McKay quiver of $G$ is obtained from that of $G'$ by replacing Nakayama translation with arrows.

4. **Examples**

In this section, we show that certain interesting examples of McKay quivers are obtained in these two ways. These quivers are very useful in mathematics,
it is interesting to know how the construction can be used in the study of the mathematical problems concerning these quivers.

4.1. **McKay quiver of the form double quiver of affine Dynkin diagram $\hat{A}_{n-1}$.** The classical McKay quiver of type $\hat{A}_{n-1}$ can be got by the theory developed in this paper. Start with $V_1 = \mathbb{C}$, let $G_1$ be a finite subgroup of $\text{SL}(1, \mathbb{C}) = \{1\}$, then $G_1$ is a trivial group, and we know that its quiver $Q_{G_1}$ is just one loop.

Now we make extension of $G_1$ with a cyclic group $H$ of order $n + 1$, get finite subgroup $G_2 \simeq H$ of $\text{GL}(1, \mathbb{C}) = \mathbb{C}^*$, the McKay quiver $Q_{G_2}$ of $G_2$ is a regular covering of $Q_{G_1}$, in fact it is a cyclic quiver of $n$ vertices.

Now construct the skew group algebra $\wedge V * G$ with $V = V_1$, its basic algebra $\Lambda(G_1)$ is a selfinjective algebra with vanishing radical square, so the Nakayama translation for vertex $i$ is just the tail of the arrow starting at $i$. Embed $V_1$ in $V_2 = \mathbb{C}^2$ in the way as above, we get the subgroup $G_3$ of $\text{SL}(2, \mathbb{C})$ isomorphic to $G_2$, whose McKay quiver is exactly the double quiver of the affine Dynkin diagram of type $\hat{A}_{n-1}$.

Since all the finite subgroup of $\mathbb{C}^*$ are abelian, we see that the double quiver of affine Dynkin diagram of type $\hat{A}_{n-1}$ are the only ones obtained from that of
subgroups of $GL(1, \mathbb{C}) = \mathbb{C}^*$ by replacing Nakayama translation with arrows. So we get the following proposition.

**Proposition 4.1.** The double quiver of an affine Dynkin diagram is obtained from the McKay quiver of subgroups of $\mathbb{C}^*$ by replacing Nakayama translation with arrows if and only if it is of type $\tilde{A}_n$.

4.2. **From double quiver of type $\tilde{A}_1$ to 3-McKay quivers with 4 vertices.**

Start with the order 2 cyclic subgroup generated by the image of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in $SL(2, \mathbb{C})$, its McKay quiver is the double quiver of type $\tilde{A}_1$.

Extend with order 2 subgroups in $GL(2, \mathbb{C})/SL(2, \mathbb{C})$ generated by the image of $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, respectively, we get two order 4 subgroups in $GL(2, \mathbb{C})$ with the following McKay quivers, respectively.

Note the second one is exactly the double $\tilde{A}_3$ quiver, which is the same as the McKay quiver of order 4 cyclic subgroup of $SL(2, \mathbb{C})$. The difference between the two lies in their Nakayama translations. While Nakayama translation for a subgroup of $SL(2, \mathbb{C})$ is identity, it sends each vertex to the opposite one in our second McKay quiver. This also shows that the Nakayama translation should be essential ingredient when consider McKay quiver for finite subgroups of $GL(m, \mathbb{C})$.

Now embedding these subgroups in $SL(3, \mathbb{C})$, using the map $f$ defined above, we add an arrow for each vertex from its Nakayama translation back to itself and get respectively the following McKay quivers for them. They are just the only 3-McKay quivers with 4 vertex for finite subgroups of $GL(m, \mathbb{C})$ given in [25].
4.3. **An example of a quiver in the study of D-branes.** In their work [16], Govindarajan and Jayaraman use two McKay quivers and Beilinson quivers to describe the D-branes at the orbifold point. Here we show how one of their quivers are constructed from some lower dimensional ones, using the theory of this paper, the other one also has a similar construction. The McKay quiver for $\mathbb{P}^{1,1,1,2}$ in [16] can be obtained as follows. Start with the trivial subgroup of $\text{SL}(4, \mathbb{C})$, one gets McKay quiver with one vertex and four loops, extending it with the subgroup of $\text{GL}(4, \mathbb{C})$ generated by the scalar matrix with scalar $\xi_6$, a 6th primitive root of the unit. One gets the McKay quiver with oriented cyclic quiver with 6 vertices, indexed by $\mathbb{Z}_6$, with 4 arrows from $i$ to $i + 1$ for each $i \in \mathbb{Z}_6$.

In this case, the Nakayama translation sending $i$ to $i + 4$, embedding in $(4, \mathbb{C})$, the new McKay quiver adds for each $i$ an arrow from $i + 4$ to $i$, this is exactly the McKay quiver for $\mathbb{P}^{1,1,1,2}$ in [16].
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