Algebraic and Analytic Aspects of Soliton Type Equations

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Abstract. This is a review of two of the fundamental tools for analysis of soliton equations: i) the algebraic ones based on Kac-Moody algebras, their central extensions and their dual algebras which underlie the Hamiltonian structures of the NLEE; ii) the construction of the fundamental analytic solutions (FAS) of the Lax operator and the Riemann-Hilbert problem (RHP) which they satisfy. The fact that the inverse scattering problem for the Lax operator can be viewed as a RHP gave rise to the dressing Zakharov-Shabat, one of the most effective ones for constructing soliton solutions. These two methods when combined may allow one to prove rigorously the results obtained by the abstract algebraic methods. They also allow to derive spectral decompositions for non-self-adjoint Lax operators.

1. Introduction

We start with three examples of integrable nonlinear evolution equations (NLEE). The first one is the well known $N$-wave equation \[ [49, 48, 35]: \]
\[
 i[I,Q_t] - i[J,Q_t] + [[I,Q],[J,Q]](x,t) = 0, \quad \lim_{x \to \pm \infty} Q(x,t) = 0,
\]
where $Q(x,t)$ is a smooth $n \times n$ matrix-valued function, $Q(x,t) = -BQ_B$ and $I$ and $J$ are constant diagonal matrices; $B_{ij} = \delta_{ij} \epsilon_j$, $\epsilon_j = \pm 1$.

The second example is the 2-dimensional affine Toda chain \[41]:
\[
 \frac{\partial^2 Q_k}{\partial x \partial t} = e^{Q_{k+1} - Q_k} - e^{Q_k - Q_{k-1}}, \quad k = 1, \ldots, n,
\]
where we assume that $e^{Q_{n+1}} \equiv e^{Q_1}$.

The third example belongs to the same family as (1.2) and is of the form:
\[
 i \frac{\partial \psi_k}{\partial t} + \gamma \coth \frac{\pi k}{n} \frac{\partial^2 \psi_k}{\partial x^2} + i \gamma \sum_{p=1}^{n-1} \frac{d}{dx} (\psi_p \psi_{k-p}) = 0, \quad k = 1, \ldots, n,
\]
and $k - p$ is understood modulo $n$ and $\psi_0 = \psi_n = 0$.

The integrability of these equations is based on their Lax representations. This means that each of the NLEE (1.2), (1.3) can be represented as the compatibility condition
\[
 [L(\lambda), M(\lambda)] = 0,
\]
of two linear matrix differential operators depending on the spectral parameter $\lambda$. Below we will use as Lax operator $L(\lambda)$

$$L(\lambda)\psi(x, t, \lambda) = \left( i \frac{d}{dx} + q(x, t) - \lambda J \right) \psi(x, t, \lambda) = 0;$$

as examples of $M(\lambda)$-operators we use:

(1.6a) $M(\lambda)\psi = \left( i \frac{d}{dt} + V_0(x, t) + \lambda I \right) \psi(x, t, \lambda) = \lambda \psi(x, t, \lambda)I$;

(1.6b) $M_1(\lambda)\psi = \left( i \frac{d}{dt} + V_0(x, t) + \lambda V_1(x, t) + \lambda^2 V_2 \right) \psi(x, t, \lambda) = \lambda^2 \psi(x, t, \lambda)V_2^{as};$

(1.6c) $M_2(\lambda)\psi = \left( i \frac{d}{dt} + V_0(x, t) + \frac{1}{\lambda} V_{-1}(x, t) \right) \psi(x, t, \lambda) = \frac{1}{\lambda} \psi(x, t, \lambda)V_{-1}^{as};$

where $V_2^{as} = \lim_{x \to \pm \infty} V_2(x, t)$ and $V_{-1}^{as} = \lim_{x \to \pm \infty} V_{-1}(x, t)$.

Choosing the form of $L(\lambda)$ in (1.4) we fixed up the gauge by assuming that $J$ is constant diagonal matrix and $q(x, t) = [J, Q(x, t)]$; i.e. $q_{ij} = 0$.

The system (1.4) with $q(x, t)$ and $J$ $2 \times 2$-matrices (i.e., $\mathfrak{g} \simeq sl(2))$ is known as the Zakharov-Shabat (ZSs) system; the same system with $n \times n$ matrices will be referred to below as the generalized Zakharov-Shabat system (GZSs).

The Lax representation of the $N$-wave equation is provided by $L(\lambda)$ (1.3) and $M(\lambda)$ (1.6a). If the potentials in these operators are $n \times n$-matrix ones we may assume that the Lie algebra underlying the Lax representation is $\mathfrak{g} \simeq sl(n)$. The set of independent fields $Q_{ij}(x, t)$ equals $n(n - 1)$ and may be restricted by the involutions $[48, 52, 48]$.

$$q(x, t) = Bq^\dagger(x, t)B^{-1}, \quad J = J^\dagger,$$

Often by $N$-wave equations in the literature people mean eq. (1.1) with the involutions (1.7). Such systems with $n = 3$ and $n = 4$ find applications in describing wave-wave interactions, see [48, 19, 55].

The Lax representation of the $Z_n$-NLS eq. (1.3) is provided by (1.3) and (1.6b) but with rather specific restrictions imposed on $q(x, t)$ and $J$:

$$\bar{q}(x, t) = i \sum_{k=1}^{n} \psi_k(x, t) K_k^0, \quad \bar{J} = c_0 \sum_{k=1}^{n} \omega^{-k+1/2} E_{kk}, \quad K_0 = \sum_{k=1}^{n} E_{kk},$$

Here and below we will denote by $E_{kk}$ the $n \times n$-matrices equal to $(E_{kk})_{mn} = \delta_{im}\delta_{kn}$; the indices should be taken modulo $n$, i.e. $E_{n,n+1} = E_{n,1}$ and the constant $c_0$ will be defined below.

The affine Toda chain (1.2) has several equivalent Lax representations. We mention here two of them. Their Lax operators are:

$$\tilde{L}_{\text{Toda}} = i \frac{d}{dx} - i \sum_{k=1}^{n} \frac{dQ_k}{dx} E_{kk} + i\lambda \sum_{k=1}^{n} e^{(Q_{k+1} - Q_k)/2} E_{kk},$$

and its gauge equivalent:

$$\tilde{\bar{L}}_{\text{Toda}} = i \frac{d}{dx} - i \sum_{k=1}^{n} \frac{dQ_k}{dx} E_{kk} + i\lambda K_0.$$
The corresponding $M$-operators are of the form (1.6). Both choices (1.9) and (1.10) are not of the form (1.3), but are adjusted to the grading of the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ we introduce in the next subsection, see formulae (1.24)–(1.25) below.

The operator $\tilde{L}_{\text{Toda}}$ (1.11) after a similarity transformation with the constant matrix $u_0$, such that $u_0^{-1}K_0u_0 = \sum_{k=1}^{n} \omega^k E_{kk}$ can be cast into the form of (1.3) in which both $q(x, t)$ and $J$ have a special form:

\[
\hat{q}(x, t) = -i \sum_{k=1}^{n} \frac{dQ_k}{dx} \omega^{kp} K_0^p, \quad \hat{J} = c_0 \sum_{j=1}^{n} \omega^{k-1/2} E_{kk}.
\]

where $\omega = \exp(2\pi i/n)$. The special form of $q(x, t)$ and $J$ in both (1.8) and (1.11) shows that both models have only $n - 1$ independent fields. This special form can also be made compatible with the structure of the graded and Kac-Moody algebras [11, 33, 31] and is best understood with the method of the reduction group proposed by Mikhailov [41].

The idea of the ISM is based on the possibility to linearize the NLEE [53, 2, 3, 13, 49, 48, 35]. To this end we consider the solution to the NLEE $q(x, t)$ as a potential in $L(\lambda)$ (1.3). In order to solve the direct scattering problem for $L(\lambda)$ we introduce the Jost solutions $\psi_{\pm}(x, t, \lambda)$ and the scattering matrix $T(\lambda, t)$ as follows:

\[
\begin{align*}
\frac{i}{\hbar} \frac{d\psi_{\pm}}{dx} + \left( q(x, t) - \lambda J \right) \psi_{\pm}(x, t, \lambda) &= 0, \\
\lim_{x \to \pm \infty} \psi_{\pm}(x, t, \lambda) e^{i\lambda Jx} &= \mathbb{1}, \\
T(\lambda, t) &= \psi_{-}^{-1}(x, t, \lambda) \psi_{+}(x, t, \lambda).
\end{align*}
\]

The Jost solutions of $L(\lambda)$ are also eigenfunctions of the operator $M(\lambda)$. We can use this fact to determine the $t$-dependence of the scattering matrix:

\[
\begin{align*}
\frac{i}{\hbar} \frac{dT}{dt} + [f(\lambda), T(\lambda, t)] &= 0,
\end{align*}
\]

which can be easily solved as follows:

\[
T(\lambda, t) = e^{i f(\lambda)t} T(\lambda, 0) e^{-i f(\lambda)t}.
\]

By $f(\lambda) \in \mathfrak{h}$ above we mean the dispersion law of the NLEE; for the $N$-wave system we have $f_{N-\omega}(\lambda) = \lambda I$.

Thus the solution of the NLEE for a given initial condition $q(x, t)|_{t=0} = q_0(x)$ can be performed in three steps, see [18, 3, 13]:

1. insert $q(x, 0)$ as a potential in $L(\lambda)$ and determine the corresponding scattering matrix $T(\lambda, 0)$;
2. Given $T(\lambda, 0)$ and the dispersion law $f(\lambda)$ find $T(\lambda, t)$ from eq. (1.16);
3. Given $T(\lambda, t)$ reconstruct the corresponding potential $q(x, t)$ of $L(\lambda)$ which will be also the solution of the NLEE.

Step 2 is trivial. Steps 1 and 3 involve solving the direct and inverse scattering problem for (1.3) which can be reduced to linear integral equations. The most difficult step 3 provided for the name of the method. The most effective method to solve it for operators like $L(\lambda)$ is based on the equivalence to a RHP [41].

Along with solving the inverse scattering problem in step 3) we will construct also the minimal set of scattering data $\mathfrak{T}$. Indeed the scattering matrix $T(\lambda, t)$ has $n^2$ matrix elements with only one obvious constraint $\det T(\lambda, t) = 1$ while
the potential \( q(x,t) \) has only \( n(n-1) \) matrix elements. Therefore there must be additional interrelations between the matrix elements of \( T(\lambda, t) \).

The analysis of the mapping between \( q(x,t) \) and \( \mathcal{T} \) allows one to interpret it as a generalized Fourier transform \( \mathcal{F} \). The proof of all these facts and the effective solving of the ISP for the GZSs (1.5) is based on the possibility to construct fundamental solutions of (1.5) which are section-analytic functions of the spectral parameter \( \lambda \).

**Algebraic structures: Kac-Moody and graded Lie algebras**

Let us now briefly outline the first basic tool inherent in the Lax representation – its algebraic structure. Indeed, \( L(\lambda) \) and \( M(\lambda) \) above are polynomial in \( \lambda \) and/or \( 1/\lambda \) whose coefficients take values in some simple Lie algebra \( g \).

Let us take generic Lax operators in the form:

\[
L(\lambda) \psi \equiv \left( i \frac{dx}{dt} + \sum U_k(x,t) \lambda^k \right) \psi(x,t,\lambda) = 0,
\]

\[
M(\lambda) \psi \equiv \left( i \frac{dt}{dx} + \sum V_k(x,t) \lambda^k \right) \psi(x,t,\lambda) = \psi(x,t,\lambda) V^m(\lambda),
\]

where the potentials \( U(x,t,\lambda) \) and \( V(x,t,\lambda) \) are polynomials in \( \lambda \) and/or \( 1/\lambda \). Such potentials can be viewed as elements of a Kac-Moody algebra \( \hat{g}_C \). Roughly speaking the construction of \( \hat{g}_C \) involves a simple Lie algebra \( g \) and an automorphism \( C \) of finite order, i.e. there exist such an integer \( h \) that \( C^n = 1 \). Then we can split \( g \) into a direct sum of linear subspaces

\[
g = \bigoplus_{k=0}^{h-1} g^{(k)},
\]

which are eigensubspaces of \( C \), i.e. if

\[
X^{(k)} \in g^{(k)} \iff C(X^{(k)}) = \omega^{-k} X^{(k)},
\]

where \( \omega = \exp(2\pi i/h) \). The decomposition (1.20) satisfies the grading condition:

\[
[X^{(k)}, X^{(m)}] = X^{(k+m)} \in g^{(k+m)}.
\]

where the superscript \( k + m \) in \( g^{(k+m)} \) is understood modulo \( h \). Then the elements of the corresponding Kac-Moody algebra \( \hat{g}_C \) have the form:

\[
X(\lambda) = \sum_{k \leq N_1} \lambda^k X^{(k)}, \quad X^{(k)} \in g^{(k)}.
\]

Obviously due to (1.22) the commutator of any two elements \( X(\lambda), Y(\lambda) \) of the form (1.23) will also have the form (1.23).

The classification and the theory of the Kac-Moody algebras can be found in [33, 31]. Their simplest realization can be obtained from a pair \( (g,C) \) with a few special choices of the automorphism of finite order \( C \), namely:

a) \( C = 1 \); then each of the subspaces \( g^{(k)} \approx g \). This leads to a generic GZS system if \( J \) is real and to a generic CBC system if \( J \) is complex.
b) $C^h = 1$ where $C$ is the Coxeter automorphism of $\mathfrak{g}$ and $h$ is the Coxeter number. This leads to a CBC system with $\mathbb{Z}_n$-reduction and will be used in analyzing the NLEE (1.2) and (1.3).

c) $CV$ where $V$ is a nontrivial external automorphism of $\mathfrak{g}$. Such gradings also lead to interesting NLEE but will not be used in this paper.

The Kac-Moody algebras are obtained from the constructions a)–c) with additional central extensions; they are split into three classes: of height 1 (cases a) and b)) and of height 2 and 3 depending on the order of $V$.

The idea to use finite order automorphisms for the reductions of the NLEE was proposed first by Mikhailov [41] who introduced also the notion of the reduction group. The $\mathbb{Z}_n$-reduction condition according to [41] is introduced by:

$$C(X) = C_0XC_0^{-1}, \quad C_0 = \sum_{k=1}^n \omega^k E_{kk}, \quad \omega = e^{2\pi i/n},$$

where $C$ obviously satisfies $C^n = \mathbb{1}$. With this choice of $C$ we can easily check that the linear subspaces $\mathfrak{g}^{(k)}$ are spanned by

$$\mathfrak{g}^{(k)} \equiv \text{l.c.} \left\{ E_{j,j+k}, \quad j, k = 1, \ldots, n \right\},$$

and $j + k$ is considered modulo $n$. Comparing (1.8), (1.11) with (1.25) below we find that $\tilde{q}(x,t) \in \mathfrak{g}^{(0)}$ and $J \in \mathfrak{g}^{(1)}$. Note that now the condition $X^{(k)} \in \mathfrak{g}^{(k)}$ imposes a set of nontrivial constraints on $X^{(k)}$.

The potential $\tilde{U}(x,t,\lambda)$ for the $N$-wave equations equals $[J, Q(x,t)] - \lambda J$ belongs to a Kac-Moody algebra with $\mathfrak{g} \simeq sl(n)$ and $C = \mathbb{1}$. The potential $\tilde{U}(x,t,\lambda) = \tilde{q}(x,t) - \lambda J$ of the form (1.8) and (1.11) gives rise to the NLEE (1.2) and (1.3) is related to Kac-Moody algebra of the class b) with $\mathfrak{g} \simeq sl(n)$. The Coxeter number then is $h = n$; the Coxeter automorphism can be realized as inner automorphism of the form:

$$C(X) = C_0XC_0^{-1}, \quad C_0 = \sum_{k=1}^n \omega^k E_{kk}, \quad \omega = e^{2\pi i/n},$$
Important role for the Hamiltonian formulation of the NLEE is played by the dual algebras \( \hat{\mathfrak{g}}^* \), \( \tilde{\mathfrak{g}}^* = \hat{\mathfrak{g}}^* \oplus c \) and their splittings into direct sums of Borel-like subalgebras. These splittings for \( \hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_- \) look like:

\[
\begin{align*}
\hat{\mathfrak{g}}_+ & \equiv \left\{ \sum_{k=0}^{N_1} u_k(x) \lambda^k \right\}, \\
\hat{\mathfrak{g}}_- & \equiv \left\{ \sum_{k=-\infty}^{-1} u_k(x) \lambda^k \right\},
\end{align*}
\]

and for the dual \( \hat{\mathfrak{g}}^* = \hat{\mathfrak{g}}^*_+ \oplus \hat{\mathfrak{g}}^*_- \):

\[
\begin{align*}
\hat{\mathfrak{g}}^*_+ & \equiv \left\{ \sum_{k=-N_1}^{p} u_k(x) \lambda^k \right\}, \\
\hat{\mathfrak{g}}^*_- & \equiv \left\{ \sum_{k=p+1}^{\infty} u_k(x) \lambda^k \right\}.
\end{align*}
\]

The co-adjoint orbits of \( \tilde{\mathfrak{g}} \) on \( \tilde{\mathfrak{g}}^* \) in fact are isomorphic to the space of coefficients for which the NLEE is written. Thus they are natural candidate for the phase space of these equations. The freedom provided by the parameter \( p \) is directly related to the existence of hierarchy of Hamiltonian structures for the NLEE.

Fundamental analytic solutions

The second important tool in this scheme is the fundamental analytic solution (FAS) of \( L(\lambda) \). We will see that using the FAS one is able to:

- reduce the solving of the ISP for \( L(\lambda) \) to an equivalent Riemann-Hilbert problem (RHP) for the FAS \([45, 48, 52]\);
- construct the kernel of the resolvent for \( L(\lambda) \) and derive the spectral decomposition for \( L(\lambda) \) \([26, 45, 30]\);
- construct the ‘squared’ solutions of \( L(\lambda) \) which allow the interpretation of the ISM as a generalized Fourier transform (GFT) \([2, 34, 36, 25, 28, 29, 30]\);
- construct the Green function for the recursion operators \( \Lambda_\pm \) and prove the completeness relation for the ‘squared’ solutions. This property ensures the uniqueness of the solution of the ISM \([25, 27, 49, 30]\).

The existence of FAS is ensured by the analytic dependence of both \( U(x, t, \lambda) \) and \( V(x, t, \lambda) \) on \( \lambda \). The properties of FAS depend crucially on the boundary conditions imposed on the potential \( q(x, t) \). For simplicity here we consider the class of potentials \( q(x, t) \) that are sufficiently smooth functions of \( x \) and tend to zero fast enough for \( x \to \pm \infty \) for any fixed value of \( t \).

The FAS for the Zakharov-Shabat system (i.e. \( \mathfrak{g} \simeq sl(2) \)) can easily be constructed due to the fact that each of the columns of the Jost solutions

\[
L(\lambda) \psi_{\pm}(x, t, \lambda) = 0, \quad \lim_{x \to \pm \infty} e^{iJ\lambda x} \psi_{\pm}(x, t, \lambda) = \mathbf{1},
\]

allow analytic extension either for \( \lambda \in \mathbb{C}_+ \) or for \( \lambda \in \mathbb{C}_- \), see \([2]\).

If we analyze the analyticity properties of the Jost solutions \( \psi_{\pm}(x, t, \lambda) \) related to algebras of higher rank one finds that only the first and the last columns of \( \psi_{\pm}(x, t, \lambda) \) allow analytic extensions off the real \( \lambda \)-axis. An important result of Zakharov and Manakov \([13, 18]\) consisted in showing that a FAS for the GZS with \( \mathfrak{g} \simeq sl(n) \) and real-valued \( J \) can be constructed by taking proper linear combinations of the columns of \( \psi_{\pm}(x, t, \lambda) \).

The construction is more complicated for the Caudrey-Beals-Coifman (CBC) systems when the eigenvalues of \( J \) are complex \([3, 6, 8]\). The generalization of this construction for CBC systems related to any simple Lie algebra \( \mathfrak{g} \) was done in \([30]\).
We make attempt to outline the construction and the properties of each of these tools. Then we show how the FAS can be used to construct the kernel of the resolvent of \( L(\lambda) \) and to exhibit its spectral properties and the structure of its discrete spectrum. Finally, we illustrate how these tools can be used in the analysis of the NLEE and their fundamental properties and finish with some conclusions.

Both these aspects are rather broad; they have been widely discussed in hundreds of papers. Therefore inevitably the list of references consists mainly of reviews and monographs and bears an illustrative character. The thorough reader is advised to consult also the papers referred to in these references.

2. Construction of the FAS

Preliminaries: Jost solutions and scattering matrix

The direct and the inverse scattering problem for the Lax operator \((1.13)\) will be done for fixed \( t \) and in most of the corresponding formulae \( t \) will be omitted.

The crucial fact that determines the spectral properties of the operator \( L \) is the choice of the class of functions where from we shall choose the potential \( q(x) \). Below for simplicity we assume that the potential \( q(x) \) is such that the corresponding operator \( L \) has only a finite number of simple discrete eigenvalues.

Below we will use along with \( L\psi(x, \lambda) = 0 \) also the following equivalent formulations of the system \((1.3):\)

\[
\begin{align*}
(2.1) \quad & i\frac{d\xi}{dx} + q(x, t)\xi(x, \lambda) - \lambda[J, \xi(x, \lambda)] = 0, \quad \xi(x, \lambda) = \psi(x, \lambda)e^{i\lambda J x}, \\
(2.2) \quad & i\frac{d\hat{\psi}}{dx} - \hat{\psi}(x, \lambda)q(x, t) + \lambda\hat{\psi}(x, \lambda)J = 0, \quad \hat{\psi}(x, \lambda) = (\psi(x, \lambda))^{-1}, \\
(2.3) \quad & i\frac{d\hat{\xi}}{dx} - \hat{\xi}(x, \lambda)q(x, t) + \lambda\hat{\xi}(x, \lambda), J = 0, \quad \hat{\xi}(x, \lambda) = e^{-i\lambda J x}\hat{\psi}(x, \lambda),
\end{align*}
\]

where by ‘hat’ we denote the inverse matrix, \( \hat{X} = X^{-1} \). The Jost solutions \( \xi_{\pm}(x, \lambda) \) and \( \hat{\xi}_{\pm}(x, \lambda) \) for the systems \((2.1), (2.3)\) can be introduced by:

\[
\lim_{x \to \pm\infty} \xi_{\pm}(x, \lambda) = 1, \quad \lim_{x \to \pm\infty} \psi_{\pm}(x, \lambda)e^{-i\lambda J x} = 1, \quad \lim_{x \to \pm\infty} \hat{\xi}_{\pm}(x, \lambda) = 1,
\]

in analogy to \((1.13)\); then their scattering matrices are:

\[
T_2(\lambda) \equiv e^{i\lambda J x}(\xi_{-}(x, \lambda))^{-1}\xi_{+}(x, \lambda)e^{-i\lambda J x} = T(\lambda),
\]

\[
T_3(\lambda) \equiv \hat{\psi}_{+}(x, \lambda)(\hat{\psi}_{-}(x, \lambda))^{-1} = T^{-1}(\lambda),
\]

\[
T_4(\lambda) \equiv e^{i\lambda J x}\hat{\xi}_{+}(x, \lambda)(\hat{\xi}_{-}(x, \lambda))^{-1}e^{-i\lambda J x} = T^{-1}(\lambda),
\]

Below we will consider two specific reductions of the Lax operator: the GZSs with \( \mathbb{Z}_2 \)-reduction:

\[
B(U^t(x, t, \epsilon\lambda^*)B^{-1} = U(x, t, \lambda), \quad B^2 = 1, \quad \epsilon = \pm 1.
\]
The first possible choice for \( B = \text{diag}(\epsilon_1, \ldots, \epsilon_n), \epsilon_j = \pm 1 \) with \( \epsilon = 1 \) leads to the classical \( N \)-wave equations \(^{49, 48}\) with
\[
\begin{align*}
& (2.5) & q_{kj}(x, t) = \epsilon_k \epsilon_j q_{jk}(x, t), \quad J = \text{diag}(a_1, \ldots, a_n), \quad a_k = \epsilon a_k^*.
\end{align*}
\]
Since all eigenvalues of \( J \) are real (\( \epsilon = 1 \)), or purely imaginary (\( \epsilon = -1 \)), the Lax operator becomes a GZSs. The second choice for \( B \):
\[
\begin{align*}
& (2.6) & B = \sum_{k=1}^{n} E_{kk}, \quad \bar{k} = n + 1 - k, \quad \epsilon = -1,
\end{align*}
\]
will be used in combination with the \( \mathbb{Z}_n \)-reduction:
\[
\begin{align*}
& (2.7) & \tilde{C}_0(U(x, t, \omega \lambda)) \tilde{C}_0^{-1} = U(x, t, \lambda), \quad \tilde{C}^n = \mathbb{1}.
\end{align*}
\]
which leads to the CBC system. For the sake of convenience in doing the spectral problem of CBCs we choose \( C_0 = \sum_{k=1}^{n} E_{k,k+1} \); then \( L(\lambda) \) has the form \(^{(1.5)}\) with diagonal complex-valued \( J \) given by \(^{(1.8)}\) or \(^{(1.11)}\) where \( c_0 = 1 \) (resp. \( c_0 = i \)) if \( \epsilon = 1 \) (resp. \( \epsilon = -1 \)). Both Lax operators will have similar spectral properties.

In solving the NLEE \(^{(1.2)}\) and \(^{(1.3)}\) we will need to apply both reductions \(^{(2.4)}\) and \(^{(2.7)}\) simultaneously. An attempt for classification of the \( \mathbb{Z}_2 \)-reductions is made in \(^{(2.3)}\).

**The FAS of the GZSs with \( \mathbb{Z}_2 \)-reduction.**

Let us outline without proofs the construction of the FAS for the GZSs with real \( J \), see \(^{49, 48, 5, 8, 30}\). For definiteness we assume that the real eigenvalues of \( J \) are pair-wise different and ordered as follows:
\[
\begin{align*}
& (2.8) & J = \text{diag}(a_1, \ldots, a_n), \quad a_1 > a_2 > \cdots > a_n.
\end{align*}
\]

**Proposition 2.1.** Let the potential of \(^{(1.3)}\) \( q(x) \in M_s \) satisfies conditions \(^{(C.1)}, (C.2)\) and \(^{(2.3)}\). Then:

a) the Jost solutions \( \xi_{\pm}(x, \lambda) \) and \( \hat{\xi}_{\pm}(x, \lambda) \) of \(^{(2.4)}, (2.2)\) exist and are well defined functions for \( \lambda \in \mathbb{R} \);

b) the matrix elements of the scattering matrix \( T(\lambda) \) and its inverse \( \hat{T}(\lambda) \) are Schwartz-type functions of \( \lambda \) for \( \lambda \in \mathbb{R} \).

**Remark 2.2.** The proposition 2.1 concerns the Jost solutions as fundamental solutions. One can prove that the first and the last columns \( \xi_{\pm}^{[1]}(x, \lambda) \) and \( \xi_{\pm}^{[n]}(x, \lambda) \) of the Jost solutions allow analytic extension with respect to \( \lambda \) as follows:

| Column | \( \xi_{\pm}^{[1]}(x, \lambda) \) | \( \xi_{\pm}^{[n]}(x, \lambda) \) | \( \xi_{\pm}^{[1]}(x, \lambda) \) | \( \xi_{\pm}^{[n]}(x, \lambda) \) |
|--------|-----------------|-----------------|-----------------|-----------------|
| Analytic for | \( \lambda \in \mathbb{C}_- \) | \( \lambda \in \mathbb{C}_+ \) | \( \lambda \in \mathbb{C}_- \) | \( \lambda \in \mathbb{C}_+ \) |

Analogously the first and the last rows of the Jost solutions \( \hat{\xi}_{\pm}^{[1]}(x, \lambda) \) and \( \hat{\xi}_{\pm}^{[n]}(x, \lambda) \) allow analytic extension with respect to \( \lambda \) as follows:

| Row | \( \hat{\xi}_{\pm}^{[1]}(x, \lambda) \) | \( \hat{\xi}_{\pm}^{[n]}(x, \lambda) \) | \( \hat{\xi}_{\pm}^{[1]}(x, \lambda) \) | \( \hat{\xi}_{\pm}^{[n]}(x, \lambda) \) |
|-----|-----------------|-----------------|-----------------|-----------------|
| Analytic for | \( \lambda \in \mathbb{C}_+ \) | \( \lambda \in \mathbb{C}_- \) | \( \lambda \in \mathbb{C}_- \) | \( \lambda \in \mathbb{C}_+ \) |

All the other columns of \( \xi_{\pm}(x, \lambda) \) and rows of \( \hat{\xi}_{\pm}(x, \lambda) \) are defined only for \( \lambda \in \mathbb{R} \) and do not allow analytic extensions off the real axis.
We start by explaining the construction of the FAS $\chi^\pm(x, \lambda)$ or rather of the solutions
\begin{equation}
\xi^\pm(x, \lambda) = \chi^\pm(x, \lambda)e^{i\lambda Jx}
\end{equation}
to equation (2.1) which allow analytic extensions for $\lambda \in \mathbb{C}_\pm$. Skipping the details (see \textsuperscript{[45, 48, 49]} we formulate the answer and determine $\xi^\pm(x, \lambda)$ as the solution of the following set of integral equations:
\begin{align}
(2.10a) \xi^+_{ij}(x, \lambda) &= \delta_{ij} + i \int_{-\infty}^{x} dy e^{-i\lambda(a_i-a_j)(x-y)} \sum_{p=1}^{h} q_{ip}(y) \xi^+_p(y, \lambda), \quad i \geq j; \\
(2.10b) \xi^+_{ij}(x, \lambda) &= i \int_{-\infty}^{x} dy e^{-i\lambda(a_i-a_j)(x-y)} \sum_{p=1}^{h} q_{ip}(y) \xi^+_p(y, \lambda), \quad i < j;
\end{align}
Analogously we define $\xi^-(x, \lambda)$ as the solution of the set of integral equations:
\begin{align}
(2.11a) \xi^-_{ij}(x, \lambda) &= i \int_{-\infty}^{x} dy e^{-i\lambda(a_i-a_j)(x-y)} \sum_{p=1}^{h} q_{ip}(y) \xi^-_p(y, \lambda), \quad i > j; \\
(2.11b) \xi^-_{ij}(x, \lambda) &= \delta_{ij} + i \int_{-\infty}^{x} dy e^{-i\lambda(a_i-a_j)(x-y)} \sum_{p=1}^{h} q_{ip}(y) \xi^-_p(y, \lambda), \quad i \leq j;
\end{align}

\textbf{Theorem 2.3.} Let $q(x) \in \mathfrak{M}_S$ satisfies conditions (C.1), (C.2) and let $J$ satisfy (2.9). Then the solution $\xi^+(x, \lambda)$ of the eqs. (2.14) (resp. $\xi^-(x, \lambda)$ of the eqs. (2.11)) exists and allows analytic extension for $\lambda \in \mathbb{C}_+$ (resp. for $\lambda \in \mathbb{C}_-$).

\textbf{Remark 2.4.} Due to the fact that in eq. (2.10) we have both $\infty$ and $-\infty$ as lower limits the equations are rather of Fredholm than of Volterra type. Therefore we have to consider also the Fredholm alternative, i.e. there may exist finite number of values of $\lambda = \lambda^\pm_k \in \mathbb{C}_\pm$ for which the solutions $\xi^\pm(x, \lambda)$ have zeroes and pole singularities in $\lambda$. The points $\lambda^\pm_k$ in fact are the discrete eigenvalues of $L(\lambda)$ in $\mathbb{C}_\pm$.

The reduction condition (2.4) with $\epsilon = 1$ means that the FAS and the scattering matrix $T(\lambda)$ satisfy:
\begin{equation}
(2.12) B \left( \chi^+(x, \lambda^*) \right)^\dagger B^{-1} = (\chi^-(x, \lambda))^{-1}, \quad B (T(\lambda^*))^\dagger B^{-1} = (T(\lambda))^{-1}.
\end{equation}
Each fundamental solution of the Lax operator is uniquely determined by its asymptotic for $x \to \infty$ or $x \to -\infty$. Therefore in order to determine the linear relations between the FAS and the Jost solutions for $\lambda \in \mathbb{R}$ we need to calculate the asymptotics of FAS for $x \to \pm\infty$. Taking the limits in the right hand sides of the integral equations (2.11) and (2.11) we get:
\begin{align}
(2.13a) \lim_{x \to -\infty} \xi^+_{ij}(x, \lambda) &= \delta_{ij}, \quad \text{for } i \geq j; \quad \lim_{x \to -\infty} \xi^+_{ij}(x, \lambda) = 0, \quad \text{for } i < j; \\
(2.13b) \lim_{x \to -\infty} \xi^-_{ij}(x, \lambda) &= \delta_{ij}, \quad \text{for } i \leq j; \quad \lim_{x \to -\infty} \xi^-_{ij}(x, \lambda) = 0, \quad \text{for } i > j;
\end{align}
This can be written in compact form using (2.9):
\begin{equation}
\chi^\pm(x, \lambda) = \psi^-(x, \lambda)S^\pm(\lambda) = \psi^+(x, \lambda)T^\pm(\lambda)D^\pm(\lambda),
\end{equation}
where the matrices \( S^\pm(\lambda), D^\pm(\lambda) \) and \( T^\pm(\lambda) \) are of the form:

\[
\begin{align*}
(2.15a) \quad S^+(\lambda) &= \begin{pmatrix} 1 & S_{12}^+ & \cdots & S_{1n}^+ \\ 0 & 1 & \cdots & S_{2n}^+ \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, &
T^+(\lambda) &= \begin{pmatrix} 1 & T_{12}^+ & \cdots & T_{1n}^+ \\ 0 & 1 & \cdots & T_{2n}^+ \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \\
(2.15b) \quad D^+(\lambda) &= \text{diag}(D_1^+, D_2^+, \ldots, D_n^+), &
D^- (\lambda) &= \text{diag}(D_1^-, D_2^-, \ldots, D_n^-), \\
(2.15c) \quad S^-(\lambda) &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ S_{21}^- & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ S_{n1}^- & S_{n2}^- & \cdots & 1 \end{pmatrix}, &
T^- (\lambda) &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ T_{21}^- & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ T_{n1}^- & T_{n2}^- & \cdots & 1 \end{pmatrix},
\end{align*}
\]

Let us now relate the factors \( T^\pm(\lambda), S^\pm(\lambda) \) and \( D^\pm(\lambda) \) to the scattering matrix \( T(\lambda) \). Comparing \( (2.14) \) with \( (1.14) \) we find

\[
(2.16) \quad T(\lambda) = T^-(\lambda)D^+(\lambda)\hat{S}^+(\lambda) = T^+(\lambda)D^-(\lambda)\hat{S}^-(\lambda),
\]

i.e. \( T^\pm(\lambda), S^\pm(\lambda) \) and \( D^\pm(\lambda) \) are the factors in the Gauss decomposition of \( T(\lambda) \).

It is well known how given \( T(\lambda) \) one can construct explicitly its Gauss decomposition, see the Appendix A. Here we need only the expressions for \( D^\pm(\lambda) \):

\[
(2.17) \quad D^+_j(\lambda) = \frac{m^+_j(\lambda)}{m^+_{j-1}(\lambda)}, \quad D^-_j(\lambda) = \frac{m^-_{n-j+1}(\lambda)}{m^-_{n-j}(\lambda)},
\]

where \( m^\pm_j \) are the principal upper and lower minors of \( T(\lambda) \) of order \( j \).

**Corollary 2.5.** The upper (resp. lower) principal minors \( m^\pm_j(\lambda) \) (resp. \( m^-_j(\lambda) \)) of \( T(\lambda) \) are analytic functions of \( \lambda \) for \( \lambda \in \mathbb{C}_+ \) (resp. for \( \lambda \in \mathbb{C}_- \)).

**Proof.** Follows directly from theorem 2.3 from the limits:

\[
(2.18) \quad \lim_{x \to \infty} \xi^+_j(x, \lambda) = D^+_j(\lambda), \quad \lim_{x \to -\infty} \xi^-_j(x, \lambda) = D^-_j(\lambda),
\]

and from \( (2.15b) \) and \( (2.17) \). \( \square \)

**Corollary 2.6.** The following relations hold:

a) \( \lim_{\lambda \to \infty} \xi^\pm(x, \lambda) = \#; \quad b) \lim_{\lambda \to \infty} m^\pm_j(\lambda) = 1. \)

**Proof.** a) follows from the integral equations \( (2.10), (2.11) \) taking into account that the integrands in their right hand sides vanish for \( \lambda \to \infty \). b) follows from a), \( (2.18) \) and \( (2.15b) \). \( \square \)

In what follows we will also assume that the set of minors \( m^\pm_k(\lambda) \) have only finite number of simple zeroes located at the points

\[
(2.19) \quad \mathcal{Z} = \{\lambda^\pm_j \in \mathbb{C}_\pm, \ j = 1, \ldots, N.\}
\]

Generically each of the \( \lambda^\pm_j \) can be a zero of a string of minors, e.g.:

\[
(2.20) \quad m^\pm_j(\lambda) = (\lambda - \lambda^+_j)m^\pm_{k,j} + \mathcal{O}((\lambda - \lambda^+_j)^2),
\]

for \( 1 \leq I_j < F_j \leq n \). Let us introduce the quantities \( b_{jk} \) as follows:

\[
(2.21) \quad b_{jk} = \begin{cases} 1 & \text{if } \lambda^+_j \text{ is a zero of } m^\pm_k(\lambda); \\
0 & \text{if } \lambda^+_j \text{ is not a zero of } m^\pm_k(\lambda). \end{cases}
\]
and note that the reduction (2.4) means that the Gauss factors of $T(\lambda)$ satisfy $(\epsilon = 1)$:

(2.22a) $B^\dagger \left( \hat{S}^+(\lambda^*) \right) B^{-1} = S^-(\lambda)$, $B^\dagger \left( \hat{T}^+(\lambda^*) \right) B^{-1} = T^-(\lambda),$

(2.22b) $\left( \hat{D}^+(\lambda^*) \right)^\dagger = D^-(\lambda).$

The relations (2.22a) are strictly valid only for $\lambda \in \mathbb{R}$ while (2.22b) together with (2.15b) and (2.17) leads to the following constraints on the minors $m_k^\pm(\lambda)$:

(2.23) $(m_k^+(\lambda^*))^* = m_{-k}^-(\lambda).$

Thus if $\lambda_k^+$ is a zero of $m_k^+(\lambda)$ then $\lambda_k^- = (\lambda_k^+)^*$ is a zero of $m_{-k}^-(\lambda).$

**The FAS of the CBCs with $Z_n$-reduction.**

The crucial difference with the $\mathbb{Z}_2$-case treated above consists in the fact that now $J$ is given by (1.8) or (1.11) and has complex eigenvalues. Skipping the details (see [3, 6, 8, 10]) we just outline the procedure of constructing the FAS.

First we have to determine the regions of analyticity. For potentials $q(x)$ satisfying the conditions (C.1) and (C.2) and subject to the $Z_n$-reduction (2.7) these regions are the $2n$ sectors $\Omega_\nu$, separated by the rays $l_\nu$ on which $\Im \lambda(a_j - a_k) = 0$. We remind that if we assume also the $Z_2$-reduction (2.4) with $c_0^\nu = \epsilon c_0$ then $a_k = q_0 \omega^{k-1/2}$. Then the rays $l_\nu$ are given by:

(2.24) $l_\nu: \arg(\lambda) = \phi_0 + \frac{\pi(\nu - 1)}{n}, \quad \nu = 1, \ldots, 2n,$

where $\phi_0 = \pi/(2n)$ only if $\epsilon = 1$ and $n$ is odd; in all other cases $\phi_0 = 0$. Thus the neighboring rays $l_\nu$ and $l_{\nu + 1}$ close angles equal to $\pi/n$.

The next step is to construct the set of integral equations analogous to (2.10) whose solution will be analytic in $\Omega_\nu$. To this end we associate with each sector $\Omega_\nu$ the relations (orderings) $\nu \geq \nu$ and $\nu < \nu$ by:

(2.25) $\nu \geq \nu$ if $\Im \lambda(a_i - a_j) < 0$ for $\lambda \in \Omega_\nu,$ $\nu < \nu$ if $\Im \lambda(a_i - a_j) > 0$ for $\lambda \in \Omega_\nu.$

Then the solution of the system (2.10)

(2.26a) $\xi^{\nu}_{ij}(x, \lambda) = \delta_{ij} + i \int_0^\infty dy e^{-i\lambda(a_i - a_j)(x-y)} \sum_{p=1}^h q_{ip}(y) \xi^{\nu}_{pj}(y, \lambda), \quad i \geq j;$

(2.26b) $\xi^{\nu}_{ij}(x, \lambda) = i \int_0^\infty dy e^{-i\lambda(a_i - a_j)(x-y)} \sum_{p=1}^h q_{ip}(y) \xi^{\nu}_{pj}(y, \lambda), \quad i < j;$

will be the FAS of the CBCs in the sector $\Omega_\nu$. The asymptotics of $\xi^{\nu}(x, \lambda)$ and $\xi^{\nu-1}(x, \lambda)$ along the ray $l_\nu$ can be written in the form:

(2.27a) $\lim_{x \to -\infty} e^{i\lambda J x} \xi^{\nu}(x, \lambda e^{i0}) e^{-i\lambda J x} = S^+_\nu(\lambda), \quad \lambda \in l_\nu,$

(2.27b) $\lim_{x \to -\infty} e^{i\lambda J x} \xi^{\nu-1}(x, \lambda e^{i0}) e^{-i\lambda J x} = S^-_\nu(\lambda), \quad \lambda \in l_\nu,$

(2.27c) $\lim_{x \to \infty} e^{i\lambda J x} \xi^{\nu}(x, \lambda e^{i0}) e^{-i\lambda J x} = T^+_\nu D^+_\nu(\lambda), \quad \lambda \in l_\nu,$

(2.27d) $\lim_{x \to \infty} e^{i\lambda J x} \xi^{\nu-1}(x, \lambda e^{i0}) e^{-i\lambda J x} = T^-_\nu D^-_\nu(\lambda), \quad \lambda \in l_\nu,$
where the matrices $S^+_\nu$, $T^+_\nu$ (resp. $S^-\nu$, $T^-\nu$) are upper-triangular (resp. lower-triangular) with respect to the $\nu$-ordering. As in the previous case they provide the Gauss decomposition of the scattering matrix with respect to the $\nu$-ordering, i.e.:

\[(2.28) \quad T^\nu_\nu(\lambda) = T^-_\nu(\lambda)D^+_\nu(\lambda)\hat S^+_\nu(\lambda) = T^+_\nu(\lambda)D^-_\nu(\lambda)\hat S^-_\nu(\lambda), \quad \lambda \in \Omega_\nu.\]

More careful analysis shows that in fact $T^\nu_\nu(\lambda)$ belongs to a subgroup $\mathfrak{G}_\nu$ of $SL(n, \mathbb{C})$. Indeed, with each ray $l_\nu$ one can relate a subalgebra $\mathfrak{g}_\nu \subset sl(n, \mathbb{C})$.

If $Z_\nu$-symmetry is present each of these subalgebras $\mathfrak{g}_\nu$ is a direct sum of $sl(2)$-subalgebras. Each such $sl(2)$-subalgebra can be specified by a pair of indices $(k, s)$ and is generated by:

\[(2.29) \quad h^{(k,s)} = E_{kk} - E_{ss}, \quad e^{(k,s)} = E_{ks}, \quad f^{(k,s)} = E_{sk}, \quad k < s.\]

Then the scattering matrix $T^\nu(\lambda)$ will be a product of mutually commuting matrices $T^{\nu(k,s)}_\nu$ of the form:

\[(2.30) \quad T^{\nu(k,s)}_\nu = 1 + (a^+_\nu k_\nu(\lambda) - 1)E_{kk} + (a^-_\nu k_\nu(\lambda) - 1)E_{ss} - b^-_\nu k_\nu(\lambda)E_{ks} + b^+_\nu k_\nu(\lambda)E_{sk},\]

where $k < s$, with only 4 non-trivial matrix elements, just like the ZS (or AKNS) system.

The $Z_\nu$-symmetry imposes the following constraints on the FAS and on the scattering matrix and its factors:

\[(2.31a) \quad C_0\xi^{\nu}(x, \lambda) C_0^{-1} = \xi^{\nu-2}(x, \lambda), \quad C_0 T^\nu_\nu(\lambda) C_0^{-1} = T^{\nu-2}_\nu(\lambda),\]

\[(2.31b) \quad C_0 S^\nu_\nu(\lambda) C_0^{-1} = S^{\nu-2}_\nu(\lambda), \quad C_0 D^\nu_\nu(\lambda) C_0^{-1} = D^{\nu-2}_\nu(\lambda),\]

where the index $\nu - 2$ should be taken modulo $2n$. Consequently we can view as independent only the data on two of the rays, e.g. on $l_1$ and $l_{2n} \equiv l_0$; all the rest will be recovered from (2.31).

If in addition we impose the $Z_2$-symmetry (2.4), (2.6) with $\epsilon = -1$ then we will have also $a_k = i\omega^{-k-1/2}$ and:

\[(2.32) \quad B(\xi^\nu(x, -\lambda^\nu))B^{-1} = (\xi^{n+1-\nu}(x, \lambda))^{-1}, \quad B(S^{\nu}_{\nu}(\lambda^\nu))B^{-1} = (S^{n+1-\nu}_{\nu}(\lambda))^{-1},\]

and analogous relations for $T^\nu_\nu(\lambda)$ and $D^\nu_\nu(\lambda)$. Another interesting subcase takes place for even values of $n$ and $Z_2$-reduction (2.4), (2.6) with $\epsilon = 1$; then $a_k = \omega^{k-1/2}$ and the FAS satisfy:

\[(2.33) \quad B(\xi^\nu(x, \lambda^\nu))B^{-1} = (\xi^{2n+1-\nu}(x, \lambda))^{-1}, \quad B(S^{\nu}_{\nu}(\lambda^\nu))B^{-1} = (S^{2n+1-\nu}_{\nu}(\lambda))^{-1},\]

In both cases the rays $l_\nu$ are defined by (2.24) with $\phi_0 = 0$. The pairs of indices $\{k_\nu, m_\nu\}$ specifying the imbeddings of the $sl(2)$-subalgebras related to the ray $l_\nu$ are defined as follows:

\[(2.34) \quad a) \quad \text{for } \epsilon = 1 \quad k_\nu + m_\nu = \left\lfloor \frac{n}{2} \right\rfloor + 2 - \nu(\mod n),\]

\[(2.34) \quad b) \quad \text{for } \epsilon = -1 \quad k_\nu + m_\nu = 2 - \nu(\mod n),\]

One can prove also that $D^\nu_\nu(\lambda)$ (resp. $D^-\nu_\nu(\lambda)$) allows analytic extension for $\lambda \in \Omega_\nu$ (resp. for $\lambda \in \Omega_{\nu-1}$), compare with corollary 2.7. Another important fact is [30] that $D^\nu_\nu(\lambda) = D^-\nu_{\nu+1}(\lambda)$ for all $\lambda \in \Omega_\nu$. 

The inverse scattering problem and the Riemann-Hilbert problem.

The next important step is the possibility to reduce the solution of the ISP for the GZSs to a (local) RHP. Indeed the relation \(2.14\) can be rewritten as:

\[
\begin{align*}
(2.35a) & \quad \xi^+(x, t, \lambda) = \xi^-(x, t, \lambda)G(x, t, \lambda), \quad \lambda \in \mathbb{R}, \\
(2.35b) & \quad G(x, t, \lambda) = e^{-i(\lambda Jx - f(\lambda)t)}G_0(\lambda)e^{i(\lambda Jx - f(\lambda)t)}, \\
(2.35c) & \quad G_0(\lambda) = \left[ \hat{S}^- (\lambda) \hat{S}^+ (\lambda) \right]_{t=0};
\end{align*}
\]

in other words the sewing function \(G(x, t, \lambda)\) satisfies the equations:

\[
(2.36) \quad i \frac{dG}{dx} - \lambda [J, G(x, t, \lambda)] = 0, \quad i \frac{dG}{dt} + [f(\lambda), G(x, t, \lambda)] = 0,
\]

Here \(f(\lambda) \in \mathfrak{h}\) determines the dispersion law of the NLEE. Together with

\[
(2.37) \quad \lim_{\lambda \to \infty} \xi^\pm (x, \lambda) = 1,
\]

eq. \(2.35\) is known as the RHP with canonical normalization.

**Theorem 2.7 (15).** Let \(\xi^+(x, t, \lambda)\) and \(\xi^-(x, t, \lambda)\) be solutions to the RHP \(2.35, 2.37\) allowing analytic extension in \(\lambda\) for \(\lambda \in \mathbb{C}_\pm\) respectively. Then \(\chi^\pm (x, t, \lambda) = \xi^\pm (x, t, \lambda)e^{i\lambda Jx}\) are fundamental analytic solutions of both operators \(L\) and \(M\), i.e. satisfy eqs. \(1.3, 1.4\) with

\[
(2.38) \quad g(x, t) = \lim_{\lambda \to \infty} \lambda \left( J - \xi^\pm (x, t, \lambda) J \hat{\xi}^\pm (x, t, \lambda) \right).
\]

**Proof.** Let us assume that \(\xi^\pm (x, t, \lambda)\) are regular solutions to the RHP and let us introduce the function:

\[
(2.39) \quad g^\pm (x, t, \lambda) = i \frac{d\xi^\pm}{dx}(x, t, \lambda) + \lambda \xi^\pm (x, t, \lambda) J \hat{\xi}^\pm (x, t, \lambda).
\]

If \(\xi^\pm (x, t, \lambda)\) are regular then neither \(\xi^\pm (x, t, \lambda)\) nor their inverse \(\hat{\xi}^\pm (x, t, \lambda)\) have singularities in their regions of analyticity \(\lambda \in \mathbb{C}_\pm\). Then the functions \(g^\pm (x, t, \lambda)\) also will be regular for all \(\lambda \in \mathbb{C}_\pm\). Besides:

\[
(2.40) \quad \lim_{\lambda \to \infty} g^+(x, t, \lambda) = \lim_{\lambda \to \infty} g^-(x, t, \lambda) = \lambda J.
\]

The crucial step in the proof of \(52\) is based on the chain of relations:

\[
\begin{align*}
g^+(x, t, \lambda) & \overset{(2.35)}{=} i \frac{d(\xi^\pm G)}{dx} \hat{G} \hat{\xi}^-(x, t, \lambda) + \lambda \xi^- G \hat{G} \hat{\xi}^+ (x, t, \lambda) \\
& = i \frac{d\xi^-}{dx} \hat{\xi}^- (x, t, \lambda) + \xi^- \left( i \frac{dG}{dx} \hat{G} + \lambda G \hat{G} \right) \hat{\xi}^- (x, t, \lambda) \\
& \overset{(2.36)}{=} i \frac{d\xi^-}{dx} \hat{\xi}^- (x, t, \lambda) + \xi^- \left( [J, G] \hat{G} + \lambda G \hat{G} \right) \hat{\xi}^- (x, t, \lambda) \\
& = i \frac{d\xi^-}{dx} \hat{\xi}^- (x, t, \lambda) + \xi^- J \hat{\xi}^- (x, t, \lambda) \\
& = g^-(x, t, \lambda), \quad \lambda \in \mathbb{R}.
\end{align*}
\]

Thus we conclude that \(g^+(x, t, \lambda) = g^-(x, t, \lambda)\) is a function analytic in the whole complex \(\lambda\)-plane except in the vicinity of \(\lambda \to \infty\) where \(g^+(x, t, \lambda)\) tends to \(\lambda J\), \(2.40\). Next from Liouville theorem we conclude that the difference \(g^+(x, t, \lambda) - \lambda J\) is a constant with respect to \(\lambda\); if we denote this ‘constant’ by \(-q(x, t)\) we get:

\[
(2.42) \quad g^+(x, t, \lambda) - \lambda J = -q(x, t).
\]
It remains to remember the definition of $g^+(x, t, \lambda)$ (2.33) to find that $\xi^\pm(x, t, \lambda)$ satisfy (2.1), i.e. that $\chi^\pm(x, t, \lambda)$ is a fundamental solution to $L$. The relation between $q(x, t)$ and $\xi^\pm(x, t, \lambda)$ (2.38) can be obtained by taking the limit of the left-hand sides of (2.42) for $\lambda \to \infty$.

Arguments along the same line applied to the functions $h^\pm(x, t, \lambda)$

\[ h^\pm(x, t, \lambda) = i \frac{d\xi^\pm}{dt}(x, t, \lambda) - \xi^\pm(x, t, \lambda)f(\lambda)\xi^\pm(x, t, \lambda), \]

can be used to prove that $\chi^\pm(x, t, \lambda)$ are fundamental solutions also of the operator $M$; equivalently it satisfies ($V'(x, t, \lambda) = V(x, t, \lambda) - f(\lambda)$):

\[ i \frac{d\xi^\pm}{dt} + V'(x, t, \lambda)\xi^\pm(x, t, \lambda) + [f(\lambda), \xi^\pm(x, t, \lambda)] = 0, \]

and one finds that $h^+(x, t, \lambda) = h^-(x, t, \lambda)$ is a function analytic everywhere in $\mathbb{C}$ except at $\lambda \to \infty$ where it has a polynomial behavior of order $N - 1$. Denoting the polynomial as $V(x, t, \lambda)$ we derive (2.43).

To conclude the proof of the theorem we have to account for possible zeroes and pole singularities of $\xi^\pm(x, t, \lambda)$ at the points $\lambda$ (2.19). Below we derive the structure of these singularities which is such that they do not influence the functions $g^\pm(x, t, \lambda)$ and $h^\pm(x, t, \lambda)$. The theorem is proved.

The analyticity properties of $m^\pm_k(\lambda)$ allow one to reconstruct them from the sewing function $G(\lambda)$ (2.35c) and from the locations of their zeroes through (see Appendix B):

\[ D_k(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} \ln \left\{ \begin{array}{l} 1, 2, \ldots, k \\ 1, 2, \ldots, k \end{array} \right\} _{G(\mu)} + \sum_{j=1}^{N} b_{jk} \ln \frac{\lambda - \lambda_j^+}{\lambda - \lambda_j^-}, \]

where

\[ D_k(\lambda) = \begin{cases} \ln m^+_k(\lambda), & \lambda \in \mathbb{C}_+ \\ -\ln m^-_{n-k}(\lambda), & \lambda \in \mathbb{C}_- \end{cases} \]

One can view $D_k(\lambda)$ as generating functionals of the conserved quantities for the related $N$-wave-type equations; the relevant expressions for them in terms of the scattering data can be obtained from the right hand sides of (2.43).

Quite analogously we can treat also the CBCs with $\mathbb{Z}_m$-symmetry. More precisely, we have:

\[ \xi^\nu(x, t, \lambda) = \xi^{\nu-1}(x, t, \lambda)G_\nu(x, t, \lambda), \quad \lambda \in l_\nu, \]

\[ G_\nu(x, t, \lambda) = e^{-i\lambda J x + if(\lambda)t}G_{0,\nu}(\lambda)e^{i\lambda J x - if(\lambda)t}, \quad G_{0,\nu}(\lambda) = \hat{S}_\nu^-(\lambda)S^0_\nu(\lambda) \bigg|_{t=0} \]

The collection of all relations (2.47) for $\nu = 1, 2, \ldots, 2n$ together with

\[ \lim_{\lambda \to \infty} \xi^\nu(x, t, \lambda) = 1, \]

can be viewed as a local RHP posed on the collection of rays $\Sigma \equiv \{l_\nu\}_{\nu=1}^{2n}$ with canonical normalization. Rather straightforwardly we can reformulate the results for the GZSs for the CBCs and prove that if $\xi^\nu(x, \lambda)$ is a solution of the RHP (2.47), (2.48) then $\chi^\nu(x, \lambda) = \xi^\nu(x, \lambda)e^{i\lambda J x}$ satisfy the CBC with potential

\[ q(x, t) = \lim_{\lambda \to \infty} \left( J - \xi^\nu(x, t, \lambda)J\xi^\nu(x, t, \lambda) \right). \]
We finish this subsection by formulating the dispersion relations for the functions \( \ln m_{\nu,k}^+(\lambda) \) which allows us to reconstruct them from their analyticity properties:

\[
(2.50) \quad \ln m_{\nu,k}^+(\lambda) = \sum_{\eta=1}^{2n} \frac{(-1)^{\eta}}{2\pi i} \int_{\gamma_\nu} \frac{d\mu}{\mu - \lambda} \ln \left\{ \begin{array}{c} 1 \\
1 \ldots k \\
1 \ldots k \end{array} \right\}^\eta_G + \sum_{j=1}^{N} b^j \ln \frac{\lambda - \lambda_{j,k}^+ \omega^\eta}{\lambda - \lambda_{j,k}^- \omega^\eta},
\]

where \( \lambda \in \Omega_\nu \) and the superscript \( \eta \) in the integrand shows that we should use the ordering characteristic for the sector \( \Omega_\eta \); \( b^j \) are the analogs for \( b_{kj} \) \((2.21)\) in \( \Omega_\eta \).

Both dispersion relations \((2.45)\) and \((2.50)\) can be used to derive the so-called trace identities (see \([48, 13]\)) for the GZSs and CBCs respectively. Indeed, \( \mathfrak{D}_k(\lambda) \) and \( \ln m_{\nu,k}^+(\lambda) \) allow asymptotic expansions

\[
(2.51) \quad \mathfrak{D}_k(\lambda) = \sum_{s=1}^{\infty} \mathfrak{D}_k^{(s)} \lambda^{-s}, \quad \ln m_{\nu,k}^+(\lambda) = \sum_{s=1}^{\infty} M_{\nu,k}^{(s)} \lambda^{-s}.
\]

The expansion coefficients \( \mathfrak{D}_k^{(s)} \) and \( M_{\nu,k}^{(s)} \) are local integrals of motion, i.e. their densities depend only on \( q(x,t) \) and its derivatives with respect to \( x \). Their explicit calculation is done via recurrent procedure. We illustrate it by the two first integrals of motion of the \( \mathbb{Z}_n\)-NLS equation \((1.3)\):

\[
(2.52) \quad M_{1,1}^{(1)} = \frac{1}{2\omega} \int_{-\infty}^{\infty} dx \sum_{p=1}^{n} \psi_p \psi_{n-p}(x,t),
\]

\[
(2.53) \quad M_{1,1}^{(2)} = \frac{1}{2\omega^2} \int_{-\infty}^{\infty} dx \left\{ \sum_{p=1}^{n} i \cotan \left( \frac{\pi p}{n} \right) \left( \frac{d\psi_p}{dx} \psi_{n-p} - \psi_p \frac{d\psi_{n-p}}{dx} \right) - \frac{2}{3} \sum_{p+k+l=n} \psi_p \psi_k \psi_l(x,t) \right\}.
\]

One can also expand the right hand sides of the dispersion relations \((2.45)\) and \((2.50)\) over the inverse powers of \( \lambda \) which allows to express \( \mathfrak{D}_k^{(s)} \) and \( M_{\nu,k}^{(s)} \) also in terms of the scattering data of GZSs and CBCs.

The dressing Zakharov-Shabat method

One of the most fruitful ideas for the explicit constructing of the NLEE’s solutions is based on the possibility starting from a given regular solutions \( \xi_0^\pm(x,t,\lambda) \) to the RHP to construct new singular solutions \( \xi^\pm(x,t,\lambda) \) having zeroes and pole singularities at the prescribed points \( \lambda_j^\pm \in \mathbb{C}_\pm \). The structure of these singularities are determined by the dressing factor \( u_j(x,t,\lambda) \):

\[
(2.54) \quad \xi^\pm(x,t,\lambda) = u_j(x,t,\lambda) \xi_0^\pm(x,t,\lambda) u_j^{-1}(\lambda),
\]

which for the \( SL(n) \)-group has a simple fraction-linear dependence on \( \lambda \):

\[
(2.55) \quad u_j(x,t,\lambda) = 1 + (c_j(\lambda) - 1)P_j(x,t), \quad c_j(\lambda) = \frac{\lambda - \lambda_j^+}{\lambda - \lambda_j^-},
\]

\[
(2.56) \quad u_j^{-1} = \lim_{x \to -\infty} u_j(x,t,\lambda).
\]
Here $P_j(x, t)$ is a projector $P_j^2 = P_j$ which for simplicity is chosen to be of rank 1; then it can be written down as:

\[(2.57) \quad P_j(x) = \frac{|n_j\rangle\langle m_j|}{\langle m_j|n_j\rangle}, \]

where the bra- and ket- eigenvectors $\langle m_j|$ and $|n_j\rangle$ are the ‘left’ and ‘right’ eigenvectors of the projector.

From (2.54) there follows that the dressing factor $u(x, t, \lambda)$ satisfies the equation:

\[(2.58) \quad i\frac{du}{dx} + q(x, t)u(x, t, \lambda) - u(x, t, \lambda)q_0(x, t) - \lambda[J, u(x, t, \lambda)] = 0. \]

The main advantage of the dressing method is in the fact that one can determine the $x$ and $t$-dependence of $\langle m_j|$ and $|n_j\rangle$ through the regular solution $\chi_0^\pm(x, t, \lambda)$ as follows:

\[(2.59)|n_j\rangle = \chi_{0j}^+(x, t)|n_j|^0\rangle, \quad \langle m_j| = \langle m_j^0|\chi_{0j}^-(x, t), \quad \chi_{0j}^\pm(x, t) = \chi_0^\pm(x, t, \lambda_j^\pm) \]

equivalently these vectors are solutions to the equations:

\[(2.60) \quad i\frac{d|m_j\rangle}{dx} + U(0)(x, t, \lambda_j^+)|n_j\rangle = 0, \quad i\frac{d|n_j\rangle}{dt} + V(0)(x, t, \lambda_j^-)|m_j\rangle = 0, \]
\[(2.61) \quad i\frac{d|m_j\rangle}{dx} - \langle m_j|U(0)(x, t, \lambda_j^-)|n_j\rangle = 0, \quad i\frac{d|n_j\rangle}{dt} - \langle m_j|V(0)(x, t, \lambda_j^-)|n_j\rangle = 0, \]
\[(2.62) \quad U(0)(x, t, \lambda) = q_0(x, t) - \lambda J, \quad V(0)(x, t, \lambda) = V(x, t, \lambda)|q=q_0. \]

Here $q_0(x, t)$ is the potential corresponding to the regular solutions $\chi_0^\pm(x, t, \lambda)$ to the RHP and $V(0)(x, t, \lambda)$ is obtained from $V(x, t, \lambda)$ (see (3.3), (3.30)) replacing $q(x, t)$ by $q_0(x, t)$. This construction is well defined also in the case when $\chi_0^\pm(x, \lambda)$ are singular solutions to the RHP, provided they are regular for $\lambda = \lambda_j^\pm$.

If $q(x, t)$ is the potential corresponding to the singular solution $\chi_j^\pm(x, t, \lambda)$ then:

\[(2.63) \quad q(x, t) = q_0(x, t) + \lim_{\lambda \to \pm\infty} \lambda(J - u_j(x, t, \lambda)J\hat{u}_j(x, t, \lambda)) \]
\[= q_0(x, t) - (\lambda_j^+ - \lambda_j^-)[J, P_j(x, t)]. \]

Thus starting from a given regular solution of the RHP (and related solution $q_0(x, t)$ to the NLEE) we can construct a singular solution to the RHP and a new solution $q(x, t)$ of the NLEE depending on the $\lambda_j^\pm$ and on the eigenvectors of $P_j(x)$. If we start from the trivial solution $q_0(x, t) = 0$ of the NLEE then we will get the one-soliton solution of the NLEE. Repeating the procedure $N$ times we can get the $N$-soliton solution of the NLEE.

With the explicit formulae for $P_j(x)$ and using (2.54) we can establish the relationship between the scattering data of the regular RHP and the corresponding singular one. The dressing factor $u_j(x, \lambda)$ is determined by the constant vectors $\langle m_j^0|$ and $|n_j^0\rangle$ can not be quite arbitrary. The condition that $q(x)$ vanishes for $x \to \pm\infty$ requires that if $\langle n_j^0|$s = 0 for all $1 \leq s < I_j$ and $F_s < s \leq n$ then also $(m_j^0)$s = 0 for all $1 \leq s < I_1$ and $F_1 < s \leq n$. Thus we derive that:

\[(2.64) \quad \lim_{x \to \pm\infty} P_j(x) = E_{I_1}I_j, \quad \lim_{x \to -\infty} P_j(x) = E_{F_s}F_j, \]

and therefore

\[(2.65) \quad u_{j, +}(\lambda) = 1 + (c_j(\lambda) - 1)E_{I_1}I_j, \quad u_{j, -}(\lambda) = 1 + (c_j(\lambda) - 1)E_{F_s}F_j. \]
Comparing these last relations with (2.17) we find for the principal minors of (2.66) and (2.67).

The interrelation between the Gauss factors of the corresponding scattering matrices are:

\[ S^±(\lambda) = u_{j,-}(\lambda)S_0^±(\lambda)u_{j,-}^{-1}(\lambda), \quad T^±(\lambda) = u_{j,+}(\lambda)T_0^±(\lambda)u_{j,+}^{-1}(\lambda), \]

and

\[ D^±(\lambda) = u_{j,+}(\lambda)D_0^±(\lambda)u_{j,-}^{-1}(\lambda). \]

Comparing these last relations with (2.17) we find for the principal minors of \( T(\lambda) \) and \( T_0(\lambda) \):

\[ m^±_s(\lambda) = \frac{\lambda - \lambda_j^+}{\lambda - \lambda_j^+}m^+_0, \quad \text{for} \quad I_j \leq s < F_j, \quad \lambda \in \mathbb{C}_+ \cup \mathbb{R}, \]

and \( m^±_s(\lambda) = m^+_0, \lambda \) for the other values of \( s \). Thus the result of the dressing is that the string of upper principle minors \( m^+_0(\lambda), I_j \leq s < F_j \) acquire simple zero at \( \lambda = \lambda_j^+ \) while the string of lower principle minors \( m^-_0(\lambda), n - F_j < s \leq n - I_j \) acquire simple zero at \( \lambda = \lambda_j^- \).

Obviously if we impose on \( L(\lambda) \) the \( \mathbb{Z}_2 \)-reduction then we should restrict also the dressing factor by:

\[ B(u(x,t,\epsilon\lambda^*))^\dagger B^{-1} = u(x,t,\lambda). \]

The ansatz (2.55) satisfies (2.69) if \( \lambda_j^- = \epsilon(\lambda_j^+)^* \) and the vectors \( |n_0) \), \( |m_0) \) are related by:

\[ \langle m_0 | = B|n_0^+). \]

If we impose the \( \mathbb{Z}_n \)-reduction (2.7) then \( u(x,t,\lambda) \) must satisfy:

\[ C_0u(x,t,\omega\lambda)C_0^{-1} = u(x,t,\lambda). \]

Such conditions require generalizations of the ansatz (2.55) [11]:

\[ u_j(x,t,\lambda) = 1 + \sum_{s=0}^{n-1} (c_j(\omega^s\lambda) - 1) C_0^sP_j(x)C_0^{-s}. \]

A slightly different approach treating also multi-soliton solutions of the \( \mathbb{Z}_n \)-symmetric NLEE is given in [3].

Up to now we dealt with the algebra \( g \simeq sl(n,\mathbb{C}) \). Treating the other simple Lie algebras (orthogonal or symplectic) needs additional care especially in constructing the dressing factors [51, 23].

In fact \( u_j(x,\lambda) \) (2.55) must be an element of the corresponding group. From the ansatz (2.55) it follows that \( u_j(x,\lambda) \) belongs to \( GL(n,\mathbb{C}) \), but one can always multiply \( u(x,\lambda) \) by an appropriate \( x \)- and \( t \)-independent scalar and to adjust its determinant to 1. Such a multiplication goes through the whole scheme outlined above but is adequate only for the \( sl(n,\mathbb{C}) \) case. However the ansatz (2.55) cannot be used, e.g. for the case \( so(n,\mathbb{C}) \). The adequate ansatz is formulated below [23].
Theorem 2.8. Let $g \sim B_r$ or $D_r$ and let the dressing factor $u(x, \lambda)$ be of the form:

\begin{equation}
(2.73)
\end{equation}

\begin{equation}
(2.74)
\end{equation}

where $S_0$ is introduced in (A.11) and $P_j(x)$ is a rank 1 projector (2.77). Let the constant vectors $|n_0\rangle$ and $\langle m_0|$ satisfy the condition

\begin{equation}
(2.74)
\end{equation}

Then $u_j(x, \lambda)$ (2.74) satisfies the equation (2.58) with a potential

\begin{equation}
(2.75)
\end{equation}

Proof. Due to the fact that $\chi^\pm(x, \lambda)$ take values in the corresponding orthogonal group we find that from (2.74) it follows $\langle m|S|m\rangle = 0$, $\langle m|JS|m\rangle = 0$ and analogous relations for the vector $|n\rangle$. As a result we get that

\begin{equation}
(2.76)
\end{equation}

Let us now insert (2.74) into (2.58) and take the limit of the r.h.side of (2.58) for $\lambda \to \infty$. This immediately gives eq. (2.75). In order that Eq. (2.58) be satisfied identically with respect to $\lambda$ we have to put to 0 also the residues of its r.h.side at $\lambda \to \lambda^+_j$ and $\lambda \to \lambda^-_j$. This gives us the following system of equation for the projectors $P_j(x)$ and $P_{-j}(x)$:

\begin{equation}
(2.77)
\end{equation}

\begin{equation}
(2.78)
\end{equation}

where we have to keep in mind that $q$ is given by (2.77). Taking into account (2.76) and the relation between $P_j(x)$ and $P_{-j}(x)$ eq. (2.77) reduces to:

\begin{equation}
(2.79)
\end{equation}

One can check by a direct calculation that (2.57) satisfies identically (2.79). The theorem is proved.

\[\square\]

3. THE RESOLVENT AND SPECTRAL PROPERTIES OF GZSSS AND CBCS

The FAS $\chi^\pm(x, \lambda)$ of $L(\lambda)$ allows one to construct the resolvent of the operator $L$ and then to investigate its spectral properties. By resolvent of $L(\lambda)$ we understand the integral operator $R(\lambda)$ with kernel $R(x, y, \lambda)$ which satisfies

\begin{equation}
(3.1)
\end{equation}

where $f(x)$ is an $n$-component vector function in $\mathbb{C}^n$ with bounded norm, i.e. $\int_{-\infty}^{\infty} dy f^T(y) f(y) < \infty$.

From the general theory of linear operators [4, 12, 46] we know that the point $\lambda$ in the complex $\lambda$-plane is a regular point if $R(\lambda)$ is a bounded integral operator. In each connected subset of regular points $R(\lambda)$ is analytic in $\lambda$. 

The points \( \lambda \) which are not regular constitute the spectrum of \( L(\lambda) \). Roughly speaking the spectrum of \( L(\lambda) \) consist of two types of points:
- i) the continuous spectrum of \( L(\lambda) \) consists of all points \( \lambda \) for which \( R(\lambda) \) is an unbounded integral operator;
- ii) the discrete spectrum of \( L(\lambda) \) consists of all points \( \lambda \) for which \( R(\lambda) \) develops pole singularities.

Let us now show how the resolvent \( R(\lambda) \) can be expressed through the FAS of \( L(\lambda) \). Indeed, if we write down \( R(\lambda) \) in the form:

\[
R(\lambda)f(x) = \int_{-\infty}^{\infty} R(x, y, \lambda)f(y),
\]

the kernel \( R(x, y, \lambda) \) of the resolvent is given by:

\[
R(x, y, \lambda) = \begin{cases} 
R^+(x, y, \lambda) & \text{for } \lambda \in \mathbb{C}^+, \\
R^-(x, y, \lambda) & \text{for } \lambda \in \mathbb{C}^-,
\end{cases}
\]

where

\[
R^\pm(x, y, \lambda) = \pm i\chi^\pm(x, \lambda)\Theta^\pm(x - y)\hat{\chi}^\pm(y, \lambda),
\]

\[
\Theta^\pm(z) = \theta(z^+\Pi_0 - \theta(z)(1 - \Pi_0)), \quad \Pi_0 = \sum_{s=1}^{k_0} E_{ss},
\]

where \( k_0 \) is the number of positive eigenvalues of \( J \); namely:

\[
a_1 > a_2 > \cdots > a_{k_0} > 0 > a_{k_0+1} > \cdots > a_n.
\]

Due to the condition \( \text{tr} J = \sum_{s=1}^{n} a_s = 0 \), \( k_0 \) is fixed up uniquely.

The next theorem establishes that \( R(x, y, \lambda) \) is indeed the kernel of the resolvent of \( L(\lambda) \).

**THEOREM 3.1.** Let \( q(x) \) satisfy conditions (C.1) and (C.2) and let \( \lambda^\pm_j \) be the simple zeroes of the minors \( m^\pm_j(\lambda) \). Then

1. \( R^\pm(x, y, \lambda) \) is an analytic function of \( \lambda \) for \( \lambda \in \mathbb{C}_\pm \) having pole singularities at \( \lambda^\pm_j \in \mathbb{C}_\pm; \)
2. \( R^\pm(x, y, \lambda) \) is a kernel of a bounded integral operator for \( \text{Im} \lambda \neq 0; \)
3. \( R(x, y, \lambda) \) is uniformly bounded function for \( \lambda \in \mathbb{R} \) and provides a kernel of an unbounded integral operator;
4. \( R^\pm(x, y, \lambda) \) satisfy the equation:

\[
L(\lambda)R^\pm(x, y, \lambda) = \delta(x - y).
\]

**IDEA OF THE PROOF.**

1. is obvious from the fact that \( \chi^\pm(x, \lambda) \) are the FAS of \( L(\lambda) \);
2. Assume that \( \text{Im} \lambda > 0 \) and consider the asymptotic behavior of \( R^+(x, y, \lambda) \) for \( x, y \to \infty \). From equations (2.9) we find that

\[
R^+_i(x, y, \lambda) = \sum_{p=1}^{n} \xi^+_p(x, \lambda)e^{-i\lambda a_p(x-y)}\Theta^+_p(x-y)\hat{\xi}^+_p(y, \lambda)
\]

Due to the fact that \( \chi^+(x, \lambda) \) has triangular asymptotics for \( x \to \infty \) and \( \lambda \in \mathbb{C}_+ \), and for the correct choice of \( \Theta^+(x - y) \) we check that the right hand side of (3.7) falls off exponentially for \( x \to \infty \) and arbitrary choice of \( y \). All other possibilities are treated analogously.
3. For \( \lambda \in \mathbb{R} \) the arguments of 2) can not be applied because the exponentials in the right hand side of (3.7) \( \text{Im} \lambda = 0 \) only oscillate. Thus we conclude that \( R^\pm(x, y, \lambda) \) for \( \lambda \in \mathbb{R} \) is only a bounded function and thus the corresponding operator \( R(\lambda) \) is an unbounded integral operator.

4. The proof of eq. (3.6) follows from the fact that \( L(\lambda)\chi^+(x, \lambda) = 0 \) and

\[
\frac{d\Theta^\pm(x - y)}{dx} = \mp \delta(x - y).
\]

PROPOSITION 3.2. Let \( q(x) \) satisfy the conditions (C.1) and (C.2), let \( \Im \lambda = 0 \) only oscillate. Thus we conclude that \( R^\pm(x, y, \lambda) \) for \( \lambda \in \mathbb{R} \) is only a bounded function and thus the corresponding operator \( R(\lambda) \) is an unbounded integral operator.

\[
\frac{d\Theta^\pm(x - y)}{dx} = \mp \delta(x - y).
\]

Proof. Let \( \chi_0^\pm(x, \lambda) \) be the FAS of \( L_0(\lambda) \) with potential \( q_0(x) \); then \( \chi_0^\pm(x, \lambda) \) are regular for \( \lambda = \lambda_j^\pm \). Now we apply the dressing method choosing \( \chi_j^\pm(x, \lambda) \) as the locations of the singularities and construct the projector \( P_j(x) \) using the constant vectors \( |n_{j,0} \rangle \) and \( |n_{0,j} \rangle \). The normalizing factor \( u_{j,-1}(\lambda) \) in the right hand side of (2.54) is a diagonal matrix that commutes with \( \Pi_0 \). Then we insert \( \chi_j^\pm(x, \lambda) = u_j(x, \lambda)\chi_0^\pm(x, \lambda) \) in (3.4) and note that the pole singularity of \( R^+(x, y, \lambda) \) at \( \lambda = \lambda_j^+ \) (resp. \( R^-(x, y, \lambda) \) at \( \lambda = \lambda_j^- \)) can come up only from the factor \( u_{j,-1}(y, \lambda) \) (resp. \( u(x, \lambda) \)). To derive the expressions in (3.10) one needs the explicit form of the projectors \( P_j(x) \) and \( P_j(y) \) (2.57) and (2.58).

The right hand sides of (3.10) do not vanish if the following conditions hold. In other words if (3.10) hold then the residues (2.9) do not vanish, \( R^\pm(x, y, \lambda) \) have simple poles at \( \lambda = \lambda_j^\pm \) and by definition \( \lambda_j^\pm \) are discrete eigenvalues of \( L(\lambda) \).

Eq. (3.10) is equivalent to the condition \( I_j \leq k_0 < F_j \). Indeed violating this condition we get either \( (\Pi_0 - \Pi_0)n_{0,j} \rangle = 0 \) or \( \Pi_0|n_{0,j} \rangle \rangle = 0 \) and as a result – vanishing right hand sides in (3.9).

To finish the proof one must check that from the definitions (3.9b) the relations (2.68) follow. Besides \( |n_j^\pm \rangle \) and \( \langle m_j^\pm | \) satisfy:

\[
\frac{d|n_j^\pm \rangle}{dx} + (q(x) - \lambda_j^\pm J)|n_j^\pm \rangle = 0, \quad \frac{d\langle m_j^\pm |}{dx} - \langle m_j^\pm |(q(x) - \lambda_j^\pm) = 0,
\]
where $q(x)$ is given by (2.63).

**Corollary 3.3.** The discrete spectrum of the Lax operator (1.5) consists of the zeroes of the principal minors $m^+_j(\lambda)$ for $\lambda \in \mathbb{C}_+$ and $m^-_j(\lambda)$ for $\lambda \in \mathbb{C}_-$ provided the conditions (3.10) are satisfied.

Now we can derive the completeness relation for the eigenfunctions of the Lax operator (1.5) by applying the contour integration method (see e.g. [26, 27, 2]) to the integral:

$$J(x,y) = \frac{1}{2\pi i} \oint_{\gamma_+} d\lambda R^+(x,y,\lambda) - \frac{1}{2\pi i} \oint_{\gamma_-} d\lambda R^-(x,y,\lambda),$$

where the contours $\gamma_\pm$ are shown on the Figure 1. Skipping the details we get:

$$\delta(x-y) \sum_{s=1}^n \frac{1}{a_s} E_{ss} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \left\{ \sum_{s=1}^{k_0} |\chi^{[s]+}(x,\lambda)|^2 \frac{\zeta^+_s(\lambda)}{\zeta^-_s(\lambda)} - \sum_{s=k_0+1}^n |\chi^{[s]-}(x,\lambda)|^2 \frac{\zeta^-_s(\lambda)}{\zeta^+_s(\lambda)} \right\}$$

$$+ 2i \sum_{j=1}^N \nu_j \left\{ |n_j^+(x)|^2 \langle m_j^+(y) | n_j^-(x) \rangle - |n_j^-(x)|^2 \langle m_j^-(y) | n_j^+(x) \rangle \right\}.$$

This relation (3.13) allows one to expand any vector-function $|z(x)\rangle \in \mathbb{C}^n$ over the eigenfunctions of the system (1.3). Indeed, let us multiply (3.13) on the right by $J \mid z(y)\rangle$ and integrate over $y$. This gives:

$$|z(x)\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \left\{ \sum_{s=1}^{k_0} |\chi^{[s]+}(x,\lambda)|^2 \cdot \zeta^+_s(\lambda) - \sum_{s=k_0+1}^n |\chi^{[s]-}(x,\lambda)|^2 \cdot \zeta^-_s(\lambda) \right\}$$

$$+ \sum_{j=1}^N \nu_j \left\{ |n_j^+(x)|^2 \langle m_j^+(y) | n_j^-(x) \rangle - |n_j^-(x)|^2 \langle m_j^-(y) | n_j^+(x) \rangle \right\},$$

Figure 1. The contours $\gamma_\pm = \mathbb{R} \cup \gamma_{\pm\infty}$. □
where the expansion coefficients are of the form:

\[ (3.15) \]
\[ \zeta_\nu^+(\lambda) = -i \int_{-\infty}^{\infty} dx \langle \tilde{\chi}^{|s|+}(x,\lambda) | J | z(x) \rangle, \quad \zeta_\nu^-(\lambda) = -i \int_{-\infty}^{\infty} dx \langle m_\nu^+ | J | z(x) \rangle. \]

**Remark 3.4.** If \( q(x) \simeq 0 \) then \( \chi^+(x,\lambda) \simeq \chi^-(x,\lambda) \simeq \exp(-i\lambda J x) \) the set \( \mathfrak{Z} \) is empty and \( (3.14) \) goes into the usual Fourier transform for the space \( \mathbb{C}^n \).

**Remark 3.5.** Here we used also the fact that all eigenvalues of \( J \) are non-vanishing. In the case when one (or several) of them vanishes we can prove completeness of the eigenfunctions only in a certain subspace of \( \mathbb{C}^n \).

The resolvent for the CBCs is defined quite analogously:

\[ (3.16) \]
\[ R(x, y, \lambda) = R_\nu(x, y, \lambda), \quad \lambda \in \Omega_\nu, \]
\[ R_\nu(x, y, \lambda) = i\chi_\nu(x, \lambda) \Theta^\nu(x - y) \hat{\chi}_\nu(x, \lambda), \]
\[ \Theta^\nu(z) = \theta(-z) \Pi^0_\nu - \theta(z)(\mathbb{1} - \Pi^0_\nu), \quad \Pi^0_\nu = \sum_{s \leq k_{0,\nu}} E_{ss}, \]

where \( \chi_\nu(x, \lambda) = \xi_\nu(x, \lambda) e^{i\lambda J x} \) and \( k_{0,\nu} \) is the number of positive eigenvalues of \( \text{Im} (\lambda J) \) in the \( \nu \)-th ordering.

The following theorem is a specific case of one in \( [34] \).

**Theorem 3.6.** Let \( q(x) \) satisfy the conditions \( (C.1) \) and \( (C.2) \) and let \( \mathfrak{Z} = \bigcup_{p=1}^{n} (\mathfrak{Z}_{2p-1} \cup \mathfrak{Z}_{2p}) \) where

\[ (3.17) \]
\[ \mathfrak{Z}_{2p-1} \equiv \{ \lambda_j^+ \omega^{p-1} \in \Omega_{2p-1}, \quad j = 1, \ldots, N \}, \]
\[ \mathfrak{Z}_{2p} \equiv \{ \lambda_j^- \omega^p \in \Omega_{2p}, \quad j = 1, \ldots, N \}, \]

are the sets of zeroes and poles of the minors \( M_{\nu,k}(\lambda) \) in the sectors \( \Omega_\nu \). Then

1. \( R_\nu(x, y, \lambda) \) is an analytic function of \( \lambda \) for \( \lambda \in \Omega_\nu \) having pole singularities at \( \mathfrak{Z}_\nu \);
2. \( R_\nu(x, y, \lambda) \) is a kernel of a bounded integral operator for \( \lambda \in \Omega_\nu ; \)
3. For \( \lambda \in \mathcal{L}_\nu \cup \mathcal{L}_{\nu+1} \) \( R_\nu(x, y, \lambda) \) is an uniformly bounded function which is a kernel of an unbounded integral operator;
4. \( R_\nu(x, y, \lambda) \) satisfies the equation:

\[ (3.18) \]
\[ L(\lambda) R_\nu(x, y, \lambda) = \mathbb{1} \delta(x - y). \]

The next natural step is to establish the structure of the singularities of \( R_\nu(x, y, \lambda) \) at the points of \( \mathfrak{Z} \). This is done quite analogously by using the dressing factor \( (2.17) \). Note that in these matters the symmetry complicates the calculations.

One of the effects of the \( \mathbb{Z}_n \)-symmetry is that the sets \( \mathfrak{Z}_\nu \) are determined uniquely by \( \mathfrak{Z}_1 \) and \( \mathfrak{Z}_0 \):

\[ (3.19) \]
\[ \mathfrak{Z}_1 = \{ \lambda_j^+ \in \Omega_1, \quad j = 1, \ldots, N \}, \quad \mathfrak{Z}_0 = \{ \lambda_j^- \in \Omega_{2n}, \quad j = 1, \ldots, N \}. \]

The residue of \( R_\nu(x, y, \lambda) \) at the point \( \lambda = \lambda_j^\nu \) can be cast into the form:

\[ (3.20) \]
\[ \text{Res}_{\lambda = \lambda_j^\nu} R_1(x, y, \lambda) = -2i \text{Im} \lambda_j^+ | m_j(x) \rangle \langle m_j^+(x) |, \]
\[ \text{Res}_{\lambda = \lambda_j^\nu} R_{2n}(x, y, \lambda) = 2i \text{Im} \lambda_j^- | m_j(x) \rangle \langle m_j^-(x) |. \]
where \(|n_j^\pm(x)|\) and \((m_j^\pm(x))\) are properly normalized eigenvectors of the Lax operator corresponding to the eigenvalues \(\lambda_j^\mp \in \Omega_{+1}\). The residues in the other sectors \(\Omega_{\nu}\) with \(\nu \neq 0, 1\) (mod 2n) are evaluated from (3.21) by employing eq. (2.31). Here we also have the analog of the condition (3.10).

The derivation of the completeness relation of the eigenfunctions for CBCs with \(Z_n\) reduction follows the same lines but needs some modifications. Instead of \(\tilde{J}(x, y)\) (3.12) we should consider

\[
\tilde{J}(x, y) = \sum_{\nu=1}^{2n} \frac{(-1)^{\nu-1}}{2\pi i} \oint_{\gamma_{\nu}} d\lambda R_{\nu}(x, y, \lambda),
\]

where the contours \(\gamma_{\nu}\) are defined by:

\[
\gamma_{2\nu-1} = l_{2\nu-1} \cup \gamma_{2\nu-1}^\infty \cup \bar{l}_{2\nu}, \quad \gamma_{2\nu} = \bar{l}_{2\nu} \cup l_{2\nu+1}^\infty \cup \bar{\gamma}_{2\nu}.
\]

Here \(l_{\nu}\) are the rays (2.24) oriented from 0 to \(\infty\); \(\gamma_{\nu}^\infty\) is the ‘infinite’ arc \(R_0 e^{i\phi_0}\) with \(R_0 \gg 1\) and \(\pi(\nu - 1)/n \leq \phi_0 \leq \pi\nu/n\); by overbar we denote the same contour with opposite orientation. Thus all the contours \(\gamma_{2\nu-1}\) (resp. \(\gamma_{2\nu}\)) are positively (resp. negatively) oriented.

Now we apply again the contour integration method and get two answers for \(\tilde{J}(x, y)\). The first, according to Cauchy residue theorem is:

\[
\tilde{J}(x, y) = \sum_{\nu=1}^{2n} \sum_{j=1}^{N} \left( \text{Res}_{\lambda = \lambda_j^+} R_{2\nu+1}(x, y, \lambda) + \text{Res}_{\lambda = \lambda_j^-} R_{2\nu}(x, y, \lambda) \right).
\]

Integration along the contours taking into account that \(\lim_{\lambda \to \infty} \chi'(x, \lambda) = 0\) gives:

\[
\tilde{J}(x, y) = \sum_{\nu=1}^{2n} \frac{(-1)^{\nu-1}}{2\pi i} \oint_{\gamma_{\nu}} dx (R_{\nu}(x, y, \lambda) - R_{\nu-1}(x, y, \lambda)) + J^{-1}\delta(x - y).
\]

The completeness relation follows after equating both expressions and taking into account that (compare with (3.20) and (2.31)):

\[
\text{Res}_{\lambda = \lambda_j^+} R_{2\nu+1}(x, y, \lambda) = -2i\text{Im} \lambda_j^+ |n_j^{(2\nu+1)}(x)\rangle\langle m_j^{(2\nu+1)}(x)|,
\]

\[
\text{Res}_{\lambda = \lambda_j^-} R_{2\nu}(x, y, \lambda) = 2i\text{Im} \lambda_j^- |n_j^{(2\nu)}(x)\rangle\langle m_j^{(2\nu)}(x)|,
\]

where \(|n_j^{(2\nu)}(x)\rangle\rangle\) (resp. \(|m_j^{(2\nu)}(x)\rangle\rangle\) and \((m_j^{(2\nu+1)}(x))\)) are properly normalized discrete eigenfunctions of the CBCs (1.3) (resp. of the adjoint CBCs (2.2)) corresponding to the discrete eigenvalues \(\lambda_j^+\omega^{2p}\) and \(\lambda_j^-\omega^{2p}\). For the lack of space we can not provide all the details of the calculations. The final result is similar to the one for GZSs. Namely, any vector-function \(|z(x)\rangle\rangle \in \mathbb{C}^n\) can be expanded over the eigenfunctions of the CBCs as follows:

\[
\sum_{\nu=1}^{2n} \frac{(-1)^{\nu-1}}{2\pi} \oint_{\gamma_{\nu}} dx \left\{ \sum_{s < k_0, \nu} \zeta_{\nu, s}^+(\lambda) |\chi^{(s)}(x, \lambda)\rangle - \sum_{s > k_0, \nu} \zeta_{\nu, s}^-\langle \chi^{(s)}(x, \lambda)| \right\} + \sum_{j=1}^{N} \sum_{\nu=1}^{2n} \text{Im} \lambda_j^+ \left[ \zeta_{\nu, j}^+(|n_j(x)\rangle\langle m_j^{(\nu, +)}) - \zeta_{\nu, j}^-|n_j(x)\rangle\langle m_j^{(\nu, -)}| \right],
\]

where \(|n_j^\pm(x)|\) and \((m_j^\pm(x))\) are properly normalized eigenvectors of the Lax operator corresponding to the eigenvalues \(\lambda_j^\pm \in \Omega_{+1}\).
where the expansion coefficients are given by:

\[
\zeta_{\nu,s}^+(\lambda) = -i \int_{-\infty}^{\infty} dx \langle \hat{\chi}_{\nu,s}^+(x, \lambda) | J | z(x) \rangle,
\]

\[
\zeta_{\nu,s}^-(\lambda) = -i \int_{-\infty}^{\infty} dx \langle \hat{\chi}_{\nu,s}^-(x, \lambda) | J | z(x) \rangle,
\]

\[
\zeta_{\nu,j}^+(\lambda) = -i \int_{-\infty}^{\infty} dx \langle \hat{m}_{\nu,j}^+(x) | J | z(x) \rangle, \quad \zeta_{\nu,j}^-(\lambda) = -i \int_{-\infty}^{\infty} dx \langle \hat{m}_{\nu,j}^-(x) | J | z(x) \rangle.
\]

The completeness relations derived for GZSs and CBCs above can be viewed as the spectral decompositions for the generically non-self-adjoint operators \( L(\lambda) \).

**Remark 3.7.** The special case of a CBCs with \( \mathbb{Z}_n \)-symmetry is equivalent to \( n \)-th order scalar differential operator \([11]\). Indeed, one can easily check that the system \( L(1.3), (1.8) \) can be written down as:

\[
L \chi \equiv i \left[ \frac{d}{dx} + \sum_{k=1}^{n} \psi_k(x) K_k^0 + i\lambda c_0 \omega^{-1/2} \sum_{k=1}^{n} \omega^k E_{kk} \right] \chi(x, \lambda) = 0.
\]

After similarity transformation with \( u_0 = \sum_{s,j=1}^{n} \omega^{sj} E_{sj} \) goes into:

\[
L' \tilde{\chi} \equiv \frac{1}{k} u_0^{-1} L u_0 \tilde{\chi} \equiv \left[ \frac{d}{dx} + \sum_{s=1}^{n} \phi_s(x) E_{ss} - \tilde{\lambda} \sum_{s=1}^{n} E_{s,s+1} \right] \tilde{\chi}(x, \lambda) = 0,
\]

\[
\phi_s(x) = \sum_{k=1}^{n} \psi_k(x) \omega^{ks}, \quad \tilde{\lambda} = i\lambda c_0 \omega^{-1/2},
\]

and can be rewritten as the scalar operator

\[
L^{(n)} \chi_1 \equiv d_1 d_{n-1} \cdots d_2 d_1 \chi_1(x, \lambda) = \tilde{\lambda}^n \chi_1(x, \lambda),
\]

where \( d_k X(x, \lambda) = dX/dx + \phi_k(x) X(x, \lambda) \). If \( \phi_k(x) \) are real functions (additional \( \mathbb{Z}_2 \)-reduction of the type \([2.4]\) ensures this) then \( L^{(n)} \) is a self-adjoint operator.

**Remark 3.8.** The author is aware that these type of derivations need additional arguments to be made rigorous. One of the real difficulties is to find explicit conditions on the potential \( q(x) \) that are equivalent to the condition \((C.2)\) or equivalently, to the conditions that \( m_k^\pm(\lambda) \) have only finite number of simple zeroes. Nevertheless there are situations (e.g., the reflectionless potentials) when all these conditions are fulfilled and all eigenfunctions of \( L(\lambda) \) can be explicitly calculated. Another advantage of this approach is the possibility to apply it to Lax operators with more general dependence on \( \lambda \), e.g., quadratic or polynomial in \( \lambda \).

The ‘diagonal’ of the resolvent

By the diagonal of the resolvent one usually means \( R(x, y, \lambda) \) evaluated at \( y = x \). However the definition \([3.3]\) is not continuous for \( y = x \) and needs regularization. The simplest possibility is to consider as the diagonal of the resolvent:

\[
R(x, \lambda) = \frac{1}{2} \left( R(x + 0, x, \lambda) + R(x, x + 0, \lambda) \right).
\]

In fact we will consider as the a somewhat more general expression:

\[
R_P(x, \lambda) = i \chi^\pm(x, \lambda) P \hat{\chi}^\pm(x, \lambda),
\]

where the expansion coefficients are given by:

\[
\zeta_{\nu,s}^+(\lambda) = -i \int_{-\infty}^{\infty} dx \langle \hat{\chi}_{\nu,s}^+(x, \lambda) | J | z(x) \rangle,
\]

\[
\zeta_{\nu,s}^-(\lambda) = -i \int_{-\infty}^{\infty} dx \langle \hat{\chi}_{\nu,s}^-(x, \lambda) | J | z(x) \rangle,
\]

\[
\zeta_{\nu,j}^+(\lambda) = -i \int_{-\infty}^{\infty} dx \langle \hat{m}_{\nu,j}^+(x) | J | z(x) \rangle, \quad \zeta_{\nu,j}^-(\lambda) = -i \int_{-\infty}^{\infty} dx \langle \hat{m}_{\nu,j}^-(x) | J | z(x) \rangle.
\]
where $P$ is a constant diagonal matrix. Obviously $R_P(x, \lambda)$ satisfies

$$
(3.32) \quad i \frac{dR_P(x, \lambda)}{dx} + [q(x) - \lambda J, R_P(x, \lambda)] \equiv [L(\lambda), R_P(x, \lambda)] = 0.
$$

Thus $R_P(x, \lambda)$ belongs to the kernel of the operator $[L(\lambda), \cdot]$. Due to the fact that $\chi^{\pm}(x, \lambda)$ is the FAS and satisfies a RHP with canonical normalization we find:

$$
(3.33) \quad R_P(x, \lambda) = iP + \sum_{k=1}^{\infty} R_P^{(k)}(x) \lambda^{-k}.
$$

The coefficients $R_P^{(k)}(x)$ can be expressed through $q(x)$ using the recursion relations generalizing the ones of AKNS \[2, 18, 37, 27\]. These relations are solved by the recursion operators $\Lambda_{\pm}$ which have the form:

$$
(3.34) \quad \Lambda_{\pm} X = \text{ad}_J^{-1} \left( \frac{dX}{dx} + P_0([q_x], X(x)) + i \left[q(x), \int_{-\infty}^{\infty} dy[q(y), X(y)]\right] \right),
$$

where $P_0$ is the projector onto the off-diagonal part of the matrix $P_0 X = X_0^t$, the matrix $X(x)$ in (3.34) satisfies $X \equiv P_0 X$ and

$$
(\text{ad}_J^{-1} X_0^t)_{ij} = \frac{(X_0^t)_{ij}}{a_i - a_j}.
$$

The coefficients $R_P^{(k)}(x)$ can be expressed in compact form through $\Lambda_{\pm}$ as follows:

$$
(3.35) \quad R_P^{k+1} = \Lambda_{\pm} R_P^k = -\Lambda_{\pm} \text{ad}_J^{-1}[iP, q(x, t)],
$$

$$
(3.36) \quad R_P^{(k)d} = i \int_{-\infty}^{\infty} dy (\mathbb{1} - P_0) \left[q(y, t), R_P^{(k)d}\right] + \lim_{x \to \pm\infty} R_P^{(k)d}(x, t).
$$

Quite naturally these coefficients, or rather the diagonal of the resolvent generates \[17, 14, 18\]:

- the class of NLEE. Given the dispersion law, e.g., $f(\lambda) = \lambda N P$ of the NLEE, we can write down the equation itself by:

$$
(3.37) \quad -\frac{dq}{dt} + i \left( \frac{dR_P^{(N)}(x)}{dx} \right)^f + P_0([q(x, t), R_P^{(N)}(x, t)]) = 0.
$$

- the corresponding Lax representations, or in other words, the $M$-operators for each of these NLEE as follows:

$$
(3.38) \quad \Lambda_P^{(N)}(x, \lambda) = \sum_{k=0}^{N} R_P^{(k)}(x) \lambda^{N-k}.
$$

- the integrals of motion of the corresponding NLEE. This follows from

**Theorem 3.9** (\[18\]). The quantities

$$
(3.39) \quad R_{\Pi^{(k)}}^{\pm}(x, \lambda) = i \chi^{\pm}(x, \lambda) \Pi^{(k)} \chi^{\pm}(x, \lambda), \quad \Pi^{(k)} = \sum_{s=1}^{k} E_{ss} - \frac{k}{n} \mathbb{1},
$$

satisfy the relations

$$
(3.40) \quad \int_{-\infty}^{\infty} dx \text{tr} \left(R_{\Pi^{(k)}}^{\pm}(x, \lambda) J - i \Pi^{(k)} J\right) = -\frac{d}{d\lambda} \mathcal{D}_k(\lambda),
$$

where $\mathcal{D}_k^{\pm}(\lambda)$ is defined by \[2, 4\].
Combined with the (3.34) we can deduce that the diagonal of the resolvent and the recursion operator

\[(\Lambda_\pm - \lambda)R^\pm_P(x, \lambda) = i[P, \text{ad}_J^{-1}q(x)],\]

directly reproduce the generating functionals of the conserved quantities.

The term ‘squared’ solutions and recursion operator do not reflect properly the algebraic properties of these objects. The recursion operators \(\Lambda_\pm\) can be understood as the Lax operator \(L(\lambda)\) in the adjoint representation. One of the definitions of the adjoint representation means that we should replace each element \(U(x, \lambda) \in \mathfrak{g}\) by \(\text{ad}_{\text{ad}}U(x, \lambda) = [U(x, \lambda), \cdot]\). Therefore due to (3.32) we can view the diagonal of the resolvent \(R^\pm_P(x, \lambda)\) as the eigenfunction of \(L(\lambda)\) in the adjoint representation. It remains to project out the kernel of \(\text{ad}_J\) in order to derive \(\Lambda_\pm\) from \(L(\lambda)\).

The ‘squared’ solutions are eigenfunctions of \(\Lambda_\pm\) and belong to a linear space, which is the co-adjoint orbit of \(\hat{\mathfrak{g}}^*_a\) determined by \(J\). The gauge covariant way to introduce them involves the FAS of \(L(\lambda)\) and is:

\[(3.41)\]
\[e_{ij}^\pm(x, \lambda) = P_0 \left( \chi^\pm(x, \lambda)E_{ij}\chi^\pm(x, \lambda) \right), \quad h_{ij}^\pm(x, \lambda) = P_0 \left( \chi^\pm(x, \lambda)H_{ij}\chi^\pm(x, \lambda) \right),\]

where \(\chi^\pm(x, \lambda)\) are the FAS of \(L(\lambda)\) GZSSs. The similarity transformation by \(\chi^\pm(x, \lambda)\) is the adjoint action of the group \(\mathfrak{g}\) on the algebra \(\mathfrak{g}\); therefore \(e_{ij}^\pm(x, \lambda)\) and \(h_{ij}^\pm(x, \lambda)\) are elements again of \(\mathfrak{g}\). The projection \(\Pi_0 = \text{ad}_J^{-1}\text{ad}_J\) is a natural linear operator on \(\mathfrak{g}\). Besides the ‘squared’ solutions are analytic functions of \(\lambda\) having both poles and zeroes at \(\lambda^\pm\).

More detailed analysis based on the Wronskian relations reveals several other important aspects [36, 19, 30] of the ‘squared’ solutions of GZSSs. First, the sets

\[\{e_{ij}^\pm(x, \lambda), e_{ji}^\pm(x, \lambda)\}, \quad e_{ij;k}^\pm(x), e_{ji;k}^\pm(x), e_{ij;k}^\pm(x), e_{ji;k}^\pm(x), \quad i < j, k = 1, \ldots N\}

and

\[\{e_{ji}^\pm(x, \lambda), e_{ij}^\pm(x, \lambda)\}, \quad e_{ji;k}^\pm(x), e_{ij;k}^\pm(x), e_{ji;k}^\pm(x), e_{ij;k}^\pm(x), \quad i < j, k = 1, \ldots N\}

form complete sets of functions on \(\mathfrak{M}\) that realize the mapping \(\mathfrak{M} \leftrightarrow \mathfrak{T}\). Here by \(e_{ji;k}^\pm(x)\) and \(\dot{e}_{ji;k}^\pm(x)\) we have denoted:

\[e_{ij;k}^\pm(x) = \left. \frac{d\epsilon_{ij}^\pm(x, \lambda)}{d\lambda} \right|_{\lambda = \lambda^\pm} \epsilon_{ij;k}^\pm(x).

Second, it is possible to expand the potential \([P, \text{ad}_J^{-1}q(x, t)]\) and its variation \(\text{ad}_J^{-1}\delta q(x)\) over each of the complete sets shown above. The corresponding expansion coefficients are expressed through \(\mathfrak{T}\) and their variations. These facts constitute the grounds on which one can show that the ISM can be understood as a generalized Fourier transform. The important difference as compare to the standard Fourier transform is in the fact that the operator \(L\) (as well as the operators \(\Lambda_\pm\)) allows for discrete eigenvalues. Therefore the completeness relations involve both integrals along the continuous spectrum and sum over the discrete eigenvalues. In the usual Fourier transform the discrete eigenvalues are absent.
Hamiltonian properties of the NLEE

Here we briefly formulate the Hamiltonian properties of the NLEE paying more attention to its algebraic structure. This has been widely studied problem, see \[3, 11, 39, 14, 17, 10, 13, 18, 18, 19\] and the numerous references therein.

In doing so we follow mainly the ideas of \[39\] with a natural generalization from \(sl(2)\) to \(sl(n)\)-algebras. The main idea in these papers is the possibility to write down the Lax equation \[1.4\] in explicitly Hamiltonian form as the co-adjoint action of \(\tilde{g}\) on its dual \(\tilde{g}^*\). Obviously any non-trivial grading in \(g\) (resp. \(\tilde{g}, \tilde{g}^*\)) will reflect into a corresponding grading of the dual algebra \(g^*\) (resp. \(\tilde{g}^*, \tilde{g}^*\)).

Below we will need also the Cartan-Weyl basis of \(sl(n)\). Choosing for definiteness the typical \(n \times n\) representation we fix it up by:

\[
(3.43) \quad h \equiv \text{l.c.} \{ H_i = E_{ii} - E_{i+1,i+1}, \quad i = 1, \ldots, n - 1 \}, \quad \{ E_{ij}, \quad i \neq j \}.
\]

As invariant bilinear form we can use \(\langle X, Y \rangle = \text{tr} \{XY\}\). Then the commutation relations can be written in the form:

\[
(3.44) \quad [H_i, E_{jk}] = (e_i - e_{i+1}, e_j - e_k)E_{jk}, \quad j \neq k,
\]

\[
[H_i, E_{kl}] = E_{il}, \quad [E_{jk}, E_{ki}] = -E_{kl}, \quad l \neq j,
\]

\[
[H_i, E_{kl}] = \sum_{s=j}^{k-1} H_s, \quad j < k.
\]

By \(e_k\) above we mean an orthonormal basis in the \(n\)-dimensional Euclidean space with a standard scalar product: \((e_j, e_k) = \delta_{jk}\). Those, who are familiar with Lie algebras will recognize \(e_i - e_{i+1}\) as the simple roots of \(sl(n)\) and the set of \(e_j - e_k\), \(j \neq k\) as the root system of \(sl(n)\).

If \(C = 1\) (i.e. with the trivial grading) each of the matrices \(U_k(x)\) in \[1.19\] is of generic form:

\[
(3.45) \quad U_k(x) = \sum_{j=1}^{n-1} u_j^{(k)} H_j + \sum_{j \neq p} u_{jp}^{(k)} E_{jp}.
\]

The coefficients \(u_j^{(k)}(x), u_{jp}^{(k)}(x)\) can be viewed as linear functionals on \(u_k(x)\) and thus they belong to \(g^*\). Using the bilinear form \[1.27\] they can be interpreted as linear functionals on \(\tilde{g}\) and thus as elements also of \(\tilde{g}^*\). The algebraic structure on \(\tilde{g}^*\) can be introduced in analogy with the commutation relations \[3.44\], namely:

\[
(3.46) \quad \left\{ u_i^{(s)}(x), u_{j+m+k}^{(m)}(y) \right\}_p = (e_i - e_{i+1}, e_j - e_k)u_{j+m+k}^{(s+m-p)}(x)\delta(x - y),
\]

\[
\left\{ u_{i,i+k}^{(s)}(x), u_{i+k,j}^{(m)}(y) \right\}_p = u_{i,j}^{(s+m-p)}(x)\delta(x - y),
\]

\[
\left\{ u_{i,i+k}^{(s)}(x), u_{j+k,i}^{(m)}(y) \right\}_p = -u_{j+k,i}^{(s+m-p)}(x)\delta(x - y),
\]

\[
\left\{ u_{i,i+k}^{(s)}(x), u_{i+k,k}^{(m)}(y) \right\}_p = \sum_{l=i}^{i+k-1} u_l^{(s+m-p)}(x)\delta(x - y) + i\delta_{s+m,p}\delta'(x - y).
\]

The derivation of these relations follows \[33\] in a rather straightforward manner; though a bit tedious, it can be generalized also to any simple Lie algebra.

Note that if \(p = -1\) then the term with \(\delta'(x - y)\) disappears and the Poisson brackets \[3.46\] become ultralocal. Then we can rewrite them in a compact form
using the classical $r$-matrix \[13\]:

\[
(3.47) \quad \left\{ U(x, \lambda) \otimes U(y, \mu) \right\}_{-1} = [r(\lambda - \mu), U(x, \lambda) \otimes \mathbb{1} + \mathbb{1} \otimes U(x, \mu)]\delta(x - y),
\]

\[
(3.48) \quad r(\lambda - \mu) = \frac{\Pi_0}{\lambda - \mu}, \quad \Pi_0 = \sum_{i,j=1}^n E_{ij} \otimes E_{ji}.
\]

The left hand side of \[3.47\] has the structure of the usual tensor product of $n \times n$ matrices, but instead of taking the product one should rather take the Poisson bracket between the matrix elements of the scattering matrix.

To do this we need to ‘integrate’ \[3.47\] which needs to take into account also the boundary conditions. For periodic boundary conditions on $q(x)$ this gives:

\[
(3.49) \quad \left\{ T(\lambda) \otimes T(\mu) \right\}_{-1} = [r(\lambda - \mu), T(\lambda) \otimes T(\mu)].
\]

For vanishing boundary conditions on $q(x)$ and $J = J^*$ the calculations need some additional considerations with the result (see \[13\]):

\[
(3.50) \quad r_\pm(\lambda - \mu) = \frac{1}{\lambda - \mu} \sum_{j=1}^n E_{jj} \otimes E_{jj} \mp i\pi\delta(\lambda - \mu) \sum_{i \neq j=1}^n E_{ij} \otimes E_{ji}.
\]

From both relations \[3.44\] and \[3.50\] there follows that the principal minors $m_k^\pm(\lambda)$ commute with respect to the Poisson brackets \[3.46\] \[19\], i.e.:

\[
(3.51) \quad \{ \mathcal{D}_k(\lambda), \mathcal{D}_j(\mu) \}_{-1} = 0.
\]

Since $\mathcal{D}_k(\lambda)$ are the generating functionals of integrals of motion $\mathcal{D}_k^{(s)}$ (see eq. \[2.51\]), then eq. \[3.51\] means that all these integrals are in involution with respect to these Poisson brackets.

The $\mathbb{Z}_n$-symmetry may modify substantially some of the above results. Indeed, it can be viewed as a set of constraints on the phase space $\mathcal{M}$ and on the generic Poisson brackets \[3.46\]. Then one should evaluate the corresponding Dirac brackets on the reduced phase space. However in the case of the $\mathbb{Z}_n$-NLS equation \[1.3\] with Lax operator $L$ given by \[1.7\], \[1.8\] somewhat surprisingly the approach of \[39\] gives us directly the correct answer. If we define $\psi_j(x, t)$ as linear functionals of $U(x, t, \lambda) = q(x, t) - \lambda J$ by:

\[
(3.52) \quad \psi_j(x, t) = \frac{1}{n} \text{tr} \left( U(x, t, \lambda) K^{n-j} \right),
\]

and make use of \[1.8\] then the set of Poisson brackets in \[3.46\] simplify to

\[
(3.53) \quad \{ \psi_j(x, t), \psi_k(y, t) \} = \delta_{k+j-n} \delta'(x - y).
\]
Together with the Hamiltonian $H = \omega^2 M_{1,1}^{(2)}$ they provide the Hamiltonian formulation of (1.3). Unfortunately this Poisson brackets are not ultra-local and the corresponding Lax operator does not allow classical $r$-matrix of the form (1.47).

For the affine Toda chain (1.2) the simplest Poisson brackets are provided by:

$$\left\{ \frac{dQ_j}{dx}, Q_k(y, t) \right\} = \delta_{kj} \delta(x - y).$$

(3.54)

The corresponding Lax operator (1.9) unlike the previous case allows classical $r$-matrix satisfying (3.47) which however has more complicated dependence on $\lambda - \mu$; it is known as the trigonometric $r$-matrix [38].

Another special property of the $\mathbb{Z}_n$-symmetric CBCs concerns the existence of the so-called symplectic basis [25]. The elements of these bases are special linear combinations of the ‘squared solutions’ (3.42) which are also complete in $\mathcal{M}$ and which are such that the expansion coefficients of $\delta q(x, t)$ over it produce the variations of the action-angle variables of the corresponding set of NLEE. In [25] this basis was worked out for the Zakharov-Shabat system related to the $sl(2)$ algebra. For GZSs related to algebras of higher rank such basis is yet unknown although it must exist since the action-angle variables for them are known [10, 7].

For the $\mathbb{Z}_n$-symmetric CBCs the construction of the symplectic basis is very much like the one in [25] due to the fact that the subalgebras $g_\nu$ related to each of the rays $l_\nu$ are direct sums of $sl(2)$ subalgebras. It is a complete set of functions on the phase space of the corresponding $\mathbb{Z}_n$-symmetric NLEE (1.1) and (1.2). Skipping the details we just give the explicit expressions for the set $\mathfrak{A}$ of action-angle variables of the $\mathbb{Z}_n$-NLS equation in terms of the scattering data of its Lax operator. Obviously $\mathfrak{A}$ will consists of two sets of functions $\mathfrak{A} = \mathfrak{A}_0 \cup \mathfrak{A}_1$ each set defined on the ray $l_0$ and $l_1$ respectively:

$$\mathfrak{A}_0 = \{ \pi_{ij}(\lambda), \kappa_{ij}(\lambda), \quad \lambda \in l_0, i + j = 2( \text{mod} \ n) \},$$

$$\mathfrak{A}_1 = \{ \pi_{ij}(\lambda), \kappa_{ij}(\lambda), \quad \lambda \in l_1, i + j = 1( \text{mod} \ n) \},$$

where

$$\pi_{ij}(\lambda) = -\frac{1}{\pi} \ln (1 + \rho_{ij}^{+} \rho_{ij}^{-}), \quad \kappa_{ij}(\lambda) = -\frac{i}{2} \frac{b_{ij}^{+}(\lambda)}{b_{ij}^{-}(\lambda)}, \quad \rho_{ij}^{\pm}(\lambda) = \frac{b_{ij}^{\pm}(\lambda)}{a_{ij}(\lambda)}.$$  

(3.55)

and the coefficients $a_{ij}^{+}(\lambda), b_{ij}^{+}(\lambda)$ were introduced in (2.30).

Quite analogous are the expressions for the action-angle variables for the two-dimensional affine Toda chain provided we use the scattering data of the Lax operator (1.10).

4. Conclusion

The restricted space did not allow us to give more details or explanations on these and related problems. We only mention some of them below.

One such important to our mind result is the interpretation of the ISM as a generalized Fourier transform. In its derivation for the GZSs and CBCs [27, 19, 30] both algebraic methods and analytic ones were used. As a result the pair-wise equivalence of the symplectic structures in the hierarchy becomes obvious.

The approach based on the Kac-Moody algebras is a natural basis for the Hamiltonian hierarchies. If one can derive a bi-Hamiltonian formulation of a given...
NLEE then there is a whole hierarchy of them related by a recursion operator $\Lambda$. Here we mention the paper \[15\] where the operator $\Lambda$ was derived as the ‘ratio’ of two such Hamiltonian structures for the $N$-wave equations. The result, of course coincides with the natural expression for $\Lambda$ obtained with the AKNS recursion method and whose spectral theory was constructed by other means in \[27, 18\].

The method based on the diagonal of the resolvent of the Lax operator started by Gel’fand and Dickey \[17, 10\] can be viewed also as a formal algebraic one. The authors studied by algebraic means the ring of operators, commuting with $L$. They expressed most of the quantities, including the diagonal of the resolvent of $L$, as series over fractional powers of $L$ and did not investigate the existence and convergence of these series. Once identified with the expression \[3.31\] in terms of the FAS these problems find their natural and positive solution.

Besides the classical $r$-matrix corresponding to the ultralocal Poisson brackets there exist also dynamical $r$-matrices depending on the fields $q_{ij}(x)$ in the NLEE. One of the problems, that is still not solved is to find the interrelation between the dynamical $r$-matrices, $r$ and the recursion operator $\Lambda$.

Finally, we should mention that both approaches have been further generalized. For example, the analytic approach was generalized from a local RHP to a nonlocal RHP and to $\bar{\partial}$-bar problem (also local and nonlocal), see \[1, 50, 37\]. This allowed to treat NLEE of soliton type in $2 + 1$ dimensions.

Another direction is to study Lax operators with more general $\lambda$-dependence such as polynomial, or rational \[51\]. Obviously all results concerning spectral decompositions can be formulated in a gauge covariant way thus allowing to treat also gauge equivalent NLEE \[28, 29, 19\].

The algebraic approach was also generalized to use as a basis infinite dimensional algebras such as Virasoro algebra, $W_{1+\infty}$ etc. which lead to the important construction of the Japanese $\tau$-function and its relation to the soliton theory, see \[32, 14\].

Thus we just outlined the beginning of all this and so it is time to stop.

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Appendix A. Gauss decompositions

The Gauss decompositions mentioned above have natural group-theoretical interpretation and can be generalized to any semi-simple Lie algebra. It is well known that if given group element allows Gauss decompositions then its factors are uniquely determined. Below we write down the explicit expressions for the matrix elements of $T^{\pm}(\lambda)$, $S^{\pm}(\lambda)$, $D^{\pm}(\lambda)$ through the matrix elements of $T(\lambda)$:

\[
T^{\pm}_{pj}(\lambda) = \frac{1}{m^\pm_j(\lambda)} \left\{ \begin{array}{c} 1, \ldots, j - 1, p \end{array} \right\}^{(j)}_{T(\lambda)}, \quad (A.1)
\]

\[
\hat{T}^{\pm}_{jp}(\lambda) = \frac{(-1)^{j+p}}{m^{j-1}_{p}(\lambda)} \left\{ \begin{array}{c} 1, \ldots, p, \ldots, j \end{array} \right\}^{(j-1)}_{T(\lambda)}, \quad (A.2)
\]
(A.3) \[ S_{pj}^+(\lambda) = \frac{(-1)^{p+j}}{m_{j-1}^+(\lambda)} \left\{ 1, 2, \ldots, p, \ldots, j - 1 \right\}_{T(\lambda)}^{(j)} \]

(A.4) \[ S_{jp}^+(\lambda) = \frac{1}{m_{j}^+(\lambda)} \left\{ 1, 2, \ldots, j - 1, j \right\}_{T(\lambda)}^{(j+1)} \]

(A.5) \[ T_{pj}^+(\lambda) = \frac{1}{m_{n-j+1}^+(\lambda)} \left\{ p, j + 1, \ldots, n - 1, n \right\}_{T(\lambda)}^{(n-j+1)} \]

(A.6) \[ T_{jp}^+(\lambda) = \frac{(-1)^{p+j}}{m_{n-j}^+(\lambda)} \left\{ j, j + 1, \ldots, p, \ldots, n \right\}_{T(\lambda)}^{(n-j)} \]

(A.7) \[ S_{pj}^-(\lambda) = \frac{(-1)^{p+j}}{m_{n-j}^-(\lambda)} \left\{ j + 1, j + 2, \ldots, p, \ldots, n \right\}_{T(\lambda)}^{(n-j)} \]

(A.8) \[ S_{jp}^-(\lambda) = \frac{1}{m_{n-j+1}^-(\lambda)} \left\{ j, j + 1, \ldots, n - 1, n \right\}_{T(\lambda)}^{(n-j+1)} \]

where

\[ \left\{ i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \right\}_{T(\lambda)}^{(k)} = \text{det} \begin{vmatrix} T_{i_1,j_1} & T_{i_1,j_2} & \cdots & T_{i_1,j_k} \\ T_{i_2,j_1} & T_{i_2,j_2} & \cdots & T_{i_2,j_k} \\ \vdots & \vdots & \ddots & \vdots \\ T_{i_k,j_1} & T_{i_k,j_2} & \cdots & T_{i_k,j_k} \end{vmatrix} \]

is the minor of order \( k \) of \( T(\lambda) \) formed by the rows \( i_1, i_2, \ldots, i_k \) and the columns \( j_1, j_2, \ldots, j_k \); by \( \bar{p} \) we mean that \( p \) is missing.

From the formulae above we arrive to the following

**Corollary A.1.** In order that the group element \( T(\lambda) \in SL(n, \mathbb{C}) \) allows the first (resp. the second) Gauss decomposition (2.14) is necessary and sufficient that all upper- (resp. lower-) principle minors \( m_k^+(\lambda) \) (resp. \( m_k^-(\lambda) \)) are not vanishing.

These formulae hold true also if we need to construct the Gauss decomposition of an element of the orthogonal \( SO(n) \) group. Here we just note that if \( T(\lambda) \in SO(n) \) then

\[ S_0(T(\lambda))^T S_0^{-1} = T^{-1}(\lambda), \]

where

\[ S_0 = \sum_{k=1}^{n_0} (-1)^{k+1} (E_{k,n+1-k} + E_{n+1-k,k}), \quad \text{if} \quad n = 2n_0, \]

\[ S_0 = \sum_{k=1}^{n_0} (-1)^{k+1} (E_{k,n+1-k} + E_{n+1-k,k}) + (-1)^{n_0} E_{n_0+1,n_0+1}, \quad \text{if} \quad n = 2n_0 + 1. \]

One can check that if \( T(\lambda) \) satisfies (A.14) then each of the factors \( T^\pm(\lambda), S^\pm(\lambda) \) and \( D^\pm(\lambda) \) also satisfy (A.10) and thus belong to the same group \( \Theta \). In addition we have the following interrelations between the principal minors of \( T(\lambda) \):

\[ m_j^+(\lambda) = m_{n-j}^+(\lambda), \quad \text{for} \quad SO(n), \]

\[ m_j^+(\lambda) = m_{n-j}^-(\lambda), \quad \text{for} \quad SP(n), \]
Appendix B. Dispersion relations for $\mathcal{D}_k(\lambda)$ and $\ln m_{\nu,k}(\lambda)$

Let us introduce the functions $f^\pm_k(\lambda)$:

$$f^+_k(\lambda) = \frac{m^+_k(\lambda)}{R_k(\lambda)}, \quad f^-_{n-k}(\lambda) = R_k(\lambda)m^-_{n-k}(\lambda), \quad R_k(\lambda) = \prod_{j=1}^{N} \left( \frac{\lambda - \lambda^+_j}{\lambda - \lambda^-_j} \right)^{b_{jk}},$$

which like $m^\pm_k(\lambda)$ are: i) analytic for $\lambda \in \mathbb{C}_\pm$; ii) satisfy $\lim_{\lambda \to \infty} f^\pm_k(\lambda) = 1$. Besides, $f^+_k(\lambda)$ have no zeroes in their regions of analyticity and therefore the functions $\ln f_k(\lambda)$ are analytic for $\lambda \in \mathbb{C}_\pm$ and tend to 0 for $\lambda \to \infty$. This allows one to apply the Plemelj-Sokhotzky formula with the result:

$$\mathcal{D}_k(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} \ln \left( f^+_k(\mu)f^-_{n-k}(\mu) \right),$$

where

$$\mathcal{D}_k(\lambda) = \begin{cases} 
\ln f^+_k(\lambda), & \lambda \in \mathbb{C}_+ \\
-\ln f^-_{n-k}(\lambda), & \lambda \in \mathbb{C}_-
\end{cases}$$

It remains to insert the above definitions of $f^\pm_k(\lambda)$ into (B.1) and (B.3) to derive Eqs. (2.45), (2.46).

The dispersion relation (2.50) is derived analogously treating the integral

$$\tilde{\mathcal{D}}(\lambda) = \sum_{\nu=1}^{2n} \frac{(-1)^{\nu-1}}{2\pi i} \int_{\gamma_i} \frac{d\mu}{\mu - \lambda} \ln \left\{ m^+_{\nu,k}(\mu) \prod_{\eta=1}^{N} \prod_{j=1}^{N} \left( \frac{\mu - \lambda^+_j}{\mu - \lambda^-_j} \right)^{b^\nu_{j,k}} \right\},$$

with $\lambda \in \Omega_{\nu}$ and the contours $\gamma_i$ as in (3.22).

References

[1] M. J. Ablowitz, A. S. Fokas, R. Anderson. The direct linearising transform and the Benjamin–Ono equation. Phys. Lett. 93A, n.8, 375–378, 1983.
[2] M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur. The inverse scattering transform – Fourier analysis for nonlinear problems. Studies in Appl. Math. 53, n. 4, 249–315, 1974.
[3] M. Adler. On a trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg-de Vries equations. Inv. Math. 50, 219-248 (1979).
[4] N. I. Akhiezer, I. M. Glazman. Theory of Linear operators in Hilbert space. Translated from Russian, New York, F. Ungar (1961-1963).
[5] R. Beals, R. R. Coifman. Scattering and inverse scattering for first order systems. Commun. Pure & Appl. Math. 37, 39 (1984).
[6] R. Beals, R. R. Coifman. Inverse scattering and evolution equations. Commun. Pure & Appl. Math. 38, 29 (1985).
[7] R. Beals, D. H. Sattinger. On the complete integrability of completely integrable systems. Commun. Math. Phys. 138, 409 (1991).
[8] P. J. Caudrey. The inverse problem for the third order equation $u_{xxx} + q(x)u_x + r(x)u = -i\xi^3u$. Phys. Lett. A 79A, 264 (1980); — The inverse problem for a general $n \times n$ spectral equation. Physica D 66, 56 (1982).
[9] F. Calogero, A. Degasperis. Spectral transform and solitons. Vol. I. North Holland, Amsterdam, 1982.
[10] L. A. Dickey. Soliton equations and Hamiltonian systems. Advanced series in Math. Phys., 12, World Scientific, (1991).
[11] V. Drinfeld, V. V. Sokolov. Lie Algebras and equations of Korteweg - de Vries type. Sov. J. Math. 30, 1975–2036 (1985).
[12] N. Dunford, J. T. Schwartz. Linear operators, vol. 2, Spectral theory. Self-adjoint operators in Hilbert space. (1963), Interscience Publishers, Inc., NY.
V. S. Gerdjikov, M. I. Ivanov. Expansions over the “squared” solutions and the inhomogeneous nonlinear evolution equations. Lett. Math. Phys. 6, n. 6, 315–324, (1982).

V. S. Gerdjikov. Generalized Fourier transforms for the soliton equations. Gauge covariant formulation. Inverse Problems 2, n. 1, 51–74, 1986.

— Generating operators for the nonlinear evolution equations of soliton type related to the semi-simple Lie algebras. Doctor of Sciences Thesis, 1987, JINR, Dubna, USSR, (In Russian).

V. S. Gerdjikov. $Z_2$-reductions and new integrable versions of derivative nonlinear Schrödinger equations. In Nonlinear evolution equations: integrability and spectral methods, Ed. A. P. Fordy, A. Degasperis, M. Lakshmanan, Manchester University Press, (1981), p. 367–379.

V. S. Gerdjikov. Generalized Fourier transforms for the soliton equations. Gauge covariant formulation. Inverse Problems 2, n. 1, 51–74, (1986).

V. S. Gerdjikov. Complete integrability, gauge equivalence and Lax representations of the inhomogeneous nonlinear evolution equations. Theor. Math. Phys. 92, 374–386 (1992).

V. S. Gerdjikov, G. G. Grahovski, R. I. Ivanov and N. A. Kostov. N-wave interactions related to simple Lie algebras.

— $Z_2$- reductions and soliton solutions. Inverse Problems 17, 999-1015 (2001).

V. S. Gerdjikov, M. I. Ivanov. Expansions over the “squared” solutions and the inhomogeneous nonlinear Schrödinger equation. Inverse Problems 8, 831–847 (1992).

V. S. Gerdjikov, E. Kh. Khristov. On the evolution equations solvable with the inverse scattering problem. I. The spectral theory. Bulgarian J. Phys. 7, No.1, 28–41, (1980). (In Russian); — On the evolution equations solvable with the inverse scattering problem. II. Hamiltonian structures and Backlund transformations. Bulgarian J. Phys. 7, No.2, 119–133, (1980) (In Russian).

V. S. Gerdjikov, P. P. Kulish. Complete integrable Hamiltonian systems related to the non-self-adjoint Dirac operator. Bulgarian J. Phys. 5, No.4, 337–349, (1978). (In Russian).

V. S. Gerdjikov, P. P. Kulish. The generating operator for the $n \times n$ linear system. Physica D, 3D, n. 3, 549–564, 1981.

V. S. Gerdjikov, A. B. Yanovsky. Gauge covariant formulation of the generating operator. II. Systems on homogeneous spaces. Phys. Lett. A, 110A, n. 1, 53–58, 1985.

V. S. Gerdjikov, A. B. Yanovsky. Gauge covariant formulation of the generating operator. I. Commun. Math. Phys. 103A, n. 4, 549–568, 1986.

V. S. Gerdjikov, A. B. Yanovsky. Completeness of the eigenfunctions for the Caudrey–Beals–Coifman system. J. Math. Phys. 35, no. 7, 3687–3725 (1994).

S. Helgasson. Differential geometry, Lie groups and symmetric spaces. Academic Press, 1978.

M. Jimbo, T. Miwa. Solitons and infinite dimensional algebras. Publications RIMS 19, 943–1000, 1983.

V. G. Kac. Infinite dimensional Lie algebras. Progress in Mathematics, vol. 44, Boston, Birkhauser, 1983.

V. G. Kac, A. K. Raina. Bombay lectures on highest weight representations of infinite dimensional Lie algebras. Advanced series in Math. Phys. vol. 2, (1987).

D. J. Kaup. Closure of the squared Zakharov–Shabat eigenstates. J. Math. Annal. Appl. 54, n. 3, 849–864, 1976.

D. J. Kaup. The three-wave interaction – a non-dispersive phenomenon. Studies in Appl. Math. 55, 9–44 (1976); D. J. Kaup, A. Reiman, A. Bers. Rev. Mod. Phys. Space-time evolution of nonlinear three-wave interactions. I. Interaction in a homogeneous medium. 51, 275-310 (1979).

D. J. Kaup, A. C. Newell. Soliton equations, singular dispersion relations and moving eigenvalues. Adv. Math. 31, 67–100, 1979.
[37] B. G. Konopelchenko Solitons in Multidimensions. Inverse Spectral Transform Method. World Scientific, Singapore, 1993.

[38] P. P. Kulish. Quantum difference nonlinear Schrodinger equation. Lett. Math. Phys. 5, 191–197, 1981.

[39] P. P. Kulish, A. G. Reiman Hamiltonian structure of polynomial bundles. Sci. Notes. LOMI seminars 123 67 - 76, (1983) (In Russian); Translated in J. Sov. Math. 28, 505-513 (1985);

M. A. Semenov-Tyan-Shanski. Classical r-matrices and the method of orbits. Sci. Notes. LOMI seminars 123 77 - 91, (1983) (In Russian); Translated in J. Sov. Math. 28, 513-523 (1985).

[40] S. V. Manakov. An example of a completely integrable nonlinear wave field with non-trivial dynamics (Lee model). Teor. Mat. Phys. 28, 172-179 (1976).

[41] A. V. Mikhailov The reduction problem and the inverse scattering problem. Physica D, 3D, n. 1/2, 73–117, 1981.

[42] R. Miura, (editor). Backlund transformations. Lecture Notes in Math., vol. 515, Berlin, Springer (1979).

[43] A. G. Reymann, M. A. Semenov-Tian Shanski. The jets algebra and nonlinear partial differential equations. DAN USSR (Reports of the USSR Academy), 251, No 6, p.1310-1314, (1980) (In Russian).

G. Segal, G. Wuilson. Loop groups and equations of KdV type. Publ. IHES, vol. 61, 5-65 (1985).

[44] A. B. Shabat. The inverse scattering problem for a system of differential equations. Functional Annal. & Appl. 9, n.3, 75 (1975) (In Russian);

— The inverse scattering problem. Diff. Equations 15, 1824 (1979) (In Russian).

[45] E. C. Titchmarsch. Eigenfunctions expansions associated with second order differential equations. Part I. Expansions over the eigenfunctions of \((d/dx)^2 + \lambda - q(x)\). (Oxford, Clarendon Press, 1958).

[46] N. P. Vekua. Systems of singular integral equations. Translated from Russian by A. G. Gibbs and G. M. Simmons, (Gröningen, P. Noordhoff Ltd., The Netherlands, 1967).

[47] V. E. Zakharov, S. V. Manakov, S. P. Novikov, L. I. Pitaevskii. Theory of solitons: the inverse scattering method. (Plenum, N.Y.: Consultants Bureau, 1984).

[48] V. E. Zakharov, S. V. Manakov, A. V. Mikhailov. Theory of resonant interaction of wave packets in nonlinear media. Sov. Phys. JETP 69, 1654 (1975) (In Russian).

[49] V. E. Zakharov, S. V. Manakov. Multidimensional nonlinear integrable systems and methods for constructing their solutions. Sci. Notes. LOMI seminars 133 77 - 91, (1984) (In Russian); Translated in J. Sov. Math. 31, 3307-3316 (1985).

[50] V. E. Zakharov, V. E. Zakharov, A. V. Mikhailov. On the integrability of classical spinor models in two–dimensional space–time. Commun. Math. Phys. 74, n. 1, 21–40, 1980;

— Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method. Zh. Eksp. Teor. Fiz. 74 1953, (1978).

[51] V. E. Zakharov, A. B. Shabat. A scheme for integrating nonlinear equations of mathematical physics by the method of the inverse scattering transform. I. Funct. Annal. and Appl. 8, no. 3, 43–53 (1974);

— A scheme for integrating nonlinear equations of mathematical physics by the method of the inverse scattering transform. II. Funct. Anal. Appl. 13(3) 13-23, (1979).

[52] S. Zhang. Classical Yang-Baxter equation and low-dimensional triangular Lie bialgebras. Phys. Lett. A 246 71–81 (1998).

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