In the paper, we provide the construction of a coincidence degree being a homotopy invariant detecting the existence of solutions of equations or inclusions of the form $Ax \in F(x)$, $x \in U$, where $A: D(A) \to E$ is an $m$-accretive operator in a Banach space $E$, $F: K \to E$ is a weakly upper semicontinuous set-valued map constrained to an open subset $U$ of a closed set $K \subset E$. Two different approaches are presented. The theory is applied to show the existence of non-trivial positive solutions of some nonlinear second-order partial differential equations with discontinuities.

This article is part of the theme issue ‘Topological degree and fixed point theories in differential and difference equations’.

1. Introduction

We study the existence of a solution to the problem

$$0 \in -Au + F(u), \quad u \in U,$$

where $A: D(A) \to E$ is a quasi-$m$-accretive operator in a (real) Banach space $E$, $F: K \to E$ is a set-valued map, subject to constraints in an open set $U \subset K \subset E$ of the closed set $K$ of state constraints. Such problems appear as an abstract setting of steady states of nonlinear evolution processes of the reaction–diffusion type, where $A$ corresponds to the (nonlinear) diffusion, while a (possibly set-valued) $F$ represents a reaction term. It is reasonable to speak of the solution to (1.1) as an $A$-equilibria of $F$. As an example,
we will discuss a boundary-value problem for a coupled PDE system

\[
\begin{aligned}
-\Delta (\rho \circ u)(x) &\in \varphi(u(x)), \quad u \in \mathbb{R}^M, \ x \in \Omega, \\
u_i(x) &\geq 0, \quad x \in \Omega, \ i = 1, \ldots, M, \\
u|_{\partial \Omega} &= 0,
\end{aligned}
\tag{1.2}
\]

where \( \Omega \subset \mathbb{R}^N \) is open, the unknown \( u = (u_1, \ldots, u_M) \in \mathbb{R}^M \), \( \Delta \) denotes the vectorial Laplace operator, \( \rho: \mathbb{R}^M \to \mathbb{R}_+^M \) and \( \varphi: \mathbb{R}_+^M \to \mathbb{R}_+^M \) is a set-valued perturbation.

If \( A \equiv 0 \) (resp. \( A = I \) is the identity) and \( K = E \), i.e. in the absence of constraints, problems (1.1), concerning the existence of the equilibria, i.e. zeros (resp. fixed points) of single- or set-valued maps are major topics of research that have long been studied by numerous authors. Classical invariants of Brouwer, Leray–Schauder and their relatives, i.e. topological degrees or the fixed point indices, are major topics of research that have long been studied by numerous authors. Classical invariants concerning the existence of the equilibria, i.e. zeros (resp. fixed points) of single- or set-valued maps are major topics of research that have long been studied by numerous authors. Classical invariants concerning the existence of the equilibria, i.e. zeros (resp. fixed points) of single- or set-valued maps are major topics of research that have long been studied by numerous authors. Classical invariants concerning the existence of the equilibria, i.e. zeros (resp. fixed points) of single- or set-valued maps are major topics of research that have long been studied by numerous authors. Classical invariants concerning the existence of the equilibria, i.e. zeros (resp. fixed points) of single- or set-valued maps are major topics of research that have long been studied by numerous authors.

The general problem (1.1) without constraints was studied in many different authors (e.g. [6–8]), but these methods fail in the presence of constraints. The constrained problem involving a resolvent compact m-accretive operator \( A \) and a continuous single-valued \( F \) was studied in [9], and a degree, having its roots in [10], detecting coincidences of \( A \) and \( F \) was introduced. An attitude for usc, i.e. upper semicontinuous set-valued \( F \) with compact convex values, started in [5], via single-valued approximations leads to a strict generalization of results from [9]. Here, however, we come across another difficulty since when studying partial differential inclusions (1.2), even if the perturbation \( \varphi \) is usc with compact convex values, the Nemytskii operator associated with \( \varphi \) is only weakly usc and never has compact values.

This motivates us to consider (1.1) with a resolvent compact quasi-m-accretive \( A: D(A) \to E \) (see a comment in remark 3.3 (iv)), an \( \mathcal{L} \)-retract \( K \) as a set of state constraints, an open (in \( K \)) \( U \subset K \) and a weakly usc map \( F: U \to E \) with convex weakly compact values. Such assumptions are optimal from the viewpoint of applications. We look for \( A \)-equilibria of \( F \), i.e. vectors \( u \in U \cap D(A) \) such that \( 0 \in -Au + F(u) \) (equivalently \( Au \cap F(u) \neq 0 \)). Let us briefly explain the idea hidden behind our approach. In cases without constraints it has been started in [10]: equation (1.1) is replaced by an equivalent fixed-point problem

\[
u \in (I + \lambda A)^{-1}(u + \lambda F(u)), \quad u \in U, \ \lambda > 0,
\tag{1.3}
\]

where \((I + \lambda A)^{-1}\) is the resolvent of \( A \). In order to deal appropriately with constraints, we suppose that \( K \) is invariant with respect to resolvents of \( A \), and we employ the so-called weak tangency of \( F \) and an approximation approach combined with some properties of \( \mathcal{L} \)-retracts.

The paper is organized as follows. After this introduction, we provide preliminaries concerning terminology, notation and auxiliary results. The third section provides the construction of a constrained topological degree; the fourth and fifth sections are devoted to the discussion of (1.2) and its relatives. It seems that even though the construction is quite long and tedious, the degree we present is a convenient tool for various problems in the field of differential equations and/or inclusions and variational inequalities involving set-valued maps that arise in the evolution processes.
2. Preliminaries

(a) Notation

The notation used throughout the paper is standard. In particular, \( (x, y) \) is the scalar product of \( x, y \in \mathbb{R}^N \) and \( |x| := \sqrt{x^t x} \) stands for the norm of \( x \). The use of function spaces (\( L^p \), Sobolev etc.), linear (unbounded in general) operators in Banach spaces, \( C_0 \) semigroups is standard. Given a metric space \((X,d)\), \( K \subset X \) and \( x \in X \), \( d_k(x) = \text{dist}(x,K) := \inf_{y \in K} d(x,y) \); by \( \overline{K} \) and \( dK \) we denote the closure and the boundary of \( K \); if \( x \in K \), then we write \( y \to x \) if \( y \to x \) and \( y \in K \).

\((E, \| \cdot \|)\) is a real Banach space, \( B := \{ x \in E \mid \| x \| < 1 \} \) (resp \( \overline{B} \)) is the unit open (resp. closed) ball in \( E \); \( E^* \) stands for the dual of \( E \); if \( x \in E \), \( p \in E^* \), then \((x,p) := p(x)\); by default \( E^* \) is normed. \( E_{w}^* \) stands for \( E^* \) endowed with the weak topology; \( \rightharpoonup \) denotes the weak convergence. \( B(u,r) \) (resp. \( D(u,r) \)) is the open (resp. closed) ball around \( u \in E \) of radius \( r > 0 \). Note that \( B(u,r) = u + rB \).

(b) Set-valued maps

Terminology in set-valued analysis is as in [11, Sec. 3.1] or [12, Sec. 1.3]: \( F : X \to E \) is a set-valued map \( F \) defined on a metric space \( X \) with at least closed values in \( E \); \( \text{Gr}(F) := \{(x,y) \in X \times E \mid y \in F(x)\} \) is the \text{graph} of \( F \). A map \( F : X \to E \) is \( H \)-usc (\( H \) stands for ‘Hausdorff’ and usc abbreviates ‘upper semicontinuous’) if for any \( x \in X \) and \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( F(y) \subset F(x) + \varepsilon B = \{ z \in E \mid d(z,F(x)) < \varepsilon \} \) if \( y \in X \) and \( d(y,x) < \delta \). We collect some well-known facts concerning \( H \)-usc maps:

(a) A usc map is \( H \)-usc and an \( H \)-usc map with compact values is usc.

(b) An \( H \)-usc map with weakly compact values is weakly usc, i.e. usc with respect to \( E_{w}^* \).

(c) If \( F \) is weakly usc, then it is uhc (i.e. upper hemicontinuous, see [11, Sec 3.2]). A uhc map with convex weakly compact values is weakly usc.

(d) If \( F \) has convex weakly compact values, then it is weakly usc if and only if given a sequence \( y_n \to x \) in \( X \) and \( y_n \in F(x_n) \) for all \( n \in \mathbb{N} \), there is a subsequence \( y_{n_k} \to y \in F(x) \).

In particular, such maps are \text{locally bounded}, i.e. for any \( x \in X \), there is \( \varepsilon > 0 \) such that \( \sup \{ \| v \| \mid v \in F(y), d(x,y) < \varepsilon \} < \infty \).

Below, we rely strongly on the following lemmata.

\textbf{Lemma 2.1 (comp. [3, Lemma 3.2]).} Let \( X \) be a metric space, \( \Phi : X \to E \) an \( H \)-usc set-valued map with convex values and let \( \xi : X \times E \to \mathbb{R} \) be such that for any \( (x,v) \), \( \xi(\cdot,v) \) is usc (as a real function) and \( \xi(x,\cdot) \) is convex. If for \( x \in X \), there is \( v_x \in \Phi(x) \) such that \( \xi(x,v_x) \leq 0 \), then for any \( \varepsilon > 0 \) and a continuous function \( \eta : X \to (0,\infty) \) there is a continuous \( f : X \to E \) such that \( f(x) \in \Phi(B(x,\eta(x))) + \eta(x)B \) and \( \xi(x,f(x)) \leq \varepsilon \) for \( x \in X \).

\textbf{Proof.} For any \( y \in X \) let

\[ V(y) := B \left( y, \frac{\eta(y)}{2} \right) \cap \left\{ z \in X \mid \Phi(z) \subset \Phi(y) + \frac{\eta(y)}{2} B, \eta(y) < 2\eta(z) \right\}. \]

Clearly, \( V := \{ V(y) \}_{y \in X} \) is an open cover of \( X \). Let \( W \) of \( X \) be an open cover of \( X \) star refining \( V \). For \( x \in X \) let \( \nu_x \in \Phi(x) \) with \( \xi(x,\nu_x) \leq 0 \). For \( W \in W \) and \( x \in W \), \( T_W(x) := \{ y \in W \mid \xi(y,v_x) < \varepsilon \} \). Clearly, \( x \in T_W(x) \) and \( T_W(x) \) is open since \( \xi(\cdot,v_x) \) is usc. Hence \( T := \{ T_W(x) \} \) is an open cover of \( X \). Let \( \{ \lambda_s \}_{s \in S} \) be a partition of unity subordinated to \( T \), i.e. for any \( s \in S \), there is \( W_s \in \mathcal{W} \), \( x_s \in W_s \) with \( \text{supp} \lambda_s \subset T_s := T_{W_s}(x_s) \). Let

\[ f(x) := \sum_{s \in S} \lambda_s(x)v_s, \quad u \in U, \]

where \( v_s := v_{x_s} \). Then \( f : X \to E \) is well-defined and continuous. For \( x \in X \) let \( S(x) := \{ s \in S \mid \lambda_s(x) \neq 0 \} \). If \( s \in S(x) \), then \( x \in T_s \), i.e. \( \xi(x,v_s) < \varepsilon \). Since \( \xi(\cdot,v_s) \) is convex, \( \xi(x,f(x)) < \varepsilon \). For any \( s \in S(x), \nu_s, x \in \)
$T_s \subset W_s$. Since $W$ star refines $V$, for $s \in S(x)$ points $x, x_s$ belong to the star of $x$ with respect to $W$, i.e.

$$x, x_s \in \bigcup_{W \in V : x \in W} W \subset V(y)$$

for some $y \in X$. Thus $\|x - y\| < \frac{n(y)}{\rho_n} < \eta(x)$, i.e. $y \in B(x, \eta(x))$ and $v_s \in \Phi(x_s) \subset \Phi(y) + \frac{n(y)}{2}B$ for $s \in S(x)$. This together with the convexity of $\Phi(y)$ shows that

$$f(x) \in \Phi(y) + \eta(x)B \subset \Phi\left(B(x, \eta(x))\right) + \eta(x)B. \quad (2.1)$$

**Lemma 2.2.** If $\Phi : X \times [0, 1] \rightarrow E$ is $H$-usc, then for any $\varepsilon > 0$ there is a continuous function $\delta : X \rightarrow (0, 1)$ such that

$$\Phi\left(B(x, \delta(x)) \times [0, \delta(x))\right) \subset \Phi(B(x, \varepsilon) \times [0, \varepsilon)) + \varepsilon B, \ x \in X.$$

**Proof.** For $x \in X$, there is $0 < \delta_x < \min(1, \varepsilon)$ such that $\Phi(B(x, 2\delta_x) \times [0, \delta_x)) \subset \Phi(x, 0) + \varepsilon B$. Let $\{\lambda_s\}_{s \in S}$ be a partition of unity subordinated to the open cover $\{B(x, \delta_x)\}_{x \in X}$. Hence for any $s \in S$, there is $x_s \in X$ such that $supp \lambda_s \subset B(x_s, \delta_x)$, where $\delta_s := \delta_{x_s}$. Let $\delta(x) := \sum_{s \in S} \lambda_s(x) \delta_x$ for $x \in X$. Then $\delta : X \rightarrow (0, 1)$ is well-defined and continuous. If $x \in X$, then there is $t \in S$ with $\lambda_t(x) \neq 0$, i.e. $x \in B(x_t, \delta_{x_t})$, and $\delta(x) \leq \delta_t$. Then $\Phi(B(x_t, \delta_t) \times [0, \delta_t)) \subset \Phi(B(x_t, 2\delta_t) \times [0, \delta_t)) \subset \Phi(x_t, 0) + \varepsilon B \subset \Phi(B(x_t, \varepsilon) \times [0, \varepsilon)) + \varepsilon B.$

(c) **Accretive operators (e.g. [13])**

A (possibly set-valued) operator $A : D(A) \rightarrow E$, where $D(A) \subset E$, is called **accretive**, if for all $x, y \in D(A)$, $u \in Ax$, $v \in Ay$ and $\lambda > 0$, $\|x - y\| \leq \|x - y + \lambda(u - v)\|$. $A$ is $m$-accretive if it is accretive and Range$(I + \lambda A) = E$ for some (equivalently for all) $\lambda > 0$. $A$ is $\omega$-accretive (resp. $\omega$-$m$-accretive), where $\omega \in \mathbb{R}$, if $\omega A + A$ is accretive (resp. $m$-accretive). The following facts are of importance.

(a) If $A : D(A) \rightarrow E$ is accretive and only if for any $(u, v), (u', v') \in Gr(A)$, there is $p \in J(I(u - u'))$, where $J : E \rightarrow E^*$ is the (normalized) duality map (see [13, eqn. (1.1)]), such that $\langle v - v', p \rangle \geq 0$.

(b) By the Lumer–Phillips Theorem [14, Theorem 3.15], a densely defined linear $A : D(A) \rightarrow X$ is $\omega$-$m$-accretive if and only if $-A$ is the generator of a $C_0$ semigroup $\{e^{-tA}\}_{t \geq 0}$ such that $\|e^{-tA}\| \leq e^\omega t$ for $t \geq 0$.

(c) If $A$ is $\omega$-$m$-accretive, $\lambda > 0$ and $\lambda \omega < 1$, then $f^A_\lambda := (I + \lambda A)^{-1} : E \rightarrow D(A)$ is well-defined single-valued, $\|f^A_\lambda u - f^A_{\lambda_0} u\| \leq (1 - \lambda \omega)^{-1}\|u - w\|$. The map $E \times (0, \lambda_0) \ni (u, \lambda) \mapsto f^A_\lambda u$, where $\lambda_0 := \infty$ if $\omega \leq 0$ and $\lambda_0 = \omega^{-1}$ otherwise, is continuous. If $u \in D(A)$, then $\lim_{\lambda \rightarrow 0^+} f^A_\lambda u = x$.

(d) If $A$ is $\omega$-$m$-accretive, then the following resolvent identities are satisfied

$$f^A_{\lambda_i} = f^A_{\lambda_2} \left(\lambda_2 \lambda^{-1}_{i} I + (1 + \lambda_2 \lambda^{-1}_{i})f^A_{\lambda_i}\right), \ \lambda_i > 0, \ \lambda_i \omega < 1, \ i = 1, 2; \quad (2.2)$$

and

$$f^A_{\lambda} u = f^A_{\lambda(1 - \omega^{-1})} \left((1 - \lambda \omega)^{-1} u\right), \ \lambda > 0, \ \lambda \omega < 1. \quad (2.3)$$

(d) **Resolvent compact accretive operators**

An $\omega$-$m$-accretive operator $A$ is said to be **resolvent compact** if $f_{\lambda} = f^A_{\lambda}$ is compact for any $\lambda > 0$ with $\lambda \omega < 1$ (1). If $A$ is resolvent compact, $0 < \lambda_1 \leq \lambda_2$ and $\lambda_i \omega$ for $i = 1, 2$, then a map $E \times [\lambda_1, \lambda_2] \ni (u, \lambda) \mapsto f_{\lambda} u$ is compact.

---

1 A continuous map is compact if it maps bounded sets into compact ones.
A family \( \mathcal{A} = \{A(t)\}_{t \in [0,1]} \), where \( A(t) \) is an \( \omega(t)\)-\( m\)-accretive for \( t \in [0,1] \), is resolvent continuous (resp. resolvent compact) if \( \omega: [0,1] \to \mathbb{R} \) is continuous and for \( \lambda > 0 \), \( \sup_{t \in [0,1]} \omega(t) < 1 \), the map
\[
E \times [0,1] \ni (u,t) \mapsto f^A(t)u
\]
is continuous (resp. compact). If \( \mathcal{A} \) is resolvent compact, then \( A(t) \) is resolvent compact for all \( t \). If \( A \) is \( m\)-accretive, then the family \( \mathcal{A}(t) := \omega(t)I + \mu(t)A \), where \( \omega(t), \mu(t): [0,1] \to \mathbb{R} \) are continuous, if \( \mu > 0 \), is resolvent continuous; it is resolvent compact if so is \( A \) (see [9, Example 2.6]). The following closedness result is of importance.

**Proposition 2.3.** Let \( \mathcal{A} = \{A(t)\}_{t \in [0,1]} \) be a resolvent continuous family of \( \omega-m\)-accretive operators.

(i) For sequences \( (u_n) \) in \( E \), \( (t_n) \) in \( [0,1] \) and \( (v_n) \) in \( E \) such that \( u_n \in D(A(t_n)) \), \( v_n \in A(t_n)x_n \) for all \( n \geq 1 \), if \( u_n \to u_0 \in M \), \( t_n \to t_0 \in [0,1] \) and \( v_n \to v_0 \in E \) (resp. \( E^* \) is uniformly convex and \( v_n \to v_0 \), then \( (u_0, v_0) \in \text{Gr}(A) \), i.e. \( u_0 \in D(A(t_0)) \) and \( v_0 \in A(t_0)u_0 \).

(ii) If \( \mathcal{A} \) is resolvent compact, then for any bounded sequences \( (u_n) \) in \( M \), \( (v_n) \) in \( E \) and \( (t_n) \) in \( [0,1] \) such that \( u_n \in D(A(t_n)) \) and \( v_n \in A(t_n)u_n \), the set \( \{u_n\} \) is relatively compact.

**Proof.** The ‘strong’ version of (i) and (ii) follows from [9, Prop. 2.7]. We have only to show (i) when \( v_n \to v_0 \). Assume first that \( \omega(t) = 0 \) for all \( t \), i.e. \( A \) consists of \( m\)-accretive operators. For \( n \geq 0 \), let \( y_n := \int_1^0 (u_0 + v_0) \). Then \( u_0 + v_0 = y_n + z_n \), where \( z_n \in A(t_n)y_n \), for any \( n \geq 0 \). The resolvent continuity implies that \( y_n \to y_0 \) and hence \( z_n \to z_0 \). Since \( E^* \) is uniformly convex, the duality mapping \( J \) is single valued and continuous. In view of (a) on p. 4, \( \langle u_n - z_n, J(u_n - y_n) \rangle \geq 0 \) for all \( n \geq 1 \). Passing with \( n \to \infty \) one has
\[
0 \leq \langle v_0 - y_0, J(u_0 - y_0) \rangle = \langle y_0 - u_0, J(u_0 - y_0) \rangle = \|y_0 - u_0\|^2.
\]
Hence \( u_0 = y_0 \) and \( v_0 = z_0 \), i.e. \( (u_0, v_0) \in \text{Gr}(A(t_0)) \).

Suppose \( \omega(t) \) is continuous. Then for \( t \in [0,1] \), \( B(t) := \omega(t)I + A(t) \) is \( m\)-accretive and the family \( \{B(t)\} \) is resolvent continuous (see (2.3)). Clearly, \( w_n := \omega(t_n)u_n + v_n \in B(t_n)u_n \) and \( w_n - w_0 = \omega(t_0)u_0 + v_0 \). By the above argument \( (u_0, v_0) \in \text{Gr}(B) \); hence \( (u_0, v_0) \in \text{Gr}(A) \).

**Corollary 2.4.** If \( A \) is \( \omega\)-\( m\)-accretive and \( E^* \) is uniformly convex, then \( \text{Gr}(A) \) is closed as the subset of \( E \times E^*_w \), i.e. given a sequence \( (u_n, v_n) \in \text{Gr}(A) \) if \( u_n \to u_0 \) and \( v_n \to v_0 \), then \( (u_0, v_0) \in \text{Gr}(A) \). If \( A \) is resolvent compact, \( (u_n, v_n) \in \text{Gr}(A) \) and sequences \( (u_n), (v_n) \) are bounded, then \( (u_n) \) has a convergent subsequence.

(e) **Tangent cones and \( \mathcal{L}\)-retracts**

(a) If \( K \subset E \) is closed and \( x \in K \), then the Clarke tangent cone is defined by
\[
T_K(x) := \left\{ u \in E \mid \limsup_{y \to x, h \to 0^+} \frac{d_K(y + hu)}{h} = 0 \right\}.
\]

Observe that
\[
\limsup_{y \to x, h \to 0^+} \frac{d_K(y + hu)}{h} = d^*_K(x, u)
\]
is the Clarke derivative of \( d_K(\cdot) \) at \( x \) in the direction of \( u \) (e.g. [15] or [11, Proposition 7.3.10] for details). Hence, the map \( K \times E \ni (x, u) \mapsto d^*_K(x, u) \) is upper semicontinuous (as a real function) and for \( x \in K, d^*_K(x, \cdot) \) is convex. If \( K \) is closed convex, then (see [12, Section 4.2])
\[
T_K(x) = \bigcup_{h > 0} h^{-1}(K - x) = \left\{ u \in E \mid \lim_{h \to 0^+} \frac{d_K(x + hu)}{h} = 0 \right\}
\]
is the tangent cone in the sense of convex analysis. Some other facts and details are to be found in [12, Section 4.1] or [11, Section 4.1]. Let us only mention that if \( K = D(x_0, R) \) and \( \|x - x_0\| = R \),
then \( u \in T_K(x) \) if and only if \( [x - x_0, u]_+ \leq 0 \), where the semi-inner product (e.g. [16, Section 1.6]),

\[
[x, y]_+ := \lim_{h \to 0^+} \frac{\|x + hy\| - \|x\|}{h}, \quad x, y \in E.
\]

Observe that if \( E \) is a Hilbert space, then \( \|x\| [x, y]_+ = \langle x, y \rangle \) is the inner product.

**Corollary 2.5.** Let \( K \subseteq E \) be closed and \( U \subseteq K \). If \( \Phi : U \times [0, 1] \to E \) is \( H \)-usc with convex values and \( \Phi(u, t) \cap T_K(u) \neq \emptyset \), \( u \in U, t \in [0, 1] \), then for any \( \varepsilon > 0 \) there is a continuous \( f : U \times [0, 1] \to E \) such that for \( u \in U, t \in [0, 1] \),

\[
d^*_{K}(u, f(u, t)) < \varepsilon \quad \text{and} \quad f(u, t) \in \Phi(B(u, \varepsilon) \times (t - \varepsilon, t + \varepsilon)) + \varepsilon B
\]

and

\[
f(u, i) \in \Phi_i(B(u, \varepsilon)) + \varepsilon B, \quad \text{where} \quad \Phi_i := \Phi(\cdot, i), \quad i = 0, 1.
\]

**Proof.** By lemma 2.2, we find a continuous \( \delta : U \to (0, 1) \) such that

\[
\Phi\left(B(u, \delta(u)) \times [0, \delta(u))\right) \subset \Phi\left(B(u, \varepsilon) \times [0])\right) + \varepsilon B
\]

and

\[
\Phi\left(B(u, \delta(u)) \times (1 - \delta(u), 1)\right) \subset \Phi\left(B(u, \varepsilon) \times (1]\right) + \varepsilon B.
\]

Letting \( \xi((u, t), v) := d^*_{K}(u, v) \) and \( \eta(u, t) := \min(\delta(u), \varepsilon) \) for \( u \in U, t \in [0, 1] \). In view of lemma 2.1, we find a continuous \( f : U \times I \to E \) such that \( d^*_{K}(u, f(u, t)) < \varepsilon \) and

\[
f(u, t) \in \Phi\left(B(u, \eta(u, t)) \times (t - \eta(u, t), t + \eta(u, t))\right) + \eta(u, t)B \subset \Phi\left(B(u, \varepsilon) \times (t - \varepsilon, t + \varepsilon)\right) + \varepsilon B.
\]

In particular,

\[
f(u, 0) \in \Phi\left(B(u, \eta(u, 0)) \times [0, \eta(u, 0))\right) \subset \Phi\left(B(u, \delta(u)) \times [0, \delta(u))\right) \subset \Phi_0(B(u, \varepsilon)) + \varepsilon B
\]

and

\[
f(u, 1) \in \Phi\left(B(u, \eta(u, 1)) \times (1 - \eta(u, 0), 1]\right) \subset \Phi\left(B(u, \delta(u)) \times (1 - \delta(u), 1]\right) \subset \Phi_1(B(u, \varepsilon)) + \varepsilon B.
\]

(b) Let \( \varepsilon > 0 \). In the setting of corollary 2.5, we say that \( f : U \times [0, 1] \to E \) is an \( \varepsilon \)-tangent (graph-) approximation of \( \Phi : U \times [0, 1] \to E \) (writing \( f \in a(\Phi, \varepsilon) \)) if condition (2.5) is satisfied. Similarly, we say that \( f : U \to E \) is an \( \varepsilon \)-tangent approximation of \( F : U \to E \) (writing \( f \in a(F, \varepsilon) \)), if for all \( u \in U \),

\[
d^*_{K}(u, f(u)) < \varepsilon \quad \text{and} \quad f(u) \in F(B(u, \varepsilon)) + \varepsilon B.
\]

(c) The closed set \( K \subseteq E \) is an \( L \)-retract if there is a \( \eta > 0 \), a constant \( L \geq 1 \) and a retraction \( r : B(K, \eta) \to K \) such that \( \|x - r(x)\| \leq Ld_K(x) \) for all \( x \in B(K, \eta) \). The class of \( L \)-retracts is broad; among others it contains closed convex sets, the so-called epi-Lipschitz sets, \( C^2 \)-manifolds in \( E \) and others (see [1]). It is clear that an \( L \)-retract \( K \) belongs to the class of metric ANRs (absolute neighbourhood retracts).

(f) Fixed point index on ANRs (see [17] or [18])

Let \( M \) be a metric ANR. A compact map \( \phi : U \to M \), where \( U \subseteq M \) is open, is admissible if the fixed point set \( \text{Fix}(\phi) := \{ x \in U \mid \phi(x) = x \} \) is compact. By an admissible homotopy, we mean a compact map \( \phi : U \times [0, 1] \to M \) such that the set \( \{ x \in U \mid \phi(x, t) = x \text{ for some } t \in [0, 1]\} \) is compact. To any admissible map \( \phi : U \to M \), there corresponds an integer \( \text{ind}_M(\phi, U) \) such that:

(i) (Existence) If \( \text{ind}_M(\phi, U) \neq 0 \), then \( \text{Fix}(\phi) \neq \emptyset \).

(ii) (Additivity) Let \( \phi : U \to M \) be an admissible map and \( U_1, U_2 \subseteq U \) be open and disjoint sets such that \( \{ x \in U \mid \phi(x) = x \} \subseteq U_1 \cup U_2 \). Then

\[
\text{ind}_M(\phi, U) = \text{ind}_M(\phi|_{U_1}, U_1) + \text{ind}_M(\phi|_{U_2}, U_2).
\]
(iii) (Homotopy invariance) If \( \psi : U \times [0, 1] \rightarrow M \) is an admissible homotopy, then \( \text{ind}_M(\psi(\cdot, 0), U) = \text{ind}_M(\psi(\cdot, 1), U) \).

(iv) (Contraction) If \( N \subset M \) is closed and \( N \in \text{ANR}, \phi(U) \subset N \), then \( \text{ind}_M(\phi, M) = \text{ind}_N(\psi, U \cap N) \), where \( \psi : U \cap N \rightarrow N \) is the contraction of \( \phi \).

(v) (Local normalization) If \( \Theta_{x_0} : U \rightarrow M \), where \( x_0 \in M \setminus \partial U \) is given by \( \Theta_{x_0}(x) := x_0 \) for \( x \in U \), then

\[
\text{ind}_M(\Theta_{x_0}, U) = \begin{cases} 1 & \text{if } x_0 \in U \\ 0 & \text{if } x_0 \in M \setminus \text{cl } U. \end{cases}
\]

(vi) (Global normalization) If \( \phi : M \rightarrow M \) is compact at large (i.e. \( \phi(M) \) is relatively compact), then \( \phi \) is a Lefschetz map (see [18, Theorem (7.1)]), its generalized Lefschetz number \( \Lambda(\phi) \) is defined \( \text{ind}_M(\phi, M) = \Lambda(\phi) \).

3. The constrained coincidence degree

We now define an invariant detecting solutions to (1.1), where \( K \subset E \) is an \( \mathcal{L} \)-retract. This assumption is general; in applications, however, convex closed constraining sets are usually encountered (recall that convex closed sets are \( \mathcal{L} \)-retracts).

**Assumption 3.1.** A pair \((A, F)\) is admissible in the sense that:

1. an \( \omega \)-\( m \)-accretive operator \( A : D(A) \rightarrow E \) is resolvent compact and \( K \) is resolvent-invariant, i.e. \( J_\lambda(K) \subset K \) for \( 0 < \lambda \leq \hat{\lambda} \), where \( \hat{\lambda} \omega < 1 \);
2. a map \( F : U \rightarrow E \), where \( U \subset K \) is open (in \( K \)), is weakly tangent, i.e. \( F(u) \cap T_K(u) \neq \emptyset \) for \( u \in K \);
3. the set \( C = \text{Coin}(A, F; U) := \{ u \in U \mid u \in D(A), Au \cap F(u) \neq \emptyset \} \) is compact.

**Assumption 3.2.** A pair \((\{A(t)\}_{t \in [0, 1]}, \Phi)\) is an admissible homotopy if:

1. the family of \( \omega(t) \)-\( m \)-accretive operators \( \{A(t)\}_{t \in [0, 1]} \) is resolvent compact; \( J_\lambda^{A(t)}(K) \subset K \) for all \( 0 < \lambda \leq \hat{\lambda} \), where \( \hat{\lambda} \sup_{t \in [0, 1]} \omega(t) < 1 \);
2. a map \( \Phi : U \times [0, 1] \rightarrow E \) is weakly tangent, i.e. \( \Phi(u, t) \cap T_K(u) \neq \emptyset \) for \( u \in U, t \in [0, 1] \);
3. the set \( \tilde{C} := \bigcup_{t \in [0, 1]} \text{Coin}(A(t), \Phi(\cdot, t)) \) is compact.

**Remark 3.3.** Let us briefly comment on the assumptions.

(i) The structural assumption that \( K \) is an \( \mathcal{L} \)-retract is necessary in a sense. It has been thoroughly discussed in [5], where examples of neighbourhood retracts and tangent maps without equilibria were shown. It is moreover clear that without tangency condition (which, in fact, prescribes directions of values taken by \( F \)) no equilibria of \( F \) exist.

(ii) The resolvent compactness of \( A \) is the only compactness condition considered in this paper. As it seems if \( E \) is a Hilbert space, \( K \) is convex, \( \omega < 0 \) and \( F \) is a set-valued map being locally Lipschitz with respect to the Hausdorff distance, then the construction of an invariant detecting \( A \)-equilibria is also possible.

(iii) Recall that \( A \)-equilibria, i.e. solutions to (1.1), correspond to stationary states of the evolution process governed by \( A \) and perturbed by \( F \). It is, therefore, reasonable to assume that the 'unperturbed' evolution determined by \( A \) is viable in \( K \), i.e. if \( u \in K \), then \( S_A(t)u \in K \) for \( t \geq 0 \), where \( \{S_A(t)\}_{t \geq 0} \) is the (nonlinear) semigroup generated by \( A \). The viability condition (i.e. the semigroup invariance of \( K \)) is slightly weaker than resolvent invariance (due to the so-called Crandall–Liggett formula). The invariance of this type is a subject of intensive research (e.g. [3]).

---

2 Interesting and important computations of the index on closed convex sets are provided in [19,20].
(iv) Let us also address the following question. By supposition $A$ is $\omega$-m-accretive, i.e. $B := A + \omega I$ is $m$-accretive. It is easy to see that (1.1) is equivalent to $0 \in -Bu + G(u)$, $u \in K$, where $G = F + \omega I$. It is clear that $B$ is resolvent compact. Moreover, if $K$ is convex, $\omega < 0$ and $0 \in K$, then $K$ is invariant with respect to resolvents of $B$ (see (2.3)) and $G$ is weakly tangent. Therefore, under these circumstances one may discuss $m$-accretive operators without loss of generality. This is not, however, the case in a general situation: the assumption of quasi-$m$-accretivity can not, in general, be replaced by $m$-accretivity.

We shall provide two constructions: in the first one we assume that the admissible perturbation $F$ is $H$-usc with weakly compact convex values; in the second one weakly usc admissible perturbations will be considered. As mentioned above the ideas behind our construction rely on an approach started by [10] (see also [21]) in the unconstrained and single-valued situation and [9], where the perturbation was single-valued, too. It is to be mentioned that a different, ‘dynamical’, approach has been presented in [22] (see also [23]) for single-valued locally Lipschitz perturbations.

(a) $H$-upper semicontinuous perturbations

In this subsection, we assume additionally that

(D$_1$) the dual space $E^*$ is uniformly convex;
(D$_2$) $(A, F)$ is admissible and $F : U \to E$ is $H$-usc with compact convex values;
(D$_3$) $(\{A(t)\}_{t \in [0, 1]}, \Phi)$ is admissible and $\Phi : U \times [0, 1] \to E$ is $H$-usc with convex weakly compact values.

**Remark 3.4.** (1) Condition 3.1 (resp. 3.2 (3)) holds true if and only if $C$ and $F(C)$ (resp. $\overline{\bar{C}} \times [0, 1]$) are bounded. Indeed, if $C$ is compact, then $F(C)$ is weakly compact and, hence, bounded. If $C$ and $F(C)$ are bounded, then $\overline{C}$ is relatively compact since $C \subset \{J_\lambda(u + \lambda v) \mid u \in C, v \in F(u)\}$ and $J_\lambda$ is compact. To see that $\overline{C}$ is closed take a sequence $(u_n)$ in $C$, $u_n \to u_0$ and $v_n \in Au_n \cap F(u_n)$, $n \geq 1$. Condition (D$_2$) implies that (a subsequence) $v_{n_k} \to v_0 \in F(u_0)$. In view of corollary 2.4, $v_0 \in D(A)$ and $v_0 \in Au_0$, i.e. $u_0 \in C$.

(2) Observe that if $F$ is usc with compact values, the provisional assumption concerning the uniform convexity of $E^*$ is not necessary because in this case the graph $\text{Gr}(F)$ is closed.

(3) In view of corollary 2.5 if $F$ (resp. $\Phi$) is admissible and (D$_2$) (resp. (D$_3$)) holds true, then $a(F, \varepsilon) \neq \emptyset$ (resp. $a(\Phi, \varepsilon) \neq \emptyset$). Moreover, there are $\varepsilon$-tangent approximations of $\Phi$ satisfying condition (2.6).

(b) The construction

Let us first briefly describe the idea behind. Instead of (1.1), we study a problem

$$u \in J_\lambda \circ r(\mu + \lambda f(u)), \ u \in U,$$

where $\lambda > 0$ is small enough, $r$ is an $L$-retraction and $f$ is a sufficiently close tangent graph-approximation of $F$. Since the map in the right-hand side of (3.1) is compact, the fixed point index can be used. We shall proceed in several steps.

**Step 1:** Since $C$ is compact and $F$ is locally bounded, there are an open (in $K$) bounded set $W \subset K$ such that

$$C \subset W \subset \overline{W} \subset U,$$

and $x_0 > 0$ such that $F(W)$ and $f(W)$ are bounded if $f \in a(F, \varepsilon), 0 < \varepsilon \leq x_0$ ($\overline{W}$ is the closure of $W$ in $K$).
Step 2: Take $W$ as in Step 1. For any $M > 0$, there is $0 < \varepsilon_0 = \varepsilon_0(M) \leq \varepsilon$ depending on $M$ (and $W$) such that if $0 < \varepsilon \leq \varepsilon_0$ and $f^1, f^2 \in a(F, \varepsilon)$, then for $u \in \partial W \cap D(A)$, $t \in [0, 1]$ and $v \in A u$

$$\|v - f(u, t)\| \geq M\varepsilon_0, \quad (3.3)$$

where $f(\cdot, t) := (1 - t)f^1 + t f^2$, $t \in [0, 1]$. Suppose to the contrary that there are sequences $\varepsilon_n \to 0$, $0 < \varepsilon_n \leq \varepsilon$, $(u_n)$ in $\partial W \cap D(A)$ and $v_n \in A u_n$ such that $\|v_n - f(u_n, t_n)\| \ll M\varepsilon_0$, where $f(u_n, t) := (1 - t)f^1_n + t f^2_n$ and $f^1_n, f^2_n \in a(F, \varepsilon_n)$, $n \geq 1$. The sequence $(v_n)$ is bounded. In view of corollary 2.4 ($u_n$) has a convergent subsequence; assume without loss of generality that $u_n \to u_0 \in \partial W$ and that $t_n \to t_0$. Since $f^1_n \in a(F, \varepsilon)$, we get $f^1_n(u_n) \in F(B(u_n, \varepsilon_n)) + \varepsilon_n B$, there is $\bar{u}^n \in L_{x_0} | u_n - u_0 | < \varepsilon_n$ and $\bar{u}^n \in F(\bar{u}^n)$ such that $\|f^1_n(u_n) - \bar{u}^n\| < \varepsilon_n$, $i = 1, 2$. Clearly, $u^n \to u_0$ and (after passing to subsequence) $\bar{u}^n \to \bar{u}_0 \in F(u_0)$ since $F$ is weakly use with convex weakly compact values (see p. 4). Thus $\bar{u}_n := (1 - t_n)\bar{u}^n + t^n \bar{u}^n \to 0 := (1 - t_0)\bar{u}^0 + t^n \bar{u}^2 \in F(u_0)$ and $v_n \to v_0$, too. Therefore, $v_0 \in A u_0$ by corollary 2.4, i.e. $u_0 \in \text{Coin}(A, F; U)$, a contradiction.

Step 3: Take a retraction $r$: $B(K, \eta) \to K$ with $\|x - r(x)\| \leq Ld_K(x)$ for some $L \geq 1$. Let $f^1, f^2 \in a(F, \varepsilon)$, where $0 < \varepsilon \leq \varepsilon_0(L)$. We claim that there is $0 < \lambda \leq \lambda_0$ (see assumption 3.1 (1)) depending on $W$ such that $0 < \lambda \leq \lambda_0$, then for any $u \in \bar{W}$, $t \in [0, 1]$, $u + \lambda f(u, t) \in B(K, \eta)$, where $f(\cdot, t) := (1 - t)f^1 + t^2, \quad \text{and the set}$

$$C_\lambda := \left\{ u \in \bar{W} \mid u = f_\lambda \circ r(u + \lambda f(u, t)) \right\}$$

is contained in $W$.

Without loss of generality we may suppose $\lambda$ is small enough that for $0 < \lambda \leq \lambda$, $\lambda \|f(u, t)\| < \eta$ for $u \in \bar{W}$. Hence if $0 < \lambda \leq \lambda$, then $u + \lambda f(u, t) \in B(K, \eta)$ for $u \in \bar{W}$ and $t \in [0, 1]$. Thus $C_\lambda$ is well defined. Suppose to the contrary that there are sequences $\lambda_n \to 0$, $0 < \lambda_n \leq \lambda_n$, $(u_n)$ in $\partial W \cap D(A)$ and $(t_n)$ in $[0, 1]$ such that

$$u_n = f_{\lambda_n} \circ r(u_n + \lambda_n f(u_n, t_n)), \quad n \geq 1. \quad (3.4)$$

Hence, there is $v_n \in A u_n$ with $r(u_n + \lambda_n f(u_n, t_n)) = u_n + \lambda_n v_n$, $n \geq 1$. By the definition of an $L$-retract, we see that

$$\lambda_n \|v_n - f(u_n, t_n)\| = \|r(u_n + \lambda_n f(u_n, t_n)) - (u_n + \lambda_n f(u_n, t_n))\|
\leq L d_K(u_n + \lambda_n f(u_n, t_n)), \quad n \geq 1. \quad (3.5)$$

Observe that $d_K(u_n \lambda_n f(u_n, t_n)) \leq \lambda_n \|f(u_n, t_n)\|$ since $u_n \in K$. In view of (3.5) $\|v_n\| \leq (L + 1)\|f(u_n, t_n)\|$, i.e. the sequence $(v_n)$ is bounded. This, together with the boundedness of $(u_n)$ and corollary 2.4, implies that, after passing to a subsequence, $u_n \to u_0 \in \partial W$. We also let without loss of generality $t_n \to t_0$. In view of (3.3) and (3.5), for all $n \geq 1$

$$L \varepsilon_0 \leq \|v_n - f(u_n, t_n)\| \leq L d_K(u_n + \lambda_n f(u_n, t_n)) \frac{\lambda_n}{\lambda_n} + L \|f(u_n, t_n) - f(u_0, t_0)\|. \quad (3.6)$$

Passing with $n \to \infty$ in (3.6), property (2.7) of $f^1$ and $f^2$ yields

$$L \varepsilon_0 \leq \limsup_{n \to \infty} \|v_n - f(u_n, t_n)\| \leq L d_K(u_0, f(u_0, t_0))
\leq L \left[(1 - t_0)d_K(u_0, f^1(u_0)) + t_0 d_K(u_0, f^2(u_0)) \right] < L \varepsilon,
$$

i.e. $\varepsilon_0 < \varepsilon$: a contradiction.

Step 4: Take $0 < \varepsilon < \varepsilon_0 = \varepsilon_0(L + 1)$ and $0 < \lambda \leq \lambda_0$. Let $g^\varepsilon(\lambda): \bar{W} \to K$ be given by

$$g^\varepsilon(\lambda) := f_{\lambda_0} \circ r(u + \lambda f(u)), \quad u \in \bar{W}, \quad (3.7)$$

where $f \in a(F, \varepsilon)$. Then $g^\varepsilon(\lambda)$ is well defined. Moreover, $g^\varepsilon(\lambda)$ is compact and so is $f_{\lambda_0}$. In view of Step 3, the set $\text{Fix} g^\varepsilon(\lambda) := \{u \in \bar{W} \mid u = g^\varepsilon(\lambda)(u)\}$ of fixed points of $g^\varepsilon(\lambda)|_{\bar{W}}$ is contained in $W$. Therefore, we are in a position to consider the fixed point index $\text{ind}_K(g^\varepsilon(\lambda), W)$. We claim that
if \(0 < \varepsilon_1 \leq \varepsilon_2 \leq \varepsilon_0, 0 < \lambda_1 \leq \lambda_2 \leq \lambda_0, f_1 \in a(F, \varepsilon_1), f_2 \in a(F, \varepsilon_2)\) and \(g_i(u) := I_{\lambda_i} \circ r(u + \lambda_i f_i(u))\)
for \(u \in \bar{W}, i = 1, 2\), then maps \(g_i : \bar{W} \to K\) are compact and
\[
\text{ind}_K(g_1, W) = \text{ind}_K(g_2, W).
\] (3.8)

To see this consider a homotopy \(g : \bar{W} \times [0, 1] \to K\) given by
\[
g(u, t) = I_{\lambda(t)} \circ r(u + \lambda(t)f(u, t)), \quad u \in \bar{W}, t \in [0, 1],
\]
where \(\lambda(t) := (1 - t)\lambda_1 + \lambda_2, \lambda(t)(t) = (1 - t)f_1 + tf_2, t \in [0, 1]\). It is clear that \(g\) is compact (see page 5) and \([u \in \bar{W} \mid u = g(u, t)\) for some \(t \in [0, 1]\] \(\subset W\). Therefore, (3.8) follows from the homotopy invariance of the index.

**Step 5:** Let open (in \(K\)) and bounded sets \(W_1, W_2\) be such that \(W_1 \subset W_2, C \subset W_i \subset \bar{W}_i \subset U\) and \(f(\bar{W}_i)\) be bounded if \(f \in a(F, \varepsilon)\), where \(\varepsilon > 0\) is sufficiently small. Take a sufficiently small \(\lambda > 0\), too. Then arguing as in Step 4, we show that \(g_{\lambda}(u) \neq u\) for \(u \in \bar{W}_2 \setminus W_1\). The localization property of the fixed point index implies that \(\text{ind}_K(g_{\lambda}, W_1) = \text{ind}_K(g_{\lambda}, W_2)\).

**Step 6:** Finally: if \(W\) is taken as in Step 1, \(0 < \varepsilon \leq \varepsilon_0\) and \(\lambda > 0\) are sufficiently small, then the index \(\text{ind}_K(g_{\lambda}, W)\) does not depend on the choice of a neighbourhood retraction \(r\).

To this end suppose that \(\eta_i : B(K, \eta_i) \to K\) are retractions with \(\|r_i(u) - u\| \leq L_i d_K(u)\) for \(u \in B(K, \eta_i), i = 0, 1, \lambda := \min[\eta_0, \eta_1]\) and \(L := \max[L_0, L_1]\). Consider
\[
g_i(u) := I_{\lambda_i} \circ r_i(u + \lambda f_i(u)), \quad u \in \bar{W}, i = 0, 1,
\]
where \(f \in a(F, \varepsilon)_0 < \varepsilon \leq \varepsilon_0(L(L + 2))\) and \(0 < \lambda \leq \lambda_0\). We show that \(\text{ind}_K(g_0, W) = \text{ind}_K(g_1, W)\) provided \(\lambda\) is small enough. To this aim consider a map
\[
g(u, t) = I_{\lambda(t)} \circ r_i (u + \lambda f_i(u)), \quad u \in \bar{W}, t \in [0, 1].
\]
It is easy to see that \(d_K((1 - t)(u + \lambda f(u)) + t r_i(u + \lambda f_i(u))) \leq \lambda(1 + t)\|f(u)\|\). Hence \(g\) is well defined when \(\lambda\) is small. Clearly, \(g(i, \lambda) = g_i\) for \(i = 0, 1\). As in Step 5, \(g\) is compact. To conclude the proof by the homotopy invariance, we check that if \(\lambda\) is sufficiently small, then \(u \neq g(u, t)\) for \(u \in \partial W, t \in [0, 1]\). Suppose to the contrary that there are sequences \(\lambda_n \searrow 0, (u_n)\) in \(\partial W\) and \((t_n) \subset [0, 1]\) such that \(u_n = g(u_n, t_n)\) for \(n \geq 1\), i.e.
\[
u_n + \lambda_n v_n = r_0 \left( (1 - t_n) (u_n + \lambda_n f(u_n)) + t_n r_1 (u_n + \lambda_n f(u_n)) \right)
= r_0 \left( u_n + \lambda_n f(u_n) + t_n \left( r_1 (u_n + \lambda_n f(u_n)) - (u_n + \lambda_n f(u_n)) \right) \right),
\]
where \(v_n \in Au_n, n \geq 1\). By a straightforward computation \(\lambda_n \|v_n - f(u_n)\| \leq L(L + 2)d_K(u_n + \lambda_n f(u_n))\) and hence
\[
L(L + 2)\varepsilon_0 \leq \limsup_n \|v_n - f(u_n)\| \leq L(L + 2)d_K^2(u_0, f(u_0)) < L(L + 2)\varepsilon.
\]
A contradiction shows the assertion.

**(c) Conclusion**

We define the constrained degree of coincidence of the pair \((A, F)\) by the following formula:
\[
\deg_K(A, F; U) := \lim_{\lambda \searrow 0} \text{ind}_K(g_{\lambda}^F, W),
\] (3.9)

where \(\varepsilon > 0\) is sufficiently small, \(W\) is a neighbourhood of \(\text{Coin}(A, F; U)\) (in \(K\)) and \(g_{\lambda}^F\) is given by (3.7) with \(f \in a(F, \varepsilon)\). Arguments from Steps 1 to 7 justify this construction and show that the sequence in the right hand of the definition stabilizes and its limit does not depend on any auxiliary objects used to define it.

**Theorem 3.5.** _Let a pair \((A, F)\) be admissible. The degree defined by (3.9) has the following properties:

(i) (Existence) If \(\deg_K(A, F; U) \neq 0\) then \(\text{Coin}(A, F; U) \neq \emptyset\)._
(ii) (Additivity) If \( U_1, U_2 \subset U \) are open disjoint and \( \text{Coin}(A, F; U) \subset (U_1 \cup U_2) \setminus U_1 \cap U_2 \), then
\[
\text{deg}_K(A, F; U) = \text{deg}_K(A, F; U_1) + \text{deg}_K(A, F; U_2).
\]

(iii) (Homotopy invariance) If \((|A(t)|)_{t \in [0,1]}, \Phi)\) is an admissible homotopy, then
\[
\text{deg}_K(A(0), F(\cdot, 0); U) = \text{deg}_K(A(1), F(\cdot, 1); U).
\]

(iv) (Normalization) If \( K \) is bounded, \( F: K \to E \) and \( F(K) \) is bounded in \( E \), then the Euler characteristic \( \chi(K_A) \), where \( K_A := K \cap \partial D(A) \), is well-defined and \( \text{deg}_K(A, F; K) = \chi(K_A) \).

**Proof.** (i) Suppose to the contrary that \( \text{Coin}(A, F; U) = \emptyset \) and take open \( W \subset K \) and \( \bar{\varepsilon} > 0 \) as in Step 1. Arguing as in Step 2, we get \( 0 < \varepsilon_0 \leq \bar{\varepsilon} \) such that \( \|v - f(u)\| \geq L_{\varepsilon_0} \) for any \( u \in \overline{W} \cap \partial D(A) \) and \( v \in Au \), where \( f \in a(F, \varepsilon) \) with \( 0 < \varepsilon < \varepsilon_0 \). If \( 0 < \varepsilon < \varepsilon_0 \) is small enough and \( \lambda_\varepsilon \downarrow 0 \), then \( 0 \neq \text{deg}_K(A, F; U) = \text{deg}_K(g_A, W) \), where \( g_A(u) := j_{u} \circ r(u + \lambda_\varepsilon f(u)), u \in \overline{W} \), and \( f \in a(F, \varepsilon) \). Arguing as in Step 3, we find sequences \((u_n)\) in \( W \), \( v_n \in Au_n \) such that (after passing to a subsequence) \( u_n \to u_0 \) and
\[
L_{\varepsilon_0} \leq \limsup_{n \to \infty} \|v_n - f(u_n)\| \leq \limsup_{n \to \infty} \left( L_{\varepsilon_0} \frac{\text{deg}_K(u_n, \lambda_\varepsilon f(u_0))}{\lambda_\varepsilon} + L_{\varepsilon_0} \right) < L_{\varepsilon_0},
\]
a contradiction.

(ii) The additivity property follows immediately from the additivity property of the index \( \text{ind}_K \).

(iii) We argue as follows. Choose an open \( W \subset K \) and \( \bar{\varepsilon} > 0 \) such that \( \Phi(\overline{W} \times [0,1]) \) and \( f(\overline{W} \times [0,1]) \) are bounded, where \( f \in a(\Phi, \varepsilon) \) with \( 0 < \varepsilon \leq \bar{\varepsilon} \) and \( \bigcup_{t \in [0,1]} \text{Coin}(A(t), \Phi(\cdot, t); U) \subset W \subset \overline{W} \subset U \).

Using arguments similar to those from Step 2 we get \( \varepsilon_0 \leq \bar{\varepsilon} \) such that \( \|v - f(u, t)\| \geq L_{\varepsilon_0} \) for any \( t \in [0,1], u \in \partial W \cap \partial D(A(t)) \) and \( v \in A(t, u) \), where \( f \in a(\Phi, \varepsilon) \), \( 0 < \varepsilon < \varepsilon_0 \). Next, we show that there is \( \lambda_0 > 0 \) such that for every \( f \in a(\Phi, \varepsilon) \), \( 0 < \varepsilon < \varepsilon_0 \), the set \( \{u \in \overline{W} \mid \text{deg}_K(u, f) = 0\} \) is compact, \( \lambda_0 > 0 \) is sufficiently small.

(iv) By definition \( \text{deg}_K(A, F, K) = \text{deg}_K(g, K) \), where \( g: K \to K \) is compact, \( g(u) = g_A(u) := j_{u} \circ r(u + \lambda_\varepsilon f(u)), u \in K \), where \( f \in a(F, \varepsilon) \) and \( \varepsilon, \lambda_\varepsilon > 0 \) are sufficiently small. Actually \( g(K) \subset K_A \).

Let \( g': K \to K_A \) be the contraction of \( g, j: K_A \to K \) be the inclusion and \( g_A := g' | K_A = g' \circ j_A \).

Clearly, \( g_A \) is compact. Consider \( h: K_A \times [0,1] \to K \) given by
\[
h(u, t) := \begin{cases} 
  g_B(u) & u \in K_A, \ t \in (0,1] \\
  u & u \in K_A, \ t = 0.
\end{cases}
\]

In view of (c) on p. 4, we see that \( h \) is a well-defined continuous homotopy joining \( g_A \) to the identity \( \text{id}_{K_A} \). In view of the normalization property of \( \text{ind}_K \), \( g \) is a Lefschetz map and \( \Lambda(g) = \text{ind}_K(g, K) \). The commutativity of the diagram
\[
\begin{array}{ccc}
K & \xrightarrow{g'} & K_A \\
\downarrow g & \downarrow j & \downarrow g_A \\
K & \xrightarrow{g'} & K_A
\end{array}
\]
along with [18, Lemma (3.1)] implies \( g_A \) is a Lefschetz map, i.e. \( H_*(g_A) \) is a Leray endomorphism (see [18, Section 2]) and \( \Lambda(g_A) = \Lambda(g) \odot \Lambda(J_{K_A}) \). Since \( g_A \) is homotopic to \( \text{id}_{K_A} \), we have that \( H_*(g_A) = H_*(\text{id}_{K_A}) = \text{id}_{H_*(K_A)} \). Therefore, \( \text{id}_{H_*(K_A)} \) is a Leray endomorphism, i.e. the graded vector space
\footnote{\( H_*(\cdot) \) stands for the singular homology functor with the rational coefficients.}
H_{s}(K_A) is of finite type and the Euler characteristic \( \chi(K_A) := \sum_{q \geq 0} (-1)^{q} \dim_{\mathbb{Q}} H_{q}(K_A) \) is a well-defined integer number. Moreover, \( \chi(K_A) = \lambda(id_{K_A}) = \Lambda(id_{K_A}) = \Lambda(\varphi_{A}) = \Lambda(\varphi) \), where \( \lambda(id_{K_A}) \) is the ordinary Lefschetz number. □

Let us now derive a series of results that can be treated as constrained generalizations of the Schaeffer or Leray–Schauder nonlinear alternatives.

**Proposition 3.6.** In addition to our standing assumptions, suppose that \( U = K \) (i.e. \( F : K \rightarrow E \)), \( K \) is closed convex, \( A \) is densely defined or \( E \) is uniformly convex. Let \( C := \{ u \in K \cap D(A) \mid Au \cap tF(u) \neq \emptyset \} \) for some \( t \in [0, 1] \). If \( F(C) \) is bounded and

(a) \( K \) is bounded; or

(b) \( C \) is bounded, \( A \) is \( \omega \)-m-accretive with \( \omega \leq 0 \) and \( 0 \in Au_{0} \) for some \( u_{0} \in D(A) \), then \( \deg_{K}(A, F; K) = 1 \).

**Proof.** Under our assumption \( \overline{D(A)} = E \) or, if \( E \) is uniformly convex (recall that so is \( E^{*} \) by assumption), then \( \overline{D(A)} \) is convex in view of [13, Proposition 3.5]. Let \( \Phi : K \times [0, 1] \rightarrow E \) be given by \( \Phi(u, t) := tu, u \in K \). It is easy to see that \( \Phi \) is admissible. Hence by the homotopy invariance and normalization properties we get \( \deg_{K}(A, F; K) = \deg_{K}(A, 0; K) = \chi(K_A) = 1 \) because \( K_A \) is convex.

Suppose now that \( K \) is not bounded but \( C \) is. The map \( \Phi \) defined above provides an admissible homotopy showing that \( \deg_{K}(A, F; K) = \deg_{K}(A, 0; K) \). Take \( R > 0 \) such that \( C \subset B(u_{0}, R) \), let \( W := B(u_{0}, R) \cap K \) and \( K' := \overline{W} = K \cap D(u_{0}, R) \). Then \( K' \) is closed convex; hence \( K' \) is ANR. By definition \( \deg_{K}(A, 0; K) = \ind_{K}(J_{\lambda}, W) \) for sufficiently small \( \lambda > 0 \). Since \( J_{\lambda} \) is non-expansive and \( J_{\lambda}(u_{0}) = u_{0} \) we get \( J_{\lambda}(K') = J_{\lambda}(\overline{W}) \subset \overline{W} \subset K' \). Hence, by the contraction and normalization properties of the index we get that

\[
\ind_{K}(J_{\lambda}, W) = \ind_{K}(J_{\lambda}, W) = \ind_{K}(J_{\lambda}, K') = \chi(K' \cap \overline{D(A)}) = 1. \tag{3.10}
\]

As a consequence, we get the counterpart of the local normalization property of the index.

**Corollary 3.7.** Suppose that \( A \) is \( \omega \)-m-accretive with \( \omega < 0 \), \( A \) is densely defined or \( E \) is uniformly convex, \( K \subset E \) is closed convex. If \( 0 \in Au_{0} \) for some \( u_{0} \in D(A) \), then for any \( U \) open in \( K \),

\[
\deg_{K}(A, 0; U) = \begin{cases} 
1 & \text{if } x_{0} \in U, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** For the proof, it is sufficient to observe that \( A \) is invertible since \( A = -\omega(I + (-\omega)^{-1}(A + \omega I)) \) and appeal to part (b) of the above Proposition. □

**Proposition 3.8.** Again, in addition to the standing assumptions, let \( K \subset E \) be closed convex, \( U = K \), \( A \) an \( \omega \)-m-accretive operator with \( \omega \leq 0 \), \( 0 \in Au_{0} \) for some \( u_{0} \in D(A) \) and \( A \) is densely defined or \( E \) is uniformly convex. Suppose that \( F \) is bounded on bounded sets and

\[
\lim_{\|u\| \to \infty} \sup_{u \in K} \sup_{v \in F(u)} \|u - u_{0} - v\| < 0.
\]

Then \( C = \text{Coin}(A, F; K) \neq \emptyset \) is unbounded or \( \deg_{K}(A, F; K) = 1 \).

**Proof.** Assume that \( C \) is bounded and take \( R > 0 \) such that \( C \subset W := B(u_{0}, R) \) and \( \sup_{u \in F(u)} \|u - u_{0} - v\| \leq 0 \) for all \( u \in E \) with \( \|u - u_{0}\| = R \). This implies that \( F(u) \subset D(u_{0}, R)(u) \) for any \( u \in D(u_{0}, R) \). If \( R \) is large enough, then \( B(u_{0}, R) \cap K \neq \emptyset \) and, by [11, Theorem 4.1.16], \( T_{K \cap D(u_{0}, R)}(u) = T_{K}(u) \cap T_{D(u_{0}, R)}(u) \) for all \( u \in K \cap D(u_{0}, R) \). Therefore, \( (A, F) \) is admissible with respect to the \( L \)-retract \( K' := K \cap D(0, R) \). Observe also that since \( J_{\lambda} \) is non-expansive, we get \( J_{\lambda}(K') \subset K' \) for any \( \lambda > 0 \). Therefore, we are in a position to define \( \deg_{K}(A, F; K) = \deg_{K}(A, F; W) \) and \( \deg_{K}(A, F; K') \). By the normalization property we see that \( \deg_{K}(A, F; K') = 1 \). We now show that both degrees are equal. To see this take a tangent \( \varepsilon \)-approximation of \( f : K \rightarrow E \). If \( \varepsilon > 0 \) is small
enough that $\deg_K(A,F;K') = \text{ind}_K(g'_\omega, K')$ where $g'_\omega(u) = J_\omega \circ r'(u + \lambda f(u))$ for $u \in K'$, $r'$ is an $L$-retraction (defined on $E$) onto $K'$ and $\lambda > 0$ sufficiently small. On the other hand, $\deg_K(A,F;W) = \text{ind}_K(g_\omega, W)$, where $g_\omega(u) = J_\omega \circ r(u + \lambda f(u)), u \in W$. Consider the map $h: W \times [0, 1] \to K$ given by

$$h(u, t) = J_\omega \circ \left( u + \lambda f(u) + t \left( r'(u_\omega f(u)) - (u + \lambda f(u)) \right) \right), \quad u \in W, \ t \in [0, 1].$$

Exactly as in Step 6 of our construction, we show that $h(u, t) \neq u$ for $u \in \partial W$ and $t \in [0, 1]$. Hence $\deg_K(A,F;W) = \text{ind}_K(h(\cdot, 1), W)$, where $h(\cdot, 1): W \to K$ is given by $h(u, 1) = J_\omega \circ r'(u + \lambda f(u))$. This, in view of the contraction property of the index, concludes the proof. 

**d) Weakly upper semicontinuous perturbations**

Now, in addition to assumptions 3.1 and 3.2, we assume that

$$(D_4) \ A \text{ is a densely defined linear operator (see (b) on p. 4)};$$

$$(D_5) \ (A,F) \text{ is admissible and } F: U \to E \text{ is weakly usc with convex weakly compact values.}$$

Recall that $A$, as the generator of a $C_0$ semigroup, is closed and densely defined.

**e) Construction**

The idea of this construction is similar to that above. However, under new assumptions tangent graph-approximations are not available. Therefore, we consider (3.1) with $f$ being an ‘approximation’ of a different type (comp. [24,25]). Namely, it is possible to find a continuous field $q(u) \in E^*$, $u \in U$, such that if $Au \notin F(u)$, then $q(u)$ separates $Au$ from $F(u)$, i.e. $Au$ and $F(u)$ lie in different half-spaces determined by the hyperplane induced by $q(u)$, and $f$ is a continuous map such that $f(u)$ lies in the same half-space as $F(u)$ does. As above we proceed in several steps.

**Step 1:** Take a bounded open (in $K$) set $W$ such that

$$C := \text{Coin}(A,F;U) = \{ u \in U \cap D(A) \mid Au \in F(u) \} \subset W \subset \overline{W} \subset U$$

(3.11)

and $F$ is bounded on $\overline{W}$, i.e. $\sup_{v \in F(u), u \in \overline{W}} \|v\| < \infty$.

**Step 2:** We need a technical lemma.

**Lemma 3.9.** There are bounded continuous maps $q: \overline{W} \to E^*$, $w = w_q: \overline{W} \to \mathbb{R}$ and $\varepsilon_0 > 0$ such that

$$w(u) = \langle Au, q(u) \rangle \text{ for } u \in \overline{W} \cap D(A), \ q(u) = 0 \text{ for } u \in C \text{ and}$$

$$\inf_{v \in F(u)} \langle v, q(u) \rangle > w(u) \text{ for } u \in \overline{W} \setminus C,$$

$$\inf_{v \in F(u)} \langle v, q(u) \rangle > w(u) + \varepsilon_0 \text{ for } u \in \partial W.$$  

(3.12)

For any $0 < \varepsilon < \varepsilon_0$ and $\gamma > 0$, there is a continuous bounded map $f: \overline{W} \to E$ such that for $u \in \partial W$

$$d_K^w(u,f(u)) < \gamma \varepsilon \text{ and } \langle f(u), q(u) \rangle > w(u) + \varepsilon.$$  

(3.13)

The form $q(u)$ separates $Au$ and $F(u)$ if $u \notin C$ (see second and third condition in (3.12)). Condition (3.13) means that $f$ is an ‘approximation’ (in the above-mentioned sense) and is ‘almost’ tangent to $K$.

**Proof.** Define $G(u) := f_0(u + \lambda_0 F(u)) - u, \ u \in \overline{W}$, where $f_0 := f_{\lambda_0}$ with $\lambda_0 > 0$ such that $\lambda_0(\omega + 1) \leq 1$. Since $f_0$ is linear, it is weak-to-weak continuous. Therefore, $G$ is weakly usc and has convex weakly compact values. Moreover, $\{u \in \overline{W} \mid 0 \in G(u)\} = C := \text{Coin}(A,F;U)$. Hence for any $u \in \overline{W} \setminus C, 0 \notin G(u)$, i.e. $\inf_{v \in G(u)} \|v\| > \beta_u > 0$. As it is not difficult to see, there is $\varepsilon_0 > 0$ such that $\inf_{v \in G(u)} \|v\| > \varepsilon_0$ for $u \in \partial W$ (in the proof, similar to that provided in Step 2 of the previous
construction, the ‘strong × weak’ closedness of \( \text{Gr}(A) \) plays a role). Given \( u \in \overline{W} \setminus C \), we have

\[
\inf_{u \in \overline{W}} \|u\| = \inf_{v \in \overline{W}} \sup_{q \in E^*, \|q\| \leq 1} \langle v, q \rangle = \sup_{q \in E^*, \|q\| \leq 1} \inf_{u \in \overline{W}} \langle v, q \rangle
\]

in view of the Sion version of the von Neumann min–max equality. Hence for any \( u \in \overline{W} \), there is \( q_u \in E^* \), \( \|q_u\| \leq 1 \), such that \( \inf_{v \in \overline{W}} \langle v, q_u \rangle > \beta_u \) and, for \( u \in \partial W \), \( \inf_{v \in \overline{W}} \langle v, q_u \rangle > \varepsilon_0 \). For \( u \in W \), let \( V(u) := \{ y \in \overline{W} \setminus C \mid \inf_{v \in \overline{W}} \langle v, q_u \rangle \geq \beta_u \} \) and for \( u \in \partial W \) let \( V(u) = \{ y \in \overline{W} \setminus C \mid \inf_{v \in \overline{W}} \langle v, q_u \rangle > \varepsilon_0 \} \). Evidently, \( \{V(u)\}_{u \in \overline{W}} \) is an open covering of \( \overline{W} \) since \( G \) is uhc (see p. 4). Let \( \{\lambda_s\}_{s \in S} \) be a partition of unity subordinated to this cover, i.e. for any \( s \in S \), there is \( u_s \in \overline{W} \setminus C \) such that \( \supp \lambda_s \subset V(u_s) \). Additionally take a continuous \( \mu : \overline{W} \to [0, 1] \) such that \( C = \mu^{-1}(0) \), \( \partial W = \mu^{-1}(1) \) and define \( q_0 : U \to E^* \) by

\[
q_0(u) := \begin{cases} 
\mu(u) \sum_{s \in S} \lambda_s(u) q_{u_s} & \text{for } u \in \overline{W} \setminus C, \\
0 & \text{for } u \in C.
\end{cases}
\]

It is easy to see that \( q_0 \) is continuous and so is \( q : U \to E^* \) given by \( q := \lambda_0 f_0^* \circ q_0 ; \) evidently \( \|q(u)\| \leq \lambda_0 \|f_0\| \|q_0(u)\| \leq 1 \) for \( u \in \overline{W} \). Let \( w : U \to \mathbb{R} \) be given by \( w(u) := \langle u - f_0(u), q_0(u) \rangle \), \( u \in U \). Then \( w \) is continuous,

\[
w(u) = \langle f_0((u + \lambda_0 Au) - u), q_0(u) \rangle = \langle Au, q(u) \rangle \text{ for } u \in D(A) \cap \overline{W},
\]

for \( u \in \overline{W} \setminus C \)

\[
\inf_{v \in F(u)} \{ \langle v, q(u) \rangle - w(u) \} \geq \mu(u) \sum_{s \in S} \lambda_s(u) \inf_{v \in F(u)} \{ f_0(u + \lambda_0 v) - u, q_{u_s} \} > \mu(u) \sum_{s \in S} \lambda_s(u) \beta_{u_s} > 0
\]

and \( \inf_{v \in F(u)} \langle v, q(u) \rangle - w(u) > \varepsilon_0 \) for \( u \in \partial W \).

Take any \( \gamma > 0 \) and \( 0 < \varepsilon < \varepsilon_0 \). For \( u \in \partial W \) take \( v_u \in F(u) \cap T_K(u) \). Then \( \langle v_u, q(u) \rangle > w(u) + \varepsilon_0 > w(u) + \varepsilon \). Let \( T(u) := \{ y \in \partial W \mid d_K^\gamma(y, v_u) < \gamma \varepsilon, \langle v_u, q(y) \rangle > w(y) + \varepsilon \} \). Then \( \{T(u)\}_{u \in \partial W} \) is an open cover of \( \partial W \). Le \( \{\lambda_s\}_{s \in S} \) be a partition of unity subordinated to this cover, i.e. for \( s \in S \) there is \( u_s \in \partial W \) such that \( \supp \lambda_s \subset T(u_s) \). Define \( \bar{f} : \partial W \to E \)

\[
\bar{f}(u) = \sum_{s \in S} \lambda_s(u) v_{u_s}, \ u \in \partial W.
\]

Then \( \bar{f} \) is continuous, bounded and for any \( u \in \partial W \)

\[
\langle \bar{f}(u), q(u) \rangle = \sum_{s \in S} \lambda_s(u) \langle v_{u_s}, q(u) \rangle > w(u) + \varepsilon, \ d_K^\gamma(u, \bar{f}(u)) < \gamma \varepsilon.
\]

Finally, let \( f : \overline{W} \to E \) be an arbitrary continuous bounded extension of \( \bar{f} \).

Step 3: Take \( \gamma = \frac{1}{L} \), where \( L \) is the constant from the definition of an \( L \)-retract, \( 0 < \varepsilon < \varepsilon_0 \) and \( f \) satisfying condition (3.13). For any \( \lambda > 0 \) with \( \lambda \omega < 1 \) define \( g_\lambda : \overline{W} \to K \) by

\[
g_\lambda(u) := f_\lambda \circ r \circ f(u), \ u \in \overline{W},
\]

where \( r : B(K, \eta) \to K \) is an \( L \)-retraction (with the constant \( L \)). It is clear that without loss of generality we may assume that this definition is correct for \( 0 < \lambda < \lambda_0 \) (see assumption 3.1 (1)). We claim that there is \( \lambda > 0 \), \( \lambda \omega < 1 \) such that for \( 0 < \lambda < \lambda_0 \)

\[
\text{Fix} g_\lambda = \{ u \in \overline{W} \mid u = g_\lambda(u) \} \subset W.
\]

Suppose the contrary. As in Step 3 of the previous construction, we get sequences \( \lambda_n \to 0, 0 < \lambda_n < \lambda_0, \ u_n \in \partial W \cap D(A) \) such that (after passing to a subsequence) \( u_n \to u_0 \in \partial W \) and

\[
\|A u_n - f(u_n)\| \leq L d_K(u_n + \lambda_n f(u_n)) \leq L d_K(u_n + \lambda_n f(u_0)) + L \|f(u_n) - f(u_0)\|.
\]
On the other hand,
\[ \varepsilon_0 < \langle f(u_n), q(u_n) \rangle - w(u_n) = \langle f(u_n), q(u_n) \rangle - \langle Au_n, q(u_n) \rangle \]
\[ \leq \sup_{u \in \mathcal{W}} \|q(u)\| \|f(u_n) - Au_n\| \leq \|f(u_n) - Au_n\|. \]
Passing with \( n \to \infty \), we get
\[ \varepsilon \leq \limsup_{n \to \infty} \|Au_n - f(u_n)\| \leq Ld_K^2(u_0, f(u_0)) < \varepsilon. \]
A contradiction concludes the proof. \[\blacksquare\]

**Step 4:** We are in a position to define \( \text{ind}_K(g_\lambda, W) \) if \( \lambda \) and \( g_\lambda \) are as above and then let
\[ d_K(A, F; U) = \lim_{\lambda \to 0} \text{ind}_K(g_\lambda, W). \quad (3.14) \]

It is relatively easy to show, by the use of arguments similar to those used in the previous construction, that this definition is correct since it does not depend on the choice of \( W \) satisfying condition (3.11), a separating map \( q \) satisfying condition (3.12), an ‘approximation’ \( f \) satisfying condition (3.13) and an \( L \)-retraction \( r \).

**Theorem 3.10.** The function \( \text{deg}_K \) defined by (3.14) has the properties enlisted in theorem 3.10. We leave the detailed formulation to a reader.

**Proof.** One follows (at least from the conceptual viewpoint) the arguments of the proof of theorem 3.5. For instance, in order to get the existence, we assume to the contrary that \( C = \emptyset \) and construct a separating map \( q \) and an ‘approximation’ \( f \) such that conditions (3.12), (3.13) hold on the whole \( \mathcal{W} \). In this case, as it is easy to see, \( \text{ind}_K(g_\lambda, W) = 0 \) for all sufficiently small \( \lambda \). \[\blacksquare\]

**Remark 3.11.** Let us now observe that if \( A \) is a resolvent compact linear quasi-\( m \)-accretive operator and \( F \) is weakly tangent and \( H \)-usc, then two degree theories are available. The degree \( \text{deg}_K(A, F; U) \) may be defined via formulae (3.9) or (3.14). It can be easily seen by observing that if \( f \) is a tangent \( \eta \)-(graph)-approximation of \( F \), then the condition (3.13) is satisfied provided that \( \eta > 0 \) is sufficiently small.

In particular, if \( A \) is \( m \)-accretive (resp. linear and \( m \)-accretive) and \( F \) is a single-valued map, then the degree defined via (3.9) (resp. (3.14)) coincides with the degree considered in [9]. Hence, apart from from results being direct consequences of propositions 3.6, 3.8 and corollary 3.7, we get the following.

**Proposition 3.12 (see [9, Proposition 4.2]).** Assume that \( K \) is a closed convex cone, \( \lambda_1 > 0 \) is the smallest real eigenvalue of \( A \) to which there corresponds an eigenvector \( u_1 \in K \setminus \{0\} \) such that \( (A - \lambda I)^{-1}u_1 \cap K = \emptyset \) and \( \ker(A - \lambda I) \cap K = \{0\} \) for all \( \lambda > \lambda_1 \). Then
\[ \text{deg}_K(A, \lambda I, K) = \begin{cases} 1 & \text{if } \lambda < \lambda_1 \\ 0 & \text{if } \lambda > \lambda_1. \end{cases} \quad (3.15) \]

**4. Nonlinear reaction–diffusion equation**

Now we shall apply the introduced degree to discuss the existence of solutions to problem (1.2), i.e. the \( M \)-dimensional system
\[ \begin{align*}
-\Delta(\rho \circ u)(x) &\in \varphi(u(x)), & u = (u_1, \ldots, u_M) &\in \mathbb{R}^M, & x &\in \Omega, \\
\frac{\partial u_i}{\partial \nu} &\geq 0, & x &\in \Omega, & i &\in \{1, \ldots, M\}, \\
u |_{\partial \Omega} &= 0, & x &\in \partial \Omega.
\end{align*} \quad (4.1) \]
\( \Omega \subset \mathbb{R}^N \) is open bounded with smooth boundary \( \partial \Omega \) and \( \Delta \) denotes the vectorial Laplace operator (1).

**Assumption 4.1.** Let us make the following standing assumptions.

1. \( \varphi: \mathbb{R}^M_+ \to \mathbb{R}^M \) is usc with compact convex values and sublinear linear growth, i.e. for some \( c > 0 \)
   \[ \sup_{v \in \varphi(u)} \|v\| \leq c(1 + |u|), \quad u \in \mathbb{R}^M_+; \quad (4.2) \]

2. For any \( u \in \mathbb{R}^M \) there is \( v \in \varphi(u) \) such that \( u_i \geq 0 \) if \( u_i = 0 \);
3. \( \rho = (\rho_1, \ldots, \rho_M): \mathbb{R}^M_+ \to \mathbb{R}^M_+ \) is a homeomorphism and \( |\rho(u)| \geq \alpha |u| \) for some \( \alpha > 0 \);
4. For any \( I \subset \{1, \ldots, M\} \) the face \( \{ y \in \mathbb{R}^M \mid y_i = 0 \text{ for } i \in I \} \) of \( \mathbb{R}^M_+ \) is invariant with respect to \( \rho \).

We say that a function \( u: \Omega \to \mathbb{R}^M_+ \) is a (weak) solution to (4.1) if \( \rho \circ u \in H^1_0(\Omega, \mathbb{R}^M) \) and there is a function \( v \in L^2(\Omega, \mathbb{R}^M) \) such that \( v(x) \in \varphi(u(x)) \) a.e. on \( \Omega \) and

\[ \int_{\Omega} \langle v(x), \phi(x) \rangle_{\mathbb{R}^M} \, dx = \int_{\Omega} \sum_{i=1}^{M} \langle \nabla \rho_i(u(x)), \nabla \phi_i(x) \rangle_{\mathbb{R}^N} \, dx \text{ for any } \phi \in H^1_0(\Omega, \mathbb{R}^M). \]

It is easy to see that if \( u \) is a solution, then \( u \in L^2(\Omega, \mathbb{R}^M) \) and \( u|_{\partial \Omega} = 0 \) in the trace sense, i.e. there is a sequence \((u_n)\) in \( C^\infty_0(\Omega, \mathbb{R}^M) \) such that \( \|u_n - u\|_{L^2(\Omega, \mathbb{R}^M)} \to 0 \).

**Remark 4.2.** In the case of a single-valued \( \varphi \), i.e. if \( \varphi = (\varphi_1, \ldots, \varphi_M) \) with \( \varphi_i: \mathbb{R}^M \to \mathbb{R} \), problem (4.1) is a reaction–diffusion system describing equilibria of the distribution of \( M \) substances subject to chemical reactions and diffusion; for \( 1 \leq j \leq M \), \( \varphi_j(u) \) corresponds to the reaction/ perturbation term acting on the \( j \)-th substance, \( u_j(x) \) is the concentration of this substance. The above problem corresponds to the situation of the problem under control, with uncertainties or discontinuities. The constraint \( u_j(x) \geq 0, \ x \in \Omega, \ 1 \leq j \leq M \) is physically justified and means that the concentration \( u_j \) is non-negative. It is worth pointing out here, that contrary to many other results, we avoid the assumption of ‘non-negativity’ of \( \varphi \), implying that all substances are only produced. This is unrealistic since during the process some reactants vanish or are transformed. Instead, we assume 4.1 (2). It actually means that if some reactant vanishes in some area, its amount (in this area) cannot decrease. The reaction–diffusion problem was studied by many authors, e.g. [26] and references therein, but the case under constraints is still not very well recognized. In [9], (see also [27]) problem (4.1) was considered in a one-dimensional case, i.e. if \( M = 1 \), and with a single-valued \( \varphi \). The situation \( M > 1 \) with a single-valued perturbation was treated in [28].

Let \( E := L^2(\Omega, \mathbb{R}^m) \) and let

\[ K := \{ u \in E \mid u_i(x) \geq 0 \text{ for a.a. } x \in \Omega \}. \]

By \( A: E \ni D(A) \to E \) we denote the self-adjoint Dirichlet \( L^2 \)-realization of the classical vectorial Laplacian \( -\Delta \) (cf. e.g. [29, Theorem 4.27]), i.e. it arises from the Lions–Lax–Milgram construction (cf. [29, Theorem 12.18]) and let \( F: K \to E \) be the Nemytskii operator \( N_{\varphi \circ \gamma} \) determined by \( \varphi \circ \gamma \), i.e.

\[ F(u) = N_{\varphi \circ \gamma}(u) := \{ v \in E \mid v(x) \in \varphi \circ \gamma(u(x)) \text{ for a.a. } x \in \Omega \}, \]

where \( \gamma := \rho^{-1}: \mathbb{R}^M_+ \to \mathbb{R}^M \). It is clear that \( u \in E \) solves (4.1) if and only if \( w := \rho(u) \in E \) is a solution of the problem
\[ Aw = F(w), \ w \in K. \quad (4.3) \]

**Proposition 4.3.** The following conditions are satisfied:

\[ \text{If a function } v = (v_1, \ldots, v_M): \Omega \to \mathbb{R}^M \text{ is twice differentiable (or } v \in H^2(\Omega, \mathbb{R}^M)) \text{, then } \Delta v = (\Delta v_1, \ldots, \Delta v_M) \text{ and } \Delta v_i = \sum_{j=1}^{N_A} a_{ij}^2 v_i, i = 1, \ldots, M. \]
(a) $A$ is a resolvent compact linear densely defined $m$-accretive operator in $E$, $D(A) = H^1_0(\Omega, \mathbb{R}^M) \cap H^2(\Omega, \mathbb{R}^M)$;
(b) $K$ is resolvent invariant, i.e. $P^A_\lambda(K) \subset K$ for any $\lambda > 0$;
(c) $F$ is $H$-use with weakly compact convex values and weakly tangent to $K$.

Proof. It is well known that $-A$ generates a $C_0$ semigroup of contractions, hence $A$ is $m$-accretive. The compactness of resolvents follows immediately from the compactness of the Sobolev embedding $H^1_0(\Omega, \mathbb{R}^M) \subset E$. The invariance follows from [3, Proposition 4.3].

It is evident that values of $F$ are convex and closed and bounded, hence, weakly compact. The $H$-upper semicontinuity of $F$ follows from assumptions 4.1 (1), (3) and [3, Lemma 4.2].

It is clear that if $u \in \mathbb{R}^M$, then $T_{R^M}(u) = \{v \in \mathbb{R}^M | v_j \geq 0 \text{ if } u_j = 0, j = 1, \ldots, M\}$. Conditions (2) and (4) from assumption 4.1 imply that $\varphi \circ \gamma(u) \cap T_{R^M}(u)$ for any $u \in \mathbb{R}^M$. Let now $u \in K$. The map $T_{R^M} : \mathbb{R}^M \to \mathbb{R}^M$ is lower semicontinuous (cf. [12, Th. 4.2.2]), $\varphi \circ \gamma$ is measurable; hence $\Omega \ni x \to \varphi \circ \gamma(u(x)) \cap T_{R^M}(u(x)) \subset \mathbb{R}^M$ is measurable with non-empty values. By the Kuratowski–Ryll-Nardzewski theorem, there is a measurable $v : \Omega \to \mathbb{R}^M$ such that $v(x) \in \varphi \circ \gamma(u(x)) \cap T_{R^M}(u(x))$ for a.e. $x \in \Omega$. Clearly, $v \in E$ and $v \in T_K(u) \cap F(u)$ since in view of [12, Cor. 8.5.2] $T_K(u) = \{v \in E | v(x) \in T_{R^M}(u(x)) \text{ a.e. in } \Omega\}$. 

Example 4.4. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}, \varphi(u) = [u - 1, u + 1], u \in \mathbb{R}^+$. For any $n \in \mathbb{N}, s_n \in N_\varphi(0)$ (the Nemitskii operator associated with $\varphi$), where $s_n(x) = \sin(nx), x \in \Omega = (-\pi/2, \pi/2)$, but $(s_n)$ has no $L^2$-convergent subsequence. Moreover, $C = \{(1 + 1/n)s_n | n \in \mathbb{N}\}$ is closed in $L^2(\Omega)$ but $N_\varphi^{-1}(C)$ is not. This shows that $N_\varphi$ is neither usc nor has compact values.

(a) Existence
In what follows we are going to find conditions for the non-triviality of the difference $\deg_K(A, F; U_0) - \deg(A, F, U_\infty)$, where $U_0$ (resp. $U_\infty$) is a ‘small’ (resp. ‘large’) ball around 0 and, thus, establish the existence of non-trivial solutions to problem (4.3).

(i) Linear perturbations
Let $G : E \to E$ be the Nemyskii operator determined by a matrix $D = [d_{ij}]_{i,j=1}^M$, i.e. $G(u)(x) = Du(x)$ for $u \in E$ and $x \in \Omega$. From [30, Example 3.1], it follows that $Du \in T_{R^M}(u)$ for $u \in \mathbb{R}^M$ if and only if $D$ is quasi-non-negative, i.e. the off-diagonal entries $d_{ij} \geq 0, 1 \leq i \neq j \leq M$.

We, therefore, assume that $D$ is quasi-non-negative. Hence, $G(u) \in T_K(u)$ for any $u \in K$. Let $\sigma_+(D) := \{\lambda \in \mathbb{R} | \text{there exists } 0 \neq u \in \mathbb{R}^M \text{ such that } Du = \lambda u \subset \sigma(D)\}$. In view of [30, p. 2241], it immediately follows that $\sigma_+(D)$ is non-empty and $s(D) := \max \text{Re}(\sigma(D)) = \max \sigma_+(D)$.

Let $\lambda_1$ be the first eigenvalue of the (scalar) Laplace operator $-\Delta$ on $\Omega$ with Dirichlet boundary condition and $\phi \geq 0$ be an eigenfunction related to $\lambda_1$. Then $\phi > 0$ on $\Omega$ and $\lambda_1$ is simple, i.e. $\phi$ is unique up to a positive factor (cf. e.g. [31] for details).

Proposition 4.5. If $\lambda_1 \notin \sigma_+(D)$, then $\ker(A - G) \cap K = \{0\}$. If, additionally, $s(D) > \lambda_1$, then there is $\bar{u} \in K$ such that $(A - G)^{-1}(\bar{u}) \cap K = \{0\}$.

Proof. Step 1: Let $E_1 := \{u \in E | Au = \lambda_1 u\}$ be the eigenspace corresponding to $\lambda_1$ and let $E_1^\perp$ be the orthogonal complement of $E_1$. Clearly, both $E_1$ and $E_1^\perp$ are invariant with respect to $A$; a direct computation shows that they are also invariant with respect to $G$. Let us collect some other properties:

1. $E_1^\perp \cap K = \{0\}$. For if $u \in E_1^\perp \cap K$ then for every $j = 1, \ldots, M$ we have $\int_\Omega u_j \phi \, dx = \langle u, e_j \rangle_E = 0$, where $e_j \in \mathbb{R}^M$ is the $j$-th vector from the standard basis of $\mathbb{R}^M$. Since the integrand is
non-negative, thus it is zero almost everywhere. Therefore, \( u_j = 0 \), as \( \phi \) is strictly positive on \( \Omega \).

(2) If \( u = u^1 + u^\perp \in K \), where \( u^1 \in E_1 \), \( u^\perp \in E_\perp \), then \( u^1 \in K \). To see this, we show that for \( i = 1, \ldots, M \), \( 0 \leq \langle u, \phi_i \rangle_E = \langle u^1, \phi_i \rangle = \int_\Omega u^1 \phi_i \, dx \). Now, since \( u^1_j = \alpha_j \phi \) for some \( \alpha_j \in \mathbb{R} \), we get \( \alpha_j \geq 0 \), which yields \( u^1_j \geq 0 \), and consequently \( u^1 \in K \).

(3) If \( \lambda_1 \notin \sigma_+(D) \), then \( \ker(A - G) \cap K \cap E_1 = \{0\} \). Indeed : let \( u \in \ker(A - G) \cap K \cap E_1 \). Therefore, \( G(u) = Au = \lambda_1 u \). If \( u \neq 0 \) then there is \( x \in \Omega \) such that \( u(x) \) is a non-negative non-zero vector in \( \mathbb{R}^M \) satisfying the equality \( Du(x) = \lambda_1 u(x) \) and therefore \( \lambda_1 \in \sigma_+(D) \). The contradiction shows our assertion.

Step 2: If \( C \) is an \( M \times M \) square matrix, \( \ker C \cap \mathbb{R}^M_+ = \{0\} \), \( v, w \in \mathbb{R}^M_+ \), \( Cv = \mu v \) with \( \mu < 0 \) and \( Cw = v \), then \( C(\mu w) = \mu v \). Hence \( v - \mu w \in \ker C \). Consequently \( v = \mu w \). This implies that \( v = w = 0 \).

Step 3: Let \( u = u^1 + u^\perp \in \ker(A - G) \cap K \), where \( u^1 \in E_1 \), \( u^\perp \in E_\perp \). The invariance of \( E_1 \) and \( E_\perp \) under \( A \) and \( G \) yields \( u^1 \in \ker(A - G) \). From (2) above it follows that \( u^1 \in K \), therefore, by (3), \( u^1 = 0 \). Finally, by (1) in Step 1, \( u^\perp \in E_\perp \cap K = \{0\} \). This proves the first part of proposition 4.5.

Step 4: Let \( \mu := \sigma(D) > \lambda_1 \), let \( 0 \neq v \in \mathbb{R}^M_+ \) be an eigenvector of \( D \) corresponding to \( \mu \) and \( \tilde{u} = \phi v \). Then \( \tilde{u} \in K \). Suppose \( (A - G)u = \tilde{u} \) for some \( u = u^1 + u^\perp \in K \). The invariance of \( E_1 \) and \( E_\perp \) under \( A \) and \( G \) shows that \( (A - G)u^1 = \tilde{u} \), where \( u^1 \in K \) as in (2) above. Since \( \lambda_1 \) is a simple eigenvalue, there is \( w \in \mathbb{R}^M_+ \) such that \( u^1 = \phi w \). Moreover, \( Au^1 = \lambda_1 \phi w \) and \( G(u^1) = \phi Dw \). Since \( \phi > 0 \) on \( \Omega \), this implies that \( \lambda_1 I - D w = v \). Applying Step 2 to \( C := \lambda_1 I - D \) and \( \lambda := \lambda_1 - \mu \) we see that \( w = v = 0 \). This contradicts that \( v \neq 0 \).

Theorem 4.6. Let \( D, G \) be as above and let \( \lambda_1 \notin \sigma_+(D) \). Then \( Au = G(u) \), \( u \in K \) if and only if \( u = 0 \) and for any \( r > 0 \)

\[
\deg_K(A, G, B_K(0, r)) = \begin{cases} 1 & \text{if } s(D) < \lambda_1 \\ 0 & \text{if } s(D) > \lambda_1 \end{cases}
\]

where \( B_K(0, r) := B(0, r) \cap K \).

Proof. The first part follows directly from proposition 4.5. Let \( s(D) < \lambda_1 \). Then \( C = \{u \in K \cap D(A) \mid Au = tG(u) \text{ for some } t \in [0, 1] \} = \{0\} \); since if \( u \in C \) and \( u \neq 0 \), then \( \lambda_1 \|u\|^2 \leq \langle Au, u \rangle_E = t \|(G(u), u) \| \leq ts(D) \|u\|^2 \), which is impossible. Therefore, we have \( 1 = \deg_K(A, G; K) = \deg_K(A, G, B_K(0, r)) \) in view of proposition 3.6 (b).

Let now \( s(D) > \lambda_1 \). From proposition 4.5, it follows that there exists \( \tilde{u} \in K \) such that the \( \{u \in K \cap D(A) \mid Au = G(u) + t\tilde{u}, t \in [0, 1] \} = \{0\} \). By the homotopy invariance \( \deg_K(A, G; B_K(0, r)) = \deg_K(A, G + \tilde{u}, B_K(0, r)) \). But there are no \( u \in K \cap D(A) \) with \( Au = G(u) + \tilde{u} \). Hence \( \deg_K(A, G + \tilde{u}, B_K(0, r)) = 0 \).

(ii) Linearization

Now we generalize results from the previous section to the case of a nonlinear perturbation \( \varphi \).

Instead, we require that \( \varphi \) admits linearization of sorts.

Assumption 4.7. In addition to assumption 4.1, we suppose that

(\( F_0 \)) there is an \( M \times M \) matrix \( D_0 \) such that \( \varphi(u) = D_0 u + \varphi_0(u) \) for \( u \in \mathbb{R}^M_+ \), where \( \varphi_0(0) = 0 \) and

\[
\lim_{u \to 0} \sup_{\|v\| \leq \varphi_0(u)} \frac{\|v\|}{\|u\|} = 0. \tag{4.5}
\]

(\( F_\infty \)) there is an \( M \times M \) matrix \( D_\infty \) such that \( \varphi(u) = D_\infty u + \varphi_\infty(u) \) for \( u \in \mathbb{R}^M_+ \), where

\[
\lim_{\|u\| \to \infty} \sup_{\|v\| \leq \varphi_\infty(u)} \frac{\|v\|}{\|u\|} = 0.
\]
(R) there are $M \times M$ matrices $R_0$ and $R_\infty$ such that $\rho(u) = R_0 u + \rho_0(u)$, $\rho(u) = R_\infty u + \rho_\infty(u)$ for $u \in \mathbb{R}^M_+$, where

$$\lim_{u \to 0} \frac{\rho_0(u)}{|u|} = 0, \quad \lim_{|u| \to \infty} \frac{\rho_\infty(u)}{|u|} = 0.$$ 

**Remark 4.8.** (1) Observe that if $\phi$ is single-valued then assumption $(F_0)$ (resp. $(F_\infty)$) means that $\phi$ is differentiable at 0 (resp. at infinity) with respect to the cone $\mathbb{R}^M_+$ and and $\phi'(0) = D_0$.

(2) Condition $(4.2)$ together with $(F_0)$ imply that there is $M > 0$ such that

$$\sup_{v \in \phi(u)} |v| \leq M|u|. \tag{4.6}$$

(3) For all $u \in \mathbb{R}^n_+$, we have $D_0 u, D_\infty u \in T_{\mathbb{R}^n_+}(u)$. Indeed, let $u \in \mathbb{R}^n_+$. Then $\phi(tu) = tD_0 u + \phi_0(tu)$ for all $t > 0$. Since $t^{-1} \phi(tu) \cap T_{\mathbb{R}^n_+}(u) \neq \emptyset$, passing with $t \to 0$ we obtain that $D_0 u \in C_{\mathbb{R}^n_+}(u)$. The tangency of $D_\infty$ can be proved in a similar manner.

(4) The growth condition $|\rho(u)| \geq \alpha |u|$ shows that matrices $R_0, R_\infty$ are invertible and:

$$\lim_{u \to 0} \frac{\gamma(u) - R_0^{-1} u}{|u|} = 0, \quad \lim_{|u| \to \infty} \frac{\gamma(u) - R_\infty^{-1} u}{|u|} = 0. \tag{4.7}$$

Let $\Phi: K \to E$ be the Nemytskii operator induced by $\phi$, i.e. $\Phi(u) = \{v \in E | v(x) = \phi(u(x)) \}$ for a.a. $x \in \Omega$ for $u \in K$. Assumption 4.7 implies that $\Phi$ admits linearizations in the sense of Hadamard (5). Namely, it is easy to see that

**Proposition 4.9.** For any sequences $(s_n)$, $s_n \to 0^+$ (resp. $s_n \to \infty$), $(u_n)$ in $K$, $u_n \to u \in K$ and $v_n \in \Phi(s_n u_n)$, one has that $s_n^{-1} v_n \to G_0(u)$ (resp. $s_n^{-1} v_n \to G_\infty(u)$), where $G_0$ (resp. $G_\infty$) is the Nemytskii of $D_0$ (resp. $D_\infty$). Consequently maps $H_0, H_\infty: K \times [0, 1] \to E$, given for $u \in K$ by

$$H_0(u, t) := \begin{cases} t^{-1} \Phi(tu), & t \in (0, 1] \\ \{G_0(u)\}, & t = 0, \end{cases} \quad H_\infty(u, t) := \begin{cases} t\Phi(t^{-1} u), & t \in (0, 1] \\ \{G_\infty(u)\}, & t = 0, \end{cases} \tag{4.8}$$

are $H$-usc with weakly compact convex values.

**Proof.** Suppose $s_n \searrow 0$, $u_n \to u$ in $K$ and $v_n \in \Phi(s_n u_n)$. We may assume that $u_n \to u$ a.e. and $|u_n| \leq \bar{u} \in L^2(\Omega)$ a.e. on $\Omega$. Let $z_n := s_n^{-1} v_n$. By (4.6), $|z_n| \leq M|u_n| \leq \bar{u}$ a.e. Take $x \in \Omega$ from the full measure set. If $u(x) \neq 0$, then for large $n \in \mathbb{N}$,

$$z_n(x) - D_0 u_n(x) \in \frac{\rho_0(s_n u_n(x))}{s_n u_n(x)}.$$ 

Thus $z_n(x) \to D_0 u(x)$. If $u(x) = 0$, then $|z_n(x)| \leq M|u_n(x)| \to 0$. Therefore, the assertion follows in view of the Lebesgue dominated convergence theorem. The proof of the second part is analogous. $

**Remark 4.10.** In view of assumption 4.7 (R) maps $\widehat{H}_0, \widehat{H}_\infty: K \times [0, 1] \to E$ given for $u \in K$ by

$$\widehat{H}_0(u, t) := \begin{cases} t^{-1} \Gamma(tu), & t \in (0, 1] \\ \{\Gamma_0(u)\}, & t = 0, \end{cases} \quad \widehat{H}_\infty(u, t) := \begin{cases} t\Gamma(t^{-1} u), & t \in (0, 1] \\ \{\Gamma_\infty(u)\}, & t = 0, \end{cases}$$

where $\Gamma$, $\Gamma_0$ and $\Gamma_\infty$ are Nemytskii operators induced by $\gamma$, $R_0^{-1}$ and $R_\infty^{-1}$, respectively, are continuous.

---

5Simple examples show that $\Phi$ admits neither the linearization at 0 nor at $\infty$, i.e. it is not necessarily true that if $u_n \to 0$ and $w_n \in \Phi(u_n)$, then $\|u_n\|^{-1} \|w_n - G_0(u_n)\| \to 0$, where $G_0$ is the Nemytskii operator induced by $D_0$ (see [32]).
Theorem 4.11. (i) Under assumption 4.1 and 4.7 suppose that \( \lambda_1 \notin \sigma_+(D_0 \circ R_0^{-1}) \). Then there is \( r > 0 \) such that 0 is the unique solution to \( Au \in F(u) \) in \( B_K(0, \delta) \) and

\[
\deg_K(A, F; B_K(0, r)) = \begin{cases} 
1 & \text{if } s(D_0 \circ R_0^{-1}) < \lambda_1 \\
0 & \text{if } s(D_0 \circ R_0^{-1}) > \lambda_1.
\end{cases}
\]

(ii) If \( \lambda_1 \notin \sigma_+(D_0 \circ R_0^{-1}) \), then there is \( R > 0 \) such that if \( u \in K \cap D(A) \) and \( A(u) \in F(u) \), then \( \|u\| < R \) and

\[
\deg_K(A, F; B_K(0, R)) = \begin{cases} 
1 & \text{if } s(D_0 \circ R_0^{-1}) < \lambda_1 \\
0 & \text{if } s(D_0 \circ R_0^{-1}) > \lambda_1.
\end{cases}
\]

Proof. (i) Define \( H: K \times [0, 1] \to E \) by \( H(u, t) := H_0 \circ \tilde{H}_0(u, t), u \in K, t \in [0, 1], \) where \( H_0 \) and \( \Gamma_0 \) are given by (4.8). It is clear that \( H \) is well defined, \( H \)-usc with convex weakly compact values. We shall show that there is \( r > 0 \) such that 0 is the unique solution of \( Au \in H(u, t) \) in \( B_K(0, r) \).

Suppose to the contrary that there is a sequence \( (u_n) \) in \( K \) such that \( u_n \neq 0 \) for all \( n \in \mathbb{N} \) and \( v_n := Au_n \in F(u_n) \). From (4.6), we see that the sequence \( \|u_n\|^{-1}v_n \) is bounded. Therefore, the sequence \( (w_n) \), where \( w_n := \int_1^1(\|u_n\|^{-1}u_n + v_n) = \|u_n\|^{-1}u_n \), has the convergent subsequence, without loss of generality we may assume that \( w_n \to w \).

On the other hand \( v_n \in F(\|u_n\|\|w_n\|) \). Proposition 4.9 yields that \( \|u_n\|^{-1}v_n \to G_0 \circ \Gamma_0(w) \) and, consequently, \( w = \int_1^1(w + G_0 \circ \Gamma_0(w)) \), i.e. \( Aw = G_0 \circ \Gamma_0(w) \). Thus \( w = 0 \) in view of the first part of proposition 4.5. This a contradiction with \( \|w\| = 1 \).

The homotopy invariance implies that \( \deg_K(A, F; B_K(0, r)) = \deg_K(A, G_0 \circ \Gamma_0, B_K(0, r)) \). To complete the proof it remains to apply theorem 4.6.

(ii) The proof is similar to the above one; one considers the homotopy \( H_\infty \circ \tilde{H}_\infty \) and shows that there is \( R > 0 \) such that all solutions to problem \( Au \in H_\infty \circ \tilde{H}_\infty(u, t) \), \( u \in K \), are contained in a large ball \( B(0, R) \).

Now we are prepared to establish the main theorem of this section.

Theorem 4.12. Under assumption 4.1 and 4.7 suppose that

\[
\lambda_1 \notin \sigma_+(D_0 R_0^{-1}) \cup \sigma_+(D_\infty R_\infty^{-1}) \quad \text{and} \quad (s(D_0 R_0^{-1}) - \lambda_1) \cdot (s(D_\infty R_\infty^{-1}) - \lambda_1) < 0,
\]

then there exists at least one non-trivial solution to problem (4.3) and (4.1).

Proof. We see that there are \( 0 < r < R \) such that \( \deg_K(A, F; B(0, r)) \neq \deg_K(A, F; B(0, R)) \). The additivity property of the degree implies that there is \( u_0 \in K \) such that \( Au_0 \in F(u_0) \) and \( r < \|u_0\| < R \).

Remark 4.13. Instead of condition (2) from assumption 4.1 assume that, for \( u \in \mathbb{R}_+^M \), if \( v \in \varphi(u) \) and \( u_i = 0 \), then \( v_i \geq 0, i = 1, \ldots, M \). Then \( \varphi \) is strongly tangent, i.e. \( \varphi(u) \cap T_{\mathbb{R}_+^M}(u) \) for any \( u \in \mathbb{R}_+^M \). Moreover, in this situation, conditions (F0) and (F+) from assumption 4.7 may be slightly relaxed. Namely instead of (F0) and (F+) suppose that there are two \( M \times M \) quasi-non-negative matrices \( D_0, D_\infty \) such that

\[
\lim_{u \to 0} \frac{\text{dist}(D_0 u, \varphi(u))}{|u|} = 0, \quad \lim_{|u| \to \infty} \frac{\text{dist}(D_\infty u, \varphi(u))}{|u|} = 0.
\]

Then the conclusion of theorem 4.12 remains true.

Indeed, in view of (4.9) there are a continuous function \( \alpha: [0, \infty) \to [0, \infty) \) and numbers \( \xi < \tilde{t} \) such that

\[
\lim_{t \to 0} \frac{\alpha(t)}{t} = \lim_{t \to \infty} \frac{\alpha(t)}{t} = 0, \quad \text{dist}(D_0 u, \varphi(u)) < \alpha(|u|) \text{ if } |u| < \xi \quad \text{and}
\]

\[
\text{dist}(D_\infty u, \varphi(u)) < \alpha(|u|) \text{ if } |u| > \tilde{t}.
\]
The map \( \psi : K \to \mathbb{R}^M \) given for \( u \in \mathbb{R}^M_+ \) by
\[
\psi(u) = \begin{cases} 
\varphi(u) \cap (D_0(u) + \alpha(|u|)B) & |u| < \frac{t}{\lambda} \\
\varphi(u) & \frac{t}{\lambda} \leq |u| \leq \frac{t}{\lambda} \\
\varphi(u) \cap (D_\infty(u) + \alpha(|u|)B) & |u| > \frac{t}{\lambda}
\end{cases}
\]
is upper semicontinuous with non-empty convex compact values and satisfies conditions \((F_0)\) and \((F_\infty)\) with \( \varphi \) replaced by \( \psi \). Therefore, problem \((4.1)\) with \( \varphi \) replaced by \( \psi \) has a non-trivial solution \( u_0 \in K \). Since for all \( u \in K \) we have \( \psi(u) \subset \varphi(u) \), \( u_0 \) is the solution to the original problem.

5. Reaction–diffusion equations with discontinuities

Results of the above section allow us to study the existence of solutions of reaction–diffusion systems with a discontinuous right-hand side.

Consider a possibly discontinuous function \( f : \mathbb{R}^M_+ \to \mathbb{R}^M \) and the following reaction–diffusion system
\[
\begin{aligned}
- \left( \Delta(\rho \circ u)(x) \right) &= f(u(x)) \\
u_i(x) &\geq 0, \ x \in \Omega, \ i = 1, \ldots, m \\
u|_{\partial \Omega} &= 0.
\end{aligned}
\tag{5.1}
\]

In order to apply the above approach, one needs the notion of a regularization allowing the removal of discontinuities. With \( f \) the so-called Krasovski regularization
\[
\rho_K(f)(x) := \bigcap_{\varepsilon > 0} \text{cl conv} \left( B^+(x, \varepsilon) \right), \ x \in \mathbb{R}^M_+,
\]
where \( B^+(x, \varepsilon) = \{ y \in \mathbb{R}^M_+ \mid |x - y| < \varepsilon \} \), and the Filippov regularization
\[
\rho_F(f)(x) := \bigcap_{\varepsilon > 0} \bigcap_{\mu(N) = 0} \text{cl conv} \left( B^+(x, \varepsilon) \setminus N \right),
\]
where \( \mu \) stand for the Lebesgue measure, are associated. Obviously, \( f(x) \), \( \rho_F(f)(x) \subset \rho_K(f)(x) \) and \( \rho_F(f)(x) = \rho_K(f)(x) = \{ f(x) \} \) if \( f \) is continuous in \( x \). It is important to show that \( \rho_K(f) \) and \( \rho_F(f) \) are use with compact convex values.

When solving \((5.1)\) the common practice is to consider solutions in terms of regularization. Namely, a non-negative function \( u \) is a solution of \((5.1)\) if it is the solution of the inclusion \((4.1)\), where \( \varphi \) is a regularization of \( f \) (comp. e.g. \([7,33–36]\)).

Let conditions \((4)\) from assumption \(4.1\), condition \((R)\) from assumption \(4.7\) be satisfied and
\[
\begin{align*}
\text{(a) } & \text{there is } c > 0 \text{ such that } |f(u)| \leq c(1 + |u|) \text{ for } u \in \mathbb{R}^M_+, \\
\text{(b) } & f(u) \in T_{\mathbb{R}^M_+}(u) \text{ for } u \in \mathbb{R}^M_+, \\
\text{(c) } & f(u) = D_0u + \beta_0(u) = D_\infty u + \beta_\infty(u), \text{ where } D_0, D_\infty \text{ are } M \times M \text{ quasi-non-negative matrices and} \\
& \beta_0(0) = 0, \lim_{u \to 0} \frac{|\beta_0(u)|}{|u|} = 0, \lim_{|u| \to \infty} \frac{|\beta_\infty(u)|}{|u|} = 0.
\end{align*}
\]

In view \((c)\) and the definition of regularizations we see that \( \varphi = \rho_K(f) \) or \( \varphi = \rho_F(f) \) satisfies the conditions \((F_0)\) and \((F_\infty)\) from assumption \(4.7\). In the case of the Krasovski regularization condition \((b)\) above implies the tangency of \( \varphi \) to \( \mathbb{R}^m_+ \), i.e. condition \((2)\) in assumption \(4.1\). In the case of Filippov regularization this implication may not be true. To overcome this difficulty one can assume additionally the continuity of \( f \) at points from the boundary of \( \mathbb{R}^m_+ \).

Assuming that the regularization \( \varphi \) is tangent to \( \mathbb{R}^m_+ \) we get the following result.

**Theorem 5.1.** Assume that \( \lambda_1 \notin \sigma_+(D_0R_0^{-1}) \cup \sigma_+(D_\infty R_\infty^{-1}) \). If \( (s(D_0R_0^{-1}) - \lambda_1) \cdot (s(D_\infty R_\infty^{-1}) - \lambda_1) < 0 \), then there exists at least one non-trivial solution of the problem \((5.1)\).
Above solutions to (5.1) were understood in the sense of the Krasowski or Filippov regularizations. The natural question is whether \( u \) such that \(-\Delta(u) \in \varphi(u)\) satisfies \(-\Delta(u) = f(u)\) almost everywhere. Such solutions will be called primitive. It is not difficult to find examples showing that in general solutions are not primitive. However, we have the following.

**Theorem 5.2.** If the set of discontinuities of \( f \) is at most countable and

\[
0 \not\in \varphi(u) \setminus \{f(u)\}
\]  

(5.2)

for all \( u \in \mathbb{R}^M_+ \), then any solution \( u \) of (5.1) in the sense of regularization \( \varphi = r_K(f) \) or \( r_F(f) \) is a primitive solution.

**Proof.** The assumption implies that \( \varphi(u) = \{f(u)\} \) fails to hold for at most countably many \( u \in \mathbb{R}^M_+ \). Let \( u : \Omega \rightarrow \mathbb{R}^M_+ \) be a solution. It is known (see [31]) that \( u \in D(\Delta) = H^2(\Omega, \mathbb{R}^M) \cap H^1_0(\Omega, \mathbb{R}^M) \).

Let \( u_0 \in \mathbb{R}^M_+ \) be a fixed point satisfying the relation \( \varphi(u_0) \neq \{f(u_0)\} \). Put \( \Omega_{u_0} := \{x \in \Omega \mid u(x) = u_0\} \).

The function \( u \) is constant on \( \Omega_{u_0} \), therefore ([37, Lemma 7.7]) \( \forall u \in H^1(\Omega, \mathbb{R}^m) \) gives that \(-\Delta u(x) = 0\) for a.a. \( x \in \Omega_{u_0} \). On the other hand, \( 0 = -\Delta u(x) = \varphi(u(x)) = \varphi(u_0) \) for a.a. \( x \in \Omega_{u_0} \). The assumption (5.2) implies that the set \( \Omega_{u_0} \) is of measure zero or \( f(u_0) = 0 \). In both cases, equation (5.1) is satisfied for almost all \( x \in \Omega_{u_0} \).

Summarizing, equation (5.1) is satisfied for almost all \( x \in \Omega \) for which \( \varphi(u(x)) \neq \{f(u(x))\} \). For the remaining points, the equation is trivially satisfied. \( \blacksquare \)

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