Optimal Charging of Electric Vehicles in Smart Grid: Characterization and Valley-Filling Algorithms

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Abstract

Electric vehicles (EVs) offer an attractive long-term solution to reduce the dependence on fossil fuel and greenhouse gas emission. However, a fleet of EVs with different EV battery charging rate constraints, that is distributed across a smart power grid network requires a coordinated charging schedule to minimize the power generation and EV charging costs. In this paper, we study a joint optimal power flow (OPF) and EV charging problem that augments the OPF problem with charging EVs over time. While the OPF problem is generally nonconvex and nonsmooth, it is shown recently that the OPF problem can be solved optimally for most practical power networks using its convex dual problem. Building on this zero duality gap result, we study a nested optimization approach to decompose the joint OPF and EV charging problem. We characterize the optimal offline EV charging schedule to be a valley-filling profile, which allows us to develop an optimal offline algorithm with computational complexity that is significantly lower than centralized interior point solvers. Furthermore, we propose a decentralized online algorithm that dynamically tracks the valley-filling profile. Our algorithms are evaluated on the IEEE 14 bus system, and the simulations show that the online algorithm performs almost near optimality (< 1% relative difference from the offline optimal solution) under different settings.

Index Terms

Optimal power flow, electric vehicle charging, valley-filling, online algorithm, convex optimization.

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I. INTRODUCTION

Electric vehicles (EVs) are getting more popular as a long-term vehicular technology to reduce the dependence on fossil fuel and the emission of greenhouse gases. However, as EVs can consume a relatively high amount of power during charging and recharging the batteries, a large fleet of EVs charging at the same time may strain the power grid. In fact, with an increase in EV penetration, uncoordinated charging can lead to additional power losses and unacceptable voltage variation that overload the power grid [1]. One way to tackle this problem is to increase the power delivery capacity to deal with new EV peak demands, but this will lead to significant infrastructure cost. Another way is to adopt a “smart grid” solution, which allows EVs to communicate with the utility that coordinates their charging activities. As wireless infrastructure is becoming ubiquitous and more readily available, this solution can be economically viable. Furthermore, a coordinated EV charging can improve frequency regulation [2], smooth out the generation intermittency from renewable sources, and increase the efficiency in electricity usage [3], [4].

A smart grid is a next-generation power grid that has an overlay of smart meters and communication network to enable real-time data exchange between consumers and power providers. As the smart grid is purported to achieve better reliability, and greater efficiency in power production and consumption [5], upgrading conventional grids to smart grids has captured wide attention in the past few years. The design of a smart grid thus needs new fundamental theories as well as new technologies in networking [6]. In particular, some residential loads, like washing machines and EVs, might be equipped with smart devices that can communicate with the utility company over a network, and dynamically adjust their power consumption using feedback control. In this setting, we consider two types of load that are connected to the power network:

- **Price-inelastic load**: The exact power requested by this type of load must be provided. This corresponds to standard loads in a conventional grid such as lighting and heating.
- **Price-elastic load**: The power delivered to this type of load can vary depending on the current cost and a deadline, at which the total energy demand must be satisfied. An example is the charging and recharging of EV batteries.

Considering these two types of loads, the two key problems that we study are: What is the optimal charging schedule for EVs to minimize the total power generation cost and EV charging cost? How to find a near optimal online algorithm if the future price-inelastic load is uncertain (due to the realistic causality constraint)? To formulate these problems, we leverage the well-known optimal power flow (OPF) problem and consider its time-dependent extension.

The solution of the OPF problem optimizes the operation of a power grid, and in general is NP-hard and nonconvex [7]. As the OPF is hard to solve optimally, a number of work are based on local optimization
methods, but these often cannot guarantee global optimality [8]. Recently, the authors of [7], [9] show that most practical power configurations surprisingly exhibit a useful property that guarantees zero duality gap between the OPF problem and its convex dual relaxation. In particular, [7] suggests solving a convex relaxation problem, which is formulated as a semidefinite programming (SDP) problem. When the zero duality gap property holds, the relaxation solution coincides with the global optimal solution of the original OPF problem. The zero duality gap condition derived in [7] is shown to be satisfied by IEEE systems with 14, 30, 57, 118, and 300 buses. This makes efficient polynomial time algorithm to solve the OPF problem possible. However, these work look at OPF only as a static optimization problem.

To incorporate time-varying electricity demand and price-elastic load, we extend the OPF problem to a joint OPF-EV charging time-dependent problem that spans over a scheduling period. It consists of a finite number of OPF subproblems coupled with one another by the constraints associated with price-elastic load, e.g., the EV charging constraints. Similar OPF-related problem formulation has also been given in [10]. The joint OPF-EV charging problem is as hard as the OPF problem. In [10], the authors prove that the joint OPF-EV charging problem has zero duality gap if the power grid network satisfies the zero duality gap condition in [7]. The authors propose to solve the convex dual of the joint OPF-EV charging optimization, which is a SDP that can be solved optimally in polynomial time. However, this approach that uses SDP solvers in [10] is offline, since it assumes that all the future demand for the price-inelastic load is known in advance at the start of the scheduling period. Also, for a large number of time intervals, this solution is not computationally efficient as the size of a SDP problem grows with the number of time intervals. Note that the EV charging problem alone has a scalability barrier, because for a long scheduling time, the number of control variables and the possible control sequence can be prohibitively large. The dynamic programming approach may not be computationally practical because of the huge number of decision states (state explosion). Hence, recent works in EV charging have looked at approximation techniques such as approximate dynamic programming in [11].

In this paper, we leverage the zero duality gap result in [7] to develop both offline and online algorithms that solve the joint OPF-EV charging optimization problem. To this end, we propose a nested optimization approach that decomposes the joint OPF-EV charging problem into separable subproblems, and then solve the decoupled problem using a nonsmooth separable programming technique. We give a novel valley-filling profile characterization to the optimal offline solution, and leverage it to design an online algorithm for the joint OPF-EV charging problem. The main contribution of the paper are as follows:

1) For time invariant EV charging cost, we characterize the offline optimal solution to be valley-filling. The valley-filling characterization holds true for all network configurations that guarantee the zero duality gap condition in the OPF problem.
2) With this characterization, we give an optimal EV charging schedule that follows the valley-filling characteristic when all the load information are present. This allows us to propose an offline algorithm that can solve the joint OPF problem with a computational complexity lower than centralized interior point solvers.

3) To account for the causality constraint from the price-inelastic load, we propose an online algorithm that dynamically tracks this valley-filling characteristic. The online algorithm can be easily implemented in a decentralized manner: at each time interval, charging decision is made locally by each vehicle after comparing its battery’s charging rate limit to an estimated valley level set by the utility. We evaluate the performance of our algorithms under different settings of demand and EV penetration, and demonstrate that the performance of the online algorithm is near optimal.

The rest of the paper is organized as follows. The system model is introduced in Section II, and the problem formulation is given in Section III. The main analytical results and valley-filling algorithms are given in section IV and V. Numerical evaluations are given in Section VI. Finally, the conclusions are given in Section VII. We use lower case letters for scalars, blocked lower case letters for vectors, blocked upper case letter for matrices, and use \((x[t])_j\) to represent the value of the \(j^{th}\) component of a vector \(x[t]\) at iteration \(t\). For clarity of presentation, all the proofs are given in the appendix.

II. System Model

We consider a discrete-time model where the time-slot interval matches the timescale at which the power grid adjusts its power generation. The goal is to optimize the operation of the power grid over a time-interval of interest \(t \in \{0, 1, \ldots, T\}\), where \(T\) is the scheduling period duration. In practice, \(T\) could be a day and a slot \(t\) could be in the order of minutes. In addition, we assume that price-inelastic loads are fixed over each time interval \([t, t + 1]\).

A. Network Structure

Consider a power network with a set of buses \(\mathcal{N} := \{1, \ldots, n\}\), a set of generator buses \(\mathcal{G} \subseteq \mathcal{N}\), and a set of flow lines \(\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}\). Let \(z_{lm}\) and \(y_{lm}\) be the complex impedance and admittance between bus \(l\) and \(m\), respectively, and we have \(y_{lm} = \frac{1}{z_{lm}}\). Denote \(Y = (Y_{lm}, 1 \leq l, m \leq n)\) as the admittance matrix, where

\[
Y_{lm} = \begin{cases} 
  y_{ll} + \sum_{n(l)} y_{lk} & \text{if } l = m, \\
  -y_{lm} & \text{if } (l, m) \in \mathcal{L}, \\
  0 & \text{otherwise},
\end{cases}
\]
where \( y_{ll} \) is the admittance to ground, and \( n(l) \) denotes the set of bus \( k \) such that \((k, l) \in \mathcal{L} \). Now let column vectors \( \mathbf{v} = (V_1, V_2, \ldots, V_{|\mathcal{N}|})^T \in \mathbb{C}^{|\mathcal{N}|} \) where \( V_k \) is the complex voltage at bus \( k \in \mathcal{N} \) and \( \mathbf{i} = (I_1, I_2, \ldots, I_{|\mathcal{N}|})^T \in \mathbb{C}^{|\mathcal{N}|} \) be the voltage and current vectors, respectively. By Ohm’s Law and Kirchoff’s Current Law, we have \( \mathbf{i} = \mathbf{Yv} \).

From [7], [9], it is shown that the zero duality gap condition always holds for radial networks, i.e., distribution networks. For general networks including transmission networks, the zero duality gap condition may hold under mild conditions, e.g., load oversatisfaction [7], [9]. In addition, this zero duality gap condition has been numerically verified for most practical IEEE test cases [7], [9]. In this paper, we assume that the zero duality gap condition is satisfied for a given power grid configuration.

**B. The OPF optimization problem**

The OPF problem finds the optimal operating point that minimizes an appropriate cost function, for example generation cost or power loss, subject to certain physical constraints on the power and voltage variables [12]. Let us introduce the key notations as follows:

- \( p_{d_k}^j + q_{d_k}^j \): Apparent power of the load connected to bus \( k \in \mathcal{N} \) (this value is zero whenever bus \( k \) is not connected to any load).
- \( p_{k} + q_{k}^j \): Output apparent power of generator \( k \in \mathcal{G} \).
- \( S_{lm} \): Apparent power transferred from bus \( l \in \mathcal{N} \) to the rest of the network through the line \((l, m) \in \mathcal{L} \).
- \( f_k(p_k) \): A convex function modeling the power generation cost for generator \( k \in \mathcal{G} \).

Define \((e_1, e_2, \ldots, e_{|\mathcal{N}|})\) as the standard basis vectors in \( \mathbb{R}^{|\mathcal{N}|} \). Then the power balance equation is given by

\[
(p_k - p_{d_k}^j) + (q_k - q_{d_k}^j)_j = V_k I_k^*
= (e_k^* \mathbf{v})(e_k^* \mathbf{i})^* = \text{Tr}\{\mathbf{vv}^* \mathbf{Y}^* e_k e_k^*\}. \tag{1}
\]

By introducing an auxiliary variable \( \mathbf{W} \in \mathbb{H}^{|\mathcal{N}| \times |\mathcal{N}|} \) to replace the rank-one matrix \( \mathbf{vv}^* \), the OPF problem can be equivalently formulated as
\[
\min_{W \in \mathcal{H}^{|\mathcal{N}| \times |\mathcal{N}|}} \sum_{k \in \mathcal{N}} f_k(p_k) \tag{2a}
\]

\[
\text{s.t. } P_{k}^{\text{min}} \leq p_k \leq P_{k}^{\text{max}}, \tag{2b}
\]
\[
Q_{k}^{\text{min}} \leq q_k \leq Q_{k}^{\text{max}}, \tag{2c}
\]
\[
(V_k^{\text{min}})^2 \leq W_{kk} \leq (V_k^{\text{max}})^2, \tag{2d}
\]
\[
(W_{ll} - W_{lm}) y_{lm}^* \leq S_{lm}^{\text{max}}, \tag{2e}
\]
\[
\text{Tr}(WY_{ek}^* e_k^*) = p_k - p_d^k + (q_k - q_d^k) j, \tag{2f}
\]
\[
W \succeq 0, \tag{2g}
\]
\[
\text{rank}(W) = 1 \tag{2h}
\]

for every \(k \in \mathcal{N}\) and \((l, m) \in \mathcal{L}\). In the above, the physical limits \(P_{k}^{\text{min}}, P_{k}^{\text{max}}, Q_{k}^{\text{min}}, Q_{k}^{\text{max}}, V_{k}^{\text{min}}, V_{k}^{\text{max}}\) and \(S_{lm}^{\text{max}}\) (where \(S_{lm}^{\text{max}}\) is equal to \(S_{ml}^{\text{max}}\)) are given constants. The \(f_k(\cdot)\) in the objective function is the power generation cost function at bus \(k\). Note that (2b), (2c) are the power generation constraints, (2d) is the voltage magnitude constraint, (2e) is the line flow constraint, and (2f) is the power balance constraint given in (1).

Without loss of generality, we assume that \(\mathcal{G} = \mathcal{N}\) (since when a bus \(k \in \mathcal{N}\) is not connected to a generator, we let \(P_{k}^{\text{min}}\) and \(P_{k}^{\text{max}}\) be both zero). Now, this problem can be relaxed by removing the rank constraint (2h). The relaxed problem without (2h) is, in fact, an SDP, which is equivalent to the Lagrange dual problem of the OPF [7]. It has been shown in [7] that, for a large class of practical networks, including IEEE 14, 30, 56, 118, 300 bus test systems, the zero duality gap condition is satisfied, and the relaxed problem has a rank-one solution.

C. EV Battery model

Suppose that each bus \(k \in \mathcal{N}\) can connect to a price-inelastic load and a price-elastic EV battery. Furthermore, we assume that each EV battery can absorb or inject only active power at an adjustable rate (this can be realized via a smart outlet). For simplicity, we consider that each bus is connected to only one EV battery. However, our results can be generalized to the case when multiple EV batteries are co-located at the same bus.

Let \((\hat{p}[t])_k\) be the control variables corresponding to the active power of the price-elastic load connected to bus \(k \in \mathcal{N}\) at time \(t\) (these numbers are zero when bus \(k\) is not connected to any EV battery). Due to the battery specification, an EV at bus \(k\) is constrained to charge within a given range \([\underline{r}_k, \bar{r}_k]\). To model
these charging constraints, we let \(r_k\) and \(\bar{r}_k\) be time-dependent and set \((r[t])_k = (\bar{r}[t])_k = 0\) for \(t\) that lies outside the allowable charging period of an EV at bus \(k\) [13]. As such, we have

\[
(r[t])_k \leq (\hat{p}[t])_k \leq (\bar{r}[t])_k \tag{3}
\]

for all \(k \in \mathcal{N}\) and all \(t \in \{1, 2, \ldots, T\}\). Let \(B_k, s_k(0),\) and \(\eta_k\) denote the battery capacity, initial state of charge (SOC), and charging efficiency of an EV connected to bus \(k\), respectively. By the deadline \(T\), the EV should be fully charged and this is captured by the total energy stored over the whole time horizon \(\eta_k \sum_{t=1}^{T-1} (\hat{p}[t])_k \Delta t = B_k (1 - s_k(0))\). Let \(c_k := B_k (1 - s_k(0))/(\eta_k \Delta t)\), then the EV charging constraint is equivalent to [13]:

\[
\sum_{t=1}^{T-1} (\hat{p}[t])_k = c_k \tag{4}
\]

for all \(k \in \mathcal{N}\).

Moreover, the battery cannot be discharged when there is no energy available in the battery and cannot be charged over its capacity. Hence, for all \(\tau \in \{1, \ldots, T-1\}\), we have \(-B_k s_k(0) \leq \eta_k \sum_{t=1}^{\tau} (\hat{p}[t])_k \Delta t \leq B_k (1 - s_k(0))\). By denoting \(c_k = -B_k s_k(0)/(\eta_k \Delta t)\), the EV charging constraint becomes

\[
c_k \leq \sum_{t=1}^{\tau} (\hat{p}[t])_k \leq c_k \tag{5}
\]

for all \(k \in \mathcal{N}\), and \(\tau \in \{1, \ldots, T-1\}\). Note that if \(r_k \geq 0\) for all \(k\), then (5) is naturally satisfied. In the following, we consider the case when \(r_k \geq 0\) for all \(k\) [13].

III. PROBLEM FORMULATION

A. Joint OPF-EV Charging Optimization Problem

Using the EV battery model and constraints in Sec. II we formulate a joint OPF-EV charging problem that optimizes the OPF problem with additional EV charging over discrete time intervals. This is a time-dependent OPF charging problem with a coupling of the EV charging constraints over time. Let \((\hat{p}[t])_k\) and \((\hat{q}[t])_k\) be, respectively, the active and reactive parts of the price-inelastic load connected to bus \(k \in \mathcal{N}\) at time \(t\). In the online scenario, we only know the values of \(\hat{p}[t]\) at the beginning of time \(t\).

Given \(t \in \{1, \ldots, T-1\}\), we define \(\tilde{p}[t]\) to be the vector of power consumed by the price-inelastic
demand, and the joint OPF-EV charging problem can be formulated as

\[
\min_{\{W[t], \hat{p}[t]\}} \sum_{t=1}^{T-1} \sum_{k \in N} f_k((p[t]_k) + \sum_{t=1}^{T-1} \sum_{k \in N} (\alpha[t]_k)(\hat{p}[t]_k)
\]

subject to

\[
P^\text{min}_k \leq (p[t]_k) \leq P^\text{max}_k,
\]

\[
Q^\text{min}_k \leq (q[t]_k) \leq Q^\text{max}_k,
\]

\[
(V^\text{min}_k)^2 \leq W[t]_{kk} \leq (V^\text{max}_k)^2,
\]

\[
(W[t]_{ll} - W[t]_{lm})y^*_m \leq S^\text{max}_{lm},
\]

\[
\text{Tr}\{W[t]_Y^*e_k e_k^*\} = ((p[t]_k) - (\hat{p}[t]_k) - (\tilde{p}[t]_k) + ((q[t]_k) - (\tilde{q}[t]_k)),
\]

\[
W[t] \succeq 0,
\]

\[
\text{rank}(W[t]) = 1,
\]

\[
\sum_{t=1}^{T-1} \hat{p}[t] = c,
\]

\[
\vec{r}[t] \leq \hat{p}[t] \leq \vec{r}[t],
\]

where \((\alpha[t]_k)\) is a known scalar accounting for the linear charging cost of the EV battery connected to bus \(k\) at time \(t\), subject to a set of physical limits and power balance constraints. It is shown in [10] that, if the duality gap of an OPF problem is zero for a given network, then the duality gap of the joint OPF-EV charging problem is also zero in the same network. Hence, solving the relaxed dual problem gives the optimal solution to the joint OPF-EV charging problem, provided that \(\tilde{p}[t]_k\) is known for all \(t\). However, there will be \(O(|\mathcal{N}| + |\mathcal{L}|T)\) variables in the dual SDP problem, and so the computational complexity even when solved by centralized interior point solvers may be prohibitive for large \(T\) (the computational bottleneck of interior point solvers lies in the inversion of the Hessian matrix when the problem size scales up).

It should be noted that the joint OPF-EV charging problem is coupled through time because of constraint (6i). If we can find the optimal \(\hat{p}[t]\), then the problem becomes separable instances of the OPF problem. Under the zero duality gap assumption, we will characterize the optimal solution of the joint OPF-EV charging problem under the constant price scheme and propose an offline valley-filling algorithm. In addition, we propose an online algorithm that solves the joint OPF-EV charging problem by dynamically tracking the optimal valley-filling profile. To do this efficiently, we first show how to decouple the OPF problem space, i.e., finding \(W[t]\), and the EV scheduling problem space, i.e., finding \(\hat{p}[t]\).
B. Decoupling power dispatching from EV scheduling

While the optimization variables in the Joint OPF-EV charging problem are $W[t]$ and $\hat{p}[t]$, we see that if the charging decision $\hat{p}[t]$ is known, then all the remaining variables $W[t]$ become separable in time $t$. Hence, solving the Joint OPF-EV charging problem at time $t$ is the same as solving the OPF problem with a total demand given by $(\hat{p}[t] + \tilde{p}[t])$ as follows:

$$F(\hat{p}[t] + \tilde{p}[t]) := \min_{W[t]} \left( \sum_{k \in \mathcal{N}} f_k((p[t])_k) \right)$$

s.t. (6b), (6c), ..., (6h).

Since the convex dual problem of the OPF can be efficiently solved [7] [9], we can decouple the power dispatching, i.e., finding $W[t]$, from the EV scheduling, i.e., finding $\hat{p}[t]$, and focus on the following EV scheduling problem:

$$\min_{\hat{p}[t]} \sum_{i=1}^{T-1} F(\hat{p}[t] + \tilde{p}[t]) + \alpha[t]^T \hat{p}[t]$$

s.t. $\underline{r}[t] \leq \hat{p}[t] \leq \overline{r}[t] \quad \forall t \in [1, T-1]$, $\sum_{i=1}^{T-1} \hat{p}[t] = c$, (8c)

where $F(\hat{p}[t] + \tilde{p}[t])$ in (7) returns the optimal value of the OPF problem for a total demand $(\hat{p}[t] + \tilde{p}[t])$. Note that $\alpha[1], \ldots, \alpha[T-1] \in \mathbb{R}^{|\mathcal{N}|}$ correspond to the nodal charging cost for each EV at different time intervals. In the following, we consider the case when the nodal charging cost is invariant, i.e., $\alpha[1] = \alpha[2] = \ldots = \alpha[T-1] = \alpha$.

IV. Optimal Offline Algorithm for EV Scheduling Problem

In this section, we show that $F(\cdot)$ is convex and the optimal solution to EV power scheduling follows a valley-filling profile. This characterization allows us to optimally determine the charging rates for EVs in $O(1)$ time. This result therefore reduces the computational complexity of solving (8) even when $T$ is very large. In addition, we propose an online decentralized algorithm that follows this valley-filling characterization and discuss its performance.

A. Convexity of $F(\cdot)$

Recall that the function $F(\cdot)$ given in (7) returns the optimal value of the OPF problem for a given load demand at each bus. Now, it is not immediately clear whether $F(\cdot)$ is convex or not, because the argument to $F(\cdot)$ not only appears in the objective function, but it also appears as part of the constraint set. However, by using the zero duality gap condition (when it exists), $F(\cdot)$ can also be expressed as the
optimal value of its dual problem. It turns out that for this alternate definition of $F(\cdot)$, the total demand $(\hat{p}[t] + \tilde{p}[t])$ only appears in the objective function, which leads to the following result.

**Theorem 1.** If the zero duality gap condition holds in (7), then $F : \mathbb{R}^{|\mathcal{N}|} \to \mathbb{R}$ is a convex function.

**Remark 1.** The convexity of $F(\cdot)$ is a direct consequence of the zero duality gap condition of OPF.

**Corollary 1.** For the EV scheduling problem in (8), since the price-inelastic load $(\tilde{p}[1], \ldots, \tilde{p}[T-1])$ is constant, (8) is a convex problem in $(\hat{p}[1], \ldots, \hat{p}[T-1])$, as its objective function is the sum of convex functions, and the feasible set is also convex.

Although $F(\cdot)$ is convex, it can be nonsmooth. In fact, $F(\cdot)$ is a pointwise supremum function in general (cf. (14) in the appendix). In the following, we characterize the optimal solution of (8) by making use of the convexity of $F(\cdot)$ only, and does not make any assumption on its smoothness.

### B. Characterization of optimal offline solution

Suppose that the charging rate limits are inactive. Then (8) becomes:

$$
\min_{\hat{p}[t]} \sum_{t=1}^{T-1} F(\hat{p}[t] + \tilde{p}[t]) + \alpha^T \hat{p}[t]
$$

s.t. $\hat{p}[1] + \hat{p}[2] + \ldots + \hat{p}[T-1] = c$. \hfill (9)

By convexity of $F$, we can apply Jensen’s inequality and get the following result:

**Lemma 1.** If $\forall t, r[t] = -\infty, \bar{r}[t] = \infty$, then the EV scheduling problem (8) has an optimal solution $\hat{p}[1] + \tilde{p}[1] = \hat{p}[2] + \tilde{p}[2] = \ldots \hat{p}[T-1] + \tilde{p}[T-1] = p_E + p_I$, where $p_E = \left(\sum_{t=1}^{T-1} \hat{p}[t]\right) / (T-1)$ and $p_I = \left(\sum_{t=1}^{T-1} \tilde{p}[t]\right) / (T-1)$.

When the charging rate limits are inactive, the optimal solution is a flat profile, i.e., $\forall t, \hat{p}[t] + \tilde{p}[t]$ is constant. Next, we consider the case where the charging rate constraints can be active. The optimal solution will then no longer be flat, but valley-filling as defined in the following:

**Definition 1.** A charging profile is valley-filling, if there exists a unique vector $a$, such that $\hat{p}[t] = [a - \tilde{p}[t]]_{r[t]}^{\bar{r}[t]}$, $\forall t$, where $[x]_l^u = \max(l, \min(x, u))$.

In the definition, $a$ can be seen as a valley level that $\hat{p}[t] + \tilde{p}[t]$ tries to reach unless $\hat{p}[t]$ is constrained by its charging rate limits. A similar definition of valley-filling for EV scheduling can be found in [14]. However, it should be noted that the valley level in [14] is the aggregation of all EV demands and is a scalar. On the other hand, the valley level in Definition 1 is defined for each individual EV and is a vector. Interestingly, the valley-filling characterization is reminiscent of the water-filling notion for power allocation to maximize capacity in information theory [15].
The next result demonstrates the optimality of a valley-filling profile. It can be proved by using a substitution argument, i.e., if there is an optimal charging profile that is not valley-filling, then by convexity of $F$, we can always construct a valley-filling profile that has the same or lower objective value.

**Theorem 2.** For a general convex function $F(\cdot)$, a valley-filling profile is optimal to the EV Scheduling problem (8).

**Corollary 2.** A valley-filling profile is a minimizer for any convex function $F(\cdot)$, whether smooth or nonsmooth, not just the $F(\cdot)$ defined in (7). For example, $F(\cdot)$ can be the $\ell_2$ norm of the aggregate load demand, i.e., $F(\bar{p}[t] + \tilde{p}[t]) = (\sum_{k \in \mathcal{N}}((\bar{p}[t])_k + (\tilde{p}[t])_k))^2$. As the total load demand given by $\sum_{t=1}^{T-1} \sum_{k \in \mathcal{N}}(\bar{p}[t])_k + (\tilde{p}[t])_k$ is constant, we can see that the valley-filling profile is also minimizing the load demand variance.

Next, we show the uniqueness of the valley level $a$. Note that $a$ must satisfy the following for $j = 1, \ldots, |\mathcal{N}|$:

$$\min_t \{(\bar{p}[t])_j + (r[t])_j\} \leq a_j \leq \max_t \{(\bar{p}[t])_j + (r[t])_j\}, \quad (10a)$$

$$\sum_{t=1}^{T-1} \bar{p}[t] = \sum_{t=1}^{T-1} [a - (\bar{p}[t])_j]_g[t] = c. \quad (10b)$$

If we look at (10b) component-wise, it is a continuous and strictly increasing function of $a_j$ for $a_j$ in the box constraint (10a). Since (10b) is continuous and strictly increasing, we can find a unique $a$ via the bisection method for the offline case. This is presented in the following algorithm with $\varepsilon$ as an error tolerance level.

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**Algorithm 1** Valley-filling Bisection

```
1: \forall j, u_j \leftarrow \max_t \{(\bar{p}[t])_j + (r[t])_j\};
    l_j \leftarrow \min_t \{(\bar{p}[t])_j + (r[t])_j\};
2: \textbf{for } j = 1 \rightarrow |\mathcal{N}| \textbf{ do}
3: \textbf{ while } (\|u_j - l_j\| \geq \varepsilon) \textbf{ do}
4: \qquad m_j \leftarrow \frac{1}{2}(u_j + l_j);
5: \qquad \textbf{if } (\sum_{t=1}^{T-1} [m_j - (\bar{p}[t])_j]_g[t])_j > c_j \textbf{ then}
6: \qquad \qquad u_j \leftarrow m_j;
7: \qquad \textbf{ else}
8: \qquad \qquad l_j \leftarrow m_j;
9: \quad a \leftarrow m.
```
Remark 2. Algorithm 1 determines \( a \) in a component-wise manner. Each iteration in the while loop will halve the search space for \( a_j \), and therefore the computational complexity of the bisection algorithm is low.

Once we have determined the value of \( a \), the scheduling of EV vehicle (solving \( \hat{p}[t] \)) can now be done optimally in \( O(1) \) time following the valley-filling characterization in Definition 1. Given the optimal \( \hat{p}[t] \), the remaining optimization variables \( W[t] \) become separable. The offline algorithm for (8) is shown in the following.

Algorithm 2 Offline EV Scheduling

1: Calculate the valley level \( a \) using Algorithm 1;
2: for \( t = 1 \rightarrow T - 1 \) do
3: \( \hat{p}[t] \leftarrow [a - \tilde{p}[t]]_{\text{E}[t]} \);
4: Solve the OPF problem with the active load demand set to \( (\hat{p}[t] + \tilde{p}[t]) \);

Remark 3. Algorithm 2 decomposes the joint OPF-EV charging problem (6) from a SDP with \( O(|N| + |L|)(T - 1) \) variables to \( (T - 1) \) SDPs each with \( O(|N| + |L|) \) variables. Since the complexity of SDP interior point algorithms grows superlinearly with respect to the number of variables \([16], [17]\), this decomposition leads to a lower computational complexity.

V. Online Algorithm for EV Scheduling Problem

Under a causality constraint, we do not assume any knowledge of \( \tilde{p}[t] \) until time \( t \). Therefore, we cannot use the previous bisection algorithm to find \( a \) in an online manner. Instead, we propose an algorithm that estimates the valley level, which is denoted by \( a'[t] \) and adjusts it dynamically in an online fashion (Algorithm 3 below).

For Algorithm 3 line 1 initializes the first estimation of the (unknown) valley level \( a \). From Lemma 1, the ideal valley level is indeed \((p_E + p_I)\) if the charging rate constraints are not active. We know the value of \( p_E \), which is just the charging target \( c \) divided by time \((T - 1)\), but the value of \( p_I \) has to be estimated, possibly by learning from historical record of the price-inelastic load. It will be shown later that, the more accurate the first estimation is, the better Algorithm 3 performs. Line 3 follows the valley-filling characterization outlined in the previous section. However, as the valley level estimation is not perfect, we need to take extra steps from line 5 to line 8 to ensure the feasibility of the solution. The rationale for line 5 to 6 is to ensure that the charging profile \((\hat{p}[1], \ldots, \hat{p}[T - 1])\) will not overcharge the EV batteries at any point in time. Roughly speaking, it means that “if after this instance, even charging at the minimal rate will eventually overcharge the EV batteries, then slow down the current charging rate.”
Algorithm 3 Online EV Scheduling

1: \( a'[1] \leftarrow p_E + \hat{p}_I \); \((\hat{p}_I \text{ is an estimation of } p_I)\)

2: \( \text{for } t = 1 \rightarrow T - 1 \) \( \text{do} \)

3: \( \hat{p}[t] \leftarrow [a'[t] - \tilde{p}[t]]_{\mathbb{Z}}; \)

4: \( \text{for } j = 1 \rightarrow |N| \) \( \text{do} \)

5: \( \text{if } (\sum_{k=1}^{t} \hat{p}[k])_j > c_j - \sum_{l=t+1}^{T-1} (\mathbb{r}[l])_j \) \( \text{then} \)

6: \( (\hat{p}[t])_j \leftarrow c_j - \sum_{l=t+1}^{T-1} (\mathbb{r}[l])_j - \sum_{k=1}^{t-1} (\hat{p}[k])_j; \) \((\text{prevent overcharging})\)

7: \( \text{if } (\sum_{k=1}^{t} \hat{p}[k])_j < c_j - \sum_{l=t+1}^{T-1} (\mathbb{r}[l])_j \) \( \text{then} \)

8: \( (\hat{p}[t])_j \leftarrow c_j - \sum_{l=t+1}^{T-1} (\mathbb{r}[l])_j - \sum_{k=1}^{t-1} (\hat{p}[k])_j; \) \((\text{prevent undercharging})\)

9: \( \text{if } (t < T - 1) \) \( \text{then} \)

10: \( (a'[t+1])_j \leftarrow (a'[t])_j + \frac{(a'[t])_j - (\hat{p}[t])_j - (\tilde{p}[t])_j}{T-1-t}; \)

11: Solve the OPF problem with the active load demand set to \((\hat{p}[t] + \tilde{p}[t])\);
inelastic load exceeds the current estimation; it increases when the sum is below the estimation, and it stays the same when the sum meets the estimated level. This behavior is dictated by line 10 which updates the estimation in order to spread out the current error to subsequent estimated valley levels. The result below shows that with this dynamic adjustment, the “if” conditions in line 5 and line 7 will be inactive in most cases when the first estimation of the valley level is sufficiently good.

**Theorem 4.** Assuming that the estimation of \( p_1 \) is perfect, and \( \hat{p}[T - 1] - \tilde{p}[T - 1] = \alpha'[T - 1] \), then the charging profile \((\hat{p}[1], \ldots, \hat{p}[T - 1])\) obtained from Algorithm 3 without line 5 to 8 is a feasible solution to (8).

**Remark 6.** Theorem 4 shows that as long as \( \alpha'[T - 1] - \tilde{p}[T - 1] \) falls within the range of \([\bar{r}[T - 1], r[T - 1]]\), Algorithm 3 will produce a feasible charging profile without triggering the “if” conditions in line 5 and 7. Assuming that the charging rate window \([\bar{r}[T - 1], r[T - 1]]\) is proportional to the total EV load, this suggests that, when there is a high EV penetration level, the dynamic adjustment of the valley level \( \alpha'[t] \) alone is enough to produce a feasible result.

**Remark 7.** As the online adjustment of the valley level (line 9 to line 10) of Algorithm 3 is trying to make \( \hat{p}[1] + \tilde{p}[1] + \ldots + \hat{p}[T - 1] + \tilde{p}[T - 1] = (T - 1)\alpha'_1 \), the accuracy in estimating \( \alpha'_1 \) has a direct impact on the performance of Algorithm 3.

Fig. 3 compares the different EV charging profiles produced by Algorithm 2 and Algorithm 3. As shown, both exhibit the valley-filling characteristic, but the valley level of Algorithm 3 (grey bar) is changing dynamically.

**VI. NUMERICAL RESULTS**

Consider the IEEE 14-bus system depicted in Fig. 4 where the circuit specifications and the physical limits are given in the library of the toolbox MATPOWER [18]. The system has five generators connected to buses 1, 2, 3, 6, and 8. Assume that each of the nongenerator bus 4, 5, 7, 9, 10, 11, 12, 13, and 14 is connected to an EV load. Enumerate the batteries of these vehicles as 1, 2, \ldots, 9. Consider that all the batteries are plugged in at time \( t = 1 \) and must be fully charged by time \( t = 25 \), the charging rate of each battery can be controlled only at the discrete time instants \( 1, 2, \ldots, 24 \).

Aside from the elastic EV loads, suppose that each bus \( k \in \{1, 2, \ldots, 14\} \) is also connected to a price-inelastic load given by

\[
(\tilde{P}[t])_k = \frac{l(t) \times P_k}{l(t)}, \quad t = 1, 2, \ldots, 24,
\]
where \((P_1, \ldots, P_{14})\) is equal to the load profile for a IEEE 14-bus system \([19]\), \(l(t)\) follows the average residential load in the service area of SCE at different times of the day, (cf. SCE website \([20]\)), and \(\overline{l}(t) = \sum_{t=1}^{24} l(t)\). The goal is to optimize the controllable parameters of the power network such as the active power supplied by a generator or the charging rate of a battery, which can be modified only at the time instants 1, 2, \ldots, 24. To this end, we aim to minimize the following cost function

\[
\sum_{t=1}^{24} \sum_{k \in \mathcal{N}} (p[t])_k + \sum_{t=1}^{24} \sum_{k \in \mathcal{N}} \alpha(\hat{p}[t])_k.
\]

This cost function has the following features:

- The generation cost is the total active power generated by all the generators over the time horizon from \(t = 1\) to \(t = 24\).
- The pricing vector of each battery is assumed to be independent of its bus number and invariant over time, and we let \(\alpha = 2\) in the following.

To compare Algorithm 3 with the optimal solution of (12) solved by the SDP relaxation approach in \([10]\), we used CVX \([21]\), which is a computational package for solving SDP. Three scenarios will be considered in the sequel, and a working example will be shown in Section VI-D.

A. Effect of EV penetration

In this example, we vary the EV load to be from 10\% to 100\% of the price-inelastic load, and compute the percentage difference. The percentage difference is given by \((p_{\text{online}}^* - p_{\text{offline}}^*)/p_{\text{offline}}^* \times 100\), where \(p_{\text{offline}}^*\) and \(p_{\text{online}}^*\) are the optimal values obtained by Algorithm 2 and Algorithm 3 respectively. Fig. 5 shows the simulation results using three different 24-hour load profiles taken randomly from the SCE residential load data \([20]\). Firstly, we observe that Algorithm 3 solves the joint OPF-EV charging problem (6) almost optimally. From the three randomly chosen load profile, the worst performance is less than 0.016\% different from \(p_{\text{offline}}^*\). Secondly, we can see all three plots go up initially and then eventually decrease. This is because at the initial, the EV load is relatively insignificant, and thus Algorithm 2 and 3 perform almost the same as there is little to optimize. As the EV load becomes more significant, the performance gap grows because Algorithm 3 lacks perfect knowledge. However, a higher EV penetration will also lead to a larger room for optimization. Hence, the performance gap decreases and eventually approaches zero as the EV penetration increases.

B. Effect of controllable charging window size

In this example, we set the charging rate constraint \(\bar{r}[t] = 0, \forall t\), hence, the charging window size corresponds to the upper charging rate limit. We plot the graph for the cases when \(\bar{r}[t]\) equal to 5\%, 10\%, and 20\%
of the actual EV load respectively. Here, only the residential load profile from 15:00 on Aug. 27th to 14:00 on Aug. 28th is used. From Fig. 6 we can see that a larger allowable charging rate window leads to better results, as the curves corresponding to 10% and 20% charging rate window is below that of 5% charging rate window. However, once the charging rate window is larger than 10%, any additional increase in charging rate window size does not add significantly to performance.

C. Effect of average load prediction accuracy

In Fig. 7, we let the initial estimation of the total average load to be $a'[1] = (\tilde{p_E} + \tilde{p_I}) \times (1 + 10^i\%), i = 1, \ldots, 10$, which corresponds to 10%, 20%, \ldots, 100% error in estimation of the total average electricity load. We set the charging rate window to be 30% of the total EV load. Here, only the residential load profile from 15:00 on Aug. 27th to 14:00 on Aug. 28th is used. We can see from Fig. 7 that the more inaccurate the initial estimation is, the worse is the performance of Algorithm 3. Also, with a higher EV penetration level, Algorithm 3 becomes more sensitive to error in initial estimation. It should also be noted that, when the EV penetration level is 80%, an initial estimation error greater than 90% in this case causes Algorithm 3 to schedule excessive EV load to be charged at the initial first few time slots that leads to the OPF in (7), i.e., finding $W[t]$, becoming infeasible.

D. A working example

In this example, the price-inelastic load variation (see Fig. 8) is based on the residential load profile from 15:00 on Aug. 27th to 14:00 on Aug. 28th (taken from the SCE website [20]). Given the price-inelastic load, the EV charging profiles produced by Algorithm 2 and Algorithm 3 are shown in Fig. 9 and Fig. 10 respectively. The EV penetration level is set to 50%, and the charging rate window is 20% of the EV load, and we assume that there is 10% error in overestimating the initial valley level for Algorithm 3. We can make several observations from Fig. 9 and Fig. 10:

1) The charging rate of each EV is high when the inelastic demand (shown in Fig. 8) is low, which corresponds to the valley-filling characterization. The charging profile for EV 3 is flat because there is no price-inelastic load present in that bus in the IEEE 14 bus test case specification.

2) An over-estimation of the initial valley level causes Algorithm 3 to charge the EV batteries faster than optimum. As a result, all the EV batteries are fully charged two time slots before the deadline.

3) The result of Algorithm 3 is still very close to the optimal value, as the percentage difference is small: $(p_{\text{online}}^* - p_{\text{offline}}^*) / p_{\text{offline}}^* \times 100\% = 0.0627\%$. Hence, in terms of power loss minimization, Algorithm 3 performs nearly optimally at this setting.
E. Runtime comparisons

In this section, we compare the computational time for the SDP optimization approach that uses SDP interior point algorithm to solve the Joint OPF-EV charging problem in [10] with that of Algorithms 2 and 3. The simulation is run on the IEEE 14 bus system, for 6, 12, 24, 48 timeslots respectively. The computational time measured is the average of running the respective algorithm for ten times.

| $T$ | SDP Optimization | Algorithm 2 | Algorithm 3 |
|-----|------------------|-------------|-------------|
| 6   | 6.0 s            | 5.8 s       | 5.9 s       |
| 12  | 13.1 s           | 11.6 s      | 11.6 s      |
| 24  | 31.5 s           | 22.9 s      | 22.8 s      |
| 48  | 84.0 s           | 45.8 s      | 45.7 s      |
| 96  | 262.6 s          | 87.5 s      | 87.4 s      |

From Table I, we can see that the time complexity of Algorithm 2 and Algorithm 3 are comparable. Also, both Algorithm 2 and Algorithm 3 have lower time complexity as compared to the SDP optimization method in [10], and the saving in computational time from using Algorithm 2 or Algorithm 3 is more significant as $T$ increases. This demonstrates the advantage of the nested optimization approach to solve the joint OPF-EV charging problem.

VII. CONCLUSION

We studied a time-dependent OPF charging problem that optimized jointly the operation of the power grid and the charging activity of Electric Vehicles. We proved that this problem is convex with respect to the total electricity demand, characterized the valley-filling charging profile to be optimal under constant electricity price, and proposed a decentralized online algorithm that followed this characterization. At each iteration of the online algorithm, each EV calculated its own charging rate according to the valley level broadcast by the utility, and the utility guided their charging rate by updating the valley level. The online algorithm can be decentralized and thus requires low communication and computation capability. Simulation results showed that the online algorithm performed almost optimally in minimizing power loss, and the optimal value of the online algorithm approached to that of the offline solution as the penetration of EVs increased. However, a higher EV penetration will also lead to a higher sensitivity on the accuracy of estimating the average price-inelastic load.

In this paper, the online algorithm considers a time invariant pricing scheme. That is, the nodal electricity price remains constant throughout the scheduling period. However, when there are renewable sources,

\footnote{1We used MATLAB version 7.6.0.324 (R2008a) and Windows 7 OS, and ran the programs on an Intel Xeon CPU 2.80GHz machine.}
electricity prices can vary in real time. In addition, EVs may require charging at different times in a more dynamical setting. Incorporating real time pricing, modeling EV arrivals as random events and accounting for the additional uncertainties with these extensions are interesting directions for future research. Furthermore, our results can be generalized to more complex joint OPF-EV charging problems with security constraints, e.g., ramp rate constraints on the power generators formulated along the line in [22].

VIII. ACKNOWLEDGEMENT

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IX. Appendix

A. Analysis of Valley-filling Characterization

In this section, we prove Theorem 1, Lemma 1 and Theorem 2 respectively.

Proof of Theorem 1: Let \( p_E[t] + p_I[t] \) be the argument to \( F(\cdot) \) in (7). From [7], with the zero duality gap condition, \( F(\hat{p}[t] + \tilde{p}[t]) \) is equal to the optimal value of the following problem:

\[
\begin{align*}
\text{maximize} & \quad h(\Theta, \hat{p}[t] + \tilde{p}[t]) \\
\text{subject to} & \quad A(\Theta) \succeq 0,
\end{align*}
\]

where \( \Theta \) is the set of dual variables corresponding to the different physical constraints in (6) without the rank constraint (6b). \( h(\Theta, \hat{p}[t] + \tilde{p}[t]) \) is the Lagrange dual function (in fact, an affine function in \( \hat{p}[t] + \tilde{p}[t] \)) for every feasible \( \Theta \), and \( A(\Theta) \) is a linear matrix inequality constraint in \( \Theta \) (cf. the definition of Optimization 4 in [7]). Let \( \mathbb{D} \) denote the feasible set of \( \Theta \) that satisfies the constraint (13b). Then,

\[ F(\hat{p}[t] + \tilde{p}[t]) = \sup_{\Theta \in \mathbb{D}} (h(\Theta, \hat{p}[t] + \tilde{p}[t])), \]

which is a convex function in \( (\hat{p}[t] + \tilde{p}[t]) \) as it is a pointwise supremum of convex functions [23]. □

Proof of Lemma 1: \[ \sum_{t=1}^{T-1} \alpha^T \hat{p}[t] = \alpha^T c, \] which is a constant. Also, by Jensen’s inequality, we have

\[
\frac{F(\hat{p}[1] + \tilde{p}[1]) \ldots + F(\hat{p}[T-1] + \tilde{p}[T-1])}{T-1} \geq F \left( \frac{\hat{p}[1] + \tilde{p}[1] + \ldots + \hat{p}[T-1] + \tilde{p}[T-1]}{T-1} \right) = F \left( \frac{c + d}{T-1} \right),
\]

where the equality holds when \( \hat{p}[1] + \tilde{p}[1] = \ldots = \hat{p}[T-1] + \tilde{p}[T-1] = (c + d)/(T-1). \) □

To prove Theorem 2 we start with the following definition and lemma:

Definition 2. Denote \( a > (b, c) > d \) to mean both \( a > \max(b, c) \) and \( d < \min(b, c) \).

Lemma 2. Let \( f \) be a convex function, and \( a + d = b + c \) and \( a > (b, c) > d \), then \( f(a) + f(d) \geq f(b) + f(c) \).
Proof: Without loss of generality, assume that \(a > b > c > d\), as \(a + d = b + c\), then there exists a \(k \in (0, 1)\) such that \(ka + (1 - k)c = b\), and \(kd + (1 - k)b = c\). By convexity of \(f\), we have
\[
kf(a) + (1 - k)f(c) \geq f(b), \quad (15a)
\]
\[
kf(d) + (1 - k)f(b) \geq f(c). \quad (15b)
\]
Summing \((15a)\) and \((15b)\) completes the proof. \(\square\)

Proof of Theorem 2: Consider all the arguments to \(F(\cdot)\) to be scalars first, we can get the proof for the vector case by repeating the argument for each component of the vector. Suppose \((\hat{p}'[1], \ldots, \hat{p}'[T-1])\) is an optimal solution to the scalar case of \((8)\), and let \((\hat{p}[1], \ldots, \hat{p}[T-1])\) be the valley-filling load.

Suppose we have some \(\hat{p}'[j] \neq \hat{p}[j]\), without loss of generality, assume \(\hat{p}'[j] > \hat{p}[j]\), as both charging profiles are feasible, they need to sum up to the same battery capacity \(c\) according to \((8c)\), hence there must exist some \(k\), such that \(\hat{p}'[k] < \hat{p}[k]\). From Definition 1, we can observe that the valley-filling profile has the minimal deviation from the flat profile, i.e., for \((8)\), if \((\hat{p}[1], \ldots, \hat{p}[T-1])\) is a valley-filling profile, and \((\hat{p}'[1], \ldots, \hat{p}'[T-1])\) is any feasible charging profile, then \(a_j - ((\hat{p}[i])_j + (\hat{p}[i])_j) \leq a_j - ((\hat{p}'[i])_j + (\hat{p}'[i])_j), \quad \forall i, j\). Hence,
\[
((\hat{p}[i])_j + (\hat{p}[i])_j) < ((\hat{p}'[i])_j + (\hat{p}'[i])_j) \Rightarrow ((\hat{p}'[i])_j + (\hat{p}'[i])_j) < a_j, \quad (16)
\]
\[
((\hat{p}[i])_j + (\hat{p}[i])_j) > ((\hat{p}'[i])_j + (\hat{p}'[i])_j) \Rightarrow ((\hat{p}'[i])_j + (\hat{p}'[i])_j) > a_j. \quad (17)
\]
From \((16)\) and \((17)\), we have
\[
\hat{p}'[j] + \hat{p}[j] > a, \quad \hat{p}'[k] + \hat{p}[k] < a. \quad (18)
\]
We consider two cases, \(\hat{p}'[j] - \hat{p}[j] \leq p_E[k] - \hat{p}'[k]\) and \(\hat{p}'[j] - \hat{p}[j] > \hat{p}[k] - \hat{p}'[k]\) separately.

Case 1, suppose that \(\hat{p}'[j] - \hat{p}[j] \leq \hat{p}[k] - \hat{p}'[k]\), meaning that \(\hat{p}'[j]\) deviates from the valley-filling load by a smaller amount. In this case, we swap \(\hat{p}'[j]\) and \(\hat{p}'[k]\) by \(\hat{p}''[j]\) and \(\hat{p}''[k]\) defined as follows:
\[
\hat{p}''[j] = \hat{p}[j], \quad \hat{p}''[k] = \hat{p}'[k] + \hat{p}'[j] - \hat{p}[j].
\]
Now, \(\hat{p}''[k]\) is feasible because \(\hat{p}'[k] \leq \hat{p}''[k] \leq \hat{p}[k]\) by assumption. Furthermore, \(\hat{p}''[j] + \hat{p}''[k] = \hat{p}'[j] + \hat{p}'[k]\).
From \((16)\), \((17)\), and \((18)\), we have
\[
\hat{p}'[k] + \hat{p}[k] \leq (p_E'[k] + \hat{p}[k], p_E'[j] + \hat{p}[j]) \leq \hat{p}'[j] + \hat{p}[j].
\]
Hence, by Lemma 2, we have
\[
F(\hat{p}[j]'') + F(\hat{p}[k]'') + \hat{p}[k]) \leq F(\hat{p}'[j] + \hat{p}[j]) + F(\hat{p}'[k] + \hat{p}[k]),
\]
which means that applying such swapping procedure will not increase the objective value of \((8)\).
Case 2, suppose that \( \hat{p}'[j] - \hat{p}[j] > \hat{p}[k] - \hat{p}'[k] \). Then in this case apply the swapping procedure as follows:

\[
\hat{p}''[j] = \hat{p}'[j] + \hat{p}'[k] - \hat{p}[k], \quad \hat{p}''[k] = p_E[k].
\]

It can be shown via a symmetric argument to case 1 that, applying such a swapping procedure will not decrease the objective value of (8) as well.

Note that each time a swapping is applied, there will be at least one fewer \( \hat{p}'[j] \) that deviates from the valley-filling load. Applying a finite number of this swapping procedure completes the proof.

For the vector case of (8), repeat the above argument to each component of the vectors \((\hat{p}[1], \ldots, \hat{p}[T-1])\).

\[\Box\]

B. Analysis of Algorithm 3

In this section, we prove Theorem 3 and Theorem 4 respectively.

**Proof of Theorem 3**: Firstly, the charging profile \((\hat{p}[1], \ldots, \hat{p}[T-1])\) will not overcharge EV batteries, i.e., \( \sum_{k=1}^{t} \hat{p}[k] \leq c \quad \forall t \), by line 3 to 6. Secondly, it will not undercharge, i.e., \( \sum_{t=1}^{T-1} \hat{p}[t] \geq c \) because of line 5 to 6. Hence, \( \sum_{t=1}^{T-1} \hat{p}[t] = c \), the charging sum constraint is satisfied. Thirdly, assuming the charging rate constraints permits a feasible solution, then \( r[t] \leq \hat{p}[t] \leq \bar{r}[t] \), \( \forall t \). This can be proved by contradiction, suppose the charging profile \((\hat{p}[1], \ldots, \hat{p}[T-1])\) obtained from the above algorithm has some \( i, j \), such that \( \hat{p}[t])_j < (r[t])_j \), note that \((\hat{p}[t])_j\) can only go below \((r[t])_j\) if the condition in line 5 is true. Pick the smallest \( t \) where the charging rate constraint is violated. If \( t = 1 \), then that means \( \sum_{t=1}^{T-1} (r[t])_j > c_j \) by line 5, i.e., the charging limit does not permit a feasible solution, which is a contradiction. If \( t > 1 \), then by line 6 we have \((\hat{p}[t])_j = c_j - \sum_{l=t+1}^{T-1} (r[l])_j - \sum_{k=1}^{t-1} (\hat{p}[k])_j < (r[t])_j \), and after rearranging, we have \( \sum_{k=1}^{t-1} (\hat{p}[k])_j > c_j - \sum_{l=t}^{T-1} (r[l])_j \), but that is not possible, because at the end of iteration \( t - 1 \), line 5 to line 6 will always make sure that \( \sum_{k=1}^{t-1} (\hat{p}[k])_j \leq c_j - \sum_{l=t}^{T-1} (r[l])_j \). The other case where the upper charging rate constraint is violated can be proved in a similar way. Therefore, when both the charging sum and the charging rate constraints are satisfied, the charging profile produced by Algorithm 3 is a feasible solution to (8).

\[\Box\]

**Proof of Theorem 4**: To prove that \((\hat{p}[1], \ldots, \hat{p}[T-1])\) is a feasible solution to (8), we only need to show that \( \sum_{t=1}^{T-1} (\hat{p}[t] + \bar{p}[t]) = (T-1)(p_E + p_I) = (T-1)a'[1] \). Note that \( \hat{p}[T-1] + \bar{p}[T-1] = a'[T-1] \).
If $T = 2$ then we are done, otherwise by line 10 of Algorithm 3 we have

$$\hat{p}[T-2] + \hat{p}[T-2] + \tilde{p}[T-1] + \hat{p}[T-1] = 2\alpha'[T-2],$$

$$\hat{p}[T-3] + \hat{p}[T-3] + \ldots + \hat{p}[T-1] + \hat{p}[T-1] = 3\alpha'[T-3],$$

$$\vdots$$

$$\hat{p}[1] + \hat{p}[1] + \ldots + \hat{p}[1] + \hat{p}[T-1] + \hat{p}[T-1] = (T-1)\alpha'[1].$$

Hence $\sum_{t=1}^{T-1} (\hat{p}[t] + \tilde{p}[t]) = (T-1)\alpha'[1]$, which is what we set out to prove. 

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Fig. 1: Base demand curve is the average residential load in the service area of Southern California Edison from 20:00 on Feb. 13th to 19:00 on Feb 14th, 2011 [20].
Fig. 2: Illustration of dynamic estimation of valley level.

Fig. 3: The base demand curve is the average residential load in the service area of SCE from 20:00 on Feb. 13th to 19:00 on Feb 14th, 2011 [20]. The valley-filling curve is obtained using Algorithm 2. The valley-filling online curve is obtained using Algorithm 3.
Fig. 4: IEEE 14-bus system studied in Section VI taken from the IEEE power system test archive [19].

Fig. 5: Profile 1: residential load from 10:00 on Jul. 6th to 9:00 on Jul. 7th; Profile 2: average residential load from 15:00 on Aug. 27th to 14:00 on Aug. 28th; Profile 3: residential load from 1:00 on Mar. 11th to 0:00 on Mar. 12th. All load profiles are taken from SCE website [20].
Fig. 6: Effect of varying the charging window size on the performance of Algorithm 3.

Fig. 7: Effect of percentage error in total average load prediction.
Fig. 8: Variation of price-inelastic load in a 24 hour time frame based on the residential load profile from 15:00 on Aug. 27th to 14:00 on Aug. 28th (taken from the SCE website [20]).

Fig. 9: Optimal EV charging rates for the offline case, $p_{\text{offline}}^* = 79.5348$pu.
Fig. 10: Optimal EV charging rates for the online case, $p^*_{\text{online}} = 79.5847 \text{pu}$. 