Large N transition in the 2D SU(N)xSU(N) nonlinear sigma model.

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We consider the characteristic polynomial associated with the smoothed two point function in two dimensional large N principal chiral model. We numerically show that it undergoes a transition at a critical distance of the order of the correlation length. The transition is in the same universality class as two dimensional large N QCD.
Large N transition in the 2D SU(N)xSU(N) nonlinear sigma model.

Rajamani Narayanan

1. Two dimensional SU(N) X SU(N) principal chiral model

The two dimensional SU(N) X SU(N) principal chiral model is similar to four dimensional SU(N) gauge theory in many respects\[1\]. The continuum action is given by

\[ S = \frac{N}{T} \int d^2x Tr \partial_\mu g(x) \partial_\mu g^\dagger(x) \] (1.1)

where \( g(x) \in SU(N) \). The global symmetry group \( SU(N)_L \times SU(N)_R \) reduces down to a single SU(N) “diagonal subgroup” if we make a translation breaking “gauge choice”, \( g(0) = 1 \). This model is asymptotically free and there are \( N-1 \) particle states with masses

\[ M_R = M \frac{\sin \left( \frac{R\pi}{N} \right)}{\sin \left( \frac{\pi}{N} \right)}, \quad 1 \leq R \leq N-1. \] (1.2)

The states corresponding to the \( R \)-th mass are a multiplet transforming as an \( R \) component antisymmetric tensor of the diagonal symmetry group.

The two point function \( W = g(0)g^\dagger(x) \) plays the role of Wilson loop with the separation \( x \) playing the role of area. We expect the behavior to be perturbative for small \( x \). On the other hand, non-perturbative effects become important for large \( x \).

One expects

\[ G_R(x) = \langle \chi_R(g(0)g^\dagger(x)) \rangle \sim C_R \left( \frac{N}{R} \right) e^{-M_R|x|} \] (1.3)

where \( \chi_R \) is the trace in the \( R \)-antisymmetric representation. Comparison with the heat-kernel representation of the characteristic polynomial associated with the Wilson loop operator in two dimensional large \( N \) QCD \[2\] suggests the following connections:

- The two point correlator, \( W(d) = g(0)g^\dagger(d) \), is analogous to the Wilson loop operator.
- \( M|x| \) is analogous to the dimensionless area, \( t \).

Based on this analogy, we hypothesize \[3\] that the characteristic polynomial, \( \det(z - g(0)g^\dagger(d)) \), will undergo a transition at some value \( d_c \). The universal behavior at this transition will be in the same universality class as two dimensional large \( N \) QCD.

2. Setting the scale

Numerical measurement of the correlation length using the lattice action

\[ S_L = -2Nb \sum_{x,\mu} \Re Tr [g(x)g^\dagger(x+\mu)] \] (2.1)

and

\[ \xi_G^2 = \frac{1}{4} \sum_i x_i^2 G_1(x) \] (2.2)

yields the following continuum result \[4\]:

\[ M\xi_G = 0.991(1) \] (2.3)
Large N transition in the 2D SU(N)xSU(N) nonlinear sigma model. Rajamani Narayanan

We use $\xi_G$ to set the scale and it is well described by

$$\xi_G = 0.991 \left[ \frac{e^{\frac{\pi}{16\pi}}}{16\pi} \right] \sqrt{E} \exp \left( \frac{\pi}{E} \right)$$  \hspace{1cm} (2.4)

in the range $11 \leq \xi_G \leq 20$ with

$$E = 1 - \frac{1}{N} \text{Re} \langle Tr [g(0)g^\dagger(\hat{1})] \rangle = \frac{1}{8b} + \frac{1}{256b^2} + \frac{0.000545}{b^3} - \frac{0.00095}{b^4} + \frac{0.00043}{b^5}$$  \hspace{1cm} (2.5)

The above equations will be used to find a $b$ for a given $\xi$.

3. Smeared SU(N) matrices

Well defined operators are obtained using smeared matrices. We start with $g(x) \equiv g_0(x)$ and one smearing step takes us from $g_t(x)$ to $g_{t+1}(x)$ using the following procedure. Define $Z_{t+1}(x)$ by:

$$Z_{t+1}(x) = \sum_{\pm\mu} [g_{t}^\dagger(x)g_t(x+\mu) - 1].$$  \hspace{1cm} (3.1)

Construct anti-hermitian traceless $SU(N)$ matrices $A_{t+1}(x)$

$$A_{t+1}(x) = Z_{t+1}(x) - Z^\dagger_{t+1}(x) - \frac{1}{N} \text{Tr}(Z_{t+1}(x) - Z^\dagger_{t+1}(x)) \equiv -A^\dagger_{t+1}(x).$$  \hspace{1cm} (3.2)

Set

$$L_{t+1}(x) = \exp[fA_{t+1}(x)].$$  \hspace{1cm} (3.3)

g_{t+1}(x) is defined in terms of $L_{t+1}(x)$ by:

$$g_{t+1}(x) = g_t(x)L_{t+1}(x).$$  \hspace{1cm} (3.4)

This procedure is iterated till we reach $g_n(x)$ and the smearing parameter is defined by $\tau = nf$. For a fixed $\xi_G$, the parameter $\tau$ is fixed such that $\tau/\xi_G^2$ remains unchanged. We set $n = 30$ in our numerical simulations and this was found sufficiently large to eliminate a dependence on the two factors, $f$ and $n$, individually.

4. Numerical details

We need $L/\xi_G > 7$ to minimize finite volume effects. We worked in the range $11 \leq \xi_G \leq 20$ and therefore we chose $L = 150$. We used a combination of Metropolis and over-relaxation at each site $x$ for our updates. The full SU(N) group was explored. 200-250 passes of the whole lattices were sufficient to thermalize starting from $g(x) \equiv 1$. 50 passes per step were enough to equilibrate if $\xi_G$ was increased in steps of 1.

The test of the universality hypothesis proceeds in the same manner as for the three dimensional large $N$ gauge theory. We defined the characteristic polynomial, $F(y,d)$, as

$$F(y,d) = \langle \det(e^{y/2} + e^{-y/2}W(d)) \rangle$$  \hspace{1cm} (4.1)
Large $N$ transition in the 2D SU($N$)×SU($N$) nonlinear sigma model.

Rajamani Narayanan

Figure 1: Behavior of $\Omega$ as a function of $\alpha$ in the scaling region.

We perform a Taylor expansion,

$$F(y,d,N) = C_0(d,N) + C_2(d,N)y^2 + C_4(d,N)y^4 + \ldots \quad (4.2)$$

since $F(y,d)$ is an even function of $y$. It is useful to define

$$\Omega(d,N) = \frac{C_0(d,N)C_4(d,N)}{C_2^2(d,N)} \quad (4.3)$$

which resembles a Binder cumulant.

As $N \to \infty$, $\Omega(d,\infty)$ is a step function with $\Omega = \frac{1}{6}$ for short distances $d < d_c$ and $\Omega = \frac{1}{2}$ for long distances, $d > d_c$. Zooming in on the step function as $N \to \infty$ in the vicinity of $d = d_c$ using the scaling variable $\alpha = \sqrt{N}(d - d_c)$, we obtain Fig. 1.

We use $\Omega(\alpha = 0) = 0.364739936$ to obtain the critical size $d_c$ in the following manner. Given an $N$ and a $\xi$, we find the $d_c$ that makes the Binder cumulant $\Omega(d_c,N) = 0.364739936$ as shown in Fig. 2. We look at $d_c$ as a function of $\xi$ for a given $N$. This gives us the continuum value of $d_c/\xi$ for that $N$. This extrapolation is shown in Fig. 3 for $N = 30$. We then take the large $N$ limit as shown in Fig. 4 and it gives us

$$\left. \frac{d_c}{\xi} \right|_{N=\infty} = 0.885(3) \quad (4.4)$$

Further substantiation of the universal behavior can be given by comparing the eigenvalues distribution in the model to the Durhuus-Olesen eigenvalue distributions in two dimensional QCD. This is shown for one example each on either side of the critical point in Fig. 5 and very close to the critical point in Fig. 6. We use $2k = t$ to match with the notation in [5].
Figure 2: Plot of $\Omega(d)$ after the subtraction of $\Omega(\alpha = 0) = 0.364739936$ as a function of $d/\xi_G$.

Figure 3: Extrapolation to continuum of $d_c/\xi$ for $N = 30$. 

fit: $d_c/\xi = 1543/\xi^4 - 26.10/\xi^2 + 0.7682$

With error estimation: $(d/\xi)_{critical,\,continuum} = 0.768(2)$
Large $N$ transition in the 2D SU($N$)$\times$SU($N$) nonlinear sigma model.

Figure 4: Extrapolation of the continuum $d_c/\xi$ to infinite $N$.

Figure 5: Examples of eigenvalue distribution for one small and one large distance.
Large N transition in the 2D SU(N) x SU(N) nonlinear sigma model.

Rajamani Narayanan

Figure 6: An example of an almost critical eigenvalue distribution.

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