Principles for the design of a fully-resourced, coherent, research-informed school mathematics curriculum

Colin Foster, Tom Francombe, Dave Hewitt and Chris Shore
Mathematics Education Centre, Loughborough University, Loughborough, UK

ABSTRACT
The curriculum resources used for teaching secondary mathematics vary considerably from school to school. Some schools base their teaching largely on a single published scheme, while others design their own schemes of learning, curating their resources from a range of (often free) online sources. Both approaches seem problematic from the perspective of experiencing the mathematics curriculum as a coherent story, and neither seems likely to take best advantage of the accumulated body of knowledge in the education research literature about effective didactics for mathematics. In this position paper, as we embark on the collaborative, research-informed design of a complete, fully-resourced, free-to-access mathematics curriculum for students aged 11–14, we use the conceptual framework of mathematics curriculum as a story to draw out five key curriculum design principles. A mathematics curriculum should harness and develop the skills and expertise of teachers; balance the teaching of fluency, reasoning and problem solving; give explicit attention to important errors and misconceptions; compare and contrast alternative methods; and engineer coherence through strategic use of consistent representations and contexts. We use these five principles to set out our vision for the next step in research-informed mathematics curriculum design.

1. Introduction
The content of the mathematics taught in secondary schools in England is determined almost entirely by the national curriculum (Department for Education (DfE), 2013), particularly as this is expanded on and exemplified through the published specifications for public examinations. However, the sequencing of this content for teaching, and the balance of time allocated to different mathematics topics—which together we term the curriculum—is largely up to schools (Askew et al., 2010; Leung, 2014). Schools also have autonomy over their choice of curriculum resources—the entire set of learning tasks that students experience within the curriculum, along with any associated teacher guidance materials. This presents schools with broadly two options for designing and resourcing their curricula—either to rely on a published scheme, or to curate their own—both of which, we will argue, are seriously suboptimal.

In contrast to these approaches, in this position paper we outline what a fully-resourced, coherent, research-informed school mathematics curriculum for students aged 11–14 would entail. As we embark on this ambitious project, we choose to take a slow-design approach (Burkhardt, 2006; Burkhardt & Schoenfeld, 2003), and begin by seeking to establish in this paper the research-informed
design principles that will guide the design and development of the curriculum and its resources. To do this, we begin by outlining Dietiker’s (2015) *mathematics curriculum as story* conceptual framework, which we use to frame our research synthesis, in which we draw on the educational design literature, along with the best-available international evidence on the teaching and learning of mathematics. In so doing, we combine research findings with the craft knowledge (Polanyi, 2009) and expertise in educational design and mathematics education of the authors of this paper to establish five overarching mathematics curriculum design principles. Finally, we use these principles to set out our vision for research-informed mathematics curriculum design.

### 2. Conceptual framework: mathematics curriculum as story

We take as our theoretical framework for conceptualizing how ideas unfold across a mathematics curriculum the notion of the *mathematics curriculum as story* (Dietiker, 2013, 2015). As with stories in literature, a *mathematical story* can be thought of as an ordered sequence of connected ideas, with a beginning, a middle and an end, and consisting of story elements such as characters, action, setting and plot. This narrative lens helps to frame how posing and resolving a sequence of mathematical tasks could contribute to the students’ mathematical experience. As Dietiker (2015) explained:

> interpreting mathematics curriculum as a story draws attention to the sequence of content (i.e., how the story unfolds in the moment) and its aesthetic dimensions (i.e., surprising turns). In addition, the metaphor story also brings with it new descriptive language for what happens along the way (i.e., action), environment (i.e., where the story is set), objects of the sequence (i.e., its characters), as well as the interrelationships between parts of the sequence (e.g., foreshadowing). (p. 286)

As an example, Dietiker (2015) applies this framework to show how two different orderings of three tasks concerning the roots of quadratic equations can lead to completely different experiences for the student, emerging from two different stories with different conclusions. In her framework, Dietiker (2015) describes a mathematical *character* as a mathematical object, such as an expression like $x^2 - 6x + 3$. She points out that appreciating that this expression could also be written in completed square form as $(x - 3)^2 - 6$ corresponds, within the mathematical story framework, to character development—learning a new name or property for an existing mathematical character. Mathematical *actions* are manipulations of mathematical characters, such as transformations and operations, which might introduce new characters or highlight relationships between them. Some actions move the story forward (e.g. completing the square as a prelude to solving an equation) while others may not (e.g. routine exercises on completing the square). Mathematical *settings* denote the mathematical representations used as the ‘space’ for the action. For example, if the setting is the coordinate $xy$ plane, then the expression $(x - 3)^2 - 6$ could be conceptualized as a *function* $y = (x - 3)^2 - 6$, and this might allow additional character development, such as appreciating the shape of a parabola, or actions, such as finding the zeroes by observing where the curve cuts the $x$-axis. Finally, mathematical *plot* refers to a sequence of events and takes account of the (incomplete) knowledge of the student at each point, and the likely questions that might come to mind as they anticipate what is to come. In this way, a well-constructed plot may reliably engineer twists and turns and stimulate feelings such as intrigue, curiosity, surprise or satisfaction.

From the perspective of this framework, it is tempting to observe that the majority of the mathematical stories told in commonly-used mathematics curriculum resources consist of a very large number of very similar *characters*, highly repetitive *action*, and virtually no sustained drama or noticeable *plot*. No wonder students so often describe learning mathematics as ‘boring’ (Brown et al., 2008). In contrast, Dietiker (2015, p. 300) suggested that curriculum designers might use the conceptual tools from her framework to ask themselves, ‘How can this sequence build excitement or suspense for what is to come?’

An important consideration for any design of a curriculum and its resources is coherence (Howard & Hill, 2020). Richman et al. (2019, p. 4) defined ‘*story coherence* as the extent to which the events and
mathematical ideas of the mathematical story (i.e. a lesson) are connected to each other for a reader’. This perspective on coherence ties in with other calls to sequence the mathematics curriculum and its resources in a coherent, logical order—particularly important for a hierarchically-organized subject such as mathematics. Howard and Hill (2020, p. 61) noted the ‘need to sequence our curriculum to reduce cognitive load by drawing on prior knowledge and logically sequencing episodes of learning so they accumulate in small stages, securing understanding at one stage before moving on to the next’. However, they also commented, more closely in line with Dietiker (2015), that ‘Sequencing of the curriculum is clearly more than the ordering of its component parts—it is about the relationships and connections between them, and the deeper understanding that the sequence allows our students to access’ (p. 63). To do this well is a major challenge of mathematics curriculum and resource design, and a feature that would seem to us to be notable by its absence from most current curriculum resources used in schools in England, as we will now discuss.

3. Problems with current approaches to curriculum design and resourcing

Schools currently face two broad alternative approaches to designing and resourcing their mathematics curricula, both of which suffer from serious problems, particularly when viewed from Dietiker’s (2015) perspective of the mathematics curriculum as a story.

3.1 Adopt one of the many commercial schemes available, as paper-based textbooks and/or online materials

Published schemes are often attractive to schools because they are endorsed by examination awarding bodies, but they are also often perceived to be expensive and of low quality (Askew et al., 2010). In England, published resources are typically designed to tight publication deadlines, led by publishers’ economic imperatives and the necessity of responding to often short-notice changes to curriculum specifications in time for the start of a new academic year. Textbooks sell based on the first impressions of a teacher flicking through them, rather than on how effectively they enable students to learn mathematics over the long term in the classroom. They tend to be judged quickly by busy middle leaders in schools, who are unlikely to have time even to read them through, let alone carry out a detailed analysis or try them out extensively with students (see Oates, 2014). Superficial inspection, even by an experienced teacher, is an unreliable way to distinguish effective from ineffective resources; the most reliable way to judge the quality of a resource being for one or more teachers to try it out systematically with students.3

At first glance, a textbook scheme may look polished, but the consistent page design may conceal a less coherent didactical approach under the surface, with different chapters often written by different authors, with little opportunity for serious discussion or redrafting (see Van Steenbrugge et al., 2013). It is likely to be only after a school has made their purchase that they discover the less-than-ideal didactical design choices—and at that point there may be little they can do to ameliorate this. Schools are consequently faced with choosing among a large range of published schemes which, though superficially different, may all suffer from similar problems of a lack of underpinning didactical coherence. Oates (2014, p. 6) described a ‘chronic market failure’ in which high-stakes assessments and examination league tables have pressurised teachers into an assessment-focused teaching style, resulting in them seeking out resources that support this, and consequently this is what publishers are incentivized to supply. And, since published schemes are designed to be all-encompassing solutions to a school’s mathematics curriculum resourcing, making edits is difficult and time-consuming for schools to contemplate.

The reluctance of mathematics teachers in England to adopt textbook series (Oates, 2014; Ollerton, 2002; Ruthven, 2014) contrasts sharply with the situation in countries such as Japan, in which textbooks are mandated and widely seen as embodying many decades of wisdom in didactical design (Seino & Foster, 2019; Takahashi, 2006). In Japan, the national specification of the
Curriculum content is changed infrequently, and textbooks consequently have the time to develop their sequencing and resources slowly through multiple editions, with any changes made being incremental and carefully considered (Takahashi, 2006). The situation is similar in Finland, where, although textbooks are not mandated, the vast majority of mathematics teachers use them, because they are perceived to be of high quality (Oates, 2014). Accompanying teacher guides are also provided, produced by collaborative teams of teachers and teacher educators (Niemi et al., 2012).

### 3.2 Curate a detailed, bespoke scheme of learning

Alternatively, and perhaps to some extent in response to the problems set out above, schools, or groups of schools, may decide to take matters into their own hands and curate their own bespoke schemes of learning, by assembling and sequencing a set of teaching resources from a wide range of sources (see Cooper et al., 2020). Home-grown solutions would seem to offer many advantages, including variety, benefiting from the best ideas available, and the ability to tailor the resources to a school’s particular students’ needs and the preferred teaching styles of their teaching staff.

Assembling a bespoke scheme of learning may seem like an exciting prospect for an enthusiastic head of mathematics. The existence of enormous quantities of free classroom resources on websites—for example, 700,000 resources on the Tes website⁴—can be enticing. However, many free online resources are of very low quality, in terms of didactical design, and finding the rare diamonds among the glass may feel like more work than designing from scratch.⁵ Although, in recent years, some websites have begun to address the quality issue by providing carefully-chosen collections of resources (e.g. http://www.resourceaholic.com/), only schools in the most favourable circumstances can hope to find the considerable amount of time needed to turn these into a fully-resourced curriculum, and, in practice, this work is often undertaken in teachers’ own time, with detrimental effects on their wellbeing. It is unlikely that even large groups of schools working collaboratively have the means to do this job as well as they would wish, and reinventing the curriculum resource wheel multiple times across the country seems highly inefficient. The limited time available for teachers to work together in school may be better spent on developing existing resources and improving them, and carefully planning their effective use in their school context (i.e. planning lessons rather than resourcing them, McCrea, 2019). Curriculum and resource design is a highly complex undertaking and may involve skills quite different from those of even a highly-expert teacher (Wittmann, 1995; Lemov & Badillo, 2020; Remillard, 2005). Good doctors are not expected to invent their own medicines; the greatest actors are not necessarily playwrights.

Crucially, it may also be the case that a collection of great tasks does not necessarily make a great collection of tasks. However carefully individual resources are selected, they will not necessarily add up to a coherent sequence of curriculum resources, because each separate resource is likely to be designed from different, perhaps conflicting, principles, and hence may pull the curriculum in contradictory ways.

### 3.3 Problems with these approaches

It seems unlikely that either published schemes, under the prevailing commercial pressures, or school-curated ones, with the inevitable constraints to that process, will lead to the best possible quality of mathematics curriculum resources. It also seems unlikely that either approach is well placed to take full advantage of the dramatic increase over the last few years in the knowledge base of the research literature about effective didactics for learning mathematics (for a recent review, see Hodgen et al., 2018).

From the perspective of the mathematics curriculum as story framework (Dietiker, 2015), these two models of curriculum design and resourcing both suffer from fundamentally the same problems (see Askew et al., 2010). Both models rely on an author or experienced teacher finding or creating suitable resources and integrating these either into a textbook or into a school’s scheme of learning. This
constitutes what Swan and Burkhardt (2014) termed a *craft-based* (as opposed to ‘research-based’) model, where knowledge from research on the didactics of mathematics and findings from the educational design literature are unlikely to be drawn on. It also follows the *authorship model* of curriculum design and resourcing, which Swan and Burkhardt described as: ‘produce a draft; gather comments; revise; publish’ (2014, np), as opposed to a design research project, which involves systematic classroom trialling and iterated re-design and improvement before publication (Swan & Burkhardt, 2014).

Both models contrast with the rare curriculum development projects (see Brown & Hodgen, 2014), such as the SMP (School Mathematics Project) of the 1960s (Little, 1993), SMILE (Secondary Mathematics Individualised Learning Experiment) in the 1970s, CAME (Cognitive Acceleration in Maths Education) in the 1990s (Adey & Shayer, 2002), the Standards Unit in the 2000s (Swan, 2005), and, to some extent, the Key Stage 3 Framework for Teaching Mathematics (particularly the Supplement of Examples) (Department for Education and Skills (DfES), 2001). However, only SMP and SMILE could claim to have produced both complete and research-informed curriculum resources, and at least four decades of research in mathematics education has been published since the time of their creation. CAME did not cover a full curriculum—it was presented as a supplement, consisting of ‘Thinking Maths’ lessons—and the Standards Unit and other Shell Centre materials, such as the Mathematics Assessment Project, focused on particular key concepts, and did not attempt complete curriculum coverage. The National Strategy (Department for Education and Skills (DfES), 2001) claimed to be evidence-based, but the extent to which this was the case has been questioned (Brown et al., 1998). We are not aware of any modern attempt to design a free, fully-resourced, coherent, research-informed school mathematics curriculum.

### 4. Our proposed approach to curriculum design and resourcing

High-quality coherent curriculum resources must be developed through cycles of iterated trialling and refining, following detailed classroom feedback (Swan & Burkhardt, 2014). In our model, lessons will be observed 3–5 times during each of two cycles of development. This scale affords rich, detailed feedback, while making it possible to distinguish between general implementation issues and more idiosyncratic variations made by particular teachers. Inspired by recent calls to base the teaching of mathematics more securely on sound evidence (e.g. Royal Society, 2016), in 2019 we established the Loughborough University Mathematics Education Network as a partnership between a higher education institution with a particularly strong mathematics education research centre and local schools and colleges. Initially, supportive teachers within this network will work with us to trial and give feedback on draft lessons, allowing obvious problems to be ironed out. This will then be followed by more extensive trialling, with more ‘typical’ teachers in ‘typical’ school settings (see Burkhardt, 2006).

A key aspect of the curriculum resources is that they will be designed ready to use as provided, so as to cater for the many schools currently without the capacity to make extensive in-house adaptations. However, by releasing the resources in a completely editable format, teachers and schools will be free to select and adapt elements as they wish. Schools ready for a big change might want to take our resources as a starting point, select the units that they wish to use, and add to or delete from these as appropriate. At the other extreme, schools who are already largely happy with the resources that they have in place might wish to examine the approaches taken in ours, to see where there may be areas where they might want to make modifications to their current approaches—or simply borrow ideas from anywhere and incorporate them into their own scheme of learning. Excitingly, from a research standpoint, there would also be opportunities for schools to create alternative designs of the same unit and conduct simple controlled experiments (i.e. A/B test) with different classes, and begin to acquire their own evidence about ‘what works’ in their particular contexts. We see teachers being involved beyond the initial design phase as critical friends to the
project, by providing ongoing feedback on what happens in the classroom when using the resources, as well as by making suggestions of productive adaptations. Since the resources will be hosted online, they can be continually updated into the future—an extremely valuable affordance.

Two current factors combine to make this a particularly opportune time to construct a free, fully-resourced, coherent, research-informed school mathematics curriculum:

(1) England currently faces a serious teacher retention and recruitment crisis (Allen & McInerney, 2019), with young teachers, often promoted quickly to head of department roles, burning themselves out seeking to manage the many responsibilities placed on them (see Newmark, 2019). Key among these responsibilities is often setting up a viable scheme of learning for the department, a task many young heads of mathematics may feel (and be) inadequately prepared for. Some notions of teacher professionalism prioritize in-house solutions to curriculum resourcing, but other conceptualizations do not demand that every teacher, or even every school, must reinvent the wheel of their curriculum and its resources for themselves. A free, fully-resourced curriculum would be of enormous value, especially for trainee and newly-qualified teachers, as well as to address workload concerns for all teachers (Department for Education (DfE), 2016, 2018).

(2) The last few years have seen escalating interest among school/college mathematics teachers in England in making school mathematics teaching practices more evidence-informed, as they appear to be in some higher-performing jurisdictions (see Askew et al., 2010). This has led to the rise of grassroots organizations, such as ResearchED, which is focused on improving research literacy among teachers, and the growing interest shown in the research syntheses and reports published by the Education Endowment Foundation and disseminated through the Research Schools Network. This impetus has coincided with a recent proliferation of teacher-facing blogs, podcasts and books, and the rapid rise of ‘eduTwitter’ (see Watson, 2020). In the last few years, there has also been an explosion of resources (books, websites, conferences, blogs) devoted to making use of evidence from psychology and cognitive science (e.g. Gilmore et al., 2018), within education (e.g. Christodoulou, 2014; Didau & Rose, 2016; Kirschner & Hendrick, 2020; Rosenshine, 2012; Weinstein et al., 2018; Willingham, 2010), and, particularly, mathematics education (e.g. Barton, 2018, 2020; McCourt, 2019; McCrea, 2019), as well as several reports synthesizing the best-available evidence on teaching mathematics (e.g. Anthony & Walshaw, 2009; Hodgen et al., 2018). Specifically, there have been calls for research into theory-informed task design (Thompson et al., 2007; Wake, 2018) and detailed attempts to apply cognitive science to curriculum and resource design (Kirschner et al., 2017), alongside developments within the field of design-based research (e.g. see Bakker, 2018; Mckenney & Reeves, 2018, and issues of Educational Designer). Many writers have drawn attention to the centrality of didactics, by which is meant subject-specific knowledge about teaching, as distinct from pedagogy, which is more generic (Andrews, 2007; McCourt, 2019, p. 117).

An in-depth synthesis of the knowledge about mathematics teaching in the research literature has recently been carried out (e.g. Hodgen et al., 2018), but curating the very large body of tasks for mathematics teaching, and using these as a basis for coherent design, is a considerable undertaking that has so far not been attempted. The mathematics curriculum as story framework makes clear the impossibility of assembling a meaningful story by simply juxtaposing individual resources, however good, from different sources—this would be akin to attempting to construct a satisfying narrative by combining chapters taken from disparate different novels. To design and resource a curriculum that has the potential to build rich connections across and within topics, and facilitate students’ progress in a structured way, it seems necessary to set out clear design principles for building and resourcing a curriculum that makes coherent mathematical sense (Schmidt et al., 2005). According to Almond (2020):
the role of curriculum is to take learners’ shallow knowledge and understanding of a topic and make it deeper. For this to happen, the curriculum needs to be planned carefully so that key ideas and concepts are continually revisited, remembered and built upon. (p. 61)

An effective curriculum cannot simply be a consensus curriculum, where experts come together and each contribute their own preferred tasks and approaches. This would be almost guaranteed not to achieve a coherent whole. Instead, difficult issues and choices need to be surfaced, considered and resolved; some good lesson resources will find no place, because, however strong, they do not fit coherently within an overall agreed trajectory. Ideas and approaches that might be effective for one standalone lesson, but which fit less well into an overall progression, will need to be rejected. So, hard decisions need to be made about directions to take. We frame this work in terms of the collaborative refinement of ‘curriculum and resource design principles’, which we now turn to.

5. Deriving principles for curriculum and resource design

We now outline how we arrived at the five principles for curriculum design which we will discuss in more detail in Section 6, and which we have distilled from the research literature, as well as from professional writings embodying the tacit, craft knowledge (Polanyi, 2009) of teachers and expert designers. These principles are the outcomes of our analysis of the relevant literature, followed by intensive discussions among the authors of this paper and critical comments from highly-experienced designers of mathematics resources. We have sought to be guided by well-known and widely-respected previous designs and designers of mathematics curriculum resources, and creative and insightful ideas from practitioners (e.g. Ollerton, 2002). Throughout the design process, attention to detail is critical at every stage (see Bakker, 2018; McKenzie & Reeves, 2018), since ‘the devil is in the detail’ (Seino & Foster, 2019), and small changes to tasks can have big effects (Prestage & Perks, 2001). We believe that this is the case not just when contemplating the nitty-gritty of specific lessons but also at this early stage of setting out the principles and direction for the curriculum and its resources.

Our starting point was the observation that teachers will be the end users of the resources. The rise of ResearchED (mentioned in Section 4) highlights teachers’ commitment to linking research with classroom practice, which is essential for effective curriculum and resource design. Teacher agency, through a dynamic role in determining their work, has increasingly been seen as key to meaningful professional development (Biesta et al., 2015; Imants & Van Der Wal, 2020). Previous experience of successful curriculum initiatives, such as SMILE (mentioned in Section 3.3), also led us to principle #1: to harness and develop the skills and expertise of teachers.

The principles for the design and resourcing of a mathematics curriculum are inevitably dependent on the aims of that curriculum. Curricula from around the world have broadly similar goals (Burkhardt, 2014), and we take as the aims for our curriculum the stated aims of the National Curriculum for England. These are:

- to ensure that all pupils:
  - become fluent in the fundamentals of mathematics . . .
  - reason mathematically . . .
  - can solve problems . . . (Department for Education (DfE), 2013, p. 2)

To be relevant to schools, any set of curriculum resources has to address these aims, and these aims are fundamental aspects of doing mathematics (see Cuoco et al., 1996). Hence, we established principle #2: to balance the teaching of fluency, reasoning and problem solving.

Balancing principle #1, with its focus on teachers, students must always lie at the heart of teaching. Teaching is a dynamic between what a teacher might offer and what previous understandings students bring. As Ausubel (1968, p. iv) remarked: ‘The most important single factor influencing learning is what the learner already knows; ascertain this and teach [them] accordingly’. Thus, a set of curriculum resources must take careful account of students’ previous learning,
especially when those understandings are either incorrect or not known well enough to be applied appropriately in novel contexts (Shulman, 1987). Teachers need to work with students’ current understandings, and so it is important to identify current student thinking and not to allow important errors and misconceptions to go unaddressed (Ball et al., 2008). However, many curriculum resources fail to take account of common misconceptions and sometimes actively promote them (Sewell, 2002). This strand of literature led to principle #3: to give explicit attention to important errors and misconceptions.

A set of curriculum resources themselves can only take us so far; the way in which the teacher works with developing the mathematical ideas is critical. Research literature points to the value of comparing and contrasting alternative methods as a key aspect of teaching, which is common practice in successful educational systems in Japan and Shanghai, for instance, (see Kullberg et al., 2017; Takahashi, 2006; Watson, 2017). This literature led us to principle #4: to compare and contrast alternative methods.

Finally, a strong repeated theme within the research literature concerns the use of representations and contexts (e.g. Carbonneau et al., 2013; Van Den Heuvel-Panhuizen, 2020), and this ties in with reports of effective practice from widely-varied countries, including Finland and Japan (Askew et al., 2010). This led us to principle #5: to engineer coherence through strategic use of consistent representations and contexts.

Together, these five principles represent the chief features of our reading of the literature and synthesis of craft knowledge in the design of mathematics curriculum and resources.

6. Implications for curriculum and resource design

We now discuss each of our five design principles and, in particular, we consider each principle from the point of view of mathematics curriculum as story (Dietiker, 2015), in order to identify ways in which character, action, setting and/or plot may be exploited to enhance students’ experience of the curriculum. We explain below under each design principle how we see it relating to this framework.

6.1 Harness and develop the skills and expertise of teachers

Within our mathematics curriculum as story framework, the teacher has the critical role of storyteller, on whom everything depends. We are mindful of Bruner’s (1960) remark that:

A curriculum is more for teachers than it is for pupils. If it cannot change, move, perturb, inform teachers, it will have no effect on those whom they teach. It must be first and foremost a curriculum for teachers. If it has any effect on pupils, it will have it by virtue of having an effect on teachers. (p. xv)

Any set of resources is totally dependent on the skills of effective teachers (Watson et al., 2003); it is the teacher who will ‘make or break it’. Effective teaching of mathematics is dependent on teachers’ mathematical knowledge for teaching (Ball et al., 2008), including their deep knowledge of fundamental mathematics (Ma, 2010) and their pedagogical content knowledge (Shulman, 2004). Teachers’ awarenesses need to be tuned to noticing (Mason, 2002) and to observing their students carefully. ‘Teacher-proof’ resources (see Taylor, 2013), where the teacher is reduced to merely following instructions or a script, would be diametrically opposed to our intentions (Howard & Hill, 2020); on the contrary, we wish to build strong ongoing collaborations with teachers, which place them at the heart of the design process, informing the design at every stage, through trialling, feedback and ongoing modifications (Burkhardt, 2006).

The overriding strategy is for us as designers to make big-picture design decisions while leaving the optimal amount of flexibility for the teacher, so as to encourage ‘productive adaptations’ (Burkhardt & Schoenfeld, 2003) but avoid ‘lethal mutations’, which kill the design purpose of the task. Having clear guidance need not diminish teachers’ creativity; on the contrary, if done well, it provides fertile ground for it (compare with the actor in a play performing predetermined lines under
a director, yet showing boundless creativity: the most creative actors—or even directors—do not have to write their own stories). To this end, we will provide detailed guidance in teacher notes accompanying every teaching unit, intended to be extensive without being excessive, so supporting, rather than usurping, teachers’ professional judgements. Our intention is to serve teachers by providing what their limited time might make it harder for them to do. This includes offering answers and suggested solutions to student tasks, with key difficulties highlighted and possible teacher responses suggested in ‘key issues tables’ (see Wake et al., 2016), and giving links in the teacher notes to sources of relevant research. Above all, the notes to teachers must explicate the mathematical story arcs of the unit, so that the teacher is clear about where surprise or drama, for instance, is intended, and how to support students at moments of resolution and consolidation—and avoid unintended ‘spoilers’.

Guidance documents that are too lengthy will not be looked at by busy teachers, and a measure of success will be that teachers feel they save time by reading the guidance, because it forewarns them of difficulties students may encounter and offers suggestions of how to respond. A well-designed set of curriculum resources should support teachers’ wellbeing, drastically reducing the time they need to spend in their evenings hunting on the internet for ideas for the next day’s lessons (Cooper et al., 2020; Department for Education (DfE), 2016, 2018; Foster, 2020).

The text in the resources will be carefully constructed so as to offer a steer to less experienced teachers on how they might present new ideas, but would not be a straitjacket for the more experienced teacher who feels that they do not need it. In many cases, the text might describe a scenario in which students and the teacher do various things, and, where appropriate, a confident teacher might choose to actually do these things, rather than merely read through the text describing it. However, where this is not practicable or desired, the ‘second-hand’ approach can be taken, in which the text is simply read and thought about. Within the mathematical story framework, this would correspond to doing a read-through in contrast to acting out a play. Over time the materials might support teachers in taking a more active role.

A teacher needs to engage students in careful arguing and conjecturing (e.g. Lampert, 1990; Stein et al., 2008), and strategic questioning, and we are clear that without sensitive use by a skilled teacher the resources will achieve nothing. At the same time, we intend that use of these resources, and discussion with colleagues, will assist teachers to make incremental changes to their teaching practice (Star, 2016), and complete designed resources such as these should be ideal for trainee and newly-qualified teachers to use to develop their craft.

### 6.2 Balance the teaching of fluency, reasoning and problem solving

As mentioned earlier, the design principles of the curriculum and its resources are intrinsically linked to the aims of the curriculum, which we stated in Section 5. We see the first two of these aims (fluency in facts, procedures and concepts; and mathematical reasoning) not as ends in themselves, but as means to supporting students in the principal (third) aim of solving problems. Consequently, we seek to value facts and procedures neither as the ultimate goal, nor as lesser aspects of learning, but as fundamental to the development of truly relational understanding of concepts (Skemp, 1976). In this way, procedures are not the enemy of sensemaking but an essential feature of the learning process. While it is true that an excessive focus on algorithms can stop students from thinking (Stein, 1987) and that poorly-conceived rules ‘expire’ (Dougherty et al., 2017), we emphasize the value of integrating facts, procedures and concepts. We wish to avoid setting up an unhelpful opposition between procedural and conceptual understanding, which we regard as being interrelated and intertwining (Star, 2005).

The resources will present a variety of opportunities for students to work mathematically and develop mathematical habits of mind (Cuoco et al., 1996); i.e. ‘developing the mathematician as well as the mathematics’ (Francome & Hewitt, 2017). Within the mathematics curriculum as story framework (Dietiker, 2015), this involves ensuring that a variety of different story forms are used, involving
not just minimally different characters in similar scenarios but substantively varied plots. The Cockcroft report (1982, para 243) stated “that there are certain elements which need to be present in successful mathematics teaching to pupils of all ages: Mathematics teaching at all levels should include opportunities for:

- exposition by the teacher;
- discussion between teacher and pupils and between pupils themselves;
- appropriate practical work;
- consolidation and practice of fundamental skills and routines;
- problem solving, including the application of mathematics to everyday situations;
- investigational work.” (p. 71)

Providing worthwhile tasks in which students operate in these varied ways is a major challenge of curriculum and resource design. We seek to benefit from a flexible approach, having a variety of different formats or templates for lesson structures (McCrea, 2019; Swan, 2005).

Mathematics can be seen as essentially about relationships (Gattegno, 1971), and seeing mathematical structure enables relationships to be noticed and appreciated. Within the mathematics curriculum as story framework this is conceptualized as exploring the interactions between different mathematical characters. Therefore, we consider it important to focus on helping students develop a sense of structure within mathematics (Hewitt, 2019). This means that the emphasis is on developing a sense of generality and seeing the particular through the general.

In our explanations of mathematical content, we will take full advantage of findings from cognitive load theory and multimedia learning (Mayer, 2019; Sweller et al., 2019), such as the modality/multimedia principle, that words and graphics are more effective when used together than either is when used alone, and the split attention principle, that keeping words and pictures in close spatial proximity aids learning (Kirschner & Hendrick, 2020). We will also make strategic, deliberate and consistent use of colour in text and diagrams to attempt to make mathematical structure visible, rather than for merely decorative purposes.

We see fluency as the availability of particular mathematical procedures and concepts when they are needed within mathematical tasks. There are two aspects to this. The first is the awareness that such a procedure or concept is relevant to the situation at hand. The second is the ability to carry out that procedure, or apply that particular concept, in an efficient and accurate way. The first aspect highlights for us the fact that awareness should be at the centre of the curriculum and its resources (Hewitt, 2009). It is of no use for students to know by rote certain procedures if they do not have the mathematical awareness of when such procedures are appropriate or inappropriate. For example, an awareness that ‘two negatives make a positive’ can be applied not only to correct situations, but also to inappropriate situations, such as \((-3) + (-4)\).

We see the knowledge within the mathematics curriculum and its resources as being divided into two categories: the arbitrary and the necessary (Hewitt, 1999). The arbitrary are those things where there is choice, and a decision has been made by the mathematical community. These are names and socially-agreed conventions, and they do need to be memorized, since there is no reason why they must be the way they are. The necessary are those things which have to be the way they are, and for which there can be justification and proof. We feel that it is important that students come to know necessary things through awareness of their interconnectivity with other known aspects of mathematics, rather than memorized as isolated facts. The arbitrary and necessary require different teaching approaches, and thus the curriculum and resource design needs to reflect this. What is appropriate to be introduced at any point will depend on the point in the mathematical story that has been reached.

As mentioned above, while facts, procedures, and concepts (which we collectively refer to as mathematical content) are fundamental to what mathematics is, this knowledge must not remain ‘inert’ (Howard & Hill, 2020). The purpose of knowing these things and having facility with them is to be able to operate mathematically in a variety of situations, including unanticipated ones; in
other words, to be able to tackle mathematical problems (see Foster, 2019). Consequently, the resources need to go beyond teaching imitative procedures, even when these are practised in contexts, to equip students to engage in authentic problem solving, in which students engage with problems that they do not initially know how to solve—what Halmos (1980) called the ‘heart of mathematics’. Within the mathematics curriculum as story framework, problem solving is one of the most dramatic and compelling narrative styles possible, motivating the actions of characters and driving the plot.

Problem solving will feature in the curriculum in three broad ways. Firstly, a problem-solving approach will be taken in some places to introduce new ideas of mathematical content. This could entail students exploring a new situation to identify patterns and form conjectures before these are formalized in a teaching sequence. In our story framework, this has commonalities with the literary device of foreshadowing, building anticipation for and hinting at events later in the story. Secondly, open-ended investigative tasks will be incorporated towards the end of each content unit, based on the content of that unit and preceding ones. Thirdly, some curriculum units will be entirely focused on developing students’ problem-solving abilities, actively teaching problem-solving heuristics and giving students experience tackling unfamiliar problems (Foster, 2019). The mathematical content for these units may draw largely on pre-secondary-school knowledge, following the rationale of the ‘many-year rule’: that if serious focus on mathematical processes is required, then, where the problem-solving demands are high, other (content) demands must be low. It is doubtful that students can engage productively in creative problem solving with content that they have only just met (Foster, 2019). This also has the advantage that students are thereby not cued into thinking that the mathematical content required to solve the problem must be the thing that they have been taught most recently, and hence they have the opportunity to practise selecting appropriate procedures and concepts.

In any set of curriculum resources, differentiation is inevitable and must be planned for (Ollerton, 2014), depending on students’ prior competencies and how these might be most effectively built on and developed (Wheeler, 2001). The approach to differentiation within content units will be to offer a common set of content for all students, with progression/selection through this to be judged by the school/teacher as appropriate for the students and the class. Some students will need more time and scaffolding (see Holton & Clarke, 2006) than others to achieve success. Every unit will begin with a common starting point, with some students accelerating faster onto more demanding and open-ended, low-threshold-high-ceiling tasks (see Kiddle, 2020), which have the potential to deepen their understanding (see Barton, 2020). The emphasis is on a strong conceptual understanding for all students, rather than any incentive to race through, getting answers to short, closed procedural questions. As Oates (2014, p. 12) characterized the attitude in Shanghai, ‘All children are assumed to be capable of understanding, and ideas are elaborated in different ways in order to encourage individual understanding’.

Every effort will be made to avoid unhelpful path-smoothing (Wigley, 1992), that makes things as easy as possible for the students in the short term but does not support their long-term growth. Within the mathematics curriculum as story framework, we see path-smoothing as destructive to the power of narrative, in short-circuiting dramatic arcs and reducing the story to short, trite episodes. Instead, we seek to challenge students appropriately and engage them in productive struggle (Kilpatrick et al., 2001) over longer story arcs. Breaking down a procedure into small steps can be extremely valuable, provided that those small steps can be reassembled again. Teacher expertise is needed to judge when it might be appropriate to atomize a procedure further (Francome, 2020). At other times, there is merit in deliberately offering a certain level of complexity, in order to develop students’ ability to abstract relevant parts to work on, so that what they learn is seen to be connected to a greater whole, and not just an isolated element. This ability to extract what is relevant from relative complexity is one of the powers of the mind (Hewitt, 2015) that all students possess and have used throughout their daily learning in a complex world.
Wherever possible, we will take advantage of whole-task sequencing (Van Merriënboer & Kirschner, 2017, p. 113), so that learners ‘quickly acquire a complete view of the whole skill that is gradually embellished during training’. Within the mathematics curriculum as story framework, this could correspond to a trailer or synopsis giving an overview of what is to come. In cases where the entire procedure would be too demanding to encounter all at once, we will use backward chaining (Van Merriënboer & Kirschner, 2017, p. 123), in which parts of a task are learned in the opposite order to the order in which they will eventually be performed (i.e. students begin by doing the last step). This resonates with the way in which pilots do not begin to learn to fly by learning how to take off (Sangwin, 2019)! Overlearning is needed for students to develop a very high level of automaticity (Van Merriënboer & Kirschner, 2017, p. 266)—what the Cockcroft Report (1982) described as an ‘at-homeness’ with a concept—and this can often be achieved in ways that avoid laborious drill and practice (see Foster, 2018; Francom, 2020; Francom & Hewitt, 2017).

6.3 Give explicit attention to important errors and misconceptions

Many compelling narratives turn on situational irony, in which there is a misapprehension about a character or situation. In the mathematics curriculum as story framework, we see important errors and misconceptions as opportunities to create plot twists that provoke and then resolve confusions about mathematical characters. While recognizing that it is natural that students will develop from their more informal and intuitive concept images towards more precise and formal concept definitions (Tall & Vinner, 1981), we nevertheless will emphasize mathematical correctness from the first time that an idea is met (Lockhart, 2009), so that concepts build up rigorously, without the need for ‘unlearning’ problematic interpretations (see McCourt, 2019). For example, we will avoid defining trigonometric functions as ratios within right-angled triangles, and instead build from a unit circle model, since this avoids embedding the idea that the angles must be less than 90° (Hewitt, 2007).

We will justify or prove the main results used in the text, or, where this is not possible, or might take too long, we will at least be explicit that a proof is required, but not yet available. In particular, we will eschew ‘backward-looking methods (that may deliver answers, but which hinder progression)’ (Gardiner, 2014, p. 4) in favour of ‘forward-looking methods (that may at first seem unnecessarily difficult, but which hold the key to future progression)’ (Gardiner, 2014, p. 4; see also McCourt, 2019, p. 115). For example, teaching percentage increase by finding the increase and adding it on does not allow students to reverse the process and find the original amount before a specified percentage change, whereas an approach based on scaling factors (such as multiplying by 1.2 for a 20% increase) is more powerful and forward-looking. We are mindful of the phenomenon of backward transfer in the learning of mathematics (Hohensee, 2016), where learning something new transforms a student’s understanding of something that they had previously learned, and we seek to plan for this always to be a supportive rather than a disruptive experience.

While believing that prevention is better than cure, we also recognize that many misconceptions are likely to be inevitable waypoints along the journey of learning mathematics (Hart et al., 1981; Smith Ill et al., 1994), and perhaps even desirable, if they allow pertinent discussions to be had that move the plot forwards. Rather than focusing on attempting to identify which particular students might show evidence of which particular misconceptions, in a ‘diagnose and treat’ model, we will instead largely adopt a ‘treat all’ approach, in which we will address the most common difficulties head-on with all students, such as through refutation texts (see Prinz et al., 2019). By raising common or important possible misconceptions as discussion points for all students, in ‘Discuss’ boxes, where they are often placed in the mouths of fictional students, or in fictitious designed student responses (Evans & Dawson, 2017), we will support all students in being aware of common misconceptions, and understanding why they are incorrect. We believe that it is desirable for all students to consider common malrules (incorrect rules leading to repeated similar errors) and misconceptions (incorrect application of rules) (Van Merriënboer & Kirschner, 2017, p. 219)—and to be able to explain why these things are incorrect. We think this is preferable to spending time diagnosing particular
misconceptions for particular students, since the remediation needed to follow these up at individual or small-group level may not be practicable in a typical classroom, and an error that a student does not seem inclined to make today might be one that they make on a future occasion. Indeed, even where a student might never make a certain error, there may still be considerable value in including discussion of it within the mathematical narrative, as it may provide a vehicle for deeper appreciation of the features of a mathematical character (teaching what it is not as well as what it is), corresponding to the character development in a novel that is achieved when a character is suspected of being one thing but turns out to be another.

### 6.4 Compare and contrast alternative methods

An important feature of narrative construction is variation of setting—we understand the features of a character by seeing how they operate in a variety of situations and comparing their behaviour. Variation theory draws attention to the importance of students comparing and contrasting similar examples of mathematical objects in order to appreciate mathematical structure through examining similarities and differences (Kullberg et al., 2017; Watson, 2017). One way to implement this in the curriculum is to engineer frequent opportunities to compare and contrast alternative solution methods to problems. In this, we draw on practices common within the style of the Japanese problem-solving lesson (Takahashi, 2006), and the rationale embodied in the statement attributed to George Pólya, that ‘it is better to solve one problem five different ways, than to solve five problems one way’. We will make use of contrasting cases (Durkin et al., 2017), such as two correct but alternative methods, or a correct method and an incorrect one. Where there are multiple methods to solve a problem, we will generally present them side by side and ask students to consider what is the same and what is different about them (Watson & Mason, 1998).

An effective way to generate powerful thinking about alternative methods can be through discussion and especially through tasks such as always-sometimes-never (see Swan, 2002). We particularly note the importance of students generating their own examples and non-examples as a fundamental feature of genuine mathematical activity (Watson & Mason, 2006). Gattegno had a strong position of always doing the inverse process at the same time as the direct process; for example, when multiplying by 6, the inverse of multiplying by one-sixth would also be worked on (Cuisenaire & Gattegno, 1959, p. 8). This has parallels with the approaches taken by Chinese curriculum resources, which deliberately build coherence through this kind of variation (see Sun, 2013). As Gardiner (2014, p. 4) stated, ‘Whilst fluency in the direct operation [e.g. addition, multiplying out brackets] is essential, its main purpose is to serve as a foundation for solving the harder, and more important inverse problems [subtraction, factorising]. In particular, resist the temptation to break harder inverse problems into manageable (direct) steps.’ As concepts become clarified through examining alternative processes, we see many opportunities for productive use of surprise (Movshovits-Hadar, 1988; ‘surprise and delight’ is one of the criteria for the International Society for Design and Development in Education prize 19), and supporting students’ satisfaction in learning mathematics (Lockhart, 2009). Within the mathematics curriculum as story framework (Dietiker, 2015), engineering moments of surprise within the narrative arc has the potential to generate interest and insight, since ‘enabling expectation allows both surprise if an expectation is violated and relief and satisfaction when an anticipated result . . . is met’ (p. 298).

A final important facet of this principle is comparing and contrasting commonly confused ideas, such as area and perimeter, and factors and multiples. There is a large body of evidence that interleaving helps students distinguish among similar concepts (e.g. Rohrer et al., 2020). This is important for practice tasks and is helpful for practising selecting appropriate strategies in ‘mixed exercises’. However, it is also important for learning the ‘arbitrary’ in mathematics (Hewitt, 1999).

So, we will draw on variation theory to interleave examples and non-examples (such as ‘triangle’ and ‘not triangle’), and we benefit from evidence from perceptual learning and instructional design regarding when and how to do this (Engelmann & Carnine, 1982). Addressing, for example,
area and perimeter together means that students learn to differentiate at the point of learning, rather than later on, when just one of these things is called on. Many narratives function by tracking diverging differences between two characters who start out in a similar situation but end up in dramatically different circumstances. Curriculum and resource design as story might exploit superficially similar or similar-sounding objects to create surprise by uncovering important differences between them.

6.5 Engineer coherence through strategic use of consistent representations and contexts

A satisfying story must cohere as a whole, and too often this aspect is overlooked within mathematics curriculum and resource design, with the narrative taking the form of a seemingly arbitrary ‘list of things students must be able to do’, rather than a meaningful story. In the UK, curricular coherence has recently become the major focus of the Office for Standards in Education, Children’s Services and Skills, the non-ministerial department of the UK government responsible for inspecting schools (Office for Standards in Education [Ofsted], 2019), placing it high on every school’s agenda (Howard & Hill, 2020).

We earlier used the definition of ‘story coherence’ as the extent to which the events and mathematical ideas of the mathematical story (i.e. a lesson) are connected to each other for a reader’ (Richman et al., 2019, p. 4). Understood this way, coherence means much more than superficial consistency (such as of choice of terminology and conventions). Instead, we see coherence as a vehicle for drawing attention to connections across the curriculum, such as between graphical and algebraic solutions to equations. We recognize that schools may wish to lift particular units out of our curriculum and use them within their own schemes of learning. However, we will encourage schools to use the units in the order in which they are presented, so as to preserve the coherence of the big-picture mathematical storyline (Newmann et al., 2001; Schmidt et al., 2005).

A particularly important aspect of curriculum coherence in mathematics would seem to be the stance adopted with regard to representations and models—the ‘setting’ aspect within the mathematics curriculum as story framework (Dietiker, 2015). We know that visual representations and physical and virtual manipulatives can play a critical role in students’ awareness of mathematical concepts (Carbonneau et al., 2013), and we wish to engineer the effective use of multiple representations (Ainsworth, 2006; Rau & Matthews, 2017). However, the representational dilemma (Watanabe, 2015) highlights the cost of using multiple representations, and hence we choose to restrict the total number of representations of number, privileging, in particular, number lines (Frykholm, 2010). This will be operationalized, for example, in the units on fractions, by an absence of circular models, such as cakes and pizza (see McCourt, 2019). Where fraction walls or bar models are used, this will always be adjacent to a number line (normally with zero marked). We do not have evidence that other representations of number are harmful, and we recognize that students are likely to have met a variety of representations previously in school, and it may be important to validate these experiences. However, we believe that one way to build coherence across the curriculum and its resources may be by stressing the idea of a number line, as it develops into a double number line, and subsequently into two perpendicular number lines intersecting at their zeroes (i.e. Cartesian axes). Our approach to representations may be characterized as ‘less is more’.

This raises the issue of to what extent coherence is desirable, and whether a carefully-controlled measure of incoherence may at times be helpful. It could be argued, for example, that learners may benefit from exposure to a wide variety of representations and models for mathematical actions, as, when learners meet problems, they appear in a variety of forms. However, as noted previously, we have a principle to focus on forward-looking methods. Chekhov argued that a gun ought not to be on stage unless it is used later in the play; in parallel to this, limited representations that are not used later should perhaps be removed from the story. It seems to us that the curricula resources currently in use are more problematic through their incoherence than through any excessive coherence, and so our emphasis will be on building in as much coherence as possible. Inevitably, incoherences will
remain, and during trialling we hope to explore the value of keeping or removing these on a case-by-case basis.

A final issue is that of situating mathematics within relatable real-life contexts (Van Den Heuvel-panhuizen, 2020)—another aspect of ‘setting’ within the mathematics curriculum as story framework (Dietiker, 2015). We will avoid the use of pseudo-real-life contexts masquerading as realistic, aware of their dangers, particularly for disadvantaged students (Cooper & Harries, 2002). We may indulge in occasional comically-absurd ‘real’ contexts, where it is made clear that everyday assumptions are not relevant (see Wiliam, 1997), and sometimes we will make serious use of contexts that are of real importance for students in their lives (Gutstein, 2006). In such cases, the intention will be for students to think about necessary simplifying assumptions and engage with the modelling cycle (Stillman et al., 2020).

7. Conclusion

The problem of sequencing and resourcing the mathematics curriculum is as yet unsolved, with no free, fully-resourced, coherent, research-informed school mathematics curriculum yet in existence. In setting out here five principles that could underlie the design of such a curriculum and its resources, we are conscious of the demanding nature of the task, and the impossibility of anyone ever producing a perfect set of curriculum resources—and we recognize that many possible stories could be told based on these principles. However, success in school mathematics acts as a gatekeeper qualification to students’ subsequent prospects, especially in STEM-related areas, and not achieving a mathematics qualification is associated with worse employment and economic opportunities and negative effects on health and wellbeing (Royal Society, 2016). We believe that a completely free, fully-resourced, coherent, research-informed school mathematics curriculum has the potential to raise student achievement, since the tasks that students undertake in the classroom are the basis for both their learning of mathematics (Sullivan et al., 2013) and their appreciation of what mathematics is.

A major challenge beyond the initial design will be the implementation. ‘Realizing a planned curriculum change is an unsolved problem in most school systems’ and requires ‘a well-engineered mixture of pressure and support’ (Burkhardt, 2014, p. 32). Our partnership with teachers and schools is essential as we seek to move forwards. As Schoenfeld (2014) comments:

Curricula are tools, to be used in the hands of teachers. Like any tools, their effectiveness depends on the preparation of those who will be using them. Those who construct curricula make assumptions about the people who will be using them, and construct the curricula accordingly. Thus, a particular curriculum may work well in certain contexts and be problematic in others. (p. 50)

By using the framework of the mathematics curriculum as story (Dietiker, 2015), we have set out design principles that we believe could achieve a step-change in mathematics curriculum and resource design. The rise of the Covid-19 pandemic during the preparation of this paper has only served to make more urgent this task, as schools, at the time of writing, face the likelihood of repeated periods of school closure and home learning. In such circumstances, a freely available, high-quality set of curriculum resources would seem to be of enormous potential value to schools, teachers, parents, and, ultimately, the students who stand to benefit from them.

Notes

1. Note that, in England, ‘academies’ (see https://www.gov.uk/types-of-school/academies) do not have to follow the national curriculum, although in practice almost all of them do.
2. This contrasts with many other jurisdictions which have mandatory textbooks and programmes of study, including many that perform to a high standard in large-scale international assessments, such as the Programme for International Student Assessment (PISA) and the Trends in International Mathematics and
Science Study (TiMSS). It seems to be the curricular coherence, rather than the national control, that contributes to the relative successes (Schmidt & Prawat, 2006).

3. Ideally, such trialling would be conducted in a systematic way, with independent observers working to a carefully-constructed protocol. In the absence of this, informal trials in the teacher’s own classroom are likely to be informative, but even doing this at the scale required for an entire set of curriculum resources is a daunting endeavour for any school. In practice, teachers may lack even the time to try out the mathematics of the tasks themselves before selecting them for inclusion.

4. https://www.tes.com/teaching-resources
5. In a US context, for English Language Arts, rather than mathematics, a review of over 300 of the most popular resources from three of the most visited supplemental curriculum websites found that expert reviewers rated the vast majority of the materials as ‘mediocre’ or ‘probably not worth using’ (Polikoff & Dean, 2019).
6. Some of the original materials have recently been made freely available here: http://generic.wordpress.soton.ac.uk/smp2/access-to-smp/
7. Materials are available here: https://www.stem.org.uk/cx5r4
8. Materials are available here: https://www.map.mathshell.org/
9. https://www.lboro.ac.uk/lumen/blinded
10. For example, as Microsoft Word documents, released under the CC BY 4.0 Creative Commons licence (see https://creativecommons.org/licenses/by-nc-sa/4.0/). This means that third-party users can use, reproduce and share the resources in whole or in part, provided they give appropriate credit to the author.
11. Although this could constitute a threat to the coherence of a well-designed curriculum, it seems an important feature of professional courtesy to give schools the option to do whatever they judge best.
12. https://researched.org.uk/
13. https://educationendowmentfoundation.org.uk/ and https://researchschool.org.uk/
14. Notable here is the important work of Cambridge Maths (https://www.cambridgemaths.org/) and the National Centre for Excellence in the Teaching of Mathematics (https://www.ncetm.org.uk/); in particular the Maths Hubs.
15. See also Teaching for Robust Understanding (TRU) https://www.map.mathshell.org/trumath.php
16. https://www.educationaldesigner.org/ed/index.htm
17. For example, the pioneering work of the Shell Centre at the University of Nottingham under Professor Malcolm Swan (https://www.mathshell.com/), the many publications of the Association of Teachers of Mathematics (http://www.atm.org.uk/), the beautiful tasks from NRICH (https://nrich.maths.org/), and the skill and creativity of many wonderful designers, including Don Steward (https://donsteward.blogspot.com/), Mike Ollerton (https://mikeollerton.com/publications-resources/ideas-for-classrooms/) and Dietmar Küchemann (http://www.mathsmed.co.uk/).
18. Anne Brown, in her work on design research, coined the term ‘lethal mutations’.
19. https://www.isdde.org/2020-prizes-call-for-nominations/

Acknowledgments

We would like to thank the editor and the reviewers for their extremely helpful comments on previous versions of this paper, which we believe have improved it considerably.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by the Economic and Social Research Council [grant number ES/S014292/1].

Notes on contributors

Colin Foster is a Reader in Mathematics Education in the Mathematics Education Centre at Loughborough University, UK. His research interests in mathematics education focus on the learning and teaching of mathematics in ways that support students’ conceptual understanding, and he is particularly interested in finding ways to enable students to develop the necessary fluency in mathematical processes to support them in solving mathematical problems.

Tom Francome is a Senior Fellow in the Centre for Mathematical Cognition at Loughborough University, UK, following working in mathematics education at the University of Birmingham and teaching in schools for many years. He is
interested in how mathematics education research connects to practice, particularly equitable approaches to teaching mathematics and developing expertise.

**Dave Hewitt** is Director of Mathematics PGCE at Loughborough University, UK. His research interests concern the principle of economic use of personal time and effort in the teaching and learning of mathematics, particularly with regard to algebra and the dynamic use of technology.

**Chris Shore** is a Senior Enterprise Fellow at the Mathematics Education Centre at Loughborough University, UK, and a secondary school mathematics teacher. He is interested in learners developing connected, conceptual understanding, the use of technology in the classroom, and the pedagogical ideas which teachers bring to their lessons.

**ORCID**

Colin Foster [http://orcid.org/0000-0003-1648-7485](http://orcid.org/0000-0003-1648-7485)  
Tom Francome [http://orcid.org/0000-0003-2334-0238](http://orcid.org/0000-0003-2334-0238)  
Dave Hewitt [http://orcid.org/0000-0002-1993-4749](http://orcid.org/0000-0002-1993-4749)  
Chris Shore [http://orcid.org/0000-0002-8953-5225](http://orcid.org/0000-0002-8953-5225)

**References**

Adey, P. S., & Shayer, M. (2002). Cognitive acceleration comes of age. In M. Shayer & P. S. Adey (Eds.), *Learning intelligence: Cognitive acceleration across the curriculum* (pp. 1–17). Open University Press.

Ainsworth, S. (2006). DeFT: A conceptual framework for considering learning with multiple representations. *Learning and Instruction*, 16(3), 183–198. [https://doi.org/10.1016/j.learninstruc.2006.03.001](https://doi.org/10.1016/j.learninstruc.2006.03.001).

Allen, B., & McInerney, L. (2019). *The recruitment gap: Attracting teachers to schools serving disadvantaged communities*. Sutton Trust.

Almond, N. (2020). *Curriculum coherence: How best to do it?* In C. Sealy (Ed.), *The ResearchED Guide to: The curriculum* (pp. 59–69). John Catt Educational.

Andrews, P. (2007). Conditions for learning: A footnote on pedagogy and didactics. *Mathematics Teaching*, 204, 22. [https://eva.fing.edu.uy/pluginfile.php/118226/mod_resource/content/1/PedagogyDidactics.pdf](https://eva.fing.edu.uy/pluginfile.php/118226/mod_resource/content/1/PedagogyDidactics.pdf).

Anthony, G., & Walshaw, M. (2009). *Effective pedagogy in mathematics* (Vol. 19). International Academy of Education.

Askew, M., Hodgen, J., Hossain, S., & Bretscher, N. (2010). *Values and variables: Mathematics education in high-performing countries*. Nuffield Foundation.

Ausubel, D. (1968). *Educational psychology*. Holt, Rinehart & Winston.

Bakker, A. (2018). *Design research in education*. Routledge. [https://doi.org/10.4324/9781315797397](https://doi.org/10.4324/9781315797397).

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: what makes it special?. *Journal of Teacher Education*, 59(5), 389–407. [https://doi.org/10.1177/0022487108324554](https://doi.org/10.1177/0022487108324554).

Barton, C. (2018). *How I wish I'd taught maths: Lessons learned from research, conversations with experts, and 12 years of mistakes*. John Catt Educational Ltd.

Barton, C. (2020). *Reflect, expect, check, explain: Sequences and behaviour to enable mathematical thinking in the classroom*. John Catt Educational Ltd.

Biesta, G., Priestley, M., & Robinson, S. (2015). The role of beliefs in teacher agency. *Teachers and Teaching*, 21(6), 624–640. [https://doi.org/10.1080/13540602.2015.1044325](https://doi.org/10.1080/13540602.2015.1044325).

Brown, M., & Hodgen, J. (2014). Curriculum, teachers and teaching: Experiences from systemic and local curriculum change in England. In Y. Li & G. Lappan (Eds.), *Mathematics curriculum in school education* (pp. 377–389). Springer. [https://doi.org/10.1007/978-94-007-7560-2_18](https://doi.org/10.1007/978-94-007-7560-2_18).

Brown, M., Askew, M., Baker, D., Denvir, H., & Millett, A. (1998). Is the national numeracy strategy research-based?. *British Journal of Educational Studies*, 46(4), 362–385. [https://doi.org/10.1111/1467-8527.00090](https://doi.org/10.1111/1467-8527.00090).

Brown, M., Brown, P., & Bibby, T. (2008). "I would rather die": reasons given by 16-year-olds for not continuing their study of mathematics. *Research in Mathematics Education*, 10(1), 3–18. [https://doi.org/10.1080/14794800801915814](https://doi.org/10.1080/14794800801915814).

Bruner, J. (1960). *The process of education*. Harvard University Press.

Burkhardt, H. (2006). From design research to large-scale impact: Engineering research in education. In J. Van Den Akker, K. Gravemeijer, S. McKenney, & N. Nieveen (Eds.), *Educational design research* (pp. 185–228). Routledge.

Burkhardt, H. (2014). Curriculum design and systemic change. In Y. Li & G. Lappan (Eds.), *Mathematics curriculum in school education* (pp. 13–34). Springer. [https://doi.org/10.1007/978-94-007-7560-2_2](https://doi.org/10.1007/978-94-007-7560-2_2).

Burkhardt, H., & Schoenfeld, A. H. (2003). Improving educational research: toward a more useful, more influential, and better-funded enterprise. *Educational Researcher*, 32(9), 3–14. [https://doi.org/10.3102/0013189X032009003](https://doi.org/10.3102/0013189X032009003).

Carbonneau, K. J., Marley, S. C., & Selig, J. P. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology*, 105(2), 380. [https://doi.org/10.1037/a0031084](https://doi.org/10.1037/a0031084).

Christodoulou, D. (2014). *Seven Myths About Education*. Routledge. [https://doi.org/10.4324/9781315797397](https://doi.org/10.4324/9781315797397).
Cockcroft Report. (1982). *Mathematics counts: Report of the committee of inquiry into the teaching of mathematics in schools under the chairmanship of Dr WH Cockcroft.* Her Majesty’s Stationery Office. [http://www.educationengland.org.uk/documents/cockcroft/cockcroft1982.html](http://www.educationengland.org.uk/documents/cockcroft/cockcroft1982.html)

Cooper, B., & Harries, T. (2002). Children’s responses to contrasting realistic mathematics problems: just how realistic are children ready to be?. *Educational Studies in Mathematics, 49*(1), 1–23. [https://doi.org/10.1023/A1016013332659](https://doi.org/10.1023/A1016013332659)

Cooper, J., Osher, S., & Yerushalmy, M. (2020). Didactic metadata informing teachers’ selection of learning resources: boundary crossing in professional development. *Journal of Mathematics Teacher Education, 23*(4), 363–384. [https://doi.org/10.1007/s10857-019-09428-1](https://doi.org/10.1007/s10857-019-09428-1)

Cuisenaire, G., & Gattegno, C. (1959). *Numbers in colour: A new method of teaching arithmetic in primary schools.* Heinemann.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: an organizing principle for mathematics curricula. *The Journal of Mathematical Behavior, 15*(4), 375–402. [https://doi.org/10.1016/S0732-3123(96)90023-1](https://doi.org/10.1016/S0732-3123(96)90023-1)

Department for Education (DfE). (2013). *Mathematics programmes of study: Key stage 3: National curriculum in England.* DfE.

Department for Education (DfE). (2016). *Eliminating unnecessary workload around planning and teaching resources: Report of the Independent Teacher Workload Review Group.* DfE. [https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/511257/Eliminating-unnecessary-workload-around-planning-and-teaching-resources.pdf](https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/511257/Eliminating-unnecessary-workload-around-planning-and-teaching-resources.pdf)

Department for Education (DfE). (2018). *Addressing teacher workload in initial teacher education (ITE): Advice for ITE providers.* DfE. [https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/753502/Addressing_Workload_in_ITE.pdf](https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/753502/Addressing_Workload_in_ITE.pdf)

Department for Education and Skills (DFES). (2001). *Key Stage 3 National Strategy – Framework for Teaching Mathematics: Years 7, 8 and 9.* DFES.

Didau, D., & Rose, N. (2016). *What every teacher needs to know about psychology.* John Catt Educational.

Dietiker, L. (2013). Mathematical texts as narrative: rethinking curriculum. *For the Learning of Mathematics, 33*(3), 14–19. [http://www.jstor.com/stable/43894855](http://www.jstor.com/stable/43894855)

Dietiker, L. (2015). Mathematical story: A metaphor for mathematics curriculum. *Educational Studies in Mathematics, 90*(3), 285–302. [https://doi.org/10.1007/s10649-015-9627-x](https://doi.org/10.1007/s10649-015-9627-x)

Dougherty, B. J., Bush, S. B., & Karp, K. S. (2017). Circumventing high school rules that expire. *Mathematics Teacher, 111*(2), 134–139. [https://doi.org/10.5951/mathteacher.111.2.0134](https://doi.org/10.5951/mathteacher.111.2.0134)

Durkin, K., Star, J. R., & Rittle-Johnson, B. (2017). Using comparison of multiple strategies in the mathematics classroom: Lessons learned and next steps. *ZDM Mathematics Education, 49*(4), 585–597. [https://doi.org/10.1007/s11858-017-0853-9](https://doi.org/10.1007/s11858-017-0853-9)

Engelmann, S., & Carnine, D. (1982). *Theory of instruction: Principles and applications.* Irvington Pub.

Evans, S., & Dawson, C. (2017). Orchestrating productive whole class discussions: the role of designed student responses. *Mathematics Teacher Education and Development, 19*(2), 159–179. [https://files.eric.ed.gov/fulltext/EJ1160838.pdf](https://files.eric.ed.gov/fulltext/EJ1160838.pdf)

Foster, C. (2018). Developing mathematical fluency: comparing exercises and rich tasks. *Educational Studies in Mathematics, 97*(2), 121–141. [https://doi.org/10.1007/s10649-017-9788-x](https://doi.org/10.1007/s10649-017-9788-x)

Foster, C. (2019). The fundamental problem with teaching problem solving. *Mathematics Teaching, 265*, 8–10. [https://www.foster77.co.uk/MT26503.pdf](https://www.foster77.co.uk/MT26503.pdf)

Foster, C. (2020). Stop planning lessons! *Teach Secondary, 9*(1), 80–81. [https://www.foster77.co.uk/Foster,%20Teach%20Secondary,%20Stop%20planning%20lessons.pdf](https://www.foster77.co.uk/Foster,%20Teach%20Secondary,%20Stop%20planning%20lessons.pdf)

Francombe, T. (2020). Practising purposefully: adding to your exercises by taking away. *Mathematics Teaching, 271*, 6–8. [https://www.atm.org.uk/Mathematics-Teaching-Journal-Archive/157182](https://www.atm.org.uk/Mathematics-Teaching-Journal-Archive/157182)

Francombe, T., & Hewitt, D. (2017). *Practising mathematics: Developing the mathematician as well as the mathematics.* Association of Teachers of Mathematics.

Frykholm, J. (2010). *Learning to think mathematically with the number line: A resource for teachers, a tool for young children.* Math Learning Center.

Gardiner, A. D. (2014). Teaching mathematics at secondary level. *The De Morgan Gazette, 6*(1), 1–215. [http://education.lms.ac.uk/wp-content/uploads/2014/07/DMG_6_no_1_2014.pdf](http://education.lms.ac.uk/wp-content/uploads/2014/07/DMG_6_no_1_2014.pdf)

Gattegno, C. (1971). *What we owe children. The subordination of teaching to learning.* Routledge and Kegan Paul Ltd.

Gilmore, C., Göbel, S. M., & Inglis, M. (2018). *An introduction to mathematical cognition.* Routledge. [https://doi.org/10.4324/9781315684758](https://doi.org/10.4324/9781315684758)

Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice.* Taylor & Francis.

Halmos, P. R. (1980). The heart of mathematics. *The American Mathematical Monthly, 87*(7), 519–524. [https://doi.org/10.1080/00029890.1980.11995081](https://doi.org/10.1080/00029890.1980.11995081)

Hart, K. M., Brown, M. L., Kuchemann, D. E., Kerslake, D., Ruddock, G., & McCartney, M. (1981). *Children’s understanding of mathematics: 11–16.* John Murray.

Hewitt, D. (1999). *Arbitrary and necessary part 1: A way of viewing the mathematics curriculum.* For the *Learning of Mathematics, 19*(3), 2–9. [https://www.jstor.org/stable/40248303](https://www.jstor.org/stable/40248303)
Hewitt, D. (2007). Canonical images. Mathematics Teaching Incorporating Micromath, 205, 6–11. https://www.atm.org.uk/write/MediaUploads/Journals/MT205/Non-Member/ATM-MT205-06-11.pdf

Hewitt, D. (2009). Towards a curriculum in terms of awareness. In S. Lerman & B. Davis Eds., Mathematical action & structures of noticing: Studies on John Mason’s contribution to mathematics education (pp. 89–100). Sense Publishers. https://doi.org/10.1163/9789460910319_008

Hewitt, D. (2015). The economic use of time and effort in the teaching and learning of mathematics. In S. Oesterle & D. Allan (Eds.), Proceedings of the 2014 annual meeting of the canadian mathematics education study group (pp. 3–23). Edmonton, Canada: University of Alberta.

Hewitt, D. (2019, “Never carry out any arithmetic”: the importance of structure in developing algebraic thinking. In U. T. Jankvist, M. Van Den Heuvel-panhuizen, & M. Veldhuis (Eds.), Proceedings of the eleventh congress of the european society for research in mathematics education (pp. 558–565). Utrecht, the Netherlands.

Hodgen, J., Foster, C., Marks, R., & Brown, M. (2018). Evidence for review of mathematics teaching: Improving mathematics in key stages two and three. Education Endowment Foundation.

Hohensee, C. (2016). Teachers’ awareness of the relationship between prior knowledge and new learning. Journal for Research in Mathematics Education, 47(1), 17–27. https://doi.org/10.5951/jresmatheduc47.1.0017

Holton, D., & Clarke, D. (2006). Scaffolding and metacognition. International Journal of Mathematical Education in Science and Technology, 37(2), 127–143. https://doi.org/10.1080/00220272.2009.9781103

Howard, K., & Hill, C. (2020). Symbiosis: The curriculum and the classroom. John Catt Educational Ltd.

Imants, J., & Van Der Wal, M. M. (2020). A model of teacher agency in professional development and school reform. Journal of Curriculum Studies, 52(1), 1–14. https://doi.org/10.1080/00220272.2019.1604809

Kiddle, A. (2020). An introduction to low threshold high ceiling tasks. Mathematics in School, 49(2), 2–3.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). Adding it up: Helping children learn mathematics. National Academies Press.

Kirschner, P. A., & Hendrick, C. (2020). How learning happens: Seminal works in educational psychology and what they mean in practice. Routledge. https://doi.org/10.4324/9780429061523

Kirschner, P. A., Verschaffel, L., Star, J., & Van Dooren, W. (2017). There is more variation within than across domains: an interview with Paul A. Kirschner about applying cognitive psychology-based instructional design principles in mathematics teaching and learning. ZDM Mathematics Education, 49(4), 637–643. https://doi.org/10.1007/s11858-017-0875-3

Kullberg, A., Kempe, U. R., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics?. ZDM Mathematics Education, 49(4), 559–569. https://doi.org/10.1007/s11858-017-0858-4

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: mathematical knowing and teaching. American Educational Research Journal, 27(1), 29–63. https://doi.org/10.3102/00028312027001029

Lemov, D., & Badillo, E. (2020). On writing a knowledge-driven English curriculum. In C. Sealy (Ed.), The ResearchED Guide to: The curriculum (pp. 85–94). John Catt Educational.

Leung, F. K. (2014). What can and should we learn from international studies of mathematics achievement?. Mathematics Education Research Journal, 26(3), 579–605. https://doi.org/10.1007/s13394-013-0109-0

Little, C. (1993). The school mathematics project: Some secondary school assessment initiatives in England. In M. Niss Ed., Cases of Assessment in Mathematics Education (pp. 85–97). Springer. https://doi.org/10.1007/978-94-017-0980-4_8

Lockhart, P. (2009). A mathematician’s lament: How school cheats us out of our most fascinating and imaginative art form. Bellevue Literary Press.

Ma, L. (2010). Knowing and teaching elementary mathematics: Teachers’ understanding of fundamental mathematics in China and the United States. Routledge. https://doi.org/10.4324/9780203856345

Mason, J. (2002). Researching your own practice: The discipline of noticing. Routledge. https://doi.org/10.4324/9780203471876

Mayer, R. E. (2019). How multimedia can improve learning and instruction. In J. Dunlosky & K. A. Rawson Eds., The Cambridge handbook of cognition and education (pp. 460–479). Cambridge University Press. https://doi.org/10.1017/9781082356319.019

McCourt, M. (2019). Teaching for mastery. John Catt Educational Ltd.

McCrea, E. (2019). Making every maths lesson count: Six principles to support great maths teaching. Crown House Publishing.

McKenney, S., & Reeves, T. (2018). Conducting educational design research. Routledge. https://doi.org/10.4324/9781315105642

Movshovits-Hadar, N. (1988). School mathematics theorems: an endless source of surprise. For the Learning of Mathematics, 8(3), 34–40. https://www.jstor.org/stable/40248150

Newmann, F. M., Smith, B., Allensworth, E., & Bryk, A. S. (2001). Instructional program coherence: what it is and why it should guide school improvement policy. Educational Evaluation and Policy Analysis, 23(4), 297–321. https://doi.org/10.3102/01623737023004297

Newmark, B. (2019). Why teach?. John Catt Educational Ltd.
Stillman, G. A., Kaiser, G., & Lampen, C. E. (2020). *Mathematical modelling education and sense-making*. Springer. [https://doi.org/10.1007/978-3-030-37673-4](https://doi.org/10.1007/978-3-030-37673-4).

Sullivan, P., Clarke, D., & Clarke, B. (2013). *Teaching with tasks for effective mathematics learning*. Springer. [https://doi.org/10.1007/978-1-4614-4681-1](https://doi.org/10.1007/978-1-4614-4681-1).

Sun, X. (2013). The structures, goals and pedagogies of “variation problems” in the topic of addition and subtraction of 0-9 in Chinese textbooks and reference books. *Eighth congress of european research in mathematics education (CERME 8)*, February 6-10, 2013, Antalya, Turkey.

Swan, M. (2002). Always, sometimes or never true?. *Mathematics Teaching*, 181, 32–33. [https://www.atm.org.uk/Mathematics-Teaching-Journal-Archive/3880](https://www.atm.org.uk/Mathematics-Teaching-Journal-Archive/3880).

Swan, M. (2005). Standards Unit – Improving learning in mathematics: Challenges and strategies. Department for Education and Science.

Swan, M., & Burkhardt, H. (2014). Lesson design for formative assessment. *Educational Designer*, 2(7). [https://www.educationaldesigner.org/ed/volume2/issue7/article24/](https://www.educationaldesigner.org/ed/volume2/issue7/article24/).

Sweller, J., Van Merriënboer, J., & Paas, F. (2019). Cognitive architecture and instructional design: 20 years later. *Educational Psychology Review*, 31(2), 261–292. [https://doi.org/10.1007/s10648-019-09465-5](https://doi.org/10.1007/s10648-019-09465-5).

Takahashi, A. (2006). Characteristics of Japanese mathematics lessons. *Tsukuba Journal of Educational Study in Mathematics*, 25(1), 37–44. [http://e-archives.cried.tsukuba.ac.jp/data/doc/pdf/2007/06/Akhiiko%20Takahashi.pdf](http://e-archives.cried.tsukuba.ac.jp/data/doc/pdf/2007/06/Akhiiko%20Takahashi.pdf).

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151–169. [https://doi.org/10.1007/BF00305619](https://doi.org/10.1007/BF00305619).

Taylor, M. W. (2013). Replacing the ‘teacher-proof’ curriculum with the ‘curriculum-proof’ teacher: toward more effective interactions with mathematics textbooks. *Journal of Curriculum Studies*, 45(3), 295–321. [https://doi.org/10.1080/00220272.2012.710253](https://doi.org/10.1080/00220272.2012.710253).

Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers’ development of coherent mathematical meanings. *Journal of Mathematics Teacher Education*, 10(4–6), 415–432. [https://doi.org/10.1007/s10857-007-9054-8](https://doi.org/10.1007/s10857-007-9054-8).

Van Den Heuvel-panhuizen, M. (Ed.). (2020). *International reflections on the Netherlands didactics of mathematics: Visions on and experiences with realistic mathematics education*. Springer Nature. [https://doi.org/10.1007/978-3-030-20223-1](https://doi.org/10.1007/978-3-030-20223-1).

Van Merriënboer, J. J., & Kirschner, P. A. (2017). Ten steps to complex learning: A systematic approach to four-component instructional design. Routledge. [https://doi.org/10.4324/9781315113210](https://doi.org/10.4324/9781315113210).

Van Steenbrugge, H., Valcke, M., & Desoete, A. (2013). Teachers’ views of mathematics textbook series in flanders: does it (not) matter which mathematics textbook series schools choose? *Journal of Curriculum Studies*, 45(3), 322–353. [https://doi.org/10.1080/00220272.2012.713995](https://doi.org/10.1080/00220272.2012.713995).

Wake, G., Swan, M., & Foster, C. (2016). Professional learning through the collaborative design of problem-solving lessons. *Journal of Mathematics Teacher Education*, 19(2–3), 243–260. [https://doi.org/10.1007/s10857-015-9332-9](https://doi.org/10.1007/s10857-015-9332-9).

Wake, G. C. (2018). A case study of theory-informed task design: What might we, as designers, learn? In L. J. Rodríguez-Muñiz, L. Muñiz-Rodríguez, A. Aguilar-González, P. Alonso, F. J. García, & A. Bruno (Eds.), *Investigación en Educación Matemática XXII* (pp. 94–109). Universidad de Oviedo.

Watanabe, T. (2015). Visual reasoning tools in action. *Mathematics Teaching in the Middle School*, 21(3), 152–160. [https://doi.org/10.5951/mattheamchildsco.21.3.0152](https://doi.org/10.5951/mattheamchildsco.21.3.0152).

Watson, A. (2017). Pedagogy of variations: Synthesis of various notions of variation pedagogy. In R. Huang & Y. Li (Eds), *Teaching and Learning Mathematics through Variation* (pp. 85–106). Rotterdam: Sense. [https://doi.org/10.1007/978-94-6300-782-5_5](https://doi.org/10.1007/978-94-6300-782-5_5).

Watson, A., De Geest, E., & Prestage, S. (2003). *Deep progress in mathematics*. University of Oxford.

Watson, A., & Mason, J. (1998). *Questions and prompts for mathematical thinking*. Association of Teachers of Mathematics.

Watson, A., & Mason, J. (2006). *Mathematics as a constructive activity: Learners generating examples*. Lawrence Erlbaum Associates. [https://doi.org/10.1007/BF02655890](https://doi.org/10.1007/BF02655890).

Watson, S. (2020). New right 2.0: teacher populism on social media in England. *British Educational Research Journal*, Advance online. [https://doi.org/10.1002/berj.3664](https://doi.org/10.1002/berj.3664).

Weinstein, Y., Sumeracki, M., & Caviglioli, O. (2018). *Understanding how we learn: A visual guide*. Routledge. [https://doi.org/10.4324/9780203710463](https://doi.org/10.4324/9780203710463).

Wheeler, D. (2001). A mathematics educator looks at mathematical abilities. *For the Learning of Mathematics*, 21(2), 4–12. [https://flm-journal.org/Articles/3A68B151356F711BC3624473C1020F.pdf](https://flm-journal.org/Articles/3A68B151356F711BC3624473C1020F.pdf).

Wigley, A. (1992). Models for teaching mathematics. *Mathematics Teaching*, 141(4), 7. [https://www.atm.org.uk/write/mediauploads/journals/mt141/non-member/atm-mt141-04-07.pdf](https://www.atm.org.uk/write/mediauploads/journals/mt141/non-member/atm-mt141-04-07.pdf).

Willam, D. (1997). Relevance as macGuffin in mathematics education. *Chreods*, 12, 8–19. [https://mr Bartonmaths.com/resources/new/8.%20Research/Real%20Life/Relevance%20as%20MacGuffin%20(BERA%2097).pdf](https://mr Bartonmaths.com/resources/new/8.%20Research/Real%20Life/Relevance%20as%20MacGuffin%20(BERA%2097).pdf).

Williams, D. T. (2010). Why don’t students like school? A cognitive scientist answers questions about how the mind works and what it means for the classroom. Jossey Bass. [https://doi.org/10.1002/9781118269527](https://doi.org/10.1002/9781118269527).

Wittmann, E. C. (1995). Mathematics education as a ‘design science’. *Educational Studies in Mathematics*, 29(4), 355–374. [https://doi.org/10.1007/BF01273911](https://doi.org/10.1007/BF01273911).