Rankin–Selberg local factors modulo-\(\ell\)

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Abstract

We associate Rankin–Selberg local \(L\)-factors and \(\gamma\)-factors to pairs of \(\ell\)-modular representations of Whittaker type, of general linear groups over locally compact non-archimedean local fields of residual characteristic different to \(\ell\). We compare the \(\ell\)-modular local factors defined by two reduction modulo-\(\ell\) maps: one which associates to certain pairs of \(\ell\)-adic representations local factors and then reduces these local factors modulo-\(\ell\), the other which reduces these \(\ell\)-adic representations modulo-\(\ell\) and then associates local factors to this pair of \(\ell\)-modular representations.

1 Introduction

Let \(F\) be a locally compact non-archimedean local field of residual characteristic \(p\) and residual cardinality \(q\), and let \(R\) be an algebraically closed field of characteristic \(\ell\) prime to \(p\). In this article, following Jacquet–Piatetski-Shapiro–Shalika in [7] for complex representations, we associate local Rankin–Selberg integrals to pairs of \(R\)-representations of Whittaker type \(\rho\) and \(\rho'\) of \(\text{GL}_n(F)\) and \(\text{GL}_m(F)\), and show that they define \(L\)-factors \(L(\rho,\rho',X)\) and satisfy a functional equation defining local \(\gamma\)-factors.

In particular, we define local factors for \(\ell\)-modular representations. The theory of \(\ell\)-modular representations of \(\text{GL}_n(F)\) was developed by Vignéras in [15], culminating in her \(\ell\)-modular local Langlands correspondence for \(\text{GL}_n(F)\), c.f. [18], which is characterised initially on supercuspidal \(\ell\)-modular representations by compatibility with the \(\ell\)-adic local Langlands correspondence. The possibility of characterising such a correspondence with \(\ell\)-modular invariants forms part of the motivation for this work. Indeed, already for \(\text{GL}_2(F)\) this is an interesting question, answered in this special case by Vignéras in [17].

We show that an \(L\)-factor attached to \(\ell\)-adic representations of Whittaker type is equal to the inverse of a polynomial with coefficients in \(\mathbb{Z}_\ell\), allowing us to define a natural reduction modulo-\(\ell\) map on the set of \(\ell\)-adic \(L\)-factors. Furthermore, for \(\ell\)-modular representations \(\pi\) and \(\pi'\) of Whittaker type of \(\text{GL}_n(F)\) and \(\text{GL}_m(F)\), there exist \(\ell\)-adic representations \(\tau\) and \(\tau'\) of Whittaker type of \(\text{GL}_n(F)\) and \(\text{GL}_m(F)\) which stabilise natural \(\mathbb{Z}_\ell\)-lattices \(\Lambda\) and \(\Lambda'\) in their respective spaces such that the \(\ell\)-modular representations induced by the actions of \(\tau\) and \(\tau'\)

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on $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{F}_\ell$ and $A' \otimes_{\mathbb{Z}_p} \mathbb{F}_\ell$ are isomorphic to $\pi$ and $\pi'$. Our first main result is a comparison between the $L$-factors and local $\gamma$-factors defined by these two reduction modulo-$\ell$ maps.

**Theorem.** 1. The $\ell$-modular $L$-factor $L(\pi, \pi', X)$ divides the reduction modulo-$\ell$ of the $\ell$-adic $L$-factor $L(\tau, \tau', X)$. Furthermore, the division of $L$-factors may not be an equality.

2. Let $\theta$ be an $\ell$-adic character of $F$. The local $\gamma$-factor associated to $L(\pi, \pi', X)$ and to the reduction modulo-$\ell$ of $\theta$ is equal to the reduction modulo-$\ell$ of the local $\gamma$-factor associated to $L(\tau, \tau', X)$ and $\theta$.

A particularly interesting case is the $L$-factor associated to a pair of irreducible cuspidal representations of $GL_n(F)$. For $R$-representations $\rho$ and $\rho'$ of $GL_n(F)$, we write $n(\rho, \rho')$ for the number of unramified characters $\chi$ of $GL_n(F)$ such that $\rho \simeq \chi \otimes (\rho')^{\vee}$. Let $\tau$ and $\tau'$ be integral cuspidal $\ell$-adic representations of $GL_n(F)$ and let $\pi$ and $\pi'$ denote their reductions modulo-$\ell$.

In our second main result we examine the $L$-factor $L(\pi, \pi', X)$.

**Theorem.** 1. If $q \equiv 1[\ell]$, then $L(\pi, \pi', X) = 1$.

2. If $n(\pi, \pi') = 0$, then $L(\pi, \pi', X) = 1$.

3. If $q^n \not\equiv 1[\ell]$, then $L(\pi, \pi', X) = r_\ell(L(\tau, \tau', X))$.

4. If $q^n(\pi, \pi') \equiv 1[\ell]$ and $n$ is prime to $\ell$, then $L(\pi, \pi', X) = 1$.

This work further develops the theory of $\ell$-modular local $L$-factors of Mínguez in [10]. In particular, we use his results on Tate $L$-factors modulo-$\ell$. Recently, Moss in [13] has studied $L$-factors attached to representations of $GL_n(F) \times GL_l(F)$ in a more general setting, and has given partial results concerning the $GL_n(F) \times GL_{n-1}(F)$ convolution in [14]. In a further investigation, we intend to study the local factors associated to generic segments, in terms of the local factors associated to cuspidal representations, as well as the inductivity relation satisfied by the local factors.

## 2 Preliminaries

Before embarking on the study of local $L$-factors in positive characteristic, we introduce the basic theory and background on representations of the general linear group. In particular, starting with results given in the standard reference [15], we show how integration behaves with respect to group decompositions. Indeed, this deserves checking as not all formulae follow from mimicking the proofs in the characteristic zero setting, due to the presence of compact open subgroups of measure zero. Additionally, we review the theory of $\ell$-adic and $\ell$-modular representations of Whittaker type and reduction modulo-$\ell$, drawing on results originally in [16], but our exposition will be influenced by the recent generalisation to inner forms of general linear groups in [11].

### 2.1 Notations

Let $F$ be a locally compact non-archimedean local field of residual characteristic $p$ with absolute value $| |$. Let $\mathfrak{o}$ denote the ring of integers in $F$, $p = \mathfrak{o}\mathfrak{o}$ the unique maximal ideal of $\mathfrak{o}$, and $q$ the cardinality of $k = \mathfrak{o}/p$.

Let $R$ be a commutative ring with identity of characteristic $\ell$ not equal to $p$. If $R$ contains a square root of $q$, we fix such a choice $q^{1/2}$. 

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Let $\mathcal{M}_{n,m} = \text{Mat}(n,m,F)$, $\mathcal{M}_n = \text{Mat}(n,n,F)$, $\eta$ be the row vector $(0,\ldots,0,1) \in \mathcal{M}_{1,n}$, $G_n = \text{GL}(n,F)$. We write $\nu$ for the character $| \cdot | \circ \text{det}$. Let $G^k_n = \{g \in G_n, |g| = q^{-k}\}$ (and more generally $X^k = X \cap G^k_n$ for $X \subset G_n$), $B_n$ the Borel subgroup of upper triangular matrices, $A_n$ the diagonal torus, $N_n$ the unipotent radical of $B_n$.

We fix a character $\theta$ from $(F,\cdot)$ to $R^\infty$ and, by abuse of notation, we will denote by $\theta$ the character $x \mapsto \theta(\sum_{i=1}^{n-1} n_{i,i+1})$ of $N_n$.

If $\lambda$ is a partition of $n$, $P_\lambda$ is the standard parabolic subgroup of $G_n$ attached to it, $M_\lambda$ the standard Levi factor of $P_\lambda$, and $N_\lambda$ its unipotent radical. If $t + r = n$, we let

$$U_{t,r} = \left\{ \begin{pmatrix} I_t & x \\ y & \end{pmatrix} : x \in M_{t,r}, y \in N_r \right\},$$

and $H_{t,r} = G_t U_{t,r}$. By restriction, $\theta$ defines a character of $U_{t,r}$. We let $P_n = H_{n-1,1}$ denote the mirabolic subgroup of $G_n$.

We denote by $w_n$ the antidiagonal matrix of $G_n$ with ones on the second diagonal, and if $n = r+t$, we denote by $w_{t,r}$ the matrix $\text{diag}(I_t, w_r)$. Notice that our notations are different from those of $\mathcal{S}$ for $U_{t,t}$, $H_{t,t}$, and $w_{t,t}$.

If $\phi \in C_c^\infty(F^n)$, we denote by $\widehat{\phi}$ its Fourier transform with respect to the $\theta$-self-dual $R$-Haar measure $dx$ on $F^n$ satisfying $dx(\phi) = q^{-s/2}$, where $s$ satisfies that $\theta |_{p^s}$ is trivial, but $\theta |_{p^{s-1}}$ is not.

For $G$ a locally profinite group, we let $\mathcal{R}_R(G)$ denote the abelian category of smooth $R$-representations of $G$. All $R$-representations henceforth considered are assumed to be smooth. For $\pi$ an $R$-representation with central character, for example an $R$-representation of $G_n$ parabolically induced from an irreducible $R$-representation, we denote its central character by $c_\pi$.

Let $\overline{\mathbb{Q}_\ell}$ be an algebraic closure of the $\ell$-adic numbers, $\overline{\mathbb{Z}_\ell}$ its ring of integers, and $\overline{\mathbb{F}_\ell}$ its residue field which is an algebraic closure of the finite field of $\ell$ elements. By an $\ell$-adic representation of $G$ we mean a representation of $G$ on a $\overline{\mathbb{Q}_\ell}$-vector space, and by an $\ell$-modular representation of $G$ we mean a representation of $G$ on a $\overline{\mathbb{F}_\ell}$-vector space. For $H$ a closed subgroup of $G$, we write $\text{Ind}_H^G$ for the functor of normalised smooth induction from $\mathcal{R}_R(H)$ to $\mathcal{R}_R(G)$, and write $\text{ind}_H^G$ for the functor of normalised smooth induction with compact support.

We assume that our choice of square roots of $q$ in $\overline{\mathbb{F}_\ell}$ and $\overline{\mathbb{Q}_\ell}$ are compatible; in the sense that the former is the reduction modulo-$\ell$ of the latter, which is chosen in $\mathbb{Z}_\ell$.

### 2.2 $R$-Haar Measures

Let $R$ be a commutative ring with identity of characteristic $\ell$ and let $G$ be a locally profinite group which admits a compact open subgroup of pro-order invertible in $R$. We let

$$C_c^\infty(G,R) = \{ f : G \to R : f \text{ is locally constant and compactly supported} \},$$

(we sometimes write this, more simply, as $C_c^\infty(G)$ according to the context). A left (resp. right) $R$-Haar measure on $G$ is a non-zero linear form on $C_c^\infty(G,R)$ which is invariant under left (resp. right) translation by $G$. If $\mu$ is a left (or right) $R$-Haar measure on $G$ and $f \in C_c^\infty(G,R)$, we write

$$\mu(f) = \int_G f(g) \mu.$$
By [15 I 2.4], for each compact open subgroup $K$ of $G$ of pro-order invertible in $R$ there exists a unique left $R$-Haar measure $\mu$ such that $\mu(K) = 1$. The volume $\mu(K') = \mu(1_{K'})$ of a compact open subgroup $K'$ of $G$ is equal to zero if and only if the pro-order of $K'$ is equal to zero in $R$. In the present work, the modulus character of $G$ is the unique character $\delta_G : G \rightarrow R^\times$ such that, if $\mu$ is a left $R$-Haar measure on $G$, $\delta_G \mu$ is a right $R$-Haar measure on $G$. More generally, if $H$ is a closed subgroup of $G$, we let $\delta = \delta_G^{-1} |_H \delta_H$, and

$$C^\infty_c(H \setminus G, \delta, R)$$

be the space of functions from $G$ to $R$, fixed on the right by a compact open subgroup of $G$, compactly supported modulo $H$, and which transform by $\delta$ under $H$ on the left (we sometimes write this as $C^\infty_c(H \setminus G, \delta)$). For $f \in C^\infty_c(G, R)$, we denote by $f^H$ the function in $C^\infty_c(H \setminus G, \delta, R)$ defined by

$$f^H(g) = \int_H f(h)\delta^{-1}(h)dh,$$

for $dh$ a right $R$-Haar measure on $H$. It is proved in [15 I 2.8] that the map $f \mapsto f^H$ is surjective, and that there is a unique, up to an invertible scalar, non-zero linear form $d_{H \setminus G}g$ on $C^\infty_c(H \setminus G, \delta, R)$, which is right invariant under $G$. We call such a non-zero linear form on $C^\infty_c(H \setminus G, \delta, R)$ a $\delta$-quasi-invariant quotient measure on $H \setminus G$ and, for $f \in C^\infty_c(H \setminus G, \delta, R)$, we write

$$d_{H \setminus G}g(f) = \int_{H \setminus G} f(g)d_{H \setminus G}g.$$

With the correct normalisation, for all $f \in C^\infty_c(G, R)$, we have

$$\int_G f(g)dg = \int_{H \setminus G} f^H(g)d_{H \setminus G}g.$$

**Remark 2.1.** Let $H$ be a closed subgroup of $G$. Let $C^\infty_c(\hat{H})$ denote the subspace of functions in $C^\infty_c(\hat{H}, \mathbb{Q}_l)$ which take integral values. Up to a correct normalisation of the $\mathbb{Q}_l$-Haar measure $dh$ on $H$, $\int_H f(h)dh$ belongs to $\mathbb{Q}_l$. Suppose $K$ is a closed subgroup of $H$, for which there is a $\delta$-quasi-invariant quotient measure $d_{K \setminus H}h$ on $K \setminus H$. We write $C^\infty_c(\hat{K}, \delta)$ for the subspace of functions in $C^\infty_c(\hat{K}, \delta)$ which take integral values. Similarly, up to correct normalisation of the quotient measure, the value of $\int_{K \setminus H} f(h)d_{K \setminus H}h$ belongs to $\mathbb{Z}_l$ when $f \in C^\infty_c(\hat{K}, \delta)$. Moreover, for all $f$ in $C^\infty_c(\hat{K}, \delta)$, we have

$$r_{\ell} \left( \int_{K \setminus H} f(h)d_{K \setminus H}h \right) = \int_{K \setminus H} r_{\ell}(f(h))d_{K \setminus H}h.$$

We write $C^\infty_c(F^m)$ for the subspace of functions in $C^\infty_c(F^m)$ which take values in $\mathbb{Z}_l$. Note that if $\phi \in C^\infty_c(F^m)$, then $\hat{\phi} \in C^\infty_c(F^m)$.

For the remainder of this section, let $G$ denote a unimodular locally profinite group. Suppose that $B$ is a closed subgroup of $G$, $K$ is a compact open subgroup of $G$ such that $G = BK$, and $K_1$ is a normal compact open subgroup of $K$ with pro-order prime to $\ell$.

**Lemma 2.2.** Let $dg$ be an $R$-Haar measure on $G$. There exist a right $R$-Haar measure $db$ on $B$ and a right $K$-invariant measure $dk$ on $K \cap B \setminus K$ such that, for all $f \in C^\infty_c(G, R)$, we have

$$\int_G f(g)dg = \int_{K \cap B \setminus K} \int_B f(bk)\delta_B(b)^{-1}dbdk.$$
Proof. We observe first that the map $\phi \mapsto \phi |_K$ is a vector space isomorphism between $C_c^\infty(B \setminus G, \delta_B, R)$ and $C_c^\infty((K \cap B) \setminus K, R)$. It is injective because $G = BK$. To show surjectivity, we recall that the characteristic functions $1_{(K \cap B) \setminus U}$, with $U$ a compact subgroup of $K$ of pro-order invertible in $R$ and $k \in K$, span $C_c^\infty((K \cap B) \setminus K, R)$. Moreover, $1_{kU}^B$ belongs to $C_c^\infty(B \setminus G, \delta_B, R)$, and a computation shows $1_{kU}^B |_K = db(B \cap kUk^{-1})1_{(K \cap B) \setminus U}$. Surjectivity follows as $db(B \cap kUk^{-1})$ is invertible in $R$. In particular, if $dk$ is a right $K$-invariant measure on $C_c^\infty((K \cap B) \setminus K, R)$, the map $\mu : \phi \mapsto dk(\phi |_K)$ is a right $G$-invariant measure on $B \setminus G$. The result follows from the formula $\int_G f(g)dg = \mu(f^B)$.

Remark 2.3. Let $K_n = \text{GL}_n(\mathfrak{o}_F)$. By the Iwasawa decomposition, we have $G_n = B_nK_n$. Let $\mu_{G_n}$ be an $R$-Haar measure on $G_n$. If $\ell = 0$, or more generally $\ell | q - 1$, then, for all $f \in C_c^\infty(G_n, R)$, we have

$$\int_{G_n} f(g)dg = \int_{B_n} \int_{K_n} f(bk)dbdk,$$

for good choices of a left $R$-Haar measure $db$ on $B_n$ and an $R$-Haar measure $dk$ on $K_n$. As noticed by Minguez in [10], this is no longer true in general. More precisely, it is not true when $\ell | q - 1$ as the restriction of an $R$-Haar measure on $K$ to $C_c^\infty((K_n \cap B_n) \setminus K_n)$ is zero. That is why we use a right invariant measure on $K \cap B \setminus K$ in Lemma 2.2.

Let $K$ be a compact group, $K_1$ an open subgroup of $K$ of pro-order invertible in $R$ and $P$ be a closed subgroup of $K$.

Lemma 2.4. Let $\mu$ be a right $K$-invariant measure on $P \setminus K$. Then $\mu(P \setminus K) = \mu(1_K)$ is zero if and only if the integer $|P \setminus K/K_1|$ is zero in $R$.

Proof. To prove this, we introduce the $R$-Haar measure $\lambda : f \mapsto \mu(f^P)$ on $K$. By computation, as in the proof of Lemma 2.2, we have

$$1^P_{sK_1} = dp(P \cap sK_1s^{-1})1_{PsK_1},$$

for $s$ in $K$. In particular, with $t = 1/dp(P \cap K_1)$, we have

$$1_K = \sum_{s \in P \setminus K/K_1} 1_{PsK_1} = t \sum_{s \in P \setminus K/K_1} 1^P_{sK_1}|_K.$$

This implies

$$\mu(1_K) = t \sum_{s \in P \setminus K/K_1} \mu(1^P_{sK_1}) = t \sum_{s \in K \setminus K/K_1} \lambda(sK_1) = t' |P \setminus K/K_1|,$$

where $t' = t\lambda(K_1) \in R^*$. And the result follows.

Remark 2.5. An example of this we need is when $K = K_n$, $K_1 = K_{n,1}$ the pro-$p$ unipotent radical of $K_n$, $P = P_n \cap K_n$, and $R$ is of positive characteristic $\ell$ not equal to $p$. In this case, $P \setminus K/K_1 \simeq k^n - \{0\}$ so that $\mu(P \setminus K) = 0$ if and only if $q^n \equiv 1[\ell]$.

Let $Q$, $L$ and $N$ be closed subgroups of $G$ such that $Q = LN$ and $L$ normalises $N$. Suppose that there exists a compact open subgroup $K_1$ of $G$ of pro-order invertible in $R$ such that $Q \cap K_1 = (N \cap K_1)(M \cap K_1)$. Let $dl$ be an $R$-Haar measure on $L$ and $dn$ be an $R$-Haar measure on $N$.

Lemma 2.6. Let $f \in C_c^\infty(Q, R)$. There exists a unique right $R$-Haar measure $dq$ on $Q$ such that

$$\int_Q f(q)dq = \int_L \int_N f(nl)dldn.$$
From Iwasawa decomposition, we also have \( G = G_n, Q = LN \) is a standard parabolic subgroup of \( G_n \), and \( K_1 = K_{n,1} \).

We have the following corollary to Lemmata 2.2 and 2.6.

**Corollary 2.8.** Let \( dg \) be an \( R \)-Haar measure on \( G \). There exist an \( R \)-Haar measure \( da \) on \( A_n \), and a right \( K_n \)-invariant measure \( dk \) on \( (K_n \cap B_n) \backslash K_n \) such that, for all \( f \in C_c^\infty(N_n \backslash G_n, R) \), we have

\[
\int_{N_n \backslash G_n} f(g)dg = \int_{(K_n \cap B_n) \backslash K_n} \int_{A_n} f(ak)\delta_{B_n}(a)^{-1}dak.
\]

**Proof.** Let \( f \in C_c^\infty(N_n \backslash G_n, R) \), then \( f = h^{N_n} \) for \( h \in C_c^\infty(G_n, R) \). By Lemmata 2.2 and 2.6,

\[
\int_{G_n} h(g)dg = \int_{K_n \cap B_n \backslash K_n} \int_{A_n} \int_{N_n} h(nak)\delta_{B_n}(a)^{-1}dn dak,
\]

However,

\[
\int_{G_n} h(g)dg = \int_{N_n \backslash G_n} f(g)dg
\]

and

\[
\int_{N_n} h(nak)dn = h^{N_n}(ak) = f(ak).
\]

From Iwasawa decomposition, we also have \( G_n = P_nZ_nK_n \). We use the following integration formula, which is proved in a similar fashion.

**Corollary 2.9.** Let \( dg \) be an \( R \)-Haar measure on \( G \). There exist an \( R \)-Haar measure \( dz \) on \( Z_n \), a \( \delta \)-quasi-invariant quotient measure \( dp \) on \( N_n \backslash P_n \), and a right \( K_n \)-invariant measure \( dk \) on \( (K_n \cap B_n) \backslash K_n \) such that, for all \( f \in C_c^\infty(N_n \backslash G_n, R) \),

\[
\int_{N_n \backslash G_n} f(g)dg = \int_{(K_n \cap P_n) \backslash K_n} \int_{Z_n} \int_{N_n \backslash P_n} f(pzk)\det(p)^{-1}dp dz dk.
\]

Henceforth, equalities involving integrals will be true only up to the correct normalisation of measures.

### 2.3 Derivatives

Henceforth, we suppose that \( R \) is an algebraically closed field. Following [2], we define the following exact functors:

1. \( \Psi^+ : \mathfrak{R}_R(G_{n-1}) \rightarrow \mathfrak{R}_R(P_n) \), extension by the trivial representation twisted by \( \nu^\frac{1}{n} \).
2. \( \Psi^− : \mathfrak{R}_R(P_n) \rightarrow \mathfrak{R}_R(G_{n-1}) \), the functor of \( U_{m-1}(F) \)-coinvariants twisted by \( \nu^{-\frac{1}{n}} \).
3. \( \Phi^+ : \mathfrak{R}_R(P_{n-1}) \rightarrow \mathfrak{R}_R(P_n) \), the functor \( \Phi^+(X) = \text{ind}^{P_n}_{P_{n-1}}U_{m}(X \otimes \theta) \).
4. $\Phi^{-} : \mathcal{R}_{R}(P_n) \to \mathcal{R}_{R}(P_{n-1})$, the functor of $(U_{m-1}, \psi)$-coinvariants twisted by $\nu^{-\frac{1}{2}}$.

5. $\Phi_{nc}^{+} : \mathcal{R}_{R}(P_{n-1}) \to \mathcal{R}_{R}(P_n)$, the functor $\Phi_{nc}^{+}(X) = \text{Ind}_{P_{n-1}U_n}^{P_n(F)}(X \otimes \theta)$.

**Theorem 2.10** ([1] & [2] (c.f. [15] III 1.3)).

1. We have $\Psi^{-}\Phi^{+} = \Psi^{-}\Phi_{nc}^{+} = \Phi^{-}\Psi^{+} = 0$, and $\Psi^{-}$, resp. $\Phi^{+}$, resp. $\Phi^{-}$, is left adjoint to $\Psi^{+}$, resp. $\Phi^{-}$, resp. $\Phi_{nc}^{+}$.

2. The identity functor $1 : \mathcal{R}_{R}(P_n) \to \mathcal{R}_{R}(P_n)$ admits a filtration

$$0 = T_n \subset T_{n-1} \subset \cdots \subset T_0 = 1$$

such that $T_k = (\Phi^{+})^{k}(\Phi^{-})^{k}$ and $T_{k-1}/T_k = (\Phi^{+})^{k-1}\Psi^{+}\Psi^{-}(\Phi^{-})^{k-1}$.

Let $\tau$ be an $R$-representation of $P_n$. Let $\tau_{(k)} = (\Phi^{-})^{k-1}(\tau)$. The $k$-th derivative $\tau^{(k)}$ of $\tau$ is defined by $\tau^{(k)} = \Psi^{-}(\tau_{(k)})$. We recall the classification of irreducible $R$-representations of $P_n$.

**Lemma 2.11** ([1] (c.f. [15] III 1.5)). Let $\pi$ be an irreducible $R$-representation of $P_n$. There exists a unique non-zero derivative of $\tau$. Furthermore, for $k$ in $\{1, \ldots, n\}$, if $\tau^{(k)} \neq 0$ then $\pi = (\Phi^{+})^{k}\Psi^{+}(\tau^{(k)})$. Conversely, if $\rho$ is an irreducible $R$-representation of $G_{n-k}$, then $\pi = (\Phi^{+})^{k}\Psi^{+}(\rho)$ is irreducible and $\rho$ is the $k$-th derivative of $\pi$.

Let $\pi$ be an $R$-representation of $G_n$. The zeroth derivative $\pi^{(0)}$ of $\pi$ is $\pi$. Let $\tau = \pi|_{P_n}$ and set $\pi_{(k)} = \tau_{(k)}$ for $k = 0, 1, \ldots, n$. Define the $k$-th derivative $\pi^{(k)}$ of $\pi$ by $\pi^{(k)} = \tau^{(k)}$ for $k = 1, \ldots, n$.

**Lemma 2.12** ([1]). Let $\pi$ be an $R$-representation of finite length. Then the dimension of $\pi^{(n)}$ is finite and equal to the dimension of $\text{Hom}_{N_n}(\pi, \theta)$.

The derivatives of a product are given by the Leibniz rule.

**Lemma 2.13** ([2] (c.f. [15] III 1.10)). Suppose $\pi$ is an $R$-representation of $G_n$ and $\rho$ is an $R$-representation of $G_m$, then $(\pi \times \rho)^{(k)}$ has a filtration with successive quotients $\pi^{(i)} \times \rho^{(k-i)}$, for $0 \leq i \leq k$.

Finally, we will use several times the following proposition.

**Proposition 2.14** ([2] Proposition 3.7). Let $\rho$ and $\rho'$ be $R$-representations of $G_n$ and $\tau$ and $\tau'$ be $R$-representations of $P_m$, we have

1. $\text{Hom}_{P_{n+1}}(\Phi^{+}(\rho) \otimes \Phi^{+}(\rho'), R) = \text{Hom}_{G_n}(\rho \otimes \rho', R)$;
2. $\text{Hom}_{G_n}(\Phi^{+}(\tau) \otimes \Phi^{+}(\tau'), R) = \text{Hom}_{P_n}(\tau \otimes \tau', R)$;
3. $\text{Hom}_{G_n}(\Psi^{+}(\rho) \otimes \Phi^{+}(\tau), R) = \{0\}$.

### 2.4 Parabolic induction, integral structures and reduction modulo-$\ell$

Let $Q$ be a parabolic subgroup of $G_n$ with Levi factor $L$. We write $i_{Q}^{G_n}$ for the functor of normalised parabolic induction from $\mathcal{R}_{R}(L)$ to $\mathcal{R}_{R}(G_n)$. If $\tau = \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_r$ is a smooth $R$-representation of $L_{(m_1, \ldots, m_r)}$ with $\sum_i m_i = n$, we will use the product notation $\pi_1 \times \pi_2 \times \cdots \times \pi_r$,
for the induced representation $i_{F}^{G_{n}}_{(m_{1},...,m_{r})}(\tau)$. An $R$-representation of $G_{n}$ is called 	extit{cuspidal} if it is irreducible and it does not appear as a subrepresentation of any parabolically induced representation.

An $\ell$-adic representation $(\pi, V)$ of $G$ is called 	extit{integral} if it has finite length, and if $V$ contains a $G$-stable $\mathbb{Z}_{\ell}$-lattice $\Lambda$. Such a lattice is called an integral structure in $\pi$. A character is integral if and only if it takes values in $\mathbb{Z}_{\ell}$. By [15, II 4.12], a cuspidal representation is integral if and only if its central character is integral.

If $\pi$ is an integral $\ell$-adic representation with integral structure $\Lambda$, then $\pi$ defines an $\ell$-modular representation on the space $\Lambda \otimes_{\mathbb{Z}_{\ell}} \mathbb{F}_{\ell}$. By the Brauer-Nesbit principle [19 Theorem 1], the semisimplification, in the Grothendieck group of finite length $\ell$-modular representations, of $(\pi, \Lambda \otimes_{\mathbb{Z}_{\ell}} \mathbb{F}_{\ell})$ is independent of the choice of integral structure in $\pi$ and we call this semisimple representation $r_{\ell}(\pi)$ the reduction modulo-$\ell$ of $\pi$. We say that an $\ell$-modular representation $\pi$ lifts to an integral $\ell$-adic representation $\pi$ if $r_{\ell}(\pi) \simeq \pi$, we will only really use this notion of lift when $\pi$ is irreducible.

Let $H$ be a closed subgroup of $G$, $\sigma$ be an integral $\ell$-adic representation of $H$, and $\Lambda$ be an integral structure in $\sigma$. By [15 I 9.3], $\text{ind}_{H}^{G}(\Lambda)$ is a lattice in $\text{ind}_{G}^{G}(\sigma)$. Suppose $Q$ is a parabolic subgroup of $G$ with Levi decomposition $Q = LN$, $\sigma$ is an integral representation of $L$, $\Lambda$ is an integral structure in $\sigma$. Then $i_{Q}^{G}(\Lambda)$ is an integral structure in $i_{Q}^{G}(\sigma)$. Moreover, we have $i_{Q}^{G}(\Lambda \otimes_{\mathbb{Z}_{\ell}} \mathbb{F}_{\ell}) \simeq i_{Q}^{G}(\Lambda) \otimes_{\mathbb{Z}_{\ell}} \mathbb{F}_{\ell}$.

2.5 Representations of Whittaker type

Before defining representations of Whittaker type we recall that the irreducible $R$-representations of $G_{n}$ satisfying $\text{Hom}_{N_{n}}(\pi, \theta) \neq 0$ are called 	extit{generic}.

We now recall facts from [11 Section 7]. A segment $\Delta = [a, b]_{\rho}$ is a sequence $(\nu^{a}\rho, \nu^{a+1}\rho, \cdots, \nu^{b}\rho)$ with $\rho$ a cuspidal $R$-representation of $G_{m}$ for some $m \geq 1$, and $a, b \in \mathbb{Z}$ with $a \leq b$. Its length is, by definition, $b - a + 1$. Two segments $\Delta = [a, b]_{\rho}$ and $\Delta' = [a', b']_{\rho}$ are said to be equivalent if they have the same length, and $\nu^{a}\rho \simeq \nu^{a'}\rho'$. Hence, as noticed in [11 7.2], the segment $[a, b]_{\rho}$ identifies with the cuspidal pair

$$r_{\Delta} = (M_{(m,...,m)}, \nu^{a}\rho \otimes \nu^{a+1}\rho \otimes \cdots \otimes \nu^{b}\rho),$$

and the equivalence relation on segments is the restriction of the classical isomorphism equivalence relation on cuspidal pairs. To such a segment $\Delta$, in [11 Definition 7.5] the authors associate a certain quotient $L(\Delta)$ of $\nu^{a}\rho \times \nu^{a+1}\rho \times \cdots \times \nu^{b}\rho$. The representation $L(\Delta)$ in fact determines $\Delta$, as its normalised Jacquet module with respect to the opposite of $N_{(m,...,m)}$ is equal to $r_{\Delta} = (M_{(m,...,m)}, \nu^{a}\rho \otimes \nu^{a+1}\rho \otimes \cdots \otimes \nu^{b}\rho)$ according to [11 Lemma 7.14]. The conclusion of this is that the objects $\Delta$, $L(\Delta)$, and $r_{\Delta}$ determine one another, and hence we call $L(\Delta)$ a segment and, by abuse of notation, we write $\Delta$ for $L(\Delta)$, in order to lighten notations. We say that $\Delta$ precedes $\Delta'$ if we can extract from the sequence $(\nu^{a}\rho, \ldots, \nu^{b}\rho, \nu^{a'}\rho', \ldots, \nu^{b'}\rho')$ a subsequence which is a segment of length strictly larger than both the length of $\Delta$ and the length of $\Delta'$. We say $\Delta$ and $\Delta'$ are linked if $\Delta$ precedes $\Delta'$ or $\Delta'$ precedes $\Delta$.

Let $\rho$ be a cuspidal $R$-representation of $G_{m}$. In [11], a positive integer $e(\rho)$ is attached to $\rho$,

$$e(\rho) = \begin{cases} +\infty & \text{if } R = \overline{Q}_{\ell}; \\ |\{\nu^{r}\rho, r \in \mathbb{Z}\}| & \text{if } R = \overline{F}_{\ell} \text{ and } |\{\nu^{r}\rho, r \in \mathbb{Z}\}| > 1; \\ \frac{\ell}{\ell} & \text{if } R = \overline{F}_{\ell} \text{ and } |\{\nu^{r}\rho, r \in \mathbb{Z}\}| = 1. \end{cases}$$
For $k \in \mathbb{N}$, we denote by $\text{St}(k, \rho)$ the normalised generalised Steinberg representation associated with $\rho$, i.e. the unique generic subquotient of $\nu^{(1-k)/2}\rho \times \cdots \times \nu^{(k-1)/2}\rho$. By [11] Remarque 8.14, the representation $\text{St}(k, \rho)$ is equal to the segment $\Delta$ associated to the sequence $[(1-k)/2, (k-1)/2]$ if and only if $k < e(\rho)$. In this case, we say that $\Delta = \text{St}(k, \rho)$ is a generic segment. Note that, our notation $\text{St}(k, \rho)$ differs from that used in [11] and [16] (more precisely, our $\text{St}(k, \rho)$ corresponds to $\text{St}(k, \nu^{(1-k)/2}\rho)$ in those references).

By [15] III 5.10, if $\rho$ is a cuspidal $\ell$-modular representation of $G_r$, then it is the reduction modulo-$\ell$ of an integral cuspidal $\ell$-adic representation $\sigma$ of $G_r$. Let $\Delta = \text{St}(k, \rho)$ be a generic segment, then $\Delta$ is the reduction modulo-$\ell$ of $D = \text{St}(k, \sigma)$ ([16] V.7.]).

**Definition 2.15.** A representation $\pi$ of $G_n$ is called of Whittaker type if it is parabolically induced from generic segments, i.e. if $\pi = \Delta_1 \times \cdots \times \Delta_t$ with $\Delta_i$, for $1 \leq i \leq t$, generic segments.

We will study $L$-factors of representations of Whittaker type.

Let $\pi$ be a representation of Whittaker type. By Lemmata 2.12 and 2.13, the space $\text{Hom}_{\mathcal{N}_n}(\pi, \theta)$ is of dimension 1, and we denote by $W(\pi, \theta)$ the Whittaker model of $\pi$, i.e. $W(\pi, \theta)$ denotes the image of $\pi$ in $\text{Ind}_{\mathcal{N}_n}^{G_n}(\theta)$. Note that, a representation of Whittaker type may not be irreducible, however, it is of finite length. In fact, thanks to the results of Zelevinsky (c.f. [20]) in the $\ell$-adic setting, and by [16] Theorem 5.7 (a more detailed proof of which can be found in [11] Theorem 9.10 and Corollary 9.12) in the $\ell$-modular setting, the irreducible representations of Whittaker type of $G_n$ are exactly the generic representations.

**Remark 2.16.** According to [16] Theorem V.7, if $\pi = \Delta_1 \times \cdots \times \Delta_t$ is a representation of Whittaker type of $G_n$, then $\pi$ is irreducible if and only if the segments $\Delta_i$ and $\Delta_j$ are unlinked, for all $i, j \in \{1, \ldots t\}$ with $i \neq j$.

If $\pi$ is a smooth representation of $G_n$, we denote by $\tilde{\pi}$ the representation $g \mapsto \pi(g^{-1})$ of $G_n$. Let $\tau$ be an $\ell$-adic irreducible representation of $G_n$, then $\tilde{\tau} \simeq \tau^\vee$, by [6]. Hence when $\Delta$ is an $\ell$-modular generic segment of $G_n$, it lifts to an $\ell$-adic segment $D$ according to the discussion before Definition 2.15, and as $\tilde{D} \simeq D^\vee$, we deduce by reduction modulo-$\ell$, c.f. [15] I 9.7, that $\tilde{\Delta} \simeq \Delta^\vee$. If $\pi = \Delta_1 \times \cdots \times \Delta_t$ is a representation of $G_n$ of Whittaker type, we have $\tilde{\pi} = \tilde{\Delta_1} \times \cdots \times \tilde{\Delta_t} = \Delta_1^\vee \times \cdots \times \Delta_t^\vee$, and we deduce that $\tilde{\pi}$ is also of Whittaker type. In order to state the functional equation for $L$-factors of representations of Whittaker type, we will need the following lemma:

**Lemma 2.17.** Let $\pi$ be a representation of Whittaker type of $G_n$, then $\tilde{\pi}$ is of Whittaker type and the map $W \mapsto \tilde{W}$, where $\tilde{W}(g) = W(w_n g^{-1})$, is an $R$-vector space isomorphism between $W(\pi, \theta)$ and $W(\tilde{\pi}, \theta^{-1})$.

### 3 Rankin–Selberg local factors for representations of Whittaker type

The theory of derivatives ([1] and [2]) being valid in positive characteristic (see Subsection 2.3) and equipped with the theory of $R$-Haar measures (see Subsection 2.2), means we can now safely follow [7] to define $L$-factors and $\varepsilon$-factors. However, as we do not have a Langlands’ quotient theorem at our disposal, which would allow us to associate to an irreducible representation of $G_n$, a unique representation with an injective Whittaker model lying above it, we restrict our attention to representations of Whittaker type (see Subsection 2.5).
3.1 Definition of the L-factors

We first recall the asymptotics of Whittaker functions obtained in [8, Proposition 2.2]. We write $Z_i$ for subgroup $\{\text{diag}(tI_i, I_{n-i}), t \in F^\times\}$ of $G_n$. The diagonal torus $A_n$ of $G_n$ is the direct product $Z_1 \times \cdots \times Z_n$.

**Lemma 3.1.** Let $\pi$ be a representation of Whittaker type of $G_n$. For each $i$ between 1 and $n-1$, there is a finite family $X_i(\pi)$ of characters of $Z_i$, such that if $W$ is a Whittaker function in $W(\pi, \theta)$, then its restriction $W(z_1, \ldots, z_{n-1})$ to $A_{n-1} = Z_1 \times \cdots \times Z_{n-1}$ is a sum of functions of the form

$$
\phi(z) \prod_{i=1}^{n-1} \chi_i(z_i)v(z_i)^{m_i}
$$

for $\chi_i \in X_i(\pi)$, integers $m_i \geq 0$, and $\phi \in C_c^\infty(F^{n-1})$.

The proof of Jacquet–Piatetskii-Shapiro–Shalika in [op. cit.] applies *mutatis mutandis* for $\ell$-modular representations.

**Remark 3.2.** For $1 \leq i \leq n-1$, we can take $X_i(\pi)$ to be the family of central characters (restricted to $Z_i$) of the irreducible components of the (non-normalised) Jacquet module $\pi_{N_i,n-i}$. We denote by $X_i(\pi)$ the singleton $\{\omega_i\}$. We denote by $E_i(\pi)$, the family of central characters (restricted to $Z_i$) of the irreducible components of the normalised Jacquet module $\pi_{N_i,n-i}$, for $1 \leq i \leq n-1$, and let $E_i(\pi) = X_i(\pi)$. The family $E_i(\pi)$ is obtained from $X_i(\pi)$ by multiplication by an unramified character of $Z_i$, in particular, if $R = \mathbb{Q}_p$, the characters in $E_i(\pi)$ are integral if and only if those in $X_i(\pi)$ are integral.

**Proposition 3.3.** Let $\pi$ be a representation of Whittaker type of $G_n$, and $\pi'$ a representation of Whittaker type of $G_m$, for $m \leq n$.

- The case $n = m$. Let $W \in W(\pi, \theta)$, $W' \in W(\pi', \theta^{-1})$, and $\phi \in C_c^\infty(F^n)$. Then, for every $k \in \mathbb{Z}$, the coefficient

$$
c_k(W, W', \phi) = \int_{N_n \backslash G_n} W(g)W'(g)\phi(\eta g)dg
$$

is well-defined, and vanishes for $k << 0$.

- The case $m \leq n-1$. For $0 \leq j \leq n-m-1$, let $W \in W(\pi, \theta)$, and $W' \in W(\pi', \theta^{-1})$. Then for every $k \in \mathbb{Z}$, the coefficient

$$
c_k(W, W'; j) = \int_{M_{j,m}} \int_{N_m \backslash G_m} W\left(\begin{bmatrix} g & \ast \\ x & I_j \\ \ast & I_{n-m-j}\end{bmatrix}\right)W'(g)dgdx
$$

is well-defined, and vanishes for $k << 0$. When $m = n - 1$, we will simply write $c_k(W, W')$ for $c_k(W, W'; 0)$.

**Proof.** The only thing to check is that the coefficients in the statement are well defined, i.e. finite sums, and zero for $k$ negative enough. This is a consequence of Iwasawa decomposition together with Corollary 2.8, and Lemma 3.1 for the case $m \geq n - 1$, and, in the case $m \leq n-2$, that the map $W\left(\begin{bmatrix} g & \ast \\ x & I_j \\ \ast & I_{n-m-j}\end{bmatrix}\right)$ has compact support with respect to $x$, independently of $g$, by [7, Lemma 6.2].
Following Proposition 3.3, we now can define our Rankin–Selberg formal series.

**Definition 3.4.**

- The case $n = m$. Under the same notation as Proposition 3.3, we define the following formal Laurent series
  \[
  I(W, W', \phi, X) = \sum_{k \in \mathbb{Z}} c_k(W, W', \phi)X^k \in R((X)).
  \]

- The case $m \leq n - 1$. Under the same notation as Proposition 3.3, we define the following formal Laurent series
  \[
  I(W, W', X; j) = \sum_{k \in \mathbb{Z}} c_k(W, W'; j)q^{k(n-m)/2}X^k \in R((X)).
  \]

When $m = n - 1$, we will simply write $I(W, W', X)$ for $I(W, W', X; 0)$.

The $L$-factors we study are defined by the following theorem.

**Theorem 3.5.** Let $\pi$ be a representation of Whittaker type of $G_m$, and $\pi'$ a representation of Whittaker type of $G_m$, for $1 \leq m \leq n$.

- If $n = m$, the $R$-submodule spanned by the Laurent series $I(W, W', \phi, X)$ as $W$ varies in $W(\pi, \theta)$, $W'$ varies in $W(\pi', \theta^{-1})$, and $\phi$ varies in $C^\infty_c(F^n)$, is a fractional ideal of $R[X^\pm 1]$, and it has a unique generator which is an Euler factor $L(\pi, \pi', X)$.

- If $1 \leq m \leq n - 1$, fix $j$ between 0 and $n - m - 1$. The $R$-submodule spanned by the Laurent series $I(W, W', X; j)$ as $W$ varies in $W(\pi, \theta)$, $W'$ varies in $W(\pi', \theta^{-1})$, is a fractional ideal of $R[X^\pm 1]$, is independent of $j$, and it has a unique generator which is an Euler factor $L(\pi, \pi', X)$.

**Proof.** We treat the case $m \leq n - 2$, the case $m \geq n - 1$ is totally similar. First we want to prove that our formal series in fact belong to $R(X)$. In this case, the coefficient $c_k(W, W'; j)$ is equal to
  \[
  \int_{(K_m \cap B_m) \setminus K_m} \int_{A_m^h} \int_{M_{j,m}} W\left(\begin{array}{cc} a & x \\ \mu_j & I_{n-m-j} \end{array} \right) W'(a)\delta_{B_m}(a)^{-1} dx da dz,
  \]
which, by smoothness of $W$ and $W'$, we can write as a finite sum
  \[
  \sum_i \int_{A_m^h} W_i\left(\begin{array}{c} a \\ I_{n-m} \end{array} \right) W'_i(a)\delta_{B_m}(a)^{-1} da,
  \]
with functions $W_i \in W(\pi, \theta)$ and $W'_i \in W(\pi', \theta^{-1})$. For $W$ and $W'$ in $W(\pi, \theta)$, let
  \[
  b_k(W, W') = \int_{A_m^h} W\left(\begin{array}{c} a \\ I_{n-m} \end{array} \right) W'(a)\delta_{B_m}(a)^{-1} da,
  \]
it is thus enough to check that
  \[
  J(W, W') = \sum_{k \in \mathbb{Z}} b_k(W, W')q^{(n-m)/2}X^k
  \]
belongs to $R(X)$. Following the proof of [7] we see that, by Lemma 3.1, they belong to $P(X)^{-1}R[X^\pm 1]$, where $P(X)$ is a suitable power of the product over the unramified characters $\chi_i$’s in $E_i(\pi)$ for $1 \leq i \leq n$ and the unramified characters $\mu_j$’s in $E_j(\pi')$ for $1 \leq j \leq m$ of the Tate $L$ factors $L(\chi_i\mu_j, X)$. By [10], this factor is equal to 1 if $R = \mathbb{F}_\ell$, and $q \equiv 1[\ell]$, and is equal to $1/(1 - \chi_i\mu_j(\pi)X)$ otherwise. The other properties follow immediately from [7].
The proof of Theorem 3.5 implies the following corollary.

**Corollary 3.6.**
- If $\pi$ and $\pi'$ are $\ell$-adic representations of Whittaker type of $G_n$ and $G_m$, then $1/L(\pi, \pi', X) \in \overline{\mathbb{Z}[X]}$.
- If $\pi$ and $\pi'$ are $\ell$-modular representations of Whittaker type of $G_n$ and $G_m$, and $q \equiv 1[\ell]$, then $L(\pi, \pi', X) = 1$.

**Proof.** By our assertion at the end of the proof of Theorem 3.5, the polynomial $Q = 1/L(\pi, \pi', X)$ divides $(R[X^\pm 1])$, hence in $R[X]$) a power of the product $P$ of the polynomials $1/L(\chi_i \mu_j, X)$ over the set of unramified characters $\chi_i$, $1 \leq i \leq n$ and unramified characters $\mu_j$'s in $E_j(\pi')$ for $1 \leq j \leq m$. We already noticed that $P$ must be 1 if $R = \mathbb{F}_q$ and $q \equiv 1[\ell]$, which proves our assertion in this case. In general, $P$ belongs to $\overline{\mathbb{Z}_q}[X]$, with constant term 1, as so do the polynomials $1/L(\chi_i \mu_j, X)$. Let $A$ be the quotient $P/Q$ in $R[X]$. We have $P = aQ$ in $R[X]$, with $P(0) = Q(0) = 1$, and $P \in \overline{\mathbb{Z}_q}[X]$. This equality in fact takes place in $K[X]$, for $K$ a finite extension of $\mathbb{Q}_\ell$ such that $Q \in \mathcal{O}_K[X]$ (with $\mathcal{O}_K$ the ring of integers in $K$). As $\mathcal{O}_K[X]$ is a unique factorisation domain, the fact that $P(0) = Q(0) = A(0) = 1$ implies that $A$ and $Q$ are in fact in $\overline{\mathbb{Z}_q}[X]$. \hfill $\square$

### 3.2 The functional equation

We have defined Rankin–Selberg $L$-factors of pairs of representations of Whittaker type, we now need to show that these satisfy a local functional equation. By identifying $F^n$ with $\mathcal{M}_{1,n}$, the space $C_c^\infty(F^n)$ provides a smooth representation $\rho$ of $G_n$, with $G_n$ acting by right translation. We also denote by $\rho$ the action by right translation of $G_n$ on any space of functions. For $a \in R[X^\pm 1]$, we denote by $\chi_a$ the character in $\text{Hom}(G_n, R[X^\pm 1]^\times)$ defined as: $g \mapsto a^{\nu(det(g))}$, in particular $\nu = \chi_{1/2}$ is the absolute value of the determinant.

Let $\pi$ be a representation of Whittaker type of $G_n$, and $\pi'$ be a representation of Whittaker type of $G_m$. If $m = n$, we write

$$D(\pi, \pi', C_c^\infty(F^n)) = \text{Hom}_{G_n}(\pi \otimes \pi' \otimes C_c^\infty(F^n), \chi X),$$

for the space of $R$-linear maps, $\mathcal{L} : \pi \times \pi' \times C_c^\infty(F^n) \rightarrow R[X^\pm 1]$, satisfying

$$\mathcal{L}(\rho(h)W, \rho(h)W', \rho(h)\phi) = X^k \mathcal{L}(W, W', \phi)$$

for all $W \in W(\pi, \theta)$, $W' \in W(\pi', \theta^{-1})$, $\phi \in C_c^\infty(F^n)$, and $h \in G_n$. If $m \leq n - 1$, we write

$$D(\pi, \pi') = \text{Hom}_{G_n U_{m+1,n-n-1}}(\pi \otimes \pi', \chi_{q(n-m)/2} X \otimes \theta),$$

for the space of $R$-linear maps, $\mathcal{L} : \pi \times \pi' \rightarrow R[X^\pm 1]$, satisfying

$$\mathcal{L}(\rho(h)W, \rho(h)W') = q^{k(n-m)/2} X^k \mathcal{L}(W, W', \phi), \quad \mathcal{L}(\rho(u)W, W') = \theta(u) \mathcal{L}(W, W')$$

for all $W \in W(\pi, \theta)$, $W' \in W(\pi', \theta^{-1})$, $h \in G_m$, and $u \in U_{m+1,n-n-1}$. We denote by $C_c^\infty_{c,0}(F^n)$ the subspace of $C_c^\infty(F^n)$ which is the kernel of the evaluation map $E_{\nu_0} : \phi \mapsto \phi(0)$.

**Proposition 3.7.** The spaces $D(\pi, \pi', C_c^\infty(F^n))$ and $D(\pi, \pi')$ are free $R[X^\pm 1]$-modules of rank 1.

The proof in the complex case of Jacquet–Piatetski-Shapiro–Shalika in [7] is long. Some results obtained in the complex case [op. cit.] using invariant distributions can be obtained quicker using derivatives which is how we proceed.
Proof. We start with the case \( n = m \). The map

\[
(W, W', \phi) \mapsto I(W, W', \phi, X)/L(\pi, \pi', X)
\]

is a nonzero element of \( D(\pi, \pi', C_\infty^{\infty}(F^m)) \), hence we only need to show that \( D(\pi, \pi', C_\infty^{\infty}(F^m)) \) is a free \( R[X^\pm 1] \)-module of rank at most 1.

We have an exact sequence of representations of \( G_n \)

\[
0 \to C^{\infty}_{c,0}(F^n) \to C^{\infty}_c(F^n) \to 1 \to 0.
\]

We tensor this sequence by \( \pi \otimes \pi' \) and, as \( \pi \otimes \pi' \) is flat as an \( R \)-vector space, we obtain

\[
0 \to \pi \otimes \pi' \otimes C^{\infty}_{c,0}(F^n) \to \pi \otimes \pi' \otimes C^{\infty}_c(F^n) \to \pi \otimes \pi' \to 0.
\]

By considering central characters, it is clear that the space \( \text{Hom}_{G_n}(\pi \otimes \pi', \chi_X) = 0 \). Applying \( \text{Hom}_{G_n}(\pi \otimes \pi', \chi_X) \) which are left exact, we obtain that \( \text{Hom}_{G_n}(\pi \otimes \pi' \otimes C^{\infty}_c(F^n), \chi_X) \) is an irreducible representation of \( G_n \) acting via right translation and \( \text{Ind}_{P_n}(\delta^{1/2}_{P_n}) \), hence we have

\[
\text{Hom}_{G_n}(\pi \otimes \pi' \otimes C^{\infty}_{c,0}(F^n), \chi_X) \simeq \text{Hom}_{G_n}(\pi \otimes \pi', \chi_X \text{Ind}_{P_n}(\delta^{1/2}_{P_n})),
\]

\[
\simeq \text{Hom}_{P_n}(\pi \otimes \pi', \chi_X \delta_{P_n}^{1/2}).
\]

by Frobenius reciprocity. Now, by the theory of derivatives (see Subsection 2.3), \( \pi \) and \( \pi' \), as \( P_n \)-modules, are of finite length, with irreducible subquotients of the form \( (\Phi^+)^{k}\Psi^{\pm}(\rho) \), for \( \rho \) an irreducible representation of \( G_{n-k-1} \) and \( k \) between 0 and \( n-1 \). Moreover, \( (\Phi^+)^{n-1}\Psi^{+}(1) \) appears with multiplicity 1, as a submodule. By Proposition 2.14, the space

\[
\text{Hom}_{P_n}( (\Phi^+)^{k}\Psi^{+}(\rho) \otimes (\Phi^+)^{j}\Psi^{+}(\rho'), \chi_X \delta_{P_n}^{1/2}) = \text{Hom}_{P_n}( (\Phi^+)^{k}\Psi^{+}(\rho) \otimes (\Phi^+)^{j}\Psi^{+}(\chi_X \delta_{P_n}^{1/2}), 1)
\]

is zero, except when \( j = k \), in which case it is isomorphic to \( \text{Hom}_{G_n}(\rho \otimes \rho', \chi_X \nu^{-1}) \). If \( \rho \) and \( \rho' \) are irreducible and \( k \geq 1 \), by considering central characters, the space \( \text{Hom}_{G_n}(\rho \otimes \rho', \chi_X \nu^{-1}) \) is zero. Thus \( \text{Hom}_{P_n}(\pi \otimes \pi', \chi_X \delta_{P_n}^{1/2}) \) is a \( R[X^\pm 1] \)-submodule of \( \text{Hom}_{G_n}(1 \otimes 1, \chi_X) \simeq R[X^\pm 1] \). This ends the proof in the case \( n = m \), as \( R[X^\pm 1] \) is principal.

We now consider the case \( m \leq n-1 \). Again, the space \( D(\pi, \pi') \) is nonzero as it contains the map \( (W, W') \mapsto I(W, W', X)/L(\pi, \pi', X) \), we will show that it injects into \( R[X^\pm 1] \), which will prove the statement. Let \( L \) be in \( D(\pi, \pi') \), by definition, the map \( L \) factors through \( \tau \times \pi' \), where \( \tau \) is the quotient of \( \pi \) by its subspace spanned by \( \pi(u)W - \theta(u)W \) for \( u \in U_{m+1,n-m-1} \) and \( W \in W(\pi, \theta) \). Hence \( \tau \) is nothing other than the space of \( (\Phi^+)^{n-m-1}(\pi) \). Taking into account the normalisation in the definition of the derivatives, we obtain the following injection:

\[
\text{Hom}_{G_m}(U_{m+1,n-m-1}(\pi \otimes \pi', \chi_{q^{(n-m)/2}X} \otimes \theta) \hookrightarrow \text{Hom}_{G_m}((\Phi^+)^{n-m-1}(\pi) \otimes \pi', \chi_{q^{1/2}X}).
\]

We next prove the following lemma.

**Lemma 3.8.** If \( \sigma \) is an irreducible representation of \( P_m \), then

\[
\text{Hom}_{G_m}(\Phi^+(\sigma) \otimes \pi', \chi_{q^{1/2}X}) \simeq \text{Hom}_{P_m}(\sigma \otimes \pi', \chi_X).
\]

If \( \sigma \) is an irreducible representation of \( G_m \), then

\[
\text{Hom}_{G_m}(\Phi^+(\sigma) \otimes \pi', \chi_{q^{1/2}X}) \simeq \text{Hom}_{G_m}(\sigma \otimes \pi', \chi_X) = \{0\}.
\]
Proof of the Lemma. We first prove the second assertion. By definition of $\Psi^+$, the $R[X^{\pm 1}]$-module $\text{Hom}_{G_m}(\Psi^+(\sigma) \otimes \pi', \chi_{q^{1/2}X})$ is equal to $\text{Hom}_{P_m}(\Psi^+(\sigma) \otimes \pi', \chi_X)$, which is itself isomorphic to $\text{Hom}_{G_m}(\sigma \otimes \pi', \chi_X)$ by Proposition [2.13]. As $\sigma$ is irreducible, it has a central character, and thus $\text{Hom}_{G_m}(\sigma, \chi_X) = \{0\}$.

For the first assertion, as $P_{m+1} = G_m(P_m U_{m+1})$, we have an isomorphism $\Phi^+(\sigma) \mid_{G_m} \simeq \text{ind}_{P_m}^{G_m}(\sigma)$ by Mackey theory (c.f. [2, Theorem 5.2]). Hence, we obtain

$$\text{Hom}_{G_m}(\Psi^+(\sigma) \otimes \pi', \chi_{q^{1/2}X}) \simeq \text{Hom}_{G_m}(\text{ind}_{P_m}^{G_m}(\sigma) \otimes \pi', \chi_{q^{1/2}X}) \simeq \text{Hom}_{P_m}(\sigma \otimes \pi', \chi_X),$$

the last isomorphism by Frobenius reciprocity.

Now, $(\Phi^-)^{n-m-1}(\pi)$ is a $P_{m+1}$-module of finite length, and as $\pi$ is of Whittaker type, it contains $(\Phi^+)^{m-1}\Psi^+(\pi)$ as a submodule, the latter’s multiplicity being 1 as a composition factor. By the theory of derivations, all of the other irreducible subquotients are either of the form $\Psi^+(\sigma)$, with $\sigma$ an irreducible representation of $G_m$, or of the form $\Phi^+(\sigma)$, with $\sigma$ an irreducible representation of $P_m$ of the form $(\Phi^+)^{m-j-1}\Psi^+(\sigma')$, with $\sigma'$ a representation of $G_j$, for some $j \geq 1$. By Lemma [3.8] $\text{Hom}_{G_m}(\Psi^+(\sigma) \otimes \pi', \chi_X)$ is zero. For all subquotients of the form $\Phi^+(\sigma)$, we have $\text{Hom}_{G_m}(\Phi^+(\sigma) \otimes \pi', \chi_{q^{1/2}X}) \simeq \text{Hom}_{P_m}(\sigma \otimes \pi', \chi_X)$ by Lemma [3.8].

Lemma 3.9. The $R[X^{\pm 1}]$-module $\text{Hom}_{P_m}(\sigma \otimes \pi', \chi_X)$ is zero if $\sigma$ is an irreducible representation of $P_m$ of the form $(\Phi^+)^{m-j-1}\Phi^+(\sigma')$, with $\sigma'$ a representation of $G_j$, for some $j \geq 1$, whereas $\text{Hom}_{P_m}(((\Phi^+)^{m-j-1}\Phi^+(1) \otimes \pi', \chi_X)) \not\simeq R[X^{\pm 1}].$

Proof of the Lemma. As $\pi'$ is of Whittaker type, its restriction to $P_m$ is of finite length, with irreducible subquotients of the form $(\Phi^+)^{m-k-1}\Psi^+(1)$, for $\mu$ an irreducible representation of $G_k$. Moreover, the representation $(\Phi^+)^{m-k-1}\Psi^+(1)$ occurs with multiplicity 1, and is a submodule. If $\sigma$ is an irreducible representation of $P_m$ of the form $(\Phi^+)^{m-j-1}\Psi^+(\sigma')$, with $\sigma'$ a representation of $G_j$, for some $j \geq 1$, then $\text{Hom}_{P_m}((\Phi^+)^{m-j-1}\Phi^+(\sigma') \otimes (\Phi^+)^{m-k-1}\Phi^+(\mu), \chi_X)$ is zero by Proposition [2.14] if $j \neq k$ (in particular if $k = 1$). Moreover, if $k = j$, by the same Proposition, we have $\text{Hom}_{P_m}((\Phi^+)^{m-j-1}\Phi^+(\sigma') \otimes (\Phi^+)^{m-k-1}\Phi^+(\mu), \chi_X)$ is isomorphic to $\text{Hom}_{G_j}(\sigma' \otimes \mu, \chi_X)$, which is zero, by considering central characters. Hence we have proved the first part of the lemma. If $\sigma = (\Phi^+)^{m-1}\Psi^+(1)$, reasoning as above, we see at once that $\text{Hom}_{P_m}((\Phi^+)^{m-1}\Phi^+(1) \otimes \pi', \chi_X)$ injects into $\text{Hom}_{P_m}((\Phi^+)^{m-1}\Phi^+(1) \otimes (\Phi^+)^{m-1}\Phi^+(1), \chi_X) \simeq \text{Hom}_{G_0}(1 \otimes 1, \chi_X)$, the latter space being isomorphic to $R[X^{\pm 1}]$, and this completes the proof of the lemma.

All in all, we deduce that $\text{Hom}_{G_m}((\Phi^-)^{n-m-1}(\pi) \otimes \pi', \chi_{q^{1/2}X})$ injects as a $R[X^{\pm 1}]$-submodule into $\text{Hom}_{P_m}((\Phi^+)^{n-1}\Phi^+(1) \otimes \pi', \chi_X)$, which itself injects into $R[X^{\pm 1}]$, and this ends the proof of the proposition.

Remark 3.10. Notice that all the injections defined in the proof of Proposition [3.7] are in fact isomorphisms. This could be viewed directly, or we can simply see that after composing all of them we obtain an isomorphism.

We are now in a position to state the local functional equation and define the Rankin–Selberg $\varepsilon$-factor of a pair of representations of Whittaker type. We recall that an invertible element of $R[X^{\pm 1}]$ is an element of the form $cX^k$, for $c$ in $R^\times$, and $k$ in $Z$.

Corollary 3.11. Let $\pi$ be a representation of Whittaker type of $G_m$, and $\pi'$ be a representation of Whittaker type of $G_m$.  

\[\text{Hom}_{G_m}(\Psi^+(\sigma) \otimes \pi', \chi_{q^{1/2}X}) \simeq \text{Hom}_{P_m}(\sigma \otimes \pi', \chi_X),\]

the last isomorphism by Frobenius reciprocity.
1. If \( m = n \), there is an invertible element \( \varepsilon(\pi, \pi', \theta, X) \) of the ring \( R[\mathcal{X}] \) such that for any \( W \in W(\pi, \theta) \), any \( W' \in W(\pi', \theta^{-1}) \), and any \( \phi \in C_c(\mathbb{F}_n) \), we have:

\[
I(W, W', \phi, qX^{-1})/L(\pi, \pi', qX^{-1}) = c_{\pi'}(-1)^{m-1}\varepsilon(\pi, \pi', \theta, X)I(W, W', \phi, X)/L(\pi, \pi', X).
\]

2. If \( m \leq n - 1 \), there is an invertible element \( \varepsilon(\pi, \pi', \theta, X) \) of the ring \( R[\mathcal{X}] \) such that, for any \( W \in W(\pi, \theta) \), any \( W' \in W(\pi', \theta^{-1}) \), and any \( 0 \leq j \leq n - m - 1 \), we have:

\[
\frac{I(\rho(w_{m-n}W, W', \phi, qX^{-1}; n-m-1-j)}{L(\pi, \pi', qX^{-1})} = c_{\pi'}(-1)^{m-1}\varepsilon(\pi, \pi', \theta, X)I(W, W', X, j)/L(\pi, \pi', X).
\]

**Proof.** It is a consequence of Proposition 3.7 if \( n = m \), and if \( m \leq n - 1 \) with \( j = 0 \), as the functionals on both sides of the equality belong, respectively, to \( D(\pi, \pi', C_c(\mathbb{F}_n)) \) and \( D(\pi, \pi') \).

For \( j \neq 0 \), it follows from the case \( j = 0 \) as in the complex setting, c.f. [8].

We call \( \varepsilon(\pi, \pi', \theta, X) \) the *local \( \varepsilon \)-factor* associated to \( \pi, \pi', \) and \( \theta \), and we write

\[
\gamma(\pi, \pi', \theta, X) = \frac{\varepsilon(\pi, \pi', \theta, X)L(\pi, \pi', qX^{-1})}{L(\pi, \pi', X)},
\]

for the *local \( \gamma \)-factor* associated to \( \pi, \pi', \) and \( \theta \).

### 3.3 Compatibility with reduction modulo-\( \ell \)

Let \( \pi = \Delta_1 \times \cdots \times \Delta_t \) be an \( \ell \)-modular representation of Whittaker type of \( G_n \). As in Section 2.5 for \( 1 \leq i \leq t \), we can choose integral \( \ell \)-adic segments \( D_i \) and integral structures \( \Lambda_i \) in \( D_i \) such that \( \Lambda_i \otimes_{\mathbb{Q}_\ell} \mathbb{F}_\ell \simeq \Delta_i \).

Moreover, the \( \ell \)-adic representation \( \tau = D_1 \times \cdots \times D_t \) is an integral representation of Whittaker type, and \( \Lambda = \Lambda_1 \times \cdots \times \Lambda_t \) is an integral structure in \( \tau \) satisfying \( \Lambda \otimes_{\mathbb{Q}_\ell} \mathbb{F}_\ell \simeq \pi \).

We denote by \( W_e(\tau, \theta) \) the functions in \( W(\tau, \theta) \) with integral values. We will need the following result concerning integral structures in \( \ell \)-adic representations of Whittaker type.

**Theorem 3.12.** There exists a sublattice \( W \subset W_e(\tau, \theta) \) in \( W(\tau, \theta) \), such that \( W(\pi, r_\ell(\theta)) = W \otimes_{\mathbb{Q}_\ell} \mathbb{F}_\ell \).

**Proof.** If \( \tau \) is generic, it is shown in [19] Theorem 2] that \( W_e(\tau, \theta) \) is an integral structure in \( W(\tau, \theta) \), such that \( W(\pi, r_\ell(\theta)) = W_e(\tau, \theta) \otimes_{\mathbb{Q}_\ell} \mathbb{F}_\ell \).

We use this result together with the properties of parabolic induction with respect to lattices, and a result from [4] about the explicit description of Whittaker functionals on induced representations.

For each \( 1 \leq i \leq t \), each \( D_i \) is generic and hence \( W_e(D_i, \theta) \) is an integral structure in \( W(D_i, \theta) \). By [13] 1.9.3, \( \mathcal{L} = W_e(D_1, \theta) \times \cdots \times W_e(D_t, \theta) \) is a lattice in \( \tau \). The space \( \mathcal{L} \) consists of all smooth functions from \( G \) to \( G(\pi, \tau(\theta)) = W_e(D_1, \theta) \otimes \cdots \otimes W_e(D_t, \theta) \), with the tensor product over \( \mathbb{Z}_\ell \), which transform on the left by \( D_1 \otimes \cdots \otimes D_t \). We recall that \( W(D, \theta) = W(D_1, \theta) \otimes \cdots \otimes W(D_t, \theta) \), as well as \( W(\Delta, \theta) = W(\Delta_1, \theta) \otimes \cdots \otimes W(\Delta_t, \theta) \), is a representation of a standard parabolic subgroup \( Q = M U \) of \( G_n \), trivial on \( U \).

A function \( f \) in \( \tau = W(D_1, \theta) \times \cdots \times W(D_t, \theta) \), by definition of parabolic induction, is a map from \( G_n \) to \( W(D, \theta) \), i.e. for \( g \in G_n \), \( f(g) \in W(D, \theta) \) identifies with a map from \( M \) to \( \mathbb{Q}_\ell \), so we can view \( f \) as a map of two variables from \( G_n \times M \) to \( \mathbb{Q}_\ell \), and similarly, we can view the
elements in $\pi = W(\Delta_1, \theta) \times \cdots \times W(\Delta_i, \theta)$ as maps from $G_n \times M$ to $\overline{\mathbb{F}}_\ell$. In [4, Corollary 2.3], it is shown (for minimal parabolics, but their method works for general parabolics), that there is a Weyl element $w$ in $G_n$, such that if one takes $f \in \tau$, then there is a compact open subgroup $U_f$ of $U$ which satisfies that for any compact subgroup $U'$ of $U$ containing $U_f$, the integral $\int_{U_f} f(wu, 1_M)\theta^{-1}(u)du$ is independent from $U'$. We will write $\lambda(f) = \int_{U_f} f(wu, 1_M)\theta^{-1}(u)du$.

This assertion is also true for $\pi$ with the same proof, for the same choice of $w$, we write $\mu(h) = \int_{U_f} h(wu, 1_M)\tau^{-1}(u)du$ for $h \in \pi$. Both $\lambda$ and $\mu$ are nonzero Whittaker functionals on $\tau$ and $\pi$, respectively, and $\lambda$ sends $L$ to $\mathbb{Z}[\ell]$ for a correct normalisation of $du$. We can moreover suppose, for correct normalisations of the $\ell$-adic and the $\ell$-modular Haar measures $du$, that $\mu = r_\ell(\lambda)$. The surjective map $w : \tau \rightarrow W(\tau, \theta)$ which takes $f$ to $W_f$, defined by $W_f(g) = \lambda(\tau(f)g)$, sends $L$ to $W_e(\tau, \theta)$. Similarly, for $h \in \pi$, if we set $W_h(g) = \mu(\pi(g)h)$, then the map $\pi \rightarrow W(\pi, r_\ell(\theta))$, taking $h$ to $W_h$, is surjective, and we have $r_\ell(W_f) = W_{r_\ell(f)}$. From this, we obtain that $\mathbb{W} = w(L)$ is a sublattice of $W_e(\tau, \theta)$ (see [15, I 9.3], and $r_\ell(\mathbb{W}) = W(\pi, \theta)$. $\square$

Let $\pi$ and $\pi'$ are integral $\ell$-adic representations of Whittaker type of $G_n$ and $G_m$. By Corollary 3.6, we already know that $L(\pi, \pi', X)$ is the inverse of a polynomial with integral coefficients, even without the integrality assumption. With the integrality assumption, we now consider the associated $\varepsilon$-factor.

**Lemma 3.13.** The factor $\varepsilon(\pi, \pi', \theta, X)$ is of the form $cX^k$, for $c$ a unit in $\mathbb{Z}_\ell$.

**Proof.** We only do the case $n = m$, the case $m \leq n - 1$ follows mutatis mutandis. By Remark 2.1, we can write $L(W, W', \phi)$ have integral values, the Laurent series $I(W, W', \phi, qX^{-1})$ and $I(W, W', \phi, qX^{-1})$ belong, respectively, to $\mathbb{Z}[\ell](X)$ and $\mathbb{Z}[\ell]((X^{-1}))$. As $L(\pi, \pi', X)$ and $L(\pi, \pi', qX^{-1})$ are the inverse of polynomials in $\mathbb{Z}[\ell][X^{\pm 1}]$, such that $F(X)$ belongs to $\mathbb{Z}[\ell][X^{\pm 1}]$, and similarly for the quotient $I(W, W', \phi, \psi)/L(\pi, \pi', X)$. The functional equation then implies that the scalar $c$ is in $\mathbb{Z}_\ell$, and applying it twice shows that it is in $\mathbb{Z}_\ell^\times$. $\square$

If $P$ is an element of $\mathbb{Z}[\ell][X]$ with nonzero reduction modulo-$\ell$, we write $r_\ell(P^{-1})$ for $(r_\ell(P))^{-1}$. We now prove our first main result.

Let $\pi$ and $\pi'$ be $\ell$-modular representations of Whittaker type of $G_n$ and $G_m$. Let $\tau$ and $\tau'$ be $\ell$-adic representations of Whittaker type $\tau$ and $\tau'$ of $G_n$ and $G_m$ with integral structures $W_e(\tau, \theta)$ and $W_{\pi'}(\tau', \theta)$, as in Lemma 3.12.

**Theorem 3.14.** We have

$$L(\pi, \pi', X) | r_\ell(L(\tau, \tau', X)),$$

and

$$\varepsilon(\pi, \pi', r_\ell(\theta), X) = r_\ell(\varepsilon(\tau, \tau', \theta, X)).$$

**Proof.** We give the proof for $m \leq n - 1$, and $j = 0$, the other cases being similar. By definition, one can write $L(\pi, \pi', X)$ as a finite sum $\sum_i I(W_i, W'_i, X)$, for $W_i \in W(\pi, \theta)$ and $W'_i \in W(\pi', \theta^{-1})$. By Theorem 3.12, there are Whittaker functions $W_{\pi, e} \in W_e(\tau, \theta, \mu)$ and $W'_{\pi, e} \in W_{\pi'}(\tau', \mu^{-1})$, such that $W_i = r_\ell(W_{\pi, e})$, and $W'_i = r_\ell(W'_{\pi, e})$. By Remark 2.1, we have $L(\pi, \pi', X) = r_\ell(\sum_i I(W_{\pi, e}, W'_{\pi, e}, X))$. As $I(W_{\pi, e}, W'_{\pi, e}, X)$ belongs to

$$L(\tau, \tau', X)\mathbb{Z}[\ell][X^{\pm 1}] \cap \mathbb{Z}[\ell]((X)) = L(\tau, \tau', X)\mathbb{Z}[\ell][X^{\pm 1}],$$

we obtain that $L(\pi, \pi', X)$ belongs to $\mathbb{Z}[\ell][X^{\pm 1}]$. This proves the first assertion. The equality for $\varepsilon$ factors follows the functional equation, and Remark 2.1. $\square$
Remark 3.15. As for the Godement–Jacquet $L$-functions (see \cite{10}), we do not always have an equality between $r_\ell(L(\tau,\tau',X))$ and $L(\pi,\pi',X)$. For instance when $q \equiv 1[\ell]$, we always have $L(\pi,\pi',X) = 1$. But already, for unramified $\ell$-adic characters $\tau$ and $\tau'$ of $G_1$, we have $r_\ell(L(\tau,\tau',X)) = 1/(1 - X)$.

4 $L$-factors of pairs of cuspidal representations.

We introduce the terminology of \cite{5} on exceptional poles of Rankin–Selberg $L$-functions. We will not, however, make full use of this machinery in the following, as we will specialise to $L$-factors of pairs of cuspidal representations. In this case, following \cite{7} or \cite{5} is completely equivalent. However, for a further inquiry of $L$-factors of generic segments, we will use the full theory. As we intend to study in more detail the $L$-factors of representations of Whittaker type in the near future, we already introduce the terminology in this section.

4.1 Exceptional poles

We now recall the notion of exceptional pole, due to Cogdell and Piatetski-Shapiro (\cite{5}). Let $R$ be an algebraically closed field. Let $\pi$ and $\pi'$ be a pair of $R$-representations of Whittaker type of $G_n$, $W \in W(\pi,\theta)$ and $W' \in W(\pi',\theta^{-1})$, and $\phi \in C^\infty_c(F^n)$. Suppose that $x$ as is a pole of order $d$ of $L(\pi,\pi',X)$. As $R[X]$ is a principal ideal domain, we can write the partial fraction expansion of $I(W,W',\phi,X)$ at $x$ as

$$I(W,W',\phi,X) = \frac{T_x(W,W',\phi)}{(X-x)^d} + \text{higher order terms.}$$

The map $T_x$ is a non-zero trilinear from $W(\pi,\theta) \times W(\pi',\theta^{-1}) \times C^\infty_c(F^n)$ to $R$.

Definition 4.1. Let $\pi$ and $\pi'$ be $R$-representations of Whittaker type of $G_n$ and $G_m$, and let $W$ and $W'$ belong to $W(\pi,\theta)$ and $W(\pi',\theta^{-1})$, respectively.

- If $n = m$, we say that a pole $x$ in $R$ of the rational map $L(\pi,\pi',X)$ is an exceptional pole if and only if the trilinear form $T_x$, vanishes on $W(\pi,\theta) \times W(\pi',\theta^{-1}) \times C^\infty_c(F^n)$, i.e. admits a factorisation of the form $T_x(W,W',\phi) = B_x(W,W')\phi(0)$.

- If $m < n$, we say the local factor $L(\pi,\pi',X)$ has no exceptional poles.

Thanks to a change of variable in $I(W,W',\phi,X)$, one sees that if $x$ is an exceptional pole of $L(\pi,\pi',X)$, then $B_x$ satisfies

$$B_x(\pi(g)W,\pi'(g)W') = \chi_x(g)B_x(W,W')$$

for any $g$ in $G_n$. We deduce the following property.

Proposition 4.2. If $\pi$ and $\pi'$ are irreducible $R$-representations of Whittaker type of $G_n$ and $G_m$, i.e. generic $R$-representations, and $x$ is an exceptional pole of $L(\pi,\pi',X)$, then $\pi' \vee \simeq \chi_x^{-1}\pi$.

Remark 4.3. When $\pi$ and $\pi'$ are complex or $\ell$-adic representations, the converse of Proposition 4.2 is true, and is proved in \cite{9}, Proposition 4.6]. This is no longer the case for $\ell$-modular representations, in cases when $q^\ell \equiv 1[\ell]$, as we shall see later. In fact, we can already see this when $q \equiv 1[\ell]$, as $L(\pi,\pi',X)$ is always equal to 1 in this case. This makes the case $q^\ell \equiv 1[\ell]$ pathological for the computation of $L$ factors of pairs.
We now introduce an auxiliary Euler factor.

**Definition 4.4.** Let $\pi$ and $\pi'$ be $R$-representations of Whittaker type of $G_n$, and let $W$ and $W'$ belong to $W(\pi, \theta)$ and $W(\pi', \theta^{-1})$. We define formally

$$I_0(W, W', X) = \sum_{k \in \mathbb{Z}} \left( \int_{N_{n-1} \setminus G_{n-1}^P} W(\text{diag}(g, 1))W'(\text{diag}(g, 1))dg \right) q^k X^k.$$ 

As before, we have the following result.

**Proposition 4.5.** With the hypothesis of Definition 4.4, the integrals $I_0(W, W', X)$ are Laurent series, and in fact span, as $W$ and $W'$ vary, a fractional ideal $I_0(\pi, \pi')$ of $R[X^\pm 1]$ which has a unique generator $L_0(\pi, \pi', X)$ which is an Euler factor.

**Remark 4.6.** When $\pi$ is cuspidal, as $W \mid_{P_n}$ has compact support modulo $N_n$ (hence $W \mid_{G_{n-1}}$ has compact support modulo $N_{n-1}$), the Laurent series $I_0(W, W', X)$ only has finitely many nonzero terms, hence is a Laurent polynomial. In particular, the factor $L_0(W, W', X)$ is equal to 1.

The following property follows from Corollary 2.9.

**Proposition 4.7.** The ideal $I_0(\pi, \pi')$ is also the ideal spanned by the integrals $I(W, W', \phi, X)$, for $\phi$ in $C^\infty_0(F^n)$. In particular, the factor $L_0(\pi, \pi', X)$ divides $L(\pi, \pi', X)$, and we denote by $L_0(\pi, \pi', X)$ their quotient. It is an Euler factor which divides the Tate $L$-factor $L(c_\pi c_{\pi'}, X^n)$, and its poles are exactly the exceptional poles of $L(\pi, \pi', X)$.

**Proof.** We denote by $f_\phi$ the function on $F$ defined as $f_\phi : t \mapsto \phi(0, \ldots, 0, t)$. Using the integration formula of Corollary 2.9 can write

$$I(W, W', \phi, X) = \int_{(K_n \cap P_n) \setminus K_n} I_0(\rho(k)W, \rho(k)W', X)I(c_\pi, c_{\pi'}, f_\rho(k)\phi, X^n)dk,$$

where $\rho$ denotes right translation. This proof now follows mutatis mutandis the proof of [9, Proposition 4.3].

**Remark 4.8.** When $\pi$ and $\pi'$ are complex or $\ell$-adic representations, the factor $L_0(\pi, \pi', X)$ has simple poles, as $L(c_\pi c_{\pi'}, X^n)$ does. This latter assertion is not true for $\ell$-modular representations, when $n$ is not prime to $\ell$. The first assertion is not true either, as we shall see for example when $q^n \neq 1[\ell]$. In any case, we always have $L(\pi, \pi', X) = L_0(\pi, \pi', X)L_0(\pi, \pi', X)$, and $L_0(\pi, \pi', X)$ can be expressed in terms of the $L$-factors of the derivatives of $\pi$ and $\pi'$ (see [4], at least for generic segments, this fact remains true modulo-$\ell$). So in the characteristic zero case, the factor $L_0(\pi, \pi', X)$ is well understood, as it has simple poles, and these are the exceptional poles, which can be determined thanks to Proposition 4.2 and Remark 4.3. In the $\ell$-modular setting, we already said that in Remark 4.3 that the converse of Proposition 1.2 is no longer true in general. Moreover, $L_0(\pi, \pi', X)$ might have poles which are not simple anymore. These are the two sources of complications modulo-$\ell$, the first being the most problematic.

### 4.2 L-factors of pairs for cuspidal representations

We will study in more detail the $L$-factors of pairs of cuspidal representations. We will express such factors in terms of the Tate $L$-factors of the unramified characters fixing these cuspidal representations. Before we can do this, we need to recall the following result of Bernstein in [3] for $\ell$-adic representations.
Theorem 4.9. Let $\pi$ and $\pi'$ be irreducible $\ell$-adic representations of $G_n$, if $\pi' \simeq \pi^\vee$ (i.e. $\text{Hom}_{G_n}(\pi \otimes \pi', \mathbb{Q}_\ell) \neq \{0\}$), then we have $\text{Hom}_{P_n}(\pi \otimes \pi', \mathbb{Q}_\ell) = \text{Hom}_{G_n}(\pi \otimes \pi', \overline{\mathbb{Q}}_\ell)$, i.e. any $P_n$-invariant bilinear pairing between $\pi$ and $\pi'$ is in fact $G_n$-invariant.

Let $\rho$ and $\rho'$ be cuspidal representations of $G_n$ and $G_m$, with $m \leq n$. First, we observe that if $m < n$, as the restriction to $P_n$ of any $W$ in $W(\rho, \theta)$ has compact support modulo $N_m$ (c.f. [17]), the integrals of the form $I(W, W', X)$ for $W' \in W(\rho', \theta)$ are in fact in $R[X^{\pm 1}]$. In particular, if $m < n$, then $L(\rho, \rho', X)$ is trivial.

Proposition 4.10. If $m < n$, then $L(\rho, \rho', X)$ is equal to 1.

Hence the interesting case is when $n = m$. We will use the following $\ell$-modular version of Bernstein’s result.

Proposition 4.11. Let $\pi$ be a cuspidal $\ell$-modular representation of $G_m$, then the bilinear map

$$B : (W, W') \mapsto \int_{N_m \backslash P_m} W(p)W'(p)dp$$

from $W(\pi, \theta) \times W(\pi^\vee, \theta^{-1})$ to $\mathbb{C}^\ell$ is $G_m$-invariant.

Proof. Let $\tau$ be a cuspidal $\ell$-adic representation of $G_m$ with reduction modulo-$\ell$ equal to $\pi$. Any $W$ and $W'$ lift to $V$ and $V'$ in $W(\tau, \theta)$ and $W(\tau^\vee, \theta^{-1})$. As $C : (V, V') \mapsto \int_{N_m \backslash P_m} V(p)V'(p)dp$ is $P_m$-invariant, it is $G_m$-invariant by theorem 4.9. But $C$ takes integral values on $W(\tau, \theta) \times W(\tau^\vee, \theta^{-1})$, and $r_\ell(C) = B$, hence $B$ is $G_m$-invariant.

Let $R$ be $\mathbb{C}^\ell$ or $\mathbb{F}_\ell$, and let $R_u(G_1)$ denote the set of unramified $R$-characters of $G_1$. Let $\tau$ be a integral cuspidal $\ell$-adic representation of $G_m$, and $\pi$ be the reduction modulo-$\ell$ of $\tau$. Denote by $R(\tau)$ and $R(\pi)$ the respective cyclic subgroups of $R_u(G_1)$ fixing $\tau$ and $\pi$ by twisting, and denote by $n(\tau)$ and $n(\pi)$ their respective orders. We recall, that by looking at the central characters of $\tau$ and $\pi$, the integers $n(\tau)$ and $n(\pi)$ both divide $n$. It follows, from the Bushnell–Kutzko construction of all irreducible cuspidal representations via types given in [15] III 5), that the map $r_\ell$ from $R(\tau)$ to $R(\pi)$ is surjective, with kernel the $\ell$-part $R_\ell(\tau)$ of $R(\tau)$. Hence we can write $\ell^{d_\tau} = |R(\tau)|/|R(\pi)|$, with $d_\tau$ the multiplicity of $\ell$ as a factor of $|R(\tau)|$, which is independant from $\tau$ (c.f. [12] Remark 3.21) for more details about these assertions).

More generally, if $\tau$ and $\tau'$ be integral cuspidal $\ell$-adic representations of $G_m$, and $\pi$ and $\pi'$ denote their reductions modulo-$\ell$, we denote by $R(\tau, \tau')$ (resp. $R(\pi, \pi')$) the set of unramified characters $\chi$ of $G_1$ such that $\tau \simeq \chi\tau'$ (resp $\pi \simeq \chi\pi'$), and by $n(\tau, \tau')$ and $n(\pi, \pi')$ their cardinality. In particular $R(\tau, \tau') = R(\tau)$ and $R(\pi, \pi') = R(\pi)$. Notice that $R(\tau, \tau')$ (resp. $R(\pi, \pi')$), if not empty, is an element of $R(\tau)\backslash R_u(G_1)$ (resp. $R(\pi)\backslash R_u(G_1)$). In particular, the reduction modulo-$\ell$ map from $R(\tau, \tau')$ to $R(\pi, \pi')$ is surjective. Thus $R(\tau, \tau')$ is non-empty if and only if $R(\pi, \pi')$ is non-empty, in which case $n(\tau, \tau') = n(\tau) = \ell^{d_\tau}n(\pi) = \ell^{d_\tau}n(\pi, \pi')$, and $r_\ell$ induces a bijection $R_\ell(\tau)\backslash R(\tau, \tau') \simeq R(\pi, \pi')$.

Lemma 4.12. Let $\tau$ and $\tau'$ be integral $\ell$-adic cuspidal representations of $G_m$, with reduction modulo-$\ell$ equal to $\pi$ and $\pi'$, then

$$r_\ell(L(\tau, \tau', X)) = 1$$

if $R(\pi, \pi')$ is empty, and

$$r_\ell(L(\tau, \tau', X)) = \prod_{\chi \in R(\pi, \pi')} 1/(1 - \chi(\pi)X)^{\ell d_\chi}$$

otherwise.
Proof. As in [7 Proposition 8.1], we have $L(\tau, \tau^\vee, X) = \prod_{\chi \in \mathcal{R}(\tau, \tau^\vee)} 1/(1 - \chi(\varpi)X)$. The result follows from the discussion before the lemma. \hfill \square

Note that, for $q \neq 1[\ell]$, with the correct normalisation of $R$-Haar measure on $F^\times$, the Tate Laurent series $I(1, 1_{p^k}, X)$ is equal to $X^{-k}L(1, X)$. In some cases, we can describe the $\ell$-modular $L$-factors associated to a pair of cuspidal representations of $G_n$ more precisely than in Theorem 3.14. This leads us to the main theorem of this section.

**Theorem 4.13.** Let $\pi$ and $\pi'$ be cuspidal $\ell$-modular representations of $G_n$, and $\tau$ and $\tau'$ be cuspidal $\ell$-adic representations of $G_n$ lifting $\pi$ and $\pi'$.

1. If $\text{ord}(\pi, \pi') = 0$, then $L(\pi, \pi', X) = 1$.
2. If $q^n \neq 1[\ell]$, then $L(\pi, \pi', X)$ is equal to $r(\tau, \tau', X)$.
3. If $q^n \equiv 1[\ell]$, and $n$ is a power of $\ell$, which is equivalent to $q \equiv 1[\ell]$, then $L(\pi, \pi', X) = 1$.
4. If $q^n(\pi, \pi') \equiv 1[\ell]$, and $n$ is prime to $\ell$, then $L(\pi, \pi', X) = 1$.

**Proof.** As $\pi$ is cuspidal, we have $L(\pi, \pi', X) = 1$, hence all poles of $L(\pi, \pi', X)$ are exceptional. If $R(\pi, \pi')$ is empty, then $L(\pi, \pi', X) = L(\pi, \pi', X) = 1$ by Propositions 4.7 and 4.2 giving case 1. If $q \equiv 1[\ell]$, we have already seen that $L(\pi, \pi', X)$ is equal to 1, so cases 1. and 3. have been proved. We thus suppose that $R(\pi, \pi')$ is non empty, and $q \neq 1[\ell]$, which implies that $L(1, X^n)$ is equal to $1/(1 - X^n)$.

By Proposition 4.2 we know that if $x$ is a pole of $L(\pi, \pi', X)$, then it will be of the form $x = \chi(\varpi)$ for some $\chi \in R(\pi, \pi')$. In this case, we want to compute the order of $x$ as a pole of $L(\pi, \pi', X)$. We can suppose that $\pi' \simeq \pi^\vee$, and look at the pole at $x = 1$.

Let $\phi = 1_{(p^k)n}$, so that $f_\phi = 1_{p^k}$ is fixed under the action of $K_n$, and $I(1, 1_{p^k}, X^n)$ is equal to $X^{-kn}L(1, X^n)$. Now let $\phi$ belong to $\mathcal{C}_\varnothing(F^n)$, for $k$ large enough, we can write it $\phi = \phi_01_{(p^k)n} + \phi_0$, where $\phi_0 \in \mathcal{C}_\varnothing(F^n)$. In particular, for $W$ and $W'$ in $W(\pi, \theta)$ and $W(\pi^\vee, \theta^{-1})$, we have

$$ I(W, W', \phi, X) = \phi(0)I(W, W', 1_{(p^k)n}, X) + I(W, W', \phi_0, X). $$

The Laurent series $I(W, W', \phi, X)$ belongs to $R[X^{\pm 1}]$, and as in the proof of Proposition 4.7 we see that $I(W, W', 1_{(p^k)n}, X)$ is equal to

$$ \phi(0)X^{-kn}L(1, X^n) \int_{(P_n \cap K_n) \setminus K_n} I(0)(\pi(k)W, \pi(k)W', X)dk. $$

In particular, the order of 1 as a pole of $L(\pi, \pi', X)$ will either be 0, or its order as a pole of

$$ F(X) = X^{-kn}L(1, X^n) \int_{(P_n \cap K_n) \setminus K_n} I(0)(\pi(k)W, \pi(k)W', X)dk; $$

if the latter is positive. Thus, the whole point is to understand this order. To do this, we use the functional equation, taking $\phi = 1_{(\ell)n} = \hat{\phi}$. The functional equation together with the Iwasawa decomposition gives us

$$ \gamma(\pi, \pi', X)F(X) = (qX)^{kn}L(1, (qX)^{-n})Q(X), $$

where

$$ Q(X) = \int_{(P_n \cap K_n) \setminus K_n} I(0)(\pi(k)W, \pi(k)W', (qX)^{-1})dk. $$

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However, $\gamma(\pi, \pi', X) = r_\ell(\gamma(\tau, \tau', X))$, and moreover, according to Proposition 4.11 we know that the Laurent polynomial $Q$ evaluated at $X = 1$ is equal to

$$dk(\!(P_n \cap K_n)\!ackslash K_n)_I(0)(W, W', q^{-1}).$$

The case $q^n \not\equiv 1[\ell]$ now appears naturally. If $q^n \not\equiv 1[\ell]$, then $dk((P_n \cap K_n)\!\backslash K_n) \neq 0$, and the factor $L(1, (qX)^{-n}) = 1/(1 - q^{-n}X^{-n})$ has no pole at 1. Hence, by definition of the $\gamma$-factor, the order of 1 as a pole of $F$ is equal to $n_1 - n_2$, where $n_1$ is the order of 1 as a pole of $r_\ell(L(\tau, \tau', X))$, and $n_2$ is its order as a pole of $r_\ell(L(\tau, \tau', (qX)^{-1}))$. Now, thanks to Lemma 4.12 we see that 1 can’t be a pole of $r_\ell(L(\tau, \tau', (qX)^{-1}))$, otherwise we would have $\chi(\varpi) = q^{-1}$ for some character in $R(\pi)$, but this is absurd as $\gamma(n)$ is trivial. Finally, the order of 1 as a pole of $F$ is equal to its order as a pole of $r_\ell(L(\tau, \tau', X))$. This proves case 2.

Hence we now suppose that $q^n(\pi) \equiv 1[\ell]$, and $n$ is prime to $\ell$. The factor $L(1, (qX)^{-n}) = 1/(1 - q^{-n}X^{-n})$ has a pole of order $n_0 = 1$ at 1. On the other hand, with the same notations as before, $n_2$ is equal to zero, except if there is $\chi$ in $R(\pi)$ such that $\chi(\varpi) = q^{-1}$, i.e. if and only if $q^n(\pi) = 1$, which is the case by assumption. Hence $n_2 = n_1 = \ell d_s$. Let $n_3$ be the order of 1 as a pole of $Q$, we know that $Q(1) = 0$ because $q^n \equiv 1[\ell]$, hence $n_3 < 0$. Finally, the order of 1 as a pole of $F$ being equal to $n_0 + n_1 - n_2 + n_3 = 1 + n_3 \leq 0$, and case 4, follows.

**Remark 4.14.** We can now give examples of the sort we alluded to in Remarks 4.3 and 4.8. In case 2 of Theorem 4.13 take $n(\pi, \pi') \neq 0$, and suppose that we have $d_\pi > 0$ (which is possible, as is easily seen for level zero cuspidal representations), then $L^0(\pi, \pi', X) = L(\pi, \pi', X)$ does not have simple poles. In case 4 of Theorem 4.13 take $\pi' = \pi^\vee$, then $L^0(\pi, \pi', X) = L(\pi, \pi', X) = 1$, and 1 is not an exceptional pole of $L(\pi, \pi', X)$.

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