Laguerre and Touchard Polynomials for Linear Volterra Integral and Integro Differential Equations

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Abstract. In this paper, efficient numerical methods are given to solve linear Volterra integral (VI) equations and Volterra Integro differential (VID) equations of the first and second types with exponential, singular, regular and convolution kernels. These methods based on Laguerre polynomials (LPs) and Touchard polynomials (TPs) that convert these equations into a system of linear algebraic equations. The results are compared with one another method and with each other. The results show that these methods are applicable and efficient. In addition, the accuracy of solution is presented and also the calculations and Graphs are done by using matlab2018 program.

Keywords: Volterra integral and integro differential equation, Laguerre polynomials, Touchard polynomials, approximate numerical solutions

Introduction:

The idea of this work is to illustrate the results of the solutions for linear Volterra integral (VI) equations and linear Volterra integro differential (VID) equations in two methods using the (LPs) and (TPs). Such equations are model of problems in many applications, like, heat conduction, dynamics of viscoelastic, electrodynamics [1]. The solutions of integral and integro differential equations have an essential role in several applied areas which include “mechanics, chemistry, physics, biology, astronomy and potential theory” [2].

The general formulas of the linear (VI) equations of the 2nd and 1st types [3, 4] respectively are defined by:

\begin{equation}
Q(\alpha) = w(\alpha) + \gamma \int_{a}^{\alpha} Y(\alpha, \tau) Q(\tau) d\tau, \quad b_{1} \leq \alpha \leq b_{2}
\end{equation}

\begin{equation}
-w(\alpha) = \gamma \int_{a}^{\alpha} Y(\alpha, \tau) Q(\tau) d\tau, \quad b_{1} \leq \alpha \leq b_{2}
\end{equation}

Also the general formula of linear Abel’s singular of the 1st type [4, 5 and 6] is defined as follows:

\begin{equation}
W(\alpha) = \gamma \int_{a}^{\alpha} \frac{1}{\sqrt{\alpha - \tau}} Q(\tau) d\tau, \quad b_{1} \leq \alpha \leq b_{2}
\end{equation}

The general formula of the linear (VID) equation of the 1st order and 2nd type [4] is defined as follows:

\begin{equation}
Q'(\alpha) = w(\alpha) + \gamma \int_{0}^{\alpha} Y(\alpha, \tau) Q(\tau) d\tau, \quad b_{1} \leq \alpha \leq b_{2}
\end{equation}

with initial condition \( Q(0) = Q_{0} \).
where $Q' (\alpha) = \frac{dQ}{d\alpha}, b_1, b_2$ are constants, $Q (\alpha)$ is the unknown function that must be determined, $\gamma$ is a known constant, it represents the physical meaning of the material, and $Y (\alpha, \tau)$ is a kernel of the Integral equations (IEs), which is a known continuous or dis-continuous function holds characteristic or property of the material, $w (\alpha)$ is a known function represents the integration surface and $Q(0) = Q_0$ is a constant initial condition for eq. (3).

There are many approximate numerical methods used and developed by the scientific researchers to obtain the approximate numerical solutions for the (VI) equations and (VID) equations, mentioned as follows: [7] proposed numerical methods to solve weakly (VI) equations of the 1st type. [8] gave numerical method for the approximation of the (VI) equations with oscillatory Bessel kernels. [9] applied Chebyshev wavelet method to solve the (VI) equations with weakly singular of kernels. [10] used the standard Galerkin polynomial method to solve weakly singular kernels for the (VI) equations. [11] extended the single step pseudo spectral method to the multi step pseudo spectral method for the (VI) equations of 2nd type. [12] applied the Galerkin weight residual method and (LPs) as a trial function for solving the (VI) equations of the 1st, 2nd type with singular and regular kernels. [13] used the (LPs) for solving system of generalized Abel integral equations. [14] used iterative methods to solve the (VID) equations with singular kernel. [15] applied collocation method to solve the (VID) equations. [16] applied “Galerkin the weight residual method” with the (TPs) as a trial function to get numerical solutions to (IEs).

This article is arranged as follows: Laguerre polynomials, function of approximation using the (LPs), Touchard polynomials, function of approximation using the (TPs), solution the (VI) equation using the (LPs) method, accuracy of solutions, convergence rate, illustrative examples, tables and figures are provided, summary of conclusions and recommendations. Finally the references are mentioned.

1. Laguerre Polynomials [12 and 13]:

This section, begin with definition of the (LPs) which was studied in 1782 by Adrien-Marie Legendre. The (LPs) consisting of the polynomial sequence of binomial type, it’s defined on $[0, \infty)$ as follows:

$$V_k (\alpha) = \sum_{s=0}^{k} (-1)^s \frac{1}{s!} \binom{k}{s} \alpha^s = \sum_{s=0}^{k} \frac{(-1)^s}{(s!)^2 (k-s)!} \alpha^s, k = 0, 1, 2, ... n \text{ and } \alpha \in [0, \infty) \cdots (4)$$

where $k$ and $s$ represent the degree and the index for the (LPs) respectively.

The first five polynomials of the (LPs) are given below:

$V_0 (\alpha) = 1$

$V_1 (\alpha) = 1 - \alpha$

$V_2 (\alpha) = \frac{1}{2} (2 - 4\alpha + \alpha^2)$.

$V_3 (\alpha) = \frac{1}{6} (6 - 18\alpha + 9\alpha^2 - \alpha^3)$

$V_4 (\alpha) = \frac{1}{24} (24 - 96\alpha + 72\alpha^2 - 16\alpha^3 + \alpha^4)$

3. Function of Approximation using the (LPs):

Suppose that the function $Q_k (\alpha)$ is approximated using the (LPs) as follows:

$$Q_k (\alpha) = \vartheta_0 V_0 (\alpha) + \vartheta_1 V_1 (\alpha) + \cdots + \vartheta_k V_k (\alpha) = \sum_{s=0}^{k} \vartheta_s V_s (\alpha) \quad 0 \leq \alpha < \infty, \quad \cdots (5)$$
for \( s \geq 0 \), the function \( \{ V_s(\alpha) \}_{s=0}^{k} \) denotes the Laguerre basis polynomials of \( k \)th degree, as defined in Eq. (4). \( \Theta_s \) \((s = 0, 1, \ldots, k)\) are the unknowns Laguerre coefficients that calculate later. Writing Eq. (5) as a dot product:

\[
Q_0(\alpha) = [V_0(\alpha) V_1(\alpha) \ldots V_k(\alpha)] \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_k \end{bmatrix},
\]

Eq. (6) can be written in the following form:

\[
Q_k(\alpha) = \begin{bmatrix} 1 & \alpha & \alpha^2 & \ldots & \alpha^k \end{bmatrix} \begin{bmatrix} \theta_{00} & \theta_{01} & \theta_{02} & \ldots & \theta_{0k} \\ 0 & \theta_{11} & \theta_{12} & \ldots & \theta_{1k} \\ 0 & 0 & \theta_{22} & \ldots & \theta_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \theta_{kk} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_k \end{bmatrix},
\]

where \( \theta_{\rho \rho} (\rho = 0, 1, 2, \ldots, k) \) are known values of the power basis that are used to find the (LPs), also the square matrix is an upper triangular and non-singular. For example, if \( k = 1, 2 \), the operational matrices are shown as in Eqs. (8) and (9) respectively:

\[
Q_1(\alpha) = [1 \ \alpha], \begin{bmatrix} \theta_0 \\ 1 \end{bmatrix},
\]

\[
Q_2(\alpha) = [1 \ \alpha^2], \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}.
\]

Since the derivative of Eq. (4) is:

\[
V'_k(\alpha) = \frac{d}{d\alpha} \sum_{s=0}^{k} (-1)^s \frac{1}{s!} (k^s \alpha^s) = \sum_{s=1}^{k} \frac{(-1)^s}{(s!)^2 (k-s)!} s \alpha^{s-1}, \text{ } k = 1, 2, \ldots, n, \text{ and } \alpha \in [0, \infty)
\]

so, the derivative of Eqs. (7), (8) and (9) is respectively:

\[
Q'_k(\alpha) = \begin{bmatrix} 0 & 1 & \alpha & \ldots & k\alpha^{k-1} \end{bmatrix}, \begin{bmatrix} \theta_{00} & \theta_{01} & \theta_{02} & \ldots & \theta_{0k} \\ 0 & \theta_{11} & \theta_{12} & \ldots & \theta_{1k} \\ 0 & 0 & \theta_{22} & \ldots & \theta_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \theta_{kk} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_k \end{bmatrix},
\]

\[
Q'_1(\alpha) = [0 \ 1], \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \theta_0 \\ 1 \end{bmatrix},
\]

\[
Q'_2(\alpha) = [0 \ 1 \ 2\alpha], \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix}, \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix}.
\]

4. Touchard Polynomials [16, 17, 18 and 19]:
(TPs) were first studied by the French mathematician Jacques Touchard 1885–1968, consisting of the polynomial sequence of binomial type, it’s defined on [0, 1] as following:

\[ O_k(\alpha) = \sum_{s=0}^{k} A(k,s)\alpha^s = \sum_{s=0}^{k} \binom{k}{s} \alpha^s , \quad A(k,s) = \frac{k!}{s!(k-s)!} , \quad \ldots (11) \]

where \( k \) and \( s \) represent the degree and the index for the (TPs) respectively.

The first five polynomials of the (TPs) are written below:

\[
\begin{align*}
O_0(\alpha) &= 1 \\
O_1(\alpha) &= 1 + \alpha \\
O_2(\alpha) &= 1 + 2\alpha + \alpha^2 \\
O_3(\alpha) &= 1 + 3\alpha + 3\alpha^2 + \alpha^3 \\
O_4(\alpha) &= 1 + 4\alpha + 6\alpha^2 + 4\alpha^3 + \alpha^4.
\end{align*}
\]

5. Function of Approximation using the (TPs):

Suppose that the function \( Q_\alpha(\alpha) \) is approximated using the (TPs) as follows:

\[
Q_\alpha(\alpha) = \vartheta_0 O_0(\alpha) + \vartheta_1 O_1(\alpha) + \cdots + \vartheta_k O_k(\alpha) = \sum_{s=0}^{k} \vartheta_s O_s(\alpha), \quad 0 \leq \alpha \leq 1 \quad \ldots (12)
\]

for \( s \geq 0 \), the function \( \{O_s(\alpha)\}_{s=0}^{k} \) denotes the Touchard basis polynomials of \( k \)th degree, as defined in Eq. (11). \( \vartheta_s (s = 0,1,\ldots,k) \) are the unknowns Touchard coefficients that determine later.

Writing Eq. (12) as a dot product:

\[
Q_\alpha(\alpha) = [O_0(\alpha) \ O_1(\alpha) \ldots O_k(\alpha)] \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ \vdots \\ \vartheta_k \end{bmatrix} \quad \ldots (13)
\]

Eq. (13) can be written as follows:

\[
Q_\alpha(\alpha) = \begin{bmatrix} \varepsilon_{00} & \varepsilon_{01} & \cdots & \varepsilon_{0k} \\
0 & \varepsilon_{11} & \cdots & \varepsilon_{1k} \\
0 & 0 & \cdots & \varepsilon_{kk} \\
0 & 0 & 0 & \cdots \varepsilon_{kk} \end{bmatrix} \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ \vdots \\ \vartheta_k \end{bmatrix} \quad \ldots (14)
\]

where \( \varepsilon_{\rho\rho} (\rho = 0, 1, 2,\ldots, k) \) are known constants of the power basis that are used to find the (TPs), also the square matrix is an upper triangular and non-singular. For instance, if \( k=2 \) and \( 3 \), the operational matrices are shown in Eqs. (15) and (16) respectively:

\[
Q_2(\alpha) = \begin{bmatrix} 1 & 1 & 1 \\
0 & 1 & 2 \\
0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ \vartheta_2 \end{bmatrix} \quad \ldots (15)
\]
\[ Q_3(\alpha) = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \] ... (16)

Since, the derivative of Eq. (11) is:

\[ O'_{k}(\alpha) = \frac{d}{d\alpha} \sum_{s=0}^{k} A(k,s)\alpha^s = \sum_{s=1}^{k} \binom{k}{s}s\alpha^{s-1} \text{, where } \binom{k}{s} = \frac{k!}{s!(k-s)!} \] ... (17)

then, the derivative of Eqs. (14), (15) and (16) respectively is:

\[ Q'_k(\alpha) = \begin{bmatrix} \epsilon_{00} & \epsilon_{01} & \ldots & \epsilon_{0k} \\ 0 & \epsilon_{11} & \ldots & \epsilon_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \epsilon_{kk} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \ldots \\ \theta_k \end{bmatrix} \] ... (17a)

\[ Q'_2(\alpha) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} \] ... (17b)

\[ Q'_3(\alpha) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \] ... (17c)

6. Solution the (VI) Equation of the 2nd type using the (LPs):

In this section, the (LPs) is used to find the solutions for the (VI) equation. Since Eq. (1) is:

\[ Q(\alpha) = w(\alpha) + \gamma \int_{a}^{\alpha} Y(\alpha, \tau) Q(\tau) d\tau , \quad b_1 \leq \alpha \leq b_2 \] ... (18)

by using Eq. (5), suppose that:

\[ Q(\alpha) \equiv Q_{k}(\alpha) = \sum_{s=0}^{K} \theta_s V_s(\alpha) \] ... (19)

now, substituting Eq. (19) into Eq. (18), gives:

\[ \sum_{s=0}^{K} \theta_s V_s(\alpha) = w(\alpha) + \gamma \int_{a}^{\alpha} Y(\alpha, \tau) \sum_{s=0}^{K} \theta_s V_s(\tau) d\tau , \] ... (20)

By using Eq. (6), then Eq. (20) becomes:

\[ [V_0(\alpha) \quad V_1(\alpha) ... V_k(\alpha)] \begin{bmatrix} \theta_0 \\ \theta_1 \\ \ldots \\ \theta_k \end{bmatrix} = w(\alpha) + \gamma \int_{a}^{\alpha} Y(\gamma, \tau)[V_0(\tau) \quad V_1(\tau) ... V_k(\tau)] d\tau , \] ... (21)

And by using Eq. (7), so, Eq. (21) is converted to:
Now, after simplifying Eq. (22), the unknown Laguerre coefficients \( \theta_0, \theta_1, \ldots, \theta_k \) are obtained by selecting points \( \alpha_\beta (\beta = 0, 1, \ldots, k) \) in the interval \( [b_1, b_2] \). Consequently, Eq. (22) converts to a system of \((k+1)\) linear algebraic equations in \((k+1)\) unknown coefficients, this system can be solved using “Gauss elimination method” to obtain these coefficients, which have the unique solutions. These coefficients are substituted into Eq. (5), to get the approximate numerical solution for Eq. (1).

The same procedure can be applied to Eqs. (1a) and (2) when using the (TPs).

7. Solution the (VID) Equation of the 1st order and 2nd type using the (LPs):
In this section, the (TPs) is used to find the solutions for the (VID) equation. Since Eq. (3) is:

\[ Q'(\alpha) = w(\alpha) + \gamma \int_a^\alpha Y(\alpha, \tau) Q(\tau) d\tau, \quad b_1 \leq \alpha \leq b_2, \quad \ldots \quad (23) \]

\[ Q(0) = Q_0, \quad \ldots \quad (23a) \]

by using Eqs.(7) and (10a), suppose that:

\[
Q(\alpha) \cong Q_k(\alpha) = \begin{bmatrix} 1 & \alpha & \ldots & \alpha^k \end{bmatrix} \begin{bmatrix} \theta_{00} & \theta_{01} & \theta_{02} & \ldots & \theta_{0k} \\ 0 & \theta_{11} & \theta_{12} & \ldots & \theta_{1k} \\ 0 & 0 & \theta_{22} & \ldots & \theta_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \theta_{kk} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_k \end{bmatrix}
\]

\[ Q(\alpha) \cong Q_k(\alpha) = \begin{bmatrix} 1 & \alpha & \ldots & \alpha^k \end{bmatrix} \begin{bmatrix} \theta_{00} & \theta_{01} & \theta_{02} & \ldots & \theta_{0k} \\ 0 & \theta_{11} & \theta_{12} & \ldots & \theta_{1k} \\ 0 & 0 & \theta_{22} & \ldots & \theta_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \theta_{kk} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_k \end{bmatrix}
\]

\[ Q'(\alpha) \cong Q'_k(\alpha) = \begin{bmatrix} 0 & 1 & \alpha & \ldots & \alpha^{k-1} \end{bmatrix} \begin{bmatrix} \theta_{00} & \theta_{01} & \theta_{02} & \ldots & \theta_{0k} \\ 0 & \theta_{11} & \theta_{12} & \ldots & \theta_{1k} \\ 0 & 0 & \theta_{22} & \ldots & \theta_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \theta_{kk} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_k \end{bmatrix}
\]

now, by substituting Eqs. (24) and (25) into Eq. (23), gives:
So, after simplifying Eq. (26), the unknown Touchard coefficients ($\theta_0$, $\theta_1$, ..., $\theta_k$) are obtained by selecting points $\alpha_\beta$ ($\beta = 0, 1, ..., k$) in the interval \([b, b]\), with the initial condition Eq. (23a). Therefore, Eq. (26) converts to a system of \((k+1)\) linear algebraic equations in \((k+1)\) unknown coefficients, this system can be solved using “Gauss elimination method” to obtain theses coefficients, which have unique solutions. These coefficients are substituted into Eq. (5), to get the approximate numerical solution for Eq. (3).

The same procedure can be applied when using the (TPs).

8. Accuracy of Solutions:

In this section, the accuracy of the proposed methods is tested.

8.1: For the (VI) equation:

Since Eq. (20) has the following formula:

$$\sum_{s=0}^{k} \theta_s V_s(\alpha) = w(\alpha) + \gamma \int_{a}^{\alpha} Y(\alpha, \tau) \sum_{s=0}^{k} \theta_s V_s(\tau) d\tau,$$

Since Eq. (5) has the following form:

$$Q(\alpha) = \sum_{s=0}^{k} \theta_s V_s(\alpha),$$

And the unknown Laguerre coefficients ($\theta_0, \theta_1, ..., \theta_k$) were determined by using Eq. (22). Also, by using Eq. (19), we have:

$$Q(\alpha) \equiv Q_k(\alpha) = \sum_{s=0}^{k} \theta_s V_s(\alpha),$$

then, Eq. (28) is the unique approximate solution for Eq. (27), and it’s substituted into Eq. (27). Now, suppose that $\alpha = \alpha_\theta \in [0, 1]$, $\theta = 0, 1, 2, ..., k$, and then, the error function:

$$AR(\alpha_\theta) = \left| \sum_{s=0}^{k} \theta_s V_s(\alpha_\theta) - w(\alpha_\theta) - \gamma \int_{a}^{\alpha} Y(\alpha_\theta, \tau) \sum_{s=0}^{k} \theta_s V_s(\tau_\theta) d\tau \right| \leq \epsilon,$$

Then, the difference for error function $AR(\alpha_\theta)$ at each point $\alpha_\theta$ will be smaller than any positive integer $\epsilon > 0$. Thus, the error function $AR(\alpha)$ can be estimated using the relation:
\[
AR_k (\alpha) = \sum_{s=0}^{k} \theta_s V_s(\alpha) - w(\alpha) - \gamma \int_{a}^{\alpha} Y(\alpha, \tau) \sum_{s=0}^{k} \theta_s V_s(\tau) d\tau,
\]

then, \(AR_k (\alpha) \leq \epsilon\).

This procedure is suitable for Eqs. (1a) and (2). Also this procedure can be applied using the (TPs).

8.2 For the (VID) equation:

Since Eq. (3) with initial condition is:
\[
Q'(\alpha) = w(\alpha) = \gamma \int_{0}^{\alpha} Y(\alpha, \tau) Q(\tau) d\tau, \quad b_1 \leq \alpha \leq b_2, \quad \ldots (29)
\]
\[Q(0) = Q_0,\]

since Eq. (5) has the following form:
\[
Q_k(\alpha) = \sum_{s=0}^{k} \theta_s V_s(\alpha),
\]
and the unknown Laguerre coefficients \((\theta_0, \theta_1, \ldots, \theta_k)\) were determined by using Eq. (26). Also, by using Eq. (19), we have:
\[
Q(\alpha) \equiv Q_k(\alpha) = \sum_{s=0}^{k} \theta_s V_s(\alpha), \quad \ldots (30)
\]
is the approximate numerical solution for Eq. (29) also, Eq. (30) and its derivative is substituted into Eq. (29). Now, suppose that \(\alpha = \alpha_0 \in [0, 1], \theta = 0, 1, 2, \ldots, k, \) and then, the error function:
\[
AR (\alpha_0) = \left| \left( \sum_{s=0}^{k} \theta_s V_s(\alpha_0) \right)' - w(\alpha_0) - \gamma \int_{a}^{\alpha} Y(\alpha_0, \tau_0) \sum_{s=0}^{k} \theta_s V_s(\tau_0) d\tau_0 \right| \equiv 0, \text{ then}
\]
\[AR (\alpha_0) \leq \epsilon, \text{ for each } \alpha_0 \text{ in [0, 1] and } \epsilon > 0.
\]

Then, the difference for error function \(AR (\alpha_0)\) at each point \(\alpha_0\) will be smaller than any positive integer \(\epsilon > 0\).

Thus, the error function \(AR (\alpha_0)\) can be estimated using the relation:
\[
AR_k (\alpha) = \left( \sum_{s=0}^{k} \theta_s V_s(\alpha_0) \right)' - w(\alpha) - \gamma \int_{a}^{\alpha} Y(\alpha, \tau) \sum_{s=0}^{k} \theta_s V_s(\tau) d\tau,
\]
then, \(AR_k (\alpha) \leq \epsilon\).

Note: This procedure can be applied using the (TPs) for Eq. (3).

9. Convergence Rate:

In this section, the error function can be defined by the following relation [20]:
\[
\|AR_k(\alpha)\| = \left( \int_{0}^{1} AR_k^2 (\alpha) d\alpha \right)^{1/2} \approx \left( \frac{1}{k} \sum_{s=0}^{k} ER_k^2 (\alpha_s) \right)^{1/2},
\]
where $\|AR_k(\alpha)\|$ is an arbitrary vector norm of error function, $AR_k(\alpha) = Q(\alpha) - Q_k(\alpha)$, where $Q(\alpha)$ and $Q_k(\alpha)$ are the exact and approximate numerical solutions respectively.

10. Illustrative Examples:
In this section, the (LPs) and (TPs) are used to solve linear (VI) and (VID) equations. These two polynomials have been applied to six numerical examples, and the convergence of solutions using the error function is given.

Example 1: Solve the linear (VI) equation of 1st type with the exponential kernel [20]:
\[
\int_0^\alpha e^{(\alpha - \tau)} Q(\tau) \, d\tau = \sin(\alpha), \quad \alpha \in [0, 1],
\]
where $Q(\alpha) = \cos(\alpha) - \sin(\alpha)$ is the exact solution.

For $k = 2, 3, 4, 5, 6$, the approximate results using:
1. The (LPs) are:
   - $Q_2(\alpha) = -0.8489V_0(\alpha) + 2.6884V_1(\alpha) - 0.8392V_2(\alpha)$.
   - $Q_3(\alpha) = 0.1621V_0(\alpha) - 0.4997V_1(\alpha) + 2.5137V_2(\alpha) - 1.1761V_3(\alpha)$.
   - $Q_4(\alpha) = 0.7515V_0(\alpha) - 2.9774V_1(\alpha) + 6.4215V_2(\alpha) - 3.9166V_3(\alpha) + 0.7210V_4(\alpha)$.
   - $Q_5(\alpha) = -0.2178V_0(\alpha) + 2.1162V_1(\alpha) - 0.4997V_2(\alpha) + 2.5137V_3(\alpha) - 1.1761V_5(\alpha)$.
   - $Q_6(\alpha) = -0.6581V_0(\alpha) + 4.8926V_1(\alpha) - 11.5853V_2(\alpha) + 17.5776V_3(\alpha) - 13.2746V_4(\alpha) + 4.9573V_5(\alpha)$.

2. The (TPs) are:
   - $Q_2(\alpha) = 1.5906O_0(\alpha) - 0.1707O_1(\alpha) - 0.4196O_2(\alpha)$.
   - $Q_3(\alpha) = 1.2960O_0(\alpha) + 0.6034O_1(\alpha) - 1.0954O_2(\alpha) + 0.1960O_3(\alpha)$.
   - $Q_4(\alpha) = 1.3568O_0(\alpha) + 0.3984O_1(\alpha) - 0.8371O_2(\alpha) + 0.0519O_3(\alpha) + 0.0300O_4(\alpha)$.
   - $Q_5(\alpha) = 1.3872O_0(\alpha) + 0.2747O_1(\alpha) - 0.6368O_2(\alpha) - 0.1097O_3(\alpha) + 0.0950O_4(\alpha) - 0.0104O_5(\alpha)$.
   - $Q_6(\alpha) = 1.3835O_0(\alpha) + 0.2923O_1(\alpha) - 0.6713O_2(\alpha) - 0.0738O_3(\alpha) + 0.0741O_4(\alpha) - 0.0039O_5(\alpha) - 8.2750E^{-4}O_6(\alpha)$.

The solutions were approximated in five different degrees. The comparison of error functions of the proposed methods and those in [20] is shown in Table 1, showing the (LPs) and (TPs) methods having a higher accuracy than in [20] with the same degrees, and that both proposed methods having the same accuracy.

Figure 1 shows the comparison of result for $k=6$ with exact solution. They seem to be identical.

| $k$ | $\|AR_k\|$ Method in [20] | (LPs) Method | (TPs) Method |
|-----|--------------------------|--------------|--------------|
| 2   | 5.06401E-02              | 1.1940E-01   | 1.1940E-01   |
| 3   | 2.07936E-03              | 6.7190E-03   | 6.7191E-03   |
| 4   | 6.14967E-04              | 1.2897E-03   | 1.2897E-03   |
| 5   | 1.42477E-04              | 3.3798E-05   | 3.3775E-05   |
| 6   | 5.41139E-05              | 3.7027E-06   | 3.5964E-06   |
Example 2: Solve the Abel’s (IEs) (linear (VI) equation of 1st type with singular kernel) [20]:
\[
\int_{0}^{\alpha} \frac{1}{\sqrt{\alpha - \tau}} Q(\tau) d\tau = \frac{2\sqrt{\alpha}}{105} (105 - 56\alpha^2 + 48\alpha^3), \quad \alpha \in [0, 1],
\]
where \( Q(\alpha) = \alpha^3 - \alpha^2 + 1 \) is the exact solution.

For \( k = 2, 3 \) and 4, the approximate results using:

1. The (LPs) are:
   \[
   Q_2(\alpha) = -0.0441V_0(\alpha) + 2.0183V_1(\alpha) - 0.9714V_2(\alpha),
   \]
   \[
   Q_3(\alpha) = 5V_0(\alpha) - 14V_1(\alpha) + 16V_2(\alpha) - 6V_3(\alpha),
   \]
   \[
   Q_4(\alpha) = 5V_0(\alpha) - 14V_1(\alpha) + 16V_2(\alpha) - 6V_3(\alpha) + 4.5324E - 12V_4(\alpha).
   \]

2. The (TPs) are:
   \[
   Q_2(\alpha) = 0.5925 O_0(\alpha) + 0.8960 O_1(\alpha) - 0.4857O_2(\alpha).
   \]
   \[
   Q_3(\alpha) = -O_0(\alpha) + 5 O_1(\alpha) - 4 O_2(\alpha) + O_3(\alpha),
   \]
   \[
   Q_4(\alpha) = -O_0(\alpha) + 5 O_1(\alpha) - 4 O_2(\alpha) + O_3(\alpha) + 1.8885E - 13 O_4(\alpha).
   \]

The solutions were approximated in three different degrees. The comparison of error functions of the proposed methods and those in [20] is shown in Table 2, showing the (LPs) and (TPs) methods having a higher accuracy than in [20] with the same degrees, and that both proposed methods having the same accuracy.

Figure 2 shows the comparison of result for \( k=4 \) with exact solution. They seem to be identical.

| Table 2. Comparison of the Error Function of Example 2. |
|---|---|---|
| \( k \) | \([\|AR_k\|]_\text{Method of [20]}\) | \([\|AR_k\|]_\text{(LPs) Method}\) | \([\|AR_k\|]_\text{(TPs) Method}\) |
| 2 | 6.39819E−02 | 5.1892E−01 | 5.1892E−01 |
| 3 | 2.42366E−02 | 4.2274E−07 | 5.4209E−07 |
| 4 | 3.26226E−03 | 3.6611E−07 | 4.6947E−07 |
Example 3: Solve the linear (VI) equation of 2nd type with the regular kernel [4]:

\[ Q(\alpha) = \alpha + \alpha^2 + \frac{1}{2} \alpha^3 + \frac{1}{5} \alpha^4 - \int_{0}^{\alpha} Q(\tau) \, d\tau, \quad \alpha \in [0, 1], \]

where the exact solution is \( Q(\alpha) = \alpha + \alpha^4 \).

For \( k = 2, 3, 4, 5 \) and 6, the approximate results using:

1. The (LPs) are:

   \begin{align*}
   Q_2(\alpha) &= 1.4402 V_0(\alpha) - 1.9327 V_1(\alpha) + 0.4959 V_2(\alpha), \\
   Q_3(\alpha) &= 6.3321 V_0(\alpha) - 17.5993 V_1(\alpha) + 17.2449 V_2(\alpha) - 5.9799 V_3(\alpha), \\
   Q_4(\alpha) &= 25 V_0(\alpha) - 97 V_1(\alpha) + 144 V_2(\alpha) - 96 V_3(\alpha) + 24 V_4(\alpha), \\
   Q_5(\alpha) &= 25 V_0(\alpha) - 97 V_1(\alpha) + 144 V_2(\alpha) - 96 V_3(\alpha) + 24 V_4(\alpha) - 4.2929E - 11 V_5(\alpha), \\
   Q_6(\alpha) &= 25 V_0 - 97 V_1 + 144 V_2 - 96 V_3 + 24 V_4 + 4.4567E - 09 V_5 - 7.9628E - 10 V_6.
   \end{align*}

2. The (TPs) are:

   \begin{align*}
   Q_2(\alpha) &= -0.68955 O_0(\alpha) + 0.44500 O_1(\alpha) + 0.24796 O_2(\alpha), \\
   Q_3(\alpha) &= -2.3957 O_0(\alpha) + 4.7342 O_1(\alpha) - 3.3374 O_2(\alpha) + 0.99666 O_3(\alpha), \\
   Q_4(\alpha) &= -2.8232E - 13 O_0(\alpha) - 3 O_1(\alpha) + 6 O_2(\alpha) - 4 O_3(\alpha) + O_4(\alpha), \\
   Q_5(\alpha) &= -1.5673E - 12 O_0(\alpha) - 3 O_1(\alpha) + 6 O_2(\alpha) - 4 O_3(\alpha) + O_4(\alpha) + 3.5774E - 13 O_5(\alpha), \\
   Q_6(\alpha) &= -7.9198E - 12 O_0 - 3 O_1 + 6 O_2 - 4 O_3 + O_4 + 9.3103E - 12 O_5 - 1.1059E - 12 O_6.
   \end{align*}

The solutions were approximated in five different degrees. The comparison of error functions of the (LPs) method and those in the (TPs) method is shown in Table 3, showing the (TPs) method having a higher accuracy than in the (LPs) method with the same degrees. Figure 3 shows the comparison of result for \( k=6 \) with exact solution. They seem to be identical.
Example 4: Solve the linear (VI) equation of 2nd type with the convolution kernel [4]:

\[ Q(\alpha) = \alpha + \int_0^\alpha (\tau - \alpha)Q(\tau)\,d\tau, \alpha \in [0, 1], \]

where \( Q(\alpha) = \sin(\alpha) \) is the exact solution.

For \( k = 2, 3, 4 \) and 5, the approximate results using:

1. The (LPs) are:
   \[
   \begin{align*}
   Q_2(\alpha) &= 0.8187 V_0(\alpha) - 0.6212 V_1(\alpha) - 0.1985 V_2(\alpha) \\
   Q_3(\alpha) &= 0.0277V_0(\alpha) + 1.9126 V_1(\alpha) - 2.9081V_2(\alpha) + 0.9677 V_3(\alpha) \\
   Q_4(\alpha) &= 0.2570V_0(\alpha) + 0.9372V_1(\alpha) - 1.3506V_2(\alpha) - 0.1387V_3(\alpha) + 0.2950V_4(\alpha) \\
   Q_5(\alpha) &= 0.9497V_0(\alpha) - 2.7377V_1(\alpha) + 6.4523V_2(\alpha) - 8.4278V_3(\alpha) + 4.7009V_4 \\
   &\quad - 0.9374V_5.
   \end{align*}
   \]

2. The (TPs) are:
   \[
   \begin{align*}
   Q_2(\alpha) &= -1.1184 O_0(\alpha) + 1.2167 O_1(\alpha) - 0.0993 O_2(\alpha) \\
   Q_3(\alpha) &= -0.8416 O_0(\alpha) + 0.5215 O_1(\alpha) + 0.4814 O_2(\alpha) - 0.1613 O_3(\alpha) \\
   Q_4(\alpha) &= -0.8121O_0(\alpha) + 0.4262O_1(\alpha) + 0.5963O_2(\alpha) - 0.2227O_3(\alpha) + 0.0123 O_4(\alpha)
   \end{align*}
   \]

Table 3. Comparison of the Error Function of the (LPs) and (TPs) of Example 3.

| k  | \( ||AR_k|| \) (LPs) Method | \( ||AR_k|| \) (TPs) Method |
|----|----------------------------|----------------------------|
| 2  | 7.1586E−01                 | 7.1586E−01                 |
| 3  | 2.0767E−01                 | 2.0767E−01                 |
| 4  | 3.3525E−06                 | 1.2191E−06                 |
| 5  | 2.9986E−06                 | 1.0904E−06                 |
| 6  | 2.7373E−06                 | 9.9539E−07                 |

**Figure 3(a).** The (LPs) of Example 4 for \( k=6 \)

**Figure 3(b).** The (TPs) of Example 4 for \( k=6 \).
\[ Q_5(\alpha) = -0.84020O_0(\alpha) + 0.5358O_1(\alpha) + 0.4262O_2(\alpha) - 0.0911O_3 - 0.0385O_4 + 0.0078O_5 \]

The solutions were approximated in five different degrees. The comparison of error functions of the (LPs) method and those in the (TPs) method is shown in Table 4, showing the (LPs) method having a higher accuracy than in the (TPs) method with the same degrees. Figure 4 shows the comparison of results for \(k=4\) and 5 with exact solution. They seem to be identical.

Table 4. Comparison of the Error Function of the (LPs) and (TPs) of Example 4.

| k  | (LPs) Method | (TPs) Method |
|----|--------------|--------------|
| 2  | 7.0865E-02  | 7.0865E-02  |
| 3  | 3.2692E-03  | 3.2693E-03  |
| 4  | 6.3587E-04  | 6.3589E-04  |
| 5  | 1.6865E-05  | 1.6908E-05  |

Figure 4(a). The (LPs) of Example 4 for \(k=4\) and 5.

Figure 4(b). The (TPs) of Example 4 for \(k=4\) and 5.

Example 5: Solve the (VID) equation of the 2nd type with constant kernel [4]:

\[ Q'(\alpha) = 6 - 3\alpha^2 + \int_0^\alpha Q(\tau) d\tau, \quad Q(0) = 0, \]

where \(Q(\alpha) = 6\alpha\) is the exact solution.

For \(k=2, 3\) and 4, the same exact solution is obtained, so, using the (LPs), we have:

\[ Q_2(\alpha) = 6V_0(\alpha) - 6V_1(\alpha) = Q(\alpha) = 6\alpha \]
\[ Q_3(\alpha) = 6V_0(\alpha) - 6V_1(\alpha) = Q(\alpha) = 6\alpha \]
\[ Q_4(\alpha) = 6V_0(\alpha) - 6V_1(\alpha) = Q(\alpha) = 6\alpha \]
Also using the (TPs), we have:

\[ Q_2(\alpha) = -6 \, O_{1}(\alpha) + 6 \, O_{2}(\alpha) = Q(\alpha) = 6\alpha \]
\[ Q_3(\alpha) = -6 \, O_{1}(\alpha) + 6 \, O_{2}(\alpha) = Q(\alpha) = 6\alpha \]
\[ Q_4(\alpha) = -6 \, O_{1}(\alpha) + 6 \, O_{2}(\alpha) = Q(\alpha) = 6\alpha \]

The solutions were approximated in three different degrees and the exact solution was obtained the same and this shows that the error function is zero in this case. Figure 5 displays the comparison of results for \(k=2, 3\) and \(4\) with exact solution. They seem to be identical.

**Example 6:** Solve the generalized Abel’s integro-differential equation of the 2\textsuperscript{nd} type \([2]\):

\[ Q'(\alpha) = -Q(\alpha) - \alpha + 0.2 \int_{0}^{\alpha} \frac{Q'(\tau) + 1}{\sqrt{\alpha - \tau}} \, d\tau, \quad 0 < \alpha \leq 1, \quad Q(0) = 1, \]

where \(Q(\alpha) = 1 - \alpha\), is the exact solution.

For \(k= 2, 3\) and \(4\), the same exact solution is obtained, then, using the (LPs) and (TPs), the results are respectively:

\[ Q_2(\alpha) = Q_3(\alpha) = Q_4(\alpha) = Q(\alpha) = 1 - \alpha \quad \text{and} \quad Q_2(\alpha) = Q_3(\alpha) = Q_4(\alpha) = Q(\alpha) = 1 - \alpha. \]

The solutions were approximated in three different degrees and the exact solution was obtained the same and this shows that the error function is zero in this case. Figure 6 displays the comparison of results for \(k=2, 3\) and \(4\) with exact solution. They seem to be identical.
11. Conclusions and Recommendations:

In this work, two effective approximate numerical methods base on the (LPs) and (TPs) have been used to get approximate numerical solutions for four examples of linear (VI) equation and two examples of the linear (VID) equation. The error function of these methods were established and appeared its accuracy. The results of both proposed methods in Tables 1 and 2 were better than in [20]. The results of error function for example 3 in Table 3 were decreasing with increased polynomials degrees, also, the results in Table 4 have shown that the (LPs) method is better than the (TPs) method. In examples 5 and 6, the approximate solutions were exactly the same as exact solution, so, the error functions were zero in these cases for both proposed methods. In general, all results indicate that the errors function decreasing with increasing the degree of polynomials as shown in the relevant Tables and Figures. Therefore, the methods used in this article can be applied to other types of integral equations, like, nonlinear integral and integro differential equations.

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