Entropy and anisotropy

Francisco J. Hernández and Hernando Quevedo

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México
P.O. Box 70-543, 04510 México D.F., México

(Dated: October 31, 2018)

Abstract

We address the problem of defining the concept of entropy for anisotropic cosmological models. In particular, we analyze for the Bianchi I and V models the entropy which follows from postulating the validity of the laws of standard thermodynamics in cosmology. Moreover, we analyze the Cardy-Verlinde construction of entropy and show that it cannot be associated with the one following from relativistic thermodynamics.

PACS numbers: 04.20.Jb, 04.70.Bw, 98.80.Hw

*Electronic address: fmoreno, quevedo@nucleares.unam.mx
I. INTRODUCTION

Entropy is a very important concept in physics. As a thermodynamic variable it should be present in any physical system to which energy, in any of its forms, can be associated. The origin of thermodynamics is purely phenomenological and, consequently, is based on certain laws which are the result of experiments usually performed on simple systems. It is therefore a difficult task to generalize these laws to include cases in which experiments are not available. For instance, it is not completely clear how to handle relativistic systems from the point of view of thermodynamics. This is especially difficult in the case of gravitational systems in which the concept of energy is not well-defined. Nevertheless, under certain assumptions gravitational fields can be treated as thermodynamical systems, i.e. systems in which the standard laws of thermodynamics are supposed to be valid. For a review of the main aspects of relativistic thermodynamics see, for instance, [1]. In the case of a simple Friedman-Robertson-Walker (FRW) universe, relativistic thermodynamics gives certain values for the thermodynamic variables which are in agreement with physical expectations [2].

Nevertheless, it is quite possible to find cosmological configurations in which mathematically well-defined state and thermodynamic variables predict unphysical behaviors. Indeed, it has been shown [3] that the thermodynamic variables obtained by applying the laws of relativistic thermodynamics to inhomogeneous cosmological models lead to unphysical temperature evolution laws, and that physical relevant behaviors can be expected only when all the inhomogeneities vanish, i.e., in the FRW limit. This result obviously points out to an inconsistency between cosmology in general relativity and the first law of thermodynamics, at least when inhomogeneities are present. The natural question arises whether this inconsistency persists in the case of homogeneous (anisotropic) cosmological models. In this work we will show that cosmological anisotropies can be treated in a consistent manner in the context of relativistic thermodynamics.

Recently, Verlinde [4] proposed an alternative approach to the concept of entropy, based on a formal analogy between the field equations for a FRW cosmology and the thermodynamic formulas of conformal field theory (CFT). This analogy seems to be related to the holographic principle according to which for a given volume one can associate a maximal amount of entropy which corresponds to the entropy of the largest black hole that can be
fitted inside this volume \cite{5, 6}. Furthermore, the entropy of black holes has been calculated recently \cite{7, 8} by counting microscopic states. Although these computations start from very different physical concepts, they make use of techniques known in two-dimensional CFT. At the first sight this seems to be only a useful trick – the Cardy entropy formula \cite{9} allows to easily count states in two-dimensional CFT – but some recent results \cite{10, 11} suggest that CFT’s offer a model-independent description of black hole thermodynamics at low energies. The universality of this description might be related to the simple common feature that the algebra of diffeomorphisms at the black hole horizon has a conformal structure \cite{12}. In light of these results it seems reasonable to expect that the Cardy entropy formula could have some applications in cosmology as proposed by Verlinde \cite{4} for FRW models. The Cardy-Verlinde procedure for defining entropy has been generalized to anisotropic Bianchi IX cosmologies in \cite{13}.

In this paper, we analyze the entropy of homogeneous cosmological models with a perfect fluid as source. We first review in Section II an alternative approach to classical thermodynamics which is based upon the contact structure of the thermodynamic phase space. This geometric approach allows a simple generalization to the case of relativistic systems, and makes it particularly easy to interpret relativistic thermodynamics. We find the sufficient and necessary conditions which need to be fulfilled in order to determine entropy and temperature for a given cosmological model, and show in Section IV that in the case of homogeneous configurations they are identically satisfied. This result is used to compute explicit expressions for the entropy and temperature of homogeneous models and we show that they correspond to an adiabatic process, and evolve in a physically reasonable manner when compared with their FRW counterparts. Then in Section V we find for Bianchi I and V models the corresponding CFT entropy by using the Cardy-Verlinde formula. It is shown that this entropy does not satisfy the adiabatic property which follows from energy conservation and relativistic thermodynamics. We conclude in Section VI that this result is not enough to dismiss Cardy’s entropy as unphysical.

II. CLASSICAL THERMODYNAMICS OF EQUILIBRIUM STATES

In this section we find the conditions which must be satisfied in order to define the entropy for a given thermodynamic system. For the sake of simplicity, we limit ourselves
here to the case of a monocomponent simple system. According to Hermann’s geometric approach \cite{15}, one usually begins with the introduction of a suitable thermodynamic phase space $\mathcal{T}$ which in this case is a 5-dimensional manifold, topologically equivalent to $\mathbb{R}^5$. In $\mathcal{T}$ we can introduce coordinates $\{U, T, S, P, V\}$ which correspond to the thermodynamic variables of internal energy, temperature, entropy, pressure, and volume, respectively. If we demand smoothness, at each point $x$ of $\mathcal{T}$ we can construct the tangent $T_x\mathcal{T}$ and cotangent $T^*_x\mathcal{T}$ manifolds in the standard manner so that vectors, tensors, and differential forms are well-defined geometric objects. In particular, we introduce the fundamental Gibbs 1-form 

$$
\Theta = dU - TdS + PdV ,
$$

where $d$ represents the operator of exterior derivative. This is a very general construction in which all simple thermodynamic systems can be represented. To differentiate one thermodynamic system from another, one usually specifies an equation of state which is a relationship between different thermodynamic variables. Alternatively, one can specify the fundamental equation from which all the equations of state can be derived \cite{16}. In the energy representation we are using for the thermodynamic phase space, the fundamental equation relates the internal energy $U$ with the state thermodynamic variables. In principle one can take any pair of variables $\{T, S, P, V\}$ as state variables. The only condition is that they must be well-defined in the corresponding submanifold of $\mathcal{T}$. For later use we choose a fundamental equation of the form $U = U(P, V)$.

Although the following definition makes use of a specific fundamental equation, it can be shown that it does not depend on it \cite{17}. A simple thermodynamic equilibrium system corresponds to a two-dimensional submanifold $\mathcal{E} \subset \mathcal{T}$ defined by the smooth mapping $\varphi : \mathcal{E} \to \mathcal{T}$ with

$$
\varphi : (P, V) \mapsto [U(P, V), T(P, V), S(P, V), P, V] .
$$

such that the pull-back $\varphi^*$ of the Gibbs 1-form vanishes, i.e.

$$
\varphi^* (\Theta) = 0 ,
$$

and the convexity condition is satisfied:

$$
\frac{\partial^2 U}{\partial X^A X^B} \geq 0 ,
$$

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where $X^4 = (P, V)$. In the energy representation we are using here, the convexity condition leads to the second law of thermodynamics and reduces to its standard form in the entropy representation.

Using Eq. (1), condition (3) reads

$$\frac{\partial U}{\partial P} = T \frac{\partial S}{\partial P}, \quad \frac{\partial U}{\partial V} = T \frac{\partial S}{\partial V} - P.$$  

Consequently, on the space of equilibrium states of a given thermodynamic system we obtain the first law of thermodynamics [23]

$$TdS = dU + PdV,$$  

with $U = U(P, V)$. This construction shows that if a thermodynamic system is considered by means of its fundamental equation $U = U(P, V)$ as a submanifold $E$ of the general thermodynamic phase space $\mathcal{T}$, then the variables $T$ and $S$ on $E$ are determined through the differential relationship (6). The question arises whether in general it is possible to integrate the first law of thermodynamics as derived in (6). The answer can easily be found by using Frobenius’ theorem according to which for the differential 1-form

$$\Omega := dU + PdV$$  

(7)

to be integrable it is necessary and sufficient that [18]

$$\Omega \wedge d\Omega = 0,$$  

(8)

where the wedge represents the exterior product. On the 2-dimensional manifold $E$ this condition is trivially satisfied since any 3-form on $E$ vanishes identically. Consequently, it must be always possible to find functions $T = T(P, V)$ and $S = S(P, V)$ such that (6) is satisfied. This is in accordance with the definition of the embedding mapping $\varphi$ as given in (2). Notice that the sufficient condition $d\Omega = 0$ is not satisfied in general; if it were satisfied, we could find a function, say $Q = Q(P, V)$, such that $dQ = \Omega = dU + PdV$, an expression which is obviously not true. This relationship is often written as $dQ = dU + PdV$ to emphasize that the right-hand side is not an exact 1-form.

Another important element of thermodynamics is the Euler identity which is a consequence of the existence of extensive thermodynamic variables which can be used as coordinates in the space of equilibrium states. In the case of the simple system we are studying
here the extensive variables are entropy and volume. Consider a mapping \( \tilde{\varphi} : \mathcal{E} \rightarrow \mathcal{T} \) with

\[
\tilde{\varphi} : (S, V) \mapsto [U(S, V), T(S, V), S, P(S, V), V], \tag{9}
\]

so that in these variables the condition \( \tilde{\varphi}^*(\Theta) = 0 \) yields

\[
\frac{\partial U}{\partial S} = T, \quad \frac{\partial U}{\partial V} = -P. \tag{10}
\]

The thermodynamical potential \( U = U(S, V) \) satisfies the homogeneity condition

\[
U(\lambda S, \lambda V) = \lambda^\beta U(S, V) \]

for constants \( \lambda \) and \( \beta \). The variables \( S, V, \) and \( U \) are called extensive, sub-extensive or supra-extensive if \( \beta = 1, \beta < 1 \) or \( \beta > 1 \), respectively. From the homogeneity condition we obtain

\[
\frac{\partial U(\lambda S, \lambda V)}{\partial (\lambda S)} \frac{\partial (\lambda S)}{\partial \lambda} + \frac{\partial U(\lambda S, \lambda V)}{\partial (\lambda V)} \frac{\partial (\lambda V)}{\partial \lambda} = \beta \lambda^{\beta - 1} U(S, V). \tag{11}
\]

Putting \( \lambda = 1 \) and using the relations (10), from the last equation we get the Euler identity

\[
\beta U - TS + PV = 0. \tag{12}
\]

Furthermore, calculating the exterior derivative of the Euler identity and using the first law (6), we obtain the Gibbs-Duhem relation

\[
SdT - VdP + (1 - \beta) dU = 0. \tag{13}
\]

The above geometric approach to thermodynamics is based only on the embedding structure of the thermodynamic phase space and the space of equilibrium states. A more general structure can be obtained by introducing Riemannian metrics on both spaces and comparing them by imposing invariance with respect to Legendre transformations. The resulting geometrothermodynamical approach allows to handle certain aspects of thermodynamic systems in terms of geometric objects \[19\].

### III. RELATIVISTIC THERMODYNAMICS

The generalization of thermodynamics to gravitational systems is a delicate procedure which must take into account the invariance of certain thermodynamic variables with respect to measurements carried out by different observers (see, for instance, \[1\], for a lucid introductory review). In its final form, relativistic thermodynamics consists in imposing the
fulfillment in curved spacetimes of the first law of thermodynamics (\(6\)), the Euler identity (\(12\)) and the Gibbs-Duhem relation (\(13\)), whereas the second law is imposed in the form of an entropy current \([1]\) (see below).

In the geometric language introduced in the last section the passage to the relativistic generalization is quite simple. The first law of thermodynamics in a curved spacetime corresponds to assuming the validity of (\(6\)) with the exterior derivative operator acting on functions which depend on the spacetime coordinates, i.e. we assume that \(P = P(x^\mu)\) and \(V = V(x^\mu)\) with \(\mu = 0, 1, 2, 3\). Then the first law of relativistic thermodynamics reads

\[
TS_{\mu} \, dx^\mu = (U_{,\mu} + PV_{,\mu}) \, dx^\mu ,
\]

where the comma represents partial derivatives. The first thing we notice now is that the integrability condition (\(8\)) is no longer identically satisfied since it represents now a 3-form on a 4-dimensional manifold. In fact, if \(\Omega_{\mu}\) represents the components of the 1-form \(\Omega\) in the coordinate basis \(\{dx^\mu\}\), the integrability condition is equivalent to

\[
\Omega_{[\mu} \Omega_{\nu,\tau]} = 0 ,
\]

where the square brackets indicate antisymmetrization. If this condition is not satisfied, there are no functions \(T(x^\mu)\) and \(S = S(x^\mu)\) such that the first law of thermodynamics (\(6\)) is fulfilled. Consequently, in relativity theory one could in principle find gravitational systems which cannot be treated as thermodynamic systems. In some sense, this is not surprising since this approach implies that the corresponding system must be in equilibrium as a thermodynamic system, and it is not difficult to imagine gravitational systems with no equilibrium states at all. On the other hand, the question arises whether there are gravitational systems for which the integrability condition is satisfied. If the answer is positive, Frobenius’ theorem guarantees the existence of functions \(T(x^\mu)\) and \(S(x^\mu)\) which fulfill Eq. (\(14\)); then, these functions can be considered as physically meaningful, i.e., as temperature and entropy of the system, if they satisfy the remaining thermodynamic laws. This is exactly the question that was analyzed in a series of works \([3]\) with the result that there exists gravitational configurations where mathematically well-defined thermodynamic variables predict unphysical behaviors.

In this work we will focus on gravitational systems corresponding to cosmological models with a perfect fluid source

\[
T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} ,
\]

where \(u^\mu\) is the four-velocity of the fluid, \(\rho\) its energy density, \(p\) its pressure, and \(g_{\mu\nu}\) the metric tensor of the spacetime.

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where $p$, $\rho$ and $u_\mu$ are the pressure, energy density and 4-velocity, respectively. The conservation law for this energy-momentum tensor can be written as

$$\dot{\rho} + (\rho + p)\Theta = 0, \quad h^\mu_\nu\rho_\mu + (\rho + p)u_\mu = 0,$$

where $\dot{\rho} = u^\mu \rho_\mu$, $\Theta = u^\mu_\mu$ is the expansion, $\dot{u}^\mu = u_\nu u^\nu$ is the 4-acceleration and $h^\mu_\nu = \delta^\mu_\nu + u_\nu u^\mu$ is the projection tensor. The internal energy is $U = \rho V$ and

$$\Omega = [V \rho_\mu + (\rho + p)V_\mu]dx^\mu.$$

The further assumption that $\Omega = TS_\mu dx^\mu$ corresponds to the first law of thermodynamics for the perfect fluid. A straightforward calculation shows that the integrability condition (15) implies that

$$\rho_{[t} p_{,i} V_{,j]} = 0, \quad \rho_{[i} p_{,j} V_{,k]} = 0,$$

where $t$ is the time-coordinate and small latin indices denote spatial coordinates. Since the thermodynamic variables $\rho$ and $p$ are related through Einstein’s equations, it is clear that Eqs.(19) are not necessarily identically satisfied. Notice that the fundamental equation $U = U(P,V)$ in the case of the perfect fluid under consideration reduces to $\rho = \rho(p)$, i.e. to a barotropic equation of state.

The second law is postulated for the entropy current $(su^\mu)$ with $s = S/V$ in the form

$$(su^\mu)_{,\mu} \geq 0$$

where the equality holds in the case of no entropy production. Usually, the second law is considered together with the condition of conservation of the particle number density $n = 1/V$:

$$(nu^\mu)_{,\mu} = 0.$$

**IV. HOMOGENEOUS COSMOLOGICAL MODELS**

Let us consider an non-rotational, homogeneous perfect fluid. The 4-velocity can be shown to be hypersurface orthogonal and therefore there exist local comoving coordinates $(t, x^i)$ such that

$$ds^2 = N^2 dt^2 - g_{ij} dx^i dx^j, \quad u^\mu = N^{-1} \delta^\mu_t, \quad \dot{u}_i = (\ln N)_,\mu \delta^\mu_i, \quad h_{\mu\nu} = g_{ij} \delta^i_\mu \delta^j_\nu.$$
Considering that all metric coefficients are independent of the spatial coordinates, the conservation law (17) reduces to

$$\rho, t + (\rho + p)(\ln \sqrt{\Delta}), t = 0, \quad p, i = 0,$$

(23)

where $\Delta = \det(g_{ij})$. It is possible to introduce a new time coordinate so that the lapse function $N = 1$. We choose such a time coordinate and denote by a dot the derivative with respect to it. For simplicity we denote this new time coordinate again as $t$. Moreover, we will limit ourselves to perfect fluids that satisfy a barotropic equation of state, i.e. $p = \omega \rho$, where $\omega$ is a constant. Then, Eq. (23) can be integrated and yields

$$p = p(t), \quad \rho = \rho_0 \Delta^{-(1+\omega)/2},$$

(24)

where $\rho_0$ is a positive constant.

We now analyze the thermodynamic variables. According to Eq. (19), the integrability conditions to determine temperature and entropy are identically satisfied for time-dependent functions $p$ and $\rho$. This means that there must exist mathematical expressions for $T$ and $S$ satisfying Eq. (18), i.e.

$$T \dot{S} = V \left[ \dot{\rho} + (p + \rho) \dot{V}/V \right].$$

(25)

The physical volume in the case of the metric (22) can be defined as $V = \int \sqrt{\det(g_{ij})}d^3x = \kappa \sqrt{\Delta}$ where $\kappa$ is a constant which can be chosen as $\kappa = 1$, without loss of generality. It follows then from (25) and (23) that the expansion in this cosmological model corresponds to an adiabatic process, i.e.

$$\dot{S} = 0.$$

(26)

On the other hand, from the Euler identity (12) with $U = \rho V$ and $\beta = 1$ we have that

$$S = \frac{p + \rho}{T} \frac{V}{T} = \frac{(1 + \omega)\rho_0}{T} \sqrt{\Delta} = \frac{(1 + \omega)\rho_0}{T \Delta^{\omega/2}}$$

(27)

for a barotropic state equation. Consequently, the adiabaticity condition (26) implies that

$$S = \frac{(1 + \omega)\rho_0}{T_0}, \quad T = T_0 \Delta^{-\omega/2},$$

(28)

where $T_0$ is a positive constant. The physical relevance of these expressions can be derived by comparison with the corresponding expressions in FRW cosmologies. We see that the entropy for anisotropic models is a constant that always can be made to coincide with the
corresponding FRW value. The temperature evolves as the physical volume $\Delta$ in the same way as in the FRW case. Consequently, the behavior of the anisotropic temperature will coincide with the limiting FRW case if the physical volume behaves similarly. This can be shown to be true in general homogeneous models. In particular, for the Bianchi IX model it was shown in \[13\] that the different anisotropies essentially do not affect the dynamical behavior of the physical volume which turns out to be determined only by the different scale factors. This shows that the mathematical expressions for entropy and temperature (28), which follow from the laws of relativistic thermodynamics, are physically meaningful.

V. THE CARDY ENTROPY

In two-dimensional CFT, the Cardy formula allows to count microscopic states in a particularly easy manner and leads to an explicit value for the entropy \[9\]

$$S_C = 2\pi \sqrt{\frac{c}{6} \left( L_0 - \frac{c}{24} \right)},$$

(29)

where $c$ is the central charge and $L_0$ the eigenvalue of the Virasoro operator. Verlinde \[4\] postulated the universal validity of this formula and found a surprising link with Friedman’s equations. In \[13\] it was shown that the Cardy formula can be generalized to the case of closed anisotropic cosmologies described by the Bianchi IX models. To see if this generalization can be extended to include plane and open anisotropic cosmologies, let us consider the Bianchi I and V models whose metric can be written as

$$ds^2 = dt^2 - a_1(t)^2 dx^2 - a_2(t)^2 e^{2\alpha x} dy^2 - a_3(t)^2 e^{2\alpha x} dz^2.$$  

(30)

The metric for the Bianchi I geometry formally corresponds to the case $\alpha = 0$, while for the Bianchi V case we have $\alpha = 1$. To study the correspondence between Cardy’s formula and these cosmological models we only need the Hamiltonian constraint which can be expressed as

$$H_1 H_2 + H_1 H_3 + H_2 H_3 + F(a_1, a_2, a_3) = 8\pi G \rho$$

(31)

with the directional Hubble parameters defined as $H_i = \dot{a}_i/a_i$. Here $G$ is Newton’s gravitational constant and $\rho$ is the energy density of the perfect fluid. For convenience we introduced the function $F(a_1, a_2, a_3)$ which takes different values for the different Bianchi
models we are considering here (see Table I). It is also convenient to introduce a constant factor \( k \) into the Cardy formula in order to include all different cases. Then

\[
S_C = 2\pi \sqrt{\frac{c}{6}} \left( L_0 - k \frac{c^2}{24} \right),
\]

(32)

where \( k = 0, -1, 1 \) for Bianchi I, V and IX models, respectively. An equivalent relationship was proposed by Youm \[14\] in an attempt to generalize Cardy formula to include the different types of FRW cosmologies. Clearly, in the isotropic limit of the Bianchi I, V and IX models we recover the corresponding spatially flat, open an closed FRW universes analyzed in \[14\].

Now, the idea is to identify \( S_C, L_0, \) and \( c \) such that the Cardy formula (32) reduces formally to the Hamiltonian constraint (31). It is easy to see that the identification is unique and corresponds to

\[
S_C = \frac{1}{2\sqrt{3G}} \sqrt{H_1 H_2 + H_1 H_3 + H_2 H_3} \ V,
\]

(33)

\[
L_0 = \frac{1}{3} \tilde{a} E \quad c = \frac{3}{\pi G} \frac{V}{\tilde{a}},
\]

(34)

where \( V \) is the physical volume, \( V = a_1 a_2 a_3 \), the total energy is \( E = \rho V \), and \( \tilde{a} \) is a function of the directional scale factors. We conclude that Cardy’s formula can be generalized to include the cases of plane and open cosmologies. The explicit form of Cardy’s entropy remains the same for all cases considered and differences appear only at the level of the central charge and Virasoro operator. In Table I we present the explicit values of these quantities for all cases investigated. We see that Cardy’s formula postulates an explicit value for the CFT’s entropy of homogeneous cosmologies. This has been shown explicitly only for Bianchi I, V, and IX models, but the generality of our results seems to indicate that they are valid for any Bianchi cosmology.

The modifications due to the presence of anisotropies can be derived from Eqs.(33) and (34). In fact, these relationships can be obtained by introducing an effective Hubble parameter \( H \rightarrow \tilde{H} = \sqrt{(H_1 H_2 + H_1 H_3 + H_2 H_3)/3} \) and an effective scale factor \( a \rightarrow \tilde{a} \) (see Table I) in the original FRW values, which correspond to the limiting case \( a_1 = a_2 = a_3 = a \). We conclude that in the case of anisotropic cosmologies the identification of the Cardy entropy with the Hamiltonian constraint can be performed just by introducing effective anisotropic parameters.

We now turn back to the study of the compatibility of Cardy’s entropy with the laws of relativistic thermodynamics. If we try to identify Cardy’s entropy \( S_C \) with the entropy
TABLE I: **CFT’s entropy for Bianchi cosmologies.** In all cases Cardy’s formula gives a definite value for the entropy $S_C$ as given in Eq.(33), while the values of parameters $L_0$ and $c$ are used to reproduce the Hamiltonian constraint. In the case of Bianchi IX models the explicit form of $\epsilon = \epsilon(a_1, a_2, a_3)$ is given in [13]. Notice that the chosen value of $\tilde{a}$ seems to single out the scale factor $a_1$, i.e., the anisotropy in the $x-$direction. This is only a matter of convention because similar expressions can be written for $a_2$ and $a_3$, with the corresponding changes in the expressions for $\epsilon$ and $\tilde{a}$. Therefore, any direction could have been chosen for writing down the explicit expressions.

S =const we obtained from the thermodynamic approach of Section IV, the first thing we can notice is that Cardy’s entropy does not satisfy the adiabaticity condition $\dot{S}_C = 0$. At the first sight, this could be a sufficient reason for considering Cardy’s cosmological entropy as unphysical since it is not compatible with the laws of relativistic thermodynamics. However, we believe that it is necessary to perform a deeper analysis, before trying to make definite conclusions.

On the one hand, one can try to take into account other properties of CFT’s entropy to see if it is possible to arrive to an adiabatic expression. As pointed out by Verlinde [4], Cardy’s entropy is characterized by its sub-extensive nature. Euler’s identity (12) for CFT’s entropy $S_C$ turns out to be valid only for $\beta = 1/3$, i.e.

$$\frac{1}{3} U_C - T S_C + PV = 0 ,$$

where $U_C$ is proportional to Casimir’s energy $E_C$. Then, the total energy $E$ should contain a sub-extensive term: $E = E_E + E_C/2$. The analysis of the entropy with this total energy was performed for FRW models in [4] and for Bianchi IX models in [13]. For the cases
considered here one can perform similar computations by allowing an additive constant term in the entropy (28), i.e. \( S \rightarrow S + S_0 \) with \( S_0 = \text{const} \), and then identifying \( S_0 \) with the Casimir energy. Then we obtain

\[
S = \left[ \frac{2\pi}{3} \sqrt{2E_E E_C} \right]^{3/(2+3\omega)}.
\]  

(36)

It is then possible to show that this expression for the total entropy still corresponds to an adiabatic process \( \dot{S} = 0 \), if the dynamical behavior of \( E_E \) and \( E_C \) is chosen correspondingly. So we see that the sub-extensive character of Casimir’s energy is not sufficient for generating an entropy which would resemble the non-adiabatic character of Cardy’s entropy. Notice that the expression for the entropy of the universe resembling the Cardy formula in terms of the different kinds of energy, as given in Eq.(36), takes the special square-root form, originally postulated by Verlinde [4], only in the case of a radiation dominated universe \( (\omega = 1/3) \). A similar result was obtained in [14] in the limiting FRW cosmologies.

On the other hand, the adiabaticity condition for \( S \) was obtained under the assumption that the laws of relativistic thermodynamics are valid. According to the geometric approach to thermodynamics described in Sections II and III, relativistic thermodynamics is obtained from its classical version by assuming that the thermodynamic variables explicitly depend on the coordinates used in the spacetime manifold that describes the gravitational field. This procedure seems to lead to reasonable results in the case of FRW and homogeneous (anisotropic) cosmologies, where the integrability conditions (19) are trivially satisfied. However, in the case of inhomogeneous cosmologies the resulting thermodynamic variables are characterized by a very unphysical behavior [3]. It is then natural to ask whether the present version of relativistic thermodynamics is a definite and correct procedure. At least in the case of inhomogeneous fields a modification seems to be necessary. The connection and curvature appear in the formalism only indirectly through the spacetime coordinates. Perhaps one needs a generalized formalism which also takes into account the dynamics of curvature and additional gravitational degrees of freedom.

Furthermore, a closer look at the first law of relativistic thermodynamics (14) in the case of cosmological models considered here reveals that \( U = E \) includes only the energy \( \rho \) corresponding to the perfect fluid and no “gravitational energy” is taken into account. Of course, this is a completely different problem since there is no general definition of energy for gravitational fields. Nevertheless, it seems reasonable to demand that for a correct
VI. CONCLUSIONS

In this work we used two different procedures to define the entropy of anisotropic cosmological models in the case of Bianchi models of type I and V. First, we presented a geometric approach to classical thermodynamics which allows us to derive the relativistic generalization in a very simple manner. The relativistic version of the laws of thermodynamics was then used here to derive an expression for the entropy which is in accordance with an adiabatic cosmological expansion. This entropy shows a physical dynamical behavior and leads to the FRW case in the corresponding limit.

Secondly, we use the Cardy formula for the entropy of two-dimensional conformal field theories and its generalization to any dimensions. It turns out that for homogeneous cosmologies it is possible to choose the eigenvalue of the Virasoro operator and the central charge in such a way that Cardy’s entropy formula coincides with the Hamiltonian constraint for closed, plane, and open models. The resulting expression for Cardy’s entropy, however, does not correspond to an adiabatic expansion, even if it is considered as a sub-extensive thermodynamic variable. It is in this sense that we conclude that it is not possible to interpret Cardy’s entropy as the entropy of a cosmological model. Nevertheless, we believe that this negative result is not sufficient to dismiss Cardy’s entropy as unphysical. The relativistic version of thermodynamics used to derive the adiabaticity condition leads to unphysical results, as soon as inhomogeneities are taken into account, so that we cannot exclude the possibility of searching for a different approach to relativistic thermodynamics. Also, this version does not include all gravitational degrees of freedom which intuitively are expected to affect the behavior of a thermodynamic system, especially its entropy. This would imply that there is no physical reason for demanding that Cardy’s entropy should be comparable with the thermodynamic entropy of a perfect fluid.

An additional point that one should examine critically is the assumption of a perfect fluid as the source of the Bianchi anisotropic models. A more realistic source must necessarily take into account dissipative processes due to anisotropic expansions. This would imply the analysis of more general fluids which include viscosity terms, a task which is beyond the
scope of the present work.

We conclude that it is necessary to perform a more detailed analysis in order to interpret Cardy’s entropy as a thermodynamic variable which is in accordance with the laws of relativistic thermodynamics.

VII. ACKNOWLEDGEMENTS

It is a great pleasure to dedicate this work to Octavio Obregón on the occasion of his 60-th birthday. This work was supported in part by CONACyT grant 48601-F.

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[23] In fact, this is Gibbs’ equation which in cosmology is mistakenly referred to as the first law.