Spin-lattice models: inhomogeneity and diffusion

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In spin-lattice models with order parameter conserved, we generalize the idea of spin diffusion incorporating a variety of factors as possible driving forces, including external field and temperature. The Kawasaki dynamics in the Gaussian model and the one-dimensional Ising model are studied as specific examples. In the obtained diffusion equation, the term describing the diffusion induced by the inhomogeneity of the magnetization itself is unaffected and is believed to vanish near the critical point. Meanwhile the nonvanishing diffusion induced by the inhomogeneity of the environment may be coupled to the spin configuration and weakened by thermal noise. Interesting dynamic behavior is observed as a result of a competition of internal and external inhomogeneities and time scales. Several interesting examples are visualized, and the concept of local hysteresis is proposed in this spin-conserved dynamics.

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I. INTRODUCTION

In the past decade, great interest has been aroused by the response of a cooperative system to external perturbations, e.g. an external field varying with time or a scanning temperature (see Ref. \cite{1–7} and references therein). In particular, significant effort has been made in the dynamic phase transition and hysteresis of the Ising model subject to an oscillating field \cite{1}, using Monte Carlo simulation and Mean-Field approximation based on Glauber’s dynamics. Yet the dynamics with conservation law\textsuperscript{1}, i.e. a model B in the classification of Hohenberg and Halperin \cite{10}, has been much less studied. Such dynamics may describe, for example, the phase separation of a binary mixture subject to inhomogeneous temperature or gravity.

Generally speaking, in spin-lattice models with the order parameter conserved, it is possible to describe a diffusion process with a simple diffusion equation. In homogeneous environment, it is natural to guess that the spin flow is proportional to the gradient of the spin, and the equation will be

\[
\frac{\partial q(\mathbf{r}, t)}{\partial t} = D \nabla^2 q(\mathbf{r}, t),
\]

where \(q(\mathbf{r}, t)\) denotes the local magnetization and \(D\) is a diffusion coefficient. This equation has been exactly obtained in the Gaussian model and \(D\) is found to vanish at the critical point (see Ref. \cite{11}, and the following section). It has also been studied in other systems, including the Ising model \cite{12}.

Diffusion is always a result of inhomogeneities. The diffusion process characterized by Eq. (1) can be viewed as being caused by the inhomogeneity of the magnetization itself, while diffusion driven by external inhomogeneities remains to be an unsolved interesting problem. Many studies on the dynamic response of the Ising model deal with a uniform field varying with time \cite{1}. However, such a field has no effect on the Kawasaki dynamics \cite{12} with spin conservation. So here is a more difficult task, which should incorporate both spatial and temporal inhomogeneities. In relevant studies several interesting problems have been treated, e.g., the random-field spin-conserving Ising model \cite{13–16} and, more recently, the domain growth in a temporally constant field varying linearly in space \cite{17,18}. In the present study, we narrow our scope to the Gaussian model and the one-dimensional Ising model, and focus on the modification of the above diffusion equation by the inhomogeneity of field and temperature.

As is well known, the Ising model has wide and important applications in various fields. Unfortunately, as we shall see below, the obtained diffusion equation can not be solved. Meanwhile, it is analytically tractable in an idealization of the Ising model, the kinetic Gaussian model, which may serve as a starting point and a test field of new ideas.

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\textsuperscript{1}In the absence of driving force, the Ising model with spin conserved is relatively well understood: Quenched from a high temperature disordered state, some kind of order rises with the appearance of coarsening and competing up-and-down domains, of which the length scale grows as a power law with time \cite{8,9}.
In this article, the rigorous treatment of the Gaussian model will be combined with the less quantitative discussions and Monte Carlo simulations of the Ising model, along the same line, for a comparison. Section II deals with the diffusion induced by inhomogeneous external field, and in Sec. III we extend the scope to inhomogeneous temperature. (Although here we treat these two problems separately, it is possible to put them together.) Section IV is the summarization with some discussions.

II. INHOMOGENEITY OF THE EXTERNAL FIELD

In this section, based on the master equation, we shall study diffusion process in an inhomogeneous external field. Such a process is subject to two factors acting together, the inhomogeneity of the field and that of the magnetization itself. This competition may lead to interesting and rich dynamic behavior, as we shall see below. Before we move to the specific models, we briefly review the governing dynamic mechanism, spin-pair redistribution mechanism, which is a natural generalization of Kawasaki’s spin-pair exchange mechanism [12].

In the Ising model, spins can only take two opposite values, +1 and −1, and spin-exchange is simply a natural choice; but it turns to be not as capable when applied to other more complex systems. A natural generalization is as follows: a pair of nearest neighbors selected at random, σ_i and σ_j, may take any values, ˆ{σ_i} and ˆ{σ_j}, as long as their sum keeps conserved, and this is called spin-pair redistribution (clearly in the Ising model it reduces to the spin-exchange). We list several important equations in Appendix A, the details of which can be found in Ref. [11].

A. The Gaussian model

First we will study the three-dimensional kinetic Gaussian model in an inhomogeneous external field, which might also be varying with time (e.g., a traveling wave). In lattice models, a traditional way is to assign to each vertex, σ_i, a reduced field, H_i(t) [1]. In the kinetic Gaussian model,

$$-\beta H = K \sum_{i,j} \sigma_i \sigma_j + \frac{1}{k_B T} \sum_i H_i(t) \sigma_i,$$

where $K = J/k_B T$, and J is the coupling strength. The spins take continuous value from $-\infty$ to $+\infty$, and the probability of finding a given spin between $\sigma_i$ and $\sigma_i + d\sigma_i$ is assumed to be the Gaussian-type distribution $f(\sigma_i) d\sigma_i = \sqrt{\frac{b}{2\pi}} \exp \left( -\frac{b}{2} \sigma_i^2 \right) d\sigma_i$, where b is a distribution constant independent of temperature. Thus, a summation over the spin values will be in fact an integration, $\sum_{\sigma} \to \int_{-\infty}^{\infty} f(\sigma) d\sigma$.

Now, we substitute the Hamiltonian (2) to the evolving equation of single spin, Eq. (A3). A linear equation can be obtained (three-dimensional)

\[
\frac{d}{dt} q_{ijk}(t) = \frac{1}{2(b+K)} \left[ \frac{1}{2} \right] \left( \begin{array}{c}
(q_{i+1,j+1,k} - 2q_{ijk} + q_{i-1,j,k}) \\
(q_{i+1,j+1,k} - 2q_{ijk} + q_{i+1,j,k-1}) \\
+ K[2(2q_{i-1,j,k} - q_{i-1,j+1,k} - q_{i-1,j-1,k}) + (2q_{i-1,j,k} - q_{ijk} - q_{i-2,j,k})] \\
+ 2(2q_{i+1,j+1,k} - q_{i+1,j,k-1} - q_{i+1,j-1,k}) + (2q_{i+1,j,k} - q_{ijk} - q_{i+2,j,k})] \\
+ 2(q_{i,j,k-1} - q_{i,j,k-1} - q_{i,j,k-1}) + (2q_{i,j,k-1} - q_{ijk} - q_{i,j,k-2})] \\
+ 2(q_{i,j,k+1} - q_{i,j,k+1} - q_{i,j,k+1}) + (2q_{i,j,k+1} - q_{ijk} - q_{i,j,k+2})] \\
+ 1 \frac{1}{k_B T} \left[ \frac{1}{2} \right] \left( \begin{array}{c}
(2H_{ijk} - H_{i+1,j,k} - H_{i-1,j,k}) \\
(2H_{ijk} - H_{i,j+1,k} - H_{i,j-1,k}) + (2H_{ijk} - H_{i,j,k+1} - H_{i,j,k-1}) \end{array} \right) \\
\end{array} \right] \]
\]

This equation is in fact not as complex as it seems to be. Each term in a bracket is a second-order derivative either of the magnetization or of the external field, and they will cancel each other if a summation is taken, guaranteeing the conservation of spin. With lattice constant a we can transform the above equation (and similar equations for lower dimensions) into
where $D$ is the dimensionality. It reveals an important feature that, surprisingly, the field and the spins are not coupled, which is mainly a result of the integration taken from $-\infty$ to $+\infty$. The influence of the inhomogeneous external field rigorously takes the form of a second order derivative, with a prefactor getting weaker at higher temperature (weakened by thermal fluctuations).

Discussions: (1) If the external field is spatially homogeneous, the system behavior can be described by a simple diffusion equation [11].

(2) By contrast, if the external field varies solely in space, an equilibrium state can be obtained by setting $\partial q (r, t)/\partial t = 0$. We find that $\nabla^2 q (r) \propto \nabla^2 H (r)$, except at the critical point $K_c = b/2D$. The susceptibility, $\chi \sim \nabla^2 q (r)/\nabla^2 H (r)$, can be obtained. To visualize this process, we suppose that the shape of the external field is a Gaussian packet: Above the critical point, the magnetization will reach the stable equilibrium also in the shape of a Gaussian packet, and $\chi$ is finite and positive. As the system is cooling and approaching the critical point, we shall observe that the peak of the magnetization (as well as $\chi$) tends to positive infinity. With the scaling hypothesis $\chi \sim |T - T_c|^{-\gamma}$, the critical exponent $\gamma = 1$ can be (in fact generally) obtained. When the temperature is below $T_c$, the equilibrium value of the magnetization (and $\chi$) will suddenly become negative infinity. As the temperature continues to decrease, the magnetization becomes flatter, and $\chi$ is finite and negative. In this region the equilibrium is obviously unstable.

A comparison of the two dynamic versions, Kawasaki’s and Glauber’s, of the Gaussian model shall be interesting. In Glauber’s version, we can obtain [20]

$$
\frac{d}{dt} \langle \sigma_i(t) \rangle = -\langle \sigma_i(t) \rangle + \frac{1}{b} \sum_{\langle i,j \rangle} K_{i,j} \langle \sigma_j(t) \rangle + \frac{1}{bk_BT} H_i(t)
$$

(5)

and

$$
\frac{dM(t)}{dt} = -\left(1 - \frac{2D}{bK}\right) M(t) + \frac{H(t)}{bk_BT},
$$

(6)

where $M = \sum_k \langle \sigma_k \rangle / N$, and $H = \sum_k H_k / N$. The external field does not change the critical point either, and we can also get the equilibrium magnetization by setting $dM/dt = 0$. The situation here is quite similar: Above the critical point, the equilibrium is stable, and it becomes unstable below the critical point, and $\chi$ changes sign. In other words, in both versions, the observable state below $T_c$ is a fast growing dynamic process.

(3) From Eq. (4) we are able to obtain the whole dynamic process, and here we discuss two interesting one-dimensional examples.

The first example: We assume a time-independent field $H = H_0 \cos (kx)$, and the initial magnetization is zero. Substitution into Eq. (4) yields a solution,

$$
q(x, t) = \frac{A}{D} \left(1 - \exp \left(-Dk^2t\right)\right) H_0 \cos (kx),
$$

(7)

where $D = a^2(b/2 - K) / (b + K)$, and $A = a^2/[2(b + K) k_BT]$. When $K < K_c = b/2$, $D$ is positive, and $q$ will approach the equilibrium value, $q_{eq} = (A/D) H_0 \cos (kx)$, with the relaxation time $\tau = 1/Dk^2$. A typical critical slowing down phenomenon is obvious when $K \to K_c^-$, since $\tau \to \infty$. Interestingly, the speed, $dq/dt \sim \exp (-Dk^2t)$, does not really "slow down" because of the diverging prefactor $A/D$, and this is against our common knowledge about this phenomenon.

The second example: We assume a time-dependent field $H = H_0 \cos (kx - \omega t)$, which is a traveling wave, and we obtain a solution of Eq. (4) as

$$
q(x, t) = \frac{Ak^2}{\sqrt{D^2k^4 + \omega^2}} H_0 \cos (kx - \omega t + \varphi), \ \varphi = \arccot \left(Dk^2/\omega\right).
$$

(8)

It describes a spin wave, which lags in phase with a temperature-dependent factor $\varphi$. As a counterpart of the hysteresis loop studied previously, formed by the average magnetization and the spatially uniform field, here we introduce the concept of local hysteresis loop, which is formed by local magnetization and local field. It is a useful tool characterizing the response to oscillating perturbations in a simple way. Generally, the local hysteresis loop may vary in space, while in this specific example it has an elliptical shape that is spatially uniform and varies with temperature in the way illustrated in Fig. 1.
We assume that the variance of the external field between two nearest-neighboring vertices is small (i.e., when the wavelength is much longer than the lattice constant), then

\[
\sum_{\sigma_{k,\sigma_{k\pm1}}} \delta_k W_{k,k\pm1} \left( \sigma_k \sigma_{k\pm1} \to \delta_k \sigma_{k\pm1} \right)
\]

\[
= \sigma_k e^{K(\sigma_{k\pm1} \sigma_k + \sigma_k \sigma_{k\pm2}) + \frac{1}{k_B T} (H_k \sigma_k + H_{k\pm1} \sigma_{k\pm1}) + \sigma_{k\pm1} e^{K(\sigma_{k\mp1} \sigma_{k\pm1} + \sigma_{k\pm1} \sigma_{k\pm2}) + \frac{1}{k_B T} (H_k \sigma_{k\pm1} + H_{k\pm1} \sigma_k)}}
\]

\[
= \left\{ \begin{array}{ll}
\frac{1}{2} (1 - \sigma_k \sigma_{k\pm1}) \tanh \left[ K (\sigma_{k\mp1} - \sigma_{k\pm2}) + \frac{1}{k_B T} (H_k - H_{k\pm1}) \right], & \text{if } \sigma_k = \sigma_{k\pm1} \\
\frac{1}{2} (1 - \sigma_k \sigma_{k\pm1}) \tanh \left[ K (\sigma_{k\pm1} - \sigma_{k\pm2}) + \frac{1}{k_B T} (H_k - H_{k\pm1}) \right], & \text{if } \sigma_k = -\sigma_{k\pm1}
\end{array} \right.
\]

We assume that the variance of the external field between two nearest-neighboring vertices is small (i.e., when the wavelength is much longer than the lattice constant), then

\[
\tanh \left[ K (\sigma_{k\mp1} - \sigma_{k\pm2}) + \frac{1}{k_B T} (H_k - H_{k\pm1}) \right]
\approx \frac{1}{2} (1 - \sigma_k \sigma_{k\pm1}) \tanh 2K + \frac{1}{k_B T} (H_k - H_{k\pm1}) \left[ 1 - \frac{1}{2} (1 - \sigma_k \sigma_{k\pm1}) \tanh^2 2K \right]
\]

Substituting them into Eq. (A3), we get the evolving equation of single spins,

\[
\frac{d}{dt} q_k = \frac{a^2}{2} \left[ \frac{(q_{k+1} - q_k)}{a} - \frac{(q_k - q_{k-1})}{a} \right] / a
\]

\[
- \frac{3a^2}{4} \left\{ \frac{\langle (q_k \sigma_{k+1} + q_k \sigma_{k+2}) - (q_{k-1} \sigma_{k+1} + q_k \sigma_{k+2}) \rangle}{a} \right\} \tanh 2K
\]

\[
+ \frac{a^2}{4} \left\{ \frac{\langle (q_k \sigma_{k+1} + q_k \sigma_{k+2}) - (q_{k-1} \sigma_{k+1} + q_k \sigma_{k+2}) \rangle}{a} \right\} \tanh 2K
\]

\[
- \frac{1}{2} \frac{1}{k_B T} (H_{k+1} - H_k) \left[ 1 - \sigma_k \sigma_{k+1} \right] - \frac{1}{2} \frac{1}{k_B T} (H_{k-1} - H_k) \left[ 1 - \sigma_k \sigma_{k-1} \right] - \frac{1}{2} \frac{1}{k_B T} (H_{k+1} - H_{k+2}) \left[ 1 - \sigma_k \sigma_{k+1} \right] - \frac{1}{2} \frac{1}{k_B T} (H_{k-1} - H_{k-2}) \left[ 1 - \sigma_k \sigma_{k-1} \right] \tanh^2 2K
\]

If the external field is spatially uniform, we use \( \langle \sigma^{(3)}_k \rangle \) to denote \( \langle \sigma_{k-1} \sigma_k \sigma_{k+1} \rangle \), and

\[
\left( \frac{\partial q_k}{\partial t} \right)_M = \frac{1}{2} a^2 \left( 1 - \frac{3}{2} \tanh 2K \right) \frac{\partial^2 q}{\partial x^2} + \frac{1}{4} a^2 \tanh 2K \frac{\partial^2 \langle \sigma^{(3)} \rangle}{\partial x^2}.
\]

In Ref. [12], Kawasaki has used local equilibrium approximation and concluded that

\[
\left( \frac{\partial q}{\partial t} \right)_M = D \frac{\partial^2 q}{\partial x^2},
\]

where \( D \propto 1/\chi \) is the diffusion coefficient vanishing near the critical point. If this is correct in the Ising model (\( \tanh 2K_c = 1 \) for the one-dimensional case), it requires,

\[
\frac{d^2}{dx^2} \langle \sigma^{(3)}_k \rangle = \frac{d^2 q}{dx^2},
\]

(11)

to be true at least near the critical point, otherwise the diffusion process in the Ising model cannot be described by such a simple equation.

In the following we will incorporate the inhomogeneity of the external field in our discussion. Let the function \( f(ka + a/2) \) denote the whole term

\[
f(ka + a/2) = 1 - \langle \sigma_k \sigma_{k+1} \rangle - \frac{1}{2} \langle (1 - \sigma_k \sigma_{k+1}) (1 - \sigma_{k+1} \sigma_{k+2}) \rangle \tanh^2 2K
\]
and we get

$$\frac{\partial q}{\partial t} = \left( \frac{\partial q}{\partial t} \right)_M - \frac{1}{2k_B T} \left( f(x) \frac{\partial^2 H}{\partial x^2} - \frac{\partial f(x)}{\partial x} \right).$$

(12)

Compared with the Gaussian model, we find both similarities and differences. First, the diffusion term induced by the inhomogeneity of the magnetization itself retains the same form as Eq. (9), and this is also true for the Gaussian model. Second, the influence of the external field is also weakened by a prefactor $1/T$. However, here the field is coupled to the spins. And, a somewhat surprising prediction of Eq. (12) is that the driving force may partly come from the gradient of the field, which is at the same time coupled to the inhomogeneity of the spin configuration. For example, there will still be field-induced "diffusion" if the field varies with space linearly and thus $\frac{\partial^2 H}{\partial x^2} = 0$, provided that the magnetization is inhomogeneous and thus $\frac{\partial f}{\partial x} \neq 0$, and this case has been particularly studied in Ref. [17,18]. Beside this case, a lot of issues will make us even more curious, including, as studied in the Gaussian model, the effect of a travelling wave: Shall there also be a similar spin wave with the same time period and a phase lag? Shall there be new dynamic phases? What will be the shape of the local hysteresis loop? Here we provide a tentative answer with Monte Carlo (MC) simulations.

In a one-dimensional model of $N$ vertices with periodic boundary condition, different initial conditions are chosen while the total spin is kept around zero. We randomly pick a pair of nearest neighboring spins, $\sigma_i$ and $\sigma_j$, and exchange them with a probability

$$w_{ij} = \min \{1, \exp(-\Delta E_{ij}/k_B T)\},$$

where $\Delta E_{ij}$ is the change in energy if $\sigma_i$ and $\sigma_j$ are exchanged. A MC step consists of $N$ such actions. Time is measured in MC steps and the system is assumed to be of unity length. A travelling wave, $H = H_0 \cos(kx - \omega t)$ is applied.

Fig. 2 shows the local hysteresis loops of the 1st spin in a 100-spin chain at several typical temperatures, with $\omega = 2\pi/1000$, $k = 2\pi$ and $H_0 = 4$ (in units of $J = k_B$). (The shape of a loop is found to be independent of the position of the spin it describes.) Each point on a loop actually denotes the expected value of the spin, i.e., $p_+ - p_-$, where $p_+$ is the probability that the spin takes upward direction and $p_-$ the contrary. The results are to a great degree similar to the rigorous ones obtained in the Gaussian model. At $K = 0.001$ (Fig. 2(a)), the spin fluctuates around zero (actually it may be a very narrow ellipse with a tiny phase lag). At $K = 0.05$ (Fig. 2(b)), the loop is a tilted ellipse with a finite phase lag between 0 and $\pi/2$. At about $K = 0.4 \sim 0.6$ ($K = 0.5$ in Fig. 2(c)), the loop evolves into a state that has two symmetries. It is symmetric about the horizontal axis, and the average magnetization over a period is zero. At the same time it is also symmetric about the vertical axis, and this indicates a phase lag of approximately $\pi/2$. In the Gaussian model it agrees with the critical point, and here it also corresponds to a region where a dynamic symmetry loss occurs. First, the loss of the vertical symmetry: as temperature continues to decrease, the loops begin to transform into a shape which we are familiar with ($K = 2.5$ in Fig. 2(d)). It means that, though the spin value still varies with the same period, it no longer varies sinusoidally. Second, the loss of the horizontal symmetry: for a given spin, the average value over a period may increasingly deviate from zero as temperature decreases, while the extent and direction of the deviation are influenced by fluctuations and initial conditions. This may be accepted as an inherent property of the one-dimensional Ising model and is observed in the quasi-static limit. We find that the temperature at which a noticeable symmetry loss occurs is lowered by the magnetic wave.

Here, we tentatively reveal the interesting features that result from a competition of the external and the internal inhomogeneities and time scales. A thorough study is needed to answer questions such as: What is the effect of wave speed, frequency and amplitude? What is the relationship between the area of the local hysteresis loop and the parameters (previous works in non-conserved dynamics have revealed several interesting scaling laws)? In higher dimensions, the effect of a magnetic wave on the formation and destruction of domains also deserves further investigation. In the case of a plane wave, we may expect to observe two coexisting length scales, one parallel to the wave direction, and the other perpendicular to it. This requires extensive MC simulations which are beyond the scope of this article.

III. INHOMOGENEITY OF THE TEMPERATURE

In Sec. II, we have investigated the field-induced diffusion. As mentioned in the Introduction, the inhomogeneity, as the driving force of the diffusion, may be actually of a broad meaning. In this section we shall focus on the influence of inhomogeneous temperature. Inspired by the dynamic response to a periodically altered temperature experimentally observed in ferroelectric systems [4] and charge-density-wave systems [5], there has been theoretical effort in simple
spin models [6,7]. With spin-nonconserved dynamics, interesting behavior and thermal hysteresis have been reported. In the following we present our treatment of the Kawasaki dynamics of the Gaussian model and the Ising model.

For simplicity we limit our scope to an one-dimensional model that consists of $N$ spins. There is no (or homogeneous) external field and each vertex, $\sigma_k$, is in contact with a heat reservoir of temperature $T_k$. Thus the system’s effective Hamiltonian

$$-\beta \mathcal{H}_{\text{eff}}(\{\sigma\}) = \sum_{\langle i,j \rangle} \frac{K_i + K_j}{2} \sigma_i \sigma_j. \quad (13)$$

where $K_i = J/k_B T_i$. With

$$\frac{\nabla K}{K} = - \frac{\nabla T}{T} \quad (14)$$

and

$$\frac{\nabla^2 K}{K} = - \frac{\nabla^2 T}{T} + 2 \left(\frac{\nabla T}{T}\right)^2, \quad (15)$$

we shall treat $K$ for convenience in the following.

A. The Gaussian model

In the Gaussian model, we substitute the new Hamiltonian, Eq. (13), into the evolving equation of single spin, Eq. (A3), and get

$$\frac{d}{dt} q_k(t) = -2q_k + \frac{1}{2|b + (K_k + K_{k+1})/2|} \left[ \left( \frac{K_k + K_{k-1}}{2} \right) q_{k-1} + \left( \frac{K_k + K_{k+1}}{2} \right) q_{k+1} - \left( \frac{K_{k+1} + K_{k+2}}{2} \right) q_{k+2} + b(q_k + q_{k+1}) \right]$$

$$+ \frac{1}{2|b + (K_k + K_{k-1})/2|} \left[ \left( \frac{K_k + K_{k+1}}{2} \right) q_{k+1} + \left( \frac{K_k + K_{k-1}}{2} \right) q_{k-1} - \left( \frac{K_{k-1} + K_{k-2}}{2} \right) q_{k-2} + b(q_k + q_{k-1}) \right]. \quad \text{(16)}$$

Substituting

$$K_{k\pm 1} \approx K_k \pm a \frac{dK}{dx} + \frac{1}{2} a^2 \frac{d^2 K}{dx^2}, \quad K_{k\pm 2} \approx K_k \pm 2a \frac{dK}{dx} + 2a^2 \frac{d^2 K}{dx^2}, \quad (17)$$

and

$$q_{k\pm 1} \approx q_k \pm a \frac{dq}{dx} + \frac{1}{2} a^2 \frac{d^2 q}{dx^2}, \quad q_{k\pm 2} \approx q_k \pm 2a \frac{dq}{dx} + 2a^2 \frac{d^2 q}{dx^2}, \quad \text{(18)}$$

into Eq. (16), we get

$$\frac{1}{q} \frac{\partial q}{\partial t} = a^2 \frac{3b}{K'} - a^2 \frac{dK'}{K'} \left( \frac{1}{q} \frac{\partial^2 q}{\partial x^2} \right) - a^2 \frac{(2K' + 3b)}{2K'} \left( \frac{1}{q} \frac{\partial q}{\partial x} \right) \left( \frac{1}{K'} \frac{dK'}{dx} \right)$$

$$+ a^2 \left( \frac{1}{K'} \frac{dK'}{dx} \right)^2 - a^2 \left( \frac{1}{K'} \frac{d^2 K'}{dx^2} \right), \quad \text{(19)}$$

where $K' = K + b$. The first term on the right hand side denotes the diffusion induced by the inhomogeneity of the magnetization itself, which we are familiar with. The influence of the inhomogeneity of $K$ is described by the other three terms. The temperature and spin are partially coupled. It is well established that the magnetization–induced diffusion will vanish near the critical point (in the one-dimensional model $K_c = b/2$), but the $K$-induced diffusion will not (neither will the field-induced one). With Eqs. (14) and (15), we can see that the role played by the thermal
noise is more difficult to analyze in $K$-induced diffusion, compared with the case of field-induced diffusion. Besides, the third term on the right hand side is always positive, yet, surprisingly, this is still not against the conservation of the order-parameter. We can use a simple example to show it. Assume that at one moment the magnetization is

\[ \int_{-\infty}^{+\infty} \frac{\partial q}{\partial t} \, dx = a^2 q \int_{-\infty}^{+\infty} \left[ \left( \frac{1}{K'} \frac{dK'}{dx} \right)^2 - \left( \frac{1}{K'} \frac{d^2 K'}{dx^2} \right) \right] \, dx = -a^2 q \int_{-\infty}^{+\infty} d \left( \frac{1}{K'} \frac{dK'}{dx} \right) = 0. \]

The previous studies cited above mainly considered non-conserved dynamics when the temperature is varying with time but spatially homogeneous. Although a similar study of local thermal hysteresis is possible, here we turn to treat the contrary situation: spatially modulated but temporally fixed temperature. In the following we visualize the equilibrium state (if there is one) and the evolving process in three typical examples. The numerical simulations are based on Eq. (16), and the system consists of 100 spins with periodic boundary condition. We set $b = 1$ and assume that when $t = 0$, all the spins take the value of unity.

**The first example:** The overall system is above the critical point ($\bar{K} < K_c = 1/2$) and $K_i = 1/4 + (1/8) \sin (2i\pi/100)$, $i = 1, 2, ..., 100$. The evolving process is shown in Fig. 3. We can clearly see that the system is approaching an equilibrium state and is slowing down as time passes. Changing the parameters, but keeping $q_i(0) = 1$ and $K_{\text{max}} < K_c$, we find that, interestingly, the spins keep to be positive in the evolution.

**The second example:** The whole system is below the critical point and $K_i = 1 + (1/8) \sin (2i\pi/100)$, $i = 1, 2, ..., 100$. Here we may expect the system to be fast growing. The results show an interesting self-organizing process, which can be characterized by two succeeding dynamic phases, a steady one, and a growing one. Fig. 4(a) shows a typical spin configuration during the period between $t = 0$ and approximately $t = 400$. The magnetization takes a sinusoidal-like shape and the amplitude is growing very slowly. After the amplitude reaches a threshold value, we can observe the self-organization as shown in Fig. 4(b). Note that part of the magnetization, $q_{50}$ to $q_{100}$, still seems to be smooth. In fact, self-organization also occurs in this part, only later and weaker. After this period, the system moves into a fast growing phase with a fixed shape, as shown in Fig. 4(c). The contrast in the intensity of motion exhibited by the two regions, $q_1$ to $q_{50}$ and $q_{50}$ to $q_{100}$, persists if we change the temperature to, for example, $K_i = 2 + (1/8) \sin (2i\pi/100)$. So it may be caused not by the absolute value of the temperature, but by the mutual influence between the relatively high temperature region and the relatively low temperature one.

**The third example:** Here we study a rather interesting case, where $K_i = 1/2 + (1/4) \sin (2i\pi/100)$. We can divide the system into two regions, $q_1 \sim q_{50}$ and $q_{51} \sim q_{100}$. In the first region, the observed evolution is similar to the second example, but with less peaks and lower growing speed (it is because the temperature is higher here). In the second region, however, the system behavior is similar to the first example, i.e. the magnetization is smooth and approaching equilibrium. We do not observe any strange phenomenon at $K_{50} = K_c = 1/2$. This is not surprising, and can be predicted with Eq. (19).

**B. The Ising model**

Now we turn to treat the one-dimensional Ising model. Similarly, by substituting Eq. (13) into Eq. (A3), we get

\[ \sum_{\sigma_k, \sigma_{k+1}} \tilde{\sigma}_k W_{k, k+1} (\sigma_k \sigma_{k+1} \rightarrow \tilde{\sigma}_k \tilde{\sigma}_{k+1}) = \frac{1}{2} \left( \sigma_k + \sigma_{k+1} \right) + \frac{1}{2} (1 - \sigma_k \sigma_{k+1}) \tanh \left( \frac{K_{\text{max}} + K_k}{2} - \frac{K_{\text{max}} + K_{k+1}}{2} \right) \]

\[ = \frac{1}{2} \left( \sigma_k + \sigma_{k+1} \right) + \frac{1}{4} (1 - \sigma_k \sigma_{k+1} - \sigma_k \sigma_{k+2} - \sigma_{k+1} \sigma_{k+2}) \tanh \left( \frac{(K_{\text{max}} + K_k + K_{k+1} + K_{k+2})}{2} \right) \]

\[ + \frac{1}{4} (1 - \sigma_k \sigma_{k+1} - \sigma_k \sigma_{k+2} - \sigma_{k+1} \sigma_{k+2}) \tanh \left( \frac{(K_{\text{max}} + K_k + K_{k+1} + K_{k+2})}{2} \right). \]

Applying Eqs. (17)-(18) and similar equations for $\sigma^{(3)}$, we obtain

\[ \frac{\partial q}{\partial t} = \left( \frac{\partial q}{\partial t} \right)_M + a^2 \left[ 1 - \tanh^2 (2K) \right] \frac{\partial K}{\partial x} \left( \frac{1}{2} \frac{\partial \sigma^{(3)}}{\partial x} - \frac{3}{2} \frac{\partial q}{\partial x} \right) + a^2 \frac{\partial^2 K}{\partial x^2} \left( \sigma^{(3)} - q \right). \]

\[ \text{(21)} \]
Compared with the rigorous results of the Gaussian model, there are similarities and differences. Similar to the Gaussian model, the first term at the right hand side, \((\partial q/\partial t)_M\), is the diffusion induced by the magnetization itself (see Eq. (9)). It will vanish near the zero-temperature critical point. Besides that we can observe the nonvanishing \(K\)-diffusion, which will, in contrast, get infinitely sensitive near this point. A major difference is that, the terms of Eq. (21) do not take such relative forms as in Eq. (19). Thus, although the influence of \(dK/dx\) and \(d^2K/dx^2\) still depend on the spin configuration, the dependence is in a different way. The system will get less sensitive when the temperature is increasing, and we may attribute this to the increasing influence of the thermal noise.

IV. SUMMARY

The present research is based on the Kawasaki-type spin-pair redistribution mechanism. We generalize the idea of spin diffusion incorporating a variety of factors as possible driving forces, including external field and temperature. Two models are selected for a detailed investigation: the Ising model, which is widely applicable, and the Gaussian model, which is idealized but provides a good basis for an analytical treatment. It is found that the two models are similar in principle, and the features are likely to be shared by other models and experimental systems.

Generally speaking, the diffusion equation can be written as

\[
\frac{\partial q}{\partial t} = \left( \frac{\partial q}{\partial t} \right)_M + \left( \frac{\partial q}{\partial t} \right)_e. 
\]

The right-hand side of the equation consists of two juxtaposed terms: One describes the diffusion induced by the inhomogeneity of the magnetization itself. It retains the same form as that obtained without any external inhomogeneities, and is believed to be vanishing near the critical point. The other one is the "general diffusion" induced by the inhomogeneity of the environment, which may be coupled to the spin and contain both first and second derivatives. The latter, of which a more appropriate name is environment-induced self-organization, generally does not vanish near the critical point, and strongly depends on spin configuration and may be weakened by thermal noise. The concept of local hysteresis is proposed in this spin-conserved dynamics as a convenient tool to characterize the response of the system to oscillating external perturbations. The response of both the Gaussian model and the one-dimensional Ising model to a travelling electromagnetic wave is specifically studied.

About the Kawasaki dynamics with external inhomogeneities, there remain a number of questions worthy of further investigations. For example, the domain growth in an Ising model subject to a travelling electro-magnetic wave\(^2\), the response to the temperature varying with time, and the dynamics in other models with continuous symmetry, such as the \(XY\) model [3] and the Heisenberg model, etc. We hope the interesting features revealed in this article may stimulate future research, probably with extensive numerical simulations.

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APPENDIX A: THE SPIN-PAIR REDISTRIBUTION MECHANISM

In spin-pair redistribution mechanism [11,19], two neighboring spins, \(\sigma_j\sigma_l\), may change to any possible values, \(\hat{\sigma}_j\hat{\sigma}_l\), as long as their sum are conserved. The master equation is

\[
\frac{d}{dt} P(\{\sigma\}; t) = \sum_{j(l)} \sum_{\hat{\sigma}_j, \hat{\sigma}_l} \left[ -W_{jl} (\sigma_j\sigma_l \rightarrow \hat{\sigma}_j\hat{\sigma}_l) P(\{\sigma\}; t) \\
+ W_{jl} (\hat{\sigma}_j\hat{\sigma}_l \rightarrow \sigma_j\sigma_l) P(\{\sigma_i \neq j, \sigma_l \neq k\} \hat{\sigma}_j, \hat{\sigma}_l; t) \right]. \tag{A1}
\]

The redistribution probability is in a normalized form determined by a heat Boltzmann factor,

\(^2\)Such processes are subject to several factors acting together: the inherent properties of the model, the modulating effect of the field, and a competition of time scales, namely the relaxation time of the system and the time period of the local field.
\[
W_{jl} (\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l) = \frac{1}{Q_{jl}} \delta_{\sigma_j+\sigma_l, \hat{\sigma}_j+\hat{\sigma}_l} \exp \left[ -\beta \mathcal{H}_{jl} (\hat{\sigma}_j, \hat{\sigma}_l, \{\sigma_m\}_{m \neq j,l}) \right],
\]

where the normalization factor \(Q_{jl}\) is
\[
Q_{jl} = \sum_{\sigma_j, \hat{\sigma}_l} \delta_{\sigma_j+\sigma_l, \hat{\sigma}_j+\hat{\sigma}_l} \exp \left[ -\beta \mathcal{H}_{jl} (\hat{\sigma}_j, \hat{\sigma}_l, \{\sigma_m\}_{m \neq j,l}) \right].
\]

For single spins, the time expectation, \(q_k \equiv \langle \sigma_k \rangle \equiv \sum_{\{\sigma\}} \sigma_k P (\{\sigma\}; t)\), is
\[
\frac{d}{dt} q_k (t) = -2D q_k (t) + \sum_w \sum_{\{\sigma\}} \sum_{\hat{\sigma}_k, \hat{\sigma}_{k+w}} \hat{\sigma}_k W_{k,k+w} (\sigma_k \sigma_{k+w} \rightarrow \hat{\sigma}_k \hat{\sigma}_{k+w}) P (\{\sigma\}; t),
\]

where \(D\) is the spatial dimensionality and \(\sum_w\) denotes a summation taken over the nearest neighbors.

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**Caption of figures**

Fig. 1. Schematic plots of typical local hysteresis loops in the Gaussian model.

Fig. 2. Typical local hysteresis loops for different temperatures in the Ising model. System size \(N = 100\), and parameters \(\omega = 2\pi/1000\), \(k = 2\pi\) and \(H_0 = 4\).

Fig. 3. The evolution of the system above the critical point. The magnetization is approaching the equilibrium with its shape similar to the external field. Squares correspond to the system being at \(t = 50s\), circles correspond to \(t = 300s\), and triangles correspond to \(t = 1000s\).

Fig. 4. The evolution of the system below the critical point. (a) The magnetization at \(t = 350s\) is in the steady phase and keeps a sinusoidal shape while growing very slowly. (b) The self-organization begins, and the magnetization at \(t = 400s\) is shown. (c) The system is in the growing phase and the magnetization is fast growing while keeping an fixed shape. Squares correspond to \(t = 500s\), circles correspond to \(t = 510s\), and triangles correspond to \(t = 520s\).
(a) $t = 350s$

\[ q(x,t) \]

(b) $t = 400s$

\[ q(x,t) \]

(c) $t = 500s$, $t = 510s$, $t = 520s$

\[ q(x,t) \]