WAVE BREAKING FOR SHALLOW WATER MODELS WITH TIME DECA YING SOLUTIONS

Igor Leite Freire$^{1,2}$

$^1$Mathematical Institute, Silesian University in Opava, Na Rybníčku, 1, 74601, Opava, Czech Republic

$^2$Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, Avenida dos Estados, 5001, Bairro Bangu, 09.210 – 580, Santo André, SP - Brasil,  
igor.freire@ufabc.edu.br, igor.leite.freire@gmail.com

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ABSTRACT

Abstract: A family of Camassa-Holm type equations with a linear term and cubic and quartic nonlinearities is considered. Local well-posedness results are established via Kato’s approach. Conserved quantities for the equation are determined and from them we prove that the energy functional of the solutions is a time-dependent, monotonically decreasing function of time, and bounded from above by the Sobolev norm of the initial data under some conditions. The existence of wave breaking phenomenon is investigated and necessary conditions for its existence are obtained. In our framework the wave breaking is guaranteed, among other conditions, when the coefficient of the linear term is sufficiently small, which allows us to interpret the equation as a linear perturbation of some recent Camassa-Holm type equations considered in the literature.

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1 Introduction

Indubitably the Camassa-Holm (CH) equation is one of the cutting edge topics in mathematics and mathematical physics in general, and differential equations and applied analysis in particular.

The work of Camassa and Holm [4], where the equation was derived as a model for shallow water waves, can be considered as of fundamental nature and is the cornerstone of an active field of investigation of non-local integrable equations. The discovery of the CH equation was the prelude of a myriad of other non-local integrable equations, such as the Degasperis-Procesi (DP) [16, 17] and the Novikov equations [24, 33]. They share the following properties with the CH equation: bi-Hamiltonian structure, integrability, existence of peakon and multipeakon solutions, and infinite hierarchy of conservation laws, to name a few.

The CH equation was firstly deduced by Fuchsteiner and Fokas in [20]. It was, in fact, a formal deduction, without physical motivation. However, on the grounds of physics, in [4] the equation was recovered as a unidirectional model for describing the height of water’s free surface above the flat bottom for a shallow water system, see also [26].

One of the interesting properties of the CH equation, and other similar equations not necessarily among the ones mentioned before, is the fact that they have many conservation laws. This is very useful and vital to establish qualitative results about the solutions of equations and partially explains why this sort of equations are particularly fashion in the field of differential equations.

We would like to recall some simple, but extremely important concepts in differential equations where time is involved. Given a differential equation and an initial data (that is, a Cauchy problem), some questions are of capital importance:

1. Does the problem have a solution? This is the question of existence.
2. If there is a solution, is it unique? This is the problem of uniqueness.
3. If there is a solution, does it depend continuously on the initial data?

These three essential questions are central when one is investigating qualitative behaviour of the solutions of equations subject to the restrictions imposed by a certain condition. In this paper, the equation we shall study depends on $t$ and $x$ and the condition we shall consider is an initial condition, that is, we know how the solution behaves at $t = 0$. In case these questions are satisfied, the problem under consideration is said to be well-posed (in the sense of Hadamard).

Although the questions above are basic and fundamental, they are not necessarily simple or easy to be addressed. In particular, very often the solution $u$ of a problem like the one we shall deal with in this paper belongs to $C^0([0, T), E)$, meaning that $u(\cdot, x) \in C^0([0, T))$ and $u(t, \cdot) \in E$, where $E$ is a usually a Banach space, whereas $T > 0$ is the maximal time of existence of the solution, or lifespan, which usually depends on the initial data and the space $E$.

The value of the lifespan add another ingredient to our menu: If the problem is well-posed, what happens in case $T < \infty$? What if $T = \infty$? In the first case we have a local well-posed problem: it is well-posed provided that $t \in [0, T)$, while in the second we have a global well-posed problem, meaning that the solution exists for any $t$.

In case we only have local well-posedness, we say that the solution $u$ of the problem develops a finite time blow-up. This happens just because the solution cannot be described for all values of $t$. 

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The blow-up phenomena can have different manifestations, depending on the problem. It can arise, for example, if the solution of the problem becomes unbounded when $t$ approximates $T$. More specifically, a blow-up in finite time turns out if $T < \infty$ and

$$\limsup_{t \to T} \|u(t, \cdot)\|_E = \infty.$$  

A blow-up can also arise in the following situation: Assume that the Banach space $E$ is a subspace of $C^1(\mathbb{R})$ and $T < \infty$. Then we have another sort of blow-up if

$$\sup_{(t, x) \in [0, T) \times \mathbb{R}} |u(t, x)| < \infty, \quad \text{but} \quad \limsup_{t \to T} \left( \sup_{x \in \mathbb{R}} |u_x(t, x)| \right) = \infty.$$  

This kind of blow-up, which takes the $x-$derivative of the solution into account, is best known as wave breaking. From a geometrical viewpoint, it means that the tangent line to the curve $x \mapsto (x, u(t, x))$ tends to the perpendicular line to the $x-$direction when $t$ approaches $T$.

### 1.1 Notation and conventions

Throughout this paper, if $u$ is a function depending on two variables, both $u_t$ and $\partial_t u$ denote the derivative of $u$ with respect to its first argument, while $u_x$ and $\partial_x u$ mean the derivative with respect to the second independent variable.

The norm in a Banach space $E$ is denoted by $\| \cdot \|_E$, whereas $\langle \cdot, \cdot \rangle_H$ means the inner product in a Hilbert space $H$. If $E$ and $F$ are two Banach spaces, the set of bounded linear operators from $E$ into $F$ is denoted by $\mathcal{L}(E, F)$, or $\mathcal{L}(E)$ in the case $F = E$.

The space of test functions $\varphi : \mathbb{R} \to \mathbb{R}$ is denoted by $S(\mathbb{R})$, whereas its dual topological space is refereed as $S'(\mathbb{R})$. A member of $S(\mathbb{R})$ is known as rapidly decreasing smooth function, while members of $S'(\mathbb{R})$ are called tempered distributions.

### 1.2 Main results of the paper and its outline

Let us introduce the subject of investigation of the present paper. Our main interest here is the equation

$$u_t - u_{txx} + 3uu_x + \lambda(u - uu_x) = 2u_x u_{xx} + uu_{xxx} + u_x + \beta u^2 u_x + \gamma u^3 u_x + \Gamma u_{xxx}. \quad (1.1)$$

It reduces to the CH equation if $\lambda = \alpha = \beta = \gamma = \Gamma = 0$, whereas the Dullin-Gottwald-Holm equation \cite{13,18} is recovered when $\lambda = \beta = \gamma = 0$ and $\alpha \Gamma \neq 0$. If $\alpha = \beta = \gamma = \Gamma = 0$ and $\lambda > 0$ we have the weakly dissipative CH equation \cite{41,42}, whereas if $\lambda > 0$, $\alpha \Gamma \neq 0$ and $\beta = \gamma = 0$ we have the weakly dissipative DGH equation \cite{34,35,43}.

If $\lambda = 0$, equation (1.1) includes a shallow water model with Coriolis effects proposed in \cite{5,22,23,36}, see also \cite{14,15}.

Making use of the auxiliary variable $m = u - uu_x$ we reformulate the object of investigation of this paper as the following: We aim at investigating properties of the initial value problem,

$$\begin{cases}
m_t + um_x + 2u_x m + \lambda m = \alpha u_x + \beta u^2 u_x + \gamma u^3 u_x + \Gamma u_{xxx}, & t > 0, \ x \in \mathbb{R}, \ \lambda \neq 0, \\
u(0, x) = u_0(x).
\end{cases} \quad (1.2)$$
Our first aspiration is to address the three questions listed in the beginning of the paper. We prove that (1.1) is locally well-posed if the initial data belongs to the Sobolev space $H^s(\mathbb{R})$, with $s > 3/2$. Namely, we have the following result:

**Theorem 1.1.** Given $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, then there exist a maximal time $T = T(u_0) > 0$ and a unique solution $u$ to (1.1) satisfying the initial condition $u(0, x) = u_0(x)$, such that $u = u(\cdot, u_0) \in C^0([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}))$. Moreover, the solution depends continuously on the initial data, in the sense that the map $u_0 \mapsto u(\cdot, u_0) : H^s(\mathbb{R}) \to C^0([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}))$ is continuous and $T$ does not depend on $s$.

Theorem 1.1 is proved using Kato’s approach [27] and its demonstration is presented in Section 2.

In order to enlighten further qualitative analysis of the solutions of (1.1), the next proposition gives us some differential identities.

**Proposition 1.1.** Let $v = v(t, x)$ be a function such that $v \in C^1(\mathbb{R}^2; \mathbb{R})$, its second and third order derivatives exist, $v_{tx} = v_{xt}$, and

$$E := v_t - v_{xxx} + \lambda(v - v_{xx}) - 2v_x v_{xx} - vv_{xxx} - (\alpha + \beta v^2 + \gamma v^3 - 3v)v_x + \Gamma v_{xxx},$$

where $\alpha$, $\beta$, $\gamma$ and $\Gamma$ are constants. Then the following formal identities holds:

$$E = \lambda v + \partial_t v + \partial_x \left( \frac{3}{2} v^2 - v_{tx} - v v_{xx} - \frac{v^2}{2} - \alpha v - \frac{\beta}{3} v^3 - \frac{\gamma}{4} v^4 - \Gamma v_{xx} - \lambda t \right), \quad (1.3)$$

and

$$vE = \lambda(v^2 + \frac{v^2}{2}) + \partial_t \left( \frac{v^2 + \frac{v^2}{2}}{2} \right) + \partial_x \left( v^3 - v^2 v_{xx} - vv_t x + \Gamma \frac{v^2}{2} - \Gamma vv_{xx} - \frac{v^2}{2} - \frac{\beta}{4} v^4 - \frac{\gamma}{5} v^5 - \lambda v v_x \right). \quad (1.4)$$

In the very particular case $\beta = \gamma = 0$ and $\Gamma = -\alpha$, we have a third identity:

$$\frac{1}{2}(v - v_{xx})^{-1/2}E = \frac{\lambda}{2}(v - v_{xx})^{1/2} + \partial_t (v - v_{xx})^{1/2} + \partial_x [(-\alpha)(v - v_{xx})^{1/2}].$$

The proof of Proposition 1.1 is straightforward and, for this reason, is omitted. However, these identities, when considered on the solutions of the equation (1.2), give us very useful invariants.

**Theorem 1.2.** Assume that $u \in C^1(\mathbb{R}^2; \mathbb{R})$ is a solution of (1.1) such that $u, u_x \to 0$ as $x \to \pm \infty$, $u_{tx} = u_{xt}$ and its second order derivatives are bounded. Let

$$\mathcal{H}_0(t) := \int_\mathbb{R} u(t, x) \, dx$$

and

$$\mathcal{H}_1(t) := \frac{1}{2} \int_\mathbb{R} (u(t, x)^2 + u_x(t, x)^2) \, dx. \quad (1.5)$$

Then, for any $t$, we have $\mathcal{H}_0(t) = e^{-\lambda t}\mathcal{H}_0(0)$ and $\mathcal{H}_1(t) = e^{-2\lambda t}\mathcal{H}_1(0)$. In particular, $\|u\|_{H^1(\mathbb{R})}^2 = 2\mathcal{H}_1(t)$, and if $\lambda \geq 0$, then $\|u\|_{H^1(\mathbb{R})} = e^{-\lambda t}\|u_0\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})}$. 

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We shall refer to (1.5) generically as energy (functional) and it is nothing but a time dependent $H^1(\mathbb{R})$-norm of the solution $u$ (which we shall simply refer as Sobolev norm for convenience). If $\lambda < 0$ we easily see that its norm increases as the increasing of $t$, leading to unbounded solutions on $[0, \infty) \times \mathbb{R}$, whereas the case $\lambda > 0$ is more interesting, since in this situation the Sobolev norm for any $t$ is bounded from above by the Sobolev norm of the initial data.

Another qualitative property of the solutions is given in the next result.

**Theorem 1.3.** Let $u$ be a solution to (1.1) such that $u_{tx} = u_{xt}$, and $u, u_x, u_{xx}$ are integrable and vanishing as $x \to \pm \infty$ for any value of $\lambda$. If $u_0(x) := u(0, x)$, $m = u - u_{xx}$ and $m_0 := u_0 - u_0''$, then

\[
e^{-\lambda t} \int_\mathbb{R} m_0 dx = \int_\mathbb{R} m dx = \int_\mathbb{R} u dx = e^{-\lambda t} \int_\mathbb{R} u_0 dx. \tag{1.6}
\]

Theorems 1.2 and 1.3 are proved in Section 3.

A natural question is whether (1.1) admits wave breaking, that can be assured in the following scenario, which will be proved in Section 4.

**Theorem 1.4.** Let $u_0 \in H^3(\mathbb{R})$, $u = u(t, x)$ be the corresponding solution to (1.2), $y(t) := \inf_{x \in \mathbb{R}} u_x(t, x)$, and $\kappa := \max\{|\alpha|, |\beta|/3, |\gamma|/4, |\Gamma|\}$. Assume that there exists $\theta \in (0, \theta_0]$ and $x_0 \in \mathbb{R}$ such that $\theta u_0'(x_0) < \min\{-\|u_0\|^{1/2}_{H^1(\mathbb{R})}, -\|u_0\|^2_{H^1(\mathbb{R})}\}$, where $\theta_0 := \sqrt{2/(1 + 12\kappa)}$.

If

\[\lambda \in \left(0, -\frac{y(0)}{4} \frac{\theta^2 u_0'(x_0)^2 - \max\{\|u_0\|_{H^1(\mathbb{R})}, \|u_0\|^4_{H^1(\mathbb{R})}\}}{\theta^2 u_0'(x_0)^2}\right) , \]

then wave breaking for (1.2) occurs.

Theorem 1.4 can only foresee the emergence of wave breaking of the solutions of (1.2) for limited, but positive, values of $\lambda$. We observe that if $\lambda$ in (1.2) is small, then we can interpret the term $\lambda m$ as a perturbation of the equation

\[
\begin{cases}
m_t + um_x + 2u_{xx}m = \alpha u_x + \beta u^2 u_x + \gamma u^3 u_x + \Gamma u_{xxx}, & t > 0, \ x \in \mathbb{R}, \\
u(0, x) = u_0(x),
\end{cases}
\tag{1.7}
\]

which was studied in [14,15], and also in [5,22,23,36] for specific choices of the parameters $\alpha, \beta, \gamma$ and $\Gamma$, as already mentioned.

Although we may unravel the presence of the term $\lambda m$ in (1.1) as a perturbation of (1.7), we cannot underrate it, since its presence brings structural and substantial changes in the behaviour of the solutions of equation (1.1). As we shall show in section Section 3 it implies that if global solutions of (1.2) exist, as well as their corresponding energy functional, then they vanish as $t \to \infty$. Moreover, for $\lambda$ sufficiently small, the conditions for wave breaking of the solutions (1.2) are unaltered when compared to (1.7), as one can observe comparing our results with those proved in [15] regarding this matter.

The proof of local well-posedness naturally leads to the question of whether the problem (1.2) has global solutions. While we have an accurate description of the manifestation of blow-up phenomena as wave breaking, the question of global solvability of equation (1.1) is unclear. In Section 5 we bring some light to this problem and we show the limitations and open problems that lead us to fail in giving a complete description for the global existence of solutions of (1.1).
We discuss our results in Section 6, while in Section 7 we present our conclusions.

1.3 Novelty and challenges of this paper

The presence of the cubic and quartic nonlinearities in (1.2) brings some difficulties in the qualitative analysis of the solutions of (1.2) when compared with similar works dealing with (1.7). This is somewhat expected, since the presence of these higher order nonlinearities introduces substantial modifications on the behaviour of the solutions of (1.2) in comparison with (1.7), as one can infer by comparing the results in [6–9, 13, 19, 28, 30, 37, 39, 40] with those in [5, 14, 15, 22, 23, 36]. In particular, the equation (1.7) does not seem to be integrable unless \( \beta = \gamma = 0 \), see [14, 15]. Turning back to our case, if \( \lambda \beta \neq 0 \) or \( \lambda \gamma \neq 0 \) we have a dramatically different situation to be considered regarding both global existence and wave breaking phenomena. Besides, if one of these conditions is satisfied, then we cannot reduce the analysis of equation (1.2), with \( \lambda \neq 0 \), to equation (1.7) as it is possible for some dissipative CH type equations, as pointed out in [28].

In case \( \alpha = -\Gamma, \lambda > 0 \) and \( \beta = \gamma = 0 \), the equation (1.7) is reduced to

\[
mt + um_x + 2u_x m + \lambda m = \alpha m_x,
\]

which, up to notation, was investigated in [34, 43]. The change \( (u(t, x) \mapsto u(t, x + \alpha t) \) transform the last equation into the weakly dissipative CH equation, which was investigated in [41, 42].

It was pointed out by Lenells and Wunsch [28] that the analysis of the latter equation can be reduced to the CH equation through the change of variables

\[
u(t, x) \mapsto e^{-\lambda t} u \left( \frac{1 - e^{-\lambda t}}{\lambda}, x \right).
\]

In our case, if \( \alpha \neq -\Gamma \) or \( (\beta, \gamma) \neq (0, 0) \), then we cannot use the results in [28] to reduce the analysis of equation (1.1) to other previously known.

The clever observation made by Lenells and Wunsch [28] can be applied to a large class of relevant equations in the field of shallow water models and integrable systems, in particular, to some of the references cited above (and perhaps in others in the list) regarding (weakly) dissipative equations. As one can infer from the results in [28], the transformation connecting equations are of the type

\[
u(t, x) \mapsto e^{-\lambda t} u \left( \frac{1 - e^{-(p-1)\lambda t}}{\lambda}, x \right),
\]

where \( p \) is a certain parameter. The reason for this transformation works is the following (we pay attention only to the scalar cases, but the explanation for systems follows the same argument): The equations considered in [28] are of the form \( mt + \lambda m = F[u, m] \), where \( F \) is a homogeneous polynomial in \( u, m \) and their derivatives with respect to \( x \). This means that if \( \mu \) is a parameter, then \( F[\mu u, \mu m] = \mu^p F[u, m] \), for some \( p \). It is well known that the CH, DP, Novikov and other similar equations are scale-invariant, see [18, 11, 12] and references therein. In particular, \( p = 2 \) for the CH equation and its corresponding dissipative equation. The homogeneity of \( F[u, m] \) allows us to use the substitution (1.8) to reduce the equations mentioned in [28] to their counterparts with \( \lambda = 0 \).

In our case, if \( \alpha \neq -\Gamma \) we cannot eliminate the term \( \alpha u_x + \Gamma u_{xxx} \). On the other hand, if \( (\beta, \gamma) \neq (0, 0) \) then we have the presence of cubic and quartic terms. In any situation, we lose homogeneity

\footnote{It is worth mentioning that in all of these works there is no linear term in the equations involved.}
and, therefore, the elegant transformation introduced in [28] is no longer admissible. Moreover, these higher order terms not only prevent us to invoke the results aforementioned, but also bring difficulties to completely describe conditions for the global existence of solutions for the Cauchy problem (1.2).

Another peculiarity of the present paper is the fact that we construct some invariants for the so-

2 Local well-posedness

Here we establish the existence and uniqueness of solutions of the Cauchy problem (1.2). More precisely, we respond questions 1, 2 and 3 in the Introduction of the paper. These results are established at the local level, meaning that we guarantee the existence of \( T > 0 \) such that the solution exists on \([0, T) \times \mathbb{R}\).

2.1 Overview of functional analysis

We present a short overview of Sobolev spaces, embeddings and mappings between Sobolev spaces. For further readings about these subjects, the reader is referred to [2, 21, 25, 29, 38].

If \( \phi \) is a tempered distribution, its Fourier transform \( \mathcal{F}(\phi) \) and its corresponding inverse are, respectively, given by

\[
\hat{\phi}(\xi) := \mathcal{F}(\phi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(x) e^{-ix\xi} dx \quad \text{and} \quad \phi(x) = \mathcal{F}^{-1}(\hat{\phi})(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{\phi}(\xi) e^{ix\xi} d\xi.
\]

Given \( s \in \mathbb{R} \), the Sobolev space of order \( s \) is given by \( H^s(\mathbb{R}) = \Lambda^{-s}(L^2(\mathbb{R})) \), where \( L^2(\mathbb{R}) (= H^0(\mathbb{R})) \) denotes the usual Hilbert space of the squared integrable functions and \( \Lambda^s u := \mathcal{F}((1 + |\xi|^2)^{s/2} \hat{u}) \).

The operator \( \Lambda^s \) is a unitary isomorphism between \( H^t(\mathbb{R}) \) and \( H^{t-s}(\mathbb{R}) \), for any \( t \in \mathbb{R} \). Of particular importance is the case \( s = 2 \), in which the operator can be identified with the differential operator \( \Lambda^2 := 1 - \partial_x^2 \) (also known as Helmholtz operator) and its inverse is given by

\[
\Lambda^{-2} u = g * u = \frac{1}{2} \int_{\mathbb{R}} g(x - y) u(y) dy,
\]

where * denotes the convolution and \( g(y) = e^{-|y|}/2 \) is the Green function of the equation \((1 - \partial_x^2) u = \delta(x)\), and \( \delta(x) \) is the Dirac delta distribution.

We note that \( H^s(\mathbb{R}) \), for each \( s \in \mathbb{R} \), is a Hilbert space when endowed with the inner product

\[
\langle u, v \rangle_{H^s} := \int_{\mathbb{R}} (1 + |\xi|^2)^s \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi.
\]

We also note that if \( u \in H^1(\mathbb{R}) \), then its norm is given by \( \| u \|^2_{H^1(\mathbb{R})} = \| u \|^2_{L^2(\mathbb{R})} + \| u_x \|^2_{L^2(\mathbb{R})} \).

For each \( s \in \mathbb{R} \), \( \partial_x \in \mathcal{L}(H^s(\mathbb{R}), H^{s-1}(\mathbb{R})) \), where \( u \mapsto u_x \), and if \( s \) and \( t \) are real numbers such that \( s \geq t \), then \( \mathcal{S}(\mathbb{R}) \subseteq H^s(\mathbb{R}) \subseteq H^t(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R}) \). We, indeed, shall use the estimates \( \| \partial_x f \|_{H^{s-1}(\mathbb{R})} \leq \| f \|_{H^s(\mathbb{R})} \), \( \| \Lambda^{-2} f \|_{H^s(\mathbb{R})} \leq \| f \|_{H^{s+2}(\mathbb{R})} \) and \( \| \partial_x \Lambda^{-2} f \|_{H^s(\mathbb{R})} \leq \| f \|_{H^{s-1}(\mathbb{R})} \).
Some useful results to our purposes are:

**Lemma 2.1. [Algebra Property]** For $s > 1/2$, there is a constant $c_s > 0$ such that
\[ \| fg \|_{H^s(\mathbb{R})} \leq c_s \| f \|_{H^s(\mathbb{R})} \| g \|_{H^s(\mathbb{R})}. \]

*Proof.* See [29] or [38], pages 51 and exercise 6 on page 320, respectively.

**Lemma 2.2. [Sobolev Embedding Theorem]** If $s > 1/2$ and $u \in H^s(\mathbb{R})$, then $u$ is bounded and continuous. Moreover, in case we have $s > 1/2 + k$, then $H^s(\mathbb{R}) \subseteq C^k_0(\mathbb{R})$.

*Proof.* See [29] or [38], pages 47 and 317, respectively.

As a consequence of the Sobolev Embedding Theorem, if $s > 1/2$, then $u \in L^\infty(\mathbb{R})$. Moreover, if $u \in H^s(\mathbb{R})$, with $s > 1/2 + k$, for a certain non-negative number $k$, then $u \in C^k_0$. In any case, $\| u \|_{L^\infty(\mathbb{R})} \leq c_s \| u \|_{H^s(\mathbb{R})}$, for some positive constant $c_s$ depending on $s$.

We recall that if $E$ and $F$ are Banach spaces, a mapping $f : E \rightarrow F$ is called Lipschitz if there exists $k > 0$ such that $\| f(u) - f(v) \|_F \leq k \| u - v \|_E$, for all $u, v \in E$.

**Lemma 2.3.** Let $F \in C^\infty(\mathbb{R})$ such that $F(0) = 0$. If $f \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $s \geq 0$, then $\| F(f) \|_s \leq K \| f \|_s$, where $K$ depends only on $\| f \|_{L^\infty(\mathbb{R})}$, and $\| f \|_s = \| f \|_{H^s(\mathbb{R})} + \| f \|_{L^\infty(\mathbb{R})}$.

*Proof.* See Proposition on page 1065 of [10].

Note that for $s > 1/2$ and in view of the Sobolev Embedding Theorem, $H^s(\mathbb{R}) \cap L^\infty(\mathbb{R}) = H^s(\mathbb{R})$. We conclude this subsection with a very useful differential inequality.

**Lemma 2.4. [Gronwall Inequality]** Let $y \in C^1(I, \mathbb{R})$, where $I$ is an interval on $\mathbb{R}$ containing 0. Assume that $y'(t) \leq \beta(t)y(t)$, where $\beta$ is a smooth function. Then $y(t) \leq y(0)e^{\int_0^t \beta(s)ds}$.

*Proof.* See [25], page 56.

We note that the Gronwall inequality also implies $y(t) \geq y(0)e^{\int_0^t \beta(s)ds}$ if $y'(t) \geq \beta(t)y(t)$.

### 2.2 Proof of Theorem 1.1

Our main ingredient to prove the local-well posedness of equation (1.1) with initial data $u(0, x) = u_0(x)$ is the following result, due to Tosio Kato [27].

**Lemma 2.5. (Kato’s theorem)** Let $A(u)$ be a linear operator and consider the problem
\[
\begin{align*}
\frac{du}{dt} + A(u)u &= f(u) \in X, & t \geq 0, \\
u(0) &= u_0 \in Y.
\end{align*}
\]

Assume that the following conditions are satisfied:

**C1** Let $X$ and $Y$ be reflexive Banach spaces, such that $Y \subseteq X$ and the inclusion $Y \hookrightarrow X$ is continuous and dense. In addition, there exists an isomorphism $S : Y \rightarrow X$ such that $\| u \|_Y = \| Su \|_X$;
**C2** There exist a ball $W$ of radius $R$ such that $0 \in W \subseteq Y$ and a family of operators $(A(u))_{u \in W} \subseteq \mathcal{L}(X)$ such that $-A(u)$ generates a $C_0$ semi-group in $X$ with $\|e^{-sA(u)}\|_{\mathcal{L}(X)} \leq e^{\beta s}$, for any $u \in W$, $s \geq 0$, for a certain real number $\beta$.

**C3** Let $S$ be the isomorphism in Condition C1. Then $B(u) := [S, A(u)]S^{-1} \in \mathcal{L}(X)$. Moreover, there exists constants $c_1$ and $c_2$ such that $\|B(u)\|_{\mathcal{L}(X)} \leq c_1$, $\|B(u) - B(v)\|_{\mathcal{L}(X)} \leq c_2\|u - v\|_{Y}$, for all $u, v \in W$.

**C4** For any $w \in W$, $Y \subseteq \text{dom}(A(w))$ and $\|A(u) - A(v)\|_{\mathcal{L}(Y, X)} \leq c_3\|u - v\|_X$, for any $u, v \in W$.

**C5** The function $f : X \to Y$ satisfies the following conditions:

(a) $f|_W : W \to Y$ is bounded, that is, there exists a constant $c_4$ such that $\|f(w)\|_Y \leq c_4$, for all $w \in W$;

(b) $f|_W : W \to X$ is Lipschitz when taking the norm of $X$ into account, that is, there is another constant $c_5$ such that $\|f(u) - f(v)\|_X \leq c_5\|u - v\|_X$, for all $u, v \in W$.

If $u_0 \in W$, then there is $T > 0$ such that (2.1) has a unique solution $u \in C^0([0, T), W) \cap C^1([0, T), X)$, with $u(0) = u_0$.

The constants mentioned in the conditions in the lemma above depend on the radius $R$ of $W$, see [19, 27, 31, 37].

At first sight Lemma 2.5 does not seem to be applicable to (1.1), since in (2.1) we have an evolution equation. This difficulty can be easily overcome by making use of the isometric isomorphism $\Lambda^2 : H^s(\mathbb{R}) \to H^{s-2}(\mathbb{R})$ and its inverse. More precisely, we have the following:

**Proposition 2.1.** Equation (1.1) can be rewritten as

$$u_t + (u + \Gamma)u_x = \Lambda^{-2}\partial_x h(u) - \Lambda^{-2}\partial_x \left(u^2 + \frac{u_x^2}{2}\right) - \lambda u,$$

where

$$h(u) := (\alpha + \Gamma)u + \frac{\beta}{3}u^3 + \frac{\gamma}{4}u^4.$$  

**Proof.** Applying the operator $\Lambda^2$ into (2.2) we obtain the equation in (1.2), which is nothing but (1.1). \qed

**Remark 2.1.** Equation (2.2) explains why very often in the literature of CH type equations they are referred as non-local evolution equations. Firstly, note that it is now an evolution equation. However, the price to transform the non-evolution equation (1.1) into an evolution one is the arising of non-local terms given by the action of the operator $\Lambda^{-2}$, which is nothing but a convolution and brings non-local terms to (2.2).

In view of Proposition 2.1 the Cauchy problem (1.2) is equivalent to

$$\begin{cases}
    u_t + (u + \Gamma)u_x = \Lambda^{-2}\partial_x h(u) - \Lambda^{-2}\partial_x \left(u^2 + \frac{u_x^2}{2}\right) - \lambda u, & t > 0, \ x \in \mathbb{R}, \\
    u(0, x) = u_0(x),
\end{cases}$$

where $h$ is given by (2.3).
We observe that in the case \( \lambda \) we note that (2.4) is an equation in the variables \( u \) for a suitable choice of the initial data. More specifically, we shall show that the quantities this means that it makes the Sobolev norm \( H^s(\mathbb{R}) \) of the corresponding solution \( (1.1) \) satisfying \( u(0, x) = u_0(x) \) is time-dependent and decreases along time if \( u_0 \neq 0 \). Then, by the Sobolev Embedding Theorem, \( \|u\|_{H^s(\mathbb{R})} \leq 2H_i(t) \leq 2H_i(0) = \|u_0\|_{H^s(\mathbb{R})} < \infty \).

Assume that \( u_0 \neq 0 \) and \( \lambda < 0 \). Theorem 1.2 implies that if \( T = \infty \), then \( |H_i(t)| \to \infty \) as \( t \to \infty \), meaning that the solution of (1.2) is unbounded.

3 Time dependent conserved quantities

In this section we shall find some quantities that are conserved on the solutions of the equation in (1.2) for a suitable choice of the initial data. More specifically, we shall show that the quantities

\[
e^\lambda t \int_\mathbb{R} u(t, x) dx \quad \text{and} \quad e^{2\lambda t} \int_\mathbb{R} (u(t, x))^2 + u_x(t, x)^2 \, dx
\]

are constants. This is an immediate consequence of Theorem 1.2 that will play vital role in the investigation of the existence of global solutions and blow-up phenomena.

The relations above show that the result of the integrals has exponential decaying and, in particular, it makes the Sobolev norm \( \| \cdot \|_{H^1(\mathbb{R})} \) of the solutions of (1.1) go to 0 if \( \lambda > 0 \) and \( t \to \infty \), while for \( \lambda < 0 \) it assures in a very simple, but elegant, way the presence of unbounded solutions.

**Proof of Theorem 1.2** Observe that \( u \) satisfies the requirements in Proposition 1.1. Substituting \( v = u \) into (1.3) and (1.4), noticing that \( E|_{v=u} \equiv 0 \) and integrating, we obtain the results. □

**Corollary 3.1.** Equation (1.1) conserves energy if and only if \( \lambda = 0 \). If \( \lambda < 0 \), then both \( H_i(t) \) and \( \mathcal{H}_i(t) \) are unbounded. If \( \lambda > 0 \), then \( \mathcal{H}_i(t) \to 0 \) as \( t \to \infty \).
We summarise the comments above in the next result.

**Corollary 3.2.** Assume that the initial data $u_0$ of the problem (1.2) belongs to $H^s(\mathbb{R})$, $s > 3/2$. If $u_0 = 0$, then the solution is globally defined and $u(t, x) \equiv 0$, for any $(t, x) \in [0, \infty) \times \mathbb{R}$ and any value of $\lambda$. If $u_0 \neq 0$ and $\lambda < 0$, then the solution is finite on each $[0, T]$, for any $T < \infty$, but is not bounded on $[0, \infty)$, while if $\lambda > 0$, then the solution $u$ is bounded from above by $\|u_0\|_{H^s(\mathbb{R})}$, for any $(t, x) \in [0, T) \times \mathbb{R}$.

From Corollary 3.2 we infer the presence of a necessary ingredient to the rising of wave breaking: if $\lambda > 0$, then the solution $u$ is spatially bounded for $t < T$. In Section 4 we shall retake this fact to find conditions to figure out wave breaking of the solutions of the problem (1.2).

**Proof of Theorem 1.3.** Consider the identity (1.3) with $v = u$. We can rewrite it as

$$0 = \lambda m + \frac{\partial m}{\partial t} + \frac{\partial}{\partial x} \left( \frac{3}{2} u^2 - uu_{xx} - u_x^2 - \alpha u - \frac{\beta}{3} u^3 - \frac{\gamma}{4} u^4 - \Gamma u_{xx} \right).$$

Integrating the expression above over $\mathbb{R}$, we have

$$\frac{d}{dt} \int_{\mathbb{R}} m \, dx + \lambda \int_{\mathbb{R}} m \, dx = 0.$$ 

Solving the ODE above and taking into account that at $t = 0$ we have $m(0, x) = m_0(x)$ and obtain the first equality in (1.6).

The last equality is obtained in a similar way by integrating directly (1.3) with $v$ replaced by $u$.

It remains to be proved the middle equality, but it is a direct consequence of the fact $m = u - u_{xx}$ and $u_x \to 0$ as $|x| \to \infty$.

\[ \square \]

### 4 Blow-up scenario

Here we investigate the conditions for the occurrence of wave breaking of the solutions of (1.2). Our main influence here is the text by Escher [19], the works by Constantin and Escher [6–9], and Mustafa [31]. However, in view of the presence of the function $h(u)$ and the term $\lambda u$ in (2.4), their ideas are not directly applicable to our problem. In fact, we can deal with the term $\lambda u$ following similar procedures presented in [41,42]. The main issue in our case is the term $h(u)$ in (2.4), which we need to control in order to put our problem in a suitable place to be tractable.

#### 4.1 Preliminary results

Here we determine when the solution of the equation possesses finite $H^3(\mathbb{R})$–norm. We begin with the following result:

**Theorem 4.1.** Let $u$ be a solution of (1.1) with initial data $u(0, x) = u_0(x)$ and $m_0 = u_0(x) - u_0''(x)$. Assume that $m_0 \in H^1(\mathbb{R})$ and there exists a positive constant $k$ such that $u_x > -k$. Then there exist a function $\sigma \in C^1(\mathbb{R})$ such that $\|u\|_{H^3(\mathbb{R})} \leq \sigma(t) \|m_0\|_{H^1(\mathbb{R})}$. In particular, $u$ does not blow up in finite time.

**Proof.** We begin with recalling that $\Lambda^{-2}$ is an isometric isomorphism between $H^s$ and $H^{s+2}$. Moreover, since $m = u - u_{xx}$, then $u = \Lambda^{-2} m$. Our strategy in the present demonstration is to prove
that  \( \|m\|_{H^1(\mathbb{R})} \leq \sigma(t) \|m_0\|_{H^1(\mathbb{R})} \), for a certain \( \sigma \in C^1(\mathbb{R}) \). The result is then obtained from the relations \( \|u\|_{H^3(\mathbb{R})} = \|\Lambda^{-2} m\|_{H^3(\mathbb{R})} = \|m\|_{H^1(\mathbb{R})} \).

Note that
\[
\frac{d}{dt} \|m\|_{H^1(\mathbb{R})}^2 = \frac{d}{dt} \|m\|_{L^2(\mathbb{R})}^2 + \frac{d}{dt} \|m_x\|_{L^2(\mathbb{R})}^2 = 2 \left( \langle m, m_t \rangle_{L^2(\mathbb{R})} + \langle m_x, m_{tx} \rangle_{L^2(\mathbb{R})} \right). \tag{4.1}
\]

Let us find the parcels of the right hand side of (4.1). From (1.7) and (2.3) we have
\[
m_t = -(u + \Gamma)m_x - 2u_x m - \lambda m + \partial_x h(u). \tag{4.2}
\]

Therefore,
\[
\langle m, m_t \rangle_{L^2(\mathbb{R})} = -\langle m, um_x \rangle_{L^2(\mathbb{R})} - \Gamma \langle m, m_x \rangle_{L^2(\mathbb{R})} - 2\langle u_x, m^2 \rangle_{L^2(\mathbb{R})} - \lambda \langle m, m \rangle_{L^2(\mathbb{R})} + \langle m, \partial_x h(u) \rangle_{L^2(\mathbb{R})}
= -\frac{3}{2} \langle u_x, m^2 \rangle - \lambda \|m\|_{L^2(\mathbb{R})}^2 + \langle m, \partial_x h(u) \rangle,
\]
where we used the relations \( \langle m, u_x m \rangle = \langle u_x, m^2 \rangle, \langle m, um_x \rangle = \langle u, mm_x \rangle = -\langle u_x, m^2 \rangle/2 \) and
\[
\langle m, m_x \rangle = \int_{\mathbb{R}} mm_x dx = \frac{1}{2} \int_{\mathbb{R}} \partial_x m^2 dx = 0.
\]

Deriving (4.2) with respect to \( x \) and substituting the result into \( \langle m_x, m_{tx} \rangle_{L^2(\mathbb{R})} \) yield
\[
\langle m_x, m_{tx} \rangle_{L^2(\mathbb{R})} = -\langle m_x, um_{xx} \rangle_{L^2(\mathbb{R})} - \Gamma m_x, m_{xx} \rangle_{L^2(\mathbb{R})} - \langle m_x, u_x m_x \rangle_{L^2(\mathbb{R})}
- 2\langle m_x, u_{xx} m \rangle_{L^2(\mathbb{R})} - 2\langle m_x, u_x m_x \rangle_{L^2(\mathbb{R})}
- \lambda \langle m_x, m_x \rangle_{L^2(\mathbb{R})} + \langle m_x, \partial_x^2 h(u) \rangle_{L^2(\mathbb{R})}
= -\frac{5}{2} \langle u_x, m^2 \rangle_{L^2(\mathbb{R})} - 2\langle u_{xx}, mm_x \rangle_{L^2(\mathbb{R})}
- \lambda \|m_x\|_{L^2(\mathbb{R})}^2 + \langle m_x, \partial_x^2 h(u) \rangle_{L^2(\mathbb{R})}
= -\frac{5}{2} \langle u_x, m^2 \rangle_{L^2(\mathbb{R})} + \langle u_x, m^2 \rangle_{L^2(\mathbb{R})}
- \lambda \|m_x\|_{L^2(\mathbb{R})}^2 + \langle m_x, \partial_x^2 h(u) \rangle_{L^2(\mathbb{R})},
\]
where we used \( \langle u_{xx}, mm_x \rangle_{L^2(\mathbb{R})} = \langle u, \partial_x m^2 \rangle_{L^2(\mathbb{R})}/2 = \langle m, mm_x \rangle_{L^2(\mathbb{R})} = -\langle u_x, m^2 \rangle_{L^2(\mathbb{R})}/2 \).

From (4.3), (4.4) and after some manipulation, we have
\[
\frac{d}{dt} \|m\|_{H^1(\mathbb{R})}^2 = -\langle u_x, m^2 \rangle_{L^2(\mathbb{R})} - 5\langle u_x, m^2 \rangle_{L^2(\mathbb{R})} - 2\lambda \|m\|_{H^1(\mathbb{R})}^2
+ 2 \left( \langle m, \partial_x^2 h(u) \rangle_{L^2(\mathbb{R})} + \langle m_x, \partial_x^2 h(u) \rangle_{L^2(\mathbb{R})} \right). \tag{4.5}
\]
Let \( I := \langle m, \partial_x^2 h(u) \rangle_{L^2(\mathbb{R})} + \langle m_x, \partial_x^2 h(u) \rangle_{L^2(\mathbb{R})} \). Then

\[
I = \langle m, \partial_x^2 h(u) \rangle_{L^2(\mathbb{R})} + \langle m_x, \partial_x^2 h(u) \rangle_{L^2(\mathbb{R})}
= \langle m, \partial_x^2 h(u) - \partial_x^2 h(u) \rangle_{L^2(\mathbb{R})} = \langle m, \Lambda^2 \partial_x h(u) \rangle_{L^2(\mathbb{R})} \leq \|m\|_{L^2(\mathbb{R})} \|\Lambda^2 \partial_x h(u)\|_{L^2(\mathbb{R})}.
\]

Since \( 2ab \leq a^2 + b^2 \), for any real numbers \( a \) and \( b \), we have

\[
I \leq \frac{\|\Lambda^2 \partial_x h(u)\|_{L^2(\mathbb{R})}^2 + \|m\|_{L^2(\mathbb{R})}^2}{2}.
\]

We still have the inequality \( \|\Lambda^2 \partial_x h(u)\|_{L^2(\mathbb{R})} \leq \|\partial_x h(u)\|_{H^2(\mathbb{R})} \leq \|h(u)\|_{H^3(\mathbb{R})} \).

**Claim.** There exists a constant \( c_1 \) depending only on \( \|u_0\|_{H^3(\mathbb{R})} \) such that \( \|h(u)\|_{H^3(\mathbb{R})} \leq c_1 \|u\|_{H^3(\mathbb{R})} \).

We note that if our claim is true, then \( \|h(u)\|_{H^3(\mathbb{R})} \leq c_1 \|u\|_{H^3(\mathbb{R})} = c_1 \|\Lambda^{-2} u\|_{H^1(\mathbb{R})} = c_1 \|m\|_{H^1(\mathbb{R})} \) and since \( \|m\|_{L^2(\mathbb{R})} \leq \|m\|_{H^1(\mathbb{R})} \) we conclude that

\[
\langle m, \partial_x^2 h(u) \rangle_{L^2(\mathbb{R})} + \langle m_x, \partial_x^2 h(u) \rangle_{L^2(\mathbb{R})} \leq c \|m\|_{H^1(\mathbb{R})}^2,
\]

for some positive constant \( c \). In addition, we have

\[
- \langle u_x, m^2 \rangle_{L^2(\mathbb{R})} - 5 \langle u_x, m_x^2 \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} (-u_x)(m^2 + 5m_x^2)dx \leq k \int_{\mathbb{R}} (m^2 + 5m_x^2)dx \leq 5k \|m\|_{H^1(\mathbb{R})}^2.
\]

Substitution of (4.6) and (4.7) into (4.5) reads

\[
\frac{d}{dt} \|m\|_{H^1(\mathbb{R})}^2 \leq (5k + c) \|m\|_{H^1(\mathbb{R})}^2.
\]

Using the Gronwall inequality, we conclude that \( \|m\|_{H^1(\mathbb{R})}^2 \leq e^{(5k + c + 2\lambda)t} \|m_0\|_{H^1(\mathbb{R})}^2 =: \sigma(t)^2 \|m_0\|_{H^1(\mathbb{R})}^2 \), which is sufficient to have the result proved.

To conclude the proof, we must now prove the claim. To do it, note that

\[
h(u) = \int_0^1 uh'(su)ds.
\]

From the last equality we easily conclude that \( |h(u)| \leq |u| \sup\{|h'(r)|, |r| \leq \|u\|_{L^\infty(\mathbb{R})}\} \). Moreover, we also know that \( \|u\|_{L^\infty(\mathbb{R})} \leq \|u\|_{H^1(\mathbb{R})} \) (see, for instance, [31], page 1396). This, combined with the fact that \( \|u\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})} \) implies that \( |h(u)| \leq |u| \sup\{|h'(r)|, |r| \leq \|u_0\|_{H^1(\mathbb{R})}\} \).

Let \( c_1 := \sup\{|h'(r)|, |r| \leq \|u_0\|_{H^1(\mathbb{R})}\} \). Then \( |h(u)| \leq c_1 |u| \), which yields the desired result.

Observe that \( H^3(\mathbb{R}) \subseteq H^s(\mathbb{R}) \), for any \( s \leq 3 \). Moreover, this embedding is dense and continuous. This proves the following

**Corollary 4.1.** Let \( u_0 \in H^s \), \( s \geq 3/2 \), and \( u \) be the corresponding solution to (1.1) with initial data \( u(0, x) = u_0(x) \). Assume that \( u_x > -k \), for some positive constant \( k \). Then \( \|u\|_{H^s(\mathbb{R})} \leq \sigma(t)\|u_0\|_{H^s(\mathbb{R})} \), for a certain positive function \( \sigma \in C^1(\mathbb{R}) \).
4.2 Wave breaking criteria

We recall that wave breaking occurs when

\[
\sup_{(t,x) \in [0,T) \times \mathbb{R}} |u(t,x)| < \infty, \quad \text{and} \quad \limsup_{t \to T} \left( \sup_{x \in \mathbb{R}} |u_x(t,x)| \right) = \infty.
\]

Corollary 3.1 implies that if \( \lambda > 0 \), then \( u \) is bounded, which is a necessary condition for the appearance of wave breaking. On the other hand, Theorem 4.1 assures that if \( u \) is a solution of (1.2) and \( u_x \) is bounded from below, then we do not have wave breaking. The consequence of these facts is: If we want to investigate wave breaking, then we must look for solutions having \( u_x \) with no lower bound. We observe that if \( u_x(t,x) < 0 \), then we should replace \( \sup \) and \( \infty \) by \( \inf \) and \(-\infty\), respectively, in the conditions above whenever we replace \( |u_x(t,x)| \) by \( u_x(t,x) \). To understand what we are bound to do, let \( y(t) := \inf \{u_x(t,x), x \in \mathbb{R}\} \), which we simply write as \( y(t) := \inf_x u_x(t,x) \) from now on. Then, the wave breaking will occur if

\[
-\infty = \liminf_{t \to T} \left( \inf_{x \in \mathbb{R}} u_x(t,x) \right) = \liminf_{t \to T} y(t) = \liminf_{t \to T} y(s).
\]

**Lemma 4.1.** Let \( T > 0 \) and \( v \in C^1([0,T), H^2(\mathbb{R})) \) be a given function. Then, for any \( t \in [0,T) \), there exists at least one point \( \xi(t) \in \mathbb{R} \) such that

\[
y(t) = \inf_{x \in \mathbb{R}} v_x(t,x) = v_x(t,\xi(t)) \tag{4.8}
\]

and the function \( y \) is almost everywhere differentiable (a.e) in \((0,T)\), with \( y'(t) = v_{tx}(t,\xi(t)) \) a.e. on \((0,T)\).

**Proof.** See Theorem 2.1 in [7] or Theorem 5 in [19]. \( \square \)

Let us consider equation (2.2), with \( h \) given by (2.3). Differentiating (2.2) with respect to \( x \), we obtain

\[
u_{xx} + \frac{u_x^2}{2} + (u + \Gamma)u_{xx} + \lambda u_x = u^2 - \Lambda^{-2} \left( u^2 + \frac{u_x^2}{2} \right) - h(u) - \Lambda^{-2}h(u), \tag{4.9}
\]

were we used the identity \( \partial_x^2 \Lambda^{-2} = \Lambda^{-2} - 1 \).

We observe that (4.9) holds to any \((t,x)\) where \( u \) is defined. If \( u_0 \in H^3(\mathbb{R}) \), then \( u(t, \cdot) \in C^1(\mathbb{R}) \) and satisfies the conditions required in Lemma 4.1. Defining

\[
y(t) = \inf_{x \in \mathbb{R}} u_x(t,x) = u_x(t,\xi(t)),
\]

evaluating equation (4.9) at \((t,\xi(t))\) and noticing that \( u_{xx}(t,\xi(t)) = 0 \), we arrive at the following ordinary differential equation to \( y \):

\[
y'(t) + \frac{y(t)^2}{2} + \lambda y(t) = u(t)^2 - F(u(t)) - G(u(t)), \tag{4.10}
\]

where \( u(t) := u(t,\xi(t)) \),

\[
F(u) := \Lambda^{-2} \left( u^2 + \frac{u_x^2}{2} \right), \tag{4.11}
\]

and

\[
G(u) := h(u) + \Lambda^{-2}h(u). \tag{4.12}
\]
Proposition 4.1. Let \( F \) and \( G \) be given by (4.11) and (4.12), respectively. If \( u \in H^1(\mathbb{R}) \), then \( F(u(t)) \geq u^2(t)/2 \) and \( |G(u(t))| \leq 3\|h(u)\|_{H^1(\mathbb{R})} \).

Proof. The proof that \( F(u(t)) \geq u^2(t)/2 \) can be found in [19], pages 106 and 107 and, therefore, is omitted. Let us estimate \( |G(u(t))| \). We first note that \( \|h(u)\|_{L^\infty(\mathbb{R})} \leq \|h(u)\|_{H^1(\mathbb{R})} \), \( h(0) = 0 \) and by Lemma 2.3 if \( u \in H^1(\mathbb{R}) \), then \( h(u) \in H^1(\mathbb{R}) \). Therefore,

\[
|\Lambda^{-2}h(u(t))| = \left| \int_{-\infty}^{\infty} e^{-|\xi(t)|-y}h(u(t,y))dy \right| \leq \int_{-\infty}^{\infty} e^{-|\xi(t)|-y}|h(u(t,y))|dy \leq 2\|h(u)\|_{H^1(\mathbb{R})}.
\]

It is then easy to find the upper bound \( |G(u(t))| \leq 3\|h(u(t))\|_{H^1(\mathbb{R})} \). \( \square \)

Let

\[
U_0 := \max\{\|u_0\|_{H^1(\mathbb{R})}, \|u_0\|^2_{H^1(\mathbb{R})} \}.
\]

Then \( \|h(u)\|_{H^1(\mathbb{R})} \leq \kappa U_0 \), for a certain positive constant \( \kappa \) depending on \( \alpha, \beta, \gamma \) and \( \Gamma \). Moreover,

\[
2u^2(t) = 2 \left( \int_{-\infty}^{\xi(t)} u(t,y)u_x(t,y)dy - \int_{\xi(t)}^{\infty} u(t,y)u_x(t,y)dy \right)
\]

\[
\leq \int_{-\infty}^{\xi(t)} (u^2(t,y) + u_x^2(t,y))dy + \int_{\xi(t)}^{\infty} (u^2(t,y) + u_x^2(t,y))dy = \|u(t)\|^2_{H^1(\mathbb{R})}.
\]

From the comments above, Proposition 5.1 and equation (4.10) we have the following inequality

\[
y'(t) + \frac{y(t)^2}{2} + \lambda y(t) \leq \frac{u(t)^2}{2} + |G(u(t))| \leq \frac{1}{4}\|u_0\|^2_{H^1(\mathbb{R})} + 3\kappa U_0. \tag{4.13}
\]

Suppose that \( u_0 \in H^3(\mathbb{R}) \) be an initial data of (1.2) such that

\[
\theta u'_0(x_0) < \min\{-\|u_0\|^1_{H^1(\mathbb{R})}, -\|u_0\|^2_{H^1(\mathbb{R})} \}, \tag{4.14}
\]

for some constant \( \theta > 0 \) and some point \( x_0 \in \mathbb{R} \). Since \( y(t) = \inf_x u_x(t,x) \), then \( y(0) \leq u_x(0,x_0) = u'_0(x_0) \) and

\[
\theta y(0) \leq \theta u'_0(x_0) < \min\{-\|u_0\|^1_{H^1(\mathbb{R})}, -\|u_0\|^2_{H^1(\mathbb{R})} \},
\]

which implies

\[
U_0 < \theta^2 u'_0(x_0)^2 \leq \theta^2 y(0)^2.
\]

Suppose that we might be able to find \( \epsilon \) such that

\[
U_0 \leq (1 - \epsilon)\theta^2 u'_0(x_0)^2 \leq (1 - \epsilon)\theta^2 y(0)^2,
\]

and noting that \( \|u_0\|^2_{H^1(\mathbb{R})} \leq U_0 \), we obtain the upper bound

\[
\frac{1}{4}\|u_0\|^2_{H^1(\mathbb{R})} + 3\kappa U_0 \leq \left( \frac{1}{4} + 3\kappa \right) (1 - \epsilon)\theta^2 y(0)^2.
\]

If we could choose \( \theta \leq \sqrt{2/(1 + 12\kappa)} \), then the inequality (4.13) would read

\[
y'(t) + \frac{y(t)^2}{2} + \lambda y(t) \leq \frac{1 - \epsilon}{2} y(0)^2. \tag{4.15}
\]

We are bound to find conditions for the occurrence of wave breaking. We only need two very technical observations that will make easier the proof of Theorem 1.4.
Remark 4.1. Let \( u_0 \in H^3(\mathbb{R}) \) be an initial data to the problem (1.2), \( y(0) := \inf_x u_x(0, x) \). If \( u_0 \) satisfy the condition (4.14) for some \( \theta > 0 \), let us define
\[
\epsilon_0 := \frac{\theta^2 u_0'(x_0)^2 - \max\{\|u_0\|_{H^1(\mathbb{R})}, \|u_0\|_{H^1(\mathbb{R})}^4\}}{\theta^2 u_0'(x_0)^2}.
\] (4.16)
Clearly \( \epsilon_0 \in (0, 1) \) and if \( \epsilon \in (0, \epsilon_0) \), then
\[
\max\{\|u_0\|_{H^1(\mathbb{R})}, \|u_0\|_{H^1(\mathbb{R})}^4\} \leq (1 - \epsilon) \theta^2 u_0'(x_0)^2.
\]

Remark 4.2. Under the same conditions in Remark 4.1, let
\[
\lambda_0 := -\frac{y(0)}{4} \epsilon_0,
\] (4.17)
where \( \epsilon_0 \) is given by (4.16). If \( \lambda \in (0, \lambda_0) \), then
\[
\frac{\epsilon_0}{4\lambda} + \frac{1}{y(0)} > 0.
\] (4.18)

Proof of Theorem 1.4. In view of Corollary 4.1 we only need to show that \( u_x \) does not have any lower bound.

Under the conditions of the theorem, we note that \( y(t) = \inf_x u_x(t, x) \) satisfies the inequality (4.15).

It follows from [19], page 108, that \( y(t)^2 > (1 - \epsilon/2)y(0)^2 \). This implies that (4.15) can be rewritten as \( y'(t) + \lambda y(t) < -\epsilon y(0)^2/4 \). Moreover, it is also immediate that \( y(0)^2 < 2y(t)^2/(2-\epsilon) < 2y(t)^2 \).

Taking all of these inequalities into account, substituting them into (4.15) and taking \( \epsilon = \epsilon_0/2 \), where \( \epsilon_0 \) is given in Remark 4.1, we obtain
\[
y'(t) + \lambda y(t) < -\frac{\epsilon_0}{4} y(t)^2.
\]
We observe that the last inequality shows that \( y(\cdot) \) is a decreasing function and since \( y(0) < 0 \), then \( y(t) < 0 \) for any \( t > 0 \). Moreover,
\[
\frac{d}{dt} \left( \frac{1}{e^{\lambda t} y(t)} \right) = -\frac{e^{\lambda t} y' + \lambda y}{y^2} \geq \frac{\epsilon_0}{4} e^{-\lambda t}
\]
and a direct integration followed by simple manipulation yield
\[
e^{\lambda t} \left( \frac{\epsilon_0}{4\lambda} + \frac{1}{y(0)} \right) \leq \frac{\epsilon_0}{4\lambda} + \frac{1}{y(t)} \leq \frac{\epsilon_0}{4\lambda}.
\]
Due to \( \lambda < \lambda_0 \), we conclude that
\[
0 < e^{\lambda t} \left( \frac{\epsilon_0}{4\lambda} + \frac{1}{y(0)} \right) \leq \frac{\epsilon_0}{4\lambda},
\]
which forces \( t \) to be finite. Therefore, the solution \( u \) cannot be defined for all values of \( t \), and we then conclude the existence of a finite lifespan \( T > 0 \). Since \( m_0 \in H^1(\mathbb{R}) \), Theorem 4.1 implies that \( u_x \) has no lower bound, that is, \( \liminf_{t \to T} \left( \inf_{x \in \mathbb{R}} u_x(t, x) \right) = -\infty \). □
5 Comments on the limitations to ensure global existence of solutions

In this section we show the limitations to completely describe the scenario for global existence.
We begin with a technical result proving the existence of a diffeomorphism.

**Proposition 5.1.** Let \( u \in C^1([0, T), H^2(\mathbb{R})) \) be a solution of (1.1). Then the problem

\[
\begin{aligned}
q_t(t, x) &= u(t, q) + \Gamma, \\
q(0, x) &= x
\end{aligned}
\]

has a unique solution \( q(t, x) \) and \( q_x(t, x) > 0 \), for any \( (t, x) \in [0, T) \times \mathbb{R} \). Moreover, \( q(t, \cdot) \) is an increasing diffeomorphism of the line.

**Proof.** We prove that the system has a unique solution \( q(t, x) \) and that \( q_x(t, x) > 0 \), for any \( (t, x) \in [0, T) \times \mathbb{R} \). The proof that this function is a diffeomorphism is the same as for the \( \text{CH} \) equation and it can be found in [9], Theorem 3.1.

Since \( u \in C^1([0, T), H^2(\mathbb{R})) \) and \( H^2(\mathbb{R}) \subset C^1(\mathbb{R}) \), then \( u \in C^1([0, T) \times \mathbb{R}, \mathbb{R}) \), which means that problem (5.1) has a unique solution. If we differentiate (5.1) with respect to \( x \), and noticing that \( \partial_x u(t, q) = u_x q_x \), we have another IVP, given by

\[
\begin{aligned}
\partial_t q_x(t, x) &= u_x(t, q)q_x(t, y), \\
q_x(0, x) &= 1
\end{aligned}
\]

The solution of (5.2) is

\[
q_x(t, x) = \exp \left( \int_0^t u_x(s, q(s, x)) ds \right),
\]

which completes the proof. \( \square \)

We note that Theorem 4.1 describes a sufficiency condition for the global existence of solutions, but it does not give us any information about whether this condition is satisfied.

**Theorem 5.1.** Let \( u_0 \in H^s(\mathbb{R}), s > 3/2, \) and \( m_0(x) = u_0(x) - u_0''(x) \). Assume that:

1. there exists a point \( x_0 \in \mathbb{R} \) such that \( m_0(x) \leq 0 \), if \( x \in (-\infty, x_0] \), and \( m_0(x) \geq 0 \), if \( x \in [x_0, \infty) \);
2. \( \text{sgn}(m_0) = \text{sgn}(m) \).

Then the solution \( u \) of (1.1) possesses bounded from below \( x \)-derivative.

**Proof.** Since \( u = p * m \), where \( * \) denotes the convolution and \( p = e^{-|x|}/2 \), we have

\[
u(t, x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-\xi|} m(t, \xi) d\xi = \frac{1}{2} e^{-x} \int_{-\infty}^{x} e^{\xi} m(t, \xi) d\xi + \frac{1}{2} e^{x} \int_{x}^{\infty} e^{-\xi} m(t, \xi) d\xi.
\]

Differentiating this representation of \( u \) with respect to \( x \) gives

\[
u_x(t, x) = -\frac{1}{2} e^{-x} \int_{-\infty}^{x} e^{\xi} m(t, \xi) d\xi + \frac{1}{2} e^{x} \int_{x}^{\infty} e^{-\xi} m(t, \xi) d\xi.
\]
Since \( \text{sgn} \, m = \text{sgn} \, m_0 \), then \( m(t, x) \leq 0 \) if \( x \leq q(t, x_0) \) and \( m(t, x) \geq 0 \) if \( x \geq q(t, x_0) \), where \( q \) is the function in Proposition 5.1. Therefore,

\[
  u_x(t, x) = -\frac{1}{2} e^{-t} \int_{-\infty}^{x} e^\xi m(t, \xi) d\xi - \frac{1}{2} e^{-t} \int_{x}^{\infty} e^{-\xi} m(t, \xi) d\xi + \frac{1}{2} e^{x} \int_{-\infty}^{\infty} e^\xi m(t, \xi) d\xi + \frac{1}{2} e^{-x} \int_{x}^{\infty} e^{-\xi} m(t, \xi) d\xi.
\]

As a consequence, if \( x \geq q(t, x_0) \), then \( u_x(t, x) \geq -u(t, x) \). On the other hand, a similar calculation reads

\[
  u_x(t, x) = u(t, x) - e^{-x} \int_{-\infty}^{x} e^\xi m(t, \xi) d\xi,
\]

which implies that \( u_x(t, x) \geq u \), provided that \( x \leq q(t, x_0) \).

These two facts are enough to assure that \( u_x(t, x) \geq -\|u\|_{L^\infty(\mathbb{R})} \). Since \( \|u\|_{L^\infty(\mathbb{R})} \leq \|u\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})} \) we conclude that \( -\|u_0\|_{H^1(\mathbb{R})} \leq u_x(t, x) \).

We observe that if the conditions of Theorem 5.1 are satisfied and if \( m_0 \in H^1(\mathbb{R}) \), then \( u \) does not blow up in finite time in view of Theorem 4.1.

**Theorem 5.2.** Let \( u \) be a solution of (1.1) with initial data \( u(0, x) = u_0(x), m_0(x) := u_0 - u_0'' \). Then

\[
  m(t, q(t, x)) q_x^2(t, x) = m_0(x) e^{-\lambda t} + \int_0^t e^{\lambda(s-t)} q_x^2(s, x) \partial_x h(u(s, q)) ds,
\]

where \( q \) is the solution of the problem (5.1) and \( h(u) \) is the function given in (2.3).

**Proof.** Differentiating \( m(t, q(t, x)) q_x^2(t, x) \) with respect to \( t \) and using (4.2), we conclude that

\[
  \frac{d}{dt} (mq_x^2) = [m_t + (u + \Gamma)m_x + 2u x m] q_x^2 = -\lambda mq_x^2 + \partial_x h(u)q_x^2,
\]

which is a linear ODE to \( mq_x^2 \). Integrating (5.4) and taking (5.1) and (5.2) into account we conclude that its solution is (5.3). \( \square \)

**Remark 5.1.** In Theorem 5.2 we observe how the presence of the function \( h(u) \) affects the investigation of global existence of solutions to the Cauchy problem (1.2). If \( h(u) = 0 \), from (5.3) we would then immediately conclude that \( \text{sgn} \, (m) = \text{sgn} \, (m_0) \) and the second condition in Theorem 5.1 would be a consequence of the first.

**Remark 5.2.** From Theorem 5.1 we see that the condition \( \text{sgn} \, (m_0) = \text{sgn} \, (m) \) is essential, at least in the venue we followed, to prove the global existence of solutions. However, we are unable to describe completely whether such condition is satisfied, which is an open problem in the study global solvability of the equation (1.2).

A possible direction to have a complete description of the existence of global solutions is the following: Let \( h(u) \) be the function given by (2.3). From (4.5) we have

\[
  m(t, q(t, x)) q_x^2(t, x) = e^{\lambda t} \left( m_0(x) + \int_0^t e^{\lambda s} q_x^2(s, x) \partial_x h(u(s, q)) ds \right).
\]
If we could determine a function $\xi = \xi(x) \geq 0$ such that
\[
\left| \int_0^t e^{\lambda(s)} q_x^2(s, x) \partial_q h(u(s, q)) \, ds \right| \leq \xi(x)
\]
and if $|m_0(x)| \geq \xi(x)$, it would then follow from (5.5) that $\text{sgn}(m) = \text{sgn}(m_0)$. Unfortunately we have not succeed to find such a function for arbitrary values of the parameters $\alpha, \beta, \gamma,$ and $\Gamma$ in (2.3).

In line with the remarks above, we have the following corollary.

**Corollary 5.1.** If $h(u) \equiv 0$, $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, and the first condition in Theorem 5.1 holds, then the solutions of the Cauchy problem (1.2) exists globally.

**Proof.** It follows from (5.5) that $\text{sgn}(m) = \text{sgn}(m_0)$ and the conclusion is a consequence of theorems 4.1 and 5.1.

Corollary 5.1 recover global existence results proved in [34, 35, 41–43] regarding weakly dissipative Camassa-Holm and Dullin-Gottwald-Holm equations.

### 6 Discussion

Very recently [5, 22, 23, 36], an equation of the type (1.1) was deduced as a model for shallow water waves with Coriolis effect. The mentioned equation has its coefficients depending on physical parameters related to the rotation of the Earth. On the other hand, in [14] we considered (1.1) with $\lambda = 0$ and we investigated it from a complementary point of view (taking the results in [5, 22, 23, 36] into account). This equation revealed to be mathematically very rich, as one can see by the multitude of travelling waves [14, 23] it possesses, local well-posedness [5, 14] and wave breaking [15].

The main difference of our results and those established in [14, 15] is the presence of the term $\lambda m$ in (1.1) or $\lambda u$ in (2.4). From a different perspective, we can also argue that the main difference between the problem we treated here with those treated in [32, 34–35, 41–43] is the presence of the cubic and quartic nonlinearities in (1.1).

In some parts of the text we mentioned *dissipation*. Let us explain the term and why this is our case. Let $u_0$ be an initial data of (1.2) such that $0 \neq u_0 \in H^s(\mathbb{R})$, with $s > 3/2$, and $\lambda > 0$. From Theorem 1.2 we have the energy $\mathcal{H}_1(t) = e^{-2\lambda t} \mathcal{H}_1(0)$, and $\mathcal{H}_1(0) = \|u_0\|_{H^1(\mathbb{R})}^2 > 0$. Then we can easily infer
\[
\frac{d\mathcal{H}_1(t)}{dt} = -2\lambda \mathcal{H}_1(t) = -2\lambda e^{-2\lambda t} \mathcal{H}_1(0) < 0.
\]

From (6.1) we observe that $\mathcal{H}_1(t)$ decreases with time, which means that the energy is not conserved along time, or better, we have loss, or *dissipation*, of energy. Moreover, we observe that the energy of the solution is a monotonic decreasing function of $t$.

The presence of the function $h(u)$ in (2.4) or, more precisely, the cubic and quartic nonlinearities in (1.1), brings some complexity to the problem when compared with similar results of CH and DGH equations. For example, the condition we found for the existence of wave breaking are affected by the values of the parameters, as one can see by the range of $\theta$ in the condition (4.14) given by Theorem 1.4. Moreover, the value of $\epsilon_0$ also depends on powers of the norm $\|u_0\|_{H^1(\mathbb{R})}$, see (4.16),
as well as the upper bound to $\lambda$, as shown in (4.17). If $\|u_0\|_{H^1(\mathbb{R})} \leq 1$ then (4.16) and (4.17) reduce to

$$
\epsilon_0 = \frac{\theta^2 u_0'(x_0)^2 - \|u_0\|_{H^1(\mathbb{R})}}{\theta^2 u_0'(x_0)^2} \quad \text{and} \quad \lambda_0 = -\frac{y(0) \theta^2 u_0'(x_0)^2 - \|u_0\|_{H^1(\mathbb{R})}}{4 \theta^2 u_0'(x_0)^2},
$$

where $y(0) = \inf_x u_x(0, x)$ and the possible values for $\theta$ are given in Theorem 1.4. However, in case $\|u_0\|_{H^1(\mathbb{R})} > 1$ and $\gamma \neq 0$, then these constants changes to

$$
\epsilon_0 = \frac{\theta^2 u_0'(x_0)^2 - \|u_0\|_{H^1(\mathbb{R})}}{4 \theta^2 u_0'(x_0)^2} \quad \text{and} \quad \lambda_0 = -\frac{y(0) \theta^2 u_0'(x_0)^2 - \|u_0\|_{H^1(\mathbb{R})}}{\theta^2 u_0'(x_0)^2},
$$

evidencing how the higher order nonlinearities affects these parameters and the wave breaking as well.

The wave breaking phenomena of the solutions of (1.2) is assured by Theorem 1.4 provided that $\lambda \in (0, \lambda_0)$. We would like to point out the following comments about the parameter $\lambda$:

1. The presence of the term $\lambda m$ in (1.2), with $\lambda > 0$, is enough to guarantee the existence of time-decaying solutions to the equation with sufficient regularity, no matter the value of $\lambda$, as can infer from equation (6.1). In particular, larger values of $\lambda$ imply a faster decaying of energy than the small ones.

2. Although larger values of $\lambda$ result into fast decaying of the energy and, consequently, the fast vanishing of the solutions, they cannot guarantee the existence of wave breaking of the solutions. Actually, our results, namely Theorem 1.4, can only assure the appearance of wave breaking under restrictive conditions, which among then, small values of $\lambda$. By small values of $\lambda$ we mean those smaller than $\lambda_0$, see (4.17).

3. Theorem 1.4 does not give any information to us if wave breaking may or not occur if $\lambda \geq \lambda_0$. Actually, this is an open question.

4. In line with the previous comments, only very small values of $\lambda$ surely allow the wave breaking of solutions of (1.2). We observe that for small values of $\lambda$, the term $\lambda m$ in (1.2) can be interpreted as a perturbation in the equation in (1.7). This perturbation, no matter how small it is, is enough to bring the vanishing of the solutions with enough regularity of the equation.

5. The conditions for wave breaking phenomena for equation (1.7) were investigated in [15]. We note that if $\lambda$ is small (in the sense mentioned above), the conditions for wave breaking of (1.2) are essentially unaltered when compared to those for (1.7), see [15].

7 Conclusion

In the present work we investigated equation (1.1) and, with more emphasis, its corresponding Cauchy problem (1.2).

Our main results can be summarised as follows:

1. We prove the local well posedness to the Cauchy problem (1.2), see Theorem 1.1

2. We established conservation laws and conserved quantities for the equation and the problem, see Theorem 1.2. In particular, we proved that the solutions of (1.2) are bounded from above by the Sobolev norm of the initial data.
3. We also obtain sufficient condition for the appearance of wave breaking of the solutions of (1.2), see Theorem 1.4.

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