Ambiguities pertaining to quark-lepton complementarity

C. Jarlskog

Division of Mathematical Physics
LTH, Lund University
Box 118, S-22100 Lund, Sweden

Abstract

Recently the possible origin of the so-called quark-lepton complementarity relations has received a considerable amount of attention. We point out some of the inherent ambiguities in such analyses.

1 Introduction

Recently the so-called Quark-Lepton Complementarity (hereafter abbreviated by QLC) has attracted much attention. The current status of this topic is eloquently described in a recent review by Minakata [1] who gives a large number of references to relevant papers. Minakata, except in a few cases, does not convey what each listed paper has actually contributed, an impossible task indeed. The main message is, however, crystal clear: the QLC is considered to be remarkable and a large number of papers has been devoted to it.

First of all a brief account of what QLC is and why it has attracted so much attention is in order. For a more detailed discussion the reader is again referred to Ref. [1].

The QLC relates the angles of the quark mixing matrix to that of the lepton mixing matrix. The empirical relation

\[ \theta_{12}^{\text{quark}} + \theta_{12}^{\text{lepton}} = \frac{\pi}{4} \]  

valid to within a couple of degrees is considered to be the most remarkable of the two QLC relations. The second QLC relation reads

\[ \theta_{23}^{\text{quark}} + \theta_{23}^{\text{lepton}} = \frac{\pi}{4} \]  

This is not as interesting because \( \theta_{23}^{\text{quark}} \) is only about two degrees and therefore this equation would have been satisfied, within the errors, even if this angle had been say zero. Finally, a third analogous relation

\[ \theta_{13}^{\text{quark}} + \theta_{13}^{\text{lepton}} = \frac{\pi}{4} \]

is badly violated as the sum of the angles in the LHS is less than 10 degrees.

The reason for the popularity of QLC is that already since three decades theorists have been almost desperately looking for some clues that may help them understand the pattern of masses and mixings of fundamental fermions. All models proposed so far for this purpose have severe drawbacks, either by containing too many assumptions or free parameters. A hint from experiments would be very welcome indeed. The empirical values of the quark masses and the elements of the quark mixing matrix have been known for quite sometime. Yet we have failed to produce
a reasonable model which could explain them. The corresponding lepton sector has exhibited a remarkably different pattern, with large mixing angles. It would be no less than fantastic if neutrino oscillation measurements could provide the solution to the long-standing problem of masses and mixings in the quark sector. It is argued that QLC may actually serve such a purpose.

Evidently any measurement could have a message or more to convey. But how these messages are to be interpreted is not always trivial. In this article we emphasise a few of the uncertainties that could easily invalidate the QLC analyses. The point of this paper is by no means to discourage the reader from working on the eminently fundamental problem of masses and mixings of fundamental fermions but to bring to his/her attention some of the pitfalls so that this outstanding topic is not degraded to the level of "playing QLC games".

2 Ambiguities in QLC

A straight-forward issue of concern has to do with mixing angles and phases. These are by no means "invariant" quantities and therefore their definitions, signs and magnitudes are convention dependent. Not even the conventions adopted by the PDG [3] may be taken as sacred. As an example, the CP phase in one convention generally contains mixing angles of a different convention. The invariants of the quark mixing matrix are known to be the magnitudes of the elements of that matrix. Even the quantity $J$, introduced as a measure of CP violation, is a function of these magnitudes as it is twice the area of any of the six unitarity triangles [4]. Suppose for a moment that the parameterisation of the quark and lepton mixing matrices had been unique. Even then there would still be ambiguities when we add angles from these two different sectors. As an example one may read the message from experiments on $\theta_{12}^{\text{quark}} + \theta_{12}^{\text{lepton}}$ as being $12 + 33 = 45$ degrees but equally well as $-12 + 33 = 21$ degrees. Additional phases, such as Majorana phases of neutrinos, would give us other possible interpretations.

Another serious source of theoretical error has to do with the fact that with massive neutrinos lepton-hadron universality is generally lost in a substantial way. In other words, the leptonic sector is not a copy of the quark sector.

The simplest way to accommodate massive neutrinos is to keep the symmetry as well as the family structure of the Minimal Standard Model and to introduce a number of passive (right-handed) neutrinos ($\nu_R$'s). The neutrino mass matrix is then of the familiar form

$$\mathbf{M}_\nu = \begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix}$$

(4)

where $m_D$ and $M_R$ are the so-called Dirac and Majorana mass matrices and the $T$ stands for transposition. With three left-handed and $n$ right-handed neutrinos the matrix $m_D$ is 3-by-n and $M_R$ is n-by-n. Furthermore, the matrix $M_R$ is symmetric by construction but otherwise arbitrary. There is no justification for taking $M_R$ to be diagonal. However, one can start by diagonalising $M_R$ through

$$M_R = U_R D_R U_R^T$$

(5)

where $U_R$ is the required unitary matrix such that $D_R$ is diagonal. With three right-handed
neutrinos one has

\[ D_R = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix} \] (6)

where the \( M_j \), j=1-3 are real parameters. The mass matrix can now be written in the form

\[ \mathcal{M}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & U_R \end{pmatrix} \begin{pmatrix} 0 & m_D U_R^* \\ U_R^T m_D^T & D_R \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & U_R^T \end{pmatrix} \] (7)

and the matrices \( U_R \) and \( U_R^T \) can be absorbed into the definition of the right-handed neutrinos. For a detailed discussion of this point see, for example, Ref. [2]. However \( U_R \) leaves its mark by modifying the definition of the Dirac mass matrix

\[ m_D \rightarrow m_D' = m_D U_R^* \] (8)

\( m_D' \), being the effective Dirac mass matrix and as such it contains parameters stemming from the right-handed sector. Since the matrix \( U_R \) is unknown to us there is no justification in assuming that the resulting effective Dirac mass matrix would have any resemblance to the mass matrices of the quarks.

In the see-saw type models, the \( M_i \), in Eq.(6), are assumed to be huge as compared to the weak scale. In such scenarios the effective three-by-three neutrino mass matrix for the light neutrinos is given by

\[ m_\nu = -m_D'^T \frac{1}{D_R} m_D' \] (9)

To further elucidate this point we now consider a very simple case where \( M_3 \) is much larger than the other two mass scales \( M_1 \) and \( M_2 \) so that we can neglect \( 1/M_3 \). We denote the three columns of the effective Dirac mass matrix, which we take to be real, by \( \vec{a}, \vec{b} \) and \( \vec{c} \) respectively,

\[ m_D' \equiv \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} \] (10)

Using this notation the neutrino mass matrix, up to an overall normalisation factor, is given by

\[ m_\nu = \begin{pmatrix} \vec{a}^T \\ \vec{b}^T \\ \vec{c}^T \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} \] (11)

Here \( r = M_1/M_2 \), is a free parameter. The eigenvalues, \( \lambda \), of this matrix are solutions of the equation

\[ \lambda^3 - \lambda^2 \text{tr} m_\nu + \frac{\lambda}{2} \{(\text{tr} m_\nu)^2 - \text{tr}(m_\nu^2)\} - \text{det} m_\nu = 0 \] (12)

Since the determinant of \( m_\nu \) vanishes one of the eigenvalues is zero. The third column of the effective Dirac mass matrix thus decouples and we can write the mass matrix in the following very simple form

\[ m_\nu = |a><a| + r|b><b| \] (13)
where \((|a><a|)_{jk} \equiv a_j a_k,\) etc. Note also that one could absorb the parameter \(r\) into the definition of the vector \(b\), if one so wishes. The traces in Eq.\((12)\) are easily deduced from Eq.\((13)\) and the corresponding formula for the square of the mass matrix, i.e.,

\[
m^2_{\nu} = a^2 |a><a| + r \vec{a}.\vec{b} \left\{|a><b| + |b><a|\right\} + r^2 b^2 |b><b| \tag{14}
\]

where \(a^2 = \vec{a}.\vec{a}\) and \(b^2 = \vec{b}.\vec{b}\). We find

\[
tr m_{\nu} = a^2 + rb^2 \tag{15}
\]

and

\[
tr m^2_{\nu} = a^4 + 2r(\vec{a}.\vec{b})^2 + r^2 b^4 \tag{16}
\]

Thus the non-zero eigenvalues are given by

\[
\lambda_{\pm} = \frac{1}{2} \left\{ a^2 + rb^2 \pm \sqrt{(a^2 + rb^2)^2 - 4r|\vec{a} \times \vec{b}|^2} \right\} \tag{17}
\]

To find the matrix \(V\) that diagonalises \(m_{\nu}\) we must compute the corresponding eigenvectors.

As an example, let us consider the case where vectors \(a\) and \(b\) are orthogonal to each other. In this case, the two non-zero eigenvalues are given by

\[
\lambda_+ = a^2, \quad \lambda_- = rb^2 \tag{18}
\]

and the eigenvectors are unit vectors along \(\vec{a}, \vec{b}\) and \(\vec{a} \times \vec{b}\) respectively. Thus we have

\[
V = \begin{pmatrix}
\frac{a_1}{a} & \frac{b_1}{b} & \frac{a_2 b_3 - a_3 b_2}{ab} \\
\frac{a_1}{a} & \frac{b_2}{b} & \frac{a_3 b_1 - a_1 b_3}{ab} \\
\frac{a_2}{a} & \frac{b_3}{b} & \frac{a_1 b_2 - a_2 b_1}{ab}
\end{pmatrix} \tag{19}
\]

If we wish to reproduce the popular bimaximal mixing matrix \([1]\) and none of the components of vectors \(a\) and \(b\) are zero we can take one of the components of the cross product to be zero. Taking the first component to be zero gives

\[
\frac{a_3}{a_2} = \frac{b_1}{b_2} \equiv \alpha \tag{20}
\]

The matrix \(V\) is then given by

\[
V = \begin{pmatrix}
\cos \phi_1 & -\sin \phi_1 & 0 \\
\sin \phi_1 \cos \phi_2 & \cos \phi_1 \cos \phi_2 & -\sin \phi_2 \\
\sin \phi_1 \sin \phi_2 & \cos \phi_1 \sin \phi_2 & \cos \phi_2
\end{pmatrix} \tag{21}
\]

where

\[
\cos \phi_1 = \frac{a_1}{a}, \quad \sin \phi_1 = -\frac{b_1}{b} \tag{22}
\]

\[
\cos \phi_2 = \frac{1}{\sqrt{1+\alpha^2}}, \quad \sin \phi_2 = \frac{-\alpha}{\sqrt{1+\alpha^2}} \tag{23}
\]
The matrix in Eq. (21) can also be written in the form

\[
V = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos\phi_2 & -\sin\phi_2 \\
0 & \sin\phi_2 & \cos\phi_2
\end{pmatrix}
\begin{pmatrix}
\cos\phi_1 & -\sin\phi_1 & 0 \\
\sin\phi_1 & \cos\phi_1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  

(24)

For \(\phi_1 = \phi_2 = \pi/4\) this matrix is exactly the bimaximal matrix discussed by many authors (see [1]). By putting \(\phi_1 = \pi/6\) and \(\phi_2 = \pi/4\) we obtain the currently popular matrix

\[
V = \begin{pmatrix}
\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\
\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]  

(25)

as discussed, for example, by Petcov [5]. Furthermore, by tuning the parameter \(r\) we may change the mass-splitting of the two massive neutrinos and by adjusting \(a\) we may vary the scale of these masses. Thus by making assumptions we can get the required results. There is no sign that the Dirac mass matrices of the quarks play any role here.

3 Conclusions

In this article we have pointed out that in general adding mixing angles from the quark and lepton sectors does not make sense. Furthermore, not knowing the right-handed Majorana mass matrix \(M_R\) makes it impossible to relate the quark and lepton mixing matrices. Any assumption about \(M_R\) is a source of uncertainty which easily invalidates the subsequent conclusions.

A further uncertainty arises as the matrix \(V\), discussed above is still not the lepton mixing matrix. It has to be multiplied with the (to us unknown) unitary matrix that diagonalises the mass matrix of charged leptons.

Note that the hypothesis that leptogenesis is responsible for the observed baryon asymmetry of the universe brings \(M_R\), at least partially, into the realm of observability. However, current models contain many assumptions and free parameters (see, for example, a recent review by T. Hambye [6]). The upshot of these studies is that leptogenesis looks very promising, as a mechanism for generating the observed baryon asymmetry of the universe. But the specific models are far from being sufficiently pinned-down to be useful for connecting quark and lepton mixing angles as is done in QLC. Further progress in this domain would be highly welcome.

References

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