Stochastic processes on non-Archimedean spaces. I. Stochastic processes on Banach spaces.

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Abstract

Non-Archimedean analogs of Markov quasimeasures and stochastic processes are investigated. They are used for the development of stochastic antiderivations. The non-Archimedean analog of the Itô formula is proved.

1 Introduction.

Stochastic differential equations on real Banach spaces and manifolds are widely used for solutions of mathematical and physical problems and for construction and investigation of measures on them [1, 14, 17, 31, 32, 34]. In particular stochastic equations can be used for the constructions of quasi-invariant measures on topological groups. In the cases of real Banach-Lie groups, some simplest cases of diffeomorphisms groups and free loop spaces of real manifolds such stochastic equations and measures were investigated[1, 14, 17, 31, 32, 34].
Stochastic processes on geometric loop groups and diffeomorphism groups of wide classes of real and complex manifolds were investigated in [29]. On the other hand, non-Archimedean functional analysis develops fastly in recent years and also its applications in mathematical physics [3, 9, 39, 40, 38, 42, 19, 18]. Wide classes of quasi-invariant measures including analogous to Gaussian type on non-Archimedean Banach spaces, loops and diffeomorphisms groups were investigated in [20, 22, 24, 27, 28]. Quasi-invariant measures on topological groups and their configuration spaces can be used for the investigations of their unitary representations (see [25, 26, 27, 28] and references therein).

In view of this developments non-Archimedean analogs of stochastic equations and diffusion processes need to be investigated. Some steps in this direction were made in [3, 10]. There are different variants for such activity, for example, $p$-adic parameters analogous to time, but spaces of complex-valued functions. At the same time measures may be real, complex or with values in a non-Archimedean field.

In the classical stochastic analysis indefinite integrals are widely used, but in the non-Archimedean case the field of $p$-adic numbers $\mathbb{Q}_p$ has not linear order structure apart from $\mathbb{R}$. For elements $f$ in the space of $m-1$ times continuously differentiable functions $C^{m-1}$ there are antiderivation operators $P_m : C^{m-1} \rightarrow C^m$ such that $(P_m f)' = f$ [39, 40]. Therefore, in this paper these indefinite integrals are used, but they are transformed for more complicated needed cases. In the classical case for the investigations of stochastic processes nuclear, Hilbert-Schmidt and of the class $L_q$ operators are used [3, 39]. In the non-Archimedean case the operator theory differs from that of classical and the corresponding definitions and propositions was necessary to give anew in this article.

This work treats the case which was not considered by another authors and that is suitable and helpful for the investigation of stochastic processes and quasi-invariant measures on non-Archimedean topological groups. These investigations are not restricted by the rigid geometry class [13], since it is rather narrow. Wider classes of functions and manifolds are considered. This is possible with the use of Schikhof’s works on classes of functions $C^n$ in the sense of difference quotients, which he investigated few years later the published formalism of the rigid geometry. Here are considered spaces of functions with values in Banach spaces over non-Archimedean local fields, in particular, with values in the field $\mathbb{Q}_p$ of $p$-adic numbers. For this
non-Archimedean analogs of stochastic processes are considered on spaces of functions with values in the non-Archimedean infinite field with a non-trivial valuation such that a parameter analogous to the time is $p$-adic (see §§4.1, 4.2). Their existence is proved in Theorem 4.3. Specific antiderivation operators generalizing Schikhof antiderivation operators on spaces of functions $C^n$ are investigated (see §2). Their continuity and differentiability properties are given in Lemmas 2.2, 2.3 and Theorem 2.14. Also operators analogous to nuclear operators are studied (see Definition 2.10 and Propositions 2.11, 2.12). In §3 non-Archimedean analogs of Markov quasimeasures are defined and Propositions 3.3.1 and 3.3.2 about their boundedness and unboundedness are proved. The non-Archimedean stochastic integral is defined in §4.4. Its continuity as the operator on the corresponding spaces of functions is proved in Proposition 4.5. In Theorems 4.6, 4.8 and Corollary 4.7 analogs of the Itô formula are proved. Spaces of analytic functions lead to simpler expressions of the Itô formula analog, but the space of analytic functions is very narrow and though it is helpful in non-Archimedean mathematical physics it is insufficient for solutions of all mathematical and physical problems. For example, in many cases of topological groups for non-Archimedean manifolds spaces of analytic functions are insufficient. On the other hand, for spaces $C^n$ rather simple formulas are found. This work was started five years ago, but because of lack of free time it was finished only recently. All results of this paper are obtained for the first time.

2 Specific antiderivations of operators.

2.1. Let $X := c_0(\alpha, K_p)$ be a Banach space over a local field (see [43]) $K_p$ such that $K_p \supset Q_p$, $\{e_j : j \in \alpha\}$ denotes the standard orthonormal base in $c_0(\alpha, K_p)$ where $\alpha$ is an ordinal $\mathbb{N}$, $e_j = (0, ..., 0, 1, 0, ...)$ with the unit on the $j$-th place, $j \in \alpha$. The space $c_0(\alpha, K_p)$ consists of vectors $x = (x_j : x_j \in K_p, j \in \alpha)$ such that for each $\epsilon > 0$ a set $\{j : j \in \alpha; |x_j| > \epsilon\}$ is finite. The norm in it is the following: $\|x\| := \sup_j |x_j|$. It is convenient to supply the set $\alpha$ with the ordinal structure due to the Kuratowski-Zorn lemma. Let $F$ be a continuous function on $B_r \times C^0(B_r, X)^{\otimes k}$ with values in $C^0(B_r, X)$:

\[
(1) \ F \in C^0(B_r \times C^0(B_r, X)^{\otimes k}, C^0(B_r, X)),
\]

where $Z^{\otimes k} = Z \otimes ... \otimes Z$ is the product of $k$ copies of a normed space $Z$. 

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and $Z^\otimes k$ is supplied with the box (maximum norm) topology [33], $B_r := B(K_p,t_0,r)$ is a ball in $K_p$ containing $t_0$ and of radius $r$, Banach spaces $C^t(M,X)$ of mappings $f : M \rightarrow X$ from a $C^\infty$-manifold $M$ with clopen charts modelled on a Banach space $Y$ over $K_p$ into $X$ of class of smoothness $C^t$ with $0 \leq t < \infty$ are the same as in [23, 27, 30, 28] with the supremum-norm, when $M$ is closed and bounded in the corresponding Banach space.

Such mappings can be written in the following form:

$$\sum_{j\in\alpha} F_j^i(v,\xi_1,\ldots,\xi_k) e_j,$$

where $F_j^i \in C^0(B_r \times C^0(B_r, X)^\otimes k, K_p)$ for each $j \in \alpha$. In particular let

$$F(v;\xi_1,\ldots,\xi_k) = G(v;\xi_1,\ldots,\xi_l). (A_{l+1}(v)\xi_{l+1},\ldots,A_k(v)\xi_k),$$

where $L(X,Y)$ denotes a Banach space of continuous linear operators $A : X \rightarrow Y$ supplied with the operator norm $\|A\| := \sup_{0 \neq x \in X} \|Ax\|_Y / \|x\|_X$ and $L(X) := L(X,Y)$, $A_i(v)$ are continuous linear operators for each $v \in B_r$ such that $A_i \in C^0(B_r, L(X))$, $G(v;\xi_1,\ldots,\xi_l) \in L_{k-l}(X^\otimes (k-l); X)$ for each fixed $v \in B_r$ and $\xi_1,\ldots,\xi_l \in C^0(B_r, X)$, that is, $F$ is a $(k-l)$-linear operator by $\xi_{l+1},\ldots,\xi_k$, where $G = G(v;\xi_1,\ldots,\xi_l)$ is the short notation of $G(v;\xi_1(v),\ldots,\xi_l(v))$, $L_k(X_1,\ldots,X_k;Y)$ denotes the Banach (normed) space of $k$-linear continuous operators from $X_1 \otimes \ldots \otimes X_k$ into $Y$ for Banach (normed) spaces $X_1,\ldots,X_k,Y$ over $K$ and $L_k(X^\otimes k;Y) := L_k(X_1,\ldots,X_k;Y)$ for the particular case $X_1 = \ldots = X_k = X$. When $l = 0$ put $G = G(v)$. There exists the following antiderivation of operators given by equation (3):

$$\hat{\gamma}_{(\xi_{l+1},\ldots,\xi_k)} [G(s;\xi_1,\ldots,\xi_l) \circ (A_{l+1} \otimes \ldots \otimes A_k)](v) := \sum_{n=0}^{\infty} G(v_n;\xi_1,\ldots,\xi_l). (A_{l+1}(v_n)[\xi_{l+1}(v_{n+1})-\xi_{l+1}(v_n)],\ldots,A_k(v_n)[\xi_k(v_{n+1})-\xi_k(v_n)]),$$

where $v_n = \sigma_n(t)$, $\{\sigma_n : n = 0,1,2,\ldots\}$ is an approximation of the identity in $B_r$. By its definition the approximation of the identity satisfies the following conditions:

(i) $\sigma_0(t) = t_0$,

(ii) $\sigma_m \circ \sigma_n = \sigma_n \circ \sigma_m$ for each $m \geq n$ and there exists $0 < \rho < 1$ such that from

(iii) $|x - y| < \rho^n$ it follows $\sigma_n(x) = \sigma_n(y)$ and
(iv) $|\sigma_p(x) - x| < \rho^n$ (see §62 and §79 in [39]).

**2.2. Lemma.** (1) If $G \in C^0(B_r \times X^{\otimes l}, L_{k-l}(X^{\otimes(k-l)}, X))$, $\xi_i \in C^0(B_r, X)$ for each $i = 1, \ldots, k$ and $A_{l+i} \in C^0(B_r, L(X))$ for each $i = 1, \ldots, k - l$, then $\hat{P}(\xi_{l+1}, \ldots, \xi_k)[G(s; \xi_1, \ldots, \xi_l \circ (A_{l+1} \otimes \ldots \otimes A_k)](v) \in C^0(B_r \times C^0(B_r, X)^{\otimes l}, C^0(B_r, X))$ as the function by $v, \xi_1, \ldots, \xi_l$ for each fixed $\xi_{l+1}, \ldots, \xi_k$ and $\hat{P}$ is of class $C^\infty$ by $\xi_{l+1}, \ldots, \xi_k$.

(2). Moreover, if $G$ is of class of smoothness $C^m$ by arguments $\xi_1, \ldots, \xi_l$, then $\hat{P}(\xi_{l+1}, \ldots, \xi_k)G$ is also in class of smoothness $C^m$ by $\xi_{l+1}, \ldots, \xi_l$.

**Proof.** Since $B_r$ is compact, then $\xi_i$ are uniformly continuous together with $A_{l+i}(v)[\xi_{l+i}(v)]$. There is the following inequality

$$
|\hat{P}(\xi_{l+1}, \ldots, \eta_k)[G(v; \eta_1, \ldots, \eta_l \circ (A_{l+1} \otimes \ldots \otimes A_k)](x) -
\hat{P}(\xi_{l+1}, \ldots, \xi_l)[G(v; \xi_1, \ldots, \xi_l \circ (A_{l+1} \otimes \ldots \otimes A_k)](y)|
\leq \max(|\hat{P}(\xi_{l+1}, \ldots, \eta_k)[G(v; \eta_1, \ldots, \eta_l \circ (A_{l+1} \otimes \ldots \otimes A_k)](x) -
\hat{P}(\xi_{l+1}, \ldots, \xi_k)[G(v; \eta_1, \ldots, \eta_l \circ (A_{l+1} \otimes \ldots \otimes A_k)](x)|,
|\hat{P}(\xi_{l+1}, \ldots, \xi_k)[G(v; \eta_1, \ldots, \eta_l \circ (A_{l+1} \otimes \ldots \otimes A_k)](x) -
\hat{P}(\xi_{l+1}, \ldots, \xi_k)[G(v; \xi_1, \ldots, \xi_l \circ (A_{l+1} \otimes \ldots \otimes A_k)](x)|,
|\hat{P}(\xi_{l+1}, \ldots, \xi_k)[G(v; \xi_1, \ldots, \xi_l \circ (A_{l+1} \otimes \ldots \otimes A_k)](x) -
\hat{P}(\xi_{l+1}, \ldots, \xi_k)[G(v; \xi_1, \ldots, \xi_l \circ (A_{l+1} \otimes \ldots \otimes A_k)](y)|).
$$

In addition $\hat{P}$ is the linear operator by $\xi_{l+1}, \ldots, \xi_k$. From this and Conditions 2.1.(i – iv) the first statement follows. The last statement follows from the linearity of $\hat{P}$ by $G$ and applying the operator of difference quotients $\hat{P}^m$ by $\xi_1, \ldots, \xi_l$ (see [25, 28]).

**2.3. Lemma.** If $\xi_i \in C^1(B_r, X)$ for each $i = 1, \ldots, k$ and Conditions (1) of Lemma 2.2 are satisfied, then

$$
\hat{P}(\xi_{l+1}, \ldots, \xi_k)[G(s; \xi_1, \ldots, \xi_l \circ (A_{l+1} \otimes \ldots \otimes A_k)](x) \in C^1(B_r, X)
$$

as a function by the argument $x \in B_r$ and

$$
\partial/\partial x(\hat{P}(\xi_{l+1}, \ldots, \xi_k)[G(s; \xi_1, \ldots, \xi_l \circ (A_{l+1} \otimes \ldots \otimes A_k)](x) =
\sum_{q=l+1}^{k} \hat{P}(\xi_{l+1}, \ldots, \xi_{q-1}, \xi_{q+1}, \ldots, \xi_k)G(x; \xi_1, \ldots, \xi_l \circ (A_{l+1}(x) \xi_{l+1}(x), \ldots, A_{q-1}(x) \xi_{q-1}(x),$$

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such that

$$
\| \hat{P}_{(\xi_{l+1}, ..., \xi_k)}[G(s; \xi_1, ..., \xi_l) \circ (A_{l+1} \otimes ... \otimes A_k)](x) \|_{C^1(B_r, \mathbb{R})} \leq \\
\| G \|_{C^0(\mathbb{R}^+ \times \mathbb{R}^d, L_{k-1}(\mathbb{R}^d; \mathbb{R}))} \prod_{i=l+1}^k \| A_i \|_{C^0(B_r, \mathbb{R})} \| \xi_i \|_{C^1(B_r, \mathbb{R})}.
$$

**Proof.** Let $\gamma := \hat{P}_{(\xi_{l+1}, ..., \xi_k)}[G(z; \xi_1, ..., \xi_l) \circ (A_{l+1} \otimes ... \otimes A_k)](y) - \hat{P}_{(\xi_{l+1}, ..., \xi_k)}[G(z; \xi_1, ..., \xi_l) \circ (A_{l+1} \otimes ... \otimes A_k)](y) - (x-y) \sum_{q=l+1}^k \hat{P}_{(\xi_{l+1}, ..., \xi_{q-1}, \xi_{q+1}, ..., \xi_k)}[G(y; \xi_1, ..., \xi_l) A_{l+1}(y) \xi_{l+1}(y), ..., A_{q-1}(y) \xi_{q-1}(y), A_q(y) \xi_q(y), A_{q+1}(y) \xi_{q+1}(y), ..., A_k(y) \xi_k(y)]$

and $\rho^* \leq |x - y| < \rho^*$, where $s \in \mathbb{N}$. Therefore, $x_0 = y_0, ..., x_s = y_s, x_{s+1} \neq y_{s+1}$ and

$$
\gamma = \left[ \sum_{q=l+1}^k E(x_q)(v_{l+1}, ..., v_{q-1}, h_q, z_{q+1}, ..., z_k) + E(x_q)(h_{l+1}, h_{l+2}, z_{l+3}, ..., z_k) + E(x_q)(h_{l+1}, v_{l+2}, h_{l+3}, z_{l+4}, ..., z_k) + ... + E(x_q)(v_{l+1}, ..., v_{k-2}, h_{k-1}, h_k) + ...

+ E(x_q)(h_{l+1}, ..., h_k) + \sum_{j=s+1}^{\infty} \{ E(x_j)(\xi_{l+1}(x_{j+1}) - \xi_{l+1}(x_j), ..., (\xi_k(x_{j+1}) - \xi_k(x_j))

- E(y_j)(\xi_{l+1}(y_{j+1}) - \xi_{l+1}(y_j), ..., (\xi_k(y_{j+1}) - \xi_k(y_j))

- (x-y) \sum_{q=l+1}^k \hat{P}_{(\xi_{l+1}, ..., \xi_{q-1}, \xi_{q+1}, ..., \xi_k)}[E(y)(\xi_{l+1}(y), ..., \xi_{q-1}(y), \xi_q(y), \xi_{q+1}(y), ..., \xi_k(y)),

$$

where $v_j = \xi_j(x_{s+1}) - \xi_j(x_s), h_j = \xi_j(x_{s+1}) - \xi_j(y_{s+1}), z_j = \xi_j(y_{s+1}) - \xi_j(y_s)$ for $j = l + 1, ..., k$ and

(i) $E := E(x) := G(x; \xi_1, ..., \xi_l). (A_{l+1}(x) \otimes ... \otimes A_k(x))$ and

(ii) $E(x)(\xi_{l+1}, ..., \xi_k) := G(x; \xi_1, ..., \xi_l). (A_{l+1}(x) \xi_{l+1}, ..., A_k(x) \xi_k)$

in accordance with Formula 2.1.(3). On the other hand, $||\xi_i(y_{j+1}) - \xi_i(y_{j}) - (y_{j+1} - y_j)\xi_i(y)|| = ||(y_{j+1} - y_j)[(\Phi^1 \xi_i)(y_j; 1; y_{j+1} - y_j) - \xi_i(y_j)]|| \leq |y_{j+1} - y_j||\xi_i||_{C^1(B_r, \mathbb{R})} \text{ and } E(x). (a_{l+1} + b_{l+1}, ..., a_k + b_k) - E(y). (a_{l+1}, ..., a_k) = E(x). (a_{l+1} + b_{l+1}, ..., a_k + b_k) - E(x). (a_{l+1}, ..., a_k) + [E(x) - E(y)]. (a_{l+1}, ..., a_k) = E(x). (b_{l+1}, a_{l+2}, ..., a_k) + ... + E(x). (a_{l+1}, ..., a_k - 1, b_k) + E(x). (b_{l+1}, b_{l+2}, a_{l+3}, ..., a_k) + \ldots$
... 

\[ E(x). (a_{l+1}, ..., a_{k-2}, b_{k-1}, b_k) + \ldots + E(x). (b_{l+1}, ..., b_k) + [E(x) - E(y)]. (a_{l+1}, ..., a_k) \]

for each \( a_{l+1}, ..., a_k, b_{l+1}, ..., b_k \in C^0(B_r, X) \), hence

\[ \| [\sum_{q=l+1}^k E(x_q) (v_{l+1}, ..., v_{q-1}, h_q, z_{q+1}, ..., z_k)] - (x-y) \sum_{q=l+1}^k \hat{P}(\xi_{l+1}, \ldots, \xi_{q-1}, \ldots, \xi_k) \]

\[ E(y)(\xi_{l+1}(y), \ldots, \xi_{q-1}(y), \xi_q(y), \xi_{q+1}(y), \ldots, \xi_k(y)) \| \]

\[ \leq \| E \| C^0 \rho^s \prod_{q=l+1}^k \| \xi_q \| C^1 \alpha(s) \] and

\[ \| E(x_j)(\xi_{l+1}(x_{j+1}) - \xi_{l+1}(x_j)), \ldots, (\xi_k(x_{j+1}) - \xi_k(x_j)) \]

\[ - E(y_j)(\xi_{l+1}(y_{j+1}) - \xi_{l+1}(y_j)), \ldots, (\xi_k(y_{j+1}) - \xi_k(y_j)) \| \]

\[ \leq \| E \| C^0 \rho^s \prod_{q=l+1}^k \| \xi_q \| C^1 \alpha(s) \] for each \( j \geq s+1 \), where \( \lim_{s \to \infty} \alpha(s) = 0 \), consequently, \( \lim_{x-y \to 0} \gamma = 0 \) and \( \Phi^1(\hat{P}(\xi_{l+1}, \ldots, \xi_k) E)(x) \in C^0(B_r, X) \), where

\[ \Phi^1 \eta(x; h; \xi) = \{ \eta(x + \xi h) - \eta(x) \} / \xi \] for \( 0 \neq \xi \in K, h \in H, \eta \in C^1(U, Y) \),

\( U \) is open in \( X, X \) and \( Y \) are Banach spaces over \( K, \Phi^1 \eta \) is a continuous extension of \( \Phi^1 \eta \) on \( U \times V \times B(K, 0, 1) \) for a neighbourhood \( V \) of \( 0 \) in \( X \) (see §2.3 [22] or §I.2.3 [22]). Then

\[ (iii) \ (\hat{P}(\xi_{l+1}, \ldots, \xi_k) E)(x) = \sum_{n=0}^{\infty} (x_{n+1} - x_n)^{k-1} G(x_n; \xi_1, \ldots, \xi_l) (A_{l+1}(x_n)(\Phi^1 \xi_{l+1}))(x_n; \]

\[ 1; x_{n+1} - x_n), \ldots, (A_k(x_n)(\Phi^1 \xi_k)(x_n; 1; x_{n+1} - x_n)). \]

Let \( \eta := (\hat{P}_E \xi_E)(x) - (\hat{P}_E \xi_E)(y) \), then \( \eta = E(x)(w(x_{s+1}) - w(y_{s+1})) + \sum_{n=s+1}^{\infty} \{ E(x_n)(w(x_{n+1}) - w(x_n)) - E(y_n)(w(y_{n+1}) - w(y_n)) \} \), consequently,

\[ \| \eta \| \leq \| E \| C^0(B_r \times X^{s+1}, L_{\text{comp}}(X^{(k-1)}; X)) (\prod_{j=l+1}^k \| \xi_j \| C^1 |x - y|), \]

since \( E \) is polylinear mappings by \( \xi_{l+1}(z), \ldots, \xi_k(z) \in X, |x_{s+1} - y_{s+1}| \leq |x - y| \) and \( |x_{n+1} - x_n| \leq |x - y| \) and \( |y_{n+1} - y_n| \leq |x - y| \) for each \( n > s \), where \( \rho^s + 1 \leq |x - y| < \rho^s \),

\( w = (\xi_{l+1}, \ldots, \xi_k) \).

Note. In particular, when \( X = K, l = 0, k = 1, A_1 = 1 \) and \( \xi(x) = x \) this gives the usual formula \( d[\hat{P}_E G(x)](x)/dx = G(x) \).

2.4. Suppose that \( X \) and \( Y \) are Banach spaces over a (complete relative to its uniformity) local field \( K \). Let \( X \) and \( Y \) be isomorphic with the Banach spaces \( c_0(\alpha, K) \) and \( c_0(\beta, K) \) and there are given the standard orthonormal bases \( \{ e_j : j \in \alpha \} \in X \) and \( \{ q_j : j \in \beta \} \in Y \) respectively, then each \( E \in L(X, Y) \) has its matrix realisation \( E_{j,k} := q_k^* E e_j \), where \( \alpha \) and \( \beta \) are ordinals, \( q_k^* \in Y^* \) is a continuous \( K \)-linear functional \( q_k^* : Y \to K \) corresponding to \( q_k \) under the natural embedding \( Y \hookrightarrow Y^* \) associated with the chosen basis, \( Y^* \) is a topologically conjugated or dual space of \( K \)-linear functionals on \( Y \).

2.5. Let \( A \) be a commutative Banach algebra and \( A^+ \) denotes the Gelfand space of \( A \), that is, \( A^+ = Sp(A) \), where \( Sp(A) \) in another words spectrum
of $A$ was defined in Chapter 6 [38]. Let $C_\infty(A^+, K)$ be the same space as in [38]. This means the following. For a locally compact Hausdorff totally disconnected topological space $E$ the vector space $C_\infty(E, K)$ is a subspace of a space $C(E, K)$ of bounded continuous functions $f : E \to K$ such that for each $\epsilon > 0$ there exists a compact subset $V \subset E$ for which $|f(x)| < \epsilon$ for each $x \in E \setminus V$. When $E$ is not locally compact and have an embedding into $B(K, 0, 1)^\gamma$ (for example, when $K$ is not locally compact) such that $E \cup \{x_0\} = \text{cl}(E)$ we put $C_\infty(E, K) := \{f \in C(E, K) : \lim_{x \to x_0} f(x) = 0\}$, where $B(X, x, r) := \{y \in X : d(x, y) \leq r\}$ is a ball in the metric space $(X, d)$, the closure cl$(E)$ is taken in $B(K, 0, 1)^\gamma$, $\gamma$ is an ordinal, $x_0 \in B(K, 0, 1)^\gamma$.

**Definition** (see also Ch. 6 in [38]). A commutative Banach algebra $A$ is called a $C$-algebra if it is isomorphic with $C_\infty(X, K)$ for a locally compact Hausdorff totally disconnected topological space $X$, where $f + g$ and $fg$ are defined pointwise for each $f, g \in C_\infty(X, K)$).

**2.6.** Let $H = c_0(\alpha, K)$ and $X$ be a topological space with the small inductive dimension $\text{ind}(X) = 0$, where $K$ is a complete feld as the uniform space. A strong operator topology in $L(H, Y)$ (see §2.1) is given by a base $V_{\epsilon;E;x_1,\ldots,x_n} := \{Z \in L(H, Y) : \sup_{1 \leq j \leq n} \|(E - Z)x_j\| < \epsilon\}$, where $0 < \epsilon$, $E \in L(H, Y)$, $x_j \in H$; $j = 1, \ldots, n$; $n \in \mathbb{N}$. An $H$-projection-valued measure on an algebra $L$ of subsets of $X$ is a function $P$ on $L$ assigning to each $A \in L$ a projection $P(A)$ on $H$ and satisfying the following conditions:

(i) $P(X) = 1_H$,

(ii) for each sequence $\{A_n : n \in \mathbb{N}\}$ of pairwise disjoint sets in $L$ there are pairwise orthogonal projections $P(A_n)$ and $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$, where $L \supset Bco(X)$, $Bco(X)$ is an algebra of clopen (closed and open at the same time) subsets of $X$, the convergence on the right hand side is unconditional in the strong operator topology and the sum is equal to the projection onto the closed linear span of $\bigcup_n \{\text{range}(P(A_n)) : n \in \mathbb{N}\}$ such that $P(\emptyset) = 0$.

If $\eta \in H^*$ and $\xi \in H$, then $A \mapsto \eta(P(A)\xi)$ is a $K$-valued measure on $L$. Then by the definition $P(A) \leq P(B)$ if and only if $\text{range}(P(A)) \subset \text{range}(P(B))$.

There are many projection operators on $H$, but for $P$ there is chosen some such fixed system.

A subset $A \subset X$ is called $P$-null if there exists $B \subset L$ such that $A \subset B$ and $P(B) = 0$, $A$ is called $P$-measurable if $A \triangle B$ is $P$-null, where $A \triangle B := (A \setminus B) \cup (B \setminus A)$. A function $f : X \to K$ is called $P$-measurable, if $f^{-1}(D)$ is $P$-measurable for each $D$ in the algebra $Bco(K)$ of clopen subsets of $K$. It is essentially bounded, if there exists $k > 0$ such that $\{x : |f(x)| > k\}$
is \( P \)-null, \( \| f \|_\infty \) is by the definition the infimum of such \( k \). Then \( \mathcal{F} := \text{span}_{\mathbb{K}} \{ Ch_B : B \in \mathcal{L} \} \) is called the space of simple functions, where \( Ch_B \) denotes the characteristic function of \( B \). The completion of \( \mathcal{F} \) relative to \( \| \cdot \|_\infty \) is the Banach algebra \( L_\infty(P) \) under the pointwise multiplication.

For each \( f \in L_\infty(P) \) there exists the unique linear mapping \( I : \mathcal{F} \to L(H) \) by the following formula:

\[
(iii) \quad \| \sum_{i=1}^n \lambda_i Ch_{B_i} \| = \sum_{i=1}^n \lambda_i \| P(B_i) \|,
\]

where \( n \in \mathbb{N}, B_i \in \mathcal{L}, \lambda_i \in \mathbb{K} \). Since

\( (iv) \quad \| I(f) \| = \| f \|_\infty \), then \( I \) extends to a linear isometry (also called \( I \)) of \( L_\infty(P) \) onto \( L(H) \).

If \( f \in L_\infty(P) \), then the operator \( I(f) \) in \( L(H) \) is called the spectral integral of \( f \) with respect to \( P \) and it is denoted by

\( (v) \quad \int_X f(x)P(dx) := I(f) \).

From this definition using Chapter 7 we get the following statement (compare with the classical case §II.11.8).

**Proposition.** (I). \( \int_X f(x)P(dx) = \int_X g(x)P(dx) \) if and only if \( f \) and \( g \) differ only on a \( P \)-null set.

(II). \( \int_X f(x)P(dx) \) is linear in \( f \).

(III). \( \int_X f(x)g(x)P(dx) = (\int_X f(x)P(dx)) (\int_X g(x)P(dx)) \) for each \( f \) and \( g \in L_\infty(P) \).

(V). \( \| \int_X f(x)P(dx) \| = \| f \|_\infty \).

(VI). If \( A \in \mathcal{L} \), then \( \int_X Ch_A(x)P(dx) = P(A) \), in particular \( \int_X P(dx) = P(X) = 1_H \).

(VII). For each pair \( \xi \in H \) and \( \eta^* \in H^* \), let \( \mu_{\xi,\eta} := \eta^*(P(A)\xi) \) for each \( A \in \mathcal{L} \). If \( E = \int_X f(x)P(dx) \) then \( \eta^*(E \xi) = \int_X f(x)\mu_{\xi,\eta}(dx) \).

(VIII). If \( A \in \mathcal{L} \), then \( P(A) \) commutes with \( \int_X f(x)P(dx) \).

An \( H \)-projection-valued measure \( P \) on \( \text{Bco}(X) \) is called an \( H \)-projection-valued measure on \( X \). We call \( P \) regular if

\( (v) \quad P(A) = \sup \{ P(C) : C \subset A \text{ and } C \text{ is compact} \} \) for each \( A \in \text{Bco}(X) \), where sup is the least closed subspace of \( H \) containing range \( P(C) \) and to it corresponds projector on this subspace. Indeed, \( P(A)H \) is closed in \( H \), since \( P^2(A) = P(A) \). Therefore,

\( (vi) \quad P(A) = \inf \{ P(U) : U \text{ is open and } U \supset A \} = I - \sup \{ P(C) : C \subset X \setminus A \text{ and } C \text{ is compact} \} \), hence

\( (vii) \quad \text{the infimum corresponds to the projection on } \bigcap_{U \supset A} \text{ is open } P(U)H \).

A measure \( \mu : \text{Bco}(X) \to \mathbb{K} \) is called regular, if for each \( \epsilon > 0 \) and each \( A \in \text{Bco}(X) \) with \( \| A \|_\mu < \epsilon \) there exists a compact subset \( C \subset A \)
such that \( \| A \setminus C \|_\mu < \epsilon \). Since \( \| P(X) \| = 1 \), then \( \| \mu_{\xi, \eta} \| \leq \| \xi \|_H \| \eta \|_{H^*} \).

For the space \( H \) over \( K \) measures \( \mu_{\xi, \eta} \) on the locally compact Hausdorff totally disconnected topological space \( X \) are tight for each \( \xi, \eta \) in a subset \( J \subset H \hookrightarrow H^* \) separating points of \( H \) if and only if \( P \) is defined on \( Bco(X) \); \( P \) is regular if and only if \( \mu_{\xi, \eta} \) are regular for each \( \xi, \eta \in J \) due to Conditions \((vi)\) and \((vii)\). We can restrict our consideration by \( \mu_{\xi, \xi} \) instead of \( \mu_{\xi, \eta} \) with \( \xi, \eta \in \text{span}_K J \), since \( (+)^2 \mu_{\xi, \eta} = \mu_{\xi, \eta} = \mu_{\xi, \xi} - \mu_{\xi, \xi} - \mu_{\xi, \eta} \).

By the closed support of an \( H \)-projection-valued measure \( P \) on \( X \) we mean the closed set \( D \) of all those \( x \in X \) such that \( P(U) \neq 0 \) for each open neighbourhood \( x \in U \), \( \text{supp} \ (P) := D \).

2.7. Remark. We fix a locally compact totally disconnected Hausdorff space \( X \) and a Banach space \( H \) over \( K \) and let \( T : C_\infty(X, K) \to L(H) \) be a linear continuous map from the \( C \)-algebra \( C_\infty(X, K) \) of functions \( f : X \to K \) such that:

(i) \( T_f g = T_f T_g \) for each \( f \) and \( g \in C_\infty(X, K) \),
(ii) \( T_1 = I \).

From this definition it follows, that \( \| T \| \leq 1 \), since \( T_{fn} = T^n_f \) for each \( n \in \mathbb{Z} \) and \( f \in C_\infty(X, K) \). If \( X \) is not compact and it is locally compact, then \( X_\infty := X \cup \{ x_\infty \} \) be its one-point Alexandroff compactification. Each \( f \in C(X_\infty, K) \) can be written just in one way in the form \( f = \lambda 1 + g \), where \( g \in C_\infty(X, K) \) and \( 1 \) is the unit function on \( X_\infty \). Therefore, we can extend \( T : C_\infty(X, K) \to L(H) \) to a linear map \( T' : C(X_\infty, K) \to L(H) \) by setting \( T'(\lambda 1 + g) = \lambda 1_H + T_g \) such that \( T'_1 = 1_H \).

Therefore, \( f \mapsto \eta^*(T_f \xi) := \mu_{\xi, \eta}(f) \) is a continuous \( K \)-linear functional on \( C_\infty(X, K) \), where \( \xi \in H \) and \( \eta^* \in H^* \). In view of the Theorems 7.18 and 7.22 \[8\] about correspondence between measures and continuous linear functionals (the non-Archimedean analog of the F. Riesz representation theorem) there exists the unique measure \( \mu_{\xi, \eta} \in M(X) \) such that

(I) \( \eta^*(T_f \xi) = \int_X f(x) \mu_{\xi, \eta}(dx) \) for each \( f \in C_\infty(X, K) \). Since \( T_1 = I \), then \( \mu_{\xi, \eta}(X) = \eta^*(\xi) = \xi^*(\eta) \). Then for each \( A \in Bco(X) \) we have \( \| A \| \mu_{\xi, \eta} \leq \| \eta \| \| \eta \| \sup_{f \neq 0} \| T_f \| \leq \| \xi \| \| \eta \| \). Since \( H \) considered as a subspace of \( H^* \) separates points in \( H \), then for each \( A \in Bco(X) \) there exists the unique linear operator \( P(A) \in L(H) \) such that:

(II) \( \| P(A) \| \leq 1 \) and \( \eta^*(P(A) \xi) = \mu_{\xi, \eta}(A) \), since \( \mu_{\xi, \eta}(A) \) is a continuous bilinear \( K \)-valued functional by \( \xi \) and \( \eta \in H \). From the existence of a \( H \)-projection-valued measure in the case of compact \( X \) we get a projection-
valued measure $P'$ on $X_\infty$ such that $T'_f = \int_{X_\infty} f(x)P'(dx)$ for each $f \in C(X_\infty, K)$. Suppose further in the locally compact non-compact case of $X$, that

(iv) $\text{span}_K \{ T_\xi : f \in C_\infty(X, K), \xi \in H \}$ is dense in $H$. From this last condition it follows, that

$$
(III) \quad P = P'|_{Bco(X)} \text{ (see also } [36, 38]).
$$

2.8. Note. A particular case of $H = C_\infty(X, K)$ for locally compact totally disconnected Hausdorff space $X$ and $T_f = f$ for each $f \in C_\infty(X, K)$ can be considered independently. Each such $f$ is a limit of a certain sequence by $n \in \mathbb{N}$ of finite sums $\sum_j f(x_{j,n})Ch_{V_{j,n}}(x)$, where $\{V_{j,n} : j \in \Lambda_n\}$ is a finite partition of $X$ into the disjoint union of subsets $V_{j,n}$ clopen in $X$, $x_{j,n} \in V_{j,n}, \Lambda_n \subset \mathbb{N}$, since $\text{Range} (f)$ is bounded. If to take $P(V) = Ch_V$ for each $V \in Bf(X)$, then $T_g = \lim_{n \to \infty} \sum_j f(x_{j,n})Ch_{V_{j,n}}(x)g = \int_X f(x)P(dx)g$ for each $g \in H$, so there is the bijective correspondence between elements $f \in A$ of a $C$-algebra $A$ realised as $C_\infty(X, K)$ with $X = Sp(A)$ and their spectral integral representations. It can be lightly seen that $P(V_1 \cap V_2) = Ch_{V_1 \cap V_2} = Ch_{V_1}Ch_{V_2} = P(V_1)P(V_2) = P(V_2)P(V_1)$ for each $V_j \in Bco(X)$. If $\{V_j : V_j \in Bco(X), j \in \mathbb{N}\}$ is a disjoint family, then $P(\bigcup_j V_j)g = \sum_j Ch_{V_j}g = \sum_j P(V_j)g$ for each $g \in H$. Also $P(\emptyset)H = Ch_\emptyset H = \{0\}$ and $P(X)g = Ch_X g = g$ for each $g \in H$. Therefore, $P$ is indeed the $H$-projection-valued measure.

Suppose now that $X$ is not locally compact, for example, $X = c_0(\omega_n, S)$ with an infinite residue class field $k$ of a field $S$. Then there are $f \in C_\infty(X, K)$ for which convergence of finite or even countable or of the cardinality $\text{card} (k)$ (which may be greater or equal to $\text{card} (\mathbb{R})$) sums $\sum_j f(x_{j,n})Ch_{V_{j,n}}$ becomes a problem for a disjoint family $\{V_{j,n} : j\}$ of clopen in $X$ subsets, since $\|Ch_{V_{j,n}}\|_{C(X, K)} = 1$ for each $j$ and $n$.

2.9. Remark. Fix a Banach space $H$ over a non-Archimedean complete field $F$, as above $L(H)$ denotes the Banach algebra of all bounded $F$-linear operators on $H$. If $b \in L(H)$ we write shortly $Sp(b)$ instead of $Sp_{L(H)}(b) := cl(Sp(span_F\{b^n : n = 1, 2, 3, \ldots\}))$ (see also $[36]$).

It was proved in Theorem 2 $[36]$ in the case of $F$ with the discrete valuation group, that each continuous $F$-linear operator $A : E \to H$ with $\|A\| \leq 1$ from one Banach space $E$ into another $H$ has the form

$$
A = U \sum_{n=0}^\infty \pi^n P_{n,A},
$$

11
where $P_n := P_{n,A}$, $\{P_n : n \geq 0\}$ is a family of projections and $P_n P_m = 0$ for each $n \neq m$, $\|P_n\| \leq 1$ and $P_n^2 = P_n$ for each $n$, $U$ is a partially isometric operator, that is, $U|_{d(\sum_n P_n(E))}$ is isometric, $U|_{E \ominus \overline{d(\sum_n P_n(E))}} = 0$, $\ker(U) \supset \ker(A)$, $\text{Im}(U) = \overline{\text{Im}(A)}$, $\pi \in \mathbf{F}$, $|\pi| < 1$ and $\pi$ is the generator of the valuation group of $\mathbf{F}$.

For $\mathbf{F}$ not necessarily with the discrete valuation group and a completely continuous operator $A$ it was proved the Fredholm alternative for the operator $I + A$ [15].

We restrict our attention to the case of the local field $\mathbf{F}$, consequently, $\mathbf{F}$ has the discrete valuation group. If $\|A\| > 1$ we get

$$(i) \quad A = \lambda_A U \sum_{n=0}^{\infty} \pi^n P_{n,A},$$

where $\lambda_A \in \mathbf{F}$ and $|\lambda_A| = \|A\|$. In view of §2.6-2.8 this is the particular case of the spectral integration on the discret topological space $X$. Evidently, for each $1 \leq r < \infty$ there exists $J \in L(H)$ for which

$$(ii) \quad \left\{ \sum_{n \geq 0} s_n^r \text{dim}_\mathbf{F} P_{n,J(H)} \right\}^{1/r} < \infty$$

for $1 \leq r < \infty$, where $J$ has the spectral decomposition given by Formula $(i)$, $s_n := |\lambda_J| |\pi^n| \|P_n\|$. Using this result it is possible to give the following definition.

**2.10.1. Definition.** Let $E$ and $H$ be two normed $\mathbf{F}$-linear spaces, where $\mathbf{F}$ is an infinite spherically complete field with a nontrivial non-Archimedean valuation. The $\mathbf{F}$-linear operator $A \in L(E, H)$ is called of class $L_q(E, H)$ if there exists $a_n \in E^*$ and $y_n \in H$ for each $n \in \mathbf{N}$ such that

$$(i) \quad \left( \sum_{n=1}^{\infty} \|a_n\|_{E^*}^q \|y_n\|_H^q \right) < \infty$$

and $A$ has the form

$$(ii) \quad Ax = \sum_{n=1}^{\infty} a_n(x) y_n$$

for each $x \in E$, where $1 \leq q < \infty$. For each such $A$ we put

$$(iii) \quad \nu_q(A) = \inf \left\{ \sum_{n=1}^{\infty} \|a_n\|_{E^*}^q \|y_n\|_H^q \right\}^{1/q},$$
where the infimum is taken by all such representations (ii) of $A$,

$$(iv) \quad \nu_\infty(A) := \|A\|$$

and $L_\infty(E, H) := L(E, H)$.

**2.10.2. Proposition.** $L_q(E, H)$ is the normed $F$-linear space with the norm $\nu_q$.

**Proof.** Let $A \in L_q(E, H)$ and $1 \leq q < \infty$, since the case $q = \infty$ follows from its definition. Then $A$ has the representation 2.10.1.(ii). Then due to the ultrametric inequality

$$\|Ax\|_H \leq \|x\|_E \sup_{n \in \mathbb{N}} (\|a_n\|_{E^*} \|y_n\|_H)^{1/q},$$

hence $\sup_{x \neq 0} \|Ax\|_H/\|x\|_E =: \|A\| \leq \nu_q(A)$.

Let now $A, S \in L_q(E, H)$, then there exists $0 < \delta < \infty$ and two representations $Ax = \sum_{n=1}^{\infty} a_n(x)y_n$ and $Sx = \sum_{m=1}^{\infty} b_m(x)z_m$ for which

$$(\sum_{n=1}^{\infty} \|a_n\|_{E^*} \|y_n\|_H)^{1/q} \leq \nu_q(A) + \delta$$

and

$$(\sum_{n=1}^{\infty} \|b_n\|_{E^*} \|z_n\|_H)^{1/q} \leq \nu_q(S) + \delta,$$

hence

$$(A + S)x = \sum_{n=1}^{\infty} (a_n(x)y_n + b_n(x)z_n)$$

and

$$\nu_q(A + S) \leq (\sum_{n=1}^{\infty} \|a_n\|^q \|y_n\|^q)^{1/q} + (\sum_{n=1}^{\infty} \|b_n\|^q \|z_n\|^q)^{1/q} \leq \nu_q(A) + \nu_q(S) + 2\delta$$

due to the Hölder inequality.

**2.11. Proposition.** If $J \in L_q(H)$, $S \in L_r(H)$ are commuting operators, the field $F$ is with the discrete valuation group and $1/q + 1/r = 1/v$, then $JS \in L_v(H)$, where $1 \leq q, r, v \leq \infty$.

**Proof.** Since $F$ is with the discrete valuation, then $J$ and $S$ have the decompositions given by Formula 2.9(i). Certainly each projector $P_{n,J}$ and $P_{m,S}$ belongs to $L_1(H)$ and have the decomposition given by Formula 2.10.1.(ii). The $F$-linear span of $\bigcup_{n,m}$ range$(P_{n,J}P_{m,S})$ is dense in $H$. In particular, for each $x \in \text{range}(P_{n,J}P_{m,S})$ we have $J^kS^lx = \lambda_j^k\lambda_s^lm^k + mlP_{n,J}P_{m,S}x$. Applying §2.9 to commuting operators $J^k$ and $S^l$ for each $k, l \in \mathbb{N}$ and using
the base of $H$ we get projectors $P_{n,J}$ and $P_{m,S}$ which commute for each $n$ and $m$, consequently, $JS = U_{JS} P_{n,J} P_{m,S}$, hence $U_{JS} = U_{JS} P_{n,J} P_{m,S}$. In view of the Hölder inequality $\nu_r(JS) = \inf(\sum_{n=0}^{\infty} s_{n,JS}^r \dim F P_{n,JS}(H))^{1/r} \leq \nu_q(J) \nu_T(S)$ (see §IX.4 [37]).

2.12.1. Proposition. If $E$ is the normed space and $H$ is the Banach space over the field $F$ (complete relative to its uniformity), then $L_r(E, H)$ is the Banach space such that if $J, S \in L_r(E, H)$, then
\[
\|J + S\|_r \leq \|J\|_r + \|S\|_r;
\]
\[
\|bJ\|_r = |b| \|J\|_r
\]
for each $b \in K$; $\|J\|_r = 0$ if and only if $J = 0$, where $1 \leq r \leq \infty$, $\|\cdot\|_q := \nu_q(\cdot).

Proof. In view of Proposition 2.10.2 it remains to prove that $L_r(E, H)$ is complete, when $H$ is complete. Let $\{T_n\}$ be a Cauchy net in $L_r(E, H)$, then there exists $T \in L(E, H)$ such that $\lim_n T_n x = Tx$ for each $x \in E$, since $L_r(E, H) \subset L(E, H)$ and $L(E, H)$ is complete. We demonstrate that $T \in L_r(E, H)$ and $T_n$ converges to $T$ relative to $\nu_r$ for $1 \leq r < \infty$. Let $\alpha_k$ be a monotone subsequence in $\{\alpha\}$ such that $\nu_r(T_{\alpha_k} - T_\beta) < 2^{-k-2}$ for each $\alpha, \beta \geq \alpha_k$, where $k \in \mathbb{N}$. Since $T_{\alpha_k+1} - T_{\alpha_k} \in L_r(E, H)$, then $(T_{\alpha_k+1} - T_{\alpha_k}) x = \sum_{n=1}^{\infty} a_{n,k}(x) y_{n,k}$ with $\sum_{n=1}^{\infty} a_{n,k}^r \|y_{n,k}\|_r \leq 2^{-k-2}$. Therefore, $(T_{\alpha_k+p} - T_{\alpha_k}) x = \sum_{n=1}^{\infty} a_{n,h}(x) y_{n,h}$ for each $p \in \mathbb{N}$, consequently, using convergence while $p$ tends to $\infty$ we get $(T - T_{\alpha_k}) x = \sum_{n=1}^{\infty} a_{n,h}(x) y_{n,h}$. Then $\nu_r(T - T_{\alpha_k}) \leq \sum_{n=1}^{\infty} a_{n,h}^r \|y_{n,h}\|_r \leq 2^{-k-1}$, hence $T - T_{\alpha_k} \in L_r(E, H)$ and inevitably $T \in L_r(E, H)$. Moreover, $\nu_r(T - T_{\alpha_k}) \leq \nu_r(T - T_{\alpha_k} + \nu_r(T_{\alpha_k} - T_{\alpha_k})) \leq 2^{-(k-1)/r}$ for each $\alpha \geq \alpha_k$.

2.12.2. Proposition. Let $E, H, G$ be normed spaces over spherically complete $F$. If $T \in L(E, H)$ and $S \in L_r(H, G)$, then $ST \in L_r(E, G)$ and $\nu_r(ST) \leq \nu_r(S) \|T\|$. If $T \in L_r(E, H)$ and $S \in L(H, G)$, then $ST \in L_r(E, G)$ and $\nu_r(ST) \leq \|S\| \nu_r(T).

Proof. For each $\delta > 0$ there are $b_n \in H^*$ and $z_n \in G$ such that $Sy = \sum_{n=1}^{\infty} b_n(y) z_n$ for each $y \in H$ and $\sum_{n=1}^{\infty} \|b_n^r\| z_n^r \leq \nu_r(S) + \delta$. Therefore, $ST x = \sum_{n=1}^{\infty} T^* b_n(x) z_n$ for each $x \in E$, hence $\nu_r(ST) \leq \sum_{n=1}^{\infty} \|T^* b_n\| z_n^r \leq \|T\| \|S\| + \delta$, since $\|T^* b_n\| = \|b_n(Tx)\| \leq \|b_n\| \|T\| \|x\|$, where $T^* \in L(H^*, E^*)$ is the adjoint operator such that $b(Tx) = (T^* b)(x)$.
for each \( b \in H^* \) and \( x \in E \). The operator \( T^* \) exists due to the Hahn-Banach theorem for normed spaces over the spherically complete field \( \mathbf{F} \).

**2.12.3. Proposition.** If \( T \in L_r(E, H) \), then \( T^* \in L_r(H^*, E^*) \) and \( \nu_r(T^*) \leq \nu_r(T) \), where \( E \) and \( H \) are over the spherically complete field \( \mathbf{F} \).

**Proof.** For each \( \delta > 0 \) there are \( a_n \in E^* \) and \( y_n \in H \) such that \( Tx = \sum_{n=1}^{\infty} a_n(x) y_n \) for each \( x \in E \) and \( \sum_{n=1}^{\infty} \|a_n\|_r \|y_n\|_r \leq \nu_r^*(T) + \delta \). Since \( (T^*b)(x) = b(Tx) = \sum_{n=1}^{\infty} a_n(x) b(y_n) \) for each \( b \in H^* \) and \( x \in E \), then \( T^*b = \sum_{n=1}^{\infty} y_n(b) a_n \), where \( y_n^*(b) := b(y_n) \), that is correct due to the Hahn-Banach theorem for \( E \) and \( H \) over the spherically complete field \( \mathbf{F} \). Therefore, \( \nu_r^*(T^*) \leq \sum_{n=1}^{\infty} \|y_n\|_r \|a_n\|_r \leq \nu_r^*(T) + \delta \), since \( \|y^*\|_{H^*} = \|y\|_H \) for each \( y \in H \).

**2.13.** For a space \( L_k(H_1, ..., H_k; H) \) of \( k \)-linear mappings of \( H_1 \otimes ... \otimes H_k \) into \( H \) we have its embedding into \( L(E, H) \), where \( E \) is a normed space \( H_1 \otimes ... \otimes H_k \) in its maximum norm topology for normed spaces \( H_1, ..., H_k, H \) over \( \mathbf{F} \) (see §§2.1, 2.10). Therefore, we can define the following normed space \( L_{r,k}(H_1, ..., H_k; H) := L_k(H_1, ..., H_k; H) \cap L_r(E; H) \) in particular

\[
L_{r,k}(H^\otimes k; H) := L_k(H^\otimes k; H) \cap L_r(H^\otimes k; H)
\]

and

\[
L_{\infty,k}(H_1, ..., H_k; H) := L_k(H_1, ..., H_k; H)
\]

with the norm \( \nu_r(J) =: \|J\|_r \), where \( 1 \leq r \leq \infty \). Certainly, \( L_{r,k} \subset L_{q,k} \) for each \( 1 \leq r < q \leq \infty \).

Suppose that \( (\Omega, \mathbf{B}, \lambda) \) is a probability space (with non-negative measure \( \lambda \)), where \( \mathbf{B} \) is a \( \sigma \)-algebra of subsets of \( \Omega \). We define a \( \mathbf{K} \)-linear Banach space \( L^q(\Omega, \mathbf{B}, \lambda; L_{r,k}(H_1, ..., H_k; H)) \) and \( L^q(\Omega, \mathbf{B}, \lambda; L_k(H_1, ..., H_k; H)) \) as a completion of a family of mappings \( \sum_{j=1}^{n} A_j \mathcal{C}_{W_j} \) with \( A_j \in L_{r,k}(H_1, ..., H_k; H) \) or \( A_j \in L_k(H_1, ..., H_k; H) \) respectively and \( W_j \in \mathbf{B} \) and \( n \in \mathbf{N} \). That is, as consisting of those mappings \( \Omega \ni \nu \mapsto A(\nu) \in L_{r,k}(H_1, ..., H_k; H) \) for which \( \|A(\nu)\|_r \) is \( \lambda \)-measurable and

\[
\|A\|_{L^q} := \left\{ \int_{\Omega} \|A(\nu)\|_r^q \lambda(d\nu) \right\}^{1/q} < \infty,
\]

where \( 1 \leq q < \infty \);

\[
\|A\|_{L^\infty} := ess - \sup_{\lambda} \|A(\nu)\|_r.
\]

**2.14.** We consider a \( C^\infty \)-manifold \( X \) with an atlas \( \Lambda X = \{(U_j, \phi_j) : j \in \Lambda X\} \), where \( \bigcup_j U_j = X \), \( \phi_j(U_j) \) are open in \( c_0(\alpha, \mathbf{K}) \) and \( U_j \) are open in \( X \), \( \phi_j : U_j \rightarrow \phi_j(U_j) \) are homeomorphisms, \( \phi_i \circ \phi_j^{-1} \in C^\infty \) for each \( U_i \cap U_j \neq \emptyset \) and \( \|\phi_i \circ \phi_j^{-1}\|_{C^m} < \infty \) for each \( m \in \mathbf{N} \), \( \phi_j(U_j) \) are bounded in \( c_0(\alpha, \mathbf{K}) \) for each \( j \in \Lambda X \), \( \Lambda X \) is a set, \( C^m_b(X, H) \) is a completion of a set of
all functions \( f : X \to H \) such that \( f \circ \phi_j^{-1} \in C^n(\phi_j(U_j), H) \) for each \( j \in \Lambda_X \) and \( \sup_j \| f \circ \phi_j^{-1} \|_{C^n} =: \| f \|_{C^n(X, H)} < \infty \), where \( H \) is a Banach space over \( K \). Then \( C^n(X, H) \) is the set of all functions \( f : X \to H \) such that for each \( x \in X \) there exists a neighbourhood \( x \in U \subset X \) for which \( f|_U \in C^n_b(U, H) \).

By \( L^s(\Omega, B, \lambda; C^n(X, H)) \) we denote a completion of a space of simple functions \( \sum_{j=1}^n \xi_j(x)Ch_{W_j}(\nu) \) with \( \xi_j(x) \in C^n(X, H), W_j \in B \) and \( n \in \mathbb{N} \), relative to the following norm

\[
\| \xi \|_{L^s} := \left\{ \int_{\Omega} \| \xi(x, \nu) \|_{C^n(X, H)}^s \lambda(\nu) \right\}^{1/s} < \infty
\]

for each \( 1 \leq s < \infty \) or

\[
\| \xi \|_{L^\infty} := ess \sup_{\lambda} \| \xi(x, \nu) \|_{C^n(X, H)} < \infty,
\]

where \( X \) is the \( C^\infty \) Banach manifold on \( c_0(\alpha, K) \), \( \| \xi(x, \nu) \|_{C^n(X, H)} \) is attached to \( \xi \) as a function by \( x \in X \) with parameter \( \nu \in \Omega \) such that \( \| \xi(x, \nu) \|_{C^n(X, H)} \) is a measurable function by \( \nu \).

**Theorem.** Let \( G \in L^s(\Omega, B, \lambda; C^0(B_R \times H^{\otimes l}, L_{k-l}(H^{\otimes(k-l)}; H)), \xi_1, ..., \xi_k \in L^q(\Omega, B, \lambda; C^0(B_R, H)), A_{l+1} \in C^0(B_R, L(H)) \) for each \( i = 1, ..., k - l \) (see \( \S 2.1 \)), where \( B_R = B(K, 0, R), G = G(x; \xi_1, ..., \xi_l; \nu), \xi_i = \xi_i(x, \nu) \) with \( x \in B_R, \nu \in \Omega, 1/r + 1/q = 1/s \) with \( 1 \leq r, q, s \leq \infty \). Then \( (\hat{P}_{\xi_1, ..., \xi_k} \circ G) \) is a measurable function by \( \nu \).

**Proof.** In \( L^q(\Omega, F, \lambda; C^0(B_R \times V, W)) \) the family of step functions \( f(t, x, \omega) = \sum_{j=1}^n Ch_{U_j}(\omega)f_j(t, x) \) is dense, where \( f_j \in C^0(B_R \times V, W) \), \( Ch_{U} \) is the characteristic function of \( U \in F \), \( n \in \mathbb{N} \), \( V \) and \( W \) are Banach spaces over \( K \), \( t \in B_R, x \in V, \omega \in \Omega \), since \( \lambda(\Omega) = 1 \) and \( \lambda \) is nonnegative \( \mathbb{[}\mathbb{]} \). Each matrix element \( F_{h,b}(x, \nu) \) is in \( L^q(\Omega, B, \lambda; C^0(B_R, K)) \) and \( \xi_j \in L^q(\Omega, B, \lambda; C^0(B_R, K)) \), where \( F(x, \nu) := G(x; a_1, ..., a_l; \nu), (A_{l+1}a_{l+1}(x), ..., A_ka_k(x)) \), \( h \in H^*, b \in H, F_{h,b} := h(Fb), a_i \in C^0(B_R, H) \) for each \( i = 1, ..., k \). Since \( \| \xi_j(x, \nu) \|_{C^n(X, H)} \in L^q(\lambda), \| F_{a,b}(x, \nu) \|_{C^n(X, H)} \in L^r(\lambda), \) then \( F(x, \nu), w(x, \nu) \in L^s(\Omega, B, \lambda; C^0(B_R, H)) \), where \( w = (\xi_1, ..., \xi_k) \) (see \( \S \mathbb{IX.4 \mathbb{[}37\mathbb{]} \)). The operator \( \hat{P}_w F \) is linear by \( w \) and \( F \), hence it is defined on simple functions. In view of Lemma 2.2

\[
\| \hat{P}_w F(x, \nu) \|_H \leq \| F(x, \nu) \|_{C^0(B_R \times H^{\otimes l}, L_{k-l}(H^{\otimes(k-l)}; H))}
\]
\[
\prod_{i=l+1}^{k} \|A_i\|_{C^0(B_R,L^q(H))} \|\xi_i(x,\nu)\|_{C^0(B_R,H)}
\]
for \(\lambda\)-a.e. \(\nu \in \Omega\), hence \(\|P_\nu F(x,\nu)\|_{L^q} \leq \|G\|_{L^q} \prod_{i=l+1}^{k} \|A_i\|_{C^0} \|\xi_i\|_{L^q}\).

**Corollary.** If in suppositions of Theorem 2.14 \(\xi_i \in L^q(\Omega, B, \lambda; C^1(B_R, H))\) for each \(i = 1, \ldots, k\), then \((P_\nu F) \in L^q(\Omega, B, \lambda; C^1(B_R, H))\) and

\[
(i) \quad \|P_\nu G((A_{l+1} \ldots \otimes A_k))\|_{L^q(\lambda; C^1(B_R, H))} \leq \|G\|_{L^q(\lambda; C^0(B_R \times H^\otimes l, L_{k-l}(H^{\otimes (k-l)}; H)))}
\]

\[
\prod_{i=l+1}^{k} \|A_i\|_{C^0(B_R,L^q(H))} \|\xi_i\|_{L^q(\lambda; C^0(B_R, H))}.
\]

**Proof.** In view of Lemma 2.3 and Theorem 2.14

\[
\|P_\nu F(x,\nu)\|_{C^1(B_R,H)} \leq \|G(x; \xi_1, \ldots, \xi_k; \nu)\|_{C^0(B_R \times H^\otimes l, L_{k-l}(H^{\otimes (k-l)}; H))}
\]

\[
\prod_{i=l+1}^{k} \|A_i\|_{C^0(B_R,L^q(H))} \|\xi_i(x,\nu)\|_{C^1(B_R,H)}
\]
for \(\lambda\)-almost each \(\nu \in \Omega\). From this Formula \(i\) follows.

### 3 Markov quasimeasures for a non-Archimedean Banach space.

**3.1. Remark.** Let \(H = C_0(\alpha, K)\) be a Banach space over a local field \(K\) with an ordinal \(\alpha\) and the standard orthonormal base \(\{e_j : j \in \alpha\}, e_j = (0, \ldots, 0, 1, 0, \ldots\}\) with 1 on the \(j\)-th place. Let \(U^p\) be a cylindrical algebra generated by projections on finite-dimensional over \(K\) subspaces \(F\) in \(H\) and Borel \(\sigma\)-algebras \(Bf(F)\). Denote by \(U\) the minimal \(\sigma\)-algebra \(\sigma(U^p)\) generated by \(U^p\). When \(\text{card}(\alpha) \leq \aleph_0\), then \(U = Bf(H)\), where \(\text{card}(A)\) denotes the cardinality of a set \(A\). Each vector \(x \in H\) is considered as continuous linear functional on \(H\) by the formula \(x(y) = \sum_j x^j y^j\) for each \(y \in H\), so there is the natural embedding \(H \hookrightarrow H^* = l^\infty(\alpha, K)\), where \(x = \sum_j x^j e_j\), \(x^j \in K\).

**3.2. Notes and definitions.** Let \(T = B(K, t_0, r)\) be a ball in the field \(K\) of radius \(r > 0\) and containing a point \(t_0\) and \(X_t = X\) be a locally \(K\)-convex space for each \(t \in T\). Put \((X_T, \mathcal{U}) := \prod_{t \in T}(X_t, \mathcal{U}_t)\) be a product...
of measurable spaces, where $\mathcal{U}_t$ are $\sigma$-algebras of subsets of $X_t$, $\tilde{U}$ is the $\sigma$-algebra of cylindrical subsets of $\tilde{X}_T$ generated by projections $\pi_q : \tilde{X}_t \to X^q$, $X^q := \Pi_{t \in q} X_t$, $q \subset T$ is a finite subset of $T$ (see §I.1.3 [3]). A function $P(t_1, x_1, t_2, A)$ with values in $C$ for each $t_1 \neq t_2 \in T$, $x_1 \in X_{t_1}$, $A \in \mathcal{U}_{t_2}$ is called a transition measure if it satisfies the following conditions:

(i) the set function $\nu_{x_1, t_1, t_2}(A) := P(t_1, x_1, t_2, A)$ is a $\sigma$-additive measure on $(X_{t_2}, \mathcal{U}_{t_2})$;

(ii) the function $\alpha_{t_1, t_2, A}(x_1) := P(t_1, x_1, t_2, A)$ of the variable $x_1$ is $\mathcal{U}_{t_1}$-measurable;

(iii) $P(t_1, x_1, t_2, A) = \int_{X_{s}} P(t_1, x_1, s, dy)P(s, y, t_2, A)$ for each $t_1 \neq t_2 \in T$.

A transition measure $P(t_1, x_1, t_2, A)$ is called normalised if

(iv) $P(t_1, x_1, t_2, X_{t_2}) = 1$ for each $t_1 \neq t_2 \in T$.

For each set $q = (t_0, t_1, \ldots, t_{n+1})$ of pairwise distinct points in $T$ there is defined a measure in $X^s := \prod_{t \in s} X_t$ by the formula

$$(v) \quad \mu_{x_0}^q(E) = \int_E \prod_{k=1}^{n+1} P(t_{k-1}, x_{k-1}, t_k, dx_k), \ E \in \mathcal{U}^s := \prod_{t \in s} \mathcal{U}_t,$$

where $s = q \setminus \{t_0\}$, variables $x_1, \ldots, x_{n+1}$ are such that $(x_1, \ldots, x_{n+1}) \in E$, $x_0 \in X_{t_0}$ is fixed.

Let $E = E_1 \times X_{t_j} \times E_2$, where $E_1 \in \prod_{t \in 1} \mathcal{U}_t$, $E_2 \in \prod_{t \in 2} \mathcal{U}_t$, then

$$(vi) \quad \mu_{x_0}^q(E) = \int_{E_1 \times X_{t_j} \times E_2} \prod_{k=1}^{n-1} P(t_{k-1}, x_{k-1}, t_k, dx_k) \times \int_{X_{t_j}} P(t_{t_1}, x_{t_1}, t_j, dx_j) P(t_{k-1}, x_{k-1}, t_k, dx_k) = \mu_{x_0}^r(E_1 \times E_2),$$

where $r = q \setminus \{t_j\}$. From Equation (vi) it follows, that

$$(vii) \quad [\mu_{x_0}^q]^{\pi_q}_s = \mu_{x_0}^q$$

for each $v < q$ (that is, $v \subset q$), where $\pi^q_v : X^s \to X^w$ is the natural projection, $s = q \setminus \{t_0\}$, $w = v \setminus \{t_0\}$. If the transition measure $P(t, x_1, t_2, dx_2)$ is normalised and $F = E \times X_{t_{n+1}}$ with $E \in \mathcal{U}^s$, then

$$(viii) \quad \mu_{x_0}^v(E) := \mu_{x_0}^q(F) = \int_F \prod_{j=1}^{n} P(t_{j-1}, x_{j-1}, t_j, dx_j),$$

where $q = (t_0, \ldots, t_{n+1})$, $v = (t_0, \ldots, t_n)$, points $t_0, \ldots, t_n$ are pairwise distinct in $T$. If $\nu$ is the complex-valued measure on $(X, \mathcal{U})$, then $\nu = \nu_1 - \nu_2 + \nu_3$. 

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$i\nu_3 - i\nu_4$, where $\nu_j$ is a nonnegative measure on $(X, U)$ for each $j = 1, \ldots, 4$, $i := (-1)^{1/2} \in \mathbb{C}$, $U$ is a $\sigma$-algebra of subsets of $X$. By the definition $\|\nu\| := \sum_{j=1}^{4} \nu_j(X)$ and it is called the variation of the measure $\nu$ on $X$. Therefore, due to Conditions (iv, v, vii) : \{ $\mu_0^q \gamma^q_1; \pi^q_1; \Upsilon_T$ \} is the consistent family of measures, which induce the quasimeasure $\tilde{\mu}_0$ on $(X, \tilde{U})$ such that $\tilde{\mu}_0(\pi^q_1^{-1}(E)) = \mu_0^q(E)$ for each $E \in \mathcal{U}$, where $\Upsilon_T$ is the family of all finite subsets $q$ in $T$ such that $t_0 \in q \subset T$, $v \leq q \in \Upsilon_T$. $\pi_q : \tilde{U} \to X^*$ is the natural projection, $s = q \setminus \{t_0\}$.

The quasimeasures given by Equations (i - v, vii) are called Markov quasimeasures.

3.3. Proposition. 1. If a normalized transition measure $P$ satisfies the condition

$$(i) \ C := \sup_{q}(\sup_{x} \ln(\|\nu_{x,t_{k-1},t_{k}}\|)) < \infty,$$

where $q = (t_0, t_1, \ldots, t_n)$ with pairwise distinct points $t_0, \ldots, t_n \in T$ and $n \in \mathbb{N}$, then the Markov quasimeasure $\tilde{\mu}_0$ is bounded.

3.3.2. Proposition. If

$$(ii) \ C_x := \sup_{q}(\sup_{k=1}^{n} \ln(\|\nu_{x,t_{k-1},t_{k}}\|)) = \infty$$

for each $x$, where $q = (t_0, t_1, \ldots, t_n)$ with pairwise distinct points $t_0, \ldots, t_n \in T$ and $n \in \mathbb{N}$, then the Markov quasimeasure $\tilde{\mu}_0$ has the unbounded variation on each nonvoid set $E \in \mathcal{U}$.

Proof. (1). If $E \in \tilde{U}$, then $E \in \mathcal{U}$ for some set $q = (t_0, t_1, \ldots, t_n)$ with pairwise distinct points $t_0, \ldots, t_n \in T$ and $n \in \mathbb{N}$ and $s = q \setminus \{t_0\}$, consequently, $|\mu_0^q(E)| \leq \prod_{k=1}^{n} \sup_{x} \|\nu_{x,t_{k-1},t_{k}}\| \leq \exp(C)$, since $t_k \in T$ for each $k = 0, 1, \ldots, n$.

(2). For each $(t_1, t_2, x)$ there exists a compact set $\delta(t_1, t_2, x) \in \mathcal{U}_{t_2}$ such that $P(t_1, x_1, t_2, \delta(t_1, t_2, x)) > 1 + \epsilon(t_1, t_2, x_1, x)$, where $\epsilon(t_1, t_2, x) > 0$.

In view of Condition (ii) for each $R > 0$ and $x$ we choose $q$ such that $\sum_{k=1}^{n} \epsilon(t_k, x, t_{k+1}) > R$. For chosen $u \neq u_1 \in T$ and $x \in X_u$ we represent the set $\delta(u, u_1, x)$ as a finite union of disjoint subsets $\gamma_{j_1}$ such that for each $\gamma_{j_1}$ and $u_2 \neq u_1$ there is a set $\delta_{j_1}$ satisfying $P(u_1, x_1, u_2, \delta_{j_1}) \geq 1 + \epsilon(u_1, u_2, x_1, x)$ for each $x \in \gamma_{j_1}$. Then by induction $\delta_{j_1 \ldots j_n} = \bigcup_{j_{n+1}}^{\infty} \gamma_{j_1 \ldots j_{n+1}}$ so that for $u_{n+2} \neq u_{n+1} \in T$ there is a set $\delta_{j_1 \ldots j_{n+1}}$ for which $P(u_{n+1}, x_{n+1}, u_{n+2}, \delta_{j_1 \ldots j_{n+1}}) \geq 1 + \epsilon(u_{n+1}, u_{n+2}, x_{n+1}, x)$ for each $x \in \gamma_{j_1 \ldots j_{n+1}}$. Put $\Gamma_{j_1 \ldots j_n} = \{x : x(u) =
\(x_0, x(u_1) \in \gamma_{j_1}, ..., x(u_n) \in \delta_{j_1, ..., j_n}, x(u_{n+1}) \in \gamma_{j_1, ..., j_n}\) and \(\Gamma_{u,x_0} := \bigcup_{j_1, ..., j_n} \Gamma_{u,x_0}^{j_1, ..., j_n}\). Then \(\bar{\mu}_{x_0}(\Gamma_{u,x_0}) = \sum_{j_1, ..., j_n} \int_{\delta_{j_1, ..., j_n}} \int_{\gamma_{j_1, ..., j_n}} ... \int_{\gamma_{j_1}} \prod_{k=1}^{n+1} P(u_{k-1}, x_{k-1}, u_k, dx_k) \geq \prod_{k=1}^{n} [1 + \epsilon(u_{k-1}, u_k, x_{k-1}, x_k)] > R\).

3.3.3 Evidently Condition (i) of Proposition 3.3.1 is satisfied for the nonnegative normalized transition measure.

3.4. Let \(X_t = X\) for each \(t \in T\), \(\tilde{X}_{t_0,x_0} := \{x \in \tilde{X}_T : x(t_0) = x_0\}\). We define a projection operator \(\bar{\pi}_q : x \mapsto x_q\), where \(x_q\) is defined on \(q = (t_0, ..., t_{n+1})\) such that \(x_q(t) = x(t)\) for each \(t \in q\), that is, \(x_q = x|_q\). For every \(F : \tilde{X}_T \to C\) there corresponds \((S_qF)(x) := F(x_q) = F(y_0, ..., y_n)\), where \(y_j = x(t_j)\). \(F_q : X^q \to C\). We put \(F := \{F|F : \tilde{X}_T \to C, S_qF\) are \(U^q\) measurable\}. If \(F \in F\), \(\tau = t_0 \in q\), then there exists an integral

\[
(i) \ J_q(F) = \int_{X^q} (S_qF)(x_0, ..., x_n) \prod_{k=1}^{n+1} P(t_{k-1}, x_{k-1}, t_k, dx_k).
\]

Definition. A function \(F\) is called integrable with respect to the Markov quasimeasure \(\mu_{x_0}\) if the limit

\[
(ii) \ \lim_q J_q(F) =: J(F)
\]

along the generalized net by finite subsets \(q\) of \(T\) exists. This limit is called a functional integral with respect to the Markov quasimeasure:

\[
(iii) \ J(F) = \int_{\tilde{X}_{t_0,x_0}} F(x)\mu_{x_0}(dx).
\]

3.5. Remark. Consider a complex-valued measure \(P(t, A)\) on \((X, U)\) for each \(t \in T := B(K, 0, R)\) such that \(A - x \in U\) for each \(A \in U\) and \(x \in X\), where \(A \in U\), \(X\) is a locally \(K\)-convex space, \(U\) is a \(\sigma\)-algebra of \(X\). Suppose \(P\) be a spatially homogeneous transition measure (see also §3.2), that is,

\[
(i) \ P(t_1, x_1, t_2, A) = P(t_2 - t_1, A - x_1)
\]

for each \(A \in U\), \(t_1 \neq t_2 \in T\) and every \(x_1 \in X\), where \(P(t, A)\) satisfies the following condition:

\[
(ii) \ P(t_1 + t_2, A) = \int_X P(t_1, dy)P(t_2, A - y).
\]

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Such a transition measure $P(t_1, x_1, t_2, A)$ is called homogeneous. In particular for $T = \mathbb{Z}_p$ we have

$$(iii) \ P(t + 1, A) = \int_X P(t, dy) P(1, A - y).$$

If $P(t, A)$ is a continuous function by $t \in T$ for each fixed $A \in U$, then Equation $(iii)$ defines $P(t, A)$ for each $t \in T$, when $P(1, A)$ is given, since $Z$ is dense in $Z_p$.

**3.6. Notes and definition.** Let $X$ be a locally $K$-convex space and $P$ satisfies Conditions 3.2($i - iii$). For $x$ and $z \in Q^\wedge_p$ we denote by $(x, z)$ the following sum $\sum_{j=1}^n x_jz_j$, where $x = (x_j : j = 1, ..., n)$, $x_j \in Q_p$. Each number $y \in Q_p$ has a decomposition $y = \sum_{l = a_l}$, where $a_l \in (0, 1, ..., p - 1)$, $\min(l : a_l \neq 0) = ord_p(y) > -\infty$ ($ord(0) := \infty$) [33, 39], we define a symbol $(y) := \sum_{l < 0} a_lP_l$ for $|y|_p > 1$ and $(y) = 0$ for $|y|_p \leq 1$. We consider a character of $X$, $\chi^\gamma : X \to \mathbb{C}$ given by the following formula:

$$(i) \ \chi^\gamma(x) = e^{z^{-1}((e, \gamma(x)))_p}$$

for each ${((e, \gamma(x)))_p \neq 0}$, $\chi^\gamma(x) := 1$ for $\{(e, \gamma(x))\}_p = 0$, where $e = 1^z$ is a root of unity, $z = p^{ord((e, \gamma(x)))_p}$, $\gamma \in X^*$, $X^*$ denotes the topologically conjugated space of continuous $K$-linear functionals on $X$, the field $K$ as the $Q_p$-linear space is $n$-dimensional, that is, $dim_{Q_p}K = n$, $K$ as the Banach space over $Q_p$ is isomorphic with $Q^\wedge_p$, $e = (1, ..., 1) \in Q^\wedge_p$ (see [42] and [22]). Then

$$(ii) \ \phi(t_1, x_1, t_2, y) := \int_X \chi_y(x) P(t_1, x_1, t_2, dx)$$

is the characteristic functional of the transition measure $P(t_1, x_1, t_2, dx)$ for each $t_1 \neq t_2 \in T = B(K, t_0, R)$ and each $x_1 \in X$. In the particular case of $P$ satisfying Conditions 3.5.($i, ii$) with $t_0 = 0$ its characteristic functional is such that

$$(iii) \ \phi(t_1, x_1, t_2, y) = \psi(t_2 - t_1, y) \chi_y(x_1), \ \text{where}$$

$$(iv) \ \psi(t, y) := \int_X \chi_y(x) P(t, dx) \ \text{and}$$

$$(v) \ \psi(t_1 + t_2, y) = \psi(t_1, y) \psi(t_2, y)$$

for each $t_1 \neq t_2 \in T$ and $y \in X^*$, $x_1 \in X$.  

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4 Non-Archimedean stochastic processes.

4.1. Remark and definition. A measurable space $(\Omega, F)$ with a normalised non-negative measure $\lambda$ on a $\sigma$-algebra $F$ of a set $\Omega$ is called a probability space and is denoted by $(\Omega, F, \lambda)$. Points $\omega \in \Omega$ are called elementary events and values $\lambda(S)$ probabilities of events $S \in F$. A measurable map $\xi : (\Omega, F) \to (X, \mathcal{B})$ is called a random variable with values in $X$, where $\mathcal{B}$ is the $\sigma$-algebra of a locally $K$-convex space $X$. The random variable $\xi$ induces a normalized measure $\nu_\xi(A) := \lambda(\xi^{-1}(A))$ in $X$ and a new probability space $(X, \mathcal{B}, \nu_\xi)$. We take $X = C^0(T, H)$ (see §2.1) and the $\sigma$-algebra $\mathcal{B}$ which is the subalgebra of the Borel $\sigma$-algebra $Bf(X)$ of $X$, where $H$ is a Banach space over $K$, $T = B(K, t_0, R) =: B_R$, $0 < R < \infty$, $K$ is the local field. A random variable $\xi : \omega \mapsto \xi(t, \omega)$ with values in $(X, \mathcal{B})$ is called a (non-Archimedean) stochastic process on $T$ with values in $H$.

Events $S_1, ..., S_n$ are called independent in total if $P(\Pi_{k=1}^n S_k) = \Pi_{k=1}^n P(S_k)$. $\sigma$-Subalgebras $F_k \subset F$ are said to be independent if all collections of events $S_k \in F_k$ are independent in total, where $k = 1, ..., n, n \in \mathbb{N}$. To each collection of random variables $\xi_\gamma$ on $(\Omega, F)$ with $\gamma \in \Upsilon$ is related the minimal $\sigma$-algebra $F_\Upsilon \subset F$ with respect to which all $\xi_\gamma$ are measurable, where $\Upsilon$ is a set. Collections $\{\xi_\gamma : \gamma \in \Upsilon_j\}$ are called independent if such are $F_{\Upsilon_j}$, where $\Upsilon_j \subset \Upsilon$ for each $j = 1, ..., n, n \in \mathbb{N}$.

4.2. Definition. We define a (non-Archimedean) stochastic process $w(t, \omega)$ with values in $H$ as a stochastic process such that:

(i) the differences $w(t_3, \omega) - w(t_2, \omega)$ and $w(t_2, \omega) - w(t_1, \omega)$ are independent for each chosen $\omega$, $(t_1, t_2)$ and $(t_3, t_4)$ with $t_1 \neq t_2, t_3 \neq t_4$, either $t_1$ or $t_2$ is not in the two-element set $\{t_3, t_4\}$, where $\omega \in \Omega$;

(ii) the random variable $\omega(t, \omega) - \omega(u, \omega)$ has a distribution $\mu^{F_{t,u}}$, where $\mu$ is a probability measure on $C^0(T, H)$, $\mu^g(A) := \mu(g^{-1}(A))$ for $g \in C^0(T, H)^*$ and each $A \in \mathcal{B}$, a continuous linear functional $F_{t,u}$ is given by the formula $F_{t,u}(w) := w(t, \omega) - w(u, \omega)$ for each $w \in L^q(\Omega, F, \lambda; C^0_0(T, H))$, where $1 \leq q \leq \infty$, $C^0_0(T, H) := \{f : f \in C^0(T, H), f(t_0) = 0\}$ is the closed subspace of $C^0(T, H)$.

(iii) we also put $w(0, \omega) = 0$, that is, we consider a Banach subspace $L^q(\Omega, F, \lambda; C^0_0(T, H))$ of $L^q(\Omega, F, \lambda; C^0(T, H))$, where $\Omega \neq \emptyset$.

This definition is justified by the following theorem.

4.3. Theorem. There exists a family of pairwise inequivalent (non-Archimedean) stochastic processes on $C^0_0(T, H)$ of the cardinality $\mathfrak{c}$, where
c := card(\mathbb{R}).

**Proof.** Since H is over the local field, then H has a projection \( \pi_0 \) on its Banach subspace \( H_0 \) of separable type over K (see its definition in [38]), that is, \( H_0 \) is isomorphic with \( c_0(\alpha, K) \) with countable \( \alpha \). Therefore, a \( \sigma \)-additive measure \( \mu_0 \) on \((H_0, Bf(H_0))\) induces a \( \sigma \)-additive measure \( \mu \) on \((H, \pi_0^{-1}[Bf(H_0)])\), where \( \pi_0^{-1}[Bf(H_0)] := \{ \pi_0^{-1}(A) : A \in Bf(H_0) \} \). Therefore, it is sufficient to consider the case of \( H \) of separable type over K.

If \( \omega \) is the real-valued nonnegative Haar measure on \( K \) with \( \omega(B(K, 0, 1)) = 1 \), then \( \omega \) has not any atoms, since it is defined on \( Bf(K) \), each singleton \( \{x\} \) is the Borel subset and \( \omega(y + A) = \omega(A) \) for each \( A \in Bf(K) \).

Indeed, if \( \omega \) would have some atom \( E \), then \( \omega \) would be a singleton, since \( K \) is the complete separable metric space and for each disjoint \( \omega \)-measurable subsets \( A \) and \( S \) in \( E \) either \( \omega(A) = \omega(E) > 0 \) with \( \omega(S) = 0 \) or \( \omega(S) = \omega(E) > 0 \) with \( \omega(A) = 0 \). But \( \sum_{y \in K} \omega(y + \{x\}) = \infty \), when \( \omega(\{x\}) > 0 \) for a singleton \( \{x\} \) (see Chapter VII in [4]). Therefore, each measure \( \mu_j(dx^j) = f_j(x^j)\omega(dx^j) \) on \( K \) has not any atom, since \( \omega \) has not any atom, where \( f_j \in L^1(K, Bf(K), \omega, R) \) (that is, \( f_j \) is \( \omega \)-measurable and \( \|f_j\|_{L^1(K)} := \int_K |f_j(x)|\omega(dx) < \infty \)) and \( \mu_j(K) = 1 \). Hence each measure \( \mu \) on \( C_0^0(T, H) \) has not any atom, when \( \mu(dx) = \bigotimes_{j=1}^\infty \mu_j(dx^j) \), where \( C_0^0(T, H) \) is isomorphic with \( c_0(\omega_0, K) \), \( x \in C_0^0(T, H) \), \( x = (x^j : j \in \omega_0) \), \( x^j \in K \), \( x = \sum_j x^je_j \), \( e_j \) is the standard orthonormal base in \( c_0(\omega_0, K) \), \( \omega_0 \) is the first countable ordinal, since \( K \) is the local field (see [38] and [22]).

Let on the Banach space \( c_0 := c_0(\omega_0, K) \) there is given an operator \( J \in L_1(c_0) \) such that \( Je_i = \nu_i e_i \) with \( \nu_i \neq 0 \) for each \( i \) and a measure \( \nu(dx) := f(x)\omega(dx) \), where \( f : K \to [0, 1] \) is a function belonging to the space \( L^1(K, \omega, R) \) such that \( \lim_{|x| \to \infty} f(x) = 0 \) and \( \nu(K) = 1 \), \( \nu(S) > 0 \) for each open subset \( S \) in \( K \), for example, when \( f(x) > 0 \) \( \omega \)-almost everywhere. In view of the Prohorov theorem there exists a \( \sigma \)-additive product measure \( \nu(dx) := \prod_{|i| = 1}^\infty \nu_i(dx^i) \) on the \( \sigma \)-algebra of Borel subsets of \( c_0 \), since the Borel \( \sigma \)-algebras defined for the weak topology of \( c_0 \) and for the norm topology of \( c_0 \) coincide, where \( \nu_i(dx^i) := f(x^i/v_i)\nu(dx^i/v_i) \) (see [4] [22]).

Let \( Z \) be a compact subset without isolated points in a local field \( K \), for example, \( Z = B(K, t_0, 1) \). Then the Banach space \( C^0(Z, K) \) has the Amice polynomial orthonormal base \( Q_m(x) \), where \( x \in Z \), \( m \in N \) := \{0, 1, 2, ...\} [4]. Each \( f \in C^0 \) has a decomposition \( f(x) = \sum_m a_m(f)Q_m(x) \) such that \( \lim_{m \to \infty} a_m = 0 \), where \( a_m \in K \). These decompositions establish the isometric isomorphism \( \theta : C^0(T, K) \to c_0(\omega_0, K) \) such that \( \|f\|_{C^0} = \max_m |a_m(f)| = \|f\|_{C^0} = \max_m |a_m(f)| = \infty \)
\|\theta(f)\|_{c_0}.

If \( H = c_0(\omega_0, K) \), then the Banach space \( C^0(T, H) \) is isomorphic with the tensor product \( C^0(T, K) \otimes H \) (see §4.R [38]). If \( J_i \in L_1(Y_i) \) is nondegenerate for each \( i = 1, 2 \), that is, \( \ker(J_i) = \{0\} \), then \( J := J_1 \otimes J_2 \in L_1(Y_1 \otimes Y_2) \) is nondegenerate (see also Theorem 4.33 [38]). If \( u_i \) are roots of basic polynomials \( Q_m \) as in [3], then \( Q_m(u_i) = 0 \) for each \( m \geq i \). The set \( \{u_i : i\} \) is dense in \( T \).

Put \( Y_1 = C^0(T, K) \) and \( Y_2 = H \) and \( J := J_1 \otimes J_2 \in L_1(Y_1 \otimes Y_2) \), where \( J_1 Q_m := \alpha_m Q_m \) such that \( \alpha_m \neq 0 \) for each \( m \) and \( \sum_i |\alpha_i| < \infty \). Take \( J_2 \) also nondegenerate. Then \( J \) induces a product measure \( \mu \) on \( C^0(T, H) \) such that \( \mu = \mu_1 \otimes \mu_2 \), where \( \mu_i \) are measures on \( Y_i \) induced by \( J_i \) due to Formulas (i, ii). Analogously considering the following Banach subspace \( C^0(T, H) := \{f \in C^0(T, H) : f(t_0) = 0\} \) and operators \( J := J_1 \otimes J_2 \in L_1(C^0(T, K) \otimes H) \) we get the measures \( \mu \) on it also, where \( t_0 \in T \) is a marked point.

For each finite number of points \( (t_1, ..., t_n) \subset T \) and \( (z_1, ..., z_n) \subset H \) there exists a closed subset \( C^0(T, H; (t_1, ..., t_n); (z_1, ..., z_n)) := \{f \in C^0(T, H) : f(t_i) = z_i; i = 1, ..., n\} \) such that \( C^0(T, H; (t_1, ..., t_n); (z_1, ..., z_n)) = (z_1, ..., z_n) + C^0(T, H; (t_1, ..., t_n); (0, ..., 0)) \) is the Banach subspace of finite codimension \( n \) in \( C^0(T, H) \). Therefore, (iii) \( \sigma \)-algebras \( F_{t_2, t_1}(Bf(H)) \) and \( F_{t_4, t_3}^{-1}(Bf(H)) \) are independent subalgebras in the Borel \( \sigma \)-algebra \( Bf(C^0(T, H)) \), when \( (t_1, t_2) \) and \( (t_3, t_4) \) satisfy Condition 4.2.(i).

Put \( P(t_1, x_1, t_2, A) := \mu(\{f : f(t_1) = x_1, f(t_2) \in A\}) \) for each \( t_1 \neq t_2 \in T \), \( x_1 \in H \) and \( A \in Bf(H) \). In view of (iii) we get, that \( P \) satisfies Conditions 3.2.(i—iv). By the above construction (and Proposition 3.3.1 also) the Markov quasimeasure \( \tilde{\mu}_{x_0} \) induced by \( \mu \) is bounded, since \( \mu \) is bounded, where \( x_0 = 0 \) for \( C^0(T, H) \). Let \( \Omega \) be a set of elementary events \( \omega := \{f : f \in C^0(T, H), f(t_i) = x_i, i \in \Lambda_\omega\} \), where \( \Lambda_\omega \) is a countable subset of \( N \), \( x_i \in H \), \( (t_i : i \in \Lambda_\omega) \) is a subset of \( T \) of pairwise distinct points. There exists the algebra \( \mathcal{U} \) of cylindrical subsets of \( C^0(T, H) \) induced by projections \( \pi_s : C^0(T, H) \to H^s \), where \( H^s := \prod_{t \in s} H_t \), \( s = (t_1, ..., t_n) \) are finite subsets of \( T \), \( H_t = H \) for each \( t \in T \). In view of the Kolmogorov theorem [3, 34, 22, 23] \( \tilde{\mu}_{x_0} \) on \( ((C^0(T, H), \tau_\omega), \mathcal{U}) \) induces the probability measure \( \lambda \) on \( (\Omega, Bf(\Omega)) \), where \( \tau_\omega \) is the weak topology in \( C^0(T, H) \).

Therefore, using product of measures we get examples of such measures \( \mu \) for which stochastic processes exist (see also Theorem 3.23, Lemmas 2.3, 2.5, 2.8 and §3.30 in [22]). Hence to each such measure on \( C^0(T, H) \) there corresponds the stochastic process. Considering all operators \( J := J_1 \otimes \)}
\( J_2 \in L_1(Y_1 \otimes Y_2) \) and the corresponding measures as above we get \( c_{\aleph_0} = c \) inequivalent measures by the Kakutani theorem II.4.1 [5] for each chosen \( f \).

**Note.** Evidently, this theorem is also true for \( C^0(T,H) \), that follows from the proof. If to take \( \nu \) with \( \text{supp}(\nu) = B(K,0,1) \), then repeating the proof it is possible to construct \( \mu \) with \( \text{supp}(\mu) \subset B(C^0(T,K),0,1) \times B(H,0,1) \).

In the weak topology inherited from \( C^0(T,H) \) the set \( B(C^0(T,K),0,1) \times B(H,0,1) \) is compact and the condition \( J \in L_1 \) may be dropped. Certainly such measure \( \mu \) can not be quasi-invariant relative to shifts from a dense \( K \)-linear subspace in \( C^0(T,H) \), but it can be constructed quasi-invariant relative to a dense additive subgroup \( G' \) of \( B(C^0(T,K),0,1) \times B(H,0,1) \), moreover, there exists \( \mu \) for which \( G' \) is also \( B(K,0,1) \)-absolutely convex.

**4.4.** We consider stochastic processes \( E \in L^r(\Omega,F,\lambda;C^0(T,L_0(H))) \) such that \( E = E(t,\omega) \), where \( 1 \leq v \leq \infty, 1 \leq r \leq \infty, t \in T = B(K,t_0,R) \) and \( \omega \in \Omega \) (see §2.14 and §4.2).

**Definition.** For \( L^r(\Omega,F,\lambda;C^0(T,L_0(H))) \) the non-Archimedean stochastic integral is defined by the following equation:

\[
(i) \ 1(E)(t,\omega) := (\hat{\omega} E(t,\omega) = \sum_{j=0}^{\infty} E(t_j,\omega)[w(t_{j+1},\omega) - w(t_j,\omega)],
\]

where \( w = w(t,\omega), t_j = \sigma_j(t) \) (see §2.1).

**4.5. Proposition.** The non-Archimedean stochastic integral is the continuous \( K \)-bilinear operator from \( L^r(\Omega,F,\lambda;C^0(T,L_0(H))) \otimes L^q(\Omega,F,\lambda;C^0(T,H)) \) into \( L^s(\Omega,F,\lambda;C^0(T,H)) \), where \( 1/q + 1/r = 1/s \) and \( 1 \leq r,q,s \leq \infty \).

**Proof.** It follows from Theorem 2.14, since \( (\hat{\omega} aE) + (\hat{\omega} bV) \) and \( (\hat{\omega} (aE + bV)) = (\hat{\omega} aE) + (\hat{\omega} bV) \) for each \( a,b \in K \), each \( w,y \in L^q(\Omega,F,\lambda;C^0(T,H)) \) and each \( E,V \in L^r(\Omega,F,\lambda;C^0(T,L_0(H))) \).

**4.6.** Consider a function \( f \) from \( T \times H \) into \( Y = c_0(\beta,K) \) satisfying conditions:

\( a \) \( f \in C^1(T \times H,Y) \);
\( b \) \( (\hat{f}^n)(t,x;h_1,...,h_n) \in C^0(T \times H^{n+1} \times K^n,Y) \) for each \( n \leq m \),
\( c \) \( (\hat{f}^{n+1})(t,x;h_1,...,h_n;\zeta_1,...,\zeta_n) = 0 \) for \( n = m+1 \),
\( d \) \( f(t,x) - f(0,x) = (\hat{g}(t,x) \) with \( g \in C^0(T \times H,Y) \), where \( 2 \leq m \in N, f = f(t,h), t \in T, x \in H; h_1,...,h_n \in H, \zeta_1,...,\zeta_n \in K; \hat{\omega} \) is the antiderivation operator on \( C^0(T,Y) \), \( (\hat{\omega} g(t,x) \) is defined for each fixed \( x \in H \) by \( t \in T \) such that \( (\hat{\omega} g(t,x) = \hat{\omega} g(u,x)|_{u=t} \) with \( u \in T \) (see §2.1 and
also about difference quotients ($\Phi^n f$) and spaces of functions of smoothness class $C^n$ in \cite{25, 28}.

Suppose $a \in L^r(\Omega, F; C^0(T, H))$, $w \in L^q(\Omega, F; C^0(T, H))$ and $E \in L^r(\Omega, F; C^0(T, L(H)))$, where $1/r + 1/q = 1$, $1 \leq r, q \leq \infty$, $a = a(t, \omega)$, $E = E(t, \omega)$, $t \in T$, $\omega \in \Omega$. A stochastic process of the type

\[(i) \quad \xi(t, \omega) = \xi_0(\omega) + (\tilde{P}_u a)(u, \omega)|_{u=t} + (\tilde{P}_w(u, \omega))E(u, \omega)|_{u=t}\]

is said to have a stochastic differential

\[(ii) \quad d\xi(t, \omega) = a(t, \omega)dt + E(t, \omega)dw(t, \omega), \text{ since } (\tilde{P}_t g)\prime(t) = g(t) \text{ for each } g \in C^0(T, H), \text{ where } \xi_0 \in L^s(\Omega, F; \lambda; H), t_0, t \in T, w(t_0, \omega) = 0. \text{ In view of Lemma 2.3, Theorem 2.14 and Proposition 4.5 } \xi \in L^s(\Omega, F; \lambda; C^0(T, H)).\]

Let $\tilde{P}_{u, \omega}$ denote the antiderivation operator $\tilde{P}_{(\xi_1, ..., \xi_{b+h})}$ given by Formula 2.1(4), where $\xi_1 = u, ..., \xi_b = u$, $\xi_{b+1} = w, ..., \xi_{b+h} = w$. Henceforth, it is used the notation

\[(iii) \quad \tilde{P}^m_{a, E} f(u, \xi(u, \omega)) := \sum_{k=1}^n (k!)^{-1} \sum_{l=0}^k \left(\begin{array}{c} k \\ t \end{array}\right) (\tilde{P}_{u^{k-l}}(u, \omega) f(x, \xi(u, \omega)) [((\tilde{\partial}^k f/\tilde{\partial}^k x)(u, \xi(u, \omega)) \circ (a_{\otimes(k-l)} \otimes E_{\otimes l}))])\]

for such operator, when it exists (see the conditions above and below), where $n \in \mathbb{N}$ or $n = \infty$.

**Theorem.** Let Conditions 4.6.(a – d), (i, ii) be satisfied, then

\[(iv) \quad f(t, \xi(t, \omega)) = f(t_0, \xi_0) + (\tilde{P}_u f_t(u, \xi(u, \omega))|_{u=t} + \tilde{P}^m_{a, E} f(u, \xi(u, \omega))|_{u=t}.\]

**Proof.** Let $\{u_k : k = 0, 1, ..., n\}$ be a finite $|\pi|^l$ net in $T$, that is, for each $t \in T$ there exists $k$ such that $|u_k - t| \leq |\pi|$, where $n = n(k) \in \mathbb{N}$, $\pi \in K$, $p^{-1} \leq |\pi| < 1$ and $|\pi|$ is the generator of the valuation group of $K$, since the ball $T$ is compact. We choose $t = u_n$ and $t_0 = u_0$. Denote by $\eta(t)$ a stochastic process $f(t, \xi(t, \omega))$. Then by the Taylor formula (see Theorem 29.4 \cite{39} and Theorem 2.9 \cite{21})

\[(v) \quad f(t, \xi(t)) - f(u, \xi(u)) = f_t^1(u, \xi(u))(t-u) + f^2_{u}(u, \xi(u)).(\Delta \xi) + (1/2)f''_{t,t}(u, \xi(u))(t-u)^2 + f''_{t,x}(u, \xi(u))((t-u), \Delta \xi) + (1/2)f^{''}_{x,x}(u, \xi(u))...(\Delta \xi, \Delta \xi) + \{(\Phi^2 f)(u, \xi(u);(t-u), (t-u); 1, 1) - (1/2)f''_{t,t}(u, \xi(u))(t-u)^2\}

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so the problem reduces to the consideration of functions $f$ independent from $t$ for each $x, x'$ for each $\xi$.

This theorem is accomplished for each $\xi$ since $\lim_{v \to \infty} \omega(t, \xi, \omega)\xi^{(\infty)}$ where $\Delta = \xi(t) - \xi(u)$, for a brevity we denote $\xi(t) = \xi(t, \omega)$ and $w(t) := w(t, \omega)$ for a chosen $\omega$. If $t_n = \sigma_n(t)$ for each $n = 0, 1, 2, \ldots$, then by Formulas (i) and 2.1.(4):

$$(vi) \xi(t_{n+1}, \omega) - \xi(t_n, \omega) = a(t_n, \omega)(t_{n+1} - t_n) + E(t_n, \omega)(w(t_{n+1}, \omega) - w(t_n, \omega)),$$

where $\{\sigma_n : n = 0, 1, 2, \ldots\}$ is the approximation of the identity in $T$.

From Condition (d) it follows that $(\partial f(t, x)/\partial t) = g(t, x) = (P_t g)'_t$ and $P_t f' t(x) = f(t, x) - f(0, x)$, which also leads to dissappearance of terms $\partial^{n+m} f(t, x) / \partial t^n \partial x^m$ from Formula (iv) for each $b, m$ such that $1 \leq b$ and $2 \leq m + b$. Now we approximate $f(t, x)$ by functions of the form $\sum_j \phi_j(t) \psi_j(x)$, so the problem reduces to the consideration of functions $f(x)$ which are independent from $t$. Due to Conditions (i, ii) it is possible to put $\xi(t, \omega) = \xi_0(\omega) + a(\omega)(t - t_0) + E(\omega)[w(t) - w(t_0)]$. By the Taylor formula:

$$(vii) f(x) = f(x_0) + \sum_{n=1}^{m} (n!)^{-1} f^{(n)}(x_0) (x - x_0)^{\otimes n}$$

for each $x, x_0 \in H$, since $\Phi^{m+1} f = 0$. Put $t_k = \sigma_k(t)$ for each $k = 0, 1, 2, \ldots$, then $\eta(t) - \eta(t_0) = \sum_{j=0}^{\infty} f(\xi_{j+1}) - f(\xi_j)$, where $\xi_j := \xi(t_j)$, since $\lim_{j \to \infty} \xi_j = \xi$. Then each term $f(\xi_{j+1}) - f(\xi_j)$ can be expressed by Formula (vii) due to Condition (b). On the other hand, $(\xi_{j+1} - \xi_j) = a(\omega)(t_{j+1} - t_j) + E(\omega)[w(t_{j+1}) - w(t_j)]$ as the particular case of Formula (vi).

From Formulas 2.1.(4), (v - vii) and Theorem 2.14 we get the statement of this theorem.

**4.7 Corollary.** If Conditions 4.6(a, d, i, ii) are satisfied, 4.6(b) is accomplished for each $n \in \mathbb{N}$ and

$$(c') \lim_{n \to \infty} \| (\tilde{\Phi}_n f)(t, x; h_1, \ldots, h_n; \zeta_1, \ldots, \zeta_n) \|_{C^0(T \times B(H, 0, R_1))^{n+1} \times B(K^{n+1}, 0, R_1), Y} = 0$$

for each $0 < R_1 < \infty$, then

$$(i) f(t, \xi(t, \omega)) = f(t_0, \xi_0) + (P_t f' t(u, \xi(t, u, \omega)))|_{u=t} + (\tilde{P}_a^{\infty} f(u, x))|_{u=t}.$$

**Proof.** From the proof of Theorem 4.6 we get a function $f(x)$ for which

$$(ii) f(x) = f(x_0) + \sum_{n=1}^{\infty} (n!)^{-1} f^{(n)}(x_0) (x - x_0)^{\otimes n}$$

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due to Condition \((c')\). In view of Theorem 2.14

\[
\lim_{m \to \infty} (m!)^{-1} \sum_{l=0}^{m} \binom{m}{l} (\tilde{P}_{u_{m-l},w(u,\omega)}^m)(\tilde{P}_{u_{m-l},w(u,\omega)}^m)(\xi(u,\omega))_0^n \times E^{\otimes l}) = 0.
\]

Approximating \(f(x)\) by the Taylor formula up to terms \(\tilde{\Phi}^m f\) by finite sums and taking the limit while \(m\) tends to the infinity one deduces Formula (iv) from Formula 4.6(iv), since for each chosen \(\omega \in \Omega\) functions \(a(t,\omega)\) and \(w(t,\omega)\) are bounded on the compact ball \(T\).

**4.8. Theorem.** Let \(f(u,x) \in C^\infty (T \times H, Y)\) and

\[(i) \lim_{n \to \infty} \max_{0 \leq l \leq n} \| (\tilde{\Phi}^n f)(t,x; h_1, ..., h_n); \]

\[
\zeta_1, ..., \zeta_n \| C^0(T \times B(K,0,r) \times B(H,0,1)^{n-1} \times T(0, R^n), Y) = 0
\]

for each \(0 < R_1 < \infty\), where \(h_j = e_1 \) and \(\zeta_j \in B(K,0,r)\) for variables corresponding to \(t \in T = B(K, t_0, r)\) and \(h_j \in B(H,0,1), \zeta_j \in B(K,0,R_1)\) for variables corresponding to \(x \in H\), then

\[(ii) f(t,\xi(t,\omega)) = f(t_0,\xi_0) + \sum_{m+b \geq 1, 0 \leq m \in \mathbb{Z}, 0 \leq b \in \mathbb{Z}} (m+b)!^{-1} \sum_{l=0}^{m+b} \binom{m+b}{l} \binom{m}{l} \]

\[
(\tilde{P}_{u_{m+b-l},w(u,\omega)}^m)(\partial^{m+b} f/\partial u^b \partial x^m)(u,\xi(u,\omega)) \circ (I^{\otimes b} \otimes a^{\otimes (m-l)} \otimes E^{\otimes l}))|_{u=t}.
\]

**Proof.** In view of the Taylor formula we have (see [21, 39, 40])

\[(iii) f(t,x) = f(t_0,x_0) + \sum_{m+b=1}^{k} \binom{m+b}{m} (\tilde{\Phi}^{m+b} f/\partial u^b \partial x^m)(t_0,x_0)
\]

\[
(t-t_0)^b.(x-x_0)^{\otimes m} + \sum_{m+b=k+1} \binom{k+1}{m} (\tilde{\Phi}^{k+1} f/\partial u^b \partial x^m)(t_0,x_0)(t-t_0)^b.(x-x_0)^{\otimes m}, \]

\[
-((k+1)!)^{-1}(\partial^{k+1} f/\partial u^b \partial x^m)(t_0,x_0)(t-t_0)^b.(x-x_0)^{\otimes m}
\]

for each \(k \in \mathbb{N}\). In view of Condition (i), Formulas (iii), 2.1.(4), 4.6.(vi) we get Formula (ii) (see the proof of Theorem 4.6).
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