Quantum-classical equivalence and ground-state factorization

JAHANFAR ABOUIE\textsuperscript{1,3(a)(c)} and REZA SEPEHRINIA\textsuperscript{2,3(b)(c)}

\textsuperscript{1} Department of Physics, Institute for Advanced Studies in Basic Sciences (IASBS) - Zanjan 45137-66731, Iran
\textsuperscript{2} Department of Physics, University of Tehran - Tehran 14395-547, Iran
\textsuperscript{3} School of Physics, Institute for Research in Fundamental Sciences, IPM - Tehran 19395-5531, Iran

received 8 November 2015; accepted in final form 15 February 2016
published online 29 February 2016

PACS 75.10.Pq – Spin chain models
PACS 75.10.Hk – Classical spin models
PACS 03.67.Mn – Entanglement measures, witnesses, and other characterizations

Abstract – We have performed an analytical study of quantum-classical equivalence for quantum \textit{XY}-spin chains with arbitrary interactions to explore the classical counterpart of the factorizing magnetic fields that drive the system into a separable ground state. We demonstrate that the factorizing line in the parameter space of a quantum model is equivalent to the so-called natural boundary that emerges in mapping the quantum \textit{XY}-model onto the two-dimensional classical Ising model. As a result, we show that the quantum systems with the non-factorizable ground state could not be mapped onto the classical Ising model. Based on the presented correspondence we suggest a promising method for obtaining the factorizing field of quantum systems through the commutation of the quantum Hamiltonian and the transfer matrix of the classical model.

Copyright © EPLA, 2016

Introduction. – One of the essential challenges in condensed-matter physics is that of solving many-body interacting systems. One major class of these systems are quantum spin models which, despite their simplicity, capture various complex physical phenomena. Even among these idealized models, we rarely and only in low dimensions encounter solvable cases. Therefore, in many systems, it is very valuable to find even a single eigenstate of the Hamiltonian. It is found that in general non-exactly solvable models admit an exact \textit{factorized} ground state for special values of Hamiltonian parameters, e.g., external transverse magnetic field [1–6]. The analysis of quantum entanglement contained in the ground state has revealed additional features of the factorization point. Unlike the standard magnetic observable the entanglement displays an anomalous behavior at the factorizing field (in addition to the critical field) and vanishes at this point. Across this point the system undergoes an entanglement phase transition [7,8]. Furthermore, the quantum discord, another measure of quantum correlations, exhibits scaling behavior close to the factorization point [9–12] which is of collective nature but different from a quantum phase transition which is accompanied by a change of symmetry. For finite systems, it is also shown that the ground state remains entangled as the factorizing field is approached and undergoes a parity transition across this point [13–15].

A fairly general approach which allows us to make precise statements about the ground state of quantum systems is mapping the $d$-dimensional quantum system onto a $(d + 1)$-dimensional classical system. Then the different properties of the quantum model like orders, correlations and response functions and scaling behavior near quantum-critical points can be studied through the relation to their classical counterparts. However, it turns out that mapping onto a specific classical model holds only for a restricted region of the parameter space of the quantum model. For instance, the ground state of the spin-$\frac{1}{2}$ quantum \textit{XY}-chain with uniform interactions in the presence of a magnetic field is equivalent to a two-dimensional (2D) rectangular Ising model but this equivalence is restricted to the region outside a circle which is called natural boundary [16]. This circular boundary has been also observed in studying rather different aspects of the \textit{XY}-model. One of them is the spin-spin correlation of the quantum \textit{XY}-chain which exhibits non-oscillatory (classical) asymptotic behavior outside this circle, whereas it has oscillatory (quantum) asymptotic behavior inside it [17]. Another property of this circle was found in the investigation of the ground-state entanglement of the quantum \textit{XY}-model. Along this circle, the entanglement vanishes...
in correspondence with an exactly factorized state \[7\] and across it an entanglement phase transition from odd- to even-parity ground state occurs \[8\].

In this letter, motivated by these observations, we perform a detailed investigation to find out the relationship between the factorization of the ground state of quantum XY-chains with arbitrary interactions and their mappability onto the classical Ising model. We show that the above-mentioned properties are not restricted to the particular case of uniform couplings and are maintained for the general XY-chains with arbitrary interactions. We obtain the natural boundary for the general case and show that this boundary coincides with the factorizing line of the quantum chain. We apply our results to several examples which do or do not have a factorizing field. We find that there is no equivalent Ising model for the quantum models which do not possess a factorized ground state.

**General XY-model and ground-state factorization.** – The Hamiltonian of a spin-$\frac{1}{2}$ quantum XY-chain in the presence of a magnetic field can be written as

\[
H_q = - \sum_{i,r} (J^x_{i,i+r} \sigma^x_i \sigma^x_{i+r} + J^y_{i,i+r} \sigma^y_i \sigma^y_{i+r}) - \sum_i h_i \sigma^z_i, \tag{1}
\]

where $\sigma^{x,y}$ are Pauli matrices and $h_i$ is a general transverse magnetic field. The interaction between two spins depends on the position $i$ and the distance $r$ via $J^{x,y}_{i,i+r}$, where $\mu = x, y$ and they are assumed to be ferromagnetic ($J^{x,y}_{i,i+r} > 0$) or antiferromagnetic ($J^{x,y}_{i,i+r} < 0$) such that there is no frustration in the system. The above Hamiltonian possesses a factorized state as $|FS\rangle = \otimes |\theta_i\rangle$ at the factorizing fields \[1,4\]:

\[
h^F_i = \cot \theta_i \sum_r (J^x_{i,i+r} \sin \theta_{i+r} + J^y_{i,i+r} \sin \theta_{i-r}), \tag{2}
\]

where $|\theta_i\rangle = \cos \frac{\theta_i}{2} |+\rangle + \sin \frac{\theta_i}{2} |->\rangle$, and $\sigma^z_i |\pm\rangle = |\pm\rangle$.

Here, $\theta_i \in (-\pi/2, \pi/2)$ is the acute angle of the spin $\sigma_i$ with the $+z$-axis given by relation

\[
\cos \theta_i \cos \theta_{i+r} = J^y_{i,i+r}/J^x_{i,i+r}. \tag{3}
\]

Equation (3) requires $|J^y_{i,i+r}/J^x_{i,i+r}| \leq 1$ which ensures that the factorized state $|FS\rangle$ is the ground state \[4\]. For chains with nearest-neighbor interactions, $J^y_{i,i+r} = J^x_{i,i+r} = \delta_{i,1}$, the factorizing field (2) is simplified to

\[
h^F_i = \cot \alpha_i (\sin \alpha_{i-1} |J^x_i| + \sin \alpha_i |J^x_{i-1}|), \tag{4}
\]

where we have defined $\theta_i = \xi_i \alpha_i$, with $\alpha_i \in (0, \pi/2)$ and $\xi_i$ is the sign of $\theta_i$ satisfying $\xi_i = \text{sgn} J^x_{i-1} \xi_{i-1}$. The Hamiltonian (1) has the global $z$-parity symmetry and the factorized ground state has a degenerate partner state, $| - FS\rangle = \otimes | - \theta_i\rangle$.

**Mapping onto 2D Ising model.** – By relating a transfer matrix associated with the classical system to the Hamiltonian of its corresponding quantum model, it has been proved by Suzuki \[16\] that the two-dimensional Ising model is equivalent to the ground state of the quantum XY-chain with uniform interactions in the presence of a magnetic field, under appropriate relations among coupling parameters appearing in the two Hamiltonians. His remarkable observation was that the Hamiltonian of the XY-chain commutes with the transfer matrix of a rectangular Ising model for a certain range of parameters. The two-dimensional classical Ising model is expressed by the Hamiltonian

\[
H_I = - \sum_{i,j} (J^h_{i,j+1} \sigma^x_i \sigma^x_{i+1} + J^v_{i+1,j} \sigma^y_i \sigma^y_{i+1}), \tag{5}
\]

where $J^h$ and $J^v$ are horizontal and vertical couplings, respectively, and $\sigma^z_i = \pm 1$. For the uniform couplings $J^h_{i,j+1} = J^h$ and $J^v_{i+1,j} = J^v$, the transfer matrix commutes with the Hamiltonian of the XY-chain ($H_q$) with nearest-neighbor interactions, $J^y_{i,i+r} = J^x_{i,i+r}$, and uniform magnetic field $h_i = h$, if the coupling parameters are related via the following equations:

\[
J^y/J^x = \tan \theta^2 K^{s*}, \quad h = 2J^x \tanh K^{s*} \coth 2K^h, \tag{6}
\]

where $K^{v,h} = J^{v,h}/k_B T$, tanh $K^{s*} = \exp(-2K^v)$ and $T$ is the temperature of the Ising system. As a result of the commutation of $H_q$ and $V$, they can be diagonalized simultaneously and have a common set of eigenvectors. In particular, the ground state of the quantum Hamiltonian coincides with the eigenvector of the transfer matrix with the maximum eigenvalue. This brings out several interesting results: i) The critical temperature $T_c$ of the Ising model corresponds to the critical field of the XY-model.

![Fig. 1: (Color online) The mapping of the quantum XY-chain with uniform interactions onto a classical Ising model. The region outside the circle is equivalent to a 2D square lattice Ising model with nearest-neighbor interactions. The region inside the circle may correspond to the Ising model with complex interactions. The unit circle is the factorizing line which is mapped onto the Ising model with an infinite horizontal coupling ($K^h \to \pm \infty$). The critical fields $h_c = \pm 2J$ correspond to critical lines, sinh $2K^{s*} \sinh 2K^h = \pm 1$, in the Ising model. The blue lines represent the Ising model in a transverse field (ITF).]
Quantum-classical equivalence and ground-state factorization

Boundary of equivalence and factorization. – The above results, eqs. (6), have been generalized recently to the XY-chains with randomness and free boundary [19]. The conditions for the commutation of the Hamiltonian of the XY-chain, eq. (1) with nearest-neighbor interactions, and the transfer matrix of the Ising model

\[ V = V_1^{1/2}V_2V_1^{1/2}, \]

where

\[ V_1 = \Pi_i(e^{K^{y^z}_i} / \cosh K^{x^z}_i)e^{\sum_i K^{x^z}_i \sigma_i^z}, \quad V_2 = e^{\sum_i K^{x^z}_i \sigma_i^z \sigma_{i+1}^z}, \]

with \( K^{y^z}_i = J^{y^z}_i / k_B T \) and \( \tanh K^{x^z}_i = \exp(-2K^{x^z}_i) \), are given in the appendix (eqs. (A.1)–(A.5)). The solutions for these equations have been presented in ref. [19] for special cases. Here we determine the domain of the existence of a solution for them. For this purpose it is instructive to deduce the following equations by adding and subtracting eqs. (A.1) and (A.2) or eqs. (A.3) and (A.4),

\[ h_i = \pm J_{i-1}^{\pm} \coth(K^h_i \pm K^h_i), \]

where

\[ J_{i-1}^{\pm} = \sinh K^{x^z}_i (\text{sech}K^{x^z}_i \pm J^{x^z}_i \text{sech}K^{x^z}_{i-1}J^{x^z}_{i-1}). \]

We have substituted \( J^y \) in terms of \( J^x \) using the following relation:

\[ \tanh K^{x^z}_i \tanh K^{x^z}_{i+1} = J^y_i / J^{x^z}_i, \]

obtained from eq. (A.5). The first constraint imposed on the parameters of the XY-chain, as is clearly seen from eq. (10) and the positivity of \( K^{x^z}_i \), is that \( J^y_i \) and \( J^{x^z}_i \) should have the same sign and \( |J^y_i / J^{x^z}_i| \leq 1 \). For given exchange couplings \( J^z \) and \( J^y \) the vertical couplings of the Ising model, \( K^y_i \), will be fixed by eq. (10), while the horizontal couplings, \( K^h_i \), can be varied to find the allowed values of the magnetic field. However, the couplings of the neighboring links should satisfy the constraint

\[ J_{i-1}^{\pm} / J_{i-1}^{\pm} = \coth(K^h_i - K^h_i), \]

obtained by eliminating \( h_i \) in eqs. (8). The range of the hyperbolic functions in eqs. (8) and (11), place the following bounds on the values of the magnetic field at each site:

\[ |h_i| \geq \begin{cases} |J_{i-1}^{\pm}|, & \text{if } |J^{x^z}_{i-1}| \geq |J_{i-1}^{\pm}|, \\ |J_{i-1}^{+}|, & \text{if } |J^{x^z}_{i-1}| \leq |J_{i-1}^{\pm}|. \end{cases} \]

The equality in (12) gives the boundary of the equivalence; however a further care is needed on the signs of magnetic fields. By applying eq. (11) for two successive sites \( i \) and \( i + 1 \) of the XY-chain we find that there must be a relation between the signs of the magnetic fields \( h_i \) and \( h_{i+1} \) as \( \text{sgn} h_{i+1} = \text{sgn} h_i \). Finally the equation for the boundary reads as follows:

\[ h_i^+ = \sinh K^{x^z}_{i+1}(\text{sech}K^{x^z}_{i+1}J^{x^z}_i + \text{sech}K^{x^z}_{i-1}J^{x^z}_{i-1}). \]
In the special case of the uniform XY-chain $K_u^v = K^{uv}$ and the above equation simplifies to $h_b^i = 2J\sqrt{1 - \gamma^2}$ which is the circular boundary shown in fig. 1.

Now we show that the corresponding coefficients in eqs. (13) and 4 and, therefore, $h_b^i$ and $h_f^i$ are identical. Since, $K_u^v > 0$, sinh $K_u^v \in (0, \infty)$, and sech $K_u^v \in (0, 1)$ we can rewrite the hyperbolic functions in (13) in terms of trigonometric functions as sech $K_u^v = \sinh a_i$ and sinh $K_u^v = \cos a_i$ with $a_i \in (0, \pi/2)$. According to (10) the angles $a_i$ satisfy the equation $\cos a_i \cos a_{i+1} = J_p^i/J_e^i$ which is the same as the equation of $\alpha_i$s and, therefore, we conclude $a_i = \alpha_i$. In other words, the equivalence boundary is the factorizing line of the quantum XY-chain,

$$h_b^i = h_f^i. \quad (14)$$

In the following we give more explicit results and solutions to the general equations within specific examples.

Uniform factorizing field. In addition to the uniform XY-chain, there are other spin models which also have a uniform factorizing field even though they have non-uniform exchange couplings. Among them the spin-(1/2) chains with ferro-antiferro ($f-a$) and antiferro-antiferro ($a-a$) bond alternations have been extensively investigated in the field of quantum magnetism and quantum information due to their rich field-induced quantum phases such as the Luttinger liquid phase [20-22], the dimerized phase and the symmetry protected topological phases [14,23]. The exchange interactions in these systems are $J_{u}^{f} = J_{u}^{a} < 0$ and $J_{u}^{f+1} = J_{u}^{f} > 0$ for $a-f$ and $J_{u}^{a} = J_{u}^{a} < 0$ and $J_{u}^{a+1} = J_{u}^{a} < 0$ for the $a-a$ models. For systems with $a-f$ bond alternations, the factorized ground state is $|FS\rangle = |\alpha, \alpha, -\alpha, -\alpha, \ldots\rangle$ with $\cos \alpha = \sqrt{J_{u}^{f}/J_{u}^{f}}$ at the factorizing field,

$$h_f^i = \sqrt{J_{u}^{f}\ J_{u}^{f}}(1 + |J_{u}^{f}/J_{u}^{f}|). \quad (15)$$

In systems with $a-a$ bond alternations, the factorized ground state is the Néel state, $|FS\rangle = |\alpha, -\alpha, \alpha, -\alpha, \ldots\rangle$, with $\cos \alpha = \sqrt{J_{u}^{a}/J_{u}^{a}}$ and the factorizing field is

$$h_f^i = \sqrt{J_{u}^{a}\ J_{u}^{a}}(1 + |J_{u}^{a}/J_{u}^{a}|). \quad (16)$$

The bond alternating quantum spin chains are mapped to a 2D Ising model with uniform vertical interaction, $K_u^v = K^v$, and horizontal $a-f$ bond alternations, $K_{u}^{2i} = K^u$ and $K_{u}^{2i+1} = K^f$, or $a-a$ bond alternations, $K_{u}^{2i} = K^u$ and $K_{u}^{2i+1} = K^a$. In these cases the equivalence boundary (13) is simplified to

$$h_b^i = \tanh K^{uv} |J_{u}^{a}|(1 + |J_{u}^{a}/J_{u}^{a}|), \quad (17)$$

where $\tanh K^{uv} = \sqrt{J_{u}^{a}/J_{u}^{a}}$ and $J_{u}^{a} = J_{u}^{f}$ or $J_{u}^{a}$. These boundaries exactly coincide with the factorizing lines, (15) and (16) of the XY-chains with bond alternations.

Non-uniform factorizing field. In many cases, we cannot get a factorized ground state by applying a uniform magnetic field. Some of them, for example, trimerized and tetramerized spin chains [24,25], have a factorized ground state in the presence of a non-uniform field which is the sum of a uniform field and a staggered field [6]. In a tetramerized spin chain with the exchange couplings $J_{u}^{f} = J_{u}^{f+1} = J_{u}^{a} < 0$ and $J_{u}^{f+2} = J_{u}^{a+3} = J_{u}^{f} > 0$, the factorizing fields are given by $h_b^i = h_b^{i+2} = h_f^i$, $h_b^i = h_f^{i+1} + h_f^i$, and $h_f^{i+2} = h_f^{i} - h_f^{i+1}$, where

$$h_f^i = \sqrt{J_{u}^{a}\ J_{u}^{f}}(1 + |J_{u}^{a}/J_{u}^{a}|), \quad \quad h_b^i = \sqrt{J_{u}^{a}\ J_{u}^{f}}(1 - |J_{u}^{a}/J_{u}^{a}|). \quad (18)$$

The corresponding factorized ground state in this case is $|\alpha, \alpha, \alpha, -\alpha, \ldots\rangle$, where $\cos \alpha = \sqrt{J_{u}^{a}/J_{u}^{a}}$. The equivalent classical model is a 2D tetramerized Ising model with $K_u^v = K_{u}^{2i+1} = K_{u}^{2i+2} = K_{u}^{2i+3} = K_{u}^{f}$ and $K_u^v = K^v$. Using eq. (13) we find that $h_b^i = h_b^{i+2} = \tanh K^v ((J_{u}^{a}) + J_{u}^{f})$, $h_b^i = 2 \tanh K^v |J_{u}^{f}||$, and $h_b^i = 2 \tanh K^v |J_{u}^{f}||$. By replacing $\tanh K^v$ with $\sqrt{J_{u}^{a}/J_{u}^{a}}$ from eq. (10) and using eqs. (18) one can see that these boundary fields are exactly the factorizing fields of the tetramerized spin chain.

Non-factorizable models. There are models in which it is not possible to factorize the ground state by applying any values of the magnetic fields. For example, the model with exchange interactions $J_{u}^{f} = 1 - (1)^{i}\delta$ and $J_{u}^{a} = 1 - (1)^{i}\delta$, where $0 < \delta < 1$ does not possess a factorizing field because $|J_{u}^{f}| < |J_{u}^{a}|$ at even links while $|J_{u}^{a}| > |J_{u}^{a}|$ for odd links. This property does not permit a real solution for angles $\theta_i$ in eq. (3). On the other hand, eq. (10) imposes a similar limitation in mapping this model onto the classical Ising model. This means that there is a relationship between the non-factorizability of the ground state and the impossibility of the mapping onto the Ising model. It should be mentioned that the model with $|J_{u}^{f}| > |J_{u}^{a}|$ in all links could be transformed into the one with $|J_{u}^{f}| < |J_{u}^{a}|$ by a global rotation of $\pi/2$ around the z-axis and, therefore, it does have a factorizing field. This transformation is not possible even if there is only one link with $|J_{u}^{f}| > |J_{u}^{a}|$ while all other links have $|J_{u}^{f}| < |J_{u}^{a}|$. In conclusion, the non-factorizability of the ground state of the quantum model implies the impossibility of the mapping onto the classical Ising model and, therefore, the absence of the equivalence boundary for such models.

Summary and conclusion. – In summary, we have provided a connection between the ground-state factorization of quantum XY-chains with arbitrary interactions and their mappability onto a two-dimensional classical Ising model. By making use of Suzuki’s criterion for equivalence, i.e. the commutativity of $H_q$ and $V$, we obtained the natural boundary for the general XY-model and showed that it is identical to the factorizing line
in the parameter space. This allows us to discuss the counterparts of the factorization and the vanishing of the entanglement of the ground state in the behavior of the equivalent classical model. We have applied our results to several examples and shown that in the cases with a factorizing field, this line coincides with the boundary obtained from mapping to the classical model while the cases with no factorizing field do not have an equivalent Ising model. Here we have considered the infinite chains; however, the results are also applicable to finite chains with different boundary conditions. The presented correspondence suggests a promising approach for obtaining the factorizing field of quantum systems. Unlike the standard approach [1] this method is not restricted to the systems with two-particle interactions and frustration. Another benefit of this approach for the future work is that it might lead to a criterion for factorizability of the ground state of the Hamiltonians based on their symmetries since the commutation with the transfer matrix indicates some hidden symmetry of the Hamiltonian. The generalization of the results to systems with XYZ interactions, frustration, and higher dimensions will be investigated in a future work.

***

RS would like to acknowledge the financial support of University of Tehran for this research under grant No. 28957/01/1.

Appendix: mapping conditions. – The conditions for commutation of the Hamiltonian of the XY-chain, eq. (1) with nearest-neighbor interactions, and the transfer matrix of the Ising model, eq. (7), are given by [19]

\[ h_0 \partial_{x}^i - 2[J_1^{-1}(K_{i-1}^h \bar{q}_i^h + K_{i}^h \bar{q}_i^h) + J_1^+(K_{i-1}^h \bar{q}_i^h + K_{i}^h \bar{q}_i^h)] = h_i, \]
\[ h_0 \partial_{y}^i - 2[J_1^{-1}(K_{i-1}^h \bar{q}_i^h + K_{i}^h \bar{q}_i^h) + J_1^+(K_{i-1}^h \bar{q}_i^h + K_{i}^h \bar{q}_i^h)] = 0, \]
\[ J_2^{-1}(\bar{p}_i^h + 1) - 2h_0(K_{i-1}^h \bar{q}_i^h + K_{i}^h \bar{q}_i^h) + J_1^+ \bar{p}_i^h = 0, \]
\[ J_2^+(\bar{p}_i^h + 1) - 2h_0(K_{i-1}^h \bar{q}_i^h + K_{i}^h \bar{q}_i^h) + J_1^- \bar{p}_i^h = 0, \]
\[ p_i^+ J_i^+ - p_i^- J_i^- = 0, \]

where

\[ p_i^\pm = (\cosh(K_{i}^h + K_{i+1}^h) + \cosh(K_{i}^h - K_{i+1}^h))/2, \]
\[ q_i^\pm = (\sinh(K_{i}^h + K_{i+1}^h) + \sinh(K_{i}^h - K_{i+1}^h))/2, \]
\[ \bar{p}_i^\pm = (\cosh(2(K_{i-1}^h + K_{i}^h) + \cosh(2(K_{i-1}^h - K_{i}^h)))/2, \]
\[ \bar{q}_i^\pm = (\sinh(2(K_{i-1}^h + K_{i}^h) + \sinh(2(K_{i-1}^h - K_{i}^h)))/2, \]

and \( \sinh x = \sinh x/x \).

REFERENCES

[1] Kurmann J., Thomas H. and Müller G., Physica A. Stat. Mech. Appl., 112 (1982) 235.
[2] Giampaolo S. M., Adesso G. and Illuminati F., Phys. Rev. Lett., 100 (2008) 197201.
[3] Giampaolo S. M., Adesso G. and Illuminati F., Phys. Rev. B, 79 (2009) 224344.
[4] Rossignoli R., Canosa N. and Matera J., Phys. Rev. A, 80 (2009) 062325.
[5] Rezai M., Langari A. and Abouie J., Phys. Rev. B, 81 (2010) 060401.
[6] Abouie J., Rezai M. and Langari A., Prog. Theor. Phys., 127 (2012) 315.
[7] Amico L. and Fazio R., J. Phys. A: Math. Theor., 42 (2009) 504001.
[8] Roscilde T., Verrucchi P., Fubini A., Haas S. and Tognetti V., Phys. Rev. Lett., 93 (2004) 167203.
[9] Chiberti L., Rossignoli R. and Canosa N., Phys. Rev. A, 82 (2010) 042316.
[10] Tomaselbo B., Rossini D., Hamma A. and Amico L., EPL, 96 (2011) 27002.
[11] Campbell S., Richens J., Gullo N. L. and Busch T., Phys. Rev. A, 88 (2013) 062305.
[12] Sarandy M. S., De Oliveira T. R. and Amico L., Int. J. Mod. Phys. B, 27 (2013) 1345030.
[13] Rossignoli R., Canosa N. and Matera J., Phys. Rev. A, 77 (2008) 052322.
[14] Canosa N., Rossignoli R. and Matera J. M., Phys. Rev. B, 81 (2010) 054415.
[15] Giorgi G. L., Phys. Rev. B, 79 (2009) 060405.
[16] Suzuki M., Prog. Theor. Phys., 46 (1971) 1337.
[17] Babouch E. and McCoy B. M., Phys. Rev. A, 3 (1971) 786.
[18] den Nijs M. in Phase Transitions and Critical Phenomena, edited by Domb C. and Lebowitz J. L., Vol. 12 (Academic Press) 1988, p. 264.
[19] Minami K., EPL, 106 (2014) 30001.
[20] Sakai T., J. Phys. Soc. Jpn., 64 (1995) 251.
[21] Abouie J. and Mahdavifar S., Phys. Rev. B, 78 (2008) 184437.
[22] Mahdavifar S. and Abouie J., J. Phys.: Condens. Matter, 23 (2011) 246002.
[23] Wang H. T., Li B. and Cho S. Y., Phys. Rev. B, 87 (2013) 054402.
[24] Gu B., Su G. and Gao S., Phys. Rev. B, 73 (2006) 134427.
[25] Rachel S. and Greiter M., Phys. Rev. B, 78 (2008) 134415.