Need for fully unintegrated parton densities

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Abstract
Associated with the use of conventional integrated parton densities are kinematic approximations on parton momenta which result in unphysical differential distributions for final-state particles. We argue that it is important to reformulate perturbative QCD results in terms of fully unintegrated parton densities, differential in all components of the parton momentum.

1 Introduction
Conventional parton densities are defined in terms of an integral over all transverse momentum and virtuality for a parton that initiates a hard scattering. While such a definition of an integrated parton density is appropriate for very inclusive quantities, such as the ordinary structure functions $F_1$ and $F_2$ in DIS, the definition becomes increasingly unsuitable as one studies less inclusive cross sections. Associated with the use of integrated parton densities are approximations on parton kinematics that can readily lead to unphysical cross sections when enough details of the final state are investigated.

We propose that it is important to the future use of pQCD that a systematic program be undertaken to reformulate factorization results in terms of fully unintegrated densities, which are differential in both transverse momentum and virtuality. These densities are called “doubly unintegrated parton densities” by Watt, Martin and Ryskin \cite{1,2}, and “parton correlation functions” by Collins and Zu \cite{3}; these authors have presented the reasoning for the inadequacy, in different contexts, of the more conventional approach. The new methods have their motivation in contexts such as Monte-Carlo event generators where final-state kinematics are studied in detail. Even so, a systematic reformulation for other processes to use unintegrated densities would present a unified methodology.

These methods form an extension of $k_T$-factorization, which has so far been applied in small-$x$ processes and, as the CSS formalism, in the transverse-momentum distribution of the Drell-Yan and related processes.

2 Inadequacy of conventional pdfs
The problem that is addressed is nicely illustrated by considering photoproduction of $c\bar{c}$ pairs. In Figs. 1 we compare three methods of calculation carried out within the CASCADE event generator \cite{4}:

- Use of a conventional gluon density that is a function of parton $x$ alone.
- Use of a $k_T$ density that is a function of parton $x$ and $k_T$. These are the objects usually called “unintegrated parton densities”.
- Use of a doubly unintegrated density that is a function of parton $x$, $k_T$ and virtuality, that is, of the complete parton 4-momentum.

The partonic subprocess in all cases is the lowest order photon-gluon-fusion process $\gamma + g \rightarrow c + \bar{c}$ (Fig. 2). Two differential cross sections are plotted: one as a function of the transverse momentum of the $c\bar{c}$ pair, and the other as a function of the $x_\gamma$ of the pair. By $x_\gamma$ is meant the fractional momentum of the photon carried by the $c\bar{c}$ pair, calculated in the light-front sense as

$$x_\gamma = \sum_{i=c,\bar{c}} (E_i - p_z i) / 2y E_\gamma = p_{c\bar{c}} / q^-.$$
Fig. 1: (a) and (b): Comparison between use of simple LO parton model approximation and of the use of $k_T$ densities for the $p_T$ of $c \bar{c}$ pairs in photoproduction, and for the $x_g$. (c) and (d): Comparison of use of $k_T$ densities and full simulation.

Fig. 2: Photon-gluon fusion.
Here $E_e$ is the electron beam energy and the coordinates are oriented so that the electron and proton beams are in the $-z$ and $+z$ directions respectively.

In the normal parton model approximation for the hard scattering, the gluon is assigned zero transverse momentum and virtuality, so that the cross section is restricted to $p_{T\bar{c}c} = 0$ and $x_\gamma = 1$, as shown by the solid lines in Fig. 1(a,b). When a $k_T$ dependent gluon density is used, quite large gluonic $k_T$ can be generated, so that the $p_{T\bar{c}c}$ distribution is spread out in a much more physical way, as given by the dashed line in Fig. 1(a). But as shown in plot (b), $x_\gamma$ stays close to unity. Neglecting the full recoil mass $m_{rem}$ (produced in the shaded subgraph in Fig 2) is equivalent of taking $k_T = -k_T^2/(1-x)$ with $k^2$ being the virtuality of the gluon in Fig. 2, $k_T$ its transverse momentum and $x$ its light cone energy fraction. This gives a particular value to the gluon’s $k^-$. When we also take into account the correct virtuality of gluon, there is no noticeable change in the $p_{T\bar{c}c}$ distribution — see Fig. 1(c) (dashed line) — since that is already made broad by the transverse momentum of the gluon. But the gluon’s $k^-$ is able to spread out the $x_\gamma$ distribution, as in Fig. 1(d) with the dashed line. This is equivalent with a proper treatment of the kinematics and results in $k^2 = -(k_T^2 + x m_{rem}^2)/(1-x)$, where $m_{rem}$ is the invariant mass of the beam remnant, the part of the final state in the shaded blob in Fig. 2. This change can be particularly significant if $x$ is not very small.

Note that if partons are assigned approximated 4-momenta during generation of an event in a MC event generator, the momenta need to be reassigned later, to produce an event that conserves total 4-momentum. The prescription for the reassignment is somewhat arbitrary, and it is far from obvious what constitutes a correct prescription, especially when the partons are far from a collinear limit. A treatment with fully unintegrated pdfs should solve these problems.

If, as we claim, an incorrect treatment of parton kinematics changes certain measurable cross sections by large amounts, then we should verify directly that there are large discrepancies in the distributions in partonic variables themselves. We see this in Fig. 3. Graph (a) plots the gluonic transverse momentum divided by the charm-pair mass. As is to be expected, the typical values are less than one, but there is a long tail to high values. But the use of full parton kinematics does not have much of an effect, the unintegrated parton distributions already providing realistic distributions in transverse momentum.

On the other hand, a simple collinear approximation for showering sets the remnant mass, $m_{rem}$, to zero. As can be seen from the formulae for the gluon virtuality, this only provides a good approximation to the gluon kinematics if $m_{rem}$ is much less than $k_T$. In reality, as we see from graph (b), there is a long tail to large values of $m_{rem}/k_T$, and the tail is much bigger when correct kinematics are used. A more correct comparison uses $x m_{rem}^2$, with an extra factor of $x$. Even then, there is a large effect, shown in graph (c). The vertical scale is logarithmic, so the absolute numbers of events are relatively small, but the tail is broad. Finally, graph (d) shows that the distribution in $m_{rem}$ itself is very broad, extending to many tens of GeV. This again supports the argument that unless a correct treatment of parton kinematics is made, very incorrect results are easily obtained.

It is important to note that, for the cross sections themselves, the kinematic variables used in Fig. 1 are normal ones that are in common use. Many other examples are easily constructed. Clearly, the use of the simple parton-model kinematic approximation gives unphysically narrow distributions. The correct physical situation is that the gluon surely has a distribution in transverse momentum and virtuality, and for the considered cross sections neglect of parton transverse momentum and virtuality leads to wrong results. It is clearly better to have a correct starting point even at LO, for differential cross sections such as we have plotted.

3 Kinematic approximations

The standard treatment of parton kinematics involves replacing the incoming parton momentum $k$ by its plus component only: $k^\mu \rightarrow k^\mu \equiv (k^+, 0, 0_T)$. There are actually two parts to this. The first is to neglect the $-$ and transverse components of $k$ with respect to the large transverse momenta in the calculation of
Fig. 3: Comparison of distributions in partonic variables between calculations with full parton kinematics and with ordinary unintegrated pdfs.
the numerical value of the hard-scattering amplitude; this is a legitimate approximation, readily corrected by higher order terms in the hard scattering. The second part is to change the kinematics of the final-state particles, \( p_1 \) and \( p_2 \), so that their sum is \( q \) plus the approximated gluon momentum. It is this second part that is problematic, for it amounts to the replacement of the momentum conservation delta function \( \delta^{(4)}(k + q - p_1 - p_2) \) by \( \delta^{(4)}(\hat{k} + q - p_1 - p_2) \). These delta-functions are infinitely different, point-by-point. Only when integrated with a sufficiently smooth test function can they be regarded as being approximately the same, as in a fully inclusive cross section.

In an event generator, the effect is to break momentum conservation, which is restored by an ad hoc correction of the parton kinematics. Note that the change of parton kinematics is only in the hard scattering, i.e., in the upper parts of the graphs. Parton kinematics are left unaltered within the parton density part, and the integrals over \( k_T \) and virtuality are part of the standard definition of integrated pdfs.

The situation is ameliorated by inclusion of NLO terms, and perhaps also by some kind of resummation. But these do not correct the initial errors in the approximation, and lead to a very restricted sense in which the derivation of the cross section can be regarded as valid. Furthermore, when much of the effect of NLO terms is to correct the kinematic approximations made in LO, this is an inefficient use of the enormous time and effort going into NLO calculations. A case in point is the BFKL equation, where 70\% of the (large) NLO corrections are accounted for [5] by the correction of kinematic constraints in the LO calculation.

4 Conclusions

The physical reasoning for the absolute necessity of fully unintegrated densities is, we believe, unquestionable. Therefore it is highly desirable to reformulate perturbative QCD methods in terms of doubly unintegrated parton densities from the beginning. A full implementation will be able to use the full power of calculations at NLO and beyond.

Among other things, a full implementation, as in [3], will provide extra factorization formulae for obtaining the values of the unintegrated densities at large parton transverse momentum and virtuality. This will incorporate all possible perturbatively calculable information, so that the irreducible nonperturbative information, that must be obtained from data, will be at low transverse momentum and virtuality. In addition, the implementation will quantify the relations to conventional parton densities. With the most obvious definitions, the integrated pdfs are simple integrals of the unintegrated densities. However, in full QCD a number of modifications are required [3, 6], so that the relations between integrated and unintegrated pdfs are distorted.

The fact that we propose new and improved methods does not invalidate old results in their domain of applicability. The work of Watt, Martin and Ryskin, and of Collins and Zu provides a start on this project; but much remains to be done to provide a complete implementation in QCD; for example, there is as yet no precise, valid, and complete gauge-invariant operator definition of the doubly unintegrated densities in a gauge theory.

The outcome of such a program should have the following results:

1. Lowest order calculations will give a kinematically much more realistic description of cross sections. This may well lead to NLO and higher corrections being much smaller numerically than they typically are at present, since the LO description will be better.

2. It will also obviate the need for separate methods (resummation or the CSS technique), which are currently applied to certain individual cross sections like the transverse-momentum distribution for the Drell-Yan process. All these and others will be subsumed and be given a unified treatment.

3. A unified treatment will be possible for both inclusive cross sections using fixed order matrix element calculations and for Monte-Carlo event generators.

4. For a long-term theoretical perspective, the doubly unintegrated distributions will interface to methods of conventional quantum many-body physics much more easily than regular parton den-
ities, whose definitions are tuned to their use in ultra-relativistic situations.

This program is, of course, technically highly nontrivial if it is to be used in place of conventional methods with no loss of predictive power. A start is made in the cited work.

Among the main symptoms of the difficulties are that the most obvious definition of a fully unintegrated density is a matrix element of two parton fields at different space-time points, which is not gauge-invariant. It is often said that the solution is to use a light-like axial gauge \( A^+ = 0 \). However, in unintegrated densities, this leads to divergences — see [6] for a review — and the definitions need important modification, in such a way that a valid factorization theorem can be derived.

We also have to ask to what extent factorization can remain true in a generalized sense. Hadron-hadron collisions pose a particular problem here, because factorization needs a quite nontrivial cancellation arising from a sum over final-state interactions. This is not compatible with simple factorization for the exclusive components of the cross section, and makes a distinction between these processes and exclusive components of DIS, for example.

Acknowledgments
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