Solvable random matrix ensemble with a logarithmic confining potential

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(Dated: November 15, 2022)

This work identifies a solvable, rotationally invariant random matrix ensemble with a logarithmic confining potential. The ensemble, which can be interpreted as a transformed Jacobi ensemble, is in the thermodynamic limit characterized by a Lorentzian eigenvalue density. It is shown that spectral correlation functions can be expressed in terms of Gegenbauer polynomials $C_n^{(3)}(x)$ with $\lambda = -1/2$ and $n \geq 2$. A procedure to sample matrices from the ensemble is outlined and used to provide a numerical verification for some of the analytical results. This newly proposed ensemble is pointed out to potentially provide a useful ingredient in physical applications.

I. INTRODUCTION

Random matrix theory plays a major role in the analysis of various types of complex quantum systems [1, 2], with applications in for example nuclear physics [3, 4], mesoscopic physics [5, 6], high-energy physics [7, 8], and quantum chaos [9, 10]. One of the main challenges in physically-motivated random matrix theory is to construct random matrix models that are on the one hand simple enough to be tractable analytically, and on the other hand provide a reasonably good description of the system of interest. Notable progress in the search for such ensembles has been made in the last decade [11–14].

The central building blocks of random matrix theory are the three classical (Gaussian, Wishart-Laguerre, and Jacobi) random matrix ensembles (see e.g. Refs. [15, 16]). These ensembles are rotationally invariant (i.e., basis-independent) and solvable in the sense that spectral correlation functions can be expressed in terms of orthogonal polynomials. For these ensembles, the joint probability distribution for the eigenvalues is known explicitly. Expressing this distribution as the Boltzmann probability distribution allows one to study the thermodynamic limit using tools from statistical mechanics.

This work identifies a solvable, rotationally invariant random matrix ensemble with a logarithmic confining potential. In the thermodynamic limit, the eigenvalue densities of the Gaussian, Wishart-Laguerre, and Jacobi ensembles are given by respectively the Wigner semicircle, Marčenko-Pastur, and Wachter laws [17]. Using the Coulomb gas technique, the eigenvalue density in the thermodynamic limit corresponding to the logarithmic potential is found to be given by a Lorentzian. Random matrices with a Lorentzian eigenvalue density appeared very recently in the context of ergodicity breaking in quantum many-body systems in Ref. [18].

The orthogonal polynomials in terms of which spectral correlation functions can be expressed are identified as the Gegenbauer polynomials $C_n^{(3)}(x)$ with $\lambda = -1/2$ and $n \geq 2$, from which it is deduced that the ensemble can be interpreted as a transformed Jacobi ensemble. A procedure to (numerically) sample eigenvalue spectra from the ensemble is outlined and demonstrated by verifying some of the analytical results obtained.

The outline of this work is as follows. Sec. II considers the Coulomb gas picture for the joint probability distribution of the eigenvalues. Here, it is discussed how the Lorentzian density of states emerges from the logarithmic potential. Sec. III identifies the associated orthogonal polynomials, and outlines how the spectral correlation functions can be obtained. Here, also the relation with the Jacobi ensemble is discussed. Sec. IV outlines how the eigenvalue spectra can be obtained from the spectra of random matrices. Sec. V provides a summary of the findings and proposes suggestions for further investigations.

II. COULOMB GAS PICTURE

For the classical random matrix ensembles, the joint probability distribution $P(H)$ of the entries of matrices $H$ can be written as

$$P(H) \propto \exp[-\text{Tr}V(H)],$$

(1)

where $V(x)$ is a function referred to as the potential (see e.g. Chaps. 4 and 5 of Ref. [16]). Let $N$ denote the dimension of the matrices. As $P(H)$ depends only on (powers of) the trace of $H$, the ensembles are rotationally invariant. The joint probability distribution $P(x_1, \ldots, x_N)$ of the eigenvalues $x_n$ is in terms of the potential given by

$$P(x_1, \ldots, x_N) = \frac{1}{Z_N^\beta} e^{-\beta V(x_1, \ldots, x_N)},$$

(2)

$$V = \frac{1}{\beta} \sum_{n=1}^N V(x_n) + \frac{1}{2} \sum_{n,m=1}^N \ln |x_n - x_m|.$$  

(3)

Here, $\beta \in \{1, 2, 4\}$ is the Dyson index giving the number of degrees of freedom per (real, imaginary, or quaternionic) matrix element. Next, $Z_N^\beta$ is a normalization constant fixing the integrated probability to unity.

Eq. (2) can be interpreted as the Boltzmann factor of a Coulomb gas with the energy of a given configuration
given by Eq. (3). This interpretation allows one to use tools from statistical mechanics to study the thermodynamic limit. After making a continuum and saddle-point approximation, the eigenvalue density $\rho_N(x)$ for a potential $V(x)$ can be shown to satisfy the integral equation

$$\Pr \int_{-\infty}^{\infty} \frac{\rho_N(y)}{x-y} \, dy = \frac{1}{\beta} \frac{dV}{dx},$$  \hspace{1cm} (4)$$

where $\Pr$ denotes the principal value. This integral equation is subject to the constraint $\int \rho_N(x) \, dx = N$. Eq. (4) is generically difficult to solve, and only for a limited number of potentials the corresponding eigenvalue density has been found (see e.g. Sec. 3.2 of Ref. [19]).

In this work, the focus is on the random matrix ensemble associated with the logarithmic potential

$$V(x) = \frac{\beta N}{2} \ln(1+x^2).$$  \hspace{1cm} (5)$$

The prefactor $N$ ensures that the first and second term in Eq. (3) are of the same order, namely $N^2$. For $|x| \gg 1$, this potential approximates $\beta N \ln(x)$. At a technical level, the motivation to consider this particular form is as follows. Substituting Eq. (5) in Eq. (4) and dividing both sides by $N$, one recognizes the left-hand side as the Hilbert transform (see e.g. Chap. 5 of Ref. [20]) $\mathcal{H}(f(y))$ of some function $f(y)$,

$$\mathcal{H}(f(y)) = \frac{1}{\pi} \Pr \int_{-\infty}^{\infty} \frac{f(y)}{x-y} \, dy.$$  \hspace{1cm} (6)$$

By comparing the right-hand side $x/(1+x^2)$ of Eq. (4) for $V(x)$ as given in Eq. (5) with known Hilbert transforms, one deduces that $\rho_N(x)$ is given by

$$\rho_N(x) = \frac{N}{\pi(1+x^2)}.$$  \hspace{1cm} (7)$$

An explicit derivation can be found e.g. in Example 5.17 of the Reference cited above. Indeed, it follows that this function obeys the normalization condition $\int \rho_N(x) \, dx = N$ by noting that $\rho_N(x)/N$ gives the normalized (to unity) probability density for the Cauchy distribution.

It can be of interest to note that random matrix ensembles with logarithmic or squared-logarithmic potentials (although without prefactor $N$) have been proposed as models for the intermediate level spacing statistics and multifractality at the Anderson localization transition [21] (Chap. 12); [22–25].

### III. ORTHOGONAL POLYNOMIALS

Spectral correlation functions for the classical random matrix ensembles at finite dimension can be expressed in terms of orthogonal polynomials (see e.g. Chap. 10 of Ref. [16]). In view of the discussion below, the main ideas are introduced using the Jacobi ensemble as an example. The Jacobi ensemble is known historically to be relevant in physics in the context of quantum conductance [26]. In recent years, new applications appeared in the computation of eigenstate entanglement of random free fermionic models [27–30] and the spectral form factor of a self-dual kicked Ising model [31].

As before, let $N$ and $\beta$ denote respectively the dimension of the matrices and the Dyson index. The eigenvalues $x_n \in [-1,1]$ of samples from the Jacobi ensemble are distributed according to

$$P(x_1, \ldots, x_N) = \frac{1}{Z_{abN\beta}} \prod_{n=1}^{N} w(x_n) \prod_{m<k} |x_m - x_k|^\beta \hspace{1cm} (8)$$

with the weight function $w(x)$ characterized by parameters $a > -1$ and $b > -1$ given by

$$w(x) = (1-x)^{a\beta/2}(1+x)^{b\beta/2}.$$  \hspace{1cm} (9)$$

Similar to the above, $Z_{abN\beta}$ is a normalization constant fixing the integrated probability to unity. For probability distributions of the form (8), spectral correlation functions can be expressed for $\beta = 2$ in terms of the kernel

$$K_N(x_1, x_2) = e^{-\frac{1}{2}(V(x_1)+V(x_2))} \sum_{n=0}^{N-1} p_n(x_1) p_n(x_2),$$  \hspace{1cm} (10)$$

where for notational convenience the potential $V(x)$ satisfying $w(x) = e^{-V(x)}$ is re-introduced. The functions $p_n(x)$ are polynomials orthogonal with respect to the weight function. For the Jacobi ensemble, thus

$$\int_{-1}^{1} (1-x)^{a\beta/2}(1+x)^{b\beta/2} p_n(x) p_m(x) \, dx = \delta_{nm}.$$  \hspace{1cm} (11)$$

For Eq. (11), the polynomials $p_n(x)$ are given by the Jacobi polynomials $P_n^{(a,b)}(x)$ (up to normalization) with the same parameters (see e.g. Sec. 9.8 of Ref. [32]). In terms of the kernel, the normalized (to unity) spectral density $n_N(x) = \rho_N(x)/N$ is given by

$$n_N(x) = \frac{1}{N} K_N(x, x).$$  \hspace{1cm} (12)$$

Two-point correlation functions can be expressed in terms of the kernel as

$$\rho_N^{(2)}(x_1, x_2) = C_N \det \begin{pmatrix} K_N(x_1, x_1) & K_N(x_1, x_2) \\ K_N(x_2, x_1) & K_N(x_2, x_2) \end{pmatrix}$$  \hspace{1cm} (13)$$

with $C_N = 1/[N(N-1)]$.

For the ensemble proposed in this work, the polynomials $p_n(x)$ have to be orthogonal with respect to the weight function $w(x) = e^{-\frac{1}{2}\ln(1+x^2)}$ on the full real line $x \in (-\infty, \infty)$. A fruitful strategy to find these polynomials turns out to perform a transformation of variables $x \rightarrow y$ on the orthogonality relation

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\ln(1+y^2)} p_n(x) p_m(x) \, dx = \delta_{nm}.$$  \hspace{1cm} (14)$$
such that the orthogonality relation for a known set of orthogonal polynomials results. Suppose that the transformed polynomials are orthogonal with respect to a weight function \( w(y) \) on a domain \([a, b]\). The transformation \( x \to y \) can be obtained by solving the differential equation

\[
\frac{dy}{dx} w(y) = e^{-\frac{1}{2} \ln(1+x^2)}
\]

subject to the constraints \( y(-\infty) = a \), \( y(\infty) = b \), and \( y'(x) > 0 \) to ensure that the transformation \( x \to y \) is one-to-one. Notice that this differential equation is over-constrained. Taking \( a = -1 \), \( b = 1 \), and \( w(y) = 1/(1 - y^2) \) results in a solution of Eq. (15) given by

\[
y(x) = \frac{x}{\sqrt{1+x^2}}.
\]

For this choice of \( a \), \( b \), and \( w(y) \), one recognizes the orthogonality condition for the Gegenbauer polynomials \( C_n^{(\lambda)}(y) \) with \( \lambda = -1/2 \), which are Jacobi polynomials \( P_n^{(a, b)}(y) \) with \( a = b = \lambda = -1/2 \) [Eq. (11)]. Having found the orthogonal polynomials, the ensemble proposed in this work can be considered as being solvable.

For \( \lambda = -1/2 \), the first two Gegenbauer polynomials \((n = 0 \text{ and } n = 1)\) are not normalizable. See Appendix A for details. In the evaluation of the kernel [Eq. (10)], the counting thus starts at \( n = 2 \), due to which the summation runs up to \( n = N + 1 \). A numerical verification is presented in the Figures below.

Aiming to illustrate the above results, here some numerical evaluations of \( n_N(x) \) [Eq. (12)] and \( \rho^2(0, x) \) [Eq. (13)] are presented. The variable \( x \) is expressed in terms of \( y \) by the inverse of Eq. (16) as

\[
x(y) = \frac{y}{\sqrt{1+y^2}}.
\]

Fig. 1 shows \( n_N(x) - n(x) \), where \( n(x) \) gives the normalized (to unity) eigenvalue density for \( N \to \infty \), \( n(x) = 1/|\pi(1 + x^2)| \), for \( N = 25 \), \( N = 50 \), and \( N = 100 \). The difference becomes smaller with increasing \( N \). The data for each of the Figures presented in this work can be generated in \( \sim 10 \) minutes of computational time on a mid-range laptop.

Fig. 2 compares \( \rho^2(0, x) \) obtained through Eq. (13) for \( N = 25 \), \( N = 100 \), and \( N = 1000 \) with the evaluation for Wigner-Dyson level statistics at \( N \to \infty \) for unit mean level spacing given by

\[
\rho^2(0, x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2,
\]

see e.g. Ref. [2]. The finite-\( N \) results have been scaled and transformed such that the mean level spacing is unity at \( x = 0 \) for \( N \to \infty \) (see the caption for details). One observes that the curves become closer to each other with increasing \( N \). For \( N = 25 \), effects due to the non-uniform eigenvalue density (decaying with increasing \( x \)) are clearly visible. For \( N = 1000 \), the curves are almost visually indistinguishable.

![Fig. 1. The difference between the normalized (to unity) eigenvalue density for \( N \) finite and \( N \to \infty \) for \( N = 25 \), \( N = 50 \), and \( N = 100 \). Finite-\( N \) results have been obtained through a numerical evaluation of Eq. (12). One observes that the difference tends to become smaller with increasing values of \( N \).](image-url)

### IV. RANDOM MATRIX CONSTRUCTION

In Sec. III, it was found that the spectral correlation functions for the random matrix ensemble proposed in this work can be obtained from the spectral correlation functions for the Jacobi ensemble with parameters \( a = b = -1 \) through the transformation given in Eq. (17). This mapping of the spectral properties of a classical ensemble to the spectral properties of the ensemble of interest allows one to sample spectra by diagonalizing random matrices.

Let \( X_1 \) and \( X_2 \) denote respectively \( M_1 \times N \) and \( M_2 \times N \) matrices with independent sampled Gaussian entries. The real \((\beta = 1)\), imaginary \((\beta = 2)\), or quaternion-valued \((\beta = 4)\) entries \( x \), \( z \), or \( w \) are sampled from respectively the probability densities

\[
\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2}, \quad \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}, \quad \frac{2}{\sqrt{2\pi}} e^{-2|w|^2},
\]

see e.g. Sec. 6.3 of Ref. [15]. Next, let \( W_1 = X_1^\dagger X_1 \) and \( W_2 = X_2^\dagger X_2 \). A spectrum \( \{y_n\} \) from the Jacobi ensemble of dimension \( N \) with parameters \( a = N - M_1 + 1 - 2/\beta \) and \( b = N - M_2 + 1 - 2/\beta \) is obtained by transforming the eigenvalues \( \{z_n\} \) of the double-Wishart matrix

\[
W = W_1(W_1 + W_2)^{-1},
\]

which obey \( z_n \in [0, 1] \), as \( y_n = 1 - 2z_n \). Notice that there are no issues with sampling for \( a = b = -1 \), which can be accomplished by choosing \( M_1 = M_2 = N + 1 \).

Given an eigenvalue spectrum \( \{y_n\} \) sampled from the Jacobi ensemble, a spectrum \( \{x_n\} \) of the ensemble proposed in this work can be obtained by applying the transformation \( y_n \to x_n \) as given in Eq. (17). Fig. 3 compares \( n_N(x) \) as given in Eq. (12) with a properly normalized...
FIG. 2. A comparison between $\rho_N^{(2)}(0,x)$ (solid lines) obtained through Eq. (13) and the result for Wigner-Dyson level statistics at $N \to \infty$ of Eq. (18) (dashed lines) for $N = 25$, $N = 100$, and $N = 1000$. The finite-$N$ results have been scaled by $1/[n(0)]^2 = \pi^2$ and the transformation $x \to n(0)x/n = \pi x/n$ has been applied in order to set the mean level spacing at $x = 0$ to unity for $N \to \infty$. The curves tend towards each other with increasing values of $N$.

FIG. 3. A plot of $n_N(x)$ as given in Eq. (12) (“analytical”) compared with a normalized [to unity on $x \in (-\infty,\infty)$] histogram of the eigenvalues for the random matrix ensemble proposed in this work for $N = 10$ and $\beta = 2$ ($10^6$ samples, “numerical”). Perfect agreement can be observed.

to be given by a Lorentzian. A procedure to sample numerically from this random matrix ensemble has been outlined, and used to verify some of the analytical result discussed in this work.

As the random matrix ensemble proposed in this work can be interpreted as a transformed Jacobi ensemble, properties of the ensemble that have not been discussed here, such as extreme value statistics [33] or the extension to a continuous $\beta$-ensemble [34], could in principle be established in a straightforward way. It would be of interest to see how the ensemble proposed in this work appears in physical settings. For example, in the spirit of Ref. [18] and other generalizations [35–39], this ensemble could potentially serve as a building block for improved generalizations of the Rosenzweig-Porter ensemble [11].

ACKNOWLEDGMENTS

Useful discussions with Vladimir Gritsev and Ward L. Vleeshouwers are gratefully acknowledged. This work is supported by the Kreitman School of Advanced Graduate Studies at Ben-Gurion University.

Appendix A: Gegenbauer polynomials

The Gegenbauer polynomials $C_n^{(\lambda)}(x)$ are Jacobi polynomials $P_n^{(a,b)}(x)$ with $a = b = \lambda - 1/2$ for $\lambda > -1/2$ (see e.g. Sec. 9.8.1 of Ref. [32]). The Gegenbauer polynomials are orthogonal with respect to the weight function $(1 + x^2)^{\lambda - 1/2}$ on $x \in [-1,1]$. They obey the recurrence

V. CONCLUSIONS AND OUTLOOK

In this work, a solvable (in the sense that spectral correlation functions can be expressed in terms of orthogonal polynomials), rotationally invariant random matrix ensemble with a logarithmic confining potential has been identified. This ensemble is found to be a transformed Jacobi ensemble. Using the Coulomb gas technique, the eigenvalue density in the thermodynamic limit is found

histogram of the eigenvalues for $N = 10$ and $\beta = 2$ from the ensemble proposed in this work, obtained through diagonalizations of matrices $W$ as given in Eq. (20). A small value of $N$ has been chosen in order to ensure the visibility of finite-$N$ effects. One observes perfect agreement.

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relation
\[ 2(n + \lambda) x C_n^{(\lambda)}(x) = (n + 1) C_{n+1}^{(\lambda)}(x) + (n + 2\lambda - 1) C_{n-1}^{(\lambda)}(x) \]  
(A1)

with \( C_0^{(\lambda)}(x) = 1 \) and \( C_1^{(\lambda)}(x) = 2x \). For illustrational purposes, the next two polynomials can be obtained as
\[ C_2^{(\lambda)}(x) = -\lambda + 2\lambda(1 + \lambda)x^2, \]  
(A2)
\[ C_3^{(\lambda)}(x) = -2\lambda(1 + \lambda)x + \frac{4}{3}\lambda(1 + \lambda)(2 + \lambda)x^3. \]  
(A3)

In view of the discussion below, notice that these polynomials are well-defined for \( \lambda = -1/2 \).

The normalization condition for the Gegenbauer polynomials reads
\[ \int_{-1}^{1} (1 - x^2)^{\lambda-1/2} [C_n^{(\lambda)}(x)]^2 \, dx = \mathcal{N}_n \]  
(A4)

with
\[ \mathcal{N}_n = \frac{\pi\Gamma(n + 2\lambda)2^{2\lambda-1}}{n!(n + \lambda)\Gamma(\lambda)^2}. \]  
(A5)

For \( \lambda = -1/2 \), \( \mathcal{N}_n \) reduces to \( 1/[n(n-1)/2(n-1)] \), meaning that the polynomials with indices \( n = 0 \) and \( n = 1 \) can not be properly normalized. Consequently, in this case the counting starts at \( n = 2 \). The polynomials satisfying the orthogonality condition of Eq. (11) are given by \( p_n(x) = C_n^{(\lambda)}(x)/\sqrt{\mathcal{N}_n} \). A recent generalization of the orthogonality condition to the complex plane has been obtained in Ref. [40].

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