Euler-Poincaré obstruction for pretzels
with long tentacles à la Cantor-Nyikos

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Abstract. We present an avatar of the Euler obstruction to foliated structures on certain non-metric surfaces. This adumbrates (at least for the simplest 2D-configurations) that the standard mechanism—to the effect that the devil of algebra sometimes barricades the existence of angelic geometric structures (obstruction theory more-or-less)—propagates slightly beyond the usual metrical proviso. Alas, the game is much more conservative than revolutionary: in particular we enjoyed retrospecting at Poincaré’s argument of 1885 (announced in 1881).

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1 Introduction

1.1 Statement and nomenclature

In a previous paper [11], we explored foliated surfaces at the large scale by contemplating (rather passively) how several classic paradigms (like those of Poincaré-Bendixson or Haefliger-Reeb) transpose non-metrically. A thermodynamical metaphor was found convenient to synthesize a body of disparate results (including the irrational toric windings of Kronecker, the allied labyrinths of Dubois-Violette, Franks, Rosenberg and the surgeries of Peixoto-Blohin). The metaphor implicates the familiar solid-liquid-gaseous phases as follows:

• If one warms sufficiently the fundamental group (by increasing its rank $r$, say via iterated punctures in a closed surface), then as the temperature is high enough ($r \geq 4$) metric surfaces are always transitively foliated (by a dense leaf). (Observationally, such random motions are expected to occur when tracing out a curve following some primate’s fingerprints leading quickly to some complicated ‘labyrinth’ filling almost all of the hand epiderm.)

• Conversely at low temperatures ($r \leq 1$), e.g., in the simply-connected case, the situation is completely frozen: any foliated structure is intransitive, regardless of the metric proviso.
• Between the first gaseous-volatile regime \((r \geq 4)\) and the second solid-frozen state \((0 \leq r \leq 1)\), we observe for \(2 \leq r \leq 3\) an intermediate liquid-phase, where the transitivity issue truly depends on the detailed topology.

In particular, each closed surface has a critical temperature at which it starts an ebullition, namely the least integer \(k\) such that the \(k\)-times punctured surface permits a transitive foliation (i.e., with a dense leaf). (This boiling temperature is computed in \([11]\) for all surfaces except the Klein bottle. We presume the answer to be more-or-less implicit in the works of Hellmuth Kneser from the 1920’s \([24], [25]\), but as yet we have not assembled the details.)

Besides, still in \([11]\), the question of an avatar of the Euler-Poincaré obstruction was left dormant. The present note aims to remedy this gap by showing:

**Proposition 1.1** An \(\omega\)-bounded surface of negative Euler characteristic \((\chi < 0)\) cannot be foliated.

• Here, a surface is as usual a 2-dimensional manifold, that is a Hausdorff space everywhere locally homeomorphic to the number-plane \(\mathbb{R}^2\), yet without imposing (a priori) a global metric compatible with the topological structure.

• A space is \(\omega\)-bounded if any countable subset has a compact closure. This concept turned out to be a vivid substitute to compactness in the non-metric realm, especially in view of Nyikos’ bagpipe theorem \([30]\). The latter extrapolates widely the classification of compact surfaces initiated by Möbius circa 1860 published 1863 \([29]\) (after loosing an international Parisian contest for linguistic issues) and subsequently revisited (and generalized) by a long list of workers including Jordan, Klein, Dyck 1888 \([7]\), Dehn-Heegaard 1907, Brahana 1921. If some compact manifold should geometrize the myth \(^1\) of a finite universe, \(\omega\)-boundedness posits the scenario of a possibly infinite, yet inextensible universe. It is indeed easy to convince that \(\omega\)-boundedness implies sequential-compactness, implying in turn manifold, i.e., a manifold which cannot be continued to a strictly larger (connected) manifold of the same dimensionality. This little remark may help to grasp the substance (at least the aesthetical value) of \(\omega\)-boundedness. The simplest (non-metric) prototype of an \(\omega\)-bounded manifold is the long line \(L\) or, in the bordered case, the closed long ray \(L_{\geq 0}\) (discovered by Cantor in 1883). Those spaces are both grandiose and claustrophobic: one cannot find a Fluchtweg to infinity in denumerable time.

• Foliations refer as usual to those geometric structures microscopically modelled after the partition of the plane into parallel lines (we focus on the surface case for simplicity). Regular family of curves is an older synonym, employed by Kerékjártó 1929 \([22, \text{p. 111}]\) and Whitney 1933 \([37]\). Gauss in 1839 uses the term Liniensystem (in Allgemeine Theorie des Erdmagnetismus \([12, \text{p. 135}]\)) and the German speaking literature (especially Kerékjártó 1923 \([21, \text{p. 249}]\), Kneser 1921 \([24]\), 1924 \([25]\)) used the jargon Kurvenschar. Later Ehresmann and Reeb coined feuilletage, from which the modern nomenclature was derived. For our concern, the real issue is that the foliated concept (and likewise the manifold idea), being purely local in nature, are not imprisoned in the metric realm.

• Finally, in the \(\omega\)-bounded context the (singular) homology is a priori known to be of finite type (cf. optionally the discussion in \([10]\)). In particular the Euler characteristic \((\chi=\text{alternating sum of Betti numbers starting with } b_0\text{ signed positively}^2)\) is finite (unambiguously defined despite the severe—but not so dramatic—absence of triangulation). Actually a simple argument (reproduced below) shows the characteristic of the surface to coincide with that of Nyikos’ bag (which is a compact bordered surface).

Admittedly, the theory of nonmetric manifolds is far from popular, yet its bad reputation seems to be slightly overdone, since ironically much of the game is just a matter of tranposing metrical truths. The present note is no exception.

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\(^1\) Courtesy of Claude Weber for pointing out a text of Borges, establishing a one-to-one correspondence between conceptions of times through the ages and 1D-manifolds; e.g. \(S^1 \leftrightarrow "l’ételret retour"\).

\(^2\) In the older literature it seems that the sign of \(\chi\) was the opposite one! Courtesy of M. Kervaire’s lecture notes borrowed from L. Bartholdi.
In practice, there is often an automatic transfusion (of truths) from Lindelöf subregions to the whole manifold (e.g., for Jordan separation or the Schoenflies bounding disc property, orientability, etc., cf. optionally [8] and [11]), whereas in the present case we rather use a dévissage into metric sub-pieces (cf. Fig. 2) where the obstruction is classical.

1.2 Metric background and historiographical links

Specifically, our proof of (1.1) relies on the following (metrical) background, reviewed subsequently so as not to relegate our issue behind a mountain of preliminaries:

1. Poincaré’s index formula or the variant thereof for foliations. (Main contributors: Poincaré 1880 [31], 1881 [32], 1885 [33], 1885, p. 203–8], Dyck 1885 [6] p. 317–320], 1888 [7] p. 462–3; 499–501], Brouwer and Hadamard 1910 [13], Hamburger 1924 [15] p. 58–62], Hopf 1926 [18] and Lefschetz about the same period. It is perhaps fair to recall that this circle of ideas had been partly anticipated by Gauss and Kronecker 1869, e.g., via the Curvatura integra, just to name two among several other forerunners carefully listed in Dyck 1888 [7, p. 463] (e.g., Gauss 1839, Reech 1858, Möbius 1863, Poincaré 1881–86, Klein 1882, Betti 1885, etc.). See also the recent historiography in Mawhin 2000 [28], reminding in particular Hermite’s rôle, as an interface from Kronecker to Poincaré.)

2. Bendixson-Hamburger’s index formula computing the Poincaré index in terms of the local phase-portrait (already (???) in Enrico Poincaré 1885 [33, p. 203], Bendixson 1901 [5, p. 39], Hamburger 1922 [14], [15], [16], Kerékjártó 1925 [22], etc.) To elucidate the above question marks, we may agree perfectly with the following comment of Mawhin [28, p. 118]:

“It is of interest to notice that Poincaré introduces here a further definition for the index of a cycle, without worrying about proving its equivalence or relation with the previous definitions. This time, the index is defined as \( E - I - 2 \), where \( E \) (resp. \( I \)) is the number of exterior (resp. interior) tangency points of the vector field to the cycle. Proving the equivalence between this new definition and the previous one is essentially the contents of what is called today the Poincaré-Bendixson index theorem [Bendixson, 1901].”

3. Some geometric topology à la Schoenflies circa 1906, plus some variants due to R. Baer 1928, R. J. Cannon 1969. Compare Gabard-Gauld 2010 [8] for a pedestrian exposition of the fact that any null-homotopic Jordan curve in a surface bounds a disc. This holds universally (without metric proviso).

1.3 Related phenomenology and an application à la Kaplan-Haefliger-Reeb

For comparison, non-singular flows (fixed-point free \( \mathbb{R} \)-actions) are also regulated by an Euler obstruction namely \( \chi \neq 0 \) (Gabard 2011 [10]). This weaker numerical condition is not suited to foliations, as the long plane \( L^2 \) (Cartesian square of the long line \( L \)), despite having non-zero \( \chi = b_0 - b_1 + b_2 = 1 - 0 + 0 = 1 \), still foliates (e.g., in the trivial fashion by parallel long lines).

The converse of (1.1) fails: a surface with \( \chi \geq 0 \) does not necessarily foliate. The simplest example is probably the sphere \( S^2 \) (as well-known since Poincaré 1880 [31]). For a non-metric prototype, we may consider a long cigar \( S^1 \times L_{\geq 0} \) (=circle crossed by the closed long ray) capped off by a 2-disc. The resulting surface (resembling a long glass) lacks foliations (cf. Baillif et al. 2009 [3]). This derives from the super-massive black hole scenario materialized by Cantor’s long ray, according to which a (finally violent) aspiration of leaves in the semi-long cylinder \( S^1 \times L_{\geq 0} \) occurs either as an ultimately vertical collection of straight long rays or as a compulsive infinite repetition of horizontal circle leaves parametrized by a closed unbounded set (compare Fig. 1a,b). Thus for many toy

3Little joke to imitate Enriques-Chisini.
Figure 1: Super-massive black hole and plumbing with spaghetti

examples of $\omega$-bounded surfaces, say those constructed by inserting long cylindrical pipes into a pretzel (Fig. 1c). Proposition 1.1 boils down to the classic compact Euler obstruction. With some little more work (plumbing) the same trick applies to planar long pipes modelled after the long plane (compare Fig. 1d showing how to “plumb” with a replica the bagpipe surface suitably truncated according to the 6 possible asymptotic patterns described in [3]; arithmetical details left to the indulgent readers). In short, the little innovation of the present result is that, while presupposing no explicit knowledge of the pipes (whose biodiversity overwhelms any classification scheme), it still affords a qualitative prediction in close accordance to our metric intuition.

An application of (1.1) can be given to foliated structures on simply-connected $\omega$-bounded surfaces. By a (non-metric) extension of Kaplan/Haefliger-Reeb’s theory (cf. [11]), any leaf in a simply-connected surface separates the surface (à la Jordan). Thus given a configuration of 3 leaves they can either (compare Fig. 1d) be parallel (with one central leaf separating the other two) or bound an amoeba (if no leaf disconnects the other two). In the $\omega$-bounded case, the amoebic option cannot occur since the doubled amoeba yields a long pants (Fig. 1e) with $\chi = -1$, hence not foliable by (1.1). It follows that the leaf-space is necessarily a Hausdorff 1-manifold (since given 2 leaves, any leaf chosen in between imposes a separation à la Jordan, implying a separation à la Hausdorff in the quotient leaf-space).

Perhaps it is reasonable to expect higher-dimensional extensions of (1.1) for foliations of dimension- or codimension-one. Maybe, one should first concretize
Nyikos-Gauld’s grand programme of a 3D-bagpipe philosophy (probably by now “harpoonable” via the Poincaré conjecture of Perelman).

2 Proof of the proposition

The proof of (1.1) can be given two slightly different flavors by arguing either with foliations or (the allied) flows (continuous \(\mathbb{R}\)-actions).

First, we may reduce to the case of an oriented foliation by passing to the double cover orienting the foliation (cf. e.g., [11, 4.1]). This doubles the Euler characteristic \(\chi\), thereby preserving its negativity. The (fundamental) theorem of Kerékjártó-Whitney (1925 [22], 1933 [37])—to the effect that an oriented foliation admits a compatible flow—fails non-metrically ([9]), but applies to Lindelöf (equivalently metric) subregions. Hence the theory of flows can still be advantageously exploited after some precautions. In its foliated variant, the proof below is pure but uses the index formula for line-fields (involving semi-integral indices), whereas working with flows requires (beside Kerékjártó-Whitney) some ad hoc (but classical) mechanisms for slowing down flow lines (Beck’s technique 1958 [4]), as we shall discuss at the appropriate moment.

Proof of (1.1). Here is the common core of the argument (quite regardless of which viewpoint is adopted, and concretely which version of the index formula is applied). The first ingredient is Nyikos’ theorem [30] according to which an \(\omega\)-bounded surface has a bagpipe decomposition

\[
M = B \cup \bigcup_{i=1}^{n} P_i.
\]

This is to mean that there is a compact bordered surface \(B \subset M\) (referred to as the bag) such that the components of \(M \setminus \text{int}B\) are bordered surfaces \(P_i\) (the pipes) which filled by a disc become simply-connected surfaces \(\Pi_i\). Hence \(\chi(\Pi_i) = 1 - 0 + 0 = 1\) (recall the vanishing of the top-dimensional homology of an open Hausdorff manifold, cf. e.g., Samelson 1965 [35]). It follows by additivity of the characteristic (formally Mayer-Vietoris) that \(\chi(P_i) = 0\) and in turn that

\[\chi(M) = \chi(B).\]  

(1)

As \(B\) is compact its boundary \(\partial B\) consists of finitely many circles \(C_i\) (say \(n\)), which we call the contours of \(B\). Each contour \(C_i\) has a tubular neighborhood \(U_i \approx S^1 \times [-1, 1]\) (also interpretable as a bicollar) whose border \(\partial U_i\) splits in 2 circles \(C_i^+\) and \(C_i^-\). We agree that the plus version \(C_i^+\) is the one lying in the pipe \(P_i\), whereas the minus \(C_i^-\) are all in the bag \(B\) (compare Fig. 2 top part).

Collapse this bagpipe configuration \(M = B \cup \bigcup_{i=1}^{n} P_i\) in essentially 3 ways:

(a) shrink the outer contours \(C_i^+\) to points \(p_i\), to produce a closed surface \(F\) homeomorphic to the bag \(B\) capped off by \(n\) discs (Fig. 2a), hence

\[\chi(F) = \chi(B) + n;\]  

(2)

(b) shrink the inner contours \(C_i^-\) to points \(q_i\), to get \(n\) surfaces \(\Pi_i\) respectively homeomorphic to the pipe \(P_i\) capped off by a disc \(D_i\) (Fig. 2b), hence simply-connected;

(c) shrink both contours \(C_i^+, C_i^-\) simultaneously to points \(p_i\), resp. \(q_i\), to get \(n\) spaces homeomorphic to the sphere \(S^2\) (Fig. 2c).

In each case it is understood that the foliated structure undergoes the same shrinkages, thereby creating isolated singularities precisely at those points where circles are collapsed. If one prefers to argue with flows, first choose a compatible flow on some open (metric) neighborhood of the bag \(B\), and slow it down to be at rest on the circles \(C_i\). This is achieved via Beck’s technique (1958 [4]). Hence the points (=collapsed circles) are the unique rest points of the flow, which can therefore be dissociated into several flows over the elementary pieces of the dissection given by Fig. 2.
Applying the Poincaré index formula (cf. e.g., (3.3) below) in those varied surfaces, we get from collapse (a)
\[ \sum_{i=1}^{n} i(p_i) = \chi(F), \]  
whereas collapse (c) gives
\[ i(p_i) + i(q_i) = \chi(S^2) = 2. \]  
(Claim 2.1) In the filled pipes \( \Pi_i = P_i \cup D_i \) generated by operation (b), the following estimate holds:
\[ i(q_i) \leq 1. \]  
We postpone the verification of (2.1) to complete first the proof of (1.1).

Assembling those five equations (1)–(5), we get
\[ \sum_{i=1}^{n} (2 - i(q_i)) \geq \sum_{i=1}^{n} i(p_i) \geq \chi(F) = \chi(B) + n \geq \chi(M) + n, \]
violating the assumption \( \chi(M) < 0. \)

Proof of Claim 2.1. As the filled pipe \( \Pi_i := P_i \cup D_i \) is simply-connected, let us imagine it pictured in the plane (yet a spicy version thereof going at infinity in a funny way). This depiction has no intrinsic meaning, except reminding us that the Schoenflies theorem holds true in every simply-connected surface.
(Recall from [8] that any Jordan curve in a simply-connected surface bounds a 2-disc, regardless from any metric assumption.)

The key trick (quite omnipresent in Bendixson 1901 [5], or Mather 1982 [27]) is to pay special attention at leaves both of whose ends converge to the ‘origin’ \( q_i \). Call such leaves loops, for short.

![Diagram](image.png)

**Figure 3:** Blocking the proliferation of loops in a big circle \( K \)

Any loop can be completed to a Jordan curve by adding the point \( q_i \). Hence, according to the (universal) Schoenflies theorem [8], any loop \( L \) bounds a unique disc \( D_L \) in \( \Pi \) (Fig. 3b). Thus there is a partial order on the set \( \mathcal{L} \) of all loops by decreeing \( L \leq L' \) whenever the inclusion \( D_L \subset D_{L'} \) holds for the corresponding bounding discs. (Note: \( D_L \) is essentially what Bendixson calls a nodal region.)

Now observe that a well-ordered chain of loops \( (L_\alpha) \) has at most countable ‘height’, i.e. cardinality. Otherwise looking at the successive symmetric differences \( \text{int}D_{L_{\alpha+1}} - D_{L_\alpha} \) inside the disc \( D_i \) gives uncountably many pairwise disjoint open sets in the disc, against its separability (compare Fig. 3c).

Likewise there is at most countably many loops pairwise incomparable w.r.t. the order \( \leq \) on \( \mathcal{L} \). Thus we can find a countable sequence of loops \( (L_n)_{n<\omega} \) cofinal in \( (\mathcal{L}, \leq) \), i.e. for each \( L \in \mathcal{L} \) there is an integer \( n < \omega \) such that \( L \leq L_n \). Then the union of all loops \( \Lambda := \bigcup_{L \in \mathcal{L}} L \) and the union \( \Delta := \bigcup_{n<\omega} D_{L_n} \) have the same closures. Since the latter set, \( \Delta \), is \( \sigma \)-compact (hence Lindelöf), it has a compact closure by \( \omega \)-boundedness.

So the closure \( \overline{\Lambda} \) is a compactum in a simply-connected surface (which is not the sphere), hence contained in a large disc \( D \) (compare Gabard 2011 [11, Lemma 2.34]) whose boundary contour \( \partial D = K \) is a large circle enclosing \( \overline{\Lambda} \) in its interior. By construction the circle \( K \) does not encounter any loop. Further we may assume that \( K \) encloses also the disc \( D_i \) in its interior, and so (by Schoenflies again) \( K \) is freely homotopic to \( C_i \) (in fact the difference \( D - \text{int}D_i \) is an annulus). Accordingly, one may compute the index \( i(q_i) = i(q_i, C_i) \) w.r.t. to the curve \( K \). Finally, classical index theory (cf. Lemma 3.1 below) shows that \( i(q_i, K) \leq 1 \). This is the desired estimate.

## 3 Memento of 2D-index theory

### 3.1 Index of an isolated singularity (Poincaré, Bendixson, Hamburger, etc.)

Without developing the full theory in a coherent fashion, we just recall enough background to establish the next lemma required to complete our argument.
From now on, we switch (as it is quite customary) the generic notation $i$ for the (Poincaré) index to $j \in \frac{1}{2}\mathbb{Z}$ to emphasize its semi-integral nature.

**Lemma 3.1** If a foliation (or a flow) on the punctured plane $\mathbb{R}^2_*$ admits a circle $K$ enclosing the origin through which no leaf is a loop (i.e., a leaf both of whose ends converge to the puncture). Then the index $j(0, K)$ of the origin w.r.t. the curve $K$ is $\leq 1$.

We shall derive this from the classical formula for the Poincaré index due to Bendixson [5, p. 39], which Kerékjártó [22, p. 108–9] assigns to Hamburger 1922 [14]. (Having no access to Hamburger’s paper, we tried to reconstruct a proof, although the original is surely more readable than what to be found bellow.)

Added in proof. Meanwhile several presentations in book forms are available, e.g. Lefschetz 1957 [26, p. 222], Hartman 1964 [17, p. 173, Thm 9.2] and Andronov et al. 1973 [2, p. 511]. Little warning (for ‘googlers’): what is called below Hamburger’s formula (after Kerékjártó [22, p. 108]) is more commonly known as Bendixson’s index formula, and less frequently also as the Poincaré-Bendixson index formula, which is quite fair in view of the formula $J = E - I - 2$ to be found in Poincaré 1885 [33, 1885, p. 203].

**Lemma 3.2** Given a foliation of the plane with an isolated singularity $p$, there is always a polygonal circuit $P$ enclosing the singularity composed of a finite number of leaf-arcs and of cross-arcs. Call a vertex of this polygon convex or concave according as the leaf through it can instantaneously move outside the polygon or stays in its interior (cf. Fig. 4a, where the shaded region depicts the residual component of $P$ containing $p$). Let $c$, $c'$ be the numbers of convex resp. concave vertices. Then the Poincaré index of the singularity $p$ is given by

$$j(p) = 1 - \frac{c - c'}{4}. \quad (6)$$

**Figure 4:** Naive approach to Hamburger’s index formula

**Proof of 3.2**. By general position, we may assume that the circle $K$ has only finitely many tangencies with the foliation of the Morse-type or naively speaking $U$-shaped (cf. Fig. 4b). The “U” may go either inside or outside
the circle $K$ (locally at least), providing a division in internal vs. external tangencies. Choose about each U-shaped leaf-arc tangent to $K$ a little tube (foliated box) resembling a ‘horseshoe’ (Fig. 4). By pushing the circle $K$ inside resp. outside for each external resp. internal tangencies gives the existence of a circuit $K' =: P$ of the desired type (again Fig. 4, thick line).

We aim to compute the Poincaré index, which by definition is the total angular variation of the tangent during a complete circulation around any circle enclosing the singularity (up to division by $2\pi$).

To this end we shall for simplicity assume (yet without justification) that modulo a homeomorphism the configuration can be normalized to one of the geometric type where the horseshoes are so to speak in polar coordinates (Fig. 4). On this picture it is further assumed that the foliation is radial throughout the circular segments.

As on Fig. 4, let $\alpha_i$, resp. $\varepsilon_j$ be the angles swept out by internal tangencies resp. external ones, and $\delta_k$ be the remaining angles corresponding to the arcs of $K\cap K'$ which are radially foliated. We have trivially $\sum \alpha_i + \sum \varepsilon_j + \sum \delta_k = 2\pi$.

An internal tangency, whose horseshoe sweeps out an angle $\alpha_i$, causes an angular variation of $\pi + \alpha_i$ for the turning tangent (compare Fig. 4). Likewise an external tangency of angle $\varepsilon_j$ implies a variation of $-\pi + \varepsilon_j$. Finally a circular portion of the circuit offering an angle of $\delta_k$ contributes to a variation of $\delta_k$.

Thus the total angular variation of the turning tangent(s) is

$$\sum \pi + \alpha_i + \sum (-\pi + \varepsilon_j) + \sum \delta_k = I\pi - E\pi + \sum \alpha_i + \sum \varepsilon_j + \sum \delta_k = 2\pi$$

where $I$, $E$ are the number of internal (resp. external) tangencies. Since the Poincaré index $j$ is the angular variation (divided by $2\pi$), it follows that

$$j = 1 + \frac{I - E}{2}.$$  

Finally, from Fig. 4, we have $2I = c'$, as each internal tangency produces 2 concavities on the circuit $K'$ deformed by the horseshoes, and likewise $2E = c$ as each external tangency contributes to 2 convexities. This proves formula (6).

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With Hamburger’s formula (6) we are ready to complete the proof of (3.1):

**Proof of Lemma 3.1.** Choose a polygonal circuit $K'$ as in (3.2), cf. also Fig. 4 (thick line). We shall describe two surgical processes on the circuit $K'$ diminishing the number $c'$ of concavities so as to make it eventually equal to 0, thereby proving our claim ($j \leq 1$) in the light of Hamburger’s formula

$$j = 1 - \frac{c'}{2}.$$  

As usual the proof involves some pictures. The goal is to kill by surgery the 2 concavities generated by an inner tangency.

If there is no inner tangencies, we are done. Else, fix an inner tangency point $p$. If we extend the U-shaped arc of leaf emanating from $p$, then by assumption (no loops) one at least of both ends must go to infinity. Otherwise a Poincaré-Bendixson argument implies that the leaf starts spiraling towards an asymptotic circle (cycle limite) which by Schoenflies must enclose the puncture. In this case the index computed w.r.t. this circle is clearly 1 (either via Hamburger’s formula or by the very definition of the index). Hence the appropriate semi-leaf emanating from $p$ must eventually leave the bounded domain interior to $K'$ at some escape-point, say $e$. Close the segment of semi-leaf $pe$ by the (unique) arc $A$ of $K'$ joining $p$ to $e$ such that $pe + A$ is not null-homotopic (in the punctured plane). This defines a new Jordan curve $K'' =: J$.

As shown on Fig. 4 two scenarios are possible depending on whether the leaf-arc $pe$ closed by the sub-arc of $K'$ circulator along the clockwise orientation of the circle $K$ or $K'$ encloses or not the puncture.

In the first case (Scenario A on Fig. 4), we observe that the new Jordan curve $J$ is an admissible polygonal circuit where $p$ is concave and $e$ is convex.

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Hence the 2 concavities near $p$ on $K'$ are traded against 1 convexity (at $e$) and 1 concavity at $p$. Of course during the process $K' \rightarrow J$ we may lose several tangencies. In any event, the new numbers $(c_1, c'_1)$ of convex resp. concave vertices (w.r.t. $J$) satisfy $c_1 \leq c + 1$ and more importantly $c'_1 \leq c' - 1$.

In the second case (Scenario B on Fig.5), we observe that the 2 concavities are traded against 2 convexities, yielding thereby $c_1 \leq c + 2$ and $c'_1 \leq c' - 2$.

In both scenarios the number of concavities $c'$ decreases under the surgery $K' \rightarrow J$, and after finitely many iterations reaches (inevitably) the value 0. This completes the proof of $j \leq 1$ in view of Hamburger’s formula.

### 3.2 Foliated index formula (Poincaré, von Dyck, Brouwer, Hadamard, Hamburger, Kneser, Hopf, Lefschetz)

For a smooth vector field with isolated singularities on a closed manifold, there is a well-known (remarkable) identity between the total sum of the indices at the singularities and the Euler characteristic of the manifold. This is known as the *Poincaré-Hopf index formula*. Its intricate history may additionally involve Gauss, Cauchy, Kronecker 1869, Poincaré 1881–1885 [32, 33], Dyck 1888 [7 p. 501], Brouwer and Hadamard circa 1910 [13], etc., up to Hopf 1926 [18], not forgetting Lefschetz for closely related works and the exposition in Alexandroff-Hopf 1935 [1]. (Again we may refer to Mawhin 2000 [28] for a thorough historical discussion, cf. also Hopf 1966 [20].)

For our application we need only the surface case. Besides, there is a well-known formulation of the index formula for line-fields or foliations (compare e.g., Hopf 1946–56 [19] p. 113, 2.2 Thm II or Spivak 1975–79 [36, p. 331]):

**Theorem 3.3** For a foliated closed surface $F$ with isolated singularities, the total sum of the indices is equal to the Euler characteristic of the surface:

$$\sum_{p \in F} j(p) = \chi(F).$$ (8)
We present 2 proofs of this superb theorem (in Lefschetz’s 1967 appreciation [28, p. 120]). The first is mostly inspired from Poincaré’s original paper of 1885. The second (easier to find alone) gives only a reduction to the vector field case, thus not very insightful, so safely to be skipped.

**Historical quiz.** It is not perfectly clear to the writer, who first formulated the index formula for foliations (3.3). Of course, loosely speaking it is Poincaré 1885 [33, p. 203–8], since the foliated case is very akin to the flow case, either by passing to a double cover orienting the foliation or by noticing that Poincaré’s argument transposes better than mutatis mutandis to the foliated situation, as practically nothing must be changed to it (compare the next section). Strictly speaking, the first source might be Hamburger 1924 [15, p. 58–62], where a simple proof under very general assumptions is given (according to Hamburger 1940 [16, Footnote 4, p. 64]). Much of this quiz is clarified by reading Hamburger 1924 [15, Fußnote 17, p. 57] (reproduced below as [Ham]), where Kneser is mentioned as being also well aware of the foliated index formula, cf. Kneser 1921 [24, p. 83] (= [Kne] below), where however [by a little inadvertence?] the semi-integral nature of the index is not emphasized, and the reader is “just” referred to Dyck 1888 [7]. Here are the relevant extracts:

[Ham] 17) Nach Abschluß meines Manuskriptes erfuhr ich durch eine freundliche briefliche Mitteilung von Herrn Helmut Kneser, daß dieser auch im Besitz eines Beweises für die Poincaré’sche Formel in sehr allgemeinen Fällen ist. Seine Methode, den Index \( i \) der Singularität zu bestimmen, ist der Methode der vorliegenden Note sehr ähnlich. Vergleiche die Andeutungen der Kneserschen Methoden und Ergebnisse in dem Verhandlungsberichte des Naturforschertages in Jena. Jahresb. d. D. M. V. 30 (1921), S. 84.

[Kne] Bei Kurvenscharen mit Singularitäten wird jedem Ausnahmepunkt eine Zahl zugeordnet, und \( k \) ist gleich der Summe dieser Zahlen, vgl. Dyck, Math. Ann. 32.

### 3.3 Poincaré’s argument of 1885

The proof presented below is, apart from minor variations (to suit modern conventions), a ‘condensed’ copy of Poincaré’s original argument (compare [33, 1885, p. 203–8]), which is already announced in a *Comptes Rendus* Note of 1881 [32]. In fact the latter assumes orientability of the surface and of the foliation (i.e., argues with vector fields), yet his argument works without those provisos. (Overlooking the details of Poincaré’s argument, one gets the wrong impression that he only establishes the formula for special singularities (cols, nauds and foyers), yet on p. 208 (of *loc. cit.*) the general case is established.)

**Proof of 3.3.** We aim to compute the sum of the indices at the singularities, relating this to the Euler characteristic \( \chi = \sigma_0 - \sigma_1 + \sigma_2 \), where \( \sigma_i \) is the number of \( i \)-simplices (\( i = 0, 1, 2 \)) of any triangulation. (As usual such simplices are resp. termed vertices, edges and triangles.) By general position, we may triangulate the surface \( F \) so that each singularity lies in the interior of a triangle, and each circuit bounding a triangle has only finitely many tangencies with the foliation. By invariance under deformation (homotopy), we can extend the summation to all triangles (precisely their boundaries), as those lacking singularities will not contribute (to the total sum of indices). (Of course, to triangulate concrete pretzels (spheres with handles) as well as their non-orientable avatars (cross-capped spheres), one does not need Rado’s triangulation theorem [34], but it is still somehow implicitly used to classify surfaces.)

Recall, from Eq. (7), that the index is given by Hamburger’s formula (1922) (already in Bendixson 1901 [5, p. 39], and even in Poincaré 1885 [33, p. 203]):

\[
j = 1 + \frac{I - E}{2},
\]

where \( I, E \) are the number of internal resp. external tangencies (or contacts as Poincaré calls them).

Poincaré observes [33, p. 207] that each tangency occurring along an edge of the triangulation is counted twice, once as an internal and once as an external contact, hence does not contribute to the total sum of the indices.

---

\(^{4}\)This standard technique is implicit (already) in Kneser 1921 [24, p. 85]

\(^{5}\)Jargon coined by Dehn–Heegaard in 1907 (*sauf erreur!*).
Hence he needs only to worry about contacts occurring at vertices (of the triangulation). Fix a vertex and assume it to be the apex of \( \nu \) triangles. A leaf (trajectoire in his context) through the vertex will traverse 2 triangles, while having an external contact with the \( \nu - 2 \) remaining triangles. Thus,

\[
\sum_{\text{triangles}} (I - E) = \sum_{\text{vertices}} (2 - \nu) = 2\sigma_0 - 3\sigma_2,
\]

(10)
since \( \sum \nu = 3\sigma_2 \) as each 2-simplex is counted thrice (once for each of its apex).

Next, recall the following relation holding in any triangulated closed surface

\[
3\sigma_2 = 2\sigma_1.
\]

(11)
(This is easily verified by a double counting argument of the incidence relation \( I \) among pairs of triangles given by adjacency: mapping an element of \( I \) to the common edge yields a 2-to-1 surjection to the 1-simplices, whereas projecting on the (first) factor yields a 3-to-1 map to the set of all 2-simplices.)

Thus, we find for the total sum of indices (computed along the contours of all triangles of the triangulation):

\[
\sum_{\text{triangles}} j = \sum_{\text{triangles}} (1 + \frac{I - E}{2}) = \sigma_2 + \frac{2\sigma_0 - 3\sigma_2}{2} = \sigma_0 - \sigma_1 + \sigma_2 = \chi(F).
\]

3.4 Reduction of the index formula to the flow case

Another proof of 3.3. Now, we just recall a formal reduction to the classical index formula for flows (taking the latter’s validity for granted via some external source of your preference).

If the foliation \( \mathcal{F} \) is orientable, then there is a compatible flow by Kerékjártó-Whitney [22, 37], and we assume this case settled.

If not, \( \mathcal{F} \) determines a double cover \( \mathcal{F}^* \rightarrow \mathcal{F} \) over which the lifted foliation \( \mathcal{F}^* \) is orientable. More slowly, one first removes the singular set \( S \subset \mathcal{F} \) of the foliation, and local orientations of leaves defines a double cover \( \Sigma \rightarrow \mathcal{F} - S \), which compatifies as a branched cover \( \pi: \Sigma^* \rightarrow \mathcal{F} \) by filling over the punctures via Riemann’s trick (cf. e.g. [11, 2.16]). Then by the Riemann-Hurwitz formula:

\[
\chi(\Sigma^*) = 2\chi(\mathcal{F}) - \deg(R),
\]

(12)
where \( \deg(R) \) counts the ramification.

Each singularity of the foliation \( \mathcal{F} \) is either orientable (\( O \)) or not (\( N \)), yielding a partition \( S = O \sqcup N \). Singularities in \( N \) (non-orientable) are precisely those responsible of the ramification so that \( \#(N) = \deg(R) \). For \( p \in O \), let \( \pi^{-1}(p) = \{q_1, q_2\} \) and for \( p \in N \), let \( \pi^{-1}(p) = \{p^*\} \).

For a non-orientable singularity, recall the following relation

\[
j = \frac{i + 1}{2},
\]

between its index \( j \) and the index \( i \) of the lifted orientable foliation (compare Lemma 3.4 below)

Finally, computing the total sum of the indices we find

\[
\sum_{p \in S} j(p, \mathcal{F}) = \sum_{p \in O} j(p, \mathcal{F}) + \sum_{p \in N} j(p, \mathcal{F})
\]

\[
= \frac{1}{2} \left( \sum_{q \in \pi^{-1}O} i(q, \mathcal{F}^*) \right) + \sum_{p^* \in \pi^{-1}N} \frac{i(p^*, \mathcal{F}^*) + 1}{2} \quad \text{(Lemma 3.4)}
\]

\[
= \frac{1}{2} \left( \chi(\Sigma^*) + \deg(R) \right) \quad \text{(Poincaré’s index formula)}
\]

\[
= \chi(F) \quad \text{(by Riemann-Hurwitz (12)).}
\]
Lemma 3.4 Given a non-orientable isolated singularity \( p \) of a foliation \( F \) on an elementary (piece of) surface \( F \cong \mathbb{R}^2 \), let \( \pi : F^* \to F \) be the double cover orienting the foliation and let \( F^* \) be the lifted orientable foliation. Then the index \( i(p^*, F^*) \) of the lifted foliation at the unique point \( p^* \) lying above \( p \) and the original index \( j(p, F) \) downstairs are related by:

\[
j(p, F) = \frac{i(p^*, F^*) + 1}{2}
\]

(13)

Proof. As we already used it (twice!), we permit us to deduce this from Hamburger’s formula (although a proof from the scratch definition of the index might also be possible, cf. for this Spivak [36, Lemma 19, p. 328]). Indeed choose \( P \) a polygonal circuit as in (3.2) enclosing the point \( p \), and denote again by \( c \) resp. \( c' \) the number of convex vs. concave vertices. By covering space theory, the map \( \pi \) is topologically equivalent to \( z \mapsto z^2 \) in complex coordinates on the punctured complex plane \( \mathbb{C}^* \). Lifting the circuit \( P \) to the covering \( F^* \) yields an admissible circuit \( P^* = \pi^{-1}(P) \) having doubled quantities of convexities resp. concavities, i.e. \( c_* = 2c \) and \( c'_* = 2c' \). Plugging into Hamburger’s formula (3.2) gives the asserted relation:

\[
i(p^*, F^*) = 1 - \frac{c_* - c'_*}{4} = 1 - \frac{2c - 2c'}{4} = 2\left(1 - \frac{c - c'}{4}\right) - 1 = 2j(p, F) - 1.
\]

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