A DIAGRAMMATIC CALCULUS FOR CATEGORICAL QUANTUM PROTOCOLS

DUŠAN DORDEVIĆ, ZORAN PETRIĆ, AND MLADEN ZEKIĆ

Abstract. We propose an explicit realisation of a category with enough structure to check validity of certain class of quantum protocols. This category is based on the category of 1-dimensional cobordisms. A coherence result proves that it fulfils all desired conditions.

Mathematics Subject Classification (2020): 18M10, 18M40, 18D20, 81P45, 57Q20

Keywords: dagger compact closed category, biproducts, cobordism, entanglement, coherence

1. Introduction

This paper offers a mathematical result and its application in the field of quantum information. The goal is to provide a minimal graphical language sufficient for verification of categorical quantum protocols. This is achieved through a category $1\text{Cob}^\oplus_G$ based on the category of 1-dimensional cobordisms. We will work out in some details a few of quantum protocols and apply a technique of their verification based on the category $1\text{Cob}^\oplus_G$. This work contains both pure mathematical results, as well as results in the applied field of quantum information, which are related to both physics and computer science. Therefore, we will try to make our exposition sufficiently detailed, in order to make the results available to a larger group of scientists from different areas.

There are several diagrammatic calculi proposed earlier (e.g. [4, 14]) appropriate for verification of categorical quantum protocols. Our intention is to make one within a frame of a category relevant for quantum mechanics—the category of cobordisms. This is achieved by relying on a result from [8] just by relaxing (i.e. neglecting some components of) the arrows of a category constructed in that paper. Moreover, we want to get rid of boxes with several ingoing and outgoing wires attached, which are ubiquitous in such calculi. However, some labels of connected components of the underlying 1-manifolds remain in our calculus, but we have minimized their role. One way to get rid of all labels is to increase the dimension of cobordisms in question. We discuss this suggestion in Section 9.

Ever since its formulation in the first half of the 20-th century, quantum mechanics is naturally set to live in a separable Hilbert space. This enables one to talk about notions as entanglement and measurement in an almost trivial way. However, they are far from being understood by a scientific community. While the basic mathematical formalism is easy to understand, its physical meaning is much less clear and various interpretations of quantum mechanics are possible, without currently any basis on which we could select only one correct. This means that it could be fruitful to reconsider some basic notion about quantum mechanics, and to try to formulate it in terms of a different mathematical structure. Similar motivation led authors of [1] to develop the so called categorical approach to quantum mechanics in terms of dagger compact closed categories with biproducts. Of course, the starting point is naturally the case where standard Hilbert space formalism leads to the consideration of finite dimensional spaces (for example, consideration of spin $1/2$
gives rise to the two dimensional Hilbert space; on the other hand, it is not too hard to construct classical theories whose Hilbert space after quantization turns out to be finite dimensional, although they are much less known for an average physicist). For simplicity, and for the practical application, we restrict to the case of two dimensional Hilbert spaces (qubits). Basis vectors are $|0⟩$ and $|1⟩$ (we use Dirac notation for this part). If we denote $H = \text{span} (|0⟩, |1⟩)$, then the total Hilbert space of a composite bipartitive system is $H \otimes H$. As we use tensor product, there are no projections to $H$, and we can introduce the notion of an entangled state. For example, we can form a basis in $H \otimes H$ from the entangled states as

$$|\beta_1⟩ = \frac{1}{\sqrt{2}}(|0⟩ \otimes |0⟩ + |1⟩ \otimes |1⟩),$$
$$|\beta_2⟩ = \frac{1}{\sqrt{2}}(|0⟩ \otimes |1⟩ + |1⟩ \otimes |0⟩),$$
$$|\beta_3⟩ = \frac{1}{\sqrt{2}}(|0⟩ \otimes |1⟩ - |1⟩ \otimes |0⟩),$$
$$|\beta_4⟩ = \frac{1}{\sqrt{2}}(|0⟩ \otimes |0⟩ - |1⟩ \otimes |1⟩).$$

Basis $\{β_i\}^4_{i=1}$ is usually referred to as Bell basis (see e.g. [12]). Of course, this construction is made on a few assumptions. The first one is a linear structure of vector spaces. Historically, linear structure was a natural guess based on an intuition about wavelike properties of particles (electron). For example, it was known long before the birth of quantum mechanics that light can be described in terms of oscillating electric and magnetic field, and that for those fields superposition principle holds (linearity of Maxwell’s equations). Moreover, intensity of a wave is proportional to $|E|^2$ (where $E$ stands for a complex representative of electric field), and this further motivates the Born rule for a probabilities associated with the measurement outcome. Despite its success, it is still intriguing to consider theories without vector space structure.

Quantum mechanics can be considered as a special case of quantum field theories for 0 + 1 dimensions. On the other hand, it is well known that one approach to quantum field theories (especially to the case of topological quantum field theories) is using cobordisms to represents space-time evolution processes. This opens another natural question, and that is to what extent one can use the category $\mathcal{C}ob$ (of 1-dimensional cobordisms) to simulate quantum mechanical processes.

In order to obtain this correspondence, we will introduce the notion of $\mathcal{G}$-cobordisms. They correspond to a regular cobordisms (for a precise definition of 1-dimensional cobordisms see the following sections), but with additional structure, such that each connected component has an element of a group $\mathcal{G}$ attached. This introduces a notion of a $\mathcal{G}$-segment or a $\mathcal{G}$-circle. Group elements will play the role of (unitary) transformations that can be done on a quantum state.

On the other hand, a motivation for our work can be purely mathematical. In category theory and its application, it is of great importance to establish whether a certain diagram commutes. Usually, this is done by inspection, using a set of equalities (for example as those from Appendix A). Though in principle a straightforward task, it usually consumes a non-negligible amount of time. For this reason, it is practical to prove certain coherence results. Such a result enables one to check the commutativity of diagrams, consisting of canonical arrows of a certain categorical structure, just by drawing pictures in an appropriate graphical language. A detailed explanation of our approach to coherence is given in [13 Introduction], where also results akin to those proven here are presented. Briefly, we start with a freely generated category built out of syntax material, whose objects are formulae
and arrows are equivalence classes of terms in an equational system. Then we show a completeness result with respect to a model in a form of a graphical category.

In Section 2 we review some basic categorical notions relevant for this paper. In Section 3 we further discuss the category 1Cob. Section 4 introduces two compact closed categories with some additional structure both freely generated by a free group considered as a category. The isomorphism of these two categories is established in that section. Next two sections (Sections 5 and 6) are technical necessity,

forthcoming in that section. A comparison of our work with other similar approaches is given in Section 9, together with some suggestions for a future work. Appendix A contains an equational presentation of dagger compact closed categories with dagger biproducts and Appendix B discusses a categorical approach to scalars and probability amplitudes.

2. Closed categories and biproducts

Some notions from category theory relevant for this paper are introduced in this section. A symmetric monoidal category is a category $\mathcal{A}$ equipped with a distinguished object $I$, a bifunctor $\otimes: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ (we abbreviate $1_a \otimes f$ and $f \otimes 1_a$ by $a \otimes f$ and $f \otimes a$, respectively) and the natural isomorphisms $\alpha$, $\lambda$, and $\sigma$ with components $\alpha_{a,b,c}:a \otimes (b \otimes c) \to (a \otimes b) \otimes c$, $\lambda_a:I \otimes a \to a$ and $\sigma_{a,b}:a \otimes b \to b \otimes a$. (Note that in [1], $A$ denotes the inverse of our $\lambda$ and due to the presence of symmetry, we do not introduce a name for the isomorphism $a \cong a \otimes I$.) Moreover, the coherence conditions concerning the arrows of $\mathcal{A}$ (see the equalities A.6-A.8 in Appendix A) hold. A symmetric monoidal category is monoidally strict when the operation $\otimes$ on its objects is associative with $I$ being the neutral, and moreover, the arrows $\alpha$ and $\lambda$ are identities.

A compact closed category is a symmetric monoidal category in which every object $a$ has its dual $a^*$. This means that there are units $\eta_a:I \to a^* \otimes a$ and counits $\varepsilon_a:a \otimes a^* \to I$ such that the equalities A.9 of Appendix A hold. If a functor between two compact closed categories preserves this structure “on the nose”, then we say that it strictly preserves the compact closed structure, and we use the same terminology in other cases.

It is straightforward to conclude that the following isomorphisms hold in every compact closed category.

$$u_{a,b}: (a \otimes b)^* \cong b^* \otimes a^*, \quad v: I^* \cong I, \quad w_a:a^{**} \cong a$$

(In a monoidally strict compact closed category

$$u_{a,b} = (a^* \otimes a^* \otimes \varepsilon_{a \otimes b}) \circ (a^* \otimes \eta_b \otimes b \otimes (a \otimes b)^*) \circ (\eta_b \otimes (a \otimes b)^*)$$

$v = \varepsilon_I$ and $w_a = (\varepsilon a^* \otimes a) \circ (\sigma_{a^{**},a^*} \otimes a) \circ (a^{**} \otimes \eta_a)$.) A compact closed category is strict when it is monoidally strict and $(a \otimes b)^* = b^* \otimes a^*$, $I^* = I$ and $a^{**} = a$, while $u_{a,b}$, $v$ and $w_a$ are identities.

For quantum protocols discussed below, the following derived operations on arrows of a compact closed category are frequently used. For $f: a \to b$ its name $\gamma f^*: I \to a^* \otimes b$ and its coname $\psi f: a \otimes b^* \to I$ are defined as

$$\gamma f^* = (a^* \otimes f) \circ \eta_a, \quad \psi f = \varepsilon_b \circ (f \otimes b^*)$$

The function $*$ on objects of a compact closed category $\mathcal{A}$, extends to a functor $*: \mathcal{A}^{op} \to \mathcal{A}$ in the following way. For $f: a \to b$, let $f^*: b^* \to a^*$ be

$$\lambda_{a^*} \circ \sigma_{a^*,I} \circ (a^* \otimes \varepsilon_b) \circ \alpha_{a^*,b,b}^{-1} \circ ((a^* \otimes f) \otimes b^*) \circ (\eta_b \otimes b^*) \circ \lambda_{b^*}^{-1}.$$
A dagger category is a category $\mathcal{A}$ equipped with a functor $\dagger: \mathcal{A} \to \mathcal{A}^{op}$ such that for every object $a$ and every arrow $f$ of this category $a^\dagger = a$, and $f^{\dagger\dagger} = f$. (For more details see [11, 2].) A dagger compact closed category is a compact closed category $\mathcal{A}$, which is also a dagger category satisfying the equalities $A.16 - A.19$ of Appendix A.

By composing the functors $\dagger$ and $^*$ one obtains the functor $\ast = \ast \circ \dagger: \mathcal{A} \to \mathcal{A}$ ($a_\ast = a^\ast$, $f_\ast = (f^\dagger)^*$). For a strict dagger compact closed category $\mathcal{A}$, the functor $\ast$ satisfies

$$f_{\ast\ast} = f, \quad (f_\ast)^\ast = (f^\ast)_\ast.$$ 

A zero-object is an object which is both initial and terminal. For a category with a zero-object $0$ there is a composite $0_{a,b}: a \to 0 \to b$ for every pair $a, b$ of its objects, and for every other zero-object $0'$ of this category, the composite $a \to 0' \to b$ is equal to $0_{a,b}$. A biproduct of $a_1$ and $a_2$ in a category with a zero-object consists of a coproduct and a product diagram

$$a_1 \xleftarrow{i_1} a_1 \oplus a_2 \xrightarrow{i_2} a_2, \quad a_1 \xleftarrow{\pi_1} a_1 \oplus a_2 \xrightarrow{\pi_2} a_2$$

for which

$$\pi^j \circ i^j = \begin{cases} 1_{a_i}, & i = j, \\ 0_{a_i, a_j}, & \text{otherwise}, \end{cases} (2.1)$$

where $i, j \in \{1, 2\}$ (cf. the equalities $A.16 - A.19$ in Appendix A). For arrows $f_1: a_1 \to c$ and $f_2: a_2 \to c$, the unique arrow $h: a_1 \oplus a_2 \to c$ for which $h \circ i^j = f_i$, $i \in \{1, 2\}$ is denoted by $[f_1, f_2]$, and for arrows $g_1: c \to a_1$ and $g_2: c \to a_2$, the unique arrow $h: c \to a_1 \oplus a_2$ for which $\pi^j \circ h = g_i, i \in \{1, 2\}$ is denoted by $\langle f_1, f_2 \rangle$.

More generally, a biproduct of a family of objects $\{a_j \mid j \in J\}$ consists of a universal cocone (coproduct diagram) and a universal cone (product diagram)

$$\{i^j: a_j \to \bigoplus_{j \in J} a_j \mid j \in J\}, \quad \{\pi^j: \bigoplus_{j \in J} a_j \to a_j \mid j \in J\}$$

for which the equality $2.1$ holds for all $i, j \in J$. A category with biproducts is a category with a zero-object and biproducts for every pair of objects. A biproduct is a dagger biproduct when for every pair $a, b$ of objects the equalities $A.21$ of Appendix A hold.

For $f, g: a \to b$ in a category with biproducts whose codiagonal and diagonal maps are $\mu_b: b \oplus b \to b$ and $\mu_a: a \to a \oplus a$ one defines $f + g$ as $\mu_b \circ (f \oplus g) \circ \mu_a$. This operation on the set $\Hom(a, b)$ of arrows from $a$ to $b$ is commutative and has $0_{a,b}$ as neutral. Moreover, the composition distributes over $+$. Hence, every category with biproducts may be conceived as a category enriched over the category $\mathsf{Cmd}$ of commutative monoids.

Alternatively, to define biproducts in a category enriched over $\mathsf{Cmd}$ it suffices to assume the existence of a bifunctor $\oplus$, a special object $0$, and for every pair of objects $a, b$ the arrows $\tau_{a,b}: a \oplus b \to a$, $\tau_{a,b}: a \oplus b \to b$, $\tau_{a,b}: a \to a \oplus b$ and $\tau_{a,b}: b \to a \oplus b$, for which the equalities $A.14 - A.19$ of Appendix A hold. As a justification of this approach see the proof of Corollary 5.3 below.

In a compact closed category with biproducts, tensor distributes over $\oplus$, i.e. there exist distributivity isomorphisms $\tau_{a,b,c}: a \otimes (b \oplus c) \to (a \otimes b) \oplus (a \otimes c)$ and $\nu_{a,b,c}: (a \otimes b) \oplus c \to (a \otimes c) \oplus (b \otimes c)$ explicitly given by

$$\tau_{a,b,c} = \langle 1_a \otimes 1_{b,c}, 1_a \otimes \pi_{b,c}^1 \rangle, \quad \tau_{a,b,c}^{-1} = \langle [1_a \otimes 1_b, 1_a \otimes 1_{b,c}] \rangle,$$

$$\nu_{a,b,c} = \langle \pi_{a,b}^1 \otimes 1_c, \pi_{a,b}^2 \otimes 1_c \rangle, \quad \nu_{a,b,c}^{-1} = \langle 1_{a,b} \otimes 1_c, \pi_{a,b,c}^2 \otimes 1_c \rangle.$$ (2.2) (2.3)

(We are aware that it is hard to distinguish between the Latin letter $v$, which is reserved for the isomorphism from $I^*$ to $I$ and the Greek letter $\nu$ denoting the isomorphism of the form $(a \oplus b) \otimes c \to (a \otimes c) \oplus (b \otimes c)$, but we decided to follow
the notation from [S] relevant for the strict compact closed structure, and from [I] which is relevant for categorical quantum protocols.)

In a compact closed categories with biproducts, the scalars, i.e. the endomorphisms from \( I \) to \( I \) form a commutative semiring \( \text{Hom} (I, I) \). The multiplication in this semiring is given by composition, for which \( 1_I \) is the neutral, and the addition is defined as above. (We will omit \( \circ \) when we compose, i.e. multiply, scalars.) For a scalar \( s : I \to I \) and an object \( a \) of such a category, one defines the arrow \( s_a : a \to a \) as the composition

\[
a \xrightarrow{\lambda_a^{-1}} I \otimes a \xrightarrow{s \otimes a} I \otimes a \xrightarrow{\lambda_a} a,
\]

and the operation \( s \ast \) on arrows such that for \( f : a \to b \), the arrow \( s \ast f \) is \( f \circ s_a \). It is straightforward to check that this new operation satisfies the following equalities.

\[
(2.4) \quad a \otimes (s \ast f) = s \ast (a \otimes f), \quad (s \ast f) \otimes a = s \ast (f \otimes a),
\]

\[
(2.5) \quad (s_2 \ast f_2) \circ (s_1 \ast f_1) = s_2 s_1 \ast (f_2 \circ f_1),
\]

\[
(2.6) \quad (s \ast f_1, \ldots, s \ast f_n) = s \ast (f_1, \ldots, f_n).
\]

**Example 1.** As a paradigm for dagger compact closed category with dagger biproducts we use the category \( \text{fdHilb} \) of finite dimensional Hilbert spaces over the field \( \mathbb{C} \) of complex numbers. The objects of this category are finite dimensional Hilbert spaces (finite dimensional vector spaces with inner product). The arrows of this category correspond to (bounded) linear maps between vector spaces. Dagger is given by an adjoint map. Since every vector space over \( \mathbb{C} \) of dimension \( n \) is isomorphic to \( \mathbb{C}^n \), we can pass from \( \text{fdHilb} \) to its skeleton consisting of objects of the form \( \mathbb{C}^n \). By choosing orthogonal bases of such objects, the linear maps are envisaged as matrices. In this case, dagger corresponds to the usual adjoint of matrices (conjugation and transposition), and the operation \( * \) on arrows corresponds to the complex conjugation of an operator (matrix). Also, the operation \( \ast \) on arrows is given by a matrix transpose.

3. **The category \( 1\text{Cob} \)**

The category \( 1\text{Cob} \) of 1-dimensional cobordisms has as objects closed oriented 0-dimensional manifolds i.e. finite (possibly empty) sequences of points together with their orientation (either + or −). For example, an object of \( 1\text{Cob} \) is +++−−−−−. Since there will be several roles of \( \emptyset \) in this paper, we denote the empty sequence of points by \( o \).

A compact oriented 1-dimensional topological manifold with boundary, i.e. a finite collection of oriented circles and line segments is called here 1-manifold. For objects \( a \) and \( b \) of \( 1\text{Cob} \), a 1-cobordism from \( a \) to \( b \) is a triple \((M, f_0 : a \to M, f_1 : b \to M)\), where \( M \) is a 1-manifold and \( f_0, f_1 \) are embeddings. The boundary of \( M \) is \( \Sigma_0 \sqcup \Sigma_1 \) and its orientation is induced from the orientation of \( M \). The embedding \( f_0 \) is orientation preserving and its image is \( \Sigma_0 \), while the embedding \( f_1 \) is orientation reversing and its image is \( \Sigma_1 \). Two cobordisms \((M, f_0, f_1)\) and \((M', f_0', f_1')\) from \( a \) to \( b \) are equivalent, when there is an orientation preserving homeomorphism \( F : M \to M' \) such that the following diagram commutes.

\[
\begin{array}{ccc}
\text{a} & \searrow^{f_0} & \downarrow^M & \nearrow_{f_1} & \text{b} \\
\downarrow_{f_0'} & \text{F} & & \downarrow_{f_1'} & \text{M'}
\end{array}
\]
The equivalence classes of 1-cobordisms are the arrows of $1\text{Cob}$. The identity $1_a: a \to a$ is the equivalence class of $(a \times \mathcal{I}, x \mapsto (x, 0), x \mapsto (x, 1))$, which in the case $a = o$ stands for the empty 1-cobordism from the empty sequence of oriented points $o$ to itself. Two cobordisms $(M, f_0, f_1): a \to b$ and $(N, g_0, g_1): b \to c$ are composed by “gluing”, i.e. by making the pushout of $M \xleftarrow{f_1} b \xrightarrow{g_0} N$. All the arrows of $1\text{Cob}$ are illustrated so that the source of an arrow is at the top, while its target is at the bottom of the picture. Therefore, the direction of pictures is top to bottom, a convention used in [13]. Note that some authors use a different convention, left to right, or bottom to top [10, 17]. The latter is presumably the most popular in the physics literature.

The category $1\text{Cob}$ is dagger strict compact closed. We have symmetric monoidal structure on $1\text{Cob}$ in which $\otimes$ on objects is defined by concatenation, the empty sequence $o$ is the neutral and serves as the unit object $I$, while $\otimes$ on arrows is given by putting two cobordisms “side by side”. The arrows $\alpha$ and $\lambda$ are identities and symmetry $\sigma$ is generated by transpositions $++ \to ++$, $+- \to -+$, $-+ \to +-$, and $-\to -+$. These transpositions are illustrated as follows:

For example, the transposition $+-+- \to -++$ is given by a manifold consisting of two oriented segments and two embeddings of the source $+-$ and the target $-++$ into its boundary.

The dual $a^*$ of an object $a$ is the reversed sequence of points with reversed orientation. For example, if $a = ++--$, then $a^* = +++-$. (Note that this definition differs from the one given in [13] where just the orientation was reversed—both definitions are correct in presence of symmetry.) The arrows $\eta: o \to a^* \otimes a$ and $\varepsilon: a \otimes a^* \to o$, for $a$ as above are the following cobordisms:

It is not difficult to check that the arrows $u_{a,b}, v$ and $w_a$ are all identities.

Let $f: a \to b$ be an arrow of $1\text{Cob}$ represented by a triple $(M, f_0: a \to M, f_1: b \to M)$. Its name $\Gamma f: o \to a^* \otimes b$ is represented by the triple $(M, g_0: o \to M, g_1: a^* \otimes b \to M)$, where for every point $x$ of $a$ and the corresponding point $\bar{x}$ of $a^*$ we have $g_1(\bar{x}) = f_0(x)$ and for every point $y$ of $b$ we have $g_1(y) = f_1(y)$. The coname $\gamma f$ of $f$ is defined in $1\text{Cob}$ analogously.
The arrow \( f^* : b^* \to a^* \) is represented by the triple \((M, h_0 : b^* \to M, h_1 : a^* \to M)\), where for every point \( x \) of \( a \) and the corresponding point \( \bar{x} \) of \( a^* \) we have \( h_1(\bar{x}) = f_0(x) \), and for every point \( y \) of \( b \) and the corresponding point \( \bar{y} \) of \( b^* \) we have \( h_0(\bar{y}) = f_1(y) \).

The cobordism \( f^! : b \to a \) is obtained by reversing the orientation of the 1-manifold representing the cobordism \( f : a \to b \). It is not hard to check that the equalities A.21-A.23 of Appendix A hold.

\[
\begin{align*}
\text{f:} & \quad \text{r f\text{\textsuperscript{\dagger}}:} \\
\text{\text{\text{\textsuperscript{\dagger}}}} f & \quad \text{\text{\textsuperscript{\dagger}}} f \quad \text{f:}
\end{align*}
\]

By the above definitions of \( f^* \) and \( f^! \) for a cobordism \( f : a \to b \) it is straightforward to reconstruct the cobordism \( f_* = (f^!)^*: a^* \to b^* \).

4. A PAIR OF FREE CATEGORIES

We start with a construction of a dagger compact closed category \( \mathcal{F}^! \) with dagger biproducts freely generated by a single object \( p \) and a set \( \Gamma \) of unitary endomorphisms on this object. An arrow \( f : a \to b \) in a dagger category is unitary when \( f^! : b \to a \) is its both-sided inverse. The universal property of \( \mathcal{F}^! \) is the following: for every dagger compact closed category \( \mathcal{C} \) with dagger biproducts, and a function \( \varphi \) from the set \( \Gamma \) to the set of unitary endomorphisms of an object \( c \) of \( \mathcal{C} \), there exists a unique functor \( F : \mathcal{F}^! \to \mathcal{C} \) strictly preserving the whole structure, such that \( Fp = c \) and for every \( \gamma \in \Gamma \), \( F\gamma = \varphi(\gamma) \).

Our construction of this category is syntactical; it is akin to the construction of the category \( F\mathcal{A} \) from [8, Sections 3-4], and it follows the construction of the category \( \mathcal{F}p \) from [13, Section 4]. As noted in [8], it is “perfectly general, applying to categories with any explicitly-given equational extra structure”. The objects of \( \mathcal{F}^! \) are the formulae built out of a single letter \( \dagger \) and \( \uparrow \), with \( \uparrow \) a unary operational symbol \( \dagger \) and \( \uparrow \), and for every \( \gamma \in \Gamma \), \( \gamma^\dagger = \gamma^{-1} \).

The arrows of \( \mathcal{F}^! \) are obtained as equivalence classes of terms built in the following manner. We start with primitive terms, which are of the form \( \gamma, \gamma^{-1} \) for every \( \gamma \in \Gamma \), or \( 1_a, \alpha_{a,b,c}, \alpha_{a,b,c}^1, \lambda_{a}, \lambda_{a}^{-1}, \sigma_{a,b}, \eta_{a}, \varepsilon, \pi_{a,b}^1, \pi_{a,b}^2, \iota_{a,b}^1, \iota_{a,b}^2, 0_{a,b} \), for all objects \( a, b \) and \( c \) of \( \mathcal{F}^! \). The terms are built out of primitive terms with the help of one unary operational symbol \( \dagger \) and four binary operational symbols \( \odot, \odot, +, \odot \). (Each such term is equipped with the source and the target, which are objects of \( \mathcal{F}^!, \) e.g. the source and the target of every \( \gamma \in \Gamma \) is \( p \), and constructions of terms with \( + \) and \( \odot \) are restricted to appropriate sources and targets.) The equivalence classes of these terms, i.e. the arrows of \( \mathcal{F}^! \), are obtained modulo the congruence generated by the equalities A.1-A.2 of Appendix A and, for every \( \gamma \in \Gamma \), the equalities A.1-A.2 below.

\[
\begin{align*}
(\text{4.1}) \quad \gamma \circ \gamma^{-1} & = 1_p = \gamma^{-1} \circ \gamma, \\
(\text{4.2}) \quad \gamma^\dagger & = \gamma^{-1}.
\end{align*}
\]
On the other hand, consider the category $F$ with the same objects and the same primitive terms as $F^\dagger$, just the terms of $F$ are constructed without the unary operational symbol $i$. The arrows of $F$, are the equivalence classes of these terms, modulo the congruence generated by the equalities $[1.1]$ and $[1.2]$. The category $F$ is a compact closed category with biproducts freely generated by the group (envisioned as a category with one object) freely generated by the set $\Gamma$. The universal property of $F$ is the following: for every compact closed category $C$ with biproducts, and a function $\varphi$ from the set $\Gamma$ to the set of automorphisms of an object $c$ of $C$, there exists a unique functor $F : F \to C$ strictly preserving the whole structure, such that $Fp = c$ and for every $\gamma \in \Gamma$, $F\gamma = \varphi(\gamma)$.

**Proposition 4.1.** The categories $F^\dagger$ and $F$ are isomorphic.

**Proof.** From the equalities $[1.20]-[1.26]$ it follows that every arrow of $F^\dagger$ (as an equivalence class of terms) contains a $\dagger$-free term. Also, every equality assumed for $F^\dagger$ in which $\dagger$ appears boils down to the trivial identity after $\dagger$-elimination at both sides. Thus, the identity on objects and the function on arrows that maps the equivalence class of a term in $F$ to the equivalence class of the same term in $F^\dagger$ is an isomorphism between these two categories. \hfill \Box

5. Injections and Projections

For the functor $*: F^{op} \to F$ defined as in Section 2 the unit $\eta$ and the counit $\xi$ become dinatural, i.e. for $f : a \to b$ the following equalities hold:

$$ (a^* \otimes f) \circ \eta_a = (f^* \otimes b) \circ \eta_b, \quad \xi_a \circ (a \otimes f^*) = \xi_b \circ (f \otimes b^*). $$

Also, for arrows $f, g : a \to b$ in $F$, the following equality holds,

$$ (f + g)^* = f^* + g^*. $$

**Definition 5.1.** Let $a$ be an object of $F$. By induction on complexity of $a$, we define two finite sequences $I_a = (i_a^0, \ldots, i_a^{n-1})$ (the injections of $a$) and $\Pi_a = (\pi_a^0, \ldots, \pi_a^{n-1})$ (the projections of $a$) of arrows of $F$ in the following way. If $a$ is the letter $p$ or either $I$ or 0, then $n = 1$ and $I_a = (I_a) = \Pi_a$. Let us assume that $I_{a_1} = (i_{a_1}^0, \ldots, i_{a_1}^{n_1-1})$, $\Pi_{a_1} = (\pi_{a_1}^0, \ldots, \pi_{a_1}^{n_1-1})$ and $I_{a_2} = (i_{a_2}^0, \ldots, i_{a_2}^{n_2-1})$, $\Pi_{a_2} = (\pi_{a_2}^0, \ldots, \pi_{a_2}^{n_2-1})$ are already defined. For $|x|$ being the floor function of a real $x$, i.e. the greatest integer less than or equal to $x$, and $i \mod n$ being the residue of $i$ modulo $n$, we have the following.

- If $a = a_1 \odot a_2$, then $n = n_1 \cdot n_2$, and for $0 \leq i < n_1 \cdot n_2$,

$$ i_a^i = i_1^{i/n_2} \odot i_2^{i \mod n_2}, \quad \pi_a^i = \pi_1^{i/n_2} \odot \pi_2^{i \mod n_2}. $$

- If $a = a_1^*$, then $n = n_1$, and for $0 \leq i < n_1$,

$$ i_a^i = (i_1^n)^* \quad \pi_a^i = (i_1^n)^*. $$

- If $a = a_1 \odot a_2$, then $n = n_1 + n_2$, and for $0 \leq i < n_1 + n_2$, $s_i = \frac{\min\{i, n_1\}}{n_1}$

$$ i_a^i = \begin{cases} i_1 + i_2, & 0 \leq i < n_1, \\ i_2, & n_1 \leq i < n_1 + n_2, & \text{ otherwise,} \end{cases} $$

$$ \pi_a^i = \begin{cases} \pi_1 \circ \pi_2, & 0 \leq i < n_1, \\ \pi_2, & n_1 \leq i < n_1 + n_2, & \text{ otherwise.} \end{cases} $$

**Example 2.** Let $a = (p \oplus I) \oplus 0$ and $b = ((p \oplus 0) \oplus p) \oplus (I \oplus p)^*$. Then $i_a^i$ and $\pi_a^i$ for $0 \leq i < 3$ as well as $i_b^i$ and $\pi_b^i$ for $0 \leq j < 6$ are given in the following tables.
Then we have projections. For every object $a$, the target of $i_a^i$ and the source of $\pi_a^i$ are both equal to $a$, while the source $a^i$ of $i_a^i$ is equal to the target of $\pi_a^i$, and $a^i$ is $\oplus$-free. Moreover, if $a$ is $\oplus$-free, then $I_a = (1_a) = \Pi_a$.

The following proposition establishes the desired properties of injections and projections.

**Proposition 5.2.** For every object $a$ of $\mathcal{F}$

$$\pi_a^i \circ i_a^i = \begin{cases} 1_{a^i}, & i = j, \\ 0_{a^{i,i'}}, & \text{otherwise}, \end{cases} \quad \sum_{i=0}^{n-1} i_a^i \circ \pi_a^i = 1_a.$$  

**Proof.** We proceed by induction on complexity of $a$. When $a$ is $p, I$ or 0, all injections and projections are identities, and the claim holds. For the inductive step, we consider the following three cases.

1. Suppose that $a = a_1 \otimes a_2$, where $|I_{a_1}| = |\Pi_{a_1}| = n_1$ and $|I_{a_2}| = |\Pi_{a_2}| = n_2$. Then we have

$$\pi_a^i \circ i_a^i = (\pi_1^i \otimes \pi_{n_2}^{j \mod n_2}) \circ (i_1^i \otimes i_{n_2}^{j \mod n_2})$$

$$= (\pi_1^i \otimes i_1^i) \otimes (\pi_{n_2}^{j \mod n_2} \circ i_{n_2}^{j \mod n_2}),$$

and the first claim follows by the inductive hypothesis. Also, using the inductive hypothesis and the equality $[A.27]$ of Appendix $A$, we have

$$\sum_{i=0}^{n-1} i_a^i \circ \pi_a^i = \sum_{i=0}^{n_1-1} (i_1^i \circ \pi_1^{k \mod n_2}) \circ (i_2^{k \mod n_2} \circ \pi_{n_2}^{k \mod n_2})$$

$$= \sum_{i=0}^{n_1-1} (i_1^i \circ \pi_1^{k \mod n_2}) \otimes (i_2^{k \mod n_2} \circ \pi_{n_2}^{k \mod n_2})$$

$$= \sum_{k=0}^{n_1-1} (i_1^k \circ \pi_1^{k \mod n_2}) \otimes a_2 = 1_{a_1} \otimes a_2 = 1_{a_1 \otimes a_2}.$$

2. When $a = a_1^i$, we have

$$\pi_a^i \circ i_a^i = (i_a^i)^* \circ (\pi_1^i)^* = (\pi_1^i \circ i_a^i)^*,$$

and the first claim follows by the inductive hypothesis. Also, using the inductive hypothesis and the equality $[A.27]$ we have

$$\sum_{i=0}^{n-1} i_a^i \circ \pi_a^i = \sum_{i=0}^{n-1} (\pi_1^i)^* \circ (i_a^i)^* = \sum_{i=0}^{n-1} (i_a^i \circ \pi_1^i)^* = \left(\sum_{i=0}^{n-1} i_a^i \circ \pi_1^i\right)^* = (1_{a_1})^* = 1_a.$$
(3) Suppose that \( a = a_1 \oplus a_2 \), and again \( |I_{a_1}| = |\Pi_{a_1}| = n_1 \) and \( |I_{a_2}| = |\Pi_{a_2}| = n_2 \). We have

\[
\pi_a^i \circ t_a^i = \pi_{a_1}^{i-n_1 \cdot s_i} \circ t_{a_1}^{1+s_i} \circ \pi_{a_2}^{1+s_i} \circ t_{a_2}^{1+n_1 \cdot s_i}.
\]

Since \( i \neq j \) implies \( 1 + s_i \neq 1 + s_j \) or \( i - n_1 \cdot s_i \neq j - n_1 \cdot s_j \), the first claim follows according to the inductive hypothesis. For the second claim, we have

\[
\sum_{i=0}^{n-1} t_a^i \circ \pi_a^i = \sum_{i=0}^{n_1-1} t_{a_1}^i \circ \pi_{a_1}^i \circ \pi_{a_2}^i + \sum_{j=0}^{n_2-1} t_{a_1}^j \circ \pi_{a_1}^j \circ \pi_{a_2}^j = 1_{a_1 \oplus a_2}.
\]

As a corollary of Proposition 5.2, we have the following.

**Corollary 5.3.** For every object \( a \) of \( F \), the cocone \((a, I_a)\) together with the cone \((a, \Pi_a)\) make a biproduct.

**Proof.** In order to show that the cocone \((a, I_a)\) is universal, consider for \( 0 \leq i < n \) arrows \( f^i: a^i \rightarrow c \) of \( F \) and define \( h: a \rightarrow c \) to be \( \sum_{i=0}^{n-1} f^i \circ \pi_a^i \). For every \( 0 \leq i < n \), by the left-hand side equality of Proposition 5.2, we have that \( h \) satisfies \( h \circ t_a^i = f^i \).

Assume that \( h': a \rightarrow c \) for every \( 0 \leq i < n \) satisfies \( h' \circ t_a^i = f^i \). We conclude that

\[
h' \circ \sum_{i=0}^{n-1} t_a^i \circ \pi_a^i = \sum_{i=0}^{n-1} f^i \circ \pi_a^i = h,
\]

and by the right-hand side equality of Proposition 5.2, we have \( h' = h \). That \((a, \Pi_a)\) is a universal cone is proved analogously. \( \square \)

6. A NORMALISATION

Our normalisation of arrows of the category \( F \) is a procedure derived from the one developed in [13] Section 5]. The goal is to represent every arrow of \( F \), whose source and target are \( \oplus \)-free, by a term free of occurrences of \( \oplus \), \( \iota \) and \( \pi \).

For every arrow \( u: a \rightarrow b \) of \( F \), where \( I_a = (I_{a_1}^{n_1}, \ldots, I_{a_n}^{n_n}) \), \( \Pi_b = (\pi_b^{m_1}, \ldots, \pi_b^{m_r}) \), let \( M_u \) be the \( m \times n \) matrix whose \( ij \)-entry is \( \pi_b^j \circ u \circ \pi_a^i \); \( a^j \rightarrow b^i \). Let \( X \) be an \( m \times n \) matrix whose \( ij \)-entry is an arrow of \( F \) from \( a^j \) to \( b^i \) and let \( Y \) be a \( q \times r \) matrix whose \( ij \)-entry is an arrow of \( F \) from \( c^i \) to \( d^j \). We define \( X \otimes Y \) as the Kronecker product of matrices over a field, save that the multiplication in the field is replaced by the tensor product of arrows in \( F \). For example,

\[
\begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \end{pmatrix} \otimes \begin{pmatrix} y_{00} & y_{01} \\ y_{10} & y_{11} \end{pmatrix}
\]

is

\[
\begin{pmatrix} x_{00} \otimes y_{00} & x_{00} \otimes y_{01} & x_{01} \otimes y_{00} & x_{01} \otimes y_{01} & x_{02} \otimes y_{00} & x_{02} \otimes y_{01} \\ x_{00} \otimes y_{10} & x_{00} \otimes y_{11} & x_{01} \otimes y_{10} & x_{01} \otimes y_{11} & x_{02} \otimes y_{10} & x_{02} \otimes y_{11} \\ x_{10} \otimes y_{00} & x_{10} \otimes y_{01} & x_{11} \otimes y_{00} & x_{11} \otimes y_{01} & x_{12} \otimes y_{00} & x_{12} \otimes y_{01} \\ x_{10} \otimes y_{10} & x_{10} \otimes y_{11} & x_{11} \otimes y_{10} & x_{11} \otimes y_{11} & x_{12} \otimes y_{10} & x_{12} \otimes y_{11} \end{pmatrix}.
\]

Also, we define \( X \oplus Y \) as the matrix of arrows of \( F \), schematically presented as

\[
\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.
\]
More precisely, for \( X = (x_{ij})_{m \times n} \) and \( Y = (y_{ij})_{q \times r} \) as above, \( X \oplus Y \) is the \((m + q) \times (n + r)\) matrix whose \( ij\)-entry is:

1. \( x_{ij} \), when \( i < m, j < n \),
2. \( y_{i(m-j) - n} \), when \( i \geq m, j \geq n \),
3. \( 0, \) when \( i \geq m, j < n \),
4. \( 0, \) when \( i < m, j \geq n \).

If \( m = q, n = r \), for every \( 0 \leq j < n, a^j = c^j \), and for every \( 0 \leq i < m, b^i = d^i \), i.e. \( X \) and \( Y \) are of the same type having the corresponding elements in the same hom-sets, then \( X + Y \) is the matrix of the same type whose \( ij\)-entry is \( x_{ij} + y_{ij} \). If \( n = q \) and for every \( 0 \leq k < n, a^k = d^k \), i.e. \( X \) is an \( m \times n \) matrix, \( Y \) is an \( n \times r \) matrix and for every \( 0 \leq i < m, 0 \leq j < r, 0 \leq k < n \) the composition \( x_{ik} \circ y_{kj} \) is defined, then we define \( X \circ Y \) as the \( m \times r \) matrix whose \( ij\)-entry is \( \sum_{k=0}^{n-1} x_{ik} \circ y_{kj} \) (this sum is defined since every \( x_{ik} \circ y_{kj} \) is from \( c^i \) to \( b^j \)).

Just by omitting the case (2) of [13, Proposition 5.1] we obtain the following.

**Proposition 6.1.** For \( \bullet \in \{ \otimes, \oplus, +, \circ \} \), we have

\[
M_{u_1 \bullet u_2} = M_{u_1} \bullet M_{u_2}.
\]

Our next proposition is related to [13, Propositions 5.2, 8.2].

**Proposition 6.2.** If \( u \) is a primitive term of \( F \), then all the entries of the matrix \( M_u \) are primitive terms of \( F \), not of the form \( \pi \circ \iota \), whose indices are \( \oplus \)-free.

**Proof.** We illustrate just a couple of cases. If \( u = \gamma \), for \( \gamma \in \Gamma \), then \( M_u \) is a \( 1 \times 1 \) matrix whose only entry is \( \gamma \). The same holds when \( \gamma \) is replaced by \( \gamma^{-1} \). If \( u = \alpha_{a,b,c}^{-1} \), then for some \( i_1, i_2, i_3 \) and \( j_1, j_2, j_3 \)

\[
(M_u)_{i,j} = \alpha_{a,b,c}^{-1} \odot \alpha_{a,b,c}^{-1} \odot \iota^{i_1}_{(a \otimes b) \otimes c} = \left( \iota^{i_1}_{a} \otimes \iota^{i_2}_{b} \otimes \iota^{i_3}_{c} \right) \odot \left( \iota^{j_1}_{a} \otimes \iota^{j_2}_{b} \otimes \iota^{j_3}_{c} \right) = \left\{
\begin{array}{ll}
\alpha_{a^{i_1},b^{i_2},c^{i_3}}, & i_1 = j_1, \ i_2 = j_2, \ i_3 = j_3,
0_{(a^{i_1} \otimes b^{i_2} \otimes c^{i_3}) \circ (b^{i_2} \otimes c^{i_3})}, & \text{otherwise},
\end{array}
\right.
\]

If \( u = \sigma_{a,b} \), then for some \( i_1, i_2 \) and \( j_1, j_2 \)

\[
(M_u)_{i,j} = \sigma_{a,b} \odot \sigma_{a,b} \odot \iota^{i_1}_{a \otimes b} = \left( \iota^{i_1}_{a} \otimes \iota^{i_2}_{b} \right) \odot \left( \iota^{j_1}_{a} \otimes \iota^{j_2}_{b} \right) = \left\{
\begin{array}{ll}
\sigma_{a^{i_1},b^{i_2}}, & i_1 = j_1, \ j_2 = j_2, \\
0_{(a^{i_1} \otimes b^{i_2}) \circ (a^{j_2} \circ b^{j_2})}, & \text{otherwise},
\end{array}
\right.
\]

If \( u = \varepsilon_{a} \), then \( M_u \) is a row matrix and for some \( j_1, j_2 \) we have

\[
(M_u)_{1,j} = \varepsilon_{a} \odot \iota^{j_1}_{a \otimes a^{*}} = \varepsilon_{a} \odot \left( \iota^{j_1}_{a} \otimes \iota^{j_2}_{a} \odot (a^{j_2})^{*} \right) = \varepsilon_{a^{j_2}} \odot \left( \iota^{j_2}_{a} \circ \iota^{j_1}_{a} \right) = \left\{
\begin{array}{ll}
\varepsilon_{a^{j_1}}, & j_1 = j_2, \\
0_{a^{j_1} \circ (a^{j_2})^{*}}, & \text{otherwise},
\end{array}
\right.
\]

If \( u = \pi_{a,b} \), then \((M_u)_{i,j} = \pi_{a,b} \odot \pi_{a,b} \circ \iota^{i_1}_{a \otimes b} \circ \iota^{i_2}_{a \otimes b} \circ \iota^{i_3}_{a \otimes b} \circ \iota^{j_1}_{a \otimes b} \circ \iota^{j_2}_{a \otimes b} \circ \iota^{j_3}_{a \otimes b}) \) which is either \( \pi_{a,b} \odot \pi_{a,b} \circ \iota_{a,b} \circ \iota_{a,b} \) for some \( j_1 \), or \( \pi_{a,b} \odot \pi_{a,b} \circ \iota_{a,b} \circ \iota_{a,b} \circ \iota_{a,b} \) for some \( j_2 \). The statement holds since

\[
\pi_{a,b} \odot \pi_{a,b} \circ \iota_{a,b} \circ \iota_{a,b} = \left\{
\begin{array}{ll}
1_{a^{i_1}}, & i = j_1, \\
0_{a^{j_1},a^{i_1}}, & \text{otherwise},
\end{array}
\right.
\]

We proceed analogously when \( u \) is \( 1_{a}, \alpha_{a,b,c}, \lambda_{a}, \lambda_{a}^{-1}, \eta_{a}, \pi_{a,b}^{2}, \iota_{a,b}^{1}, \iota_{a,b}^{2}, \) or \( 0_{a,b} \).
Corollary 6.3. For every arrow $u$ of $\mathcal{F}$, every entry of $M_u$ is expressible free of $\oplus$, $\iota$, and $\pi$.

As a consequence of Remark 5.1 and Corollary 6.3 we have the following.

Corollary 6.4. Every arrow of $\mathcal{F}$ whose source and target are $\oplus$-free is expressible free of $\oplus$, $\iota$, and $\pi$.

7. The category $1\text{Cob}_G$ and coherence

The aim of this section is to introduce a category providing a diagrammatical checking of validity of quantum protocols. We start with a set $\Gamma$ (usually finite and non-empty) and a group $G$ freely generated by $\Gamma$. The category $1\text{Cob}$ and the group $G$ deliver the category $1\text{Cob}_G$ through the following construction. The objects of $1\text{Cob}_G$ are the objects of $1\text{Cob}$ and in order to define the arrows of $1\text{Cob}_G$ we introduce the notions of $G$-components and $G$-cobordisms first.

A $G$-component is a connected, oriented 1-manifold possibly with boundaries, together with an element of $G$. When a $G$-component is closed, we call it a $G$-circle, otherwise it is a $G$-segment. We call the element of $G$ associated to a component the label of this component.

A $G$-cobordism from $a$ to $b$ is a finite collection of $G$-components whose underlying manifold is $M$, together with two embeddings $f_0: a \to M$ and $f_1: b \to M$ such that $(M, f_0, f_1)$ is a 1-cobordism from $a$ to $b$. Two $G$-cobordisms are equivalent, when the underlying 1-cobordisms are equivalent and the homeomorphism $F$ witnessing this equivalence satisfies:

1. every segment and its $F$-image are labeled by the same element of $G$;
2. the labels of a circle and its $F$-image could differ only in a circular permutation, i.e. if one is of the form $g_2 \cdot g_1$, the other could be $g_1 \cdot g_2$.

The operation $\dagger$ on $G$-cobordisms is defined so that it is applied to the underlying cobordism and every label is replaced by its inverse.

The category $1\text{Cob}_G$ has the equivalence classes of $G$-cobordisms as arrows. The identity $1_a: a \to a$ is the ordinary identity cobordism in which every segment is labeled by the neutral $e$ of $G$. Two $G$-cobordisms are composed so that the underlying 1-cobordisms are composed in the ordinary manner. It remains to label the resulting segments and circles: if the segments $l_1, \ldots, l_k$ with labels $g_1, \ldots, g_k$ respectively, are glued together in a segment or a circle of the resulting 1-cobordism so that the terminal point of $l_i$ is identified with the initial point of $l_{i+1}$, then $g_k \cdot \cdots \cdot g_1$ is the ("a" in the case of a circle) label of the resulting component. The category $1\text{Cob}_G$ has dagger strict compact closed structure inherited from $1\text{Cob}$ (all segments in canonical arrows $\sigma$, $\eta$ and $\varepsilon$ are labeled in $1\text{Cob}_G$ by the neutral $e$ of $G$).

Let us compare the above construction with the construction of the category $G\mathcal{A}$ given in [8], for $\mathcal{A}$ being the groupoid $\mathcal{G}$, i.e. the category with a single object $p$ whose arrows are the elements of $\mathcal{G}$ and the composition is the multiplication in $\mathcal{G}$. The main theorem of [8] claims that $G\mathcal{G}$ is a compact closed category freely generated by the category $\mathcal{G}$. This means that for every compact closed category
C and a function \( \varphi \) from the set \( \Gamma \) to the set of automorphisms of an object \( c \) of \( C \), there exists a unique functor

\[ F : G\mathfrak{C} \rightarrow C \]

that strictly preserves compact closed structure, and such that \( Fp = c \) and for every \( \gamma \in \Gamma \), \( F\gamma = \varphi(\gamma) \).

One could easily conclude that \( 1\text{Cob}_\mathfrak{G}^\circ \) is a strict compact closed version of \( G\mathfrak{C} \).

More precisely, the functor \( F^\circ : G\mathfrak{C} \rightarrow 1\text{Cob}_\mathfrak{G}^\circ \) obtained by the above universal property of \( G\mathfrak{C} \) is defined as follows. It maps every object \( X \) of \( G\mathfrak{C} \) to the sequence of signs corresponding to the signed set \( P(X) \) (see [8, Section 3]). On arrows it is defined just by replacing the source and the target by the corresponding sequences of signs. Namely, an arrow of \( G\mathfrak{C} \) (see [8, Section 3]) is represented by a triple, which is essentially contained in the notion of \( \mathfrak{G} \)-cobordism. Hence, \( F^\circ \) maps an arrow (neglecting its source and target) to itself. It is straightforward to see that we have the following.

**Proposition 7.1.** The functor \( F^\circ : G\mathfrak{C} \rightarrow 1\text{Cob}_\mathfrak{G}^\circ \) is faithful.

In another words, to pass from \( 1\text{Cob}_\mathfrak{G}^\circ \) to \( G\mathfrak{C} \) one has to “decorate” the objects of \( 1\text{Cob}_\mathfrak{G}^\circ \) with propositional formulae built in the language including single propositional letter, constant \( I \), unary connective \( * \) and binary connective \( \otimes \). However, this just disguises strict compact closed nature of \( 1\text{Cob}_\mathfrak{G}^\circ \), which is intrinsic to this category.

Let \( 1\text{Cob}_\mathfrak{G}^\circ \) be the category with the same objects as \( 1\text{Cob}_\mathfrak{G} \), while the arrows of \( 1\text{Cob}_\mathfrak{G}^\circ \) from \( a \) to \( b \) are the formal sums of arrows of \( 1\text{Cob}_\mathfrak{G} \) from \( a \) to \( b \). These formal sums may be represented by finite (possibly empty) multisets of \( \mathfrak{G} \)-cobordisms from \( a \) to \( b \). Formally, a multiset of elements of a set \( X \) is a function from \( X \) to the set of natural numbers (including zero). Less formally, it is a set in which elements may have multiple occurrences.

We abuse the notation by using the set brackets \( \{, \} \) for multisets and by denoting a singleton sequence \( \{f\} \) by \( f \). Note that in this notation \( \emptyset + \emptyset \), i.e. \( \{\emptyset, \emptyset\} \) is not equal to \( \emptyset \), i.e. \( \{\emptyset\} \), where \( \emptyset \) is a circular component with arbitrary label.

The identity arrow \( 1_a : a \rightarrow a \) is the singleton multiset \( 1_a \). For example, \( 1_a \) is a multiset with one entry equal to \( a \) and we denote it by \( 0 \). (Here, according to our convention, \( \emptyset \) denotes the empty multiset.) Again, because of two many roles of \( \emptyset \) in this paper, we denote the empty multiset of \( \mathfrak{G} \)-cobordisms from \( a \) to \( b \) by \( 0_{a,b} \), and call it zero-arrow. The existence of zero-arrows implies that every hom-set in \( 1\text{Cob}_\mathfrak{G}^\circ \) is inhabited. The category \( 1\text{Cob}_\mathfrak{G}^\circ \) is enriched over the category \( \text{Cmd} \). The addition in \( \text{Hom}(a,b) \) is the operation + (disjoint union) on multisets and the neutral is \( 0_{a,b} \).

Let \( 1\text{Cob}_\mathfrak{G}^\oplus \) be the biproduct completion of \( 1\text{Cob}_\mathfrak{G}^\circ \) constructed as follows (see [13, Section 5.1]). The objects of \( 1\text{Cob}_\mathfrak{G}^\oplus \) are finite (possibly empty) sequences \((a_0, \ldots, a_{n-1}) \), \( n \geq 0 \), of objects \( a_0, \ldots, a_{n-1} \) of \( 1\text{Cob}_\mathfrak{G}^\circ \). (We abuse the notation by denoting a singleton sequence \( (a_0) \) by \( a_0 \).) For example, \((+,+,+,\odot,+,+,\odot)\) is an object of \( 1\text{Cob}_\mathfrak{G}^\circ \). (Here, according to our convention, \( \odot \) denotes the empty sequence of oriented points.) The empty sequence of objects of \( 1\text{Cob}_\mathfrak{G}^\circ \) plays the role of zero-object in \( 1\text{Cob}_\mathfrak{G}^\circ \), and for the above reasons we denote it by \( \emptyset \) and not by \( \emptyset \). Note the distinction between this object and the object presented by the singleton sequence \( o \) whose only member is the empty sequence of oriented points.

The arrows of \( 1\text{Cob}_\mathfrak{G}^\oplus \) from \((a_0, \ldots, a_{n-1}) \) to \((b_0, \ldots, b_{m-1}) \) are the \( m \times n \) matrices whose \( ij \)-entry is an arrow of \( 1\text{Cob}_\mathfrak{G}^\circ \) from \( a_j \) to \( b_i \). If \( m = 0 \), i.e. \((b_0, \ldots, b_{m-1}) = \emptyset \), then the empty matrix is the unique arrow from \( a = (a_0, \ldots, a_{n-1}) \) to \( \emptyset \), and we denote it by \( 0_{a,\emptyset} \). We proceed analogously when \( n = 0 \).
The identity arrow $1_a$ on $a = (a_0, \ldots, a_{n-1})$ in $\mathsf{1Cob}^{\oplus}$ is the $n \times n$ matrix with corresponding identity arrows of $\mathsf{1Cob}^+$ in the main diagonal and corresponding zero-arrows of $\mathsf{1Cob}^\oplus$ elsewhere. The arrows are composed by the rule of matrix multiplication, save that the addition and multiplication in a field are replaced by addition in hom-sets and composition in the category $\mathsf{1Cob}^{\oplus}$. For $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{m-1})$, we denote by $0_{a,b}$, or simply $0_{m \times n}$, the $m \times n$ matrix whose $ij$-entry is the zero-arrow $0_{a_i,b_j}$ of $\mathsf{1Cob}^{\oplus}$. In the limit cases, when we compose the empty matrices $0_{a,0}$ and $0_{0,b}$, we define the result as the zero-matrix $0_{a,b}$.

**Proposition 7.2.** The category $\mathsf{1Cob}^{\oplus}$ has the structure of strict compact closed category with biproducts. The group of automorphisms of the object $+$ in this category is isomorphic to $\Phi$. Moreover, $\dagger$ is definable in $\mathsf{1Cob}^{\oplus}$, which makes it dagger strict compact closed category with dagger biproducts, while the automorphisms of $+$ are unitary.

**Proof.** We define the compact closed structure on $\mathsf{1Cob}^{\oplus}$ as follows. The tensor product of objects $(a_0, \ldots, a_{n-1})$ and $(b_0, \ldots, b_{m-1})$ is the object $(a_0 \otimes b_0, \ldots, a_0 \otimes b_{m-1}, \ldots, a_{n-1} \otimes b_{m-1})$ of $\mathsf{1Cob}^{\oplus}$. If either $n = 0$ or $m = 0$, the result is zero-object $0$. The unit object is $1$. The tensor product of arrows from $\mathsf{1Cob}^{\oplus}$ is defined as the Kronecker product of matrices over a field, save that this time the multiplication when we compose the empty matrices $0_{a,0}$, $0_{0,b}$, we define the result as the zero-matrix $0_{a,b}$.

The arrows $\alpha$ and $\lambda$ are identities. For $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{m-1})$, the $(n \times m)$ matrix $\sigma_{a,b}$ (an arrow of $\mathsf{1Cob}^{\oplus}$) is defined as the permutation matrix representing the isomorphism between $V \otimes W$ and $W \otimes V$ for $V$ being $n$-dimensional and $W$ being $m$ dimensional vector space, save that instead of the entries $1$, we have arrows $\sigma$ from $\mathsf{1Cob}^+$, with corresponding indices, and instead of entries $0$, we have zero-arrows (i.e. empty multisets) of $\mathsf{1Cob}^{\oplus}$ with corresponding indices. For example, if $a = (a_0, a_1, a_2)$ and $b = (b_0, b_1)$, the matrix $\sigma_{a,b}$ (with indices of zero-arrows omitted) is

$$
\begin{pmatrix}
\sigma_{a_0,b_0} & 0_{a_0 \otimes b_1} & 0_{a_0 \otimes a_0} & 0 & 0 & 0 & 0 \\
0 & 0 & \sigma_{a_1,b_0} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sigma_{a_2,b_0} & 0 & 0 & 0 \\
\sigma_{a_0,b_1} & 0 & 0 & 0 & \sigma_{a_1,b_1} & 0 & 0 \\
0 & 0 & 0 & 0 & \sigma_{a_2,b_1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sigma_{a_2,b_1}
\end{pmatrix}.
$$

The operation $*$ on objects of $\mathsf{1Cob}^{\oplus}$ is defined componentwise. The arrow $\eta_a$ for $a = (a_0, \ldots, a_{n-1})$ is the $n^2 \times 1$ matrix with the singleton multiset $\varepsilon_{a_k}$ in the $k \cdot (n + 1)$-th row, for $0 \leq k < n$, and zero-arrows of $\mathsf{1Cob}^+$, with corresponding indices, elsewhere. The arrow $\varepsilon_a$ is the $1 \times n^2$ matrix having $\varepsilon_{a_k}$ in the $k \cdot (n + 1)$-th column, for $0 \leq k < n$, and zero-arrows of $\mathsf{1Cob}^+$, with corresponding indices, elsewhere. One can verify that the equalities $\square_{A.1} \square_{A.12}$ hold in $\mathsf{1Cob}^{\oplus}$. Moreover, the arrows $u_{a,b}$, $v$, and $w_a$ defined in Section 2 are identities. Hence $\mathsf{1Cob}^{\oplus}$ is a strict compact closed category.

The operation $+$ on arrows from $(a_0, \ldots, a_{n-1})$ to $(b_0, \ldots, b_{m-1})$ is defined componentwise and zero-matrices are the neutrals for this operation. The equations $\square_{A.10} \square_{A.12}$ hold, which guarantees that $\mathsf{1Cob}^{\oplus}$ is enriched over $\mathsf{Cmd}$.

For objects $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{m-1})$ the object $a \oplus b$ is the sequence $(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1})$. 


The object 0 is the zero-object of $1\text{Cob}^\oplus_\mathcal{G}$ and it is the neutral for $\oplus$. For arrows $A_{m \times n}$ and $B_{p \times q}$ of $1\text{Cob}^\oplus_\mathcal{G}$ its direct sum $A \oplus B$ is the $(m + p) \times (n + q)$ matrix
\[
\begin{pmatrix}
A & 0_{m \times q} \\
0_{p \times n} & B
\end{pmatrix}.
\]
For $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{m-1})$, the arrows $\pi_{a,b}^1$, $\pi_{a,b}^2$, $\iota_{a,b}^1$ and $\iota_{a,b}^2$ are defined as
\[
\begin{align*}
\pi_{a,b}^1 &= \begin{pmatrix} 1_a & 0_{n \times m} \end{pmatrix}, & \pi_{a,b}^2 &= \begin{pmatrix} 0_{m \times n} & 1_b \end{pmatrix}, \\
\iota_{a,b}^1 &= \begin{pmatrix} 1_a & 0_{m \times n} \end{pmatrix}, & \iota_{a,b}^2 &= \begin{pmatrix} 0_{m \times n} & 1_b \end{pmatrix}.
\end{align*}
\]
After checking that the equalities A.13-A.19 hold in $1\text{Cob}^\oplus_\mathcal{G}$, one concludes that this category is strict compact closed with biproducts.

That the group of automorphisms of the object $+$ in $1\text{Cob}^\oplus_\mathcal{G}$ is isomorphic to $\mathcal{G}$ is shown as follows. Every arrow from $+$ to itself is a $1 \times 1$ matrix whose entry is a multiset of arrows of $1\text{Cob}_\mathcal{G}$ from the singleton sequence of oriented points $+$ to itself. This multiset is a singleton in the case of an isomorphism, which follows from the fact that the composition in $1\text{Cob}_\mathcal{G}$ of a multiset of cardinality $n$ with a multiset of cardinality $m$ is a multiset of cardinality $n \cdot m$, and an isomorphism must be canceled to $1_+: + \rightarrow +$, which is the singleton multiset $1_+$. Hence, every isomorphism from $+$ to $+$ in $1\text{Cob}_\mathcal{G}$ is of the form $\psi: + \rightarrow +$, for $\psi$ an arrow of $1\text{Cob}_\mathcal{G}$. Moreover, $\psi$ must be an isomorphism in $1\text{Cob}_\mathcal{G}$. An arrow of $1\text{Cob}_\mathcal{G}$ from $+$ to $+$ consists of a single $\mathcal{G}$-segment and several (possibly zero) $\mathcal{G}$-circles. Since $\psi$ is an isomorphism and $\mathcal{G}$-circles are not cancelable, there are no $\mathcal{G}$-circles in $\psi$ and it could be identified with the underlying $\mathcal{G}$-segment. The label of this segment is the element of $\mathcal{G}$ corresponding to the initial isomorphism of $1\text{Cob}_\mathcal{G}$. It is evident that this correspondence is a one-to-one homomorphism.

The operation $\dagger$ on arrows of $1\text{Cob}^\oplus_\mathcal{G}$ is defined as
\[
\{ (f_j: a \rightarrow b) \mid j \in J \} \dagger = \{ f_j^\dagger: b \rightarrow a \mid j \in J \}.
\]
The operation $\dagger$ on a matrix representing an arrow of $1\text{Cob}^\oplus_\mathcal{G}$ is defined by transposing this matrix, and by applying the operation $\dagger$, defined above, to each of its entries. In order to verify that $1\text{Cob}^\oplus_\mathcal{G}$ is dagger strict compact closed with dagger biproducts, it remains to check that the equalities A.20-A.24 of Appendix A hold. The definition of $\dagger$ in $1\text{Cob}_\mathcal{G}$ guarantees that the automorphisms of $+$ are unitary.

**Remark 7.3.** For our purposes it is useful to have a direct presentation of $\Gamma f$, $\iota f$, $f_\ast$, $f_+$ and $(f_1, \ldots, f_n)$ at least for arrows $f, f_1, \ldots, f_n$ of $1\text{Cob}_\mathcal{G}$. The first three operations are defined as in $1\text{Cob}$ (the labels of $\mathcal{G}$-components remain the same in the first two cases, while in the case of $f_\ast$ the labels become the inverses of the initial labels). The last operation (see the definition of biproducts in Section 2) produces the $n \times 1$ matrix
\[
\begin{pmatrix}
f_1 \\
\vdots \\
f_n
\end{pmatrix}.
\]

**Remark 7.4.** By relying on the equalities 2.23 it is not difficult to show that the left distributivity isomorphism $\nu_{a,b,c}: (a \oplus b) \odot c \rightarrow (a \odot c) \oplus (b \oplus c)$ is the identity in the category $1\text{Cob}^\oplus_\mathcal{G}$. Similarly, if we assume that $a$ is a singleton sequence, then by relying on the equalities 2.22 we can show that the right distributivity isomorphism $\tau_{a,b,c}: a \odot (b \oplus c) \rightarrow (a \odot b) \oplus (a \odot c)$ is also the identity in $1\text{Cob}^\oplus_\mathcal{G}$. 
Remark 7.5. By the universal property of the category $F$ from Section 4 there exists a unique functor $H : F \to \mathbf{1Cob}_\mathfrak{G}$ that strictly preserves the compact closed structure with biproducts, for which $H \pi = +$ and for every $\gamma \in \Gamma$, $H \gamma$ is the $\mathfrak{G}$-cobordism from $+$ to $+$ given by one $\mathfrak{G}$-segment labeled by $\gamma$. The isomorphism of $F$ and $F'$ from the proof of Proposition 4.1 enables one to consider $H$ as a functor from $F'$ to $\mathbf{1Cob}_\mathfrak{G}$ that strictly preserves the dagger compact closed structure with dagger biproducts.

Proposition 7.6. The functor $H : F \to \mathbf{1Cob}_\mathfrak{G}$ is faithful.

Proof. Let $f, g : a \to b$ be two arrows of $F$ such that $H f = H g$, and let $I_a = (i_a^0, \ldots, i_a^{n-1})$ and $\Pi_b = (\pi_b^0, \ldots, \pi_b^{m-1})$. By Corollary 5.3 and properties of biproducts it suffices to show that, for every $0 \leq i < m$ and $0 \leq j < n$, 

\[ \pi_b^i \circ f \circ i_a^j = \pi_b^i \circ g \circ i_a^j. \]

(7.1)

By Corollary 6.4 both sides of (7.1) are expressible free of $\oplus, \iota$ and $\pi$. By relying on the equalities A.11, A.12, A.27, and A.28 both sides are expressible as sums of terms, which are all free of $\oplus, +$ and 0, $\iota, \pi$-arrows. Here, the empty sum is denoted by $0_{\iota, \pi}$. If one side of the above equality is equal to $0_{\iota, \pi}$, then it is mapped by $H$ to the empty multiset. By functorial properties of $H$, the sum at the other side must be mapped by $H$ to the empty multiset too, which means that this sum is empty, i.e. it is $0_{\iota, \pi}$.

It remains the case when for $n, m \geq 1$ the left-hand side of (7.1) is equal to $\sum_{k=1}^n f_k$ and the right-hand side of this equality is equal to $\sum_{k=1}^m g_k$, for $f_k, g_k$ free of $\oplus, +$ and 0, $\iota, \pi$-arrows. We have that 

\[ \{H f_k \mid 1 \leq k \leq n\} = H \sum_{k=1}^n f_k = H \sum_{k=1}^m g_k = \{H g_k \mid 1 \leq k \leq m\}, \]

which means that $n = m$, and modulo some permutation of elements of these multisets, for every $1 \leq k \leq n$, $H f_k = H g_k$. The terms $f_k$ and $g_k$ belong entirely to the compact closed fragment generated by $\mathfrak{G}$. Hence, these terms represent arrows of a compact closed category $F \mathfrak{G}$ freely generated by $\mathfrak{G}$ (see [8, Section 4]). The functor $H$ restricts to $F \mathfrak{G}$ as the composition of an isomorphism (from $F \mathcal{A}$ to $G \mathcal{A}$, for $\mathcal{A}$ being $\mathfrak{G}$; see [8, Sections 3-4]) and the faithful functor $F^{q_{\mathfrak{G}}}$ of Proposition 7.1 which means that this restriction is faithful. We conclude that $f_k$ and $g_k$ represent the same arrow of $F \mathfrak{G}$, and hence of $F$. \qed

8. Validity of categorical quantum protocols

It was suggested in [1] that compact closed categories with biproducts provide a generalisation of von Neumann’s presentation of quantum mechanics in terms of Hilbert spaces, [11]. Such an approach is called categorical quantum mechanics. For a survey of theory of categorical quantum mechanics, we recommend [2, 18] and references therein.

In this section we use Proposition 7.6 to establish commutativity of diagrams in the category $F$, which provides a verification of the corresponding protocols from the realm of categorical quantum mechanics. All the protocols verified in [1] require a compact closed category with dagger biproducts, possessing some additional structure. For the first two protocols below, this extra structure consists of an object $Q$ (the qubit), an arrow from $4 \cdot_I I$ to $Q^* \otimes Q$ and a scalar $s$ satisfying some conditions listed in [1, Section 9]. (Here we abbreviate $((I \oplus I) \oplus I) \oplus I$ by $4 \cdot_I I$, and more generally, $n \cdot a$ and $n \cdot f$ abbreviate the $n$-fold biproducts, associated to the left, of an object $a$ and an arrow $f$ respectively.)
However, the only additional structure upon a compact closed structure with
dagger biproducts important for the verification diagrams consists of four unitary
isomorphisms $\beta_1, \beta_2, \beta_3, \beta_4 : Q \to Q$. Hence, to establish that the verification
diagrams are commutative in an arbitrary such category, i.e. that the categorical
quantum protocols are correct, it suffices to establish their commutativity in the
compact closed category $\mathcal{F}$ with biproducts freely generated by the free group $\mathbb{S}$
on four generators. (Since $\beta_1$ is standardly taken to be identity, a group with three
generators suffices.) The role of the generator $p$ for objects of $\mathcal{F}$ (see Section 4)
belong now to the qubit $Q$.

Our Proposition 7.6 enables one to check the commutativity of diagrams in $\mathcal{F}$
by “drawing pictures” and this is the style of verification given below. The qubit $Q$
is interpreted in $1\text{Cob}_G$ as $\star$. At some points we have to draw matrices of pictures
and this is done in the first example below, otherwise just the $ij$ element of such a
matrix is described.

8.1. Quantum teleportation. Quantum teleportation is a well-known quantum
protocol [1, 12]. Assume that Alice has a qubit in some state $|\psi\rangle$, and wishes to
sent this state do Bob, without any knowledge of what this state is. This is done
by taking an entangled pair of qubits (EPR-pair, $|\beta_1\rangle$) and sending one to Alice
and another to Bob. Then, Alice measures (in the Bell basis) her qubit and the
qubit that is entangle with the one Bob has. In the next step, she communicates
the result of the measurement to Bob, who applies unitary corrections to his qubit,
depending on the Alice’s outcome. The final result is that Bob’s qubit is in the
same state as Alice’s qubit originally was (Alice does not have a qubit in state $|\psi\rangle$
after this protocol is done).

\[
\begin{align*}
\Delta^a_{bc} : & \quad Q_a \\
& \xrightarrow{\sigma_{1,a} \circ \lambda^{-1}_a} \text{import unknown state} \\
& \xrightarrow{Q_a \otimes I} \\
& \xrightarrow{1_a \otimes \Gamma_{bc}^{-1}} \text{produce EPR-pair} \\
& \xrightarrow{Q_a \otimes (Q_b^* \otimes Q_c)} \\
& \xrightarrow{\alpha_{a,b,c}} \text{spatial delocation} \\
& \xrightarrow{(Q_a \otimes Q_b^*) \otimes Q_c} \\
& \xrightarrow{\langle i, \beta^a_{ib} \rangle_{i=1}^4 \otimes 1_c} \text{teleportation observation} \\
& \xrightarrow{(4 \cdot I) \otimes Q_c} \\
& \xrightarrow{(4 \cdot \lambda_c) \circ \nu_{4L,c}} \text{classical comunication} \\
& \xrightarrow{4 \cdot Q_c} \\
& \xrightarrow{\bigoplus_{i=1}^4 (\beta^i_c)^{-1}} \text{unitary correction} \\
& \xrightarrow{4 \cdot Q_c} 
\end{align*}
\]

The correctness of the quantum teleportation protocol is expressed by commuta-
tivity of the diagram given in [1 Theorem 9.1]. One can easily factor the scalars out
of both legs in this diagram just by appealing to the equalities 2.4-2.6. This makes the commutativity of Diagram 8.1 sufficient for the correctness of the protocol. We follow the terminology and notation introduced in [1] in this diagram.

Note that we treat $Q_a$, $Q_b$ and $Q_c$ as three instances of the same object $Q$ of $\mathcal{F}$. Also, $\Delta_{e,c}$ is an abbreviation for $\langle 1_Q, 1_Q, 1_Q, 1_Q \rangle$. For example, producing the EPR-pair, means to apply the arrow $1_Q \otimes \downarrow \downarrow$, which is interpreted in $1\text{Cob}^0_\mathcal{G}$ as:

$e$ $e$

This is the first nontrivial step in the diagram 8.1. (Note that since $1\text{Cob}^0_\mathcal{G}$ is a strict compact closed category, the steps called “import unknown state” and “spatial delocation” are interpreted as identities in this category.)

In drawings of $\mathcal{G}$-cobordisms, when we interpret the arrows of the diagram 8.1 and the diagrams below, the orientation and the label $e$ (denoting the neutral of $\mathcal{G}$) will be omitted. As we noted at the beginning of this section, our group $\mathcal{G}$ is generated by the set $\Gamma = \{ \beta_1, \beta_2, \beta_3, \beta_4 \}$. The second nontrivial step in the diagram 8.1 is the teleportation observation, given by $\langle \downarrow \downarrow \rangle_i^{i=1} \otimes 1_Q$, or in terms of arrows of $1\text{Cob}^0_\mathcal{G}$:

$\uparrow \uparrow \uparrow \uparrow$

$\beta_1$ $\beta_2$ $\beta_3$ $\beta_4$

$\downarrow \downarrow \downarrow \downarrow$

By composing this $4 \times 1$ matrix with the $1 \times 1$ matrix representing production of EPR-pair, we get

At this point, Alice had performed her measurement, and communicated the result to Bob using classical interchange of bits. By Remark 7.4, we know that the distributivity isomorphism $\nu_{44,c}$ is the identity in $1\text{Cob}^0_\mathcal{G}$. This, together with the strictness of this category, makes the step named “classical communication” trivial, i.e. it is interpreted as identity.

Next, Bob applies unitary corrections, given by $\oplus_{i=1}^{i=4} (\beta_i)^{-1}$. In our matrix representation, $\oplus$ correspond to the direct sum of matrices. We therefore have the unitary correction

$\oplus_{i=1}^{i=4} (\beta_i)^{-1}$
By composing the last two matrices, we get the final result

\[
\begin{pmatrix}
\beta_1^{-1} & 0 & 0 & 0 \\
0 & \beta_2^{-1} & 0 & 0 \\
0 & 0 & \beta_3^{-1} & 0 \\
0 & 0 & 0 & \beta_4^{-1}
\end{pmatrix}
\]

By stretching the diagrams the group elements cancel out, and we are left with the diagonal \(\Delta^4 = \langle 1_Q, 1_Q, 1_Q, 1_Q \rangle\).

8.2. Entanglement Swapping. The idea of this protocol is to, starting with two pairs of mutually entangled qubits in EPR-states, obtain again two pairs of entangled states, but with different pairing. Assume Alice, as well as Bob, share a single EPR-pair with a third person, named Charlie. Then Charlie performs a measurement on his qubits, and via classical communication transfers information on his outcomes to other parties, upon which a unitary correction is applied. Net result of this protocol is that Alice and Bob share an entangled EPR-pair, while Charlie is left with another EPR-pair. We thus say that the entanglement is swapped.

A complete description of this protocol in terms of categorical quantum mechanics is presented in [1, Theorem 9.3]. Again, as in Section 8.1, by relying on the equalities 2.4-2.6, one may completely neglect the role of scalars and just check the commutativity of the diagram 8.2 below for the correctness of this protocol.

Let \(\tau: Q_a^* \otimes (4 \cdot ((Q_a \otimes Q_b^*) \otimes Q_c)) \to 4 \cdot (Q_d^* \otimes ((Q_a \otimes Q_b^*) \otimes Q_c))\) and \(\upsilon: (4 \cdot ((Q_a \otimes Q_b^*) \otimes Q_c) \to 4 \cdot ((Q_a \otimes Q_b^*) \otimes Q_c)\) be distributivity isomorphisms, and let

\[\gamma_i = (\beta_i)_*, \quad P_i = \tau \gamma_i \circ \iota \beta_i, \quad \zeta_{ia} = \bigoplus_{i=1}^{4} (1^*_a \otimes \beta_i) \otimes (1^*_d \otimes \beta_i^{-1}), \quad \Theta_{ab} = 1^*_a \otimes (P_i)_{i=1}^{4} \otimes 1_c, \quad \varphi = (4 \cdot ((\sigma_{ab} \otimes 1_{dc}) \circ \alpha_{ab,d,c}^{-1} \circ (\sigma_{d,ab} \otimes 1_c) \circ \alpha_{d,ab,c}) \circ \tau \circ (1_d \otimes \upsilon), \quad \Omega_{ab} = (1_{ba} \gamma_{i=1}^{4} \circ (1_{dc} \gamma_{i=1}^{4})).\]

The commutativity of the diagram from [1, Theorem 9.3] justifies the correctness of the entanglement swapping protocol. By factoring the scalars out from the legs, it reduces to the following diagram.
The right-hand side of this diagram is represented in $1\text{Cob}_G$ by the $4 \times 1$ matrix whose $i1$-entry is the following $G$-cobordism (note that we ignore associativity and distributivity isomorphisms since they are identities).

By stretching the above diagram and cancelling $\beta_i$ and $\beta_i^{-1}$, we are left with the following $G$-cobordism.
On the other side, $\Omega_{ab}$ is represented in $1\text{Cob}^\oplus$ by $4 \times 1$ matrix, whose $i1$-entry is exactly the above $\mathcal{G}$-cobordism. Due to Proposition 7.6 this proves the commutativity of the diagram 8.2.

8.3. Superdense coding. In this section, we will apply our diagrammatic verification to another protocol, called a superdense coding, [12] (sometimes referred to as a dense coding). This quantum algorithm can be considered as an opposite of the quantum teleportation. The idea is to transfer some amount of classical information, using qubits. A review of this protocol can be found in [18], where its validity was shown in a similar manner.

The validity of this protocol is expressed in the categorical setting by the commutativity of a diagram in which some special scalars, namely traces of some arrows, occur. Every compact closed category can be lifted to the traced category by a suitable definition of a categorical trace. This can be achieved as follows. Let $f : a \to a$ be an arrow in a compact closed category. The scalar $\text{Tr}(f) : I \to I$ is defined as

\begin{equation}
\text{Tr}(f) = \varepsilon_a \circ (f \otimes a^*) \circ \sigma_{a^*,a} \circ \eta_a.
\end{equation}

In terms of diagrams, we have

A category appropriate for the superdense coding requires the same structure as in the first two protocols. Moreover, the following conditions must be satisfied. If $i \neq j$, then $\text{Tr}(\beta_i \beta_j^\dagger) = 0_{I,I}$, and $\text{Tr}(1_Q) \neq 0_{I,I}$ (see Appendix B for the details why we demand this condition to be satisfied). With this in mind the arrow $\Xi : I \to 16 \cdot I$ defined as

\begin{equation}
\langle \text{Tr}(\beta_1 \beta_1^\dagger), \text{Tr}(\beta_1 \beta_2^\dagger), \text{Tr}(\beta_1 \beta_3^\dagger), \ldots, \text{Tr}(\beta_4 \beta_1^\dagger), \text{Tr}(\beta_4 \beta_2^\dagger), \text{Tr}(\beta_4 \beta_3^\dagger) \rangle
\end{equation}

is actually $(t,0,0,0,0,t,0,0,0,0,t,0,0,0,0,t)$, for $t = \text{Tr}(1_Q)$. The assumption above also enables Bob to make a distinction between the four quadruples of scalars in this row.

\footnote{More generally, categorical trace corresponds to the partial trace in Hilbert space picture, though we will not review this here, as our interest lies only in pure states.}
Our task is to show that the following diagram, which verifies the superdense coding protocol, commutes.

\[
\begin{array}{c}
I \\
\uparrow 1_{ab} \quad \text{preparation of EPR-pair} \\
Q_a^* \otimes Q_b \\
\downarrow (\langle \beta_i \rangle_{i=1}^{\lambda=4} \otimes 1_b) \quad \text{selection of classical information} \\
(4 \cdot Q_a^*) \otimes Q_b \\
\downarrow (4 \cdot \sigma_{ab}) \circ \nu_{4a,b} \quad \text{spatial delocation} \\
4 \cdot (Q_b \otimes Q_a^*) \\
\downarrow \langle \beta_{ab}^{-1} \rangle_{i=1}^{\lambda=4} \quad \text{observation} \\
16 \cdot I
\end{array}
\]

Here, again, the first step is the EPR-pair production, achieved by a cap diagram.

One qubit is located at Alice’s point, and another at Bob’s. Alice then applies an unitary transformation to her qubit, depending on the classical information she wants to communicate. This is achieved by \(\langle \beta_i \rangle_{i=1}^{\lambda=4} \otimes 1_b\). By composing the first two arrows, we get a \(4 \times 1\) matrix, whose i1-entry is given by a following arrow.

Spatial delocation is represented by a transposition, and after its application we obtain a matrix with i1-entry given by

Finally, Alice sends her qubit to Bob, who performs an entangled state measurement, given by a suitable coname. The result is a \(16 \times 1\) matrix, whose \((4(i-1)+j)1\)-element is given by the \(\Theta\)-circle
and the same matrix is obtained by interpreting the arrow $\Xi: I \to 16 \cdot I$ in the category $\mathcal{Cob}^{\oplus}_{\mathfrak{G}}$. The additional assumptions on the compact closed structure, listed in the paragraph where the arrow $\Xi$ is defined, enable Bob to distinguish between different Alice’s messages.

9. Concluding remarks

In recent years, different type of graphical calculi, used in the context of categorical quantum mechanics, have been developed. Presumably, the most famous is the so called ZX calculus (see [5]). The fact is that dagger compact closed categories admit various diagrammatic calculi with “boxes” (see e.g. [4,14]), and one could in principle use such an approach to check the validity of quantum protocols. It is, however, not fully satisfactory that one has to introduce the elements of string diagrams that can have various number of external legs, and that can be used in various contexts. Here, we reduced the role of boxes to labels of connected components of 1-manifolds underlying our graphical language.

In order to completely eliminate such external elements from the calculus, one has to increase the dimension of cobordisms by two, to replace the finite sequences of oriented points by finite sequences of a closed connected orientable surface $\Sigma_g$ of genus $g \geq 1$, and to replace the segments labeled by the elements of $\mathfrak{G}$ by manifolds obtained from $\Sigma_g \times [0,1]$ by some surgery. At this point one has to rely on the mapping class group of $\Sigma_g$. Such a calculus would be less practical at some points, but it could bring some new insight to the subject through the Kirby calculus (see [9]) for surgery data. Our plan for a future work is to investigate this 3-dimensional calculus.

After the introduction of categorical quantum mechanics, it is natural to seek for a different dagger compact closed categories with biproducts, in order to check whether they can sustain quantum protocols, as quantum teleportation. The possible complication is the existence of a base. Abstractly, base can be defined using biproducts: we demand existence of unitary arrows $I \oplus I \to Q$. The problem with the category of cobordisms is that there does not seem to be enough options to construct the desired unitary morphism. This was alluded, in a slightly different context, in [3]. Luckily, in order to verify protocols as quantum teleportation, it is not mandatory to use the described morphism.

Furthermore, in low dimensions, it is hopeless to try to accommodate different unitary transformations present in quantum protocols as different cobordisms. In order to heal this problem, we introduced a group structure $\mathfrak{G}$. We believe that the approach suggested at the end of the first paragraph of this section could provide a solution for these problems.

Of course, this raises some conceptual questions. First, by identifying the qubit state space with $\{+\}$ (or $\{-\}$), we are not able to use our graphical language to define states, i.e. morphisms of the form $I \to Q$. In all mentioned quantum protocols, this was not an issue, as we used names to create entangled states, and this can be seen in $\mathcal{Cob}^{\oplus}_{\mathfrak{G}}$ language. In order to circumvent this issue, one could increase the dimension of cobordisms as suggested in the first paragraph, or to take zero-dimensional spheres, i.e. the two element sequences $\{+,+\}$ to represent state spaces. Then, one has the possibility to introduce morphisms that define states. Also, we can use this new type of qubits to define measurements on a single qubit, not just on an entangled pair. Considerations of this type could be of interest when dealing with single-particle protocols [6].

We conclude this section with a comment concerning the generality of quantum protocols brought by replacing the Hilbert spaces by objects of a compact closed category. It is known that (with minor provisos) all the 1-dimensional topological
quantum field theories, i.e. functors from the category $\mathbf{1Cob}$ to the category of finite dimensional vector spaces over a field, are faithful according to [15]. However, this does not mean that the whole $\mathbf{1Cob}^\delta$ could be faithfully represented by matrices over a field. On the other hand, since protocols do not use the full strength of $\mathbf{1Cob}^\delta$, one could expect that some could be verified by relying on the matrix calculus (working again in the skeleton of $\mathbf{fdHilb}$ with chosen bases). This could be an advantage concerning computational issues of the problem.

**APPENDIX**

**A. The language and the equations for dagger compact closed categories with dagger biproducts**

Our choice of a language for dagger compact closed categories with dagger biproducts is the one in which enrichment over $\mathbf{Cmd}$ is primitive and not derived from the biproduct structure. Such a language is suitable for the proofs of our results. A dagger compact closed category with dagger biproducts $\mathcal{A}$ consists of a set of objects and a set of arrows. There are two functions (source and target) from the set of arrows to the set of objects of $\mathcal{A}$. For every object $a$ of $\mathcal{A}$ there is the identity arrow $1_a : a \to a$. The set of objects includes two distinguished objects $I$ and $0$. Arrows $f : a \to b$ and $g : b \to c$ compose to give $g \circ f : a \to c$, and arrows $f_1,f_2 : a \to b$ add to give $f_1 + f_2 : a \to b$. For every object $a$ of $\mathcal{A}$, there is the object $a^*$, and for every pair of objects $a$ and $b$ of $\mathcal{A}$, there are the objects $a \otimes b$ and $a \oplus b$. Also, for every arrow $f : a \to b$, there is the arrow $f^\dagger : b \to a$, and for every pair of arrows $f : a \to a'$ and $g : b \to b'$ there are the arrows $f \otimes g : a \otimes b \to a' \otimes b'$ and $f \oplus g : a \oplus b \to a' \oplus b'$. In $\mathcal{A}$ we have the following families of arrows indexed by its objects.

\[
\begin{align*}
\alpha_{a,b,c} : a \otimes (b \otimes c) &\to (a \otimes b) \otimes c, & \alpha_{a,b,c}^{-1} : (a \otimes b) \otimes c &\to a \otimes (b \otimes c), \\
\lambda_a : I \otimes a &\to a, & \lambda_a^{-1} : a &\to I \otimes a, \\
\sigma_{a,b} : a \otimes b &\to b \otimes a, \\
\eta_a : I &\to a^* \otimes a, & \varepsilon_a : a \otimes a^* &\to I, \\
\pi_{a,b}^1 : a \oplus b &\to a, & \iota_{a,b}^1 &\to a \oplus b, \\
\pi_{a,b}^2 : a \oplus b &\to b, & \iota_{a,b}^2 &\to a \oplus b, \\
0_{a,b} : a &\to b.
\end{align*}
\]

The arrows of $\mathcal{A}$ should satisfy the following equalities:

\[
\begin{align*}
(A.1) & \quad f \circ 1_a = f = 1_{a'} \circ f, \quad (h \circ g) \circ f = h \circ (g \circ f), \\
(A.2) & \quad 1_a \otimes 1_b = 1_{a \otimes b}, \quad (f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \circ f_1) \otimes (g_2 \circ g_1), \\
(A.3) & \quad ((f \otimes g) \otimes h) \circ \alpha_{a,b,c} = \alpha_{a',b',c'} \circ (f \otimes (g \otimes h)), \\
(A.4) & \quad f \circ \lambda_a = \lambda_{a'} \circ (I \otimes f), \quad \lambda_a^{-1} \circ \lambda_a = 1_{I \otimes a}, \quad \lambda_a \circ \lambda_a^{-1} = 1_a, \\
(A.5) & \quad (g \otimes f) \circ \sigma_{a,b} = \sigma_{a',b'} \circ (f \otimes g), \quad \sigma_{b,a} \circ \sigma_{a,b} = 1_{a \otimes b}, \\
(A.6) & \quad \alpha_{a \otimes b,c,d} \circ \alpha_{a,b,c \otimes d} = (\alpha_{a,b,c} \otimes d) \circ (a \otimes \alpha_{b,c,d}).
\end{align*}
\]
The following equalities are derivable from (A.1)-(A.24):

\( \lambda_{a \otimes b} = (\lambda_a \otimes b) \circ \alpha_{f,a,b} \),

\( \alpha_{c,a,b} \circ \sigma_{a \otimes b,c} \circ \alpha_{a,b,c} = (\sigma_{a,c} \otimes b) \circ \alpha_{a,c,b} \circ (a \otimes \sigma_{b,c}) \),

\( (a^* \otimes \varepsilon) \circ \alpha_{a^*,a,a^*} \circ (\eta \otimes a^*) = \sigma_{f,a^*}, \quad (\varepsilon \otimes a) \circ \alpha_{a,a^*,a} \circ (a \otimes \eta) = \sigma_{a,I} \),

\( f_1 + (f_2 + f_3) = (f_1 + f_2) + f_3, \quad f_1 + f_2 = f_2 + f_1, \quad f + 0_{a,a'} = f \),

\( (g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f, \quad g \circ (f_1 + f_2) = g \circ f_1 + f \circ f_2 \),

\( 0_{a',b} \circ f = 0_{a,b}, \quad f \circ 0_{b,a} = 0_{b,a'} \),

\( 1_a \otimes 1_b = 1_{a \otimes b}, \quad (f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \circ f_1) \otimes (g_2 \circ g_1) \),

\( (f \otimes g) \circ \varepsilon_{a,a'} = (f_2 \otimes g_2) \circ \varepsilon_{a,a'}, \quad (f \circ g) \circ \iota_{a,b} = (f_2 \circ g_2) \circ \iota_{a,b} \circ g \),

\( f \circ \pi_{a,b} = \pi_{a',b'} \circ (f \otimes g), \quad g \circ \pi_{a,b} = \pi_{a',b'} \circ (f \otimes g) \),

\( \pi_{a,b} \circ \iota_{a,b} = 1_a, \quad \pi_{a,b} \circ \iota_{a,b} = 1_b \),

\( \pi_{a,b} \circ \iota_{a,b} = 0_{a,b}, \quad \pi_{a,b} \circ \iota_{a,b} = 0_{b,a} \),

\( \iota_{a,b} \circ \pi_{a,b} + \iota_{a,b} \circ \pi_{a,b} = 1_{a \otimes b} \),

\( 0_{a,0} = 1_0 \),

\( 1_{a}^\dagger = 1_a, \quad (g \circ f)^\dagger = f^\dagger \circ g^\dagger, \quad f^{\dagger \dagger} = f \),

\( (f \otimes g)^\dagger = f^\dagger \otimes g^\dagger \),

\( \alpha_{a,b,c}^\dagger = \alpha_{a,b,c}^{-1}, \quad \lambda_{a}^\dagger = \lambda_{a}^{-1}, \quad \sigma_{a,b}^\dagger = \sigma_{b,a} \),

\( \varepsilon^\dagger = \sigma_{a^*,a} \circ \eta \),

\( (\pi_{a,b}^1)^\dagger = \iota_{a,b}^1, \quad (\pi_{a,b}^2)^\dagger = \iota_{a,b}^2 \).
Appendix B. Scalars and Probability Amplitudes

As firmly laid, quantum mechanics is based on complex vector spaces (Hilbert spaces to be more precise). Implied in this structure is the notion of scalars, that correspond here to the field of complex numbers. In categorical language, one can define scalars more abstractly [8, 1]. A scalar is a morphism \( s : I \to I \). It can be proved that the hom-set \( \text{Hom}(I, I) \), for a compact closed category, is a commutative monoid, therefore justifying further this structure’s name.

In 1Cob, the scalars correspond to closed, one-dimensional manifolds, and the only candidate for such a structure is a finite collection of circles \( S^1 \) (as denoted on the left-hand side of the following picture). In 1Cob\(_G\), we have \( G \)-circles; topological circles dressed with group elements (right-hand side of the following picture). Due to the compact closed structure of this category, there is a natural interpretation of those circles. Namely, any compact closed category can be lifted to a traced category by a suitable definition of a categorical trace (see Section 8.3 for the definition).

That closed loops should be connected with traces in not limited to a categorical approach to quantum mechanics. Even when considering Feynman diagrams in quantum electrodynamics, fermions loops are accompanied by a trace in spinorial indices. Moreover, in TQFT, we are customed to the fact that closing manifold by gluing the outward future to inward past (if possible), results in a trace, that for a cylinder, i.e. the identity, it simply gives the dimension of the respective Hilbert space.

Furthermore, as explained in [4], these traces correspond to the probability weights of different branches. This is further confirmed by a Hilbert space picture computations. Recall that one reason we have scalars (different from the multiplicative unit) is normalisation on states. In order to get the probabilistic interpretation, according to the Born rule, we must insist on normalised states. For a state

\[
|\beta_{00}\rangle \otimes |0\rangle + |\beta_{01}\rangle \otimes |1\rangle + |\beta_{10}\rangle \otimes |0\rangle + |\beta_{11}\rangle \otimes |1\rangle,
\]

we have its norm squared

\[
|\beta_{00}|^2 + |\beta_{01}|^2 + |\beta_{10}|^2 + |\beta_{11}|^2 = \text{Tr}(\beta^\dagger \beta),
\]

where \( \beta \) is a \( 2 \times 2 \) matrix whose components are \( \beta_{ij} \) constants. Therefore, we consider significance of traces in the usual sense.

When dealing with quantum protocols, one usually takes \( \beta \) to be proportional to Pauli sigma matrices. (Extended) Pauli matrices are defined as

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

We see that those matrices are unitary, self-adjoint and satisfy \( \text{Tr}(\sigma_i \sigma_j) = 2\delta_{ij} \), where \( \delta_{ij} \) is a Kronecker delta symbol (equal to one if \( i = j \) and zero otherwise). In order to make the connection with the Bell basis, introduced in Section 1 we take \( \beta_1 = \sigma_0 \), \( \beta_2 = \sigma_1 \), \( \beta_3 = \sigma_3 \) and \( \beta_4 = -i\sigma_2 \). This implies that we have

\[
\text{Tr}(\beta_i \beta_j^\dagger) = 2\delta_{ij},
\]

with the usual definition of matrix adjoint.

However, in order to check whether two diagrams commute, it is usually straightforward to include scalars into consideration. One can then just neglect this issue of scalars and work without explicitly using them (as done previously). They are, of course, needed if one is to obtain probabilities for different outcomes of a measurement, but in this work (and related work of [1, 4]) this is not a primary task.
A DIAGRAMMATIC CALCULUS FOR CATEGORICAL QUANTUM PROTOCOLS

Acknowledgements

Zoran Petrić and Mladen Zekić were supported by the Science Fund of the Republic of Serbia, Grant No. 7749891, Graphical Languages - GWORDS. Dušan Đorđević was supported by the Faculty of Physics, University of Belgrade, through the grant of the Ministry of Education, Science, and Technological Development of the Republic of Serbia (Contract No. 451-03-68/2022-14/200162).

References

[1] S. Abramsky and B. Coecke, A categorical semantics of quantum protocols, *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, LICS 2004*, IEEE Computer Society Press, 2004, pp. 415-425

[2] ———, Categorical Quantum Mechanics, (K. Engesser, D.M. Gabbay and D. Lehmann editors) *Handbook of Quantum Logic and Quantum Structures*, vol. 2, Elsevier, 2008, pp. 261-323

[3] J.C. Baez, Quantum Quandaries: A Category Theoretic Perspective, (D. Rickles, S. French, and J. Saatsi editors) *The Structural Foundations of Quantum Gravity*, Oxford University Press, Oxford, 2006, pp. 240-265

[4] B. Coecke, *Kindergarten Quantum Mechanics: Lecture Notes*, (G. Adenier, A. Khrennikov and T.M. Nieuwenhuizen editors) *Quantum Theory: Reconsiderations of the Foundations - 3*, AIP Conference Proceedings, vol. 810, 2005, pp. 81-98

[5] B. Coecke and R. Duncan, Interacting quantum observables: categorical algebra and diagrammatics, *New Journal of Physics*, vol. 13 (2011), 043016.

[6] F. Del Santo and B. Dakić, Two-Way Communication with a Single Quantum Particle, *Physical Review Letters*, vol. 120 (2018), 060503

[7] C. Heunen, Categorical Quantum Models and Logics, Pallas Publications—Amsterdam University Press, 2009

[8] G.M. Kelly and M.L. Laplaza, Coherence for compact closed categories, *Journal of Pure and Applied Algebra*, vol. 19 (1980), pp. 193-213

[9] R. Kirby, A calculus for framed links in $S^3$, *Inventiones Mathematicae*, vol. 45 (1978), pp. 35-56

[10] J. Kock, *Frobenius Algebras and 2D Topological Quantum Field Theories*, Cambridge University Press, Cambridge, 2003

[11] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Springer, Berlin, 1932 (English translation: *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, 1955)

[12] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition*, Cambridge University Press, Cambridge, 2010

[13] Z. Petrić and M. Zekić, Coherence for closed categories with biproducts, *Journal of Pure and Applied Algebra*, vol. 225 (2021), 106533

[14] P. Selinger, Dagger compact closed categories and completely positive maps, *Quantum Programming Languages*, Electronic Notes in Theoretical Computer Science, vol. 170, Elsevier, 2007, pp. 139-163

[15] S. Telebaković Onić, On the Faithfulness of 1-dimensional Topological Quantum Field Theories, *Glasnik Matematički*, vol. 55 (2020), pp. 67-83

[16] D. Tong and K. Wong, Monopoles and Wilson Lines, *Journal of High Energy Physics*, vol. 2014 (2014), 48

[17] V.G. Turaev, *Quantum Invariants of Knots and 3-Manifolds*, De Gruyter, Berlin/New York, 2010

[18] J. Vicary, Higher Quantum Theory, arXiv preprint, 2012, available at https://arxiv.org/abs/1207.4563

Facility of Physics, Studentski trg 12, 11001 Belgrade, Serbia
Email address: dušan.djordjevic@ff.bg.ac.rs

Mathematical Institute SANU, Knez Mihailova 36, p. F. 367, 11001 Belgrade, Serbia
Email address: zpetric@mi.sanu.ac.rs

Mathematical Institute SANU, Knez Mihailova 36, p. F. 367, 11001 Belgrade, Serbia
Email address: mzekic@mi.sanu.ac.rs