Matterwave localization in disordered cold atom lattices

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We propose to observe Anderson localization of ultracold atoms in the presence of a random potential made of atoms of another species and trapped at the nodes of an optical lattice, with a filling factor less than unity. Such systems enable a nearly perfect experimental control of the disorder, while the possibility of modelling the scattering potentials by a set of point-like ones allows an exact theoretical analysis. This is illustrated by a detailed analysis of the one-dimensional case.

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Anderson localization \(^1\) is an interference phenomenon occurring in waves propagating in a static disorder: rather than spreading, the wave remains localized in a portion of space. For classical waves, localization was observed in 2D water waves \(^2\), and in 1D \(^3\) and 3D \(^4\) light beams. When the wave corresponds to a quantum particle wavefunction, localization in a disordered potential (of infinite spatial extension) corresponds to the existence of square integrable stationary states at energies for which the classical motion is not bounded \(^5\).

Indirect evidence of Anderson localization of electrons in condensed matter systems (e.g., the conductance dependence on the temperature) was obtained in 2D and 3D and also in thin wires, with an interpretation made difficult by interaction effects and the presence of a thermal bath \(^6\). Truly 1D condensed matter systems are ordinarily subjected to strong interactions, and there is presently an active debate about the role of interactions in 2D initiated by the observation of a metal-insulator transition \(^6\) in 2D electron gases.

On the other hand, ultracold atomic gases appear as very favorable systems for experimental study of Anderson localization of matterwaves. These systems have the advantage of being very flexible: due to a very weak coupling to the environment they are virtually immune to unwanted decoherence while enabling the possibility of coupling to a specifically engineered thermal bath \(^7\) or an effective magnetic field \(^8\) for the aim of probing their effect on the localization. The Feshbach resonance \(^8\) allows a controllable introduction of interactions whose strength can be chosen at will \(^9\), and therefore opens the possibility of experimental tests for interaction-localization effects as appeared in models \(^10\) which, although analytically solvable, were considered unrealistic until now. Finally, the dimensionality of the gas is also adjustable by the use of tailorable optical potentials \(^12\).

A natural way of producing a disordered potential for atoms is by using a laser speckle \(^13, 14\). This requires however a very careful control of the speckle, to ensure that the absence of spreading of a matterwave is not due to a trapping of an atom in a local potential minimum, which is particularly challenging in 3D where only the lowest energy states may be localized. Also, comparison with theory generally requires a numerical, rather than analytical, solution of Schrödinger’s equation.

In this work, we propose a way to create an almost perfectly well controlled disordered potential for an atomic matterwave, which can even be determined by a direct measurement. Moreover, this potential can be justifiably modelled by point-like scatterers which allows an analytical, and often exact \(^11, 17\) study of the localization and its observability. Although we focus on 1D, our scheme is applicable to any dimension. In fact, in 2D it can be subjected to an exact analysis \(^18\) even in the presence of a magnetic field.

We consider a gas of atoms trapped at the nodes of an optical lattice of spatial period \(b\) (Fig. 1): each atom is cooled down to the ground vibrational state of the local micro-well; each node is occupied by an atom with probability \(p\) independently of the other nodes, multiple occupancies assumed not to occur; tunneling between neighboring sites is made negligible by choosing the modulation depth of the optical lattice much larger than the energy \(\hbar^2(\pi/b)^2/m_s\), where \(m_s\) is the trapped atom mass, to ensure that the spatial configuration is static \(^19\).

The set of trapped atoms, designated below as ‘scatterers’, will act as a random potential for atoms of another species or of the same species but in another internal state, denoted as ‘test particles’. One should ensure that the test particles (unlike the scatterers) will not be trapped by the optical lattice. This can be achieved by using two different species with sufficiently different resonance frequencies \(^15\). In what follows, we assume that the test particles experience as an external potential only the interaction potential with the scatterers. To ensure elastic scattering we also require that the incoming kinetic energy of a test particle is less than the level spacing \(\hbar\omega_s\) of a trapped scatterer,

\[
\frac{\hbar^2 k^2}{2m_t} \ll \hbar\omega_s\tag{1}
\]

where \(k\) and \(m_t\) are the wavevector and the mass of the
test particle. \( \omega_s \) is the oscillation frequency of a scatterer in the local micro-well of the lattice. Eq. (1) ensures, by energy conservation, that at the end of a scattering event the scatterer is not left in an excited vibrational level: hence, the disordered potential is static.

Focusing on 1D we assume that the test particle is strongly trapped in a matterwave guide, with a quantum of vibrational energy \( h \omega \) much larger than the longitudinal kinetic energy, so that its transverse \( y-z \) motion is frozen in the ground vibrational state of the guide. We then introduce the model Hamiltonian for the quantum motion of a test particle along the lattice direction \( x \):

\[
H = \frac{p_x^2}{2m_t} + \sum_{j=1}^{N} g \delta(x - x_j).
\]

Here the \( x_j \) are the positions of the occupied micro-wells, all integer multiple of the lattice spacing \( b \). The effect of each scatterer is represented by a Dirac delta potential with a coupling constant \( g \). This assumption is reasonable when the wavevector \( k \) of the test particle is small enough: assuming for simplicity that \( m_s = m_t = m \), \( 1/k \) should be larger than the sizes \( a_s^{ho} = (h/m \omega_{s,z})^{1/2} \) of the harmonic oscillator ground state of a scatterer in a micro-well and of the transverse guide ground state of the test particle conditions already ensured by the elasticity condition Eq. (1) and by the 1D nature of the motion of the test particle. A scatterer may then be modelled by a zero range potential. When the 3D scattering length, \( a \), describing the free space interaction between a scatterer and a test particle is much smaller than the harmonic oscillator lengths \( a_s^{ho} \) one is in the so-called Born regime and the 1D coupling constant \( g \) is given by:

\[
g = 4\hbar\frac{\omega_z}{\omega_s + \omega_t} a.
\]

Our model Hamiltonian was shown in [21] to lead to localization as a consequence of a theorem derived in Ref. [22]. An acceptable quantitative measure of localization, that we adopt in this paper, is the decay length of the transmission coefficient: taking all the scatterers to be in the half space \( x \geq 0 \) and introducing the transmission amplitude \( t_N(k) \) of an incoming plane wave of momentum \( k > 0 \) through a set of \( N \) scatterers, we define the localization constant \( \kappa(k) \) as [23]:

\[
\kappa(k) = \lim_{N \to \infty} \frac{\log |t_N(k)|}{x_N - x_1}.
\]

The average \( \langle . \rangle \) is over all possible realizations of the disorder; although, strictly speaking it is not required since \( \log |t_N(k)| \), contrarily to \( |t_N(k)| \) itself, is a self-averaging quantity for \( N \to +\infty \) [24, 25]. The transmission and reflection amplitudes are given by

\[
t_N = 1/(R_N)_{11} \quad r_N/t_N = -(R_N)_{21},
\]

where \( R_N \) is related to the transfer matrix of the matterwave through \( N \) scatterers, \( \tilde{R}_N \), through the relation

\[
R_N(x_1, \ldots, x_N) \equiv T(x_N)^{-1} \tilde{R}_N T(x_1) = G_0 T(x_N - x_{N-1})^{-1} G_0 \ldots T(x_2 - x_1)^{-1} G_0.
\]

\( G_0 \) is the transfer matrix of a single scatterer at \( x = 0 \):

\[
G_0 = \begin{pmatrix} 1 - i\alpha & -i\alpha \\ i\alpha & 1 + i\alpha \end{pmatrix}
\]

with \( \alpha = mg/(\hbar^2 k) \), and \( T(x) \) is the transfer matrix corresponding to a free propagation over an abscissa \( x \):

\[
T(x) = \begin{pmatrix} e^{-ikx} & 0 \\ 0 & e^{ikx} \end{pmatrix}.
\]

The use of \( R_N \) instead of \( \tilde{R}_N \) in Eq. (6) does not affect \( |t_N(k)| \), but simplifies the calculations, since \( R_N \) depends only on the variables \( x_{i+1} - x_i \), which are independent random variables with a common probability distribution given by: \( P(s_i) = p(1-p)^{s_i-1} \) where \( s_i = (x_{i+1} - x_i)/b \).

We calculated the localization constant \( \kappa \) numerically, by a Monte Carlo averaging over the disorder, taking a large enough number of scatterers to ensure convergence in Eq. (4). When expressed in units of \( 1/b \), \( \kappa \) depends on three dimensionless parameters: the filling factor \( p \), the reduced momentum \( kb \) and the reduced coupling constant \( mgb/h^2 \). Assuming for simplicity that \( \omega_s = \omega_t = \omega \), so that the harmonic oscillator lengths also coincide, \( \alpha^{ho} = \alpha_t^{ho} = \alpha_s^{ho} \), and introducing the recoil energy \( E_R = h^2(\pi/b)^2/2m \), we find \( mgb/h^2 = 2\pi(a/\alpha^{ho})(\hbar \omega/2E_R)^{1/2} \). Typically, \( \hbar \omega < 10E_R \). Since we required \( a/\alpha^{ho} \ll 1 \), one should therefore have \( mgb/h^2 \ll 15 \). In all our numerical calculations \( mgb/h^2 = 2.278 \).

Figure 2 shows the localization constant dependence on the momentum \( k \): Fig. 2a for filling factor \( p = 0.9 \) (solid lines) and \( p = 1 \) (dashed line), and Fig. 2b for filling factor \( p = 0.1 \) (solid lines). In the case \( p = 1 \) where the scatterers form a finite periodic chain, allowed bands (where \( \kappa = 0 \) corresponding to infinite localization length of the Bloch waves) are separated by forbidden gaps. The \( p = 0.9 \) case corresponds to a perturbation of the periodic chain by an occasional appearance of empty sites. Now \( \kappa \) takes non-zero values in the former allowed
bands while its value in the former forbidden gaps is only weakly affected by the disorder. For \( p = 0.1 \) the band structure of the periodic chain is washed out, except for the vanishing of \( \kappa \) in the points where \( kb \) is an integer multiple of \( \pi \) (corresponding to band edges in the periodic case). The peaks and steps on the \( \kappa \) curves (marked by arrows) will be interpreted analytically below, in terms of a phase shift being a rational multiple of \( \pi \).

At a given scattering energy, measuring a finite value of \( \kappa \) is, strictly speaking, not a proof of localization but may be due simply to the presence of a spectral gap as is the case in periodic systems. To confirm the existence of localized states one should calculate the density of states and check that it is also finite. We performed such a numerical check, (shown in the upper frames of Figs. 2a and 2b), by imposing periodic boundary conditions in a box of size \( x_N - x_1 \), and averaging over disorder.

We also performed an analytic calculation of \( \kappa \) in several limiting cases. The first limiting case is the \( p \to 0 \) limit for a fixed value of \( kb \) and can be treated along the lines of Ref. 8. By expanding the matrix product involving the two factors \( G_0 T(x_N - x_{N-1})^{-1} \) and \( R_{N-1} \) in Eq. (10), one obtains the recursion relation:

\[
\log |t_N| = \log |t_0 t_{N-1}| - \log \left| 1 + r_0^* r_{N-1} t_{N-1}^* t_{N-1} e^{-2i(kb x_N - x_{N-1})} \right| \tag{9}
\]

where \( r_0 \) and \( t_0 \) are the reflection and transmission amplitudes for the transfer matrix \( G_0 \). When \( p \) tends to zero, the accumulated phase shift \( kb s \) (with \( P(s) = p(1-p)s^{-1} \) as above) between two successive scatterers is uniformly distributed between 0 and \( 2\pi \) (modulo \( 2\pi \)), as long as \( kb \) is not a rational multiple of \( \pi \). Noting that \( \int_0^{2\pi} d\theta \log |1 - z e^{i\theta}| = 0 \) for \( |z| < 1 \), we obtain

\[
\langle \log |t_N| \rangle = \log |t_0| + \langle \log |t_{N-1}| \rangle \tag{10}
\]

which leads to

\[
\kappa b = (1/2)p \log (1 + \alpha^2). \tag{11}
\]

Fig. 2b shows a good agreement of Eq. (11) with numerics, except for the peaks at \( kb = \pi/(s+1) \) and the steps at \( kb = 2\pi/(2s+1) \), for \( s \) integer. The calculation is readily extended to the \( p \to 1 \) limit, by considering the defects in the chain as scatterers on top of a periodic background: the propagation of the matter waves in between two consecutive defects of distance \( sb \) is given by the transfer matrix \( T(sb) = (T(b))^{-1} G_0 \), and the scattering over a defect corresponds to the transfer matrix \( G_0 \). Assuming that the incoming energy is in an allowed band of the periodic lattice, the matrix \( T(b)^{-1} G_0 \) has eigenvalues of the form \( e^{\pm i \theta} \). When \( \theta \) is not a rational multiple of \( \pi \), the procedure of the previous paragraph can be reused, replacing \( G_0 \) by \( G_0 \) and \( T(sb) \) by \( T(sb) \). We then get a formula similar to Eq. (11), with \( t_0 \) replaced by \( t_0 \), which is the transmission amplitude for the transfer matrix \( \bar{G}_0 \) in the basis where \( \bar{T}(b) \) is diagonal. This leads to

\[
\kappa b \sim (1 - p) \log \left| \exp(ikb) - \rho \exp(-ikb) \right|/1 - \rho \tag{12}
\]

where \( \rho = |\exp(i\theta) - (1 - i\alpha) \exp(ikb)|^2/\alpha^2 \) and where \( \theta \) is the solution of \( \cos \theta = \cos(kb) + \alpha \sin(kb) \) for \( \rho < 1 \). This expression agrees well with numerics, see Fig. 2, except for the peaks: as shown in this figure, these indeed correspond to values of \( \theta/\pi \) that are rationals.

The third limit we investigated analytically is that of a narrow distribution of the phase shift \( \theta = skb \) between two consecutive scatterers: denoting its average by \( \langle \theta \rangle \), it is assumed that the probability of finding \( \theta + \pm \langle \theta \rangle \) out of an interval of size \( \ll 1 \) is small so that \( kb < 1 \) and that the variance \( \Delta \theta^2 = (1 - p)(kb/p)^2 < 1 \). Along the lines of 24 we expand the relation (derived in 24):

\[
\kappa b = p \int_0^{2\pi} d\phi \mu(\phi) |\langle \log |M_{11}(\theta)| e^{i\phi} + M_{12}(\theta) e^{-i\phi} \rangle| \tag{13}
\]
between $\kappa$ and the invariant (Haar) measure $\mu(\phi)$ which is a solution to the Dyson-Schmidt equation:

$$\mu(\phi) = \langle \mu[\gamma(\phi, \theta)] \partial \theta \gamma(\phi, \theta) \rangle$$  \hspace{1cm} (14)

where $M(\theta) = G_0 T(\theta k^{-1})^{-1}$. $\gamma(\phi, \theta)$ is the argument of the complex number $M_{11}(\theta)e^{i\phi} + M_{12}(\theta)e^{-i\phi}$. It is useful to note \[26\] that Eqs. \[13,14\] are invariant under any $\theta$ independent $SU(1,1)$ similarity transformation $M(\theta) \rightarrow D^{-1}M(\theta)D$. Assuming the energy is in an allowed band of a periodic chain of scatterers of period $\theta/k = \theta/p$, the transfer matrix $M(\theta)$ has then unimodular eigenvalues and we choose $D$ so that $D^{-1}M(\theta)D$ is diagonal. We then expand all functions of $\theta$ in powers of $(\theta - \langle \theta \rangle)$ up to fourth order, which requires an expansion of $\mu(\phi)$ and $\kappa$ to first and second orders in $\Delta \theta^2$:

$$\mu(\phi) = \mu^{(0)}(\phi) + \mu^{(1)}(\phi) + \ldots ; \quad \kappa = \kappa^{(0)} + \kappa^{(1)} + \kappa^{(2)} + \ldots$$  \hspace{1cm} (15)

One has $\mu^{(0)}(\phi) = 1/(2\pi)$ and $\kappa^{(0)} = 0$. Denoting $a_i \equiv Re(e^{ikb/p}(1-i\alpha))$, $a_i \equiv Im(e^{ikb/p}(1-i\alpha))$, we get \[27\]:

$$\kappa^{(1)}b = \frac{p(1-p)a_2}{1-a_2^2} \left( \frac{kb}{p} \right)^2$$  \hspace{1cm} (16)

$$\kappa^{(2)}b = -p(1-p)a_2^2 \frac{a_2^2 + a_2^2/2}{1-a_2^2} \left( \frac{kb}{p} \right)^4$$  \hspace{1cm} (17)

In conclusion, we proposed a well controlled way of producing a disordered potential for atomic matter waves, by the scattering of test particles on scatterers trapped at the nodes of an optical lattice. We showed how a transmission experiment through such a one-dimensional disordered chain provides a clear direct evidence of Anderson localization at energies where the density of states is appreciable. The proposed experiment is one of several possibilities, such as the measurement of the (absence of) spreading of a wavepacket initially prepared inside the disordered medium which is linked to other exactly calculable \[10,17\] aspects of localization: the finite return probability and finite inverse participation ratio. The proposed scheme is extendable to higher dimensions and enables a controlled experimental study of how the localization is affected by the introduction of engineered thermal-like bath \[11\], effective magnetic field \[8\], interactions among test particles \[11\], or several coupled channels for the transverse motion of the test particle \[28\].

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