Numerical solution of Showalter–Sidorov and Cauchy problems of ion–acoustic waves propagation mathematical model

E V Bychkov and A A Zamyshlyaeva
South Ural State University, 76 Lenin ave, 454080 Chelyabinsk, Russia
E-mail: bychkovev@susu.ru

Abstract. The paper deals with the problem of numerical investigation of semilinear mathematical model of ion-acoustic waves propagation in plasma. The research is based on the previous study of solvability of the Cauchy problem for an abstract semilinear Sobolev type equation of higher order. The theory of relatively polynomially bounded operator pencils, the theory of differentiable Banach manifolds, and the phase space method are used for analytical study of the model. Projectors splitting spaces into direct sums of subspaces are constructed. Given equation is reduced to a system of two equations. One of them determines the phase space of the initial equation, and its solution is a function with values from the eigenspace of the operator at the highest time derivative. The solution of the second equation is the function with values from the image of the projector. Moreover, in the second section, the sufficient conditions for the solvability of the abstract problem under study are presented. These results are applied to the mathematical model of ion-acoustic waves in plasma which is based on the fourth-order equation with a singular operator at the highest time derivative. Reducing the mathematical model to an abstract problem, we obtain sufficient conditions for the existence of unique solution. The results of analytical investigation of the Showalter – Sidorov problem which is more natural for Sobolev type equations are also presented in this section. The Galerkin method is used for numerical study of the model. An algorithm for the numerical solution of the Showalter – Sidorov problem for the model of ion-acoustic waves in plasma is described in the last section.

1. Introduction
Let \( \Omega = (0, a) \times (0, b) \times (0, c) \subset \mathbb{R}^3 \). In a cylinder \( \Omega \times \mathbb{R} \) consider equation which arose in a theory of ion-acoustic waves in plasma [1]

\[
(\Delta - \lambda)u_{tttt} + (\Delta - \lambda')u_{tt} + \alpha \frac{\partial^2 u}{\partial x^2} = \Delta(u^3)
\]

(1)

with the Dirichlet condition

\[
u(x, t) = 0, \quad (x, t) \in \partial \Omega \times \mathbb{R}.
\]

(2)

and the Cauchy conditions

\[
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x),
\]

\[
u_{tt}(x, 0) = u_2(x), \quad u_{ttt}(x, 0) = u_3(x), \quad x \in \Omega.
\]

(3)
or the Showalter–Sidorov conditions [2]

\[
P(u(x,0) - u_0(x)) = 0, \quad P(u(x,0) - u_1(x)) = 0,
\]

\[
P(u_{tt}(x,0) - u_2(x)) = 0, \quad P(u_{ttt}(x,0) - u_3(x)) = 0, \quad x \in \Omega.
\] (4)

In suitable Banach spaces Ω and F mathematical model (1) – (3) can be reduced to the Cauchy problem

\[
u^{(k)}(0) = u_k, \quad k = 0, 1, \ldots, n - 1
\] (5)

(mathematical model (1), (2), (4) can be reduced to the Showalter–Sidorov problem

\[
P(u^{(k)}(0) - u_k) = 0, \quad k = 0, 1, \ldots, n - 1
\] (6)

for a semilinear Sobolev type equation of higher order [3]

\[
Au^{(n)} = B_{n-1}u^{(n-1)} + B_{n-2}u^{(n-2)} + \ldots + B_0u + N(u),
\] (7)

where \(u^{(k)}\) is the time derivative of order \(k\), the operators \(A, B_{n-1}, B_{n-2}, \ldots, B_0 \in \mathcal{L}(\Omega; \mathcal{F})\), \(N \in C^\infty(\Omega; \mathcal{F})\), \(P\) is some projector. By Sobolev type equations [4] we mean those equations that are not solvable with respect to the highest time derivative [5] in particular when the operator \(A\) is not invertible. Such situations often arise when \(\ker A \neq \{0\}\). Mathematical models representable in form (5), (7) (or (6), (7)) will be called Sobolev type mathematical models of higher order.

It is known that the Cauchy problem for Sobolev type equations is unsolvable in principle for arbitrary initial data. In our opinion, the most fruitful approach to the study of such equations is the phase space method developed by G.A. Sviridyuk and T.G. Sukacheva [6] for the study of semilinear Sobolev type equations of the first order. The essence of this method consists in reducing the singular equation to a regular one, defined, however, not on the entire space, but on some subset containing admissible initial values, understood as the phase space of the original equation.

The aim of the work is to develop a method of numerical investigation of the mathematical model of waves propagation in plasma with Cauchy or Showalter–Sidorov conditions.

2. Theory Relatively Polynomially Bounded Operator Pencils

Let Ω, F be Banach spaces and operators \(A, B_0, B_1, \ldots, B_{n-1} \in \mathcal{L}(\Omega; F)\). By \(\tilde{B}\) denote the pencil formed by operators \(B_{n-1}, \ldots, B_1, B_0\). The sets \(\rho^A(\tilde{B}) = \{\mu \in \mathbb{C} : (\mu^n A - \mu^{n-1} B_{n-1} - \ldots - \mu B_1 - B_0)^{-1} \in \mathcal{L}(F; \Omega)\}\) and \(\sigma^A(\tilde{B}) = \mathbb{C} \setminus \rho^A(\tilde{B})\) are called an A-resolvent set and an A-spectrum of the pencil \(\tilde{B}\) respectively. The operator-function of a complex variable \(R^A_{\mu}(\tilde{B}) = (\mu^n A - \mu^{n-1} B_{n-1} - \ldots - \mu B_1 - B_0)^{-1}\) with the domain \(\rho^A(\tilde{B})\) is called an A-resolvent of the pencil \(\tilde{B}\).

**Definition 2.1** The operator pencil \(\tilde{B}\) is called polynomially bounded with respect to an operator \(A\) (or polynomially A-bounded) if \(\exists a \in \mathbb{R}_+ \forall \mu \in \mathbb{C} \quad (|\mu| > a) \Rightarrow (R^A_{\mu}(\tilde{B}) \in \mathcal{L}(F; \Omega)).\)

**Remark 1** If there exists an operator \(A^{-1} \in \mathcal{L}(\mathcal{F}; \Omega)\) then the pencil \(\tilde{B}\) is A-bounded.

In [3] A.A. Zamyshtyaveva received the necessary condition for the construction of projectors

\[
\int_\gamma \mu^k R^A_{\mu}(\tilde{B}) d\mu \equiv 0, \quad k = 0, 1, \ldots, n - 2,
\] (8)

where the circuit \(\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}\).
Lemma 2.1 [3] Let the operator pencil $\vec{B}$ be polynomially $A$-bounded and condition (8) be fulfilled. Then the operators

$$
P = \frac{1}{2\pi i} \int \overline{R}_\mu^A(\vec{B}) \mu^{n-1} d\mu, \quad Q = \frac{1}{2\pi i} \int \mu^{n-1} AR_\mu^A(\vec{B}) d\mu
$$

are projectors in spaces $\mathfrak{U}$ and $\mathfrak{F}$ respectively.

Denote $\mathfrak{U}^0 = \ker P$, $\mathfrak{F}^0 = \ker Q$, $\mathfrak{U}^1 = \text{im } P$, $\mathfrak{F}^1 = \text{im } Q$. According to lemma 2.1 $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$, $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$. By $A^k (B^k)$ denote restriction of operators $A (B_i)$ on $\mathfrak{U}^k$, $k = 0, 1; \ l = 0, 1, \ldots, n-1$.

Theorem 2.1 [3] Let the operator pencil $\vec{B}$ be polynomially $A$-bounded and condition (8) be fulfilled. Then

(i) $A^k \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k)$, $k = 0, 1$;
(ii) $B^k_l \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k)$, $k = 0, 1$, $l = 0, 1, \ldots, n-1$;
(iii) operator $(A^1)^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$ exists;
(iv) operator $(B^0_0)^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$ exists.

Using theorem 2.1 construct operators $H_0 = (B^0_0)^{-1}A^0 \in \mathcal{L}(\mathfrak{U}^0)$, $H_1 = (B^0_0)^{-1}B^0_0 \in \mathcal{L}(\mathfrak{U}^0)$, $H_{n-1} = (B^0_0)^{-1}B^0_{n-1} \in \mathcal{L}(\mathfrak{U}^0)$ and $S_0 = (A^1)^{-1}B^1_0 \in \mathcal{L}(\mathfrak{U}^1)$, $S_1 = (A^1)^{-1}B^1_1 \in \mathcal{L}(\mathfrak{U}^1)$, ..., $S_{n-1} = (A^1)^{-1}B^1_{n-1} \in \mathcal{L}(\mathfrak{U}^1)$.

Definition 2.2 Define the family of operators $\{K_0^1, K_2^1, \ldots, K^n_q\}$ as follows:

$$
K_0^1 = \mathbb{O}, \ s \neq n; \ K_0^n = \mathbb{I},
K_1^1 = H_0, \ K_2^1 = -H_1, \ldots, K_{s}^1 = -H_{s-1}, \ldots, K_n^1 = H_{n-1},
K_0^q = K_{q-1}^nH_0, \ K_2^q = K_{q-1}^n - K_{q-1}^{n-1}H_1, \ldots, K_q^q = K_{q-1}^{n-1} - K_{q-1}^{n-2}H_{s-1}, \ldots,
K_q^q = K_{q-1}^{n-1} - K_{q-1}^{n-2}H_{n-1}, \ q = 1, 2, \ldots,
$$

Definition 2.3 The point $\infty$ is called

- a removable singular point of the $A$-resolvent of the pencil $\vec{B}$, if $K_1^s \equiv \mathbb{O}, s = 1, 2, \ldots, n$;
- a pole of order $p \in \mathbb{N}$ of the $A$-resolvent of the pencil $\vec{B}$, if $\exists p$ such that $K^s_p \neq \mathbb{O}, s = 1, 2, \ldots, n$, but $K^s_{p+1} \equiv \mathbb{O}, s = 1, 2, \ldots, n$;
- an essential singular point of the $A$-resolvent of the pencil $\vec{B}$, if $K^q_n \neq \mathbb{O}$ for all $q \in \mathbb{N}$.

Further a removable singularity of an $A$-resolvent of the pencil $\vec{B}$ will be called a pole of order 0 for brevity. If the operator pencil $\vec{B}$ is polynomially $A$-bounded and the point $\infty$ is a pole of order $p \in \{0\} \cup \mathbb{N}$ of an $A$-resolvent of the pencil $\vec{B}$ then the operator pencil $\vec{B}$ is called polynomially $(A, p)$-bounded.

Theorem 2.2 [7] Let $A, B_0, B_1, \ldots, B_n \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$ and $A$ be a Fredholm operator. Then the following statements are equivalent:

(i) The lengths of all chains of the $\vec{B}$-adjoined vectors of the operator $A$ are bounded by number $(p + n - 1) \in \{0\} \cup \mathbb{N}$ and the chain of length $(p + n - 1)$ exists.
(ii) The operator pencil $\vec{B}$ is polynomially $(A, p)$-bounded.
3. The Abstract Problem

First of all consider problem (7), (5) and give definition of its solution.

Definition 3.1 If a vector-function \( u \in C^\infty((-\tau, \tau); \mathcal{U}) \), \( \tau \in \mathbb{R}_+ \) satisfies equation (7) then it is called a solution of this equation. If the vector-function satisfies additionally condition (5) then it is called a solution of (5), (7).

Definition 3.2 The set \( \mathcal{P} \) is called a phase space of (7), if

(i) for all \((u_0, u_1, \ldots, u_{n-1}) \in T^{n-1} \mathcal{P}\) (tangent bundle of order \( n-1 \) of set \( \mathcal{P} \)) there exists a unique solution of (7), (5);

(ii) the solution \( u = u(t) \) of (7) lies in \( \mathcal{P} \) as a trajectory, i.e. \( u(t) \in \mathcal{P} \) for all \( t \in (-\tau, \tau) \).

If \( \ker A = \{0\} \) then equation (7) can be reduced to an equivalent equation

\[
u^{(n)} = F(u, \dot{u}, \ldots, u^{(n-1)}),
\]

where \( F(u, \dot{u}, \ldots, u^{(n-1)}) = A^{-1}(B_{n-1}u^{(n-1)} + B_{n-2}u^{(n-2)} + \ldots + B_0u + N(u)) \) is a mapping of class \( C^\infty \) by construction. The existence of a unique local solution \( u \) of (5), (7) for all \((u_0, u_1, \ldots, u_{n-1}) \) follows from classical theorem [8].

Let \( \ker A \neq \{0\} \) and operator pencil \( \bar{B} \) be \((A, 0)\)-bounded, then by theorem 1.1 equation (7) can be reduced to an equivalent system of equations

\[
\begin{align*}
0 &= (I - Q)(B_0 + N)(u^0 + u^1), \\
\frac{d^n}{dt^n} u^1 &= A^{-1}Q(B_{n-1}u^{(n-1)} + B_{n-2}u^{(n-2)} + \ldots + B_0u + N)(u^0 + u^1),
\end{align*}
\]

(9)

where \( u^1 = Pu, u^0 = (I - P)u \).

Now consider a set \( \mathfrak{M} = \{u \in \mathcal{U} : (I - Q)(B_0u + N(u)) = 0\} \). Let the set \( \mathfrak{M} \) be not empty, i.e. there is a point \( u_0 \in \mathfrak{M} \). Denote \( u_0^1 = Pu \in \mathcal{U}_1 \).

Let following condition be fulfilled

\[
(I - Q)(B_0 + N'_{u_0}): \mathcal{U}_0^1 \to \mathcal{F}_0^1 \text{ is a toplinear isomorphism.}
\]

(10)

According to the implicit function theorem [9] there exist neighborhoods \( \mathcal{O}_0^1 \subset \mathcal{U}_0^1 \) and \( \mathcal{O}_1^1 \subset \mathcal{U}_1^1 \) of points \( u_0^0 = (I - P)u_0, u_0^1 = Pu_0 \) respectively and the operator \( B \in C^\infty(\mathcal{O}_1^1, \mathcal{O}_0^1) \) such that \( u_0 = B(u_0^0) \). Lets construct an operator \( \delta = I + B : \mathcal{O}_1^1 \to \mathfrak{M}, \delta(u_0^0) = u_0 \). Then the operator \( \delta^{-1} \) together with the set \( \mathcal{O}_1^1 \) makes a map of \( \mathfrak{M} \) and is a restriction of \( P \) on \( \delta[\mathcal{O}_1^1] = \mathcal{O} \subset \mathfrak{M} \). Thus we prove

Lemma 3.1 The set \( \mathfrak{M} = \{u \in \mathcal{U} : (I - Q)(B_0u + N(u)) = 0\} \) under condition (10) is a \( C^\infty \)-manifold at point \( u_0 \).

Let act with the Frechet derivative \( \delta^{(n)}_{u_0^0, u_1, \ldots, u_{n-1}} \) of order \( n \) on the second equation of system (9). Since \( \delta(u^1) = u \) and

\[
\delta^{(n)}_{u_0^0, u_1, \ldots, u_{n-1}} u^{1(n)} = \frac{d^n}{dt^n} (\delta(u^1))
\]

we obtain equation \( u^{(n)} = F(u, \dot{u}, \ldots, u^{(n-1)}) \), where

\[
F(u, \dot{u}, \ldots, u^{(n-1)}) = \delta^{(n)}_{u_0^0, u_1, \ldots, u_{n-1}} A^{-1}Q(B_{n-1}u^{(n-1)} + B_{n-2}u^{(n-2)} + \ldots + B_0u + N(u)) \in C^\infty(\mathcal{U}).
\]

By virtue of the classical theorem [8], we get
Theorem 3.1 Let the operator pencil \( \vec{B} \) be \((A,0)\)-bounded, \( N \in C^\infty(\Omega, \mathfrak{F}) \) and condition (10) be fulfilled. Then for any \( (u_0, u_1, \ldots, u_{n-1}) \in T^{n-1}\mathfrak{M} \) there exists a unique solution of (5), (7) lying in \( \mathfrak{M} \) as trajectory.

The condition \( (u_0, u_1, \ldots, u_{n-1}) \in T^{n-1}\mathfrak{M} \) is satisfied for the problem (6), (7), automatically, and from previous theorem we have

**Theorem 3.2** Let the operator pencil \( \vec{B} \) be \((A,0)\)-bounded, \( N \in C^\infty(\Omega, \mathfrak{F}) \) and condition (10) be fulfilled. Then for any \( (u_0, u_1, \ldots, u_{n-1}) \in \mathfrak{M} \) there exists a unique solution of (7), (6) lying in \( \mathfrak{M} \) as trajectory.

4. Mathematical Model of Ion-Acoustic Waves in Plasma

In order to reduce (1)–(3) to (5), (7) (and (1), (2), (4) to (6), (7)) set

\[ \mathfrak{U} = \{ u \in W_2^{p+2}(\Omega) : u(x) = 0, x \in \partial \Omega \}, \quad \mathfrak{F} = W_2^p(\Omega), \quad p \in \{ 0 \} \cup \mathbb{N}. \]

Define operators \( A = \Delta - \lambda, B_2 = (\lambda' - \Delta), B_0 = -\alpha \frac{\partial^2}{\partial x_3^2}, B_3 = B_1 = \emptyset \). Operators \( A, B_3, B_2, B_1, B_0 \in \mathcal{L}(\mathfrak{U}; \mathfrak{F}) \) for all \( l \in \{ 0 \} \cup \mathbb{N} \).

Denote the eigenfunctions of the Dirichlet problem (2) for the Laplace operator by \( \varphi_{k,m,l} \). The spectrum \( \sigma(\Delta) \) is negative, discrete, finite and tends only to \(-\infty\). Since \( \{ \varphi_{k,m,l} \} \subset C^\infty(\Omega) \) we obtain

\[
\mu^4 A - \mu^3 B_3 - \mu^2 B_2 - \mu B_1 - B_0 = \sum_{k,m,l=1}^{\infty} [(\lambda_{k,m,l} - \lambda)\mu^4 + (\lambda_{k,m,l} - \lambda')\mu^2 - \alpha \left( \frac{\pi l}{c} \right)^2] < \varphi_{k,m,l}, \cdot > \varphi_{k,m,l},
\]

where \( < \cdot, \cdot > \) is the inner product in \( L^2(\Omega) \).

**Remark 2** In case (i) when \( \lambda \not\in \sigma(\Delta) \) the \( A \)-spectrum of pencil \( \vec{B} \sigma^A(\vec{B}) = \{ \mu^4_{k,m,l} : k, m, l \in \mathbb{N}, j = 1, \ldots, 4 \} \), where \( \mu^4_{k,m,l} \) are the roots of equation

\[
(\lambda_{k,m,l} - \lambda)\mu^4 + (\lambda_{k,m,l} - \lambda')\mu^2 - \alpha \left( \frac{\pi l}{c} \right)^2 = 0, \quad (11)
\]

In case (ii) when \( (\lambda \in \sigma(\Delta)) \land (\lambda \neq \lambda') \) the \( A \)-spectrum of pencil \( \vec{B} \sigma^A(\vec{B}) = \{ \mu^4_{k,m,l} : k, m, l \in \mathbb{N}, j = 1, \ldots, 4 \} \), where \( \mu^4_{k,m,l} \) the roots of equation (11) with \( \lambda \neq \lambda_{k,m,l} \). In case (iii) when \( (\lambda \in \sigma(\Delta)) \land (\lambda = \lambda') \) the \( A \)-spectrum of pencil \( \vec{B} \sigma^A(\vec{B}) = \{ \mu^4_{k,m,l} : k, m, l \in \mathbb{N}, j = 1, \ldots, 4 \} \).

Check condition (8). In case (i) there exists \( A^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U}) \) therefore condition (8) is fulfilled.

In case (ii)

\[
\frac{1}{2\pi i} \int_{\gamma} \sum_{k=1}^{\infty} \frac{\mu^r < \varphi_{k,m,l}, \cdot > \varphi_{k,m,l} d\mu}{(\lambda_{k,m,l} - \lambda)\mu^4 + (\lambda_{k,m,l} - \lambda')\mu^2 - \alpha \left( \frac{\pi l}{c} \right)^2} = \frac{1}{2\pi i} \int_{\gamma} \sum_{k,m,l=1}^{\infty} \frac{\mu^r < \varphi_{k,m,l}, \cdot > \varphi_{k,m,l} d\mu}{(\lambda_k - \lambda')\mu^2 - \alpha \left( \frac{\pi l}{c} \right)^2} \neq 0,
\]

when \( r = 1 \), therefore condition (8) is not fulfilled and this case is excluded from further considerations. In case (iii) \( (\lambda \in \sigma(\Delta)) \land (\lambda = \lambda') \) condition (8) is fulfilled.

**Lemma 4.1** Let (i) \( \lambda \not\in \sigma(\Delta) \) or (ii) \( (\lambda \in \sigma(\Delta)) \land (\lambda = \lambda') \). Then pencil \( \vec{B} \) is polynomially \((A,0)\)-bounded.
Proof.
In case (i) $\ker A = \{0\}$ that is, the operator $A$ has no eigenvectors and, by remark 1 the pencil $\tilde{B}$ is $(A,0)$-bounded.
In case (ii) $\lambda \in \sigma(\Delta)$ and $\lambda = \lambda'$ construct the chain of $\tilde{B}$-adjoined vectors of an eigenvector $\varphi_0 = \sum_{\lambda=\lambda_{kmn}} a_{kmn} \varphi_{kmn} \in \ker A \setminus \{0\}$. Since $B_3 = B_1 = 0$ the first three $\tilde{B}$-adjoined vectors can be taken equal to zero. On the fourth we obtain

$$B_0 \varphi_0 = B_0 (\sum_{\lambda=\lambda_{kmn}} a_{kmn} \varphi_{kmn}) = -\alpha \left( \frac{\pi l}{c} \right)^2 \sum_{\lambda=\lambda_{kmn}} a_{kmn} \varphi_{kmn} \not\in \text{im} A,$$

since $\sum_{\lambda=\lambda_{kmn}} |a_{kmn}| > 0$.

Therefore the eigenvector $\varphi_0$ doesn’t have a $\tilde{B}$-adjoined vector of order four, the length of the chains of $\tilde{B}$-adjoined vectors of operator $A$ is bounded by three, and the chain of length three exists.

Construct projectors. In case (i) $P = I$ and $Q = I$. In case (ii)

$$P = I - \sum_{\lambda=\lambda_{kmn}} <\varphi_{kmn}, \cdot > \varphi_{kmn},$$

and the projector $Q$ has the same form but it is defined on space $\mathfrak{F}$. Construct the set

$$\mathfrak{M} = \{ u \in \mathfrak{U} : \sum_{\lambda=\lambda_{kmn}} <\alpha \left( \frac{\pi l}{c} \right)^2 u + \Delta(u^2), \varphi_{kmn} > \varphi_{kmn} = 0 \}.$$

By theorem 3.1 we have

**Theorem 4.1** (i) Let $\lambda \not\in \sigma(\Delta)$, $(u_0, u_1, u_2, u_3) \in \mathfrak{U}^4$. Then for some $\tau = \tau(u_0, u_1, u_2, u_3) > 0$ there exists a unique solution $u \in C^\alpha((-\tau, \tau), \mathfrak{U})$ of problem (1)–(3).

(ii) Let $(\lambda \in \sigma(\Delta)) \land (\lambda = \lambda')$, $(u_0, u_1, u_2, u_3) \in T^3 \mathfrak{M}$ the condition (10) be fulfilled. Then for some $\tau = \tau(u_0, u_1, u_2, u_3) > 0$ there exists a unique solution $u \in C^\alpha((-\tau, \tau), \mathfrak{M})$ of problem (1)–(3).

By theorem 3.2 we have

**Theorem 4.2** Let $(\lambda \in \sigma(\Delta)) \land (\lambda = \lambda')$, $(u_0, u_1, u_2, u_3) \in T^3 \mathfrak{M}$ and condition (10) be fulfilled. Then for some $\tau = \tau(u_0, u_1, u_2, u_3) > 0$ there exists a unique solution $u \in C^\alpha((-\tau, \tau), \mathfrak{M})$ of problem (1), (2), (4).

5. Algorithm of the numerical solution
Now consider the algorithm of numerical solution to mathematical model (1)–(3). The algorithm for numerical solution of (1)–(4) will differ in the absence of verification of belonging of the initial data to phase space.

We look for a numerical solution using the Galerkin method. Let us choose the eigenfunctions of the problem (2) for the operator $\Delta$ as $\{\varphi_k(x)\}$ without loss of generality.

Generate approximate solution $u^s$ in form of Galerkin sum

$$u^s = \sum_{k,m,l=1}^{s} a_{k,m,l}^s(t) \varphi_{k,m,l},$$
Substitute $u^s$ into equation (1) and multiply scalarly by basis functions $\{\varphi_{k,m,l}\}_{k,m,l=1}^s$, we get

$$(\lambda k,m,l - \lambda)u^s_{ttt}, \varphi_{k,m,l} + ((\lambda k,m,l - \lambda')u^s_{tt}, \varphi_{k,m,l}) + \alpha((\frac{\partial^2 u^s}{\partial x^2}), \varphi_i) = ((\Delta((u^s)^3)), \varphi_{k,m,l}). \quad (12)$$

The system of equations (12), generally speaking, can be algebraic-differential. For the Cauchy problem (3) to be solvable, the initial functions $u_0$, $u_1$, $u_2$ and $u_3$ must belong to the phase space of the equation, while problem (4) can be solvable for any initial functions.

Using the expansions of the initial functions in a series in basis functions, we obtain the initial conditions for the system of algebraic-differential equations (12)

$$a^s_{k,m,l}(0) = \beta^s_{k,m,l}, \quad a^s_{ttk,m,l}(0) = \gamma^s_{k,m,l}, \quad 1 \leq k, m, l \leq s, \quad (13)$$

where

$$u^s_0 = \sum_{k,m,l=1}^s \beta^s_{k,m,l} \varphi_{k,m,l} \rightarrow u_0, \quad u^s_1 = \sum_{k,m,l=1}^s \gamma^s_{k,m,l} \varphi_{k,m,l} \rightarrow u_1,$$

$$u^s_2 = \sum_{k,m,l=1}^s \theta^s_{k,m,l} \varphi_{k,m,l} \rightarrow u_2, \quad u^s_3 = \sum_{k,m,l=1}^s \theta^s_{k,m,l} \varphi_{k,m,l} \rightarrow u_3 \quad \text{for} \quad m \rightarrow \infty.$$

If the differential operator $A$ has a nontrivial kernel, then system (12) will be algebraic-differential. A necessary condition for the solvability of such a system is that the initial values (3) belong to the phase space, or, which is the same, the initial data (13) corresponding to algebraic equations satisfy them. Check the condition of Lemma 3.1.

If $\lambda$ doesn’t belong to the spectrum of the Laplace operator then system (12) with initial data (13) step is solved by Runge – Kutta method.

If $\lambda$ belongs to the spectrum of the Laplace operator and $\lambda = \lambda_k$ then system (12) is algebraic-differential. Solve the algebraic equations with respect to $a^s_{k,m,l}$. Substitute this solution into differential equations. Exclude the initial data for $a^s_{k,m,l}$ from the set of initial conditions. By virtue of the classical Cauchy theorem, the system of differential equations has a unique solution on some interval $(0, t_s)$. ODE system with the initial data is solved by Runge – Kutta method.

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