FOURIER COEFFICIENTS OF SIEGEL CUSP FORMS OF Degree 2
IN THE ATKIN–LEHNER TYPE NEWSPACE

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ABSTRACT. We prove that a non–zero \( F \) in the Atkin–Lehner type Siegel newspace of
degree 2 and an odd level \( N \) is determined by fundamental Fourier coefficients up to a
divisor of \( N \). In particular when the level is odd and square–free we show that \( F \) is
determined by its fundamental Fourier coefficients.

1. INTRODUCTION

The theory of modular forms has a central place in number theory for various reasons. One of these is that the Fourier coefficients of a modular form contain very crucial arith-
etic information. As an example, let us mention that the odd and square–free Fourier
coefficients of a half–integral weight newform are related to the twisted central L–values
of an integral weight Hecke eigenform through Waldspurger’s formula. The study of these
central values, in particular their vanishing or non–vanishing properties and growth, oc-
cupies the forefront of analytic number theory. Quite recently, it has transpired that the
non–vanishing of odd, square–free Fourier coefficients of integral weight and half–integral
weight modular forms imply similar properties for the Fourier coefficients of Hermitian
modular forms (see [1]), Siegel modular forms (see [3], [8], [13]) of higher degrees. As will
be briefly discussed below, these results have profound implications to the theory of auto-
morphic representations. For related application to (the restrictions of) Hilbert Eisenstein
series one can look at [6]. This makes the study of non–vanishing of Fourier coefficients a
highly fruitful and interesting problem.

As we mentioned above, the question of non–vanishing of Fourier coefficients can be asked
for modular forms on the classical tube domains and has important implications. This
includes the holomorphic Siegel, and Hermitian modular forms. We begin by mentioning
a result of Zagier (see [14]), which says that only the primitive Fourier coefficients (see
section 2.1 for definition) are enough to determine a Siegel cusp form of degree 2. This was
later generalized to higher degrees and level by Yamana (see [13]) over the classical tube
domains. In [4], Ibukiyama and Katsurada introduced an Atkin–Lehner type subspace of
newforms (with respect to the Fourier coefficients, see section 2.3), and using the result of
Yamana, proved that the forms in this subspace are determined by their primitive Fourier
coefficients.

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Improving upon the result of Zagier, Saha in [8] proved that fundamental Fourier coefficients (see section 2.1) determine a Siegel cusp of degree 2 and full level. In [9], for even weights \( k > 2 \), Saha and Schmidt generalized this result to include square–free levels with an additional assumption of forms being eigenfunctions for certain Hecke operators. Boecherer and Das in [3] generalized the result in [8] to vector–valued Siegel modular forms of any degree \( \geq 2 \) with full level. These results are tied closely to the corresponding questions on elliptic modular forms of integral and half–integral weight discussed in the first paragraph. Moreover, the above results are crucial in applications such as transfer from \( \text{GSp}(2) \) to \( \text{GL}(4) \) (see [8] for more detail), obtaining the functional equation of spinor \( L \)–function attached to a degree 3 Siegel cuspidal eigenform (see [3]) etc.

Apart from the subset of fundamental Fourier coefficients one can also look for other arithmetically interesting subset of Fourier coefficients that determines a form. For example, in [7], Martin proved that a Siegel cusp form of degree 2 with square–free level and primitive nebentypus is determined by certain diagonal Fourier coefficients. However, one should note that this subset is disjoint from the subset of fundamental Fourier coefficients. Motivated by these results, in this article, we take up the problem of determining a Siegel cusp form in the Atkin–Lehner type subspace of newforms mentioned above.

Now we state our main result. For positive integers \( k \) and \( N \), let \( S_k^2(N) \) denote the space of Siegel cusp forms of degree 2, weight \( k \) and level \( N \). \( S_k^{2,\text{new}}(N) \subseteq S_k^2(N) \) be the subspace of Atkin–Lehner type newforms (see section 2.3). The Fourier coefficients \( A(F,T) \) of any \( F \in S_k^{2,\text{new}}(N) \) are supported on \( \Lambda_2^+ \), the set of \( 2 \times 2 \) half–integral, symmetric, positive definite matrices.

Let \( N = p_1^{\alpha_1}p_2^{\alpha_2}...p_t^{\alpha_t} \) be an odd integer. For any \( X \geq 1 \), let \( S_F(X) \) denote the set of odd and square–free integers \( n \leq X \) such that \( (n,N) = 1, \prod_{j=1}^t p_j^{r_j}n = 4\det(T) \) for some \( T \in \Lambda_2^+ \) and \( A(F,T) \neq 0 \).

**Theorem 1.** Let \( k > 2 \) and \( F \in S_k^{2,\text{new}}(N) \) is non–zero. Then there exist infinitely many \( \text{GL}(2,\mathbb{Z})–\text{inequivalent} \ T \in \Lambda_2^+ \) such that

1. \( A(F,T) \neq 0 \).
2. \( 4\det(T) \) is of the form \( \prod_{j=1}^t p_j^{r_j}n \) where \( n \) is odd and square–free, \( (n,N) = 1 \) and \( 0 \leq r_j \leq \alpha_j \).

Moreover, for any \( \epsilon > 0 \), \( S_F(X) \gg_{F,\epsilon} X^{5/8–\epsilon} \).

In particular, when \( N \) is an odd, square–free integer we see that \( F \) is in fact determined by its fundamental Fourier coefficients. As mentioned above, it was proved by Saha and Schmidt in [9] that Siegel cusp forms of even weight and square–free level \( N \) that are eigenfunctions of the operators \( U(p) \) for all \( p|N \) are determined by their fundamental Fourier coefficients. In this case, the fundamental \( T \in \Lambda_2^+ \) that determine the form are away from the level \( N \), i.e., \( (4\det(T),N) = 1 \). But in our case, as seen from Theorem 1, \( 4\det(T) \) could contain primes that divide \( N \). As far as we could see, none of these results imply the other.
The proof of Theorem 1 uses the quite useful technique of passing to the case of half–integral weight cusp forms by using the Jacobi forms as an intermediate bridge. The passage to the case of Jacobi forms is aided by the Fourier–Jacobi expansion. To get to the desired half–integral case cusp form, instead of using the Eichler–Zagier map, whose theory seems not adequately developed to handle our purpose, we directly use the theta components of the Jacobi forms as in [3]. Although this passage to half–integral weight forms is inspired by the calculations in [3], we are in a somewhat more technically challenging situation due to the existence of level and thus requires a fresh set of calculations. The use of theta components instead of the Eichler–Zagier map (as in [9]) lets us prove the result for all weights \( k > 2, \ k \text{ odd} \) included. We prove the following result on the non–vanishing of theta components of Jacobi forms with level (see section 2.4 for definition) to facilitate this passage.

**Theorem 2.** Let \( N \) be an odd integer and \( m_1, m_2 \) be square–free integers such that \( m_1 | N \) and \( (N, m_2) = 1 \). For any non–zero \( \phi \in J_{k,m_1m_2}(N) \) there exists a \( \mu \mod 2m_1m_2 \) with either \( (\mu, 2m_1m_2) = 1 \) or \( (\mu, 2m_1m_2) \nmid 2m_2 \) such that the theta component \( h_\mu \neq 0 \).

In fact we will make use of the following corollary of Theorem 2 when \( m_1 = 1 \).

**Corollary 1.1.** Let \( N \) be an odd integer and \( m \) be a square–free integer with \( (N, m) = 1 \). For any non–zero \( \phi \in J_{k,m}(N) \) there exists a \( \mu \mod 2m \) with \( (\mu, 2m) = 1 \) such that the theta component \( h_\mu \neq 0 \).

The non–vanishing of theta components of a Jacobi form is in itself an interesting problem. We mention a few results in this direction. In [11], non–vanishing of certain theta components was proved for Jacobi forms of full level and index \( p, \ p^2 \) or \( pq \), where \( p \) and \( q \) are primes. In [3], non–vanishing of primitive theta components (see section 2.4 for definition) was proved for Jacobi forms of full level and square–free index. Theorem 2 is a generalization of this result for Jacobi forms with level in degree 1. It will be interesting to investigate this for higher degree Jacobi forms.

For Jacobi forms of square–free level \( N \) and index \( p \), a prime with \( (p, 2N) = 1 \), it was proved by Martin in [7] that all the theta components are non–zero. Note here that in this case of prime index and square–free level the result in [7] is stronger than Theorem 2. But Martin’s result is not enough in our case since we deal with non square–free levels also. Moreover, Theorem 2 could be of use when proving similar results for Siegel modular forms of higher degree with level (including the vector valued case).

By virtue of Corollary 1.1, the problem reduces to half–integral weight cusp forms on the congruence subgroup \( \Gamma_1(4L) \) for some \( L \geq 1 \). We then make use of the following result to finish the proof of Theorem 1.

**Theorem 3.** Let \( \kappa \geq 5/2 \) be a half–integer and \( L = \prod_{i=1}^{t} p_i^a_i \). Suppose \( f \in S_\kappa(\Gamma_1(4L)) \) is non–zero and \( a(f,n) = 0 \) for all \( (n, L_f) > 1 \), for an even divisor \( L_f \) of \( 4L \). Then there exist infinitely many odd and square–free integers \( n \) such that

1. For \( p_i | L \), \( (n, p_i) = 1 \).
Moreover, for any $\epsilon > 0$

$$\#\{n \leq X : n \text{ square–free }, (n, 4L) = 1, a(f, p_i^1 ... p_i^r n) \neq 0\} \gg_{f, \epsilon} X^{5/8-\epsilon}.$$ 

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2. **Notations and preliminaries**

For any $z \in \mathbb{C}$ and $m \in \mathbb{Z}$, we write $e(z) := e^{2\pi iz}$, $e^m(z) := e^{2\pi imz}$ and $e_m(z) := e^{2\pi iz/m}$.

2.1. **Siegel modular forms.** The Siegel’s upper half space of degree 2 is given by

(2.1) \[ \mathbb{H}_2 = \{ Z \in M(2, \mathbb{C}) | Z = Z^\dagger, \text{Im}(Z) > 0 \} \]

The symplectic group $\text{Sp}(2, \mathbb{R})$ acts on $\mathbb{H}_2$ by $Z \mapsto \gamma(Z) = (AZ + B)(CZ + D)^{-1}$. For any integer $k$, the stroke operator on functions $F : \mathbb{H}_2 \to \mathbb{C}$ is defined as

(2.2) \[ (F|_k \gamma)(Z) := \det(CZ + D)^{-k} F(\gamma(Z)). \]

For any positive integer $N$, the congruence subgroup $\Gamma_0^2(N) \subset \text{Sp}(2, \mathbb{Z})$ is given by

(2.3) \[ \Gamma_0^2(N) = \{ (A \ B \ C \ D) \in \text{Sp}(2, \mathbb{Z}) | C \equiv 0 \mod N \} \]

A Siegel modular form of degree 2, weight $k$ and level $N$ is a holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}$ satisfying

(2.4) \[ F|_k \gamma = F \quad \text{for all } \gamma \in \Gamma_0^2(N). \]

Any such form has a Fourier expansion of the form

(2.5) \[ F(Z) = \sum_{T \in \Lambda_2} A(F, T)e(\text{Tr}(TZ)), \]

where $\Lambda_2$ is the set of $2 \times 2$ half–integral, symmetric, positive semi–definite matrices. Denote the subset of positive definite matrices by $\Lambda_2^+$. If $A(F, T)$ non–zero for only $T \in \Lambda_2^+$ we say that $F$ is a cusp form. The space of Siegel modular forms of degree 2, weight $k$ and level $N$ is denoted by $M_k^2(N)$ and the subspace of cusp forms is denoted by $S_k^2(N)$.

For a $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in \Lambda_2$, define the content $c(T)$ as below:

(2.6) \[ c(T) = \max\{ d \in \mathbb{N} | d^{-1}T \text{ is half–integral}\}. \]

When $c(T) = 1$, we say that $T$ is primitive. If $-D = b^2 - 4ac (< 0)$ is a fundamental discriminant, then we say that $T$ is fundamental and the corresponding Fourier coefficient $A(F, T)$ is called a fundamental Fourier coefficient. The same analogy applies to primitive Fourier coefficients.
2.2. Fourier–Jacobi expansion. For any \( Z \in \mathbb{H}_2 \) consider the following decomposition
\[
Z = (\tau \ w \ z);
\]
with \( w \in \mathbb{C}, \tau, z \in \mathbb{H} \) (the complex upper half plane).

Any \( F \in M^2_k(N) \) has a Fourier–Jacobi expansion with respect to the above decomposition as below.
\[
F(Z) = \sum_{m \geq 0} \phi_m(\tau, w) e(mz).
\]
Here \( \phi_m \) are the Jacobi forms of weight \( k \), index \( m \) and level \( N \) defined in section 2.4. If \( F \) is a cusp form then \( \phi_m \) are also cusp forms.

2.3. A newspace of Siegel cusp forms. In this article we deal with the Atkin-Lehner type space of newforms introduced in [4]. We give a brief description of the space below.

Let \( N \) be any positive integer and \( d(>1) \), \( M \) be positive integers such that \( dM|N \). Consider any \( F \in S^2_k(M) \), then \( F(dZ) \in S^2_k(N) \). Denote by \( S^2_k\text{-}old(N) \) the subspace of \( S^2_k(N) \) spanned by
\[
\{ F(dZ) | F \in S^2_k(M), dM|N \}.
\]

The orthogonal complement of \( S^2_k\text{-}old(N) \) with respect to the Petersson inner product is denoted by \( S^2_k\text{-}new(N) \) and this is the subspace of \( S^2_k(N) \) that we refer to as the Atkin–Lehner type newspace in this article.

2.4. Jacobi forms with level. In this article we encounter Jacobi forms with level. For more details about Jacobi forms for congruence subgroups, see [5].

A holomorphic function \( \phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C} \) is a Jacobi form of weight \( k \), index \( m \) and level \( N \) if:

1. \( \phi \left( \frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d} \right) (c\tau + d)^{-k} e^m \left( \frac{cz^2}{cd} \right) = \phi(\tau, z) \) for all \( \gamma = (a \ b \ c \ d) \in \Gamma_0(N) \).
2. \( \phi(\tau, z + \lambda \tau + \mu) e^m(\lambda^2 \tau + 2\lambda z) = \phi(\tau, z) \) for all \( (\lambda, \mu) \in \mathbb{Z}^2 \).
3. \( \phi \) satisfies the boundedness condition at the cusps.

We denote by \( J_{k,m}(N) \) the space of Jacobi forms of weight \( k \), index \( m \) and level \( N \). As a consequence of condition (3) above, any \( \phi \in J_{k,m}(N) \) has a Fourier expansion of the form (at the cusp \( i\infty \))
\[
\phi(\tau, w) = \sum_{4nm - r^2 \geq 0} c(n, r) e(n\tau) e(rw).
\]
A similar Fourier expansion holds at other cusps too. If the Fourier coefficients corresponding to \( 4nm - r^2 = 0 \) vanish at all the cusps, then we say that \( \phi \) is a Jacobi cusp form. We denote the space of Jacobi cusp forms by \( J^\text{cusp}_{k,m}(N) \).

For a \( \phi \in J_{k,m}(N) \) that appears in the Fourier–Jacobi expansion of a Siegel modular form \( F \) as in (2.8), the Fourier coefficients of \( \phi \) can be written in terms of Fourier coefficients of
$F$ as follows:

\[(2.11) \quad c_\phi(n, r) = A \left(F, \left( n \frac{r/2}{m} \right) \right).\]

Any $\phi \in J_{k,m}(N)$ admits a theta expansion:

\[(2.12) \quad \phi(\tau, w) = \sum_{\mu \mod 2m} h_\mu(\tau) \cdot \Theta_{\mu,m}(\tau, w),\]

where $\Theta_{\mu,m}$ is the theta series given by

\[(2.13) \quad \Theta_{\mu,m}(\tau, w) = \sum_{n \in \mathbb{Z}} e\left(\frac{(2mn - \mu)^2}{4m} - (2mn - \mu)w\right)\]

and $h_\mu$ are modular forms of weight $k - 1/2$ for the congruence subgroup $\Gamma(4mN)$ (see [3] for a proof) and have a Fourier expansion as below.

\[(2.14) \quad h_\mu(\tau) = \sum_{n \geq 0, 4mn - \mu^2 \geq 0} c(n, \mu) e\left(\frac{4mn - \mu^2}{4m} \tau\right).\]

As in [3], we refer to the theta components corresponding to $(\mu, 2mN) = 1$ as primitive theta components.

For $0 \leq \mu < 2m$, $\Theta_{\mu,m}$ and $h_\mu$ satisfy the following transformation properties (see [5]). Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \text{SL}(2, \mathbb{Z})$, then

\[(2.15) \quad \Theta_{\mu,m}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right)(c\tau + d)^{-1/2} e\left(\frac{-mcz^2}{c\tau + d}\right) = \sum_{\nu \mod 2m} \varepsilon_m(\nu, \mu; \gamma) \Theta_{\nu,m}(\tau, z).\]

Here we take the branch of square root having argument in $(-\pi/2, \pi/2]$.

Now from (2.15) and the transformation properties of $\phi$ under $\Gamma_0(N)$ (see conditions (1) and (2) above),

\[(2.16) \quad (-N\tau + 1)^{-k+1/2} h_\mu\left(-\frac{\tau}{N\tau + 1}\right) = \sum_{\nu \mod 2m} \varepsilon_m(\nu, \mu; \gamma) h_\nu(\tau),\]

where $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)$ and $\varepsilon_m(\nu, \mu; \gamma)$ is the generalized quadratic Gauss sum and is given by

\[(2.17) \quad \varepsilon_m(\nu, \mu; \gamma) = \frac{1}{2m} \sum_{\eta \mod 2m} \varepsilon_{4m}(N\eta^2 + 2\eta(\nu - \mu)).\]

For the sake of simplicity we write $\varepsilon_m(\nu, \mu; \gamma) = \varepsilon_m(\nu, \mu)$. 

2.5. Generalized quadratic Gauss sums. The generalized quadratic Gauss sum is defined as

\[ G(a, b, c) := \sum_{l \mod c} e_c(al^2 + bl), \quad \text{where } a, b, c \in \mathbb{Z}. \]

\( G(a, b, c) \) has the following properties (see [2] for more details):

\[ G(a, b, c_1c_2) = G(c_2a, b, c_1)G(c_1a, b, c_2), \quad \text{when } (c_1, c_2) = 1. \]

Moreover, \( G(a, b, c) = 0 \) if \( (a, c) > 1 \) and \( (a, c) \) does not divide \( b \) and when \( (a, c)|b \), we have

\[ G(a, b, c) = (a, c)G\left(\frac{a}{(a, c)}, \frac{b}{(a, c)}, \frac{c}{(a, c)}\right) \]

and for \( (a, c) = 1 \)

\[ G(a, b, c) = \begin{cases} 
\varepsilon_c\sqrt{c}(-\psi(a)b^2) & \text{if } c \equiv 1 \mod 2, 4a\psi(a) \equiv 1 \mod c; \\
2G(2a, b, c/2) & \text{if } c \equiv 2 \mod 4, b \equiv 1 \mod 2; \\
0 & \text{if } c \equiv 0 \mod 4, b = 0; \\
(1 + i)\epsilon_a^{-1}\sqrt{c} & \text{if } c \equiv 0 \mod 4, b \equiv 1 \mod 2.
\end{cases} \]

Here \( \epsilon_m = 1 \) or \( i \) depending on \( m \equiv 1 \mod 4 \) or \( 3 \mod 4 \) respectively.

3. Proof of main result

Assuming the corresponding results for Jacobi forms (Theorem 2) and half integral weight forms (Theorem 3) we give the proof of Theorem 1. First we need the following result due to Ibukiyama and Katsurada (see [4]).

\textbf{Theorem 3.1.} Let \( F \in S_{2k}^\text{new}(N, \chi) \) be non-zero. Then there exists a primitive \( T \) such that \( A(F, T) \neq 0 \).

For any \( M \in \text{GL}(2, \mathbb{Z}) \), we have that \( A(F, T) = \det(M)^k A(F, M^t TM) \). Thus if \( A(F, T) \neq 0 \), we can say that \( A(F, M^t TM) \neq 0 \). Now coming to the case at hand, the quadratic form associated to \( T \) represents infinitely many primes (see [12]). Since \( T \) is primitive, we can choose \( M \) such that the matrix \( T_0 = M^t TM \) has right lower entry to be an odd prime, say \( p \) (see [8, Lemma 2.1] for exact arguments). Since the quadratic form defined by \( T \) represents infinitely many odd primes, we can choose \( p \) such that \( (p, N) = 1 \). Let \( T_0 = \left( \begin{array}{cc} n_0 & \mu_0/2 \\ \mu_0/2 & p \end{array} \right) \)
and we have $A(F, T_0) \neq 0$. Thus we get a non-zero Fourier–Jacobi coefficient $\phi_p \in J_{k,p}(N)$ of $F$. Let

$$
(3.1) \quad \phi_p(\tau, z) = \sum_{n, r: 4pn > r^2} c(n, r)e(n\tau)e(rz),
$$

where $c(n, r) = A \left( F, \left( \frac{n \cdot r/2}{p} \right) \right)$. Since $A(F, T_0) \neq 0$, we see that $c(n_0, \mu_0) \neq 0$.

**Proposition 3.1.** Let $F \in S^2_{k, new}(N)$ be non-zero. Then there exists an odd prime $p$ with $(p, N) = 1$ such that the Fourier–Jacobi coefficient $\phi_p \in J_{k,p}(N)$ of $F$ is non-zero.

Write $N = p_1^{\alpha_1}p_2^{\alpha_2}...p_t^{\alpha_t}$. Then from Proposition 3.1 and Corollary 1.1 we get a $\mu$ mod $2p$ with $(\mu, 2p) = 1$ such that $h_\mu \neq 0$. Let $f(\tau) = h_\mu(4p\tau)$, then $f \in S_{k-\frac{1}{2}}(\Gamma_1((4p)^2N))$. From the Fourier expansion of $h_\mu$ in (4.1), we see that

$$
(3.2) \quad f(\tau) = \sum_{n \geq 1} a(f, n)q^n,
$$

where $a(f, n) = c \left( (n + \mu^2)/4p, \mu \right)$.

Now using the Theorem 3 for level $4L = 16p^2N$, we get primes $p_i|N$ and $0 \leq r_i \leq \alpha_i$ and infinitely many odd and square-free integers $n$ with $(n, 2pN) = 1$ such that $a(f, p_1^{r_1}p_2^{r_2}...p_t^{r_t}n) \neq 0$. Thus for infinitely many odd and square-free integers $n$ co-prime to $N$ we see that

$$
\begin{align*}
 a(f, p_1^{r_1}p_2^{r_2}...p_t^{r_t}n) &= c \left( \frac{p_1^{r_1}p_2^{r_2}...p_t^{r_t}n + \mu^2}{4p}, \mu \right) = A \left( F, \left( \frac{p_1^{r_1}p_2^{r_2}...p_t^{r_t}n + \mu^2}{4p}, \mu/2 \right) \right) \\
 &\neq 0.
\end{align*}
$$

If we denote by $T = \left( \frac{p_1^{r_1}p_2^{r_2}...p_t^{r_t}n + \mu/2}{4p}, \mu/2 \right)$, then $4 \det(T) = p_1^{r_1}p_2^{r_2}...p_t^{r_t}n$. Thus we get that $A(F, T) \neq 0$ for infinitely many $T$ such that $4 \det(T)$ is of the form $p_1^{r_1}p_2^{r_2}...p_t^{r_t}n$, where $n$ is odd and square-free and co-prime to $N$.

The quantitative result $S_F(X) \gg F^{5/8-\epsilon}X^{5/8-\epsilon}$ follows from the corresponding quantitative result for half-integral weight cusp forms. \hfill $\Box$

**Remark 3.1.** The condition $k > 2$ on the weight comes from the corresponding condition on the half-integral weight modular forms. For a discussion on this see [3, Remark 4.7].

**Remark 3.2.** Theorem 1 is not true in $S^2_{k, odd}(N)$. As an example, let $N$ be square-free and let $d$ be a proper divisor of $N$. Then the Fourier coefficients of $F(dZ)$, where $F \in S^2_{k, new}(N/d)$, are supported on $T \in \Lambda^2_*$ for which $4 \det(T)$ is of the form $d^2n$.

4. **Theta components of Jacobi forms with level**

Let $N$ be an odd integer and $m_1$, $m_2$ be square-free integers such that $m_1|N$ and $(N, m_2) = 1$. For any $\phi \in J_{k, m_1m_2}(N)$, let $h_\mu$ denote the theta components of $\phi$ (see
First note that the RHS of (4.3) is zero whenever $m_1 | (\nu - \mu)$ (see section 2.5). On the other hand we also have that $(\mu, 2m_1 m_2) = 1$ and $(\nu, 2m_1 m_2) > 1$. Now if $(\nu, m_1) > 1$, since $(\mu, m_1) = 1$, we have $m_1 | (\nu - \mu)$. Thus $\varepsilon_{m_1 m_2}(\nu, \mu) = 0$ in this case. □
Now coming back to (4.2), using the transformation $\tau \mapsto \tau + t$ we get

\[ \sum_{\nu \text{ mod } 2m_1m_2} \varepsilon_{m_1m_2}(\nu, \mu) e_{4m_1m_2}(-\nu^2 t) h_\nu(\tau) = 0. \]

Using the same argument as in [3] we see that it is enough to consider $\nu_0$ with $(\nu_0, 2m_1m_2) > 1$. Thus from (2.17) and Lemma 4.1 we get the following:

\[ \sum_{\nu \text{ mod } 2m_1m_2} e_{4m_2} \left( -\frac{M(\nu - \mu)^2}{m_1^2} \right) h_\nu(\tau) = 0. \]

For $(\nu_0, m_1) = 1$ and any $l$ with $(l, 2m_2) = 1$, consider the matrix

\[ M_{\nu_0} = (\varepsilon_{m_1m_2}(\nu, \mu))_{\nu, \mu}, \]

where $\varepsilon_{m_1m_2}(\nu, \mu) = e_{4m_2} \left( \frac{l(\nu - \mu)^2}{m_1^2} \right)$ when $m_1 | \nu - \mu$ and is zero otherwise, $(\mu, 2m_1m_2) = 1$ and $\nu$ runs over the set

\[ S_{\nu_0} = \{ \nu \text{ mod } 2m_1m_2 : \nu^2 \equiv \nu_0^2 \text{ mod } 4m_1m_2 \}. \]

We prove that the only possible solution to (4.7) is the trivial solution (i.e., $h_\nu = 0$) by showing that the matrix $M_{\nu_0}$ has maximal rank. When $N = 1$, this reduces to the arguments presented in [3].

**Lemma 4.2.** Let $(\nu_0, m_1) = 1$. Then the matrix $M_{\nu_0}$ has maximal rank.

**Proof.** Let $m_1 = p_1p_2...p_r$ and $m_2 = q_1q_2...q_s$. Write $t = r + s$. The set

\[ \{ \nu \text{ mod } 2m_1m_2 : \nu^2 \equiv \nu_0^2 \text{ mod } 4m_1m_2 \} \]

has the cardinality $2^{t'}$, where $t'$ is the number of primes dividing $m_1m_2$ but not $\nu_0$. But the additional condition $(\nu_0, m_1) = 1$ gives us that the cardinality of the above set is at least $2^r$. We prove by induction on $t$.

When $t = 0$, $|S_{\nu_0}| = 1$ and the lemma is trivially true. Consider the case $t = 1$. Then either $r = 1$ and $s = 0$ or $r = 0$ and $s = 1$.

Case 1: $r = 1$ and $s = 0$. In this case $|S_{\nu_0}| = 2$ with $\nu_0$ and $-\nu_0$ being the solutions. Choose $\mu_1 = \nu_0 + 2p_1$ ($\nu_0 + p_1$ if $\nu_0$ is even) and $\mu_2 = -\nu_0 + 2p_1$ ($-\nu_0 + p_1$ if $\nu_0$ is even). Then clearly $(\mu_i, 2p_1) = 1$ for $i = 1, 2$. Clearly the diagonal matrix

\[ \begin{pmatrix} e_4 \left( \frac{(\nu_0 - \mu_1)^2}{p_1^2} \right) & 0 \\ 0 & e_4 \left( \frac{(\nu_0 - \mu_2)^2}{p_1^2} \right) \end{pmatrix} \]

has non-zero determinant. Thus $M_{\nu_0}$ has rank 2.
Case 2: \( r = 0 \) and \( s = 1 \). If \( q_1 | v_0 \), then \( |S_{v_0}| = 1 \) and the matrix \( M_{v_0} \) is a non-zero column and thus the lemma follows. If \( (v_0, q_1) = 1 \), then \( |S_{v_0}| = 2 \). Determinant of the matrix

\[
(4.12) \begin{pmatrix} e_{4q_1}(l(\nu - 1)^2) & e_{4q_1}(l(\nu + 1)^2) \\ e_{4q_1}(l(\nu - \mu)^2) & e_{4q_1}(l(\nu + \mu)^2) \end{pmatrix}
\]

is given by \( e_{4q_1}(l(2\nu^2 + \mu^2 + 2\nu(\mu - 1))) (1 - e_{4q_1}(l\nu(\mu - 1))) \). Choosing \( \mu \) mod 2\( q_1 \) with \( (\mu, 2q_1) = 1 \) and different from 1, we see that this determinant is non-zero. Thus the matrix \( M_{v_0} \) has rank 2.

Now coming to the induction step \( t \implies t + 1 \), first we assume that \( r > 0 \) and write \( m_1 = m_3 p \) and decompose \( v_0 \mod 2m_1m_2 \) as \( v_0 = 2m_3m_2v_0' + pv_0'' \) with \( v_0' \mod p \) and \( v_0'' \mod 2m_3m_2 \). Decompose \( v' \) and \( \mu \) similarly. Note that

1. \( m_1|\nu - \mu \iff m_3|\nu'' - \mu'' \) and \( p|\nu' - \mu' \).
2. \( \nu'' \equiv v_0'' \mod 4m_1m_2 \) if \( \nu' \equiv v_0' \mod p \) and \( \nu'' \equiv v_0'' \mod 4m_1m_2 \).

Moreover, we also have

\[
(4.13) \quad \frac{(\nu - \mu)^2}{4m_1^2m_2} = \frac{m_2(\nu' - \mu')^2}{p^2} + \frac{(\nu'' - \mu'')^2}{4m_3^2m_2} \mod Z.
\]

Let

\[
(4.14) \quad A := (\varepsilon'(\nu', \mu')), v', \mu' \mod p \quad \text{and} \quad B := (\varepsilon''(\nu'', \mu''))v'', \mu'' \mod 2m_3m_2,
\]

where \( \varepsilon'(\nu', \mu') = e^{\frac{t_\nu l_\nu (\nu' - \mu')^2}{p^2}} = 1 \) if \( p|\nu' - \mu' \) and is zero otherwise and \( \varepsilon''(\nu'', \mu'') = e_{4m_2}^{\frac{t_\nu l_\nu (\nu'' - \mu'')^2}{m_3^2}} \) if \( m_3|\nu'' - \mu'' \) and zero otherwise. Also, \( \nu' \) runs over the set \( \{\nu' \mod p : \nu'' \equiv v_0'' \mod p\} \) and \( \nu'' \) runs over the set \( \{\nu'' \mod 2m_3m_2 : \nu'' \equiv v_0'' \mod 4m_3m_2\} \).

From the induction hypothesis, \( B \) has maximal rank. To see that \( A \) also has maximal rank, first note that \( (v_0', p) = 1 \) since \( (v_0, m_1) = 1 \). Thus \( \nu'' \equiv v_0'' \mod p \) has two solutions (i.e., the rank of \( A \) is at most 2). Denote them by \( \nu'_1 = \nu_0' \) and \( \nu'_2 = -\nu_0' \). Let \( \mu'_1 = \nu'_1 + p \) and \( \mu'_2 = \nu'_2 + p \). Clearly \( (\mu'_i, p) = 1 \) and \( p|\mu'_i - \mu'_j \iff i = j \). Thus \( A \) has a sub-matrix of the form \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). This shows that \( A \) is indeed of rank 2. The lemma is true in this case by noting that \( M_{v_0} = A \otimes B \).

Now we turn to the case when \( r = 0 \). Here \( s = t + 1 \) and we write \( m_2 = m_4 q \), where \( m_4 \) has \( t = s - 1 \) prime factors. Note that in this case \( m_1 = 1 \). As in the previous case decompose \( v_0 \mod 2m_2 \) as \( v_0 = 2m_4v_0' + qv_0'' \) with \( v_0' \mod q \) and \( v_0'' \mod 2m_4 \). Decompose \( v \) and \( \mu \) similarly. Note that \( \nu'' \equiv v_0'' \mod 4m_2 \) if \( \nu'' \equiv v_0'' \mod 4m_2 \). We also have

\[
(4.15) \quad \frac{(\nu - \mu)^2}{4m_2} = \frac{m_4(\nu' - \mu')^2}{q} + \frac{q(\nu'' - \mu'')^2}{4m_4} \mod Z.
\]
As before, we write
\[ (4.16) \quad A := (\varepsilon'(\nu', \mu'))_{\nu', \mu' \mod q} \quad \text{and} \quad B := (\varepsilon''(\nu'', \mu''))_{\nu'', \mu'' \mod 2m_4}, \]
where \( \varepsilon'(\nu', \mu') = e_q(lm_4(\nu' - \mu')^2) \) and \( \varepsilon''(\nu'', \mu'') = e_{4m_4}(lq(\nu'' - \mu'')^2) \). As in the previous case, \( \nu' \) runs over the set \( \{ \nu' \mod q : \nu'^2 \equiv \nu'^2_0 \mod q \} \) and \( \nu'' \) runs over the set \( \{ \nu'' \mod 2m_4 : \nu''^2 \equiv \nu''^2 \mod 4m_4 \} \).

Since \( (l, 2m_4) = 1 \), the matrix \( B \) has maximal rank by induction hypothesis. To see that \( A \) has maximal rank, note that \( A \) has determinant \( \nu \mod \mu \)
\[ \text{we fail to get primitive theta components (cf. Remark 4.1).} \]

As before, we write \( \varepsilon'(lq, \nu_0^2) \mod q \) has two solutions. The matrix
\[ (4.17) \quad \left( e_q(lm_4(\nu' - 1)^2) \quad e_q(lm_4(\nu' + 1)^2) \right) \]
has determinant \( e_q(lm_4(2\nu'^2 + \mu^2 + 1 + 2\nu'(\mu' - 1))) (1 - e_q(4lm_4\nu'(\mu' - 1))) \). Choosing \( \mu \mod q \) with \( (\mu, q) = 1 \) and different from 1, we see that the above determinant is non-zero. Thus the matrix \( A \) has rank 2. By noting that \( M_{\lambda_0} = A \otimes B \), the lemma is now complete.

Now define the relation \( \nu \sim \nu' \) iff \( \nu'^2 \equiv \nu'^2 \mod 4m_1m_2 \). This defines an equivalence relation among \( \nu \mod 2m_1m_2 \) with \( (\nu, m_1) = 1 \) and \( (\nu, 2m_1m_2) | 2m_2 \). Thus from the above discussion we see that \( h_\nu = 0 \) for all such indices. This gives us the proof of Theorem 2.

**Remark 4.1.** In this article we use the Corollary of Theorem 2 when \( m_1 = 1 \). Note that we fail to get primitive theta components (cf. [3, Proposition 3.5]). But it ensures that we get a theta component \( h_\mu \neq 0 \) such that \( (\mu, 2p) = 1 \) (\( p \) is a prime such that \( \phi_p \neq 0 \)). This is crucial in getting hold of Fourier coefficients of the resulting half–integral weight form that are not dependent on the prime \( p \).

5. HALF–INTEGRAL WEIGHT FORMS FOR \( \Gamma_1(4L) \)

For any positive integer \( L \) and a positive half–integer \( \kappa \) (i.e., \( \kappa = l/2 \) for some odd \( l > 0 \)) denote by \( S_{\kappa}(\Gamma_1(4L)) \) the space of half–integral weight cusp forms for the congruence subgroup \( \Gamma_1(4L) \).

Let \( L \) be as above and \( L_f \) an even divisor of \( 4L \). Consider a non–zero \( f \in S_{\kappa}(\Gamma_1(4L)) \) such that \( a(f, n) = 0 \) for all \( (n, L_f) > 1 \). Write
\[ (5.1) \quad L = \prod_{i=1}^{t} p_i^{\alpha_i} \quad , \quad L' := \prod_{i=1}^{t} p_i \quad , \quad L'_f = \prod_{p | L_f} p \quad , \quad L' = M_f L'_f. \]

Our aim is to construct a new modular form \( g \) such that \( a(g, n) = 0 \) for all \( (n, L') > 1 \). To proceed further we need the following two results. First one is an analogue of Lemma.
Proof. Put $L_0 = L/p$. Denote by $B_p$ the matrix $\begin{pmatrix} \frac{a}{p} & 0 \\ b & p \end{pmatrix}$. Then
\[ g(\tau) = h(\tau/p) = \sum_{n \geq 1} a(h, np)q^n = p^{k/4}h|_k(B_p, p^{1/4}). \]

We consider the following subgroup of $\Gamma_1(4L_0)$.
(5.2) \[ \Gamma_p = \{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_1(4L_0) : b \equiv 0 \mod p, d \equiv 1 \mod p \}. \]

Let $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_p$ and $\alpha = \left( \begin{array}{cc} a/bp \\ cp \end{array} \right)$. Then $\alpha \in \Gamma_1(4L)$. Using the notations as in [10], write $\gamma^* = (\gamma, \epsilon^{-1}_d \left( \frac{a}{d} \right) (cz + d)^{1/2})$ and $\alpha^* = (\alpha, \epsilon^{-1}_d \left( \frac{b}{d} \right) (pcz + d)^{1/2})$. Then we have
(5.3) \[ (B_p, p^{1/4}) \gamma^* = (1, \chi_p(d)) \alpha^*(B_p, p^{1/4}). \]

But since $d \equiv 1 \mod 4L$, we have $\chi_p(d) = 1$. Thus
(5.4) \[ g|_k \gamma^* = p^{k/4}h|_k(B_p, p^{1/4}) = g. \]

Also $g|T = g$, where $T = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$. Now we claim that $\Gamma_1(4L_0)$ is generated by $\Gamma_p$ and $T$. Thus $g \in S_\kappa(\Gamma_1(4L_0))$. This completes the proof of lemma.

Proof of claim: First consider the case when $p|L_0$. Let $\gamma = \left( \begin{array}{cc} a \\ c \end{array} \right) \in \Gamma_1(4L_0)$. Then we have $\gamma T^n = \left( \begin{array}{cc} a + bn & b + dn \\ c + dn & d + cn \end{array} \right)$. Since $p|L_0$ and $4L_0|c$, we see that $d + cn \equiv 1 \mod p$ for any $n \in \mathbb{Z}$. Since $(p, a) = 1$, we get $x, y \in \mathbb{Z}$ such that $px + ay = 1$. Thus $b + a(-by) \equiv 0 \mod p$, which gives us that $\gamma T^{-by} \in \Gamma_p$. The claim is thus true in the case when $p|L_0$.

Next consider the case when $(p, L_0) = 1$. Let $\gamma = \left( \begin{array}{cc} a \\ c \end{array} \right) \in \Gamma_1(4L_0)$. Here we further split into two cases as below.

Case 1: Here we consider $(p, c) = 1$. We get $x, y \in \mathbb{Z}$ such that $px + cy = 1$. That is $d + cy(1 - d) \equiv 1 \mod p$. In other words, there exists an $n \in \mathbb{Z}$ such that $d \equiv 1 \mod p$ in $\gamma T^n$. Thus WLOG we can assume that $d \equiv 1 \mod p$ in $\gamma$. With this additional assumption consider $T^m \gamma = \left( \begin{array}{cc} a + cm & b + dm \\ c & d \end{array} \right)$. Since $(p, d) = 1$, it is possible to choose an $m \in \mathbb{Z}$ (as in the first case) such that $T^m \gamma \in \Gamma_p$. Thus the claim is true in this case.

Case 2: Here $p|c$. Let $\alpha = \left( \begin{array}{cc} 1 & 0 \\ 4L_0 \end{array} \right) \in \Gamma_p$. Then $\alpha \gamma = \left( \begin{array}{cc} a \\ c + 4L_0an \end{array} \right)$. Since $p|c$, we see that $(p, a) = 1$. Moreover $(p, 4L_0) = 1$. Thus choosing any $n$ such that $(p, n) = 1$ we have $(p, c + 4L_0an) = 1$. Now from Case 1 we get that $T^m \alpha \gamma \in \Gamma_p$ for some $m \in \mathbb{Z}$. This completes the proof of claim.

Second result is the following lemma from [3, Lemma 4.2].

Lemma 5.2. Let $h \in S_\kappa(\Gamma_1(4L))$. Suppose $a(h, n) = 0$ for all $(n, p) = 1$ for an odd prime $p \nmid N$, then $h = 0$. 

We now construct the new cusp form \( g \) using Lemma 5.1 and 5.2 and an inductive argument. These arguments are similar in essence to those used in [8] for the congruence subgroup \( \Gamma_0(4N) \) and in [1] for the integral weight cusp forms.

Let \( M_f, L_f' \) be the co–prime square–free integers as in (5.1). Write \( M_f = p_1 p_2 \ldots p_r \).

(1) Consider
\[
g_0(\tau) = \sum_{(n,L_f')=1} a(f,n)q^n.
\]
Clearly \( g_0 \neq 0 \) and \( g_0 \in S_\kappa(\Gamma_1(4LL_f'p_1^2)) \). We construct \( g_1 \) such that \( a(g_1,n) = 0 \) for \((n,p_1L_f') > 1\).

(a) Let
\[
g_{1,0}(\tau) = \sum_{(n,p_1)=1} a(g_0,n)q^n.
\]
Then \( g_{1,0} \in S_\kappa(\Gamma_1(4LL_f' p_1^2)) \).

(b) If \( g_{1,0} = 0 \), then we set \( g_{1,0}'(\tau) = g_0(\tau/p_1) \). Clearly \( g_{1,0}' \neq 0 \), since \( g_0 \neq 0 \). Since \( g_{1,0} = 0 \), we have from Lemma 5.1 that \( g_{1,0}' \in S_\kappa(\Gamma_1(4LL_f' p_1)) \). Now we set
\[
g_{1,1}(\tau) = \sum_{(n,p_1)=1} a(g_{1,0}',n)q^n
\]
and we have \( g_{1,1} \in S_\kappa(\Gamma_1(4LL_f' p_1)) \).

(c) Now suppose \( g_{1,i} \in S_\kappa(\Gamma_1(4LL_f' p_1^{2^i})) \) has been constructed as in step (b) for some \( 0 \leq i \leq \alpha_1 \). If \( g_{1,i} = 0 \), then we go back to step (b) and construct \( g_{1,i+1} \in S_\kappa(\Gamma_1(4LL_f' p_1^{2^{(i+1)}})) \) with \( g_{1,0} \) and \( g_0 \) replaced by \( g_{1,i} \) and \( g_{1,i-1} \) respectively. The first \( i_0 \) for which \( g_{1,i_0} \neq 0 \), we set \( g_1 = g_{1,i_0} \).

If all of \( g_{1,0}, \ldots, g_{1,\alpha_1} \) are zero, since \( g_{1,\alpha_1+1} = S_\kappa(\Gamma_1(4LL_f' p_1^{2^{\alpha_1}})) \) and \( p_1^{\alpha_1} || L \), we see from the Fourier coefficients of \( g_{1,\alpha_1} \) and Lemma 5.2 that \( g_{1,\alpha_1+1} = 0 \). This gives us that \( g_0 = 0 \), a clear contradiction. Hence the process must stop for some \( 1 \leq i \leq \alpha_1 \). Thus we get \( g_1 \in S_\kappa(\Gamma_1(4LL_f' p_1^{2^{\alpha_1}})) \) for some \( 1 \leq i_0 \leq \alpha_1 \) such that \( a(g_1,n) = 0 \) for \((n,p_1L_f') > 1 \) and \( a(g_1,n) = a(g_0,p^{i_0}n) \).

(2) Now suppose \( g_j \in S_\kappa(\Gamma_1(4LL_f' p_1^{2^{-j}} p_2^{2^{-j}} \ldots p_j^{2^{-j}})) \) has been constructed for some \( 1 \leq j < r \), then we construct \( g_{j+1} \) as in step (1) with \( g_0, L_f' \) and \( M_f \) replaced by \( g_j, L_f' p_1 \ldots p_j \) and \( p_{j+1} \ldots p_r \) respectively.

(3) Finally we set \( g = g_r \) and we have \( g \in S_\kappa(\Gamma_1(4LL_f' p_1^{2^{-i_1}} p_2^{2^{-i_2}} \ldots p_r^{2^{-i_r}})) \). Moreover \( a(g,n) = 0 \) if \((n,L_f' p_1 \ldots p_r) > 1 \) and \( a(g,n) = a(f,p_1^{i_1} \ldots p_r^{i_r} n) \) otherwise.

We summarize the discussion in the following proposition.

Proposition 5.1. Let \( L = \prod_{i=1}^r p_i^{s_i} \) and \( L_f \) be an even divisor of \( 4L \). Suppose \( f \in S_\kappa(\Gamma_1(4L)) \) is non–zero and \( a(f,n) = 0 \) for all \((n,L_f) > 1 \). Then there exists a \( g \in S_\kappa(\Gamma_1(4LL_f' p_1^{2^{-i_1}} p_2^{2^{-i_2}} \ldots p_r^{2^{-i_r}})) \) with the following properties.
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(1) \( a(g, n) = 0 \) whenever \((n, 4L) > 1\).
(2) \( a(g, n) = a(f, p_1^{i_1} ... p_r^{i_r} n) \) when \((n, 4L) = 1\).
(3) \( 0 \leq i_j \leq \alpha_j \) for \(1 \leq j \leq r\).

Now we use the following result for half-integral weight cusp forms from [3, Theorem 4.5].

**Theorem 5.1.** Let \( \kappa \geq 5/2 \) be a half-integer and \( L \geq 1 \) be an integer. Suppose \( g \in S_\kappa(\Gamma_1(4L)) \) is such that \( a(g, n) = 0 \) whenever \((n, 4L) > 1\). Then there exists infinitely odd and square-free integers \( n \) such that \( a(g, n) \neq 0 \). More precisely, for any \( \epsilon > 0 \)

\[
\# \{n \leq X, n \text{ square-free} : a(g, n) \neq 0\} \gg_{g, \epsilon} X^{5/8-\epsilon}.
\]

To complete the proof of Theorem 3, first note that for any such \( f \) as in the statement of Theorem 3 we get from Proposition 5.1 a \( g \in S_\kappa(\Gamma_1(4L)) \) whose Fourier coefficients are supported away from the level. Now using Theorem 5.1 for \( g \) and noting the relation between the Fourier coefficients of \( f \) and \( g \) we get Theorem 3. \( \square \)

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