Bridge spectra of twisted torus knots

R. Sean Bowman, Scott Taylor, and Alexander Zupan*

Abstract

We compute the genus zero bridge numbers and give lower bounds on the genus one bridge numbers for a large class of sufficiently generic hyperbolic twisted torus knots. As a result, the bridge spectra of these knots have two gaps which can be chosen to be arbitrarily large, providing the first known examples of hyperbolic knots exhibiting this property. In addition, we show that there are Berge and Dean knots with arbitrarily large genus one bridge numbers, and as a result, we give solutions to problems of Eudave-Muñoz concerning tunnel number one knots.

1 Introduction

The bridge spectrum \([30]\) of a knot records the minimum bridge numbers of a knot with respect to Heegaard surfaces of all possible genera in a 3-manifold. In particular for a knot \(K \subset S^3\), the bridge spectrum is the sequence

\[
b(K) = (b_0(K), b_1(K), b_2(K), \ldots),
\]

where \(b_g(K)\) is the minimal bridge number of \(K\) with respect to a genus \(g\) Heegaard surface in \(S^3\). The process of meridional stabilization (see below) shows that, for all \(g\) such that \(b_g(K) > 0\),

\[
b_{g+1}(K) \leq b_g(K) - 1
\]

It is natural to consider the gaps in the bridge spectrum: that is, the values of \(g\) for which \(b_g(K) - b_{g+1}(K)\) (the order of the gap) is greater than one. Zupan [30] showed that for each \(n\) there is an infinite family of iterated torus knots with exactly \(n\) gaps of arbitrarily large order. Those examples are, of course, non-hyperbolic. Rieck and Zupan [15, 30] have asked if there are hyperbolic knots having more than one gap in their bridge spectrum. We answer the question affirmatively by exhibiting a class of twisted torus knots with two gaps of arbitrarily large order. We show:

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Theorem 1.1. Given $C > 0$, there are integers $p$, $q$, and $r$ with and $1 < r < p < q$ such that whenever $|s| > 18p$, the twisted torus knot $K = T(p,q,r,s)$ is hyperbolic and satisfies

$$b_0(K) = p$$

and

$$C \leq b_1(K) \leq \frac{1}{2}p.$$ 

Moreover, these knots may be chosen to have tunnel number one.

Consequently, such knots have two gaps, of arbitrarily large order, in their bridge spectra:

Corollary 1.2. Given $C > 0$, there are hyperbolic knots in $K \subset S^3$ with

$$b_0(K) - b_1(K) \geq b_1(K) - b_2(K) \geq C.$$ 

Twisted torus knots are an interesting class of knots which have been studied in several contexts. These knots represent many different knot types. Morimoto and Yamada [24] and Lee [18] have constructed twisted torus knots which are cables. Morimoto has also shown that infinitely many twisted torus knots are composite [21]. Guntel has shown that infinitely many twisted torus knots are torus knots [12]. Lee [19] has characterized twisted torus knots which are actually the unknot. Moriah and Sedgwick have shown that certain hyperbolic twisted torus knots have minimal genus Heegaard splittings which are unique up to isotopy [20]. It is also known that twisted torus knots have arbitrarily large volume [7]. Little, however, has been proved about the bridge numbers of twisted torus knots apart from Morimoto, Sakuma, and Yokota’s result that a certain infinite family is not one bridge with respect to a genus one splitting of the 3–sphere [23].

Twisted torus knots are also interesting from the point of view of Dehn surgery. Berge constructed examples of knots in $S^3$ with lens space surgeries, infinitely many of which are twisted torus knots [4]. Later, Dean constructed twisted torus knots with small Seifert fibered surgeries [8].

The knots constructed by Berge are knots lying on a genus two splitting of $S^3$ which represent a primitive element in the fundamental group of the handlebody on either side. Such knots are called doubly primitive, and it is not difficult to see that surgery along the slope determined by the splitting yields a lens space. An open question is whether the list Berge gives in [4] is a complete list of all knots in $S^3$ with lens space surgeries (problem 1.78 in Kirby’s list [16]). Many of the knots in Berge’s list have bounded genus one bridge number. However, we show the following theorem:

Theorem 1.3. There are hyperbolic Berge knots with arbitrarily large genus one bridge number. These are Berge knots of type VII and VIII, knots which lie in the fiber of a genus one fibered knot in $S^3$. 

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A similar result holds for the Dean knots mentioned above:

**Theorem 1.4.** There are hyperbolic Dean knots with arbitrarily large genus one bridge number.

We remark that Theorem 1.3 has been known to Ken Baker and Jesse Johnson for some time [1, 14].

Finally, we note that in Problems 2.1 and 2.3 of [11], Eudave-Muñoz proposed the following:

**Problem 1.5.** Give explicit examples of tunnel number one knots with arbitrarily large genus one bridge number.

**Problem 1.6.** Give explicit examples of tunnel number one knots $K$ with arbitrarily large genus one bridge number such that, in addition, a minimal genus Heegaard surface for the exterior $M_K$ has Hempel distance two.

Relevant definitions may be found in [11]. As a corollary to Theorem 1.3 we provide a solution to these problems:

**Corollary 1.7.** For any $m > 1$, the family of Berge knots $K_n = T(mn + 1, mn + n + 1, n, \pm 1)$ has the property that for sufficiently large $n$, $M_{K_n}$ has a minimal genus Heegaard surface of distance two, and

$$\lim_{n \to \infty} b_1(K_n) = \infty.$$ 

The plan of the paper is as follows: In section 2, we introduce relevant terminology and important results used in the rest of the paper. Next, we determine the genus zero bridge numbers for twisted torus knots with certain parameters in section 3. In section 4, we construct collections of twisted torus knots and establish properties which will be used to prove the main theorems in section 5.

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## 2 Preliminaries

For convenience, we work in $M = S^3$, although some of our results hold in more general 3-manifolds. For a link $L \subset M$, we denote the exterior $M_L = M \setminus \overline{N(L)}$ of $L$ in $M$ by $M_L$ (and use $N(L)$ to denote an closed regular neighborhood of $L$ in $M$).

A collection of arcs $\tau$ properly embedded in a handlebody $H$ is *trivial* if each arc $t \in \tau$ cobounds a disk $\Delta$ with an arc in $\partial H$ such that $\Delta \cap \tau = t$. We call such $\Delta$ a *bridge disk*. A $(g, b)$-bridge splitting (or $(g, b)$-splitting) is the decomposition of
Suppose now that $S$ sects $L$ containing $\delta$. Alternatively, we may isotope $\Sigma$ along $\Delta$ embedded disk such that $D$ contains $\delta$.

Let $b$ be essential in $S$. In this case, there is a component $\delta$ of strongly irreducible surfaces. Suppose further that for any pair of compressing or boundary-compressing disks $D$ and $D'$ on opposite sides of $S_L$, we have $D \cap D' \neq \emptyset$. In this case we call $S$ strongly irreducible. The following result (a special case of a theorem of Hayashi and Shimokawa) reveals the significance of strongly irreducible surfaces.

(4) $M, L$ as $(V, \alpha) \cup_{\Sigma} (W, \beta)$, where $V$ and $W$ are genus $g$ handlebodies containing nonempty collections $\alpha$ and $\beta$ of $b$ trivial arcs, $M = V \cup_{\Sigma} W$ is a Heegaard splitting, and $L = \alpha \cup \beta$. We call the closed surface $\Sigma$ a $(g, b)$-bridge surface and let $\Sigma_L$ denote $\Sigma \cap M_L$. The surface $\Sigma_L$ has $2b$ meridional boundary components. It is well known that for any link $L \subset M$ and any genus $g$, $(M, L)$ admits a $(g, b)$-bridge splitting for some $b$. Thus, for each $g$ the link $L$ has a genus $g$ bridge number $b_g(L)$, where

$$b_g(L) = \min\{b : L \text{ admits a } (g, b)-\text{splitting}\}$$

These invariants, first defined by Doll in [9], generalize the classical bridge number $b(L) = b_0(L)$ due to Schubert [27].

Given a $(g, b)$-bridge surface $\Sigma$ for a link $L$, we can create a $(g + 1, b - 1)$-bridge surface for $L$ by tubing $\Sigma$ along any one of the arcs $L \setminus \Sigma$. Thus, $b_{g+1}(L) \leq b_g(L) - 1$. This process is called meridional stabilization and is described in detail in [26, Lemma 3.2].

Let $S$ be a surface properly embedded in $M_L$. A simple closed curve in $S$ is called essential if it is not parallel to a component of $\partial S$ and does not bound a disk in $S$. A properly embedded arc in $S$ is called essential if it does not cobound a disk in $S$ with an arc in $\partial S$. A compressing disk $D$ for $S$ is an embedded disk such that $D \cap S = \partial D$ and $\partial D$ is an essential curve in $S$. A $\partial$-compressing disk is an embedded disk $\Delta$ such that $\Delta \cap S$ is an arc $\gamma$ which is essential in $S$, $\Delta \cap \partial M_L$ is an arc $\delta$, and $\partial \Delta$ is the endpoint union of $\gamma$ and $\delta$. We say that $S$ is incompressible if there does not exist a compressing disk $D$ for $S$. An incompressible surface $S$ is said to be essential if it is not $\partial$–parallel and, in addition, there is no boundary compressing disk for $S$. As $\partial M_L$ is a collection of tori, $S$ is essential if and only if it is incompressible and is not a $\partial$–parallel annulus or torus.

Suppose that $\Sigma$ is an $n$-bridge sphere for a link $L$ in $M = S^3$, and suppose further that there exist bridge disks $\Delta_1$ and $\Delta_2$ on opposite sides of $\Sigma$ with the property that $\Delta_1 \cap \Delta_2$ is one or two points contained in $L$. If $|\Delta_1 \cap \Delta_2| = 1$, then we may reduce the number of bridges of $L$ with respect to $\Sigma$, and we say $\Sigma$ is perturbed. On the other hand, if $|\Delta_1 \cap \Delta_2| = 2$, we say $\Sigma$ is cancellable. In this case, there is a component $L' \subset L$ contained in $\partial \Delta_1 \cup \partial \Delta_2$, and the discs $\Delta_1 \cup \Delta_2$ may be used to isotope $L'$, in the complement of $L \setminus L'$, into $\Sigma$. Alternatively, we may isotope $\Sigma$ along $\Delta_1 \cup \Delta_2$ to a bridge surface $\Sigma'$ for $L \setminus L'$ containing $L'$. We call $\Sigma'$ the result of cancelling $\Sigma$.

Suppose now that $S \subset M$ is a connected, separating surface which intersects $L$ transversely, and $S_L = S \setminus N(L)$ is either compressible or boundary-compressible to both sides in $M_L$. Suppose further that for any pair of compressing or boundary-compressing disks $D$ and $D'$ on opposite sides of $S_L$, we have $D \cap D' \neq \emptyset$. In this case we call $S$ strongly irreducible. The following result (a special case of a theorem of Hayashi and Shimokawa) reveals the significance of strongly irreducible surfaces.
Theorem 2.1 ([13], Theorem 1.2). Suppose that \( L \) is a link in \( S^3 \) and \( \Sigma \) is a bridge sphere for \( L \). Then \( \Sigma \) is perturbed, cancellable, or for each component \( L' \) of \( L \), there is strongly irreducible 2-sphere \( \Sigma' \) which meets \( L' \) and which intersects every component of \( L \) at most as many times as \( \Sigma \).

It is well known that any two essential surfaces in an irreducible link exterior may be isotoped so that all arcs of intersection are essential in both surfaces. This property extends to a strongly irreducible surface and an essential surface. This result is also likely well known. A proof can be found in [5, Proposition 6.1]; it is also implicit in [30, Lemma 5.2].

Lemma 2.2. Suppose \( M_L \) is an irreducible link exterior containing an essential surface \( S \) and a strongly irreducible surface \( S' \). There exists an isotopy after which all arcs of \( S \cap S' \) are essential in both surfaces and \( |\partial S \cap \partial S'| \) is minimal up to isotopy.

Once we know that all arcs of intersection of two surfaces are essential in both of them, we can utilize the next lemma, which is based on similar results proved by Gordon-Litherland [10], Rieck [25], and Torisu [29].

Lemma 2.3. Let \( L \subset M \) be a link such that \( M_L \) is irreducible, \( \partial \)-irreducible, and an annular. Suppose that \( F \) and \( G \) are connected, orientable surfaces with nonempty boundary properly embedded in \( M_L \) such that \( \chi(F), \chi(G) < 0 \), \( G \) is essential in \( M_L \), \( |\partial F \cap \partial G| \) is minimal up to isotopy, and \( F \) and \( G \) intersect in a nonempty collection of \( n \) essential arcs. Then

\[
  n \leq 9\chi(F)\chi(G).
\]

Proof. Let \( S \) be a connected, orientable surface properly embedded in \( M_L \) such that \( \chi(S) < 0 \). Let \( \Lambda \) be a collection of properly embedded essential arcs in \( S \) such that no two are parallel in \( S \). Then \( \Lambda \) can be completed to an ideal triangulation of the interior of \( S \) by adding more edges between components of \( \partial S \) if necessary. Let the new collection of edges be \( \Lambda' \) and the set of faces of the ideal triangulation be \( F \). Then we have \( 3|F| = 2|\Lambda'| \) as well as \( \chi(S) = -|\Lambda'| + F \), and so \(|\Lambda| \leq |\Lambda'| \leq -3\chi(S)\).

Viewing the intersection \( F \cap G \) as a graph \( \Lambda_F \subset F \), let \( \Lambda'_F \) be the reduced graph (obtained by combining all sets of parallel edges into a single edge), and let \( m_F \) be the maximal number of mutually parallel edges in \( \Lambda_F \), so that each edge in \( \Lambda'_F \) corresponds to at most \( m_F \) edges of \( \Lambda_F \). As \( \Lambda'_F \) has no parallel edges,

\[
  n/m_F \leq -3\chi(F).
\]

and so

\[
  n/(-3\chi(F)) \leq m_F.
\]

Let \( \Lambda_G \) be the graph in \( G \) consisting of a collection of \( m_F \) arcs of \( F \cap G \) which are mutually parallel in \( F \). If \( \Lambda_G \) has no parallel edges, then \( m_F \leq -3\chi(G) \); hence

\[
  n \leq 9\chi(F)\chi(G).
\]
Otherwise, there are arcs \( \lambda_1, \lambda_2 \subset F \cap G \) which are parallel in both \( F \) and \( G \), chosen to be adjacent in \( G \). Then \( \lambda_1 \) and \( \lambda_2 \) cobound rectangles \( R_F \subset F \) and \( R_G \subset G \) with arcs in \( \partial F \) and \( \partial G \), and we have that \( A = R_F \cup R_G \) is a properly embedded annulus or Möbius band. Suppose first that \( A \) is a Möbius band. If \( \partial A \) is inessential in \( \partial M_L \), then \( M_L \) contains a properly embedded \( \mathbb{RP}^2 \), contradicting that \( M_L \) is irreducible. The case in which \( \partial A \) is essential is ruled out by Lemma 5.1 of [25]. Next, suppose that \( A \) is an annulus. If one component of \( \partial A \) is inessential, then compressing along this component yields an embedded disk; hence, the other component of \( \partial A \) must be inessential as \( \partial M_L \) is incompressible. In this case, \( R_F \) is isotopic to \( R_G \) in \( M_L \). However, such an isotopy would allow us to reduce the number of arcs of \( F \cap G \), contradicting the minimality of \( \partial F \cap \partial G \). Thus, we may assume that both components of \( \partial A \) are essential. As such, \( A \) is incompressible. If \( A \) is boundary parallel, then \( \lambda_1 \) cobounds a boundary compressing disk for \( G \), which contradicts the assumption that \( G \) is essential. It follows that \( A \) is an essential annulus, contradicting the assumption that \( M_L \) is an annular.

We make one more definition before proceeding. Let \( K \) be a knot in \( M = S^3 \). Then \( H_1(\partial N(K)) \) has a natural basis \( ([\mu], [\lambda]) \), where \( \mu \) bounds a meridian disk of \( N(K) \) and \( [\lambda] = 0 \) in \( H_1(M_K) \). We parameterize a given curve \( \gamma \subset \partial M_K \) as a fraction (or slope) \( \frac{a}{b} \), where \( [\gamma] = a[\mu] + b[\lambda] \). Given such a \( \gamma \), we may construct a new manifold \( M_K(\gamma) \) by gluing a solid torus \( V \) to \( \partial M_K \) so that a curve bounding a meridian disk of \( V \) is glued to \( \gamma \). We say that the resulting manifold \( M_K(\gamma) \) is the result of \( \gamma \) Dehn surgery on \( K \).

3 Genus zero bridge numbers of some twisted torus knots

To begin this section, we define the class of knots known as twisted torus knots. Assume that \( p, q > 1 \) are relatively prime integers. Consider a \((p,q)\) torus knot \( T_{p,q} \) which lies on a Heegaard torus \( T \) for \( S^3 \). Let \( C \) be a curve bounding a disk \( D \) such that \( D \) meets \( T \) in a single arc and the interior of \( D \) meets \( K \) in \( 0 \leq r \leq p + q \) points of the same sign. We say that the result of doing \(-1/s\) Dehn surgery on \( C \) is the twisted torus knot \( T(p,q,r,s) \). Informally, this is the knot obtained from \( T_{p,q} \) by twisting \( r \) parallel strands by \( s \) full twists. Note that we leave open the possibility that \( r \leq 1 \), in which case the resulting knot is clearly a torus knot. Note also that various alternate definitions of twisted torus knots exist in the literature. For a discussion of these variations, see (for instance) [7].

For the remainder of this section, we let \( K \) denote \( T_{p,q} \). If the link \( K \cup C \) is hyperbolic, we know from results of Thurston that the resulting knot will be hyperbolic for all but finitely many surgeries. The following proposition of Lee says that, for most values of \( p, q, \) and \( r \), this is indeed the case.
Proposition 3.1 ([19], Proposition 5.7). When \( r > 1 \) is not a multiple of \( p \) or \( q \), the link \( K \cup C \) is hyperbolic.

Next, we exhibit an incompressible surface in \( M_{K \cup C} \) which will play the role of \( G \) in Lemma 2.3 above. Since \( K \) is a torus knot, \( M_K \) contains an essential annulus \( G' \) which intersects \( C \) transversely in two points. We let \( G \) denote \( G' \cap M_{K \cup C} \), so that \( G \) may be regarded at a twice-punctured annulus (or a 4-punctured sphere).

Lemma 3.2. The surface \( G \) is essential in \( M_{K \cup C} \).

Proof. Since \( G \) is a 4-punctured sphere, it suffices to show that \( G \) is incompressible. Suppose by way of contradiction that \( \gamma \subset G \) bounds a compressing disk \( D \) in \( M_{K \cup C} \). As a curve in the annulus \( G' \), \( \gamma \) must be inessential since \( K \) is a nontrivial torus knot. Thus, \( \gamma \) bounds a twice-punctured disk in \( G \), and compressing \( G \) along \( D \) yields an essential annulus contained in \( M_{K \cup C} \), contradicting Proposition 3.1. \( \square \)

In the link manifold \( M_{K \cup C} \), let \( T_K = \partial N(K) \) and \( T_C = \partial N(C) \). In addition, for each \( s \), let \( K_s = T(p, q, r, s) \) and \( L_s = K_s \cup C_s \), where \( C_s \) is the core of the solid torus which results from performing \( 1/s \) surgery on the twisting curve \( C \). With this notation, \( K = K_0 \) and \( C = C_0 \). In the following lemma, we compare the \( b_0(K_s) \) to \( b_0(K) \), where

\[
b_0(K) = \min\{p, q\}
\]

by a theorem of Schubert [27] with a modern proof given by Schultens [28].

Lemma 3.3. Given \( p, q, \) and \( r \) satisfying \( 1 < r < p < q \), if \( |s| > 18p \), we have

\[
b_0(K_s) = p.
\]

Proof. The torus knot \( K \) lies on a Heegaard torus \( T \) for \( S^3 \). We may consider \( T \) as the boundary of a regular neighborhood of an unknot in 1-bridge presentation with respect to a bridge sphere \( \Sigma \) for \( S^3 \). In fact, we may arrange for \( \Sigma \) to be a \( p \)-bridge sphere for \( K \). The bridge sphere \( \Sigma \) then realizes the minimal bridge number \( b_0(K) = p \) of \( K \). Since \( r < p \), it is not hard (though somewhat tedious) to show that \( C \) may be isotoped in the exterior of \( K \) to lie on \( \Sigma \). Performing \(-1/s\) surgery on \( C \) is equivalent to performing \(-s\) Dehn twists on an annular neighborhood of \( C \subset \Sigma \). The bridge sphere \( \Sigma \) is then a \( p \)-bridge sphere for \( K_s \), showing that

\[
b_0(K_s) \leq p,
\]

for any integer \( s \). Observe also that \( \Sigma \) is a \((p + 1)\)-bridge sphere for \( L_s \) with \(|\Sigma \cap L_s| = 2p + 2\). See Figure 1.

By Proposition 3.1, the exterior \( M_{L_s} = M_L \) is hyperbolic, and thus we can apply Theorem 1.1 of [17], which asserts that \( K_s \) is hyperbolic whenever \(|s| > 5\).
Figure 1: An example of a link $L_s$. There is a horizontal plane which is a
bridge sphere $\Sigma$ for $K_s$ containing $C_s$; by perturbing $C_s$ slightly, we see that
$|\Sigma \cap L_s| = 2p + 2$.

Suppose $|s| > 5$. In this case, the exterior of $K_s$ is then irreducible, $\partial$-irreducible,
and an annular.

Let $\Sigma_s$ be a minimal bridge sphere for $K_s$ minimizing the pair $(|\Sigma_s \cap K_s|, |\Sigma_s \cap C_s|)$ lexicographically, so that $\Sigma_s$ is not perturbed. If $\Sigma_s$ is cancellable, then $C_s$ can be isotoped onto $\Sigma_s$, as $K_s$ is not the unknot. As before, performing $-1/s$ surgery on $C_s$ is equivalent to performing $-s$ twists along $C_s \subset \Sigma_s$. After twisting, we see that $\Sigma_s$ is a bridge sphere for $K$ and so $p = b_0(K) \leq b_0(K_s)$. Hence, if $\Sigma_s$ is cancellable, $b_0(K_s) = p$.

Suppose, therefore, that $\Sigma_s$ is not cancellable. By Theorem 2.1, there is a
strongly irreducible surface $\Sigma'_s$ such that $\Sigma'_s$ satisfies $2 \leq |\Sigma'_s \cap C_s| \leq |\Sigma_s \cap C_s|$, and $|\Sigma'_s \cap K_s| \leq |\Sigma_s \cap K_s| \leq 2p$, intersects $G$ in a non-empty collection of arcs
essential on both surfaces, and minimizes $|\Sigma'_s \cap G \cap \partial M_L|$ up to isotopy of $\Sigma'_s$. Each component of $\Sigma'_s \cap T_C$ intersects each component of $G \cap T_C$ in exactly $|s|$ points, and since $|G \cap T_C| = 2$, it follows that

$$|\Sigma'_s \cap G \cap T_C| = 2|s| \cdot |\Sigma'_s \cap C_s|.$$ 

Thus, $\Sigma'_s \cap G$ contains at least $|s| \cdot |\Sigma'_s \cap C_s|$ arcs of intersection. By Lemma 2.3,

$$|s| \cdot |\Sigma'_s \cap C_s| \leq 9 \chi(\Sigma'_s \cap M_L) \chi(G) 
\leq 18(|\Sigma'_s \cap L_s| - 2) 
\leq 18(|\Sigma'_s \cap K_s| + |\Sigma'_s \cap C_s| - 2) 
\leq 36p + 18|\Sigma'_s \cap C_s| - 36.$$ 

Consequently,

$$|s| \leq \frac{36p + 18|\Sigma'_s \cap C_s| - 36}{|\Sigma'_s \cap C_s|} = \frac{36p - 36}{|\Sigma'_s \cap C_s|} + 18 \leq 18p.$$
We conclude that if $|s| > 18p$, then $\Sigma_n$ is cancellable and $b_0(K_s) = p$, as desired.

\section{Construction of the knots}

In this section, we construct collections $\{K^n\}$ of twisted torus knots by twisting one curve $\alpha$ about another curve $\beta$ on a genus two Heegaard surface $\Sigma$ for $S^3$. We establish properties of an associated link exterior in order to use the machinery developed in [2]. In section 5, we will use this machinery to show that the knots $\{K^n\}$ have unbounded genus one bridge number.

The knot $K = T(p, q, r, s)$ has a natural position on $\Sigma$; hence $b_2(K) = 1$. See Figure 2 (a picture of the knotted component of Figure 1).

![Figure 2: The twisted torus knot $T(3, 2, 2, 1)$ and its position on a genus two splitting of $S^3$.](image)

Observe that $K$ meets an obvious disk system for the inside handlebody $|p|$ and $|r|$ times, respectively, and meets an obvious disk system for the outside handlebody $|q|$ and $|rs|$ times, respectively. Define the \textit{surface slope} of $K$ with respect to $\Sigma$ to be the isotopy class of $\partial N(K) \cap \Sigma$ in $\partial N(K)$. Dean computed the surface slope of $T(p, q, r, s)$:

\textbf{Lemma 4.1} ([8], Proposition 3.1). \textit{The surface slope of $T(p, q, r, s)$ is $pq + r^2s$}.

Now consider two curves $\alpha$ and $\beta$ on a genus two Heegaard surface $\Sigma$ for $S^3$. Choose $\alpha$ to be a knot of type $T(a, b, 0, 0)$, without additional restrictions. This definition requires that $\gcd(a, b) = 1$, but $\alpha$ may be unknotted in $S^3$. Choose $\beta$ to be a hyperbolic knot or a nontrivial torus knot of type $T(c, d, e, f)$, where $\gcd(c, d) = 1$, $1 < e < c < d$, $f \neq 0$, and $|ad - be| \neq 0$. If $\beta$ is a torus knot of type $(x, y)$, we require two additional properties:

- the knot types of $\alpha$ and $\beta$ are different when $\alpha$ and $\beta$ are considered as knots in $S^3$, and
- the surface slope of $\beta$ with respect to $\Sigma$ is different from $xy$. 

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These requirements may seem burdensome, but they are not difficult to satisfy in practice. For example, the knots $T(c, d, f)$, $f \neq 0$, are torus knots of type $(c, d)$ whose surface slope differs from $cd$ by Lemma 4.1. Our construction of Berge and Dean knots below uses this form for $\beta$. Moreover, Proposition 3.1 shows that most twisted torus knots are hyperbolic, so we do not need to worry about knot types or surface slopes in this case.

Note that after orienting $\alpha$ and $\beta$, we may isotope them to meet in $\Delta = |ad-bc|$ points of the same intersection sign. See Figure 3 for an example in which $\alpha$ is $T(1, 1, 0, 0)$ and $\beta$ is $T(2, 3, 1, 1)$.

Dehn twisting $\alpha$ along $\beta$ results in the knot $T(a+n\Delta c, b+n\Delta d, n\Delta e, f)$ where $n$ is the number of twists. If we twist $\alpha$ along $\beta$ to the left $n$ times in Figure 3, we obtain the knot $T(2n+1, 3n+2, n, 1)$ since in this case $\Delta = 1$.

Now let $K = \alpha$, and let $L_1$ and $L_2$ be two copies of $\beta$ pushed along the positive and negative normal directions, respectively, of $\Sigma$. Then $L_1 \cup L_2$ bounds an annulus $\hat{R}$ which meets $K$ exactly $\Delta$ times. Let $L$ denote the link $L_1 \cup L_2 \cup K$, and let $T_1$, $T_2$, and $T_K$ be the boundary components of $M_L$ arising from $L_1$, $L_2$, and $K$, respectively. Let $R = \hat{R} \cap M_{L_1 \cup L_2}$ and note that the slope of $R$ on $T_1$ and $T_2$ is the surface slope of $\Sigma$ on $\beta$.

By orienting $M$ and $\hat{R}$, we obtain orientations on $L_1$, $L_2$, and their meridians $\mu_1$ and $\mu_2$. With respect to coordinates on $T_1$ and $T_2$ given by $\mu_1$, $\mu_2$, and $\partial R$, performing $1/n$ surgery on $L_1$ and $-1/n$ surgery on $L_2$ has the effect of “twisting $K$ along the annulus $\hat{R}$,” an operation described in [2 Definition 1.1]. The resulting knot $K''$ is the same knot obtained by twisting $\alpha$ along $\beta$ in the appropriate direction $n$ times.

We first show that for large enough $n$, the knots $K''$ satisfy the hypotheses of Lemma 3.3.

**Lemma 4.2.** Let $p = a+n\Delta c$, $q = b+n\Delta d$, $r = n\Delta e$, and $s = f$ so that
For sufficiently large $n$, 

$$1 < r < p < q.$$ 

**Proof.** Clearly $r > 1$ when $n$ is large. We also have 

$$a + n\Delta c = a + n\Delta d + n\Delta (c - d) < b + n\Delta d$$ 

for large $n$ since $c - d < 0$. 

In addition, 

$$n\Delta e = n\Delta c + n\Delta (e - c) < a + n\Delta c$$ 

for large $n$ since $e - c < 0$. 

Next, we exhibit a catching surface for $(\hat{R}, K)$. The precise definition is given in [2]. For our purposes an orientable, connected, properly embedded surface $Q \subset M_L$ catches $(\hat{R}, K)$ if $\chi(Q) < 0$, $\partial Q \cap T_i$ is a nonempty collection of coherently oriented parallel curves on $T_i$, and $\partial Q$ intersects $T_i$ in slopes different from $\partial R$, $i = 1, 2$. 

**Lemma 4.3.** The pair $(\hat{R}, K)$ is caught by a surface $Q$ with 

$$\chi(Q) = 1 - (|a| - 1)(|b| - 1) - |bc| - |ad|.$$ 

Furthermore, $\partial Q$ is meridional on $T_1$ and $T_2$ and meets a meridian of $T_K$ exactly once. 

**Proof.** We may consider $K$ as lying in a Heegaard torus $T$ for $S^3$. The standard Seifert surface $Q$ for $K$ can be constructed by taking $|a|$ disks on one side of $T$, $|b|$ disks on the other side, and banding them together with $|ab|$ bands. Such a surface has genus $\frac{1}{2}(|a| - 1)(|b| - 1)$, so we must determine how many times $L_1$ and $L_2$ meet $Q$. 

Observe that the value of $f$ has no effect on $|L_i \cap Q|$ for $i = 1, 2$. Thus we need only compute $|L_i \cap Q|$ when $f = 0$, and in this case we consider $L_i$ to be pushoffs of $(c, d)$ curves contained in $T$. 

We may arrange that $L_1$ and $L_2$ meet only the disks on their respective sides, and all with the same sign of intersection. On one side, we see that $L_1$ meets $|a|$ disks $|d|$ times each. On the other side, $L_2$ meets $|b|$ disks $|c|$ times each. This gives the claimed Euler characteristic, and $\partial Q$ is meridional on $T_1$ and $T_2$. 

An example (with $K$ the unknot) appears in Figure 4. Performing $-1$ surgery on the unknotted curve at right and twisting along the annulus bounded by $L_1 \cup L_2$ gives the knots in Figure 3. In this case, the catching surface is a planar surface.
with one longitudinal boundary component on $T_K$ and 5 meridional boundary components on $T_1 \cup T_2$.

The remainder of this section is devoted to establishing several topological properties about $\hat{R}$ and the manifolds $M_{L_1 \cup L_2}$ and $M_C$ in order to employ the tools of [2].

Lemma 4.4. The annulus $\hat{R}$ is not isotopic into a genus one splitting of $S^3$.

Proof. We will show that $\beta$, the core of $\hat{R}$, is not isotopic into a genus one splitting of $S^3$ such that the surface slope of $\beta$ with respect to this splitting matches the surface slope of $\beta$ with respect to $\Sigma$.

Recall that $\beta$ is knotted by hypothesis. If $\beta$ is a torus knot of type $(x, y)$, then our additional hypothesis that $xy$ differ from the surface slope of $\beta$ with respect to $\Sigma$ ensures that the surface slopes differ.

If $\beta$ is not a torus knot, then the core of $\hat{R}$ is not isotopic into a genus one splitting for $S^3$, and so the conclusion certainly holds. $\square$

Lemma 4.5. The spaces $M_{L_1 \cup L_2}$ and $M_C$ are irreducible.

Proof. From the proof of Lemma 4.3 we see that the linking numbers of $K$ with $L_1$ and $L_2$ are nonzero. Therefore $M_C$ is irreducible.

If $M_{L_1 \cup L_2}$ is reducible, a reducing sphere must intersect the annulus $R$ in an essential curve. However, this shows that $R$ is compressible and therefore $L_1$ or $L_2$ is trivial. Since this is not true, $M_{L_1 \cup L_2}$ is irreducible. $\square$

The annulus $R \subset M_{L_1 \cup L_2}$ has one boundary component on each of $T_1$ and $T_2$. Let $R' = R \cap M_C$. This is a planar surface with one longitudinal boundary com-
ponent on \( T_1 \), one on \( T_2 \), and \(|ad - bc|\) coherently oriented meridional boundary components on \( T_K \).

**Lemma 4.6.** There is no essential annulus \( A \subset M_\mathcal{L} \) with one boundary component on \( T_1 \) and the other on \( T_2 \).

**Proof.** Suppose for contradiction that \( A \) is such an annulus. If the slopes of \( \partial A \) and \( \partial R \) agree on \( T_1 \) and \( T_2 \), then we may assume that \( \partial A \cap \partial R' = \emptyset \) after an isotopy. In \( S^3 \), gluing \( A \) to \( R \) along subannuli of \( \partial T_1 \) and \( \partial T_2 \) yields an immersed torus \( T \), where \( T \) intersects \( \alpha \) transversely in \(|ad - bc| > 0 \) points of the same orientation. However, this contradicts that \( T \) is homologically trivial in \( S^3 \); hence, there is no such annulus \( A \).

We see that the slopes of \( \partial A \) and \( \partial R \) cannot agree. Note that \( A \) is also an essential annulus in \( M_\mathcal{L} \cup L_1 \cup L_2 \) with one boundary component on \( T_1 \) and the other on \( T_2 \). We will show that \( M_\mathcal{L} \cup L_1 \cup L_2 \) contains no such essential annuli whose boundary slopes differ from \( \partial R \).

Suppose that it does, and choose an \( A \subset M_\mathcal{L} \cup L_1 \cup L_2 \) which minimizes \(|A \cap R|\). A cut and paste argument shows that \( A \cap R \) consists of arcs essential in both \( A \) and \( R \). Therefore the slopes of \( \partial A \) differ from those of \( \partial R \) on both \( T_1 \) and \( T_2 \), and these slopes meet each other in points of the same intersection sign. Let \( M' = M_{\mathcal{L} \cup L_1 \cup L_2} \setminus N(R) \), so that \( M' \) is homeomorphic to \( M_\beta \), and let \( A_1 \) and \( A_2 \) be the two components of the frontier of \( N(R) \) in \( M_{\mathcal{L} \cup L_1 \cup L_2} \); thus \( A_i \subset \partial M' \).

Since \( A \cap R \) is a collection of essential arcs oriented in the same direction, \( A \cap M' = A \setminus N(R) \) is a collection of disks, each of which intersects \( A_1 \) and \( A_2 \) once. This implies that each disk component of \( A \cap M' \) has essential boundary, contradicting the incompressibility of \( \partial M' \). \( \square \)

## 5 Genus one bridge number bounds and proof of the main theorems

In this section, we give a lower bound on the genus one bridge numbers of the knots \( K_n \), following which we prove the main theorems of the paper. As mentioned above, we will utilize a theorem from \([2]\). Here we state a version of the theorem specialized to our needs.

**Theorem 5.1** ([2], Theorem 1.2). Let \( \mathcal{L} = K \cup L_1 \cup L_2 \) be a link in \( S^3 \), and let \( \bar{R} \) be an annulus in \( M = S^3 \) with \( \partial \bar{R} = L_1 \cup L_2 \). Assume \((\bar{R}, K)\) is caught by the surface \( Q \) in \( M_\mathcal{L} \) with \( \chi(Q) < 0 \). Let \( T_K \), \( T_1 \), and \( T_2 \) be the components of \( \partial M_\mathcal{L} \) corresponding to \( K \), \( L_1 \), and \( L_2 \), respectively. Suppose that \( \partial Q \) is meridional on \( T_1 \) and \( T_2 \) and meets a meridian of \( T_K \) exactly once. Let \( K^n \) be \( K \) twisted \( n \) times along \( \bar{R} \). If \( H_1 \cup H_2 \) is a genus \( g \) Heegaard splitting of \( S^3 \), then either

1. \( \bar{R} \) can be isotoped to lie in \( \Sigma \),
2. there is an essential annulus properly embedded in $M_L$ with one boundary component in each of $T_1$ and $T_2$, or

3. for each $n$, 
   
   \[ b_g(K^n) \geq \frac{1}{2} \left( \frac{n}{-36\chi(Q)} - 2g + 1 \right) \]

Now we bound the genus one bridge number of the knots constructed in the previous section. Recall that $K^n$ is the twisted torus knot $T(a + n\Delta c, b + n\Delta d, n\Delta e, f)$, where $\Delta$ is the intersection number of $\alpha$ and $\beta$ in $\Sigma$.

**Proposition 5.2.** For each $n$ we have

\[ b_1(K^n) \geq \frac{1}{2} \left( \frac{n}{-36(|ad| + |bc| + (|a| - 1)(|b| - 1) - 1)} - 1 \right) \]

**Proof.** This follows from Theorem 5.1. The pair $(\tilde{R}, K)$ is caught by Lemma 4.3. The twisting annulus $\tilde{R}$ is not isotopic into a genus one Heegaard splitting of $S^3$ by Lemma 4.4. Finally, by Lemma 4.6 there is no essential annulus as in case 2. Therefore, the stated bound holds. \qed

In order to compute a bound on $b_0(K^n) - b_1(K^n)$, we need an upper bound on $b_1(K^n)$. Recall that for large enough $n$, $K^n$ is a $T(p,q,r,s)$ twisted torus knot with $1 < r < p < q$, and consider the Heegaard torus $\Sigma_1$ for $S^3$ depicted in Figure 5. Here we have cut the torus along a meridian disk and arranged the parts vertically. After pulling $p - r$ strands of the pictured twisted torus braid through $\Sigma_1$ in the first case, or $r$ strands in the second case, $\Sigma_1$ becomes a bridge surface for $T(p,q,r,s)$. This gives the upper bound of the following lemma.

**Lemma 5.3.** We have

\[ b_1(T(p,q,r,s)) \leq \min\{r, p - r\} \]

We need the next lemma, which is implicit in the work of Dean [8] and appears explicitly in Lemma 3.2 of [19], to prove that the knots in Theorem 1.1 may be taken to have tunnel number one. Later, we will use it to give examples of Berge knots with arbitrarily large genus one bridge number. Let the genus 2 splitting $\Sigma$ bound handlebodies $H_1$ and $H_2$ in $S^3$, and think of $H_1$ as the “inside” handlebody in Figure 3.

**Lemma 5.4.** The knot $T(p,q,r,s)$ (with notation as above and $p, q > 1$) is primitive in $H_1$ if and only if $r \equiv \pm 1 \pmod{p}$ or $r \equiv \pm q \pmod{p}$. This knot is primitive in $H_2$ if and only if

1. $s = \pm 1$, and
2. $r \equiv \pm 1 \pmod{q}$ or $r \equiv \pm p \pmod{q}$. 

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We can now prove Theorem 1.1.

Proof of Theorem 1.1. Choose curves $\alpha$ and $\beta$ as in section 4 and let $K^n = T(p,q,r,s)$ be the knot defined by twisting $\alpha$ along $\beta$ in the genus 2 splitting $\Sigma$. Given $C > 0$, we may choose $n$ large enough so that Lemma 4.2 is satisfied and $b_1(K^n) \geq C$ by Proposition 5.2. Fix this $n$ (so that $p$, $q$, and $r$ are also fixed) and note that $s$ does not appear in the bound for $b_1(K^n)$ given by Proposition 5.2. By Proposition 3.1 and Lemma 3.3 the knots $K_s = T(p,q,r,s)$ are hyperbolic and have $b_0(K_s) = p$ for $|s| > 18p$. Furthermore, $b_1(K_s) \leq \min(r,p-r) \leq \frac{1}{2}p$ by Lemma 5.3.

To see that we may choose such examples to have tunnel number one, note that we may choose $\alpha$ and $\beta$ so that the knots $K^n$ are primitive in $H_1$ according to Lemma 5.4. For example, choosing $\alpha$ to be $T(1,1,0,0)$ and $\beta$ to be $T(m,m+1,1,1)$ for $m > 1$ results in the family $K^n = T(mn+1,mn+n+1,n,s)$ (cf. proof of Theorem 1.3). A knot which is primitive on one side of $\Sigma$ is isotopic to a core of the handlebody on that side, and so has tunnel number one.

Now we examine special cases of twisted torus knots.

Proof of Theorem 1.3. Recall the curves $\alpha$ and $\beta$ of Figure 3. Twisting $n$ times around $\beta$ we obtain the knots $T(3n+1,2n+1,n,1)$, which are Berge knots by Lemma 5.4.

More generally, let $\alpha$ be a $T(1,1,0,0)$ twisted torus knot and $\beta$ be a $T(m,m+1,1,1,\pm1)$ twisted torus knot for $m > 1$. Since $\Delta = 1$, we obtain the knot $K^n = T(mn+1,mn+n+1,n,\pm1)$. Letting $p = mn+1$, $q = mn+n+1$, $r = n$, $s = \pm1$.
and $s = \pm 1$, an easy calculation shows that $K^n$ is primitive on both sides of $\Sigma$, so these are Berge knots.

Applying the bound of Proposition 5.2 to these knots, we see that they have arbitrarily large genus one bridge number. We use the argument of Proposition 4.3 from [20] to show that these knots are hyperbolic: Because $K^n$ is primitive on at least one side of $\Sigma$, it is a tunnel number one knot. Toroidal tunnel one knots are classified by Morimoto and Sakuma [22] and have genus one bridge number one. Therefore, for large enough $n$, $K^n$ is atoroidal and not a torus knot; thus it is hyperbolic.

To see that these are Berge knots of type VII and VIII (knots which lie in the fiber of a trefoil or figure eight knot, see [3]), we note that $K^n$ lies in a neighborhood of $\alpha \cup \beta$ in $\Sigma$ and examine the boundary of this neighborhood in $S^3$. Introduce an unknotted curve $c$ with surgery coefficient $\mp 1$ according to whether $s = \pm 1$ as in Figure 6. Before surgery on $c$, we see $N(\alpha \cup \beta)$ as the neighborhood of a $a = (1, 1)$ curve and a $b_m = (m, m+1)$ curve on a genus one splitting $T$ of $S^3$. It is not difficult to see that $\partial N(a \cup b_1)$ becomes the figure 8 knot after $+1$ surgery on $c$ and a trefoil knot after $-1$ surgery. But $N(a \cup b_m)$ is isotopic to $N(a \cup b_1)$ in $T$ since we can “untwist” along $a$. Therefore $N(a \cup b_m)$ is a genus one Seifert surface for $\partial N(a \cup b_1)$, which becomes a trefoil or figure eight after the appropriate surgery on $c$, and so $K^n$ is a Berge knot of type VII or VIII.

\[ \text{Figure 6: The curves } a \text{ and } b_1 \text{ lying on a genus 1 splitting of } S^3, \text{ together with the twisting curve } c. \]

Next, we consider the knots studied by Dean in [8]. These are knots which lie in a genus two splitting of $S^3$ so that they are primitive on one side and Seifert fibered on the other (in the sense that attaching a 2–handle to the handlebody along the knot results in a Seifert fibered space). Such knots have small Seifert fibered surgeries, so in this sense Dean knots are a generalization of Berge knots. Dean shows that the knots $T(p, q, 2q-p, \pm 1)$, $(p+1)/2 < q < p$, and $T(p, q, p-kq, \pm 1)$, $1 < q < p/2$, $2 \leq k \leq (p-2)/q$ are primitive/Seifert fibered [8 Theorem 4.1].

Proof of Theorem 1.4. Let $\alpha$ be $T(2, 1, 0, 0)$ and $\beta$ be $T(2m-1, m, 1, \pm 1)$. Twisting $\alpha$ around $\beta$ yields $T(2mn - n + 2, mn + 1, n, \pm 1)$. With $p = 2mn - n + 2$,
q = mn + 1, r = n, and s = ±1, we see that r = 2q − p and \( \frac{p+1}{2} < q < p \) for large enough n. These are Dean knots of the first type in [8, Theorem 4.1].

Similarly, if we let \( \alpha \) be \( T(l, 1, 0, 0) \) and \( \beta \) be \( T(lm + 1, m, 1, \pm 1) \) for \( l \geq 2, m \geq 2 \) we obtain the knot \( T(p, q, r, s) \) with \( p = (lm + 1)n + l, q = mn + 1, r = n, \) and \( s = \pm 1 \) after twisting. It is clear that \( r = p - lq, 1 < q < p/2, \) and \( 2 \leq l \leq \frac{p - 2}{q} \) for large enough n, and so these are Dean knots of the second type in [8, Theorem 4.1].

The same arguments used in the proof of Theorem 1.3 apply to show that these knots are hyperbolic and have arbitrarily large genus one bridge number.

Finally, we show that the knots from Theorem 1.3 have minimal genus Heegaard splittings of Hempel distance two.

**Proof of Corollary 1.7.** As in the proof of Theorem 1.3, for sufficiently large \( n \) the knots \( K^n = T(mn + 1, mn + n + 1, n, \pm 1) \) are hyperbolic Berge knots with unbounded genus one bridge number. Since each \( K^n \) has a doubly primitive representative on a genus two Heegaard surface \( \Sigma' \) for \( S^3 \), there are compressing disks \( D \) and \( D' \) on opposite sides of \( \Sigma' \) which are disjoint from \( K^n \). Pushing \( K^n \) off of \( \Sigma' \) into one of the handlebodies yields a genus two Heegaard surface \( \Sigma \) for \( M_{K^n} \), where \( D \) and \( D' \) are compressing disks for \( \Sigma \) and the distance between \( \partial D \) and \( \partial D' \) in the curve complex of \( \Sigma \) is at most 2. On the other hand, the distance of \( \Sigma \) cannot be zero or one by [6] because \( K^n \) is hyperbolic. Therefore, the distance of \( \Sigma \) is exactly two.

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