On Locating Chromatic Number ofDisconnected Graph with Path, Cycle, Stars or Double Stars as its Components

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Abstract. Concept of the locating-chromatic number of graphs is introduced firstly by Chartrand et al. in 2002. In this concept only limited for connected graphs. In 2004, Welyyanti et al. generalized the concept and notion of the locating-chromatic number of a graph such that it can be applied for a disconnected graph. In this paper, we determine the locating-chromatic number for disconnected graphs with path, stars, cycle, or double stars as its components.

1. Introduction

The locating chromatic number of a graph is introduced by Chartrand et al. in 2002\cite{4}. Chartrand et al.\cite{4} determine locating chromatic number of some class of graphs. There are path, cycle, double stars, complete graph and graphs with the locating-chromatic number \( n - 1 \).

For some class of graphs, the locating-chromatic number of firecrackers\cite{1}, amalgamation stars\cite{2}, and trees which have the locating-chromatic number 3\cite{3} are determined by Asmiati. Furthermore, Syofyan et al. determined the locating-chromatic number of homogenous lobster\cite{5}. Another results for a connected graph, the locating-chromatic number of graph with dominant vertices is determined by Welyyanti et al.\cite{7} in 2015.

Since the concept locating-chromatic number of a graph only limited for a connected graph, Welyyanti et al.\cite{6} generalized that concept such that can be applied for a disconnected graph in 2014. They determined upper bounds and lower bounds for the locating-chromatic number of a disconnected graph. They also determined the locating-chromatic number of a graph with two components\cite{8} and two homogenous components\cite{9}.

There is some notation for concept of the locating-chromatic number of disconnected a graph as follows. Let \( H \) be a disconnected graph. Let \( c \) be a locating \( k \)-coloring of a disconnected graph \( H \). The locating-chromatic number of \( H \), denoted by \( \chi'_L(H) \), is the smallest \( k \) such that \( H \) admits a locating \( k \)-coloring\cite{6}.

In this paper, we determine the locating-chromatic number of disconnected graphs with path, star, cycle, or double stars as its components, denoted by \( H = xP_l \cup yK_{1,m} \cup zC_n \) and derive some conditions under which the locating of \( H \) is finite.

2. Results

For a finite locating-chromatic number, we already determined the boundaries for \( \chi'_L(H) \)[6], as follows.
Theorem 2.1.[5] Let $G_i$ be a connected graph and let $H = \bigcup_{i=1}^m G_i$. If $\chi'_L(H) < \infty$, then $q \leq \chi'_L(H) \leq r$, where $q = \max \{\chi'_L(G_i): i \in [1, m] \}$ and $r = \min \{|V(G_i)|: i \in [1, m] \}$.

Another results in [6], we also determine the locating-chromatic number of disconnected graphs that stars or paths as its components.

Theorem 2.2.[6] Let $H = \bigcup_{i=1}^t K_{1,n_i}$ and $n_i \geq 2$, then
$$\chi'_L(H) = \begin{cases} n + 1, & \text{for } n_1 = n_2 = \cdots = n_t = n \text{ and } t \leq n + 1, \\ \infty, & \text{otherwise.} \end{cases}$$

Theorem 2.3.[6] Let $P_n$ be a path on $n$ vertices. Let $H = rP_l$, for $r \geq 1$ and $l \geq 3$. If $\chi'_L(H) \leq l$, then $r \leq \frac{n}{2m}$, where $m = \left\lfloor \frac{l}{2} \right\rfloor$.

A path $P_n$ with $l$ vertices, a star $K_{1,m}$ with $m + 1$ vertices, a cycle $C_n$ with $n$ vertices, and double stars $S_{m,n}$ with $m + n + 2$ vertices. There are some theorem and corollary for the locating-chromatic number of disconnected graphs with path, star, cycle, or double stars as its components.

Theorem 2.4. If $H = P_l \cup K_{1,m} \cup C_n$ and $\chi'_L(H) < \infty$, then $l, n \geq m + 1$.

Proof. Let $H = P_l \cup K_{1,m} \cup C_n$ and $\chi'_L(H) < \infty$. By Theorem 2.1, we have max \{\chi'_L(G_i): i \in [1, m] \} \leq \chi'_L(H) \leq \min \{|V(G_i)|: i \in [1, m] \}$. Let $q = \max \{\chi'_L(G_i): i \in [1, m] \}$ and $r = \min \{|V(G_i)|: i \in [1, m] \}$. Since $P_l, K_{1,m}$ and $C_n$ are components of $H$, we have $q = m + 1$. Consequently, $\chi'_L(H) \geq m + 1$. Its means $H$ has a locating-coloring with $m + 1$ colors. Since $\chi'_L(H) < \infty$ and every vertex in a component, $K_{1,m}$ must has different color. Therefore, another component in $H$ must at least have $m + 1$ vertices. Thus, $l, n \geq m + 1$.

Furthermore, we determine the locating-chromatic number for disconnected graphs with a star $K_{1,3}$, one or two copies of cycles and paths.

Theorem 2.5. If $H = xP_l \cup K_{1,3} \cup zC_n, x = z = 1,2$, and $l, n \geq 4$, then $\chi'_L(H) = 4$.

Proof. Let $H = xP_l \cup K_{1,3} \cup zC_n$ be disconnected graph with a star $K_{1,3}$, one or two copies of cycles $C_n$ and paths $P_l$. Let $H = xP_l \cup K_{1,3} \cup zC_n$ be a disconnected graph with a star $K_{1,3}$, cycle $C_n$ and path $P_l$. By Theorem 2.1, we have max \{\chi'_L(G_i): i \in [1, m] \} \leq \chi'_L(H) \leq \min \{|V(G_i)|: i \in [1, m] \} and max\{\chi'_L(G_i): i \in [1, m] \} = 4. Consequently, $\chi'_L(H) \geq 4$.

Figure 1. A disconnected graph $H = P_l \cup K_{1,3} \cup C_n$, with $n$ even.
Next, for $H = xP_l \cup K_{1,3} \cup zC_n$, for $x = z = 1,2$, and $l, n \geq 4$, we will show $\chi'_L(H) \leq 4$.

For that proof, we have four cases.

- **Case 1.** $x = 1$ and $z = 1$. Let $H = xP_l \cup K_{1,3} \cup zC_n$ be disconnected graph. Let $V(H) = \{V(P_l) \cup V(K_{1,3}) \cup V(C_n)\}$ where $V(P_l) = \{a_1, a_2, a_3, ..., a_{l-1}, a_l\}$, $V(K_{1,3}) = \{b_1, b_2, b_3, b_4\}$, and $V(C_n) = \{c_1, c_2, c_3, ..., c_{n-1}, c_n\}$ (see Figure 1 and 2). Define a coloring $c: V(H) \to \{1,2,3,4\}$ such that
  \[c(a_1) = c(b_1) = c(c_1) = 1\]
  \[c(a_2) = c(b_2) = c(c_2) = 2\]
  \[c(a_3) = c(b_3) = c(c_3) = 3\]
  \[c(a_4) = c(b_4) = 4\]

For $r \in [5, l]$, define
  \[c(a_r) = \begin{cases} 
  3, & \text{if } r \text{ is odd}, \\
  4, & \text{if } r \text{ is even}.
  \end{cases}\]

For $s \in [4, n]$ and $n$ even, define
  \[c(a_r) = \begin{cases} 
  3, & \text{if } s \text{ is odd}, \\
  4, & \text{if } s \text{ is even}.
  \end{cases}\]

For $s \in [4, n]$ and $n$ odd, define
  \[c(a_r) = \begin{cases} 
  2, & \text{if } s \text{ is even}, \\
  3, & \text{if } s \text{ is odd}, \\
  4, & \text{s = n}.
  \end{cases}\]
Let $\Pi = \{C_1, C_2, C_3, C_4\}$ be partition in $V(H)$ which induced by $c$. Now, we will determined that every vertex in $V(H)$ have distinct color code (see Figure 3 and 4). We have $c_\Pi(a_1) = (0,1,2,3,4)$, $c_\Pi(a_2) = (1,0,2,3)$, and $c_\Pi(a_3) = (2,1,0,1)$. For $r \geq 3$, $c_\Pi(a_r) = (r - 1, r - 2, 0, 1)$ if $r$ is odd and $c_\Pi(a_r) = (r - 1, r - 2, 1, 0)$ if $r$ is even.

Next, we have $c_\Pi(b_1) = (0,1,1,2)$, $c_\Pi(b_2) = (1,0,2,2)$, $c_\Pi(b_3) = (1,2,0,0)$ and $c_\Pi(b_4) = (1,2,2,0)$. For even cycle, we have $c_\Pi(c_1) = (0,1,1,2)$, $c_\Pi(c_2) = (1,0,2,1)$, $c_\Pi(c_3) = (1,2,0,1)$, and $c_\Pi(c_4) = (2,1,1,0)$. For $s \geq 5$, the color code of vertices, $c_\Pi(c_s)$ will different by a partition of $V(H)$ which contains vertices $c_1$ or $c_2$.

For odd cycle, we have $c_\Pi(c_1) = (0,1,1,\frac{n}{2})$, $c_\Pi(c_2) = (1,0,1,\frac{n}{2} - 1)$, $c_\Pi(c_3) = (1,0,1,\frac{n}{2} - 2)$, and $c_\Pi(c_n) = (\frac{n}{2}, 1,1,0)$. For $s \geq 3$, the color code of vertices, $c_\Pi(c_s)$ will different by a partition of $V(H)$ which contains vertices $c_1$ or $c_n$. Therefore, $\forall v \in V(H)$ have different color codes.

The prof for Case 2,3, and 4 similar with Case 1. In Case 2,3, and 4, we will determine a 4-locating-coloring for every case.
- **Case 2.** $x = 1$ and $z = 2$. Let $H = P_1 \cup K_{1,3} \cup 2C_n$ be disconnected graph. In Figure 5 and 6, we see a 4-locating-coloring for $H$. Therefore, $\forall v \in V(H)$ have different color codes.

![Figure 5. 4-Locating-Coloring of $H = P_1 \cup K_{1,3} \cup 2C_n$, with $n$ even](image)

- **Case 3.** $x = 2$ and $z = 1$. Let $H = 2P_1 \cup K_{1,3} \cup C_n$ be disconnected graph. In Figure 7 and 8, we see a 4-locating-coloring for $H$. Therefore, $\forall v \in V(H)$ have different color codes.

![Figure 6. 4-Locating-Coloring of $H = P_1 \cup K_{1,3} \cup 2C_n$, with $n$ odd](image)

![Figure 7. 4-Locating-Coloring of $H = 2P_1 \cup K_{1,3} \cup C_n$, with $n$ even](image)
Figure 8. 4-Locating-Coloring of $H = 2P_1 \cup K_{1,3} \cup C_n$, with $n$ odd

- **Case 4.** $x = 2$ and $z = 2$. Let $H = 2P_1 \cup K_{1,3} \cup 2C_n$ be disconnected graph. In Figure 9 and 10, we see a 4-locating-coloring for $H$. Therefore, $\forall v \in V(H)$ have different color codes.

Figure 9. 4-Locating-Coloring of $H = 2P_1 \cup K_{1,3} \cup 2C_n$, with $n$ even

Figure 10. 4-Locating-Coloring of $H = 2P_1 \cup K_{1,3} \cup 2C_n$, with $n$ odd

Consequently, $\chi'_L(H) \leq 4$. 
Now, we study a condition for disconnected graphs, $H = P_l \cup K_{1,m} \cup C_n$, having finite locating-chromatic number.

**Theorem 2.6.** Let $H = xP_l \cup yK_{1,m} \cup C_n$, $l = m + 1$, $n \geq m + 1$, and $\chi'_L(H) < \infty$, then $x \leq \frac{n}{2u}$, where $u = \left\lfloor \frac{l}{2} \right\rfloor$ and $y \leq m + 1$. In particular $\chi'_L(H) = m + 1$.

**Proof.** Let $H = xP_l \cup yK_{1,m} \cup C_n$, $l = m + 1$, $n \geq m + 1$, and $\chi'_L(H) < \infty$. Since $P_l$ and $K_{1,m}$ are components of $H$, by Theorem 2.2, we have $x \leq \frac{n}{2u}$, where $u = \left\lfloor \frac{l}{2} \right\rfloor$. Furthermore, by Theorem 2.3, we get $y \leq m + 1$.

Now, we will determine $\chi'_L(H) = m + 1$. Since $K_{1,m}$ are components of $H$, by Theorem 2.1, we have $\chi'_L(H) \leq m + 1$ and $\chi'_L(H) \geq m + 1$. Its means $\chi'_L(H) = m + 1$.

From previous theorem, we have some corollary above

**Corollary 2.7.** Let $H = xP_l \cup yK_{1,3} \cup C_n$, $l = 4$, $n \geq 4$, and $\chi'_L(H) = 4$, then $x \leq 6$ and $y \leq 4$.

**Corollary 2.8.** Let $H = xP_l \cup yK_{1,4} \cup C_n$, $l = 5$, $n \geq 5$, and $\chi'_L(H) = 5$, then $x \leq 30$ and $y \leq 5$.

For next corollary, we will study about the locating-chromatic number of a disconnected graph that a path $P_l$, star $K_{1,m}$, and cycle $C_n$ as it components.

**Corollary 2.9.** Let $H = P_l \cup K_{1,m} \cup C_n$ and $l, n \geq m + 1$, then $\chi'_L(H) = m + 1$.

Furthermore, we will determine the locating-chromatic number of a disconnected graph that a path $P_l$, star $K_{1,m}$, and double stars $S_{m,n}$ as it components in above theorem.

**Theorem 2.10.** Let $H = P_l \cup K_{1,m} \cup S_{m,n}$, then $\chi'_L(H) = m + 1$ for $l \geq m + 1$ and $m = n$.

**Proof.** By Theorem 2.1, we have $\chi'_L(H) \geq m + 1$. Let $V(H) = \{V(P_l) \cup V(K_{1,m}) \cup V(S_{m,n})\}$ where $V(P_l) = \{a_1, a_2, ..., a_l\}$, $V(K_{1,m}) = \{b_1, b_2, ..., b_m, b_{m+1}\}$, and $V(S_{m,n}) = \{x_1, x_{11}, x_{12}, ..., x_{1m}, x_{21}, x_{22}, ..., x_{2n}\}$. Define a coloring $c: V(H) \to \{1,2, ..., m + 1\}$ such that

\[
c(a_i) = i, \text{ for } 1 \leq i \leq m + 1
\]

\[
c(a_r) = \begin{cases} m, & \text{if } r \in \{m + 2, m + 4, ..., l\}, \\ m + 1, & \text{if } r \in \{m + 3, m + 5, ..., l\}. \end{cases}
\]

\[
c(b_1) = 1,
\]

\[
c(b_i) = i, \text{ for } i \in [2, m + 1].
\]

\[
c(x_1) = 2,
\]

\[
c(x_{1i}) = i, \text{ for } i \in \{1,3,4,5,6, ..., m + 1\}.
\]

\[
c(x_2) = 3,
\]

\[
c(x_{2i}) = i, \text{ for } i \in \{1,2,4,5,6, ..., m + 1\}.
\]
Every color code of vertex in $H$ will be distinguished by dominant vertex see Figure 11. So, $\chi'_L(H) \leq m + 1$. Thus, $\chi'_L(H) = m + 1$.

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