CONVEXITY PROPERTIES OF GRADIENT MAPS

PETER HEINZNER AND PATRICK SCHÜTZDELLER

Abstract. We consider the action of a real reductive group $G$ on a Kähler manifold $Z$ which is the restriction of a holomorphic action of the complexified group $G^C$. We assume that the induced action of a compatible maximal compact subgroup $U$ of $G^C$ on $Z$ is Hamiltonian. We have an associated gradient map $\mu_p: Z \rightarrow p$ where $g = k \oplus p$ is the Cartan decomposition of $g$. For a $G$-stable subset $Y$ of $Z$ we consider convexity properties of the intersection of $\mu_p(Y)$ with a closed Weyl chamber in a maximal abelian subspace $a$ of $p$. Our main result is a Convexity Theorem for real semi-algebraic subsets $Y$ of $Z = P(V)$ where $V$ is a unitary representation of $U$.

1. Introduction

Let $U$ be a compact Lie group and $U^C$ its complexification. Then the map map $U \times iu \rightarrow U^C$, $(u, \xi) \mapsto u \exp \xi$ is a diffeomorphism. A closed subgroup $G$ of $U^C$ with Lie algebra $g$ is said to be compatible if the restriction $K \times p \rightarrow G$ is a diffeomorphism where $K = G \cap U$ and $p = iu \cap g$. In the rest of this paper we fix a compatible $G$ a compact complex manifold $Z$ and a holomorphic action $U^C \times Z \rightarrow Z$. We also assume that there is a $U$-invariant Kähler form $\omega$ and a $U$-equivariant momentum map $\mu: Z \rightarrow u^*$. We fix a $U$-invariant inner product $\langle \cdot, \cdot \rangle$ on $u \cong iu$ and view $\mu$ as a map from $Z$ into $iu$. Since $p \subset iu$ the composition of $\mu$ with the orthogonal projection of $iu$ onto $p$ defines a $K$-equivariant map $\mu_p: Z \rightarrow p$ which we call the $G$-gradient map. Then we have grad $\mu_p^\xi = \xi_Z$ for $\xi \in p$ where grad is computed with respect to the Riemannian structure given by $\omega$, $\mu_p^\xi := \langle \mu_p, \xi \rangle$ and $\xi_Z$ is the vector field induced by the action. For a maximal dimensional Lie subalgebra $a$ of $g$ which is contained in $p$ and a $G$-stable subset $Y$ of $Z$ we have the set $A(Y) := \mu_p(Y) \cap a$. In section 5 we prove the following

Theorem. If $Y$ is closed, then $A(Y)$ is a finite union of convex polytopes. Each of the polytopes is the convex hull of $\mu_p$-images of a fixed points in $Y$ where $A = \exp a$.

Let $a_+$ be a positive Weyl chamber of $a$ and set $A(Y)_+ := A(Y) \cap a_+$. The main result of this paper is the following

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Convexity Theorem. Let \( Z = \mathbb{P}(V) \) be the complex projective space of a unitary representation of \( U \) and \( \omega \) the induced Kähler structure on \( \mathbb{P}(V) \) with standard momentum map. Let \( Y \) be a closed real semi-algebraic subset of \( Z \) whose real algebraic Zariski closure is irreducible. Then \( A(Y)_+ \) is a convex polytope.

Corollary. Let \( Z = U^C/Q \) be a complex flag manifold endowed with a \( U \)-invariant Kähler structure and \( G \) a real form of \( U^C \). Then \( A(G \cdot x)_+ \) is a convex polytope for any \( x \in Z \).

All of the above are well known if \( G \) is a complex subgroup of \( U^C \) and \( Y \) is an irreducible complex analytic subset of \( Z \). In this entirely holomorphic setup the Convexity Theorem holds for any compact Kähler manifold \( Z \) (see e.g. [GS05] for more on the history of the subject). On the other hand very little is known for a non complex group \( G \) and general \( Z \). In this generality convexity of \( A(Y)_+ \) is known only in very special cases. See e.g. [Ko73] and [OS00]

2. Basic properties of the gradient map

As before we assume that \( U^C \) acts holomorphically on a compact Kähler manifold \( Z \) and that the Kähler form \( \omega \) is \( U \)-invariant. It is also assumed that there is a \( U \)-equivariant momentum map \( \mu \) and we denote the associated \( G \)-gradient map by \( \mu_p \). For the convenience of the reader we recall here several known basic facts which will be needed later.

For a subspace \( m \) of \( g \) and \( z \in Z \) let \( m \cdot z := \{ \xi_Z(z) \mid \xi \in m \} \). The following elementary fact is shown in [HSch07].

**Lemma 2.1.** We have \( \ker \mu_p(z) = (p \cdot z)\perp \) for all \( z \in Z \).

Let \( G = K \exp p \) be a compatible closed subgroup of \( U^C \). For \( \beta \in p \) we set \( M_\beta := \mu^{-1}_p(\beta) \subset Z \) and \( M := M_0 \). For \( z \in M \) the isotropy group \( G_z = K_z \exp p_z \) is a compatible subgroup of \( U^C \) ([HSch07, 5.5]). Since the \( G_z \)-representation on \( T_z(Z) \) is completely reducible ([HSch07, 14.9]), there is a \( G_z \)-stable decomposition \( T_z(Z) = g \cdot z \oplus W \). We have the following general Slice Theorem ([HSch07, 14.10, 14.21]):

**Theorem 2.2** (Slice Theorem). Let \( z \in M \). Then there exists a \( G_z \)-stable open neighborhood \( S \) of \( 0 \in W \), a \( G \)-stable open neighborhood \( \Omega \) of \( z \in Z \) and a \( G \)-equivariant diffeomorphism \( \Psi: G \times ^{G_z} S \to \Omega \) where \( \Psi([e,0]) = z \) and \( G \times ^{G_z} S \) denotes the \( G \)-bundle associated with the principal bundle \( G \to G/G_z \).

Actually, we have a Slice Theorem at every \( z \in Z \). Set \( \beta := \mu_p(z) \) and let \( G^\beta = \{ g \in G : \text{Ad}g \cdot \beta = \beta \} \) denote the centralizer of \( \beta \). Then we have a slice for the action of \( G^\beta \), as follows.

The centralizer \( G^\beta \) is a compatible subgroup of \( U^C \) with Cartan decomposition \( G^\beta = K^\beta \exp(p^\beta) \) where \( K^\beta = K \cap G^\beta \) and \( p^\beta = \{ \xi \in p : \text{ad}(\xi)\beta = 0 \} \).
The group $G^\beta$ is also compatible with the Cartan decomposition of the centralizer $(U^C)^\beta = (U^\beta)^C$ and $\beta$ is fixed by the action of $U^\beta$ on $u^\beta$. This implies that the $u^\beta$-component of $\mu$ defines an $U^\beta$-equivariant shifted momentum map $\mu_{u^\beta}: Z \rightarrow u^\beta$, $\mu_{u^\beta}(z) = \mu_{u^\beta}(z) - \beta$. The associated $G^\beta$-gradient map is given by $\mu_{p^\beta}: Z \rightarrow p^\beta$, $\mu_{p^\beta}(z) = \mu_{p^\beta}(z) - \beta$. This shows that the Slice Theorem applies to the action of $G^\beta$ at every point $z \in (\mu_{p^\beta})^{-1}(0) = \mathcal{M}_{p^\beta}(\beta)$. In particular, if $G$ is commutative, then we have a Slice Theorem for $G$ at every point of $Z$.

3. Orbit-type stratification

Let $Y$ be a closed $G$-stable subset of $Z$. Our initial goal is to show that $A(Y) = \mu_p(Y) \cap \mathfrak{a}$ is a finite union of convex polytopes. The proof is completed in section 5. It depends on the orbit type stratification of $Z$ with respect to compatible commutative subgroups of $G$ which we explain now.

Let $\mathfrak{b}$ be a Lie subalgebra of $\mathfrak{g}$ which is contained in $\mathfrak{p}$. Note that $B = \exp \mathfrak{b}$ is compatible with the Cartan decomposition of $U^C$ and that $\exp: \mathfrak{b} \rightarrow B$ is an isomorphism of commutative Lie groups. Let $Z^B$ denote the set of $B$-fixed points in $Z$. Since $B = \exp \mathfrak{b}$ we have $Z^B = Z^\mathfrak{b} = \{x \in Z : \xi_z(x) = 0 \text{ for all } \xi \in \mathfrak{b}\}$. Further let $\mu_\mathfrak{b}: Z \rightarrow \mathfrak{b}$ be the composition of $\mu$ with the orthogonal projection of $u^\beta$ onto $\mathfrak{b}$.

**Lemma 3.1.** For $B = \exp \mathfrak{b}$ we have

1. the set $Z^B$ of $B$-fixed points is a smooth complex submanifold of $Z$ and
2. the $B$-gradient map $\mu_{\mathfrak{b}}|_{Z^B}: Z^B \rightarrow \mathfrak{b}$ is locally constant.

**Proof.** Since $B$ acts on $Z$ by holomorphic transformations the set $Z^B$ is a complex subspace of $Z$. The isotropy representation defines a linear $B$-action on $T_z(Z)$. By the Slice Theorem a $B$-stable open neighborhood of $x$ is $B$-equivariantly diffeomorphic to an open neighborhood of $0$ in $T_z(Z)$. Since the set of fixed points of a linear action is a linear subspace the set $Z^B$ is smooth. This shows (1). The second assertion follows from Lemma 2.1. □

For a connected subgroup $B$ of $A = \exp(\mathfrak{a})$ let $Z^{(B)} := \{z \in Z : A_z = B\}$. The group $B$ is compatible and we have $Z^{(B)} = Z^{(\mathfrak{b})} := \{z \in Z : \mathfrak{a}_z = \mathfrak{b}\}$. A connected component $S$ of $Z^{(B)}$ is called an $A$-stratum of type $A/B$ or alternatively an $\mathfrak{a}$-stratum of type $\mathfrak{a}/\mathfrak{b}$.

**Lemma 3.2.** Let $S$ be an $\mathfrak{a}$-stratum of type $\mathfrak{a}/\mathfrak{b}$, $q \in \mu_\mathfrak{a}(S)$ and $\mathfrak{a}(S) := q + \mathfrak{b}^\perp$. Then we have:

1. $S$ is open in $Z^B$.
2. $\mu_\mathfrak{a}(S)$ is an open subset of $\mathfrak{a}(S)$.
3. $\mu_\mathfrak{a}: S \rightarrow \mathfrak{a}(S)$ is a submersion.

**Proof.** Any $x \in Z^{(B)}$ has an open $A$-stable neighborhood $\Omega$ which is $A$equivariantly diffeomorphic to an $A$-stable neighborhood of $[e, 0]$ in $A \times^B W$.
where $W$ is a $B$-representation space and $e$ is the neutral element in $A$ (Slice Theorem). The $A$-stratum in $A \times B W$ of type $A/B$ is given by $A \times B W_B$ and coincides with the set of $B$-fixed points in $A \times B W$. This shows that $Z(B)$ is open in $Z^B$. In particular $S$ is open in $Z^B$ and we have (1).

With respect to the orthogonal decomposition $a = b \oplus b^\perp$ we have $q = q_b + q_{b\perp} \in \mu_a(S)$ and $\mu_a = \mu_b \oplus \mu_{b\perp}$. We may also replace $Z$ by $Z^B$ and $U^C$ by the analytic Zariski closure of $A$ in $U^C$ without changing our assumptions. With this in mind we have $\ker d\mu_a(x) = (a \cdot x)^\perp$ for all $x \in Z = Z^B$. This implies that $d\mu_{b\perp}(x) : T_{z}(Z(B)) \to b^\perp$ is a bijection for all $x \in Z(B)$. Since $\mu_b : Z^B \to b$ is locally constant on $Z^B$ this shows (3) and also (2). \hfill $\Box$

Let $A^c$ be the analytic Zariski closure of $A$ in $U^C$ and $a^c$ its Lie algebra. The group $A^c$ is a complex reductive compatible subgroup of $U^C$ with maximal compact subgroup $T = A^c \cap U$. We have $A^c = T \exp(it)$ where $t$ denotes the Lie algebra of $T$. In the following $S$ denotes the topological closure of a subset $S$ of $Z$ in $Z$. The same notation is used for a subset of $a$ or more generally for subsets of a given topological space.

**Lemma 3.3.** Let $S$ be an $a$-stratum of type $a/b$ in $Z$. Then

1. $S$ is an $A^c$-stable locally closed complex submanifold of $Z$
2. $\mu_a(S) = \mu_{a^c}(S)$ is a convex polytope.
3. every $y$ in $S$ which is mapped by $\mu_a$ onto a vertex of $\mu_a(S)$ is an $A$-fixed point.

**Proof.** Since $A^c$ is connected and $A_{g,y} = (A^c)_{g,y} \cap A = (A^c)_y \cap A = A_y$ holds for all $g \in A^c$ and $y \in Z$ we have (1).

Note that any $A$-stratum $S$ is an $A^c$-stable Kählerian submanifold of $Z$. Let $\mu : S \to t^*$ denote the momentum map on $Z$ given by restricting $\mu : Z \to u^*$ to $t$. In [HI96] it is shown that $\mu(S)$ is a convex polytope in $t^*$. Equivalently $\mu(S) = \mu_{a}(S)$ is a convex polytope, where $\mu_{a} : Z \to t$ is the $A^c$-gradient map given by $\mu$. Since $\mu_{a}$ is the composition of $\mu_{t}$ and the orthogonal projection of it onto $a$ this shows that $\mu_{a}(S)$ is a convex polytope in $a$. Finally it is shown in [HSS07] that every $y \in S$ whose image is a vertex of $\mu_{a}(S)$ has to be an $A$-fixed point. \hfill $\Box$

For the following Lemma we recall that we assume the $G$-action and therefore also the $A$-action on $Z$ to be effective.

**Lemma 3.4.** (1) There are only finitely many $A$-strata.

2. The $A$-stable subset of $Z$ where $A$ acts freely is the unique open $A$-stratum is given by $S_0 = \{z \in Z : a_z = \{0\}\}$ and is open and dense in $Z$.

3. $Z$ is the disjoint union of $A$-strata.

4. The boundary $\overline{S} \setminus S$ of an $A$-stratum is a finite union of $A$-strata $\tilde{S}$ such that $\dim \tilde{S} < \dim S$ holds.

**Proof.** This follows from compactness of $Z$ and the Slice Theorem. \hfill $\Box$
Lemma 3.5. Let \( S \neq S_0 \) be an \( \mathfrak{a} \)-stratum of type \( \mathfrak{a}/\mathfrak{b} \) and \( y \in S \). Then there are \( \mathfrak{a} \)-strata \( S_j \) of type \( \mathfrak{a}/\mathfrak{a}_j \), \( j = 1, \ldots, r \) such that \( y \in \overline{S_j} \), \( \dim \mathfrak{a}_j = 1 \) and \( \mathfrak{b} = \mathfrak{a}_1 + \cdots + \mathfrak{a}_r \) hold.

Proof. We fix a point \( y \in S \) and apply the Slice Theorem to the \( A \)-action on \( Z \) at \( y \). This means that we find an \( A \)-stable open neighborhood \( \Omega \) of \( y \), a \( B \)-stable neighborhood \( \Omega_W \) of \( 0 \in W \) and an \( A \)-equivariant diffeomorphism \( \Psi: A \times_B \Omega_W \rightarrow \Omega \) such that \( \Psi([e, 0]) = y \). Since the \( A \)-action on \( Z \) is assumed to be effective and since it is real analytic the \( B \)-action on \( W \) is effective. We view \( A \times_B \Omega_W \) as an open subset of \( A \times_B W \) and note that a \( \mathfrak{b} \)-stratum \( S(W) \in W \) of type \( \mathfrak{b}/\mathfrak{c} \) determines uniquely the \( \mathfrak{a} \)-stratum \( A \times_B S(W) \subseteq A \times_B W \) of type \( \mathfrak{a}/\mathfrak{c} \). This implies that we may restrict our attention to the \( \mathfrak{b} \) representation \( W \).

The image of \( B \) in \( \text{GL}(W) \) is real diagonalizable since \( B \) acts on \( T_x(Z) \) by selfadjoint operators ([HSc07]). Let \( W = W_{\chi_0} \oplus \cdots \oplus W_{\chi_r} \) be the isotypical decomposition of \( W \) where for any linear function \( \chi: \mathfrak{b} \rightarrow \mathbb{R} \) we set \( W_{\chi} = \{ w \in W : \xi \cdot w = \chi(\xi) w \text{ for all } \xi \in \mathfrak{b} \} \) and \( \chi_0 \) denotes the zero map. We have \( W_{\chi_0} = W^\mathfrak{b} \). The open \( \mathfrak{b} \)-stratum in \( \mathfrak{b} \) and contains \( W^\mathfrak{b} \times (W_{\chi_1} \setminus \{0\}) \times \cdots \times (W_{\chi_r} \setminus \{0\}) \). Then

a) Any \( \mathfrak{b} \)-stratum has 0 in its closure and

b) if 0 does not lie in the open \( \mathfrak{b} \)-stratum, then there are \( \mathfrak{b} \)-strata \( S_j(W) \)

of type \( \mathfrak{b}/\mathfrak{c}_j \), \( j = 1, \ldots, l \) such that \( \dim \mathfrak{c}_j = 1 \) and \( \mathfrak{b} = \mathfrak{c}_1 \oplus \cdots \oplus \mathfrak{c}_l \).

This follows from the fact that the \( B \)-representation \( W \) is diagonalizable. Since the \( B \)-action on \( W \) is effective the open \( \mathfrak{b} \)-stratum of \( W \) is of type \( \mathfrak{b} \).

\[ \square \]

4. Decomposition of the gradient map image

As in the previous section let \( \mathfrak{a} \) be a linear subspace of \( \mathfrak{p} \) which is a subalgebra of \( \mathfrak{g} \) and \( A = \exp \mathfrak{a} \) the corresponding commutative compatible subgroup of \( G \). Let \( S \) be an \( \mathfrak{a} \)-stratum of type \( \mathfrak{a}/\mathfrak{b} \). We set \( \sigma := \mu_\mathfrak{a}(S) \) and let \( \mathfrak{a}(\sigma) := \mathfrak{a}(\sigma) \) be the unique affine subspace of \( \mathfrak{a} \) which contains \( \mathfrak{a} \) as an open subset (Lemma 3.3). We have \( \mathfrak{a}(S) = \mathfrak{a}(\sigma) = q + \mathfrak{b}^\perp \) for any \( q \in \mu_\mathfrak{a}(S) \) where \( \mathfrak{a} = \mathfrak{b} \oplus \mathfrak{b}^\perp \). Since \( \mathfrak{b} \) only depends on \( \mathfrak{a}(\sigma) \) we will also use the notation \( \mathfrak{a}_\sigma = \mathfrak{b} \). Formulated in more geometric terms \( \mathfrak{a}_\sigma \) is the linear subspaces of \( \mathfrak{a} \) which is perpendicular to the affine linear space \( \mathfrak{a}(\sigma) \) and coincides with the isotropy Lie algebra of any point \( z \in S \).

Let \( \Sigma := \{ \mathfrak{a}(\sigma) : S \text{ is an A-stratum and } \sigma = \mu_\mathfrak{a}(S) \} \) denote the set of all affine subspaces of \( \mathfrak{a} \) obtained in this way. For the open \( A \)-stratum \( S_0 \) we have \( \mathfrak{a} = \mathfrak{a}(\sigma_0) \) and \( \sigma_0 \) is the interior of \( P := \mu_\mathfrak{a}(Z) \). Let \( \Sigma_1 := \{ \sigma \in \Sigma : \text{codim}_\mathfrak{a} \mathfrak{a}(\sigma) = 1 \} \) and \( P_0 := P \setminus \bigcup_{\sigma \in \Sigma_1} P \cap \mathfrak{a}(\sigma) \).

Lemma 4.1. The set \( P_0 \) is open in \( \mathfrak{a} \).

Proof. It is sufficient to show that every face \( F \) of \( P = \mu_\mathfrak{a}(Z) \) of codimension one is contained in \( \mathfrak{a}(\sigma) \) for some \( \sigma \in \Sigma_1 \).
The image \( \mu_a(S) = \sigma \) of any \( a \)-stratum is open in \( a(\sigma) \). If we apply this to the open stratum \( S_0 \) we see that for any face \( F \neq P \) of \( P \) this implies that \( S_0 \cap \mu_a^{-1}(F) = \emptyset \). Since we have only finitely many \( a \)-strata this shows for a face \( F \) with \( \text{codim}_a F = 1 \) that there is an \( a \)-stratum \( S_F \) with \( \sigma_F \in \Sigma_1 \) and that \( \sigma_F \) is open in \( F \). We have \( F \subset a(\sigma_F) \) and therefore \( P \setminus \bigcup_{\sigma_F} a(\sigma_F) \) is open in \( a \) where the union is over all faces of \( P \) which are of codimension one. This implies that \( P_0 \) is open in \( a \).

As in the previous section let \( S_0 \) denote the unique open \( a \)-stratum in \( Z \).

**Lemma 4.2.** We have \( \mu_a^{-1}(P_0) \subset S_0 \) or equivalently \( a_y = \{0\} \) for all \( y \in \mu_a^{-1}(P_0) \).

**Proof.** Assume that there is a \( y \in \mu_a^{-1}(P_0) \) such that \( a_y \neq 0 \). Let \( \tilde{S} \) be the \( a \)-stratum which contains \( y \). Since \( \tilde{S} \) is not the open \( a \)-stratum there is an \( a \)-stratum \( S \) of type \( a/a_1 \) where \( \dim a_1 = \dim a - 1 \) such that \( \tilde{S} \subset \overline{S} \). This shows that \( \mu_a(y) \in \overline{\sigma} \subset a(\sigma) \) for \( \sigma = \mu_a(S) \). Since \( \sigma \in \Sigma_1 \) this contradicts the definition of \( P_0 \). \( \square \)

Let \( C(P_0) \) denote the set of connected components of \( P_0 \). For \( \gamma \in C(P_0) \) let \( P(\gamma) \) be the closure of the connected component \( \gamma \). The set \( P(\gamma) \) is a convex polytope with non-empty interior \( \text{int}_a(P(\gamma)) \) in \( a \). Let \( \mathcal{F}(P_0) := \{ F : F \text{ is a face of } P(\gamma) \text{ where } \gamma \in C(P_0) \} \) be the set of faces which are determined by \( P_0 \). We have \( P_0 = \bigcup_{\gamma \in C(P_0)} \text{int}_a(P(\gamma)) \) and \( P = \bigcup_{\gamma \in C(P_0)} P(\gamma) \).

More importantly every face \( F \in \mathcal{F}(P_0) \) of codimension one is given by \( P(\gamma) \cap a(\sigma) \) for some \( \sigma \in \Sigma_1 \) and \( \gamma \in C(P_0) \).

For a convex polytope \( F \) in \( a \) we introduce the following notation. The affine span of \( F \) is denoted by \( a(F) \) and \( \text{int}(F) = \text{int}_a(F)(F) \) denotes the interior of \( F \) as a subspace of \( a(F) \). The linear subspace of \( a \) which is perpendicular to \( a(F) \) is denoted by \( a_F \). The dimension of \( F \) is denoted by \( \dim F \) as is by definition the dimension of \( a(F) \). Similarly \( \text{codim} F \) means the codimension of \( a(F) \) as a subspace of \( a \) and coincides with the dimension of \( a_F \).

For \( \gamma \in C(P_0) \) let \( \Sigma_1(\gamma) \) denote the set of codimension one faces of \( P(\gamma) \).

**Proposition 4.3.** Let \( \gamma \in C(P_0) \) and let \( F \) be a face of \( P(\gamma) \) of codimension \( k \). Then there are \( \sigma_1, \ldots, \sigma_k \in \Sigma_1(\gamma) \) such that

1. \( F = P(\gamma) \cap a(\sigma_1) \cap \cdots \cap a(\sigma_k) \) and
2. \( a_y \subset a_F = a_{\sigma_1} + \cdots + a_{\sigma_k} \) for all \( q \in \text{int}(F) \) and \( y \in \mu_a^{-1}(q) \).

**Proof.** Property (1) follows from the definition of \( P(\gamma) \).

Let \( q \in \text{int}(F) \) and \( y \in \mu_a^{-1}(q) \). Since \( \sigma_1, \ldots, \sigma_k \in \Sigma_1(\gamma) \) we have \( a(F) = a(\sigma_1) \cap \cdots \cap a(\sigma_k) \). We have to show that \( a_y \subset a_{\sigma_1} + \cdots + a_{\sigma_k} \). Since \( q \in \text{int}(F) \) we have \( \sum_{\sigma \in \Sigma_1, q \in a(\sigma)} a_{\sigma} = a_{\sigma_1} + \cdots + a_{\sigma_k} \). The Slice Theorem implies that \( a_y = \sum_{\sigma \in \Sigma} a_{\sigma} \) for some subset \( \Sigma \subset \Sigma_1 \) (Lemma 3.5). This gives \( a_y \subset a_{\sigma_1} + \cdots + a_{\sigma_k} \). \( \square \)
By the construction of the polytopes $\mathcal{P}(\gamma)$ the set $\mathcal{F}(P_0)$ is closed under intersection and we have the following

**Remark 4.4.** Let $D$ be the finite union of elements in $\mathcal{F}(P_0)$. Then the set of all points $\xi \in D$ such that $D$ is non convex in any neighborhood of $\xi$ is again a finite union of elements in $\mathcal{F}(P_0)$.

5. Semistable points and convexity

In this section we show that convexity of $A(Y)_+$ is closely related to the behavior of semistable points after shifting.

Let $\beta$ be a point in $\mathfrak{p}$. The $U$-orbit $U \cdot \beta \subset \mathfrak{u}$ can be identified with the coadjoint orbit $U \cdot i\beta \subset \mathfrak{u}$ and is a complex flag manifold $O := U^C/Q$, where $Q := \{ g \in U^C \mid \lim_{t \to -\infty} \exp(t\beta) \cdot g \cdot \exp(-t\beta) \in U^C \}$. In particular, this induces a Kähler structure and a holomorphic $U^C$-action on $U \cdot \beta$. We denote this action by $(g, x) \mapsto g \cdot x$. The $G$-gradient map on $U^C \cdot \beta$ is then just given by the projection of $O = U \cdot \beta \subset \mathfrak{u}$ onto $\mathfrak{p}$.

**Proposition 5.1.** ([HSt05]) For $\beta \in \mathfrak{p}$ we have $G \cdot \beta = K \cdot \beta$ in $O$.

The $G$-gradient map $\mu_{p, \beta} : Z \times U^C \cdot \beta \to \mathfrak{p}, (z, \xi) \mapsto \mu_{p}(z) - \pi_p(\xi)$ is called the shifting of $\mu_p$ with respect to $\beta$ where $\pi_p : \mathfrak{u} \to \mathfrak{p}$ denotes the orthogonal projection. In particular, $\beta$ is contained in the image of $\mu_p : Z \to \mathfrak{p}$ if and only if $0$ is contained in the image of $\mu_{p, \beta} : Z \times U^C \cdot \beta \to \mathfrak{p}$. The set of semistable points in $Y \times G \cdot \beta \subset Y \times U^C \cdot \beta$ with respect to the value $\alpha$ is by definition the set

$$S_G(M_{p, \alpha})(Y \times G \cdot \beta) := \{(y, \xi) \in Y \times G \cdot \beta \mid \overline{G \cdot (y, \xi)} \cap (\mu_{p, \beta})^{-1}(\alpha) \neq \emptyset\}$$

for any $\alpha \in \mathfrak{a}$. For $\alpha = 0$ we set $S_G(M_{p, 0})(Y \times G \cdot \beta) = S_G(M_p)(Y \times G \cdot \beta)$. With this notation we have the following.

**Theorem 5.2.** Let $Y$ be a closed $G$-stable subset of $Z$ such that the intersection

$$S_G(M_{p, \alpha_1})(Y \times G \cdot \beta) \cap S_G(M_{p, \alpha_2})(Y \times G \cdot \beta)$$

is nonempty for any $\alpha_j \in A_+(Y)$ and $\beta \in \mathfrak{a}$ with $S_G(M_{p, \alpha_j})(Y \times G \cdot \beta) \neq \emptyset$. Then $A_+(Y)$ is a convex polytope.

For the proof of the theorem we need some preparation.

**Lemma 5.3.** Let $\mathfrak{a}_+$ be a closed Weyl-chamber in $\mathfrak{a}$ and $q, p \in \mathfrak{a}_+$. Then

$$\|k \cdot q - p\|^2 \geq \|q - p\|^2$$

holds for all $k \in K$.

**Proof.** Since the inner product on $\mathfrak{p}$ is $K$-invariant we have

$$\|k \cdot q - p\|^2 - \|q - p\|^2 = -2 \cdot <k \cdot q - q, p> .$$

We have $<k \cdot q, p> = <\pi_{\mathfrak{a}}(k \cdot q), p>$ where $\pi_{\mathfrak{a}}$ is the orthogonal projection of $\mathfrak{p}$ onto $\mathfrak{a}$. But $\pi_{\mathfrak{a}}(k \cdot q)$ is contained in the convex hull of the orbit of the Weyl group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ through $q$ ([Ko73]) and therefore it suffices
Remark 5.4.  

(1) If one reads through our paper in the case that $G = A$, then one obtains a proof of the result we needed from Kostant’s paper [Ko73]. Thus our results are independent of [Ko73].

(2) Kostant’s fundamental paper was the first paper containing convexity results in the spirit presented here.

Proposition 5.5. Let $p_0 \in a$ and assume that $q_0 \in A(Y) := \mu_p(Y) \cap a$ is a minimum of the function $\psi_{p_0} : A(Y) \to \mathbb{R}$, $q \mapsto \|q - p_0\|$. Then

\[(\mu_p|_Y)^{-1}(q_0) \subset Y_\xi := \{ y \in Y \mid \xi_y(y) = 0 \}\]

for $\xi := q_0 - p_0$.

Proof. We claim that $q_0$ is also a minimum of the function $\tilde{\psi}_{p_0} : \mu_p(Y) \to \mathbb{R}$, $q \mapsto \|q - p_0\|$. Let $a_+$ be a Weyl chamber such that $p_0 \in a_+$. Since $\mu_p(Y)$ is $K$-stable we have $\mu_p(Y) = K \cdot A(Y) = K \cdot (A(Y) \cap a_+)$. Let $\tilde{q} \in \mu_p(Y)$ and $k \in K$ such that $\tilde{q} = k \cdot q$ where $q \in a_+$. This implies $\|\tilde{q} - p_0\| \geq \|q - p_0\| \geq \|q_0 - p_0\|$ (Lemma 5.3).

Let $y \in (\mu_p|_Y)^{-1}(q_0)$ and $\xi = q_0 - p_0$. Then $y$ is a critical point of the function $\eta : G \cdot y \to \mathbb{R}$, $g \cdot y \mapsto \frac{1}{2} \|\mu_p(g \cdot y) - q_0\|^2$. Now $0 = d\eta(y) = \langle d\mu_p(y), \mu_p(y) - p_0 \rangle = d\mu(y)$ implies $\xi_y(y) = 0$. □

Proposition 5.6. We have $A(Y) \cap F = F$ for all faces $F \in \mathcal{F}(P_0)$ such that $\text{int}(F) \cap A(Y) \neq \emptyset$.

We will prove the proposition recursively by arguing by dimension of the faces of $\mathcal{F}(P_0)$ and starting with those faces which are of maximal dimension. In order to carry this out we note the following

Lemma 5.7. Let $F^* \in \mathcal{F}(P_0)$. Then $F^* \cap A(Y) = F^*$ implies that $F \cap A(Y) = F$ for all $F \in \mathcal{F}(P_0)$ which are contained in $F^*$.

Proof of Proposition 5.6. Let $F$ be an arbitrary face such that $\text{int}(F) \cap A(Y) \neq \emptyset$. By the above indicated induction we may assume that our Proposition holds for all faces $F^* \in \mathcal{F}(P_0)$ with $\dim F^* > \dim F$. Lemma 5.7 implies that we additionally may assume that $\text{int}(F^*) \cap A(Y) = \emptyset$ for all faces $F^*$ which properly contain our given face $F$. The advantage of this assumption is that for any $q_1 \in \text{int}(F) \cap A(Y)$ we can find a $r > 0$ such that $A(Y) \cap \Delta_r(q_1) = \text{int}(F) \cap A(Y) \cap \Delta_r(q_1)$ holds. For any $p_1 \in \Delta_r(q_1)$ such that $\xi_1 := p_1 - q_1 \in a_F$ is perpendicular to our face $F$ we obtain $\|p_1 - q_1\| \leq \|p_1 - q\|$ for all $q \in \Delta_r(q_1) \cap A(Y)$. Proposition 5.5 shows that
(ξ₁)_{Z}(y) = 0 for all such ξ₁ and y ∈ μ_{a}^{-1}(q₁). Since q₁ ∈ int(F) ∩ A(Y) was arbitrary this shows a_F ⊂ a_y for all y ∈ μ_{a}^{-1}(int(F) ∩ A(Y)). Now Proposition 4.3 implies a_y = a_F for all y ∈ μ_{a}^{-1}(int(F) ∩ A(Y)).

We will now argue that this leads to a contradiction. Assume that there is a q₁ ∈ bd_{int(F)}(int(F) ∩ A(Y)) := (int(F) ∩ A(Y)) \ int_{a(F)}(F ∩ A(Y)) and let r > 0 such that Δ_r(q₁) ∩ F ⊂ int(F) and Δ_r(q₁) ∩ A(Y) = F ∩ Δ_r(q₁) ∩ A(Y) hold. Here we use the assumption that int(F) ∩ A(Y) ≠ ∅ and r > 0 such that Δ_r(q₁) ∩ F ⊂ int(F) and Δ_r(q₁) ∩ A(Y) = ∅ for all faces F∗ which properly contain F. Then there is a p₀ ∈ Δ_r(q₁) ∩ F with p₀ ∉ A(Y) and therefore a q₀ ∈ A(Y) ∩ Δ_r(q₁) with satisfies ||p₀ − q₀|| ≤ ||p₀ − q|| for all q ∈ A(Y) ∩ int(F) ∩ Δ_r(q₁). Since A(Y) ∩ int(F) ∩ Δ_r(q₁) = A(Y) ∩ Δ_r(q₁) we may apply Proposition 5.5. This gives ξ_{Z}(y) = 0 where ξ = p₀ − q₀ and y ∈ μ_{a}^{-1}(q₀). Since ξ ≠ 0 and ξ ∉ a_y this contradicts a_F = a_y.

**Corollary 5.8.** The set A(Y) is a union of faces F ∈ F(P₀) and is therefore a finite union of convex polytopes each of it the convex hull of images of fixed points of T in Y.

For the proof of Theorem 5.2 we also need the the fact that a subset D of an Euclidian vector space which is a finite union of convex polytopes and is not convex has the property that for any sufficiently small r > 0 there exists a point β ∈ a such that the closed ball of radius r and center β meets D in precisely two points α₁ and α₂. This geometric input has also been used in Kirwan’s proof of her convexity result ([Kir84b]).

**Proof of Theorem 5.2.** The set A⁺(Y) is a finite union of convex polytopes (Corollary 5.8). Assume that A⁺(Y) is not convex. Then there exist r > 0 and β ∈ a such that the closed ball of radius r and center β meets A⁺(Y) in precisely two points α₁ and α₂ which are on the boundary of this ball. Now if for α ∈ a the value ||α|| is critical for the function ||µ_{p,β}||² : Z × G • β → ℝ then there is an associated pre-stratum S_{α} = {w ∈ Z × G • β : ||α|| = min{||µ_{p,β}(g • w)|| : g ∈ G}) for the G-action on Z × G • β in the sense of [HSS807]. The values ||α|| are critical points of ||µ_{p,β}||² : Z × G • β → ℝ and define two non empty G-pre-strata for the G-action on Z × G • β. We have S_{α₁} ∩ Y = S_{G}(M_{p,α₁})(Y × G • β). Since α₁ ≠ α₂ and α_j ∈ a⁺ these pre-strata are disjoint. Consequently we have

S_{G}(M_{p,α₁})(Y × G • β) ∩ S_{G}(M_{p,α₂})(Y × G • β) = ∅.

This contradicts the assumption of Theorem 5.2. □

**Remark 5.9.** In the case where G = U^C and Y is an irreducible U^C-stable complex subspace of Z the set Y × U^C • β is a Kählerian space and the pre-strata in the above proof are locally closed complex subspaces of Y × U^C • β. This implies that there is a unique open U^C-stratum which is dense in Z × U^C • β. The above proof then gives Kirwan’s convexity theorem for actions of complex reductive groups on compact U^C-stable irreducible...
complex subspaces of Kähler manifolds. This is a rather special case of the more general convexity result in [HH96].

6. The projective case

We fix now a finite dimensional unitary representation space \( V \) of the compact group \( U \) and consider \( Z = \mathbb{P}(V) \). The action of \( U \) on \( V \) extends to a holomorphic linear action of \( U^\mathbb{C} \) and induces an algebraic \( U^\mathbb{C} \)-action on the associated complex projective space \( \mathbb{P}(V) \). There are \( G \)-gradient maps \( \mu_{p,V}: V \to p, \mu^\xi_{p,V}(v) = \langle \xi, v \rangle \) on \( V \) with respect to the Kähler structure induced by the unitary one on \( V \) and a \( G \)-gradient map \( \mu_{p,\mathbb{P}(V)}: \mathbb{P}(V) \to p, \mu^\xi_{p,\mathbb{P}(V)}([v]) = \frac{\langle \xi, v \rangle}{\|v\|^2} \) on \( \mathbb{P}(V) \) with respect to the induced Fubini-Study Kählerian structure on \( \mathbb{P}(V) \). Here we denote the fixed positive Hermitian structure on \( V \) by \( \langle \, , \, \rangle \) and \([v] \in \mathbb{P}(V)\) denotes the line through \( v \in V \setminus \{0\} \). Note that the Fubini-Study form on \( \mathbb{P}(V) \) is given by symplectic reduction and is up to a positive constant the unique Kähler form on \( \mathbb{P}(V) \) which is invariant with respect to the natural action of the special unitary group \( SU(V) \).

In order to simplicity the notation we set \( \mu_p := \mu_{p,\mathbb{P}(V)} \). We view \( \mathbb{P}(V) \) as a real algebraic variety and fix a \( G \)-stable closed real semialgebraic subset \( Y \) of \( \mathbb{P}(V) \). We say that \( Y \) is irreducible if the real Zariski closure of \( Y \) in \( \mathbb{P}(V) \) is a real irreducible subvariety. Our main result is

**Theorem 6.1.** The set \( A_+(Y) := \mu_p(Y) \cap a_+ \) is a convex polytope.

We have the following consequences which are shown below.

**Corollary 6.2.** Let \( Z = U^\mathbb{C}/Q \) be a complex flag manifold with \( G \)-gradient map \( \mu_p : Z \to p \). Then the sets \( A_+(Z) \) and \( A_+(\mathfrak{g} \cdot \mathbf{x}) \) are convex polytopes.

Using this fact we also have

**Corollary 6.3.** Let \( Z = U^\mathbb{C}/Q \) be a complex flag manifold and assume that \( G \) is a real form of \( U^\mathbb{C} \) which is given as the set of fixed points of an anti-holomorphic involution commuting with the given Cartan involution on \( U^\mathbb{C} \). Let \( \xi \in A_+(Z) \) be the unique closest point to the origin. The set of semistable points \( S_G(\mathcal{M}_p\xi)(Z) := \{ z \in Z \mid \overline{G \cdot z \cap \mu_p^{-1}(\xi) \neq \emptyset} \} \) coincides with the union of all open \( G \)-orbits in \( Z \). Moreover, the closed \( K^\mathbb{C} \)-orbits have the same image under the \( G \)-gradient map \( \mu_p \).

To prove Theorem 6.1 we use the same strategy as in the proof of Theorem 5.2. For this we need the notion of quasi-rational points in \( a \). Let \( \mathfrak{h} \) be a maximal abelian subalgebra of the centralizer \( \mathfrak{z}_I(a) \). Then \( \mathfrak{s}_u := \mathfrak{h} \oplus i\mathfrak{a} \) is maximal torus in \( \mathfrak{u} \). We call a point \( \alpha \in a \simeq i\mathfrak{a} \) quasi-integral if \( \alpha \) is the projection of an integral element \( \alpha' \) in the compact torus \( \mathfrak{s}_u = \mathfrak{h} \oplus i\mathfrak{a} \) onto \( i\mathfrak{a} \). We denote this projection by \( \pi_{i\mathfrak{a}} \). A point \( \beta \in i\mathfrak{a} \) is called quasi-rational if it is a rational multiple of an quasi-integral element \( \alpha \in a \). The following lemma allows us to make a reduction to quasi-rational points.
Lemma 6.4. The set $A(Y)_+$ is a finite union of quasi-rational polytopes, i.e. the polytopes are convex hulls of finitely many quasi-rational points. In particular, the quasi-rational points are dense in $A(Y)_+$. 

Proof. Corollary 5.8 says that $A(Y)_+$ is the intersection of a positive Weyl chamber with a finite union of convex polytopes which are given by the convex hull of images of sets of $A$-fixed points in $Y$. If $v \in Y^A$. Then $v$ is contained in a weight space $V^\chi := \{ v \in V \mid \xi \cdot v = \chi(\xi) \cdot v \ \forall \ \xi \in \mathfrak{a} \}$ of the $\mathfrak{a}$-representation $V$. Here $\chi : \mathfrak{a} \rightarrow \mathbb{R}$ is a linear function with $\chi = i\varphi \cdot |_{\mathfrak{a}}$ for some character $\varphi : SU \rightarrow S^1$. Here $SU$ denotes the maximal torus of $U$ with Lie algebra $\mathfrak{s}_u$ and $S^1$ is the maximal compact subgroup of $\mathbb{C}^*$. For every $\xi \in \mathfrak{a}$ we therefore have $\mu_a([v])(\xi) = \chi(\xi)$. So $\mu_a([v])$ is an quasi-integral element in $\mathfrak{a}$ in the sense of the appendix. Consequently $A(Y)_+$ is a finite union of quasi-rational convex polytopes. □

Any semialgebraic set $Y$ has a finite semialgebraic stratification, i.e. $Y$ can be decomposed into real analytic locally closed semialgebraic submanifolds $A_i$ such that for $A_i \cap A_j \neq \emptyset$ we have $A_j \subset A_i$ and $\dim A_j < \dim A_i$. We choose one decomposition, define $\hat{Y}$ to be the union of the maximal dimensional strata and set $\dim Y := \dim \hat{Y}$. If $Y$ is a real algebraic set this gives the Krull dimension of $Y$. In general we have $\dim Y = \dim \text{cl}(Y)$ where $\text{cl}(Y)$ denotes the real Zariski closure of $Y$. For detailed proofs see e.g. [BR] or [C00].

Let $\hat{Y}$ be a closed $G$-stable irreducible semialgebraic subset of $\mathbb{P}(V)$. The technical part of the proof of Theorem 6.1 is the following

Proposition 6.5. Let $\alpha \in \mathfrak{a}_+$ and $\beta \in \mathfrak{a}$ such that $S_G(M_{p,\alpha})(Y \times G \cdot \beta)$ is non empty. Further assume that $\beta$ is quasi-rational. Then $S_G(M_{p,\alpha})(Y \times G \cdot \beta) \cap (\hat{Y} \times G \cdot \beta)$ is open and dense in $\hat{Y} \times G \cdot \beta$.

Actually the assumption that $\beta$ is quasi-rational is not necessary but simplifies the proof.

Proof of Theorem 6.1. Assume $A_+(Y)$ is non convex. As in the proof of Theorem 5.2 we get points $\alpha_1, \alpha_2 \in \mathfrak{a}_+$ and $\beta \in \mathfrak{a}$ such that $S_G(M_{p,\alpha_1})(Y \times G \cdot \beta) \cap S_G(M_{p,\alpha_2})(Y \times G \cdot \beta) = \emptyset$.

Using Lemma 6.4 and Remark 4.4 one can choose the point $\beta$ to be quasi-rational. See [Kir84b] for the explicit construction. But this contradicts Proposition 6.5 and shows the assertion. □

Proof of Proposition 6.5. Since $\beta$ is quasi-rational, there exists an integral element $\delta' \subset \mathfrak{s}_u$ such that $\beta = \frac{1}{n} \cdot \pi_{ia}(\delta')$ for some $n \in \mathbb{N}$. Therefore we
have a $G$-equivariant diffeomorphism $\varphi : Y \times G \cdot \beta' \to Y' \times G \cdot \delta'$ given by multiplication with $n$ in the second component. Here $\beta' = \frac{1}{n} \cdot \beta'$ and $Y' = Y$ but seen as a subset of $\mathbb{P}(V)$ endowed with the Fubini Study form multiplied with $n$. In particular, we have

$$\mathcal{M}_{p, \alpha}(Y \times G \cdot \beta) = \mathcal{M}_{p, \alpha}(Y \times G \cdot \beta') = \varphi^{-1}(\mathcal{M}_{p, n, \alpha}(Y' \times G \cdot \delta)).$$

Since $\varphi$ is $G$-equivariant, we get the analog equation for the sets of semistable points. Since $Y' \times G \cdot \delta'$ is a closed $G$-stable irreducible semialgebraic set of some projective space it suffices to prove the proposition for $\beta = 0$.

So let $\alpha \in a_+$ such that $\mathcal{M}_{p, \alpha}(Y)$ is non empty. Then there exists a point $y \in Y$ such that $\mu_p(y) = \alpha$. Since the quasi-rational points are dense in $A_+(Y)$ there exists a sequence $(y_n)_{n \in \mathbb{N}}$ such that $\mu_p(y_n) = \alpha_n$ are quasi-rational and $\lim_{n \to \infty} \alpha_n = \alpha$. In particular $\mathcal{M}_{p, \alpha_1}(Y)$ is non empty.

Let assume that the open set $S_G(\mathcal{M}_{p, \alpha})(Y) \cap \hat{Y}$ is not dense in $\hat{Y}$. Then there exists a $G$-stable open subset $U$ in the complement and an $r > 0$ such that $\|\mu_p(y') - \alpha\|^2 \geq r$ for all $y' \in U$. Moreover, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\|\mu_p(y') - \alpha_n\|^2 \geq \frac{r}{2}$ for all $y' \in$ the $G$-stable subset $U$. Therefore, for $n \geq N$, the set $S_G(\mathcal{M}_{p, \alpha_n})(Y) \cap \hat{Y}$ is a non empty open subset of $\hat{Y}$ which is not dense. Therefore it suffices to prove the assertion for quasi-rational $\alpha$.

If $\alpha \in a_+$ is a quasi-rational point, i.e. $\alpha = \frac{p}{q} \cdot \gamma \in a_+$ for some quasi-integral element $\gamma$ and some coprime integers $p$ and $q$, it follows from the results given in the second part of the appendix that $Y'_\alpha = \Phi_\alpha(Y \times K \cdot \gamma)$ is a closed $G$-stable irreducible semialgebraic subset of $\mathbb{P}(V_\alpha)$. Here $\gamma'$ again is an integral element in $\mathfrak{a}_q$ over $\gamma$. Note that the projection of $\Phi_\gamma^{-1}(S_G(\mathcal{M}_{p})(Y_\gamma) \cap \hat{Y}_\gamma) \subset Y \times \{v_{\gamma'}\}$ is just $S_G(\mathcal{M}_{p}(\alpha))(Y) \cap \hat{Y}$ where $\hat{Y}_\alpha = \Phi_\alpha(\hat{Y} \times K \cdot \{v_{\gamma'}\})$. Therefore we can also restrict to the case $\alpha = 0$.

We have $\mathcal{M}_p(Y) \neq \emptyset$ and $S_G(\mathcal{M}_p)(Y) = Y \setminus (Y \cap \pi(N_G))$ where $N_G := \{v \in V \mid 0 \in G \cdot v\}$ is the null cone in $V$. Since the null cone is a real algebraic subset of $V$ (Lemma 7.1) and $c_d(Y)$ is irreducible, the intersection $\pi(N_G) \cap c_d(Y)$ is either $c_d(Y)$ or a proper algebraic subset of lower dimension in $c_d(Y)$. Since $\dim \hat{Y} = \dim Y = \dim c_d(Y)$ and $\mathcal{M}_p(Y) \neq \emptyset$ the set $\hat{Y} \cap \pi(N_G)$ is a proper semialgebraic subset of lower dimension in $\hat{Y}$. In particular, its complement $S_G(\mathcal{M}_p(Y)) \cap \hat{Y}$ is open and dense in $\hat{Y}$ (see e.g. [BR] for basic properties of semialgebraic sets).

We now prove the two corollaries 6.2 and 6.3 about complex flag manifolds. Note first that every complex flag manifold $Z = U^C/Q$ can be identified with an orbit $U \cdot \beta$ where $\beta$ is contained in the cone $C_Q := \{\lambda \in a_+ \mid \lambda = \sum c_j \cdot \alpha_j, c_j \in \mathbb{R}^+ \}$ where $\Pi'$ is the subset of the simple roots which define $Q$. Using the notation of the second part of the appendix we have the following
Corollary 6.6. Let \( Z = U^C/Q \) be a complex flag manifold and let \( Y \) be a closed \( G \)-stable subset such that \( \varphi_\beta(Y) \) is an irreducible semialgebraic subset of \( \mathbb{P}(\mathbb{T}_\beta) \) for every integral \( \beta \) in the cone \( C_Q \). Then \( A_+(Y) \) is a convex polytope for every \( G \)-gradient map \( \mu_p: Z \to p \) on \( Z \).

Proof. Every momentum map on the complex flag manifold \( Z \) with respect to the \( U \)-action is of the form \( \mu: Z \to u, x \mapsto \alpha_x \), where \( \alpha \) is an element in the cone \( C_Q \). By Theorem 6.1, the lemma holds if \( \alpha \) is an integral element in \( C_Q \). If \( \alpha \) is a rational point, it can be written in the form \( \alpha = \frac{1}{n} \cdot \beta \) for some integral element \( \beta \) in \( s_u \) and some \( n \in \mathbb{N} \). In particular, \( \mu_p(Y) = \frac{1}{n} \cdot \tilde{\mu}_p(Y) \), where \( \tilde{\mu} \) is given by \( \tilde{\mu}(x) = \beta_x \). This proves the rational case. Since the rational points are dense in each cone \( C_Q \), we can construct a sequence \( (\alpha_n)_{n \in \mathbb{N}} \) of rational points \( \alpha_n \in s_u \) such that \( Q_-(\alpha_n) \) coincides with \( Q \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \alpha_n = \alpha \). This gives a sequence of momentum maps \( \left( \mu^{(n)} \right)_{n \in \mathbb{N}} \) such that the assertion holds for every element of this sequence. This gives the corollary for the limit point \( \mu \).

In particular, the above corollary can be applied to the complex flag manifold \( Z \) itself and closures of \( G \)-orbits in \( Z \) which gives Corollary 6.2.

Proof of Corollary 6.3. Since \( \xi \) is the unique closest point to the origin in \( A_+(Z) \), a point \( x_0 \in (\mu_p)^{-1}(\xi) \) is a global minimum of \( \eta_p \). Therefore, the \( G \)-orbit through \( x_0 \) is open and the \( K^C \)-orbit through \( x_0 \) is closed and coincides with the \( K \)-orbit through \( x_0 \) (see \([BL02]\) and \([MUV92]\)). So we need to show that all open \( G \)-orbits are contained in the set \( S_G(M_p(\xi)) \). By Proposition 6.5, we know that \( S_G(M_p(\xi)) \) is open and dense in \( Z \) in the integral case. This can be extended to the non integral case as in the proof of Proposition 6.5. Consequently, the set \( S_G(M_p(\xi)) \) contains all open \( G \)-orbits and the closed \( K^C \)-orbits in these open \( G \)-orbits are contained in \( (\mu_p)^{-1}(K: \xi) \).

7. Appendix

7.1. Algebraicity of the null cone. In the proof of Proposition 6.5 we use the algebraicity of the null cone \( \mathcal{N}_G = \{ v \in V \mid 0 \in G \cdot v \} \). Here we give a proof of this fact.

Lemma 7.1. The null cone \( \mathcal{N}_G \) is an algebraic subset of \( V \).

Proof. Using the decomposition of \( G \) into its semisimple part \( G_s \) and its center \( Z(G) \) (see \([HS05]\)) we first prove the algebraicity of the null cone \( \mathcal{N}_{G_s} \) with respect to \( G_s \) by using results of \([B]\) and \([RS]\). All of them appear as special cases in \([HSch07]\).

Let \( G_s^C \) be the complexification of \( G_s \) and let \( V^C \) the corresponding complexified representation space. Further let \( F_1, ..., F_d \) be the generators of the algebra \( \mathbb{C}[V^{C}]^{G_s^C} \) of \( G_s^C \)-invariant polynomials on \( V^C \). Then the map \( F = (F_1, ..., F_d): V^C \to \mathbb{C}^d \), parameterizes the Zariski closed \( G_s^C \)-orbits in
$V^C$. Without loosing generality we can assume that $F_j(0) = 0$ for all generators $F_j$.

By a result of [B], for every $v \in V$ the orbit $G_s^C \cdot v$ is Zariski closed in $V^C$ if $G_s \cdot v$ is closed in $V$. Therefore, for every closed orbit $G_s \cdot v$, $v \neq 0$, there exists a function $F_j$ from the list of generators of the algebra $\mathbb{C}[V^C]^G_s^C$ such that $0 \neq F_j|_{G_s \cdot v}$. Since $G_s$ is Zariski dense in $G^C_s$, we can assume that the polynomials $F_j$ are extensions of real polynomials $f_1, \ldots, f_d$ which generate the algebra $\mathbb{R}[V]^G_s$. In particular, we have $0 \neq f_j|_{G_s \cdot v}$ which shows that the null cone is given as the real algebraic subset \{ $v \in V \mid f_j(v) = 0$, $j = 1, \ldots, d$ \} of $V$.

To prove the general case let $V_j$ denote the weight spaces of $Z(G)$ in $V$. Since $G_s$ and $Z(G)$ commute, these subspaces are stable under $G_s$. In particular, we have $\mathbb{R}[V]^G_s = \bigotimes \mathbb{R}[V_j]^G_s$. Each factor $\mathbb{R}[V_j]^G_s$ has finitely many generators which can be chosen to be homogeneous polynomials. Let $f : V \to \mathbb{R}^k$ be the polynomial map which is given by all these generators. Then $f$ is invariant with respect to $G_s$ and equivariant with respect to $Z(G)$. Here the action of $Z(G)$ on $\mathbb{R}^k$ is given by the action on each generator. The corresponding null cone of $Z(G)$ in $\mathbb{R}^k$ is a finite union of linear subspaces $H_j \subset \mathbb{R}^k$ (see [HSch07] Corollary 15.5). Therefore, the preimage of this null cone under $f$ is an algebraic subset of $V$ which we call $\mathcal{N}'$. We show that the null cone $\mathcal{N}_G$ coincides with this algebraic set $\mathcal{N}'$.

Let $G_U$ denote the maximal compact subgroup of $G_s^C$ and let $H$ be a $G_U$-invariant positive definite Hermitian form on $V^C$ such that the alternating part vanishes on $V$. Such a form exists by [RS] and we get a momentum map $\mu_{G_U} : V^C \to g_U^*$ for the $G_U$ action on $V^C$. By construction, $\tilde{\mathcal{M}}_p = \mathcal{M} \cap V$ where $\mathcal{M} := (\mu_{G_U})^{-1}(0)$.

We have $\mathcal{N}_G \subset \mathcal{N}'$. For the opposite inclusion let $v \in \mathcal{N}'$. By definition, there exists a sequence $(g_n)_{n \in \mathbb{N}} \subset Z(G)$ such that $\lim_{n \to \infty} g_n \cdot f(v) = 0$. For every $g_n \cdot f(v) \in \mathbb{R}^k$ let $\alpha_n$ be a point in a closed $G_s$-orbit in $f^{-1}(g_n \cdot f(v))$. By [RS] every closed $G_s$-orbit intersects the set $\tilde{\mathcal{M}}_p$ and we may choose $\alpha_n \in \tilde{\mathcal{M}}_p$. Let $F$ be the complex extension of $f$ to $V^C$. The map $f$ is a proper map when restricted to $\mathcal{M}$ (see [RS]). This is also true for the restriction of $F$ to $\tilde{\mathcal{M}}_p$ since $\tilde{\mathcal{M}}_p = \mathcal{M} \cap V$. Consequently the sequence $(\alpha_n)_{n \in \mathbb{N}}$ has a convergent subsequence with limit point $\alpha \in \tilde{\mathcal{M}}_p$. But $F(\alpha) = 0$ which implies $\alpha \in \mathcal{N}_G$ and consequently $v \in \mathcal{N}_G$.

7.2. Shifting with respect to quasi-rational points. Since there is in general no symplectic embedding of the orbit $U \cdot \beta \subset \mathbb{P}(W)$, equipped with the Fubini-Study metric coming from a $U$-invariant Hermitian form on $W$, the set $Y \times G \cdot \beta$ is in general not contained in the class of examples we are considering in section 6 of this paper. Therefore, we have to introduce a slight modification of the shifting procedure.
Given an integral element \( \alpha' \in \mathfrak{s}_u \subset \mathfrak{s} \) we get an associated character \( \chi_{\alpha'} : Q \rightarrow \mathbb{C}^* \) on the parabolic subgroup and a \( U^\mathbb{C} \)-homogeneous line bundle 

\[
L^\alpha = U^\mathbb{C} \times \chi_{\alpha'}, \quad \mathbb{C} = (U^\mathbb{C} \times \mathbb{C})/Q
\]

over \( U^\mathbb{C}/Q \). Let \( \Gamma_{\alpha'} : = \Gamma(U^\mathbb{C}/Q \cdot L^{\alpha'}) \) denote the space of holomorphic sections of the line bundle \( L^{\alpha'} \) and let \( \Gamma_{\alpha'}^* \) denote its dual space. By the theorem of Borel and Weil, the space \( \Gamma_{\alpha'} \) is an irreducible \( U^\mathbb{C} \)-representation space with highest weight \( \alpha' \) and it follows that the projective space \( \mathbb{P}(\Gamma_{\alpha'}^*) \) contains a unique complex \( U \)-orbit \( U \cdot \{v_{\alpha'}\} \) with \( U \cdot \{v_{\alpha'}\} \simeq U^\mathbb{C}/Q \). In particular this gives an embedding of \( U \cdot \alpha' \) into a projective space. For detailed proofs see e.g. [Akh95] or [Huc01].

Now, let \( \beta \in \mathfrak{a} \) be a quasi-rational point and let \( \alpha \) be the unique minimal quasi-integral element in \( \mathbb{R}^+ \cdot \beta \). Then \( \beta = \frac{p}{q} \alpha \) for coprime natural numbers \( p \) and \( q \). Let \( \alpha' \) be an integral element in \( \mathfrak{s}_u \) with \( \pi_{\mathfrak{a}_0}(\alpha') = \alpha \). The momentum map \( \mu_{\mathbb{P}(\Gamma_{-\alpha'})} \) restricted to the orbit \( U \cdot \{v_{-\alpha}\} \) is given by \( \mu_{\mathbb{P}(\Gamma_{-\alpha'})}([v]) = \mu_{\mathbb{P}(\Gamma_{-\alpha'})}(u \cdot [v_{\alpha}]) = -u \cdot \alpha' = -\alpha_{[v]} \). Since \( \beta \) and \( \alpha \) are related by the fixed numbers \( p \) and \( q \), we can define a unique embedding 

\[
\Phi_\beta : Y \times U \cdot \alpha' \hookrightarrow \mathbb{P}(V) \times \mathbb{P}(\Gamma_{-\alpha'}) \hookrightarrow \mathbb{P}(V^\otimes q \otimes (\Gamma_{-\alpha'})^\otimes p) =: \mathbb{P}(V_\beta)
\]

using the Segre embedding. Define \( Y_\beta := \Phi_\beta(Y \times G \cdot \alpha') \). Since \( G \cdot \alpha' \) is a real algebraic set we may choose \( Y_\beta \) to be \( \Phi_\beta(Y \times G \cdot \alpha') \). We get a \( K \)-equivariant map 

\[
\mu_{\mathbb{P},\beta} : Y_\beta \rightarrow \mathfrak{p}^*, \quad \Phi_\beta((y, \xi)) \mapsto q \cdot \mu_p(y) - p \cdot \xi
\]

which we call the shifting of \( \mu_p \) with respect to the quasi-rational point \( \beta \in \mathfrak{a} \). In particular, this is contained in the class of examples which we consider in section 6.

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**Fakultät und Institut für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum**

*E-mail address*: heinzner@cplx.rub.de

**Institut für Mathematik, Universität Paderborn, D-33095 Paderborn**

*E-mail address*: schuetzd@math.upb.de