Non-Abelian Brill–Noether theory
and Fano 3-folds

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The Jacobian variety of an algebraic curve $C$ is one connected component of the moduli space of line bundles over $C$; the moduli space of vector bundles can be viewed as a non-Abelian generalization of this. Here we fix a line bundle $\xi$, and consider the moduli space of rank 2 vector bundles with fixed determinant:

$$M_C(2, \xi) = \left\{ \text{stable rank 2 vector bundles } E \text{ over } C \mid \bigwedge^2 E \cong \xi \right\} / (\text{isomorphism}).$$

The number $h^0(L)$ of linearly independent section of a line bundle $L$ can be used to define subschemes of $\text{Jac} C$, called the Brill–Noether loci. These have been studied since the 19th century, since they reflect properties of an individual curve that are beyond the control of the Riemann–Roch theorem. In this article, we recall this theory briefly in §2, then generalize it to the moduli spaces $M_C(2, \xi)$, and give applications to Fano manifolds and curves on K3s.

There are various types of generalizations; here we treat Types II and III. For Type I, see, for example, [36], [44].

| algebraic group | moduli space of principal bundles | type | measure | expected codimension |
|-----------------|----------------------------------|------|---------|---------------------|
| $\mathbb{C}^*$  | $\text{Jac} C$                   | classic | $h^0(L)$ | $h^0(L)h^1(L)$     |
| $\text{GL}(r, \mathbb{C})$ | $M_C(r, d)$                          | Type I | $h^0(E)$ | $h^0(E)h^1(E)$     |
| $\text{SL}(2, \mathbb{C})$ | $M_C(2, \xi)$                        | Type II | $\text{hom}(F, E)$ | $\left(\frac{\text{hom}(F, E)}{2}\right)$ |
| $\text{SL}(2, \mathbb{C})$ | $M_C(2, K_C)$                        | Type III | $h^0(E)$ | $\left(\frac{h^0(E) + 1}{2}\right)$ |

The moduli spaces $\text{Jac} C$ and $M_C(2, \xi)$ are Kähler manifolds, but their underlying structures of symplectic manifolds parametrize unitary representations of the fundamental group $\pi_1(C)$. Here $\text{Jac} C$ parametrizes representations in $\text{U}(1)$, and if $\xi$ has even degree then $M_C(2, \xi)$ parametrizes irreducible representations in $\text{SU}(2)$ (see Narasimhan and Seshadri [33]).

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From now on, we let $C$ be a curve of genus $\geq 2$. As is well known, the universal cover of $C$ is the upper half-plane $H = \{\text{Im } z > 0\}$. We fix a subgroup $\Gamma \subset \text{SL}(2, \mathbb{R})$ which maps isomorphically to $\pi_1(C)$ under the natural surjective homomorphism $\text{SL}(2, \mathbb{R}) \to \text{Aut} H$, and consider the space $S_1(\Gamma, \rho)$ of automorphic forms of weight 1 with coefficients in $\rho$. In the classic case, when $d = g - 1$, the Brill–Noether locuses $W_{g-1}^r$ are nothing other than the theta divisor $\Theta \subset \text{Jac} C$ of the Jacobian and its singular sets; these parametrize representations $\rho: \Gamma \to U(1)$ for which $\dim S_1(\Gamma, \rho) \geq r + 1$.

In a similar way, the Type III Brill–Noether locuses

$$M_C(2, K, n) = \{E \mid h^0(E) \geq n + 2\}$$

parametrize irreducible representations $\rho: \Gamma \to \text{SU}(2)$ for which the space

$$S_1(\Gamma, \rho) = \left\{ \begin{pmatrix} f(z) \\ g(z) \end{pmatrix} \middle| \begin{array}{l}
 f(z) \text{ and } g(z) \text{ are holomorphic functions} \\
 \text{on } H, \text{ and for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \\
 f\left(\frac{az+b}{cz+d}\right) = (cz+d)\rho(\gamma) f(z) \\
 g\left(\frac{az+b}{cz+d}\right) = (cz+d)\rho(\gamma) g(z)
\end{array} \right\}$$

of vector valued automorphic forms of weight 1 has dimension $\geq n + 2$.

The study of Brill–Noether locuses in practical cases is an illustration of the power of determinantal formalism for defining equations. §1 recalls the general theory of determinantal locuses. It is interesting to note that the 3 different types of non-Abelian Brill–Noether locuses have different types of determinantal forms. Brill–Noether locuses of Type III are defined by conditions on the dimension of intersection of two Lagrangian subspaces in a symplectic vector space (see §5). Similarly, Brill–Noether locuses of Type II are defined as subschemes as the set of points where a skewsymmetric homomorphism between two dual (or twisted dual) vector bundles drops rank (see §6).

\footnote{This is the same thing as a theta characteristic or spin structure on $C$, that is, the choice of an element in $\frac{1}{2} K_C$.}
One of the original examples inspiring our interest in non-Abelian Brill–Noether locuses was the linear section theorem for a general curve of genus 8:

\[ C = \text{Grass}(2, 6) \cap H_1 \cap H_2 \cap \cdots \cap H_7 \]

(see [21], [22]); to prove this, we had to construct an element \( E \in M_C(2, K) \) such that \( h^0(E) \geq 6 \). The theorem itself has its origins in the classification of Fano 3-folds, and Brill–Noether locuses have many other nice applications to Fano 3-folds. The point is that certain Fano 3-folds are realized as special cases of Brill–Noether locuses of Type III and Type II. Thus for a curve of genus 7, the Brill–Noether locus \( M_C(2, K, 3) = \{ h^0(E) \geq 5 \} \) is a Fano 3-fold of genus 7 (see §8). Similarly, for a given stable vector bundle \( F \) over a curve \( C \) of genus 3, the locus \( M_C(2, K: 3F) = \{ \text{hom}(F, E) \geq 3 \} \) is a Fano 3-fold of genus 9 (see §9). In this sense, the current article follows on from [24].

Non-Abelian Brill–Noether locuses are also closely related to vector bundles over K3 surfaces. It is known that any 2-dimensional component of the moduli space of these is again a K3 surface. The search for this kind of phenomenon over curves is a further inspiration for our study of non-Abelian Brill–Noether locuses. In this vein, we are following on from [20]. For moduli spaces of vector bundles \( E \) over a surface, we obtain varieties of various dimensions by changing the specified value of the second Chern class \( c_2(E) \); for curves, changing the number of global sections \( h^0(E) \) or the number of homomorphisms \( \text{hom}(F, E) \) can be viewed as a substitute for the choice of \( c_2(E) \). In the final section §10, as an application of this idea, we try out a non-Abelian version of the Albanese morphism; when a curve \( C \) is contained in a K3 surface \( S \), we show how to recover \( S \) from \( C \) in terms of double moduli, that is, a twice iterated moduli construction.

**Notation and terminology** We consider algebraic varieties over the complex numbers \( \mathbb{C} \). We write \( V^\vee \) or \( E^\vee \) to denote the dual of a vector space \( V \) or a vector bundle \( E \), and \( \chi(E) = \sum (-1)^i h^i(E) \) for the Euler–Poincaré characteristic.

## 1 Determinantal subschemes

It frequently happens that an algebraic variety can be defined by a system of equations given as the minors of a matrix. The most basic case of this is the degeneracy locus of a homomorphism \( f: E \to F \) of vector bundles over a variety \( X \).

**Definition 1.1** We write \( D_k(f) \) for the set of zeros of the \( k \)th exterior power of \( f \)

\[ \bigwedge^k f: \bigwedge^k E \to \bigwedge^k F; \]

more precisely, \( D_k(f) \) is the zero subscheme of \( \bigwedge^k f \). We also write \( D^k(f) = D_{r-k}(f) \), where \( r = \text{rank } E \).
The homomorphism $f$ is locally expressed as a matrix with entries regular functions on $X$, so that $D_k(f)$ is the locus (subscheme) of common zeros of all the $k \times k$ minors of this matrix. Thus as subsets of $X$, $D_k(f)$ and $D^k(f)$ are equal to
\[
\{ x \in X \mid \text{rank}(f_k : E_x \to F_x) \leq k \} \quad \text{and} \quad \{ x \in X \mid \dim \text{ker}(f_k : E_x \to F_x) \geq k \}.
\]

**Proposition 1.2** (**[7]**) Suppose that $X$ is nonsingular. Then

1. $\dim D_k(f) \geq \dim X - (r-k)(s-k)$, where $r$ and $s$ are the ranks of $E$ and $F$; we assume here that $D_k(f)$ is nonempty.

2. $D_{k-1}(f)$ is contained in the singular locus of $D_k(f)$.

3. Write $\rho$ for the right hand side of the inequality of (1). Then $D_k(f)$ is nonsingular of dimension $\rho$ at a point $x \in D_k(f) \setminus D_{k-1}(f)$ if and only if the linear map $\ker f_x \otimes (\text{coker} f_x)^\vee \to m_x/m_x^2$ corresponding to $x$ is injective.

4. The Giambelli–Thom–Porteous formula: define the Chern polynomial of the homomorphism $f$ by

\[
\sum_{i \geq 0} c_i(f) t^i = \left( \sum_{i \geq 0} c_i(E) t^i \right)^{-1} \left( \sum_{i \geq 0} c_i(F) t^i \right).
\]

where $c_i(f) \in H^{2i}(X)$. Assume that equality holds in the inequality of (1). Then the fundamental cohomology class of $D_k(f)$ is given by the following Schur polynomial in the Chern classes of $f$:

\[
[D_k(f)] = \Delta_{s-k, \ldots, s-k}(c(f)) \in H^{2N-2\rho}(X),
\]

where $N = \dim X$.

Here in (4), the Schur polynomial $\Delta_{\lambda_1, \ldots, \lambda_m}(c)$ is defined quite generally for $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ as the determinant of the matrix having $(i, j)$th entry $c_{\lambda_i-i+j}$. In other words,

\[
\Delta_{\lambda_1, \ldots, \lambda_m}(c) = \det \begin{bmatrix} c_{\lambda_1} & c_{\lambda_1+1} & c_{\lambda_1+2} & \cdots \\ c_{\lambda_2-1} & c_{\lambda_2} & c_{\lambda_2+1} & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ c_{\lambda_m} & & & \ddots \end{bmatrix}.
\]

Assume that $D_{k-1} = \emptyset$, and that the assumption in (3) holds for all $x \in D_k(f)$. In this case, the conormal bundle to $D_k(f)$ in $X$ is isomorphic to

\[
\ker(f_{|D(f)}) \otimes \text{coker}(f_{|D(f)})^\vee.
\]
In cases when the matrix representing a homomorphism of vector bundles is symmetric or skewsymmetric, the dimension of the degeneracy locus is bigger than the $\rho$ in Proposition 1.2 so that (3) and (4) are not meaningful as they stand. However, we can modify them to make them meaningful. Consider first the symmetric case. Perhaps the obvious way to get symmetry is to take a homomorphism $f$ from a vector bundle $E$ to its dual $E^\vee$ satisfying $f^\vee = f$. In this case, the degeneracy locus $D^k(f)$ satisfies the following.

**Proposition 1.4**

1. $\dim D^k(f) \geq \dim X - k(k+1)/2$. Here we assume that $D^k(f)$ is nonempty.
2. $D^{k+1}(f)$ is contained in the singular locus of $D^k(f)$.
3. Write $\sigma$ for the right hand side of the inequality in (1). Then $D^k(f)$ is nonsingular of dimension $\sigma$ at a point $x \in D^k(f) \setminus D^{k+1}(f)$ if and only if the linear map

$$S^2 \ker f_x \to m_x/m_x^2$$

corresponding to $x$ is injective.
4. The Harris–Tu formula [11]: If equality holds in the inequality of (1) then the fundamental class of $D^k(f)$ is given by the Schur polynomial:

$$[D^k(f)] = 2^k \Delta_{k,k-1,\ldots,2,1}(c(E^\vee)) \in H^{2N-k(k+1)}(X),$$

where $N = \dim X$.

A case when the appearance of symmetry is perhaps slightly less obvious is that of a pair of Lagrangian subbundles. As a warming up exercise, consider a finite dimensional vector space $V$ with a skew inner product, that is, a nondegenerate skewsymmetric bilinear form

$$\langle \ , \ \rangle : V \times V \to \mathbb{C}$$

A vector subspace $L \subset V$ with $2 \dim L = \dim V$ such that the skew inner product $\langle \ , \ \rangle$ restricts to zero on $L$ is called a Lagrangian subspace. Note that the quotient space $V/L$ is then dual to $L$ under the skew inner product $\langle \ , \ \rangle$. Given a pair $(L_1, L_2)$ of Lagrangian subspaces, the composites

$$L_1 \hookrightarrow V \twoheadrightarrow V/L_2 \cong L_2^\vee \quad \text{and} \quad L_2 \hookrightarrow V \twoheadrightarrow V/L_1 \cong L_1^\vee$$

provide us with two linear maps. Now clearly, these are dual to one another, and both have the same kernel $L_1 \cap L_2$.

**Example 1.5** If $L_1$ is a vector space and $L_1^\vee$ its dual, we can give the direct sum $V = L_1 \oplus L_1^\vee$ the standard skew inner product

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \text{ where } n = \dim L_1.$$
The two direct summands $L_1$ and $L_1^\vee$ are both Lagrangian subspaces. Now let $L_2$ be an $n$-dimensional subspace of $V$ with $L_1^\vee \cap L_2 = 0$; any such $L_2$ is the graph of a linear map $f: L_1^\vee \to L_1$,

$$\Gamma_f = \{ (f(x), x) \mid x \in L_1^\vee \}.$$

We easily check the following:

1. $L_1 \cap L_2 \cong \ker f$;
2. $L_2 = \Gamma_f$ is a Lagrangian subspace if and only if $f$ is symmetric.

To globalize the above construction, we now consider a vector bundle $V$ together with a skew inner product $\langle , \rangle: V \times V \to \Phi$ with values in a line bundle $\Phi$. A Lagrangian vector subbundle $L \subset V$ is defined as above, and we consider a pair $(L_1, L_2)$ of such.

**Definition 1.6** Consider the composite of inclusion and quotient maps

$$L_1 \hookrightarrow V \to V / L_2.$$

We write $D^k(L_1, L_2)$ for its degeneracy locus; as a set, it is the locus of $x \in X$ such that $\dim L_{1,x} \cap L_{2,x} \geq k$.

**Example 1.7** Consider the direct sum $V$ of a vector bundle $L$ and its dual $L^\vee$, with the standard skew inner product. The graph of a symmetric homomorphism $f: L^\vee \to L$ is a Lagrangian subbundle $\Gamma_f \subset V$, and for every $x \in X$ we have $L_x \cap \Gamma_{f,x} \cong \ker f_x$. Thus $D^k(L, \Gamma_f)$ coincides with $D^k(f)$.

Locally, any pair $(L_1, L_2)$ of Lagrangian vector subbundles can be written in terms of a symmetric homomorphism as in Example 1.7 so that $D^k(L_1, L_2)$ is the set of common zeros of the minors of a symmetric matrix.

**Proposition 1.8**

1. $\dim D^k(L_1, L_2) \geq \dim X - k(k+1)/2$; here we assume that $D^k(L_1, L_2)$ is nonempty.

2. $D^{k+1}(L_1, L_2)$ is contained in the singular locus of $D^k(L_1, L_2)$.

3. Write $\sigma$ for the right hand side of the inequality of (1). Then $D^k(L_1, L_2)$ is nonsingular of dimension $\sigma$ at a point $x \in D^k(L_1, L_2) \setminus D^{k+1}(L_1, L_2)$ if and only if the linear map

$$\Phi_x^{-1} \otimes S^2(L_{1,x} \cap L_{2,x}) \to m_x / m_x^2$$

is injective.
Formulas for the fundamental class of the degeneracy locus $D^k(L_1, L_2)$ have been obtained by Pragacz and Fulton ([38], [39]); the former uses the Schur $Q$-polynomials.

Assume that $D^{k+1}(L_1, L_2) = \emptyset$, and that the assumption in (3) of the Proposition holds for every $x \in D^k(L_1, L_2)$; then, as in (1.3), we get an exact sequence

$$0 \to \Omega_{D(L_1, L_2)} \to \Omega_{X|D(L_1, L_2)} \to \Phi^{-1}|_{D(L_1, L_2)} \otimes S^2(L_1|_{D(L_1, L_2)} \cap L_2|_{D(L_1, L_2)}) \to 0.$$  

We finally consider the skewsymmetric case. For an $n \times n$ skewsymmetric matrix $A = (a_{ij})_{1 \leq i, j \leq n}$, with $a_{ij} + a_{ji} = 0$, it is well known that

$$\det A = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ (\text{Pfaff } A)^2 & \text{if } n \text{ is even.} \end{cases}$$

Here the Pfaffian of $A$ is

$$\text{Pfaff } A = \frac{1}{2^n \cdot n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2n-1)\sigma(2n)}.$$ 

Let $f: E \to E^\vee$ be a skewsymmetric homomorphism of vector bundles, that is, $f^\vee + f = 0$. The usual way of defining the degeneracy locus of $f$ in terms of minors is unsuitable, because it introduces an unwanted nilpotent structure. The right method is to replace the minors by the Pfaffians of principal submatrixes of even order. Note first that the rank of $f_x: E_x \to E_x^\vee$ is automatically even for any $x \in X$. Write $r = \text{rank } E$; then for an integer $\nu \geq 0$ with $\nu \equiv r \mod 2$, we define

$$P^\nu(f) = \{x \in X \mid \text{rank } f_x \leq r - \nu\};$$

$$\text{locus of common zeros of the Pfaffians of}$$

$$(r - \nu) \times (r - \nu)$$

$$\text{principal submatrixes}$$

$$\text{of a skewsymmetric matrix representing } f.$$ 

In this display, the first line defines $P^\nu(f)$ as a point set; the second line only makes sense locally, but provides a system of defining equations for $P^\nu(f)$. Thus putting the two together defines the subscheme structure of $P^\nu(f)$.

**Proposition 1.11**  
(1) If $P^\nu(f)$ is nonempty then

$$\dim P^\nu(f) \geq \dim X - \nu(\nu - 1)/2.$$ 

(2) $P^{\nu+2}(f)$ is contained in the singular locus of $P^\nu(f)$.

(3) Write $\tau$ for the right hand side of the inequality of (1). Then $P^\nu(f)$ is nonsingular of dimension $\tau$ at a point $x \in P^\nu(f) \setminus P^{\nu+2}(f)$ if and only if the linear map

$$\wedge^2 \ker f_x \to m/m_x^2$$

corresponding to $x$ is injective.
(4) The Harris–Tu formula [11]: If equality holds in (1) then the fundamental class of $P^\nu(f)$ is given by

$$[P^\nu(f)] = \Delta_{\nu-1,\nu-2,\ldots,2,1}(c(E^\vee)) \in H^{2N-\nu(\nu-1)}(X),$$

where $N = \dim X$.

**Remark 1.12** For a line bundle $\Phi$, we say that a homomorphism $f: E \to E^\vee \otimes \Phi$ is twisted skew-symmetric if $f^\vee \otimes 1 \Phi + f = 0$; then (4) holds on replacing $c(E)$ by $c(E \otimes \sqrt{\Phi}^{-1})$ (the squaring principle of Harris and Tu, see [11]).

Assume that $P^{\nu+2} = \emptyset$ and that the assumption of (3) holds for every $x \in P^\nu(f)$. Then, as in (1.3) and (1.9), we get the exact sequence

$$0 \to \Omega_{P(f)} \to \Omega_X|_{P(f)} \to \Phi^{-1}|_{P(f)} \otimes \Lambda^2 \ker(f|_{P(f)}) \to 0. \quad (1.13)$$

**Remark 1.14** In (4) of Proposition 1.11, the right hand side is a polynomial in the Chern characters $\text{ch}_i(E)$ of $E$, and this formula contains only the Chern characters of odd degree. For example, when $\nu = 3$ or 4, we get

$$\Delta_{2,1}(c(E^\vee)) = - \begin{vmatrix} c_2 & c_3 \\ 1 & c_1 \end{vmatrix} = \frac{c_1^3}{3} - 2\text{ch}_3,$$

$$\Delta_{3,2,1}(c(E^\vee)) = \begin{vmatrix} c_3 & c_4 & c_5 \\ c_1 & c_2 & c_3 \\ 0 & 1 & c_1 \end{vmatrix} = \frac{1}{45} c_1^6 - \frac{1}{3} c_1^3 \text{ch}_3 + 24 c_1 \text{ch}_5 - 4 c_1^2.$$

**Remark 1.15** The 3 types of degeneracy locus in Propositions 1.2, 1.8 and 1.11 correspond to the bounded symmetric domains of Type I$_{r,s}$, Type III (Siegel upper half-space) and Type II.

### 2 Brill–Noether theory

The set $H^1(\mathcal{O}_C^*)$ of isomorphism classes of all holomorphic line bundles over a curve $C$ has a natural complex structure. This is called the *Picard variety* of $C$, denoted by Pic$C$. The degree of a line bundle decomposes Pic$C$ as the disjoint union of its connected components:

$$\text{Pic}^d C = \{ \xi \mid \deg \xi = d \} / \text{isomorphism}.$$  

The short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2\pi \sqrt{-1}} \mathcal{O}_C \xrightarrow{\exp} \mathcal{O}_C^* \xrightarrow{} 1$$

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of sheaves over $C$ induces the long exact cohomology sequence

$$
\cdots \to H^1(C, \mathbb{Z}) \to H^1(O_C) \to H^1(O_C^*) \to H^2(C, \mathbb{Z}) \to 0,
$$

and one sees from this that $\text{Pic}^d C$ is a $g$-dimensional complex torus, where $g$ is the genus of $C$. The classic Brill–Noether locuses for curves are defined inside $\text{Pic}^d C$ by setting

$$
W^r_d = \{ \xi \mid \deg \xi = d \text{ and } h^0(\xi) \geq r + 1 \} \subset \text{Pic}^d C.
$$

The number $h^0(\xi)$ of linearly independent global sections of $\xi$ is an upper-continuous function of $\text{Pic}^d C$, so that the $W^r_d$ are closed subsets of $\text{Pic}^d C$. We leave the definition of the $W^r_d$ as subschemes until later, and list their known properties (compare \[1\]).

(2.1) $\dim W^r_d \geq g - (r + 1)(r - d + g)$. The right hand side is called the Brill–Noether number, and is usually denoted by $\rho$.

(2.2) $W^{r+1}_d$ is contained in the set of singular points of $W^r_d$.

(2.3) Suppose that the line bundle $\xi$ is in $W^r_d \setminus W^{r+1}_d$, in other words, that $H^0(\xi)$ is exactly $(r + 1)$-dimensional. Then $W^r_d$ is nonsingular of dimension $\rho$ at the point $[\xi]$ if and only if the multiplication map (called the Petri map)

$$
H^0(\xi) \otimes H^0(K_C \xi^{-1}) \to H^0(K_C) \cong H^1(O_C)^\vee
$$

is injective.

(2.4) If equality holds in the inequality of (2.1) then the fundamental class of $W^r_d$ is given by

$$
[W^r_d] = \lambda(r, d, g) \Theta^{g - \rho} \in H^{2g - 2\rho}(\text{Pic}^d C).
$$

Here $\Theta$ is the theta divisor, and we set

$$
\lambda(r, d, g) = \prod_{i=0}^{r} \frac{i!}{(g - d + r + i)!}.
$$

(2.5) If the Brill–Noether number $\rho$ is nonnegative then $W^r_d$ is nonempty (see \[13\], \[14\]). Also, if $\rho > 0$ then $W^r_d$ is connected (\[8\]). Conversely, for a general curve $C \in \mathcal{M}_g$, if $\rho < 0$ then $W^r_d = \emptyset$.

We give the point set $W^r_d \subset \text{Pic}^d C$ a subscheme structure as the degeneracy locus of a homomorphism of vector bundles. We choose distinct points $P_1, \ldots, P_N$ on $C$, and define a divisor $D$ as their sum, $D = \sum_{i=1}^{N} P_i$. We choose $N$ sufficiently large so that $H^1(\xi(D)) = 0$ for every $\xi \in \text{Pic}^d C$. Now consider the short exact sequence of sheaves on $C$

$$
0 \to \xi \to \xi(D) \to \bigoplus_{i=1}^{N} \xi(D)|_{P_i} \to 0
$$
and the induced cohomology long exact sequence

$$0 \to H^0(\xi) \to H^0(\xi(D)) \xrightarrow{\alpha_\xi} \bigoplus_{i=1}^N H^0(\xi(D)|_{P_i}) \to H^1(\xi) \to 0.$$  \hspace{1cm} (2.6)

We obviously have \(h^0(\xi) = h^0(\xi(D)) - \text{rank } \alpha_\xi\). Now globalizing this, we get a homomorphism of vector bundles over \(\text{Pic}^d C\) of the form

$$A = \coprod_{\xi \in \text{Pic}^d C} \alpha_\xi: E = \coprod_{\xi \in \text{Pic}^d C} H^0(\xi(D)) \longrightarrow F = \bigoplus_{i=1}^N F_i,$$  \hspace{1cm} (2.7)

where \(F_i = \coprod_{\xi \in \text{Pic}^d C} H^0(\xi(D)|_{P_i})\). To say this in a more precise way, let \(L\) be a Poincaré line bundle. That is, \(L\) is a line bundle on the product \(C \times \text{Pic}^d C\) such that \(L|_{C \times [\xi]} \cong \xi\) for every \(\xi \in \text{Pic}^d C\). Consider the exact sequence of sheaves

$$0 \to L \to L \otimes \mathcal{O}_C \mathcal{O}_C(D) \to L \otimes \mathcal{O}_C \mathcal{O}_D(D) \to 0$$

on the direct product; then taking direct images with respect to the second projection \(\pi: C \times \text{Pic}^d C \to \text{Pic}^d C\) induces the cohomology long exact sequence

$$0 \to \pi_* L \to \pi_*(L \otimes \mathcal{O}_C \mathcal{O}_C(D)) \to \pi_* (L \otimes \mathcal{O}_C \mathcal{O}_D(D)) \to R^1 \pi_* L \to 0;$$

the homomorphism \(A\) appearing here gives the precise definition of (2.7). In fact, \(A\) restricted to the fiber over each point \([\xi]\) \(\in \text{Pic}^d C\) is exactly the \(\alpha_\xi\) of (2.7).

**Definition 2.8** We define the subscheme structure of the Brill-Noether locus \(W^r_d \subset \text{Pic}^d C\) as the degeneracy locus \(D^{r+1}(A)\) of the above homomorphism of vector bundles

$$A: \pi_*(L \otimes \mathcal{O}_C \mathcal{O}_C(D)) \to \pi_*(L \otimes \mathcal{O}_C \mathcal{O}_D(D)).$$

The assertions (2.1), (2.2), (2.3) follow easily from this definition and Proposition 1.3. (But to get \(W^r_d \neq \emptyset\) when \(\rho \geq 0\), we also have to prove that \(\lambda(r,d,g) \neq 0\).) By the Grothendieck–Riemann–Roch theorem, one sees that the Chern class polynomial \(\sum_{i \geq 0} c_i(A)t^i\) of the vector bundle homomorphism \(A\) is equal to \(\exp(t\Theta)\); therefore a calculation based on Proposition 1.2, (4) gives that the fundamental class of \(W^r_d\) is

$$[W^r_d] = \Delta_{r-d+g, \ldots, r-d+g} \left( c_i = \Theta^i/i! \right)$$

\[= \lambda(r,d,g)\Theta^{g-\rho} \in H^{2g-2\rho}(\text{Pic}^d C).\]

This is (2.4).
Remark 2.9 The scheme structure of $W_d$ just defined is independent of the choice of the auxiliary divisor $D = \sum_{i=1}^{N} P_i$. In fact, by definition, $W_d$ is the subscheme defined by a Fitting ideal of the cokernel of the homomorphism $A$, which is the first higher direct image sheaf $R^1 \pi_* L$; however, the Fitting ideal of a module is independent of its realization as the cokernel of a homomorphism of vector bundles (see [37]).

3 The moduli space of stable vector bundles

The set of isomorphism classes of all holomorphic vector bundles of rank $r$ over a curve $C$ is equal to the cohomology set $H^1(\text{GL}(r, O_C))$. However, in contrast to the case of line bundles ($r = 1$), before we can give this set a complex structure, we first have to restrict the set of vector bundles under study.

Definition 3.1 (Mumford [27]) A rank 2 vector bundle $E$ over a curve $C$ is stable if

$$\text{deg } \xi < \frac{1}{2} \text{deg } E$$

for every line subbundle $\xi \subset E$. Moreover, if the inequality holds in the weaker form $\leq$, we say that $E$ is semistable.

We consider the moduli space

$$M_C(2, \xi) = \{ \text{stable rank 2 vector bundles } E \text{ over } C \mid \Lambda^2 E \cong \xi \} / \text{(isomorphism)}.$$ 

of stable vector bundles over $C$ with fixed determinant line bundle $\xi$. The first thing to note is that the set of all first order infinitesimal deformations of $E$ is parametrized by the vector space $H^1(\text{End} E)$. Here $\text{End} E$ is the sheaf formed by the (local) endomorphisms of $E$, or in other words, $\text{End} E = E^\vee \otimes E$. However, $\text{End} E$ splits as a direct sum

$$O_C \cdot \text{id} \oplus \mathfrak{s}l E,$$

where the first summand consists of scalar multiplication by functions, and the second of endomorphisms having zero trace. Simple considerations lead to the following facts:

(3.2) Infinitesimal deformations that leave $\text{det } E$ invariant are parametrized by the subspace $H^1(\mathfrak{s}l E)$. Stability of $E$ is an open condition, preserved by small perturbations; thus $H^1(\mathfrak{s}l E)$ is the tangent space to the moduli space $M_C(2, \xi)$ at the point $[E]$. Obstructions to deformations live in the space $H^2(\mathfrak{s}l E)$, which is zero for reasons of dimension. Also, by the Riemann–Roch theorem, $H^1(\mathfrak{s}l E)$ has dimension $3g - 3$, so that we obtain the following result:

(3.3) $M_C(2, \xi)$ is nonsingular and of dimension $3g - 3$.

We now consider the global structure. By Geometric Invariant Theory ([28]), $M_C(2, \xi)$ is a quasiprojective algebraic variety (see [37], [12]). Up to
isomorphism, it only depends on the parity of \( \deg \xi \). For odd degree, \( M_C(2, \xi) \) is projective, and in particular compact; its second cohomology group is isomorphic to \( \mathbb{Z} \), and its first Chern class \( c_1 \) equal to twice the positive generator. Thus \( M_C(2, \xi) \) is a Fano manifold of dimension \( 3g - 3 \) and index 2 (see [40]). For a hyperelliptic curve \( C \), it has the following concrete description.

**Theorem 3.4 (Desale–Ramanan [4])** Let \( C \) be the hyperelliptic curve defined by the equation
\[
y^2 = \prod_{i=1}^{2g+2} (x - \lambda_i).
\]
Then the odd moduli space \( M_C(2, \xi) \) is the closed subset of the Grassmannian variety \( \text{Grass}(g, \mathbb{P}^{2g+1}) \) consisting of \( (g-1) \)-planes of \( \mathbb{P}^{2g+1} \) contained in the complete intersection of two quadrics
\[
V_4 \subset \mathbb{P}^{2g+1}
\]
defined by
\[
V_4 : \left( \sum_{i=1}^{2g+2} X_i^2 = \sum_{i=1}^{2g+2} \lambda_i X_i^2 = 0 \right) \subset \mathbb{P}^{2g+1}.
\]

The even moduli space \( M_C(2, \xi) \), that is, when \( \xi \) has even degree, is not compact. However, it is contained as an open set in a projective algebraic variety \( \overline{M}_C(2, \xi) \), with points of the boundary parametrizing the isomorphism classes of bundles that are direct sums of the form
\[
L \oplus \xi L^{-1} \quad \text{for} \quad L \in \text{Pic}^d C, \quad \text{where} \quad d = \deg \xi/2.
\]

In more rigorous terms, \( \overline{M}_C(2, \xi) \) is the moduli space of so-called \( S \)-equivalence classes of semistable bundles. It is singular along the boundary (the case of genus 2 is an exception); however, the anticanonical line bundle exists, and \( \overline{M}_C(2, \xi) \) is a singular Fano manifold of index 4 (see (4.4) below).

We can construct a natural map from the even moduli space \( M_C(2, \xi) \) to projective space as follows. Write \( d = \deg \xi/2 \), and set
\[
D_E = \{ \eta \mid H^0(E \otimes \eta^{-1}) \neq 0 \} \subset \text{Pic}^{d+1-g} C \quad \text{for} \quad E \in M_C(2, \xi).
\]
This is a codimension 1 subvariety, and as a divisor, it is an element of the linear system \( |2\Theta| \); we thus obtain a morphism
\[
M_C(2, \xi) \to \mathbb{P}^{2g-1} = |2\Theta|, \quad \text{given by} \quad E \mapsto D_E.
\]
This morphism extends naturally to the compactification \( \overline{M}_C(2, \xi) \) (see [30] or [49]).

**Examples 3.5**

1. For a genus 2 curve \( C \), the above morphism is an isomorphism \( \overline{M}_C(2, \xi) \overset{\approx}{\to} \mathbb{P}^3 \), and the boundary of the compactification maps to the Kummer quartic surface associated with the Jacobian variety \( \text{Jac} C \).

2. Consider the above morphism
\[
\overline{M}_C(2, \xi) \to \mathbb{P}^7
\]
for a genus 3 curve $C$. If $C$ is hyperelliptic, then it is a double covering of the quadric $Q^6$. For nonhyperelliptic $C$, that is, when $C$ is a plane quartic, it is an embedding, with image a quartic hypersurface (see [32]). This quartic hypersurface is singular along the Kummer 3-fold of $\text{Jac} C$, and its defining equation can be derived from this fact (see [3]).

We now explain the Hecke correspondence between the even and odd moduli space (compare [31] and [49]). For a rank 2 vector bundle $E$ and a point $p$, suppose that we are given a 1-dimensional subspace of the fiber $E_p$; write $E_p \rightarrow k(p)$ for the linear map with this as kernel. Now consider its composite $E \rightarrow E_p \rightarrow k(p)$

with the natural projection map $E \rightarrow E_p = E/m_pE$. Its kernel $F$ is a vector bundle over $C$; we say that $F$ is obtained by pinching $E$ at $p$. We have $\wedge^2 F \cong (\wedge^2 E)(-p)$, so that $\deg F = \deg E - 1$. Taking the dual of the exact sequence defining $F$

$$0 \rightarrow F \rightarrow E \rightarrow k(p) \rightarrow 0$$

gives an exact sequence

$$0 \rightarrow E^\vee \rightarrow F^\vee \rightarrow k(p) \rightarrow 0.$$
obtained in this way is called the Hecke correspondence. In what follows, we refer to $Z$ as the Hecke graph; it can alternatively be viewed as a moduli space of stable parabolic bundles (compare [11], [15]).

(3.7) Over the open subset $M_C(2, \xi(-p))$, every fiber of $\varphi$ is isomorphic to the projective line $\mathbb{P}^1$. (Over boundary points, the behaviour of $\varphi$ provides interesting examples of extremal rays.)

(3.8) There exists a vector bundle $\mathcal{F}$ over the direct product $C \times Z$ with $\mathcal{F}|_{C \times z} \cong F_z$ for each point $z \in Z$. This is obtained from the universal bundle $\mathcal{E}$ over $C \times M_C(2, \xi(-p))$ by taking the pullback $(1 \times \pi)^* \mathcal{E}$ and then pinching it as follows: because $Z = \mathbb{P}(\mathcal{E}_p)$, it has a tautological line bundle $\mathcal{O}_Z(H)$, together with a tautological surjective sheaf homomorphism $\pi^* \mathcal{E}_p \to \mathcal{O}_Z(H) \to 0$. Noting that $\mathcal{E}_p = \mathcal{E}|_{p \times M}$, we need only take $\mathcal{F}$ to be the kernel of the composite homomorphism

$$(1 \times \pi)^* \mathcal{E} \to \pi^* \mathcal{E}|_{Z \times p} \to \mathcal{O}_Z(H).$$
4 Non-Abelian Brill–Noether locuses of Type III

The even moduli space admits two particular cases, when $\xi$ is equal to the trivial bundle $O_C$ or the canonical line bundle $K_C$.

Theorem 4.1 (Narasimhan–Seshadri [33], Donaldson [6]) For a rank 2 vector bundle $E$ over $C$, the following 2 conditions are equivalent.

1. $E \in \mathcal{M}_C(2, O_C)$, or in other words, $E$ is stable and $\bigwedge^2 E \cong O_C$.

2. $E$ is a flat bundle obtained from an irreducible $SU(2)$ representation of the fundamental group $\pi_1(C)$.

In particular, $\mathcal{M}_C(2, O_C)$ can be identified with the set of equivalence classes of irreducible representations of $\pi_1$ in $SU(2)$.

The other particular case, the moduli space $\mathcal{M}_C(2, K)$ of bundles with the canonical bundle as determinant, parametrizes representations $\rho : \Gamma \to SU(2)$, where $\Gamma \subset SL(2, \mathbb{R})$ is a lift of $\pi_1$ (see the introduction).

Proposition 4.2 The space $H^0(E)$ of global sections of a bundle $E \in \mathcal{M}_C(2, K)$ is isomorphic to the space $S_1(\Gamma, \rho_E)$ of automorphic forms with coefficients in the corresponding representation $\rho_E$.

We use the dimension of this vector space to define the Brill–Noether locuses:

$\mathcal{M}_C(2, K, n) = \{[E] \mid h^0(E) \geq n + 2\} \subset \mathcal{M}_C(2, K)$.  \hfill (4.3)

In the case $n = -1$, we write $\Xi$ for $\mathcal{M}_C(2, K, -1)$. This is a codimension 1 subvariety, and we call the corresponding line bundle $\mathcal{O}_M(\Xi)$ the determinant line bundle of $\mathcal{M}_C(2, K)$. Its fiber at the point $[E]$ is naturally isomorphic to the 1-dimensional vector space

$\det H^0(E) \otimes \det H^1(E)^\vee$.

Moreover, the canonical divisor of $\mathcal{M}_C(2, K)$ is given by

$K_M \sim -4\Xi$, \hfill (4.4)

and this extends as a linear equivalence of Cartier divisors to the compactification $\overline{\mathcal{M}}_C(2, K)$.

We give $\mathcal{M}_C(2, K, n)$ a subscheme structure in essentially the same way as for $W^r_d$. We fix an effective divisor $D$ of degree $\geq g-1$; then for every $E \in \mathcal{M}_C(2, K)$, we have

$H^1(E(D)) = H^0(E(-D)) = 0$.

The short exact sequence of sheaves on $C$

$0 \to E \to E(D) \to E(D)|_D \to 0$ \hfill (4.5)
induces a cohomology long exact sequence
\[
0 \to H^0(E) \to H^0(E(D)) \xrightarrow{\alpha_E} H^0(E(D)\vert_D) \to H^1(E) \to 0, \tag{4.6}
\]
so that \( h^0(E) = 2 \deg D - \text{rank} \alpha_E \). Next, we globalize \( \alpha_E \) to a homomorphism
\[
A: \bigoplus_E H^0(E(D)) \to \bigoplus_E H^0(E(D)\vert_D)
\]
of vector bundles over the moduli space \( \mathcal{M}_C(2, K) \), and define the subscheme structure of \( \mathcal{M}_C(2, K, n) \) as its degeneracy locus \( D^{n+2}(A) \). (We omit the technical question of how to deal with the fact that the universal bundle does not exist.)

Applying Proposition 1.2 to \( D^{n+2}(A) \) gives the estimate: if \( \mathcal{M}_C(2, K, n) \) is nonempty, then
\[
\dim \mathcal{M}_C(2, K, n) \geq 3g - 3 - (n + 2)^2.
\]
However, this is pretty feeble; the right estimate is the following:

**Theorem 4.7**

1. If \( \mathcal{M}_C(2, K, n) \) is nonempty then
   \[
   \dim \mathcal{M}_C(2, K, n) \geq 3g - 3 - (n + 2)(n + 3)/2.
   \]

2. Write \( \sigma \) for the right hand side of this inequality. Moreover, suppose that for some \( E \in \mathcal{M}_C(2, K) \), the space \( H^0(E) \) has dimension exactly \( n + 2 \). Then \( \mathcal{M}_C(2, K, n) \) is nonsingular at \( [E] \) of dimension \( \sigma \) if and only if the linear map defined by multiplication (which we call the Petri map)
   \[
   S^2H^0(E) \to H^0(S^2E) \cong H^1(\mathfrak{sl}E) \cong \mathfrak{sl}E
   \]
is injective (see (3.2) for \( H^1(\mathfrak{sl}E) \)).

To prove the theorem, we need to consider \( \mathcal{M}_C(2, K, n) \) with a different structure. We explain this in the following section. The analog of the classic existence result (2.5) for \( \mathcal{M}_C(2, K, n) \) is an unsolved problem:

**Problem 4.8**

1. Suppose that in Theorem 4.7, (1), the right hand side is \( \geq 0 \); then is it true that \( \mathcal{M}_C(2, K, n) \neq \emptyset \)? (Here we put \( \mathcal{M} \) to allow semistable bundles.)

2. Does equality holds in (1) of the theorem for a general curve \( C \in \mathcal{M}_g \)?

The cases in which \( \mathcal{M}_C(2, K, n) \) is well understood are shown in the following table (compare also [50], [52], [54]). Here the curve \( C \) is assumed to satisfy suitable generality assumptions, which we omit. Case (6) of the table is described
in more detail in §8.

|   | $g(C)$ | $\dim M_{C}(2, K)$ = $3g - 3$ | n+2 | dimension | $M_{C}(2, K, n)$ and its boundary |
|---|---|---|---|---|---|
| (1) | 3 | 6 | 2 | 3 | cone over Veronese $\mathbb{P}^{2}$, the boundary is $\cong C$ |
| (2) | 3 | 6 | 3 | 0 | single point |
| (3) | 4 | 9 | 3 | 3 | singular cubic 3-fold |
| (4) | 5 | 12 | 4 | 2 | $\mathbb{P}^{2} \supset$ quintic curve |
| (5) | 6 | 15 | 5 | 0 | single point |
| (6) | 7 | 18 | 5 | 3 | Fano 3-fold of genus 7 |
| (7) | 8 | 21 | 6 | 0 | single point |
| (8) | 9 | 24 | 6 | 3 | singular quartic 3-fold |
| (9) | 11 | 30 | 7 | 2 | polarized K3 of genus 11 |

5 Lagrangian formalism for Brill–Noether locuses

Let $D$ be an effective divisor of degree $\geq g - 1$. Instead of the exact sequence (4.5), we consider

$$0 \to E(-D) \to E(D) \to A_{E} \to 0.$$  \hfill (5.1)

The cokernel $A_{E}$ is an Artinian sheaf with the same support as $D$, and $V_{E} = H^{0}(A_{E})$ is a vector space of dimension $4 \deg D$. Consider the skewsymmetric bihomomorphism

$$E(D) \times E(D) \to \bigwedge^{2} E(D) \cong K_{C}(2D),$$  \hfill (5.2)

and the induced map

$$A_{E} \times A_{E} \to K(2D)/K.$$  \hfill (5.3)

We take global sections, and compose with the residue map

$$H^{0}(K(2D)/K) \to H^{1}(K_{C}) \cong \mathbb{C}$$

with

$$(\omega_{p})_{p} \mapsto \sum_{p \in \text{Supp}D} \text{Res}_{p} \omega_{p}$$

to obtain a skewsymmetric bilinear map

$$\langle\ ,\ \rangle : V_{E} \times V_{E} \to H^{1}(\bigwedge^{2} E) \cong \mathbb{C},$$

$$(f_{p}, g_{p}) \mapsto \sum_{p \in \text{Supp}D} \text{Res}_{p} f_{p} \wedge g_{p}.$$  \hfill (5.4)

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Now the original pairing (5.2) was nondegenerate, so that \( \langle , \rangle \) is also nondegenerate. Thus \( V_E \) is a symplectic vector space; moreover, it contains two naturally occurring Lagrangian subspaces. One of these is
\[
L^{\text{loc}} := H^0(E/E(-D)),
\tag{5.5}
\]
which is obviously Lagrangian by definition. The other is
\[
L^{\text{rat}} := \text{im} [H^0(E(D)) \to V_E = H^0(E(D)/E(-D))].
\tag{5.6}
\]
First, the Riemann–Roch formula gives
\[
\dim L^{\text{rat}} = h^0(E(D)) - h^0(E(-D)) = \chi(E(D)) = 2 \deg D,
\]
that is, exactly half the dimension of \( V_E \). The fact that the skewsymmetric pairing \( \langle , \rangle \) restricts to zero on \( L^{\text{rat}} \) is a consequence of the following standard result.

**Residue Theorem 5.7** If \( \omega \) is a rational (Abelian) differential on a curve \( C \) then the sum of its residues is zero: \( \sum_p \text{Res}_p \omega = 0 \).

The following statement is obvious:

**Proposition 5.8 ([23])** The image of \( H^0(E) \to V_E \) is exactly the intersection of the two Lagrangian subspaces described above:
\[
\text{im} [H^0(E) \to V_E] = L^{\text{loc}} \cap L^{\text{rat}}.
\]

Now \( L^{\text{loc}} \cap L^{\text{rat}} \) is the kernel of the composite homomorphism
\[
L^{\text{rat}} \to V \to V/L^{\text{loc}} \cong (L^{\text{loc}})^\vee,
\]
but we note that this map is nothing other than \( \alpha_E \) in (4.6).

Globalizing the above argument is routine. We assume for the sake of convenience that the universal bundle exists. Then we can construct a vector bundle \( \mathcal{V} \) over the moduli space \( M_C(2,K) \), together with a skewsymmetric bihomomorphism \( \langle , \rangle : \mathcal{V} \times \mathcal{V} \to \Phi \) and two Lagrangian subbundles \( \mathcal{L}^{\text{loc}}, \mathcal{L}^{\text{rat}} \subseteq \mathcal{V} \), so that at each point \( [E] \in M_C(2,K) \), the fiber is \( V_E \), together with \( L^{\text{loc}}, L^{\text{rat}} \) as above. Taking \( \deg D \) to be sufficiently large, we get the following:

**Proposition 5.9** At each point \( [E] \) of the moduli space \( M_C(2,K) \), we have
\[
H^0(E) \cong L^{\text{loc}}_{[E]} \cap L^{\text{rat}}_{[E]};
\]
therefore the Brill–Noether locus \( M_C(2,K,n) \) coincides with the degeneracy locus \( D^{n+2}(L^{\text{loc}}, L^{\text{rat}}) \) of the pair \( (L^{\text{loc}}, L^{\text{rat}}) \) of Lagrangian subbundles of \( \mathcal{V} \).

Thus Theorem 4.7 follows from Proposition 1.8. Moreover, from the exact sequence (1.9) and a calculation, we get the following result.
Proposition 5.10 If \( n \) is odd then \( M_C(2, K, n + 1) = \emptyset \). Assume in addition that the Petri map \( S^2 H^0(E) \to H^0(S^2E) \) is injective for every \( E \in M_C(2, K, n) \). Then the canonical bundle of \( M_C(2, K, n) \) is isomorphic to the restriction of \( O_M((n - 5)/2) \); here \( O_M(1) \) is the determinant line bundle of the moduli space \( M_C(2, K) \).

6 Non-Abelian Brill–Noether locuses of Type II

We say that a pair \((F, E)\) of rank 2 vector bundles over a curve \( C \) has canonical difference if \( \bigwedge^2 F \otimes K_C \cong \bigwedge^2 E \). We want to consider Brill–Noether locuses for such pairs, defined in terms of the space of homomorphisms \( \text{Hom}(F, E) \). In this vein, we could of course allow both of \( F \) and \( E \) to vary, but here we work with a fixed \( F \). To start with, we write the dual bundle \( F^{\vee} \) any-old-how as an extension of two line bundles, that is, we fix an exact sequence

\[
0 \to M \to F^{\vee} \to N \to 0.
\] (6.1)

Now tensoring this with \( E \) and taking global sections gives a long exact sequence

\[
0 \to H^0(M \otimes E) \to \text{Hom}(F, E) \to H^0(N \otimes E)
\]

\[
\delta_E \to H^1(M \otimes E) \to \text{Ext}^1(F, E) \to H^1(N \otimes E) \to 0.
\] (6.2)

The natural bihomomorphism

\[
(N \otimes E) \times (M \otimes E) \to N \otimes M \otimes \bigwedge^2 E \cong K_C
\]

induces a cup product

\[
\langle \ , \ \rangle : H^0(N \otimes E) \times H^1(M \otimes E) \to H^1(K_C) \cong \mathbb{C},
\]

which is nondegenerate by Serre duality. On the other hand, a simple calculation with Čech cocycles shows that

\[
\langle s, \delta_E(s) \rangle = 0 \quad \text{for all } s \in H^0(N \otimes E).
\]

Thus the connecting homomorphism \( \delta_E \) in (6.2) is skewsymmetric. In particular, rank \( \delta_E \) is even, so that we obtain the following.

Proposition 6.3 Let \((F, E)\) be a pair of rank 2 vector bundles over a curve \( C \) with canonical difference; then

\[
\dim \text{Hom}(F, E) \equiv \deg F \mod 2.
\]

Now, for any integer \( \nu \geq 0 \) with \( \nu \equiv \deg F \mod 2 \), we define a subset of \( M_C(2, \bigwedge^2 F \otimes K_C) \) by

\[
M_C(2, K: \nu F) := \{ E \mid \dim \text{Hom}(F, E) \geq \nu \}.
\]
We can put a subscheme structure on $M_C(2, K: \nu F)$ by using the globalized form of (6.2). For convenience, we explain this under the pretense that there is a universal bundle $E$ over $C \times M_C(2, \Lambda^2 F \otimes K_C)$, although of course no such thing exists when $\deg F$ is even. Then pulling back the exact sequence (6.1) to the direct product and tensoring with $E$ gives a sequence

$$0 \to \mathcal{E} \otimes_{\mathcal{O}_C} M \to \mathcal{E} \otimes_{\mathcal{O}_C} F^\vee \to \mathcal{E} \otimes_{\mathcal{O}_C} N \to 0$$

over $C \times M_C(2, \Lambda^2 F \otimes K_C)$. Taking direct images by the second projection $\pi$ gives

$$0 \to \pi_* (\mathcal{E} \otimes_{\mathcal{O}_C} M) \to \pi_* (\mathcal{E} \otimes_{\mathcal{O}_C} F^\vee) \to \pi_* (\mathcal{E} \otimes_{\mathcal{O}_C} N) \to 0.$$

(6.4)

The connecting homomorphism $\Delta$ restricted to every fiber equals the $\delta_E$ of (6.2), which is skewsymmetric with respect to Serre duality. Twisting by a suitable line bundle $\Phi$, we can make $\Delta$ itself skewsymmetric with respect to Serre duality (relative to $\pi$):

$$R^1 \pi_* (\mathcal{E} \otimes_{\mathcal{O}_C} M) \cong \pi_* (\mathcal{E} \otimes_{\mathcal{O}_C} N)^\vee \otimes \Phi;$$

come the start of §7 for the twist $\Phi$.

**Definition 6.5** We write $M_C(2, K: \nu F) \subset M_C(2, \Lambda^2 F \otimes K_C)$ for the subscheme defined as the Pfaffian locus $P^\nu(\Delta)$ (see (1.10)) of the twisted skewsymmetric vector bundle homomorphism $\Delta: \pi_* (\mathcal{E} \otimes_{\mathcal{O}_C} N) \to R^1 \pi_* (\mathcal{E} \otimes_{\mathcal{O}_C} M)$, and call it a non-Abelian Brill–Noether locus of Type II.

**Remark 6.6** To prove that this definition is independent of the choice of the extension (6.1) seems not to be all that easy, but it can be proved using a Lagrangian formalism for inner product bundles.

By applying Proposition 1.11 to the above definition, we get the following result.

**Theorem 6.7** (1) If $M_C(2, K: \nu F) \neq \emptyset$ then

$$\dim M_C(2, K: \nu F) \geq 3g - 3 - \nu(\nu - 1)/2.$$

(2) Write $\tau$ for the right hand side of this inequality. Assume also that $E \in M_C(2, \Lambda^2 F \otimes K_C)$ is such that $\text{Hom}(F, E)$ is exactly $\nu$-dimensional. Then $M_C(2, K: \nu F)$ is nonsingular of dimension $\tau$ at $[E]$ if and only if the linear map (called the Petri map)

$$\Lambda^2 \text{Hom}(F, E) \to \text{Hom}(\Lambda^2 F, S^2 E)$$

is injective.
Moreover, from the exact sequence (1.13), we get the following result.

**Proposition 6.8** Suppose that $\nu$ is odd, and $M_C(2, K; (\nu + 2)F) = \emptyset$; suppose also that the Petri map is injective for all $E \in M_C(2, K; \nu F)$. Then the canonical class of $M_C(2, K; \nu F)$ is isomorphic to the restriction of $\mathcal{O}_M((\nu - 5)/2)$. Here $\mathcal{O}_M(1)$ is the positive generator of the Picard group of the moduli space.

7 Computing the fundamental class of a non-Abelian Brill–Noether locus

We want to generalize to $M_C(2, K, n)$ and $M_C(2, K; \nu F)$ the formula (2.4) for the fundamental class of the locus $W_{r \delta}$ of special line bundles. For this, we explain some facts we need concerning the cohomology of the moduli space $M_C(2, \xi)$. What follows is taken from Newstead [34].

First consider the odd moduli space $M_C(2, \xi)$, which is compact, and for which the universal bundle $E$ exists. For each point $[E] \in M_C(2, \xi)$ we have $\bigwedge^2 \xi \cong \bigwedge^2 E \cong \xi$, so that $\bigwedge^2 E = \Phi \boxtimes \xi$ for some line bundle $\Phi$. Write $\phi$ for the first Chern class of $\Phi$. Write out the Chern classes of $E$ separated into their Künneth components:

$$c_1(E) = \phi + df \in H^2(M) \oplus H^2(C);$$
$$c_2(E) = \chi + \psi + \omega \otimes f$$

for some $\chi \in H^3(M)$, $\psi \in H^3(M) \otimes H^1(C)$ and $\omega \in H^2(M)$, where $d = \text{deg} \xi$ and $f \in H^2(C)$ is the fundamental class. The only ambiguity in the choice of the universal bundle $E$ is tensoring with a line bundle on $M_C(2, \xi)$, so that the 2 cohomology classes

$$\alpha = 2\omega - d\phi \in H^2(M);$$
$$\beta = \phi^2 - 4\chi \in H^4(M)$$

are independent of the choice of $E$. Also, $\psi$ is not itself a cohomology class of $M_C(2, \xi)$, but setting

$$\psi^2 = \gamma \otimes f$$

gives a well-defined cohomology class $\gamma \in H^6(M)$. For our purposes in what follows, the 3 classes $\alpha, \beta, \gamma$ are all we need. The moduli space has complex dimension $3g - 3$, so that the intersection number $(\alpha^m \cdot \beta^n \cdot \gamma^p)$ is defined when $m + 2n + 3p = 3g - 3$. These numbers have been computed explicitly, to give the following result.

**Theorem 7.1 (Thaddeus [43], Zagier [48])**

$$(\alpha^m \cdot \beta^n \cdot \gamma^p) = (-1)^{p-g} \frac{g! m!}{(g - p)! q!} 2^{2g - 2 - p} (2^q - 2) B_q,$$
where \( q = m + p + 1 - g \) and \( B_q \) is the \( q \)th Bernoulli number, defined by
\[
\frac{x}{e^x - 1} = 1 - \frac{1}{2} x + \sum_{q \text{ even}} B_q \frac{x^q}{q!}.
\]
We set \( B_0 = 1 \) and \( B_q = 0 \) for \( q < 0 \).

The cohomology class \( \alpha \) is the first Chern class of the positive generator of the Picard group of \( \text{Pic}_{\text{M}}(2, \xi) \); thus when \( n = p = 0 \), the number
\[
(\alpha^{3g-3}) = (-1)^g \frac{(3g-3)!}{(2g-2)!} 2^{2g-2} (2^{2g-2} - 2) B_{2g-2}
\]
is the degree of the odd moduli space.

We now compute the fundamental class of a Brill–Noether locus of Type II, under the assumption that it has the expected codimension. The whole point is the connecting homomorphism \((\ref{eq:6.4})\)
\[
\Delta : \pi_*(E \otimes O_N) \to R^1 \pi_*(E \otimes O_M).
\]
This vector bundle homomorphism is skewsymmetric under a nondegenerate pairing
\[
\pi_*(E \otimes O_N) \times R^1 \pi_*(E \otimes O_M) \to \Phi
\]
with values in the line bundle \( \Phi \) defined above. We can determine the Chern characters of the direct image by applying the Grothendieck–Riemann–Roch formula to the virtual vector bundle \( E \otimes O_M \sqrt{\Phi}^{-1} \), obtaining
\[
\text{ch}_{2n-1}(\pi_*(E \otimes O_N) \otimes \sqrt{\Phi}^{-1}) = \left\{ \begin{array}{ll}
-\alpha/2 & \text{for } n = 1, \\
\left( \frac{\beta}{4} \right)^{n-2} \left( \frac{-\alpha\beta}{8} + \frac{n-1}{2} \gamma \right) / (2n-1)! & \text{for } n \geq 2.
\end{array} \right. \tag{7.2}
\]

The evenly numbered Chern characters can also be calculated, but are not required because of Remark \((\ref{rem:1.14})\). By Proposition \((\ref{prop:1.11})\), \((\ref{eq:4})\) we get the following result.

**Proposition 7.3** Suppose that the Type II Brill–Noether locus \( \text{Pic}_{\text{M}}(2, K: 3F) \) has codimension 3 in \( \text{Pic}_{\text{M}}(2, K_{\text{C}} \wedge^2 F) \). Then its fundamental class equals
\[
(\alpha^3 - \alpha\beta + 4\gamma)/24.
\]

**Example 7.4** If the genus of \( C \) is 2, 3, 4, 5, then under the assumption of Proposition \((\ref{prop:7.3})\), \( \text{Pic}_{\text{M}}(2, K: 3F) \) has degree (relative to \( \alpha \)) equal to 1, 16, 2544, 1231616.
Next, we consider the even moduli space $M_C(2, \xi(-p))$. Rather than the Brill–Noether locus inside this, we take its pullback to the Hecke graph $Z$ of (3.6), and determine the fundamental class of this pullback. By definition, we have $c_1(E_p) = \alpha$ and $c_2(E_p) = (\alpha^2 - \beta)/4$. Thus writing $H$ for the tautological divisor of $Z = \mathbb{P}(E_p)$, we find that $H^2 = \alpha H - (\alpha^2 - \beta)/4$. (We continue to write $\alpha, \beta, \gamma$ for their pullbacks to $Z$.) Using this, together with Theorem 7.1 and the recurrence formula for the Bernoulli numbers, we get the following result.

**Proposition 7.5** For $m + 2n + 3p = 3g - 3$, the intersection number in the Hecke graph $Z$ is given by

$$(H \cdot \alpha^m \cdot \beta^n \cdot \gamma^p) = \begin{cases} (-1)^{p+q} \frac{g^{lm} l!}{(g-p)! q!} B_q & \text{if } q \neq 0, \\ (-1)^{p+q+1} \frac{g^{lm} m!}{(g-p)!} & \text{if } q = 0; \end{cases}$$

here $q$ and $B_q$ are as in Theorem 7.1.

By (3.8), we get

$$c_1(F) = \alpha + (d-1)f \in H^2(C \times Z),$$

$$c_2(F) = \chi + \psi + (H + (d-1)\alpha/2) \otimes f \in H^4(C \times Z),$$

for the Chern classes of the universal vector bundle $F$ over $C \times Z$. As in (7.2), we get

$$\text{ch}_{2n-1}(\pi_*(F(-\alpha/2) \otimes \mathcal{O}_C N)) = \begin{cases} -H & \text{for } n = 1, \\ \left(\frac{\beta}{4}\right)^{n-2} \left(-\frac{\beta H}{4} + \frac{n-1}{2} \gamma\right) / (2n-1)! & \text{for } n \geq 2. \end{cases}$$

(7.6)

(Thus in effect, we just substitute $H$ for $\alpha/2$ in (7.2).) Similarly to Proposition 7.3, we get the following result.

**Proposition 7.7** Suppose that the Type II Brill–Noether locus $M_C(2, K: 4F)$ has codimension 6 in $M_C(2, K_C \wedge^2 F)$. Then its pullback to $Z$ has fundamental class equal to

$$h^6 + \frac{h^3 \gamma}{18} - \frac{\gamma^2 + h^4 \beta}{36} - \frac{h \beta \gamma}{45} + \frac{h^2 \beta^2}{180}.$$

The pullback to $Z$ of the linear system $|\alpha|$ on $M_C(2, \xi)$ defines a map from $Z$ to $M_C(2, \xi(-p))$ which is a double cover over the generic point. Thus by Proposition 7.3, one can calculate the degree of the Brill–Noether locus with respect to the determinant line bundle.

**Example 7.8** If the genus of $C$ is 4, 5, 6, then under the assumption of the Proposition 7.7, the degree of $M_C(2, K: 4F)$ equals 6, 256, 28640. 23
The fundamental class of Brill–Noether locus of Type III can be determined by applying Pragacz’s formula \[38\] to their Lagrangian representation (see §5).

**Proposition 7.9** For \(n = 0, 1\) or \(2\), suppose that \(M_C(2, K, n)\) has codimension exactly \(\binom{n+3}{2}\) in the moduli space \(M_C(2, K)\). Then its pullback to the Hecke graph \(Z\) has fundamental class given by

1. \(\frac{h^3}{6} - \frac{h\beta}{6} + \frac{\gamma}{3}\) when \(n + 2 = 2\);
2. \(\frac{h^6}{360} - \frac{h^4\beta}{72} + \frac{h^3\gamma}{36} + \frac{h^2\beta^2}{90} - \frac{2h\beta\gamma}{45} - \frac{\gamma^2}{18}\) when \(n + 2 = 3\);
3. \(\frac{h^{10}}{302400} - \frac{h^8\beta}{20160} + \frac{h^7\gamma}{10080} - \frac{17h^4\beta^3}{60480} + \frac{17h^3\beta^2\gamma}{10080} + \frac{h^2\beta\gamma^2}{720}\)
   \(+ \frac{h^2\beta^4}{8400} + \frac{h\beta^3\gamma}{216} + \frac{h\beta^2\gamma^2}{1050} - \frac{3\beta^2\gamma^2}{2800} + \frac{h^6\beta^2}{4800} - \frac{h^5\beta\gamma}{1200}\) when \(n + 2 = 4\).

**8 Fano 3-folds of genus 7**

The Brill–Noether locus \(M_C(2, K, 3) \subset M_C(2, K)\), that is, the set of \(E\) such that \(h^0(E) \geq 5\), has expected codimension 15. By Proposition 5.10, if the codimension is as expected, the anticanonical line bundle is the restriction of the determinant line bundle of \(M_C(2, K)\). In particular, it is a Fano variety. We consider the case when \(C\) has genus 7.

**Theorem 8.1** Let \(C\) be a curve of genus 7 not having a special line bundle \(g_1^4\). (This just means that \(C\) cannot be expressed as a cover of the Riemann sphere of degree \(\leq 4\).) Then the Type III Brill–Noether locus \(M_C(2, K, 3)\) is a Fano 3-fold of Picard number 1 and genus 7.

We study this variety from the following two points of view:

1. the construction of birational maps and the intermediate Jacobian variety;
2. the degeneration of Fano 3-folds and moduli spaces.

There are a number of ways of constructing birational maps; here we explain the simplest, which is obtained from 2 points \(p, q \in C\). Note that for an element \([E] \in M_C(2, K, 3)\), we have \(h^0(E(-p - q)) \geq 5 - 4 = 1\), that is, \(E\) has a nonzero global section with zeros at \(p\) and \(q\). If we assume that this section has no other zeros, we obtain an exact sequence

\[0 \to \mathcal{O}_C(p + q) \to E \to K_C(-p - q) \to 0. \quad (8.2)\]

Conversely, consider the question of when a vector bundle \(E\) obtained as an extension of the line bundles \(K_C(-p - q)\) and \(\mathcal{O}_C(p + q)\) has \(h^0(E) \geq 5\). By
Serre duality, the vector space of extensions $\text{Ext}^1(K_C(-p - q), O_C(p + q))$ is dual to $H^0(K_C^2(-2p - 2q))$. Thus if we write
\[ \Phi: C \hookrightarrow \mathbb{P}^{13} = \mathbb{P}^* H^0(K_C^2(-2p - 2q)) \]
for the embedding defined by $K_C^2(-2p - 2q)$, a nonzero extension class $e \in \text{Ext}^1(K_C(-p - q), O_C(p + q))$ determines a point of this $\mathbb{P}^{13}$. We write
\[ \delta_e: H^0(K_C(-p - q)) \rightarrow H^1(O_C(p + q)). \]
for the connecting homomorphism in the cohomology sequence of the exact sequence (8.2) determined by $e$. The condition $h^0(E) \geq 5$ is equivalent to saying that $\delta_e$ has rank \leq 1. Moreover, the linear map
\[ \text{Ext}^1(K_C(-p - q), O_C(p + q)) \rightarrow \text{Hom}(H^0(K_C(-p - q)), H^1(O_C(p + q))) \]
given by $e \mapsto \delta_e$ is the Serre dual of the multiplication map $\mu: H^0(K_C(-p - q)) \otimes H^0(K_C(-p - q)) \rightarrow H^0(K^2(-2p - 2q))$.

Now consider the diagram
\[ \begin{array}{ccc}
C & \xrightarrow{\psi} & \mathbb{P}^{13} \\
\Phi_{K_C(-p-q)} \downarrow & & \downarrow \mathbb{P}^* \mu \\
\mathbb{P}^4 & \xrightarrow{\text{Veronese map}} & \mathbb{P}^{14}
\end{array} \]

The condition $\text{rank} \delta_e = 1$ means that the point $\mathbb{P}^* \mu(p)$ is in the image of the Veronese map. Thus the set of nontrivial extensions $E$ with $h^0(E) \geq 5$ is parametrized by the quadric hypersurface $Q^3 \subset \mathbb{P}^4$ which passes through the image $C_{10} = \Phi_{K_C(-p-q)}(C)$. The vector bundle $E$ corresponding to a general point of this quadric $Q^3$ is stable, and we obtain a birational map
\[ M_C(2, K, 3) \dashrightarrow Q^3. \]

Taking a closer look at this correspondence, we find that this birational map factors as follows.

(8.3) Write $J_{p,q}$ for the set of $E \in M_C(2, K, 3)$ with $h^0(E(-p - q)) \geq 2$; it is a rational curve of degree 2 with respect to the anticanonical class (or the determinant line bundle). $J_{p,q}$ is clearly contained in the indeterminacy locus of the birational map $M_C(2, K, 3) \dashrightarrow Q^3$, so that we blow up $M_C(2, K, 3)$ along it, writing $\tilde{M}_C(2, K, 3)$ for the blowup.

(8.4) A vector bundle $E$ obtained as the extension corresponding to a point $x$ of the curve $C_{10} \subset Q^3 \subset \mathbb{P}^4$ contains the line subbundle $K_C(-p - q - x)$, and is not stable. Therefore, it is contained in the indeterminacy locus of the inverse birational map $Q^3 \dashrightarrow M_C(2, K, 3)$. Thus we write $\tilde{Q}^3$ for the variety obtained by blowing up the quadric hypersurface $Q^3$ along $C_{10}$. 

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The two blowups $\tilde{M}_C(2, K, 3)$ and $\tilde{Q}^3$ are isomorphic outside a subset of codimension 2. In fact, $\tilde{M}_C(2, K, 3)$ is obtained from $\tilde{Q}^3$ by flopping the birational transforms of the lines contained in $Q^3$ which are trisecant lines of $C_{10}$ (there are 14 of these in general):

$$\begin{array}{cccc}
\text{blowup} & \downarrow & \text{flops} & \rightarrow \\
\tilde{M}_C(2, K, 3) & \cdots & \tilde{Q}^3 \\
M_C(2, K) \supset M_C(2, K, 3) \supset J_{p,q} & C_{10} \subset Q^3 \subset \mathbb{P}^4
\end{array}$$

Here $X_6$ is the common anticanonical model of $\tilde{M}_C(2, K, 3)$ and $\tilde{Q}^3$, so that these are two different small resolutions of the singular Fano 3-fold $X_6$.

Because flops do not change the intermediate Jacobian, we get the following result.

**Theorem 8.6** $M_C(2, K, 3)$ is rational, and its intermediate Jacobian is isomorphic as a principally polarized Abelian variety to the Jacobian of $C$.

In particular, the Torelli theorem holds for $M_C(2, K, 3)$. Moreover, it can be shown that every nonsingular Fano 3-fold of genus 7 and Picard number 1 is isomorphic to a $M_C(2, K, 3)$. By the above theorem, their moduli space is isomorphic to the moduli space of curves of genus 7 not having a $g^1_4$.

Although we restricted Theorem 8.1 to the nonsingular case, even when singular, the moduli space $M_C(2, K, 3)$ extends in a flat family, provided it remains 3-dimensional. Moreover, its singularities are all Gorenstein. There are two main directions of degenerations:

(a) when $C$ has an ordinary double point;

(b) when $C$ remains nonsingular, but acquires a $g^1_4$, that is, it specializes to a 4-fold cover of the Riemann sphere.

In either case, we obtain a Fano 3-fold of genus 7 having an ordinary double point. In the two cases, the (algebraic) local ring of the double point is a unique factorization domain in Case (a), and not so in (b).

We take a closer look at Case (a). A curve $C$ of genus 7 with an ordinary double point is obtained by identifying two points $p, q$ on a nonsingular curve $\tilde{C}$ of genus 6. In the moduli space $M_{\tilde{C}}(2, K(p+q))$ of stable rank 2 bundles over $\tilde{C}$, consider the locus $M_{\tilde{C}}(2, K(p+q), 3)$ of bundles having $h^0(E) \geq 5$.

**Proposition 8.7** Suppose that the genus 6 curve $\tilde{C}$ is not trigonal, and write $\tilde{C}_8 \subset \mathbb{P}^3$ for the image of $\tilde{C}$ under $\Phi_{K(-p-q)}$. Then $M_C(2, K(p+q), 3)$ is obtained by blowing up $\mathbb{P}^3$ in $\tilde{C}_8$, then flopping the birational transforms of its 5 quadrisecants.
The curve $\tilde{C}_8 \subset \mathbb{P}^3$ is contained in a cubic surface. After the flops, its birational transform becomes isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and can be contracted to an ordinary double point. The singular manifold so obtained is the Brill–Noether locus of the curve in (a). However, the node itself corresponds not to a vector bundle over $C$, but to a torsion free sheaf.

## 9 Fano 3-folds of genus 9

A nonhyperelliptic curve of genus 3 is isomorphic to a plane quartic curve.

**Theorem 9.1** Let $C$ be a nonsingular plane quartic curve, and $F$ a rank 2 vector bundle over $C$ of odd degree such that any section $S$ of the associated $\mathbb{P}^1$ bundle $\mathbb{P}(F)$ has selfintersection number $S^2 \geq 3$. Then the Brill–Noether locus of Type II $M_C(2, K: 3F)$ is a nonsingular Fano 3-fold of index 1 and genus 9. Here the fact that $M$ is Fano follows from Proposition 6.8, and genus 9 from Example 7.4.

**Remark 9.2** The vector bundle $F$ is stable; in fact $F$ is stable if and only if every section $S$ has selfintersection number $S^2 \geq 1$.

If we twist $F$ by a line bundle to get $F \otimes L$, the isomorphism class of $M_C(2, K: 3F)$ does not change. Thus the Fano 3-folds $M_C(2, K: 3F)$ depend only on the ruled surface $\mathbb{P}(F)$.

As for the Fano 3-folds of genus 7 in the preceding section, the various properties of a Fano 3-fold of genus 9 can be studied via vector bundles in terms of moduli theory. We first construct a birational map. Let $S$ be a section of $\mathbb{P}(F)$ with selfintersection $S^2 = 3$; in terms of the exact sequence

$$0 \to \xi \to F \to \eta \to 0$$

associated with $S$, this means that $\text{deg} \eta - \text{deg} \xi = 3$.

**Proposition 9.4** The locus of $E \in M_C(2, K: 3F)$ such that $H^0(\eta^{-1}E)$ is a rational curve of degree 1 with respect to the anticanonical divisor.

We call this locus the line associated with $S$, and denote it by $l_S$. Next, write $C_7 \subset \mathbb{P}^4$ for the image of the embedding $\Phi: C \to \mathbb{P}^7$. By Serre duality, $\text{Ext}^1(\eta, \xi) \cong H^0(K\xi^{-1}\eta)^\vee$, so that the class of the extension (9.3) determines a point $p \in \mathbb{P}^4$. Since $F$ is stable, $p$ is not on $C_7$. Furthermore, since $\mathbb{P}(F)$ does not have a section of selfintersection 1, it is not contained on the secant hypersurface of $C_7$. We write $C_S \subset \mathbb{P}^4$ for the nonsingular degree 7 space curve obtained by projecting $C_7$ from $p$.

**Theorem 9.5** Let $C, F$ and $S$ be as above. Write $\tilde{M}_C(2, K: 3F)$ for the Brill–Noether locus $M_C(2, K: 3F)$ with the line $l_S$ blown up and $\mathbb{P}^3$ for $\mathbb{P}^3$ with the
curve $C_S$ blown up. Then these are isomorphic outside a set of codimension 2, and are obtained from one another by flops:

\[
\begin{array}{cccccc}
\tilde{M}_C(2, K : 3F) & \xleftarrow{\text{flops}} & \cdots & \xrightarrow{\text{flops}} & \mathbb{P}^3 \\
\text{blowup} \downarrow & & & & \downarrow \text{blowup} \\
\mathbb{P}^3 \supset C_S \subset M_C(2, K : 3F) & & & & \mathbb{P}^3 \supset C_S
\end{array}
\]

Here $X_{12}$ is the common anticanonical model, as in (8.5).

A point $x \in \mathbb{P}^3$ of the projective space containing the degree 7 curve $C_S$ determines a 3-dimensional vector subspace $U_x \subset H^0(K_{\xi}^{-1} \eta)$. If $x \notin C_S$ then the evaluation homomorphism

\[ ev_x : U_x \otimes \mathcal{O}_C \to K_{\xi}^{-1} \eta \]

is surjective. Now we consider the kernel of

\[ ev_x \otimes \xi^{-1} : U_x \otimes \xi^{-1} \to K_{\xi}^{-2} \eta, \]

writing $E_x$ for its dual. We always have $\dim \text{Hom}(F, E_x) \geq 3$. Moreover, if $x$ is not on any quadrisecant of $C$ then $E_x$ is stable. This defines a rational map

\[ \mathbb{P}^3 \to M_C(2, K : 3F) \quad \text{given by} \quad x \mapsto [E_x]. \quad (9.6) \]

The inverse can also be constructed, proving the rationality of $M_C(2, K : 3F)$, but we omit the details.

Similarly to Theorem 8.6, we obtain the following result.

**Corollary 9.7** The intermediate Jacobian of $M_C(2, K : 3F)$ is isomorphic to the Jacobian of the curve $C$.

There are 3 main ways in which $M_C(2, K : 3F)$ degenerates. The first two of these are similar to the degenerations of $M_C(2, K, 3)$ in the preceding section.

(a) when $C$ has an ordinary double point;

(b) when $C$ specializes to a hyperelliptic curve;

(c) finally, when the bundle $F$ is no longer general: namely, suppose $F$ is stable, but its projectivization $\mathbb{P}(F)$ has a section with selfintersection 1.

In any of the 3 cases, $M_C(2, K : 3F)$ acquires a single ordinary double point. Its local ring is a unique factorization domain in Case (a) only. We discuss Case (c) briefly. In this case, $F$ can be written as an extension

\[ 0 \to \mathcal{O}_C \to F \to \Lambda \to 0, \]

where $\Lambda$ is a line bundle of degree 1. Suppose that $h^0(F) = 1$; then by Serre duality, $h^0(K_C F^\vee) = 4$. Also, $K^2 \Lambda^{-1}$ embeds $C$ into $\mathbb{P}^4$. Write $C_7 \subset \mathbb{P}^4$ for its image. The Grassmann map determined by the sections of $K_C F^\vee$ determines a quadric hypersurface $Q^3$ containing $C_7$. 28
Proposition 9.8 In Case (c), $M_C(2, K : 3F)$ is isomorphic to the anticanonical model of the blowup of $C_7$ in $Q^3$.

$C_7 \subset Q^3$ has a single trisecant line. The birational transform of this can be contracted to the ordinary double point of the anticanonical model, and $M_C(2, K : 3F)$ is nonsingular outside this point.

10 The non-Abelian Albanese map of a K3 surface

We give an application of Brill–Noether locuses of Type III to K3 surfaces, and to the relation between curves and K3s. Harris and Mumford have computed the Kodaira dimension of the moduli space of curves, and obtained the following result:

Theorem 10.1 (10) For sufficiently large $g$, a general curve $C \in M_g$ is not isomorphic to a hyperplane section of a surface $S \subset \mathbb{P}^N$.

Here we consider the case when $S$ is a K3.

Problem 10.2 Determine whether a given curve $C$ can be embedded in a K3 surface; if so, analyze the set of all such embeddings.

Definition 10.3 If some embedding of $C$ into a K3 exists, we say that $C$ is a K3 curve or a K3 section.

If a curve $C$ of genus $g \geq 2$ is contained in a K3 surface $S$, then as a divisor, it is nef and big on $S$. In fact $|C|$ is base point free, and $C^2 = 2g - 2 > 0$. Write $h$ for the linear equivalence class of the image of $C \hookrightarrow S$. In what follows we consider the case that $h$ is primitive in $\text{Pic} S$, that is, not divisible by a natural number $\geq 2$.

A K3 surface $(S, h)$ together with a (primitive) nef and big divisor is called a polarized K3, and the natural number $g = h^2/2 + 1$ its genus. We write $\mathcal{F}_g$ for the moduli space of polarized K3 surfaces $(S, h)$ of genus $g$. It is a 19-dimensional quasiprojective algebraic variety. We also write $\mathcal{P}_g$ for the moduli space of pairs $(S, C)$ consisting of a K3 surface together with a genus $g$ curve on it; this has natural projection maps to $\mathcal{F}_g$ and $M_g$, which we write $\Psi_g$ and $\Phi_g$:

$$\mathcal{P}_g = \{(S, C)\}/\text{iso} \rightarrow \bigsqcup_{(S, h) \in \mathcal{F}_g} |h|$$

$$\mathcal{M}_g \supset \mathcal{K}_g$$

K3 curves

$$\mathcal{F}_g = \{(S, h)\}/\text{iso}$$

polarized K3s

By definition, the image of $\Phi_g$ is the locus of K3 curves; we write $\mathcal{K}_g$ for it, and $\varphi_g$ for the restriction of $\Phi_g$ over it. The moduli space $\mathcal{P}_g$ is contained as an
open subset in the \(\mathbb{P}^g\)-bundle over \(\mathcal{F}_g\) having fibers the complete linear systems \(|h| \cong \mathbb{P}^g\). Therefore, its dimension is \(19 + g\). In particular, we get the following:

\[
\dim \mathcal{P}_g \leq \dim \mathcal{M}_g \iff 19 + g \leq 3g - 3 \iff g \geq 11.
\]

(10.4)

**Main Conclusion 10.5** For a curve of odd genus \(g \geq 11\), we can construct the inverse of \(\varphi_g\) as a rational map. See (10.14)–(10.18) for details; for \(g = 11\), see also [26].

**Corollary 10.6** For odd \(g \geq 11\), the locus of K3 curves \(K_g\) is birationally equivalent to a \(\mathbb{P}^g\)-bundle over \(\mathcal{F}_g\).

**Remark 10.7** (1) In spite of what inequality (10.4) may suggest, the projection \(\varphi_{12}: \mathcal{P}_{12} \to K_{12} \subset \mathcal{M}_{12}\) is not even generically finite. The reason behind this phenomenon is the existence of the Fano 3-folds \(V_{12} \subset \mathbb{P}^{13}\) of genus 12, discovered by Iskovskikh. In fact, the general element of \(\mathcal{K}_{12}\) can be written as a double linear section \(C = V_{12} \cap H_1 \cap H_2\), so that \(C\) is contained in infinitely many K3s, namely the pencil \(V_{12} \cap (a_1H_1 + a_2H_2)\) for \((a_1 : a_2) \in \mathbb{P}^1\).

More generally, the values of \(g\) for which \(\varphi_g\) is not birational are exactly the values for which there exists a Fano 3-fold \(V\) with \(\text{Pic} V = \mathbb{Z}(-K_V)\) (explicitly, all odd numbers \(g \leq 9\) and all even numbers \(g \leq 12\)).

(2) It was proved in [10] that \(\varphi_g\) is generically finite for odd \(g \geq 11\). For genus 11, Corollary 10.6 also follows from the theorem of Ciliberto and Miranda [3] on the irreducibility of the fiber of \(\varphi_{11}\).

(3) We do not treat this here, but there are theorems of J. Wahl ([13], [14]), saying that the Gauss map \(\bigwedge^2 H^0(K_C) \to H^0(K_C^2)\) is not surjective for K3 curves \(C\).

The construction of the inverse, that is, recovering a K3 surface from a K3 section \(C \in \mathcal{K}_g\) is a non-Abelian version of the Albanese map. We first recall the theory of the dual of an Abelian variety. If \(A\) is an Abelian variety, the dual Abelian variety \(\hat{A}\) is the identity component \(\text{Pic}^0 A\) of the moduli space \(\text{Pic} A\) of line bundles on \(A\); there is a universal Poincaré line bundle \(\mathcal{P}\) over the direct product \(A \times \hat{A}\). By definition, \(\hat{A} \ni \alpha \mapsto \mathcal{P}|_{A \times \hat{A}} \in \text{Pic}^0 A\) is an isomorphism. We can normalize \(\mathcal{P}\) by the condition that \(\mathcal{P}|_{0 \times \hat{A}}\) is trivial, then consider its restriction to the fibers in the other direction \(a \times \hat{A}\).

**Duality Theorem 10.8** ([28], §13) The correspondence \(a \mapsto \mathcal{P}|_{a \times \hat{A}}\) defines an isomorphism \(A \overset{\approx}{\to} \hat{A}\).
Corollary 10.9 If $X$ is a nonsingular projective algebraic variety, the dual Abelian variety of the Picard variety $\text{Pic}^0 X$ is the Albanese variety of $X$. That is, there exists a morphism $\alpha : X \to (\text{Pic}^0 X)^\wedge$, which has the universal mapping property for maps from $X$ to an Abelian variety.

Remark 10.10

1. The Albanese variety is constructed analytically by picking a basis $\omega_1, \ldots, \omega_n \in H^0(X, \Omega^1_X)$ for regular differential 1-forms on $X$, and then taking the integral

$$X \to \mathbb{C}^n/(\text{periods})$$

$$x \mapsto \left( \int_x \omega_1, \ldots, \int_x \omega_n \right)$$

2. In positive characteristic, if $\text{Pic}^0 X$ is nonreduced, we have to replace $\text{Pic}^0 X$ by its reduced Abelian variety before taking the dual.

We define a non-Abelian version of the above duality theorem for K3 surfaces (see [51] for details). Let $(S, h)$ be a polarized K3 surface of odd genus $g = 2n + 1$. Consider the moduli space

$$M_S(2, h, s) := \left\{ E \mid E \text{ is a rank 2 stable sheaf, with } c_1(E) = h \text{ and } \chi(E) = s + 2 \right\} / (\text{isomorphism}) .$$

This is nonsingular and of dimension $2(g - 2s)$ (see [15], [19], [20]). In particular, when $s = n = (g - 1)/2$, it is a surface; we denote it by $\hat{S} = M_S(2, h, n)$. We assume in what follows that $n$ is also odd, so that $g \equiv 3 \mod 4$.

Proposition 10.11 Assume that $\hat{S}$ is compact. (This is equivalent to saying that all semistable sheaves are stable.)

1. $\hat{S}$ is a K3 surface.
2. There exists a polarization $\hat{h}$ of $\hat{S}$ having the same degree as $h$.
3. There exists a universal vector bundle $E$ over the direct product $S \times \hat{S}$ with $c_1(E) = \pi_S^* h + \pi_{\hat{S}}^* \hat{h}$; (we call this the normalized universal vector bundle).

By definition, the correspondence $\alpha \mapsto E|_{S \times \alpha}$ defines an isomorphism $\hat{S} \cong M_S(2, h, n)$; restricting to the fibers in the other direction gives the following.

Duality Theorem 10.12 Let $(S, h) \in \mathcal{F}_g$ be a general element, with $g \equiv 3 \mod 4$, and let $E$ be the normalized universal vector bundle. Then the correspondence $s \mapsto E|_{s \times \hat{S}}$ defines an isomorphism from $S$ to $\hat{S}$.
Remark 10.13  (1) The correspondence \((S, h) \mapsto (\hat{S}, \hat{h})\) extends to an automorphism of the moduli space \(\mathcal{F}_g\). When \(g = 3\) (that is, for space quartics), it is the identity map, for \(g \geq 7\), a map of order 2.

(2) In general, we have \(\hat{S} \not\cong S\) for \(g \geq 7\), but the derived categories of coherent sheaves on \(S\) and \(\hat{S}\) are equivalent under an integral transformation. That is, as in [7], there is an equivalence of categories \(\mathcal{D}(\text{Coh} \ S) \cong \mathcal{D}(\text{Coh} \ \hat{S})\).

The construction of the inverse of \(\varphi_g: \mathcal{P}_g \to \mathcal{K}_g\) is as follows. We assume that \(C \in \mathcal{K}_g\) is a general K3 curve of genus \(g \geq 11\) with \(g \equiv 3 \mod 4\).

(10.14) Consider the moduli space \(\mathcal{M}_C(2, K)\) of vector bundles over \(C\), and the Brill–Noether locus of Type III \(\mathcal{M}_C(2, K, n)\), where \(n = (g - 1)/2\). Then \(M_C(2, K, n)\) is a K3 surface, and the restriction to it of the determinant line bundle of \(M_C(2, K)\) is a polarization of genus \(g\). We write \((T, h_{\text{det}})\) for this pair.

(10.15) There exists a universal vector bundle \(E\) over the direct product \(C \times T\) such that \(c_1(E) = \pi_C^* K_C + \pi_T^* h_{\text{det}}\).

(10.16) For every \(p \in C\), the restriction of \(E\) to \(p \times T\) is a stable bundle, and has \(\chi = n + 2\).

(10.17) The correspondence \(p \mapsto E|_{p \times T}\) defines a classifying map \(C \to \hat{T} = \mathcal{M}_T(2, h_{\text{det}}, n)\), which is an embedding from \(C\) to the K3 surface \(\hat{T}\).

(10.18) Every K3 surface containing \(C\) is isomorphic to \(\hat{T}\).

To summarize what we have said above, there is a double moduli construction, starting from a moduli space of vector bundles over \(C\), giving a non-Abelian Albanese map \(C \to \hat{T}\); namely, \(\hat{T}\) is the dual K3 surface of a Brill–Noether locus of \(C\).

When \(g \equiv 1 \mod 4\), we find in a similar way that \(T = M_C(2, K, n)\) is a K3 surface. But now \(h_{\text{det}}\) is divisible by 2, and the universal vector bundle \(E\) does not exist globally over it. However, there does exist a universal \(\mathbb{P}^1\)-bundle corresponding to \(\mathbb{P}(E)\), and the K3 surface containing \(C\) can be recovered as a moduli space of suitable \(\mathbb{P}^1\)-bundles over \(T\) (we can equally well say principal SO(3)-bundles).

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