Abstract

In this article, we calculated the refined topological vertex for the one parameter case using the Jack symmetric functions. Also, we obtain the partition function for elliptic N=2 models, the results coincide with those of Nekrasov instanton counting partition functions for the $N = 2^*$ theories.

1. Introduction

The study of refined topological vertex sheds lights on many other physical or mathematical problems in recent years. For physical interests, we are interested in the $N = 2$ gauge system\cite{1-10}, which can be realized as a IIA string theory compactifying on certain toric Calabi-Yau three-folds\cite{CY-3folds} by the geometric engineering. The instanton part of the N=2 theory is captured by the topological string amplitude. This amplitude, can be calculated by the refined topological vertex\cite{11} formulation. Otherwise, the $N = 2$ systems can also have the standard NS5-D4 brane configurations and show a great many interesting properties, such as the S-duality, the natural confinement, the integrability and so on and so forth. Recently, Alday, Gaitto and Tachikawa\cite{AGT}\cite{13} showed that for an arbitrary $N = 2$ $SU(2)$ superconformal gauge system, there is a dual 2d theory which is a Liouville theory living on the moduli space of the 4d gauge theory\cite{13-31}. The moduli space (Seiberg-Witten curve) of the $N = 2$ theory is also the ramification of the Riemann surface M5 branes wrapping on. Later on, Dijkgraaf and Vafa\cite{5} proved this 2d-4d relation using intrinsic correspondences of topological string-matrix model-Liouville theory. Recently, Cheng, Dijkgraaf and Vafa\cite{6} extended this proof to more detailed cases, they showed that the instanton part of Nekrasov partition function, which duals to the conformal block of the 2d conformal field theory\cite{CFT}, is in fact a linear combination of the non-perturbative string partition functions.
The NS5-D4 brane configuration of a given $N = 2$ gauge theory has a rather simple translation to topological string. Roughly speaking, the brane configuration diagram can be seen as the singular version of the related toric diagram of the related CY-3fold. For example, if we consider there are $N$ coincided D4 branes truncated by 2 separated NS5 branes, which low energy theory is a 4d $N = 2 U(N)$ gauge theory with $N_f = 2N$ fundamental matters. Classically, there are two $N$-fold singularities located at the two intersection points of the branes. However, while lift this to M-theory, these singularities can be "blown-up" to an sequence of $S^2$s due to the quantum effect. Then the brane diagram becomes an $N$-ramified Riemann sphere on which M5 branes wrapping on. On the topological string side, to get the same gauge theory, by the standard geometric engineering, one can identify the "blow-up" process that the conifold singularities transit to resoloved ones. Fig.1 shows the simple case for $N = 2 U(2)$ gauge theory.

The topological string partition function on a given toric CY3-fold is expected to correspond to instanton sums in the related gauge theory. The instanton part of the partition function of a certain $N = 2$ gauge theory can also have a brane expression, the D0-D4 configuration. In this configuration, the D0 branes dissolve into the D4 branes as the instanton background of the gauge theory. These D0 branes come from the M-theory compactification as the Kaluza-Klein modes. Apart from M5 branes, there are M2 branes which are the magnetic dual of the M5 branes. Then if there are M2 branes intersecting with M5 branes, then after the M-theory compactification, they become the D2-D0 bound states. If these D2 branes wrap on some nontrivial Lagrangian 2-cycles in the CY3-fold, they behave just like D0 branes for the observer living on the D4 branes, thus they also contribute to the instanton counting of the $N = 2$ gauge theories. So the instantons of the $N = 2$ theory are expected to relate to the D2-D0 bound states. Besides, as noted in [12], the instanton calculation has more refined information. This information comes from the Nekrasov $\Omega$ deformation of $N = 2$ gauge theories. From the M-theory point of view, the CY3 compactification gives a 5d gauge theory living on $\mathbb{C}^2 \times S^1$ with $S^1$ the M-theory cycle. The BPS spectrum of the theory corresponds to the little group
representation of the motion group of $\mathbb{C}^2 \times S^1$, which is, obviously, the $SO(4) = SU(2)_L \times SU(2)_R \in SO(5)$. The $\Omega$ deformation is a $T^2$ action on $\mathbb{C}^2$:

$$T^2 : (z_1, z_2) \mapsto (e^{i\epsilon_1 z_2}, e^{i\epsilon_2 z_2}). \quad (1.1)$$

This deformation has a direct impact on the definition of the topological amplitude, which can be easily calculated by the topological vertex formulism. Now the fundamental vertices change to the refined ones which are the two-parameter generalization of the original topological vertices. This change is due to the fact that topological string amplitude counts the holomorphic maps from string world sheet to Langrangian submanifolds of the toric CY3-fold. On the other hand, the maps also correspond to BPS bound states\cite{35, 36} of M2 branes, which are representations of $SO(4) = SU(2)_L \times SU(2)_R$, now twisted by the $\Omega$ deformation. Thus the refined topological vertex is a fundamental block for building the $N = 2$ theories.

It is crucial that this observation also implies the AGT relation, and reveals the essential net dualities between these topological string-matrix model and $N = 2$ 4d gauge theories-2d conformal field theories. However, for generic $N = 2$ theories, it is hard to verify these dualities, since there are many ambiguities in all of these theories. The simpler cases are the so called ”$N = 2^*$ theory”, which only involves an adjoint matter, and the ”necklace” quiver $N = 2$ theories which only have bifundamental matters. In this article, we will concentrate on these theories. Their brane configurations are just the elliptic models, which have punctured torus $T_{M,1}$ as Seiberg-Witten curves. The related integrable system is the two dimensional elliptic Calogero-Sutherland(eCS) model, from which one can easily read off the Liouville/Toda theories living on torus. From either the topological string or the eCS model, one can have a rather simple description of the one parameter refined topological vertex and further the instanton counting by invoking the Jack symmetric functions. We found that the refined topological vertex have a simple description by using the Jack polynomials. The instanton counting of $N = 2$ quiver theories can also be computed by the same Jack polynomials. The main result of this article is the following closed formulae for $N = 2 U(N) M$-node($M \geq 2$) necklace quiver gauge theories(see Fig2).

\footnote{For convenience, we call all the elliptic $N = 2$ models $N = 2^*$ theories.}
Figure 2 \(M\)-node necklace quiver \(U(N)\) gauge theory

\[
Z_{4D \text{ inst}}^{\text{M-necklace}}(\vec{a}_\ell, \vec{\lambda}_\ell; m_\ell) = \prod_{m,n=1}^{N} \langle E_{m,n}^{(\ell,\ell+1)}(E^*)^{\beta-m(\ell,\ell+1)} J_{\lambda_m,\lambda_{\ell+1}} \rangle_\beta (1.2)
\]

\[
Z_{4D \text{ inst}}^{\text{adj}}(\vec{a}_\ell, \vec{\lambda}_\ell; m_\ell) = Z_{4D \text{ inst}}^{\text{bifund}}(\vec{a}_\ell, \vec{\lambda}_\ell)
\]

\[
Z_{4D \text{ inst}}^{\text{vec}}(\vec{a}, \vec{\lambda}) = 1/Z_{4D \text{ inst}}^{\text{adj}}(\vec{a}, \vec{\lambda}; 0)
\]

\[
Z_{U(N) \text{ inst}}^{M-\text{necklace}} = \sum_{\vec{\lambda}_1, \ldots, \vec{\lambda}_M} \prod_{i,\ell=1}^{M} \tilde{Q}_i^{\vec{\lambda}_i} Z_{\text{vec}}(\vec{a}_i, \vec{\lambda}_i) Z_{\text{bifund}}(\vec{a}_\ell, \vec{\lambda}_\ell, \vec{a}_{\ell+1}, \vec{\lambda}_{\ell+1}; m_\ell)
\]

\[
= \sum_{\vec{\lambda}_1, \ldots, \vec{\lambda}_M} \prod_{i,\ell=1}^{M} (\tilde{Q}_i)^{|\vec{\lambda}_i|} \prod_{j,k=1}^{N} \left[ \langle E_{a_{j,k}}^{(i)}(E^*)^{\beta-a_{j,k}^{(i)}-1} J_{\lambda_{ij}}, J_{\lambda_{ik}} \rangle_\beta \right]^{-1} \langle J_{\lambda_{ij}}, J_{\lambda_{ij}} \rangle_\beta \prod_{m,n=1}^{N} \langle E_{m,n}^{(\ell,\ell+1)}(E^*)^{\beta-m(\ell,\ell+1)} J_{\lambda_m,\lambda_{\ell+1}} \rangle_\beta (1.3)
\]

Here \(\vec{a}_\ell = \{a_{\ell,1}, \ldots, a_{\ell,N}\}\) and \(\vec{\lambda}_\ell = \{\lambda_{\ell,1}, \ldots, \lambda_{\ell,N}\}\) defines the Coulomb parameter vector and instanton partition Young tableau vector of the \(\ell\)-th \(U(N)\) gauge group, respectively. \(m_\ell = \frac{\tilde{m}}{\epsilon_2}\) denotes the mass of the \(\ell\)-th bifundamental matter.

\[
\tilde{Q}_i = \exp(2\pi i \tau_{UV}^i), \quad \tau_{UV} = \frac{4\pi i}{g_{UV}^2} + \frac{\theta_{UV}}{2\pi}
\]

are the sewing parameters. \(a_{j,k}^{(i)} = a_{i,j} - a_{i,k}, a_{i,j} = \frac{\tilde{a}_{i,j}}{\epsilon_2}, \quad m_{m,n}^{(\ell,\ell+1)} = a_{\ell+1,n} - a_{\ell,m} - m_\ell\).

\[
E = 1 + e[1] + e[2] + \cdots = \exp\left(\sum_{n>0} \frac{(-1)^n}{n} p_n\right)
\]

is related to Dijkgraaf-Vafa’s topological B-brane background and this will be shown explicitly in the second part of this note\([46]\), \(e[m], p_n\) is elementary and power sum polynomials respectively. \(E^*\) is the adjoint action under the inner product of Jack polynomials \([34]\)

\[
\langle E_J, J_{\mu} \rangle_\beta = \langle J_{\lambda}, (E^*) J_{\mu} \rangle_\beta, \quad \beta = -\frac{\epsilon_1}{\epsilon_2}.
\]
The inner product is defined and proved in [34] as following
\[
\langle E^m(E^*)^{β-1}J_λ, J_μ \rangle_β = (-1)^{λ-|λ|-|μ|} \prod_{s∈λ}(m + a_λ(s) + 1 + βl_μ(s)) \prod_{t∈μ}(m - a_μ(t) - β(l_λ(t) + 1)),
\]

(1.4)

here

\[a_λ(s) = λ_i - j, \quad l_λ(s) = λ_j^t - i\]

are hook arm-length and leg-length of box \(s = (i, j)\) of the Young tableau respectively.

The structure of this article is as following. In section 2, we review the refined topological vertex formulation in the A-model setup and its applications to instanton counting problems of \(N = 2^*\) theories. The eCS model and its spectrum which is captured by Jack symmetric functions, are described in section 3. In section 4, we show that the Jack symmetric functions exactly reproduce the Nekrasov instanton partition function as expected. This computation confirms the relation between topological string theory, which geometric engineers the \(N = 2^*\) theory, and the 2d eCS theory, which relates to the 4d theory by the AGT relation[1, 3]. Section 5 is left for conclusions and further interests.

2. Refined Topological Vertex and instanton counting in \(N = 2^*\) theories

The refined topological vertex(RTV) is a two-parameter generalization of the ordinary topological vertex. In the topological vertex formulation, one can easily get the partition function of an A-model which generates an \(N = 2\) gauge theory by geometric engineering. On the other hand, the same \(N = 2\) theory can be obtained by the NS5-D4 brane setup of IIA string theory. The bridge between these two apparently different configurations is the large \(n\) transition. On the field theory side, the nonperturbative part of the partition function is captured by the Nekrasov instanton counting, which involves the so-called \(Ω\) deformation of \(\mathbb{R}^4\). On the topological A-model side, the \(Ω\) deformation relates to the two-parameter generalization of the topological vertex, which is the RTV. The refined partition function of topological string is equivalent to the Nekrasov partition function of \(N = 2\) theories[12, 38, 37, 41, 42].

Since we will frequently use the relation between these two procedures, it is necessary to review the refined topological vertex and its connection with Nekrasov’s partition function.

2.1 Brane setup and toric diagram

The brane setup of \(N = 2\) theories can be translated into toric diagrams of topological A-model as following. One draws the brane intersection diagram of a desired \(N = 2\) theory as in Fig.3a,
Figure 3  

**a. NS5-D4 configuration of $N = 2$ $U(2)$ theory with $N_f = 4$. b. The related toric diagram**

then blows up every 4-vertex as two 3-vertices, adjusts the toric diagram to match with the geometric engineering procedure\[33\] as showing in Fig.3b.

From the NS5-D4 intersection branes configuration, one can immediately read off its low energy effective theory is just the $N = 2$ gauge theory. The pure gauge part of the theory comes from the coincided D4 branes, the matters are due to the truncation of the two NS5 branes\[3\]. However, the topological string realization of the $N = 2$ theory is totally different. The pure gauge part comes from the blowup of the singularities of the ALE space in Calabi-Yau. The matters correspond to D-branes wrapping on Lagrangian submanifolds in Calabi-Yau.

The detailed relation of these two realizations of $N = 2$ gauge theories were considered in Dijkgraaf and Vafa’s article\[5\] which we will now briefly review in the following. Instead of the A-model, they considered the mirror B-model realization. The Coulomb parameters of the gauge theory which are positions of D4 branes in their transverse directions, relate to the large $n$ limit of the condensation of D2 branes, or equivalently, the condensation of the screen charges in the 2d CFT language of the B-model. The matters are related to the insertions of stacks of D2 branes which can be written as vertex operators in 2d CFT. Their masses correspond to the numbers of branes. The Nekrasov $\Omega$ deformation is translated to a phase changing of the complex coordinate of the spectral curve. We will come back to these points in the second part of this note.

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\[2\]For a toric CY-3fold related to a gauge theory, there should exists a preferred direction in which all gluing legs of the toric diagram are parallel.

\[3\]Our main considerations in the present article do not involve fundamental matters. In the brane setup they do not only from the infinity D4 branes ending on the left or the right of NS5 branes, but also can be alternatively realized as the addition of D6 branes.
2.2 The refined topological vertex

The refined topological vertex is defined as [12]

\[ C_{\lambda\mu\nu}(t, q) = \left( \frac{q}{t} \right)^{\frac{\|\mu\|^2 + \|\nu\|^2}{2}} t^{\|\nu\|^2} P_\nu(t^{-\rho}; q, t) \]

\[ \times \sum_{\eta} \left( \frac{q}{t} \right)^{\frac{|\mu| + |\nu| - |\lambda|}{2}} s_{\lambda\mu/\eta}(t^{-\rho}q^{-\nu}) s_{\mu/\eta}(t^{-\nu\eta}q^{-\rho}) \]

\[ P_\nu(t^{-\rho}; q, t) = t^{\frac{\|\nu\|^2}{2}} \tilde{Z}_\nu(t, q) = \prod_{s \in \nu} (1 - t^{l_\nu(s)+1} q^{a_\nu(s)})^{-1} \]

\[ t = e^{\beta e_1}, \quad q = e^{-\beta e_2}, \quad \| \mu \|^2 = \sum_i \mu_i^2, \quad \rho = \{-1, -\frac{3}{2}, -\frac{5}{2}, \cdots\} \]

where \( \lambda, \mu, \nu \) denote Young tableaus of partitions of instantons. \( s_\lambda \) and \( s_{\lambda/\eta} \) are the Schur and the skew Schur function which is briefly reviewed in Appendix A. \( P_\nu(t^{-\rho}; q, t) \) is the Macdonald function.

For a toric diagram describing a chosen CY-3fold, the refined partition function can be calculated by gluing all topological vertices. For \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \mapsto \mathbb{P}^1 \) as in Fig. 4a, the refined

\[ \text{Figure 4} \quad a. \text{Toric diagram for } \mathcal{O}(-1) \oplus \mathcal{O}(-1) \mapsto \mathbb{P}^1, \quad b. \text{Toric diagram of } U(1) \text{ with a single adjoint matter} \]
partition function can be written as

$$Z(t, q, Q) = \sum_{|\nu|} Q^{|\nu|} (-1)^{|\nu|} C_{\substack{0\nu}}(t, q) C_{\substack{0
\nu'}}(q, t)$$

$$= \sum_{|\nu|} Q^{|\nu|} (-1)^{|\nu|} \frac{q^{\|\nu\|^2}}{2} \tilde{Z}_{\nu}(t, q) \tilde{Z}_{\nu'}(q, t)$$

$$= \sum_{|\nu|} \prod_{s<|\nu|} (1 - q_{\nu}^{-1} q_{\nu}^s q_{\nu}^s + (1 - q_{\nu}^{-1} q_{\nu}^s q_{\nu}^s + 1)).$$

For more complicated toric diagram, the calculation principle is the same.

### 2.3 Refined partition functions for 5D $N = 2^*$ theories

#### 2.3.1 $U(1)$ theory

The simplest 5D $N = 2^*$ theory is the $U(1)$ gauge theory with a single adjoint hypermultiplet\[39\][38]. The toric diagram looks the same as the $O(-1) + O(-1) \mapsto \mathbb{P}^1$ but partially compactifying the two external legs as shown in Fig.4b. Now the refined partition function reads

$$Z_{5D}(Q, Q_m, t, q) = \sum_{\mu, \nu} (-Q)^{|\nu|} (-Q_m)^{|\mu|} C_{\mu\nu}(t, q) C_{\nu\mu'}(q, t)$$

$$= \sum_{\lambda, \mu, \eta, \eta'} (-Q)^{|\nu|} (-Q_m)^{|\mu|} \frac{q^{|\nu|}}{2} \left(\frac{q}{t}\right) \frac{q^{|\mu|}}{2} \frac{t^{|\mu|}}{2} \left(\frac{t}{q}\right) \prod_{s<|\nu|} \frac{1}{1 - q_{\nu}^{s+1} q_{\nu}^{s} + 1}. $$

This 5D refined partition function contains a perturbative part which is just the zero-instanton part $Z_{5D}(Q, Q_m, t, q)$, thus the pure instanton part is given by

$$Z_{\text{inst}}(Q_m, t, q) = \frac{Z_{5D}(Q, Q_m, t, q)}{Z_{5D}(Q, Q_m, t, q)} = \sum_{|\nu|} (-Q)^{|\nu|} \left(\frac{q}{t}\right)^{|\nu|} \prod_{(i, j) \in \nu} \frac{1 - Q_m q^{-|\nu|} q^{1-|\nu|}}{1 - t^{q-|\nu|}}$$

#### 2.3.2 $U(2)$ theory

We now consider the $U(2)$ theory using the same RTV formulation. The toric diagram is showed in Fig. 6, the 5D refined partition function is given by
Figure 5 Toric diagram for $U(2)$ $N=2^*$ theory

\[ Z_{\nu_1,\nu_2}^{5D}(U(2)) = \sum_{\nu_1,\nu_2,\nu_1} \prod_{i=1}^{2} (-Q_i) |\nu_i|^1 (-Q_m) |\mu_i|^1 (-Q)^|\lambda| \]

\[ \times C_{\mu_1\nu_1}(t, q) C_{\lambda\nu_2}(t, q) C_{\mu_2\nu_2}(t, q) C_{\nu_1\nu_2}(q, t) \]

\[ = \sum_{\nu_1,\mu_1,\nu_2} \prod_{i=1}^{2} (-Q_i) |\nu_i|^1 (-Q_m) |\mu_i|^1 (-Q)^|\lambda| \left( \frac{q}{t} \right)^{\frac{\|\nu_1\|+\|\mu_1\|}{2}} \frac{\|\nu_2\|^2+\|\nu_2\|^2}{q} \frac{\|\nu_3\|^2+\|\nu_3\|^2}{q} \]

\[ \times Z_{\nu_1}(t, q) Z_{\nu_2}(t, q) Z_{\nu_3}(q, t) s_{\mu_1/\eta_1} (t-q) s_{\mu_1/\eta_1} (t-q) s_{\lambda/\eta_2} (t-q) s_{\lambda/\eta_2} (t-q) \]

\[ = \sum_{\nu_1,\nu_2} t^{\frac{\|\nu_1\|^2+\|\nu_2\|^2}{2}} q^{\frac{\|\nu_1\|^2+\|\nu_2\|^2}{2}} Z_{\nu_1}(t, q) Z_{\nu_2}(t, q) Z_{\nu_3}(q, t) \]

\[ \times \prod_{i,j=1}^{\infty} \frac{1 - Q t^{\rho_i-\nu_i,j} q^{-\rho_i-\nu_i,j}}{1 - Q Q_{m_1} t^{i-1-\nu_i,j} q^{j-\nu_j}} \frac{1 - Q Q_{m_2} t^{i-1-\nu_i,j} q^{j-\nu_j}}{1 - Q Q_{m_2} t^{i-1-\nu_i,j} q^{j-\nu_j}} \]

\[ \times \frac{1 - Q Q_{m_1} t^{i-1-\nu_i,j} q^{j-\nu_j}}{1 - Q Q_{m_2} t^{i-1-\nu_i,j} q^{j-\nu_j}} \frac{1 - Q Q_{m_2} t^{i-1-\nu_i,j} q^{j-\nu_j}}{1 - Q Q_{m_2} t^{i-1-\nu_i,j} q^{j-\nu_j}} \]
The instanton part of the refined partition reads

\[
Z_{\text{inst}}^{5D}(U(2)) = \frac{Z_{\nu,\nu',\nu''}^{5D}(U(2))}{Z_{\rho,\rho',\rho''}^{5D}(U(2))} = \sum_{\nu_1,\nu_2}^2 \left( -\sqrt{\frac{q}{t}} Q_i \right)^{|\nu_i|} \\
\times \prod_{(j,k)\in \nu_i} (1 - Q_{m_1} t^{-\rho_j - \nu_{i,k} \rho} q^{-\rho_j - \nu_{i,j}}) (1 - Q_{m_1} t^{\rho_j + \nu_{i,k} \rho} q^{\rho_j + \nu_{i,j}}) \\
\times \prod_{(j,k)\in \nu_1} (1 - Q'_{m_1} t^{-\rho_j + \nu_{i,k} \rho} q^{\rho_j + \nu_{i,j}}) \prod_{(j,k)\in \nu_2} (1 - Q'_{m_1} t^{-\rho_j - \nu_{i,k} \rho} q^{\rho_j - \nu_{i,j}}) \\
\times \prod_{(j,k)\in \nu_1} (1 - Q Q_{m_2} t^{-j + \nu_j \rho - k} q^{\nu_j - k}) \prod_{(j,k)\in \nu_2} (1 - Q Q_{m_2} t^{j - \nu_j \rho + k} q^{\nu_j + k})^{-1} \\
\times \prod_{(j,k)\in \nu_1} (1 - Q Q_{m_2} t^{-j + \nu_j \rho + 1} q^{\nu_j - k}) \prod_{(j,k)\in \nu_2} (1 - Q Q_{m_2} t^{j - \nu_j \rho - 1} q^{\nu_j + k + 1})^{-1} \\
\times \prod_{s\in \nu_i} (1 - t^{-\nu_i(s)-1} q^{-a_{\nu_i(s)}}) (1 - t^{\nu_i(s)} q^{a_{\nu_i(s)}} + 1)^{-1},
\]

here we define \( Q'_1 = Q \), \( Q'_2 = QQ_{m_1} Q_{m_2} \).

### 2.3.3 \( U(2) \times U(2) \) theory

The toric diagram for \( N = 2^* U(2) \times U(2) \) theory is given in Fig. 6a. Using the gluing rule\[^{[37],[38]}\], one can truncate the toric diagram into two separate ones denoted by \( T_1 \) and \( T_2 \) (as showing in Fig. 6b). The 5D refined partition for \( T_1 \) and \( T_2 \) are
After an elementary calculation, we get:

\[
Z^{5D}_{\nu_1, \nu_3; \nu_2, \nu_4} = \sum_{\{\nu_i\}} \left( -\sqrt{\frac{q}{t}} Q_i \right)^{\nu_i} \prod_{\{r,s\} \atop (j,k) \in \nu_r} (1 - Q_{m_r} t^{-\rho_k - \nu_{i,j}^k} q^{-\rho_j - \nu_{r,k}})(1 - Q_{m_r} t^{\rho_k + \nu_{i,j}^k} q^{\rho_j + \nu_{r,k}}) \\
\times \prod_{m=1}^2 \prod_{\{j,k\} \in \nu_2} (1 - \dot{Q}_{1,m} t^{-\rho_j - \nu_{i,k}^j} q^{-\rho_j - \nu_{2,j}}) \prod_{\{j,k\} \in \nu_4} (1 - \dot{Q}_{2,m} t^{\rho_j + \nu_{i,k}^j} q^{\rho_j + \nu_{2,j}}) \\
\times \prod_{n=1}^2 \prod_{\{j,k\} \in \nu_4} (1 - \dot{Q}_{2,n} t^{-\rho_j - \nu_{i,k}^j} q^{-\rho_j - \nu_{4,j}}) \prod_{\{j,k\} \in \nu_1} (1 - \dot{Q}_{2,n} t^{\rho_j + \nu_{i,k}^j} q^{\rho_j + \nu_{4,j}}) \\
\times \prod_{s \in \nu_i} \left( 1 - t^{-l_{\nu_i}(s)} q^{-a_{\nu_i}(s)} \right)^{-1} \prod_{s \in \nu_i} \left( 1 - t^{l_{\nu_i}(s)} q^{a_{\nu_i}(s)+1} \right)^{-1} \\
\times \prod_{\{p,q\} \atop (j,k) \in \nu_p} (1 - \dot{Q}_{p} t^{j-\nu_{i,k}^j} q^{-\nu_{p,j} + k-1}) \prod_{\{j,k\} \in \nu_q} (1 - \dot{Q}_{q} t^{\nu_{p,k}^j - j+1} q^{\nu_{q,j} - k} q^{-\nu_{p,j} + k-1})^{-1},
\]

respectively. The instanton part is given by

\[
Z^{5D}_{\text{inst}}(U(2) \times U(2)) = \frac{Z^{5D}_{\nu_1, \nu_3; \nu_2, \nu_4} Z^{5D}_{\nu_3, \nu_1; \nu_4, \nu_2}}{Z^{5D}_{\nu_1, \nu_3; \nu_2, \nu_4} Z^{5D}_{\nu_3, \nu_1; \nu_4, \nu_2}}.
\]
If one defines the following identities

\[ Z^{\text{bifund}, 5D}_{\nu_a^{(\ell)}, \nu_b^{(\ell+1)}} (Q_{ab}^{(\ell, \ell+1)}, t, q) = \prod_{(i,j) \in \nu_a^{(\ell)}} (1 - Q_{ab}^{(\ell, \ell+1)} t^{\nu_b^{(\ell+1)} j - i - \nu_{a,i}^{(\ell)} + 1} q^{\nu_{a,i}^{(\ell)} - j + 1}) \times \prod_{(i,j) \in \nu_b^{(\ell+1)}} (1 - Q_{ab}^{(\ell, \ell+1)} t^{\nu_{a,j}^{(\ell)} - i - \nu_{b,i}^{(\ell+1)} + 1} q^{\nu_{b,i}^{(\ell+1)} + j}) \]

\[ Z^{\text{vec}, 5D}_{\nu_a^{(\ell)}, \nu_b^{(\ell)}} (Q_{ab}^{\ell}, t, q) = \left[ \prod_{(i,j) \in \nu_a^{(\ell)}} (1 - Q_{ab}^{(\ell)} t^{\nu_b^{(\ell)} j - i} q^{\nu_{a,i}^{(\ell)} - j + 1}) \prod_{(i,j) \in \nu_b^{(\ell)}} (1 - Q_{ab}^{(\ell)} t^{\nu_{a,j}^{(\ell)} - i - 1} q^{\nu_{b,i}^{(\ell)} + j}) \right]^{-1} \]

the above refined 5D instanton partition can be written as

\[ Z^{\text{inst}}_{5D} = \sum_{\{\nu_i\}} \prod_{i=1}^{4} \left( -\sqrt{\frac{q}{t}} Q_{m_i}^{\nu_i} \right) \prod_{a,b=1}^{2} Z^{\text{bifund}, 5D}_{\nu_a^{(\ell)}, \nu_b^{(\ell+1)}} (Q_{ab}^{(\ell, \ell+1)}, t, q) \times Z^{\text{vec}, 5D}_{\nu_a^{(\ell)}, \nu_b^{(\ell)}} (Q_{ab}^{\ell}, t, q) Z^{\text{vec}, 5D}_{\nu_a^{(\ell)}, \nu_b^{(\ell)}} (t, q). \] (2.15)

This formula coincides with Nekrasov’s instanton partition function for 5D $N = 2$ $U(2) \times U(2)$ gauge theory.

### 2.3.4 $U(N)$ M-node necklace quiver theory

The generalization to $U(N)$ M-node necklace quiver theory (Fig.9) is straightforward, and the result has the following expression

\[ Z^{U(N) \text{ inst}}_{M \text{-necklace}} = \prod_{i=1}^{N,M} \left( -\sqrt{\frac{q}{t}} Q_i^{(\ell)} \nu_i^{(\ell)} \right) \prod_{a,b=1}^{N} Z^{\text{bifund}, 5D}_{\nu_a^{(\ell)}, \nu_b^{(\ell+1)}} (Q_{ab}^{(\ell, \ell+1)}, t, q) \times Z^{\text{vec}, 5D}_{\nu_a^{(\ell)}, \nu_b^{(\ell)}} (Q_{ab}^{\ell}, t, q) Z^{\text{vec}, 5D}_{\nu_a^{(\ell)}, \nu_b^{(\ell)}} (t, q). \] (2.17)

This can be easily proved by using the RTV formulation and the mathematical induction.

### 2.4 4D field theory limit

To compare the refined partition functions with the real 4D theories, one could shrink the perimeter of the cyclic 5-th dimension to zero, that is, $\beta \rightarrow 0$.

For $U(1)$ theory, the parameters are set as

\[ Q_m = \sqrt{\frac{t}{q}} e^{\beta (-\bar{\bar{m}})} = e^{\beta (\epsilon_+ / 2 - \bar{\bar{m}})} \quad Q = \sqrt{\frac{t}{q}} e^{\beta (-\bar{\bar{a}} + \bar{\bar{m}})} = e^{\beta (-\bar{\bar{a}} + \bar{\bar{m}} - \epsilon_+/2)}, \]
Figure 7 Toric diagram for elliptic M-node $U(N)$ theory

where

$$m = \frac{\tilde{m}}{\epsilon_2}, \quad a = \frac{\tilde{a}}{\epsilon_2}.$$  

Then the U(1) instanton partition function reads

$$Z_{D,U}^{4,(1)} = \lim_{\beta \to 0} Z_{D}^{5}(Q_m, t, q)$$  

$$= \sum_{\nu} \left( \sqrt{\frac{q}{t} Q_{\nu}} \right)^{|\nu|} \prod_{s \in \nu} \left( -m + \beta l(s) + a(s) + 1 \right) \left( -m - \beta (l(s) + 1) - a(s) \right) \left( \beta (a(s) + 1) + l(s) \right) \left( \beta a(s) + 1 + l(s) \right)$$  

(2.18)

For $U(2)$ theory, the parameters are set as

$$Q_{m_1} = Q_{m_2} = \sqrt{\frac{t}{q}} e^{\beta (-\tilde{m})}, \quad Q Q_{m_1} = Q Q_{m_2} = e^{\beta (-\tilde{a})}.$$  

The instanton partition function for this theory is

$$Z_{D,U}^{4,(2)} = \lim_{\theta \to 0} Z_{D}^{5}(U(2))$$  

$$= \sum_{\nu_1, \nu_2} \prod_{i=1,j=1}^{2} \left( \sqrt{\frac{q}{t} Q_i} \right)^{|\nu_i|}$$  

$$\times \prod_{s \in \nu_1} \left( -m_{i,s} + a_{\nu_i}(s) + 1 + \beta l_{\nu_i}(s) \right) \prod_{t \in \nu_j} \left( -m_{i,j} - a_{\nu_j}(t) - \beta (l_{\nu_j}(t) + 1) \right)$$  

$$\times \left[ \prod_{i \neq j} \prod_{s \in \nu_i} \left( a_{i,s} + a_{\nu_i}(s) + 1 + \beta l_{\nu_i}(s) \right) \prod_{t \in \nu_j} \left( a_{i,j} - a_{\nu_j}(t) - \beta (l_{\nu_j}(t) + 1) \right) \right]^{-1}$$  

$$\times \left[ \prod_{s \in \nu_1} \left( \beta (l_{\nu_1}(s) + 1) + a_{\nu_i}(s) \right) \left( \beta l_{\nu_1}(s) + 1 + a_{\nu_i}(s) \right) \right]^{-1}.$$
here
\[ m_{i,j} = a_i - a_j - m, \quad a_{i,j} = a_i - a_j, \quad a_{1,2} = -a_{2,1} = a. \]

Now one can immediately read off the expression (1.3) we proposed in the introduction section. This is just a substitution
\[ \tilde{Q}_i = \sqrt{q} Q_i, \quad \lambda_{\ell,i} = \nu^{(\ell)}_i. \]

The \( U(2) \times U(2) \) instanton partition function is just a product of two \( U(2) \) ones,
\[
Z_{\text{inst}}^{4D,U(2) \times U(2)} = \lim_{\theta \to 0} Z_{\text{inst}}^{5D}(U(2) \times U(2)) = \sum_{\vec{\nu}_1, \vec{\nu}_2} \prod_{i,\ell=1}^{2} \left( \langle E^{a(i)}_{j,k}(E^*)^{\beta-a(j)}_{j,k} J_{\nu_{i,j}}, J_{\nu_{i,k}} \rangle_{\beta} \right)_{j \neq k}^{-1} \prod_{j,k=1}^{2} \langle J_{\nu_{i,j}}, J_{\nu_{i,k}} \rangle_{\beta}^{-1} \times \prod_{m,n=1}^{2} \langle E^{m(\ell+1)}_{m,n}(E^*)^{\beta-m(\ell+1)}_{m,n} J_{\nu_{\ell,m}}, J_{\nu_{\ell+1,n}} \rangle_{\beta}. \tag{2.19}
\]

It is easy to generalize to the M-node quiver \( U(N) \) theory. The result is given in Eq. (1.3).

3. Jack symmetric functions and eCS models

Since the \( N = 2^* \) theories are all superconformal field theories, according to the 4d-2d relation proposed by Alday, Gaiotto and Tachikawa [13], there are 2D Liouville/Toda integrable systems corresponding to these gauge theories. The \( N = 2^* \) theories are related to the elliptic Calogero-Sutherland(eCS) models. The CS model plays an important role in many subjects in physics and mathematics. Such as conformal field theory(CFT), unitary matrix models, fractional quantum hall effects(FQHE), etc. Its spectrum can be totally released from the so-called Jack polynomials. In principle, the total system can be solved by using the properties of Jack polynomials, from the CFT point of view, Jack polynomials have natural meanings of characters of the symmetry which drives the model. For instance, the Jack polynomials related to certain Young tableaux are believed to correspond to the singular vectors of \( W \)-algebra, this algebra reflects the hidden \( W_{1+\infty} \) symmetry of CS model.

The instanton counting of the \( N = 2^* \) theories should be related to the counting of the BPS spectrums in 4D gauge theories. In 2D point of view, this can be seen as the counting of the admissible representations of the eCS model, that is, the counting of singular vectors in the model. As pointed out in Awata, Sakamoto’s works [40, 43] on singular vectors in CS model, Jack polynomials and skew Jack polynomials define the singular vector space under \( W \)-algebra.

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5. The eCS model can be seen as the analytic continuation of the original CS model.
6. Actually, the Jack polynomials associated with rectangular Young tableaux are singular vectors of Virasoro
3.1 Jack polynomials and Calogero-Sutherland model

The Hamiltonian of Calogero-Sutherland model is given by:

\[ H = P_i^2 + \beta (\beta - 1) \sin^2 \left( \frac{1}{2} (x_i - x_j) \right) = (-i \partial_i - iA_i)(-i \partial_i + iA_i) = -\partial_i^2 + A_i^2 - \sum_i \partial_i A_i \]  \hspace{1cm} (3.20)

here \( \partial_i = \partial_{x_i}, A_i = \sum_i \beta \cot g x_{ij} \) its ground state captures by the equation of motion:

\[ (-i \partial_i + iA_i) \psi_0 = 0, \]

the general solution gives

\[ \psi_0 = \sum_{i<j} \sin^\beta (x_i - x_j). \]

Define the excitation state as \( \psi_\lambda = J_\lambda \psi_0 \), thus it should satisfy

\[ [-\partial_i^2 + \beta \sum \cot g x_{ij} (\partial_i - \partial_j)] J_\lambda = c_\lambda J_\lambda. \]

It is not hard to get the operator formulism for this \( \hat{J}_\lambda \). However, there are very simple vertex operator maps from Calogero-Sutherland model to CFT. Denote

\[ \psi_0 = \langle k_f | V_k(z_1) \cdots V_k(z_n) | k_i \rangle \]

\[ z_i = e^{ix_i} \]

\[ V_k(z_i) = e^{ik\phi(z_i)} \]

and choose proper vacuum momentum \( k_i \) as that

\[ \psi_0 \sim \prod_{i<j}^N (z_i - z_j)^{k^2} \prod_{i=1}^N z_i^{k_{i,k}}, \]

then the excitation state is just as following

\[ \psi_\lambda = \langle k_f | \hat{J}_\lambda V_k(z_1) \cdots V_k(z_n) | k_i \rangle \]

\[ \hat{J}_\lambda = \sum_n d^{[n]}_\lambda \hat{P}^{[n]} = \sum_n d^{[n]}_\lambda \frac{\hat{a}^{[n]}_\lambda}{(\sqrt{\beta})^{l(\lambda)}} \]

\[ \hat{a}^{[n]}_\lambda = \hat{a}_{n_1} \cdots \hat{a}_{n_l}, \]

algebra. The non-rectangular ones are related to W-algebra.
here $\hat{P}_{[n]}$ is the operator formulism of Newton polynomial, $\ell(\lambda)$ is the total number of rows in $\lambda$. $d^{[n]}_\lambda$ is the normalization factor such that the normalization of $J_\lambda(z^i)$ (for the partition $\lambda = \{j^k_j\}$)

$$\langle J_\lambda, J_\mu \rangle_\theta = \delta_{\lambda\mu}d^{[n]}_\lambda d^{[n]}_\mu \langle k_f | \frac{\hat{a}_\lambda \hat{a}_\mu}{\beta \ell(\lambda) \beta \ell(\mu)} | k_i + Nk \rangle$$

(3.23)

$$j_\lambda = \prod_{s \in \lambda} (a_\lambda(s) + \beta(l_\lambda(s) + 1))(\beta l_\lambda(s) + a_\lambda(s) + 1),$$

$$z_\lambda^j = \prod_{j=1}^{\infty} j^{k_j} k_j!,$$

here $k_i \rightarrow k_i + Nk$ reflects the action of the zero modes of vertex operators. By the mode expansion of the free boson field $\phi(z)$, one reaches

$$\phi(z) = \hat{q} + \hat{p} \ln z + \sum_{n \in \mathbb{Z}, n \neq 0} \frac{\hat{a}_n}{n} z^n$$

$$V_k(z) = e^{k \cdot \phi(z)}.$$

Substitute these to Eq.(3.22), it is easy to show that

$$\psi_\lambda = J_\lambda \psi_0$$

$$= \langle k_f | d^{[n]}_\lambda \frac{\hat{a}[n]}{(\sqrt{\beta} \ell(\lambda))} \prod_{i < j}^N (z_i - z_j)^{k^2} \prod_{i=1}^N z_i^{k_i} \sum_{m \in \mathbb{Z}^+} k^{\frac{a_m-m}{m}} \frac{z^m}{\sqrt{\beta}} | k_i + Nk \rangle$$

$$= \langle k_f | e^{\hat{a}[n]} \sum_i k^{z_{i1}^{n1}} \frac{1}{\sqrt{\beta}} \sum_i k^{z_{i2}^{n2}} \frac{1}{\sqrt{\beta}} \cdots \sum_i k^{z_{il}^{nl}} \frac{1}{\sqrt{\beta}} \psi_0(z_i) | k_i + Nk \rangle$$

$e^{\sum_{m \in \mathbb{Z}^+} k^{\frac{a_m-m}{m}} \frac{z^m}{\sqrt{\beta}}}$ is the remaining $\prod_i^N V_k^+$ after normal ordering, we have used the relation $\hat{a} e^{\hat{a}^+ \alpha} | 0 \rangle = \alpha e^{\hat{a}^+ \alpha} | 0 \rangle$ from the second step to the third step of the above expression. If $k = \sqrt{\beta}$, we see that the Jack polynomial $J_\lambda(z)$ is actually can be seen as the excitation state of the CS model.

3.2 Screening charges and singular vectors

It is shown in Dijkgraaf and Vafa’s article[5] the screening charges are related to instanton insertions. The screening charges of CS model are defined as in[40, 45]

$$\alpha_+ = k, \quad \alpha_- = -\frac{1}{k^2}$$

then by the Felder’s cohomology[47] and using Thorn’s method[40, 44, 45], one can easily prove that the singular vector $| \chi_{s^r, r-s}^+ \rangle$ associated with rectangular Young tableau $\lambda = \{s^r\}$ can
be written as

\[ |\chi_{r,s}^+\rangle = \oint \frac{dz_j}{2\pi i} \prod_{i=1}^{r} e^{\alpha + \phi(z_i)} : |\alpha_{r,s}\rangle \] (3.24)

\[ = \oint \frac{dz_j}{2\pi i} \prod_{i \neq j} (z_i - z_j)^{2\beta} \prod_{i=1}^{r} z_i^{(1-r)\beta - s} \prod_{j=1}^{r} e^{\alpha + \phi(z_j)} |\alpha_{r,s}\rangle \]

The integration contours have been chosen as the Felder’s contours as in Fig.8

\[ |\alpha\rangle = e^{\hat{q}}|0\rangle, \quad \alpha_{r,s} = \frac{(1-r)\alpha_+ + (1-s)\alpha_-}{2} \]

Similarly, as proved in [40], the Jack polynomials associated with non-rectangular Young tableaux are related to the singular vectors for \( W_N \)-algebra.

Figure 8: Felder’s Integration contours

The Jack polynomial can be identified with the following expression

\[ \mathcal{N}_{r,s}^+ \mathcal{N}_{(s')}^+ J_{(s')} (x) = \langle \alpha_{r,s} | C_k | \chi_{r,s}^+ \rangle \]

\[ = \oint \frac{dz_j}{2\pi i} \prod_{i \neq j} (z_i - z_j)^{2\beta} \prod_{i=1}^{r} z_i^{(1-r)\beta - s} \prod_{j=1}^{r} e^{\alpha + \phi(z_j)} |\alpha_{r,s}\rangle \]

\[ C_k = e^{R \sum_{n>0} \frac{1}{\pi} a_n p_n} = \prod_{i} V_k^- (w_i), \quad V_k^- (w_i) = e^{-k\phi_-(w_i)} \]

where the normalization constants \( \mathcal{N}_{\lambda}^+ \) [48] and \( \mathcal{N}_{r,s}^+ \) [40] are given by

\[ \mathcal{N}_{\lambda}^+ = \prod_{s \in \lambda} \frac{(\ell_{\lambda}(s) + 1)\beta + a_\lambda(s)}{\ell_{\lambda}(s)\beta + a_\lambda(s) + 1}, \quad \mathcal{N}_{r,s}^+ = \frac{1}{r!} \prod_{j=1}^{r} \frac{\sin \pi j \beta}{\sin \pi \beta} \cdot \frac{\Gamma(r \beta + 1)}{\Gamma(\beta + 1)^r}. \] (3.26)

Similarly, as proved in [40], the Jack polynomials associated with non-rectangular Young tableaux are related to the singular vectors for \( W_N \)-algebra.

\[ |\chi_{r,s}^-\rangle = \oint \frac{dz_j}{2\pi i} \prod_{a=1}^{N-1} \prod_{j=1}^{s^a} e^{\alpha + \phi(z_j^a)} : |\lambda_{r,s}^- - \alpha_+ \sum_{a=1}^{N-1} \tilde{r}^a \tilde{\alpha}^a \rangle \] (3.27)
with $s^1 > \cdots > s^{N-1}$. The corresponding Young tableau is showed as follows.

$$\lambda = \begin{array}{cccc}
  s^1 & s^2 & \cdots & s^{N-2} \\
  r^1 & r^2 & \cdots & r^{N-2} \\
  & & & r^{N-1}
\end{array}.$$ 

The operator formalism of generic Jack polynomial can be identified with the insertion between the left and the right vacuum denote by $\langle \lambda_{\vec{r},\vec{s}} |$ and $\lambda_{\vec{r},\vec{s}}^- - \alpha_+ \sum_{a=1}^{N-1} r^a \alpha^a \rangle$. It follows

$$\hat{J}_\lambda \sim \prod_i V_k^{-}(w_i) \oint \prod_{a=1}^{N-1} \frac{dz^a}{2\pi i} \prod_{a=1}^{N-1} : e^{\alpha_+ \phi(z^a)} :.$$ 

(3.28)

### 4. Jack polynomial and Nekrasov’s instanton partition function

The existence of singular vectors implies that correlation functions in CS model can be split into conformal blocks. These conformal blocks, due to the AGT relation, should be exact the instanton partition function of the related $\mathcal{N} = 2^*$ theory. However, for the M-node necklace quiver gauge theory, the related 2D correlation function is still hard to calculate. However, we can read off that there should have a more simple description of this correlation function from the result we obtained in present paper. It is just simple multiplications of two point functions within the insertion of two Jack polynomials! This is a factorization formulism rather than a summation, the combinatorial properties of conformal blocks are totally determined by the Jack polynomials. Now we extract these information at the level of result. We will explain the hidden physics using Dijkgraaf-Vafa’s mirror B-model picture in the second part of this note.

The deformation parameters of the eCS model can be written as

$$\epsilon_1 = ig_s k, \quad \epsilon_2 = \frac{ig_s}{k}, \quad Q = k + \frac{1}{k} = \frac{\epsilon_1 + \epsilon_2}{ig_s}.$$ 

(4.29)

The bifundamental part is the building block of the instanton partition function. The expression reads

$$Z_{\text{bifund}}^{4D \ \text{inst}}(\vec{a}_\ell, \vec{\lambda}_\ell, \vec{a}_{\ell+1}, \vec{\lambda}_{\ell+1}; m_\ell) = \prod_{m,n=1}^N \langle E^{m(\ell,\ell+1)}_{m_n} (E^*)^{\beta-m(\ell,\ell+1)}_{m_n} -1 J_{\lambda_{\ell,m}, \lambda_{\ell+1,n}} \rangle_\beta.$$ 

(4.30)
The insertion of $E^{m_{m,n}}_{\ell_1,\ell_2+1}$, $m_{m,n} = a_{\ell_1,n} - a_{\ell_2,m} - m_{\ell}$ can be rewritten as follows

$$
\langle 0 | C_{k} \exp \left( \frac{-\epsilon_{2}}{m_{\ell}} (-1)^{n} p_{n} \right) = \langle 0 | C_{k} \exp \left( \frac{im}{g_{a}} (-1)^{n} a_{n} \right) \tag{4.31} \\
= \langle 0 | C_{k} \prod_{\ell_{m}} \Gamma (1) \\
= \langle 0 | s_{\mu} (-1,-1,-1,...) C_{k} \\
= \sum_{\mu} \langle \mu | (-1)^{\mu} | C_{k} \rangle.
$$

The conjugate state induced by the insertion of $(E^*)^{\beta-m-1}$ has the same express except the conjugate charge is given by $\epsilon_{+} - m$ as expectation. The whole expression now is given by

$$
\sum_{\mu} \langle \mu | (-1)^{\mu} | \hat{J}_{\lambda} \hat{J}_{\nu} \sum_{\mu'} (-1)^{\mu'} | \mu' \rangle \tag{4.32}
$$

When one expands the $s_{\mu}$ as the monomial symmetric function $m_{\mu}$, and writes the Jack polynomial as the complete homogeneous symmetric function $h_{\lambda}$, one immediately gets the right expression of the inner product. The insertions of $E$ and $E^*$ have a explanation that $m$ Wilson loops translated between the associated branes, also, this will be shown in the second part of this note.

### 5. Conclusions and discussions

We calculate in this note the instanton partition function of the elliptic N=2 M-node quiver gauge theory using the refined topological vertex formulation. The result exactly coincident with Nekrasov’s instanton partition. We find the instanton counting of $N = 2^*$ theories has a neat expression in terms of Jack polynomials as expected[3]. We give a explanation of the expression at the level of result. This result implies that the AGT duality between 4D $N = 2$ supersymmetric gauge theories and the 2D conformal field theories has more refined structures such as the physical reason of the factorization of conformal blocks.

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7This can be done as that in Stanley’s article[48] and Macdonald’s textbook[49].
References

[1] R. Donagi, "Seiberg-Witten Integrable Systems" "Surveys in Differential Geometry", arxiv: alg-geom/9705010

[2] N. Nekrasov "Seiberg-Witten prepotential from instanton counting" "Proceedings of the ICM, Beijing 2002", vol. 3, 477–496 arxiv: hep-th/0206061

[3] N. Nekrasov, S. Shatashvili "Quantum integrability and supersymmetric vacua" arXiv:0901.4748 [hep-th]

[4] N. Seiberg, E. Witten,"Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory", Nuclear Phys. B 426 (1): 19C52. N. Seiberg, E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD", Nuclear Phys. B 431 (3): 484C550 W. Lerche "Introduction to Seiberg-Witten Theory and its Stringy Origin"Nucl.Phys.Proc.Suppl. 55B (1997) 83-117; Fortsch.Phys. 45 (1997) 293-340; arxiv: hep-th/9611190

[5] R. Dijkgraaf, C. Vafa, "Toda Theories, Matrix Models, Topological Strings, and N=2 Gauge Systems", arXiv:0909.2453v1 [hep-th]

[6] M. Cheng, R. Dijkgraaf, C. Vafa, "Non-Perturbative Topological Strings And Conformal Blocks", arXiv:1010.4573v1 [hep-th]

[7] E. Witten, "Solutions Of Four-Dimensional Field Theories Via M Theory", Nucl.Phys.B 500:3-42,1997

[8] R. Donagi, E. Witten, "Supersymmetric Yang-Mills Systems And Integrable Systems", Nucl.Phys.B 460:299-334,1996

[9] P. Argyres, N. Seiberg, "S-duality in N=2 supersymmetric gauge theories", JHEP 0712:088,2007

[10] D. Gaiotto, "N = 2 Dualities", arXiv:0904.2715v1 [hep-th]

[11] M. Aganagic, A. Klemm, M. Marino, C. Vafa, "The Topological Vertex", Commun. Math. Phys. 254, 425-478(2005)

[12] A. Iqbal, C. Kozcaz, C. Vafa "The Refined Topological Vertex", JHEP 0910:069,2009

[13] L.F. Alday, D. Gaiotto, Y. Tachikawa, "Liouville Correlation Functions from Four-dimensional Gauge Theories" Lett. Math. Phys. 91 (2010) 167-197
[14] R. Szabo, "Instantons, Topological Strings and Enumerative Geometry ", arXiv:0912.1509

[15] N. Drukker, D. Morrison, T. Okuda, "Loop operators and S-duality from curves on Riemann surfaces ", arXiv:0907.2593. N. Drukker, J. Gomis, T. Okuda, J. Teschner "Gauge Theory Loop Operators and Liouville Theory", arXiv:0909.1105

[16] N. Drukker, D. Gaiotto, J. Gomis "The Virtue of Defects in 4D Gauge Theories and 2D CFTs", arXiv:1003.1112

[17] T. Eguchi, K. Maruyoshi, "Penner Type Matrix Model and Seiberg-Witten Theory", arXiv:0911.4797

[18] T. Eguchi, K. Maruyoshi, "Seiberg-Witten theory, matrix model and AGT relation", arXiv:1006.0828

[19] N. Nekrasov, E. Witten, "The Omega Deformation, Branes, Integrability, and Liouville Theory", arXiv:1002.0888

[20] Y. Nakayama "Refined Cigar and Omega-deformed Conifold", arXiv:1004.2986

[21] K. Maruyoshi, M. Taki, "Deformed Prepotential, Quantum Integrable System and Liouville Field Theory", arXiv:1006.1214

[22] H. Liu, "Notes On U(1) Instanton Counting On A_{1−1} ALE Spaces ", arXiv:009.3324

[23] L. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa, H. Verlinde "Loop and surface operators in N=2 gauge theory and Liouville modular geometry ", arXiv:0909.0945

[24] D. Nanopoulos, D. Xie "Hitchin Equation, Singularity, and N=2 Superconformal Field Theories ", arXiv:0911.1990

[25] A. Marshakov, A. Mironov, A. Morozov, "On Combinatorial Expansions of Conformal Blocks," arXiv:0907.3946 [hep-th]. A. Mironov, S. Mironov, A. Morozov, A. Morozov, "CFT exercises for the needs of AGT," arXiv:0908.2064 [hep-th]. A. Mironov, A. Morozov, "The Power of Nekrasov Functions," arXiv:0908.2190 [hep-th]. A. Mironov, A. Morozov, "On AGT relation in the case of U(3)," arXiv:0908.2569 [hep-th]. A. Marshakov, A. Mironov, A. Morozov, "On non-conformal limit of the AGT relations," arXiv:0909.2052 [hep-th]. A. Marshakov, A. Mironov, A. Morozov, "Zamolodchikov asymptotic formula and instanton expansion in N=2 SUSY N_f = 2N_c QCD," arXiv:0909.3338 [hep-th]. A. Mironov and A. Morozov, "Proving AGT relations in the large-c limit," arXiv:0909.5531 [hep-th]. A. Mironov and A. Morozov, "Nekrasov Functions and Exact Bohr-Zommerfeld Integrals," arXiv:0910.5670 [hep-th]. V. Alba and
A. Morozov, “Non-conformal limit of AGT relation from the 1-point torus conformal block,” [arXiv:0911.0363][hep-th].

[26] K. Maruyoshi, M. Taki, S. Terashima, F. Yagi, “New Seiberg Dualities from N=2 Dualities”, JHEP 0909:031,2009 [arXiv:0907.2625][hep-th].

[27] D. Gaiotto, “Asymptotically free N=2 theories and irregular conformal blocks”, arXiv:0908.0307 [hep-th].

[28] S. M. Iguri, C. A. Nunez, “Coulomb integrals and conformal blocks in the AdS3-WZNW model,” arXiv:0908.3460 [hep-th].

[29] R. Poghossian, “Recursion relations in CFT and N=2 SYM theory,” [arXiv:0909.3412][hep-th].

[30] G. Bonelli, A. Tanzini, “Hitchin systems, N=2 gauge theories and W-gravity,” arXiv:0909.4031 [hep-th].

[31] S. Giombi, V. Pestun, “The 1/2 BPS ’t Hooft loops in N=4 SYM as instantons in 2d Yang-Mills,” [arXiv:0909.4272][hep-th].

[32] H. Ooguri, C. Vafa, ”Knots Invariants and Topological Strings”, Nucl.Phys.B 577:419-438,2000

[33] S. Katz, A. Klemm, C. Vafa “Geometric Engineering of Quantum Field Theories”, Nucl.Phys. B 497 (1997) 173-195

[34] E. Carlsson, A. Okounkov ”Exts and Vertex Operators”, [arXiv:0801.2565]

[35] R. Gopakumar, C. Vafa ”M theory and Topological Strings I”, [arXiv:hep-th/9809187]

[36] R. Gopakumar, C. Vafa ”M theory and Topological Strings II”, [arXiv:hep-th/9812127]

[37] M. Taki, ”Surface Operator, Bubbling Calabi-Yau and AGT Relation”, [arXiv:1007.2524]

[38] T. Dimofte, S. Gukov, L. Hollands ”Vortex Counting and Lagrangian 3-manifolds”, [arXiv:1006.0977]

[39] A. Iqbal, C. Kozcaz, T. Sohail ”Periodic Schur Process, Cylindric Partitions and N=2* Theory”, [arXiv:0903.0961]

[40] H. Awata, Y. Matsuo, S. Odake, ”Excited States of Calogero-Sutherland Model and Singular Vectors of the W_N Algebra”, Nucl.Phys. B449 (1995) 347-374
[41] H. Awata, H. Kanno, "Instanton counting, Macdonald function and the moduli space of D-branes", JHEP 0505 (2005) 039

[42] H. Awata, Y. Yamada, "Five-dimensional AGT Conjecture and the Deformed Virasoro Algebra", JHEP 1001:125,2010

[43] R. Sakamoto, J. Shiraishi, D. Arnaudon, L. Frappat, E. Ragoucy, "Correspondence between conformal field theory and Calogero-Sutherland model", Nucl.Phys. B704 (2005) 490-509

[44] R. Brower, C. Thorn, "Eliminating Spurious States from the Dual Resonance Model", Nucl.Phys. B31, 163-182 (1971)

[45] M. Yu, J. F. Wu, Y. Y. Xu, "Singular Vectors in Calogero-Sutherland Models and a New Approach to Skew Jack Polynomials" (in preparing)

[46] J. F. Wu, "Note on Refined Topological Vertex, Jack Polynomials and Instanton Counting(II)", (to appear)

[47] G. Felder, "BRST approach to minimal models", Nucl.Phys. B 317 (1989) 215-236

[48] R. Stanley, "Some combinatorial properties of Jack symmetric functions", Advances in Mathematics 77, 76-115(1989)

[49] I. Macdonald, "Symmetric functions and Hall polynomials" 2nd Edition, Camb. Univ. Press(1995)