Stability of mKdV breathers on the half-line

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Abstract
In this paper we study the stability problem for mKdV breathers on the left half-line. We are able to show that leftwards moving breathers, initially located far away from the origin, are strongly stable for the problem posed on the left half-line, when assuming homogeneous boundary conditions. The proof involves a Lyapunov functional which is almost conserved by the mKdV flow once we control some boundary terms which naturally arise.

Keywords Modified KdV equation · Breather solution · Cauchy Problem · Orbital stability · Half-line

Mathematics Subject Classification Primary 35Q55

1 Introduction

1.1 Setting of the problem
This paper deals with the nonlinear stability of breathers of the focusing modified Korteweg-de Vries (mKdV) equation introduced in [17] and posed on the left half-line \( \mathbb{R}^- \) := \((-\infty, 0)\):

\[
\partial_t u + \partial_x (\partial_x^2 u + u^3) = 0, \quad u(x, t) \in \mathbb{R}, \quad (x, t) \in \mathbb{R}^- \times (0, T).
\] (1.1)
The focusing\(^1\) mKdV equation (1.1) in the whole real line \(\mathbb{R}\), is an integrable and canonical non-linear dispersive equation, originally describing shallow water wave dynamics [18], and therefore appearing as a good approximation of different physical problems. A few examples are the motion of the curvature of some geometric fluxes [9, 15,16], vortex patches, ferromagnetic vortices [19], traffic models, anharmonic lattices, hyperbolic surfaces, among others. As a consequence of its integrability in \(\mathbb{R}\), it is possible to get explicit solutions. For instance, the simplest one is the (real-valued) mKdV soliton solution which, to be more precise, has the form

\[
u(x,t) = \tilde{Q}_c(x-ct-x_0), \quad \tilde{Q}_c(s) := \sqrt{c} \tilde{Q}(\sqrt{c} s), \quad c > 0, \ x_0 \in \mathbb{R}, \tag{1.2}\]

where

\[
\tilde{Q}(s) = \frac{\sqrt{2}}{\cosh(s)} = 2\sqrt{2} \partial_s [\arctan(e^s)], \tag{1.3}
\]

with \(c\) the propagation speed of the wave. The real-line soliton \(\tilde{Q}_c\) satisfies the following “boundary value problem” (BVP) on \(\mathbb{R}\),

\[
\begin{aligned}
\tilde{Q}''_c - c \tilde{Q}_c + \tilde{Q}^3_c &= 0, \quad x \in \mathbb{R}, \\
\lim_{x \to \pm \infty} \tilde{Q}(x) &= 0, \tag{1.4}
\end{aligned}
\]

and it is the unique positive \(H^1(\mathbb{R})\)-solution of (1.1) up to translations in space.

The Cauchy theory for the initial value problem (IVP) for the focusing mKdV posed on the real axis,

\[
\begin{aligned}
\partial_t u + \partial_x (\partial_x^2 u + u^3) &= 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \\
u(x,0) &= u_0(x), \quad x \in \mathbb{R}, \tag{1.5}
\end{aligned}
\]

has been extensively studied in the last years. In the case of real-valued initial data, the IVP for (1.5) is globally well posed for initial data in \(H^s(\mathbb{R})\) for any \(s > 1/4\); see [11] and [7]. Moreover, the (real-valued) flow map is not uniformly continuous if \(s < 1/4\) (see [12]). This was proved by using a special family of solutions of (1.5) called breathers and discovered by Wadati [17]. Explicitly, the mKdV breather is defined as follows.

**Definition 1.1** (See e.g. [13, 17]) Let \(\alpha, \beta > 0\) and \(x_1, x_2 \in \mathbb{R}\) be fixed parameters. The focusing mKdV breather is a smooth solution of (1.5) given by the formula

\[
\begin{aligned}
\tilde{B}_{\alpha,\beta}(x,t; x_1, x_2) := 2\sqrt{2} \partial_x \left[ \arctan \left( \frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right] \\
= 2\sqrt{2} \beta \operatorname{sech}(y_2) \left[ \cos(\alpha y_1) - (\beta/\alpha) \sin(\alpha y_1) \tanh(\beta y_2) \over 1 + (\beta/\alpha)^2 \sin^2(\alpha y_1) \operatorname{sech}^2(\beta y_2) \right], \tag{1.6}
\end{aligned}
\]

where

\[
y_1 := x + \delta t + x_1, \quad y_2 := x + \gamma t + x_2 \tag{1.7}
\]

and

\[
\delta := \alpha^2 - 3\beta^2, \quad \gamma := 3\alpha^2 - \beta^2. \tag{1.8}
\]

\(^1\) Focusing or defocusing means \(\pm u^3\) respectively in the equation.
Observe that this wave-like solution of (1.5) is periodic in variable $y_1$ and localized in variable $y_2$. Also, note that $\gamma \neq \delta$ for any $\alpha, \beta \neq 0$, which implies that the traveling wave arguments$^2$:

$$y_1 = x + \delta t \quad \text{and} \quad y_2 = x + \gamma t$$

are always different. Currently, $\beta$ and $\alpha$ are called amplitude and frequency parameters of the breather, and $-\gamma$ will be the velocity of the mKdV breather solution (1.6). Note that this corresponds to the speed of the sech envelope of the breather profile, dragging to the left or to the right (depending on its sign) the corresponding inner oscillations of the breather. In [2] it was proved that breather solutions of the focusing mKdV equation (1.5) in $\mathbb{R}$ are actually globally stable in a natural $H^2$ topology. In the proof, the authors introduced a new Lyapunov functional, at the $H^2$ level, which allowed to describe the dynamics of small perturbations, including oscillations induced by the periodicity of the solution, as well as a direct control of the corresponding instability modes. In particular, degenerate directions were controlled using low-regularity conservation laws. Finally, we point out that in [5] the soliton resolution for the focusing mKdV equation on the real line $\mathbb{R}$, was established for initial conditions in some weighted Sobolev spaces, where one should realize that general solution to the focusing mKdV will consist of solitons moving to the right, breathers traveling to both directions and a radiation term. Moreover, the authors obtained the asymptotic stability of nonlinear structures involving solitons and breathers.

Note that (regular) breather solutions only appear in some particular PDEs. For instance, in gKdV models, they only arise in the mKdV (1.1) but, they do not appear in the KdV case, as it was recently proved [14]. Therefore, profiting its existence in the mKdV model (1.5), our main aim in this work will be to approach the stability analysis of focusing mKdV breathers in the left half-line $\mathbb{R}^-$. As a direct consequence, we present two main contributions: firstly, we go a step further, in comparison with [4], where the stability analysis for simpler solutions, like KdV solitons in the half-line was presented. Secondly, we extend previous stability results of mKdV breathers in the real line $\mathbb{R}$ (see [2]), by adapting these techniques to the case of boundary conditions as it corresponds to a $\mathbb{R}^-$ domain, which is more realistic case for experimental purposes.

In this work, we consider the mKdV equation on the left half-line and we will deal with mKdV breather solutions (1.6) moving leftwards in space (see Fig. 1), and therefore when its velocity $-\gamma < 0$ or equivalently, from (1.8), when $\beta < \sqrt{3} \alpha$. In this situation we can impose two boundary conditions for the IBVP (1.9).

It remains as an interesting open problem to study the stability properties of these mKdV breathers on the right hand side $\mathbb{R}^+ = (0, +\infty)$. In fact, a few differences with respect to the left hand side arise in that case. For instance, the case of rightwards moving breathers, implies that $\beta > \sqrt{3} \alpha$. Unfortunately, in this situation, we can not impose a second boundary condition $u_x(0, t) = 0$ of the corresponding IBVP. This fact prevent us from constructing a suitable Lyapunov functional, almost conserved and well defined on $H^2(\mathbb{R}^+)$ (See Remark 3.1).

From another point of view, many physical problems naturally arise as initial boundary value problems (IBVP), because of the local character of the corresponding phenomenon [20]. However, the IBVP for the mKdV equation has been considerably less studied than the corresponding IVP (1.5). For example, there are at least two interesting IBVP for mKdV still in unbounded domains: the one posed on the right half-line, and a second one posed on the left portion of the line, which we consider in this work.

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2 Assuming the simplest case $x_1 = x_2 = 0$. 
Fig. 1 The evolution of the mKdV breather (1.6), with $\alpha = 3$, $\beta = 1$, $x_1 = 0$ and $x_2 = 30$ at times $0$, $0.5$ and $1$ (full, dashed, dotted lines, respectively). In this case, $-\gamma = \beta^2 - 3\alpha^2 < 0$ and hence the breather moves leftwards.

1.2 Unbounded initial boundary value problems

The IBVP for the focusing mKdV equation posed on the left half-line is the following: for $\mathbb{R}^- := (-\infty, 0)$ and $T > 0$, we look for solutions $u$ of the model

$$
\begin{aligned}
\partial_t u + \partial_x (\partial_x^2 u + u^3) &= 0, \quad (x, t) \in \mathbb{R}^- \times (0, T), \\
\partial_x u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^-, \\
u(0, t) &= f(t), \quad t \in (0, T), \\
\partial_x u(0, t) &= f_1(t), \quad t \in (0, T).
\end{aligned}
$$

In the recent literature, the mathematical study of IBVP (1.9) is usually considered in the following setting

$$
(u_0, f, f_1) \in H^s(\mathbb{R}^-) \times H^{(s+1)/3}(\mathbb{R}^+) \times H^{s/3}(\mathbb{R}^+).
$$

These assumptions are in some sense sharp because of the following localized smoothing effect for the linear evolution [11]

$$
\| \psi(t)e^{-t\partial_x^3}\phi(x)\|_{C(\mathbb{R}_t; H^{(s+1)/3}(\mathbb{R}_x))} \lesssim \| \phi\|_{H^s(\mathbb{R})},
$$

and

$$
\| \psi(t)\partial_x e^{-t\partial_x^3}\phi(x)\|_{C(\mathbb{R}_x; H^{s/3}(\mathbb{R}_t))} \lesssim \| \phi\|_{H^s(\mathbb{R})},
$$

where $\psi(t)$ is a smooth cutoff function and $e^{-t\partial_x^3}$, denoting the linear homogeneous solution group on $\mathbb{R}$ associated to the linear part of the equation in (1.9). Therefore, and hereafter, we will follow the setting (1.10).
Other classical IBVP is the mKdV on the right half-line given by
\begin{align*}
\partial_t u + \partial_x (\partial_x^2 u + u^3) &= 0, \quad (x, t) \in \mathbb{R}^+ \times (0, T), \\
u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^+, \\
u(0, t) &= f(t), \quad t \in (0, T).
\end{align*}
(1.11)

The presence of one boundary condition in (1.11) versus two boundary conditions in the left half-line problem (1.9) for the KdV-component of the system is justified in [10]. The local well-posedness was considered in [6] on the Sobolev Spaces \( H^{\frac{3}{4}} \left( \mathbb{R}^+ \right) \). It was recently shown in [4] that solitons initially posed far away from the origin are strongly stable for the problem posed on the right half-line, assuming homogeneous boundary conditions. The proof of this stability result involved the construction of two almost conserved quantities adapted to the evolution of the KdV soliton, in the particular case of the half-line.

With respect to previous advances, Faminskii showed global well-posedness for the following IBVP associated to the classical KdV equation (see [8]):
\begin{align*}
\partial_t u + \partial_x (\partial_x^2 u + u^2) &= 0, \quad (x, t) \in \mathbb{R}^- \times (0, T), \\
u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^- , \\
u(0, t) &= f(t), \quad t \in (0, T), \\
\partial_x u(0, t) &= f_1(t), \quad t \in (0, T).
\end{align*}
(1.12)

In the current work, we consider the solution \( u \) posed on the space
\[ u \in C\left( \mathbb{R}^+ ; H^2 \left( \mathbb{R}^- \right) \right) \quad \text{and} \quad \partial_x^j u \in C\left( \mathbb{R}^+ ; H^{\left(3-j\right)/3} (0, T) \right) \quad \text{for} \quad j = 0, 1, 2, 3. \quad (1.13)\]

**Remark 1.1 (Well-posedness)** Concerning the well-posedness theory for the IBVP (1.9) at the level \( H^2 (\mathbb{R}^-) \), we remark the following:

(a) **(Local Theory).** The approach used by Faminskii in [8] to solve a similar problem by considering the quadratic nonlinearity can be applied to our current problem to get a local theory. In fact, local solutions in \( C\left( [0, T] ; H^2 (\mathbb{R}^-) \right) \) for the IBVP (1.9), with conditions (1.10) at the regularity level \( s = 2 \), can be constructed by using the contraction principle. In such a case, the main difficulty is to get the fundamental trilinear estimate needed to solve (1.9) on the modified Bourgain spaces adapted to the corresponding problem posed on the half-line. This is a technical argument and it can be obtained by using similar ideas contained in [3], where the modified Kawahara equation with cubic nonlinearities was studied. There, the key point was to obtain the corresponding trilinear estimates (see Theorem 1.1 in [3]).

(b) **(Global Theory).** Local solutions obtained in (a) can be extended globally in time from apriori estimates presented in Sect. 3 (see Corollary 3.2).

**1.3 Main result**

We consider a breather solution on the left half-line as the restriction on \( \mathbb{R}^- \) of classical breathers posed on the whole line (1.6), i.e.
\[ B_{\alpha, \beta} = \tilde{B}_{\alpha, \beta} \bigg|_{\mathbb{R}^-}. \quad (1.14)\]

We highlight that the above breather on the left half-line is not an exact solution for the IBVP (1.9), except for very particular boundary conditions \( f(t) \) and \( f_1(t) \). More precisely,
restricted breathers \( B = B(x; t; x_1, x_2) \) induce the natural traces given by

\[
f(t) = B(x = 0, t; x_1, x_2) \quad \text{and} \quad f_1(t) = \partial_x B(x = 0, t; x_1, x_2).
\] (1.15)

In this work we will prove that any classical mKdV breather solution, restricted to the left half-line \( \mathbb{R}^- \) posed on the left half-line, and placed far enough from the origin \( x = 0 \), is stable in \( H^2(\mathbb{R}^-) \) under perturbations that preserve the zero boundary conditions. More precisely, we prove the following:

**Theorem 1.2** (Nonlinear \( H^2 \) stability of mKdV breathers on the left half-line) Let \( \alpha, \beta > 0 \), and \( B_{\alpha, \beta} \) a restricted breather (1.14). Assuming that \( \beta \leq \alpha \) (breathers moving leftwards), there exist parameters \( \eta_0, A_0 \) and \( L_0 \), depending on \( \alpha \) and \( \beta \), such that for all \( L > L_0 \) and \( \eta \in (0, \eta_0) \) the following holds: consider \( u_0 \in H^2(\mathbb{R}^-) \) such that

\[
\|u_0 - B_{\alpha, \beta}(\cdot, 0; 0, L)\|_{H^2(\mathbb{R}^-)} \leq \eta.
\] (1.16)

Then there exist continuous functions \( \rho_1(t), \rho_2(t) \in \mathbb{R} \) such that the solution \( u(\cdot, t) \) of the IBVP (1.9) with initial data \( u_0 \) and homogeneous boundary conditions \( f(t) = f_1(t) = 0 \), satisfies

\[
\sup_{t \in \mathbb{R}} \|u(t) - B_{\alpha, \beta}(\cdot, t; \rho_1(t), \rho_2(t) + L)\|_{H^2(\mathbb{R}^-)} \leq A_0 \eta + Ke^{-\beta L}.
\] (1.17)

for some constant \( K > 0 \).

This result shows that leftwards moving breathers posed initially far away from the origin are strongly stable for the IVBP problem (1.9) posed on the left half-line, assuming homogeneous boundary conditions.

Our proof involves an almost conserved Lyapunov functional, for which we have to control some boundary terms. In addition, we have some error contributions that appear because the restricted breather (1.14) is not an exact solution for the initial boundary value problem (1.9).

**Remark 1.2** Some points deserve to be enlightened:

(a) (On the zero boundary condition). Note that conditions \( u(x = 0, t) = u_x(x = 0, t) = 0 \) are assumed to avoid bad trace higher order functions on the energy identities, which are the fundamental ingredients to construct the almost conserved Lyapunov functional. The case with non-homogeneous boundary conditions raises as an interesting open problem.

(b) (Right half-line). The case of the IBVP on the right half-line remains as a challenging open problem. This problem imposes several new conditions with respect to the left half-line case, as for instance, that the breather speed \( -\gamma > 0 \) or that we cannot impose a second boundary condition to the corresponding IBVP.

(c) (Applications). We think that the developed techniques and ideas presented in this work can be applied, with minor changes but with more involved computations, to the Gardner equation posed on the left half-line

\[
w_t + (w_{xx} + 3\mu w^2 + w^3)_x = 0, \quad \mu \in \mathbb{R} \setminus \{0\}, \quad w(x; t) \in \mathbb{R},
\]

\[
(x, t) \in \mathbb{R}^- \times (0, T).
\] (1.18)

This model can be thought as a perturbed focusing mKdV equation, by a small parameter \( \mu \in \mathbb{R} \setminus \{0\} \) controlling the strength of the quadratic nonlinear part or KdV term \( w^2 \). The Gardner eq. (1.18) also bears breather solutions, and they can be interpreted as perturbed mKdV breathers. See [1] for further details.
1.4 Organization of this paper

After some preliminaries in Sect. 2, we show restricted functionals to the left half-line in Sect. 3. Afterwards, in Sect. 4 we prove the main Theorem 1.2. Finally in Appendix A and B we explicitly prove some technical previous results.

2 Preliminaries

In this section we summarize some useful facts obtained in [2] about breather profiles on \( \mathbb{R} \).

Lemma 2.1 The mKdV breather \( \tilde{B} := \tilde{B}_{\alpha, \beta} (1.6) \), with \( \alpha, \beta > 0 \), satisfies the following properties:

(i) \( \tilde{B} = \mathfrak{B}_x \), with \( \mathfrak{B} = \mathfrak{B}_{\alpha, \beta} \) given by the smooth \( L^\infty \)-function,

\[
\mathfrak{B}(x, t) := 2\sqrt{2} \arctan \left( \frac{\beta \sin (\alpha y_1)}{\alpha \cosh (\beta y_2)} \right).
\]

(ii) For any fixed \( t \in \mathbb{R} \), we have \( \mathfrak{B}_t \) well-defined in the Schwartz class, satisfying

\[
\tilde{B}_{xx} + \mathfrak{B}_t + \tilde{B}^3 = 0.
\]

(iii) For all \( t \in \mathbb{R} \), \( \tilde{B} \) satisfies

\[
\tilde{B}_{xt} + 2 \left( \mathcal{M}_{\alpha, \beta} \right)_t \tilde{B} = 2 \left( \beta^2 - \alpha^2 \right) \mathfrak{B}_t + \left( \alpha^2 + \beta^2 \right)^2 \tilde{B},
\]

where

\[
\mathcal{M}_{\alpha, \beta}(x, t) := \frac{1}{2} \int_{-\infty}^{x} \tilde{B}_{\alpha, \beta}^2 (s, t; x_1, x_2) ds
\]

\[
= \frac{2\beta \left[ \alpha^2 + \beta^2 + \alpha \beta \sin (2\alpha y_1) - \beta^2 \cos (2\alpha y_1) + \alpha^2 (\sinh (2\beta y_2) + \cosh (2\beta y_2)) \right]}{\alpha^2 + \beta^2 + \alpha^2 \cosh (2\beta y_2) - \beta^2 \cos (2\alpha y_1)}.
\]

(iv) Also, for all \( t \in \mathbb{R} \), \( \tilde{B} \) satisfies the nonlinear stationary equation

\[
G[\tilde{B}] := \tilde{B}_{x(4x)} - 2 \left( \beta^2 - \alpha^2 \right) \left( \tilde{B}_{xx} + \tilde{B}^3 \right) + \left( \alpha^2 + \beta^2 \right)^2 \tilde{B}
+ 5\tilde{B}_x^2 + 5\tilde{B}^2 \tilde{B}_{xx} + \frac{3}{2} \tilde{B}^5 = 0.
\]

Another important ingredient defined in [2] is the fourth order linear operator

\[
\mathcal{L}[z](x; t) := z_{(4x)}(x) - 2 \left( \beta^2 - \alpha^2 \right) z_{xx}(x) + \left( \alpha^2 + \beta^2 \right)^2 z(x) + 5\tilde{B}^2 z_{xx}(x)
+ 10\tilde{B}_x z_x(x) + \left[ 5\tilde{B}_x^2 + 10\tilde{B}_x \tilde{B}_{xx} + \frac{15}{2} \tilde{B}^4 - 6 \left( \beta^2 - \alpha^2 \right) \tilde{B}_x^2 \right] z(x).
\]
and its associated quadratic form:
\[
\mathcal{Q}[z] := \int_{\mathbb{R}} z \mathcal{L}[z] = \int_{\mathbb{R}} z_{xx}^2 + 2(\beta^2 - \alpha^2) \int_{\mathbb{R}} z_x^2 + (\alpha^2 + \beta^2)^2 \int_{\mathbb{R}} z^2 - 5 \int_{\mathbb{R}} B_z^2 z_x^2 \]
\[+ 5 \int_{\mathbb{R}} B_x^2 z_x^2 + 10 \int_{\mathbb{R}} B B_{xx} z_x^2 + \frac{15}{2} \int_{\mathbb{R}} B^4 z_x^2 - 6(\beta^2 - \alpha^2) \int_{\mathbb{R}} B^2 z_x^2. \tag{2.7}\]

Now we introduce two important directions associated to spatial translations. Let \(\tilde{B}_{\alpha,\beta}\) as in (1.6). We define
\[
\tilde{B}_1(x, t; x_1, x_2) := \frac{\partial}{\partial x_1} \tilde{B}_{\alpha,\beta}(x, t; x_1, x_2) \quad \text{and} \quad \tilde{B}_2(x, t; x_1, x_2) := \frac{\partial}{\partial x_2} \tilde{B}_{\alpha,\beta}(x, t; x_1, x_2). \tag{2.8}\]

It is clear that, for all \(t \in \mathbb{R}, \alpha, \beta > 0\) and \(x_1, x_2 \in \mathbb{R}\), both \(\tilde{B}_1\) and \(\tilde{B}_2\) are real-valued functions in the Schwartz class, exponentially decreasing in space. Moreover, it is not difficult to see that they are linearly independent as functions of the \(x\)-variable, for all time \(t\) fixed. The following result in [2] will be useful:

**Proposition 2.2** Let \(\tilde{B} = \tilde{B}_{\alpha,\beta}\) be any mKdV breather, and \(\tilde{B}_1, \tilde{B}_2\) the corresponding kernel of the associated operator \(\mathcal{L}\). There exists \(\mu_0 > 0\), depending only on \(\alpha, \beta\), such that, for any \(z \in H^2(\mathbb{R})\) satisfying
\[
\int_{\mathbb{R}} \tilde{B}_1 z = \int_{\mathbb{R}} \tilde{B}_2 z = 0,
\]
one has
\[
\mathcal{Q}[z] \geq \mu_0 \|z\|_{H^2(\mathbb{R})}^2 - \frac{1}{\mu_0} \left(\int_{\mathbb{R}} z \tilde{B}\right)^2.
\]

In what follows we denote
\[
B(x, t; x_1, x_2) := B_{\alpha,\beta}(x, t; x_1, x_2), \tag{3.1}
\]
\[
B_j(x, t; x_1, x_2) := \frac{\partial}{\partial x_j} B_{\alpha,\beta}(x, t; x_1, x_2), \quad j = 1, 2, \tag{3.2}
\]
with \(B_{\alpha,\beta}\) defined in (1.14), in order to simplify future computations.

### 3 Almost conserved Lyapunov functional

In this section we will define a suitable Lyapunov functional in the spirit of [2], keeping in mind the boundary terms.

The following functionals (obtained from the first three conserved quantities of (1.1)) will be important to understand the dynamics of the solutions \(u(\cdot, t)\) of the IBVP (1.9) close to breathers,
\[
M[u](t) := \frac{1}{2} \int_{\mathbb{R}} u^2(x, t) \, dx, \quad \text{(mass)} \tag{3.1}
\]
\[
E[u](t) := \int_{\mathbb{R}} \left(\frac{1}{2} u_x^2(x, t) - \frac{1}{4} u^4(x, t)\right) \, dx, \quad \text{(energy)} \tag{3.2}
\]
and
\[ F[u](t) := \int_{\mathbb{R}} \left( \frac{1}{2} u_{xx}^2(x, t) - \frac{5}{2} u^2(x, t)u_x^2(x, t) + \frac{1}{4} u^6(x, t) \right) dx, \quad \text{(second order energy)} \]  

(3.3)

which are well-defined for solutions in \( C(\mathbb{R}; H^2(\mathbb{R}^-)) \).

Before presenting some key functional estimates, we define the following nonlinear terms which will appear in the computations. Explicitly, in the current context of half-line domains, they arise as additional factors associated to boundary terms. Namely

\[ \tau_M(x, t) := \frac{1}{2} u_x^2 - u_{xx}u - \frac{3}{4} u^4, \]  

(3.4)

\[ \tau_E(x, t) := \frac{1}{2} u^6 + u^3 u_{xx} + \frac{1}{2} u_{xx}^2 - u_{xxx}u_x - 3u^2 u^2_x \]  

(3.5)

\[ \tau_F(x, t) := -u_t(u^3)_x - \frac{9}{2} u^4 u_x^2 + \frac{1}{2} u_{xxx} + u_{xx}u_{xt} \]  

\[ -u^2 u^2_{xx} - 2u_t u^2 u_x + \frac{3}{2} u^4 u_x^2 - \frac{1}{4} u_x^4 \]  

\[ +uu_{xx}^2 - \frac{3}{2} u^5 u_{xx} - \frac{9}{16} u^8. \]  

(3.6)

Note that the above trace terms \( \tau_M(0, t), \tau_E(0, t) \) and \( \tau_F(0, t) \) are well-defined for solutions \( u \) on the space

\[ \mathcal{U}_T(\mathbb{R}^-) := \left\{ u \in C(\mathbb{R}^+; H^2(\mathbb{R}^-)) : \partial^j_x u \in C \left( \mathbb{R}^+_x; H^{(3-j)/3}(0, T) \right), \; j = 0, 1, 2, 3 \right\}. \]

Moreover, note the following:

**Lemma 3.1** Let \( u = u(x, t) \) be the solution of the IBVP (1.9) with initial data \( u_0 \in H^2(\mathbb{R}^-) \). Then, the following identities are satisfied:

\[ M[u](t) = M[u_0] + \int_0^t \tau_M(0, s) ds, \]  

(3.7)

\[ E[u](t) = E[u_0] + \int_0^t \tau_E(0, s) ds \]  

(3.8)

and

\[ F[u](t) = F[u_0] + \int_0^t \tau_F(0, s) ds, \]  

(3.9)

for all \( t \geq 0 \). Moreover, under homogeneous boundary conditions

\[ u(0, t) = 0 \quad \text{and} \quad u_x(0, t) = 0 \]  

(3.10)

we have

\[ M[u](t) = M[u_0], \]  

(3.11)

\[ E[u](t) \geq E[u_0] \]  

(3.12)

and

\[ F[u](t) = F[u_0], \]  

(3.13)

for all \( t \geq 0 \).
Thus, using (3.17) in (3.16), combined with Young’s inequality, we have the estimate:

\[ \|u(\cdot, t)\|_{L^2(\mathbb{R}^-)} + \|u_{xx}(\cdot, t)\|_{L^2(\mathbb{R}^-)}, \]

by using the conservation of the functionals (3.11) and (3.13).

In view of the conservation (3.11) we only need to get a control of the \( \|u_{xx}(\cdot, t)\|_{L^2(\mathbb{R}^-)} \).

To proceed, we first note that from (3.13) we have

\[
\int_{\mathbb{R}^-} \left( \frac{1}{2} u_{xx}^2 + \frac{1}{4} u^6 \right) = F[u_0] + \frac{5}{2} \int_{\mathbb{R}^-} u^2 u_x^2.
\]

(3.14)

Now using integration by parts and the homogeneous boundary conditions one gets

\[
\psi [u] := \frac{5}{2} \int_{\mathbb{R}^-} u^2 u_x^2 = -\frac{5}{2} \int_{\mathbb{R}^-} u^3 u_{xx} dx - 2 \psi [u].
\]

(3.15)

So, from (3.15) it follows that

\[
\psi [u] = \frac{-5}{6} \int_{\mathbb{R}^-} u^3 u_{xx} \leq \frac{5}{6} \|u\|_{L^6(\mathbb{R}^-)}^3 \|u_{xx}\|_{L^2(\mathbb{R}^-)}.
\]

(3.16)

On the other hand, by using a Gagliardo-Nirenberg inequality and (3.11), we get that

\[
\|u\|_{L^2(\mathbb{R}^-)} \lesssim \|u_{xx}\|_{L^2(\mathbb{R}^-)}^{1/6} \|u_0\|_{L^2(\mathbb{R}^-)}^{5/6}.
\]

(3.17)

Thus, using (3.17) in (3.16), combined with Young’s inequality, we have the estimate:

\[
\psi [u] = \frac{5}{2} \int_{\mathbb{R}^-} u^2 u_x^2 \leq C_1 \|u_{xx}\|_{L^2(\mathbb{R}^-)}^{3/2} \|u_0\|_{L^2(\mathbb{R}^-)}^{5/2}
\]

\[
\leq \frac{1}{4} \|u_{xx}\|_{L^2(\mathbb{R}^-)}^2 + C_2 \|u_0\|_{L^2(\mathbb{R}^-)}^{10},
\]

(3.18)

for some positive constants \( C_1 \) and \( C_2 \). Finally, putting the estimate (3.18) in (3.14) we obtain

\[
\frac{1}{4} \|u_{xx}\|_{L^2(\mathbb{R}^-)}^2 \leq \int_{\mathbb{R}^-} \left( \frac{1}{4} u_{xx}^2 + \frac{1}{4} u^6 \right) \leq F[u_0] + C_2 \|u_0\|_{L^2(\mathbb{R}^-)}^{10},
\]

and we have the desired a priori control for the \( \|u_{xx}\|_{L^2(\mathbb{R}^-)} \). Then, the proof is finished. □

**Remark 3.1** *(About breathers moving rightwards)* It is important to note that in the case of the right half-line (\( \mathbb{R}^+ \)), the corresponding trace terms would be \( -\tau_M(0, t), -\tau_E(0, t) \) and \( -\tau_F(0, t) \). Hence, since the homogeneous boundary condition \( u_x(0, t) = 0 \) is not allowed in the corresponding IBVP on \( \mathbb{R}^+ \), we see that, only by using the homogeneous condition \( u(x, 0) = 0 \), the term \( \left( \frac{1}{4} u_x^4 - u_x u_{xx} \right) \) remains in \( -\tau_F(x, 0) \), and this nonlinear term is difficult to control. This fact prevents us from building a Lyapunov functional on the right half-line. This is the main reason to not address here the case of breathers moving rightwards.
Now, we are able to introduce an almost conserved Lyapunov functional, specifically related to the breather function $B_{\alpha, \beta}$ on $\mathbb{R}^-$ (1.14). Let $t > 0$ and $M[u]$, $E[u]$ and $F[u]$ the conserved quantities defined in (3.1)–(3.3). Based on the work [2] we define the restricted Lyapunov functional

$$\mathcal{H}[u](t) := F[u](t) + 2(\beta^2 - \alpha^2) E[u](t) + (\alpha^2 + \beta^2)^2 M[u](t).$$  \hfill (3.19)

Note that, by using Lemma 3.1 with $0 < \beta \leq \alpha$, the functional $\mathcal{H}$ is well defined for initial conditions $u_0 \in H^2(\mathbb{R}^-)$ and homogeneous boundary conditions. Therefore, $\mathcal{H}$ has the following monotonicity property

$$\mathcal{H}[u](t) \leq \mathcal{H}[u](0), \text{ for all } t > 0.$$  \hfill (3.20)

**Remark 3.2** *(About breather’s parameters)* The condition $\beta \leq \alpha$ is consistent with the first hypothesis $\beta \leq \sqrt{3}\alpha$ imposed in order to treat the case of mKdV breathers moving leftwards. However, we can not use (3.20) in the case $\alpha < \beta \leq \sqrt{3}\alpha$ because we do not control the right sign in $\mathcal{H}$ (3.19), a contradiction with the energy growth. In fact, in this interval, the stability question remains open.

Let $z \in H^2(\mathbb{R}^-)$, and $B = B_{\alpha, \beta}$ be any restricted mKdV breather (1.14). We define, the corresponding restriction to $\mathbb{R}^-$ of the quadratic form associated to $\mathcal{L}$ (see (2.7)):

$$Q[z] := \int_{\mathbb{R}^-} z \mathcal{L}[z]$$

$$= \int_{\mathbb{R}^-} z^2_{xx} + 2(\beta^2 - \alpha^2) \int_{\mathbb{R}^-} z^2_x + (\alpha^2 + \beta^2)^2 \int_{\mathbb{R}^-} z^2 - 5 \int_{\mathbb{R}^-} B^2 z_x^2$$

$$+ 5 \int_{\mathbb{R}^-} B_x^2 z^2 + 10 \int_{\mathbb{R}^-} B B_{xx} z^2 + \frac{15}{2} \int_{\mathbb{R}^-} B^4 z^2 - 6(\beta^2 - \alpha^2) \int_{\mathbb{R}^-} B^2 z^2.$$  \hfill (3.21)

Now, in the spirit of [2] we have the following result.

**Lemma 3.3** Let $z \in H^2(\mathbb{R}^-)$ be any function with sufficiently small $H^2$-norm, and $B = B_{\alpha, \beta}$ be any breather function (1.14). Then, for all $t \in \mathbb{R}$, one has that $\mathcal{H}$ (3.19) verifies

$$\mathcal{H}[B + z] - \mathcal{H}[B] = \frac{1}{2} Q[z] + N[z] + B_{xx}(x = 0, t) z_x(x = 0, t)$$

$$- B_{3x}(x = 0, t) z(x = 0, t) - 5 B^2 B_x(x = 0, t) z(x = 0, t)$$

$$+ B_x(x = 0, t) z(x = 0, t),$$  \hfill (3.22)

with $Q$ being the quadratic form defined in (3.21) and $N[z]$ satisfying $|N[z]| \leq K \|z\|^3_{H^2(\mathbb{R}^-)}$, where $K$ is a positive constant.

**Proof.** Just following [2, Lemma 5.2], we skip the details. Namely, expanding $\mathcal{H}[B + z] - \mathcal{H}[B]$ and collecting terms proportional to $z$, the only difference is that some trace terms appear as a consequence of the integration by parts. Indeed,

$$\frac{1}{2} \int_{\mathbb{R}^-} 2 B_{xx} z_{xx} = \int_{\mathbb{R}^-} B_{4x} z + B_{xx}(x = 0, t) z_x(x = 0, t) - B_{3x}(x = 0, t) z(x = 0, t),$$

$$- \frac{5}{2} \int_{\mathbb{R}^-} 2 B^2 B_x z_x = - \frac{5}{2} \int_{\mathbb{R}^-} (2 B^2 B_{xx} z - 4 B B_x^2 z).$$
\[-5B^2B_x(x = 0, t)z(x = 0, t),\]

and

\[
\frac{1}{2} \int_{\mathbb{R}^-} 2B_xz_x = \frac{1}{2} \int_{\mathbb{R}^-} -2B_{xx}z + B_x(x = 0, t)z(x = 0, t).
\]

Notice that the restricted breather $B$ in (1.14) also satisfies the differential identities given in Lemma 2.1, and hence the following fundamental identity

\[
G[B] := B(4x) - 2(\beta^2 - \alpha^2)(B_{xx} + B^3) + (\alpha^2 + \beta^2)^2 B
\]
\[+ 5B^2B_x + 5B^2B_{xx} + \frac{3}{2}B^5 = 0,
\]

it was used in the above expansion (see Lemma 2.1- (iv) for details). \qed

4 Proof of Theorem 1.2

The proof follows some ideas developed in [2] and [4]. These ideas allow us to avoid some problems caused by the trace terms $\tau_M(0, t), \tau_E(0, t)$ and $\tau_F(0, t)$. In our proof, we adapted these previous arguments to the restricted breather $B$. The control of the shift function $\rho_2$, obtained in Lemma 4.1, will be a key step in the proof.

4.1 Starting of the proof of Theorem 1.2

Take $\alpha$ and $\beta$ satisfying $0 < \beta \leq \alpha$ and fix $L > L_0$, where $L_0$ will be taken larger enough. Assume that

\[
\|u_0 - B(\cdot, 0; 0, L)\|_{H^2(\mathbb{R}^-)} \leq \eta
\]

(4.1)
is satisfied for $u_0$ and for $\eta \leq \eta_0$ with $\eta_0$ small enough to be chosen later.

Let $u(\cdot, t) \in C(\mathbb{R}^+, H^2(\mathbb{R}^-))$ be the associated solution of the IBVP (1.9) with initial data $u(x, 0) = u_0$ and homogeneous boundary conditions. By using the continuity of the flow (see Remark 1.1), given $\eta > 0$ there exist a small time $T_0$ and continuous parameter functions $\rho_j(t)$ ($j = 1, 2$) such that

\[
\sup_{0 \leq t \leq T_0} \|u(\cdot, t) - B(\cdot, t, \rho_1(t), \rho_2(t) + L)\|_{H^2(\mathbb{R}^-)} \leq 2\eta
\]

(4.2)

for all $0 \leq t \leq T_0$.

Let $K_0 > 2$ a constant to be fixed later and consider the maximal time of stability, defined as follows:

\[
T_* := \sup \left\{ T > 0 : \text{ for all } t \in [0, T] \text{ there exist } \rho_1(t), \rho_2(t) \in \mathbb{R} \text{ such that} \right. \]
\[
\sup_{0 \leq t \leq T} \|u(\cdot, t) - B(\cdot, t, \rho_1(t), \rho_2(t) + L)\|_{H^2(\mathbb{R}^-)} \leq K_0(\eta + e^{-\beta L/2}) \right\}.
\]

(4.3)

Notice that from (4.2) we have that $T_*$ is well-defined.

By choosing $L$ and $K_0$ large, with $\eta \leq \eta_0$, we will prove that $T_* = \infty$. The idea is to use a contradiction argument under the assumption $T_* < \infty$. Indeed, as we will see, a bootstrap
type argument will ensure the inequality
\[ \|u(\cdot, t) - B(\cdot, t; \rho_1(t), \rho_2(t) + L)\|_{H^2(\mathbb{R}^-)} \leq \frac{1}{2} K_0(\eta + e^{-\beta L/2}), \] (4.4)
for all \(0 \leq t \leq T_*\), which is a contradiction with the definition of \(T_*\) (if it is finite).

We split the proof of (4.4) in the following steps: first of all in sect. 4.2 we establish the modulation theory and exponential decays for the modulated breather on the boundary. Next, in sect. 4.3 we give some error estimates for the evolution in time of the restricted Lyapunov functional (3.19). Finally, in sect. 4.4, we derive the desired inequality (4.4) to complete the proof.

### 4.2 Modulation

Using the notation introduced in (2.9)–(2.10) we have the following result.

**Lemma 4.1** Let \(T_*\) defined in (4.3). There exist constants, \(\eta_0 > 0\) small enough and \(L_0\) large enough such that, for all \(\eta \in (0, \eta_0)\) and \(L > L_0\), the following holds. There exist continuous functions \(\rho_1 : [0, T_*] \rightarrow \mathbb{R}\) and \(\rho_2 : [0, T_*] \rightarrow (-\frac{L}{2}, \frac{L}{2})\), such that
\[ z(x, t) := u(x, t) - B(x, t; \rho_1(t), \rho_2(t) + L) \] (4.5)
satisfies the orthogonality conditions
\[ \int_{\mathbb{R}^-} B_j(x, t; \rho_1(t), \rho_2(t) + L) z(x, t) dx = 0, \quad j = 1, 2, \] (4.6)
for all \(t \in [0, T_*]\). Moreover, there exist a positive constant \(K > 0\), independent of \(K_0\), ensuring the following estimates:
\[ \|z(\cdot, t)\|_{H^2(\mathbb{R}^-)} \leq K K_0(\eta + e^{-\beta L/2}), \] (4.7)
\[ \|z(\cdot, 0)\|_{H^2(\mathbb{R}^-)} \leq K(\eta + e^{-\beta L/2}). \] (4.8)

**Proof** Let \(K_0\) and \(T_*\) as defined in (4.3). We first define the set
\[ V_1[K_0] := \left\{ v \in H^2(\mathbb{R}^-) : \inf_{\rho_1, \rho_2 \in \mathbb{R}} \|v - B(\cdot, t; \rho_1, \rho_2 + L)\|_{H^2(\mathbb{R}^-)} \leq K_0(\eta + e^{-\beta L/2}) \right\} \] (4.9)
and we note that
\[ \inf_{\rho_1, \rho_2 \in \mathbb{R}} \|u(\cdot, t) - B(\cdot, t; \rho_1, \rho_2 + L)\|_{H^2(\mathbb{R}^-)} \leq K_0(\eta + e^{-\beta L/2}), \] (4.10)
for all \(0 \leq t \leq T_*\) with \(K_0\) large enough. Hence,
\[ u(\cdot, t) \in V_1[K_0], \quad 0 \leq t \leq T_. \] (4.11)

The idea is to apply the Implicit Function Theorem. Firstly, we define the functional operator:
\[ \mathcal{J} = (\mathcal{J}_1, \mathcal{J}_2) : H^2(\mathbb{R}^-) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2, \quad j = 1, 2, \]
with
\[ \mathcal{J}_j[v; \rho_1, \rho_2] := \int_{\mathbb{R}^-} (v(x) - B(x, t; \rho_1, \rho_2 + L)) B_j(x, t; \rho_1, \rho_2 + L) dx. \] (4.12)
We can check that \( \mathcal{J}_j (j = 1, 2) \) are of class \( C^1 \) and also satisfy
\[
\mathcal{J}_j[B(\cdot, t; \rho_1, \rho_2 + L); \rho_1, \rho_2] = 0, \tag{4.13}
\]
for all \( \rho_1, \rho_2 \in \mathbb{R} \). In what follows we use the notation \( \partial_k := \partial_\rho_k \) and \( \partial^2_{kj} := \partial^2_{\rho_k \rho_j} \). So, for \( j, k = 1, 2 \), one has
\[
\partial_k \mathcal{J}_j[v; \rho_1, \rho_2] = - \int_{\mathbb{R}^-} B_k(x, t; \rho_1, \rho_2 + L) B_j(x, t; \rho_1, \rho_2 + L) dx + \int_{\mathbb{R}^-} (v - B(x, t; \rho_1, \rho_2 + L)) \partial_k B(x, t; \rho_1, \rho_2 + L) dx. \tag{4.14}
\]
Hence, we have
\[
\mathfrak{J}_{jk} := \partial_k \mathcal{J}_j[v; \rho_1, \rho_2] \bigg|_{v = B(\cdot, t; 0, L), 0, 0} = - \int_{\mathbb{R}^-} B_k(x, t; 0, L) B_j(x, t; 0, L) dx. \tag{4.15}
\]
and we define \( \mathfrak{J} \) as the \( 2 \times 2 \) matrix with components
\[
\mathfrak{J} = (\mathfrak{J}_{jk})_{j,k=1,2}. \tag{4.16}
\]
As in [2], putting \( B_j(x, t) := B_j(x, t; 0, L) \) we have from Cauchy-Schwarz inequality and the fact that \( B_1 \) and \( B_2 \) are not parallel for all time that
\[
\det \mathfrak{J} = - \left[ \int_{\mathbb{R}^-} B_1^2(x, t) dx \int_{\mathbb{R}^-} B_2^2(x, t) dx - \left( \int_{\mathbb{R}^-} B_1(x, t) B_2(x, t) dx \right)^2 \right] (t; 0, L) \neq 0
\]
for all \( 0 \leq t \leq T_\ast \).

Therefore, in a small neighbourhood \( U_t \times I_t \times J_t \subset H^2(\mathbb{R}^-) \times \mathbb{R} \times \mathbb{R} \) of the point \( (B(t; 0, 0, L), 0, 0) \), and for \( t \in [0, T_\ast] \) (given by the definition of (4.3)), it is possible to write the decomposition (4.5) satisfying
\[
\mathcal{J}[u(\cdot, t), \rho_1(t; u(\cdot, t)), \rho_2(t; u(\cdot, t))] = 0, \quad 0 \leq t \leq T_\ast \tag{4.17}
\]
for \( \eta_0 \) small enough, \( L \) larger enough and for unique functions \( \rho_1 := \rho_1(t, u(\cdot, t)) \in I_t \) and \( \rho_2 := \rho_2(t, u(\cdot, t)) \in J_t \subset (-L^\frac{1}{2}, L^\frac{1}{2}) \).

This directly implies that \( \rho_2(t) > -\frac{L}{2} \). We choose this in order to control the traces of the modulated breather (see Corollary 4.2). The uniqueness of the functions \( \rho_1 \) and \( \rho_2 \) is a consequence of the uniqueness coming from the Implicit Function Theorem in each \( U_t \times I_t \times J_t \). Finally, bounds (4.7) and (4.8) follow from (4.1) and (4.11). \( \square \)

The following result shows an estimate for the trace terms of the breather solution \( B \) which is localized far away from the origin (i.e at distance \( L \)).

**Corollary 4.2** (Boundary values of \( B \)) Let \( B = B(x, t; \rho_1(t), \rho_2(t) + L) \) a restricted breather given by (1.14). Then the following estimate holds:
\[
|\partial_j \partial^2_{jk} B(0, t; \rho_1(t), \rho_2(t) + L)| \leq C e^{-\beta L^2/2}, \quad j = 1, 2, \quad k = 0, 1, 2, 3. \tag{4.18}
\]

**Proof** See Appendix B for the proof of this result. \( \square \)
4.3 Error estimate

Applying Lemma 3.3 to the solution \( u \in C(\mathbb{R}^+; H^2(\mathbb{R}^-)) \) with homogeneous boundary conditions \( u(0, t) = \partial_x u(0, t) = 0 \) and by using the smallness of \( z(x, t) := u(x, t) - B(x, t) \) in (4.5), we get

\[
\mathcal{H}[u](t) = \mathcal{H}[B](t) + \frac{1}{2} Q[z](t) + N[z](t) + B_{xx}(x = 0, t)z(x = 0, t) - B_{3x}(x = 0, t)z(x = 0, t)
\]

(4.19)

Now, by using that \( u(0, t) = u_x(0, t) \equiv 0 \) we get

\[
B_{xx}(x = 0, t)z(x = 0, t) - B_{3x}(x = 0, t)z(x = 0, t) - 5B^2B_x(x = 0, t)z(x = 0, t) + B_x(x = 0, t)z(x = 0, t)
\]

(4.20)

We now fix the following notation: \( \tilde{\mathcal{H}} \) is the extension of \( \mathcal{H}(3.19) \) to the whole line \( \mathbb{R} \).

Lemma 4.3 Let \( B(x, t; \rho_1(t), \rho_2(t) + L) \) given by (1.6). Then the following error estimate holds

\[
|\mathcal{H}[B](t) - \mathcal{H}[B](0)| \lesssim e^{-\beta L/2}.
\]

(4.21)

Proof As introduced above, we have that \( \tilde{\mathcal{H}}[\tilde{B}](t) = \tilde{\mathcal{H}}[\tilde{B}](0) \). By using the localization on the left size of the breather, far away from the origin, we get the result.

Now we continue with the proof. By using Lemmas 3.3 and 4.3 and Corollary 4.2, we get for \( t \leq T_n \) that

\[
Q[z](t) \leq C Q[z](0) + K \|z(t)\|_{H^2(\mathbb{R}^-)}^3 + K \|z(0)\|_{H^2(\mathbb{R}^-)}^3 + Ke^{-\beta L/2}
\]

\[
\leq C \|z(0)\|_{H^2(\mathbb{R}^-)}^3 + C \|z(t)\|_{H^2(\mathbb{R}^-)}^3 + Ke^{-\beta L/2}
\]

(4.22)

\[
\leq C \eta^2 + C K_0^3(\eta + e^{-\beta L})^3 + Ce^{-\beta L/2},
\]

where the term \( \|z(0)\|_{H^2(\mathbb{R}^-)}^3 \) was absorbed by \( \|z(0)\|_{H^2(\mathbb{R}^-)}^2 \).

4.4 End of the Proof of Theorem 1.2

The final step in the proof of Theorem 1.2 consists of making a suitable extension of the required functions and functionals to the whole line.

Definition 4.4 (Zero extension, left half-line case). Let \( v \in H^2(\mathbb{R}^-) \) such that \( v(x = 0) = 0 \) and \( v_x(x = 0) = 0 \). We define its (zero) extension \( \tilde{v} \) as the function

\[
\tilde{v}(x) := \begin{cases} v(x), & x \leq 0, \\ 0, & x > 0. \end{cases}
\]

(4.23)

Note that \( \tilde{B} \) cannot be considered as the zero extension of \( B \), since this function and its derivative does not vanish at the origin. Therefore we consider the natural extension of the
breather $\tilde{B}$ as given in (1.6). This interesting difference will be important for the stability proof.

Let $\tilde{u}$ the extension of the solution $u$ defined in (4.23) and consider the function $\tilde{z} \in C([0, \infty); H^2(\mathbb{R}))$ by

$$\tilde{z} = \tilde{u} - \tilde{B}.$$  

(4.24)

Then, we write

$$E[\tilde{z}] = \tilde{Q}[\tilde{z}(t)] - \tilde{Q}[z(t)],$$

where $E$ is the quadratic error functional restricted to $\mathbb{R}^+$, namely

$$E[\tilde{z}] := \int_{\mathbb{R}^+} z \mathcal{L}[z] = \int_{\mathbb{R}^+} \frac{z^2}{2} + 2 (\beta^2 - \alpha^2) \int_{\mathbb{R}^+} \frac{z^2}{2}$$

$$+ (\alpha^2 + \beta^2)^2 \int_{\mathbb{R}^+} z^2 - 5 \int_{\mathbb{R}^+} B^2 \frac{z^2}{2}$$

$$+ 5 \int_{\mathbb{R}^+} B^2 z^2 + 10 \int_{\mathbb{R}^+} B B_{xx} z^2$$

$$+ \frac{15}{2} \int_{\mathbb{R}^+} B^4 z^2 - 6 (\beta^2 - \alpha^2) \int_{\mathbb{R}^+} B^2 z^2. $$

(4.25)

From the above definitions, we have the following result on the error control.

**Lemma 4.5** Let $\tilde{z}$ given by (4.24). Then for any $t > 0$

$$E[\tilde{z}] \lesssim e^{-\beta L/2}.$$

**Proof** Follows directly from the fact that $\|\tilde{z}(t)\|_{H^2(\mathbb{R}^+)} \lesssim e^{-\beta L/2}$. \qed

Now we are able to treat the term $\tilde{Q}[z(t)]$. To proceed, we use Proposition 2.2 and Lemma 4.5 to get

$$\tilde{Q}[z(t)] = \tilde{Q}[\tilde{z}(t)] - E[\tilde{z}] \geq \mu_0 \|\tilde{z}\|_{H^2(\mathbb{R})}^2 - \frac{1}{\mu_0} \left( \int_{\mathbb{R}} \tilde{z} \tilde{B} \right)^2 - C e^{-\beta L/2}$$

$$= \mu_0 \left( \|z(t)\|_{H^2(\mathbb{R}^+)}^2 + \|z(t)\|_{H^2(\mathbb{R}^-)}^2 \right) - \frac{1}{\mu_0} \left( \int_{\mathbb{R}} \tilde{z} \tilde{B} \right)^2 - C e^{-\beta L/2}$$

(4.26)

$$\geq \mu_0 \|z(t)\|_{H^2(\mathbb{R}^-)}^2 - \frac{1}{\mu_0} \left( \int_{\mathbb{R}} \tilde{z} \tilde{B} \right)^2 - C e^{-\beta L/2}.$$

Now by using the conservation of the mass (3.7) we get

$$\|u(t)\|_{L^2(\mathbb{R}^-)}^2 = \|(B + z)(t)\|_{L^2(\mathbb{R}^-)}^2 = \|B(t)\|_{L^2(\mathbb{R}^-)}^2$$

$$+ \|z(t)\|_{L^2(\mathbb{R}^-)}^2 + 2 \int_{\mathbb{R}^-} B(t) z(t) dx$$

$$= \|B(0)\|_{L^2(\mathbb{R}^-)}^2 + \|z(0)\|_{L^2(\mathbb{R}^-)}^2$$

$$+ 2 \int_{\mathbb{R}^-} B(0) z(0) dx = \|u_0\|_{L^2(\mathbb{R}^-)}. $$

(4.27)
It follows that
\[
\left| \int_{\mathbb{R}^-} B(t)z(t)dx \right| \leq C \left( \int_{\mathbb{R}^-} B(0)z(0)dx \right) + \|z(t)\|_{L^2(\mathbb{R}^-)}^2 \\
+ \|z(0)\|_{L^2(\mathbb{R}^-)}^2 + Ce^{-\beta L/2} \\
\leq C \left( \eta + K_0^2(\eta + e^{-\beta L/2})^2 + \eta^2 \right) + Ce^{-\beta L/2},
\]
for any \( t \in [0, T^*_0]. \) Replacing (4.22), (4.28) in (4.26) we get
\[
\mu_0 \|z(t)\|_{H^2(\mathbb{R}^-)}^2 \leq \frac{C}{\mu_0} \left( \eta + K_0(\eta + e^{-\beta L/2}) + \eta^2 \right)^2 + C\eta^2 + C(K_0(\eta + e^{-\beta L/2})^3 \\
+ Ce^{-\beta L/2}.
\]
Taking \( K_0 \) large enough and \( \eta \) small enough, we finally get
\[
\|z(t)\|_{H^2(\mathbb{R}^-)}^2 \leq \frac{1}{2} K_0(\eta + e^{-\beta L})^2
\]
as we stated in (4.4). Then, the proof is finished.

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Data availability no datasets were generated during and/or analysed during the current study, except to formal computations and one graphic, which it is shown in the paper. But for any doubt or feedback, the corresponding author is available to clarify them on reasonable request.

Declaration

Conflicts of interest All authors declare that they have no conflicts of interest.

Appendix A: Proof of Lemma 3.1

Multiplying the equation of the IBVP (1.9) and integrating by parts one get
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^-} |u|^2 dx = \int_{\mathbb{R}^-} u_{xx}u_x dx - [uu_{xx}](0, t) - 3 \int_{\mathbb{R}^-} u^3 u_x dx \\
= \left[ \frac{1}{2} u_x^2 - u_{xx}u_x - \frac{3}{4} u^4 \right](0, t) \tag{A.1}
\]
which implies (3.7) by integration in time of (A.1). Under the hypotheses in (3.10) we see that \( \tau_M(0, t) = 0 \) and then we have (3.11).

Now we derive with respect to \( x \) the equation of the IBVP (1.9) and after that we multiply by \( u_x \) to get
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^-} u_x^2 dx = - \int_{\mathbb{R}^-} \left( u_{xxx} + (u^3)_x \right) u_x dx \\
= - \int_{\mathbb{R}^-} u_{xxx}u_x dx + \int_{\mathbb{R}^-} (u^3)_x u_x dx - [u_{xxx}u_x + 3u^2 u_x^2](0, t) \\
= \int_{\mathbb{R}^-} u^3 (u_x + (u^3)_x) dx + \left[ \frac{1}{2} u_{xx}^2 + u^3 u_x - u_{xxx}u_x - 3u^2 u_x^2 \right](0, t)
\]
\[ = \frac{1}{4} \frac{d}{dt} \int_{\mathbb{R}^-} u^4 dx + \left[ \frac{1}{2} u^6 + \frac{1}{2} u_x^2 + u^{3} u_{xx} - u_{xxxx} u_x - 3u^2 u_x^2 \right](0, t). \]  

(A.2)

Thus,

\[ \frac{d}{dt} E[u](t) = \tau_E(0, t) \]  

(A.3)

which give us (3.8). In the case of the homogeneous boundary condition (3.10), we see that

\[ \tau_E(0, t) = \frac{1}{2} u_{xx}^2 \]

and hence the lower bound for the energy (3.12) is obtained.

Now we are going to prove the identity (3.9). We compute separately, the derivative in time for the integral terms \( \int_{\mathbb{R}^-} u_{xx}^2 \, dx \) and \( \int_{\mathbb{R}^-} u^6 \, dx \). By using the structure of the mKdV equation and integration by parts we get:

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^-} u_{xx}^2 \, dx = \int_{\mathbb{R}^-} u_{xx} u_{xxt} \, dx = - \int_{\mathbb{R}^-} u_{xxxx} u_{xt} \, dx + [u_{xx} u_{xt}](0, t) \\
= \int_{\mathbb{R}^-} u_{xxx} \left[ u_{xxx} + (u^3)_x \right] \, dx + [u_{xx} u_{xt}](0, t) \\
= \frac{1}{2} u_{xxx}^2(0, t) + \int_{\mathbb{R}^-} u_{xxx} (u^3)_x \, dx + [u_{xx} u_{xt}](0, t) \\
= - \int_{\mathbb{R}^-} u_t (u^3)_x \, dx - \int_{\mathbb{R}^-} (u^3)_x (u^3)_x \, dx + \frac{1}{2} u_{xxx}^2(0, t) + [u_{xx} u_{xt}](0, t). \]  

(A.4)

On the other hand we have

\[
\frac{1}{4} \frac{d}{dt} \int_{\mathbb{R}^-} u^6 \, dx = - \frac{3}{2} \int_{\mathbb{R}^-} u^5 [u_{xxx} + (u^3)_x] \, dx \\
= \frac{3}{2} \int_{\mathbb{R}^-} (u^5)_x u_{xx} \, dx - \frac{3}{2} u^5 u_{xx}(0, t) - \frac{9}{2} \int_{\mathbb{R}^-} u^7 u_x \, dx \\
= \frac{3}{2} \int_{\mathbb{R}^-} (u^5)_x u_{xx} \, dx - \frac{3}{2} u^5 u_{xx}(0, t) - \frac{9}{16} u^8(0, t). \]  

(A.5)

with

\[
I = - \frac{3}{2} \int_{\mathbb{R}^-} [u_t + u_{xxx}] u^2 u_{xx} \, dx + \frac{3}{2} \int_{\mathbb{R}^-} u^3 (u^2)_x u_{xx} \, dx \\
= - \frac{3}{2} \int_{\mathbb{R}^-} u_t u^2 u_{xx} \, dx - \frac{3}{2} \int_{\mathbb{R}^-} u_{xxx} u^2 u_{xx} \, dx + \frac{2}{5} I \\
= \frac{3}{2} \int_{\mathbb{R}^-} (u_t u^2)_x u_{xx} \, dx - \frac{3}{2} [u_t u^2 u_x](0, t) - \frac{3}{4} \int_{\mathbb{R}^-} u^2 (u_{xx})_x \, dx + \frac{2}{5} I; \]

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so

\[
I = \frac{5}{4} \int_{\mathbb{R}^-} u^2(u_x^2)_t dx + \frac{5}{2} \int_{\mathbb{R}^-} (u^2)_t u_x^2 dx - \frac{5}{4} \int_{\mathbb{R}^-} u^2(u_{xx})_x dx - \frac{5}{2}[u_t u_x^2]_x(0, t)
\]

\[
= \frac{5}{4} \int_{\mathbb{R}^-} u^2(u_x^2)_t dx + \frac{5}{2} \int_{\mathbb{R}^-} (u^2)_t u_x^2 dx + \frac{5}{2} \int_{\mathbb{R}^-} uu_x u_{xx}^2 dx
\]

(A.6)

\[
- \frac{5}{4}[u^2 u_{xx}^2]_x(0, t) - \frac{5}{2}[u_t u_x^2]_x(0, t).
\]

The integral \( J \) can be compute as follows:

\[
J = \frac{5}{2} \int_{\mathbb{R}^-} uu_x u_{xx} u_{xxx} dx
\]

\[
= -\frac{5}{2} \int_{\mathbb{R}^-} u_x^2 u_{xx} dx - J - \frac{5}{2} \int_{\mathbb{R}^-} uu_x^2 u_{xxx} dx + \frac{5}{2}[u u_x^2 u_{xx}](0, t);
\]

thus,

\[
J = -\frac{5}{16} u_x^4(0, t) - \frac{5}{4} \int_{\mathbb{R}^-} uu_x^2 u_{xxx} dx + \frac{5}{4}[u u_x^2 u_{xx}](0, t)
\]

\[
= \frac{5}{8} \int_{\mathbb{R}^-} (u^2)_x u_x^2 dx + \frac{15}{4} \int_{\mathbb{R}^-} u_x^4 u_x^2 dx - \frac{5}{16} u_x^4(0, t) + \frac{5}{4}[u u_x^2 u_{xx}](0, t).
\]

It is not difficult to check that

\[
\int_{\mathbb{R}^-} u_x^3 u_x^2 dx = -\frac{1}{15} I + \frac{1}{4}[u u_x^2 u_{xx}](0, t),
\]

then we have

\[
J = \frac{5}{8} \int_{\mathbb{R}^-} (u^2)_x u_x^2 dx - \frac{1}{4} I + \frac{15}{16}[u u_x^2 u_{xx}](0, t) - \frac{5}{16} u_x^4(0, t) + \frac{5}{4}[u u_x^2 u_{xx}](0, t)
\]

and putting the expression of \( J \) in (A.6) one get

\[
I = \int_{\mathbb{R}^-} u^2(u_x^2)_t dx + \frac{5}{2} \int_{\mathbb{R}^-} (u^2)_t u_x^2 dx
\]

\[
+ \left[-u_x^2 u_{xx}^2 - 2u_t u_x^2 u_x + \frac{3}{4} u_x^4 u_x^2 - \frac{1}{4} u_x^4 + uu_x^2 u_{xx}\right](0, t).
\]

Now, by using the expression obtained for \( I \) and adding (A.4) with (A.5) we get

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^-} u_{xx}^2 dx + \frac{1}{4} \frac{d}{dt} \int_{\mathbb{R}^-} u^6 dx = \frac{5}{2} \frac{d}{dt} \int_{\mathbb{R}^-} u_x^2 u_{xx}^2 dx + \tau_F(0, t),
\]

(A.7)

with \( \tau_F \) defined in (3.6).

Finally (3.9) is obtained by integrating (A.7) and under the homogeneous condition (3.10) we have \( \tau_F(0, s) = 0 \) that yields (3.13).

**Appendix B: Proof of Corollary 4.2**

The proof of (4.18) in the case \( k = 0 \) follows directly from (1.14) and the fact that

\[
\gamma t + \rho_2 + L > \frac{L}{2} \quad \forall t \in [0, T_*],
\]
which is a consequence of \( \rho_2(t) \in (-\frac{L}{2}, \frac{L}{2}) \) obtained in Lemma 4.1, and the hypothesis \( \gamma > 0 \). Now we prove (4.18). By direct computation, we obtain

\[
\partial_x B(x, t; \rho_1(t), \rho_2(t)) = \frac{4\sqrt{2}a\beta h_1(x, t)}{(\alpha^2 + \beta^2 + \alpha^2 \cosh(2\beta y_2) - \beta^2 \cos(2\alpha y_1))^2} \tag{B.1}
\]

where

\[
h_1(x, t) := - (\alpha^2 + \beta^2) \cosh(\beta y_2) \sin(\alpha y_1) \left[ \alpha^2 + \beta^2 + \alpha^2 \cosh(2\beta y_2) - \beta^2 \cos(2\alpha y_1) \right] - 2\alpha\beta [\alpha \cos(\alpha y_1) \cosh(\beta y_2) - \beta \sin(\alpha y_1) \sinh(\beta y_2)] \left[ \beta \sin(2\alpha y_1) + \alpha \sinh(2\beta y_2) \right],
\]

where \( y_1 := x + \rho_1(t) + x_1, \ y_2 := x + \rho_2(t) + x_2 \) and \( \delta := \alpha^2 - 3\beta^2, \ \gamma := 3\alpha^2 - \beta^2 \). Now, we control the function \( h_1 \) as follows

\[
|h_1(x, t)| \leq C_{\alpha, \beta} \cosh(\beta y_2) [1 + \cosh(2\beta y_2)] + C_{\alpha, \beta} [\cosh(\beta y_2) + 1 + |\sinh(\beta y_2)|] \left[ (1 + |\sinh(2\beta y_2)|) \right] \tag{B.2}
\]

\[
\leq \cosh^3(\beta y_2).
\]

We also have that

\[
\frac{1}{(\alpha^2 + \beta^2 + \alpha^2 \cosh(2\beta y_2) - \beta^2 \cos(2\alpha y_1))^2} \leq \frac{K_{\alpha, \beta}}{\cosh(2\beta y_2)^2}. \tag{B.3}
\]

Thus, by using (B.2), (B.3) in (B.1)

\[
|\partial_x B(x, t; \rho_1(t), \rho_2(t))| \leq C_{\alpha, \beta} \sech(\beta y_2). \tag{B.4}
\]

Finally, combining (B.4) and the fact \( \rho_2(t) \in (-\frac{L}{2}, \frac{L}{2}) \) we get the case \( k = 1 \). The case \( k = 2 \) follows from (2.2). The case \( k = 3 \) just follows from the PDE (1.1).

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