Extending the Applicability Condition in the Formal System $\lambda\delta$

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Abstract

The formal system $\lambda\delta$ is a typed $\lambda$-calculus derived from $\Lambda_\infty$, aiming to support the foundations of Mathematics that require an underlying theory of expressions (for example the Minimal Type Theory).

The system is developed in the context of the Hypertextual Electronic Library of Mathematics as a machine-checked digital specification, that is not the formal counterpart of previous informal material. The first version of the calculus appeared in 2006 and proved unsatisfactory for some reasons.

In this article we present a revised version of the system and we prove three relevant desired properties: the confluence of reduction, the strong normalization of an extended form of reduction, known as the “big tree” theorem, and the preservation of validity by reduction. To our knowledge, we are presenting here the first fully machine-checked proof of the “big tree” theorem for a calculus that includes $\Lambda_\infty$.

1 Introduction

The formal system $\lambda\delta$ is a typed $\lambda$-calculus aiming to support the foundations of Mathematics that require an underlying theory of expressions (for example mTT of [Maietti, 2009] and its predecessors). The system is developed in the context of the HELM project of [Asperti et al., 2003] as a machine-checked digital specification, that is not the formal counterpart of some previous informal material. The first version of the calculus [Guidi, 2006], formalized in the proof management system (p.m.s.) Coq [Coq development team, 2002] and published by [Guidi, 2009], proved unsatisfactory for some reasons. So a revision of the calculus is ongoing since April 2011 and includes a brand new formalization [Guidi, 2014] in the p.m.s. Matita of [Asperti et al., 2011]. Firstly, the revision aims at this problem: the calculus of [Guidi, 2009] comes from $\Lambda_\infty$ [van Benthem Jutting, 1994b], a language of the Automath family [Nederpelt et al., 1994], and yet it cannot type every term typed by $\Lambda_\infty$ since it lacks the “pure” type inference rule for function application [de Bruijn, 1991]. If $\Gamma \vdash M : N$ is a type assignment judgment and $\Gamma \vdash M !$ is the corresponding validity judgment, this rule states:

$$\frac{\Gamma \vdash f : F \quad \Gamma \vdash F(t) !}{\Gamma \vdash f(t) : F(t)}_{\oplus\text{--pure}}$$ (1)

This rule is redundant when the terms have three degrees (objects, classes, and sorts) as in Pure Type Systems (PTS’s) [Barendregt, 1993] and their derivatives. On the contrary it becomes effective when more degrees are available, as in the Aut-4 family [de Bruijn, 1994b] or in $\Lambda_\infty$, since $\Gamma \vdash f : F$ and $\Gamma \vdash F : \mathcal{F}$ do not imply that $\mathcal{F}$ is a sort. In this case $f$ can be a function, $F$ a function space, and $\mathcal{F}$ a family of function spaces. If we take $t$ in the domain of $f$, we might want $\Gamma \vdash f(t) !$ even when $f$ and $F$ are given abstractly as variables declared in $\Gamma$. Rule (1) is designed to realize this situation.

In the mathematical language we express a large variety of concepts, each with its own requirements. When we translate this language to typed $\lambda$-calculus, a widely accepted policy suggests that expressions
denoting concepts with different requirements should correspond to \(\lambda\)-terms with different degrees. Consider typical concepts of interest: sets, elements, propositions and proofs. While well-established similarities between elements and proofs support their representation with terms of the same degree, significant differences arise as well, playing in favor of representing them differently.

Mainly, identifying two proofs of a proposition (also known as “proof irrelevance”) is sensible, while identifying two elements of a set generally is not. \cite{de Bruijn, 1994b} approaches this problem by advocating a calculus in which two terms inhabiting the same type of degree 3 are definitionally equal. This is to say that terms of degree 4 are provided for representing irrelevant proofs. Similarly, subtle differences can be found in the requirements for sets and propositions. So it seems that a calculus with many degrees for its terms, may allow flexible interpretations of the mathematical language.

We note that \(\lambda\delta\) has a disadvantage in this sense because of its “isotropy”, by which we mean that the features of its terms do not depend on their degree.

Secondly, the revised \(\lambda\delta\) aims at other improvements some of which were advocated already by \cite{Guidi, 2009}. Simpler “arities” make the arity assignment judgment decidable for all values of the sort hierarchy parameter. The reaxiomatized step of environment-dependent parallel reduction allows to remove the substitution operator and provides for the long-awaited Rule (2). Tait-style reducibility candidates \cite{Tait, 1975} in place of Girard-style ones \cite{Girard et al., 1989} simplify the strong normalization theorem. Simpler environments allow to remove some ancillary operators.

\[
\begin{align*}
\Gamma \vdash f_1 & \Rightarrow f_2 & \Gamma \vdash t_1 & \Rightarrow t_2 \\
\Gamma \vdash f_1(t_1) & \Rightarrow f_2(t_2) \quad \text{appl}
\end{align*}
\]

The main contributions of this article are the so-called “big tree” theorem \cite{de Vrijer, 1994} for \(\lambda\delta\), which yields the subject reduction theorem for its stratified validity.

The “big tree” theorem states that valid terms are strongly normalizing with respect to a relation comprising reduction steps, type steps, subtraction steps, and more. It generalizes ordinary strong normalization and gives a very powerful induction principle for proving properties on valid terms. We are confident that this tool may prove useful in systems other than \(\lambda\delta\) as well.

Stratified validity (i.e., validity up to a specified degree) replaces type assignment as a primitive notion in the revised \(\lambda\delta\). This choice is motivated by the subject reduction theorem, which, in presence of Rule (1), is proved more easily for validity (the property of having an unspecified type) than for type assignment (the property of having a specified type) since types in \(\lambda\delta\), as well as in other systems, are not specified uniquely but up to conversion. The same situation arises for \(\Lambda\infty\) \cite{van Daalen, 1994}.

At this stage the revised \(\lambda\delta\) does not include a type judgment and the exclusion binder \(\chi\) of \cite{Guidi, 2009}, however we conjecture that our notion of validity implies Rule (1).

The revised \(\lambda\delta\) is defined in Section 2 and its properties are presented in Section 3. Our conclusions are in Section 4 Appendix A gives a summary of the notation we introduce, while Appendix B gives the pointers to the digital version of our results.

We agree that the symbol ▲ terminates our definitions and our proofs in the text.

## 2 Definition of \(\lambda\delta\)

In this section we define the revised \(\lambda\delta\) from scratch presenting its language (Section 2.1), its reduction rules (Section 2.3), and its validity rules (Section 2.6). These rules depend on some ancillary notions: relocation (Section 2.2), static type assignment (Section 2.4), and degree assignment (Section 2.5). Other notions are introduced to state or prove the main theorems of this article: closures (Section 2.7), extended reduction (Section 2.8), atomic arity assignment (Section 2.9), reducibility candidates (Section 2.10), lazy equivalence (Section 2.11), and an extension of “big trees” termed here “very big trees” (Section 2.12).

We shall use some logical constants: \(\forall\) (universal quantification), \(\exists\) (existential quantification), \(\Rightarrow\) (implication), \& (conjunction), and natural numbers with standard operators: \(\leq\), \(<\), \(+\), and \(\sim\). We shall need
natural number \(i, j, k\) starting at 0

term \(T, U, V, W\) \(::= \star k \ | \ #i \ | \ \delta V.T \ | \ \lambda W.T \ | \ @V.T \ | \ \odot W.T\)

environment \(K, L\) \(::= \star \ | \ L.\delta V \ | \ L.\lambda W\)

Figure 1: Terms and environments.

\[@\pi.T = T \quad \ominus(V_\pi V).T = \ominus V.(\ominus V.T)\]

Figure 2: Multiple application.

lists for the normalization theorem. Metavariables for lists will be overlined, like \(\overline{c}\). The empty list will be \(\overline{c}\), and the infix semicolon will denote concatenation, like \(c; c\).

Contrary to [Guidi, 2009], in this presentation we want to follow the digital specification of the calculus strictly, especially in the treatment of variables, and we make some notational changes with respect to that article. The reader will find a summary of the revised notation in Appendix A.

### 2.1 Language

The grammar of \(\lambda\delta\) features two syntactic categories: terms and environments, and uses natural numbers. Terms are presented in the “item notation” of [Kamareddine and Nederpelt, 1996b], and include sorts, variable occurrences, abbreviations, typed abstractions, applications, and type annotations. Contrary to [Guidi, 2009], environments contain just (nonrecursive) definitions, and typed declarations.

**Definition 2.1** (terms and environments). Terms and environments are defined in Figure 1. \(\star k\) is the sort of index \(k\), \(#i\) is the reference to the variable introduced at depth \(i\) [de Bruijn, 1994a] (so \(i\) is a “de Bruijn index”), \(\delta V.T\) is the abbreviation “let \(#0 = V\) in \(T\)”, \(\lambda W.T\) is the function “\((\#0 : W) \rightarrow T\)”, \(@V.T\) is the application “\(T(V)\)”, and \(\odot W.T\) is the type annotation “\((T : W)\)”. \(\star\) is the empty environment, \(L.\delta V\) is \(L\) with the definition “let \(#0 = V\)”, and \(L.\lambda W\) is \(L\) with the declaration “\((\#0 : W)\)”.

Convention: the symbol \(\delta/\lambda\) means: “either \(\delta\), or \(\lambda\)”. If the symbol occurs many times in a statement, it means: “either \(\delta\) in every occurrence, or \(\lambda\) in every occurrence”. The same convention holds for similar symbols we will use, like \(\star/#\) and \(\odot/\odot\).

The application can be extended to take a list \(\overline{V}\) of arguments.

**Definition 2.2** (multiple application). \(\ominus\overline{V}.T\) defined in Figure 2 denotes the application of \(T\) the arguments in the list \(\overline{V}\) starting from the rightmost term in \(\overline{V}\).

Environments are lists so some standard operators can be defined on them.

**Definition 2.3** (length). Figure 3 defines the length \(|L|\) of an environment \(L\).

**Definition 2.4** (concatenation). Figure 4 defines the concatenation \(K.L\) of \(L\) before \(K\). In particular we write \(\delta/\lambda W.L\) for \((\star.\delta/\lambda W).L\).

Normalization requires two predicates: see Definition 2.33 and Theorem 3.5.7.

**Definition 2.5** (neutrality). \(S(T)\) states that the term \(T\) is simple (or neutral) as defined in Figure 4. Specifically, \(T\) is neither an abbreviation, nor an abstraction.

**Definition 2.6** (top structure). \(T_1 \equiv T_2\) states that the terms \(T_1\) and \(T_2\) have the same top structure as defined in Figure 6. Specifically, \(T_1\) and \(T_2\) are the same atomic term or start with the same constructor.
\[ \left| \cdot \right| = 0 \quad \left| L.\delta/\lambda W \right| = |L| \]

Figure 3: Length of an environment.

\[ K.\star = K \quad K.(L.\delta/\lambda W) = (K.L).\delta/\lambda W \]

Figure 4: Concatenation of two environments.

### 2.2 Relocation

Managing variables referred by depth requires a well-known function \( \uparrow^{(l,m)} T \) connected to the function \( \tau_m(T) \) of \([\text{de Bruijn}, 1994a]\). In particular, when the term \( T \) enters the scope of \( m \) binders, \( \uparrow^{(0,m)} T \) relocates the indexes of its free variables. The composition of such functions is of interest as well.

**Definition 2.7** (relocation). The relation \( \uparrow^{(l,m)} T_1 = T_2 \) defined in Figure 7 states that \( T_2 \) is the relocation of \( T_1 \) at level \( l \) with depth (or “height”) \( m \).

We term the pair \((l,m)\) a “relocation pair”.

**Definition 2.8** (vector relocation). The relation \( \uparrow^{(l,m)} T_1 = T_2 \) defined in Figure 8 applies \((l,m)\) to the components of the list \( T_1 \) preserving their order in the list \( T_2 \).

**Definition 2.9** (multiple relocation). The relation \( \uparrow^c T_1 = T_2 \) defined in Figure 9 applies the list \( c \) of relocation pairs to \( T_1 \) starting from the leftmost pair in \( c \).

If \( \uparrow^{(l,m)} T_1 = T_2 \), notably, \( T_2 \) does not refer to the variables introduced at depth \( i \) with \( l \leq i < l + m \). So a relation \( \downarrow^{(l,m)} L_1 = L_2 \) is provided for removing the \( i \)-th entries of \( L_1 \) such that \( l \leq i < l + m \), while relocating the \( i \)-th entries such that \( i < l \). The relation is defined only when this relocation is possible, that is when an \( i \)-th entry with \( i < l \) does not refer to an \( i \)-th entry with \( l \leq i < l + m \). The 0-th entry of \( L_1 \) is the head of \( L_1 \). We term this relation “drop” as opposed to relocation, which is sometimes termed “lift”.

Notice that if \( \downarrow^{(0,i)} L_1 = L_2 \), then the head of \( L_2 \) contains the \( i \)-th entry of \( L_1 \).

**Definition 2.10** (drop). The relation \( \downarrow^{(l,m)} L_1 = L_2 \) defined in Figure 10 states that \( L_2 \) is \( L_1 \) without the \( i \)-th entries such that \( l \leq i < l + m \), and with the \( i \)-th entries such that \( i < l \) relocated accordingly.

Figure 10 (atom) generalizes “drop” of \([\text{Guidi}, 2006]\) allowing \( \downarrow^{(l,0)} L = L \) when \( |L| \leq l \).

**Definition 2.11** (multiple drop). The relation \( \downarrow^c L_1 = L_2 \) defined in Figure 11 applies the list \( c \) of relocation pairs to \( L_1 \) starting from the leftmost pair in \( c \).

The next equivalence relation appears in Theorem 3.9(3).

**Definition 2.12** (ranged equivalence). The relation \( L_1 \equiv \sim^{(l,m)} L_2 \) defined in Figure 12 states that \( L_1 \) and \( L_2 \) have the same length and the same \( i \)-th entries for \( l \leq i < l + m \).

### 2.3 Reduction

\( \lambda\delta \) features a transition system with five schemes of reducible expressions (redexes). Care is taken to design a deterministic and confluent system with disjoint redex schemes, in which the call-by-value \( \beta \)-reduction is broken into its basic components.
Definition 2.13 (transitions). Figure 13 defines the redexes and their transitions $\beta$, $\delta$, $\epsilon$, $\zeta$, and $\theta$, which depend on an environment $L$. The $\beta$-reduction is delayed (call-by-name style), the $\delta$-expansion expands a definition in $L$, the $\epsilon$-contraction removes a type annotation, the $\zeta$-contraction removes an unreferenced abbreviation, and the $\theta$-reduction [Curien and Herbelin, 2000] swaps an application and an abbreviation. $\blacktriangleleft$

Notice that the $\beta$-redex contains a type annotation $W$ that, contrary to [Guidi, 2009], remains in the $\beta$-reductum. This choice is connected with the revised form of the normalization theorem. Also notice that $\delta$-expansion, contrary to [Guidi, 2009], does not mention substitution. In the light of next Definition 2.14, delayed parallel substitution is seen as a special case of reduction.

Following [Guidi, 2009], we present parallel reduction to ease the proof of the confluence theorem, but here we take environment-dependent reduction as primitive.

Definition 2.14 (parallel reduction for terms). The relation $L \vdash T_1 \Rightarrow T_2$ defined in Figure 14 indicates one step of parallel reduction from $T_1$ to $T_2$ in $L$. $\blacktriangleleft$

We compute a call-by-value $\beta$-reduction in two steps, as we illustrate by computing the term $\Delta(T)$. In particular we set $\Delta_T = \lambda T. \epsilon \# \neq 0 \neq 0$ and we agree that $\epsilon^{(0,1)} T = U$.

$$
\begin{align*}
\beta & \quad L \vdash \epsilon \Delta_T. \Delta_T \Rightarrow \delta(\epsilon T. \Delta_T). \epsilon \# \neq 0 \neq 0 \\
\epsilon, \delta, \zeta & \quad L \vdash \delta(\epsilon T. \Delta_T). \epsilon \# \neq 0 \neq 0 \Rightarrow \epsilon \Delta_T. \Delta_T \\
& \quad \text{by } L \delta(\epsilon T. \Delta_T) \epsilon \# \neq 0 \Rightarrow \Delta_U \text{ and } L \vdash \epsilon T. \Delta_T \Rightarrow \Delta_T
\end{align*}
$$

The advantage of environment-dependent parallel reduction over the approach of [Guidi, 2009] lies in the increased parallelism of $\delta$-expansions, which we need for the “big tree” theorem. Suppose that $[m \vdash V] T$ replaces with $V$ some references in $T$ to the variable introduced at depth $m$, and compare Figure 14 (bind) and Figure 14 (parallel computation and conversion) (i.e., their environment-free counterpart). When we replace many variable instances in one step with this rule, each instance receives the same reduct $V_2$ of $V_1$. Whereas, by Figure 14 (parallel reduction for environments) each instance may receive a different reduct of $V_1$.

$$
\begin{align*}
V_1 \Rightarrow V_2, T_1 \Rightarrow T, [0 \vdash V_2] T & \Rightarrow T_2 & \Rightarrow V_1, T_1 \Rightarrow \delta V_2, T_2 & \Rightarrow \delta V_1, T_1 \Rightarrow \delta V_2, T_2 & \Rightarrow \delta V_1, T_1 \Rightarrow T_2 & \Rightarrow T_2 \Rightarrow T_2 & \Rightarrow T_2 \Rightarrow T_2 & \Rightarrow T_2 & \Rightarrow T_2 & \Rightarrow T_2
\end{align*}
$$

Notice that the subsystem of rules: Figure 14 (bind), Figure 14 (flat), Figure 14 (atom), and Figure 14 (parallel computation and conversion) axiomatizes environment-dependent parallel substitution.

We derive several notions from parallel reduction: an extension for environments needed in the confluence theorem, and some transitive closures. In this setting we agree that a “computation” is a reduction sequence consisting of zero or more steps.

Definition 2.15 (parallel reduction for environments). The relation $L_1 \vdash \Rightarrow^* L_2$ defined in Figure 15 indicates one step of parallel reduction from $L_1$ to $L_2$. $\blacktriangleleft$

Definition 2.16 (parallel computation and conversion). The relation $L \vdash T_1 \Rightarrow^* T_2$ (computation) is the transitive closure of $L \vdash T_1 \Rightarrow T_2$, while $L \vdash T_1 \Rightarrow^* T_2$ (conversion) is the symmetric and transitive closure of $L \vdash T_1 \Rightarrow T_2$. Moreover $L \vdash \Rightarrow^* L_2$ (computation) is the transitive closure of $L_1 \vdash \Rightarrow L_2$. Figure 16 defines $L \vdash T_1 \Rightarrow^* T_2$ for reference. The other notions are defined in the same manner. $\blacktriangleleft
The transitive closures we just defined are reflexive, because so is \( L \vdash T_1 \Rightarrow T_2 \). Therefore the symbol \( * \) in their notation is justified as a Kleene star meaning “zero or more”.

A characteristic feature of \( \lambda \delta \) is the use of reflexive relations for environments termed here “refinements”, invoked when proving that reduction preserves some property. Specifically, they are invoked in the case of Figure 14 excluding the use of reflexive relations for environments termed here “refinements”, invoked when proving that reduction preserves some property. Specifically, they are invoked in the case of Figure 14 excluding the use of reflexive relations for environments termed here “refinements”. In fact we are not interested in full reflexivity (i.e., \( L \vdash T \bullet^{(n)} U \) for each \( T \)). On the contrary, we wish to ensure that \( L \vdash T \bullet^{(n)} U \) holds iff the head variable occurrence of \( T \) is hereditarily closed in \( L \) regardless of \( n \), hence even for \( n = 0 \). As a matter of fact, differentiating the cases for instance 0 for the sake of reflexivity, yields a less elegant definition of \( L \vdash T \bullet^{(n)} U \).

According to our type policy, the sort of index \( k \) is typed by the sort of index \( h(k) \) where \( h \) is function chosen at will as long as an extension condition is satisfied.

**Definition 2.17** (refinement for preservation of reduction). Figure 17 defines the relation \( L_1 \leq L_2 \) stating that \( L_1 \) refines \( L_2 \) for preservation of reduction. ▲

The main results on reduction, conversion, and refinement are in Section 3.1.

### 2.4 Iterated Static Type Assignment

The “static” type assignment defined in this section is our counterpart of the so-called “de Bruijn” type assignment of the Automath tradition [van Daalen, 1994]. As such, it plays a central role in our definition of validity. Its name recalls that we can compute it without the help of \( \beta\zeta\theta \)-reductions.

Intuitively, the term \( T \) has a static type \( U \) in the environment \( L \) iff the head variable occurrence of \( T \) is hereditarily closed in \( L \). In that case, \( U \) is just a candidate type for \( T \). However, when \( T \) is valid, its static type serves as the “canonical” type [Kamareddine and Nederpelt, 1996a] or as the “inferred” type [Coscoy, 1996].

The “static type iterated \( n \) times” is related to the notion of validity implied by Rule 1 and it will be convenient to define it as a primitive notion (denoted by \( L \vdash T \bullet^{(n)} U \)), that will not be the reflexive and transitive closure of the “static type iterated one time”. In fact we are not interested in full reflexivity (i.e., \( L \vdash T \bullet^{(n)} U \) for each \( T \)). On the contrary, we wish to ensure that \( L \vdash T \bullet^{(n)} U \) holds iff the head variable occurrence of \( T \) is hereditarily closed in \( L \) regardless of \( n \), hence even for \( n = 0 \). As a matter of fact, differentiating the case \( n = 0 \) for the sake of reflexivity, yields a less elegant definition of \( L \vdash T \bullet^{(n)} U \).

According to our type policy, the sort of index \( k \) is typed by the sort of index \( h(k) \) where \( h \) is function chosen at will as long as an extension condition is satisfied.

**Definition 2.18** (iterated static type assignment). A “sort hierarchy parameter” is any function \( h \) satisfying the strict extension condition: \( k < h(k) \). Moreover \( h^n \) will denote \( h \) composed \( n \) times. For a natural number

\[
\begin{align*}
\text{natural number} & \quad l, m \quad \text{starting at 0} \\
\text{relocation pair} & \quad c \quad \{\ldots\}h_l m
\end{align*}
\]
The "degree" of a term $T$ is a number $d$ indicating the position of $T$ in a type hierarchy. A well-established definition assigns degree 1 to the bottom sort (for instance $\tau$) and degree $d+1$ to $T$ such that $\Gamma \vdash T : U$ when $U$ has degree $d$. In $\lambda\delta$, as in ECC [Luo, 1990], there is no top sort and the degree is an integer. So this definition prevents from reasoning by induction on the degree.

According to our policy, the degree of a sort is a natural number given by a function $g$ that can be chosen at will as long as a compatibility condition is satisfied.

Once sorts are assigned a degree, the assignment extends to terms accordingly.

**Definition 2.19 (degree assignment).** Given a sort hierarchy parameter $h$, a “sort degree parameter” is any function $g_h$ satisfying the compatibility condition: if $g_h(k) = d$ then $g_h(h(k)) = d - 1$. The relation $L \vdash T \bullet_{h,g}^d$ defined in Figure 19 indicates that $T$ has degree $d$ in $L$ according to $h$ and $g_h$. ▲

As we see, the term $T$ has a degree in $L$ iff the head variable occurrence of $T$ is hereditarily closed in $L$. So having a degree, is equivalent to having a static type.

The refinement given next occurs in the proof of the preservation theorem and is needed to prove that the reduction of valid terms preserves their degree.

**Definition 2.20 (refinement for preservation of degree).** Figure 20 defines the relation $L_1 \hat{\vDash}_{h,g} L_2$ stating that $L_1$ refines $L_2$ for preservation of degree. ▲

The main results on degree assignment and on its refinement are in Section 3.2.

### 2.5 Degree Assignment

The “degree” of a term $T$ is a number $d$ indicating the position of $T$ in a type hierarchy. A well-established definition assigns degree 1 to the bottom sort (for instance $\tau$ in $\Lambda_\infty$ or $\ast$ in the $\lambda$-Cube [Barendregt, 1993]) and degree $d+1$ to $T$ such that $\Gamma \vdash T : U$ when $U$ has degree $d$. In $\lambda\delta$, as in ECC [Luo, 1990], there is no top sort and the degree is an integer. So this definition prevents from reasoning by induction on the degree.

According to our policy, the degree of a sort is a natural number given by a function $g$ that can be chosen at will as long as a compatibility condition is satisfied.

Once sorts are assigned a degree, the assignment extends to terms accordingly.

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As we see, the term $T$ has a degree in $L$ iff the head variable occurrence of $T$ is hereditarily closed in $L$. So having a degree, is equivalent to having a static type.

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The main results on degree assignment and on its refinement are in Section 3.2.

### 2.6 Stratified Validity

Our validity rules for a term, say $X$, in an environment $L$, are designed to ensure that:
\[ L_1 = \emptyset \quad \downarrow \quad \downarrow_L L_1 = L \quad \downarrow_L L_2 = L_2 \quad \downarrow \quad \downarrow \]

Figure 11: Multiple Drop.

\[
\begin{align*}
L_1 & \equiv \langle 0,0 \rangle L_2 \\
L_1 & \equiv \langle (l,m) \rangle L_2 \\
L_1 & \equiv \langle 0,m \rangle L_2 \\
L_1 & \equiv \langle l,m \rangle L_2
\end{align*}
\]

Figure 12: Ranged equivalence.

1. a variable occurrence is closed in \( L \); the expected type of a declared variable occurrence is valid in its environment; the expansion of a defined variable occurrence is valid in its environment;
2. every subterm of \( X \) is valid in its environment;
3. for a type annotation \( \oplus W.V \), the inferred type of \( V \) converts to \( W \) in \( L \);
4. for an application \( @V.T \), the inferred type of \( T \) iterated enough times converts to the form \( \lambda W.U \), and the inferred type of \( V \) converts to \( W \) in \( L \).

Clause (4) is our extension of the “applicability condition”, which in a PTS is:

\[ @V.T, \text{ the inferred type of } T \text{ iterated one time converts to the form } \Pi W.U, \text{ and the inferred type of } V \text{ converts to } W \text{ in } L. \]

In [Guidi, 2009] we took by mistake the latter condition replacing \( \Pi \) with \( \lambda \), rather than Clause (4).

The idea of Clause (3) and Clause (4) is that a valid term is typable and its types are the valid terms that convert to its inferred type. Notice that this property holds for the calculus of [Guidi, 2009]. As for \( \Lambda_\infty \) [van Daalen, 1994], the preservation theorem for \( \lambda \) (stating that validity is preserved by reduction) requires an induction on the degree motivated by its extended applicability condition.

So we define a “stratified” validity depending on a degree assignment in that we require a positive degree for \( V \) in Clause (3) and Clause (4), and in that the inferred type of \( T \) is not iterated more times than the degree of \( T \) in Clause (4). Intuitively, this is validity up to a degree. The next ancillary relations are needed in the formal statement of Clause (3) and Clause (4).

Definition 2.21 (decomposed computation and conversion). The relation \( L \vdash T_1 \mapsto_{h,g}^*(n) T_2 \) defined in Figure 21 concatenates a degree-guarded iterated static type assignment and a computation, and the corresponding conversion \( L \vdash T_1 \mapsto_{h,g}^* T_2 \).

Definition 2.22 (stratified validity). The relation \( L \vdash T !_{h,g} \) defined in Figure 22 states that the term \( T \) is valid in \( L \) with respect to the parameters \( h \) and \( g_h \).

The refinement given next is needed to prove the preservation Theorem 3.14

Definition 2.23 (refinement for preservation of validity). Figure 23 defines the relation \( L_1 \sqsubseteq_{h,g} L_2 \) stating that \( L_1 \) refines \( L_2 \) for preservation of stratified validity.

The main results on stratified validity and on its refinement are in Section 3.8.
Definition 2.24 (direct subclosure). The relation \( \langle L_1, T_1 \rangle \rightarrow \langle L_2, T_2 \rangle \) defined in Figure 12 states that \( \langle L_2, T_2 \rangle \) is a “direct subclosure” of \( \langle L_1, T_1 \rangle \).

The relation \( \langle L_1, T_1 \rangle \rightarrow^* \langle L_2, T_2 \rangle \) is its reflexive closure.

The symbol \( \rightarrow^* \) in \( \langle L_1, T_1 \rangle \rightarrow^* \langle L_2, T_2 \rangle \) means “one or none” as for regular expressions.

Definition 2.25 (subclosure and proper subclosure). Figure 12 defines the relation \( \langle L_1, T_1 \rangle \rightarrow^* \langle L_2, T_2 \rangle \) (subclosure) as the (reflexive and) transitive closure of \( \langle L_1, T_1 \rangle \rightarrow^* \langle L_2, T_2 \rangle \). While the proper subclosure is the transitive closure of \( \langle L_1, T_1 \rangle \rightarrow \langle L_2, T_2 \rangle \).

We want to remark that generalizing the constant 0 in Figure 12 (drop), invalidates the commutation property between the direct subclosure and the parallel reduction, which is crucial for the preservation theorem. Moreover the proper subclosure is well founded, as we see by observing that each step of direct subclosure decreases the sum of the term constructors in the closure.

2.8 Extended Reduction

Having introduced subclosures, we can take a glance at the strong normalization of “rst-reduction” [de Vrijer, 1994], informally known as the “big tree” theorem.
and we are interested in proving that any sequence of such steps starting from

These considerations lead us to define the “extended reduction” such that:

This is the strong normalization of a relation

An infinite sequence would lead to an infinite cluster, contradicting either strong normalization of reduction

We would like to commute adjacent steps until the steps of the same kind are clustered. At that point, an

To prove the “pentagon” (i.e., a proposition on five closures connected by five relations) of Rule (5), in which the reduction for environments emerges in the case

These considerations lead us to define the “extended reduction” such that:

1. it extends ordinary reduction (i.e., a →r step) by supporting a →t step;

2. it preserves strong normalization “smoothly” in that little effort is expected in updating the proof that works for ordinary reduction [Guidi, 2009];

3. it preserves the commutation with subclosures in the form of Rule (5).
Definition 2.26 (extended transitions). Figure 20 defines the extended redexes and their associated transitions, \( t, l, \) and \( e \), which depend on a sort degree parameter \( g_h \) and on an environment \( L \). The transitions \( t, l \) and \( e \) respectively replace a sort, a declared variable, and a type annotation with their expected type. ▲

The transitions \( t \) and \( l \) provide the support for the \( t \)-step of (4), while the transition \( e \) allows the “smooth” update of the parallel proof advocated by Clause (2), as we shall see. We present extended reduction in its parallel form to extend Definition 2.14, with respect to which we add the rules for the transitions \( t \) and \( e \). Rule \( \delta \) is modiﬁed as well to include the support for transition \( l \). Definition 2.15 and Definition 2.16 are extended accordingly. The point of extended reduction compared to static type assignment, is that its context rules allow to compute the static type in every subterm and not just along the “spine”.

Definition 2.27 (extended parallel reduction for terms). The relation \( L \vdash T_1 \Rightarrow_{h,g} T_2 \) of Figure 27 indicates one step of extended parallel reduction from \( T_1 \) to \( T_2 \) in \( L \). ▲

Definition 2.28 (extended parallel reduction for environments). Figure 28 defines \( L \vdash \Rightarrow_{h,g} L_2 \) indicating one step of extended parallel reduction from \( L_1 \) to \( L_2 \). ▲

Definition 2.29 (extended parallel computation). The relation \( L \vdash T_1 \Rightarrow_{h,g}^* T_2 \) is the transitive closure of \( L \vdash T_1 \Rightarrow_{h,g} T_2 \), while \( L_1 \vdash \Rightarrow_{h,g}^* L_2 \) is the transitive closure of \( L_1 \vdash \Rightarrow_{h,g} L_2 \). Figure 29 defines \( L \vdash T_1 \Rightarrow_{h,g}^* T_2 \) for reference. \( L_1 \vdash \Rightarrow_{h,g}^* L_2 \) is defined in the same manner. ▲

The main results on extended reduction are in Section 3.3.
2.9 Atomic Arity Assignment

Atomic arities are simple types representing the abstract syntax of our reducibility candidates, introduced in the next Section 2.10 and replace in this role the more complex “binary arities” used by [Guidi, 2009]. Such arities are assigned to terms according to well-established rules. The term “atomic” indicates that the base constructor of these arities is not structured.

**Definition 2.30 (atomic arities and their assignment).** Atomic arities are the simple types defined in Figure 30. ⋆ is the base type, and \( B \supset A \) is the arrow type. Moreover the relation \( \vdash T \vdash A \), defined in Figure 30 as well, assigns the arity \( A \) to \( T \) in \( L \).

As a type assignment, \( L \vdash T : A \) has two interpretations: either \( A \) is the simple type of the object \( T \), or \( A \) is the simple type associated to the type \( T \). In this respect, consider the map \( T \mapsto T^* \) that turns a term of \( \lambda \delta \) into a term of \( \lambda \rightarrow \delta \) by operating the necessary \( \delta \epsilon \zeta \)-reductions on \( T \) and by replacing every abstraction in \( T \), say \( \lambda W \) in the environment \( K \), with the abstraction \( \lambda B \) such that \( K \vdash W : B \). Moreover, extend this map to environment entries. Then the rules of Figure 30 clearly show that \( L \vdash T : A \) implies \( L^* \vdash T^* : A \) in \( \lambda \rightarrow \) (we did not prove this fact formally yet).

We need the next refinement in order to prove the preservation of atomic arity.

**Definition 2.31 (refinement for preservation of atomic arity).** Figure 31 defines the relation \( L_1 \preceq_{h,g} L_2 \) stating that \( L_1 \) refines \( L_2 \) for preservation of atomic arity.

Our results on atomic arity assignment and on its refinement are in Section 3.4.

2.10 Reducibility Candidates

The “reducibility candidates” are subsets of \( \lambda \)-terms satisfying certain “saturation” conditions used to establish properties of some typed \( \lambda \)-calculi. In this article we use subsets of closures, closed under the next seven conditions, to prove that every term having an atomic arity in an environment, is strongly normalizing with respect to extended reduction. We start by defining the normal terms and the strongly normalizing terms. These definitions take into account the fact that extended reduction is reflexive and forbids single-step cycles.

**Definition 2.32 (normal terms and strongly normalizing terms).** Figure 32 defines \( L \vdash N(T) \) and \( L \vdash N(T) \), stating respectively that \( T \) in \( L \) is normal, and that \( T \) in \( L \) is strongly normalizing, for extended reduction with respect to \( h \) and \( g_h \).
\[
\begin{align*}
n \leq d & \quad L \vdash T_1 \bullet_{h,g} d \quad L \vdash T_1 \bullet_{h,g}^*(n) T_2 \quad L \vdash T \Rightarrow^* T_2 \quad \text{scpds} \\
L \vdash T_1 \bullet_{h,g}^* \Rightarrow^*(n) T_2 & \\
L \vdash T_1 \bullet_{h,g}^* \Rightarrow^*(n_1) T \quad L \vdash T_2 \bullet_{h,g}^* \Rightarrow^*(n_2) T & \quad \text{scpres}
\end{align*}
\]

Figure 21: Stratified decomposed computation and conversion.

\[
\begin{align*}
L \vdash *k ! \ h,g_{\text{sort}} & \quad \downarrow(0,i)\ L \vdash K \cdot \lambda W \cdot \delta ! \ L \vdash W ! \ h,g_{\text{ref}} \\
& \quad L \vdash \#i ! \ h,g_{\text{bind}} \quad L \vdash \delta ! \lambda W . T ! \ h,g \\
& \quad L \vdash U ! \ h,g_{\text{cast}} \quad L \vdash T ! \ h,g_{\text{cast}} \\
& \quad L \vdash U \bullet_{h,g}^*(0) U_0 \quad L \vdash T \bullet_{h,g}^*(1) U_0 \\
& \quad L \vdash V ! \ h,g_{\text{appl}} \quad L \vdash V \bullet_{h,g}^*(1) W_0 \\
& \quad L \vdash \emptyset V . T ! \ h,g
\end{align*}
\]

Figure 22: Stratified validity.

Notice that \( L \vdash \Rightarrow^*_h T_1 \) is inductively defined with base case \( L \vdash \Rightarrow^*_h N(T_1) \). In fact, \( L \vdash \Rightarrow^*_h N(T_1) \) implies \( L \vdash \Rightarrow^*_h T_1 \) since \( L \vdash \Rightarrow^*_h T_2 \) holds by “ex falso quodlibet”.

Given a property \( \mathcal{R} \) on closures, a reducibility candidate \( \mathcal{C} \) for \( \mathcal{R} \) is a subset of closures satisfying \( \mathcal{R} \), that we describe constructively as a relation. So we may write \( \mathcal{C}(T, L) \) for \((L, T) \in \mathcal{C} \). Our reducibility theorem states that if \( \mathcal{R} \) is a reducibility candidate, then every closure with an atomic arity belongs to some \( \mathcal{C} \) and therefore, satisfies \( \mathcal{R} \). In formal words we can prove that \( L \vdash T : A \) implies \( \mathcal{R}(T, L) \). Strong normalization follows from choosing \( L \vdash \Rightarrow^*_h T \) as \( \mathcal{R}(T, L) \).

We are going to present Tait-style reducibility candidates \cite{Tait1975}, which differ from the Girard-style reducibility candidates \cite{Girardetal1989} used by \cite{Guidi2009}, in that condition “CR2” is not required (i.e., \((L, T_1) \in \mathcal{C} \) and \( L \vdash T_1 \Rightarrow_{h,g} T_2 \) imply \((L, T_2) \in \mathcal{C} \), and notably, in that closures without an arity are allowed in \( \mathcal{C} \). This simplification gives us more freedom for constructing elements of \( \mathcal{C} \).

**Definition 2.33** (reducibility candidate). Given a subset \( \mathcal{R} \) of closures satisfying Rule (S) and Rule (S0) of Figure 35 a reducibility candidate \( \mathcal{C} \) for \( \mathcal{R} \) is a subset of closures satisfying Rule (S1) to Rule (S7) of Figure 36. The notation “\( \langle L, V \rangle \in \mathcal{R} \)” means “\( \{L, V \} \in \mathcal{C} \) for each component \( V \) of \( V \)”.

Compound reducibility candidates are built through well-established constructions. For now we are interested just in the “functional” construction introduced next.

**Definition 2.34** (function subset). If \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are subsets of closures, then the subset \( \mathcal{C}_1 \Rightarrow \mathcal{C}_2 \) is defined in Figure 34.

Notice that the environment \( L \) of \( W \) possibly extends the environment \( K \) of \( T \), as is required to prove Figure 33 (S6), in which \( L \) and \( L . \delta V \) have different length.

**Definition 2.35** (interpretation of an atomic arity). For a subset of closures \( \mathcal{R} \), the subset of closures \([A] \mathcal{R}\) associated to the atomic arity \( A \), is defined in Figure 37.

The refinement given next is needed to state the general form of the reducibility theorem. In particular it expresses in \( \lambda \delta \) a simultaneous substitution like the one occurring in the reducibility theorem for System F, which is stated using the “parametric” reducibility of \cite{Girardetal1989}.

\[\]
\[
L \vdash U \downarrow h,g \quad L \vdash T \downarrow h,g \quad \forall n, n \leq d \Rightarrow L \vdash U \bullet \leftrightarrow^{(n, n+1)} (n, n+1) T
\]

\[
L \vdash (\circ U, T) \downarrow h,g^{(d)}
\]

\[
L_1 \not\models h,g \quad L_2 
\]

\[
\begin{array}{c}
L_1 \not\models h,g \quad L_2 \\
\ast \not\models h,g \ast \quad L_1, \delta/\lambda W \not\models ! h,g \quad L_2, \delta/\lambda W
\end{array}
\]

\[
\begin{array}{c}
L_1 \not\models h,g \quad L_2 \\
L_1 \not\models (\circ W, V) \downarrow h,g^{(d)} \\
L_2 \vdash W \downarrow h,g \\
L_1 \vdash V \bullet h,g \quad d + 1 \\
L_2 \vdash W \bullet h,g \quad d
\end{array}
\]

\[
L_1, \delta/(\circ W, V) \not\models ! h,g \quad L_2, \lambda W
\]

\[
L_3, \delta((\circ W, V)) \not\models ! h,g \quad L_2, \lambda W
\]

Figure 23: Refinement for preservation of stratified validity.

\[
\begin{array}{c}
(L, \delta/\lambda/\circ/\# V, T) \models (L, V)^{\text{pair}} \\
(L, \circ/\# V, T) \models (L, T)^{\text{flat}} \\
(L, \delta/\lambda W, T) \models (L, \delta/\lambda W, T)^{\text{bind}}
\end{array}
\]

\[
\begin{array}{c}
\downarrow^{(0, m+1)} L = K \\
\downarrow^{(0, m+1)} T = U
\end{array}
\]

\[
\begin{array}{c}
\left\langle K, \delta/\lambda W, \#0 \right\rangle \models (L, U) \\
\left\langle K, T \right\rangle
\end{array}
\]

Figure 24: Direct subclosure.

**Definition 2.36** (refinement for reducibility). The relation \( L_1 \not\models_{\mathcal{R}} L_2 \) defined in Figure 36 states that \( L_1 \) refines \( L_2 \) for reducibility. ▲

The main results on candidates and on their refinement are in Section 3.10

**2.11 Lazy Equivalence**

In Section 2.10 we defined the normalization of a term \( T \) in the environment \( L \) that, by Theorem 5.7.5, is implied by \( L \vdash T : A \). Now we would like to define the normalization of an environment \( L \) in such a way that \( L \vdash T : A \) implies it as well. However, we notice from Figure 30[lref] that \( L \vdash T : A \) constrains just the entries of \( L \) hereditarily referred by \( T \). Thus, following the paradigm of Figure 32[csx], we need to replace \( T_1 \neq T_2 \) with the negated equivalence \( L_1 \not\models_T L_2 \) stating that \( L_1 \) and \( L_2 \) differ in one entry hereditarily referred by \( T \). The corresponding equivalence is defined next. Working under the assumption that every entry of \( L \) has an arity, simplifies the development significantly, but we aim at showing that this assumption is redundant.

**Definition 2.37** (lazy equivalence for environments). The relation \( L_1 \equiv_T L_2 \) defined in Figure 37 states that the environments \( L_1 \) and \( L_2 \) are equal in the entries hereditarily referred by the term \( T \) at level \( l \). ▲

This relation is an equivalence that we term “lazy” since we check for equality just the entries of \( L_1 \) and \( L_2 \) hereditarily referred by \( T \). Its nonrecursive definition 3 uses “hereditarily free” variables. We say that a variable is “hereditarily free” in \( (L, T) \) when it is free in \( T \) or in an entry of \( L \) hereditarily referred by \( T \). This idea is expressed formally by the next definition. Alternatively, we can say that a variable is hereditarily free in \( (L, T) \) when it is free in a \( \delta \)-reduct of \( T \) in \( L \) (see Definition 2.13 and Definition 2.26 for \( \delta \)-reducts and \( l \)-reducts respectively).

**Definition 2.38** (hereditarily free variables). Figure 38 defines \( i \in P_i(L, T) \), stating that the variable introduced at depth \( i \) is hereditarily free at level \( l \) in \( (L, T) \). ▲

We need the level \( l \) to reason about hereditarily free variables in the scope of binders.

For example we can prove that \( i + 1 \in P_{i+1}(L, \delta/\lambda W, U) \) implies \( i \in P_i(L, \delta/\lambda W, U) \).

An ancillary operation that we term “pointwise union” at level \( l \) of \( L_1 \) and \( L_2 \) with respect to \( T \) (notation: \( L_1 \uplus_T L_2 \)), leads to important properties connecting lazy equivalence and parallel reduction for environments such as Theorem 5.9[6]. The environment \( L_1 \uplus_T L_2 \) is defined when \( |L_1| = |L_2| \) and its \( i \)-th entry is taken from \( L_2 \) if \( l \leq i \) and \( i \in P_i(L_1, T) \), or else it is taken from \( L_1 \).
Definition 2.39 (pointwise union). The partial operation $L_1 \uplus^T L_2$ defined in Figure 39 constructs the “pointwise union” at level $l$ of $L_1$ and $L_2$ with respect to $T$. ▲

Lazy equivalence yields environments $L$ normalizing with respect to $T$ (notation $\blacklozenge^*_{h,g,l} L$) such that $L \vdash h,g T$ implies $\blacklozenge^*_{h,g,l} L$ for every level $l$. See Theorem 3.10(6).

Definition 2.40 (strongly normalizing environments). Figure 50 defines the relation $\blacklozenge^*_{h,g,l} L$, stating that $L$ is strongly normalizing at level $l$ for extended reduction with respect to the parameters $h$ and $gh$, and with respect to $T$. ▲

Notice the common structure of Figure 40 (lsx) and Figure 52 (csx).

An ancillary predicate on environments $\sim_{h,g,l} L$ is needed in Theorem 3.10(11). It serves $\blacklozenge^*_{h,g,l} L$ as, for instance, $L_1 \vdash L_2$ serves $L \vdash T_1 \vdash T_2$ in Theorem 3.11(6).

Definition 2.41 (strongly co-normalizing environments). The predicate $\sim_{h,g,l} L$ defined in Figure 41 states that the $L$ is “co-normalizing” at level $l$ with respect to $h$ and $gh$. This means that every $i$-th entry of $L$ such that $i < l$, is strongly normalizing according to Definition 2.40. “Co-normalizing” refers to “$i < l$” as opposed to “$l \leq i$”. ▲

The main results on lazy equivalence, pointwise union, and strongly normalizing environments are in Section 5.6. Comparing Section 2.2 with Section 2.11 the reader should notice that the notions defined here depend just on the component $l$ of the relocation pair $(l,m)$. In this perspective, the given definitions are the general ones instantiated for $m = \infty$. We present them in this form because the parameter $m$ turns out to be unnecessary for now.

2.12 Very Big Trees

With the help of lazy equivalence, we can finally define our counterpart of “rst-reduction” de Vrijer, 1994, which we informally introduced in Section 2.8. This counterpart is actually an extension that operates on closures. We term it “qrst-reduction” because we add a “q-step” of lazy equivalence.

Definition 2.42 (qrst-reduction). The relation $(L_1, T_1) \triangleright h,g (L_2, T_2)$ defined in Figure 42 denotes one step of qrst-reduction from the closure $(L_1, T_1)$ to the closure $(L_2, T_2)$ with respect to the parameters $h$ and $gh$. This means that every $i$-th entry of $L$ such that $i < l$, is strongly normalizing according to Definitions 2.40. “Co-normalizing” refers to “$i < l$” as opposed to “$l \leq i$”. ▲

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Figure 42 (fquq) is the “s-step”, Figure 42 (cpx) is the “rt-step” for terms, Figure 42 (lpx) is the “rt-step” for environments, and Figure 42 (leq) is our new “q-step”. Because of it, our “big” trees are actually “very big” with respect to de Vrijer, 1994. Formally, the “very big” tree rooted at $(L, T)$ comprises the qrst-computations starting at $(L, T)$. Our “very big tree” theorem states that if $T$ has an atomic arity in $L$ (Section 2.4), then the nonreflexive rst-steps in this tree are finite.
In order to state the theorem, the next definition highlights the proper (i.e., nonreflexive) \(rst\)-steps and the \(qrst\)-computations containing them.

**Definition 2.43** (proper \(rst\)-reduction and proper \(qrst\)-computation). Figure 43 defines the relation \(\langle L_1, T_1 \rangle \succcurlyeq_{h,g} \langle L_2, T_2 \rangle\), denoting one step of proper \(rst\)-reduction from \(\langle L_1, T_1 \rangle\) to \(\langle L_2, T_2 \rangle\) with respect to the parameters \(h\) and \(g\), and the relation \(\langle L_1, T_1 \rangle \succcurlyeq_{h,g} \langle L_2, T_2 \rangle\), denoting a proper \(qrst\)-computation. ▲

Theorem 3.11(3) shows that a step of proper \(rst\)-reduction is never reflexive, but a proper \(qrst\)-computation may be. Consider the term \(\uplus\Delta_{k,T}\Delta_{k,T}\) where \(\Delta_{k,T} = \lambda T.\uplus k.\uplus \#0\). Following the example of \(\uplus\Delta_T\Delta_T\) in Section 2.3, we can prove \(L \vdash \uplus\Delta_{k,T}\Delta_{k,T} \rightarrow \uplus k.\uplus\Delta_T.\Delta_{k,T}\) (proper \(r\)-step), and then \(\langle L, \uplus k.\uplus\Delta_T.\Delta_{k,T} \rangle \succcurlyeq \langle L, \uplus\Delta_{k,T}.\Delta_{k,T} \rangle\) (\(s\)-step) by Figure 24(flat). Moreover by Theorem 3.11(4), starting a proper \(qrst\)-computation with a proper step, is not restrictive.

Now we can define the closures whose “very big” tree contains a finite number of nonreflexive \(rst\)-steps. This is achieved by standard means with the next definition.

**Definition 2.44** (\(q\)-equivalence and strongly \(rst\)-normalizing closures). Figure 44 defines the relation \(\langle L_1, T_1 \rangle \equiv_{h,g} \langle L_2, T_2 \rangle\) (\(q\)-equivalence) that extends lazy equivalence to closures, and the predicate \(\exists_{h,g}(L,T)\) stating that \(\langle L, T \rangle\) is strongly normalizing for \(qrst\)-reduction with respect to the parameters \(h\) and \(g\). ▲

Theorem 3.11(2) and Theorem 3.11(1) show that \(\langle L_1, T_1 \rangle \succcurlyeq_{h,g} \langle L_2, T_2 \rangle\) is equivalent to \(\langle L_1, T_1 \rangle \equiv_{h,g} \langle L_2, T_2 \rangle\) \& \(\langle L_1, T_1 \rangle \not\equiv_{h,g} \langle L_2, T_2 \rangle\), so we can rephrase Figure 44(sb) following the pattern of Figure 32(csx) and Figure 30(lx). Moreover \(\exists_{h,g}(L,T)\) can be generated by Rule 11, which is Figure 44(sb) with \(\langle L_1, T_1 \rangle \succcurlyeq_{h,g} \langle L_2, T_2 \rangle\) in place of \(\langle L_1, T_1 \rangle \succcurlyeq_{h,g} \langle L_2, T_2 \rangle\). So \(\langle L, T \rangle\) is strongly \(rst\)-normalizing if it is strongly \(qrst\)-normalizing.

Our results on \(qrst\)-computations and \(qrst\)-normalization are in Section 3.7.

### 3 Propositions on \(\lambda\delta\)

In this section we present the main properties of reduction (Section 3.1), of degree assignment (Section 3.2), of \(rt\)-reduction (Section 3.3), of atomic arity assignment (Section 3.4), of reducibility candidates (Section 3.5),
\[ L \vdash T_1 \Rightarrow T_2 \quad \Rightarrow^*_{h,g} \quad L \vdash T_1 \Rightarrow h,g T_2 \quad \Rightarrow^*_{h,g} \quad L \vdash T_1 \Rightarrow T_2 \]

Figure 29: Extended parallel computation for terms (multi-step).

\[
\begin{align*}
L \vdash \ast & \text{sort} & L \vdash W ; B & \text{abst} & L \vdash V ; B & \text{app} & L \vdash U ; A & \text{cast} \\
L \vdash k ; \ast & \text{sort} & L \vdash K \delta \lambda W & \text{lref} & L \vdash V ; B & \text{app} & L \vdash \delta V . T ; A & \text{abstr} \\
L \vdash W ; B & \text{abst} & L \vdash V ; B & \text{app} & L \vdash U ; A & \text{cast} \\
L \vdash \lambda W . T ; B \supset A & \text{abstr} & L \vdash V ; B & \text{app} & L \vdash \delta V . T ; A & \text{abstr} \\
L \vdash \lambda W . T ; B \supset A & \text{abst} & L \vdash V ; B & \text{app} & L \vdash \delta V . T ; A & \text{abstr} \\
L \vdash W ; B & \text{abst} & L \vdash V ; B & \text{app} & L \vdash U ; A & \text{cast} \\
L \vdash \lambda W . T ; B \supset A & \text{abstr} & L \vdash V ; B & \text{app} & L \vdash \delta V . T ; A & \text{abstr} \\
\end{align*}
\]

Figure 30: Atomic arities and their assignment.

of lazy equivalence (Section 3.6), of 
qrst-reduction (Section 3.7), and finally of stratified validity (Section 3.8) respecting the dependences between these properties.

We aim at reaching our versions of the “three problems” [Nederpelt et al., 1994]: Theorem 3.2(1) (confluence of computation), Theorem 3.12(2) (strong 
qrst-normalization of valid terms), and Theorem 3.14(6) (subject reduction of stratified validity).

The detailed theory of \(\lambda\delta\) (1416 proofs) exists only in the digital form of [Guidi, 2014]. In this article we just outline the proofs of the presented statements by reporting on the proof strategy and on the main dependences of each proof. Most proofs are by induction on the height of a derivation or by cases on the last step of a derivation. Very often both techniques are applied together.

Appendix B lists the pointers to the digital proofs outlined in the article.

3.1 Results on Reduction

The relevant properties of reduction, conversion, and their refinement are listed next.

**Theorem 3.1** (reduction and its refinement).

1. **(transitivity of refinement)**
   
   If \(L_1 \subseteq L\) and \(L_2 \subseteq L\), then \(L_1 \subseteq L_2\).

2. **(transitivity of reduction for terms through refinement)**
   
   If \(L_1 \subseteq L_2\) and \(L_2 \vdash T_1 \Rightarrow T_2\), then \(L_1 \vdash T_1 \Rightarrow T_2\).

3. **(confluence of reduction for terms with itself, diamond property, general form)**
   
   If \(L_0 \vdash T_0 \Rightarrow T_1\) and \(L_0 \vdash T_0 \Rightarrow T_2\) and \(L_0 \vdash T_1 \Rightarrow T_2\) and \(L_0 \vdash T_2 \Rightarrow T_1\) and \(L_0 \vdash T_2 \Rightarrow T_1\) then there exists \(T\) such that \(L_1 \vdash T_1 \Rightarrow T\) and \(L_2 \vdash T_2 \Rightarrow T\).

4. **(confluence of reduction for environments with itself, diamond property)**
   
   If \(L_0 \vdash L_1\) and \(L_0 \vdash L_2\), then there exists \(L\) such that \(L_1 \vdash L\) and \(L_2 \vdash L\).

**Proof.** Clause (1) is proved by induction on its first premise and by cases on its second premise. Clause (2) is proved by induction on its second premise. Clause (3) is proved by induction on the proper subclosures of \(\langle L_0, T_0 \rangle\) (Section 2.7) and by cases on its four premises. Reduction for environments emerges when considering Figure 14(\(\delta\)) and when a binder in the “spine” of \(T_0\) is pushed into \(L_0\) in the cases of Figure 14(bind), Figure 14(\(\beta\)), and Figure 14(\(\theta\)). Moreover, Clause (2) and Figure 14(\(\beta\)) are invoked when Figure 14(\(\beta\)) is considered. Clause (4) is proved by induction on \(|L_0|\) and by cases on its two premises with the help of Clause (3). ▲
\[ L_1 \trianglelefteq L_2 \quad L_1 \trianglelefteq L_2 \quad \text{atom} \\
L_1.\delta/\lambda W \trianglelefteq L_2.\delta/\lambda W \quad \text{pair} \\
L_1.\delta(\bowtie W.V) \trianglelefteq L_2.\lambda W \quad \text{beta} \]

Figure 31: Refinement for preservation of atomic arity.

\[ \forall T_2, (L \vdash T_1 \Rightarrow h,g T_2) \Rightarrow (T_1 = T_2) \quad \text{ctx} \]

\[ L \vdash \Rightarrow h,g \text{N}(T_1) \]

\[ \forall T_2, (L \vdash T_1 \Rightarrow h,g T_2) \Rightarrow (T_1 \neq T_2) \Rightarrow (L \vdash \Rightarrow^* h,g T_2) \quad \text{ctx} \]

Figure 32: Normal terms and strongly normalizing terms for extended reduction.

Theorem 3.2 (computation and conversion).

1. (confluence of computation for terms with itself, Church-Rosser property)
   If \( L \vdash T_0 \Rightarrow^* T_1 \) and \( L \vdash T_0 \Rightarrow^* T_2 \) then there exists \( T \) such that \( L \vdash T_1 \Rightarrow^* T \) and \( L \vdash T_2 \Rightarrow^* T \).

2. (confluence of computation for environments with itself, Church-Rosser property)
   If \( L_0 \vdash \Rightarrow^* L_1 \) and \( L_0 \vdash \Rightarrow^* L_2 \) then there exists \( L \) such that \( L_1 \vdash \Rightarrow^* L \) and \( L_2 \vdash \Rightarrow^* L \).

3. (formulation of conversion as a pair of confluent computations)
   If \( L \vdash T_1 \rightsquigarrow^* T_2 \) then there exists \( T \) such that \( L \vdash T_1 \Rightarrow^* T \) and \( L \vdash T_2 \Rightarrow^* T \).

Proof. Clause (1) and Clause (2) are proved by induction on their first premise by invoking the corresponding “strip” lemmas [Barendregt, 1993] from Theorem 3.1(3) and Theorem 3.1(4) respectively. Clause (3) is proved by induction on its premise with the help of the “strip” lemma from Theorem 3.1(3).

The main result on reduction is Church-Rosser property, also known as the confluence theorem and one of the so-called “three problems” in the Automath tradition. The main result on conversion is its formulation as a pair of confluent computations: one direction is Theorem 3.2(3), the reverse is straightforward. Using this formulation, Theorem 3.1(9) and Theorem 3.2(11), give the generation lemma on abstraction, a desired property mentioned by [van Daalen, 1994]. This lemma states that \( L \vdash \lambda W_1. T_1 \rightsquigarrow^* \lambda W_2. T_2 \) implies \( L \vdash W_1 \rightsquigarrow^* W_2 \) and \( L. \lambda W_1 \vdash T_1 \rightsquigarrow^* T_2 \).

3.2 Results on Degree Assignment

The relevant properties of degree assignment and of its refinement are listed next.

Theorem 3.3 (degree assignment and its refinement).

1. (equivalence of degree assignment and iterated static type assignment, left to right)
   If \( L \vdash T \triangleleft h,g d \) then for each \( n \) there exists \( U \) such that \( L \vdash T \triangleleft^*_h U \) and \( L \vdash U \triangleleft h,g d - n \).

2. (equivalence of degree assignment and iterated static type assignment, right to left)
   If \( L \vdash T \triangleleft^*_h U \) then for each \( g_h \) there exists \( d \) such that \( L \vdash T \triangleleft h,g d \) then \( L \vdash U \triangleleft h,g d - n \).
3. (equivalence of degree assignment and iterated static type assignment, variant)
   If $L \vdash^{(n)} U$ then for every $n_0$ there exist $g_k$ and $d \geq n_0$
   such that $L \vdash T \ast_{h,g} d$ and $L \vdash U \ast_{h,g} d - n$.

4. (inclusion of refinement)
   If $L_1 \leq_{h,g} L_2$ then $L_1 \leq_{h,g} L_2$.

5. (transitivity of degree assignment through refinement)
   If $L_1 \leq_{h,g} L_2$ and $L_2 \vdash T \ast_{h,g} d$ then $L_1 \vdash T \ast_{h,g} d$.

6. (confluence of refinement and degree assignment)
   If $L_1 \leq_{h,g} L_2$ and $L_1 \vdash T \ast_{h,g} d$ then $L_2 \vdash T \ast_{h,g} d$.

7. (transitivity of refinement)
   If $L_1 \leq_{h,g} L$ and $L \leq_{h,g} L_3$ then $L_1 \leq_{h,g} L_2$.

Proof. Clause (1), Clause (2), Clause (3), and Clause (4) are proved by induction on the premise. Clause (5)
and Clause (6) are proved by induction on the second premise and by cases on the first premise. Clause (7)
is proved by induction on its first premise and by cases on its second premise by invoking Clause (5) and
Clause (6). ▲

Theorem 3.3[1] and Theorem 3.3[3] are the main properties of degree assignment, from which we derive
the next Theorem 3.4[1] (notice that in [Guidi, 2009] we were able to prove it just for $n = 0$).

Theorem 3.4 (iterated static type assignment).

1. (uniqueness of iterated static type assignment)
   If $L \vdash T \ast_{h}^{(n)} U_1$ and $L \vdash T \ast_{h}^{(n)} U_2$ then $U_1 = U_2$.

2. (irreflexivity of static type assignment iterated at least once)
   $L \vdash T \ast_{h}^{(n+1)} T$ is contradictory.

Proof. Clause (1) is proved by induction on its first premise and by cases on its second premise. Clause (2)
is proved directly with the help of Theorem 3.3[3]. ▲
\[ \llbracket \ast \rrbracket_R = R \quad \llbracket B \triangleright A \rrbracket_R = \llbracket B \rrbracket_R \triangleright \llbracket A \rrbracket_R \]

Figure 35: Interpretation of an atomic arity as a subset of closures.

\[
\begin{array}{c}
\text{atom} \\
\ast \in_R \ast
\end{array}
\quad
\begin{array}{c}
L_1 \in_R L_2 \\
L_1, \delta/\lambda W \in_R L_2, \delta/\lambda W
\end{array}
\quad
\begin{array}{c}
L_1 \in_R L_2 \quad (L_1, W) \in \llbracket B \rrbracket_R \quad (L_1, V) \in \llbracket B \rrbracket_R \quad L_2 \vdash W : B
\end{array}
\]

\[ \text{beta} \]

Figure 36: Refinement for reducibility.

### 3.3 Results on Extended Reduction

The relevant properties of extended reduction are listed next.

**Theorem 3.5** (extended reduction).

1. (transitivity of extended reduction for terms through refinement)
   \( L_1 \not\in L_2 \) and \( L_2 \vdash T_1 \vdash_{h,g} T_2 \) then \( L_1 \vdash T_1 \vdash_{h,g} T_2 \).

2. (inclusion of reduction, “\( \ast \)-step”)
   \( L \vdash T_1 \vdash T_2 \) then \( L \vdash T_1 \vdash_{h,g} T_2 \).

3. (inclusion of static type assignment, “\( \alpha \)-step”)
   \( L \vdash T_1 \vdash_{h,\beta} T_2 \) and \( L \vdash T_1 \vdash_{h,g} \beta + 1 \) then \( L \vdash T_1 \vdash_{h,g} T_2 \).

4. (commutation of direct subclosure with extended reduction for terms)
   If \((L, T_1) \vdash (K, V_1) \) and \( K \vdash V_1 \vdash_{h,g} V_2 \) then there exists \( T_2 \)
   such that \( L \vdash T_1 \vdash_{h,g} T_2 \) and \((L, T_2) \vdash (K, V_2) \).

5. (commutation of extended reduction for environments with direct subclosure)
   If \( L_1 \vdash \vdash_{h,g} L_2 \) and \((L_2, T_2) \vdash (K_2, V) \) then there exist \( K_1 \) and \( T \)
   such that \( L = T_1 \vdash_{h,g} T \) and \((L_1, T) \vdash (K_1, V) \) and \( K_1 \vdash \vdash_{h,g} K_2 \).

6. (absorption of extended reduction for environments)
   If \( L_1 \vdash \vdash_{h,g} L_2 \) and \( L_2 \vdash T_1 \vdash_{h,g} T_2 \) then \( L_1 \vdash T_1 \vdash_{h,g} T_2 \).

7. (extended computation from a \( \ast \)-redex)
   If \( L \vdash \pi\lambda W.T \vdash_{h,g} T_2 \) then either \( \pi\lambda W.T \vdash_{h,g} T_2 \) or \( L \vdash \delta(\pi\lambda W.V) \vdash_{h,g} T_2 \).

**Proof.** Clause (1) is proved by induction on its second premise and by cases on its first premise. For the reference to a declaration, Figure 27(δ), we have \( T_1 = \#i \), \( \downarrow_{(0, i)} L_2 = K_2, \lambda W_1, \) and \( K_2 \vdash W_1 \vdash_{h,g} W_2, \) and \( \uparrow^{(0, i+1)} W_2 = T_2. \) It may be the case, not occurring with ordinary reduction, that \( \downarrow_{(0, i)} L_1 = K_1, \delta(\pi\lambda W_1.V_1) \) and \( K_1 \vdash K_2 \) for some \( K_1 \) and \( V_1 \) by Figure 17(b) beta. In that event the induction hypothesis yields \( K_1 \vdash W_1 \vdash_{h,g} W_2 \) and Figure 27(ε) gives \( K_1 \vdash (\pi\lambda W_1.V_1 \vdash_{h,g} W_2 \) so Figure 27(δ) concludes \( L_1 \vdash \#i \vdash_{h,g} T_2. \) Here we see the purpose of \( e \)-reduction and of the expected type \( W_1 \) in the \( \ast \)-reduced item \( \delta(\pi\lambda W_1.V_1) \). The untyped \( \beta \)-reduced item \( \delta V_1 \) of [Guidi, 2009] shows here its weakness causing Clause (7) to fail. Clause (2) is proved by induction on its premise. Clause (3) is proved by induction on its first premise and by cases on its second premise after replacing \( L \vdash T_1 \vdash_{h,\beta} T_2 \) with \( L \vdash T_1 \vdash_{h,\beta} T_2 \) and \( n = 1. \) Clause (4) is proved by cases on its first premise. Clause (5) is proved by cases on its second premise and then by cases on its first premise. Clause (6) is proved by induction on its second premise and by cases on its first premise. Clause (7) is proved directly with the help of Clause (1) and Figure 17(b) beta. \( \Box \)
which improves the “pentagon” of Rule (5). Notice that a transition normalizing closures. Another interesting property of extended reduction is the “square” of Theorem 3.5(4), Theorem 3.5(7), and thus preserves the saturation condition of Figure 33(S3) for the subset of strongly update of the strong normalization proof advocated in Section 2.8. In particular, extended reduction preserves holds for ordinary reduction and that extended reduction must preserve in order to guarantee the “smooth” resembles Figure 28. The same holds for ordinary computation.

The “transitivity through refinement”, Theorem 3.1[12] and Theorem 3.5[11], is the crucial property that holds for ordinary reduction and that extended reduction must preserve in order to guarantee the “smooth” update of the strong normalization proof advocated in Section 2.8. In particular, extended reduction preserves Theorem 3.5[17], and thus preserves the saturation condition of Figure 33(S3) for the subset of strongly normalizing closures. Another interesting property of extended reduction is the “square” of Theorem 3.5[14], which improves the “pentagon” of Rule (5). Notice that a transition \( l \) makes the fifth “side” disappear.

Unfortunately, the “pentagon” remains in Theorem 3.5[14], where the extended reduction for terms is needed in the case \( L_2 = K_2, \delta / \lambda V \) and \( T_2 = \#0 \).

Theorem 3.5[10] shows that extended computation for environments is generated by the next rules that resemble Figure 28. The same holds for ordinary computation.

\[
\begin{align*}
& \forall T, i \uparrow^{(1,1)} T \neq U \quad i \in \mathbb{P}(L, U) \\
& l \leq j \quad j < i \quad (\forall T, i \uparrow^{(j,1)} T \neq U) \quad i - j - 1 \in \mathbb{F}_0(K, W) \quad \text{be}
\end{align*}
\]

\[
\begin{align*}
& \text{atom} \quad L_1 \vdash W_2 \quad L_1 \vdash W_1 \vdash \text{pair} \quad L_1, \delta / \lambda W_1 \vdash \text{pair}
\end{align*}
\]

Figure 37: Lazy equivalence for environments.

Figure 38: Hereditarily free variables.

3.4 Results on Atomic Arity Assignment

The properties of atomic arity assignment and of its refinement are listed next.

**Theorem 3.6** (arity assignment and its refinement).

1. (inclusion of refinement)
   
   If \( L_1 \subseteq L_2 \) then \( L_1 \subseteq L_2 \).

2. (transitivity of assignment through refinement)
   
   If \( L_1 \subseteq L_2 \) and \( L_2 \vdash T : A \) then \( L_1 \vdash T : A \).

3. (confluence of refinement and assignment)
   
   If \( L_1 \subseteq L_2 \) and \( L_1 \vdash T : A \) then \( L_2 \vdash T : A \).

4. (transitivity of refinement)
   
   If \( L_1 \subseteq L \) and \( L \subseteq L_2 \) then \( L_1 \subseteq L_2 \).
\[
\begin{align*}
\text{atom} & \iff * \\
L_1 \uplus^I L_2 = L & \iff |L_1| \in \mathcal{E}_I^\ast(\delta_1/\lambda_1W_1,L_1,U) \\
\delta_1/\lambda_1W_1 \uplus^I L_1 \uplus^I \delta_2/\lambda_2W_2L_2 = \delta_1/\lambda_1W_1L_1 & \iff \delta_2/\lambda_2W_2L_2 = \delta_2/\lambda_2W_2L_2 \\
L_1 \uplus^I L_2 = L & \iff |L_1| \in \mathcal{E}_I^\ast(\delta_1/\lambda_1W_1,L_1,U) \\
\delta_1/\lambda_1W_1 \uplus^I L_1 \uplus^I \delta_2/\lambda_2W_2L_2 = \delta_2/\lambda_2W_2L_2 & \iff \delta_2/\lambda_2W_2L_2
\end{align*}
\]

Figure 39: Pointwise union of environments.

\[
\forall L_1. (L_1 \leadsto h,g L_2) \Rightarrow (L_1 \not\leadsto h,g T_2) \Rightarrow (\mathcal{N}_{h,g,d}^* \Gamma_2) \leq \mathcal{N}_{h,g,d}^* \Gamma_1
\]

Figure 40: Strongly normalizing environments for extended reduction.

5. (uniqueness of atomic arities)
   If \( L \vdash T : A_1 \) and \( L \vdash T : A_2 \) then \( A_1 = A_2 \).

6. (inclusion of assignment)
   If \( L \vdash T : A \) then for each \( h \) and \( n \) then there exists \( U \) such that \( L \vdash T \bullet^\ast_n U \) and \( L \vdash U : A \).

7. (preservation of atomic arity through extended reduction, general form)
   If \( L \vdash T_1 : A \) and \( L \vdash T_1 \leadsto_{h,g} T_2 \) and \( L \vdash \leadsto_{h,g} L_2 \) then \( L_2 \vdash T_2 : A \).

Proof. Clause 1 and Clause 6 are proved by induction on their premise. Clause 2 and Clause 3 are proved by induction on their second premise and by cases on their first premise. Clause 4 is proved by induction on its first premise and by cases on its second premise with the help of Clause 2 and Clause 3. Clause 4 is proved by induction on its first premise and by cases on its second premise. Clause 7 is proved by induction on its first premise by cases on its second premise and then by cases on its third premise. As for Theorem 3.7, reduction for environments emerges when considering Figure 27(\( \delta \)) and when a binder in the “spine” of \( T_1 \) is pushed into \( L_1 \) in the cases of Figure 27(bind), Figure 27(\( \beta \)), and Figure 27(\( \delta \)). Moreover, Clause 2 and Figure 31(beta) are invoked when Figure 27(\( \beta \)) is considered. ▲

Theorem 3.6(7) (proposition 500 of Guidi, 2014) states the “subject reduction” property of the arity assignment, a prerequisite for the preservation Theorem 3.14.

3.5 Results on Reducibility Candidates

The properties of reducibility candidates and of their refinement are listed next.

**Theorem 3.7** (reducibility candidates and their refinement).

1. (the candidate of strongly normalizing closures for extended reduction)
   For any \( h \) and \( g \), the subset \( \{ (L,T) \mid L \vdash \mathcal{N}_{h,g}^* T \} \) is a reducibility candidate for itself.

2. (the candidate associated to an atomic arity)
   If \( R \) is a reducibility candidate for itself then \( [A]_R \) is a reducibility candidate for \( R \).
\[
\begin{align*}
\begin{array}{lll}
\text{atom} & \sim_{h,g,0} & L \\
\text{skip} & \sim_{h,g,l} & (L,\delta/\lambda W) \\
\text{pair} & \sim_{h,g,l+1} & (L,\delta/\lambda W)
\end{array}
\end{align*}
\]

Figure 41: Strongly co-normalizing environments for extended reduction.

\[
\begin{align*}
\frac{L \vdash T_1 \Rightarrow_{h,g} T_2}{(L,T_1) \geq_{h,g} (L,T_2)} & \quad \text{cpx} \\
\frac{(L_1,T_1) \cong (L_2,T_2)}{(L_1,T_1) \geq_{h,g} (L_2,T_2)} & \quad \text{tseq} \\
\frac{(L_1,T_1) \geq_{h,g} (L_2,T_2)}{(L_1,T_1) \geq_{h,g} (L_2,T_2)} & \quad \text{step}
\end{align*}
\]

Figure 42: \texttt{qrst}-reduction and \texttt{qrst}-computation.

3. (reducibility theorem for extended reduction, general form)
If \( R \) is a reducibility candidate for itself then
\[ L_1 \odot \subseteq R L_2 \] and
\[ \downarrow c L_2 = K_2 \]
and
\[ K_2 \vdash T : A \] and
\[ \uparrow c T = U \]
imply \( \langle L_1, U \rangle \in \llbracket A \rrbracket_R \).

4. (reducibility theorem for extended reduction)
If \( R \) is a reducibility candidate for itself then
\[ L \vdash T : A \]
implies \( \langle L, T \rangle \in R \).

5. (strong normalization theorem for extended reduction)
If \( L \vdash T : A \) then \( L \vdash \Diamond_{h,g} T \).

6. (inclusion of refinement)
If \( L_1 \odot \subseteq R L_2 \) then \( L_1 \odot \subseteq L_2 \).

7. (inverse inclusion of refinement)
If \( R \) is a reducibility candidate for itself then
\[ L_1 \odot \subseteq \vdash: L_2 \]
implies \( L_1 \odot \subseteq R L_2 \).

Proof. Clause (1) is proved directly by invoking Theorem 3.5(7) and similar propositions (one for each extended redex). Clause (2) is proved by induction on \( A \). Clause (3) is proved by induction on \( K_2 \vdash T : A \) and by cases on the other premises by invoking Clause (2). Multiple relocation emerges from Figure 34(cfun), while the refinement emerges since Figure 36(beta) is needed when \( T \) is a \( \lambda \)-abstraction. Theorem 3.6(5) is invoked when \( T \) is a reference to a declaration in the case of Figure 36(beta). Clause (1) is a corollary of Clause (3) and of Figure 33(S1). Clause (6) is a corollary of Clause (4) and of Clause (1). Clause (7) is proved by induction on its premise with the help of Clause (3).

Theorem 3.7(1) is the most relevant property of strongly normalizing terms. Moreover the relation \( L \vdash \Diamond_{h,g} T \) is generated by the next rule resembling Figure 32(csf).

\[
\forall T_2, (L \vdash T_1 \Rightarrow_{h,g} T_2) \Rightarrow (T_1 \vdash T_2) \Rightarrow (L \vdash \Diamond_{h,g} T_2) \\
L \vdash \Diamond_{h,g} T_1
\]

3.6 Results on Lazy Equivalence
The relevant properties of pointwise union and lazy equivalence are listed next.
Theorem 3.8 (pointwise union).

1. (construction lemma for tail binder, positive case) If \(|L_1| \in \mathbb{E}_t^\ell(\delta_1/\lambda_1W_1).L_1.\) and \(l \leq |L_1|\) then

\[
L_1 \updownarrow l L_2 = L \quad \text{implies} \quad (\delta_1/\lambda_1W_1).L_1. \updownarrow l (\delta_2/\lambda_2W_2.L_2) = \delta_2/\lambda_2W_2.L.
\]

2. (construction lemma for tail binder, negative case) If \(|L_1| \notin \mathbb{E}_t^\ell(\delta_1/\lambda_1W_1).L_1.\) then

\[
L_1 \updownarrow l L_2 = L \quad \text{implies} \quad (\delta_1/\lambda_1W_1).L_1. \updownarrow l (\delta_2/\lambda_2W_2.L_2) = \delta_1/\lambda_1W_1.L.
\]

3. (existence lemma) If \(|L_1| = |L_2|\) then for each \(T\) and \(l\) there exists \(L\) such that \(L_1 \updownarrow l L_2 = L\).

Proof. Clause (1) and Clause (2) are proved by cases on their last premise. Clause (3) is proved by induction on \(|L_1|\) with the help of Clause (1) and Clause (2). ▲

Theorem 3.9 (lazy equivalence).

1. (left operand lemma) If \(L_1 \equiv l L_2 \quad \text{and} \quad L_2 \vdash_{h,g} K_2 \quad \text{and} \quad L_1 \updownarrow l K_2 = K_1 \quad \text{then} \quad L_1 \vdash_{h,g} K_1.\)

2. (right operand lemma) If \(L_1 \equiv l L_2 \quad \text{and} \quad L_2 \vdash_{h,g} K_2 \quad \text{and} \quad L_1 \updownarrow l K_2 = K_1 \quad \text{then} \quad K_2 \equiv l K_1.\)
3. (transitivity with ranged equivalence) If \( L_1 \equiv_i^T L \) and (\( \forall m. L \equiv_{(i, m)}^T L_2 \) then \( L_1 \equiv_i^T L_2 \).

4. (transitivity with direct subclosure) If \( L_1 \equiv_0^T L_2 \) and \( \langle L_2, T \rangle \vdash (K_2, U) \) then there exists \( K_1 \) such that \( \langle L_1, T \rangle \vdash (K_1, U) \) and \( K_1 \equiv_0^T K_2 \).

5. (transitivity with extended reduction for terms) If \( L_1 \equiv_0^T L_2 \) and \( L_2 \vdash T_1 \Rightarrow h, g T_2 \) then \( L_1 \vdash T_1 \Rightarrow h, g T_2 \).

6. (transitivity with extended reduction for environments) If \( L_1 \equiv_0^T L_2 \) and \( L_2 \vdash \Rightarrow h, g K_2 \) then there exists \( K_1 \) such that \( L_1 \vdash \Rightarrow h, g K_1 \) and \( K_1 \equiv_0^T K_2 \).

7. (confluence with extended reduction for terms) If \( L_1 \vdash T_1 \Rightarrow h, g T_2 \) then \( L_1 \equiv_0^T L_2 \) implies \( L_1 \equiv_0^T L_2 \).

Proof. Clause (1) and Clause (2) are proved directly by accessing to lazy equivalence through Rule (8). Clause (3) is proved by induction on its first premise. Clause (4) and Clause (5) are proved by induction on their second premise and by cases on their first premise. Clause (6) follows from Clause (1) and Clause (2) by taking \( K_1 = L_1 \uplus f^\perp K_2 \), which results from Theorem 3.8(3). Here we see the purpose of pointwise union. Clause (7) is proved by induction on its first premise and by cases on its second premise with the help of Clause (6) when Figure 27(bind), Figure 27(3), and Figure 27(θ) are considered. Here we see the purpose of ranged equivalence.

The shape of the second premise in Theorem 3.9(3) is due the implicit instantiation of \( m \) with \( \infty \) in Definition 2.44 (lazy equivalence) as noted at the end of Section 2.11. Theorem 3.9(6) and Theorem 3.9(7) (proposition 1000 of [Guidi, 2014]) are the most interesting properties of lazy equivalence with respect to extended reduction. Their proofs were the most demanding of this set.

Theorem 3.10 (strongly normalizing environments).

1. (transitivity of strong normalization for environments through extended reduction) If \( \sim_{h, g, l} T \) and \( L \vdash T_1 \Rightarrow h, g T_2 \), then \( \sim_{h, g, l}^* L \) implies \( \sim_{h, g, l}^* T_1 \).

2. (construction lemma for variable reference, general form) If \( l \leq i \) and \( K_1 \vdash \sim_{h, g}^* W \) and \( K_1 \vdash \Rightarrow_{h, g}^* K_2 \), then \( \theta_{h, g, i} L_2 = K_2, \delta \lambda W \) and \( \sim_{h, g, 0}^* K_2 \) imply \( \sim_{h, g, l}^* L_2 \).

3. (strong normalization for terms implies strong normalization for environments) If \( L \vdash \sim_{h, g}^* T \) then \( \sim_{h, g, i}^* L \) for every \( l \).

Proof. Clause (1) is proved by induction on its second premise and by cases on its third premise. Strongly co-normalizing environments (Definition 2.41) emerge when \( T_1 = \# i \) with \( i < l \) and Figure 27(δ) is considered. Every construction lemma is needed except for Clause (2), which is proved by induction on \( K_1 \vdash \sim_{h, g}^* W \) using Rule (7) and then by induction on \( \sim_{h, g, 0}^* K_2 \) with the help of Clause (1) and of Theorem 3.5(6). Clause (3) is proved by induction on the proper subclosures of \( \langle L, T \rangle \) with the help of every construction lemma including Clause (2). \( \blacksquare \)

Theorem 3.10(3) is the most relevant property of strongly normalizing environments. Notice that \( \sim_{h, g, i}^* L \) is generated by the next rule resembling Figure 10(lxs).

\[
\forall L_2, (L_1 \vdash \sim_{h, g}^* L_2) \Rightarrow (L_1 \equiv_i^T L_2) \Rightarrow (\sim_{h, g, l}^* T_1 \ L_2)_{\text{lex}}
\]

(10)
3.7 Results on Very Big Trees

The properties of qrst-computations and strong qrst-normalization are listed next.

**Theorem 3.11** (qrst-computations).

1. (decomposition property for qrst-computation)
   If \( (L_1, T_1) \geq_{h,g} (L_2, T_2) \) then there exist \( L_0, L \) and \( T \) such that
   \( L_1 \vdash T_1 \Rightarrow^* h,g T \) and \( (L_1, T) \Rightarrow^* (L_0, T_2) \) and \( L_0 \vdash T_3 \Rightarrow^* h,g T \) and \( L \Rightarrow^* T_3, L_2 \).

2. (formulation of proper rst-reduction with q-equivalence, left to right)
   If \( (L_1, T_1) \succ_{h,g} (L_2, T_2) \) then \( (L_1, T_1) \geq_{h,g} (L_2, T_2) \) and \( (L_1, T_1) \neq_0 (L_2, T_2) \).

3. (formulation of proper rst-reduction with q-equivalence, right to left)
   If \( (L_1, T_1) \geq_{h,g} (L_2, T_2) \) and \( (L_1, T_1) \neq_0 (L_2, T_2) \) then \( (L_1, T_1) \succ_{h,g} (L_2, T_2) \).

4. (transitivity of proper rst-reduction through lazy equivalence)
   If \( K_1 \equiv^L_0 K_2 \) and \( (K_2, T) \succ_{h,g} (L_2, U) \) then there exists \( L_1 \) such that \( (K_1, T) \succ_{h,g} (L_1, U) \) and \( L_1 \equiv^L_0 L_2 \).

5. (transitivity of proper qrst-computation through qrst-reduction, left case)
   If \( (L_1, T_1) \geq_{h,g} (L, T) \) and \( (L, T) \succ_{h,g} (L_2, T_2) \) then \( (L_1, T_1) \succ_{h,g} (L_2, T_2) \).

6. (transitivity of proper qrst-computation through qrst-computation, left case)
   If \( (L_1, T_1) \geq_{h,g} (L, T) \) and \( (L, T) \succ_{h,g} (L_2, T_2) \) then \( (L_1, T_1) \succ_{h,g} (L_2, T_2) \).

**Proof.** Clause (1) is proved by induction on its premise rearranging the qrst-steps with Theorem 3.5(6), Theorem 3.5(6), Theorem 3.9(6), Theorem 3.9(4), and Theorem 3.9(7). Clause (2) is proved by cases on its premise. Clause (3) is proved by cases on its first premise. Clause (4) is proved on its second premise with the help of Theorem 3.9(6), Theorem 3.9(5), Theorem 3.9(6), and Theorem 3.9(7). Clause (5) is a corollary of Clause (2) and Clause (3). Clause (6) is proved by induction on its first premise with the help of Clause (5). ▲

Notice that the reverse of Theorem 3.11(5) is straightforward. Also notice that Theorem 3.11(6) implies the transitivity of proper qrst-computation. The “right case” of the transitivity, that is: \( (L_1, T_1) \succ_{h,g} (L, T) \) \& \( (L, T) \geq_{h,g} (L_2, T_2) \) \Rightarrow \( (L_1, T_1) \succ_{h,g} (L_2, T_2) \), comes immediately from the transitivity of qrst-computation (defined as a transitive closure in Section 2.12). Another important corollary of Theorem 3.11(6) is that the relation \( \geq_{h,g} (L, T) \) is generated by the next rule:

\[
\forall L_2, T_2. (L_1, T_1) \succ_{h,g} (L_2, T_2) \Rightarrow \geq_{h,g} (L_2, T_2)_{lab}
\]

The induction principle for \( \geq_{h,g} (L, T) \) derived from this rule, gives a very strong induction hypothesis that takes advantage of the generality of proper qrst-computation (Definition 2.13). We need such a strength to prove the preservation Theorem 3.14.

**Theorem 3.12** (strongly qrst-normalizing closures).

1. (strong normalization implies strongly qrst-normalization, general form)
   If \( L_1 \vdash^* h,g T_1 \) and \( (L_1, T_1) \geq_{h,g} (L_2, T_2) \) then \( \geq_{h,g} (L_2, T_2) \).

2. (very big tree theorem)
   If \( L \vdash T : A \) then \( \geq_{h,g} (L, T) \) for each \( h \) and \( gnh \).

**Proof.** Clause (1) is proved by induction on its first premise and then by induction on the proper subclosures of \( (L_2, T_2) \) by invoking Theorem 3.10(6) and the the reverse of Theorem 3.11(1). Clause (2) is a corollary of Clause (1) and Theorem 3.7. ▲
3.8 Results on Stratified Validity

The relevant properties of stratified validity and of its refinement are listed next.

**Theorem 3.13** (stratified validity and its refinement).

1. (inclusion of validity)
   If $L \vdash T ! h, g$ then there exists $A$ such that $L \vdash \top : A$.

2. (validity implies strong qrst-normalization)
   If $L \vdash T ! h, g$ then $\vdash_{h, g} (L, T)$.

3. (first inclusion of refinement)
   If $L_1 \vdash_{h, g} L_2$ then $L_1 \vdash_{h, g} L_2$.

4. (second inclusion of refinement)
   If $L_1 \vdash_{h, g} L_2$ then $L_1 \vdash L_2$.

5. (transitivity of degree-guarded iterated static type assignment through refinement)
   If $n \vdash d$ and $L_2 \vdash T \cdot_{h, g} d$ then $L_1 \vdash_{h, g} L_2$ and $L_2 \vdash T \cdot_{h}^* U_2$
   imply $L_1 \vdash T \cdot_{h}^* U_1$ and $L_1 \vdash \top 1 \vdash^* U_2$ for some $U_1$.

6. (transitivity of stratified decomposed computation through refinement)
   If $L_1 \vdash_{h, g} L_2$ and $L_2 \vdash T_1 \cdot^* \cdot_{h, g} T_2$
   then $L_1 \vdash T_1 \cdot^* \cdot_{h, g} T$ and $L_1 \vdash T_2 \cdot^* T$ for some $T$.

7. (transitivity of validity through refinement)
   If $L_1 \vdash_{h, g} L_2$ and $L_2 \vdash T ! h, g$ then $L_1 \vdash T ! h, g$.

**Proof.** Clause (1) is proved by induction on its premise by invoking Theorem 3.6(5) and Theorem 3.6(7). Here we see that preservation of validity requires preservation of atomic arity. Clause (2) is a corollary of Clause (1) and of Theorem 3.12(2). Clause (3) is proved by induction on its premise. Clause (4) is proved by induction on its premise by invoking Theorem 3.6(5) and Theorem 3.6(7). When Figure 23(beta) is considered. Clause (5) is proved by induction on its premise. Clause (6) is proved by induction on its second premise, by cases on its first premise, by invoking Clause (6) and Theorem 3.2(3), and Theorem 3.3(5). Clause (7) is proved by induction on its second premise and by cases on its first premise, by invoking Clause (6) and Theorem 3.2(1) when Figure 22(appl) and Figure 22(cast) are considered. ▲

We introduce some abbreviations in the style of van Daalen, 1994 to state the preservation theorem. With respect to van Daalen, 1994, our PVR is connected to his CL, and our PT is connected to his P*T.

**Definition 3.1** (preservation properties). Figure 47 defines four properties of the closure $\langle L_1, T_1 \rangle$ with respect to $h$ and $g_r$. They are: preservation of degree by reduction (PD), preservation of validity by reduction (PVR), preservation of validity by static type (PVT), and preservation of static type by reduction (PT). ▲

**Theorem 3.14** (preservation properties).

1. (conditional preservation of degree by reduction)
   $(\forall L_1, T_1. (L, T) \Rightarrow_{h, g} (L_1, T_1) \Rightarrow PD_{h, g} (L_1, T_1))$ and
   $(\forall L_1, T_1. (L, T) \Rightarrow_{h, g} (L_1, T_1) \Rightarrow PVR_{h, g} (L_1, T_1))$ and
   $(\forall L_1, T_1. (L, T) \Rightarrow_{h, g} (L_1, T_1) \Rightarrow PVT_{h, g} (L_1, T_1))$ imply $PD_{h, g} (L, T)$.
2. (conditional preservation of validity by reduction)

\((\forall L, T_1. (L, T) \triangleright h,g \langle L_1, T_1 \rangle \Rightarrow PD_{h,g} \langle L_1, T_1 \rangle)\) and

\((\forall L, T_1. (L, T) \triangleright h,g \langle L_1, T_1 \rangle \Rightarrow PVR_{h,g} \langle L_1, T_1 \rangle)\) and

\((\forall L, T_1. (L, T) \triangleright h,g \langle L_1, T_1 \rangle \Rightarrow PVT_{h,g} \langle L_1, T_1 \rangle)\) and

\((\forall L, T_1. (L, T) \triangleright h,g \langle L_1, T_1 \rangle \Rightarrow PT_{h,g} \langle L_1, T_1 \rangle)\) imply PVR_{h,g} \langle L, T \rangle.

3. (conditional preservation of validity by static type)

\((\forall L, T_1. (L, T) \triangleright h,g \langle L_1, T_1 \rangle \Rightarrow PD_{h,g} \langle L_1, T_1 \rangle)\) and

\((\forall L, T_1. (L, T) \triangleright h,g \langle L_1, T_1 \rangle \Rightarrow PVR_{h,g} \langle L_1, T_1 \rangle)\) and

\((\forall L, T_1. (L, T) \triangleright h,g \langle L_1, T_1 \rangle \Rightarrow PVT_{h,g} \langle L_1, T_1 \rangle)\) and

\((\forall L, T_1. (L, T) \triangleright h,g \langle L_1, T_1 \rangle \Rightarrow PT_{h,g} \langle L_1, T_1 \rangle)\) imply PVT_{h,g} \langle L, T \rangle.

4. (conditional preservation of static type by reduction)

\((\forall L, T_1. (L, T) \triangleright h,g \langle L_1, T_1 \rangle \Rightarrow PD_{h,g} \langle L_1, T_1 \rangle)\) and

\((\forall L, T_1. (L, T) \triangleright h,g \langle L_1, T_1 \rangle \Rightarrow PVR_{h,g} \langle L_1, T_1 \rangle)\) and

\((\forall L, T_1. (L, T) \triangleright h,g \langle L_1, T_1 \rangle \Rightarrow PVT_{h,g} \langle L_1, T_1 \rangle)\) and

\((\forall L, T_1. (L, T) \triangleright h,g \langle L_1, T_1 \rangle \Rightarrow PT_{h,g} \langle L_1, T_1 \rangle)\) imply PT_{h,g} \langle L, T \rangle.

5. (preservation theorem, general form)

If \(L \vdash T \upharpoonright h,g\) then PD_{h,g} \langle L, T \rangle and PVR_{h,g} \langle L, T \rangle and PVT_{h,g} \langle L, T \rangle and PT_{h,g} \langle L, T \rangle.

6. (preservation of validity by computation)

If \(L \vdash T_1 \uparrow T_2\) then \(L \vdash T_1 \upharpoonright h,g\) implies \(L \vdash T_2 \upharpoonright h,g\).

7. (preservation of conversion by static type)

If \(L \vdash T_1 \upharpoonright h,g\) and \(L \vdash T_2 \upharpoonright h,g\) and \(n \leq d_1\) and \(n \leq d_2\) and \(L \vdash T_1 \bullet_{h,g} d_1\) and \(L \vdash T_2 \bullet_{h,g} d_2\) and \(L \vdash T_1 \bullet^{*} h,g U_1\) and \(L \vdash T_2 \bullet^{*} h,g U_2\) then \(L \vdash T_1 \equiv T_2\) implies \(L \vdash U_1 \equiv U_2\).

Proof. Clause (1), Clause (2), Clause (3), and Clause (4) are proved by cases on \(T\), and then by cases on the other premises. When Figure (14β) is considered, Clause (1) invokes Theorem 3.3(3) and Figure 20 beta), Clause (2) invokes Theorem 3.13(4) and Figure 23 beta), while Clause (3) invokes Theorem 3.13(4) and Figure 23 beta). Moreover Clause (2) needs Theorem 3.3(1) in the cases of Figures (14 flat) (already noted by van Daalen, 1994) and of Figure (14δ), while Clause (3) needs Theorem 3.3(1) in the case of Figure (14δ). Clause (5) is proved by induction on the proper rst-reducts of \((L, T)\) by invoking Clause (1), Clause (2), Clause (3), and Clause (4). The induction is assured by Theorem 3.3(2) and by Rule (1). Clause (6) is proved by induction on its first premise by invoking PVR from Clause (5). Clause (7) is a corollary of Clause (6), Theorem 3.3(3), and Theorem 3.3(4), given PVT and PD from Clause (5). ▲
Theorem 3.14(5) sums up the most significant propositions discussed in this article.

4 Conclusion and Future Work

We presented in Section 2 a revised version of the formal system $\lambda\delta$ to be termed “$\lambda\delta$ version 2A1”, and we proved in Section 3 that this calculus enjoys three relevant desired properties: confluence of reduction (Theorem 3.2), strong normalization along $qrst$-computations (Theorem 3.7), and preservation of validity by reduction (Theorem 3.14).

Notably, the matter of this article was entirely developed by the author with the unavoidable help of the proof management system Matita of [Asperti et al., 2011], which mechanically validated the resulting formalization of [Guidi, 2014] in full. The development took 42 months, producing 143 definitions and 1416 propositions. More data is available at $\lambda\delta$ Web site <http://lambdadelta.info/>.

We wish to stress that, to our knowledge, we are presenting as Theorem 3.12(2) the first fully machine-checked proof of the so-called “big tree” theorem [de Vrijer, 1994] for a calculus that includes $\Lambda_{\infty}$. It is also important to point out that the proof of this theorem is harder in $\lambda\delta$ than in $\Lambda_{\infty}$ since the latter system does not have environments.

The long time we needed to take $\lambda\delta$ to this stage, played in favor of presenting the development as is, while the revision of the calculus is far from being complete. In particular the present treatment lacks the type assignment judgment $L \vdash T: h(U)$ and its desired properties found in [Guidi, 2009]. Anyway, it is a design feature of $\lambda\delta$, the fact that a term is typed iff it is valid, so the preservation theorem presented here is the crux of the “subject reduction” property of this judgment.

Moreover, we are interested in relating the present notion of validity, based on an extended (i.e., $\alpha$-like) applicability condition, with the one implied by [Guidi, 2009] which is based on a restricted (i.e., PTS-like) applicability condition (see Section 2.6). It might happen that every valid closure in the extended sense has an $\eta$-equivalent formulation that is valid in the restricted sense. We support this conjecture by noting that a typical case in which we need extended validity, is the next:

$$L.\lambda z.\lambda y.(\lambda k. k) \cdot \lambda x y \vdash \@z. x ! h, g$$

(12)

where named variables improve the readability. If we $\eta$-expand $y$ (i.e., the expected type of $x$) to $\lambda w. k. @w. y$, restricted validity suffices.

It is important to stress that the above transformation looks like an $\eta$-expansion because of the notation, but it might have a different logical meaning. We see such a case considering Landau’s “Grundlagen der Analysis” (GdA) formalized in the system Aut-QE [van Benthem Jutting, 1994a], where Automath’s unified binder $[x:w]$ stands either for $\lambda x W$, or for $\Pi_{\eta} W$. The GdA validates just in the extended sense because a situation like (12) occurs in the definition of the constant $\text{ande2}^{\text{1-r}}$, but four formal $\eta$-expansions assure its validity in the restricted sense as well. Each one takes an expected type $b$, that is the $y$ of (12), and turns it into $[x:a] \times b$ (\times is our applicator $@x$). We must note that the expected type of $b$ is $[x:a]'\text{prop}'$, whereas the expected type of $[x:a] \times b$ is ‘$\text{prop}$’. So this expansion is not type-preserving, especially if we accept the statement of [Brown, 2011] on the GdA that every unified binder of degree one stands for a $\Pi$. This means that the expansion is indeed a $\Pi$-introduction. Interestingly, [Brown, 2011] states that formal $\eta$-expansions, whose logical meaning should be investigated, solve all incompatibilities preventing the GdA to validate in a PTS.

Theorem 3.13(1) shows that a valid closure can be typed by a simple type. Using $\lambda\delta$ as a logical framework is not a priority, but if we wish to do so (say, for validating the GdA), we need the additional expressive power given by universes (say, ‘$\ast$’ in the $\lambda$-Cube, or ‘$\text{type}$’ and ‘$\text{prop}$’ in the GdA). However, adding universes to $\lambda\delta$ while preserving its properties is challenging because the naive extension of $\Lambda_{\infty}$ with “type inclusion” (the device with which universes are built in the Automath tradition) is not conservative, since either confluence or uniqueness of types is lost.
Other additions to \(\lambda\delta\) we shall consider, include: “global” variables referred by level (while the current variables referred by depth would be “local”), term-like environments with projections as we advocated in [Guidi, 2009], and metavariables. Furthermore, we are interested in improving multiple relocation (Definition 2.9), which we introduced since the set of the functions \(\uparrow^{(l,m)}\) is not closed for composition, by considering the functions \(\uparrow\) as primitive, and by representing a multiple relocation more conveniently than with a list of pairs. As the reader can see, our long-term aim is to make \(\lambda\delta\) a fully fledged and elegant type system suitable for many purposes.

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# A Summary of Notation

The ongoing revision of $\lambda\delta$ includes an update of the notational conventions of [Guidi, 2009]. This Appendix summarizes the revised notation we introduced in Section 2.

| Symbol | Description | Section |
|--------|-------------|---------|
| $A, B$ | atomic arity | 2.30 |
| $C$ | reducibility candidate | 2.33 |
| $K, L$ | environment | 2.1 |
| $R$ | generic property on closures | 2.33 |
| $T, U, V, W$ | term | 2.1 |
| $\triangledown$ | list of arguments | 2.2 |
| $c$ | relocation pair | 2.7 |
| $\mathcal{C}$ | list of relocation pairs | 2.9 |
| $d$ | degree | 2.19 |
| $g$ | sort degree parameter | 2.19 |
| $h$ | sort hierarchy parameter | 2.18 |
| $i, j$ | variable reference depth | 2.1 |
| $k$ | sort index | 2.1 |
| $l$ | relocation level | 2.7 |
| $m$ | relocation depth | 2.7 |
| $n$ | number of iterations | 2.18 |
| $B \supset A$ | functional atomic arity | 2.30 |
| $C_1 \supset C_2$ | function subset | 2.34 |
| $K.L$ | concatenation | 2.4 |
| $L.\delta V$ | definition | 2.1 |
| $L.\lambda W$ | declaration | 2.1 |
| $L_1 \equiv_{(l,m)} L_2$ | ranged equivalence | 2.12 |
| $L_1 \vdash L_2$ | parallel reduction for environments | 2.16 |
| $L_1 \Rightarrow_{h,g} L_2$ | extended parallel reduction for env.’s | 2.25 |
| $L \Rightarrow_{h,g}^n T$ | normal term for extended reduction | 2.32 |
| $L_1 \Rightarrow L_2$ | parallel computation for environments | 2.16 |
| $L_1 \Rightarrow_{h,g,n_1,n_2} L_2$ | extended parallel computation for env.’s | 2.29 |
| $L \Rightarrow_{h,g}^{*} T$ | strongly norm. term for ext. reduction | 2.32 |
| $L \vdash T !_{h,g} (d)$ | stratified validity | 2.22 |
| $L \vdash T !_{h,g} (d)$ | stratified higher validity | 2.23 |
| $L \vdash T : A$ | atomic arity assignment | 2.30 |
| $L \vdash T_1 \Rightarrow T_2$ | sequential reduction | 2.13 |
| $L \vdash T_1 \Rightarrow_{h,g} T_2$ | extended sequential reduction | 2.26 |
| $L \vdash T_1 \Rightarrow T_2$ | parallel reduction for terms | 2.14 |
| $L \vdash T_1 \Rightarrow_{h,g} T_2$ | extended parallel reduction for terms | 2.27 |
| $L \vdash T_1 \Rightarrow^{*} T_2$ | parallel computation for terms | 2.16 |
| $L \vdash T_1 \Rightarrow_{h,g} T_2$ | extended parallel computation for terms | 2.29 |
| $L \vdash T_1 \Rightarrow^* T_2$ | parallel conversion for terms | 2.18 |
| $L \vdash T_1 \Rightarrow_{h,g}^{(n)} U$ | iterated static type assignment | 2.18 |
| $L \vdash T_1 \Rightarrow_{h,g}^{(n)} T_2$ | stratified decomposed computation | 2.24 |
| $L \vdash T_1 \Rightarrow_{h,g}^{(n_1,n_2)} T_2$ | stratified decomposed conversion | 2.21 |
| $L \vdash T_1 \Rightarrow_{h,g}^{(n)} T_2$ | degree assignment | 2.19 |
| $L_1 \subseteq L_2$ | refinement for preservation of reduction | 2.11 |
| $L_1 \subseteq_{\mathcal{R}} L_2$ | refinement for reducibility | 2.30 |
$L_1 \subseteq L_2$ refinement for preservation of atomic arity (Definition 2.31)
$L_1 \refinement_{h,g} L_2$ refinement for preservation of degree (Definition 2.30)
$L_1 \refinement_{h,g} L_2$ refinement for preservation of strat. validity (Definition 2.23)
$L_1 \equiv^T L_2$ lazy equivalence for environments (Definition 2.37)
$L_1 \uplus^T L_2 = L$ pointwise union (Definition 2.30)
$T_1 \approx T_2$ same top structure (Definition 2.6)
$h^n$ iterated composition (Definition 2.18)
i \in \mathcal{F}^*_{\langle L, U \rangle}$ hereditarily free variable (Definition 2.38)
\emptyset empty list (Section 2)
\emptyset empty environment (Definition 2.1)
\emptyset base atomic arity (Definition 2.30)
\emptyset k sort (Definition 2.1)
\#i variable reference (Definition 2.1)
$\llbracket A \rrbracket_R$ interpretation of the atomic arity (Definition 2.35)
$S(T)$ simple (or neutral) term (Definition 2.5)
$PD_{h,g} \langle L_1, T_1 \rangle$ preservation of degree by reduction (Definition 3.1)
$PT_{h,g} \langle L_1, T_1 \rangle$ preservation of static type by reduction (Definition 3.1)
$PVR_{h,g} \langle L_1, T_1 \rangle$ preservation of validity by reduction (Definition 3.1)
$PVT_{h,g} \langle L_1, T_1 \rangle$ preservation of validity by static type (Definition 3.1)
\texttt{;} list concatenation (Section 2)
\forall, \exists, \Rightarrow, \& shared notation (Definition 2.1)
\begin{itemize}
  \item end of definition, end of proof (Section 4)
\end{itemize}
B Pointers to the Certified Proofs

At the moment of writing this article, the certified specification of the revised $\lambda\delta$ is available just as a bundle of script files for the latest version of the proof management system Matita. The bundle is available on the Web at: <http://lambdadelta.info/download/lambdadelta_2A1.tar.gz>. For each proposition stated in the article, we give a pointer consisting of a path with three components: a directory inside the directory basic_2 of the bundle, a file name inside this directory, and a proved statement inside this file.

Notice that the notation in the files may differ from Appendix A because of incompatibilities between the characters available for $\LaTeX$ and for Matita.

Moreover, the given pointers might be modified in the forthcoming revisions of $\lambda\delta$.

Here we are referring to the directory <trunk/matita/contribs/lambdadelta/> in revision 12964 of HELM repository at <http://helm.cs.unibo.it/software/index.html>.

1. Path for Theorem 3.1(1): static/lsubr/lsubr/lsubr_trans
2. Path for Theorem 3.1(2): reduction/cpr/lsubr_cpr_trans
3. Path for Theorem 3.1(3): reduction/lpr/lpr/lpr_conf_lpr
4. Path for Theorem 3.1(4): reduction/lpr/lpr/lpr_conf
5. Path for Theorem 3.2(1): computation/cprs_cprs/cprs_conf
6. Path for Theorem 3.2(2): computation/lprs_lprs/lprs_conf
7. Path for Theorem 3.2(3): equivalence/cpcs_cpcs/cpcs_inv_cprs
8. Path for Theorem 3.3(1): unfold/lstas_da/da/lstas
9. Path for Theorem 3.3(2): unfold/lstas_da/lstas_inv_da
10. Path for Theorem 3.3(3): unfold/lstas_da/lstas_inv_da_ge
11. Path for Theorem 3.3(4): static/lsubd/lsubd_fwd/lsubr
12. Path for Theorem 3.3(5): static/lsubd_da/lsubd_da_trans
13. Path for Theorem 3.3(6): static/lsubd_da/lsubd_da_conf
14. Path for Theorem 3.3(7): static/lsubd/lsubd/lsubd_trans
15. Path for Theorem 3.3(8): unfold/lstas_lstas/lstas_mono
16. Path for Theorem 3.3(9): unfold/lstas_da/lstas_inv_refl_pos
17. Path for Theorem 3.5(1): reduction/cpx/lsubr_cpx_trans
18. Path for Theorem 3.5(2): reduction/cpx/cpr_cpx_trans
19. Path for Theorem 3.5(3): reduction/cpx_lift/sta_cpx
20. Path for Theorem 3.5(4): reduction/cpx_lift/fqu_cpx_trans
21. Path for Theorem 3.5(5): reduction/lpx_drop/lpx_fqu_trans
22. Path for Theorem 3.5(6): computation/lpxs/lpxs/lpx_cpx_trans
23. Path for Theorem 3.5(7): computation/cpxs_tsts/cpxs_fwd_beta

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24. Path for Theorem 3.6(1): static/lsuba/lsubafwd_lsubr
25. Path for Theorem 3.6(2): static/lsuba/aaa/lsuba_ana_trans
26. Path for Theorem 3.6(3): static/lsuba/aaa/lsuba_ana_conf
27. Path for Theorem 3.6(4): static/lsuba/lsuba/lsuba_trans
28. Path for Theorem 3.6(5): static/aaa/aaa/aaa_mono
29. Path for Theorem 3.6(6): unfold/lastas/aaa/aaa_lastas
30. Path for Theorem 3.6(7): reduction/lpx_aaa/cpx_lpx_ana_conf
31. Path for Theorem 3.7(1): computation/csxtst_vector/csx_gcr
32. Path for Theorem 3.7(2): computation/gcp_cr/acr_gcr
33. Path for Theorem 3.7(3): computation/gcp_ana/acr_ana_csbc_lifts
34. Path for Theorem 3.7(4): computation/gcp_ana/gcr_ana
35. Path for Theorem 3.7(5): computation/cs_ana/aaa_csx
36. Path for Theorem 3.7(6): computation/lsubc/lsubc_fwd_lsubr
37. Path for Theorem 3.7(7): computation/lsubc_lsuba/lsubc_lsubc
38. Path for Theorem 3.8(1): multiple/llor_alt/lor_tail_frees
39. Path for Theorem 3.8(2): multiple/llor_alt/lor_tail_cofrees
40. Path for Theorem 3.8(3): multiple/llor_drop/llor_total
41. Path for Theorem 3.9(1): multiple/llpx_snllor/llpx_snllor_fwd_sn
42. Path for Theorem 3.9(2): multiple/lleq_llor/llpx_snllor_dx
43. Path for Theorem 3.9(3): multiple/lleq_lreq/llreq_lreq_trans
44. Path for Theorem 3.9(4): multiple/lleq_fqus/llreq_fqi_trans
45. Path for Theorem 3.9(5): reduction/cpx_llreq/llreq_cpx_trans
46. Path for Theorem 3.9(6): reduction/lpx_llreq/llreq_lpx_trans
47. Path for Theorem 3.9(7): reduction/cpx_llreq/cpx_llreq_conf_sn
48. Path for Theorem 3.10(1): computation/lcosx_cpx/lsx_cpx_trans_lcosx
49. Path for Theorem 3.10(2): computation/lsx_csx/lsx_ref_be_lpxs
50. Path for Theorem 3.10(3): computation/lsx_csx/csx_lsx
51. Path for Theorem 3.11(1): computation/fpbs_alt/fpbs_inv_alt
52. Path for Theorem 3.11(2): reduction/fpbq_alt/fpbq_fpbq_alt
53. Path for Theorem 3.11(3): reduction/fpbq_alt/fpbq_inv_fpbq_alt
54. Path for Theorem 3.11(4): reduction/fpb_llreq/llreq_fpb_trans
55. Path for Theorem \textbf{3.11(5)}: computation/fpbg\_fpbs/fpbq\_fpbg\_trans
56. Path for Theorem \textbf{3.11(6)}: computation/fpbg\_fpbs/fpbs\_fpbg\_trans
57. Path for Theorem \textbf{3.12(1)}: computation/fsb\_csx/csx\_fsb\_fpbs
58. Path for Theorem \textbf{3.12(2)}: computation/fsb\_aaa/aaa\_fsb
59. Path for Theorem \textbf{3.13(1)}: dynamic/sn v\_aaa/sn v\_fwd\_aaa
60. Path for Theorem \textbf{3.13(2)}: dynamic/sn v\_fsb/sn v\_fwd\_fsb
61. Path for Theorem \textbf{3.13(3)}: dynamic/lsubsv\_lsubd/lsubsv\_fwd\_lsubd
62. Path for Theorem \textbf{3.13(4)}: dynamic/lsubsv\_lsuba/lsubsv\_fwd\_lsuba
63. Path for Theorem \textbf{3.13(5)}: dynamic/lsubsv\_lstas/lsubsv\_lstas\_trans
64. Path for Theorem \textbf{3.13(6)}: dynamic/lsubsv\_scpds/lsubsv\_scpds\_trans
65. Path for Theorem \textbf{3.13(7)}: dynamic/lsubsv\_lsubv\_snv\_trans
66. Path for Theorem \textbf{3.14(1)}: dynamic/sn v\_da\_lpr/da\_cpr\_lpr\_aux
67. Path for Theorem \textbf{3.14(2)}: dynamic/sn v\_lpr/sn v\_cpr\_lpr\_aux
68. Path for Theorem \textbf{3.14(3)}: dynamic/sn v\_lstas/sn v\_lstas\_aux
69. Path for Theorem \textbf{3.14(4)}: dynamic/sn v\_lstas\_lpr/lstas\_cpr\_lpr\_aux
70. Path for Theorem \textbf{3.14(5)}: dynamic/sn v\_preserve/sn v\_preserve
71. Path for Theorem \textbf{3.14(6)}: dynamic/sn v\_preserve/sn v\_cprs\_lpr
72. Path for Theorem \textbf{3.14(7)}: dynamic/sn v\_preserve/lstas\_cpcs\_lpr