INTERNAL MODE-INDUCED GROWTH
IN 3D NONLINEAR KLEIN-GORDON EQUATIONS

TRISTAN LÉGER AND FABIO PUSATERI

Abstract. This note complements the paper [19] by proving a scattering statement for solutions of nonlinear Klein-Gordon equations with an internal mode in 3d. We show that small solutions exhibit growth around a one-dimensional set in frequency space and become of order one in $L^\infty$ after a short transient time. The dynamics are driven by the feedback of the internal mode into the equation for the field (continuous spectral) component.

The main part of the proof consists of showing suitable smallness for a “good” component of the radiation field. This is done in two steps: first, using the machinery developed in [19], we reduce the problem to bounding a certain quadratic normal form correction. Then we control this latter by establishing some refined estimates for certain bilinear operators with singular kernels.

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1. Introduction

1.1. Background. The study of the small data regime for dispersive PDEs with quadratic nonlinearities has attracted a lot of attention in recent years, and is closely related to several open questions about the asymptotic stability of coherent structures, such as (topological) solitons. This has turned out to be a challenging problem, since quadratic nonlinearities are strong enough to influence the long-time behavior of solutions and may lead to blow-up even in a weakly nonlinear regime.

In recent years, several general methods have been developed to tackle problems about global existence and long-time asymptotics of nonlinear (and quasilinear) dispersive equations with low power nonlinearities. Without trying to be exhaustive, we mention works by Germain-Masmoudi-Shatah [8] and [9] and the development of the ‘space-time resonance’ method, parallel work by Gustafson-Nakanishi-Tsai [12] on Gross-Pitaevski, Guo-Ionescu-Pausader [11] and Deng-Ionescu-Pausader [5] on the Euler-Maxwell system, Ionescu and the second author on 2d water waves [13, 14], and Deng-Ionescu-Pausader and the second author [6] on 3d gravity-capillary waves. All of these results can be interpreted as proving asymptotic stability of the zero or ‘trivial’ solution.

F.P. was supported in part by a start-up grant from the University of Toronto, and NSERC grant RGPIN-2018-06487.
To prove similar results for non-trivial solutions, it is natural to consider corresponding linearized problems. This generically introduces a potential term in the equation. A typical model example is $i\partial_t u + L(\sqrt{-\Delta + V + m^2}) = u^2$, for some real-valued dispersion relation $L$ and potential $V = V(x)$. Consequently, a new line of active research has emerged seeking to understand the global behavior of small solutions to such equations.

A first result was obtained for the quadratic NLS $i\partial_t u + (-\Delta + V)u = \bar{u}^2$ in 3d by Germain-Hani-Walsh in [7]. The first author of this note dealt with the case of a $u^2$ nonlinearity with small, time-dependent $V$ [17]. Note that while a $\bar{u}^2$ nonlinearity is non-resonant (for any type of evolution), a $u^2$ nonlinearity is space-time resonant at the origin even in the case $V = 0$. See also [18] for the case of electromagnetic perturbations. More recently, Soffer and the second author [22] treated a $u^2$ nonlinearity with a large potential, and developed a general approach based on the distorted Fourier transform, that is, the transform adapted to the Schrödinger operator $H = -\Delta + V(x)$. The starting point of this is a refined study of the so-called ‘nonlinear spectral distribution’ (NSD)

$$\mu(\xi, \eta, \sigma) := (2\pi)^{-9/2} \int_{\mathbb{R}^3} \psi(x, \xi) \psi(x, \eta) \psi(x, \sigma) dx,$$  

(1.1)

where $\psi = \psi(x, \xi)$, $\xi \in \mathbb{R}^3$ are the generalized eigenfunctions of the Schrödinger operator, $(-\Delta + V(x))\psi = |\xi|^2 \psi$. In particular, a precise knowledge of the structure and singularities of $\mu$ allows the exploitation of oscillations in (distorted) frequency space, which is a key step to extend the techniques from the works on the case $V = 0$ cited above.

We also mention that there have been several recent works in 1d on equations with potentials and low power nonlinearities. Since this is not the main focus of this note, we refer the reader to the introductions of [10] and [21] for extensive discussions and references.

Up until now, global dispersive solutions for low powers (in particular quadratic) in $d \leq 3$ had only been constructed in the case of ‘trivial’ spectrum, $\sigma(-\Delta + V) = \sigma(-\Delta)$. In [19] we dealt with a problem where an internal mode is present. More precisely, we considered the following initial value problem:

$$\begin{align*}
\partial^2_t u + (-\Delta + V(x) + 1)u &= u^2, \\
u(0, x) &= u_0, \quad \partial_t u(0, x) &= u_1, 
\end{align*}$$  

(1.2)

where $u : \mathbb{R}_+ \times \mathbb{R}^3_+ \to \mathbb{R}$, with a sufficiently regular and decaying generic external potential $V : \mathbb{R}^3 \to \mathbb{R}$, and smooth, localized initial data $u_0$ and $u_1$. We will make the assumptions on $V$ and the initial data more precise below.

In this setting, the spectrum of the Schrödinger operator $H := -\Delta + V$ consists of the purely absolutely continuous part $[0, \infty)$ and a finite number of negative eigenvalues, with corresponding smooth and fast decaying eigenfunctions [25]. We further assume that the operator $L^2 := -\Delta + V + 1$ has a unique strictly positive eigenvalue with a corresponding normalized eigenfunction $\phi$:

$$(-\Delta + V + 1)\phi = \lambda^2 \phi, \quad 1/2 < \lambda < 1, \quad ||\phi||_{L^2} = 1.$$  

(1.3)

The eigenvalue $\lambda$ is usually called the “internal frequency of oscillation”, and $\phi$ is referred to as an “internal mode” of the dynamics.

[13] gives rise to a two parameter family of solutions to the linear equation $\partial^2_t u + (-\Delta + V + 1)u = 0$ of the form:

$$\phi_{A, \theta}(t, x) = A \cos(\lambda t + \theta) \phi(x), \quad A, \theta \in \mathbb{R}.$$  

(1.4)

These solutions - referred to as “bound states”, or “internal modes” with a slight abuse - are time-periodic, oscillating and spatially localized. In this weakly nonlinear regime, we expect nonlinear solutions to retain features of the linear system. Therefore, a natural question is whether such periodic solutions persist under the nonlinear flow. Our main result in [19] answered in the negative: in a

\[\text{In 1d we mention important recent works [15] and [4] on the stability under odd perturbations of the kink for the } \phi^4 \text{ model. The linearization at the kink gives exactly a 1d quadratic Klein-Gordon equation with an internal mode like [12] satisfying the Fermi Golden rule [16].}\]
neighborhood of zero, the bound states \([1.4]\) are destroyed by the quadratic nonlinearity, and do not continue to quasi-periodic or other non-decaying solutions. Moreover, solutions exhibit weak dispersive features and obey decay estimates, although with a rate that is much slower compared to solutions of \((\partial_t^2 - \Delta + 1 + V)u = 0\).

The first result of this kind was obtained by Sigal \([24]\), who showed instability of bound states for very general classes of equations, introducing the “nonlinear Fermi Golden Rule” (FGR). Soffer and Weinstein \([26]\) then derived decay and asymptotics in the case of cubic Klein-Gordon equations, that is, \([1.2]\) with a \(u^3\) nonlinearity. See also the recent work \([29]\) on the case \(\lambda \in (1/3, 1/2)\). The problem of meta-stability in the presence of multiple bound states in 3d was treated by Bambusi-Cuccagna \([1]\). For the nonlinear Schrödinger equation (NLS) the problem of meta-stability (for excited states) was continues of \([19]\). As a consequence we will obtain that solutions of \((1.2)\) with a regular and decaying \(V\) continue to quasi-periodic or other non-decaying solutions. Moreover, solutions exhibit weak dispersive features and obey decay estimates, although with a rate that is much slower compared to solutions of \((\partial_t^2 - \Delta + 1 + V)u = 0\).

Our main result in this note complements \([19]\) with precise scattering statements and asymptotics. In particular, we will show scattering in \(L^\infty_\xi\), i.e., pointwise in (distorted) frequency space, for the ‘good’ component of the radiation identified in \([19]\). As a consequence we will obtain that solutions of \((1.2)\) become of \(O(1)\) in \(L^\infty_\xi\) after a short time, and exhibit a certain oscillatory behavior over time. This appears to be the first rigorous proof for this type of asymptotic dynamics.

1.2. The main result of \([19]\). Set-up and assumptions. Let \(P_c\) denote the projection onto the continuous spectral subspace, namely, for every \(\psi \in L^2(\mathbb{R}^3, \mathbb{R})\)
\[
P_c \psi := \psi - (\phi, \psi)\phi, \quad (\psi_1, \psi_2) := \int_{\mathbb{R}^3} \psi_1 \psi_2 \, dx. \tag{1.5} \]
We make the following assumptions:

- **Regularity and decay of \(V\):** We assume \(V \in \mathcal{S}\), but a finite amount of regularity and decay (measured in weighted Sobolev spaces) would suffice.

- **Coupling to continuous spectrum:** \(\lambda \in (1/2, 1)\) so that \(\lambda \notin \sigma_{ac}(-\Delta + V + 1)\) and \(2\lambda \in \sigma_{ac}(-\Delta + V + 1)\).

- **Fermi Golden Rule:** The “Fermi Golden Rule” resonance condition holds:
\[
\Gamma := \frac{\pi}{2\lambda} \left( P_c \phi^2, \delta(L - 2\lambda) P_c \phi^2 \right) > 0, \tag{1.6} \]
where \(L := \sqrt{-\Delta + V + 1}\).

- **Genericity of \(V\):** the 0 energy level is regular for \(H := -\Delta + V\), that is, 0 is not an eigenvalue, nor a resonance, i.e., there is no \(\tilde{\psi} \in L^{1/2+}(\mathbb{R}^3)\) such that \(H \tilde{\psi} = 0\). Such a potential \(V\) is said to be ‘generic’.

*Distorted Fourier transform.* For a regular and decaying \(V\) (as in our assumptions above) we can decompose \(L^2\) into absolutely continuous and pure point subspaces: \(L^2(\mathbb{R}^d) = L^2_{ac}(\mathbb{R}^d) \oplus L^2_{pp}(\mathbb{R}^d)\) where \(L^2_{pp}(\mathbb{R}^d) = \text{span}(\phi)\). Let \(\psi(x, \xi)\) denote the generalized eigenfunctions solving for \(\xi, x \in \mathbb{R}^3\)
\[
(-\Delta + V)\psi(x, \xi) = |\xi|^2 \psi(x, \xi), \quad \text{with} \quad |\psi(x, \xi) - e^{ix\xi}| \longrightarrow 0, \quad \text{as} \quad |x| \to \infty, \tag{1.7} \]
with the Sommerfeld radiation condition \(r(\partial_t - i|k|)\psi(x,k) - e^{ixk} \to 0\), for \(r = |x| \to \infty\). We can then define a unitary operator \(\tilde{F}\), the distorted Fourier Transform (dFT), as
\[
\tilde{F} f(\xi) := \tilde{f}(\xi) = \frac{1}{(2\pi)^{3/2}} \lim_{R \to +\infty} \int_{|x| \leq R} f(x) \psi(x, \xi) dx, \tag{1.8} \]
with inverse
\[
(\tilde{F}^{-1} f)(x) = \frac{1}{(2\pi)^{3/2}} \lim_{R \to +\infty} \int_{|\xi| \leq R} f(\xi) \psi(x, \xi) d\xi. \tag{1.9} \]
In particular, for any $g \in L^2$ we can write $g(x) = \mathcal{F}^{-1}(\mathcal{F}_c g) + (g, \phi)\phi$. Moreover $\mathcal{F}$ diagonalizes the Schrödinger operator (on the continuous spectrum), $\mathcal{F}\mathcal{P}_c H = |\xi|^2 \mathcal{F}$.

**Continuous-Discrete decomposition.** Next, we decompose a solution $u$ into a discrete and a continuous component:

$$u(t) = a(t)\phi + v(t), \quad \text{with} \quad (v(t), \phi) = 0 \quad \text{for all } t. \quad (1.10)$$

Then $(1.2)$ with $(1.3)$ reads

$$\begin{cases}
\ddot{a} + \lambda^2 a = ((a\phi + v)^2, \phi) \\
\partial_t^2 v + L^2 v = \mathcal{P}_c ((a\phi + v)^2), \quad L := \sqrt{-\Delta + V + 1}.
\end{cases} \quad (1.11)$$

We call $v$ the radiation or field component of the solution $u$, and will call $a$ the (amplitude of the) discrete component or internal mode of the solution.

**Main result on Radiation Damping.**

**Theorem 1.1** ([19]). Consider $(1.2)$, with initial data $u(0, x) = u_0(x)$, $u_t(0, x) = u_1(x)$ such that:

$$\begin{align*}
&|(u_0, \phi)| + |(u_1, \phi)| \leq \varepsilon_0, \\
&\|((\nabla)u_0, u_1)\|_{L^2} + \|((\nabla)v_0, u_1)\|_{H^1} \leq \varepsilon_0
\end{align*} \quad (1.12)$$

for $N \gg 1$. Then, there exists $\varepsilon_0 \in (0, 1)$ such that, for all $\varepsilon_0 \leq \varepsilon$,

- The equation $(1.2)$ under the assumptions stated above has a unique global solution $u \in C(\mathbb{R}; H^{N+1}(\mathbb{R}^3))$ such that the following hold: $u = a(t)\phi + v(t)$ with

$$|a(t)| \approx \varepsilon_0 (1 + \varepsilon_0^2 t)^{-1/2}, \quad (1.13)$$

and

$$\begin{align*}
\langle t \rangle \|\partial_t v + iLv\|_{L^\infty} &\approx 1, \quad t \gtrsim \varepsilon_0^{-2}, \\
\|\partial_t v + iLv\|_{H^N} &\lesssim \varepsilon_0^{1-\delta},
\end{align*} \quad (1.14, 1.15)$$

for arbitrarily small $\delta > 0$.

- Furthermore, we have the following asymptotic behavior: Let $f := e^{-itL}(\partial_t + iL)v$. Then, there exists $f_\infty \in \varepsilon_0^{1-\delta} H^N_x$, with $\delta > 0$ arbitrarily small as above, such that

$$\|f(t) - f_\infty\|_{H^N_x} \lesssim \varepsilon_0^{1-\delta}(t)^{-\delta'}, \quad (1.16)$$

for some small $\delta' > 0$. Finally, we have the asymptotic growth

$$\|\partial_x f(t)\|_{L^2} \gtrsim \langle t \rangle^{1/2}, \quad t \gtrsim \varepsilon_0^{-2}. \quad (1.17)$$

**Set-up and notation.** In this note we work on the solutions constructed in Theorem 1.1 and will use notation which is consistent with [19]. The starting point for our analysis is the natural definition of a ‘good’ component, $h$, and a ‘bad’ component, $g$, of the profile $f = e^{-itL}(\partial_t + iL)v$:

$$f = h - g, \quad \tilde{g}(t, \xi) := -\chi_C(\xi) \int_0^t B^2(s)e^{-is(\xi) - 2\lambda} ds \mathcal{P}_c \phi^2(\xi), \quad (1.18)$$

where $\chi_C(\xi) := \varphi_{\leq -C(|\xi| - 2\lambda)}$, with $\varphi_{\leq -C}(x)$ a bump function supported on $|x| \leq 2^{-C+1}$ for $C = C(\lambda)$ large enough, and where $B = B(s)$ can be defined through the profile of the amplitude of the internal oscillations given by

$$A(t) := \frac{1}{2i\lambda} e^{-i\lambda t}(\dot{a} + i\lambda a). \quad (1.19)$$
We refer the reader to Lemma 3.5 in [19] for the details of the definition of $B$ which are not necessary here, and only recall the fact that (see (1.21) below for the definition of $\rho$)
\begin{equation}
|A(s) - B(s)| = O(|A(s)|^2) \approx \rho(t),
\end{equation}
\begin{equation}
B(t) = \rho(t)^{3/2} + \rho(t)^{3/2-a(t)^{-1}}, \quad a \in (1/2,1).
\end{equation}
We also define two useful quantities
\begin{equation}
\rho(t) := \varepsilon_0(1 + \varepsilon_0^2(\Gamma/\lambda)t)^{-1}, \quad Z := Z_\beta(\varepsilon, m) := (2^m \rho(2^m))^{1-\beta}(2^m \varepsilon)^\beta
\end{equation}
for some absolute constant $0 < \beta \ll 1$, and $\varepsilon := C_0 \varepsilon_0$ for some large constant $C_0 > 0$.

1.3. **New result**: Scattering in $L_\xi^\infty$. As already mentioned, our goal is to provide additional and more precise information on the asymptotic behavior of solution of (1.2). In particular, we identify a growth phenomenon due to the presence of the internal mode, and describe the asymptotic behavior of the solution in $L_\xi^\infty$. This allows us to see that after a short time the solution becomes of size $O(1)$ around a certain frequency, despite arising from small initial data. Our main result can be stated as follows:

**Proposition 1.2.** There exist $F_\infty \in \varepsilon^\delta L_t^\infty L_\xi^\infty$ and a real-valued $\Psi_\infty = O(\varepsilon^\delta)$, with some small $\delta > 0$, such that
\begin{equation}
\bar{f}(t, \xi) = \int_0^t \exp \left( \frac{i s(2\lambda - \langle \xi \rangle - 2\lambda) + i \frac{2c_2\lambda}{\Gamma} \log \left( 1 + \frac{1}{\chi Y_0^2 s} \right)}{\Gamma} y(s)^2 ds \right) e^{2\varepsilon_\infty \Phi_\infty \rho \phi^2(\xi)}
\end{equation}
\begin{equation}
+F_\infty(t, \xi) + O_{L_\xi^\infty} (\varepsilon^\delta \langle t \rangle^{-\delta}),
\end{equation}
where $c_2$ denotes a non-zero numerical constant, and $y(s) := Y_0 (1 + (\Gamma/\lambda)sY_0^2)^{-1/2}$ with $Y_0 = O(\varepsilon_0)$ a function of the initial data $(a(0), \dot{a}(0))$.

In particular, for $t \gtrsim \varepsilon_0^{-2}$ and $|\langle \xi \rangle - 2\lambda| \leq \delta_0 t^{-1}$ where $\delta_0$ denotes a sufficiently small constant, we have
\begin{equation}
\bar{f}(t, \xi) = \frac{1}{2ic_2} \left[ e^{2i c_2 \log \left( 1 + \frac{Y_0^2 s}{\chi} \right)} - 1 \right] e^{2\varepsilon_\infty \Phi_\infty \rho \phi^2(\xi)} + O_{L_\xi^\infty} (\delta_0).
\end{equation}

**Remark 1.3.** The leading term in (1.23) exhibits an oscillatory behavior, and is $O(1)$ for most times. Note that this does not contradict (1.16) since this only happens around a one-dimensional set.

**Remark 1.4.** Although the equation we study is quadratic, we expect the same behavior to take place in many other settings, typically when an internal mode is present.

We can reduce the proof of the main Proposition 1.2 to a scattering statement for the field $h$ plus a quadratic (normal form) correction through the following:

**Lemma 1.5.** Assume there exists a correction $N(\bar{f}) \in \varepsilon^\delta L_t^\infty L_\xi^\infty$ and $h_\infty \in \varepsilon^\delta L_\xi^\infty$ such that
\begin{equation}
\| \bar{h}(t) - N(\bar{f})(t) - h_\infty \|_{L_\xi^\infty} \lesssim \varepsilon^\delta \langle t \rangle^{-\delta}
\end{equation}
for some $\delta > 0$. Then (1.22) holds true.

**Proof of Proposition 1.2 using Lemma 1.5.** Recall (3.64) in [19]:
\begin{equation}
\bar{g}(t, \xi) = -\int_0^t \exp \left( is(2\lambda - \langle \xi \rangle) + i \frac{2c_2\lambda}{\Gamma} \log \left( 1 + \frac{1}{\chi Y_0^2 s} \right) \right) y(s)^2 ds \right) e^{2i \varepsilon_\infty \Phi_\infty \rho \phi^2(\xi)}
\end{equation}
\begin{equation}
+F_\infty(t, \xi) + O_{L_\xi^\infty} (\varepsilon^\delta \langle t \rangle^{-\delta}),
\end{equation}
for some $g_\infty \in \varepsilon^\delta L_\xi^\infty$. Therefore, assuming that there exist $N(\bar{f})$ as in Lemma 1.5, the result (1.22) follows by letting $F_\infty(t) := N(\bar{f})(t) + h_\infty - g_\infty \in \varepsilon^\delta L_t^\infty L_\xi^\infty$ since $f = h - g$. 
Lemma 1.6. For all \( g \),

\[
\begin{align*}
\| \partial_t^k \tilde{h}(t) \|_{L^2} & \lesssim Z, \quad k \leq m, \quad (1.26a) \\
\| G(t) \|_{H^N} & \lesssim \varepsilon^{-1}, \quad G \in \{g, h\}, \quad (1.26b) \\
\| e^{itL} G(t) \|_{L^q} & \lesssim 2^{(3/2 - k_1) \cdot m} Z, \quad G \in \{g, h\}, \quad q \in [2, 6], \quad (1.26c) \\
\| \varphi_k G(t) \|_{L^1_t} & \lesssim 2^{(5/2 - k_1) \cdot m} Z, \quad G \in \{g, h\}, \quad (1.26d)
\end{align*}
\]

where \( \delta_N := 5/N \).

For the component \( g \) we also have, for \( t \approx 2^m \),

\[
\begin{align*}
\| \tilde{g}(t, \xi) \|_{L^\infty_t} & \lesssim 2^m \rho(2^m m), \quad (1.27a) \\
\| \partial_\xi \tilde{g}(t, \xi) \|_{L^1_t} & \lesssim 2^m \rho(2^m m)^2, \quad (1.27b) \\
\| \partial_\xi \tilde{g}(t, \xi) \|_{L^2} & \lesssim 2^{3m/2} \rho(2^m m)^2. \quad (1.27c)
\end{align*}
\]

Finally we have estimates for the time-derivative of the two components: with \( M_0 := \delta_N m \)

\[
\begin{align*}
\| e^{itL} \partial_t g(t)(\xi) \|_{L^p} & \lesssim \rho(t), \quad p \geq 1, \quad (1.28a) \\
\| e^{itL} \partial_t h(t)(\xi) \|_{L^q} & \lesssim 2^{-m + 30 M_0} Z^2, \quad q > (3/2) -, \quad (1.28b)
\end{align*}
\]

Proof. \((1.26a)\) is one of the bootstrap bounds established on \( h \) as part of the proof of Theorem 1.1 see (1.50) in [19]. \((1.26b)\) is a consequence of the second a priori assumption in (1.50) of [19] and a direct estimation of the \( H^N \) norm of \( g \), see for example (1.57) in [19]. \((1.26c)\) is a consequence of dispersive estimates for the free Klein-Gordon semi-group and our bootstrap estimates, see Lemma 4.5 in [19]. \((1.26d)\) is a consequence of Bernstein’s and Hardy’s inequality, the bounds on \( h \) and the \( L^1_t \) bound on \( g \), see [1.27a].

\((1.27)\) and \((1.28a)\) are directly borrowed from Lemmas 4.7 and 4.6 in [19]. Only \((1.28b)\) requires a proof. It follows from the identity

\[
\partial_t h(s) = e^{-isL}(L^{-1} \Im w)^2 + e^{-isL}O(s)\theta + e^{-isL}(a(s)\phi \cdot L^{-1} \Im w),
\]

and we can conclude using \((1.26c)\) and interpolation with \((1.26b)\).
2. Preliminaries

We start by recording a consequence of (1.25) which improves (1.27a):

**Lemma 2.1.** We have \( \| \tilde{g}(t) \|_{L^\infty} \lesssim 1. \)

*Proof.* It suffices to show that the time integral in (1.25) is \( O(1) \). We first assume \( |\langle \xi \rangle - 2\lambda|^{-1} \leq t \) and split the time interval into an integral from 0 to \( |\langle \xi \rangle - 2\lambda|^{-1} \) and its complement. For the first piece, recalling the definition of \( y(s) = Y_0(1 + (\Gamma / \lambda)sY_0^2)^{-1/2} \), we write

\[
\left| \int_{0}^{t} \exp \left( is(\langle \xi \rangle - 2\lambda) + \frac{2c_2 \lambda}{\Gamma} \log \left( 1 + \frac{\Gamma Y_0^2 s}{\lambda} \right) \right) y^2(s) \, ds \right| = \frac{1}{2c_2} \exp \left( is(\langle \xi \rangle - 2\lambda) \right) \partial_s \exp \left( \frac{2c_2 \lambda}{\Gamma} \log \left( 1 + \frac{\Gamma Y_0^2 s}{\lambda} \right) \right) \left| \int_{0}^{t} y^2(s) \, ds \right| \lesssim 1,
\]

where the last inequality can be seen integrating by parts in \( s \). For the other piece we instead write

\[
\left| \int_{|\langle \xi \rangle - 2\lambda|^{-1}}^{t} \exp \left( is(\langle \xi \rangle - 2\lambda) + \frac{2c_2 \lambda}{\Gamma} \log \left( 1 + \frac{\Gamma Y_0^2 s}{\lambda} \right) \right) y^2(s) \, ds \right| = |\langle \xi \rangle - 2\lambda|^{-1} \left| \exp \left( is(\langle \xi \rangle - 2\lambda) \right) \right| \left( \partial_s \exp \left( \frac{2c_2 \lambda}{\Gamma} \log \left( 1 + \frac{\Gamma Y_0^2 s}{\lambda} \right) \right) \right) \left| \int_{|\langle \xi \rangle - 2\lambda|^{-1}}^{t} y^2(s) \, ds \right|.
\]

Integrating by parts gives the boundary term

\[
|\langle \xi \rangle - 2\lambda|^{-1} \left| \exp \left( is(\langle \xi \rangle - 2\lambda) \right) \right| \left( \partial_s \exp \left( \frac{2c_2 \lambda}{\Gamma} \log \left( 1 + \frac{\Gamma Y_0^2 s}{\lambda} \right) \right) \right) \left| \int_{|\langle \xi \rangle - 2\lambda|^{-1}}^{t} y^2(s) \, ds \right| \bigg|_{s=|\langle \xi \rangle - 2\lambda|^{-1}}^{s=t} \lesssim 1,
\]

and a bulk term which is also \( O(1) \) since \( |(d/ds)y^2(s)| \lesssim y^4(s) \). Finally, if \( |\langle \xi \rangle - 2\lambda|^{-1} \geq t \) one can estimate as we did for the first integral above.

\( \square \)

The control of the \( L^\infty_\xi \) norm and the proof of the main scattering statement for \( \tilde{h} \), see (1.24), is based on a decomposition obtained from Duhamel’s formula. This is done in two steps below, Lemma \ref{Lemma2.2} and Lemma \ref{Lemma2.6} (see also \ref{Lemma2.7}).

We introduce a partition of unity of the interval \([0, t]\) to decompose our time integrals. Let \( \tau_0, \tau_1, \cdots, \tau_{L+1} : \mathbb{R} \to [0, 1] \) denote cut-off functions, where the integer \( L \) is chosen so that \( |L - \log_2(t + 2)| < 2 \), with the following properties:

\[
\sum_{n=0}^{L+1} \tau_n(s) = 1_{[0, t]}(s), \quad \text{supp}(\tau_0) \subset [0, 2], \quad \text{supp}(\tau_{L+1}) \subset [t/4, t],
\]

\( \text{and} \quad \text{supp}(\tau_n) \subset [2^{n-1}, 2^{n+1}], \quad |\tau'_n(t)| \lesssim 2^{-n}, \quad \text{for} \quad n = 1, \ldots, L. \)

Recall the definitions \ref{Eq1.18} and \ref{Eq1.19}-\ref{Eq1.20}. Lemma 4.1 in \cite{19} gives us the following:

**Lemma 2.2.** We have

\[
\tilde{h} = \tilde{f}_0 + F + \sum_{m=0}^{L+1} (S_m + M_m),
\]

where the ‘source terms’ are (recall the definition of \( \chi_C \) below \ref{Eq1.18})

\[
S_m = S_{1,m} + S_{2,m} + \{ \text{similar and better terms} \},
\]

\( S_{1,m} := \int_{0}^{t} e^{-i\theta(s)-2\lambda}(A^2(s) - B^2(s)) \tilde{\theta}(\xi) \tau_m(s) \, ds, \)

\( S_{2,m} := (1 - \chi_C(\xi)) \int_{0}^{t} e^{-i\theta(s)-2\lambda}B^2(s) \tilde{\theta}(\xi) \tau_m(s) \, ds, \)

\( \text{Scubic} \)

\( \text{Si-chi} \)
the ‘mixed terms’ are
\[
M_m = -\frac{i}{2} M_{1,m} + \{\text{similar and better terms}\},
\]
\[
M_{1,m} := \int_0^t B(s) \int_{\mathbb{R}^3} e^{-is(\xi - m\eta)} \langle \eta \rangle^{-1} \tilde{f}(s, \eta)\nu(\xi, \eta)d\eta \tau_m(s)ds,
\]
\[
\nu(\xi, \eta) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \tilde{\psi}(x, \xi)\psi(x, \eta)\phi(x)dx,
\]
and the ‘continuous/field’ self interactions are
\[
F := -\frac{1}{4} \sum_{\epsilon_1, \epsilon_2 \in \{+,-\}} \epsilon_1 \epsilon_2 F_{\epsilon_1, \epsilon_2,m}(f_{\epsilon_1}, f_{\epsilon_2}), \quad f_+ = f, \quad f_- = \tilde{f},
\]
\[
F_{\epsilon_1, \epsilon_2}(a, b) := \int_0^t \int_{\mathbb{R}^6} e^{-is\Phi_{\epsilon_1, \epsilon_2}(\xi, \eta, \sigma)} \tilde{a}(s, \eta)\tilde{b}(s, \sigma)\mu(\xi, \eta, \sigma)d\eta d\sigma ds,
\]
\[
\mu(\xi, \eta, \sigma) := \frac{1}{(2\pi)^{9/2}} \int_{\mathbb{R}^3} \tilde{\psi}(x, \xi)\psi(x, \eta)\psi(x, \sigma)dx,
\]
where the phases are defined by
\[
\Phi_{\epsilon_1, \epsilon_2}(\xi, \eta, \sigma) = \langle \xi \rangle - \epsilon_1 \langle \eta \rangle - \epsilon_2 \langle \sigma \rangle.
\]
We split the \(F\) terms from the above lemma into low and high frequencies by letting
\[
F := F_L + F_H, \quad F_L(t, \xi) := \int_0^t \varphi_{\leq M_0(s)}(\xi) e^{-is\xi} F \left( \frac{P_{\leq M_0(s)} \mathcal{M}}{L} \right)^2 ds,
\]
\[
M_0(s) := \delta_N \log \langle s \rangle, \quad \delta_N := 5/N.
\]
We will also let \(F_{H,m}\) denote the term \(F_H\) where the integral in time is localized to \(s \approx 2^m\) using the cutoffs \(\tau_m\) (as in \([2.4]\) for example).

We need to further decompose \(F_L\) using some more refined information on the measure \(\mu\) contained in Proposition 5.6 of \([19]\), which we reproduce below.

**Lemma 2.3.** Let \(M_0\) be a fixed parameter, and let \(N_2\) be a sufficiently large integer. We can decompose \(\mu\) into a singular part (apex \(S\)), a regular part (apex \(R\)) and a remainder part (index \(\text{Re}\))
\[
\mu = \mu^S + \mu^R + \mu^\text{Re}.
\]
These three components satisfy the following structural properties (up to irrelevant constants):
\[
\mu^S(\xi, \eta, \sigma) := \delta_0(\xi - \eta - \sigma) + \mu_1^S(\xi, \eta, \sigma) + \mu_2^S(\xi, \eta, \sigma) + \mu_3^S(\xi, \eta, \sigma),
\]
\[
\mu^R(\xi, \eta, \sigma) := \mu_1^R(\xi, \eta, \sigma),
\]
where the right-hand sides are defined as follows:
- \(\mu_1^S\) and \(\mu_1^R\) are given by the formulas
\[
\mu_1^S(\xi, \eta, \sigma) := \nu_1^S(-\xi + \eta, \sigma) + \nu_1^S(-\xi + \sigma, \eta) + \nu_1^S(-\eta - \sigma, \xi), \quad \nu_2^S(\xi, \eta, \sigma), \quad \sigma \in \{S, R\}
\]
with
\[
\nu_1^S(p, q) := \varphi_{\leq -M_0 - 5}(|p| - |q|) \left[ \nu_0(p, q) + \frac{1}{|p|} \sum_{a=1}^{N_2} \sum_{J \in \mathbb{Z}} b_{a,J}(p, q) \cdot 2^J \tilde{K}_a(2^J(|p| - |q|)) \right]
\]
\[\text{\textsuperscript{2}}\text{The parameter } N_2 \text{ can be fixed large enough as a fraction of } N.\]
where $K_\alpha$ are Schwartz functions, $b_{\alpha,j}$ are symbols satisfying the bounds
\begin{equation}
\sum_{j \in \mathbb{Z}} \left| \varphi_P(p) \varphi_Q(q) \nabla^{\alpha}_p \nabla^{\beta}_q b_{\alpha,j}(p,q) \right| \lesssim 2^{-|\alpha|} P \left( 2^{|\alpha|} Q + 2^{(1 - |\beta|)Q} \right) \cdot 1_{\{|P-Q|<2}\},
\end{equation}
for all $P, Q \leq M_0$, $|\alpha| + |\beta| \leq N_2$, and \footnote{We are using the notation $b_0(p/|p|,q)$ here which slightly differs from the one in Proposition 5.1 of \cite{22} (or Proposition 5.6 in \cite{19}); in this latter the symbol appearing in the analogous formulas (5.11)-(5.12) was denoted $b_0(p,q)$ instead. The symbol in (5.13) actually denotes (up to a constant) the symbol $g(-p/|p|,q)$ which appears in Lemma 5.2 of \cite{22}. The relation between the two symbols in \cite{22} is (up to a constant) $b_0(p,q) = g(-p/|p|,q)$, consistently with \cite{21} \cite{19}.}
\begin{equation}
\nu_0(p,q) := \frac{b_0(p/|p|,q)}{|p|} \left[ i \pi \delta(|p| - |q|) + \text{p.v.} \frac{1}{|p| - |q|} \right]
\end{equation}
with
\begin{equation}
\left| \varphi_P(p) \varphi_Q(q) \nabla^{\beta}_q b_0(p/|p|,q) \right| \lesssim 2^{(1 - |\beta|)Q} \cdot 1_{\{|P-Q|<5\}}, \quad Q := \min(Q,0), \quad 1 \leq |\beta| \leq N_2.
\end{equation}
• $\mu^S_2$ is given by the formulas
\begin{equation}
\mu^S_2(\xi, \eta, \sigma) := \nu^S_2(\xi, \eta, \sigma) + \nu^{S,2}_2(\xi, \eta, \sigma) + \nu^{S,2}_2(\xi, \eta, \sigma),
\end{equation}
with
\begin{equation}
\nu^{S,1}_2(\xi, \eta, \sigma) := \frac{1}{|\eta|} \sum_{\epsilon \in \{1,-1\}} \sum_{i=1}^{N_2} \sum_{j \in \mathbb{Z}} \varphi_{\xi-M_0-5}(|\xi| - |\eta| - |\sigma|)
\end{equation}
\begin{equation}
\times b_{i,j}(\xi, \eta, \sigma) \cdot K_i(2^j(|\xi| - |\eta| - |\sigma|)),
\end{equation}
and
\begin{equation}
\nu^{S,2}_2(\xi, \eta, \sigma) = \frac{1}{|\eta|} \sum_{\epsilon \in \{1,-1\}} \sum_{i=1}^{N_2} \sum_{j \in \mathbb{Z}} \varphi_{\xi-M_0-5}(|\xi| + \epsilon|\eta| - |\sigma|)
\end{equation}
\begin{equation}
\times b_{i,j}(\xi, \eta, \sigma) \cdot K_i(2^j(|\xi| + \epsilon|\eta| - |\sigma|)),
\end{equation}
where $K_i$ are Schwartz functions and $b_{i,j}$ and $b^{\pm}_{i,j}$ are symbols satisfying the bounds
\begin{equation}
\left| \varphi_k(\xi) \varphi_k(\eta) \varphi_k(\sigma) \nabla^a_\xi \nabla^b_\eta \nabla^c_\sigma b_{i,j}(\xi, \eta, \sigma) \right|
\lesssim 2^{-|\alpha|} \left( 2^{|\alpha|} \max(k_1,k_2) + 2^{(1 - |\beta|)k_1} 2^{(1 - |\beta|)k_2} \right) 1_{\{|k-\max(k_1,k_2)|<5\}},
\end{equation}
and
\begin{equation}
\left| \varphi_k(\xi) \varphi_k(\eta) \varphi_k(\sigma) \nabla^a_\xi \nabla^b_\eta \nabla^{b_{i,j}}_\sigma(\xi, \eta, \sigma) \right|
\lesssim 2^{-|\alpha|} \left( 2^{|\alpha|} \max(k_1,k_2) + 2^{(1 - |\beta|)k_1} 2^{(1 - |\beta|)k_2} \right) 1_{\{|k-\max(k_1,k_2)-\text{med}(k_1,k_2)|<5\}},
\end{equation}
for all $k, k_1, k_2 \leq M_0$, and $|\alpha| + |\beta| + |\gamma| \leq N_2$, respectively.
• $\mu^S_3$ is given by the formula
\begin{equation}
\mu^S_3(\xi, \eta, \sigma) := \sum_{i=0}^{N_2} \sum_{j \in \mathbb{Z}} \varphi_{\xi-M_0-5}(|\xi| - |\eta| - |\sigma|)
\end{equation}
\begin{equation}
\times b_{i,j}(\xi, \eta, \sigma) \cdot K_i(2^j(|\xi| - |\eta| - |\sigma|)),
\end{equation}
where $K_i$ are Schwartz functions and $b_{i,j}$ are symbols satisfying the bounds stated in \eqref{2.12}.
Lemma 2.4. Let $M_0$ and $N_2$ be as in Lemma 2.3. For the measure $\nu(\xi, \eta)$ we have
\[
|\varphi_k(\xi)\varphi_{k_1}(\eta)\nabla^a_\xi \nabla^b_\eta \nu(\xi, \eta)| \lesssim 1, \quad |a|, |b| \geq 1. \tag{2.21}
\]
For $\mu^{\text{Re}}$ we have, for all $|a| + |a| + |b| \leq N_2$,
\[
|\varphi_k(\xi)\varphi_{k_1}(\eta)\varphi_{k_2}(\sigma)\nabla^a_\xi \nabla^b_\eta \nabla^c_\sigma \mu^{\text{Re}}(\xi, \eta, \sigma)| \lesssim 2^{-(|a|+|b|) \max \{k_1, k_2, k_3\}} \cdot 2^{-|a| \min \{k_1, k_2, k_3\}} \cdot 2^{N_0(|a|+|b|+|c|+1)}, \tag{2.22}
\]
where $\alpha_i$ denotes the order of differentiation of the variable that is located at the smallest frequency, $\alpha_e$ is similarly defined for the middle frequency and $\alpha_a$ for the largest one.

Finally for $\nu^1_\text{R}$, we have for all $|a| + |b| \leq N_2$,
\[
|\varphi_{k_1}(\eta)\varphi_{k_2}(\sigma)\nabla^a_\eta \nabla^b_\sigma \nu^1_\text{R}(\eta, \sigma)| \lesssim \begin{cases} 
2^{-2k_1} 2^{-(|a|+|b|)k_2} \cdot 2^{2(|a|+|b|)N_0} & \text{if } |k_1 - k_2| < 5 \\
2^{-2(k_1 \lor k_2)} \cdot 2^{-|a|(k_1 \lor k_2)} \max \{1, 2^{1-|b|k_2}\} \cdot 2^{(|a|+|b|+2)N_0} & \text{if } |k_1 - k_2| \geq 5.
\end{cases} \tag{2.23}
\]

We now adopt bilinear operators associated with the measures above.

Definition 2.5. For a general symbol $b = b(\xi, \eta, \sigma)$ let us denote
\[
T_0[b](G, H)(x) := \tilde{T}_1^{-1} \int_{\mathbb{R}^x \times \mathbb{R}^3} G(\eta)H(\sigma) b(\xi, \eta, \sigma) \delta_0(\xi - \eta - \sigma) \, d\eta d\sigma, \tag{2.24a}
\]
\[
T_{1,1}^1[b](G, H)(x) := \tilde{T}_1^{-1} \int_{\mathbb{R}^x \times \mathbb{R}^3} G(\xi - \eta)H(\sigma) b(\xi, \eta, \sigma) \nu^1_{\text{R}}(\eta, \sigma) \, d\eta d\sigma, \quad * \in \{S, R\}, \tag{2.24b}
\]
\[
T_{1,2}^1[b](G, H)(x) := \tilde{T}_1^{-1} \int_{\mathbb{R}^x \times \mathbb{R}^3} G(\xi - \eta - \sigma)H(\sigma) b(\xi, \eta, \sigma) \nu^1_{\text{R}}(\eta, \sigma) \, d\eta d\sigma, \quad * \in \{S, R\}, \tag{2.24c}
\]
\[
T_2^3[b](G, H)(x) := \tilde{T}_1^{-1} \int_{\mathbb{R}^x \times \mathbb{R}^3} G(\eta)H(\sigma) b(\xi, \eta, \sigma) \mu^3(\xi, \eta, \sigma) \, d\eta d\sigma, \tag{2.24d}
\]
\[
T_3^3[b](G, H)(x) := \tilde{T}_1^{-1} \int_{\mathbb{R}^x \times \mathbb{R}^3} G(\eta)H(\sigma) b(\xi, \eta, \sigma) \mu^3(\xi, \eta, \sigma) \, d\eta d\sigma, \tag{2.24e}
\]
\[
T^{\text{Re}}[b](G, H)(x) := \tilde{T}_1^{-1} \int_{\mathbb{R}^x \times \mathbb{R}^3} G(\eta)H(\sigma) b(\xi, \eta, \sigma) \mu^{\text{Re}}(\xi, \eta, \sigma) \, d\eta d\sigma. \tag{2.24f}
\]

We also adopt the following notation: for a general symbol $b$ we let (omitting the dependence on $k_1, k_2 \in \mathbb{Z}$)
\[
b(\xi, \eta, \sigma) := b(\xi, \eta, \sigma)\varphi_{k_1}(\eta)\varphi_{k_2}(\sigma), \quad b_1(\xi, \eta, \sigma) := b(\xi, \xi - \eta, \sigma)\varphi_{k_1}(\eta)\varphi_{k_2}(\sigma), \quad b'_1(\xi, \eta, \sigma) := b(\xi, \sigma - \eta, \sigma)\varphi_{k_1}(\eta)\varphi_{k_2}(\sigma), \quad b_2(\xi, \eta, \sigma) := b(\xi, -\eta - \sigma, \sigma)\varphi_{k_1}(\eta)\varphi_{k_2}(\sigma). \tag{2.25}
\]

With the building blocks (2.21) and (2.23) we can further decompose $F_L$ into ‘singular’ and ‘regular’ parts. The singular parts (denoted with an apex $S$) are defined so that the phases (2.6) are lower bounded on their support, and we can apply a normal form transformation; this will give rise to boundary terms that are quadratic (see (2.24) below) and cubic bulk terms (see (2.28) below). The regular parts (see (2.30) and (2.32) below) cannot be transformed but will satisfy better bilinear estimates. This singular vs. regular decomposition is the content of the next lemma, and we refer the reader to §5.4 of [17] for more details.

Lemma 2.6. We have (up to irrelevant constants)
\[
F_L(t) = \sum_{\epsilon_1, \epsilon_2 \in \{+, -\}} F_{\epsilon_1, \epsilon_2}^{S(1)}(t) - F_{\epsilon_1, \epsilon_2}^{S(1)}(0) + \sum_{m=0}^{L+1} F_{\epsilon_1, \epsilon_2, m}^{S(2)}(t) + F_{\epsilon_1, \epsilon_2, m}^{R}(t) + F_{\epsilon_1, \epsilon_2, m}^{\text{Re}}(t) \tag{2.26}
\]
Finally (2.27) with, using again the notation since these should create no confusion (and they will often play no role).

\[
F_{\epsilon,\theta,m}^{(1)}(t) = e^{-it\langle \xi \rangle} \sum_{k_1,k_2 \leq M_0(t) \cap G,H \in \{g,h\}} \sum_{i=2,3} \tilde{\mathcal{F}}_{x-\xi} T_0[b\{e^{it\langle \xi \rangle - i\epsilon \theta} G_{\epsilon i}, e^{it\langle \xi \rangle - i\epsilon \theta} H_{\epsilon 2}\}],
\]

and (disregarding analogous symmetric terms)

\[
F_{\epsilon,\theta,m}^{(2)}(t) = \int_0^t e^{-is\langle \xi \rangle} \sum_{k_1,k_2 \leq M_0(s) \cap G,H \in \{g,h\}} \sum_{i=2,3} \tilde{\mathcal{F}}_{x-\xi} T_0[b\{e^{is\langle \xi \rangle - i\epsilon \theta} G_{\epsilon i}, e^{is\langle \xi \rangle - i\epsilon \theta} H_{\epsilon 2}\}] \tau_m(s)ds,
\]

where, using the notation (2.25), we define the symbols through

\[
b(\xi,\eta,\sigma) = \frac{\varphi_{\leq M_0(\langle \xi \rangle)}}{\langle \eta \rangle \langle \sigma \rangle \Phi_{\epsilon_1,\theta}^0(\xi,\eta,\sigma)}.
\]

Moreover

\[
F_{\epsilon,\theta,m}^R := F_{\epsilon,\theta,m}^{(1)} + F_{\epsilon,\theta,m}^{(2)}
\]

\[
F_{\epsilon,\theta,m}^{R,(1)}(t) := \int_0^t e^{-is\langle \xi \rangle} \sum_{k_1,k_2 \leq M_0(s) \cap G,H \in \{g,h\}} \sum_{i=2,3} \tilde{\mathcal{F}}_{x-\xi} T_0^R[m_1\{e^{is\langle \xi \rangle - i\epsilon \theta} G_{\epsilon i}, e^{is\langle \xi \rangle - i\epsilon \theta} H_{\epsilon 2}\}] \tau_m(s)ds,
\]

\[
F_{\epsilon,\theta,m}^{R,(2)}(t) := \int_0^t e^{-is\langle \xi \rangle} \sum_{k_1,k_2 \leq M_0(s) \cap G,H \in \{g,h\}} \sum_{i=2,3} \tilde{\mathcal{F}}_{x-\xi} T_0^R[m_2\{e^{is\langle \xi \rangle - i\epsilon \theta} G_{\epsilon i}, e^{is\langle \xi \rangle - i\epsilon \theta} H_{\epsilon 2}\}] \tau_m(s)ds,
\]

with, using again the notation (2.25),

\[
m(\xi,\eta,\sigma) := \varphi_{\leq M_0(\langle \xi \rangle)}\langle \eta \rangle^{-1}\langle \sigma \rangle^{-1}.
\]

Finally

\[
F_{\epsilon,\theta,m}^{Re}(t) := \int_0^t \int_{\mathbb{R}^6} e^{-is\Phi_{\epsilon_1,\theta}^0(\xi,\eta,\sigma)} \tilde{f}_{\epsilon_1}(s,\eta) \tilde{f}_{\epsilon_2}(s,\sigma) \mu_{Re}(\xi,\eta,\sigma) d\eta d\sigma \tau_m(s)ds.
\]

In what follows, we will often omit the indexes \(\epsilon_1,\epsilon_2\) in the notation for the bilinear terms above, since these should create no confusion (and they will often play no role).

Next we recall Hölder type bounds satisfied by the operators in (2.24):
Let \( p, q, \in [1, \infty) \), and \( r \geq 1 \) with
\[
\frac{1}{p} + \frac{1}{q} = \frac{1}{r},
\]
and assume there is \( 10A \leq D \leq 2^{4/10} \) such that
\[
\mathcal{D}(G, H) := \|G\|_{L^2} \|H\|_{L^2} + \min (\|\partial_{\xi} G\|_{L^2} \|H\|_{L^2}, \|G\|_{L^2} \|\partial_{\xi} H\|_{L^2}) \leq 2^D. \tag{2.34} \]
Then, there exists an absolute constant \( C_0 \) (for example, \( C_0 := 65 \) is suitable), such that the following bilinear bounds hold for the operators defined in (2.24):
\[
\|P_K T^{S,i}_1 b(G, H)\|_{L^r} \lesssim \|\hat{G}\|_{L^p} \|\hat{H}\|_{L^q} \cdot 2^{C_0 M_0} + 2^{-D} \mathcal{D}(G, H), \quad i = 1, 2,
\]
\[
\|P_K T^{S,i}_2 b(G, H)\|_{L^r} \lesssim \|\hat{G}\|_{L^p} \|\hat{H}\|_{L^q} \cdot 2^{C_0 M_0}, \quad i = 2, 3,
\]
\[
\|P_K T^{R,i}_1 b(G, H)\|_{L^r} \lesssim \|\hat{G}\|_{L^p} \|\hat{H}\|_{L^q} \cdot 2^{C_0 A}, \quad i = 1, 2. \tag{2.35c}
\]

Remark 2.8 (Bilinear estimates). Note that the bilinear estimates in (2.35) differ from those of [19] since we allow for the endpoint \( r = 1 \). Note that it is possible at the expense of adding a projection \( P_K \) in front, see (6.35) in [22]. This is harmless since our estimates are done in \( L^\infty \).

Remark 2.9 (Estimating \( D \)). In the bilinear bound (2.35a) the quantity \( D \) appears. This is a lower order remainder term in all our estimates, and is already adequately estimated in [19]; therefore we can disregard these terms in our proof.

3. Bilinear estimates

This section contains the new bilinear estimates we need for the proof of Proposition 12.

Lemma 3.1. Let \( m = m(\xi, \eta), \xi, \eta \in \mathbb{R}^3 \) be a bounded multiplier localized at frequencies \( |\xi| \approx 2^K, |\eta| \approx 2^L \) such that
\[
|\nabla_{\xi}^{\alpha} \nabla_{\eta}^{\beta} m(\xi, \eta)| \lesssim 2^{-|\alpha| K - |\beta| L}, \quad 0 \leq |\alpha| + |\beta| \leq 4, \tag{3.1}
\]
and let \( b_0 = b_0(\omega, \xi, \omega \in \mathbb{S}^2, \xi \in \mathbb{R}^3, b \) be such that
\[
|\nabla_{\xi}^{\beta} b_0(\omega, \xi)| \lesssim 1 + (|\xi|/|\xi|)^{1-|\beta|}, \quad 0 \leq |\beta| \leq 4. \tag{3.2}
\]
We have (note the frequency localization on the profile \( h \))
\[
\left\| \int_{\mathbb{R}^6} \tilde{f}(\xi - \eta)(\varphi_M \tilde{h})(\sigma) m(\xi, \eta) b_0(\eta/|\eta|, \sigma) p.v. \frac{1}{|\eta| - |\xi|} d\eta d\sigma \right\|_{L^\infty_{\xi}} \lesssim 2^{L+M} (2^M)^{-2} \|m\|_{L^\infty_{\xi, \sigma}} \|f\|_{L^2_{\xi}} \|h\|_{H^2}, \tag{3.3}
\]
and
\[
\left\| \int_{\mathbb{R}^6} \tilde{f}(\eta + \sigma) \tilde{h}(\sigma) m(\eta, \sigma) p.v. \frac{1}{|\eta| - |\xi|} b_0(\eta/|\eta|, \eta) d\eta d\sigma \right\|_{L^\infty_{\xi}} \lesssim 2^{2K} (2^K)^{-2} \|f\|_{H^2} \|h\|_{H^2}. \tag{3.4}
\]

Proof. Let \( \epsilon \in (0, 1) \). We stress that the implicit constants in the argument below do not depend on \( \epsilon \). Denote \( \varphi_\epsilon \) a radial cut-off function supported on the region \( \epsilon < r < \epsilon^{-1} \).

We first prove the more challenging inequality (3.4). We start by inserting a cut-off \( \varphi_k(\xi) \) and pass to polar coordinates \( \xi = \rho \phi \) and \( \eta = \rho \omega, \phi, \omega \in \mathbb{S}^2 \), and estimate the left-hand side of (3.4) by
\[
\sup_{\epsilon > 0} \sup_{k \in \mathbb{Z}} \sup_{\rho > 0} \left| \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \int_0^\infty \int_0^\infty \tilde{f}(r(\omega + \sigma) m(\omega, \sigma) \tilde{h}(\sigma) d\sigma \varphi_k(\rho - \rho) \varphi_k(\rho) b_0(\omega, \rho \phi) r^2 d\rho d\omega \right|. \tag{3.5}
\]
Let us define \( f_1 := L^2 f \) and \( h_1 := L^2 h \), and
\[
F(r, \omega) := \int_{\mathbb{R}^3} \tilde{f}_1(r(\omega + \sigma) m_1(\omega, \sigma) \tilde{h}_1(\sigma) d\sigma, \quad m_1(\eta, \sigma) := \frac{m(\eta, \sigma)}{(\eta + \sigma)^2 (\sigma)^2}.
\]
We then take an inverse Fourier transform in $\rho$ in (3.5); using the fact that the Fourier transform of the Hilbert transform kernel is bounded, we can bound (3.5) by a constant times

$$\sup_{\epsilon > 0} \sup_{k \in \mathbb{Z}} \sup_{\phi \in \mathbb{S}^2} \left\| \int_{\mathbb{R}^2} \mathcal{F}^{-1}_{\rho \rightarrow x} \left\{ \int_{\mathbb{R}} r^2 F(r, \omega) 1_{[0,\infty)}(r) \frac{\varphi_k(r-\rho)}{r-\rho} dr \right\} \varphi_k(\rho) b_0(\omega, \rho \phi) \right\|_{L^2_r(\mathbb{R})} \lesssim \sup_{\epsilon > 0} \sup_{k \in \mathbb{Z}} \sup_{\omega \in \mathbb{S}^2} \left\| \mathcal{F}^{-1}_{\rho \rightarrow x} \left( \frac{\varphi_k(\cdot r)}{r} \ast (r^2 F(r, \omega) 1_{[0,\infty)}(r)) \right) \right\|_{L^1_r(\mathbb{R})} \lesssim \left\| \mathcal{F}^{-1}_{\rho \rightarrow x} \left( \varphi_k(\cdot r) b_0(\omega, \cdot \phi) \right) \right\|_{L^1_r(\mathbb{R})},$$

(3.6) pr2

To bound the second term in (3.6) we use the pointwise bound

$$\left| \mathcal{F}^{-1}_{\rho \rightarrow x}(\varphi_k(\cdot) b_0(\omega, \cdot \phi))(x) \right| \lesssim \frac{2^k}{(1 + |x|)(1 + 2^k/\langle 2^k \rangle |x|)^2},$$

which follows by integration by parts in $\rho$ and our assumptions (3.2) on $b_0$. This yields

$$\left\| \mathcal{F}^{-1}_{\rho \rightarrow x}(\varphi_k(\cdot) b_0(\omega, \cdot \phi)) \right\|_{L^2_{\omega, \phi} L^1_x} \lesssim \langle 2^k \rangle.$$

We then focus on the first term. We denote the wave operator by $\mathcal{W} := \mathcal{F}^{-1} \mathcal{F}$, $\mathcal{W}^* := \mathcal{F}^{-1} \mathcal{F}$, and recall that it is bounded on Sobolev spaces under our assumptions. We change coordinates by denoting $R_\omega$ the rotation that sends $\omega$ to $e_1 := (1, 0, 0)$ and denoting $g^\omega := g \circ R_\omega^{-1}$, so that we find (omitting irrelevant constants)

$$\mathcal{F}^{-1}_{r \rightarrow x}(r^2 F(r, \omega) 1_{[0,\infty)}(r)) = \int_{\mathbb{R}} e^{ixr} \int_{\mathbb{R}^3} \tilde{f}_1(\sigma + r \omega) m_1(r \omega, \sigma) \tilde{h}_1(\sigma) d\sigma r^2 1_{[0,\infty)}(r) dr$$

$$= \int_{\mathbb{R}^6} \int_{\mathbb{R}^4} e^{iyr} e^{-i\sigma \cdot (y+z)} m_1(r \omega, \sigma) r^2 1_{[0,\infty)}(r) dr d\sigma \mathcal{W} f_1(y) \mathcal{W} h_1(z) dydz$$

$$:= \int_{\mathbb{R}^6} T[m_1](x - y, \mathcal{W}_1^{-1} y + \mathcal{W}_1^{-1} z)(\mathcal{W}^* f_1)^\omega(y) (\mathcal{W}^* h_1)^\omega(z) dydz.$$

Note that the last identity defines $T[m_1]$. We then observe that, given the localization of $m_1(\eta, \sigma)$ to $|\eta| \approx 2^K$ and $|\sigma| \approx 2^L$, using integration by parts and (3.1) we have

$$|T[m_1](y_1, z)| = \left| \int_{\mathbb{R}^4} e^{iy_1 r} e^{-i\sigma \cdot z} m_1(r \omega, \sigma) r^2 1_{[0,\infty)}(r) dr d\sigma \right| \lesssim \frac{2^{3K} (2^K)^{-2}}{(1 + 2^K |y_1|)^2} \frac{2^{3L} (2^L)^{-2}}{(1 + 2^L |z|)^2}.$$  

This yields the bound

$$\|T[m_1]\|_{L^1(\mathbb{R} \times \mathbb{R}^3)} \lesssim 2^{2K} (2^K)^{-2}.$$

Changing variables $z \mapsto y - z, y \mapsto (x - y_1, y_2, y_3)$ and using the Cauchy-Schwarz inequality as well as the boundedness of wave operators on $L^2$, we have

$$\left| \mathcal{F}^{-1}_{r \rightarrow x}(r^2 F(r, \omega) 1_{[0,\infty)}(r)) \right| \lesssim \int_{\mathbb{R}^4} |T[m_1](y_1, \mathcal{W}_1^{-1} z)| dy_1 dz$$

$$\times \sup_{(y_1, z) \in \mathbb{R}^4} \int_{\mathbb{R}^3} |(\mathcal{W}^* f_1)^\omega(x - y_1, y_2, y_3) \cdot (\mathcal{W}^* h_1)^\omega(x - y_1 - z, y_2 - z_2, y_3 - z_3)| dx dy_2 dy_3$$

$$\lesssim 2^{2K} (2^K)^{-2} \| f_1 \|_{L^2} \| h_1 \|_{L^2} \lesssim \| f \|_{H^2} \| h \|_{H^2}.$$

We now prove the easier inequality (3.3). Passing to polar coordinates $\eta = r \omega$ and $\sigma = \rho \phi$, $\omega, \phi \in \mathbb{S}^2$, we estimate using the boundedness of the Hilbert transform on $L^2$, and recalling that on the support
of the integral $|\xi| \approx 2^K$, $|\eta| \approx 2^L$ and $|\sigma| \approx 2^M$:

$$\left| \int_0^\infty \int_{S^2} \tilde{f}(\xi - r\omega)m(\xi, r\omega) \int_0^\infty \int_{S^2} b_0(\omega, \rho \phi) \frac{\varphi_0(r - \rho)}{r - \rho} h(\rho \phi) \varphi_M(\rho) \rho^2 d\rho d\phi r^2 dr d\omega \right|$$

$$\lesssim \int_{S^2} \|\tilde{f}(\xi - r\omega)m(\xi, r\omega)\|_{L^\infty_t(\nu^2 d\nu)} d\omega \cdot \sup_{\omega \in S^2} \left\| \frac{\varphi_0(r)}{r} \ast \left( \int_{S^2} b_0(\omega, r \phi) \tilde{h}(r \phi) \varphi_M(r) r^2 d\phi \right) \right\|_{L^2_t}$$

$$\lesssim \|m\|_{L^\infty_{\xi, n}} \cdot 2^L \|\varphi_L(\cdot) \tilde{f}(\xi - \cdot)\|_{L^2} \cdot 2^M \|\varphi_M \tilde{h}\|_{L^2(S^3)}.$$

$$\square$$

4. Asymptotic behavior of $h$

In this section we begin the verification of the hypotheses in Lemma 1.5. We start by reducing these to proving a set of bounds for the bilinear and trilinear terms appearing in Lemma 2.6.

**Lemma 4.1.** Recall the decomposition for $h$ in (2.24) - (2.27) (see also the notation for $F_{H,m}$ below (2.27)), and for $F_L$ in (2.26) - (2.32). Let (see (2.27))

$$\mathcal{N}(\tilde{f})(t, \xi) := F^{S, (1)}(t, \xi).$$

Assume a priori that, for all $t \geq 0$,

$$\|\tilde{h}(t)\|_{L^\infty_{\xi}} \leq \varepsilon^\delta,$$  

(4.2) bootLinfty

and that the following bounds hold for all $m = 0, 1, \ldots, L + 1$ (recall (2.1)):

$$\|S_m\|_{L^\infty_{\xi}}, \|M_m\|_{L^\infty_{\xi}}, \|F_{H,m}\|_{L^\infty_{\xi}}, \|F^{S, (2)}_m\|_{L^\infty_{\xi}}, \|F^R_m\|_{L^\infty_{\xi}}, \|F^{Re}_m\|_{L^\infty_{\xi}} \lesssim 2^{-\delta m} \varepsilon^\delta.$$  

(4.3) scattering

Then the following hold true:

(i) There exists $h_\infty \in \varepsilon^\delta L^\infty_{\xi}$ such that (1.24) holds true.

(ii) We have

$$\|\mathcal{N}(\tilde{f})(t)\|_{L^\infty_{\xi} L^\infty_{\xi}} \leq \varepsilon^\delta.$$  

(4.4) corrh

(iii) The a priori assumption (4.2) can be improved to

$$\|\tilde{h}(t)\|_{L^\infty_{\xi}} \leq \frac{1}{2} \varepsilon^\delta.$$  

(4.5) bootLinfty

The main goal of this section is to prove the estimates (4.3); these proofs rely on the machinery developed in [19]. We will also use the following:

**Remark 4.2.** Note that obtaining a bound by $2^{-\delta m}$ for all the quantities on the left-hand side of (4.3) is sufficient since, as in [19], we may assume that we are working past the local time of existence of $O(1/\varepsilon)$.

Below we show how (i) and (iii) in Lemma 4.1 follow from the assumptions (4.2) and (4.3), using (ii) as well. The bound (4.1) in (ii) is proved in Section 5 relying on the new bilinear bounds from Section 3 and the a priori assumption (4.2).

**Proof of (i) and (iii) in Lemma 4.1.** From the formulas (2.2), (2.7) (see also the notation for $F_{H,m}$ below that), and (2.26), we see that (4.3) implies, for all $t_1 < t_2$,

$$\|\tilde{h}(t_1) - \mathcal{N}(\tilde{f})(t_1) - (\tilde{h}(t_2) - \mathcal{N}(\tilde{f})(t_2))\|_{L^\infty_{\xi}} \leq \varepsilon^\delta (t_1)^{-2\delta}.$$  

Then $\tilde{h}(t) - \mathcal{N}(\tilde{f})(t)$ is Cauchy in time with values in $L^\infty_{\xi}$ and the first claim follows.

For the bound (4.5) we use again the formula for $h$ given by (2.2) with (2.7) and (2.26), and the bounds on each of the terms appearing in this formula provided by (4.3) (under the assumption that $2^m \geq \varepsilon^{-1}$, see Remark 4.2 and (4.4) (see the definition (4.1)).
To summarize, in order to prove Lemma 4.1 and obtain our main result in Proposition 1.2, it suffices to prove all the bounds in (1.3), and the bound (4.4) under the assumption (4.2).

**Remark 4.3.** We remark that the uniform bound (4.2) for $\overline{h}$ is not proven in [19]. Moreover, as a consequence of (4.2) and (2.1) we also have, for all $t \geq 0$,

$$\|\tilde{f}(t)\|_{L^\infty_t} \lesssim 1. \quad (4.6)$$

**Proof of (4.3).** We estimate each of the terms on the right-hand side of (4.3) by $2^{-3km}$. We will constantly use the bilinear bounds stated in Theorem 2.7 (see also Remark 2.9) without referring to them at each application.

**Step 1:** Bounding $F^{S,(2)}_m$. From (2.28) we have that

$$\|F^{S,(2)}_m\|_{L^\infty_t} \lesssim 2^m \sup_{k \in \mathbb{Z}} \sum_{G,H \in \{g,h\}} \sum_{k_1,k_2 \leq \delta \nu} \sup_{s \in 2^m} \left[ \|\varphi_k \tilde{F}_{x \rightarrow \xi} T_0^{S,(0)} [b_0(e^{is\xi_1(\xi)} \partial G_{\xi_1}, e^{is\xi_2(\xi)} \overline{H}_{\xi_2})]\|_{L^\infty_t} \right.$$  

$$+ \|\varphi_k \tilde{F}_{x \rightarrow \xi} T_1^{S,(1)} [b_1(e^{is\xi_1(\xi)} \partial \overline{G}_{\xi_1}, e^{is\xi_2(\xi)} \overline{H}_{\xi_2})]\|_{L^\infty_t} + \|\varphi_k \tilde{F}_{x \rightarrow \xi} T_1^{S,(1)} [b'_1(e^{is\xi_1(\xi)} \overline{H}_{\xi_2}, e^{is\xi_2(\xi)} \partial G_{\xi_1})]\|_{L^\infty_t} \right.$$  

$$+ \left. \|\varphi_k \tilde{F}_{x \rightarrow \xi} T_1^{S,(2)} [b_2(e^{is\xi_1(\xi)} \partial \overline{G}_{\xi_1}, e^{is\xi_2(\xi)} \overline{H}_{\xi_2})]\|_{L^\infty_t} + \sum_{i=2,3} \|\varphi_k \tilde{F}_{x \rightarrow \xi} T_1^{S,(0)} [b_1(e^{is\xi_1(\xi)} \partial \overline{G}_{\xi_1}, e^{is\xi_2(\xi)} \overline{H}_{\xi_2})]\|_{L^\infty_t} \right].$$

Since the sum over $k_1, k_2$ will only contribute $O(m)$ terms we will often disregard them.

**Case 1:** $G = g$. We estimate using (2.35a), the decay estimates (1.26a) and (1.25a), that

$$\|\varphi_k \tilde{F}_{x \rightarrow \xi} T_1^{S,(1)} [b_1(e^{is\xi_1(\xi)} \partial \overline{G}_{\xi_1}, e^{is\xi_2(\xi)} \overline{H}_{\xi_2})]\|_{L^\infty_t} \lesssim \|P_k T_1^{S,(1)} [b_1(e^{is\xi_1(\xi)} \partial \overline{G}_{\xi_1}, e^{is\xi_2(\xi)} \overline{H}_{\xi_2})]\|_{L^\infty_t} \lesssim \|e^{isL} \partial \overline{g}\|_{L^{5/4}_t} \cdot \|e^{isL} P_k H\|_{L^2_t} \cdot 2^{C_m} \lesssim \rho(2^m) \cdot 2^{-m+10M} Z \cdot 2^{C_m}.$$  

This is an acceptable bound.

**Case 2:** $G = h$. Using (2.35a), (1.26c) and (1.25b), we estimate similarly that

$$\|P_k T_1^{S,(1)} [b_1(e^{is\xi_1(\xi)} \partial \overline{h}_{\xi_1}, e^{is\xi_2(\xi)} \overline{H}_{\xi_2})]\|_{L^\infty_t} \lesssim \|e^{isL} \partial \overline{h}\|_{L^{5/2}_t} \cdot \|e^{isL} P_k H\|_{L^2_t} \cdot 2^{C_m} \lesssim 2^{-m+10M} Z^2 \cdot 2^{-m/2+10M} Z \cdot 2^{C_m},$$  

which yields a sufficient bound.

**Step 2:** Bounding $F^{R}_m, F^{Re}_m$. From (2.22) and (2.23) one can see that $F^{Re}_m$ satisfies better estimates than $F^{R}_m$, and is therefore strictly easier to treat. Therefore we only explain how to deal with $F^{R}_m$. Starting with the first part we write that, using (2.30),

$$\|F^{R,(1)}_m\|_{L^\infty_t} \lesssim 2^m \sum_{G,H \in \{g,h\}} \sum_{k_1,k_2 \leq \delta \nu} \sup_{s \in 2^m} \|\varphi_k \tilde{F}_{x \rightarrow \xi} T_1^{R,(1)} [b_1](e^{is\xi_1(\xi)} \overline{G}_{\xi_1}, e^{is\xi_2(\xi)} \overline{H}_{\xi_2})\|_{L^\infty_t} \lesssim 2^m \cdot 2^{-m+10M} Z \cdot 2^{C_m}.$$  

Recall the definition for the symbols (2.28) with the notation (2.23), and note that these always localize the integration variables $\eta, \sigma$ in the formulas (2.24) to $|\eta| \approx 2^k_1, |\sigma| \approx 2^k_2$, but do not localize the inputs of the operators necessarily. It suffices to estimate the term $T_1^{R,(1)} [b_1]$.

**Case 1:** $k_2 \leq -m/2 + 6m$. Then integrating directly we find that, using (1.26d) and (2.23),

$$2^m \sup_{s \in 2^m} \|\varphi_k \tilde{F}_{x \rightarrow \xi} T_1^{R,(1)} [b_1](e^{is\xi_1(\xi)} \overline{G}_{\xi_1}, e^{is\xi_2(\xi)} \overline{H}_{\xi_2})\|_{L^\infty_t} \lesssim 2^m \|\varphi_{k_1}(-\eta) \overline{G}(s,\eta)\|_{L^\infty_t} \cdot \|\varphi_{k_2} \overline{H}\|_{L^1_\eta} \cdot \sup_{|\eta| \approx 2^k_1, |\sigma| \approx 2^k_2} \nu^R(\eta, \sigma) \lesssim 2^m \cdot 2^{(5/2-k_1)z} Z \cdot 2^{(5/2-k_2)z} Z \cdot 2^{-2(k_1 \vee k_2)2^{10M}}.$$
This gives the desired bound.

**Case 2:** $k_2 \geq -m/2 + \delta m$ and $k_1 \leq -m/10$.

**Case $H = g.$** Integrating by parts in $\sigma$ and using (1.26d) and (2.23) (note that the case where the derivative hits the cut-offs or measure are better and can be disregarded) we obtain the bound

$$
2^m \sup_{s \approx 2^m} \left\| \varphi_k \tilde{F}_x \cdot \xi T_1^{R,1}[b_1](e^{isc_1(\xi)} \tilde{G}_{c_1}(s, \xi), e^{isc_2(\xi)} \tilde{H}_{c_2}) \right\|_{L^\infty_\xi}
$$

$$
\lesssim \sup_{s \approx 2^m} \|\varphi_k (\cdot - \eta) \tilde{G}(s, \eta)\|_{L^1_\eta} \cdot \left( \|\varphi_k \sigma(\cdot)\|_{L^1_{|\sigma|}} \cdot \left( \sup_{|\eta| \approx 2^{k_1}, |\sigma| \approx 2^{k_2}} \left| \nu^R(\eta, \sigma) \right| \right) \right)
$$

$$
\lesssim 2^{(5/2) - k_1} Z \cdot 2^{k_2/2} Z \cdot 2^{-2(k_1 \vee k_2)} 2^{10M_0},
$$

which is sufficient to conclude.

**Case $H = g.$** In this case note first that $k_2 \approx 0$. We let $\ell_0 := [-m + \delta m]$ and insert cut-offs $\varphi_\ell(0) ((\sigma) - 2\lambda)$ defined for $\ell \geq \ell_0$ that are such that $\varphi_\ell(0) = \varphi_\ell(0)$ and $\varphi_\ell(0) = \varphi_\ell$ if $\ell > \ell_0$. In the case where $\ell = \ell_0$ we integrate directly and find that, using (1.27a), (1.26d) and (2.23)

$$
2^m \sup_{s \approx 2^m} \left\| \varphi_k \tilde{F}_x \cdot \xi T_1^{R,1}[b_1](e^{isc_1(\xi)} \tilde{G}_{c_1}(s, \xi) e^{isc_2(\xi)} \varphi_\ell(0) ((\xi) - 2\lambda) \tilde{g}_{c_2}) \right\|_{L^\infty_\xi}
$$

$$
\lesssim 2^m \cdot \sup_{s \approx 2^m} \|\varphi_k (\cdot - \eta) \tilde{G}(s, \eta)\|_{L^1_\eta} \cdot 2^{10M_0} \lesssim 2^m \cdot 2^{(5/2) - k_1} Z \cdot 2^{k_2} (2m)^2 \cdot 2^{10M_0},
$$

which yields an acceptable bound provided $\delta > 0$ is chosen small enough. If $\ell > \ell_0$ we can integrate by parts in $\sigma$ and using (1.27b), (1.26d) and (2.23) we obtain

$$
2^m \sup_{s \approx 2^m} \left\| \varphi_k \tilde{F}_x \cdot \xi T_1^{R,1}[b_1](e^{isc_1(\xi)} \tilde{G}_{c_1}(s, \xi) e^{isc_2(\xi)} \varphi_\ell(0) ((\xi) - 2\lambda) \tilde{g}_{c_2}) \right\|_{L^\infty_\xi}
$$

$$
\lesssim \sum_{\ell > \ell_0} \sup_{s \approx 2^m} \|\varphi_k (\cdot - \eta) \tilde{G}(s, \eta)\|_{L^1_\eta} \cdot \left( \sup_{|\eta| \approx 2^{k_1}, |\sigma| \approx 2^{k_2}} \left| \nu^R(\eta, \sigma) \right| \right)
$$

$$
\lesssim 2^{(5/2) - k_1} Z \cdot 2^{k_2} (2m)^2 \cdot 2^{10M_0},
$$

which is an acceptable bound.

**Case 3:** $k_1 \geq -m/10, k_2 \geq -m/2 + \delta m$. In this case we have from (2.23) that

$$
\sup_{|\eta| \approx 2^{k_1}, |\sigma| \approx 2^{k_2}} \left| \nu^R(\eta, \sigma) \right| \lesssim 2^{m/10 + 10M_0}, \quad (4.8)
$$

Next we insert another cut-off $\varphi_{\ell_3}(\xi - \eta)$ to localize the frequency of the input $G$.

**Subcase 3.1:** $k_3 \leq -m/2 + \delta m$. We can integrate directly as in Case 1 and obtain, using (1.26d) and (4.8),

$$
2^m \sup_{s \approx 2^m} \left\| \varphi_k \tilde{F}_x \cdot \xi T_1^{R,1}[b_1](e^{isc_1(\xi)} \varphi_{\ell_3}(\xi) \tilde{G}_{c_1}(s, \xi) e^{isc_2(\xi)} \varphi_{\ell_3}(\xi) \tilde{H}_{c_2}) \right\|_{L^\infty_\xi}
$$

$$
\lesssim 2^m \cdot 2^{(5/2) - (k_1 \wedge k_3)} Z \cdot 2^{(5/2) - k_2} Z \cdot 2^{m/10 + 10M_0}.
$$

**Subcase 3.2:** $k_3 \geq -m/2 + \delta m$. In this case we integrate by parts in $\eta$ and $\sigma$. Treating first the case where $(G, H) = (h, h)$ we write, using (1.26a) and (4.8),

$$
2^{-m} \sup_{s \approx 2^m} \left\| \varphi_k \tilde{F}_x \cdot \xi T_1^{R,1}[b_1](e^{isc_1(\xi)} \varphi_{\ell_3}(\xi) (\xi) e^{isc_2(\xi)} \varphi_{\ell_3}(\xi) \tilde{H}_{c_2}) \right\|_{L^\infty_\xi}
$$

$$
\lesssim 2^{-m} \cdot 2^{k_2} (2^{(3/2) - (k_3 \wedge k_1)} Z \cdot 2^{k_2} Z \cdot 2^{m/10 + 10M_0}.
$$

The desired bound follows.

Moving on to the $(g, g)$ case, we insert cutoffs $\varphi_\ell((\xi - \eta) - 2\lambda)$ and $\varphi_\ell((\sigma) - 2\lambda)$ for $\ell_0 = [-m + \delta m]$ as above. Note that $k_3, k_2 \approx 0$. First we treat the case where $\ell_1, \ell_2 > \ell_0$. Then we can
integrate by parts in both $\eta$ and $\sigma$ to obtain, using $\text{(1.27d), (1.8)}$

\[
2^m \sup_{s \ll 2^m} \left\| \varphi_{k, \xi} \right\|_{L^\infty_{\xi}} T^{R,1}_{1,h_1}[b_1] \left( e^{i \epsilon \xi} \varphi \right)_{\eta,0} \left( \xi \right) \tilde{g}_{e_1}, e^{i \epsilon \xi} \varphi \left( \xi \right)_{\eta,0} \tilde{g}_{e_2} \right)_{L^\infty_{\xi}} \ll 2^{-m} \sum_{l_1, l_2 > l_0} \left\| \varphi_{l_1}^{(l_0)} \left( \langle \eta \rangle - 2\lambda \right) \nabla \eta \tilde{g}_{e_1} \right\|_{L^1_{\eta}} \cdot \left\| \varphi_{l_2}^{(l_0)} \left( \langle \sigma \rangle - 2\lambda \right) \nabla \sigma \tilde{g}_{e_2} \right\|_{L^1_{\sigma}} \cdot 2^{m/10 + 10M_0} \ll 2^{-m} \cdot (2^m \rho(2^m) m^2)^2 \cdot 2^{m/10 + 10M_0}.
\]

In the case where one of the indices $l$ equals $l_0$, we can integrate directly instead of integrating by parts. We skip the details.

Similarly for the mixed cases $(G, H) = (h, g)$ or $(g, h)$ we can combine the approaches above to obtain the desired bound.

Next, we deal with the piece $F^R_{m,2}$.

**Case 1**: $k_2 \leq -m/2 + m/100$. Note that we can add a cut-off $\varphi_{k,1, k_2, 5}(\sigma)$ on the profile $H$. Then we can write, using $\text{(1.26d)}$ and $\text{(2.23)}$, that

\[
2^m \sup_{s \ll 2^m} \left\| \varphi_{k, \xi} \right\|_{L^\infty_{\xi}} T^{R,2}_{1, h_1}[b_2] \left( e^{i \epsilon \xi} \tilde{G}_{e_1}, e^{i \epsilon \xi} \tilde{H}_{e_2} \right)_{L^\infty_{\xi}} \ll 2^m \cdot \left\| \varphi_{k_1} \left( \eta, \eta \right) \tilde{G}_{\sigma} \right\|_{L^\infty_{\eta}} \cdot \left\| \varphi_{k_2} \tilde{H}_{\sigma} \right\|_{L^1_{\sigma}} \cdot 2^{m/2 - k_1} \cdot 2^{10M_0} \ll 2^m \cdot 2^{(5/2 - k_1)} Z \cdot 2^{(5/2 - k_2)} Z \cdot 2^{-2k_1} 2^{10M_0}.
\]

This is sufficient to conclude.

**Case 2**: $k_2 > -m/2 + m/100$.

**Subcase 2.1**: $k_1 \leq -m/2 + m/200$. In this case $|\eta + \sigma| \approx 2^{k_2}$, and we can add a cutoff $\varphi_{\approx k_2}(\eta + \sigma)$ to the expression.

**Subcase 2.1.1**: $(\epsilon_1, \epsilon_2) \neq (+, -), (-, +)$. In this case we note that $|\nabla \xi \varphi_{\xi, \epsilon_1, \epsilon_2} | \gtrsim 2^{k_2} (2^{k_2})^{-1}$ and, therefore, we can integrate by parts in $\sigma$. We may assume that $k_2 < 0$, as otherwise the bound is easier to obtain. Integration by parts gives two types of main terms, one when the derivative hits $H$ and another when it hits $G$. In the first case we integrate directly, in the second we change variables $\eta \mapsto -\eta - \sigma$ and then integrate. Overall, using $\text{(1.26d, 1.26a), (1.27b), (1.27a), (2.23)}$ we get,

\[
2^m \sum_{G, H \in (g, h)} \sup_{s \ll 2^m} \left\| \varphi_{k, \xi} \right\|_{L^\infty_{\xi}} T^{R,2}_{1, h_1}[b_2] \left( e^{i \epsilon \xi} \tilde{G}_{e_1}, e^{i \epsilon \xi} \tilde{H}_{e_2} \right)_{L^\infty_{\xi}} \ll \sum_{H \in (g, h)} 2^m \cdot \left\| \varphi_{k_1} \left( \eta, \eta \right) \tilde{G} \right\|_{L^\infty_{\eta}} \cdot L^1_{\xi} \cdot 2^{m} \cdot 2^{-k_2} \cdot \left\| \varphi_{\approx k_2} \nabla \tilde{h} \right\|_{L^1_{\sigma}} \cdot 2^{30} \cdot 2^{10M_0} \\ll \sum_{H \in (g, h)} 2^m \cdot \left\| \varphi_{k_1} \left( \eta, \eta \right) \tilde{G} \right\|_{L^\infty_{\eta}} \cdot L^1_{\xi} \cdot 2^{-m} \cdot 2^{-k_2} \cdot \left\| \varphi_{\approx k_2} \nabla \tilde{h} \right\|_{L^1_{\sigma}} \cdot 2^{30} \cdot 2^{10M_0} \\ll 2^{(5/2 - k_1)} Z \cdot 2^{k_2} Z \cdot 2^{2k_1} 2^{10M_0}.
\]

which is an acceptable bound.

**Subcase 2.1.2**: $(\epsilon_1, \epsilon_2) \in \{(+, -), (-, +)\}$. In this case we note that $|\Phi_{\epsilon_1, \epsilon_2} | \gtrsim 1$. Therefore we can proceed as for $F^{R}_{m}$ and integrate by parts in time to produce terms of the type $F^{S}_{m}$. Since the operator $T^{R,2}_{1}$ satisfies the same bilinear estimates as $T^{S,2}_{1}$, the bounds are the exact same; we then skip the details and refer the reader to the bounds for $T^{S,2}_{1}$ in the next section.
Subcase 2.2: \( k_2 > -m/2 + m/100 \) and \( k_1 > -m/2 + m/200 \). We insert an additional cutoff \( \varphi_{k_3}(\eta + \sigma) \) to localize the frequency of the input \( G \). We fix \( \delta_0 \in (0, 1/200) \).

Subcase 2.2.1: \( k_3 \leq -m/2 + \delta_0 m \). In this case \( |\eta| \approx |\sigma| \approx 2^{k_1} \approx 2^{k_2} \). Integrating directly we get, using (1.26a), (2.23),

\[
2^m \sum_{G, H \in \{g, h\}} \sup_{s \leq 2^m} \| \varphi_k \tilde{F}_{x \rightarrow \xi} T_1^{R, 2}[b_2](e^{i \xi \epsilon_1(\xi)} \tilde{G}_{\epsilon_1}, e^{i \xi \epsilon_2(\xi)} \tilde{H}_{\epsilon_2}) \|_{L_\xi^\infty} \lesssim 2^m \cdot 2^{(5/2 -)k_3} Z \cdot 2^{(5/2 -)k_1} Z \cdot 2^{-2k_1} 2^{10M_0},
\]

which is acceptable.

Subcase 2.2.2: \( k_3 > -m/2 + \delta_0 m \). Here we integrate by parts in both \( \sigma \) and \( \eta \) after changing variables \( \eta \rightarrow -\eta - \sigma \). We can then bound, using (1.26a), (1.27), (1.27a), (2.23) (note that the worse terms correspond to the derivative falling on the profile)

\[
2^m \sum_{G, H \in \{g, h\}} \sup_{s \leq 2^m} \| \varphi_k \tilde{F}_{x \rightarrow \xi} T_1^{R, 2}[b_2](e^{i \xi \epsilon_1(\xi)} \tilde{G}_{\epsilon_1}, e^{i \xi \epsilon_2(\xi)} \tilde{H}_{\epsilon_2}) \|_{L_\xi^\infty} \lesssim 2^m [2^{k_2/2} Z + 2^{\delta_m} 2^m \rho(2^m) + 2^m \rho(2^m)3^m] \cdot [2^{k_2/2} Z + 2^{\delta_m} 2^m \rho(2^m) + 2^m \rho(2^m)3^m] \cdot 2^{-2k_1} 2^{10M_0},
\]

which is acceptable provided \( \delta, \delta_0 \) and \( \delta_N \) are small enough.

Step 3: Bounding \( M_{1,m} \). Taking an inverse Fourier transform of \( M_{1,m} \) and relying on the dispersive estimates (1.26a), we have

\[
\| M_{1,m} \|_{L_\xi^\infty} \lesssim \left\| \int_0^t B(s) \left[ L^{-1} e^{i s L} f \right] \phi \tau_m(s) ds \right\|_{L_\xi^2} \lesssim 2^m \sup_{s \leq 2^m} \| B(s) \|_{L_0^\infty} \| L^{-1} e^{i s L} f \|_{L_0^\infty} \lesssim 2^m \cdot 2^{-m/2} \cdot 2^{-m + 10\delta_N} Z,
\]

which yields the desired result.

Step 4: Bounding \( F_H \). From the definition of \( F_H = \sum_m F_{H,m} \) given through (2.7), we see that when we express this term in distorted Fourier space at least one of the three frequencies (the output \( \xi \), and the two input frequencies \( \eta, \sigma \)) is of size \( \gtrsim 2^{M_0(s)} \approx s^{\delta_N} \). Then we can split \( F_{H,m} \) into seven terms by inserting frequency projections \( P_+ \) and \( P_- \) associated to cutoffs \( \varphi_+ := \varphi_{< M_0} \) and \( \varphi_- := \varphi_{> M_0} \) (see (2.7)). We denote these terms by \( F_{H,m}^{(i,j,k)} \) where the signs \( i, j, k \in \{+,-\} \) correspond to the presence of a cutoff \( \varphi_+ \) or \( \varphi_- \), and \( (i, j, k) \neq (+, +, +) \).

Case 1: \( j = - \) or \( k = - \). Without loss of generality we let \( j = - \). Then we write, for \( k \in \{+,-\} \)

\[
\| F_{H,m}^{(i,-k)} \|_{L_\xi^\infty} \lesssim \left\| \tilde{F}_{\tilde{\xi} \rightarrow \tilde{x}} \left[ \varphi_+(\tilde{\xi}) \int_0^t \int_{\mathbb{R}^6} e^{-i s \Phi_{k_1 \rightarrow 2} (\tilde{\xi}, \eta, \sigma)} \varphi_-(s) \varphi_k(s, \sigma) \mu(\tilde{\xi}, \eta, \sigma) d\eta d\sigma \tau_m(s) ds \right] \right\|_{L_\xi^2} \lesssim 2^m \left\| L^{-1} e^{i t L} P_\xi \varphi_{k_1} (s) \right\|_{L_\xi^2} \left\| L^{-1} e^{i t L} P_\xi \varphi_{k_2} (s) \right\|_{L_\xi^2} \lesssim 2^m \cdot 2^{-N M_0} \| f_{\varphi_{k_1}} \|_{H^N} \| f_{\varphi_{k_2}} \|_{L^2},
\]

which is more than sufficient since \( M_0 = \delta_N m = \frac{\delta}{N} m \).

Case 2: \( (i, j, k) = (-, +, +) \). In this case using the boundedness of wave operators on \( L^1 \) and distributing derivatives, we can estimate

\[
\| F_{H,m}^{(-, +, +)} \|_{L_\xi^\infty} \lesssim 2^{-N M_0} \left\| \int_0^t \int_{\mathbb{R}^6} e^{-i s \Phi_{k_1 \rightarrow 2} (\tilde{\xi}, \eta, \sigma)} \varphi_-(s) \varphi_k(s, \sigma) \mu(\tilde{\xi}, \eta, \sigma) d\eta d\sigma \tau_m(s) ds \right\|_{L_\xi^2} \lesssim 2^{-N M_0} \sum_{N_1 + N_2 \leq N} \| D^{N_1} L^{-1} e^{i t L} f_{\varphi_{k_1}} \|_{L_\xi^2} \| D^{N_2} L^{-1} e^{i t L} f_{\varphi_{k_2}} \|_{L_\xi^2},
\]

we then conclude as we did above using (1.26a).
Step 5: Bounding $S_m$. Recall the definitions \([2.3]\). To treat $S_{2,m}$ we integrate by parts in time and then use the estimate for $\hat{B}$ in \((1.20)\) to obtain

$$
\|S_{2,m}(t)\|_{L^\infty_\xi} \lesssim \|\tau_m(t)B^2(t)\frac{e^{-it(\xi-2\lambda)}(1 - \chi_C(\xi))\theta(\xi)}{\chi_C(\xi) - 2\lambda}\|_{L^\infty_\xi} + \|\frac{1 - \chi_C(\xi)}{(\xi - 2\lambda)^2} \int_0^t e^{-is(\xi-2\lambda)}(2\hat{B}(s)B(s)\tau_m(s) + B^2(s)2^{-m}\tau_m'(s))ds \theta(\xi)\|_{L^2} \\
\lesssim \rho(2^m) + \int_{s=2^m} \rho(s)ds \lesssim \rho(2^m).
$$

For the term $S_{1,m}$ we note that $A^2 - B^2 = O(A^3)$, see \((1.20)\), and therefore we can bound directly $\|S_{1,m}\|_{L^\infty_\xi} \lesssim 2m \rho(2^m)^{3/2}$, which suffices. The proof of \((4.3)\) is complete. \hfill \Box

5. Estimating the correction

We conclude the proof by estimating the normal form correction defined in the previous section, namely showing \((4.4)\), under the a priori assumption \((4.2)\). Recalling \((4.1)\) and \((2.27)\), we have to estimate $T_{1,2}^{S,1}$ and $T_{2}^{S,2}$, $i = 1, 2, 3$.

Step 1: Bounding $T_{1,2}^{S,1}$. Using the identities \((2.11)\), \((2.13)\) and \((2.14)\) we can reduce matters to estimating the following three operators:

$$
T_{\delta}^{S,1}[b](G,H) := \mathcal{F}^{-1}_{\xi \to x} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \tilde{G}(\xi - \eta)\tilde{H}(\sigma) b(\xi,\eta,\sigma) \frac{b_0(\eta,\sigma)}{|\eta|} \delta(|\eta| - |\sigma|) d\eta d\sigma,
$$

$$
T_{p.v.}^{S,1}[b](G,H) := \mathcal{F}^{-1}_{\xi \to x} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \tilde{G}(\xi - \eta)\tilde{H}(\sigma) b(\xi,\eta,\sigma) \frac{b_0(\eta,\sigma)}{|\eta|} \text{p.v.} \frac{\varphi_{M_0-5}(|\eta| - |\sigma|)}{|\eta| - |\sigma|} d\eta d\sigma,
$$

where we denoted $\hat{\eta} := -\eta/|\eta|$, and, for $a = 1, \ldots, N_2$,

$$
T_{a}^{S,1}[b](G,H) := \mathcal{F}^{-1}_{\xi \to x} \int_{\mathbb{R}^3 \times \mathbb{R}^3} G(\xi - \eta)H(\sigma) b(\xi,\eta,\sigma) \varphi_{M_0-5}(|\eta| - |\sigma|) \frac{1}{|\eta|} \times \sum_{J \in \mathbb{Z}} b_{a,J}(\eta,\sigma) \cdot 2^J K_a(|\eta| - |\sigma|) d\eta d\sigma.
$$

Recall the definition of $b$ in \((2.29)\). Notice that in the formulas above the inputs $(G,H)$ play the role of either $e^{\pm iLt}h$ or $e^{\pm iLt}g$. In particular the $H_{\delta}$ and $\mathcal{F}^{-1}_{\xi \to x}$ norms of $(G,H)$ are those of $g$ and $h$.

We can deal with $T_{\delta}^{S,1}, T_{a}^{S,1}$ by direct integration, using the lower bound $|\Phi(\xi,\eta,\sigma)| \gtrsim (\langle \xi \rangle + \langle \eta \rangle + \langle \sigma \rangle)^{-1}$ see \((5.47)\), Lemma 5.10 in \([19]\). More precisely,

$$
\|T_{\delta}^{S,1}[b](G,H)\|_{L^\infty_\xi} \lesssim \left\| \frac{\varphi_{k_1}(\eta)}{\Phi(\xi,\xi - \eta,\eta)\langle \xi - \eta \rangle}\right\|_{L^\infty_{\xi,\eta}} \|b_0(\omega,\xi)\|_{L^\infty_{\omega,\xi}} \times \sup_{\eta \in \mathbb{R}^3} \left| \int_{\mathbb{R}^3} \tilde{H}(\sigma)\varphi_{k_2}(\sigma) \frac{\delta(|\eta| - |\sigma|)}{|\eta|} d\sigma \right| \cdot \|G(\xi - \eta)\varphi_{k_1}(\eta)\|_{L^\infty_{\xi,\xi}} \\
\lesssim \langle k_1 \rangle^{-1} \cdot 2^{-k_1} \|\tilde{H}\|_{L^\infty_{\xi}} \cdot 2^{3k_1/2}\|G\|_{L^2} \lesssim 2^{k_1/2}(2^{k_1})^{-1} \epsilon^{-1}.
$$
We can then sum over $k_1$ and get a bound consistent with (4.4). Note that we have used (1.26b). Similarly we can bound
\[
\left\| T_1^{S,1}(b)(G, H) \right\|_{L_x} \lesssim \left\| \frac{\varphi_{k_1}(\eta)\varphi_{k_2}(\sigma)}{\Phi(\xi - \eta, \sigma)} \right\|_{L_x} \left\| \frac{\tilde{G}(\xi - \eta)}{\eta} \right\|_{L_x} \left\| \frac{b_0(\eta, \sigma)}{\tilde{H}(\sigma)} \right\|_{L_x} \sum_{J \in \mathbb{Z}} \left| \varphi_{-k_1}(\eta) \varphi_{-k_2}(\sigma) b_{\alpha, J}(\eta, \sigma) \right|
\]
\[
\lesssim \langle k_1 \rangle^{-1} \langle k_2 \rangle^{-1} \left\| \tilde{H} \right\|_{L_x} \cdot \langle k_1 \rangle \langle k_2 \rangle \left\| G \right\|_{H^2} \sum_{J \in \mathbb{Z}} \left| \varphi_{-k_1}(\eta) \varphi_{-k_2}(\sigma) b_{\alpha, J}(\eta, \sigma) \right|
\]
\[
\lesssim \langle k_1 \rangle / (2k_2)^{-1} \varepsilon^{-1}.
\]
Note that here we have used again (1.26b), and also the a priori assumption (4.2) and Lemma 21 to bound $\tilde{H}$ in $L_x^\infty$. We can then sum over $k_1, k_2$ using $|k_1 - k_2| < 5$, see (2.11).

We then look at the principal value part and bound
\[
|T_1^{S,1}(b)(G, H)| \leq |I(\xi)| + |II(\xi)|,
\]
\[
I(\xi) := \int_{\mathbb{R}^3} \frac{\tilde{G}(\xi - \eta) \varphi_{k_1}(\eta)}{\eta} \int_{\mathbb{R}^3} \frac{\Phi_{\epsilon_1, \epsilon_2}(\xi - \eta, \eta)}{\eta} \frac{b_0(\eta, \sigma)}{\tilde{H}(\sigma)} \varphi_{k_2}(\sigma) d\sigma d\eta,
\]
\[
II(\xi) := \int_{\mathbb{R}^3} \frac{\tilde{G}(\xi - \eta)}{\eta} \frac{\varphi_{k_1}(\eta)}{\Phi_{\epsilon_1, \epsilon_2}(\xi - \eta, \eta)} \frac{b_0(\eta, \sigma)}{\tilde{H}(\sigma)} \varphi_{k_2}(\sigma) d\sigma d\eta.
\]
The term $I$ can be treated using direct integration as $T_1^{S,1}$ and $T_0^{S,1}$ since the singularity is canceled by the difference of the phases. For $II$ we use (3.3) and the fact that $|\Phi_{\epsilon_1, \epsilon_2}(\xi - \eta, \eta)| \lesssim 1$, and find
\[
\left\| II \right\|_{L_x^\infty} \lesssim \langle k_1 \rangle \cdot \langle k_2 \rangle \varepsilon^{-1}.
\]

**Step 2: Bounding $T_2^{S,2}$.** We can use similar argument and eventually rely on (3.4) for the hardest term. Defining $T_2^{S,2}, T_{p,v}^{S,2}$, and $T_0^{S,2}$ in an analogous way to what was done above, i.e., replacing the measure $\mu_1^S$ in the expression $T_2^{S,2}$ (see (2.24a)) by the three pieces in (2.11), (2.13).

We only focus on the principal value part since the other terms can be bounded by direct integration as above. We start by writing that
\[
|T_2^{S,2}(b)(G, H)| \leq |III(\xi)| + |IV(\xi)|,
\]
\[
III(\xi) := \int_{\mathbb{R}^6} \frac{\tilde{G}(\eta + \sigma) \varphi_{k_1}(\eta) \varphi_{k_2}(\sigma)}{\Phi_{\epsilon_1, \epsilon_2}(\eta, \eta + \sigma, \sigma)} \frac{\tilde{H}(\sigma) b_0(\eta, \xi)}{|\eta|} \frac{1}{|\eta| - |\xi|} d\eta d\sigma d\eta,
\]
\[
IV(\xi) := \int_{\mathbb{R}^6} \frac{\tilde{G}(\eta + \sigma) \varphi_{k_1}(\eta) \varphi_{k_2}(\sigma)}{\Phi_{\epsilon_1, \epsilon_2}(\eta, \eta + \sigma, \sigma)} \frac{\tilde{H}(\sigma) b_0(\eta, \xi)}{|\eta|} \frac{1}{|\eta| - |\xi|} d\eta d\sigma d\eta.
\]
Since the term $IV$ is not singular we can treat it by direct integration as above; we skip the details. To bound the main term $III$ we apply (3.4) to the multiplier
\[
m(\eta, \sigma) := \frac{\varphi_{k_1}(\eta) \varphi_{k_2}(\sigma)}{\Phi_{\epsilon_1, \epsilon_2}(\eta, \eta + \sigma, \sigma)(\eta + \sigma)^5(\sigma)^5} \langle \eta \rangle / \langle \xi \rangle.
\]
and, using again (1.26b), we obtain (here $K = k_1$)
\[
\|III\|_{L^\infty} \lesssim 2^{k_1} (2^{k_1})^{-2} \|G\|_{H^6} \|P_{k_2} H\|_{H^6} \\
\lesssim 2^{k_1} (2^{k_1})^{-2} \varepsilon^{1-} \cdot \min \left\{ \langle 2^{k_2} \rangle^{-1} \|H\|_{H^7}, 2^{3k_2/2} \|H\|_{L^\infty} \right\}.
\]

We can then use Lemma 2.1 and (4.2) to control the last $L^\infty$ norm and sum over $k_1, k_2$ to conclude.

**Step 3: Bounding $T^S_i$.** Using (2.15), (2.16), (2.17) and (2.20), we can treat these terms by direct integration as we did above for $T^S_\delta$. Estimates are easier in this case, therefore we omit the details.

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Tristan Léger, Princeton University, Department of Mathematics, Fine Hall, Washington Road, Princeton, NJ, 08544, United States.

Email address: tleger@princeton.edu

Fabio Pusateri, University of Toronto, Department of Mathematics, 40 St George Street, Toronto, ON, M5S 2E4, Canada.

Email address: fabiop@math.toronto.edu