Analysis of the variable step method of Dahlquist, Liniger and Nevanlinna for fluid flow

William Layton | Wenlong Pei | Yi Qin | Catalin Trenchea

1 Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania, USA
2 School of Arts and Sciences, Shaanxi University of Science and Technology, Xi’an, China

Correspondence
Wenlong Pei, Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA.
Email: wep17@pitt.edu

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Abstract
The one-leg, two-step time discretization proposed by Dahlquist, Liniger and Nevanlinna is second order and variable step G-stable. G-stability for systems of ordinary differential equations (ODEs) corresponds to unconditional, long time energy stability when applied to the Navier–Stokes equations (NSEs). In this report, we analyze the method of Dahlquist, Liniger and Nevanlinna as a variable step, time discretization of the Navier–Stokes equations. We prove that the kinetic energy is bounded for variable time-steps, show that the method is second-order accurate, characterize its numerical dissipation and prove error estimates. The theoretical results are illustrated by several numerical tests.

KEYWORDS
adaptive time steps, Navier–stokes equations, G-stability

1 | INTRODUCTION

The accurate numerical simulation of flows of an incompressible, viscous fluid, with the accompanying complexities occurring in practical settings, is a problem where speed, memory and accuracy never seem sufficient. For time discretization (considered herein), many longer time simulations use constant step low order methods, and (with few exceptions noted in Section 1.1) the remainder use the constant time step implicit midpoint or the trapezoidal schemes, for example, [2, 3, 8, 31, 39], (often combined with fractional steps, or with ad hoc fixes to correct for oscillations due to lack of $L$-stability [6, 38, 44]), or the backward differentiation formula 2 (BDF2) method [1, 18, 22, 33, 34, 46]. Time accuracy requires time step adaptivity within the computational, space and cognitive complexity limitations of computational fluid dynamics (CFD). Beyond accuracy, adaptivity has the
secondary benefit (depending on implementation) of reducing memory requirements and decreasing the number of floating point operations.

The richness of scales of higher Reynolds number flows and the cost per step of their solution suggest a preference for $A$-stable (or even $L$-stable) multi-step methods, called Smart Integrators in Gresho, San and Engelman [20, section 3.16.4] For constant time steps, a complete analysis of the general (2 parameter family) 2-step, $A$-stable linear multi-step method is performed in the 1979 book Girault and Raviart [19] but there is no analogous stability or convergence analysis for the important case of variable time steps. As an example of the challenges involved in variable steps, BDF2 (a popular member of that $A$-stable family) loses $A$-stability for increasing time steps, allowing non-physical energy growth. This BDF2 variable step instability is weak since 0-stability is preserved for smoothly varying time steps [4, 40]. Similarly, the trapezoidal method is unstable [16] [41, pp. 181–182] when used with variable steps. Specifically, Nevanlinna and Liniger [35, 36] gave a simple example where the Trapezoidal Rule (multistep method) is unstable, but the implicit midpoint rule (its one-leg “twin”) is stable, for all problems of the form $y'(t) = \lambda(t)y(t)$, Re$\lambda(t) \leq 0$ and any step size sequence.

In Reference [16], Dahlquist, Liniger and Nevanlinna give a family of one-leg 2-step methods that are $G$-stable for any sequence of time steps. This paper builds on and is related to work in References [12–14, 23, 29, 30]. Consider the differential equation $y'(t) = g(t, y)$, with $t \in [0, T]$, where $y(t) \in X$, a Banach space, and $g : [0, T] \times X \to X'$ is a sufficiently smooth function. The family of one-leg, 2-step methods proposed by Dahlquist, Liniger and Nevanlinna (DLN) takes the form

$$\sum_{\ell=0}^{2} \alpha_\ell y_{n-1+\ell} = \hat{k}_n g \left( \sum_{\ell=0}^{2} \rho_\ell(n) y_{n-1+\ell} \right), \quad n = 1, \ldots, N - 1,$$

(DLN)

where $\hat{k}_n$ is an weighted average of the time steps $k_n, k_{n-1}$, and the generating polynomials are $\rho(\zeta) = \sum_{\ell=0}^{2} \alpha_\ell \zeta^\ell$, $\sigma_\ell(\zeta) = \sum_{\ell=0}^{2} \beta_\ell(\zeta) \zeta^\ell$ (see Equation (1) in Section 1.2).

Let $\Omega$ be the flow domain in $\mathbb{R}^d$ ($d = 2$ or 3), and denote by $u(x, t)$ the fluid velocity, $p(x, t)$ the pressure, and $f(x, t)$ body force. Herein we analyze an application of the (DLN) method for the Navier–Stokes Equations:

$$u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = f, \quad x \in \Omega, \quad 0 < t \leq T,$$

(NSE)

$$\nabla \cdot u = 0, \quad x \in \Omega \text{ for } 0 < t \leq T, \quad u(x, 0) = u_0(x), \quad x \in \Omega,$nabla \cdot u = 0 \text{ on } \partial\Omega, \quad \int_{\Omega} p dx = 0 \text{ for } 0 < t \leq T.$$

Section 1.2 presents (DLN)’s critical property of variable step $G$-stability with the $G$-matrix independent of the time-step ratio. Notations and preliminaries are presented in Section 1.3. Section 1.4 gives a proof of variable time step, unconditional, long time, bound of the energy for the one-leg (DLN) method for (NSE). This analysis shows that the natural kinetic energy, $E(t_n)$, and numerical dissipation rate, $D(t_n)$, of the DLN approximation are

$$E(t_n) = \frac{1}{4} (1 + \theta) \| u_n^h \|^2 + \frac{1}{4} (1 - \theta) \| u_{n-1}^h \|^2,$$

$$D(t_n) = \frac{1}{k_n} \sum_{\ell=0}^{2} \alpha_\ell(\Theta) \| u_{n-1+\ell}^h \|^2,$$

where $\| \cdot \|$ denotes the $L^2(\Omega)$ norm. Section 2 provides the variable step error analysis. The DLN method is proven second-order for any sequence of time steps. Numerical tests are presented in Section 3. The first example confirms the theoretical prediction of second order accuracy. The second
test shows that DLN has stability advantages over BDF2 for variable time steps. There is a recent idea of Capuano, Sanderse, Angelis and Coppola [10] to adapt the time step to control the ratio of numerical to physical dissipation. Rather than testing a standard approach to error estimation and adaptivity, we also test this idea, which uses the above explicit formula for the method’s numerical dissipation, in Section 3.

Remark 1 We focus herein on the variable step DLN time discretization. It is impossible to draw clear conclusions when varying more than one thing. Thus, at each non time discretization decision we select the most classical one, for example, standard Galerkin, well-known finite element spaces, standard nonlinearity, no stabilizations, no turbulence models and so on. Each of these can be further optimized using the properties of DLN, developed here.

1.1 Related work

The number of papers studying time-stepping methods for flow problems is very large. The general (2 parameter two-legs) linear 2-step $A$-stable method was analyzed for the NSE for constant time steps in Girault and Raviart [19], and developed further in [27]. Time adaptive discretizations of the NSE have been limited by the Dahlquist barrier, storage limitations and the cognitive complexity of extending to the NSE many of the standard methods for systems of ordinary differential equations [7, 9, 45]. One early and important work is Kay, Gresho, Griffiths and Silvester [28]. It presents an adaptive algorithm based on the trapezoid scheme/linearized midpoint rule (with error estimation done using an explicit AB2 type method) that is memory and computation efficient. It is well-known [32] for systems of ODEs that variable step, variable order methods are efficient choices, and have been considered for the NSE in References [17, 24].

1.2 The variable step DLN method

The (DLN) method is a 1-parameter ($0 \leq \theta \leq 1$) family of $A$-stable, one-leg 2-step $G$-stable methods. When $\theta = 1$, (DLN) reduces to the one-step, one-leg implicit midpoint scheme [9], in which case is also symplectic, and conserves all linear and quadratic Hamiltonians. The (DLN)’s key property is that its $G$-matrix depends on the parameter $\theta$, but not on the time-step ratio. For a comparison in terms of stability regions, in the constant time step case, Figure 1 shows the DLN and the BDF2 root locus curves.

Let $\{t_n\}_{n \geq 0}$ denote the time mesh points, $k_n = t_{n+1} - t_n$ the time step, and $\varepsilon_n = (k_n - k_{n-1}/k_n + k_{n-1})$ the step size variability (notice $\varepsilon_n \in (-1, 1)$). Then the $\{\alpha, \beta\}_{r=0,1,2}$ coefficients in (DLN) are

$$
\begin{pmatrix}
\alpha_2 & \beta_2^{(n)} \\
\alpha_1 & \beta_1^{(n)} \\
\alpha_0 & \beta_0^{(n)}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2}(\theta + 1) & \frac{1}{4} \left(1 + \frac{1-\theta^2}{(1+\varepsilon_n\theta)^2} + \varepsilon_n^2 \frac{\theta(1-\theta^2)}{(1+\varepsilon_n\theta)^2} + \theta \right) \\
-\theta & \frac{1}{2} \left(1 - \frac{1-\theta^2}{(1+\varepsilon_n\theta)^2} \right) \\
\frac{1}{2}(\theta - 1) & \frac{1}{4} \left(1 + \frac{1-\theta^2}{(1+\varepsilon_n\theta)^2} - \varepsilon_n^2 \frac{\theta(1-\theta^2)}{(1+\varepsilon_n\theta)^2} - \theta \right)
\end{pmatrix},
$$

and the averaged time step $\hat{k}_n$ is

$$
\hat{k}_n = \alpha_2 k_n - \alpha_0 k_{n-1} = \frac{1}{2}(1+\theta)k_n + \frac{1}{2}(1-\theta)k_{n-1} = \theta \frac{k_n - k_{n-1}}{2} + \frac{k_n + k_{n-1}}{2}.
$$
The $\alpha_\ell$-coefficients are independent of the time-step ratio, but $\beta_\ell$ depend on the time-step ratios via the variability coefficients $\varepsilon_n$. For expression of the numerical dissipation, we also define

$$
d_1^{(n)} = -\frac{\sqrt{\theta(1 - \theta^2)}}{\sqrt{2}(1 + \varepsilon_n \theta)}, \quad d_2^{(n)} = -\frac{1 - \varepsilon_n}{2} d_1^{(n)}, \quad d_0^{(n)} = -\frac{1 + \varepsilon_n}{2} d_1^{(n)},$$

which also depend on the time-step ratios through the variability coefficients $\varepsilon_n$.

**Definition 1**  For $\theta \in [0, 1]$, we define the symmetric semi-positive definite $G(\theta)$ matrix

$$G(\theta) = \begin{pmatrix} \frac{1}{4}(1 + \theta) I_d & 0 \\ 0 & \frac{1}{4}(1 - \theta) I_d \end{pmatrix},$$

and the corresponding G-norm

$$\left\|\begin{bmatrix} u \\ v \end{bmatrix}\right\|_{G(\theta)}^2 = \left[u^T \ v^T\right] G(\theta) \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{4}(1 + \theta) \|u\|_{\mathbb{R}^d}^2 + \frac{1}{4}(1 - \theta) \|v\|_{\mathbb{R}^d}^2, \quad \forall u, v \in \mathbb{R}^d,$$

where $tr$ means transpose of vectors. Recall the following result [16] related to the $G$-stability of the (DLN) method.

**Lemma 1**  Let $0 \leq \theta \leq 1$. The variable step, one-leg (DLN) methods are $G$-stable, and for any $n \geq 1$, with $a_\ell^{(n)}$ ($\ell = 0, 1, 2$) given above (2), we have

$$\left(\sum_{\ell=0}^2 a_\ell y_{n-1+\ell}, \sum_{\ell=0}^2 \beta_\ell^{(n)} y_{n-1+\ell}\right)_{\mathbb{R}^d} = \left\|y_{n+1}\right\|_{G(\theta)}^2 - \left\|y_n\right\|_{G(\theta)}^2 + \left\|\sum_{\ell=0}^2 a_\ell^{(n)} y_{n-1+\ell}\right\|_{\mathbb{R}^d}^2.$$

**Proof.** The proof (implicit in Dahlquist, Liniger and Nevanlinna [16]) is an algebraic calculation.
1.3 Notation and preliminaries

The discussion of the DLN method connects to stability theory in numerical ODEs. Much of this theory addresses the response of the method when applied to \( y' = \lambda y, y(0) = 1 \), for example, Reference [21]. Recall that a method is 0-stable if, when applied to \( y' = \lambda y \), \( y(0) = 1 \), the approximate solution does not grow. For more general problems, 0-stability allows exponential growth but excludes rate constants that blow up as \( \Delta t \to 0 \). A method is A-stable if, when applied to \( y' = \lambda y, y(0) = 1 \), the constant time step approximation \( y_n \to 0 \) as \( n \to \infty \) for any \( \Delta t \) and any \( \lambda \) with \( \text{Re}(\lambda) < 0 \). A-stability addresses time asymptotics (\( t_n \to \infty \)) and is thus stronger than 0-stability. A method is L-stable if it is A-stable and satisfies the additional condition that \( y_n \to 0 \) for \( n \) fixed for any \( \Delta t \) as (real) \( \lambda \to -\infty \). L-stability means that the method will not experience the ±1 type oscillation (called ringing) of the trapezoid rule for constant \( \Delta t \) and large, negative \( \lambda \). G-stability is an extension of A-stability to unconditional, long time, energetic stability of nonlinear problems.

Let \( \Omega \) be any domain in \( \mathbb{R}^d \) (\( d = 2 \) or 3). For \( 1 \leq p < \infty \) and \( r \in \mathbb{N} \), \( \| \cdot \|_{L^p} \) and \( \| \cdot \|_{W^r_p} \) are norms on function spaces \( L^p(\Omega) \) and \( W^r_p(\Omega) \) respectively. When \( p = 2 \), we denote \( \| \cdot \| \) be \( L^2(\Omega) \) norm, and the \( L^2(\Omega) \) inner product \((\cdot, \cdot)\). Moreover, \( H^r(\Omega) \) denotes the Sobolev space \( W^r_2(\Omega) \) with norm \( \| \cdot \|_r \). For the velocity \( u \) and pressure \( p \), we define the spaces

\[
X = \{ v : \Omega \to \mathbb{R}^d : v \in L^2(\Omega), \nabla v \in L^2(\Omega) \text{ and } v = 0 \text{ on } \partial \Omega \},
\]

\[
Q = \left\{ q : \Omega \to \mathbb{R} : v \in L^2(\Omega) \text{ and } \int_{\Omega} q \, dx = 0 \right\}.
\]

The space of divergence free functions is denoted

\[
V = \{ v \in X : (\nabla \cdot v, q) = 0, \quad \forall q \in Q \}.
\]

The spaces \( X^* \) is the dual space of \( X \) with the dual norm

\[
\| f \|_* = \sup_{\substack{v \in X \\ v \neq 0}} \frac{(f, \nabla v)}{\| \nabla v \|}, \quad \forall f \in X^*.
\]

For convenience, we denote

\[
\| v \|_{p,r} = \| v \|_{L^p(0,T,H^r)},
\]

for any functions \( v(t,x) \) and \( 1 \leq p \leq \infty \). For \( u, v, w \in X \), we define the explicitly skew symmetrized trilinear form

\[
b(u, v, w) := \frac{1}{2} (u \cdot \nabla v, w) - \frac{1}{2} (u \cdot \nabla w, v).
\]

This is the most common form of the nonlinearity in use so it is selected herein. In passing, we note that the energy, momentum and angular momentum conserving (EMAC) form of the nonlinearity from Olshanskii and Rebholz [37], although less common, provides better results. Obviously we have \( b(u,v,w) = 0 \) and by the divergence theorem, we have \( b(u,v,w) = (u \cdot \nabla v, w) \) for all \( u \in V \) and \( v, w \in X \). Moreover, \( b(u,v,w) \) satisfies the bounds (see e.g., Reference [43])

\[
b(u,v,w) \leq C(\Omega)\| \nabla u \| \| \nabla v \| \| \nabla w \|, \quad b(u,v,w) \leq C(\Omega)\| u \|^{1/2} \| \nabla u \|^{1/2} \| \nabla v \| \| \nabla w \|. \quad (6)
\]

For the spatial discretization we use the finite element method. The approximate solutions for the velocity and pressure are in the finite element spaces \( X_h \subset X, Q_h \subset Q \), based on an edge to edge triangulation \( \Omega \) (with maximum triangle diameter \( h \)). We assume that \( X_h \) and \( Q_h \) satisfy the usual
discrete inf-sup condition \((LBB^h\) condition). The Taylor–Hood elements, which satisfy the condition, are used in the numerical tests. The discretely divergence-free subspace of \(X_h\) is

\[
V_h := \{ v_h \in X_h : (\nabla \cdot v_h, q_h) = 0, \forall q_h \in Q_h \}.
\]

We also assume that \(X_h\) and \(Q_h\) have degree \(r\) and \(s\), respectively \((r, s \in \mathbb{N})\), and the following interpolation error estimate for the velocity \(u\) and pressure \(p\) holds (see e.g., Reference [5, p. 108]):

\[
\|u - I^h u\|_m \leq C h^{r+1-m} \|u\|_{r+1}, \quad u \in H^{r+1}(\Omega)^d, \quad 0 \leq m \leq r
\]

\[
\|p - I^h p\|_m \leq C h^{s+1-m} \|p\|_{s+1}, \quad p \in H^{s+1}(\Omega), \quad 0 \leq m \leq s, \quad m \in \mathbb{N},
\]

where \(I^h u\) and \(I^h p\) are the \(L^2\) projection of \(u\) and \(p\) into \(V^h\) and \(Q^h\) respectively. (These estimates can follow, for example, from standard ones for the Stokes projection for \(H^2\) regular domains.

For any given sequence \(\{t_n\}_{n \geq 1}\), we denote by

\[
z_{n,\beta} = \beta_2^{(n)} z_{n+1} + \beta_1^{(n)} z_n + \beta_0^{(n)} z_{n-1}
\]

the convex combination of the three adjacent terms in the sequence. For example, \(\{t_{n,\beta}\}\) is the set of time-values and \(u_{n,\beta}\) are the implicit values where the equation is evaluated

\[
t_{n,\beta} = \beta_2^{(n)} t_{n+1} + \beta_1^{(n)} t_n + \beta_0^{(n)} t_{n-1}, \quad u_{n,\beta} = \beta_2^{(n)} u_{n+1} + \beta_1^{(n)} u_n + \beta_0^{(n)} u_{n-1}.
\]

The variational formulation of the one-leg (DLN) method for the NSE is as follows. Given \(u_{n,\beta}^h, u_{n-1,\beta}^h \in X_h\) and \(p_{n,\beta}^h, p_{n-1,\beta}^h \in Q_h\), find \(u_{n+1,\beta}^h\) and \(p_{n+1,\beta}^h\) satisfying

\[
\frac{1}{k_n}(\alpha_2 u_{n+1}^h + \alpha_1 u_n^h + \alpha_0 u_{n-1}^h, \nu^h) + (\nabla v_{n,\beta}, \nabla \nu^h) + (b(u_{n,\beta}, \nabla u_{n,\beta}, \nabla \nu^h) - (p_{n,\beta}, \nabla \cdot \nu^h) = 0,
\]

for all \(v^h \in X^h\), \(q^h \in Q^h\). Under the discrete inf-sup condition, (8) is equivalent to

\[
\frac{1}{k_n}(\alpha_2 u_{n+1}^h + \alpha_1 u_n^h + \alpha_0 u_{n-1}^h, \nu^h) + (\nabla v_{n,\beta}, \nabla \nu^h) + (b(u_{n,\beta}, \nabla u_{n,\beta}, \nabla \nu^h) = \left(\sum_{\ell=0}^{2} \beta_{\ell}^{(n)} f(t_{n-1+\ell}, \nu^h), \nu^h\right), \quad \forall \nu^h \in V^h.
\]

### 1.4 Stability of DLN method for the NSE

In this section, we prove the unconditional, long time, variable time step bound of the energy for (8), using the \(G\)-stability property (5) of the method.

**Theorem 1** The one-leg 2-step (DLN) methods (8) satisfy unconditionally the following long-time energy bounds: for any integer \(M > 1\),

\[
\frac{1}{4} (1 + \theta) \|u_{M,\beta}^h\|^2 + \frac{1}{4} (1 - \theta) \|u_{M-1,\beta}^h\|^2 + \sum_{n=1}^{M-1} \left\| \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u_{n+1-\ell,\beta}^h \right\|^2 + \frac{M-1}{2} \sum_{n=1}^{M-1} \|\nabla u_{n,\beta}^h\|^2
\]

\[
\leq \frac{1}{2} \nu \sum_{n=1}^{M-1} \|f(t_{n,\beta})\|^2 + \frac{1}{4} (1 + \theta) \|u_0^h\|^2 + \frac{1}{4} (1 - \theta) \|u_0^h\|^2,
\]

where \(\{a_i^{(n)}\}_{i=0,1,2}\) are defined in (2).
Proof. For \( n \geq 1 \), set \( v^h = u^h_{n,\beta} \) in (8). Using the skew-symmetry property of \( b \) and the Cauchy–Schwarz inequality we obtain

\[
\left( \sum_{\ell=0}^{2} \alpha_{\ell} u_{n-1+\ell}^h, u_{n,\beta}^h \right) + \frac{v}{2} \hat{\kappa}_n \| \nabla u_{n,\beta}^h \|^2 \leq \frac{1}{2v} \hat{\kappa}_n \| f(t_n) \|^2. 
\]

Then the \( G \)-stability relation (5) implies

\[
\left\| u_{n+1}^h \right\|_{G(\theta)}^2 - \left\| u_n^h \right\|_{G(\theta)}^2 + \frac{1}{2} \sum_{\ell=0}^{2} \| a_{\ell}^{(n)} u_{n-1+\ell}^h \|^2 + \frac{v}{2} \hat{\kappa}_n \| \nabla u_{n,\beta}^h \|^2 \leq \frac{1}{2v} \hat{\kappa}_n \| f(t_n) \|^2. 
\]

Finally, summation over \( n \) from 1 to \( M-1 \) and the definition (4) yield the conclusion.

The above energy equality result identifies the DLN method’s kinetic energy and numerical energy dissipation rates:

\[
E_n = \frac{1}{4}(1+\theta)\| u_n^h \|^2 + \frac{1}{4}(1-\theta)\| u_{n-1}^h \|^2, \quad D_n = \frac{1}{k_n} \sum_{\ell=0}^{2} \| a_{\ell}^{(n)} u_{n-1+\ell}^h \|^2. 
\]

## 2 VARIABLE TIME-STEP ERROR ANALYSIS

In this section, we analyze the error in the approximate solutions by the one-leg DLN method for variable time steps. The discrete time error analysis requires norms that are discrete time analogs of the norms used in the continuous time case. As before, let \([0, T]\) denote the whole time interval, \( P = \{ t_n \}_{n=0}^{M} \) be a partition on \([0, T]\) and \( \{ k_n \}_{n=0}^{M-1} \) be the set of time step sizes. For a function \( v(x,t) \) and \( 1 \leq p < \infty, r \in \mathbb{N} \), we define

\[
\|v\|_{\infty,r} = \max_{0 \leq n \leq M} \|v(t_n)\|_r, \quad \|v\|_{\infty,\infty,r} = \max_{1 \leq n \leq M-1} \|v(t_n)\|_r, 
\]

\[
\|v\|_{p,r} = \left( \sum_{n=1}^{M-1} (k_{n-1} + k_n)^{1/p} \|v(t_{n,\beta})\|_r^p \right)^{1/p}, \quad \|v\|_{p,\infty} = \left( \sum_{n=1}^{M-1} (k_{n-1} + k_n)^{1/p} \|v(t_{n,\beta})\|_r^p \right)^{1/p}. 
\]

In the above definitions, \( \|v\|_{p,r} \) and \( \|v\|_{p,\infty} \) are forms of Riemann sums in which the function \( v \) is evaluated at point \( t_{n,\beta} \in [t_{n-1}, t_{n+1}] \).

Now we recall the following consistency results [15, 29, 30] on the interpolation and differentiation defects.

**Lemma 2** Let \( u(t) \) be any continuous function on \([0, T]\) such that \( u_{\infty} \in L^2(0, T) \). Then for \( \theta \in [0, 1] \)

\[
\| \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u_{n-1+\ell}(t_{n,\beta}) - u(t_{n,\beta}) \|_r^2 \leq C(\theta)(k_n + k_{n-1})^3 \int_{t_{n-1}}^{t_{n+1}} \| u_{\infty} \|^2 dt. 
\]

Moreover, for \( \theta \in [0, 1] \), if \( u_{\infty} \in L^2(0, T) \), then also

\[
\left\| \frac{\alpha_2 u(t_{n+1}) + \alpha_1 u(t_n) + \alpha_0 u(t_{n-1})}{k_n} - u_{t}(t_{n,\beta}) \right\|_r^2 \leq C(\theta)(k_n + k_{n-1})^3 \int_{t_{n-1}}^{t_{n+1}} \| u_{\infty} \|^2 dt. 
\]
Proof. The proof for smooth functions is simply Taylor expansion with integral reminder after expanding function $u(t_{n+1})$, $u(t_{n-1})$ and $u(t_{n,\beta})$ at $t_n$. For less smooth functions it then follows by a density argument.

Now we introduce the main theorem about error analysis under the following time step condition:

$$\sum_{\ell=0}^{2} \left( \frac{C(\theta)}{\sqrt{v}} (k_{\text{max}}^\ell \|\nabla u_t\|_{4,0}^4 + \|\nabla u\|_{p,\infty,0}^4) + 1 \right) \hat{k}_{n-1+\ell} \leq 1, \quad \text{for all} \quad 2 \leq n \leq M - 2. \quad (11)$$

**Theorem 2** Let $(u(t), p(t))$ be a sufficiently smooth, strong solution of the (NSE). Under the time step condition (11), there exists a constant $C > 0$ such that the solution to the DLN algorithm (8) satisfies the following error estimates

$$\|u - u^h\|_{\infty,0} \leq Ch^{r+1}\|u\|_{\infty,r+1} + F(h,k_{\text{max}}), \quad (12)$$

and

$$\left( \frac{\sum_{n=1}^{M-1} \|\nabla (u(t_n,\beta) - u^h_{n,\beta})\|^2}{\sqrt{v}} \right)^{1/2} \leq C(\theta) \sqrt{h^2 \|\nabla u_t\|_{2,0}} + F(h,k_{\text{max}}), \quad (13)$$

where $k_{\text{max}} = \max \{k_n\}_{n=0}^{M-1}$ and

$$F(h,k_{\text{max}}) = C(\theta) \sqrt{h^{r+1/2}} \|u_t\|_{2,r+1} + C(\theta) \sqrt{h^r} \|u\|_{\beta,2,r+1}$$

$$\begin{align*}
&+ C(\theta) \left( \frac{h^4}{\sqrt{v}} \left[ k_{\text{max}}^2 \|u_t\|_{4,r+1}^2 + \|\nabla u_t\|_{4,0}^2 \right] + \|u_t\|_{\beta,4,r+1}^2 + \|\nabla u\|_{\beta,4,0}^2 \right) \\
&+ C(\theta) \left( \frac{h^2}{\sqrt{v}} \left[ \frac{1}{\sqrt{v}} \|f\|_{\beta,2,\ast} + \frac{1}{\sqrt{v}} \|u^h_t\| + \frac{1}{\sqrt{v}} \|u^h_t\| + k_{\text{max}}^4 \|u_t\|_{4,r+1}^2 + \|u\|_{\beta,4,r+1}^2 \right] \right) \\
&+ C(\theta) \frac{1}{\sqrt{v}} \left( \|u_t\|_{2,0} + \|f_t\|_{2,0} + \frac{1}{\sqrt{v}} \|p_t\|_{2,0} + \sqrt{v} \|\nabla u_t\|_{2,0} \right) \\
&+ \frac{1}{\sqrt{v}} \|\nabla u\|_{\beta,4,0} + \frac{1}{\sqrt{v}} \|\nabla u^h\|_{\beta,4,0} \right). \quad (14)
\end{align*}$$

**Remark 2** Time step restrictions (11) like $\Delta t \leq O(v^{-3})$ arise from the discrete Gronwall inequality in the analysis of fully implicit methods. To bound the error above the discrete Gronwall inequality requires the linearized discrete problem to be positive definite. If a suitable linearly implicit (semi-implicit) method is used instead, the discrete problem automatically satisfies this and no similar time step restriction occurs, see the treatment of the implicit midpoint (Crank–Nicolson) method in Ingram [47] for details. We comment below in Remark 3 on what this linearly implicit realization is for DLN.

**Proof.** For $\theta = 1$, the one-leg 2-step (DLN) method becomes the implicit midpoint rule and the conclusions of the theorem have been proved in many places, hence we will examine the case $\theta \in (0, 1)$. The proof is relatively long, thus we separate the proof in the following steps:
1. Combining NSE at time $t_{n,\beta}$ and the DLN algorithm \((9)\) to derive the equation of pointwise error $e_n := u(t_n) - u_n$ and the truncation error $\tau$ in \((16)\).

2. Decomposing $e_n$ by sum of $\eta_n$ (the difference of $u(t_n)$ and its $L^2$ projection onto $V_h$) and $\phi_n^h$ and then transferring the error equation derived in step 1 into the equation of $\eta_n$ and $\phi_n^h$.

3. Giving bound for $\phi_n^h$ in the error equation obtained in step 2

4. Combining interpolant approximation theorem in \((7)\) and conclusion in step 3 to obtain the convergence of the DLN solution in $L^2$-norm and $H^1$-norm.

\[\text{Step 1} \quad \text{Consider (NSE) at time } t_{n,\beta}(1 \leq n \leq M - 1). \text{ For any } v^h \in V_h, \text{ we have}\]

\[
\frac{1}{\kappa_n} \left( \sum_{\ell=0}^{2} \alpha_{\ell} u(t_{n-1+\epsilon}, v^h) \right) + b \left( \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\epsilon}), \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\epsilon}), v^h \right) + \frac{1}{2} \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\epsilon}), \nabla v^h) - \left( \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} p(t_{n-1+\epsilon}), \nabla \cdot v^h \right) = \frac{1}{\kappa_n} \left( \sum_{\ell=0}^{2} \alpha_{\ell} \eta_{n-1+\epsilon}, \phi_{n,\beta}^h \right) + \frac{1}{2} \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} p(t_{n-1+\epsilon}), \phi_{n,\beta}^h - \tau(u, p, v^h), \quad \forall v^h \in V_h.
\]

\[\text{Step 2} \quad \text{As usual, let } U_n \text{ be } L^2 \text{ projection of } u(t_n) \text{ onto } V_h, \text{ and we decompose } e_n \text{ as }\]

\[e_n = u(t_n) - U_n - (u_n^h - U_n) := \eta_n - \phi_n^h.\]

Setting $v^h = \phi_{n,\beta}^h \equiv \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} \phi_{n-1+\epsilon}^h$, then \((17)\) writes

\[
\frac{1}{\kappa_n} \left( \sum_{\ell=0}^{2} \alpha_{\ell} \phi_{n-1+\epsilon}^h, \phi_{n,\beta}^h \right) + \left( \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\epsilon}), \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\epsilon}), \phi_{n,\beta}^h \right) - b \left( \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\epsilon}), \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\epsilon}), \phi_{n,\beta}^h \right) = e_n^h = u(t_n) - U_n - (u_n^h - U_n) := \eta_n - \phi_n^h.
\]
Using \( (q^h, \nabla \cdot \phi_{n,\beta}^h) = 0 \) for any \( q^h \in Q^h \) and multiplying the above equation by \( \hat{k}_n \), we obtain

\[
\left( \sum_{\ell=0}^{2} \alpha_{\ell} \phi_{n-1+\ell}^h , \phi_{n,\beta}^h \right) + \nu \hat{k}_n \| \nabla \phi_{n,\beta}^h \| \leq \frac{11}{16} \frac{1}{\nu} \hat{k}_n \| \nabla \phi_{n,\beta}^h \|^2 + \left( \sum_{\ell=0}^{2} \alpha_{\ell} \eta_{n-1+\ell} , \phi_{n,\beta}^h \right) - \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\ell}) - \hat{k}_n b(u_{n,\beta}^h, u_{n,\beta}^h) + \hat{k}_n b \left( \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\ell}) , \phi_{n,\beta}^h \right) + \nu \hat{k}_n (\nabla \eta_{n,\beta}, \nabla \phi_{n,\beta}^h) - \hat{k}_n \left( \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} p(t_{n-1+\ell}) - q^h \cdot \nabla \phi_{n,\beta}^h \right) - \hat{k}_n \tau(u, p, \phi_{n,\beta}^h), \quad \forall q^h \in Q^h. \tag{18}
\]

Now we analyze the terms on the right-hand side of (18). By the property of projection operators and the linearity of inner products, we have

\[
\left( \sum_{\ell=0}^{2} \alpha_{\ell} \eta_{n-1+\ell} , \phi_{n,\beta}^h \right) = 0.
\]

Using the skew-symmetry of the trilinear form \( b \), using (6), the Cauchy–Schwarz and Young inequalities, and the \( G \)-stability relation (5), we obtain in a typical manner

\[
\begin{align*}
& \| \phi_{n+1}^h \|_{G(\delta)}^2 - \| \phi_n^h \|_{G(\delta)}^2 + \| \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\ell}) \|^2 + \| \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} \phi_{n-1+\ell}^h \|^2 \\
& \leq C \hat{k}_n \| \eta_{n,\beta} \| \| \nabla \eta_{n,\beta} \| \| \nabla \phi_{n,\beta}^h \|^2 + C \nu \hat{k}_n \| \nabla \phi_{n,\beta}^h \|^2 \\
& + C \frac{\nu}{\hat{k}_n} \left( \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\ell}) \right) - q^h \|^2 + \hat{k}_n \tau(u, p, \phi_{n,\beta}^h). \\
\end{align*}
\]

Summing up from \( n = 1 \) to \( n = M - 1 \), we get

\[
\begin{align*}
& \| \phi_M^h \|_{G(\delta)}^2 - \| \phi_1^h \|_{G(\delta)}^2 + \sum_{n=1}^{M-1} \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} \phi_{n-1+\ell}^h \|^2 + \frac{\nu}{\hat{k}_n} \sum_{n=1}^{M-1} \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} \phi_{n-1+\ell}^h \|^2 \\
& \leq \sum_{n=1}^{M-1} C \frac{\nu}{\hat{k}_n} \| \eta_{n,\beta} \| \| \nabla \eta_{n,\beta} \| \| \nabla \phi_{n,\beta}^h \|^2 + C \frac{\nu}{\hat{k}_n} \left( \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\ell}) \right) - q^h \|^2 + \hat{k}_n \tau(u, p, \phi_{n,\beta}^h). \\
\end{align*}
\]

(19)
We set the approximate solution of \( u \) at two initial time-steps \( t_0 \) and \( t_1 \) to be \( L^2 \) projection of \( u \) into \( V^h \):

\[
\phi^h_i = u^h_i - U_i = 0, \quad i = 0, 1.
\]

Using the definition of the \( G \)-norm (4), the estimate (19) becomes

\[
\begin{align*}
\frac{1}{4} (1 + \theta) \| \phi_{M}^h \|^2 &+ \frac{1}{4} (1 - \theta) \| \phi_{M-1}^h \|^2 \\
&+ \sum_{n=1}^{M-1} \left( \sum_{\ell = 0}^{2} \alpha_{\ell} \| \phi_{n-1+\ell}^h \|^2 + \frac{1}{2} \sum_{n=1}^{M-1} \| \nabla \phi_{n,\rho} \|^2 \right) \\
&\leq \sum_{n=1}^{M-1} C \beta_n \| \eta_{n,\rho} \| \| \nabla \eta_{n,\rho} \| \left( \| \phi_{n,\rho} \|^2 \right) \\
&+ \sum_{n=1}^{M-1} C \beta_n \| u_{n,\rho} \| \| \nabla u_{n,\rho} \| \| \nabla \eta_{n,\rho} \|^2 \\
&+ \sum_{n=1}^{M-1} C \beta_n \| \eta_{n,\rho} \| \| \nabla \eta_{n,\rho} \| \left( \| \phi_{n,\rho} \|^2 \right) \\
&+ \sum_{n=1}^{M-1} C \beta_n \| u_{n,\rho} \| \| \nabla u_{n,\rho} \| \| \nabla \eta_{n,\rho} \|^2 \\
&+ \sum_{n=1}^{M-1} \| \sum_{\ell = 0}^{2} \beta_{\ell} \| p(t_{n-1+\ell}) - q^h \| \|^2 + \sum_{n=1}^{M-1} \beta_n | \tau (u, \rho, \phi_{n,\rho}) |.
\end{align*}
\]

**Step 3**  We now bound each term in the right-hand side.

1. For the first term, we use the linearity of the \( L^2 \) projection, the interpolation error estimates (7) and Young’s inequality to obtain

\[
\begin{align*}
\sum_{n=1}^{M-1} C \beta_n \| \eta_{n,\rho} \| \| \nabla \eta_{n,\rho} \| \left( \| \phi_{n,\rho} \|^2 \right) \\
&\leq C(\theta) k^{2+1} \sum_{n=1}^{M-1} (k_{n-1} + k_n) \left( \| \phi_{n,\rho} \|^2 \right) \\
&\leq C(\theta) k^{2+1} \sum_{n=1}^{M-1} (k_n + k_{n-1}) \left( \| \phi_{n,\rho} \|^2 \right).
\end{align*}
\]

By the triangle inequality

\[
\| \sum_{\ell = 0}^{2} \beta_{\ell} \| u(t_{n-1+\ell}) \|_r^4 \leq C \left( \| \sum_{\ell = 0}^{2} \beta_{\ell} \| u(t_{n-1+\ell}) \|_r^4 \right).
\]

Then by Lemma 2 and Hölder’s inequality

\[
\| \sum_{\ell = 0}^{2} \beta_{\ell} \| u(t_{n-1+\ell}) - u(t_{n,\rho}) \|_r^4 \leq C(\theta)(k_n + k_{n-1})^6 \left( \int_{t_{n-1}}^{t_{n+1}} 1 \cdot \| u_t \|_r^2 dt \right)^2 \\
\leq C(\theta)(k_n + k_{n-1})^7 \int_{t_{n-1}}^{t_{n+1}} \| u_t \|_r^4 dt.
\]

Thus by the definition of the discrete norm (10)

\[
\sum_{n=1}^{M-1} (k_n + k_{n-1}) \| \sum_{\ell = 0}^{2} \beta_{\ell} \| u(t_{n-1+\ell}) \|_r^4 \leq C(\theta) k_n^8 \| u_t \|_r^4 + C \| u \|_r^4.
\]
Similarly
\[
\sum_{n=1}^{M-1} (k_n + k_{n-1})\|\nabla \sum_{\ell=0}^{2} \beta^{(n)}_\ell u(t_{n-1+\ell})\|^2 \leq C(\theta) k_{\text{max}}^8 \|\nabla u_n\|_{4,0}^4 + C \|\nabla u\|_{p,4,0}^4.
\] (24)

Combining (21), (23) and (24), we obtain
\[
\sum_{n=1}^{M-1} \frac{\hat{k}_n}{v} \|\eta_{n,\beta}\| \|\nabla \eta_{n,\beta}\| \|\nabla \sum_{\ell=0}^{2} \beta^{(n)}_\ell u(t_{n-1+\ell})\|^2 \leq C(\theta) \frac{h^{2\gamma+1}}{v} \{ k_{\text{max}}^8 (\|u_n\|_{4,r+1}^4 + \|\nabla u_n\|_{4,0}^4) + \|u\|_{\beta,4,r+1}^4 + \|\nabla u\|_{\beta,4,0}^4 \}. 
\] (25)

2. For the second term we use the linearity of the projection, the interpolation error estimates (7), Lemma 2, and the definition of the discrete norm (10)
\[
Cv \sum_{n=1}^{M-1} \frac{\hat{k}_n}{v} \|\eta_{n,\beta}\|^2 \leq vCh^2r \sum_{n=1}^{M-1} \frac{\hat{k}_n}{v} \|\sum_{\ell=0}^{2} \beta^{(n)}_\ell u(t_{n-1+\ell})\|^2 \leq vC\theta h^2r \sum_{n=1}^{M-1} (k_{n-1} + k_n) \left( \|\sum_{\ell=0}^{2} \beta^{(n)}_\ell u(t_{n-1+\ell}) - u(t_{n,\beta})\|_{r+1}^2 + \|u(t_{n,\beta})\|_{r+1}^2 \right) 
\leq C(\theta) vh^2r \sum_{n=1}^{M-1} (k_{n-1} + k_n) \int_{t_{n-1+1}}^t \|u_n\|_{r+1}^2 dt + C(\theta) vh^2r \sum_{n=1}^{M-1} (k_{n-1} + k_n) \|u(t_{n,\beta})\|_{r+1}^2 
\leq C(\theta) vh^2r k_{\text{max}}^4 \|u_n\|_{2,r+1}^2 + C(\theta) vh^2r |\|u\|_{\beta,2,r+1}^2 |. (26)
\]

3. Using the a priori bounds from Theorem 1, we have
\[
\sum_{n=1}^{M-1} \frac{\hat{k}_n}{v} \|u_{n,\beta}\| \|\nabla u_{n,\beta}\| \|\nabla \eta_{n,\beta}\|^2 \leq \frac{Ch^2r}{v} \sum_{n=1}^{M-1} \frac{\hat{k}_n}{v} \|\nabla u_{n,\beta}\| \|\sum_{\ell=0}^{2} \beta^{(n)}_\ell u(t_{n-1+\ell})\|^2 \leq \frac{Ch^2r}{v} \sum_{n=1}^{M-1} \frac{\hat{k}_n}{v} \left( \|\nabla u_{n,\beta}\|^2 + \|\sum_{\ell=0}^{2} \beta^{(n)}_\ell u(t_{n-1+\ell})\|^4 \right) 
\leq \frac{C(\theta) h^2r}{v^2} \left( \|f\|_{2,\alpha}^2 + \|u_{h}\|^2 + \frac{1}{v} \|u_{h}\|^2 + \frac{1}{v} \|u_{h}\|^2 + C(\theta) k_{\text{max}}^8 \|u_n\|_{4,r+1}^4 + C \|u\|_{\beta,4,r+1}^4 \right). (27)
\]

4. We bound We bound the fourth term in a similar way to the derivation of (25)
\[
\sum_{n=1}^{M-1} \frac{\hat{k}_n}{v} \|\phi_{n,\beta}\| \|\nabla \sum_{\ell=0}^{2} \beta^{(n)}_\ell u(t_{n-1+\ell})\|^4 \leq \sum_{n=1}^{M-1} C(\theta) \frac{\hat{k}_n}{v^2} \|\phi_{n,\beta}\|^2 \left( (k_{n-1} + k_n) \int_{t_{n-1+1}}^t \|\nabla u_{n,\beta}\|^4 dt + \|\nabla u(t_{n,\beta})\|^4 \right) 
\leq \sum_{n=1}^{M-1} C(\theta) \frac{\hat{k}_n}{v^2} \|\phi_{n,\beta}\|^2 \left( k_{\text{max}}^8 \|\nabla u_{n,\beta}\|_{4,0}^4 + \|\nabla u\|_{\beta,\infty,0}^4 \right). (28)
\]

5. Using the triangle inequality again
\[
\|\sum_{\ell=0}^{2} \beta^{(n)}_\ell p(t_{n-1+\ell}) - q^h\|^2 \leq C \left( \|\sum_{\ell=0}^{2} \beta^{(n)}_\ell p(t_{n-1+\ell}) - p(t_{n,\beta})\|^2 + \|p(t_{n,\beta}) - q^h\|^2 \right).
\]
Using the interpolation error estimate in (7) and the consistency errors Lemma 2 for pressure $p$, we have

$$\sum_{n=1}^{M-1} \frac{\hat{k}_n}{v} \left\| \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} p(t_{n-1+\ell}) - q^h \right\|^2 \leq \frac{C(\theta)}{v} (k_{\text{max}}^4 \| p_{\text{ref}} \|_{2,0}^3 + h^{2+\ell} \| p \|_{2,\ell+1}^2).$$

(29)

6. Let us now treat the truncation error $|\tau(u, p, \phi_{n, \beta}^h)|$. Using the Cauchy–Schwarz inequality, we have

$$\left( \frac{1}{k_{n, \beta}} \sum_{\ell=0}^{2} \alpha_{\ell} u(t_{n-1+\ell}) - u_{t}(t_{n, \beta}), \phi_{n, \beta}^h \right) \leq \frac{1}{2} \| \phi_{n, \beta}^h \|^2 + \frac{1}{2} \left\| \frac{1}{k_{n, \beta}} \sum_{\ell=0}^{2} \alpha_{\ell} u(t_{n-1+\ell}) - u_{t}(t_{n, \beta}) \right\|^2,$$

and applying again Lemma 2, for $\theta \in [0, 1)$ to the last term above

$$\sum_{n=1}^{M-1} \frac{\hat{k}_n}{\theta} \left\| \frac{2}{k_{n, \beta}} \sum_{\ell=0}^{2} \alpha_{\ell} u(t_{n-1+\ell}) - u_{t}(t_{n, \beta}) \right\|^2 \leq C(\theta) k_{\text{max}}^4 \| u_{\text{ref}} \|_{2,0}^2.$$

Thus we have

$$\sum_{n=1}^{M-1} \frac{\hat{k}_n}{\theta} \left( \frac{2}{k_{n, \beta}} \sum_{\ell=0}^{2} \alpha_{\ell} u(t_{n-1+\ell}) - u_{t}(t_{n, \beta}), \phi_{n, \beta}^h \right) \leq \frac{1}{2} \sum_{n=1}^{M-1} \frac{\hat{k}_n}{\theta} \| \phi_{n, \beta}^h \|^2 + C(\theta) k_{\text{max}}^4 \| u_{\text{ref}} \|^2_{2,0}.$$

Similarly,

$$\sum_{n=1}^{M-1} \frac{\hat{k}_n}{\theta} \left( f(t_{n, \beta}) - \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} f(t_{n-1+\ell}), \phi_{n, \beta}^h \right) \leq \frac{1}{2} \sum_{n=1}^{M-1} \frac{\hat{k}_n}{\theta} \| \phi_{n, \beta}^h \|^2 + C(\theta) k_{\text{max}}^4 \| f_{\text{ref}} \|^2_{2,0},$$

and also

$$\sum_{n=1}^{M-1} \frac{\hat{k}_n}{\theta} \left( \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} p(t_{n-1+\ell}) - p(t_{n, \beta}), \nabla \cdot \phi_{n, \beta}^h \right) \leq \frac{\epsilon v}{\theta} \sum_{n=1}^{M-1} \frac{\hat{k}_n}{\theta} \| \nabla \phi_{n, \beta}^h \|^2 + \frac{C(\epsilon, \theta)}{v} k_{\text{max}}^4 \| p_{\text{ref}} \|^2_{2,0},$$

$$\sum_{n=1}^{M-1} \frac{\hat{k}_n}{\theta} \left( \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\ell}) - u(t_{n, \beta}), \nabla \phi_{n, \beta}^h \right) \leq \frac{\epsilon v}{\theta} \sum_{n=1}^{M-1} \frac{\hat{k}_n}{\theta} \| \nabla \phi_{n, \beta}^h \|^2 + C(\epsilon, \theta) v k_{\text{max}}^4 \| \nabla u_{\text{ref}} \|^2_{2,0}.$$

Moreover

$$b \left( \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\ell}), \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\ell}), \phi_{n, \beta}^h \right) - b(u_{t}(t_{n, \beta}), u(t_{n, \beta}), \phi_{n, \beta}^h)$$

$$= b \left( \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\ell}) - u_{t}(t_{n, \beta}), \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\ell}), \phi_{n, \beta}^h \right)$$

$$+ b \left( u(t_{n, \beta}), \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\ell}) - u_{t}(t_{n, \beta}), \phi_{n, \beta}^h \right)$$

$$\leq C \left\| \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\ell}) - u(t_{n, \beta}) \right\| \| \nabla \phi_{n, \beta}^h \| + \left\| \sum_{\ell=0}^{2} \beta_{\ell}^{(n)} u(t_{n-1+\ell}) \right\| + \| \nabla u(t_{n, \beta}) \|.$$
\[
\leq \epsilon v \| \nabla \phi^h_{n,\beta} \|^2 + \frac{C(\epsilon)}{v} \left\| \nabla \left( \sum_{\ell=0}^2 \beta^{(n)}_{\ell} u(t_{n-1+\ell}) - u(t_{n,\beta}) \right) \right\|^2
\times \left( \sum_{\ell=0}^2 \beta^{(n)}_{\ell} u(t_{n-1+\ell}) \right)^2 + \| \nabla u(t_{n,\beta}) \|^2 \right).
\]

Then by triangle inequality again
\[
\left\| \nabla \left( \sum_{\ell=0}^2 \beta^{(n)}_{\ell} u(t_{n-1+\ell}) - u(t_{n,\beta}) \right) \right\|^2 \leq C \left( \sum_{\ell=0}^2 \beta^{(n)}_{\ell} u(t_{n-1+\ell}) - u(t_{n,\beta}) \right) + C \left\| \nabla \left( \sum_{\ell=0}^2 \beta^{(n)}_{\ell} u(t_{n-1+\ell}) - u(t_{n,\beta}) \right) \right\|^2 \| \nabla u(t_{n,\beta}) \|^2.
\]

Similar to (22) we have that
\[
\left\| \nabla \left( \sum_{\ell=0}^2 \beta^{(n)}_{\ell} u(t_{n-1+\ell}) - u(t_{n,\beta}) \right) \right\|^4 \leq C(\theta) (k_n + k_{n-1}) \int_{t_n}^{t_{n+1}} \| \nabla u_t \|^4 dt,
\]
and also using Lemma 2 and Young's inequality we obtain
\[
\left\| \nabla \left( \sum_{\ell=0}^2 \beta^{(n)}_{\ell} u(t_{n-1+\ell}) - u(t_{n,\beta}) \right) \right\|^2 \leq C(\theta) (k_n + k_{n-1}) \int_{t_n}^{t_{n+1}} \| \nabla u(t_{n,\beta}) \|^2 \| \nabla u_t \|^2 dt
\]
\[
\leq C(\theta) (k_n + k_{n-1}) \int_{t_n}^{t_{n+1}} \| \nabla u(t_{n,\beta}) \|^4 + \| \nabla u_t \|^4 dt \leq C(\theta) (k_n + k_{n-1}) \| \nabla u(t_{n,\beta}) \|^4
\]
\[
+ C(\theta) (k_n + k_{n-1}) \int_{t_n}^{t_{n+1}} \| \nabla u_t \|^4 dt.
\]

Thus
\[
\sum_{n=1}^{M-1} \hat{k}_n \left[ \left( \sum_{\ell=0}^2 \beta^{(n)}_{\ell} u(t_{n-1+\ell}) - u(t_{n,\beta}) \right) - b(u(t_{n,\beta}), u(t_{n,\beta}), \phi^h_{n,\beta}) \right]
\leq \epsilon v \sum_{n=1}^{M-1} \hat{k}_n \| \nabla \phi^h_{n,\beta} \|^2 + \frac{C(\epsilon, \theta)}{v} k_\beta^8 \| \nabla u_t \|_{L^4}^4 + \frac{C(\epsilon, \theta)}{v} \| \nabla u_t \|_{L^4}^4 \| \nabla u_t \|_{L^4}^4.
\]

Setting \( \epsilon = 1/16 \), we obtain the following estimate for the truncation error term
\[
\sum_{n=1}^{M-1} \hat{k}_n \| \tau(u, p, \phi^h_{n,\beta}) \| \leq \sum_{n=1}^{M-1} \hat{k}_n \| \nabla \phi^h_{n,\beta} \|^2 + \frac{3v}{16} \sum_{n=1}^{M-1} \hat{k}_n \| \nabla \phi^h_{n,\beta} \|^2 + \frac{C(\theta)}{v} k_\max^8 \| \nabla u_t \|_{L^4}^4
\]
\[
+ C(\theta) k_\max^4 \left\{ \| u_t \|_{L^4}^4 + \| f_t \|_{L^4}^4 + \frac{1}{v} \| p_t \|_{L^4}^4 + \frac{1}{v} \| \nabla u_t \|_{L^4}^4 + \frac{1}{v} \| \nabla u \|_{L^4}^4 + \frac{1}{v} \| \nabla u_t \|_{L^4}^4 + \| \nabla u \|_{L^4}^4 + \| \nabla u_t \|_{L^4}^4 \right\}.
\]

Now collecting the terms from (25), (26), (27), (28), (29) and (30), the inequality (20) becomes
\[
\leq \sum_{n=1}^{M-1} \left( \frac{C}{v^3} (k_n^n \| \nabla u_t \|_{L^4}^4 + \| \nabla u \|_{L^4}^4, \phi^h_{n,\beta}) + 1 \right) \hat{k}_n \| \nabla \phi^h_{n,\beta} \|^2 + \hat{\Phi}_n (h, \hat{k}_n).
\]
where

\[ \tilde{F}(h, k_{\text{max}}) = C(\theta) \nu h^{2r} k_{\text{max}}^4 ||u_{\text{tr}}||^2_{2,r+1} + C(\theta) \nu h^{2r} ||u||^2_{2,2,r+1} + \frac{C(\theta)}{v} k_{\text{max}}^h ||\nabla u_{\text{tr}}||^4_{2,0} \]

\[ + C(\theta) \frac{h^{2r+1}}{v} \left( k_{\text{max}}^h ||u_{\text{tr}}||^4_{2,r+1} + ||\nabla u_{\text{tr}}||^4_{2,0} \right) \]

Step 4 Under the time step condition (11), using the discrete Gronwall inequality (see e.g., Reference [25]), the inequality (31) yields

\[ \|\phi_{n}^M\|^2 + v \sum_{n=1}^{M-1} k_n^h ||\nabla \phi_{n,h}||^2 \leq C(\theta) \tilde{F}(h, k_{\text{max}}). \]
TABLE 4  The errors and convergence order of the DLN scheme at time $T = 1$ for the velocity and pressure of $L^\infty$-norm with $\theta = 0.5$

| $h = \Delta t$ | $\|e_u\|_\infty$ | $R$ | $\|\nabla e_u\|_\infty$ | $R$ | $\|e_p\|_\infty$ | $R$ |
|----------------|----------------|-----|----------------|-----|----------------|-----|
| $\frac{1}{16}$ | 0.00110053     | -   | 0.0898315      | -   | 0.00236018    | -   |
| $\frac{1}{32}$ | 0.000147375    | 2.90| 0.0241532      | 1.89| 0.000595252   | 1.99|
| $\frac{1}{64}$ | 2.3207e-05     | 2.67| 0.001716777    | 1.75| 0.000153932   | 1.95|
| $\frac{1}{128}$| 2.91792e-06    | 2.99| 0.00178222     | 2.01| 3.79152e-05   | 2.02|
| $\frac{1}{256}$| 3.34876e-07    | 3.12| 0.000409336    | 2.12| 9.32863e-06   | 2.02|

TABLE 5  The errors and convergence order of the DLN scheme at time $T = 1$ for the velocity and pressure of $L^2$-norm with $\theta = 0.7$

| $h = \Delta t$ | $\|e_u\|_{2,0}$ | $R$ | $\|\nabla e_u\|_{2,0}$ | $R$ | $\|e_p\|_{2,0}$ | $R$ |
|----------------|----------------|-----|----------------|-----|----------------|-----|
| $\frac{1}{16}$ | 0.000689478    | -   | 0.0560293      | -   | 0.00127634    | -   |
| $\frac{1}{32}$ | 8.45301e-05    | 3.03| 0.0133912      | 2.06| 0.000296992   | 2.10|
| $\frac{1}{64}$ | 1.22087e-05    | 2.79| 0.00371588     | 1.85| 7.46964e-05   | 1.99|
| $\frac{1}{128}$| 1.45411e-06    | 3.07| 0.000883034    | 2.07| 1.79769e-05   | 2.05|
| $\frac{1}{256}$| 1.62107e-07    | 3.17| 0.000197972    | 2.16| 4.32679e-06   | 2.05|

Finally, combining (7) and (34) we obtain (12)

$$
\| u - u^h \|_{\infty,0} := \max_{0 \leq n \leq M} \| u(t_n) - u_n^h \| \leq \max_{0 \leq n \leq M} \| \eta_n \| + \max_{0 \leq n \leq M} \| \phi_n^h \|
$$

$$
\leq \max_{0 \leq n \leq M} Ch^{r+1} \| u_n \|_{r+1} + C(\theta) \sqrt{F(h, k_{max})} \leq Ch^{r+1} \| u \|_{\infty, r+1} + F(h, k_{max}),
$$

concluding the proof of the first part of the theorem.

For the second part, in order to prove (13), we begin by noticing that

$$
\sum_{n=1}^{M-1} \hat{k}_n \| \nabla (u(t_n, \beta) - u_n^h) \|^2 \leq \sum_{n=1}^{M-1} \hat{k}_n \| \nabla \left( u(t_n, \beta) - \sum_{\ell=0}^{2} \rho_{\ell}^{(n)} u(t_{n-1+\ell}) \right) \|^2
$$

$$
+ \sum_{n=1}^{M-1} \hat{k}_n \| \nabla \left( u_n^h - \sum_{\ell=0}^{2} \rho_{\ell}^{(n)} u(t_{n-1+\ell}) \right) \|^2.
$$

We then apply Lemma 2 to the first term in the right-hand side

$$
\nu \sum_{n=1}^{M-1} \hat{k}_n \| \nabla \left( u(t_n, \beta) - \sum_{\ell=0}^{2} \rho_{\ell}^{(n)} u(t_{n-1+\ell}) \right) \|^2 \leq C(\theta) \nu k_{max}^4 \| \nabla u_n \|_{2,0}^2,
$$

and use the triangle inequality, (26) and (32) for the second term

$$
\nu \sum_{n=1}^{M-1} \hat{k}_n \| \nabla \left( u_n^h - \sum_{\ell=0}^{2} \rho_{\ell}^{(n)} u(t_{n-1+\ell}) \right) \|^2
$$

$$
\leq C \nu \sum_{n=1}^{M-1} \hat{k}_n \left( \| \nabla \eta_n, \beta \|^2 + \| \nabla \phi_n, \beta \|^2 \right) \leq C(\theta) \tilde{F}(h, k_{max}),
$$

to obtain (13) and complete the proof.
Convergence test (constant time step size)

\[
\frac{b(u_n^h, u_{n-1,\theta}^h, v) - b(\tilde{u}_n^h, u_{n-1,\theta}^h, v)}{b(\tilde{u}_n^h, u_{n-1,\theta}^h, v)} \leq K_1(\theta) \quad \text{if } n \geq 3
\]

\[
\tilde{u}_n^h = \begin{cases} 
\left(1 + \frac{t_{n,\theta} - t_{n-1,\theta}}{t_{n-1,\theta} - t_{n-2,\theta}}\right) u_{n-1,\theta}^h - \frac{t_{n,\theta} - t_{n-1,\theta}}{t_{n-1,\theta} - t_{n-2,\theta}} u_{n-2,\theta}^h & \text{if } n \geq 3 \\
\beta_2^{(n)} \left(1 + \frac{k_2}{k_{n-1}}\right) + \beta_1^{(n)} u_n + \left(\beta_0^{(n)} - \beta_2^{(n)} \frac{k_2}{k_{n-1}}\right) u_{n-1} & \text{if } n = 1, 2
\end{cases}
\]

Then we replace the non-linear term \(b(u_n^h, u_{n,\theta}^h, v)\) by \(b(\tilde{u}_n^h, u_{n-1,\theta}^h, v)\) in the DLN algorithm (8) and (9). One issue behind the linear DLN algorithm is that we need \((t_{n,\theta} - t_{n-1,\theta}) > 0\) for all \(n\). For \(\theta = 0, 1\), this condition always holds and it is easy to check that for \(\theta \in (0, 1)\), there exist upper bound \(K_1(\theta) \geq 3\) and lower bound \(0 < K_2(\theta) \leq (1/2)\) for ratio of steps \(k_n/k_{n-1}\) such that \((t_{n,\theta} - t_{n-1,\theta}) > 0\) for all \(n\). Under this simple step restriction, we have stability and second order accuracy (in time) of the solutions. Moreover for constant time-stepping case, \((t_{n,\theta} - t_{n-1,\theta}) > 0\) for all \(\theta \in [0, 1]\) and thus no time step restriction is needed for stability and error analysis.

3 | NUMERICAL TESTS

For numerical tests we use FreeFem++ and Taylor–Hood (P2 – P1) finite elements. We verify the second-order convergence and stability of the DLN algorithm with variable time steps through three numerical experiments.

3.1 | Convergence test (constant time step size)

The second order convergence of DLN algorithm is verified on the Taylor–Green benchmark problem, see for example, [42]. In the domain \(\Omega = (0, 1) \times (0, 1)\), the true solution is

\[
u_1(x, y, t) = -\cos(w \pi x) \sin(w \pi y) \exp(-2w^2 \pi^2 t/\tau),
\]

\[
u_2(x, y, t) = \sin(w \pi x) \cos(w \pi y) \exp(-2w^2 \pi^2 t/\tau),
\]

\[
p(x, y, t) = -\frac{1}{4} (\cos(2w \pi x) + \cos(2w \pi y)) \exp(-4w^2 \pi^2 t/\tau),
\]

and we take the final time \(T = 1\), \(w = 1\) and \(\tau = Re = 100\). The body force \(f\), initial condition, and boundary condition are determined by the true solution. Setting \(\Delta t = h\) to calculate the convergence order \(R\) by the error \(e\) at two successive values of \(\Delta t\) via

\[
R = \ln(e(\Delta t_1)/e(\Delta t_2))/ \ln(\Delta t_1/\Delta t_2).
\]
Tables 1, 2, Tables 3, 4 and Tables 5, 6 correspond to $\theta = 0.2, 0.5, 0.7$, respectively. The results illustrate that the DLN algorithm has second-order convergence for both velocity and pressure. In the tests, the convergence of velocity is better. It is quite common to observe higher rate of convergence for velocity than for velocity gradient. However proving this result requires a fairly long additional duality argument and its details are open.

### 3.2 2D offset circles problem (with preset variable time step size)

This is a test problem from Jiang [26] that is inspired by flow between offset cylinders. The domain is a disk with a smaller off center obstacle inside. Let $\Omega_1 = \{(x, y) : x^2 + y^2 \leq 1\}$ and $\Omega_2 =$
with no-slip boundary conditions imposed on both circles. The body force $f = 0$ on the outer circle. The flow rotates about $(0,0)$ and the inner circle induces a von Kármán vortex street which re-interacts with the immersed circle creating more complex structures. Figures 2 and 3 show this situation.

For this test, we set $Re = 200$, the number of mesh points around the inner circle and the mesh points around the outer circle to be 10 and 40 respectively. The parameter $\theta = 0.5$, and the number of computations is $n = 1000$. We test the stability of different methods by varying the time step using the
FIGURE 4  Energy, $\|u\|$ and $\|\nabla u\|$ of DLN, BDF2, BDF3 and BDF4 with variable time step size

function defined in Chen, Layton and McLaughlin [11]:

$$k_n = \begin{cases} 
0.05 & 0 \leq n \leq 10, \\
0.05 + 0.002 \sin(10t_n) & n > 10.
\end{cases}$$

For comparison, we solve this problem with the variable step DLN, BDF2, BDF3 and BDF4 time discretizations, and compute the $(1/2)\|u\|^2$ (energy), $\|u\|$ and $\|\nabla u\|$. Let the number of mesh points on boundary of outside circle and inner circle be 160 and 40, respectively, and time step be $k_0 = 0.05$ and $k_n = k_{n-1} + 0.001$. We stop the simulation when the time step size reaches 0.5, since a greater time step size yields inaccurate solutions. In Figure 4b,c, the energy and $\|u\|$ by BDF2 have the trend to increase with increasing time step, while the energy and $\|u\|$ by DLN remain at low level. The BDF3 and BDF4 (more accurate in time) have the energy and $\|u\|$ values between those of BDF2 and DLN. For the $H^1$-norm, Figure 4d shows that the values of $\|\nabla u\|$ by the four algorithms are at the same level, while DLN and BDF4 have larger oscillations than BDF2 and BDF3, as the time step size increases. In conclusion, this test shows that the DLN algorithm has greater stability for the energy than BDF2.

3.3  Adapting the time step

Finally, we use this example to perform a simple adaptivity experiment. For this test, we adapt the time step using the minimum dissipation criterion of Capuano, Sanderse, De Angelis and Coppola
Our goal is to test if adapting the time step produces a significant difference in the solution. Other criteria/estimators are under study. Their idea is to adapt the time step to keep the numerical dissipation, $\epsilon_{DLN}$ from the dominating physical dissipation, $\epsilon^v$. Thus we adapt for

$$\chi = \left| \frac{\epsilon_{DLN}}{\epsilon^v} \right| < \delta.$$ 

Here $\epsilon_{DLN}$ is the numerical dissipation and $\epsilon^v$ is the viscous dissipation. These are given by:

$$\epsilon_{DLN} = \left\| \sum_{\ell=0}^{2} \alpha^0_{\ell} u_{n-1+\ell} \right\|^2 \sqrt{k_n}, \quad \epsilon^v = v \| \nabla u_{n,\beta} \|^2.$$ 

In the test, we set the tolerance for the dissipation ratio $\delta$ to be 0.002. The time step size is then adapted by the simplest strategy of halving or doubling according to

$$\Delta t^{n+1} = \min\{2 \ast \Delta t^n, \ 0.5\}; \quad \text{if} \quad \chi < \delta, \quad \Delta t^n = \max\{0.5 \ast \Delta t^n, \ 0.01\}; \quad \text{if} \quad \chi \geq \delta.$$ 

We adapted the next time step when the dissipation ratio was out of range. Naturally, other strategies for varying $\Delta t$ could be tested, such as formula (16) of Capuano, Sanderse, De Angelis and Coppola [10, p. 2317] (which is $\Delta t^{n+1} = \Delta t^n |\delta/\chi|^{1/2}$). We select the final time $T = 60$, minimal time step size and maximal time step size to be 0.01 and 0.5, respectively. The adaptive algorithm completed in 5687 steps. Figures 5 and 6 are line diagrams of time step size $k_n$, energy $(1/2)\|u\|^2$, numerical dissipation $\sqrt{\epsilon_{DLN}}$ and ratio $\chi$ changing with time $t$, respectively.

Then we select the same final time $T = 60$, the same calculated steps 5687 and use the constant time step $k = T/5687$ to calculate to obtain the line diagram of energy $(1/2)\|u\|^2$, numerical dissipation $\sqrt{\epsilon_{DLN}}$ and ratio $\chi$ changing with time $t$, see Figures 7 and 8.

We now compare the constant time step size results in Figures 7 and 8 with the adaptive results in Figures 5 and 6. Time step size under adaptivity reaches maximum value 0.5 in a few steps then goes down sharply to the minimum step size 0.01 thereafter. In the test represented in Figure 5a, the time step alternates between the minimum step size and twice minimum. This is due to the preset algorithmic choice. DLN under constant step size takes 777 time steps to reach a kinetic energy of approximately 23, a level which adaptive DLN algorithm reaches in 544 time steps. In comparison of numerical dissipation, Figure 6b and 8b show the numerical dissipation with adaptive time step size.
FIGURE 6  The energy \((1/2)||u||^2\) and numerical dissipation \(\sqrt{\varepsilon_{DLN}}\) changing with adaptive time step size

FIGURE 7  The time step size \(k\) and ratio \(\chi\) changing with constant time step size

FIGURE 8  The energy \((1/2)||u||^2\) and numerical dissipation \(\sqrt{\varepsilon_{DLN}}\) changing with constant time step size
evolves smoothly with a peak value below 0.35. Similarly the ratio $\chi$ has an order of magnitude smaller for adaptive time step size, Figure 5b, than constant time step size, Figure 8b. Comparing Figures 6 and 8, we can see that energy patterns are similar for both variable and constant step algorithms but numerical dissipation by the variable step DLN algorithm is much smaller than that of constant case.

4 | CONCLUSIONS

Based on the theory and the simple numerical tests for time discretization of flow problems the 2-step DLN method is to be preferred over the common BDF2 method. It is second order, unconditionally, long time, nonlinearly stable. For increasing step sizes, BDF2 injects nonphysical kinetic energy in the discrete solution (disrupting long time behavior and statistical equilibrium) while DLN does not. Important open questions include how to select the DLN parameter $\theta$. At this point we have no systematic (either universal or application specific) method to choose an optimal DLN parameter $\theta$ balancing stability and accuracy. How to perform error estimation in a memory and computationally efficient (and effective) way is also an important open problem. In particular, finding a memory efficient estimator, as was done in Gresho, Sani and Engelman [20] for the trapezoid rule, is a necessary step. It would be useful if the DLN method could be embedded in a family of different orders with good properties or if it could be induced from simpler methods by added time filters. Both are open problems.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

ORCID

William Layton https://orcid.org/0000-0002-2539-4478
Catalin Trenchea https://orcid.org/0000-0002-8135-5836

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