Background field method in the Wilson formulation

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Abstract

A cutoff regularization for a pure Yang-Mills theory is implemented within the background field method keeping explicit the gauge invariance of the effective action. The method has been applied to compute the beta function at one loop order.

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1 Introduction

The Wilson, or exact, renormalization group (RG) approach regards an interacting field theory as an effective theory, i.e. the higher modes of the fields, with respect to some scale \( \Lambda \), generate effective interactions for the lower modes \([1]\). This division of momenta conflicts with gauge invariance as it is easy to see in the case of an homogeneous gauge transformation acting on some matter field \( \phi(x) \):

\[
\phi(x) \rightarrow \Omega(x)\phi(x).
\]  

In the momentum space the gauge transformed field is mapped into a convolution with the gauge transformation and then any division of momenta is lost. From the functional point of view this conflict appears as a breaking term in the Slavnov-Taylor (ST) identities for the Wilsonian effective action \([2, 3]\), which originates from the introduction of a cutoff function into the propagators. Although one can rewrite this as modified Slavnov-Taylor \([4, 5]\), it is necessary to prove that the physical effective action satisfies the ST identities. As shown in refs. \([2, 6]\) this can be achieved, in perturbation theory, by properly fixing the boundary conditions for the non-invariant couplings in the Wilsonian effective action at the ultraviolet (UV) scale \( \Lambda_0 \). This is the so-called fine tuning procedure and it is equivalent to solve the modified Slavnov-Taylor identities at the scale \( \Lambda_0 \).

It has been proved \([7]\) for a pure Yang Mills theory that by using a covariant gauge fixing function depending on some classical external field, named the background field, it is possible to define a gauge invariant effective action. In \([7]\) explicit calculations at two loops order have been performed using the dimensional regularization. A definite proof that the S-matrix elements can be obtained from the background gauge invariant effective action has been recently established \([8]\).

These results have led to apply this background field method to the RG formulation, with the aim of keeping the gauge invariance explicit \([9]\). To implement this requirement one allows the cutoff function to depend on the background field in a covariant way. However, it is easy to realize, by simple one loop computations, that the cutoff vertices are not well regularized. A possible way out consists in adding a mass term for the quantum gauge fields and for the ghost \([10]\). In this way the BRS invariance is lost and the need of restoring the ST identities requires a fine tuning procedure. Therefore the advantages coming from the background gauge method are partially lost when the cutoff regularization is used.

The aim of this paper is to show, by an explicit example, how the cutoff regularization can be successfully implemented within the background field method. In section 2 we give the notation. In section 3 we work out the cutoff regularization preserving the explicit gauge invariance and we determine the Feynman rules needed to compute the one-loop two point background amplitude. In section 4 we show the transversality of this amplitude and we compute the beta function at one loop order. The detailed calculation of the background field wave function renormalization is given in the Appendix. Section 5 contains some remarks and final comments.
2 The Background Field

The gauge-fixed classical action for a pure Yang-Mills theory is given by

\[ S = S_{cl} + S_{gf} + S_{FP}, \] (2)

where \( S_{cl} = -\frac{1}{4} \int d^4x F_{\mu\nu}^a F_{\mu\nu}^a \) is the classical action, \( F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c \) is the field strength tensor and \( f^{abc} \) are the structure constants of the group. As it has been proposed in \[7\], one can choose the gauge fixing term depending on a non-dynamical field

\[ S_{gf} = -\frac{1}{2} \int d^4x (\overline{D}_{ab}^\mu (A_\mu^b - \overline{A}_\mu^b))^2, \] (3)

where \( \overline{A}_\mu^a \) is the background field and the covariant derivative is

\[ \overline{D}_{ab}^\mu = \partial_\mu \delta_{ab} + gf^{acb} \overline{A}_\mu^c. \] (4)

As a consequence of (3) the ghost term also depends on the background field and reads

\[ S_{FP} = -\int d^4x \overline{c}_a \overline{D}_{ab}^\mu D_{\mu}^{bd} c_d, \] (5)

with \( D_{ab}^\mu = \partial_\mu \delta_{ab} + gf^{abc} A_\mu^c \). The particular choice of the gauge fixing term makes the action (2) invariant under the following background gauge transformation:

\[ \delta A_\mu^a = D_{ab}_\mu \lambda^b, \quad \delta \overline{A}_\mu^a = \overline{D}_{ab}^\mu \lambda^a, \] (6)

\[ \delta c^a = gf^{abc} \overline{c}_b \lambda^c, \quad \delta \overline{c}_a = \eta \overline{D}_{ab}^\mu (A_\mu - \overline{A}_\mu) \] (7)

where \( \lambda = \lambda(x) \) is the infinitesimal gauge parameter. The invariance of the classical action under the gauge transformation of the field \( A_\mu \) is implemented at the quantum level by the BRS symmetry:

\[ \delta A_\mu^a = \eta D_{ab}^\mu c_b, \quad \delta c^a = -\frac{1}{2} \eta g f^{abc} \overline{c}_b c^c, \quad \delta \overline{c}_a = \eta \overline{D}_{ab}^\mu (A_\mu - \overline{A}_\mu) \] (8)

where \( \eta \) is a Grassmann parameter. Adding to the action (2) the source term associated to the BRS variations (8) of \( A_\mu \) and \( c \) one has

\[ S_{BRS}[\Phi, \gamma] = S + \int d^4x (u_\mu^a D_{\mu}^{ab} c_b - \frac{q}{2} f^{abc} v_\mu^a c^b c^c), \] (9)

where we have denoted by \( \Phi = (A_\mu^a, c^a, \overline{c}_a) \) and \( \gamma = (u_\mu^a, v_\mu^a) \) the fields and the BRS sources.

In the conventional functional approach one defines the generating functional

\[ Z[J, \gamma] = e^{iW} = \int D\Phi \ e^{iS_{BRS}[\Phi, \gamma] + i(J, \Phi)}, \] (10)

where \( J = (j_\mu^a, \eta^a, -\eta^a) \) are the field sources and we introduced the notation

\[ (J, \Phi) = \int d^4x (j_\mu^a A_\mu^a + \eta^a \overline{c}_a + \overline{c}_a \eta^a). \] (11)

\(^1\text{We set the gauge fixing parameter } \alpha = 1 \text{ corresponding to the Feynman gauge.}\)
The symmetry of the BRS action with respect to the background gauge transformation (8) translates for the effective action

$$\Gamma[\Phi_{cl}, \gamma] = W[J, \gamma] - (J\Phi_{cl}), \quad \Phi_{cl} = \frac{\delta W}{\delta J},$$  

(12)

into

$$G^a \Gamma[\Phi_{cl}, \gamma] + \bar{G}^a \Gamma[\Phi_{cl}, \gamma] = 0,$$

(13)

where

$$G^a = D^{ab}_{\mu} \frac{\delta}{\delta A^{b}_{\mu, cl}} + g f^{abc} c^b_{cl} \frac{\delta}{\delta c^a_{cl}} + g f^{abc} \bar{c}^b_{cl} \frac{\delta}{\delta \bar{c}^a_{cl}},$$

(14)

and

$$\bar{G}^a = \bar{D}^{ab}_{\mu} \frac{\delta}{\delta \bar{A}^{b}_{\mu, cl}}.$$

(15)

Similarly the BRS transformation translates into the ST identities. The Ward identity (13) expresses the gauge invariance of the effective action $\Gamma[\Phi_{cl}, \gamma]$ for $\bar{A}^a_{\mu} = A^a_{\mu, cl}$.

In the generating functional (10) one can set $Q^a_{\mu} = A^a_{\mu} - \bar{A}^a_{\mu}$ obtaining the new functional $\tilde{W}$ related to $W$ by

$$\tilde{W}[J, \gamma; \bar{A}] = W[J, \gamma] - \int d^4x \eta^a_{\mu} \bar{A}^a_{\mu},$$

(16)

and the corresponding effective actions are related by

$$\tilde{\Gamma}[\tilde{\Phi}_{cl}, \gamma; \bar{A}] = \Gamma[\Phi_{cl}, \gamma], \quad \tilde{A}^a_{\mu, cl} = A^a_{\mu, cl} - \bar{A}^a_{\mu}.$$

(17)

From this relation it is evident that the gauge invariant effective action $\Gamma[\Phi_{cl}, \gamma]_{A = A_{cl}}$ can be obtained from $\tilde{\Gamma}[\tilde{\Phi}_{cl}, \gamma; \bar{A}]_{\bar{A}_{cl} = 0}$, i.e. by evaluating 1PI Green functions with background fields on the external legs and the quantum fields $Q, c$ and $\bar{c}$ inside loops. A rigorous proof of this background gauge equivalence has been recently given in [8].

One has to realize that the derivation of the Ward identities (13) is purely formal since the regularization procedure has not been taken into account. If one uses the dimensional regularization all the symmetries are preserved (for a pure YM theory) and the Ward identity also holds for the renormalized background effective action. In particular one can derive the $\beta$-function from the two point background amplitude, since the gauge coupling and wave function renormalizations are related [7].

In the following sections we will use the cutoff regularization, by adjusting the Wilson-Polchinski RG approach to the background field method. To maintain the background gauge invariance we will introduce a covariant cutoff function as proposed in [9]. Moreover we will be forced to add a mass term to the bare action, to properly regularize the one loop contribution.

### 3 Covariant regularization

In the following we will specify to the SU(2) case and, to simplify the notation, we will use the dot and wedge SU(2) products. The BRS action $S_{BRS}$ expressed in terms of the
quantum gauge field \( Q \) is
\[
S = \int_x \left\{ -\frac{1}{4} \bar{F}_{\mu\nu} \cdot F_{\mu\nu} + \frac{1}{2} Q_\mu \cdot \bar{D}^2 Q_\mu + Q_\mu \cdot \bar{F}_{\mu\nu} \wedge Q_\nu = \bar{c} \cdot \bar{D}^2 c \right\} + S_{\text{int}}[Q, c, \bar{A}; \gamma], \tag{18}
\]
where \( \bar{F}_{\mu\nu} \) is the field strength tensor of the background field and we have explicitly written the terms quadratic in the quantum field \( Q_\mu \) and in the ghosts. The remaining terms, which have been collected in \( S_{\text{int}} \), do not contribute to the one-loop vertex functions with background fields on the external legs and for this reason it is not necessary to work out their expression. Notice that the field \( Q \) transforms according to the adjoint representation under the background gauge transformation \( \bar{A} \).

To select the modes of the quantum fields below the UV cutoff \( \Lambda_0 \) we introduce a cutoff function \( K_{\Lambda_0} \) and make the following change of variables in the generating functional \( \tilde{W} \)
\[
Q_\mu \rightarrow K_{\Lambda_0} Q_\mu, \quad c \rightarrow K_{\Lambda_0} c, \tag{19}
\]
(it is not necessary to introduce a new \( \bar{c} \) field). The invariance of the action with respect to the background gauge transformation \( \bar{A} \) can be maintained if the regularized fields transform according to the adjoint representation. This can be achieved if \( K \) is a function of an appropriate covariant operator, such as \( \bar{D}^2 \) and then we choose
\[
K_{\Lambda_0} \equiv K \left[ -\bar{D}^2 / \Lambda_0^2 \right] = \sum_{n=0}^{\infty} \frac{1}{n!} K^{(n)}(0) \left( -\frac{\bar{D}}{\Lambda_0} \right)^n = K(-\bar{D}^2 / \Lambda_0^2) + \cdots, \tag{20}
\]
where \( K^{(n)}(0) = \left[ \frac{d^n K(x)}{dx^n} \right]_{x=0} \) and the dots refer to the terms containing the \( \bar{A} \) field. After the substitution \( \bar{D}^2 = \bar{D}^2 / \Lambda_0^2 \) (or its derivatives, see later) the gauge propagator is multiplied by the factor \( K(p^2 / \Lambda_0^2)^{-2} \) while the ghost one by the factor \( K(p^2 / \Lambda_0^2)^{-1} \). As usual in the Wilson RG formulation, one chooses the cutoff function such as to suppress the propagation of the modes with \( p^2 > \Lambda_0^2 \). However, the choice \( (20) \) of the cutoff function produces new interactions among the quantum and the background fields which are multiplied by the cutoff function \( K(p^2 / \Lambda_0^2) \) (or its derivatives, see later). Thus the loop momenta which are suppressed by the inverse of the cutoff function in the propagator are enhanced by the cutoff function in these new vertices and the regularization fails. This fact is not surprising and is essentially a consequence of the Ward identity, which relates the vertex with the inverse of the propagator.\(^2\)

To overcome this obstacle we introduce in the action a mass terms for the quantum fields
\[
S_m = \int_x \left[ -\frac{1}{2} Q_\mu \cdot M_Q Q_\mu + \bar{c} \cdot M_c c \right], \tag{21}
\]
where the matrices \( M_Q \) and \( M_c \) depend on the cutoff function and must satisfy the requirement that this mass term does not generate relevant interactions in the \( \Lambda_0 \rightarrow \infty \) limit. For instance, by choosing the exponential covariant cutoff function:
\[
K \left[ -\bar{D}^2 / \Lambda_0^2 \right] = e^{\frac{p^2}{\Lambda_0^2}}, \tag{22}
\]
the matrices \( M_Q \) and \( M_c \) have the structure
\[
M_Q^{ab} = \Lambda_0^2 \left( \delta^{ab} - (K^2)^{ab} + K^{ac} \frac{(D^2)^{cd} - K^{db}}{2 \Lambda_0^2} K^{cb} \right), \quad M_c^{ab} = 2 \Lambda_0^2 \left( \delta^{ab} - K^{ab} + \frac{(D^2)^{ac} - K^{cb}}{2 \Lambda_0^2} K^{ac} \right). \tag{23}
\]
\(^2\)We thanks Prof. Carlo Becchi for fruitful discussions on this point.
Notice that this structure holds for every cutoff function satisfying the conditions
\[ K(0) = 1 \quad K'(0) = -1/2. \] (24)

After using (18), (19) and (21), the regularized action becomes
\[
S_{\Lambda_0} = \int_x \left[ -\frac{1}{4} \bar{F}_{\mu\nu} \cdot F_{\mu\nu} - \frac{1}{2} \Lambda_0^2 Q_\mu \cdot (1 - K^2) Q_\mu - g \bar{F}_{\mu\nu} \cdot (KQ_\mu) \wedge (KQ_\nu)
+ 2\Lambda_0^2 \bar{c} \cdot (1 - K) c + S_{\text{int}}[KQ, Kc, \bar{A}; \gamma] \right].
\] (25)

This action preserves the background gauge invariance (6) but breaks the BRS symmetry (7) and a fine tuning procedure must be imposed in order to restore the ST identities. This analysis can be performed as in the standard Wilsonian approach for gauge theories [2, 3]. First one introduces a cutoff dependent BRS transformation, studies the modified ST identities and determines the non-invariant couplings which compensate the breaking introduced by the cutoff. From the quantum action principle these couplings are order \( \hbar \) and therefore affect the background field amplitudes starting from the second loop order. In this paper we are only interested in one-loop computations and therefore this fine-tuning problem can be ignored. However, this procedure is unavoidable in the complete analysis.

From (25) the gauge and ghost propagators read
\[
D^{ab}_{\mu\nu}(q) = \frac{\delta^{ab} g_{\mu\nu}}{\Lambda_0^2 (1 - K^2(q))}, \quad D^{ab}(q) = -\frac{\delta^{ab}}{2\Lambda_0^2 (1 - K(q))}, \] (26)

where
\[
K(q) \equiv K(q^2/\Lambda_0^2) = e^{-q^2/2\Lambda_0^2}.
\] (27)
Notice that for large \( q \) (i.e. \( q >> \Lambda_0 \)) these propagators become constant, and the UV finiteness of the loop integral will be ensured by the cutoff function in the vertices.

In order to evaluate the Feynman rules coming from (25), we first expand \((KQ)_\mu^a\) in terms of the \( \bar{A} \) field. For instance the terms with one and two \( \bar{A} \) fields are given by
\[
(KQ)_\mu^a \big|_{1\bar{A}} = \sum_{n=1}^{\infty} \frac{1}{n!^2 \Lambda_0^{2n}} \sum_{k=0}^{n-1} (\partial^2)^k \Delta_1^{ab} (\partial^2)^{n-k-1} Q_\mu^b \] (28)
and
\[
(KQ)_\mu^a \big|_{2\bar{A}} = \sum_{n=1}^{\infty} \frac{1}{n!^2 \Lambda_0^{2n}} \sum_{k=0}^{n-1} (\partial^2)^k \Delta_2^{ab} (\partial^2)^{n-k-1} Q_\mu^b
+ \sum_{n=2}^{\infty} \frac{1}{n!^2 \Lambda_0^{2n}} \sum_{k=0}^{n-2} \sum_{l=0}^{n-k-2} (\partial^2)^k \Delta_1^{ac} (\partial^2)^l \Delta_1^{db} (\partial^2)^{n-k-l-2} Q_\mu^b, \] (29)

where
\[
\Delta_1^{ab} = g\epsilon^{acb} (\bar{A}_\rho^c \partial_\rho + \partial_\rho \bar{A}_\rho^c) \] (30)
and
\[
\Delta_2^{ab} = g^2 \epsilon^{ace} \epsilon^{deb} \bar{A}_\rho^c \bar{A}_\rho^d. \] (31)
In the following section we will compute the one loop two point function for the background field. For this purposes we need to evaluate the Feynman rules with at most two $A_\mu$ fields.

By inserting (28) in (25) the $Q^a_\mu(q)-\bar{A}^{a_1}_{\mu_1}(p_1)-Q^b_\nu(p)$-vertex is given by

$$V^{a_1 a_2 b}_{\mu_1 \mu_2 \nu}(q, p_1, p_2, p) = i g e^{a_1 b} \left[ 2 K(q) K(p) \left( g_{\mu_1 \nu} p_{\mu_1} - g_{\mu_1 \mu_2} p_{\mu_2} \right) + \Lambda_0^2 \left( K(q) + K(p) \right) F(q, p) g_{\mu_1 \mu_2} \right]$$

where

$$F(q, p) = \frac{\left( K(q) - K(p) \right)}{(q^2 - p^2)}.$$

The vertex with two $Q_\mu$ and two $\bar{A}_\mu$ fields receives contribution from the term $\bar{F}_{\mu \nu} \cdot K Q_\mu \wedge K Q_\nu$ in (25) and from the covariant cutoff functions (29). We do not need to compute the former since it does not contribute to the one loop two point functions (i.e. to the tadpole diagram). The remaining terms of the $Q^a_\mu(q)-\bar{A}^{a_1}_{\mu_1}(p_1)-\bar{A}^{a_2}_{\mu_2}(p_2)-Q^b_\nu(p)$-vertex are

$$V^{a_1 a_2 b}_{\mu_1 \mu_2 \nu}(q, p_1, p_2, p) = g^2 \Lambda_0^2 g_{\mu_1 \mu_2} \left( \delta^{a_1 a_2} \delta^{ab} - \delta^{a_1 b} \delta^{a_2 b} \right) \left( K(q) + K(p) \right) F(q, p) g_{\mu_1 \mu_2}$$

$$- \left[ \frac{F(q, p) - F(q, p + p_2)}{p^2 - (p + p_2)^2} K(q) + \frac{F(q, p) - F(p, q + p_1)}{q^2 - (q + p_1)^2} K(p) \right]$$

$$+ F(q, q + p_1) F(p, p + p_2) \left\{ (2q + p_1)_{\mu_1} (2p + p_2)_{\mu_2} \right\} + 1 \leftrightarrow 2.$$  

The interactions of the ghosts with the background fields can be obtained from (25) and expanding $K(\bar{D}^2/\Lambda_0^2)c$ in powers of $\bar{A}_\mu$ as we have done for $K Q_\mu$. The $\bar{c}^a(q)-\bar{A}^{a_1}_{\mu_1}(p_1)-\bar{c}^b(p)$-vertex is given by

$$V^{a_1 b}_{\mu_1}(q, p_1, p) = -2i g e^{a_1 b} (q - p)_{\mu_1} F(q, p).$$

The $\bar{c}^a(q)-\bar{A}^{a_1}_{\mu_1}(p_1)-\bar{A}^{a_2}_{\mu_2}(p_2)-\bar{c}^b(p)$-vertex is given by

$$V^{a_1 a_2 b}_{\mu_1 \mu_2}(q, p_1, p_2, p) = 2 \Lambda_0^2 g^2 \left\{ \left( \delta^{a_1 a_2} - \delta^{a_1 b} \delta^{a_2 b} \right) \left[ (2q + p_1)_{\mu_1} (2p + p_2)_{\mu_2} \right] \right.$$ 

$$\times \frac{F(q, p) - F(q, p + p_2)}{p^2 - (p + p_2)^2} - g_{\mu_1 \mu_2} F(q, p) \right\} + 1 \leftrightarrow 2.$$  

Though the vertices (32)-(35) have been computed using the cutoff function (22), their expression in term of the functions $K(q)$ and $F(q, p)$ holds for every cutoff function satisfying (24).

From the above expression for the vertices and the propagators, it is clear that the UV finiteness of the loop integrals is ensured if the function $K(q^2/\Lambda^2)$ decreases rapidly enough in the region $q^2 >> \Lambda^2$.

4 One loop computations

As briefly discussed in Section 2, one is only interested to discuss vertices with background external fields. We will explicitly compute the two point amplitude for the background
field and we will verify that is transverse, as it must be since the regularization preserves
the background gauge invariance. There are four Feynman graphs contributing to this
amplitude, which are depicted in figure 1a-1d. The corresponding loop integrals are

\begin{align}
G^{[1]}_{\mu\nu}(p; \Lambda_0) &= g^2 \delta^{ab} \int_q \left\{ 16 K(q) F(q, q) g_{\mu\nu} + \left[ (2q + p)_\mu (2q + p)_\nu \left( 4F^2(q, q + p) \\
+ 8 \frac{F(q, q) - F(q, q + p)}{q^2 - (q + p)^2} K(q) \right) + p \rightarrow -p \right] \right\} \frac{1}{1 - K^2(q)}, \\
G^{[2]}_{\mu\nu}(p; \Lambda_0) &= g^2 \delta^{ab} \int_q \left\{ 4(2q + p)_\mu (2q + p)_\nu F^2(q, q + p) [K(q) + K(q + p)]^2 \\
+ 8 \left( g_{\mu\nu} p^2 - p_\mu p_\nu \right) \frac{K^2(q)K^2(q + p)}{\Lambda_0^4} \right\} \frac{1}{(1 - K^2(q))(1 - K^2(q + p))}, \\
G^{[3]}_{\mu\nu}(p; \Lambda_0) &= -2g^2 \delta^{ab} \int_q \left\{ 2F(q, q) g_{\mu\nu} + \left[ (2q + p)_\mu (2q + p)_\nu \right] \frac{F(q, q) - F(q, q + p)}{q^2 - (q + p)^2} \right\}
\end{align}

Figure 1: Graphical contribution to the two point function with background external legs which
are depicted as curly lines. The wavy and the full lines in the loops refer to the quantum gauge
and ghost field, respectively.

\begin{align}
G^{[1]}_{\mu\nu}(p; \Lambda_0) &= g^2 \delta^{ab} \int_q \left\{ 16 K(q) F(q, q) g_{\mu\nu} + \left[ (2q + p)_\mu (2q + p)_\nu \left( 4F^2(q, q + p) \\
+ 8 \frac{F(q, q) - F(q, q + p)}{q^2 - (q + p)^2} K(q) \right) + p \rightarrow -p \right] \right\} \frac{1}{1 - K^2(q)}, \\
G^{[2]}_{\mu\nu}(p; \Lambda_0) &= g^2 \delta^{ab} \int_q \left\{ 4(2q + p)_\mu (2q + p)_\nu F^2(q, q + p) [K(q) + K(q + p)]^2 \\
+ 8 \left( g_{\mu\nu} p^2 - p_\mu p_\nu \right) \frac{K^2(q)K^2(q + p)}{\Lambda_0^4} \right\} \frac{1}{(1 - K^2(q))(1 - K^2(q + p))}, \\
G^{[3]}_{\mu\nu}(p; \Lambda_0) &= -2g^2 \delta^{ab} \int_q \left\{ 2F(q, q) g_{\mu\nu} + \left[ (2q + p)_\mu (2q + p)_\nu \right] \frac{F(q, q) - F(q, q + p)}{q^2 - (q + p)^2} \right\}
\end{align}

7
\[ p \rightarrow -p \] 

\[
\mathcal{G}_{\mu
u}^{(a)ab}(p; \Lambda_0) = -2g^2 \delta^{ab} \int_q (2q + p)_\mu (2q + p)_\nu \frac{F^2(q, q + p)}{(1 - K(q))(1 - K(q + p))},
\]  

(39)

where in the first two contribution the symmetry factor \(1/2\) has been included and

\[
F(q, q) = \lim_{p \to 0} F(q, q + p) = \frac{d}{dq^2} K(q^2/\Lambda_0^2) \equiv \frac{K'(q)}{\Lambda_0^2}.
\]

One can easily verify the transversality of the two-point function. Indeed form the gauge field loop (fig.1a and 1b) one finds

\[
p_\mu p_\nu (\mathcal{G}_{\mu\nu}^{(a)ab} + \mathcal{G}_{\mu\nu}^{(b)ab}) = 4g^2 \delta^{ab} \int_q \left[ -2 \frac{K^2(q) - K^2(q + p)}{1 - K^2(q)} + \frac{(K^2(q) - K^2(q + p))^2}{(1 - K^2(q))(1 - K^2(q + p))} \right]
= 8g^2 \delta^{ab} \int_q \frac{K^2(q) - K^2(q + p)}{1 - K^2(q)} \left[ \frac{K^2(q)}{1 - K^2(q + p)} - 1 \right] = 0,
\]

where the last two equalities has been obtained by performing the change of the integration variable \(q \to -p - q\). Similarly from the ghost loop (fig.1c and fig 1d) one finds

\[
p_\mu p_\nu (\mathcal{G}_{\mu\nu}^{(c)ab} + \mathcal{G}_{\mu\nu}^{(d)ab}) = 2g^2 \delta^{ab} \int_q \left[ -2 \frac{K(q) - K(q + p)}{1 - K(q)} + \frac{(K(q) - K(q + p))^2}{(1 - K(q))(1 - K(q + p))} \right] = 0.
\]

Therefore the four graphs (36)-(39) sum up to

\[
\mathcal{G}_{\mu\nu}^{ab}(p, \Lambda_0) = \delta^{ab} (p^2 g_{\mu\nu} - p_\mu p_\nu) \mathcal{G}(p, \Lambda_0)
\]

where

\[
\mathcal{G}(p, \Lambda_0) = \frac{1}{6p^2} \mathcal{G}_{\mu\nu}^{aa}(p, \Lambda_0).
\]

(40)

The renormalized background two point amplitude

\[
\Gamma_{\mu\nu}^{ab}(p) = \delta^{ab} (p^2 g_{\mu\nu} - p_\mu p_\nu) \Sigma(p), \quad \Sigma(p)|_{p^2=\mu^2} = 0
\]

at one loop level is given by

\[
\Sigma(p) = \lim_{\Lambda_0 \to \infty} (\mathcal{G}(p, \Lambda_0) - \sigma_1 (\mu/\Lambda_0)),
\]

where the relevant coupling \(\sigma_1\) is

\[
\sigma_1 (\mu/\Lambda_0) = \mathcal{G}(p, \Lambda_0)|_{p^2=\mu^2}
\]

(41)

and \(\mu\) is the renormalization scale. This relevant coupling, which depends on the regularization, will be computed in the appendix in the case of a polynomial cutoff function.
4.1 One-loop beta function

In the Wilsonian approach one can show \[1\] that the beta function can be obtained from the bare coupling constant \( g_B \) by

\[
\beta(g) \equiv \mu \partial_\mu g = \frac{\Lambda_0 \partial_{\Lambda_0} g_B}{\partial g g_B}. \tag{42}
\]

Due to the background gauge invariance the bare coupling constant \( g_B \) is related to the relevant coupling \( \sigma_1 \) by

\[
g_B = g(1 + \sigma_1)^{-1/2}
\]

from which, at one-loop order, one obtains

\[
\beta(g) = -\frac{1}{2} \Lambda_0 \frac{\partial}{\partial \Lambda_0} \sigma_1 (\mu/\Lambda_0). \tag{43}
\]

In the Appendix we will compute this coupling by using a polynomial cutoff function. However the first coefficient of the beta function is independent of the regularization and therefore it should be computed without specifying the cutoff function.

In order to determine the beta function we first consider the contribution to \[1\] originating from the two diagrams with the gauge field loop (see Figs. 1a and 1b). From \[37\] one obtains

\[
G_{\mu\mu}^{[1a]aa} + G_{\mu\mu}^{[1b]aa} = 16g^2 \int_q \left\{ \frac{3p^2 K^2(q)K^2(q+p)}{\Lambda_0^4 (1 - K^2(q))(1 - K^2(q+p)))} + \frac{8K(q)K'(q)}{\Lambda_0^2 (1 - K^2(q))} \right. \\
- \frac{(2q+p)^2}{\Lambda_0^2 p(2q+p)} \left( \frac{K(q)K'(q)}{1 - K^2(q)} - \frac{K(q+p)K'(q+p)}{1 - K^2(q+p)} \right). \tag{44}
\]

Similarly, from the two diagrams with the ghost loop (see Figs. 1c and 1d) one gets

\[
G_{\mu\mu}^{[1c]aa} + G_{\mu\mu}^{[1d]aa} = -\frac{4g^2}{\Lambda_0^2} \int_q \left\{ \frac{8K'(q)}{1 - K(q)} - \frac{(2q+p)^2}{p(2q+p)} \left( \frac{K'(q)}{1 - K(q)} - \frac{K'(q+p)}{1 - K(q+p)} \right) \right\}. \tag{45}
\]

Because of the infrared divergence \[1\] one can not set \( p^2 = \mu^2 = 0 \) in these integrals, however as far as the beta function is concerned, in \[12\] one can take \( \mu/\Lambda_0 \to 0 \) \[13\]. Therefore, from \[10\] and \[13\], the ghost contribution to the beta function is

\[
-\frac{1}{2} \Lambda_0 \frac{\partial}{\partial \Lambda_0} \left( \frac{1}{6\mu^2} \left( G_{\mu\mu}^{[1c]aa} + G_{\mu\mu}^{[1d]aa} \right) \right) |_{\mu^2 = 0}
\]

and, by Taylor expanding in \( p \) the integrand of \[45\], it becomes

\[
\frac{1}{3} \frac{g^2}{16\pi^2} \int_0^\infty dq^2 q^2 \Lambda_0 \frac{\partial}{\partial \Lambda_0} \left( \frac{1}{\Lambda_0^4} \left[ k'(x) + 3xk''(x) + \frac{2}{3} x^2 k'''(x) \right] \right), \tag{47}
\]

where \( k(x) \equiv [K'(x)/(1 - K(x))] \) and \( x = q^2/\Lambda_0^2 \). After integrating by parts and using \[24\], this contribution is

\[
-\frac{g^2}{16\pi^2} \frac{2}{3}.
\]

Notice that the mass term \[24\] does not change the infrared behavior of the propagators.
Performing a similar analysis for the gauge loop contribution, one obtains

\[ -\frac{1}{2} \Lambda_0 \frac{\partial}{\partial \Lambda_0} \left( \frac{1}{6 \mu^2} \left( G_{\mu\mu}^{[1a]aa} + G_{\mu\mu}^{[1b]aa} \right) \right) \bigg|_{\mu^2=0} = -\frac{4}{3} \frac{g^2}{16 \pi^2} \int_0^\infty dq^2 q^2 \Lambda_0 \frac{\partial}{\partial \Lambda_0} \left\{ \frac{1}{\Lambda_0^4} \left[ \frac{3 K^4(x)}{(1-K^2(x))^2} + h'(x) + 3x h''(x) + \frac{2}{3} x^2 h'''(x) \right] \right\} = -\frac{20}{3} \frac{g^2}{16 \pi^2}, \]

(48)

with \( h(x) \equiv [K(x)K'(x)/(1-K^2(x))] \). Summing up the two contributions one obtains the well-known one-loop result

\[ \beta(g) = -\frac{22}{3} \frac{1}{16 \pi^2} g^3. \]

5 Conclusions

In this paper we have implemented a cutoff regularization which maintains the gauge invariance explicit. It is based on the introduction of a background field and a cutoff regulator covariantly depending on this field. The finiteness of the loop integrals is ensured by the presence of a cutoff function which multiplies all interactions and suppresses the loop momenta in the UV region. Moreover, by adding a mass term for the quantum fields the propagators remain finite in this region. This mass term is necessary if one wants to keep gauge invariance and, at the same time, to have an efficient regularization. The only request we made is that this mass term does not introduce relevant interactions into the action when removing the cutoff. The explicit gauge invariance of the effective action has been exploited to compute the beta function at one loop order from the wave function renormalization of the background field. The fact that this result is independent of the cutoff function choice is a check of the consistency of our computation.

All the background amplitudes can be made finite to all loops by using an appropriate cutoff function, such as the exponential one (22). The main problem one has to address before extending this analysis to higher loops lies on the explicit BRS symmetry breaking introduced by the mass term. Although this symmetry only affects the quantum fields, the ST identities are necessary to show the background gauge equivalence [8]. Therefore a complete analysis requires the determination of the symmetry breaking counterterms. As in the standard (i.e. without the background field) Wilsonian formulation of gauge theories, these counterterms can be calculated by introducing a generalized BRS symmetry, dependent on the cutoff function, and studying the corresponding ST identities. Also in this case one can show that by imposing these modified ST identities, the renormalized effective action satisfies the ST, in the limit in which the cutoff is removed. The solution of this fine tuning problem is outside the aim of the present work and is left to a further publication. We only remarque that the background gauge invariance greatly simplifies the determination of the symmetry breaking counterterms. Moreover, in order to determine background gauge amplitudes only few of these counterterms are needed. For instance for computing the second coefficient of the beta function one only needs to determine the couplings of the interactions with at most two quantum gauge fields.
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Appendix

In this appendix we compute the relevant coupling $\sigma_1$ given in (11). To performe this calculation one needs to specify the cutoff function. We use the polynomial cutoff function

$$K(q) = \frac{1}{\left(1 + \frac{q^2}{4\Lambda_0^2}\right)^2},$$

(49)

which satisfies the conditions (24) and allows to evaluate the Feynman integrals by exploiting the Feynman parameterization.

We first concentrate on the ghost loop contribution (45), which becomes

$$G^{[1]aa}_{\mu \mu} + G^{[1d]aa}_{\mu \mu} = 128 g^2 \Lambda_0^4 \int_q \left\{ \frac{8}{q^2 (q^2 + 4\Lambda_0^2)(q^2 + 8\Lambda_0^2)} - (2q + p)^2 \left[ \frac{1}{q^2 (q^2 + 4\Lambda_0^2)(q^2 + 8\Lambda_0^2)} \right] \right\};$$

(49)

where $\int_q \equiv \int_0^1 dz \int_0^{1-z} dy \int_0^{1-x-y} dx$. After integrating with respect to $q$ and the Feynman parameters the ghost loop contribution gives

$$1024 g^2 \Lambda_0^4 \int_q \frac{1}{q^2 (4\Lambda_0^2 + q^2)(8\Lambda_0^2 + q^2)} = \frac{16 \Lambda_0^2}{\pi^2} g^2 \log 2.$$  

(50)

For the remaining terms one uses the Feynman parameterization. For instance one has

$$\int_q \frac{(2q + p)^2}{q^2 (4\Lambda_0^2 + q^2)(8\Lambda_0^2 + q^2)(q + p)^2} = 3! \int_q \int_{x,y,z} \frac{[2q + p(1 - 2z)]^2}{q^2 + z (1 - z) p^2 + 4\Lambda_0^2 (x + 2y)},$$

(51)

where $\int_{x,y,z} \equiv \int_0^1 dz \int_0^{1-z} dy \int_0^{1-x-y} dx$. After integrating with respect to $q$ and the Feynman parameters the ghost loop contribution gives

$$- \frac{p^2}{4\pi^2} \log \frac{p^2}{\Lambda_0^2} + \frac{p^2}{\pi^2} \left( \frac{2}{3} + \frac{1}{4} \log 2 \right) + O(p^4/\Lambda_0^2).$$

(52)

The gauge loop contribution (14), can be evaluated in a similar way. The presence of the square of the cutoff function makes the Feynman integrals more involved. By inserting the cutoff function (49) in (14) one obtains

$$G^{[1a]aa}_{\mu \mu} + G^{[1b]aa}_{\mu \mu} = 512 g^2 \Lambda_0^8 \int_q \left\{ 6144 \frac{p^2 \Lambda_0^4}{(4\Lambda_0^2 + q^2)^4 - (4\Lambda_0^2 + (q + p)^2)^4 - (4\Lambda_0^2 + q^2)^4} \right\}$$

$$- \frac{1}{8} \frac{6144 p^2 \Lambda_0^4}{q^2 (4\Lambda_0^2 + q^2)(8\Lambda_0^2 + q^2)(32\Lambda_0^4 + 8\Lambda_0^2 q^2 + q^4)}.$$


\[
\frac{(2q + p)^2}{(8\lambda_0^2 + q^2)} \left[ \frac{1}{(4\lambda_0^2 + q^2)(32\lambda_0^4 + 8\lambda_0^2 q^2 + q^4)(q + p)^2(4\lambda_0^2 + (q + p)^2)} \right.
+ \frac{q^2(4\lambda_0^2 + q^2)(32\lambda_0^4 + 8\lambda_0^2 q^2 + q^4)(32\lambda_0^4 + 8\lambda_0^2(q + p)^2 + (q + p)^4)}{1}
+ \frac{(32\lambda_0^4 + 8\lambda_0^2 q^2 + q^4)(q + p)^2(4\lambda_0^2 + (q + p)^2)(8\lambda_0^2 + (q + p)^2)}{q^2(q + p)^2(4\lambda_0^2 + q^2)(32\lambda_0^4 + 8q^2\lambda_0^2 + q^4))} \right].
\]

One can see that this integral is UV finite and that it vanishes for \( p = 0 \). We only compute the contribution originating from the first and the last terms since the others do not generate \( \log p^2/\Lambda_0^2 \), as can be seen by taking the \( \Lambda_0 \to \infty \) limit in the integrand. After applying the Feynman parameterization and integrating on the whole set of variables one obtains

\[
- \frac{5p^2}{2\pi^2} \log \frac{p^2}{\Lambda_0^2} + \mathcal{O}(p^2).
\]

Adding the two contributions from the ghost and the gauge loop one obtains

\[
\mathcal{G}(p/\Lambda_0) = - \frac{g^2}{16\pi^2} \frac{22}{3} \log(p^2/\Lambda_0^2) + \mathcal{O}(1).
\]

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