IWASAWA THEORY OF RUBIN-STARK UNITS AND CLASS GROUP

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Abstract. Let $K$ be a totally real number field of degree $r = [K : \mathbb{Q}]$ and let $p$ be an odd rational prime. Let $K_\infty$ denote the cyclotomic $\mathbb{Z}_p$-extension of $K$ and let $L_\infty$ be a finite extension of $K_\infty$, abelian over $K$. In this article, we extend results of [Bu 09] relating characteristic ideal of the $\chi$-quotient of the projective limit of the ideal class groups to the $\chi$-quotient of the projective limit of the $r$-th exterior power of units modulo Rubin-Stark units, in the non semi-simple case, for some $\mathbb{Q}_p$-irreductible characters $\chi$ of $\text{Gal}(L_\infty/K_\infty)$.

1. Introduction

Let $K$ be a totally real number field of degree $r = [K : \mathbb{Q}]$. Fix a rational odd prime $p$ and let $K_\infty$ denote the cyclotomic $\mathbb{Z}_p$-extension of $K$; fix a finite extension $L_\infty$ of $K_\infty$, abelian over $K$. Fix also a decomposition of $\text{Gal}(L_\infty/K_\infty) = \text{Gal}(L_\infty/K_\infty) \times \Gamma$, $\Gamma \simeq \mathbb{Z}_p$.

Then the fields $L := L_\Gamma$ and $K_\infty$ are linearly disjoint over $K$.

If $F/K$ is a finite abelian extension of $K$, we write $A(F)$ for the $p$-part of the class group of $F$, and $\mathcal{E}(F)$ for the group of global units of $F$. For a $\mathbb{Z}$-module $M$, let $\hat{M} = \lim_{\leftarrow} \frac{M}{p^n M}$ denote the $p$-adic completion of $M$. Let

$$A_\infty = \lim_{\leftarrow} A(F), \quad \mathcal{E}_\infty = \lim_{\leftarrow} \mathcal{E}(F),$$

where the projective limit is taken over all finite sub-extensions of $L_\infty$, with respect to the norm maps. Let $\Delta = \text{Gal}(L_\infty/K_\infty)$ and let

$$\chi : \Delta \longrightarrow \overline{\mathbb{Q}}_p^\times$$

be a non-trivial $\overline{\mathbb{Q}}_p$-irreducible character of $\Delta$. Let $\mathcal{O} := \mathbb{Z}_p[\chi]$ be the ring generated by the values of $\chi$ over $\mathbb{Z}_p$ and let $\mathcal{O}(\chi)$ denote the ring $\mathcal{O}$ on which $\Delta$ acts via $\chi$. For any $\mathbb{Z}_p[\Delta]$-module $M$, we define the $\chi$-quotient $M_\chi$ by

$$M_\chi := M \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}(\chi).$$

For any profinite group $\mathcal{G}$, we define the Iwasawa algebra

$$\mathcal{O}[[\mathcal{G}]] := \lim_{\leftarrow} \mathcal{O}[[\mathcal{G}]/\mathcal{H}],$$

where the projective limit is over all finite quotient $\mathcal{G}/\mathcal{H}$ of $\mathcal{G}$. In case $\mathcal{G} = \Gamma$, we shall write $\Lambda := \mathcal{O}[[\Gamma]]$.

2000 Mathematics Subject Classification. 11R23, 11R27, 11R29, 11R42.

Key words and phrases. Iwasawa theory, Euler systems, local conditions.
It is well known that \( \Lambda \) is a complete noetherian regular local domain, with finite residue field of characteristic \( p \). The structure theorem of \( \Lambda \)-modules shows that for a finitely generated torsion \( \Lambda \)-module \( M \), there exists an injective pseudo-isomorphism

\[
\bigoplus_i \Lambda / f_i \Lambda \rightarrow M
\]

with \( f_i \in \Lambda \), and we define the characteristic ideal of \( M \)

\[
\text{char}(M) = \prod_i f_i \Lambda.
\]

We denote by \( \text{St}_\infty := \varinjlim St_n \) the \( \mathbb{Z}[[\text{Gal}(L_\infty/K)]] \)-module constructed by the Rubin-Stark elements (see Definition 4.5). Our objective in this paper is to compare the characteristic ideal of \( (A_\infty)_\chi \) with the characteristic ideal of \( \left( \left( \bigwedge_r \mathbb{Z}_p[[\text{Gal}(L_\infty/K)]] \bigotimes_{\text{St}_\infty} \mathbb{E}_\infty \right) / \hat{\text{St}}_{\infty} \right)_\chi \). Let \( \Sigma_\infty \) be the set of infinite places of \( K \) and let \( L_\chi \) be the fixed field of \( \ker(\chi) \). Let \( K(1) \) be the maximal \( p \)-extension inside the Hilbert class field of \( K \). In the sequel we will assume (for simplicity) that

\[
L = L_\chi \quad \text{and} \quad K = L \cap K(1).
\]

For a \( p \)-adic prime \( p \) of \( K \), let \( \text{Frob}_p \) denote a Frobenius element at \( p \) inside the absolute Galois group of \( K \). Assume that

\[
(\mathcal{H}_0) \quad \text{The number field } L \text{ is totally real.}
\]

\[
(\mathcal{H}_1) \quad \text{The extension } L/\mathbb{Q} \text{ is unramified at } p.
\]

\[
(\mathcal{H}_2) \quad \chi(\text{Frob}_p) \neq 1 \text{ for any } p \text{-adic prime } p \text{ of } K.
\]

\[
(\mathcal{H}_3) \quad \text{The Leopoldt conjecture holds for every finite extension } F \text{ of } L \text{ in } L_\infty.
\]

In the semi-simple case Büyükboduk proved

**Theorem 1.1.** (Büyükboduk [Bu 09 Theorem A]) Assume that the hypotheses \( (\mathcal{H}_0) - (\mathcal{H}_3) \) hold. If \( p \nmid [L : K] \), then

\[
\text{char}((A_\infty)_\chi) = \text{char} \left( \left( \bigwedge_r \mathbb{E}_\infty \right) / \hat{\text{St}}_{\infty} \right)_\chi.
\]

There is at least two ideas behind the proof of such a theorem. On the one hand, the result may be stated in terms of Selmer groups. On the other hand, Rubin-Stark elements give rise to Euler systems for the \( p \)-adic representation \( T = \mathbb{Z}_p \otimes \mathcal{O}(\chi^{-1}) \). Mazur and Rubin developed in [MR 04] an Euler system and Kolyvagin system machinery so as to determine the structure of the associated Selmer groups, in the case where a certain cohomological invariant, called the Selmer core rank, is one. As an application to Iwasawa theory Büyükboduk obtains a divisibility relation between the characteristic ideals of the projective limit of these Selmer groups, which transforms into equality if the corresponding Kolyvagin system is primitive. For more detail, see [Bu 07]. He then applied this theory to the proof of Theorem [1] by constructing a primitive Kolyvagin system from Rubin-Stark elements and Selmer groups of core rank one.

**Remark 1.2.** Mazur and Rubin introduced in [MR 16] the notion of Stark system/Kolyvagin system of rank \( r \). They used this notion to determine the structure of Selmer groups, when the core rank is greater than one.

In this paper we prove
Theorem 1.3. Assume that the hypotheses \((H_0) - (H_3)\) hold. Then
\[
\text{char}(A_\infty, \chi) \quad \text{divides} \quad P^d \text{char}\left(\left(\bigwedge^r \mathcal{E}_\infty / \text{St}_\infty, \chi\right)\right)
\]
where \(d = \max_{p|p} \{v_p(1 - \chi(Frob_p))\} = \max_{p|p} \{v_p(1 - \chi(Frob_p))\}.

We take our inspiration from [Bi 09]. But if \(p \mid [L : K]\) the results of [MR 04], [MR 16] and [Bi 07] do not apply, since the notion of core rank is not defined. Therefore, we are led to use the theory of Euler systems exposed in [Ru 00]. In particular, we construct an ad-hoc Selmer structure and an associated Kolyvagin system. We have to use the structure of the semi-local units, cf. Theorem 5.1. But this is already known, thanks to Greither who applied Coleman’s theory in [Gr 96] to determine this structure. In the semi-simple case Büyükboduk used a weak structure theorem of Colmez-Cherbonnier, obtained by using the theory of \((\phi, \Gamma)\)-modules.

2. Selmer structures

In this section we recall some definitions concerning the notion of Selmer structure introduced by Mazur and Rubin in [MR 04] and [MR 16]. For any field \(k\) and a fixed separable algebraic closure \(\overline{k}\) of \(k\), we write \(G_k := \text{Gal}(\overline{k}/k)\) for the Galois group of \(\overline{k}/k\). Let \(\mathcal{O}\) be the ring of integers of a finite extension \(\Phi\) of \(\mathbb{Q}_p\) and let \(D\) denote the divisible module \(\Phi/\mathcal{O}\). For a \(p\)-adic representation \(T\) with coefficients in \(\mathcal{O}\), we define
\[
D(1) = D \otimes \mathbb{Z}_p(1), \quad T^* = \text{Hom}_\mathcal{O}(T, D(1)),
\]
where \(\mathbb{Z}_p(1) := \lim_{\to} \mu_p^n\) is the Tate module.

Let \(F\) be a number field, and for a place \(w\) of \(F\), let \(F_w\) denote the completion of \(F\) at the place \(w\). Let us recall the local duality theorem c.f. [Mi 86] Corollary I.2.3]: For \(i = 0, 1, 2\), there is a perfect pairing
\[
H^{2-i}(F_w, T) \times H^i(F_w, T^*) \xrightarrow{(\cdot, \cdot)_w} H^2(F_w, D(1)) \cong D, \text{ if } v \text{ is finite,}
\]
\[
\hat{H}^{2-i}(F_w, T) \times \hat{H}^i(F_w, T^*) \xrightarrow{(\cdot, \cdot)_w} \hat{H}^2(F_w, D(1)), \text{ if } v \text{ is infinite}
\]

where \(\hat{H}^*(F_w, \cdot)\) denotes the Tate cohomology group.

Definition 2.1. Let \(T\) be \(p\)-adic representation of \(G_F\) with coefficients in \(\mathcal{O}\) and let \(w\) be a non \(p\)-adic prime of \(F\). A local condition \(\mathcal{F}\) at the prime \(w\) on \(T\) is a choice of an \(\mathcal{O}\)-module \(H^1_{\text{Sel}}(F_w, T)\) of \(H^1(F_w, T)\). For the \(p\)-adic primes, a local condition at \(p\) will be a choice of an \(\mathcal{O}\)-submodule \(H^1_{\text{Sel}}(F_p, T)\) of the semi-local cohomology group
\[
H^1(F_p, T) := \bigoplus_{w|p} H^1(F_w, T).
\]

Let \(I_w\) denote the inertia subgroup of \(G_{F_w}\). We say that \(T\) is unramified at \(w\) if the inertia subgroup \(I_w\) of \(w\) acts trivially on \(T\). We assume in the sequel that \(T\) is unramified outside a finite set of places of \(F\).

Definition 2.2. A Selmer structure \(\mathcal{F}\) on \(T\) is a collection of the following data:
- a finite set \(\Sigma(\mathcal{F})\) of \(F\), including all infinite places and primes above \(p\), and all primes where \(T\) is ramified;
- for every \(w \in \Sigma(\mathcal{F})\), a local condition on \(T\).
If $\mathcal{F}$ is a Selmer structure on $T$, we define the Selmer group $H^1_{tf}(F,T) \subset H^1(F,T)$ to be the kernel of the sum of the restriction maps

$$H^1(G_{\Sigma(\mathcal{F})}(F), T) \to \bigoplus_{w \in \Sigma(\mathcal{F})} (H^1(F_w, T)/H^1_{tf}(F_w, T)),$$

where $G_{\Sigma(\mathcal{F})}(F) := \text{Gal}(F_{\Sigma(\mathcal{F})}/F)$ is the Galois group of the maximal algebraic extension of $F$ which is unramified outside $\Sigma(\mathcal{F})$.

A Selmer structure $\mathcal{F}$ on $T$ determines a Selmer structure $\mathcal{F}^*$ on $T^*$. Namely,

$$\Sigma(\mathcal{F}) = \Sigma(\mathcal{F}^*), \quad H^1_{tf}(F_w, T^*) := H^1_{tf}(F_w, T)^{\perp}, \quad \text{if } w \in \Sigma(\mathcal{F}^*) - \Sigma_p$$

under the local Tate pairing $(\cdot, \cdot)_w$ and

$$H^1_{tf}(F_p, T^*) := H^1_{tf}(F_p, T)^{\perp}$$

under the pairing $\sum_{w|p}(\cdot, \cdot)_w$.

**Example 2.3.** Let $w$ be a place of $F$ and let $F_w^{ur}$ denote the maximal unramified extension of $F_w$. Define the subgroup of universal norm

$$H^1(F_w, T)^u = \bigcap_{F_w \subset k \subset F_w^{ur}} \text{cor}_{k,F}(k,T),$$

where the intersection is over all finite unramified extensions $k$ of $F_w$. Let $H^1(F_w, T)^u, \text{sat}$ denote the $\mathcal{O}$-saturation of $H^1(F_w, T)^u$ in $H^1(F_w, T)$, i.e., $H^1(F_w, T)/H^1_{ur}(F_w, T)$ is a free $\mathcal{O}$-module and $H^1_{ur}(F_w, T)/H^1(F_w, T)^u$ has finite length. Following [MR 16, Defintition 5.1], we define the unramified local condition $F_{ur}$ by

$$H^1_{ur}(F_w, T) = H^1(F_w, T)^u, \text{sat}, \quad \text{if } w \nmid p, \quad \text{and} \quad H^1_{ur}(F_p, T) = \bigoplus_{p|\mathfrak{p}} H^1(F_p, T)^{u,\text{sat}}$$

(3)

For any $G_{F_{ur}}$-module $M$, we define the subgroup of unramified cohomology classes $H^1_{ur}(F_w, M) \subset H^1(F_w, M)$ by

$$H^1_{ur}(F_w, M) = \ker( H^1(F_w, M) \to H^1(I_w, M) ).$$

For future use, we record here the following well-known properties of unramified cohomology:

(i) $H^1_{ur}(F_w, T^*) = H^1_{ur}(F_w, T^*)^{\text{div}}$, $H^1_{ur}(F_p, T^*) = \bigoplus_{p|\mathfrak{p}} H^1_{ur}(F_p, T^*)^{\text{div}}$.

(ii) If $w \nmid p$ and $T$ is unramified at $w$, then

$$H^1_{ur}(F_w, T) = H^1_{ur}(F_w, T) \quad \text{and} \quad H^1_{ur}(F_p, T^*) = H^1_{ur}(F_p, T^*).$$

where for an abelian group $A$, $A^{\text{div}}$ denotes the maximal divisible subgroup of $A$.

The assertion (i) follows from [PR 92, §2.1.1, Lemme] and the assertion (ii) can be deduced from [Ru 00, Lemma 1.3.5].

3. Iwasawa Theory

Fix a totally real number field $K$. Let $r = [K : \mathbb{Q}]$ and $K_{\infty} = \bigcup_{n \geq 0} K_n$ denote the cyclotomic $\mathbb{Z}_p$-extension of $K$. Assume that all algebraic extensions of $K$ are contained in a fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. If $F$ is a finite extension of $K$ and $w$ is a place of $F$, fix a place $\mathfrak{w}$ of $\overline{\mathbb{Q}}$ lying above $w$. The decomposition (resp. inertia) group of $\mathfrak{w}$ in $\overline{\mathbb{Q}}/F$ is denoted by $D_w$ (resp. $I_w$). If $v$ is a
prime of $K$ and $F$ is a Galois extension of $K$, we denote the decomposition group of $v$ in $F/K$ by $D_v(F/K)$. Recall that

$$
\chi : G_K \longrightarrow \mathcal{O}^\times
$$

is a non-trivial $\mathbb{Q}_p$-irreducible character, factoring through a finite abelian extension $L$ of $K$. Assume that $L$ and $K_\infty$ are linearly disjoint over $K$. Let $L_n = LK_n$ and let $L_\infty = LK_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $L$. In the sequel, we will denote by $T$ the $p$-adic representation

$$
T = \mathbb{Z}_p(1) \otimes \mathcal{O}(\chi^{-1}).
$$

Let $\Sigma$ be a finite set of places of $K$ containing all infinite places, all $p$-adic places and all places where $T$ is ramified. If $F$ is an extension of $K$, we denote also by $\Sigma$ the set of places of $F$ lying above places in $\Sigma$.

**Definition 3.1.** Let $F$ be a finite extension of $K$ and let $\Sigma_p$ denote the set of $p$-adic primes of $K$. Following [MR 16, Example 5.1], we define the canonical and the strict Selmer structures $\mathcal{F}_{\text{can}}$ and $\mathcal{F}_{\text{str}}$ on $T$, by

- $\Sigma(\mathcal{F}_{\text{can}}) = \Sigma(\mathcal{F}_{\text{str}}) = \Sigma$.
- if $w | p$, $H^1(\mathcal{F}_{\text{can}}(F_w,T), H^1(\mathcal{F}_{\text{str}}(F_w,T) := H^1_{\mathcal{F}_{\text{str}}}(F_w,T)$.
- $H^1(\mathcal{F}_{\text{can}}(F_p,T) := H^1(\mathcal{F}_{\text{str}}(F_p,T)$, and $H^1(\mathcal{F}_{\text{str}}(F_p,T) = 0$.

where $\mathcal{F}_{\text{str}}$ is the unramified local condition defined in [3].

Let $F'/F$ be a finite extension of $F$. Remark that for $\mathcal{F} = \mathcal{F}_{\text{can}}, \mathcal{F} = \mathcal{F}_{\text{str}}$ or $\mathcal{F} = \mathcal{F}_{\text{str}}$, we have

$$
cor_{F_{w'},F_w}(H^1_{\mathcal{F}}(F_{w'},T)) \subset H^1_{\mathcal{F}}(F_w,T) \quad \text{and} \quad \text{res}_{F_{w'},F_w}(H^1_{\mathcal{F}}(F_{w'},T')) \subset H^1_{\mathcal{F}}(F_{w'},T')
$$

where $w' | w$ and $\text{cor}_{F_{w'},F_w}$ (resp. $\text{res}_{F_{w'},F_w}$) denote the corestriction (resp. restriction) map. For these local conditions, we write

$$
H^1_{\mathcal{F}}(FK_\infty,T) := \lim_{\longleftarrow} H^1_{\mathcal{F}}(FK_n,T), \quad H^1_{\mathcal{F}}(FK_\infty,T') := \lim_{\longleftarrow} H^1_{\mathcal{F}}(FK_n,T')
$$

where the projective (resp. injective) limit is taken with respect to the corestriction (resp. restriction) maps.

Recall that for any Galois extension $F/F'$ of number fields, $K \subset F' \subset F$, the restriction map induces an isomorphism

$$
\text{res} : H^1(F',T) \xrightarrow{\sim} H^1(F,T)_{\text{Gal}(F/F')},
$$

c.f. [AMO, Lemme 4.3].

**Lemma 3.2.** We have

$$
H^1(\mathcal{F}_{\text{str}}(K_\infty,T) = 0.
$$

**Proof.** This equality is implicit in [Bü 09, Proposition 2.12]. It is a consequence of the weak Leopoldt conjecture. Indeed, using (4) and passing to the inverse limit, we obtain

$$
\lim_{\longleftarrow} H^1(G_{\Sigma}(K_n),T) \cong (\lim_{\longleftarrow} H^1(G_{\Sigma}(L_n),T)_{\text{Gal}(L_\infty/K_\infty)},
$$

then $H^1_{\mathcal{F}_{\text{str}}}(K_\infty,T) \cong H^1_{\mathcal{F}_{\text{str}}}(L_\infty,T)_{\text{Gal}(L_\infty/K_\infty)}$. This shows that it suffices to prove $H^1_{\mathcal{F}_{\text{str}}}(L_\infty,T) = 0$. Indeed, for $w \in \Sigma - \Sigma_p$, Proposition B.3.2 of [Ru 00] and (i) of example [2.3] show that

$$
\lim_{\longleftarrow} H^1(L_{n,w},T) \cong \lim_{\longleftarrow} H^1_{\text{str}}(L_{n,w},T) \quad \text{and} \quad H^1_{\text{str}}(L_{n,w},T) = H^1_{\text{str}}(L_{n,w},T).
$$
Then by definition of $H^1_{\text{str}}(L_{\infty}, T)$, we have an exact sequence

$$0 \longrightarrow H^1_{\text{str}}(L_{\infty}, T) \longrightarrow \varprojlim_n H^1(G_{\Sigma}(L_n, T)) \longrightarrow \lim_{\leftarrow} H^1(L_{n, p}, T).$$

Since $L_{\infty}$ is the cyclotomic $\mathbb{Z}_p$-extension of $L$, then the weak Leopold conjecture is true for $L_{\infty}/L$, e.g. [NSW 91, Theorem 10.3.25]. Therefore, using $\chi(G_L) = 1$, we get

$$\lim_{\leftarrow} H^1_{\text{str}}(L_n, T) = 0.$$ 

□

Let $n$ be a nonnegative integer, we write $A_n$ for the $p$-part of the class group of $L_n$ and $E'_n$ for the group of $p$-units of $L_n$. Let

$$A_{\infty} := \varprojlim_n A_n \quad \text{and} \quad \widehat{E}'_{\infty} := \varprojlim_n \widehat{E}'_n,$$

where all inverse limits are taken with respect to norm maps. It is well known that

$$\lim_{\leftarrow} H^1(G_{\Sigma}(L_n), \mathbb{Z}_p(1)) \cong \widehat{E}'_{\infty}.$$ 

Then, by (4), we have

$$H^1_{\text{can}}(K_{\infty}, T) \cong \lim_{\leftarrow} H^1(K_n, T) \cong (\widehat{E}'_{\infty} \otimes \mathbb{Z}_p \mathcal{O}(\chi^{-1}))^{\text{Gal}(L_{\infty}/K_{\infty})}.$$  \ (5)

**Proposition 3.3.** Suppose that every infinite place of $K$ is completely decomposed in $L/K$. Then the $\Lambda$-module $H^1_{\text{can}}(K_{\infty}, T)$ is free of rank $r = [K : \mathbb{Q}]$.

**Proof.** Using Dirichlet’s unit theorem and the assumption $(H_0)$, we get

$$\text{rank}_{\mathcal{O}}((\widehat{E}'_n \otimes \mathbb{Z}_p \mathcal{O}(\chi^{-1}))^{\text{Gal}(L_n/K_n)}) = r, p^n + t$$

where $t$ is a nonnegative integer independent of $n$. Then by [Gr 94, Theorem], we see that

$$\text{rk}_{\Lambda}((\widehat{E}'_{\infty} \otimes \mathbb{Z}_p \mathcal{O}(\chi^{-1}))^{\text{Gal}(L_{\infty}/K_{\infty})}) = r.$$ 

This finishes the proof of the proposition. □

**Proposition 3.4.** The $\mathcal{O}[\text{Gal}(L_n/K)]$-modules $H^1_{\text{can}}(L_n, T^*)$ and $\text{Hom}(A_n, T^*)$ are isomorphic.

**Proof.** See §6.1 of [MR 04]. □

For an $\mathcal{O}$-module $M$, we denote by $M^\vee := \text{Hom}_{\mathcal{O}}(M, D) \cong \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ its Pontryagin dual.

**Proposition 3.5.** The $\Lambda$-modules $H^1_{\text{can}}(K_{\infty}, T^*)^\vee$ and $(H^1_{\text{can}}(L_{\infty}, T^*)^\vee)^{\text{Gal}(L_{\infty}/K_{\infty})}$ are pseudo-isomorphic.

**Proof.** This is [AMO, Proposition 3.8]. □

**Lemma 3.6.** If one of the hypotheses $(H_2)$ or $(H_3)$ holds then

$$\text{char}(A_{\infty}) \mid \text{char}(H^1_{\text{can}}(K_{\infty}, T^*)^\vee).$$
Proof. Consider the exact sequence
\[
H_{\mathcal{F}_{ur}}^1(L_{n,p}, T^*)^\vee \longrightarrow H_{\mathcal{F}_{ur}}^2(L_n, T^*)^\vee \longrightarrow H_{\mathcal{F}_{can}}^2(L_n, T^*)^\vee \longrightarrow 0.
\]
Since
\[
H_{\mathcal{F}_{ur}}^1(L_{n,p}, T^*) \cong \bigoplus_{w \mid p} \text{Hom}(D_w / I_w, T^*)
\]
Then
\[
H_{\mathcal{F}_{ur}}^1(L_{n,p}, T^*)^\vee \cong \bigoplus_{v \mid p} \mathcal{O}(\chi^{-1})[\text{Gal}(L_n / K) / D_v(L_n / K)].
\]
Passing to the projective limit and taking the $\Delta$-co-invariants, we get
\[
(\mathcal{O}(\chi^{-1})[\mathcal{G} / D_v(L_\infty / K)])_{\Delta} \simeq \begin{cases} \text{finite}, & \text{if } \chi(D_v(L/K)) \neq 1; \\ \mathcal{O}[\text{Gal}(K_\infty / K) / D_v(K_\infty / K)], & \text{if } \chi(D_v(L/K)) = 1. \end{cases}
\]
where $\Delta = \text{Gal}(L_\infty / K_\infty)$. Using Proposition 3.4 and Proposition 3.5, we obtain that
\[
\text{char}((A_\infty)_L) \text{ divides } \mathcal{J}^* \text{char}(H_{\mathcal{F}_{can}}^1(K_\infty, T^*)^\vee)
\]
where $\mathcal{J}$ is the augmentation ideal of $\Lambda$ and $s = \# \{ v \mid p ; \chi(Frob_v) = 1 \}$. It is well known that $\text{char}((A_\infty)_L)$ is prime to $\mathcal{J}$ in the cyclotomic $\mathbb{Z}_p$-extension if ($H_3$) is satisfied. This concludes the proof of the lemma. \qed

4. Euler systems of Rubin-Stark units

In this section, we construct an Euler system in the sense of [Ru 00, Definition 2.1.1] for the $p$-adic representation $\mathbb{Z}_p(1) \otimes \mathcal{O}(\chi^{-1})$, coming from the elements predicted by Rubin-Stark conjecture [Ru 96, Conjecture $B^\prime$].

We set some notation. Let $K$ be a number field and let $F$ be a finite abelian extension of $K$. Fix a finite set $S$ of places of $K$ containing all infinite places and all places ramified in $F / K$, and a second finite set $T$ of places of $K$, disjoint from $S$. Let $G = \text{Gal}(F / K)$ and $\hat{G} = \text{Hom}(G, \mathbb{C}^*)$. If $\rho \in \hat{G}$ we define the modified Artin $L$-function attached to $\rho$ by
\[
L_{S,T}(s, \rho) = \prod_{p \not\in S} (1 - \rho(Frob_p)Np^{-s})^{-1} \prod_{p \in T} (1 - \rho(Frob_p)Np^{1-s})
\]
where $Frob_p \in G$ is the Frobenius of the (unramified) prime $p$.

For each $\rho \in \hat{G}$, there is an idempotent
\[
e_{\rho} = \frac{1}{|G|} \sum_{\sigma \in G} \rho(\sigma)\sigma^{-1} \in \mathbb{C}[G].
\]
Following [Ta 84] we define the Stickberger element
\[
\Theta_{S,T}(s) = \Theta_{S,T,F/K}(s) = \sum_{\rho \in \hat{G}} L_{S,T}(s, \rho^{-1})e_{\rho}
\]
which we view as a $\mathbb{C}[G]$-valued meromorphic function on $\mathbb{C}$. Let $\rho \in \hat{G}$ and let $r_S(\rho)$ be the order of vanishing of $L_{S,T}(s, \rho)$ at $s = 0$. Recall that
\[
r_S(\rho) = \text{ord}_{s=0}L_{S,T}(s, \rho) = \begin{cases} |\{ v \in S : \rho(D_v(F/K)) = 1 \}|, & \rho \neq 1; \\ |S| - 1, & \rho = 1. \end{cases}
\]
Remark also that the module \( D_v(F/K) \) is the decomposition group of \( v \) relative to \( F/K \).

Before stating the Rubin-Stark conjecture we record some hypotheses \( H(F/K, S, T, r) \):

1. \( S \) contains all the infinite primes of \( K \) and all the primes which ramify in \( F/K \);
2. \( S \) contains at least \( r \) places which split completely in \( F/K \);
3. \( |S| \geq r + 1 \);
4. \( T \neq \emptyset \), \( S \cap T = \emptyset \) and \( U_{S,T}(F) \) is torsion-free,

here \( U_{S,T}(F) \) is the group of \( S \)-units of \( F \) which are congruent to \( 1 \) modulo all the primes in \( T \).

**Remark 4.1.** Conditions (2) and (3) ensure that \( s^{-\tau} \Theta_{S,T}(s) \) is holomorphic at \( s = 0 \). Condition (4) is easily satisfied. For example, if \( T \) contains primes of two different residue characteristics.

We will identify \( \text{Hom}_{\mathbb{Z}[G]}(U_{S,T}(F), \mathbb{Z}[G]) \) with a submodule of \( \text{Hom}_{\mathbb{C}[G]}(C \otimes U_{S,T}(F), \mathbb{C}[G]) \). For any \( r \)-tuple \((\phi_1, \cdots, \phi_r) \in \text{Hom}_{\mathbb{Z}[G]}(U_{S,T}(F), \mathbb{Z}[G])^r \), we define a \( \mathbb{C}[G] \)-morphism

\[
\mathbb{C} \otimes \bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(F) \xrightarrow{\phi_1 \wedge \cdots \wedge \phi_r} \mathbb{C}[G]
\]

by

\[
\phi_1 \wedge \cdots \wedge \phi_r(u_1 \wedge \cdots \wedge u_r) := \det_{1 \leq i,j \leq r} (\phi_i(u_j)),
\]

for any \( u_1, \cdots, u_r \in U_{S,T}(F) \).

For any \( \mathbb{Z}[G] \)-module \( M \) with trivial \( \mathbb{Z} \)-torsion and any positive integer \( r \), we let

\[
M_{r,S} := \{ x \in M \mid e_{\rho}x = 0 \text{ in } \mathbb{C} \otimes M \text{ for all } \rho \in \hat{G} \text{ such that } r_S(\rho) > r \}.
\]

Assuming that \((F/K, S, T, r)\) satisfies hypotheses \( H(F/K, S, T, r) \) and \( r \geq 1 \), let \( \Lambda_{S,T} \) the \( \mathbb{Z}[G] \)-submodule of \( \mathbb{Q} \otimes \bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(F) \) defined by

\[
\Lambda_{S,T} := \left\{ x \in (\mathbb{Q} \otimes \bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(F))_{r,S} \mid (\phi_1 \wedge \cdots \wedge \phi_r)(x) \in \mathbb{Z}[G] \right\}.
\]

**Remark 4.2.** It is immediate that for \( r = 1 \), we have \( \Lambda_{S,T} = (U_{S,T}(F))_{1,S} \), where for a \( \mathbb{Z}[G] \)-module \( M \), \( \tilde{M} \) denotes the image of \( M \) via the canonical morphism \( M \to \mathbb{Q} \otimes M \). For a general \( r \geq 1 \), we have inclusions

\[
|G|^n \Lambda_{S,T} \subset \left( \bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(F) \right)_{r,S} \subset \Lambda_{S,T},
\]

for sufficiently large positive integer \( n \). Since \( U_{S,T}(F) \) has finite index in \( U_S(F) \),

\[
\mathbb{Q} \otimes \Lambda_{S,T} = (\mathbb{Q} \otimes \bigwedge_{\mathbb{Z}[G]}^r U_{S,T}(F))_{r,S} = (\mathbb{Q} \otimes \bigwedge_{\mathbb{Z}[G]}^r U_S(F))_{r,S}.
\]

Remark also that the module \((\mathbb{Q}U_S(F))_{r,S}\) is isomorphic to \((\mathbb{Q}[G]_{r,S})^r\) over \( \mathbb{Q}[G] \). Therefore, every element \( x \in (\mathbb{Q} \otimes \bigwedge_{\mathbb{Z}[G]}^r U_S(F))_{r,S} \) can be written as \( x_1 \wedge \cdots \wedge x_r \), with \( x_i \in (\mathbb{Q} \otimes U_S(F))_{r,S} \) for all \( i \).
Let $V = \{v_1, \cdots, v_r\}$ be a set of $r$ places in $S$, which split completely in $F/K$ and let $W = \{w_1, \cdots, w_r\}$ be a set of places of $F$ such that $w_i$ lies above $v_i$, for all $i = 1, \cdots, r$. For any place $w_i$, we define the $G$-equivariant map:

$$
\lambda_{w_i} : U_{S,T}(F) \xrightarrow{x} \mathbb{C}[G] \xrightarrow{- \sum_{\sigma \in G} \log(|\sigma(x)|_{w_i})\sigma^{-1}}.
$$

Rubin’s $\mathbb{C}[G]$-linear regulator

$$
\text{Reg}^{w_1, \cdots, w_r}_{S,T} : \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge
$$

is defined by

$$
\text{Reg}^{w_1, \cdots, w_r}_{S,T} := \lambda_{w_1} \wedge \cdots \wedge \lambda_{w_r}.
$$

Let $\Theta^{(r)}_{S,T}(0)$ be the coefficient of $s^r$ in the Taylor series of $\Theta_{S,T}$:

$$
\Theta^{(r)}_{S,T}(0) := \lim_{x \to 0} s^{-r} \Theta^{(r)}_{S,T}(s).
$$

**Conjecture 1.** RS$(F/K, S, T, r)$. Suppose that $(F/K, S, T, r)$ satisfies hypotheses $H(F/K, S, T, r)$. Then, for any choice of $V$ and $W$, there exists a unique element $\varepsilon_{F,S,T} \in \Lambda_{S,T}$ such that

$$
\text{Reg}^{w_1, \cdots, w_r}_{S,T}(\varepsilon_{F,S,T}) = \Theta^{(r)}_{S,T}(0).
$$

**Remark 4.3.** The truth of the Rubin-Stark conjecture RS$(F/K, S, T, r)$ does not depend on the particular choice of $V$ and $W$.

Now fix a totally real number field $K$ and let $r = [K : \mathbb{Q}]$. Recall that

$$
\chi : G_K \longrightarrow \mathcal{O}^\times
$$

is a non-trivial $\mathbb{Q}_p$-irreducible character, factoring through a finite abelian extension $L$ of $K$. Assume that $L$ and $K_\infty$ are linearly disjoint over $K$, and $L$ is the fixed field of $\ker \chi$. We continue to assume that our representation is

$$
T = \mathbb{Z}_p(1) \otimes \mathcal{O}(\chi^{-1}).
$$

For a cycle $\tau$ of $K$, let $K(\tau)$ be the maximal $p$-extension inside the ray class field of $K$ modulo $\tau$. Let $L_n = LK_n$ and let $f_n$ denote the finite part of the conductor of $L_n/K$. Remark that $f_n$ has the form $f_n = s_a$, where $s$ is prime to $s_n$ for all $n$ and does not depend on $n$; moreover $s_n$ is divisible only by those prime ideals of $\mathcal{O}_K$ which ramify in $L_\infty/L$. For any extension $F$ of $K$, we define $F(\tau)$ as the composite of $K(\tau)$ and $F$, and for any ideal $a$, we denote the product of all distinct prime ideals dividing $a$ by $\mathfrak{a}$. Let us also assume that $S$ contains the set $\Sigma_\infty$ and at least one finite place, but does not contain any $p$-adic prime of $K$. For any Galois extension $F$ of $K$, we denote the set of ramified primes in $F/K$ by $\text{Ram}(F/K)$. Let

$$
S_F = S \cup \text{Ram}(F/K).
$$

Following Büyükboduk [Bu 09] we choose $T = \{q_0\}$ where $q_0$ is a prime such that $p \mid Nq_0 - 1$ and $q_0 \nmid 2$, for such $T = \{q_0\}$ we have

$$
U^{S, \tau}_{S_F, \tau}(F) = U^{S_F}(F)
$$

Moreover, if $F$ is totally real then the hypothesis $H(F/K, S_F, \{q_0\}, r)$ is satisfied. Let

$$
K_0 = \{L_n, L_n(\tau) : \tau \text{ is a finite cycle of } K \text{ prime to } q_0f_a, \mathfrak{g} \mid \mathfrak{h} \text{ and } n \in \mathbb{Z}_{\geq 0}\}.$$
where $L_{n,g}$ is the maximal subextension of $L_n$ whose conductor is prime to $\mathfrak{h}p^{-1}$, and $f_\chi$ is the conductor of $\chi$. In the rest of this paper we assume

\[(H_0) : \text{the number field } L \text{ is totally real,}\]

and that

the conjecture $\text{RS}(F/K, S_F, \{q_0\}, r)$ holds, for all $F \in K_0$.

**Remark 4.4.** Let $\varepsilon_{n,g} = \varepsilon_{L_{n,g}, S_{L_{n,g}}(q_0), r}$ be the Rubin-Stark element for the conjecture $\text{RS}(L_{n,g}/K, S_{L_{n,g}}, \{q_0\}, r)$. Since $|S_{L_{n,g}}| > r + 1$ for $n \geq 1$, the element $\varepsilon_{n,g}$ lies in $\mathbb{Q} \otimes \mathbb{Z}^r \mathcal{E}_n$, where $\mathcal{E}_n$ denotes the group of global units of $L_{n,g}$.

**Definition 4.5.** Let $n$ be a nonnegative integer. We denote by $\text{St}_n$ the $\mathbb{Z}[	ext{Gal}(L_n/K)]$-module generated by the inverse images of $\varepsilon_{n,g}$ under the map $\mathbb{Q} \otimes \mathbb{Z}^r \mathcal{E}_n$ for all $g | \mathfrak{h}$.

Recall that for any number field $F$, Kummer theory gives a canonical isomorphism

$$H^1(F, \mathbb{Z}_p(1)) \cong F^{\times,\wedge} := \varprojlim F^\times / (F^\times)^{p^n}.$$ 

Since $\chi(G_{L_n(t)}) = 1$ for every $n \geq 0$,

$$H^1(L_n(t), \mathbb{Z}_p(1)) \otimes \mathcal{O}(\chi^{-1}) \cong H^1(L_n(t), \mathbb{Z}_p(1) \otimes \mathcal{O}(\chi^{-1})).$$

Therefore

$$L_n(t)^{x,\wedge} \otimes \mathcal{O}(\chi^{-1}) \cong H^1(L_n(t), \mathbb{Z}_p(1) \otimes \mathcal{O}(\chi^{-1})). \quad (6)$$

Let $\varepsilon_n(t) = \varepsilon_{L_n(t), S_{L_n(t)}, \{q_0\}}$ be the Rubin-Stark element for $\text{RS}(L_n(t)/K, S_{L_n(t)}, \{q_0\}, r)$. By Remark 4.2, $\varepsilon_n(t)$ can be uniquely written as $\varepsilon_1 \wedge \cdots \wedge \varepsilon_r$, with $\varepsilon_i \in \mathbb{Q} \otimes L_n(t)^{x}$. Let us note

$$\varepsilon_{n,\chi}(t) := \bar{\varepsilon}_1 \otimes 1_{\chi^{-1}} \wedge \cdots \wedge \bar{\varepsilon}_r \otimes 1_{\chi^{-1}} \quad (7)$$

where $\bar{\varepsilon}_i$ is the image of $\varepsilon_i$ by the natural map $\mathbb{Q} \otimes L_n(t)^{x} \longrightarrow \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_n(t)^{x,\wedge}$. Then under the isomorphism (6), we can view each

$$\varepsilon_{n,\chi}(t)$$

as an element of $\mathbb{Q}_p \otimes \bigwedge^r H^1(L_n(t), \mathbb{Z}_p(1) \otimes \mathcal{O}(\chi^{-1})).$

For any cycle $r$ which is prime to $q_0f_\chi p$, and every $n \geq 0$, we define

$$c_n(t) = \text{cor}_{L_{n+1}(t), K_n(t)}(\varepsilon_{n+1,\chi}(t)), \quad c_n = \text{cor}_{L_{n+1}, K_n}(\varepsilon_{n+1,\chi}) \quad (8)$$

where $\text{cor}_{L_{n+1}(t), K_n(t)}$ is the map

$$\mathbb{Q}_p \otimes \bigwedge^r H^1(L_{n+1}(t), T) \longrightarrow \mathbb{Q}_p \otimes \bigwedge^r H^1(K_n(t), T)$$

induced by the corestriction map

$$\text{cor}_{L_{n+1}(t), K_n(t)} : H^1(L_{n+1}(t), T) \longrightarrow H^1(K_n(t), T).$$

For the convenience of the reader we recall that for any finite group $G$ and any $\mathcal{O}[G]$-module $M$, we have a map

$$i_M : \bigwedge^{r-1}_\mathcal{O}[G] \text{Hom}_{\mathcal{O}[G]}(M, \mathcal{O}[G]) \longrightarrow \text{Hom}_{\Phi[G]}(\Phi \otimes \bigwedge^{r}_\mathcal{O}[G] M, \Phi \otimes M). \quad (9)$$
Indeed, the natural map $\text{Hom}_{O[G]}(M, O[G]) \longrightarrow \text{Hom}_{\Phi[G]}(\Phi \otimes M, \Phi[G])$ gives a morphism

$$\bigwedge_{O[G]}^{r-1}\text{Hom}_{O[G]}(M, O[G]) \longrightarrow \bigwedge_{\Phi[G]}^{r-1}\text{Hom}_{\Phi[G]}(\Phi \otimes M, \Phi[G]).$$

On the other hand, the map

$$f : \text{Hom}_{\Phi[G]}(\Phi \otimes M, \Phi[G]) \longrightarrow \text{Hom}_{\Phi[G]}(\bigwedge_{\Phi[G]}^{s} \Phi \otimes M, \bigwedge_{\Phi[G]}^{s-1} \Phi \otimes M)$$

defined by $f(\psi)(m_1 \wedge \cdots \wedge m_s) = \sum_{i=1}^{s} (-1)^{i+1}\psi(m_i)m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \cdots \wedge m_s$, and iterated from $s = r$ to $s = 2$ gives a morphism

$$\bigwedge_{\Phi[G]}^{r-1}\text{Hom}_{\Phi[G]}(\Phi \otimes M, \Phi[G]) \longrightarrow \text{Hom}_{\Phi[G]}(\bigwedge_{\Phi[G]}^{r} \Phi \otimes M, \Phi \otimes M).$$

Let us recall also that for any finite Galois extensions $F \subset F'$ of $K$, the norm map from $F'$ to $F$ induces a homomorphism

$$\bigwedge_{O[\Delta_{F'}]}^{r-1}\text{Hom}_{O[\Delta_{F'}]}(H^1(F', T), O[\Delta_{F'}]) \longrightarrow \bigwedge_{O[\Delta_{F}]}^{r-1}\text{Hom}_{O[\Delta_{F}]}(H^1(F, T), O[\Delta_{F}])$$

where $\Delta_F = \text{Gal}(F/K)$. In particular, the collection

$$\left( \bigwedge_{n, \tau}^{r-1}\text{Hom}_{O[\Delta_{K_n(\tau)}]}(H^1(K_n(\tau), T), O[\Delta_{K_n(\tau)}]) \right)_{n, \tau}$$

is a projective system for the maps given in (10).

**Definition 4.6.** Let $\Psi = \{\psi_{n, \tau}\}_{n, \tau}$ be an arbitrary element of

$$\lim_{\overset{\longrightarrow}{n, \tau}} \bigwedge_{n, \tau}^{r-1}\text{Hom}_{O[\Delta_{K_n(\tau)}]}(H^1(K_n(\tau), T), O[\Delta_{K_n(\tau)}]).$$

Let us identify $\psi_{n, \tau}$ with its image $\psi_M(\psi_{n, \tau})$ under (9), $M = H^1(K_n(\tau), T)$. We define

$$\varepsilon_{n, \Psi}(\tau)^X := \psi_{n, \tau}(c_n(\tau)) \quad \text{and} \quad \varepsilon_{n, \Psi}^X := \psi_n(c_n),$$

where $c_n(\tau)$ is given in Definition (8).

By the defining integrality property of the elements $\varepsilon_n(\tau)$ and Corollary 1.3 in [Ru 96], we get

$$\varepsilon_{n, \Psi}(\tau)^X \in H^1(K_n(\tau), T) \quad \text{and} \quad \varepsilon_{n, \Psi}^X \in H^1(K_n, T).$$

Remark that for every $n \geq 0$, we have $\text{cor}_{K_n, K_n}(\varepsilon_{n, \Psi}(1)^X) = \varepsilon_{n, \Psi}^X$.

**Proposition 4.7.** The collection $\{\varepsilon_{n, \Psi}(\tau)^X\}_{n \geq 0, \tau}$ is an Euler system for the $G_K$-representation $T$, in the sense of [Ru 00] Definition 2.1.1.

**Proof.** This is a consequence of Proposition 6.2 in [Ru 96]. □
5. Modifying the local condition at $p$.

In this section, we modify the classical local conditions at the primes above $p$ to obtain a Selmer structure $\mathcal{L}$ on $T$. To this end we assume the hypotheses $(H_1)$ and $(H_2)$ all throughout. The following theorem is crucial for our purpose. It is a direct consequence of [Gr 96, Theorem 2.2].

**Theorem 5.1.** For any $p$-adic prime $v$ of $K(\tau)$, the $O[[\text{Gal}(K_{\infty}(\tau)/\mathbb{Q}_p)]]$-module

$$H^1_{i_w}(K(\tau)_v, T) := \lim_{n} H^1(K_n(\tau)_v, T)$$

is free of rank one.

**Proof.** Let $F = L(\tau)$ and $F_n = L_n(\tau)$. Fix a place $w$ of $F$ lying above $v$. By [Gr 96] Theorem 2.2, we have an exact sequence of $\mathbb{Z}_p[[\text{Gal}(F_w(\mu_{p\infty})/\mathbb{Q}_p)]]$-modules

$$0 \rightarrow \mathbb{Z}_p(1) \rightarrow \prod_{n} U^1(F_{n,w}(\mu_p)) \rightarrow V(1) \rightarrow \mathbb{Z}_p(1) \rightarrow 0$$

where the module $V(1)$ is free cyclic over $\mathbb{Z}_p[[\text{Gal}(F_w(\mu_{p\infty})/\mathbb{Q}_p)]]$. Taking the $\text{Gal}(F_w(\mu_{p\infty})/F_{\infty,w})$-cohomology of the exact sequences

$$0 \rightarrow \mathbb{Z}_p(1) \rightarrow \prod_{n} U^1(F_{n,w}(\mu_p)) \rightarrow h \rightarrow 0,$$

and remarking that

$$H^i(F_w(\mu_{p\infty})/F_{\infty,w}, \mathbb{Z}_p(1)) = 0, \quad \text{for} \quad i \geq 0$$

we obtain

$$H^0(F_w(\mu_{p\infty})/F_{\infty,w}, \prod_{n} U^1(F_{n,w}(\mu_p)) \cong H^0(F_w(\mu_{p\infty})/F_{\infty,w}, V(1))$$

It follows that

$$\prod_{n} U^1(F_{n,w}) \cong H^0(F_w(\mu_{p\infty})/F_{\infty,w}, V(1)).$$

Since $\chi(\text{Frob}_v) \neq 1$, we obtain

$$H^1_{i_w}(K(\tau)_v, T) \cong (\prod_{n} U^1(F_{n,w}) \otimes O(\chi^{-1}))^{\text{Gal}(F_{\infty,w}/K_{\infty}(\tau)_v)} \cong (H^0(F_w(\mu_{p\infty})/F_{\infty,w}, V(1)) \otimes O(\chi^{-1}))^{\text{Gal}(F_{\infty,w}/K_{\infty}(\tau)_v)}.$$ 

Then $H^1_{i_w}(K(\tau)_v, T)$ is a free $O[[\text{Gal}(K_{\infty}(\tau)/\mathbb{Q}_p)]]$-module of rank one. \hfill $\square$

**Corollary 5.2.** The $O[[\text{Gal}(K_{\infty}(\tau)/K)]]$-module

$$H^1_{i_w}(K(\tau)_p, T)$$

is free of rank $[K : \mathbb{Q}]$. In particular, $H^1_{i_w}(K(\tau)_p, T)$ is a free $O$-module of rank $[K : \mathbb{Q}]$.

**Proof.** Let $v$ be a $p$-adic prime of $K(\tau)$. By Theorem 5.1 we see that $H^1_{i_w}(K(\tau)_v, T)$ is a free $O[[\text{Gal}(K_{\infty}(\tau)/K_w)]]$-module of rank $[K_w : \mathbb{Q}_p]$, where $w$ is a place of $K$ lying below $v$. Then

$$H^1_{i_w}(K(\tau)_p, T) = \oplus_{v \mid p} H^1_{i_w}(K(\tau)_v, T)$$

is a free $O[[\text{Gal}(K_{\infty}(\tau)/K)]]$-module of rank $\sum_{w \mid p} [K_w : \mathbb{Q}_p] = [K : \mathbb{Q}]$. \hfill $\square$

Let $\mathcal{K} = \bigcup_{n} K_n(\tau)$ and let $\mathcal{G}$ denote the Galois group $\text{Gal}(K/K)$.
**Corollary 5.3.** The $\mathcal{O}[[G]]$-module $\mathbb{V} := \varprojlim_{n \to \infty} H^1(K_n(v)_p, T) = \varprojlim_{n \to \infty} H^1_w(K(v)_p, T)$ is free of rank $[K : \mathbb{Q}]$.

**Proof.** Immediate after Corollary 5.2. □

**Remark 5.4.** This corollary is a generalization of [Bü09, Corollary 3.10].

If $F \subset F'$ is an extension of fields, we will write $F \subset F'$ to indicate that $[F' : F]$ is finite.

**Proposition 5.5.** Let $K \subset F \subset \mathbb{K}$ be a finite extension and let $G_F$ denote the Galois group $\text{Gal}(\mathbb{K}/F)$. Let $d = \max\{v_p(1 - \chi(\text{Frob}_p))\}$, where $p$ is a prime of $K$. Then the canonical map

$$
\mathbb{V} \rightarrow H^1(F_p, T)
$$

is injective, with cokernel annihilated by $p^d$.

**Proof.** For any $p$-adic place $v$ of $F$, we fix a place $\mathfrak{p}$ of $K$ lying above $v$. One has

$$
\mathbb{V}_{G_F} \cong \bigoplus_{v \mid p} \left( \varprojlim_{F_v \subset F_w \subset K_{\mathfrak{p}}} H^1(F_w', T) \right)_{D_v(K/F)}
$$

where $D_v(K/F)$ is the decomposition subgroup of $v$ in $K/F$. By dualizing the inflation-restriction exact sequence

$$
H^1(D_v(K/F), (T^*)^{G_{K/F}}) \rightarrow H^1(F_v, T^*) \rightarrow H^1(K_{\mathfrak{p}}, T^*)^{D_v(K/F)} \rightarrow H^2(D_v(K/F), (T^*)^{G_{K/F}}),
$$

we obtain an exact sequence

$$
H^2(D_v(K/F), (T^*)^{G_{K/F}})^\vee \rightarrow (H^1(K_{\mathfrak{p}}, T^*)^{D_v(K/F)})^\vee \rightarrow H^1(F_v, T^*)^\vee \rightarrow (H^1(D_v(K/F), (T^*)^{G_{K/F}}))^\vee
$$

Since $H^2(D_v(K/F), (T^*)^{G_{K/F}})^\vee$ is a torsion $\mathcal{O}$-module and

$$(H^1(K_{\mathfrak{p}}, T^*)^{D_v(K/F)})^\vee \cong \varprojlim_{F_v \subset F_w \subset K_{\mathfrak{p}}} H^1(F_w', T)_{D_v(K/F)}$$

is a torsion-free $\mathcal{O}$-module, then we have an exact sequence:

$$
0 \rightarrow (H^1(K_{\mathfrak{p}}, T^*)^{D_v(K/F)})^\vee \rightarrow H^1(F_v, T^*)^\vee \rightarrow (H^1(D_v(K/F), (T^*)^{G_{K/F}}))^\vee.
$$

Since $T^* = D(\chi)$, then for every $K \subset F \subset \mathbb{K}$, $H^0(K_p, T) = H^0(F_p, T^*)$. Therefore

$$
p^d(T^*)^{G_{K/F}} = 0
$$

where $d = \max\{v_p(1 - \chi(\text{Frob}_p))\}$, $p$ is a prime of $K$. It follows that the canonical map

$$
\mathbb{V}_{G_F} \rightarrow H^1(F_p, T)
$$

is injective, with cokernel annihilated by $p^d$. □

As Büyükboduk did in [Bü09], we fix a free $\mathcal{O}[[\text{Gal}(\mathbb{K}/K)]]$-direct summand $L$ inside of

$$
\mathbb{V} = \varprojlim_{K \subset F \subset \mathbb{K}} H^1(F_p, T) = \varprojlim_{K \subset F \subset \mathbb{K}} \mathbb{V}_{G_F}
$$

which is free of rank one as $\mathcal{O}[[\text{Gal}(\mathbb{K}/K)]]$-module. Recall that $\Sigma$ is a finite set of places of $K$ containing all infinite places, all $p$-adic places and all places where $T$ is ramified.
Definition 5.6. Define the modified Selmer structure $\mathcal{L}$ on $T$ by

- $\Sigma(\mathcal{L}) = \Sigma,$
- if $w \nmid p$, $H^1_{\mathcal{L}}(F_w, T) = H^1_{\mathcal{L}_{\text{can}}}(F_w, T),$
- $H^1_{\mathcal{L}}(F, T) \subset H^1(F, T)$ as the $O$-saturation of $\mathcal{L}_{\mathcal{L}}$ in $H^1(F, T)$.

5.1. Choosing homomorphisms. Let us keep the same notation as above. In this subsection, we show the existence of a homomorphism $\Psi' = (\Psi'_{\mathcal{L}})_{\mathcal{L} \subset F \subset \mathcal{K}}$ such that

$$\Psi'_{\mathcal{L}}(\bigwedge^r H^1(F, T)) \subset H^1_{\mathcal{L}}(F, T).$$

Fix a basis $\{\psi_{\mathcal{L}}^{(i)}\}_{i=1}^{r-1}$ of the free $O[[\text{Gal}(\mathcal{K}/\mathcal{K})]]$-module

$$\text{Hom}_O[[\text{Gal}(\mathcal{K}/\mathcal{K})]](V/\mathcal{L}, O[[\text{Gal}(\mathcal{K}/\mathcal{K})]])$$

of rank $r - 1$. This in return fixes a basis $\{\psi_{\mathcal{L}}^{(i)}\}_{i=1}^{r-1}$ for the free $O[\Delta_F]$-module

$$\text{Hom}_{O[\Delta_F]}(V_{\mathcal{L}F}/\mathcal{L}_{\mathcal{L}F}, O[\Delta_F])$$

for all $\mathcal{L} \subset F \subset \mathcal{K}$, such that the homomorphisms $\{\psi_{\mathcal{L}F}^{(i)}\}_{i=1}^{r-1}$ are compatible with respect to the maps

$$\text{Hom}_{O[\Delta_F]}(V_{\mathcal{L}F}/\mathcal{L}_{\mathcal{L}F}, O[\Delta_F]) \longrightarrow \text{Hom}_{O[\Delta_F]}(V_{\mathcal{L}F}/\mathcal{L}_{\mathcal{L}F}, O[\Delta_F]).$$

for $\mathcal{L} \subset F'$. Let $\psi_{\mathcal{L}}^{(i)}$ denote the image of $\psi_{\mathcal{L}F}^{(i)}$ under the canonical injection

$$\text{Hom}_{O[\Delta_F]}(V_{\mathcal{L}F}/\mathcal{L}_{\mathcal{L}F}, O[\Delta_F]) \longrightarrow \text{Hom}_{O[\Delta_F]}(V_{\mathcal{L}F}, O[\Delta_F]).$$

Remark that the map

$$\Psi_{\mathcal{L}F} := \bigoplus_{i=1}^{r-1} \psi_{\mathcal{L}F}^{(i)} : V_{\mathcal{L}F} \longrightarrow O[\Delta_F]^{r-1}$$

is surjective and $\ker(\Psi_{\mathcal{L}F}) = \mathcal{L}_{\mathcal{L}F}$.

Define

$$\Psi_F := \psi_{\mathcal{L}F}^{(1)} \wedge \psi_{\mathcal{L}F}^{(2)} \wedge \cdots \wedge \psi_{\mathcal{L}F}^{(r-1)} \in \bigwedge^{r-1} \text{Hom}_{O[\Delta_F]}(V_{\mathcal{L}F}, O[\Delta_F]).$$

We may therefore regard $\Psi := (\Psi_F)_{\mathcal{L} \subset F \subset \mathcal{K}}$ as an element of the module

$$\lim_{\mathcal{L} \subset F \subset \mathcal{K}} \bigwedge^{r-1} \text{Hom}_{O[\Delta_F]}(V_{\mathcal{L}F}, O[\Delta_F]).$$

Proposition 5.7. Let $\Psi := (\Psi_F)_{\mathcal{L} \subset F \subset \mathcal{K}}$ be as above. Then for every $\mathcal{L} \subset F \subset \mathcal{K}$, $\Psi_F$ induces an isomorphism

$$\Psi_F : \bigwedge^r V_{\mathcal{L}F} \longrightarrow \mathcal{L}_{\mathcal{L}F} = \ker(\Psi_{\mathcal{L}F}).$$

Proof. The proof is identical to the proof of [Bü 09 Proposition 3.17], which follows the proof of Lemma B.1 of [MR 04].

Proposition 5.8. There exists an element

$$\Psi' = (\Psi'_{\mathcal{L}})_{\mathcal{L} \subset F \subset \mathcal{K}} \in \lim_{\mathcal{L} \subset F \subset \mathcal{K}} \bigwedge^{r-1} \text{Hom}_{O[\Delta_F]}(H^1(F, T), O[\Delta_F]).$$
such that for any $K \subset_f F \subset \mathcal{K}$,
\[ \psi_p(\bigwedge^r H^1(F_p, T)) \subset H^1_F(F_p, T). \]

**Proof.** Thanks to Proposition 5.5, the canonical map
\[ \mathcal{V}_{\mathcal{G}_p} \longrightarrow H^1(F_p, T) \]
is injective, with cokernel annihilated by $p^d$. Then the cokernel of the canonical map
\[ \text{Hom}_{\mathcal{O}[\Delta_F]}(H^1(F_p, T), \mathcal{O}[\Delta_F]) \longrightarrow \text{Hom}_{\mathcal{O}[\Delta_F]}(\mathcal{V}_{\mathcal{G}_p}, \mathcal{O}[\Delta_F]) \]
is annihilated by $p^d$. Hence the cokernel of
\[ \lim_{\substack{\longrightarrow \kappa \in F \subset \mathcal{K}}} \bigwedge^{r-1} \text{Hom}_{\mathcal{O}[\Delta_F]}(H^1(F_p, T), \mathcal{O}[\Delta_F]) \longrightarrow \lim_{\substack{\longrightarrow \kappa \in F \subset \mathcal{K}}} \bigwedge^{r-1} \text{Hom}_{\mathcal{O}[\Delta_F]}(\mathcal{V}_{\mathcal{G}_p}, \mathcal{O}[\Delta_F]) \] \hspace{1cm} (11)
is annihilated by $p^{d(r-1)}$. Therefore $p^{d(r-1)} \Psi$ is an element of the image of the map (11), where $\Psi$ is defined in Proposition 5.7. Then
\[ (p^{d(r-1)} \Psi)(\bigwedge^r H^1(F_p, T)) \subset L_{\mathcal{G}_p} \subset H^1_F(F_p, T). \]

5.2. Kolyvagin systems for $(T, \mathcal{L})$. In this subsection, we show that the Kolyvagin’s derivative class associated to the Euler system of Rubin-Stark elements defines a Kolyvagin system for the modified Selmer structure $\mathcal{L}$.

Let $c_{K, \infty} = \{ \varepsilon_{n, \psi} \} \in H^1_F(K, T)$ denote the element corresponding to the Euler system $\{ \varepsilon_{n, \psi}(\tau) \}^{\infty}_{n, \tau}$ in $H^1_F(K, T) = \lim_{\tau} H^1(K, T)$.

**Proposition 5.9.** Let $\mathbb{H}_L$ be the set of the maps $\Psi = (\Psi_F) \in \lim_{\substack{\longrightarrow \kappa \in F \subset \mathcal{K}}} \bigwedge^{r-1} \text{Hom}_{\mathcal{O}[\Delta_F]}(\mathcal{V}_{\mathcal{G}_p}, \mathcal{O}[\Delta_F])$ such that $\Psi_F(\bigwedge^r \mathcal{V}_{\mathcal{G}_p}) = L_{\mathcal{G}_p}$. Let $\text{loc}_p$ denote the localization map at $p$;
\[ \text{loc}_p : H^1(K, T) \longrightarrow H^1(K_p, T). \]

Then
\[ \{ \text{loc}_p(\varepsilon_{n, \psi}^r) : \Psi \in \mathbb{H}_L \} = [\bigwedge^r \mathcal{V}_G : \mathcal{O} \circ \text{loc}_p^r(c_0)] L_{\mathcal{G}_p}. \]

**Proof.** The proof is identical to the proof of Corollary 3.5 of [Bu 08] line by line. \hfill \Box

**Remark 5.10.** If the localization map $\text{loc}_p : H^1(K, T) \longrightarrow H^1(K_p, T)$ is injective, then by Proposition 6.6 (ii), we see that $[\bigwedge^r \mathcal{V}_G : \mathcal{O} \circ \text{loc}_p^r(c_0)] < \infty$.

Let $F$ be a finite extension of $K$ in $\mathcal{K}$ and let $\mathcal{F}$ be a Selmer structure on $T$. For any cycle $w$ of $K$, we write $\mathcal{F}^w$ for the Selmer structure defined by
- $\Sigma(\mathcal{F}^w) = \Sigma(\mathcal{F}) \cup \Sigma_w$
- $H^1_{\mathcal{F}^w}(F_w, T) = \begin{cases} H^1_F(F_w, T), & w \in \Sigma(\mathcal{F}) - \Sigma; \\ H^1(F_w, T), & w \in \Sigma_w. \end{cases}$
where $\Sigma = \{ w \subset F : w \mid \frak{r} \}$.
Let $M$ be a power of a uniformizer of $\mathcal{O}$ and let $W_M = T/MT$. Recall that for any Euler system $c$ of $T$ we can associate a Kolyvagin derivative class $\kappa_{[K_n, r, M]}$, see [Ru 00, §4.4]. Recall also that
$$\kappa_{[K_n, r, M]} \in H^1_{\text{can}}(K_n, W_M)$$
c.f. [Ru 00, Theorem 4.5.1]. Next, we construct an Euler system $c$ of $T$ such that
$$\kappa_{[K_n, r, M]} \in H^1_{L}(K_n, W_M)$$
For this we need some facts about the local condition $L$.

**Lemma 5.11.** Let $L_F := \mathcal{L}_F$, then
$$p^d.(H^1_{L}(F_p, T)/L_F) = 0$$
where $d = \max_{p \mid p}\{v_p(1 - \chi(\text{Frob}_p))\}.$

**Proof.** First consider, the exact sequence
$$0 \longrightarrow H^1_{L}(F_p, T)/L_F \longrightarrow H^1(F_p, T)/L_F \longrightarrow H^1(F_p, T)/H^1_{L}(F_p, T)$$
By definition, the $\mathcal{O}$-module $H^1(F_p, T)/H^1_{L}(F_p, T)$ is torsion-free, then
$$\text{tor}_{\mathcal{O}}(H^1(F_p, T)/L_F) = H^1_{L}(F_p, T)/L_F.$$  
Second the facts that $\mathcal{V}_{\mathcal{G}_F}/\mathcal{L}_F$ is $\mathcal{O}$-torsion-free and $H^1(F_p, T)/\mathcal{V}_{\mathcal{G}_F}$ is $\mathcal{O}$-torsion, and the exact sequence
$$0 \longrightarrow \mathcal{V}_{\mathcal{G}_F}/\mathcal{L}_F \longrightarrow H^1(F_p, T)/L_F \longrightarrow H^1(F_p, T)/\mathcal{V}_{\mathcal{G}_F}$$
show that
$$\text{tor}_{\mathcal{O}}(H^1(F_p, T)/L_F) \cong H^1(F_p, T)/\mathcal{V}_{\mathcal{G}_F}.$$  
Then, by Proposition 5.5 we get
$$p^d.(H^1_{L}(F_p, T)/L_F) = 0. \quad \Box$$

**Proposition 5.12.** Let $G_{n, r} = \text{Gal}(K_n(r)/K_n)$. Then the cokernel of
$$H^1_{L}(K_n, W_M) \longrightarrow H^1_{L}(K_n(r)\cdot W_M)_{G_{n, r}}$$
is annihilated by $p^d$.

**Proof.** Let $F = K_n(r)$ or $F = K_n$. Consider the exact sequence
$$0 \longrightarrow T \overset{M}{\longrightarrow} T \longrightarrow W_M \longrightarrow 0.$$  
Using [MR 04, Lemma 3.7.1], [MR 04, Lemma 1.1.5] and the fact that $H^1(F_p, T)$ is torsion-free $\mathcal{O}$-module, we get an exact sequence
$$0 \longrightarrow H^1_{L}(F_p, T) \overset{M}{\longrightarrow} H^1_{L}(F_p, T) \longrightarrow H^1_{L}(F_p, W_M) \longrightarrow 0.$$
Since the restriction \( \text{res} : H^1(K_{n,p}, T) \rightarrow H^1(K_n(t), T) \) is an isomorphism and the \( \mathcal{O} \)-module \( H^1(K_{n,p}, T)/H^1_{\varepsilon}(K_{n,p}, T) \) is torsion-free, the commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & H^1_{\varepsilon}(K_{n,p}, T) & \rightarrow & H^1(K_{n,p}, T) & \rightarrow & H^1(K_{n,p}, T)/H^1_{\varepsilon}(K_{n,p}, T) & \rightarrow & 0 \\
\downarrow & & \downarrow \text{res} & & \downarrow & & \downarrow \\
0 & \rightarrow & H^1_{\varepsilon}(K_n(t), T)^{G_{n,t}} & \rightarrow & H^1(K_n(t), T)^{G_{n,t}} & \rightarrow & (H^1(K_n(t), T)/H^1_{\varepsilon}(K_n(t), T)^{G_{n,t}}) \\
\end{array}
\]

shows that the map \( H^1_{\varepsilon}(K_{n,p}, T) \xrightarrow{\text{res}} H^1_{\varepsilon}(K_n(t), T)^{G_{n,t}} \) is an isomorphism. Therefore, we have an exact commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & H^1_{\varepsilon}(K_{n,p}, T) & \rightarrow & H^1_{\varepsilon}(K_{n,p}, T) & \rightarrow & H^1_{\varepsilon}(K_{n,p}, W_{M}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^1_{\varepsilon}(K_n(t), T)^{G_{n,t}} & \rightarrow & H^1_{\varepsilon}(K_n(t), T)^{G_{n,t}} & \rightarrow & H^1_{\varepsilon}(K_n(t), T)^{G_{n,t}} & \rightarrow & 0 \\
\end{array}
\]

Then by the snake lemma, we get

\[
\text{coker}(H^1_{\varepsilon}(K_{n,p}, W_{M}) \rightarrow H^1_{\varepsilon}(K_n(t), T)^{G_{n,t}}) \cong \text{coker}(H^1_{\varepsilon}(K_n(t), T)^{G_{n,t}} \rightarrow H^1_{\varepsilon}(K_n(t), W_{M})^{G_{n,t}}) = H^1(G_{n,t}, H^1_{\varepsilon}(K_n(t), T))[M]
\]

where \( H^1(G_{n,t}, H^1_{\varepsilon}(K_n(t), T))[M] \) is the submodule of \( H^1(G_{n,t}, H^1_{\varepsilon}(K_n(t), T)) \) annihilated by \( M \). Therefore it suffices to prove that

\[
p^d \cdot H^1(G_{n,t}, H^1_{\varepsilon}(K_n(t), T)) = 0.
\]

For this, consider the exact sequence

\[
0 \rightarrow \mathcal{L}_{K_n(t)} \rightarrow H^1_{\varepsilon}(K_n(t), T) \rightarrow H^1_{\varepsilon}(K_n(t), T)/\mathcal{L}_{K_n(t)} \rightarrow 0.
\]

By cohomology we obtain the exact sequence

\[
H^1(G_{n,t}, \mathcal{L}_{K_n(t)}) \rightarrow H^1(G_{n,t}, H^1_{\varepsilon}(K_n(t), T)) \rightarrow H^1(G_{n,t}, H^1_{\varepsilon}(K_n(t), T)/\mathcal{L}_{K_n(t)}) \rightarrow H^2(G_{n,t}, \mathcal{L}_{K_n(t)})
\]

Since \( \mathcal{L}_{K_n(t)} \) is a free summand of \( \mathbb{V}_{\text{Gal}(K/K_n(t))} \) as \( \mathcal{O}[\text{Gal}(K_n(t)/K)] \)-module, it follows that

\[
H^i(G_{n,t}, \mathcal{L}_{K_n(t)}) = 0 \quad \text{for} \quad i \geq 1.
\]

Hence

\[
H^1(G_{n,t}, H^1_{\varepsilon}(K_n(t), T)) \cong H^1(G_{n,t}, H^1_{\varepsilon}(K_n(t), T)/\mathcal{L}_{K_n(t)}).
\]

Since \( p^d \cdot H^1_{\varepsilon}(K_n(t), W_{M})/\mathcal{L}_{K_n(t)} = 0 \) (see Lemma 5.11), we get

\[
p^d \cdot H^1(G_{n,t}, H^1_{\varepsilon}(K_n(t), T)) = 0.
\]

Let \( c_n(t) \in \mathbb{Q}_p \otimes \bigwedge^r H^1(K_n(t), T) \) be the element defined in \( \mathcal{S} \) and let \( \text{loc}_p \) denote the localization map into the semi-local cohomology at \( p \)

\[\text{loc}_p : \mathbb{Q}_p \otimes H^1(K_n(t), T) \rightarrow \mathbb{Q}_p \otimes H^1(K_n(t), T).\]

Since \( \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{V}_{G_{K_n(t)}} \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^1(K_n(t), T), \) it follows that

\[
\text{loc}_p(c_n(t)) \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge^r \mathbb{V}_{G_{K_n(t)}}.
\]
The defining (integrality) property of the Rubin-Stark elements shows that, for any
\[ \psi = \psi_1 \wedge \cdots \wedge \psi_r \in \bigwedge^r \text{Hom}(\mathcal{V}_{\mathcal{G}_{K_n(r)}}, \mathcal{O}[\text{Gal}(K_n(r)/K)]) \]
we have
\[ \psi(\text{loc}^p_{c_n}(t)) \in \mathcal{O}[\text{Gal}(K_n(r)/K)]. \]
Hence, by Example 1 following Proposition 1.2 in [Ru 96], we get
\[ \text{loc}^p_{c_n}(t) \in \bigwedge^r \mathcal{V}_{\mathcal{G}_{K_n(r)}}. \]
Let \( F \) be a finite extension of \( K \) in \( \mathcal{K} \), the map
\[ \text{loc}_p : H^1(F,T) \longrightarrow H^1(F_p,T). \]
induces a map
\[ \lim_{K \subseteq F \subseteq \mathcal{K}} \bigwedge^{r-1} \text{Hom}_{\mathcal{O}[\Delta_F]}(H^1(F_p,T), \mathcal{O}[\Delta_F]) \longrightarrow \lim_{K \subseteq F \subseteq \mathcal{K}} \bigwedge^{r-1} \text{Hom}_{\mathcal{O}[\Delta_F]}(H^1(F,T), \mathcal{O}[\Delta_F]). \]
The image of \( \Psi \in \lim_{K \subseteq F \subseteq \mathcal{K}} \bigwedge^{r-1} \text{Hom}_{\mathcal{O}[\Delta_F]}(H^1(F_p,T), \mathcal{O}[\Delta_F]) \) will still be denoted by \( \Psi \). Let \( \Psi' \) be the element constructed in Proposition 4.8 and let \( \{ \varepsilon_{n,\psi}(t)^{\chi} \}_{n,\tau} \) be the Euler system for \( T \) associated to \( \Psi' \) (see Proposition 4.7). Using Proposition 4.8 and the fact that \( \text{loc}^p_{c_n}(t) \in \bigwedge^r \mathcal{V}_{\mathcal{G}_{K_n(r)}} \), we see that
\[ \text{loc}_p(\varepsilon_{n,\psi}(t)^{\chi}) \in H^1_\mathcal{G}(K_n(t), T). \]
Let \( \mathcal{G} \) and \( \mathcal{F} \) be Selmer structures on \( T \). Following [MR 04] §2.1, we say that \( \mathcal{G} \leq \mathcal{F} \) if
\[ H^1_\mathcal{G}(K_v,T) \subset H^1_\mathcal{F}(K_v,T) \quad \text{for all prime } v. \]
If \( \mathcal{G} \leq \mathcal{F} \) we have an exact sequence [MR 04] Theorem 2.3.4]
\[ H^1_\mathcal{G}(K,T) \longrightarrow H^1_\mathcal{F}(K,T) \longrightarrow \bigoplus_v H^1_\mathcal{F}(K_v,T)/H^1_\mathcal{G}(K_v,T) \longrightarrow H^1_\mathcal{F}(K,T^*) \longrightarrow H^1_\mathcal{F}(K,T^*) \]
The following lemma is crucial for our purpose

**Lemma 5.13.** Let \( \kappa_{[K_n,r,M]} \) denote the Kolyvagin’s derivative class, associated to the Euler system \( c = \{ p^d, \varepsilon_{n,\psi}(t)^{\chi} \}_{n,\tau} \), constructed in [Ru 00] Chap IV, §4. Then
\[ \kappa_{[K_n,r,M]} \in H^1_{\mathcal{G}_{r}}(K_n,W_M) \]

**Proof.** Since \( \mathcal{G} \leq \mathcal{F}_{\text{can}} \), we have an exact sequence
\[ H^1_{\mathcal{G}_{r}}(K_n,W_M) \longrightarrow H^1_{\mathcal{F}_{\text{can}}}(K_n,W_M) \longrightarrow H^1(K_{n,p},W_M)/H^1_{\mathcal{G}_{r}}(K_{n,p},W_M). \]

Theorem 4.5.1 of [Ru 00] shows that \( \kappa_{[K_n,r,M]} \in H^1_{\mathcal{F}_{\text{can}}}(K_n,W_M) \). Then it suffices to prove that
\[ \text{loc}_p(\kappa_{[K_n,r,M]}) \in H^1_{\mathcal{F}_{\text{can}}}(K_{n,p},W_M) \]
where \( \text{loc}_p \) is the localization map into the semi-local cohomology at \( p \). Let \( D_\chi \) denote the derivative operator, defined as in [Ru 00] Definition IV.4.1. Since
\[ \text{loc}_p : H^1(K_n(t), T) \longrightarrow H^1(K_n(t)_p, T) \]
is Galois equivariant,
\[
\text{loc}_p(D_{\epsilon}p^d \cdot \varepsilon_n \cdot \Psi(t)^\chi) = D_{\epsilon} \text{loc}_p(p^d \cdot \varepsilon_n \cdot \Psi(t)^\chi).
\]
Furthermore, using \([12]\), we get
\[
\text{loc}_p(p^d \cdot \varepsilon_n \cdot \Psi(t)^\chi) \in H^1_L(K_n(t)_p, T).
\]
On the other hand, by \([Ru 00\, Lemma 4.4.2]\), \(D_{\epsilon}p^d \cdot \varepsilon_n \cdot \Psi(t)^\chi \mod M\) is fixed by \(\text{Gal}(K_n(t)/K_n)\), then
\[
\text{loc}_p(D_{\epsilon}p^d \cdot \varepsilon_n \cdot \Psi(t)^\chi) \mod M \in (H^1_L(K_n(t)_p, T) / MH^1_L(K_n(t)_p, T))^{\text{Gal}(K_n(t)/K_n)}.
\]
By \([Ru 00\, Lemma 4.4.13]\) and Proposition \(5.12\) we have
\[
\text{loc}_p(\kappa_{[K_n, \tau, M]}) \in H^1_L(K_{n,p}, W_M).
\]
This finishes the proof of the lemma. \(\square\)

6. PROOF OF THEOREM 1.3

Recall that we proved in Proposition \(5.8\) the existence of an element
\[
\Psi' = \{\psi'_F\}_F \in \bigcap_{K \leq F \leq K} \text{Hom}_O[H^1(F_p, T), O[\Delta_F]]
\]
such that \(\Psi'_F(A^r H^1(F_p, T)) \subset H^1_L(F_p, T)\). Let \(c_{K, \infty} = \{p^d \cdot \varepsilon_n \cdot \Psi\}_n \in H^1_{Iw}(K, T)\) denote the element corresponding to the Euler system \(\{p^d \cdot \varepsilon_n \cdot \Psi(t)^\chi\}_n\) in \(H^1_{Iw}(K, T) := \varprojlim_n H^1(K_n, T)\).

Remark that under the Leopoldt conjecture for \(L\), the localization map
\[
\text{loc}_p : H^1(K, T) \longrightarrow H^1(K_p, T)
\]
is injective. Then by Remark \(5.10\) and Proposition \(5.9\) we can find an element \(\Psi'\) such that \(c_{K, \infty} \neq 0\).

One of the keys of the proof of the main theorem of this article is the following result

**Theorem 6.1.** Suppose the hypotheses \((H_0)\) and \((H_3)\) hold. Then
\[
\text{char}(H^1_L(K_\infty, T^\vee)) \text{ divides } \text{char}(H^1_L(K_\infty, T) / \lambda_c c_{K, \infty})
\]

**Proof.** Remark that for any place \(v \nmid p\)
\[
H^1_L(F_v, T) = H^1_{F_{can}}(F_v, T).
\]
Then the proof of this theorem is similar to the proof of \([Ru 00\, Theorem 2.3.3]\) if we replace
\[
S_{\Sigma_p}(F, W_M^*) \text{ by } H^1_L(T, T^*[M])
\]
and
\[
S_{\Sigma_{\epsilon}}(F, W_M^*) \text{ by } H^1_{L\cdot}(F, W_M).
\]
We only need to justify the following facts:
(i) \(\kappa_{[K_n, \epsilon, M]} \in H^1_L(K_n, W_M)\).
(ii) \((H^1_L(K_\infty, T^\vee))_{\Gamma_n}\) and \(\Lambda_{\Gamma_n} / \text{char}(H^1_L(K_\infty, T^\vee))\) are finite.
The assertion (i) is Lemma 5.13. For the assertion (ii), on the one hand, we have a surjective map
\[ H_{F_{str}}^1(K_{\infty}, T^*)^\vee \longrightarrow H_{E_{str}}^1(K_{\infty}, T^*)^\vee. \]

On the other hand, the kernel of the restriction
\[ H_{F_{str}}^1(K_{\infty}, T^*) \longrightarrow H_{F_{str}}^1(L_{\infty}, T^*) \]
is finite. Therefore, taking the inverse limit in Lemma 5.11 we deduce that the \( \Lambda \)-modules
\[ H_{F_{str}}^1(K_{\infty}, T^*)^\vee \]
and
\[ H_{E_{str}}^1(K_{\infty}, T^*)^\vee \]
are pseudo-isomorphic. The \( \Lambda \)-modules
\[ H_{I_{w,L}}^1(K_\infty, T) \]
are also pseudo-isomorphic, thanks to Proposition 5.25. Therefore, by Proposition 5.7 we conclude that the \( \Lambda \)-modules
\[ H_{I_{w,L}}^1(K_p, T) \]
are pseudo-isomorphic. (13)

Let \( \iota \) denote the composite of the natural maps
\[ \bigwedge_n \lim_{n} H^1(K_n, T) \longrightarrow \lim_{n} \bigwedge_n H^1(K_n, T) \longrightarrow \lim_{n} (Q_p \otimes \mathbb{Z}_p \bigwedge_n H^1(K_n, T)) \]
and let \( c_\infty \) be the maximal abelian \( \mathbb{Z}_p \)-extension of \( K \), which is unramified outside of the primes above \( p \) and \( \mathcal{X}_{\infty} = \text{Gal}(M_{\infty}/L_{\infty}). \)

We may identify \( H_{I_{w,L}}^1(K_{\infty}, T) \) with \( \mathcal{X}_{\infty} \) c.f. [Ru 00 §I.6.3].

The Leopoldt conjecture for \( L_1 \) shows that \( \mathcal{X}_{\infty} = 0 \) e.g. [NSW 91 Proposition 11.3.3]. Then
\[ H_{F_{str}}^1(L_{\infty}, T^*)^\vee \]
is finite. □

**Proposition 6.2.** Suppose that the hypotheses \((H_0)\) and \((H_3)\) hold. Then
\[ \text{char}(H_{F_{str}}^1(K_{\infty}, T^*)^\vee) \]
divides \( \text{char}(H_{I_{w,L}}^1(K_p, T)/\text{loc}_p(c_{K,\infty})) \)

**Proof.** Since \( F_{str} \leq L \), we have an exact sequence
\[ H_{F_{str}}^1(K_{\infty}, T)^\vee \longrightarrow H_E^1(K_{\infty}, T) \longrightarrow H_{I_{w,L}}^1(K_p, T) \longrightarrow H_{F_{str}}^1(K_{\infty}, T^*)^\vee \longrightarrow H_{E_{str}}^1(K_{\infty}, T^*)^\vee. \]

Lemma 5.2 shows that \( H_{F_{str}}^1(K_{\infty}, T) = 0 \). Then we have an exact sequence
\[ 0 \longrightarrow H_{I_{w,L}}^1(K_{\infty}, T)/\Lambda, c_{K,\infty} \longrightarrow H_{I_{w,L}}^1(K_p, T)/\text{loc}_p(c_{K,\infty}) \longrightarrow H_{F_{str}}^1(K_{\infty}, T^*)^\vee \longrightarrow H_{E_{str}}^1(K_{\infty}, T^*)^\vee. \]

Theorem 6.1 permits to conclude. □
Theorem 6.3. Let $c$ be an element in $i^{-1}(p^d,c_\infty)$. Under the hypotheses $(H_0)$ and $(H_3)$,
\[
\text{char}(H^1_{\text{str}}(K_\infty,T^*)^\vee) \mid \text{char}(\bigwedge^r H^1_{\text{can}}(K_\infty,T))/\Lambda.c)
\]
Proof. Since $\mathcal{F}_{\text{str}} \leq \mathcal{F}_{\text{can}}$ and $H^1_{\mathcal{F}_{\text{str}}}(K_\infty,T) = 0$ (see Proposition 3.2), we have an exact sequence
\[
H^1_{\mathcal{F}_{\text{can}}}(K_\infty,T) \xrightarrow{\text{loc}_p} H^1_{\text{Iw}}(K_p,T) \rightarrow H^1_{\mathcal{F}_{\text{str}}}(K_\infty,T^*)^\vee \rightarrow H^1_{\mathcal{F}_{\text{can}}}(K_\infty,T^*)^\vee.
\]
Proposition 3.3 (resp. Lemma 5.2) shows that the $\Lambda$-module $H^1_{\mathcal{F}_{\text{can}}}(K_\infty,T)$ (resp. $H^1_{\text{Iw}}(K_p,T)$) is free of rank $r$, then the injection $H^1_{\mathcal{F}_{\text{can}}}(K_\infty,T) \xrightarrow{\text{loc}_p} H^1_{\text{Iw}}(K_p,T)$ induces an exact sequence:
\[
0 \rightarrow (\bigwedge^r H^1_{\mathcal{F}_{\text{can}}}(K_\infty,T))/\Lambda.c \rightarrow (\bigwedge^r H^1_{\text{Iw}}(K_p,T))/\Lambda.(\text{loc}_p^r(c)) \rightarrow \text{coker}(\text{loc}_p^r(c)),
\]
where $\text{loc}_p^r$ denotes the map induced on the $r$-th exterior power. Hence
\[
\text{char}(\bigwedge^r H^1_{\mathcal{F}_{\text{can}}}(K_\infty,T))/\Lambda.c = \text{char}(\bigwedge^r H^1_{\text{Iw}}(K_p,T))/\Lambda.c).\text{char}(\text{coker}(\text{loc}_p^r(c))).
\]
On the other hand, using (13), we see that the $\Lambda$-modules $\bigwedge^r H^1_{\text{Iw}}(K_p,T)/\Lambda.\text{loc}_p^r(c))$ and $H^1_{\mathcal{F}_{\text{can}}}(K_\infty,T)/\Lambda.\text{loc}_p^r(c)_{K,\infty}$ are pseudo-isomorphic. Since
\[
\text{char}(H^1_{\mathcal{F}_{\text{str}}}(K_\infty,T^*)^\vee) = \text{char}(H^1_{\mathcal{F}_{\text{can}}}(K_\infty,T^*)^\vee).\text{char}(\text{coker}(\text{loc}_p^r(c))),
\]
it follows that
\[
\text{char}(H^1_{\mathcal{F}_{\text{str}}}(K_\infty,T^*)^\vee) \mid \text{char}(H^1_{\mathcal{F}_{\text{can}}}(K_\infty,T^*)^\vee).\text{char}(\text{coker}(\text{loc}_p^r(c)))
\]
(see Proposition 6.2). Hence the result follows from the fact that
\[
\text{char}(\text{coker}(\text{loc}_p^r(c))) = \text{char}(\text{coker}(\text{loc}_p^r(c)))
\]
see [B3] page 258. \hfill \Box

Proposition 6.4. Under the assumption $(H_3)$, the cokernel of
\[
(N_{\Delta})(\hat{E}_\infty \otimes_{\mathbb{Z}_p} \mathcal{O}(\chi^{-1})) \Delta \rightarrow (\hat{E}_\infty \otimes_{\mathbb{Z}_p} \mathcal{O}(\chi^{-1}))^\Delta
\]
is pseudo-null, where $\Delta = \text{Gal}(L_\infty/K_\infty)$.

Proof. This is Theorem 5.13 of [AMO]. \hfill \Box

Corollary 6.5. Under the assumption $(H_3)$, the cokernel of
\[
\bigwedge^r (\hat{E}_\infty \otimes_{\mathbb{Z}_p} \mathcal{O}(\chi^{-1})) \Delta \rightarrow \bigwedge^r (\hat{E}_\infty \otimes_{\mathbb{Z}_p} \mathcal{O}(\chi^{-1}))^\Delta
\]
is pseudo-null.

Proof. Let $p$ be a prime ideal of $\Lambda$ of height $\leq 1$. By Proposition 6.4, the cokernel of
\[
(N_{\Delta})(\hat{E}_\infty \otimes_{\mathbb{Z}_p} \mathcal{O}(\chi^{-1})) \Delta \rightarrow (\hat{E}_\infty \otimes_{\mathbb{Z}_p} \mathcal{O}(\chi^{-1}))^\Delta
\]
is pseudo-null, then
\[
\text{Im}(N_{\Delta})_p \cong ((\hat{E}_\infty \otimes_{\mathbb{Z}_p} \mathcal{O}(\chi^{-1}))^\Delta)_p,
\]
Since $\chi_{\text{null}}$, so
\[ \hat{\text{coker}}(N_{\Delta}) = \bigcap \text{Im}(N_\Delta) = \bigcap ((\hat{E}_\infty \otimes \mathbb{Z}_p \mathcal{O}(\chi^{-1}))^\Delta)_p. \]

It follows that the cokernel of
\[ \bigwedge^r (\hat{E}_\infty \otimes \mathbb{Z}_p \mathcal{O}(\chi^{-1}))^\Delta \xrightarrow{N_{\Delta}^{(r)}} \bigwedge^r (\hat{E}_\infty \otimes \mathbb{Z}_p \mathcal{O}(\chi^{-1}))^\Delta \]

is pseudo-null. \hfill $\square$

Recall that $\text{St}_n$ denotes the $\mathbb{Z}[\text{Gal}(L_n/K)]$-module generated by the Rubin-Stark elements (see Definition 4.5). Recall also that $c_n$ denotes the element defined in (8). Let $\text{St}_\infty := \lim_n \text{St}_n$ and let $\bar{\text{St}}_{\infty,\chi} := \{\bar{\text{St}}_{n,\chi}\}_{n \geq 1}$. Since for $n \geq 1$, $c_n = \text{cor}_{L_{n+1},K_n}(\bar{\text{St}}_{n,\chi})$, it follows that
\[ \text{res}_{K_n,L_n}(c_n) = \text{res}_{K_n,L_n}(\text{cor}_{L_n,K_n}(\bar{\text{St}}_{n,\chi})) = |\Delta|^{r-1}N_\Delta(\bar{\text{St}}_{n,\chi}) \]

Therefore, using the fact that the restriction map
\[ \text{res}_{K_n,L_n} : H^1(K_n,T) \to H^1(L_n,T)_{\text{Gal}(L_n/K_n)} \]

is an isomorphism, see (4). We obtain
\[ |\Delta|^{r-1}N_\Delta((\text{St}_\infty)_{\chi}) = \Lambda c, \]

where $c$ is the inverse image of $|\Delta|^{r-1}N_\Delta(\bar{\text{St}}_{\infty,\chi})$ under the composite
\[ \bigwedge^r \lim_n H^1(K_n,T) \to \lim_n (\mathbb{Q}_p \otimes \mathbb{Z}_p \bigwedge^r H^1(K_n,T)) \].

Recall that
\[ H^1_{F_an}(K_{\infty},T) \cong (\hat{E}_\infty \otimes \mathbb{Z}_p \mathcal{O}(\chi^{-1}))^\Delta \quad \text{(see (5)).} \]

**Proof Theorem 1.3** Consider the commutative exact diagram
\[
\begin{array}{ccc}
(\text{St}_\infty)_{\chi} & \longrightarrow & \bigwedge^r (\hat{E}_\infty)_{\chi} \longrightarrow (\bigwedge^r \hat{E}_\infty / \text{St}_\infty)_{\chi} \\
0 & \longrightarrow & |\Delta|^{r-1}N_\Delta(c) \longrightarrow \bigwedge^r (\hat{E}_\infty)^\chi \longrightarrow \bigwedge^r (\hat{E}_\infty)^\chi / |\Delta|^{r-1}N_\Delta(c) \\
& & \downarrow \text{coker}(N_{\Delta}^{(r)}) \\
& & \text{coker}(N_{\Delta}^{(r)})
\end{array}
\]

where $(\hat{E}_\infty)^\chi = (\hat{E}_\infty \otimes \mathbb{Z}_p \mathcal{O}(\chi^{-1}))^\Delta$. Corollary 6.5 shows that the $\Lambda$-module $\text{coker}(N_{\Delta}^{(r)})$ is pseudo-null, so
\[ \text{char}(\bigwedge^r (\hat{E}_\infty)^\chi / |\Delta|^{r-1}N_\Delta(c)) \text{ divides } \text{char}((\bigwedge^r \hat{E}_\infty / \text{St}_\infty)_{\chi}). \]

Since $\chi(D_v(L/K)) \neq 1$ for any $p$-adic prime of $K$, then
\[ (\hat{E}_\infty \otimes \mathbb{Z}_p \mathcal{O}(\chi^{-1}))^\Delta \cong (\hat{E}_\infty \otimes \mathbb{Z}_p \mathcal{O}(\chi^{-1}))^\Delta. \]
Hence the theorem follows from Theorem 6.3 and Lemma 3.6:
\[ \text{char}( \langle A_\infty \rangle_\chi ) \text{ divides } p^d \cdot \text{char} \left( \left( \bigwedge^r \widehat{\mathcal{E}}_\infty / \widehat{\mathcal{S}}_\infty \right)_\chi \right). \]

**References**

[AMO] Assim, J. Mazigh, Y. Oukhaba, H. *Théorie d’Iwasawa des unités de Stark et groupe de classes.* to appear in International Journal of Number Theory.

[Bo] Bourbaki, N *Algèbre commutative, chapitre 7, Diviseurs,* Hermann, 1965.

[Bü 07] Büyükboduk, K. *Λ-adic Kolyvagin systems.* Int. Math. Res. Not. IMRN 2011, no. 14, 3141-3206.

[Bü 08] Büyükboduk, K. *Kolyvagin systems of Stark units.* J. Reine Angew. Math. 631 (2009), 85-107.

[Bü 09] Büyükboduk, K. *Stark units and the main conjectures for totally real fields.* Compos. Math. 145 (2009), no. 5, 1163-1195.

[Gr 94] Greither, C. *Sur les normes universelles dans les \( \mathbb{Z}_p \)-extensions.* Journal de Théorie des Nombres de Bordeaux 6 (1994), 205-220.

[Gr 96] Greither, C. *On Chinburg’s second conjecture for abelian fields.* J. Reine Angew. Math. 479 (1996), 1-37.

[Mi 86] Milne, J. *Arithmetic Duality theorems.* Acad. Press, Boston, 1986.

[MR 04] Mazur, B.; Rubin, K. *Kolyvagin systems.* Mem. Amer. Math. Soc., 168(799):viii+96, 2004.

[MR 16] Mazur, B.; Rubin, K. *Controlling Selmer groups in the higher core rank case.* J. Théor. Nombres Bordeaux 28 (2016), no. 1, 145-183.

[NW 91] Neukirch, J.; Schmidt, A.; Wingberg, K. *Cohomology of Number Fields.* Springer (1991).

[Ne 06] Nekovář, J. *Selmer complexes.* Astérisque 310 (2006).

[PR 92] B. Perrin-Riou. *Théorie d’Iwasawa et hauteurs p-adiques.* Invent. Math. 109 (1992) 137-185.

[Po 02] Popescu, C. *Base change for Stark-type conjectures "over \( \mathbb{Z} \)."* J. Reine Angew. Math, 524:85-111, 2002.

[Ru 96] Rubin, K. *A Stark conjecture "over \( \mathbb{Z} \)" for abelian L-functions with multiple zeros.* Ann. Inst. Fourier (Grenoble), 46(1):33-62, 1996.

[Ru 00] Rubin, K. *Euler systems.* Annals of Mathematics Studies, 147. Hermann Weyl Lectures. The Institute for Advanced Study. Princeton University Press, Princeton.

[Ta 84] Tate, J. *Les conjectures de Stark sur les fonctions L d’Artin en s=0.* Birkhäuser Boston Inc, 1984. Lecture notes edited by Dominique Bernardi and Norbert Schappacher.

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