STABILITIES OF HOMOTHETICALLY SHRINKING YANG-MILLS SOLITONS

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Abstract. In this paper we introduce entropy-stability and F-stability for homothetically shrinking Yang-Mills solitons, employing entropy and the second variation of the $F$-functional respectively. For a homothetically shrinking soliton which does not descend, we prove that entropy-stability implies F-stability. These stabilities have connections with the study of Type-I singularities of the Yang-Mills flow. Two byproducts are also included: We show that the Yang-Mills flow in dimension four cannot develop a Type-I singularity, and we obtain a gap theorem for homothetically shrinking solitons.

1. Introduction

In this paper we introduce entropy-stability and F-stability for homothetically shrinking (Yang-Mills) solitons. Let $E$ be a trivial $G$-vector bundle over $\mathbb{R}^n$ and of rank $r$. Here the gauge group $G$ is a Lie subgroup of $SO(r)$. A homothetically shrinking soliton, centered at the space-time point $(x_0 = 0, t_0 = 1)$, is a connection $A(x)$ on $E$ such that

$$(d^\nabla)^* F + \frac{1}{2} i_x F = 0,$$

where $F$ is the curvature of $A(x)$, $(d^\nabla)^*$ denotes the formal adjoint of the covariant exterior differentiation $d^\nabla$, and $i_x$ stands for the interior product by the position vector $x$.

A homothetically shrinking soliton $A(x)$ gives rise to a special solution of the Yang-Mills flow. In fact in the exponential gauge of $A(x)$,

$$A(x,t) = A_j(x,t) dx^j := (1 - t)^{-\frac{1}{2}} A_j((1 - t)^{-\frac{1}{2}} x) dx^j$$

is a solution to the Yang-Mills flow. On the other hand, homothetically shrinking solitons are closely related to Type-I singularities of the Yang-Mills flow. Weinkove \cite{22} proved that Type-I singularities of the Yang-Mills flow are modelled by homothetically shrinking solitons whose curvatures do not vanish identically. Examples of homothetically shrinking solitons have been found in \cite{10,22}. In this paper, we restrict ourselves to homothetically shrinking solitons which have uniform bounds on $|\nabla^k A(x)|$ for each $k \geq 1$. In fact, Weinkove showed in \cite{22} that Type-I singularities of the Yang-Mills flow can be modelled by such solitons.

Recently, Colding and Minicozzi \cite{8} discovered two functionals for immersed surfaces in Euclidean space, i.e. the $F$-functional and the entropy. Critical points of
both functionals are self-shrinkers of the mean curvature flow. Colding and Minicozzi introduced entropy-stability and F-stability for self-shrinkers. Inspired by their work, in this paper we aim to introduce corresponding stabilities for homothetically shrinking Yang-Mills solitons. In fact there are many aspects in common concerning the entropy-stability and F-stability for self-similar solutions to various geometric flows, which includes mean curvature flow, Ricci flow, harmonic map heat flow, and Yang-Mills flow. For the entropy-stability and linearly stability of Ricci solitons, see for instance [4, 9]; for the entropy-stability and F-stability of self-similar solutions to the harmonic map heat flow, see [23].

We begin with the definition of an $\mathcal{F}$-functional. Let $x_0$ be a point in $\mathbb{R}^n$ and $t_0$ a positive number. The $\mathcal{F}$-functional with respect to $(x_0,t_0)$, defined on the space of connections on $E$, is given by

\begin{equation}
\mathcal{F}_{x_0,t_0}(A) = t_0^2 \int_{\mathbb{R}^n} |F|^2 (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}} \, dx.
\end{equation}

The functional $\mathcal{F}_{x_0,t_0}$ can trace back to the monotonicity formula of the Yang-Mills flow. For the monotonicity formula see [7, 12, 18]. Let $A(x,t)$ be a solution to the Yang-Mills flow on $E$ and let

$$
\Phi_{x_0,t_0}(A(x,t)) = (t_0 - t)^2 \int_{\mathbb{R}^n} |F|^2 [4\pi (t_0 - t)]^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4(t_0 - t)}} \, dx.
$$

Along the Yang-Mills flow, $\Phi_{x_0,t_0}$ is non-increasing in $t$. Moreover $\Phi_{x_0,t_0}$ is preserved if and only if $A(x,0)$ is a homothetically shrinking soliton centered at $(x_0,t_0)$. Here a homothetically shrinking soliton centered at $(x_0,t_0)$ is a connection on $E$ satisfying the following differential equation

\begin{equation}
(d\nabla)^* F + \frac{1}{2t_0} i_{x-x_0} F = 0.
\end{equation}

The $\mathcal{F}$-functional leads to another characterization of homothetically shrinking solitons: Critical points of $\mathcal{F}_{x_0,t_0}$ are exactly homothetically shrinking solitons centered at $(x_0,t_0)$; moreover, $(x_0,t_0,A_0)$ is a critical point of the function $(x,t,A) \mapsto \mathcal{F}_{x,t}(A)$ if and only if $A_0$ is a homothetically shrinking soliton centered at $(x_0,t_0)$.

The $\lambda$-entropy of a connection $A(x)$ on the bundle $E$ is defined by

\begin{equation}
\lambda(A) = \sup_{x_0 \in \mathbb{R}^n, t_0 > 0} \mathcal{F}_{x_0,t_0}(A).
\end{equation}

A crucial fact is the following

**Proposition 1.1.** Let $A(x,t)$ be a solution to the Yang-Mills flow on the bundle $E$. Then the entropy $\lambda(A(x,t))$ is non-increasing in $t$.

The entropy is a rescaling invariant. More precisely, let $A(x)$ be a connection on $E$ and $A^c$ a rescaling of $A(x)$ given by $A^c_i(x) = c^{-1} A_i(c^{-1}x)$, $c > 0$. Then $\mathcal{F}_{x_0,c^2 t_0}(A^c) = \mathcal{F}_{x_0,t_0}(A)$ and hence $\lambda(A^c) = \lambda(A)$. In particular the entropy of each time-slice of the homothetically shrinking Yang-Mills flow, induced from a homothetically shrinking soliton, is preserved. The entropy is also invariant under translations of a connection. Let $A(x)$ be a connection on $E$, $x_1 \in \mathbb{R}^n$ a given point, and $A_i(x) = A_i(x + x_1)$. Then we have $\mathcal{F}_{x_0-x_1,t_0}(A) = \mathcal{F}_{x_0,t_0}(A)$ and hence $\lambda(A) = \lambda(A)$. 
In general the entropy $\lambda(A)$ of a connection $A(x)$ is not attained by any $F(x_0,t_0)(A)$. However if $A(x)$ is a homothetically shrinking soliton centered at $(x_0,t_0)$, then $\lambda(A) = F(x_0,t_0)(A)$. In fact we prove the following

**Proposition 1.2.** Let $A(x)$ be a homothetically shrinking soliton centered at $(0,1)$ such that $i_V F \neq 0$ for any non-zero $V \in \mathbb{R}^n$. Then the function $(x_0,t_0) \mapsto F(x_0,t_0)(A)$ attains its strict maximum at $(0,1)$.

Note that if $i_V F = 0$ for some non-zero vector $V$, then $A(x)$ can be viewed as a connection on a $G$-vector bundle over any hyperplane perpendicular to $V$ and we say $A(x)$ descends (to $V$).

Entropy-stability and F-stability are defined for homothetically shrinking solitons.

**Definition 1.1.** A homothetically shrinking soliton $A(x)$ is called entropy-stable if it is a local minimum of the entropy, among all perturbations $\tilde{A}(x)$, such that $||\tilde{A} - A||_{C^1}$ is sufficiently small.

Entropy-stability of homothetically shrinking solitons has direct connections with Type-I singularities of the Yang-Mills flow. For example, given an entropy-unstable homothetically shrinking soliton $A(x)$, by definition we can find a perturbation $\tilde{A}(x)$ of $A(x)$ such that $||\tilde{A} - A||_{C^1}$ is arbitrarily small and has less entropy. Then by comparing the entropy, the Yang-Mills flow starting from $\tilde{A}$ cannot converge back to a rescaling of $A(x)$. Moreover, the Yang-Mills flow cannot develop a Type-I singularity modelled by $A(x)$, due to the fact that the entropy is a rescaling invariant.

Let $A_0(x)$ be a homothetically shrinking soliton centered at $(x_0,t_0)$. For a 1-parameter family of deformations $(x_s,t_s,A_s)$ of $(x_0,t_0,A_0)$, let $V = \frac{dA_s}{ds}|_{s=0}, q = \frac{dt_s}{ds}|_{s=0}, \theta = \frac{dR_s}{ds}|_{s=0}$.

**Definition 1.2.** $A_0(x)$ is called F-stable if for any compactly supported $\theta$, there exist a real number $q$ and a vector $V$ such that

$$F''_{x_0,t_0}(q,V,\theta) := \frac{d^2}{ds^2}|_{s=0}F_{x_s,t_s}(A_s) \geq 0.$$

Entropy-stability has an apparent connection with the singular behavior of the Yang-Mills flow; however the F-stability is more practical when we are trying to do classification. The classification of entropy-stable homothetically shrinking solitons can be relied on the classification of F-stable ones. In fact we have the following relation for entropy-stability and F-stability.

**Theorem 1.3.** Let $A(x)$ be a homothetically shrinking soliton such that $i_V F \neq 0$ for any non-zero $V \in \mathbb{R}^n$. If $A(x)$ is entropy-stable, then it is F-stable.

Let $A_0(x)$ be a homothetically shrinking soliton centered at $(0,1)$. Denote

$$L \theta = -[d(d^*) + \mathcal{R}(\theta) + i\bar{z}d^\theta],$$

where $\mathcal{R}(\theta)(\bar{\eta}) := [F_{ij}, \theta_i]$. For the homothetically shrinking soliton $A_0(x)$, we have

$$L(d^*) F = (d^*) F$$

and

$$L_i V F = \frac{1}{2} i_V F, \quad \forall V \in \mathbb{R}^n.$$
The second variation of the $F$-functional and at $A_0$ is given by

$$\frac{1}{2} F''_{0,1}(q, V, \theta) = \int_{\mathbb{R}^n} \langle -L \theta + 2q(d\nabla)^*F - iVF, \theta \rangle Gdx$$

(1.6)

where $G(x) = (4\pi)^{-\frac{n}{2}} \exp(-\frac{|x|^2}{4})$. Denote the space of $\theta$ satisfying $L\theta = -\lambda \theta$ by $E_\lambda$. We have the following characterization for $F$-stability.

**Theorem 1.4.** $A_0(x)$ is $F$-stable if and only if the following properties are satisfied:

- $E_{-1} = \{c(d\nabla)^*F, \ c \in \mathbb{R}\}$;
- $E_{-\frac{1}{2}} = \{iVF, \ V \in \mathbb{R}^n\}$;
- $E_\lambda = \{0\}$, for any $\lambda < 0$ and $\lambda \neq -1, -\frac{1}{2}$.

Theorem 1.4 amounts to saying that $A_0(x)$ is $F$-stable if and only if $-L$ is non-negative definite modulo the vector space spanned by $(d\nabla)^*F$ and $iVF$. This is actually the reflection of the invariance property of the $F$-functional and the entropy under rescalings and translations. Since Colding-Minicozzi’s work [8], classification problem of $F$-stable self-shrinkers of the mean curvature flow has drawn much attention, see for instance [1,2,15,16].

We have two simple byproducts regarding homothetically shrinking solitons. We show the non-existence of homothetically shrinking solitons in dimensions four and lower, and a gap theorem. Let $A(x)$ be a homothetically shrinking soliton centered at $(0,1)$. Then we have the identity

$$\int_{\mathbb{R}^n} |x|^2 |F|^2 G(x)dx = 2(n-4) \int_{\mathbb{R}^n} |F|^2 G(x)dx.$$ 

It immediately implies the following

**Proposition 1.5.** When $n = 2, 3$, or $4$, there exists no homothetically shrinking soliton such that $|F|$ is uniformly bounded and not identically zero.

Råde [19] proved that the Yang-Mills flow, over a compact Riemannian manifold of dimension $n = 2$ or $3$, exists for all time and converges to a Yang-Mills connection. However if the base manifold has dimension five or above, Naito [18] showed that the Yang-Mills flow can develop a singularity in finite time, see also [11]. It is unclear yet whether the Yang-Mills flow over a four-dimensional manifold develops a singularity in finite time. For partial results in this dimension, see [9,13,20] and the references therein. Together with Weinkove’s blowup analysis for Type-I singularities of the Yang-Mills flow, Proposition 1.5 shows that the Yang-Mills flow cannot develop a singularity of Type-I. This was actually a known fact, see for instance [12].

Gap theorems for Yang-Mills connections over spheres was considered in [3]. Gap theorems for various kinds of self-similar solutions have also been obtained, see for instance [5,14,23]. By (1.5), we have the following gap result for homothetically shrinking solitons.

**Theorem 1.6.** Let $A(x)$ be a homothetically shrinking soliton centered at $(0,1)$. If $|F|^2 < \frac{n}{2(n-1)}$, then $(E, A)$ is flat.
analysis for Type-I singularities of the Yang-Mills flow. In Section 3, we consider the $\mathcal{F}$-functional and its first variation. Section 4 is devoted to the calculation of the second variation of the $\mathcal{F}$-functional, i.e. (1.6). In Section 5, we study the $\mathcal{F}$-stability of homothetically shrinking solitons and prove Theorem 1.4 and Theorem 1.6. In Section 6, we introduce the $\lambda$-entropy and prove Proposition 1.2. In the last section, we prove that entropy-stability implies $\mathcal{F}$-stability, i.e. Theorem 1.3.

We would like to point out that although we assume, for simplicity, that the homothetically shrinking solitons have uniform bounds on $|\nabla^k A|$, our statements except Theorem 1.3 are still straightforwardly valid if $|\nabla^k A|$ has polynomial growth.

Many results in this paper have also been obtained by Kelleher and Streets [17].

2. Preliminaries

In this section we briefly introduce the Yang-Mills flow and its singularity. We shall introduce the blowup analysis for Type-I singularities, which was carried out by Weinkove [22]. It leads to the main object in this paper, i.e. the homothetically shrinking soliton.

Let $(M, g)$ be a closed $n$-dimensional Riemannian manifold. Let $G$ be a compact Lie group and $P(M, G)$ a principle bundle over $M$ with the structure group $G$. We fix a $G$-vector bundle $E_M = P(M, G) \times_{\rho} \mathbb{R}^r$, associated to $P(M, G)$ with a faithful representation $\rho : G \to SO(r)$. Let $\mathfrak{g}$ denote the Lie algebra of $G$. A connection on $E_M$ is locally a $\mathfrak{g}$-valued 1-form. Using Latin letters for the manifold indices, one may write a connection $A$ in the form of $A = A_i dx^i$, where $A_i \in \text{so}(r)$. Using Greek letters for the bundle indices, one may also write $A = A_{ij} dx^i$. The curvature of the connection $A$ is locally a $\mathfrak{g}$-valued 2-form $F = \frac{1}{2} F_{ij} dx^i \wedge dx^j = \frac{1}{2} F_{ij}^\alpha dx^i \wedge dx^j$, and

$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$

The Yang-Mills functional, defined on the space of connections, is given by

$YM(A) = \frac{1}{2} \int_M |F|^2 d\mu_g,$

where

$|F|^2 = \frac{1}{2} g^{ik} g^{jl} (F_{ij}, F_{kl}) = \frac{1}{2} g^{ik} g^{jl} F_{ij}^{\alpha} F_{kl}^{\alpha},$

and

$F_{ij}^{\alpha} = \partial_i A_{j\beta}^{\alpha} - \partial_j A_{i\beta}^{\alpha} + A_{r\gamma}^{\alpha} A_{j\beta}^{\gamma} - A_{r\gamma}^{\alpha} A_{i\beta}^{\gamma}.$

Let $\nabla$ denote the covariant differentiation on $\Gamma(E_M)$ associated to the connection $A$, and also the covariant differentiation on $\mathfrak{g}$-valued $p$-forms induced by $A$ and the Levi-Civita connection of $(M, g)$. Curvature $F$ satisfies the Bianchi identity $d\nabla F = 0$, where $d\nabla$ denotes the covariant exterior differentiation. Let $(d\nabla)^*$ denote the formal adjoint of $d\nabla$. A connection $A$ is a critical point of the Yang-Mills functional, called a Yang-Mills connection, if and only if it is a solution of the Yang-Mills equation $(d\nabla)^* F = 0$. The Yang-Mills equation can also be written as

$\nabla^p F_{ij}^{\alpha} = 0.$

In normal coordinates of $(M, g)$, we have $\nabla^p F_{ij}^{\alpha} = \partial_p F_{ij}^{\alpha} + A_{r\gamma}^{\alpha} F_{ij}^{\gamma} - F_{ij}^{\alpha} A_{r\gamma}^{\gamma}.$
As the $L^2$-gradient flow of the Yang-Mills functional, the Yang-Mills flow is defined by

$$
\frac{dA}{dt} = -(d\nabla)^* F.
$$

(2.2)

Assume $A(x, t)$ is a smooth solution to the Yang-Mills flow for $0 \leq t < T$, and as $t \to T$ the curvature blows up, i.e. $\limsup_{t \to T} \max_{x \in M} |F(x, t)| = \infty$. If there exists a positive constant $C$ such that

$$
|F(x, t)| \leq \frac{C}{T - t},
$$

(2.3)

one says that the Yang-Mills flow develops a Type-I singularity, or a rapidly forming singularity. Otherwise one says that the Yang-Mills flow develops a Type-II singularity. If (2.3) is satisfied and $x_0$ is a point such that $\limsup_{t \to T} |F(x_0, t)| = \infty$, we call $(x_0, T)$ a Type-I singularity.

Let $A(x, t)$ be a smooth solution to the Yang-Mills flow and $(x_0, T)$ a Type-I singularity. We now follow [22] introducing the blowup procedure around $(x_0, T)$.

Let $B_r(x_0)$ be a small geodesic ball centered at $x_0$ and of radius $r$ over which $E_M$ is trivial. For simplicity we identify $B_r(x_0)$ with the ball $B_r(0)$ in $\mathbb{R}^n$. Let $\lambda_i$ be a sequence of positive numbers tending to zero. For each $i$, one gets a Yang-Mills flow $A^{\lambda_i}(y, s)$ by setting

$$
A^{\lambda_i}(y, s) = \lambda_i A_p(\lambda_i y, T + \lambda_i^2 s)dy^p, \quad y \in B_r/\lambda_i(0), s \in [-\lambda_i^{-2}T, 0).
$$

(2.4)

(An alternative way of obtaining a sequence of blowups of $A(x, t)$ is to rescale the metric around the singular point $x_0$.) Let $x = \lambda_i y$ and $t = T + \lambda_i^2 s$. By the assumption (2.3), the curvature of $A^{\lambda_i}$ satisfies

$$
|F^{\lambda_i}(y, s)| = \lambda_i^2 |F(x, t)| = |s|^{-1}(T - t)|F(x, t)| \leq C|s|^{-1}.
$$

Let $h = h^\alpha_\beta$ be a gauge transformation which acts on connections by

$$
h^* \nabla = h^{-1} \circ \nabla \circ h,
$$

or equivalently,

$$
h^* A = h^{-1} dh + h^{-1} A h.
$$

Note that gauge transformations preserve Yang-Mills flows. Hence $h^* A^{\lambda_i}(y, s)$ defines a solution to the Yang-Mills flow. Weinkove [22] proved the following

**Theorem 2.1.** Let $(x_0, T)$ be a Type-I singularity of the Yang-Mills flow $A(x, t)$ over $M$. Then there exists a sequence of blowups $A^{\lambda_i}(y, s)$ defined by (2.4) and a sequence of gauge transformations $h_i$ such that $h^*_i A^{\lambda_i}(y, s)$ converges smoothly on any compact set to a flow $\tilde{A}(y, s)$. Here $\tilde{A}(y, s)$, defined on a trivial $G$-vector bundle over $\mathbb{R}^n \times (-\infty, 0)$, is a solution to the Yang-Mills flow, which has non-zero curvature and satisfies

$$
\nabla^p \tilde{F}_{pj} - \frac{1}{2|s|^p} y^p \tilde{F}_{pj} = 0.
$$

(2.5)

In Theorem 2.1, $h_i$ are chosen as suitable Coulomb gauge transformations so that for any $s < 0$ and $k \geq 1$, $|\nabla^k h_i^* A^{\lambda_i}|$ is uniformly bounded. The bounds do not depend on $i$. Hence for any $s < 0$ and $k \geq 1$, $|\nabla^k \tilde{A}|$ is uniformly bounded.
A solution $A(y, s)$ to the Yang-Mills flow, defined on a trivial bundle over $\mathbb{R}^n \times (-\infty, 0)$, is called a homothetically shrinking soliton if it satisfies

$$
(2.6) \quad A_{\alpha}^\nu(y, s) = \frac{1}{\sqrt{|s|}} A_{\alpha}^\nu(y \sqrt{|s|}, -1)
$$

for any $y \in \mathbb{R}^n$ and $s < 0$; for more details see [22]. The limiting Yang-Mills flow $\bar{A}(y, s)$ is actually a homothetically shrinking soliton. In fact via an exponential gauge for $\bar{A}(y, s)$, in which $y^\mu \bar{A}_{\mu}^\alpha = 0$, (2.5) and (2.6) are equivalent for the Yang-Mills flow $\bar{A}(y, s)$.

One of the main ingredients of Theorem 2.1 is the monotonicity formula for the Yang-Mills flow; see [7,12,18]. In the simplest case that $A(x,t)$ is a solution to the Yang-Mills flow over $\mathbb{R}^n$, one can define

$$
(2.7) \quad \Phi_{x_0,t_0}(A(x,t)) = (t_0 - t)^2 \int_{\mathbb{R}^n} |F(x,t)|^2 G_{x_0,t_0}(x,t) dx.
$$

Here $t_0 > 0$, $t \in [0, \min\{T,t_0\})$, and $G_{x_0,t_0}(x,t) = [4\pi(t_0 - t)]^{-\frac{n}{2}} \exp(-\frac{|x-x_0|^2}{4(t_0 - t)})$ is the backward heat kernel. The monotonicity formula of the Yang-Mills flow reads

$$
(2.8) \quad \frac{d}{dt} \Phi_{x_0,t_0}(A(x,t)) = -2(t_0 - t)^2 \int_{\mathbb{R}^n} |\nabla^p F_{pj} - \frac{1}{2(t_0 - t)} (x - x_0)^p F_{pj}|^2 G_{x_0,t_0}(x,t) dx.
$$

The monotonicity $\Phi_{x_0,t_0}$ is non-increasing in $t$, and is preserved if and only if

$$
(2.9) \quad \nabla^p F_{pj} - \frac{1}{2(t_0 - t)} (x - x_0)^p F_{pj} = 0.
$$

For the limiting Yang-Mills flow $\bar{A}(y, s)$ obtained in Theorem 2.1 and any $(x_0, t_0) \in \mathbb{R}^n \times (0, +\infty)$, one can translate it into

$$
(2.10) \quad A(x,t) = A_p(x,t) dx^p = \bar{A}_p(x-x_0, t-t_0) dx^p;
$$

then $A(x,t)$ is a solution to the Yang-Mills flow and (2.9) is satisfied. On the other hand if a connection $A(x)$ on a trivial $G$-vector bundle over $\mathbb{R}^n$ satisfies

$$
\nabla^p F_{pj} - \frac{1}{2t_0} (x - x_0)^p F_{pj} = 0,
$$

then, in the exponential gauge for $A(x)$, i.e. a gauge such that $(x - x_0)^p A_p(x) = 0$, the flow of connections given by

$$
A_p(x,t) := \sqrt{\frac{t_0}{t_0 - t}} A_p(x_0 + \sqrt{\frac{t_0}{t_0 - t}} (x - x_0))
$$

is a solution to the Yang-Mills flow which satisfies (2.9). All these amount to saying that limiting flows $\bar{A}(y, s)$, homothetically shrinking solitons $A(x)$ and homothetically shrinking Yang-Mills flows are the same thing.

From now on we assume that $E$ is a trivial $G$-vector bundle over $\mathbb{R}^n$.

**Definition 2.1.** A connection $A(x)$ on $E$ is called a homothetically shrinking soliton centered at $(x_0, t_0)$ if it satisfies

$$
(2.11) \quad \nabla^p F_{pj} - \frac{1}{2t_0} (x - x_0)^p F_{pj} = 0.
$$
Let $A(x)$ be a homothetically shrinking soliton centered at $(x_0, t_0)$ and $A(x, t)$ the Yang-Mills flow initiating from $A(x)$. In an exponential gauge such that $(x - x_0)^p A_p(x, t) = 0$, we have for any $\lambda > 0$ and any $t < t_0$ that $A_j(x, t) = \lambda A_j(x - x_0) + x_0, \lambda^2 (t - t_0) + t_0$.

3. $F$-FUNCTIONAL AND ITS FIRST VARIATION

In this section we define the $F$-functional of connections on the trivial $G$-vector bundle $E$ over $\mathbb{R}^n$. Homothetically shrinking solitons are critical points of the $F$-functional. We shall prove necessary integral identities for homothetically shrinking solitons. As a corollary of one of these identities, we give a proof of the fact that the Yang-Mills flow in dimension four cannot develop a Type-I singularity.

For convenience, we set two $\mathfrak{g}$-valued 1-forms $J$ and $X$, respectively, by

$$J := \nabla^p F_{pj} dx^j, \quad X := i_{x-x_0} F = (x - x_0)^p F_{pj} dx^j.$$  

According to (2.11), $A(x)$ is a homothetically shrinking soliton centered at $(x_0, t_0)$ if and only if

$$J = \frac{1}{2t_0} X.$$  

We also set

$$(3.1) \quad S_{x_0, t_0} = \{A(x) : A \text{ is a homothetically shrinking soliton centered at } (x_0, t_0) \text{ and with } \sup |\nabla^k A| < \infty, \forall k \geq 1 \}.$$  

Note that for any $k \geq 1$, any time-slice $\tilde{A}(:, s)$ in Theorem 2.1 satisfies $\sup |\nabla^k \tilde{A}(:, s)| < \infty$.

**Definition 3.1.** For any $x_0 \in \mathbb{R}^n, t_0 > 0$, the $F$-functional with respect to $(x_0, t_0)$ is defined by

$$(3.2) \quad F_{x_0, t_0}(A) = t_0^2 \int_{\mathbb{R}^n} |F|^2 (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}} dx.$$  

We now compute the first variation of the $F$-functional. Consider a differentiable 1-parameter family $(x_s, t_s, A_s)$, where $A_0 = A$. Denote

$$i_s = \frac{d}{ds} t_s, \quad \dot{x}_s = \frac{d}{ds} x_s, \quad \theta_s = \frac{d}{ds} A_s,$$

and

$$G_s(x) = (4\pi t_s)^{-\frac{n}{2}} e^{-\frac{|x-x_s|^2}{4t_s}}.$$  

**Proposition 3.1.** Assume $|\nabla^k A_s| < \infty$ for any $k \geq 1$ and $\int_{\mathbb{R}^n} (|\theta_s|^2 + |\nabla \theta_s|^2) G_s dx < \infty$. The first variation of the $F$-functional is given by

$$\frac{d}{ds} F_{x_s, t_s}(A_s) = \int_{\mathbb{R}^n} i_s \left( \frac{4-n}{2} - t_s + \frac{1}{4} |x-x_s|^2 |F_s|^2 G_s(x) dx + \int_{\mathbb{R}^n} \frac{1}{2} t_s (\dot{x}_s, x - x_s) |F_s|^2 G_s(x) dx - \int_{\mathbb{R}^n} 2t_s^2 (\theta_s, J_s - \frac{X_s}{2t_s}) G_s(x) dx. \right.$$  

$$(3.3)$$
Proof. Note that
\[
\frac{\partial}{\partial s} G_s(x) = \left(-\frac{n}{2} \frac{i_s}{t_s} + \frac{i_s}{4t_s^2} |x - x_s|^2 + \frac{\langle \dot{x}_s, x - x_s \rangle}{2t_s} \right) G_s(x),
\]
and
\[
\frac{\partial}{\partial s} |F_s|^2 = F_{ij\beta}^\alpha (\nabla_i \theta_{j\beta}^\alpha - \nabla_j \theta_{i\beta}^\alpha),
\]
so we have
\[
\frac{d}{ds} F_{x, t_s}(A_s) = \int_{\mathbb{R}^n} 2t_s^2 F_{ij\beta}^\alpha (\nabla_i \theta_{j\beta}^\alpha - \nabla_j \theta_{i\beta}^\alpha) G_s(x) dx
\]
\[
+ \int_{\mathbb{R}^n} t_s^2 |F_s|^2 \left(-n \frac{i_s}{2t_s} + \frac{i_s}{4t_s^2} |x - x_s|^2 + \frac{\langle \dot{x}_s, x - x_s \rangle}{2t_s} \right) G_s(x) dx
\]
\[
= \int_{\mathbb{R}^n} 2t_s^2 F_{ij\beta}^\alpha \nabla_i \theta_{j\beta}^\alpha G_s(x) \eta(x) dx
\]
\[
+ \int_{\mathbb{R}^n} t_s^2 |F_s|^2 \left(-n \frac{i_s}{2t_s} + \frac{i_s}{4t_s^2} |x - x_s|^2 + \frac{\langle \dot{x}_s, x - x_s \rangle}{2t_s} \right) G_s(x) dx.
\]
Let \( \eta(x) \) be a cutoff function on \( \mathbb{R}^n \). By integration by parts, we have
\[
\int_{\mathbb{R}^n} 2t_s^2 F_{ij\beta}^\alpha \nabla_i \theta_{j\beta}^\alpha G_s(x) \eta(x) dx
\]
\[
= \int_{\mathbb{R}^n} -2t_s^2 \theta_{j\beta}^\alpha \nabla_i F_{ij\beta}^\alpha G_s \eta + F_{ij\beta}^\alpha \partial_i (G_s) \eta + F_{ij\beta}^\alpha \partial_i \eta G_s dx
\]
\[
(3.4)
\]
\[
= \int_{\mathbb{R}^n} -2t_s^2 \theta_{j\beta}^\alpha \nabla_i F_{ij\beta}^\alpha \eta - \frac{(x - x_s)^i}{2t_s} F_{ij\beta}^\alpha \eta + F_{ij\beta}^\alpha \partial_i \eta G_s dx.
\]

Let \( \eta(x) = 1 \) for \( |x| \leq l \), and cut off to zero linearly on \( B_{l+1} \setminus B_l \). Taking \( \eta = \eta_l \) in (3.4) and applying Lebesgue’s dominated convergence theorem, we get
\[
\int_{\mathbb{R}^n} 2t_s^2 F_{ij\beta}^\alpha \nabla_i \theta_{j\beta}^\alpha G_s(x) dx = \int_{\mathbb{R}^n} \theta_{j\beta}^\alpha \left[-2t_s^2 \nabla_i F_{ij\beta}^\alpha + t_s (x - x_s)^i F_{ij\beta}^\alpha \right] G_s dx.
\]

Hence we get
\[
\frac{d}{ds} F_{x, t_s}(A_s) = \int_{\mathbb{R}^n} 2t_s^2 F_{ij\beta}^\alpha \nabla_i \theta_{j\beta}^\alpha G_s(x) dx
\]
\[
+ \int_{\mathbb{R}^n} t_s^2 |F_s|^2 \left(-n \frac{i_s}{2t_s} + \frac{i_s}{4t_s^2} |x - x_s|^2 + \frac{\langle \dot{x}_s, x - x_s \rangle}{2t_s} \right) G_s(x) dx
\]
\[
= \int_{\mathbb{R}^n} t_s (4 - n \frac{1}{2} t_s + \frac{1}{4} |x - x_s|^2) |F_s|^2 G_s(x) dx
\]
\[
+ \int_{\mathbb{R}^n} \frac{1}{2} t_s \langle \dot{x}_s, x - x_s \rangle |F_s|^2 G_s(x) dx
\]
\[
- \int_{\mathbb{R}^n} 2t_s^2 \langle \theta_s, J_s - X_s \rangle G_s(x) dx.
\]
From Proposition 3.1, we have the following

**Corollary 3.1.** A connection \( A(x) \) is a critical point of \( \mathcal{F}_{x_0,t_0} \) if and only if \( A(x) \) is a homothetically shrinking soliton centered at \((x_0, t_0)\).

We shall check that \((A(x), x_0, t_0)\) is a critical point of the \( \mathcal{F} \)-functional \( \tilde{A}(x, t) \mapsto F_{x,t}(\tilde{A}) \) if and only if \( A(x) \) is a homothetically shrinking soliton centered at \((x_0, t_0)\). To check this we need some identities for homothetically shrinking solitons. We also need such identities in the calculation of the second variation of the \( \mathcal{F} \)-functional in the next section. Denote \( G(x) = (4\pi t_0)^{-\frac{n}{2}}e^{-\frac{|x-x_0|^2}{4t_0}} \).

**Lemma 3.2.** Let \( A(x) \) be a homothetically shrinking soliton centered at \((x_0, t_0)\) such that \( \sup |F(x)| < \infty \). Let \( \varphi = \varphi^p \partial_p \) be a vector field on \( \mathbb{R}^n \) such that \( |\varphi| \) is a polynomial in \( |x-x_0| \), and \( V \) a vector in \( \mathbb{R}^n \). Then we have

\[
\int_{\mathbb{R}^n} \varphi^p(x-x_0)^p |F|^2 G(x) dx = \int_{\mathbb{R}^n} [2t_0 \partial_p(\varphi^p)|F|^2 - 4t_0 \partial_i \varphi^p F^\alpha_{pj\beta} F^\alpha_{ij\beta}] G(x) dx.
\]

In particular,

(a) \( \int_{\mathbb{R}^n} |x-x_0|^2 |F|^2 G(x) dx = \int_{\mathbb{R}^n} 2(n-4)t_0 |F|^2 G(x) dx \);

(b) \( \int_{\mathbb{R}^n} (x-x_0)^k |F|^2 G(x) dx = 0 \);

(c) \( \int_{\mathbb{R}^n} |x-x_0|^4 |F|^2 G(x) dx = \int_{\mathbb{R}^n} [4(n-2)(n-4)t_0^2] (32) G dx \);

(d) \( \int_{\mathbb{R}^n} |x-x_0|^2 < V, x-x_0 > |F|^2 G(x) dx = 0 \);

(e) \( \int_{\mathbb{R}^n} (x-x_0, V)^2 |F|^2 G dx = \int_{\mathbb{R}^n} (2t_0 |V|^2 |F|^2 - 4t_0 < V F_{ij}, V F_{pj} > ) G dx \).

**Proof.** Let \( \eta(x) \) be a cutoff function on \( \mathbb{R}^n \). By integration by parts, we get

\[
\int_{\mathbb{R}^n} \varphi^p(x-x_0)^p |F|^2 G(x) \eta(x) dx = \int_{\mathbb{R}^n} -2t_0 \varphi^p |F|^2 \partial_p G(x) \eta(x) dx
\]

\[
= \int_{\mathbb{R}^n} 2t_0 [\partial_p(\varphi^p)|F|^2 \eta + \varphi^p \partial_p(|F|^2) \eta + \varphi^p |F|^2 \partial_p \eta] G(x) dx.
\]

By integration by parts we have

\[
\int_{\mathbb{R}^n} 4t_0 \varphi^p F^\alpha_{pj\beta} F^\alpha_{ij\beta} G \eta dx = \int_{\mathbb{R}^n} 4t_0 \varphi^p [\nabla_i (F^\alpha_{pj\beta} F^\alpha_{ij\beta}) - \nabla_i F^\alpha_{pj\beta} F^\alpha_{ij\beta}] G \eta dx
\]

\[
= \int_{\mathbb{R}^n} -4t_0 F^\alpha_{pj\beta} F^\alpha_{ij\beta} \partial_i \varphi^p - \frac{(x-x_0)^i}{2t_0} \varphi^p G \eta dx
\]

\[
- \int_{\mathbb{R}^n} 2t_0 \varphi^p (\nabla_i F^\alpha_{pj\beta} F^\alpha_{ij\beta} + \nabla_j F^\alpha_{ip\beta} F^\alpha_{ij\beta}) G \eta dx
\]

\[
- \int_{\mathbb{R}^n} 4t_0 \varphi^p F^\alpha_{pj\beta} F^\alpha_{ij\beta} G \partial_i \eta dx.
\]
It then follows from the Bianchi identity that

\[
\int_{\mathbb{R}^n} 4t_0 \varphi^p F^\alpha_{p j \beta} J^\alpha_{j \beta} G \eta dx
= \int_{\mathbb{R}^n} -4t_0 F^\alpha_{p j \beta} F^\alpha_{i j \beta} [\partial_i \varphi^p] - \frac{(x - x_0)^i}{2t_0} \varphi^p |G \eta dx
- \int_{\mathbb{R}^n} 2t_0 \varphi^p \nabla_p F^\alpha_{p j \beta} F^\alpha_{i j \beta} G \eta dx - \int_{\mathbb{R}^n} 4t_0 \varphi^p F^\alpha_{p j \beta} F^\alpha_{i j \beta} G \eta dx
= \int_{\mathbb{R}^n} -4t_0 F^\alpha_{p j \beta} F^\alpha_{i j \beta} [\partial_i \varphi^p] - \frac{(x - x_0)^i}{2t_0} \varphi^p |G \eta dx
- \int_{\mathbb{R}^n} 2t_0 \varphi^p \partial_p (|F|^2) G \eta dx - \int_{\mathbb{R}^n} 4t_0 \varphi^p F^\alpha_{p j \beta} F^\alpha_{i j \beta} G \eta dx,
\]
i.e.

\[
\int_{\mathbb{R}^n} 2t_0 \varphi^p \partial_p (|F|^2) G \eta dx = - \int_{\mathbb{R}^n} 4t_0 \varphi^p F^\alpha_{p j \beta} J^\alpha_{j \beta} G \eta dx
- \int_{\mathbb{R}^n} 4t_0 F^\alpha_{p j \beta} F^\alpha_{i j \beta} [\partial_i \varphi^p] - \frac{(x - x_0)^i}{2t_0} \varphi^p |G \eta dx
- \int_{\mathbb{R}^n} 4t_0 \varphi^p F^\alpha_{p j \beta} F^\alpha_{i j \beta} G \eta dx.
\]

Thus we have

\[
\int_{\mathbb{R}^n} \varphi^p (x - x_0)^p |F|^2 G(x) \eta(x) dx
= \int_{\mathbb{R}^n} 2t_0 [\partial_p (\varphi^p)] |F|^2 \eta + \varphi^p \partial_p (|F|^2) \eta + \varphi^p |F|^2 \partial_p \eta] G(x) dx
= \int_{\mathbb{R}^n} 2t_0 \varphi^p \partial_p (|F|^2) G \eta dx - \int_{\mathbb{R}^n} 4t_0 \varphi^p F^\alpha_{p j \beta} J^\alpha_{j \beta} G \eta dx
- \int_{\mathbb{R}^n} 4t_0 F^\alpha_{p j \beta} F^\alpha_{i j \beta} [\partial_i \varphi^p] - \frac{(x - x_0)^i}{2t_0} \varphi^p |G \eta dx
- \int_{\mathbb{R}^n} 4t_0 \varphi^p F^\alpha_{p j \beta} F^\alpha_{i j \beta} G \partial_i \eta dx + \int_{\mathbb{R}^n} 2t_0 \varphi^p |F|^2 G \partial_p \eta dx
= \int_{\mathbb{R}^n} [2t_0 \partial_p (\varphi^p)] |F|^2 - 4t_0 \partial_i \varphi^p F^\alpha_{p j \beta} F^\alpha_{i j \beta} G \eta dx
- \int_{\mathbb{R}^n} 4t_0 \varphi^p F^\alpha_{p j \beta} J^\alpha_{j \beta} - \frac{1}{2t_0} X^\alpha_{j \beta} G \eta dx
- \int_{\mathbb{R}^n} 4t_0 \varphi^p F^\alpha_{p j \beta} F^\alpha_{i j \beta} G \partial_i \eta dx + \int_{\mathbb{R}^n} 2t_0 \varphi^p |F|^2 G \partial_p \eta dx.
\]

Therefore for a homothetically shrinking soliton centered at \((x_0, t_0)\),

\[
\int_{\mathbb{R}^n} \varphi^p (x - x_0)^p |F|^2 G \eta dx = \int_{\mathbb{R}^n} [2t_0 \partial_p (\varphi^p)] |F|^2 - 4t_0 \partial_i \varphi^p F^\alpha_{p j \beta} F^\alpha_{i j \beta} G \eta dx
- \int_{\mathbb{R}^n} 4t_0 \varphi^p F^\alpha_{p j \beta} J^\alpha_{j \beta} - \frac{1}{2t_0} X^\alpha_{j \beta} G \eta dx
- \int_{\mathbb{R}^n} 4t_0 \varphi^p F^\alpha_{p j \beta} F^\alpha_{i j \beta} G \partial_i \eta dx + \int_{\mathbb{R}^n} 2t_0 \varphi^p |F|^2 G \partial_p \eta dx.
\]

(3.6)
Applying to (3.6) with \(\eta(x) = \eta_l(x)\), where \(\eta_l(x) = 1\) for \(|x| \leq l\) and is cut off to zero linearly on \(B_{l+1} \setminus B_l\), we get

\[
(3.7) \quad \int_{\mathbb{R}^n} \phi^p(x - x_0)^p |F|^2 G dx = \int_{\mathbb{R}^n} [2t_0 \partial_p(\phi^p)|F|^2 - 4t_0 \partial_i \phi^p F^\alpha_{ij} F^\alpha_{ij}] G dx.
\]

Taking \(\phi^p = (x - x_0)^p\), by (3.7) we get

\[
\int_{\mathbb{R}^n} |x - x_0|^2 |F|^2 G(x) dx = \int_{\mathbb{R}^n} 2(n - 4)t_0 |F|^2 G(x) dx.
\]

Taking \(\phi^p = \delta^p_k\), by (3.7) we get for any \(k = 1, \ldots, n\),

\[
\int_{\mathbb{R}^n} (x - x_0)^k |F|^2 G(x) dx = 0.
\]

Taking \(\phi^p = |x - x_0|^2 (x - x_0)^p\), by (3.7) and (a) we get

\[
\int_{\mathbb{R}^n} |x - x_0|^4 |F|^2 G(x) dx = \int_{\mathbb{R}^n} [2t_0 (n + 2)|x - x_0|^2 |F|^2 - 8t_0 |x - x_0|^2 |F|^2 - 8t_0 |X|^2] G dx
\]

\[
= \int_{\mathbb{R}^n} [4(n - 2)(n - 4) t_0^2 |F|^2 - 32 t_0^2 |J|^2] G dx.
\]

Taking \(\phi^p = |x - x_0|^2 V^p\), by (3.7) and (b) we get

\[
\int_{\mathbb{R}^n} |x - x_0|^2 \langle V, x - x_0 \rangle |F|^2 G(x) dx = \int_{\mathbb{R}^n} -16 t_0^2 \langle J, V^p F_{ij} \rangle G dx.
\]

On the other hand taking \(\phi^p = \langle V, x - x_0 \rangle (x - x_0)^p\), by (3.7) and (b) we get

\[
\int_{\mathbb{R}^n} |x - x_0|^2 \langle V, x - x_0 \rangle |F|^2 G(x) dx = \int_{\mathbb{R}^n} -8 t_0^2 \langle J, V^i F_{ij} \rangle G dx.
\]

Thus we have

\[
\int_{\mathbb{R}^n} |x - x_0|^2 \langle V, x - x_0 \rangle |F|^2 G(x) dx = \int_{\mathbb{R}^n} \langle J, V^p F_{ij} \rangle G dx = 0.
\]

Taking \(\phi^p = \langle V, x - x_0 \rangle V^p\), by (3.7) we get

\[
\int_{\mathbb{R}^n} \langle x - x_0, V \rangle^2 |F|^2 G dx = \int_{\mathbb{R}^n} (2t_0 |V|^2 |F|^2 - 4t_0 \langle V^i F_{ij}, V^p F_{ij} \rangle) G dx.
\]

□

By the first variation formula (3.3), and (a) and (b) of Lemma 3.2 we get the following

**Corollary 3.2.** \((A(x), x_0, t_0)\) is a critical point of the \(F\)-functional if and only if \(A(x)\) is a homothetically shrinking soliton centered at \((x_0, t_0)\).

**Corollary 3.3.** When \(n = 2, 3,\) or 4, there exists no homothetically shrinking soliton such that \(|F|\) is uniformly bounded and not identically zero. In particular in dimension four, the Yang-Mills flow on \(E_M\) cannot develop a singularity of Type-I.
Proof. The first part follows from Lemma 3.2 (a). By Weinkove’s result [22] (see also Section 2) at a Type-I singularity of a Yang-Mills flow one can obtain a homothetically shrinking soliton on a trivial $G$-vector bundle over $\mathbb{R}^n$ whose curvature is uniformly bounded and non-zero. Therefore in dimension four if a Type-I singularity occurs, it would contradict with the non-existence of such a homothetically shrinking soliton.

4. SECOND VARIATION OF $F$-FUNCTIONAL

We now compute the second variation of the $F$-functional at a homothetically shrinking soliton $A(x)$. Let $d^\nabla$ denote the covariant exterior differentiation on $\mathfrak{g}$-valued forms and $(d^\nabla)^*$ denote the formal adjoint of $d^\nabla$. For a $\mathfrak{g}$-valued 1-form $\theta$, let

\begin{equation}
(4.1) \quad R(\theta_j) = R(\theta)(\partial_j) := [F_{ij}, \theta_i],
\end{equation}

and

\begin{equation}
(4.2) \quad L\theta := -t_0[(d^\nabla)^*d^\nabla \theta + R(\theta) + i \frac{1}{2t_0}(x-x_0)d^\nabla \theta].
\end{equation}

We also introduce the space

\begin{equation}
(4.3) \quad W_{C}^{2,2} := \{ \theta : \int_{\mathbb{R}^n} (|\theta|^2 + |\nabla \theta|^2 + |L\theta|^2)G(x)dx < \infty \}.
\end{equation}

Denote

\begin{equation*}
\frac{d}{ds}|_{s=0} A_s, \quad \frac{d^2}{ds^2}|_{s=0} F_{x_0, t_0}(q, V, \theta) = \frac{d^2}{ds^2}|_{s=0} F_{x, t_x}(A_s).
\end{equation*}

Proposition 4.1. Let $A(x)$ be a homothetically shrinking soliton in $S_{x_0, t_0}$; see (3.1). Then for any $\theta \in W_{C}^{2,2}$, we have

\begin{equation}
(4.4) \quad \frac{1}{2t_0} F_{x_0, t_0}''(q, V, \theta) = \int_{\mathbb{R}^n} (-L\theta - 2qJ - i_V F, \theta)Gdx - \int_{\mathbb{R}^n} (q^2|J|^2 + \frac{1}{2}|i_V F|^2)Gdx.
\end{equation}

Proof. Recall that

\begin{equation}
\frac{d}{ds} F_{x, t_x}(A_s) = \int_{\mathbb{R}^n} \left[ i_s \left( \frac{4-n}{2} t_s + \frac{1}{4} |x-x_s|^2 \right) |F_s|^2 G_s(x) dx 
\right. 
\left. + \int_{\mathbb{R}^n} \frac{1}{2} t_s (\dot{x}_s - |x-x_s|^2) |F_s|^2 G_s(x) dx 
\right. 
\left. - \int_{\mathbb{R}^n} 2t_s^2 (J_s - \frac{X_s}{2t_s}, \theta_s) G_s(x) dx \right].
\end{equation}

By the assumption that $A(x) \in S_{x_0, t_0}$ and Lemma 3.2 (a), (b), we have

\begin{equation}
F_{x_0, t_0}''(q, V, \theta) = \int_{\mathbb{R}^n} \left[ q \left( \frac{4-n}{2} q + \frac{1}{2} \left( x - x_0, V \right) \right) + \frac{1}{2} t_0 \left( V, - V \right) \right]|F|^2 Gdx 
\end{equation}

\begin{equation}
+ \int_{\mathbb{R}^n} \left[ q \left( \frac{4-n}{2} t_0 + \frac{1}{4} |x-x_0|^2 \right) + \frac{1}{2} t_0 \left( V, x - x_0 \right) \right] \partial |F_s|^2 \partial s |_{s=0} Gdx 
\end{equation}

\begin{equation}
+ \int_{\mathbb{R}^n} \left[ q \left( \frac{4-n}{2} t_0 + \frac{1}{4} |x-x_0|^2 \right) + \frac{1}{2} t_0 \left( V, x - x_0 \right) \right]|F|^2 \partial G_s \partial s |_{s=0} dx 
\end{equation}

\begin{equation}
- \int_{\mathbb{R}^n} 2t_0^2 \left( \frac{\partial}{\partial s} |_{s=0} (J_s - \frac{X_s}{2t_s}, \theta) \right) Gdx.
\end{equation}
Note that
\[
\frac{\partial |F_s|^2}{\partial s} \big|_{s=0} = F_{ij\beta}^\alpha (\nabla_i \theta_{j\beta}^\alpha - \nabla_j \theta_{i\beta}^\alpha) = 2 F_{ij\beta}^\alpha \nabla_i \theta_{j\beta}^\alpha,
\]
\[
\frac{\partial G_s}{\partial s} \big|_{s=0} = (-\frac{n q}{2 t_0} + \frac{q |x - x_0|^2}{4 t_0^2} + \frac{\langle V, x - x_0 \rangle}{2t_0}) G(x),
\]
\[
\frac{\partial}{\partial s} \big|_{s=0} \alpha_{ij\beta} = \nabla_i \nabla_j \alpha_{\beta} - \nabla_j \nabla_i \alpha_{\beta} + \theta_\alpha \gamma F_{ij\beta}^\gamma - F_{ij\beta}^\gamma \theta_\gamma.
\]
Thus we get
\[
F''_{x_0, t_0} (q, V, \theta)
= \int_{\mathbb{R}^n} \left[ q \left( \frac{4 - n}{2} - \frac{1}{2} \langle x - x_0, V \rangle \right) - \frac{1}{2} t_0 |V|^2 \right] |F|^2 Gdx
+ \int_{\mathbb{R}^n} \left[ q \left( \frac{4 - n}{2} t_0 + \frac{1}{4} |x - x_0|^2 \right) + \frac{1}{2} t_0 \langle V, x - x_0 \rangle \right] 2 F_{ij\beta}^\alpha \nabla_i \theta_{j\beta}^\alpha Gdx
+ \int_{\mathbb{R}^n} \left[ q \left( \frac{4 - n}{2} t_0 + \frac{1}{4} |x - x_0|^2 \right) + \frac{1}{2} t_0 \langle V, x - x_0 \rangle \right] |F|^2
\times \left( -\frac{n q}{2 t_0} + \frac{q |x - x_0|^2}{4 t_0^2} + \frac{\langle V, x - x_0 \rangle}{2t_0} \right) Gdx
- \int_{\mathbb{R}^n} 2 t_0^2 \nabla_i (\nabla_i \theta_{j\beta}^\alpha - \nabla_j \theta_{i\beta}^\alpha) + \theta_\alpha \gamma F_{ij\beta}^\gamma - F_{ij\beta}^\gamma \theta_\gamma \theta_{ij\beta}^\gamma Gdx
- \int_{\mathbb{R}^n} 2 t_0^2 \left( \frac{q}{2 t_0} X_{ij\beta}^\alpha + \frac{1}{2 t_0} V^k F_{ij\beta}^\alpha - \frac{1}{2 t_0} (x - x_0)^k (\nabla_k \theta_{ij\beta}^\alpha - \nabla_j \theta_{i\beta}^\alpha) \right) \theta_j^\alpha Gdx.
\]
By integration by parts, we have
\[
\int_{\mathbb{R}^n} \left[ q \left( \frac{4 - n}{2} t_0 + \frac{1}{4} |x - x_0|^2 \right) + \frac{1}{2} t_0 \langle V, x - x_0 \rangle \right] 2 F_{ij\beta}^\alpha \nabla_i \theta_{j\beta}^\alpha Gdx
= \int_{\mathbb{R}^n} -2 \left[ q \left( \frac{4 - n}{2} t_0 + \frac{1}{4} |x - x_0|^2 \right) + \frac{1}{2} t_0 \langle V, x - x_0 \rangle \right] \langle J - \frac{1}{2 t_0} X, \theta \rangle Gdx
- \int_{\mathbb{R}^n} 2 \left[ \frac{1}{2} q (x - x_0)^i - t_0 V^i \right] F_{ij\beta}^\alpha \theta_{j\beta}^\alpha Gdx
= \int_{\mathbb{R}^n} \left[ -q (x - x_0)^i - t_0 V^i \right] F_{ij\beta}^\alpha \theta_{j\beta}^\alpha Gdx.
Then by using Lemma 3.2, we have
\[
\mathcal{F}_{x_0,t_0}''(q,V,\theta) = \int_{\mathbb{R}^n} \left[ \frac{4-n}{2} q^2 - \frac{1}{2} t_0 |V|^2 \right] |F|^2 G dx \\
+ \int_{\mathbb{R}^n} [-q(x-x_0)^i - t_0 V^i] F_{ij\beta} \theta_j^\alpha \theta_{\beta}^\alpha G dx \\
+ \int_{\mathbb{R}^n} \left[ \frac{n-4}{2} q^2 |F|^2 - 2t_0 q^2 |J|^2 \right] G dx \\
+ \frac{1}{4} \left( 2t_0 |V|^2 |F|^2 - 4t_0 \langle V^i F_{ij}, V^p F_{pj} \rangle \right) G dx \\
- \int_{\mathbb{R}^n} 2t_0 [\nabla_p (\nabla_p \theta_j^\alpha - \nabla_j \theta_p^\alpha) + \theta_p^\gamma F_{pj\gamma} - F_{pj\gamma} \theta_p^\alpha] \theta_j^\alpha \theta_{\beta}^\alpha G dx \\
- \int_{\mathbb{R}^n} [q(x-x_0)^i + t_0 V^i] F_{ij\beta} \theta_j^\alpha \theta_{\beta}^\alpha G dx \\
+ \int_{\mathbb{R}^n} t_0 (x-x_0)^k (\nabla_k \theta_j^\alpha - \nabla_j \theta_k^\alpha) \theta_j^\alpha \theta_{\beta}^\alpha G dx.
\]

Thus,
\[
\mathcal{F}_{x_0,t_0}''(q,V,\theta) = \int_{\mathbb{R}^n} \left[ -2q(x-x_0)^i - 2t_0 V^i \right] F_{ij\beta} \theta_j^\alpha \theta_{\beta}^\alpha G dx \\
- \int_{\mathbb{R}^n} 2t_0 q^2 |J|^2 G dx - \int_{\mathbb{R}^n} t_0 \langle V^i F_{ij}, V^p F_{pj} \rangle G dx \\
- \int_{\mathbb{R}^n} 2t_0 [\nabla_p (\nabla_p \theta_j^\alpha - \nabla_j \theta_p^\alpha) + \theta_p^\gamma F_{pj\gamma} - F_{pj\gamma} \theta_p^\alpha] \theta_j^\alpha \theta_{\beta}^\alpha G dx \\
+ \int_{\mathbb{R}^n} t_0 (x-x_0)^k (\nabla_k \theta_j^\alpha - \nabla_j \theta_k^\alpha) \theta_j^\alpha \theta_{\beta}^\alpha G dx.
\]

Note that
\[
(d^V)^* d^V \theta_j = -\nabla_p (\nabla_p \theta_j - \nabla_j \theta_p), \\
\mathcal{R}(\theta_j) = [F_{pj}, \theta_p] = F_{pj} \theta_p - \theta_p F_{pj},
\]
so we have
\[
\mathcal{F}_{x_0,t_0}''(q,V,\theta) = \int_{\mathbb{R}^n} 2t_0 \langle (d^V)^* d^V \theta_j + \mathcal{R}(\theta_j) + i \frac{1}{2t_0} (x-x_0)^k \nabla_k \theta_j \theta_j \rangle G dx \\
- \int_{\mathbb{R}^n} 2t_0 \langle 2q J_j + V^i F_{ij}, \theta_j \rangle G dx \\
-2t_0 \int_{\mathbb{R}^n} (q^2 |J|^2 + \frac{1}{2} |i_V F|^2) G dx.
\]

Let
\[
L = -t_0 [(d^V)^* d^V + \mathcal{R} + i \frac{1}{2t_0} (x-x_0)^k \nabla_k];
\]
then we have
\[
\frac{1}{2t_0} \mathcal{F}_{x_0,t_0}''(q,V,\theta) = \int_{\mathbb{R}^n} \langle -L \theta - 2q J - i_V F, \theta \rangle G dx - \int_{\mathbb{R}^n} (q^2 |J|^2 + \frac{1}{2} |i_V F|^2) G dx.
\]

□
5. F-stability and its characterization

In this section we define the F-stability for homothetically shrinking solitons in $\mathcal{S}_{x_0,t_0}$. The operator $L$ admits eigenfields $J$ and $i_V F$ of eigenvalues $-1$ and $-\frac{1}{2}$, respectively. F-stability is equivalent to the semi-positiveness of $-L$ modulo the vector space spanned by $J$ and $i_V F$. Let $C^\infty_0(\Omega^1 \otimes \mathfrak{g})$, or simply $C^\infty_0$, denote the space of $\mathfrak{g}$-valued 1-forms with compact support on $\mathbb{R}^n$. The space $C^\infty_0$ is dense in $W^{2,2}_G$.

**Definition 5.1.** A homothetically shrinking soliton $A \in \mathcal{S}_{x_0,t_0}$ is called F-stable if for any $\theta$ in $C^\infty_0$, or equivalently in $W^{2,2}_G$, there exist a real number $q$ and a vector $V$ such that

$$F'_{x_0,t_0}(q,V,\theta) \geq 0.$$ 

Given a homothetically shrinking soliton $A \in \mathcal{S}_{x_0,t_0}$ with an exponential gauge, the rescaling

$$\tilde{A}_i(x) = \sqrt{t_0} A_i(\sqrt{t_0} x + x_0)$$

is a homothetically shrinking soliton in $\mathcal{S}_{0,1}$. Without loss of generality, in the remainder of this section we let $x_0 = 0$ and $t_0 = 1$. Then

$$G(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}$$

and

$$L \theta = -[(d^\nabla)^* d^\nabla \theta + \mathcal{R}(\theta) + i_\nabla d^\nabla \theta].$$

The operator $L$ is self-adjoint in the following sense: for any $\theta, \eta \in W^{2,2}_G$,

$$\int_{\mathbb{R}^n} \langle L \theta, \eta \rangle G dx = -\int_{\mathbb{R}^n} [(d^\nabla \theta, d^\nabla \eta) + \langle \mathcal{R}(\theta), \eta \rangle] G dx = \int_{\mathbb{R}^n} \langle \theta, L \eta \rangle G dx.$$

A $\mathfrak{g}$-valued 1-form $\theta \in W^{2,2}_G$ is called an eigenfield of $L$ and of eigenvalue $\lambda$ if $L \theta = \lambda \theta$. We denote the eigenfield space of eigenvalue $\lambda$ by $E_\lambda$.

**Proposition 5.1.** Let $A$ be a homothetically shrinking soliton in $\mathcal{S}_{0,1}$. Then

$$(5.2) \quad L J = J,$$

and

$$(5.3) \quad L (i_V F) = \frac{1}{2} i_V F, \quad \forall V \in \mathbb{R}^n.$$ 

**Proof.** Note that

$$J_j = \nabla_p F_{pj} = \frac{1}{2} x^p F_{pj},$$

$$L = -(d^\nabla)^* d^\nabla - \mathcal{R} - i_\nabla d^\nabla,$$

and

$$L J_j = \nabla_p \nabla_p J_j - \nabla_p \nabla_j J_p - [F_{pj}, J_p] - \frac{1}{2} (d^\nabla J)(x^p \partial_p, \partial_j)$$

$$= \nabla_p \nabla_p J_j - \nabla_p \nabla_j J_p - [F_{pj}, J_p] - \frac{1}{2} x^p (\nabla_p J_j - \nabla_j J_p).$$

We have

$$\nabla_p J_j = \nabla_p \left( \frac{1}{2} x^q F_{qj} \right) = \frac{1}{2} F_{pj} + \frac{1}{2} x^q \nabla_p F_{qj}.$$ 

Then

$$\nabla_p \nabla_j J_p = \nabla_p \left( -\frac{1}{2} F_{pj} + \frac{1}{2} x^q \nabla_j F_{qp} \right) = -\frac{1}{2} J_j - \frac{1}{2} x^q \nabla_p \nabla_j F_{pq}.$$
Proof. Note that
\[ \nabla_p \nabla_p J_j = \nabla_p F_{pq} + \frac{1}{2} x^q \nabla_p \nabla_p F_{pq} \]
\[ = \nabla_p F_{pq} + \frac{1}{2} x^q \nabla_p (-\nabla_q F_{jp} - \nabla_j F_{pq}) \]
\[ = \nabla_p F_{pq} - \frac{1}{2} x^q (\nabla_q \nabla_p F_{jp} + F_{pq} F_{jp} - F_{jp} F_{pq}) - \frac{1}{2} x^q \nabla_p \nabla_j F_{pq} \]
\[ = J_j + \frac{1}{2} x^q \nabla_q J_j + [J_p, F_{jp}] - \frac{1}{2} x^q \nabla_p \nabla_j F_{pq}. \]
Hence
\[ L J_j = \frac{3}{2} J_j + \frac{1}{2} x^p \nabla_j J_p. \]
The identity \((5.2)\) then follows from
\[ \frac{1}{2} x^p \nabla_j J_p = \frac{1}{2} \nabla_j (x^p J_p) - \frac{1}{2} J_j = \frac{1}{2} \nabla_j (x^p \frac{1}{2} x^q F_{qp}) - \frac{1}{2} J_j = -\frac{1}{2} J_j. \]

We now prove \((5.3)\). By using the Bianchi identity and the Ricci formula, we get
\[ \nabla_p \nabla_p (V^q F_{qj}) = V^q \nabla_p (-\nabla_q F_{jp} - \nabla_j F_{pq}) \]
\[ = -V^q (\nabla_q \nabla_p F_{jp} + F_{pq} F_{jp} - F_{jp} F_{pq}) - V^q \nabla_p \nabla_j F_{pq} \]
\[ = V^q \nabla_q \left( \frac{1}{2} x^p F_{pj} \right) + [V^q F_{qp}, F_{jp}] + \nabla_p \nabla_j (V^q F_{qp}); \]
hence
\[ L (V^q F_{qj}) = \nabla_p \nabla_p (V^q F_{qj}) - \nabla_p \nabla_j (V^q F_{qp}) - [F_{pj}, V^q F_{qp}] \]
\[ - \frac{1}{2} x^p [\nabla_p (V^q F_{qj}) - \nabla_j (V^q F_{qp})] \]
\[ = V^q \nabla_q \left( \frac{1}{2} x^p F_{pj} \right) - \frac{1}{2} x^p [\nabla_p (V^q F_{qj}) - \nabla_j (V^q F_{qp})] \]
\[ = \frac{1}{2} V^q F_{qj} + \frac{1}{2} x^p V^q (\nabla_q F_{pj} + \nabla_p F_{jq} + \nabla_j F_{pq}) \]
\[ = \frac{1}{2} V^q F_{qj}. \]
\[ \square \]

**Corollary 5.1.** Let \( A \) be a homothetically shrinking soliton in \( S_{0,1} \). If \( |F|^2 < \frac{n}{2(n-1)} \), then \((E, A)\) is flat.

**Proof.** Note that \( J_j = \nabla^p F_{pj} = \frac{1}{2} x^p F_{pj} \). By integration by parts, we have
\[ \int_{\mathbb{R}^n} ((d\nabla)^* d\nabla J + i \frac{1}{2} d\nabla J, J) G dx = \int_{\mathbb{R}^n} |d\nabla J|^2 G dx. \]

On the other hand by \((5.2)\), we have
\[ \int_{\mathbb{R}^n} ((d\nabla)^* d\nabla J + i \frac{1}{2} d\nabla J, J) G dx = \int_{\mathbb{R}^n} \langle -L J - R(J), J \rangle G dx \]
\[ = -\int_{\mathbb{R}^n} |J|^2 G dx - \int_{\mathbb{R}^n} \langle [F_{ij}, J_i], J_j \rangle G dx. \]
For any $B, C \in \text{so}(r)$, we have $||B, C|| \leq |B||C|$; see Lemma 2.30 in [3]. Hence
\[
|\langle [F_{ij}, J_i], J_j \rangle| \leq 2\sum_{i<j} |F_{ij}||J_i||J_j| 
\leq 2 \sqrt{\sum_{i<j} |F_{ij}|^2} \left( \frac{1}{2} |J_i|^4 - \sum_k |J_k|^4 \right) 
\leq 2 |F| \sqrt{\frac{1}{2}(1 - \frac{1}{n})}|J|^4 
= \sqrt{\frac{2(n-1)}{n}} |F||J|^2 
\]
and
\[
\int_{\mathbb{R}^n} d^\n J^2 G dx \leq \int_{\mathbb{R}^n} \left( \frac{2(n-1)}{n} |F| - 1 \right) |J|^2 G dx. 
\]
If $|F|^2 < \frac{n}{2(n-1)}$, one then gets $J = 0$. Note that if $A \in S_{0.1}$ has $J = 0$, then for any $t_0 > 0$ we have $J = \frac{1}{2t_0}X$. Hence Lemma 3.2 (a) holds for any $t_0 > 0$ and $F$ vanishes.

Theorem 5.2. Let $A$ be a homothetically shrinking soliton in $S_{0.1}$. Then it is $F$-stable if and only if the following properties are satisfied:

1. $E_{-1} = \{ cJ, c \in \mathbb{R} \}$;
2. $E_{-\frac{1}{2}} = \{ iV F, V \in \mathbb{R}^n \}$;
3. $E_\lambda = \{ 0 \}$, for any $\lambda < 0$ and $\lambda \neq -1, -\frac{1}{2}$.

Proof. Let $\theta$ be a $g$-value 1-form in $W_{G}^{2,2}$ of the form
\[
\theta = aJ + iW F + \tilde{\theta}, \quad a \in \mathbb{R}, W \in \mathbb{R}^n, 
\]
and satisfying
\[
\int_{\mathbb{R}^n} \langle \tilde{\theta}, J \rangle G dx = \int_{\mathbb{R}^n} \langle \tilde{\theta}, iV F \rangle G dx = 0, \quad \forall V \in \mathbb{R}^n. 
\]
Then it follows from Proposition 4.1, Proposition 5.1 and (5.1) that
\[
\frac{1}{2} F_{0,1}''(q,V,\theta) = \int_{\mathbb{R}^n} \langle -L \theta - 2qJ - iV F, \theta \rangle G dx - \int_{\mathbb{R}^n} (q^2 |J|^2 + \frac{1}{2}|iV F|^2) G dx 
= \int_{\mathbb{R}^n} \langle -aJ - \frac{1}{2} iW F - L \tilde{\theta}, aJ + iW F + \tilde{\theta} \rangle G dx 
+ \int_{\mathbb{R}^n} \langle -2qJ - iV F, aJ + iW F + \tilde{\theta} \rangle G dx 
- \int_{\mathbb{R}^n} (q^2 |J|^2 + \frac{1}{2}|iV F|^2) G dx 
= -(a + q)^2 \int_{\mathbb{R}^n} |J|^2 G dx - \frac{1}{2} \int_{\mathbb{R}^n} |iV + W F|^2 G dx 
+ \int_{\mathbb{R}^n} \langle -L \tilde{\theta}, \tilde{\theta} \rangle G dx. 
\]
Letting $q = -a$, $V = -W$, one has the equivalence. □
6. Entropy and entropy-stability

We now introduce \( \lambda \)-entropy of connections on the trivial \( G \)-vector bundle \( E \) over \( \mathbb{R}^n \). We shall show that along the Yang-Mills flow, the entropy is non-increasing. We also prove that the entropy of a homothetically shrinking soliton \( A(x) \in S_{x_0, t_0} \) is achieved exactly by \( F_{x_0, t_0}(A) \), provided that \( i_V F \neq 0 \) for any non-zero vector \( V \in \mathbb{R}^n \).

**Definition 6.1.** Let \( A(x) \) be a connection on \( E \). We define the entropy by

\[
\lambda(A) = \sup_{x_0 \in \mathbb{R}^n, t_0 > 0} F_{x_0, t_0}(A).
\]

We first consider the invariance property of the entropy.

**Proposition 6.1.** The entropy \( \lambda \) is invariant under translations and rescalings.

**Proof.** Let \( A(x) \) be a connection on \( E \). A translation of \( A(x) \) is a new connection, denoted by \( \tilde{A}(x) \), of the form

\[
\tilde{A}_i(x) = A_i(x + x_1),
\]

where \( x_1 \) is a point in \( \mathbb{R}^n \). For any \( x_0 \in \mathbb{R}^n \) and \( t_0 > 0 \), we have

\[
F_{x_0 - x_1, t_0}(\tilde{A}) = F_{x_0, t_0}(A).
\]

Hence

\[
\lambda(\tilde{A}) = \lambda(A).
\]

A rescaling of \( A(x) \) is a new connection, denoted by \( A^c(x) \), of the form

\[
A^c_i(x) = c^{-1} A_i(c^{-1} x),
\]

where \( c \) is a positive number. Then, by setting \( y = c^{-1} x \), we have

\[
F_{c x_0, c^2 t_0}(A^c) = (c^2 t_0)^2 \int_{\mathbb{R}^n} |F^c(x)|^2 (4\pi c^2 t_0)^{-\frac{n}{2}} e^{-\frac{|x - cx_0|^2}{4\pi c^2 t_0}} dx,
\]

\[
= (c^2 t_0)^2 \int_{\mathbb{R}^n} c^{-4}|F(c^{-1} x)|^2 (4\pi c^2 t_0)^{-\frac{n}{2}} e^{-\frac{|x - cx_0|^2}{4\pi c^2 t_0}} dx,
\]

\[
= (c^2 t_0)^2 \int_{\mathbb{R}^n} c^{-4}|F(y)|^2 (4\pi c^2 t_0)^{-\frac{n}{2}} e^{-\frac{|y - cx_0|^2}{4\pi c^2 t_0}} c^n dy
\]

\[
= t_0^2 \int_{\mathbb{R}^n} |F(y)|^2 (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{|y - x_0|^2}{4\pi t_0}} dy
\]

\[
= F_{x_0, t_0}(A).
\]

Hence

\[
\lambda(A^c) = \lambda(A).
\]

□

In the case that \( A(x) \) is a homothetically shrinking soliton, Proposition 6.1 explains why in Theorem 5.2 \( J \) and \( i_V F \) do not violate the F-stability.

**Proposition 6.2.** Let \( A(x, t) \) be a solution to the Yang-Mills flow on \( E \). Then the entropy \( \lambda(A(x, t)) \) is non-increasing in \( t \).
Proof. Let \( t_1 < t_2 < T \). Here \( T \) denotes the first singular time of the Yang-Mills flow. By (6.1), for any given \( \epsilon > 0 \) there exists \((x_0, t_0)\) such that
\[
\lambda(A(x, t_2)) - \epsilon \leq \mathcal{F}_{x_0, t_0}(A(x, t_2)).
\]
Note that for any \( c > 0 \) and \( 0 \leq t < T \), we have
\[
F_{x_0, c}(A(x, t)) = \Phi_{x_0, c+t}(A(x, t)).
\]
By (6.3), the monotonicity formula (2.8), and the definition of entropy, we have
\[
F_{x_0, t_0}(A(x, t_2)) = \Phi_{x_0, t_0+t_2}(A(x, t_1)) \leq \lambda(A(x, t_1)).
\]
Together with (6.2), we see that \( \lambda(A(x, t_2)) \leq \lambda(A(x, t_1)). \)

**Definition 6.2.** A homothetically shrinking soliton \( A(x) \) is called entropy-stable if it is a local minimum of the entropy, among all perturbations \( \tilde{A}(x) \), such that \( ||\tilde{A} - A||_{C^1} \) is sufficiently small.

In general the entropy \( \lambda(A) \) is not attained by any \( F_{x_0, t_0}(A) \). However if \( A \in S_{x_0, t_0} \) and \( i_V F \neq 0 \) for any \( V \in \mathbb{R}^n \), we will show that \( \lambda(A) \) is attained exactly by \( F_{x_0, t_0}(A) \). We first examine the geometric meaning of \( i_V F = 0 \) in the case that \( A(x) \) is a homothetically shrinking soliton.

**Proposition 6.3.** If \( A(x) \) is a homothetically shrinking soliton satisfying \( i_V F = 0 \) for some non-zero vector \( V \), then \( A(x) \) is defined on a hyperplane perpendicular to \( V \).

**Proof.** Without loss of generality we assume \( A(x) \) is centered at \((0, 1)\) and let \( A(x, t) \) be the homothetically shrinking Yang-Mills flow with \( A(x, 0) = A(x) \). In the exponential gauge, i.e. a gauge such that \( x^j A_j(x) = 0 \), we have for any \( t < 1 \) and \( \lambda > 0 \) that
\[
A_j(x, t) = \lambda A_j(\lambda x, \lambda^2(t - 1) + 1) = \frac{1}{\sqrt{1-t}} A_j(\frac{x}{\sqrt{1-t}}, 0) = \frac{1}{\sqrt{1-t}} A_j(\frac{x}{\sqrt{1-t}})
\]
and
\[
F_{ij}(x, t) = \frac{1}{1-t} F_{ij}(\frac{x}{\sqrt{1-t}}).
\]
Moreover the exponential gauge is uniform for all \( t < 1 \), i.e. \( x^j A_j(x, t) = 0 \).

By assumption we have \( i_V F(x) = 0 \). For simplicity let \( V = \frac{\partial}{\partial x^j} \). Then by (6.5) we have
\[
F_{jl}(x, t) = 0, \quad \forall j.
\]
Note that \( A(x, t) \) is a homothetically shrinking Yang-Mills flow; hence
\[
J_l(x, t) = \frac{1}{2(1-t)} x^j F_{jl}(x, t) = 0.
\]
Then
\[
\frac{\partial}{\partial t} A_l(x, t) = J_l(x, t) = 0.
\]
In particular,
\[
A_l(x, t') = A_l(x, t), \quad \forall t, t' < 1.
\]
Then by (6.3), we have for any $\lambda > 0$ and $t < 1$ that

$$A_t(x, t) = \lambda A_t(\lambda x, \lambda^2(t - 1) + 1) = \lambda A_t(\lambda x, t).$$

Letting $\lambda \to 0$, we see that

(6.6) $A_t(x, t) = 0$.

Note that

$$0 = F_j(x, t) = \partial_t A_j - \partial_j A_t + A_t A_j - A_j A_t = \partial_t A_j(x, t),$$

so for any $j$, we have

$$\partial_t A_j(x, t) = 0$$

and

(6.7) $A_j(x + cV, t) = A_j(x, t), \forall c \in \mathbb{R}$.

For example if $V = \frac{\partial}{\partial x^n}$, then by (6.6) and (6.7) we have

$$A(x^1, \cdots, x^{n-1}, x^n) = A_1(x^1, \cdots, x^{n-1}, 0, t)dx^1 \cdots + A_{n-1}(x^1, \cdots, x^{n-1}, 0)dx^{n-1}.$$

In particular for $V = \frac{\partial}{\partial x^n}$ and in the exponential gauge, we have

(6.8) $A(x^1, \cdots, x^{n-1}, x^n) = A_1(x^1, \cdots, x^{n-1}, 0)dx^1 \cdots + A_{n-1}(x^1, \cdots, x^{n-1}, 0)dx^{n-1}$.

This means that $A(x)$ is defined on a hyperplane perpendicular to $V$, i.e. $A(x)$ descends to a trivial $G$-vector bundle over a hyperplane $V^\perp$.

The following proposition is analogous to a corresponding result for self-shrinkers of the mean curvature flow; see [5]. We follow closely the arguments given in [5].

**Proposition 6.4.** Let $A(x)$ be a homothetically shrinking soliton centered at $(0, 1)$ such that $iVF \neq 0$ for any non-zero $V$. Then the function $(x_0, t_0) \mapsto \mathcal{F}_{x_0, t_0}(A)$ attains its strict maximum at $(0, 1)$. In fact for any given $\epsilon > 0$, there exists a constant $\delta > 0$ such that

(6.9) $\sup\{\mathcal{F}_{x_0, t_0}(A) : |x_0| + |\log t_0| \geq \epsilon\} < \lambda(A) - \delta$.

In particular, the entropy of $A$ is achieved by $\mathcal{F}_{0, 1}(A)$.

**Proof.** We first show that $(0, 1)$ is a local maximum of the function $(x_0, t_0) \mapsto \mathcal{F}_{x_0, t_0}(A)$. That is, to show

$$\mathcal{F}_{0, 1}'(q, V, 0) = 0, \forall q, V,$$

and

$$\mathcal{F}_{0, 1}''(q, V, 0) < 0, \forall (q, V) \neq (0, 0).$$

In fact by the first variation formula (3.3) and Lemma 3.2 (a), (b), we have

$$\frac{d}{ds}\big|_{s=0}\mathcal{F}_{x_s, t_s}(A) = \mathcal{F}_{0, 1}'(q, V, 0) = 0.$$

Let $x_s = sV, t_s = 1 + sq$. Note that $J \neq 0$. Otherwise $F$ would be vanishing, as shown in the proof of Corollary 5.1 which violates the assumption that $iVF \neq 0$ for any non-zero $V$. Then by the second variation formula (4.4), we have for any $(q, V) \neq (0, 0)$ that

$$\frac{1}{2}\mathcal{F}_{0, 1}''(q, V, 0) = -\int_{\mathbb{R}^n} (q^2|J|^2 + \frac{1}{2}|iVF|^2)Gdx < 0.$$
For any fixed \((y, T)\), where \(y \in \mathbb{R}^n\) and \(T > 0\), we set 
\[ x_s = sy, \quad t_s = 1 + (T - 1)s^2. \]
Note that \((x_s, t_s), s \in [0, 1]\), is a path from \((0, 1)\) to \((y, T)\). Let 
\[ g(s) = F_{x_s, t_s}(A). \]
The remainder of the proof is to show that \(g'(s) \leq 0\) for \(s \in [0, 1]\).

By the first variation formula \([3.3]\), we have 
\[
g'(s) = \int_{\mathbb{R}^n} \dot{t}_s \left( \frac{4 - n}{2} t_s + \frac{1}{4} |x - x_s|^2 \right) |F|^2 G_s(x) dx 
+ \int_{\mathbb{R}^n} \frac{1}{2} t_s (\dot{x}_s, x - x_s) |F|^2 G_s(x) dx.
\]
In the same way as in the proof of Lemma \([3.2]\), for vector fields \(\varphi\) on \(\mathbb{R}^n\) we have 
\[
\int_{\mathbb{R}^n} \varphi^p (x - x_s)^p |F|^2 G_s(x) dx 
= \int_{\mathbb{R}^n} \left[ 2t_s \partial_p (\varphi^p) |F|^2 - 4t_s \partial_i \varphi^p F^\alpha_{pj\beta} F^\alpha_{ij\beta} \right] G_s(x) dx 
- \int_{\mathbb{R}^n} 4t_s \varphi^p F^\alpha_{pj\beta} (\dot{J}^\alpha_{j\beta} - \frac{1}{2t_s} X^\alpha_{j\beta}) G_s(x) dx,
\]
where 
\[ X^\alpha_{j\beta} = (x - x_s)^p F^\alpha_{pj\beta}. \]

Taking \(\varphi = \frac{\partial}{\partial x^p}\) and noting that \(J_j = x^p F^p_{pj}\), we get 
\[
\int_{\mathbb{R}^n} (x - x_s)^p |F|^2 G_s(x) dx 
= - \int_{\mathbb{R}^n} 4t_s F^\alpha_{pj\beta} (\dot{J}^\alpha_{j\beta} - \frac{1}{2t_s} X^\alpha_{j\beta}) G_s(x) dx 
= - \int_{\mathbb{R}^n} 4t_s F^\alpha_{pj\beta} (\frac{1}{2} x^i - \frac{1}{2t_s} (x - x_s) x^i) F^\alpha_{ij\beta} G_s(x) dx.
\]
Taking \(\varphi(x) = x - x_s\), we get 
\[
\int_{\mathbb{R}^n} |x - x_s|^2 |F|^2 G_s(x) dx 
= \int_{\mathbb{R}^n} [2(n - 4)t_s |F|^2] G_s(x) dx - \int_{\mathbb{R}^n} 4t_s X^\alpha_{j\beta} (\dot{J}^\alpha_{j\beta} - \frac{1}{2t_s} X^\alpha_{j\beta}) G_s(x) dx 
= \int_{\mathbb{R}^n} [2(n - 4)t_s |F|^2 + 2|X|^2] G_s(x) dx - \int_{\mathbb{R}^n} 2t_s X^\alpha_{j\beta} x^i F^\alpha_{ij\beta} G_s(x) dx.
\]
Hence we have 
\[
g'(s) = - \int_{\mathbb{R}^n} \frac{n - 4}{2} t_s \dot{t}_s |F|^2 G_s(x) dx 
+ \frac{1}{4} t_s \left[ \int_{\mathbb{R}^n} [2(n - 4)t_s |F|^2 + 2|X|^2] G_s(x) dx - \int_{\mathbb{R}^n} 2t_s X^\alpha_{j\beta} x^i F^\alpha_{ij\beta} G_s(x) dx \right] 
- t_s y^p \int_{\mathbb{R}^n} t_s F^\alpha_{pj\beta} (x^i - \frac{1}{t_s} (x - x_s)^i) F^\alpha_{ij\beta} G_s(x) dx 
= \frac{1}{2} t_s \int_{\mathbb{R}^n} |X|^2 G_s(x) dx - \int_{\mathbb{R}^n} t_s X^\alpha_{j\beta} x^i F^\alpha_{ij\beta} G_s(x) dx 
- t_s y^p \int_{\mathbb{R}^n} t_s F^\alpha_{pj\beta} (x^i - \frac{1}{t_s} (x - x_s)^i) F^\alpha_{ij\beta} G_s(x) dx.
\]
Set \( z = x - x_s = x - sy \). We have \( x = z + sy \) and \( X_j = z^i F_{ij} \). Then we get

\[
g'(s) = \frac{1}{2} t_s \int_{\mathbb{R}^n} (1 - t_s) |X|^2 G_s dx - \int_{\mathbb{R}^n} t_s X_{j\beta} \frac{1}{t_s} z^i F_{ij\beta} G_s dx
\]

\[
- t_s y^p \int_{\mathbb{R}^n} t_s F^\alpha_{pj\beta}(z^i + sy^j) F_{ij\beta} G_s dx
\]

\[
= \frac{1}{2} t_s \int_{\mathbb{R}^n} (1 - t_s) |X|^2 G_s dx - \int_{\mathbb{R}^n} t_s X_{j\beta} \frac{1}{t_s} z^i F_{ij\beta} G_s dx
\]

\[
- t_s y^p \int_{\mathbb{R}^n} (t_s - 1) F^\alpha_{pj\beta} X_j^\alpha G_s dx - t_s^2 \int_{\mathbb{R}^n} y^p F^\alpha_{pj\beta} y^i F_{ij\beta} G_s dx
\]

\[
= \frac{1}{2} t_s (1 - t_s) \int_{\mathbb{R}^n} |X|^2 G_s dx - \left( \frac{1}{2} st_s t_s + t_s (t_s - 1) \right) \int_{\mathbb{R}^n} \langle X_j, y^i F_{ij} \rangle G_s dx
\]

\[
- st_s^2 \int_{\mathbb{R}^n} |y^i F_{ij}|^2 G_s dx.
\]

For \( t_s = 1 + (T - 1)s^2 \), we have

\[
g'(s) = -s \left( (T - 1)^2 s^2 \int_{\mathbb{R}^n} |X|^2 G_s dx + 2(T - 1) s t_s \int_{\mathbb{R}^n} \langle X_j, y^i F_{ij} \rangle G_s dx \right)
\]

\[
+ t_s^2 \int_{\mathbb{R}^n} |y^i F_{ij}|^2 G_s dx
\]

\[
= -s \int_{\mathbb{R}^n} |(T - 1)s X_j + t_s y^i F_{ij}|^2 G_s dx
\]

\[
\leq 0.
\]

\[
\square
\]

7. Entropy-stability and F-stability

In this section we shall show that the entropy-stability of a homothetically shrinking soliton such that \( i_V F \neq 0 \) for any non-zero \( V \) implies F-stability.

**Theorem 7.1.** Let \( A(x) \) be a homothetically shrinking soliton in \( S_{0,1} \) such that \( i_V F \neq 0 \) for any non-zero \( V \). If \( A(x) \) is entropy-stable, then it is F-stable.

**Proof.** We argue by contradiction. Assume that \( A(x) \) is F-unstable. By the definition of F-stability there exists a 1-parameter family of connections \( A_s(x), s \in [-\epsilon, \epsilon] \), with \( \theta_s(x) := \frac{d}{ds} A_s(x) \in C^\infty_{0} \), such that for any deformation \((x_s, t_s)\) of \((x_0 = 0, t_0 = 1)\), we have

\[
(7.1) \quad \frac{d^2}{ds^2} |_{s=0} F_{x_s, t_s}(A_s) < 0.
\]

We start from this to show that \( A \) is entropy-unstable. Let

\[
H : \mathbb{R}^n \times \mathbb{R}^+ \times [-\epsilon, \epsilon], \quad H(y, T, s) = F_{y, T}(A_s).
\]
In fact we will show that there exists $\epsilon_0 > 0$ such that for $s$ with $0 < |s| \leq \epsilon_0,$
\begin{equation}
(7.2) \quad \sup_{y, T} H(y, T, s) < H(0, 1, 0).
\end{equation}

Hence for $s$ with $0 < |s| \leq \epsilon_0,$ $\lambda(A_s) < \lambda(A),$ which contradicts our assumption.

**Step 1.** We prove that there exists $\epsilon_1 > 0$ such that for any $s$ with $0 < |s| \leq \epsilon_1,$
\begin{equation}
(7.3) \quad \sup\{H(y, T, s) : |y| \leq \epsilon_1, |\log T| \leq \epsilon_1\} < H(0, 1, 0).
\end{equation}

By the assumption that $A(x) \in S_{0,1}$ and Corollary 3.2, we have
\[ \nabla H(0, 1, 0) = 0. \]

For any $y \in \mathbb{R}^n, a \in \mathbb{R}$ and $b \in \mathbb{R}, (sy, 1 + as, bs)$ is a curve through $(0, 1, 0).$ In the case of $b \neq 0,$ by (7.1) we have
\[
\frac{d^2 H}{ds^2}\bigg|_{s=0}(sy, 1 + as, bs) = \frac{d^2}{ds^2}\bigg|_{s=0}\mathcal{F}_{sy, 1+as}(A bs) \notag = b^2 \frac{d^2}{ds^2}\bigg|_{s=0}\mathcal{F}_{sy, 1+as}(A s) < 0.
\]

For $b = 0$ and $(a, y) \neq (0, 0),$ we have
\[
\frac{d^2 H}{ds^2}\bigg|_{s=0}(sy, 1 + as, 0) = \frac{d^2}{ds^2}\bigg|_{s=0}\mathcal{F}_{sy, 1+as}(A) \notag = -2 \int_{\mathbb{R}^n} (a^2 |J|^2 + \frac{1}{2} |i_y F|^2) G dx < 0,
\]
where we used the assumption that $i_y F \neq 0$ for $y \neq 0$ and its implication that $J \neq 0.$ Hence the Hessian of $H$ at $(0, 1, 0)$ is negative definite and $H$ has a local strict maximum at $(0, 1, 0).$ Thus there exists $\epsilon_1 \in (0, \epsilon]$ such that if $0 < |y| + |\log T| + |s| \leq 3\epsilon_1,$ then $H(y, T, s) < H(0, 1, 0).$ In particular for any $s$ with $0 < |s| \leq \epsilon_1,$ we have
\[ \sup\{H(y, T, s) : |y| \leq \epsilon_1, |\log T| \leq \epsilon_1\} < H(0, 1, 0). \]

**Step 2.** We prove that there exists $R_0 > 0$ such that
\begin{equation}
(7.4) \quad \sup_{T, s} H(y, T, s) < H(0, 1, 0), \quad \text{for} \quad |y| \geq R_0.
\end{equation}

Denote the support of $\theta_s$ by $\Omega_s$ and $\Omega = \bigcup_{s \in [-\epsilon, \epsilon]} \Omega_s.$ Then on $\mathbb{R}^n \setminus \Omega, F_s = F.$ Hence
\[
H(y, T, s) = T^2 \int_{\Omega} |F_s|^2 (4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} dx + T^2 \int_{\mathbb{R}^n \setminus \Omega} |F|^2 (4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} dx \notag \leq T^2 \int_{\Omega} |F_s|^2 (4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} dx + H(y, T, 0).
\]
Note that for $|y| \geq 1$, there exists $\delta > 0$ such that $H(y, T, 0) \leq H(0, 1, 0) - \delta$; see Proposition [6.4]. Let $M = \sup \{|F_s(x)|^2 : s \in [-\epsilon, \epsilon], x \in \mathbb{R}^n\}$, $D = \sup_{x \in \Omega} |x|$ and $|\Omega| = \int_{\Omega} dx$. Then for $|y| \geq D + R$ with $R \geq 1$, we have

$$H(y, T, s) \leq M|\Omega|T^2(4\pi T)^{-\frac{n}{2}} e^{-\frac{|y|^2}{4T}} + H(0, 1, 0) - \delta.$$ 

Let $f(r) = r^{-\frac{n-4}{2}} e^{-\frac{4}{nT}}, r > 0$, which is uniformly bounded. Note that $n \geq 5$. Then as $R \to \infty$, $T^{\frac{n}{4-n}} e^{-\frac{R^2}{4T}} = f(T^{-\frac{2}{n}})R^{4-n} \to 0$, uniformly in $T > 0$. Hence we can choose sufficiently large $R$ such that for $|y| \geq D + R := R_0$, we have $H(y, T, s) \leq H(0, 1, 0) - \frac{\delta}{2}$.

**Step 3.** We prove that there exists $T_0 > 0$ such that

$$\sup_{y,s} H(y, T, s) < H(0, 1, 0), \quad \text{for} \quad |\log T| \geq T_0. \quad (7.5)$$

We first consider the case that $T$ is large. Note that for any $T > 0$,

$$\begin{align*}
H(y, T, s) &= T^2 \int_{\Omega} |F_s|^2(4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} dx + T^2 \int_{\mathbb{R}^n \setminus \Omega} |F|^2(4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} dx \\
&\leq T^2 \int_{\Omega} |F_s|^2(4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} dx + H(y, T, 0) \\
&\leq M|\Omega|T^2(4\pi T)^{-\frac{n}{2}} + H(y, T, 0).
\end{align*}$$

By Proposition [6.4] there exists $\delta > 0$ such that $H(y, T, 0) \leq H(0, 1, 0) - \delta$ when $T \geq 2$. Hence there exists $T_1 \geq 2$ such that

$$H(y, T, s) \leq H(0, 1, 0) - \frac{\delta}{2}, \quad \text{for} \quad T \geq T_1. \quad (7.6)$$

Note that $M = \sup \{|F_s(x)|^2 : s \in [-\epsilon, \epsilon], x \in \mathbb{R}^n\}$. Hence for any $T > 0$, we have

$$H(y, T, s) = F(y, T)(A_s) = T^2 \int_{\mathbb{R}^n} |F_s|^2(4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} dx \leq MT^2.$$ 

Thus there exists $T_2 > 0$ such that

$$\sup_{y,s} H(y, T, s) < H(0, 1, 0), \quad \text{for} \quad T \leq T_2. \quad (7.7)$$

Combing (7.6) and (7.7), we get (7.5).

**Step 4.** Set

$$U = \{(y, T) : |y| \leq R_0, |\log T| \leq T_0\} \setminus \{(y, T) : |y| < \epsilon_1, |\log T| < \epsilon_1\}.$$ 

We now prove that there exists $\epsilon_0 \leq \epsilon_1$ such that for any $s$ with $|s| \leq \epsilon_0$,

$$\sup_{U} \{H(y, T, s) : (y, T) \in U\} < H(0, 1, 0). \quad (7.8)$$

Note that $U$ is a compact set which does not contain $(0,1)$. By Proposition [6.4] there exists $\delta > 0$ such that

$$\sup_{U} H(y, T, 0) \leq H(0, 1, 0) - \delta.$$
By the first variation formula (3.3) of the $\mathcal{F}$-functional, we have
\[ \frac{d}{ds}H(y,T,s) = -2T^2 \int_{\mathbb{R}^n} \langle J(A_s) - \frac{1}{2T} i_{x-y} F(A_s), \theta_s \rangle G_{y,T}(x) dx. \]
Since $\theta_s$ is compactly supported, $\frac{\partial}{\partial s} H$ is continuous in all three variables $y,T,s$. Therefore there exists $0 < \epsilon_0 \leq \epsilon_1$ such that if $|s| \leq \epsilon_0$, then
\[ \sup_U H(y,T,s) \leq H(0,1,0) - \frac{\delta}{2}. \]
This proves (7.8). Combining (7.3), (7.4), (7.5) and (7.8), we get (7.2) and complete the proof. □

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