Bilayer Graphene as a Platform for Bosonic Symmetry Protected Topological States

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Bosonic symmetry protected topological (BSPT) states, the bosonic analogue of topological insulators, have attracted enormous theoretical interest in the last few years. Although BSPT states have been classified by various approaches, there is so far no successful experimental realization of any BSPT state in two or higher dimensions. In this paper, we propose that a two dimensional BSPT state with $U(1) \times U(1)$ symmetry can be realized in bilayer graphene in a magnetic field. Here the two $U(1)$ symmetries represent total spin $S^z$ and total charge conservation respectively. The Coulomb interaction plays a central role in this proposal – it gaps out all the fermions at the boundary, so that only bosonic charge and spin degrees of freedom are gapless and protected at the edge. Based on the bosonic nature of the boundary states, we derive the bulk wave function for the bosonic charge and spin degrees of freedom, which takes exactly the same form as the desired BSPT state. We also propose that the bulk quantum phase transition between the BSPT and trivial phase, could become a “bosonic phase transition” with interactions. That is, only bosonic modes close their gap at the transition, which is fundamentally different from all the well-known topological insulator to trivial insulator transitions which occur for free fermion systems. We discuss various experimental consequences of this proposal.

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ferromagnetic quantum Hall state, since the bulk is fully spin polarized. In order to avoid a canted antiferromagnetic phase, one also needs an inplane magnetic field to increase the Zeeman coupling [21][22], which will be discussed in detail in the supplementary material [55]). In a bilayer, this possesses at the Hartree-Fock level two channels of counter-propagating spin-filtered helical fermionic edge states [22][24]. However, when the Coulomb interaction is included, we will demonstrate that (as illustrated in Fig. 1), the behavior is qualitatively modified to correspond precisely to that of the BSPT theories, Eq. (1) with $k = 1$, so that, although it is built from electrons, it is a proper BSPT state in the following senses:

1. The Coulomb interaction, which is expected to play an important role in this system, induces a gap for all fermionic excitations at the boundary, while bosonic charge and spin excitations remain gapless and protected by the two $U(1)$ symmetries (Fig. 1);

2. After the fermions are gapped out at the boundary by the Coulomb interaction, using the correlation functions of the boundary states, and following the procedure in Ref. [24], one can derive the bulk wave function for the bosonic charge and spin, which takes exactly the form as the BSPT wave function constructed using the mutual flux attachment picture in Ref. [25];

3. Using the Chalker-Coddington picture, the bulk quantum phase transition between the nontrivial SPT phase ($k = 1$) and trivial ($k = 0$) phase (hereafter phrased as “topological to trivial transition”) can be described by percolation of domains and the corresponding network of interface/boundary states. Because the boundary only has gapless bosonic modes, such a topological quantum phase transition can occur while preserving the bulk gap for fermionic quasiparticles. The topological to trivial transition can be driven by varying competing magnetic and electric fields, and we propose that the bosonic scenario for this quantum phase transition could occur with sufficiently strong interactions. This is a qualitatively different situation from the well-known topological to trivial transitions in weakly correlated systems, such as the plateau transition between integer quantum Hall states, or the transition between normal and topological band insulators – these transitions have a free fermion description which involves the fermion gap closing in the bulk. The above statement is supported by recent numerical studies of a similar model on the bilayer honeycomb lattice [26][27].

We now proceed to an exposition of these results. In this work we will focus on the boundary states and the bulk wave function of the BSPT state, we will defer the detailed analysis of the bulk topological transition to future study. For non-interacting bilayer graphene, there are two channels of helical edge states, described by the Hamiltonian

$$H_0 = \int dx \sum_{l=1}^{2} \left[ \frac{v}{2K} (\partial_x \phi_l)^2 + \frac{vK}{2} (\partial_x \phi_l)^2 \right], \quad (2)$$

where $l = 1, 2$ labels the channels, $L, R$ denote the left and right moving fermions respectively, which also correspond to electrons with spin-up and down, and $v$ is the Fermi velocity [56]. The presence of some counter-propagating edge states was deduced experimentally from non-local transport signatures [22]. When the Coulomb interaction is ignored, the boundary is a free fermion conformal field theory (CFT) with central charge $c = 2$.

The free fermion edge states can be bosonized into two flavors of free bosons:

$$H_0 = \int dx \sum_{l=1}^{2} \left[ \frac{v}{2K} (\partial_x \phi_l)^2 + \frac{vK}{2} (\partial_x \phi_l)^2 \right], \quad (3)$$

where $[\phi_l(x), \partial_x \phi_l(x')] = i\delta(x - x')\delta\phi_l$, and $\psi_{1, L/R} \sim e^{i\delta x \pm \pi \phi_l}$. For free 1$d$ fermions without interaction, the Luttinger parameter $K = \pi$.

Coulomb interactions $H_{int}$ are conveniently handled in the bosonization framework. Using the representation of the fermion density $n_l \sim \partial_x \phi_l$, one obtains:

$$H_{int} = \int dx \sum_{l=1}^{2} \left[ \frac{U_{\text{intra}}}{2} (\partial_x \phi_l)^2 + U_{\text{inter}} \partial_x \phi_1 \partial_x \phi_2 + H_v \right], \quad (4)$$

where $U_{\text{intra}}$ and $U_{\text{inter}}$ represent intralayer and interlayer forward-scattering interactions, respectively. $H_v$ is an anharmonic vertex term, and will play a central role here [57]:

$$H_v \sim \alpha \cos(2\pi \phi_1 - 2\pi \phi_2). \quad (5)$$

Here we have assumed that the long range Coulomb interaction is screened to a short range one, but this is not essential. Physically $H_v$ describes the backscattering between two channels of edge states: $H_v \sim \psi_{1, L} \psi_{1, R} \psi_{2, R} \psi_{2, L}$. The anharmonic $H_v$ is relevant in the
When it is relevant, \( H_\nu \) will "pin" the bosonic mode \( \phi_\pm = (\phi_1 + \phi_2)/2 \), causing large fluctuations of \( \theta_\pm = \theta_1 - \theta_2 \), leading to a gap in this antisymmetric sector, and also a gap for all fermions at the boundary. The symmetric edge modes \( \phi = (\phi_1 + \phi_2)/2 \) and \( \theta = \theta_1 + \theta_2 \), however, remain gapless, because \( \theta \) transforms under symmetry \( U(1)_c \), while \( \phi \) transforms under \( U(1)_s \). It is straightforward to show — see below — that only physical operators which create bosonic excitations can be built from the gapless \( \phi, \theta \) fields, consistent with the statement that the boundary has symmetry protected gapless bosonic modes. The size of the fermion gap at the boundary state is estimated in detail in the supplementary material.

The effective low energy theory that describes the canonical conjugate modes \( \phi \) and \( \theta \) reads

\[
\tilde{H} = \int dx \left( \frac{\tilde{v}}{2K} (\partial_x \phi)^2 + \frac{\tilde{v}_K}{2} (\partial_x \theta)^2 \right). \tag{6}
\]

Hence interaction reduces the central charge of the system from \( c = 2 \) to \( c = 1 \). Because \( \theta \) and \( \phi \) transform nontrivially (i.e., shift under \( U(1)_c \) and \( U(1)_s \) symmetries respectively), there are no anharmonic vertex operators allowed by symmetry in Eq. (6). Because \( \theta \) and \( \phi \) are "dual" to each other, a unit soliton of \( \phi \) at the 1d boundary carries charge-2\( e \), and a unit soliton of \( \theta \) carries spin \( S^z = 1 \). The gaplessness of the boundary state is protected by the \( U(1)_c \times U(1)_s \) symmetry alone: even if the translation symmetry of the boundary is broken by disorder (which is inevitable in any real system), as long as the \( U(1)_c \times U(1)_s \) symmetry is preserved, the boundary must still remain gapless. The edge state in our system is also very different from the cases studied in Ref. [23, 29] since in those systems the states localized at the domain wall is unstable to disorder.

Here we note that although the bosonization of the edge states of bilayer graphene in a magnetic field was also studied in Ref. [30, 31] in these works only the spin symmetry was considered in the bosonization, and the conclusion of Ref. [30, 31] was that the system is equivalent to a 1d spin model. Here we stress that, both the \( U(1)_s \) and \( U(1)_c \) symmetries are crucial to define the BSPT state: \( i.e.\) if either of the \( U(1) \) symmetries is broken (for example if the bulk forms a canted antiferromagnetic order), the system will become a trivial state. With both \( U(1) \) symmetries in our system, the boundary theory Eq. (6) must remain gapless, and it can never be realized as a 1d system, but rather only as the boundary of a 2d system, which is an essential property of all SPT states.

Let us discuss the operator content further. Assuming \( \phi_- \) is pinned and \( \theta_- \) fluctuates strongly, one can obtain the low energy components of the four component vector \( \mathbf{n} \) in Eq. (1)

\[
\begin{align*}
&n_1 + in_2 \sim \epsilon_{\alpha\beta} \psi_{1,\alpha} \psi_{2,\beta} \sim e^{i\theta}, \\
&n_3 + in_4 \sim \sum_l (-1)^l \psi_{1,\alpha}^l \sigma^+ \psi_{1,\alpha} \sim e^{i2\pi \phi}. \tag{7}
\end{align*}
\]

Here \( n_1 + in_2 \) corresponds to an interlayer spin-singlet \( (S^z = 0) \) Cooper pair, while \( n_3 \) and \( n_4 \) correspond to in-plane magnetic order. All components of the vector \( \mathbf{n} \) have power-law correlations at the boundary, and their scaling dimensions are \( \Delta[\epsilon_{\alpha\beta} \psi_{1,\alpha} \psi_{2,\beta}] = \frac{K}{4\pi} \), \( \Delta[\sum_l (-1)^l \psi_{1,\alpha}^l \sigma^+ \psi_{1,\alpha}] = \frac{c}{K} \). Thus we see that indeed the low energy correlations at the edge all correspond to bosonic fields, which could be built from elementary bosons of even charge and integer spin. The presence of four distinct “primary fields” is characteristic of the Wess-Zumino-Witten (WZW) SU(2) CFT, which is well-known to be expressible in terms of a single gapless boson and has \( c = 1 \) CFT. The model in Eq. (6) is a deformation of the usual SU(2) theory which reduces the symmetry to \( U(1)_c \times U(1)_s \). It is also equivalent to a (deformed) O(4) NLSM with \( k = 1 \) WZW term — see e.g. Ref. [34].

Eq. (7) identified the effective bosonic degrees of freedom that form a bosonic SPT state in the bulk. There are two flavors of bosons carrying charge and spin quantum numbers respectively. Following the method of Ref. [23], we can derive the wave function of the bosons in the bulk, by calculating the following correlation function of the boundary conformal field theory:

\[
\Psi(z_1, z_2 \cdots w_1, w_2 \cdots) \sim \prod_j e^{i\theta(z_j)} \prod_k e^{2\pi i \phi(w_k)} \mathcal{O}_{bg}, \tag{8}
\]

where \( z_j \) and \( w_k \) are the complex coordinates in the \( 2d \) plane for the two flavors of bosons. This is equivalent to calculating the partition function of a \( 2d \) Coulomb gas with both electric and magnetic charges \([35, 36]\), and \( \mathcal{O}_{bg} \) represents a neutralizing background charge operator. The correlation function in Eq. (8) can be evaluated with either Eq. (3) or Eq. (6) and the result will be qualitatively the same:

\[
\Psi(z_1, z_2 \cdots w_1, w_2 \cdots) \sim \text{Norm}(z_j, w_k) \prod_{j,k} (z_j - w_k), \tag{9}
\]

where \( \text{Norm}(z_j, w_k) \) only depends on the norm of \( z_j - w_k \), \( z_i - z_j \) and \( w_i - w_j \), and contains all the dependence upon the Luttinger parameters in Eq. (3) and Eq. (6). This wave function indeed represents a bosonic SPT state: it is symmetric under interchange of identical \( z_i \) or \( w_j \) bosons, and the two flavors of bosons view each other as a \( 2\pi \) flux. This mutual “flux attachment” picture is the very essence of the BSPT state [23].
term at level $k = 1$, the bulk theory can be constructed with the Chalker-Coddington network model, and as was shown in Ref. [13, 38] the bulk theory obtained by this construction is precisely Eq. (1) with $\Theta = 2\pi$. The physical meaning of this topological $\Theta$-term is that, a vortex of $(n_1, n_2)$, i.e. a vortex of the superconductor order parameter, which traps magnetic flux $\phi_0/2$, would carry spin $S^z = 1$, which is perfectly consistent with the physics of the bilayer QSH state.

It is worth contrasting with the case of a single layer QSH insulator, in which the boundary cannot be driven into a state with gapped fermions but gapless bosonic modes, as long as the $U(1)_c$ and time-reversal (or $U(1)_s$) symmetry of the QSH insulator are preserved [39, 40]. The mapping between fermionic QSH insulator and BSPT is only valid for two copies of QSH insulators (which mathematically is equivalent to four copies of $p \pm ip$ topological superconductors), as was shown in Ref. [41].

By varying competing electric and magnetic fields normal to the layer, a quantum phase transition can occur between the BSPT and the trivial state in the 2d bulk. Using the Chalker-Coddington network picture, one may construct a theory for the 2d bulk phase transition which involves only gapless bosonic modes and retains the single-fermion gap. In the field theory Eq. (1) this transition occurs when $\Theta$ is tuned to $\pi$. Although directly analyzing the bulk field theory at $\Theta = \pi$ is difficult, recent unbiased determinant quantum Monte Carlo simulation on a similar bilayer honeycomb lattice interacting fermion model confirms that this purely bosonic topological-trivial quantum phase transition can indeed happen [26, 27], which is fundamentally different from the ordinary topological to trivial transition in any free fermion system. Maintaining the single particle gap requires strong interactions, and other less interesting possibilities are possible in experiment, such as other intermediate phases between the BSPT phase and the trivial phase. Nevertheless, a direct second order “bosonic” transition like the one found in Ref. [26, 27] seems allowed and a quite interesting prospect.

**Experimental Implications**

The central prediction of our theory is that in a bilayer graphene in the quantum spin Hall phase [22], the gapless boundary modes are bosonic rather than fermionic. The low energy charge carriers on the edge are Cooper pairs $e_{\alpha\beta} c_{\alpha1} c_{\alpha2\beta}$, with charge $2e$. Tunnelling from a normal metal electrode or tip is predicted to show a hard gap, despite ballistic, dissipationless in-plane resistance. Conversely, tunnelling from a superconducting tip should show zero gap.

A purely transport measurement is also possible using shot noise, which has previously been used to probe fractional charges in quantum Hall edge states [12, 35]. By introducing a quantum point contact, either using electrostatic gates or a nano-constriction, edge-to-edge backscattering is possible at that contact, with a finite transmission probability [14]. Individual tunneling events will carry charge $\pm 2e$, which is directly observable in the noise spectrum. The detailed calculation about the shot noise in a quantum point contact geometry has been presented in a follow-up paper by some of the current authors [16].

Here we propose a different method to measure the carrier charge at the boundary. Compared with the point-contact geometry, our current proposal is easier to implement experimentally, and more convenient to analyze theoretically, as it only involves one edge instead of two opposite edges. Our proposal is based on the dual-gated geometry that has been used in experiments Ref. [22]. The screened Coulomb interaction in our system can be tuned by its distance $d$ to the gates due to screening. The competition between interaction and the Zeeman energy can lead to a rich phase diagram, and when the interaction is dominant, the system develops a canted antiferromagnetic (CAF) order [22]. The size of the fermion gap at the boundary, as well as the magnetic field required to realize the BSPT state in this setup will be discussed in detail in the supplementary material.

The stability of the edge states of our system relies on the conservation of $S^z$, and if locally the $S^z$ conservation is broken, the edge modes encounters backscattering, and hence leads to noise of the current. We propose to screen the Coulomb interaction for most of the sample, while leaving a region close to the edge unscreened, in order to develop a local CAF order, which serves as a local “magnetic impurity” that breaks the $S^z$ conservation. We calculate the quantum shot noise in the supplementary material with the proposed set-up Fig. 2, and recover the CAF order [22, 30].

**FIG. 2:** Our proposed set-up for measuring the carrier charge at the boundary of our system. Most of the sample are screened by the inner symmetric gates, while the unscreened region has a stronger interaction which leads to a CAF order, and induces backscattering of the edge states. We also add a pair of outer gates to control the strength of interaction in the CAF region.
the expected result:

\[ \tilde{S}(\omega = 0) = 2e^* (I) \coth \frac{e^* V}{2k_B T}. \]  

(10)
e^* = 2e is the smoking gun signature of the BSPT state proposed in our work.

If a direct second order quantum phase transition between the BSPT and trivial phase found in Ref. [26, 27] indeed happens in a real system, then at the transition, which corresponds to a (2 + 1)d CFT, the bulk conductivity should be a universal value \( \sigma = D \epsilon^2 / h \), where \( D \) is an order-1 universal constant [17, 48]. Moreover the transition should be accompanied by a closing of the spin gap, with observable consequences for spin susceptibility as well as thermal transport measurements.

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to renormalizations of short-range (finite) interactions on scales where the relativistic approximation applies. This has been discussed theoretically in several contexts [49]. This effect enhances the anisotropic short-range interactions. For bilayer graphene, this is some range of energy scales between the inter-layer hopping scale and some fraction of order one of the bandwidth. At scales between the inter-layer hopping energy and the cyclotron energy, there is a quadratic dispersion, and the problem is even more complex, but further renormalization certainly occurs. Ultimately the renormalizations are cutoff, in the absence of external screening, by the magnetic length $l_B$ (or in energy, the cyclotron energy).

Here we consider the effects of symmetric gates, placed a distance $d$ above and below the bilayer. Charges in the bilayer experienced a screened interaction due to an infinite array of alternating image charges created by the gates. In the limit of large separation of those charges, the screening becomes exponential. One can show that two electric charges $e$ sandwiched symmetrically between metal plates is given by

$$V(r) \sim \frac{2e^2}{\sqrt{dr}} \exp\left(-\frac{\pi r}{2d}\right),$$

when $r \gg d$.

This screening effect can cut off the renormalization of the anisotropic interactions, so the renormalization is cut off by the smaller of $l_B$ and the screening length of order $d$. The amount of renormalization is determined by the ratio of this cut-off length to the lattice scale $a_0$. Very roughly, then, the renormalized anisotropic interaction is of order

$$V_{ani}^R \sim \frac{e^2 a_0}{d l_B} \left( \text{Min}(l_B, A d) \right)^\eta$$

$$\sim \frac{e^2 a_0}{d l_B} \left( \frac{A^2 d^2 l_B^2}{a_0^2 (A^2 d^2 + l_B^2)} \right)^{\eta/2},$$

where $\epsilon$ is the dielectric constant, and $\eta$ is a positive exponent proportional to the bare effective fine-structure constant, which we will view as an unknown parameter. In the second line we replace the min function with a smoother interpolation.

The critical total magnetic field $B_T^c$ (including both in-plane field $B_{in}$ and out-of-plane field $B_z$) is determined just by the balance of the anisotropy energy and the Zeeman energy. Hence, $B_T^c \propto V_{ani}^R$ in Eq. (12). Making the ad hoc assumptions $\eta = 1$ and $A = 1$, we then have the functional form of the critical field versus $d$,

$$B_T^c = C \frac{e^2}{d l_B} \left( \frac{d^2}{d^2 + l_B^2} \right)^{1/2},$$

where $C$ is a constant independent of $d$ and $l_B$ which can be fit to data, in for instance Ref. [22] where the gates are
FIG. 3: The estimated critical $B_T^c$ at given $d$, with $B_z = 4T$. The constant $C$ in Eq. (13) is fixed by experiment Ref. [22] where the screening from the gates is weak. So if the distance $d$ from the sample to the screening gates is 8 nm, a 14T total field will produce the BSPT (ferromagnetic state) in the screened region in Fig.2 of the main text.

FIG. 4: The estimated fermion gap at the boundary at given $d$, with $B_z = 4T$. We have taken $\epsilon = 11$ [50] and the constant $\tilde{C} = 1$ in Eq. (20).

far enough compared with the magnetic length, hence the screening from the gates is weak.

Now we turn to the edge gap. This is more difficult and less universal because it depends upon the actual structure of the edge, which depends upon the nature of the boundary, the potential there, etc. For example, if it is a physical edge of a graphene sheet, the result will be different than for an edge defined electrostatically. What one has to do in principle is to write the screened Coulomb interaction, taking into any renormalizations due to the aforementioned Coulombic effects at energies larger than the cyclotron energy, and then project it into the manifold of single-particle states defining the gapless edge modes near the Fermi energy. From this we obtain the interactions at the edge, which include a forward-scattering contribution and the inter-channel backscattering term (anharmonic term in Eq.6) of the main text. Then we should use the theory of the one-dimensional edge including both these terms to estimate the gap.

The main difficulty is with the first step. We do not expect the Coulombic “high energy” renormalizations to be critical at the edge, and will neglect that to a first approximation. The SU(4) symmetry of the system is anyway broken by the edge so that the dominant long-range part of the Coulomb interaction will presumably contribute there to both the forward-scattering and the anharmonic terms of the 1d theory. The forward-scattering terms $U_{\text{intra}}$ and $U_{\text{inter}}$ in Eq.(4) of the main text reflect long-distance interactions of the smooth part of the electron density in the two channels. Consequently, we expect them to be given just by the zero momentum Fourier transform of the screened Coulomb interaction. We write

$$U \sim \int dx V(x),$$

and use the interpolating form

$$V(x) = \frac{e^2}{\epsilon \sqrt{l_B^2 + x^2}} \left( \frac{d^2 + 16x^2}{d^2} \right)^{1/4} e^{-\pi x/(2d)},$$

which matches Eq. (11) when $r \gg d$, shows pure Coulomb behavior for $l_B < x < d$, and is regularized by the magnetic length at short distances. This gives in the two limits (with similar forms for both $U$’s)

$$U \sim \begin{cases} \frac{2e^2}{\epsilon \ln(d/l_B)} & \text{for } d \gg l_B \\ 2.02e^2/\epsilon \times d/l_B & \text{for } d \ll l_B \end{cases}$$

So a reasonable interpolation is

$$U \approx \frac{2e^2}{\epsilon} \frac{d}{d + l_B},$$

up to the weak logarithmic factor.

The anharmonic term $\alpha$ in Eq. (5) of the main text reflects short-distance physics which allows scattering between channels. It is much more sensitive to the structure of the edge state wavefunctions. We can proceed purely dimensionally by noting the $\alpha$ is dimensions of energy divided by length. Assuming that it is primarily determined by distances of order $l_B$ itself, we see that $\alpha \sim 4e^2/(\epsilon l_B^2)$, with an unknown prefactor $A$ which depends on all the edge details, if we neglect screening. Screening will reduce this if $d < l_B$. The form of this screening is not really clear, and again how it affects the value of $\alpha$ must depend upon edge details. So we propose a simple ad-hoc form for $\alpha$:

$$\alpha = A \frac{e^2}{\epsilon l_B^2} \left( \frac{d}{d + l_B} \right).$$

Now we are in a position to estimate the edge fermion gap. The scaling dimension of the anharmonic term is determined by the Luttinger parameter as

$$\Delta_{[\alpha]} = 2 - \frac{2\pi}{K_{\epsilon,\text{eff}}},$$
where $K_{eff}$ is the effective Luttinger parameter including the forward-scattering interactions. Using the interpolation for $U$ in Eq. (17), we then estimated

$$K_{eff} = \sqrt{\left(\pi + \frac{\tilde{C}}{\nu} U / \hbar \nu\right) \pi}, \quad (20)$$

where $\tilde{C}$ is another order-1 constant. $\nu$ is the velocity of the edge states, $\hbar \nu / l_B$ can be taken as the cyclotron energy $\hbar \omega_c$ of bilayer graphene: $\hbar \omega_c \approx 3.2 B_z[T]$ meV, which is also the ultraviolet energy cut-off of the 1d edge system.

Taking the bare value of $\alpha$ in Eq. (18), for small $d$ we use standard methods to determine the fermion gap $\Delta_f$ in the corresponding sine-Gordon model:

$$\Delta_f \sim \frac{\hbar \nu}{l_B} \left(\frac{\alpha}{\hbar \nu}\right)^{1/\Delta[\alpha]}. \quad (21)$$

This scaling form is given by both the exact solution of the mass gap of the sine-Gordon model, and also the standard renormalization group flowing of the anharmonic term. At large $d$ we simply find a gap of order $\hbar \nu / l_B$ or of order the cyclotron energy. It is probably somewhat smaller, but this reduction is hard to estimate, and comes from the unknown prefactors, e.g. $A$ in Eq. (18).

If we fix $B_z$ and increase $d$, since the interaction energy scale will increase monotonically with $d$, so will the fermion gap at the boundary.

For $B_z$ fixed at 4 Tesla, at given $d$, the estimate of critical total field $B_T$, and also the corresponding fermion gap at the boundary, are plotted in Fig. 3 and Fig. 4. So if the screening gates are made at 8 nm from the sample, a 14T total field will produce the BSPT (ferromagnetic order) in the screened region. In plot Fig. 4 we have taken $\epsilon = 11$ [20] and $\tilde{C} = 1$ in Eq. (20).

We also need another pair of outer gates to tune the strength of the local CAF, to ensure the local CAF is not too strong to completely block the transmission current. The strength of the CAF can also be tuned by the distance $d'$ to the outer gates. The renormalized anisotropic interaction inside the CAF domain can be estimated using the same equation Eq. (12) where now $d$ is replaced by the distance to the outer gates $d'$. For example for $B_z$ fixed at 4 Tesla, and $d = 8\,\text{nm}$, we need $d'$ greater than 13 nm to create the CAF in the region not screened by the inner gates.

**SINGLE PARTICLE BACKSCATTERING AND INSTANTON TUNNELLING**

Without interaction, our system reduces to two channels of quantum spin Hall (QSH) insulator. A local CAF order at the edge of this system will induce single particle backscattering. At the edge of the system, the most general form of single particle backscattering (SPB) is

$$T_{i_1, i_2} = t_{i_1, i_2} \psi_{i_1, L}^\dagger \psi_{i_2, R} + \text{h.c.}$$

$$\sim t_{i_1, i_2} \cos[\pi(\phi_{i_1} + \phi_{i_2}) + (\theta_{i_1} - \theta_{i_2})], \quad (22)$$

where $i_1, i_2 \in \{1, 2\}$ labels the two channels of the fermion modes. As an example, we take $i_1 = i_2 = l$, and

$$T_{l, l} = t_{l, l} \cos 2\pi(\phi \pm \phi_\pm). \quad (23)$$

The sign of $\pm$ depends on whether $l$ takes $l = 1$ or $l = 2$. In the strong scattering limit (or when the backscattering is relevant), we find that

$$\phi = \frac{(n + m + 1)}{2}, \quad \phi_\pm = \frac{(n - m)}{2}. \quad (24)$$

i.e. $\phi$ and $\phi_\pm$ are pinned to discrete values with $m, n \in \mathbb{Z}$. Quantum tunnelling generally happens between different minima of the cosine potential, which is accompanied by a non-zero change $\Delta n$ or $\Delta m$. This is known as the instanton tunnelling events. The instanton tunnellings of $\phi$ and $\phi_\pm$ also corresponds to “charge” currents associated with fields $\phi$ and $\phi_\pm$ across the local CAF region. In particular, $\phi$ carries the physical electric charge and thus the minimal instanton tunnelling of $\phi$ characterizes the current of the elementary electric charge of the system. In the above SPB process, the minimal instanton tunnelling event of $\phi$ field is given by $\Delta \phi = \frac{1}{2}$ or equivalently $\Delta m + \Delta n = 1$, which corresponds to the physical electric current

$$I_\phi = I_{\phi_1} + I_{\phi_2} = -e \partial_t (\phi_1 + \phi_2) = -2e \partial_t \phi. \quad (25)$$

Therefore, the electric charge transport during this minimal instanton event is given by

$$\Delta Q = \int dt I_\phi = -2e \int dt \partial_t \phi = -2e \times \frac{1}{2} = -e. \quad (26)$$

This result reveals that the elementary charge of fermionic QSH system is $e$ electric charge.

**MINIMAL BACKSCATTERING ON A BSPT EDGE**

When the Coulomb interaction is turned on, the boundary of the system becomes purely bosonic, as we showed in the main text. In mathematical terms, the Coulomb interaction pins $\phi_\pm$ to a fixed value (Here we choose $\phi = 0$ or equivalently $n = m$). As a result, the backscattering takes the following form:

$$T = t \cos 2\pi \phi. \quad (27)$$
In the strong scattering limit, \( \phi \) is pinned to \( \frac{(2n+1)}{2} \) and the minimal instanton tunnelling event is

\[
\Delta \phi = 1. \tag{28}
\]

The elementary electric charge transport now is thus given by

\[
\Delta Q = -2e\Delta \phi = -2e. \tag{29}
\]

The 2\( e \)-charge instanton event is a direct consequence of the bosonic nature of the BSPT edge physics.

The instanton tunnelling charge can be visualized in Fig. 5, where we plot the bosonic field configuration space of \( \phi \) and \( \phi_- \). We start with an initial configuration shown by a dashed circle with \( \phi \) and \( \phi_- \) of Fig. 5, where we plot the bosonic field configuration of the BSPT limit, however, \( \phi_- \) is pinned and does not enter the instanton tunnelling process. This results in an extra constraint,

\[
\Delta m = \Delta n. \tag{30}
\]

As a consequence, the previous minimal instanton tunnelling is prohibited and the new minimal instanton tunnelling \( (\Delta m, \Delta n) = (1, 1) \) is depicted by the green arrow in Fig. 5, which corresponds to a 2\( e \)-charge tunnelling event.

In realistic systems, the rate of the instanton tunnelling event \( \tilde{t} \), also known as the fugacity of the instanton gas, is rather small. As a result, any instanton event that involves multiple instanton tunnellings will be suppressed. Therefore, we expect that the minimal instanton tunnelling which induces 2\( e \) charge transport will be dominant in experimental detection.

### Current Noise Spectrum

The charge of minimal instanton tunnelling can be directly read out by measuring the noise spectrum of tunnelling current. As shown in Ref. \[46, 51, 52\], the non-equilibrium current noise spectrum at zero frequency \( \tilde{S}(\omega = 0) \) is defined as the anti-commutator of tunneling current operator \( I_\eta \),

\[
\tilde{S}(\omega = 0) = \int dt - t' \sum_\eta \langle \{T_K \{I_\eta(t), I_{-\eta}(t')\} \rangle + O(t_{\tilde{t},1}^2) \tag{31}
\]

where we have defined the Keldysh contour as \( K \) and the index \( \eta = \pm \) to characterize the forward (+) and backward (−) branch. In the BSPT limit, the instanton tunnelling current is given by

\[
I_\eta(t) = 4\tilde{t}_{\tilde{t},1} \sin(2\tilde{\phi}(t) - 2eVt), \tag{32}
\]

where we have performed the duality transformation and introduced a new dual field \( \tilde{\phi} \). Here \( V \) is the applied voltage bias and \( \tilde{t}_{\tilde{t},1} \) is the fugacity of the instanton gas. The non-equilibrium current is,

\[
\langle I(t) \rangle = \frac{1}{2} \sum_\eta \langle T_K I_\eta e^{-2i \int K dt \tilde{t}_{\tilde{t},1} \cos(2\tilde{\phi} - 2eVt) \rangle. \tag{33}
\]

By evaluating the Keldysh contour integral exactly, we find that the relation between current noise spectrum \( \tilde{S}(\omega = 0) \) and electric current \( I \) due to instanton tunnelling is given by

\[
\tilde{S}(\omega = 0) = 2e^* \langle I \rangle \coth \frac{e^* V}{2k_BT}, \tag{34}
\]

with \( e^* = 2e \). In the low-temperature/high-bias limit \( (e^* V >> k_BT) \), the Schottky relation of quantum shot noise is recovered,

\[
\tilde{S}(\omega = 0) = 2e^* \langle I \rangle. \tag{35}
\]