A DIMENSION GAP FOR CONTINUED FRACTIONS WITH INDEPENDENT DIGITS - THE NON STATIONARY CASE

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Abstract. We show there exists a constant $0 < c_0 < 1$ such that the dimension of every measure on $[0,1]$, which makes the digits in the continued fraction expansion independent, is at most $1 - c_0$. This extends a result of Kifer, Peres and Weiss from 2001, which established this under the additional assumption of stationarity. For $k \geq 1$ we prove an analogous statement for measures under which the digits form a $*$-mixing $k$-step Markov chain. This is also generalized to the case of $f$-expansions. In addition, we construct for each $k$ a measure, which makes the continued fraction digits a stationary and $*$-mixing $k$-step Markov chain, with dimension at least $1 - 2^{3-k}$.

1. Introduction

Let $X$ denote the set of irrational numbers in $(0,1)$. It is well known each $x \in X$ has a unique continued fraction expansion of the form

$$x = \frac{1}{A_1(x) + \frac{1}{A_2(x) + \frac{1}{A_3(x) + \cdots}}} ,$$

where $A_1(x), A_2(x), \ldots$ are positive integers. Given a probability measure $\nu$ on $X$, each $A_n$ defines a random variable on $(X, \nu)$ and the digits $\{A_n\}_{n=1}^\infty$ form a discrete time stochastic process.

In 1966, Chatterji [Ch] has shown every probability measure $\nu$ on $[0,1]$, which makes the digits in the continued fraction expansion independent variables, is singular with respect to the Lebesgue measure. In 2001, Kifer, Peres and Weiss [KPW] have proven that $\dim_H \nu \leq 1 - c$, if in addition the digits are identically distributed. Here $0 < c < 1$ is a global constant, independent of $\nu$, and $\dim_H \nu$ denotes the Hausdorff dimension of $\nu$, which is defined in Section 2 below. In this paper we show the result from [KPW] remains true, even if the digits are independent but not necessarily identically distributed.

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Assuming $A_1, A_2, \ldots$ are i.i.d. with $E[\log A_1] < \infty$ and $H(A_1) < \infty$, where $H(A_1)$ is the entropy of $A_1$, Kinney and Pitcher [KP] have proven that
\begin{equation}
\dim_H \nu = \frac{H(A_1)}{\int_0^1 \log x^2 \, d\nu}.
\end{equation}
The Gauss measure
\[ \mu_G(E) = \frac{1}{\log 2} \int_E \frac{dx}{1 + x} \]
is the unique equilibrium state of the Gauss map $Tx = \frac{1}{x}$ (mod 1) with respect to the function $x \to \log x^2$. This follows from the thermodynamic formalism approach of Walters [Wa1]. Hence under the i.i.d. assumption
\[ 0 = \int_0^1 \log x^2 \, d\mu_G(x) + h_{\mu_G}(T) > \int_0^1 \log x^2 \, d\nu(x) + h_\nu(T), \]
where $h_\nu(T)$ is the entropy of $T$ with respect to a $T$-invariant measure $\eta$. Since $h_\nu(T) = H(A_1)$, we get from (1.1) that $\dim_H \nu < 1$ in this case. When $A_1, A_2, \ldots$ are not identically distributed the formula (1.1) is no long er valid, and so it is not even clear that $\dim_H \nu$ is strictly less than 1. As mentioned above, we shall show that there exists a global constant $c_0 > 0$ such that $\dim_H \nu \leq 1 - c_0$, assuming $A_1, A_2, \ldots$ are independent.

We actually prove more generally that for every integer $k \geq 0$ there exists $0 < c_k < 1$, which depends only on $k$, such that $\dim_H \nu \leq 1 - c_k$ if the digits form a $k$-step Markov chain which is $\ast$-mixing. This is the main result of this paper. The $\ast$-mixing condition was introduced in [BHK], and is a bit less restrictive than the more familiar $\psi$-mixing condition. The definitions are given in Section 2. In the last section we generalize our main result to the case of $f$-expansions.

Given $k \geq 0$ it was shown in [KPW] that there exists $0 < c'_k < 1$, for which $\dim_H \nu \leq 1 - c'_k$ whenever $\nu$ makes the digits a stationary and ergodic $k$-step Markov chain. Our proof is a modification of the argument given there for this result. We shall also construct for each $k$ a measure $\nu_k$, under which the digits form a stationary and $\psi$-mixing $k$-step Markov chain, with $\dim_H \nu_k \geq 1 - 2^{3-k}$. This of course shows $c_k$ and $c'_k$ are at most $2^{3-k}$.

The rest of the paper is organized as follows. In Section 2 we give some necessary definitions and state our results. In Section 3 we establish a uniform bound on the dimension of subsets of $X$, which are defined via certain digit frequencies. This is the key ingredient in the proof of our main result, which is carried out in Section 4. In Section 5 we construct the measures $\nu_k$ mentioned above. In Section 6 we generalize our main result to the setup of $f$-expansions.

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2. Preliminaries and results

First, we define the mixing conditions mentioned above. Given random variables \( \{A_i\}_{i \in I} \), all defined on the same probability space, denote by \( \sigma\{A_i\}_{i \in I} \) the smallest \( \sigma \)-algebra with respect to which each \( A_i \) is measurable.

**Definition 2.1.** A sequence of random variables \( \{A_n\}_{n=1}^{\infty} \) is called *-mixing if there exist an integer \( N \geq 1 \) and a real valued function \( f \), defined on the integers \( n \geq N \), such that

- \( f \) is non-increasing with \( \lim_{n \to \infty} f(n) = 0 \), and
- if \( n \geq N, m \geq 1 \), \( E \in \sigma\{A_1, ..., A_m\} \) and \( F \in \sigma\{A_{m+n}, A_{m+n+1}, ...\} \) then
  \[ |P(E \cap F) - P(E)P(F)| \leq f(n)P(E)P(F). \]

If such an \( f \) exists for \( N = 1 \) the sequence is said to be \( \psi \)-mixing.

**Remark 2.2.** A sequence of independent random variables is clearly \( \psi \)-mixing. It is not hard to show that the \( \psi \)-mixing condition is satisfied for a finite state Markov chain \( \{A_n\}_{n=1}^{\infty} \), with state space \( S \), for which

\[ \inf \{ P(A_{n+1} = j \mid A_n = i) : n \geq 1 \text{ and } i, j \in S \} > 0. \]

Examples of *-mixing countable state Markov chains can be found in Section 3 of [BHK]. Another important example of a \( \psi \)-mixing sequence is obtained by the continued fraction digits with respect to the Gauss measure \( \mu_G \) (see [Ad] or [He]).

Set \( X = (0, 1) \setminus \mathbb{Q} \) and for each \( x \in X \) and \( i \geq 1 \) let \( \alpha_i(x) \in \mathbb{N} := \{1, 2, ..., \} \) be the \( i \)'th digit in the continued fraction expansion of \( x \), i.e.

\[ x = \frac{1}{\alpha_1(x) + \frac{1}{\alpha_2(x) + \frac{1}{\alpha_3(x) + ...}}}. \]

Given \( a_1, a_2, ..., \in \mathbb{N} \) denote by \([a_1, a_2, ...]\) the unique \( x \in X \) with \( \alpha_i(x) = a_i \) for \( i \geq 1 \). For \( E \subset X \) write \( \dim_H(E) \) for the Hausdorff dimension of \( E \). Given a Borel probability measure \( \nu \) on \( X \) its Hausdorff dimension is defined by

\[ \dim_H(\nu) = \inf \{ \dim_H(E) : E \subset X \text{ is a Borel set with } \nu(E) = 1 \}. \]

The following theorem is our main result.

**Theorem 2.3.** Let \( \{A_n\}_{n=1}^{\infty} \) be \( \mathbb{N} \)-valued random variables and let \( k \geq 0 \). Assume \( \{A_n\}_{n=1}^{\infty} \) is a \( k \)-step Markov chain (when \( k = 0 \) this means \( A_1, A_2, ... \) are independent) which is *-mixing. Let \( \nu \) be the distribution of the random variable \([A_1, A_2, ...]\). Then \( \dim_H(\nu) \leq 1 - c_k \), where \( 0 < c_k < 1 \) is a constant depending only on \( k \).
Remark 2.4. As mentioned in the introduction, it was shown in [KPW] that there exists $0 < c'_k < 1$, for which $\dim_H \nu \leq 1 - c'_k$ whenever $\nu$ makes the continued fraction digits a stationary and ergodic $k$-step Markov chain.

It might be desirable to estimate $c_k$ and $c'_k$. The next claim shows these constants are at most $2^{3-k}$.

Claim 2.5. For each $k \geq 3$ there exits an $\mathbb{N}$-valued $k$-step stationary and $\psi$-mixing Markov chain $\{A_n\}_{n=1}^{\infty}$ with $\dim_H(\nu) \geq 1 - 2^{3-k}$, where $\nu$ is the distribution of $[A_1, A_2, \ldots]$.

The main ingredient in the proof of Theorem 2.3 is Theorem 2.6 stated below, for which we need some more notations. Let $T : X \to X$ be the Gauss map, which is defined by

$$Tx = \frac{1}{x} \pmod{1} \text{ for } x \in X.$$ Denote by $\mu_G$ the Gauss measure, which satisfies

$$\mu_G(E) = \frac{1}{\log 2} \int_E \frac{dx}{1 + x} \text{ for every Borel set } E \subset X.$$

It is well known that $\mu_G$ is invariant and ergodic with respect to $T$. For $(a_1, \ldots, a_k) = a \in \mathbb{N}^k$ set

$$I_a = \{x \in X : \alpha_i(x) = a_i \text{ for each } 1 \leq i \leq k\},$$

and define $\mathbb{I}_a : X \to \{0, 1\}$ by

$$\mathbb{I}_a(x) = \begin{cases} 1, & \text{if } x \in I_a \text{ for } x \in X, \\ 0, & \text{if } x \notin I_a \end{cases}.$$

Given $L > 1$ denote by $Q_L$ the set of maps $q : \mathbb{N} \to \mathbb{N}$ with

$$q(n+1) > q(n) \text{ for each } n \in \mathbb{N}$$

and

$$\liminf_{n \to \infty} \frac{q(n)}{n} < L.$$

For $q \in Q_L$, $a \in \cup_{k=1}^{\infty} \mathbb{N}^k$ and $\delta > 0$ define

$$\Gamma_{q, a}^\delta = \{x \in X : \liminf_{n \to \infty} \left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_a(T^q(i)x) - \mu_G(I_a) \right| > \delta \}.$$

Theorem 2.6. For every $L > 1$ and $\delta > 0$ there exists $0 < c_{L, \delta} < 1$ with

$$\sup\{\dim_H(T_{q, a}^\delta) : q \in Q_L, a \in \cup_{k=1}^{\infty} \mathbb{N}^k\} \leq 1 - c_{L, \delta}.$$

Remark 2.7. The proof of Theorem 2.6 resembles the proof of the main result (Theorem 2.1) of [KPW]. There an upper bound, which depends only on $\delta$, is obtained.
for the dimension of sets of the form

\[(2.1) \quad \{ x \in X : \lim_{n \to \infty} \left| \frac{1}{n} \sum_{i=1}^{n} I_a(T^i x) - \mu_G(I_a) \right| > \delta \}. \]

Here we need to consider the families $\mathcal{Q}_L$, and the more general averages

\[
\frac{1}{n} \sum_{i=1}^{n} I_a(T^{q(i)} x),
\]

due to the lack of stationarity. As a result we must define $\Gamma_{q,a}^\delta$ with $\lim \inf$, as opposed to the sets (2.1) which are defined with $\lim \sup$.

3. Proof of Theorem 2.6

The following large deviations estimate will be needed. Its proof is almost identical to the proof of Lemma 3.1 from [KPW], but we include it here for completeness.

**Lemma 3.1.** Suppose $S = \{ \eta_n \}_{n=1}^{\infty}$ is a stationary and $\ast$-mixing sequence of random variables. Let $k \geq 1$ and $F : \mathbb{R}^k \to \{0, 1\}$, set

\[ p = \mathbb{P}\{ F(\eta_1, ..., \eta_k) = 1 \}, \]

and let $r : \mathbb{N} \to \mathbb{N}$ be strictly increasing. Then for every $\delta > 0$ there exists a constant $M = M(S, \delta, k) > 1$, independent of $q$ and $F$, such that for every $n \geq 1$,

\[ \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} F(\eta_{q(i)}) - p \right| > \delta \right\} \leq M \cdot e^{-n/M}. \]

**Proof.** Fix $\delta > 0$, then since $S$ is $\ast$-mixing there exists $M \in \mathbb{N}$ with

\[ (3.1) \quad |\mathbb{P}(E_1 \cap E_2) - \mathbb{P}(E_1)\mathbb{P}(E_2)| \leq \frac{\delta^2}{2} \mathbb{P}(E_1)\mathbb{P}(E_2) \]

for each $m \geq 1$, $E_1 \in \sigma\{\eta_1, ..., \eta_{m+k-1}\}$ and $E_2 \in \sigma\{\eta_{m+M}, \eta_{m+M+1}, \ldots\}$. For $i \geq 1$ set $\xi_i = F(\eta_i, ..., \eta_{i+k-1})$, fix $n \geq M$, and write

\[ A_n = \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} \xi_{q(i)} - p \right| > \delta \right\}. \]

Let $N$ be the integral part of $n/M$, and for $1 \leq j \leq M$ set

\[ B_{n,j} = \left\{ \left| \frac{1}{N} \sum_{i=0}^{N-1} \xi_{q(i+jM)} - p \right| > \delta - \frac{1}{N} \right\}. \]

Clearly $A_n \subset \bigcup_{j=1}^{M} B_{n,j}$, hence

\[ (3.2) \quad \mathbb{P}(A_n) \leq \sum_{j=1}^{M} \mathbb{P}(B_{n,j}). \]
Fix $1 \leq j \leq M$, and for $\epsilon_0, \ldots, \epsilon_{N-1} \in \{0,1\}$ write
\[ C_{\epsilon_0,\ldots,\epsilon_{N-1}} = \{\xi_{q(j+iM)} = \epsilon_i \text{ for each } 0 \leq i < N\} \, . \]
Let $\zeta_0, \zeta_1, \ldots$ be independent $\{0,1\}$-valued random variables with mean $p$. Since $q$ is strictly increasing it follows easily from (3.1) that,
\[ \mathbb{P}(C_{\epsilon_0,\ldots,\epsilon_{N-1}}) \leq (1 + \frac{\delta^2}{2})^N \cdot \prod_{i=0}^{N-1} \mathbb{P}(\xi_{q(j+iM)} = \epsilon_i) \]
\[ \leq e^{\delta^2 N/2} \cdot \mathbb{P}\{\zeta_i = \epsilon_i \text{ for each } 0 \leq i < N\} \, . \]
Set $Z = \sum_{i=0}^{N-1} \zeta_i$, then $Z$ is a binomial random variable with parameters $N$ and $p$, and
\[ \mathbb{P}(B_{n,j}) = \sum_{|\sum_{i=0}^{N-1} \epsilon_i - Np| > N\delta - 1} \mathbb{P}(C_{\epsilon_0,\ldots,\epsilon_{N-1}}) \]
(3.3)
\[ \leq e^{\delta^2 N/2} \cdot \mathbb{P}\{|Z - Np| > N\delta - 1\} \, . \]
By the exponential estimate for the binomial distribution (see e.g. Cor. A.1.7 in [AS]) we have for $N \geq 4/\delta$,
\[ \mathbb{P}\{|Z - Np| > N\delta - 1\} \leq 2e^{-N\delta^2} \, . \]
This together with (3.3) gives,
\[ \mathbb{P}(B_{n,j}) \leq 2e^{-\delta^2 N/2} \text{ for each } 1 \leq j \leq M \, . \]
The lemma now follows from (3.2). \hfill \Box

As mentioned in Remark 2.2 the sequence $\{\alpha_i\}_{i=1}^\infty$ is $\psi$-mixing with respect to $\mu_G$. From this and Lemma 3.1 we get the following corollary.

**Corollary 3.2.** Given $k \geq 1$ and $\delta > 0$ there exists a constant $M = M(\delta,k) > 1$, such that for every strictly increasing $q : \mathbb{N} \to \mathbb{N}$, $a \in \mathbb{N}^k$ and $n \geq 1$,
\[ \mu_G \left\{ x \in X : \frac{1}{n} \sum_{i=1}^{n} \|T_i q(x) - \mu_G(I_a)\| > \delta \right\} \leq M \cdot e^{-n/M} \, . \]

Given $x \in X$ and $n \geq 1$ write $J_n(x) = I_{(\alpha_1(x),\ldots,\alpha_n(x))}$. Let $\mathcal{L}$ be the Lebesgue measure, and write $|I| = \mathcal{L}(I)$ for $I \subset X$. For $s \geq 0$ let $\mathcal{H}^s$ be the $s$-dimensional Hausdorff measure on $X$. For $\eta > 0$ and $E \subset X$ write
\[ \mathcal{H}^s_{\eta}(E) = \inf\left\{ \sum_{i=1}^{\infty} |I_i|^s : E \subset \cup_{i=1}^{\infty} I_i \text{ and } |I_i| \leq \eta \right\} \, , \]
then
\[ \lim_{\eta \downarrow 0} \mathcal{H}^s_{\eta}(E) = \mathcal{H}^s(E) \, . \]
Given $n \geq 1$ write
\[ \beta_n = \sup\{|I_a| : a \in \mathbb{N}^n\}, \]
then $\beta_n \to 0$.

**Proof of Theorem 2.6.** Let $\delta > 0$, $L > 1$, $q \in \mathcal{Q}_L$, $k \geq 1$ and $a \in \mathbb{N}^k$. Given $\lambda > 0$ set,
\[ \mathcal{E}_\lambda := \cap_{j=1}^{\infty} \cup_{n=j}^{\infty} \{ x \in X : |J_n(x)| \leq e^{-\lambda n} \}. \]
By Theorem 4.1 in [KPW] there exists $\lambda > 0$ with $\dim_H \mathcal{E}_\lambda < 1$. For $N \geq 1$ set
\[ \Gamma_{q,a}^{\delta,N} := \left\{ x \in X : \left| \frac{1}{n} \sum_{i=1}^{n} I_a(T^{q(i)}x) - \mu_G(I_a) \right| > \delta, \right. \]
\[ \left. \left| J_{n+1+k}(x) \right| \geq e^{-\lambda(q(n)+k)} \text{ for all } n \geq N \right\}, \]
then
\[ (3.4) \quad \Gamma_{q,a}^{\delta} \setminus \mathcal{E}_\lambda \subset \bigcup_{N=1}^{\infty} \Gamma_{q,a}^{\delta,N}. \]
Fix $N \geq 1$ and for $n \geq 1$ set
\[ \Upsilon_{q,a}^{\delta,n} := \left\{ x \in X : \left| \frac{1}{n} \sum_{i=1}^{n} I_a(T^{q(i)}x) - \mu_G(I_a) \right| > \delta, \right. \]
\[ \left. \left| J_{n+1+k}(x) \right| \geq e^{-\lambda(q(n)+k)} \right\}, \]
then $\Gamma_{q,a}^{\delta,N} \subset \Upsilon_{q,a}^{\delta,n}$ for all $n \geq N$.

Let $M = M(\delta,k) > 1$ be as in Corollary 3.2 set $s = 1 - \frac{1}{\lambda M}$ and let $\eta > 0$. From $q \in \mathcal{Q}_L$ we get $\liminf_{n \to \infty} \frac{q(n)}{n} < L$. From this and $\beta_n \to 0$ it follows that there exists $n \geq N$ such that $\beta_n < \eta$ and $q(n) < nL$. By the definition of $\Upsilon_{q,a}^{\delta,n}$ there exists $B_n \subset \mathbb{N}^{q(n)+k}$ with $\Upsilon_{q,a}^{\delta,n} = \cup_{b \in B_n} I_b$. From Corollary 3.2 we get
\[ \mu_G(\Upsilon_{q,a}^{\delta,n}) \leq M \cdot e^{-n/M}. \]
Since
\[ r := \min_{x \in [0,1]} \frac{d\mu_G}{d\mathcal{L}}(x) > 0, \]
it follows
\[ (3.5) \quad \sum_{b \in B_n} |I_b| = L(\Upsilon_{q,a}^{\delta,n}) \leq r^{-1} \cdot \mu_G(\Upsilon_{q,a}^{\delta,n}) \leq r^{-1} M \cdot e^{-n/M}. \]
From
\[ \Gamma_{q,a}^{\delta,N} \subset \Upsilon_{q,a}^{\delta,n} = \cup_{b \in B_n} I_b \]
and since $|I_b| \leq \beta_n < \eta$ for every $b \in B_n$,
\[ (3.6) \quad \mathcal{H}_a^s(\Gamma_{q,a}^{\delta,N}) \leq \sum_{b \in B_n} |I_b|^s \leq \left( \inf_{b \in B_n} |I_b| \right)^{s-1} \cdot \sum_{b \in B_n} |I_b|. \]
By the definition of $\Upsilon_{q,a}^{\delta,n}$,
\[ |I_b| \geq e^{-\lambda(q(n)+k)} \text{ for every } b \in B_n. \]
Hence from (3.6), (3.5), \( q(n) < nL \) and \( s = 1 - \frac{1}{\lambda L M} \),
\[
\mathcal{H}_n^s(\Gamma_{q,n}^q) \leq e^{\lambda(q(n)+k)(1-s)} \cdot r^{-1} M \cdot e^{-n/M} \leq r^{-1} M e^{\lambda k} \cdot \exp(n(\lambda L(1-s)-M^{-1})) = r^{-1} M e^{\lambda k}.
\]
As this holds for every \( \eta > 0 \)
\[
\mathcal{H}^s(\Gamma_{q,a}^q) = \lim_{\eta \downarrow 0} \mathcal{H}_n^s(\Gamma_{q,n}^q) \leq r^{-1} M e^{\lambda k} < \infty,
\]
and so
\[
\dim_H(\Gamma_{q,a}^q) \leq s = 1 - \frac{1}{\lambda L M}.
\]
As this holds for every \( N \geq 1 \) it follows from (3.4) that,
\[
(3.7) \quad \dim_H(\Gamma_{q,a}^q) \leq \sup_{N \geq 1} \dim_H(\Gamma_{q,n}^q) \leq 1 - \frac{1}{\lambda L \cdot M(\delta,k)}.
\]
We shall now complete the proof of the theorem. We continue to fix \( \delta > 0 \) and \( L > 1 \). Let
\[
k_\delta = \inf\{ k \geq 1 : \sup_{a \in \mathbb{N}^k} \mu_G(I_a) < \frac{\delta}{2} \},
\]
then clearly \( k_\delta < \infty \). For \( q \in \mathcal{Q}_L \), \( k \geq k_\delta \) and \( (a_1, \ldots, a_k) = a \in \mathbb{N}^k \),
\[
(3.8) \quad \Gamma_{q,a}^{\delta/2} \supset \left\{ x \in X : \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} I_a(T^{q(i)}x) > \delta \right\} \supset \Gamma_{q,a}^\delta.
\]
Set \( a_\delta = (a_1, \ldots, a_{k_\delta}) \), then since \( I_{a_\delta} \geq I_a \) it follows from (3.8) that \( \Gamma_{q,a_\delta}^{\delta/2} \supset \Gamma_{q,a}^\delta \), and so
\[
\dim_H(\Gamma_{q,a_\delta}^{\delta/2}) \geq \dim_H(\Gamma_{q,a}^\delta).
\]
This together with (3.7) gives
\[
\sup\{ \dim_H(\Gamma_{q,a}^\delta) : q \in \mathcal{Q}_L, a \in \bigcup_{k=1}^\infty \mathbb{N}^k \} \leq \max\{ \dim_H(\mathcal{E}_\lambda), \max_{1 \leq k \leq k_\delta} \left( 1 - \frac{1}{\lambda L \cdot M(\delta/2,k)} \right) \} < 1,
\]
which completes the proof of the theorem. \( \square \)

4. PROOF OF THE MAIN RESULT

Proof of Theorem 2.3. Fix \( k \geq 0 \), let \( \{ A_n \}_{n=1}^\infty \) an \( N \)-valued \( k \)-step Markov chain which is \( \ast \)-mixing, and let \( \nu \) be the distribution of \( [A_1, A_2, \ldots] \). Given words \( a \in \mathbb{N}^m \) and \( b \in \mathbb{N}^l \) we denote by \( ab \in \mathbb{N}^{m+l} \) their concatenation. As noted in observation 2.2 in [KPW], the continued fraction digits under \( \mu_G \) do not form a \( k \)-step Markov chain. It follows that there exist \( m \in \mathbb{N}, a \in \mathbb{N}^k, b \in \mathbb{N}^m \) and \( c \in \mathbb{N} \) with
\[
\frac{\mu_G(I_{bac})}{\mu_G(I_{ba})} \neq \frac{\mu_G(I_{ac})}{\mu_G(I_{a})},
\]
and so
\begin{align}
\delta := \left| \mu_G(I_{bac}) - \frac{\mu_G(I_{ac}) \cdot \mu_G(I_{ba})}{\mu_G(I_a)} \right| > 0.
\end{align}

If \( k = 0 \), i.e. when \( A_1, A_2, \ldots \) are independent, \( a \) is the empty word and \( I_a = X \). Let \( \mu_G(I_a) > \epsilon > 0 \) be such that if \( p_1, p_2, p_3 \in [0,1] \) satisfy
\[ |p_1 - \mu_G(I_{ac})|, |p_2 - \mu_G(I_{ba})|, |p_3 - \mu_G(I_a)| \leq \epsilon, \]
then
\begin{align}
|\frac{p_1 \cdot p_2 - \mu_G(I_{ac}) \cdot \mu_G(I_{ba})}{\mu_G(I_a)}| < \frac{\delta}{2}.
\end{align}

For each \( i \geq 1 \) and \( d \in \mathbb{N}^k \) denote by \( E_{d,i} \) the event
\[ \{A_i \cdots A_{i+|d|-1} = d\}, \]
where \( |d| \) stands for the length of \( d \), and set \( p_{d,i} := \mathbb{P}(E_{d,i}) \). Let \( d \in \mathbb{N}^k \) and assume
\[ \limsup_n \frac{1}{n} \# \{ 1 \leq i \leq n : p_{d,i} < \mu_G(I_d) - \epsilon \} > \frac{1}{10}, \]
then there exists \( q \in \mathbb{Q}_{10} \) with
\begin{align}
p_{d,q(i)} < \mu_G(I_d) - \epsilon \quad \text{for all} \quad i \geq 1.
\end{align}
Since \( \{A_n\}_{n=1}^\infty \) is \( * \)-mixing it is evident from the definition that \( \{1_{E_{d,q(i)}}\}_{i=1}^\infty \) is also \( * \)-mixing, where \( 1_E \) denotes the indicator of the event \( E \). By the law of large numbers for sums of \( * \)-mixing bounded random variables (see Theorem 2 in [BHK]),
\[ \lim n^{-1} \sum_{i=1}^n (1_{E_{d,q(i)}} - p_{d,q(i)}) = 0 \]
almost surely.

Hence for \( \nu \text{-a.e.} \ x \in X \),
\[ \lim n^{-1} \left| \sum_{i=1}^n \mathbb{I}_d(T^{q(i)}x) - \sum_{i=1}^n p_{d,q(i)} \right| = 0. \]
From this and (4.3) we get that for \( \nu \text{-a.e.} \ x \in X \),
\[ \liminf_{n \to \infty} \left| \sum_{i=1}^n \mathbb{I}_d(T^{q(i)}x) - \mu_G(I_d) \right| = \liminf_{n \to \infty} \left| \sum_{i=1}^n p_{d,q(i)} - \mu_G(I_d) \right| \geq \epsilon, \]
which implies \( \nu(T^{\epsilon/2}) = 1 \). Now by Theorem 2.6
\[ \dim_H(\nu) \leq \dim_H(1_{q,d}T^{\epsilon/2}) \leq 1 - c_{10,\epsilon/2}. \]
In a similar manner it can be shown that \( \dim_H(\nu) \leq 1 - c_{10,\epsilon/2} \) if
\[ \limsup_n \frac{1}{n} \# \{ 1 \leq i \leq n : p_{d,i} > \mu(I_d) + \epsilon \} > \frac{1}{10}. \]
It follows that we can assume
\[
\liminf_n \frac{1}{n} \# \left\{ 1 \leq i \leq n : \begin{array}{l}
|p_{ba,i} - \mu(I_{ba})| \leq \epsilon, \\
|p_{ac,i+m} - \mu(I_{ac})| \leq \epsilon, \\
|p_{a,i+m} - \mu(I_a)| \leq \epsilon
\end{array} \right\} > \frac{1}{10},
\]
and so there exists \( q \in \mathbb{Q}_{10} \) with
\[
|p_{ba,q(i)} - \mu_G(I_{ba})|, |p_{ac,q(i)+m} - \mu_G(I_{ac})|, |p_{a,q(i)+m} - \mu_G(I_a)| \leq \epsilon
\]
for all \( i \geq 1 \). Since \( \{A_n\}_n \) is a Markov chain of order \( k \)
\[
p_{bac,q(i)} = \frac{p_{ba,q(i)} \cdot p_{ac,q(i)+m}}{p_{a,q(i)+m}} \text{ for } i \geq 1,
\]
where \( p_{a,q(i)+m} > 0 \) by (4.4) and \( \mu_G(I_a) > \epsilon \). The sequence \( \{E_{bac,q(i)}\}_{i=1}^\infty \) is \( \ast \)-mixing, so by the law of large numbers for sums of \( \ast \)-mixing random variables,
\[
\lim_n \frac{1}{n} \sum_{i=1}^n (1_{E_{bac,q(i)}} - p_{bac,q(i)}) = 0 \text{ almost surely}.
\]
It follows that for \( \nu \)-a.e. \( x \in X \),
\[
\lim_n \left| \frac{1}{n} \sum_{i=1}^n \mathbb{I}_{bac}(T^{q(i)}x) - \frac{1}{n} \sum_{i=1}^n p_{bac,q(i)} \right| = 0.
\]
From this, (4.4), (4.5), (4.2) and (4.3) we get that for \( \nu \)-a.e. \( x \in X \),
\[
\liminf_n \frac{1}{n} \sum_{i=1}^n \mathbb{I}_{bac}(T^{q(i)}x) - \mu_G(I_{bac})
\]
\[
= \liminf_n \left| \frac{1}{n} \sum_{i=1}^n p_{bac,q(i)} - \mu_G(I_{bac}) \right|
\]
\[
\geq \left| \mu_G(I_{bac}) - \frac{\mu_G(I_{ac}) \cdot \mu_G(I_a)}{\mu_G(I_a)} \right| - \limsup_n \left| \frac{p_{ba,q(i)} \cdot p_{ac,q(i)+m}}{p_{a,q(i)+m}} - \frac{\mu_G(I_{ac}) \cdot \mu_G(I_a)}{\mu_G(I_a)} \right| \geq \delta/2.
\]
Hence \( \nu(T_{q,bac}^{3/4}) = 1 \), and so by Theorem 3.0
\[
\dim_H(\nu) \leq \dim_H(T_{q,bac}^{3/4}) \leq 1 - c_{10}\delta/4.
\]
This completes the proof of the theorem.

5. CONSTRUCTION OF THE MEASURES \( \nu_K \)

In the proof below we use the notation for the Kolmogorov-Sinai entropy from Chapter 4 of [Wa2]. In particular the entropy of a Borel probability measure \( \theta \) on \( X \), with respect to a countable Borel partition \( \xi \) of \( X \), is denoted by \( H_\theta(\xi) \). If \( \mathcal{F} \) is a sub-\( \sigma \)-algebra of the Borel \( \sigma \)-algebra of \( X \), then \( H_\theta(\xi \mid \mathcal{F}) \) is the entropy of \( \theta \) with
respect to $\xi$ conditioned on $F$. If $\theta$ is $T$-invariant the entropy of $T$ with respect to $\theta$ is denoted by $h_\theta$. If $\theta$ is also ergodic we write $\gamma_\theta$ for the Lyapunov exponent of the system $(X, T, \theta)$, i.e.
\[
\gamma_\theta = \int_X \log |T'(x)| \, d\theta(x) = -2 \int_X \log x \, d\theta(x) .
\]
Given $a_1, \ldots, a_m \in \mathbb{N}$ we denote by $[a_1, \ldots, a_m]$ the finite continued fraction which lies in $(0, 1)$ and has coefficients $a_1, \ldots, a_m$, i.e.
\[
[a_1, \ldots, a_m] = \frac{1}{a_1 \frac{1}{a_2 \frac{1}{\ddots \frac{1}{a_m}}} .
\]

In order to establish the $\psi$-mixing property in the proof of Claim 2.5 we shall
need the following proposition. It follows directly from Theorem 1 in [Br].

**Proposition 5.1.** Let $\{A_n\}_{n=1}^\infty$ be a stationary and mixing sequence of random variables. Assume there exists a constant $0 < C < \infty$ with
\[
C^{-1} \leq \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(E)\mathbb{P}(F)} \leq C
\]
for all $l \geq 1$, $E \in \sigma\{A_1, \ldots, A_l\}$ and $F \in \sigma\{A_{l+1}, A_{l+2}, \ldots\}$. Then $\{A_n\}_{n=1}^\infty$ is $\psi$-mixing.

**Proof of Claim 2.5.** Fix $k \geq 3$ and for every $a \in \mathbb{N}^k$ and $c \in \mathbb{N}$ set
\[
p_a = \mu_G(I_a) \text{ and } p_{a,c} = \frac{\mu_G(I_{ac})}{\mu_G(I_a)} .
\]
Then $\sum_{c \in \mathbb{N}} p_{a,c} = 1$ for each $a \in \mathbb{N}^k$ and $p = \{p_a\}_{a \in \mathbb{N}^k}$ is a probability vector. Let $\{A_n\}_{n=1}^\infty$ be the k-step $\mathbb{N}$-valued Markov chain corresponding to the transition probabilities $\{p_{a,c}\}_{(a,c) \in \mathbb{N}^{k+1}}$ and initial distribution $\{p_a\}_{a \in \mathbb{N}^k}$. For each $b \in \mathbb{N}^{k-1}$ and $d \in \mathbb{N}$
\[
\sum_{c \in \mathbb{N}} p_{a,b,c} \cdot p_{c,d} = \sum_{c \in \mathbb{N}} \mu_G(I_{ab}) \cdot \frac{\mu_G(I_{ac,d})}{\mu_G(I_{ab})} = \mu_G(T^{-1}(I_{bd})) = p_{bd},
\]
hence $\{A_n\}_{n=1}^\infty$ is stationary. Considering $\{A_n\}_{n=1}^\infty$ as a $1$-step Markov chain on the state space $\mathbb{N}^k$, it is easy to see it is irreducible and aperiodic. From this and Theorem 8.6 in [Bl] it follows $\{A_n\}_{n=1}^\infty$ is mixing.

Let us show $\{A_n\}_{n=1}^\infty$ is in fact $\psi$-mixing. From (3.22) in chapter 3 of [EW] it follows there exists a constant $1 < C < \infty$ with,
\[
C^{-1} \leq \frac{\mu_G(I_{a_1, \ldots, a_l})}{\mu_G(I_{a_1}) \cdot \ldots \cdot \mu_G(I_{a_l})} \leq C^l \text{ for } l \geq 1 \text{ and } a_1, \ldots, a_l \in \mathbb{N} .
\]
For $l, m > k$, $(a_1, \ldots, a_l) = a \in \mathbb{N}^l$ and $(b_1, \ldots, b_m) = b \in \mathbb{N}^m$ set
\[
R := \frac{\mathbb{P}\{A_1 \ldots A_{l+m} = ab\}}{\mathbb{P}\{A_1 \ldots A_l = a\}\mathbb{P}\{A_1 \ldots A_m = b\}} .
\]
then
\[ R = \frac{1}{\mu_G(I_{b_1,...,b_k})} \cdot \prod_{j=1}^{k} \frac{\mu_G(I_{a_1-1,b_j,...,a_k,b_j})}{\mu_G(I_{a_1-1,b_j,...,a_k,b_j-1})}. \]

This together with (5.1) gives
\[ C^{-2k(k+1)} \leq R \leq C^{2k(k+1)}. \]

From Proposition 5.1 combined with a monotone class argument, it now follows that \( \{A_n\}_{n=1}^{\infty} \) is \( \psi \)-mixing.

Let \( \nu \) be the distribution of \( [A_1, A_2, ...] \), then \( \nu \) is \( T \)-invariant and ergodic. In order to prove the claim it remains to show that \( \dim H \nu \geq 1 - 2^{3-k} \). Set
\[ \xi = \{I_c : c \in \mathbb{N}\}, \]
then it is easy to check that
\[ H(\xi) = H(\mu_G(\xi)) < \infty \]
and
\[ \sum_{c \in \mathbb{N}} \nu(I_c) \log c = \sum_{c \in \mathbb{N}} \mu_G(I_c) \log c < \infty, \]
which shows \( h(\nu), \gamma(\nu), h(\mu_G) \) and \( \gamma(\mu_G) \) are all finite. From this and Section 2 of [BH] it follows that
\[ \dim H \nu = \frac{h(\nu)}{\gamma(\nu)} \quad \text{and} \quad 1 = \dim H \mu_G = \frac{h(\mu_G)}{\gamma(\mu_G)}. \]

Moreover, it is well known
\[ \gamma(\mu_G) = -\frac{2}{\log 2} \int \frac{\log x}{1 + x} dx = \frac{\pi^2}{6 \log 2} > 2. \]

By an argument similar to the one given in Theorem 4.27 in [Wa2],
\[ h(\nu) = -\sum_{a \in \mathbb{N}^k} \sum_{c \in \mathbb{N}} p_ap_{a,c} \log p_{a,c}. \]

From this and the definition of conditional entropy,
\[ h(\nu) = H(\mu_G(\vee_{j=0}^{k-1} T^{-j} \xi)) - H(\mu_G(\vee_{j=1}^{k} T^{-j} \xi)) \]
\[ \quad = H(\mu_G(\xi)) - H(\mu_G(\vee_{j=1}^{k} T^{-j} \sigma(\xi))) \geq H(\mu_G(\xi)) - H(\mu_G(\vee_{j=1}^{k} T^{-j} \sigma(\xi))) = h(\mu_G). \]

Assume \( k \) is even for the moment, then
\[ |a_1, ..., a_k| \leq x \leq |a_1, ..., a_k + 1| \]
for every \((a_1, \ldots, a_k) = a = N^k\) and \(x \in I_a\). It follows that,
\[
\gamma_\nu - \gamma_{\mu_G} = -2 \int_X \log x \, d\nu(x) + \frac{1}{2} \int_X \log x \, d\mu_G(x)
\]
\[
= 2 \sum_{a \in N^k} \left( \int_{I_a} \log x \, d\nu(x) + \int_{I_a} \log x \, d\mu_G(x) \right)
\]
\[
\leq 2 \sum_{a_1, \ldots, a_k \in \mathbb{N}} \left( \int_{I_{(a_1, \ldots, a_k)}} \log \frac{1}{[a_1, \ldots, a_k]} \, d\nu(x) + \log[a_1, \ldots, a_k + 1] \, d\mu_G(x) \right)
\]
\[
= 2 \sum_{a_1, \ldots, a_k \in \mathbb{N}} \mu_G(I_{(a_1, \ldots, a_k)}) \cdot \log \frac{[a_1, \ldots, a_k + 1]}{[a_1, \ldots, a_k]}.
\]
Fix \(a_1, \ldots, a_k \in \mathbb{N}\), then
\[
\log \frac{[a_1, \ldots, a_k + 1]}{[a_1, \ldots, a_k]} \leq \frac{[a_1, \ldots, a_k + 1] - [a_1, \ldots, a_k]}{[a_1, \ldots, a_k]}.
\]
Let \(p, q \in \mathbb{N}\) be with \(\gcd(p, q) = 1\) and \(\frac{p}{q} = [a_1, \ldots, a_k]\). From inequalities (3.6), (3.7) and (3.14) in [EW] it follows that \(q, p \geq 2^{(k-2)/2}\) and
\[
[a_1, \ldots, a_k + 1] - [a_1, \ldots, a_k] \leq q^{-2}.
\]
Hence
\[
\log \frac{[a_1, \ldots, a_k + 1]}{[a_1, \ldots, a_k]} \leq \frac{1}{q^2} \leq \frac{1}{pq} \leq 2^{2-k},
\]
and so \(\gamma_\nu - \gamma_{\mu_G} \leq 2^{3-k}\). By exchanging between \(\gamma_{\mu_G}\) and \(\gamma_\nu\) it can be shown that \(\gamma_{\mu_G} - \gamma_\nu \leq 2^{3-k}\). From \(k \geq 3\) and (5.3) we get \(\gamma_\nu \geq 1\), hence
\[
1 \leq \frac{\gamma_{\mu_G}}{\gamma_\nu} + \frac{2^{3-k}}{\gamma_\nu} \leq \frac{\gamma_{\mu_G}}{\gamma_\nu} + 2^{3-k}.
\]
A similar argument shows (5.5) holds when \(k\) is odd. From (5.2), (5.4) and (5.5) we now get
\[
\dim_H \nu = \frac{\gamma_{\mu_G}}{\gamma_\nu} \cdot h_\nu \geq (1 - 2^{3-k}) \cdot h_{\mu_G} \gamma_{\mu_G} = 1 - 2^{3-k},
\]
which completes the proof of the claim. \(\square\)

6. Extension of results for \(f\)-expansions

With almost no change, Theorems 2.3 and 2.6 extend to the more general setup of \(f\)-expansions, which we now define. Let \(M \in \{2, 3, \ldots \} \cup \{\infty\}\). Let \(f\) be either a strictly decreasing continuous function defined on \([1, M + 1]\) with \(f(1) = 1\) and \(f(M + 1) = 0\), or a strictly increasing continuous function defined on \([0, M]\) with \(f(0) = 0\) and \(f(M) = 1\). For \(x \in (0, 1)\) set \(r_0(x) = x\) and \(r_{i+1}(x) = \{f^{-1}(r_i(x))\}\) for \(i \geq 0\), where \(\{\cdot\}\) denotes the fractional part of a number. Let \(X\) be the set of all
$x \in (0, 1)$ with $r_i(x) \neq 0$ for every $i \geq 0$, then $(0, 1) \setminus X$ is clearly countable. Write

$$\mathcal{N} = \{[y] : y \in f^{-1}(0, 1)\},$$

where $[\cdot]$ is the integer part of a number. For $x \in X$ and $i \geq 1$ set

$$\alpha_i(x) = [f^{-1}(r_{i-1}(x))],$$

then $\alpha_i(x) \in \mathcal{N}$. We shall assume that

$$(6.1) \quad x = f(\alpha_1(x) + f(\alpha_2(x) + f(\alpha_3(x) + \ldots)))$$

for all $x \in X$, and call the expression on the right hand side the $f$-expansion of $x$. Regularity conditions on $f$ were given by Rényi [R], which ensure that $\text{(6.1)}$ is satisfied. The main example of the decreasing case is $f(x) = 1/x$, which leads to the continued fraction expansion, and of the increasing case is $f(x) = x/M$, which leads to the base-$M$ expansion. For more details on $f$-expansions see [R], [KP], [He] and the references therein.

We use the notation $I_a$ and $I_a$, introduced in Section 2 with $X$ and $\alpha_i$ as defined in this section and $a \in \cup_{k=1}^{\infty} \mathcal{N}$. For $x \in (0, 1)$ set $Tx = f^{-1}x - \lfloor f^{-1}x \rfloor$, then $\alpha_i(Tx) = \alpha_{i+1}(x)$ for $x \in X$. We shall assume that

1. the restriction of $T$ to $f(a, a + 1)$ is $C^2$ for each $a \in \mathcal{N}$;
2. there exists $\ell \in \mathbb{N}$ and $\beta > 0$ with $|T^\ell(x)| \geq \beta$ for all $x \in X$;
3. there exists $1 \leq Q < \infty$ with $\left| \frac{T''(x)}{T'(y)T'(z)} \right| \leq Q$ for all $a \in \mathcal{N}$ and $x, y, z \in I_a$.

Then by Theorem 22 in [Wa], there exists an absolutely continuous-$T$-invariant mixing probability measure $\mu_T$ on $X$, such that $0 < \frac{\text{det} T}{\text{det} I} \in C[0, 1]$. Here, as above, $\mathcal{L}$ is the Lebesgue measure.

For $q \in \mathcal{Q}_L$ with $\mathcal{Q}_L$ defined in Section 2, $a \in \cup_{k=1}^{\infty} \mathcal{N}$ and $\delta > 0$ let

$$\Gamma_{q, a}^\delta = \{x \in X : \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} I_a(T^q(i)x) - \mu_T(I_a) \geq \delta\}.$$

The following theorem is an analogue of Theorem 2.6 and can be proven in exactly the same manner.

**Theorem 6.1.** Suppose that $T$ satisfies conditions (1)-(3) and assume, in addition, that for some $t < 1$,

$$(6.2) \quad \sup_{x \in X} \sum_{y: Ty = x} |T'(y)|^{-t} < \infty.$$

Then for every $L > 1$ and $\delta > 0$ there exists $0 < c_{f, L, \delta} < 1$ with

$$\sup\{\dim_H(\Gamma_{q, a}^\delta) : q \in \mathcal{Q}_L, a \in \cup_{k=1}^{\infty} \mathcal{N} \} \leq 1 - c_{f, L, \delta}.$$

**Remark 6.2.** The condition (6.2) is needed in order to apply Theorem 4.1 from [KPW], as we did at the beginning of the proof of Theorem 2.6. Since $\{\alpha_i\}_{i=1}^{\infty}$
is a \( \psi \)-mixing sequence with respect to \( \mu_T \) (see [Ad] or [He]), the large deviations estimate from Corollary 3.2 is valid for \( \mu_T \). Now the proof of Theorem 6.1 follows almost verbatim the proof of Theorem 2.6.

An important ingredient in the proof of Theorem 2.3 is the fact that, for any \( k \geq 0 \), the continued fraction digits under \( \mu_G \) do not form a \( k \)-step Markov chain. Hence, in order to generalize Theorem 2.3 to the case of \( f \)-expansions we shall need the following lemma. For \( t \in [0,1] \) set \( F(t) = \mu_T([0,t]) \), and let \( S = F \circ T \circ F^{-1} \).

Since \( F' = \frac{dF}{dt} \in C[0,1] \) with \( \frac{dF}{dt} > 0 \), \( F \) is a diffeomorphism of \([0,1]\) onto itself. Given \( a \in \mathcal{N} \) write \( \tilde{I}_a := f(a,a+1) \).

**Lemma 6.3.** Assume the digits \( \{\alpha_i\}_{i=1}^\infty \) of the \( f \)-expansion are not independent under \( \mu_T \). Then \( \{\alpha_i\}_{i=1}^\infty \) do not form a \( k \)-step Markov chain under \( \mu_T \) for any \( k \geq 1 \).

**Proof.** Note that \( F\mu_T = \mathcal{L} \) and \( S\mathcal{L} = \mathcal{L} \). From the chain rule it follows that for every \( a \in \mathcal{N} \) and \( x \in F\tilde{I}_a \),

\[
S'(x) = F'(TF^{-1}x)T'(F^{-1}x)\left(F'(F^{-1}x)\right)^{-1},
\]

and so \( S' \) is continuous on \( F\tilde{I}_a \). Let \( \beta_1 : FX \to \mathcal{N} \) be such that \( \beta_1(x) = a \) for \( a \in \mathcal{N} \) and \( x \in F\tilde{I}_a \). For \( i \geq 1 \) set \( \beta_i = \beta_1 \circ S^{i-1} \), then \( \beta_i = \alpha_i \circ F^{-1} \). Given \( (a_1,\ldots,a_l) = a \in \mathcal{N}^l \) let

\[
J_a = \{x \in FX : \beta_i(x) = a_i \text{ for } 1 \leq i \leq l\},
\]

then \( J_a = F\tilde{I}_a \). Note that

\[
\mathcal{L}(J_a) = \mu_T(I_a) \text{ for every } l \geq 1 \text{ and } a \in \mathcal{N}^l.
\]

Let \( k \geq 1 \) and assume by contradiction that \( \{\alpha_i\}_{i=1}^\infty \) forms a \( k \)-step Markov chain under \( \mu_T \). From this, since \( \{\alpha_i\}_{i=1}^\infty \) are not independent, and from (6.3), it follows that under \( \mathcal{L} \) the variables \( \{\beta_i\}_{i=1}^\infty \) form a stationary \( k \)-step Markov chain but are not independent. Since \( \{\beta_i\}_{i=1}^\infty \) is a stationary \( k \)-step Markov chain,

\[
\mathcal{L}\{\beta_1 = c \mid S^{-1}(J_b)\} = \mathcal{L}\{\beta_1 = c \mid S^{-1}(J_{bf})\}
\]

for every \( c \in \mathcal{N} \), \( b \in \mathcal{N}^k \), \( l \geq 1 \) and \( f \in \mathcal{N}^l \). It follows there exist \( c \in \mathcal{N} \) and \( b,d \in \mathcal{N}^k \) with,

\[
\mathcal{L}\{\beta_1 = c \mid S^{-1}(J_b)\} \neq \mathcal{L}\{\beta_1 = c \mid S^{-1}(J_d)\},
\]

otherwise it would hold that \( \{\beta_i\}_{i=1}^\infty \) are independent under \( \mathcal{L} \).

It is not hard to see that for \( \mathcal{L} \)-a.e. \( x \in J_c \),

\[
\mathcal{L}\{\beta_1 = c \mid \sigma(\beta_2,\beta_3,\ldots)\}(x) = (S'(x))^{-1},
\]
where the left hand side is the conditional \( \mathcal{L} \)-probability of the event \( \{ \beta_1 = c \} \) with respect to the \( \sigma \)-algebra \( \sigma \{ \beta_2, \beta_3, \ldots \} \). Let \( a \in \mathbb{N}^k \). Then since \( \{ \beta_i \}_{i=1}^\infty \) is a \( k \)-step Markov chain under \( \mathcal{L} \), it follows for \( \mathcal{L} \)-a.e. \( x \in J_{ca} \) that

\[
\mathcal{L} \{ \beta_1 = c \mid \sigma \{ \beta_2, \beta_3, \ldots \} \}(x) = \mathcal{L} \{ \beta_1 = c \mid \sigma \{ \beta_2, \ldots, \beta_{k+1} \} \}(x) = \mathcal{L} \{ \beta_1 = c \mid S^{-1}(J_a) \} .
\]

This together with (6.5) shows that

\[
(6.6) \quad (S'(x))^{-1} = \mathcal{L} \{ \beta_1 = c \mid S^{-1}(J_a) \} \text{ for } \mathcal{L} \text{-a.e. } x \in J_{ca}.
\]

Since \( S' \) is continuous on \( F \tilde{I}_c \) and

\[
F(\tilde{I}_c \cap X) = \bigcup_{a \in \mathbb{N}^k} J_{ca},
\]

it follows easily from (6.6) that \( S' \) must be constant on \( F \tilde{I}_c \). On the other hand, by (6.4) and (6.6) this is not possible. We have thus reached a contradiction, which shows that \( \{ \alpha_i \}_{i=1}^\infty \) does not form a \( k \)-step Markov chain under \( \mu_T \). \( \Box \)

**Remark 6.4.** In Proposition 7.1 from [KPW] it is shown that \( \{ \alpha_i \}_{i=1}^\infty \) are independent under \( \mu_T \) if and only if \( S \) is linear on \( F \tilde{I}_a \) for each \( a \in \mathcal{N} \). From this and Lemma 6.3 it follows that if \( S \) is not linear on \( F \tilde{I}_a \) for some \( a \in \mathcal{N} \), then \( \{ \alpha_i \}_{i=1}^\infty \) do not form a \( k \)-step Markov chain under \( \mu_T \) for any \( k \geq 0 \).

The following theorem is an analogue, for the case of \( f \)-expansions, of Theorem 2.3 above and Corollary 2.3 from [KPW]. It can be derived from Theorem 6.1, Theorem 2.1 in [KPW], and Lemma 6.3 by an argument similar to the one given in the proof of Theorem 2.3. Given \( a_1, a_2, \ldots \in \mathcal{N} \) denote by \( [a_1, a_2, \ldots] \) the unique \( x \in X \) with \( \alpha_i(x) = a_i \) for \( i \geq 1 \).

**Theorem 6.5.** Suppose that \( T \) satisfies the conditions (1)-(3) and, in addition, that (6.2) holds for some \( t < 1 \). Assume the digits \( \{ \alpha_i \}_{i=1}^\infty \) of the \( f \)-expansion are not independent under \( \mu_T \). Let \( k \geq 0 \) and let \( \{ A_n \}_{n=1}^\infty \) be an \( \mathcal{N} \)-valued \( k \)-step Markov chain (when \( k = 0 \) this means \( A_1, A_2, \ldots \) are independent). Assume \( \{ A_n \}_{n=1}^\infty \) is \( \ast \)-mixing or that it is stationary and ergodic. Let \( \nu \) be the distribution of the random variable \( [A_1, A_2, \ldots] \). Then \( \dim_H(\nu) \leq 1 - c_{f,k} \), where \( 0 < c_{f,k} < 1 \) is a constant depending only on \( f \) and \( k \).

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