BLACK RING SOLUTIONS IN 5D HETEROTIC STRING THEORY: A FULL FIELD CONFIGURATION

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We show how to obtain a non–supersymmetric black ring configuration in the framework of 5D Heterotic String Theory using the matrix Ernst potential (MEP) formalism, which enables us to include non–trivial dilaton, gauge and antisymmetric tensor fields.

Keywords: Black ring; matrix Ernst potentials; heterotic string theory.

1. 3D effective action and matrix Ernst potentials

We shall begin with the 5D effective action of the heterotic string at tree level

\[ S^{(5)} = \int d^5x \left| G^{(5)} \right| \frac{1}{4} e^{-\phi^{(5)}} \left( B^{(5)} + \phi^{(5)} - \frac{1}{12} H^{(5)MNP} H^{(5)MNP} - \frac{1}{4} F^{(5)I} F^{(5)I} \right), \]

where \( G^{(5)} \) is the 5D metric, \( \phi^{(5)} \) is the 5D dilaton, \( B^{(5)}_{MNP} = \partial_M A_N^{(5)I} - \partial_N A_M^{(5)I} \) and \( H^{(5)MNP} = \partial_M B_N^{(5)I} - \frac{1}{2} A_M^{(5)J} F_{NP}^{(5)I} + \) cycl. perms. of M,N,P. Here \( B^{(5)}_{MN} \) is the anti-symmetric Kalb–Ramond field and \( A_M^{(5)I} \) \((I = 1, 2, ..., n)\) is a set of \( U(1) \) vector fields. In Ref. 1 it was shown that after imposing two commuting Killing vectors, the resulting stationary theory possesses the \( U–\)duality symmetry group \( SO(3, 3 + n) \) and describes 3D gravity

\[ g_{\mu\nu} = e^{-2\phi} \left( G^{(5)}_{\mu\nu} - G^{(5)}_{m+n,3+\mu} G^{(5)}_{n+3,\nu} G^{(5)}_{mn} \right), \]

coupled to the following set of three–fields: a) scalar fields

\[ G \equiv G_{mn} = G^{(5)}_{m+n,3+n+3}, \]
\[ B \equiv B_{mn} = B^{(5)}_{m+n,3+n+3}, \]
\[ A \equiv A_m^I = A^{(5)I}_{m+3}, \]
\[ \phi = \phi^{(5)} - \frac{1}{2} \ln|\det G|, \]

b) tensor field (we set it to zero in this report)

\[ B_{\mu\nu} = B^{(5)}_{\mu\nu} - 4B_{mn} A_m^m A_n^n \]
\[ - 2(A_m^m A_n^{2+m} - A_n^m A_m^{2+m}) = 0, \]

c) vectors \( A_m^{(5)} = (A_1^m, (A_2^m)^{2+m}, (A_3^m)^{3+m}) \)

where the subscripts \( m,n = 1,2; \) and \( a = 1, ..., 4+n. \) All vector fields in three dimensions can be dualized on–shell as follows:

\[ \nabla \times A_1 = \frac{1}{2} e^{2\phi} e^{G^{(5)}_{12}} \frac{1}{A_1}, \]
\[ \nabla \times A_2 = \frac{1}{2} e^{2\phi} e^{G^{(5)}_{13}} \frac{1}{A_2}, \]
\[ \nabla \times A_3 = \frac{1}{2} e^{2\phi} e^{G^{(5)}_{13}} \frac{1}{A_3}. \]

Thus, the effective stationary theory describes gravity \( g_{\mu\nu} \) coupled to the scalars \( G, \) \( B, \) \( A, \) \( \phi \) and pseudoscalars \( u, v, s. \) These matter fields can be arranged in the following pair of matrix Ernst potentials 2:

\[ \mathcal{X} = \begin{pmatrix} -e^{-2\phi} + v^T X v + v^T A s + \frac{1}{2} s^T s & v^T X - u^T \\ X v + u + A s & X \end{pmatrix}, \]
\[ A = \begin{pmatrix} s^T + v^T A \end{pmatrix}, \]
where \( X = G + B + \frac{1}{2} AA^T \). These matrices have dimensions \( 3 \times 3 \) and \( 3 \times n \), respectively. Thus, the effective stationary theory adopts the following form in terms of the MEP:

\[
S^3 = \int d^3 x \left| g \right|^\frac{1}{2} \left\{ -R + \text{Tr} \left[ \frac{1}{2} \nabla A^T G^{-1} \nabla A + \frac{1}{4} \left( \nabla X - \nabla A A^T \right) G^{-1} \left( \nabla X^T - A \nabla A^T \right) G^{-1} \right] \right\},
\]

with \( X = G + B + \frac{1}{2} AA^T \), and hence, \( G = \frac{1}{2} (X - X^T) \) and \( B = \frac{1}{2} (X^T - AA) \).

In \(^2\) it was also shown that there exist a map between the stationary actions of the heterotic string and Einstein–Maxwell theories:

\[
X \leftrightarrow -E, \quad A \leftrightarrow F,
\]

where \textit{matrix transposition} is also interchanged with \textit{complex conjugation}; \( E \) and \( F \) are the gravitational and electromagnetic potentials of the stationary Einstein–Maxwell theory \(^3\).

In the language of the MEP the latter stationary action possesses a set of isometries which has been classified according to their charging properties in Ref. \(^4\). Among them, the so-called normalized Harrison transformation (NHT) acts in a non–trivial way on a seed spacetime: it allows us to construct charged string vacua from neutral ones preserving the asymptotical values of the initial fields. Namely, the matrix transformation

\[
A \to (1 + \frac{1}{2} \Sigma \lambda \lambda^T) \left( 1 - A_0 \right)^{-1} \times \left( 1 + \frac{1}{2} \Sigma \lambda \lambda^T \right)^{-1} \times \left[ A_0 + \left( A_0 - \frac{1}{2} \Sigma \lambda \lambda^T \right) \lambda^T \Sigma \right] + \frac{1}{2} \Sigma \lambda \lambda^T \Sigma,
\]

where \( \Sigma = \text{diag}(-1, -1, 1, 1, \ldots, 1) \) and \( \lambda \) is an arbitrary constant \( 3 \times n \)–matrix parameter, generates charged string solutions (with non–zero potential \( A \)) from the neutral seed ones \( A_0 \neq 0 \), and \( A_0 = 0 \). This solution–generating technique allows us to generate the \( U(1)^n \) electromagnetic spectrum of the effective heterotic string theory starting with just the bosonic spectrum of string theory.

2. **Charging a neutral black ring**

We can apply the NHT on a seed solution corresponding to the 5D \textit{neutral black ring} of Emparan and Reall \(^5\) (see also Ref. \(^6\)):

\[
ds^2 = -\frac{F(x)}{F(y)} \left( dt + R \sqrt{\nu(1 + y)} d\psi \right)^2 + \frac{R^2}{(x - y)^2} \left[ -F(x) \left( G(y) d\psi^2 + \frac{F(y)}{\nu(x)} dy^2 \right) \right] + \frac{y}{2} \left( \frac{dy^2}{y^2} + \frac{G(y)}{\nu(x)} d\phi^2 \right)
\]

with

\[
F(\xi) = 1 - \lambda \xi, \quad G(\xi) = (1 - \xi^2)(1 - \nu \xi).
\]

Here \( R \) is the radius of the ring, \( F(\xi) \) has the root \( \xi_1 = \lambda^{-1} \), and \( G(\xi) \) possesses three roots: \( \xi_2 = -1 \), \( \xi_3 = +1 \), \( \xi_4 = \frac{1}{\nu} \). Thus, the solution has two dimensionless parameters \( \lambda \) and \( \nu \).

The variables \( x \) and \( y \) take values in

\(-1 \leq x \leq 1, \quad -\infty < y \leq -1, \quad \lambda^{-1} < y < \infty \).

In order to balance forces in the ring, one must identify \( \psi \) and \( \phi \) with equal period

\[
\Delta \phi = \Delta \psi = \frac{4 \pi \sqrt{F(-1)}}{|G'(1)|} = \frac{2 \pi \sqrt{1 + \lambda}}{1 + \nu}.
\]

This eliminates the conical singularities at the fixed-point sets \( y = -1 \) and \( x = -1 \) of the Killing vectors \( \partial_\psi \) and \( \partial_\phi \), respectively.

For avoiding conical singularities at \( x = +1 \), we have two cases:

1) By fixing

\[
\lambda = \lambda_c = \frac{2 \nu}{1 + \nu} \quad \text{(black ring)}
\]

makes the circular orbits of \( \partial_\phi \) close off smoothly also at \( x = +1 \). Then \((x, \phi)\) parameterize a two–sphere, \( \psi \) parameterizes a circle, and the solution describes a \textbf{black ring}.

2) If we set

\[
\lambda = 1 \quad \text{(black hole)}
\]

the orbits of \( \partial_\phi \) do not close at \( x = +1 \). Then \((x, \phi, \psi)\) parameterize an \( S^2 \) at constant \( t, y \). The solution describes the \textbf{black hole} of Myers and Perry with a single rotation parameter \(^7\).
Both for black holes and black rings, $|y| = \infty$ is an ergosurface, $y = 1/\nu$ is the event horizon, and as $y \to \lambda^{-1}$ an inner, spacelike singularity is reached from above.

The parameter $\nu$ varies in the range

$$0 \leq \nu < 1.$$  \hfill (14)

As $\nu \to 0$ we recover a non–rotating black hole, or a very thin black ring. At the opposite limit, $\nu \to 1$, both the black hole and the black ring result into the same solution with a naked ring singularity. If one considers $\lambda$ as an independent parameter which can be eventually fixed to the equilibrium value $\lambda_c$. If $\lambda$ adopts values different from (12) and (13), then whenever $\nu < \lambda < 1$ one finds a black ring solution regular on and outside the horizon, except for a conical singularity on the disk bounded by the inner rim of the ring ($x = +1$). If $\nu < \lambda < \lambda_c$ then the ring is rotating faster than the equilibrium value, and there is a conical deficit balancing the excess centrifugal force. If instead $\lambda_c < \lambda < 1$ then the rotation is too slow and a conical excess appears. If $\lambda \leq \nu$ the horizon is replaced by a naked singularity. Finally, the solution with $\lambda = \lambda_c$ is the equilibrium, or balanced black ring.

The mass, spin, area, temperature and the angular velocity at the horizon for the black ring are given by

$$M_0 = \frac{3\pi R^2 \lambda(\lambda + 1)}{4G} \nu + 1,$$  \hfill (15)

$$J_0 = \frac{\pi R^3 \sqrt{\lambda \nu(\lambda + 1)^{5/2}}}{2G} \left(1 + \nu^2\right),$$  \hfill (16)

$$A_0 = \frac{8\pi^2 R^3 \lambda^{1/2}(\lambda + 1)(\lambda - \nu)^{3/2}}{(1 + \nu^2)(1 - \nu)},$$  \hfill (17)

$$T_0 = \frac{1}{4\pi R \lambda^{1/2}(\lambda - \nu)^{1/2}},$$  \hfill (18)

$$\Omega_0 = \frac{1}{R} \sqrt{\frac{\nu}{\lambda(1 + \lambda)}},$$  \hfill (19)

Looking at the dimensionless quantity

$$\frac{27\pi j_0^2}{32G M_0^3} = \begin{cases} \frac{2\nu}{\nu + 1} & \text{(black hole)} \\ \frac{(1 + \nu)^3}{8\nu} & \text{(balanced black ring)} \end{cases}$$  \hfill (20)

one sees that for black holes it grows monotonically from 0 to 1, while for (equilibrium) black rings it is infinite at $\nu = 0$, decreases to a minimum value $27/32$ at $\nu = 1/2$, and then grows to 1 at $\nu = 1$. This implies that in the range

$$\frac{27}{32} \leq \frac{27\pi j_0^2}{32G M_0^3} < 1$$  \hfill (21)

there exist one black hole and two black rings with the same value of the spin for fixed mass.

This regime of non–uniqueness occurs when the parameter $\nu$ takes values in

$$\sqrt{5} - 2 \leq \nu < 1$$  \hfill (22)

for equilibrium black rings, and in

$$\frac{27}{37} \leq \nu < 1$$  \hfill (23)

for black holes.

3. Solution–generating Technique

This 5D neutral black ring constitutes a solution of the heterotic string theory when

$$\phi^{(5)} = 0, \quad A_M^{(5)} = 0, \quad B_{MN}^{(5)} = 0,$$  \hfill (24)

and the corresponding MEP read

$$\mathcal{X} = \begin{pmatrix} -e^{-2\phi} - u^T \\ u \\ G \end{pmatrix}, \quad A = 0.$$  \hfill (25)

After applying the NHT one obtains a new field configuration that corresponds to a charged black ring with non–trivial dilaton, Kalb–Ramond and electromagnetic fields. Similar results have been applied to 5D and 4D theories in Refs. 8–11. These fields can be extracted from the respective components of the generating matrix Ernst potentials

$$\mathcal{X} = \begin{pmatrix} \chi_1 & \chi_2 \\ \chi_{12} & \chi_{22} \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$  \hfill (26)
Thus, the 5D metric adopts the form
\[ ds^2 = G_{MN} dx^M dx^N = G_{mn} (dx^{m+3} + \omega^{(m)} d\varphi) (dx^{n+3} + \omega^{(n)} d\varphi) + e^{2\phi} g_{\mu\nu} dx^\mu dx^\nu, \]
where the metric components \( G \equiv G_{mn} \) and \( e^{-2\phi} \) read
\[ G = \frac{1}{2} \left( X_{22} + X_{22}^T - A_{2n} A_{2n}^T \right), \]
and
\[ e^{-2\phi} = -X_{11} + \frac{1}{2} A_{1n} A_{1n}^T + \frac{1}{8} \left( X_{12} + X_{12}^T - A_{1n} A_{2n}^T \right) G^{-1} \times \left( 2G - A_{2n} A_{2n}^T \right) G^{-1} \left( X_{21} + X_{21}^T - A_{2n} A_{1n}^T \right), \]
and \( g_{\mu\nu} \) is the spatial 3D metric.

The remaining 5D fields can be obtained from the matrix expressions
\[ B = \frac{1}{2} \left( X_{22} - X_{22}^T \right), \]
\[ A = A_{2n}, \]
\[ v = \frac{1}{2} G^{-1} \left( X_{21} + X_{12}^T - A_{2n} A_{1n}^T \right), \]
\[ u = \frac{1}{2} X_{22} G^{-1} \left( X_{21} + X_{12}^T - A_{2n} A_{1n}^T \right) - X_{12}^T, \]
\[ s = A_{1n}^T \frac{1}{2} A_{2n}^T G^{-1} \left( X_{21} + X_{12}^T - A_{2n} A_{1n}^T \right), \]
remembering that in order to recover the non-trivial components of the 3D vector fields one must make use of the dualization relations (6) for the pseudoscalar fields.

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References

1. J. Maharana and J.H. Schwarz, Nucl. Phys. B390, 3 (1993); A. Sen, Nucl. Phys. B434, 179 (1995).
2. A. Herrera–Aguilar and O. Kechkin, Int. J. Mod. Phys. A13, 393 (1998); Int. J. Mod. Phys. A14, 1345 (1999).
3. F.J. Ernst, Phys. Rev. 167, 1175 (1968).
4. A. Herrera–Aguilar and O. Kechkin, Phys. Rev. D59, 124006 (1999).
5. R. Emparan and H.S. Reall, Phys. Rev. Lett. 88, 101101 (2002).
6. H. Elvang and R. Emparan, JHEP 0311, 035 (2003).
7. M.C. Myers and M.J. Perry, Annals Phys. 172, 704 (1986).
8. A. Herrera–Aguilar, Rev. Mex. Fís. 49S2, 141 (2003).
9. A. Herrera–Aguilar, Mod. Phys. Lett. A19, 2299 (2004).
10. A. Herrera–Aguilar and M. Nowakowski, Class. Quantum Grav. 21, 1015 (2004).
11. R. Becerril and A. Herrera–Aguilar, J. Math. Phys. 46, 052503 (2005).