ERGODICITY AND CONSERVATIVITY OF PRODUCTS OF INFINITE TRANSFORMATIONS AND THEIR INVERSES

JULIEN CLANCY, RINA FRIEDBERG, INDRANEEL KASMALKAR, ISAAC LOH, TUDOR PĂDURARIU, CESAR E. SILVA, AND SAHANA VASUDEVAN

Abstract. We construct a class of rank-one infinite measure-preserving transformations such that for each transformation $T$ in the class, the cartesian product $T \times T$ of the transformation with itself is ergodic, but the product $T \times T^{-1}$ of the transformation with its inverse is not ergodic, and examples where all products of distinct positive powers of $T$ are ergodic but $T \times T^{-1}$ is not ergodic. We also prove that the product of any rank-one transformation with its inverse is conservative, while there are infinite measure-preserving conservative ergodic Markov shifts whose product with their inverse is not conservative.

1. Introduction

The notion of weak mixing for finite measure-preserving transformations has many equivalent characterizations. Several of these characterizations, however, do not remain equivalent in the infinite measure-preserving case. The first examples showing that some of the properties are different in the infinite measure case were given by Kakutani and Parry [11], who constructed, for each positive integer $k$, an infinite measure-preserving Markov shift $T$ such that the $k$-fold cartesian product of $T$ with itself is ergodic but its $k + 1$-fold product is not (such a transformation is said to have ergodic index $k$). Later, Adams, Friedman and Silva [3] constructed a rank-one infinite measure-preserving transformation $T$ with infinite ergodic index (i.e., all finite cartesian products with itself are ergodic) but such that $T \times T^2$ is not conservative, hence not ergodic. Bergelson then asked if there existed an example of a transformation $T$ of infinite ergodic index but such that $T \times T^{-1}$ is not ergodic. This question appears as problem P10 in [8].
history and other examples, the reader may refer to [9]; more recently though, ergodic index $k$ transformations have been constructed in rank-one in [4]. In this paper we partially answer Bergelson’s question by constructing an infinite measure-preserving rank-one transformation $T$ such that $T \times T$ is ergodic, but $T \times T^{-1}$ is not ergodic (Theorem 5.2).

In addition, we construct a rank-one transformation $T$ such that for each $\alpha_1, \ldots, \alpha_k$ distinct positive integers, $T^{\alpha_1} \times \cdots \times T^{\alpha_k}$ is ergodic but $T \times T^{-1}$ is not ergodic (Theorem 6.3). We also prove that for all rank-one transformations $T$, the transformation $T \times T^{-1}$ is conservative (Theorem 4.3), while this is not the case in general (Corollary 7.6). In this context we note that it was already known that there exist rank-one transformations $T$ such that $T \times T$ is not conservative [2]. Also, whenever $T$ is a rigid transformation (i.e., there is an increasing sequence $\{n_i\}$ such that the limit the measure of $T^{n_i}(A) \triangle A$ tends to 0 for all sets $A$ of finite measure) one can verify that $T \times T^{-1}$ is conservative, and as the class of rigid transformations is generic in the group of invertible infinite measure-preserving transformations of a Lebesgue space under the weak topology [5], it follows that the property of $T \times T^{-1}$ being conservative is a generic property; this fact also follows from Theorem 5.2 and the fact that infinite measure-preserving rank-ones are generic [6]. As we show later, however, there are other transformations, in particular conservative ergodic Markov shifts, where the product $T \times T^{-1}$ is not conservative (Corollary 7.6). A consequence of the properties of our rank-one examples in Theorem 5.2 is that these transformations are not isomorphic to their inverse. Also, it follows from Theorem 4.3 that if a rank-one transformation $T$ satisfies that $T \times T$ is not conservative, then $T$ is not isomorphic to its inverse.

The methods that we use are combinatorial and probabilistic in nature. Propositions 2.8 and 2.9 use the notion of descendants, as introduced in [7], to turn the dynamics of the rank-one system into combinatorial characterizations.

We let $(X, \mu, \mathcal{B})$ denote a Lebesgue measurable subset of the real line with Lebesgue measure, and consider $T : X \to X$ an invertible measure-preserving transformation; we are interested in the case when $X$ is of infinite measure. The transformation $T$ is ergodic if whenever $T^{-1}(A) = A$, then $\mu(A) = 0$ or $\mu(A^c) = 0$, and conservative if $A \subset \bigcup_{n=1}^{\infty} T^{-n}(A) \mod \mu$. As $(X, \mu)$ is nonatomic and $T$ is invertible, when $T$ is ergodic, it is conservative.

We briefly review rank-one cutting-and-stacking transformations. A column or tower $C$ is an ordered collection of pairwise disjoint intervals (called the levels of $C$) in $\mathbb{R}$, each of the same measure. We think of the levels in a column as being stacked on top of each other,
so that the \((j + 1)\)-st level is directly above the \(j\)-th level. Every column \(C = \{I_j\}\) is associated with a natural column map \(T_C\) sending each point in \(I_j\) to the point directly above it in \(I_{j+1}\) (note that \(T_C\) is undefined on the top level of \(C\)). A rank-one cutting-and-stacking construction for \(T\) consists of a sequence of columns \(C_n\) such that:

1. The first column \(C_0\) consists only of the unit interval.
2. Each column \(C_{n+1}\) is obtained from \(C_n\) by cutting \(C_n\) into \(r_n \geq 2\) subcolumns of equal width, adding any number \(s_{n,k}\) of new levels (called spacers) above the \(k\)th subcolumn, \(k \in \{0, r_n - 1\}\), and stacking every subcolumn under the subcolumn to its right. In this way, \(C_{n+1}\) consists of \(r_n\) copies of \(C_n\), possibly separated by spacers.
3. The collection of levels \(\bigcup_n C_n\) forms a generating subring for \(B\).

Observing that \(T_{C_{n+1}}\) agrees with \(T_{C_n}\) everywhere that \(T_{C_n}\) is defined, we then take \(T\) to be the pointwise limit of \(T_{C_n}\) as \(n \to \infty\). For further details the reader may refer to [13] and [6].

Given any level \(I\) from \(C_n\) and any column \(C_n\) of \(T\) with \(m \leq n\), we define the descendants of \(I\) in \(C_n\) to be the collection of levels in \(C_n\) whose disjoint union is \(I\). We denote this set by \(D(I, n)\). By abuse of notation (and not to complicate the notation further), we will also use \(D(I, n)\) to refer to the heights of the descendants of \(I\) in \(C_n\).

Write \(h_{j,k} = h_j + s_{j,k}\). Suppose that \(I\) is a level in \(C_i\) of height \(h(I)\), where the heights in the column are 0-indexed. Then \(I\) splits into \(r_i\) levels in \(C_{i+1}\) of heights

\[
\{h(I)\} \cup \{h(I) + \sum_{k=0}^{i} h_{j,k} \mid 0 \leq i < r_j - 1\}
\]

Letting \(H_j = \{0\} \cup \{\sum_{k=0}^{i} h_{j,k} \mid 0 \leq i < r_j - 1\}\), it follows inductively that

\[
(1) \quad D(I, n) = h(I) + H_i \oplus H_{i+1} \oplus \cdots \oplus H_{n-1}
\]

Instead of describing a rank-one transformation by cutting and spacer parameters, we can describe it by specifying the descendant sets of the unit interval \([0, 1]\). For instance, given \(D([0, 1], n)\) for every \(n\) (assuming that they are “compatible”, that is, specify an actual rank-one transformation), we can easily extract the cutting and spacer sequence. The converse direction is given in equation (1). If one wishes to construct a rank-one transformation, then, one needs only to specify its descendant sets and ensure that they are “compatible”. One way to do this is to create sets \(H_k \subset \mathbb{N}\) for \(k \in \mathbb{N}\) and define \(D([0, 1], n)\) as above, that is,
\[ D([0,1], n) = H_0 \oplus \ldots \oplus H_{n-1}. \] The only compatibility restrictions, as is easily seen, are that \( 0 \in H_k \) for all \( k \), and that any two elements of \( H_k \) are further apart than \( h_{j-1} \), the height of column \( C_{k-1} \).

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2. **Preliminaries**

Throughout this paper, let \( T^{(k)} = T \times \cdots \times T \), and \( U = T \times T^{-1} \). We first have some necessary and sufficient conditions for these transformations to be ergodic. From here forward, we will use the notation \( A \subset_\delta X \) to mean \( \mu(A \cap X) > (1 - \delta) \mu(A) \); we will call this property almost-containment. This notion has some obvious properties, whose verification is left to the reader. First, if \( A \subset_\delta B \) and \( A = \sqcup_{i=1}^{m} A_i \), then \( A_i \subset_\delta B \) for some \( i \). Also, if \( \mu A_i = \mu A_j \) for every \( i, j \), then \( A_i \subset_{n\delta} \) for every \( i \).

**Lemma 2.1.** Let \( T_1, \ldots, T_k \) be rank-one transformations on \( X_1, \ldots, X_k \), and let \( T := T_1 \times \cdots \times T_k \) and \( X = X_1 \times \cdots \times X_k \). Let \( \mathcal{D} \) be the sufficient semiring consisting of rectangles of the form \( R_1 \times \cdots \times R_k \), where \( R_i \) is a level of some column of \( T_i \). Then, \( T \) is conservative ergodic if and only if for all \( A, B \in \mathcal{D} \), we have

\[ A \subset \bigcup_n T^n B \pmod{\mu} \]

**Proof.** Assume, for the sake of contradiction, that there are sets of positive measure \( E, F \subset X \) such that \( T^n E \cap F = \emptyset \) for every \( n \in \mathbb{Z} \). Because \( \mathcal{D} \) is a sufficient semiring, we can find \( A, B \in \mathcal{D} \) such that \( A \subset_{01} E \) and \( B \subset_{01} F \). Let \( A = L_1 \times \cdots \times L_k \) and \( B = L'_1 \times \cdots \times L'_k \). By the properties of almost-containment, for any \( n \in \mathbb{N} \), dividing \( A \) into \( n \) pieces we must have that at least one of them is \( \subset_{01} E \). Since the descendants of a level are a partition into equal-measure parts, for any \( i \), by passing to descendants of \( L_i \) and \( L'_i \), we can assume that \( L_i \) and \( L'_i \) lie in the same column, and therefore that they have the same measure for each \( i \). Hence, we may assume that \( A \) and \( B \) have the same measure. By assumption, given any \( \varepsilon > 0 \) we can find \( m \in \mathbb{N} \) such that \( \mu(A \setminus \bigcup_{-m}^{m} T^n B) < \varepsilon \mu A \) and \( \mu(B \setminus \bigcup_{-m}^{m} T^n A) < \varepsilon \mu B \). Fix \( 0 < \varepsilon < .01 \).

Now, partition \( A \) and \( B \) into products of sublevels, all of the same size, such that all the intersections \( T^n B \cap A \), for \( -m \leq n \leq m \), are
unions of such rectangles except for a set of measure \( \varepsilon \mu A = \varepsilon \mu B \). Let \( K \) be the number of sub-rectangles of \( A \) and \( B \); since \( \mu(A) = \mu(B) \) and we can choose the rectangles to be of the same measure, we can take the number of rectangles in \( A \) to be the same as the number of rectangles in \( B \). Since \( A \subset_{01} E \) and all the rectangles that make up \( A \) are of equal measure, for \( .9K \) of them, call such a one \( R \), we have \( R \subset_{.5} E \). Similarly, for \(.9K \) of the rectangles in \( B \), call such a one \( R \), we have \( R \subset_{.5} F \).

Now, except for a measure \( \varepsilon \mu(A) \), \( A \subset \bigcup_{-m}^{m} T^{n}B \), both sides being almost a union of rectangles. We claim that fewer than \(.3K \) of the rectangles of \( B \) can be used in the covering in \( A \). For, if \(.3K \) were used in the covering, then at least \(.2K \) of those would have to be \( \subset_{.5} F \). In turn, only \(.1K \) of the rectangles of \( A \) are not \( \subset_{.5} E \), so that there is a nontrivial intersection of \( T^{n}R_{b} \) and \( R_{a} \), where the rectangles involved are \( \subset_{.5} F \) and \( \subset_{.5} E \), respectively. But because those must cover \( A \) under \( T^{n} \) and levels are sent to levels under \( T^{n} \) (recall that \( -m \leq n \leq m \) and we chose our levels to be more than \( m \) spaces from the bottom or the top of their column), we must have that \( T^{n}R_{b} = R_{a} \). But of course \( T^{n}R_{b} \subset_{.5} T^{n}F \) so that \( \mu(T^{n}F \cap E) > 0 \), a contradiction. Thus at most \(.3K \) of the \( B \)-rectangles can be mapped to \( A \) under \( T^{n} \) for \( -m \leq n \leq m \). But by symmetry, that means at most \(.3K \) of the \( B \)-rectangles can be covered by \( \bigcup_{-m}^{m} T^{n}A \), which for \( \varepsilon < .01 \) is a contradiction. Hence \( T \) is conservative ergodic. The converse direction is clear, and is left to the reader.

We note that Lemma 2.1 does not hold in general, although the authors have verified that the lemma holds for rank-two transformations. A counterexample for the general case can be constructed as in [13] using a set \( K \) in \( X = \mathbb{R} \) such that every positive-length interval \( I \) in \( \mathbb{R} \) intersects both \( K \) and \( K^{c} \) in positive measure. Then choose conservative ergodic transformations \( T_{0} \) and \( T_{1} \) on \( K \) and \( K^{c} \) respectively, and define \( T \) on \( \mathbb{R} \) to be the disjoint union of \( T_{0} \) and \( T_{1} \). Then \( T \) is not ergodic but satisfies that for every positive measure \( A \) in the dense algebra of intervals \( \bigcup_{n} T^{n}A = X \).

To prove that \( T^{(k)} \) is ergodic, we will use the following method. We show that \( A \subset \bigcup_{n=-\infty}^{\infty} (T^{(k)})^{n} B \) holds for all rectangles \( A \) and \( B \).

\(^{1}\)It is not guaranteed that all of \( T^{n}B \cap A \) can be written as a union of such rectangles. For instance, if one of the levels that is the side of one of the rectangles is less than \( m \) spaces from the top of its column, we cannot guarantee (and in general it is not true) that its image under \( T^{n} \) is also a level. However, this only happens for the levels fewer than \( m \) places from the bottom or top of the column, and their measure becomes arbitrarily small as the size of the columns grows, hence only a very small fraction of the rectangles making up \( B \) are not also rectangles in \( D \) under \( T^{n} \) for \( n \in \{-m, \ldots, m\} \).
By cutting the rectangles if necessary, \( T^{(k)} \) is ergodic if and only if \( A \subset \bigcup_{n=-\infty}^{\infty} (T^{(k)})^{n} B \) for all rectangles \( A \) and \( B \) with sides from the same column. If the column is \( C_i \) with base \( I \) we write \( A = T^{a_0} I \times T^{a_1} I \times \cdots \times T^{a_{k-1}} I \) and \( B = T^{b_0} I \times T^{b_1} I \times \cdots \times T^{b_{k-1}} I \). We can transform both sides of \( A \subset \bigcup_{n=-\infty}^{\infty} (T^{(k)})^{n} B \) by \( T^{a_0} \times T^{-a_1} \times \cdots \times T^{-a_{k-1}} \) so that it suffices to have \( A \) of the type \( A = I \times I \times \cdots \times I \). Finally, since the union ranges over all powers of \( T^{(k)} \) we can simply translate the union so that \( b_0 \) can be taken to be 0, i.e., \( B = I \times T^{b_1} I \times \cdots \times T^{b_{k-1}} I \). The same reduction can be done for \( U = T \times T^{-1} \).

**Lemma 2.2.** Let \( T \) be a rank-one transformation. Fix \( k \in \mathbb{N} \). Let \( A = I \times \cdots \times I \) be the product of \( k \) copies of \( I \), and \( B = I \times T^{b_1} I \times \cdots \times T^{b_{k-1}} I \). Then \( T^{(k)} \) is conservative ergodic if and only if for every \( \varepsilon > 0 \), there is \( j \) such that for at least \( (1 - \varepsilon) |D(I, j)|^k \) tuples of descendants \((a_0, a_1, \ldots, a_{k-1}) \in D(I, j)^k\), we have \( T^a J \times \cdots \times T^{a_{k-1}} J \subset (T^{(k)})^n B \cap A \) for some \( n \).

**Proof.** Fix \( \varepsilon > 0 \). First, if \( T^{(k)} \) is conservative ergodic, we can find \( m \) such that \( A \) is covered by \( \bigcup_{m}^{m} (T^{(k)})^n B \) except for measure \( \varepsilon \mu(A) \). We can choose \( j \) large enough that, except for a measure \( \varepsilon \mu(A) \), all the intersections \( (T^{(k)})^n B \cap A \) are composed of unions of rectangles whose sides are levels of \( C_j \). These levels must be descendants of \( A \), that is, in \( D(I, j) \), so we have that at least \( (1 - \varepsilon) \left| \frac{1}{2} \right| |D(I, j)|^k = (1 - \varepsilon)|D(I, j)|^k \) of the rectangles are contained in \( (T^{(k)})^n B \cap A \).

Now we will show the converse. If \( T^{(k)} \) satisfies that condition, then choose \( m \) so large that almost all of the \( (1 - \varepsilon) |D(I, j)|^k \) of the rectangles are contained in \( \bigcup_{m}^{m} (T^{(k)})^n B \cap A \). Because \( B \) and \( A \) have the same measure, we must have that \( A \) is covered up to measure \( \varepsilon \mu(A) \) by \( \bigcup_{m}^{m} (T^{(k)})^n B \cap A \). By Lemma 2.1, \( T^{(k)} \) is ergodic.

The same proof gives the same result for \( U = T \times T^{-1} \):

**Lemma 2.3.** \( U \) is conservative ergodic if and only if for every \( \varepsilon > 0 \), there is \( j \) such that for at least \( (1 - \varepsilon) |D(I, j)|^2 \) tuples of descendants \((a_0, a_1) \in D(I, j)^2\), we have \( T^{a_0} J \times T^{a_1} J \subset U^n B \cap A \) for some \( n \).

To control the differences between \( T^{(k)} \) and \( U \) (and especially between \( T^{(2)} \) and \( U \)), we express the conditions of Lemma 2.2 in combinatorial terms involving their descendant sets. To do that, we need the following two technical lemmas.

**Lemma 2.4.** Let \( T \) be an infinite rank-one transformation. Then the spacer sequence of \( T \) is unbounded.

\(^2\)See the previous footnote for why this can’t be all of \( A \).
Lemma 2.5. Let \( \{r_n\} \) be the cut sequence of \( T \) and let \( \{s_{n,k}\} \) be the spacer sequence of \( T \). Suppose that the spacer sequence is bounded, say \( s_{n,k} \leq B \). Given the total measure of \( C_n \), the total measure of \( C_{n+1} \) is the measure of \( C_n \) plus the total mass of the spacers placed above \( C_n \). The number of such spacers is bounded above by \( r_nB \), and their width is \( (r_0 \ldots r_{n-1}r_n)^{-1} \), hence their total mass is \( \frac{B}{r_0 \ldots r_{n-1}} \). Hence the mass of \( C_n \) is bounded above by

\[
\mu(C_0) + B \sum_{n=1}^{\infty} \frac{1}{r_0 \ldots r_{n-1}}
\]

But as \( r_i \geq 2 \) this quantity is bounded above by \( 1 + 2B < \infty \), hence \( T \) cannot be infinite.

Proof. Let \( \{r_n\} \) be the cut sequence of \( T \) and let \( \{s_{n,k}\} \) be the spacer sequence of \( T \). Suppose that the spacer sequence is bounded, say \( s_{n,k} \leq B \). Given the total measure of \( C_n \), the total measure of \( C_{n+1} \) is the measure of \( C_n \) plus the total mass of the spacers placed above \( C_n \). The number of such spacers is bounded above by \( r_nB \), and their width is \( (r_0 \ldots r_{n-1}r_n)^{-1} \), hence their total mass is \( \frac{B}{r_0 \ldots r_{n-1}} \). Hence the mass of \( C_n \) is bounded above by

\[
\mu(C_0) + B \sum_{n=1}^{\infty} \frac{1}{r_0 \ldots r_{n-1}}
\]

But as \( r_i \geq 2 \) this quantity is bounded above by \( 1 + 2B < \infty \), hence \( T \) cannot be infinite.

Proof. Let \( T \) be an infinite rank-one transformation. Fix \( n \in \mathbb{Z}, i, j \in \mathbb{N} \) such that \( j \leq i \). Let \( I \) be the base of \( C_i \) and \( J \) be the base of \( C_j \). Let \( T^aJ \) be a level in \( C_j \), with \( 0 \leq a < h_j \). Then \( T^aJ \subset T^nI \) if and only if \( a \in n + D(I, j) \).

Proof. If \( a \in n + D(I, j) \), \( T^aJ \subset T^nI \) trivially. We prove the converse direction by cases.

First, assume \( n \geq 0 \). The first subcase is \( n \leq a \). Then, \( T^{a-n}J \subset I \), and \( T^{a-n}J \) is a level of \( C_j \), so \( a - n \in D(I, j) \), or \( a \in n + D(I, j) \). Now, suppose that \( n > a \). Then \( T^{a-n}J \) is not a level of \( C_j \), so if it is contained in \( I \), it is not immediate. In fact, we will show that it is impossible. Because the spacer sequence of \( T \) is unbounded, we can find some \( r \) such that \( C_{j+r} \) contains some descendant \( T^dK \) of \( J \) with a solid block of more than \( n \) spacers immediately below it.\(^3\) Further, we can assume that this descendant is more than \( a \) spaces from the top of \( C_{j+r} \), moving forward one column if necessary. Now, \( T^{d+a}K \) has \( a \) levels below it, followed by more than \( n \) spacers. If \( T^nI \supset T^aJ \supset T^{d+a}K \) then there are two possibilities: either there is some level \( L \) of \( C_{j+r} \), a descendant of \( I \), such that \( T^nL = T^{d+a}K \), or some descendants of \( I \) in \( C_{j+r} \) “overflow” (the top interval of a column maps into the bottom level of that same column under \( T \), and perhaps into spacers, “overflowing” into the base the column) under \( T^n \) from the top of the column. The former case is impossible; as \( n > a \) would have to lie in the sequence of \( n \) spacers, so it cannot be a descendant of \( I \). The latter case is also impossible;

\(^3\)As \( J \) is the base of \( C_j \), the base of every column \( C_k \) with \( k > j \) is a descendant of \( J \). Choose some \( r \) such that \( C_{j+r} \) has some subcolumn with more than \( n \) spacers on top of it, say it is subcolumn \( l \). If \( l \neq r_{j+r} - 1 \), then the \( l + 1 \)-st part of the base of \( C_{j+r} \) will have those spacers below it in \( C_{j+r+1} \). If \( l = r_{j+r} \), then \( C_{j+r+1} \) has more than \( n \) spacers on top of it, and so the second part of the base of \( C_{j+r+1} \) will have more than \( n \) spacers below it in \( C_{j+r+2} \).
if some subset of $C_{j+r}$ overflows to the bottom of the column under $T^n$, its image under $T^n$ can intersect at most the bottom $n$ levels of $C_{j+r}$. But $T^{d+a}K$ is more than $n$ levels from the bottom because of the spacers, so $T^{d+a}I \cap T^nI = \emptyset$, thus $T^aJ \not\subset T^nI$. Hence the $n \geq 0$ case is concluded.

Now assume $n < 0$, and write $m = -n > 0$. The first subcase is $m < h_j - a$. Then $m + a < h_j$, so that $T^{m+a}J = T^{-n}T^aJ \subset I$ is a level of $C_j$, so that $m + a \in D(I, j)$, that is, $a \in n + D(I, j)$. The second subcase is $m \geq h_j - a$, or, in the form we need later, $m > h_j - 1 - a$. Again, $T^{a+m}J$ is not a level of $C_j$, and again it is impossible that $T^aJ \subset T^nI$; the argument is almost identical to the one above. Because $T$ is infinite measure-preserving and $T^{a+(h_j-1-a)}J = T^{h_j-1}J$ is the top level of $C_j$, there is some $r$ and a descendant $T^dK$ of $T^{h_j-1}J$ in $C_{j+r}$ that has a block of more than $n$ spacers immediately above it, where $K$ is the base of $C_{j+r}$. We can further assume that this descendant is more than $h_j - 1 - a$ levels from the bottom of $C_{j+r}$: moving forward one column if necessary. By definition, $T^dK \subset T^{h_j-1}J$. Letting $d' = d - h_j + 1 + a$, $T^dK$ is a descendant of $T^aJ$. Further, its place in $C_{j+r}$ is like the place of the descendant of $a$ in the previous argument: it has $h_j - 1 - a$ levels above it, followed (again above it) by a block of more than $n$ spacers. Now, if $T^{-m}I$ contains $T^dK$, then either there is a $j + r$-descendant $L$ of $I$ above $T^dK$ such that $T^{-m}L = T^dK$, or some $j + r$-descendants of $I$ underflow (in the reverse of the process of overflow) from the top of $C_{j+r}$. The former case is impossible; as $m > h_j - 1 - a$, $L$ would have to lie in the block of spacers, hence it could not be a descendant of $I$. The latter case, too, is impossible, as it was above; if some subset of $C_{j+r}$ underflows under $T^{-m}$, its image under $T^{-m}$ can intersect only the top $n$ levels of $C_{j+r}$. But there are more than $n$ spacers above $T^dK$, hence more than $n$ levels. So $T^dK \cap T^nI = \emptyset$, thus $T^aJ \not\subset T^nI$. □

Notice that in the finite case, Lemma 2.5 will not hold. Let $T$ be the binary odometer [13]. Fix $n \in \mathbb{Z}$, let $i = j = 2$ and consider the base levels $I = [0, \frac{1}{2}), J = [0, \frac{1}{2})$. Clearly, by the construction of the binary odometer, $T^{2n}I = I$, and $D(J, 2) = \{0\}$. Consider $a = 0$, where $T^aJ$ is a level in $C_2$. Then, clearly $T^aJ = J = T^{2n}I \subset T^{2n}I$, but $0 \notin 2n + D(I, 2)$ for $n > 0$.

The following are immediate consequences of Lemma 2.5 and their proofs are left to the reader.

**Lemma 2.6.** Let $A = I \times \cdots \times I$ and $B = I \times T^{b_1}I \times \cdots \times T^{b_k-1}I$. Let $J$ be the base of $C_j$, and let $D = D(I, j)$. Suppose that $T^{a_0}J, \ldots, T^{a_k-1}J$
are levels of \( C_j \). Then \( T^{a_0} J \times \cdots \times T^{a_{k-1}} J \subset (T^{(k)})^n B \cap A \) if and only if \( a_0 \in (n+D) \cap D, a_1 \in (n+b_1+D) \cap D, \ldots, a_{k-1} \in (n+b_{k-1}+D) \cap D. \) \( \Box \)

**Lemma 2.7.** Let \( A = I \times I \) and \( B = I \times T^b I \). Let \( J \) be the base of \( C_j \), and let \( D = D(I, j) \). Suppose that \( T^{a_0} J \) and \( T^{a_1} J \) are levels of \( C_j \). Then we have that \( T^{a_0} J \times T^{a_1} J \subset U^n B \cap A \) if and only if \( a_0 \in (n+D) \cap D \) and \( a_1 \in (n+D) \cap D. \) \( \Box \)

The following two propositions form the link between the dynamics of rank-one transformations and the combinatorics of the locations of its levels, which we will exploit later in the paper.

**Proposition 2.8.** Let \( A = I \times \cdots \times I \) and \( B = I \times T^{b_1} I \times \cdots \times T^{b_{k-1}} I \), where \( I \) is the base of \( C_i \) and the \( b_i \) are (heights of) levels of \( C_i \). Let \( J \) be the base of \( C_j \) and suppose that \( T^{a_0} J \times \cdots \times T^{a_{k-1}} J \subset A, \) that is, \( a_0, \ldots, a_{k-1} \in D(I, j). \) Then \( T^{a_0} J \times \cdots \times T^{a_{k-1}} J \subset (T^{(k)})^n B \cap A \) for some \( n \) if and only if there are \( d_0, \ldots, d_{k-1} \in D(I, j) \) such that \( a_0 - d_0 = a_0 - d_0 - b_1, \) for each \( l = 2 \ldots (k-1). \)

**Proof.** We prove the \( k = 2 \) case for simplicity; the other cases are identical.

First, suppose \( T^{a_0} J \times T^{a_1} J \subset (T^{(2)})^n B \cap A \) for some \( n. \) By Lemma 2.6 we must have \( a_0 - a_1 = n - d_0 - n - b - d_1 = d'_1. \) Subtracting those equations, we get \( a_0 - a_1 = n + d_0 - n - b - d_1, \) or \( a_0 - a_1 = d_0 - d_1 - b. \)

Now we will show the converse. Suppose that \( a_0 - a_1 = d_0 - d_1 - b. \) Let \( n = a_0 - d_0; \) then \( n = a_1 - d_1 - b \) as well. But that means that \( a_0 = n + d_0 \) and \( a_1 = n + d_1 + b, \) and we already know \( a, a_0 \in D, \) so by Lemma 2.6 we get that \( T^{a_0} J \times T^{a_1} J \subset (T^{(2)})^n B \cap A. \) \( \Box \)

**Proposition 2.9.** Let \( A = I \times I \) and \( B = I \times T^b I, \) where \( T^b I \) is a level of \( C_i. \) Suppose that \( a_0, a_1 \) are (heights of) levels of \( C_j \) such that \( T^{a_0} J \times T^{a_1} J \subset A, \) that is, \( a_0, a_1 \in D(I, j). \) Then \( T^{a_0} J \times T^{a_1} J \subset U^n B \cap A \) for some \( n \) if and only if there are \( d_0, d_1 \in D(I, j) \) such that \( a_0 + a_1 = d_0 + d_1 + b. \)

**Proof.** First, suppose \( T^{a_0} J \times T^{a_1} J \subset U^n B \cap A \) for some \( n. \) Then by 2.7 we have that \( a_0 = d_0 = n + d_0 \) and \( a_1 = d_1 = -n + b + d'_1. \) Adding them, we get that \( a_0 + a_1 = d_0 + d_1 + b. \)

Now we address the converse; suppose that \( a_0 + a_1 = d_0 + d_1 + b. \) Defining \( n = a_0 - d_0, \) we get that \( n = -a_1 + d_1 + b, \) or \( a_0 = n + d_0 \) and \( a_1 = -n + d_1 + b, \) which combined with \( a_0, a_1 \in D \) gets us (by Lemma 2.7) that \( T^{a_0} J \times T^{a_1} J \subset S^n B \cap A. \) \( \Box \)

The following two lemmas are a generalization and a restatement of previous lemmas, with identical proofs.
Lemma 2.10. Let \( T \) be a rank-one transformation, and let \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k \). Let \( I \) be the base of \( C_i \) and \( J \) the base of \( C_j \), with \( j \geq i \). Let \( A = I \times \ldots \times I \) and let \( B = T^{b_1}I \times \ldots \times T^{b_k}I \), where \( b_i \in C_i \). Let \( a_1, \ldots, a_k \in D(I, j) \). Then \( \Gamma^{a_1}J \times \ldots \times \Gamma^{a_k}J \subset (\Gamma^\alpha)^n B \cap A \) for some \( n \) if and only if there are \( d_1, \ldots, d_k \in D(I, j) \) such that

\[
\frac{a_1 - d_1 - b_1}{\alpha_1} = \frac{a_i - d_i - b_i}{\alpha_i}
\]

for every \( i \).

Lemma 2.11. Let \( T \) be a rank-one transformation. Let \( I \) be the base of \( C_i \) and \( J \) the base of \( C_j \), with \( j \geq i \). Let \( A = I \times I \) and let \( B = I \times T^{b}I \), where \( b \in C_i \). Let \( a_1, a_2 \in D(I, j) \). Then \( \Gamma^{a_1}J \times \Gamma^{a_2}J \subset (\Gamma \times T^{-1})^n B \cap A \) for some \( n \) if and only if there are \( d_1, d_2 \in D(I, j) \) such that

\[
a_1 + a_2 = d_1 + d_2 + (b_1 + b_2)
\]

3. Combinatorics

The following lemma is used to construct the sets \( H_k \).

Lemma 3.1. Let \( M, \Gamma, \gamma \in \mathbb{N} \). Then there are sets of nonnegative integers \( H(U), H(L) \), where \( H(U) = \{\{V_1, W_1\}, \ldots, \{V_\Gamma, W_\Gamma\}\} \) and \( H(L) = \{\{v_1, w_1\}, \ldots, \{v_\Gamma, w_\Gamma\}\} \), and letting

\[
H = \{V_i, W_j, v_k, w_\ell | 1 \leq i, j \leq \Gamma, 1 \leq k, \ell \leq \gamma\},
\]

\( H \) satisfies the following properties:

1. For every \( \{V, W\} \in H(U) \) and \( \{v, w\} \in H(L) \) we have \( V + W = v + w - 1 \).
2. If \( x_1, x_2, x_3, x_4 \) are in \( H \) and \( |x_1 + x_2 - x_3 - x_4| < M \), then precisely one of the following holds:
   - \( \{x_1, x_2\} = \{x_3, x_4\} \)
   - \( \{x_1, x_2\} \neq \{x_3, x_4\} \) but \( x_1 + x_2 = x_3 + x_4 \), in which case \( \{x_1, x_2\} \) and \( \{x_3, x_4\} \) are both in either \( H(U) \) or \( H(L) \),
   - \( x_1 + x_2 = x_3 + x_4 - 1 \), in which case \( \{x_1, x_2\} \in H(U) \) and \( \{x_3, x_4\} \in H(L) \), or
   - \( x_1 + x_2 = x_3 + x_4 + 1 \), in which case \( \{x_1, x_2\} \in H(L) \) and \( \{x_3, x_4\} \in H(U) \).

Proof. We proceed by finding a set \( H \) such that \( V_r + W_r = v_s + w_s \) for all \( r \in \{1, \ldots, \Gamma\} \) and \( s \in \{1, \ldots, \gamma\} \), and such that \( |x_1 + x_2 - x_3 - x_4| < 1 \) with distinct summands implies \( \{x_1, x_2, x_3, x_4\} \) is one of \( \{v_s, w_s, v_s', w_s'\}, \{v_r, w_r, v_r', w_r'\} \), or \( \{V_r, W_r, V_r', W_r'\} \). For this construction of \( H \) when \( M = 1 \), choose \( n \gg 2^{(\Gamma + \gamma)} \) and even, and let

\[
H := \{2, \ldots, 2^{\Gamma + \gamma}, n - 2^{\Gamma + \gamma}, \ldots, n - 2\},
\]
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where $H(U) = \{\{2, n - 2\}, \ldots, \{2^\Gamma, n - 2^\Gamma\}\}$ and $H(L) = \{\{2^{\Gamma+1}, n - 2^{\Gamma+1}\}, \ldots, \{2^{\Gamma+\gamma}, n - 2^{\Gamma+\gamma}\}\}$. For $r \in \{1, \ldots, \Gamma\}$ let $V_r = 2^r$, $W_r = n - 2^r$ and for $s \in \{1, \ldots, \gamma\}$ let $v_s = 2^{\Gamma+s}$ and $w_s = n - 2^{\Gamma+s}$.

Now, partition $H$ into sets $R_1 = \{2, \ldots, 2^{\Gamma+\gamma}\}$ and $R_2 = \{n - 2^{\Gamma+\gamma}, \ldots, n - 2\}$. Note that, given four elements $x_1, x_2, x_3, x_4 \in H$ with $|x_1 + x_2 - x_3 - x_4| < M = 1$, we have $x_1 + x_2 = x_3 + x_4$. Suppose that $x_1, x_2 \in R_1$: that is, $x_1 = 2^z$ and $x_2 = 2^z$, for integers $0 \leq z_1, z_2 \leq \Gamma + \gamma$. Then $x_1 + x_2 \leq 2^{\Gamma+\gamma+1} \ll n - 2^{\Gamma+\gamma}$, so $x_3$ and $x_4$ are also both in $R_1$. By unique binary expansion, either $z_1 = z_3$ and $z_2 = z_4$ or $z_1 = z_4$ and $z_2 = z_3$. Then $\{x_1, x_2\} = \{x_3, x_4\}$, so we obtain the first subcase above. Suppose that $x_1 \in R_1$, $x_2 \in R_2$. Then $x_3$ and $x_4$ are not both in $R_1$ and the size of $n$ dictates that precisely one of $\{x_3, x_4\}$ is in $R_1$. Without loss of generality write $x_1 = 2^{z_1}, x_2 = n - 2^{z_2}, x_3 = 2^{z_3}$, and $x_4 = n - 2^{z_4}$ where $z_1, z_2, z_3, z_4 \in \{1, \ldots, \Gamma + \gamma\}$. Then we obtain $2^{z_1} + 2^{z_2} = 2^{z_3} + 2^{z_4}$, implying that either $z_1 = z_2$ and $z_3 = z_4$ or $z_1 = z_3$ and $z_2 = z_4$. In the former case $x_1, x_2$ are a pair $\{v, w\}$ or $\{V, W\}$ and $x_3, x_4$ also form such a pair; in the latter case because then $x_1 = x_3$ and $x_2 = x_4$, so $\{x_1, x_2\} = \{x_3, x_4\}$. Symmetry addresses the case where $x_1 \in R_2, x_2 \in R_1$. Finally, if $x_1, x_2 \in R_2$ then both $x_3$ and $x_4$ are in $R_2$; setting $x_1 = n - 2^{z_1}$, $x_2 = n - 2^{z_2}$, $x_3 = n - 2^{z_3}$, and $x_4 = n - 2^{z_4}$, we see that $2^{z_1} + 2^{z_2} = 2^{z_3} + 2^{z_4}$, which again implies the first subcase. Hence, $H$ conforms to its stated condition.

Fix a $M \in \mathbb{N}$ with $M \geq 2$. Multiply every element in $H$ by $M$, and then subtract 1 from all of the elements obtained from multiplying $M$ with a $V_r$. Call $V'_r = M \cdot V_r - 1$, $W'_r = M \cdot W_r$, and so on. Call the set containing these new pairs $H'$. Suppose that $y_1, y_2, y_3, y_4$ are distinct elements in $H'$ with $|y_1 + y_2 - y_3 - y_4| < M$. Let $x_1, x_2, x_3, x_4$ be their corresponding elements in $H$. By adding 1 to all $y$ terms of the form $V'_r$, we obtain that $|Mx_1 + Mx_2 - Mx_3 - Mx_4| < M + 2$, whence $|x_1 + x_2 - x_3 - x_4| < 1 + \frac{2}{M} \leq 2$. So $|x_1 + x_2 - x_3 - x_4| = 0$ or 1. But recall that $n$ was chosen to be even, so $|x_1 + x_2 - x_3 - x_4| = 0$. Thus, the pairs $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are either both in $H(U)$ or $H(L)$ or are split evenly between them, which implies the same for $\{y_1, y_2\}$ and $\{y_3, y_4\}$ in $H(U)'$ and $H(L)'$. Hence, $H'$ is our desired set for any given $M$, when we let $H(U)'$ be the set of pairs $\{V'_r, W'_r\}$ and $H(L)'$ be the set of pairs $\{v'_s, w'_s\}$.

**Remark 1.** Using Lemma 3.1, we can construct the height sets $H_k$ of our transformation inductively. Choose

$$M_k \gg 2 \max D(I, k) = 2 \max (H_0 \oplus H_1 \oplus \cdots \oplus H_{k-1})$$
(the only restriction on \(H_k\) is that the difference between any two elements of \(H_k\) should be larger than \(h_{k-1}\), which can easily be ensured). Let \(\{\Gamma_k\}, \{\gamma_k\} \subset \mathbb{Z}\), and in the input to the above lemma let the number of pairs in \(H_k(U)\) be \(\Gamma_k\) and the number of pairs in \(H_k(L)\) be \(\gamma_k\). The output \(H'\) of Lemma 3.7 will be our height set \(H_k\) for column \(C_k\). As of yet, let \(\{\Gamma_k\}\) and \(\{\gamma_k\}\) remain unspecified; we’ll choose them towards the end of our construction, for clarity.

For reasons that will become more clear in the following section, we need to categorize the pairs of elements of \(H_k\).

**Definition 1.** Let \(H\) be as in Lemma 3.7. A pair \(\{x, y\} \in H \times H\) is called mixed if \(x = V_i\) or \(W_j \in H(U)\), and \(y = v_k\) or \(w_l \in H(L)\), or vice-versa. A mixed pair is called positive if it is of the form \((w_j, W_i), (w_j, V_i), (v_j, V_i)\) or \((v_j, W_i)\). A pair is called negative if it is of the form \((V_i, v_j), (V_i, w_l), (W_i, v_j)\) or \((W_i, v_j)\). A pair \(\{x, y\} \in H\) is called pure if \(\{x, y\} \in H(L)\) or \(\{x, y\} \in H(U)\). Notice that the pure pairs are unordered, whereas the mixed pairs are ordered (and are positive or negative depending upon the order of the elements).

The use of the words “positive” and “negative” is meant to be evocative. Let \(a, a' \in D(I, j)\), and let \(b\) be fixed. We can write \(a = \sum_{k=1}^{j-1} a_k\) where \(a_k \in H_k\), by the decomposition \(D(I, j) = H_i \oplus \cdots \oplus H_j-1\). As established in the preceding lemmas, we are interested in necessary and sufficient conditions for, for instance, the existence of \(d, d' \in D(I, j)\) such that \(a - d = a' - d' - b\). If there are \(b\) indices \(k\) such that \(\{a_k, a'_k\}\) is negative mixed, then we can satisfy this condition; choose \(d_k, d'_k\) to be the corresponding positive mixed pair to get \(a_k - d_k = a'_k - d'_k - 1\) for those \(b\) indices, and for the remainder set \(d_k = a_k\) and \(d'_k = a'_k\). There is a similar idea for dealing with the condition relating to \(U\), that is, \(a + a' = d + d' = b\).

**Lemma 3.2.** Let \(n\) be fixed and \(M_k\) an increasing sequence with \(M_0 > 1\). Let \(I\) be the base level of \(C_i\), where \(i < n\), and suppose that \(a + a' = d + d' + 1\), with \(a, a', d, d' \in D(I, n)\). Write \(a = \sum_{k=1}^{n-1} a_k\) with \(a_k \in H_k\), and similarly for \(d, d', a'\). Then there is a \(k\) in \(\{i, \ldots, n - 1\}\) such that \(\{a_k, a'_k\} \in H_k(U)\) and \(\{d_k, d'_k\} \in H_k(L)\), or vice-versa.

**Proof.** We clearly cannot have \(a_k + a'_k = d_k + d'_k\) for each \(k\), so choose the largest \(k\) such that equality does not hold. Recall that \(M_k\) is the constant used to construct \(H_k\) in Lemma 3.1 and was chosen to be \(\geq 2 \max D(I, k)\) in Remark 1. The first case is \(|a_k + a'_k - d_k + d'_k| < M_k\). So, we have that \(\{a_k, a'_k\}\) and \(\{d_k, d'_k\}\) must be pairs in \(H_k(U)\) and \(H_k(L)\). So we have \(\{a_k, a'_k\} \in H_k(U)\) and \(\{d_k, d'_k\} \in H_k(L)\) or \(\{a_k, a'_k\} \in H_k(L)\) and \(\{d_k, d'_k\} \in H_k(U)\).
The second case is when $|a_k + a'_k - d_k - d'_k| \geq M_k \gg 2 \max D(I, k)$. We have

$$|a + a' - d - d'| = \left| \sum_{j=1}^{n-1} a_j + \sum_{j=1}^{n-1} d_j - \sum_{j=1}^{n-1} a'_j - \sum_{j=1}^{n-1} d'_j \right|$$

$$= \left| \sum_{j=1}^{k} (a_j + d_j - a'_j - d'_j) \right|$$

$$\geq |a_k + d_k - a'_k - d'_k| - \sum_{j=1}^{k-1} |(a_j + d_j - a'_j - d'_j)|$$

$$\geq M_k - 2 \sum_{j=1}^{k-1} \max H_j$$

$$= M_k - 2 \max D(I, k) \gg 1,$$

which contradicts the initial assumption, concluding the lemma. \qed

4. For each rank-one $T$, $T \times T^{-1}$ is conservative

We note that there exist rank-one transformations $T$ such that $T \times T$ is not conservative [2], as well as infinite measure-preserving transformations where $T \times T^{-1}$ is not conservative (Corollary 7.6). The following lemma, and its proof, are similar to Lemma 2.1 which provides a sufficient condition for ergodicity of products of rank-one transformations. Its analogue for more products is also true, but we leave that proof to the reader, giving only the two-fold products case to highlight the difference between the conservativity and ergodicity proofs.

**Lemma 4.1.** Let $T_1, \ldots, T_k : X \to X$ be rank-one transformations and let $D$ be the sufficient semiring of rectangles whose sides are levels of $T_1, \ldots, T_k$. Suppose that $S = T_1 \times \cdots \times T_k$ is conservative on $D$, that is, for every $A \in D$ we have $A \subset \cup_{n \neq 0} S^n A \pmod{\mu}$. Then $S$ is conservative.

**Proof.** This proof is almost the same as the corresponding lemma for ergodicity on levels, Lemma 2.1 and we only prove the $k = 2$ case, leaving the general case to the reader; it is identical to the proof provided in Lemma 2.1. Notice that we use the same reduction as in this lemma to prove our result only on $D$. Let $S = T_1 \times T_2$. Suppose, by way of contradiction, that there is a set $E$ such that $E \cap S^n E = \emptyset$ for each $n \neq 0$. Choose $A \in D$ with $A \subset_{99} E$. Given some $\varepsilon > 0$, choose $m$ so large that $A \subset \cup_{n \in \Lambda_m} S^n A$ except for a measure of at most $\varepsilon$, where $\Lambda_m = \{-m, \ldots, -1, 1, \ldots, m\}$. Divide $A$ into sub-rectangles of
the same measure such that all intersections of the form \( A \cap S^n A \) for \( n \in \Lambda_m \) are composed of such rectangles, except for measure at most \( \epsilon \). Let \( K \) be the number of such rectangles. Then \( 0.9K \) of the rectangles \( R \) are such that \( R \subset_{0.5} E \). I claim that at most \( 0.3K \) of the rectangles are used in the covering of \( A \) by \( \cup_{n \in \Lambda_m} S^n A \). For, if \( 0.3K \) of them are used, then \( 0.2K \) of them must be \( \subset_{0.5} E \), whence because only \( 0.1K \) of the rectangles of not \( \subset_{0.5} E \), for some rectangle \( R, R' \) of \( A \) such that \( R, R' \subset_{0.5} E \) we have that \( S^n R = R' \). This is a contradiction. But because only \( 0.3K \) of the rectangles of \( A \) are used in the covering of \( A \) by \( \cup_{n \in \Lambda_m} S^n A \), by symmetry this covering must cover at most \( 0.3K \) of the rectangles of \( A \), which is a contradiction for small \( \epsilon \). \( \square \)

The following lemma is almost identical to Lemma 4.2, which provides the analogous condition for ergodicity of products of rank-one transformations.

**Lemma 4.2.** Let \( T \) be a rank-be transformation, and let \( A = I \times I \), where \( I \) is the base of some column \( C_i \). Then, for \( T_1 \) and \( T_2 \) equal to \( T \) or \( T^{-1} \), \( S = T_1 \times T_2 \) is conservative if and only if for every \( \epsilon > 0 \) there is \( j \) such that at for at least \( (1 - \epsilon)|D(I, j)|^2 \) of the pairs \( (a_0, a_1) \in D(I, j)^2 \), \( T^{a_0} J \times T^{a_1} J \subset S^n A \cap A \) for some \( n \neq 0 \).

**Proof.** Fix \( \epsilon > 0 \). First, if \( S \) is conservative, we can find some \( m \) such that \( A \) is covered by \( \bigcup_{m,m \neq 0} S^n A \) except for measure \( \frac{\epsilon}{2} \mu(A) \). Then, we may choose \( j \) large enough that, up to measure \( \frac{\epsilon}{2} \mu(A) \), all intersections \( S^n A \cap A \) are composed of unions of rectangles with sides that are levels of \( C_j \). Clearly these levels must be descendants of \( A \), that is, in \( D(I, j) \). This gives us that at least \( (1 - \left( \frac{\epsilon}{2} \right)^k |D(I, j)|^k \) of the rectangles are contained in \( S^n A \cap A \). That is to say, at least \( (1 - \epsilon)|D(I, j)|^k \) of the pairs \( (a_0, a_1) \in D(I, j)^k \) will satisfy \( T^{a_0} J \times \cdots \times T^{a_{k-1}} J \subset S^n A \cap A \) for some \( n \neq 0 \).

Now, suppose the conditions of the lemma hold for \( S \). Then we may choose \( m \) so large that, up to measure \( \epsilon \mu(A) \), all of the \( (1 - \epsilon)|D(I, j)|^k \) of the rectangles are contained in \( \bigcup_{m,m \neq 0} S^n A \cap A \). Then, clearly \( A \) is covered, up to measure \( \epsilon \mu(A) \), by \( \bigcup_{m,m \neq 0} S^n A \). By Lemma 4.1 then, \( S \) is ergodic. \( \square \)

**Theorem 4.3.** Let \( T \) be a rank-one transformation. Then \( T \times T^{-1} \) is conservative.

**Proof.** Let \( A = I \times I \), where \( I \) is the base of a column \( C_i \). It suffices to show (by Proposition 2.9) that for every \( \epsilon > 0 \) there is \( j \) such that with probability at least \( 1 - \epsilon \), a pair \( (a_0, a_1) \in D(I, j) \) has a corresponding pair \( (d_0, d_1) \in D(I, j) \) such that \( a_0 \neq d_0 \) and \( a_0 + a_1 = d_0 + d_1 \).
Suppose that \( a_0 \neq a_1 \). Let \( d_0 = a_1 \) and \( d_1 = a_0 \). Then \( d_0 \neq a_0 \) and \( d_0 + d_1 = a_0 + a_1 \), as required. The number of pairs such that \( a_0 = a_1 \) is \( |D(I, j)| \), hence the probability that a pair \((a_0, a_1)\) has a corresponding pair \((d_0, d_1)\) is at least
\[
1 - \frac{|D(I, j)|}{|D(I, j)|^2}
\]
and this quantity goes to 1 as \( j \to \infty \), which concludes the proof. \( \square \)

5. \( T \times T \) ergodic but \( T \times T^{-1} \) not ergodic

In this section we construct a class of rank-one transformations \( T \) such that \( T \times T \) is ergodic but \( T \times T^{-1} \) is not ergodic. To obtain ergodicity of the cartesian square we just need \( \gamma_k = \Gamma_k \) for all \( k \) with arbitrary \( \Gamma_k \).

**Theorem 5.1.** Let \( T \) be defined using \( \gamma_k = \Gamma_k \). Then \( T^{(2)} \) is ergodic.

**Proof.** We will apply Lemma 2.2. To do so, we must show that for any \( \varepsilon > 0 \) there is \( j \in \mathbb{N} \) such that \((1 - \varepsilon)|D(I, j)|^2\) of the pairs \( \{a, a'\} \in D(I, j)^2 \) satisfy \( T^a J \times T^{a'} J \subset (T^{(2)})^n B \cap A \) for some \( n \). By Lemma 2.6, the latter happens if and only if \( a - a' = d - d' - b \) (recall that \( b \) is a constant depending on \( B \)). So, by Lemma 2.2 what we must show is that there is \( j \) such that the probability that some pair \( \{a, a'\} \in D(I, j)^2 \) satisfies
\[
a - a' = d - d' - b
\]
for some \( \{d, d'\} \in D(I, j)^2 \) is at least \( 1 - \varepsilon \).

For such an \( a, a' \), we can use the decomposition \( D(I, j) = H_l \oplus \cdots \oplus H_{l-1} \) to write \( a = \sum_{k=l}^{j-1} a_k \) and \( a' = \sum_{k=l}^{j-1} a'_k \), with \( a_k, a'_k \in H_k \). Suppose that there are \( b \) mixed negative pairs \( (a_\ell, a'_\ell) \). Then by definition for each there are \( d_\ell, d'_\ell \in H_l \) such that \( a_\ell - a'_\ell = d_\ell - d'_\ell - 1 \). For \( b \) of those \( l \) such that \( (a_\ell, a'_\ell) \) are negative mixed, set let \( d_\ell, d'_\ell \) be as in the above equation, and for the others, let \( d_\ell = a_\ell, d'_\ell = a'_\ell \); we’ll clearly have \( a - a' = d - d' - b \). So, the probability that \( a, a' \) is a pair satisfying \( a - a' = d - d' - b \) is at least the probability that the expansions of \( a, a' \) contain \( b \) negative mixed pairs.

We are then interested in computing that probability. Write \( H_l(U) = \{(A_1, B_1), \ldots, (A_{\gamma_l}, B_{\gamma_l})\} \) and \( H_l(L) = \{(a_1, b_1), \ldots, (a_{\gamma_\ell}, b_{\gamma_\ell})\} \). The total number of pairs in \( H_l \) is \((4\gamma_\ell)^2 = 16\gamma_\ell^2 \), and the number of negative mixed pairs is \( \gamma_\ell \cdot 2\gamma_\ell + \gamma_\ell \cdot 2\gamma_\ell = 4\gamma_\ell^2 \), hence the probability that some pair in \( H_l \) is negative mixed is 1/4. Let \( E_\ell \) be the event that \( (a_\ell, a'_\ell) \) is negative mixed. Let \( S_j = \{i, \ldots, j-1\} \). The probability that there are
at least $b$ negative mixed pairs (NMP) in the decompositions of $a, a'$ is then

$$
P\{\text{at least } b \text{ NMP}\} = 1 - P\{\text{fewer than } b \text{ NMP}\}
= 1 - \sum_{n=0}^{b-1} \sum_{\Lambda \subseteq S_j, |\Lambda| = n} \left( \prod_{\ell \in \Lambda} \mathbb{P}E^c_{\ell} \right) \left( \prod_{\ell \notin \Lambda} \mathbb{P}E_{\ell} \right)
= 1 - \sum_{n=0}^{b-1} \sum_{\Lambda \subseteq S_j, |\Lambda| = n} \left( \frac{1}{4} \right)^n \left( \frac{3}{4} \right)^{j-1-i-n}
= 1 - \sum_{n=0}^{b-1} \binom{j-i-1}{n} \left( \frac{1}{4} \right)^n \left( \frac{3}{4} \right)^{j-1-i-n}
= 1 - \left( \frac{3}{4} \right)^{j-i-1} \sum_{n=0}^{b-1} p_n(j)
= 1 - \left( \frac{3}{4} \right)^{j-i-1} \lim_{j \to \infty} p_b(j)
\to 1
$$

where $p_n$ is a degree-$n$ polynomial, and $p_b$ is a degree-$(b-1)$ polynomial. Geometric growth is faster than polynomial growth, hence the last line.

\begin{proof}

\begin{theorem}
Let $T$ be a rank-one transformation constructed using a sequence $0 < \{\gamma_\ell\}$ that satisfies

$$0 < \prod (1 - 1/4\gamma_\ell)$$

and $\gamma_k = \Gamma_k$ for all $k$. Then $T \times T$ is ergodic but $U = T \times T^{-1}$ is not ergodic.

\end{theorem}

\begin{proof}
Ergodicity of $T \times T$ follows from Theorem 5.1. We will proceed by contradiction by supposing that $U$ is conservative ergodic. Letting $I$ be the base of an arbitrary column $C_i$, let $A = I \times I$ and $B = I \times TI$ (that is, choose $b = 1$). Then for every $\varepsilon > 0$ there exists $j$ such that for at least $(1 - \varepsilon)|D(I, j)|^2$ pairs of descendants $a, a' \in D(I, j)^2$, we have ($T^a J, T^{a'} J) \subset U^n B \cap A$ for some $n$. By Proposition 2.9, the latter happens if and only if there are $d, d' \in D(I, j)$ such that $a + a' = d + d' + 1$

As in Lemma 3.2 there must be some $k \in \{i, \ldots, n-1\}$ such that $(a_k, a'_k)$ is a pure pair. But there are only $2(2\gamma_k)$ possible pure pairs in

H_k out of 16γ_k^2 total pairs. So we may write
\[ P\{\text{at least one pure pair}\} = 1 - P\{\text{no pure pairs}\} = 1 - \prod_{i=1}^{j-1} \left( 1 - \frac{1}{4\gamma_i} \right) \]
Since this quantity is strictly less than 1, for small ε, this contradicts ergodicity of U.

Regarding ergodicity of higher products, we note that T × T × T ergodic is equivalent to the statement that for any b_1, b_2, b_3 and I the base of some column, the probability that some triple (a_1, a_2, a_3) ∈ D(I, j)^3 has a corresponding triple (d_1, d_2, d_3) ∈ D(I, j)^3 with
\[ a_1 - d_1 - b_1 = a_2 - d_2 - b_2 = a_3 - d_3 - b_3 \]
goes to 1 as j → ∞. We can write this in a slightly nicer form, letting b_1 = b and b_2 = b_3 = 0, as
\[ a_1 + d_2 = a_2 + d_1 + b \\
 a_1 + d_3 = a_3 + d_1 + b. \]
While we can obtain transformations with this condition, it seems that it is not compatible with the corresponding condition for T × T^{-1} not ergodic.

6. Higher Products

In this section, we construct a T such that for any α = (α_1, . . . , α_k) with all α_i distinct, T^α = T^{α_1} × . . . × T^{α_k} is ergodic, but T × T^{-1} is not ergodic. It suffices to take α_1 = 1, for we can simply pass to a higher product to obtain the more general result: if T × T^{α_2} × . . . × T^{α_k} is ergodic, then so is T^{α_2} × . . . × T^{α_k}. In this case, the condition in Lemma 2.10 specializes to
\[ α_i a_1 - α_i d_1 = a_i - d_i + α_i b_1 - b_i \]
for each i. By Lemma 2.10, T^α is ergodic if and only if the probability that given some k-tuple (a_1, . . . , a_k) ∈ D(I, j)^k there is a k-tuple (d_1, . . . , d_k) ∈ D(I, j)^k with that system of equations satisfied.

**Lemma 6.1.** Let M > 4 be a natural number, let α be a k-tuple of natural numbers with all entries distinct and α_1 = 1, and let δ ∈ \{0, 1, -1\}^k. Then there is a set H = {V, w, v_2, W_2, . . . , v_k, W_k} such that
\[ α_i V - α_i w + δ_1 = v_i - W_i + δ_i \]
In case 4, large enough \( N \) the right hand side is on the order of \( 2^H \). Assume distinctness, let \( \alpha_i \) where uniqueness of \( a, b, c, d \) again guarantees that \( \{a, b\} = \{c, d\} \).

Proof. We first find a set \( H = \{V, w, v_2, W_2, \ldots, v_k, W_i\} \) such that

\[
\alpha_i V + W_i = v_i + \alpha_i w
\]

for each \( i \), and \( a + b = c + d \) implies \( \{a, b\} = \{c, d\} \). Let \( N \) be a large power of \( \alpha_1 \cdots \alpha_k \), let \( V = \alpha_1 \cdots \alpha_k \), let \( w = (\alpha_1 \cdots \alpha_k)^3 \), let \( W_i = N - \alpha_i(\alpha_1 \cdots \alpha_k) \), and let \( v_i = N - \alpha_i(\alpha_1 \cdots \alpha_k)^3 \). Suppose that \( a, b, c, d \in H \) and \( a + b = c + d \). If two elements of \( a, b, c, d \) are equal, then without loss of generality either \( a = c \) or \( a = b \). In the first case we have \( b = d \) so \( \{a, b\} = \{c, d\} \). In the second case, \( 2a = c + d \), and it’s easy to check that this cannot happen unless \( c = d = a \). Thus, we can take all \( a, b, c, d \) to be distinct. Now, define \( L = \{V, w\} \) and \( R = \{W_i, v_i\} \). Roughly, they are the “small” and “large” elements of \( H \). Up to obvious symmetries, there are five cases:

1. \( a, b, c \in L, d \in R \)
2. \( a, b \in L, c, d \in R \)
3. \( a, c \in L, b, d \in R \)
4. \( a \in L, b, c, d \in R \)
5. \( a, b, c, d \in R \)

where uniqueness of \( a, b, c, d \) eliminates the case where all elements are in \( L \). Cases 1 and 2 can be dismissed out of hand, as then \( c + d \gg a + b \).

In case 4, large enough \( N \) again guarantees that \( a + b \ll c + d \), as the right hand side is on the order of \( 2N \) while the left is only \( N \). In case 5, uniqueness of base-(\( \alpha_1 \cdots \alpha_k \)) expansion of integers gives that \( \{a, b\} = \{c, d\} \). The only remaining case is case 3. As we can assume distinctness, let \( a = V = \alpha_1 \cdots \alpha_k \), \( c = w = (\alpha_1 \cdots \alpha_k)^3 \), \( b = N - \alpha_i(\alpha_1 \cdots \alpha_k)^p_1 \), \( d = N - \alpha_i(\alpha_1 \cdots \alpha_k)^p_2 \). Our equality then reduces to

\[
\alpha_1 \cdots \alpha_k + N - \alpha_i(\alpha_1 \cdots \alpha_k)^p_1 = (\alpha_1 \cdots \alpha_k)^3 + N - \alpha_i(\alpha_1 \cdots \alpha_k)^p_2
\]

with \( i_1 \neq i_2 \) or \( p_1 \neq p_2 \), or rather,

\[
\alpha_1 \cdots \alpha_k + \alpha_i(\alpha_1 \cdots \alpha_k)^p_2 = (\alpha_1 \cdots \alpha_k)^3 + \alpha_i(\alpha_1 \cdots \alpha_k)^p_1
\]

but again, as \( \alpha_i > 1 \) for all \( i \) and all \( \alpha_i \) are different (and different from 1), uniqueness of base-(\( \alpha_1 \cdots \alpha_k \)) expansion gives a contradiction.

Now, let \( M \in N \). Let \( V' = M \cdot V, W'_i = M \cdot W_i, v'_i = M \cdot v_i - \delta_i + \delta_1 \), and \( w' = M \cdot w \). Let \( H' = \{V', w', v'_2, W'_2, \ldots, v'_k, W'_k\} \). Suppose that \( a', b', c', d' \in H' \) with \( |a' + b' - c' - d'| \leq M \). Let \( a, b, c, d \) be the elements
of $H$ corresponding to $a', b', c', d'$, respectively. Then $|a + b - c - d| \leq (M + 4)/M = 1 + 4/M$, using the fact that $|\delta_i| \leq 1$ for every $i$. As $M > 4$ we must have that $|a + b - c - d|$ is either 0 or 1. But every element of $H$ is divisible by $\alpha_1 \cdots \alpha_k$ so that $a + b = c + d$, hence $\{a, b\} = \{c, d\}$, hence $\{a', b'\} = \{c', d'\}$. \hfill \Box$

Now, construct the sets $H_k$ inductively, using Lemma 6.1. The input value $M$ to that lemma will always be $M_k \gg 2 \max D(I, k) = 2 \max (H_0 \oplus \ldots \oplus H_{k-1})$. Now, let $A_k$ be the set of $\alpha \in \mathbb{N}^k$ such that all coordinates of $\alpha$ are different and the first coordinate is 1. Clearly $A_k$ is countable. Let $D_k = \{-1, 0, 1\}^k$, a finite set. Then $S_k = A_k \times D_k$ is countable as well, and hence $S = \bigcup_{k=2}^\infty S_k$ is countable as well. This is simply the set of all possible multi-indices $\alpha$ with all different coordinates and first coordinate 1, paired with all possible $-1-0-1$ vectors $\delta$, that is, the set of all possible inputs to Lemma 6.1. Let $\{\alpha^{(i)}, \delta^{(i)}\}_{i=1}^\infty$ be an enumeration of $S$. We construct the sets $H_k$ through Lemma 6.1 as follows. For $k$ that are multiples of 2, input $(\alpha^{(1)}, \delta^{(1)})$. For $k$ that are multiples of 3 but not of 2, input $(\alpha^{(2)}, \delta^{(2)})$. In general, for $k$ that are multiples of $p_n$ (the $n$th prime) but not of any smaller prime, input $(\alpha^{(n)}, \delta^{(n)})$. The effect of this is that for any fixed $\alpha$ and $\delta$, the set of $k$ such that $H_k$ is constructed using $\alpha$ and $\delta$ as inputs forms an infinite arithmetic progression in $\mathbb{N}$. This will be very important for showing that $T^a$ is ergodic.

The next lemma allows us to show that for the $T$ just constructed, $T \times T^{-1}$ is not ergodic.

**Lemma 6.2.** Let $n$ be fixed and let $M_k$ be an increasing sequence with $M_0 > 2$. Let $I$ be the base of $C_i$, where $i < n$. If $a, a', d, d' \in D(I, n)$, it is not possible that $a + a' = d + d' + 1$.

**Proof.** Suppose that indeed $a + a' = d + d' + 1$. Decompose $a = \sum_{k=i}^{j-1} a_k$, where $a_k \in H_k$, and similarly for $a', d, d'$. As $a + a' \neq d + d'$, there is some largest index $k$ such that $a_k + a'_k \neq d_k + d'_k$. By Lemma 6.1 we
must have \(|a_k + a'_k - d_k - d'_k| > M_k\). In that case,

\[
|a + a' - d - d'| = \left| \sum_{j=1}^{n-1} a_j + \sum_{j=1}^{n-1} d_j - \sum_{j=1}^{n-1} a'_j - \sum_{j=1}^{n-1} d'_j \right|
\]

\[
= \left| \sum_{j=1}^k (a_j + d_j - a'_j - d'_j) \right|
\]

\[
\geq |a_k + d_k - a'_k - d'_k| - \left| \sum_{j=1}^{k-1} (a_j + d_j - a'_j - d'_j) \right|
\]

\[
> M_k - 2 \max D(I, k)
\]

\[\Rightarrow 2 \max D(I, k) - 2 \max D(I, k) = 0\]

which contradicts the initial assumption, concluding the lemma. \(\square\)

We finally have the main result of this section.

**Theorem 6.3.** Let \(T\) be defined as above. Then for any \(k\)-tuple \(\alpha = (\alpha_1, \ldots, \alpha_k)\) with \(\alpha_i \geq 1\), all \(\alpha_i\) unique, and \(\alpha_1 = 1\), \(T^\alpha\) is ergodic, but \(T \times T^{-1}\) is not ergodic.

**Proof.** Let \(\alpha\) be fixed as in the hypotheses, let \(I\) be the base of \(C_i\), let \(B = T^{b_1} I \times \ldots \times T^{b_k} I\), and let \(A = I \times \ldots \times I\). We will show that for any \(\varepsilon > 0\) there is \(j \in \mathbb{N}\) such that with probability at least \(1 - \varepsilon\), a tuple \((a_1, \ldots, a_k) \in D(I, j)^k\) has a corresponding tuple \((d_1, \ldots, d_k) \in D(I, j)^k\) such that

\[
\alpha_i a_1 - \alpha_i d_1 = a_i - d_i + \alpha_i b_1 - b_i
\]

for every \(i\). Let \(b'_i = \alpha_i b_1 - b_i\), and without loss of generality, assume that \(b'_i \geq 0\) for every \(i\). The argument in the general case simply involves more notation when dealing with the \(\delta_i\).

For a tuple \((a_1, \ldots, a_k)\), we can use the decomposition \(D(I, j) = H_i \oplus \ldots \oplus H_{j-1}\) to write (for each \(i\)) \(a_i = \sum_{k=i}^{j-1} a_{i,k}\) with \(a_{i,k} \in H_k\). Let \(S_{p,j}\) be the set of indices \(\ell\) such that \(H_{\ell}\) was constructed using \(\alpha\) and \(\delta = (\delta_{ip})_i\) (where the second \(\delta\) is the Kronecker symbol), for \(p = 2 \ldots k\). Recall that \(S_{p,j}\) is an arithmetic progression. Suppose that for \(p\) there are at least \(b'_p\) indices \(\ell\) in \(S_{p,j}\) such that \((a_{1,\ell}, \ldots, a_{k,\ell}) = (V, v_2, \ldots, v_k)\) (the elements in the tuple being those of \(H_k\)). Call such index **good** for \(p\). Then setting \(d_1 = w, d_i = W_i\) for those \(b'_p\) indices and \(d_i = a_i\) otherwise, we get that

\[
\alpha_i a_1 - \alpha_i d_1 = a_i - d_i + b'_i
\]
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Table 1. Decompositions of the $a_n$.

\[
\begin{array}{cccc}
    a_{1,i} & a_{2,i} & \ldots & a_{k,i} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{1,i+r-1} & a_{2,i+r-1} & \ldots & a_{k,i+r-1} \\
    a_{1,i+r} & a_{2,i+r} & \ldots & a_{k,i+r} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{1,i+2r-1} & a_{2,i+2r-1} & \ldots & a_{k,i+2r-1} \\
    a_{1,i+2r} & a_{2,i+2r} & \ldots & a_{k,i+2r} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{1,j-1} & a_{2,j-1} & \ldots & a_{k,j-1} \\
\end{array}
\]

$B_1$

$B_2$

$B_3$

$B_k$

for $i = p$, and

$$\alpha_i a_1 - \alpha_i d_1 = a_i - d_i$$

for $i \neq p$. If there are disjoint sets of $b'_p$ such indices for each $p$, with high probability over the $k$-tuples $(a_1, \ldots, a_k)$, then we can conclude the proof.

Let $r = \frac{j-i}{k-1}$; we can assume that $j - i$ is divisible by $k - 1$ by incrementing $j$ if necessary. We divide the indices $\{i, \ldots, j - 1\}$ into $k - 1$ blocks, each of size $r$. Let $B_1$ be the first block, $B_2$ the second block, and so on, up to $B_{k-1}$. It suffices to show that with arbitrarily high probability there are at least $b'_p$ indices $\ell$ that are good for $p$ among $B'_{p-1} = S_{p,j} \cap B_{p-1}$, for each $p = 2, \ldots, k$. Let $\ell \in B'_{p-1}$ be fixed. The probability that it is good for $p$ is the probability that $(a_{1,\ell}, \ldots, a_{k,\ell}) = (V, v_2, \ldots, v_k)$, which is $1/(2k)^k = c > 0$. Hence, the probability that
there are at least $b_p'$ indices good for $p$ in $B'_p-1$ is
\[
P\{\text{at least } b_p' \text{ good indices}\} = 1 - P\{\text{fewer than } b_p' \text{ good indices}\}
\]
\[
= \sum_{m=0}^{b_p'-1} \sum_{|\Lambda|=m} \left( \prod_{\ell \in \Lambda} c \right) \left( \prod_{\ell \notin \Lambda} 1 - c \right)
\]
\[
= \sum_{m=0}^{b_p'-1} \sum_{|\Lambda|=m} (c)^m ((1 - c))^{B'_p-1-m}
\]
\[
= \sum_{m=0}^{b_p'-1} \binom{B'_p-1}{m} (c)^m (1 - c)^{B'_p-1-m}
\]
\[
= (1 - c)^{B'_p-1} f_{b_p}(|B'_p-1|)
\]
\[
\rightarrow 1
\]

where $f_{b_p}$ is a polynomial of degree $b_p - 1$. The last line follows because $|B'_p-1|$ increases without bound as $j \to \infty$, since $S_{p,j}$ is an arithmetic progression, and $1 - c < 1$. As there are a finite number of blocks, this concludes the argument that $T_\alpha$ is ergodic.

Now, for $T \times T^{-1}$ not ergodic. Let $I$ be the base of $C_i$ with $I$ arbitrary, let $A = I \times I$, and let $B = I \times TI$. As we have seen, $T \times T^{-1}$ is ergodic if and only if the probability that a pair $(a, a') \in D(I, j)^2$ has a corresponding pair $(d, d') \in D(I, j)$ such that $a + a' = d + d' + 1$ goes to 1 as $j \to \infty$. But by Lemma [6.2], this equation does not have any solutions for any $j$, hence $T \times T^{-1}$ cannot possibly be ergodic. \[\square\]

7. A Markov shift with $T \times T^{-1}$ not conservative

In this section we construct a conservative ergodic Markov shift $T$ such that $T \times T^{-1}$ is not conservative. This is based on the examples of Kakutani and Parry [1]. For further background and terms not defined below regarding Markov shifts, the reader is referred to [1].

7.1. Preliminaries on Markov shifts. We briefly recall some properties of infinite measure-preserving countable state Markov shifts. Let $S$ be a countable set, which in our case will be $\mathbb{Z}$, and let $P$ be a stochastic matrix over $S$. Let $\lambda$ be a vector indexed by $S$ that is a left-eigenvector of $P$ with eigenvalue 1, so $\lambda P = \lambda$, and assume that $\sum_{s \in S} \lambda_s = \infty$. Let $X = S^\mathbb{Z}$, let $B$ be the Borel $\sigma$-algebra generated by
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cylinder sets of the form

\[ [s_0 \ldots s_n]_k = \{ x \in X \mid x_{j+k} = s_j \text{ for all } k = 0, \ldots, n \}. \]

Define a measure on these sets by

\[ \mu_\lambda([s_0 \ldots s_n]_k) = \lambda s_0 p_{s_0,s_1} p_{s_1,s_2} \cdots p_{s_{n-1},s_n} \]

and let \( T \) be the left shift on \( X \). Then \( T \) preserves \( \mu_\lambda \). The tuple \( (X, \mathcal{B}, \mu, T) \) is called a \( \sigma \)-finite Markov shift.

Let \( P^n \) be the matrix \( P \) taken to the \( n \)th power, and let \( p^{(n)}_{s,t} \) be the \((s,t)\)-th entry of \( P^n \). A Markov shift is called irreducible if for each \( s, t \in S \), we have that \( p^{(n)}_{s,t} > 0 \) for some \( n \). The following can be found in [1].

**Theorem 7.1.** Let \( T \) be an irreducible Markov shift. If there is \( s \in S \) such that \( \sum_{n=1}^{\infty} p^{(n)}_{s,s} = \infty \), then \( T \) is conservative. Conversely, if there is \( s \) such that \( \sum_{n=1}^{\infty} p^{(n)}_{s,s} < \infty \), then \( T \) is not conservative. Furthermore, if \( T \) is irreducible and conservative, then it is ergodic.

We will use the following theorem of Kakutani and Parry.

**Theorem 7.2 ([11]).** The following conditions hold if and only if \( T^{(k)} = T \times \cdots \times T \) is ergodic:

- **I.** If \( s_1, \ldots, s_k, t_1, \ldots, t_k \in S \), there is \( n \) with \( p^{(n)}_{s_1,t_1}, \ldots, p^{(n)}_{s_k,t_k} > 0 \)
- **II.** \( \sum_{n=1}^{\infty} p^{(n)}_{0,0} = \infty \).

In [11], the authors construct a family of Markov shifts that have ergodic index \( k \) as follows. For some \( \varepsilon > 0 \) (the choice of which determines the ergodic index of the shift), they let \( p_{i,i+1} = (1 - \varepsilon/i)/2 \), \( p_{i,i-1} = (1 + \varepsilon/i)/2 \) if \( i \neq 0 \), \( p_{0,1} = p_{0,-1} = 1/2 \), and \( p_{ij} = 0 \) if \( j \neq i+1 \) and \( j \neq i-1 \). They also define, for \( i \) positive,

\[ \lambda_i = \frac{i \cdot \Gamma(i + \varepsilon) \Gamma(i - \varepsilon)}{\Gamma(1 - \varepsilon) \Gamma(i + 1 + \varepsilon)} \]

and define \( \lambda_i = 0 \) and \( \lambda_i = \lambda_{-i} \) if \( i < 0 \). They note that \( \lambda P = \lambda \), and \( \sum_{-\infty}^{\infty} \lambda_i = \infty \). Lastly, using a particular \( \varepsilon = \varepsilon(k) \), they show that \( Q = P \cdot P \) has ergodic index \( k \).

**7.2. Reversible shifts.**

**Proposition 7.3.** Let \( T \) be a Markov shift defined by the matrix \( P \) with 1-eigenvalue \( \lambda \). If \( P \) is reversible, that is, if \( P \) satisfies

\[ \lambda_i p_{ij} = \lambda_j p_{ji} \]

then \( T \) is isomorphic to its inverse.
Proof. Define \( \phi: X \rightarrow X \) by \( \phi(x)_i = x_{-i} \). Clearly, \( T \circ \phi = \phi \circ T^{-1} \). Now, \( \phi^{-1}([s_0 \ldots s_n]) = \phi([s_0 \ldots s_n]) = [s_n \ldots s_0] \) where \( l \) is some integer. Now,

\[
\mu_\lambda[s_n \ldots s_0] = \lambda s_n p_{s_n,s_{n-1}} \ldots p_{s_1,s_0} = p_{s_{n-1},s_n} \lambda s_{n-1} p_{s_{n-1},s_{n-2}} \ldots p_{s_1,s_0} = p_{s_{n-2},s_{n-1}} \lambda s_{n-2} \ldots p_{s_1,s_0} = \ldots = p_{s_0,s_1} \ldots p_{s_{n-1},s_n} \lambda s_0 = \mu_\lambda[s_0 \ldots s_n]
\]

Thus \( \phi \) is a measure isomorphism. \( \square \)

**Proposition 7.4.** Let \( P \) and \( Q \) be reversible stochastic matrices defining Markov shifts, with the same 1-eigenvector \( \lambda \), and where \( P \) and \( Q \) commute. Then \( P \cdot Q \) is reversible.

Proof. By assumption, \( \lambda_i p_{i,j} = \lambda_j p_{j,i} \) and \( \lambda_i q_{i,j} = \lambda_j q_{j,i} \) for every \( i, j \). Now,

\[
\lambda_i(pq)_{i,j} = \sum_k \lambda_i p_{i,k} q_{k,j} = \sum_k \lambda_k p_{k,i} q_{j,k} = \sum_k \lambda_j p_{j,k} q_{k,i} = \lambda_j (qp)_{j,i} = \lambda_j (pq)_{j,i}
\]

so that \( P \cdot Q \) is reversible. \( \square \)

In specific, if \( P \) is reversible, then \( P \cdot P \) is reversible, because it has the same 1-eigenvector.

### 7.3. Main construction.

**Proposition 7.5.** The stochastic matrix \( P \) defined by Kakutani and Parry is reversible.

Proof. We wish to show that \( \lambda_i/\lambda_j = p_{j,i}/p_{i,j} \). Now,

\[
\frac{p_{i,i+1}}{p_{i+1,i}} = \frac{p_{i,i+1}}{p_{i+1,(i+1)-1}} = \frac{1 - \varepsilon/i}{1 + \varepsilon/(i + 1)}
\]
so long as \( i, i + 1 \neq 0 \). If \( i = 0 \), we have
\[
\frac{p_{i,i+1}}{p_{i+1,i}} = \frac{p_{0,1}}{p_{1,0}} = \frac{1}{1 + \varepsilon}
\]
and if \( i = -1 \), we have
\[
\frac{p_{i,i+1}}{p_{i+1,i}} = \frac{p_{-1,0}}{p_{0,-1}} = 1 + \varepsilon
\]
Recall that \( \lambda \) is defined as
\[
\lambda_i = \frac{i \cdot \Gamma(1 + \varepsilon)\Gamma(i - \varepsilon)}{\Gamma(1 - \varepsilon)\Gamma(i + 1 + \varepsilon)}
\]
if \( i > 0 \), \( \lambda_0 = 1 \), and \( \lambda_i = \lambda_{-i} \) if \( i < 0 \). We need only check that the reversibility equality holds if \( j = i + 1 \) or \( i - 1 \), as the other entries in \( P \) are all zero. If \( i > 0 \), we have
\[
\frac{\lambda_{i+1}}{\lambda_i} = \frac{(i + 1) \cdot \Gamma(1 + \varepsilon)\Gamma(i + 1 - \varepsilon)}{\Gamma(1 - \varepsilon)\Gamma(i + 2 + \varepsilon)} \cdot \frac{\Gamma(1 - \varepsilon)\Gamma(i + 1 + \varepsilon)}{i \cdot \Gamma(1 + \varepsilon)\Gamma(i - \varepsilon)}
\]
whereas
\[
\frac{p_{i,i+1}}{p_{i+1,i}} = \frac{1 - \varepsilon/i}{1 + \varepsilon/(i + 1)} = \frac{i + 1}{i} \cdot \frac{i - \varepsilon}{i + 1 + \varepsilon}
\]
which is the same. The \( i < -1 \) case is a similar calculation. This concludes unless \( i = 0, -1 \). If \( i = 0 \), we have
\[
\frac{\lambda_1}{\lambda_0} = \frac{\Gamma(1 + \varepsilon)\Gamma(1 - \varepsilon)}{\Gamma(1 - \varepsilon)\Gamma(2 + \varepsilon)} = \frac{1}{1 + \varepsilon} = \frac{p_{0,1}}{p_{1,0}}
\]
and if \( i = -1 \), we get
\[
\frac{\lambda_0}{\lambda_{-1}} = \frac{\lambda_0}{\lambda_1} = (1 + \varepsilon) = \frac{p_{-1,0}}{p_{0,-1}}
\]
as required. \( \Box \)

**Corollary 7.6.** For any \( k \), there exists a conservative ergodic Markov shift \( T \), isomorphic to its inverse, such that \( T^{(k)} \) is conservative ergodic and \( T^{(k)} \times T^{-1} \) is neither.

**Proof.** Kakutani and Parry show that by suitable choice of \( \varepsilon \), the Markov shift \( T \) defined by \( P \cdot P \) is such that \( T^{(k)} \) is conservative ergodic but \( T^{(k+1)} \) is not ergodic, hence not conservative. By the above, \( T \) is isomorphic to its inverse, so clearly \( T^{(k)} \times T^{-1} \) is not conservative (and hence not ergodic). \( \Box \)

In particular, choosing \( k = 1 \), this gives us a transformation \( T \) such that \( T \) is conservative ergodic, but \( T \times T^{-1} \) is neither.
7.4. **Power Weak Mixing is Generic.** An invertible transformation $T$ is said to be **power weakly mixing** if for every sequence of numbers $k_1, \ldots, k_r \in \mathbb{Z} \setminus \{0\}$, the product transformation $T^{k_1} \times \ldots \times T^{k_r}$ is ergodic. In finite measure this is equivalent to weak mixing, but in infinite measure it is stronger than infinite ergodic index [3]. As we will show in this section, under the weak topology in the group of invertible measure-preserving transformations, the set of transformations that are power weak mixing is a residual set, so we say this property is **generic**.

It follows that the set of transformations $T$ such that $T \times T^{-1}$ is not ergodic is meagre. Sachdeva [12] showed that infinite ergodic index is generic in the weak topology. Ageev, at the time of [5] mentioned to one of the authors that he had a proof that power weak mixing is generic, but it has not been published as far as we know. Following the proof of genericity of infinite ergodic index in [9] we include below a proof of genericity of power weak mixing, as we are interested in showing that the properties of the transformations of Section 5 are topologically rare.

We recall the weak topology defined on the group $G = G(X, \mu)$ of invertible measure-preserving transformations on a $\sigma$-finite Lebesgue measure space $(X, \mathcal{B}, \mu)$. The topology on $G$ is inherited from the strong operator topology so that a sequence $T_n$ converges to $T$ if and only if 

$$
\mu (T_n(A) \triangle T(A)) + \mu (T_n^{-1}(A) \triangle T^{-1}(A)) \to 0,
$$

for all sets of finite measure $A$. This topology is called the **weak topology** on $G$, and is completely metrizable through a natural metric [12].

We will use the following lemma from [12].

**Lemma 7.7.** The conjugacy class of any transformation $T \in G(X, \mu)$ is dense in $G(X, \mu)$.

**Theorem 7.8.** The property of power weak mixing is generic in $G(X, \mu)$, in particular, the set of power weakly mixing transformation in $G(X, \mu)$ forms a dense $G_\delta$ subset.

**Proof.** Let $P_\infty$ be the set of power weakly mixing transformations on $(X, \mu)$. First we show that it is a $G_\delta$ set. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$, where $\alpha_i \in \mathbb{Z} \setminus \{0\}$ for each $1 \leq i \leq k$. For an invertible measure-preserving transformation $T$, define $T^\alpha = T^{\alpha_1} \times \ldots \times T^{\alpha_k}$. That $T$ is power weakly mixing is equivalent to $T^\alpha$ being ergodic for every such $\alpha$. Now, define $\phi_\alpha : G(X, \mu) \to G(X^{(k)} \mu^{(k)})$ by $\phi_\alpha(T) = T^\alpha$. As is easily checked, $\phi_\alpha$ is continuous in the weak topology. By Sachdeva [12] (see also [11]), the ergodic transformations $\mathcal{E}^{(k)}$ form a $G_\delta$ subset of $G(X^{(k)} \mu^{(k)})$, hence $\phi_\alpha^{-1}(\mathcal{E}^{(k)})$ is a $G_\delta$ subset of $G(X, \mu)$. But $\phi_\alpha^{-1}(\mathcal{E}^{(k)})$ is precisely those $T \in G(X, \mu)$ such that $T^\alpha$ is ergodic, hence $P_\alpha$, the set of $T$ such that
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$T^\alpha$ is ergodic, is $G_\delta$. Because the countable intersection of $G_\delta$ sets is $G_\delta$, $P_\infty$ is $G_\delta$ in $\mathcal{G}(X, \mu)$.

It remains to show density. Since $P_\infty$ is nonempty \([10]\), if we show that it is closed under conjugation, Lemma \([7.7]\) will give us that it is dense. To that end, let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a tuple of nonzero integers, let $S$ be a measure-preserving transformation, and suppose that $(S \circ T \circ S^{-1})^\alpha(A) = A$ for some $A$. This means

$$(S \circ T \circ S^{-1})^\alpha(A) = A$$

$$(S \circ T \circ S^{-1})^{\alpha_1} \times \ldots \times (S \circ T \circ S^{-1})^{\alpha_k} = A$$

$$(S \circ T^{\alpha_1} \circ S^{-1}) \times \ldots \times (S \circ T^{\alpha_k} \circ S^{-1}) = A$$

$$S^{(k)} \circ T^{\alpha} \circ (S^{-1})^{(k)} = A$$

$$T^{\alpha} \circ (S^{-1})^{(k)} = (S^{-1})^{(k)} A$$

hence by the ergodicity of $T^\alpha$ we have $(S^{-1})^{(k)} A$ is either null or conull, hence as $S$ is measure-preserving $A$ is either null or conull, hence $S \circ T \circ S^{-1}$ is power weakly mixing. \(\square\)

REFERENCES

[1] J. Aaronson. An introduction to infinite ergodic theory, volume 50 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997.

[2] Terrence Adams, Nathaniel Friedman, and Cesar E. Silva. Rank-one weak mixing for nonsingular transformations. Israel J. Math., 102:269–281, 1997.

[3] Terrence Adams, Nathaniel Friedman, and Cesar E. Silva. Rank-one power weakly mixing non-singular transformations. Ergodic Theory Dynam. Systems, 21(5):1321–1332, 2001.

[4] Terrence Adams and Cesar E. Silva. On infinite transformations with maximal control of ergodic two-fold product powers. http://arxiv.org/abs/1402.1818, 2014.

[5] O. N. Ageev and C. E. Silva. Genericity of rigid and multiply recurrent infinite measure-preserving and nonsingular transformations. In Proceedings of the 16th Summer Conference on General Topology and its Applications (New York), volume 26, pages 357–365, 2001/02.

[6] Francisc Bozgan, Anthony Sanchez, Cesar E. Silva, David Stevens, and Jane Wang. Subsequence rational ergodicity of rank-one transformations. http://arxiv.org/abs/1310.5084.

[7] Irving Dai, Xavier Garcia, Tudor Pădurariu, and Cesar E. Silva. On rationally ergodic and rationally weakly mixing rank-one transformations. Ergodic Theory Dynam. Systems, to appear. Available on CJO 2014 doi:10.1017/etds.2013.96.

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Clancy, Friedberg, Kasmalkar, Loh, Pădurariu, Silva, and Vasudevan

[8] Alexandre I. Danilenko. \((C,F)\)-actions in ergodic theory. In Geometry and dynamics of groups and spaces, volume 265 of Progr. Math., pages 325–351. Birkhäuser, Basel, 2008.

[9] Alexandre I. Danilenko and Cesar E. Silva. Ergodic theory: Nonsingular transformations. In Encyclopedia of Complexity and System Science, volume Part 5, pages 3055–3083. Springer, 2009.

[10] Sarah L. Day, Brian R. Grivna, Earle P. McCartney, and Cesar E. Silva. Power weakly mixing infinite transformations. New York J. Math., 5:17–24 (electronic), 1999.

[11] S. Kakutani and W. Parry. Infinite measure preserving transformations with “mixing”. Bull. Amer. Math. Soc., 69:752–756, 1963.

[12] Usha Sachdeva. On category of mixing in infinite measure spaces. Math. Systems Theory, 5:319–330, 1971.

[13] C. E. Silva. Invitation to ergodic theory, volume 42 of Student Mathematical Library. American Mathematical Society, Providence, RI, 2008.

(Julien Clancy) Yale University, New Haven, CT 06520, USA
E-mail address: julien.clancy@yale.edu

(Rina Friedberg) University of Chicago, Chicago, IL 60637, USA
E-mail address: rinafriedberg@uchicago.edu

(Indraneel Kasmalkar) University of California, Berkeley, Berkeley, CA 94720, USA
E-mail address: indraneelk@berkeley.edu

(Isaac Loh) Williams College, Williamstown, MA 01267, USA
E-mail address: il2@williams.edu

(Tudor Pădurariu) University of California, Los Angeles, CA 90095-1555, US
E-mail address: tudor.pad@yahoo.com

(Cesar E. Silva) Department of Mathematics, Williams College, Williamstown, MA 01267, USA
E-mail address: csilva@williams.edu

(Sahana Vasudevan) Harvard University, Cambridge, MA 02138, USA
E-mail address: svasudevan@college.harvard.edu