ON A CLASS OF PARACONTACT METRIC 3-MANIFOLDS

K. SRIVASTAVA* AND S. K. SRIVASTAVA**

Abstract. The purpose of this paper is to classify paracontact metric 3-manifolds $M^3$ such that the Ricci operator $S$ commutes with the endomorphism $\phi$ of its tangent bundle $\Gamma(TM^3)$.

1. Introduction

Almost paracontact metric manifolds are the well known illustrations of almost para-CR manifolds. In [7] Kaneyuki and Konzai defined the almost paracontact structure on pseudo-Riemannian manifold $M^{2n+1}$ and constructed the almost paracomplex structure on $M^{2n+1} \times \mathbb{R}$. Analogous to Blair et al.[3] if the paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is $\eta$-Einstein we do not have a widespread classification. This paper is organized as follows: In §2 we present some technical apparatus which is needed for further investigations. In §3 we first prove that for any $X$ orthogonal to $\xi$ the function $Trl$ vanishes and the function $f$ defined by $lX = fX$ is constant everywhere on a paracontact metric manifold $M^3$. We then show that the conditions, (i) the structure is $\eta$-Einstein, (ii) Ricci operator $S$ commutes with tensor field $\phi$ and (iii) $\xi$ belongs to the $k$-nullity distribution of pseudo-Riemannian metric $g$ are equivalent on $M^3$. Finally, we prove that the unit torse forming vector field in this manifold with $S\phi = \phi S$ is concircular in §4.

2. Preliminaries

A $C^\infty$ smooth manifold $M^{2n+1}$ of dimension $(2n + 1)$, is said to have triplet $(\phi, \xi, \eta)$-structure, if it admits a $(1, 1)$ tensor field $\phi$, a unique vector field $\xi$ called the characteristic vector field or Reeb vector field and a contact form $\eta$ satisfying:

$$\phi^2 = I - \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1$$

where $I$ is the identity transformation. The endomorphism $\phi$ induces an almost paracomplex structure on each fibre of $D = \ker(\eta)$, the contact subbundle that is the eigen distributions $D^{\pm 1}$ corresponding to the characteristics values $\pm 1$ of $\eta$.

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φ have equal dimension n.
From the equation (2.1), it can be easily deduce that
\[ \phi \xi = 0, \quad \eta \circ \phi = 0 \quad \text{and} \quad \text{rank}(\phi) = 2n. \] (2.2)
This triplet structure \( (\phi, \xi, \eta) \) is called an almost paracontact structure and the manifold \( M^{2n+1} \) equipped with the \( (\phi, \xi, \eta) \)-structure is called an almost paracontact manifold \(^7\). If an almost paracontact manifold admits a pseudo-Riemannian metric \(^{17}\), \( g \) satisfying:
\[ g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \] (2.3)
where signature of \( g \) is necessarily \( (n + 1, n) \) for any vector fields \( X \) and \( Y \). Then the quadruple \( (\phi, \xi, \eta, g) \) is called an almost paracontact metric structure and the manifold \( M^{2n+1} \) equipped with paracontact metric structure is called an almost paracontact metric manifold. With respect to \( g \), \( \eta \) is metrically dual to \( \xi \), that is
\[ g(X, \xi) = \eta(X) \] (2.4)
Also, equation (2.3) implies that
\[ g(\phi X, Y) = -g(X, \phi Y). \] (2.5)
\textit{Note:} The above metric \( g \) is, of course, not unique.
Further, in addition to the above properties, if the quadruple \( (\phi, \xi, \eta, g) \) satisfies:
\[ d\eta(X, Y) = g(X, \phi Y), \]
for all vector fields \( X, Y \) on \( M^{2n+1} \), then the manifold is called a paracontact metric manifold and the corresponding structure \( (\phi, \xi, \eta, g) \) is called a paracontact structure with the associated metric \( g \) \(^{17}\).
Now, for an almost paracontact metric manifold, there always exists a special kind of local pseudo-orthonormal basis \( \{X_i, X_i^*, \xi\} \): where \( X_i^* = \phi X_i \); \( \xi \) and \( X_i^* \)'s are space-like vector fields and \( X_i^* \)'s are time-like. Such a basis is called \( \phi \)-basis. Indicating by \( L \) and \( R \), the Lie differentiation operator and the curvature tensor of \( M^{2n+1} \) respectively, let us define, \( (1, 1) \) type tensor fields \( h \) and \( l \), which are symmetric as well, by \( h = \frac{1}{2} L_\xi \phi, \quad l = R(., \xi)\xi \). The basic properties followed by \( h \) and \( l \) are:
\[ h\xi = 0, \quad l\xi = 0, \quad \eta \circ h = 0, \quad \text{Tr}.h = \text{Tr} \phi h = 0 \] (2.6)
and
\[ h\phi = -\phi h \quad \text{(i.e.,} \ h \text{ anti-commutes with} \ \phi). \] (2.7)
Also, \( hX = \lambda X \Rightarrow h\phi X = -\lambda \phi X, \) \textit{i.e.,} if \( \lambda \) is an eigen value of \( h \) with the corresponding eigen vector \( X \), then \( -\lambda \) is also an eigenvalue of \( h \) corresponding
to the eigen vector $\phi X$. If $S$ denotes the Ricci operator and $\nabla$ denotes the Levi-Civita connection of the metric $g$, then using the above properties of $h$ and $l$, we can easily calculate the following formulas for a paracontact metric manifold $M^{2n+1}$:

$$\nabla_X \xi = -\phi X + \phi h X, \quad \nabla_\xi \xi = 0.$$  
(2.8)

$$\nabla_\xi \phi = 0.$$  
(2.9)

$$Tr l = g(S \xi, \xi) = Tr h^2 - 2n.$$  
(2.10)

$$\phi l \phi + l = 2 \left( h^2 - \phi^2 \right).$$  
(2.11)

$$\nabla_\xi h = -\phi - \phi l + h^2 \phi.$$  
(2.12)

If the Reeb vector field $\xi$ is Killing, i.e., equivalently $h = 0$, then the paracontact metric manifold $M^{2n+1}$ is called a $K$-paracontact manifold [11]. On a 3-dimensional Riemannian manifold, since the conformal curvature tensor vanishes identically, therefore the Riemannian curvature tensor $R$ takes the form [3]

$$R(X, Y)Z = g(Y, Z)SX - g(X, Z)SY + g(SY, Z)X$$

$$-g(SX, Z)Y - \frac{r}{2} \{g(Y, Z)X - g(X, Z)Y\}$$  
(2.13)

where $r$ is the scalar curvature of the manifold and the Ricci operator $S$ is defined by

$$g(SX, Y) = Ric(X, Y).$$  
(2.14)

3. $k$-nullity distribution

In contact geometry, the notion of $k$-nullity distribution is introduced by Tanno (1988, [14]). The $k$-nullity distribution of a Riemannian manifold $(M, g)$, for a real number $k$, is a distribution

$$N(k) : p \to N_p(k) = [Z \in \Gamma(T_pM) : R(X, Y)Z = k \{g(Y, Z)X - g(X, Z)Y\}]$$  
(3.1)

for any $X, Y \in \Gamma(T_pM)$, where $R$ and $\Gamma(T_pM)$ respectively denotes the Riemannian curvature tensor and the tangent vector space of $M^{2n+1}$ at any point $p \in M$. If the characteristic vector field $\xi$ of a paracontact metric manifold belongs to the $k$-nullity distribution then the following relation holds [14]

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y).$$  
(3.2)

**Definition 3.1.** A paracontact metric manifold is said to be $\eta$-Einstein [1], if the Ricci operator $S$ can be written in the following form:

$$S = a I + b \eta \otimes \xi,$$  
(3.3)

where $a$ and $b$ are some functions.

Now we will prove the following:
Theorem 3.1. Let $M^3(\phi, \xi, \eta, g)$ be a paracontact metric manifold with $\phi S = S\phi$. Then for any $X \in \Gamma(TM^3)$ orthogonal to $\xi$

(i) the function $Trl$ vanishes on $M^3$ and
(ii) the function $f$ defined by $lX = fX$ is constant everywhere on $M^3$.

Proof. By virtue of equations (2.1), (2.2), (2.10) and $\phi S = S\phi$, we have

$$S\xi = (Trl)\xi.$$  \hspace{1cm} (3.4)

From equation (2.13), using the definition of $l$ and (3.4), we have for any $X$

$$lX = R(X, \xi)\xi = g(\xi, \xi)SX - g(X, \xi)S\xi + g(S\xi, \xi)X - g(SX, \xi)\xi - (r/2)(g(\xi, \xi)X - g(X, \xi)\xi)$$

which gives

$$lX = SX + (Trl - r/2)X + \eta(X)(r/2 - 2Trl)\xi$$ \hspace{1cm} (3.5)

since, $\eta(SX)\xi = \eta(X)(Trl)\xi$. As a matter of fact if at a point $p \in M^3$ there exists $X \in \Gamma(T_pM^3)$ such that $lX = 0$ for $X \neq \xi$, then $l = 0$ at $p$. So, let us assume that $l \neq 0$ on a neighbourhood of a point $p$. From (3.15) and (2.7), we get

$$g(\phi X, lX) = -g(X, \phi lX) = -g(\phi X, lX) \Rightarrow g(\phi X, lX) = 0.$$

Therefore, $lX$ is parallel to $X$ for any $X$ orthogonal to $\xi$. Taking a local orthogonal frame $\{e_1, e_2, \xi\}$ where $-g(e_1, e_1) = g(e_2, e_2) = g(\xi, \xi) = 1$ and $e_1, e_2$ are orthogonal to $\xi$. Then, by definition

$$Trl = \sum_{i=1}^{3} g(\phi e_i, e_i) = g(\phi e_1, e_1) + g(\phi e_2, e_2) + g(\phi l, \xi)$$

which proves (i).

For any $X$, we can write

$$lX = f\phi^2X.$$ \hspace{1cm} (3.7)

From (3.7), (2.9) and (3.18), we have

$$\xi f = 0.$$ \hspace{1cm} (3.8)

With the help of equations (3.7) and (3.5), we find

$$lX = SX + (Trl - r/2)X + \eta(X)(r/2 - 2Trl)\xi$$

which gives

$$SX = aX + b\eta(X)\xi$$ \hspace{1cm} (3.9)
where
\[ a = (f - Trl + r/2), \quad b = (2Trl - r/2 - f). \] (3.10)

From the second identity of Bianchi, we get
\[ (\nabla_X R)(Y, \xi, Z) + (\nabla_Y R)(\xi, X, Z) + (\nabla_\xi R)(X, Y, Z) = 0. \] (3.11)

Employing (3.9) in (2.13), we have
\[ R(X, Y)Z = g(Y, Z)(aX + b\eta(X)\xi) - g(X, Z)(aY + b\eta(Y)\xi) \]
\[ + g(aY + b\eta(Y)\xi, Z)X - g(aX + b\eta(X)\xi, Z)Y \]
\[ - (r/2)(g(Y, Z)X - g(X, Z)Y) \]
\[ = ag(Y, Z)X + b\eta(X)g(Y, Z)\xi - ag(X, Z)Y - b\eta(Y)g(X, Z)\xi \]
\[ + ag(Y, Z)X + b\eta(Y)g(\xi, Z)X - ag(X, Z)Y - b\eta(X)g(\xi, Z)Y \]
\[ - (r/2)(g(Y, Z)X - g(X, Z)Y) \]
\[ = \{(2a - r/2)g(Y, Z)X + b\eta(Y)\eta(Z)X\} \]
\[ - \{(2a - r/2)g(X, Z)Y + b\eta(X)\eta(Z)Y\} \]
\[ + b\eta(X)g(Y, Z)\xi - b\eta(Y)g(X, Z)\xi \]

that is,
\[ R(X, Y)Z = \{(\gamma g(Y, Z) + b\eta(Y)\eta(Z))X - (\gamma g(X, Z) + b\eta(X)\eta(Z))Y\} \]
\[ + b\{(\eta(X)g(Y, Z) - \eta(Y)g(X, Z))\} \xi \] (3.12)

where \( \gamma = 2a - r/2 \). From (3.12) for \( Z = \xi \), we get
\[ R(X, Y)\xi = \{(\gamma \eta(Y) + b\eta(Y))X - (\gamma \eta(X) + b\eta(X))\} Y \]
\[ + b\{(\eta(X)\eta(Y) - \eta(Y)\eta(X))\} \xi \]
\[ = (\gamma + b)(\eta(Y)X - \eta(X)Y) \]

switching the values of \( b \) and \( \gamma \) in the above equation, we have
\[ R(X, Y)\xi = f(\eta(Y)X - \eta(X)Y). \] (3.13)

Using (3.13), we obtain
\[ R(Y, \xi)\xi = f(Y - \eta(Y)\xi) \] (A)
\[ R(\nabla_X Y, \xi)\xi = f(\nabla_X Y - \eta(\nabla_X Y)\xi) \] (B)
\[ R(Y, \nabla_X \xi)\xi = f(\eta(\nabla_X \xi)Y - \eta(Y)\nabla_X \xi) \] (C)
\[ R(Y, \xi)\nabla_X \xi = \{(\gamma \nabla_X \xi) + b\eta(\nabla_X \xi))Y - \gamma g(Y, \nabla_X \xi) + b\eta(Y)\eta(\nabla_X \xi)\} \xi \]
\[ + b\{(\eta(Y)\eta(\nabla_X \xi) - g(Y, \nabla_X \xi))\} \xi \] (D)
From (A), we also have

\[(\nabla_X R)(Y, \xi, \xi) + R(\nabla_X Y, \xi) \xi + R(Y, \nabla_X \xi) \xi + R(Y, \xi) \nabla_X \xi = (Xf)(Y - \eta(Y)\xi) + f(\nabla_X Y - ((\nabla_X \eta)Y) \xi - \eta(Y) \nabla_X \xi)\]

(E)

Hence for \(X, Y\) orthogonal to \(\xi\), we get from (B), (C), (D) and (E)

\[(\nabla_X R)(Y, \xi, \xi) + f(\nabla_X Y) = (Xf)Y + f(\nabla_X Y)\]

this implies that

\[(\nabla_X R)(Y, \xi, \xi) = (Xf)Y.\]  
  \[(F)\]

Applying \(Z = \xi\) in equation (3.11), we get

\[(\nabla_X R)(Y, \xi, \xi) + (\nabla_Y R)(\xi, X, \xi) + (\nabla_\xi R)(X, Y, \xi) = 0\]

(G)

From (3.13), we can write

\[(\nabla_\xi R)(X, Y, \xi) + f(\nabla_X Y) - \eta(X)\nabla_Y \xi = (\xi f)\eta(X)\]

(3.14)

Using (2.8) and (3.8) in (3.14), we get

\[(\nabla_\xi R)(X, Y, \xi) + R(\nabla_\xi X, Y) + R(X, \nabla_\xi Y)\xi = (\xi f)\eta(X)\]

that is

\[(\nabla_\xi R)(X, Y, \xi) = 0.\]  
  \[(H)\]

From (H) and (G), we find that \((\nabla_X R)(Y, \xi, \xi) = (\nabla_Y R)(X, \xi, \xi)\) and by the use of (F) this implies, \((Xf)Y = (Yf)X\), for \(X\) and \(Y\) orthogonal to \(\xi\). Therefore \(Xf = 0\), for \(X\) orthogonal to \(\xi\), but \(\xi f \neq 0\), so the function \(f\) is constant everywhere on \(M^3\) and we reached at the end of the proof. □

**Proposition 3.2.** Let \(M^3(\phi, \xi, \eta, g)\) be a paracontact metric manifold with \(S\phi = \phi S\), then we have

(i) \(l\phi = \phi l\).
(ii) \(l = h^2 - \phi^2\).
(iii) \(\xi Trl = 0\).
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**Proof.** \( S\phi = \phi S \) and \( 3.15 \) yields

\[
\phi l = l\phi \tag{3.15}
\]

Further, using (3.15), (2.11) and (2.12), we obtain that

\[
l = h^2 - \phi^2 \tag{3.16}
\]

and

\[
\nabla_{\xi} h = -\phi + h^2 \phi - \phi l
= -\phi + h^2 \phi - \phi(h^2 - \phi^2)
= -\phi + h^2 \phi - \phi hh + \phi^3
= h^2 \phi + h\phi h.
\]

that is,

\[
\nabla_{\xi} h = 0. \tag{3.17}
\]

We now differentiate equation (3.16) with respect to \( \xi \) and use equations (2.9) and (3.17), and find that

\[
\nabla_{\xi} l = 0 \Rightarrow \xi T l = 0. \tag{3.18}
\]

This ends the proof. \( \square \)

**Remark 1.** Taking \( l = 0 \) everywhere, and using (2.13), (3.4) and (3.5), we get \( R(X, Y)\xi = 0 \). This together with the theorem B of [2] gives that, \( M^3 \) is flat.

**Theorem 3.3.** If \( M^3(\phi, \xi, \eta, g) \) be a paracontact metric manifold, then the following conditions are equivalent:

(i) \( M^3 \) is \( \eta \)-Einstein.
(ii) \( S\phi = \phi S \).
(iii) \( \xi \) belongs to the \( k \)-nullity distribution.

**Proof.** (i)\( \Rightarrow \)(ii).

Let \( M^3 \) be \( \eta \)-Einstein that is \( S = aI + b\eta \otimes \xi \). Therefore \( S\phi = a\phi + b(\eta \circ \phi) \otimes \xi = a\phi \). Also, \( \phi S = a\phi + b\eta \otimes \phi \xi = a\phi \). Hence \( S\phi = \phi S \).

(ii)\( \Rightarrow \)(iii).

Let \( S\phi = \phi S \). Then, we have from (3.13)

\[
R(X, Y)\xi = f(\eta(Y)X - \eta(X)Y)
\]

By the theorem 3.1 we have \( f = \text{constant} = k \) (say), therefore,

\[
R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y)
\]

this implies that \( \xi \) belongs to the \( k \)-nullity distribution.

(iii)\( \Rightarrow \)(i).

Let \( \xi \) belongs to the \( k \)-nullity distribution. Then,

\[
R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) \tag{3.19}
\]
where \( k \) is a constant.

Contracting (3.19) with respect to \( X \), we have

\[
\text{Ric}(Y, \xi) = k (3\eta(Y) - \eta(Y)) = k (2\eta(Y)) = 2k\eta(Y)
\]

that is,

\[
S\xi = 2k\xi
\]

and so from (2.13), we find

\[
R(X, Y)\xi = \eta(Y)SX - \eta(X)SY + (2k - r/2) (\eta(Y)X - \eta(X)Y).
\]

(3.20)

Comparing (3.19) and (3.20), we get

\[
\eta(Y)\{SX + (k - r/2)X\} = \eta(X)\{SY + (k - r/2)Y\}.
\]

Taking \( Y \) orthogonal to \( \xi \) and \( X = \xi \), we have

\[
SY = (r/2 - k)Y
\]

and so for any \( Z \),

\[
SZ = (r/2 - k)Z + (k - r/2) \eta(Z)\xi.
\]

This implies that \( S = aI + b\eta \otimes \xi \), where \( a = r/2 - k \) and \( b = k - r/2 \).

Therefore \( M^3 \) is \( \eta \)-Einstein. This completes the proof of the theorem.

\[\Box\]

4. TORSE FORMING VECTOR FIELDS

**Definition 4.1.** A vector field \( U \) defined by \( g(X, U) = u(X) \) for any \( X \in \Gamma(TM^3) \) is said to be torse forming vector field \[10\] (see also \[10, 13\]) if

\[
(\nabla_X u)(Y) = sg(X, Y) + \alpha(X)u(Y),
\]

(4.1)

where \( s \) and \( \alpha \) are called the **conformal scalar** and the **generating form** of \( U \), respectively.

A torse forming vector field \( U \) is called

- **recurrent or parallel**, if \( s = 0 \),
- **concircular**, if the generating form \( \alpha \) is a gradient,
- **convergent**, if it is concircular and \( s = \text{const}. \exp(\alpha) \).

**Theorem 4.1.** Let \( M^3(\phi, \xi, \eta, g) \) be a paracontact metric manifold with \( S\phi = \phi S \). Then the unit torse forming vector field in \( M^3 \) is concircular.

**Proof.** For a unit torse forming vector field \( \hat{U} \) corresponding to \( U \), if we define \( g(X, \hat{U}) = V(X) \), then

\[
V(X) = u(X)/\sqrt{u(U)}.
\]

(4.2)

From (4.1) and (4.2), we have

\[
(\nabla_X V)(Y) = \mu g(X, Y) + \alpha(X)V(Y)
\]

(4.3)
where $\mu = s/\sqrt{u(U)}$. Using $Y = \hat{U}$ and $V(\hat{U}) = 1$, equation (4.3) gives
\[
\alpha(X) = -\mu V(X) \tag{4.4}
\]
and hence (4.3) can be expressed in the following form
\[
(\nabla_X V)(Y) = \mu \{g(X, Y) - V(X)V(Y) \} \tag{4.5}
\]
which shows that $V$ is closed. Now differentiating (4.5) covariantly and using the Ricci identity, we obtain
\[
V(R(X, Y)Z) = (Y \mu) \{(g(X, Z) - V(X)V(Z) \} - (X \mu) \{g(Y, Z) - V(Y)V(Z) \} + \mu^2 \{g(X, Z)V(Y) - g(Y, Z)V(X) \}. \tag{4.6}
\]
By the use of the theorem 3.1, equations (3.13), (2.4) and (4.2), we have from (4.6)
\[
\{\eta(X) - \eta(\hat{U})V(X) \} \{f + (\hat{U}\mu) + \mu^2 \} = 0 \tag{4.7}
\]
which gives
\[
\{\eta(X) - \eta(\hat{U})V(X) \} = 0 \tag{I}
\]
or
\[
\{f + (\hat{U}\mu) + \mu^2 \} = 0. \tag{II}
\]
If (I) holds, then putting $X = \xi$ in (I), we have $\eta(\hat{U}) = \pm 1$. This implies that $\eta(X) = \pm V(X) \tag{4.8}$
From (2.8), (4.3) and (4.8), we have $\mu = \pm A(\text{constant})$. Therefore $\alpha(X) = \mp AV(X)$. Hence the vector field $\hat{U}$ is concircular.
If (II) holds, then $\{\eta(X) - \eta(\hat{U})V(X) \} \neq 0$. From (4.6), we have
\[
-g(SX, \hat{U}) = (X \mu) + (\hat{U} \mu)V(X) + 2\mu^2 V(X). \tag{4.9}
\]
Putting $X = \xi$ and using (3.4) in (4.9), we have
\[
\xi \mu = -(\mu^2 + f) \eta(\hat{U}). \tag{4.10}
\]
In view of (4.10) and $V(\xi) = \eta(\hat{U})$ equation (4.6) yields for $Y = Z = \xi$,
\[
X \mu = -(\mu^2 + f)V(X). \tag{4.11}
\]
From (4.4) and (4.11), we get
\[
Y \alpha(X) = (\mu^2 + f)V(X)V(Y) - \mu(YV(X)). \tag{a}
\]
We can also obtain
\[
X \alpha(Y) = (\mu^2 + f)V(Y)V(X) - \mu(XV(Y)) \tag{b}
\]
and
\[
\alpha([X, Y]) = -\mu V([X, Y]). \tag{4.12}
\]
From (a), (b) and (4.12), we have
\[ d\alpha(X, Y) = -\mu((dV)(X, Y)) \]
which implies that \( \hat{U} \) is concircular. These completes the proof of the theorem. \( \square \)

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*Department of Mathematics
D. D. U. Gorakhpur University
Gorakhpur-273009
Uttar Pradesh
INDIA
E-mail address: ksriddu22@gmail.com

**Department of Mathematics
Central University of Himachal Pradesh
Dharamshala - 176215
Himachal Pradesh
INDIA
E-mail address: sachink.ddumath@gmail.com