Reconstructing small perturbations of an obstacle for acoustic waves from boundary measurements on the perturbed shape itself

Habib Zribi

Department of Mathematics, College of Science, University of Hafr Al Batin, Hafr Al Batin, Saudi Arabia

Correspondence
Habib Zribi, Department of Mathematics, College of Science, University of Hafr Al Batin, P.O. Box 1803, Hafr Al Batin 31991, Saudi Arabia.
Email: zribi.habib@yahoo.fr

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We derive relationships between the shape deformation of an impenetrable obstacle and boundary measurements of scattering fields on the perturbed shape itself. Our derivation is rigorous by using a systematic way, based on layer potential techniques and the field expansion (FE) method (formal derivation). We extend these techniques to derive asymptotic expansions of the Dirichlet-to-Neumann (DNO) and Neumann-to-Dirichlet (NDO) operators in terms of the small perturbations of the obstacle as well as relationships between the shape deformation of an obstacle and boundary measurements of DNO or NDO on the perturbed shape itself. All relationships lead us to very effective algorithms for determining lower order Fourier coefficients of the shape perturbation of the obstacle.

KEYWORDS
acoustic scattering, asymptotic expansions, boundary integral method, Helmholtz equation, small boundary perturbations

MSC CLASSIFICATION
35B30; 35R30; 35C20; 35B40

1 | INTRODUCTION AND STATEMENTS OF MAIN RESULTS

Let us consider a situation where we have an incident wave \( u^i \) propagating in a homogeneous isotropic medium \( \mathbb{R}^n \) for \( n = 2 \) or \( 3 \), containing a bounded scatterer \( D \) with \( C^2 \) boundary, which is either a sound-soft or a sound-hard impenetrable obstacle. The wave will scatter by the obstacle and we can express the total wave field around the object as the sum of \( u^i \) and a scattered wave \( u^s \). The behavior of the scattered wave will depend on both the incident wave and the shape and the physical properties of the object. The most inverse shape problems are to determine the shape of an object from measurements of scattered waves. The scattering field \( u^s \) satisfies

\[
\begin{align*}
\Delta u^s + k^2 u^s &= 0 \quad \text{in} \quad \mathbb{R}^n \setminus \bar{D}, \\
\frac{\partial u^s}{\partial \nu} &= -\frac{\partial u^i}{\partial \nu} \quad \text{on} \quad \partial D, \\
|\frac{\partial u^s}{\partial |x|} - iku^s| &= O \left( \frac{1}{|x|^{n+1}} \right) \quad \text{as} \quad |x| \to \infty, 
\end{align*}
\]

(1.1)

where the wave number \( k > 0 \) and \( \nu \) is the unit outward normal to the domain \( D \).
Let \( D_\varepsilon \) be an \( \varepsilon \)-perturbation of \( D \); that is, there is a function \( h \in C^4(\partial D) \) such that \( \partial D_\varepsilon \) is given by

\[
\partial D_\varepsilon = \{ \tilde{x} = x + \varepsilon h(x) \nu(x) \} = \Psi_\varepsilon(x) | x \in \partial D \}.
\]

Let \( u_\varepsilon^s \) be the scattered field by \( D_\varepsilon \), which satisfies

\[
\begin{cases}
\Delta u_\varepsilon^s + k^2 u_\varepsilon^s = 0 & \text{in } \mathbb{R}^n \setminus \overline{D_\varepsilon}, \\
u_\varepsilon^s = -u^\text{in} \quad \text{or } \frac{\partial u_\varepsilon^s}{\partial \nu} = -\frac{\partial u^\text{in}}{\partial \nu} & \text{on } \partial D_\varepsilon,
\end{cases}
\]

(1.2)

In this work, we consider the inverse acoustic obstacle scattering problems involve reconstructing the shape perturbation of an obstacle from measurements of scattered fields. These inverse scattering problems are considerably more difficult to solve because they are nonlinear and ill-posed: the solution has an unstable dependence on the input data. We propose a way to determine the shape perturbation of an obstacle \( D \) from boundary measurements on the perturbed obstacle \( D_\varepsilon \); we get relationships between the shape deformation \( h \) and measurements of \( u_\varepsilon^s \) and \( \partial u_\varepsilon^s / \partial \nu \) on \( \partial D_\varepsilon \). In connection with our work, we should mention Lim et al.\(^1\) on the reconstructing small perturbations of bounded scatterers from electric or acoustic far-field measurements and previous works\(^2\text{--}^4\) on the reconstructing of locally small perturbations of half plan from acoustic far-field or near-field measurements.

Let \((v, w) \in H^1(\partial D_\varepsilon) \times H^1(\partial D)\); we define

\[
[v, w, \Psi_\varepsilon, D] := \int_{\partial D} \frac{\partial w}{\partial \nu} \circ \Psi_\varepsilon(x) w(x) d\sigma(x) - \int_{\partial D} v \circ \Psi_\varepsilon(x) \frac{\partial w}{\partial \nu}(x) d\sigma(x)
\]

(1.3)

We denote by \( v^s \) the solution of the following system:

\[
\begin{cases}
\Delta v^s + k^2 v^s = 0 & \text{in } \mathbb{R}^n \setminus \overline{D}, \\
v^s = -v^\text{in} \quad \text{or } \frac{\partial v^s}{\partial \nu} = -\frac{\partial v^\text{in}}{\partial \nu} & \text{on } \partial D,
\end{cases}
\]

(1.4)

The main results of this paper is the following theorem, a rigorous derivation of the leading order term in the asymptotic expansion of \([u_\varepsilon^s, v^s, \Psi_\varepsilon, D]\) as \( \varepsilon \to 0 \), based on the FE method and layer potential techniques.

**Theorem 1.1.** Let \( u_\varepsilon^s, u_\varepsilon^v, \) and \( v^s \) be the solutions of (1.1), (1.2), and (1.4), respectively. For the case of a sound-soft obstacle, we suppose that \( k^2 \) is not an eigenvalue of \(-\Delta\) on \( D \) with Neumann boundary condition and \( u^\text{in} \in C^1(\partial D) \), while for the case of a sound-hard obstacle, we suppose that \( k^2 \) is not an eigenvalue of \(-\Delta\) on \( D \) with Dirichlet boundary condition and \( u^\text{in} \in C^2(\partial D) \). The following asymptotic expansions hold:

\[
[u_\varepsilon^s, v^s, \Psi_\varepsilon, D] = \varepsilon \int_{\partial D} h \left[ \frac{\partial u_\varepsilon^s}{\partial \nu} \frac{\partial v^s}{\partial \nu} + (n - 1) \tau \frac{\partial u_\varepsilon^s}{\partial \nu} v^s - \frac{\partial u_\varepsilon^s}{\partial \nu} \frac{\partial v^s}{\partial \nu} - k^2 u_\varepsilon^s v^s \right] d\sigma + O(\varepsilon^2),
\]

(1.5)

where \( T \) is the tangential vector to \( \partial D \) and \( \tau \) is the mean curvature of \( D \). Here the remainder \( O(\varepsilon^2) \) depends only on the \( C^2\)-norm of \( X \), the \( C^1\)-norm of \( h \), and \( k \).

The term in the left-hand side of (1.5) can be determined by measurements as follows: \( \partial v^s / \partial \nu(x_\varepsilon) \) and \( v^s(x_\varepsilon) \) are computed in some locations \( x_\varepsilon \) on the boundary \( \partial D \) which are supposed to be present before small perturbations of the shape \( D \) and their coordinates are known. We tag these locations with a specific characteristic so that the perturbed locations \( \{ \tilde{x}_\varepsilon \} \) of \( \{ x_\varepsilon \} \) under the deformation can be distinguished (their coordinates are unknown) and can be used to measure the fields \( \partial u_\varepsilon^s / \partial \nu(\tilde{x}_\varepsilon) \) and \( u_\varepsilon^s(\tilde{x}_\varepsilon) \) on \( \partial D_\varepsilon \). These techniques can be applied in applications which use electrodes or sensors to make measurements. Our asymptotic expansions are still valid in the case of small perturbations of a locally half plan \(( \tau = 0 \) \) and an obstacle of a small volume \(( \tau \sim 1/\varepsilon \) \), but more elaborate arguments are needed for proofs. We derive
relationships similar to (1.5) between the shape deformation of an obstacle and boundary measurements of DNO or NDO on the perturbed shape itself.

Assuming that the unknown object boundary is a small perturbation of a circle or a ball. The relationships between the shape deformation of an obstacle and one of boundary measurements of scattered fields, DNO, and NDO are used for determining lower order Fourier coefficients of the shape perturbation of the object.

These relationships could be used to develop effective algorithms to determine certain properties of the shape perturbation of an impenetrable obstacle based on boundary measurements on the perturbed shape itself and to design new tools for solving shape optimization problems: the idea would be to compute the gradient of some target functional using our asymptotic expansions with respect to the shape of the object. To do this, we refer to asymptotic formulae related to measurements in the same spirit, generalized polarization tensors and modal measurements that have been obtained in recent papers.5,6

In this paper, we mainly focus on the derivation of Theorem 1.1 in two dimensions by systematic way, based on the FE method (Theorem 1.1). In Section 4, based on layer potential techniques, we prove that in fact the formal expansion holds in two dimensions (Theorem 1.1). In Section 5, we rigorously derive asymptotic expansions for the NDO and DNO as well as relationships between the shape deformation and one of boundary measurements of DNO or NDO. In the last section, we present algorithms to determine the shape deformation \( h \).

\section{Formal Derivations: FE Method}

The following lemma is of use to us. See, for instance, Colton and Kress\(^8\) and Kirsch.\(^9\)

\textbf{Lemma 2.1.} Let \( v_j \) satisfy (1.4) for \( j = 1, 2 \). Then

\[
\int_{\partial D} \left( \frac{\partial v_1}{\partial \nu} v_2 - v_1 \frac{\partial v_2}{\partial \nu} \right) \, d\sigma = 0. \tag{2.1}
\]

Let \( u_\epsilon^x \) be the solution to (1.2). In order to derive a formal asymptotic expansion for \( u_\epsilon^x \), we apply the FE method (see previous studies\(^1,7,10,11\)). First, we expand \( u_\epsilon^x \) in powers of \( \epsilon \); that is,

\[
u(\epsilon)x = u_0(\epsilon)x + \epsilon u_1(\epsilon)x + \epsilon^2 u_2(\epsilon)x + \cdots, \quad x \in \mathbb{R}^d \setminus \overline{D}_\epsilon, \tag{2.2}\]

where \( u_i \) are defined on \( \mathbb{R}^d \setminus \partial D \). \( u_i \) satisfy

\[
\left\{ \begin{array}{l}
\Delta u_i + k^2 u_i = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D}_\epsilon, \\
\left\| \frac{\partial u_i}{\partial |x|} - i k u_i \right\| = O \left( |x|^{\frac{d}{2} + 1} \right) \quad \text{as} \quad |x| \to \infty.
\end{array} \right. \tag{2.3}\]

In order to justify the first equation in (2.3), we substitute (2.2) in \( (\Delta + k^2)u_\epsilon^x = 0 \) in \( \mathbb{R}^d \setminus \overline{D}_\epsilon \) to get \( \Delta u_i + k^2 u_i = 0 \) in \( \mathbb{R}^d \setminus \overline{D}_\epsilon \) for \( \epsilon > 0 \). Because \( \epsilon \) is arbitrary, we confirm that \( \Delta u_i + k^2 u_i = 0 \) in \( \mathbb{R}^d \setminus \overline{D} \).

From \( u_\epsilon^x(\tilde{x}) = -u^\infty(\tilde{x}) \) (or \( \partial u_\epsilon^x/\partial \nu(\tilde{x}) = -\partial u^\infty/\partial \nu(\tilde{x}) \)) for \( \tilde{x} \in \partial D_{\epsilon} \), we get \( u_0(x) = -u^\infty(x) \) (or \( \partial u_0/\partial \nu(x) = -\partial u^\infty/\partial \nu(x) \)) for \( x \in \partial D \). Note that \( u_0 \equiv u^\infty \) in \( \mathbb{R}^d \setminus \overline{D} \).

\textbf{Two-dimensional case:} Let \( a, b \in \mathbb{R} \), with \( a < b \) and let \( X(t) : [a, b] \to \mathbb{R}^2 \) be the arclength parametrization of \( \partial D \); namely, \( X \) is a \( C^2 \)-function satisfying \( |X'(t)| = 1 \) for all \( t \in [a, b] \) and such that

\[
\partial D := \{ x = X(t), t \in [a, b] \},
\]

with \( X'(t) = T(x) \) and \( X''(t) = \tau(x) \nu(x) \).
By \( \frac{d}{dt} \), we denote the tangential derivative in the direction of \( T(x) \). Let \( \phi(x) \in C^2([a, b]) \) for \( x = X(\cdot) \in \partial D \). We have

\[
\frac{d\phi}{dt}(x) = \frac{\partial \phi}{\partial T}(x) \left( \frac{d}{dt} \right)^2 \phi(x) = \frac{\partial^2 \phi}{\partial T^2}(x) + \tau(x) \frac{\partial \phi}{\partial \nu}(x).
\]

As a consequence, the restriction of \( \Delta + k^2 \) in \( \mathbb{R}^2 \setminus \partial D \) to a neighborhood of \( \partial D \) can be expressed as follows:

\[
\Delta + k^2 = \frac{\partial^2}{\partial \nu^2} - \tau \frac{d}{\partial \nu} + \left( \frac{d}{dt} \right)^2 k^2 \text{ on } \partial D.
\]  

We will sometimes use \( h(t) \) for \( h(X(t)) \) and \( h'(t) \) for the tangential derivative of \( h(x) \). Then, \( \tilde{x} = X(t) = X(t) + ch(t)v(x) \) is a parametrization of \( \partial D_c \).

Let \( x \in \partial D \); then \( \tilde{x} = x + ch(x)v(x) \in \partial D_c \). It was proved in Ammari et al\(^1\)\(^2\) that \( v(\tilde{x}) = v(x) - ch'(t)T(x) + O(e^2) \). Using the Taylor expansion and (2.4), we write

\[
\frac{\partial u_{\tilde{x}}^e}{\partial \nu}(\tilde{x}) = (\nabla u_e(x) + ch(x)\nabla^2 u_e(x)v(x) + e \nabla u_1(x)) \cdot (v(x) - ch'(t)T(x)) + O(e^2)
\]

\[
= \frac{\partial u^e}{\partial \nu}(x) - e \frac{d}{dt} \int_{\partial D} \frac{\partial u^e}{\partial \nu}(x) + e \frac{\partial u_1}{\partial \nu}(x) - c k^2 h(x) u^e(x) + O(e^2),
\]

and

\[
u_{\tilde{x}} = u^e(x) + ch(x) \frac{\partial u^e}{\partial \nu}(x) + eu_1(x) + O(e^2).
\]  

It follows from (2.5) that

\[
\int_{\partial D} \frac{\partial u_{\tilde{x}}^e}{\partial \nu}(x)v^e(x)d\sigma(x) = \int_{\partial D} \frac{\partial u^e}{\partial \nu}v^e d\sigma + e \int_{\partial D} \frac{\partial u_1}{\partial \nu}v^e d\sigma
\]

\[
+ e \int_{\partial D} h \left( \frac{\partial u^e}{\partial T} \frac{\partial v^e}{\partial T} + \frac{\partial u^e}{\partial \nu} \frac{\partial v^e}{\partial \nu} - k^2 u^e v^e \right) d\sigma + O(e^2).
\]  

According to (2.6), we have

\[
\int_{\partial D} u_{\tilde{x}}^e(x) \frac{\partial v^e}{\partial \nu}(x)d\sigma(x) = \int_{\partial D} u^e \frac{\partial v^e}{\partial \nu} d\sigma + e \int_{\partial D} u_1 \frac{\partial v^e}{\partial \nu} d\sigma + e \int_{\partial D} h \frac{\partial u^e}{\partial \nu} \frac{\partial v^e}{\partial \nu} d\sigma + O(e^2).
\]

Subtracting (2.8) from (2.7) yields

\[
[u_{\tilde{x}}^e, v^e, \Psi, D] = e \int_{\partial D} h \left( \frac{\partial u^e}{\partial T} \frac{\partial v^e}{\partial T} + \frac{\partial u^e}{\partial \nu} \frac{\partial v^e}{\partial \nu} - k^2 u^e v^e \right) d\sigma
\]

\[
+ \int_{\partial D} \left( \frac{\partial u^e}{\partial \nu} v^e - u^e \frac{\partial v^e}{\partial \nu} \right) d\sigma + e \int_{\partial D} \left( \frac{\partial u_1}{\partial \nu} v^e - u_1 \frac{\partial v^e}{\partial \nu} \right) d\sigma + O(e^2).
\]  

By Lemma 2.1, the second and the third integrals in the right-hand side of (2.9) vanish. Thus, Theorem 1.1 is proved formally in two dimensions. For proof, see Section 4.

**Three-dimensional case:** Let \( \partial \) be an open subset of \( \mathbb{R}^2 \). Let \( X(\varphi, \theta) \) be an orthogonal parametrization of the surface \( \partial D \); that is,

\[
\partial D := \{ x = X(\varphi, \theta), (\varphi, \theta) \in \partial \}
\]

for \( X \in C^2(\partial) \), where \( \left( X_\varphi := \frac{\partial X}{\partial \varphi}, X_\theta := \frac{\partial X}{\partial \theta} \right) = 0 \). The vectors \( T_\varphi = X_\varphi / |X_\varphi| \) and \( T_\theta = X_\theta / |X_\theta| \) form an orthonormal basis for the tangent plane to \( \partial D \) at \( x = X(\varphi, \theta) \). The tangential derivative on \( \partial D \) is defined by \( \frac{d}{dt} = \frac{\partial}{\partial T_\varphi} T_\varphi + \frac{\partial}{\partial T_\theta} T_\theta. \)
Let $G$ be the matrix of the first fundamental form with respect to the basis $\{X_\varphi, X_\theta\}$ which is given by

$$G = \begin{pmatrix} |X_\varphi|^2 & 0 \\ 0 & |X_\theta|^2 \end{pmatrix}.$$ 

For $v \in C^2(\theta)$, the gradient operator in local coordinates satisfies

$$\nabla_{\varphi, \theta} v = \frac{\partial v}{\partial T_\varphi} T_\varphi + \frac{\partial v}{\partial T_\theta} T_\theta, \quad G^{-1} \nabla_{\varphi, \theta} v = \frac{1}{\sqrt{G_{\varphi \varphi}}} \frac{\partial v}{\partial T_\varphi} T_\varphi + \frac{1}{\sqrt{G_{\theta \theta}}} \frac{\partial v}{\partial T_\theta} T_\theta,$$

and the restriction of $\Delta + k^2$ in $\mathbb{R}^3 \setminus \partial D$ to a neighborhood of $\partial D$ can be expressed as follows:

$$\Delta v + k^2 v = \frac{\partial^2 v}{\partial v^2} - 2 \tau \frac{\partial v}{\partial \nu} + \frac{1}{\sqrt{\det G}} \nabla_{\varphi, \theta} \cdot \left( \sqrt{\det G} G^{-1} \nabla_{\varphi, \theta} v \right) + k^2 v \text{ on } \partial D.$$ 

We use $h(\varphi, \theta)$ for simplifying the term $h(X(\varphi, \theta))$ and $h_\varphi(\varphi, \theta)$, $h_\theta(\varphi, \theta)$ for the tangential derivatives of $h(X(\varphi, \theta))$. Then, $\bar{x} = X(\varphi, \theta) + ch(\varphi, \theta) v(x)$ is a parametrization of $\partial D_c$. It was proved in Khelifi and Zribi$^7$ that

$$v(\bar{x}) = v(x) - \epsilon \left( \frac{h_\varphi}{\sqrt{G_{\varphi \varphi}}} T_\varphi + \frac{h_\theta}{\sqrt{G_{\theta \theta}}} T_\theta \right) + O(\epsilon^2).$$

Let $\bar{x} = x + ch(\varphi) v(x) \in \partial D_t$ for $x \in \partial D$. The following Taylor expansions hold:

$$\frac{\partial u^i}{\partial \nu}(\bar{x}) = \left( \nabla u^i(x) + ch(x) \nabla^2 u^i(x) v(x) + \epsilon \nabla u_1(x) \right) \cdot v(\bar{x}) + O(\epsilon^2)$$

$$= \frac{\partial u^i}{\partial \nu}(x) + 2 \tau(x) h(x) \frac{\partial u^i}{\partial \nu}(x) + \epsilon \frac{\partial u_1}{\partial \nu}(x) - ck^2 h(x) u^i(x)$$

$$- \frac{\epsilon}{\sqrt{\det G}} \nabla_{\varphi, \theta} \cdot \left( h(x) \sqrt{\det G} G^{-1} \nabla_{\varphi, \theta} u^i(x) \right) + O(\epsilon^2),$$

and

$$u^i(\bar{x}) = u^i(x) + ch(x) \frac{\partial u^i}{\partial \nu}(x) + \epsilon u_1(x) + O(\epsilon^2).$$

Inserting the two expansions in (2.11) and (2.12) into (1.3), we obtain

$$[u^i, v^i, \Psi_c, D] = \epsilon \int_{\partial D} h \left( 2 \tau \frac{\partial u^i}{\partial \nu} v^i - \frac{\partial u^i}{\partial \nu} \frac{\partial v^i}{\partial \nu} - k^2 u^i v^i \right) d\sigma$$

$$- \epsilon \int_{\partial D} \frac{1}{\sqrt{\det G}} \nabla_{\varphi, \theta} \cdot \left( h \sqrt{\det G} G^{-1} \nabla_{\varphi, \theta} u^i \right) v^i d\sigma$$

$$+ \int_{\partial D} \left( \frac{\partial u^i}{\partial \nu} v^i - u^i \frac{\partial v^i}{\partial \nu} \right) d\sigma + \epsilon \int_{\partial D} \left( \frac{\partial u_1}{\partial \nu} v^i - u_1 \frac{\partial v^i}{\partial \nu} \right) d\sigma + O(\epsilon^2).$$
According to Lemma 2.1, the fourth and the fifth integrals in the right-hand side of (2.13) vanish. By integrating by parts and (2.10), we find that

\[
\int_{\partial D} \frac{1}{\sqrt{\det G}} \nabla_{\varphi, \theta} \cdot \left( h \sqrt{\det G^{-1}} \nabla_{\varphi, \theta} u^t \right) v^t \, d\sigma = -\int_{\partial D} \frac{h}{\sqrt{\det G}} \nabla_{\varphi, \theta} \cdot \left( \nabla_{\varphi, \theta} u^t v^t \right) \, d\sigma \\
= -\int_{\partial D} h \left( \frac{\partial v^t}{\partial \varphi^t} \right) \, d\sigma \\
= -\int_{\partial D} h \left( \frac{\partial v^t}{\partial \varphi^t} \right) \, d\sigma.
\]

Thus, Theorem 1.1 is proved formally in three dimensions.

### 3 Layer Potentials for Helmholtz Equation

#### 3.1 Definitions and Preliminary Results

We start to review some basic facts in the theory of layer potentials. Let \( \Gamma_k(x) \) be the fundamental solution of \( \Delta + k^2 \) in \( \mathbb{R}^2 \); that is, for \( x \neq 0 \),

\[
\Gamma_k(x) = -\frac{i}{4} H^1_0(k|x|),
\]

where \( H^1_0 \) is the Hankel function of the first kind of order 0. We have the following Taylor expansion of \( H^1_0(x) \) as \( |x| \to 0 \):

\[
-\frac{i}{4} H^1_0(k|x|) = \frac{1}{2\pi} \sum_{n=0}^{\infty} (-1)^n \frac{k^{2n}}{2^{2n}(n!)^2} |x|^{2n} \left( \ln(|x|) + \ln(k) - \sum_{j=1}^{n} \frac{1}{j} \right),
\]

(3.1)

where \( 2\gamma = \phi^2 - i\pi/2 \) and \( \gamma \) is Euler's constant.

According to Leibniz's rule, the \( p \)th derivative of \( r^{2n} \ln(r) \) is given by

\[
(r^{2n} \ln(r))^{(p)} = \sum_{l=0}^{p} C_p^l (r^{2n})^{(l)} (\ln(r))^{(p-l)} = (r^{2n})^{(p)} \ln(r) + \sum_{l=0}^{p-1} C_p^l (r^{2n})^{(l)} \left( \frac{1}{r} \right)^{(p-l-1)},
\]

where \( C_p^l \) is a binomial coefficient, and then, it follows from (3.1) that

\[
-\frac{ik^p}{4} H^1_0(kr)r^p \text{ is continuous at zero for } p \geq 1.
\]

For a bounded domain \( D \) in \( \mathbb{R}^2 \) and \( k > 0 \), let \( S^k_D \) and \( D^k_D \) be the single and double layer potentials defined by \( \Gamma_k \); that is,

\[
S^k_D[\phi](x) = \int_{\partial D} \Gamma_k(x-y)\phi(y) \, d\sigma(y), \ x \in \mathbb{R}^2,
\]

\[
D^k_D[\phi](x) = \int_{\partial D} \frac{\partial \Gamma_k(x-y)}{\partial n(y)} \phi(y) \, d\sigma(y), \ x \in \mathbb{R}^2 \setminus \partial D.
\]

It is well known (see Theorem 3.1 of Colton and Kress) that

\[
\frac{\partial S^k_D[\phi]}{\partial n} \bigg|_{\pm} (x) = \left( \pm \frac{1}{2} I + (K^k_D)^+ \right) [\phi](x) \text{ a.e.} x \in \partial D,
\]

(3.2)
The following uniqueness result for the exterior Helmholtz problem holds (see Colton and Kress\textsuperscript{8} and Ammari et al\textsuperscript{15}).

\[ D^k_D[\phi] \bigg|_{+}(x) = \left( \mp \frac{1}{2} I + K^k_D \right) [\phi](x) \text{ a.e.} x \in \partial D. \quad (3.3) \]

for \( \phi \in L^2(\partial D) \), where \( K^k_D \) is the operator on \( L^2(\partial D) \) defined by

\[ K^k_D[\phi](x) = \text{p.v.} \int_{\partial D} \frac{\partial k(x - y)}{\partial \nu(y)} \phi(y) d\sigma(y), \]

and \((K^k_D)^*\) is the \( L^2 \)-adjoint of \( K^k_D \). Here, p.v. denotes the Cauchy principal value. The operator \( K^k_D \) is known to be bounded on \( L^2(\partial D) \).\textsuperscript{14}

If \( D \) has a \( C^2 \) boundary and \( \phi \in H^1(\partial D) \), then \( \partial(D^k_D[\phi]) / \partial \nu \) does not have a jump across \( \partial D \); that is,

\[ \frac{\partial D^k_D[\phi]}{\partial \nu} \bigg|_{+}(x) = \frac{\partial D^k_D[\phi]}{\partial \nu} \bigg|_{-}(x), \; x \in \partial D. \quad (3.4) \]

Recall that the operators \( \frac{\partial D^k_D[\phi]}{\partial \nu^2} \left( \frac{d}{dt} \right)^2 D^k_D[\phi] \) and \( D^k_D[\phi] \) are not continuous on \( \partial D \), but it follows from \( (\Delta + k^2)D^k_D[\phi] = 0 \) in \( \mathbb{R}^2 \setminus \partial D \) and (2.4) that \( \frac{\partial D^k_D[\phi]}{\partial \nu^2} \left( \frac{d}{dt} \right)^2 D^k_D[\phi] + k^2 D^k_D[\phi] \) is continuous on \( \partial D \) and we have

\[ \frac{\partial^2 K^k_D[\phi]}{\partial \nu^2} + \left( \frac{d}{dt} \right)^2 K^k_D[\phi] + k^2 K^k_D[\phi] = \tau \frac{\partial D^k_D[\phi]}{\partial \nu} \text{ on } \partial D. \quad (3.5) \]

If \( \phi \in C^2(\partial D) \), then we get from (3.3) and (3.5) that

\[ \frac{\partial^2 D^k_D[\phi]}{\partial \nu^2} \bigg|_{\pm} = \pm \frac{1}{2} \left( \frac{d}{dt} \right)^2 \phi \pm \frac{k^2}{2} \phi + \frac{\partial^2 K^k_D[\phi]}{\partial \nu^2} \text{ on } \partial D. \]

The following uniqueness result for the exterior Helmholtz problem holds (see Colton and Kress\textsuperscript{8} and Ammari et al\textsuperscript{15}).

**Lemma 3.1.** Let \( D \) be a bounded Lipschitz domain in \( \mathbb{R}^2 \). Let \( w \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus D) \) satisfy

\[
\begin{cases}
  \Delta w + k^2 w = 0 \text{ in } \mathbb{R}^2 \setminus D, \\
  |\frac{\partial w}{\partial r} - ikw| = O \left( \frac{1}{r^2} \right) \text{ as } r = |x| \to \infty \text{ uniformly in } \frac{c}{|x|}, \\
  w = 0 \text{ or } \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial D.
\end{cases}
\]

Then, \( w \equiv 0 \) in \( \mathbb{R}^2 \setminus D \).

The following lemma is important for us.

**Lemma 3.2.** The following properties hold:

1. Suppose that \( D \) is of class \( C^2 \). If \( k^2 \) is not an eigenvalue of \( -\Delta \) on \( D \) with Dirichlet boundary condition, then the operator \( (1/2)I + K^k_D : L^2(\partial D) \to L^2(\partial D) \) is invertible.

2. Let \( D \) be a bounded Lipschitz domain. If \( k^2 \) is not an eigenvalue of \( -\Delta \) on \( D \) with Neumann boundary condition, then the operator \( -(1/2)I + (K^k_D)^* : L^2(\partial D) \to L^2(\partial D) \) is invertible.

**Proof.** The operators \( K^k_D \) and \( (K^k_D)^* \) are compact. Therefore, we can apply the Reisz–Fredholm theory. Let \( \phi \in L^2(\partial D) \) such that \( (1/2)I + K^k_D \) \( \phi \) = 0. Then, \( v(x) = D^k_D[\phi] \) on \( D \) is a solution to \( \Delta v + k^2 v = 0 \) with the boundary condition \( v|_{-} = 0 \) on \( \partial D \). If \( k^2 \) is not an eigenvalue of \( -\Delta \) on \( D \) with Dirichlet boundary condition, then \( D^k_D[\phi] = 0 \) in \( D \). Since \( \partial(D^k_D[\phi]) / \partial \nu \) exists and has no jump across \( \partial D \), we get

\[ \frac{\partial D^k_D[\phi]}{\partial \nu} \bigg|_{+} = \frac{\partial D^k_D[\phi]}{\partial \nu} \bigg|_{-} = 0 \text{ on } \partial D. \]
One easily checks that \( \nu \) is a solution to \( \Delta \nu + k^2 \nu = 0 \) on \( \mathbb{R}^2 \setminus D \) with the boundary condition \( \partial \nu / \partial \nu |_+ = 0 \) on \( \partial D \) and satisfies the radiation condition. The uniqueness result in Lemma 3.1 implies that \( D^k_{\nu}[\phi] = 0 \) in \( \mathbb{R}^2 \setminus D \). Therefore, we conclude
\[
\phi = D^k_{\nu}[\phi] - D^k_{\nu}[\phi] |_+ = 0.
\]
Suppose now \( \left(-\frac{1}{2}I + \left( K_D^k \right)^*\right)[\psi] = 0 \). Define \( w := S^k_{\nu}[\psi] \) on \( \mathbb{R}^2 \setminus D \). Therefore, \( w \) is the solution to \( \Delta w + k^2 w = 0 \) in \( D \) with the boundary condition \( \partial w / \partial \nu |_- = 0 \) on \( \partial D \). If \( k^2 \) is not an eigenvalue of \( -\Delta \) on \( D \) with Neumann boundary condition, then \( S^k_{\nu}[\psi] = 0 \) in \( D \). Furthermore, \( w \) is continuous in \( \mathbb{R}^2 \); thus, \( w \) is a solution to \( \Delta w + k^2 w = 0 \) on \( \mathbb{R}^2 \setminus D \) with the boundary condition \( w |_+ = 0 \) on \( \partial D \) and satisfies the radiation condition. The uniqueness result of Lemma 3.1 yields that \( S^k_{\nu}[\psi] = 0 \) in \( \mathbb{R}^2 \setminus D \) and hence
\[
\psi = \frac{\partial S^k_{\nu}[\psi]}{\partial \nu} |_+ - \frac{\partial S^k_{\nu}[\psi]}{\partial \nu} |_- = 0.
\]

### 3.2 Asymptotic of layer potentials

Let \( \tilde{x}, \tilde{y} \in \partial D_c \); that is,
\[
\tilde{x} = x + \epsilon h(x) \nu(x), \quad \tilde{y} = y + \epsilon h(y) \nu(y),
\]
for \( x = X(t), y = X(s) \in \partial D \). By \( \nu(\tilde{y}) \) and \( d\sigma(\tilde{y}) \), we denote the unit outward unit normal and the length element to \( \partial D_c \) at \( \tilde{y} \), respectively. It was proved in Ammari et al\textsuperscript{12} that
\[
\nu(\tilde{y}) = \frac{\nu(y) - \epsilon \left( h(y) \tau(y) \nu(y) + h'(s) T(y) \right)}{\sqrt{(1 - \epsilon h(y) \tau(y))^2 + \epsilon^2 (h'(s))^2}}, \tag{3.6}
\]
and
\[
d\sigma(\tilde{y}) = \sqrt{(1 - \epsilon h(y) \tau(y))^2 + \epsilon^2 (h'(s))^2} d\sigma(y). \tag{3.7}
\]
Since,
\[
\tilde{y} - \tilde{x} = y - x + \epsilon \left(h(y) \nu(y) - h(x) \nu(x)\right), \tag{3.8}
\]
which yields
\[
|\tilde{y} - \tilde{x}|^2 = |y - x|^2 \left( 1 + 2 \epsilon \frac{\langle y - x, h(y) \nu(y) - h(x) \nu(x) \rangle}{|y - x|^2} + \epsilon^2 \frac{|h(y) \nu(y) - h(x) \nu(x)|^2}{|y - x|^2} \right), \tag{3.9}
\]
and hence,
\[
\frac{1}{|\tilde{y} - \tilde{x}|^2} = \frac{1}{|y - x|^2} \cdot \frac{1}{1 + 2 \epsilon F(x, y) + \epsilon^2 G(x, y)}. \tag{3.10}
\]
where
\[
F(x, y) = \frac{\langle y - x, h(y) \nu(y) - h(x) \nu(x) \rangle}{|y - x|^2}, \quad G(x, y) = \frac{|h(y) \nu(y) - h(x) \nu(x)|^2}{|y - x|^2}.
\]
One can easily see that
\[
|F(x, y)| + |G(x, y)| \leq C \|X\|_{c(\partial D)} \|h\|_{c(\partial D)} \text{ for all } x, y \in \partial D.
\]
In order to prove the asymptotic expansion of the operator \( K_D^k \), we investigate
\[
(kH_0^k(k|\tilde{y} - \tilde{x}|)|\tilde{y} - \tilde{x}|) \frac{\langle \tilde{y} - \tilde{x}, \nu(\tilde{y}) \rangle}{|\tilde{y} - \tilde{x}|^2} d\sigma(\tilde{y}).
\]
By using (3.9), we write
\[
kH_0^k(k|\tilde{y} - \tilde{x}|)|\tilde{y} - \tilde{x}| = \sum_{n=0}^{\infty} \epsilon^n \mathbb{H}_n(x, y), \tag{3.11}
\]
where the series converges absolutely and uniformly. In particular,
\[
\mathbb{H}_0(x, y) = kH_0^k(k|y - x|)|y - x|,
\]
and 
\[
\mathbb{H}_1(x, y) = \left[ k^2 H_0^{(1)}(k|x - y|) + k H_0^{(1)}(k|x - y|) \right] \frac{y - x, h(y)\nu(y) - h(x)\nu(x)}{|y - x|}.
\]

It follows from (3.6), (3.7), (3.8), and (3.10) that 
\[
\frac{\langle \bar{y} - \bar{x}, \nu(\bar{y}) \rangle}{|\bar{y} - \bar{x}|^2} d\sigma(\bar{y}) = \frac{y - x + \epsilon \left( h(y)\nu(y) - h(x)\nu(x) \right)}{|y - x|^2} \times \frac{1}{1 + 2\epsilon F(x, y) + \epsilon^2 G(x, y)} d\sigma(y) 
\]
\[
:= \sum_{n=0}^{\infty} e^n M_n(x, y) d\sigma(y),
\]

where the series converges absolutely and uniformly. In particular, one can easily see that

\[
M_0(x, y) = \frac{y - x, \nu(y)}{|y - x|^2},
\]

and

\[
M_1(x, y) = h(x) \left( -\frac{(\nu(x), \nu(y))}{|y - x|^2} + 2 \frac{y - x, \nu(y)}{|y - x|^4} \right) 
\]
\[
+ \left( \frac{h(y)}{|y - x|^2} - 2h(y) \right) \frac{(y - x, \nu(y))^2}{|y - x|^4} - \frac{y - x, h(y)\tau(y)\nu(y) + h'(s)T(y)}{|y - x|^2} \right) .
\]

Thus, we obtain from (3.11) and (3.12) that
\[
(k H_0^{(1)}(k|\bar{y} - \bar{x}|)|\bar{y} - \bar{x}| \frac{\langle \bar{y} - \bar{x}, \nu(\bar{y}) \rangle}{|\bar{y} - \bar{x}|^2} d\sigma(\bar{y}) = \sum_{n=0}^{\infty} e^n \sum_{m=0}^{\infty} M_n(x, y) H_n^{(1-m)}(x, y) d\sigma(y),
\]

with

\[
-i \frac{1}{4} \mathbb{K}_0(x, y) = -i k H_0^{(1)}(k|\bar{x}|^2) \frac{y - x, \nu(x)}{|y - x|^2} = \frac{\partial \Gamma_k(x - y)}{\partial \nu(y)},
\]

and

\[
-i \frac{1}{4} \mathbb{K}_1(x, y) = h(x) \left[ \frac{ik^2}{4} H_0^{(1)}(k|\bar{x}|^2) \frac{y - x, \nu(x)}{|y - x|^2} \right] 
\]
\[
+ \frac{ik}{4} H_0^{(1)}(k|\bar{x}|^2) \left( \frac{\nu(x), \nu(y)}{|y - x|^2} + \frac{y - x, \nu(x)}{|y - x|^4} \frac{y - x, \nu(y)}{|y - x|^4} \right) \right]
\]
\[
+ h(y) \left[ -\frac{ik}{4} H_0^{(1)}(k|\bar{x}|^2) \frac{(y - x, \nu(y))^2}{|y - x|^2} \right] 
\]
\[
- \frac{ik}{4} H_0^{(1)}(k|\bar{x}|^2) \left( \frac{1}{|y - x|^2} - \frac{(y - x, \nu(y))^2}{|y - x|^4} \right) \right]
\]
\[
+ \frac{ik}{4} H_0^{(1)}(k|\bar{x}|^2) \frac{y - x, h(y)\tau(y)\nu(y) + h'(s)T(y)}{|y - x|^2} .
\]

Note that
\[
-i \frac{1}{4} \mathbb{K}_1(x, y) = h(x) \frac{\partial^2 \Gamma_k(x - y)}{\partial \nu(x) \partial \nu(y)} + h(y) \frac{\partial^2 \Gamma_k(x - y)}{\partial \nu(y)^2} 
\]
\[
- \tau(y) h(y) \frac{\partial \Gamma_k(x - y)}{\partial \nu(y)} - h'(s) \frac{\partial \Gamma_k(x - y)}{\partial T(y)} 
\]
\[
= h(x) \frac{\partial^2 \Gamma_k(x - y)}{\partial \nu(x) \partial \nu(y)} - \frac{d}{ds} \left( h(y) \frac{\partial \Gamma_k(x - y)}{\partial s} \right) - k^2 h(y) \Gamma_k(x - y).
\]

In order to justify the last equality, we use \((\Delta + k^2)\Gamma_k(x - y) = 0\) for \(x \neq y\) and the representation of \(\Delta + k^2\) on \(\partial D\) given in (2.4).
Introduce a sequence of integral operators \( \left( D_{D,n}^k \right)_{n \in \mathbb{N}} \), defined for any \( \phi \in L^2(\partial D) \) by

\[
D_{D,n}^k[\phi](x) = -\frac{i}{4} \int_{\partial D} \|k_n(x, y)\phi(y)\,d\sigma(y) \quad \text{for} \quad n \geq 0,
\]

where \( D_{D,0}^k = K^k_D \) and for \( \phi \in C^2(\partial D) \) we have

\[
D_{D,1}^k[\phi](x) = -k^2 S_0^k[h\phi](x) + h(x) \frac{\partial D_{D,1}^k[\phi]}{\partial \nu}(x) - S_0^k \left( \frac{d}{ds} \left( h \frac{d\phi}{ds} \right) \right)(x), \quad x \in \partial D.
\]

It follows from (3.6), (3.7), and (3.10) that

\[
\text{It is easy to prove that the operator } D_{D,n}^k \text{ for } n \geq 1 \text{ with the kernel } \|k_n(x, y)\text{ is bounded in } L^2(\partial D). \text{ In fact, it is an immediate consequence of the celebrate theorem of Coifman–MacIntosh–Meyer (see Coifman et al.14).}
\]

In order to establish the asymptotic expansion of the operator \( \partial(D_{D,n}) / \partial \nu \) on \( \partial D, \) we next investigate the following terms:

\[
\left[ kH_0^\nu(k|x - y|)|x - y| \right] \frac{\langle \nu(x), \nu(y) \rangle}{|x - y|^2} d\sigma(y),
\]

and

\[
\left[ -k^2H_0^\nu(k|x - y|)|x - y|^2 + kH_0^\nu(k|x - y|)|x - y| \right] \frac{\langle \nu(x), \nu(y) \rangle}{|x - y|^2} d\sigma(y).
\]

It follows from (3.6), (3.7), and (3.10) that

\[
\frac{\langle \nu(x), \nu(y) \rangle}{|x - y|^2} d\sigma(y) = \frac{\langle \nu(x), \nu(y) \rangle}{|x - y|^2} \times \frac{1}{1 + 2 \epsilon F(x, y) + \epsilon^2 G(x, y)} \sqrt{(1 - \epsilon h(x) \tau(x))^2 + \epsilon^2 (h'(t))^2}
\]

\[
\sum_{n=0}^{\infty} \frac{\epsilon^n}{\epsilon^n} \mathbb{I}_n(x, y) d\sigma(y),
\]

with

\[
\mathbb{I}_n(x, y) = \frac{\langle \nu(x), \nu(y) \rangle}{|x - y|^2},
\]

and

\[
\mathbb{I}_1(x, y) = \tau(x) h(x) \frac{\langle \nu(x), \nu(y) \rangle}{|x - y|^2}
\]

\[
+ 2h(x) \frac{\langle y - x, \nu(x) \rangle \langle \nu(x), \nu(y) \rangle}{|x - y|^4} - \frac{\langle h(x) \tau(x) \nu(x), \nu(y) \rangle}{|x - y|^2} \]

\[
- 2h(y) \frac{\langle y - x, \nu(y) \rangle \langle \nu(x), \nu(y) \rangle}{|x - y|^4} - \frac{\langle h(y) \tau(y) \nu(y), \nu(x) \rangle}{|x - y|^2}.
\]

We get from (3.11) and (3.14) that

\[
\left[ kH_0^\nu(k|x - y|)|x - y| \right] \frac{\langle \nu(x), \nu(y) \rangle}{|x - y|^2} d\sigma(y) = \left[ kH_0^\nu(k|x - y|)|x - y| \right] \frac{\langle \nu(x), \nu(y) \rangle}{|x - y|^2} d\sigma(y)
\]

\[
+ \sum_{n=1}^{\infty} \epsilon^n \sum_{m=0}^{n} \mathbb{II}_m(x, y) \mathbb{I}_{n-m}(x, y) d\sigma(y).
\]
Using (3.6), (3.8), and (3.10), we obtain
\[
\frac{\langle y - x, \nu(x) \rangle}{|x - y|^2} = \frac{\langle y - x + \epsilon (h(y)\nu(y) - h(x)\nu(x)) + \nu(x) - \epsilon \left( h(x)\tau(x)\nu(x) + h'(t)T(x) \right) \rangle}{|x - y|^2} \times \frac{1}{1 + 2\epsilon F(x, y) + \epsilon^2 G(x, y)} \frac{1}{\sqrt{(1 - \epsilon h(x)\tau(x))^2 + \epsilon^2 h'(t)^2}}
\]

where
\[
\mathbb{N}_0(x, y) = \frac{\langle y - x, \nu(x) \rangle}{|x - y|^2},
\]
and
\[
\mathbb{N}_1(x, y) = \tau(x)h(x) \frac{\langle y - x, \nu(x) \rangle}{|x - y|^2} - \frac{\langle y - x, h(x)\tau(x)\nu(x) + h'(t)T(x) \rangle}{|x - y|^2} + h(y) \left( \frac{\langle \nu(x), \nu(y) \rangle}{|x - y|^2} - 2 \frac{\langle y - x, \nu(y) \rangle \langle y - x, \nu(x) \rangle}{|x - y|^4} \right) + h(x) \left( - \frac{1}{|x - y|^2} + 2 \frac{\langle y - x, \nu(x) \rangle^2}{|x - y|^4} \right).
\]

By the Taylor expansion and (3.9), we get
\[
k^2H_0^{(\nu)}(k|\bar{x} - \bar{y}|)|\bar{y} - \bar{x}|^2 := \sum_{n=0}^{\infty} S_n(x, y) = k^2H_0^{(\nu)}(k|y - x|)|y - x|^2 + \sum_{n=1}^{\infty} S_n(x, y),
\]
with
\[
S_1(x, y) = [k^3H_0^{(\nu)}(k|y - x|) + 2k^2H_0^{(\nu)}(k|y - x|)] \langle y - x, h(y)\nu(y) - h(x)\nu(x) \rangle.
\]

Combining (3.11), (3.14), (3.16), and (3.17) yields the expansion
\[
[-k^2H_0^{(\nu)}(k|\bar{x} - \bar{y}|)|\bar{x} - \bar{y}|^2 + kH_0^{(\nu)}(k|x - \bar{y}|)|\bar{x} - \bar{y}|] \frac{\langle \bar{y} - \bar{x}, \nu(\bar{x}) \rangle \langle \bar{y} - \bar{x}, \nu(\bar{y}) \rangle}{|\bar{x} - \bar{y}|^2} d\sigma(\bar{y})
\]
\[
= [-k^2H_0^{(\nu)}(k|x - y|)|x - y|^2 + kH_0^{(\nu)}(k|x - y|)|x - y|] \frac{\langle y - x, \nu(x) \rangle \langle y - x, \nu(y) \rangle}{|x - y|^2} d\sigma(y)
\]
\[
+ \sum_{n=1}^{\infty} e^n \sum_{m+p+q=1} S_n(x, y) \mathbb{N}_p(x, y) M_q(x, y) d\sigma(y).
\]

Thanks to (3.15) and (3.18), we write
\[
\frac{\partial^2 \Gamma_k(x - y)}{\partial \nu(x) \partial \nu(y)} d\sigma(y) := -\frac{\epsilon}{4} \sum_{n=0}^{\infty} e^n \mathbb{E}_n(x, y) d\sigma(y),
\]
where
\[
-\frac{i}{4} \mathbb{E}_0(x, y) = \frac{\partial^2 \Gamma_k(x - y)}{\partial \nu(x) \partial \nu(y)}.
\]
and

\[-i \frac{1}{4} \mathbb{B}_1(x, y) = h(x) \frac{\partial^3 \Gamma_k(x - y)}{\partial v^2(x) \partial v(y)} - h'(t) \frac{\partial^2 \Gamma_k(x - y)}{\partial T(x) \partial v(y)} \]

\[+ h(y) \frac{\partial \Gamma_k(x - y)}{\partial v(x) \partial v(y)} - \tau(y) h(y) \frac{\partial^2 \Gamma_k(x - y)}{\partial v(x) \partial v(y)} - h'(s) \frac{\partial \Gamma_k(x - y)}{\partial v(x) \partial T(y)} \]

\[= h(x) \frac{\partial \Gamma_k(x - y)}{\partial v(x) \partial v(y)} - h'(t) \frac{\partial^2 \Gamma_k(x - y)}{\partial T(x) \partial v(y)} - \frac{\partial}{\partial v(x)} \frac{d}{ds} \left( h(y) \frac{d \Gamma_k(x - y)}{ds} \right) - k^2 h(y) \frac{\partial \Gamma_k(x - y)}{\partial v(x)}. \tag{3.20} \]

Introduce a sequence of integral operators \( A_{D,n}^k \) defined for any \( \phi \in L^2(\partial D) \) by

\[ A_{D,n}^k(\phi)(x) = -i \frac{1}{4} \int_{\partial D} \mathbb{B}_n(x, y) \phi(y) d\sigma(y) \text{ for } n \geq 0, \]

with \( A_{D,0}^k = \partial(D_n^k)/\partial v \). If \( \phi \in C^2(\partial D) \), we get from (3.5) and (3.20) that

\[ A_{D,1}^k(\phi)(x) = \tau(h(x)) \frac{\partial (D_n^k(\phi))}{\partial v}(x) - k^2 (K_n^k)^*(h\phi)(x) - k^2 h(x) K_n^k(\phi)(x) \]

\[ - (K_n^k)^*(\left( \frac{d}{ds} \left( h \frac{d \phi}{ds} \right) \right)(x) - \frac{d}{dt} \left( h \frac{d (K_n^k(\phi))}{dt} \right)(x) \]

\[ = \tau(h(x)) \frac{\partial (D_n^k(\phi))}{\partial v}(x) - k^2 (S_n^k(\phi))(x) \]

\[ = -\frac{\partial S_n^k}{\partial v} \left( \left( \frac{d}{ds} \left( h \frac{d \phi}{ds} \right) \right)(x) - \frac{d}{dt} \left( h \frac{d (S_n^k(\phi))}{dt} \right)(x) \right) \text{ for } x \in \partial D. \tag{3.21} \]

The operator \( A_{D,n}^k \) is bounded in \( L^2(\partial D) \) for \( n \geq 1 \). In fact, it is an immediate consequence of the celebrate theorem of Coifman–MacIntosh–Meyer.14

The results of the above asymptotic analysis is summarized in the following theorem.

**Theorem 3.3.** Let \( N \in \mathbb{N} \). There exists \( C \) depending only on \( k, ||X||_{C^2}, \text{ and } ||h||_{C^1} \), such that for any \( \phi_e \in L^2(\partial D_e) \), we have

\[ \left\| D_{D_n}^k(\phi_e) \circ \Psi_e \right\|_{L^2(\partial D)} \leq \left\| D_{D_n}^k(\phi) \right\|_{L^2(\partial D)} \leq C e^N \| \phi \|_{L^2(\partial D)}, \tag{3.22} \]

and

\[ \left\| \frac{\partial D_{D_n}^k(\phi_e)}{\partial v} \circ \Psi_e - \frac{\partial D_{D_n}^k(\phi)}{\partial v} - \sum_{n=1}^N e^n A_{D,n}^k(\phi) \right\|_{L^2(\partial D)} \leq C e^N \| \phi \|_{L^2(\partial D)}, \tag{3.23} \]

where \( \phi := \phi_e \circ \Psi_e \).

For \( \phi \in L^2(\partial D) \), we introduce

\[ S_{D,n}^k(\phi)(x) = -S_{D_n}^k(\tau h \phi)(x) + h(K_n^k)^*(\phi)(x) + K_n^k(h \phi)(x) \]

\[ = -S_{D_n}^k(\tau h \phi)(x) + \left( \frac{\partial S_{D_n}^k(\phi)}{\partial v} + D_n^k(h \phi) \right)(x), \quad x \in \partial D, \tag{3.24} \]
and
\[
K_{D,1}^k[\phi](x) = \tau(x)h(x)(K_D^k)^*[\phi](x) - K_D^k[rh\phi](x) + \frac{\partial(D_D^k[\phi])}{\partial v}(x) - \frac{d}{dt}\left(h \frac{d(S_D^k[\phi])}{dt}\right)(x) - k^2h(x)S_D^k[\phi](x),
\]
\[
= \left(\tau h \frac{\partial(S_D^k[\phi])}{\partial v} - \frac{\partial(S_D^k[rh\phi])}{\partial v}\right)(x) - k^2h(x)S_D^k[\phi](x),
\]
\[
\pm\left(\frac{\partial(D_D^k[\phi])}{\partial v}(x) - \frac{d}{dt}\left(h \frac{d(S_D^k[\phi])}{dt}\right)(x) - k^2h(x)S_D^k[\phi](x),\right) x \in \partial D.
\]

It was proved in Zribi\cite{Zribi11} that the operators $S_{D,1}^k$ and $K_{D,1}^k$ are bounded in $L^2(\partial D)$ and the following proposition holds.

**Proposition 3.4.** There exists $C$ depending only on $k, ||X||_{C^1}$, and $||h||_{C^1}$, such that for any $\phi \in L^2(\partial D)$, we have
\[
\left\| S_{D,1}^k[\phi_c] \circ \Psi_c - S_{D,1}^k[\phi] - e S_{D,1}^k[\phi] \right\|_{L^2(\partial D)} \leq C e^2 ||\phi||_{L^2(\partial D)},
\]
and
\[
\left\| \frac{\partial S_{D,1}^k[\phi_c]}{\partial v} \circ \Psi_c - \frac{\partial S_{D,1}^k[\phi]}{\partial v} \right\|_{L^2(\partial D)} \leq C e^2 ||\phi||_{L^2(\partial D)},
\]

where $\phi_c := \phi_c \circ \Psi_c$.

### 4 PROOF OF THEOREM 1.1

The solutions of (1.1) and (1.2) are given by (see previous works\cite{8,9,16})
\[
u^c(x) = S_{D,1}^k\left(\frac{\partial u^c}{\partial v}\right)(x) - D_{D,1}^k(u^c)(x), x \in \mathbb{R}^2 \setminus \bar{D},
\]
and
\[
u^c(x) = S_{D,1}^k\left(\frac{\partial u^c}{\partial v}\right)(x) - D_{D,1}^k(u^c)(x), x \in \mathbb{R}^2 \setminus \bar{D}.
\]

The following lemma holds.

**Lemma 4.1.** Let $\nu^c$ and $\nu^c$ be the solutions of (1.1) and (1.2), respectively. For the case of a sound-soft obstacle, we suppose that $k^2$ is not an eigenvalue of $-\Delta$ on $D$ with Neumann boundary condition and $u^{in} \in C^1(\partial D)$, while for the case of a sound-hard obstacle, we suppose that $k^2$ is not an eigenvalue of $-\Delta$ on $D$ with Dirichlet boundary condition and $u^{in} \in C^2(\partial D)$. The following estimates hold:
\[
\left\| \nu^c \circ \Psi_c - u^c \right\|_{L^2(\partial D)} \leq C e,
\]
and
\[
\left\| \frac{\partial u^c}{\partial v} \circ \Psi_c - \frac{\partial u^c}{\partial v} \right\|_{L^2(\partial D)} \leq C e,
\]

with a constant $C$ independent of $e$.

**Proof.** **Sound-soft obstacle.** Let $x \in \partial D$; then $\hat{x} = \Psi_c(x) = x + ch(x)v(x) \in \partial D$. We have
\[
u^c(\hat{x}) - u^c(x) = u^{in}(x + ch(x)v(x)),
\]
from which it follows by using the mean value theorem that $||\nu^c \circ \Psi_c - u^c||_{L^2(\partial D)} \leq C e$. Then, one can see from the injection continuous $L^\infty(\partial D) \hookrightarrow L^2(\partial D)$ that (4.3) is true.

It follows from (4.1), (4.2), and the jump formula (3.2) that
\[
\left(-\frac{1}{2}I + (K_D^k)^*\right)\left(\frac{\partial u^c}{\partial v}\right)(x) = \frac{\partial D_D^k(u^c)}{\partial v}(x), x \in \partial D,
\]
\[
(4.5)
\]
and
\[
\left(-\frac{1}{2}I + (\mathcal{K}_D^k)^*\right) \left(\frac{\partial u^\varepsilon}{\partial v}\right)(\bar{x}) = \frac{\partial D^k_D(u^\varepsilon)}{\partial v}(\bar{x}), \bar{x} \in \partial D_v.
\]

The following expansion follows from (3.27), (3.23), and the above equation
\[
\left(-\frac{1}{2}I + (\mathcal{K}_D^k)^*\right) \left(\frac{\partial u^\varepsilon}{\partial v} \circ \Psi \right)(x) = \frac{\partial D^k_D(u^\varepsilon \circ \Psi)}{\partial v}(x) + O(\varepsilon), x \in \partial D. \tag{4.6}
\]

Subtracting (4.5) from (4.6) yields
\[
\left(-\frac{1}{2}I + (\mathcal{K}_D^k)^*\right) \left(\frac{\partial u^\varepsilon}{\partial v} \circ \Psi \right)(x) = \frac{\partial D^k_D(u^\varepsilon \circ \Psi - u)}{\partial v}(x) + O(\varepsilon), x \in \partial D,
\]
using the fact that \(-1/2I + (\mathcal{K}_D^k)^*\) is invertible on \(L^2(\partial D)\) and (4.3) to deduce (4.4).

**Sound-hard obstacle.** For \(\bar{x} = x + \varepsilon h(x)\nu(x) \in \partial D_v\), we have
\[
\frac{\partial u^\varepsilon}{\partial v}(\bar{x}) - \frac{\partial u^\varepsilon}{\partial v}(x) = \frac{\partial u^\text{in}}{\partial v}(x) - \frac{\partial u^\text{in}}{\partial v}(\bar{x}).
\]
Since, by using the mean value theorem and the injection continuous \(L^\infty(\partial D) \hookrightarrow L^2(\partial D)\), we get (4.4). It follows from (4.1), (4.2), and the jump formula (3.3) that
\[
\left(\frac{1}{2}I + \mathcal{K}_D^k\right) \left(u^\varepsilon\right)(x) = S^k_D \left(\frac{\partial u^\varepsilon}{\partial v}\right)(x), x \in \partial D. \tag{4.7}
\]
and
\[
\left(\frac{1}{2}I + \mathcal{K}_D^k\right) \left(u^\varepsilon \circ \Psi \right)(\bar{x}) = S^k_D \left(\frac{\partial u^\varepsilon}{\partial v}\right)(\bar{x}), \bar{x} \in \partial D_v. \tag{4.8}
\]

According to (3.23), (3.27), and (4.8), the following asymptotic expansion holds
\[
\left(\frac{1}{2}I + \mathcal{K}_D^k\right) \left(u^\varepsilon \circ \Psi \right)(x) = S^k_D \left(\frac{\partial u^\varepsilon}{\partial v} \circ \Psi \right)(x) + O(\varepsilon), x \in \partial D. \tag{4.9}
\]

From (4.7) and (4.9), we get
\[
\left(\frac{1}{2}I + \mathcal{K}_D^k\right) \left(u^\varepsilon \circ \Psi - u^\varepsilon\right)(x) = S^k_D \left(\frac{\partial u^\varepsilon}{\partial v} \circ \Psi - \frac{\partial u^\varepsilon}{\partial v}\right)(x) + O(\varepsilon), x \in \partial D.
\]
Clearly, the estimate (4.3) immediately follows from (4.4) and the fact that \(1/2I + \mathcal{K}_D^k\) is invertible on \(L^2(\partial D)\). Thus, the proof of Lemma 4.1 is complete. \(\Box\)

Now we are ready to prove Theorem 1.1. Let \(v^\delta\) be the solution of (1.4). It then follows from (2.1) that
\[
\int_{\partial D} \left(\frac{\partial S^k_D[\phi]}{\partial v} \left| v^\delta - S^k_D[\phi] \frac{\partial v^\delta}{\partial v}\right|\right) d\sigma = \int_{\partial D} \left(\frac{\partial D^k_D[\psi]}{\partial v} \left| v^\delta - D^k_D[\psi] \right| \frac{\partial v^\delta}{\partial v}\right) d\sigma = 0.
\]
and
\[
\int_{\partial D} (\mathcal{K}_{D,1}^k[\phi] - A_{D,1}^k[\psi]) v^* d\sigma - \int_{\partial D} (\mathcal{S}_{D,1}^k[\phi] - D_{D,1}^k[\psi]) \frac{\partial u^*}{\partial v} d\sigma
\]
\[
= - \int_{\partial D} h \left( \frac{\partial S_{D,1}^k[\phi]}{\partial v} \right)_+ - \frac{\partial D_{D,1}^k[\psi]}{\partial v} \frac{\partial u^*}{\partial v} d\sigma
\]
\[
+ \int_{\partial D} \left[ \tau h \left( \frac{\partial S_{D,1}^k[\phi]}{\partial v} \right)_+ - \frac{\partial D_{D,1}^k[\psi]}{\partial v} \right] - \frac{d}{dt} \left( h \frac{d}{dt} \left( S_{D,1}^k[\phi] - D_{D,1}^k[\psi] \right)_+ \right)
\]
\[
- hk^2 \left( S_{D,1}^k[\phi] - D_{D,1}^k[\psi] \right)_+ \right) v^* d\sigma.
\]

Put \( \phi = \frac{\partial u^*}{\partial v} \circ \Psi_\epsilon \) and \( \psi = u_{\epsilon}^* \circ \Psi_\epsilon \). It follows from (4.3) and (4.4) that
\[
S_{D,1}^k[\phi] - D_{D,1}^k[\psi] = u^* + O(\epsilon) \quad \text{and} \quad \frac{\partial S_{D,1}^k[\phi]}{\partial v} \bigg|_+ - \frac{\partial D_{D,1}^k[\psi]}{\partial v} = \frac{\partial u^*}{\partial v} + O(\epsilon) \quad \text{on} \quad \partial D,
\]
and then the asymptotic expansions in Theorem 1.1 of \([u_{\epsilon}^*, v^*, \Psi_\epsilon, D]\) are proved as desired.

5 ASYMPTOTIC EXPANSIONS FOR THE DNO AND NDO

For a given bounded domain \( D \) with \( C^2 \) boundary, we introduce the DNO for the exterior Helmholtz problem which is defined by
\[
\mathcal{N}_0(f) = \frac{\partial u}{\partial v} \bigg|_{\partial D},
\]
where \( u \) is the solution to
\[
\begin{aligned}
\Delta u + k^2 u &= 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D}, \\
\left| \frac{\partial u}{\partial r} - iku \right| &= O \left( \frac{1}{r^2} \right) \quad \text{as} \quad r = |x| \rightarrow +\infty \quad \text{uniformly in} \quad \frac{x}{|x|}, \\
u(x) &= f(x) \quad \text{for} \quad x \in \partial D. 
\end{aligned}
\]

(5.1)

Let \( \mathcal{N}_\epsilon(f) \) be the perturbed DNO resulting from small perturbations of \( D \), namely,
\[
\mathcal{N}_\epsilon(f)(x) = \frac{\partial u_\epsilon}{\partial v} \circ \Psi_\epsilon(x), \quad \Psi_\epsilon(x) = x + \epsilon h(x)v(x) \quad \text{for} \quad x \in \partial D,
\]
where
\[
\begin{aligned}
\Delta u_\epsilon + k^2 u_\epsilon &= 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D_\epsilon}, \\
\left| \frac{\partial u_\epsilon}{\partial r} - iku_\epsilon \right| &= O \left( \frac{1}{r^2} \right) \quad \text{as} \quad r = |x| \rightarrow +\infty \quad \text{uniformly in} \quad \frac{x}{|x|}, \\
u_\epsilon \circ \Psi_\epsilon(x) &= f(x) \quad \text{for} \quad x \in \partial D. 
\end{aligned}
\]

(5.2)

In connection with the results for rough nonperiodic surfaces\(^2\)\(^1\) and periodic interfaces,\(^1\) the following theorem holds:

**Theorem 5.1.** Suppose that \( k^2 \) is not an eigenvalue of \(-\Delta\) on \( D \) with Neumann boundary condition and \( f \in C^2(\partial D) \). The following expansion holds:
\[
\mathcal{N}_\epsilon(f)(x) = \mathcal{N}_0(f)(x) + \epsilon \left( -\frac{1}{2} I + (\mathcal{K}^k_D)^* \right)^{-1} \left( A_{D,1}^k[f] - \mathcal{K}^k_{D,1} \left[ \mathcal{N}_0(f) \right] \right)(x) + O(\epsilon^2),
\]

where the operators \( A_{D,1}^k \) and \( \mathcal{K}^k_{D,1} \) are defined in (3.21) and (3.25), respectively. Here, the remainder \( O(\epsilon^2) \) depends only on the \( C^2 \) norm of \( X \), the \( C^1 \) norm of \( h \), and \( k \).

**Proof.** Let \( u_\epsilon \) be the solution to (5.2). Then the following representation formula holds:
\[
u_\epsilon(x) = S_{D,1}^k \left[ \frac{\partial u_\epsilon}{\partial v} \right](x) - D_{D,1}^k[u_\epsilon](x), \quad x \in \mathbb{R}^2 \setminus \overline{D_\epsilon}.
\]
where

\[
\frac{\partial u_c}{\partial v} \circ \Psi_c(x) = \left( \frac{1}{2} I + \left( \kappa_D^k \right)^* \right) \left[ \frac{\partial u_c}{\partial v} \right] \circ \Psi_c(x) = \frac{\partial D_D^k [u_c]}{\partial v} \circ \Psi_c(x), \quad x \in \partial D.
\]

It then follows from (3.23) and (3.27) that

\[
\left( -\frac{1}{2} I + \left( \kappa_D^k \right)^* \right) [N_c(f)] = \frac{\partial D_D^k [f]}{\partial v} + \epsilon \left( A_{D,1}^k [f] - \kappa_D^k [N_c(f)] \right) + O(\epsilon^2) \quad \text{on} \quad \partial D.
\] (5.3)

Similarly, one can check that

\[
\left( -\frac{1}{2} I + \left( \kappa_D^k \right)^* \right) [N_0(f)] = \frac{\partial D_D^k [f]}{\partial v} \quad \text{on} \quad \partial D.
\] (5.4)

Subtracting (5.10) from (5.9) yields

\[
\left( -\frac{1}{2} I + \left( \kappa_D^k \right)^* \right) [N_c(f) - N_0(f)] = \epsilon \left( A_{D,1}^k [f] - \kappa_D^k [N_c(f)] \right) + O(\epsilon^2) \quad \text{on} \quad \partial D.
\]

If \( k^2 \) is not an eigenvalue of \(-\Delta\) on \( D \) with Neumann boundary condition, then we have from Lemma 3.2 that \(- (1/2) I + \left( \kappa_D^k \right)^*\) is invertible on \( L^2(\partial D) \). Hence,

\[
N_c(f) - N_0(f) = \epsilon \left( -\frac{1}{2} I + \left( \kappa_D^k \right)^* \right)^{-1} \left( A_{D,1}^k [f] - \kappa_D^k [N_c(f)] \right) + O(\epsilon^2) \quad \text{on} \quad \partial D.
\]

Note that

\[
\| N_c(f) - N_0(f) \|_{L^2(\partial D)} \leq C \epsilon.
\] (5.5)

This completes the proof. \( \square \)

Now, let us introduce the NtD operator for the exterior Helmholtz problem which is defined by

\[
\Lambda_0[g] = v|_{\partial D},
\]

where \( v \) is the solution to

\[
\begin{align*}
\Delta v + k^2 v &= 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \bar{D}, \\
\left| \frac{\partial v}{\partial r} - ikv \right| &= O \left( 1/r^\frac{3}{2} \right) \quad \text{as} \quad r = |x| \to +\infty \quad \text{uniformly in} \quad \frac{x}{|x|}, \\
\frac{\partial v}{\partial v}(x) &= g(x) \quad \text{for} \quad x \in \partial D.
\end{align*}
\] (5.6)

We let \( \Lambda_c[g] \) be the perturbed NtD operator caused par \( D_c \); that is,

\[
\Lambda_c[g](x) = v_c \circ \Psi(x),
\]

where

\[
\begin{align*}
\Delta v_c + k^2 v_c &= 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \bar{D}_c, \\
\left| \frac{\partial v_c}{\partial r} - ikv_c \right| &= O \left( 1/r^\frac{3}{2} \right) \quad \text{as} \quad r = |x| \to +\infty \quad \text{uniformly in} \quad \frac{x}{|x|}, \\
\frac{\partial v_c}{\partial v}(x + eh(x)v(x)) &= g(x) \quad \text{for} \quad x \in \partial D.
\end{align*}
\] (5.7)

The following theorem holds.
Theorem 5.2. Suppose that $k^2$ is not an eigenvalue of $-\Delta$ on $D$ with Dirichlet boundary condition and $g \in C^2(\partial D)$. The following asymptotic formula holds:

$$
\Lambda_e[g](x) = \Lambda_0[g](x) + \epsilon \left( \frac{1}{2}I + K_D^k \right)^{-1} \left( S^k_{D,1}[g] - D^k_{D,1} [\Lambda_0(g)] \right) (x) + O(\epsilon^2),
$$

where the operators $D^k_{D,1}$ and $S^k_{D,1}$ are defined in (3.13) and (3.24), respectively. Here, the remainder $O(\epsilon^2)$ depends only on the $C^2$ norm of $X$, the $C^1$ norm of $h$, and $k$.

Proof. The solution of (5.7) is given by

$$
v_e(x) = S^k_{D,1} \left[ \frac{\partial v}{\partial v} \right] (x) - D^k_{D,1} [v_e] (x), \quad x \in \mathbb{R}^2 \setminus \overline{D}. \tag{5.8}
$$

From the jump formula (3.3) and (5.8), we deduce

$$
v_e \circ \Psi_e(x) = S^k_{D,1} \left[ \frac{\partial v}{\partial v} \right] (x) - \left( -\frac{1}{2}I + K_D^k \right) [v_e] \circ \Psi_e(x), \quad x \in \partial D.
$$

It then follows from (3.22) and (3.26) that

$$
\left( \frac{1}{2}I + K_D^k \right) [\Lambda_e(g)] = S^k_{D}[g] + \epsilon \left( S^k_{D,1}[g] - D^k_{D,1} [\Lambda_e(g)] \right) + O(\epsilon^2) \quad \text{on} \quad \partial D. \tag{5.9}
$$

In the same way as above, we can get

$$
\left( \frac{1}{2}I + K_D^k \right) [\Lambda_0(g)] = S^k_{D}[g] \quad \text{on} \quad \partial D. \tag{5.10}
$$

Subtracting (5.10) from (5.9) yields

$$
\left( \frac{1}{2}I + K_D^k \right) [\Lambda_e(g) - \Lambda_0(g)] = \epsilon \left( S^k_{D,1}[g] - D^k_{D,1} [\Lambda_e(g)] \right) + O(\epsilon^2) \quad \text{on} \quad \partial D. \tag{5.11}
$$

If $k^2$ is not an eigenvalue of $-\Delta$ on $D$ with Dirichlet boundary condition, then we have from Lemma 3.2 that $(1/2)I+K_D^k$ is invertible on $L^2(\partial D)$. Hence,

$$
\Lambda_e(g) - \Lambda_0(g) = \epsilon \left( \frac{1}{2}I + K_D^k \right)^{-1} \left( S^k_{D,1}[g] - D^k_{D,1} [\Lambda_e(g)] \right) + O(\epsilon^2) \quad \text{on} \quad \partial D,
$$

since

$$
\|\Lambda_e(g) - \Lambda_0(g)\|_{L^2(\partial D)} \leq C \epsilon, \tag{5.12}
$$

and the theorem is proved.

Based on the same arguments given in the proofs of Theorem 1.1, the following theorem holds.

Theorem 5.3. Let $f, g \in C^2(\partial D)$. The following reconstructing formulas hold:

$$
\int_{\partial D} \left( \mathcal{N}_e(f)g - f \mathcal{N}_0(g) \right) d\sigma = \epsilon \int_{\partial D} h \left( \frac{\partial f}{\partial T} \frac{\partial g}{\partial T} + (n-1)\tau \mathcal{N}_0(f)g - \mathcal{N}_0(f)\mathcal{N}_0(g) - k^2 fg \right) d\sigma + O(\epsilon^2), \tag{5.13}
$$

where $\mathcal{N}_e(f)$ and $\mathcal{N}_0(f)$ are defined in (3.13) and (3.24), respectively.
and
\[ \int_{\partial D} (f \Lambda_0(g) - \Lambda_0(f)g) \, d\sigma = \epsilon \int_{\partial D} \left( \frac{\partial \Lambda_0(f)}{\partial T} - \frac{\partial \Lambda_0(g)}{\partial T} + (n-1)rf \Lambda_0(g) - fg - k^2 \Lambda_0(f) \Lambda_0(g) \right) \, d\sigma + O(\epsilon^2), \]  
(5.13)
where the remainder \( O(\epsilon^2) \) depends only on the \( C^2 \) norm of \( X \), the \( C^1 \) norm of \( f \), and \( k \).

6 \ RECONSTRUCTION OF THE SHAPE DEFORMATION

Formulas in (1.5), (5.12), and (5.13) can be used to reconstruct an approximation of the deformation \( h \) by choosing test functions of the integral in the right-hand side appropriately. Let us treat the formulas in (1.5). The reconstruction of the shape deformation from (5.12) and (5.13) can be done in the same way.

To illustrate this, let us consider \( D \) to be the disk centred at the origin with radius \( \rho \). For an integer \( n \), set
\[ u_n(r, \theta) = H_{|n|}^{(1)}(kr)e^{in\theta} \text{ for } r > \rho, \]
since \( u_n \) satisfies \((\Delta + k^2)u_n = 0 \) in \( \mathbb{R}^2 \setminus \overline{D} \) and the Sommerfeld condition
\[ \left| \frac{\partial u_n}{\partial r} - iku_n \right| = O\left( \frac{1}{r^3} \right) \text{ as } r \to \infty. \]

We then take
\[ u^n(r, \theta) = -H_{|n|}^{(1)}(kr)e^{in\theta} \text{ and } v^n(r, \theta) = -H_{|n|}^{(1)}(kr)e^{in\theta}, \]
on \( \partial D \)
for the case of a sound-soft obstacle and
\[ \frac{\partial u^n}{\partial \nu}(r, \theta) = -kH_{|n|}^{(1)}(kr)e^{in\theta} \text{ and } \frac{\partial v^n}{\partial \nu}(r, \theta) = -kH_{|n|}^{(1)}(kr)e^{in\theta}, \]
on \( \partial D \)
for the case of a sound-hard obstacle. One can easily see that \( u^i = u_n \) and \( v^i = u_m \). It then follows from (1.5) that
\[ [u^i, v^i, \Psi, D] = cc_{n,m}(\rho, k) \int_{\partial D} h(\theta)e^{i(n+m)\theta} \, d\theta + O(\epsilon^2), \]
(6.1)
with
\[ cc_{n,m}(\rho, k) = \left[ -nm + r\sigma_1(\rho, n, k) + \sigma_2(\rho, n, k) \sigma(\rho, m, k) - k^2 \right] |H_{|n|}^{(1)}(kr)|H_{|m|}^{(1)}(kr), \]
where \( \sigma_1 \) is given by
\[ \sigma_1(\rho, n, k) = k \frac{H_{|n|}^{(1)\prime}(kr)}{H_{|n|}^{(1)}(kr)} = -k \frac{H_{|n|+1}^{(1)}(kr)}{H_{|n|}^{(1)}(kr)} + |n|. \]

Formulas in (1.5) show that the Fourier coefficients \( h_\rho \) of \( h \) can be determined from measurements on \( \partial D \) by varying the test function \( v^n \), provided that the order of magnitude of \( |h_\rho| \) is much larger than \( \epsilon \).

If \( D \) is a ball of radius \( \rho \), hence, \( h \) can be expanded as
\[ h(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} h_l^m Y_l^m(\theta, \varphi), \]
(6.2)
where \( Y_l^m \), for \( l \geq 0 \), and \(-l \leq m \leq l \) are the spherical harmonics of order \( l \). These functions constitute an orthogonal basis of the space linear of \( L^2(\partial D) \) and satisfy \( Y_l^m = (-1)^m Y_l^{-m} \) (see Kirsch\(^9\) and Nédélec\(^{18}\)). The coefficients \( h_l^m \) in (6.2) are defined by
\[ h_l^m = \int_{\partial D} h(\theta, \varphi) Y_l^m(\theta, \varphi) \, d\sigma. \]
For two integers \( l \) and \( m \), set
\[
 u_{l,m}(r, \theta) = h^{(1)}_l(kr)Y^m_l(\theta, \varphi) \quad \text{for} \quad r > \rho,
\]
where \( h^{(1)}_l \) is the spherical Hankel function of the first kind of order \( l \), since \( u_{l,m} \) satisfies
\[
 (\Delta + k^2)u_{l,m} = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \mathcal{D}, \quad \left| \frac{\partial u_{l,m}}{\partial r} - ik u_{l,m} \right| = O(1/r^2) \quad \text{as} \quad r \to \infty.
\]

Similar to the two-dimensional case, we take
\[
 u^{in}(\rho, \theta) = -h^{(1)}_l(k\rho) \quad \text{and} \quad v^{in}(\rho, \theta) = -h^{(1)}_l(k\rho)Y^m_l(\theta, \varphi), \quad \text{on} \quad \partial \mathcal{D}
\]
for the case of a sound-soft obstacle and
\[
 \frac{\partial u^{in}}{\partial n}(\rho, \theta) = -kh^{(1)}_lY^m_l(\theta, \varphi) \quad \text{and} \quad \frac{\partial v^{in}}{\partial n}(\rho, \theta) = -kh^{(1)}_lY^m_l(\theta, \varphi), \quad \text{on} \quad \partial \mathcal{D}
\]
for the case of a sound-hard obstacle. One can easily check that \( u^r = u_{0,0} \), \( v^r = u_{l,m} \), and
\[
 \frac{\partial u^r}{\partial T} + 2\tau \frac{\partial u^r}{\partial V} - \frac{\partial u^v}{\partial V} - k^2u^v = d_{l,m}(\rho, k)Y^m_l(\theta, \varphi), \quad \text{on} \quad \partial \mathcal{D},
\]
where \( d_{l,m}(\rho, k) \) is a constant independent of \( \theta \) and \( \varphi \). Then by measuring \([u^r, v^r, \Psi^r, D]\) in (1.5), we can reconstruct \( h^{-m}_l \).

This implies that the coefficients \( h^{-m}_l \) of \( h \) can be determined by varying the test function \( v^r = u_{l,m} \), provided that the order of magnitude of \( |h^{-m}_l| \) is much larger than \( \epsilon \).

**ORCID**

Habib Zribi  
https://orcid.org/0000-0002-5665-8659

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