SPINFOAM ON LEFSCHETZ-THIMBLE:

Markov Chain Monte-Carlo Computation of Lorentzian Spinfoam Propagator

Zichang Huang, Department of Physics, Fudan University
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Collaborators: Muxin Han (Florida Atlantic University), Hongguang Liu (University of Erlangen-Nuremberg), Dongxue Qu (Florida Atlantic University), Yidun Wan (Fudan University)
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INTRODUCTION

Spinfoam model
& the sign problem

Photo Credit: R. HURT - CALTECH / JPL
SPINFOAM MODEL

- Covariant formulation of Loop Quantum Gravity in 4D
- Quantum space time
  - Boundary: spin-network states
  - Bulk: map
    \[ \langle W \rangle : |\psi\rangle \rightarrow A \]

- Discretized spacetime:
  - Minimum unit: 4-simplex amplitude (space time atom)
  - Connection = Entanglement

[T.Thiemann 2007, A.Ashtekar, J. Lewandowski 2004, C.Roveill F.Vidotto 2015, A. Perez 2003, J.Engle, E.Livine, R.Pereira, C.Roveill 2008, L.Freidel K.Karsnov 2008, etc]
SPINFOAM MODEL

- Covariant formulation of Loop Quantum Gravity in 4D
- Quantum space time
  - Boundary: spin-network states
  - Bulk: map \( \langle W | : |\psi\rangle \mapsto A \)
- Discretized spacetime:
  - Minimum unit: 4-simplex amplitude
  - Connection = Entanglement

\[ \langle x_f, t_f | x_i, t_i \rangle = \int dx_{N-1} dx_{N-2} \cdots dx_1 \times \langle x_f, t_f | x_{N-1}, t_{N-1} \rangle \langle x_{N-1}, t_{N-1} | x_{N-2}, t_{N-2} \rangle \cdots \langle x_2, t_2 | x_1, t_1 \rangle \]
SPINFOAM MODEL

• Spinfoam amplitude: (finite dimensional integral)

\[ A = \langle W | \psi \rangle = \int D[\phi] W[\phi] \psi[\phi] = \int D\phi e^{-S[\phi]} \]

Path-integral formulation of LQG

• Semi-classical limit

Discrete gravity (Regge calculus)

• Observables:

\[ \langle \hat{O} \rangle = \frac{\langle W | \hat{O} | \psi \rangle}{\langle W | \psi \rangle} = \frac{\int D\phi O[\phi] e^{-S[\phi]}}{\int D\phi e^{-S[\phi]}} \]

Penrose metric:

\[ q^{ab}(x) = \delta^{ij} E_i^a(x) E_j^b(x) \]

e.g., 2-point connected correlation function:

\[ G^{abcd}(x, y) = \langle q^{ab}(x) q^{cd}(y) \rangle - \langle q^{ab}(x) \rangle \langle q^{cd}(y) \rangle \]

Semi-classical limit

graviton propagator

[C. Rovelli 2006, A. Perez 2003, E. Bianchi, L. Modesto, C. Rovelli, and S. Speziale 2006, E. Bianchi, E. Magliaro, and C. Perini 2009, E. Bianchi and Y. Ding 2012, M. Han, Z. Huang, and A. Zipfel 2013]
SPINFOAM MODEL

• Spinfoam amplitude: (finite dimensional)
  \[ A = \langle W | \psi \rangle = \int D[\phi] W[\phi] \psi[\phi] = \int D\phi e^{-S[\phi]} \]  
  Path-integral formulation of LQG

• Semi-classical limit \( \rightarrow \) Discrete gravity (Regge calculus)

• Observables:
  \[ \langle \hat{O} \rangle = \frac{\langle W | \hat{O} | \psi \rangle}{\langle W | \psi \rangle} = \frac{\int D\phi O[\phi] e^{-S[\phi]}}{\int D\phi e^{-S[\phi]}} \]

  e.g., 2-point connected correlation function:
  \[ G^{abcd}(x, y) = \langle q^{ab}(x) q^{cd}(y) \rangle - \langle q^{ab}(x) \rangle \langle q^{cd}(y) \rangle \]

We compute \( G^{abcd} \) numerically in this work
REQUIREMENTS

• Reliability
  • Can go to high spin limit  Asymptotic results
  • Can adapt to multiple situation
    • Euclidean/Lorentzian models
    • Different values of $\gamma$
    • Different boundary states

• Utility
  • Can compute different observables:

$$\langle q^{ab}(x)q^{cd}(y) \rangle, \langle q^{ab}(x) \rangle, \langle q^{cd}(y) \rangle \cdots$$
SL2CFOAM

- sl2cfoam package
  - Can only compute amplitude (so far)
  - Boundary spin cannot be very large

[Not suitable for the task]

[P. Dona, S. Speziale, F. Gozzini, G. Sarno, etc]
Can we find an algorithm to do the computation?

Yes, we can!
PREVIEW THE RESULTS

- Results of Expectation values $\langle E_1^2 \cdot E_4^3 \cdot E_1^1 \cdot E_4^5 \rangle$

| $\lambda$ | $10^2$ | $10^3$ | $10^4$ | $5 \times 10^4$ | $10^5$ | $5 \times 10^5$ | $10^6$ | $5 \times 10^6$ | $10^7$ | $5 \times 10^7$ |
|-----------|--------|--------|--------|-----------------|--------|-----------------|--------|-----------------|--------|-----------------|
| Difference (%) | 8.71   | 0.79   | 0.12   | 0.052           | 0.036  | 0.017           | 0.0062 | 0.0018          | 0.00037 | 0.00069        |

Leading order in $1/\lambda$ expansion
Thimble + Metropolis
PREVIEW THE RESULTS

- Results of Expectation values \( \left< E_1^2 \cdot E_1^3 \right> \)

| \( \lambda \)       | \( 10^2 \) | \( 10^3 \) | \( 10^4 \times 10^4 \) | \( 10^5 \) | \( 5 \times 10^5 \) | \( 10^6 \) | \( 5 \times 10^6 \) | \( 10^7 \) | \( 5 \times 10^7 \) |
|---------------------|----------|----------|---------------------|----------|----------------|----------|----------------|----------|----------------|
| Difference (%)      | 22.32    | 2.00     | 0.31                | 0.078    | 0.022          | 0.016    | 0.012          | 0.0022   | 0.000047      | 0.0016   |
PREVIEW THE RESULTS

- Results of Expectation values $\langle E_4^1 \cdot E_4^5 \rangle$

| $\lambda$ | $10^2$ | $10^3$ | $10^4$ | $5 \times 10^4$ | $10^5$ | $5 \times 10^5$ | $10^6$ | $5 \times 10^6$ | $10^7$ | $5 \times 10^7$ |
|-----------|--------|--------|--------|----------------|--------|----------------|--------|----------------|--------|----------------|
| Difference (%) | 18.66  | 1.18   | 0.18   | 0.026         | 0.017  | 0.00054        | 0.0037 | 0.00035        | 0.00036 | 0.00083        |
PREVIEW THE RESULTS

- Results of propagator component $G_{14}^{2315}$

| $\lambda$ | $10^2$ | $10^3$ | $10^4$ | $5 \times 10^4$ | $10^5$ | $5 \times 10^5$ | $10^6$ | $5 \times 10^6$ | $10^7$ | $5 \times 10^7$ |
|-----------|--------|--------|--------|----------------|--------|----------------|--------|----------------|--------|----------------|
| Difference (%) | 37.90 | 27.00 | 13.22 | 2.76 | 10.09 | 8.86 | 1.89 | 1.13 | 3.90 | 2.06 |
How to ...... ?

Do the integral directly!

\[ \langle \hat{O} \rangle = \frac{\langle W | \hat{O} | \psi \rangle}{\langle W | \psi \rangle} = \frac{\int D\phi O[\phi] e^{-S[\phi]}}{\int D\phi e^{-S[\phi]}} \]
THE SIGN PROBLEM

• Complex valued Action:

\[ S(x) \in \mathbb{C} \quad \text{oscillatory} \]

\[ e^{-S(x)} \]

• N-dimensional case:

oscillatory:

\[ \langle \hat{O} \rangle = \frac{\int Dx O(x) e^{-S(x)}}{\int D(x) e^{-S(x)}} \quad \text{should be } O(1) \]

Oscillatory:

\[ \int Dx O(x) e^{-S(x)} \sim e^{-O(N)} \]

Monte-Carlo:

\[ O \left( \frac{1}{\sqrt{N_{\text{conf}}}} \right) \]

\[ \langle O \rangle \sim \frac{e^{-O(N)} \pm O \left( \frac{1}{\sqrt{N_{\text{conf}}}} \right)}{e^{-O(N)} \pm O \left( \frac{1}{\sqrt{N_{\text{conf}}}} \right)} \quad N_{\text{conf}} = e^{O(N)} \]

[M. Fukuma and N. Matsumoto 2020]
THE SIGN PROBLEM

- Complex valued Action:

\[ S(x) \in \mathbb{C} \quad \text{oscillatory} \]

- N-dimensional case:

\[ \langle \hat{O} \rangle = \frac{\int Dx \mathcal{O}(x)e^{-S(x)}}{\int D(x)e^{-S(x)}} \quad \text{should be } O(1) \]

Oscillatory: \[ \int Dx \mathcal{O}(x)e^{-S(x)} \sim e^{-O(N)} \]

Monte-Carlo: \[ O \left( \frac{1}{\sqrt{N_{\text{conf}}}} \right) \]

\[ \langle \mathcal{O}(x) \rangle \sim \frac{e^{-O(N)}}{e^{-O(N)} \pm O \left( \frac{1}{\sqrt{N_{\text{conf}}}} \right)} \]

\[ N_{\text{conf}} = e^{O(N)} \]

Conventional Monte-Carlo is inefficient!
ONE SOLUTION!

• Lefschetz-Thimble integration:

\[ \int Dx \mathcal{O}(x)e^{-S(x)} = \int_{\mathbb{R}^N} Dz \mathcal{O}(z)e^{-S(z)} = \int_{\Sigma} Dz \mathcal{O}(z)e^{-S(z)} \]

\[ \forall z \in \Sigma \text{ s.t. Im}(S(z)) = \text{constant} \]

\[ \forall z \in \Sigma, \mathcal{O}(z)e^{-S(z)} \text{ non-oscillatory} \]
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LEFSCHETZ-THIMBLE

A cure of the sign problem
LEFSCHETZ-THIMBLE

- Lefschetz-Thimble $\mathcal{I}_\sigma$
  - Union of steepest decent (SD) paths falling to critical point $\sigma$ when $t \to \infty$

- Steepest decent equation:
  $$\frac{dz^a}{dt} = -\frac{\partial S(z)}{\partial z^a}$$

- Imaginary part of action:
  $$\frac{dS}{dt} = \frac{\partial S}{\partial z^a} \frac{dz^a}{dt} = -\left| \frac{\partial S}{\partial z^a} \right|^2$$
  $$\text{Im}(S(z)) = \text{constant}$$
LEFSCHETZ-THIMBLE

- Lefschetz-Thimble $\mathcal{J}_\sigma$
  - Union of steepest decent (SD) paths falling to critical point $\sigma$ when $t \to \infty$

- Steepest decent equation:
  \[
  \frac{dz^a}{dt} = -\frac{\partial S(z)}{\partial z^a}
  \]

- Imaginary part of action:
  \[
  \frac{dS}{dt} = \frac{\partial S}{\partial z^a} \frac{dz^a}{dt} = -\left| \frac{\partial S}{\partial z^a} \right|^2
  \]
  \[
  \text{Im}(S(z)) = \text{constant}
  \]

SD equation is not the equation of motion!
LEFSCHETZ-THIMBLE

• Picard-Lefschetz Theory:

\[
\int_{\mathbb{R}^n} d^n z \hat{f}(\bar{z}) e^{-S(\bar{z})} = \sum_{\sigma} n_{\sigma} \int_{\mathcal{J}_\sigma} d^n z \hat{f}(\bar{z}) e^{-S(z)}
\]

\( \mathbb{R}^n \) homologically equivalent to \( \sum_{\sigma} n_{\sigma} \mathcal{J}_\sigma \)
LEFSCHETZ-THIMBLE

- Expectations values:

\[
\langle f \rangle = \frac{\int_{\mathbb{R}^n} d^n z \hat{f}(\vec{z}) e^{-\hat{S}(\vec{z})}}{\int_{\mathbb{R}^n} d^n z e^{-\hat{S}(\vec{z})}} = \frac{\sum_{\sigma} n_{\sigma} \int_{\mathcal{J}_{\sigma}} d^n z \hat{f}(\vec{z}) e^{-\hat{S}(\vec{z})}}{\sum_{\sigma} n_{\sigma} \int_{\mathcal{J}_{\sigma}} d^n z e^{-\hat{S}(\vec{z})}}
\]

- If one thimble dominate the integral:

\[
\langle f \rangle \approx \frac{n_{\sigma'} e^{-i \text{Im}(S(p_{\sigma'}))} \int_{\mathcal{J}_{\sigma'}} d^n z \hat{f}(\vec{z}) e^{-\text{Re}(\hat{S}(\vec{z}))}}{n_{\sigma'} e^{-i \text{Im}(S(p_{\sigma'}))} \int_{\mathcal{J}_{\sigma'}} d^n z e^{-\text{Re}(\hat{S}(\vec{z}))}} = \frac{\int_{\mathcal{J}_{\sigma'}} d^n z \hat{f}(\vec{z}) e^{-\text{Re}(\hat{S}(\vec{z}))}}{\int_{\mathcal{J}_{\sigma'}} d^n z e^{-\text{Re}(\hat{S}(\vec{z}))}}
\]
LEFSCHETZ-THIMBLE

• Expectations values:

\[
\langle f \rangle = \frac{\int_{\mathbb{R}^n} d^n z \hat{f}(\vec{z}) e^{-\hat{S}(\vec{z})}}{\int_{\mathbb{R}^n} d^n z e^{-\hat{S}(\vec{z})}} = \frac{\sum_{\sigma} n_{\sigma} \int_{\mathcal{J}_\sigma} d^n z \hat{f}(\vec{z}) e^{-\hat{S}(\vec{z})}}{\sum_{\sigma} n_{\sigma} \int_{\mathcal{J}_\sigma} d^n z e^{-\hat{S}(\vec{z})}}
\]

• If one thimble dominate the integral:

\[
\langle f \rangle \sim \frac{n_{\sigma', e^{-i \text{Im}(S(p_{\sigma'}))}} \int_{\mathcal{J}_{\sigma'}} d^n z \hat{f}(\vec{z}) e^{-\text{Re}(\hat{S}(\vec{z}))}}{n_{\sigma', e^{-i \text{Im}(S(p_{\sigma'}))}} \int_{\mathcal{J}_{\sigma'}} d^n z e^{-\text{Re}(\hat{S}(\vec{z}))}} = \frac{\int_{\mathcal{J}_{\sigma'}} d^n z \hat{f}(\vec{z}) e^{-\text{Re}(\hat{S}(\vec{z}))}}{\int_{\mathcal{J}_{\sigma'}} d^n z e^{-\text{Re}(\hat{S}(\vec{z}))}}
\]

Spinfoam propagator satisfies
APPROXIMATED THIMBLE

1) Steepest decent falling to $V_\sigma$
2) Steepest ascent from $V_\sigma$

SA flow:
$$\frac{dz^a}{dt} = \frac{\partial S(\hat{z})}{\partial z^a}$$

- Not initiate from the far away point
- Fluctuation of $\text{Im}(S(\hat{z}))$ is small when $V_\sigma$ is small
- Only need a portion of the thimble around the critical point

$e^{-\text{Re}(S)}$ decays very fast
OPTIMAL CHOICE OF $V_\sigma$

- Generalized eigenvalue equation:
  \[ \mathbf{H}\omega = \lambda\dot{\omega}. \]

- Eigenvectors with positive eigenvalues indicate the directions of the perturbation that can generate the approx. thimble.

- The $V_\sigma$ is a good choice of generating

$$V_\sigma = \left\{ \vec{z} \mid \vec{z} = \sum_{a=1}^{n} \hat{\omega}_i x^i + \vec{z}_\sigma, \text{ each } x^i \in \mathbb{R} \text{ is small} \right\}$$
FLOW OF THE JACOBIAN

- Volume element on $V_\sigma$: $d^n x \det(J_i^k)_0$
  
  $$(J_i^k)_0 \equiv \frac{\partial z^k}{\partial x^i}$$

- Volume element on $\tilde{\mathcal{J}}_\sigma$: $d^n z = d^n x \det((J_i^k)_T)$

  $$(J_i^k)_T \equiv \frac{\partial C_T(x)^k}{\partial x^i}$$

- Linearized SA flow:

  $$\frac{d (J_i^k)_t}{dt} = \sum_{l=1}^n \frac{\partial^2 \hat{S}}{\partial z_k \partial z_l} (J_i^l)_t$$

  $$(J_i^k)_0 \lla (J_i^k)_T$$

$\quad C_T : V_\sigma \rightarrow \tilde{\mathcal{J}}_\sigma$

$\quad (J_i^k)_0 \lla (J_i^k)_T$
INTEGRATION ON THE THIMBLE

• Integration on the thimble

\[ \int_{\tilde{\mathcal{J}}_\sigma} d^n z \psi(z) = \int_{V_\sigma} d^n x \det(J(x)) \psi(z(x)) \]

• Convert to the integral on \( \tilde{\mathcal{J}}_\sigma \) to \( V_\sigma \)

\[
\langle f \rangle \approx \frac{\int_{\tilde{\mathcal{J}}_\alpha} d^n z \hat{f}(z)e^{-\hat{S}(z)}}{\int_{\tilde{\mathcal{J}}_\alpha} d^n z e^{-\hat{S}(z)}} = \frac{\int_{V_\sigma} d^n x \det(J(x))\hat{f}(C_T(x))e^{-\hat{S}(C_T(x))}}{\int_{V_\sigma} d^n x \det(J(x))e^{-\hat{S}(C_T(x))}}
\]

\[
\frac{\langle e^{i\theta_{res}} \hat{f} \rangle_{eff}}{\langle e^{i\theta_{res}} \rangle_{eff}} = \frac{\Re(\hat{S}) - \log(\det(J)) \equiv S_{eff}}{\arg(\det(J)) - \text{Im}(\hat{S}) \equiv \theta_{res}}
\]

\[
\langle O \rangle_{eff} = \frac{\int_{V_\sigma} d^n x O e^{-S_{eff}}}{\int_{V_\sigma} d^n x e^{-S_{eff}}}
\]

[A. Alexandru, G. Basar, P. F. Bedaque 2020 2017 2016, M. Cristoforetti, F. Di Renzo, A. Mukherjee, and L. Scorzato 2014 2012, T. Kanazawa 2015 2016, etc]
INTEGRATE IN THE THIMBLE

- Convert to the integral on $\tilde{J}_\sigma$ to $V_\sigma$

$$\langle f \rangle \sim \frac{\int_{\tilde{J}_\alpha} d^n z \hat{f}(z)e^{-\hat{S}(z)}}{\int_{\tilde{J}_\alpha} d^n z e^{-\hat{S}(z)}} = \frac{\int_{V_\sigma} d^n x \det(J(x)) \hat{f}(C_T(x)) e^{-\hat{S}(C_T(x))}}{\int_{V_\sigma} d^n x \det(J(x)) e^{-\hat{S}(C_T(x))}}$$

$$= \frac{\langle e^{i\theta_{res}} \hat{f} \rangle_{eff}}{\langle e^{i\theta_{res}} \rangle_{eff}}$$

Sampling from the Boltzmann factor $e^{-S_{eff}}$

Can be computed by Markov-chain Monte Carlo Method

\text{Re}(\hat{S}) - \log(\det(J)) \equiv S_{eff}$

$\text{arg}(\det(J)) - \text{Im}(\hat{S}) \equiv \theta_{res}$

$$\langle \mathcal{O} \rangle_{eff} = \frac{\int_{V_\sigma} d^n x \mathcal{O} e^{-S_{eff}}}{\int_{V_\sigma} d^n x e^{-S_{eff}}}$$
DIFFERENTIAL EVOLUTION ADAPTIVE METROPOLIS ALGORITHM (DREAM)

• Advantages of DREAM Algorithm:
  • Good balance between acceptance and progression (Good for high-dimensional cases)
  • Adapt to multimodal distribution
  • Scan different region simultaneously
  • Run in parallel, adapt to multi-core computer processor

[J. Vrugt, C. ter Braak, C. Diks, B. Robinson, J. Hyman, and D. Higdon 2009]
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SPINFOAM PROPAGATOR

Graviton Propagator in Loop Quantum Gravity

Artist's impression of quantum space in loop quantum gravity

Picture from Thiemann's book
EPRL SPINFOAM MODEL

• 4-simplex amplitude

\[ \langle W|\Psi_0 \rangle = \sum_{j_{ab}} \psi_{j_0,\zeta_0} \int_{SL(2,\mathbb{C})^5} \prod_a d g_a \prod_{a > b} P_{ab}(g) \]

\[ P_{ab}(g) = \langle \lambda j_{ab}, -\vec{n}_{ab} | Y^\dagger g_a^{-1} g_b Y | \lambda j_{ab}, \vec{n}_{ba} \rangle \]

\[ Y \text{ maps the spin-} j \text{ } SU(2) \text{ irreducible representation } \mathcal{H}_j \text{ to the lowest level in } SL(2, \mathbb{C}) \text{ } (j, \gamma j) \text{-irreducible representation } \mathcal{H}_{(j,\gamma j)} = \oplus_{k=j}^{\infty} \mathcal{H}_k \]

We take Barbero-Immirzi parameter \( \gamma \) as 0.1.

[J.Engle, E.Livine, R.Pereira, C.Roveill 2008, L.Freidel K.Karsnov 2008]
EPRL SPINFOAM MODEL

• 4-simplex amplitude

\[
\langle W \mid \Psi_0 \rangle = \sum_{\lambda_{jab}} \psi_{\lambda_{j0},\zeta_0} \int_{SL(2,\mathbb{C})^s} \prod_a d\gamma_a \int \left( \prod_{a>b} \frac{d\lambda_{jab}}{\pi} d\tilde{z}_{ab} \right) e^{\lambda S}
\]

\[S(j, g, z) = \sum_{a>b} [2j_{ab} \log (\langle J_{\xi_{ab}} Z_{ab} \rangle \langle Z_{ba}, \xi_{ba} \rangle) - (1 + i\gamma) j_{ab} \log \langle Z_{ab}, Z_{ab} \rangle - (1 - i\gamma) j_{ab} \log \langle Z_{ba}, Z_{ba} \rangle]
\]

\[d_j = 2j + 1, \ Z_{ab} = g_a^\dagger z_{ab}, \text{ and } Z_{ba} = g_b^\dagger z_{ab}
\]

\[\xi_{ab}, \psi_{\lambda_{j0},\zeta_0}, j_a \text{ given by the boundary state}\]
EPRL SPINFOAM MODEL

• Boundary Geometry

\[ P_1 = (0, 0, 0, 0), \quad P_2 = (0, 0, 0, -2\sqrt{5}/3^{1/4}), \quad P_3 = (0, 0, -3^{1/4}\sqrt{5}, -3^{1/4}\sqrt{5}), \]
\[ P_4 = (0, -2\sqrt{10}/3^{3/4}, -\sqrt{5}/3^{3/4}, -\sqrt{5}/3^{1/4}), \quad P_5 = (-3^{-1/4}10^{-1/2}, -\sqrt{5}/2/3^{3/4}, -\sqrt{5}/3^{3/4}, -\sqrt{5}/3^{1/4}). \]

\[
\begin{array}{c|cc|ccc|ccc}
\text{area} & j_{0ab} & 1 & 2 & 3 & 4 & 5 \\
\hline
a & 1 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]

\[
\text{normal } \vec{n}_{ab} \text{ by } b \\
\begin{array}{c|c|c|c|c|c}
& & 1 & 2 & 3 & 4 \\
\hline
1 & \backslash & (1,0,0) & (-0.33,0.94,0) & (-0.33,-0.47,0.82) & (-0.33,-0.47,-0.82) \\
2 & (-1,0,0) & \backslash & (0.83,0.55,0) & (0.83,-0.28,0.48) & (0.83,-0.28,-0.48) \\
3 & (0.33,-0.94,0) & (0.24,0.97,0) & \backslash & (-0.54,0.69,0.48) & (-0.54,0.69,-0.48) \\
4 & (0.33,0.47,0.82) & (0.24,0.48,0.84) & (-0.54,0.068,0.84) & \backslash & (-0.54,0.76,-0.36) \\
5 & (0.33,0.47,0.82) & (0.24,-0.48,-0.84) & (-0.54,0.068,-0.84) & (-0.54,-0.76,-0.36) & \backslash \\
\end{array}
\]

\[ N_1 = (-1, 0, 0, 0), \quad N_2 = \left( \frac{5}{\sqrt{22}}, \sqrt{\frac{3}{22}}, 0, 0 \right), \quad N_3 = \left( \frac{5}{\sqrt{22}}, \frac{1}{\sqrt{66}}, \frac{2}{\sqrt{33}}, 0 \right), \]
\[ N_4 = \left( \frac{5}{\sqrt{22}}, -\frac{1}{\sqrt{66}}, \frac{1}{\sqrt{33}}, \frac{1}{\sqrt{11}} \right), \quad N_5 = \left( \frac{5}{\sqrt{22}}, -\frac{1}{\sqrt{66}}, -\frac{1}{\sqrt{33}}, \frac{1}{\sqrt{11}} \right). \]
EPRL SPINFOAM MODEL

- Boundary Condition

$$|\Psi_0\rangle = \sum_{\lambda_{j_{ab}}} \psi_{\lambda_{j_0}, \zeta_0} |\lambda_{j_{ab}}, \vec{n}_{ab}\rangle$$

$$\psi_{\lambda_{j_0}, \zeta_0} = \exp\left(-i\sum_{ab} \zeta_0^{ab}(\lambda_{j_{ab}} - \lambda_{j_{0ab}})\right) \exp\left(-\sum_{ab,cd} \alpha^{(ab)(cd)} \frac{\lambda_{j_{ab}} - \lambda_{j_{0ab}}}{\sqrt{\lambda_{j_{0ab}}}} \frac{\lambda_{j_{cd}} - \lambda_{j_{0cd}}}{\sqrt{\lambda_{j_{0cd}}}}\right)$$

| $\zeta_0^{ab}$ | $\lambda_{j_{0ab}}$ |
|----------------|------------------|
| $a$ | b | 2 | 3 | 4 | 5 |
| 1 | -3.14+0.36γ | 0.68+0.36γ | 5.05+0.36γ | 5.05+0.36γ |
| 2 | \ | 5.05+0.50γ | -5.93+0.50γ | -3.20+0.50γ |
| 3 | \ | \ | -2.81+0.50γ | -5.54+0.50γ |
| 4 | \ | \ | \ | -4.37+0.50γ |
EPRL SPINFOAM MODEL

• Boundary Condition

\[ |\Psi_0\rangle = \sum_{\lambda j_{ab}} \psi_{\lambda j_0, \zeta_0} |\lambda j_{ab}, \vec{n}_{ab}\rangle \]

\[ \psi_{\lambda j_0, \zeta_0} = \exp \left( -i \sum_{ab} \zeta_0^{ab} (\lambda j_{ab} - \lambda j_{0ab}) \right) \exp \left( - \sum_{ab, cd} \alpha^{(ab)(cd)} \frac{\lambda j_{ab} - \lambda j_{0ab}}{\sqrt{\lambda j_{0ab}}} \frac{\lambda j_{cd} - \lambda j_{0cd}}{\sqrt{\lambda j_{0cd}}} \right) \]

In order to get a right propagator limit, \( \alpha \) has to take specific value. But, since we just want to justify the reliability of the algorithm, we can randomly choose the \( \alpha \) and compare the results with the one gotten from asymptotic expansion based on the same \( \alpha \).
CRITICAL POINT

- Semi-classical expansion $= 1/\lambda$ expansion
  (stationary phase approximation)

- In our calculation, there is only one critical point corresponding to the geometry of the 4-simplex

| $\omega_{ab}$ | a | b   | 1          | 2      | 3          | 4          | 5          |
|--------------|---|------|------------|--------|------------|------------|------------|
| $g_{0 \alpha}$ | 1 | 0    | 0.18i 1.01i| 0.18i 0.96–0.34i| 1.01i 0.48–0.34i| -0.65i 0.48–0.34i| 0.48–0.34i 1.01i|
| 2            | 0 | 1    |            | 0.18i 0.96–0.34i| 1.01i -0.48–0.34i| -0.65i -0.48–0.34i| (stationary phase approximation) |
| 3            | 1 | 1    | 1.01i 0.18i| -0.96–0.34i 0.18i| 0.48–0.34i 0.65i| 0.48–0.34i 1.01i| 0.48–0.34i 1.01i|
| 4            | 1 | 2    |            | (1, 0.333 + 0.942i) | (1, 0.184+0.259i) | (1, 1.817+2.569i) | (1, 0.058+0.082i) |
| 5            | 1 | 3    |            | (1, 1.065+0.729i) | (1, 1.857+0.989i) | (1, 0.420+0.223i) | (1, 0.071+0.470i) |
| 6            | 1 | 4    |            | (1, 0.085+0.720i) | (1, 0.313+2.080i) | (1, 0.071+0.470i) | (1, 0.058+0.082i) |
| 7            | 1 | 5    |            | (1, 0.084+0.720i) | (1, 0.685+0.729i) | (1, 0.313+2.080i) | (1, 0.071+0.470i) |
CRITICAL POINT

• Semi-classical expansion $= 1/\lambda$ expansion
  (stationary phase approximation)

• In our calculation, there is only one critical point corresponding to the geometry of the 4-simplex

\[
\frac{\partial S}{\partial g}[g_0, z_0] = 0 \quad \frac{\partial S}{\partial z}[g_0, z_0] = 0 \quad Re(S[g_0, z_0]) = 0
\]

\[
\frac{\partial S}{\partial j_{ab}}[g_0, z_0] = \lambda i \zeta_{ab}
\]

\[
\frac{\partial S_{tot}}{\partial j_{ab}}[j_0, g_0, z_0] = 0
\]
SPINFOAM PROPAGATOR

- 2-point correlation function:

\[
G_{abcd}^{mn} = \frac{\left< W | E_n^a \cdot E_n^b E_m^c \cdot E_m^d | \Psi_0 \right>}{\left< W | \Psi_0 \right>} - \frac{\left< W | E_n^a \cdot E_n^b | \Psi_0 \right>}{\left< W | \Psi_0 \right>} \cdot \frac{\left< W | E_m^c \cdot E_m^d | \Psi_0 \right>}{\left< W | \Psi_0 \right>}
\]

Penrose metric:

\[
q^{ab}(x) = \delta^{ij} E_i^a(x) E_j^b(x)
\]

\[
\langle \lambda_{j_{ab}, -\bar{n}_{ab}} | Y_{g_{a}}^{-1} g_{b} Y (E_{b}^{a})^{i} \rangle \lambda_{j_{ab}, \bar{n}_{ba}}
\]

\[
= \langle \lambda_{j_{ab}, -\bar{n}_{ab}} | Y_{g_{a}}^{-1} g_{b} Y \lambda_{j_{ab}, \bar{n}_{ba}} \rangle \lambda_{j_{ab}, \gamma} \langle \sigma, Z_{ba}, \xi_{ba} \rangle \frac{1}{Z_{ba, \xi_{ba}}}
\]

\[
\langle \lambda_{j_{ab}, -\bar{n}_{ab}} | (E_{b}^{a})^{i} Y_{g_{a}}^{-1} g_{b} Y \lambda_{j_{ab}, \bar{n}_{ba}} \rangle
\]

\[
= \langle \lambda_{j_{ab}, -\bar{n}_{ab}} | Y_{g_{a}}^{-1} g_{b} Y \lambda_{j_{ab}, \bar{n}_{ba}} \rangle (-\lambda_{j_{ab}, \gamma}) \frac{1}{J_{\xi_{ab}} \cdot Z_{ab}}
\]

[E. Bianchi, Y. Ding, etc]
SPINFOAM PROPAGATOR

• 2-point correlation function:

\[
\begin{align*}
\langle W \left| E^a_n \cdot E^b_n E^c_m \cdot E^d_m \right| \Psi_0 \rangle &= \sum_{j_{ab}} \psi_{j_{ab},\zeta_0} \int d\phi U(j, \phi) [A_{an}(j, \phi) \cdot A_{bn}(j, \phi)] [A_{cm}(j, \phi) \cdot A_{dm}(j, \phi)] e^{\lambda S(j, \phi)} \\
\langle W \left| E^a_n \cdot E^b_n \right| \Psi_0 \rangle &= \sum_{j_{ab}} \psi_{j_{ab},\zeta_0} \int d\phi U(j, \phi) A_{an}(j, \phi) \cdot A_{bn}(j, \phi) e^{\lambda S(j, \phi)} \\
\langle W \left| E^c_m \cdot E^d_m \right| \Psi_0 \rangle &= \sum_{j_{ab}} \psi_{j_{ab},\zeta_0} \int d\phi U(j, \phi) A_{cm}(j, \phi) \cdot A_{dm}(j, \phi) e^{\lambda S(j, \phi)}
\end{align*}
\]

\[
A_{ab}^i = \gamma j_{ab} \frac{\langle \sigma^i_{ba}, \xi_{ba} \rangle}{\langle Z_{ba}, \xi_{ba} \rangle}, \quad A_{ba}^i = -\gamma j_{ab} \frac{\langle J_{\xi_{ba}}, \sigma_i Z_{ba} \rangle}{\langle J_{\xi_{ba}}, Z_{ba} \rangle},
\]
INTEGRAL OF J

- Poisson re-summation:

\[
\sum_{J \in \mathbb{Z}^+ \cup 0} f(J) = \frac{1}{2} \sum_{J \in \mathbb{Z}} f(|J/2|) + \frac{1}{2} f(0) = 2 \sum_{k \in \mathbb{Z}} \int_0^\infty dJ f(J) e^{4\pi i k J} + \frac{1}{2} f(0)
\]

Remind: \( \psi_{\lambda_{j_0}, \zeta_0} = \exp \left( -i \sum_{ab} \zeta_0^{ab} (\lambda_{j_{ab}} - \lambda_{j_{0ab}}) \right) \exp \left( - \sum_{ab,cd} \alpha_{(ab)(cd)}^{(ab)} \frac{\lambda_{j_{ab}} - \lambda_{j_{0ab}}}{\sqrt{\lambda_{j_{0ab}}}} \frac{\lambda_{j_{cd}} - \lambda_{j_{0cd}}}{\sqrt{\lambda_{j_{0cd}}}} \right) \)

The term \( \frac{1}{2} f(0) \) is negligible when \( \lambda \) is large
INTEGRAL OF J

- Poisson resummation:

\[
\langle W | \Psi_0 \rangle = (2\lambda)^{10} \sum_{\{k_{ab}\} \in \mathbb{Z}^{10}} \int_0^\infty \int_0 d^{10} j \ d\phi U e^{-\lambda S_{tot}^{(k)}}
\]

\[
\langle W | E_n^a \cdot E_n^b E_m^c \cdot E_m^d | \Psi_0 \rangle = (2\lambda)^{10} \sum_{\{k_{ab}\} \in \mathbb{Z}^{10}} \int_0^\infty \int_0 d^{10} j \ d\phi U e^{-\lambda S_{tot}^{(k)}} (A_{an} \cdot A_{bn}) (A_{cm} \cdot A_{dm})
\]

\[
\langle W | E_n^a \cdot E_n^b | \Psi_0 \rangle = (2\lambda)^{10} \sum_{\{k_{ab}\} \in \mathbb{Z}^{10}} \int_0^\infty \int_0 d^{10} j \ d\phi U e^{-\lambda S_{tot}^{(k)}} (A_{an} \cdot A_{bn})
\]

\[
\langle W | E_m^c \cdot E_m^d | \Psi_0 \rangle = (2\lambda)^{10} \sum_{\{k_{ab}\} \in \mathbb{Z}^{10}} \int_0^\infty \int_0 d^{10} j \ d\phi U e^{-\lambda S_{tot}^{(k)}} (A_{cm} \cdot A_{dm})
\]

\[
S_{tot}^{(k)} = S_{tot} + 4\pi i \sum_{a > b} j_{ab} k_{ab}
\]

with

\[
S_{tot} = i \sum_{ab} \zeta_0^{ab} (j_{ab} - j_{0b}) + \sum_{ab, cd} \alpha^{(ab)(cd)} \frac{j_{ab} - j_{0ab}}{\sqrt{j_{0ab}}} \frac{j_{cd} - j_{0cd}}{\sqrt{j_{0cd}}} - S(j, \phi),
\]
INTEGRAL OF J

- The integral around the critical point dominate the whole integral.
- By our choice of $\zeta_{ab}$, critical point can only be found when $k_{ab} = 0$
- The $k_{ab} \neq 0$ terms in the summation are exponentially suppressed when $\lambda$ is large.
INTEGRAL OF $J$

\[
\langle W \mid \Psi_0 \rangle \simeq \int_{-\infty}^{\infty} d^{10}j \int d\phi \tilde{U} e^{-\lambda S_{tot}}
\]

\[
\langle W \mid E_n^a \cdot E_m^c \mid \Psi_0 \rangle \simeq \int_{-\infty}^{\infty} d^{10}j \int d\phi \tilde{U} e^{-\lambda S_{tot}} (A_{an} \cdot A_{bn})(A_{cm} \cdot A_{dm})
\]

\[
\langle W \mid E_n^a \cdot E_n^b \mid \Psi_0 \rangle \simeq \int_{-\infty}^{\infty} d^{10}j \int d\phi \tilde{U} e^{-\lambda S_{tot}} (A_{an} \cdot A_{bn})
\]

\[
\langle W \mid E_m^c \cdot E_m^d \mid \Psi_0 \rangle \simeq \int_{-\infty}^{\infty} d^{10}j \int d\phi \tilde{U} e^{-\lambda S_{tot}} (A_{cm} \cdot A_{dm})
\]

\[
S_{tot} = i \sum_{ab} \zeta_{50}^{ab} (j_{ab} - j_{0ab}) + \sum_{ab,cd} \alpha^{(ab)(cd)} \frac{j_{ab} - j_{0ab}}{\sqrt{j_{0ab}}} \frac{j_{cd} - j_{0cd}}{\sqrt{j_{0cd}}} - S(j, \phi),
\]

Our algorithm can be applied
EPRL-SPINFOAM MODEL

- The action depends on 54 real variables
- The action is complex-valued
- The observables we want to compute are

\[
\frac{\langle W|E_n^a \cdot E_n^b E_m^c \cdot E_m^d|\Psi_0\rangle}{\langle W|\Psi_0\rangle}, \quad \frac{\langle W|E_n^a \cdot E_n^b |\Psi_0\rangle}{\langle W|\Psi_0\rangle}, \quad \frac{\langle W|E_m^c \cdot E_m^d |\Psi_0\rangle}{\langle W|\Psi_0\rangle}
\]

and

\[
G^{abcd}_{mn}
\]
PERTURBATIVE AND NON-PERTURBATIVE CORRECTIONS

- Analytical continuation leads to more critical points.
- These critical points provide exponentially suppressed corrections to the integral at large $\lambda$.
- Only compute the observables on the thimble attached to dominant critical point.
- The computation contains the perturbative $1/\lambda$ corrections to all orders on the dominant thimble and neglects all the exponentially suppressed corrections.

Suppressed corrections are caused by 1) other thimbles, 2) $k_{ab} \neq 0$ terms, 3) $f(0)$ term, 4) the integral $\int_{-\infty}^{0} dj$.
OPTIMIZATIONS

- Numerical solver of SA equation (stiffness problem)
  - Time rescaling: $t \rightarrow \frac{t}{\lambda}$
  - Tolerance of the error
- DREAM optimization (sampling on thimble)
  - Initial point optimization (same energy scheme)
  - Flow time optimization (average among convergent results)
  - Burn-in stage (adaptively adjust CR and Beta)
SPINFOAM PROPAGATOR

• Results of Expectation values $\langle E_1^2 \cdot E_1^3 E_4^1 \cdot E_4^5 \rangle$

| $\lambda$ | $10^2$ | $10^3$ | $10^4$ | $5 \times 10^4$ | $10^5$ | $5 \times 10^5$ | $10^6$ | $5 \times 10^6$ | $10^7$ | $5 \times 10^7$ |
|----------|--------|--------|--------|----------------|--------|----------------|--------|----------------|--------|----------------|
| Difference (%) | 8.71   | 0.79   | 0.12   | 0.052          | 0.036  | 0.017          | 0.0062 | 0.0018         | 0.0037 | 0.00069        |
SPINFOAM PROPAGATOR

- Results of Expectation values $\langle E_1^2 \cdot E_1^3 \rangle$

| $\lambda$ | $10^2$ | $10^3$ | $10^4$ | $5 \times 10^4$ | $10^5$ | $5 \times 10^5$ | $10^6$ | $5 \times 10^6$ | $10^7$ | $5 \times 10^7$ |
|-----------|--------|--------|--------|----------------|--------|----------------|--------|----------------|--------|----------------|
| Difference (%) | 22.32 | 2.00 | 0.31 | 0.078 | 0.022 | 0.016 | 0.012 | 0.0022 | 0.000047 | 0.0016 |
SPINFOAM PROPAGATOR

- Results of Expectation values $\langle E_4^1 \cdot E_4^5 \rangle$
SPINFOAM PROPAGATOR

- Results of propagator component $G_{14}^{2315}$

\[
\begin{array}{cccccccccc}
\lambda & 10^2 & 10^3 & 10^4 & 5 \times 10^4 & 10^5 & 5 \times 10^5 & 10^6 & 5 \times 10^6 & 10^7 & 5 \times 10^7 \\
\text{Difference (\%)} & 37.90 & 27.00 & 13.22 & 2.76 & 10.09 & 8.86 & 1.89 & 1.13 & 3.90 & 2.06 \\
\end{array}
\]
SPINFOAM PROPAGATOR

- Results of propagator component $G_{14}^{2315}$

$G^{abcd}(x, y) = \langle q^{ab}(x)q^{cd}(y) \rangle - \langle q^{ab}(x) \rangle \langle q^{cd}(y) \rangle$

Propagator is of high order of $1/\lambda$
SPINFOAM PROPAGATOR

- The propagator has 1275 components

Histogram of percentage difference to leading order in $1/\lambda$ expansion

$\lambda = 10^6$

$\lambda = 10^7$
OUTLOOK

Future Plans
OUTLOOK

• Spin foam propagators in more complicated cases (Pachner 1-5 move)
• Computing other operators (flatness problem)
• Improvement of the algorithm (hybrid monte-carlo, etc)
• Use SD flow and Metropolis algorithm to find the critical point
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Thanks!