Collapse of a Superhorizon-sized Void in the Early Universe

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In this paper, we study the collapse of a superhorizon-sized void in the early, radiation-dominated universe using an improved general relativistic code. We find that in general a relativistic or nonrelativistic void collapses via a shock at the speed of light. This is true even if the outward velocity of the void wall is enormous. As the wall thickness decreases, the shock strength increases and the collapse time decreases up to a limit set by the shock tube solution. In addition, as the outward velocity of the wall increases, the collapse time increases somewhat. When the collapse occurs in much less than a Hubble time outside the void, gravitational forces contribute negligibly to the solution at the collapse time; non-gravitational forces (caused by pressure and velocity gradients) shape the solution almost entirely. This is true even if the wall velocity is large enough that fluid in the peak shocks outward substantially during this time. Gravitational forces are expected to be dominant only after a void has collapsed.

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I. Introduction

Because the total energy density is of order the critical energy density today, if the early universe is described entirely by the big bang model, a severe fine-tuning problem exists. This problem is solved if inflation occurred in the early universe. First-order inflation is one of the most interesting classes of inflationary models\[1\]. Here, inflation ends in a particular region when a true-vacuum bubble is nucleated. Inflation ends everywhere when the universe is filled with true vacuum bubbles. At this point, scalar field dynamics occurs which creates relativistic, fundamental particles. These particles thermalize much of the universe within a few Hubble times. Due to causality however, bubbles nucleated early on during inflation cannot be thermalized as quickly. Thus, since the inside of these bubbles are virtually empty, superhorizon-sized voids are created at the end of reheating.\[1\] These voids are larger than the outside Hubble radius, $cH_{\text{out}}^{-1}(t_i)$, (i.e. the Hubble radius outside the void), and can have enormous radii $R_w \sim 10^{27}cH_{\text{out}}^{-1}(t_i)$. The beginning of the radiation-dominated epoch then, is characterized by an approximately flat Friedmann-Robertson Walker (FRW) homogeneous and isotropic universe, punctuated infrequently by nearly empty, superhorizon-sized voids.

The thermalization of the very large voids has been thought to take an enormous amount of time and therefore be in conflict with the tiny temperature fluctuations measured in the microwave background\[2\]. In a previous paper\[3\] however (hereafter referred to as SV), it was found that these former estimates could be incorrect because the first-crossing time (the time taken for a photon in the void wall to reach the origin and thus make “contact” with a photon from the other side) was calculated incorrectly. In \[2\], it was found that the first-crossing time, $\Delta t_c$, (i.e. the minimum thermalization time) is $\Delta t_c/H_{\text{out}}^{-1}(t_i) \approx (c^{-1}R_w/H_{\text{out}}^{-1}(t_i))^2$. However, it was found in SV that the first-crossing time is actually much smaller than this:

$$\Delta t_c/H_{\text{out}}^{-1}(t_i) = \Phi_{\text{in}}^{-1} \left( c^{-1}R_w/H_{\text{out}}^{-1}(t_i) \right) \left[ 1 + .5 \Phi_{\text{in}}^{-1} \left( c^{-1}R_w/H_{\text{out}}^{-1}(t_i) \right) \right],$$

(1.1)

where the void’s relative size is $c^{-1}R_w/H_{\text{out}}^{-1}(t_i) < \alpha^{-1/2}$, the log of its relative fluid potential is $\Phi_{\text{in}} \simeq \alpha^{-1/4} \gg 1$, $\alpha \simeq \rho_{\text{in}}(t_i)/\rho_{\text{out}}(t_i) \ll 1$, $\rho$ is the energy density, and the subscripts “in” and “out” refer to quantities inside and outside the void, respectively. Because of the time dilation effect, the first-crossing time is smaller than the outside Hubble time if the void is empty enough. This effect occurs because the (relativistic or nonrelativistic) void has a large, negative fluid potential with respect to the relativistic background spacetime, which is caused by relativistic fluid effects. (In a matter-dominated universe, these fluid effects are tiny when the pressure is negligible.) Because the void is underdense, the time dilation effect is opposite of that for an overdense region (e.g. a black hole)—time is dilated in an underdense region and contracted in an overdense region. If a superhorizon-sized void is empty enough then, it can collapse in less than an outside Hubble time.

In this paper, we study in more detail the collapse of a superhorizon-sized void embedded in a FRW radiation-dominated universe. By collapse, we mean the evolution up

\[\footnote{Following past convention, we loosely equate the Hubble radius with the horizon in the phrase “superhorizon-sized”. Horizon in this context is not to be confused with the particle horizon.}\]
to the time the shock (created by the void wall caving in) collides with itself at the origin. In particular, we examine how large outward wall velocities affect the solution and also how gravitational forces affect the solution at the collapse time. We find that having a large outward wall velocity does not prevent a superhorizon-sized void from collapsing in less than an outside Hubble time. In addition, we show that if a void collapses in less than an outside Hubble time, gravitational effects are not important at the collapse time. This motivates us to compare the collapse of a superhorizon-sized void with that of a special relativistic void. We therefore derive the analytic collapse solutions in the slab limit, and test our code against these solutions. At the collapse time, thermalization and homogenization has not yet occurred, however. A companion paper will examine the subsequent evolution of a superhorizon-sized void after it has collapsed.

The organization of this paper is as follows. In Section II we present the metric and equations of motion used to solve this problem. In Section III, we describe the improved numerical scheme used to accurately find the solution near the origin, and we test this new code on the FRW radiation-dominated universe. In Section IV, we study the collapse of a superhorizon-sized void both analytically and numerically. In Section V we derive the solution for the collapse of an uncompensated void in the slab limit. Finally, Section VI contains a discussion of our results.

II. Spherically Symmetric General Relativistic Fluid Equations

We are interested in evolving a void embedded in a FRW universe. Because we know the initial conditions on an initial comoving time slice, we choose to work in Lagrangian synchronous coordinates. The metric then is

$$ds^2 = -c^2\Phi^2(t, r)dt^2 + \Lambda^2(t, r)dr^2 + R^2(t, r)d\Omega^2,$$

where $t$ is the coordinate time, $r$ is the comoving radius, $2\pi R$ is the spacelike circumference of a sphere centered on the origin, and $\ln \Phi$ is the pressure gradient-induced “potential”. We consider a perfect fluid with artificial viscosity. This fluid consists of particles with mass $\mu$ and temperature $T$. The stress-energy tensor in this case is

$$T^{\alpha\beta} = c^{-2}(\rho + p + Q)u^\alpha u^\beta + (p + Q)g^{\alpha\beta},$$

where $u^\alpha = (-c\Phi^{-1}, 0, 0, 0)$ is the fluid 4-velocity, $\rho = n\epsilon/c^2$ is the energy density, $p = nT/\mu = \omega n\epsilon$ is the pressure, $n$ is the mass density, $\epsilon$ is the specific energy, $Q$ is the artificial viscosity and $\omega$ is a constant. The relativistic limit is obtained when $\epsilon/c^2 = T/(c^2\omega\mu) \gg 1$. The viscosity is non-zero only in shocks. We assume that the total number of particles per comoving volume is constant, $\nabla_\mu(nu^\mu) = 0$, so that

$$4\pi nR^2R'/\Gamma = f(r),$$

where $' \equiv \partial/\partial r$, $f(r)$ is chosen to be $r^2$, and

$$\Gamma \equiv R'/\Lambda.$$
The fully general relativistic equations can then be written as\[5\],\[6\]
\[
\dot{R} = \Phi U \tag{2.4}
\]
\[
\dot{U} = -\Phi \left( \frac{G_N M}{R^2} + \frac{4\pi G_N (p + Q) R}{c^2} \right) - \frac{\Gamma^2 \Phi (p + Q)'}{W n R'} \tag{2.5}
\]
\[
\dot{M} = -4\pi (p + Q) R^2 \Phi U / c^2 \tag{2.6}
\]
\[
\dot{n} = -\frac{n \Phi (R^2 U)'}{R^2 R'} \tag{2.7}
\]
\[
\dot{\epsilon} = -\frac{\Phi (p + Q)' (R^2 U)'}{n R^2 R'} \tag{2.8}
\]
\[
\Phi' = -\Phi \frac{(p + Q)'}{n W c^2}, \tag{2.9}
\]
where \( \dot{} \equiv \partial / \partial t \),
\[
M' \equiv 4\pi c^{-2} \rho R^2 R' \tag{2.10}
\]
\[
\Gamma^2 \equiv 1 + (U/c)^2 - 2G_N M / (Rc^2), \tag{2.11}
\]
and \( W \equiv 1 + [\epsilon + (p + Q)/n] / c^2 \) (or \( c^2 n W = \rho + p + Q \)). Here, \( U \) is the fluid “velocity”, \( M \) is the “mass-energy” and \( W \) is the relativistic enthalpy. The energy density equation can be found by combining Eqs. (2.7) and (2.8):
\[
\dot{\rho} = -\Phi (\rho + p + Q) \frac{(R^2 U)'}{R^2 R'}. \tag{2.12}
\]

Eq. (2.11) is the conservation of “energy” equation, with total, kinetic and potential “energies” of \( c^2 (\Gamma^2 - 1)/2, U^2/2 \) and \(-G_N M/R\), respectively. For \( G_N = 0 \), if a particle has velocity \( v \) in a Eulerian inertial frame, then \( \Gamma = 1/\sqrt{1 - (v/c)^2} \) and \( U = \Gamma v \) (see Eq. (A.10)); \( \Gamma \) and \( U \) represent the two non-trivial components of the 4-velocity of the fluid. We can also write down an important auxiliary equation:
\[
\dot{\Gamma} = -\Gamma U \Phi \frac{(p + Q)'}{(\rho + p + Q) R'}. \tag{2.13}
\]

Thus the “energy” \( c^2 (\Gamma^2 - 1)/2 \) can only decrease in the presence of pressure gradients. In SV, Eq. (2.13) was used to argue that if \( \Gamma \gg 1 \) in the void wall, the wall might slow down in a relatively short amount of time. We will find later that this is true in some cases.

It is necessary to have artificial viscosity in the code due to the presence of shocks—it is added to physically alter the otherwise incorrect solution. Artificial viscosity dissipates just enough energy over a shock to satisfy the (exact) jump conditions, which cannot occur in a perfect fluid. These conditions are derived in the appendix, and a demonstration of the codes ability to give the correct solution with the following form for \( Q \) will be given there (Figure 13) and later on in this paper (Figure 9).
The form of the artificial viscosity used here is a generalization of the nonrelativistic form, and allows relativistic shocks to penetrate into relativistic fluids. It does not, however, alter the solution substantially in non-shock areas.

\[
Q = k^2 n W^{-2} (U')^2 dr^2 \quad \text{for } U' < 0
\]
\[
Q = 0 \quad \text{otherwise.} \quad (2.14)
\]

The constant \( k^2 \) is of order one, and is proportional to the number of grid points in the shock. For relativistic shocks penetrating non-relativistic fluids, Eq. (2.14) becomes the expression obtained previously. The expression from this reference however, does not give enough viscosity over strong shocks when the fluid in front of the shock is relativistic, because \( Q \ll p \). (It is necessary for the artificial viscosity to be of order the pressure in the shock region, in order to dissipate enough energy). Eq. (2.14) does, however. Assuming that the fluid in front of the shock is stationary, then \( k^2 \Delta U^2 \approx \Gamma^2 - 1 \) over the shock front, where we have used the shock jump conditions Eqs. (A.26)-(A.27) in the strong shock limit. Then \( Q \approx (\rho + p)U^2/(\Gamma^2 c^2) \approx \rho(\Gamma^2 - 1)/\Gamma^2 \approx \rho \approx p \), the desired result.

If the fluid is homogeneous and isotropic, Eqs. (2.15) reduce to the FRW equations, with

\[
R = ra, \quad \Phi = 1, \quad \Gamma = \sqrt{1 - \kappa r^2/c^2} = \sqrt{1 + (RH/c)^2(1 - \Omega)},
\]
\[
M = 4\pi c^{-2} \rho R^3/3, \quad H \equiv \Phi^{-1} \dot{a}/a = U/R, \quad (2.15)
\]

where \( a(t) \) is the cosmic scale factor, \( H \) is Hubble’s “constant”, \( \Omega = 1 + \kappa/(Ha)^2 = 8\pi G_N\rho/(3c^2H^2) \), \( f = r^2/\sqrt{1 - \kappa r^2/c^2} \), and \( \kappa \) is \(-1, 0 \) or \( 1 \) for negative, zero and positive spatial curvature, respectively. In a spatially flat (\( \kappa = 0 \)) FRW universe then, \( U = c^{-1}R\sqrt{8\pi G_N\rho}/3 = \sqrt{2G_N M/R} \). Note that \( \Gamma(t, r) = 1 \) in a spatially flat universe, even though the fluid everywhere is moving.

### III. Numerical Scheme and Tests of Improved Code

In this section we describe the improved numerical method used to solve the equations given in the previous section. We also test the code on the FRW radiation-dominated solution. Readers more interested in the physical results should skip to section IIIIC. The code is greatly improved over that used in SV, because we have now implemented numerical regularization into the difference equations\(^4\). This is done to get rid of instabilities that develop at the origin from the inaccurate calculation of \( \dot{\epsilon}, \dot{n} \) and \( \dot{U}_{\text{FLUID}} \equiv -\Gamma^2 \Phi(p + Q)/(WnR') \) there. As we will see in a moment, the difference equations used in SV were only accurate to \( (\Delta R/R)^2 \), where \( \Delta R \) is the grid size, which is or order one at the first few grid points surrounding the origin. Thus, no matter how small \( \Delta R \) is chosen to be, \( \epsilon, n \) and \( U_{\text{FLUID}} \) could not be calculated correctly for these grid points. It is true that decreasing \( \Delta R \) confines the inaccuracies to a much smaller physical volume. The problem, however, is that the inaccuracies can become instable when violent fluid processes take place at or near the origin. An example of such a process is when

\(^4\)It turns out that scheme 1 from SV for the \( \dot{n} \) equation is differenced correctly, however.
a spherical shock collides at the origin. These instabilities then cause the code to fail. Numerical regularization ensures that the error made in calculating $\epsilon$, $n$ and $U_{\text{FLUID}}$ is of order $(\Delta R/l)^2$, where $l$ is the characteristic length scale of the problem. With numerical regularization then, the error made can be a small as you like by choosing $\Delta R$ sufficiently small.

A. Numerical Regularization

We start by differencing the derivative term in Eq. (2.7) and (2.8), $(R^2U)'/(R^2 R')$, at spatial grid point $j$ as $\Delta (R^2U)/(R^2 \Delta R)$, as was done for the $n$ equation in scheme 2 of SV. We approximate $R$ and $U$ at adjacent grid points $j \pm 1$ by Taylor expanding:

\[
R_{j\pm 1} = R_j \pm \Delta r (\partial_r R)_j
\]

\[
U_{j\pm 1} = U_j \pm \Delta r (\partial_r U)_j.
\]

(3.1)

If we forward difference this derivative term at grid point $j$, we find

\[
\frac{R_{j+1}^2 U_{j+1} - R_j^2 U_j}{R_{j+1}^2 (R_{j+1} - R_j)} = \frac{1}{R_j^2 (\partial_r R)_j} \left[ 2U_j R_j (\partial_r R)_j \left\{ 1 + \frac{\Delta r (\partial_r R)_j}{2R_j} \right\} \right]
\]

\[+ R_j^2 (\partial_r U)_j \left\{ 1 + \frac{2\Delta r (\partial_r R)_j}{R_j} + \frac{\Delta r^2 (\partial_r R)_j^2}{2R_j^2} \right\} \].

(3.2)

Backward differencing is obtained by substituting $\Delta r \rightarrow -\Delta r$ into Eq. (3.2). The forward (backward) differencing gives the predicted (corrected) solution, as will be described in section B. The predicted and corrected solutions are then averaged together. When this is done, the error made in calculating $\epsilon$ or $n$ at the new time step is approximately $(\Delta R/R_j)^2$. This truncation error is small for all but the first few grid points near the origin. There, the truncation error is of order one, so that inaccurate calculation of $\epsilon$ and $n$ will result when $U' \neq 0$.

If our grid is equally spaced in $R$ so that $R_1 = \Delta R/2$, $R_2 = 3\Delta R/2$, etc, then for $j = 1$, 2, 3 and 4, the truncation error in the $R_j^2 (\partial_r U)_j$ term is about 400%, 44% 16% and 8%, respectively. A similar truncation error problem occurs when differencing $\dot{U}_{\text{FLUID}}$ as in SV: $U_{\text{FLUID}} \propto \Delta (p + Q)/\Delta (r^3)$.

We can rectify these problems using numerical regularization. Near the origin, we expand $p$, $Q$, $R$ and $U$ in a Taylor series in $r$. Then we use symmetry arguments to deduce the actual form of each solution there. Using the fact that $p(-r) = p(r)$, $Q(-r) = Q(r)$, $R(-r) = -R(r)$ and $U(-r) = -U(r)$ along with the fact that $R(0) = U(0) = 0$, $p(0) \neq 0$ and $Q_j \propto (U')^2$, the lowest-order terms in the series at grid point $j$ are

\[
p_j = p_0 + p_2 r_j^2 + O(r_j^4)
\]

\[
R_j = R_1 r_j + R_3 r_j^3 + O(r_j^5)
\]

\[
U_j = U_1 r_j + U_3 r_j^3 + O(r_j^5)
\]

\[
Q_j = Q_2 (U_{k+1} - U_k)^2 = Q_2 \left[ U_1 (r_{k+1} - r_k) + U_3 (r_{k+1}^3 - r_k^3) + O(r_k^5, r_{k+1}^5) \right]^2;
\]

(3.3)

\[9\text{In SV, } \epsilon \text{ was determined via calculating } \Delta (R^2 U)/(r^2 \Delta r). \text{ It can be easily shown that this gives a truncation error of order } O((\Delta R/R_j)^2) \text{ also.}
\]
where the coefficients $p_0$, $p_2$, etc, are constants, and where $k = j$ and $k = j - 1$ for the forward and backward derivatives, respectively. Because $(R^2U)_{j+1} = R_j^2U_1r_{j+1}^3$ to lowest order, it is clear that we should instead difference $(R^2U)'/(R^2R')$ as
\[
\frac{(R^2U)'}{R^2R'} = \frac{R_{j+1}^2U_{j+1} - R_j^2U_j}{R_{j+1}^3 - R_j^3}
\]

for the forward derivative, and similarly for the backward derivative. We can verify this by plugging Eq. (3.3) into Eq. (3.4):
\[
3\frac{R_{j+1}^2U_{j+1} - R_j^2U_j}{R_{j+1}^3 - R_j^3} = \frac{3U_1}{R_1} \left[ 1 + (U_3/U_1 + 2R_3/R_1)r_j^2 \left\{ \frac{(1 + \Delta r/r_j)^5 - 1}{(1 + \Delta r/r_j)^3 - 1} \right\} \right]
\times \left\{ \frac{1 + (3R_3/R_1)r_j^2 \left\{ (1 + \Delta r/r_j)^5 - 1 \right\}}{1 + (1 + \Delta r/r_j)^3 - 1} \right\}^{-1}.
\]

The correct result from Eq. (3.3) is $3U_1/R_1$. Since the quantity in curly brackets is of order 4 or less, $U_3/U_1 \sim O((\partial_r R_j)^2l^{-2})$ and $R_3/R_1 \sim O((\partial_r R_j)^2l^{-2})$, where $l$ is the characteristic length scale of the problem, the truncation error is of order $(\Delta R/l)^2$ for $R_j \sim \Delta R$. Therefore, the error made in calculating this derivative can be made as small as desired by decreasing the grid size $\Delta R$.

Now we turn to the derivative term in $\dot{U}_{\text{FLUID}}$: $(p + Q)'/(R')$. Using Eq. (3.3), we see that the obvious way to difference $p'/R'$ is $2R_j(p_{j+1} - p_j)/[(R_{j+1} + R_j)(R_{j+1} - R_j)]$ for the forward derivative, and similarly for the backward one. However, this causes a problem at $j = 1$ for the backward derivative because $p_0 = p_1$ and $R_0 = -R_1$, but $R_1(p_1 - p_0)/[(R_1 + R_0)(R_1 - R_0)] \neq 0$. Instead, we use the simple difference scheme
\[
\frac{p'}{R'} = \frac{p_{j+1} - p_j}{R_{j+1} - R_j}
\]

for the forward derivative, and similarly for the backward one. Plugging Eq. (3.3) into the previous equation, we get $(2p_2r_j/R_1)(1 + \Delta r/(2r_j))$ and $(2p_2r_j/R_1)(1 - \Delta r/(2r_j))$ for the forward and backward derivatives, respectively. Upon averaging the forward and backward derivatives, we get the correct result, $2p_2r_j/R_1$, to lowest order.\footnote{Although $p_1$ and $R_1$ will be slightly different for the forward and backward derivatives, their variations are of higher order.}

Note that in this case, the backward derivative at $j = 1$ is zero, thereby avoiding the problem caused by the former difference scheme.

We also difference the $Q'/R'$ term in the same way as Eq. (3.3) with $p$ replaced by $Q$. Using $Q_j$ from Eq. (3.3) with $k = j$, the forward derivative is
\[
\frac{Q_{j+1} - Q_j}{R_{j+1} - R_j} = \frac{12Q_2U_1U_3\Delta r^2r_j}{R_1} \left( 1 + \frac{\Delta r}{r_j} \right)
+ \frac{36Q_2U_3^2\Delta r^2r_j^3}{R_1} \left( 1 + \frac{3\Delta r}{r_j} + \frac{10}{3} \left( \frac{\Delta r}{r_j} \right)^2 + \frac{4}{3} \left( \frac{\Delta r}{r_j} \right)^3 \right).
\]
The backward derivative can be obtained by substituting $-\Delta r$ in for $\Delta r$ in the previous expression. We can now average the forward and backward expressions and obtain to lowest order

$$
\left( \frac{Q'}{R'} \right)_j \simeq \frac{12Q_2 U_1 U_3 \Delta r^2 r_j}{R_1} \left[ 1 + (3U_3/U_1)r_j^2 \left\{1 + \frac{10}{3} \left( \frac{\Delta r}{r_j} \right)^2 \right\} \right].
$$

(3.8)

The correct result, from Eq. (3.3), is $12Q_2 U_3 \Delta r^2 r_j (U_1 + 3U_3 r_j^2)/R_1$. But because $U_3/U_1 \sim \mathcal{O}(\partial_t R)^2 l^{-2}$, where $l$ is the characteristic length scale of the problem, the quantity in brackets is $[1 + \mathcal{O}(\Delta R/l)^2]$ for $r_j \sim \Delta R$. Therefore the error can be made as small as desired by decreasing the grid size $\Delta R$.

### B. Numerical Procedures

We set up a grid with $j_B + 1$ grid points which is equally spaced in $R$ such that $\Delta R(t_i) \equiv R(t_i)_{j+1} - R(t_i)_j$ for $j \in [0, j_B]$, with $R_0 = -\Delta R(t_i)/2$ and $R_1 = \Delta R(t_i)/2$. Then we specify $\rho(R(t_i))$ (or $M(R(t_i))$ and $\Gamma(R(t_i))$ (or $U(R(t_i))$). When $\Gamma(t_i, R)$ is given, we always choose to have the fluid moving outward initially, $U > 0$, using Eq. (2.11). Assume that the fluid is initially an isentrope, so that the specific energy, $\epsilon$, can be determined via solving

$$
\rho(t_i, R) = \frac{\rho(t_i, R_B)}{1 + \epsilon(t_i, R_B)/c^2} \left( \frac{\epsilon(t_i, R)}{\epsilon(t_i, R_B)} \right)^{1/\omega} \left( 1 + \epsilon(t_i, R)/c^2 \right)
$$

(3.9)

iteratively, where the subscript “$B$” denotes the quantity at the outer boundary. In addition, $\Phi(t_i, R_B) = 1$, and the artificial viscosity is initially zero: $Q(t_i, R) = 0$. Finally, $r_j$ and $\Phi(t_i, R)$ can be determined by integrating Eqs. (2.2) and (2.9) from $R = 0$ and $R = R_B$, respectively. It is important to note that for $p = \rho/3$ and $Q = 0$, $\Phi$ can be solved exactly on the initial time slice:

$$
\Phi(t_i, R) = \Phi_{\text{out}}(t_i) \left( \frac{\rho_{\text{out}}(t_i)}{\rho(t_i, R)} \right)^{1/4} \text{ for } \epsilon(t_i, R)/c^2 > 1,
$$

(3.10)

where “out” refers to quantities outside the void at $j = j_B$. We define $\overline{\rho}$ as the energy density where the fluid becomes nonrelativistic: $\overline{\epsilon}/c^2 \simeq 1$. Since the contribution to $\Phi$ is negligible when the fluid is nonrelativistic,

$$
\Phi(t_i, R) = \Phi_{\text{out}}(t_i) \left( \frac{\rho_{\text{out}}(t_i)}{\overline{\rho}(t_i, R)} \right)^{1/4} \text{ for } \rho(t_i, R) < \overline{\rho}.
$$

(3.11)

Inside the void, $\overline{\rho} = \rho_{\text{in}}(t_i)$ for a relativistic void, and $\overline{\rho} \simeq g(c^2 \mu)^{1/4}$ for a nonrelativistic void, where $g$ is the number of degrees of freedom, $\mu$ is the particle mass, and “in” refers to quantities inside the void. For a nonrelativistic void in a relativistic background fluid then,

$$
\Phi_{\text{in}}(t_i) \simeq \Phi_{\text{out}}(t_i) \epsilon_{\text{out}}(t_i)/c^2.
$$

(3.12)

\(^1\text{Again, we ignore the variations of } U_1, U_3, Q_2 \text{ and } R_1 \text{ in the averaging, since they are of higher order.}\)
We use the MacCormack predictor-corrector method to integrate the equations of motion. Suppose we know all quantities on the \(i\)th time slice for all spatial points \(j\). Using Eq. (2.7) as an example, we first predict the new quantities (with forward differencing) for all \(j\):

\[ n_p^{i+1} = n_j^i - \Delta t \cdot 3n_j^i \Phi_j^i (R_{j+1}^i - R_{j+1}^{i+1} - R_{j+1}^i - U_j^i)/(R_{j+1}^{i+1} - R_{j+1}^i)^2. \]

After using similar methods to obtain \(U_p^{i+1}, R_p^{i+1}, M_p^{i+1}\), and \(\epsilon_p^{i+1}\), we integrate Eq. (2.7) inwards from the outer boundary using the 4th-order Runge-Kutta method with linear interpolations to determine \(\Phi_p^{i+1}\). The “predicted” viscosity (to be used in the corrector step) is:

\[ Q_{p_j} = k^2 n_j^i W_j^i (\Gamma_j^i)^{-2} (U_{j+1}^i - U_j^i)^2 \text{ from Eq. (2.14)}. \]

We then integrate again (with backward differencing) and average to obtain the corrected values

\[ n_j^{i+1} = 0.5(n_j^i + n_p^{i+1}) - \Delta t \cdot 3n_j^i \Phi_j^i (R_{j+1}^i - R_{j+1}^{i+1} - R_{j+1}^i - U_j^i)/(R_{j+1}^{i+1} - R_{j+1}^i)^2. \]

The other quantities are obtained similarly. (Note that the velocity is backward differenced also when calculating \(Q_j^{i+1}\).) After the \(n\)th corrector step, we calculate the time step for the \((n+1)\)th integration, \((\Delta t)^{n+1}\), which must satisfy the Courant condition and the condition that \(n, \epsilon, M\) change slowly enough at each grid point. (This last condition is necessary for gravitational expansion and contraction.) Thus

\[
(\Delta t)^{n+1} = \min_j \left( \frac{C}{\Gamma_j^i \Phi_j^i (c_S)^2} \left[ \frac{n_j^g}{|n_j^g|}, \frac{\rho_j^g}{|\rho_j^g|}, \frac{\epsilon_j^g}{|\epsilon_j^g|}, \frac{M_j^g}{|M_j^g|} \right]_{G_N \neq 0} \right),
\]

where \((c_S)^g_j = \sqrt{(1 + \omega)(p_j^g + Q_j^g)/(n_j^g W_j^i)}\) is the speed of sound\(^2\) \(\bar{f} < 1\) is a constant, and \(n/|n|, \epsilon, |\epsilon|\), etc., are averaged over a few surrounding grid points.

At the origin, reflecting boundary conditions are used. And at the outer boundary, \(\Phi(t, R_B) = 1, \rho' = n' = \epsilon' = 0\) and \(Q(t, R_B) = 0\), so that Eqs. (2.4)-(2.6) can be easily integrated using the MacCormack method. Then \(n, \epsilon, M\) are determined using the “free-string” condition: \(n_j = n_j - 1\) and \(\epsilon_j = \epsilon_j - 1\).

C. Void Initial Conditions

A void is defined by its radius \(R_w\), wall thickness \(\Delta R_w\), and its relative energy density \(\alpha \equiv \rho_{in}(t_i)/\rho_{out}(t_i) < 1\). If \(c^{-1} R_w/H_{out}^{-1}(t_i) > 1\), the void is said to be superhorizon-sized. The void wall can be either compensated or uncompensated. If the void is initially uncompensated, then the energy density distribution used here is

\[
\rho(t_i, R) = 0.5 \rho_{out}(t_i) [(1 + \tanh x) + \alpha(1 - \tanh x)],
\]

where \(x \equiv (R - R_w)/\Delta R_w\). Thus the energy density increases monotonically to its outside FRW value, \(\rho_{out}(t_i)\). If the void is initially compensated, then the “mass-energy” distribution used here is

\[
M(t_i, R) = 0.5 c^{-2} 4 \pi \rho_{out}(t_i) [(1 + \tanh x) + \alpha(1 - \tanh x)] R^2(t_i)/3.
\]

\(^2\) We include \(Q\) in the expression for \((c_S)^g_j\) because we want the effective pressure. In practice, it makes little difference.
In this case, \( M \) reaches its FRW value outside the void. Therefore, the excess mass-energy density in the void wall compensates for that missing from the void. Using Eq. (2.11), one finds that the energy density for a compensated void is

\[
\rho = \rho_{\text{out}}(t_i) \left\{ \frac{R}{6\Delta R_w \cosh^2 x} + \frac{1}{2} \left[ (1 + \tanh x) + \alpha(1 - \tanh x) \right] \right\}. \tag{3.16}
\]

The actual wall thickness can be found by determining the radius, \( R_{\text{inner}} \), at which the energy density is approximately that inside the void. Setting \( \rho(t_i, R_{\text{inner}}) \simeq 3\rho_{\text{in}}(t_i) \) and using Eq. (3.15), we can solve for \( R_{\text{inner}} \):

\[
\frac{R_{\text{inner}} - R_w}{\Delta R_w} \simeq \ln \sqrt{\frac{3\Delta R_w}{R_{\text{inner}}} \alpha} \simeq 1.2 \log_{10} \alpha + \ln \sqrt{\frac{3\Delta R_w}{R_{\text{inner}}}}, \tag{3.17}
\]

where we have assumed that \( 3\Delta R_w \ll R_w \). Thus the actual wall thickness is larger than \( \Delta R_w \), since it is of order or greater than \( |\log_{10} \alpha| \Delta R_w \).

The initial velocity distribution is determined via

\[
\Gamma(t_i, R) = 1 + (\Gamma_w - 1) \exp\left[-(R - R_w)^2/(2\sigma_{\Gamma}^2)\right], \tag{3.18}
\]

where \( \sigma_{\Gamma} \) and \( \Gamma_w > 0 \) are constants. For this function, \( \Gamma \) is equal to one inside and outside the void. In the void wall, \( \Gamma \) can be greater than or equal to one.

In many instances, we would like to match the position of the inner edge of the energy density distribution with that of the inner edge of the velocity distribution. At \( R = R_{\text{inner}} \), the excess kinetic energy should be no larger than the “potential energy”. Setting \( \Gamma = 1 + \beta \) with \( \beta \ll 1 \) and using Eq. (2.11), we require \( 2c^2\beta \lesssim 2G_N M/R \). This occurs when

\[
\beta = \frac{b}{2} \left( \frac{c^{-1}R_{\text{inner}}}{H_{\text{out}}^{-1}(t_i)} \right)^2 \alpha, \tag{3.19}
\]

where \( b \leq 1 \). Using Eq. (3.18), the width of the velocity distribution then is

\[
\sigma_{\Gamma} = \frac{R_w - R_{\text{inner}}}{\sqrt{2 \ln[(\Gamma_w - 1)/\beta]}}, \tag{3.20}
\]

For nearly all of the simulations in this paper, we set \( b = 1 \) to determine \( \sigma_{\Gamma} \). As an example, if \( R_w = 50, \Delta R_w = 1, \alpha = 10^{-4}, \Gamma_w = 6, H_{\text{out}}^{-1}(t_i) = 2 \) and \( b = 1 \), then \( R_{\text{inner}} = 44, \beta \simeq .024 \), and \( \sigma_{\Gamma} \simeq 1.8 \).

As a final note, a void from first-order inflation is compensated and has a large outward wall momentum\(^\ddagger\). The mass (\( M \)) outside the void then, is the same as if the void were not present, and the wall “velocity” is large: \( \Gamma_w > 1 \).

**D. FRW radiation-dominated model**

In this section, we test our improved code out on the FRW \( \kappa = 0 \) radiation-dominated model. We also apply a convergence test since the exact solution is known. The relative error in \( q \), where \( q \) denotes any quantity, is defined to be \( e_i = (q_i - \bar{q}(r_i))/\bar{q}(r_i) \), where
\( \tilde{q}(r_i) \) is the exact solution and \( q_i \) is the numerical solution. We obtain a global measure of the error by defining

\[
L_1 = \frac{1}{N} \sum_{i=1}^{N} |e_i|, \tag{3.21}
\]

where \( N \) is the total number of grid points. This error is proportional to the grid spacing to some power: \( L_1 \propto \Delta R^s \), where \( s \) is the convergence rate. If \( s \approx 2 \), the code is second-order, as desired. These tools have been used previously to test codes in other applications \[10\].

For the simulations in this section, we set \( \alpha = 1 \), \( G_N = 1 \), \( c = 1 \), \( \omega = 1/3 \), \( t_i = 1 \), \( \epsilon_{\text{out}}(t_i)/c^2 = 10^6 \) and \( k^2 = 0 \). We examine a fluid which is initially relativistic, homogeneous and isotropic, with \( \Gamma(t, R) = 1 \). The solution is \( R(t, r) = R(t_i, r)(t/t_i)^{1/2}, 4\pi G_N \rho_{\text{hom}}(t) = 3c^2/(8t^2) \) and \( U = R/(2t) \).

We set \( c^{-1}R_B/H^{-1}(t_i) = 250, \Delta R(t_i) = 2.5 \) and \( C = .3 \), and run the code to \( t = 1.1 \). These are the same initial conditions used to make Figure 4 in SV. Figure 1 shows the relative error in \( \rho \) plotted versus the scaled radius at this time for \( \mathcal{\overline{r}} = .01, .005 \) and .0025 as solid, dotted and dashed lines, respectively. We see that the relative error in \( \rho \) is not only independent of \( R \), but is also very small, as compared to Figure 4 in SV. For instance, for \( \mathcal{\overline{r}} = .01 \), the inner and outer boundaries have \(-6\%\) and \(-2\%\) errors, respectively, while it is only \( 3 \times 10^{-4}\% \) here. Thus, the energy density is no longer underpredicted for this first grid points near the origin, which is an important consequence of numerical regularization.

We now calculate the convergence rate in this model. Since \( G_N \neq 0 \), the size of the time step is determined by calculating the smallest of the following: \( \Delta t_{cs} \equiv C\Delta R/(\Gamma\Phi_{cs}) \simeq \sqrt{3C\Delta R(t_i)}\sqrt{t/t_i} \) and \( \Delta t_n \equiv \mathcal{\overline{r}} |n/\dot{n}| \simeq \mathcal{\overline{r}} 2t/(3t_i) \) from Eq. (3.13). Thus we will test our code in two limits: when \( \Delta t_{cs} < \Delta t_n \) (so that decreasing \( \Delta R(t_i) \) decreases \( \Delta t \)), and when \( \Delta t_n < \Delta t_{cs} \) (so that decreasing \( \mathcal{\overline{r}} \) decreases \( \Delta t \)). Running our code in the former limit, we set \( R_B = 40, \mathcal{\overline{r}} = .5 \) and \( C = .001 \), and run the code to \( t = 1.035 \). (Note that we have made \( \mathcal{\overline{r}} \) large and \( C \) very small in order to achieve this limit). In Table 1, we show the values of \( L_1(q) \) for \( q = n, R, U, M \) and \( \epsilon \) for various \( \Delta R(t_i) \), and calculate the convergence rate. We see that the rate is roughly 2 for all variables, although there is some variation.

We also run our code in the latter limit, and thus set \( R_B = 40, \Delta R(t_i) = 5.0 \) and \( C = .3 \), and run the code to \( t = 1.014 \). In Table 2, we show the values of \( L_1(q) \) for various \( \mathcal{\overline{r}} \). We also calculate the rate \( \hat{s} \), defined by \( L_1 \propto \mathcal{\overline{r}}^s \), in order to determine the speed with which the code converges when the time step is determined solely by gravity: \( \Delta t_n \propto \mathcal{\overline{r}} \). Again, the rate is roughly 2.

IV. Collapse of a Superhorizon-sized Void

In SV, we found that the superhorizon-sized voids examined collapsed at the speed of light. The collapse occurs because the fluid in the void wall shocks inward from the wall pressure and velocity gradients, and it does so at the speed of light because the shock is in general strong (see Eqs. (A.24) and (A.30)). In addition, it was found that if the relative energy density in the void is small enough, the collapse occurs in less than
an outside Hubble time (i.e. a Hubble time outside the void). We ran many examples for which the wall energy, \((\Gamma^2 - 1)c^2/2\), was zero. However, we only ran one example (Figure 11) for which it was non-zero and large. The void still collapsed in this case. At the collapse time, \(\Delta t_{\text{collapse}}/H_{\text{out}}(t_i) \simeq 0.7\), there was substantial pressure and velocity distortion at the peak and in the peak area. We would like to understand this example and generalize to other situations. Do all superhorizon-sized voids collapse, even if the wall energy is enormous? What causes the distortion in the peak area, and does it always occur? And how does the wall energy affect the collapse time?

In order to answer these questions, we will take two approaches. First, we calculate analytically the time scales for change of the fluid in the void wall as a function of the void radius, wall thickness, wall velocity and relative energy density, \(\alpha\). Second, we answer these questions by showing the results of many numerical simulations. We consider different evolutionary outcomes caused by changing the void’s inside energy density, wall thickness and excess wall energy. We then determine which forces (gravitational or non-gravitational) are responsible for accelerating the fluid into the void and for distorting the initial peak and peak area distributions by comparing runs of superhorizon-sized voids with those where gravitational forces are neglected. It turns out that gravitational forces are negligible if the collapse occurs in much less than an outside Hubble time. This may seem confusing, because superhorizon-sized voids obviously evolve under gravitational forces. However, the time scales for which gravitational forces substantially change the solution can be much larger than that for which fluid forces (gradient forces) change the solution. When this is the case, gravitational forces negligibly affect the solution for some time. Then, the conclusion that non-gravitational forces have caused virtually all of the observed changes in the void is a correct and important result.

**A. Time Scales for Fluid Motion in the Void Wall**

In section B, we will calculate numerically the evolution of many different superhorizon-sized voids with different initial conditions. From these simulations, detailed evolutionary results can be obtained. However, it is difficult to generalize these results to situations with different initial conditions. For this reason, we calculate analytically the time scales on which the fluid in different regions of the void wall changes. This will allow us to gain intuition over the entire parameter regime of initial conditions.

Consider a compensated superhorizon-sized void with initial energy density given by Eq. (3.16) with \(\alpha \ll 1\) and \(3\Delta R_w \ll R_w\).\(^k\) In addition, the initial velocity is determined via Eq. (3.18), and the pressure is \(p = \omega \rho\), where \(\omega = 1/3\) (\(\omega = 0\)) if the fluid is relativistic (nonrelativistic). We assume that the fluid outside the void and in the peak area of the wall is relativistic, whereas the fluid in the void may be relativistic or nonrelativistic. The wall pressure, “mass” and “velocity” at the peak, \(\bar{R} = R_w\), is then

\[
p_w(t_i) \simeq p_{\text{out}}(t_i) \frac{R_w}{6\Delta R_w} \tag{4.1}
\]

\[
M_w(t_i) \simeq \frac{1}{2} \left( \frac{4\pi \rho_{\text{out}}(t_i)}{3c^2} R_w^3 \right) \tag{4.2}
\]

\(^k\)This is not a requirement that the void wall be very thin, because the actual wall thickness is greater than or of order \(|\log_{10} \alpha| \Delta R_w\) from Eq. (3.17).
\[ U_w^2(t_i) \simeq c^2(\Gamma_w^2 - 1) + \frac{1}{2} \left( \frac{R_w}{H_{\text{out}}^{-1}(t_i)} \right)^2. \] (4.3)

Note that the wall energy, \( c^2(\Gamma_w^2 - 1)/2 \), is the major contributor to the wall velocity only if \( \Gamma_w > c^{-1}R_w/H_{\text{out}}^{-1}(t_i) \).

We now calculate the dynamical time scale on which the fluid at a particular position changes. To do this, we first calculate the time for each of the fluid variables \( R, U, M, \rho \) and \( \Gamma \) to change by assuming that all variables are constant and are given by their initial values.\[ \] The change in velocity is the sum of the gravitational \( (G_N \neq 0 \) and \( (p+Q)' = 0 \)) and “fluid” \( (G_N = 0 \) and \( (p+Q)' \neq 0 \)) contributions, which we consider separately. The fluid variable which changes the fastest then sets the time scale for change of all variables.

As an example, we calculate the time scales for which \( \rho \) and \( U_{\text{FLUID}} \) change. We set the quantities in Eqs. (2.5) and (2.12) equal to their initial values, and calculate the times at which \( \Delta t|\rho|/|\rho| \simeq 1 \) and \( \Delta t|U_{\text{FLUID}}|/|U| \simeq 1 \). The time scales for change of \( \rho \) and \( U_{\text{FLUID}} \) then are

\[
\Delta t [\rho] \simeq \frac{R_w^2 R'}{(1 + \omega)^2 (R^2 U')^2}, \quad \Delta t [U_{\text{FLUID}}] \simeq \frac{\Phi^{-1}(1 + \omega)\rho UR'}{c^2 \Gamma^2 \omega \rho'},
\] (4.4)

respectively. A similar procedure can be followed to obtain \( \Delta t[R], \Delta t[U_{\text{GRAV}}], \Delta t[M] \) and \( \Delta t[\Gamma] \) from Eqs. (2.4), (2.5), (2.6) and (2.13), respectively. The time scale for change of \( \rho \), for example, is given by the smallest \( \Delta t \) obtained, rather than \( \Delta t [\rho] \). This is because we can no longer assume that the fluid quantities are equal to their initial values if some other quantity is changing on faster time scales. If \( U_{\text{FLUID}} \) changes the fastest, for example, \( R \) and \( U \) (and therefore \( \rho \)) will also change on this shorter time scale.

We are interested in the following two regions: the fluid near the peak \( (R = R_w \pm \Delta R_w) \) and at the peak \( (R = R_w) \) of the wall. When \( x = -1 \), the energy density is \( \rho = 42 \rho_w \gtrsim \rho_{\text{out}}(t_i) \) and the spatial derivatives are \( (R^2 U')/(R^2 R') \simeq U_w/\Delta R_w \) and \( \rho'/R' \simeq \rho_w/\Delta R_w \).

If the void is larger than four times the outside Hubble radius, then the energy density changes the fastest

\[
\Delta t_{\text{peak area}} = \Delta t[\rho] = \frac{1}{1 + \omega} \Phi^{-1} \frac{\Delta R_w}{U_w} = .39 \left( \frac{R_w \Delta R_w^3}{U_w} \right)^{1/4}
\lesssim .55 H_{\text{out}}^{-1}(t_i) \left( \frac{\Delta R_w}{R_w} \right)^{3/4},
\] (4.5)

where we have used the fact that \( U_w \gtrsim R_w/(\sqrt{2}H_{\text{out}}^{-1}(t_i)) \). The same time scale for change is obtained at \( x = 1 \) as long as \( U_{x=1} \ll U_w \). As the wall gets thinner or the wall energy gets larger, the time scale for change decreases because of the steeper velocity gradient.

Thus we find the interesting result that the fluid in the peak area always moves in less than an outside Hubble time. Note that the time scale for change is independent of \( \alpha \simeq \rho_{\text{in}}(t_i)/\rho_{\text{out}}(t_i) \), which is not very surprising because the velocity gradient will hardly increase if \( \rho_{\text{in}}(t_i) \) is decreased.

\[ ^{1}\text{The time scales of change for } n \text{ and } \epsilon \text{ differ from that for } \rho \text{ by a number of order one.} \]
The second region of interest is at the peak \((x = 0)\), where \(\rho' = \Gamma' = 0\). Then \((R^2U')/(R^2R') \approx 2U_w/R_w + (R_wH_{\text{out}}(t_i))^2/(4U_w\Delta R_w)\), so that mass-energy sets the time scale for change:

\[
\Delta t_{\text{peak}} = \Delta t[M] = 3\Phi_w^{-1}\Delta R_w/U_w \approx 1.9\frac{(R_w\Delta R_w^3)^{1/4}}{U_w} < 2.7H_{\text{out}}^{-1}(t_i)\left(\frac{\Delta R_w}{R_w}\right)^{3/4},
\]

which is 5 times larger than \(\Delta t_{\text{peak area}}\). Again the time scale for change is less than the outside Hubble time. Therefore a superhorizon-sized void must collapse in at least less than a Hubble time if the fluid in the peak area is to remain virtually stationary during the collapse.

Everywhere in the void wall, the time scale for change of the energy density is \(\Delta t[\rho] \approx (1 + \omega)^{-1}\Phi^{-1}\Delta R_w/U_w\), which is greater than or equal to the time scale of change at that position. As we move from the peak area into the void, \(\Phi^{-1}\) decreases because the fluid “potential” decreases due to the time dilation effect. Thus the time scale for change decreases as \(R\) decreases, so that the fluid near the base of the void accelerates into the void on time scales faster than fluid in the peak area can move; a superhorizon-sized void cannot prevent its own collapse by having excess energy in the peak wall area.

We can compare the time scale for change of a superhorizon-sized void to a void with nearly identical initial conditions, but for which the gravitational forces are zero: \(G_N = 0\). This is a SR void. At \(x = -1\), we assume that \(\Gamma_w > \Gamma > 1\). Then \(\Delta t[U_{FLUID}]\) is the smallest only if \(\Gamma - 1 \ll 1\) and \(\Gamma - 1 < .1/(\Gamma_w^2 - 1)\). Otherwise, \(\Delta t[\rho]\) is the smallest:

\[
\Delta t_{\text{peak area}} = \Delta t[\rho] \approx \frac{1}{(1 + \omega)}\Phi^{-1}\Delta R_w/U_w = \frac{39}{1} \frac{(R_w\Delta R_w^3)^{1/4}}{U_w}. \tag{4.7}
\]

When \(U_{x=1} \ll U_w\), the time scale for change is given by Eq. (4.7) also. This agrees with the results in the GR case when \(\Gamma_w \gg R_w/(cH_{\text{out}}^{-1}(t_i))\). When \(\Gamma_w < R_w/(cH_{\text{out}}^{-1}(t_i))\), \(\Delta t_{\text{peak area}}\) is smaller in the GR case because \(U_w\) is larger.

The time scale for change at the peak is quite different in the SR case however, because the energy density now sets the time scale for change:

\[
\Delta t_{\text{peak}} = \Delta t[\rho] \approx \frac{3}{8}\Phi_w^{-1}\frac{R_w}{U_w} = .24\frac{R_w^{5/4}}{\Delta R_w^{1/4}U_w}. \tag{4.8}
\]

This is larger than the GR result by the factor \(R_w/\Delta R_w \gg 1\). Thus, it takes much longer for the fluid at the peak to move for a SR void as compared to a GR void. This has interesting consequences. Because the energy density of fluid at \(x = -1\) decreases more rapidly than that in the peak, the pressure gradient in the upper wall area increases. This causes \(\Gamma_w\) to decrease, which slows the wall down rather quickly. We will see this in section B1 numerically.

\(^{mR_w, \ \Delta R_w, \ n(t_i, R), \ \epsilon(t_i, R), \ \Phi(t_i, R), \ R(t_i, r) \text{ and } \Gamma(t_i, R)\) are the same. Only \(U(t_i, R)\) differs through Eq. (2.11).\)
As an example, suppose we consider a superhorizon-sized void with $R_w = 50$, $\Delta R_w = 1$, $\Gamma_w = 6$ and $\alpha = 10^{-4}$ embedded in a FRW $k = 0$ radiation dominated universe with $H_{\text{out}}^{-1}(t_i) = 2$. Then $U_w = 19$ so that the time for the peak and peak area energy density to change is $13 H_{\text{out}}^{-1}(t_i) = 2.7$ and $.028 H_{\text{out}}^{-1}(t_i) = 0.55$, respectively. If we now consider a SR void with the same parameters and for which $\Gamma > 1.003$, then $U_w = 5.9$ so that the time scale for the peak and peak area energy density to change is $5.4$ and $.18$, respectively, which are much larger times.

It is interesting to calculate the initial conditions a superhorizon-sized void must have in order for its peak area energy density to remain roughly constant at the collapse time. From Eq. (4.5), it is clear that the collapse must occur in less than an outside Hubble time. From Eq. (1.1), this occurs if $\Phi_{\text{in}}^{-1} c^{-1} R_w / H_{\text{out}}^{-1}(t_i) < 1$, where $\Phi_{\text{in}}$ is assumed to be nearly constant. In this case, the collapse time is

$$\Delta t_{\text{collapse}} \simeq \Phi_{\text{in}}^{-1} c^{-1} R_w. \tag{4.9}$$

In general, $\Phi_{\text{in}}$ is larger than its initial, isentropic value, because the fluid is not isentropic across the inbound shock which develops. Assume that $\Phi_{\text{in}} = (\rho_{\text{out}}(t_i)/\bar{\rho})^{p/4}$, where $\bar{\rho}$ is defined after Eq. (3.11) and $p \geq 1$. Letting $\Delta t_{\text{collapse}} < \Delta t_{\text{peak area}}$, we find that

$$\frac{\bar{\rho}}{\rho_{\text{out}}(t_i)} \lesssim \left( \frac{\Delta R_w}{R_w} \right)^{3/p} \left[ \left( \Gamma_w^2 - 1 \right) + \frac{1}{2} \left( \frac{c^{-1} R_w}{H_{\text{out}}^{-1}(t_i)} \right)^2 \right]^{-2/p}. \tag{4.10}$$

Therefore, if $\bar{\rho}/\rho_{\text{out}}(t_i)$ is small enough, then at the first-crossing time the fluid in the wall will not have moved very much. It is important to emphasize that if $\Delta R_w/R_w \ll 1$ or $U_w \simeq c \Gamma_w \gg 1$, then the collapse must occur in much less than an outside Hubble time in order for this to occur.

**B. Collapse—Numerical Results**

In sections B1 and B3, we compare the collapse of a superhorizon-sized void with that of a void with similar initial conditions, but for which we neglect gravity. (This latter void is a special relativistic (SR) void). The purpose of this is to compare which evolutionary changes in the void are due to gravitational and non-gravitational (i.e. fluid) forces, since any effects which appear similarly for both SR and GR voids cannot be due to gravitational forces. We find that if a superhorizon-sized void collapses in less than the time scale for change in the peak area, the solution can be approximated very well by the solution to a SR void with the same initial conditions. For these voids then, gravitational effects are not important during the collapse. In addition, we find that even if the collapse time is larger than the time scale for change in the peak area, as long as the collapse occurs in much less than an outside Hubble time, gravitational effects are unimportant. These numerical results are important, since they show that the gravitational contribution to the solution is negligible at the collapse time if a superhorizon-sized void collapses in much less than an outside Hubble time, regardless of the details in the wall area (e.g. wall thickness, wall energy, etc).

\[\text{From Eq. (3.11), } p = 1 \text{ only in the isentropic limit. As an example, } p = 0.366/0.25 = 1.46 \text{ for much of the collapse of a thin-walled uncompensated relativistic void, using Eq. (5.24).}\]
For Figures 2-7 and Tables 3-4, we choose $\Gamma(t_i, R)$ and $\rho(t_i, R)$ from Eqs. (3.18) and (3.19), respectively. We then integrate the equations of motion twice, starting with the same initial conditions. The first time we set $G_N = 1$ (the “GR” case), which gives the solution for the evolution of a superhorizon-sized void. The second time we set $G_N = 0$ (the “SR” case), which gives the solution for the evolution of a special relativistic void. The pressure in the GR (SR) case at time $t$ is defined to be $p_{GR}(t, R_i)$ ($p_{SR}(t, R_i)$), where $R_i \equiv R(t_i, r)$.

In all figures which follow in section IVB, we set $C = 3$, $c = 1$, $\omega = 1/3$, $\bar{f} = .01$, $t_i = 1$, $c_{out}(t_i)/c^2 = 10^6$ and $4\pi\rho_{out}(t_i) = 3/8$. Thus the initial Hubble time outside the void is $H_{out}^{-1}(t_i) = 2t_i = 2$, and all voids are relativistic.

1. Gravitational Effects during the Collapse of a Superhorizon-sized Void

We start with compensated voids for which $R_w = 50$, $\Delta R_w = 1$, $\Gamma(t_i, R) = 1$, and $\Delta R(t_i) = 25$. Note that the GR void is superhorizon-sized, since $c^{-1}R_w/H_{out}^{-1}(t_i) = 25$. For $\alpha = 10^{-4}$, $10^{-7}$ and $10^{-10}$, we set $k^2 = 1.5, 1.7$ and 1.7, respectively, and we run our code twice to $t = 4.1$, $t = 1.5$, and $t = 1.08$, respectively.

In Figure 2, we show the fractional difference in the pressure, $[p_{GR}(t, R_i) - p_{SR}(t, R_i)]/p_{GR}(t, R_i)$, as a function of the initial radius. For $\alpha = 10^{-4}$, $10^{-7}$ and $10^{-10}$, we plot solid triangles, open boxes, and a dashed line, respectively. As the inside energy density decreases, the difference between the SR and GR solutions decreases also. For $\alpha = 10^{-4}$, $10^{-7}$ and $10^{-10}$, the absolute value of this difference is at most 350%, 110% and 20%, respectively. A similar result is obtained for an initially uncompensated void.

When $\alpha = 10^{-4}$, the solutions at $t = 4.1$ are very different. Most of the fluid not directly behind the shock is moving outward in the GR case, as opposed to the SR case. In addition, the large void wall pressure, $p_w$, has redshifted away in the GR case, causing the pressure gradient to be almost nonexistent in the former wall region. This is in contrast to the SR case, which has hardly changed at all in the peak area. The fact that the solution at $t = 4.1$ has changed considerably in the GR case is not very surprising, because the time scales for change in the peak and peak area are .29 and .059, respectively, from Eqs. (4.6) and (4.7).

In Figure 3, we show voids identical to those of Figure 2 but for which $\sigma_w = 6$ and $\sigma_T = \Delta R_w$. For this value of $\sigma_w$ however, $U_w/c = 19 > \Gamma_w$, so that the velocity is about 5% higher than for Figure 2. Again, as the GR void becomes emptier, the solution looks more like the SR solution. For $\alpha = 10^{-10}$, gravitational effects can be neglected during the collapse of the superhorizon-sized void. Note that the relative pressure difference for $\alpha = 10^{-4}$ is only 50% in the shock area as opposed to 300% in Figure 2. This is because the SR shock here is weaker since more wall fluid is pushed out with the larger wall velocity. A similar result is again obtained for the uncompensated case.

In Figure 4, we plot the radii of seven comoving observers as a function of time for the voids in Figure 3 with $\alpha = 10^{-4}$ and $\alpha = 10^{-10}$. The solid and dashed lines represent the GR and SR results, respectively. The figures show that when $\alpha = 10^{-4}$, the SR

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*The only variable which differs between the GR and SR cases initially is the velocity $U(t_i, R)$. All other variables are identical.

*A relativistic (non-relativistic) void is when the fluid in the void is relativistic (non-relativistic).
and GR results deviate quickly. Note that the comoving observers move outward in the wall region in the GR case. However, when $\alpha = 10^{-10}$, the SR and GR results are quite similar for the entire collapse. In addition, the peak area looks virtually unchanged in both cases. This is easy to explain. Using Eq. (4.6) and (1.8), the peak changes on time scales $0.27$ and $5.4$ for the GR and SR cases, respectively. And in the peak area, the time scales of change for the GR and SR voids are $0.055$ and $0.18$, respectively, using Eqs. (4.5) and (4.7).

In Figure 5 we plot $\rho$ and $\Gamma$ as a function of time for comoving observers in the peak area. These voids are identical to those of Figure 3 with $\alpha = 10^{-4}$. Again, the solid and dashed lines show the results in the GR and SR cases, respectively. Note that the energy density for the GR void in the peak and peak area has decreased significantly by $t \simeq 1.1$, in agreement with the time scale $0.055$ calculated above. In addition, $\rho$ keeps decreasing for all three observers. In the SR case however, the pressure at the peak hardly changes, while that in the peak region has decreased substantially by $t \simeq 1.3$, in agreement with the time scale $0.18$ calculated above. In addition, in the SR case the energy of the peak, $c^2(\Gamma_w(t)^2 - 1)/2$, is substantially less by $t \sim 2$. In the GR case however, $\Gamma_w(t)$ starts to decrease, then stops when the peak energy density decreases substantially. Therefore, if a superhorizon-sized void collapses in more than the outside Hubble time, the wall energy probably does not decrease, as it does in the SR case.

2. Collapse Times for Superhorizon-sized Voids

We now calculate the collapse times for many different superhorizon-sized voids. These times depend on the initial wall energy, $c^2(\Gamma_w^2 - 1)/2$, the relative energy density, $\alpha$, and the wall thickness $\Delta R_w$. Again, we choose a void which is $25$ times the Hubble radius, with relative wall thickness $\Delta R_w/R_w = 1/50$. We set $\Delta R(t_i) = .25$, $k^2 = 1.7$, and determine $\sigma_f$ from Eqs. (3.20), (3.17) and (3.19) by setting $b = 1$. We present our results in Table 3. The stars following some of the collapse times mark those voids where there is significant distortion observed in the peak area just before the collapse. It is interesting to compare this with the time scales for change in the peak area. Using Eqs. (4.3) and (4.7), the time scales for change are given also. As expected, for a given value of $\Gamma_w$, the peak area is distorted unless the collapse time is smaller than the time scale for change in the peak area. In addition, if the peak is distorted and $\Gamma_w$ is increased further, the collapse time increases. For $\alpha = 10^{-7}$, $\alpha = 10^{-7}$ and $\alpha = 10^{-10}$, the collapse time increases by $21\%$, $4\%$ and $< 1\%$, respectively, for $\Gamma_w = 6$ to 50. This is because more fluid moves out of the peak area (as a shock) rather than into the void, weakening the shock. (The outgoing shocks can be observed in Figures 6-7). This increases the shock formation time, and decreases the relative potential of the void because $\Phi_{in}$ decreases slightly when an outgoing shock is present also. This can be seen by rewriting Eq. (2.11) as

$$\Phi_{in}(t) = \left(\frac{\rho_{out}}{\rho_{in}}\right)^{(\omega+1)} \frac{1}{\omega + 1} \int_0^\infty \frac{Q'dr}{\gamma\rho + Q}.$$

(4.11)

$Q$ is non-zero only in a shock, and is peaked at the center of it. Divide the shock front into two portions separated by the position where $Q' = 0$ and $p = p_Q \approx Q_Q$. Since $p \gtrsim Q$
in the shock, for an inbound shock the contribution to the integral is larger when \( p < p_Q \) \((Q' > 0)\) than when \( p > p_Q \) \((Q' < 0)\). Therefore when the shock is inbound (outgoing), \( \Phi_{in} \) increases (decreases) from its isentropic value. As an example, if we assume that both inbound and outgoing shocks are strong, using Eqs. (A.29) and (3.10) we find that the relative potential of the void is

\[
\Phi_{in} = \tilde{\alpha}^{-2.5} \left( \frac{\rho_{b,i}}{\rho_{in}} \right)^{.25} \left( \frac{\rho_{out}}{\rho_{b,o}} \right)^{.25},
\]

(4.12)

where \( \rho_{b,i} \) (\( \rho_{b,o} \)) are the energy densities behind the inbound and outgoing shocks, respectively, and \( \tilde{\alpha} \equiv \rho_{in}(t_i)/\rho_{out}(t_i) \). Thus, the outgoing shock partially cancels some of the void’s relative potential.

In the last two rows, we calculate the theoretical collapse times for each \( \alpha \) using Eq. (4.9). Note that \( \Phi_{in} = \tilde{\alpha}^{-2.5} \) is the isentropic result, and \( \Phi_{in} = .6\tilde{\alpha}^{-3.66} \) is the relativistic shock tube result from Eq. (5.24). Therefore, the numerical collapse times are nearly that for the isentropic “no-shock” result. This is because the void wall is too thick, which results in the production of weaker shocks, large shock formation times and a smaller time dilation effect.

We can show this by calculating the collapse times as a function of the width \( \Delta R_w \) for a superhorizon-sized void with \( R_w = 50, \alpha = 10^{-10}, \Gamma_w = 6, \Delta R(t_i) = \Delta R_w/4 \) and \( b = 1 \). For \( \Delta R_w = 1, .5 \) and .25, we set \( k^2 = 1.7, 2.2 \) and 3.0, respectively. The results are presented in Table 4. In the last column, we calculate the exponent \( \tilde{p} \) using Eq. (4.9) with the assumption that \( \Phi_{in} = .6\tilde{\alpha}^{-\tilde{p}} \). As \( \Delta R_w \) decreases, the collapse time decreases, and \( \tilde{p} \) increases, as expected.

As a final remark, because the time scale for change in the peak area of a SR void is greater than that for a GR void with the same initial conditions, if a GR void collapses in much less than the time scale for change in the peak area, we expect the GR and SR voids to look the same and be unchanged in the peak area at the collapse time. This is observed numerically for the voids listed in Table 3. The fluid which is in motion at the first-crossing time has been accelerated into the void by fluid forces. Therefore, it is not surprising that the solutions for these GR voids are nearly identical to those of SR voids evolved under the same initial conditions. Thus gravitational effects are not important at the first-crossing times for superhorizon-sized voids with \( \Delta t_{collapse} \ll \Delta t_{peak \ area} \).

3. **Peak Distortions: Fluid or Gravitational effects**

From the above discussion, it is clear that the wall of a void will be distorted at the collapse time if \( \Gamma_w \) is large enough, even if the collapse occurs in less than an outside Hubble time. It is therefore important to find out if this distortion is caused by non-gravitational (fluid) or gravitational effects. Thus, we compare the GR solution before collapse with that of the SR solution with the same initial conditions. We choose the initial conditions such that \( \Delta t_{peak \ area} < \Delta t_{collapse} \).

---

\( \text{It is possible to find initial conditions such that the collapse time is between the GR and SR time scales for change in the peak area. However, in order for this to happen for collapse times much less than the Hubble time, } \Delta R_w/R_w \text{ must be extremely small. For example, using Eq. (4.3), if } \Delta t_{peak \ area}/H_{out}^{-1}(t_i) \approx .001 \text{ and } .005, \text{ we require } \Delta R_w/R_w \approx 2.3 \times 10^{-4} \text{ and } 1.9 \times 10^{-3}. \text{ These voids have extremely thin walls, and will not be considered here.} \)
We evolve a GR and a SR void with $R_w = 50$, $\Delta R_w = 1$, $\alpha = 10^{-10}$, $\Gamma_w = 200 \simeq U_w/c$ and $\sigma = 1.88\Delta R_w$. In addition, $k^2 = 1.7$ and $\Delta R(t) = .25$. It is found that the collapse time for the SR void is $\Delta t = .18$, which is 31% larger than the GR collapse time of .13. In Figure 6, we plot $p$ and $\Gamma$ as a function of $R(t,r)$ for the GR void (dotted line) and SR void (dashed line) at $t = 1.06$. The initial void configuration is shown as the solid line. Note that the GR void’s shock is ahead of the SR void’s shock ($24 < R < 26$), which explains why the SR collapse time is longer. Since the time scale for change in the peak area, $0.052$, is much smaller than the collapse time, we expect the large distortions seen in the peak area ($45 < R < 60$); the peak has moved outward and is thinner. In addition, the peak energy is smaller, which is due partly to the spherical damping effect of the outbound wave. However, the important and surprising result is that the GR and SR voids look very similar at this time. It is true that the energy in the void wall should drive the fluid out regardless of whether gravitational forces are included or not. However once the fluid starts expanding outward, one can imagine gravitational effects adding or subtracting substantially to this movement. Apparently, this does not happen.

We would like to know how robust this result is. Therefore we examine voids with larger and smaller collapse times and for which peak distortions are large. We also require $\Gamma_w \gg R_w/(\sqrt{2}cH_{\text{out}}^{-1}(t_i)) \simeq 18$ so that the initial velocity in the wall is the same for both GR and SR cases: $U_w \simeq c\Gamma_w$. If we choose smaller values for $\Gamma_w$, then it is not possible to distinguish the non-gravitational from the gravitational effects in this manner.

For a void with $R_w = 50$, $\Delta R_w = 1$, $\alpha = 10^{-7}$, $\Gamma_w = 50$, $\sigma = 1.73\Delta R_w$, $k^2 = 1.7$ and $\Delta R(t_i) = .25$, the GR and SR solutions are very different at $t = 1.4$. However, a substantial fraction of an outside Hubble time has passed. We therefore choose a void which collapses much more quickly and for which $\Gamma_w$ is very large.

We choose a void with $R_w = 50$, $\Delta R_w = 1$, $\alpha = 10^{-13}$, $\Gamma_w = 500$, $\sigma = 2.05\Delta R_w$, $k^2 = 1.7$ and $\Delta R(t_i) = .25$. Figure 7 shows that the solutions are nearly identical at $t = 1.01$, even though the peak is fairly distorted. In addition, the collapse times for the GR and SR void are $\Delta t = .019$ and .021, respectively, which are not very different. Thus, the gravitational contribution to the solution is smaller than that for the superhorizon-sized void in Figure 6, even though the peak distortions are comparable. This is because the collapse time is even smaller.

In conclusion, if a superhorizon-sized void collapses in much less than an outside Hubble time, gravitational forces can be neglected in the equations of motion up until the collapse time. This is true even if the fluid in the peak area moves substantially during this time.

V. Lagrangian Shock Tube as an Approximate Solution for the Collapse of a Superhorizon-sized Void

We have found that if a superhorizon-sized void collapses in much less than an outside Hubble time, then even if the void wall moves outward significantly, the solution at the collapse time looks nearly identical to that for a SR void evolved under similar conditions. This is an important result, since it shows that the force moving the fluid out of the peak

\[ \text{If } \Gamma_w \ll R_w/(\sqrt{2}cH_{\text{out}}^{-1}(t_i)), \text{ then the wall velocity } U_w \text{ is much larger in the GR case as compared to the SR case, causing the time scale for change to be smaller.} \]
area is not due to gravity, but is due to pressure and velocity gradients which are present in both cases. (The force moving the fluid into the void is non-gravitational also, for the same reason). More importantly, it shows that gravitational forces are not important until after the void collapses. This leads us to search for collapse solutions to the special relativistic equations (i.e. $G_N = 0$).

Consider an uncompensated void with a small wall velocity. As the void starts to collapse, the solution is approximately slab (linearly) symmetric. Not until the shock nears the origin will spherical effects come into play. If in addition the wall is thin ($|\log_{10} \alpha| \Delta R_w/R_w \ll 1$), then it acts like an initial discontinuity in pressure, energy density, etc. The solution for this is called a shock tube. When the solution is assumed to be spherically symmetric with analogous initial conditions (i.e. the problem we would like to solve), a similarity solution does not exist. This is because another length scale, the void radius, $R_w$, has been introduced into the problem. Thus, the slab shock tube solution derived here will only be important for uncompensated voids with thin walls, small wall velocities, and for which $\Delta t_{\text{collapse}} \ll H^{-1}(t_i)$.

The general relativistic equations used here do admit similarity solutions. We will not solve these equations here however, because we cannot find similarity solutions for our problem. In Appendix A, we review the conditions satisfied over a contact discontinuity and shock for a general relativistic fluid. In section VA, we derive the slab similarity solution for relativistic fluids in this Lagrangian coordinate system. We then attach this solution to the contact discontinuity and shock solutions, which are also similarity solutions, when the fluid in front of the shock is relativistic and nonrelativistic. This corresponds to the fluid in the “void” being relativistic and nonrelativistic, respectively. The relativistic (shock tube) solution was derived in a Eulerian coordinate system, which can be obtained from the Lagrangian description by a spacetime coordinate transformation. Finally, we compare these solutions to the numerical solutions obtained from the collapse of relativistic or nonrelativistic voids.

### A. Relativistic Similarity Solution

In this section, we derive the similarity solution for a relativistic fluid (with no viscosity) in the slab symmetric limit. This limit corresponds to the evolution of a spherically symmetric fluid distribution which is “far” from the origin. Mathematically, we require $|U/R| \ll |U'/R'|$ to be satisfied.

We define $X(t,x) = R(t,r)$ to be the location of a comoving shell with label $x = r$. In order that the solution be a similarity solution, all functions can only depend on the variable

$$z \equiv \frac{x}{t},$$

(5.1)

where $t = 0$ is the initial time. Thus, $\rho = \rho(z)$, $p = p(z)$, $U = U(z)$, $\Phi = \Phi(z)$ and $\chi(z) \equiv X/z$. ($X = R$ and $x = r$ in the slab limit). We can then rewrite Eqs. (2.4)-(2.5), (2.7), (2.9) and (2.12) as

$$z \frac{d \rho}{dz} = \frac{\Phi(\rho + p) dU}{d\chi/dz \ dz},$$

(5.2)
\[
\Phi U = -z \frac{d\tilde{\chi}}{dz} + \tilde{\chi} \tag{5.3}
\]
\[
z \frac{dU}{dz} = \frac{c^2 \Gamma^2 \Phi}{(\rho + p) d\tilde{\chi}/dz} \frac{dp}{dz} \tag{5.4}
\]
\[
d\Phi \frac{dz}{dz} = -\frac{\Phi}{\rho + p} \frac{dU}{dz} \tag{5.5}
\]
\[
z \frac{dn}{dz} = \frac{\Phi n}{d\tilde{\chi}/dz} \frac{dU}{dz}, \tag{5.6}
\]

where \( \tilde{\chi} \equiv z\chi = X/t \) and \( \epsilon = \rho/n - c^2 \). We now assume that \( \epsilon/c^2 \gg 1 \) and that the equation of state is \( p = \omega \rho \). Eqs. (5.2)-(5.6) can then be solved to obtain

\[
\Phi U = \mp c_S \Phi \Gamma + \tilde{\chi} \tag{5.7}
\]
\[
\rho/\rho_0 = (z/z_0)^{(1+\omega)/(1-\omega)} = (t_0/t)^{(1+\omega)/(1-\omega)} \tag{5.8}
\]
\[
U/c = \frac{(\rho/\rho_0)^{2\beta} - 1}{2(\rho/\rho_0)^{2\beta}} \tag{5.9}
\]
\[
\Phi = \Phi_0 (\rho/\rho_0)^{\omega/(1+\omega)} \tag{5.10}
\]
\[
n = n_0 (\rho/\rho_0)^{1/(1+\omega)} \tag{5.11}
\]

and \( \epsilon = \rho/n \), where \( \beta \equiv \pm \sqrt{\omega}/(1 + \omega) \), \( c_S \equiv \sqrt{\omega c} \) is the speed of sound, and where the subscript "0" denotes the value a shell has upon entering the region where this solution holds. Eqs. (5.7)-(5.11) describe a rarefaction wave going to the right (left) when the upper (lower) sign is chosen. In addition, the position of a shell is given by

\[
X(t, x) = \Phi \left[ \mp 2 c_S - \frac{(c \pm c_S) \left\{ 1 - (\rho/\rho_0)^{2\beta} \right\}}{2(\rho/\rho_0)^{2\beta}} \right]. \tag{5.12}
\]

Therefore, we have solved for all functions in the rarefaction wave in terms of \( \rho/\rho_0 \).

**B. Shock Tube Solution**

In this section, we find the solution for a special relativistic fluid with an initial discontinuity in the pressure. The side with the larger pressure is relativistic, and the fluid on the other side is relativistic or nonrelativistic. These solutions will be used to model the collapse of superhorizon-sized voids in the early universe.

Suppose at \( t = t_i \), the fluid is at rest everywhere, with an initial discontinuity at position \( X_w(t_i) \). On either side of the discontinuity, \( \rho, n, p \) and \( \epsilon \) are constant. The subscripts 1 and 2 represent the fluid quantities for \( X(t_i) < X_w(t_i) \) and \( X(t_i) > X_w(t_i) \), respectively. We choose the coordinate system where \( \rho_2 > \rho_1, \epsilon_2 > \epsilon_1 \) and \( n_2 > n_1 \). The equation of state is \( p = \omega \rho \) in region 2, and \( p = \omega \rho (p = 0) \) in region 1 if the fluid there is relativistic (non-relativistic). For \( t > t_i \), a shock will be formed which travels into region 2 (undisturbed fluid), a rarefaction wave will be formed which moves into region 2 (also undisturbed), and a contact discontinuity will form in between. There will be 5 distinct regions in all. The shock is located at \( X_{sh}(t, x_{sh}) \), region 3 contains the fluid behind the shock, the contact discontinuity is located at \( X_c(t, x) \), and the rarefaction wave is located
in region 4 for \( X_b(t, x) < X < X_a(t, x) \). Figure 8 is a sketch of pressure versus distance for a relativistic shock tube, with the regions and boundaries labeled.

We will assume that the fluid in the rarefaction wave is relativistic regardless of the equation of state in region 1, so that \( \epsilon_3/c^2 \gg 1 \). Doing this allows us to use the similarity solution obtained in VA for the rarefaction wave. And because \( \epsilon_3 > \epsilon_3' \), as we will see in a moment, it therefore follows that \( \epsilon_3/c^2 \gg 1 \) in both cases. Then, the solution in region 3 is given by Eqs. (A.20)-(A.24) if region 1 is nonrelativistic, and Eqs. (A.26)-(A.30) if it is relativistic. In either case, the upper signs are used because the shock moves in the direction of decreasing \( X \). Note that these are all given in terms of \( \rho_3 = \rho_0 \). The conditions across the contact discontinuity give \( \Phi_3 = \Phi_3' \), \( p_3 = p_3' \), \( \rho_3 = \rho_3' \) and \( U_3 = U_3' \) from Eq. (A.13). Finally, the rarefaction wave solution is given by Eqs. (5.8)-(5.11) with the upper signs and \( \rho_3 \equiv \rho_0 \).

1. Relativistic, Thin-Walled Uncompensated Void

We first examine the case where the fluid in region 1 is relativistic. Since \( U_3/c = ((\rho_3/\rho_2)^{2\beta} - 1)/[2(\rho_3/\rho_2)^\beta] \) from Eq. (5.9), we can use this and Eq. (A.26) to find the equation relating \( \rho \equiv \rho_3/\rho_1 \), \( \beta = \sqrt{\omega/(1 + \omega)} \) and \( \delta \equiv (\rho_1/\rho_2)^\beta \):

\[
2\delta\beta(\rho - 1)\rho^{\beta - 1/2} = 1 - \delta^2\rho^{2\beta}.
\]

An iterative method is then used to find \( \rho \). The solution in region 3, 3' and 4 is given by Eqs. (A.20)-(A.30), Eqs. (5.9)-(5.11) with \( \rho = \rho_3 \) and \( \rho_0 = \rho_2 \), and Eqs. (5.8)-(5.11), respectively.

We first calculate the location of the boundaries between the five regions: \( X_a \), \( X_b \), \( X_c \) and \( X_d \). Set \( \Phi_2 = 1 \). We can calculate \( X_a \) by noting that the velocity is zero there and by using Eq. (5.7). \( X_b \) is found by using Eq. (5.7) and (A.26). Since \( U(z) \) and \( \Phi(z) \) are constant across the contact discontinuity and in regions 3 and 3', we can integrate Eq. (5.3) to obtain the position of a shell in region 3 or 3':

\[
X(t, x) = \Phi U(t - t_0) + X_0.
\]

Here \( X_0 \) is the location of a shell when it enters region 3 or 3' at time \( t_0 \). Eq. (5.14) also applies to the location of the shock and contact discontinuity, with \( X_0 = X_w(t_1) \) and \( t_0 = t_1 \). The locations of the boundaries are then

\[
X_a(t) = X_w(t_1) + c_s(t - t_1)
\]

\[
X_b(t) = X_w(t_1) + (t - t_1)\Phi_3\beta\left(\frac{\rho - 1}{\sqrt{\rho}}\right)\left[-c + c_s\sqrt{1 + \frac{\rho}{\beta^2(\rho - 1)^2}}\right]
\]

\[
X_c(t) = X_w(t_1) + (t - t_1)\Phi_3U_3
\]

\[
X_d(t) = X_w(t_1) + (t - t_1)\Phi_3U_{sh} = X_w(t_1) + (t - t_1)(1 + \omega)\frac{\rho}{\rho - 1}\Phi_3U_3,
\]

where the shock speed is \( \Phi_3U_{sh} \equiv R_{sh}^{-1} = X_{sh}^{-1} \) from Eq. (A.30). Note that the rarefaction wave moves out at the speed of sound. In addition, from the discussion following Eq. (A.30), if the shock is strong (i.e. \( \rho > 1 \)), the shock moves into region 1 at the speed of light: \( X_d(t) = X_w(t_1) - c(t - t_1)\Phi_1 \).
Now we trace the location of all Lagrange points in the shock tube in terms of \( z = x/(t - t_1) \). A shell with coordinate \( x \) and position \( X \) initially in region 2 \((X > X_w(t_1))\) will enter the rarefaction wave at time \( t_a = t_1 + (X - X_w(t_1))/c_S \). Using Eq. (5.12) and setting \( z_a = x/(t_a - t_1) \), its position as a function of time in that wave for \( z \leq z_a \) will be

\[
X_4(t, x) = X_w(t_1) + (t - t_1) \left( \frac{z}{z_a} \right) \left[ c_S - \frac{c_S + c}{2} \left( 1 - \left( \frac{z}{z_a} \right)^\frac{1}{a} \right) \right].
\]

(5.19)

Here, the subscript “4” shows that the shell is in region 4. The energy density of this shell decreases until it reaches \( \rho_3 \), at which point it enters region 3’. This occurs at time \( t_b = t_1 + (t_a - t_1)(\rho_2/\rho_3)^{1(1-\omega)/(1+\omega)} \). Its position at this point is \( X_b(t_b, x) \) and is given by Eq. (5.10) or Eq. (5.19) with \( t = t_b \) and \( z = z_b = x/(t_b - t_1) \). Using Eq. (5.7), its position in region 3’ for \( z \leq z_b \) is

\[
X_3'(t, x) = X_w(t_1) + (t - t_1) \left\{ \Phi_3 U_3 - \frac{z}{z_b} \left( \Phi_3 U_3 - \frac{X_b - X_w(t_1)}{t_b - t_1} \right) \right\}.
\]

(5.20)

This Lagrange shell can never cross into region 3 because the contact discontinuity moves with the fluid.

Finally, if a shell initially has the position \( X < X_w(t_1) \), then it will enter region 3 when the shock reaches it. This occurs at time \( t_d = t_1 + (X - X_w(t_1))/(\Phi_3 U_{sh}) \). For \( t > t_d \), the location of this shell in region 3 for \(|z| \leq |z_d| \) is

\[
X_3(t, x) = X_w(t_1) + (t - t_1) \left\{ \Phi_3 U_3 - \frac{z}{z_d} \left( \Phi_3 U_3 - \frac{X_d - X_w(t_1)}{t_d - t_1} \right) \right\}.
\]

(5.21)

where \( z_d = x/(t_d - t_1) \).

In the strong shock limit \((\overline{\rho} \gg 1)\), \( \overline{\rho} \) is given by an analytic expression. The solution then is

\[
\rho_3 = \rho_1 a^{2(1+\omega)} \overline{\alpha}^{-2\sqrt{\omega}/(1+\sqrt{\omega})^2} \rightarrow 1.2\rho_1 \overline{\alpha}^{-0.644}
\]

(5.22)

\[
\frac{U_3}{c} = -\sqrt{\omega} a^{(1+\omega)} \overline{\alpha}^{-\sqrt{\omega}/(1+\sqrt{\omega})^2} \rightarrow -0.47\overline{\alpha}^{-2.32}
\]

(5.23)

\[
\Phi_1 = \Phi_2 \sqrt{\omega} a^{(1-\omega)} \overline{\alpha}^{-\sqrt{\omega}/(1+\sqrt{\omega})^2} \rightarrow 0.60\Phi_2 \overline{\alpha}^{-0.666}
\]

(5.24)

\[
\Phi_3 = \Phi_2 a^{-2\omega} \overline{\alpha}^{-\sqrt{\omega}/(1+\sqrt{\omega})^2} \rightarrow 0.96\Phi_2 \overline{\alpha}^{-0.134}
\]

(5.25)

\[
\epsilon_3' = \epsilon_2(\Phi_2/\Phi_3) \rightarrow 1.01\epsilon_2 \overline{\alpha}^{0.134}
\]

(5.26)

\[
\epsilon_3 = \epsilon_1\sqrt{\overline{\omega}} \frac{\rho_3}{\rho_1} \rightarrow 6.2\epsilon_1 \overline{\alpha}^{-2.32}
\]

(5.27)

where \( \overline{\alpha} \equiv \rho_1/\rho_2 \) and \( a \equiv [(1 + \omega)/(2\sqrt{\omega})]^{1/(1+\sqrt{\omega})^2} \), and where the arrow evaluates the expression when \( \omega = 1/3 \). We note that in general \( \epsilon_3 \neq \epsilon_3' \). This is because energy is dissipated over the shock, increasing the specific energy for shells behind the shock. Suppose we take our initial state to be an isentrope so that \( \epsilon_1/\epsilon_2 = (\rho_1/\rho_2)^{25} \). Then, \( \epsilon_3/\epsilon_3' = 0.6(\rho_2/\rho_1)^{116} > 1 \). Thus the specific energy and number density will be
discontinuous across the contact discontinuity, and will diverge as \( \rho_1 \to 0 \). In order that this solution be consistent, \( \rho_1/\rho_2 \) cannot be too small from Eq. (5.26) because we assumed that \( \epsilon_3/c^2 \gg 1 \). In addition, since the temperature in region in region 3 is \( T_3 = \omega \mu \epsilon_3 \) which we require to be less than the Planck temperature, the energy density inside the void cannot be too small. However, we see no reason to restrict \( \tilde{\alpha} \) for more general solutions.

The above solution approximates that of an uncompensated superhorizon-sized void in a radiation-dominated universe. We can calculate the approximate energy density in the “void” at times much less than a Hubble time before collapse. This solution gives the theoretical maximum limit for the shock tube strength. Of course, spherical effects will cause the shock strength to increase. However, our numerical simulations suggest that this increase is proportional to the initial shock strength. The ratio of the energy density behind the shock to that in region 2 is

\[
\frac{\rho_3}{\rho_2} = \left( \frac{1}{4\beta^2} \tilde{\alpha} \right)^{1/(2\beta+1)} \to 1.2\tilde{\alpha}^{-0.536},
\]

where the arrow evaluates the expression for \( \omega = 1/3 \). In addition, the collapse time is \( \Delta t_{\text{collapse}} \approx R_\infty/\Phi_1 \approx 2R_\infty \tilde{\alpha}^{-0.37} \). Thus as the “void” empties (\( \rho_1/\rho_2 \to 0 \)), the collapse time decreases rapidly and the fraction filled before the “void” collides at the origin becomes very small, unless spherical effects are extraordinarily important.

2. Nonrelativistic, Thin-Walled Uncompensated Void

Now we calculate the solution for a shock tube when region 1 is nonrelativistic: \( \rho_1 = n_1 c^2 \) and \( \epsilon_1/c^2 \ll 1 \). We assume that the fluid behind the shock and in the rarefaction wave are relativistic, however. We use Eqs. (A.20)-(A.24) with \( \epsilon_3/c^2 \gg 1 \), as well as the same rarefaction and contact discontinuity solutions as was used for the relativistic case in the previous section. In particular, we use Eq. (5.39) with \( \rho = \rho_3 \) and \( \rho_0 = \rho_2 \) in order to obtain \( U_{3'} \). Since \( \rho_3 = n_3 \epsilon_3 = (\omega + 1)/\omega \) \( \rho_1 (\epsilon_3/c^2) \) and \( U_3 = -\epsilon_3/c \), \( U_3 = U_{3'} \) becomes

\[
2\frac{\epsilon_3}{c^2} \left[ \frac{\omega + 1}{\omega} \right] \frac{\rho_1}{\rho_2} \left( \frac{\epsilon_3}{c^2} \right)^2 = 1 - \left[ \frac{\omega + 1}{\omega} \right] \frac{\rho_1}{\rho_2} \left( \frac{\epsilon_3}{c^2} \right)^2 \right]^{2\beta}.
\]

This equation can be solved iteratively to find \( \epsilon_3/c^2 \), from which point the entire solution can be obtained. We assume now that \( \rho_3/\rho_2 \ll 1 \). In this limit, we obtain the following solution:

\[
\epsilon_3/c^2 = \left\{ 2[\tilde{\alpha} (\omega + 1)/\omega]^{\frac{1}{1+2\beta}} \right\} \to 0.50\tilde{\alpha}^{-0.232}
\]

\[
\rho_3 = \left[ (\omega + 1)/\omega \right] \rho_1 \epsilon_3^2/c^4 \to \rho_1 \tilde{\alpha}^{-0.464}
\]

\[
U_3/c = -\epsilon_3/c^2
\]

\[
\Phi_1 = 2\omega \Phi_2 \left( \frac{\omega + 1}{4\omega} \right)^{\sqrt{\omega}/(1+\sqrt{\omega})^2} \tilde{\alpha}^{\frac{3\omega}{\sqrt{\omega}(1+\sqrt{\omega})^2}} \to (2/3) \Phi_2 \tilde{\alpha}^{-0.366}
\]

\[
\Phi_3 = \Phi_1 \left[ (\omega + 1)\epsilon_3/c^2 \right] \to \Phi_2 \tilde{\alpha}^{-0.134}
\]

\[
\epsilon_{3'} = \epsilon_2(\Phi_2/\Phi_3) \to 1.0\epsilon_2 \tilde{\alpha}^{-0.134},
\]

\[
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\]
where $\tilde{\alpha} \equiv \rho_1/\rho_2$. This solution depends on $\tilde{\alpha}$ in the same way as it does in the relativistic case. The prefactors however, are somewhat different. Using Eq. (A.25), (5.31), (5.33) and (4.9), we find a rough lower bound for the collapse time

$$\Delta t_{\text{collapse}} > 1c^{-1}R_w(\mu/M_{PL})^{633}. \quad (5.36)$$

where $\mu$ is the particle mass. (In general, because $T_3 < M_{PL}$, we can estimate the lower bound on the collapse time using Eq. (3.12) to be $\Delta t_{\text{collapse}} > c^{-1}R_w \mu/M_{PL}$.) We can also trace out the boundaries between regions, and the radii of all comoving shells. $X_a$ and $X_c$ are still given by Eqs.(5.15) and (5.17), respectively. However, now

$$X_b = X_w(t_i) + (t - t_i)\Phi_3 U_3(1 - \sqrt{\omega}) \quad (5.37)$$

$$X_d = X_w(t_i) + \Phi_3 U_{sh}(t - t_i) = X_w(t_i) - c(t - t_i)\Phi_1. \quad (5.38)$$

The rarefaction wave moves out at the speed of sound, and the shock moves in at roughly the speed of light.

C. Comparison of the Collapse with the Shock Tube Solutions—Numerical Results

In this section, we find the numerical solutions to the special relativistic equations of motion with spherical symmetry and compare them to the slab-symmetric shock tube solutions derived in the last section. For Figures 9-12 and Table 5, we set $C = .3$, $c = 1$, $\omega = 1/3$, $t_i = 1$, $G_N = 0$, $U(t_i, R) = 0$ and $4\pi \rho_{\text{out}}(t_i) = 3/8$.

In Figure 9 we show the collapse of a relativistic, uncompensated void with $R_w = 20$, $\Delta R_w = .005$, $\Delta R(t_i) = .0025$, $\alpha = 10^{-2}$, $\epsilon_{\text{out}}(t_i)/c^2 = 10^6$ and $k^2 = 2.5$. The triangles show the pressure and specific energy as a function of $R$ at $t_i$ and $t = 1.5$. As usual, the void collapses via a shock moving toward the origin at the speed of light. The shock tube solution is plotted as dashed lines. The agreement is excellent, because the shock has not traveled very far and $|\log_{10} \alpha| \Delta R_w/R_w \ll 1$.

Due to time dilation in the void, light travels a farther distance inside the void than outside it in the same amount of coordinate time $t - t_i$. Therefore, the amount of time taken to form the shock is very important for relativistic fluids. If the wall is too thick, the shock will take longer to form. In addition, the shock strength will be smaller, decreasing the void’s relative potential and therefore the time dilation effect. This results in the solution not equaling the shock tube solution at any time. In Table 5, we show the location of the shock, $R_{sh}$, at $t = 1.5$ for voids with the same initial conditions as that in Figure 9 but for varying wall thicknesses and $\Delta R(t_i) = \Delta R_w/2$. In addition, the value of $\Phi_{in}$ is shown. Note that $\Phi_{in}(t_i) = 3.16$ for all voids. It is clear that for this value of $\alpha$, the wall must be thinner than $2\Delta R_w/R_w \lesssim 2 \times 10^{-3}$ in order that the solution approximate the shock tube well.

In Figure 10 we plot $p$, $\epsilon$ and $U$ as a function of time for the void in Figure 9 but at $t = 5$. Note the large distortion in pressure, specific energy and velocity behind the shock. This is due to spherical geometrical effects; fluid behind the shock is forced to occupy a smaller volume, thereby increasing the number density. This strengthens the shock, which in turn dissipates more energy so that the specific energy, energy density and velocity increase. In addition, it is seen that the shock is far ahead of the shock-tube
shock. This is a consequence of the increased energy density behind the shock, $\rho_b$. From Eqs. (A.29) and (3.10), the value of $\Phi$ inside the void is

$$\Phi_{\text{in}} = \frac{1}{\sqrt{3}} \left( \frac{\rho_{\text{out}}}{\rho_{\text{in}}^2} \right)^{1/4} \rho_b^{1/4}.$$  

(5.39)

Therefore, as the energy density behind the shock increases, the shock moves further per unit coordinate time because $dR_{\text{sh}} = c\Phi dt$.

In Figure 11(a), we show the shock tube solution for $\alpha = 10^{-3}$ and $X_w(t_i) = 10$. The dotted lines are the position of $X_a$, $X_B$, $X_c$ and $X_d$, and the solid lines are the locations of comoving observers with $X(t_i, x) = 11.0, 10.1, 10.0, 8.0, 4.0$ and $2.0$. Note that the shock moves very far per coordinate time interval, due to the time dilation effect. Note also that after entering regions 3 or 3’, all shells move with the same velocity. In Figure 11(b), we show the comoving shell positions again as solid lines. In addition, the dashed lines are the positions of comoving shells with the same initial locations, $R(t_i, r) = X(t_i, x)$, in an evolving void. For this void, $R_w = 10$, $\Delta R_w = .1$, $\Delta R(t_i) = .025$, $\alpha = 10^{-3}$, $\epsilon(t_i)/c^2 = 10^6$ and $k^2 = 1.7$. Although the void wall is not very thin, the solution approximates the shock tube solution fairly well in region 3. However, it does not do as well in regions 3’ and 4, due to effects stemming from the thick wall. Note that the numerical shock collides at the origin at $t = 2.2$, which is why the velocity of the comoving shell with $R(t_i, r) = 2.01$ changes after this time.

In Figure 12, we plot $p$, $\epsilon$ $n$ and $U$ for the collapse of a nonrelativistic void in a relativistic background: $\epsilon_{\text{in}}(t_i)/c^2 = .38$, and $\epsilon_{\text{out}}(t_i)/c^2 = 5.0$. In addition, $R_w = 20$, $\Delta R_w = .0025$, $\Delta R(t_i) = \Delta R_w/4$, $\alpha = 10^{-4}$ and $k^2 = 3.0$. The triangles are the numerical solution at $t = 1.06$. The general features are similar to the relativistic void solution: the void collapses at the speed of light as the wall fluid shocks inward. In addition, a contact discontinuity (where $p$ and $U$ are continuous, while $\epsilon$ and $n$ are discontinuous) follows behind the shock. Finally, a rarefaction wave moves into the region outside the void. Note that the shock and contact discontinuity are located at $R = 19.33$ and 19.47, respectively, while the rarefaction wave is located between $R = 19.7$ and 20.04.

The dashed line is a plot of the nonrelativistic/relativistic shock tube solution derived in section B2. It is clear that this solution agrees well with the numerical solution in the outer part of the rarefaction wave (large $R$). In addition, $p$, $\epsilon$ $n$ and $U$ are predicted fairly well behind the shock. (For example, $\rho_3/\rho_1 = 61$ and $\epsilon_3 = 4.2$ from the numerical solution, while $\rho_3/\rho_1 = 72$ and $\epsilon/c^2 = 4.2$ from Eqs. (5.31) and (5.30)). However, because the fluid in regions 3’ and part of 4 are barely relativistic ($\epsilon/c^2 \simeq 1$), the assumptions used to derive the shock tube solution are not satisfied in this case. Therefore, it is no surprise that the shock tube solution does not match the numerical solution very well. At $t = 1.06$, $\Phi_{\text{in}} = 11.2$ from the numerical solution, while $\Phi_1 = 19.4$ from Eq. (5.33). Note that $\Phi_{\text{in}}$ increases from its initial value $\Phi_{\text{in}}(t_i) \simeq \epsilon_{\text{out}}/c^2 = 5.1$ because energy is dissipated over the shock. This results in a smaller collapse time.

*Note that the numerical simulation covers the range $R(t_i, r) \in [19.2, 20.1]$. In addition, $\Delta R_w$ is chosen to be small enough that the wall acts like an initial discontinuity.*

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Thus, the relativistic shock tube solution approximates the numerical solution of a thin-walled void very well for a significant amount of time during collapse. The nonrelativistic/relativistic shock tube solution does not do as well because the fluid between the contact discontinuity and the rarefaction wave tends to be barely relativistic or slightly nonrelativistic. However, it is qualitatively (and to some degree quantitatively) correct.

VI. Discussion

In this paper, we studied in detail the non-linear collapse of a superhorizon-sized void embedded in a background radiation-dominated FRW universe. We find in general that a relativistic or nonrelativistic void collapses via an inbound shock at the speed of light. This occurs because the pressure and velocity gradients in the lower part of the void wall are enormous and therefore accelerate fluid into the void. Because of a time dilation effect, this occurs very quickly, and in particular can occur in less than an outside Hubble time if the relative energy density inside a void is small enough. This is true even when the outward velocity of the wall is enormous. A large wall velocity does drive some of the fluid in the wall out behind a shock, so that less fluid ends up in the void. This causes the strength of the inbound shock to decrease, and lessens the relative potential of the void. Thus the collapse time increases somewhat. However, the void can still collapse in less than an outside Hubble time. This is important, since it emphasizes the point that seemingly nothing stops the void from quickly collapsing in on itself.

As the void wall gets thinner, the shock strength increases and therefore the collapse time decreases. This behavior does not extrapolate to infinitely thin walls, however. When the wall becomes sufficiently thin so that it acts like an initial discontinuity, a maximum value for the shock strength (and therefore a minimum value for the collapse time) is reached. Up to the point when the void has partially collapsed, the solution can be approximated by the shock tube solution. This holds for an uncompensated, superhorizon-sized void which collapses in much less than an outside Hubble time and for which the fluid in the void is relativistic or nonrelativistic. If the fluid in the void is nonrelativistic, then the collapse takes longer to occur for the same value of $\rho_{in}(t_{i})/\rho_{out}(t_{i})$.

If the wall velocity is small and the void collapses in less than an outside Hubble time, then the solution at the collapse time is virtually the same as that when gravitational effects are neglected. This is because fluid in the lower part of the wall is accelerated into the void by pressure and velocity gradient forces, not gravitational forces, and the fluid configuration in the peak area is virtually unchanged.

When the wall velocity is large however, it is not obvious a priori whether or not gravitational forces are important in moving the peak area fluid out of the void wall. In this case, fluid in the peak area of the wall moves out in less than an outside Hubble time. In addition, the time scale for this to occur is proportional to $\Delta R_{w}^{3/4}/\Gamma_{w}$, where $\Delta R_{w}$ is the wall thickness, and $\Gamma_{w}$ is approximately the “gamma-factor” or “velocity” of the wall when $\Gamma_{w} > c^{-1}R_{w}/H_{out}^{-1}(t_{i})$. (If $G_N = 0$, $\Gamma_{w}$ is the relativistic gamma-factor of the fluid, and equals the “velocity” when $\Gamma_{w} \gg 1$). Therefore, if $\Gamma_{w}$ is large enough, fluid in the peak area will shock outward during the collapse, distorting the shape of the

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wall in the process. It is found however, that in general this does not change the collapse
time by very much, given a fixed value of $\rho_{\text{in}}(t_i)/\rho_{\text{out}}(t_i)$. In addition, it is found that
gravity is unimportant in shaping the distortion of the fluid configuration in the peak.
Therefore, gravitational contributions to the solution at the collapse time are negligible.
This means that only non-gravitational forces are driving the fluid into and out of the
void wall, even though the void is superhorizon-sized. (Although it is clear that pressure
and velocity gradient forces must contribute to driving the fluid out of the peak area, one
might envision gravitational forces changing the configuration by substantially redshifting
the wall pressure, as it does when the collapse occurs in greater than an outside Hubble
time. Apparently, gravitational forces do not act quickly enough to do this.) This sounds
surprising, especially since superhorizon-sized voids are larger than the Hubble radius and
therefore must be influenced heavily by gravitational forces. It is a question of time scales,
however. Because time is dilated in the void, if the collapse occurs in much less than an
outside Hubble time, gravity has not had a chance to substantially change the solution
(pressure distribution, shell locations, etc) at the collapse time in this case. Gravitational
forces should be extremely important after the void collapses, of course, which was not
studied in this paper. Then, they will try to pull fluid lumps together, while fluid forces
will try to pull fluid lumps apart.

There are many questions still left to answer. What happens to a superhorizon-
sized void after it collapses? Now that we have a code which implements numerical
regularization to prevent instabilities near the origin, it will be possible to explore this
late stage numerically. In SV, we found that after a special relativistic void collapses,
the pressure near the origin is much greater than the pressure outside the void, creating
an overdensity. Because this lump drops off with distance from the origin, this large
pressure gradient accelerates fluid away from the origin. What happens if gravity is
added to this problem? It appears likely that a black hole might form if the wall velocity
is not very large. If it is very large, enough fluid in the peak area might be pushed out
of the void initially so that this is prevented. In this case, a mass-energy deficit will
occur in the void initially. If this mass-energy continues to be pushed away from the
void, thermalization and homogenization of this void might take a long time to occur.
However, as this expelled fluid slows down, its mass-energy $M$ will decrease. In addition,
at the collapse time the velocity in the former void region is negative, so its mass-energy
will increase. Thus, it will be important to analyze the long-term solution to see if there
is a mass deficit created where the void was located. In addition, density waves will be
created, and must also be followed for their long-term behavior—do they become growing
density perturbations, or remain outgoing local waves? A companion paper will explore
these questions[4].

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**Figure Captions**

**Fig. 1:** Relative error in energy density, \((\rho - \rho_{\text{hom}})/\rho_{\text{hom}}\), where \(\rho_{\text{hom}}\) is the FRW solution, versus \(R\sqrt{t}/t\) for 3 different values of \(\mathcal{F}\). The improved code does much better than that used in SV.

**Fig. 2:** The relative difference in pressure as a function of initial radii for the collapse of GR and SR voids with nearly identical initial conditions. \(\Gamma_{w} = 1\) for both. The closed triangles, open squares, and dashed line show the results for \(\alpha = 10^{-4}\), \(\alpha = 10^{-7}\) and \(\alpha = 10^{-10}\). As the collapse time becomes much less than an outside Hubble radius \((\alpha \lesssim 10^{-10})\), the GR and SR solutions are very similar.

**Fig. 3:** Same as Figure 2, but for \(\Gamma_{w} = 6\).

**Fig. 4:** The evolution of six comoving observers as a function of time for the voids from Figure 3 with \(\alpha = 10^{-4}\) (top) and \(\alpha = 10^{-10}\) (bottom): \(R(t_i) = 30, 35, 40, 45, 48, 50, 55\) and 60. The solid (dashed) lines are the GR (SR) solutions. These solutions are very similar for \(\alpha = 10^{-10}\) because the collapse occurs in much less than an outside Hubble time.

**Fig. 5:** \(4\pi \rho(t)\) and \(\Gamma(t)\) for 3 comoving shells in the peak area, \(R(t_i) = 50, 49\) and 48 (top to bottom) for the voids from Figure 3 with \(\alpha = 10^{-4}\). The GR (SR) void results are shown as solid (dashed) lines. Note that the peak energy density redshifts away quite quickly for the GR, but not for the SR, void.

**Fig. 6:** \(4\pi p\) and \(\Gamma\) as a function of radius for a superhorizon-sized void 25 times the Hubble radius and with \(\alpha = 10^{-10}\). The initial distributions are shown as solid lines. The numerical GR (SR) results are plotted as dotted (dashed) lines. Note that the peak has moved outward substantially at \(\Delta t/H_{\text{out}}^{-1}(t_i) = .03\) for both voids, but that the solutions are very similar anyway.

**Fig. 7:** Same as Figure 6 but for voids with relative energy density inside the void 1000 times smaller. Note that the GR and SR solutions are nearly identical at \(\Delta t/H_{\text{out}}^{-1}(t_i) = .005\), even though the peak area has shocked outward substantially.

**Fig. 8:** Schematic diagram of the shock tube showing the different regions. The shock is located at \(X_d\), and is moving into region 1, and the outer edge of the rarefaction wave is located at \(X_a\), and is moving into region 2.
**Fig. 9:** Pressure and specific energy as a function of radius for a thin-walled SR void. The initial distributions are the solid lines, the numerical solution is the triangles, and the shock tube solution is the dashed lines. Note that the numerical solution is well approximated by the shock tube solution.

**Fig. 10:** Same void as in *Figure 9*, but at a much later time. The numerical solution is given by the solid lines in this figure. The pressure, specific energy and velocity behind the shock is much larger than the shock tube solution. This is due to spherical geometrical effects. The upper part of the rarefaction wave however, is unaffected by these effects.

**Fig. 11:**(a) Location versus time for the shock tube solution with $\alpha = 10^{-3}$ and $R_w = 10$. The position of the 4 boundary locations are the dotted lines, and the solid lines are the locations of comoving points with $X(t_i) = 2, 4, 8, 10, 10.1$ and 11. (b) Shock tube solution (solid lines) and numerical solution (dashed lines) for the same comoving points from *Figure 11(a)*. The two solutions coincide well in region 3, but not as well in the wall area ($X(t_i) = R(t_i) = 10, 10.1$) because the thick wall evolves differently than that given by the simple similarity solution.

**Fig. 12:** Solution for an uncompensated, nonrelativistic void in a relativistic background (triangles). The dashed lines show the nonrelativistic/relativistic shock tube solution.

**Fig. 13:** Percent change in $\Phi, \epsilon, \Gamma$ and $n$ as a function of shock strength for a general relativistic fluid. Note that $\epsilon$ is underestimated, while the other quantities are overestimated, at the shock.
Appendix A: Contact Discontinuity and Shock Jump Conditions

In this section, we briefly re-derive the jump conditions for general relativistic shocks. This closely follows Ref. [6]. In addition, we extend the derivation to include the conditions satisfied over a contact discontinuity. Let the variables \( a \) and \( b \) represent labels for comoving observers ahead and behind a discontinuity, respectively. If each observer measures the invariant interval separating the same two events on the world surface of this discontinuity, using Eq. (2.1), we obtain

\[
[ds^2] = [-c^2 \Phi^2 dt^2 + \Lambda^2 dr^2 + R^2 d\Omega^2] = 0,
\]

(A.1)

where \([G] \equiv G_a - G_b\). Because the discontinuity is radial, we can choose the two events to have \(dr = dt = 0\) but \(d\Omega \neq 0\). Therefore, \(R\) is continuous across the discontinuity: \([R] = 0\). Now consider two events with \(dr \neq 0\) and \(dt \neq 0\). Then from Eq. (A.1),

\[
[c^2 \Phi^2 - M_{sh}^2 / n^2] = 0,
\]

(A.2)

where \(r_{sh}\) is the position of the discontinuity in comoving coordinates, \(dr_{sh}/dt\) is the discontinuity “speed”, and \(M_{sh} \equiv f (dr_{sh}/dt)/(4\pi R^2)\) from Eqs. (2.2) and (2.3). If the discontinuity is a contact discontinuity, then \(r_{sh} = \text{const}\), or \(dr_{sh}/dt = 0\).

It can be shown that the conditions satisfied over the discontinuity for the Schwarzschild metric (but not for the comoving metric) are [4], [6]

\[
[T^\mu_\nu \partial g/\partial x^\mu] = 0,
\]

(A.3)

where \(g\) is the equation for the world surface of the discontinuity. Therefore we first find these expressions in the Schwarzschild frame and then transform back to the comoving frame. The Schwarzschild metric is

\[
ds^2 = -c^2 A^2 dT^2 + B^2 dR^2 + R^2 d\Omega^2,
\]

(A.4)

where \(R\) is now the “Eulerian” coordinate radius. Since \(R\) is a coordinate, \([R] = 0\) over the discontinuity, and using the same argument as above for the two observers,

\[
[A^2 - B^2 S^2] = 0,
\]

(A.5)

where \(S\) is the discontinuity “speed” in this frame: \(cS \equiv dR_{sh}/dT\). For the Schwarzschild metric, the solution for the metrics functions are well known: \(B^{-2} = 1 - 2MG/(Rc^2) = A^2\) where \(M(R, T) = 4\pi c^{-2} \int_0^R \rho R^2 dR\). As long as \(\rho\) is not infinite in the discontinuity, the mass is continuous across it, \([M] = 0\). Then \([B] = [A] = 0\), and from Eq. (A.3), \([S] = 0\).

We now derive these conditions in the Schwarzschild frame. If a discontinuity is located at position \(R_{sh}(T)\) at time \(T\), the equation for its world surface is \(g = R_{sh}(T) - R = 0\). In addition, the perfect fluid stress-energy tensor in the Schwarzschild frame is

\[
T^\mu_\nu = c^{-2}(\rho + p)g'_{\mu\lambda}u^\nu u^{\lambda} + pg'_{\mu\lambda}g^{\nu\lambda} = nW g'_{\mu\lambda}u^\nu u^{\lambda} + pg'_{\mu\lambda}g^{\nu\lambda},
\]

(A.6)

\(^4\)Here, “discontinuity” refers to either a contact discontinuity or a shock.

\(^5\)We denote the metric and 4-velocity in the Schwarzschild frame by primes.
and the conservation of mass equation is \( n u^\nu \partial g/\partial x^\nu \) = 0. Using this and Eq. (A.3), the junction conditions become
\[
\begin{align*}
\frac{c^2}{2} \left[ -(A^2(u^T)^2nW + p)S + nWA^2u^Tu^R \right] &= 0 \\
\left[ SnWB^2u^Tu^R - (nWB^2(u^R)^2 + p) \right] &= 0 \\
\left[ nSu^T - nu^R \right] &= 0.
\end{align*}
\]
We would like to express these equations in terms of the comoving metric functions. First, we can rewrite the speed in the Schwarzschild frame as \( cS = (R_t + R_r \hat{r}_sh)/(T_t + T_r \hat{r}_sh) \). In addition, we can relate the metric functions from the comoving frame to those in the Schwarzschild frame by \( g^{\mu \nu} = (\partial x^\mu/\partial x^\sigma) (\partial x^\nu/\partial x^\lambda) g^{\sigma \lambda} \). We then obtain \((cA)^{-2} = -(c\Phi)^{-2}T_t^2 - \Lambda^{-2}T_r^2, B^{-2} = -(c\Phi)^{-2}R_t^2 + \Lambda^{-2}R_r^2 \) and \((c\Phi)^{-2}T_tR_t = \Lambda^{-2}T_tR_r \). In addition, the “mass-energy” \( M(r, t) \) is continuous across the discontinuity, since \([M] = 4\pi c^{-2}\rho R_\ast dR \frac{dR}{dr} \neq 0 \). With this and Eq. (2.11), we find that \( [T^2 - U^2/c^2] = 0 \).

We can use these relations to calculate the 4-velocity as measured in the Schwarzschild frame. Since the 4-velocity in the comoving frame is \( u^\mu = (-c\Phi^{-1}, 0, 0, 0) \), and the velocity transforms as \( u^\lambda = (\partial x^\lambda/\partial x^\sigma) u^\sigma \), \( u^\lambda = -\Phi^{-1}(cT, \hat{T}, 0, 0) \), where \( \hat{T} = \partial T/\partial t \) and \( \hat{R} = \partial R/\partial t \). Using the normalization condition \( u^\lambda u^\lambda = -c^2 = -c^2 A^2(u^T)^2 + B^2(u^R)^2 \) along with \( U \equiv \hat{R}/\Phi \), we can rewrite the 4-velocity as \( u^\lambda = -(A^{-1}\sqrt{B^2U^2 + c^2}, U, 0, 0) \). In the special relativistic limit \( (G_N = 0), A = B = 1 \). The 4-velocity of a fluid particle is then
\[
u^\lambda = -(U^2 + c^2)^{1/2}, U, 0, 0) = -(c\Gamma, \Gamma v, 0, 0),
\]
where we have defined \( v \equiv U/\Gamma \) so that \( \Gamma = 1/\sqrt{1 - (v/c)^2} \). In the special relativistic limit, then, \( v \) is the fluid particle’s radial velocity, \( \Gamma \) is usual gamma-factor (i.e. energy per particle mass), and \( U \) is the particle momentum per particle mass.

Across a contact discontinuity, which is a discontinuity with \( \hat{r}_sh = 0 \), Eqs. (A.2), (A.7)-(A.8) and \([S] = 0\),
\[
\begin{align*}
[\Phi] &= 0 \\
[pU/\Gamma] &= 0 \\
[p] &= 0 \\
[U/\Gamma] &= 0
\end{align*}
\]
respectively.\(\) Thus the pressure is continuous across a contact discontinuity, as it is in the nonrelativistic case. Using Eq. (2.11) and (A.14), we find that \( (U_a^2 - U_b^2)(1 - 2G_N M/(Rc^2)) = 0 \). If \( 2G_N M/(Rc^2) \neq 1 \), \([U] = 0 \), or the tangential velocity is continuous across the contact discontinuity, as it is in the nonrelativistic case. If \( 2G_N M/(Rc^2) = 1 \), however, the entire formalism breaks down because then \( B = \infty \). This occurs at the Schwarzschild radius. Thus, across a contact discontinuity,
\[
\begin{align*}
[\Phi] &= 0, \\
[p] &= 0, \\
[U] &= 0.
\end{align*}
\]
We now find the jump conditions at a shock where $\dot{r}_{sh} \neq 0$. We rewrite Eqs. (A.7)-(A.9) and $[S] = 0$ in terms of the comoving metric functions using the fact that $T_r/T_t = c^2 U \Lambda / (\Gamma \Phi)$ and $[\dot{r}_{sh}] = 0$. We obtain

\begin{align}
[UM_{sh}/(nc^2) + \Phi \Gamma] &= 0 \tag{A.16} \\
[c^2M_{sh}\Gamma(1 + \epsilon/c^2) - pU\Phi] &= 0 \tag{A.17} \\
[M_{sh}U(1 + \epsilon/c^2) - p\Gamma\Phi] &= 0 \tag{A.18} \\
[M_{sh}\Gamma/n + \Phi U] &= 0 \tag{A.19}.
\end{align}

Eq. (A.2) and Eqs. (A.16)-(A.19) make up the required shock conditions. One of them however, is redundant. It can be shown that in the nonrelativistic limit, the above conditions reduce to the Lagrangian shock jump conditions given in Ref. [15].

We are interested in the case where the fluid in front of the shock is either nonrelativistic or relativistic. First, consider the nonrelativistic case, which was considered previously [6]. We set $G_N = 0$ and $\epsilon_a = p_a = U_a = 0$ and take $p_b = \omega \epsilon_b n_b$. Using Eq. (A.2) and Eqs. (A.16)-(A.19), we find that

\begin{align}
U_b/c &= \mp \sqrt{\Gamma_b^2 - 1} \tag{A.20} \\
\eta &= \frac{n_b}{n_a} = \left[ 2 + \omega + (\omega + 1)\epsilon_b/c^2 \right]/\omega \tag{A.21} \\
\Phi_b &= \Phi_a \tag{A.22} \\
\Gamma_b &= 1 + \epsilon_b/c^2 \tag{A.23} \\
R'_{sh} \dot{r}_{sh} &= \mp c \Phi_a \sqrt{\frac{\epsilon_b/c^2}{\epsilon_b/c^2 + 2} \left( \frac{2 + \omega + (\omega + 1)\epsilon_b/c^2}{1 + (1 + \omega)\epsilon_b/c^2} \right)} \tag{A.24}.
\end{align}

Thus, all quantities behind the shock can be expressed in terms of the specific internal energy behind the shock, $\epsilon_b$. Note that when the fluid behind the shock is relativistic, $\epsilon_b/c^2 \gg 1$, the shock moves at the speed of light, $R'_{sh} \dot{r}_{sh} = \mp c \Phi_a$.

The energy density ahead of the shock cannot be zero since $n_a > 0$. If the mass density were zero, then $\epsilon_b = \infty$. But this is impossible since $\omega \epsilon_b = T_b/\mu$, where $\mu$ is the particle mass, leading to an infinite temperature behind the shock. Therefore, we require $T_b < M_{PL}$ in order that the classical field equations be valid. Then, if $\epsilon_b/c^2 \gg 1$, the energy density in front of the shock is bounded from below:

$$\rho_a = n_a c^2 > [\omega^3/(\omega + 1)] \rho_b(\mu/M_{PL})^2.$$ \hspace{1cm} (A.25)

In addition, it can be shown using Eqs. (A.17) and (A.18) that setting $n_a = 0$ leads to $U_b^2 - \Gamma_b^2 = 0$, in direct contradiction with Eq. (A.20).

When the fluid in front of the shock is relativistic, we end up with a completely different set of equations. We first set $G_N = 0$ and $U_a = 0$. In addition, $p = \omega \rho$ and $\rho = n \epsilon$. Then Eq. (A.2) and Eqns. (A.16)-(A.19) yield

$$U_b = \mp \sqrt{\Gamma_b^2 - 1} = \mp \sqrt{\frac{\omega}{\omega + 1} \frac{\rho_b - \rho_a}{\sqrt{\rho_a \rho_b}}}$$ \hspace{1cm} (A.26)

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\[ \Gamma_b = \sqrt{\frac{(\omega \rho_a + \rho_b)(\rho_a + \omega \rho_b)}{(w + 1)^2 \rho_a \rho_b}} \]  
(A.27)

\[ \eta \equiv \frac{n_b}{n_a} = \frac{(1 + \omega) \Gamma_b \rho_b}{\omega \rho_b + \rho_a} \]  
(A.28)

\[ \Phi_b = \frac{\epsilon_a}{\epsilon_b} \Phi_a = \Phi_a \sqrt{\frac{(\omega \rho_a + \rho_b)\rho_a}{(\omega \rho_b + \rho_a)\rho_b}} \]  
(A.29)

\[ R'_{sh} \dot{r}_{sh} = \mp (1 + \omega) \frac{\rho_b}{\rho_b - \rho_a} \Phi_b U_b. \]  
(A.30)

We note from Eqn (A.29) that \([\Phi_w] = 0\).

In the strong shock limit, when \(\rho_b \gg \rho_a\), the shock moves at the speed of light: \(R'_{sh} \dot{r}_{sh} \simeq \pm c \Phi_a\). In addition, \(n_b/n_a \simeq \sqrt{\rho_b/\rho_a}\) and \(\epsilon_b/\epsilon_a \simeq \sqrt{\omega \rho_b/\rho_a}\). Thus, \(n\) and \(\epsilon\) do not scale as \(\rho^{\omega/(1+\omega)}\) as they do when \(Q = 0\). They always scale as \(\sqrt{\rho_b/\rho_a}\), regardless of the value of \(\omega\).

In order to see how accurate our code is at the shock, we generated shocks by evolving general relativistic voids with \(R_w = 50\), \(\epsilon_{out}(t_1)/c^2 = 10^6\), \(\omega = 1/3\), \(\alpha = 10^{-10}\), and \(\Delta R_w = .25\) and \(\Delta R_w = .5\). (In addition, \(C = .3\) and \(\mathcal{F} = .01\)). After a shock formed, we calculated \(\rho_b\) by averaging \(\rho\) for the first few grid points of \(\rho\) behind the shock. In addition, we found \(\Phi_b\), \(\epsilon_b\), \(\Gamma_b\) and \(n_b\) similarly. Then, we plugged \(\rho_b\) into Eqs.(A.26)-(A.29) to calculate the predicted values. The percent change is plotted in Figure 13. These errors are expected to decrease when the artificial viscosity is decreased, since for these runs, the number of grid points in the shock front is 5 – 9.
Table 1: Relativistic Homogeneous convergence test for $\Delta t_{cs} < \Delta t_n$

| $\Delta R(t_i)$ | $n$ | $R$ | $U$ | $M$ | $\epsilon$ |
|-----------------|-----|-----|-----|-----|---------|
| 20.0            | 28  | 2.5 | 7.6 | 18  | 2.4     |
| 10.0            | 6.9 | 0.62| 1.9 | 4.4 | 0.61    |
| 5.00            | 1.75| 0.15| 0.48| 1.1 | 0.15    |
| 2.50            | 0.43| 0.038|0.12| 0.29| 0.036   |
| 1.25            | 0.11| 0.0095|0.037|0.084|0.0074 |

Table 2: Relativistic Homogeneous convergence test for $\Delta t_n < \Delta t_{cs}$

| $\bar{f}$ | $n$ | $R$ | $U$ | $M$ | $\epsilon$ |
|-----------|-----|-----|-----|-----|---------|
| 0.02      | 24  | 2.2 | 6.7 | 16  | 2.2     |
| 0.01      | 5.1 | 0.46| 1.4 | 3.3 | 0.45    |
| 0.005     | 1.1 | 0.10| 0.34| 0.79| 0.094   |
| 0.0025    | 0.27| 0.024|0.10|0.23|0.016   |

Table 3: Collapse Times for a Superhorizon-Sized relativistic void 25 times the Hubble radius

| $\Gamma_w$ | $\sigma_\Gamma/\Delta R_w$ | $\Delta t_{\text{collapse}}/H_{\text{out}}^{-1}(t_i)$ | $\Delta t_{\text{peak area}}$ |
|------------|-----------------------------|-----------------------------------------------|-----------------------------|
| 6          | 2.05 | 0.047 | 0.0076 | 0.056 |
| 18         | 1.99 | 0.047 | 0.0076 | 0.041 |
| 50         | 1.94 | 0.049 | 0.0076 | 0.020 |
| 100        | 1.91 | 0.056 | 0.0077 | 0.010 |
| 200        | 1.88 | 0.067 | 0.0079 | 0.0052 |

| $\Phi_{in}$ | $\Delta t_{\text{collapse}}/H_{\text{out}}^{-1}(t_i)$ |
|-------------|-----------------------------------------------|
| $\tilde{\alpha}^{-25}$ | 0.44 | 0.079 | 0.014 |
| $0.6\tilde{\alpha}^{-3}$ | 0.33 | 0.042 | 0.0052 |

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### Table 4: Collapse Times versus $\Delta R_w$

| $\Delta R_w$ | $R_w - R_{\text{inner}}$ | $\Delta t_{\text{collapse}}/H_{\text{out}}(t_i)$ | $\tilde{p}$ |
|--------------|-------------------------|---------------------------------------------|------------|
| 1            | 13                      | 2.05                                       | .047       |
| .5           | 7                       | 2.13                                       | .044       |
| .25          | 3                       | 2.20                                       | .034       |

### Table 5: Shock Location versus $\Delta R_w$ for an uncompensated SR void

| $\Delta R_w$ | $R_{\text{sh}}$ | $p_b$   | $\Phi_{\text{in}}$ |
|--------------|----------------|--------|---------------------|
| .01          | 18.4           | .013   | 3.59                |
| .02          | 18.4           | .014   | 3.60                |
| .05          | 18.5           | .013   | 3.57                |
| .1           | 18.5           | .011   | 3.48                |
| .2           | 18.6           | .0085  | 3.33                |
| Shock Tube   | 18.4           | .012   | 3.62                |