A GENERALIZATION OF THOM’S TRANSVERSALITY THEOREM

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ABSTRACT. We prove a generalization of Thom’s transversality theorem. It gives conditions under which the jet map $f^*|_Y : Y \subseteq J^r(D, M) \rightarrow J^r(D, N)$ is generically (for $f : M \rightarrow N$) transverse to a submanifold $Z \subseteq J^r(D, N)$. We apply this to study transversality properties of a restriction of a fixed map $g : M \rightarrow P$ to the preimage $(j^*f)^{-1}(A)$ of a submanifold $A \subseteq J^r(M, N)$ in terms of transversality properties of the original map $f$. Our main result is that for a reasonable class of submanifolds $A$ and a generic map $f$ the restriction $g|(j^*f)^{-1}(A)$ is also generic. We also present an example of $A$ where the theorem fails.

0. INTRODUCTION

We start by reminding that for smooth manifolds $M$ and $N$ the set $C^\infty(M, N)$ of smooth maps is endowed with two topologies called weak (compact-open) and strong (Whitney) topology. They agree when $M$ is compact. We say that a subset of a topological space is residual if it contains a countable intersection of open dense subsets. The Baire property for $C^\infty(M, N)$ then guarantees that it is automatically dense. This holds for both topologies but is almost exclusively used for the strong one. Clearly every residual subset of $C^\infty(M, N)$ for the strong topology is also residual for the weak topology. The following is our main theorem in which we denote by $J^r_{\text{imm}}(D, M)$ the subspace of all jets of immersions.

Theorem A. Let $D, M, N$ be manifolds, $Y \subseteq J^r_{\text{imm}}(D, M)$ and $Z \subseteq J^r(D, N)$ submanifolds. Let us further assume that $\sigma_Y \pitchfork \sigma_Z$, where

$$\sigma_Y = \sigma|_Y : Y \subseteq J^r(D, M) \rightarrow D \quad \text{and} \quad \sigma_Z = \sigma|_Z : Z \subseteq J^r(D, N) \rightarrow D$$

are the restrictions of the source maps. For a smooth map $f : M \rightarrow N$ let $f^*|_Y$ denote the map

$$Y \subseteq J^r_{\text{imm}}(D, M) \xrightarrow{f^*} J^r(D, N).$$

Then the subset $X := \{ f \in C^\infty(M, N) \mid f^*|_Y \pitchfork Z \}$ is residual in $C^\infty(M, N)$ with the strong topology, and open if $Z$ is closed (as a subset) and the target map $\tau_Y : Y \rightarrow M$ is proper.

We are interested in the theorem mainly because of the following two applications, the first of which is the classical theorem of Thom.

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**Theorem B** (Thom’s Transversality Theorem). Let $M$, $N$ be manifolds, $Z \subseteq J^r(M, N)$ a submanifold. Then the subset

$$X := \{ f \in C^\infty(M, N) \mid j^r f \nparallel Z \}$$

is residual in $C^\infty(M, N)$. It is moreover open provided that $Z$ is closed (as a subset).

To explain the second application we need to introduce a bit of notation. Let us consider the following diagram

$$
\begin{aligned}
&f^{-1}(A) \rightarrow A \\
&\downarrow j \\
&g^{-1}(B) \Leftarrow M \rightarrow N \\
&\downarrow g \\
&B \Leftarrow \rightarrow P
\end{aligned}
$$

of smooth manifolds and smooth maps between them where \( \Leftarrow \) indicates embeddings. We assume that \( f \nparallel A \) and \( g \nparallel B \) for the two pullbacks to be defined (which we emphasize by saying that they are transverse pullbacks) and also for some technical reasons. Suppose that we fix $g$ and allow ourselves to change $f$ (but only in such a way that $f \nparallel A$). It is not hard to describe when the composition $gj = g|_{f^{-1}(A)}$ is transverse to $B$. A more general condition on $gj$ is whether it satisfies some form of jet transversality. First we have to solve the problem of not having the source of $J^r(f^{-1}(A), P)$ fixed.

**Construction.** Let $D$ be a $d$-dimensional manifold and

$$\text{Diff}(\mathbb{R}^d, 0) = \text{inv}_0 \mathbb{G}_0(\mathbb{R}^d)$$

the group of germs at 0 of local diffeomorphisms $\mathbb{R}^d \to \mathbb{R}^d$ fixing 0. Define a principal $\text{Diff}(\mathbb{R}^d, 0)$-bundle

$$\text{Charts}_D = \text{inv}_0 \mathbb{G}_0(\mathbb{R}^d, D) \xrightarrow{\text{ev}_0} D$$

of germs at 0 of local diffeomorphisms $\mathbb{R}^d \to D$. If $F$ is any manifold with a (smooth in some sense) action of $\text{Diff}(\mathbb{R}^d, 0)$ then we can construct an associated bundle

$$D[F] := \text{Charts}_D \times_{\text{Diff}(\mathbb{R}^d, 0)} F \rightarrow D.$$

Any bundle of this form is “local”. Observe that this construction is functorial in $D$ on the category of $d$-dimensional manifolds and local diffeomorphisms. As an example the bundle $J^r(D, P)$ of $r$-jets of maps $D \to P$ is a local bundle as

$$D[J^r_0(\mathbb{R}^d, P)] = \text{Charts}_D \times_{\text{Diff}(\mathbb{R}^d, 0)} J^r_0(\mathbb{R}^d, P) \cong J^r(D, P)$$

where $J^r_0(\mathbb{R}^d, P)$ is the subspace of $J^r(\mathbb{R}^d, P)$ of $r$-jets with source 0. The bijection is provided by the map

$$[u, \alpha] \mapsto \alpha \circ j^r_0(u^{-1}).$$

Having a $\text{Diff}(\mathbb{R}^d, 0)$-invariant submanifold $B \subseteq J^r_0(\mathbb{R}^d, P)$ we get an associated subbundle $D[B] \subseteq J^r(D, P)$ for any $d$-dimensional manifold $D$. This allows us to

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1If we wanted to give $\text{Diff}(\mathbb{R}^d, 0)$ a topology we could do so by inducing the topology via the map $\text{Diff}(\mathbb{R}^d, 0) \to J^\infty_0(\mathbb{R}^d, \mathbb{R}^d)_0$. Or one could replace $\text{Diff}(\mathbb{R}^d, 0)$ by its image - the subspace of invertible $\infty$-jets.
talk about jet transversality conditions on a map $D \to P$ without specifying what $D$ (and hence also $J^r(D, P)$) is.

Let $A \subseteq J^s(M, N)$ be a submanifold and $j^s f \cap A$. Then $f^* A := (j^s f)^{-1}(A)$ is a submanifold of $M$ and for any $\text{Diff}(\mathbb{R}^d, 0)$-invariant submanifold $B \subseteq J^r_0(\mathbb{R}^d, P)$ and the associated submanifold

$$f^* A[B] \subseteq J^r(f^* A, P)$$

we might ask whether $j^r(g|f^* A)$ is transverse to $f^* A[B]$. To state the theorem we make the following notation: for a map $g : M \to P$ we write $g \cap B$ in place of $g^* \cap B$ where again $g^*: J^{r+s}_0(\mathbb{R}^d, M) \to J^r_0(\mathbb{R}^d, J^s(M, N))$,

$$j^r(s_{\phi}, j^r_0(\psi)) \mapsto j^r_0(j^s(\phi \circ \psi)).$$

**Theorem C.** Let

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{g} & & \downarrow{P} \\
N & \xrightarrow{\text{Diff}(\mathbb{R}^d, 0)} & P
\end{array}
$$

be a pair of smooth maps. Let $A \subseteq J^s(M, N)$ and $B \subseteq J^r_0(\mathbb{R}^d, P)$ be smooth submanifolds with $d = \dim M - \text{codim} A$. Assuming that $B$ is $\text{Diff}(\mathbb{R}^d, 0)$-invariant, $\kappa \cap J^r_0(\mathbb{R}^d, A)$ and $g \cap B$ the subset

$$\mathcal{X} := \{f \in C^\infty(M, N) \mid j^s f \cap A, j^r(g|f^* A) \cap f^* A[B]\}$$

is residual in $C^\infty(M, N)$ with the strong topology. If either $r = 0$ or $s = 0$ the transversality condition $\kappa \cap J^r_0(\mathbb{R}^d, A)$ is automatically satisfied.

**Remark.** The transversality condition $\kappa \cap J^r_0(\mathbb{R}^d, A)$ cannot be removed as we illustrate by an example in the next section. An interesting question arises whether there is a reasonable sufficient condition on $A$ for which the transversality is automatic (an example of such is e.g. $s = 0$).

**Remark.** It is also possible to state conditions under which $\mathcal{X}$ is open.

1. **AN EXAMPLE**

We describe here a family of examples. For $k = 1$ (and partially for $k = 2$) Theorem C can be applied whereas for $k \geq 3$ the theorem fails due to $\kappa \not\cap J^r_0(\mathbb{R}^d, A)$. We will not be interested in a particular choice of $B$ since the transversality condition in question depends only on $A$. Let us start with the following diagram

$$
\begin{array}{ccc}
D^k \times M & \xrightarrow{f} & \mathbb{R} \\
\downarrow & & \downarrow \\
D^k & & 
\end{array}
$$

where we think of $f$ as a family of functions $M \to \mathbb{R}$ parametrized by a disc $D^k$. To fit into our situation we should replace $D^k$ by $\mathbb{R}^k$ but the boundary is not the issue in our example. Let $s = 1$ and let $A \subseteq J^1(D^k \times M, \mathbb{R})$ denote the subspace
of all those jets which have zero derivative in the direction of $M$. Clearly $\mathcal{A}$ has codimension $\dim M$ and thus $d = k$.

For a function $\varphi : D^k \times M \to \mathbb{R}$ we have $J^1_{(x,y)}\varphi \in \mathcal{A}$ iff the composition

$$T_y M \xrightarrow{\iota} T_x D^k \times T_y M \xrightarrow{d\varphi} \mathbb{R}$$

is zero. We express this by saying that the map

$$d|_{TM} : J^1(D^k \times M, \mathbb{R}) \to T^* M$$

describes $\mathcal{A}$ as the preimage of 0. We will now show how to describe $J^r_0(\mathbb{R}^k, \mathcal{A})$ (or $f^* \mathcal{A}$ in fact) in a similar way. Compose the defining equation with an immersion $\psi : \mathbb{R}^k \to D^k \times M$ as in (1) and differentiate to get

$$T_y M \otimes \mathbb{R}^k \xrightarrow{incl \otimes \psi_*} S^2(T_x D^k \times T_y M) \xrightarrow{d^2 \varphi} \mathbb{R}.$$

Differentiating further and putting all the maps together we get a single map

$$J^{r+1}(D^k \times M, \mathbb{R}) \times M \xrightarrow{J^r_0 \text{imm}(\mathbb{R}^k, D^k \times M)} T^* M \otimes (\mathbb{R} \oplus \mathbb{R}^k \oplus \cdots \oplus S^r \mathbb{R}^k)^*$$

describing $f^* \mathcal{A}$. Moreover $\chi$ is a submersion at $f^* \mathcal{A}$ iff $\kappa \cap J^r_0(\mathbb{R}^k, \mathcal{A})$. Fixing $j_0^0 \psi$ the $T^* M \otimes (S^r \mathbb{R}^k)^*$-coordinate is a sum of various restrictions of the derivatives of $\varphi$ with only a single term involving the highest derivative $d^{r+1} \varphi$, namely

$$T_y M \otimes S^r \mathbb{R}^k \xrightarrow{incl \otimes S^r \psi_*} S^{r+1}(T_x D^k \times T_y M) \xrightarrow{d^{r+1} \varphi} \mathbb{R}.$$

This implies that if the image of the derivative of $\psi_*$ at 0 intersects $TM$ in a subspace of dimension at most 1, the highest order term is a surjective linear map and quite easily the whole map $\chi$ is a submersion (even for a fixed $\psi$). The opposite implication also holds as considerations at pairs with $j^{r+1} \varphi = 0$ show.

For $k = 1$ we have shown that $\kappa \cap J^r_0(\mathbb{R}, \mathcal{A})$ and therefore for a generic map $f$ the fibrewise singularity set $\Sigma_f = f^* \mathcal{A}$ is a 1-dimensional submanifold for which the projection $\Sigma_f \to D^1$ is generic. This means that only regular points and folds appear. The regular points of the projection are exactly the fibrewise Morse singularities of $f$ while folds correspond to the fibrewise $A_2$-singularities (those of the form $x_1^2 \pm x_2^2 \pm \cdots \pm x_n^2$).

For $k \geq 2$ it is not the case that $\kappa \cap J^r_0(\mathbb{R}^k, \mathcal{A})$, namely at pairs $(j^{r+1}_x \varphi, j_0^0 \psi)$ where the image of the tangential map $\psi_*$ at 0 intersects the tangent space of the fibre $M$ in a subspace of dimension at least 2.

Nevertheless for $k = 2$ the conclusion of Theorem [2] still holds. This is because the condition $\kappa \cap J^r_0(\mathbb{R}^k, \mathcal{A})$ is only required at those $(j^{r+1}_x \varphi, j_0^0 \psi)$ for which $\psi(\mathbb{R}^d) \subseteq f^* \mathcal{A}$ and for a generic map $f$ such $\psi$ cannot be tangent to the fibre $M$ (if that was the case then the rank of the fibrewise Hessian of $f$ at $x$ would drop by 2 and this does not happen generically). Therefore again $\Sigma_f$ is a 2-dimensional submanifold of $D^2 \times M$ partitioned into three parts: the cusps of the projection $\Sigma_f \to D^2$ (where $f$ attains an isolated fibrewise $A_4$-singularity); the folds form a 1-dimensional submanifold (where the fibrewise $A_3$-singularities occur); and the fibrewise Morse singularities.

For $k = 3$ the conclusion of Theorem [2] fails. This is due to the fact that for a generic map between 3-manifolds the rank drops at most by 1 whereas for a generic 3-parameter family of functions $M \to \mathbb{R}$ the fibrewise Hessian can drop rank by 2.
2. Proofs

First we will show how to translate transversality conditions for a restriction of a fixed map \( g : M \to P \) to the preimage \( f^{-1}(A) \) of a submanifold along a map \( f : M \to N \) in terms of transversality conditions on \( f \) itself. At the end we prove that such properties are generic in the sense that maps satisfying them form a residual subset of \( C^\infty(M,N) \). First version does not involve any jet conditions.

**Lemma 1.** Let

\[
\begin{array}{c}
A \\
\downarrow \downarrow \downarrow
\end{array}
\]

be a diagram where we assume that \( f \cap A \) and \( g \cap B \). Then the following conditions are equivalent:

(i) \( gj \cap B \),
(ii) \( fi \cap A \),
(iii) \( f^{-1}(A) \cap g^{-1}(B) \).

**Proof.** Since (iii) is symmetric it is enough to show the equivalence of (i) and (iii). But (i) is equivalent to the map

\[
g_* : T_x f^{-1}(A) \to T_{g(x)} P/T_{g(x)} B
\]

induced by the derivative of \( g \) being surjective for every \( x \in f^{-1}(A) \cap g^{-1}(B) \).

Because of the assumption \( g \cap B \), we have a commutative diagram

\[
\begin{array}{c}
T_x f^{-1}(A) \\
\downarrow \downarrow \downarrow
\end{array}
\]

and so (i) is equivalent to (iii). \( \square \)

Now we will generalize the lemma to jet transversality conditions. Let us recall that for a \( \text{Diff}(\mathbb{R}^d,0) \)-invariant submanifold \( B \subseteq J^r_0(\mathbb{R}^d, P) \) we have constructed an associated subbundle \( D[B] \subseteq J^r(D, P) \) for any \( d \)-dimensional manifold \( D \).

**Lemma 2.** For \( h : D \to P \) the following conditions are equivalent

(i) \( h_* : J^r_{0, \text{imm}}(\mathbb{R}^d, D) \to J^r_0(\mathbb{R}^d, P) \) is transverse to \( B \),
(ii) \( j^r(h) : D \to J^r(D, P) \) is transverse to \( D[B] \).

**Proof.** Taking associated bundles (i) is clearly equivalent to the transversality of

\[
h_* : J^r_{\text{imm}}(D, D) \to J^r(D, P)
\]
to $D[B]$. Let $j^r(k) \in J^r_{\text{imm}}(D,D)$ be an $r$-jet of a diffeomorphism $k : V \to W$ between open subsets of $D$. Then we have a diagram

$$
\begin{array}{ccc}
J^r_{\text{imm}}(V,D) & \xrightarrow{h_*} & J^r(V,P) \\
\downarrow k_* & \cong & \downarrow k_* \\
J^r_{\text{imm}}(W,D) & \xrightarrow{h_*} & J^r(W,P)
\end{array}
$$

Now $j^r(k)$ in the top left corner is mapped by $k_*$ down to $j^r(y)(\text{id})$. Hence we see that it is enough (equivalent) to check the transversality only at $j^r(y)(\text{id})$’s for all $y \in D$ for which $h_*(j^r(y)(\text{id})) = j^r(y) \in D[B]$. For such $y$ the same diagram shows that every $j^r(x)$ with target $y$ is mapped by $h_*$ to $D[B]$. Thus the whole fibre over $y$ of the target map

$$
J^r_{\text{imm}}(D,D) \xrightarrow{\tau} D
$$

is mapped to $D[B]$. The target map $\tau$ has a section

$$
\tau^*(\text{id}) : D \to J^r_{\text{imm}}(D,D)
$$

and so (i) is finally equivalent to the composite

$$
D \xrightarrow{\tau^*(\text{id})} J^r_{\text{imm}}(D,D) \xrightarrow{h_*} J^r(D,P)
$$

being transverse to $D[B]$. This is (ii).

We say that a map $g : M \to P$ is transverse to a $\text{Diff}(\mathbb{R}^d,0)$-invariant submanifold $B \subseteq J^0_0(\mathbb{R}^d,P)$, denoted $g \pitchfork B$, if

$$
g_* : J^0_{\text{imm}}(\mathbb{R}^d,M) \to J^0_0(\mathbb{R}^d,P)
$$

is transverse to $B$. When $r = 0$ this is equivalent to the usual transversality of a map to a submanifold. Let $f \pitchfork A$ where $f : M \to N$ and $A \subseteq N$ is a submanifold. Then we have the following diagram

$$
\begin{array}{ccc}
\xrightarrow{J^0_{\text{imm}}(\mathbb{R}^d,f^{-1}(A))} & \xrightarrow{J^0_0(\mathbb{R}^d,f^{-1}(A))} & \xrightarrow{J^0_0(\mathbb{R}^d,A)} \\
\downarrow j_* & \downarrow j_* & \downarrow j_* \\
\xrightarrow{J^0_{\text{imm}}(\mathbb{R}^d,M)} & \xrightarrow{J^0_0(\mathbb{R}^d,M)} & \xrightarrow{J^0_0(\mathbb{R}^d,N)}
\end{array}
$$

where both squares are transverse pullbacks. This can be easily seen in local coordinates. Combining Lemma 1 with Lemma 2 we get:

**Lemma 3.** Given a diagram

$$
\begin{array}{ccc}
f^{-1}(A) & \xrightarrow{f} & A \\
\downarrow j & & \downarrow j \\
M & \xrightarrow{f} & N \\
\downarrow g & & \downarrow g \\
P & & P
\end{array}
$$

assume that $f \pitchfork A$ and $g \pitchfork B$, where $B \subseteq J^0_0(\mathbb{R}^d,P)$ is a $\text{Diff}(\mathbb{R}^d,0)$-invariant submanifold with $d = \dim M + \dim A - \dim N$. Then the following conditions are equivalent:
(i) \( j^*(gj) \cap (f^{-1}(A))[B] \), where
\[
j^*(gj) : f^{-1}(A) \rightarrow J^r(f^{-1}(A), P)
\]
is the jet prolongation,
(ii) \( f_*|_Y \cap J^*_0(\mathbb{R}^d, A) \), where \( Y = (g_*)^{-1}(B) \) is defined by a pullback diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{\zeta} & J^r_0(\mathbb{R}^d, M) \\
\downarrow & & \downarrow g_* \\
B & \xrightarrow{\zeta} & J^*_0(\mathbb{R}^d, P)
\end{array}
\]

Proof. Applying Lemma 1 to the diagram
\[
\begin{array}{ccc}
J^r_0(\mathbb{R}^d, f^{-1}(A)) & \xrightarrow{j_*} & J^r_0(\mathbb{R}^d, A) \\
\downarrow & & \downarrow f_* \\
Y & \xrightarrow{\zeta} & J^r_0(\mathbb{R}^d, M) \\
\downarrow & & \downarrow g_* \\
B & \xrightarrow{\zeta} & J^*_0(\mathbb{R}^d, P)
\end{array}
\]
gives an equivalence of (ii) with the transversality of
\( (gj)_* : J^r_0(\mathbb{R}^d, f^{-1}(A)) \rightarrow J^*_0(\mathbb{R}^d, P) \)
to \( B \). By Lemma 2 this is equivalent to (i). \( \square \)

Now that we know how \( f \) controls the transversality of a map defined on the preimage \( f^{-1}(A) \) of some submanifold, we would like to see that this transversality condition (any of the two equivalent conditions in Lemma 3) is generic. This is indeed the case. We first prove a more general result which at the same time happens to generalize the Thom Transversality Theorem.

Proof of Theorem A. This is an application of Lemma 5 from Section 3. We have a map
\( \alpha : C^\infty(M, N) \rightarrow C^\infty(Y, J^r(D, N)) \)
sending \( f \) to \( f_*|_Y \). This map is continuous for the weak topology on the target and clearly \( \mathcal{X} = \{ f \in C^\infty(M, N) \mid \alpha(f) \cap Z \} \). We have to verify the conditions of Lemma 5.

Let \( f_0 \in C^\infty(M, N) \) and \( K \subseteq Y, L \subseteq Z \) compact discs. We can assume that \( \tau(K) \) lies in a coordinate chart \( \mathbb{R}^m \cong U \subseteq M \) and that \( \tau(L) \) lies in a coordinate chart \( \mathbb{R}^n \cong V \subseteq N \). We use these charts to identify \( U \) with \( \mathbb{R}^m \) and \( V \) with \( \mathbb{R}^n \) when needed. Let \( \lambda : U \rightarrow \mathbb{R} \) be a compactly supported function such that \( \lambda = 1 \)
on a neighborhood of \( \tau(K) \cap f_0^{-1}(\tau(L)) \) and such that \( \lambda = 0 \) on \( U - f_0^{-1}(V) \). This is summarized in the picture above where we put \( W = \text{int } \lambda^{-1}(1) \).

We set \( Q := J^r_0(\mathbb{R}^m, \mathbb{R}^n) \) and identify it both with \( J^r_+(U, V) \), where + stands for an arbitrary point in \( U \), and also with the space of polynomial mappings \( U \to V \). Then we get a map

\[
\beta : Q \to C^\infty(M, N)
\]

sending \( q \) to the function \( f_0 + \lambda q \) where the operations are interpreted inside \( V \) via the chart. It is continuous (in the strong topology) and the adjoin t map

\[
\gamma = (\alpha \beta)^\delta : Q \times Y \to J^r(D, N)
\]

is smooth. Thus it is enough to show that (after a suitable restriction) \( \gamma \pitchfork Z \). Clearly \( \gamma \) sends \((q, J^r_x(h)) \) to \( J^r_x((f_0 + \lambda q)h) \). Suppose now that \( h(x) \in W \) so that this equals to \( J^r_x(f_0h + qh) \). By restriction we get a map

\[
\delta : Q \cong Q \times \{ J^r_x(h) \} \xrightarrow{\sim} J^r_x(D, V).
\]

In the affine structure on \( J^r_x(D, V) \) inherited from the chart, \( \delta \) is clearly affine. Identifying \( Q \) with \( J^r_{h(x)}(U, V) \) the linear part of \( \delta \) is just a precomposition with \( h \)

\[
h^* : J^r_{h(x)}(U, V) \to J^r_x(D, V).
\]

The map \( h \), being an immersion, has (locally - near \( x \)) a left inverse \( \pi \) which then gives a right inverse \( \pi^* \) of \( h^* \) and so the linear part of \( \delta \) is surjective and hence it is a submersion.

In the horizontal direction our transversality condition \( \sigma_Y \pitchfork \sigma_Z \) applies and so \( \gamma \pitchfork Z \) on \( Q \times \tau_Y^{-1}(W) \). If \( f : M \to N \) is close enough to \( f_0 \) then

\[
\tau(K) \cap f^{-1}(\tau(L)) \subseteq W
\]

(equivalently \( f(\tau(K) - W) \subseteq N - \tau(L) \) which is one of the basic open sets for the compact-open topology) and in particular there is a neighbourhood \( Q' \) of \( 0 \) in \( Q \) such that \( \beta(Q') \) consists only of such maps. Therefore the restriction of \( \gamma \) to

\[
Q' \times K \longrightarrow J^r(D, N)
\]

is transverse to \( L \) and hence the same is also true in some neighbourhoods of \( K \) and \( L \) which then form the coverings required in Lemma.

If \( \tau \gamma \) happens to be proper then \( \alpha \) is continuous even in strong topologies and \( X \) is a preimage of the open subset of maps \( f : Y \to J^r(D, N) \) transverse to \( Z \). \( \square \)

Now we prove two corollaries of the previous theorem.

**Proof of Theorem** We apply Theorem to \( D = M \) and

\[
Y = M \xrightarrow{j^r(id)} J^r(M, M).
\]

As \( \sigma_Y = \text{id} = \tau_Y \) it is both proper and transverse to \( \sigma_Z \) for any \( Z \). \( \square \)

**Corollary 4.** Let \( M, N \) be manifolds, \( Y \subseteq J^r_{0,\text{imm}}(\mathbb{R}^d, M) \) and \( Z \subseteq J^r_0(\mathbb{R}^d, N) \) submanifolds. Then the subset

\[
\left\{ f \in C^\infty(M, N) \mid f_x|Y \pitchfork Z \right\}
\]

is residual in \( C^\infty(M, N) \) with the strong topology, and open if \( Z \) is closed (as a subset) and \( \tau_Y : Y \to M \) proper.
Proof. Under the natural identification $\mathbb{R}^d \times J^r_0(\mathbb{R}^d, N) \cong J^r(\mathbb{R}^d, N)$ we can apply Theorem \([\text{A}]\) to $D = \mathbb{R}^d$, the same $M$, $N$ and $Y$ but with $\mathbb{R}^d \times Z \subseteq J^r(\mathbb{R}^d, N)$ in place of $Z$. \(\Box\)

Now we are finally able to prove our main theorem.

Proof of Theorem \([\text{C}]\) Consider first the special case $s = 0$ and

$$A = M \times A_0 \subseteq M \times N = J^0(M, N).$$

Then $j^0 f \pitchfork A$ if $f \pitchfork A_0$ and under this assumption Lemma \([\text{B}]\) provides a translation between $g|_{f^* A} \pitchfork f^* A[B]$ and $f_*|_Y \pitchfork Z$, where $Z = J^r_0(\mathbb{R}^d, A_0) \subseteq J^r_0(\mathbb{R}^d, N)$. The genericity of $f \pitchfork A_0$ is the usual transversality theorem while the genericity of $f_*|_Y \pitchfork Z$ is the last corollary.

The next step is to replace the transversality to $A_0 \subseteq N$ by a jet transversality condition. Let $A \subseteq J^s(M, N)$ be a submanifold and consider

$$\begin{CD}
  f^* A @>>> A \\
  @VVjV @VVV \\
  M @>>> J^s(M, N) \\
  @VVgV @VVV \\
  P
\end{CD}$$

It is possible to apply Lemma \([\text{B}]\) to this situation and translate $g|_{f^* A} \pitchfork f^* A[B]$ to the transversality of the composition

$$Y \looparrowright J^r_{0, \text{imm}}(\mathbb{R}^d, M) \overset{(j^s f)_*}{\longrightarrow} J^r_0(\mathbb{R}^d, J^s(M, N))$$

to $J^r_0(\mathbb{R}^d, A)$. We cannot however apply Theorem \([\text{A}]\) directly since we are not interested in all maps $M \rightarrow J^s(M, N)$ but only in the holonomic sections (those of the form $j^s f$). This means that in our proof of Theorem \([\text{A}]\) $Q = J^s(\mathbb{R}^m, J^s(\mathbb{R}^m, \mathbb{R}^n))$ has to be replaced by its subspace $J^s_+ i_s(\mathbb{R}^m, \mathbb{R}^n)$ and in general there is no guarantee that the new map $\gamma$ (see \([\text{B}]\)) will be transverse (after restriction) to $Z = J^r_0(\mathbb{R}^d, A)$. This is however easily implied by $\kappa \pitchfork J^r_0(\mathbb{R}^d, A)$ since again we can arrive at the linear part of $\gamma$ being a composition map (as in \([\text{B}]\)) which is then easily identified with $\kappa$. \(\Box\)

3. A general transversality lemma

We will formulate a basic lemma for deciding whether a given family of maps contains a dense subset of maps with a particular transversality property. In a sense this is just the essence of any proof of such a statement. We will be considering maps $\varphi : R \rightarrow C^\infty(S, T)$. We denote by $\varphi^\sharp$ its adjoint

$$\varphi^\sharp : R \times S \rightarrow T.$$ 

Lemma 5. Let $S$, $T$ be smooth manifolds and $Z \subseteq T$ a submanifold. Let there be given two open coverings: $\mathcal{U}$ of $S$ and $\mathcal{V}$ of $T$. Let $R$ be a topological space and $\varphi : R \rightarrow C^\infty(S, T)$ a continuous map where $C^\infty(S, T)$ is given the weak topology.
Assume that for every \( r_0 \in \mathbb{R} \) and every \( U \in \mathcal{U}, V \in \mathcal{V} \) there is a finite dimensional manifold \( Q \) and a continuous map \( k : Q \to \mathbb{R} \) with \( r_0 \) in its image such that

\[
Q \times U \xrightarrow{k \times \text{incl}} R \times S \xrightarrow{\varphi^s} T
\]

is smooth and transverse to \( V \). Then the subset

\[
\mathcal{X} := \{ r \in \mathbb{R} \mid \varphi(r) \cap Z \} \subseteq R
\]

is residual in \( R \).

**Proof.** Following the proof of the Theorem 4.9. of Chapter 4 of \([GG]\), let us cover \( S \) by a countable family of compact discs \( K_i \) that have a neighbourhood \( U_i \in \mathcal{U} \) and at the same time we choose a covering of \( Z \) by a countable family of compact discs \( L_j \) that have a neighbourhood \( V_j \in \mathcal{V} \). Then the set \( \mathcal{X} \) is a countable intersection of the sets

\[
\mathcal{X}_{ij} := \{ r \in \mathbb{R} \mid \varphi(r) \cap L_j \text{ on } K_i \}
\]

and it is enough to show that each \( \mathcal{X}_{ij} \) is open and dense. The set \( \hat{\mathcal{X}}_{ij} \) of maps \( S \to T \) transverse to \( L_j \) on \( K_i \) is open in \( C^\infty(S,T) \) and \( \mathcal{X}_{ij} = \varphi^{-1}(\hat{\mathcal{X}}_{ij}) \) so it is also open.

To prove the denseness we fix \( r_0 \in \mathbb{R} \) and choose a map \( k : Q \to \mathbb{R} \) with \( r_0 = k(q_0) \) such that the map

\[
l : Q \times U_i \to T
\]

from the statement is smooth and transverse to \( V_j \). By the parametric transversality theorem (see e.g. Theorem 2.7., Chapter 3 in \([Hir]\)) the points \( q \in Q \) for which \( l(q, -) \cap V_j \) is dense in \( Q \). In particular \( q_0 \) lies in the closure of this set and hence \( r_0 \) lies in the closure of its image in \( R \). But this image certainly lies in \( \mathcal{X}_{ij} \). \( \square \)

**References**

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