Local bulk operators in AdS/CFT and the fate of the BTZ singularity

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Abstract. This paper has two parts. First we review the description of local bulk operators in Lorentzian AdS in terms of non-local operators in the boundary CFT. We discuss how bulk locality arises in pure AdS backgrounds and how it is modified at finite $N$. Next we present some new results on BTZ black holes: local operators can be defined inside the horizon of a finite $N$ BTZ black hole, in a way that suggests the BTZ geometry describes an average over black hole microstates, but with finite $N$ effects resolving the singularity.

1. Introduction

Quantum gravity in asymptotically anti-de Sitter space is dual to a conformal field theory on the boundary of AdS [Ma98]. One of the most interesting questions raised by this duality is: how does approximately local bulk gravitational physics emerge from the CFT?

To address this one needs some way of probing local physics in the bulk. We are mostly interested in the semiclassical limit of small Planck length. In this limit we should be able to recover the traditional results of quantum field theory in curved space [BD82]. So it’s natural to ask: how can a local quantum field in the bulk of AdS be represented in the boundary CFT?

This question was addressed in [BDHM98, BGL99, Ben00] and was further developed by the present authors in [HKLL06a, HKLL06b, HKLL07]. In the latter works it was shown that local operators in the bulk could be represented as non-local operators in the CFT. The CFT operators turn out to have support on a compact region of the complexified boundary. This representation makes several properties manifest. It makes it clear why bulk locality is exact at large $N$, but breaks down at finite $N$, in exactly the manner required by holography. It also

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provides a simple CFT description of the horizon and singularity of a BTZ black hole in the large $N$ limit.

An outline of this paper is as follows. In section 2 we review the representation of local bulk operators in terms of operators on the complexified boundary. In section 3 we use these boundary operators to discuss bulk locality and holography from the point of view of the CFT. In section 4 we extend the construction to the BTZ black hole and discuss the horizon and singularity in the large $N$ limit. We conclude in section 5 with some speculation on the fate of the horizon and singularity at finite $N$. Sections 2–4 are a review; the results in section 5 are new.

2. Local operators in the semiclassical limit

In Poincaré coordinates the metric on Lorentzian AdS$_D$ is

$$ds^2 = \frac{R^2}{Z^2} (-dT^2 + |dX|^2 + dZ^2) .$$

Here $R$ is the AdS radius. The Poincaré horizon is at $Z = \infty$, while the CFT$_{d=D-1}$ lives on the boundary at $Z = 0$. Consider a scalar field of mass $m$ in AdS, with normalizable fall-off near the boundary.

$$\phi(T, X, Z) \sim Z^\Delta \phi_0(T, X) \quad \text{as } Z \to 0$$

The parameter $\Delta$ is related to the mass of the field by

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 R^2} .$$

We will refer to $\phi_0$ as the boundary field. It’s dual to an operator of dimension $\Delta$ in the CFT.

$$\phi_0(T, X)_{\text{SUGRA}} \leftrightarrow \mathcal{O}(T, X)_{\text{CFT}}$$

The question is, can we express $\phi$ in terms of $\phi_0$? If so, then we can use (2.2) to find the CFT operator dual to a local operator in the bulk.

For now we’ll study this in the semiclassical limit

$$\ell_s, \ell_p \to 0 \quad \text{in the bulk}$$

$$N, \lambda \to \infty \quad \text{on the boundary}$$

Here $\ell_s$ and $\ell_p$ are the bulk string and Planck lengths, while $N$ and $\lambda$ are parameters for some kind of ’t Hooft large-$N$ expansion in the CFT whose details won’t matter for us. The basic idea is to represent

$$\phi(T, X, Z) = \int dT' d^{d-1}X' K(T', X'|T, X, Z) \phi_0(T', X')$$

using a kernel or smearing function $K$. Since AdS has a timelike boundary, this is not a standard Cauchy problem, and neither existence nor uniqueness of $K$ is guaranteed. Indeed in [HKLL06a] we discuss examples where both existence and uniqueness are violated.

A cure for these problems, at least in a pure semiclassical AdS background, is to make a Wick rotation to de Sitter space. Define a new set of boundary spatial coordinates by setting $X = iY$. This turns the AdS metric (2.1) into

$$ds^2 = \frac{R^2}{Z^2} (-dT^2 - |dY|^2 + dZ^2) .$$
Figure 1. The field at a bulk point in de Sitter space can be expressed in terms of data on the past de Sitter boundary. The slice $Y = 0$ also describes a region in AdS. So we can also regard this as expressing the field in AdS in terms of data on the complexified AdS boundary.

This is de Sitter space in flat FRW coordinates, with $Z$ playing the role of conformal time. The past boundary of de Sitter space is at $Z = 0$.

In de Sitter space we have a standard Cauchy problem. As shown in Fig. 1 we can use a retarded de Sitter Green’s function to solve for the bulk field in terms of data on the past boundary. The explicit analytic expressions are pretty simple: the field at a point in AdS can be expressed as

$$\phi(T, X, Z) = \frac{\Gamma(\Delta - \frac{d}{2} + 1)}{\pi^{d/2} \Gamma(\Delta - d + 1)} \int_{T, Y, |Y'|^2 < Z^2} dT' d^{d-1} Y' \left( \frac{Z^2 - T'^2 - |Y'|^2}{Z} \right)^{\Delta - d} \phi_0(T + T', X + iY').$$

(2.4)

Note that we have to integrate over a compact region of the de Sitter boundary (the region inside the past light-cone of the bulk point). Equivalently we integrate over a compact region of the complexified AdS boundary (the region spacelike separated from the bulk point).

By construction this lets us reproduce bulk correlation functions in the semiclassical limit.

$$\langle \phi(x_1, Z_1) \phi(x_2, Z_2) \rangle_{\text{SUGRA}} = \int dx'_1 dx'_2 K(x'_1|x_1, Z_1) K(x'_2|x_2, Z_2) \langle O(x'_1) O(x'_2) \rangle_{\text{CFT}}$$

(2.5)

This is guaranteed to work, just because $\phi_0$ and $O$ have identical correlators. Although somewhat trivial, this result has an interesting corollary. In the semiclassical limit of vanishing Planck length, bulk operators will commute at spacelike separation. Therefore the corresponding smeared boundary operators will commute with each other. This is true even though, as shown in Fig. 2, the smearing functions might overlap on the boundary.
Figure 2. Smearing functions for two bulk points separated only in the $Z$ direction. The smearing functions overlap on the boundary, nonetheless the smeared operators commute at infinite $N$.

3. Bulk locality and holography at finite $N$

What we’ve done so far is exact in the semiclassical limit; it can be regarded as a set of statements about free wave equations in a pure AdS background. In this section we’ll remain in a pure AdS background, but ask what happens at finite $N$.

First we need to decide what smearing functions to use. One possibility is to use the same smearing functions at finite $N$. For example, in $\mathcal{N} = 4$ Yang-Mills the operator

$$\Phi(T, X, Z) = \int dT' d^3X' K(T', X'|T, X, Z) \text{Tr} F^2$$

can be defined at any $N$, where $K$ is the kernel appearing in (2.4). In a pure AdS background we believe these must be the right operators to use at finite $N$, just because the construction is singled out by the symmetries. To see this, introduce a distance function on AdS.

$$\sigma(T, X, Z|T', X', Z') = \cosh \left( \frac{\text{geodesic distance}}{R} \right) = \frac{Z^2 + Z'^2 + |X - X'|^2 - (T - T')^2}{2ZZ'}$$

$\sigma$ is invariant under AdS isometries. In Poincaré coordinates the isometry

$$(T, X, Z) \to \lambda(T, X, Z)$$

acts as a scale transformation on the boundary. $Z$ has conformal weight $-1$, and since $K \sim \lim_{Z' \to 0}(\sigma Z')^{\Delta - d}$, we see that $K$ transforms covariantly under AdS isometries with conformal weight $d - \Delta$. But given an operator of dimensions $\Delta$, this is exactly what we need for $\int d^d x K O$ to behave like a scalar field in the bulk!
That is, *the smearing functions we have defined provide the unique covariant way to map a primary field in the CFT to a scalar field in the bulk.*

This might seem very strange from the point of view of holography. In (3.1) we’ve defined a continuous infinity of bulk operators. How can this be compatible with the holographic bound [SW98], which should only allow a finite number of degrees of freedom in any given region in the bulk?

We believe the resolution is that at finite $N$ not all the operators defined in (3.1) commute at spacelike separation. To see this, consider a fixed-$T$ hypersurface in the bulk, with operators placed at some radial position $Z$. The smearing functions we have defined have an extent in the time direction $\Delta T = Z$. As shown in Fig. 3 the operators will be spacelike separated on the boundary provided the bulk operators have a spatial separation $|\Delta X| > Z$. In this case the operators are guaranteed to commute, just by locality of the boundary theory. At finite $N$ we do not expect operators separated by $|\Delta X| < Z$ to commute. (This is unlike the semiclassical situation discussed in section 2, where operators could commute even though they overlapped.) So we expect $1/Z^{d-1}$ commuting operators per coordinate area on the boundary. Using the AdS metric (2.1), this means we expect $1/R^{d-1}$ commuting operators per proper area in the bulk. Equivalently we expect one commuting operator per unit cell, where the cell volume $\sim R^{d-1}$. At finite $N$, bulk locality breaks down on distances set by the AdS radius of curvature!

This is a bit disturbing, in that it seems the commuting operators we can build from $\text{Tr} F^2$ aren’t sufficient to describe a local bulk dilaton on distances less than an AdS radius. We don’t have a complete resolution of this puzzle. It’s probably too strong a condition to require that all operators describing the dilaton commute exactly. Take the set of commuting operators built from $\text{Tr} F^2$. Perhaps there are

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1. Thus the smearing functions should have a purely group-theoretic interpretation in terms of representations of the (complexified) isometry group $SO(d, 2)$, along the lines of [VK91]. We are grateful to Djordje Minic for discussions on this point.

2. To avoid a possible confusion: by construction, the operators defined in (3.1) satisfy a free wave equation. So they do not have the right interactions to be identified with the bulk dilaton field at finite $N$. Nevertheless, they should serve as good probes of local physics in the bulk.

3. They also have an extent in the imaginary spatial directions, but as shown in [HKLL07] it’s only the extent in time that matters here.
additional operators which can be used to describe the dilaton, which do not all 
commute, but whose commutators are so small at low energies that they can be 
ignored. The operators built from $\text{Tr} F^2$ using our smearing functions are good 
first candidates for the job, since their commutators do vanish in the large-$N$ limit. 
But it could also be that operator mixing is important, so that operators besides 
$\text{Tr} F^2$ can contribute.

One might also wonder about the holographic bound. Here things work out 
very nicely. The holographic bound in AdS states that the entropy per longitudinal 
coordinate area is bounded by $s \leq N^2/Z^{d-1}$. Each operator in the CFT gives 
an entropy density $\sim 1/Z^{d-1}$, so bound is saturated if we have $N^2$ commuting 
operators in the CFT. This seems quite reasonable, as $N = 4$ Yang-Mills involves 
$N \times N$ matrices and has a central charge $\sim N^2$.

4. Semiclassical BTZ black hole

We now turn to excitations of AdS$_3$, in particular we will study non-extremal 
BTZ black holes. But first let’s consider AdS$_3$ in accelerating or Rindler-like coor-
dinates.

$$ds^2 = -\frac{r^2 - r_0^2}{R^2} dt^2 + \frac{R^2}{r^2 - r_0^2} dr^2 + r^2 d\phi^2$$

$-\infty < t, \phi < \infty$ \hspace{1cm} $0 < r < \infty$ (4.1)

Here $r_0$ is an arbitrary parameter with units of length. We’ll frequently work in 
terms of the rescaled coordinates

$$\hat{t} = r_0 t / R^2 \hspace{1cm} \hat{\phi} = r_0 \phi / R.$$ (4.1a)

It’s straightforward to construct smearing functions in these coordinates. The Wick 
rotation $\phi = iy$ turns the AdS$_3$ metric (4.1) into de Sitter space, now expressed in 
static coordinates.

$$ds^2 = -\frac{r^2 - r_0^2}{R^2} dt^2 + \frac{R^2}{r^2 - r_0^2} dr^2 - r^2 dy^2$$

$-\infty < t < \infty$ \hspace{1cm} $y \approx y + 2\pi R / r_0$ \hspace{1cm} $0 < r < \infty$ (4.2)

(The periodicity in $y$ is necessary to avoid a singularity at $r = 0$.) One can use a 
retarded de Sitter Green’s function to construct smearing functions in AdS$_3$. Alter-
atively, one can just translate our previous result (2.4) into Rindler coordinates, 
to find\footnote{More precisely, we expect matrix elements of the commutators between low-energy states 
 to be small.}

$$\phi(t, r, \phi) = \int dt' dy' K_{\text{Rindler}}(t + t', \phi + iy' | t, r, \phi) \phi_0(t + t', \phi + iy')$$

$$K_{\text{Rindler}}(t, r, \phi) = \frac{(\Delta - 1)^2 \Delta - 2}{\pi R^3} \lim_{r' \to \infty} \left( \frac{\sigma}{r'} \right)^{\Delta - 2}$$

where $\sigma$ is the invariant distance between $(t, r, \phi)$ and $(t + t', r', \phi + iy')$, and the 
integral is over spacelike separated points on the complexified AdS boundary. A 
BTZ black hole can be obtained from AdS$_3$ by identifying $\phi \approx \phi + 2\pi$. This produces

\footnote{We denoted this parameter $r_+$ in our previous work.}

\footnote{The boundary field $\phi_0$ is defined slightly differently in Rindler coordinates: $\phi(t, r, \phi) \sim 
\phi_0(t, \phi) / r^{\Delta}$ as $r \to \infty$.}
an orbifold singularity at $r = 0$. But making this identification doesn’t change the
smearing functions at all: if the boundary field has the necessary periodicity, so will the bulk field. So we can use the same smearing functions \([4.3]\) in a BTZ
background.

In the semiclassical limit this gives a clear picture of the BTZ horizon and
singularity. First, the horizon. The integral in \([4.3]\) is over points on the complex-
ified boundary that are spacelike separated from the bulk point. As the bulk point
approaches the (future, past) horizon the integration region extends to $(t = +\infty,$
$t = -\infty)$. Thus to probe the horizon requires an infinite time from the boundary
point of view. This fits nicely with the bulk definition of a horizon, as bounding
the region where light rays cannot escape to infinity.\footnote{One can also describe bulk points that are located inside the horizon \([HKLL07]\). However this requires the use of smearing functions with support on both the left and right boundaries of the extended Kruskal diagram. For a bulk point inside the future horizon the smearing function extends to $t = +\infty$ on the right boundary and $t = -\infty$ on the left boundary, where time has the same orientation on the left and right. From the boundary point of view this means we are using operators that act on both copies of the thermofield-doubled CFT.}

What about the BTZ singularity? With scalar fields as probes we cannot
directly study the bulk geometry. However it turns out that the orbifold singularity
generates a divergence in scalar correlators. To see this, we use the fact that in
the semiclassical limit we can make the bulk correlator periodic with an image sum
\([4.4]\).

\[
\langle \phi(t, r, \phi) \phi(t', r', \phi') \rangle_{\text{BTZ}} = \sum_{n=-\infty}^{\infty} \langle \phi(t, r, \phi) \phi(t', r', \phi' + 2\pi n) \rangle_{\text{AdS}}
\]

But $r = 0$ is a fixed point of the isometry $\phi \rightarrow \phi + \text{const.}$, so correlators in AdS are
$\phi$-independent at $r = 0$. If we compute a bulk correlator in a BTZ background, the
image sum diverges when one of the bulk points is located at the singularity.

The same divergence arises from the boundary point of view. The CFT dual
to $\text{AdS}_3$ in Rindler coordinates lives on $\mathbb{R}^{1,1}$, while the CFT dual to BTZ lives on
$\mathbb{R} \times S^1$. In the semiclassical limit the BTZ boundary correlator can be given the
necessary periodicity with an image sum.

\[
\langle \phi_0(t, \phi) \phi_0(t', \phi') \rangle_{\text{BTZ}} = \sum_{n=-\infty}^{\infty} \langle \phi_0(t, \phi) \phi_0(t', \phi' + 2\pi n) \rangle_{\text{AdS}}
\]

To recover a bulk correlator we convolve this with our smearing functions as in
\([2.5]\). Again the image sum diverges when one of the bulk points is located at the singularity. So we also get a divergent correlator at $r = 0$ from the boundary point
of view.

For future reference it’s useful to study the divergence in a little more detail. Consider a point in AdS located near $r = 0$, at

\[(t, r, \phi) \quad \text{with} \quad t = 0, r \rightarrow 0, \phi = 0\]

and a second point in AdS located near the boundary, at

\[(t', r', \phi') \quad \text{with} \quad r' \rightarrow \infty.\]

The invariant distance \([3.2]\) between these points is approximately

\[
\sigma \approx \frac{r'}{r_0} \left( \frac{r}{r_0} \cosh \hat{\phi}' + \sinh \hat{t}' \right).
\]
The AdS bulk correlation function \[ IS95 \] decays exponentially at large \( \phi' \).

\[
\langle \phi(t, r, \phi) \phi(t', r', \phi') \rangle_{\text{AdS}} = \frac{1}{4\pi R \sqrt{\sigma^2 - 1}} \frac{1}{(\sigma + \sqrt{\sigma^2 - 1})^{\Delta - 1}}
\]

\[
\approx \frac{(r_0/2r')^{\Delta}}{2\pi R \left( \frac{r_0}{r} \cosh \phi' + \sinh \phi' \right)^{\Delta}}
\]

\[
\sim e^{-\Delta|\phi'|} \quad \text{for } |\phi'| > \phi_{\text{max}} \sim \log(r_0/r)
\]

This behavior means the BTZ image sum (4.4) is cut off at \( |n| \sim \frac{R}{2\pi} \log \frac{r_0}{r} \), which in turn means the BTZ correlator diverges logarithmically as \( r \to 0 \).

5. BTZ at finite \( N \)

What happens to the BTZ black hole at finite \( N \)? Although definitive statements are hard to come by, there are a few interesting observations to make. First, there’s the issue of what smearing functions to use. In a pure AdS background we gave a symmetry argument that the same smearing functions should be used at any value of \( N \). For excited states such as BTZ, where the symmetries are broken, this argument is not valid.\(^8\) Nonetheless one might be tempted to use the same semiclassical smearing functions for BTZ even at finite \( N \). This is indeed a reasonable prescription for bulk points outside the horizon. However as the bulk point approaches the (future) horizon the semiclassical smearing functions extend to \( t = +\infty \) on the boundary, and for points inside the horizon they grow exponentially with time.\(^9\)

\[
K(\hat{t}, \phi) \sim e^{(\Delta - d)\hat{t}} \quad \text{as } t \to \infty
\]

In the semiclassical limit this causes no problems. CFT correlators at infinite \( N \) decay exponentially \[ BSS02 \], making the convolution of a boundary correlator with a smearing function well-defined.

\[
\langle \phi_0(\hat{t}, \phi) \phi_0(0, 0) \rangle_{\text{BTZ}} \sim e^{-\Delta \hat{t}}
\]

But when \( N \) is finite such behavior cannot persist indefinitely. At finite \( N \), the CFT on \( \mathbb{R} \times S^1 \) has a finite thermal entropy. Eventually the discrete spectrum of the CFT becomes important and causes correlators to oscillate quasi-periodically rather than decay exponentially \[ DLS02 \]. So we could not hope to use our semiclassical smearing functions to reproduce sensible bulk correlators. It seems there are a couple options.

1. Perhaps this supports the fuzzball picture of Mathur and collaborators \[ Mat05 \], in which the geometry inside the horizon differs radically from what one would expect based on the traditional black hole metric. In this

\(8\) This corrects a normalization error in \[ HKLL07 \].

\(9\) From the boundary point of view BTZ is related to a CFT on a Euclidean 2-torus, and the bulk-to-boundary map could depend on the modular parameter of the torus in a way that cannot be determined from modular invariance.

\(10\) This is the behavior on the right boundary. For points inside the future horizon the smearing function also extends to \( t = -\infty \) on the left boundary, and grows as \( e^{-(\Delta - d)\hat{t}} \) in the far past.
case one would need to know the exact microstate of the black hole to make sense of the interior of the horizon.

(2) Maybe there is some modification to the smearing functions which gives sensible bulk correlators even for points inside the horizon. “Sensible” means with small $1/N$ corrections to the semiclassical result, except near $r = 0$, where the divergence \( \frac{1}{\theta} \) should be smoothed out. This would support the picture that the semiclassical geometry is a good description, even inside the horizon, but with quantum gravity effects resolving the singularity.

(3) It could be that options 1 and 2 are compatible, if one is able to recover the semiclassical BTZ metric from the fuzzball picture by a suitable averaging procedure [AdBM06, BCL+07].

We conclude by presenting a prototype construction to show how option 3 might be realized. The difficulty is that for points inside the horizon the semiclassical smearing functions grow exponentially with time. A simple cure is to define a modified smearing function $\tilde{K}$ which vanishes if $\hat{t}$ is larger than some cutoff time $\hat{t}_{\text{max}}$:

$$\tilde{K}(\hat{t}, \hat{\phi}|\cdot) = \begin{cases} K(\hat{t}, \hat{\phi}|\cdot) & \text{if } |\hat{t}| < \hat{t}_{\text{max}} \\ 0 & \text{otherwise} \end{cases}$$

(We impose the cutoff on both the left and right boundaries of the black hole.) We can use this modified smearing function to define – purely within the CFT – a set of “bulk operators.”

$$\tilde{\phi}(t, r, \phi) \overset{\text{def}}{=} \int dt' d\phi' \tilde{K}(t', \phi'|t, r, \phi) \mathcal{O}(t', \phi')$$

With the cutoff in place, the operators $\tilde{\phi}$ have finite correlation functions. Imposing a cutoff in this way might seem very arbitrary. But in fact there is a good physical motivation for the cutoff which sets an upper bound on $\hat{t}_{\text{max}}$. With $n(E) = (\# \text{ CFT states with energy } < E) = e^{S(E)}$ the density of states

$$\frac{dn}{dE} = \beta e^S$$

implies a spacing between energy levels

$$\Delta E = \frac{1}{\beta e^S}.$$  \(5.2\)

This spacing corresponds to a time (the Heisenberg time [Sre99])

$$t_H = \beta e^S.$$  \(5.3\)

By this time CFT correlators begin to oscillate quasi-periodically rather than decay exponentially. [BR03, BR04]. So imposing a cutoff on the smearing functions at $t_{\text{max}} < t_H$ is the minimal change necessary to obtain well-defined bulk correlators. \(\text{\footnote{In general } t_{\text{max}} \text{ should be set by the time at which CFT correlators begin to behave quasi-periodically. This could occur before } t_H, \text{ so really } t_H \text{ is an upper bound on } t_{\text{max}}. \text{ The main point is that at finite } N \text{ the upper bound is finite. We are grateful to Hong Liu and Massimo Porrati for discussions on this topic.}}\)}$

Moreover, such a cutoff has a nice physical interpretation. From the CFT point of view measurements with a duration exceeding $t_H$ can resolve individual microstates.
of the CFT. So putting a cutoff at $t_{\text{max}} \lesssim t_H$ implies an average over microstates. This means the smearing functions we constructed based on the classical BTZ geometry break down at the horizon when $N$ is finite, unless one averages over microstates. This suggests that the region inside the horizon of the classical BTZ geometry isn’t a good description of any individual microstate of the black hole. Rather the region inside the horizon only provides a good description of ensemble averages over black hole microstates.

With a cutoff at $t_{\text{max}} \lesssim t_H$, $\tilde{K}$ represents the minimal modification to the smearing functions necessary to plausibly represent BTZ correlators at finite $N$. In the semiclassical limit we expect $t_{\text{max}} \to \infty$, so the bulk operators we have defined have the correct semiclassical limit.

Ideally at this point we would compute correlation functions of the operators $\tilde{\phi}$ in the CFT at finite $N$. Such a calculation might be within reach [Wit07]. But for the time being we will regard $t_{\text{max}}$ as a fixed ad hoc cutoff and study correlators of the operators $\phi$ in the large-$N$ limit of the CFT. Up to small $1/N$ corrections, this should be a good guide to behavior at finite $N$.

We first work in Rindler coordinates on AdS$_3$ and consider the correlation function

\begin{equation}
\langle \tilde{\phi}(t, r, \phi) \tilde{\phi}(t', r', \phi') \rangle_{\text{AdS}}
\end{equation}

between a bulk operator located at $(t = 0, r, \phi = 0)$ and an operator near the boundary at $(t', r', \phi')$ with $r' \to \infty$. In the semiclassical limit the boundary correlator

\begin{equation}
\langle \phi_0(\hat{t}, \hat{\phi}) \phi_0(0, 0) \rangle_{\text{AdS}} = \frac{(r_0^2/2)^{\Delta}}{2\pi R \left( \cosh \hat{\phi} - \cosh \hat{t} \right)^{\Delta}}
\end{equation}

decays exponentially at spacelike separation. So the bulk-boundary correlator (5.3) will be exponentially small provided the boundary point is spacelike separated from the support of the smearing function. As shown in Fig. 4 with the modified smearing functions the cutoff at $t_{\text{max}}$ means that – even for a bulk point inside the Rindler horizon – the correlator will decay exponentially at large $\phi'$.

To extend this discussion to BTZ we use the image charge construction (4.4). For points inside the horizon $\hat{t}_{\text{max}}$ serves to cut off the AdS correlator at $|\hat{\phi}'| \approx \hat{t}_{\text{max}}$. However as shown in (4.5) the AdS correlator is already exponentially small when $|\hat{\phi}'| > \hat{\phi}_{\text{max}}$. Thus when we perform the image sum there are two possible regimes.

1. For $\hat{t}_{\text{max}} > \hat{\phi}_{\text{max}}$, or equivalently for $r > r_0 e^{-\hat{t}_{\text{max}}}$, the additional cutoff at $\hat{t}_{\text{max}}$ isn’t important. So using $\tilde{K}$ rather than $K$ makes a negligible change to the BTZ correlator away from $r = 0$. We can therefore probe a large region inside the BTZ horizon, roughly the region

\[ r_0 e^{-\hat{t}_{\text{max}}} < r < r_0 , \]

using only a finite time interval on the boundary, and without seeing significant deviations from the semiclassical result. Strictly speaking this means there is no horizon, at least not in the sense of section 4 where the horizon corresponded to integration over infinite time.

2. However for $\hat{t}_{\text{max}} < \hat{\phi}_{\text{max}}$, or equivalently for

\[ 0 < r < r_0 e^{-\hat{t}_{\text{max}}} , \]
Figure 4. Penrose diagram of the \((t, \phi)\) plane. The support of \(\tilde{K}\) is indicated by the jagged line. Points in the shaded region are spacelike separated from the support of \(\tilde{K}\). When the smearing function extends from \(-\hat{t}_{\text{max}}\) to \(+\hat{t}_{\text{max}}\) the shaded region is characterized by \(|\hat{\phi}| > \hat{t}_{\text{max}} + |\hat{t}|\).

The additional cutoff at \(\hat{t}_{\text{max}}\) is crucial. It serves to regulate the image sum, cutting it off at \(|n| \approx \frac{R}{2\pi r_0} \hat{t}_{\text{max}}\) so that

\[
\langle \tilde{\phi}(t, r, \phi) \tilde{\phi}(t', r', \phi') \rangle_{\text{BTZ}} \approx \frac{\hat{t}_{\text{max}}}{2\pi^2 r_0} \left( \frac{r_0}{2r' \sinh \hat{t}'} \right) \Delta .
\]

Note that the correlator is independent of \(r\), and the divergence at \(r = 0\) has been eliminated!

We find it appealing that the same effect that eliminates the horizon also gets rid of the divergence. Note that the effects we have discussed are very robust: they are independent of any details of the CFT and only rely on the generic thermal behavior (5.2). Our results are compatible with the detailed study of extremal BTZ black holes in [B+07].

To summarize: we’ve defined a set of operators in the CFT (5.1) which should have well-defined correlation functions even at finite \(N\). As a guide to the behavior of these operators we studied their correlation functions in the large \(N\) limit. For bulk points well outside the horizon the cutoff at \(t_{\text{max}}\) has no effect on the smearing functions. For bulk points inside the horizon but well away from the singularity the cutoff at \(t_{\text{max}}\) makes an exponentially small change in correlators. But for points very near \(r = 0\) the cutoff at \(t_{\text{max}}\) becomes important and renders correlation functions finite. It seems reasonable that working with the true finite-\(N\) correlation functions of the CFT, rather than their semiclassical large-\(N\) limit, should only make a small change in these results. If so, this would support option 3: after a
suitable average over microstates, enforced by a cutoff at $t_{\max} \lesssim t_H$, the semiclassical BTZ geometry becomes a good description, even inside the horizon, but with quantum gravity effects resolving the singularity.

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