GLOBAL AND EXPONENTIAL ATTRACTORS FOR THE 3D KELVIN-VOIGT-BRINKMAN-FORCHHEIMER EQUATIONS

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ABSTRACT. The dynamics of three dimensional Kelvin-Voigt-Brinkman-Forchheimer equations in bounded domains is considered in this work. The existence and uniqueness of strong solution to the system is obtained by exploiting the $m$-accretive quantization of the linear and nonlinear operators. The long-term behavior of solutions of such systems is also examined in this work. We first establish the existence of an absorbing ball in appropriate spaces for the semigroup associated with the solutions of the 3D Kelvin-Voigt-Brinkman-Forchheimer equations. Then, we prove that the semigroup is asymptotically compact, which implies the existence of a global attractor for the system. Next, we show the differentiability of the semigroup with respect to the initial data and then establish that the global attractor has finite Hausdorff and fractal dimensions. Furthermore, we establish the existence of an exponential attractor and discuss about its fractal dimensions for the associated semigroup of such systems. Finally, we discuss about the inviscid limit of the 3D Kelvin-Voigt-Brinkman-Forchheimer equations to the 3D Navier-Stokes-Voigt system and then to the simplified Bardina model.

1. Introduction. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $\partial \Omega$. The three dimensional Kelvin-Voigt-Brinkman-Forchheimer equations are given by (see [2])
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \mu \frac{\partial \Delta u}{\partial t} + \nu \Delta u + (u \cdot \nabla)u + \alpha u + \beta |u|^{r-1}u + \nabla p &= f, \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot u &= 0, \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot u &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
\int_{\Omega} p(t, x)dx &= 0, \quad \text{for } t \geq 0,
\end{aligned}
\]
where $\alpha \in \mathbb{R}$, $\beta > 0$ and $r \in [1, \infty)$. Here $u(t, x) \in \mathbb{R}^3$ represents the velocity field at time $t$ and position $x$, $p(t, x) \in \mathbb{R}$ denotes the pressure field, $f(t, x) \in \mathbb{R}^3$ is an external forcing. The final condition in (1) is introduced for uniqueness of the pressure $p$. For $\mu = \alpha = \beta = 0$, we obtain the classical 3D Navier-Stokes equations (see [25, 33, 34, 13, 32, 18], etc) and $\alpha = \beta = 0$ gives the Navier-Stokes-Voigt...
equations (see [22, 37], etc). Note also that $\mu = 0$ provides the Navier-Stokes-Brinkman-Forchheimer equations (see [24] for Brinkman-Forchheimer equations). Furthermore, the case of $\alpha \neq 0$ and $\beta > 0$ can be considered as the Navier-Stokes-Voigt equations with damping, or the tamed Navier-Stokes-Voigt equations.

The authors in [2] considered the initial-boundary value problem for the Kelvin-Voigt-Brinkman-Forchheimer equations in 3D domains satisfying the Poincaré inequality. The existence and uniqueness of a weak solution to the system (1) is proved by using a Faedo-Galerkin technique. The authors also showed the existence of a unique pullback attractor for the process associated to the system with respect to a large class of nonautonomous forcing terms. The exponential behavior and stabilizability of the stochastic 3D Kelvin-Voigt-Brinkman-Forchheimer equations is investigated in [3]. The asymptotic behavior of solutions of the 3D Brinkman-Forchheimer equation is explored in [24, 36], etc. The authors in [22] examined the long-term dynamics of the 3D Navier-Stokes-Voigt model of viscoelastic incompressible fluid by establishing the existence of a global attractor. The long-term behavior of the same model by establishing the existence of an exponential attractor of optimal regularity and the behavior as the regularization parameter vanishes is investigated in [37]. The existence of a global as well as exponential attractor for the 3D Kelvin-Voigt fluid flow equations with “fading memory” is examined in [30]. The global attractors of 2D and 3D Navier-Stokes-Voigt equations and related problems are discussed in [9, 23], etc. In this work, we consider the three dimensional Kelvin-Voigt-Brinkman-Forchheimer equations (with $\alpha = 0$ and $r \in [1, 4]$) in bounded domains and examine the asymptotic behavior of solutions. The existence of global as well as exponential attractors for the semigroup associated with such systems is obtained. We also establish that such attractors have finite Hausdorff and fractal dimensions.

The organization of the paper is as follows. The three dimensional Kelvin-Voigt-Brinkman-Forchheimer equations with $r \in [1, 5]$ and $\alpha = 0$ is considered in section 2. The necessary functional settings needed to obtain the global solvability results of the system (1) is obtained in the same section. After taking Helmholtz-Hodge orthogonal projection and defining linear and nonlinear operators suitably, we obtain the abstract formulation of the system (1). The existence and uniqueness of a strong solution to the system (13) (see section 2 below) in a suitable function space is obtained by using the $m$-accretive quantization of the linear and nonlinear operators (Theorem 2.2). As we are not using the compactness arguments, the results obtained in this section hold true even in unbounded domains like Poincaré domains. The regularity results for the weak solution is also obtained in this section (Remark 3). The unique weak solution for the system (13) can be represented through a one parameter family of continuous semigroup (for the autonomous case) $S(t)$ as $u(t) = S(t)u_0$. In sections 3 and 4, we consider the system (1) with $r \in [1, 4]$, due to technical reasons. In section 3, we first establish the existence of an absorbing ball in $V_1^\mu$ for the semigroup $S(t), t \geq 0$ defined for the autonomous system corresponding to (13) (Proposition 1, see section 2 for the function spaces definition). Furthermore, we prove that $S(t), t \geq 0$ is an asymptotically compact semigroup in $V_1^\mu$ (Proposition 3), and hence we show the existence of a global attractor for the semigroup (Theorem 3.2). Moreover, we establish that $S(t)$ defined on $V_2^\mu$ has an absorbing ball in $V_2^\mu$ in section 3.3. The differentiability of the semigroup with respect to the initial data is established in section 4 (Theorem 4.1). Then, we prove that the global attractor for the 3D Kelvin-Voigt-Brinkman-Forchheimer system has
finite Hausdorff and fractal dimensions (Theorem 4.2). For \( r = 1, 2 \) and \( r \in [3, 4] \),
the existence of an exponential attractor in \( \mathcal{V}_u^\mu \) with finite fractal dimensions for
the semigroup \( S(t) \) associated with the autonomous system corresponding to (13)
is established in section 5 (Theorems 5.8 and 5.9). The inviscid limit of the 3D Kelvin-Voigt-Brinkman-Forchheimer equations as \( \beta \to 0 \) (to the 3D Navier-Stokes-Voigt system) as well as \( \nu \to 0 \) (to the simplified Bardina model) is discussed in
section 6 (Lemmas 6.1 and 6.2).

2. **Mathematical formulation.** The necessary function spaces needed to obtain
the global solvability results of the system (1) is provided in this section. For
simplicity, we take \( \alpha = 0 \) and we restrict ourselves to \( r \in [1, 5] \). Note that global
solvability results for the system (1) is known in the literature for any \( r \in [1, \infty) \)
(cf. Theorem 3.1, [2]). After giving an abstract formulation of the system (1), we
discuss the well-posedness by making use of the quasi-\( m \)-accretive property of the
linear and nonlinear operators.

2.1. **Function spaces.** Let \( C_0^\infty(\Omega; \mathbb{R}^3) \) be the space of all infinitely differentiable
functions (\( \mathbb{R}^3 \)-valued) with compact support in \( \Omega \subset \mathbb{R}^3 \). Let us denote \( \mathcal{V} := \{ u \in C_0^\infty(\Omega, \mathbb{R}^3) : \nabla \cdot u = 0 \} \). Let \( \mathbb{H} \) and \( \mathcal{V} \) denote the completion of \( \mathcal{V} \) in \( L^2(\Omega; \mathbb{R}^3) \)
and \( H^1(\Omega; \mathbb{R}^3) \) norms, respectively. Then under some smoothness assumptions on
the boundary, we characterize the spaces \( \mathbb{H} \) and \( \mathcal{V} \) as \( \mathbb{H} := \{ u \in L^2(\Omega) : \nabla \cdot u = 0, u \cdot n|_{\partial \Omega} = 0 \} \), with norm \( \| u \|_{\mathbb{H}} := \int_\Omega |u(x)|^2 \, dx \), where \( n \) is the outward normal
to \( \partial \Omega \) and \( \mathcal{V} := \{ u \in H^1_0(\Omega) : \nabla \cdot u = 0 \} \), with norm \( \| u \|^2_{\mathcal{V}} := \int_{\Omega} |\nabla u(x)|^2 \, dx \). Let \( (\cdot, \cdot) \) denotes the inner product in the Hilbert space \( \mathbb{H} \) and \( (\cdot, \cdot) \) denotes the induced
duality between the spaces \( \mathcal{V} \) and its dual \( \mathcal{V}' \). Note that \( \mathbb{H} \) can be identified with
its dual \( \mathbb{H}' \). Since \( \mathcal{V} \) is densely and continuously embedded into \( \mathbb{H} \), we obtain the
following **Gelfand triple:** \( \mathcal{V} \subset \mathbb{H} \equiv \mathbb{H}' \subset \mathcal{V}' \). For bounded domains, the embedding of
\( \mathcal{V} \subset \mathbb{H} \) is compact. Moreover, \( H^2(\Omega) := H^2(\Omega; \mathbb{R}^3) \) denotes the second order Sobolev spaces.

2.2. **Linear operator.** Let \( \text{P}_\mathbb{H} : L^2(\Omega) \to \mathbb{H} \) denotes the **Helmholtz-Hodge orthog-
onal projection** (see [25, 10]). Let us define
\[
\begin{cases}
A u := -P_\mathbb{H} \Delta u, & u \in D(A), \\
D(A) := \mathcal{V} \cap H^2(\Omega).
\end{cases}
\]
It can be easily seen that the operator \( A \) is a non-negative self-adjoint operator in
\( \mathbb{H} \) with \( \mathcal{V} = D(A^{1/2}) \) and
\[
( Au, u ) = \| u \|^2_{\mathcal{V}}, \quad \text{for all } u \in \mathcal{V}, \quad \text{so that } \| Au \|_{\mathcal{V}} \leq \| u \|_{\mathcal{V}}. \tag{2}
\]
For a bounded domain \( \Omega \), the operator \( A \) is invertible and its inverse \( A^{-1} \) is bounded,
self-adjoint and compact in \( \mathbb{H} \). Thus, the spectrum of \( A \) consists of an infinite sequence
\( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots \), with \( \lambda_k \to \infty \) as \( k \to \infty \) of eigenvalues.
The behavior of these eigenvalues is well known in the literature (for example see
Theorem 2.2, Corollary 2.2, [20] and for asymptotic behavior, see [13, 34, 21], etc).
For all \( k \geq 1 \) and \( n \in \mathbb{N} \), we have
\[
\lambda_k \geq \widetilde{C} k^{2/n}, \quad \text{where } \widetilde{C} = \frac{n}{2} \left( \frac{(2\pi)^n}{\omega_n (n-1)|\Omega|} \right)^{2/n}, \quad \omega_n = \pi^{n/2} \Gamma(1+n/2), \tag{3}
\]
and \( |\Omega| \) is the \( n \)-dimensional Lebesgue measure of \( \Omega \). For \( n = 3 \), we find \( \widetilde{C} = \frac{5^{5/3} \pi^{1/3} k^{2/3}}{3^{3/5} \pi^{1/5} k^{2/3}} \). Moreover, there exists an orthogonal basis \( \{ e_k \}_{k=1}^\infty \) of \( \mathbb{H} \) consisting of
eigenvectors of $A$ such that $Ae_k = \lambda_k e_k$, for all $k \in \mathbb{N}$. We know that $u$ can be expressed as $u = \sum_{k=1}^{\infty} (u, e_k) e_k$ and $Au = \sum_{k=1}^{\infty} \lambda_k (u, e_k) e_k$. Thus, it is immediate that
\[
\|\nabla u\|^2_{L^2} = \langle Au, u \rangle = \sum_{k=1}^{\infty} \lambda_k |(u, e_k)|^2 \geq \lambda_1 \sum_{k=1}^{\infty} |(u, e_k)|^2 = \lambda_1 \|u\|^2_{H^1}.
\] (4)

In this paper, we also need to define the fractional powers of $A$. For $u \in \mathbb{H}$ and $\alpha > 0$, we define $A^\alpha u = \sum_{k=1}^{\infty} \lambda_k^\alpha (u, e_k) e_k$, $u \in D(A^\alpha)$, where the domain of $A^\alpha$ is given by $D(A^\alpha) = \{ u \in \mathbb{H} : \sum_{k=1}^{\infty} \lambda_k^\alpha |(u, e_k)|^2 < +\infty \}$. Here $D(A^\alpha)$ is equipped with the norm
\[
\|A^\alpha u\|_{\mathbb{H}^\alpha} = \left( \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |(u, e_k)|^2 \right)^{1/2}.
\] (5)

It can be easily seen that $D(A^0) = \mathbb{H}$, $D(A^{1/2}) = \mathcal{V}$. We set $\mathcal{V}_\alpha = D(A^{\alpha/2})$ with $\|u\|_{\mathcal{V}_\alpha} = \|A^{\alpha/2} u\|_{\mathbb{H}}$. Using Rellich-Kondrachov compactness embedding theorem, we know that for any $0 \leq s_1 < s_2$, the embedding $D(A^{s_2}) \subset D(A^{s_1})$ is also compact. Applying Hölder’s inequality on the expression (5), one can obtain the following interpolation estimate:
\[
\|A^s u\|_{\mathbb{H}} \leq \|A^{s_1} u\|_{\mathbb{H}}^\theta \|A^{s_2} u\|_{\mathbb{H}}^{1-\theta},
\] (6)

for any real $s_1 \leq s \leq s_2$ and $\theta$ is given by $s = s_1(1-\theta) + s_2 \theta$. Let us denote by $D(A^{-\alpha})$, the dual space of $D(A^\alpha)$ and we have the following dense and continuous inclusions, for $\alpha > 1/2$,
\[
D(A^\alpha) \subset \mathcal{V} \subset \mathbb{H} \equiv \mathbb{H}' \subset \mathcal{V}' \subset D(A^{-\alpha}).
\]

For negative powers, we also define $(u, v)_{\mathcal{V}' - \alpha} = (A^{-\alpha/2} u, A^{-\alpha/2} v)$ and $\|u\|_{\mathcal{V}' - \alpha} = \|A^{-\alpha/2} u\|_{\mathbb{H}}$.

**Remark 1.** 1. Note that $\mathcal{V}_s$ is an algebra for $s > 3/2$, i.e., if $u, v \in \mathcal{V}_s$, then $uv \in \mathcal{V}_s$ and $\|uv\|_{\mathcal{V}_s} \leq \|u\|_{\mathcal{V}_s} \|v\|_{\mathcal{V}_s}$.

2. The space $\mathcal{V}_s$ can be continuously embedded in $L^\infty(\Omega)$, for $s > 3/2$, i.e., $\|u\|_{L^\infty} \leq C\|u\|_{\mathcal{V}_s}$, for $u \in \mathcal{V}_s$, $s > 3/2$.

3. One can show that $\|\nabla u\|_{\mathcal{V}} = \|A^{1/2} u\|_{\mathbb{H}} = \|\nabla u\|_{\mathbb{H}}$ and $\|(I + \mu A)^{1/2} u\|_{\mathbb{H}}$ are equivalent norms in $\mathcal{V}$. It can be easily seen that
\[
\|(I + \mu A)^{1/2} u\|^2_{\mathbb{H}} = \|(I + \mu A) u, u\| = \|u\|^2_{\mathbb{H}} + \mu \|\nabla u\|^2_{\mathbb{H}} \leq \left( \frac{1}{\lambda_1} + \mu \right) \|u\|^2_{\mathcal{V}},
\] (7)

using the Poincaré inequality. From (7), it is also clear that
\[
\mu \|u\|^2_{\mathcal{V}} \leq \|u\|^2_{\mathbb{H}} + \mu \|u\|^2_{\mathcal{V}} = \|(I + \mu A)^{1/2} u\|^2_{\mathbb{H}}
\] (8)

We combine (7) and (8) to obtain the required result.

2.3. **Nonlinear operator.** Let us define the trilinear form $b(\cdot, \cdot, \cdot) : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ by
\[
b(u, v, w) = \int_\Omega (u(x) \cdot \nabla) v(x) \cdot w(x) dx = \sum_{i,j=1}^{3} \int_\Omega u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x) dx.
\]
If \( \mathbf{u}, \mathbf{v} \) are such that the linear map \( b(\mathbf{u}, \mathbf{v}, \cdot) \) is continuous on \( \mathcal{V} \), the corresponding element of \( \mathcal{V}' \) is denoted by \( B(\mathbf{u}, \mathbf{v}) \). We also denote (with an abuse of notation) \( B(\mathbf{u}) = B(\mathbf{u}, \mathbf{u}) = P_{B}(\mathbf{u} \cdot \nabla) \mathbf{u} \). An integration by parts gives

\[
\begin{cases}
    b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \text{ for all } \mathbf{u}, \mathbf{v} \in \mathcal{V}, \\
    b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}), \text{ for all } \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}.
\end{cases}
\tag{9}
\]

In the trilinear form, an application of H"{o}lder's inequality yields

\[
|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| = |b(\mathbf{u}, \mathbf{w}, \mathbf{v})| \leq ||\mathbf{u}||_{L^{1}}||\nabla \mathbf{w}||_{H^{1}}||\mathbf{v}||_{L^{1}}, \text{ for all } \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V},
\]

and thus \( ||B(\mathbf{u}, \mathbf{v})||_{\mathcal{V}'} \leq ||\mathbf{u}||_{L^{4}} ||\mathbf{v}||_{L^{4}} \). Hence, the trilinear map \( b : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R} \) has a unique extension to a bounded trilinear map from \( L^{4}(\Omega) \times (L^{4}(\Omega) \cap \mathbb{H}) \times \mathcal{V} \) to \( \mathbb{R} \). It can also be seen that \( B \) maps \( L^{4}(\Omega) \cap \mathbb{H} \) (and so \( \mathcal{V} \)) into \( \mathcal{V}' \) and

\[
|b(\mathbf{u}, \mathbf{v})| = |b(\mathbf{u}, \mathbf{v}, \mathbf{u})| \leq ||\mathbf{u}||_{L^{4}}^{2} ||\nabla \mathbf{v}||_{H^{1}} \leq 2 ||\mathbf{u}||_{H^{1}}^{1/2} ||\nabla \mathbf{u}||_{H^{1}}^{3/2} ||\mathbf{v}||_{\mathcal{V}}, \tag{10}
\]

for all \( \mathbf{v} \in \mathcal{V} \), so that

\[
||B(\mathbf{u})||_{\mathcal{V}'} \leq 2 ||\mathbf{u}||_{H^{1}}^{1/2} ||\nabla \mathbf{u}||_{H^{1}}^{3/2} \leq \frac{2}{\lambda_{1}^{1/4}} ||\mathbf{u}||_{\mathcal{V}}^{2}, \text{ for all } \mathbf{u} \in \mathcal{V}, \tag{11}
\]

using the Poincaré inequality (see (4)). Note that \( B \) also maps \( L^{4}(\Omega) \cap \mathbb{H} \) into \( \mathcal{V}' \) and

\[
|b(\mathbf{u}, \mathbf{v}, \mathbf{u})| = |b(\mathbf{u}, \mathbf{v}, \mathbf{u})| \leq ||\mathbf{u}||_{L^{2}} ||\mathbf{u}||_{L^{6}} ||\nabla \mathbf{v}||_{H^{1}} \leq C ||\mathbf{u}||_{L^{6}}^{2} ||\mathbf{v}||_{\mathcal{V}},
\]

so that once again, we get (11). Using (11), we also have

\[
||B(\mathbf{u}) - B(\mathbf{v})||_{\mathcal{V}'} \leq ||B(\mathbf{u}, \mathbf{u} - \mathbf{v})||_{\mathcal{V}'} + ||B(\mathbf{v}, \mathbf{u} - \mathbf{v})||_{\mathcal{V}'} \leq \frac{2}{\lambda_{1}^{1/4}} ((||\mathbf{u}||_{\mathcal{V}} + ||\mathbf{v}||_{\mathcal{V}})||\mathbf{u} - \mathbf{v}||_{\mathcal{V}}, \tag{12}
\]

and hence \( B(\cdot) \) is a locally Lipschitz operator.

2.4. Abstract formulation and weak solution. We take the Helmholtz-Hodge orthogonal projection \( P_{\mathbb{H}} \) in (1) to obtain the abstract formulation for \( t \in (0, T) \) as:

\[
\begin{cases}
    \frac{d}{dt}[(I + \mu A)\mathbf{u}(t)] + \nu \mathbf{u}(t) + B(\mathbf{u}(t), \mathbf{u}(t)) + \beta P_{\mathbb{H}}(|\mathbf{u}(t)|^{-1} \mathbf{u}(t)) = \mathbf{f}(t), \\
    \mathbf{u}(0) = \mathbf{u}_{0} \in \mathcal{V},
\end{cases}
\tag{13}
\]

where \( \mathbf{f} \in L^{2}(0, T; \mathcal{V}') \). Strictly speaking one should use \( P_{\mathbb{H}} \mathbf{f} \) instead of \( \mathbf{f} \), for simplicity, we use \( \mathbf{f} \). The system (13) is also equivalent to the following system for \( t \in (0, T) \):

\[
\begin{cases}
    \frac{d}{dt}[\mu \mathbf{u}(t)] + (I + \mu A)^{-1}[\nu \mathbf{u}(t) + B(\mathbf{u}(t), \mathbf{u}(t)) + \beta P_{\mathbb{H}}(|\mathbf{u}(t)|^{-1} \mathbf{u}(t))]
    = (I + \mu A)^{-1} \mathbf{f}(t), \\
    \mathbf{u}(0) = \mathbf{u}_{0} \in \mathcal{V}.
\end{cases}
\tag{14}
\]

Let us now give the definition of weak solution of the system (13).
Definition 2.1. A function $u \in C([0, T]; \mathbb{V})$ with $\partial_t u \in L^2(0, T; \mathbb{V})$, is called a weak solution to the system (13), if for $f \in L^2(0, T; \mathbb{V}')$, $u_0 \in \mathbb{V}$ and $v \in \mathbb{V}$, $u(\cdot)$ satisfies:

$$\begin{cases}
\langle \partial_t [I + \mu A]u(t) \rangle + \nu Au(t) + B(u(t)) + \beta P_B(|u(t)|^{r-1}u(t)), v \rangle = \langle f(t), v \rangle, \\
\lim_{t \to 0} \int_\Omega u(t)v dx = \int_\Omega u_0 v dx,
\end{cases}$$

and the energy equality:

$$\frac{1}{2} \frac{d}{dt}(\|u(t)\|^2 + \mu \|u(t)\|_X^2) + \nu \|u(t)\|_X^2 + \beta \|u(t)\|_{L^r}^{r+1} = \langle f(t), u(t) \rangle. \quad (16)$$

Let us now state a result on existence and uniqueness of weak solution to the system (13). A proof using the Faedo-Galerkin method can be obtained from Theorem 3.1, [2].

Theorem 2.2 (Theorem 3.1, [2]). There exists a unique weak solution $u(\cdot)$ to the system (13) in the sense of Definition 2.1.

Let us now provide the definition of accretive operators and obtain the global solvability results of the system (13) using quasi-$m$-accretive property of linear and nonlinear operators.

Definition 2.3 (Chapter 2, [4]). Let $X$ be a real Hilbert space with inner product $(\cdot, \cdot)_X$. An operator $F: D \subset X \to X$, is said to be

(i) **accretive** if $(F(x) - F(y), x - y)_X \geq 0$, for all $x, y \in D$,
(ii) **strongly accretive** if $(F(x) - F(y), x - y)_X \geq C \|x - y\|_X^2$, for some $C > 0$ and all $x, y \in D$,
(iii) **maximal accretive** if there is no accretive operator that properly contains it, i.e., if for $x \in X$ and $w \in X$, the inequality $(w - F(x), x - y)_X \geq 0$, for all $y \in X$ implies $w = F(x)$ (F is dissipative if $-F$ is accretive),
(iv) **$m$-accretive** (or hypermaximal accretive) if it is accretive and $R(I + F) = X$ or equivalently $R(I + F \varphi) = X$, for all $\varphi > 0$,
(v) **$\omega$-accretive** (respectively $\omega$-$m$-accretive), where $\omega \in \mathbb{R}$, if $F + \omega I$ is accretive (respectively $m$-accretive),
(vi) **quasi-accretive** (respectively quasi-$m$-accretive) if $F$ is $\omega$-accretive (respectively $\omega$-$m$-accretive), for some $\omega \in \mathbb{R}$,
(vii) **demicontinuous** if for all $x \in D$, the functional $x \mapsto (F(x), y)_X$ is continuous, or in other words, $x_k \to x$ in $X$ implies $F(x_k) \xrightarrow{w} F(x)$ in $X$.

It should be noted that in Hilbert spaces, ‘accretive’ is also known as ‘monotone’. It was established in [27] that in Hilbert spaces, the notions of ‘maximal accretive’ and ‘$m$-accretive’ are equivalent. Note also that every demicontinuous monotone operator with dense domain is maximal accretive (see [7]). Interested readers are referred to see [4, 5], etc for more details on accretive operators.

Let us now present a proof of the existence of a strong solution to the system (13) by exploiting the $m$-accretive quantization of the linear and nonlinear operators. This method also avoids the tedious Galerkin approximation scheme used in Theorem 3.1, [2]. We apply the similar techniques used in [6] for the Navier-Stokes equations as well as in [29] for the Kelvin-Voigt fluid flow equations to obtain the well-posedness of the system (13).

Theorem 2.4. Let us assume that $u_0 \in \mathbb{V}$ and $f \in W^{1,1}(0, T; D(A))$. Then, there exists a unique strong solution $u(\cdot)$ to the system (13) such that $u \in W^{1,\infty}([0, T]; \mathbb{V})$. \hfill \Box
Proof. We make use of the $m$-accretive quantization of the linear and nonlinear operators to get the required result. Since, the norms $\|u\|_V$ and $\|(I + \mu A)^{1/2}u\|_H$ are equivalent (see Remark 1), we can define an another inner product on $V$ as

$$
(\langle u, v \rangle)_V = \langle (I + \mu A)^{1/2}u, (I + \mu A)^{1/2}v \rangle = (u, v) + \mu(u, v),
$$

for all $u, v \in V$.

Let us first define the quantized nonlinearity $\tilde{B}_N(\cdot) : V \to V'$ as

$$
\tilde{B}_N(u) := \left\{ \begin{array}{ll}
B(u) & \text{if } \|u\|_V \leq N, \\
\left( \frac{\mu}{\|u\|_V} \right)^2 B(u) & \text{if } \|u\|_V > N.
\end{array} \right.
$$

We also define the following quantized nonlinearity $B_N(\cdot) : V \to V$:

$$
B_N(u) := \left\{ \begin{array}{ll}
(I + \mu A)^{-1}B(u) & \text{if } \|u\|_V \leq N, \\
\left( \frac{\mu}{\|u\|_V} \right)^2 (I + \mu A)^{-1}B(u) & \text{if } \|u\|_V > N,
\end{array} \right.
$$

that is, $B_N(u) = (I + \mu A)^{-1}B_N(u)$. Let us define the operator $\Gamma_N : V \to V$ by

$$
\Gamma_N = \nu(I + \mu A)^{-1}A + \beta(I + \mu A)^{-1}P_H(|u|^{r-1}u) + B_N, \quad \text{with } D(\Gamma_N) = V.
$$

We obtain the required result in the following steps.

**Step (1).** $\Gamma_N$ defines an operator on $V$, that is, $\Gamma_N : V \to V$. It can be easily seen that

$$
\|(I + \mu A)^{-1}A u\|_V^2 = \|(I + \mu A)^{-1}A^{3/2}u\|_H^2 = \sum_{j=1}^{\infty} \frac{\lambda_j^3}{(1 + \mu \lambda_j)^2} |(u, e_j)|^2 \\
\leq \frac{1}{\mu^2} \sum_{j=1}^{\infty} \lambda_j |(u, e_j)|^2 \leq \frac{1}{\mu^2} \|u\|_V^2.
$$

Next, we show that $B_N(\cdot) : V \to V$. For $\|u\|_V \leq N$ and all $v \in V$, using Hölder’s, Poincaré and Ladyzhenskaya inequalities, we get

$$
\|(B_N(u), v)\|_V = \|(I + \mu A)^{-1}B(u), v\|_V = \|(I + \mu A)^{-1/2}B(u), (I + \mu A)^{1/2}v\|_V \\
= \langle B(u), v \rangle = -\langle B(u), v \rangle \leq \|u\|_V^2 \|v\|_V \leq \frac{1}{\lambda_1^{1/4}} \|u\|_V^2 \|v\|_V.
$$

Since $v \in V$ is arbitrary, we obtain $\|B_N(u)\|_V \leq \frac{2}{\lambda_1^{1/4}} \|u\|_V^2$ and hence $B_N(\cdot) : V \to V$, if $\|u\|_V \leq N$. A similar calculation holds for $\|u\|_V > N$ also. Moreover, using Hölder’s and Gagliardo-Nirenberg inequalities (see (A.6)), we get

$$
\beta\langle (I + \mu A)^{-1}P_H(|u|^{r-1}u), v \rangle_V = \beta\langle P_H(|u|^{r-1}u), v \rangle_V \leq \beta \|u\|_L^{r-1} \|v\|_L^{r+1} \leq \beta \|u\|_L^{r+1} \|v\|_L^{r+1} \leq \frac{C}{\lambda_1^r} \beta \|u\|_V^r \|v\|_V,
$$

for all $u, v \in V$ and $r \in [1, 5]$. Thus, the operator $\beta(I + \mu A)^{-1}P_H(|u|^{r-1}) : V \to V$, for $r \in [1, 5]$. Furthermore, comparing second and final inequalities, we have

$$
\|P_H(|u|^{r-1})\|_V \leq \frac{C}{\lambda_1^r} \|u\|_V^r.
$$

**Step (2).** $\Gamma_N + \lambda I$ is strongly accretive, for some $\lambda > 0$. First note that

$$
\langle (\nu(I + \mu A)^{-1}A(u - v), u - v) \rangle_V = \nu \langle A(u - v), u - v \rangle = \nu \|u - v\|_V^2.
$$
Let us now consider the operator $B_N(\cdot)$. Without loss of generality, we assume that $\|v\|_V \geq \|u\|_V$. Let us take $\|u\|_V, \|v\|_V \leq N$. Then using (9), (10), Ladyzhenskaya and Young’s inequalities, we estimate $(B_N(u) - B_N(v), u - v)_V$ as
\[
|(B_N(u) - B_N(v), u - v)_V| = |(B(u - v, v), u - v)| \leq \|u - v\|_L^2 \|\nabla v\|_H \\
\leq 2\|u - v\|_{L^4} \|u - v\|_{H^1/2} \|v\|_V \leq \frac{\nu}{2} \|u - v\|_V^2 + \frac{27N^4}{2\nu^3} \|u - v\|_H^2 \\
\leq \frac{\nu}{2} \|u - v\|_V^2 + \frac{27N^4}{2\nu^3} \|u - v\|_V^2.
\]

Next, we consider the case, $\|u\|_V, \|u\|_V > N$. Using Hölder’s, Ladyzhenskaya, Poincaré and Young’s inequalities, we estimate $(B_N(u) - B_N(v), u - v)_V$ as
\[
|(B_N(u) - B_N(v), u - v)_V| = \left| \left( \frac{N}{\|v\|_V} \right)^2 B(u) - \left( \frac{N}{\|v\|_V} \right)^2 B(v), u - v \right| \leq N^2 \|v\|_V + \|u\|_V \|B(u) - B(v), u - v\| \leq \frac{N^2 \|u\|_V^2}{\|v\|_V} \|B(u) - B(v), u - v\| \\
\leq \frac{2N^2 \|v\|_V + \|u\|_V^2}{\lambda_1^{1/4} \|v\|_V^2} \|u - v\|_V^2 + \frac{27N^4}{\lambda_1^{1/4} \|v\|_V^2} \|u - v\|_{H^1/2}^2 \\
\leq \frac{4N}{\lambda_1^{1/4} \|v\|_V} \|u - v\|_V^2 + \left[ \frac{1}{2\nu^2} \left( \frac{14}{\nu} \right)^7 N^8 + \left( \frac{12}{\nu^3} \right)^3 N^4 \right] \|u - v\|_H^2 \\
\leq \frac{\nu}{2} \|u - v\|_V^2 + N^4 \left[ \frac{1}{2\nu^2} \left( \frac{14}{\nu} \right)^7 N^8 + \left( \frac{12}{\nu^3} \right)^3 \right] \|u - v\|_V^2.
\]

For $\|u\|_V > N$ and $\|v\|_V \leq N$, we estimate $(B_N(u) - B_N(v), u - v)_V$ as
\[
|(B_N(u) - B_N(v), u - v)_V| = \left| \left( \frac{N}{\|u\|_V} \right)^2 B(u) - B(v), u - v \right| \leq \left( \frac{N}{\|u\|_V} \right)^2 \|B(u) - B(v), u - v\| \leq \left( \frac{N}{\|u\|_V} \right)^2 \|B(u - v, u) - B(u, u - v)\| + \frac{\|u\|_V^2 - N^2}{\|v\|_V^2} \|B(u - v, u)\| \\
\leq \frac{N^2}{\|u\|_V^2} \|u - v\|_{L^4} + \frac{\|u\|_V^2 - \|v\|_V^2}{\|u\|_V^2} \|v\|_V \|u - v\|_{L^4}.
Combining all the above cases, we have
\[ \Gamma \in \mathbb{W} \]
where we used (7). Let us now consider (B
\[ \tau \]
for some N.

Since the function \( f(x) = x|x^{r-1} \) is differentiable with derivative \( f'(x) = r|x^{r-1} \), one can use Taylor’s formula to obtain
\[ u|u|^{r-1} = v|v|^{r-1} + r(u - v)\theta u + (1 - \theta)v|v|^{r-1}, \]
for some 0 < \( \theta < 1 \). Finally, for any \( r \in [1, \infty) \), using (23), we have
\[ \beta((I + \mu A)^{-1}P_B(u|u|^{r-1}) - (I + \mu A)^{-1}P_B(v|v|^{r-1}), u - v)_V \]
\[ = \beta(P_B(u|u|^{r-1}) - P_B(v|v|^{r-1}), u - v)_V \]
\[ = \beta(r(u - v)\theta u + (1 - \theta)v|v|^{r-1}, u - v)_V \]
\[ = \beta r(\|\theta u + (1 - \theta)v|v|^{r-1}, |u - v|^2) \geq 0, \]
for all \( u, v \in L^r(\Omega) \cap V \).

Combining (21)-(24), we infer that \( \Gamma \) satisfies
\[ ((\Gamma + \lambda I)u - (\Gamma + \lambda I)v, u - v)_V \geq \frac{\nu}{2}\|u - v\|_V^2, \]
for all \( u, v \in \mathbb{D}(\Gamma + \lambda I) \).

and \( \lambda \geq C_N \), and hence \( \Gamma + \lambda I \) is an accretive operator.

**Step (3).** \( \Gamma \) is demicontinuous. Let \( \{u_n\} \) be a sequence in \( V \) such that \( u_n \to u \) in \( V \). In order to show \( \Gamma \) is demicontinuous, we need to prove that \( ((\Gamma + \lambda I)u_n - (\Gamma + \lambda I)u, v)_V \to 0 \) as \( n \to \infty \). Let us consider
\[ ((\Gamma + \lambda I)u_n - (\Gamma + \lambda I)u, v)_V \]
\[ = \nu((I + \mu A)^{-1}A(u_n - u), v)_V + ((B_N(u_n) - B_N(u), v)_V \]
\[ + ((I + \mu A)^{-1}(P_B|u|^{r-1}u_n) - (I + \mu A)^{-1}(P_B|u|^{r-1}u), v)_V \]
\[ + \lambda((I + \mu A)^{1/2}(u^n - u), (I + \mu A)^{1/2}v)_V \]
\[ \leq \nu((I + \mu A)^{-1}A(u_n - u), v)_V + ((B_N(u_n) - B_N(u), v)_V \]
\[ + \beta(P_B(u^n|u^n|^{r-1}) - P_B(u|u|^{r-1}), v)_V \]
\[ + \lambda((I + \mu A)^{1/2}(u^n - u), (I + \mu A)^{1/2}v)_V \]
\[ \leq \left[ \nu + \lambda \left( \frac{\mu + \frac{1}{\lambda^2}}{2} \right) \right] \|u_n - u\|_V \|v\|_V + ((B_N(u_n) - B_N(u), v)_V \]
\[ + \beta(P_B(u^n|u^n|^{r-1}) - P_B(u|u|^{r-1}), v)_V, \]
where we used (7). Let us now consider \( ((B_N(u_n) - B_N(u), v)_V \) and estimate it using the definition of \( B_N(\cdot) \) by considering the following different cases.

**Case (1).** \( \|u\|_V, \|u_n\|_V \leq N \). Using Cauchy-Schwarz, Ladyzhenskaya and Poincaré inequalities, we estimate \( ((B_N(u_n) - B_N(u), v)_V \) as
\[ ((B_N(u_n) - B_N(u), v)_V \]
\[
\langle B(u_n) - B(u), v \rangle = \langle B(u_n - u, u_n), v \rangle + \langle B(u, u_n - u), v \rangle \\
= -\langle B(u_n - u, u_n), v \rangle - \langle B(u, v), u_n - u \rangle \\
\leq \|u_n - u\|_L^4\|u_n\|_L^4\|v\| + \|u\|_L^4\|u_n - u\|_L^4\|v\| \\
\leq 2\|u_n - v\|^{1/4}\|u_n - u\|^{3/4}\left[\|u_n\|_L^{1/4}\|u_n\|_V^{3/4} + \|u\|_L^{1/4}\|u\|_V^{3/4}\right]\|v\| \\
\leq \frac{2}{\lambda_1^{1/4}}\|u_n - u\|\|v\|\|u_n\|_V + \|u\|_V\|u\|_V\|v\| \\
\leq \frac{4N}{\lambda_1^{1/4}}\|u_n - u\|_V\|v\|_V. \tag{27}
\]

**Case (2).** \(\|u_n\|_V \leq N, \|u\|_V > N\). Using (11) and (27), we estimate \(\langle (B_N(u_n) - B_N(u), v)\rangle_V\) as
\[
\langle (B_N(u_n) - B_N(u), v)\rangle_V \\
= \left\langle B(u_n) - \left(\frac{N}{\|u\|_V}\right)^2 B(u), v \right\rangle \\
= \left\langle B(u_n) - B(u) + \left[1 - \left(\frac{N}{\|u\|_V}\right)^2\right] B(u), v \right\rangle \\
= \langle B(u_n - u, u_n), v \rangle + \langle B(u, u_n - u), v \rangle + \frac{1}{\|u\|_V} (\|u\|_V^2 - N^2) \langle B(u), v \rangle \\
\leq \frac{2}{\lambda_1^{1/4}}(\|u_n\|_V + \|u\|_V)\|u_n - u\|_V\|v\| + \frac{1}{\|u\|_V} (\|u\|_V - N)(\|u\|_V + N)\|u\|_V^2\|v\|_V \\
\leq \frac{4N}{\lambda_1^{1/4}}(\|u\|_V)\|u_n - u\|_V\|v\|_V. \tag{28}
\]

**Case (3).** \(\|u_n\|_V > N, \|u\|_V \leq N\). Once again using (11) and (27), we estimate \(\langle (B_N(u_n) - B_N(u), v)\rangle_V\) as
\[
\langle (B_N(u_n) - B_N(u), v)\rangle_V = \left\langle \left(\frac{N}{\|u_n\|_V}\right)^2 B(u_n) - B(u), v \right\rangle \\
= \left\langle \left[\left(\frac{N}{\|u_n\|_V}\right)^2 - 1\right] B(u_n) + B(u_n) - B(u), v \right\rangle \\
= \frac{1}{\|u\|_V^4} (N - \|u\|_V)(N + \|u_n\|_V) \langle B(u_n), v \rangle \\
+ \langle B(u_n - u, u_n), v \rangle + \langle B(u, u_n - u), v \rangle \\
\leq \frac{1}{\|u\|_V^4} (\|u_n\|_V - N)(\|u_n\|_V + N)\|u_n\|_V^2\|v\|_V \\
+ \frac{2}{\lambda_1^{1/4}}(\|u_n\|_V + \|u\|_V)\|u_n - u\|_V\|v\|_V \\
\leq \frac{4N}{\lambda_1^{1/4}}(\|u_n\|_V + N)\|u_n - u\|_V\|v\|_V. \tag{29}
\]

**Case (4).** \(\|u_n\|_V > N, \|u\|_V > N\). Using (11) and (27), we estimate the term \(\langle (B_N(u_n) - B_N(u), v)\rangle_V\) as
\[
\langle (B_N(u_n) - B_N(u), v)\rangle_V
Let us now estimate \( \beta (\mathcal{P}_{3} (u^n) \| u^n \|^{-1} - \mathcal{P}_{3} (u) \| u \|^{-1}) , v ) \) as
\[
\beta (\mathcal{P}_{3} (u^n) \| u^n \|^{-1} - \mathcal{P}_{3} (u) \| u \|^{-1}) , v ) = \beta ((u^n - u) \| u \|^{-1} , v + \beta (u^n \| u \|^{-1} - \| u \|^{-1}) , v ) ,
\]
for all \( u^n , u , v \) \( \in L^r (\Omega) \cap V \) and using Hölder’s inequality, we estimate the term \( \langle (u^n - u) \| u \|^{-1} , v \rangle \) as
\[
\langle (u^n - u) \| u \|^{-1} , v \rangle \leq \| u^n - u \|_{L^{r+1}} \| u^n \|_{L^{r+1}}^{-1} \| v \|_{L^{r+1}} \leq C \| u^n - u \|_{V} \| u^n \|_{V}^{-1} \| v \|_{V} ,
\]
for \( r + 1 \in [2, 6] \). For \( r = 1 \), the second term in the right hand side of the equality (31) is zero.

**Case (1).** \( r \in (1, 2) \). Note that \( (a + b)^p \leq a^p + b^p \), for \( 0 < p < 1 \). For \( a, b > 0 \), this can be shown in the following way. For \( f(a) = (a + b)^p - a^p - b^p \), \( f'(a) = p((a + b)^{p-1} - a^{p-1}) \), thus, since \( a > 0 \), we have \( f(a) \leq f(0) = 0 \) and hence \( (a + b)^p \leq a^p + b^p \), for \( 0 < p < 1 \). Moreover, for \( 0 < p < 1 \), we also have
\[
| x^p | = | x - y + y^p | \leq | x - y |^p + | y |^p \Rightarrow | x |^p - | y |^p \leq | x - y |^p .
\]

Interchanging the roles of \( x \) and \( y \), we also get \( | x |^p - | y |^p \leq | x - y |^p \), for all \( 0 < p < 1 \). Using this, Hölder’s and Gagliardo-Nirenberg inequalities, we estimate \( \langle u^n \| u^n \|^{-1} - | u |^{-1} , v \rangle \) as
\[
\langle u^n \| u^n \|^{-1} - | u |^{-1}, v \rangle \leq \langle | u | \| u^n - u |^{-1} , v \rangle \leq \| u \|_{L^{2r}} \| | u^n - u |^{-1} \|_{L^{2r}} \| v \|_{L^{2r}} \leq \| u^n - u \|_{L^{2r}} \| v \|_{L^{2r}} \leq C \frac{1}{\lambda^{2r}} \| u^n - u \|_{V}^{-1} \| v \|_{V} .
\]

**Case (2).** \( r \in [2, 3] \). For \( r \in [2, 3] \), i.e., \( r = 2 + p , p \in [0, 1) \), we use (33), Hölder’s and Gagliardo-Nirenberg inequalities to estimate \( \langle u^n \| u^n \|^{-1} - | u |^{-1} , v \rangle \) as
\[
\langle u^n \| u^n \|^{-1} - | u |^{-1}, v \rangle = \langle u^n | u^n |^{1+p} - | u |^{1+p}, v \rangle = \langle u^n | | u^n |^p - | u |^p , v \rangle + \langle (u^n - | u |) | u |^p , v \rangle \leq \langle | u | | u^n | - | u |^p, | u | + \langle u^n - | u | | u |^{p+1} , | u | \rangle.
\]
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Since $f$ is differentiable and the derivative is given by $f'(x) = (r-1)x|x|^{r-3}$. We use Taylor's formula (for $0 < \theta < 1$), Holder’s and Gagliardo-Nirenberg inequalities to estimate $\langle u(|u^n|^{r-1} - |u|^{r-1}), v \rangle$ as

$$\langle u(|u^n|^{r-1} - |u|^{r-1}), v \rangle = (r-1)\langle u^n - u \rangle \cdot (\theta u^n + (1-\theta)u)|\theta u^n + (1-\theta)u|^{r-3, v}$$

$$\leq (r-1)\langle |u^n - u| |\theta u^n + (1-\theta)u|^{r-2, |v|} \rangle$$

$$\leq 2^{r-3}(r-1)\langle |u^n|_{L^{r+1}} |u^n - u|^{r-2, |v|} \rangle$$

$$\leq 2^{r-3}(r-1)\langle |u^n|_{L^{r+1}} |u^n - u|^{r-2, |v|} \rangle$$

$$\leq \frac{C}{\lambda_1^{1+}} 2^{r-3}(r-1)\langle |u^n|_{L^{r+1}} |u^n - u|^{r-2, |v|} \rangle$$

Since $|u_n - u| \rightarrow 0$ as $n \rightarrow \infty$, we know that $|u_n|_V \rightarrow |u|_V$ as $n \rightarrow \infty$. Combining (32)-(36), and substituting it in (31), we find that

$$\langle ((I + \mu A)^{-1}\beta u^n|u^n|^{r-1} - (I + \mu A)^{-1}\beta u|u|^{r-1}), v \rangle \rightarrow 0 \text{ as } n \rightarrow \infty,$$

for $r \in [1, 5]$. Let us now combine (27)-(30) and substitute it in (26), we find that $((\Gamma_N + \lambda I)u_n - (\Gamma_N + \lambda I)u, v) \rightarrow 0$ as $n \rightarrow \infty$, whenever $|u_n - u|_V \rightarrow 0$ as $n \rightarrow \infty$. Hence, the operator $\Gamma_N + \lambda I$ is demicontinuous on $V$.

Step (4). Local strong solution to the system (13). Since all demicontinuous, accretive operators are maximal accretive, $\Gamma_N + \lambda I$ is a maximal accretive operator on $V$. Remember that maximal accretive is equivalent to $m$-accretive on $V$. Thus, the operator $\Gamma_N$ is quasi-$m$-accretive on $V$. Then using Theorem 4.5, [5] (see Remark 4.1, Theorem 4.8, [5] also), for each $u_0 \in V$ and $f \in W^{1,1}(0, T; \mathbb{V})$, there exists a time $T^* = T(u_0) \in [0, T]$ and a unique solution $u \in C([0, T^*]; \mathbb{V})$ that satisfies:

$$u \in W^{1,\infty}((0, T^*]; \mathbb{V}), \text{ and } a.e. \ t \in (0, T^*)$$

$$\left\{ \begin{array}{l}
\frac{du(t)}{dt} + \nu(I + \mu A)^{-1}A u(t) + B_N(u(t)) + \beta(I + \mu A)^{-1}P_{H}(|u(t)|^{r-1}u(t)) = f(t),
\quad u(0) = u_0.
\end{array} \right.$$  

(37)

For $f \in W^{1,1}(0, T; D(A))$ and a.e. $t \in (0, T^*)$, the system (37) can also be written as:

$$\left\{ \begin{array}{l}
\frac{d}{dt}(I + \mu A)u(t) + \nu Au(t) + \tilde{B}_N(u(t)) + \beta P_{H}(|u(t)|^{r-1}u(t)) = (I + \mu A)f(t),
\quad u(0) = u_0,
\end{array} \right.$$  

(38)

where $\tilde{B}_N$ is defined in (17). Thus, it turns out that for large enough $N$, $u_N$ is the solution of the system (38) on a certain interval $(0, T^*)$.

Step (5). Global strong solution to the system (13). Finally, we show that the strong solution we obtained is global. Let us take inner product with $u_N(\cdot)$ to the
first equation in (38) to obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \|u_N(t)\|_{H}^2 + \mu \|u_N(t)\|_{H}^2 \right) + \nu \|u_N(t)\|_{V}^2 + \beta \|u_N(t)\|_{L^2+1}^{r+1} \\
= ((I + \mu A)f(t), u_N(t)).
\]
(39)
But, we know that
\[
|(I + \mu A)^{1/2}f, (I + \mu A)^{1/2}u_N) \leq \|(I + \mu A)^{1/2}f\|_{H} \|(I + \mu A)^{1/2}u_N\|_{H}
\leq \nu \|u_N\|_{V}^2 + \frac{1}{2\nu} \left( \frac{1}{\lambda_1} + \mu \right)^2 \|f\|_{V}^2.
\]
From (39), it is immediate that
\[
\|u_N(t)\|_{H}^2 + \mu \|u_N(t)\|_{H}^2 + \nu \int_{0}^{t} \|u_N(s)\|_{V}^2 ds + \beta \int_{0}^{t} \|u_N(s)\|_{L^2+1}^{r+1} ds
\leq \left( \frac{1}{\lambda_1} + \mu \right) \|u_0\|_{V}^2 + \frac{1}{\nu} \left( \frac{1}{\lambda_1} + \mu \right)^2 \int_{0}^{t} \|f(s)\|_{V}^2 ds,
\]
so that \|u_N(t)\|_{V} \leq C, for all \(t \in [0, T]\), where \(C\) is independent of \(N\). Thus the solution of (38) exists for all time \(t \in [0, T]\). Choose \(N \geq C\), so that we have \(\overline{B}_N(u_N) = B(u_N)\), and hence \(u = u_N\) is a strong solution of (38) on \([0, T]\).

**Remark 2.** Note that we have not used compactness arguments in the proof of Theorem 2.4. Thus, the well-posedness for the system (13) is valid even in unbounded domains like Poincaré domains.

Let us now discuss about the regularity of the weak solution to the system (13).

**Remark 3.** For \(f \in L^2(0, T; \mathbb{H})\) and \(u_0 \in D(A)\), we get further regularity of the weak solution of (13) as \(u \in C([0, T]; D(A))\). Let us take inner product with \(Au(\cdot)\) to the first equation in (13) to obtain
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{H}^2 + \mu \|Au(t)\|_{H}^2 + \nu \|Au(t)\|_{H}^2
= -(B(u(t)), Au(t)) - \beta(P_{\mathbb{H}}(|u(t)|^{-1}u(t)), Au(t)) + (f(t), Au(t)).
\]
(41)
We use Hölder’s, Agmon’s (see (A.15)) and Young’s inequalities to estimate the term \(|(B(u), Au)|\) as
\[
|(B(u), Au)| \leq \|u\|_{L^\infty} \|\nabla u\|_{H} \|Au\|_{H} \leq C \|\nabla u\|_{H}^{3/2} \|Au\|_{H}^{3/2}
\leq \frac{\nu}{4} \|Au\|_{H}^2 + \frac{27C}{4\nu^3} \|u\|_{V}^2.
\]
(42)
Using Cauchy-Schwarz and Young’s inequalities, we estimate \(|(f, Au)|\) as
\[
|(f, Au)| \leq \|f\|_{H} \|Au\|_{H} \leq \frac{\nu}{4} \|Au\|_{H}^2 + \frac{1}{\nu} \|f\|_{H}^2.
\]
(43)
For \(r \in [1, 3]\), we estimate \(\beta \|P_{\mathbb{H}}(u|^{-1}u), Au)| as
\[
\beta \|P_{\mathbb{H}}(u|^{-1}u), Au) \leq \beta \|u|^{-1}u\|_{H} \|Au\|_{H} \leq \beta \|u\|_{L^{r^{-1}}} \|Au\|_{H} \leq \frac{C_{\beta} u}{\lambda_1^{r^{-1}}} \|u\|_{V} \|Au\|_{H}
\leq \frac{\nu}{4} \|Au\|_{H}^2 + \frac{C \beta^2}{\nu \lambda_1^{2r}} \|u\|_{V}^2.
\]
(44)
For \( r \in (3, 5) \), we use Hölder’s, Gagliardo-Nirenberg (see (A.4)), Agmon’s (see (A.15)) and Young’s inequalities to estimate \( \beta(\mathcal{P}_\mathbb{H}(u^{r-1}u), Au) \) as

\[
\beta(\mathcal{P}_\mathbb{H}(u^{r-1}u), Au) \leq \beta\|u^{r-1}u\|_{\mathbb{H}} \|Au\|_{\mathbb{H}} \leq \beta\|u^3\|_{L^2} \|u^{r-3}\|_{L^\infty} \|Au\|_{\mathbb{H}} \\
\leq \beta\|u\|_{L^2}^3 \|u\|_{L^\infty}^{r-3} \|Au\|_{\mathbb{H}} \leq C\beta\|u\|_{\mathbb{V}}^{2r+1} \|Au\|_{\mathbb{H}}^{\frac{r-1}{r}} \\
\leq \frac{\nu}{4}\|Au\|_{\mathbb{H}}^2 + C\beta^{\frac{2}{r+1}} \left( \frac{5 - r}{4} \right) \left( \frac{r - 1}{\nu} \right)^{\frac{r-1}{r}} \|u\|_{\mathbb{V}}^{2(r+\nu)}. \quad (45)
\]

For \( r = 5 \), we estimate \( \beta(\mathcal{P}_\mathbb{H}(u^4u), Au) \) using Hölder’s, Ladyzhenskaya and Agmon’s inequalities as

\[
\beta(\mathcal{P}_\mathbb{H}(u^4u), Au) \leq \beta\|u^4u\|_{\mathbb{H}} \|Au\|_{\mathbb{H}} \leq \beta\|u^3\|_{L^2} \|u\|_{L^\infty} \|Au\|_{\mathbb{H}} \\
\leq \beta\|u\|_{L^2}^3 \|u\|_{L^\infty} \|Au\|_{\mathbb{H}} \leq C\beta\|u\|_{\mathbb{V}} \|Au\|_{\mathbb{H}}^2. \quad (46)
\]

For \( r \in [1, 3] \), let us combine (42)-(44) and use it in (41) to find

\[
\frac{1}{2} \frac{d}{dt} \left[ \|\nabla u(t)\|_{\mathbb{H}}^2 + \mu\|Au(t)\|_{\mathbb{H}}^2 \right] + \frac{\nu}{2}\|Au(t)\|_{\mathbb{H}}^2 \\
\leq 27C \frac{1}{4\nu^3}\|u(t)\|_{\mathbb{V}}^6 + C\beta^2 \frac{1}{\nu^2}\|u(t)\|_{\mathbb{V}}^4 \|f(t)\|_{\mathbb{H}}^2 + \frac{1}{\nu}\|f(t)\|_{\mathbb{H}}^2. \quad (47)
\]

Integrate the inequality from 0 to \( t \) to obtain

\[
\|\nabla u(t)\|_{\mathbb{H}}^2 + \mu\|Au(t)\|_{\mathbb{H}}^2 + \frac{\nu}{2} \int_0^t \|Au(s)\|_{\mathbb{H}}^2 ds \\
\leq \|u_0\|_{\mathbb{H}}^2 + \mu\|Au_0\|_{\mathbb{H}}^2 + C\beta^2 \frac{1}{\nu^2} \left( \frac{5 - r}{2} \right) \left( \frac{r - 1}{\nu} \right)^{\frac{r-1}{r}} \int_0^t \|u(s)\|_{\mathbb{V}}^{2(3+\nu)} ds \\
+ \frac{2}{\nu} \int_0^t \|f(s)\|_{\mathbb{H}}^2 ds. \quad (48)
\]

Since \( u \in C([0, T]; \mathbb{V}) \), \( u_0 \in D(A) \) and \( f \in L^2(0, T; \mathbb{H}) \), one can easily see that the right hand side of the inequality (48) is finite and hence we easily obtain \( u \in C([0, T]; D(A)) \), since \( u \in W^{1,\infty}(0, T; D(A)) \). Similarly, for \( r \in [3, 5] \), we get

\[
\|u(t)\|_{\mathbb{V}}^2 + \mu\|Au(t)\|_{\mathbb{H}}^2 + \frac{\nu}{2} \int_0^t \|Au(s)\|_{\mathbb{H}}^2 ds \\
\leq \|u_0\|_{\mathbb{V}}^2 + \mu\|Au_0\|_{\mathbb{H}}^2 + C\beta^2 \frac{1}{\nu^2} \left( \frac{5 - r}{2} \right) \left( \frac{r - 1}{\nu} \right)^{\frac{r-1}{r}} \int_0^t \|u(s)\|_{\mathbb{V}}^{2(3+\nu)} ds \\
+ \frac{27C}{2\nu^3} \int_0^t \|u(s)\|_{\mathbb{V}}^6 ds + \frac{2}{\nu} \int_0^t \|f(s)\|_{\mathbb{H}}^2 ds. \quad (49)
\]

and the result follows. Finally, for \( r = 5 \), combining (42), (43) and (46), substituting it in (41) and then integrating from 0 to \( t \), we obtain

\[
\|u(t)\|_{\mathbb{V}}^2 + \mu\|Au(t)\|_{\mathbb{H}}^2 + \nu \int_0^t \|Au(s)\|_{\mathbb{H}}^2 ds \\
\leq \|u_0\|_{\mathbb{V}}^2 + \mu\|Au_0\|_{\mathbb{H}}^2 + \frac{27C}{2\nu^3} \int_0^t \|u(s)\|_{\mathbb{V}}^6 ds + \frac{2}{\nu} \int_0^t \|f(s)\|_{\mathbb{H}}^2 ds \\
+ C\beta \int_0^t \|u(s)\|_{\mathbb{V}} \|Au(s)\|_{\mathbb{H}}^2 ds. \quad (50)
\]
An application for Gronwall’s inequality in (50) yields
\[ \|u(t)\|_{V}^2 + \mu\|Au(t)\|_{H}^2 + \nu \int_{0}^{t} \|Au(s)\|_{H}^2 ds \leq \left(\|u_0\|_{V}^2 + \mu\|A[u_0]\|_{H}^2 + \frac{27C}{2\nu^3} \int_{0}^{t} \|u(s)\|_{V}^2 ds + \frac{2}{\nu} \int_{0}^{t} \|f(s)\|_{H}^2 ds\right) \times \exp\left(\frac{C\beta}{\mu} \int_{0}^{t} \|u(s)\|_{V}^2 ds\right), \]
for all \( t \in [0, T] \). Since \( u \in C([0, T]; V) \), the required result follows.

For \( f(\cdot) \) independent of \( t \), using Theorem 2.2, we know that there exists a unique weak solution to the system (13) and the solution can be represented through a one parameter family of continuous semigroup. Thanks to Theorem 2.2, we can define a continuous semigroup \( \{S(t)\}_{t \geq 0} \) in \( V \) by
\[ S(t)u_0 = u(t), \quad t \geq 0, \]
where \( u(\cdot) \) is the unique weak solution of the system (13) with \( f(t) = f \in H \) and \( u(0) = u_0 \in V \).

3. **Global attractor.** This section is devoted for proving the existence of a global attractor for the semigroup \( S(t), t \geq 0 \), for autonomous version of the 3D Kelvin-Voigt-Brinkman-Forchheimer system given in (13). We need the following notation also for the rest of the sections. For \( \mu \geq 0 \) and \( \alpha \in \mathbb{R} \), we define the scale of Hilbert spaces, \( V_{\alpha}^\mu = D(A^{\alpha/2}) \) endowed with the scalar product
\[ (u, v)_{V_{\alpha}^\mu} = (A^{(\alpha-1)/2}u, A^{(\alpha-1)/2}v) + \mu(A^{\alpha/2}u, A^{\alpha/2}v), \]
and norm
\[ \|u\|_{V_{\alpha}^\mu} = \|A^{(\alpha-1)/2}u\|_{H}^2 + \mu\|A^{\alpha/2}u\|_{H}^2. \]
A simple application of Poincaré inequality yields
\[ \mu\|u\|_{V_{\alpha}^\mu} = \|u\|_{V_{\alpha-1}^\mu}^2 + \mu\|u\|_{V_{\alpha}^\mu}^2 = \left(\frac{1}{\lambda_1} + \mu\right)\|u\|_{V_{\alpha}^\mu}, \]
so that both norms are equivalent on \( V_{\alpha} \) (see Remark 1 also).

For \( r \in [1, 4] \), our first aim is to establish the existence of an absorbing ball in \( V_{\alpha}^\mu \) for \( S(t), t \geq 0 \) and then show that it is an asymptotically compact semigroup in \( V_{\alpha}^1 \).

Using Theorem 2.3.5, [11], we show the existence of a global attractor \( S(t), t \geq 0 \) in \( V_{\alpha}^\mu \). Moreover, we establish that the semigroup \( S(t) \) defined on \( V_{\alpha}^\mu \) has an absorbing ball in \( V_{\alpha}^1 \).

Let us prove the following lemma on Lipschitz continuity of the semigroup \( S(t) \).

**Lemma 3.1.** The map \( S(t) : V_{\alpha}^\mu \to V_{\alpha}^\mu, \) for \( t \geq 0 \), is Lipschitz continuous on bounded subsets of \( V_{\alpha}^\mu \).

**Proof.** Let us take \( S(t)u_0 = u(t) \) and \( S(t)v_0 = v(t) \), for all \( t \geq 0 \). Then \( w(t) := u(t) - v(t) \) satisfies:
\[ \begin{cases} \frac{d}{dt}[(I + \mu A)w(t)] + \nu Aw(t) + B(w(t), u(t)) + B(u(t), w(t)) \\ + \beta(P_{H}(u(t))^{-1}u(t)) - P_{H}(v(t))^{-1}v(t)) = 0, \quad t \in (0, T), \\ u(0) = u_0 \in V, \end{cases} \]
Taking inner product with \( w(t) \) in the first equation in (54), we find
\[
\frac{1}{2} \frac{d}{dt} (\|w(t)\|_H^2 + \mu \| \nabla w(t) \|_H^2) + \nu \| \nabla w(t) \|_H^2 + \beta (P_\mathbb{H} (|u(t)|^{-1} u(t)) - P_\mathbb{H} (|v(t)|^{-1} v(t))), w(t)) = -b(w(t), u(t), w(t)),
\]
where we used the fact that \( b(u, v, w) = 0 \). From (24), we know that \(|u|^{-1} u - |v|^{-1} v, w| \geq 0 \). Using Hölder’s, Ladyzhenskaya and Poincaré inequalities, we estimate \(|b(w, u, w)|\) as
\[
|b(w, u, w)| \leq \|w\|_H^2 \|\nabla u\|_H \leq 2\|w\|_H^{1/2} \|\nabla w\|_H^{3/2} \|\nabla u\|_H \leq \frac{2}{\lambda_1^{1/2}} \|u\|_V \|w\|_V^2
\]
(56)
Using (56) in (55) and then integrating the from 0 to \( t \), we get
\[
\|w(t)\|_H^2 + \mu \|w(t)\|_V^2 + \nu \int_0^t \|w(s)\|_V^2 ds
\leq \|w_0\|_H^2 + \mu \|w_0\|_V^2 + \frac{4}{\nu \lambda_1^{1/2}} \int_0^t \|u(s)\|_V^2 \|w(s)\|_V^2 ds.
\]
(57)
An application of Gronwall’s inequality in (57) yields
\[
\|w(t)\|_V^2 \leq \|w_0\|_V^2 \exp\left( \frac{4}{\nu \mu \lambda_1^{1/2}} \int_0^t \|u(s)\|_V^2 ds \right).
\]
(58)
Thus, we have
\[
\|S(t)u_0 - S(t)v_0\|_{\mathbb{V}_1^\mu} \leq \exp\left( \frac{2}{\nu \mu \lambda_1^{1/2}} \int_0^t \|u(s)\|_V^2 ds \right) \|u_0 - \nu_0\|_{\mathbb{V}_1^\mu},
\]
(59)
and hence the map \( S(t) : \mathbb{V}_1^\mu \rightarrow \mathbb{V}_1^\mu \), for \( t \geq 0 \), is Lipschitz continuous on bounded subsets of \( \mathbb{V}_1^\mu \).

3.1. Absorbing ball in \( \mathbb{V}_1^\mu \). Let \( u(t) \), \( t \geq 0 \) be the unique weak solution of the system (13) with \( f(t) = f \in \mathbb{V}' \). We show that \( S(t) \) has an absorbing ball in \( \mathbb{V}_1^\mu \).

Proposition 1. The set
\[
\mathcal{B}_1 := \left\{ v \in \mathbb{V}_1^\mu : \|v\|_{\mathbb{V}_1^\mu} \leq M_1 = \frac{1}{\nu} \sqrt{\frac{2(1 + \mu \lambda_1)}{\lambda_1}} \|f\|_V \right\},
\]
(60)
is a bounded absorbing set in \( \mathbb{V}_1^\mu \) for the semigroup \( S(t) \). That is, given a bounded set \( \mathbb{B} \subset \mathbb{V}_1^\mu \), there exists an entering time \( t_B > 0 \) such that \( S(t)B \subset \mathcal{B}_1 \), for all \( t \geq t_B \).

Proof. Let us take inner product with \( u(\cdot) \) to the first equation in (13) to obtain
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 + \mu \| \nabla u(t) \|_H^2 + \nu \| \nabla u(t) \|_H^2 + \beta \|u(t)\|_H^{\gamma+1} = \langle f, u(t) \rangle.
\]
(61)
Using Cauchy-Schwarz and Young’s inequalities, we estimate \( \langle f, u(t) \rangle \) as
\[
\|f\|_V \|u\|_V \leq \frac{\nu}{2} \|u\|_V^2 + \frac{1}{2\nu} \|f\|_V^2.
\]
(62)
Substituting (62) in (61), we find
\[
\frac{1}{2} \frac{d}{dt} \left[ \|u(t)\|_{H^1}^2 + \mu \|\nabla u(t)\|_{L^2}^2 \right] + \frac{\nu}{2} \|\nabla u(t)\|_{L^2}^2 + \beta \|u(t)\|_{L^{r+1}_t}^2 \leq \frac{1}{2\nu} \|f\|_{V'}^2.
\] (63)
Thus using (53) in (63), we get
\[
\frac{d}{dt} \left[ \|u(t)\|_{H^1}^2 + \mu \|\nabla u(t)\|_{L^2}^2 \right] + \frac{\nu}{2} \|\nabla u(t)\|_{L^2}^2 + \beta \|u(t)\|_{L^{r+1}_t}^2 \leq \frac{1}{\nu} \|f\|_{V'}^2.
\] (64)
An application of Gronwall’s inequality yields
\[
\|u(t)\|_{H^1}^2 + \mu \|\nabla u(t)\|_{L^2}^2 \leq e^{-\left(\frac{\nu t}{2\mu} \right)} \left(\|u_0\|_{H^1}^2 + \mu \|\nabla u_0\|_{L^2}^2\right) + \left(1 + \mu \lambda_1\right) \frac{j}{\nu^2 \lambda_1} \|f\|_{V'}^2,
\] (65)
which implies
\[
\|u(t)\|_{H^1}^2 \leq e^{-\left(\frac{\nu t}{2\mu} \right)} \|u_0\|_{H^1}^2 + \left(1 + \mu \lambda_1\right) \frac{j}{\nu^2 \lambda_1} \|f\|_{V'}^2.
\] (66)
Taking limit as \(t \to \infty\), we find
\[
\limsup_{t \to \infty} \|u(t)\|_{V'}^2 \leq \left(1 + \mu \lambda_1\right) \frac{j}{\nu^2 \lambda_1} \|f\|_{V'}^2.
\] (67)
Integrating the inequality (63) from 0 to \(t\), we also obtain
\[
\nu \int_0^t \|u(s)\|_{V'}^2 ds \leq \|u_0\|_{V'}^2 + \frac{t}{\nu} \|f\|_{V'}^2, \text{ for all } t \geq 0.
\] (68)
The inequality (67) implies that the semigroup \(S(t) : V_1^\mu \to V_1^\mu, t \geq 0\) associated with the weak solution to the problem (13) has an absorbing ball given in (60).
Hence, the following uniform estimate is valid:
\[
\|u(t)\|_{V_1^\mu} \leq M_1, \text{ where } M_1 = \frac{1}{\nu} \sqrt{\frac{2(1 + \mu \lambda_1)}{\lambda_1}} \|f\|_{V'},
\] (69)
for \(t\) large enough (\(t \gg 1\)) depending on the initial data. \( \Box \)

3.2. **Asymptotic compactness.** Let us now show that the semigroup \(S(t), t \geq 0\) defined in \(V_1^\mu\) is asymptotically compact. The following proposition can be proved using a Faedo-Galerkin approximation.

**Proposition 2.** Let \(s \in \mathbb{R}\). If \(z_0 \in V_s, g \in L^2(0,T;V_{s-2})\), then the linear problem
\[
\begin{cases}
\frac{d}{dt} \left[ (I + \mu A)z(t) \right] + \nu A z(t) = g(t), \\
z(0) = z_0,
\end{cases}
\] (70)
has a unique weak solution \(z \in C([0,T];V_s)\). The weak solution also belongs to \(C([0,T];V_s)\) and the following estimate holds:
\[
\sup_{t \in [0,T]} \|z(t)\|_{V_s} \leq C \|g\|_{L^2(0,T;V_{s-2})}, \text{ } s \in \mathbb{R}.
\] (71)

**Proof.** Using a Faedo-Galerkin approximation, the existence of a weak solution to the system (70) can be established. The uniqueness of weak solution follows from linearity. Let us now establish the regularity result. We take inner product with \(A^{s-1}z(\cdot)\) in the first equation in (70) to obtain
\[
\frac{1}{2} \frac{d}{dt} \left[ \|A^{s-1}z(t)\|_{H^1}^2 + \mu \|A^{s}z(t)\|_{L^2}^2 \right] + \nu \|A^{s}z(t)\|_{L^2}^2 \leq (A^{s-1}g(t), A^{s}z(t))
\]
Let \( \mathcal{H} \) be a Hilbert space and \( u \) be an asymptotically compact semigroup in the space \( \mathcal{H} \). Hence, the semigroup \( R(t) \) for all \( t \) is obtained from the solution operator of the Cauchy problem:

\[
\begin{align*}
\frac{d}{dt}[(I + \mu A)y(t)] + \nu Ay(t) &= 0, \\
y(0) &= u_0,
\end{align*}
\]

where \( u_0 \) is the initial data. Taking supremum over \( t \in [0, T] \), the estimate (71) follows.

**Proposition 3.** Let \( f \in \mathcal{H} \) be time independent. Then the semigroup \( S(t), t \geq 0 \) is an asymptotically compact semigroup in \( \mathcal{V}^1 \).

**Proof.** Let \( f \in \mathcal{H} \) and \( u_0 \in \mathcal{V} \) be given. We rewrite the semigroup \( S(t) \) as

\[
S(t)u_0 = R(t)u_0 + T(t)u_0,
\]

where \( R(t) \) is the semigroup generated by the solution operator of the linear problem

\[
\begin{cases}
\frac{d}{dt}[(I + \mu A)y(t)] + \nu Ay(t) = 0, \\
y(0) = u_0,
\end{cases}
\]

and \( z(t) = T(t)(u_0) \) is the solution of the problem:

\[
\begin{cases}
\frac{d}{dt}[(I + \mu A)z(t)] + \nu Az(t) = f - B(u(t)) - \beta P_{\mathcal{H}}(|u(t)|\nu^{-1}u(t)), \\
z(0) = 0,
\end{cases}
\]

where \( u(t) = y(t) + z(t) \) is the unique weak solution of the system (13) with the initial data \( u_0 \). Let us take inner product with \( y(\cdot) \) to the first equation in (75) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|y(t)\|_\mathcal{H}^2 + \mu \|\nabla y(t)\|_\mathcal{H}^2 + \nu \|\nabla y(t)\|_\mathcal{H}^2 = 0.
\]

Using (53) and then applying Gronwall’s inequality, we find

\[
\|y(t)\|_\mathcal{H}^2 + \mu \|\nabla y(t)\|_\mathcal{H}^2 \leq e^{-\left(\frac{\mu_1}{\nu + \lambda_1}\right)t} (\|y_0\|_\mathcal{H}^2 + \mu \|\nabla y_0\|_\mathcal{H}^2),
\]

for all \( t \geq 0 \). That is, we have

\[
\|y(t)\|_\mathcal{V}^1 \leq e^{-\left(\frac{\mu_1}{\nu + \lambda_1}\right)t} \|y_0\|_\mathcal{V}^1.
\]

Hence, the semigroup \( R(t) : \mathcal{V}^1 \to \mathcal{V}^1 \) is exponentially contractive.

Using Hölder’s and Gagliardo-Nirenberg inequalities (see (A.13) and (A.6)), for \( 0 < \varepsilon \leq 3 \) (one can take \( \varepsilon \) sufficiently small also), we estimate \( \|A^{-\frac{3(4-\varepsilon)}{4-\varepsilon}} B(u, u)\|_\mathcal{H} \) as

\[
\|A^{-\frac{3(4-\varepsilon)}{4-\varepsilon}} B(u, u)\|_\mathcal{H} = \sup_{\varphi \in \mathcal{V}, \|A^{\frac{3(4-\varepsilon)}{4-\varepsilon}} \varphi\|_\mathcal{H} = 1} b(u, u, \varphi)
\]

\[
\leq \frac{2(4-\varepsilon)}{4-\varepsilon} \sum_{\|u\|_\mathcal{H} = 1} \|u\|_\mathcal{L}^{2(4-\varepsilon)} \|u\|_\mathcal{V} \|\varphi\|_\mathcal{H}^{4-\varepsilon}
\]

\[
\leq C \sup_{\varphi \in \mathcal{V}, \|A^{\frac{3(4-\varepsilon)}{4-\varepsilon}} \varphi\|_\mathcal{H} = 1} \|u\|_2 \|\varphi\|_2 \sum_{\|u\|_\mathcal{H} = 1} \|A^{\frac{3(4-\varepsilon)}{4-\varepsilon}} \varphi\|_\mathcal{H}
\]

\[C = \frac{1}{\lambda_1^{\frac{3(4-\varepsilon)}{4-\varepsilon}}} \sup_{\|A^{\frac{3(4-\varepsilon)}{4-\varepsilon}} \varphi\|_\mathcal{H} = 1} \|\varphi\|_2 \|A^{\frac{3(4-\varepsilon)}{4-\varepsilon}} \varphi\|_\mathcal{H}.\]
Proof. Remember that $\mathbf{u}(t) = \mathbf{y}(t) + \mathbf{z}(t) \in \mathcal{A}_1^\mu$. Let us take inner product with $\Lambda^{\frac{d}{2}} \mathbf{z}(-)$ to the first equation in (76) to obtain

$$
\frac{1}{2} \frac{d}{dt} \left( \| \Lambda^{\frac{d}{2}} \mathbf{z}(t) \|_{H}^{2} + \mu \| \Lambda^{\frac{d}{2}} \mathbf{z}(t) \|_{L}^{2} \right) + \nu \Lambda^{\frac{d}{2}} \mathbf{z}(t) \|_{H}^{2}
$$

and hence we get $B(\mathbf{u}, \mathbf{u}) \in L^\infty(0, T; \mathcal{V}_{\frac{2(4-s)}{6-s}})$. In order to estimate the term $\| \Lambda^{-\frac{3(4-s)}{4(6-s)}} |u|^{-1} u \|_{H}$, let us first consider the case $r \in [2\left(\frac{5-s}{6-s}\right), 6\left(\frac{5-s}{6-s}\right)]$. For this, we estimate $\| \Lambda^{-\frac{3(4-s)}{4(6-s)}} |u|^{-1} u \|_{H}$ using Hölder’s inequality, (A.13) and (A.6) as

$$
\| \Lambda^{-\frac{3(4-s)}{4(6-s)}} |u|^{-1} u \|_{H} = \sup_{\varphi \in \mathcal{V}, \| \Lambda^{-\frac{3(4-s)}{4(6-s)}} \varphi \|_{H} = 1} \langle |u|^{-1} u, \varphi \rangle
$$

$$
\leq \sup_{\varphi \in \mathcal{V}, \| \Lambda^{-\frac{3(4-s)}{4(6-s)}} \varphi \|_{H} = 1} \| |u|^{-1} u \|_{L^{\frac{r(6-s)}{6-s}}} \| \varphi \|_{L^{r-s}}
$$

$$
\leq C \sup_{\varphi \in \mathcal{V}, \| \Lambda^{-\frac{3(4-s)}{4(6-s)}} \varphi \|_{H} = 1} \| u \|_{L^{\frac{r(6-s)}{6-s}}} \| \Lambda^{-\frac{3(4-s)}{4(6-s)}} \varphi \|_{H}
$$

$$
\leq \frac{C}{\Lambda^{\frac{3(4-s)}{4(6-s)} - \frac{d}{2}}} \| u \|_{\mathcal{V}}.
$$

(81)

For $r \in [1, 2\left(\frac{5-s}{6-s}\right)]$, we have $\frac{r(6-s)}{6-s} \in [\frac{5-s}{6-s}, 2)$. Thus, one can use the fact that $L^{2}(\Omega) \subset L^{\frac{r(6-s)}{6-s}}(\Omega)$ and Poincaré inequality to get

$$
\| u \|_{L^{\frac{r(6-s)}{6-s}}} \leq C \| u \|_{H} \leq C \Lambda^{\frac{d}{2}} \| u \|_{\mathcal{V}}.
$$

Thus, for $r \in [1, 6\left(\frac{5-s}{6-s}\right)]$, we have $|u|^{-1} u \in L^{\infty}(0, T; \mathcal{V}_{\frac{2(4-s)}{6-s}})$. Since $\mathbf{u}(t)$ is the weak solution of the system (13) with $\mathbf{u}_0 \in \mathcal{V}$, we know that $\mathbf{u} \in L^{\infty}(0, T; \mathcal{V})$. Using Proposition 2, (80) and (81), we also know that the solution $\mathbf{z}(t)$ of the problem (76) belongs to $C([0, T]; \mathcal{V}_{\frac{12-s}{12}})$. Thus the operator $T(t)$ maps $\mathcal{V}$ to $\mathcal{V}_{\frac{12-s}{12}}$. Combining this result with (79), we infer that the semigroup $S(t)$ satisfies the conditions of the Theorem 3.3, [26] (see also Theorem 2.1, [22]), and hence $S(t)$, $t \geq 0$ is an asymptotically compact semigroup on $\mathcal{V}^\mu_1$, which completes the proof. \hfill \Box

**Theorem 3.2.** Let $r \in [1, 6\left(\frac{5-s}{6-s}\right)]$, for some $\varepsilon > 0$ and $f \in \mathcal{H}$. Then the semigroup $S(t) : \mathcal{V}^\mu_1 \to \mathcal{V}^\mu_1$ has an absorbing ball $\mathcal{B}_1 = \left\{ \mathbf{v} \in \mathcal{V}^\mu_1 : \| \mathbf{v} \|_{\mathcal{V}^\mu_1} \leq M_1 \right\}$ and a global attractor $\mathcal{A}^\mu_1 \subset \mathcal{V}^\mu_1$. Moreover, the attractor $\mathcal{A}^\mu_1$ is compact, connected and invariant.

**Proof.** An application of the Theorem 2.3.5, [11] gives the required result. \hfill \Box

Our next aim is to establish that the global attractor $\mathcal{A}^\mu_1$ is a bounded subset of $\mathcal{V}^\mu_2$. Due to technical reasons, we are able to establish the boundedness result for $r \in [1, 4]$ only, (i.e., we take $\varepsilon = 3$ in (80) and (81)).

**Proposition 4.** For $r \in [1, 4]$, the global attractor $\mathcal{A}^\mu_1$ is a bounded subset of $\mathcal{V}^\mu_2$.

**Proof.**
\( = (f, A^{\frac{1}{2}}z(t)) - (B(u(t), A^{\frac{1}{2}}z(t)) - \beta(P_{H}(|u(t)|^{-1} u(t)), A^{\frac{1}{2}}z(t)). \)  

(82)

The first term from the right hand side of the equality (82) can be estimated using Cauchy-Schwarz and Young's inequalities as

\( |(f, A^{\frac{1}{2}}z)| = |(A^{-\frac{1}{2}}f, A^{\frac{1}{2}}z)| \leq \|A^{-\frac{1}{2}}f\|_{H} \|A^{\frac{1}{2}}z\|_{H} \leq \frac{\nu}{4} \|A^{\frac{1}{2}}z\|_{H}^{2} + \frac{1}{\nu} \|A^{-\frac{1}{2}}f\|_{H}^{2}. \)  

(83)

Using (80) (by choosing \( \varepsilon = 3 \)), we estimate \( |(B(u), A^{\frac{1}{2}}z)| \) as

\( |(B(u), A^{\frac{1}{2}}z)| = |(A^{-\frac{1}{2}}B(u), A^{\frac{1}{2}}z)| \leq \|A^{-\frac{1}{2}}B(u)\|_{H} \|A^{\frac{1}{2}}z\|_{H} \leq C \|u\|_{V} \|A^{\frac{1}{2}}z\|_{H} \leq \frac{\nu}{4} \|A^{\frac{1}{2}}z\|_{H}^{2} + \frac{C}{\nu} \|u\|_{V}. \)  

(84)

Similarly, using (81), (for \( \varepsilon = 3 \)) we estimate the term \( |(P_{H}(|u|^{-1} u), A^{\frac{1}{2}}z)| = |(|u|^{-1} u, A^{\frac{1}{2}}z)| \) as

\( |(|u|^{-1} u, A^{\frac{1}{2}}z)| = |(A^{-\frac{1}{2}}(|u|^{-1} u), A^{\frac{1}{2}}z)| \leq \|A^{-\frac{1}{2}}(|u|^{-1} u)\|_{H} \|A^{\frac{1}{2}}z\|_{H} \leq \frac{C}{\lambda_{1}^{-\frac{3}{2}}} \|u\|_{V} \|A^{\frac{1}{2}}z\|_{H} \leq \frac{\nu}{4} \|A^{\frac{1}{2}}z\|_{H}^{2} + \frac{C}{\nu \lambda_{1}^{-\frac{3}{2}}} \|u\|_{V}. \)  

(85)

Combining (83)-(85) and then substituting it in (80), we find

\[
\frac{d}{dt} \left( \|A^{\frac{1}{2}}z(t)\|_{H}^{2} + \mu \|A^{\frac{1}{2}}z(t)\|_{H}^{2} \right) + \frac{\nu}{2} \|A^{\frac{1}{2}}z(t)\|_{H}^{2} \leq 2 \left( \frac{\nu}{4} \|A^{-\frac{1}{2}}f\|_{H}^{2} + \frac{C}{\nu} \|u(t)\|_{V}^{2} + \frac{C}{\nu \lambda_{1}^{-\frac{3}{2}}} \|u(t)\|_{V}. \right) 
\]

(86)

For large \( t \), using (53) and (69) in (86), we further get

\[
\frac{d}{dt} \left( \|A^{\frac{1}{2}}z(t)\|_{H}^{2} + \mu \|A^{\frac{1}{2}}z(t)\|_{H}^{2} \right) + \frac{\nu \lambda_{1}}{2(1 + \mu \lambda_{1})} \left( \|A^{\frac{1}{2}}z(t)\|_{H}^{2} + \mu \|A^{\frac{1}{2}}z(t)\|_{H}^{2} \right) \leq \frac{2}{\nu} \left( \|A^{-\frac{1}{2}}f\|_{H}^{2} + CM_{1}^{2} + \frac{CM_{2}^{2}}{\lambda_{1}^{-2}} \right),
\]

and hence we have

\[
\|A^{\frac{1}{2}}z(t)\|_{H}^{2} + \mu \|A^{\frac{1}{2}}z(t)\|_{H}^{2} \leq \left( \frac{1 + \mu \lambda_{1}}{\lambda_{1}} \right) \left( \|A^{-\frac{1}{2}}f\|_{H}^{2} + CM_{1}^{2} + \frac{CM_{2}^{2}}{\lambda_{1}^{-2}} \right) =: L_{0}. \)  

(87)

That is, \( \|z(t)\|_{V_{1}} \leq L_{0} \). Note that the global attractor \( A_{t}^{\mu} \) is invariant, that is, \( S(t)A_{t}^{\mu} = A_{t}^{\mu} \) and thanks to inequality (79), we have

\[
\|u(t) - z(t)\|_{V_{1}} \leq \|y(t)\|_{V_{1}} \leq e^{-t(\nu \lambda_{1})} \|u_{0}\|_{V_{1}} \leq e^{-t(\nu \lambda_{1})} \|u_{0}\|_{V_{1}}, \)  

(88)

for all \( t \geq 0 \). From the inequality (88), we infer that for each \( v \in A_{t}^{\mu} \), there exists a sequence \( \{z(t_{k})\} \), \( t_{k} \to \infty \), corresponding to \( u_{k}(0) \in A_{t}^{\mu} \) such that

\( v = \lim_{k \to \infty} z(t_{k}), \) \( u_{k}(0) \in A_{t}^{\mu}. \)  

(89)

The inequality (87) reveals us that the sequence \( \{z(t_{k})\} \) belongs to a ball in \( V_{\frac{1}{2}}^{u} \), with radius \( L_{0} \), depending only on \( M_{1} \) and \( \|f\|_{V_{-\frac{1}{2}}} \). Thus, the sequence \( \{z(t_{k})\} \) is precompact in \( V_{\frac{1}{2}}^{u} \). Using (87) and the lowersemicontinuity property of the \( V_{\frac{1}{2}}^{u} \) norm, that is, \( \|v\|_{V_{\frac{1}{2}}^{u}} \leq \liminf_{k \to \infty} \|z(t_{k})\|_{V_{\frac{1}{2}}^{u}} \) easily gives us that \( A_{t}^{\mu} \) is bounded in \( V_{\frac{1}{2}}^{u} \).
If we take inner product with $A^\frac{r}{2}\mathbf{z}(\cdot)$ to the first equation in (76), and use similar arguments as above, one can show that $A^H_1$ is bounded in $V^u_{\frac{3}{2}}$. The estimate (84) needs to be replaced as

$$
|\langle B(\mathbf{u}), A^\frac{r}{2}\mathbf{z} \rangle| = |\langle A^{-\frac{r}{2}}B(\mathbf{u}), A^\frac{r}{2}\mathbf{z} \rangle| \leq C\|A^{-\frac{r}{2}}B(\mathbf{u})\|_H\|A^\frac{r}{2}\mathbf{z}\|_H. \tag{90}
$$

One can estimate $\|A^{-\frac{r}{2}}B(\mathbf{u})\|_H$ using Hölder’s inequality, (A.10) and (A.12) as

$$
\|A^{-\frac{r}{2}}B(\mathbf{u})\|_H = \sup_{\varphi \in V, \|A^\frac{r}{2}\varphi\|_H = 1} b(\mathbf{u}, \mathbf{u}, \varphi)
\leq \sup_{\varphi \in V, \|A^\frac{r}{2}\varphi\|_H = 1} \|\mathbf{u}\|_L^2 \|\mathbf{u}\|_V \|\varphi\|_L^\frac{1}{r}
\leq C \sup_{\varphi \in V, \|A^\frac{r}{2}\varphi\|_H = 1} \|\mathbf{u}\|_V^\frac{r}{2} \|\mathbf{u}\|_V \|\varphi\|_V^\frac{1}{r}
\leq \frac{C}{\lambda_1^\frac{r}{2}} \sup_{\varphi \in V, \|A^\frac{r}{2}\varphi\|_H = 1} \|\mathbf{u}\|_V^\frac{r}{2} \|\mathbf{u}\|_V \|A^\frac{r}{2}\varphi\|_H = \frac{C}{\lambda_1^\frac{r}{2}} \|\mathbf{u}\|_V^\frac{r}{2} \|\mathbf{u}\|_V^\frac{1}{r}. \tag{91}
$$

Thus, from (90), we have

$$
|\langle B(\mathbf{u}), A^\frac{r}{2}\mathbf{z} \rangle| \leq \frac{C}{\lambda_1^\frac{r}{2}} \|\mathbf{u}\|_V^\frac{r}{2} \|\mathbf{u}\|_V \|A^\frac{r}{2}\mathbf{z}\|_H \leq \frac{\nu}{4} \|A^\frac{r}{2}\mathbf{z}\|_H^2 + \frac{C}{\nu \lambda_1^\frac{r}{2}} \|\mathbf{u}\|_V^\frac{r}{2} \|\mathbf{u}\|_V^\frac{1}{r}. \tag{92}
$$

Now, we estimate $\|(|u|^{-1}u, A^\frac{r}{2}\mathbf{z})\|$ as

$$
|\langle |u|^{-1}u, A^\frac{r}{2}\mathbf{z} \rangle| = |\langle A^{-\frac{r}{2}}(|u|^{-1}u), A^\frac{r}{2}\mathbf{z} \rangle| \leq \|A^{-\frac{r}{2}}(|u|^{-1}u)\|_H \|A^\frac{r}{2}\mathbf{z}\|_H
\leq \frac{\nu}{4} \|A^\frac{r}{2}\mathbf{z}\|_H^2 + \frac{1}{\nu} \|A^{-\frac{r}{2}}(|u|^{-1}u)\|_H^2. \tag{93}
$$

Let us estimate $\|A^{-\frac{r}{2}}(|u|^{-1}u)\|_H$ using Hölder’s inequality, (A.10) and (A.12) as

$$
\|A^{-\frac{r}{2}}(|u|^{-1}u)\|_H = \sup_{\varphi \in V, \|A^\frac{r}{2}\varphi\|_H = 1} \langle |u|^{-1}u, \varphi \rangle
\leq \sup_{\varphi \in V, \|A^\frac{r}{2}\varphi\|_H = 1} \|\mathbf{u}\|_V^\frac{r}{2} \|\varphi\|_V^\frac{1}{r}
\leq C \sup_{\varphi \in V, \|A^\frac{r}{2}\varphi\|_H = 1} \|\mathbf{u}\|_V^\frac{r}{2} \|\mathbf{u}\|_V^{-\frac{1}{r}} \|A^\frac{r}{2}\varphi\|_H
\leq C \|\mathbf{u}\|_V^\frac{r}{2} \|\mathbf{u}\|_V^{-\frac{1}{r}}. \tag{94}
$$

for $r \in [\frac{11}{9}, 4]$. Substituting (94) in (93), we find

$$
|\langle |u|^{-1}u, A^\frac{r}{2}\mathbf{z} \rangle| \leq \frac{\nu}{4} \|A^\frac{r}{2}\mathbf{z}\|_H^2 + \frac{C}{\nu} \|\mathbf{u}\|_V^\frac{2r}{2} \|\mathbf{u}\|_V^{-\frac{2r}{2}}. \tag{95}
$$

For $r \in [1, \frac{11}{9}]$, we use the embedding of $L^2 \subset L^\frac{4r}{2r-2}$ and Poincaré inequality to obtain $\|u\|_V^\frac{2r}{2r-2} \leq C\|u\|_H^r \leq \frac{C}{\lambda_1^r} \|u\|_V^r$, and the result follows.

Taking inner product with $A\mathbf{z}(\cdot)$ to the first equation in (76), and following similarly as above, one can show that $A^H_1$ is bounded in $V^u_{\frac{3}{2}}$ also. In this case, one has to replace (84) with

$$
|\langle B(\mathbf{u}), A\mathbf{z} \rangle| \leq \|B(\mathbf{u})\|_H \|A\mathbf{z}\|_H \leq \|\mathbf{u}\|_{L^\infty} \|\mathbf{u}\|_V \|A\mathbf{z}\|_H \leq C\|\mathbf{u}\|_{V^u_{\frac{3}{2}}} \|\mathbf{u}\|_V \|A\mathbf{z}\|_H.
$$
\[ \leq \frac{\nu}{4} \|Az\|_H^2 + \frac{C}{\nu} \|u\|_V^2 \|v\|_Y^2, \]  

(96)

where we used the fact that \( V_2 \) is continuously embedded in \( L^\infty(\Omega) \) (see Remark 1). Similarly, we estimate \(|(|u|^{-1}u, Az)|\) as

\[ |(|u|^{-1}u, Az)| \leq \|u\|_{L^\infty} \|Az\|_H \leq \frac{\nu}{4} \|Az\|_H^2 + \frac{C}{\lambda_1 \nu} \|u\|_V^{2(r-1)} \|v\|_Y^2, \]  

(97)

and this completes the proof. \( \square \)

3.3. **Absorbing ball in \( V_2^\mu \).** Let us now establish that the semigroup \( S(t) : V_2^\mu \rightarrow V_2^\mu \) has an absorbing ball in \( V_2^\mu \).

**Proposition 5.** For \( r \in [1, 3] \), the set

\[ B_2 := \left\{ v \in V_2^\mu : \|v\|_{V_2^\mu} \leq M_2 \equiv \sqrt{\frac{2(1 + \mu \lambda_1)}{\lambda_1}} \left( \|f\|_H^2 + \frac{27CM_1^6}{4\nu^2} + \frac{C_2M_2^{2r}}{\lambda_1^{1+r}} \right)^{1/2} \right\}, \]  

(98)

is a bounded absorbing set in \( V_2^\mu \) for the semigroup \( S(t) \). That is, given a bounded set \( B \subset V_2^\mu \), there exists an entering time \( t_B > 0 \) such that \( S(t)B \subset B_2 \), for all \( t \geq t_B \).

For \( r \in (3, 4] \), the set

\[ B_2 := \left\{ v \in V_2^\mu : \|v\|_{V_2^\mu} \leq M_2 \equiv \sqrt{\frac{2(1 + \mu \lambda_1)}{\lambda_1}} \left( \|f\|_H^2 + \frac{27CM_1^6}{4\nu^2} + \frac{C_2M_2^{2r}}{\lambda_1^{1+r}} \right)^{1/2} \right\}, \]  

(99)

where \( C_r = C\left(\frac{3-r}{1-r}\right)\left(\frac{r-1}{r}\right)^{\frac{\gamma}{1+r}} \) is a bounded absorbing set in \( V_2^\mu \) for the semigroup \( S(t) \).

**Proof.** For \( r \in [1, 3] \), from (47), we have

\[ \frac{d}{dt} \left[ \|A^\frac{1}{2}u(t)\|_H^2 + \mu \|Au(t)\|_Y^2 \right] + \frac{\nu}{2} \|Au(t)\|_Y^2 \leq 2 \left( \|f\|_H^2 + \frac{27CM_1^6}{4\nu^2} \|u(t)\|_Y^6 + \frac{C_2M_2^{2r}}{\lambda_1^{1+r}} \|u(t)\|_Y^2 \right), \]  

(100)

For large enough \( t \), using (69) in (100), we deduce that

\[ \frac{d}{dt} \left[ \|A^\frac{1}{2}u(t)\|_H^2 + \mu \|Au(t)\|_Y^2 \right] + \frac{\nu \lambda_1}{2(1 + \mu \lambda_1)} \left[ \|A^\frac{1}{2}u(t)\|_H^2 + \mu \|Au(t)\|_Y^2 \right] \leq 2 \left( \|f\|_H^2 + \frac{27CM_1^6}{4\nu^2} + \frac{C_2M_2^{2r}}{\lambda_1^{1+r}} \right). \]  

(101)

An application of Gronwall’s inequality in (100) yields

\[ \|A^\frac{1}{2}u(t)\|_H^2 + \mu \|Au(t)\|_Y^2 \leq e^{-\left(\frac{\nu \lambda_1}{2(1 + \mu \lambda_1)}\right)t} \left( \|A^\frac{1}{2}u_0\|_H^2 + \mu \|Au_0\|_Y^2 \right) \]
Then, from the estimates (66)-(69), we have
to the autonomous version of the system (13) belonging to the global attractor
used in [22, 37, 30], etc to get required results. Let $u$
the system has finite Hausdorff and fractal dimensions. We apply similar techniques
we show that the global attractor for the 3D Kelvin-Voigt-Brinkman-Forchheimer
the semigroup with respect to the initial data is established in this section. Then,

That is, we have

$$
\|u(t)\|_{V_2^2}^2 \leq e^{-\left(\frac{\nu_1}{3(1 + \mu_1)}\right) t} \|u_0\|_{V_2^2}^2 + \frac{(1 + \mu_1)}{\lambda_1} \left(\|f\|_{H}^2 + \frac{27CM_1^6}{4\nu^2} + \frac{C\beta^2M_1^{2\nu}}{\lambda_1^{2\nu}}\right) + \frac{(1 + \mu_1)}{\lambda_1} \left(\|f\|_{H}^2 + \frac{27CM_1^6}{4\nu^2} + \frac{C\beta^2M_1^{2\nu}}{\lambda_1^{2\nu}}\right).
$$

(102)

Hence, the inequality (102) assures the existence of an absorbing ball (98), so that
for all $t \gg 1$, we have $\|u(t)\|_{V_2^2} \leq M_2.$

Similarly, for $r \in (3, 5)$, we obtain

$$
\|u(t)\|_{V_2^2}^2 \leq e^{-\left(\frac{\nu_1}{3(1 + \mu_1)}\right) t} \|u_0\|_{V_2^2}^2 + \frac{(1 + \mu_1)}{\lambda_1} \left(\|f\|_{H}^2 + \frac{27CM_1^6}{4\nu^2} + C\beta^2 \frac{1}{\nu} \left(5 - r \frac{1}{\nu} \left(2(1 + \mu_1)\lambda_1^{-2\nu} \right)\right)\right),
$$

so that $B_2$ given in (99) is an absorbing ball in $V_2^\mu$ for the semigroup $S(t)$. □

4. Estimates of dimensions of the global attractor. The differentiability of
the semigroup with respect to the initial data is established in this section. Then,
we show that the global attractor for the 3D Kelvin-Voigt-Brinkman-Forchheimer
system has finite Hausdorff and fractal dimensions. We apply similar techniques
used in [22, 37, 30], etc to get required results. Let $u(\cdot)$ be the unique weak solution
to the autonomous version of the system (13) belonging to the global attractor $A_1^\mu$.
Then, from the estimates (66)-(69), we have

$$
\sup_{t \geq 0} \sup_{u_0 \in K_1} \|u(t)\|_{V_1^\mu} \leq M_1 = \frac{1}{\nu} \sqrt{\frac{3(1 + \mu_1)}{\lambda_1}} \|f\|_{V_1^\nu},
$$

(103)

and

$$
\limsup_{T \to \infty} \frac{1}{T} \int_0^T \|u(t)\|_{V_1^\nu}^2 dt \leq K_1 := \frac{\|f\|_{V_1^\nu}^2}{\nu^2}.
$$

(104)

**Theorem 4.1.** Let $u_0$ and $v_0$ be two members of $V_1^\mu$. Then there exists a constant $K = K(\|u_0\|_{V_1^\mu}, \|v_0\|_{V_1^\nu})$ such that

$$
\|S(t)u_0 - S(t)v_0 - \Lambda(t)(u_0 - v_0)\|_{V_1^\mu} \leq K\|u_0 - v_0\|_{V_1^\nu}^r,
$$

(105)

where the linear operator $\Lambda(t) : V_1^\mu \to V_1^\mu$, for $t > 0$ is the solution operator of the problem:

$$
\begin{aligned}
\frac{d}{dt}[(I + \mu A)\xi(t)] + \nu A\xi(t) + B(\xi(t), u(t)) + B(u(t), \xi(t)) \\
+ \beta r P_\nu(|u(t)|^{-1} \xi(t)) = 0, \quad t \in (0, T), \\
\xi(0) = \xi_0 \in V_1^\mu,
\end{aligned}
$$

(106)

$\xi_0 = u_0 - v_0$ and $u(t) = S(t)u_0$. That is, for every $t > 0$, the solution $S(t)u_0$, as
a map $S(t) : V_1^\mu \to V_1^\mu$ is Fréchet differentiable with respect to the initial data, and
its Fréchet derivative $D_{u_0}S(t)u_0|w_0 = \Lambda(t)w_0.$
Proof. Let us define $\eta(t) := u(t) - v(t) - \xi(t) = S(t)(u_0 - v_0) - \xi(t)$. Then, using Taylor’s formula, one can easily see that $\eta(t)$ satisfies:

$$
\begin{aligned}
\frac{d}{dt}[(I + \mu A)\eta(t)] + \nu A\eta(t) + B(\eta(t), u(t)) + B(u(t), \eta(t)) - B(w(t), w(t)) \\
+ \beta r P_H \left( (\theta u(t) + (1 - \theta)v(t))|^{r-1} - |u(t)|^{r-1} \right) w(t) + \beta r P_H \left( |u(t)|^{r-1} \eta(t) \right) = 0,
\end{aligned}
$$

(107)

for $t \in (0, T)$, where $w(t) = u(t) - v(t)$. Let us take inner product with $\eta(\cdot)$ to the first equation in (107) to obtain

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left( ||\eta(t)||_{L^2}^2 + \mu ||\eta(t)||_{V}^2 \right) + \nu ||\eta(t)||_{V}^2 + \beta r (P_H (|u(t)|^{r-1} \eta(t)), \eta(t)) \\
+ \beta r (P_H (|\theta u(t) + (1 - \theta)v(t)|^{r-1} - |u(t)|^{r-1}) w(t), \eta(t)).
\end{aligned}
$$

(108)

We estimate $|b(\eta, u, \eta)|$, using Hölder’s, Ladyzhenskaya and Young’s inequalities as

$$
|b(\eta, u, \eta)| \leq ||u||_V ||\eta||_{L^2}^2 \leq 2 \hat{M}_1 ||\eta||_{L^2} ||\eta||_{V}^{3/2} \leq \frac{2 \hat{M}_1}{\lambda_1^{1/2}} ||\eta||_V^3.
$$

(109)

Furthermore, applying Hölder’s, Ladyzhenskaya, Poincaré and Young’s inequalities, we estimate $|b(w, w, \eta)|$ as

$$
|b(w, w, \eta)| = |b(w, \eta, w)| \leq ||\eta||_{V} ||w||_{L^2}^2 \leq \frac{2}{\lambda_1^{1/2}} ||\eta||_{V} ||w||_V^2 \leq \frac{\nu}{4} ||\eta||_V^2 + \frac{4}{\nu \lambda_1^{1/2}} ||w||_V^4.
$$

(110)

Let us take final term from the equality (108) and estimate in the following different cases.

**Case (1).** $r \in [1, 2]$. For $r = 1$, the term is zero. For $r \in (1, 2)$, and $p = r - 1$, $0 < p < 1$, let us consider

$$
\begin{aligned}
\beta r \langle (|\theta u + (1 - \theta)v|^p - |u|^p) w, \eta \rangle \\
\leq \beta r (1 - \theta) \langle |u - v|^p |w, |\eta| \rangle \leq \beta r ||w||_{L^{1+p/2}} ||\eta||_{L^3} \\
\leq C \beta r ||w||_{L^{1+p/2}} ||\nabla u||_{L^6} ||\eta||_{L^2} \leq \frac{C \beta r}{\lambda_1^{1/4}} ||w||_{L^{1+p/2}} ||\eta||_V \\
\leq \frac{\nu}{4} ||\eta||_V^2 + \frac{C \beta^2 r^2}{\nu \lambda_1^{1/2}} ||w||_V^{2(1+p)/2} \leq \frac{\nu}{4} ||\eta||_V^2 + \frac{C \beta^2 r^2}{\nu \lambda_1^{1/2}} ||w||_V^{2(1+p)},
\end{aligned}
$$

(111)

where we used (A.6) and (33). For $r = 2$, we estimate $\beta r \langle (|\theta u + (1 - \theta)v|^2 - |u|^2) w, \eta \rangle$, using (A.6), Hölder’s and Young’s inequalities as

$$
\begin{aligned}
\beta r \langle (|\theta u + (1 - \theta)v|^2 - |u|^2) w, \eta \rangle \leq \beta r (1 - \theta) ||u - v||_V ||w||_V ||\eta||_V \\
\leq \beta r ||w||_{L^{1/2}} ||\eta||_{L^3} \leq \frac{C \beta r}{\lambda_1^{1/4}} ||w||_{L^2} ||\eta||_V \\
\leq \frac{\nu}{4} ||\eta||_V^2 + \frac{C \beta^2 r^2}{\nu \lambda_1^{1/2}} ||w||_V^4.
\end{aligned}
$$

(112)

**Case (2).** $r \in (2, 3)$. Let us now consider the case $r \in (2, 3)$, that is, we take $r = 2 + p$, for $0 < p < 1$. We estimate $\beta r \langle (|\theta u + (1 - \theta)v|^r - |u|^r) w, \eta \rangle$ using (A.6),
Hölder’s and Young’s inequalities as

\[
\beta r \langle (\theta u + (1 - \theta)v)^{1+p} - |u|^{1+p}, \eta \rangle \\
= \beta r \langle (\theta u + (1 - \theta)v)[(1 - \theta)v]^p - |u|^p, \eta \rangle \\
+ \beta r \langle (\theta u + (1 - \theta)v| |u|^p, \eta \rangle \\
\leq \beta r (1 - \theta) \|\theta u + (1 - \theta)v| |u - v|^p \| \| \eta \|_2 + \beta r (1 - \theta) \|\theta u + (1 - \theta)v - u| |^p \| \| \eta \|_2 \\
\leq \beta r (\|u\|_{L^p} + \|v\|_{L^p}) \| \eta \|_2^{1+p} + \beta r \|w\|_{L^p}^p \| \eta \|_2 \\
\leq \frac{\nu}{4} \|\eta\|_2^p + \frac{2\beta^2 r^2}{\nu \lambda_1^{1/2}} (\|u\|_{L^p}^2 + \|v\|_{L^p}^2) \| \eta \|_2^{2(p+1)} + \frac{2\beta^2 r^2}{\nu \lambda_1^{1/2}} \|u\|_{L^p}^2 \| \eta \|_2^{1+p} \\
\leq \frac{\nu}{4} \|\eta\|_2^p + \frac{4\beta^2 r^2}{\nu \lambda_1^{1/2}} \|u\|_{L^p}^2 \| \eta \|_2^{2(p+1)} + \frac{2\beta^2 r^2}{\nu \lambda_1^{1/2}} \|u\|_{L^p}^2 \| \eta \|_2^4 \\
\leq \frac{\nu}{4} \|\eta\|_2^p + \frac{8\beta^2 r^2 \lambda_1^{1/2}}{\nu \lambda_1^{1/2}} \|u\|_{L^p}^2 \| \eta \|_2^{2(p+1)} + \frac{2\beta^2 r^2 \lambda_1^{1/2}}{\nu \lambda_1^{1/2}} \|u\|_{L^p}^2 \| \eta \|_2^4. \tag{113}
\]

Case (3). \( r \in [3, 4] \). For \( r \in [3, 4] \), let us use the differentiability of \( f(x) = |x|^{r-1} \) (that is, use Taylor’s formula), (A.6), Hölder’s and Young’s inequalities as

\[
\beta r \langle (\theta u + (1 - \theta)v)^{r-1} - |u|^{r-1}, \eta \rangle \\
= \beta r (r-1)(1 - \theta) \langle (h + (1 - h)\theta)u + (1 - h)(1 - \theta)v| |h + (1 - h)\theta)u \\
+ (1 - h)(1 - \theta)v| |^{r-3} \|w\|_2^2, \eta \rangle \\
\leq \beta r (r-1)(1 - \theta) \langle (h + (1 - h)\theta)u + (1 - h)(1 - \theta)v| |^{r-2} \|w\|_2, \eta \rangle \\
\leq \beta r (r-1)2^{r-3} \langle (|u|^{r-2} + |v|^{r-2}) \|w\|_2, \eta \rangle \\
\leq \beta r (r-1)2^{r-3} \langle \|u\|_{L^2}^{r-2} + \|v\|_{L^2}^{r-2} \rangle \|w\|_2 \| \eta \|_2^{r-3} \\
\leq \frac{\nu}{4} \|\eta\|_2^p + \frac{C\beta^2 r^2 (r-1)2^{2r-6}}{\nu \lambda_1^{1/2}} (\|u\|_{L^2}^{r-2} + \|v\|_{L^2}^{r-2})^2 \|w\|_2^2 \\
\leq \frac{\nu}{4} \|\eta\|_2^p + \frac{C\beta^2 r^2 (r-1)2^{2r-5}}{\nu \lambda_1^{1/2}} (\|u\|_{L^2}^{2(r-2)} + \|v\|_{L^2}^{2(r-2)}) \|w\|_2^4 \\
\leq \frac{\nu}{4} \|\eta\|_2^p + \frac{C\beta^2 r^2 (r-1)2^{2(r-2)\lambda_1^{1/2}2^{(r-2)}}}{\nu \lambda_1^{1/2}} \|w\|_2^4. \tag{114}
\]

For \( r \in [1, 2] \), combining (108)-(112) and substituting it in (108), we find

\[
\frac{d}{dt} \left( \|\eta(t)\|_H^2 + \mu \|\eta(t)\|_2^2 \right) + \nu \|\eta(t)\|_2^2 + \beta r \|\eta(t)\|_2^{r-1} \|\eta(t), \eta(t)\|_2 \\
\leq \frac{4\lambda_1^{1/2}}{\lambda_1^{1/2}} \|\eta(t)\|_2^2 + \frac{2\beta^2 r^2}{\nu \lambda_1^{1/2}} \|w(t)\|_2^4 + \frac{2C\beta^2 r^2}{\nu \lambda_1^{1/2} + \lambda_1^{1/2}} \|w(t)\|_{L^4}^{2(1+p)}, \tag{115}
\]

for \( p \in (0, 1) \).

Note that the function \( w(t) = u(t) - v(t) = S(t)u_0 - S(t)v_0 \) satisfies the following system:
\[
\begin{aligned}
&\frac{d}{dt}[(I + \mu A)w(t)] + \nu Aw(t) + Bw(t, u(t)) + B(v(t), w(t)) - B(w(t), w(t)) \\
&+ \beta P_H[(u(t)]^{-1}u(t)) - \beta P_H[(v(t)]^{-1}v(t)) = 0, \quad t \in (0, T), \\
w(0) = u_0 - v_0 =: w_0 \in V.
\end{aligned}
\]

(116)

Taking inner product with \(w(\cdot)\) to the first equation in (116), we get
\[
\frac{1}{2} \frac{d}{dt} \left( \|w(t)\|^2_H + \mu \|w(t)\|^2_\nu \right) + \nu \|w(t)\|^2_\nu + \beta \langle P_H[(u(t)]^{-1}u(t)) - P_H[(v(t)]^{-1}v(t)), w(s) \rangle \rangle = 0.
\]

(117)

A calculation similar to (109) yields
\[
|b(w, u, w)| \leq \frac{2\|u\|_\nu}{\lambda^{1/4}} \|w\|^2_\nu \leq \frac{\nu}{2} \|w\|^2_\nu + \frac{2}{\nu \lambda^{1/2}} \|u\|^2_\nu \|w\|^2_\nu.
\]

(118)

Using (118) in (116), and integrating the inequality from 0 to \(t\), we obtain
\[
\begin{aligned}
&\|w(t)\|^2_H + \mu \|w(t)\|^2_\nu + \nu \int_0^t \|w(s)\|^2_\nu ds \\
&+ \beta \int_0^t \langle P_H[(u(s)]^{-1}u(s)) - P_H[(v(s)]^{-1}v(s)), w(s) \rangle \rangle ds \\
&\leq \|w_0\|^2_H + \mu \|w_0\|^2_\nu + \frac{4}{\nu \lambda^{1/2}} \int_0^t \|u(s)\|^2_\nu \|w(s)\|^2_\nu ds.
\end{aligned}
\]

(119)

Thus, using (24) and (53) in (119), we deduce that
\[
\|w(t)\|^2_\nu \leq \|w_0\|^2_\nu + \frac{4}{\nu \lambda^{1/2}} \int_0^t \|u(s)\|^2_\nu \|w(s)\|^2_\nu ds.
\]

(120)

An application of Gronwall’s inequality in (120) yields
\[
\begin{aligned}
\|w(t)\|^2_\nu \leq \|w_0\|^2_\nu \exp \left( \frac{4}{\nu \mu \lambda^{1/2}} \int_0^t \|u(s)\|^2_\nu ds \right) \\
&\leq \|w_0\|^2_\nu \exp \left( \frac{4}{\nu \mu \lambda^{1/2}} \left[ \|u_0\|^2_\nu + \frac{t}{\nu} \|f\|^2_\nu \right] \right),
\end{aligned}
\]

(121)

where we used (68). Using (121) in (115) and then integrating from 0 to \(t\), we find
\[
\begin{aligned}
&\|w(t)\|^2_\nu + \mu \|\eta(t)\|^2_\nu + \nu \int_0^t \|\eta(s)\|^2_\nu ds \\
&\leq \frac{4M_1}{\lambda^{1/4}} \int_0^t \|\eta(s)\|^2_\nu ds + \frac{2\beta^2 r^2}{\nu \mu \lambda^{1/2}} \|w_0\|^4_\nu \times \int_0^t \exp \left( \frac{4}{\mu \lambda^{1/4}} \left[ 2 \left( \frac{1 + \mu \lambda}{\nu \mu} \right) \|u_0\|_{\nu^p} + \frac{1}{\nu} \sqrt{\frac{1 + \mu \lambda}{\lambda}} \|f\|_{\nu^p} \right] \right) ds \\
&+ \frac{2C \beta^2 r^2}{\nu \mu \lambda^{1/2}} \|w_0\|_{\nu^{1+p}} \times \int_0^t \exp \left( \frac{2(1+p)}{\mu \lambda^{1/4}} \left[ 2 \left( \frac{1 + \mu \lambda}{\nu \mu} \right) \|u_0\|_{\nu^p} + \frac{1}{\nu} \sqrt{\frac{1 + \mu \lambda}{\lambda}} \|f\|_{\nu^p} \right] \right) ds.
\end{aligned}
\]

(122)
Thus, from (122), we infer that

$$\|\eta(t)\|_{V_t}^2 \leq \frac{4\tilde{M}_1}{\lambda_0^2 \mu^2} \int_0^t \|\eta(s)\|_{V_t}^2 \, ds + \kappa_1(t)\|w_0\|_{V_t}^4 + \kappa_2(t)\|w_0\|_{V_t}^{2(1+p)},$$

(123)

where

$$\kappa_1(t) = \frac{\beta^2 r^2}{2\lambda_1^3 \sqrt{1 + \mu \lambda_1} \|f\|_{V_t}} \times \exp \left\{ \frac{4}{\mu^{5/2} \lambda_1^{1/4}} \left[ 2 \left( \frac{1 + \mu \lambda_1}{\nu \mu} \right) \|u_0\|_{V_t} + \frac{1}{\nu} \sqrt{\frac{1 + \mu \lambda_1}{\lambda_1} \|f\|_{V_t} t} \right] \right\},$$

(124)

and

$$\kappa_2(t) = \frac{C \sqrt{\beta^2 r^2 \mu^{p - \frac{5}{2}}}}{(1 + p) \lambda_1^{2 + \frac{5}{2}} \sqrt{1 + \mu \lambda_1} \|f\|_{V_t}} \times \exp \left\{ \frac{2(1 + p)}{\mu^{5/2} \lambda_1^{1/4}} \left[ 2 \left( \frac{1 + \mu \lambda_1}{\nu \mu} \right) \|u_0\|_{V_t} + \frac{1}{\nu} \sqrt{\frac{1 + \mu \lambda_1}{\lambda_1} \|f\|_{V_t} t} \right] \right\}.$$  

(125)

An application of Gronwall’s inequality in (123) yields

$$\|\eta(t)\|_{V_t}^2 \leq \left( \kappa_1(t)\|w_0\|_{V_t}^4 + \kappa_2(t)\|w_0\|_{V_t}^{2(1+p)} \right) e^{\frac{4\tilde{M}_1 t}{\lambda_1^2 r^p}}.$$  

Thus, by the definition of $\eta(t)$, it is immediate that

$$\frac{\|u(t) - v(t) - \xi(t)\|_{V_t}}{\|u_0 - v_0\|_{V_t}} \leq \left( \kappa_1(t)\|u_0 - v_0\|_{V_t} + \kappa_2(t)\|u_0 - v_0\|_{V_t}^{(1+p)} \right)^{1/2} e^{\frac{2\tilde{M}_1 t}{\lambda_1^2 r^p}},$$

(126)

and hence the differentiability of $S(t)$ with respect to the initial data follows.

For $r \in (2, 3)$, a similar calculation as above gives

$$\frac{\|u(t) - v(t) - \xi(t)\|_{V_t}}{\|u_0 - v_0\|_{V_t}} \leq \left( \kappa_3(t)\|u_0 - v_0\|_{V_t} + \kappa_4(t)\|u_0 - v_0\|_{V_t}^{(1+p)} \right)^{1/2} e^{\frac{2\tilde{M}_1 t}{\lambda_1^2 r^p}},$$

(127)

where

$$\kappa_3(t) = \frac{C \beta^2 r^2 \tilde{M}_1^{2p}}{\mu^{2 + \frac{5}{2}} \lambda_1^{2 + \frac{5}{2}} \sqrt{1 + \mu \lambda_1} \|f\|_{V_t}} \times \exp \left\{ \frac{4}{\mu^{5/2} \lambda_1^{1/4}} \left[ 2 \left( \frac{1 + \mu \lambda_1}{\nu \mu} \right) \|u_0\|_{V_t} + \frac{1}{\nu} \sqrt{\frac{1 + \mu \lambda_1}{\lambda_1} \|f\|_{V_t} t} \right] \right\},$$

(128)

and

$$\kappa_4(t) = \frac{4C \beta^2 r^2 \mu^{p - \frac{5}{2}} \lambda_1^{2 + \frac{5}{2}} \tilde{M}_1^{2p}}{(1 + p) \lambda_1^{2 + \frac{5}{2}} \sqrt{1 + \mu \lambda_1} \|f\|_{V_t}} \times \exp \left\{ \frac{2(1 + p)}{\mu^{5/2} \lambda_1^{1/4}} \left[ 2 \left( \frac{1 + \mu \lambda_1}{\nu \mu} \right) \|u_0\|_{V_t} + \frac{1}{\nu} \sqrt{\frac{1 + \mu \lambda_1}{\lambda_1} \|f\|_{V_t} t} \right] \right\}.$$  

(129)
For $r \in [3, 4]$, we obtain
\[
\frac{\|u(t) - v(t) - \xi(t)\|_{\mathcal{V}_r}}{\|u_0 - v_0\|_{\mathcal{V}_r}} \leq \sqrt{\kappa_5(t)}\|u_0 - v_0\|_{\mathcal{V}_1} e^{\frac{2\lambda_1 t}{\sqrt{1 + \mu \lambda_1}}}, \tag{130}
\]
where
\[
\kappa_5(t) = \frac{2^{2r} C \sqrt{\mu} \beta r^2 (r - 1)^2 \hat{M}^2(\hat{r} - 2)}{\lambda_1^{\frac{r}{2r} - r} \mu^\frac{1}{2} \sqrt{1 + \mu \lambda_1} \|f\|_{\mathcal{V}_r}}
\times \exp\left\{\frac{4}{\mu^3/2 \lambda_3^{1/4}} \left[\left(\frac{1 + \mu \lambda_1}{\nu \mu}\right)\|u_0\|_{\mathcal{V}_r} + \frac{1}{\nu} \sqrt{\frac{1 + \mu \lambda_1}{\lambda_1}} \|f\|_{\mathcal{V}_r} t\right]\right\}, \tag{131}
\]
which completes the proof. \(\square\)

Let us now set $G^2 = (1 + \mu A)$, $\tilde{u} = Gu$, $\tilde{v} = Gv$. Thus, we rewrite the system (13) as
\[
\begin{cases}
\frac{d}{dt} \tilde{u}(t) = -\frac{\mu_0}{\mu} \tilde{u}(t) + \frac{\mu_0}{\mu} G^{-2} \tilde{u}(t) - G^{-1} B(G^{-1} \tilde{u}(t), G^{-1} \tilde{u}(t)) \\
- \beta G^{-1} P_\mathcal{H}(|G^{-1} \tilde{u}|^{-1} G^{-1} \tilde{u}) + G^{-1} f, \quad t \in (0, T),
\end{cases} \tag{132}
\]
where $Gu_0 \in \mathcal{H}$. Note that the systems (132) and (13) are equivalent. Remember that the system (13) is well posed in $\mathcal{V}$, while the system (132) is well posed in $\mathcal{H}$ as the norms $\|\cdot\|_{\mathcal{V}}$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent (see Remark 1). Therefore, there exists a unique weak solution $\tilde{u}(t)$ of (132) in $C([0, T]; \mathcal{H})$. Moreover, the system (132) generates an one parameter family of strongly continuous semigroup $\tilde{S}(t)$ of solution operators
\[
\tilde{S}(t) : \mathcal{H} \rightarrow \mathcal{H}, \quad \tilde{u}_0 \mapsto \tilde{u}(t) = \tilde{S}(t)\tilde{u}_0.
\]
Since $\tilde{u}(t) = Gu(t)$ and $\tilde{u}_0 = Gu_0$, the semigroup $\tilde{S}(t)$ is connected to the original semigroup $S(t)$ through the relation
\[
\tilde{S}(t) = G^{-1} S(t) G. \tag{133}
\]
Thus, it is immediate that the semigroup $\tilde{S}(t)$ possesses the global attractor $\tilde{A}^\mu_1$, where
\[
\tilde{A}^\mu_1 = GA^\mu_1,
\]
and $A^\mu_1$ is the global attractor for $S(t)$.

Our first aim is to show a bound for the fractal dimension of $\tilde{A}^\mu_1$ in $\mathcal{H}$. Furthermore, the same bound for fractal dimension of $A^\mu_1$ in $\mathcal{V}^\mu_1$ follows easily using the following argument. From Proposition 3.1, Chapter VI, [35], we know that the fractal dimension estimates are preserved under Lipschitz maps. Moreover, form Remark 3.14, [30], we infer that
\[
\dim_{L^p}(A^\mu_1) = \dim_{L^p}(G^{-1} \tilde{A}^\mu_1) = \dim_{L^p}(\tilde{A}^\mu_1).
\tag{134}
\]
Let us first consider the linear variations of the system (132). The equation of linear variations corresponding to (132) has the form
\[
\frac{dw(t)}{dt} = L(t, \tilde{u})w(t), \tag{135}
\]
where
\[ L(t, \tilde{u})w(t) = -\frac{\nu}{\mu} w(t) + \frac{\nu}{\mu} G^{-2} w(t) - G^{-1} B(G^{-1} w(t), G^{-1} \tilde{u}(t)) \]
\[ \quad - G^{-1} B(G^{-1} \tilde{u}(t), G^{-1} w(t)) - \beta r G^{-1} P_H(|G^{-1} \tilde{u}(t)|^{-1} G^{-1} w(t)). \]

The adjoint \( L^*(t, \tilde{u}) \) of \( L(t, \tilde{u}) \) is given by
\[ L^*(t, \tilde{u})w(t) = -\frac{\nu}{\mu} w(t) + \frac{\nu}{\mu} G^{-2} w(t) - G^{-1} B(G^{-1} w(t), G^{-1} \tilde{u}(t)) \]
\[ \quad - G^{-1} B(G^{-1} \tilde{u}(t), G^{-1} w(t)) - \beta r G^{-1} P_H(|G^{-1} \tilde{u}(t)|^{-1} G^{-1} w(t)). \]

Hence, \( \tilde{L}(t, \tilde{u})w(t) = L(t, \tilde{u})w(t) + L^*(t, \tilde{u})w(t) \) can be computed as
\[ \tilde{L}(t, \tilde{u})w(t) = -\frac{2\nu}{\mu} w(t) + \frac{2\nu}{\mu} G^{-2} w(t) - 2G^{-1} B(G^{-1} w(t), G^{-1} \tilde{u}(t)) \]
\[ \quad - 2G^{-1} B(G^{-1} \tilde{u}(t), G^{-1} w(t)) - 2\beta r G^{-1} P_H(|G^{-1} \tilde{u}(t)|^{-1} G^{-1} w(t)). \] (136)

Next, we derive the following important result.

**Proposition 6.** Let \( w \in H \). Then, we have
\[ (\tilde{L}(t, \tilde{u})w(t), w(t)) \leq -\overline{h}_0 |w(t)|^2_{H^2} + \overline{h}_1(t) |G^{-1} w(t)|^2_{H^2}, \] (137)

where
\[ \overline{h}_0 = \frac{\nu}{\mu} \quad \text{and} \quad \overline{h}_1(t) = 2 \left[ \frac{\nu}{\mu} + \frac{27M^2}{16\nu^3 \mu} \|u(t)\|_{\nu}^2 \right]. \] (138)

**Proof.** Let us take inner product with \( w(\cdot) \) to the first equation in (136) to find
\[ (\tilde{L}(t)w(t), w(t)) \]
\[ \quad = -\frac{2\nu}{\mu} \|w(t)\|_{H^2}^2 + \frac{2\nu}{\mu} \|G^{-1} w(t)\|_{H^2}^2 - 2b(G^{-1} w(t), G^{-1} \tilde{u}(t), G^{-1} w(t)) \]
\[ \quad - 2\beta r (P_H(|G^{-1} \tilde{u}(t)|^{-1} G^{-1} w(t), G^{-1} w(t))). \] (139)

Remember that
\[ \|G^{-1} u\|_{\nu}^2 = \|A^{1/2}(I + \mu A)^{-1/2} u\|_{H^2}^2 = \sum_{j=1}^{\infty} \frac{\lambda_j}{(1 + \mu \lambda_j)} |(u, e_j)|^2 \leq \frac{1}{\mu} \sum_{j=1}^{\infty} |(u, e_j)|^2 \]
\[ = \frac{1}{\mu} \|u\|_{H^2}^2. \]

Using Hölder’s, Ladyzhenskaya and Young’s inequalities, we estimate the term \( |b(G^{-1} w, G^{-1} \tilde{u}, G^{-1} w)| \) as
\[ |b(G^{-1} w, G^{-1} \tilde{u}, G^{-1} w)| \leq \|G^{-1} \tilde{u}\|_\nu \|G^{-1} w\|_{H^2}^2, \]
\[ \leq 2 \|u\|_\nu \|G^{-1} w\|_{H^2}^{1/2} \|G^{-1} w\|_{H^2}^{3/2}, \]
\[ \leq \frac{2}{\mu^{3/2}} \|u\|_\nu \|G^{-1} w\|_{H^2}^{1/2} \|w\|_{H^2}^{3/2}, \]
\[ \leq \frac{\nu}{\mu} \|w\|_{H^2}^2 + \frac{27}{16\nu^3 \mu} \|u\|_{\nu}^2 \|G^{-1} w\|_{H^2}^2. \] (140)

Let us use (103) and (53) to estimate \( \|u\|_\nu^2 \) as
\[ \|u\|_\nu^2 = \|u\|_\nu^2 \|u\|_{\nu}^2 \leq \frac{1}{\mu} \|u\|_{H^2} \|u\|_{\nu}^2 \leq \frac{1}{\mu} \|u\|_{H^2} \|u\|_{\nu}^2 \leq \frac{1}{\mu} \|u\|_{H^2} \|u\|_{\nu}^2. \]
Using this in (140), we get

\[ |b(G^{-1}w, G^{-1}\tilde{u}, G^{-1}w)| \leq \frac{\nu}{\mu} \|w\|_{\mathcal{H}}^2 + \frac{27\hat{M}^2}{16\nu^2\mu} \|u\|_\nu^2 \|G^{-1}w\|_{\mathcal{H}}^2. \]  

(141)

Note that

\[ \beta r(|G^{-1}\tilde{u}|G^{-1}w, G^{-1}w) \geq 0. \]  

(142)

Combining (141)-(142) and using it in (139), we obtain

\[ \langle \tilde{L}(t)w(t), w(t) \rangle \leq -\frac{\nu}{\mu} \|w(t)\|_{\mathcal{H}}^2 + 2 \left( \frac{\nu}{\mu} + \frac{27\hat{M}^2}{16\nu^2\mu} \|u(t)\|_\nu^2 \right) \|G^{-1}w(t)\|_{\mathcal{H}}^2. \]  

(143)

Comparing with Theorems 4.8 and 4.9, [26] or Theorem 2.2, [22], we find

\[ s_0 = 0, s_1 = -1, \tilde{r}_0 = \frac{\nu}{\mu}, \tilde{r}_s(t) = 2 \left[ \frac{\nu}{\mu} + \frac{27\hat{M}^2}{16\nu^2\mu} \|u(t)\|_\nu^2 \right], \]

and \( \tilde{r}_s(t) = 0 \), for all \( k \geq 2 \).

\[ \square \]

**Proposition 7.** The global attractor \( \tilde{A}_1^\mu \) has the finite fractal dimension in \( \mathbb{H} \), with

\[ \dim\mathcal{H}(\tilde{A}_1^\mu) \leq \dim\mathcal{F}(\tilde{A}_1^\mu) \leq \left\{ \frac{2}{C_\mu} \left[ 1 + \frac{81}{16\nu^2} \left( \frac{1}{\lambda_1} + \mu \right) \|f\|_\nu^2 \right] \right\}^{3/2}. \]  

(144)

**Proof.** Let \( w_{1,0}, \ldots, w_{n,0} \), for some \( n \geq 1 \), be an initial orthogonal set of infinitesimal displacements. The volume of the parallelepiped spanned by \( w_{1,0}, \ldots, w_{n,0} \), is given by

\[ V_n(0) = |w_{1,0} \wedge \ldots \wedge w_{n,0}|, \]

where \( \wedge \) denotes the exterior product. The evolution of such displacements satisfies the following evolution equation:

\[ \begin{aligned}
\frac{d}{dt} w_i(t) &= L(t, \tilde{u})w_i(t) \\
 w_i(0) &= w_{i,0},
\end{aligned} \]  

(145)

for all \( i = 1, \ldots, n \). Using Lemma 3.5, [12] (see [13] also), we know that the volume elements

\[ V_n(t) = |w_1(t) \wedge \ldots \wedge w_n(t)|, \]

satisfy

\[ V_n(t) = V_n(0) \exp \left[ \int_0^t \text{Tr}(P_n(s)L(s, \tilde{u}))ds \right]. \]  

(146)

where \( P_n(s) \) is the orthogonal projection onto the linear span of \( \{w_1(t), \ldots, w_n(t)\} \) in \( \mathbb{H} \). We also know that

\[ \text{Tr}(P_n(s)L(s, \tilde{u})) = \sum_{k=1}^n (L(s, \tilde{u})\varphi_k(s), \varphi_k(s)), \]

with \( n \geq 1 \) and \( \{\varphi_1(s), \ldots, \varphi_n(s)\} \) an orthonormal set spanning \( P_n(s)\mathbb{H} \). Next, we define

\[ [P_nL(\tilde{u})] := \limsup_{T \to \infty} \frac{1}{T} \int_0^T \text{Tr}(P_n(t)L(\tilde{u}, t))dt. \]
From (146), we have

$$V_n(t) = V_n(0) \exp \left[ t \sup_{u \in \hat{A}} \sup_{P_n(0)} \left[ P_n L(\tilde{u}) \right] \right], \quad (147)$$

for all $t \geq 0$, where the supremum over $P_n(0)$ is a supremum over all choices of initial $n$ orthogonal set of infinitesimal displacements that have taken around $\tilde{u}$. Let us now show that the volume elements $V_n(t)$ decays exponentially in time whenever $n \geq N$, with $N > 0$ to be determined later. Let us use Proposition 6 and Lemma 6.2, Chapter VI, [35] to estimate $\frac{1}{T} \int_0^T \text{Tr}(P_n(t)L(\tilde{u},t))dt$ as

$$\frac{1}{T} \int_0^T \text{Tr}(P_n(t)L(\tilde{u},t))dt = \frac{1}{T} \int_0^T \sum_{k=1}^n (L(t,\tilde{u})\varphi_k(t),\varphi_k(t)) dt$$

$$\leq \frac{1}{T} \int_0^T \sum_{k=1}^n -\tilde{h}_0 \|\varphi_j(t)\|_H^2 dt + \frac{1}{T} \int_0^T \tilde{h}_1(t) \sum_{k=1}^n \|G^{-1}\varphi_j(t)\|_H^2 dt$$

$$\leq -h_0 n + \frac{1}{T} \int_0^T \tilde{h}_1(t) dt \sum_{k=1}^n \|\varphi_j(t)\|_H^2$$

$$\leq -h_0 n + \frac{1}{T} \int_0^T \tilde{h}_1(t) dt \sum_{k=1}^n \|\varphi_j(t)\|_H^2$$

$$\leq -h_0 n + \frac{n^{1/3}}{C\mu} \frac{1}{T} \int_0^T \tilde{h}_1(t) dt, \quad (148)$$

where $\tilde{C}$ is defined in (3). One can estimate $\frac{1}{T} \int_0^T \tilde{h}_1(t) dt$ using (138), (103) and (104) as

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T \tilde{h}_1(t) dt = \limsup_{T \to \infty} \frac{2}{T} \int_0^T \left[ \frac{\nu}{\mu} + \frac{27\tilde{M}_1^2}{16\nu^3\mu} \|u(t)\|^2 \right] dt$$

$$\leq 2 \left[ \frac{\nu}{\mu} + \frac{27\tilde{M}_1^2 K_1}{16\nu^3\mu} \right]. \quad (149)$$

Substituting (138) and (149) in (148), we find

$$\|P_n L(\tilde{u})\| \leq -\frac{\nu n}{\mu} + \frac{2n^{1/3}}{C\mu} \left[ \frac{\nu}{\mu} + \frac{27\tilde{M}_1^2 K_1}{16\nu^3\mu} \right]. \quad (150)$$

We need the right hand side of the above inequality must be negative, therefore, we require

$$n \geq N := \left\{ \frac{2}{C\mu} \left[ 1 + \frac{27\tilde{M}_1^2 K_1}{16\nu^4} \right] \right\}^{3/2}, \quad (151)$$

where $\tilde{C}$ is defined in (3). Applying the definition of $\tilde{M}_1$ and $K_1$ given in (103) and (104), we obtain

$$N = \left\{ \frac{2}{C\mu} \left[ 1 + \frac{81}{16\nu^8} \left( \frac{1}{\lambda_1} + \mu \right) \|f\|_V^4 \right] \right\}^{3/2}, \quad (152)$$
which completes the proof. \hfill \square

Since $\tilde{A}_1^\mu$ has finite fractal dimension in $H$ with the bound (144), using (134), one can easily prove the following Theorem.

**Theorem 4.2.** The global attractor $A_1^\mu$ obtained in Theorem 3.2 has finite Hausdorff and fractal dimensions, which can be estimated by

$$\dim_H(A_1^\mu) \leq \dim_F(A_1^\mu) \leq \left\{ \frac{2}{C\mu} \left[ 1 + \left( \frac{3}{2\nu^2} \right)^4 \left( \frac{1}{\lambda_1} + \mu \right) \|f\|_{V'}^4 \right] \right\}^{3/2}.$$  \hfill (153)

**Remark 4.** In terms of the three-dimensional Grashof number

$G = \|f\|_{\nu^2 \lambda_1^{3/4}}$, the fractal dimension bound given in (153) can be written as

$$\dim_F(A_1^\mu) \leq \left\{ \frac{2}{C\mu} \left[ 1 + \frac{81\lambda_1^2}{16} (1 + \mu \lambda_1)\mathcal{C} \right] \right\}^{3/2},$$  \hfill (154)

where $\tilde{C}$ is defined in (3).

5. **Exponential attractors.** In this section, we establish the existence of an exponential attractor in $V_1^\mu$ for the semigroup $S(t)$ associated with the Kelvin-Voigt-Brinkman-Forchheimer equations. The results obtained in this section are true for $r = 1, 2$ and $r \in [3, 4]$ only, due to technical reasons. Similar result for 3D Navier-Stokes-Voigt models is obtained in [37] and for 3D Kelvin-Voigt fluid flow equations with “fading memory” is established in [30]. Let us first provide the definition of exponential attractors.

**Definition 5.1** (Definition 3.1, [17, 14]). A set $\mathcal{M}_\mu$ is an exponential attractor for $S(t)$ in $V_1^\mu$ if

(i) it is compact in $V_1^\mu$,
(ii) it is positive invariant, i.e., $S(t)\mathcal{M}_\mu \subset \mathcal{M}_\mu$, for all $t \geq 0$,
(iii) it has finite fractal dimension,
(iv) it attracts exponentially fast the bounded sets of initial data, i.e., there exists a monotone function $Q$ and a constant $\alpha > 0$ such that for all $B \subset V_1^\mu$ bounded

$$\text{dist}_H(S(t)B, \mathcal{M}_\mu) \leq Q(\|B\|_{V_1}^\alpha) e^{-\alpha t}, \text{ for all } t \geq 0.$$

We use Proposition 1, [16], Theorem 3.2, [17] (see Theorem A1, [38]) also to obtain the existence of an exponential attractor of our system. From the estimate (66) (see Proposition 1), it is immediate that

$$\sup_{t \geq 0} \sup_{u_0 \in B_1} \|S(t)u_0\|_{V_1^\mu} \leq \tilde{M}_1 := \frac{1}{\nu} \sqrt{3\left( \frac{1}{\lambda_1} + \mu \right) \|f\|_{V'}},$$  \hfill (155)
where $\mathcal{B}_2$ is defined in (98). For $r \in [1, 2]$, once again from (102) and (99), we also have

$$
\sup_{t \geq 0} \sup_{u_0 \in \mathcal{B}_2} \|S(t)u_0\|_{\mathcal{V}_2^\mu} \leq \tilde{M}_2 := \sqrt{3\left(1 + \mu \lambda_1 \right)} \left( \|f\|_H + \frac{27CM_6^6}{4\nu^2} + C_{\text{r}} \beta \frac{1}{\nu} M_1^{2(\beta + 2)} \right)^{1/2},
$$

(157)

where $\mathcal{B}_2$ is defined in (99).

Let us first define $D := \mathcal{B}_1 \cap \mathcal{B}_2$. Now, we take $t_e > 0$ as the entering time of $D$ in the absorbing set $\mathcal{B}_1$. Moreover, we define

$$
\mathcal{K}_\mu = \bigcup_{t \geq t_e} S(t)D^\mu_{\mathcal{V}_1^\mu}.
$$

(158)

We prove that $\mathcal{K}_\mu$ is invariant and compact. In order to establish that $D$ is invariant, we use semigroup property to obtain

$$
S(t)\mathcal{K}_\mu = S(t) \bigcup_{\tau \geq t_e} S(\tau)D^\mu_{\mathcal{V}_1^\mu} \subset \bigcup_{\tau \geq t_e} S(t + \tau)D^\mu_{\mathcal{V}_1^\mu} = \mathcal{K}_\mu.
$$

(159)

Our next aim is to show that $\mathcal{K}_\mu$ is compact set in $\mathcal{V}_1^\mu$ and bounded set in $\mathcal{V}_2^\mu$. In order to do this, it is enough to show the boundedness of $\mathcal{K}_\mu$ in $\mathcal{V}_2^\mu$, as $\mathcal{V}_2^\mu$ is compactly embedded in $\mathcal{V}_1^\mu$. Let us first take $w \in \mathcal{K}_\mu$. Then, there exists a sequence $t_n \geq t_e$ and $w_n \in S(t)D$, such that $w_n \rightharpoonup w$ strongly in $\mathcal{V}_2^\mu$.

Thanks to the estimate (156) and hence we have $\|w_n\|_{\mathcal{V}_2^\mu} \leq M_2$. Using the weak compactness and uniqueness of limits, along with the Banach-Alaoglu theorem, we can extract a subsequence of $\{w_n\}$ (still denoted by $\{w_n\}$) such that $w_n \overset{w}{\rightharpoonup} w$ weakly in $\mathcal{V}_2^\mu$. Since the norm is weakly lower semicontinuous, we finally arrive at

$$
\|w\|_{\mathcal{V}_2^\mu} \leq \liminf_{n \to \infty} \|w_n\|_{\mathcal{V}_2^\mu} \leq M_2,
$$

(160)

and hence the boundedness in $\mathcal{V}_2^\mu$ follows. The exponentially attracting property of $\mathcal{K}_\mu$ can be established similarly as in Lemma 4.6, [37].

Let us now establish our main goal of this section, that is, to show that the semigroup $S(t)$ on $\mathcal{V}_1^\mu$ associated with the system (13) admits an exponential attractor $\mathcal{M}_\mu$ contained and bounded in $\mathcal{V}_2^\mu$. Our major idea behind this is to use Proposition 1, [16], Theorem 3.2, [17] (or Theorem A1, [38]). As discussed in the previous sections, we start by decomposing the solution semigroup $S(t)$ in the following way. For $u_0 \in \mathcal{K}_\mu$, let us split the solution of (13) as (see section 3 also)

$$
S(t)u_0 = R(t)u_0 + T(t)u_0,
$$

where

$$
v(t) = R(t)u_0 \quad \text{and} \quad w(t) = T(t)u_0,
$$

solves the following systems for $t \in (0, T)$:

$$
\begin{align*}
\frac{d}{dt} \left[(1 + \mu \lambda_1) v(t)\right] + \nu A v(t) &= 0, \\
v(0) &= u_0 \in \mathcal{V},
\end{align*}
$$

(161)
and
\[
\frac{d}{dt}[(I + \mu A)w(t)] + \nu A w(t) + \beta P u(t) + B(u(t), u(t)) = f, \quad w(0) = 0, \tag{162}
\]
respectively. First of all, we show that the semigroup \(R(t)\) is an exponentially stable linear semigroup.

**Lemma 5.2.** The semigroup \(R(t)\) is linear and exponentially stable.

*Proof.* For all \(u_0 \in V_1^t\), from (79) (see Proposition 3), we easily have
\[
\|R(t)u_0\|_{V_1^t} \leq e^{-\left(\frac{\nu \lambda_1}{1+\lambda_1}\right)t}\|u_0\|_{V_1^t} \quad \text{and} \quad \|R(t)\|_{L(V_1^t, V_1^t)} \leq e^{-\left(\frac{\nu \lambda_1}{1+\lambda_1}\right)t},
\]
and hence the semigroup \(R(t)\) is exponentially stable on \(V_1^t\). \(\square\)

Since the system (161) is linear, we can prove the following result in a similar fashion.

**Lemma 5.3.** Let \(u_0 \in K_\mu\). Then
\[
\|R(t)u_0 - R(t)v_0\|_{V_1^t} \leq e^{-\left(\frac{\nu \lambda_1}{1+\lambda_1}\right)t}\|u_0 - v_0\|_{V_1^t}, \tag{163}
\]
for all \(t \geq 0\).

Let us now establish the continuous dependence results for the semigroups \(S(t)\) and \(T(t)\).

**Lemma 5.4.** Let \(u_0, v_0 \in V_1^t\). Then, whenever \(\|u_0\|_{V_1^t}, \|v_0\|_{V_1^t} \leq R\), we have
\[
\|S(t)u_0 - S(t)v_0\|_{V_1^t} \leq e^{\frac{\nu}{1+\lambda_1}(R^2 + t)}\|u_0 - v_0\|_{V_1^t}, \quad \text{where} \quad \rho = \max\left\{1, \frac{\|f\|^2_{V_t}}{\nu}\right\}. \tag{164}
\]

*Proof.* From (59), we have
\[
\|S(t)u_0 - S(t)v_0\|_{V_1^t} \leq \exp\left(\frac{2}{\nu \mu \lambda_1 t^{1/2}} \int_0^t \|u(s)\|_V^2 ds\right)\|u_0 - v_0\|_{V_1^t}. \tag{165}
\]
Using the estimate given in (68), we also obtain
\[
\|S(t)u_0 - S(t)v_0\|_{V_1^t} \leq \exp\left(\frac{2}{\nu \mu \lambda_1 t^{1/2}} \left[\frac{\|u_0\|_{V_1^t}^2 + \frac{t}{\nu} \|f\|_{V_t}^2}{\rho}\right]\right)\|u_0 - v_0\|_{V_1^t}, \tag{166}
\]
for all \(t \geq 0\). Since \(\|u_0\|_{V_1^t} \leq R\), (166) easily gives (164). \(\square\)

**Lemma 5.5.** The following inequality holds:
\[
\|T(t)u_0 - T(t)v_0\|_{V_1^t} \leq \tilde{\rho} e^{\frac{2\nu^{2/3} \lambda_1^{1/4}(R^2 + t)}{\sqrt{\rho}}}\|u_0 - v_0\|_{V_1^t}, \tag{167}
\]
where
\[
\tilde{\rho} = \sqrt{\frac{2\nu}{\rho}} \lambda_1^{1/4} \left[C\widetilde{M}_1 \widetilde{M}_2 + \frac{C\beta_2^2}{\lambda_1} \left(\widetilde{M}_1^{-1} \widetilde{M}_2^{-1} + \widetilde{M}_1 \widetilde{M}_2\right)\right]^{1/2},
\]
for \(r = 1, 2\) and
\[
\tilde{\rho} = \sqrt{\frac{2\nu}{\rho}} \lambda_1^{1/4} \left[C\widetilde{M}_1 \widetilde{M}_2 + \frac{2^{2(r-2)}C\beta_2^2(r-1)^2}{\lambda_1} \left(\widetilde{M}_1^{-1} \widetilde{M}_2^{-1}\right)\right]^{1/2},
\]
for \(r \in [3, 4]\).
Proof. Let us take $w_1(t) = T(t)u_0$ and $w_2(t) = T(t)v_0$. Then, their difference $w(t) = w_1(t) - w_2(t)$ satisfies:

$$
\begin{align*}
\frac{d}{dt}[ (I + \mu A)w(t) ] + \nu Aw(t) + B(u_1(t), u(t)) + B(u(t), u_2(t)) \\
+ \beta(P_H([u_1(t)]^{-1}u_1(t)) - P_H([u_2(t)]^{-1}u_2(t))) = 0, 
\end{align*}
$$

for $t \in (0, T)$, where $u(t) = u_1(t) - u_2(t) = S(t)u_0 - S(t)v_0$. Let us now take inner product with $Aw(t)$ in (168) to obtain

$$
\frac{1}{2} \frac{d}{dt}[\|w(t)\|_3^2 + \mu\|Aw(t)\|_3^2] + \nu\|Aw(t)\|_3^2
$$

$$
= -(B(u_1(t), u(t)), Aw(t)) - (B(u(t), u_2(t)), Aw(t))
$$

$$
- \beta(P_H([u_1(t)]^{-1}u_1(t)) - P_H([u_2(t)]^{-1}u_2(t))), Aw(t)).
$$

We use Hölder’s, Agmon’s and Young’s inequalities to estimate $(B(u_1, u), Aw)$ as

$$
|\langle B(u_1, u), Aw \rangle| \leq \|u_1\|_{L^\infty} \|u\|_V \|Aw\|_H \leq C\|u_1\|_V^{1/2} \|Au_1\|_H^{1/2} \|u\|_V \|Aw\|_H
$$

$$
\leq \frac{\nu}{4} \|Aw\|_H^2 + C\|u_1\|_V \|Au_1\|_H \|u\|_V^2.
$$

(170)

Using Hölder’s, Gagliardo-Nirenberg (see (A.2) and (A.4)), Poincaré and Young’s inequalities, we estimate $(B(u, u_2), Aw)$ as

$$
|\langle B(u, u_2), Aw \rangle| \leq \|u\|_{L^2} \|\nabla u_2\|_{L^2} \|Aw\|_H \leq C\|u\|_V \|u_2\|_V^{1/2} \|Au_2\|_H^{1/2} \|Aw\|_H
$$

$$
\leq \frac{\nu}{4} \|Aw\|_H^2 + C\|u_2\|_V \|Au_2\|_H \|u\|_V^2.
$$

(171)

Let us write $\beta((|u_1|^{-1}u_1 - |u_2|^{-1}u_2), Aw)$ as

$$
\beta((|u_1|^{-1}u_1 - |u_2|^{-1}u_2), Aw)
$$

$$
= \beta(|u_1|^{-1}(u_1 - u_2), Aw) + (|u_1|^{-1} - |u_2|^{-1})u_2, Aw) =: I_1 + I_2.
$$

(172)

Using Hölder’s, Agmon’s and Young’s inequalities, we estimate $I_1$ as

$$
I_1 \leq \beta\|u_1\|_V \|u_2\|_V \|u\|_H \|Aw\|_H \leq \beta\|u_1\|_V \|u_2\|_V \|u\|_H \|Aw\|_H
$$

$$
\leq C\beta\|u_1\|_V \|u_2\|_V \|Au_1\|_H \|u\|_H \|Aw\|_H
$$

$$
\leq \frac{\nu}{8} \|Aw\|_H^2 + \frac{2C\beta^2}{\nu \lambda_1} \|u_1\|_V \|u_2\|_V \|Au_1\|_H \|u_2\|_V^2.
$$

(173)

For $r = 1$, $I_2$ is zero. For $r = 2$, we estimate $I_2$ using Hölder’s, Agmon’s and Young’s inequalities as

$$
I_2 = \beta\langle(|u_1| - |u_2|)u_2, Aw \rangle \leq \beta\|u_1\|_H \|u_2\|_{L^\infty} \|Aw\|_H \leq C\beta \|u_2\|_V \|Au_2\|_H \|u\|_V \|Aw\|_H
$$

$$
\leq \frac{\nu}{8} \|Aw\|_H^2 + \frac{2C\beta^2}{\nu \lambda_1} \|u_2\|_V \|Au_2\|_H \|u_2\|_V^2.
$$

(174)

For $r \in [3, 4]$, we use Taylor’s formula, Hölder’s, Agmon’s and Young’s inequalities to estimate $I_2$ as

$$
I_2 = \beta(r - 1)\|\theta u_1 + (1 - \theta)u_2\|_V \|u_1\|_V \|u_2\|_V \|u\|_V \|Aw\|_H
$$

$$
\leq \beta(r - 1)2^{-3}(\|u_1\|_V^{-2} + \|u_2\|_V^{-2}) \|u_2\|_{L^\infty} \|u\|_H \|Aw\|_H
$$

$$
\leq \beta(r - 1)2^{-3}(\|u_1\|_V^{-2} + \|u_2\|_V^{-2}) \|u_2\|_{L^\infty} \|u\|_H \|Aw\|_H
$$

$$
\leq \frac{\nu}{8} \|Aw\|_H^2 + \frac{2C\beta^2}{\nu \lambda_1} \|u_2\|_V \|Au_2\|_H \|u_2\|_V^2.
$$

(174)
where we used (155), (156) and (164) also. Thus, we have

\begin{align}
\text{Proof.} & \quad \text{Let} \\
& \quad \text{Lemma 5.6.} \nonumber
\end{align}

result (120) by using (175) in (177).

\begin{align}
\int Integrating the equality (169) from 0 to \(t\) and then using (170), (176) and (171), we get

\begin{align}
\|w(t)\|_V^2 + \mu \|Aw(t)\|_H^2 + \frac{\nu}{2} \int_0^t \|Aw(s)\|_H^2 \, ds \\
\leq 2C \int_0^t (\|u_1(s)\|_V \|Au_1(s)\|_H + \|u_2(s)\|_V \|Au_2(s)\|_H) \|u(s)\|_V^2 \, ds \\
+ \frac{4C\beta^2}{\nu \lambda_1} \int_0^t (\|u_1(s)\|_V \|Au_1(s)\|_H^2 + \|u_2(s)\|_V \|Au_2(s)\|_H) \|u(s)\|_V^2 \, ds \\
\leq \left[ \frac{4C\hat{M}_1 \hat{M}_2}{\nu} + \frac{4C\beta^2}{\nu \lambda_1} \left( \hat{M}_1^{r-1} \hat{M}_2^{r-1} + \hat{M}_1 \hat{M}_2 \right) \right] \int_0^t \|u(s)\|_V^2 \, ds \\
\leq \left[ \frac{4C\hat{M}_1 \hat{M}_2}{\nu} + \frac{4C\beta^2}{\nu \lambda_1} \left( \hat{M}_1^{r-1} \hat{M}_2^{r-1} + \hat{M}_1 \hat{M}_2 \right) \right] \nu^2 \lambda_1^{1/2} \frac{2^r}{\rho} e^{2 \mu \lambda_1^{1/2} (R^2 + t)} \|u_0 - v_0\|_V^2,
\end{align}

(177)

where we used (155), (156) and (164) also. Thus, we have

\begin{align}
\|w(t)\|_V^2 \\
\leq \frac{\nu}{\beta^{1/4}} \left( C\hat{M}_1 \hat{M}_2 + \frac{4C\beta^2}{\lambda_1} \left( \hat{M}_1^{r-1} \hat{M}_2^{r-1} + \hat{M}_1 \hat{M}_2 \right) \right)^{1/2} e^{2 \mu \lambda_1^{1/2} (R^2 + t)} \|u_0 - v_0\|_V^2,
\end{align}

(178)

which completes the proof. Similarly, for \(r \in [3,4]\), one can obtain the required result (120) by using (175) in (177).

\[\square\]

**Lemma 5.6.** Let \(u_0 \in K_\mu\). Then, we have

\begin{align}
\sup_{t \geq 0} \|\dot{u}(t)\|_{V^r} \leq \frac{2}{\sqrt{\mu}} \left( \|f\|_{V^r} + \frac{1}{\mu} \left( 1 + \frac{\hat{M}_1^2}{\mu \lambda_1^{1/2}} \right) \hat{M}_2^2 + \frac{2\beta^2 C}{\nu \lambda_1^{1/2}} \hat{M}_2^2 \right)^{1/2} \|u_0\|_{V^r}.
\end{align}

(179)

**Proof.** We take inner product with \(\dot{u}(\cdot)\) to the first equation in (13) to obtain

\begin{align}
\|\dot{u}(t)\|^2_{V^r} + \mu \|\dot{u}(t)\|_{V^r}^2 = \langle f, \dot{u}(t) \rangle - \nu \langle Au(t), \dot{u}(t) \rangle - \langle B(u(t)), \dot{u}(t) \rangle \\
- \beta \langle P_H([u(t)]^{r-1} u(t)), \dot{u}(t) \rangle.
\end{align}

(180)
Using Cauchy-Schwarz and Young’s inequalities, we estimate \( \langle f, \dot{u} \rangle \) as
\[
|\langle f, \dot{u} \rangle| \leq \|f\|_V \|\dot{u}\|_V \leq \frac{2}{\mu} \|f\|_V^2 + \frac{\mu}{8} \|\dot{u}\|_V^2. \tag{181}
\]

Once again using Cauchy-Schwarz and Young’s inequalities, we estimate \( \langle Au, \dot{u} \rangle \) as
\[
|\langle Au, \dot{u} \rangle| = |\langle \nabla u, \nabla \dot{u} \rangle| \leq \|\nabla u\|_H^2 \|\nabla \dot{u}\|_H \leq \frac{\mu}{8} \|\dot{u}\|_V^2 + \frac{2}{\mu} \|u\|_H^2. \tag{182}
\]

Using Hölder’s, Ladyzhenskaya, Young’s and Poincaré inequalities, we estimate \( \langle B(u), \dot{u} \rangle \) as
\[
|\langle B(u), \dot{u} \rangle| = |\langle B(u), \dot{u} \rangle| \leq \|\dot{u}\|_H \|u\|_L^2 \leq 2 \|\dot{u}\|_V \|u\|_H^{3/2} \|u\|_H^{1/2} \leq \frac{\mu}{8} \|\dot{u}\|_V^2 + \frac{2}{\mu} \|u\|_V^2 + \frac{\mu}{8} \|u\|_H^4. \tag{183}
\]

Finally, we estimate \( \beta \langle \mu^{-1}u^{-1}, \dot{u} \rangle \) using (A.6), (A.2), Hölder’s and Young’s inequalities as
\[
\beta \langle |u|^{-1}u, \dot{u} \rangle \leq \beta \|u|^{-1}u\|_{L^3} \|\dot{u}\|_{L^3} \leq \beta \|u\|_{L^2} \|\dot{u}\|_V \leq \frac{\beta C}{\lambda_1^2} \|u\|_V^2 \|\dot{u}\|_V \leq \frac{\beta C}{\lambda_1^2} \|u\|_V^2 + \frac{4\beta^2 C}{5\mu} \|u\|_V^2. \tag{184}
\]

Combining (181)-(184) and using it in (180), we find
\[
\|\ddot{u}(t)\|_H^2 + \frac{\mu}{2} \|\ddot{u}(t)\|_V^2 \leq \frac{2}{\mu} \|f\|_V^2 + \frac{2}{\mu} \left(1 + \frac{|u(t)|_L^2}{\lambda_1^4}\right) \|u(t)\|_V^2 + \frac{4\beta^2 C}{5\mu} \|u(t)\|_V^2. \tag{185}
\]

Since \( u_0 \in \mathcal{K}_\mu \), using (155) in (185), we also infer that
\[
\|\ddot{u}(t)\|_H^2 + \mu \|\ddot{u}(t)\|_V^2 \leq \frac{4}{\mu} \|f\|_V^2 + \frac{4}{\mu^2} \left(1 + \frac{\tilde{M}_1^2}{\mu^4 \lambda_1^2}\right) \tilde{M}_1^2 + \frac{8\beta^2 C}{\mu^1 + r \lambda_1^2} \tilde{M}_2^r, \tag{186}
\]
which implies the result (179).

Our next aim is to establish that the map \( S(t) \) is Lipschitz continuous.

**Lemma 5.7.** Let \( T > 0 \) be arbitrary and fixed. Then the map
\[
(t, u_0) \mapsto S(t)u_0 : [0, T] \times \mathcal{K}_\mu \rightarrow \mathcal{K}_\mu,
\]
is Lipschitz continuous.

**Proof.** For \( u_0, v_0 \in \mathcal{K}_\mu \) and \( t_1, t_2 \in [0, T] \), we have
\[
\|S(t_1)u_0 - S(t_2)v_0\|_V \leq \|S(t_1)u_0 - S(t_1)v_0\|_V + \|(S(t_1) - S(t_2))v_0\|_V. \tag{187}
\]

Using (164), we find
\[
\|S(t_1)u_0 - S(t_1)v_0\|_V \leq e^{\frac{\sigma}{2} \mu \lambda_1^2 (\tilde{M}_1^2 + t)} \|u_0 - v_0\|_V. \tag{188}
\]
Making use of the Lemma 5.6, we also get
\[
\|(S(t_1) - S(t_2))v_0\|_V.
\]
Combining (188) and (189) and using it in (187), we obtain the required result.

We are now ready to prove the existence of an exponential attractor for the non-autonomous version of the system (13). Let us apply Proposition 1, [16], Theorem 3.2, [17] (see Theorem A1, [38] also) to obtain such an exponential attractor.

Theorem 5.8. For $r = 1, 2$ and $r \in [3, 4]$, the dynamical system $S(t)$ on $\mathbb{V}_1^\mu$ associated with the system (13) admits an exponential attractor $M_{\mu}$ contained and bounded in $\mathbb{V}_1^\mu$ in the sense of Definition 5.1.

Proof. We have already shown that $K_{\mu}$ is compact and invariant. From Lemma 5.7, we know that the operator $S(t)$ is Lipschitz continuous for any $t^* \in [0, T]$. Let us fix $t^*$ according to the estimate given in Lemma 5.3 (see for example (163)), i.e., we fix
\[
e^{-\left(\frac{\nu_1}{1 + \mu}\right)} t^* = \frac{1}{8}.
\]
Thus, we have
\[
\|R(t)u_0 - R(t)v_0\|_\mathbb{V}_1^\mu \leq \frac{1}{8}\|u_0 - v_0\|_\mathbb{V}_1^\mu,
\]
for all $t \geq 0$. From Lemma 5.5, we infer that
\[
\|T(t^*)u_0 - T(t^*)v_0\|_{\mathbb{V}_2^\mu} \leq C_0\|u_0 - v_0\|_\mathbb{V}_1^\mu,
\]
where $C_0 = \bar{C}e^{2\|\nabla\|_{\mathbb{V}_2^\mu}} \left(\bar{M}_1^2 + \bar{M}_2^2\right)$. Thus, one can apply Theorem A1, [38] with $L = R(t^*)$ and $K = T(t^*)$ to get an exponential attractor for the system (13).

Let us now discuss a result on the fractal dimension of exponential attractors associated with the Kelvin-Voigt-Brinkman-Forchheimer model (13).

Theorem 5.9. For $r = 1, 2$ and $r \in [3, 4]$, the semigroup $S(t)$ admits an exponential attractor $M_{\mu}$ whose fractal dimension satisfies the estimate:
\[
dim_F(M_{\mu}) \leq \dim_F(A_{\mu}^1) + 1,
\]
where $A_{\mu}^1$ is the global attractor for $S(t)$.

Proof. From Theorem 4.1 (see (126), (127) and (130)), we know that $S(t)$ is differentiable. Moreover, from the Lemma 5.7, we have (see (188) and (189))
\[
\|S(t_1)u_0 - S(t_2)v_0\|_{\mathbb{V}_1^\mu} \leq L(t)\left(\|u_0 - v_0\|_{\mathbb{V}_1^\mu} + |t_1 - t_2|\right),
\]
where
\[
L(t) = \max\left\{\frac{\mu}{\mu \lambda_1^2} e^{-\left(\frac{\nu_1}{1 + \mu}\right)} t^*, \frac{2}{\sqrt{r}} \left\{\|f\|_{\mathbb{V}_2^\mu} + \frac{1}{\mu} \left(1 + \frac{\bar{M}_1^2}{\mu ^{1/4}}\right) \bar{M}_1^2 + \frac{2\beta^2 C}{\mu ^{1/2}} \bar{M}_2^2\right\}^{1/2}\right\},
\]
and hence $S(t)$ is Lipschitz.

Let us now show that the flow is an $\alpha$-contraction for all $t > 0$ (see [15]). In order to prove $\alpha$-contraction, we use Remark 4.6.2, [19]. Note that $S(t)u_0 = \text{...}$. 


6. The inviscid limit. In this section, we first discuss the inviscid limit of the equation (13) as \( \beta \to 0 \) (see \cite{8, 22, 28, 29} also). Then, we consider the limit \( \nu \to 0 \).

**Lemma 6.1.** Let \((u(\cdot), p(\cdot))\) be the unique weak solution of the Kelvin-Voigt-Brinkman-Forchheimer system (1). As \( \beta \to 0 \), the weak solution \((v(\cdot), q(\cdot))\) of the system (1) tends to the weak solution of the Navier-Stokes-Voigt system:

\[
\frac{\partial}{\partial t}[(I - \mu \Delta)v(t, x)] - \nu \Delta v(t, x) + (v(t, x) \cdot \nabla)v(t, x) + \nabla q(t, x) = f(t, x) \quad \text{for} \quad x \in \Omega, \quad t > 0, \\
(\nabla \cdot v)(t, x) = 0, \quad \text{for} \quad x \in \Omega, \quad t > 0, \\
v(t, x) = 0, \quad \text{for} \quad x \in \partial \Omega, \quad t \geq 0, \\
v(0, x) = u_0(x), \quad \text{for} \quad x \in \Omega, \\
\int_{\Omega} q(t, x) dx = 0, \quad \text{for} \quad t \geq 0. 
\]

**Proof.** Let us first apply the Helmholtz-Hodge orthogonal projection \( P_H \) onto the equation (195) to obtain

\[
\frac{d}{dt}[(I + \mu A)v(t)] + \nu A v(t) + B(v(t), v(t)) = f(t), \quad t \in (0, T),
\]

\[
v(0) = u_0 \in V, 
\]
The existence and uniqueness of weak solution \( v \in C([0, T]; \mathbb{V}) \) to the system (196) is available in the literature, see for example [22, 29], etc. Let \( u(\cdot) \) denotes the unique weak solution of the system (13) in \( C([0, T]; \mathbb{V}) \) with initial data \( u_0 \in \mathbb{V} \). For \( w(\cdot) := u(\cdot) - v(\cdot) \) and \( t \in (0, T) \), \( w(\cdot) \) satisfies:

\[
\begin{aligned}
\frac{d}{dt}[(I + \mu A) w(t)] + \nu Aw(t) + B(w(t), u(t)) + B(v(t), w(t)) &= -\beta P_H(|u(t)|^{-1}u(t)) \quad \text{in } (0, T), \\
w(0) &= 0.
\end{aligned}
\]  

(197)

Let us take inner product with \( w(\cdot) \) to the first equation in (197) to obtain

\[
\frac{1}{2} \frac{d}{dt} \left[ \| w(t) \|_{\mathbb{V}}^2 + \mu \| w(t) \|_{L^2}^2 \right] + \nu \| w(t) \|_{\mathbb{V}}^2 + b(w(t), u(t), w(t)) = -\beta \langle P_H(|u(t)|^{-1}u(t)), Aw(t) \rangle.
\]  

(198)

We use Hölder’s, Ladyzhenskaya and Poincaré inequalities to estimate \( |b(w, u, w)| \) as

\[
|b(w, u, w)| \leq \| w \|_{L^4}^2 \| \nabla u \|_{L^2} \leq 2 \| \nabla w \|_{H^1}^{3/2} \| w \|_{H^1}^{1/2} \| u \|_{\mathbb{V}} \leq \frac{2}{\lambda_1^{1/4}} \| u \|_\mathbb{V} \| w \|_{\mathbb{V}}^2.
\]  

(199)

Using Cauchy-Schwarz and Young’s inequalities, and (20), we estimate the term \( \beta(\| u \|^{-1} u, Aw) \) as

\[
|b(\| u \|^{-1} u, Aw)| \leq \beta \| w \|_\mathbb{V} \| u \|^{-1} u \|_{\mathbb{V}'} \leq \frac{1}{2} \| w \|_\mathbb{V}^2 + \frac{C\beta^2}{2} \| u \|_{\mathbb{V}'}^2.
\]  

(200)

Let us combine (199)-(200), substitute it in (198) and then integrate over time from 0 to \( t \) to obtain

\[
\| w(t) \|_{\mathbb{V}}^2 + \mu \| w(t) \|_{L^2}^2 + \nu \int_0^t \| w(s) \|_{\mathbb{V}}^2 ds \leq \int_0^t \left[ \frac{4}{\lambda_1^{1/4}} \| u(s) \|_{\mathbb{V}} + 1 \right] \| w(s) \|_{\mathbb{V}}^2 ds + C\beta^2 \int_0^t \| u(s) \|_{\mathbb{V}'}^2 ds.
\]  

(201)

An application of Gronwall’s inequality in (201) gives

\[
\| w(t) \|_{\mathbb{V}}^2 \leq \left( \frac{C\beta^2}{\mu} \int_0^t \| u(s) \|_{\mathbb{V}'}^2 ds \right) e^{\beta^2 / \mu \lambda_1^{1/4}} \left\{ \int_0^t \| u(s) \|_{\mathbb{V}} ds \right\}.
\]  

(202)

Our next aim is to show that \( \int_0^t \| u(s) \|_{\mathbb{V}} ds \) depends only on \( \mu \), initial data and external forcing for each time \( t \in [0, T] \). Taking inner product with \( u(\cdot) \) to the first equation in (13) and then integrating from 0 to \( t \), we find

\[
\| u(t) \|_{\mathbb{V}}^2 + \mu \| u(t) \|_{L^2}^2 + 2\nu \int_0^t \| u(s) \|_{\mathbb{V}}^2 ds + 2\beta \int_0^t \| u(s) \|_{L^\infty}^2 ds = \| u_0 \|_{\mathbb{V}}^2 + \mu \| u_0 \|_{L^2}^2 + 2\int_0^t \langle f(s), u(s) \rangle ds.
\]  

(203)

Using Cauchy-Schwarz and Young’s inequalities, we estimate \( |\langle f, u \rangle| \) as

\[
|\langle f, u \rangle| \leq \| f \|_{\mathbb{V}'} \| u \|_{\mathbb{V}} \leq \frac{\mu}{2} \| u \|_{\mathbb{V}}^2 + \frac{1}{2\mu} \| f \|_{\mathbb{V}'}^2.
\]  

(204)
Substituting (204) in (203), we obtain

\[ \|u(t)\|_H^2 + \mu \|u(t)\|_V^2 + 2\nu \int_0^t \|u(s)\|_V^2 ds + 2\beta \int_0^t \|u(s)\|_L^2 ds \leq \|u_0\|_H^2 + \mu \|u_0\|_V^2 + \frac{1}{\mu} \int_0^t \|f(s)\|_V^2 ds + \mu \int_0^t \|u(s)\|_V^2 ds. \]  

(205)

Let us apply Gronwall’s inequality in (205) to get

\[ \mu \|u(t)\|_V^2 \leq \left( \mu + \frac{1}{\lambda_1} \right) \|u_0\|_V^2 + \frac{1}{\mu} \int_0^T \|f(t)\|_V^2 dt \right) e^{T} =: \mathcal{K}_T, \]  

(206)

for all \( t \in [0, T] \). From (202), it can be easily seen that

\[ \|w(t)\|_V^2 \leq \frac{C \beta^2 T \mathcal{K}_T}{\mu + 1} e^{\frac{T}{\mu + 1}} , \]  

(207)

for all \( t \in [0, T] \). Passing \( \beta \to 0 \) in (207), we obtain the required result. \( \square \)

The following inviscid limit to the simplified Bardina model is established in section 6, [22] (see [8, 28], etc, also).

**Lemma 6.2.** Let \((v(\cdot), q(\cdot))\) be the unique weak solution of the Kelvin-Voigt system (195). As \( \nu \to 0 \), the weak solution of the Kelvin-Voigt system (195) tends to the weak solution \((z(\cdot), r(\cdot))\) of the simplified Bardina model:

\[
\begin{aligned}
&\frac{\partial}{\partial t} \left[ (1 - \mu \Delta)z(t,x) \right] + (z(t,x) \cdot \nabla)z(t,x) + \nabla r(t,x) = f(t,x) \quad \text{for} \ x \in \Omega, \ t > 0, \\
&(\nabla \cdot z)(t,x) = 0, \ \text{for} \ x \in \Omega, \ t > 0, \\
&z(t,x) = 0, \ \text{for} \ x \in \partial \Omega, \ t \geq 0, \\
&z(0,x) = u_0(x), \ \text{for} \ x \in \Omega, \\
&\int_\Omega r(t,x)dx = 0, \ \text{for} \ t \geq 0.
\end{aligned}
\]  

(208)

**Remark 6.** One can also study the convergence (as \( \beta \to 0 \)) of global and exponential attractors for the autonomous version of the Kelvin-Voigt-Brinkman-Forchheimer equations (see (13)) towards global and exponential attractors for the autonomous version of the Navier-Stokes-Voigt equations (see (195)) obtained in [22, 37].

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**Appendix A. Important inequalities.** In this appendix, we give some important inequalities like Gagliardo-Nirenberg, Ladyzhenskaya, Agmon’s. etc.
Lemma A.1 (Gagliardo-Nirenberg inequality, Theorem 2.1, [31]). Let \( \Omega \subset \mathbb{R}^n \) and \( u \in W^{1,p}_0(\Omega; \mathbb{R}^n), p \geq 1 \). Then, for any fixed number \( q, r \geq 1 \), there exists a constant \( C > 0 \) depending only on \( n, p, q \) such that
\[
\|u\|_{L^r} \leq C \|\nabla u\|_{L^p}^{\eta} \|u\|_{L^{q,r}}^{1-\eta}, \quad \eta \in [0, 1],
\]
where the numbers \( p, q, r \) and \( \eta \) are connected by the relation
\[
\eta = \left( \frac{1}{q} - \frac{1}{r} \right) \left( \frac{1}{n} - \frac{1}{p} + \frac{1}{q} \right)^{-1}.
\]

If we take \( r = n = 3 \) and \( p = q = 2 \) in (A.1), we obtain \( \eta = \frac{1}{2} \) and
\[
\|u\|_{L^3} \leq C \|\nabla u\|_{H^1}^{1/2} \|u\|_{H^1}^{1/2},
\]
for all \( u \in V \). Next, we consider \( r = 4, n = 3 \) and \( p = q = 2 \) in (A.1) to find \( \eta = \frac{3}{4} \) and
\[
\|u\|_{L^4} \leq C \|\nabla u\|_{H^1}^{3/4} \|u\|_{H^1}^{1/4},
\]
where the constant \( C = \sqrt{2} \) (see Lemma 2, Chapter 1[25]). We also take \( r = 6, n = 3 \) and \( p = q = 2 \) in (A.1) to obtain \( \eta = 1 \) and
\[
\|u\|_{L^6} \leq C \|\nabla u\|_{H^1},
\]
where the constant \( C = 48 \) (see Lemma 2, Chapter 1, [25]). Moreover, if we take \( r = \frac{6}{3-2q}, n = 3 \) and \( p = q = 2 \) in (A.1), we obtain \( \eta = \varepsilon \), and
\[
\|u\|_{L^{\frac{6}{3-2q}}} \leq C \|u\|_{H^1}^{\frac{3}{3-2q}} \|\nabla u\|_{H^1}^{-\varepsilon}.
\]
For \( r \in [2, 6] \), we know that
\[
\|u\|_{L^r} \leq C \|\nabla u\|_{H^1}^{\frac{2}{3}} \|u\|_{H^1}^{\frac{1}{3} - \frac{1}{r}} \leq \frac{C}{\lambda^{\frac{r}{2}}} \|u\|_{V}. \tag{A.6}
\]

We need the following generalized version of the Gagliardo-Nirenberg interpolation inequality and Agmon’s inequality for higher order estimates. For functions \( u : \Omega \rightarrow \mathbb{R}^n \) defined on a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^n \), the Gagliardo-Nirenberg interpolation inequality is given by:

**Lemma A.2** (Gagliardo-Nirenberg interpolation inequality, Theorem 2.1, [31]). Let \( \Omega \subset \mathbb{R}^n \), \( u \in W^{m,p}(\Omega; \mathbb{R}^n), p \geq 1 \) and fix \( 1 \leq p, q \leq \infty \) and a natural number \( m \). Suppose also that a real number \( \eta \) and a natural number \( j \) are such that
\[
\eta = \left( \frac{j}{n} + \frac{1}{q} - \frac{1}{r} \right) \left( \frac{m}{n} - \frac{1}{p} + \frac{1}{q} \right)^{-1},
\]
and \( \frac{1}{m} \leq \eta \leq 1 \). Then for any \( u \in W^{m,p}(\Omega; \mathbb{R}^n) \), we have
\[
\|\nabla^j u\|_{L^r} \leq C \left( \|\nabla^m u\|_{L^p}^{\eta} \|u\|_{L^{q,r}}^{1-\eta} + \|u\|_{L^r} \right), \tag{A.8}
\]
where \( s > 0 \) is arbitrary and the constant \( C \) depends upon the domain \( \Omega, m, n, \) and \( j \).

If \( 1 < p < \infty \) and \( m - j - \frac{2}{q} \) is a non-negative integer, then it is necessary to assume also that \( \eta \neq 1 \). Note also that (A.8) can be written as
\[
\|\nabla^j u\|_{L^r} \leq C \|u\|_{W^{m,p}(\Omega; \mathbb{R}^n)}^{\eta} \|u\|_{L^{q,r}}^{1-\eta}, \tag{A.9}
\]
for any \( u \in W^{m,p}(\Omega; \mathbb{R}^n) \).
Let us take $j = 0$, $n = 3$, $p = q = 2$ and $m = \frac{3}{2}$ in (A.7) to get
\[ \|u\|_{L^r} \leq C\|u\|_{H^{\frac{3}{2}}} \|u\|_{V^{\frac{3}{2}}}^{1 - \frac{3}{r}} \] \hspace{1cm} (A.10)
for all $u \in V^\frac{3}{2}$ and $r \in [2, \infty)$. If we take $j = 0$, $n = 3$, $p = q = 2$ and $m = \frac{1}{2}$ in (A.7) to obtain $\eta = 1$ and
\[ \|u\|_{L^r} \leq C\|u\|_{V^{\frac{3}{2}}} = C\|A^{\frac{1}{2}}u\|_{H^{\frac{1}{2}}}, \] \hspace{1cm} (A.11)
for all $u \in V^\frac{3}{2}$. Moreover, if we take $j = 0$, $n = 3$, $p = q = 2$, $r = \frac{18}{7}$ and $m = \frac{1}{3}$ in (A.7) to find
\[ \|u\|_{L^r} \leq C\|u\|_{V^{\frac{3}{2}}} = C\|A^{\frac{1}{3}}u\|_{H^{\frac{1}{3}}}, \] \hspace{1cm} (A.12)
for all $u \in V^\frac{3}{2}$. For $j = 0$, $n = 3$, $p = 2$, $q \geq 1$ and $m = 3(\frac{1}{2} - \frac{1}{r})$, we get $\eta = 1$ and
\[ \|u\|_{L^r} \leq C\|u\|_{V^{\frac{1}{2}}}^{(\frac{3}{2} - \frac{1}{\alpha})} = C\|A^{\frac{1}{3}}u\|_{H^{\frac{1}{3}}}, \] \hspace{1cm} (A.13)
for any $r \in [2, \infty)$ and $u \in V^\frac{3}{2}(\frac{1}{2} - \frac{1}{\alpha})$

**Lemma A.3** (Agmon’s inequality, Lemma 13.2, [1]). Let $u \in H^{s_2}(\Omega; \mathbb{R}^n)$, and choose $s_1$ and $s_2$ such that $s_1 < \frac{n}{2} < s_2$. Then, if $0 < \alpha < 1$ and $\frac{n}{2} = \alpha s_1 + (1 - \alpha)s_2$, the following inequality holds
\[ \|u\|_{L^\infty} \leq C\|u\|_{H^{s_2}}^{\alpha}, \|u\|_{H^{s_1}}^{1 - \alpha}. \] \hspace{1cm} (A.14)
If we take $n = 3$, $s_1 = 1$ and $s_2 = 2$ in (A.14), we obtain $\alpha = \frac{1}{2}$ and
\[ \|u\|_{L^\infty} \leq C\|u\|_{V}^{1/2}, \|u\|_{H^{2}}^{1/2}. \] \hspace{1cm} (A.15)

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