A CONTROLLING NORM FOR ENERGY-CRITICAL
SCHRÖDINGER MAPS

BENJAMIN DODSON AND PAUL SMITH

Abstract. We consider energy-critical Schrödinger maps with target either the sphere $S^2$ or hyperbolic plane $H^2$ and establish that a unique solution may be continued so long as a certain space-time $L^4$ norm remains bounded. This reduces the large data global wellposedness problem to that of controlling this norm.

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1. Introduction

We consider the Schrödinger map equation on $\mathbb{R}^{2+1}$ with target either the sphere $S^2$ or hyperbolic plane $H^2$. With the appropriate modeling we may interpret $S^2$ and $H^2$ as submanifolds of $\mathbb{R}^3$, e.g.,

$S^2 = \{y = (y_0, y_1, y_2) \in \mathbb{R}^3 : y_0^2 + y_1^2 + y_2^2 = 1\},$

$H^2 = \{y = (y_0, y_1, y_2) \in \mathbb{R}^3 : -y_0^2 - y_1^2 + y_2^2 = 1, y_2 \geq 0\},$

with the Riemannian structures induced by the Euclidean metric $dy_0^2 + dy_1^2 + dy_2^2$ in the case of $S^2$ and by the Minkowski metric $-dy_0^2 + dy_1^2 + dy_2^2$ in the case of $H^2$. Setting $\eta_\mu := \text{diag}(1, 1, \mu)$, we define the cross product $\times_\mu$ by $v \times_\mu w := \eta_\mu \cdot (v \times w)$. The Schrödinger map initial-value problem then takes the form

$$\partial_t \phi = \phi \times_\mu \Delta \phi,$$

$$\phi(0, x) = \phi_0(x),$$

where $\phi$ is assumed to take values in $S^2$ or $H^2$ according to whether $\mu = +1$ or $\mu = -1$, respectively. Schrödinger maps admit the conserved energy

$$E(\phi) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \phi|^2_\mu dx,$$

Received by the editors February 18, 2013 and, in revised form, August 15, 2013.
2010 Mathematics Subject Classification. Primary 35Q55; Secondary 35B33.
The first author was supported by NSF grant DMS-1103914 and the second by NSF grant DMS-1103877.
and both the equation (1.1) and the energy (1.2) are invariant with respect to scalings
\[ \phi(t, x) \mapsto \phi(\lambda^2 t, \lambda x). \]
The problem we study is therefore energy critical.

When the target is \( S^2 \), Schrödinger maps arise as a Heisenberg model, i.e., a nearest-neighbor spin model, of ferromagnetism [22]. The Schrödinger map equation is also known as the Landau-Lifshitz equation. See [7,12,13,15,16,22] and the references therein.

For the local theory, see [14,22]. We use the following formulation of the local existence result from [1,2], which is proved in [14].

**Theorem 1.1** (Local existence). If \( \phi_0 \in \dot{H}^1 \cap \dot{H}^3 \), then there exists a time \( T > 0 \) such that (1.1) has a unique solution in \( L^\infty_t([0,T] : \dot{H}^1 \cap \dot{H}^3) \).

The small data problem with the sphere as target has been intensely studied; see [3–7,10,11]. Global wellposedness for data with small energy is established in [4]. In [20] this result is extended to data small in the scale invariant Besov space \( \dot{B}^1_{2,\infty} \), assuming control on a certain space-time \( L^4 \) norm (the same that we assume in this article); the result still holds even when the assumption of space-time \( L^4 \) control is dropped [18]. Large data results have been established in some special settings. Global wellposedness and scattering hold for radial Schrödinger maps into the sphere [1], though interestingly the problem is open for radial maps into \( \mathbb{H}^2 \). Global wellposedness and scattering also hold for equivariant maps into the sphere with energy less than \( 4\pi \) (see [1]); the same holds true for equivariant maps into \( \mathbb{H}^2 \) with finite energy [2].

Instead of working with (1.1) directly, we work at the level of the differentiated system. This leads to a system with coupled \( L^2 \)-critical covariant Schrödinger equations that exhibit gauge invariance. The details of this approach, first undertaken in [7] in the context of Schrödinger map wellposedness problems, are presented in the next section. The differentiated system cannot be wellposed unless the gauge freedom is eliminated by making a gauge choice. We adopt the caloric gauge, which was first introduced in [23] to study wave maps and then used for Schrödinger maps (with small energy) for the first time in [4]. Details of the construction for the target \( \mathbb{H}^2 \) are established in [24] and, for bounded geometry settings up to the ground state, in [19]. We describe the caloric gauge construction in the next section and discuss the bounds available in that gauge.

Loosely speaking, our main result is that a Schrödinger map \( \phi \) may be continued in time so long as the \( \ell^2 L^4 \) norm of \( \nabla \phi \) is finite. The \( \ell^2 L^4 \) norm for functions \( f \) is defined by
\[
\|f\|_{\ell^2 L^4} := \left( \sum_{k \in \mathbb{Z}} \|P_k f\|_{L^4_{t,x}}^2 \right)^{\frac{1}{2}},
\]
where \( P_k \) denote standard Littlewood-Paley projections to dyadic frequency shells of size \( \sim 2^k \). From this definition it is clear that we have the embedding \( \ell^2 L^4 \hookrightarrow L^4 \).

For technical reasons, our theorem applies to maps with energy less than the energy of the ground state, denoted by \( E_{\text{crit}} \), which is \( 4\pi \) when the target is \( S^2 \) and \( +\infty \) when the target is \( \mathbb{H}^2 \). Also for technical reasons, we work with maps \( \phi \) that have finite mass, i.e.,
\[
\int_{\mathbb{R}^2} |\phi - Q|^2 \, dx < \infty,
\]
where \( Q \in M \) is a fixed point of \( M \in \{ S^2, \mathbb{H}^2 \} \). This quantity is not scale invariant but is preserved by the flow.

**Theorem 1.2** (Main result). Let \( I = [t_0, t_1] \subset \mathbb{R} \) with \( t_0 < t_1 \) and let \( 0 < \varepsilon \ll 1 \). If \( \dot{H}^1 \cap \dot{H}^3 \ni \phi_0 = \phi(t_0) : \mathbb{R}^2 \to M, M \in \{ S^2, \mathbb{H}^2 \} \), and \( \phi \) is a solution of (1.1) on \( I \) with finite mass and energy \( E(\phi) < E_{\text{crit}} \) satisfying \( \| \nabla \phi \|_{L^4_t L^4_x(I \times \mathbb{R}^2)} \leq \varepsilon \ll 1 \), then there exists a time \( 0 < T = T(\varepsilon) \) such that (1.1) has a unique solution in \( L^\infty_t([t_0, t_1 + T] : \dot{H}^1 \cap \dot{H}^3) \).

In fact, we obtain much more precise control on the solution \( \phi \) than is indicated here; see \( \mathbb{H} \) for our main technical result, which implies Theorem 1.2. Note that by time divisibility Theorem 1.2 also applies to solutions with large \( \ell^2 L^4 \) norm.

## 2. The Caloric Gauge

Let \( I \times \mathbb{R}^2 \ni (t, x) \mapsto \phi(t, x) \in M \) with \( M \in \{ S^2, \mathbb{H}^2 \} \) be a smooth Schrödinger map. A gauge choice may be represented by the map \( e(t, x) \) in the diagram

\[
\begin{array}{ccc}
\mathbb{R}^2 \times \mathbb{C} & \xrightarrow{e} & \phi^* TM \\
| & & | \downarrow \pi \\
I \times \mathbb{R}^2 & \xrightarrow{id} & I \times \mathbb{R}^2 \\
\end{array}
\]

Here \( \psi_\alpha = e^* \partial_\alpha \phi \) denotes the vector \( \partial_\alpha \phi \) written with respect to the choice of orthonormal frame after canonically identifying \( \mathbb{R}^2 \) with \( \mathbb{C} \). The complex structure on \( M \) pulls back to multiplication by \( i \), and the Levi-Civita connection pulls back to the covariant derivatives \( D_\alpha := \partial_\alpha + i A_\alpha \), which generate curvatures \( F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha \). Orthonormality of the frame ensures \( A_\alpha \in \mathbb{R} \). The zero-torsion property of the connection enforces the compatibility condition \( D_\alpha \psi_\beta = D_\beta \psi_\alpha \). Using the fact that \( M \) has constant curvature \( \mu \), one may calculate directly that \( F_{\alpha\beta} = \mu \text{Im}(\bar{\psi}_\beta \psi_\alpha) \). According to our conventions, \( \mu = +1 \) corresponds to the case of the sphere and \( \mu = -1 \) to that of the hyperbolic plane. For any map \( \phi \) and any choice of frame \( e(t, x) \) it therefore holds that

\[
F_{\alpha\beta} = \mu \text{Im}(\bar{\psi}_\beta \psi_\alpha) \quad \text{and} \quad D_\alpha \psi_\beta = D_\beta \psi_\alpha.
\]

These relations are preserved by the gauge transformations

\[
(2.1) \quad \psi_\alpha \mapsto e^{-i\theta} \psi_\alpha \quad A \mapsto A + d\theta,
\]

where \( \theta(t, x) \) is a fast-decaying real-valued function. This gauge invariance corresponds precisely to the freedom we that have in the choice of frame \( e(t, x) \).

Here and throughout we use \( \partial_0 \) and \( \partial_t \) interchangeably. We also adopt the convention that Greek indices are allowed to assume values from the set \( \{-1, 0, 1, 2\} \), whereas Latin indices are restricted to \( \{1, 2\} \), corresponding only to spatial variables. Repeated Latin indices indicate an implicit sum over the spatial variables. The case \( \alpha = -1 \) will be discussed below.

The lift of the Schrödinger map equation to the level of frames is

\[
(2.2) \quad \psi_t = i D_j \psi_j.
\]

To get an evolution equation for \( \psi_j \), we covariantly differentiate with \( D_k \) and use the compatibility condition \( D_k \psi_t = D_t \psi_k \) to obtain \( D_t \psi_k = i D_k D_j \psi_j \). Next we commute \( D_k \) and \( D_j \) using the curvature relation and we invoke the compatibility
condition once more. This results in a covariant Schrödinger evolution equation for $\psi_k$. The whole gauge field system is

\[
\begin{cases}
D_t \psi_k = iD_j D_j \psi_k + F_{jk} \psi_j, \\
F_{01} = \mu \text{Re}(\bar{\psi}_1 D_j \psi_j), \\
F_{02} = \mu \text{Re}(\bar{\psi}_2 D_j \psi_j), \\
F_{12} = \mu \text{Im}(\bar{\psi}_2 \psi_1), \\
D_1 \psi_2 = D_2 \psi_1.
\end{cases}
\]

(2.3)

Note that $\psi_t$ does not appear in this formulation, but can be recovered from the $\psi_j$ by using (2.2). The system (2.3) enjoys gauge freedom, which must be eliminated in order to obtain a well-defined flow. The frame approach was first used to study Schrödinger map wellposedness problems in [7], though the system was formulated at the level of frames at least as early as [13]. In [7], the gauge freedom in (2.3) is eliminated by choosing the Coulomb gauge condition. We eliminate the gauge freedom by instead imposing the caloric gauge condition. The caloric gauge was originally introduced in the setting of wave maps [23], then subsequently applied to the Schrödinger map problem in [4].

The construction of the caloric gauge is most easily carried out at the level of maps, though in principle one could carry it out entirely at the level of frames. Let $\phi(t, x)$ be a map into $M$ defined on $I \times \mathbb{R}^2$ with energy $E_0 := E(\phi) = E(\phi(t))$. We evolve, for each fixed $t_0 \in I$, the map $\phi(t_0, x)$ under harmonic map heat flow, which is the gradient flow associated to the energy (1.2):

\[
\frac{\partial_s}{\partial s} \phi = \Delta \phi + \mu |\partial_x \phi|^2 \phi.
\]

(2.4)

We also use $\phi$ to denote the extension $\phi(s, t, x)$ along this flow. Provided that the mass of $\phi(t_0)$ is finite and the energy of $\phi(t_0)$ is less than that of the ground state, i.e., $E_0 < 4\pi$ when the target is the sphere and $E_0 < \infty$ when the target is the hyperboloid, the flow is well-defined and trivializes as $s \to \infty$, sending all of $\mathbb{R}^2$ to a single point $Q \in M$. To construct the caloric gauge, choose an orthonormal frame at $Q$, pull it back at $s = \infty$, and finally pull it back along the heat flow using parallel transport. This construction is unique modulo the one degree of freedom in the choice of frame at $Q$. The validity of this construction up to the ground state and several related quantitative estimates are established in [19], extending the work initiated for $M = \mathbb{H}^2$ in [23][24].

At the level of gauges, (2.4) assumes the form

\[
\begin{cases}
D_s \psi_k = D_j D_j \psi_k - i F_{jk} \psi_j, \\
F_{-1,1} = \mu \text{Im}(\bar{\psi}_1 D_j \psi_j), \\
F_{-1,2} = \mu \text{Im}(\bar{\psi}_2 D_j \psi_j), \\
F_{12} = \mu \text{Im}(\bar{\psi}_2 \psi_1), \\
D_1 \psi_2 = D_2 \psi_1.
\end{cases}
\]

(2.5)

Here we use $-1$ to denote the $s$-time variable. We also may introduce

\[
\psi_s = D_j \psi_j
\]

(2.6)
in analogy with $\psi_t$ given by (2.2). Pulling back by parallel transport in the $s$ direction corresponds to taking $A_s \equiv 0$. Then the $F_{-1,j}$ equations reduce to transport
equations for $A_\alpha$, and we may define $A_\alpha$ at finite times by integrating back from infinity:
\begin{equation}
A_\alpha(s) = -\int_s^\infty \Im(\bar{\psi}_\alpha D_j \psi_j)(s') ds'.
\end{equation}

Note that (2.7) is valid not only for $\alpha \in \{1, 2\}$, but also for $\alpha \in \{-1, 0\}$.

From [19, Theorem 7.4], we have several energy-type bounds for the connection coefficients $A_x$:
\begin{equation}
\sup_{s>0} s^{\frac{k+1}{2}} \|\partial^k_x A_x(s)\|_{L^\infty_x} \lesssim 1,
\end{equation}
\begin{equation}
\sup_{s>0} s^{\frac{k}{2}} \|\partial^k_x A_x(s)\|_{L^2_x} \lesssim 1,
\end{equation}
\begin{equation}
\int_0^\infty s^{\frac{k-1}{2}} \|\partial^k_x A_x(s)\|_{L^\infty_x} ds \lesssim 1,
\end{equation}
\begin{equation}
\int_0^\infty s^{\frac{k-1}{2}} \|\partial^{k+1}_x A_x(s)\|_{L^2_x} ds \lesssim 1.
\end{equation}

Also, from [19, Corollary 7.5], we have energy-type bounds for the gauge fields $\psi_x$:
\begin{equation}
\sup_{s>0} s^{\frac{k}{2}} \|\partial^{k-1}_x \psi_x\|_{L^\infty_x} \lesssim 1,
\end{equation}
\begin{equation}
\sup_{s>0} s^{\frac{k-1}{2}} \|\partial^{k-1}_x \psi_x\|_{L^2_x} \lesssim 1,
\end{equation}
\begin{equation}
\int_0^\infty s^{k-1} \|\partial^{k-1}_x \psi_x\|_{L^\infty_x}^2 ds \lesssim 1,
\end{equation}
\begin{equation}
\int_0^\infty s^{k-1} \|\partial^{k}_x \psi_x\|_{L^2_x}^2 ds \lesssim 1.
\end{equation}

In addition to (2.9), we have analogous estimates when one replaces $\partial_x \psi_x$ with $\psi_s$, $\partial^2_x$ with $\partial_s$, and/or $\partial_x$ with $D_x$. The constants in (2.8) and (2.9) are allowed to depend upon $k$ and upon the energy $E_0$.

We conclude by noting that the main evolution equations of (2.8) and (2.5) may respectively be rewritten as
\begin{equation}
(i\partial_t + \Delta)\psi_m = N_m,
\end{equation}
\begin{equation}
(\partial_s - \Delta)\psi_\alpha = U_\alpha,
\end{equation}
where
\begin{align}
N_m &:= -2iA_j \partial_j \psi_m - i(\partial_j A_j)\psi_m + (A_t + A^2_x)\psi_m - i\mu \psi_j \Im(\bar{\psi}_j \psi_m),
U_\alpha &:= 2iA_j \partial_j \psi_\alpha + i(\partial_j A_j)\psi_\alpha - A^2_x \psi_\alpha + i\mu \psi_j \Im(\bar{\psi}_j \psi_\alpha).
\end{align}

We assume that we are using the caloric gauge, which is why $A_s \equiv 0$ does not explicitly appear in the (2.11) expression for $U_\alpha$.

3. Function spaces

The main function spaces that we use were first introduced in their present form in [4], in which also is found a discussion of their development. In particular, our $X_k$ and $Y_k$ spaces correspond to the $G_k$ and $N_k$ function spaces of [4]; we have no need for and therefore do not introduce the auxiliary function space $F_k$ of [4].
Lemma 3.1 (Strichartz estimate). Let \( f \in L^2_x(\mathbb{R}^2) \) and \( k \in \mathbb{Z} \). Then the Strichartz estimate

\[
\|e^{it\Delta} f\|_{L^4_{t,x}} \lesssim \|f\|_{L^2_x}
\]

holds, as does the maximal function bound

\[
\|e^{it\Delta} P_k f\|_{L^4_t L^\infty_x} \lesssim 2^k \|f\|_{L^2_x}.
\]

The first bound is the original Strichartz estimate \cite{21} and the second follows from scaling.

For a unit length \( \theta \in S^1 \), we denote by \( H_\theta \) its orthogonal complement in \( \mathbb{R}^2 \) with the induced Lebesgue measure. Define the lateral spaces \( L^{p,q}_\theta \) as those consisting of all measurable \( f \) for which the norm

\[
\|h\|_{L^{p,q}_\theta} := \left[ \int_{\mathbb{R}} \left[ \int_{H_\theta \times \mathbb{R}} |h(t,x_1 \theta + x_2)|^q dx_2 dt \right]^{\frac{p}{q}} dx_1 \right]^{\frac{1}{p}}
\]

is finite. We make the usual modifications when \( p = \infty \) or \( q = \infty \). For proofs of the following lateral Strichartz estimates, see \cite{4, §3, §7}.

Lemma 3.2 (Lateral Strichartz estimates). Let \( f \in L^2_x(\mathbb{R}^2) \), \( k \in \mathbb{Z} \), and \( \theta \in S^1 \). Let \( 2 < p \leq \infty \), \( 2 \leq q \leq \infty \) and \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \). Then

\[
\|e^{it\Delta} P_k \theta f\|_{L^{p,q}_\theta} \lesssim 2^{k\left(\frac{2}{p} - \frac{1}{2}\right)} \|f\|_{L^2_x}, \\
\|e^{it\Delta} P_k f\|_{L^{p,q}_\theta} \lesssim 2^{k\left(\frac{2}{p} - \frac{1}{2}\right)} \|f\|_{L^2_x},
\]

\( p \geq q \)

\( p \leq q \)

In the Schrödinger map setting, local smoothing spaces were first used in \cite{10} and subsequently in \cite{3, 5, 11}.

Lemma 3.3 (Local smoothing \cite{10, 11}). Let \( f \in L^2_x(\mathbb{R}^2) \), \( k \in \mathbb{Z} \), and \( \theta \in S^1 \). Then

\[
\|e^{it\Delta} P_k \theta f\|_{L^\infty_t L^2_x} \lesssim 2^{-\frac{k}{2}} \|f\|_{L^2_x}.
\]

For \( f \in L^2_x(\mathbb{R}^d) \), the maximal function bound

\[
\|e^{it\Delta} P_k f\|_{L^2_t L^\infty_x} \lesssim 2^{-\frac{k(k-1)}{2}} \|f\|_{L^2_x}
\]

holds in dimension \( d \geq 3 \).

In \( d = 2 \), the maximal function bound fails due to a logarithmic divergence. This is overcome by exploiting Galilean invariance as in \cite{4}. For \( p, q \in [1, \infty] \), \( \theta \in S^1 \), \( \lambda \in \mathbb{R} \), define \( L^{p,q}_{\theta, \lambda} \) using the norm

\[
\|f\|_{L^{p,q}_{\theta, \lambda}} := \|T_{\lambda \theta}(f)\|_{L^{p,q}_\theta} = \left[ \int_{\mathbb{R}} \left[ \int_{H_\theta \times \mathbb{R}} |f(t, (x_1 + t\lambda) \theta + x_2)|^q dx_2 dt \right]^{\frac{p}{q}} dx_1 \right]^{\frac{1}{p}},
\]

where \( T_w \) denotes the Galilean transformation

\[
T_w(f)(t,x) := e^{-ix\frac{\lambda}{2}} e^{-it\frac{|w|^2}{4}} f(t,x+tw).
\]
With \( W \subset \mathbb{R} \) finite we define the spaces \( L^p_{θ,W} \) by
\[
L^p_{θ,W} := \sum_{λ \in W} L^p_{θ,λ}, \quad \|f\|_{L^p_{θ,W}} := \inf_{f = \sum_{λ \in W} f_λ} \|f_λ\|_{L^p_{θ,λ}}.
\]
For \( k \in \mathbb{Z}, K \in \mathbb{Z}_{≥0} \), set
\[W_k := \{λ \in [-2^k, 2^k] : 2^k+2Kλ \in \mathbb{Z}\} \]
We work on a finite time interval \([-2^{2K}, 2^{2K}]\) in order to ensure that the \( W_k \) are finite. This is still sufficient for global results provided all effective bounds are uniform in \( K \).

**Lemma 3.4** (Local smoothing/maximal function estimates). Let \( f \in L^2(\mathbb{R}^2) \), \( k \in \mathbb{Z} \), and \( \theta \in S^1 \). Then
\[
\|e^{itΔ}P_{k,θ}f\|_{L^{∞,2}_{θ,W_k}} \lesssim 2^{-\frac{k}{2}}\|f\|_{L^2}, \quad |λ| \leq 2^{k-40},
\]
and moreover, if \( T \in (0,2^{2K}] \), then
\[
\|1_{[-T,T]}(t)e^{itΔ}P_{k}f\|_{L^{2,∞}_{θ,W_k,W_{k+40}}} \lesssim 2^{\frac{k}{2}}\|f\|_{L^2}. \]

**Proof.** The first bound follows from Lemma 3.3 via a Galilean boost. The second is more involved and proven in [4 §7]. \( \square \)

Let \( I \subset \mathbb{R} \) be a time interval. For \( k \in \mathbb{Z} \), let \( \Xi_k = \{ξ \in \mathbb{R}^2 : |ξ| \in [2^{k-1}, 2^{k+1}]\} \). Let
\[L^2_k(I) := \{f \in L^2(I × \mathbb{R}^2) : \text{supp } ˆf(t,ξ) \subset I × Ξ_k\}. \]
For \( f \in L^2(I × \mathbb{R}^2) \), let
\[
\|f\|_{X_k(I)} := \|f\|_{L^∞ L^2} + \|f\|_{L^4 L^6_{t,x}} + 2^{-\frac{k}{2}}\|f\|_{L^4_{t,x} L^∞_{λ}} + 2^{-\frac{k}{2}}\sup_{\theta ∈ S^1}\|f\|_{L^2_{θ,W_k,W_{k+40}}}. \]
Define \( X_k(I), Y_k(I) \) as the normed spaces of functions in \( L^2_k(I) \) for which the corresponding norms are finite:
\[
\|f\|_{X_k(I)} := \|f\|_{X^3_k(I)} + 2^{-\frac{k}{2}}\sup_{θ ∈ S^1}\|P_{j,θ}f\|_{L^{3,6}_{θ}} + 2^{-\frac{k}{2}}\sup_{|j-k| \leq 20}\|P_{j,θ}f\|_{L^{6,3}_{θ}}, \]
\[
\|f\|_{Y_k(I)} := \inf_{f = f_1+f_2+f_3+f_4}\|f_1\|_{L^4_{t,x}} + 2^{\frac{k}{6}}\|f_2\|_{L^{2,6}_{θ}} + 2^{\frac{k}{3}}\|f_3\|_{L^{2,6}_{θ}} + 2^{-\frac{k}{6}}\sup_{θ ∈ S^1}\|f_4\|_{L^{1,2}_{θ,W_k,W_{k+40}}}, \]

where \((\hat{θ}_1, \hat{θ}_2)\) denotes the canonical basis in \( \mathbb{R}^2 \).
These spaces are related via the following linear estimate, which is proved in [4].

**Proposition 3.5** (Main linear estimate). Assume \( K ∈ \mathbb{Z}_{≥0}, I = [t_0, t_1] ⊂ (0, 2^{2K}] \) and \( k \in \mathbb{Z} \). Then for each \( u_0 ∈ L^2 \) that is frequency localized to \( Ξ_k \) and for any \( h ∈ Y_k(I) \), the solution \( u \) of
\[
(i∂_t + Δ_x)u = h, \quad u(t_0) = u_0
\]
(3.1) satisfies
\[
\|u\|_{X_k(I)} \lesssim \|u(t_0)\|_{L^2} + \|h\|_{Y_k(I)}.
\]
Lemma 3.6. For \( k, j \in \mathbb{Z}, \ h \in L^2_{t,x}, \ f \in X_j(I), \) we have the following inequalities under the given restrictions on \( k, j \):

\[
\| P_k(hf) \|_{Y_k(I)} \lesssim \begin{cases} 
2^{-|j-k|} \| h \|_{L^2_{t,x}} \| f \|_{X_j(I)} & |j-k| \leq 80, \\
2^{-|j-k|} \| h \|_{L^2_{t,x}} \| f \|_{X_j(I)} & j \leq k - 80, \\
2^{-|j-k|} \| h \|_{L^2_{t,x}} \| f \|_{X_j(I)} & k \leq j - 80.
\end{cases}
\] (3.2)

Proof. See [4, Lemma 6.3]. \qed

4. THE MAIN RESULT

In this section we state and outline the proof of our main technical result. It is shown in [20, Lemma 4.3] that

\[
\| \nabla \phi \|_{L^4(I \times \mathbb{R}^2)} \sim \| \psi_x \|_{L^4(I \times \mathbb{R}^2)},
\]

and so the small \( L^4 \) assumption of Theorem 1.2 directly lifts to the gauge formulation: we take \( 0 < \varepsilon \ll 1 \) such that

\[
\| \psi_x \|_{L^4(I \times \mathbb{R}^2)} \leq \varepsilon.
\] (4.1)

Pick \( 0 < \delta \ll 1 \). A positive sequence \( \{a_k\}_{k \in \mathbb{Z}} \) is said to be a frequency envelope provided that it belongs to \( L^2 \) and is slowly varying in the sense that \( a_k \leq a_j 2^{\delta |k-j|} \) for all \( j, k \in \mathbb{Z} \). Frequency envelopes satisfy the summation rules

\[
\sum_{k' \leq k} 2^{pk'} a_{k'} \lesssim (p-\delta)^{-1} 2^{pk} a_k, \quad p > \delta,
\]

\[
\sum_{k' \geq k} 2^{-pk'} a_{k'} \lesssim (p-\delta)^{-1} 2^{-pk} a_k, \quad p > \delta.
\]

We absorb the \( (p-\delta)^{-1} \) factor into the constant in applications since it only ever appears \( O(1) \) many times.

For \( \sigma \in \mathbb{Z}_{\geq 0}, \) and \( I = [t_0, t_1], \) define the frequency envelopes \( b_k(\sigma), \alpha_k(\sigma), \) and \( \beta_k(\sigma) \) via

\[
b_k(\sigma) = \sup_{j \in \mathbb{Z}} 2^{\sigma j} 2^{-\delta |k-j|} \| P_j \psi_x \|_{X_j(I)},
\]

\[
\alpha_k(\sigma) = \sup_{j \in \mathbb{Z}} 2^{\sigma j} 2^{-\delta |k-j|} \| P_j \psi_x \|_{L^4_{t,x}},
\]

\[
\beta_k(\sigma) = \sup_{j \in \mathbb{Z}} 2^{\sigma j} 2^{-\delta |k-j|} \| P_j \psi_x(t_0) \|_{L^2_x}.
\]

Set \( b_k = b_k(0), \ \alpha_k = \alpha_k(0), \) and \( \beta_k = \beta_k(0) \) for short. These envelopes satisfy

\[
\sum_k b_k^2 \sim \sum_k \| P_k \psi_x \|_{X_k(I)}^2
\]

and

\[
\sum_{k \in \mathbb{Z}} \alpha_k^2 \sim \| \psi_x \|_{L^2}^2, \quad \sum_{k \in \mathbb{Z}} \beta_k^2 \sim \| P_j \psi_x(t_0) \|_{L^2}^2, \quad L_0^2 \sim \| P_j \psi_x(t_0) \|_{L^2}^2.
\]

For convenience, set

\[
u_k(\sigma) := \alpha_k(\sigma) + \beta_k(\sigma).
\]
Theorem 4.1 (Main technical result). Let $I = [t_0, t_1] \subset \mathbb{R}$ with $t_0 < t_1$. Let \( H^1 \cap H^2 \ni \phi_0 = \phi(t_0) : \mathbb{R}^2 \to M, M \in \{ S^2, \mathbb{H}^2 \}, \) and let \( \phi \) be a solution of (1.1) on \( I \) with finite mass, with energy \( E(\phi) < E_{\text{crit}} \), and with caloric gauge representation \( (\psi_\alpha, A_\alpha) \). Let \( 0 < \delta, \varepsilon \ll 1, \sigma_1 \in \mathbb{Z}_{>0} \), and let frequency envelopes \( b_k(\sigma), v_k(\sigma) \) be defined as above. If (4.1) holds, then
\[
b_k(\sigma) \lesssim \varepsilon v_k(\sigma)
\]
for \( \sigma \in \{0, 1, \ldots, \sigma_1\} \).

Proof. The proof is by a standard continuity argument where we make the bootstrap hypothesis \( b_k \leq \varepsilon^{-\frac{1}{2}} v_k \). Next we apply \( P_k \) to the covariant Schrödinger equation in (2.10) and apply the main linear estimate (3.1). This reduces the problem to controlling \( P_k N_m \) in the \( Y_k(I) \) spaces. Part of \( P_k N_m \) is perturbative, in that it can be bounded in \( Y_k(I) \) by \( \varepsilon^2 b_k \). This is proved in Lemma 4.6 below. Remaining is the nonperturbative part of \( P_k N_m \). In \( \S 6 \) we provide two separate arguments that address how to deal with the nonperturbative part and close the bootstrap. In both arguments we need to control \( X_k(I) \) bounds of \( P_k \psi_x \) along the heat flow; such bounds are established in \( \S 5 \). \( \square \)

Remark 4.2. In the proofs we work with the \( \sigma = 0 \) case, which is the critical case to establish. The same proofs are valid for \( \sigma = \sigma_1 > 0 \) provided that in controlling the Littlewood-Paley decompositions we use the \( \sigma = \sigma_1 \) frequency envelope for the highest frequency term and the \( \sigma = 0 \) frequency envelopes for the remaining terms. See [18] §7 for additional related remarks.

Our main technical tool is the following bilinear estimate, established by the first author in [8]. Various precursors to this estimate appear in [4,17,18,20]. In [8], the estimate is established for the target \( \mathbb{S}^2 \), but the proof also applies to the case where the target is \( \mathbb{H}^2 \).

Theorem 4.3. Let \( \phi \) be a Schrödinger map with finite mass. If a solution \( \psi_x \) of (2.3) in the caloric gauge (with underlying map \( \phi \)) satisfies (1.1) and has \( L^2_x \) norm less than \( E_{0}^\frac{1}{2} \), then
\[
\|(P_j \psi_x(s))(P_k \psi_x(s))\|_{L^{2}_{x}(I \times \mathbb{R}^2)} \lesssim 2^{-\frac{j-k}{2}} v_j v_k (1+s2^{2j})^{-4}(1+s2^{2k})^{-4}.
\]

Related to this are the following \( L^2 \)-based estimates, established in the proof of [8] Theorem 6.3] (see equation (6.107) in that work).

Lemma 4.4. If a solution \( (\psi_x, A_\alpha) \) of (2.3) in the caloric gauge satisfies (1.1) and \( \psi_x \) has \( L^2_x \) norm less than \( E_{0}^\frac{1}{2} \), then
\[
\|A_x\|_{L^2_{t,x}}^{2} + \|\psi_x\|_{L^2_{t,x}}^{2} + \|\partial_x A_x\|_{L^2_{t,x}} + \|A_t\|_{L^2_{t,x}} \lesssim \varepsilon^2,
\]
where all norms are taken over the space-time slab \( I \times \mathbb{R}^2 \).

We have the following technical lemma from [18] Lemma 5.2:

Lemma 4.5. Let \( f \in L^2_{t,x} \). Then
\[
\|P_k(f \psi_m)\|_{Y_k(I)} \lesssim \|f\|_{L^2_{t,x}(I \times \mathbb{R}^2)}b_k.
\]

Combining Lemmas 4.5 and 4.4 yields
\[
\|P_k[-i(\partial_x A_j)\psi_m + (A_t + A_x^2)\psi_m - i\mu \psi \text{Im}(\psi_j \psi_m)]\|_{Y_k(I)} \lesssim \varepsilon^2 b_k,
\]
which proves the following lemma.
Lemma 4.6. The term $N_m + 2iA_\ell \partial_\ell \psi_m$ is perturbative.

We address the remaining nonperturbative term $2iA_\ell \partial_\ell \psi_m$ in §6. Here we show that we can return from the gauge formulation to the map formulation with the following

Lemma 4.7. It holds that
\[ \| \nabla \phi \|_{H^\sigma}^2 \lesssim \sum_{k \in \mathbb{Z}} \sum_{\sigma' = 0}^{2\sigma} b_k^2(\sigma'). \]

Proof. A stronger estimate is established in [20 §4F, Proof of (4-10)], but under a certain smallness assumption. The smallness assumption is not needed, however, for the $\sigma = 0$ case, and this carries over without modification. In fact, as can be seen from [20 §4F, Proof of (4-10)], the full strength of the $b_k(0)$ frequency envelopes is not needed, but rather only the energy space $L^\infty_t L^2_x$ of each $X^0_\ell$ function space is required.

The same argument works for $\sigma > 0$ except for certain high-low frequency interactions for $\sigma > 0$. To derive the claimed expression for $\sigma > 0$, consider $\| \nabla \phi \|_{H^\sigma}$ as an expression bilinear in $\nabla \phi$ and project the product to frequencies $\sim 2^k$. Then use a standard Littlewood-Paley decomposition, enough integrations by parts, and Cauchy-Schwarz. This modification to the argument accounts for the loss of derivatives, which we make no attempt to optimize. □

Theorem 4.1 combined with the preceding lemma and the local result stated in Theorem 1.1 establish Theorem 1.2.

5. Bounds along the heat flow

In this section we prove that $X_\ell$ bounds of $P_\ell \psi_x$ propagate along the heat flow and exhibit decay. We make frequent use of the Duhamel representation
\begin{equation}
\psi_m(s) = e^{s\Delta} \psi_m(0) + \int_0^s e^{(s-s')\Delta} U_m(s') ds'
\end{equation}
with $U_m$ as in (2.11). For frequency envelope definitions, see §4.

Theorem 5.1. Let $\phi$ be a Schrödinger map with finite mass. If a solution $\psi_x$ of (2.3) in the caloric gauge (with underlying map $\phi$) satisfies (4.1) and has $L^2_x$ norm less than $E_{01}^2$, then
\begin{equation}
\| P_\ell \psi_x(s) \|_{X^1(\ell)} \lesssim v_k(1 + s2^{2k})^{-4}.
\end{equation}

Proof. The following three inequalities play a key role in the proof and will be established in subsequent lemmas:
\begin{align}
\| P_\ell \psi_x(s) \|_{L^\infty_t L^2_x} &\lesssim v_k(1 + s2^{2k})^{-4}, \\
\| P_\ell [\psi_x(s) - e^{s\Delta} \psi_x(0)] \|_{L^2_t L^2_x} &\lesssim v_k(s^{-\frac{1}{2}} + 2^k)^{-1}(1 + s2^{2k})^{-4}, \\
\| (\partial_\ell - i\Delta) P_\ell \psi_x(s) \|_{L^2_t L^2_x} &\lesssim v_k(s^{-\frac{1}{2}} + 2^k)(1 + s2^{2k})^{-4}.
\end{align}

First we prove (5.2) for high modulations. Using (5.5), we have for $j > k + 10$ that
\[ \| P_{|\tau| \sim 2^j, |\xi| \sim 2^k} \psi_x(s) \|_{L^2_t L^2_x} \lesssim 2^{-2j}(s^{-\frac{1}{2}} + 2^k) v_k(1 + s2^{2k})^{-4}. \]
This is enough to establish (5.2) for $P_{|\tau|>2^{2k+20},|\xi|\sim 2^k}$ when $s > 2^{-2k}$, $j > k + 10$ and for $P_{|\tau|>2^{20}s^{-1},|\xi|\sim 2^k}$, $s < 2^{-2k}$, $j > k + 10$ by appropriate Sobolev embeddings. We work out the $s > 2^{-2k}$ case in more detail, taking, for notational convenience, $\theta$ chosen along a coordinate axis.

Let $s > 2^{-2k}$. By Bernstein’s inequality in $\tau$ and Sobolev embedding in $x_1$, (5.5) implies
\[
\|P_{|\tau|>2^{2k+20},|\xi|\sim 2^k} \psi_x(s)\|_{L_1^\infty L_2^s L_t^2} \lesssim 2^{3k/2} 2^{-2k} v_k (1 + s 2^k)^{-4} \lesssim 2^{-k/2} v_k (1 + s 2^k)^{-4}.
\]
By Sobolev embedding in $t$ and $x_2$, we obtain
\[
\|P_{|\tau|>2^{2k+20},|\xi|\sim 2^k} \psi_x(s)\|_{L_1^4 L_2^s L_t^4} \lesssim 2^{3k/2} \sum_{j > k+10} 2^{-3j/2} v_k (1 + s 2^k)^{-4} 
\]
\[
\lesssim v_k (1 + s 2^k)^{-4},
\]
\[
\|P_{|\tau|>2^{2k+20},|\xi|\sim 2^k} \psi_x(s)\|_{L_1^4 L_2^s L_t^4} \lesssim 2^{3k/2} \sum_{j > k+10} 2^{-j} v_k (1 + s 2^k)^{-4} \lesssim 2^{k/2} v_k (1 + s 2^k)^{-4},
\]
\[
\|P_{|\tau|>2^{2k+20},|\xi|\sim 2^k} \psi_x(s)\|_{L_1^4 L_2^s L_t^4} \lesssim 2^{3k/2} \sum_{j > k+10} 2^{-5j/3} v_k (1 + s 2^k)^{-4} \lesssim 2^{-k/6} v_k (1 + s 2^k)^{-4},
\]
\[
\|P_{|\tau|>2^{2k+20},|\xi|\sim 2^k} \psi_x(s)\|_{L_1^4 L_2^s L_t^4} \lesssim 2^{3k/2} \sum_{j > k+10} 2^{-4j/3} v_k (1 + s 2^k)^{-4} \lesssim 2^{k/6} v_k (1 + s 2^k)^{-4}.
\]
Together these inequalities establish (5.2) for the high modulation, $s > 2^{-2k}$ case. We can use a similar argument for $s \leq 2^{-2k}$, $P_{|\tau|>2^{20}s^{-1},|\xi|\sim 2^k}$.

For the low modulation case, we expand $\psi_x$ using (5.1). The bound for the linear flow follows from the translation invariance of the $X_k$ norms:
\[
\|e^{s \Delta} P_k \psi_x(0)\|_{X_k} \lesssim v_k (1 + s 2^k)^{-4}.
\]
To obtain the bound for $P_k [\psi_x(s) - e^{s \Delta} \psi_x(0)]$, we combine (5.3) with appropriate Sobolev embeddings, separately considering $s > 2^{-2k}$ and $s < 2^{-2k}$.

This completes the proof for the case $I = \mathbb{R}$. We return to the general case after establishing several lemmas.

In [8] it is established that $L^4$ control propagates along the heat flow:
\[
(5.6) \quad \|P_k \psi_x(s)\|_{L_1^4 L_2^s L_t^4} \lesssim v_k (1 + s 2^k)^{-4}.
\]
This is complemented by the following result.

**Lemma 5.2.** It holds that
\[
(5.7) \quad \|P_k A_x(s)\|_{L_1^4 L_2^s L_t^4} \lesssim v_k (1 + s 2^k)^{-4}.
\]
Proof. We have from (2.7), (2.6), and the Littlewood-Paley trichotomy that

\[ P_k A_x(s) = - \int_s^\infty P_k \text{Im}(\bar{\psi}_x \psi_s)(s') ds' \]

\[ = - \sum_{j_1 \geq k-5, |j_1-j_2| \leq 5} \int_s^\infty P_k \text{Im}(P_{j_1} \bar{\psi}_x \cdot P_{j_2} \psi_s)(s') ds' \]

\[ - \int_s^\infty P_k \text{Im}(P_{k-5} \leq k+5 \bar{\psi}_x \cdot P_{\leq k+10} \psi_s)(s') ds' \]

\[ - \int_s^\infty P_k \text{Im}(P_{\leq k-5} \bar{\psi}_x \cdot P_{k-5} \leq k+5 \psi_s)(s') ds'. \]

By (2.6), \( \psi_s = \partial_t \psi_t + iA_t \psi_t \). Therefore if \( s < 2^{-2k} \), then making use of the Sobolev embeddings \( ||P_k \psi_x||_{L^4} \lesssim 2^k ||P_k \psi_x||_{L^{4/3}} \), \( ||P_k \psi_x||_{L^\infty} \lesssim 2^k/2 ||P_k \psi_x||_{L^4} \), we get

\[ ||P_k A_x(s)||_{L^4_{t,x}} \lesssim 2^k \sum_{j_1 \geq k+5, |j_1-j_2| \leq 5} \int_s^\infty ||\text{Im}(P_{j_1} \bar{\psi}_x \cdot P_{j_2} (\partial_t \psi_t + iA_t \psi_t))(s')||_{L^4_{t,x}} ds' \]

\[ + \int_s^\infty ||P_{k-5} \leq k+5 \bar{\psi}_x(s')||_{L^4_{t,x}} ||P_{\leq k+10} (\partial_t \psi_t + iA_t \psi_t)(s')||_{L^\infty_{t,x}} ds' \]

\[ + \int_s^\infty ||P_{\leq k-5} \bar{\psi}_x||_{L^4_{t,x}} ||P_{k-5} \leq k+5 (A_t \psi_t)(s')||_{L^\infty_{t,x}} ds'. \]

Together (2.8), (2.9), (5.6), and Bernstein’s inequality imply that (5.8) satisfies the bound

\[ ||P_k A_x(s)||_{L^4_{t,x}} \lesssim \sum_{j \geq k+5} v_j 2^k \left( \int_s^\infty (1 + s'^{2j})^{-8} ds' \right)^{1/2} + v_k \]

\[ + \left( \sum_{j \leq k-5} v_j 2^{j/2} \right) ||\psi_x||_{L^2_{t,x}} ||2^{-k/2} ||\partial_x A_x||_{L^2_{t,x}} ||A_x||_{L^2_{t,x}} \lesssim v_k. \]

We work out the first term to demonstrate the technique:

\[ \sum_{j \geq k+5} 2^k \int_s^\infty ||P_j \psi_x||_{L^4_{t,x}} ||P_j (\partial_t \psi_t + iA_t \psi_t)(s')||_{L^\infty_{t,x}} ds' \]

\[ \lesssim \sum_{j \geq k+5} 2^k v_j \int_s^\infty (1 + s'^{2j})^{-4} \left( ||\partial_x \psi_x||_{L^2_x} + ||\psi_x||_{L^2_x} ||A_x||_{L^\infty_x} \right) ds' \]

\[ \lesssim \sum_{j \geq k+5} v_j 2^{k-j} \lesssim v_k. \]
Now let $s \geq 2^{-2k}$. By Bernstein’s inequality and the Sobolev embedding,

$$
\| P_k A_x(s) \|_{L^4_t L^2_x} \lesssim \sum_{j \geq k-5} \int_s^\infty \| P_j \psi_x \|_{L^4_t L^2_x} (\| \partial_x \psi_x \|_{L^\infty_x} + \| \psi_x \|_{L^\infty_x} \| A_x \|_{L^\infty_x}) ds' 
+ 2^{-17k/2} \sum_{j \leq k-5} \int_s^\infty \| P_j \psi_x \|_{L^4_t L^2_x} (\| P_{k-5} \partial_x^{10} \psi_x \|_{L^2_x}^{1/2} \| P_{k-5} \partial_x^9 \psi_x \|_{L^\infty_x}^{1/2} 
+ \| P_{k-5} \partial_x^9 (A_x \psi_x) \|_{L^2_x}^{1/2} \| P_{k-5} \partial_x^8 (A_x \psi_x) \|_{L^\infty_x}^{1/2}) ds' 
\lesssim \sum_{j \geq k-5} v_j 2^{-8j} \int_s^\infty (s')^{-5} + 2^{-17k/2} \sum_{j \leq k-5} 2^{j/2} v_j (\int_s^\infty (s')^{-5} ds') \lesssim v_k 2^{-8k} s^{-4}.
$$

\[ \]

Lemma 5.3. It holds that

$$
\| P_k \psi_x(s) \|_{L^\infty_t L^2_x} \lesssim v_k (1 + s 2^{2k})^{-4}.
$$

Proof. Using representation (5.1), we have by translation invariance that

$$
\| e^{s \Delta} P_k \psi_x(0) \|_{L^\infty_t L^2_x} \lesssim e^{-s 2^{2k}} \| P_k \psi_x(0) \|_{L^\infty_t L^2_x} \lesssim v_k (1 + s 2^{2k})^{-4},
$$

for the linear term. Next we bound the nonlinear Duhamel term in $L^2_t$. Using the Littlewood-Paley trichotomy and (2.10), we obtain

$$
\int_0^s e^{(s-s') \Delta} P_k U_x(s') ds' = \sum_{\ell=1}^6 K_{\ell},
$$

where

$$
K_1 = \int_0^{s'} e^{(s-s') \Delta} P_k [(i(\partial_j A_j) - A_x^2 + i \mu \psi_j \text{Im}(\bar{\psi}_j \psi_x))(P_{k-5 \leq \cdot \leq k+5} \psi_x)](s') ds',
$$

$$
K_2 = \int_0^{s'} e^{(s-s') \Delta} P_k [(2i A_j) (\partial_j P_{k-5 \leq \cdot \leq k+5} \psi_x)](s') ds',
$$

$$
K_3 = \int_0^{s'} e^{(s-s') \Delta} P_k [P_{k-5 \leq \cdot \leq k+5} (i(\partial_j A_j) - A_x^2 + i \mu \psi_j \text{Im}(\bar{\psi}_j \psi_x)) \cdot (P_{k-5 \leq \cdot \leq k+5} \psi_x)](s') ds',
$$

$$
K_4 = \int_0^{s'} e^{(s-s') \Delta} P_k [P_{k-5 \leq \cdot \leq k+5} (2i A_j) \cdot (\partial_j P_{k-5 \leq \cdot \leq k+5} \psi_x)](s') ds',
$$

$$
K_5 = \sum_{l_2 \geq k+5, |l_1-l_2| \leq 5} \int_0^{s'} e^{(s-s') \Delta} P_k [P_{l_1} (i(\partial_j A_j) - A_x^2 + i \mu \psi_j \text{Im}(\bar{\psi}_j \psi_x)) \cdot (P_{l_2} \psi_x)](s') ds',
$$

$$
K_6 = \sum_{l_2 \geq k+5, |l_1-l_2| \leq 5} \int_0^{s'} e^{(s-s') \Delta} P_k [P_{l_1} (2i A_j) \cdot (\partial_j P_{l_2} \psi_x)](s') ds'.
$$

We proceed to bound these terms individually:

$$
\| K_1 \|_{L^2_t} \lesssim (\sup_{s \geq 0} \| P_{k-5 \leq \cdot \leq k+5} \psi_x(s) \|_{L^2_x}) (\| \partial_x A_x \|_{L^4_t L^\infty_x} + \| A_x \|_{L^2_t L^\infty_x}^2 + \| \psi_x \|_{L^2_t L^\infty_x}^2)
$$
and
\[ \|K_2\|_{L^2_x} \lesssim 2^k \left( \sup_{s \geq 0} \|P_{k-5 \leq \cdot \leq k+5} \psi_x(s)\|_{L^2_x} \right) \|A_x\|_{L^2_x L^\infty} \left( \int_0^{s'} e^{-s s^{2k}} ds \right)^{1/2}. \]

By Bernstein’s inequality,
\[ \|K_3\|_{L^2_x} \lesssim 2^{-k} \|P_{k-5 \leq \cdot \leq k+5} \psi_x\|_{L^2_x} \| \int_0^{s'} e^{(s-s') \Delta} P_k \partial_x [P_{k-5 \leq \cdot \leq k+5} (i(\partial_j A_j) - A_x^2 + i\mu \psi_x \text{Im}(\bar{\psi}_j \psi_x))] (s') ds' \|_{L^2_x} \]
and
\[ \|K_4\|_{L^2_x} \lesssim 2^{-k} \|P_{k-5 \leq \cdot \leq k+5} \|_{L^2_x} \| \int_0^{s'} e^{-s s^{2k}} ds \|_{L^2_x}^{1/2}. \]

Finally, by Bernstein’s inequality and the Sobolev embedding,
\[ \|K_5\|_{L^2_x} \lesssim 2^k \sum_{l_2 \geq k+5, |l_1 - l_2| \leq 5} \| \int_0^{s'} e^{(s-s') \Delta} P_k P_t (i(\partial_j A_j) - A_x^2 + i\mu \psi_x \text{Im}(\bar{\psi}_j \psi_x)) (s') ds' \|_{L^1_x} \]
\[ \lesssim \sum_{l_2 \geq k+5, |l_1 - l_2| \leq 5} 2^{k-l_1} \int_0^{s'} \| \partial_x P_t (i(\partial_j A_j) - A_x^2 + i\mu \psi_x \text{Im}(\bar{\psi}_j \psi_x)) \|_{L^2_x} \]
\[ \|((P_t \psi_x))(s')\|_{L^2_x} ds' \]
and
\[ \|K_6\|_{L^2_x} \lesssim \sum_{l_2 \geq k+5, |l_1 - l_2| \leq 5} 2^{k-l_1} \left( \int_0^{s'} e^{-s s^{2k}} ds \right)^{1/2} \| \partial_x A_j \|_{L^2_x} \| P_t \psi_x \|_{L^\infty L^2_x}. \]

Therefore, using Bernstein’s inequality and the Littlewood-Paley trichotomy, we obtain
\[ \|P_k \int_0^s e^{(s-s') \Delta} U_x(s') ds' \|_{L^2_x} \lesssim K_7 \|P_{k-5 \leq \cdot \leq k+5} \psi_x\|_{L^\infty L^2_x} \]
\[ + K_8 \sum_j 2^{-|j-k|} \|P_j \psi_x\|_{L^\infty L^2_x}, \]
where
\[ K_7 := \| \nabla \cdot A \|_{L^1_x L^\infty} + 2^k \left( \int_0^\infty e^{-s s^{2k}} ds \right)^{1/2} \| A_x \|_{L^2_x L^\infty} + \| A_x \|_{L^2_x L^\infty} + \| \psi_x \|_{L^2_x L^\infty}^2 \]
and
\[ K_8 := \| \Delta A \|_{L^1_x L^2_x} + 2^k \left( \int_0^\infty e^{-s s^{2k}} ds \right)^{1/2} \| \nabla A \|_{L^2_x} + \| \nabla (A_x + \psi_x) \|_{L^2_x} \| A_x + \psi_x \|_{L^2_x L^\infty}. \]

In view of the estimates (2.34) and (2.59), we have \( K_7 + K_8 \lesssim E_0 \) and, therefore by partitioning \([0, \infty)\) into finitely many pieces we may arrange \( K_7 + K_8 \lesssim \epsilon \ll 1 \) on each piece. Iterating then yields \( \|P_k \psi_x(s)\|_{L^2_x} \lesssim v_k \).

To obtain decay in \( s \), we make the bootstrap assumption \( \|P_k \psi_x(s)\|_{L^2_x} \leq C v_k (1 + s^{2k})^{-4} \). For \( \delta > 0, N \in \mathbb{Z}_{\geq 0} \), it holds that
\[ e^{-(s-s')^{2k}} \lesssim_{\delta, N} (1 + s^{2k})^{-N}, \quad s' < (1 - \delta)s, \]
and so the integral over $[0, (1 - \delta)s]$ is controlled as follows:

$$
\| P_k \int_0^{(1-\delta)s} e^{(s-s') \Delta} U_x(s') ds' \|_{L^2_t L^2_x} \lesssim v_k (1 + s2^{2k})^{-4}.
$$

Over $[(1 - \delta)s, s]$, we have from the weighted estimates (2.8), (2.9) that

$$
\| s \nabla \cdot A \|_{L^\infty_{t,x}} + 2^k \left( \int_0^\infty e^{-s2^{2k}} ds \right)^{\frac{1}{2}} \| s^{\frac{1}{2}} A \|_{L^\infty_{s,x}} + \| s^{\frac{1}{2}} A_x \|_{L^2_{s,x}} + \| s^{\frac{1}{2}} \psi_x \|_{L^2_{s,x}} \lesssim E_0 1
$$

and

$$
\| s A \|_{L^\infty_{t,x}} + 2^k \left( \int_0^\infty e^{-s2^{2k}} ds \right)^{\frac{1}{2}} \| s^{\frac{1}{2}} \nabla A \|_{L^\infty_{t,x}}
$$

$$
+ \| s^{\frac{1}{2}} (A_x + \psi_x) \|_{L^\infty_{t,x} L^2_x} + \| s^{\frac{1}{2}} (A_x + \psi_x) \|_{L^\infty_{s,x}} \lesssim E_0 1.
$$

Choosing $\delta = \delta(E_0) > 0$ sufficiently small closes the argument. \qed

**Lemma 5.4.** It holds that

$$
\| P_k [\psi_x(s) - e^{s \Delta} \psi_x(0)] \|_{L^2_{t,x}} \lesssim v_k (s^{-\frac{1}{2}} + 2^k)^{-1} (1 + s2^{2k})^{-4}.
$$

**Proof.** To prove the estimate, we need to bound the nonlinear Duhamel term of $\psi_x$ in $L^2_{t,x}$. We proceed term by term, grouping the nonlinear Duhamel term into three terms.

**The term $A \nabla \psi_x$.** First, we have by Bernstein’s inequality, Sobolev embedding, (2.8), (2.9), and Lemma 4.4 that

$$
\| 2^k \int_0^s e^{(s-s') \Delta} P_k (A \psi_x)(s') ds' \|_{L^2_{t,x}}
$$

$$
\lesssim 2^k \int_0^s e^{(s'-s)2^{2k}} \| P_{k-5 \leq \cdot \leq k+5} \psi_x(s') \|_{L^4_{t,x}} \| A_x(s') \|_{L^4_{s,x}} ds'
$$

$$
+ 2^k \int_0^s e^{(s'-s)2^{2k}} \| P_{k-5 \leq \cdot \leq k+5} A_x(s') \|_{L^4_{t,x} L^2_x} \sum_{j \leq k-5} \| P_j \psi_x(s') \|_{L^4_{t,x}} ds'
$$

$$
+ \sum_{j > k+5} 2^{2k-j} \int_0^s e^{(s'-s)2^{2k}} \| \nabla P_j A_x(s') \|_{L^4_{t,x} L^4_{t,x}} \| P_j \psi_x(s') \|_{L^4_{t,x}} ds'.
$$

The second term we split in the following way:

$$
2^k \int_0^s e^{(s'-s)2^{2k}} \| P_{k-5 \leq \cdot \leq k+5} A_x(s') \|_{L^4_{t,x} L^2_x} \sum_{j \leq k-5} \| P_j \psi_x(s') \|_{L^4_{t,x}} ds'
$$

$$
\leq 2^k e^{-s2^{2k}/2} \int_0^{s/2} \| P_{k-5 \leq \cdot \leq k+5} A_x(s') \|_{L^4_{t,x} L^2_x} \sum_{j \leq k-5} \| P_j \psi_x(s') \|_{L^4_{t,x}} ds'
$$

$$
+ 2^k \int_0^s \| P_{k-5 \leq \cdot \leq k+5} A_x(s') \|_{L^4_{t,x} L^2_x} \sum_{j \leq k-5} \| P_j \psi_x(s') \|_{L^4_{t,x}} ds'.
$$
Altogether, we conclude that
\[
\|2^k \int_0^s e^{(s-s')\Delta} P_k(A_x)ds'\|_{L^2_{t,x}} \\
\lesssim \min(2^{-k}, s^{1/2}) e^{-(s/2)2^k} \|P_k \psi_x\|_{L^2_{t,x}} \|\nabla \cdot A_x\|_{L^2_{x}} \\
+ \min(2^{-k}, s^{1/2}) \sup_{\frac{s}{2} \leq s' \leq s} \|P_k \psi_x(s')\|_{L^2_{t,x}} \|\nabla \cdot A_x(s')\|_{L^2_{x}} \\
+ \min(2^{-k}, s^{1/2}) e^{-(s/2)2^k} \|P_k(\nabla \cdot A)\|_{L^2_{x}([0,\frac{s}{2}])} \sum_{j \leq k} 2^{j/2} \|P_j \psi_x\|_{L^4_{t,x}} \\
+ \min(2^{-k}, s^{1/2}) \|P_k(\nabla \cdot A)\|_{L^2_{x}([\frac{s}{2}, s])} \sum_{j \leq k} 2^{j/2} \|P_j \psi_x\|_{L^4_{t,x}} \\
+ e^{-(s/2)2^k} \sum_{j > k+5} 2^{k/2} \|P_j(\nabla \cdot A)\|_{L^2_{x}([\frac{s}{2}, s])} \sum_{j \leq k} 2^{\frac{j}{2}} \|P_j \psi_x\|_{L^4_{t,x}} \\
+ 2^{2^k} \sum_{j > k+5} 2^{k/2} \|P_j(\nabla \cdot A)\|_{L^2_{x}([\frac{s}{2}, s])} \|P_j \psi_x\|_{L^2_{x}([\frac{s}{2}, s])} \|P_j \psi_x\|_{L^2_{x}([\frac{s}{2}, s])}.
\]

The right hand side is controlled by \(\min(s^{\frac{1}{2}}, 2^{-k}) v_k (1 + s 2^{2k})^{-4}\).

Remark 5.5. Once again the \((1 + s 2^{2k})^{-4}\) gain comes from the decay of \(e^{(s-s')\Delta}\) when \(s' < \frac{s}{2}\) and from \(\|P_k \psi_x\|_{L^4_{t,x}} \lesssim (1 + s 2^{2k})^{-4}\) and the decay of \(P_k A_x\) for larger \(s'\).

The term \((\nabla \cdot A)\psi_x\). Next, we have
\[
\| \int_0^s e^{(s-s')\Delta} P_k(\nabla \cdot A)\psi_x ds' \|_{L^2_{t,x}} \\
\lesssim \min(2^{-k}, s^{\frac{1}{2}}) \|P_k \psi_x\|_{L^4_{t,x}} \|\nabla \cdot A_x\|_{L^2_{x}} \\
+ \min(s^{\frac{1}{2}}, 2^{-k}) \|P_k(\nabla \cdot A)\|_{L^2_{x} L^4_{x}} \sum_{j \leq k} 2^{\frac{j}{2}} \|P_j \psi_x\|_{L^4_{t,x}} \\
+ \sum_{j > k+5} 2^{\frac{j}{2}} \|P_j(\nabla \cdot A)\|_{L^2_{x} L^4_{x}} \|P_j \psi_x\|_{L^2_{x} L^4_{t,x}},
\]
and again we obtain the desired control on the right hand side.

The term \(A^2_x \psi_x + \psi^3_x\). Finally, we have
\[
\| \int_0^s e^{(s-s')\Delta} P_k(A^2_x + \psi^2_x)\psi_x ds' \|_{L^2_{t,x}} \\
\lesssim \|P_k \psi_x\|_{L^4_{t,x}} \|(A_x + \psi_x)\|_{L^4_{t,x}} \|(A_x + \psi_x)\|_{L^2_{x} L^\infty} \\
+ \sum_{j > k+5} 2^k \|P_j \psi_x\|_{L^4_{t,x}} \|P_j(A_x + \psi_x)\|_{L^2_{t,x}} \|A_x + \psi_x\|_{L^4_{t,x}} \\
+ \sum_{j \leq k} 2^\frac{j}{2} \|P_j \psi_x\|_{L^4_{t,x}} \|P_{>k}(A_x + \psi_x)\|_{L^2_{x} L^4_{t,x}} \|A_x + \psi_x\|_{L^4_{t,x}},
\]
and the desired bound follows. \(\square\)

Lemma 5.6. It holds that
\[
\|(\partial_t - i\Delta) P_k \psi_x(s)\|_{L^2_{t,x}} \lesssim v_k (s^{-\frac{1}{2}} + 2^k)(1 + s 2^{2k})^{-4}.
\]
Proof. We take advantage of the compatibility condition $D_t \psi_x = D_x \psi_t$, which upon expansion reads

\begin{equation}
\partial_t \psi_x(s) = \partial_x \psi_t(s) + i A_x \psi_t(s) - i A_t \psi_x(s).
\end{equation}

The terms on the right hand side we then expand using the Duhamel representation \([5.1]\). Our strategy will be to bound each of $A_t \psi_x$ and $A_x \psi_t$ directly, followed by $\partial_x \psi_t - i \Delta \psi_x$.

The term $A_t \psi_x$. For $A_t \psi_x$, we have by Sobolev embedding that

$$\|P_k(A_t \psi_x)\|_{L^p_{t,x}} \lesssim \sum_{j \geq k+5} 2^k \|P_j A_t\|_{L^4_{t} L^8_x} \|P_j \psi_x\|_{L^4_{t,x}} + \|P_{k-5} \leq \leq \leq k+5 \psi_x\|_{L^4_{t,x}} \|A_t\|_{L^4_{t,x}}$$

$$+ \sum_{j \leq k-5} 2^{j/2} \|P_j \psi_x\|_{L^4_{t,x}} \|P_{k-5} \leq \leq \leq k+5 \psi_x\|_{L^4_{t,x}} \|A_t\|_{L^4_{t,x}}^{1/2}.$$

By \([2.8]\), \([2.9]\),

$$\|A_t\|_{L^4_{t} L^8_x} + \|s^{1/4} A_t\|_{L^4_{t,x}} \lesssim \|\psi_t\|_{L^2_{t} L^4_{x}} (\|s \partial_x \psi_x\|_{L^4_{t,x}} + \|s^{1/4} A_x\|_{L^4_{t,x}} \|s^{1/4} \psi_x\|_{L^4_{t,x}}) \lesssim \varepsilon$$

and

$$\|A_t\|_{L^4_{t} L^8_x} \lesssim \|\psi_t\|_{L^2_{t} L^4_{x}} (\|\partial_x \psi_x\|_{L^2_{t,x}} + \|A_x\|_{L^2_{t} L^4_{x}} \|\psi_x\|_{L^2_{t,x}}) \lesssim \varepsilon,$$

thanks to \([8]\) Lem. 6.7, Cor. 6.8, Thm. 6.9. We also obtain a stronger frequency localized analogue using

$$P_{k-5} \leq \leq k+5 A_t = - \int_s^\infty P_{k-5} \leq \leq \leq k+5 \text{Im}(\bar{\psi}_t D_j \psi_j) (s') ds'$$

and a standard Littlewood-Paley decomposition of the right hand side. We conclude that $\|P_k(A_t \psi_x)\|_{L^p_{t,x}} \lesssim (s^{-\frac{1}{2}} + 2^k) v_k (1 + s 2^{2k})^{-4}$.

The term $A_x \psi_t$. Invoking \([5.7]\), we have

$$\|P_{k-5} \leq \leq k+5 A_x\|_{L^4_{t,x}} \|\psi_t\|_{L^4_{t,x}} \lesssim s^{-\frac{1}{2}} v_k (1 + s 2^{2k})^{-4}.$$

For the remaining terms, we use the following Duhamel expansion of $\psi_t$:

$$\psi_t(s) = i e^{s \Delta} \partial_t \psi_x(0) - e^{s \Delta} (A_t \psi_x)(0) + \int_0^s e^{(s-s') \Delta} U_t(s') ds'.$$

For the first term from the expansion, we have from Bernstein’s inequality that

$$\|P_{k-5} \leq \leq k+5 i e^{s \Delta} \partial_t \psi_x(0)\|_{L^4_{t,x}} \|A_x\|_{L^4_{t,x}}$$

$$+ \sum_{j \geq k+5} 2^{j-1} \|P_j i e^{s \Delta} \partial_t \psi_x(0)\|_{L^4_{t,x}} \|P_j A_x\|_{L^4_{t,x}} \|P_j A_x\|_{L^4_{t,x}} \|P_j \nabla A_x\|_{L^2_{t,x}}$$

$$\lesssim s^{-\frac{1}{2}} v_k (1 + s 2^{2k})^{-4}.$$

For the second, we use

$$\|e^{s \Delta} P_j(A_t \psi_t)(0)\|_{L^4_{t} L^8_x} \lesssim 2^{-\frac{1}{2}} s^{-\frac{1}{2}} \|\psi_x\|_{L^4_{t,x}} \|A_x\|_{L^2_x} \lesssim 2^{-\frac{1}{2}} s^{-\frac{1}{2}} \varepsilon$$

to conclude that

$$\sum_{j \leq k} 2^{j/2} \|P_j A_x\|_{L^4_{t,x}} \|P_{k-5} e^{s \Delta} (A_t \psi_t)(0)\|_{L^4_{t} L^8_x}$$

$$+ \sum_{j > k+5} 2^{j/2} \|P_j A_x\|_{L^4_{t,x}} \|P_j e^{s \Delta} (A_t \psi_t)(0)\|_{L^4_{t} L^8_x}$$

$$\lesssim s^{-\frac{1}{2}} v_k (1 + s 2^{2k})^{-4}.$$
This leaves the nonlinear Duhamel term. We can control it in $L^4_t L^2_x$ via
\[
\left\| \int_0^s e^{(s-s') \Delta} U_t(s') ds' \right\|_{L^4_t L^2_x} \lesssim \| \psi_t \|_{L^2_t L^4_{t,x}} (\| A \|_{L^\infty_t L^2_x} + \| \nabla \cdot A \|_{L^2_{t,x}} + \| A_x \|_{L^4_{t,x}} + \| \psi_x \|_{L^4_{t,x}}) \lesssim \varepsilon.
\]
Then
\[
\sum_{j \leq k-5} \| P_k \psi_t \|_{L^2_{t,x}} \lesssim (s^{-\frac{1}{2}} + 2^k) v_k (1 + s 2^{2k})^{-4}
\]
and
\[
\sum_{j \geq k+5} 2^k |P_j| \int_0^s e^{(s-s') \Delta} U_t(s') ds' \|_{L^4_t L^2_x} \lesssim 2^k v_k (1 + s 2^{2k})^{-4}.
\]
We conclude that \( \| P_k(A_x \psi_t) \|_{L^2_{t,x}} \lesssim (s^{-\frac{1}{2}} + 2^k) v_k (1 + s 2^{2k})^{-4} \).

**Remark 5.7.** It is possible to prove the $2^{-8k} s^{-4}$ decay of \( \| P_{>k-5} \psi_t \|_{L^2_{t,x}} \) by using the usual bootstrap argument (see, for example, Lemma 5.3).

The term $\partial_x \psi_t - i \Delta \psi_t$. By (5.4), we have
\[
\left\| -i \Delta \int_0^s e^{(s-s') \Delta} P_k(U_t(s')) ds' \right\|_{L^2_{t,x}} \lesssim 2^k v_k (1 + s 2^{2k})^{-4}.
\]
Next, taking advantage of $\int_0^s e^{-(s-s')2^k} ds' \lesssim 2^{-2k}$, we write
\[
\| \partial_x \int_0^s e^{(s-s') \Delta} P_k(U_t(s')) ds' \|_{L^2_{t,x}} \lesssim K_1 + K_2 + K_3 + K_4 + K_5,
\]
where here
\[
K_1 := 2^k \| s^\frac{1}{2} P_k(A_x \psi_t) \|_{L^\infty_t L^2_{t,x}},
\]
\[
K_2 := 2^{\frac{3}{2}k} \sum_{j \geq k} \| \nabla P_j A_x \|_{L^2_t L^4_t L^2_x} \| \psi_t \|_{L^2_t L^2_{t,x}}
\]
\[
+ 2^k \| s^\frac{1}{2} P_{k-5} \leq k+5(A_x \psi_t) \|_{L^\infty_t L^2_{t,x}} \| A_x \|_{L^\infty_t L^2_x},
\]
\[
K_3 := \sum_{j \leq k} 2^j \| s^\frac{1}{2} P_j(A_x \psi_t) \|_{L^\infty_t L^2_{t,x}} \| P_{>k-5} A_x \|_{L^\infty_t L^2_x}
\]
\[
+ \sum_{j > k+5} 2^{\frac{3}{2}k} \| s^\frac{1}{2} P_j(A_x \psi_t) \|_{L^\infty_t L^2_{t,x}} \| s^\frac{1}{2} P_j A_x \|_{L^\infty_t L^2_x},
\]
\[
K_4 := 2^k \| P_{k-5} \leq k+5 \psi_x \|_{L^\infty_t L^4_{t,x}} \| \psi_t \|_{L^2_t L^4_{t,x}} \| \psi_x \|_{L^2_t L^\infty_x}
\]
\[
+ \sum_{j > k+5} \| P_j \psi_x \|_{L^4_{t,x}} \| P_j \psi_t \|_{L^2_t L^4_{t,x}} \sum_{t \leq k-5} 2^t \| P_t \psi_x \|_{L^\infty_t L^2_x},
\]
\[
K_5 := 2^k \sum_{k+5 < j \leq t} \| P_j \psi_x \|_{L^4_{t,x}} \| P_t \psi_x \|_{L^2_{t,x}} \| \psi_t \|_{L^2_t L^4_{t,x}}
\]
\[
+ \sum_{t \leq j \leq k} 2^t \| P_t \psi_x \|_{L^\infty_t L^2_x} \| P_j \psi_x \|_{L^4_{t,x}} \| \psi_t \|_{L^2_t L^4_{t,x}}.
\]
Then $\sum_{j=1}^{5} K_j \leq 2^k v_k$. Through using the usual bootstrapping methods we may upgrade this estimate to

$$
\| \partial_x \int_0^s e^{(s-s')\Delta} P_k U(t) ds' \|_{L_{t,x}^2} \lesssim 2^k v_k (1 + s 2^{2k})^{-4}.
$$

Finally,

$$
\partial_x e^{s\Delta} \psi_t(0) - i e^{s\Delta} \Delta \psi_x(0) = -\partial_x e^{s\Delta} (A_t \psi_t)(0) + i \partial_t e^{s\Delta} (\partial_x \psi_t)(0) - i \Delta e^{s\Delta} \psi_x(0)
$$

$$
= i \partial_t e^{s\Delta} (D_x \psi_t)(0) - i \Delta e^{s\Delta} \psi_x(0) - \partial_x e^{s\Delta} (A_t \psi_t)(0)
$$

$$
+ \partial_t e^{s\Delta} (A_x \psi_t)(0)
$$

$$
= -\partial_t e^{s\Delta} (A_t \psi_t)(0) - \partial_x e^{s\Delta} (A_t \psi_t)(0) + \partial_t e^{s\Delta} (A_x \psi_t)(0).
$$

As

$$
\| \nabla e^{s\Delta} (A_x \psi_x) \|_{L_{t,x}^2} \lesssim 2^k \| P_{k-5 \leq k \leq 5} \| \| A_x \|_{L_{t,x}^4}
$$

$$
+ \sum_{j \leq k} 2^j \| P_j \psi_x \|_{L_{t,x}^4} \| P_k A_x \|_{L_{t}^4 L_{x}^2}
$$

$$
+ \sum_{j > k+5} 2^k \| P_j \psi_x \|_{L_{t,x}^4} \| P_j A_x \|_{L_{t}^4 L_{x}^2},
$$

with right hand side bounded by $2^k v_k (1 + s 2^{2k})^{-4}$, this completes the proof. \qed

Finally, we prove (5.2) for the case $|I| < \infty$.

**Proof of Theorem 5.1 when $|I| < \infty$**. Without loss of generality assume $|I| = 1$. Let $\chi \in \mathcal{C}^\infty$ be a nonnegative bump function that is identically 1 on $I$ and supported on an interval of length 2.

While (5.3), (5.4) are not sensitive to time cutoffs, (5.5) is. However, by combining (5.4) and (5.5), we have

$$
\| (\partial_t - i\Delta)\chi(t)(P_k \psi_x) \|_{L_{t,x}^4} \lesssim v_k 2^k (1 + s 2^{2k})^{-4} + \| P_k \psi_x \|_{L_t^6 L_x^2}
$$

$$
\lesssim v_k (2^k + 1)(1 + s 2^{2k})^{-4},
$$

which is sufficient for the case $k \geq 0$.

For low frequencies, we build up the $X_k$ bound by bounding $\psi_x$ in each function space that appears in the definition. As noted, we already have (5.3). Combining this with the Sobolev embedding, Hölder in time, and the fact that $k \leq 0$, we conclude that

$$
2^k \sup_{|j-k| \leq 20} \sup_{\theta \in S^1} \sup_{|\lambda| < 2^{-k-4}} \| P_{j,\theta} P_k \psi_x \|_{L_{t,x}^6 L_{x,\lambda}^2} \lesssim 2^k \| P_k \psi_x \|_{L_{t,x}^4} \lesssim v_k (1 + s 2^{2k})^{-4}.
$$

By the Sobolev embedding, Hölder’s inequality, and interpolating (5.3) and (5.6), we obtain

$$
2^k \| P_{j,\theta} P_k \psi_x(s) \|_{L_{t,x}^6 L_{x,\lambda}^2(I)} \lesssim 2^k \| P_{j,\theta} P_k \psi_x(s) \|_{L_{t,x}^4(I)} \lesssim 2^k \| P_{j,\theta} P_k \psi_x(s) \|_{L_{t}^4 L_{x}^2(I)}
$$

$$
\lesssim v_k (1 + s 2^{2k})^{-4}
$$

for $|j-k| \leq 20$, $\theta \in S^1$, and $k \leq 0$. It remains to prove

$$
\| P_k \psi_x(s) \|_{L_t^2 L_x^\infty(I)} \lesssim v_k (1 + s 2^{2k})^{-4}
$$

for $k \leq 0$. To prove (5.12), it suffices to prove

$$
\| \partial_t P_k \psi_x(s) \|_{L_t^2 L_x^4} \lesssim v_k (1 + s 2^{2k})^{-1}.
$$
in view of the energy estimate \((5.3)\) and the fundamental theorem of calculus. We do this in several steps. To begin, we take advantage of the compatibility relation \((5.9)\) and expand the \(\psi_k\) terms using \((5.11)\). We then control each term individually.

**The term** \(\partial_x \psi_1\). The control the linear contribution from \(\partial_x \psi_1\), we use the fact that at heat time \(s = 0\) we have the relation \((2.2)\). Hence we control

\[
\| e^{s\Delta} P_k \partial_x \partial_t \psi_1(0) \|_{L^1_t L^2_x} \lesssim v_k 2^k (1 + s 2^{2k})^{-4} \lesssim v_k (1 + s 2^{2k})^{-4}
\]

(as \(k \leq 0\)). It also holds that

\[
\| e^{s\Delta} \partial_x P_k (A_t(0) \psi_1(0)) \|_{L^1_t L^2_x} \lesssim 2^k \sum_{j \leq k} 2^j \| P_j \psi_1(0) \|_{L^\infty_t L^2_x} \| A_{x} \|_{L^\infty_t L^2_x}
\]

\[+ 2^k (1 + s 2^{2k})^{-4} \| P_{k-5 \leq j \leq k+5} \psi_1(0) \|_{L^1_{t,x}} \| A_{x} \|_{L^4_{t,x}}
\]

\[+ 2^k \sum_{j > k+5} 2^{-j} \| P_j \partial_t \psi_1(0) \|_{L^\infty_t L^2_x} \| P_j \nabla A_x \|_{L^4_{t,x}}
\]

and the right hand side is bounded by \(v_k (1 + s 2^{2k})^{-4}\). To control the nonlinear Duhamel term, we use \((5.11)\) and the fact that \(|I| = 1\) to obtain

\[
\| P_k \partial_x \int_0^s e^{(s-s')\Delta} U_t(s') ds' \|_{L^1_t L^2_x} \lesssim v_k 2^k (1 + s 2^{2k})^{-4} \lesssim v_k (1 + s 2^{2k})^{-4}.
\]

Therefore we conclude

\[
\| \partial_x P_k \psi_1 \|_{L^\infty_t L^2_x} \lesssim v_k (1 + s 2^{2k})^{-4}, \quad k \leq 0.
\]

**The term** \(A_t \psi_x\). Here we use \((5.4)\) to obtain

\[
\| P_k (A_t \psi_x) \|_{L^2_t L^4_x} \lesssim \| A_t \|_{L^\infty_t L^2_x} \sum_{j \leq k} 2^j \| P_j \psi_1 \|_{L^\infty_t L^2_x} + 2^k \| P_{k-5 \leq j \leq k+5} \psi_1 \|_{L^\infty_t L^2_x} \| A_{t} \|_{L^4_{t,x}}
\]

\[+ \sum_{j > k+5} 2^k \sup_{|I| = 20} \sup_{\theta \in \mathbb{S}^1} \| P_{I,\theta} P_j e^{s \Delta} \psi_x(0) \|_{L^6_t \| P_j A_t \|_{L^2_x}}
\]

\[+ \sum_{j > k+5} 2^k \| P_j \int_0^s e^{(s-s') \Delta} U_x(s') ds' \|_{L^2_t L^4_{t,x}} \| P_j A_t \|_{L^4_{t,x}}.
\]

The right hand side is bounded by \(v_k (1 + s 2^{2k})^{-4}\), as desired.

**The term** \(A_x \psi_1\). We again take advantage of \((5.9)\) and \((5.11)\). The nonlinear Duhamel term is bounded by

\[
\| P_k \left[ A_x \int_0^s e^{(s-s') \Delta} U_t(s') ds' \right] \|_{L^1_t L^2_x}
\]

\[\lesssim 2^k \| \int_0^s e^{(s-s') \Delta} U_t(s') ds' \|_{L^4_{t,x}} \sum_{j \leq k+5} 2^{\frac{j}{2}} \| P_j A_{x} \|_{L^4_{t,x}}
\]

\[+ 2^k \sum_{j > k+5} \| P_j A_{x} \|_{L^4_{t,x}} \| P_j \int_0^s e^{(s-s') \Delta} U_t(s') ds' \|_{L^2_{t,x}}
\]

and the right hand side is bounded by \(v_k (1 + s 2^{2k})^{-4}\).
This leaves the terms coming from the linear evolution. The first of these we control with
\[ \| P_k(A_x e^{s\Delta} \partial_t \psi_l(0)) \|_{L^2_t L^2_x} \]
\[ \lesssim \sum_{j \leq k-5} \| P_{k-4 \leq j \leq k+5} A_x \|_{L^4_t L^4_x} \| P_j e^{s\Delta} (\partial_t \psi_l(0)) \|_{L^4_t L^4_x} \]
\[ + \| P_{k-5 \leq j \leq k+5} e^{s\Delta} \partial_t \psi_l(0) \|_{L^4_t L^4_x} A_x \|_{L^4_t L^4_x} \]
\[ + 2^{\frac{j}{2}} \sum_{j \geq k+5} \| P_j A_x \|_{L^2_t L^2_x} \sup_{|p-j| \leq 20} \| e^{s\Delta} P_{j,\theta} \partial_t \psi_l(0) \|_{L^6_t L^3_x}, \]
whose right hand side is bounded by \( v_k(1 + s2^{2k})^{-4} \). For the second, we have
\[ \| P_k(e^{s\Delta} A_l(0) \psi_l(0)) A_x \|_{L^1_t L^2_x} \]
\[ \lesssim (1 + s2^{2k})^{-4} \| \psi_l(0) \|_{L^4_t L^4_x} A_x(0) \|_{L^2_t L^2_x} \sum_{j \leq k-5} 2^{\frac{j}{2}} \| P_j A_x(s) \|_{L^4_t L^4_x} \]
\[ + \| P_{k-5 \leq j \leq k+5} A_x \|_{L^4_t L^4_x} \| A_l(0) \|_{L^4_t L^4_x} \| \psi_l(0) \|_{L^4_t L^4_x} \]
\[ + \sum_{j \geq k+5} 2^j \| P_j A_x(s) \|_{L^2_t L^2_x} \left[ (1 + s2^{2k})^{-4} \| P_j \psi_l(0) \|_{L^4_t L^4_x} A_l(0) \|_{L^4_t L^4_x} \right] \]
\[ + \| \psi_l(0) \|_{L^4_t L^4_x} \| P_j A_x(0) \|_{L^4_t L^4_x} \right), \]
and the right hand side is bounded by \( v_k(1 + s2^{2k})^{-4} \).

6. CLOSING THE ARGUMENT

We present two different arguments for handling the part of the nonlinearity \( N_m \) not covered by Lemma \ref{lem5.6}. The first argument relies on a time subdivision, and the second on a discrete Gronwall-type approach.

6.1. **Argument I.** It suffices to control either \( \nabla \cdot (A_x \psi_x) \) or \( A_x \nabla \psi_x \), as they are equivalent up to a \( (\nabla \cdot A_x) \psi_x \) term, which was controlled in \([4]\).

Partition \( I \) into finitely many pieces such that
\[ \sum_k \sum_l \int_0^\infty \| (P_k \psi_x(0))(P_l \psi_x(s)) \|_{L^2_x I_j}^2 2^{(k-l)} 2^{l} ds \leq \epsilon \]
on each subinterval \( I_j \).

If
\[ \sum_k \| P_k \psi_x \|_{X_k(I_j)}^2 < \infty, \]
then there exists a frequency envelope \( a_k \) such that
\[ \| P_k \psi_x \|_{X_k(I_j)} \leq a_k \]
for all \( k \) and
\[ \sum_k a_k^2 \lesssim \sum_k \| P_k \psi_x \|_{X_k(I_j)}^2. \]

Therefore Theorem \ref{thm5.1} implies
\[ \sum_k (\sup_s \| P_k \psi_x(s) \|_{X_k(I_j)})^2 \lesssim \sum_k \| P_k \psi_x \|_{X_k(I_j)}^2. \]
We look at the term 
\[
\nabla \cdot (A_x \psi_x) = -\nabla \cdot (\psi_x \int_0^\infty \text{Im}(\bar{\psi}_x \partial_l \psi_l)(s) \, ds)
\]
and decompose it into two main pieces.

First, we consider
\[
(6.2) \quad \sum_k \|P_k((\nabla \psi_x) \cdot \int_0^\infty \text{Im}(\bar{\psi}_x \partial_l \psi_l)(s) \, ds)\|_{Y_k(I_j)}^2 \lesssim K_1 + K_2 + K_3 + K_4,
\]
where here
\[
K_1 := \sum_k 2^k \left( \sum_{l_1 \leq l_2 \leq k-10} a_{l_1} 2^{l_2} 2^{l_3} \int_0^\infty \|(P_{k-5 \leq \cdot \leq k+5} \psi_x(0))(P_2 \psi_x(s))\|_{L^4_{t,x}(I_j)}^2 \, ds \right)^2,
\]
\[
K_2 := \sum_k (2^k \sum_{j > k-10} \int_0^\infty \|P_j \psi_x(s)\|_{L^4_{t,x}}^2) a_k^2 \epsilon^2,
\]
\[
K_3 := \sum_k 2^{2k} \left( \sum_{j > k+10} a_j \sum_{j_1 > j-10} 2^{j_1} \|P_{j_1} \psi_x(s)\|_{L^4_{t,x}}^2 \right) \epsilon^2,
\]
\[
K_4 := \sum_k \left( \sum_{j < k-10} \|(P_j \psi_x)(\nabla P_{k-10 \leq \cdot \leq k+10} A_x)\|_{Y_k(I_j)} \right)^2.
\]
By (3.2),
\[
K_4 \lesssim \epsilon^4 \sum_k \left( \sum_{j < k-10} a_j 2^{\frac{j-2k}{2}} \right)^2 \lesssim \epsilon^4 \left( \sum_k a_k^2 \right).
\]
Combining (6.1) with the Cauchy-Schwarz inequality,
\[
\sum_k \|P_k((\nabla \psi_x) \cdot \int_0^\infty \text{Im}(\bar{\psi}_x \partial_l \psi_l)(s) \, ds)\|_{Y_k(I_j)}^2 \lesssim \epsilon \left( \sum_k a_k^2 \right).
\]
Now consider the term
\[
\sum_k \|P_k((\nabla \psi_x) \cdot \int_0^\infty \text{Re}(\bar{\psi}_x A_l \psi_l)(s) \, ds)\|_{Y_k(I_j)}^2.
\]
As
\[
\|P_l (A_x \psi_x)\|_{L^4_{t,x}} \lesssim 2^l \|P_{l-5 \leq \cdot \leq l+5} \psi_x\|_{L^4_{t,x}} \|A_x\|_{L^\infty_x L^2_t} + 2^{\frac{l}{2}} \sum_{l_1 \leq l-5} 2^{\frac{l_1}{2}} \|P_{l_1} \psi_x\|_{L^4_{t,x}} \|P_{l-5} A_x\|_{L^\infty_x L^2_t} + 2^{\frac{l}{2}} \sum_{j > l+5} \|P_j \psi_x\|_{L^4_{t,x}} \|P_{j} A_x\|_{L^\infty_x L^4_t}
\]
with the right hand side bounded by \(v_l(1 + s^{2l})^{-4}\), it therefore follows that terms involving \(P_{l \leq k-10}(A_x \psi_x)(s)\) or \(P_{l \geq k-10} \psi_x(s)\) can be analyzed in identical fashion to those appearing in the last two lines of (6.2). This leaves us with a term of the form
\[
\sum_k \|P_k((\nabla \psi_x) \cdot \int_{l_1 \leq t_2 \leq l-10} \text{Re}(P_{l_1} (\bar{\psi}_x P_{l_2} (A_l \psi_l))(s) \, ds)\|_{Y_k(I_j)}^2.
\]
Invoking (3.2) and by choosing $C_0(E_0)$ to be a very large, fixed constant, we bound this term by

\[
\sum_{k} 2^{k} 2^{C_0} \left( \sum_{l_1 \leq l_2 \leq k-10} a_{l_1} 2^{l_2} 2^{l_3} \cdot \int_{0}^{\infty} \|(P_{k-5 \leq \cdot \leq k+5} \psi_x(0))(P_{l_2} \psi_x(s))\|_{L_{t,x}^2(I_j)}^2 ds \right)^{1/2} + \sum_{k} 2^{k} \left( \sum_{l_1 \leq l_2 \leq k-10, l_3 \geq l_2 + C_0} (\sup_s \|(P_{k-5 \leq \cdot \leq k+5} \psi_x(0))(P_{l_1} \psi_x(s))\|_{L_{t,x}^2}) \right) \cdot 2^{\frac{1}{2}} a_{l_3} \|P_{l_3} A_x\|_{L_{t,x}^2}^2,
\]

which is controlled by $\epsilon^{2C_0} (\sum a_{l_1}^2) + 2^{-\frac{C_0}{2}} (\sum a_{l_1}^2)$.

Choosing $C_0(E_0)$ sufficiently large and $\epsilon(C_0)$ sufficiently small, then we have, since $\sum_{l} b_{l}^2 \lesssim \sum_{l} \|P_{l} \psi_x(0)\|_{X_{I(J)}}^2$,

\[
\sum_{k} \|P_{k} \psi_x(0)\|_{X_{I(J)}}^2 \lesssim_{E_0, C_0} 1.
\]

Plugging the finiteness of $\sum_{l} b_{l}^2$ back into the above yields

\[
\|P_{k} \psi_x(0)\|_{X_{I(J)}} \lesssim \epsilon k.
\]

6.2. **Argument II.** Our goal is to control $A_t \partial_t \psi_m$. To do so, we begin by decomposing it according to the usual Littlewood-Paley trichotomy:

\[
P_k(A_t \partial_t \psi_m) = P_k[(\sum_{k_1 \leq k-5} + \sum_{k_2 \leq k-5, |k_2-k| \leq 4} + \sum_{k_1, k_2 \geq k-5, |k_1-k_2| \leq 8} P_{k_1} A_t \partial_t P_{k_2} \psi_m)].
\]

**Remark 6.1.** In this section we slightly abuse asymptotic notation, as we endow it with a meaning different from its usual one when applied to indices indicating frequency projections. For instance, if $k_1$, $k_2$ are indices associated to the Littlewood-Paley projections $P_{k_1}$, $P_{k_2}$, then the expression $k_1 \lesssim k_2$ is short for $2^{k_1} \lesssim 2^{k_2}$, etc. In other circumstances, the asymptotic notation retains its usual meaning.

When $k_2 \lesssim k_1 \sim k$ as in the second sum of (6.3), we treat the derivative on $\psi_m$ as $2^{k_2}$, transfer it to $A_t$, and then use the $L_{t,x}^2$ bound on $\partial_t A_x$.

**Lemma 6.2.** It holds that

\[
\|\sum_{k_2 \leq k-5 \atop |k_1-k| \leq 4} P_{k_1} A_t \partial_t P_{k_2} \psi_m \|_{Y_k(I)} \lesssim \epsilon^2 b_k.
\]

**Proof.** By Lemmas 4.5 and 4.4 and the frequency envelope property, we have

\[
\|\sum_{k_2 \leq k-5 \atop |k_1-k| \leq 4} P_{k_1} A_t \partial_t P_{k_2} \psi_m \|_{Y_k(I)} \lesssim \sum_{k_2 \leq k-5 \atop |k_1-k| \leq 4} 2^{-|k_2-k_1|} \|P_{k_1} A_x\|_{L_{t,x}^2} \|P_{k_2} \psi_m\|_{X_{k_2}(I)} \lesssim \sum_{k_2 \leq k-5 \atop |k_1-k| \leq 4} 2^{-|k_2-k_1|} \epsilon^2 b_{k_2} \lesssim \epsilon^2 b_k.
\]

□
For the high frequencies we lose summability. This can be overcome by expanding \( A \) using the representation (2.7) and refining our analysis. Hence we consider

\[
(6.4) \quad P_k(A \partial_t \psi_m) = -P_k[\sum_{k_1, k_2, k_3} \int_0^\infty P_{k_1} \psi_\ell(s') P_{k_2} \psi_\ell(s') ds' \partial_t P_{k_3} \psi_m].
\]

In (6.4), we expand \( \psi_\ell \) using (2.2), so that \( \psi_\ell(s') = \partial_j \psi_j(s') + (iA_j \psi_j)(s') \), and then treat the \( \partial_j \psi_j(s') \) and \( (iA_j \psi_j)(s') \) terms separately.

**Lemma 6.3.** It holds that

\[
\|P_k[\sum_{k_1, k_2 \geq k - 5} P_{k_1} A_\ell \partial_t P_{k_2} \psi_m]\|_{Y_k(I)} \lesssim v_k^2 b_k.
\]

**Proof.** For the summation under consideration, in (6.4) we are restricted to the range \( k_3 \geq k \) and \( \max\{k_1, k_2\} \geq k_3 \).

First we treat only the \( \partial_x \psi_x \) portion of the \( \psi_\ell \) term. Hence we need to control

\[
K_1 := P_k[\sum_{\substack{k_3 \geq k \\
\max\{k_1, k_2\} \geq k_3}} \int_0^\infty P_{k_1} \psi_x(s') \partial_x P_{k_2} \psi_x(s') ds' \partial_x P_{k_3} \psi_x(0)].
\]

Suppose \( k_1 \geq k_2 \). Then we apply (3.2), pulling out \( P_{k_1} \psi_x \) in \( X_{k_1}(I) \) and picking up a gain of at least \( 2^{-\frac{|k-k_1|}{6}} \). To the remaining term we apply (4.2). Then

\[
\|K_1\|_{k_1 \geq k_2} \lesssim \sum_{\substack{k_3 \geq k \\\n\max\{k_1, k_2\} \geq k_3}} b_{k_1} v_{k_2} v_{k_3} 2^{-\frac{|k-k_1|}{6}} 2^{-\frac{|k_3-k_1|}{2}} 2^{k_2-k_1} 2^{k_3-k_1}.
\]

Using the frequency envelope summation properties, we can sum the geometric series. If, on the other hand, \( k_2 \geq k_1 \), then we pull out \( P_{k_3} \psi_x \) in \( X_{k_3}(I) \) using (3.2) instead. We obtain

\[
\|K_1\|_{k_2 \geq k_1} \lesssim \sum_{\substack{k_3 \geq k \\\k_2 \geq k_1}} b_{k_3} v_{k_1} v_{k_2} 2^{-\frac{|k-k_1|}{6}} 2^{-\frac{|k_2-k_1|}{2}} 2^{k_3-k_2} 2^{k_3-k_1}.
\]

Combining the two cases, we conclude that

\[
(6.5) \quad \|K_1\|_{Y_k(I)} \lesssim v_k^2 b_k.
\]

**Remark 6.4.** When \( k_2 > k_1 \), one could pull out \( P_{k_3} \psi_x \) in \( X_{k_2}(I) \) rather than \( P_{k_3} \psi_x \) in \( X_{k_3}(I) \); this strategy, however, poses additional challenges when it comes to handling the analogous \( P_{j_1} A_{j_2} P_{j_3} \psi_x \) term (see below), as the frequency localizations in the definition of the \( X_k \) norm prevent us from directly pulling out \( P_{j_1} A_x \) in \( L_t^\infty \).

Next we consider

\[
K_2 := P_k[\sum_{\substack{k_3 \geq k \\\\max\{k_1, k_2\} \geq k_3}} \int_0^\infty P_{k_1} \psi_x(s') P_{k_2} \sum_{j_1, j_2} (P_{j_1} A_{j_2} P_{j_3} \psi_x)(s') ds' \partial_x P_{k_3} \psi_x(0)].
\]

If \( k_1 \geq k_2 \), then, as above, we apply (3.2) and pull out \( P_{k_1} \psi_x \) in \( X_{k_1}(I) \). To the rest we apply (4.2), but pull out \( P_{j_1} A_x \) in \( L_t^\infty \), i.e.,

\[
\|P_{k_2}((P_{j_1} A_x P_{j_2} \psi_x)(s')) \partial_x P_{k_3} \psi_x(0)\|_{L_t^2} \lesssim \|P_{j_1} A_x(s')\|_{L_t^\infty} \|P_{j_2} \psi_x(s') \cdot P_{k_3} \partial_x \psi_x(s')\|_{L_t^2} \lesssim \|P_{j_1} A_x(s')\|_{L_t^\infty} 2^{k_3} 2^{-\frac{|k_3-j_1|}{2}} v_{j_2} v_{k_3} (1 + s_2^{2j_2})^{-1}(1 + s_2^{2k_3})^{-1}.
\]
We use (2.8) to obtain
\[ \|P_{j_1} A_x(s')\|_{L^\infty_{t,x}} \lesssim E_0 2^{j_1} (1 + s 2^{j_1})^{-1}. \]
When \( j_1, j_2 \lesssim k_2 \), the contribution in \( K_2 \) is weaker than the corresponding contribution from \( \partial_t P_{k_2} \psi_x \) in \( K_1 \). The summation of \( j_1 \) up to \( k_2 \) is taken directly and the summation of \( j_2 \) up to \( k_2 \) is achieved using the decay from the application of (4.2):
\[ \|K_2\|_{j_1, j_2 \leq k_2} \|Y_k(I)\| \lesssim \sum_{j_1, j_2 \leq k_2} \sum_{k_1 \geq k_2} b_{k_1} v_{j_2} v_{k_2} 2^{-\frac{|k-k_1|}{6}} 2^{-\frac{|k_1-j_2|}{2}} 2^{j_1+k_2-2k_1}. \]
When \( j_1 \sim j_2 \gtrsim k_2 \), we have extra decay from the heat flow and from (4.2), which enable us to sum:
\[ \|K_2\|_{j_1 \sim j_2 \gtrsim k_2} \|Y_k(I)\| \lesssim \sum_{j_1 \sim j_2 \gtrsim k_2} \sum_{k_1 \geq k_2} b_{k_1} v_{j_2} v_{k_2} 2^{-\frac{|k-k_1|}{6}} 2^{-\frac{|k_1-j_2|}{2}} 2^{j_1+k_2-2k_1}. \]
If \( k_2 > k_1 \), then we pull out \( P_{k_2} \psi_x \) in \( X_{k_3}(I) \) instead and control the remaining terms in \( L^2 \) as above. Combining the cases, we conclude
\[ \|K_2\|_{Y_k(I)} \lesssim v_k^2 b_k. \]
This leaves in (6.3) only the first term
\[ \sum_{k_1 \leq k-5} \sum_{|k_2-k| \leq 4} P_{k_1} A_x \partial_t P_{k_2} \psi_m \]
to control. The next lemma shows that part of this term enjoys bounds similar to those in the previous lemma.

**Lemma 6.5.** It holds that
\[ \|P_k[ \sum_{|k_1-k_2| \leq 8, k_1 \geq k-4} \int_0^\infty P_{k_1} \psi_x(s') P_{k_2} \psi_x(s') ds' \partial_t P_{k_3} \psi_m(0)]\|_{Y_k(I)} \lesssim v_k^2 b_k. \]

**Proof.** The proof is analogous to that of Lemma 6.3. We treat
\[ K_1 := P_k[ \sum_{|k_1-k_2| \leq 8, k_1 \geq k-4} \int_0^\infty P_{k_1} \psi_x(s') \partial_x P_{k_2} \psi_x(s') ds' \partial_t P_{k_3} \psi_x] \]
and
\[ K_2 := P_k[ \sum_{|k_1-k_2| \leq 8, k_1 \geq k-4} \int_0^\infty P_{k_1} \psi_x(s') P_{k_2} \sum_{j_1, j_2} (P_{j_1} A_x P_{j_2} \psi_x(s')) ds' \partial_t P_{k_3} \psi_x(0)] \]
separately, but in both cases apply (6.2) to pull out \( P_{k_1} \psi_x \) in \( X_{k_1} \). We can always apply estimate (8.2) in this way because here \( k_1 \gtrsim k_2 \). Next we use (4.2) and, in the case of \( K_2 \), bound \( P_{j_1} A_x \) in \( L^\infty_{t,x} \). We are left with a summable geometric series in both cases. \( \square \)
Combining Lemmas \textbf{4.6, 6.2, 6.3} and \textbf{6.5} we conclude

\textbf{Lemma 6.6.} It holds that

\[ \|P_k[N_m - 2i \sum_{k_1, k_2 \leq k \atop |k_3 - k| \leq 4} \int_0^\infty \text{Im}(\overline{P_{k_1} \psi_j P_{k_2} \psi_s}(s')ds')P_{k_3} \partial_j \psi_m(0))\|_Y_k(t) \lesssim (\varepsilon^2 + v_k^2)b_k. \]

Now we pick up with the term of $N_m$ left unaddressed by Lemma \textbf{6.6}. The term does not enjoy bounds like those above, but does still obey bounds good enough for closing a bootstrap argument. This approach to closing the argument is the same as the one used in \cite{18, 20}.

\textbf{Lemma 6.7.} It holds that

\[ \|P_k \sum_{k_1, k_2 \leq k \atop |k_3 - k| \leq 4} \int_0^\infty \text{Im}(\overline{P_{k_1} \psi_j P_{k_2} \psi_s}(s')ds')P_{k_3} \partial_j \psi_m(0))\|_Y_k(I) \lesssim v_k \sum_{j \leq k - C_1} b_j v_j. \]

\textbf{Proof.} As in the proofs of Lemmas \textbf{6.3} and \textbf{6.5}, we use (3.2) to pull out $P_{k_1} \psi_x$ in $X_{k_1}(I)$. Here, however, we need the full one-half power of decay afforded us by (3.2). As previously, to the remaining term we apply (4.2) to control it in $L^2$.

When we consider the $\partial_x \psi_x$ portion of $\psi_s$, we have the bound

\[ \sum_{k_1, k_2 \leq k \atop |k_3 - k| \leq 4} b_{k_1} v_{k_2} v_{k_3} 2^{-\frac{|k_1 - k|}{2}} 2^{-\frac{|k_3 - k_2|}{2}} 2^{k_2 + k_3} 2^{-\max\{k_1, k_2\}}. \]

In this expression $k_1$ and $k_2$ play symmetric roles and so we do not need to consider the cases $k_1 \geq k_2$ and $k_2 > k_1$ separately. The decay along the heat flow allows us to sum to the diagonal, but not all the way up to $k$: there is no extra source of decay along the diagonal $k_1 = k_2$, which is why the best bound we can achieve for this term is $v_k \sum_{j \leq k - C_1} b_j v_j$.

As for the $P_{j_1} A_x(s')$ term appearing in the expansion of $\psi_s$, we place it in $L^\infty_{t,x}$ as done previously, and recover summation to the diagonal. \hfill \Box

Combining Proposition \textbf{3.5} and Lemmas \textbf{6.6} and \textbf{6.7} we obtain

\[ \|P_k \psi_m\|_{X_k(I)} \lesssim \|P_k \psi_m(t_0)\|_{L^2_x} + (\varepsilon^2 + v_k^2)b_k + v_k \sum_{j \leq k - C_1} b_j v_j. \]

The initial data is included in the $v_k$ envelope; moreover, by using the frequency envelope property, we may absorb the $v_k^2 b_k$ term in the summation. Therefore

\[ b_k \lesssim v_k + v_k \sum_{j \leq k - C_1} b_j v_j. \]

Squaring and applying Cauchy-Schwarz yields

\[ b_k^2 \lesssim (1 + \sum_{j \leq k - C_1} b_j^2) v_k^2. \]

Setting

\[ B_k := 1 + \sum_{j \leq k} b_j^2 \]

in (6.6) leads to

\[ B_{k+1} \leq B_k (1 + C_2 v_k^2) \]
with $C_2 > 0$ independent of $k$. Therefore

\[
B_{k+m} \leq B_k \prod_{\ell=1}^{m} (1 + C_2 \nu^2_{k+\ell}) \leq B_k \exp(C_2 \sum_{\ell=1}^{m} \nu^2_{k+\ell}) \lesssim B_k.
\]

Since $B_k \to 1$ as $k \to -\infty$, we conclude that

\[
B_k \lesssim 1
\]

uniformly in $k$, so that, in particular,

\[
(6.7) \quad \sum_{j \in \mathbb{Z}} b_j^2 \lesssim 1
\]

which, joined with (6.6), implies

\[
b_k \lesssim \nu_k
\]

as desired.

ACKNOWLEDGMENTS

The second author thanks Ioan Bejenaru for helpful conversations and encouragement in connection with pursuing this project.

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Department of Mathematics, 970 Evans Hall, University of California, Berkeley, California 94720-3840

E-mail address: benjadod@math.berkeley.edu

Current address: Department of Mathematics, Johns Hopkins University, 404 Krieger Hall, 3400 N. Charles Street, Baltimore, Maryland 21218

E-mail address: dodson@math.jhu.edu

Department of Mathematics, 970 Evans Hall, University of California, Berkeley, California 94720-3840

E-mail address: smith@math.berkeley.edu

Current address: Google, 1600 Amphitheatre Parkway, Mountain View, California 94043