CHANGE OF VARIABLE FORMULA FOR LOCAL TIME OF CONTINUOUS SEMIMARTINGALE

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Abstract. In this paper we generalize a representation formula for the local time of a function of a semimartingale due to Coquet and Ouknine [3], our formula being a pointwise equality between two processes we show in addition that the equality is in fact trajectorial, finally we give an application in mathematical finance.

1. Introduction

Let \((X_t)_{t \geq 0}\) be a continuous semimartingale and let \((L^a_t(X))_{a \in \mathbb{R}, t \geq 0}\) be its family of local times continuous in \(t\) and càd-làg in \(a\), recall that this family of random variables satisfies Tanaka’s formula (theorem 1.2 in chapter 6 of [9]):

\[
|X_t - a| = |X_0 - a| + \int_0^t \text{sgn}(X_s - a) \, dX_s + L^a_t(X)
\]

with \text{sgn} is the function defined by:

\[
\text{sgn}(x) = \begin{cases} 
1 & x > 0 \\
-1 & x \leq 0 
\end{cases}
\]

Let \(\phi : \mathbb{R} \to \mathbb{R}\) be a function of class \(C^2\), according to Ito’s Formula \(\phi(X)\) is a continuous semimartingale. Following the work of Eméry and Yor [5], Coquet and Ouknine proved in [3] the following change of variable formula:

\[
L^0_t(\phi(X)) = \sum_{\phi(x)=0, \phi'(x) \neq 0} \left[ (\phi'(x))^+ L^x_t(X) + (\phi'(x))^L^x_t(X) \right]
\]

Let:

\[
dL^{x,+}_t(X) = \begin{cases} 
dL^{x}_t(X) & \phi'(x) > 0 \\
dL^{x}_t(X) & \phi'(x) < 0 
\end{cases}
\]

then we can rewrite (3) as:

\[
L^0_t(\phi(X)) = \sum_{\phi(x)=0, \phi'(x) \neq 0} \int_0^t |\phi'(x)| \, dL^{x,+}_s(X)
\]

Then:

\[
L^0_t(\phi(X)) = \sum_{\phi(x)=0, \phi'(x) \neq 0} \int_0^t |\phi'(x)| \, dL^{x,+}_s(X)
\]

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Let $V: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ a map of class $C^{1,2}$ \footnote{meaning there exists $\epsilon > 0$ such that $V \in C^{1,2}([-\epsilon, +\infty[\times \mathbb{R} \to \mathbb{R})$ i.e on this open set the partial derivatives $\frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2}, \frac{\partial^2 V}{\partial x \partial t}$ do exist and are continuous}. By Ito’s Formula $(V(t, X_t))_{t \geq 0}$ is a semimartingale, the goal of this paper is to show a similar formula, i.e there exists an increasing process that will be defined later such that for all $t \geq 0$ we have almost surely:

$$L^0_t (V(., X)) = \int_0^t \sum_{V(s,x)=0, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x} (s, x) \right| d\Lambda_s^{x+}(X)$$

(6)

This paper is organized as follows: First in the second section we define the increasing process

$$\left( \int_0^t \sum_{V(s,x)=0, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x} (s, x) \right| d\Lambda_s^{x+}(X) \right)_{t \geq 0}$$

Then in the third section we prove the main result: theorem 3.1 of section 3, in the fourth section we prove that the two processes $(L^*_t (V(., X)))_{z \in \mathbb{R}, t \geq 0}$ and $$\left( \int_0^t \sum_{V(s,x)=z, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x} (s, x) \right| d\Lambda_s^{x+}(X) \right)_{z \in \mathbb{R}, t \geq 0}$$

are indistinguishable, in the fifth section we show how to obtain Coquet-Ouknine formula via a classical change of variable, finally in the sixth section we give an application in mathematical finance.

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2. Definition of the Increasing Process

2.1. Parametrization of the set \(((t, x)|t > 0, V(t,x) = 0, \frac{\partial V}{\partial x}(t,x) \neq 0)\). Let $V: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a map of class $C^{1,2}$, suppose that $V$ satisfies the following assumption:

Assumption (1). The map $(t, x) \to \frac{\partial V}{\partial x}(t, x)$ defined on the open set $\Omega = \{(t, x)|\frac{\partial V}{\partial x}(t, x) \neq 0\}$ is locally lipschitz in $x$

Note that if $V$ of class $C^2$ then the assumption 2.1 is satisfied. We establish now a result that enables us to describe in a convienant way the elements of the set \(((t, x)|t > 0, V(t,x) = 0, \frac{\partial V}{\partial x}(t,x) \neq 0)\)

Theorem 2.1. Let $S = \{(t, x)|t > 0, V(t,x) = 0, \frac{\partial V}{\partial x}(t,x) \neq 0\}$ and suppose that $S$ is nonempty, then there exists a countable family of functions of class $C^1$ $(\Phi^0_k)_{k \geq 0}$ defined on open intervals $\mathbb{R}_+^n \left( I^0_n \right)_{n \geq 0}$ such that:

$$\left(7\right) \quad S = \bigcup_{k \geq 0} \text{Im} \Phi^0_k$$

where $\text{Im} \Phi^0_k$ is the image of the application $t \to (t, \Phi^0_k(t))$, furthermore the union in (7) is disjoint.
Proof. We start by showing that $S$ is separable: we consider the countable family of open balls $\mathcal{B}(y, r)_{y \in \mathbb{Q}^2, r \in \mathbb{Q}^*_+}$; for each ball $\mathcal{B}(y, r)$ we pick an element of $S \cap \mathcal{B}(y, r)$ if the latter intersection is non empty. Let:

$$S' = \{(t_0, x_0), (t_1, x_1), (t_2, x_2), \ldots\}$$

the set of chosen elements, let $x \in S$ and $s \in \mathbb{Q}^*_+$, let $y \in \mathbb{Q}^2 \cap \mathcal{B}(x, \frac{s}{2})$ hence $x \in \mathcal{B}(y, \frac{s}{4})$ this shows that the intersection is nonempty and it’s obvious that the distance between the chosen element of this intersection and $x$ is inferior to $\frac{s}{2}$, thus $S'$ is dense in $S$.

We fix $(t, x) \in S$, by the implicit functions theorem there exists an open ball centered at $(t, x) \mathcal{B}(t, x)$ and a function $h_{t,x}$ of $C^1$ defined on a open interval $I_{t,x}$ such that:

$$h_{t,x}(t', x') \in S \cap \mathcal{B}(t, x) \iff t' \in I_{t,x} \text{ et } x' = h_{t,x}(t')$$

For each element of $S' (t_i, x_i)$ and $r \in \mathbb{Q}^*_+$ we pick $h_{t,x}$ such that $\text{Im } h_{t,x}$ contains $\mathcal{B}((t_i, x_i), r) \cap S$ (if it exists! ) and we denote by $\mathcal{H}$ the countable set of these chosen functions.

Let $(t, x) \in S$ et $h_{t,x}, \mathcal{B}(t, x) = \mathcal{B}((t, x), r)$ obtained with implicit functions theorem with $r > 0$ is a rational, let $(t_i, x_i) \in S'$ such that $d((t_i, x_i), (t, x)) < \frac{s}{2}$ then :

$$S = \bigcup_{i \in \mathcal{H}} h_{t,x}(t')$$

We deduce that the image of $t' \rightarrow (t', h_{t,x}(t'))$ contains the ball $\mathcal{B}((t_i, x_i), \frac{r}{2})$ which implies the existence of an element $h$ of $\mathcal{H}$ satisfying this property and hence $(t, x) \in \text{Im } \mathcal{H}$. Thus we write:

$$\left\{ (t, x) \big| V(t, x) = z, \frac{\partial V}{\partial x}(t, x) \neq 0 \right\} = \bigcup_{h \in \mathcal{H}} \text{Im } h$$

Let $h_{t,x} \in \mathcal{H}$, by taking the derivative with respect to $s$ both of sides of the equality $V(s, h_{t,x}(s)) = z$ we obtain that: $h_{t,x}$ is a solution of the following Cauchy’s problem :

$$\left\{\begin{array}{c}
y'(s) = -\frac{\partial V}{\partial y}(y, g(s)) \\
y(t) = x \end{array}\right.$$

On the open set $((t, x)| \frac{\partial V}{\partial x}(t, x) \neq 0)$ the application $F : (t, x) \rightarrow -\frac{\partial V}{\partial x}(t, x)$ is locally Lipschitz continuous in $x$, hence by Cauchy-Lipschitz theorem the problem (11) has a unique maximal solution defined on an open interval of $\mathbb{R}^*_+$, in this case $h_{t,x}$ is the restriction of this solution to its interval of definition. We denote by $(\Phi^k_0)_{k \geq 0}$ the set of these maximal solutions, again by Cauchy-Lipschitz theorem if $i \neq j$ then $\Phi^k_i(t) \neq \Phi^j_i(t)$, this garanties that the union in (7) is disjoint. $\square$

Remark 2.1. The union $\bigcup_{k \geq 0} \text{Im } \Phi^k_0$ does not have to be infinite, that is way we didn’t specify to which subset of $\mathbb{N}$ belongs $k$.

Remark 2.2. The functions $\Phi^0_k$ are unique up to a permutation meaning if $(\Psi^0_k)_{k \geq 0}$ satisfy (7) and are obtained via the method used in the proof of theorem 2.1 then for all $k \geq 0$ there exists a unique $k' \geq 0$ such that $\Phi^0_k = \Psi^0_k$ and conversely for all $k \geq 0$ there exists $k' \geq 0$ such that $\Phi^0_k = \Psi^0_k = \Psi^0_{k'}$, this is obviously due to Cauchy-Lipschitz theorem.

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Remark 2.3. Evidently the theorem [2.1] is valid for each set

\[ S = \left\{ (t, x) \mid t > 0, V(t, x) = z, \frac{\partial V}{\partial x} (t, x) \neq 0 \right\} \]

2.2. Local time at curve of classe \( C^1 \).

Définition 2.1. Let \( (X_t)_{t \geq 0} \) be a continuous semimartingale and \( \gamma : I \subset \mathbb{R}_+^* \to \mathbb{R} \) a function of classe \( C^1 \), we fix \( t_0 \in I \). For each \( u \in I \) the process \( (X_t - \gamma(t))_{t \geq u, t \in I} \) is a continuous semimartingale, we denote by \( \mathbb{L}^u \) its local time at 0, similarly for all \( u \in ]0, t_0] \) the process \( (X_t - \gamma(t))_{u \leq t \leq t_0} \) is a continuous semimartingale and we denote by \( L^2,u \) its local time at 0. The increasing continuous process

\[ \Lambda^\gamma_{t_0} (X) = \begin{cases} \mathbb{L}^t_{t_0} & t \geq t_0 \\ -\mathbb{L}^t_{t_0} & t \leq t_0 \end{cases} \]

shall be called the local time of \( X \) at curve \( \gamma \) with basis point \( t_0 \) and we have almost surely for all \( t > 0 \):

\[ \Lambda^\gamma_{t_0} (X) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_0}^t 1_{0 < X_s - \gamma(s) < \epsilon} d\langle X, X \rangle_s \]

Note that for \( t \neq t' \in \mathbb{R}_+^* \) and for an arbitrary base point \( t_0 \):

\[ \Lambda^\gamma_{t_0} (X) - \Lambda^\gamma_{t_0} (X) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t}^{t'} 1_{0 < X_s - \gamma(s) < \epsilon} d\langle X, X \rangle_s \]

hence the continuous process \( \Lambda^\gamma_{t_0} (X) \) defines on the space \( (I, \mathcal{B}(I)) \) a measure that is independent of \( t_0 \) by which we will denote \( d\Lambda^\gamma_{t_0} (X) \), this measure can be extended naturally on \( (\mathbb{R}_+^*, \mathcal{B}(\mathbb{R}_+^*)) \) by setting:

\[ \int_A d\Lambda^\gamma_{t_0} (X) = \int_{A \cap t_0} d\Lambda^\gamma_{t_0} (X) \]

Proposition 2.1. Let \( (X_t)_{t \geq 0} \) be a continuous semimartingale and \( \gamma : I \subset \mathbb{R}_+^* \to \mathbb{R} \) a function of classe \( C^1 \), then \( d\Lambda^\gamma_{t} (X) \) is carried by the set \( (t \mid X_t = \gamma(t)) \), furthermore this measure is \( \sigma \) finite.

Proof. Let \( t_0 \) a basis point, by using the noations of the definition 2.1 let us first observe that for all \( t' \in I \cap - \infty, t_0 \] \n
\[ \forall t \geq t', -\mathbb{L}^t_{t_0} = \mathbb{L}^t_{t_0} - \mathbb{L}^{t'}_{t_0} \]

then using the fact that for a continuous semimartingale \( Z \int dL^0_t (Z) \) is carried by the set \( (t \mid Z_t = 0) \) we obtain: for all \( A \in \mathcal{B}(I) \) and for all \( t' \leq t_0 \)

\[ \int_{A \cap [t', +\infty[} d\Lambda^\gamma_{t} (X) = \int_{A \cap [t', +\infty[} 1_{[t_0, +\infty[} (t) \, d\mathbb{L}^{t_0}_{t} + \int_{A \cap [t', +\infty[} 1_{]-\infty, t_0]} (t) \, d\mathbb{L}^{t'}_{t} \]

\[ = \int_{A \cap [t', +\infty[} 1_{[t_0, +\infty[} (t) \times 1_{\{X_t = \gamma(t)\}} d\mathbb{L}^{t_0}_{t} \]

\[ + \int_{A \cap [t', +\infty[} 1_{]-\infty, t_0]} (t) \times 1_{\{X_t = \gamma(t)\}} d\mathbb{L}^{t'}_{t} \]

\[ = \int_{A \cap [t', +\infty[} 1_{\{X_t = \gamma(t)\}} d\Lambda^\gamma_{t} (X) \]
Thus we conclude that our measure is carried by the set \((t|Z_t = 0)\). Finally it’s should be clear that the measure is σ finite (take the sets \(E_n = [a_n, b_n]\cup[0, a]\cup[b, +\infty]\) with \(b_n \uparrow_n a\), \(a_n \downarrow n a\)).

**Remark 2.4.** If \(I = [a, b]\) with \(b < +\infty\) we conjecture that \(d\Lambda_t^\gamma(X)\) is finite. We explain our intuition: the latter statement is equivalent to showing that for \(\sup_{0 \leq s < t} L_t^\sigma(X - \gamma) < +\infty\). By the occupation times formula for \(C > 0\) we have almost surely for all \(t \in [0, b]s:\)

\[
\int_0^t 1_{x - \gamma(s) \in [-C, C]} d\langle X, X \rangle_s = \int_{-C}^C L_t^\sigma(X - \gamma) da
\]

Since \(\forall a \in [-C, C[, t \rightarrow L_t^\sigma(X - \gamma)\) is increasing then \(\forall a \in [-C, C[, \lim_{t \rightarrow b} L_t^\sigma(X - \gamma) = \sup_{0 \leq t < b} L_t^\sigma(X - \gamma)\), by sending \(t\) to \(b\) in \(\mathbb{R}\) we obtain:

\[
\langle X, X \rangle_b \geq \int_0^b 1_{x - \gamma(s) \in [-C, C]} d\langle X, X \rangle_s = \int_{-C}^C \sup_{0 \leq t < b} L_t^\sigma(X - \gamma) da
\]

This show that in \([-C, C[,\) Lebesque almost everywhere in \(a\)

\[
\sup_{0 \leq t < b} L_t^\sigma(X - \gamma) < +\infty
\]

, unfortunately it is not sufficient to conclude that is true for \(a = 0\). On the other hand there is a similar false result: Let \((f_n)\) a sequence of positive continuous functions, such that \(\forall x, (f_n(x))_{n \geq 0}\) is increasing, we suppose that \(\int_{\mathbb{R}} \lim_{n \rightarrow +\infty} f_n(x)dx = \int_{\mathbb{R}} \sup_{n \in \mathbb{N}} f_n(x)dx < +\infty\) then \(\forall x \in \mathbb{R}, \sup_{n \in \mathbb{N}} f_n(x) < +\infty\). The obvious counterexample is to take \(f_n(x) = \sum_{k=1}^n \frac{1}{n + n^2 x^2}\): By the Monotone convergence theorem

\[
\int_{\mathbb{R}} \lim_{n \rightarrow +\infty} f_n(x)dx = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}} \sum_{k=1}^n \frac{1}{n + n^2 x^2} dx = \frac{\pi}{2} \sum_{k=1}^{+\infty} \frac{1}{k^2}
\]

however \(\lim_{n \rightarrow +\infty} f_n(0) = +\infty\)

**Proposition 2.2.** Let \((X_t)_{t \geq 0}\) a continuous semimartingale, \((A_t)_{t \geq 0}\) its total variation part. Let \(V \in C^1(\mathbb{R}_+^* \times \mathbb{R} \rightarrow \mathbb{R})\) and \((\Phi_k)_{k \geq 0}\) the functions mentioned in theorem \([21]\) defined on the open intervals \(I_k\) of \(\mathbb{R}_+^*\). Recall that \(\frac{\partial V}{\partial x}(s, \Phi_k(s))\) has a constant sign, for \(k \geq 0\), we set:

\[
d\Lambda_s^{\Phi_k^+}(X) =
\]

\[
\begin{cases} 
{d\Lambda_s^{\Phi_k^+}(X) \quad \frac{\partial V}{\partial x}(s, \Phi_k(s)) > 0} \\
{d\Lambda_s^{\Phi_k^-}(X) - 2 \times 1_{\{X_s = \gamma(s)\}} dA_s + 2 \times 1_{\{X_s = \gamma(s)\}} \Phi_k(s)' ds \quad \frac{\partial V}{\partial x}(s, \Phi_k(s)) < 0}
\end{cases}
\]

\[
d\Lambda_s^{\Phi_k^-}(X) =
\]

\[
\begin{cases} 
{d\Lambda_s^{\Phi_k^-}(X) - 2 \times 1_{\{X_s = \gamma(s)\}} dA_s + 2 \times 1_{\{X_s = \gamma(s)\}} \Phi_k(s)' ds \quad \frac{\partial V}{\partial x}(s, \Phi_k(s)) > 0} \\
{d\Lambda_s^{\Phi_k^+}(X) \quad \frac{\partial V}{\partial x}(s, \Phi_k(s)) < 0}
\end{cases}
\]

Then both borelian measures \(d\Lambda_s^{\Phi_k^+}(X), d\Lambda_s^{\Phi_k^-}(X)\) are σ finite and positive, carried by the set \((s|X_s = \Phi_k(s))\)
Consequently, the two maps \( \mu_1, \mu_2 \) defined on \( \mathcal{B}(\mathbb{R}_+^*) \) by:

\[
\mu_1(A) = \sum_{k \geq 0} \int_{A} \frac{\partial V_t(s, \Phi_k(s))}{\partial x} d\Lambda^{\Phi_k}_{s} (X) = \sum_{k \geq 0} \int_{A} \frac{\partial V_t(s, \Phi_k(s))}{\partial x} d\Lambda^{\Phi_k}_{s} (X)
\]

(23)

\[
\mu_2(A) = \sum_{k \geq 0} \int_{A} \frac{\partial V_t(s, \Phi_k(s))}{\partial x} d\Lambda^{-\Phi_k}_{s} (X) = \sum_{k \geq 0} \int_{A} \frac{\partial V_t(s, \Phi_k(s))}{\partial x} d\Lambda^{-\Phi_k}_{s} (X)
\]

(24)

are two positive borelian random measures. Note that \( \int_t d\Lambda^{\Phi_k}_{s} (X) = 0 \) for all \( t > 0 \) and hence \( \mu_i([0,t]) = \mu_i[0,t[ \). We have the following definition:

**Définition 2.2.** For \( t > 0 \) we define almost surely:

\[
\int_{0}^{t} \sum_{V(s,x)=0, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x}(s, x) \right| d\Lambda^{\Phi_k}_{s} (X) = \mu_1([0,t]) = \mu_1([0,t])
\]

(25)

\[
\int_{0}^{t} \sum_{V(s,x)=0, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x}(s, x) \right| d\Lambda^{-\Phi_k}_{s} (X) = \mu_2([0,t]) = \mu_2([0,t])
\]

(26)

**Remark 2.5.** We justify the notation (25): For all \( A \in \mathcal{B}(\mathbb{R}_+^*) \) \( \mu_i(A) \) is sum of positive elements and hence invariant by every permutation of these elements, thus by remark 2.2 \( \mu_i \) is independent of the parametrization as defined in theorem 2.1.

We give now the proof of proposition 2.2.

Proof. We fix \( k \in \mathbb{N} \), it is sufficient to prove that \( d\Lambda^{\Phi_k}_{s} (X) \) is \( \sigma \) finite and positive. If \( \frac{\partial V}{\partial x}(s, \Phi_k(s)) > 0 \) there is nothing to prove, otherwise let \( t_0 \) a basis point. For all \( u \in [0, t_0] \cap I_k \) let \( (L_{1,u}^a)_{t \geq u, t \in I_k, a \in \mathbb{R}} \) be the family of local times of the continuous semimartingale \( X_t - \Phi_k(t) \). For \( t \in [u, +\infty[ \cap I_k \) we set:

\[
L_{t,u} = \lim_{a \to 0^-} L_{t,u}^{a,u}
\]

(27)

\( L_{t,u} \) is indeed an increasing process, since \( A - \Phi_k \) is the total variation part of \( X - \Phi_k \) then by theorem 1.7 of chapter 6 in [9], and using the notation of definition 2.1 we have almost surely for all \( t, u, v, t \geq u \):

\[
L_{t,u} - L_{u} = -2 \times \int_{u}^{t} \{ X_s = \Phi_k(s) \} d(A_s - \Phi_k(s))
\]

(28)

in particular we have almost surely for all \( t \geq t_0 \):

\[
L_{t,t_0} - L_{t_0} = -2 \times \int_{t_0}^{t} \{ X_s = \Phi_k(s) \} d(A_s - \Phi_k(s))
\]

(29)
and for all \( t \leq t_0 \):

\[
L_{t_0}^r - L_{t_0}^l = 2 \times \int_{t_0}^{t} 1_{\{x_s = \Phi_k^0(s)\}} d(A_s - \Phi_k^0(s))
\]

\[(30)\]

\[
= 2 \times \left( \int_{t_0}^{t} 1_{\{x_s = \Phi_k^0(s)\}} dA_s - \int_{t_0}^{t} 1_{\{x_s = \Phi_k^0(s)\}} \Phi_k^0(s)'(s) ds \right)
\]

Finally we set :

\[
(31) \quad \Lambda_{t_0}^{r,t_0,+}(X) = \begin{cases}  
L_{t_0}^r & t \geq t_0 \\
-L_{t_0}^l & t \leq t_0 
\end{cases}
\]

it’s obvious that \( \Lambda_{t_0}^{r,t_0,+}(X) \) is an increasing continuous process, for \( t, t' \) we have:

\[
(32) \quad \Lambda_{t'}^{r,t_0,+}(X) - \Lambda_{t'}^{r,t_0,+}(X) = \Lambda_{t'}^{r,t_0}(X) - \Lambda_{t'}^{r,t_0}(X) - 2 \times \left( \int_{t'}^{t} 1_{\{x_s = \Phi_k^0(s)\}} dA_s - \int_{t'}^{t} 1_{\{x_s = \Phi_k^0(s)\}} \Phi_k^0(s)' ds \right)
\]

Thus the Lebesgue-Stieljes measure associated to \( \Lambda_{t_0}^{r,t_0,+}(X) \) is independent of \( t_0 \) by which we denote \( d\Lambda_{s}^{r,t_0,+}(X) \), it satisfies \( (21) \) and it’s obvious that such measure is \( \sigma \) finite positive and carried by the set \( (s)X_s = \Phi_k^0(s) \). \( \Box \)

**Remark 2.6.** With the same method we can define

\[
\int_0^t \sum_{V(s,x) = z, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x}(s,x) \right| d\Lambda_{s}^{r,+}(X) , \int_0^t \sum_{V(s,x) = z, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x}(s,x) \right| d\Lambda_{s}^{r,-}(X)
\]

for all \( z \).

We give now some examples:

**Example 2.2.** (1) Let \( V(t,x) = x^2 - t \), the functions of theorem \( 2.1 \) are \( \Phi_1^0(t) = \sqrt{t}, \Phi_2^0(t) = -\sqrt{t} \) both are defined on \( I = \mathbb{R}_+^* \). We have:

\[
(33) \quad \int_0^t \sum_{V(s,x) = 0, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x}(s,x) \right| d\Lambda_{s}^{r,+}(X) = 2 \times \int_0^t \sqrt{s}d\Lambda_{s}^{r,+}(X)
\]

\[
+ 2 \times \int_0^t \sqrt{s}d\Lambda_{s}^{r,-}(X)
\]

(2) Let \( V(t,x) = (t-1)^2 \times x^2 - 1 \), the functions of \( 2.1 \) are \( \Phi_1^0(t) = \frac{1}{t-1}, \Phi_2^0(t) = -\frac{1}{t-1} \) defined on \( ]1, +\infty[ \), \( \Phi_3^0(t) = \frac{1}{t-1}, \Phi_4^0(t) = -\frac{1}{t-1} \) defined on \( ]0, 1[ \). We have:

\[
(34) \quad \int_0^t \sum_{V(s,x) = 0, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x}(s,x) \right| d\Lambda_{s}^{r,+}(X)
\]

\[
= \begin{cases} 
2 \times \int_0^t |s - 1| d\Lambda_{s}^{r,+}(X) + 2 \times \int_0^t |s - 1| d\Lambda_{s}^{r,-}(X) \\
+ 2 \times \int_0^t |s - 1| d\Lambda_{s}^{r,+}(X) + 2 \times \int_0^t |s - 1| d\Lambda_{s}^{r,-}(X) \\
2 \times \int_0^t |s - 1| d\Lambda_{s}^{r,+}(X) + 2 \times \int_0^t |s - 1| d\Lambda_{s}^{r,-}(X)
\end{cases} \quad t \geq 1
\]

\[
2 \times \int_0^t |s - 1| d\Lambda_{s}^{r,+}(X) + 2 \times \int_0^t |s - 1| d\Lambda_{s}^{r,-}(X) \quad 0 \leq t < 1
\]
(3) Let: $V(t, x) = \Phi(x)$ where $\Phi : \mathbb{R} \to \mathbb{R}$ is a function of class $C^2$. Let:

$$\{x | \phi(x) = 0, \phi'(x) \neq 0\} = \{x_0, x_1\}$$

Then the functions of theorem 2.1 are: $\Phi_k^0 : s \to x_k$ and we have:

$$d\Lambda_s^{\Phi_k^0+}(X) = \begin{cases} 
  dL_s^+(X) & \phi'(x) > 0 \\
  dL_s^-(X) & \phi'(x) < 0 
\end{cases}$$

Hence:

$$\int_0^t \sum_{V(s,x)=0, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x} (s, x) \right| d\Lambda_s^{x,+}(X) = \int_0^t \left( \sum_{\phi(x)=0, \phi'(x) \neq 0} |\phi'(x)| dL_s^+(X) \right)$$

3. PROOF OF THE MAIN RESULT

We state our main result:

**Theorem 3.1.** Let $(X_t)_{t \geq 0}$ a continuous semimartingale, let $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ an application of class $C^{1,2}$ satisfying assumption $\mathbb{Z}_A$. By Itô’s formula, $V(., X)$ is a semimartingale. Then for $t > 0$ we have almost surely:

$$L_t^0 (V(., X)) = \int_0^t \sum_{V(s,x)=0, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x} (s, x) \right| d\Lambda_s^{x,+}(X)$$

and

$$L_t^0 (V(., X)) = \int_0^t \sum_{V(s,x)=0, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x} (s, x) \right| d\Lambda_s^{x,-}(X)$$

The strategy of the proof is the following: First we prove the following inequalities:

$$L_t^0 (V(., X)) \geq \int_0^t \sum_{V(s,x)=0, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x} (s, x) \right| d\Lambda_s^{x,+}(X)$$

$$L_t^0 (V(., X)) \geq \int_0^t \sum_{V(s,x)=0, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x} (s, x) \right| d\Lambda_s^{x,-}(X)$$

we conclude then by showing that:

$$L_t^0 (V(., X)) + L_t^0 (V(., X))$$

$$\leq \int_0^t \sum_{V(s,x)=0} \left| \frac{\partial V}{\partial x} (s, x) \right| d\Lambda_s^{x,+}(X) + \int_0^t \sum_{V(s,x)=0, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x} (s, x) \right| d\Lambda_s^{x,-}(X)$$

We start by proving the following lemma:

**Lemma 3.1.** Let $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ an application of class $C^{1,2}$ and $I \subset \mathbb{R}_+$ an open interval such that:

$$\forall t \in I, \ h(t, 0) = 0, \frac{\partial h}{\partial x} (t, 0) \neq 0$$

8
Let \((X_t)_{t\geq 0}\) be a continuous semimartingale and \(L^0(X)\) be its local time at 0 and \(L^0_-(X) = \lim_{a\to 0^-} L^a(X)\) (it’s obvious that \(L^0_-(X)\) is continuous increasing process). Let \(a < b \in I\), and \(y > 0\) sufficiently small then there exists a positive sequence \((\epsilon_n)_{n \geq 0}\) converging to 0 such that almost surely:

\[
\lim_{n \to +\infty} \frac{1}{\epsilon_n} \int_a^b 1_{|h(s,x_s)|<\epsilon_n, |x_s|<y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d\langle X, X \rangle_s = \int_a^b \left| \frac{\partial h}{\partial x} (s, X_s) \right| dL^0_s (X) + \int_a^b \left| \frac{\partial h}{\partial x} (s, X_s) \right| dL^0_-(X)
\]

**Proof.**

**Step 1** Note that \(t \to \frac{\partial h}{\partial x} (t, 0)\) doesn’t change sign \(I\), we suppose without loss of generality that it remains positive. We know that there exists \(\beta > 0\) such that by setting \(K = ((s, x)|s \in [a, b], x \in [-\beta, \beta])\) we have \(\inf_{(s, x) \in K} \frac{\partial h}{\partial x} (s, x) = m > 0\). In the sequel we suppose that \(y \in [0, \beta]\). We fix \(\lambda \in [0, 1]\) then for all \(\epsilon\) there exists an integer \(n^\epsilon\) and a partition \((t^\epsilon_i)_{i \in [1, n^\epsilon]}\) of \([a, b]\) with \(t^\epsilon_1 = a, \ t^\epsilon_n = b\) such that \(\lim_{\epsilon \to 0} \sup_{i \in [1, n^\epsilon - 1]} |t^\epsilon_{i+1} - t^\epsilon_i| = 0\) and:

\[
\max_{i \in [1, n^\epsilon - 1]} \sup_{s \in [t^\epsilon_i, t^\epsilon_{i+1}]} \left| \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 - \left( \frac{\partial h}{\partial x} (t^\epsilon_i, X_s) \right)^2 \right| \leq \epsilon^2
\]

\[
\max_{i \in [1, n^\epsilon - 1]} \sup_{s \in [t^\epsilon_i, t^\epsilon_{i+1}]} |h(s, X_s) - h(t^\epsilon_i, X_s)| \leq \lambda \times \epsilon
\]

For \(i \in [1, n^\epsilon - 1]\) we have:

\[
\frac{1}{\epsilon} \int_{t^\epsilon_i}^{t^\epsilon_{i+1}} 1_{|h(t^\epsilon_i, X_s)|<(1-\lambda)\times \epsilon, |x_s|<y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d\langle X, X \rangle_s
\]

\[
\leq \frac{1}{\epsilon} \int_{t^\epsilon_i}^{t^\epsilon_{i+1}} 1_{|h(s, X_s)|<\epsilon, |x_s|<y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d\langle X, X \rangle_s
\]

\[
\leq \frac{1}{\epsilon} \int_{t^\epsilon_i}^{t^\epsilon_{i+1}} 1_{|h(t^\epsilon_i, X_s)|<(1+\lambda)\times \epsilon, |x_s|<y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d\langle X, X \rangle_s
\]

and so:

\[
\sum_{i=1}^{n^\epsilon-1} \frac{1}{\epsilon} \int_{t^\epsilon_i}^{t^\epsilon_{i+1}} 1_{|h(t^\epsilon_i, X_s)|<(1-\lambda)\times \epsilon, |x_s|<y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d\langle X, X \rangle_s
\]

\[
\leq \frac{1}{\epsilon} \int_{a}^{b} 1_{|h(s, X_s)|<\epsilon, |x_s|<y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d\langle X, X \rangle_s
\]

\[
\leq \sum_{i=1}^{n^\epsilon-1} \frac{1}{\epsilon} \int_{t^\epsilon_i}^{t^\epsilon_{i+1}} 1_{|h(t^\epsilon_i, X_s)|<(1+\lambda)\times \epsilon, |x_s|<y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d\langle X, X \rangle_s
\]
We deal now with righthandside in (48) by inequality (45) we have:

\[
\sum_{i=1}^{n^\epsilon-1} \frac{1}{\epsilon} \int_{t^\epsilon_i}^{t^\epsilon_{i+1}} 1_{|h(t^\epsilon_i,x)\times_x|<\lambda} \left( \frac{\partial h}{\partial x}(t^\epsilon_i, X_s) \right)^2 d\langle X, X \rangle_s \\
- \sum_{i=1}^{n^\epsilon-1} \frac{1}{\epsilon} \int_{t^\epsilon_i}^{t^\epsilon_{i+1}} 1_{|h(t^\epsilon_i,x)\times_x|<\lambda} \left( \frac{\partial h}{\partial x}(t^\epsilon_i, X_s) \right)^2 d\langle X, X \rangle_s
\]

(49)

\[
\leq \epsilon \times \langle X, X \rangle_s
\]

by the occupation times formula for \(i \in [1, n^\epsilon - 1]\) we have almost surely:

\[
\frac{1}{\epsilon} \int_{t^\epsilon_i}^{t^\epsilon_{i+1}} 1_{|h(t^\epsilon_i,x)\times_x|<\lambda} \left( \frac{\partial h}{\partial x}(t^\epsilon_i, X_s) \right)^2 d\langle X, X \rangle_s
\]

(50)

\[
= \frac{1}{\epsilon} \int_{-\infty}^{+\infty} \left| \frac{\partial h}{\partial x}(t^\epsilon_i, g^\epsilon_s(x)) \right| \times \left( L_{t^\epsilon_{i+1}}^x(X) - L_{t^\epsilon_i}^x(X) \right) dx
\]

Since the function \(x \to h(t^\epsilon_i, x)\) is of class \(C^1\) on the interval \([-y, y]\) with strictly positive derivative, it’s a diffeomorphism from \([-y, y]\) onto \([h(t^\epsilon_i, -y), h(t^\epsilon_i, y)\] with inverse \(g^\epsilon_t\), recall that \(\min_{t \in [a, b]} h(t, y) > 0\) and \(\max_{t \in [a, b]} h(t, -y) < 0\), using integration by substitution we obtain for \(\epsilon\) small enough:

\[
\frac{1}{\epsilon} \int_{-\infty}^{+\infty} \left| \frac{\partial h}{\partial x}(t^\epsilon_i, g^\epsilon_s(x)) \right| \times \left( L_{t^\epsilon_{i+1}}^x(X) - L_{t^\epsilon_i}^x(X) \right) dx
\]

(51)

Step 2 For all \(t \in [a, b]\) let \(g_t\) the inverse of the application \(x \to h(t, x)\) defined on \([-y, y]\). For \(x \in [-\epsilon(1+\lambda), +\epsilon(1+\lambda)\) \([s \in [a, b]\) we set:

\[
g^\epsilon_{1,s}(x) = \min(g^\epsilon_t(x), g_s(x))
\]

(52)

\[
g^\epsilon_{2,s}(x) = \max(g^\epsilon_t(x), g_s(x))
\]

(53)

Let \((z^\epsilon_{x} : [t^\epsilon_i, t^\epsilon_{i+1}] \times \mathbb{R} \to \mathbb{R})_{x \in [a, b], i \in [1, n^\epsilon - 1]}\) be a family of functions uniformly bounded, by Fubini’s theorem for stochastic integrals we have:

\[
\frac{1}{\epsilon} \int_{-\epsilon(1+\lambda)}^{+\epsilon(1+\lambda)} \left| \frac{\partial h}{\partial x}(t^\epsilon_i, g^\epsilon_s(x)) \right| \times \left( \int_{t^\epsilon_i}^{t^\epsilon_{i+1}} z^\epsilon_t(x, s) \times 1_{g^\epsilon_{1,s}(x) < X_s \leq g^\epsilon_{2,s}(x)} dX_s \right) dx
\]

(54)

Consider the process \(\alpha^\epsilon\) defined on \([a, b]\) by:

\[
\alpha^\epsilon_s = \sum_{i=1}^{n^\epsilon-1} \frac{1}{\epsilon} \left( \int_{-\epsilon(1+\lambda)}^{+\epsilon(1+\lambda)} z^\epsilon_t(x, s) \times \left| \frac{\partial h}{\partial x}(t^\epsilon_i, g^\epsilon_s(x)) \right| \times 1_{g^\epsilon_{1,s}(x) < X_s \leq g^\epsilon_{2,s}(x)} dX_s \right) \times 1_{s \in [t^\epsilon_i, t^\epsilon_{i+1}]}(x)
\]

(55)

It is obvious that \(\Lambda^\epsilon\) is a bounded process adapted to the filtration generated by \(X\). Let \(s \in [a, b]\) and \(\sigma_x(s)\) the integer such that \(s \in [t^\epsilon_{\sigma_x(s)}, t^\epsilon_{\sigma_x(s)+1}]\)

, using integration by substitution:
\[ \alpha^\epsilon_s = \int_{-(1+\lambda)}^{+(1+\lambda)} z^\epsilon_i(x, s) \left| \frac{\partial h}{\partial x} (t^\epsilon_{\sigma_s(s)}, g^\epsilon_{\sigma_s(s)}(x^\epsilon)) \right| \mathbf{1}_{g^1_{\sigma_s(s)}(x) < X_s \leq g^2_{\sigma_s(s)}(x)} \, dx \]

Recall that \( m = \inf_{(t,x) \in K} \frac{\partial h}{\partial x}(t,x) \). For \( t \in [a,b], x \in \mathcal{E} - y, y[i] \), we have \( h(t,x) - h(t,0) = \int_0^x \frac{\partial h}{\partial x}(t,x') \, dx' \) and so:

\[ \begin{cases} 
    h(t,x) \geq m \times x & x > 0 \\
    h(t,x) \leq m \times x & x < 0
\end{cases} \]

meaning \( |x| \leq \frac{|h(t,x)|}{m} \), in other words:

\[ \lim_{x \to 0} |g_i(x)| = 0 \]

the latter limit is uniform in \( t \) which implies:

\[ \lim_{\epsilon \to 0} \sup_{|x| \leq \epsilon \times (1+\lambda)} \left( \max_{i \in [1^n \times \mathbb{N}]} |g_i^\epsilon(x)| \right) = 0 \]

Since \( g^1_{i,s}(x^\epsilon) \times g^2_{i,s}(x^\epsilon) = g^\epsilon_{\sigma_s(s)}(x^\epsilon) \times g_s(x^\epsilon) > 0 \) pour tout \( x \in [- (1 + \lambda), + (1 + \lambda)] \), for all \( i \) and \( \epsilon \) sufficiently small, and by (59) \( \lim_{\epsilon \to 0} g^1_{i,s}(x^\epsilon) = \lim_{\epsilon \to 0} g^2_{i,s}(x^\epsilon) = 0 \) consequently for all \( s \)

\[ \lim_{\epsilon \to 0} 1 \left( g^1_{i,s}(x) < X_s \leq g^2_{i,s}(x) \right) = 0 \]

Since the term \( z^\epsilon_i(\cdot) \right) \left( \frac{\partial h}{\partial x}(\cdot, \cdot) \right) \) is bounded, by the dominated convergence theorem : forall \( s \in [a,b], \lim_{\epsilon \to 0} \alpha^\epsilon_s = 0 \). Hence by dominated convergence theorem for stochastic integration \( \lim_{\epsilon \to 0} \int_a^b \alpha^\epsilon_s dX_s = 0 \) the latter limit is in probability. We can thus write in probability:

\[ \lim_{\epsilon \to 0} \sum_{i=1}^{n^-1} \frac{1}{\epsilon} \int_{t_i^\epsilon}^{t_{i+1}^\epsilon} \left( \int_{-(1+\lambda)}^{(1+\lambda)} z^\epsilon_i(x, s) \times \left| \frac{\partial h}{\partial x} (t^\epsilon_{\sigma_i(s)}, g^\epsilon_{\sigma_i(s)}(x)) \right| \mathbf{1}_{g^1_{\sigma_i(s)}(x) < X_s \leq g^2_{\sigma_i(s)}(x)} \, dx \right) \, dX_s = 0 \]
Step 3 For $i \in [1, n^e - 1]$, by Tanaka’s formula:

(62)

$$\frac{1}{\epsilon} \int_{-\epsilon}^{+\epsilon} \frac{\partial h}{\partial x} (t^e_i, g^e_i(x)) \times \left( L_{T_{t_{i+1}}}^{g^e_i} (X) - L_{T_{t_i}}^{g^e_i} (X) \right) dx$$

$$= \frac{2}{\epsilon} \int_{-\epsilon}^{+\epsilon} \frac{\partial h}{\partial x} (t^e_i, g^e_i(x)) \times \left( \left( X_{T_{t_{i+1}}} - g^e_i(x) \right) + \int_0^{T_{t_{i+1}}} 1_{X_s > g^e_i(x)} dX_s \right) dx$$

$$- \frac{2}{\epsilon} \int_{-\epsilon}^{+\epsilon} \frac{\partial h}{\partial x} (t^e_i, g^e_i(x)) \times \left( \left( X_{T_{t_i}} - g^e_i(x) \right) + \int_0^{T_{t_i}} 1_{X_s > g^e_i(x)} dX_s \right) dx$$

$$= \frac{2}{\epsilon} \int_{-\epsilon}^{+\epsilon} \frac{\partial h}{\partial x} (t^e_i, g^e_i(x)) \times \left( \left( X_{T_{t_{i+1}}} - g^e_i(x) \right) - \left( X_{T_{t_{i+1}}} - g^e_{i+1}(x) \right) \right) dx$$

$$+ \frac{2}{\epsilon} \int_{-\epsilon}^{+\epsilon} \frac{\partial h}{\partial x} (t^e_i, g^e_i(x)) \times \left( \int_0^{T_{t_{i+1}}} 1_{X_s > g^e_i(x)}, \times \frac{\partial g^e_i(x)}{\partial s} \right) ds \right) dx$$

$$- \frac{2}{\epsilon} \int_{-\epsilon}^{+\epsilon} \frac{\partial h}{\partial x} (t^e_i, g^e_i(x)) \times \left( \int_0^{T_{t_i}} 1_{X_s > g^e_i(x)} \times \frac{\partial g^e_i(x)}{\partial s} \right) ds \right) dx$$

$$+ \frac{1}{\epsilon} \int_{-\epsilon}^{+\epsilon} \frac{\partial h}{\partial x} (t^e_i, g^e_i(x)) \times \left( \Lambda_{T_{t_{i+1}}}^{g^e_i} (X) - \Lambda_{T_{t_i}}^{g^e_i} (X) \right) dx$$

$$\left( \Lambda_{T_{t_{i+1}}}^{g^e_i} (X) \right)_{t \in [a,b]}$$ is the local time at the curve $t \to g^e_i(x)$ defined at an arbitrary base point. Recall that for all $x \to g^e_i(x)$ is of class $C^1$ and we have

$$\frac{\partial g^e_i(x)}{\partial t} = - \frac{\partial h}{\partial x} (t^e_i, g^e_i(x))$$

Since $1_{X_s > g^e_i(x)} - 1_{X_s > g^e_i(x)} = z^e_i(x, s) \times 1_{g^e_i(x) < X_s \leq g^e_{i+1}(x)}$ with:

$$z^e_i(x, s) = \begin{cases} 1 & g^e_i(x) \leq g^e_i(x) \\ -1 & g^e(x) > g^e_{i+1}(x) \end{cases}$$

then by Step 3

(63)

$$\sum_{i=1}^{n^e-1} \frac{1}{\epsilon} \int_{-\epsilon}^{+\epsilon} \frac{\partial h}{\partial x} (t^e_i, g^e_i(x)) \times \left( \int_0^{T_{t_{i+1}}} 1_{X_s > g^e_i(x)} \times 1_{g^e_i(x) < X_s \leq g^e_{i+1}(x)} \right) ds \right) dx$$

goest to 0 in probability. We have:

(64)

$$\left| \frac{2}{\epsilon} \int_{-\epsilon}^{+\epsilon} \frac{\partial h}{\partial x} (t^e_i, g^e_i(x)) \times \left( \left( X_{T_{t_{i+1}}} - g^e_i(x) \right) - \left( X_{T_{t_{i+1}}} - g^e_{i+1}(x) \right) \right) \right| dx$$

$$\leq \frac{2}{\epsilon} \int_{-\epsilon}^{+\epsilon} \frac{\partial h}{\partial x} (t^e_i, g^e_i(x)) \times \left( \int_0^{T_{t_{i+1}}} \left| \frac{\partial h}{\partial x} (t, g^e_i(x)) \right| dt \right) dx$$
By (58), for $\epsilon$ sufficiently small $\forall x \in [-\epsilon \times (1+\lambda), +\epsilon \times (1+\lambda)]$, $|\frac{\partial h}{\partial x} (t, g_t(x))| \geq \frac{\eta}{2}$, hence there exists $\eta > 0$ such that:

$$\limsup_{x \to \infty} \frac{2\eta}{\epsilon} \int_{-\epsilon \times (1+\lambda)}^{+\epsilon \times (1+\lambda)} \left| \frac{\partial h}{\partial x} (t', g_t'(x)) \right| \times \left( X_{t+1} - g_t'(x) \right)^+ - \left( X_{t+1} - g_t'(x) \right)^+ \, dx \leq 0, \quad \epsilon \to 0$$

We have $\forall t \in [a, b] h(t, 0) = 0$ by taking the derivative with respect to $t$ we obtain $\frac{\partial h}{\partial t} (t, 0) = 0$. By (58) and the dominated convergence theorem:

$$\lim_{x \to \infty} \frac{2\eta}{\epsilon} \int_{-\epsilon \times (1+\lambda)}^{+\epsilon \times (1+\lambda)} \left| \frac{\partial h}{\partial t} (t, g_t(x)) \right| \, dx = 0$$

in the same way one can show that:

$$\lim_{x \to \infty} \frac{2\eta}{\epsilon} \int_{-\epsilon \times (1+\lambda)}^{+\epsilon \times (1+\lambda)} \left| \frac{\partial h}{\partial t} (t, g_t(x)) \right| \, dx = 0$$

the conclusion of this step is the following:

$$\lim_{x \to \infty} \frac{2\eta}{\epsilon} \int_{-\epsilon \times (1+\lambda)}^{+\epsilon \times (1+\lambda)} \left| \frac{\partial h}{\partial t} (t, g_t(x)) \right| \times \left( X_{t+1} - g_t'(x) \right) \, dx = 0$$

Step 4 The application $(t, x) \to g_t(x)$ is of class $C^1$, hence it’s a family of regular curves and for $x \in ]-y, y[$ we have $\frac{\partial g_t(x)}{\partial x} = \frac{\partial h}{\partial x} (t, g_t(x)) > 0$. In [1] corollary 1.1 Ben tabe show that we can find a modification de $(\Lambda_t^{g(x)} (X))_{t \in [a, b]}$ càdlàg in $x$ (continuous on the right, limit on the left). We have almost surely:

$$\lim_{x \to \infty} \frac{2\eta}{\epsilon} \int_{-\epsilon \times (1+\lambda)}^{+\epsilon \times (1+\lambda)} \left| \frac{\partial h}{\partial x} (t, g_t(x)) \right| \times \left( X_{t+1} - g_t'(x) \right) \, dx = 0, \quad \epsilon \to 0$$

$$\lim_{x \to \infty} \frac{2\eta}{\epsilon} \int_{-\epsilon \times (1+\lambda)}^{+\epsilon \times (1+\lambda)} \left| \frac{\partial h}{\partial x} (t, g_t(x)) \right| \times \left( X_{t+1} - g_t'(x) \right) \, dx = 0, \quad \epsilon \to 0$$

where:

$$f^*(t) = \sum_{i=1}^{n-1} \left| \frac{\partial h}{\partial x} (t', g_t'(x)) \right| \times 1_{[t', t+1]} + \left| \frac{\partial h}{\partial x} (b, g_b(x)) \right| \times 1_{[b, 1]}$$
By \(58\) and due to uniform continuity of \(\frac{\partial h}{\partial x}\), we deduce that
\[
\sup_{t \in [a,b], x \in [-\varepsilon \times (1+\lambda), +\varepsilon \times (1+\lambda)]} \left| f^\varepsilon(t) - \frac{\partial h}{\partial x}(t, 0) \right|
\]
goes to 0, this enables us to say that:
\[
\sum_{i=1}^{n-1} \frac{1}{\varepsilon} \int_{t_i}^{t_{i+1}} \left| \frac{\partial h}{\partial x}(t', g_i'(x)) \right| \left( \Lambda_{t_i}^q(x) - \Lambda_{t_{i+1}}^q(x) \right) dx
= \frac{1}{\varepsilon} \int_{a}^{b} \left| \frac{\partial h}{\partial x}(t, 0) \right| d\Lambda_{t}^q(X) + o(1)
\]

By generalized portmanteau theorem (for example theorem 2.5.37 in [4])
\[
\lim_{x \to 0^+} \int_{a}^{b} \left| \frac{\partial h}{\partial x}(t, 0) \right| d\Lambda_{t}^q(X) = \int_{a}^{b} \left| \frac{\partial h}{\partial x}(t, 0) \right| d\Lambda_{t}^q(0)(X)
\]

By all the results above, we have in probability:
\[
\lim_{\varepsilon \to 0} \sum_{i=1}^{n-1} 1 \int_{t_i}^{t_{i+1}} 1_{|h(t', X_s)| < (1+\lambda) \times \varepsilon, |X_s| < y} \left( \frac{\partial h}{\partial x}(t', X_s) \right)^2 d\langle X, X \rangle_s
= (1 + \lambda) \times \left[ \int_{a}^{b} \left| \frac{\partial h}{\partial x}(t, X_t) \right| dL_{t}^0(X) + \int_{a}^{b} \left| \frac{\partial h}{\partial x}(t, X_t) \right| dL_{t}^{0-}(X) \right]
\]

hence there exists a sequence \((\epsilon_n)_{n \geq 0}\) converging to 0 such that we have almost surely:
\[
\lim_{n \to +\infty} \sum_{i=1}^{n-1} 1 \int_{t_i}^{t_{i+1}} 1_{|h(t', X_s)| < (1+\lambda) \times \epsilon_n, |X_s| < y} \left( \frac{\partial h}{\partial x}(t', X_s) \right)^2 d\langle X, X \rangle_s
= (1 + \lambda) \times \left[ \int_{a}^{b} \left| \frac{\partial h}{\partial x}(t, X_t) \right| dL_{t}^0(X) + \int_{a}^{b} \left| \frac{\partial h}{\partial x}(t, X_t) \right| dL_{t}^{0-}(X) \right]
\]

Thus by inequalities \([48], [49]\) we have:
\[
\limsup_{n \to +\infty} \frac{1}{\epsilon_n} \int_{a}^{b} 1_{|h(s, X_s)| < \epsilon_n, |X_s| < y} \left( \frac{\partial h}{\partial x}(s, X_s) \right)^2 d\langle X, X \rangle_s
\leq (1 + \lambda) \times \left[ \int_{a}^{b} \left| \frac{\partial h}{\partial x}(t, X_t) \right| dL_{t}^0(X) + \int_{a}^{b} \left| \frac{\partial h}{\partial x}(t, X_t) \right| dL_{t}^{0-}(X) \right]
\]

**Step 5** By repeating the same procedure for the right hand side in \([48]\) but instead of working with all the \(\epsilon\) we work only with the sequence \(\epsilon_n\), we can find
Corollary 3.1. Under the assumption of lemma 3.1. Let $a < b \in I$ be $y > 0$ sufficiently small then there exists a positive sequence $(\epsilon_n)_{n \geq 0}$ converging to $0$ such that:

1. If $\frac{\partial h}{\partial x} (t, 0) > 0$,

\[
\lim_{n \to +\infty} \frac{1}{\epsilon_n} \int_a^b 1_{0 \leq h(s, X_s) < \epsilon_n, 0 \leq X_s < y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s = \int_a^b \left| \frac{\partial h}{\partial x} (s, X_s) \right| dL^0_s (X)
\]

2. If $\frac{\partial h}{\partial x} (t, 0) < 0$,

\[
\lim_{n \to +\infty} \frac{1}{\epsilon_n} \int_a^b 1_{0 \leq h(s, X_s) < \epsilon_n, -y < X_s \leq 0} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s = \int_a^b \left| \frac{\partial h}{\partial x} (s, X_s) \right| dL^0_s (X)
\]

3. If $\frac{\partial h}{\partial x} (t, 0) > 0$,

\[
\lim_{n \to +\infty} \frac{1}{\epsilon_n} \int_a^b 1_{-\epsilon_n \leq h(s, X_s) < 0, -y < X_s \leq 0} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s = \int_a^b \left| \frac{\partial h}{\partial x} (s, X_s) \right| dL^0_s (X)
\]

4. If $\frac{\partial h}{\partial x} (t, 0) < 0$,

\[
\lim_{n \to +\infty} \frac{1}{\epsilon_n} \int_a^b 1_{-\epsilon_n \leq h(s, X_s) < 0, 0 \leq X_s < y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s = \int_a^b \left| \frac{\partial h}{\partial x} (s, X_s) \right| dL^0_s (X)
\]

Proof. For example, first we choose $(t^*_i)_{i \in [1, n^* - 1]}$ such that:

\[
\max_{i \in [1, n^* - 1]} \sup_{s \in [t^*_i, t^*_i + 1]} \left| \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 - \left( \frac{\partial h}{\partial x} (t^*_i, X_s) \right)^2 \right| \leq \epsilon^2
\]

\[
\forall i \in [1, n^* - 1], \forall s \in [t^*_i, t^*_i + 1], -\epsilon^2 \leq h(s, X_s) - h(t^*_i, X_s) \leq \lambda \epsilon
\]
The left handside (47) becomes: For \( i \in [1, n^\varepsilon - 1] \) we have
\[
\frac{1}{\varepsilon} \int_{t_i^\varepsilon}^{t_{i+1}^\varepsilon} 1_{0 \leq h(t_i^\varepsilon, X_s) < (1-\lambda) \times \varepsilon, 0 \leq x_s < y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]
\[
\leq \frac{1}{\varepsilon} \int_{t_i^\varepsilon}^{t_{i+1}^\varepsilon} 1_{-\varepsilon^2 \leq h(t_i^\varepsilon, X_s) < 0, 0 \leq x_s < y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]
\[
\leq \frac{1}{\varepsilon} \int_{t_i^\varepsilon}^{t_{i+1}^\varepsilon} 1_{-\varepsilon^2 \leq h(s, X_s) < 0, 0 \leq x_s < y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]
\[
+ \frac{1}{\varepsilon} \int_{t_i^\varepsilon}^{t_{i+1}^\varepsilon} 1_{0 \leq h(s, X_s) < \varepsilon, 0 \leq x_s < y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]
(84)

and:
\[
\sum_{i=1}^{n^\varepsilon-1} \frac{1}{\varepsilon} \int_{t_i^\varepsilon}^{t_{i+1}^\varepsilon} 1_{-\varepsilon^2 \leq h(s, X_s) < 0, 0 \leq x_s < y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]
\[
\leq \varepsilon \times \left( \sup_{\varepsilon > 0} \frac{1}{\varepsilon^2} \int_a^b 1_{|h(s, X_s)| \leq \varepsilon^2} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s \right)
\]
(85)

the right handside in the last inequality goes to 0. Then we choose \((t_i^\varepsilon)_{i \in [1, n^\varepsilon - 1]}\) such that:
\[
\max_{i \in [1, n^\varepsilon - 1]} \sup_{s \in [t_i^\varepsilon, t_{i+1}^\varepsilon]} \left| \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 - \left( \frac{\partial h}{\partial x} (t_i^\varepsilon, X_s) \right)^2 \right| \leq \varepsilon^2
\]
(86)
et:
\[
\forall i \in [1, n^\varepsilon - 1], \forall s \in [t_i^\varepsilon, t_{i+1}^\varepsilon], \quad -\varepsilon^2 \leq h(t_i^\varepsilon, X_s) - h(s, X_s) \leq \lambda \times \varepsilon
\]
(87)

The right handside in inequality (47) becomes: For \( i \in [1, n^\varepsilon - 1] \)
\[
\frac{1}{\varepsilon} \int_{t_i^\varepsilon}^{t_{i+1}^\varepsilon} 1_{0 \leq h(s, X_s) < \varepsilon, 0 \leq x_s < y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]
\[
\leq \frac{1}{\varepsilon} \int_{t_i^\varepsilon}^{t_{i+1}^\varepsilon} 1_{0 \leq h(s, X_s) < \varepsilon^2, 0 \leq x_s < y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]
\[
+ \frac{1}{\varepsilon} \int_{t_i^\varepsilon}^{t_{i+1}^\varepsilon} 1_{0 \leq h(s, X_s) < e^2, 0 \leq x_s < y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]
\[
\leq \frac{1}{\varepsilon} \int_{t_i^\varepsilon}^{t_{i+1}^\varepsilon} 1_{0 \leq h(s, X_s) < \varepsilon, 0 \leq x_s < y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]
\[
+ \frac{1}{\varepsilon} \int_{t_i^\varepsilon}^{t_{i+1}^\varepsilon} 1_{0 \leq h(t_i^\varepsilon, X_s) \leq (1+\lambda) \times \varepsilon, 0 \leq x_s < y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]
(88)
in the same manner we notice
\[
\sum_{i=1}^{n^\varepsilon-1} \frac{1}{\varepsilon} \int_{t_i^\varepsilon}^{t_{i+1}^\varepsilon} 1_{0 \leq h(s, X_s) < \varepsilon^2, 0 \leq x_s < y} \left( \frac{\partial h}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]
tend vers 0 \( \square \)
Remark 3.1. Evidently in corollary 3.1 the signs < (resp \leq) can be replaced by ≤ (resp <).

Let \( V : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \) of class \( C^{1,2} \) and \( \gamma : I \rightarrow \mathbb{R} \) of class \( C^1 \) such that \( \forall t, V(t, \gamma(t)) = 0, \frac{\partial V}{\partial t}(t, \gamma(t)) \neq 0 \). We set \( h(t, x) = V(t, x + \gamma(t)) \), alors \( \forall t \in I, h(t, 0) = 0, \frac{\partial h}{\partial x}(t, 0) \neq 0 \). By a diagonal argument we obtain:

**Corollary 3.2.** Let \((X_t)_{t \geq 0}\) a continuous semimartingale. Let \( V : \mathbb{R}_+ \times \mathbb{R} \) an application of class \( C^{1,2} \). Let \((\Phi^0_k)_{k \geq 0}\) the functions of class \( C^1 \) defined on intervals \( I_k \) mentioned in theorem 2.1. Pour all \( k \) we fix \( y_k > 0 \) sufficiently small, we consider \((i_k)_{k \geq 0}\) a sequence of natural integers and \(([a_k,b_k])_{k \geq 0}\) a sequence of segments such that \( \forall k, [a_k,b_k] \subset I_k \). There exists a positive sequence \((\epsilon_n)_{n \geq 0}\) converging to 0 such that:

1. If \( \frac{\partial V}{\partial x}(s, \Phi^0_{i_k}(\cdot)) > 0 \) then :
   \[
   \lim_{n \rightarrow +\infty} \frac{1}{\epsilon_n} \int_{a_k}^{b_k} 1_{0 \leq V(s,X_s) < \epsilon_n, 0 \leq X_s - \Phi^0_{i_k}(s) < y_k} \left( \frac{\partial V}{\partial x}(s, X_s) \right)^2 d\langle X, X \rangle_s
   \]
   \[
   = \int_{a_k}^{b_k} \left| \frac{\partial V}{\partial x}(s, X_s) \right| d\Lambda^{\Phi^0_{i_k}}_s(X)
   \]

2. If \( \frac{\partial V}{\partial x}(s, \Phi^0_{i_k}(\cdot)) < 0 \) then :
   \[
   \lim_{n \rightarrow +\infty} \frac{1}{\epsilon_n} \int_{a_k}^{b_k} 1_{0 \leq V(s,X_s) < \epsilon_n, y_k < X_s - \Phi^0_{i_k}(s) \leq 0} \left( \frac{\partial V}{\partial x}(s, X_s) \right)^2 d\langle X, X \rangle_s
   \]
   \[
   = \int_{a_k}^{b_k} \left| \frac{\partial V}{\partial x}(s, X_s) \right| d\Lambda^{\Phi^0_{i_k}}_-s(X)
   \]

3. If \( \frac{\partial V}{\partial x}(s, \Phi^0_{i_k}(\cdot)) > 0 \) then :
   \[
   \lim_{n \rightarrow +\infty} \frac{1}{\epsilon_n} \int_{a_k}^{b_k} 1_{-\epsilon_n < V(s,X_s) \leq 0, y_k < X_s - \Phi^0_{i_k}(s) \leq 0} \left( \frac{\partial V}{\partial x}(s, X_s) \right)^2 d\langle X, X \rangle_s
   \]
   \[
   = \int_{a_k}^{b_k} \left| \frac{\partial V}{\partial x}(s, X_s) \right| d\Lambda^{\Phi^0_{i_k}}_-s(X)
   \]

4. If \( \frac{\partial V}{\partial x}(s, \Phi^0_{i_k}(\cdot)) < 0 \) then :
   \[
   \lim_{n \rightarrow +\infty} \frac{1}{\epsilon_n} \int_{a_k}^{b_k} 1_{-\epsilon_n < V(s,X_s) \leq 0, 0 \leq X_s - \Phi^0_{i_k}(s) < y_k} \left( \frac{\partial V}{\partial x}(s, X_s) \right)^2 d\langle X, X \rangle_s
   \]
   \[
   = \int_{a_k}^{b_k} \left| \frac{\partial V}{\partial x}(s, X_s) \right| d\Lambda^{\Phi^0_{i_k}}_-s(X)
   \]

5. 
   \[
   \lim_{n \rightarrow +\infty} \frac{1}{\epsilon_n} \int_{a_k}^{b_k} 1_{|V(s,X_s)| \leq \epsilon_n, X_s - \Phi^0_{i_k}(s) < y_k} \left( \frac{\partial V}{\partial x}(s, X_s) \right)^2 d\langle X, X \rangle_s
   \]
   \[
   = \int_{a_k}^{b_k} \left| \frac{\partial V}{\partial x}(s, X_s) \right| d\Lambda^{\Phi^0_{i_k}}_-s(X) + \int_{a_k}^{b_k} \left| \frac{\partial V}{\partial x}(s, X_s) \right| d\Lambda^{\Phi^0_{i_k}}_+s(X)
   \]
3.1. Proof of inequality $L^0_\mathcal{S} (V (., X)) \geq \int \sum_{(s,x) = 0}^\infty \frac{\partial V}{\partial x} (s, x) d\Lambda^{n,+}_{s} (X)$.

Let $\Phi^0_1, \Phi^0_2, \ldots, \Phi^0_n$ are functions of class $C^1$ among those mentioned in theorem 2.1 defined on open intervals $I^0_{k} \cap ]0, t[, I^0_{k+1} \cap ]0, t[, \ldots, I^0_{n} \cap ]0, t[$. Let $[a_0, b_0] \subset I^0_{0} \cap ]0, t[, [a_1, b_1] \subset I^0_{1} \cap ]0, t[, \ldots, [a_n, b_n] \subset I^0_{n} \cap ]0, t[$, we write $I = \bigcup_{i=0}^n [a_i, b_i] = \bigcup_{k=0}^n J_k$ where the interiors of $J_k$ are pairwise disjoint. For convenience we consider that $J_k$ and its interior are identical, thus we can write:

\[
\int_I \Phi^0 (.) d \langle X, X \rangle_s = \sum_{k=0}^N \int_{J_k} \Phi^0 (.) d \langle X, X \rangle_s
\]

and for $i \in [0, n]$

\[
\int_{[a_i, b_i]} \left| \frac{\partial V}{\partial x} (s, X_s) \right| d\Lambda^{n,+}_{s} (X) = \sum_{k=0}^n \int_{J_k} \left| \frac{\partial V}{\partial x} (s, X_s) \right| d\Lambda^{n,+}_{s} (X)
\]

For $k \in [0, N]$ let $\Phi^0_{j_1}, \ldots, \Phi^0_{j_{n_k}}$ which are defined on $J_k$, note that these functions are also defined on the adherence of $J_k$. By definition of the $(\Phi^0_k)_{k \geq 0}$ for all $t \in J_k$, $\forall i \neq j \in [1, n_k]$, $|\Phi^0_{j_1} (t) - \Phi^0_{j_2} (t)| \neq 0$ hence by compactness there exists $y_k > 0$ independent of $t$ such that the intervals $[\Phi^0_{j_1} (t) - y_k, \Phi^0_{j_2} (t) + y_k] \in [1, n_k]$ are pairwise disjoints. We can always take $y_k$ sufficiently small in order to use corollary 3.2 and thus deduce the existence of a positive sequence $(\epsilon_n)_{n \geq 0}$ converging to 0 such that $\forall k \in [0, N], i \in [1, n_k]$:

1. If $\frac{\partial V}{\partial x} (., \Phi^0_{j_i} (.) ) > 0$ then:

\[
\lim_{n \to +\infty} \frac{1}{\epsilon_n} \int_{J_k} 1_{0 \leq V (s, X_s) < \epsilon_n, 0 \leq X_s - \Phi^0_{j_i} (s) < y_k} \left( \frac{\partial V}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]

(96)

2. If $\frac{\partial V}{\partial x} (., \Phi^0_{j_i} (.) ) < 0$ then:

\[
\lim_{n \to +\infty} \frac{1}{\epsilon_n} \int_{J_k} 1_{0 \leq V (s, X_s) < \epsilon_n, y_k < X_s - \Phi^0_{j_i} (s) \leq 0} \left( \frac{\partial V}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]

(97)

Since for all $s \in J_k$ the intervals $[\Phi^0_{j_1} (s) - y_k, \Phi^0_{j_1} (s) + y_k] \in [1, n_k]$ are pairwise disjoint then for all $n$:

\[
\frac{1}{\epsilon_n} \int_{J_k} 1_{0 \leq V (s, X_s) < \epsilon_n} \left( \frac{\partial V}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]

\[
\geq \sum_{i \in [1, n_k], \frac{\partial V}{\partial x} (., \Phi^0_{j_i} (.) ) > 0} \frac{1}{\epsilon_n} \times \int_{J_k} 1_{0 \leq V (s, X_s) < \epsilon_n, 0 \leq X_s - \Phi^0_{j_i} (s) < y_k} \left( \frac{\partial V}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]

\[
+ \sum_{i \in [1, n_k], \frac{\partial V}{\partial x} (., \Phi^0_{j_i} (.) ) < 0} \frac{1}{\epsilon_n} \times \int_{J_k} 1_{0 \leq V (s, X_s) < \epsilon_n, y_k < X_s - \Phi^0_{j_i} (s) \leq 0} \left( \frac{\partial V}{\partial x} (s, X_s) \right)^2 d \langle X, X \rangle_s
\]

18
By the occupations time formula, the lefthandside of the last inequality the has always a limit when $\epsilon_n$ goes to 0, by taking the limit in both sides we obtain:

$$
\lim_{n \to +\infty} \frac{1}{\epsilon_n} \int_{J_k} 1_{0 \leq V(s, X_s) < \epsilon_n} \left( \frac{\partial V}{\partial x} (s, X_s) \right)^2 \, d \langle X, X \rangle_s
$$

(99)

$$
\geq \sum_{i=1}^{n_k} \int_{J_k} \left| \frac{\partial V}{\partial x} (s, X_s) \right| \, d\Lambda_s^{\phi_i^{0,+,+}} (X)
$$

we deduce by (94) and occupation times formula that:

$$
L_t^0 (V(.), X) = \lim_{n \to +\infty} \frac{1}{\epsilon_n} \int_{0}^{t} 1_{0 \leq V(s, X_s) < \epsilon_n} \left( \frac{\partial V}{\partial x} (s, X_s) \right)^2 \, d \langle X, X \rangle_s
$$

(100)

$$
\geq \sum_{k=0}^{N} \lim_{n \to +\infty} \frac{1}{\epsilon_n} \int_{J_k} 1_{0 \leq V(s, X_s) < \epsilon_n} \left( \frac{\partial V}{\partial x} (s, X_s) \right)^2 \, d \langle X, X \rangle_s
$$

$$
\geq \sum_{k=0}^{N} \sum_{i=1}^{n_k} \int_{J_k} \left| \frac{\partial V}{\partial x} (s, X_s) \right| \, d\Lambda_s^{\phi_i^{0,+,+}} (X)
$$

$$
\geq \sum_{i=0}^{n} \sum_{k \in [0, N]} \int_{J_k} \left| \frac{\partial V}{\partial x} (s, X_s) \right| \, d\Lambda_s^{\phi_i^{0,+,+}} (X)
$$

(101)

$$
L_t^0 (V(.), X) \geq \sum_{i=0}^{n} \int_{a_i}^{b_i} \left| \frac{\partial V}{\partial x} (s, X_s) \right| \, d\Lambda_s^{\phi_i^{0,+,+}} (X)
$$

the last inequality being true for all $n$ and for all segments $[a_i, b_i] \subset I_i^0 \cap \{0, t]\$, hence the result!.

3.2. Proof of inequality $L_t^0 (V(.), X) \geq \int_{0}^{t} \sum_{V(s, x) = 0, \frac{\partial V}{\partial x} (s, x) \neq 0} \left| \frac{\partial V}{\partial x} (s, x) \right| \, d\Lambda_s^{\phi_i^{0,+,+}} (X)$. We proceed with the same method above by writing:

(102)

$$
L_t^0 (V(.), X) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{0}^{t} 1_{-\epsilon \leq V(s, X_s) \leq 0} \left( \frac{\partial V}{\partial x} (s, X_s) \right)^2 \, d \langle X, X \rangle_s
$$

3.3. Proof of inequality $L_t^0 (V(.), X) + L_t^0 (V(.), X) \leq \int_{0}^{t} \sum_{V(s, x) = 0, \frac{\partial V}{\partial x} (s, x) \neq 0} \left| \frac{\partial V}{\partial x} (s, x) \right| \, d\Lambda_s^{\phi_i^{0,+,+}} (X)$. We start by establishing the following identity:

(103)

$$
\int_{0}^{t} 1_{\frac{\partial V}{\partial x} (s, X_s) = 0} \, dL_t^0 (V(.), X) = 0
$$

By the occupation times formula, one can easily see that $L_t^0 (V(.), X) = L_t^0 (V(.), X)$, by exercise 1.16 of chapter 6 in [1] $L_t^0 (V(.), X) = 2 \times \int_{0}^{t} 1_{\{V(s, X_s) = 0\}} \, dV(s, X_s)$ hence it suffices to prove that :

(104)

$$
\int_{0}^{t} 1_{\{\frac{\partial V}{\partial x} (s, X_s) = 0\}} 1_{\{V(s, X_s) = 0\}} \, d\left( V(s, X_s) \right) = 0
$$
by Tanaka’s formula and Ito’s formula for all \( u \):

\[
\int_0^u 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0 \right\}} d\left( V(s,X_s)^+ \right) = \int_0^u 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0 \right\}} \times 1_{\{V(s,X_s)>0\}} dV(s,X_s) \\
+ \frac{1}{2} \int_0^u 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0 \right\}} dL_s^0(V(.,X)) \\
= \int_0^u 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0 \right\}} \times 1_{\{V(s,X_s)>0\}} \times \frac{\partial V}{\partial x}(s,X_s) dX_s \\
+ \int_0^u 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0 \right\}} \times 1_{\{V(s,X_s)>0\}} \frac{\partial^2 V}{\partial x^2}(s,X_s) d\langle X, X \rangle_s \\
+ \frac{1}{2} \int_0^u 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0 \right\}} \times dL_s^0(V(.,X)) \\
= \int_0^u 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0 \right\}} \times 1_{\{V(s,X_s)>0\}} \frac{\partial V}{\partial t}(s,X_s) ds \\
+ \frac{1}{2} \int_0^u 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0 \right\}} \times \frac{\partial^2 V}{\partial x^2}(s,X_s) d\langle X, X \rangle_s \\
+ \frac{1}{2} \int_0^u 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0 \right\}} \times dL_s^0(V(.,X))
\]

This proves that the process \( A_u = \int_0^u 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0 \right\}} d\left( V(s,X_s)^+ \right) \) is of finite total variation. Let \( Y_u = \int_0^u 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s) \neq 0 \right\}} d\left( V(s,X_s)^+ \right) \) (\( Y_u \) \( u \geq 0 \), it’s a continuous semimartingale and we have \( V(\cdot, X)^+ = Y + A \), thus we can use exercise 1.24 of chapter 6 in [2] to obtain:

\[
\int_0^t 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0 \right\}} d\left( V(s,X_s)^+ \right) = \int_0^t 1_{\{Y_s=0\}} dA_s = 0
\]

note that in the same way one can show that :

\[
\int_0^t 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0, V(s,X_s)=0 \right\}} dV(s,X_s) = 0
\]

hence :

\[
\int_0^t 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0 \right\}} dL_s^0(V(.,X)) - \int_0^t 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0 \right\}} dL_s^0(V(.,X)) \\
= \int_0^t 1_{\left\{ \frac{\partial V}{\partial x}(s,X_s)=0, V(s,X_s)=0 \right\}} dV(s,X_s) = 0
\]

thus :

\[
L_t^0(V(.,X)) + L_t^0(V(.,X)) \\
= \int_0^t 1_{\{V(s,X_s)=0, \frac{\partial V}{\partial x}(s,X_s) \neq 0 \}} d\left( L_s^0(V(.,X)) + L_s^0(V(.,X)) \right) \\
= \sum_{k \geq 0} \int_{k-1}^{k} 1_{\{X_s=\phi_k(s)\}} d\left( L_s^0(V(.,X)) + L_s^0(V(.,X)) \right)
\]
We fix $k \geq 0$, $[a_k, b_k] \subset I_k^0 \cap ]0, t[,$ by occupation times formula for all $s \geq 0$

\[(110)\]

\[I_k^0 (V(., X)) + L_s^0 (V(., X)) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t 1_{|V(s, X)| \leq \epsilon} \left( \frac{\partial V}{\partial x} (s, X) \right)^2 d \langle X, X \rangle_s \]

let $y_k > 0$ sufficiently small in the sense of corollary $3.2$ since the set $(s \in]a_k, b_k[||X_s - \Phi_k^0 (s) | < y_k)$ is open, by generalized portemanteau theorem and point $5$ in corollary $3.2$

\[(111)\]

\[
\int_{[a_k, b_k]} 1\{X_s = \Phi_k^0 (s)\} d \left( L_s^0 (V(., X)) + L_s^0 (V(., X)) \right)
\]
\[
\leq \int_{[a_k, b_k]} 1\{|X_s - \Phi_k^0 (s) | < y_k\} d \left( L_s^0 (V(., X)) + L_s^0 (V(., X)) \right)
\]
\[
\leq \liminf_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_{[a_k, b_k]} 1\{|V(s, X)| \leq \epsilon\} \times 1\{|X_s - \Phi_k^0 (s) | < y_k\} \left( \frac{\partial V}{\partial x} (s, X) \right)^2 d \langle X, X \rangle_s
\]
\[
\leq \int_{a_k}^{b_k} \left| \frac{\partial V}{\partial x} (s, X) \right| d \Lambda_s^{\Phi_k^0 -} (X) + \int_{a_k}^{b_k} \left| \frac{\partial V}{\partial x} (s, X) \right| d \Lambda_s^{\Phi_k^0 +} (X)
\]

the last one being true for all segments $[a_k, b_k]$ we deduce:

\[(112)\]

\[
\int_{I_k^0 \cap ]0, t[} 1\{X_s = \Phi_k^0 (s)\} d \left( L_s^0 (V(., X)) + L_s^0 (V(., X)) \right)
\]
\[
\leq \int_{I_k^0 \cap ]0, t[} \left| \frac{\partial V}{\partial x} (s, X) \right| d \Lambda_s^{\Phi_k^0 -} (X) + \int_{I_k^0 \cap ]0, t[} \left| \frac{\partial V}{\partial x} (s, X) \right| d \Lambda_s^{\Phi_k^0 +} (X)
\]

which implies the desired inequality □

4. Trajectorial identification of two processes

The goal of this section is to show that both processes $(L_t^0 (V(., X)))_{z \in \mathbb{R}, t \geq 0}$ and

\[
\left( \int_0^t \sum_{V(s, x) = z, \frac{\partial V}{\partial x} (s, x) \neq 0} \left| \frac{\partial V}{\partial x} (s, x) \right| d \Lambda_s^{z+} (X) \right)_{z \in \mathbb{R}, t \geq 0}
\]

are indistinguishable, meaning almost surely we have:

\[
\forall z \in \mathbb{R}, \forall t \geq 0, L_t^0 (V(., X)) = \int_0^t \sum_{V(s, x) = z, \frac{\partial V}{\partial x} (s, x) \neq 0} \left| \frac{\partial V}{\partial x} (s, x) \right| d \Lambda_s^{z+} (X)
\]

. First we shall establish the result for $V = \phi$ homogeneous:

4.1. Homogeneous case. This particular case is useless to prove the general case but the proof is interesting from our point of view. Recall that $(L_t^0 (X))_{a \in \mathbb{R}, t \geq 0}$ is continuous in $t$ càdlàg $a$, our goal is to show that almost surely forall $a \in \mathbb{R}, t \geq 0$

\[(113)\]

\[L_t^0 (\phi (X)) = \sum_{\phi (x) = a, \phi' (x) \neq 0} \left[ (\phi' (x))^+ L_t^+ (X) + (\phi' (x))^-- L_t^- (X) \right] \]

this follows directly from the fact that for all $a$ the function

\[
t \to \sum_{\phi (x) = a, \phi' (x) \neq 0} \left[ (\phi' (x))^+ L_t^+ (X) + (\phi' (x))^-- L_t^- (X) \right]
\]

is continuous and the following lemma:
**Lemma 4.1.** Let \( \phi : \mathbb{R} \to \mathbb{R} \) a function of class \( C^2 \) et \( \psi : \mathbb{R} \to \mathbb{R} \) une fonction càdlàg à support compact. Alors pour tout \( a \in \mathbb{R} \) l’ensemble \( B_a = (x|\phi(x) = a, \phi'(x) \neq 0) \) est au plus dénombrable et la fonction :

\[
a \to \sum_{x \in B_a} \left( (\phi'(x))^+ \psi(x) + (\phi'(x))^− \psi(x−) \right)
\]

est continue à droite

**Proof.** Let \( a \in \mathbb{R} \), then the points of \( B_a \) are isolated and hence \( B_a \) is almost countable, to see this : let \( x \in B_a \) hence \( \phi \) is strictly monotonous on a neighbourhood of \( x \) so the intersection of such neighbourhood with \( B_a \) is in fact \( \{x\} \). Let \( C > 0 \) such that \( \text{supp}(\psi) \subset ]−C, C[ \). Let \( ([a_n, b_n])_{n \geq 0} \) the path wise connected components of the open set \( \Omega = (−C < x < C|\phi'(x) \neq 0) \), in other words :

\[
\Omega = \bigcup_{n \geq 0} ]a_n, b_n[
\]

Note that \( \phi'(a_n) = \phi'(b_n) = 0 \), ( otherwise if \( \phi'(a_n) \neq 0 \) for example, it follows by continuity of \( \phi' \) that one can find another path wise connected component that contains strictly \( ]a_n, b_n[ \) which is absurd. We write :

\[
\sum_{x \in B_a} \left( (\phi'(x))^+ \psi(x) + (\phi'(x))^− \psi(x−) \right)
\]

\[
= \sum_{n=1}^{+\infty} \sum_{x \in B_a \cap ]a_n, b_n[} \left( (\phi'(x))^+ \psi(x) + (\phi'(x))^− \psi(x−) \right)
\]

**Step 1** Let \( g_n : a \to \sum_{x \in B_a \cap ]a_n, b_n[} ((\phi'(x))^+ \psi(x) + (\phi'(x))^− \psi(x−)) \). As the derivative of \( \phi \) never hits \( 0 \) on \( ]a_n, b_n[ \) \( \phi \) is strictly monotonous on \( ]a_n, b_n[ \), hence the function \( \phi_n = \phi|_{[a_n, b_n]} \) is a diffeomorphism of \( ]a_n, b_n[ \) on \( \phi_n([a_n, b_n]) \) thus \( g_n \) is given by :

\[
\begin{cases}
(\phi'(\phi^{-1}_n(a)))^+ \psi(\phi^{-1}_n(a)) + (\phi'(\phi^{-1}_n(a)))^− \psi(\phi^{-1}_n(a)−) & a \in \phi_n([a_n, b_n]) \\
0 & \text{otherwise}
\end{cases}
\]

**Step 2** The application \( g_n \) is right continuous: suppose for example that \( \phi_n \) is strictly increasing on \( ]a_n, b_n[ \) and let \( a \in \mathbb{R} \).

- If \( a \geq \phi(b_n) \) then \( g_n(a) = 0 \) and \( \lim_{t \to a+} g_n(t) = 0 = g_n(a) \)
- If \( a < \phi(b_n) \) then \( g_n(a) = 0 \) and \( \lim_{t \to a+} g_n(t) = 0 = g_n(a) \)
- If \( a = \phi(a_n) \) then \( g_n(a) = 0 \), by continuity of \( \phi^{-1} \) and \( \phi' \) \( \lim_{t \to a+} g_n(t) = 0 = g_n(a) \)
- If \( a \in ]\phi(a_n), \phi(b_n)[ \) then \( g_n(a) = \phi'(\phi^{-1}(a))\psi(\phi^{-1}(a)) \), the right continuity in this case is also evident.

If \( \phi_n \) is strictly decreasing on \( ]a_n, b_n[ \), we proceed with the same manner and will notice that the application \( x \to \psi(x−) \) is left continuous.
Step 3 The series $\sum g_n$ is normally convergent: for $a \in \phi_n([a_n, b_n])$:

$$|g_n(a)| \leq \sup_{x \in \mathbb{R}} |\psi(x)| \times |\phi'(\phi^{-1}(a))|$$

(117)

and we have:

$$\sum_{n \geq 0} \int_{a_n}^{b_n} |\phi''(x)| dx \leq \int_{-C}^{C} \phi''(x) dx < +\infty$$

(118)

This concludes the proof!

To see that for a fixed the application

$$t \to \sum_{\phi(x) = a, \phi'(x) \neq 0} \left[ (\phi'(x))^+ L_t^+ (X) + (\phi'(x))^-= L_t^- (X) \right]$$

is continuous, it suffices to use the same idea in the proof of Lemma 4.1, the series of functions

$$\sum_{\phi(x) = a, \phi'(x) \neq 0} [t \to (\phi'(x))^+ L_t^+ (X) + (\phi'(x))^-= L_t^- (X)]$$

is normally convergent on every compact.

4.2. General case. First let’s see why for $z$ fixed, almost surely the application

$$t \to \int_0^t \sum_{V(s,x) = z} \left| \frac{\partial V}{\partial x} (s, x) \right| dN_s^z+ (X)$$

is continuous: since almost surely it’s increasing, owing to Theorem 2.1, we have almost surely: $\forall t \geq 0$, $L_t^z (V(.), X) \geq \int_0^t \sum_{V(s,x) = z} \left| \frac{\partial V}{\partial x} (s, x) \right| dN_s^z+ (X)$, this means that the series of functions

$$\sum_{k \geq 0} t \to \int_{t \cap [0, t]} \left| \frac{\partial V}{\partial x} (s, \Phi_k^z(s)) \right| dN_s^z+ (X)$$

converges normally on every compact hence its sum is continuous. Before showing the right continuity in $z$ we observe that the existence of a regular version of $(L_t^z (X))_{a \in \mathbb{R}, t \geq 0}$ was necessary to prove the right continuity in $z$. The following proposition is corollary 1.1 in [1].

**Proposition 4.1.** Let $\Omega \subset \mathbb{R} \times \mathbb{R}^+$ an open set and $F : \Omega \rightarrow \mathbb{R}$ an application of class $C^1$, let $(X_t)_{t \geq 0}$ a continuous semimartingale. Then there exists a version of the family

$$\left( \Lambda_t^{F(z, .)} (X) - \Lambda_s^{F(z, .)} (X) \right)_{\{z\} \times \{s, t \subset \Omega \}}$$

such that almost surely for all $z, s, t$ such that $[s, t] \times \{z\} \subset \Omega$:

1. By compacity for $\eta$ sufficiently small $[s, t] \times [z - \eta, z + \eta] \subset \Omega$, if $\forall u \in [s, t], \frac{\partial F}{\partial x} (z, u) > 0$ the application $z \rightarrow \Lambda_t^{F(z, .)} (X) - \Lambda_s^{F(z, .)} (X)$ is càdlàg and we have:

$$\lim_{z' \rightarrow z^+} \Lambda_t^{F(z', .)} (X) - \Lambda_s^{F(z', .)} (X) = \int_s^t d\Lambda_u^{F(z', .)} (X)$$

(119)
(2) If \( \forall u \in [s, t], \frac{\partial F}{\partial z}(z, u) < 0 \) the application \( z \to \Lambda_t^F(z, \cdot) - \Lambda_s^F(z, \cdot) \) is lâd-câlg (continuous on the leftheight limit on the right)

\[
\lim_{z' \to z^+} \Lambda_t^F(z', \cdot) (X) - \Lambda_s^F(z', \cdot) (X) = \int_s^t d\Lambda_u^F(z', \cdot) (X)
\]

(3) \( \lim_{(u,v) \to (s,t)} \Lambda_u^F(z, \cdot) (X) - \Lambda_v^F(z, \cdot) (X) = \Lambda_t^F(z, \cdot) (X) - \Lambda_s^F(z, \cdot) (X) \)

**Corollary 4.1.** Under the assumptions of proposition 4.1 Let \( C \) the set of \((P, U)\) where \( P \subset \mathbb{R}_+^1 \times \mathbb{R} \) is a closed rectangle and \( U : P \to \mathbb{R} \) is an application of class \( C^1 \) satisfying:

\[
\forall (t, z) \in P \ V(t, U(t, z)) = z, \ \frac{\partial V}{\partial x}(t, U(t, z)) \neq 0
\]

Then there exists a version of the family \( \left( \Lambda_t^{U(z)}(X) - \Lambda_s^{U(z)}(X) \right) \) satisfying points 4.1.2 of proposition 4.1. In particular: we fix \( P \subset P \) with rational endpoints, let \( U_1, U_2 \) satisfying 121 meaning the set of \( \{z\} \subset P \), by portmanteau’s theorem forall \( g : [s, t] \to \mathbb{R} \) continuous we have:

\[
\lim_{z' \to z^+} \int_s^t g(u) d\Lambda_u^{U(z, \cdot),+} = \int_s^t g(u) d\Lambda_u^{U(z, \cdot),+}
\]

**Proof.** Let \((P, U) \in C\). Let \([s, t] \times \{z\} \in P \) there exists a closed rectangle \( P_0 \) with rational endpoints such that \( P \subset P_0 \subset U \), hence it suffices to consider the set of \((P_0, U)\) where \( P_0 \) is of rational endpoints, let \( U_1, U_2 \) satisfying 121 if both functions are identical in a point then it’s not hard to see that \( U_1, U_2 \) are identical on the whole rectangle (it suffices to notice that forall \( t \) the branch \( z \to U_i(t, z) \) is a solution of a regular ODE, we deduce the unicity branch by branch via Cauchy-Lipschitz theorem), note that for \((t_0, z_0) \in P_0\) the set \( \{x|V(t_0, x) = z_0, \frac{\partial V}{\partial x}(t_0, x) \neq 0\} \) is countable we deduce that for each \( P_0 \) one can define a countable set of applications \( U \) satisfying meaning the set of \((P_0, U)\) is countable, we deduce the result using the proposition 4.1.1 and some technical details left to the reader \( \square \)

**Remark 4.1.** When \( V(t, z) = z \) we recover the fact that the family \((L^a_t)_a \in \mathbb{R}_+^1 \) has a modification càd-làg in \( a \) and continuous in \( t \).

**Lemma 4.2.** Let \((X_t)_{t \geq 0} \) a continuous semimartingale, \( V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) an application of class \( C^{1.2} \). Let \((\Phi^k_t)_{k \geq 0}\) the functions mentioned in theorem 2.4 then almost surely forall \( t \geq 0 \) the application \( z \to \int_0^t \sum_{V(s, x) = z} \left| \frac{\partial V}{\partial x}(s, x) \right| d\Lambda^x_s (X) \) is right continuous.

**Proof.**

**Step 1** We start by showing that almost surely: forall \( t \geq u \in \mathbb{R}_+ \) forall \( z \in \mathbb{R} \):

\[
L^z_t (V(., X)) - L^z_u (V(., X)) \geq \int_u^t \sum_{V(s, x) = z, \frac{\partial V}{\partial x}(s, x) \neq 0} \left| \frac{\partial V}{\partial x}(s, x) \right| d\Lambda^x_s (X)
\]

It’s obvious that the last inequality is true for all \( t, u, z \in \mathbb{Q} \). We fix \( x \in \mathbb{R} \), let \( \Phi^1, \Phi^2, ..., \Phi^n \) be functions among those mentioned in theorem 2.4 defined on the open intervals \( I_0^z, I_1^z, ..., I_n^z \). Let \([a_i, b_i] \subset I_i^z \) arbitrary segments, for \( i \in [0, n] \) we have:

\[
\forall s \in [a_i, b_i], V(s, \Phi^z_i(s)) = z
\]
By compacity \( \forall \epsilon > 0 \) sufficiently small, there exists \( \eta > 0 \) such that for all \( z' \in [z - \eta, z + \eta] \) for all \( s \in [a_i, b_i] \) the equation:

\[
V(s, x) = z'
\]

has a unique solution in \( \Phi_i(z') - \Phi_i(s) + \eta \), it’s not hard to see that this defines \( n \) functions \( \Phi_0^z', \Phi_1^z', \ldots, \Phi_n^z' \) of class \( C^1 \) such that:

\[
\forall i; \forall s \in [a_i, b_i] \quad V(s, \Phi_i^z(s)) = z', \quad |\Phi_i^z(s) - \Phi_i(s)| \leq \eta
\]

and \( n \) applications of class \( C^1 U_1, \ldots, U_n \) defined on the rectangles \([a_i, b_i] \times [z - \eta, z + \eta]\). By corollary 4.1

\[
\forall i, \lim_{z' \rightarrow z^+} \int_{a_i}^{b_i} \left| \frac{\partial V}{\partial x} (s, X_s) \right| d\Lambda_s^{U,(z')^+} = \int_{a_i}^{b_i} \left| \frac{\partial V}{\partial x} (s, X_s) \right| d\Lambda_s^{U,(z)^+}
\]

Let \((z_m)_{m \geq 0}\) be a sequence of rationals converging to \( z \) on the right, then by \[27\] for all \( \epsilon > 0 \) there exists \( m_0 \) such that for all \( m \geq m_0 \):

\[
\sum_{i=0}^{n} \int_{[a_i, b_i] \cap [u, t]} \left| \frac{\partial V}{\partial x} (s, \Phi_i^z(s)) \right| d\Lambda_s^{\Phi_i^z + \epsilon} \leq \sum_{i=0}^{n} \int_{[a_i, b_i] \cap [u, t]} \left| \frac{\partial V}{\partial x} (s, \Phi_i^z(s)) \right| d\Lambda_s^{\Phi_i^z + \epsilon} + \epsilon
\]

this implies:

\[
\sum_{i=0}^{n} \int_{[a_i, b_i] \cap [u, t]} \left| \frac{\partial V}{\partial x} (s, \Phi_i^z(s)) \right| d\Lambda_s^{\Phi_i^z + \epsilon} \leq \liminf_{m \rightarrow +\infty} \int_{u}^{t} \sum_{V(s,x)=z_m} \left| \frac{\partial V}{\partial x} (s, x) \right| d\Lambda_s^{x^+} (X)
\]

the last inequality being true for all \( n \), for all segments \([a_i, b_i]\) we deduce:

\[
\int_{u}^{t} \sum_{V(s,x)=z_m} \left| \frac{\partial V}{\partial x} (s, x) \right| d\Lambda_s^{x^+} (X) \leq \liminf_{m \rightarrow +\infty} \int_{u}^{t} \sum_{V(s,x)=z_m} \left| \frac{\partial V}{\partial x} (s, x) \right| d\Lambda_s^{x^+} (X)
\]

Since \( z \rightarrow L_i^z (V(., X)) \) is right continuous the desired inequality follows.

**Step 2** Note that almost surely:

\[
\forall z \in \mathbb{R}, \forall t \geq 0, \quad \int_{0}^{t} 1_{\{\frac{\partial V}{\partial x}(s,x)=0\}} dL_s^z (V(., X)) = 0
\]

In fact : By the proof of theorem 3.1 for \( z, t \) fixed we have almost surely

\[
\int_{0}^{t} 1_{\{\frac{\partial V}{\partial x}(s,x)=0\}} dL_s^z (V(., X)) = 0
\]

consequently : by setting \( \hat{L}_i^z = \int_{0}^{t} 1_{\{\frac{\partial V}{\partial x}(s,x)=0\}} dL_s^z (V(., X)) \)

we have almost surely:

\[
\hat{L}_i^z = \int_{0}^{t} 1_{\{\frac{\partial V}{\partial x}(s,x)=0\}} dL_s^z (V(., X)) = |X_i - z| - |X_0 - z| - \int_{0}^{t} \text{sgn} (X_s - z) dX_s
\]

Since for all \( z \) the application \( t \rightarrow \hat{L}_i^z \) is continuous and increasing, hence it’s a local time for \( X \). By ( theorem 1.7 of chapter 6 in [9] ) \( \hat{L}_i^z \in \mathbb{R}, t \geq 0 \)

\[
\hat{L}_i^z \quad \text{has a version càd-làg in z and continuous in t, this implies that almost surely:}
\]

\[
\forall z, \forall t \geq 0, \quad \hat{L}_i^z = L_i^z (V(., X)) \quad \text{which is equivalent to [10]}
\]

For all \( z \) we
consider the measure ( see proposition 2.2):
\[ \mu^z(A) = \sum_{k \geq 0} \int_{I_k \cap A} \left| \frac{\partial V}{\partial x}(s, \Phi_k(s)) \right| d\Lambda s^{\Phi_k+} \]

by Step 1 almost surely for all \( z \) for all \( t \geq 0 \) the restriction of \( \mu^z \) to \( ([0, t], B([0, t])) \) is a finite measure. Fix \( t \geq 0 \), and let \( z \in \mathbb{R} \) and \( z_m \) a sequence converging to \( z \) on the right, we shall consider \( \mu^z, \mu^{z_m} \) identical with their restrictions on \( ([0, t], B([0, t])) \), hence the sequence \( \mu^{z_m} \) is obviously tight, by the generalized Prokhorov theorem ( theorem 8.6.2 in [2] ), without loss of generality ( otherwise extract a subsequence) there exists a measure \( \mu \) weak limit of the sequence \( \mu^{z_m} \).

**Step 3** First we establish \( \mu(s \in [0, t]) [\frac{\partial V}{\partial x}(s, X_s) = 0] = 0 \): Since
\[ \int_0^1 1_{\{V(s, X_s) = 0\}} dL_s^z(V(., X_s)) = 0 \]
meaning ( by the dominated convergence theorem )
\[ \lim_{\epsilon \to 0^+} \int_0^t 1_{\{\frac{\partial V}{\partial x}(s, X_s) \leq \epsilon\}} dL_s^z(V(., X_s)) = 0 \]
Hence for \( \eta > 0 \) there exists \( \epsilon_0 \) such that \( \int_0^t 1_{\{\frac{\partial V}{\partial x}(s, X_s) \leq \epsilon_0\}} dL_s^z(V(., X_s)) \leq \eta \). Since the set \( (s \in [0, T]) [\frac{\partial V}{\partial x}(s, X_s) \leq \epsilon_0] \) is closed, by the generalized portmanteau theorem:
\[ \eta \geq \int_0^t 1_{\{\frac{\partial V}{\partial x}(s, X_s) \leq \epsilon_0\}} dL_s^z(V(., X_s)) \]
\[ \geq \lim \sup_{n \to +\infty} \int_0^t 1_{\{\frac{\partial V}{\partial x}(s, X_s) \leq \epsilon_0\}} dL_s^{z_m}(V(., X_s)) \]
Since the set \( (s \in [0, t]) [\frac{\partial V}{\partial x}(s, X_s) < \epsilon_0] \) is open and thus countable union of intervals and by the result of Step 1 and generalized portmanteau theorem
\[ \mu \left( s \in [0, t] [\frac{\partial V}{\partial x}(s, X_s) = 0] \right) = \mu \left( s \in [0, t] [\frac{\partial V}{\partial x}(s, X_s) = 0] \right) \]
\[ \leq \mu \left( s \in [0, t] [\frac{\partial V}{\partial x}(s, X_s) < \epsilon_0] \right) \leq \lim \inf_{n \to +\infty} \mu^{z_m} \left( s \in [0, t] [\frac{\partial V}{\partial x}(s, X_s) < \epsilon_0] \right) \]
\[ \leq \lim \sup_{n \to +\infty} \int_0^t 1_{\{\frac{\partial V}{\partial x}(s, X_s) \leq \epsilon_0\}} dL_s^{z_m}(V(., X_s)) \]
\[ \leq \eta \]
this enables us to conclude, since \( \eta > 0 \) is arbitrary. It’s obvious that \( \mu u \) is carried by \( (s \in [0, t]) |V(s, X_s) = z| \), in fact by portmanteau’s theorem for all \( \epsilon > 0 \)
\[ \mu \left( s \in [0, t] [|V(s, X_s) - z| > \epsilon] \right) \leq \lim \inf_{n \to +\infty} \mu^{z_m} \left( s \in [0, t] [|V(s, X_s) - z| > \epsilon] \right) \]
\[ \mu \left( s \in [0, t] [|V(s, X_s) - z| > \epsilon] \right) \leq \lim \inf_{n \to +\infty} \mu^{z_m} \left( s \in [0, t] [|V(s, X_s) - z| > \epsilon] \right) \]
We conclude that \( \mu \) is carried by:

\[
\left( s \in [0, t] \left| V(s, X_s) = z, \frac{\partial V}{\partial x}(s, X_s) \neq 0 \right. \right)
\]

Let \( \Phi_n^z \) a function among those mentioned in theorem 2.1 defined on \( I_n^z \). Let \( [a, b] \subset I_n^z \cap [0, t] \) by compacity there exists \( \eta > 0, m > 0 \) such that:

\[
\forall s \in [a, b], \forall x \in [\Phi_n^z(s) - \eta, \Phi_n^z(s) + \eta], \left| \frac{\partial V}{\partial x}(s, x) \right| \geq m
\]

hence by setting:

\[
E_n = ((s, x) | s \in [a, b], x \in [\Phi_n^z(s) - \eta, \Phi_n^z(s) + \eta])
\]

we have:

\[
E_n \cap \left( (s, x) | V(s, x) = z, \frac{\partial V}{\partial x}(s, x) \neq 0 \right) = \emptyset
\]

for all \( m \) sufficiently big let \( \Phi_n^z \) the function constructed in [Step 1] this function is unique so we have:

\[
\mu^z_m(E_n) = \int_a^b \left| \frac{\partial V}{\partial x}(s, \Phi_n^z(s)) \right| d\Lambda_{\Phi_n^z}^z(X)
\]

the boundary of \( E_n \) is given by:

\[
\partial E_n = ((s, x) | s \in [a, b], x = \Phi_n^z(s) + \eta, x = \Phi_n^z(s) - \eta)
\]

By [137]

\[
\mu \left( \partial E_n \right) = 0
\]

By [140] we deduce by generalized portmanteau theorem:

\[
\mu \left( s \in [a, b] | (s, \Phi_n^z(s)) \right) = \mu \left( E_n \right)
\]

\[
= \lim_{m \to +\infty} \int_a^b \left| \frac{\partial V}{\partial x}(s, \Phi_n^z(s)) \right| d\Lambda_{\Phi_n^z}^z(X)
\]

\[
= \int_a^b \left| \frac{\partial V}{\partial x}(s, \Phi_n^z(s)) \right| d\Lambda_{\Phi_n^z}^z(X)
\]

the latter holds forall \( [a, b] \subset I_n^z \) we deduce:

\[
\mu \left( s \in I_n^z \cap [0, t] | (s, \Phi_n^z(s)) \right) = \int_{I_n^z \cap [0, t]} \left| \frac{\partial V}{\partial x}(s, \Phi_n^z(s)) \right| d\Lambda_{\Phi_n^z}^z(X)
\]

hence \( \mu \) is identical to \( A \to \int_{A \cap [0, t]} \sum_{V(s, x) = z} \left| \frac{\partial V}{\partial x}(s, x) \right| d\Lambda_{\Phi_n^z}^z(X) \), we deduce that \( \mu^z_m \) converges weakly to \( \mu \) and hence \( \mu^z ([0, t]) \) converges to \( \mu ([0, t]) \) ce which is the desired result.

\( \square \)
5. Coquet-Ouknine's formula as a change of variable formula

In this section we obtain Coquet Ouknine's formula from a classical change of variable formula: The Coera formula (theorem 2 section 3.4.4 in[8]):

We fix \( t > 0 \). Let \( V \) be the total variation part of \( X \). We have almost surely:

\[
\forall a \in \mathbb{R}, \sum_{a \in \mathbb{R}} |L_t^a(X) - L_t^{a-}(X)| = \sum_{a \in \mathbb{R}} |\int_0^t 1_{X_s = a} dV_s| \leq \sum_{a \in \mathbb{R}} \int_0^t 1_{X_s = a} |dV_s| = \int_0^t |dV_s|
\]

(146)

Hence the set \( (a||L_t^a(X) - L_t^{a-}(X)|| \neq 0) \) is countable and thus it’s Lebesgue measure is 0, in other words: for \( t \) fixed, we have almost surely:

\[
L_t^a(X) = L_t^{a-}(X) \text{ Lebesgue presque partout en } a
\]

(147)

We write the occupation times formula for \( \phi(X) \). We almost surely:

\[
\forall f \in C^\infty_c(\mathbb{R}) \int_0^t f(\phi(X_s)) \phi'(X_s)^2 d\langle X, X \rangle_s = \int_{-\infty}^{+\infty} f(a)L_t^a(\phi(X)) \, da
\]

(148)

We apply the same formula to the semimartingale \( X \) and the function \( x \to f(\phi(x))\phi'(x)^2 \):

\[
\forall f \in C^\infty_c(\mathbb{R}) \int_0^t f(\phi(X_s)) \phi'(X_s)^2 d\langle X, X \rangle_s = \int_{-\infty}^{+\infty} f(a)\phi'(a)^2L_t^a(X) \, da
\]

(149)

By (147) we obtain:

\[
\int_0^t f(\phi(X_s)) \phi'(X_s)^2 d\langle X, X \rangle_s = \int_{-\infty}^{+\infty} f(a) \times |\phi'(a)| \times [(\phi'(a))^+] \times L_t^a(X) \, da
\]

(150)

\[
\int_{-\infty}^{+\infty} f(\phi(a)) \times |\phi'(a)| \times [(\phi'(a))^+] \times L_t^a(X) + (\phi'(a)^-) L_t^{a-}(X) \, da
\]
Owing to the Coarea formula we obtain:

\[
\int_{-\infty}^{+\infty} f(\phi(a)) \times |\phi'(a)| \times ((\phi'(a))^+ L^a_t(X) + (\phi'(a))^− L^a_t(X)) da = \\
\int_{-\infty}^{+\infty} \left( \int_{\phi^{-1}(y)} f(\phi(a)) \times ((\phi'(a))^+ L^a_t(X) + (\phi'(a))^− L^a_t(X)) dH^0(a) \right) dy
\]

(151)

Recall the equality 148, we obtain: for fixed \(a,t\) we almost surely:

\[
L^a_t(\phi(X)) = \sum_{x \in B_a} [(\phi'(x))^+ L^x_t(X) + (\phi'(x))^− L^x_t(X)]
\]

By lemma 141 we deduce that we have almost surely:

\[
\forall a \in \mathbb{R}, \forall t \geq 0 \ L^a_t(\phi(X)) = \sum_{x \in B_a} [(\phi'(x))^+ L^x_t(X) + (\phi'(x))^− L^x_t(X)]
\]

(153)

which concludes the proof \(\square\).

6. Application: A Forward Formula

In finance CVA theoretically is defined on the level of portfolio or a contract as the difference between it’s value under the risk neutral measure and it’s value when one take into consideration the default by the counterparty (also known as counterparty risk). Let us consider a contract of maturity \(T\) and \((F_t)_{t \geq 0}\) the filtration representing the flow of information, in absence of counterparty risk it’s value at time \(t\) is:

\[
V(t) = \mathbb{E}(\Pi(t,T)|F_t)
\]

Where \(\Pi(t,T)\) is the sum of discounted cashflows between \(t\) and \(T\), \(\mathbb{E}\) is the expectation under risk neutral measure. In the presence of counterparty risk we denote by \(\Pi^D(t,T)\) the sum of cashflows between \(t\) and \(T\) thus:

\[
CVA_t = \mathbb{E}(\Pi(t,T)|F_t) - \mathbb{E}(\Pi^D(t,T)|F_t)
\]

Proposition 6.1.

\[
CVA_t = \alpha \times \mathbb{E}(1_{\tau \leq T} D(t, \tau) (V(\tau)^+) | F_t)
\]

With \(\tau\) is the default time, \(D\) is the discounting factor, \(\alpha\) is a marked fixed constant.

When \(D = 1\), \(\tau\) is independent of \(V\) we have:

\[
CVA_s = \int_0^T \mathbb{E}(V(t)^+|F_s) \mathbb{P}(dt)
\]

with \(\mathbb{P}\) is the law of the random variable \(\tau\), in particular:

\[
CVA_0 = \int_0^T \mathbb{E}(V(t)^+) \mathbb{P}(dt) = \int_0^T \mathbb{E}E(t) \mathbb{P}(dt)
\]
the quantity $EE(t)$ is called Expected exposure. We consider now the following EDS:

\begin{equation}
X_t = \int_0^t \mu(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dW_s
\end{equation}

Where $(W_t)_{t \geq 0}$ is a brownian motion, $\mu, \sigma : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ are two maps of class $C^\infty$ whose partial derivatives are bounded. In addition, we make the following assumption on $\sigma$ called uniform ellipticity:

\begin{equation}
\forall s \geq 0, \forall x, K_1 x \leq \sigma^2(s, x) \leq K_2 x
\end{equation}

By making another technical assumption (see for example theorem 4.5 page 141 in [7]) one infer that the equation (159) has a unique strong solution $X$ satisfying

\begin{equation}
E\left[\sup_{t \in [0,T]} |X_t|\right] < +\infty, \text{ and for all } s \text{ the law of } X_s \text{ has a density } q(s, x) \text{ of class } C^\infty. \text{ The following result is theorem 3.2 in [8]:}
\end{equation}

**Theorem 6.1.** We fix $T > 0$ a maturity and we consider a contract $V(t)$ on the underlying $X$. We work in a markovian setting and we suppose that $V(t) = V(t, X_t)$, furthermore we make the following assumptions:

1. The application $V : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is of class $C^{1,2}$.
2. There are $n$ continuous functions $l_1, l_2, \ldots, l_n : [0, T] \rightarrow \mathbb{R}$ such that:
   - $((t, x)|t \in [0, T], V(t, x) = 0) = \bigcup_{i=1}^n ((t, l_i(t))|t \in [0, T])$
   - The last union is disjoint and we have:
     \[ \forall i, \forall t \in [0, T] \frac{\partial V}{\partial x}(t, l_i(t)) \neq 0 \]
3. There exists $\beta, A$ and a constant $C$ such that for $|x| \geq A$, for all $t \in [0, T]$ $|V(t, x)| \exp(\beta x^2) \geq C$
4. $V(t, X_t)$ is a martingale

Then $\forall t \in [0, T]$:

\begin{equation}
EE(t) = EE(0) + \frac{1}{2} \sum_{i=1}^n \int_0^t \sigma^2(s, l_i(s)) \left| \frac{\partial V}{\partial x}(s, l_i(s)) q(s, l_i(s)) \right| \, ds
\end{equation}

The formula (161) is a forward formula by means of which one can accelerate the computations of the expected exposure according to [8].

**Proposition 6.2.** Let $(L^z_t (X))_{z \in \mathbb{R}, t \geq 0}$ be the family of local times of $X$. We have:

\begin{equation}
\forall z \in \mathbb{R}, \forall t \geq 0 \mathbb{E} \left( L^z_t (X) \right) = \int_0^t \sigma^2(s, z) q(s, z) \, ds
\end{equation}

**Proof.** By the occupation times formula we have almost surely: $\forall f \geq 0$ measurable $\forall t \geq 0, \forall x \in \mathbb{R}$:

\begin{equation}
\int_0^t f(X_s) \sigma^2(s, X_s) \, ds = \int_{-\infty}^{+\infty} f(x) L^x_t (X) \, dx
\end{equation}
we have:
\[ E \left( \int_0^t f(X_s) \sigma^2(s, X_s) \, ds \right) = \int_0^t E \left( f(X_s) \sigma^2(s, X_s) \right) \, ds \]
(164)
\[ = \int_0^t \int_{-\infty}^{+\infty} f(x) \sigma^2(s, x) \times q(s, x) \, dx \]
\[ = \int_{-\infty}^{+\infty} f(x) \times \left[ \int_0^t \sigma^2(s, x) q(s, x) \, ds \right] \, dx \]

By (163) we deduce that for fixed \( t \), we have Lebesgue almost everywhere:
\[ E(L^x_t(X)) = \int_0^t \sigma^2(s, x) q(s, x) \, ds \]
(165)

one can easily see that the function \( x \to E(L^x_t(X)) \) is lower semi continuous on the right, it follows that:
\[ \forall x \in \mathbb{R}, \forall t \geq 0, E(L^x_t(X)) \leq \int_0^t \sigma^2(s, x) q(s, x) \, ds \]
(166)

By Tanaka’s formula, for all \( x, t \):
\[ L^x_t(X) = 2 \times \left( (X_t - x)^+ - (-x)^+ - \int_0^t 1_{\{X_s > x\}} \sigma(s, X_s) \, dW_s - \int_0^t 1_{\{X_s > x\}} \mu(s, X_s) \, ds \right) \]
(167)

Let:
\[ \hat{W}^x_t = \int_0^t 1_{\{X_s > x\}} \sigma(s, X_s) \, dW_s \]
(168)

using the proof of theorem 1.7 of chapter 5 in [9], one can easily conclude that the map \( x \to \hat{W}^x_t \) is continuous in \( L^2 \) and thus in \( L^1 \), since the partial derivatives of \( \mu \) are bounded and owing to the fact that \( E \left[ \sup_{t \in [0,T]} |X_t| \right] < +\infty \) we have:
\[ \mathbb{E} \left( \int_0^t |\mu(s, X_s)| \, ds \right) < +\infty \]
(169)

By the dominated convergence theorem, the map \( x \to \mathbb{E} \left( \int_0^t 1_{\{X_s > x\}} \mu(s, X_s) \, ds \right) \) is right continuous, by (167) we infer thus that the map \( x \to E(L^x_t) \) right continuous for all \( t \). Finally by (165) we deduce the result. \( \square \)

In [8] the authors tried to link the formula (161) to local time theory by the following: If \( V = \phi \) is homogenous in time with \( \phi : \mathbb{R} \to \mathbb{R} \) of class \( C^2 \), by Tanaka’s formula:
\[ (\phi(X_t))^+ = (\phi(X_0))^+ + \int_0^t 1_{\{\phi(X_s) > 0\}} d\phi(X_s) + \frac{1}{2} \times L^0_t(\phi(X)) \]
(170)
Note that \( \mathbb{E} \left[ \int_0^t 1_{\{\phi(X_s) > 0\}} d\phi(X) \right] = 0 \) this is a consequence of exercise 4.10 of chapter 6 in [9]. By using Coquet-Ouknine Formula we obtain:

(171)

\[
EE(t) = EE(0) + \frac{1}{2} \times \mathbb{E} \left( \sum_{\phi(x)=0, \phi(x)\neq 0} (\phi'(x))^+ \times L_t^+ (X) + (\phi'(x))^+ \times L_t^- (X) \right)
\]

hence:

(172)

\[
\mathbb{E} \left( L_t^+ (X) - L_t^- (X) \right) = 2 \times \mathbb{E} \left( \int_0^t 1_{\{X_s=0\}} \mu (s, X_s) ds \right)
\]

We deduce by proposition 6.2 that:

(173)

\[
EE(t) = EE(0) + \frac{1}{2} \times \left[ \int_0^t \sum_{\phi(x)=0, \phi(x)\neq 0} |\Phi'(x)| \times \sigma^2(s, x) \times q(s, x) ds \right]
\]

which is the forward formula for the homogenous case on can consider this as a neat proof of the formula (161) without any technical assumptions. The same authors asked if there is an analogous formula to Coquet-Ouknine’s by means of which one can prove neatly theorem 6.1, this was the motivation of our paper and the desired formula is simply theorem 3.1. Note first that for a curve of class \( C^1 \) \( \gamma \), for all \( s \) the density of the law of \( X_s - \gamma(s) \) is \( q(s, x + \gamma(s)) \) and the volatility of \( X - \gamma \) is given by \( \sigma^2(s, x + \gamma(s)) \). Fix \( t > 0 \) and let \( \Phi_0^t \) a function of class \( C^1 \) among those mentioned in theorem 2.1 defined on \( I_n^0 \), let \( [a, b] \subset I_n^0 \), by proposition 6.2

(174)

\[
\mathbb{E} \left[ \int_a^b \right. d\Lambda_{s}^{\Phi_0^t, +} = \int_a^b \sigma^2(s, \Phi_0^t(s)) \times q(s, \Phi_0^t(s)) ds
\]

one can deduce by monotone class theorem that:

(175)

\[
\mathbb{E} \left( \int_{I_n^0 \cap [0, t]} \left| \frac{\partial V}{\partial x} \right| (s, \Phi_0^t(s)) d\Lambda_{s}^{\Phi_0^t, +} \right) = \int_{I_n^0 \cap [0, t]} \left| \frac{\partial V}{\partial x} \right| (s, \Phi_0^t(s)) \times \sigma^2(s, \Phi_0^t(s)) \times q(s, \Phi_0^t(s)) ds
\]

thus by theorem 2.1

(176)

\[
\mathbb{E} \left( \int_0^t \sum_{V(s,x)=z, \frac{\partial V}{\partial x}(s,x)\neq 0} \left| \frac{\partial V}{\partial x} \right| (s, x) d\Lambda_{s}^{\gamma^+} \right) = \int_0^t \sum_{V(s,x)=z, \frac{\partial V}{\partial x}(s,x)\neq 0} \left| \frac{\partial V}{\partial x} \right| (s, x) \sigma^2(s, x) \times q(s, x) ds
\]

Finally using Tanaka’s formula ( keep in mind that \( \mathbb{E} \left[ \int_0^t 1_{V(s,X_s)} dV(s, X_s) \right] = 0 \) as a consequence of exercise 4.10 of chapter 6 in [9]). By using Tanaka’s formula
we obtain finally the generalized forward formula for the Expected Exposure:

\[
EE(t) = EE(0) + \frac{1}{2} \times \left[ \int_0^t \left( \sum_{V(s,x)=0, \frac{\partial V}{\partial x}(s,x) \neq 0} \left| \frac{\partial V}{\partial x}(s,x) \right| \sigma^2(s,x) \times q(s,x) \right) ds \right]
\]

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