DERIVATIVE OPERATOR
AND HARMONIC NUMBER IDENTITIES

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ABSTRACT. By applying the derivative operator to the corresponding hypergeometric form of a \(q\)-series transformation due to Andrews [1, Theorem 4], we establish a general harmonic number identity. As the special cases of it, several interesting Chu-Donno type identities and Paule-Schneider type identities are displayed.

1. Introduction

For a nonnegative integer \(n\), define the harmonic numbers by

\[ H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{when} \quad n = 1, 2, \ldots. \]

For a differentiable function \(f(x)\), the derivative operator \(\mathcal{D}\) can be defined by

\[ \mathcal{D}f(x) = \left. \frac{d}{dx} f(x) \right|_{x=0}. \]

Then it is not difficult to show the following two derivatives of binomial coefficients:

\[
\mathcal{D} \binom{n+x}{r} = \binom{n}{r} \{H_n - H_{n-r}\}, \quad \mathcal{D} \binom{n-x}{r} = \binom{n}{r} \{H_{n-r} - H_n\},
\]

where \(r \leq n\) with \(r = 0, 1, \ldots\).

For a complex number \(x\), define the shifted factorial by

\[ (x)_0 = 1 \quad \text{and} \quad (x)_n = \prod_{k=0}^{n-1} (x + k) \quad \text{when} \quad n = 1, 2, \ldots. \]

The fractional form of it reads as

\[
\left[ \begin{array}{c} a, b, \ldots, c \\ \alpha, \beta, \ldots, \gamma \end{array} \right]_n = \frac{(a)_n(b)_n \cdots (c)_n}{(\alpha)_n(\beta)_n \cdots (\gamma)_n}.
\]

Then the hypergeometric series (cf. Bailey [3]) can be defined by

\[
1 + F_a \left[ \begin{array}{c} a_0, a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \right]_z = \sum_{k=0}^{\infty} \left[ \begin{array}{c} a_0, a_1, \ldots, a_r \\ 1, b_1, \ldots, b_s \end{array} \right]_k z^k,
\]

where \(\{a_i\}_{i \geq 0}\) and \(\{b_j\}_{j \geq 1}\) are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side.

For a complex sequence \(\{A_k\}_{k \geq 0}\) and two nonnegative integers \(i\) and \(j\), define the product by

\[
\prod_{k=i}^{j} A_k = \left\{ \begin{array}{ll} A_1A_{i+1} \cdots A_j, & \text{for} \quad j \geq i, \\
1, & \text{for} \quad j = i - 1. \end{array} \right.
\]

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In 1975, Andrews [1, Theorem 4] gave a beautiful q-series transformation. The corresponding hypergeometric form of it (cf. Krattenthaler et al. [3, Theorem 8] and [5, Equation 4.2]) can be stated as

\[
2m+5 F_{2m+4} \left[ \begin{array}{c}
2m+2, 1+a+a/2, a/2, \\
1 + a - a T_{2m+1}, 1 + a + n \end{array} \right] = \left[ \begin{array}{c}
1 + a, 1 + a - P_{2m+1} - P_{2m+2} \\
1 + a - P_{2m+1}, 1 + a - P_{2m+2} \end{array} \right]_n
\]

\times \sum_{0 \leq i_1 \leq \ldots \leq i_m \leq n} \prod_{r=1}^m \left[ \begin{array}{c}
1, P_{2r+1}, P_{2r+2} - a - i_r, 1 + a - P_{2r-1}, 1 + a - P_{2r} \end{array} \right]_{i_r}, \tag{1}

where \( i_{m+1} = n \) and \( m \in \mathbb{N} \). When \( m = 1 \), the last equation reduces to the famous Whipple’s transformation (cf. Bailey [3, p. 25]):

\[
_{7}F_{6} \left[ \begin{array}{c}
a, 1 + a/2, P_1, 1 + a - P_1, 1 + a - P_2, 1 + a - P_3, 1 + a - P_4, 1 + a + n \end{array} \right] = \left[ \begin{array}{c}
1 + a, 1 + a - P_3 - P_4 \\
1 + a - P_3, 1 + a - P_1 - P_2, 1 + a - P_1, 1 + a - P_2 \end{array} \right]_n
\]

By applying the derivative operator \( D \) to (1), we shall establish a general harmonic number identity in the next section. As the special cases of it, several interesting Chu-Donno type identities and Paul-Schneider type identities will be displayed.

2. Harmonic number identities

§2.1. A general harmonic number identity.

Let \( v \) be a nonnegative integer with \( 0 \leq v \leq 2m + 2 \). For two finite sets \( \{\alpha_s\}_{s=1}^v \) and \( \{\alpha_s\}_{s=v+1}^{2m+2} \), the case \( v = 0 \) corresponds to the former is empty and the latter is \( \{\alpha_s\}_{s=1}^{2m+2} \), and the case \( v = 2m + 2 \) corresponds to the former is \( \{\alpha_s\}_{s=1}^v \) and the latter is empty.

Performing the replacements \( a \rightarrow -x - n, P_s \rightarrow 1 + P_s \) with \( 1 \leq s \leq v \) and \( P_s \rightarrow -n - P_s \) with \( v + 1 \leq s \leq 2m + 2 \) for (1), we obtain the following expression:

\[
\sum_{0 \leq i_1 \leq \ldots \leq i_m \leq n} \prod_{r=1}^m \left[ \begin{array}{c}
1 - i_r, T_{2r+1}, T_{2r+2}, 1 - x - n - T_{2r-1} - T_{2r} \end{array} \right]_{i_r},
\]

where \( T_s = 1 + P_s \) with \( 1 \leq s \leq v \) and \( T_s = -n - P_s \) with \( v + 1 \leq s \leq 2m + 2 \).

Applying the derivative operator \( D \) to both sides of the last equation, we establish the following theorem.

Theorem 1. For \( 2m+2 \) nonnegative integers \( \{P_s\}_{s=1}^{2m+2} \) with \( i_{m+1} = n \) and \( m \in \mathbb{N} \), there holds the general harmonic number identity:

\[
\sum_{k=0}^n \binom{n}{k} \prod_{s=1}^v \frac{(k+P_s)}{k} \prod_{s=v+1}^{2m+2} \frac{(k+P_s)}{k} \left( 1 + (n-2k) \left( 2H_k - \sum_{s=1}^v \frac{2m+2}{s} \right) \right) = \left[ \begin{array}{c}
-n, 1 - n - T_{2m+1} - T_{2m+2} \\
1 - n - T_{2m+1}, 1 - n - T_{2m+2} \end{array} \right]_n
\]

\times \sum_{0 \leq i_1 \leq \ldots \leq i_m \leq n} \prod_{r=1}^m \left[ \begin{array}{c}
1 - i_r, T_{2r+1}, T_{2r+2}, 1 - n - T_{2r-1} - T_{2r} \end{array} \right]_{i_r},
\]

where \( T_s = 1 + P_s \) with \( 1 \leq s \leq v \) and \( T_s = -n - P_s \) with \( v + 1 \leq s \leq 2m + 2 \).

§2.2. Special cases: harmonic number identities of Chu-Donno type.

Setting \( m = 1 \) in Theorem 1 we get the following equation.
Proposition 2. For four nonnegative integers \( \{P_s\}_{s=1}^{\infty} \), there holds the harmonic number identity:

\[
\sum_{k=0}^{n} \binom{n}{k}^2 \sum_{s=1}^{v} \binom{k+P_s}{k}^{4} \sum_{s=1}^{v} \binom{n+P_s}{n} \left\{ 1 + (n - 2k) \left( 2H_k - \sum_{s=1}^{v} H_{k+P_s} + \sum_{s=1}^{v} H_{k+P_s} \right) \right\}
\]

\[
= \left[ -n, 1 - n - T_3 - T_4, 1 - n - T_3, 1 - n - T_4 \right] \times \left[ -n, T_3, T_4, 1 - n - T_3 - T_2, T_3 + T_4, 1 - n - T_1, 1 - n - T_2 \right],
\]

where \( T_s = 1 + P_s \) with \( 1 \leq s \leq v \) and \( T_s = -n - P_s \) with \( v + 1 \leq s \leq 4 \).

Letting \( P_s \to nP_s \) with \( 1 \leq s \leq 4 \) in Proposition 2 we get the following result.

Corollary 3. For four nonnegative integers \( \{P_s\}_{s=1}^{\infty} \), there holds the harmonic number identity:

\[
\sum_{k=0}^{n} \binom{n}{k}^2 \prod_{s=1}^{v} \binom{k+P_s}{k}^{4} \prod_{s=1}^{v} \binom{n+P_s}{n} \left\{ 1 + (n - 2k) \left( 2H_k - \sum_{s=1}^{v} H_{k+nP_s} + \sum_{s=1}^{v} H_{k+nP_s} \right) \right\}
\]

\[
= \left[ -n, 1 - n - T_3 - T_4, 1 - n - T_3, 1 - n - T_4 \right] \times \left[ -n, T_3, T_4, 1 - n - T_1 - T_2, T_3 + T_4, 1 - n - T_1, 1 - n - T_2 \right],
\]

where \( T_s = 1 + nP_s \) with \( 1 \leq s \leq v \) and \( T_s = -n - nP_s \) with \( v + 1 \leq s \leq 4 \).

The importance of Corollary 3 lies in that it implies eight important theorems due to Chu and Donno. The details are laid out as follows.

Taking respectively \( v = 2, 1, 0 \) in Corollary 3 and then letting \( P_1 \to b, P_2 \to c, P_3 \to \infty, P_4 \to \infty \), we gain the following three known harmonic number identities.

Example 1 (Chu and Donno [4] Theorem 5]). For two nonnegative integers \( \{b, c\} \), there holds

\[
\sum_{k=0}^{n} \binom{n}{k}^2 \prod_{s=1}^{v} \binom{k+b+c}{k}^{4} \prod_{s=1}^{v} \binom{n+c}{n} \left\{ 1 + (n - 2k) \left( 2H_k - H_{bn+k} - H_{cn+k} \right) \right\} = \left( \frac{1+bn+cn+n}{n+bn+cn} \right) \quad \text{for } n \geq 0.
\]

Example 2 (Chu and Donno [4] Theorem 6]). For two nonnegative integers \( \{b, c\} \), there holds

\[
\sum_{k=0}^{n} \binom{n}{k}^2 \prod_{s=1}^{v} \binom{k+b+c}{k}^{4} \prod_{s=1}^{v} \binom{n+c}{n} \left\{ 1 + (n - 2k) \left( 2H_k - H_{bn+k} + H_{cn+k} \right) \right\} = \left( \frac{1+bn-cn}{n+bn+cn} \right) \quad \text{for } n \geq 0.
\]

Example 3 (Chu and Donno [4] Theorem 7]). For two nonnegative integers \( \{b, c\} \), there holds

\[
\sum_{k=0}^{n} \binom{n}{k}^2 \prod_{s=1}^{v} \binom{k+b+c}{k}^{4} \prod_{s=1}^{v} \binom{n+c}{n} \left\{ 1 + (n - 2k) \left( 2H_k + H_{bn+k} + H_{cn+k} \right) \right\} = \left( \frac{1+2n+bn-cn}{n+bn+cn} \right) \quad \text{for } n \geq 0.
\]

Taking respectively \( v = 4, 3, 2, 1, 0 \) in Corollary 3 and then letting \( P_1 \to b, P_2 \to c, P_3 \to d, P_4 \to e \), we achieve the following five known harmonic number identities.

Example 4 (Chu and Donno [4] Theorem 8]). For four nonnegative integers \( \{b, c, d, e\} \), there holds

\[
\sum_{k=0}^{n} \binom{n}{k}^2 \prod_{s=1}^{v} \binom{k+b+c+d+e}{k}^{4} \prod_{s=1}^{v} \binom{n+c+d+e}{n} \left\{ 1 + (n - 2k) \left( 2H_k - H_{bn+k} - H_{cn+k} - H_{dn+k} - H_{en+k} \right) \right\} = \left( \frac{1+bn+cn+dn+en+n}{n+bn+cn+dn+en} \right) \quad \text{for } n \geq 0.
\]

Example 5 (Chu and Donno [4] Theorem 9]). For four nonnegative integers \( \{b, c, d, e\} \), there holds

\[
\sum_{k=0}^{n} \binom{n}{k}^2 \prod_{s=1}^{v} \binom{k+b+c+d+e}{k}^{4} \prod_{s=1}^{v} \binom{n+c+d+e}{n} \left\{ 1 + (n - 2k) \left( 2H_k - H_{bn+k} - H_{cn+k} - H_{dn+k} + H_{en+k} \right) \right\} = \left( \frac{1+bn+cn+dn+en+n}{n+bn+cn+dn+en} \right) \quad \text{for } n \geq 0.
\]
Example 6 (Chu and Donno [4] Theorem 10]). For four nonnegative integers \(\{b, c, d, e\}\), there holds
\[
\sum_{k=0}^{n} \binom{n}{k}^2 \frac{\binom{k}{b}}{\binom{k}{b}} \frac{\binom{k}{c}}{\binom{k}{c}} \frac{\binom{n+dn}{n+cn}}{\binom{k}{k}} \frac{\binom{n+en}{n+en}}{\binom{k}{k}} \times \left\{ 1 + (n - 2k)(2H_k - H_{bn+k} - H_{cn+k} + H_{dn+k} + H_{en+k}) \right\}
\]
\[
= \left( -1 \right)^n \sum_{i=0}^{n} (-1)^i \binom{n+bn+i}{n+i} \binom{n+cn+i}{i} \binom{n+dn+i}{i} \binom{n+en+i}{n+en-n}.
\]

Example 7 (Chu and Donno [4] Theorem 11]). For four nonnegative integers \(\{b, c, d, e\}\), there holds
\[
\sum_{k=0}^{n} \binom{n}{k}^2 \frac{\binom{k}{b}}{\binom{k}{b}} \frac{\binom{k}{c}}{\binom{k}{c}} \frac{\binom{n+dn}{n+cn}}{\binom{k}{k}} \frac{\binom{n+en}{n+en}}{\binom{k}{k}} \times \left\{ 1 + (n - 2k)(2H_k - H_{bn+k} + H_{cn+k} + H_{dn+k} + H_{en+k}) \right\}
\]
\[
= \left( -1 \right)^n \sum_{i=0}^{n} (-1)^i \binom{n+bn+i}{n+i} \binom{n+cn+i}{i} \binom{n+dn+i}{i} \binom{n+en+i}{n+en-n}.
\]

Example 8 (Chu and Donno [4] Theorem 12]). For four nonnegative integers \(\{b, c, d, e\}\), there holds
\[
\sum_{k=0}^{n} \binom{n}{k}^2 \frac{\binom{k}{b}}{\binom{k}{b}} \frac{\binom{k}{c}}{\binom{k}{c}} \frac{\binom{n+dn}{n+cn}}{\binom{k}{k}} \frac{\binom{n+en}{n+en}}{\binom{k}{k}} \times \left\{ 1 + (n - 2k)(2H_k + H_{bn+k} + H_{cn+k} + H_{dn+k} + H_{en+k}) \right\}
\]
\[
= \left( -1 \right)^n \sum_{i=0}^{n} (-1)^i \binom{n+bn+i}{n+i} \binom{n+cn+i}{i} \binom{n+dn+i}{i} \binom{n+en+i}{n+en-n}.
\]

Setting \(m = 2\) in Theorem 1 we attain the following equation.

Proposition 4. For six nonnegative integers \(\{P_s\}_{s=1}^{6}\), there holds the harmonic number identity:
\[
\sum_{k=0}^{n} \binom{n}{k}^2 \frac{\binom{k}{b}}{\binom{k}{b}} \frac{\binom{k}{c}}{\binom{k}{c}} \frac{\binom{n+dn}{n+cn}}{\binom{k}{k}} \frac{\binom{n+en}{n+en}}{\binom{k}{k}} \times \left\{ 1 + (n - 2k)(2H_k - \sum_{s=1}^{v} H_{k+P_s} + \sum_{s=v+1}^{6} H_{k+P_s}) \right\}
\]
\[
= \left[ \begin{array}{c} -n, 1 - n - T_5 - T_6, 1 - n - T_3 - T_4, 1 - n - T_5 - T_6, 1 - n - T_3 - T_4, 1 - n - T_3, 1 - n - T_4, 1 - n - T_2, 1 - n - T_1. \end{array} \right]
\times 4 F_3 \left[ \begin{array}{c} -i, -n, T_3, T_4, 1 - n - T_1 - T_2, T_3 + T_4 + n - i, 1 - n - T_1, 1 - n - T_2, 1 - n - T_1. \end{array} \right],
\]
where \(T_s = 1 + P_s\) with \(1 \leq s \leq v\) and \(T_s = -n - P_s\) with \(v + 1 \leq s \leq 6\).

Taking respectively \(v = 6, 5, 4, 3, 2, 1, 0\) in Proposition 4 and then letting \(P_1 = b, P_2 = c, P_3 = d, P_4 = e, P_5 = f, P_6 = g\), we derive the following seven harmonic number identities of Chu-Donno type.

Example 9. For six nonnegative integers \(\{b, c, d, e, f, g\}\), there holds
\[
\sum_{k=0}^{n} \binom{n}{k}^2 \frac{\binom{k}{b}}{\binom{k}{b}} \frac{\binom{k}{c}}{\binom{k}{c}} \frac{\binom{k}{d}}{\binom{k}{d}} \frac{\binom{k}{e}}{\binom{k}{e}} \frac{\binom{k}{f}}{\binom{k}{f}} \frac{\binom{n+en}{n+en}}{\binom{k}{k}} \times \left\{ 1 + (n - 2k)(2H_k - H_{bn+k} - H_{cn+k} - H_{dn+k} - H_{en+k} - H_g) \right\}
\]
\[
= \frac{\left( -1 \right)^n \sum_{i=0}^{n} (-1)^i \binom{n+bn+i}{n+i} \binom{n+cn+i}{i} \binom{n+dn+i}{i} \binom{n+en+i}{n+en-n}}{\binom{n+bn+i}{n+i} \binom{n+cn+i}{i} \binom{n+dn+i}{i} \binom{n+en+i}{n+en-n}}.
\]

Example 10. For six nonnegative integers \(\{b, c, d, e, f, g\}\), there holds
\[
\sum_{k=0}^{n} \binom{n}{k}^2 \frac{\binom{k}{b}}{\binom{k}{b}} \frac{\binom{k}{c}}{\binom{k}{c}} \frac{\binom{k}{d}}{\binom{k}{d}} \frac{\binom{k}{e}}{\binom{k}{e}} \frac{\binom{k}{f}}{\binom{k}{f}} \frac{\binom{n+en}{n+en}}{\binom{k}{k}} \times \left\{ 1 + (n - 2k)(2H_k - H_{bn+k} - H_{cn+k} - H_{dn+k} - H_{en+k} - H_{ef}) \right\}
\]
\[
= \frac{\left( -1 \right)^n \sum_{i=0}^{n} (-1)^i \binom{n+bn+i}{n+i} \binom{n+cn+i}{i} \binom{n+dn+i}{i} \binom{n+en+i}{n+en-n}}{\binom{n+bn+i}{n+i} \binom{n+cn+i}{i} \binom{n+dn+i}{i} \binom{n+en+i}{n+en-n}}.
\]
Example 11. For six nonnegative integers \{b, c, d, e, f, g\}, there holds

\[
\sum_{k=0}^{n} \binom{n}{k} 2^{\binom{n+b}{k} \binom{n+c}{k} \binom{n+d}{k} \binom{n+e}{k} \binom{n+f}{k} \binom{n+g}{k}} \times \left\{1 + (n - 2k)(2H_k - H_{b+k} - H_{c+k} - H_{d+k} - H_{e+k} - H_{f+k} + H_{g+k})\right\} = \frac{\binom{1+b+c+n}{n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{i+b}{i} \binom{i+c}{i} \binom{i+d+c+n}{i} \sum_{j=0}^{i} \binom{i}{j} \binom{i+d+j}{i} \binom{i+f+j}{i} \binom{n+g+j}{i}}{\binom{n+b+c+n}{n} \sum_{i=0}^{n} \binom{i+b+c+n}{i} \binom{i+b+c+n}{i} \binom{i+b+c+n}{i} \binom{i+b+c+n}{i}}.
\]

Example 12. For six nonnegative integers \{b, c, d, e, f, g\}, there holds

\[
\sum_{k=0}^{n} \binom{n}{k} 2^{\binom{n+b}{k} \binom{n+c}{k} \binom{n+d}{k} \binom{n+e}{k} \binom{n+f}{k} \binom{n+g}{k}} \times \left\{1 + (n - 2k)(2H_k - H_{b+k} - H_{c+k} - H_{d+k} - H_{e+k} - H_{f+k} + H_{g+k})\right\} = \frac{\binom{1+b+c+n}{n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{i+b}{i} \binom{i+c}{i} \binom{i+d+c+n}{i} \sum_{j=0}^{i} \binom{i}{j} \binom{i+d+j}{i} \binom{i+f+j}{i} \binom{n+g+j}{i}}{\binom{n+b+c+n}{n} \sum_{i=0}^{n} \binom{i+b+c+n}{i} \binom{i+b+c+n}{i} \binom{i+b+c+n}{i} \binom{i+b+c+n}{i}}.
\]

Example 13. For six nonnegative integers \{b, c, d, e, f, g\}, there holds

\[
\sum_{k=0}^{n} \binom{n}{k} 2^{\binom{n+b}{k} \binom{n+c}{k} \binom{n+d}{k} \binom{n+e}{k} \binom{n+f}{k} \binom{n+g}{k}} \times \left\{1 + (n - 2k)(2H_k - H_{b+k} - H_{c+k} - H_{d+k} - H_{e+k} - H_{f+k} + H_{g+k})\right\} = \frac{\binom{1+b+c+n}{n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{i+b}{i} \binom{i+c}{i} \binom{i+d+c+n}{i} \sum_{j=0}^{i} \binom{i}{j} \binom{i+d+j}{i} \binom{i+f+j}{i} \binom{n+g+j}{i}}{\binom{n+b+c+n}{n} \sum_{i=0}^{n} \binom{i+b+c+n}{i} \binom{i+b+c+n}{i} \binom{i+b+c+n}{i} \binom{i+b+c+n}{i}}.
\]

Example 14. For six nonnegative integers \{b, c, d, e, f, g\}, there holds

\[
\sum_{k=0}^{n} \binom{n}{k} 2^{\binom{n+b}{k} \binom{n+c}{k} \binom{n+d}{k} \binom{n+e}{k} \binom{n+f}{k} \binom{n+g}{k}} \times \left\{1 + (n - 2k)(2H_k - H_{b+k} + H_{c+k} + H_{d+k} + H_{e+k} + H_{f+k} + H_{g+k})\right\} = (-1)^n \frac{n-b}{n} \sum_{i=0}^{n} \binom{n}{i} \binom{i+b+c+n}{i} \binom{i+b+c+n}{i} \sum_{j=0}^{i} \binom{i}{j} \binom{i+j}{i} \binom{i+j}{i} \binom{n+g+j}{i}.
\]

Example 15. For six nonnegative integers \{b, c, d, e, f, g\}, there holds

\[
\sum_{k=0}^{n} \binom{n}{k} 2^{\binom{n+b}{k} \binom{n+c}{k} \binom{n+d}{k} \binom{n+e}{k} \binom{n+f}{k} \binom{n+g}{k}} \times \left\{1 + (n - 2k)(2H_k + H_{b+k} + H_{c+k} + H_{d+k} + H_{e+k} + H_{f+k} + H_{g+k})\right\} = (-1)^n \frac{n+b}{n} \sum_{i=0}^{n} \binom{n}{i} \binom{i+b+c+n}{i} \binom{i+b+c+n}{i} \binom{i+b+c+n}{i} \sum_{j=0}^{i} \binom{i}{j} \binom{i+j}{i} \binom{i+j}{i} \binom{n+g+j}{i}.
\]

It should be pointed out that Examples 11\textsuperscript{tr} are only the suitable limiting cases of Examples 14\textsuperscript{tr}. Although the latter are also crossed one another as the former, they can create numerous beautiful harmonic number identities with doubt. Further, Theorem 1 may produce more harmonic number identities of Chu-Donno type with the change of \(m\). The interested reader may write several ones of them down as exercises.

\[\text{§2.3. Special cases: harmonic number identities of Paule-Schneider type.}\]

For an integer \(u\) with \(u \neq 0\), define \(T_n^{(u)}\) by

\[
T_n^{(u)} = \sum_{k=0}^{n} \binom{n}{k} u^{k} \{1 + u(n - 2k)H_k\}.\]
Then eight known harmonic number identities can be stated as follows:

\[ T_n^{(-2)} = \frac{2(1+n)^2}{(2+n)} H_{n+1}, \]
\[ T_n^{(-1)} = (1+n) H_{n+1}, \]
\[ T_n^{(1)} = 1, \]
\[ T_n^{(2)} = 0, \]
\[ T_n^{(3)} = (-1)^n, \]
\[ T_n^{(4)} = (-1)^n \left( \frac{2n}{n} \right), \]
\[ T_n^{(5)} = (-1)^n \sum_{i=0}^{n} \binom{n}{i}^2 \left( \frac{n+i}{n} \right), \]
\[ T_n^{(6)} = (-1)^n \sum_{i=0}^{n} \binom{n}{i}^2 \left( \frac{n+i}{n} \right) \left( \frac{2n-i}{n} \right). \]

\[ \text{Eq. (1)} \] - [3] appeared first in Paule and Schneider [8]. Chu and Donno [4] offered other three ones and showed that these eight harmonic number identities just displayed can be derived by specifying the parameters in Examples [1] [3] [4] and [8]. Now, we shall bend ourselves to display the remaining results of the same type by specifying the parameters in Theorem [1].

Letting \( P_{2m+2} \to \infty, v \to 0 \) and \( P_s \to 0 \) with \( 1 \leq s \leq 2m+1 \) in Theorem [1] we obtain the equivalent form of the first equation of Krattenthaler and Rivoal [5] Proposition 1.

**Proposition 5.** For \( m \in \mathbb{N} \), there holds the harmonic number identity:

\[ T_n^{(2m+3)} = (-1)^n \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n} \binom{n}{i_m}^2 \left( \frac{n+i_1}{n} \right) \prod_{r=1}^{m-1} \binom{n}{i_r}^2 \left( \frac{n+i_{r+1} - i_r}{n} \right). \]

Setting \( v = 0 \) and \( P_s = 0 \) with \( 1 \leq s \leq 2m+2 \) in Theorem [1] we get the following equation.

**Proposition 6.** For \( i_{m+1} = n \) with \( m \in \mathbb{N} \), there holds the harmonic number identity:

\[ T_n^{(2m+4)} = (-1)^n \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n} \binom{n}{i_1} \prod_{r=1}^{m} \binom{n}{i_r}^2 \left( \frac{n+i_{r+1} - i_r}{n} \right). \]

Proposition [6] and the second equation of Krattenthaler and Rivoal [5] Proposition 1] have different versions although that the purpose of them are the same. Proposition [6] reduces to [9] exactly when \( m = 1 \). Other two results are laid out as follows.

**Example 16** (Harmonic number identity of Paule-Schneider type: \( m = 2 \) in Proposition [5].)

\[ T_n^{(8)} = (-1)^n \sum_{i=0}^{n} \binom{n}{i}^2 \left( \frac{2n-i}{n} \right) \sum_{j=0}^{i} \binom{n}{j} \left( \frac{n+j}{n} \right) \left( \frac{n+i-j}{n} \right). \]

**Example 17** (Harmonic number identity of Paule-Schneider type: \( m = 3 \) in Proposition [5].)

\[ T_n^{(10)} = (-1)^n \sum_{i=0}^{n} \binom{n}{i}^2 \left( \frac{2n-i}{n} \right) \sum_{j=0}^{i} \binom{n}{j} \left( \frac{n+i-j}{n} \right) \sum_{k=0}^{j} \binom{n}{k} \left( \frac{n+k}{n} \right) \left( \frac{n+j-k}{n} \right). \]

The open problem posed at the end of Paule and Schneider [8] states that whether \( T_n^{(u)} \) can be expressed as a definite hypergeometric single-sum for all \( u \geq 3 \). Although the equations on \( T_n^{(u)} \) with \( u \geq 3 \) have been given in Krattenthaler and Rivoal [5] Proposition 1] and this subsection, we can’t still judge that whether \( T_n^{(u)} \) can be expressed as a definite hypergeometric single-sum for all \( u \geq 6 \).

Letting \( v \to 2m+2, P_{2m+2} \to \infty \) and \( P_s \to 0 \) with \( 1 \leq s \leq 2m+1 \) in Theorem [1] we attain the following equation.

**Proposition 7.** For \( m \in \mathbb{N} \), there holds the harmonic number identity:

\[ T_n^{(1-2m)} = (1+n)^m \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n} \frac{1}{1+n-i_1} \prod_{r=1}^{m-1} \frac{(1)_{i_r} (i_{r+1})_{i_{r+1}}}{(-n)_{i_r} (1+n-i_{r+1})_{i_{r+1}}}. \]

Proposition [7] leads to [3] exactly when \( m = 1 \). Other two results are displayed as follows.
Example 18 (Harmonic number identity of Paule-Schneider type: \( m = 2 \) in Proposition 7).

\[
T_n^{(-3)} = (1 + n)^2 \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(1)_j (-i)_j}{(-n)_j (1 + n - i)_{j+1}} \frac{1}{1 + n - j}.
\]

Example 19 (Harmonic number identity of Paule-Schneider type: \( m = 3 \) in Proposition 7).

\[
T_n^{(-5)} = (1 + n)^3 \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(1)_j (-i)_j}{(-n)_j (1 + n - i)_{j+1}} \sum_{t=0}^{j} \frac{(1)_t (-j)_t}{(-n)_t (1 + n - j)_{t+1}} \frac{1}{1 + n - t}.
\]

Taking \( v = 2m + 2 \) and \( P_s = 0 \) with \( 1 \leq s \leq 2m + 2 \) in Theorem 1, we achieve the following equation.

Proposition 8. For \( i_{m+1} = n \) with \( m \in \mathbb{N} \), there holds the harmonic number identity:

\[
T_n^{(-2m)} = (1 + n)^{m+1} \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n} \prod_{r=1}^{m} \frac{1}{1 + n - i_r} \begin{bmatrix} 1, & -i_{r+1} \\ -n, & 2 + n - i_{r+1} \end{bmatrix}_{i_r}.
\]

Proposition 8 reduces to (9) exactly when \( m = 1 \). Other two results are laid out as follows.

Example 20 (Harmonic number identity of Paule-Schneider type: \( m = 2 \) in Proposition 8).

\[
T_n^{(-4)} = (1 + n)^3 \sum_{i=0}^{n} \frac{1}{1 + i} \sum_{j=0}^{i} \frac{(1)_j (-i)_j}{(-n)_j (1 + n - i)_{j+1}} \frac{1}{1 + n - j}.
\]

Example 21 (Harmonic number identity of Paule-Schneider type: \( m = 3 \) in Proposition 8).

\[
T_n^{(-6)} = (1 + n)^4 \sum_{i=0}^{n} \frac{1}{1 + i} \sum_{j=0}^{i} \frac{(1)_j (-i)_j}{(-n)_j (1 + n - i)_{j+1}} \sum_{t=0}^{j} \frac{(1)_t (-j)_t}{(-n)_t (1 + n - j)_{t+1}} \frac{1}{1 + n - t}.
\]

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