A New Meshfree Method for Solving Steady-state Modified Burgers’ Equation in Transport Problems

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ABSTRACT
A new meshfree method is presented to solve the steady-state modified Burgers’ equation (MBE) in transport problems. To solve a steady-state MBE in each iteration, a multiple-scale Pascal triangle approach is applied to generate the nonlinear algebraic equations, in which the multiple scales are automatically determined by the collocation points. It is shown that these scales can largely reduce the condition number of the coefficient matrix in each nonlinear system, such that the iteration process converges rapidly, and the obtained numerical solutions are stable and accurate against large noise. Furthermore, numerical results substantiate the validity of the current scheme for solving the steady-state MBE in transport problems with a peanut-shaped domain.

KEYWORDS
multi-scale polynomial expansion; meshfree method; steady-state modified Burgers’ equation; transport problems; large noise effect

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1. Introduction

In the past few years, many researchers proposed several schemes to solve the modified Burgers’ equation (MBE) or investigate the shock phenomena, for example, Harris [1] studied the evolution of N-waves in a medium governed by the MBE. It was presented that the general behavior when the nonlinearity was of arbitrary odd integer order. After that, a balancing argument was employed to construct the large-time asymptotic solution of the MBE with sinusoidal initial conditions [2]. Ramadan and El-Danaf [3] contemplated the solution of the MBE by utilizing the collocation approach with quartic splines. They also demonstrated that the proposed scheme was unconditionally stable by applying the Von-Neumann stability analysis approach, and obtained good numerical results.

Duan et al. [4] mentioned that the MBE was also named the nonlinear advection–diffusion equation. It maintained the strong nonlinear aspects of the governing equation in many practical transport problems such as the nonlinear waves in a medium with low-frequency pumping or absorption, turbulence transport, wave processes in thermoelastic medium, ion reflection at quasi-perpendicular shocks, sediment transport, and transport and dispersion of pollutants in rivers. The only known solution to the MBE correspond to the steady shock wave (analogous to the famous Taylor shock wave in a thixomolous fluid) or to a similarity form. It can, furthermore, be proved that there can exist no Backlund transformation of the MBE onto itself or onto any other parabolic equation, and thus, that no linearizing transformation of Cole–Hopf type can exist. They claimed that the proposed lattice Boltzmann method offered accurate results and good stability. Apart from this, Griewank and El-Danaf [5] employed a non-polynomial spline-based algorithm to acquire the numerical solutions of the non-linear MBE; however, their developed method was shown to be conditionally stable.

Roshan and Bhamra [6] utilized the Petrov–Galerkin method with a linear hat function as the trial function and a cubic B-spline function as the test function, and a linear stability analysis of the approach presented it to be unconditionally stable. The further comparison of the presented numerical results with their proposed method was better than the other cited references. Later, Karakoç et al. [7] used the quartic B-spline subdomain finite element method over which the nonlinear term was locally linearized and utilized the quartic B-spline differential quadrature method, and obtained that their numerical results were better than some earlier papers. Recently, Kutluay et al. [8] applied a cubic B-spline collocation approach to solve MBE. In the solution process, a linearization technique on the basis of the quasi-linearization has been used to cope with the non-linear term occurring in the MBE. They also computed the numerical errors and acquired accurate results. In the above-mentioned references, they were about the time-dependent equations and regular domains; however, there were few references about time-independent equations and irregular domains.

This article is summarized as follows. Section 2 shows the steady-state MBE and a modified polynomial expansion approach. Then, in Section 3 we apply the multiple-scale concept to the Pascal triangle expansion approach, which is totally determined by the collocation points. In Section 4, we demonstrate the iterative procedure for solving steady-state MBE. The numerical instances for the direct issues are resolved in Section 5. Finally, conclusions are shown in Section 6.

2. The Steady-state MBE and a Modified Polynomial Expansion Approach

We begin with the following steady-state MBE:

\[ \nu \Delta u(x, y) = u^p u_x + F(x, y, u, u_x, u_y), \ (x, y) \in \Omega, \ m > 1, \]  

(1)

\[ u(x, y) = H(x, y), \ (x, y) \in \Gamma_1, \]  

(2)

\[ u_n(x, y) = G(x, y), \ (x, y) \in \Gamma_2, \]  

(3)

where \( \Delta \) is the Laplacian operator, \( p \) is an integer, \( \nu \) is a positive parameter, and \( F, H \) and \( G \) are given functions. \( \Gamma \) is the boundary of problem domain \( \Omega \) with \( \Gamma = \Gamma_1 \cup \Gamma_2 \), and \( n \) being an unit outward normal on \( \Gamma \). Under the given Dirichlet boundary condition (2) and the Neumann boundary condition (3), we must solve Equation (1) to reveal the solution of \( u(x,y) \).

We utilize the polynomial expansion as a trial solution of partial differential equation (PDE) and derive the required algebraic equations after a suitable collocation in the issue domain. However, it is seldom employed as a major numerical tool to cope with the nonlinear PDEs. The primary cause is that the resultant nonlinear algebraic equations (NAEs) are often seriously ill-conditioned.

The elements in the following polynomial matrix:

\[
\begin{bmatrix}
1 & y & y^2 & \cdots & y^{m-1} & y^m \\
x & xy & xy^2 & \cdots & xy^{m-1} & xy^m \\
x^2 & x^2y & x^2y^2 & \cdots & x^2y^{m-1} & x^2y^m \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x^m & x^my & x^my^2 & \cdots & x^my^{m-1} & x^my^m \\
\end{bmatrix}
\]  

(4)

are usually employed to expand the solution of \( u(x,y) \). If the elements are confined in the left-upper triangle then such an expansion is known as the Pascal triangle expansion.
Therefore, the solution \( u(x, y) \) is expanded by:

\[
u(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{i} c_{ij} x^{i-j} y^{j-1},\]

where the coefficients \( c_{ij} \) are to be determined, whose number of all elements is \( n = m(m+1)/2 \). The highest order of the above polynomial is \( m-1 \).

From Equation (6), it can write directly as follows:

\[
u_x(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{i} c_{ij} (i-j)x^{i-j-1}y^{j-1},\]

\[
u_y(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{i} c_{ij} (j-1)x^{i-j}y^{j-2},\]

\[
u_{x,y}|_{u(x,y)\in \Gamma} = \sum_{i=1}^{m} \sum_{j=1}^{i} c_{ij} [(i-j)x^{i-j-1}y^{j-1}n_x + (j-1)x^{i-j}y^{j-2}n_y],\]

\[
\Delta u(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{i} c_{ij} [(i-j)(i-j-1)x^{i-j-2}y^{j-1} + (j-1)(j-2)x^{i-j}y^{j-3}].
\]

Inserting these equations into Equations (1)–(3), and selecting \( n_1 \) and \( n_2 \) collocation points on the boundary and in the area, to satisfy the boundary condition and the field equation, respectively, we can acquire a system of NAEs to resolve the \( n \) coefficients \( c_{ij} \).

### 3. A Multi-scale Pascal Triangle

Because \( x \) and \( y \) in the problem domain \( \Omega \) may be an arbitrarily large quantity, the above expansion would result in a divergence of the powers \( x^m \) and \( y^m \). To attain the accurate solution of MBE using the modified Pascal triangle polynomial expansion approach, we have to develop more accurate and effective algorithm to solve these NAEs by reducing the condition numbers. We ponder a new multiple-scale Pascal triangle expansion of \( u(x, y) \) by:

\[
u(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{i} c_{ij} s_{ij} x^{i-j} y^{j-1},\]

in which the scales \( s_{ij} \) are determined below.

The coefficients \( c_{ij} \) employed in the expansion (6) can be displayed as an \( n \)-dimensional vector \( c \) with components \( c_k, k = 1, \ldots, n \) by:

\[
k = 0 \\
\text{Do } i = 1, m \\
\text{Do } j = 1, i \\
k = k + 1 \\
c_k = c_{ij} \\
\text{Enddo.}
\]

Then, for a generic point \((x, y)\in \Omega\), the term \( u(x, y) \) can be demonstrated as an inner product of a vector \( a \) with \( c \), i.e.:

\[
u(x, y) = [1 \ x \ y \ x^2 \ xy \ y^2 \ x^3 \ x^2y \ xy^2 \ y^3 \ ...] \ [c_1 \ c_2 \ c_3 \ ... \ c_n]
\]

in which the components \( d_k \) are in the form of \( d_k = (i-j)(i-j-1)x^{i-j-2}y^{j-1} + (j-1)(j-2)x^{i-j}y^{j-3} \).

Then, while we select \( n_1 \) points \((x_i, y_i), i = 1, \ldots, n_1 \) on the boundary \( \Gamma \) to gratify the boundary condition, and \( n_2 \) points \((x_j, y_j), i = 1, \ldots, n_2 \) on the area \( \Omega \) to satisfy the field equation. For instance, for the Laplace equation we have:

\[
A = \begin{bmatrix}
a_1^T \\
\vdots \\
da_{n_1}^T \\
d_1^T \\
\vdots \\
d_{n_2}^T
\end{bmatrix},
\]

\[
b = \begin{bmatrix}
H(x_1, y_1) \\
\vdots \\
H(x_n, y_n) \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]

We can resolve a normal linear system instead of \( Ac = b \):

\[
Dc = b,
\]

where

\[
b_1 = A^Tb,
\]

\[
D = A^T A > 0.
\]
We can use the conjugate gradient method (CGM) to solve Equation (15).

If we inquire the norm of each column of the coefficient matrix of \( A \) is equal, the multiple-scale \( s_{ij} \) is equal to \( \|c_i\|/\|c_j\| \), where \( s_{11} = 1 \) and \( c_k \) indicates the \( k \)th column of \( A \) in Equation (16). Such that in the new system:

\[
Bc = b
\]  
the \( n \) column norms of the new coefficient matrix \( B \) are equal.

Let \( D_k = s_{ij} \), we can introduce a postconditioning matrix:

\[
P_k = \text{diag}(D_1, \ldots, D_n),
\]

such that the above equilibrated multiple-scale technique is equivalent to derive the new coefficient \( B \) by:

\[
B = AP_k
\]

4. The Iterative Process for Steady-state MBE

At the beginning, we give \( c_{ij} = c_{ij}^0 \) and \( (u, u_x, u_y) \) are calculated by:

\[
\begin{align*}
    u(x, y) &= \sum_{i=1}^{m} \sum_{j=1}^{i-1} c_{ij} s_{ij} x^{i-j} y^{i-j}, \\
    u_x(x, y) &= \sum_{i=1}^{m} \sum_{j=1}^{i-1} c_{ij} (i-j) s_{ij} x^{i-j-1} y^{i-j-1}, \\
    u_y(x, y) &= \sum_{i=1}^{m} \sum_{j=1}^{i-1} c_{ij} (j-1) s_{ij} x^{i-j} y^{i-j-2},
\end{align*}
\]

Then, upon collocating \( n_{ij} \) points to satisfy the boundary conditions (2) and (3) and the field equation \( Ac = b \), we

Figure 1. The geometry configuration of the MBE in transport problem is shown in the peanut shape.

Figure 2. The exact solutions for the nonlinear steady-state modified Burgers’ equation with peanut-shaped domain and \( Re = 100 \) are shown in (a), in (b) the proposed approach solution without random noise effect, and in (c) the current scheme solution with random noise.
can derive a coefficient matrix $A$ from the left-hand side of field equation with the aid of Equations (7)–(10), and then by utilizing the multiple-scale $s_{ij}$, thus $P$ via Equation (18). Meanwhile a tentative right-hand side $b_0$ is attained from the right-hand side of Equation (1) by inserting Equations (20)–(22). Therefore, we have a linear system with different right-hand side:

$$\textbf{APc} = b_k,$$

whose normal form is solved by the CGM to generate a new coefficients $c_{ij}^{k+1}$. We repeat this process until convergence.

The numerical processes of this proposed scheme used to solve steady-state MBE are summarized the following steps.

1. Give $n_1, n_2 (n_k = n_1 \times n_2)$, and $m [n = m(m + 1)/2]$.
2. Give $c_{ij}^{0}$.
3. Give collocation points $(x_i, y_j)$, $i = 1, \ldots, n_k$.
4. (IV) Only one-time:
   - Generate $s_{ij}$, $s_{ij} = \|c_1\|/\|c_0\|$.
   - Generate $P$ from $D_k = s_j$ and Equation (18):
     $$\textbf{B} = \textbf{AP}.$$  

(23)

5. For $k = 0, 1, 2, \ldots$, we repeat the iterations as follows:
   - Compute $u, u_x, u_y$ from Equations (20)–(22).
   - Generate $b_k$ from the right-hand side of Equation (1):
   - Solve the normal form of $\textbf{Bc} = b_k$ to obtain $c_{ij}^{k+1}$.  

If $c_{ij}^{k}$ converges in accordance with a given stopping criterion

$$\sum_{i=1}^{m} \sum_{j=1}^{i} (c_{ij}^{k+1} - c_{ij}^{k})^2 < \varepsilon,$$

which is named relative distance of coefficients, then stop; otherwise, go to step (iv).

5. Numerical Example

We can estimate the stability by increasing the different levels of random noise in the boundary data:

$$\hat{u}_B = u_B + s[2R(i) - 1],$$

where $u_B$ are the boundary data. We employ the function RANDOM_NUMBER given in Fortran to generate the noisy data $R(i)$, which are random numbers in $[−1, 1]$, and $s$ denotes the level of absolute noise. Then, the boundary noisy data $\hat{u}_B$ are utilized in the calculations.

Chang [9] utilized the multiple-scale Pascal triangle method to solve a quasi-linear steady-state Burgers’ model equation and obtained good results. In this study, we consider a nonlinear steady-state MBE in a peanut-shaped domain:

$$\nu \Delta u = u^2 (u_x + u_y)$$

$$+ e^x \sin \left(\frac{\pi y}{2}\right) \left\{ \nu \left(1 - \frac{\pi^2}{4}\right) - e^{2x} \sin^2 \left(\frac{\pi y}{2}\right) \right\},$$

$$\rho(\theta) = 0.3 \sqrt{\cos 2\theta + \sqrt{1.1 - \sin^2 2\theta}},$$

and the closed-form solution is:

$$u = e^x \sin \left(\frac{\pi y}{2}\right).$$

Under the Dirichlet boundary condition, for the peanut-shaped domain is shown in Figure 1, under the

![Figure 3. The numerical errors of Re = 100 without and with random noise effect are displayed in (a) and (b), respectively.](image-url)
following parameters: $n_1 = 32$, $n_2 = 2$ ($n_k = 64$), $m = 3$, $p = 2$, Reynolds number $Re = v^{-1} = 100, 200, 400$, and $n = 6$, we utilize the multiple-scale expansion approach with the CGM under the convergence criterion $\varepsilon_i = 10^{-3}$. Under $\varepsilon_2 = 10^{-3}$, the iterative process is convergence with 4, 4, and 5 iterations, respectively, and the results shown in Figure 2(a) and (b), and the numerical error is plotted in Figure 3(a), whose maximum error is $1.07 \times 10^{-2}$, $1.62 \times 10^{-2}$, and $2.98 \times 10^{-2}$, respectively. Note that the proposed algorithm is highly efficient, and it can provide the accurate solution. Furthermore, to the authors’ best knowledge, there has no report that numerical schemes can calculate this problem very well as our method.

When the boundary data are disturbed by random noise, we are interested in the stability of this approach. Under the following parameters: $n_1 = 55$, $n_2 = 2$ ($n_k = 110$), $m = 3$, $Re = 100$, $s = 0.05$, and $n = 6$, we utilize the proposed scheme with the CGM under the same convergence criterion, and the iterations are 100 and the results as displayed in Figure 2(c), and the numerical error is drawn in Figure 3(b), whose maximum error is $1.51 \times 10^{-2}$. Note that the proposed scheme still can obtain accurate result when the imposed noise is large up to over 15%.

### 6. Conclusion

We addressed a new meshfree approach to solve the forward issues for the steady-state MBEs defined in peanut-shaped domain. We reveal that the proposed method is applicable to the two-dimensional steady-state MBEs and very good computational efficient based on the numerical experiment, and even for adding the large random noise up to over 15%. The numerical errors of our method are in the order of $O(10^{-2})$. The present scheme can be extended to resolve the three-dimensional steady-state nonlinear elliptic PDEs and practical engineering problems in the future.

### Nomenclature

- $A$: coefficient matrix
- $c_{ij}$: expansion coefficient
- $c$: $n$-dimensional vector of coefficients
- $d$: a vector
- $F$: a given function
- $G$: a given function
- $H$: a given function
- $m$: the number of terms
- $n_1$: collocation points on the boundary $\Gamma$
- $n_2$: collocation points on the domain $\Omega$
- $p$: an integer
- $P$: $\text{diag}(D_1, \ldots, D_n)$
- $R(i)$: random numbers
- $Re$: Reynolds number
- $s$: noise level
- $u$: wave distribution
- $u_B$: the boundary data
- $\hat{u}_B$: the boundary noisy data
- $x$: space variable
- $y$: space variable

### Greek symbols

- $\Delta$: the Laplacian operator
- $\Omega$: the problem domain
- $\Gamma$: the boundary
- $\varepsilon$: convergence criterion
- $\rho(\theta)$: the contour of the domain

### Subscripts and superscripts

- $i$: index
- $j$: index
- $k$: index
- $T$: transpose

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