IGPR connectedness on intuitionistic topological spaces

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Abstract

The aim of this paper is to show the existence of various types of Igpr connectedness in the intuitionistic topological spaces. Also some characterisations concerning the Igpr connectedness and Igpr connected sets were studied and compared.

Keywords: Intuitionistic generalized pre regular closed sets, Igpr \( C_1 \)-connectedness \((i = 1, 2, 5, s)\), super connectedness, strongly Igpr connectedness.

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1. Introduction

The concept of intuitionistic sets and intuitionistic topological spaces(also named as intuitionistic fuzzy special topological spaces) was first introduced by Çoker [2]. He studied some properties of connectedness, compactness, continuity and separation axioms in intuitionistic topological spaces. Later, C. Duraisamy et al. [5] studied some weakly open functions in intuitionistic topological spaces. Since the number of intuitionistic subsets of a set having ’n’ elements is \(3^n\), in this paper we introduce and studied few properties of various connectedness in intuitionistic topological spaces namely Igpr connectedness, Igpr \( C_1 \)-connectedness, Igpr \( C_2 \)-connectedness, Igpr \( C_5 \)-connectedness, Igpr \( C_s \)-connectedness and strongly Igpr connectedness. Also we have compared the connectedness of Igpr open sets with intuitionistic open sets.

2. Preliminaries

We recall some definitions and results which are useful for this sequel. Throughout the present study, a space \(X\) means an intuitionistic topological space \((X, \tau)\) and \(Y\) means an intuitionistic topological space \((Y, \sigma)\) unless otherwise mentioned.
**Definition 2.1.** [2] Let $X$ be a non empty set. An intuitionistic set (IS, for short) $A$ is an object having the form $A = <X, A_1, A_2>$, where $A_1$ and $A_2$ are subsets of $X$ satisfying $A_1 \cap A_2 = \phi$. The set $A_1$ is called the set of members of $A$, while $A_2$ is called the set of non-members of $A$.

**Definition 2.2.** [2] Let $X$ be a non empty set and let $A$, $B$ are intuitionistic sets in the form $A = <X, A_1, A_2>$, $B = <X, B_1, B_2>$, respectively. Then

1. $A \subseteq B$ iff $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$.
2. $A = B$ iff $A \subseteq B$ and $B \subseteq A$.
3. $\overline{A} = <X, A_2, A_1>$.
4. $A - B = A \cap \overline{B}$.
5. $\emptyset = <X, \phi, X>$, $X = <X, X, \phi>$.
6. $A \cup B = <X, A_1 \cup B_1, A_2 \cap B_2>$.
7. $A \cap B = <X, A_1 \cap B_1, A_2 \cup B_2>$.

Furthermore, let \{$A_i : i \in J$\} be an arbitrary family of intuitionistic sets in $X$, where $A_i = <X, A_i^{(1)}, A_i^{(2)}>$. Then:

8. $\bigcap A_i = <X, \bigcap A_i^{(1)}, \bigcup A_i^{(2)}>$.
9. $\bigcup A_i = <X, \bigcup A_i^{(1)}, \bigcap A_i^{(2)}>$.

**Definition 2.3.** An intuitionistic topology (IT, for short) on a non empty set $X$ is a family $\tau$ of IS’s in $X$ containing $\emptyset$, $X$ and closed under finite infima and arbitrary suprema. The pair $(X, \tau)$ is called an intuitionistic topological space (ITS, for short). Any intuitionistic set in $\tau$ is known as an intuitionistic open set (IOS, for short) in $X$ and the complement of IOS is called intuitionistic closed set (ICS, for short).

**Definition 2.4.** [2] Let $(X, \tau)$ be an ITS and $A = <X, A_1, A_2>$ be an IS in $X$. Then the interior and closure of $A$ are defined as:

$$Icl(A) = \bigcap \{K : K \text{ is an ICS in } X \text{ and } A \subseteq K\},$$

$$Iint(A) = \bigcup \{G : G \text{ is an IOS in } X \text{ and } G \subseteq A\}.$$

It can be shown that $Icl(A)$ is an ICS and $Iint(A)$ is an IOS in $X$ and $A$ is an ICS in $X$ iff $Icl(A) = A$ and is an IOS in $X$ iff $Iint(A) = A$.

**Definition 2.5.** [2] Let $X$ be a non empty set and $a \in X$. Then the IS $\underline{a}$ defined by $\underline{a} = <X, \{a\}, \{a\}^c>$ is called an intuitionistic point (IP, for short) in $X$. The intuitionistic point $\underline{a}$ is said to be contained in $A = <X, A_1, A_2>$ (i.e., $\underline{a} \in A$) if and only if $a \in A_1$.

**Definition 2.6.** [2] Let $X$ be a non empty set and $a \in X$. Then the IS $\overline{a}$ defined by $\overline{a} = <X, \phi, \{a\}^c>$ is called the vanishing intuitionistic point (VIP, for short) in $X$. The VIP $\overline{a}$ is said to be contained in $A = <X, A_1, A_2>$ (i.e., $\overline{a} \in A$) if and only if $a \notin A_2$.

**Definition 2.7.** Let $(X, \tau)$ be an ITS. An intuitionistic set $A$ of $X$ is said to be intuitionistic regular open (intuitionistic regular closed) if $A = Iint(Icl(A))$ ($A = Icl(Iint(A))$).

**Definition 2.8.** [7] Let $(X, \tau)$ be an ITS and let $A = <X, A_1, A_2>$ be an intuitionistic set. Then $A$ is said to be intuitionistic generalized pre regular closed (Igpr-closed) if $Ipcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is intuitionistic regular open in $X$. The class of all Igpr-closed subsets of $(X, \tau)$ is denoted by IGPRC(\tau).

The complement of intuitionistic generalized pre regular closed sets are intuitionistic generalized pre regular open (Igpr-open) and the class of all Igpr-open subsets of $(X, \tau)$ is denoted by IGPRO(\tau).
Definition 2.9. [2] Let $X$, $Y$ be two non empty sets and $f : X \to Y$ be a function.

(a) If $B = \langle X, B_1, B_2 \rangle$ is an IS in $Y$, then the preimage of $B$ under $f$, denoted by $f^{-1}(B)$, is the IS in $X$ defined by $f^{-1}(B) = \langle X, f^{-1}(B_1), f^{-1}(B_2) \rangle$.

(b) If $A = \langle X, A_1, A_2 \rangle$ is an IS in $X$, then the image of $A$ under $f$, denoted by $f(A)$, is the IS in $Y$ defined by $f(A) = \langle Y, f(A_1), f(A_2) \rangle$, where $f(A_2) = (f(A_2'))^c$.

Definition 2.10. [2] Let $(X, \tau)$ and $(Y, \sigma)$ be two intuitionistic topological spaces and $f : X \to Y$ be a function. Then $f$ is said to be continuous iff the preimage of each ICS in $Y$ is intuitionistic closed in $X$.

Corollary 2.11. [3, 4] Let $A, A_i (i \in J)$ be IS’s in $X$, $B, B_j (j \in K)$ be IS’s in $Y$ and $f : (X, \tau) \to (Y, \sigma)$ be a function. Then:

1. $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$, $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$.
2. $A \subseteq f^{-1}(f(A))$ and if $f$ is injective, then $A = f^{-1}(f(A))$.
3. $f(f^{-1}(B)) \subseteq B$ and if $f$ is surjective then $f(f^{-1}(B)) = B$.
4. $f(\bigcup A_i) = \bigcup f(A_i)$, $f(\bigcap A_i) \subseteq \bigcap f(A_i)$ and if $f$ is injective then $f(\bigcap A_i) = f(\bigcap A_i)$.
5. $f^{-1}(\bigcup B_i) = \bigcup f^{-1}(B_i)$, $f^{-1}(\bigcap B_i) \subseteq \bigcap f^{-1}(B_i)$.
6. If $f$ is surjective, then $\overline{f(A)} \subseteq f(\overline{A})$. Further if $f$ is injective then $\overline{f(A)} = f(\overline{A})$.
7. $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

Proposition 2.12. [3] The following statements are equivalent:

1. $f : (X, \tau) \to (Y, \sigma)$ is continuous.
2. $f^{-1}(\text{int}(B)) \subseteq \text{int}(f^{-1}(B))$ for each IS $B$ in $Y$.
3. $\text{cl}(f^{-1}(\text{int}(B))) \subseteq f^{-1}(\text{cl}(B))$ for each IS $B$ in $Y$.

Definition 2.13. [7] Let $(X, \tau)$ be an intuitionistic topological space. Then $X$ is called disconnected if there exists an open sets $A \neq \emptyset$ and $B \neq \emptyset$ such that $A \cup B = X$ and $A \cap B = \emptyset$. $X$ is called connected, if $X$ is not disconnected.

Definition 2.14. [9] An intuitionistic topological space $X$ is called $C_5$-disconnected if there exists an intuitionistic set $A$ which is both open and closed such that $\overline{\emptyset} \neq A \neq \overline{X}$. $X$ is called $C_5$-connected, if $X$ is not $C_5$-disconnected.

Definition 2.15. [7] An intuitionistic topological space $(X, \tau)$ is said to be strongly connected, if there exists no non-empty closed sets $A$ and $B$ in $X$ such that $A \cap B = \emptyset$.

3. Igpr connectedness in intuitionistic topological spaces

Definition 3.1. An intuitionistic topological space $X$ is called Igpr-disconnected if there exists an Igpr open sets $A \neq \emptyset$ and $B \neq \emptyset$ such that $A \cup B = X$ and $A \cap B = \emptyset$. $X$ is called Igpr-connected, if $X$ is not Igpr-disconnected.

Definition 3.2. An intuitionistic topological space $X$ is called Igpr $C_5$-disconnected if there exists an intuitionistic set $A$ which is both Igpr open and Igpr closed such that $\overline{\emptyset} \neq A \neq \overline{X}$. $X$ is called Igpr $C_5$-connected, if $X$ is not Igpr $C_5$-disconnected.

Proposition 3.3. [9] Every $C_5$-connected space is connected but the converse is not true.

Proposition 3.4. Every Igpr $C_5$-connected space is $C_5$-connected.

Proof. Let $(X, \tau)$ be an Igpr $C_5$-connected space and suppose that $(X, \tau)$ is not $C_5$-connected. Then there exists a proper intuitionistic set $A$ such that $A$ is both intuitionistic open and intuitionistic closed. Since every intuitionistic open set is Igpr-open and every intuitionistic closed set is Igpr-closed, $X$ is not Igpr $C_5$-connected which is a contradiction. Hence every Igpr $C_5$-connected space is $C_5$-connected. \qed
Remark 3.5. Every $C_5$-connected space need not be Igpr $C_5$-connected which is seen from the following example.

Example 3.6. Let $X = \{a, b, c\}$ and $\tau = \{\phi, A, B, C, D, E, F, X\}$ where $A = X, \phi, \{a, b\} >$, $B = X, \{c\}, \{a, b\} >$, $C = X, \phi, \{b, c\} >$, $D = X, \{c\}, \{b\} >$, $E = X, \{a, c\}, \{b\} >$ and $F = X, \phi, \{b\} >$ which is $C_5$-connected but not Igpr $C_5$-connected.

**Proposition 3.7.** Every Igpr-connected space is connected.

**Proof.** Since every intuitionistic open set is Igpr open, the result follows. □

Remark 3.8. Every connected space need not be Igpr-connected.

Example 3.9. Let $X$ be the set of all positive integers. Consider the intuitionistic sets given by $A_1 = X, \{2, 3, \ldots\}, \phi >$, $A_2 = X, \{3, 4, \ldots\}, \{1\} >$, $A_3 = X, \{4, 5, \ldots\}, \{1, 2\} >$, etc $A_n = X, \{n + 1, n + 2, \ldots\}, \{1, 2, 3, \ldots, n - 1\} >, n \geq 2$. Then $\tau = \{X, \phi\} \cup \{A_n : n = 1, 2, 3, \ldots\}$ is an intuitionistic topological space which is connected but not Igpr connected.

**Proposition 3.10.** Every Igpr $C_5$-connected space is Igpr-connected.

**Proof.** Let $X$ be Igpr disconnected. Then there exist nonempty disjoint Igpr-open sets $A$ and $B$ in $X$ such that $A \cap B = X, A \cap B_2 = \phi, A_1 \cap B_1 = \phi$ and $A_2 \cap B_2 = X$ which implies $A = B^c$. Hence $A$ is both Igpr-open and Igpr-closed. So, $A$ is Igpr $C_5$-disconnected. □

\[
\begin{array}{c}
\text{Igpr C}_5\text{-connected} \longrightarrow \text{Igpr Connected} \\
\downarrow \quad \quad \downarrow \\
\text{C}_5\text{-connected} \longrightarrow \text{connected}
\end{array}
\]

But none of the reverse implication is true.

**Proposition 3.11.** If $IPO(X) = IPC(X)$, then $IGPRC(X) = \mathbb{P}(X)$ where $\mathbb{P}(X)$ is the set of all intuitionistic subsets of $X$.

**Proposition 3.12.** If $X$ is an intuitionistic topological space with at least two points and if $IPO(X) = IPC(X)$, then $X$ is not Igpr $C_5$-connected.

**Proof.** By Proposition 3.11, there exists a proper intuitionistic subset of $(X, \tau)$ which is both Igpr open and Igpr closed. Hence $(X, \tau)$ is not Igpr $C_5$-connected. □

**Proposition 3.13.** An intuitionistic topological space $(X, \tau)$ is Igpr $C_5$-connected if and only if there exist no nonempty Igpr open set $A$ and $B$ in $X$ such that $A = B^c$.

**Proof.** Necessity: Suppose $A$ and $B$ are Igpr open sets such that $A \neq \phi \neq B$ and $A = B^c$. Since $A = B^c$ and $B$ is an Igpr open set implies $B^c = A$ is Igpr closed and $B \neq \phi$. Also $B^c \neq X$ implies $A \neq X$. Hence there exist a proper intuitionistic Igpr open set $A (A \neq \phi \neq X)$ such that $A$ is both Igpr open and Igpr closed. But this is a contradiction to the fact that $X$ is Igpr $C_5$-connected.

Sufficiency: Let $(X, \tau)$ be an ITS and Igpr $C_5$-disconnected then there exists an intuitionistic set $A$ which is both Igpr open and Igpr closed in $X$ such that $\phi \neq A \neq X$. Let $B = A^c$. In this case $B$ is an Igpr open set and since $A \neq X$ which implies $B = A^c \neq \phi$. Hence $A \neq \phi$ which is a contradiction that there exists no nonempty intuitionistic proper subset of $X$ which is both Igpr open and Igpr closed. Therefore $X$ is Igpr $C_5$-connected. □
Proposition 3.14. An intuitionistic topological space \((X, \tau)\) is Igpr \(C_5\)-connected if and only if there exist no nonempty intuitionistic sets \(A\) and \(B\) in \(X\) such that \(B = (\text{Igprcl}(A))^c\), \(B = A^c\) and \(A = (\text{Igprcl}(B))^c\).

Proof. Necessity: Let \(A\) and \(B\) be intuitionistic sets such that \(A \neq \phi \neq B\), \(B = A^c\), \(B = (\text{Igprcl}(A))^c\), and \(A = (\text{Igprcl}(B))^c\). Then \(A\) and \(B\) are Igpr closed. Since \(A = B^c\), \(A\) and \(B\) are also Igpr open in \(X\), which is a contradiction.

Sufficiency: Let \((X, \tau)\) be Igpr \(C_5\)-disconnected, then there exist an intuitionistic set \(A\) which is both Igpr open and Igpr closed in \(X\) such that \(\phi \neq A \neq X\). Let \(B = A^c\), we obtain a contradiction as similar to that of above Proposition.

Definition 3.15. Let \((X, \tau)\) and \((Y, \sigma)\) be two intuitionistic topological spaces and \(f : X \rightarrow Y\) be a function. Then \(f\) is said to be Igpr continuous if the preimage of every intuitionistic closed set of \(Y\) is Igpr closed in \(X\).

Definition 3.16. Let \((X, \tau)\) and \((Y, \sigma)\) be two intuitionistic topological spaces and \(f : X \rightarrow Y\) be a function. Then \(f\) is said to be Igpr irresolute if the preimage of every Igpr closed set of \(Y\) is Igpr closed in \(X\).

Proposition 3.17. Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be an Igpr continuous surjective function. If \(X\) is Igpr-connected, then \(Y\) is connected.

Proof. Let \(Y\) be disconnected. Then there exist intuitionistic open sets \(C \neq \phi\), \(D \neq \phi\) in \(Y\) such that \(C \cup D = Y\), \(C \cap D = \phi\). Since \(f\) is Igpr continuous, there exists an Igpr open sets \(A = f^{-1}(C)\) and \(B = f^{-1}(D)\) in \(X\). But \(C \neq \phi\) implies \(A = f^{-1}(C) \neq \phi\) and \(D \neq \phi\) implies \(B = f^{-1}(D) \neq \phi\). Now \(C \cup D = Y\) implies \(f^{-1}(C) \cup f^{-1}(D) = f^{-1}(Y) = X\Rightarrow A \cup B = X\) and \(C \cap D = \phi\) implies \(f^{-1}(C) \cap f^{-1}(D) = f^{-1}(\phi) = \phi\Rightarrow A \cap B = \phi\). So \(X\) is Igpr-disconnected which is a contradiction to our hypothesis. Therefore \(Y\) is connected.

Proposition 3.18. Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be an Igpr irresolute and onto and \(X\) is Igpr-connected, then \(Y\) is also Igpr connected.

Proof. Suppose \(Y\) be Igpr disconnected. Then \(Y = A \cup B\) and \(A \cap B = \phi\), where \(A, B \neq \phi\) and Igpr open in \(Y\). Since \(f\) is Igpr irresolute and onto, \(X = f^{-1}(A) \cup f^{-1}(B)\) is a disjoint non-empty open subset of \(X\). This contradicts the fact that \(X\) is Igpr connected. Hence \(Y\) is Igpr connected.

Proposition 3.19. Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be an Igpr irresolute surjection and \(X\) is Igpr \(C_5\)-connected, then \(Y\) is also Igpr \(C_5\)-connected.

Proof. Let \(Y\) be Igpr \(C_5\) disconnected. Then there exist a proper intuitionistic set \(A\) of \(Y\) which is both Igpr open and Igpr closed. Since \(f\) is Igpr irresolute and surjective, \(f^{-1}(A)\) is also a proper intuitionistic set of \(X\) which is both Igpr open and Igpr closed. Hence \(X\) is not Igpr \(C_5\) connected which is a contradiction. Hence \(Y\) is Igpr \(C_5\) connected.

Definition 3.20. [9] If there exists an intuitionistic regular open set \(A\) in \(X\) such that \(\phi \neq A \neq X\), then \(X\) is called super disconnected. \(X\) is called super connected, if \(X\) is not super disconnected.

Example 3.21. Consider the ITS \(X = \{a, b\}\) and \(\tau = \{\phi, X, < X, \{a\}, \{b\} >\}\). Then the intuitionistic topological space \(X\) is super connected.

Proposition 3.22. If \(X\) is super connected, then every intuitionistic subset of \((X, \tau)\) is Igpr closed.

Proof. If \(X\) is super connected, then there does not exist an intuitionistic regular open set \(A\) in \(X\) such that \(\phi \neq A \neq X\). So the only intuitionistic regular open sets are \(\phi\) and \(X\). So \(\text{Ipcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is intuitionistic regular open. Hence \((X, \tau)\) is Igpr closed.
Proposition 3.23. If $(X, \tau)$ is super connected, then $X$ is $\text{Igpr}C_5$-disconnected.

Proof. By Proposition 3.22, if $(X, \tau)$ is super connected then every intuitionistic subset of $X$ is $\text{Igpr}$ closed. Hence there exists an intuitionistic set $A \neq \phi \neq X$ which is both $\text{Igpr}$ open and $\text{Igpr}$ closed. So $X$ is $\text{Igpr}C_5$-disconnected.

Remark 3.24. $\text{Igpr}C_5$-disconnectedness does not implies super connectedness.

Example 3.25. Consider the ITS $X = \{a, b\}$ and $\tau = \{\phi, X, < X, \{a\}, \phi >, < X, \{a\}, \{b\} >, < X, \phi, \{b\} >\}$ which is $\text{Igpr}C_5$-disconnected but it is not super connected.

Proposition 3.26. If $(X, \tau)$ is super connected, then $X$ is $\text{Igpr}$ disconnected.

Proof. Since $(X, \tau)$ is super connected then every intuitionistic subset of $X$ is $\text{Igpr}$ closed. So there exists $\text{Igpr}$ open set $A \neq \phi$, $B \neq \phi$ such that $A \cup B = X$ and $A \cap B = \phi$. Hence $X$ is $\text{Igpr}$ disconnected.

Remark 3.27. $\text{Igpr}$ disconnectedness does not implies super connectedness.

Example 3.28. Consider the Example 3.25 which is also $\text{Igpr}$-disconnected but not super connected. Hence,

\[
\begin{array}{ccc}
\text{super connected} & \rightarrow & C_5 \text{ connected} \\
\downarrow & & \downarrow \\
\text{Igpr disconnected} & \rightarrow & \text{Igpr}C_5\text{-disconnected}
\end{array}
\]

But none of the reverse implication is true.

Proposition 3.29. If $(X, \tau)$ is super connected, then there exists $\text{Igpr}$ open sets $A$ and $B$ in $X$ such that $A \neq \phi \neq B$, $B = (\text{Igprcl}(A))^c$ and $A = (\text{Igprcl}(B))^c$.

Proof. Obvious.

Definition 3.30. An intuitionistic topological space $(X, \tau)$ is said to be strongly $\text{Igpr}$ connected, if there exists no non-empty $\text{Igpr}$ closed sets $A$ and $B$ in $X$ such that $A \cap B = \phi$.

Proposition 3.31. $X$ is strongly $\text{Igpr}$ connected if and only if there exists no $\text{Igpr}$ open sets $A$ and $B$ in $X$ such that $A \neq X \neq B$ and $A \cup B = X$.

Proof. Let $A$ and $B$ be $\text{Igpr}$ open sets in $X$ such that $A \neq X \neq B$ and $A \cup B = X$. If $C = A^c$ and $D = B^c$, then $C$ and $D$ becomes $\text{Igpr}$ closed sets in $X$ and as $C \neq \phi \neq D$ and $C \cap D = \phi$ which implies $X$ is strongly $\text{Igpr}$ disconnected.

The converse part can be proved similarly.

Proposition 3.32. If $(X, \tau)$ is strongly $\text{Igpr}$ connected, then it is strongly connected.

Proof. Let $X$ be strongly disconnected. Then there exists nonempty intuitionistic closed sets $A$ and $B$ such that $A \cap B = \phi$. Since every intuitionistic closed set is $\text{Igpr}$ closed, $X$ is strongly $\text{Igpr}$ disconnected.

Remark 3.33. Strongly connected does not implies strongly $\text{Igpr}$ connected.

Example 3.34. Consider the ITS $\tau$ on $X = \{a, b\}$ where $\tau = \{\phi, X, < X, \{a\}, \phi >, < X, \phi, \{b\} >, < X, \{a\}, \{b\} >\}$. Then the ITS $(X, \tau)$ is strongly connected but not strongly $\text{Igpr}$ connected.

Proposition 3.35. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an $\text{Igpr}$ irresolute surjective function. If $X$ is strongly $\text{Igpr}$-connected, then so is $Y$.
Proof. Let $Y$ be not strongly Igpr-connected. Then there exist Igpr closed sets $C$ and $D$ in $Y$ such that $C \neq \emptyset \neq D$, $C \cap D = \emptyset$. Since $f$ is Igpr irresolute, $f^{-1}(C)$ and $f^{-1}(D)$ are Igpr closed sets in $X$ and $f^{-1}(C) \cap f^{-1}(D) = \emptyset$, $f^{-1}(C) \neq \emptyset$, $f^{-1}(D) \neq \emptyset$. So $X$ is strongly Igpr disconnected, which is a contradiction. Hence $Y$ is strongly Igpr connected [ If $f^{-1}(C) = \emptyset$, then $f(f^{-1}(C)) = C \Rightarrow f(\emptyset) = C \Rightarrow \emptyset = C$, which is a contradiction].

Remark 3.36. Strongly Igpr connectedness does not implies Igpr $C_5$ connectedness and Igpr $C_5$ connectedness does not implies strongly Igpr connectedness.

4. Igpr connected sets in intuitionistic topological spaces

Definition 4.1. Let $N$ be an intuitionistic set in the ITS $(X, \tau)$. If there exists Igpr open sets $A$ and $B$ in $X$ satisfying the following properties, then $N$ is called Igpr $C_i$-disconnected $(i = 1, 2)$.

Igpr $C_1$: $N \subseteq A \cup B$, $A \cap B \subseteq N^c$, $N \cap A \neq \emptyset$, $N \cap B \neq \emptyset$.

Igpr $C_2$: $N \subseteq A \cup B$, $A \cap B \cap N = \emptyset$, $N \cap A \neq \emptyset$, $N \cap B \neq \emptyset$.

$N$ is said to be Igpr $C_i$-connected $(i = 1, 2)$ if $N$ is not Igpr $C_i$-disconnected $(i = 1, 2)$.

Example 4.2. Let $X = \{a, b\}$ and $\tau = \{\emptyset, X, < X, \{a\}, \phi >, < X, \{b\}, \phi >, < X, \{a\}, < X, \phi, \{a\} >, < X, \phi, \phi >\}$. Then the intuitionistic set $N = < X, \{a\}, \{b\} >$ is both $C_1$-connected and Igpr $C_i$-connected $(i = 1, 2)$.

Remark 4.3. If $N$ is Igpr $C_i$-connected, then $N$ is $C_i$-connected, $(i = 1, 2)$ but the converse is not true.

Example 4.4. Let $X = \{a, b\}$ and $\tau = \{\emptyset, X, < X, \{a\}, \phi >, < X, \{b\}, \phi >, < X, \phi, \{a\} >, < X, \phi, \phi >\}$. Then the intuitionistic set $N = < X, \{a\}, \phi >$ is $C_i$-connected but Igpr $C_i$-disconnected $(i = 1, 2)$.

Remark 4.5. If $N$ is Igpr $C_1$-connected, then $N$ is Igpr $C_2$-connected but the converse need not be true.

Example 4.6. Let $X = \{a, b\}$ and $\tau = \{\tilde{\phi}, A, B, C, \tilde{X}\}$ where $A = < X, \{a\}, \phi >, B = < X, \{a\}, \{b\} >$ and $C = < X, \phi, \{b\} >$ be the intuitionistic topological space. Let $N = < X, \phi, \{a\} >$ in $X$, then $N$ is Igpr $C_2$-connected but not Igpr $C_1$-connected.

Definition 4.7. The two non-empty intuitionistic sets $A$ and $B$ in $(X, \tau)$ is said to be Igpr weakly separated if Igprcl$(A) \subseteq B^c$ and Igprcl$(B) \subseteq A^c$.

Definition 4.8. An intuitionistic topological space $(X, \tau)$ is said to be Igpr $C_s$-disconnected if there exists an Igpr weakly separated non-empty sets $A$ and $B$ in $(X, \tau)$ such that $X = A \cup B$. $(X, \tau)$ is called Igpr $C_s$-connected if it is not Igpr $C_s$-disconnected.

Theorem 4.9. If the Igpr closure of the subsets of $(X, \tau)$ are Igpr closed, then the nonempty sets $A$ and $B$ are Igpr weakly separated if and only if there exists $M$, $W \in IGPRO(X)$ such that $A \subseteq M$, $B \subseteq W$, $A \subseteq W^c$ and $B \subseteq M^c$.

Proof. Necessity: Let Igprcl$(A) \subseteq B^c$ and Igprcl$(B) \subseteq A^c$ and let $W = (Igprcl(A))^c$, $M = (Igprcl(B))^c$, which are Igpr open sets in $(X, \tau)$. Then $W^c \subseteq B^c$ and $M^c \subseteq A^c$ which implies $A \subseteq M$ and $B \subseteq W$. Also $W = (Igprcl(A))^c \subseteq A^c$ which implies $A \subseteq W^c$ and $M = (Igprcl(B))^c \subseteq B^c$ which implies $B \subseteq M^c$.

 Sufficiency: Suppose $M$, $W$ be two non empty sets of IGPRO$(X)$ such that $A \subseteq M$, $B \subseteq W$, $A \subseteq W^c$ and $B \subseteq M^c$. Since the Igpr closure of intuitionistic sets of $(X, \tau)$ are Igpr closed, Igprcl$(A) \subseteq Igprcl(W^c) = W^c$ and Igprcl$(B) \subseteq Igprcl(M^c) = M^c$. This implies Igprcl$(A) \subseteq W^c \subseteq B^c$ and Igprcl$(B) \subseteq M^c \subseteq A^c$. Hence Igprcl$(A) \subseteq B^c$ and Igprcl$(B) \subseteq A^c$. So $A$ and $B$ are Igpr weakly separated. \qed
Definition 4.10. An intuitionistic set \( N \) in the ITS \((X, \tau)\) is said to be Igpr \( C_s\)-disconnected if and only if there are two non-empty Igpr weakly separated sets \( A \) and \( B \) in \((X, \tau)\) such that \( N = A \cup B \). \( N \) is called Igpr \( C_s\)-connected if \( N \) is not Igpr \( C_s\)-disconnected.

Theorem 4.11. If \( \text{Igprcl}(A) \) is Igpr closed for every intuitionistic set \( A \) in \((X, \tau)\), then \( N \) is Igpr \( C_s\)-connected if \( N \) is Igpr \( C_1\)-connected.

Proof. Let \( N \) be Igpr \( C_s\)-disconnected. Then there exists intuitionistic nonempty sets \( A \) and \( B \) such that \( N = A \cup B \). A and \( B \) are Igpr weakly separated. So \( \text{Igprcl}(A) \subseteq B^c \) and \( \text{Igprcl}(B) \subseteq A^c \). Let \( P = (\text{Igprcl}(A))^c \) and \( Q = (\text{Igprcl}(B))^c \). Then \( P \) and \( Q \) are Igpr open sets. Since \( A \) and \( B \) are Igpr weakly separated, \( \text{Igprcl}(A) \cap \text{Igprcl}(B) \subseteq \overline{B}^c \cap A^c = (A \cup B)^c = N^c \) which implies

\[
N \subseteq (\text{Igprcl}(A) \cap \text{Igprcl}(B))^c = (\text{Igprcl}(A))^c \cup (\text{Igprcl}(B))^c = P \cup Q \Rightarrow N \subseteq P \cup Q.
\]

Now

\[
P \cap Q = (\text{Igprcl}(A))^c \cap (\text{Igprcl}(B))^c = (\text{Igprcl}(A) \cup \text{Igprcl}(B))^c \subseteq (A \cup B)^c = N^c.
\]

If \( P \cap N = \emptyset \), then

\[
P \subseteq N^c \Rightarrow N \subseteq P^c \Rightarrow N \subseteq \text{Igprcl}(A) \subseteq B^c,
\]

i.e., \( A \cup B \subseteq B^c \) which is a contradiction. Hence \( P \cap N \neq \emptyset \).

Similarly \( Q \cap N \neq \emptyset \). So, \( P \) and \( Q \) are Igpr \( C_1\)-disconnected.

Remark 4.12. Igpr \( C_1\)-connected implies Igpr \( C_s\)-connected but the converse is not true.

Example 4.13. Let \( X = \{a, b\} \) and \( \tau = \{\emptyset, A, B, X\} \) where \( A = \langle X, \{a\}, \emptyset \rangle \), \( B = \langle X, \phi \{b\} \rangle \) be the intuitionistic topological space. Let \( N = \langle X, \phi \{b\} \rangle \) in \( X \) is Igpr \( C_s\)-connected but not Igpr \( C_1\)-connected.

Theorem 4.14. If \( \text{Igprcl}(A) \) is Igpr closed for every intuitionistic set \( A \) in \((X, \tau)\), then \( N \) is Igpr \( C_2\)-connected if \( N \) is Igpr \( C_s\)-connected.

Proof. Let \( N \) be Igpr \( C_s\)-connected. Suppose \( N \) is Igpr \( C_2\)-disconnected. Then by definition there exists Igpr open sets \( A \) and \( B \) such that \( N \subseteq A \cup B \), \( A \cap B \cap N = \emptyset \), \( N \cap A \neq \emptyset \), \( N \cap B \neq \emptyset \). Let \( P = N \cap A \) and \( Q = N \cap B \). Since \( N \subseteq A \cup B \), \( N = N \cap (A \cup B) = (N \cap A) \cup (N \cap B) = P \cup Q \). Let \( P \subseteq A \), \( Q \subseteq B \). Suppose \( P \not\subseteq B^c \), then \( P \cap B \neq \emptyset \) which implies \( N \cap A \cap B \neq \emptyset \) which is a contradiction. Hence \( P \subseteq B^c \).

Similarly we can prove \( Q \subseteq A^c \). By Theorem 4.9, \( P \) and \( Q \) are Igpr weakly separated. Hence \( N \) is Igpr \( C_s\)-disconnected.

Hence, for an intuitionistic set \( N \)

\[
\begin{array}{c}
\text{C}_1 \text{ connected} \\
\downarrow \\
\text{C}_2 \text{ connected}
\end{array}
\begin{array}{c}
\text{Igpr C}_1 \text{ connected} \\
\downarrow \\
\text{Igpr C}_2 \text{ connected}
\end{array}
\begin{array}{c}
\text{Igpr C}_s \text{ connected}
\end{array}
\]

Theorem 4.15. If \( N_1 \) and \( N_2 \) are intersecting Igpr \( C_1\)-connected sets, then \( N_1 \cap N_2 \) is also Igpr \( C_1\)-connected.

Proof. Let \( N_1 \cup N_2 \) be Igpr \( C_1\)-connected. Then there exists Igpr open sets \( A \) and \( B \) such that \( N_1 \cup N_2 \subseteq A \cup B \), \( A \cap B \subseteq (N_1 \cup N_2)^c \) and \( (N_1 \cup N_2) \cap A \neq \emptyset \), \( (N_1 \cup N_2) \cap B \neq \emptyset \). Suppose \( N_1 \) and \( N_2 \) are Igpr \( C_1\)-connected then \( (N_1 \cap A = \emptyset \text{ or } N_1 \cap B = \emptyset) \) and \( (N_2 \cap A = \emptyset \text{ or } N_2 \cap B = \emptyset) \). Since \( N_1 \cap N_2 \neq \emptyset \), there exists \( p \in N_1 \cap N_2 \) of the following cases.
Case (i): Let $N_1 \cap A = \emptyset$ and $N_2 \cap A = \emptyset$. Then $(N_1 \cap A) \cup (N_2 \cap A) = (N_1 \cup N_2) \cap A = \emptyset$ which is a contradiction.

Case (ii): Let $N_1 \cap A = \emptyset$ and $N_2 \cap B = \emptyset$. Then there exists $p \notin A, p \notin B$ which is impossible since $p \in N_1 \cup N_2 \subseteq A \cup B$.

Case (iii): Let $N_1 \cap B = \emptyset$ and $N_2 \cap A = \emptyset$. Then there exists $p \notin A, p \notin B$ which is impossible as above.

Case (iv): Let $N_1 \cap B = \emptyset$ and $N_2 \cap B = \emptyset$. Then $(N_1 \cap B) \cup (N_2 \cap B) = (N_1 \cup N_2) \cap B = \emptyset$ which is a contradiction. Hence $N_1$ and $N_2$ are $\text{Igpr } C_1$-disconnected.

**Theorem 4.16.** If $N_1$ and $N_2$ are intersecting $\text{Igpr } C_2$-connected sets, then $N_1 \cup N_2$ is also $\text{Igpr } C_2$-connected.

*Proof.* Similar to Theorem 4.15.

**Theorem 4.17.** Let $(N_i)_{i \in J}$ be a family of $\text{Igpr } C_1$-connected sets such that $\bigcap N_j \neq \emptyset$. Then $\bigcup N_i$ is also $\text{Igpr } C_1$-connected.

*Proof.* Let $N = \bigcup N_i$ be $\text{Igpr } C_1$-disconnected. Then there exists $\text{Igpr}$ open sets $A$ and $B$ such that $N \subseteq A \cup B$, $A \cap B \subseteq N^c$, $N \cap A \neq \emptyset$, $N \cap B \neq \emptyset$. Consider any index $i_0 \in J$. Since $N_{i_0}$ is $\text{Igpr } C_1$-connected, we have $N_{i_0} \cap A = \emptyset$ or $N_{i_0} \cap B = \emptyset$. So we have three cases.

Case (i): If $N_i \cap A = \emptyset$ for each $i \in J_1$ and $N_i \cap A = (\bigcup N_i) \cap A = (\bigcup (N_i \cap A) = \emptyset$ which is a contradiction.

Case (ii): If $N_i \cap B = \emptyset$ for each $i \in J_1$ and $N_i \cap B = (\bigcup N_i) \cap B = (\bigcup (N_i \cap B) = \emptyset$ which is a contradiction.

Case (iii): If $N_i \cap A = \emptyset$ for each $i \in J_1$ and $N_i \cap B = \emptyset$ for each $i \in J_2$ where $J = J_1 \cup J_2$ and $J_1 \neq \emptyset, J_2 \neq \emptyset$.

Since $\bigcap N_j \neq \emptyset, p \in \bigcap N_j$. In this case $p \notin A$ and $p \notin B$, which is a contradiction since $p \in N \subseteq A \cup B$.

Hence $N$ is also $\text{Igpr } C_1$-disconnected.

**Theorem 4.18.** Let $(N_i)_{i \in J}$ be a family of $\text{Igpr } C_2$-connected sets such that $\bigcap N_j \neq \emptyset$. Then $\bigcup N_i$ is also $\text{Igpr } C_2$-connected.

*Proof.* Similar to Theorem 4.17.

**Theorem 4.19.** Let $(X, \tau)$ be an intuitionistic topological space. Then

1. $p$ is $\text{Igpr } C_1$-connected
2. $p$ is $\text{Igpr } C_2$-connected

*Proof.* (1) Suppose $p$ be $\text{Igpr } C_1$-disconnected. Then there exist $\text{Igpr}$ open sets $A$ and $B$ such that $p \subseteq A \cup B$, $A \cap B \subseteq (p)^c$, $p \cap A \neq \emptyset$, $p \cap B \neq \emptyset$ where $(p)^c = (X, \{p\}^c, \{p\})$. Since $p \cap A \neq \emptyset$ and $p \cap B \neq \emptyset$, we get $p \in A$ and $p \in B$. But $A \cap B \subseteq (p)^c$ implies $A_1 \cap B_1 \subseteq (p)^c$ and $A_2 \cup B_2 \supseteq (p)^c$ which is impossible. Hence $p$ is $\text{Igpr } C_1$-connected.

(ii) Let $p$ be $\text{Igpr } C_1$-disconnected. Then there exist $\text{Igpr}$ open sets $A$ and $B$ such that $p \subseteq A \cup B$, $A \cap B \cap p = \emptyset$, $p \cap A \neq \emptyset$, $p \cap B \neq \emptyset$. Since $p \cap A \neq \emptyset$ and $p \cap B \neq \emptyset$, we get $p \in A$ and $p \in B$ which implies $p \notin A_2, p \notin B_2$.

But $A \cap B \cap p = (X, A_1, A_2) \cap (X, B_1, B_2) \cap (X, \{p\}, \{p\})$ which implies $A_2 \cup B_2 \cup \{p\}^c$ which is impossible. Hence $p$ is $\text{Igpr } C_2$-connected.
References

[1] K. T. Atanassov. Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20(1)(1986), 87-96.
[2] D. Çoker. A note on intuitionistic sets and intuitionistic points. Turkish J. Math., 20(3)(1996), 343-351.
[3] D. Çoker. An introduction to intuitionistic topological spaces. BUSEFAL, 81(2000), 51-56.
[4] D. Çoker. An introduction to intuitionistic fuzzy topological spaces. Fuzzy Sets and Systems, 88(1)(1997), 81-89.
[5] C. Duraisamy, M. Dhavamani and N. Rajesh. On intuitionistic weakly open(closed) functions. European J. Sci. Res., 57(4)(2011), 646-651.
[6] Gnanambal Y. On generalized preregular closed sets in topological spaces. Indian J. Pure. Appl. Math., 28(3)(1997), 351-360.
[7] Gnanambal Ilango and S. Selvanayaki. Generalized preregular closed sets in intuitionistic topological spaces. Int. J. Math. Arch., 5(4)(2014), 1-7.
[8] N. Levine. Generalized closed sets in topology. Rend. Circ. Mat. Palermo, 19(1970), 89-96.
[9] Selma ozcag and D. Çoker. On connectedness in intuitionistic fuzzy special topological spaces. Int. J. Math. Math. Sci., 21(1)(1998), 33-40.
[10] Selma ozcag and D. Çoker. A note on connectedness in intuitionistic fuzzy special topological spaces. Int. J. Math. Math. Sci., 23(1)(2000), 45-54.
[11] S. S. Thakur and Jyoti pandey Bajpai. On intuitionistic Fuzzy Gpr-closed sets. Fuzzy Inf. Eng., 4(2012), 425-444.
[12] Younis J. Yaseen and Asmaa G. Raouf. On generalization closed set and generalized continuity on intuitionistic topological spaces. J. Al-Anbar Univ. Pure Sci., 3(1)(2009).