Collapse of the mean curvature flow for proper complex equifocal submanifolds

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Abstract

It is known that principal orbits of Hermann type actions on a symmetric space of non-compact type are proper complex equifocal and curvature-adapted and that, conversely, irreducible curvature-adapted proper complex equifocal submanifolds of codimension greater than one in the symmetric space occur as principal orbits of Hermann type actions. In this paper, we investigate the mean curvature flows having a curvature-adapted proper complex equifocal submanifold or its focal submanifold as initial data conceptionally without use of the second-part of the above facts. Concretely the investigation is performed by investigating the mean curvature flows for the lift of the submanifold to an infinite dimensional pseudo-Hilbert space through a pseudo-Riemannian submersion.

1 Introduction

Let \( f_t \)'s \((t \in [0, T])\) be a one-parameter \( C^\infty \)-family of immersions of a manifold \( M \) into a Riemannian manifold \( N \), where \( T \) is a positive constant or \( T = \infty \). Define a map \( \tilde{f} : M \times [0, T) \to N \) by \( \tilde{f}(x, t) = f_t(x) \) \((x, t) \in M \times [0, T)\). If, for each \( t \in [0, T) \), \( \tilde{f}_*(\frac{\partial}{\partial t})(x, t) \) is the mean curvature vector of \( f_t : M \hookrightarrow N \), then \( f_t \)'s \((t \in [0, T))\) is called a mean curvature flow. In particular, if \( f_t \)'s are embeddings, then we call \( M_t := f_t(M) \)'s \((0 \in [0, T))\) a mean curvature flow. Li-Hamilton [LT] investigated the mean curvature flow having isoparametric submanifolds (or their focal submanifolds) in a Euclidean space as initial data and obtained the following facts.

Fact 1([LT]). Let \( M \) be a compact isoparametric submanifold in a Euclidean space and \( C \) be the Weyl domain of \( M \) at \( x_0 \) \((\in M)\). Then the following statements (i) and (ii) hold:

(i) The mean curvature flow \( M_t \) having \( M \) as initial data collapses to a focal submanifold \( F \) of \( M \) in finite time. If the natural fibration of \( M \) onto \( F \) is spherical, then
the mean curvature flow $M_t$ has type I singularity, that is, 
\[ \lim_{t \to T-0} \max_{v \in S^\perp M_t} \|A_v^t\|_\infty^2 (T-t) < \infty, \]
where $A_v^t$ is the shape operator of $M_t$ for $v$, $\|A_v^t\|_\infty$ is the sup norm of $A_v^t$ and $S^\perp M_t$ is the unit normal bundle of $M_t$.

(ii) For any focal submanifold $F$ of $M$, the set of all parallel submanifolds of $M$ collapsing to $F$ along the mean curvature flow is a one-parameter $C^\infty$-family.

Fact 2 ([LT]). Let $M$ and $C$ be as in Fact 1 and $\sigma$ be a stratum of dimension greater than zero of $\partial C$. Then the following statements (i) and (ii) hold:

(i) For any focal submanifold $F$ (of $M$) through $\sigma$, the mean curvature flow $F_t$ having $F$ as initial data collapses to a focal submanifold $F'$ (of $M$) through $\partial \sigma$ in finite time. If the natural fibration of $F$ onto $F'$ is spherical, then the mean curvature flow $F_t$ has type I singularity.

(ii) For any focal submanifold $F$ (of $M$) through $\partial \sigma$, the set of all focal submanifolds of $M$ through $\sigma$ collapsing to $F$ along the mean curvature flow is a one-parameter $C^\infty$-family.

Since the focal submanifold of $M$ through the only 0-dimensional stratum of $\partial C$ is a one-point set, it follows from the statement (i) of Facts 1 and 2 that $M$ collapses to a one-point set after finitely many times of collapses along the mean curvature flows.

As a generalized notion of compact isoparametric hypersurfaces in a sphere and a hyperbolic space, and a compact isoparametric submanifolds in a Euclidean space, Terng-Thorbergsson [TT] defined the notion of an equifocal submanifold in a symmetric space as a compact submanifold $M$ satisfying the following three conditions:

(i) the normal holonomy group of $M$ is trivial,

(ii) $M$ has a flat section, that is, for each $x \in M$, $\Sigma_x := \exp^\perp(T^\perp_x M)$ is totally geodesic and the induced metric on $\Sigma_x$ is flat, where $T^\perp_x M$ is the normal space of $M$ at $x$ and $\exp^\perp$ is the normal exponential map of $M$.

(iii) for each parallel normal vector field $v$ of $M$, the focal radii of $M$ along the normal geodesic $\gamma_{v_x}$ (with $\gamma_{v_x}'(0) = v_x$) are independent of the choice of $x \in M$, where $\gamma_{v_x}'(0)$ is the velocity vector of $\gamma_{v_x}$ at $0$.

The author has recently investigated the mean curvature flow having equifocal submanifolds (or their focal submanifolds) in a symmetric space of compact type as initial data and obtained the following facts.

Fact 3 ([Koi12]). Let $M$ be an equifocal submanifold in a symmetric space $G/K$ of compact type. Then the following statements (i) and (ii) hold:

(i) If $M$ is not minimal, then the mean curvature flow $M_t$ having $M$ as initial
data collapses to a focal submanifold $F$ of $M$ in finite time. Furthermore, if $M$ is irreducible, the codimension of $M$ is greater than one and if the natural fibration $M$ onto $F$ is spherical, then $M_t$ has type I singularity.

(ii) For any focal submanifold $F$ of $M$, the set of all parallel submanifolds of $M$ collapsing to $F$ along the mean curvature flow is a one-parameter $C^\infty$-family.

Fact 4([Koi12]). Let $M$ be as in Fact 3, $C$ be the image of the fundamental domain of the Coxeter group of $M$ at $x_0 (\in M)$ by the normal exponential map and $\sigma$ be a stratum of dimension greater than zero of $\partial C$ (which is a stratified space). Then the following statements (i) and (ii) hold:

(i) For any non-minimal focal submanifold $F$ of $M$ through $\sigma$, the mean curvature flow $F_t$ having $F$ as initial data collapses to a focal submanifold $F'$ of $M$ through $\partial \sigma$ in finite time. If $M$ is irreducible, the codimension of $M$ is greater than one and if the natural fibration of $F$ onto $F'$ is spherical, then the mean curvature flow $F_t$ has type I singularity.

(ii) For any focal submanifold $F$ of $M$ through $\partial \sigma$, the set of all focal submanifolds of $M$ through $\sigma$ collapsing to $F$ along the mean curvature flow is a one-parameter $C^\infty$-family.

Since focal submanifolds of $M$ through the lowest dimensional stratum of $\partial C$ are reflective (hence minimal), it follows from the statement (i) of Facts 3 and 4 that $M$ collapses to a minimal focal submanifold of $M$ after finitely many times of collapses along the mean curvature flows. Note that any isoparametric submanifold in a Euclidean space is a principal orbit of some s-representation and any irreducible equifocal submanifold of codimension greater than one in a symmetric space of compact type is a principal orbit of some Hermann action (i.e., the action of a symmetric subgroup of the isometry group of the symmetric space). The author ([Koi1]) introduced the notion of a complex equifocal submanifold in a symmetric space $G/K$ of non-compact type as a generalized notion of an equifocal submanifold in the symmetric space and he ([Koi3]) introduced the notion of a proper complex equifocal submanifold as a complex equifocal submanifold having a good complex focal structure, where ”good complex focal structure” means that the focal structure of the complexification of the submanifold at any point $x_0$ consists of infinitely many complex hyperplanes in the normal space at $x_0$ and that the group generated by the complex reflections of order two with respect to the complex hyperplanes is discrete. Let $H$ be a symmetric subgroup of $G$. The natural action of $H$ to $G/K$ is called a Hermann type action. He has recently shown the following facts.

Fact 5([Koi3],[Koi11]). (i) Principal orbits of a Hermann type action are proper
complex equifocal and curvature-adapted.

(ii) Let \( M \) be an irreducible proper complex equifocal \( C^\omega \)-submanifold of codimension greater than one in a symmetric space \( G/K \) of non-compact type. If \( M \) is curvature-adapted, then \( M \) is a principal orbit of some Hermann type action.

Here we note that \( C^\omega \) means the real analyticity. Also, he has recently shown the following fact.

**Fact 6 ([Koi11])** Let \( M \) be a proper complex equifocal \( C^\omega \)-submanifold in a symmetric space \( G/K \) of non-compact type. If \( \text{codim} \, M = \text{rank}(G/K) \) and the root system of \( G/K \) is reduced, then \( M \) is curvature-adapted.

We can show that the only curvature-adapted proper complex equifocal submanifold admitting no focal submanifold is a totally umbilic hyperbolic hypersurface \( H^n \) in an \((n + 1)\)-dimensional hyperbolic space \( H^{n+1} \). Let \( M \) be a curvature-adapted proper complex equifocal submanifold in a symmetric space \( G/K \) of non-compact type other than a hyperbolic space. Then it has recently been shown that the lowest dimensional focal submanifold is a reflective submanifold (see the proof of Theorem A in [Koi11]). Note that the infinitely many lowest dimensional focal submanifolds are possible to exist. Let \( F_l \) be one of the lowest dimensional focal submanifold. Without loss of generality, we may assume that \( eK \in F_l \). Since \( F_l \) is reflective, \( F_l^\perp := \exp^\perp(\mathfrak{t}eK F_l) \) is also reflective. Both \( F_l \) and \( F_l^\perp \) are symmetric spaces. Set \( \mathfrak{p} := T_eK(G/K) \) and \( \mathfrak{p}^\perp := T_eK F_l \). Take a maximal abelian subspace \( \mathfrak{b} \) of \( \mathfrak{p}^\perp \) and a maximal abelian subspace \( \mathfrak{a} \) of \( \mathfrak{p} \) containing \( \mathfrak{b} \). Let \( \Delta \) be the root system of \( G/K \) with respect to \( \mathfrak{a} \) and \( \Delta^\prime \) be that of \( F_l^\perp \) with respect to \( \mathfrak{b} \). Also, let \( \mathfrak{p}_\alpha \) be the root space for \( \alpha \in \Delta \). If \( \text{rank} \, F_l^\perp = \text{rank}(G/K) \), then we have \( \mathfrak{a} = \mathfrak{b} \) and \( \Delta^\prime \subset \Delta \).

In this paper, we prove the following fact for the mean curvature flow having a curvature-adapted proper complex equifocal \( C^\omega \)-submanifold or its focal submanifold as initial data.

**Theorem A.** Let \( M \) be a curvature-adapted proper complex equifocal \( C^\omega \)-submanifold in a symmetric space \( G/K \) of non-compact type other than a hyperbolic space, \( M_t \) \((0 \leq t < T)\) the mean curvature flow having \( M \) as initial data, \( \Delta, \mathfrak{p}_\alpha \) and \( \mathfrak{p}^\prime \) be as above. Assume that \( \text{codim} \, M = \text{rank}(G/K) \) and that \( \dim(\mathfrak{p}_\alpha \cap \mathfrak{p}^\prime) \geq \frac{1}{2} \dim \mathfrak{p}_\alpha (\alpha \in \Delta) \). Then the following statements (i), (ii) and (iii) hold.

(i) \( M \) is not minimal and \( M_t \) collapses to a focal submanifold of \( M \) in finite time.

(ii) If \( M_t \) collapses to a focal submanifold \( F \) of \( M \) in finite time and if the natural fibration of \( M \) onto \( F \) is spherical, then \( M_t \) has type I singularity.

(iii) For any focal submanifold \( F \) of \( M \), the set of all parallel submanifolds of \( M \)
collapsing to $F$ along the mean curvature flow is a one-parameter $C^\infty$-family.

Remark 1.1. If codim $M = \text{rank}(G/K)$ and if $F_l$ is a one-point set, then $M$ satisfies all the conditions in Theorem A. For example, principal orbits of the isotropy action of a symmetric space $G/K$ of non-compact type are curvature-adapted and proper complex equifocal submanifolds, their codimensions are equal to the rank of $G/K$ and their lowest dimensional focal submanifolds are equal to the one-point set $\{eK\}$. Hence they satisfy all the conditions in Theorem A. Also, principal orbits of Hermann type actions in Table 1 (see Section 5) satisfy all the conditions in Theorem A.

The focal set of a curvature-adapted proper complex equifocal submanifold $M$ at any point $x(\in M)$ consists of the images of finitely many (real) hyperplanes in the normal space $T_x^\perp M$ by the normal exponential map $\exp^\perp$ of $M$ and the group generated by the reflections with respect to the hyperplanes is a (finite) Coxeter group. In [Koi6], we called this group the 

Theorem B. Let $M, M_t$ and $F_l$ be as above. Assume that codim $M = \text{rank}(G/K)$ and that $F_l$ is one-point set. Let $\tilde{\sigma}$ be a stratum of dimension greater than zero of the fundamental domain $\tilde{C}$ (which is a stratified space) of the real Coxeter group of $M$. Then, the following statements (i) and (ii) hold.

(i) Any focal submanifold $F$ of $M$ through $\exp^\perp(\tilde{\sigma})$ is not minimal and the mean curvature flow $F_t$ having $F$ as initial data collapses to a focal submanifold $F'$ of $M$ through $\exp^\perp(\partial\tilde{\sigma})$ in finite time. If the natural fibration of $F$ onto $F'$ is spherical, then $F_t$ has type I singularity.

(ii) For any focal submanifold $F$ of $M$ through $\exp^\perp(\partial\tilde{\sigma})$, the set of all focal submanifolds of $M$ through $\exp(\tilde{\sigma})$ collapsing to $F$ along the mean curvature flow is a one-parameter $C^\infty$-family.

According to the statement (i) of Theorems A and B, if $M$ is a curvature-adapted proper complex equifocal $C^\omega$-submanifold, codim $M = \text{rank}(G/K)$ and if $F_l$ is one-point set, then $M$ collapses to one-point set after finitely many times of collapses
along the mean curvature flows.

\[
\begin{align*}
M_t & \rightarrow^* F^1 \\
F^1_t & \rightarrow^* F^2 \\
\quad & \cdots \\
F^{k-1}_t & \rightarrow^* \text{one point set}
\end{align*}
\]

\[
\left( \begin{array}{c}
F^1 : \text{a focal submanifold of } M \\
F^i : \text{a focal submanifold of } F^{i-1} (i = 2, \cdots, k - 1)
\end{array} \right)
\]

2 Basic notions and facts

In this section, we briefly review the notions of a proper complex equifocal submanifold in a symmetric space \(G/K\) of non-compact type and a proper complex isoparametric submanifold in an (infinite dimensional) pseudo-Hilbert space. First we recall the notion of a complex equifocal submanifold in \(G/K\). Let \(M\) be a submanifold with flat section in \(G/K\), where ”\(M\) has flat section” means that, for each \(x = gK \in M\), \(\exp \perp (T_{\perp x}gK)\) is a flat totally geodesic submanifold in \(G/K\). Denote by \(A\) the shape tensor of \(M\) and \(R\) the curvature tensor of \(G/K\). Let \(v \in T_{x}^\perp M\) and \(X \in T_x M (x = gK)\). Set \(R(v) := R(\cdot, v)v\). Denote by \(\gamma_v\) the geodesic in \(G/K\) with \(\gamma_v(0) = v\). The strongly \(M\)-Jacobi field \(Y\) along \(\gamma_v\) with \(Y(0) = X\) (hence \(Y'(0) = -A_vX\)) is given by

\[
Y(s) = (P_{\gamma_v|_{[0,s]}} \circ (\cos(s\sqrt{R(v)}) - \frac{\sin(s\sqrt{R(v)})}{\sqrt{R(v)}} \circ A_v))(X),
\]

where \(Y'(0) = \tilde{\nabla}_vY\) and \(P_{\gamma_v|_{[0,s]}}\) is the parallel translation along \(\gamma_v|_{[0,s]}\). Since \(M\) has flat section, all focal radii of \(M\) along \(\gamma_v\) are given as zero points of strongly \(M\)-Jacobi fields along \(\gamma_v\). Hence all focal radii of \(M\) along \(\gamma_v\) coincide with the zero points of the real-valued function \(F_v\) over \(\mathbb{R}\) defined by

\[
F_v(s) := \det \left( \cos(s\sqrt{R(v)}) - \frac{\sin(s\sqrt{R(v)})}{\sqrt{R(v)}} \circ A_v \right).
\]

So we defined the notion of a complex focal radius of \(M\) along \(\gamma_v\) as the zero points of the complex-valued function \(F^c_v\) over \(\mathbb{C}\) defined by

\[
F^c_v(z) := \det \left( \cos(z\sqrt{\overline{R(v)c}}) - \frac{\sin(z\sqrt{\overline{R(v)c}})}{\sqrt{\overline{R(v)c}}} \circ A^c_v \right),
\]

\hspace{1cm}
where $R(v)^c$ (resp. $A_v^c$) is the complexification of $R(v)$ (resp. $A_v$). Also, for a complex focal radius $z$ of $M$ along $\gamma_v$, we call $\text{dim}_c \text{Ker} \left( \frac{\cos(z \sqrt{R(v)^c}) - \sin(z \sqrt{R(v)^c})}{\sqrt{R(v)^c}} \circ A_v^c \right)$ the \textit{multiplicity} of the complex focal radius $z$. Here we note that, in the case where $M$ is of class $C^\infty$, complex focal radii along $\gamma_v$ indicate the positions of focal points of the extrinsic complexification $M^c(\rightarrow G^c/K^c)$ of $M$ along the complexified geodesic $\gamma_{t,v}$, where $G^c/K^c$ is the anti-Kaehlerian symmetric space associated with $G/K$ and $t$ is the natural immersion of $G/K$ into $G^c/K^c$. See [Koi2] about the definitions of $G^c/K^c$, $M^c(\rightarrow G^c/K^c)$ and $\gamma_{t,v}^c$. Furthermore, assume that the normal holonomy group of $\bar{M}$ is trivial. Let $\bar{v}$ be a parallel unit normal vector field of $M$. Assume that the number (which may be $\infty$) of distinct complex focal radii along $\gamma_{\bar{v}}$ is independent of the choice of $x \in M$. Let $\{r_{i,x} \mid i = 1, 2, \cdots \}$ be the set of all complex focal radii along $\gamma_{\bar{v}}$, where $|r_{i,x}| < |r_{i+1,x}|$ or $"|r_{i,x}| = |r_{i+1,x}| & \text{Re} r_{i,x} > \text{Re} r_{i+1,x}","$ or $"|r_{i,x}| = |r_{i+1,x}| & \text{Re} r_{i,x} = \text{Re} r_{i+1,x} & \text{Im} r_{i,x} = -\text{Im} r_{i+1,x} < 0."$. Let $r_i$ ($i = 1, 2, \cdots$) be complex valued functions on $M$ defined by assigning $r_i$ to each $x \in M$. We call these functions $r_i$ ($i = 1, 2, \cdots$) complex focal radius functions for $\bar{v}$. If, for each parallel unit normal vector field $\bar{v}$ of $M$, the number of distinct complex focal radii along $\gamma_{\bar{v}}$ is independent of the choice of $x \in M$, each complex focal radius function for $\bar{v}$ is constant on $M$ and it has constant multiplicity, then we call $M$ a complex equi focal submanifold.

Next we recall the notion of a proper complex isoparametric submanifold in an (infinite dimensional) pseudo-Hilbert space. Let $M$ be a pseudo-Riemannian submanifold of finite codimension in a pseudo-Hilbert space $(V, \langle \cdot, \cdot \rangle)$. See [K-kyu] about this definition. We call $M$ a Fredholm pseudo-Riemannian submanifold (or simply Fredholm submanifold) if there exists an orthogonal time-space decomposition $V = V_- \oplus V_+$ such that $(V, \langle \cdot, \cdot \rangle_{V_\pm})$ is a Hilbert space and that, for each $v \in T^\bot M$, $A_v$ is a compact operator with respect to $f^\ast \langle \cdot, \cdot \rangle_{V_\pm}$, where an orthogonal time-space decomposition $V = V_- \oplus V_+$ means that $\langle \cdot, \cdot \rangle_{V_- \times V_-}$ is negative definite, $\langle \cdot, \cdot \rangle_{V_- \times V_+}$ is positive definite and that $\langle \cdot, \cdot \rangle_{V_- \times V_-} = 0$, and $\langle \cdot, \cdot \rangle_{V_\pm} := -\pi_{V_-} \langle \cdot, \cdot \rangle + \pi_{V_+} \langle \cdot, \cdot \rangle$ (resp. $\pi_{V_+}$) : the orthogonal projection of $V$ onto $V_-$ (resp. $V_+$). Since $A_v$ is a compact operator with respect to $f^\ast \langle \cdot, \cdot \rangle_{V_\pm}$, for each $v \in T^\bot M$, the operator $\text{id} - A_v$ is a Fredholm operator with respect to $f^\ast \langle \cdot, \cdot \rangle_{V_\pm}$ and hence the normal exponential map $\exp^\bot : T^\bot M \rightarrow V$ of $M$ is a Fredholm map with respect to the metric of $T^\bot M$ naturally defined from $f^\ast \langle \cdot, \cdot \rangle_{V_\pm}$ and $\langle \cdot, \cdot \rangle_{V_\pm}$, where $\text{id}$ is the identity transformation of $TM$. The set of all eigenvalues of the complexification $A_v^c$ of $A_v$ is described as $\{0\} \cup \{\mu_i \mid i = 1, 2, \cdots \}$, where $|\mu_i| > |\mu_{i+1}|$ or $|\mu_i| = |\mu_{i+1}| & \text{Re} \mu_i > \text{Re} \mu_{i+1}$" or $|\mu_i| = |\mu_{i+1}| & \text{Re} \mu_i = \text{Re} \mu_{i+1} & \text{Im} \mu_i = -\text{Im} \mu_{i+1} > 0"$. We call $\mu_i$ the $i$-th complex principal curvature for $v$. Assume that the normal holonomy group $M$ is trivial. Fix a parallel normal vector field $\bar{v}$ on $M$. Assume that the number
(which may be $\infty$) of distinct complex principal curvatures of $\tilde{v}_x$ is independent of $x \in M$. Then we define functions $\tilde{\mu}_i$ ($i = 1, 2, \ldots$) on $M$ by assigning the $i$-th complex principal curvature for $\tilde{v}_x$ to each $x \in M$. We call this function $\tilde{\mu}_i$ the $i$-th complex principal curvature function for $\tilde{v}$. A Fredholm submanifold $M$ is called a complex isoparametric submanifold if the normal holonomy group of $M$ is trivial and if, for each parallel normal vector field $\tilde{v}$, the number of distinct complex principal curvatures of direction $\tilde{v}_x$ is independent of the choice of $x \in M$ and if each complex principal curvature function of direction $\tilde{v}$ is constant on $M$.

Assume that $M$ is a complex isoparametric submanifold. If, for each $v \in T^\perp xM$, the complexified shape operator $A_c^v$ is diagonalizable with respect to a a pseudo-orthonormal base of $(T_xM)^c$ ($x$ : the base point of $v$), that is, there exists a pseudo-orthonormal base of $(T_xM)^c$ consisting of the eigenvectors of $A_c^v$, then we call $M$ a proper complex isoparametric submanifold. Then, for each $x \in M$, $A_c^v$'s ($v \in T_x^\perp M$) are simultaneously diagonalizable with respect to a pseudo-orthonormal base of $(T_xM)^c$ because $A_c^v$'s commute. There exists a family $\{E_i| i \in I\}$ of parallel subbundles of $(TM)^c$ such that, for each $x \in M$, $(T_xM)^c = \oplus_{i \in I} (E_i)_x$ holds and that this decomposition is a common-eigenspace decomposition of $A_c^v$'s ($v \in T_x^\perp M$). Also, there exist smooth sections $\lambda_i$ ($i \in I$) of $(T^\perp xM)^c$ such that $A_c^v = \lambda_i(v)$id on $(E_i)_x$ for each $v \in T_x^\perp M$, where $x$ is the base point of $v$. The subbundles $E_i$ ($i \in I$) are called complex curvature distributions of $M$ and $\lambda_i$ ($i \in I$) are called complex principal curvatures of $M$. Define a complex normal vector field $n_i$ ($i \in I$) by $\lambda_i(\cdot) = \langle n_i, \cdot \rangle^c$, where $\langle \cdot, \cdot \rangle^c$ is the complexification of $\langle \cdot, \cdot \rangle$. Note that each $n_i$ is parallel with respect to the complexification $\nabla^c$ of $\nabla^\perp$. The normal vector fields $n_i$ ($i \in I$) are called complex curvature normals of $M$.

Let $G/K$ be a symmetric space of non-compact type and $\pi : G \to G/K$ be the natural projection. The parallel transport map $\phi$ for the semi-simple Lie group $G$ is defined by $\phi(u) := g_u(1)$ ($u \in H^0([0,1], g)$), where $g_u$ is the element of $H^1([0,1], G)$ with $g_u(0) = e$ ($e$ : the identity element of $G$) and $g_u^{-1} g_{u'} = u$. Here we note that $H^0([0,1], g)$ is a pseudo-Hilbert space. See [Koi1] the detail of the definition of the pseudo-Hilbert space $H^0([0,1], g)$ and $\phi$. Let $M$ be a complex equifocal submanifold in $G/K$. Since $M$ is complex equifocal, $\tilde{M} := (\pi \circ \phi)^{-1}(M)$ is complex isoparametric. In particular, if $\tilde{M}$ is proper complex isoparametric, then $M$ is called a proper complex equifocal submanifold. Let $M$ be a proper complex equifocal submanifold in a symmetric space $G/K$ of non-compact type. Denote by $A$ (resp. $\tilde{A}$) the shape tensor of $M$ (resp. $\tilde{M}$). Since $\tilde{M}$ is proper complex isoparametric, the complexified shape operators of $\tilde{M}$ are simultaneously diagonalizable with respect to a pseudo-orthonormal base. Hence the complex focal set of $\tilde{M}$ at any point $u(\in \tilde{M})$ consists of infinitely many complex hyperplanes in the complexified normal space $(T_u^\perp M)^c$ and the group generated by the complex reflections of order two with respect to
the complex hyperplanes is discrete. Also, for any unit normal vector \( v \) of \( \tilde{M} \), the nullity spaces of complex focal radii along the normal geodesic \( \gamma_v \) with \( \gamma'_v(0) = v \) span \( (T_v\tilde{M})^c \ominus (\text{Ker } \tilde{A}_v)^c \). From this fact, it follows that, for the complex focal set of the proper complex equifocal submanifold \( M \), the following fact holds:

\( (*) \) The complex focal set of \( M \) at any point \( x \in M \) consists of infinitely many complex hyperplanes in the complexified normal space \( (T^\perp x M)^c \) and the group generated by the complex reflections of order two with respect to the complex hyperplanes is discrete. Also, for any unit normal vector \( v \) of \( M \), the nullity spaces of complex focal radii along the normal geodesic \( \gamma_v \) with \( \gamma'_v(0) = v \) span \( (T_x M)^c \ominus (\text{Ker } A_v \cap \text{Ker } R(v))^c \).

Conversely, it is conjectured that, if this fact \( (*) \) holds for a complex equifocal submanifold \( M \), then it is proper complex equifocal. Here we note that, for any equifocal submanifold \( M \) in a symmetric space of compact type, the following fact similar to the above \( (*) \) holds:

\( (\ast') \) The focal set of \( M \) at any point \( x \in M \) consists of infinitely many hyperplanes in the normal space \( T^\perp x M \) and the group generated by the reflections with respect to hyperplanes is discrete. Also, for any unit normal vector \( v \) of \( M \), the nullity spaces of focal radii along the normal geodesic \( \gamma_v \) with \( \gamma'_v(0) = v \) span \( T_x M \ominus (\text{Ker } A_v \cap \text{Ker } R(v)) \).

Let \( H \) be a symmetric subgroup of \( G \) (i.e., there exists an involution of \( G \) with \( (\text{Fix } \tau)_0 \subset H \subset \text{Fix } \tau \)), where \( \text{Fix } \tau \) is the fixed point group of \( \tau \) and \( (\text{Fix } \tau)_0 \) is the identity component of \( \text{Fix } \tau \). The natural action \( H \) on \( G/K \) is called a Hermann type action. It is shown that a principal orbit of a Hermann type action is a proper complex equifocal and curvature-adapted ([Koi3]), where the curvature-adaptedness means that, for each normal vector \( v \) of \( M \), \( R(\cdot, v)v \) preserves \( T_x M \) (\( x \) : the base point of \( v \)) invariantly and that \( [R(\cdot, v)v, A_v] = 0 \) (\( R : \) the curvature tensor of \( G/K \)). Let \( P(G, H \times K) := \{ g \in H^1([0, 1], G) \mid (g(0), g(1)) \in H \times K \} \), where \( H^1([0, 1], G) \) is a pseudo-Hilbert Lie group of all \( H^1 \)-paths in \( G \) having \([0, 1]\) as the domain. See [Koi1] about the detail of the definition of \( H^1([0, 1], G) \). This group \( P(G, H \times K) \) acts on \( H^0([0, 1], g) \) as gauge action. It is shown that orbits of the \( P(G, H \times K) \)-action are the inverse images of the \( H \)-orbits by \( \pi \circ \phi \) (see [Koi2]).
3 The regularized mean curvature vector of Fredholm submanifold with proper shape operators

In this section, we shall define the regularized mean curvature vector of a certain kind of Fredholm submanifold in a pseudo-Hilbert space. Let $M$ be a Fredholm submanifold in a pseudo-Hilbert space $(V, \langle \cdot, \cdot \rangle)$. Denote by $A$ the shape tensor of $M$. Fix $v \in T_{x}^\perp M$. If the complexified shape operator $A_v^c$ is diagonalizable with respect to a pseudo-orthonormal base, then $A_v^c$ is said to be proper. If $A_v^c$ is proper for any $v \in T^\perp_{x} M$, then we say that $M$ has proper shape operators. Assume that $M$ has proper shape operators. Fix $v \in T^\perp_{x} M$. Let $\{\mu_i \mid i = 1, 2, \cdots \} (\text{"} |\mu_i| > |\mu_{i+1}| \text{"})$ or $\{\mu_i \mid |\mu_i| = |\mu_{i+1}| \text{"} & Re \mu_i > Re \mu_{i+1} \text{"} \text{ or } \text{"} |\mu_i| = |\mu_{i+1}| \text{"} & Re \mu_i = Re \mu_{i+1} & Im \mu_i = -Im \mu_{i+1} > 0 \text{"})$ be the set of eigenvalues of $A_v^c$ other than zero and $m_i$ the multiplicity of $\mu_i$. Then we define the trace $\text{Tr } A_v^c$ of $A_v^c$ by $\text{Tr } A_v^c := \sum_i m_i \mu_i$. If there exists $\text{Tr } A_v^c$ for each $v \in T^\perp_{x} M$, then we say that $M$ is regularizable. It is shown that, if $\mu$ is an eigenvalue of $A_v^c$ with multiplicity $m$, then so is also the conjugate $\bar{\mu}$ of $\mu$. Hence we have $\text{Tr } A_v^c \in \mathbb{R}$.

Define $H_x \in T^\perp_{x} M$ by $\langle H_x, v \rangle = \text{Tr } A_v^c (\forall v \in T^\perp_{x} M)$. We call the normal vector field $H : (x \mapsto H_x)$ of $M$ the regularized mean curvature vector of $M$. Let $f_t : M \hookrightarrow V (0 \leq t < T)$ be a $C^\infty$-family of regularizable Fredholm submanifolds with proper shape operators and $H_t$ be the regularized mean curvature vector of $f_t$. Define by $\tilde{f} : M \times [0, T) \to V$ by $\tilde{f}(x, t) := f_t(x) ((x, t) \in M \times [0, T))$. If $\tilde{f}_t(0 \leq t < T)$ is $H^1$-regular, then we call $f_t (0 \leq t < T)$ the mean curvature flow.

Let $G/K$ be a symmetric space of non-compact type, $\pi : G \to G/K$ be the natural projection and $\phi : H^0([0, 1], g) \to G$ be the parallel transport map for $G$. Let $M$ be a curvature-adapted proper complex equifocal $C^\omega$-submanifold in $G/K$ and set $\tilde{M} := (\pi \circ \phi)^{-1}(M)$, which is proper complex isoparametric (hence has proper shape operators). Denote by $H$ the mean curvature vector of $M$. Then we have the following fact.

**Lemma 3.1.** The submanifold $\tilde{M}$ is regularizable and the regularized mean curvature vector $H$ is equal to the horizontal lift $H^L$ of $H$.

**Proof.** Without loss of generality, we may assume that $eK \in M$. For simplicity, set $m := T_{eK}M$ and $b := T_{eK}^\perp M$. Since $M$ is flat section (hence $b$ is abelian), the normal connection of $M$ is flat and since $M$ is curvature-adapted, the operators $R(-, v)v$'s $(v \in b)$ and $A_v$'s $(v \in b)$ commute to one another. Also they are diagonalizable with respect to an orthonormal base, respectively. Therefore they are simultaneously diagonalizable with respect to an orthonormal base. Let $m = m_0^R + \sum_{i \in I^R} m_i^R$ be the
common eigenspace decomposition of $R(\cdot, v)v$’s ($v \in b$) and $m = m_0 + \sum_{i \in I} m_i^A$ be the common eigenspace decomposition of $A_v$’s ($v \in b$), where $m_0^R := \cap_{v \in b} \ker R(\cdot, v)v$ and $m_0^A := \cap \ker A_v$. Set $m_i^R := \dim m_i^R$ and $m_i^A := \dim m_i^A$. Also, set $I_i^A := \{ j \in I^A | m_j^A \cap m_i^R \neq \{0\} \}$ ($i \in I^R \cup \{0\}$). Since $R(\cdot, v)v$ ($v \in b$) and $A_v$'s ($v \in b$) are simultaneously diagonalizable, we have $m = \sum_{i \in I^R} \sum_{j \in I_i^A} (m_j^A \cap m_i^R)$. Let $\beta_i(\geq 0)$ ($i \in I^R$) and $\lambda_i$ ($i \in I^R$) be linear functions over $b$ defined by $R(\cdot, v)v|_{m_i^R} = -\beta_i(v)\text{id}$ ($v \in b$) and $A_v|_{m_i^A} = \lambda_i(v)\text{id}$ ($v \in b$). Denote by $b_r$ the set of all $v \in b$ such that $\beta_i(v) \neq 0$, $\lambda_i(v) \neq 0$, $\beta_i(v)$'s ($i \in I^R$) are mutually distinct and that so are also $\lambda_i(v)$'s ($i \in I^A$). Note that $b_r$ is open and dense in $b$. Fix $v \in b_r$. Denote by $\bar{A}$ the shape tensor of $\bar{M}$ and Spec $\bar{A}_v$ the spectrum of $\bar{A}_v$, where $v^L$ is the horizontal lift of $v$ to the constant path $\hat{0}$ at the zero vector 0 of $g$. Set $I_{i,v,+}^A := \{ j \in I_i^A | |\lambda_j(v)| > |\beta_i(v)| \}$, $I_{i,v,-}^A := \{ j \in I_i^A | |\lambda_j(v)| < |\beta_i(v)| \}$ and $I_{i,v,0}^A := \{ j \in I_i^A | |\lambda_j(v)| = |\beta_i(v)| \}$. Since $M$ is a curvature-adapted proper complex equifocal $C^w$-submanifold, we can show that $I_{i,v,0}^A = \emptyset$ (see Theorem A of [Koi1]) and that $I_{i,v,+}^A$ and $I_{i,v,-}^A$ are at most one point sets, respectively (see the proof of Theorems B and C of [Koi6]). When $I_{i,v,+}^A \neq \emptyset$ (resp. $I_{i,v,-}^A \neq \emptyset$), denote by $j_{i,v}^+$ (resp. $j_{i,v}^-$) the only element. Set $I_{i,v,+}^R := \{ i \in I^R | I_{i,v,+}^A \neq \emptyset \}$ and $I_{i,v,-}^R := \{ i \in I^R | I_{i,v,-}^A \neq \emptyset \}$. For simplicity, set $m_{i,v}^+ := m_{j_{i,v}^+} \cap m_i^R$ and $m_{i,v}^- := \dim m_{j_{i,v}^-}$. According to [Koi1], Spec $\bar{A}_v \setminus \{0\}$ is given by
\begin{equation}
\text{Spec } \bar{A}_v \setminus \{0\} = \{ \frac{\beta_i(v)}{\arctanh(\beta_i(v)/\lambda_{j_{i,v}^+}(v)) + k\pi\sqrt{-1}} | i \in I_{i,v,+}^R, k \in \mathbb{Z} \}
\cup \{ \frac{\beta_i(v)}{\arctanh(\lambda_{j_{i,v}^-}(v)/\beta_i(v)) + (k + \frac{1}{2})\pi\sqrt{-1}} | i \in I_{i,v,-}^R, k \in \mathbb{Z} \}
\end{equation}
Hence we have
\begin{align}
\text{Tr } \bar{A}_v &= \sum_{i \in I_{i,v,+}^R} \sum_{k \in \mathbb{Z}} \frac{\beta_i(v)}{\arctanh(\beta_i(v)/\lambda_{j_{i,v}^+}(v)) + k\pi\sqrt{-1}} \times m_{i,v}^+ \\
&\quad + \sum_{i \in I_{i,v,-}^R} \sum_{k \in \mathbb{Z}} \frac{\beta_i(v)}{\arctanh(\lambda_{j_{i,v}^-}(v)/\beta_i(v)) + (k + \frac{1}{2})\pi\sqrt{-1}} \times m_{i,v}^- \\
&= \sum_{i \in I_{i,v,+}^R} m_{j_{i,v}^+}(v) + \sum_{i \in I_{i,v,-}^R} m_{j_{i,v}^-}(v) \\
&= \sum_{j \in I^A} m_j \lambda_j(v) = \text{Tr } A_v \quad (\in \mathbb{R}).
\end{align}
in terms of \( \coth \theta = \sum_{j \in \mathbb{Z}} \frac{1}{\theta + j\pi \sqrt{-1}} \) and \( \coth(\theta + \frac{\pi \sqrt{-1}}{2}) = \tanh \theta \). Hence \( \tilde{M} \) is regularizable and \( \text{Tr} A_{\nu}^c = \text{Tr} A_{\nu} \). This implies \( \langle H_0, vL \rangle = \langle H_{\nu K}, v \rangle = \langle (H^L)_0, vL \rangle \).

Since this relation holds for any \( v \in \mathfrak{b}_r \) and \( \mathfrak{b}_r \) is dense in \( \mathfrak{b} \), we obtain \( \tilde{H}_0 = (H^L)_0 \).

Similarly we can show \( \tilde{H}_u = (H^L)_u \) for any \( u \in \tilde{M} \). Thus we obtain \( \tilde{H} = H^L \).

q.e.d.

By using Lemma 3.1 and imitating the proof of Lemma 3.1 of [Koi12], we can show the following fact.

**Lemma 3.2.** The mean curvature flow \( \tilde{M}_t \) (resp. \( M_t \)) having \( \tilde{M} \) (resp. \( M \)) as initial data exists in short time and \( \tilde{M}_t = (\pi \circ \phi)^{-1}(M_t) \) holds.

### 4 Proofs of Theorems A and B

Let \( M \) be a curvature-adapted proper complex equivocal \( C^\omega \)-submanifold in a symmetric space \( G/K \) of non-compact type other than a hyperbolic space. Without loss of generality, we may assume that \( eK \in M \). We use the notations in the previous section. Denote by \( \tilde{\mathcal{F}} \) the focal set of \( M \) at \( eK \). Since \( \pi \circ \phi \) is a pseudo-Riemannian submersion, the focal set \( \tilde{\mathcal{F}} \) of \( \tilde{M} \) at \( \hat{0} \) is equal to \( \{ vL | \exp(v) \in \mathcal{F} \} \), where \( \exp \) is the normal exponential map of \( M \) and \( vL \) is the horizontal lift of \( v \) to \( \hat{0} \). In the sequel, we identify \( vL \) with \( v \) through \( (\pi \circ \phi) \ast \hat{0} \). The focal set \( \tilde{\mathcal{F}} \) is equal to \( \{ v | \ker(A_v - \text{id}) \neq \{ 0 \} \} \subset \mathfrak{b} \). The complex focal structure \( \tilde{\mathcal{F}}^c \) of \( \tilde{M} \) at \( \hat{0} \) is defined by \( \tilde{\mathcal{F}}^c := \{ v | \ker(A_v^c - \text{id}) \neq \{ 0 \} \} \subset \mathfrak{b}^c \). According to the proof of Theorems B and C of [Koi6], by using (3.1) and discussing delicately, we can show that \( \beta_i(v)/\lambda_{\nu i}^c(v) \) and \( \lambda_{\nu i}^c(v)/\beta_i(v) \) are independent of the choice of \( v \) (in the sequel, we denote these constants by \( c_i^+ \) and \( c_i^- \), respectively), \( I^R_{v, +} \) and \( I^R_{v, -} \) are independent of the choice of \( v \) (in the sequel, we denote these sets by \( I^R_+ \) and \( I^R_- \)) and that \( \tilde{\mathcal{F}}^c \) is described as follows:

\[
\tilde{\mathcal{F}}^c = \bigcup_{i \in I^R_+} \bigcup_{j \in \mathbb{Z}} \beta_i^{c_{i+}^-1}(\arctanh c_i^+ + j\pi \sqrt{-1}) \\
\bigcup_{i \in I^R_+} \bigcup_{j \in \mathbb{Z}} \beta_i^{c_{i-}^-1}(\arctanh c_i^- + (j + \frac{1}{2})\pi \sqrt{-1}) \bigcup_{i \in I^R_+} \bigcup_{j \in \mathbb{Z}} \beta_i^{c_{i+}^-1}(\arctanh c_i^+ + j\pi \sqrt{-1}) .
\]
Since $\lambda_{j,v}^+ = \frac{1}{c_i^+} \beta_i$ and $\lambda_{j,v}^- = c_i^- \beta_i$, they are independent of the choice of $v \in b$. Hence we denote $\lambda_{j,v}^\pm$ by $\lambda_i^\pm$. Therefore, $\tilde{S}$ is given by

$$\tilde{S} = \bigcup_{i \in \mathbb{R}^+} \beta_i^{-1}(\arctanh c_i^+) \tag{4.2}$$

Denote by $\Lambda$ the set of all complex principal curvatures of $\tilde{M}$. According to (3.1), $\Lambda$ is given by

$$\Lambda = \left\{ \frac{\tilde{\beta}^c_i}{\arctanh c_i^+ + j\pi \sqrt{-1}} \mid i \in \mathbb{R}^+, j \in \mathbb{Z} \right\} \cup \left\{ \frac{\tilde{\beta}^c_i}{\arctanh c_i^- + \left(j + \frac{1}{2}\right)\pi \sqrt{-1}} \mid i \in \mathbb{R}^-, j \in \mathbb{Z} \right\},$$

where $\tilde{\beta}^c_i$ is the parallel section of $((T^\perp \tilde{M})^c)^{\ast}$ with $(\tilde{\beta}^c_i)_0 = \beta_i^c$. Since $G/K$ is not a hyperbolic space and $M$ is proper complex equifocal and curvature-adapted, it admits a focal submanifold. Hence we have $I_i^R \neq \emptyset$ and $\bigcap_{i \in \mathbb{R}^+} \beta_i^{-1}(\arctanh c_i^+) \neq \emptyset$.

Fix $O \in \bigcap_{i \in \mathbb{R}^+} \beta_i^{-1}(\arctanh c_i^+)$. Set

$$\Delta^\prime := \{\beta_i \mid i \in \mathbb{R}^\ast\} \cup \{-\beta_i \mid i \in \mathbb{R}^\ast\},$$
$$\Delta^{\prime V} := \{\beta_i \mid i \in \mathbb{R}^+\} \cup \{-\beta_i \mid i \in \mathbb{R}^+\},$$
$$\Delta^{\prime H} := \{\beta_i \mid i \in \mathbb{R}^-\} \cup \{-\beta_i \mid i \in \mathbb{R}^-\}.$$

Let $a$ be a maximal abelian subspace of $p$ containing $b(= T^\perp_{eK} M)$ and $\Delta$ be the root system of $G/K$ with respect to $a$. Then we have $\Delta^\prime = \{\alpha \mid \alpha \in \Delta \text{ s.t. } \alpha|_b \neq 0\}$. Let $F_1$ be the focal submanifold of $M$ through $x_0 := \exp^\perp(O)$, which is one of the lowest dimensional focal submanifolds of $M$. It is shown that $F_1$ is totally geodesic (see the proof of Theorem A of [Koi11]). Furthermore, it is shown that $\tilde{F}_1 \perp := \exp^\perp(T_{x_0}^\perp F_1)$ is totally geodesic by imitating the proof of Lemma 1B.3 of [PoTh]. Thus $F_1$ and $\tilde{F}_1 \perp$ are symmetric spaces. It is shown that $\Delta^{\prime V}$ is the root system of the symmetric space $\tilde{F}_1 \perp$. For simplicity, we set

$$\tilde{\lambda}_i^+ := \frac{\tilde{\beta}^c_i}{\arctanh c_i^+} \quad (i \in \mathbb{R}^+)$$
$$\tilde{\lambda}_i^- := \frac{\beta_i^c}{\arctanh c_i^- + \frac{1}{2}\pi \sqrt{-1}} \quad (i \in \mathbb{R}^-)$$
and
\[\begin{align*}
 b^+_i &= \frac{\pi}{\arctanh c^+_i} (i \in I^R_+) \\
 b^-_i &= \frac{\pi}{\arctanh c^-_i + \frac{1}{2}\pi\sqrt{-1}} (i \in I^R_-).
\end{align*}\]

Then we have
\[\Lambda = \left\{ \frac{\tilde{\lambda}^+_i}{1 + b^+_i j\sqrt{-1}} | i \in I^R_+, j \in \mathbb{Z} \right\} \cup \left\{ \frac{\tilde{\lambda}^-_i}{1 + b^-_i j\sqrt{-1}} | i \in I^R_-, j \in \mathbb{Z} \right\}.\]

For simplicity, we set \(\tilde{\lambda}^\pm_{ij} := \frac{\tilde{\lambda}^\pm_{i}}{1 + b^\pm_{i} j\sqrt{-1}} (i \in I^R_\pm, j \in \mathbb{Z}).\) Take \(v \in b_r \setminus \tilde{\mathcal{F}},\) where we note that \(\tilde{\mathcal{F}} = \bigcup_{i \in I^R_\pm} (\tilde{\lambda}^\pm_i)^{-1}(1).\) We have \(\dim \ker (A^c_v - (\tilde{\lambda}^\pm_{ij})_0(v)\text{id}) = m^\pm (i \in I^R_\pm, j \in \mathbb{Z}).\)

Set \(E^\pm_{ij} := \ker (\tilde{A}^c_v - (\tilde{\lambda}^\pm_{ij})_0(v)\text{id}) (i \in I^R_\pm, j \in \mathbb{Z}),\) which are independent of the choice of \(v \in b_r \setminus \tilde{\mathcal{F}}.\) Take another \(w \in b_r \setminus \tilde{\mathcal{F}}.\) Let \(\tilde{w}\) be the parallel normal vector field of \(\tilde{M}\) with \(\tilde{w}_0 = w.\) Denote by \(\eta_{\tilde{w}}\) the end-point map for \(\tilde{w}\) and \(\tilde{M}_w := \eta_{\tilde{w}}(\tilde{M}),\) which is a parallel submanifold of \(\tilde{M}.\) We have
\[T_w \tilde{M}_w = \eta_{\tilde{w}}(T_{\tilde{0}} \tilde{M}) = \left( \bigoplus_{i \in I^R_\pm} \oplus_{j \in \mathbb{Z}} \eta_{\tilde{w}}(E^+_{ij}) \right) \oplus \left( \bigoplus_{i \in I^R_\pm} \oplus_{j \in \mathbb{Z}} \eta_{\tilde{w}}(E^-_{ij}) \right).\]

Denote by \(\tilde{A}_w\) the shape tensor of \(\tilde{M}_w\). We have
\[\left( \tilde{A}^w \right)_{\eta_{\tilde{w}}(E^\pm_{ij})} = \frac{(\tilde{\lambda}^\pm_{ij})_0(v)}{1 - (\tilde{\lambda}^\pm_{ij})_0(w)} \text{id} = \frac{(\tilde{\lambda}^\pm_{ij})_0(v)}{(1 + b^\pm_{i} j\sqrt{-1}) - (\tilde{\lambda}^\pm_{ij})_0(w)} \text{id}.\]
Hence the set $\Lambda^w$ of all complex principal curvatures of $\tilde{M}_w$ is given by

$$
\Lambda^w = \left\{ \frac{\tilde{\lambda}_i^\pm}{1 - (\lambda_{ij}^\pm)_0(w)} | i \in I^R, j \in \mathbb{Z} \right\} \\
\cup \left\{ \frac{\lambda_{ij}^\pm}{1 - (\lambda_{ij}^\pm)_0(w)} | i \in I^R, j \in \mathbb{Z} \right\} \\
= \left\{ \frac{1 + b_i^+ j \sqrt{-1} - (\lambda_{ij}^+)_0(w)}{\beta_i^c} | i \in I^R, j \in \mathbb{Z} \right\} \\
\cup \left\{ \frac{1 + b_i^- j \sqrt{-1} - (\lambda_{ij}^-)_0(w)}{\beta_i^c} | i \in I^R, j \in \mathbb{Z} \right\} \\
= \left\{ \frac{\arctanh c_i^+ + j \pi \sqrt{-1} - \beta_i(w)}{\beta_i^c} | i \in I^R, j \in \mathbb{Z} \right\} \\
\cup \left\{ \frac{\arctanh c_i^- + (j + \frac{1}{2}) \pi \sqrt{-1} - \beta_i(w)}{\beta_i^c} | i \in I^R, j \in \mathbb{Z} \right\}.
$$

Hence we have

$$\Tr(\tilde{A}_w^w) = \sum_{i \in I^R} \sum_{j \in \mathbb{Z}} \frac{\beta_i(v)}{\arctanh c_i^+ + j \pi \sqrt{-1} - \beta_i(w)} \times m_i^+ \\
+ \sum_{i \in I^R} \sum_{j \in \mathbb{Z}} \frac{\beta_i(v)}{\arctanh c_i^- + (j + \frac{1}{2}) \pi \sqrt{-1} - \beta_i(w)} \times m_i^- \tag{4.3}
$$

where we use $\sum_{j \in \mathbb{Z}} \frac{1}{\theta + j \pi \sqrt{-1}} = \coth \theta$ and $\coth(\theta + \frac{\pi \sqrt{-1}}{2}) = \tanh \theta$. Hence we have

$$\langle (\tilde{H}_w^w), v \rangle = \left( \sum_{i \in I^R} m_i^+ \coth(\arctanh c_i^+ - \beta_i(w))\beta_i^2, v \right) \\
+ \left( \sum_{i \in I^R} m_i^- \tanh(\arctanh c_i^- - \beta_i(w))\beta_i^2, v \right).$$

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where $\beta_i^\pm$ is defined by $\beta_i(\cdot) = \langle \beta_i^\pm, \cdot \rangle$ ($i \in I^R$). Since this relation holds for any $v \in b_r \setminus \mathfrak{F}$, we have

$$
(H^w)_w = \sum_{i \in I^R} m_i^+ \coth(\arctanh c_i^+ - \beta_i(w)) \beta_i^+
+ \sum_{i \in I^R} m_i^- \tanh(\arctanh c_i^- - \beta_i(w)) \beta_i^+.
$$

(4.4)

Set

$$
\tilde{C} := \{ w \in b \mid (\tilde{\lambda})_0(w) < 1 \ (i \in I^R) \}
= \{ w \in b \mid \beta_i(w) < \arctanh c_i^+ \ (i \in I^R) \},
$$

which is a fundamental domain of the real Coxeter group associated with $\tilde{M}$. Each parallel submanifold of $M$ passes through the only point of $\exp^\perp(\tilde{C})$ and each focal submanifold of $M$ passes through the only point of $\exp^\perp(\partial \tilde{C})$. Define a vector field $X$ on $\tilde{C}$ by

$$X_w := (H^w)_w \ (w \in \tilde{C}).$$

Let $\{ \psi_t \}$ be the local one-parameter transformation group of $X$. Now we prove the statements (i) and (iii) of Theorem A.

**Proof of (i) and (iii) of Theorem A.** First we shall show the statement (i). Denote by $\tilde{\sigma}_i$ ($i \in I^R$) the maximal dimensional stratum of $\partial \tilde{C}$ contained in $\beta_i^{-1}(\arctanh c_i^+)$. Fix $i_0 \in I^R$. Take $w_0 \in \tilde{\sigma}_{i_0}$ and and $w_0' \in \tilde{C}$ near $w_0$ such that $w_0 - w_0'$ is normal to $\tilde{\sigma}_{i_0}$. Set $w_0^\varepsilon := \varepsilon w_0' + (1-\varepsilon)w_0$ for $\varepsilon \in (0,1)$. Then we have $\lim_{\varepsilon \to +0} \beta_{i_0}(w_0^\varepsilon) = \arctanh c_{i_0}^+$ and $\sup_{0 < \varepsilon < 1} \beta_{i_0}(w_0^\varepsilon) < \arctanh c_i^+$ for each $i \in I^R \setminus \{i_0\}$. Hence we have

$$
\lim_{\varepsilon \to +0} \coth(\arctanh c_{i_0}^+ - \beta_{i_0}(w_0^\varepsilon)) = \infty,
$$

and

$$
\sup_{0 < \varepsilon < 1} \coth(\arctanh c_i^+ - \beta_i(w_0^\varepsilon)) < \infty \ (i \in I^R \setminus \{i_0\}).
$$

Therefore, we have $\lim_{\varepsilon \to +0} \frac{X_{w_0^\varepsilon}}{\|X_{w_0^\varepsilon}\|}$ is the outward unit normal vector of $\tilde{\sigma}_{i_0}$. Also we have $\lim_{\varepsilon \to +0} \|X_{w_0}\| = \infty$. From these facts, $X$ is as in the first figure of Fig. 1 on a sufficiently small collar neighborhood of $\tilde{\sigma}_{i_0}$.
Define a function $\rho$ over $\tilde{C}$ by

$$
\rho(w) := - \sum_{i \in I^+} m_i^+ \log \sinh(\text{arctanh } c_i^+ - \beta_i(w)) - \sum_{i \in I^-} m_i^- \log \cosh(\text{arctanh } c_i^- - \beta_i(w)) \quad (w \in \tilde{C}).
$$

From the definition of $X$ and (4.4), we have $\text{grad } \rho = X$. For simplicity, set $\partial_i := \frac{\partial}{\partial x_i}$ ($i = 1, \cdots, r$). Then we have

$$
(\partial_j \partial_k \rho)(w) = \sum_{i \in I^+} \frac{m_i^+}{\sinh^2(\text{arctanh } c_i^+ - \beta_i(w))} \beta_i(\partial_j) \beta_i(\partial_k) - \sum_{i \in I^-} \frac{m_i^-}{\cosh^2(\text{arctanh } c_i^- - \beta_i(w))} \beta_i(\partial_j) \beta_i(\partial_k).
$$

As $w \to \partial \tilde{C}$, $\frac{1}{\sinh^2(\text{arctanh } c_i^+ - \beta_i(w))} \to \infty$ for at least one $i \in I^+$ and $\frac{1}{\cosh^2(\text{arctanh } c_i^- - \beta_i(w))} \leq 1$ for all $i \in I^-$. Hence we see that $\rho$ is downward convex on a sufficiently small collar neighborhood of $\partial \tilde{C}$. Furthermore, since $\text{codim } M = \text{rank}(G/K)$ and $\dim(p_\alpha \cap p') \geq \frac{1}{2} \dim p_\alpha$ ($\alpha \in \Delta$) by the assumption, we have $I^+_R = I^-_R$, $m_i^+ \geq m_i^-$. 

Fig. 1.
and $c_i^+ = c_i^- (i \in I^R)$. The relation (4.5) is rewritten as follows:

$$(\partial_j \partial_k \rho)(w) \geq \sum_{i \in I^R \setminus I^R_m} \frac{m_i^+}{\sinh^2(\arctanh c_i^+ - \beta_i(w))} \beta_i(\partial_j) \beta_i(\partial_k)$$

$$+ \sum_{i \in I^R} \frac{4m_i^+}{\sinh^2 2(\arctanh c_i^+ - \beta_i(w))} \beta_i(\partial_j) \beta_i(\partial_k).$$

Hence we see that $\rho$ is downward convex on $\tilde{C}$. Also, it is clear that $\rho(w) \to \infty$ as $w \to \partial \tilde{C}$ and that $\rho(tw) \to -\infty$ as $t \to \infty$ for each $w \in \tilde{C}$. From these facts, $\rho$ and $X$ are as in Fig. 2. Hence $t \mapsto \psi_t(\hat{0})$ converges to a point $w_2$ of $\partial \tilde{C}$ in a finite time $T$. Therefore $M$ is not minimal and the mean curvature flow $M_t$ collapses to the focal submanifold of $M$ through $\exp^+(w_2)$ in finite time. Thus the statement (i) is shown.

Fig. 2.

Next we shall show the statement (iii) of Theorem A. Since $X$ is as in the second figure of Fig. 2, we obtain the following fact:
For each $w \in \partial \tilde{C}$, there exists $w' \in \tilde{C}$ such that the flow $\psi_t(w')$ converges to $w$.

Now we shall show that the situation as in Figure 3 cannot happen.

Let $W$ be the real Coxeter group of $\tilde{M}$ at $\hat{0}$, that is, the group generated by the reflections with respect to the (real) hyperplanes $l_i$’s ($i \in I_R^+$) in $b$ containing $\tilde{\sigma}_i$. This group $W$ is a finite Coxeter group. Set $V := \text{Span}\{\beta^2_i \mid i \in I_R^+\}$ and $\tilde{C}_V := \tilde{C} \cap V$ (see Fig. 4). This space $V$ is $W$-invariant and $W$ acts trivially on the orthogonal complement $V^\perp$ of $V$. Let $\{\phi_1, \cdots, \phi_{r'}\}$ be a base of the space of all $W$-invariant polynomial functions over $V$, where we note that $r' = \dim V$.

Set $\Phi := (\phi_1, \cdots, \phi_{r'})$, which is a polynomial map from $V$ to $\mathbb{R}^{r'}$. It is shown that
Φ is a homeomorphism of the closure $\overline{C}_V$ of $\overline{C}_V$ onto $\Phi(\overline{C}_V)$. Set $\xi_w(t) := \psi_t(w)$ and $\tilde{\xi}_w(t) := \Phi(\tilde{\psi}_t(w))$, where $w \in \tilde{C}_V$. Let $(x_1, \cdots, x_{r'})$ be a Euclidean coordinate of $V$ and $(y_1, \cdots, y_{r'})$ the natural coordinate of $\mathbb{R}^{r'}$. Set $\xi_w(t) := x_i(\xi_w(t))$ and $\tilde{\xi}_w(t) := y_i(\tilde{\xi}_w(t))$ $(i = 1, \cdots, r')$. Then we have

$$\langle \text{grad}(y_i \circ \Phi)\xi_w(t), X_{\xi_w(t)} \rangle = \sum_{i \in I^{+}_R} m^-_i \coth(\arctanh c_i^+ - \beta_i(\xi_w(t))) \beta_i(\text{grad}(y_i \circ \Phi)\xi_w(t))$$

$$+ \sum_{i \in I^{-}_R} m^+_i \tanh(\arctanh c_i^- - \beta_i(\xi_w(t))) \beta_i(\text{grad}(y_i \circ \Phi)\xi_w(t)).$$

Let $f_i$ be the $W$-invariant $C^\infty$-function over $V$ such that

$$f_i(v) := \sum_{j \in I^{+}_R} m^-_j \coth(\arctanh c_j^+ - \beta_j(v)) \beta_j(\text{grad}(y_i \circ \Phi)_v)$$

$$+ \sum_{j \in I^{-}_R} m^+_j \tanh(\arctanh c_j^- - \beta_j(v)) \beta_j(\text{grad}(y_i \circ \Phi)_v)$$

for all $v \in \tilde{C}_V$. It is easy to show that such a $W$-invariant $C^\infty$-function exists uniquely. According to the Schwarz’s theorem in [S], we can describe $f_i$ as $f_i = Y_i \circ \Phi$ in terms of some $C^\infty$-function $Y_i$ over $\mathbb{R}^r$. Set $Y := (Y_1, \cdots, Y_r)$, which is regarded as a $C^\infty$-vector field on $\mathbb{R}^{r'}$. Then we have $Y_{\Phi(w)} = \Phi_*(X_w)$ ($w \in \tilde{C}_V$), that is, $Y|_{\Phi(\tilde{C}_V)} = \Phi_*(X)$. Also we can show that $Y|_{\partial \Phi(\tilde{C}_V)}$ has no zero point. From these facts, we see that, for any $w \in \partial \tilde{C}_V$, the set $\{ w' \in \tilde{C}_V | \text{the flow } \psi_t(w') \text{ converges to } w \}$ is equal to the image of a flow of $X$ (see Fig. 5). In more general, from this fact, we obtain the following fact:

$$(*)_2$$ For any $w \in \partial \tilde{C}_V$, the set $\{ w' \in \tilde{C}_V | \text{the flow } \psi_t(w') \text{ converges to } w \}$ is equal to the image of a flow of $X$.

Thus the situation as in Fig 3 cannot happen.
Take an arbitrary focal submanifold $F$ of $M$. Let $\exp^{+}(w_1)$ be the only intersection point of $F$ and $\exp^{+}(\partial \widetilde{C})$. According to the above fact ($\ast_2$), the set of all parallel submanifolds of $M$ collapsing to $F$ along the mean curvature flow is a one-parameter $C^\infty$-family. Thus the statement (iii) of Theorem A is shown. q.e.d.

Next we prove the statement (ii) of Theorem A.

Proof of (ii) of Theorem A. Let $M$ and $F$ be as in (ii) of Theorem A. Since the natural fibration of $M$ onto $F$ is spherical, so is also the natural fibration of $\widetilde{M}$ onto $\widetilde{F}$. Hence $\widetilde{F}$ meets one of $(\partial \widetilde{C} \cap \beta_{i_0}^{-1}(\arctanh c_{i_0}^+))^0$'s ($i \in I^{R}$) (and one point). Assume that $\widetilde{F}$ meets $(\partial \widetilde{C} \cap \beta_{i_0}^{-1}(\arctanh c_{i_0}^+))^0$. Let $u_0$ be the intersection point. Let $T$ be the explosion time of the flow $M_t$. Denote by $A^t$ (rep. $\widetilde{A}^t$) the shape tensor of $M_t$ (resp. $\widetilde{M}_t$). $H^t$ the mean curvature vector of $M_t$ and $\widetilde{H}^t$ the regularized mean curvature vector of $\widetilde{M}_t$. We have

$$\text{Spec}(\widetilde{A}^t_v)^e \setminus \{0\} = \left\{ \frac{\beta_i(v)}{\arctanh c_i^+ + j\pi\sqrt{-1} - \beta_i(\psi_t(\hat{0}))} \mid i \in I^{R}, j \in \mathbb{Z} \right\}$$

for each $v \in T_{\psi_t(\hat{0})}^\perp \widetilde{M}_t (= T_{\hat{0}}\widetilde{M})$. Since $\lim_{t \to T-0} \psi_t(\hat{0}) = u_0 \in (\partial \widetilde{C} \cap \beta_{i_0}^{-1}(\arctanh c_{i_0}^+))^0$, since $\Phi(\nu) = \Phi(\nu_1) = \Phi(\nu_2)$. The extension of $\Phi(V)$

$$\text{Flows of } X|_{\widetilde{C}_V}$$

**Fig. 5.**
we have \( \lim_{t \to T^{-}} \beta_{i_0}(\psi_t(\hat{0})) = \arctanh c_{i_0}^{+} \) and \( \lim_{t \to T^{-}} \beta_t(\psi_t(\hat{0})) < \arctanh c_{i}^{+} \) (\( i \in I_{+}^{R} \setminus \{i_0\} \)). Hence we have

\[
(4.7) \quad \lim_{t \to T^{-}} \|[(\hat{A}_{v}^{t})^{e}]_{\infty}^{2}(T - t) = \lim_{t \to T^{-}} \frac{\beta_{i_0}(v)^{2}}{(\arctanh c_{i_0}^{+} - \beta_{i_0}(\psi_t(\hat{0})))^{2}(T - t)} = \frac{1}{2} \beta_{i_0}(v)^{2} \lim_{t \to T^{-}} \frac{1}{(\arctanh c_{i_0}^{+} - \beta_{i_0}(\psi_t(\hat{0})))\beta_{i_0}(\frac{d}{dt}\psi_t(\hat{0}))}.
\]

Since \( \frac{d}{dt}\psi_t(\hat{0}) = (\hat{H}^{t})_{i_0}(0) \), it follows from (4.4) that

\[
\begin{align*}
\lim_{t \to T^{-}} (\arctanh c_{i_0}^{+} - \beta_{i_0}(\psi_t(\hat{0})))\beta_{i_0}(\frac{d}{dt}\psi_t(\hat{0})) &= \lim_{t \to T^{-}} \left( \sum_{i \in I_{+}^{R}} m_{i}^{+} \coth(\arctanh c_{i}^{+} - \beta_t(\psi_t(\hat{0})))\langle \beta_{i}, \beta_{i_0}^{\sharp}\rangle (\arctanh c_{i_0}^{+} - \beta_{i_0}(\psi_t(\hat{0}))) \right) \\
&\quad + \sum_{i \in I_{-}^{R}} m_{i}^{-} \tanh(\arctanh c_{i}^{-} - \beta_t(\psi_t(\hat{0})))\langle \beta_{i}, \beta_{i_0}^{\sharp}\rangle (\arctanh c_{i_0}^{+} - \beta_{i_0}(\psi_t(\hat{0}))) \\
&= m_{i_0}^{+} \langle \beta_{i_0}^{\sharp}, \beta_{i_0}^{\sharp}\rangle \lim_{t \to T^{-}} \coth(\arctanh c_{i_0}^{+} - \beta_{i_0}(\psi_t(\hat{0}))) (\arctanh c_{i_0}^{+} - \beta_{i_0}(\psi_t(\hat{0}))) \\
&= m_{i_0}^{+} \langle \beta_{i_0}^{\sharp}, \beta_{i_0}^{\sharp}\rangle \lim_{t \to T^{-}} \cosh^{2}(\arctanh c_{i_0}^{+} - \beta_{i_0}(\psi_t(\hat{0}))) \\
&= m_{i_0}^{+} \langle \beta_{i_0}^{\sharp}, \beta_{i_0}^{\sharp}\rangle,
\end{align*}
\]

which together with (4.7) deduces

\[
\lim_{t \to T^{-}} \|[(\hat{A}_{v}^{t})^{e}]_{\infty}^{2}(T - t) = \frac{\beta_{i_0}(v)^{2}}{2 m_{i_0}^{+} \langle \beta_{i_0}^{\sharp}, \beta_{i_0}^{\sharp}\rangle},
\]

and hence

\[
(4.8) \quad \lim_{t \to T^{-}} \max_{v \in S_{\psi_t(\hat{0})} M_t} \|[(\hat{A}_{v}^{t})^{e}]_{\infty}^{2}(T - t) = \frac{1}{2 m_{i_0}^{+}}.
\]

Thus \( M_t \) has type I singularity. Denote by \( \exp_{G} \) the exponential map of \( G \) and \( \Exp \) the exponential map of \( G/K \) at \( eK \). Also, denote by \( S(1) \) the unit hypersphere in \( b \) centered at 0. Set \( g_{t} := \exp_{G}(\psi_{t}(\hat{0})) \) and \( \bar{v}_{t} := g_{t*}(v) \) for each \( v \in S(1) \). The relation \( \bar{v}_{t} = (\pi \circ \phi)_{*}\psi_{t}(\hat{0})(v) \) holds. Since \( M \) is proper complex equifocal and curvature-adapted and since \( M_t \) is a parallel submanifold of \( M \), \( M_t \) is also proper complex equifocal and curvature-adapted (see Lemma 3.4 of [Koi9]). It is easy to show that \( T_{\Exp(\psi_{t}(\hat{0}))} M_t = g_{t*}(m) \) and that \( T_{\Exp(\psi_{t}(\hat{0}))} M_t = g_{t*}(m_{0}^{R}) + \sum_{i \in I_{+}^{R}} g_{t*}(m_{i}^{R}) \)
is the common-eigenspace decomposition of $R(\cdot, \tilde{v}_t)\tilde{v}_t$'s ($v \in b$). In similar to $\beta_i$ ($i \in \mathbb{R}_+$), $\lambda_i^+$ ($i \in \mathbb{R}_+$) and $\lambda_i^-$ ($i \in \mathbb{R}_+$), we define linear functions $\beta_i^t$ ($i \in \mathbb{R}$), ($\lambda_i^t$)$^+$ ($i \in \mathbb{R}_+$) and ($\lambda_i^t$)$^-$ ($i \in \mathbb{R}_+$) on $T_{\psi_t(0)}^b M_t = g_{\psi_t}b$ by

$$
R(\cdot, \tilde{v}_t)\tilde{v}_t|_{g_{\psi_t}(m^{\psi_t})} = \beta_i^t(\tilde{v}_t)^2 \text{id} \ (v \in b),
$$
$$
\{\lambda \in \text{Spec}(A^t_{\psi_t}|_{g_{\psi_t}(m^{\psi_t})}) \mid |\lambda| > |\beta_i^t(\tilde{v}_t)|\} = \{(\lambda_i^t)^+(\tilde{v}_t)\} \ (v \in b)
$$
$$
\{\lambda \in \text{Spec}(A^t_{\psi_t}|_{g_{\psi_t}(m^{\psi_t})}) \mid |\lambda| < |\beta_i^t(\tilde{v}_t)|\} = \{(\lambda_i^t)^-(\tilde{v}_t)\} \ (v \in b).
$$

It is clear that $\beta_i = \beta_i \circ g_{\psi_t}^{-1}$ ($i \in \mathbb{R}$). The values $\beta_i^t(\tilde{v}_t)/((\lambda_i^t)^+(\tilde{v}_t)$ ($i \in \mathbb{R}_+$) and $(\lambda_i^t)^-(\tilde{v}_t)/\beta_i^t(\tilde{v}_t)$ ($i \in \mathbb{R}_+$) are independent of the choice of $v \in b$. Denote by $(c_i^t)^+$ and $(c_i^t)^-$ these constants, respectively. If $i \in \mathbb{R}_+ \cap \mathbb{R}_-$, then we have $(c_i^t)^+ = (c_i^t)^-$. Hence we shall denote $(c_i^t)^+$ ($i \in \mathbb{R}_+$) and $(c_i^t)^-$ ($i \in \mathbb{R}_+$) by $c_i^t$ for simplicity. In the sequel, we use this notation. The spectrum of $(A^t_{\psi_t})^c$ other than zero is given by

$$
\text{Spec}(A^t_{\psi_t})^c \setminus \{0\} = \left\{\frac{\beta_i(v)}{\arctanh c_i^t + j\pi\sqrt{-1}} \mid i \in \mathbb{R}_+, \ j \in \mathbb{Z}\right\} \\
\cup \left\{\frac{\beta_i(v)}{\arctanh c_i^t + (j + \frac{1}{2})\pi\sqrt{-1}} \mid i \in \mathbb{R}, \ j \in \mathbb{Z}\right\}.
$$

On the other hand, we have $\lim_{t \to T-0} \max_{v \in \mathcal{S}(1)} |(\lambda_i^c)^+(\tilde{v}_t)| = \infty$ and hence $\lim_{t \to T-0} c_i^t = 0$. Also we have $\lim_{t \to T-0} \max_{v \in \mathcal{S}(1)} |(\lambda_i^c)^-(\tilde{v}_t)| < \infty$ and hence $\lim_{t \to T-0} |c_i^t| > 0$ ($i \in \mathbb{R}_+ \setminus \{i_0\}$).

Therefore we obtain

$$
\lim_{t \to T-0} (T - t) \max_{v \in \mathcal{S}(1)} \|A^t_{\psi_t}\|_\infty^2 = \lim_{t \to T-0} (T - t) \max_{v \in \mathcal{S}(1)} \left(\frac{\beta_i(v)}{\arctanh c_i^t}\right)^2
$$

$$
= \max_{v \in \mathcal{S}(1)} \beta_i(v)^2 \lim_{t \to T-0} \frac{T - t}{\arctanh^2 c_i^t}
$$

$$
\lim_{t \to T-0} (T - t) \max_{v \in \mathcal{S}(1)} \left(\frac{T - t}{\arctanh^2(\beta_i(v)/((\lambda_i^c)^+(\tilde{v}_t)))}\right)^2 (\lambda_i^c)^+(\tilde{v}_t)^2
$$

$$
\lim_{t \to T-0} (T - t) \max_{v \in \mathcal{S}(1)} |A^t_{\psi_t}|^2 = \lim_{t \to T-0} (T - t) \max_{v \in \mathcal{S}(1)} |A^t_{\psi_t}|^2.
$$

From this relation and (4.8), we obtain

$$
\lim_{t \to T-0} (T - t) \max_{v \in \mathcal{S}(1)} |A^t_{\psi_t}|^2 = \frac{1}{2m^+_{i_0}} < \infty.
$$

Thus the mean curvature flow $M_t$ ($0 \leq t < T$) has type I singularity. q.e.d.
For each $S \subset I^R_+$, we set

$$
\tilde{\sigma}_S := \{ w \in \partial \tilde{C} \mid (\tilde{\lambda}^+_i)_0(w) < 1 \ (i \in I^R_+ \setminus S) \ \& \ \tilde{\lambda}^+_i(w) = 1 \ (i \in S) \} \\
= \{ w \in \tilde{C} \mid \beta_i(w) = \arctanh c^+_i \ (i \in I^R_+ \setminus S) \ \& \ \beta_i(w) = \arctanh c^+_i \ (i \in S) \},
$$

which is a stratum of $\tilde{C}$. Take $w \in \tilde{\sigma}_S$. Let $\tilde{w}$ be the parallel normal vector field of $\tilde{M}$ with $\tilde{w}_0 = w$. Denote by $\eta_{\tilde{w}}$ the end-point map for $\tilde{w}$ and $F_w := \eta_{\tilde{w}}(\tilde{M})$, which is a focal submanifold of $\tilde{M}$. We have

$$
T_w F_w = \left( \bigoplus_{i \in I^R_+ \setminus S} \bigoplus_{\lambda \in \mathbb{Z}} \eta_{\tilde{w}*}(E^+_{ij}) \right) \bigoplus \left( \bigoplus_{i \in I^R_-} \bigoplus_{\lambda \in \mathbb{Z}} \eta_{\tilde{w}*}(E^-_{ij}) \right).
$$

Denote by $\tilde{A}^w$ the shape tensor of $F_w$. In similar to (4.3), we have

$$
\text{Tr} (\tilde{A}^w) = \sum_{i \in I^R_+ \setminus S} m^+_i \coth(\arctanh c^+_i - \beta_i(w)) \beta^{{\tilde{w}}}_i(v) \\
+ \sum_{i \in I^R_-} m^-_i \tanh(\arctanh c^+_i - \beta_i(w)) \beta^{{\tilde{w}}}_i(v) \ (\in \mathbb{R})
$$

(4.10)

for any $v \in \mathfrak{b}$, where $\mathfrak{b}$ is regarded as a subspace of $T^w_\tilde{w} F_w$. Set $L := \tilde{M} \cap T^w_\tilde{w} F_w$, which is a focal leaf of $\tilde{M}$. For any $u \in L$, let $b_u$ be the section of $\tilde{M}$ through $u$. We can show $(\tilde{H}^w)_w \in \bigcap_{u \in L} b_u$. Hence, from (4.10), the regularized mean curvature vector $\tilde{H}^w$ of $F_w$ exists and $(\tilde{H}^w)_w$ is given by

$$
(\tilde{H}^w)_w = \sum_{i \in I^R_+ \setminus S} m^+_i \coth(\arctanh c^+_i - \beta_i(w)) \beta^{{\tilde{w}}}_i \\
+ \sum_{i \in I^R_-} m^-_i \tanh(\arctanh c^+_i - \beta_i(w)) \beta^{{\tilde{w}}}_i.
$$

(4.11)

Define a vector field $X^\tilde{\sigma}_S$ on $\tilde{\sigma}_S$ by $X^\tilde{\sigma}_S := (\tilde{H}^w)_w$ ($w \in \tilde{\sigma}_S$). This vector field $X^\tilde{\sigma}_S$ is tangent to $\tilde{\sigma}_S$. Let $\{ \psi^{{\tilde{\sigma}}}_i \}$ be the local one-parameter transformation group of $X^\tilde{\sigma}_S$.

Proof of Theorem B. First we shall show the statement (i) of Theorem B. Let $F$ be as in the statement (i) of Theorem B. Set $\tilde{F} := (\pi \circ \phi)^{-1}(F)$. Since the lowest dimensional focal submanifold $F_1$ of $M$ is a one-point set by the assumption, we have $I^R_1 = \emptyset$. Let $w_0$ be the intersection point of $\tilde{F}$ and $\tilde{\sigma}$. Set $S_0 := \{ i \in I^R_+ (= I^R) \mid \beta_i(w_0) = \arctanh c^+_i \}$. Since $\dim \tilde{\sigma} \geq 1$, we have $I^R \setminus S_0 \neq \emptyset$. According to
\[(X^\tilde{\sigma})_w = (\tilde{H}^w)_w = \sum_{i \in IR \setminus S_0} m_i^+ \coth(\text{arctanh} \, c_i^+ - \beta_i(w)) \beta_i^2 \quad (w \in \tilde{\sigma}).\]

We can show that \(X^\tilde{\sigma}\) is as in Fig. 6 on a sufficiently small collar neighborhood of each maximal dimensional stratum of \(\partial \tilde{\sigma}\). Define a function \(\rho_{\tilde{\sigma}}\) over \(\tilde{\sigma}\) by

\[\rho_{\tilde{\sigma}}(w) := - \sum_{i \in IR \setminus S_0} m_i^+ \log \sinh(\text{arctanh} \, c_i^+ - \beta_i(w)) \quad (w \in \tilde{\sigma}).\]

Easily we can show that \(\rho_{\tilde{\sigma}} = X^\tilde{\sigma}\). Let \((x_1, \cdots, x_{r'})\) be the Euclidean coordinate of \(\cap S_0 \beta_i^{-1}(\text{arctanh} \, c_i^+)\). For simplicity, set \(\beta_i := \frac{\partial}{\partial x_i} (i = 1, \cdots, r')\). Then we have

\[(\partial_j \partial_k \rho_{\tilde{\sigma}})(w) = \sum_{i \in IR \setminus S_0} \frac{m_i^+}{\sinh^2(\text{arctanh} \, c_i^+ - \beta_i(w))} \beta_i(\partial_j) \beta_i(\partial_k).\]

Hence we see that \(\rho_{\tilde{\sigma}}\) is downward convex on \(\tilde{\sigma}\). Also, it is clear that \(\rho_{\tilde{\sigma}}(w) \to \infty\) as \(w \to \partial \tilde{\sigma}\) and that \(\rho_{\tilde{\sigma}}(w) \to -\infty\) as \(t \to \infty\) for each \(w \in \tilde{\sigma}\). From these facts, it follows that \(\psi_{\tilde{\sigma}}^\tau(w_0)\) converges to a point \(w_1\) of \(\partial \tilde{\sigma}\) in a finite time. The mean curvature flow \(F_t\) collapses to the focal submanifold of \(M\) through \(\exp^{-\tau}(w_1)(\tau \in \exp^{-\tau}(\partial \tilde{\sigma}))\). This completes the proof of the first-half part of the statement (i). The second-half part of the statement (ii) is proved by imitating the proof of the statement (ii) of Theorem A.

Next we shall show the statement (ii) of Theorem B. Set \(V := \text{Span}\{\beta_i^+ | i \in I_R^+\}\) and \(\tilde{\sigma}_V := \tilde{\sigma} \cap V\). Denote by \(V_{\tilde{\sigma}}\) be the minimal dimensional affine subspace of \(V\) containing \(\tilde{\sigma}_V\). Let \(W_{\tilde{\sigma}}\) be a finite Coxeter group generated by the reflections with respect to the (real) hyperplanes \(l_i^\sigma\)'s \((i \in IR \setminus S_0)\) in \(V_{\tilde{\sigma}}\) containing \(\tilde{\sigma}_V \cap V_{\tilde{\sigma}}\). Let \(\{\phi_{\tilde{\sigma}}^1, \cdots, \phi_{\tilde{\sigma}}^r\}\) be a base of the space of all \(W_{\tilde{\sigma}}\)-invariant polynomial functions over \(V_{\tilde{\sigma}}\), where we note that \(r' = \dim V_{\tilde{\sigma}}\). Set \(\Phi_{\tilde{\sigma}} := (\phi_{\tilde{\sigma}}^1, \cdots, \phi_{\tilde{\sigma}}^r)\), which is a polynomial map from \(V_{\tilde{\sigma}}\) to \(R^{r'}\). It is shown that \(\Phi_{\tilde{\sigma}}\) is a homeomorphism of the closure \(\tilde{\sigma}_V\) of \(\tilde{\sigma}_V\) onto \(\Phi_{\tilde{\sigma}}(\tilde{\sigma}_V)\). Set \(\xi_w(t) := \psi_{\tilde{\sigma}}(w)\) and \(\tilde{\xi}_w(t) := \Phi_{\tilde{\sigma}}(\psi_{\tilde{\sigma}}^\tau(w))\), where \(w \in \tilde{\sigma}_V\). Let \((x_1, \cdots, x_{r'})\) be a Euclidean coordinate of \(V_{\tilde{\sigma}}\) and \((y_1, \cdots, y_{r'})\) the natural coordinate of \(R^{r'}\). Set \(\xi_w(t) := x_i(\xi_w(t))\) and \(\tilde{\xi}_w(t) := y_i(\tilde{\xi}_w(t))\) \((i = 1, \cdots, r')\). Then we have

\[\langle \xi_w' \rangle_i(t) = \langle \text{grad}(y_i \circ \Phi_{\tilde{\sigma}}) \xi_w(t), X^\tilde{\sigma}_{\xi_w(t)} \rangle \]

\[= \sum_{i \in IR \setminus S_0} m_i^+ \coth(\text{arctanh} \, c_i^+ - \beta_i(\xi_w(t))) \beta_i(\text{grad}(y_i \circ \Phi_{\tilde{\sigma}})_{\xi_w(t)}).\]
Let $f_i^\tilde{\sigma}$ be the $W_{\tilde{\sigma}}$-invariant $C^\infty$-function over $V_{\tilde{\sigma}}$ such that

$$f_i^\tilde{\sigma}(v) := \sum_{j \in \mathbb{R} \setminus S_0} m_j^+ \coth(\arctanh c_j^+ - \beta_j(v)) \beta_j(\text{grad}(y_i \circ \Phi_{\tilde{\sigma}})_v)$$

for all $v \in \tilde{\sigma}_V$. It is easy to show that such a $W_{\tilde{\sigma}}$-invariant $C^\infty$-function exists uniquely. According to the Schwarz’s theorem in [S], we can describe $f_i^\tilde{\sigma}$ as $f_i^\tilde{\sigma} = Y_i^\tilde{\sigma} \circ \Phi_{\tilde{\sigma}}$ in terms of some $C^\infty$-function $Y_i^\tilde{\sigma}$ over $\mathbb{R}'$. Set $Y^\tilde{\sigma} := (Y_1^\tilde{\sigma}, \ldots, Y_r^\tilde{\sigma})$, which is regarded as a $C^\infty$-vector field on $\mathbb{R}'$. Then we have $Y_{\Phi_{\tilde{\sigma}}(w)}(\tilde{\sigma}_V) = (\Phi_{\tilde{\sigma}})_*(X_w^\tilde{\sigma})$, that is, $Y^\tilde{\sigma}|_{\Phi_{\tilde{\sigma}}(\tilde{\sigma}_V)} = (\Phi_{\tilde{\sigma}})_*(X^\tilde{\sigma})$. Also we can show that $Y^\tilde{\sigma}|_{\partial \Phi_{\tilde{\sigma}}(\tilde{\sigma}_V)}$ has no zero point. From these facts and the fact that $X^\tilde{\sigma}$ is as in Fig. 6 on a sufficiently small collar neighborhood of each maximal dimensional stratum of $\partial \tilde{\sigma}_V$, we see that, for any $w \in \partial \tilde{\sigma}_V$, the set $\{ w' \in \tilde{\sigma}_V | \text{the flow } \psi_t^\tilde{\sigma}(w') \text{ converges to } w \}$ is equal to the image of a flow of $X^\tilde{\sigma}$. In more general, from these facts, it follows that, for any $w \in \partial \tilde{\sigma}$, the set $\{ w' \in \tilde{\sigma} | \text{the flow } \psi_t^\tilde{\sigma}(w') \text{ converges to } w \}$ is equal to the image of a flow of $X^\tilde{\sigma}$. From this fact, the statement (ii) of Theorem B follows. q.e.d.

![Fig. 6.](image)

We shall show that, in the statement of Theorem B, we cannot weaken the condition that $F_1$ is a one-point set to the condition $(\Delta' = \Delta$ and $\dim(p_\alpha \cap p') \geq \frac{1}{2} \dim p_\alpha$ $(\alpha \in \Delta)$) in the statement (i) of Theorem A. Assume that $M$ satisfies the condition in the statement (i) of Theorem A. Let $S_0$ be as above and $\tilde{\sigma} := \sigma_{S_0}$.
Define a function $\rho_{\bar{\sigma}}$ over $\bar{\sigma}$ by

$$\rho_{\bar{\sigma}}(w) := - \sum_{i \in I^R \setminus S_0} m_i^+ \log(\sinh(\arctanh c_i^+ - \beta_i(w)))$$
$$- \sum_{i \in I^R} m_i^- \log(\cosh(\arctanh c_i^+ - \beta_i(w))) \quad (w \in \bar{\sigma}).$$

We have $\text{grad} \rho_{\bar{\sigma}} = X_{\bar{\sigma}}$. Also, it follows from $m_i^+ \geq m_i^-$ and $c_i^+ = c_i^- (i \in I^R)$ that

$$\left(\partial_j \partial_k \rho_{\bar{\sigma}}\right)(w) \geq \sum_{i \in I^R \setminus (S_0 \cup I^H)} \frac{m_i^+}{\sinh^2(\arctanh c_i^+ - \beta_i(w))} \beta_i(\partial_j) \beta_i(\partial_k)$$
$$+ \sum_{i \in (I^R \setminus S_0) \cap I^H} \frac{4m_i^+}{\sinh^2 2(\arctanh c_i^+ - \beta_i(w))} \beta_i(\partial_j) \beta_i(\partial_k)$$
$$- \sum_{i \in S_0 \cap I^H} \frac{m_i^+}{\cosh^2(\arctanh c_i^+ - \beta_i(w))} \beta_i(\partial_j) \beta_i(\partial_k).$$

Thus we cannot conclude whether $\rho$ is downward convex or not because of the existence of the third term in the right-hand side of this relation. From this reason, in the statement of Theorem B, we cannot weaken the condition that $F_l$ is one-point set to the condition in the statement (i) of Theorem A.

5 Examples

Principal orbits of Hermann type actions on a symmetric space $G/K$ of non-compact type are proper complex equifocal and curvature-adapted. Principal orbits of the isotropy action $K \ltimes G/K$ and those of Hermann type actions $H \ltimes G/K$ as in Table 1 satisfy all the conditions in the statement (i) of Theorem A. In Table 1, $L$ is the fixed point group of $\theta \circ \tau$, where $\theta$ is a Cartan involution of $G$ with $\text{Fix} \theta_0 \subset K \subset \text{Fix} \theta$ and $\tau$ is an involution of $G$ with $\text{Fix} \tau_0 \subset H \subset \text{Fix} \tau$. Note that we may assume $\theta \circ \tau = \tau \circ \theta$ without loss of generality. Then, for a Hermann type action $H \ltimes G/K$, $F_l := H(eK)$ is one of the lowest dimensional focal submanifolds of principal orbits of $H \ltimes G/K$ (see Fig. 7). In particular, in case of the isotropy action $K \ltimes G/K$, $F_l$ is a one-point set. Hence the principal orbits of the isotropy action satisfy all the conditions in the statement (i) of Theorem B.
| $H$                     | $G/K$                               | $F_0 = H/H \cap K$                          | $F_0^\perp = L/H \cap K$ |
|-------------------------|-------------------------------------|---------------------------------------------|-----------------------------|
| $SO^*(2n)$              | $SU^*(2n)/Sp(n)$                    | $SO^*(2n)/U(n)$                             | $SL(n, \mathbb{C})/SU(n)$   |
| $SO^*(2p)$              | $SU(p, p)/SU(p \times U(p))$       | $SO^*(2p)/U(p)$                             | $Sp(p, \mathbb{R})/U(p)$   |
| $SO(n, \mathbb{C})$    | $SL(n, \mathbb{C})/SU(n)$          | $SO(n, \mathbb{C})/SO(n)$                  | $SL(n, \mathbb{R})/SO(n)$  |
| $SU^*(2p) \cdot U(1)$  | $Sp(p, p)/Sp(p) \times Sp(p)$      | $SU^*(2p)/Sp(p)$                            | $Sp(p, \mathbb{C})/Sp(p)$  |
| $SL(n, \mathbb{C}) \cdot SO(2, \mathbb{C})$ | $Sp(n, \mathbb{C})/Sp(n)$          | $SL(n, \mathbb{C})/SU(n)$                  | $Sp(n, \mathbb{R})/U(n)$   |
|                         |                                     | $\times SO(2, \mathbb{C})/SO(2)$            |                             |
| $Sp(1, 3)$              | $E_6^7/SU(6) \cdot SU(2)$          | $Sp(1, 3)/Sp(1) \times Sp(3)$              | $E_6^2/Sp(3) \cdot Sp(1)$  |
| $SU(1, 5) \cdot SL(2, \mathbb{R})$ | $E_6^{14}/Spm(10) \cdot U(1)$      | $SU(1, 5)/SU(1) \times U(5)$               | $SO^*(10)/U(5)$             |
|                         |                                     | $\times SL(2, \mathbb{R})/SO(2)$            |                             |
| $Sp(4, \mathbb{C})$    | $E_6^7/E_6$                         | $Sp(4, \mathbb{C})/Sp(4)$                  | $E_6^6/Sp(4)$               |
| $SU(2, 6)$              | $E_7^{15}/SO^*(12) \cdot SU(2)$   | $SU(2, 6)/SU(2) \times U(6)$               | $E_7^2/SU(6) \cdot SU(2)$  |
| $SL(8, \mathbb{C})$    | $E_7^2/E_7$                         | $SL(8, \mathbb{C})/SU(8)$                  | $E_7^2/SU(8)$               |
| $SO(16, \mathbb{C})$   | $E_8^8/E_8$                         | $SO(16, \mathbb{C})/SO(16)$                | $E_8^8/Sp(16)$              |
| $Sp(3, \mathbb{C}) \cdot SL(2, \mathbb{C})$ | $F_4^{0}/F_4$                      | $Sp(3, \mathbb{C})/Sp(3)$                  | $F_4^0/Sp(3) \cdot Sp(1)$  |
|                         |                                     | $\times SL(2, \mathbb{C})/SU(2)$           |                             |
| $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ | $G_2^2/G_2$                      | $SL(2, \mathbb{C})/SU(2)$                  | $G_2^2/SO(4)$               |
|                         |                                     | $\times SL(2, \mathbb{C})/SU(2)$           |                             |

Table 1.

$$F_1 = H(eK) = H/H \cap K$$

$$M = H(gK)$$

$$F_1^\perp = L/H \cap K$$

![Diagram](image-url)

Fig. 7.

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