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Lorentzian manifolds and scalar curvature invariants

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Abstract

We discuss (arbitrary-dimensional) Lorentzian manifolds and the scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives. Recently, we have shown that in four dimensions a Lorentzian spacetime metric is either $I$-non-degenerate, and hence locally characterized by its scalar polynomial curvature invariants, or is a degenerate Kundt spacetime. We present a number of results that generalize these results to higher dimensions and discuss their consequences and potential physical applications.

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1. Introduction

We address the question of when a Lorentzian manifold (in arbitrary dimensions) can be uniquely characterized (locally) by its scalar polynomial curvature invariants, which are scalars obtained by contraction from a polynomial in the Riemann tensor and its covariant derivatives. This question is not only of mathematical interest, but is also of fundamental physical import. We begin by introducing some necessary mathematical terminology and machinery.

For a spacetime $(\mathcal{M}, g)$, a (one-parameter) metric deformation, $\hat{g}_\tau$, $\tau \in [0, \epsilon)$, is a family of smooth metrics on $\mathcal{M}$ such that $\hat{g}_0 = g$, and $\hat{g}_\tau$ for $\tau > 0$ is not diffeomorphic to $g$. We define the set of all scalar polynomial curvature invariants on $(\mathcal{M}, g)$ by

$$I \equiv \{ R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma}, \ldots, C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}, \ldots, R_{\mu\nu\alpha\beta\gamma}R^{\mu\nu\alpha\beta\gamma\delta}, \ldots \}.$$ 

If there does not exist a metric deformation of $g$ having the same set of invariants as $g$, then we will call the set of invariants non-degenerate, and the spacetime metric $g$ will be called $I$-non-degenerate \cite{1}. Therefore, for a metric which is $I$-non-degenerate the invariants locally characterize the spacetime completely.
The Kundt class of spacetimes in $n$ dimensions is defined by those metrics admitting a null vector that is geodesic, expansion-free, shear-free and twist-free. A Kundt metric can be written in the canonical form ($i = 3, \ldots, n$) \[ ds^2 = 2dv [dv + H(v, u, x^i) \]dudv + Wi(v, u, x^i) dx^i] + hij(u, x^i) dx^i dx^j. \] (1)

A degenerate Kundt spacetime is a spacetime in which there exists a kinematic null frame (in which the appropriate Ricci rotation coefficients $L_{ij}$ are zero [2]) such that all of the positive boost weight (b.w.) terms of the Riemann tensor and all of its covariant derivatives $\nabla^{(k)}(\text{Riem})$ are zero (in this common frame) [4]. That is, a degenerate Kundt spacetime is an algebraically aligned special Riemann type II and algebraically aligned special $\nabla^{(k)}(\text{Riem})$ type II Kundt spacetime. In terms of metric (1), written in the canonical (kinematic) frame, the condition that the Riemann tensor is aligned and of algebraically special type II implies that $W_{vuv} = 0$, and the condition that $\nabla(\text{Riem})$ is aligned and of algebraically special type II implies that $H_{vvv} = 0$ (whence it follows that all of $\nabla^{(k)}(\text{Riem})$ ($k > 1$) are aligned and of algebraically special type II) [4]. We note that the important constant curvature invariant (CSI) spacetimes [3] and vanishing scalar invariant (VSI) spacetimes [5] are degenerate Kundt spacetimes.

Let us briefly review the results of [1], particularly those that have generalizations to higher dimensions, in which the class of four-dimensional (4D) Lorentzian manifolds that can be completely characterized by the scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives was determined. The important result that a spacetime metric is either $I$-non-degenerate or the metric is a degenerate Kundt metric was proven. This theorem was proven on a case-by-case basis, depending on the algebraic type, using a b.w. decomposition and, most importantly, by determining an appropriate set of projection operators from the Riemann tensor and its covariant derivatives. We recall that the 4D Lorentzian manifolds are characterized algebraically by their Petrov and Segre [6] types or, alternatively, in terms of their Ricci, Weyl (and Riemann) types [2, 7].

It is useful to state a number of partial results of when a spacetime metric is $I$-non-degenerate (which is how the theorem in 4D was actually proven), which will be exploited in obtaining and stating results in higher dimensions. First, it was proven that if a 4D spacetime metric is locally of Ricci type I or Weyl type I (i.e. algebraically general) the metric is $I$-non-degenerate [1]. This indicates that, in general, a spacetime metric is $I$-non-degenerate and the metric is locally determined by its curvature invariants. For the algebraically special cases the Riemann tensor itself does not provide sufficient information to determine all of the required projection operators, and it is necessary to also consider the covariant derivatives. In terms of the b.w. decomposition, for an algebraically special metric (which has a Riemann tensor with zero positive b.w. components) which is not Kundt, by taking covariant derivatives of the Riemann tensor positive boost weight components are found and a set of higher derivative projection operators are determined. Therefore, if the 4D spacetime metric is algebraically special, but $\nabla R$, $\nabla^{(2)}R$, $\nabla^{(3)}R$, or $\nabla^{(5)}R$ is of type I or more general, the metric is $I$-non-degenerate.

The remaining metrics which do not acquire positive boost weight components when taking covariant derivatives have a very special curvature structure; indeed, they are degenerate Kundt metrics [4]. This implies that metrics not determined by their curvature invariants must be of degenerate Kundt form, i.e. degenerate Kundt metrics are not $I$-non-degenerate. This exceptional property of the degenerate Kundt metrics essentially follows from the fact that they do not define a unique timelike curvature operator. The degenerate Kundt spacetimes are classified algebraically by the Riemann tensor and its covariant derivatives in the aligned kinematic frame [4].
We note parenthetically that these results are of importance to the equivalence problem of characterizing Lorentzian spacetimes (in terms of their Cartan invariants) [6]. Clearly, by knowing which spacetimes can be characterized by their scalar curvature invariants alone, the computation of the invariants (i.e., simple scalar invariants) is much more straightforward and can be done algorithmically. On the other hand, the Cartan equivalence method also contains, at least in principle, the conditions under which the classification is complete.

2. Higher dimensions

Let us now consider higher dimensions. The results are proven on a case-by-case basis in terms of algebraic types; hence, we need to utilize the higher dimensional algebraic classification of tensors [2, 7]. We note that recently a number of exact higher dimensional algebraically special spacetimes have been studied [8]. In particular, similar results to those that occur in 4D (discussed above) occur in the algebraically general cases. For Ricci type I or G the proof is essentially identical to the 4D case. We know the Segre types in higher dimensions and for Ricci type I they are of the form \{1, 11, \ldots \}, \{1, (11)1, \ldots \}, \{1, (111), \ldots \} etc, or \{z\bar{z}11, \ldots \}, \{z\bar{z}(11), \ldots \}, etc. We note that the stabilizer of a Ricci type I tensor is always contained in the compact group $O(n - 1)$. For all of these Segre types it is always possible to construct a timelike projector. Therefore, by projecting the various curvature tensors we can construct tensors with purely spatial indices and, consequently, we can construct curvature invariants invariant under the compact group $O(n - 1)$. Therefore, in arbitrary dimensions, we have proven that

**Theorem 2.1.** If a spacetime metric is of Ricci type I or G, the metric is $I$-non-degenerate.

We can provide a compelling argument that in higher dimensions a general Weyl type I or G spacetime is $I$-non-degenerate. A bivector formalism in higher dimensions is needed to characterize (invariantly) the curvature operators of the Weyl tensor [9]. In general, the Weyl tensor can be decomposed using the eigenspace projection operators as

$$C = N + \sum A \lambda_A \perp_A,$$

where $N$ is nilpotent. In the case of the Weyl tensor, each of the projection operators, $\perp_A$, are of type D with respect to a certain frame. Now, if all of the projection operators are of type D with respect to the same frame, then the Weyl tensor is of type II or simpler. Therefore, for the Weyl tensor to be of type $I/G$, these frames cannot be aligned. Each of these Weyl projection operators can be used to construct a projection operator of the tangent space of some type \{(1, 1, \ldots, 1)(1, \ldots, 1)\}. Since these are not aligned, and the fact that Weyl tensor is of type $I/G$ (and not simpler), by successive projections we can isolate a timelike direction and construct a timelike projection operator. It then follows that a Weyl type I or G spacetime is $I$-non-degenerate. Therefore, we have that

**Conjecture 2.2.** If the Weyl type is I or G, then the spacetime is $I$-non-degenerate.

The first step in providing a rigorous proof to this result is to investigate the curvature operators in higher dimensions and to classify these for the various algebraic types [9]. Indeed, with the aid of these operators it is then hoped that all of the results obtained in 4D can also be shown to be true in higher dimensions. In addition, such a formalism may lead to simpler proofs of some of the results outlined below.
Finally, the result that a spacetime is $I$-non-degenerate follows for any curvature operator (not just those constructed from the Ricci or Weyl tensors) that is of general algebraic type $I$ or $G$:

**Corollary 2.3.** If any curvature operator is of general algebraic type $I$ or $G$, then the spacetime is $I$-non-degenerate.

We have thus far presented some general results; that is, in general (defined algebraically) a spacetime is $I$-non-degenerate. It is also possible to present a very special result; namely, that a spacetime which is degenerate Kundt is not $I$-non-degenerate. By construction, all degenerate Kundt spacetimes with the same boost weight zero terms but with different negative b.w. terms will have precisely the same set of scalar curvature invariants since no negative boost weight terms can appear in any scalar polynomial invariant in a degenerate Kundt spacetime (see theorem II.7 below). Therefore, we have that

**Theorem 2.4.** A spacetime which is degenerate Kundt is not $I$-non-degenerate.

In higher dimensions the intermediate cases are much harder to deal with (than in the 4D case). Let us first present a partial result. In the analysis in 4D it was determined for which Segre types for the Ricci tensor the spacetime is $I$-non-degenerate (similar results were obtained for the Weyl tensor). In each case, it was found that the Ricci tensor, considered as a curvature operator, admits a timelike eigendirection. Therefore, if a spacetime is not $I$-non-degenerate, its Ricci tensor must be of a particular Segre type (corresponding to the non-existence of a unique timelike direction). Therefore, considering the following algebraic types for the Ricci tensor (or any other $(0,2)$ curvature operator written in ‘Segre form’):

1. $\{211\ldots, \{2(11)\ldots, \{2(111)\ldots, \ldots, \{2(1)\ldots, 2\}
2. $\{(21)\ldots, \{(21)(11)\ldots, \{(21)(111)\ldots, \ldots, \{(21)(111)\ldots, 2\}
3. $\{(211)\ldots, \{(211)(11)\ldots, \{(211)(111)\ldots, \ldots, \{(211)(111)\ldots, 2\}
4. $\{(211)\ldots, \{(211)(11)\ldots, \{(211)(111)\ldots, \ldots, \{(211)(111)\ldots, 2\}

it follows that if the Ricci tensor (or any other $(0,2)$ curvature operator—and similar results are true in terms of the Weyl tensor in the bivector form) is not of one of these types (for example, of type $\{1, 11, \ldots\}$), then the spacetime is $I$-non-degenerate. It is plausible that if the Ricci tensor and all other curvature operators are all of one of these types, then the spacetime is degenerate Kundt and not $I$-non-degenerate.

In addition, in higher dimensions suppose there exists a frame in which all of the positive b.w. terms of the Riemann tensor and all of its covariant derivatives $\nabla^{(k)}(\text{Riem})$ are zero (in this frame) (i.e. the spacetime is of ‘type II to all orders’), it is plausible that the resulting spacetime is degenerate Kundt (i.e. the appropriate Ricci rotation coefficients $L_{ij}$ are zero [2]). This is true in 4D (as a result of the theorems of [1]). It is likely true in higher dimensions, but this might be difficult to prove in general. In particular, we would like to prove that the degenerate Kundt metrics are the only metrics not determined by their curvature invariants (i.e. not $I$-non-degenerate) in any dimension. It is hoped that higher dimensional generalizations of all of the 4D theorems can be proven with the aid of Weyl operators in higher dimensions [9]. However, we do note that many of the results in [1, 4, 10] can be generalized to higher dimensions.

Let us present a number of partial results. First, two higher dimensional results were proven in [4]. Let $K_{n}$ denote the subclass of Kundt metrics such that there exists a kinematic frame in which the Riemann tensor up to and including its $n$th covariant derivative have vanishing positive b.w. components.
Theorem 2.5. In the higher dimensional Kundt class, $K_1$ implies $K_n$ for all $n \geq 2$.

Theorem 2.6. If for a spacetime, $(\mathcal{M}, g)$, the Riemann tensor and all of its covariant derivatives $\nabla^{(k)}(R_{\text{iem}})$ are simultaneously of type $D$ (in the same frame), then the spacetime is degenerate Kundt.

Note that the degenerate Kundt spacetimes can be written in the form (1), where

\[
H(v, u, x^k) = v^2 H^{(2)}(u, x^k) + v H^{(1)}(u, x^k) + H^{(0)}(u, x^k),
\]

\[
W_i(v, u, x^k) = v W_{i(1)}(u, x^k) + W_{i(0)}(u, x^k).
\]

Let us next present a new result.

Theorem 2.7. For a degenerate Kundt spacetime the boost weight 0 components of all curvature tensors are identical to the corresponding Kundt spacetime where $H^{(1)}(u, x^k) = H^{(0)}(u, x^k) = W^{(0)}(u, x^k) = 0$. Consequently, their curvature invariants will also be identical.

Proof. Let us introduce the null frame (Kundt frame):

\[
\ell = du,
\]

\[
\mathbf{n} = dv + (H^{(2)}v^2 + H^{(1)}v + H^{(0)}) du + \left(W_{i(1)}v + W^{(0)}\right) dx^i,
\]

\[
\mathbf{m}' = e'_j(u, x^k) dx^j,
\]

\[
\delta_{ij} \mathbf{m}' \mathbf{m}' = g_{ij}(u, x^k) dx^i dx^j
\]

where the functions $H^{(2)}$, $H^{(1)}$, $H^{(0)}$, $W_{i(1)}$ and $W^{(0)}$ do not depend on $v$. By calculating the Riemann tensor with respect to this null frame, we get that the Riemann tensor, $R$, has the following b.w. decomposition:

\[
R = (R_0)_0 + (R)_{-1} + (R)_{-2},
\]

where the components of $(R)_0$ do not depend on $v$ and involve only the functions $H^{(2)}$ and $W_{i(1)}$ and the transverse metric $g_{ij}(u, x^k)$.

Consider now the $n$th-covariant derivatives, symbolically written $\nabla^{(n)}R$. By using the Kundt frame, a covariant derivative of an arbitrary covariant tensor $T$ can be written symbolically:

\[
\nabla T = \partial T - \sum \Gamma \ast T,
\]

where $\partial$ are partial derivatives with respect to the frame, and $\Gamma$ are the connection coefficients. In the Kundt frame the connection coefficients of positive b.w. are all zero; consequently, the piece $\sum \Gamma \ast T$ cannot raise the b.w. Regarding the partial derivatives, $\ell^\mu \partial_\mu \equiv \partial_\ell$, and $\mathbf{n}$ raise and lower the b.w., respectively. The partial derivatives with respect to $\mathbf{m}$ are of b.w. 0. Therefore, if $T$ is of boost order 0, then the b.w. +1 and 0 components of $\nabla T$ will be

- b.w. +1: $\partial_\ell (T)_{0}$
- b.w. 0: $(\Gamma)_{0}(T)_{0}$, $\mathbf{m}(T)_{0}$, $\partial_\ell (T)_{-1}$.

First, from theorem II.5, we note that since these are $K_1$ they are $K_m$ implying $(\nabla^{(n)}R)_{b,0} = 0$; i.e. all positive b.w. components vanish. The highest b.w. terms are thus the b.w. 0 components. We thus need to consider the b.w. 0 components in more detail.

Consider first $\nabla R$. We note that the terms $(\Gamma)_{0}(R)_{0}$, $\mathbf{m}(R)_{0}$ all give the desired result. However, we need to investigate $\partial_\ell (R)_{-1}$ in more detail. This comes from the covariant derivative $\ell^\mu \nabla_{\mu}(R)_{-1} \equiv \nabla_{\ell}(R)_{-1}$. Now, we observe that the Bianchi identity $R_{\mu\nu\rho\sigma} = 0$ enables us to rewrite the troublesome derivatives $\nabla_\ell (R)_{-1}$ in terms of derivatives with respect to $\mathbf{n}$ and $\mathbf{m}$:

\[
R_{-jk;+} = -R_{-ik;j} - R_{-i++j}, \quad R_{-++;+} = -R_{-i+k;-} - R_{-;+-j}.
\]
Consequently, the components of $\nabla_+ (R)_{-1}$ can be written in terms of other well-behaving b.w.

0 terms.

Let us now show that we can always write the b.w. 0 terms, $\nabla_+ (\nabla^{(n)} R)_{-1}$, in terms of well-behaving b.w. 0 components of lower or equal order. We will show this by induction; therefore, assume that it is true for $\nabla_+ (\nabla^{(n)} R)_{-1}$, $n \geq 1$. These components have the form $R_{\mu \nu \alpha \beta : \delta_1, \ldots, \delta_n}$, where, by the induction assumption, $\delta_n = \{-, i\}$. We need to check the components of $\nabla_+ (\nabla^{(n)} R)_{-1}$:

$R_{\mu \nu \alpha \beta : \delta_1, \ldots, \delta_n} +$.

By the generalized Ricci identity we have that

$$R_{\mu \nu \alpha \beta : \delta_1, \ldots, \delta_n} - \delta_n + = R_{\mu \nu \alpha \beta : \delta_1, \ldots, \delta_n} - \delta_n + + \sum [R \ast \nabla^{(n-1)} R]_{\mu \nu \alpha \beta : \delta_1, \ldots, \delta_n} - \delta_n +.$$

We note that all of the terms on the right-hand side are well behaved; therefore, the left-hand side is well behaved also. Moreover, we also see that all of the b.w. 0 terms of the form $\nabla_+ (\nabla^{(n)} R)_{-1}$ can be written in terms of well-behaved components of lower or equal order. Therefore, by induction, components of $(\nabla^{(n)} R)_0$ are well behaved for all $n \geq 0$. The theorem is consequently proven.

Note that an alternative proof might be given by using $\tilde{\nabla} = \tilde{\nabla} + \tau$, where $\tilde{\nabla}$ is the corresponding connection with $H(1) = H(0) = W(0) = 0$ and $\tau$ is a tensor (the remaining piece). The tensor $\tau$ will be of boost order $-1$ (and so on).

Furthermore, there are a number of results that are a direct consequence of this theorem. For example, we can give conditions for when a Kundt spacetime is either I-symmetric or Kundt–CSI. We recall that for CSI metrics, I-non-degeneracy implies that the spacetime is curvature homogeneous to all orders; hence, an important corollary of the results of [1] is a proof of the CSI–Kundt conjecture in 4D [10], that for a 4D CSI spacetime then either the spacetime is locally homogeneous or a subclass of the Kundt spacetimes. It is plausible that this result generalizes to higher dimensions. In the context of string theory, it is of considerable interest to study higher dimensional Lorentzian CSI spacetimes. In particular, a number of $N$-dimensional CSI spacetimes are known to be solutions of supergravity theory when supported by appropriate bosonic fields [15].

First, we have that

**Corollary 2.8.** A degenerate Kundt spacetime is I-symmetric if and only if the corresponding spacetime with $H(1) = H(0) = W(0) = 0$ is also I-symmetric.

We can also prove the following:

**Theorem 2.9.** If the spacetime is of type $D^k$, then the components of the curvature tensors are determined by the scalar curvature invariants.

**Proof.** First we note that a type $D^k$ spacetime possesses at least one Killing vector (actually, at least three, see corollary that follows), namely a boost isotropy. Therefore, let $H = SO(1, d)$ be the isotropy group of all the curvature tensors (which necessarily must be at least of dimension 1, i.e. $d \geq 1$). Let $g_{AB}$ be the projector onto the tangent subspace of the orbits of $H$. Furthermore, let $g_{IJ}$ be defined so that $(g_{\mu \nu}) = (g_{AB}) \oplus (g_{IJ})$. This implies that any curvature tensor can be orthogonally decomposed as a tensor; thus, symbolically

$$\nabla^{(k)} R = \sum \phi \otimes R,$$

where $\phi = (\phi_{ABC, D})$ is a scalar representation of $H$, and $R = (R_{IJ, K})$ is a tensor over $g_{IJ}$. Consequently, if $1 + d + \tilde{d} = \text{dimension of spacetime}$, then $\nabla^{(k)} R$ can be considered as an $SO(\tilde{d})$-tensor. Hence, all scalar curvature invariants can be considered as $SO(\tilde{d})$-invariants.
Since this group is compact, the group action separates orbits, and thus the components are
determined by the scalar invariants. □

Note that this means that we can also say that a type $D^k$ spacetime is characterized by
its invariants (however, in a different sense than $I$-non-degeneracy) [11]. This result also
immediately implies that, for example, a spacetime which is of type $D^4$ and CSI is necessarily
Kundt and homogeneous [4]. In fact, in general, type $D^k$ spacetimes possess at least three
Killing vectors:

**Corollary 2.10.** A spacetime of type $D^k$ possesses three Killing vector fields with two-
dimensional timelike orbits.

**Proof.** We have already pointed out that such a spacetime possesses a boost isotropy.
Consider the Lie derivatives $\ell_I$ and $\ell_n I$ where $I$ is any polynomial curvature invariant. Since
this is of type $D^k$, we must have

$$\ell_I I = \ell(I) = \ell^\mu \nabla_\mu I = 0$$

(similarly for $n$). Therefore, $\ell^\mu \nabla_\mu R = 0$ for any curvature tensor $R$ and thus there exists a
Killing vector $\ell$ (similarly for $n$). Hence, the spacetime possesses three Killing vectors ($\ell$, $n$
and a boost). □

Perhaps a more useful consequence is

**Corollary 2.11.** For a spacetime of (aligned) type II to all orders, the boost weight 0
components are determined by the curvature invariants.

This corollary follows simply from the fact that when constructing a complete contraction
of an arbitrary tensor of type II, only the b.w. 0 components will contribute; i.e. if $T$ is of
type II, meaning $T = (T)_0 + (T)_{-1} + (T)_{-2} + \ldots$, then for a complete contraction

$$\text{Contr}[T] = \text{Contr}[(T)_0].$$

Consequently, the invariants of $T$ and $(T)_0$ are identical. Thus, since $(T)_0$ is of type D, the
above theorem implies that its components are determined from the invariants.

This further implies a proof of the ‘CSI’ conjecture:

**Corollary 2.12.** A (degenerate) Kundt CSI spacetime is a spacetime for which there exists
a frame with a null vector $\ell$ such that all components of the Riemann tensor and its covariant
derivatives in this frame have the property that (i) all positive boost weight components (with
respect to $\ell$) are zero and (ii) all zero boost weight components are constant.

Finally, it should be possible to prove an extension of the type $D^k$ result stated in [4]:

**Conjecture 2.13.** If the curvature tensors to all orders are type II (or simpler) and aligned
(i.e. $\nabla^{(0)} R$ is of type II), then the spacetime is Kundt.

It then follows, under some appropriate assumptions [11], that degenerate Kundt
spacetimes with $H^{(1)} = H^{(0)} = W^{(0)} = 0$ are of Riemann type $D^k$.

In the future we hope to establish all of the higher dimensional generalizations of
the results obtained in 4D. As noted above, the first step is to investigate the curvature
operators in higher dimensions and to classify these for the various algebraic types
[9].
3. Discussion

We have found that the degenerate Kundt metrics are not determined by their curvature invariants (in the sense that they are not $T$-non-degenerate). Degenerate Kundt spacetimes are also special in a number of other ways including, for example, their holonomy structure, which may lead to novel and fundamental physics. Indeed, in a degenerate Kundt spacetime it is not possible to define a unique timelike curvature operator, and hence a unique timelike direction, and a $1 + (n - 1)$ spacetime splitting of the spacetime is not possible.

Supersymmetric solutions of supergravity theories have played an important role in the development of string theory (see, for example, [12]). The existence of Killing spinors accounts for much of the interest in metrics with special holonomy in mathematical physics. Supersymmetric solutions in $M$-theory that are not static admit a covariantly constant null vector (CCNV) [2]. The isotropy subgroup of a null spinor is contained in the isotropy subgroup of the null vector, which in arbitrary dimensions is isomorphic to the spin cover of $ISO(n - 2)$. A CCNV metric is a degenerate Kundt metric (1) with $H_v = 0$ and $W_i,v = 0$. This class includes a subset of the Kundt–CSI and the VSI spacetimes as special cases. The VSI and CSI degenerate Kundt spacetimes are of fundamental importance since they are solutions of supergravity or superstring theory, when supported by appropriate bosonic fields [15].

The classification of holonomy groups in Lorentzian spacetimes is quite different from the Riemannian case. A Lorentzian manifold $M$ is either completely reducible, and so $M$ decomposes into irreducible or flat Riemannian manifolds and a manifold which is an irreducible or a flat Lorentzian manifold or $(\mathbb{R}, -dt)$, or $M$ is not completely reducible, which leads to the existence of a degenerate (one-dimensional) holonomy invariant lightlike subspace (the Lorentzian manifold decomposes into irreducible or flat Riemannian manifolds and a Lorentzian manifold with indecomposable, but non-irreducible holonomy representation), which gives rise to the recurrent null vector (RNV) and CCNV (Kundt) spacetimes [13, 14]. (A RNV metric has holonomy $Sim(n - 2)$ and the metric belongs to the class of Kundt metrics (1) with $W_i = W_i(u, x^\xi)$, but is not necessarily a degenerate Kundt metric.)

Therefore, the Kundt spacetimes that are of particular physical interest are degenerately reducible, which leads to complicated holonomy structure and various degenerate mathematical properties. Indeed, it could be argued that a complete understanding of string theory is not possible without a comprehensive knowledge of the properties of the Kundt spacetimes [14]. For example, as noted above, a degenerate Kundt spacetime is not completely classified by its set of scalar polynomial curvature invariants (i.e. they have important geometrical information that is not contained in the scalar invariants). All VSI spacetimes and CSI spacetimes that are not locally homogeneous (including the important CCNV subcase) belong to the degenerate Kundt class [2, 7]. In these spacetimes all of the scalar invariants are constant or zero. This leads to interesting problems with any physical property that depends essentially on scalar invariants, and may lead to ambiguities and pathologies in models of quantum gravity or string theory.

As an illustration, in many theories of fundamental physics there are geometric classical corrections to general relativity. Different polynomial curvature invariants (constructed from the Riemann tensor and its covariant derivatives) are required to compute different loop orders of renormalization of the Einstein–Hilbert action. In specific quantum models such as supergravity there are particular allowed local counterterms [16]. All classical corrections are zero in VSI spacetimes (and constant in CSI spacetimes). Indeed, it is possible that a Lorentzian Kundt spacetime does not even allow for a low-order perturbative expansion.
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