Massive tensor modes carry more energy than scalar modes in quadratic gravity

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(Dated: June 15, 2022)

Abstract

Over the last two decades, motivations for modified gravity have emerged from both theoretical and observational levels. $f(R)$ and Chern-Simons gravity have received more attention as they are the simplest generalization. However, $f(R)$ and Chern-Simons gravity contain only the scalar degree of freedom and, as a result, do not include other modes of modified theories of gravity. In contrast, quadratic gravity (also referred to as Stelle gravity) is the most general second-order modification to 4-D general relativity and contains massive tensor modes that are not present in $f(R)$ and Chern-Simons gravity. Using two different physical settings — the gravitational wave energy-flux measured by the detectors and the backreaction of the emitted gravitational radiation on the spacetime of the remnant black hole — we demonstrate that massive tensor modes carry more energy than scalar modes. Our analysis shows that the effects are pronounced for intermediate-mass black holes, which are prime targets for LISA.

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In quantum gravity, it is impossible to localize an event with a precision smaller than the Planck length ($\ell_{\text{Pl}}$) [1–4]. From the hoop conjecture and the uncertainty principle, we can deduce the existence of a Planck-sized ball [5]. Thus, the operational significance of the concept of space-time points is lost [6–9]. Most approaches to quantum gravity incorporate the Planck length by considering extended structures, rather than point particles, as fundamental blocks [10, 11]. Generalized Uncertainty Principle (GUP) is a phenomenological approach that introduces modifications to Heisenberg uncertainty principle in the ultraviolet regime and studies its consequences [12, 13]. Kempf et al. considered the following simplest modification to the canonical commutation relation [12]:

$$[\hat{x}, \hat{p}] = i\hbar (1 + \beta p^2) \quad \text{leading to} \quad \Delta x \geq \hbar \sqrt{\beta}.$$  

$\beta$ is a parameter characterizing the GUP whose value needs to be fixed by observations. Since the above GUP is non-relativistic, it is impossible to compute GUP corrections in relativistic field theories. Recently, Todorinov et al. [14] extended the above GUP to a generic class of covariant GUPs:

$$[x^\mu, p^\nu] = i\eta_{\mu \nu} (1 - \gamma p^\sigma p_\sigma) - 2i\gamma p^\mu p^\nu, \quad \gamma = \gamma_0 \ell_{\text{Pl}}^2,$$

and studied the phenomenological features of such GUPs for scalar, spinor, and vector fields [15, 16]. $\gamma_0(\ll 1)$ is a numerical constant to be fixed by observations. In Ref. [17], the authors established a correspondence between the GUP modified dynamics of a massless spin-2 field and quadratic gravity (QG) with suitably constrained parameters. Specifically, the authors showed that the 4-D gravity action [18, 19]

$$S_{\text{QG}} = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \alpha R_{\mu \nu} R^{\mu \nu} + \beta R^2 \right], \quad \kappa^2 = \frac{8\pi G}{c^4}.$$  

is a classical manifestation of the imposition of a momentum cutoff at the quantum gravity level when $\alpha = 2\beta = \gamma/\kappa^2$. QG or Stelle gravity — unlike $f(R)$ — has extra massive tensor and scalar modes [19]. Intriguingly, for this class of Stelle theories, the masses of the scalar and tensor modes coincide $(1/\sqrt{2\gamma})$. Also, this class of Stelle theories does not have malicious ghosts [20].

Given that the Stelle gravity is the most general QG in 4-D, we ask the following questions: What is the role of the massive tensor modes compared to the scalar modes? What are the leading order corrections of QG compared to $f(R)$ gravity? This work addresses these questions by evaluating the corrections to the radiation in QG.
Over the last two decades, motivations for modified gravity have emerged from both theoretical and observational levels \[21-25\]. Since \( f(R) \) and Chern-Simons gravity are the simplest generalization, they have received more attention. However, \( f(R) \) and Chern-Simons gravity contain only the scalar degree of freedom and hence do not contain other modes of modified theories of gravity. We explicitly show that massive tensor modes carry more energy than the scalar modes using two different physical settings: the energy flux of the gravitational waves, measured by an asymptotically placed gravitational wave detector, and the backreaction of the emitted gravitational radiation on the spacetime of the remnant black hole. Thus, we show that the leading order gravitational radiation correction in QG is linear in the coupling constant compared to \( f(R) \), where the corrections are quadratic in flat space-time \[26\].

To show this explicitly, we start with the GUP inspired version of QG action (2), where \( 17 \)
\[
\alpha = \gamma/\kappa^2, \quad \beta = \gamma/2\kappa^2.
\]
Varying action (2) with respect to the metric leads to:
\[
G_{\mu\nu} \equiv G_{\mu\nu} - 2\gamma \Box G_{\mu\nu} - 4\gamma R^{\rho\sigma}R_{\mu\rho\nu\sigma} + 2\gamma RR_{\mu\nu} + \gamma g_{\mu\nu} \left[ R^{\rho\sigma}R_{\rho\sigma} - R^2/2 \right] = 0.
\]
Note that \( R^2 \) term corresponds to an extra massive spin-0 (scalar) modes, while \( R^{\mu\nu}R_{\mu\nu} \) term corresponds to the massive tensor (spin-2) modes \[17, 18\]. To separate these extra degrees of freedom and obtain individual contributions, we linearize the field equation (3) about the Minkowski space-time (\( \eta_{\mu\nu} \)):
\[
g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \quad g^{\mu\nu} = \eta^{\mu\nu} - \epsilon h^{\mu\nu} \quad \text{and} \quad h = \eta^{\mu\nu}h_{\mu\nu},
\]
where \( h_{\mu\nu} \) is the metric perturbation and \( \epsilon \) is a book-keeping parameter. This leads to the following linearized equations:
\[
G^{(1)}_{\mu\nu} \equiv (1 - 2\gamma \Box) \left( R^{(1)}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R^{(1)} \right) = 0,
\]
where \( \Box \equiv \eta^{\rho\sigma} \partial_\rho \partial_\sigma \) and
\[
R^{(1)}_{\mu\nu} = \frac{1}{2} \left( \partial_\mu \partial_\rho h^{\rho\nu} + \partial_\nu \partial_\rho h^{\rho\mu} - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} \right),
\]
\[
R^{(1)} = \eta^{\mu\nu} R^{(1)}_{\mu\nu} = \left( \partial_\mu \partial_\rho h^{\rho\mu} - \Box h \right).
\]
As expected in higher-derivative gravity theories, Eq. (5) contains fourth-order derivatives whose trace is given by:

$$\Box R^{(1)} - R^{(1)}/2\gamma = 0. \tag{8}$$

To separate the contribution of the different spin-components from linearized field equation (5), we use the following ansatz for the metric perturbations:

$$h_{\mu\nu} = \left[ \psi_{\mu\nu} - \frac{\eta_{\mu\nu}}{2} \psi \right] + \left[ C_1 + \frac{C_2}{4} \right] R^{(1)} \eta_{\mu\nu} - C_2 \hat{R}^{(1)}_{\mu\nu}, \tag{9}$$

where \( \hat{R}^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{4} \eta^{\mu\nu} R^{(1)} \) is the traceless part of \( R^{(1)} \); \( \psi = \eta^{\alpha\beta} \psi_{\alpha\beta} \) and \( C_1, C_2 \) are arbitrary constants. To our knowledge, this is the first time an explicit separation of metric fluctuations is used in this context, and we would like to emphasize the following points:

First, in general relativity (GR), the constants \( C_1 \) and \( C_2 \) vanish because the linearized field equations only contain massless spin-2 (graviton) modes. Thus, in GR, \( \psi_{\mu\nu} \) reduces to the familiar trace reversed metric perturbations. Second, in the case of Starobinsky model, the field equations contain additional contribution from the massive scalar mode only (\( R^{(1)} \)), hence \( C_1 = -6\gamma \) and \( C_2 = 0 \) [26, 27]. Lastly, like in GR, we use the de-Donder gauge on the residual massless graviton mode (\( \psi_{\mu\nu} \)) to restrict the gauge freedom:

$$\partial^\mu \psi_{\mu\nu} = 0. \tag{10}$$

Substituting Eq. (9) in Eqs. (6,7) and using the gauge conditions (10), we get:

$$2 \mathcal{G}^{(1)}_{\mu\nu} = -\Box \psi_{\mu\nu}. \tag{11}$$

where we set \( C_1 = -2\gamma \) and \( C_2 = 4\gamma \). Taking cognizance of Eqs. (5),(11) and (8), we get the propagation equation for the massless and massive tensor modes as,

$$\Box \psi_{\mu\nu} = 0, \tag{12}$$

$$\Box \hat{R}^{(1)}_{\mu\nu} - \hat{R}^{(1)}_{\mu\nu}/2\gamma = 0, \tag{13}$$

whereas, the propagation of the massive scalar modes, \( R^{(1)} \), is governed by Eq. (8). Thus, the theory (2) is described by a massless graviton (\( \psi_{\mu\nu} \)); a massive tensor (\( \hat{R}^{(1)}_{\mu\nu} \)) and a massive scalar (\( R^{(1)} \)) [17–19]. The positivity of the parameter \( \gamma \) ensures that the massive modes are non-tachyonic. Repeating the analysis for the Ricci-flat background by using the following ansatz:

$$h_{\mu\nu} = \left( \psi_{\mu\nu} - \frac{1}{2} \psi g_{\mu\nu} \right) - g_{\mu\nu} \gamma R^{(1)} - 4\gamma \hat{R}^{(1)}_{\mu\nu}, \tag{14}$$
the equations of motion for the massless (12) and massive tensor modes (13) in the transverse traceless gauge ($\nabla^\mu \psi_{\mu\nu} = 0; \bar{g}^{\mu\nu} \psi_{\mu\nu} = 0$) are:

$$\Box \psi_{\mu\nu} + 2 \bar{R}_{\mu\alpha\nu\beta} \psi^{\alpha\beta} = 0,$$

(15)

$$\Box \hat{R}_{\mu\nu}^{(1)} + 2 \bar{R}_{\mu\alpha\nu\beta} \hat{R}^{(1)\alpha\beta} - \hat{R}_{\mu\nu}^{(1)}/(2\gamma) = 0,$$

(16)

where $\bar{g}_{\mu\nu}$ is the background metric, $\bar{R}_{\mu\alpha\nu\beta}$ is the background Riemann tensor, $\Box \equiv \nabla_\sigma \nabla^\sigma$ with $\nabla^\sigma$ being the covariant derivative due to the background spacetime and $\hat{R}_{\mu\nu}^{(1)} = R_{\mu\nu}^{(1)} - \bar{g}^{\mu\nu} R^{(1)}/4$. The propagation of the massive spin-0 mode is still governed by Eq. (8), with the above defined D’Alembertian operator. The mass degeneracy between the two massive (scalar and tensor) modes demonstrates that they are not completely independent and are related by linearized Bianchi identities:

$$4 \partial_\mu \hat{R}_{\mu\nu}^{(1)} - \bar{g}^{\mu\nu} \partial_\mu R_{\mu\nu}^{(1)} = 0.$$

(17)

Having separated the metric fluctuations into massless and massive modes in Ricci flat background, we now evaluate the energy and momentum carried by the gravitational waves in degenerate-Stelle gravity. To go about that, we expand the field equation to second-order in $\epsilon$:

$$\mathcal{G}_{\mu\nu} = \bar{G}_{\mu\nu} + \epsilon \mathcal{G}_{\mu\nu}^{(1)} + \epsilon^2 \mathcal{G}_{\mu\nu}^{(2)} = 0,$$

(18)

where $\bar{G}_{\mu\nu}$ represents the background quantity, $\mathcal{G}_{\mu\nu}^{(1)}$ are linear in perturbations ($h_{\mu\nu}$) and $\mathcal{G}_{\mu\nu}^{(2)}$ are quadratic in $h_{\mu\nu}$. Through second-order perturbations, Isaacson established a procedure to obtain an effective stress-energy tensor for gravitational radiation [28, 29]. Specifically, the effective stress-energy tensor of the emitted gravitational waves is obtained by averaging over a length-scale $l$ such that $\lambda \ll l \ll \Lambda$, where $\lambda$ is the wavelength of the fluctuations and $\Lambda$ is the system size. The short wavelength components will be averaged out, yielding a gauge-invariant measure of the effective gravitational wave (GW) stress-energy tensor [30]:

$$t_{\mu\nu}^{\text{GW}} = \frac{1}{\kappa^2} \langle \mathcal{G}_{\mu\nu} \rangle = -\frac{1}{\kappa^2} \langle \mathcal{G}_{\mu\nu}^{(2)} \rangle.$$

(19)

In the Ricci-flat background, we get

$$t_{\mu\nu}^{\text{GW}} = \frac{1}{\kappa^2} \left[ \left( \frac{1}{4} \nabla_\mu \psi^{\rho\sigma} \nabla_\nu \psi_{\rho\sigma} - \gamma \left( \nabla_\mu \psi^{\rho\sigma} \nabla_\nu \hat{R}_{\rho\sigma}^{(1)} + \nabla_\nu \psi^{\rho\sigma} \nabla_\mu \hat{R}_{\rho\sigma}^{(1)} \right) - \gamma^2 \left( 4 \nabla_\mu \hat{R}_{\nu\rho\sigma}^{(1)} \nabla_\nu \hat{R}_{\rho\sigma}^{(1)} - \nabla_\mu \hat{R}_{\nu\rho\sigma}^{(1)} \nabla_\nu \hat{R}_{\rho\sigma}^{(1)} \right) \right) + \langle A_{\mu\nu} \rangle + \gamma \langle B_{\mu\nu} \rangle + \gamma^2 \langle C_{\mu\nu} \rangle + \gamma^3 \langle D_{\mu\nu} \rangle \right].$$

(20)
where $A_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu}, D_{\mu\nu}$ are proportional to the background Riemann tensor (and are explicitly given in Appendix). This is the first key result of this work, regarding which we would highlight the following points: First, in the Minkowski limit (as in the case of GW detectors), $A_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu}, D_{\mu\nu}$ vanish and $t_{\mu\nu}^{GW}$ is proportional to partial derivatives of $\psi^{\rho\sigma}$ and $\tilde{R}(1)$. Second, the first term within the triangular bracket gives the dominant contribution — the contribution of the graviton mode as in GR [28]. However, the crucial difference is the dominant contribution of the massive tensor and scalar mode. The massive tensor mode contribution is proportional to $\gamma$ (A.38), whereas the massive scalar mode contribution is proportional to $\gamma^2$ (A.39). Thus, the above expression explicitly shows that the massive tensor mode carries more energy than the massive scalar modes. Third, while we have used Isaacson’s approach to obtain the stress-tensor, as demonstrated in Ref. [31], other approaches also give similar result. Fourth, the leading order contribution of the massive tensor modes is opposite to that of the graviton modes. Lastly, the corrections by QG to GR are much larger than $f(R)$ gravity [26, 32]. Consequently, our study demonstrates that the $f(R)$ theories overlook crucial information concerning the massive tensor modes.

In what follows, we use the GW stress-energy tensor (20) to examine the effect of the massive tensor modes under two distinct physical settings. First, we investigate the energy flux of GWs as measured by the GW detectors at asymptotic infinity. We then estimate the change in the spherically symmetric metric caused by the backreaction of the emitted GWs near the horizon.

**GW energy flux:** The energy of the gravitational wave within a spatial volume $V$ is [33, 34]

$$E_V = \int_V d^3x \ t_{00}^{GW}. \hspace{1cm} (21)$$

Using the stress-tensor conservation equation ($\partial_\mu t_{\mu\nu}^{GW} = 0$) for the far-away observer, the power carried by the gravitational waves is [31, 34]

$$\frac{dE}{d(ct)} = -\int_S n_i t_{0i}^{GW} dA \hspace{1cm} (22)$$

where $S$ is the surface enclosing the volume $V$ and $n_i$ is the unit outward normal to $S$. The negative sign signifies that an outward propagating gravitational wave carries away energy from the source. Thus, plugging the Minkowski-limit of Eq. (20) in Eq. (22) and considering $S$ to be a spherical surface at a large distance from the source, the gravitational wave energy
flux passing through the detector is:
\[
\frac{dE}{dt dA} = \frac{c^3}{8 \pi G} \left\langle \frac{1}{4} \psi^{\rho \sigma} \partial_r \psi_{\rho \sigma} - \gamma \left(1 + \frac{c}{v} \right) \psi^{\rho \sigma} \dot{\hat{R}}^{(1)}_{\rho \sigma} - \gamma^2 \frac{c}{v} \left(4 \dot{\hat{R}}^{(1)}_{\rho \sigma} \dot{\hat{R}}^{(1)}_{\rho \sigma} - \left(\dot{\hat{R}}^{(1)}\right)^2\right) \right\rangle ,
\] (23)

To make the calculations transparent, we assume that all the three (graviton, massive tensor, and scalar) modes of the following form:
\[
\psi^{\mu \nu} = \zeta^{\mu \nu} e^{i k_\mu x^\mu} + \zeta^{\mu \nu} e^{-i k_\mu x^\mu}; \quad \hat{R}^{(1)}_{\mu \nu} = \phi e^{i q_\mu x^\mu} + \phi^* e^{-i q_\mu x^\mu};
\] (24)
where \( \zeta_{\mu \nu}, \theta_{\mu \nu} \) and \( \phi \) depend on \( r \) and \( t \),
\[
|k| = \omega/c \quad ; \quad |w(q)| = \sqrt{\omega^2/c^2 - 1/2 \gamma}
\] (25)
with \( k^0 = w^0 = q^0 = \omega/c, \quad k = (k^i), \quad w = (w^i), \quad \text{and} \quad q = (q^i). \quad \omega > c/\sqrt{2 \gamma} \quad \text{and} \quad q > c/\sqrt{2 \gamma}
\]
corresponds to oscillatory solution for \( \hat{R}^{(1)}_{\mu \nu} \) and \( R^{(1)} \), respectively. A wave propagating radially outward (\( \Phi^{\mu \nu} \)) at large distances from the source can be represented to fall-off radially [34, 35]:
\[
\Phi^{\mu \nu} = \frac{1}{r} \chi^{\mu \nu}(t_r);
\] (26)
where \( \chi^{\mu \nu}(t_r) \) is an arbitrary function of the retarded time \( t_r = t - r/v, \) and \( v = c/\sqrt{1 - \frac{1}{2 \gamma} \left( \frac{c}{\omega} \right)^2} \) is the speed of the massive modes. Thus,
\[
\frac{\partial}{\partial r} \chi^{\mu \nu}(t - r/v) = -\frac{1}{v} \frac{\partial}{\partial t} \chi^{\mu \nu}(t - r/v) \implies \frac{\partial}{\partial r} \zeta^{\mu \nu} = -\frac{c}{v} \partial_0 \zeta^{\mu \nu} + O \left( \frac{1}{r^2} \right)
\] (27)
Substituting Eq. (24) in Eq. (23) and using Eq. (27) in the resultant expression leads to the energy flux on the GW detectors:
\[
\frac{dE}{dt dA} = \frac{c^3}{8 \pi G} \left\langle \frac{1}{4} \psi^{\rho \sigma} \psi_{\rho \sigma} - \gamma \left(1 + \frac{c}{v} \right) \psi^{\rho \sigma} \dot{\hat{R}}^{(1)}_{\rho \sigma} - \gamma^2 \frac{c}{v} \left(4 \dot{\hat{R}}^{(1)}_{\rho \sigma} \dot{\hat{R}}^{(1)}_{\rho \sigma} - \left(\dot{\hat{R}}^{(1)}\right)^2\right) \right\rangle ,
\] (28)
where dot denotes derivative w.r.t \( t \). Here again, we note that the dominant contribution comes from the graviton modes with leading order corrections (\( \gamma \)) from the massive tensor modes; the massive scalar modes contribute only in the second order. Hence, as expected, the measured energy energy-flux in the case of QG is lower than that predicted by GR. In other words, the massive tensor modes carry more energy than the scalar modes. Since, this analysis is for the Minkowski background, \( \gamma \mathcal{B}_{\mu \nu} \) (in Eq. 20) is zero. However, in the case of curved geometry, \( \gamma \mathcal{B}_{\mu \nu} \) contribution might be significantly larger than the linear order term in the above expression.
In the case of GW detectors, the average $\langle \ldots \rangle$ is purely a temporal average \[34\], and the total energy flowing through the unit area of the detector is:

$$\frac{dE}{dA} = \frac{c^3}{8\pi G} \int_{-\infty}^{\infty} dt \left[ \frac{1}{4} \dot{\psi}^{\rho\sigma} \dot{\psi}_{\rho\sigma} - \gamma \left( 1 + \frac{c}{\nu} \right) \dot{\psi}^{\rho\sigma} \dot{R}^{(1)}_{\rho\sigma} - \gamma^2 \frac{c}{\nu} \left( 4 \dot{R}^{(1)}_{\rho\sigma} \dot{R}^{(1)}_{\rho\sigma} - \left( \dot{R}^{(1)} \right)^2 \right) \right]. \quad (29)$$

Note that the above analysis is strictly valid for $\omega > c/\sqrt{2\gamma}$ and $q > c/\sqrt{2\gamma}$ corresponding to oscillatory solutions for the two massive modes. In the second physical setting, we will relax this condition and obtain the contribution of the massive tensor modes.

**Backreaction of the emitted GWs:** To study the backreaction of the emitted GWs on the background black hole space-time, we assume that the background space-time is spherically symmetric Ricci flat and is an exact solution of the QG action \(2\). Though, Schwarzschild solution in Stelle gravity is known to suffer from Gregory-Laflamme instability \(\ell = 0\) mode instability) when the scalar mode is non-propagating \[36–38\]; it is stable when the scalar mode is dynamical as in the present case.

The backreaction on the background space-time is obtained by evaluating the GW stress-energy tensor \(t^{GW}_{\mu\nu}\) w.r.t the Schwarzschild metric. Having obtained this, we then solve the effective Einstein’s equations:

$$G^{\nu}_{\mu} = \kappa^2 t^{GW}_{\mu\nu}, \quad (30)$$

where the Einstein tensor \(G^{\nu}_{\mu}\) is evaluated for spherically symmetric space-time in dimensionless, Eddington-Finkelstein (EF) coordinates \[33\]:

$$ds^2 = -e^{2\nu}dV^2 + 2e^{\nu+\lambda}dVd\rho + \rho^2 d\Omega^2, \quad (31)$$

$\nu \equiv \nu(V, \rho)$ and $\lambda \equiv \lambda(V, \rho)$ encode the corrections from the emitted GWs, and $d\Omega^2$ is the metric on unit 2-sphere. Note that $V$ and $\rho$ are dimensionless like $\theta$ and $\phi$. Regarding Eqs. \([30, 31]\), the following points are in order: First, the ansatz \(14\) assumes that all the metric components are dimensionless. Hence, we have rescaled all the coordinates to be dimensionless. Second, $V = \text{constant}$ hypersurfaces represent the ingoing null geodesics. As mentioned, the background metric is assumed to be Schwarzschild black hole; hence, $e^{2\nu} = e^{-2\lambda} = 1 - 2M_0/\rho$ where $M_0$ is the dimensionless mass parameter (setting $c = G = 1$). For $M(V)$, the above metric gives the Vaidya line element. Third, the ingoing EF coordinates are smooth across the horizon for ingoing null geodesics and are suitable for analyzing the gravitational waveform close to the horizon \[39, 40\] as well as the shift in the horizon radius due to the backreaction. Finally, since $t^{GW}_{\mu\nu}$ contains the contributions of graviton and
massive modes, the LHS of Eq. (30) only contains the Einstein gravity. Note that the radiation from a remnant black hole decreases its energy content, inducing a change in the metric.

Since the dominant contributions to the GW stress-energy tensor (20) come only from the tensor modes, to evaluate their effects on the metric, we use the following ansatz:

\[
\psi^{\mu\nu} = \sigma^{\mu\nu} \psi(V, \rho) Y^m_{\ell}(\theta, \phi); \quad \hat{R}^{(1)\mu\nu} = \iota^{\mu\nu} P(V, \rho) Y^m_{\ell}(\theta, \phi),
\]

(32)

where \(\sigma^{\mu\nu}, \iota^{\mu\nu}\) are constant, traceless (polarization) tensors, \(\psi(V, \rho), P(V, \rho)\) are scalar functions and \(Y^m_{\ell}(\theta, \phi)\) are spherical harmonics. The above assumptions essentially replaces the tensor modes \(\psi^{\mu\nu}, \hat{R}^{(1)\mu\nu}\) by scalar functions, where we ignored the non-linear transformation among the components of the individual tensor modes. To obtain the leading order corrections, we concentrate on the backreaction effect due to the \(\ell = 2\) and \(m = 0\) mode of the gravitational waveform and obtain the average contribution over the entire solid angle. Note that the \((\ell = 2, m = 0)\) mode contributes to the non-linear memory of the GWs, which is otherwise difficult to observe in ground-based GW detectors [41, 42] and nontrivial to extract in numerical relativity simulations. We can trivially extend the analysis to other modes.

We assume the modified black hole to be described by the generalized spherically symmetric metric ansatz proposed by Johannsen and Psaltis (JP) [43, 44]. In the (dimensionless) EF coordinates (31), we have:

\[
\begin{align*}
\varepsilon^{2\nu} &= f(\rho) \left[1 - \frac{2\tilde{M}}{\rho}\right]; \\
\varepsilon^{2\lambda} &= f(\rho) \frac{1}{1 - 2\tilde{M}/\rho}; \\
f(\rho) &= \sum_{n=0}^{\infty} \tilde{c}_n \left(\frac{\tilde{M}}{\rho}\right)^n.
\end{align*}
\]

(33)

where \(\tilde{c}_0 = 1\) and the first few coefficients of the expansion can be constrained from the PPN-like parameters [43]. In the limit of \(\tilde{c}_n = 0, (n > 0)\), JP metric reduces to the Schwarzschild metric. The event horizon of the corresponding black hole is at \(\rho = 2\tilde{M}\) and the (dimensionless) ADM mass is \(M_{\text{ADM}} = \tilde{M}(1 - \tilde{c}_1/2)\) [44]. As mentioned earlier, the remnant black hole decreases its energy content, inducing a change in the metric mass-function from the initial, dimensionless Schwarzschild value \((M_0)\) [45, 46]:

\[
\tilde{M} = M_0 + \Delta_M(V)
\]

(34)

Assuming the mode functions to be regular and slowly varying in \(\rho\) close to horizon \((\Delta = \rho - 2M_0 << 1)\) [39, 40, 47], and expanding both sides of Eq. (30) for the \(\rho - V\)
component, we get (See Appendix for details):

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{\tilde{e}_n} \frac{\Delta'_M(V)}{M_0^2} \approx \frac{M_0^4 \psi'^2}{21\pi} \sum_{(k,l)=\{(0,0),(2,3)\}} C^{kl} \left[ 1 - \frac{k^{kl} \gamma P'}{\sigma^{kl} \psi'} \right]$$  \hspace{1cm} (35)

where \(\{C^{00}, C^{23}\} = \{29(o^{00})^2, 20(o^{23})^2\}\) and \(\gamma\) denotes derivative with respect to \(V\). The expression is evaluated to the leading order in \(\Delta\) and \(M_0\). Integrating from some initial \(V = V_0\), when \(\Delta_M(V = V_0) = 0\) to some final \(V = V_1\), we get,

$$\Delta_M(V_1) = A \int_{V = V_0}^{V_1} dV \sum_{(k,l)=\{(0,0),(2,3)\}} C^{kl} \left[ 1 - \frac{k^{kl} \gamma P'}{\sigma^{kl} \psi'} \right] ; \quad A = \frac{M_0^6 \psi'^2}{21\pi} \sum_{n=0}^{\infty} \frac{\tilde{e}_n}{2^{n-1}}$$  \hspace{1cm} (36)

This is the third key result regarding which we would like to stress the following: First, the massive and massless tensor modes contribute oppositely to the change in the mass function, hence the shift in the horizon radius. Second, since \(\Delta_M\) is proportional to the sixth power of \(M_0\), it implies that the larger the mass of the perturbed black hole, the larger the corrections to the change in the mass. This assumes particular importance for intermediate-mass black holes which are prime targets for LISA. Rezzolla and Zhidenko [44] proposed an improvisation of the JP metric in which, in the near-horizon limit, the Taylor expansion is replaced by an expansion in continued fractions (CF) which has better convergence properties. Moreover, the CF expansion coefficients can be expressed in the JP parameters; hence we can trivially extend our results to the Rezzolla-Zhidenko metric [44].

Summary and Discussions: In this work, we examined gravitational radiation in QG. We explicitly decomposed the GWs in Stelle gravity into massless and massive tensor and scalar modes. We demonstrated that the dominant contribution to the GW stress-energy tensor comes from the graviton modes, with leading order corrections coming from the massive tensor modes, which are absent in \(f(R)\) gravity theories. In contrast, the effects of the massive tensor modes are the inverse of those of the GR modes. In the context of GW detectors, this results in a decreased energy flux measurement. We also provided an estimate of the backreaction effect of the GW emission on the background spacetime, where we once again observe that the massive tensor modes decrease the rate of mass-change and the rate of shift in the horizon radius. Our results indicate that intermediate-mass black holes (prime targets for LISA) might be good candidates to test these aspects of modified gravity theories. Focusing on the \(\ell = 2, m = 0\) modes, our analysis suggests that the backreaction effect may
play a crucial role in the study of nonlinear memory of GWs in modified gravity theories. These are currently under investigation.

The decrease in measured energy flux and the decreased rate of horizon shift and mass change due to massive tensor (spin-2) modes indicate the existence of quasi-bound states of massive spin-2 modes surrounding the black hole [48]. In the context of rotating black hole geometries, where this may lead to the formation of superradiantly induced spin-2 boson clouds, the question assumes greater significance. A detailed analysis of the massive spin-2 dynamics can resolve this question. However, such an analysis is beyond the scope of this paper, so we will leave it for future work.

ACKNOWLEDGMENTS

The authors thank Susmita Jana, S. Mahesh Chandran, and T. Parvez for the discussions. SX is financially supported by the MHRD fellowship at IIT Bombay. This work is supported by SERB-MATRICS grant.

APPENDIX

$A_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu}, D_{\mu\nu}$ in Eq. (20) are:

\begin{align}
A_{\mu\nu} &= -\tilde{R}_{\mu\lambda\rho\sigma}\psi_\alpha^\mu\psi_\rho^\lambda - \frac{1}{4}g_{\mu\nu}\tilde{R}_{\alpha\lambda\rho\sigma}\psi_\alpha^\mu\psi_\rho^\lambda \\
B_{\mu\nu} &= 2\tilde{R}_{\nu\alpha}^\sigma\gamma\tilde{R}_{\lambda\rho\sigma\mu}\psi_\mu^\alpha\psi_\rho^\lambda - 4\tilde{R}_{\mu\alpha\lambda}\tilde{R}_{\nu\rho\sigma\eta}\psi_\alpha^\lambda\psi_\rho^\sigma - 2\tilde{R}_{\mu\alpha\lambda}\tilde{R}_{\rho\lambda\eta\nu}\psi_\alpha^\lambda\psi_\rho^\sigma - \tilde{R}_{\mu\rho\nu\sigma}\nabla_\rho\psi_\alpha^\lambda\nabla_\sigma\psi_\alpha^\lambda \\
C_{\mu\nu} &= 8\tilde{R}_{\nu\alpha\lambda\rho}\tilde{R}_\mu^{(1)\alpha\lambda\rho} + 2\tilde{R}_{\mu\lambda\rho\sigma}(4\tilde{R}_\alpha^{(1)\rho\lambda\alpha} - 7\tilde{R}_\alpha^{(1)\rho\lambda\alpha} - 8\tilde{R}_\alpha^{(1)\rho\lambda\alpha}) - 8\tilde{R}_{\alpha\lambda\rho\sigma}\nabla_\nu\tilde{R}_\mu^{(1)\alpha\sigma\lambda} - 2\tilde{R}_{\nu\sigma\lambda\rho}\tilde{R}_\mu^{(1)\sigma\nu\rho}\nabla_\lambda\psi_\alpha^\lambda \\
D_{\mu\nu} &= 4\left(8\tilde{R}_\nu^\sigma_\alpha\eta_\gamma\tilde{R}_\lambda\rho\sigma\mu\tilde{R}_\alpha^{(1)\rho\lambda\alpha} - 16\tilde{R}_{\mu\alpha\lambda}\tilde{R}_{\nu\rho\sigma\eta}\tilde{R}_\alpha^{(1)\lambda\rho\eta} + 2\tilde{R}_{\nu\sigma\lambda\rho}\tilde{R}_\mu^{(1)\sigma\nu\rho}\right)
\end{align}

(A.37) (A.38) (A.39)
As mentioned earlier, we assume $\psi(V, \rho)$ and $P(V, \rho)$ to be slowly varying close to the event horizon ($\rho \sim 2M_0$). The leading order solid angle-averaged value (of the $\rho - V$ component) of the effective GW stress-energy tensor (upto leading order corrections in $\gamma$) used in Eq. (35) in the Schwarzschild background is given by

\[
\gamma_{\nu}^{\rho(GW)} \approx \frac{M_0}{21\pi} \rho \left\{ [821o^{00}L_{22} + 230^{23} + 33o^{33}] \gamma P(V, \rho) + [21(o^{22})^2 + 20(o^{23})^2 + 8(o^{33})^2] \psi'(V, \rho) \right\} + \frac{1}{21\pi} \frac{M_0^2}{\rho} \psi'(V, \rho) \left[-4(21o^{12}L_{02} + 10o^{13}L_{03} + 21o^{22}L_{12} + 10o^{33}L_{13}) \gamma P'(V, \rho) + \right.
\]
\[
+ \left[ 21o^{22}L_{12} + 10o^{03}L_{03} + 21o^{22}L_{12} + 10o^{33}L_{13} \right] M_0 \gamma \gamma P(V, \rho) P'(V, \rho) - P(V, \rho) \psi'(V, \rho) \right\} + \frac{1}{672M_0^2 \pi} \left[ 84(o^{11}L_{00} + o^{00}L_{11}) \gamma P(V, \rho) \psi(V, \rho) + (-21o^{00}L_{12} + 2(-42o^{02}L_{02} + 21(o^{12})^2 \right.
\]
\[
- 20o^{03}L_{13} + 10(o^{13})^2 - 360L_{22} + 380L_{23} \gamma \right\} \gamma (\psi(V, \rho))^2 + 84(o^{01})^2 \gamma (\psi'(V, \rho))^2 \right\} \gamma \right\}
\]
\[
+ \frac{1}{32M_0^3 \pi} \left[ 42(-o^{00}L_{00} + o^{00}L_{11}) P(V, \rho) \psi'(V, \rho) + \psi(V, \rho) \right\} 42(o^{11}L_{00} - o^{00}L_{11}) P'(V, \rho) + (21o^{02}L_{12} - 21(o^{12})^2 + 10o^{03}L_{13} - 10(o^{13})^2 - 240L_{23} \right.
\]
\[
- 21(o^{12})^2 + 240L_{23} \psi'(V, \rho) \right\} \gamma \right\}
\]
\[
+ \frac{1}{336M_0^3 \pi} \left[ 4(21o^{12}L_{02} + 10o^{13}L_{03} + 21o^{22}L_{12} + 10o^{33}L_{13}) \gamma P(V, \rho) \psi(V, \rho) - (21o^{02}L_{12} + 240L_{23} \right.
\]
\[
- 21(o^{02}L_{12} + 240L_{23} \psi'(V, \rho) \right\} \gamma \right\}
\]
\[
+ \left[ 2\psi'(V, \rho) \left(-44(o^{11}L_{00} + 2o^{01}L_{01} + o^{00}L_{11}) \gamma P'(V, \rho) + (21(o^{01})^2 \right.
\]
\[
+ 21o^{00}L_{11} + 252o^{02}L_{12} \gamma + 120o^{03}L_{13} - 84o^{01}L_{22} \gamma - 40o^{01}L_{33} \gamma \right\} \psi'(V, \rho) \right\} \gamma \right\}
\]
\[(A.41)\]

where $'$ denotes derivative wrt $V$.

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