Polarization of the fermionic vacuum by a global monopole with finite core

E. R. Bezerra de Mello\textsuperscript{1,*} and A. A. Saharian\textsuperscript{1,2†}

\textsuperscript{1}Departamento de Física-CCEN, Universidade Federal da Paraíba 58.059-970, Caixa Postal 5.008, João Pessoa, PB, Brazil

\textsuperscript{2}Department of Physics, Yerevan State University, 375025 Yerevan, Armenia

September 22, 2018

Abstract

We study the vacuum polarization effects associated with a massive fermionic field in a spacetime produced by a global monopole considering a nontrivial inner structure for it. In the general case of the spherically symmetric static core with finite support we evaluate the vacuum expectation values of the energy-momentum tensor and the fermionic condensate in the region outside the core. These quantities are presented as the sum of point-like global monopole and core-induced contributions. The asymptotic behavior of the core-induced vacuum densities are investigated at large distances from the core, near the core and for small values of the solid angle corresponding to strong gravitational fields. As an application of general results the flower-pot model for the monopole’s core is considered and the expectation values inside the core are evaluated.

PACS number(s): 03.07.+k, 98.80.Cq, 11.27.+d

\textsuperscript{*}E-mail: emello@fisica.ufpb.br

\textsuperscript{†}E-mail: saharian@ictp.it
1 Introduction

Symmetry braking phase transitions in the early universe have several cosmological consequences and provide an important link between particle physics and cosmology. In particular, different types of topological objects may have been formed by the vacuum phase transitions after Planck time \[1, 2\]. These include domain walls, cosmic strings and monopoles. A global monopole is a spherical symmetric gravitational topological defect created by a phase transition of a system comprised by self-coupling scalar field, \( \varphi^a \), whose original global \( O(3) \) symmetry is spontaneously broken to \( U(1) \). The matter fields play the role of an order parameter which outside the monopole’s core acquires a non-vanishing value. The global monopole was first introduced by Sokolov and Starobinsky \[3\]. A few years later, the gravitational effects associated with a global monopole have been considered in Ref. \[4\], where the authors have found that for points far from the monopole’s center, the geometry is similar to the black-hole with a solid angle deficit. Neglecting the mass term we get the point-like global monopole spacetime with the metric tensor given by the following line element

\[
\text{ds}^2 = dt^2 - dr^2 - \alpha^2 r^2 (d\theta^2 + \sin^2 \theta d\phi^2) ,
\]

where the parameter \( \alpha^2 \) is smaller than unity and depends on the energy scale where the symmetry is broken. It is of interest to note that the effective metric produced in superfluid \(^3\)He \( - \) \( \Lambda \) by a monopole is described by the line element \([1]\) with the negative angle deficit, \( \alpha > 1 \), which corresponds to the negative mass of the topological object \([5]\). The quasiparticles in this model are chiral and massless fermions.

In quantum field theory the non-trivial topology of the global monopole spacetime induces non-zero vacuum expectation values for physical observables. The quantum effects due to the point-like global monopole spacetime on the matter fields have been considered for massless scalar \([6]\) and fermionic \([7]\) fields, respectively. The influence of the non-zero temperature on these polarization effects has been discussed in Ref. \([8]\) for scalar and fermionic fields. Moreover, the calculation of quantum effects on massless scalar field in a higher dimensional global monopole spacetime has also been developed in Ref. \([9]\). The quantum effects of a scalar field induced by a composite topological defect consisting a cosmic string on a \( p \)-dimensional brane and a \((m + 1)\)-dimensional global monopole in the transverse extra dimensions are investigated in Ref. \([10]\). The combined vacuum polarization effects by the non-trivial geometry of a global monopole and boundary conditions imposed on the matter fields are investigated as well. In this direction, the total Casimir energy associated with massive scalar field inside a spherical region in the global monopole background has been analyzed in Refs. \([11, 12]\) by using the zeta function regularization procedure. Scalar Casimir densities induced by spherical boundaries have been calculated in Refs. \([13, 14]\) to higher dimensional global monopole spacetime by making use of the generalized Abel-Plana summation formula \([15]\). More recently, using also this formalism, a similar analysis for spinor fields obeying MIT bag boundary conditions has been developed in Refs. \([16, 17]\).

In general, the quantum analysis of matter fields in global monopole spacetime consider this object as been a point-like one. Because of this fact the renormalized vacuum expectation value of the energy-momentum tensor presents singularities at the monopole’s center. Of course, this kind of problem cannot be expected in a realistic model. So a procedure to cure this divergence is to consider the global monopole as having a non-trivial inner structure. In fact, in a realistic point of view, the global monopoles have a characteristic core radius determined by the symmetry braking energy scale where the symmetry of the system is spontaneously broken. A simplified model for the monopole core is presented in Ref. \([18]\). In this model the region inside the core is described by the de Sitter geometry. The vacuum polarization
effects associated with a massless scalar field in the region outside the core of this model are investigated in Ref. [19]. In particular, it has been shown that long-range effects can take place due to the non-trivial core structure. Recently the quantum analysis of a scalar field in a higher-dimensional global monopole spacetime considering a general spherically symmetric model for the core, has been considered in Ref. [20]. In the four-dimensional version of this model the induced electrostatic self-energy and self-force for a charged particle are investigated in Ref. [21]. Continuing in this direction, in the present paper we analyze the effects of global monopole core on properties of the fermionic quantum vacuum. The most important quantities characterizing these properties are the vacuum expectation values of the energy-momentum tensor and the fermionic condensate. Though the corresponding operators are local, due to the global nature of the vacuum, the vacuum expectation values describe the global properties of the bulk and carry an important information about the structure of the defect core. In addition to describing the physical structure of the quantum field at a given point, the energy-momentum tensor acts as the source of gravity in the Einstein equations. It therefore plays an important role in modelling a self-consistent dynamics involving the gravitational field. The problem under consideration is also of separate interest as an example with gravitational polarization of the fermionic vacuum, where all calculations can be performed in a closed form. The corresponding results specify the conditions under which we can ignore the details of the interior structure and approximate the effect of the global monopole by the idealized model. The exactly solvable models of this type are useful for testing the validity of various approximations used to deal with more complicated geometries, in particular in black-hole spacetimes.

The plan of this paper is as follows. Section 2 presents the geometry of the problem and the eigenfunctions for a massive spinor field. By using these eigenfunctions, in Section 3 we evaluate the vacuum expectation values of the energy-momentum tensor and the fermionic condensate in the region outside the monopole core. In Section 4 we consider the special case of the flower-pot model for the core and the vacuum expectation values are investigated in both exterior and interior regions. Various limiting cases are considered. The main results of the paper are summarized in Section 5. In Appendix A we discuss the contribution of possible bound states into the vacuum expectation values of the energy-momentum tensor and show that in the flower-pot model there are no bound states. Throughout we use the system of units with $\hbar = c = 1$.

2 Model and the eigenfunctions for the spinor field

In this section we analyze the eigenfunctions for a massive fermionic field in background of the global monopole geometry considering a nontrivial inner structure to the latter. The explicit expression for the metric tensor in the region inside the monopole core is unknown and here we consider a general model for a four-dimensional global monopole with a core of radius $a$, assuming that the geometry of the spacetime is described by two distinct metric tensors in the regions inside and outside the core. Adopting this model we can learn under which conditions we can ignore the specific details for its core and consider the monopole as being a point-like object. In spherical coordinates we will consider the corresponding line element in the interior region, $r < a$, with the form

$$
 ds^2 = e^{2u(r)} dt^2 - e^{2v(r)} dr^2 - e^{2h(r)} (d\theta^2 + \sin^2 \theta d\phi^2).
$$

(2)

In the region outside, $r > a$, the metric tensor is given by the line element (1), where the parameter $\alpha$ codifies the presence of the global monopole. For an idealized point-like global monopole the geometry is described by line element (1) for all values of the radial coordinate and it has singularity at the origin.
In Eq. (2) the functions \( u(r) \), \( v(r) \), \( h(r) \) are continuous at the core boundary, consequently they satisfy the conditions

\[
u(a) = v(a) = 0, \quad h(a) = \ln(\alpha a) .
\] (3)

If there is no surface energy-momentum tensor on the bounding surface \( r = a \), the radial derivatives of these functions are continuous as well. When the surface energy-momentum tensor is present the junctions in the radial derivatives of the components of the metric tensor are expressed in terms of the surface energy density and stresses.\(^1\) Introducing a new radial coordinate \( \tilde{r} = e^{h(r)} \) with the core center at \( \tilde{r} = 0 \), the angular part of line element (2) is written in the standard Minkowskian form. However, with this choice, in general, we will obtain non-standard form of the angular part in the exterior line element (1). In the model under consideration we will assume that inside the core the spacetime geometry is regular. In particular, from the regularity of the interior geometry at the core center one has the conditions \( u(r) \rightarrow 0 \) and \( h(r) \sim \ln \tilde{r} \) for \( \tilde{r} \rightarrow 0 \).

We are interested in quantum effects of a spinor field propagating on background of the spacetime described by line elements (1) and (2). The dynamics of the massive spinor field in curved spacetime is governed by the Dirac equation

\[
i\gamma^\mu \nabla_\mu \psi - M \psi = 0 ,
\] (4)

with the covariant derivative operator

\[
\nabla_\mu = \partial_\mu + \Gamma_\mu .
\] (5)

Here \( \gamma^\mu = e^\mu_{(a)} \gamma^{(a)} \) are the Dirac matrices in curved spacetime, and \( \Gamma_\mu \) is the spin connection given in terms of the flat \( \gamma \) matrices by the relation

\[
\Gamma_\mu = \frac{1}{4} \gamma_{(a)} \gamma^{(b)} e^\mu_{(a)} e^\nu_{(b)} \eta_{\nu \mu} .
\] (6)

In the equations above \( e^\mu_{(a)} \) is the vierbein satisfying the condition \( e^\mu_{(a)} e^\nu_{(b)} \eta^{ab} = g^{\mu \nu} \).

In the region inside the monopole core we choose the basis tetrad

\[
e^\mu_{(a)} = \begin{pmatrix}
e^{-u(r)} & 0 & 0 & 0 \\
0 & e^{-v(r)} \sin \theta \cos \varphi & e^{-h(r)} \cos \theta \cos \varphi & -e^{-h(r)} \cos \varphi / \sin \theta \\
0 & e^{-v(r)} \sin \theta \sin \varphi & e^{-h(r)} \cos \theta \sin \varphi & e^{-h(r)} \cos \varphi / \sin \theta \\
0 & e^{-v(r)} \cos \theta & e^{-h(r)} \sin \theta & 0
\end{pmatrix} ,
\] (7)

where the rows of the matrix are specified by the index \( a \) and the columns by the index \( \mu \). By using Eq. (7), for the non-vanishing components of the spin connection we find

\[
\Gamma_0 = \frac{1}{2} u' e^{u-v} \gamma^{(0)} \tilde{r} \cdot \hat{r} ,
\]

\[
\Gamma_2 = \frac{i}{2} (1 - h' e^{h-v}) \hat{\Sigma} \cdot \hat{\theta} ,
\]

\[
\Gamma_3 = -\frac{i}{2} \sin \theta (1 - h' e^{h-v}) \hat{\Sigma} \cdot \hat{\varphi} ,
\] (8)

where the prime denotes derivative with respect to the radial coordinate, \( \hat{r} \), \( \hat{\theta} \) and \( \hat{\varphi} \) are the standard unit vectors along the three spatial directions in spherical coordinates, and

\[
\hat{\Sigma} = \begin{pmatrix}
\hat{\sigma} & 0 \\
0 & \hat{\sigma}
\end{pmatrix} ,
\] (9)

\(^1\)This point will be analyzed in more detail at the end of the next section.
with $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ being the Pauli matrices. From the obtained results, for the combination entering in the Dirac equation we have

$$
\gamma^\mu \Gamma_\mu = \frac{u' e^{-v}}{2} \gamma^\mu \cdot \hat{r} - e^{-h}(1 - h' e^{h-v})\gamma^\mu \cdot \hat{r}.
$$

After some intermediate steps, the Dirac equation reads

$$
ie^{-u\gamma^0(0)}\partial_t \psi + ie^{-u/2-v-h(r)}e^{(u/2+h(\psi))} - ie^{-h}\gamma^r(\Sigma \cdot \vec{L} + 1)\psi - M\psi = 0,
$$

with $\vec{L}$ being the standard angular momentum operator.

To find the vacuum expectation value (VEV) of the energy-momentum tensor we need the corresponding eigenfunctions. For the problem under consideration there are two types of eigenfunctions with different parities which we will distinguish by the index $\sigma = 0, 1$. These functions are specified by the total angular momentum $j = 1/2, 3/2, \ldots$ and its projection $m = -j, -j + 1, \ldots, j$. Assuming the time dependence in the form $e^{-i\omega t}$, the four-component spinor fields specified by the set of quantum numbers $\beta = (\sigma k j m)$ with $k^2 = \omega^2 - M^2$, can be written in terms of two-component ones as

$$
\psi_\beta = e^{-i\omega t}
\begin{pmatrix}
f_\beta(r) \Omega_{jlm}(\theta, \varphi) \\
n_\sigma g_\beta(r) \Omega_{j'l'm}(\theta, \varphi)
\end{pmatrix},
$$

where

$$
n_\sigma = (-1)^\sigma, \quad l_\sigma = j - \frac{n_\sigma}{2}, \quad l'_\sigma = j + \frac{n_\sigma}{2},
$$

and $\Omega_{jlm}(\theta, \varphi)$ are the standard spinor spherical harmonics (see, for instance, Ref. [22]). The latter are eigenfunctions of the operator $K = \vec{\sigma} \cdot \vec{L} + I$ as shown below:

$$
K \Omega_{jlm} = -\kappa_\sigma \Omega_{jlm}, \quad \kappa_\sigma = -n_\sigma(j + 1/2).
$$

Note that we have the relation

$$
\Omega_{j'l'm} = e^{i\sigma\vec{\sigma} \cdot \vec{\hat{n}} \cdot \vec{\sigma}} \Omega_{jlm}.
$$

Substituting the function $\psi_\beta$ into the Dirac equation above, and using for the flat Dirac matrices the representation given in Ref. [22], for the radial functions we obtain a set of two coupled linear differential equations:

$$
f'_\beta + (u'/2 + h' + \kappa_\sigma e^{u-h})f_\beta = (M + e^{-u}\omega)e^v f_\beta,
$$

$$
g'_\beta + (u'/2 + h' - \kappa_\sigma e^{u-h})g_\beta = (M - e^{-u}\omega)e^v f_\beta.
$$

In the region $r > a$ for the radial functions we have the solutions

$$
f_\beta(r) = f_\beta^{(ex)}(r) = \frac{1}{\sqrt{r}} \left[c_1 J_{\nu_\sigma}(kr) + c_2 Y_{\nu_\sigma}(kr)\right],
$$

$$
g_\beta(r) = g_\beta^{(ex)}(r) = -\frac{1}{\sqrt{r} \omega + M} \left[n_\sigma k c_1 J_{\nu_\sigma + n_\sigma}(kr) + c_2 Y_{\nu_\sigma + n_\sigma}(kr)\right],
$$

where $J_{\nu_\sigma}(x)$ and $Y_{\nu_\sigma}(x)$ are the Bessel and Neumann functions of the order

$$
\nu_\sigma = \frac{j + 1/2}{\alpha} - \frac{n_\sigma}{2}.
$$
By taking into account relation (15), the corresponding eigenfunctions are written in the form

$$
\psi^{(\text{ex})}_\beta = e^{-i\omega t} \left( \begin{array}{c} f^{(\text{ex})}_\beta (r) \Omega_{j\ell m}(\theta, \varphi) \\ -ig^{(\text{ex})}_\beta (r) (\hat{n} \cdot \hat{\sigma}) \Omega_{j\ell m}(\theta, \varphi) \end{array} \right). 
$$  \hspace{1cm} (21)

In Eqs. (18) and (19), the integration constants $c_1$ and $c_2$ are determined from the matching condition with the interior solution.

The regular solution to Eqs. (16), (17) in the interior region, $r < a$, we will denote by

$$
f_\beta(r) = R_{1,n_\sigma}(r, k), \quad g_\beta(r) = -\frac{n_\sigma k}{\omega + M} R_{2,n_\sigma}(r, k). \hspace{1cm} (22)
$$

Near the core center, $\tilde{r} \to 0$, these functions behave like $R_{1,n_\sigma} \propto \tilde{r}^{j+(1-n_\sigma)/2}$ and $R_{2,n_\sigma} \propto \tilde{r}^{j+(1+n_\sigma)/2}$, where the radial coordinate $\tilde{r}$ is introduced in the paragraph after formula (3).

Note that from Eqs. (16), (17), for the values of the interior radial functions on the boundary of the core we have the following relations

$$
-n_\sigma k R_{2,n_\sigma}(r, a, k) = R_{1,n_\sigma}'(r, a, k) + \left( u_a'/2 + h_a - n_\sigma \lambda/a \right) R_{1,n_\sigma}(r, a, k), \hspace{1cm} (27)
$$

$$
n_\sigma k R_{1,n_\sigma}(r, a, k) = R_{2,n_\sigma}'(r, a, k) + \left( u_a'/2 + h_a + n_\sigma \lambda/a \right) R_{2,n_\sigma}(r, a, k), \hspace{1cm} (28)
$$

where $R_{j,n_\sigma}'(r, a, k) = \partial R_{j,n_\sigma}(r, k)/\partial r|_{r=a}$, $u_a' = du/dr|_{r=a}$, $h_a' = dh/dr|_{r=a}$, and we have introduced the notation

$$
\lambda = (j + 1/2)/\alpha. \hspace{1cm} (29)
$$

Substituting the expressions for the coefficients $c_1$ and $c_2$ into the formulae for the radial eigenfunctions in the exterior region, one finds

$$
\psi^{(\text{ex})}_\beta = \frac{\pi R_{1,n_\sigma}(r, a, k) e^{-i\omega t}}{2\sqrt{r/a}} \left( \begin{array}{c} \frac{g_{\nu_\sigma,0}(ka, kr)}{\omega + M} \Omega_{jim} \\ \frac{\alpha}{\omega + M} g_{\nu_\sigma,n_\sigma}(ka, kr) (\hat{n} \cdot \hat{\sigma}) \Omega_{jim} \end{array} \right), \hspace{1cm} (30)
$$

with the notation

$$
g_{\nu_\sigma,p}(x, y) = \tilde{Y}_{\nu_\sigma}(x) J_{\nu_\sigma+p}(y) - \tilde{\nu}_{\nu_\sigma}(x) Y_{\nu_\sigma+p}(y), \quad p = 0, n_\sigma. \hspace{1cm} (31)
$$

Here and in what follows, for a function $F_{\nu_\sigma}(x)$ with $F = J, Y$ we use the tilded notation defined by the formula

$$
\tilde{F}_{\nu_\sigma}(ka) = -n_\sigma ka \left[ F_{\nu_\sigma+n_\sigma}(ka) - F_{\nu_\sigma}(ka) R_{2,n_\sigma}(r, a, k)/R_{1,n_\sigma}(r, a, k) \right]
$$

$$
= ka F_{\nu_\sigma}'(ka) - \left[ a R_{1,n_\sigma}'(r, a, k) + \frac{1}{2} a u_a' + a h_a' - \frac{1}{2} \right] F_{\nu_\sigma}(ka). \hspace{1cm} (32)
$$
In deriving the second form we have used the relation (27) and the recurrence formula for the cylindrical functions.

The eigenfunctions are normalized by the condition

$$\int d^3x \sqrt{\gamma} \psi^{+\beta} \psi^{\beta'} = \delta_{\beta\beta'},$$  \hspace{1cm} (33)

where $\gamma$ is the determinant for the spatial metric and $\delta_{\beta\beta'}$ is understood as the Kronecker delta symbol for the discrete components of the collective index $\beta$ and as the Dirac delta function for the continuous ones. As the normalization integral is divergent for $\beta = \beta'$, the main contribution comes from large values $r$. We may replace the cylindrical functions with the argument $kr$ in Eq. (30) by the corresponding asymptotic expressions. In this way one finds

$$R_{1,\nu_\sigma}(a,k) = \frac{2k(M + \omega)}{\pi^2 a^2 \omega} \left[ J^2_{\nu_\sigma}(ka) + \tilde{Y}^2_{\nu_\sigma}(ka) \right]^{-1}. \hspace{1cm} (34)$$

This relation determines the normalization coefficient for the interior region. In addition to the modes with real $k$, modes with purely imaginary $k$ can be present. These modes correspond to the bound states. The eigenfunctions for the bound states and their normalization are discussed in Appendix A.

### 3 Vacuum expectation values in the exterior region

In this section we consider the VEVs for the energy-momentum tensor and the fermionic condensate in the region outside the global monopole core. We expand the field operator in terms of the complete set of eigenfunctions $\{\psi^{(+)\beta}, \psi^{(-)\beta}\}$:

$$\hat{\psi} = \sum_{\beta} \left[ \hat{a}_\beta \psi^{(+)\beta} + \hat{b}^{+\beta} \psi^{(-)\beta} \right], \hspace{1cm} (35)$$

where $\hat{a}_\beta$ is the annihilation operator for particles, and $\hat{b}^{+\beta}$ is the creation operator for antiparticles, $\psi^{(+)\beta} = \psi_{\beta}$, for $\omega > 0$ and $\psi^{(-)\beta} = \psi_{\beta}$, for $\omega < 0$. In order to find the VEV for the operator of the energy-momentum tensor, we substitute the expansion (35) and the analog expansion for the operator $\hat{\bar{\psi}}$ into the corresponding expression for spinor fields,

$$T_{\mu\nu}\{\hat{\psi}, \hat{\psi}\} = \frac{i}{2} [\hat{\psi} \gamma_{(\mu} \nabla_{\nu)} \hat{\psi} - (\nabla_{(\mu} \hat{\psi}) \gamma_{\nu)} \hat{\psi}] . \hspace{1cm} (36)$$

By making use of the standard anticommutation relations for the annihilation and creation operators, for the VEV one finds the following mode-sum formula

$$\langle 0 | T_{\mu\nu} | 0 \rangle = \sum_{\beta} T_{\mu\nu}\{\psi^{(-)\beta}(x), \psi^{(-)\beta}(x)\} , \hspace{1cm} (37)$$

where $|0\rangle$ is the amplitude for the corresponding vacuum state.

Substituting the eigenfunctions (30) into the mode-sum formula (37), using the formula

$$\sum_{m=-j}^{j} |\Omega_{jlm}(\theta, \varphi)|^2 = \frac{2j+1}{4\pi} , \hspace{1cm} (38)$$

and the relation (33)
and relation (33), the VEV of the energy-momentum tensor is presented in the form (no summation over \( \mu \))

\[
\langle 0 | T^\nu_\mu | 0 \rangle = \frac{\delta^\nu_\mu}{8\pi \alpha^2 a^3 r} \sum_{j=1/2}^\infty (2j + 1) \sum_{\sigma = 0,1} \int_0^\infty dx \frac{f^{(\mu)}_{r\sigma v}(x, g_{v_\sigma}(x, x r/a))}{f_{r\sigma v}(x) + Y^2_{\sigma v}(x)}. \tag{39}
\]

In formula (39) we use the notations

\[
f^{(0)}_{\sigma v}(x, g_{v_\sigma}(x, y)) = -x \left[ (\sqrt{x^2 + M^2 a^2} - Ma)g^2_{v_\sigma,0}(x, y) + (\sqrt{x^2 + M^2 a^2} + Ma)g^2_{v_\sigma,\eta}(x, y) \right], \tag{40}
\]

\[
f^{(1)}_{\sigma v}(x, g_{v_\sigma}(x, y)) = \frac{x^3}{\sqrt{x^2 + M^2 a^2}} \left[ g^2_{v_\sigma,0}(x, y) + g^2_{v_\sigma,\eta}(x, y) \right] - 2 f^{(2)}_{\sigma v}(x, g_{v_\sigma}(x, y)), \tag{41}
\]

\[
f^{(\mu)}_{\sigma v}(x, g_{v_\sigma}(x, y)) = \frac{x^3 (2\nu_\rho + \eta_\sigma)}{2y x^2 + M^2 a^2} g_{v_\rho,0}(x, y) g_{v_\rho,\eta}(x, y), \quad \mu = 2, 3, \tag{42}
\]

and the function \( g_{v_\rho}(x, y) \) is defined by Eq. (31). The expression on the right of Eq. (39) is divergent. It may be regularized introducing a cutoff function \( \psi_\mu(\omega) \) with the cutting parameter \( \mu \) which makes the divergent expressions finite and satisfies the condition \( \psi_\mu(\omega) \to 1 \) for \( \mu \to 0 \). After the renormalization the cutoff function is removed by taking the limit \( \mu \to 0 \). In the discussion below we will implicitly assume that the corresponding expressions are regularized.

The parts in the VEVs induced by the non-trivial structure of the core are finite and do not depend on the regularization scheme used.

To find the part in the VEV of the energy-momentum tensor induced by the non-trivial core structure, we subtract the corresponding components for the point-like monopole geometry. The latter are given by the expressions which are obtained from Eq. (39) replacing the integrand by \( f^{(\mu)}_{r\sigma v}(x, J_{v_\sigma}(x r/a)) \) (see Ref. [16]). In order to evaluate the corresponding difference we use the relation

\[
\langle 0 | T^\nu_\mu | 0 \rangle = \langle 0 | T^\nu_\mu | 0 \rangle_m + \langle T^\nu_\mu \rangle_c, \tag{44}
\]

where \( \langle 0 | T^\nu_\mu | 0 \rangle_m \) is the VEV for the point-like monopole and the part (no summation over \( \mu \))

\[
\langle T^\nu_\mu \rangle_c = \frac{-\delta^\nu_\mu}{16\pi \alpha^2 a^3 r} \sum_{j=1/2}^\infty (2j + 1) \sum_{\sigma = 0,1} \sum_{s=1,2} \int_0^\infty dx \frac{\tilde{J}_{v_\sigma}(x)}{H^2_{v_\sigma}(x)} f^{(\mu)}_{r\sigma v} \left[ x, H^s_{v_\sigma}(x r/a) \right], \tag{45}
\]

is induced by the non-trivial core structure. In formula (45), the integrand of the \( s = 1 \) (\( s = 2 \)) term exponentially decreases in the upper (lower) half of the complex plane \( x \). Consequently, we rotate the integration contour on the right of this formula by the angle \( \pi/2 \) for \( s = 1 \) and by the angle \( -\pi/2 \) for \( s = 2 \). This leads to the representation

\[
\langle T^\nu_\mu \rangle_c = \frac{-i\delta^\nu_\mu}{16\pi \alpha^2 a^3 r} \sum_{j=1/2}^\infty (2j + 1) \sum_{\sigma = 0,1} \int_0^\infty dx \sum_{s=1,2} \eta_s \times \frac{\tilde{J}_{v_\sigma}(\eta_s \pi i/2 x)}{H^2_{v_\sigma}(\eta_s \pi i/2 x)} f^{(\mu)}_{r\sigma v} \left[ \eta_s \pi i/2 x, H^s_{v_\sigma}(\eta_s \pi i/2 x r/a) \right], \tag{46}
\]
where $\eta_s = (-1)^{s+1}$.

First of all let us consider the part of the integral over the interval $(0, Ma)$. By using the standard properties of the Hankel functions it can be easily seen that in this interval one has

$$e^{i\pi f_{\sigma}^{(\mu)}} \left[ e^{\pi i/2 x} H_{\sigma}^{(1)}(e^{\pi i/2 x r/a}) \right] = e^{-i\pi f_{\sigma}^{(\mu)}} \left[ e^{-\pi i/2 x} H_{\sigma}^{(2)}(e^{-\pi i/2 x r/a}) \right].$$

Further, from equations (16) and (17) it follows that the interior solution can be written as

$$x = \text{const} \cdot R(r, \omega).$$

On the base of this observation for the combination entering into the VEV of the energy-momentum tensor.

From this formula, by taking into account that $\nu = \nu' \pm 1/2$, we find the relation

$$e^{-i\pi \lambda} \frac{J_{\nu}^{(\mu)}(e^{\pi i/2 x})}{H_{\nu}^{(1)}(e^{\pi i/2 x})} = e^{i\pi \lambda} \frac{\tilde{J}_{\nu}^{(\mu)}(e^{-\pi i/2 x})}{\tilde{H}_{\nu}^{(2)}(e^{-\pi i/2 x})}.$$

Combining relations (47) and (49), we see that the part of the integral in Eq. (46) over the interval $(0, Ma)$ vanishes.

To simplify the part of the integral over the interval $(Ma, \infty)$, we note that for the functions with different parities the following relation takes place:

$$e^{i\pi \lambda} f_{\sigma}^{(\mu)} \left[ e^{\pi i/2 x} H_{\sigma}^{(1)}(e^{\pi i/2 x r/a}) \right] = -e^{-i\pi \lambda f_{\sigma'}^{(\mu)}(e^{-\pi i/2 x}, H_{\sigma'}^{(2)}(e^{-\pi i/2 x r/a}) \right],$$

where $\sigma, \sigma' = 0, 1$, and $\sigma \neq \sigma'$. Further we note that the functions $R_{2,\nu}(r, -k)$ and $R_{1,\nu}(r, k)$ satisfy the same equation and, hence, $R_{2,\nu}(r, -k) = \text{const} \cdot R_{1,\nu}(r, k)$. By using relations (27) and (28), now it can be seen that

$$\frac{R_{2,\nu}(a, ke^{\pm i\pi/2})}{R_{1,\nu}(a, ke^{\mp i\pi/2})} = \frac{R_{1,\nu}(a, ke^{\pm i\pi/2})}{R_{2,\nu}(a, ke^{\mp i\pi/2})}.$$

From this formula, by taking into account that $\nu_{\nu'} = \nu_{\nu} + n_{\nu}$, we find the relation

$$e^{-i\pi \lambda} \frac{J_{\nu}^{(\mu)}(e^{\pi i/2 x})}{H_{\nu}^{(1)}(e^{\pi i/2 x})} = e^{i\pi \lambda} \frac{\tilde{J}_{\nu}^{(\mu)}(e^{-\pi i/2 x})}{\tilde{H}_{\nu}^{(2)}(e^{-\pi i/2 x})}.$$

Combining formulae (50) and (52), we see that the different parities give the same contribution to the VEV of the energy-momentum tensor.

By taking into account this result and introducing the modified Bessel functions, for the core-induced part in the VEV we find (no summation over $\mu$

$$\langle T_{\mu}^{(c)} \rangle_c = \frac{\delta_{\mu}^{(c)}}{2\pi^2 \alpha^2 r} \sum_{l=1}^{\infty} \int_{M}^{\infty} dx \frac{x^3}{\sqrt{x^2 - M^2}} \sum_{s=1,2} \frac{I_{l/\alpha-1/2}(ax)}{K_{l/\alpha-1/2}(ax)} F_{l/\alpha-1/2}^{(\mu, w)} \left[ x, K_{l/\alpha-1/2}(x r) \right].$$

Here for a given function $f_{\nu}(y)$ we use the notations

$$F_{\nu}^{(0, w)} \left[ x, f_{\nu}(y) \right] = \left( \frac{M^2}{x^2} - 1 \right) \left[ \left( 1 + \frac{i\eta M}{\sqrt{x^2 - M^2}} \right) f_{\nu}^2(y) \right] - \left[ \left. 1 - \frac{i\eta M}{\sqrt{x_0^2 - M^2}} \right] f_{\nu+1}^2(y) \right],$$

$$F_{\nu}^{(1, w)} \left[ x, f_{\nu}(y) \right] = f_{\nu}^2(y) - f_{\nu+1}^2(y) - \lambda y f_{\nu}(y) f_{\nu+1}(y),$$

$$F_{\nu}^{(\mu, w)} \left[ x, f_{\nu}(y) \right] = \lambda y f_{\nu}(y) f_{\nu+1}(y), \quad \mu = 2, 3,$$
with \( \lambda_K = -1 \) (the function \( F_{\nu}^{(\mu, \eta)}[x, I_\nu(y)] \) with \( \lambda_I = 1 \) will be used below), and the barred notation is defined by the formula

\[
\bar{f}^{(\eta, \sigma)}(x) = x f'(x) - \left[ a R_{1, n_{\alpha}}^{(1)}(a, e^{\nu_\alpha x/a}) + a u'_n/2 + a h''_\alpha - 1/2 \right] f(x).
\]

It can be checked that the core-induced part in the VEV of the energy-momentum tensor obeys the continuity equation \( \langle T_{\mu\nu}^c \rangle_{c\nu} = 0 \), which for the geometry under consideration takes the form

\[
r \frac{d}{dr} \langle T^1_c \rangle + 2 \left( \langle T^1_c \rangle - \langle T^2_c \rangle \right) = 0.
\]

In addition, for a massless spinor field this part is traceless and the trace anomaly is contained in the point-like monopole part only.

The fermionic condensate can be found from the formula for the VEV of the energy-momentum tensor by making use of the relation \( T_{\mu}^\eta = M \bar{\psi} \psi \). The condensate is presented in the form of the sum of idealized point-like monopole and core-induced parts:

\[
\langle 0 | \bar{\psi} \psi | 0 \rangle = \langle 0_m | \bar{\psi} \psi | 0_m \rangle + \langle \bar{\psi} \psi \rangle_c.
\]

By taking into account formula (63) for the components of the energy-momentum tensor, for the part coming from the non-trivial core structure one finds

\[
\langle \bar{\psi} \psi \rangle_c = \frac{1}{2 \pi^2 \alpha^2 r} \sum_{l=1}^{\infty} \int_M dx x \sum_{s=1,2} \int_0^{\infty} dx \sum_{l/\alpha-1/2}^{l/\alpha+1/2} \left( \frac{M}{\sqrt{x^2 - M^2}} - i \eta_s \right) K_{l/\alpha-1/2}(x) - \left( \frac{M}{\sqrt{x^2 - M^2}} + i \eta_s \right) K_{l/\alpha+1/2}(x) \right] \right).
\]

This formula may also be derived directly from the mode-sum formula \( \langle 0 | \bar{\psi} \psi | 0 \rangle = \sum_\beta \psi^{(-)}_\beta \psi^{(-)}_\beta \) with the eigenfunctions (21). In deriving formulae (63) and (60) we have assumed that no bound states exist. In Appendix A we show that these formulae are valid also in the case when the bound states are present.

For \( r > a \) the core-induced parts in the VEVs of the energy-momentum tensor and the fermionic condensate, given by Eqs. (63) and (60), are finite and the renormalization is necessary for the point-like monopole parts only. Of course, we could expect this result as in the region \( r > a \) the local geometry is the same in both models and, hence, the divergences are the same as well.

As it has been already mentioned, if there is no surface energy-momentum tensor on the bounding surface \( r = a \), then the radial derivatives of the metric are continuous and, hence, in Eqs. (22) and (67) one has \( u'_a = 0 \), \( h'_a = 1/a \). In models with an additional infinitely thin spherical shell located at \( r = a \), these quantities are related to the components of the corresponding surface energy-momentum tensor \( \tau_{\mu\nu} \). Denoting by \( n^\mu \) the normal to the shell normalized by the condition \( n_\mu n^\mu = -1 \) and assuming that it points into the bulk on both sides, from the Israel matching conditions one has

\[
\{ K_{\mu\nu} - K h_{\mu\nu} \} = 8 \pi G \tau_{\mu\nu}.
\]

In this formula the curly brackets denote summation over each side of the shell, \( h_{\mu\nu} = g_{\mu\nu} + n_\mu n^\nu \) is the induced metric on the shell, \( K_{\mu\nu} = h'^{\mu}_\beta h_\alpha^{\nu}\nabla_\beta n_\delta \) its extrinsic curvature and \( K = K^\mu_\mu \). For
the region $r \leq a$ one has $n_\mu = \delta^1_\mu e^{v(r)}$ and the non-zero components of the extrinsic curvature are given by the formulae

$$K^0_0 = -u'(r)e^{-v(r)}, \quad K^2_2 = K^3_3 = -h'(r)e^{-v(r)}, \quad r = a - . \quad (62)$$

The corresponding expressions for the region $r \geq a$ are obtained by taking in these formulae $u(r) = v(r) = 0$, $h(r) = \ln(\alpha r)$ and changing the signs for the components of the extrinsic curvature tensor. Now from the matching conditions (61) we find

$$u'_a = 8\pi G \left( \tau^2_2 - \frac{1}{2} \tau^0_0 \right), \quad h'_a = \frac{1}{a} + 4\pi G \tau^0_0. \quad (63)$$

Note that the combination in the square brackets in Eqs. (32) and (57) is related to the surface energy-momentum tensor by the formula

$$au'_a/2 + ah'_a - 1/2 = 2\pi Ga + 1/2. \quad (64)$$

where $\tau$ is the trace of the surface energy-momentum tensor.

4 Vacuum polarization in the flower-pot model

In this section, as an application of the general results given above we consider a simple example of the core model assuming that the spacetime inside it is flat. The corresponding model for the cosmic string core was considered in Refs. [23, 24, 25] and for the global monopole core in Ref. [20]. Following these papers we will refer to this model as flower-pot model. Taking in the region inside the core $u(r) = v(r) = 0$, from the zero curvature condition one finds $e^{h(r)} = r + \text{const}$. The value of the constant is found from the continuity condition for the function $h(r)$ at the boundary which gives $\text{const} = (\alpha - 1)a$. Hence, in the flower-pot model the interior line element has the form

$$ds^2 = dt^2 - dr^2 - \left[ r + (\alpha - 1)a \right]^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (65)$$

In terms of the radial coordinate $r$ the origin is located at $r = r_0 = (1 - \alpha)a$. From the matching conditions (63) we find the corresponding surface energy-momentum tensor with the non-zero components and the trace given by

$$\tau^0_0 = 2\tau^2_2 = 2\tau^3_3 = \frac{1/\alpha - 1}{4\pi Ga}, \quad 2\pi Ga\tau = \frac{1}{\alpha} - 1. \quad (66)$$

The corresponding surface energy density is positive for the global monopole with $\alpha < 1$. We will consider the VEVs in the exterior and interior regions separately.

4.1 Exterior region

In the region inside the core the radial eigenfunctions regular at the origin are the functions

$$R_{1,n_\sigma}(r,k) = C_\beta \frac{J_{\lambda_\sigma}(k\tilde{r})}{\sqrt{r}}, \quad R_{2,n_\sigma}(r,k) = C_\beta \frac{J_{\lambda_\sigma+n_\sigma}(k\tilde{r})}{\sqrt{r}}, \quad (67)$$

where

$$\lambda_\sigma = j + 1/2 - n_\sigma/2, \quad (68)$$
and \( \tilde{r} = r + (\alpha - 1) a \) is the standard Minkowskian radial coordinate, \( 0 \leq \tilde{r} \leq \alpha a \). Note that for an interior Minkowskian observer the radius of the core is \( \alpha a \). The normalization coefficient \( C_\beta \) is found from the condition (61):

\[
C_\beta^2 = \frac{M + \omega}{\pi^2 \alpha^2} \frac{2k J_{\alpha}^2(k \alpha a) - J_{\alpha}^2(k \alpha a) + Y_{\alpha}^2(k \alpha a)}{J_{\alpha}^2(k \alpha a) + Y_{\alpha}^2(k \alpha a)},
\]

with the tilted notation for the cylindrical functions

\[
\tilde{F}_{\nu \nu}(x) = xF'_{\nu \nu}(x) - \left[ x \frac{J_{\alpha}(ax)}{J_{\alpha}(ax)} + \frac{1}{2} \left( \frac{1}{\alpha} - 1 \right) \right] F_{\nu \nu}(x).
\]

Note that \( \tilde{J}_{\nu \nu}(x) = 0 \) for \( \alpha = 1 \). In this case for the barred notation in Eq. (57) one has \( \tilde{F}^{(1,0)}(x) = \tilde{F}^{(1,0)}(x) \). Hence, in the flower-pot model the part in the energy-momentum tensor due to the non-trivial structure of the core is given by the formula

\[
\langle T_{\mu \nu}^c \rangle_c = \frac{\delta_{\mu \nu}}{\pi^2 \alpha^2 R} \sum_{l=1}^{\infty} \int_0^\infty dx \frac{x^3}{\sqrt{x^2 - M^2}} \times \frac{C \left\{ I_{l-1/2}(ax), I_{l/\alpha-1/2}(ax) \right\}}{C \left\{ I_{l-1/2}(ax), K_{l/\alpha-1/2}(ax) \right\}} G^{(\mu)}_{l/\alpha-1/2} \left[ x, K_{l/\alpha-1/2}(x r) \right],
\]

where we have introduced the notation

\[
C \left\{ f(ax), g(x) \right\} = x f(ax) g'(x) - \left[ \frac{1}{2} \left( \frac{1}{\alpha} - 1 \right) f(ax) + x f'(ax) \right] g(x).
\]

In formula (71), \( G^{(\mu)}_{l/\alpha-1/2} [x, f_{\nu}(y)] = \tilde{F}^{(\mu,\eta)}_{\nu \nu} [x, f_{\nu}(y)] \) for \( \mu = 1, 2, 3 \), and

\[
G^{(0)}_{l/\alpha-1/2} [x, f_{l/\alpha-1/2}(y)] = \left( \frac{M^2}{x^2} - 1 \right) \left[ f_{l/\alpha-1/2}^2(y) - f_{l/\alpha+1/2}^2(y) \right].
\]

Note that in terms of notation (72) one has

\[
J_{\alpha}(ax) \tilde{F}_{\nu \nu}(x) = C \left\{ J_{\alpha}(ax), F_{\nu \nu}(x) \right\}. \tag{74}
\]

For \( \alpha = 1 \) we have \( C \left\{ I_{l-1/2}(ax), I_{l/\alpha-1/2}(ax) \right\} = 0 \) and the VEVs vanish. In the similar way, for the fermionic condensate from Eq. (60) we find

\[
\langle \tilde{\psi} \psi \rangle_c = \frac{M}{\pi^2 \alpha^2 R} \sum_{l=1}^{\infty} \int_0^\infty dx \frac{x}{\sqrt{x^2 - M^2}} \times \frac{C \left\{ I_{l-1/2}(ax), I_{l/\alpha-1/2}(ax) \right\}}{C \left\{ I_{l-1/2}(ax), K_{l/\alpha-1/2}(ax) \right\}} \left[ K_{l/\alpha-1/2}^2(x r) - K_{l/\alpha+1/2}^2(x r) \right]. \tag{75}
\]

For a massless fermionic field the core-induced part in the condensate vanishes. From formulae (71) and (75) it can be seen that for fixed values of \( r \) and \( M \), in the limit \( a \to 0 \) the core-induced parts vanish as \( a^{2/\alpha+1} \). Note that by using the recurrence relations for the modified Bessel functions, the functions \( C \left\{ f(\alpha y), g(y) \right\} \) in these formulae can also be written in the form

\[
C \left\{ I_{l-1/2}(\alpha y), f_{l/\alpha-1/2}(y) \right\} = \lambda_f y I_{l-1/2}(\alpha y) f_{l/\alpha+1/2}(y) - y I_{l+1/2}(\alpha y) f_{l/\alpha-1/2}(y), \tag{76}
\]

with \( f = I, K \) and \( \lambda_I = 1, \lambda_K = -1 \). In particular, from formula (76) with \( f = K \) it follows that \( C \left\{ I_{l-1/2}(\alpha y), K_{l/\alpha-1/2}(y) \right\} < 0 \). As we will show in Appendix A, this means that there are no bound states in the flower-pot model.
In this way it can be seen that the main contribution into the VEVs comes from the mode \(-l\) and the VEVs are suppressed by the factor $\exp\left(\frac{\alpha}{a M}\right)$ for large values of $l$. By using the uniform asymptotic expansions for the modified Bessel functions for large values of the order (see, for instance, [26]), to the leading order we find

$$
\langle T_0^0 \rangle_c \approx -2 \langle T_2^2 \rangle_c \approx \frac{2a}{r-a} \langle T_1^1 \rangle_c \approx \frac{\alpha - 1}{12\pi^2 a a(r-a)^3},
$$

(77)

$$
\langle \bar{\psi} \psi \rangle_c \approx \frac{(1 - \alpha) M}{12\pi^2 a a(r-a)}.\tag{78}
$$

As the parts corresponding to the geometry of the point-like global monopole are finite at $r = a$, from here we conclude that near the core the VEVs are dominated by the core-induced parts.

At large distances from the core, $r \gg a$, in the case of a massless fermionic field we introduce a new integration variable $y = x r$ and expand the integrand over $a/r$. The main contribution comes from the lowest mode $l = 1$ and to the leading order one has

$$
\langle T^\mu_\nu \rangle_c \approx -\frac{\delta^\mu_\nu}{3\pi^2 a a^2} \frac{(1 - \alpha) 2^{1-2/\alpha}(a/r)^{2/\alpha+5}}{(2 + \alpha)^2(1/\alpha + 1/2)} \int_0^\infty dy y^{2/\alpha+3} G^{(\mu)}_1(\alpha, K_{1/\alpha-1/2}(y)).\tag{79}
$$

Note that for a massless field the integrand does not depend on $r$. The integrals in Eq. (79) may be evaluated by using the formula for the integrals involving the product of the MacDonald functions (see Ref. [27]). For a massive field, assuming $Mr \gg 1$, the main contribution into the integral over $x$ comes from the lower limit of the integration. Replacing the functions $K_{1/\alpha-1/2}(x)$ by the corresponding asymptotic formulae for large values of the argument, to the leading order we obtain the following estimates

$$
\langle T_0^0 \rangle_c \approx -\langle T_1^1 \rangle_c \approx \frac{1}{Mr} \langle T_2^2 \rangle_c \approx \frac{\exp(-2Mr)}{4\sqrt{\pi} M a a^3 r^4} \sum_{l=1}^\infty \frac{C \{I_{l-1/2}(a M), I_{l-1/2}(a M)\}}{C \{I_{l-1/2}(a M), K_{l-1/2}(a M)\}}.\tag{80}
$$

As we see, in this limit the core-induced energy density and the radial stress are suppressed by the factor $Mr$ with respect to the azimuthal stress.

For $\alpha \ll 1$ the solid angle for the exterior geometry is small and the corresponding scalar curvature is large. In this limit we replace the modified Bessel function containing in the index $l/\alpha$ by the corresponding uniform asymptotic expansions for large values of the order and the functions containing in the argument $\alpha a x$ by the expansions for small values of the argument. In this way it can be seen that the main contribution into the VEVs comes from the mode $l = 1$ and the VEVs are suppressed by the factor $\exp[-(2/\alpha) \ln(r/a)]$.

In figure 1 we have plotted the dependence of the core-induced energy density (full curves) and radial stress (dashed curves) for a massless fermionic field as functions on the scaled radial coordinate $r/a$ for $\alpha = 0.5$ and $\alpha = 2$. The azimuthal stresses are found from the zero trace condition. In figure 2 the same quantities evaluated for $r/a = 1.5$ are presented as functions on the parameter $\alpha$.

### 4.2 Interior region

Now let us consider the vacuum polarization effects inside the core for the flower-pot model. The corresponding eigenfunctions have the form given by Eq. (24) where the functions $R_{l,n}(r,k)$
Figure 1: The core-induced energy density (full curves), $a^4 \langle T^0_0 \rangle_c$, and radial stress (dashed curves), $a^4 \langle T^1_1 \rangle_c$, for a massless fermionic field in the exterior region of the flower-pot model as functions on $r/a$. The numbers near the curves correspond to the values of the parameter $\alpha$.

Figure 2: The core-induced energy density (full curves), $a^4 \langle T^0_0 \rangle_c$, and radial stress (dashed curves), $a^4 \langle T^1_1 \rangle_c$, for a massless fermionic field evaluated at $r/a = 1.5$ as functions on the parameter $\alpha$. 
and \( R_{2,\nu}(r, k) \) are defined by formulae (67). Substituting the eigenfunctions into the mode-sum formula, for the corresponding energy-momentum tensor one finds (no summation over \( \mu \))

\[
\langle 0| T^\nu_{\mu}|0 \rangle = \frac{\delta^\nu_{\mu}}{2\pi^3 \alpha^3 r} \sum_{j=1/2}^{\infty} (2j + 1) \sum_{\sigma=0,1} \int_0^{\infty} dx \frac{f^{(\mu)}_{\sigma \lambda \sigma}[x, J_{\lambda \sigma}(x\tilde{r}/a)]}{J^2_{\lambda \sigma}(\alpha x) \left[ J^2_{\nu \sigma}(x) + \tilde{Y}^2_{\nu \sigma}(x) \right]}.
\]  

(81)

To find the renormalized VEV of the energy-momentum tensor we need to evaluate the difference between the VEV given by Eq. (81) and the corresponding VEV for the Minkowski bulk:

\[
\langle T^\nu_{\mu}\rangle_{\text{ren}} = \langle 0| T^\nu_{\mu}|0 \rangle - \langle 0_M| T^\nu_{\mu}|0_M \rangle.
\]

(82)

The appropriate form for the Minkowskian part is obtained from Eq. (39) taking \( \alpha = 1 \), replacing the integrand by \( f^{(\mu)}_{\sigma \lambda \sigma}[x, J_{\lambda \sigma}(x\tilde{r}/a)] \) and \( r \to \tilde{r} \). As a result for the subtracted VEV one finds

\[
\langle T^\nu_{\mu}\rangle_{\text{ren}} = \frac{\delta^\nu_{\mu}}{2\pi^3 \alpha^3 \tilde{r}^3} \sum_{j=1/2}^{\infty} (2j + 1) \sum_{\sigma=0,1} \int_0^{\infty} dx f^{(\mu)}_{\sigma \lambda \sigma}[x, J_{\lambda \sigma}(x\tilde{r}/a)]
\]

\[
\times \left[ \frac{J^{-2}_{\lambda \sigma}(\alpha x)/\alpha}{J^2_{\nu \sigma}(x) + \tilde{Y}^2_{\nu \sigma}(x)} - \frac{\pi^2}{4} \right].
\]

(83)

The integral in this formula is slowly convergent and the integrand is highly oscillatory.

In order to transform the expression for the subtracted VEV of the energy-momentum tensor into more convenient form, we note that the following identity takes place

\[
\frac{J^{-2}_{\lambda \sigma}(\alpha x)/\alpha}{J^2_{\nu \sigma}(x) + \tilde{Y}^2_{\nu \sigma}(x)} - \frac{\pi^2}{4} = -\frac{\pi^2}{8} \sum_{s=1,2} \frac{C\{H^{(s)}_{\lambda \sigma}(\alpha x), H^{(s)}_{\nu \sigma}(x)\}}{C\{J_{\lambda \sigma}(\alpha x), H^{(s)}_{\nu \sigma}(x)\}}.
\]

(84)

Substituting Eq. (83) into formula (83), we rotate the integration contour in the complex plane \( x \) by the angle \( \pi/2 \) for \( s = 1 \) and by the angle \( -\pi/2 \) for \( s = 2 \). Under the condition \( \tilde{r} < \alpha a \) the contributions from the semicircles with the radius tending to infinity vanish. The integrals over the segments \((0,iMa)\) and \((0,-iMa)\) of the imaginary axis cancel out and after introducing the modified Bessel functions we can see that different parities give the same contribution. Consequently, the renormalized VEV for the energy-momentum tensor can be presented in the form (no summation over \( \mu \))

\[
\langle T^\nu_{\mu}\rangle_{\text{ren}} = \frac{\delta^\nu_{\mu}}{\pi^2 \alpha^3 \tilde{r}^3} \sum_{l=1}^{\infty} \int_M l dx x^3 \sqrt{x^2 - M^2}
\]

\[
\times \frac{C\{K_{l-1/2}(\alpha x), K_{l/\alpha-1/2}(ax)\}}{C\{I_{l-1/2}(\alpha x), K_{l/\alpha-1/2}(ax)\}} G^{(\mu)}_{l-1/2}[x, I_{l-1/2}(x\tilde{r})].
\]

(85)

where the functions \( G^{(\mu)}_{l}[x, f_{\nu}(y)] \) are the same as in Eq. (71) with \( \lambda_l = 1 \) in Eqs. (55) and (56). Note that similar to Eq. (76), the function in the numerator of the integrand is also presented in the form

\[
C\{K_{l-1/2}(\alpha y), K_{l/\alpha-1/2}(y)\} = -yK_{l-1/2}(\alpha y)K_{l/\alpha+1/2}(y) + yK_{l+1/2}(\alpha y)K_{l/\alpha-1/2}(y).
\]

(86)

For \( \alpha = 1 \) one has \( C\{K_{l-1/2}(\alpha x), K_{l/\alpha-1/2}(ax)\} \) = 0 and, as we could expect, the VEV of the energy-momentum tensor vanishes. It can be explicitly checked that the components of the tensor given by formula (85) satisfy the continuity equation (58) and this tensor is traceless for a
massless field. In the way similar to that for the exterior region, for the renormalized fermionic condensate inside the core we find

$$\langle \bar{\psi}\psi \rangle_{\text{ren}} = \frac{M}{2\alpha^2 r} \sum_{l=1}^{\infty} l \int_{M}^{\infty} dx \frac{x}{\sqrt{x^2 - M^2}} \times \frac{C\{K_{l-1/2}(aax), K_{l-1/2}(aax)\}}{C\{I_{l-1/2}(aax), K_{l-1/2}(aax)\}} \left[I_{l-1/2}(x\tilde{r}) - I_{l+1/2}(x\tilde{r})\right].$$

(87)

The VEVs given by formulae (85) and (87) are finite for \( \tilde{r} < aa \) and diverge on the core boundary. For the points near the boundary the main contribution comes from large \( l \) and we replace the modified Bessel functions by the corresponding uniform expansions for large values of the order. In this way it can be seen that the leading terms in the asymptotic expansion with respect to the distance from the boundary are given by the formulae

$$\langle T_0^0 \rangle_{\text{ren}} \approx -2\langle T_2^2 \rangle_{\text{ren}} \approx -\frac{2\alpha a}{\alpha a - r} \langle T_1^1 \rangle_{\text{ren}} \approx \frac{\alpha - 1}{120\pi^2 a^2 (\alpha a - r)^3},$$

(88)

$$\langle \bar{\psi}\psi \rangle_{\text{ren}} \approx \frac{(1 - \alpha)M}{12\pi^2 a^2 (\alpha a - r)}. $$

(89)

Comparing these results with the corresponding formulae for the exterior region, we see that near the core boundary the energy density and azimuthal stresses have the same signs inside and outside the core, whereas the radial stresses have opposite signs.

Near the core center the contribution of the summand with a given \( l \) behaves like \( r^{2(l-1)} \) and the main contribution comes from the lowest mode \( l = 1 \) with the leading term (no summation over \( \mu \))

$$\langle T_{\mu}^\nu \rangle_{\text{ren}} \approx \frac{2\delta^\nu_{\mu}}{3\pi \alpha^2 a^4} \int_{Ma}^{\infty} dx \frac{x^3 C(\mu)(x)}{\sqrt{x^2 - M^2 a^2}} \frac{C\{K_{1/2}(aax), K_{1/2}(aax)\}}{C\{I_{1/2}(aax), K_{1/2}(aax)\}},$$

(90)

where we have introduced notations

$$C^{(0)}(x) = 3(M^2 a^2 / x^2 - 1), \quad C^{(1)}(x) = C^{(2)}(x) = 1.$$  

(91)

In the case of a massless field, formula (90) also gives the behavior of the vacuum energymomentum tensor in the limit when the core radius is large and \( \tilde{r} \) is fixed, \( a/\tilde{r} \gg 1 \). In the same limit, for a massive field, assuming \( aM \gg 1 \), we replace in Eq. (85) the modified Bessel functions containing in the argument \( aax \) by the corresponding asymptotic expressions for large values of the argument. By taking into account that the main contribution into the integral comes from the lower limit of the integration, to the leading order we have (no summation over \( \mu \))

$$\langle T_0^0 \rangle_{\text{ren}} \approx \frac{\alpha - 1}{16\pi a^4 \alpha^3 \tilde{r}} e^{-2\alpha a M} \sqrt{\pi a M} \sum_{l=1}^{\infty} l^2 \left[I_{l-1/2}(M\tilde{r}) - I_{l+1/2}(M\tilde{r})\right],$$

(92)

$$\langle T_{\mu}^\nu \rangle_{\text{ren}} \approx \frac{\delta^\nu_{\mu}(1 - \alpha)}{8\pi \alpha^4 a^7 \tilde{r}} e^{-2\alpha a M} \sqrt{\pi a M} \sum_{l=1}^{\infty} l^2 C_{l-1/2}^{(\mu)} [M, I_{l-1/2}(M\tilde{r})],$$

(93)

with \( \nu \neq 0 \). In this case the energy density is suppressed with respect to the vacuum stresses by an additional factor \( (aM)^{-1} \).

For small values of the parameter \( \alpha \) assuming that the core radius \( a \) for an internal Minkowskian observer is fixed, we replace the functions \( K_{1/2-1/2}(aax) \) in Eq. (85) by the corresponding uniform asymptotic expansion for large values of the order. The leading term is obtained by making use of the replacements

$$C\{f_{l-1/2}(aax), K_{l-1/2}(aax)\} \rightarrow y[\ln f_{l-1/2}(y)] + \sqrt{l^2 + y^2} + 1/2,$$

(94)
in the integrand of Eq. (85) with \( y = \alpha ax \) and \( f = I, K \). As a result, in this limit the VEVs behave like \( 1/\alpha \). Due to the factor \( \alpha^2 \) in the volume element the corresponding global quantities, such as total energy vanish as \( \alpha \).

In figure 3 we have plotted the dependence of the renormalized vacuum energy density (full curves) and radial stress (dashed curves) for a massless fermionic field as functions on \( \tilde{r}/\alpha a \) for \( \alpha = 0.5 \) and \( \alpha = 2 \). In figure 4 the same quantities evaluated for \( \tilde{r}/\alpha a = 0.5 \) are presented as functions on the parameter \( \alpha \).

Figure 3: The renormalized vacuum energy density (full curves), \((\alpha a)^4\langle T^0_0\rangle_{\text{sub}}\), and radial stress (dashed curves), \((\alpha a)^4\langle T^1_1\rangle_{\text{sub}}\), for a massless fermionic field inside the core of the flower-pot model as functions on \( \tilde{r}/\alpha a \). The numbers near the curves correspond to the values of the parameter \( \alpha \).

Figure 4: The energy density (full curves), \((\alpha a)^4\langle T^0_0\rangle_{\text{sub}}\), and radial stress (dashed curves), \((\alpha a)^4\langle T^1_1\rangle_{\text{sub}}\), for a massless fermionic field evaluated at \( \tilde{r}/\alpha a = 0.5 \) as functions on the parameter \( \alpha \).
In the present paper we have considered the polarization of the fermionic vacuum by the gravitational field of the global monopole with a non-trivial core structure. The previous investigations in this direction are concerned with the idealized point-like model, where the curvature has singularity at the origin. In a realistic point of view, the global monopole has a characteristic core radius determined by the symmetry braking energy scale. For a general spherically symmetric static model of the core with finite thickness, we have evaluated the VEVs of the energy-momentum tensor and the fermionic condensate for a massive spinor field. These quantities are among the most important characteristics of the vacuum properties, which carry an information about the core structure. In the region outside the core we have presented the VEVs as a sum of two contributions. The first one corresponds to the geometry of a point-like global monopole and the second one is induced by the non-trivial structure of the monopole core. In the general spherically symmetric static model for the core, we have derived closed analytic expressions for the core-induced parts given by formula (53) for the energy-momentum tensor and by formula (60) for the fermionic condensate, where the properties of the core are codified by the coefficient in the square brackets in the notation (57). In Appendix A we show that the formulae for the core-induced parts in the VEVs of the energy-momentum tensor and fermionic condensate are valid also in the case when bound states are present. For points away from the core boundary these parts are finite and the renormalization is reduced to that for the point-like monopole geometry. Of course, we could expect this result as in the model under consideration the exterior geometry is the same as that for the point-like global monopole. For the points on the boundary the VEVs contain surface divergences well-known in quantum field theory with boundaries.

As an example of the application of the general results, in section 4 we have considered a simple model with a flat spacetime inside the core. The corresponding model for the cosmic string core is known in literature as flower-pot model and here we use the same terminology for the global monopole. To have matching between the exterior and interior metrics, in this model we need the surface energy-momentum tensor located on the boundary of the core and having components given by Eq. (66). In the model with solid angle deficit (α < 1) the corresponding surface energy density is positive. The core-induced parts of the exterior VEVs in the flower-pot model are obtained from the general results taking as the interior radial functions in the eigenmodes the functions (67). These parts are given by formula (71) for the energy-momentum tensor and by formula (75) for the fermionic condensate. We have investigated the core-induced parts in various asymptotic regions of the parameters. In the limit when the core radius tends to zero, $a \to 0$, for fixed values $r$ and $M$, these parts behave like $a^{2/\alpha+1}$. For points near the core boundary the leading terms in the corresponding asymptotic expansions are given by formulae (77), (78). In this region the total VEVs are dominated by the core-induced parts. At large distances from the core these parts tend to zero as $(a/r)^{2/\alpha+5}$ for a massless field and are exponentially suppressed by the factor $\exp(-Mr)$ for a massive field. We have also investigated the limit of strong gravitational fields corresponding to small values of the parameter $\alpha$. In this limit the main contribution into the VEVs comes from the lowest mode $l = 1$ and the VEVs are suppressed by the factor $\exp[-(2/\alpha)\ln(r/a)]$.

For the flower-pot model we have also investigated the VEVs of the energy-momentum tensor and the fermionic condensate inside the core. Though the corresponding spacetime geometry is Minkowskian, the non-trivial topology of the exterior region induces vacuum polarization effects in this region as well. The renormalization is achieved by the subtraction from the mode-sums the corresponding quantities for the Minkowski spacetime. By making use of identity (84), after an appropriate deformation of the integration contour, we have presented the renormalized
VEV of the energy-momentum tensor in the form (85) and the fermionic condensate in the form (87). These quantities are finite for strictly interior points and diverge on the boundary of the core with the leading divergences given by formulae (88), (89). In particular, near the core boundary the energy density and azimuthal stresses have the same signs inside and outside the core, whereas the radial stresses have opposite signs. Near the core center the main contribution comes from the mode $l = 1$ and the VEVs tend to a finite limiting value with isotropic vacuum stresses. Although the exact behavior for the fermionic field is unknown for a realistic model of the global monopole spacetime, the flower-pot model considered here presents some expected results as, for example, finite vacuum polarization effects at the monopole’s center. For large values of the core radius the renormalized VEV of the energy-momentum tensor inside the core vanishes as $a^{-4}$ for a massless field and as $e^{-2\alpha aM}$ for a massive one. In the limit $\alpha \ll 1$, assuming that the core radius $\alpha a$ for an internal Minkowskian observer is fixed, the vacuum densities in the interior region behave as $1/\alpha$.

Note that in this paper we have considered quantum vacuum effects in a prescribed background, i.e. the gravitational back-reaction of quantum effects is not taken into account. This back-reaction could have important effects on the dynamical evolution of the bulk model. We do not consider this extension of the theory, but note that the results presented here constitute the starting point for such investigations.

**Acknowledgement**

AAS was supported by PVE/CAPES Program and in part by the Armenian Ministry of Education and Science Grant No. 0124. ERBM thanks Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and FAPESQ-PB/CNPq (PRONEX) for partial financial support.

**A Bound states**

In this appendix we consider the changes in the procedure described in the main text when bound states are present. For these states the quantity $k$ is purely imaginary, $k = i\gamma$, and the corresponding exterior eigenfunctions have the form

$$\psi_{b\beta}^{(ex)}(r) = C_b \frac{e^{-i\omega t}}{\sqrt{r}} \left( \frac{i\gamma}{\omega+M} K_{\nu_\sigma}(\gamma r) \Omega_{j_\alpha m}(\theta,\varphi) \right),$$

with $\omega^2 = M^2 - \gamma^2$. To have a stable ground state we will assume that $\gamma < M$. From the continuity of the eigenfunctions at $r = a$ one has

$$\sqrt{a} R_{1,n_\sigma}(a,i\gamma) = C_b K_{\nu_\sigma}(\gamma a), \quad \sqrt{a} R_{2,n_\sigma}(a,i\gamma) = -i n_\sigma C_b K_{\nu_\sigma+n_\sigma}(\gamma a).$$

Excluding from these relations the normalization coefficient $C_b$ we see that for possible bound states $\gamma$ is a solutions of the equation

$$\bar{K}^{(1,\sigma)}_{\nu_\sigma}(\gamma a) = 0,$$

with the barred notation from Eq. (57).

The coefficient $C_b$ in Eq. (95) is found from the normalization condition (33). To derive a formula for the normalization integral, we rewrite equations (16), (17) in terms of the functions $F_\beta(r) = e^{u/2+h} f_\beta(r)$ and $G_\beta(r) = e^{u/2+h} g_\beta(r)$ and differentiate both equations with respect to $\omega$. Further we multiply the first equation by $G_\beta(r)$, the second one by $-F_\beta(r)$ and add them.
Combining the resulting equation with Eqs. (100), (17), it can be seen that the following relation takes place

\[ e^{u-u} \left( F^2 + G^2 \right) = e^{u+2h} \frac{d}{dr} \left( G \frac{\partial F}{\partial \omega} - F \frac{\partial G}{\partial \omega} \right). \]  

(98)

Integrating this relation we obtain the formula

\[ \int_{\rho_0}^{r} dr e^{v+2h} \left[ f^2 \left( r \right) + g^2 \left( r \right) \right] = e^{u+2h} \left[ g \left( r \right) \frac{\partial f \left( r \right)}{\partial \omega} - f \left( r \right) \frac{\partial g \left( r \right)}{\partial \omega} \right], \]  

(99)

where \( r_0 \) is the value of the radial coordinate corresponding to the center of the core. By using this formula, in the case of bound states for the normalization integral in Eq. (33) one finds

\[ \int d^3 x \sqrt{\gamma} \psi^\dagger \psi = \int_{\rho_0}^{\infty} dr e^{v+2h} \left[ f^2 \left( r \right) + g^2 \left( r \right) \right] = e^{u+2h} \left[ g \left( r \right) \frac{\partial f \left( r \right)}{\partial \omega} - f \left( r \right) \frac{\partial g \left( r \right)}{\partial \omega} \right]_{r=a-}^{r=a+}. \]  

(100)

The expression on the right can be further simplified using the continuity of the eigenfunctions on the boundary of the core. In this way for the normalization coefficient one finds the formula

\[ C^2 \left( K_\nu \left( \gamma a \right) \right) = \frac{M + \omega}{\alpha^2 K_\nu \left( \gamma a \right) (\partial / \partial \omega) K_\nu \left( \gamma a \right)}, \]  

(101)

As a result, for the contribution of the bound state with \( k = i \gamma \) to the VEV of the energy-momentum tensor we have the formula (no summation over \( \mu \))

\[ \langle T^{\mu} \rangle_b = \frac{\delta^{\nu}}{2\pi \alpha} \int d^3 r e^{v+2h} \left[ f^2 \left( r \right) + g^2 \left( r \right) \right] = e^{u+2h} \left[ g \left( r \right) \frac{\partial f \left( r \right)}{\partial \omega} - f \left( r \right) \frac{\partial g \left( r \right)}{\partial \omega} \right] \]  

(102)

where \( \nu_\sigma = l/\alpha - n_\sigma / 2 \) and we have introduced the notations

\[ B^{(0)} [\gamma, K_{\nu_\sigma} (y)] = \left( 1 - \frac{M^2}{\gamma^2} \right) \left[ \left( \frac{M}{\sqrt{M^2 - \gamma^2}} - 1 \right) K^2_{\nu_\sigma} (y) \right], \]  

(103)

\[ B^{(1)} [\gamma, K_{\nu_\sigma} (y)] = K^2_{\nu_\sigma} (y) + \frac{2l_\nu_\sigma}{\alpha y} K_{\nu_\sigma} (y) K_{\nu_\sigma + n_\sigma} (y) - K^2_{\nu_\sigma + n_\sigma} (y), \]  

\[ B^{(\mu)} [\gamma, K_{\nu_\sigma} (y)] = -\frac{l_\nu_\sigma}{\alpha y} K_{\nu_\sigma} (y) K_{\nu_\sigma + n_\sigma} (y), \mu = 2, 3. \]

In deriving Eq. (102) we have used the relation \( K_\nu \left( \gamma a \right) = 1 / F_\nu \left( \gamma a \right) \), which directly follows from the Wronskian relation for the modified Bessel functions in combination with Eq. (97). In the case when several bound states are present the sum of their separate contributions should be taken.

The VEV of the energy-momentum tensor is the sum of the part coming from the modes with real \( k \) given by Eq. (15) and of the part coming from the bound states given by Eq. (102). In order to transform the first part we again rotate the integration contour in Eq. (15) by the angle \( \pi / 2 \) for \( s = 1 \) and by the angle \( -\pi / 2 \) for \( s = 2 \). But now we should take into account that the integrand has poles at \( x = \pm i \gamma a \) which are zeroes of the functions \( \bar{F}_{\nu_\sigma} \left( e^{\gamma a_\mu / 2} x \right) \) in accordance with Eq. (97). Rotating the integration contour we will assume that the pole \( e^{\gamma a_\mu / 2} \gamma a, s = 1, 2 \), on the imaginary axis is avoided by the semicircle \( C^{(s)} \) in the right half plane with small radius
\( \rho \) and with the center at this pole. The integration over these semicircles will give an additional contribution

\[
-\frac{\delta^{\nu}}{16\pi^{2}a^{3}} \sum_{j=1/2}^{\infty} (2j + 1) \sum_{\sigma=0,1} \sum_{s=1,2} \int_{c^{(s)}} dx \frac{\tilde{J}_{\nu_{\sigma}}(x)}{H_{\nu_{\sigma}}^{(s)}(x)} \int_{\sigma_{\nu_{\sigma}}}^{(\mu)} f_{\sigma_{\nu_{\sigma}}} \left[ x, H_{\nu_{\sigma}}^{(s)}(x) \right].
\]

By evaluating the integrals in this formula it can be seen that this term cancels the contribution \((102)\) coming from the corresponding bound state. Hence, we conclude that the formulae given above for the core-induced parts in the VEVs are valid in the case of the presence of bound states as well.

In the flower-pot model the equation \((97)\) for the bound states takes the form

\[
C\{I_{l-1/2}(\alpha a \gamma), K_{l/\alpha-1/2}(\alpha \gamma)\} = 0.
\]

In Section \(4\) we have shown that the function on the left of this equation is always negative and, hence, in the flower-pot model no bound states exist.

References

[1] T.W. Kibble, J. Phys. A 9, 1387 (1976).
[2] A. Vilenkin and E.P.S. Shellard, Cosmic Strings and Other Topological Defects (Cambridge University Press, Cambridge, England, 1994).
[3] D.D. Sokolov and A.A. Starobinsky, Dokl. Akad. Nauk USSR 234, 1043 (1977) [Sov. Phys. Doklady 22, 312 (1977)].
[4] M. Barriola and A. Vilenkin, Phys. Rev. Lett. 63, 341 (1989).
[5] G.E. Volovik, Pisma Zh. Eksp. Teor. Fiz. 67, 666 (1998) [JETP Lett. 67, 698 (1998)].
[6] F.D. Mazzitelli and C.O. Lousto, Phys. Rev. D 43, 468 (1991).
[7] E.R. Bezerra de Mello, V.B. Bezerra, and N.R. Khusnutdinov, Phys. Rev. D 60, 063506 (1999).
[8] F.C. Carvalho and E.R. Bezerra de Mello, Class. Quantum Grav. 18, 1637 (2001); Class. Quantum Grav. 18, 5455 (2001).
[9] E.R. Bezerra de Mello, J. Math. Phys. 43, 1018 (2002).
[10] E.R. Bezerra de Mello and A.A. Saharian, Phys. Lett. B 642, 129 (2006).
[11] M. Bordag, K. Kirsten, and S. Dowker, Commun. Math. Phys. 128, 371 (1996).
[12] E.R. Bezerra de Mello, V.B. Bezerra, and N.R. Khusnutdinov, J. Math. Phys. 42, 562 (2001).
[13] A.A. Saharian and M.R. Setare, Class. Quantum Grav. 20, 3765 (2003).
[14] A.A. Saharian and M.R. Setare, Int. J. Mod. Phys. A 19, 4301 (2004).
[15] A.A. Saharian, "The Generalized Abel-Plana Formula. Applications to Bessel functions and Casimir effect," Report No. IC/2000; [hep-th/0002239].
[16] A.A. Saharian and E.R. Bezerra de Mello, J. Phys. A 37, 3543 (2004); A.A. Saharian and E.R. Bezerra de Mello, Int. J. Mod. Phys. A 20, 2380 (2005).

[17] E.R. Bezerra de Mello and A.A. Saharian, Class. Quantum Grav. 23, 4673 (2006).

[18] D. Harari and C. Lousto, Phys. Rev. D 42, 2626 (1990).

[19] J. Spinelly and E.R. Bezerra de Mello, Class. Quantum Grav. 22, 3247 (2005).

[20] E.R. Bezerra de Mello and A.A. Saharian, JHEP 10, 049 (2006).

[21] E.R. Bezerra de Mello and A.A. Saharian, ”Electrostatic self-interaction in the spacetime of a global monopole with finite core”, hep-th/0612134.

[22] V.B. Berestetskii, E.M. Lifshits, and L.P. Pitaevskii, Quantum Electrodynamics (Pergamon Press, 1982).

[23] B. Allen and A.C. Ottewill, Phys. Rev. D 42, 2669 (1990).

[24] B. Allen, B.S. Kay, and A.C. Ottewill, Phys. Rev. D 53, 6829 (1996).

[25] E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, and A.S. Tarloyan, Phys. Rev. D 74, 025017 (2006).

[26] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions (National Bureau of Standards, Washington DC, 1964).

[27] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, Integrals and Series (Gordon and Breach, New York, 1986), Vol.2.