$N$–graphs, modular Sidon and sum–free sets, and partition identities

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Abstract

Using a new graphical representation for partitions, the author obtains a family of partition identities associated with partitions into distinct parts of an arithmetic progression, or, more generally, with partitions into distinct parts of a set that is a finite union of arithmetic progressions associated with a modular sum–free Sidon set. Partition identities are also constructed for sets associated with modular sum–free sets.

1 $N$–graphs for partitions

The standard form of a partition $n = a_1 + a_2 + \cdots + a_k$ is

$$\pi = (a_1, \ldots, a_k),$$

where the parts $a_1, \ldots, a_k$ are positive integers arranged in descending order. The standard form of a partition is unique.

Associated to a partition $\pi = (a_1, \ldots, a_k)$ of $n$ is an array of dots, called the Ferrers graph of $\pi$. This consists of $n$ dots arranged in $k$ rows, with $a_1$ dots on the first row, $a_2$ dots on the second row, \ldots, and $a_k$ dots on the $k$–th row. The rows are aligned on the left. The Durfee square $D(\pi)$ of the graph is the largest square array of dots that appears in the upper left corner of the Ferrers graph. We denote by $d(\pi)$ the number of dots on a side of the Durfee square, or, equivalently, the number of dots on a diagonal of $D(\pi)$.

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The Ferrers graph can be decomposed into a disjoint union of right angles, called hooks. The corners of the hooks are the dots on the main diagonal of the Durfee square, and so the number of hooks is $d(\pi)$. Counting the number of dots on the hooks, we obtain the hook numbers of the partition $\pi$. Since the sum of the hook numbers is $n$, we obtain a new partition of $n$, denoted $h(\pi)$ and called the hook number partition.

For example, the partition $\pi = (7, 6, 6, 5, 4)$ of 28 has $d(\pi) = 4$. The hook number partition is $h(\pi) = (11, 8, 6, 3)$.

MacMahon [2] introduced a beautiful arithmetic generalization of the Ferrers graph of a partition. Let $\pi = (a_1, \ldots, a_k)$ be a partition of $n$ in standard form. For each positive integer $m$ we shall construct the MacMahon modular $m$–graph of the partition $\pi$. By the division algorithm, we can write each part $a_i$ uniquely in the form

$$a_i = u(a_i)m + s(a_i) \quad \text{where} \quad u(a_i) \geq 0 \quad \text{and} \quad 1 \leq s(a_i) \leq m.$$

The $m$–graph of $\pi$ consists of $k$ rows. The $i$–th row has $u(a_i) + 1$ entries, where the first $u(a_i)$ entries are $m$, and the last entry is $s(a_i)$. In the special case $m = 1$, we have $u(a_i) = a_i - 1$ and $s(a_i) = 1$ for $i = 1, \ldots, k$. The 1–graph is exactly the Ferrers graph with each dot replaced by 1.

The Durfee square of the $m$–graph, denoted $D_m(\pi)$, is the largest square array of integers contained in the upper left corner of the graph. The number of dots a side of the Durfee square is denoted $d_m(\pi)$. The hook number partition associated with the $m$–graph is the partition $h_m(\pi)$ obtained by adding the numbers on the hooks of the $m$–graph. The hook number partition has $d_m(\pi)$ parts.

For example, if $\pi = (9, 8, 6, 4)$, then the $m$–graphs of $\pi$ for $m = 1, 2$, and 3 are

$$\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 1 \\
2 & 2 & 2 & 2 & 2 & 1 \\
2 & 2 & 2 & 2 & 2 & 1 \\
3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 2 & 2 & 2 & 2 \\
3 & 3 & 2 & 2 & 2 & 2 \\
3 & 3 & 2 & 2 & 2 & 2 \\
3 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 \\
\end{array}$$

Note that $d_1(\pi) = 4$, $d_2(\pi) = 3$, and $d_3(\pi) = 2$. The hook number partitions associated with these graphs are $h_1(\pi) = (12, 9, 5, 1)$, $h_2(\pi) = (15, 10, 2)$, and $h_3(\pi) = (18, 9)$. 

2
In this paper I introduce a generalization of MacMahon’s $m$–graphs. Let $m \geq 2$, and let $S = \{s_1, \ldots, s_\ell\}$ be an ordered, nonempty set of positive integers that are pairwise incongruent modulo $m$. We do not assume that $1 \leq s \leq m$ for $s \in S$. Let $A$ be the set of integers of the form $um + s$, where $u \geq 0$ and $s \in S$. Then $A$ is a finite union of arithmetic progressions with difference $m$, and every element $a \in A$ has a unique representation in the form $a = u(a)m + s(a)$, where $u(a)$ is a nonnegative integer and $s(a) \in S$.

A partition $\pi$ into parts belonging to $A$ can be written uniquely in the form $$\pi = (a_1, \ldots, a_k)_N,$$
where $$a_i = u(a_i)m + s(a_i) \in A,$$
and, if $u(a_i) = u(a_{i+1})$, $s(a_i) = s_j$, and $s(a_{i+1}) = s_{j+1}$, then $j_i \leq j_{i+1}$. We shall call this the standard $N$–form for a partition with parts in the set $A$. Note that if $(a_1, \ldots, a_k)_N$ is the standard $N$–form of a partition, then it is not necessarily true that $a_i \geq a_{i+1}$ for all $i = 1, \ldots, k-1$.

The $N$–graph of the partition $\pi = (a_1, \ldots, a_k)_N$ will consist of $k$ rows. The $i$–th row has $u(a_i) + 1$ entries, where the first $u(a_i)$ entries are $m$ and the last entry is $s(a_i)$. In particular, we obtain MacMahon’s modular $m$–graphs for partitions in the special case $\ell = m$ and $s_j = m + 1 - j$ for $j = 1, \ldots, m$.

The Durfee square $D_N(\pi)$ of the $N$–graph is the largest square array of integers contained in the upper left corner of the graph, and the hook number partition $h_N(\pi)$ associated with the $N$–graph is the partition obtained by adding the numbers on the hooks of the $N$–graph. If $d_N(\pi)$ is the number of integers on a side of the Durfee square $D_N(\pi)$, then the hook number partition has $d_N(\pi)$ parts.

For example, let $m = 13$ and $S = \{3, 2, 20\}$, where $\ell = 3$, and $s_1 = 3, s_2 = 2, s_3 = 20$. Then

\[
A = \{3, 16, 29, 42, 55, \ldots\} \cup \{2, 15, 28, 41, 54, \ldots\} \cup \{20, 33, 46, 59, \ldots\} \\
= \{2, 3, 15, 16, 20, 28, 29, 33, 41, 42, 46, 54, 55, 59, \ldots\}.
\]

Consider the partition $193 = 55 + 41 + 33 + 29 + 20 + 15$. The standard form for this partition is $$\pi = (55, 41, 33, 29, 20, 15),$$
and the standard $N$–form is $$\pi = (55, 41, 29, 15, 33, 20)_N.$$

The corresponding $N$–graph is

\[

\text{3}
\]
The Durfee square contains 9 points, \( d_N(\pi) = 3 \), and the hook number partition associated with the \( N \)-graph is \( h_N(\pi) = (127, 63, 3) \).

## 2 Sum–free Sidon sets

Let \( S = \{s_1, \ldots, s_\ell\} \) be a nonempty finite set of integers, and let \( 2S = \{s + s' : s, s' \in S\} \). The set \( S \) is a **sum–free** if \( S \cap 2S = \emptyset \). The set \( S \) is a **Sidon set** if every integer has at most one representation as a sum of two elements of \( S \), that is, \( s_i + s_{i_2} = s_{j_1} + s_{j_2} \) if and only if \( \{i_1, i_2\} = \{j_1, j_2\} \). For example, \( \{1, 6, 19\} \) is a sum–free Sidon set, and \( \{s\} \) is a sum–free Sidon set for every \( s \neq 0 \).

Let \( m \geq 2 \). The set \( S \) is **sum–free modulo** \( m \) if the elements of \( S \) are pairwise incongruent modulo \( m \) and the congruence \( s_{i_1} + s_{i_2} \equiv s_j \pmod{m} \) has no solution with \( s_{i_1}, s_{i_2}, s_j \in S \). The set \( S \) is a **Sidon set modulo** \( m \) if every congruence class modulo \( m \) has at most one representation as a sum of two elements of \( S \), that is, \( S \) is a set of pairwise incongruent integers such that \( s_{i_1} + s_{i_2} \equiv s_{j_1} + s_{j_2} \pmod{m} \) if and only if \( \{i_1, i_2\} = \{j_1, j_2\} \). For example, \( \{1, 6, 19\} \) is a sum–free Sidon set modulo 15, but not modulo 11, since \( 6 + 6 \equiv 0 + 1 \pmod{11} \). The set \( \{s\} \) is a sum–free Sidon set modulo \( m \) for every integer \( s \) and every modulus \( m \) that does not divide \( s \).

Let \( m \geq 2 \), and let \( S = \{s_1, \ldots, s_\ell\} \) be a set of positive integers that is a sum–free Sidon set modulo \( m \). Associated with \( S \) is the set \( A \) of positive integers of the form \( um + s \), where \( u \geq 0 \) and \( s \in S \). If \( a = um + s \in A \), we define \( u(a) = u \) and \( s(a) = a \). The integers \( u(a) \) and \( s(a) \) are uniquely determined by \( a \). For every positive integer \( n \), let \( A(n) \) denote the set of partitions of \( n \) in the form \( n = a_1 + \cdots + a_k \), where \( a_i \in A \) and \( u(a_i) > u(a_{i+1}) \) for \( i = 1, \ldots, k - 1 \). Then \( \pi = (a_1, \ldots, a_k)_N \) is the standard \( N \)-form of the partition. Let \( \mathcal{H}(n) \) denote the set of hook number partitions associated with the \( N \)-graphs of the partitions in \( A(n) \). The map that sends \( \pi \in A(n) \) to the hook number partition \( h_N(\pi) \in \mathcal{H}(n) \) is not, in general, one–to–one. For example, let \( m = 15 \) and \( S = \{1, 6, 19\} \). The partitions \( \pi^{(1)} = (96, 61, 64, 21)_N \) and \( \pi^{(2)} = (96, 66, 64, 16)_N \) have the same hook number partition \( h_N(\pi^{(1)}) = h_N(\pi^{(2)}) = (141, 67, 34) \). The standard \( N \)-graphs of the partitions \( \pi^{(1)} \) and \( \pi^{(2)} \) are
Even though the map $\pi \mapsto h_N(\pi)$ is not one-to-one, there is a partition identity that relates the sets $A(n)$ and $H(n)$.

**Theorem 1** Let $m \geq 2$ and let

$$S = \{s_1, \ldots, s_\ell\}$$

be a set of positive integers that is a sum–free Sidon set modulo $m$. Let

$$A = \{um + s : u \geq 0 \text{ and } s \in S\}.$$ 

Let $A(n)$ be the set of partitions of $n$ in the form

$$n = a_1 + \cdots + a_k,$$

where

$$a_i = u(a_i)m + s(a_i) \in A$$

and

$$u(a_1) > \cdots > u(a_k) \geq 0.$$  \hspace{1cm} (1)

Let $p_A(n)$ denote the number of partitions in the set $A(n)$.

Let

$$B = \{vm + s + s' : v \geq 1 \text{ and } s, s' \in S\}$$

and

$$H = A \cup B.$$ 

Since $S$ is a sum–free Sidon set modulo $m$, each element $h \in H$ can be written uniquely in the form

$$h = v(h)m + t(h),$$

where $v(h) \geq 0$ and $t(h) \in S \cup 2S$. Let $H(n)$ be the set of partitions of $n$ of the form

$$\pi' = (h_1, \ldots, h_d),$$

where

$$h_i \in H \quad \text{ for } i = 1, \ldots, d,$$

$$v(h_i) - v(h_{i+1}) \geq 3 \quad \text{ for } i = 1, \ldots, d - 1,$$
and
\[ v(h_i) - v(h_{i+1}) \geq 4 \quad \text{if } h_{i+1} \in B. \]

Let
\[ B' = \{ vm + s + s' : v \geq 1 \text{ and } s, s' \in S, s \neq s' \}. \]

For each partition \( \pi' = (h_1, \ldots, h_d) \in \mathcal{H}(n) \), let \( e'(\pi') \) denote the number of \( i \in \{1, \ldots, d\} \) such that \( h_i \in B' \). Then
\[ p_A(n) = \sum_{\pi' \in \mathcal{H}(n)} 2^{e'(\pi')}. \quad (3) \]

**Proof.** Let \( \pi = (a_1, \ldots, a_k)_N \) be the standard \( N \)-form of a partition in \( A(n) \). Then \( a_1, \ldots, a_k \) are elements of the set \( A \) that satisfy conditions (1) and (2). Let \( h_N(\pi) \) be the hook number partition determined by the \( N \)-graph of \( \pi \). Then \( h_N(\pi) = (h_1, \ldots, h_d) \), where \( d = d_N(\pi) \) is the number of integers on the side of the Durfee square of the \( N \)-graph of \( \pi \). We shall show that \( h_N(\pi) \) is a partition in \( \mathcal{H}(n) \).

Each hook in the \( N \)-graph of \( \pi \) consists of a horizontal row of numbers and a vertical column of numbers; the corner of the hook lies on the diagonal of the Durfee square. The row consists of a sequence of \( m \)'s, and ends with an element of \( S \). The column consists of a sequence of \( m \)'s, and ends either with an \( m \) or with an element of \( S \). In the first case the hook number is an element of \( A \); in the second case the hook must contain an \( m \) on the diagonal, and the hook number is an element of \( B \). Therefore, each hook number in the partition \( h_N(\pi) \) belongs to the set \( H = A \cup B \).

For \( i = 1, \ldots, d \), we let \( x_i \) denote the number of integers on the row of the \( i \)-th hook, and \( y_i \) denote the number of integers in the column below the corner of the \( i \)-th hook. Let \( 1 \leq i \leq d - 1 \). Since each row ends in an element of \( S \), it follows from (2) that
\[ x_{i+1} \leq x_i - 2, \]

and so the row in hook \( i \) contains at least two more elements equal to \( m \) than the row in hook \( i + 1 \). Similarly,
\[ y_{i+1} \leq y_i - 1, \]

and the column in hook \( i \) contains at least one more element equal to \( m \) than the column in hook \( i + 1 \). Therefore, \( v(h_i) - v(h_{i+1}) \geq 3 \). If \( h_{i+1} \in B \), then the column of hook \( i + 1 \) ends in an element of \( S \), hook \( i \) has an \( m \) to the left of this number, and \( v(h_i) - v(h_{i+1}) \geq 4 \). This proves that the map \( \pi \mapsto h_N(\pi) \) sends a partition in \( A(n) \) to a partition in \( \mathcal{H}(n) \).

Let \( \pi' = (h_1, \ldots, h_d) \) be a partition in \( \mathcal{H}(n) \) and let \( e'(\pi') \) denote the number of integers \( i \in \{1, \ldots, d\} \) such that \( h_i \in B' \). We shall prove that there exist exactly \( 2^{e'(\pi')} \) partitions \( \pi \in A(n) \) such that \( h_N(\pi) = \pi' \), and we shall explicitly construct these partitions.
The partition $\pi' = (h_1, \ldots, h_d) \in \mathcal{H}(n)$ immediately determines the shape of the $N$–graph of any partition $\pi \in \mathcal{A}(n)$ such that $h_N(\pi) = \pi'$. First, the Durfee square of $D_N(\pi)$ must satisfy $d_N(\pi) = d$. Second, we let $e$ denote the number of hook numbers $h_i$ that belong to $B$. Each of these hook numbers is of the form $vm + s + s'$, where $s, s' \in S$, and the corresponding hook in the $N$–graph of $\pi$ must contain two elements of $S$. Each of the remaining $d - e$ hook numbers is of the form $vm + s$, and the corresponding hook in the $N$–graph of $\pi$ contains only one element of $S$. Therefore, the $N$–graph of $\pi$ contains

$$2e + (d - e) = d + e = k$$

elements of $S$. Since each row of the $N$–graph of a partition contains exactly one element of $S$, it follows that the partition $\pi$ must contain exactly $k$ parts. Thus, if $\pi' = h_N(\pi)$, then $\pi'$ determines the number of parts in $\pi$.

Corresponding to the $e$ hook numbers $h_i \in B$ are integers $1 \leq j_1 < \cdots < j_e \leq d$ such that $h_{j_i} \in B$ for $i = 1, \ldots, e$. Then row $k$ in the standard $N$–graph of $\pi$ consists of $j_i - 1$ entries equal to $m$ followed by an element of $S$. Similarly, row $k - 1$ in the standard $N$–graph of $\pi$ consists of $j_2 - 1$ entries equal to $m$ followed by an element of $S$. In general, for $i = 0, 1, \ldots, e - 1$, row $k - i$ in the standard $N$–graph of $\pi$ consists of $j_{i+1} - 1$ entries equal to $m$ followed by an element of $S$. This determines the shape of the bottom $e$ rows of the $N$–graph. Then the hook numbers $h_1, \ldots, h_d$ determine the shape of the top $d$ rows of the $N$–graph. The only ambiguity concerns the elements of $S$ that are at the ends of the rows. If $h_i \equiv s \pmod{m}$ for some $s \in S$, then the integer at the right end of row $i$ is $s$. If $h_i \equiv 2s \pmod{m}$ for some $s \in S$, then the integer at the right end of row $i$ is $s$ and the integer at the bottom of column $i$ is $s$. If $h_i \in B'$ and $h_i \equiv s + s' \pmod{m}$ for $s, s' \in S$ with $s \neq s'$, then either the integer at the right end of row $i$ is $s$ and the integer at the bottom of column $i$ is $s'$, or the integer at the right end of row $i$ is $s'$ and the integer at the bottom of column $i$ is $s$. The set $S$ is a Sidon set modulo $m$, and so these are the only ways to put elements of $S$ at the ends of the $i$–th hook of the $N$–graph of $\pi$ to obtain the hook number $h_i$. Since there are $e'(\pi')$ hook numbers $h_i$ that belong to $B'$, it follows that there are exactly $2^{e'(\pi')}$ partitions $\pi \in \mathcal{A}(n)$ such that $h(\pi) = \pi'$.

This completes the proof.

**Theorem 2** Let $m \geq 2$ and let $s$ be a positive integer not divisible by $m$. Let

$$A = \{um + s : u \geq 0\},$$

$$B = \{vm + 2s : v \geq 1\},$$

and

$$H = A \cup B.$$

Let $q_A(n)$ denote the number of partitions of $n$ as a sum of distinct elements of $A$. Let $q_H(n)$ denote the number of partitions of $n$ in the form

$$n = h_1 + \cdots + h_d,$$
where
\[ h_i = v(h_i)m + t(h_i) \in H \quad \text{for } i = 1, \ldots, d, \]
\[ t(h_i) \in \{s, 2s\}, \]
\[ v(h_i) - v(h_{i+1}) \geq 3 \quad \text{for } i = 1, \ldots, d - 1, \]
and
\[ v(h_i) - v(h_{i+1}) \geq 4 \quad \text{if } h_{i+1} \in B. \]
Then
\[ q_A(n) = p_H(n). \]

**Proof.** This follows immediately from Theorem 1, applied to the sum–free Sidon set \( S = \{s\} \) modulo \( m \).

In the special case \( m = 2 \) and \( s = 1 \), the set \( A \) consists of all odd positive numbers, and we obtain the following result of Alladi [1]: The number of partitions \( n \) into distinct odd parts is equal to the number of partitions of the form \( n = h_1 + \cdots + h_d \), where \( h_d \neq 2 \), \( h_i - h_{i+1} \geq 6 \) for \( i = 1, \ldots, d - 1 \), and \( h_i - h_{i+1} \geq 7 \) if \( h_{i+1} \) is even.

### 3 Partition identities for sum–free sets

In the proof of Theorem 1, the assumption that the sum–free set \( S \) was a Sidon set modulo \( m \) implied that the cardinality of the “inverse image” of a hook number \( h_i \) was at most two. This produced the simple form of the partition identity (3). We can also derive partition identities for sets \( A \) that are finite unions of arithmetic progressions constructed from certain sets \( S \) that are sum–free modulo \( m \), but not necessarily Sidon sets modulo \( m \). For example, we can consider sets \( S \) that are sum–free modulo \( m \) and have the property that if \( s_{i_1}, s_{i_2}, s_{i_3}, s_{i_4} \in S \) and \( s_{i_1} + s_{i_2} \equiv s_{i_3} + s_{i_4} \pmod{m} \), then \( s_{i_1} + s_{i_2} = s_{i_3} + s_{i_4} \).

The set of all odd numbers \( s \) such that \( 1 \leq s \leq m/2 \) has this property.

**Theorem 3** Let \( m \geq 2 \) and let \( S \) be a set of positive integers that is sum–free modulo \( m \) and has the property that if \( s_{i_1}, s_{i_2}, s_{i_3}, s_{i_4} \in S \) and \( s_{i_1} + s_{i_2} \equiv s_{i_3} + s_{i_4} \pmod{m} \), then \( s_{i_1} + s_{i_2} = s_{i_3} + s_{i_4} \). Let
\[ A = \{um + s : u \geq 0 \text{ and } s \in S\}. \]
Let \( A(n) \) be the set of partitions of \( n \) in the form
\[ n = a_1 + \cdots + a_k, \]
where
\[ a_i = u(a_i)m + s(a_i) \in A \]
and
\[ u(a_1) > \cdots > u(a_k) \geq 0. \]
Let \( p_A(n) \) denote the number of partitions in the set \( A(n) \).
Let
\[ B = \{ um + s + s' : u \geq 1 \text{ and } s, s' \in S \} \]
and
\[ H = A \cup B. \]

Each element \( h \in H \) can be written uniquely in the form
\[ h = v(h)m + t(h), \]
where \( v(h) \geq 0 \) and \( t(h) \in S \cup 2S \). Let \( \mathcal{H}(n) \) denote the set of partitions of \( n \) of the form
\[ \pi' = (h_1, \ldots, h_d), \]
where
\[ h_i \in H \quad \text{for } i = 1, \ldots, d, \]
\[ v(h_i) - v(h_{i+1}) \geq 3 \quad \text{for } i = 1, \ldots, d - 1, \]
and
\[ v(h_i) - v(h_{i+1}) \geq 4 \quad \text{if } h_{i+1} \in B. \]

For \( h \in H \), let \( r(h) \) denote the number of representations of \( h \) as a sum of two elements of \( S \), that is, \( r(h) \) is the number of ordered pairs \((s, s')\) such that \( s + s' = h \) and \( s, s' \in S \). Then
\[ p_A(n) = \sum_{\pi' \in \mathcal{H}(n)} \prod_{i=1}^{d} r(h_i). \]

**Proof.** The proof is the same as the proof of Theorem \[.\]

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