Discrete Ramanujan-Fourier Transform of Even Functions (mod $r$)

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Abstract. An arithmetical function $f$ is said to be even (mod $r$) if $f(n) = f((n, r))$ for all $n \in \mathbb{Z}^+$, where $(n, r)$ is the greatest common divisor of $n$ and $r$. We adopt a linear algebraic approach to show that the Discrete Fourier Transform of an even function (mod $r$) can be written in terms of Ramanujan’s sum and may thus be referred to as the Discrete Ramanujan-Fourier Transform.

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1 Introduction

By an arithmetical function we mean a complex-valued function defined on the set of positive integers. For a positive integer $r$, an arithmetical function $f$ is said to be periodic (mod $r$) if $f(n + r) = f(n)$ for all $n \in \mathbb{Z}^+$. Every periodic function $f$ (mod $r$) can be written uniquely as

$$f(n) = r^{-1} \sum_{k=1}^{r} F_f(k) \epsilon_k(n),$$  \hspace{1cm} (1)
where
\[ F_f(k) = \sum_{n=1}^{r} f(n)\epsilon_k(-n) \quad (2) \]
and \( \epsilon_k \) denotes the periodic function \( \pmod{r} \) defined as
\[ \epsilon_k(n) = \exp(2\pi i kn/r). \]

The function \( F_f \) in (2) is referred to as the Discrete Fourier Transform (DFT) of \( f \), and (1) is the Inverse Discrete Fourier Transform (IDFT).

An arithmetical function \( f \) is said to be even \( \pmod{r} \) if \[ f(n) = f((n, r)) \]
for all \( n \in \mathbb{Z}^+ \), where \( (n, r) \) is the greatest common divisor of \( n \) and \( r \).

It is easy to see that every even function \( \pmod{r} \) is periodic \( \pmod{r} \). Ramanujan’s sum \( C(n, r) \) is defined as
\[ C(n, r) = \sum_{k \pmod{r}} \exp(2\pi i kn/r) \]
and is an example of an even function \( \pmod{r} \).

In this paper we show that the DFT (2) and IDFT (1) of an even function \( f \pmod{r} \) can be written in a concise form using Ramanujan’s sum \( C(n, r) \), see Section 3. We also review a proof of (1) and (2) for periodic functions \( \pmod{r} \), see Section 2, and review (1) and (2) for the Cauchy product of periodic functions \( \pmod{r} \), see Section 4. The Cauchy product of periodic functions \( f \) and \( g \pmod{r} \) is defined as
\[ (f \circ g)(n) = \sum_{a+b\equiv n\pmod{r}} f(a)g(b). \]

The results of this paper may be considered to be known. They have not been presented in exactly this form and we hope that this paper will provide a clear approach to the elementary theory of even functions \( \pmod{r} \).

The concept of an even function \( \pmod{r} \) originates from Cohen [2] and was further studied by Cohen in subsequent papers [3,4,5]. General accounts of even functions \( \pmod{r} \) can be found in the books by McCarthy [8] and Sivaramakrishnan [10]. For recent papers on even functions \( \pmod{r} \) we refer to [9,11]. Material on periodic functions \( \pmod{r} \) can be found in the book by Apostol [1].
2 Proof of (1) and (2)

Let \( P_r \) denote the set of all periodic arithmetical functions \((\mod r)\). It is clear that \( P_r \) is a complex vector space under the usual addition and scalar multiplication. In fact, \( P_r \) is isomorphic to \( \mathbb{C}^r \). Further, \( P_r \) is a complex inner product space under the Euclidean inner product given as

\[
\langle f, g \rangle = \sum_{n=1}^{r} f(n)g(n) = (f \mathcal{F} \circ \zeta)(r), \quad (3)
\]

where \( \zeta \) is the constant function 1. The set \( \{r^{-1/2}\varepsilon_k : k = 1, 2, \ldots, r\} \) is an orthonormal basis of \( P_r \). Thus, every \( f \in P_r \) can be written uniquely as

\[
f(n) = \sum_{k=1}^{r} \langle f, r^{-1/2}\varepsilon_k \rangle r^{-1/2}\varepsilon_k(n),
\]

where

\[
\langle f, r^{-1/2}\varepsilon_k \rangle = \sum_{n=1}^{r} f(n)r^{-1/2}\varepsilon_k(n) = r^{-1/2} \sum_{n=1}^{r} f(n)\varepsilon_k(-n).
\]

This proves (1) and (2).

3 DFT and IDFT for even functions \((\mod r)\)

Let \( E_r \) denote the set of all even functions \((\mod r)\). The set \( E_r \) forms a complex vector space under the usual addition and scalar multiplication. In fact, \( E_r \) is a subspace of \( P_r \). Thus (1) and (2) hold for \( f \in E_r \). We can also present (1) and (2) for \( f \in E_r \) in terms of Ramanujan’s sum as is shown below.

Note that Ramanujan’s sum \( C(n, r) \) is an integer for all \( n \) and can be evaluated by addition and subtraction of integers. In fact, \( C(n, r) \) can be written as \( C(n, r) = \sum_{d|\langle n, r \rangle} d\mu(r/d) \), where \( \mu \) is the Möbius function.

An arithmetical function \( f \in E_r \) is completely determined by its values \( f(d) \) with \( d|r \). Thus \( E_r \) is isomorphic to \( \mathbb{C}^{\tau(r)} \), where \( \tau(r) \) is the number of divisors of \( r \). The inner product (3) in \( P_r \) can be written in \( E_r \) in terms of the Dirichlet convolution. In fact, we have

\[
\sum_{k=1}^{r} 1 = \sum_{j=1}^{r/d} 1 = \phi(r/d), \quad (4)
\]
where \( \phi \) is Euler’s totient function, and thus (3) can be written for \( f, g \in E_r \) as

\[
\langle f, g \rangle = \sum_{d \mid r} f(d)g(d)\phi(r/d) = (fg \ast \phi)(r),
\]

where * is the Dirichlet convolution.

**Theorem 3.1.** The set

\[
\{(r\phi(d))^{-\frac{1}{2}}C(\cdot, d) : d \mid r\}
\]

is an orthonormal basis of the inner product space \( E_r \).

**Proof** As the dimension of the inner product space \( E_r \) is \( \tau(r) \) and the number of elements in the set (5) is \( \tau(r) \), it suffices to show the set (5) is an orthonormal subset of \( E_r \). This follows easily from the relation

\[
\sum_{e \mid r} C(r/e, d_1)C(r/e, d_2)\phi(e) = \begin{cases} r\phi(d_1) & \text{if } d_1 = d_2, \\ 0 & \text{otherwise}, \end{cases}
\]

where \( d_1 \mid r \) ja \( d_2 \mid r \) (see [8, p. 79]). \( \square \)

We now present (1) and (2) for \( f \in E_r \).

**Theorem 3.2.** Every \( f \in E_r \) can be written uniquely as

\[
f(n) = r^{-1} \sum_{d \mid r} R_f(d)C(n, d),
\]

where

\[
R_f(d) = \phi(d)^{-1} \sum_{n=1}^{r} f(n)C(n, d).
\]

**Proof** On the basis of Theorem 3.1

\[
f(n) = \sum_{d \mid r} \langle f, (r\phi(d))^{-\frac{1}{2}}C(\cdot, d) \rangle (r\phi(d))^{-\frac{1}{2}}C(n, d).
\]

Applying (3) to (8) we obtain (6) and (7). \( \square \)

The function \( R_f \) in (7) may be referred to as the Discrete Ramanujan-Fourier Transform of \( f \), and (5) may be referred to as the Inverse Discrete Ramanujan-Fourier Transform. Cf. [8].

Another expression of (7) can be obtained easily. Namely, applying (4) to (8) and then applying

\[
\phi(e)C(r/e, d) = \phi(d)C(r/d, e)
\]
(see [8, p. 93]) we obtain

\[ R_f(d) = \sum_{e|d} f(r/e)C(r/d, e). \tag{9} \]

Note that (6) can also be derived from (1). In fact, if \( f \in E_r \), then (2) can be written as

\[
F_f(k) = \sum_{n=1}^{r} f(n) \exp(-2\pi i kn/r) \\
= \sum_{e|r} \sum_{n=1}^{r} f(e) \exp(-2\pi i kn/r) \\
= \sum_{e|r} f(e) \sum_{m=1}^{r/e} \exp\left(-2\pi ikm/(r/e)\right) \\
= \sum_{e|r} f(e)C(k, r/e).
\]

A similar argument shows (6) with \( R_f(d) = F_f(r/d) \). We omit the details.

4 The Cauchy product

It is well known that if \( h \) is the Cauchy product of \( f \in P_r \) and \( g \in P_r \), then \( F_h = F_fF_g \). This follows from the property

\[
\sum_{a+b\equiv n \ (\text{mod} \ r)} \epsilon_k(a)\epsilon_j(b) = \begin{cases} r\epsilon_k(n) & \text{if } k \equiv j \ (\text{mod} \ r), \\ 0 & \text{otherwise}. \end{cases}
\]

Analogously, if \( h \) is the Cauchy product of \( f \in E_r \) and \( g \in E_r \), then \( R_h = R_fR_g \). This follows from the property

\[
\sum_{a+b\equiv n \ (\text{mod} \ r)} C(a, d_1)C(b, d_2) = \begin{cases} rC(a, d_1) & \text{if } d_1 = d_2, \\ 0 & \text{otherwise}, \end{cases}
\]

where \( d_1 \mid r \)ja \( d_2 \mid r \) (see [10, p. 333]).

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