Generalized $3x + 1$ Mappings: convergence and divergence

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Abstract
Discussion about the convergence and divergence of trajectories generated by certain functions derived from generalized $3x + 1$ mappings.

1 Introduction
In previous papers [1, 2], we analyzed the trajectories generated by the iterative application of several functions derived from generalized $3x + 1$ mappings. We have developed an algorithm that allows us to determine the necessary conditions for the existence or not of loops (cycles). From the periodicity property associated with the different trajectories for a given length, it has been possible to demonstrate that the number of cycles is finite. This led us to the notion of convergence or not of the trajectories towards these cycles which can be either closed or open. The function that gives rise to the original Collatz problem produces nine closed cycles and, since the number of cycles is limited, all the integers not belonging to these cycles are in infinite trajectories. The function that gives rise to the $3x+1$ problem produces four opened cycles with the negative integers and the zero, and only one appears for positive integers. All other natural numbers seem to converge towards this cycle. The function generating the $5x + 1$ problem seems to lead to trajectories convergent and divergent.

In this paper we will analyze the convergence and the divergence of trajectories associated with the $3x + 1$ and $5x + 1$ problems. The approach we use here is independent of the results previously obtained if we accept the conjecture which states that the number of cycles is finite ([4],[5]). We will use an intrinsic property, called periodicity, resulting from the iterative application of the functions. A priori, the behavior of trajectories generated for these functions seems chaotic. In fact, if we make appropriate groupings of trajectories we quickly observe a regularity in their distribution. In this context, we will be able to provide answers to the questions of convergence or not of these trajectories.

In this new version of this paper, we have added a section in the problem $3x + 1$ called “Distribution function $F(k)$”. We start with the approach undertaken by Riho Terras [3] and taken up by several authors including Lagarias [4] which allows to develop a density function $F(k)$. Terras proves that this function is well defined and has very interesting properties. We use his reasoning until the remarkable result which appears with the theorem of periodicity. Thereafter we use a completely different path which includes the properties generated by only two theorems developed in our paper. Our reasoning is much simpler and leads to results which coincide with those expected. In our opinion, this last exercise reinforces the power of the use of diophantine equations.
for the solution of problems like the two treated in this paper.

2 Mathematical stools

Mappings can be defined on integers represented by functions such that each element of the set $\mathbb{Z}$ is connected to a single element of this set. The iterative application of these functions produces a sequence of integers called trajectories. In this paper, we will use indifferently the term trajectory or simply sequence to indicate a sequence of integers generated by these functions. We can easily construct the equations connecting any two integers of the sequence. The result can always be expressed in the general form $c = ax + by$, where $x$ represents any integer and $y$ is the integer resulting from the iterative application of the function. The parameters $a$ and $b$ depend on the function itself. So, we have diophantine equations of first degree at two unknowns. From a well-known result of this theory, we have the theorem

Theorem 2.1 Let the diophantine equation $c = ax + by$ of first degree at two unknowns. If the coefficients $a$ and $b$ of $x$ and $y$ are prime to one another (if they have no divisor other than 1 and $-1$ in common), this equation admits an infinity of solutions to integer values. If $(x_0, y_0)$ is a specific solution, the general solution will be $(x = x_0 + bq, y = y_0 - aq)$, where $q$ is any integer, positive, negative or zero.

Proof

References: on the web and [6]

The distribution of integers resulting from the different groupings of trajectories will follow different progressions, and several of them will be of geometric type. Write a general geometric series as

$$
\sum_{k=1}^{n} ar^{k-1} = ar^0 + ar^1 + ar^2 + ar^3 + \cdots + ar^{n-1}, \tag{1}
$$

where $ar^{n-1}$ is the $n$th term of the series.

The sum of first $n$ terms is given by

$$
S_n = \frac{a(1 - r^n)}{1 - r}. \tag{2}
$$

The terms of the geometric series will be represented by multinomials. We have the binomial formula

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \tag{3}$$

where

$$BC_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}, \tag{4}$$

are called binomial coefficients $BC_{n,k}$. 

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The multinomial formula (Wikipedia) is

\[(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{t=1}^{m} x_t^{k_t}, \tag{5}\]

where

\[MC = \binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!k_2!\cdots k_m!}, \tag{6}\]

is a multinomial coefficient. The sum is taken over all combinations of nonnegative integers indices \(k_1\) through \(k_m\) such that the sum of all \(k_i\) is \(n\). That is, for each term in the expansion, the exponents of the \(x_i\) must add up to \(n\).

For example, the third power of the trinomial \(a + b + c\) is given by

\[(a + b + c)^3 = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3b^2a + 3b^2c + 3c^2a + 3c^2b + 6abc,\]

where

\[a^2b^0c^1 \text{ has the coefficient } \binom{3}{2, 0, 1} = \frac{3!}{2!\cdot 0!\cdot 1!} = 3,\]

and,

\[a^1b^1c^1 \text{ has the coefficient } \binom{3}{1, 1, 1} = \frac{3!}{1!\cdot 1!\cdot 1!} = 6.\]

### 3 Problem \(3x + 1\)

In first, we present three functions which encode the \(3x + 1\) problem.

Let the **Collatz function** \(C(n)\) be defined as follow

\[C(n) = \begin{cases} 
3n + 1, & \text{if } n \equiv 1 \pmod{2} \\
\frac{n}{2}, & \text{if } n \equiv 0 \pmod{2}.
\end{cases} \tag{7}\]

Even though this function, as well as the next two, is valid for all integers \(n\), positive, negative or zero, we will use the set of natural numbers in most of the examples that follow.

In a 2012 paper \[7\], Delahaye gives a very good introduction to the \(3x + 1\) problem. He produced a figure, that he called a tree, giving the directional tracking of several positive integers leading to the number 1. We see that each integer has 1 antecedent or 2, never more. Some odd integers such as \(3, 9, 15, \ldots\), are not preceded by any odd integers. Several representations of trees are found on the web, in particular that giving the trajectories less than 20 before reaching the number 1 \[8\]. In fact, the root of these trees is 1, and they are built from inverse algorithms to those generated by the Collatz function. If the conjecture that states that all natural numbers end on the cycle \(\langle 1, 4, 2 \rangle\) is true, these trees must cover all natural numbers.

Kontorovich and Lagarias \[9\] in a paper produced in 2009 work with two other functions with iterations faster than the Collatz function.

The first is the \(3x + 1\) function \(T(n)\) (or \(3x + 1\) map)
\[ T(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod{2} \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod{2}. \end{cases} \quad (8) \]

This function results from the fact that the \(3x+1\) operation applied to any odd integer always give an even integer.

The second function, the \textit{accelerated} \(3x+1\) \textit{function} \(U(n)\), is defined on the domain of all odd integers, and removes all powers of 2 at each step. It is given by

\[ U(n) = \frac{3n+1}{2^{\text{ord}_2(3n+1)}} \quad (9) \]

in which \(\text{ord}_2(n)\) counts the number of powers of 2 dividing \(n\). The function \(U(n)\) was studied by Crandall in 1978 [10].

3.1 Groupings (in triplets) of trajectories generated by the function \(U(n)\)

Let us represent the \textit{accelerated} \(3x+1\) \textit{function} \(U(n)\) by the following 3 operations

\[ U(n) = \begin{cases} \frac{3n+1}{4} & \text{if } n \equiv 1 \pmod{8} \\ \frac{3n+1}{2} & \text{if } n \equiv 3 \pmod{4} \\ \frac{3n+1}{8} & \text{if } n \equiv 5 \pmod{8} \end{cases} \quad (10) \]

The first two operations result in odd integers and the last operation ends with an alternation of even and odd integers \(2 \pmod{3}\). The table[4] we give the three groupings resulting of these operations including the intermediate (abreviate 'int') operation \((3x+1)/2\). In each grouping we have a sequence of three integers ('triplets'). The intermediate integer is repeated in the second grouping. The reason will appear on its own.

Even if the results contained in the table are easily deduced during its construction, it is possible to find them by writing the diophantine equations and, using the theorem 2.1.

For example, the first operation leads to the equation

\[ y = \frac{3x + 1}{4} \quad \text{or} \quad 1 = 4y - 3x. \]

The couple of values \((x_0 = 1; \ y_0 = 1)\) is a solution. According to the theorem 2.1 the general solution will be \(x = 1 + 4q_1; \ y = 1 + 3q_1\), where \(q_1\) is any integer, positive, negative or zero. If this first operation result in odd integers, we can write \(y = 2q_2 - 1\). Then,

\[ y = 1 + 3q_1 = 2q_2 - 1 \quad \text{and} \quad q_1 = \frac{2q_2 - 2}{3}. \]

The solutions to the integer values are \(q_2 = 1, 4, 7, 10, \cdots\) and \(q_1 = 0, 2, 4, 6, \cdots\).

We will have, as expected,

\[ x = 1 + 4q_1 = 1, 9, 17, 25, \cdots \quad \text{or} \quad x \equiv 1 \pmod{8}, \]

and
The same procedure is used to find the solutions of the other operations. If we analyze the decomposition of trajectories in triplets, it is possible to perceive a certain regularity. We will not further develop the search for the regularity by this way. We will instead define another function, so \( f(n) \), that apply to all integers. This new function produces trajectories whose triplet decomposition will be such that each of the integers of these triplets are in correspondence one-to-one with each of the elements of one and only one triplet produced by the function \( U(n) \). We can then build a tree that, \textit{a priori}, will include all the integers and bring out the regularity.

### 3.2 New function \( f(n) \) and one-to-one correspondence with the function \( U(n) \)

We define the function \( f(n) \)

\[
 f(n) = \begin{cases} 
 \frac{3n+1}{4}, & \text{if } n \equiv 1 \pmod{4} \\
 \frac{3n}{2}, & \text{if } n \equiv 2 \pmod{2} \\
 n+1, & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

This function does not come from generalized \( 3x+1 \) mappings. The table gives three groupings resulting of these operations. In fact, this table brings together the trajectories of all integers and the integer resulting from a single application of the function \( f(n) \) (duos). We form triplets by repeating the first integer. We can easily verify that all the integers \( n \) of triplets in the table \( f(n) \) (function \( U(n) \)) are calculated using two simple operations on all the integers \( n_{(\text{new})} \), where \( n_{(\text{new})} \) is any integer beginning the triplets in the table \( f(n) \) (function \( f(n) \)),

\[
 n = 2n_{(\text{new})} - 1 \tag{11}
\]

and

\[
 n = 3n_{(\text{new})} - 1. \tag{12}
\]

All intermediate integers \( 2 \pmod{3} \) of triplets of \( U(n) \) and the last of the third grouping come from the operation \( (3n_{(\text{new})} - 1) \). All other integers (odd integers) come from the operation \( (2n_{(\text{new})} - 1) \). Then, the correspondence between the results of the application of the function \( f(n) \) on all integers is one-to-one with the application of the function \( U(n) \) on all odd integers.

As we will see quite rapidly, this new function will show the regularity in the distribution of sequences of integers. It will allow us to build a tree in which all the branches fit into each other and, possibly containing all integers.

### 3.3 Tree generated by the function \( f(n) \) and distribution of integers

Let us define the branches as the sequence of the integers that are connected to each other by the operations \( (3n+1)/4 \) or \( 3n/2 \) starting with \( 2 \pmod{3} \) and ending with an integer resulting from the operation \( (n+1)/4 \). The branches are interrelated, thus forming a tree (table \( f(n) \) whose the
trunk is represented by the sequence $2 \rightarrow 3 \rightarrow 1$. If there is only one cycle $(1 \rightarrow 1)$ for the natural numbers, then all the integers of a branch are different two by two, and also are two by two different from all the integers of any other branch, except of course, the end-of-branch integer that connect to another branch.

We have retained the first and the last integer of the branch of length 19 starting with 14 and ending with 56, at the top of the tree. For the branch starting with 56 and whose length is 13, we selected 4 values, namely the first (56) and the last three.

The search for the convergence of all integers towards 1 is to determine if all the integers are in branches (distribution) interlocking into each other and if, sooner or later, they converge to smaller integers.

We will prove in a first time that all integers are found in sequences with an end of all possible lengths and their distribution is characterized by a geometric progression.

Using the operation $(3n_{new} - 1)$ on all integers in branch ends of the tree in the table 6 and the operation $(2n_{new} - 1)$ on all the other integers of this tree, we build the tree represented in the table 7 generated by the function $U(n)$. Nevertheless, it is easier to analyze the function $U(n)$ which applies to all integers and which naturally reveals the regularity in the distributions of the branches.

### 3.3.1 Distribution of integers in sequences with an end

The solution of diophantine equations used below is easily verified from the groupings of table 5. The sequences that we will analyze end with an integer resulting the operation $(n + 1)/4$, but the beginning of the sequences is any integer, not necessarily a branch beginner integer $2 \pmod{3}$.

Let’s first look the sequences of two integers whose second is obtained from the first by the operation $(n + 1)/4$ (end of branch). The length of the sequences is $L = 2$. The diophantine equation generating these sequences is $y = (x + 1)/4$, or $1 = 4y - x$ (general form is $c = by + ax$), of which a particular solution is $(x_0 = 3, y_0 = 1)$. By the theorem 2.1 the general solution will be $(x = 3 + 4q, y = 1 + 1q)$. $b = 4$ and $-a = 1$ are the increments which are added to $x_0$ and $y_0$ to get the general solution. In notation with modulo we will write $x \equiv 3 \pmod{4}$ and $y \equiv 1 \pmod{1}$. This result tells us that at every 4 consecutive integers there is one that starts a sequence of a first type, and this is repeated periodically to infinity. The distribution is $1/4$, so $1/4$. Each integer $x$ in this group of length 2 goes to a smaller integer. In resume, for the sequences of length $L = 2$ we build a single diophantine equation and the distribution of the first integers $x$ of these sequences is $1/d$ with $d = 4$ the denominator in the first expression of the diophantine equation.

Let the sequences of three integers (length $L = 3$). Two cases are possible, whether the operations $3n/2$ (first type) or $(3n + 1)/4$ (second type) is applied to an integer followed by the operation $(n + 1)/4$. The resolution of two diophantine equations leads to two solutions, $x \equiv 2 \pmod{8}$ ($y \equiv 1 \pmod{3}$) and $x \equiv 9 \pmod{16}$ ($y \equiv 2 \pmod{3}$). The distributions are $1/8$ and $1/16$, meaning respectively that at each 8 consecutive integers $x$ there is one that starts a sequence of a first type, and at each 16 consecutive integers there is one that starts a sequence of a second type and this is repeated periodically to infinity.

We can do this for $L = 4, 5, 6, \cdots$.

The results below are valid for $L \geq 2$.

The total number of diophantine equations $#(DE)_L$ is

$$#(DE)_L = 2^{L-2}. \quad (13)$$
The different increments (or modulo) for the integers $x$ starting these sequences are

$$\text{Increment} = d_1 \ast d_2 \ast d_3 \ast \cdots \ast d_{L-1},$$  \hspace{1cm} (14)

where $d_1, d_2, d_3, \ldots, d_{L-2} = 2, 4$ and $d_{L-1} = 4$.

The increment (or modulo) for the last integers $y$ is $3^{L-2}$.

The different distributions of first integers $x$ are calculated from the following binomial

$$D_L = \frac{1}{4} \left( \frac{1}{2} + \frac{1}{4} \right)^{L-2},$$  \hspace{1cm} (15)

which is the $n$th term ($n = L - 1$) of a geometric progression. From equation [1] which is the general form for this series, we put $a = \frac{1}{4}$ and $r = \frac{3}{4}$.

The binomial coefficients $BC_{n,k}$ of this binomial give the number of different diophantine equations for a given increment.

The sum of $n$ first terms is given by the equation

$$S_L = \frac{1}{4} \left( \left( \frac{1}{2} + \frac{3}{4} \right)^{L-1} \right) = 1 - \left( \frac{3}{4} \right)^{L-1}.$$  \hspace{1cm} (16)

When $L$ goes to infinity the sum tends towards one, meaning that all integers $x$, without exception, start sequences with end.

In table 8 and the table 10 we give the sequences and distributions of first integers $x$ generated by the function $\Omega(n)$ for the lengths $L = 2$ until $L = 6$ (and until $L = 7$ in the table 10).

According to the equation [15] a large part of natural integers is included in the first sequences. The sequences with $L = 2$ until $L = 20$ include almost 99.6\% of all natural integers.

The next step in our approach is to determine the part of natural integers that go to smaller integers.

The first sequences where the first integers $x$ are smaller than the final integers $y$ ($x < y$) are those starting with $x \equiv 48 \pmod{64}$ for the lengths $L = 6$. In fact, these sequences are the only ones of all those for the lengths $L = 2$ until $L = 6$ with this behavior. All other integers for sequences covering these lengths end with smaller integers ($x > y$). The distribution $1/64$ meaning that at each 64 consecutive integers there is one that starts a sequence which end with a larger integer.

For the sequences of length $L = 7$ and more we observe the same type of behavior. Some sequences with the smaller modulo will have the beginning integer smaller than that the integer of the end of the sequence. For all others we have the inverse behavior. It was quite predictable.

**Theorem 3.1** Let the sequences (of length $L$) of the integers that are connected to each other by the operations $(3n+1)/4$ or $3n/2$ and ending with an integer resulting from the operation $(n+1)/4$.

The diophantine equation connecting the first integer $x$ and the last integer $y$ of a sequence can be expressed in the general form $c = by - ax$ where the parameters $a$, $b$, and $c$, always positive, depend on the operations themselves and in which orders they are applied. If $b > a$, $x \geq y$ and, if $b < a$, $x < y$.

**Proof**

Let $k_1, k_2 = 0, 1, 2, \ldots$ and $k_3 = k_1 + k_2 = L - 2$, with $L \geq 2$.

Then, $a = 3^{k_3}$, $b = 4 \cdot 2^{k_1} \cdot 4^{k_2}$ and $c > 0$. 

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As the factors \( a \) and \( b \) of \( x \) and \( y \) are prime to one another, the diophantine equation admits a infinity of solutions to integer values. If \((x_0, y_0)\) is a specific solution, the general solution will be \((x = x_0 + bq, y = y_0 + aq)\), where \( q \) is any integer, positive, negative or zero.

Two cases are possible, \( b > a \) or \( b < a \).

**First case : \( b > a \)**

Suppose that a particular solution \((x_0, y_0)\) is such that \(x_0 < y_0\). We have the general solution

\[
y = y_0 + aq \quad \text{and} \quad x = x_0 + bq,
\]

where \( q \) is any integer, positive, negative or zero. As \( b > a \) and \( x_0 < y_0 \), beyond a certain value of \( q \), we will have \( x > y \). The equation

\[
c = by - ax,
\]

eventually lead to a negative \( c \) value. But, the parameter \( c \) must always be positive. Therefore \( x > y \) when \( b > a \).

For example, for \( L = 2 \), we have \( k_1 + k_2 = L - 2 = 0, k_1 = k_2 = 0 \) and \( k_3 = k_1 + k_2 = 0 \). Then, \( a = 3^{k_3} = 3^0 = 1 \) and \( b = 4 \cdot 2^{k_1} \cdot 4^{k_2} = 4 \). We write the diophantine equation

\[
y = \frac{x + 1}{4} \quad \text{or} \quad 1 = 4y - x.
\]

where \( b = 4, a = 1 \) and \( c = 1 \).

The couple of values \((x_0 = 3, y_0 = 1)\) is a particular solution. The general solution will be \((x = 3 + 4q; \; y = 1 + q)\).

**Second case : \( b < a \)**

Let the equation

\[
c = by - ax.
\]

As \( c \) is always positive and \( b < a \), \( x \) must necessarily always be smaller than \( y \) (\( x < y \)).

The sequences of length \( L = 6 \) beginning with \( x \equiv 48 \mod 64 \) treated in the following subsection are typical examples of this case. We have \( b = 64, a = 81 \) and \( c = 16 \). A particular solution is \((x_0 = 48, y_0 = 61)\), and the general solution is \((x = 48 + 64q, y = 61 + 81q)\). Therefore \( b < a \) and \( x < y \). The table [1] gives the values of \( b \) and \( a \) for \( L = 2 \) until \( L = 8 \) (function \( \mathcal{U}(n) \)).

In table [2] we give the first distributions of the natural numbers whose first of the sequence is smaller than the last (\( x < y \)). For sequences lengths \( L = 2 \) until \( L = 20 \) there is around 12\% of all integers (actually 99.58\%) behaving this way. On the other hand, around 88\% of all natural numbers have their first numbers larger than the last of the sequence (\( x > y \)). This first result is quite remarkable because it tells us that not only 88\% of all natural numbers go to smaller integers, but this is done in very specific slices (increments).

We have all the elements necessary to build the algorithm of a function \( \mathfrak{f} \) determining the fraction \( f \) (or the \%) of natural integers such as \( x < y \). For \( L = 2 \) to \( 20, 30, 40, 50, 60 \),

\[
f = 0.1198839, 0.1236245, 0.1238451, 0.1238577, 0.1238584.
\]
If we take consecutive integer slices of 5,000, for example, we can already see that the number of integers whose first of the sequence is smaller than the last is close to 12% (using an appropriate algorithm). The results (by slices) are indeed those anticipated by the function $\mathcal{F}$.

A good part of the natural integers seem to go towards smaller integers. We will reinforce this fact by analyzing a little more in detail the some 12% of integers with $x < y$.

### 3.3.2 Distribution of integers in sequences with an end at the second level and more

Let us now the sequences whose beginning integer $x$ is smaller than the end integer $y$, that is 12% of all integers. The first sequences with this behavior are those of length 6, so $x \equiv 48 \pmod{64}$. This sequences are carried out by applying 4 times the operation $3n/2$ and ending with the operation $(n+1)/4$. The diophantine equation is written as

$$\left(\left(\frac{3}{2}\right)^4 x + 1\right) \cdot \frac{1}{4} = y \quad \text{or} \quad 16 = -81x + 64y.$$  

Let $(x_0 = 48, y_0 = 61)$ be a particular solution. Then, $(x = 48 + 64q, y = 61 + 81q)$ is the general solution, where $q$ is any integer, positive, negative or zero. If we continue the iteration of the function $\mathcal{U}(n)$ on the last integer of the sequence until reaching a second end of branch, we will have for example,

$$(48 \to 72 \to 108 \to 162 \to 243 \to 61) \to (61 \to 46 \to 69 \to 52 \to 78 \to 88 \to 132 \to 198 \to 297 \to 223 \to 56) \quad \text{PP}$$

$$(112 \to 168 \to 252 \to 378 \to 567 \to 142) \to (142 \to 213 \to 160 \to 240 \to 360 \to 540 \to 810 \to 1215 \to 304) \quad \text{PP}$$

$$(176 \to 264 \to 396 \to 594 \to 891 \to 223 \to 56) \quad \text{PG}$$

$$(240 \to 360 \to 540 \to 810 \to 1215 \to 304) \to (304 \to 456 \to 684 \to 1026 \to 1539 \to 385) \quad \text{PP}$$

$$(176 \to 264 \to 396 \to 594 \to 891 \to 223 \to 56) \quad \text{PG}$$

$$(240 \to 360 \to 540 \to 810 \to 1215 \to 304) \to (304 \to 456 \to 684 \to 1026 \to 1539 \to 385) \quad \text{PP}$$

... where $\text{PP}$ ("Plus Petit") indicates that the beginning integer $x_1$ of the first sequence is smaller than the last integer $y_2$ of the second sequence, and $\text{PG}$ means larger ("Plus Grand").

The first sequence will be called the first level sequence and the second, the second level sequence. Then, $x_1 \equiv 48 \pmod{64}$ and $y_1 \equiv 61 \pmod{81}$. $y_1$ becomes then the beginning integer $x_2$ of the second sequence.

In table 14 we have the first second level sequences $x_2 \equiv 61 \pmod{81}$. We find a distribution identical to that obtained for the sequences of first level. All integers 61 (mod 81) start sequences with an end, of all possible lengths and their distribution is characterized by a geometric progression.

For example, let’s take sequences of length $L = 2$. The diophantine equation is

$$\left(\frac{x_2 + 1}{4}\right) = y_2,$$

where $x_2 = 61 + 81q$. Then,

$$62 = -81q + 4y_2.$$

A specific solution at integer values is $(q_0 = 2, y_{20} = 56)$ and the general solution is $(q = 2 + 4k, y_2 = 56 + 81k)$. We replace $q = 2 + 4k$ in $x_2 = 61 + 81q$. Then,

$$x_2 = 223 + 324k,$$
as expected.

We proceed in the same way for all possible trajectories starting with \( x_2 = 61 + 81q \). In writing the diophantine equations \( c = ax + by \), we will have the coefficient \( a = -81 \) and the coefficient \( b \) resulting from the products of 2 and 4. According to theorem 2.1, since \( a \) and \( b \) are prime between them, the equations always have an infinity of solutions to the integers values. All trajectories starting with \( x_2 \equiv 61 \pmod{81} \) are possible and their distribution will be such that nearly 88% of them will see the beginning integer \( x_2 \) larger than the last integer \( y_2 \).

An identical result would have been obtained for each of the 12% of sequences where \( x_1 < y_1 \). Then, around 88% of all these sequences have their first integers \( y_1 = x_2 \) larger than \( y_2 \). This does not mean that 88% of \( x_1 \) is larger than \( y_2 \). For example, the sequence starting with \( x_1 = 48 \) end with \( y_1 = 61 \) at the first level and \( y_2 = 56 \) at the second level. At the second level, \( x_2 = y_1 = 61 \) and \( y_2 = 56 \) (\( x_2 > y_2 \), but \( x_1 < y_2 \)). In fact, around 60% of 12% of \( x_1 \) is larger than \( y_2 \).

As the sequences of integers at the second level develop similarly to those of the first level, we can use the function \( F \) that we noted at the end of the previous section and call it recursively. For sequences up to lengths 30 at the first level (for \( L_1 = 2 \) to 30), we had

\[
f = 0.1236245. 
\]  
(17)

At the second level (for \( L_1, L_2 = 2 \) to 30) \( f \) becomes

\[
f = 0.05112079. 
\]  
(18)

With the third level (for \( L_1, L_2, L_3 = 2 \) to 30),

\[
f = 0.024040812. 
\]  
(19)

We have programmed the function \( F \) in VBA (Visual Basic for Applications) with a standard laptop (standard computer). With the level 4, the operations preformed by calling the function increase very rapidly and the integers appearing there become very large (over 29 digits). By using a more powerful computer, we could obtain the results of the function for levels higher than 3.

With the second level, we now have around 95% of all natural numbers that end up on a smaller integer and, 97.6% with the third level. This percentage increases with the fourth level and more. Not only 97.6%, or more, of all natural numbers go to smaller, but this is done in very specific slices.

In table 14 we give the results by slices (1 to 5 000, 10 000, 100 000, 1 000 000, 1 000 000), so the fraction of integers with \( x < y \) for the first 7 levels, beginning with the first natural number 1. We can verify that the results would have been similar if we had chosen slices starting with any integer other than 1. In the last column we find the first three values of \( f \) obtained from the programmed function \( F \) (equations 17, 18, 19). Once again, the results (by slices) are indeed those anticipated by the function \( F \).

### 3.4 Distribution of odd integers in sequences with an end (function \( U(n) \))

We use a method similar to the one previously used to find the distribution of odd integers in the trajectories generated by the function \( U(n) \).

In table 9 we give the sequences of first odd integers \( x \) generated by the function \( U(n) \) for the lengths \( L = 2 \) to \( L = 6 \). These results can be found by writing the different diophantine equations and solving them. Nevertheless, and because the two functions (\( U(n) \) and \( U(n) \)) are one-to-one
correspondence, we can simply use the integers of the trajectories constituting the branches given in the previous table [8] and apply the transformation $3n - 1$ to the end integers of the sequences and the transformation $2n - 1$ to all other integers.

The first trajectories for which the first odd integer $x$ is smaller than the last integer $y$ ($x < y$) is $x \equiv 15 \pmod{64}$ and $y \equiv 20 \pmod{81}$ for $L = 5$, corresponding to $x \equiv 8 \pmod{32}$ and $y \equiv 7 \pmod{27}$ for the function $U(n)$ where $x$ is larger than $y$ ($x > y$).

The complete distribution of all odd integers at the first level will be slightly different from the distribution of all integers with the function $U(n)$ in the table [12]. Nearly 20% (instead of 12%) of all odd integers $x$, without exception, start sequences with end, such as $x < y$. Then, 80% of all odd integers go to smaller integers at the first level.

At the second level, we have about 60% of 20% going to smaller integers (whose $x > y$). With the second level, we now have around 92% of all odd integers that end up on a smaller integer and this percentage increases with the third level and more. Not only 92%, or more, of all odd integers go to smaller, but this is done in very specific slices.

In table [15] we give the results by slices, so the fraction of integers with $x < y$, beginning with the first odd integer 1. In the last column we find the first three values of $f$ obtained from the programmed function $\mathcal{F}$.

### 3.5 Distribution function $F(k)$

Let us define the distribution function $F(k)$ as

$$F(k) = \lim_{m \to \infty} \frac{1}{m} \mu\{n \leq m \mid \chi(n) \geq k\}, \quad (20)$$

where $\mu$ is the number of positive integers $n \leq m$ with $m$ that tends towards infinity. $\chi(n)$ is called the "stopping time", and corresponds to the smallest positive integer such that the iterative application ($k$ times) of $3x + 1$ function $T$ (equation [5]) on a integer $n$ gives the result $T^k n < n$.

Terras [3] proves that this function is well defined for any value of $k$ and that it tends towards 0 for $k$ tending towards infinity.

Lagarias [4] redoes the demonstration using the function we will call $G(k)$,

$$G(k) = \lim_{x \to \infty} \frac{1}{x} \#\{n : n \leq x \text{ and } \sigma(n) \leq k\}, \quad (21)$$

where $\sigma(n)$ is the "stopping time". This function $G(k)$ is in away almost the reciprocal of the function $F(k)$, and tends towards 1 when $k$ tends towards infinity. The properties inherent in these functions will be clarified in the following examples.

Terras [3] and Everett [11] have independently demonstrated an important theorem (periodicity) bringing out the regularity in the trajectories generated by the $3x + 1$ function $T(n)$. We extended this property to the original Collatz problem [2] and the correspondent function $g(n)$. To do this, we used a fundamental property of the diophantine equations instead of the induction used by the cited authors.

The periodicity theorem can be interpreted as follows.

Let $k$ be a number of iterations applied to any $2^k$ consecutive integers. We will have all possible combinations $2^k$ of operations $n/2$ on the even integers and $(3n + 1)/2$ on the odd integers of the diadic sequences generated by the function $T(n)$ and each combination appears only once. All the integers $m$ of the form $m = n + 2^k$ will have the same combination of operations. The distribution of different combinations is then binomial versus the operations.
For example, let \( k = 1 \) and the \( 2^k = 2^1 = 2 \) consecutive positive integers 1 and 2. The sequences of length \( k + 1 = 2 \) generated by the function \( T(n) \) will be

\[
(1, 2) (2, 1) (3, 5) (4, 2) (5, 8) (6, 3) \cdots ,
\]

where we have added the numbers 3, 4, 5 and 6 after the two consecutive numbers so as to bring out the periodicity.

If we use the diadic sequences of the 0 and 1 representing respectively the even and odd operations, we will have

\[
(1) (0) (1) (0) (1) (0) \cdots ,
\]

all repeating periodically for every two consecutive sequences. This result follows from the fact that the all integers alternate between the even and odd integers.

Let’s write the first diophantine equation for the even integers which give the first integer \( x \) of the sequence versus the last integer (here the second) of the sequence,

\[
\frac{x}{2} = y; \quad 0 = 2y - x; \quad \text{and the general solution} \quad (2 + 2q, 1 + 1q).
\]

The second equation is the same as the first, but in the form \( c = by - ax \) with \( c = 0, b = 2 \) and \( a = 1 \). Since the parameters \( b \) and \( a \) are prime to each other, the theorem 2.1 allows us to write the general solution \((x_0 + bq, y_0 + aq)\), where \((x_0, y_0)\) is a particular solution and \( q \) is any integer, positive, negative or zero. According to the theorem 3.1 as \( b > a \) then \( x \geq y \). In this case, the stopping time is equal to the number of iterations \( k = 1 \).

The second diophantine equation for the odd integers is,

\[
\frac{3x + 1}{2} = y; \quad 1 = 2y - 3x; \quad \text{and the general solution} \quad (1 + 2q, 2 + 3q).
\]

Here, \( c = 1, b = 2 \) and \( a = 3 \). According to the theorem 3.1 as \( b < a \) then \( x < y \). In this case, the stopping time is greater than the number of iterations \( k = 1 \). Then, all sequences starting with an odd positive integer contribute to the distribution function \( F(k) \) because \( \chi > k \). As expected, \( F(k = 1) = 1/2 \).

Let another example. Take \( k = 2 \) and the \( 2^k = 2^2 = 4 \) consecutive positive integers 3, 4, 5 and 6. The sequences of length \( k + 1 = 3 \) generated by the function \( T(n) \) will be

\[
(3, 5, 8) (4, 2, 1) (5, 8, 4) (6, 3, 5) (7, 11, 17) (8, 4, 2) \cdots ,
\]

where we have added the numbers 7 and 8 after the four consecutive numbers so as to bring out the periodicity.

The diadic sequences are

\[
(1, 1) (0, 0) (1, 0) (0, 1) (1, 1) (0, 0) \cdots ,
\]

all repeating periodically for every four consecutive sequences.

We can write the \( 2^k = 2^2 = 4 \) diophantine equations in the same way as before. But, we will do it differently here. In fact the diadic sequences we will help to deduce whether or not the stopping time is greater or less than the number of iterations \( k = 2 \).
In the general case, the parameter \( b = 2^k \) and the parameter \( a = 3^{k_2} \cdot 1^{k_1} = 3^{k_2} \) with \( k \) the total number of iterations, \( k_1 \) the number of operations on the even integers, and \( k_2 = k - k_1 \) the number of operations on the odd integers.

| diadic sequences | \( k_1 \) | \( k_2 \) | \( b = 2^k \) | \( b = 3^{k_2} \) | \( b \ vs \ a \) | \( x \ vs \ y \) | stopping time \( \chi(n) \) |
|------------------|--------|--------|----------------|----------------|----------------|----------------|-----------------|
| (0, 0)           | 2      | 0      | 4              | 1              | \( b > a \)   | \( x > y \)   | –               |
| (0, 1)           | 1      | 1      | 4              | 3              | \( b > a \)   | \( x > y \)   | –               |
| (1, 0)           | 1      | 1      | 4              | 3              | \( b > a \)   | \( x > y \)   | \( \chi = k \)  |
| (1, 1)           | 0      | 2      | 4              | 9              | \( b < a \)   | \( x < y \)   | \( \chi > k \)  |

Table 1: Stopping time for \( k = 2 \)

As the first two sequences start with an even integer, we do not count them in \( F(k) \). The third sequence, so \((1, 0)\) which is generated by the integers \( 5 + 4q \) is such that \( \chi = k = 2 \). Unlike Terras, we will not count them because we have reached the condition \( T^k n < n \), which will create a slight gap with the results of Terras. Then, the distribution function \( F(k) \) with \( \chi > k \) instead \( \chi \geq k \) really becomes the reciprocal of the function \( G(k) \) defined by Lagarias. As expected, \( F(k = 2) = 1/4 \).

And so on for different values of the number of iterations \( k \).

The number of different sequences is given by \( b = 2^k \), and the number of different parameters \( a \) is calculated by the binomial coefficients \( \binom{k}{k_2} \). Binomial coefficients can be represented in a Pascal triangle,

| \( k_2 \backslash k \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
|------------------------|---|---|---|---|---|---|---|---|-----|
| 0                      | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | ... |
| 1                      |   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
| 2                      |   |   | 1 | 3 | 6 | 10| 15| 21| ... |
| 3                      |   |   |   | 1 | 4 | 10| 20| 35| ... |
| 4                      |   |   |   |   | 1 | 5 | 15| 35| ... |
| 5                      |   |   |   |   |   | 1 | 6 | 21| ... |
| 6                      |   |   |   |   |   |   | 1 | 7 | ... |
| 7                      |   |   |   |   |   |   |   | 1 | ... |
| 8                      |   |   |   |   |   |   |   |   | ... |

Table 2: Pascal triangle - Binomial coefficients

We use a similar table which will contain the number of integers \( n(i, j) \) by \( 2^k \) consecutive integers which satisfy the condition that the the stopping time \( \chi \) is greater than the number of iterations \( k \). We have
Table 3: Pascal triangle - Number of integers \( n(i, j) \) by \( 2^k \) consecutive integers with \( \chi > k \)

The index \( j \) for the columns of the table is the exponent \( k \) (the number of iterations) of 2 in the parameter \( b = 2^k \). The index \( i \) for the rows is the exponent \( k \) of 3 in the parameter \( a = 3^{k_2} \). As \( k_2 \) correspond to the number of operations on the odd integers, this value is in fact the number of 1 in the diadic sequences and varies of 0 to \( k \). The various data in this table are calculated recursively.

The first data is trivial and indicates that all the integers satisfy the condition \( \chi > k \) and this, because the number of iterations is \( k = 0 \). The cas \( k = 1 \) has ready be analyzed and we perform the following initialization, so \( n(0, 1) = 0 \) and \( n(0, 1) = 1 \). After 1 iteration, all positive even integer go to a smaller integer (\( n = 0 \)) and, all positive odd integer go to a greater integer (\( n = 1 \)).

From \( k = 2 \) we proceed recursively in the calculation of \( n(i, k) \).

We use the principle that each sequence is generated so that the new parameter \( b \) (for \( k \)) is the precedent (for \( k - 1 \)) time 2, and the new parameter \( a \) is the precedent time 1 or 3.

For example, for \( k = 2 \), we have two \( n \) which precede (for \( k = 1 \)), so \( n(0, 1) = 0 \) and \( n(1, 1) = 1 \). As \( n(0, 1) = 0 \), the sequences starting with a even positive integer for \( k = 2 \) will not contribute to \( F(k) \) and \( n(0, 2) = 0 \). On the other hand, the sequences generated by the integers with \( n(1, 1) = 1 \) can contribute to \( n(1, 2) \) and \( n(2, 2) \). The new parameter \( b \) will be \( b = 2 \cdot 2 \) and the new parameter \( a \) will be \( a = 3 \cdot 1 \) or \( a = 3 \cdot 3 \) (table 1). In the first case, \( b > a \), \( x \geq y \) and \( \chi = k \). Then \( n(1, 2) = 0 \). In the second case, \( b < a \), \( x < y \) and \( \chi > k \). Then \( n(2, 2) = 1 \). And so on for different values of \( k \).

The sum on the index \( i \) of \( n(i, k)/2^k \) for a given \( k \) gives the value of the distribution function \( F(k) \) for this number of iterations \( k \),

\[
F(k) = \sum_{i=0}^{k} \frac{n(i, k)}{2^k}. \tag{22}
\]

It is then easy to build the computer programs starting from the recursive function worked out by Terras and by the previous process which makes it possible to fill the table 3. The results of these two programs are compiled in the table 16. We have also extended the programs to the distribution function \( F_5(k) \) generated by the \( 5x + 1 \) function \( T_5 \) which we will deal with in the next section.

Unlike to the function \( F_5(k) \), the distribution function \( F(k) \) decreases constantly, monotonically and asymptotically. On the other hand, \( F(k) \) never become equal to zero, which is easily verified.
during the construction of table\textsuperscript{3}. There therefore always remain positive integers which satisfy to $T_k(n) > n$, but there is an increasingly restricted although infinite set.

The process we used in the previous sections (problem $3x + 1$) is somewhat different. We also use the periodicity property there, but we analyze everything in terms of geometric series and not of the evolution of a certain parameter $b$ (built from a product of 2) versus another parameter $a$ (built from a product of 3 and the 1). With groupings of integer sequences into geometric series, we can precisely determine the percentage of these which are such that $T_k(n) < n$. By continuing this process iteratively with the remaining sequences, we have shown that this percentage (always calculable) becomes smaller and smaller, and that the new remaining sequences (with $T_k(n) < n$) start with larger and larger integers.

It should not be forgotten that these exercises have never made it possible to exclude cycles other than the trivial cycle. On this last subject, we refer readers to the two previous papers \textsuperscript{1, 2}.

4 Problem $5x + 1$

Similar to the $3x + 1$ problem, we present three functions \textsuperscript{9} which encode the $5x + 1$ problem.

Let the Collatz $5x + 1$ function $C_5(n)$ be defined as follow

\[
C_5(n) = \begin{cases} 
5n + 1 & \text{if } n \equiv 1 \pmod{2} \\
\frac{n}{2} & \text{if } n \equiv 0 \pmod{2}.
\end{cases}
\]

(23)

Define the $5x + 1$ function $T_5(n)$ (or $5x + 1$ map)

\[
T_5(n) = \begin{cases} 
\frac{5n+1}{2} & \text{if } n \equiv 1 \pmod{2} \\
\frac{n}{2} & \text{if } n \equiv 0 \pmod{2}.
\end{cases}
\]

(24)

The third function, the accelerated $5x + 1$ function $U_5(n)$, is defined on the domain of all odd integers, and removes all powers of 2 at each step. It is given by

\[
U_5(n) = \frac{5n + 1}{2^{ord_2(5n + 1)}}
\]

(25)

in which $ord_2(n)$ counts the number of powers of 2 dividing $n$.

The function $T_5$ is known to have 5 cycles \textsuperscript{5}, with starting values 0, 1, 13, 17, $-1$.

4.1 Groupings (in triplets) of trajectories generated by the function $U_5(n)$

Let us represent the accelerated $5x + 1$ function $U_5(n)$ by the following 5 operations
\[ U_5(n) = \begin{cases} \frac{5n+1}{4}, & \text{if } n \equiv 7 \pmod{8} \\ \frac{5n+1}{2}, & \text{if } n \equiv 1 \pmod{4} \\ \frac{5n+1}{16}, & \text{if } n \equiv 3 \pmod{32} \\ \frac{5n+1}{8}, & \text{if } n \equiv 11 \pmod{16} \\ \frac{5n+1}{32}, & \text{if } n \equiv 19 \pmod{32} \end{cases} \] (26)

The first four operations result in odd integers and the last operation ends with a alternation of even and odd integers 3 (mod 5). The table [17] we give the five groupings resulting of these operations including the intermediate (abbreviate 'int') operation \((5x + 1)/2\). In each grouping we have a sequence of three integers ('triplets').

### 4.2 New function \(U_5(n)\) and one-to-one correspondence with the function \(U_5(n)\)

We define the function \(U_5(n)\)

\[ \Omega_5(n) = \begin{cases} \frac{5n}{4}, & \text{if } n \equiv 4 \pmod{4} \\ \frac{5n-1}{2}, & \text{if } n \equiv 1 \pmod{2} \\ \frac{5n+6}{16}, & \text{if } n \equiv 2 \pmod{16} \\ \frac{5n+2}{8}, & \text{if } n \equiv 6 \pmod{8} \\ \frac{n+6}{16}, & \text{if } n \equiv 10 \pmod{16} \end{cases} \]

This function does not come from generalized \(3x + 1\) mappings. The table [18] we give the five groupings resulting of these operations. In fact, this table brings together the trajectories of all integers and the integer resulting from a single application of the function \(U_5(n)\) (duos). We form triplets by repeating the first integer. We can easily verify that all the numbers \(n\) of triplets in the table [17] (function \(U_5(n)\)) are calculated using two simple operations on all the integers \(n_{\text{new}}\), where \(n_{\text{new}}\) is any integer beginning the triplets in the table [18] (function \(U_5(n)\)),

\[ n = 2n_{\text{new}} - 1 \] (27)

and

\[ n = 5n_{\text{new}} - 2. \] (28)

All intermediate integers 3 (mod 5) of triplets of \(U_5(n)\) and the last of the fifth grouping come from the operation \((5n_{\text{new}} - 2)\). All other integers (odd integers) come from the operation \((2n_{\text{new}} - 1)\). Then, the correspondence between the results of the application of the function \(U_5(n)\) on all integers is one-to-one with the application of the function \(U_5(n)\) on all odd integers.
4.3 Tree generated by the function $U_5(n)$ and distribution of integers

Let us define the branches as the sequence of the integers that are connected to each other by the operations $(5n)/4$, $(5n - 1)/2$, $(5n + 6)/16$, $(5n + 2)/8$ starting with $3 \mod 5$ and ending with an integer resulting from the operation $(n + 6)/16$. The branches are interrelated, thus forming a tree. The $U_5(n)$ function has 5 cycles as the function $T_5$. An infinity of integers enter these cycles, but there may be others that belong to the divergent trajectories.

The search for the convergence towards the cycles is to determine if all the integers are in branches (distribution) interlocking into each other and if, sooner or later, they converge to smaller integers, otherwise there is divergence.

As for the function $U(n)$, all integers are found in sequences with an end of all possible lengths and their distribution is characterized by a geometric progression.

Using the operation $(5n_{(new)} - 2)$ on all integers in branch ends of the tree and the operation $(2n_{(new)} - 1)$ on all the other integers of this tree, we build the tree generated by the function $U_5(n)$. Nevertheless, it is easier to analyze the function $U_5(n)$ which applies to all integers and which naturally reveals the regularity in the distributions of the branches.

The sequences that we will analyze end with an integer resulting the operation $(n + 6)/16$, but the beginning of the sequences is any integer, not necessarily a branch beginner integer $3 \mod 5$.

We can write the different diophantine equations, as for the $3x+1$ problem, and find the solutions of the latter for all the possible lengths. This approach leads us quickly to trajectories of all integers whose distribution follows a geometric progression. We will have the next results.

The results below are valid for $L \geq 2$.

The total number of diophantine equations $(DE)_L$ is

$$\#(DE)_L = 4^{L-2}. \quad (29)$$

The different increments (or modulo) for the integers $x$ starting these sequences are

$$\text{Increment} = d_1 * d_2 * d_3 * \cdots * d_{L-1}, \quad (30)$$

where $d_1, d_2, d_3, \cdots, d_{L-2} = 2, 4, 8, 16$ and $d_{L-1} = 16$.

The increment (or modulo) for the last integers $y$ is $5^{L-2}$.

The different distributions of first integers $x$ are calculated from the following quadrinomial.

$$D_L = \frac{1}{16} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right)^{L-2}, \quad (31)$$

which is the $n$th term ($n = L - 1$) of a geometric progression. From equation [1] which is the general form for this series, we put $a = \frac{1}{16}$ and $r = \frac{15}{16}$.

The quadrinomial coefficients $QC$ (equation [2]) of this quadrinomial give the number of different diophantine equations for a given increment.

The sum of $n$ first terms is given by the equation [2]

$$S_L = \frac{1}{16} \left(\frac{1 - (\frac{15}{16})^{L-1}}{1 - \frac{15}{16}}\right) = 1 - \left(\frac{15}{16}\right)^{L-1}. \quad (32)$$

When $L$ goes to infinity the sum tends towards one, meaning that all integers $x$, without exception, start sequences with end.
According to the equation, a large part of natural integers is included in the first sequences. The sequences with $L = 2$ until $L = 20$ include almost 70.7% of all natural integers. For this range of lengths, we covered 99.6% of all integers with the function $U(n)$. To reach this percentage with the function $U_5(n)$, it is necessary to cover the lengths $L = 2$ until $L = 85$.

The first sequences where the final integers $y$ are larger than the first integers are those starting with $x \equiv 155 \pmod{256}$, $x \equiv 367 \pmod{512}$, $x \equiv 412 \pmod{512}$, $x \equiv 435 \pmod{512}$ and $x \equiv 453 \pmod{512}$, for the lengths $L = 6$. In fact, these sequences are the only ones of all those for the lengths $L = 2$ until $L = 6$ with this behavior. All other integers for sequences covering these lengths end with smaller integers. The distributions $1/256$ (or $1/512$) meaning that at each 256 (or 512) consecutive integers there is one that starts a sequence which end with a larger integer.

For the sequences of length $L = 7$ and more we observe the same type of behavior. Some sequences with the smaller modulo will have the beginning integer smaller than that of the end of the sequence. For all others we have the inverse behavior.

If we take consecutive integer slices of 5,000, for example, we can already see that the number of integers whose last of the sequence is larger than the first is close of 60%, meaning than 60% of all integers behaving this behavior. This percentage was around 12% for the problem $3x + 1$ and the function $U(n)$. We can use the programmed function $\mathcal{F}$ determining the fraction $f$ (or the %) of natural integers such as $x < y$. For $L = 2$ to 20, 30, 40, 50, 60, 70, 80, 85, we have

$$f = 0.3092148 \ (70.7 \%), \ 0.4455945 \ (84.6 \%), \ 0.5184832 \ (91.9 \%), \ 0.5568361 \ (95.8 \%),$$

$$0.5769612 \ (97.8 \%), \ 0.5875168 \ (98.8 \%), \ 0.5930529 \ (99.4 \%), \ 0.5947369 \ (99.6 \%).$$

Between parentheses we have the proportion of all natural numbers for all sequences of length 2 until $L$ calculated with the equation. Recall that we have programmed the function $\mathcal{F}$ in VBA (Visual Basic for Applications) with a standard laptop (standard computer). From around $L = 53$ the integers appearing in the function become very large; for example, the quadrinomial coefficients have around 29 digits. By using a more powerful computer, we could obtain the results for $L > 53$.

Unlike the $3x + 1$ problem, the divergence seems quite possible. For example, the trajectory generated by the function $U_5(n)$ starting with 4 diverges quickly. Here are the first branches

$$4 \rightarrow 5 \rightarrow 12 \rightarrow \ldots \rightarrow 3 \ 978 \ 842 \rightarrow 248 \ 678 \ \ \ L = 30 \ \ PP$$
$$248 \ 678 \rightarrow 155 \ 424 \rightarrow \ldots \rightarrow 86 \ 277 \ 722 \rightarrow 5 \ 392 \ 358 \ \ \ L = 22 \ \ PP$$
$$5 \ 392 \ 358 \rightarrow 3 \ 370 \ 224 \rightarrow \ldots \rightarrow 957 \ 875 \ 242 \rightarrow 59 \ 867 \ 203 \ \ \ L = 19 \ \ PP$$

Using the function $U_5(n)$ applied to odd integers the percentage of their distribution with $x < y$ is still higher than 60% at the first level. The value is around 68%.

5 Conclusion

By properly grouping the trajectories generated by the functions at the origin of the $3x+1$ and $5x+1$ problems, or rather by their accelerated functions, we could bring out the regularity that was hidden there. This is allowed us to gather the trajectories in groups of all possible lengths. Subsequently, we have been able to determine the distribution of integers $x$ that start these trajectories and, finally, to find the proportions of them that are smaller (or larger) than the integer $y$ ending the trajectory. The distribution of integers following a geometric progression has given us the opportunity to treat all integers, without exception, until infinity. In no case did we use the notions of probabilities
(or stochastic process), which are widely used in the literature. The path we followed is clearly different from all that has been done nowadays to resolve the conjectures and brings a whole new light. We believe that a new way is open for the comprehension and the solution of the behavior of many other functions that are part of the generalized $3x + 1$ mappings, or others.

We conclude this exercise by quoting a sentence taken from the conclusion from a 1985 paper written by Lagarias [4], namely: "Of course there remains the possibility that someone will find some hidden regularity in the $3x + 1$ problem that allows some of the conjectures about it to be settled". The theorem on the periodicity bring out the regularity in the trajectories generated by the $3x + 1$ function $T(n)$, the $5x + 1$ function $T_5(n)$ and many other functions. The application of this theorem to the various problems generated by the functions derived from generalized $3x + 1$ mappings is surely an essential key to the implementation of their solutions together with the basic properties of the diophantine equations.

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\[
\begin{array}{ccc}
\text{x}_0 \equiv 1 \pmod{8} & \text{x}_0 \equiv 3 \pmod{4} & \text{x}_0 \equiv 5 \pmod{8} \\
\cdots & \cdots & \cdots \\
1 & \rightarrow & 2 & \rightarrow & 1 & 3 & \rightarrow & 5 & \rightarrow & 5 & 5 & \rightarrow & 8 & \rightarrow & 2 \\
9 & \rightarrow & 14 & \rightarrow & 7 & 7 & \rightarrow & 11 & \rightarrow & 11 & 13 & \rightarrow & 20 & \rightarrow & 5 \\
17 & \rightarrow & 26 & \rightarrow & 13 & 11 & \rightarrow & 17 & \rightarrow & 17 & 21 & \rightarrow & 32 & \rightarrow & 8 \\
25 & \rightarrow & 38 & \rightarrow & 19 & 15 & \rightarrow & 23 & \rightarrow & 23 & 29 & \rightarrow & 44 & \rightarrow & 11 \\
33 & \rightarrow & 50 & \rightarrow & 25 & 19 & \rightarrow & 29 & \rightarrow & 29 & 37 & \rightarrow & 56 & \rightarrow & 14 \\
41 & \rightarrow & 62 & \rightarrow & 31 & 23 & \rightarrow & 35 & \rightarrow & 35 & 45 & \rightarrow & 68 & \rightarrow & 17 \\
49 & \rightarrow & 74 & \rightarrow & 37 & 27 & \rightarrow & 41 & \rightarrow & 41 & 53 & \rightarrow & 80 & \rightarrow & 20 \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{y}_{\text{int}} \equiv 2 \pmod{12} & \text{y}_{\text{int}} \equiv 5 \pmod{6} & \text{y}_{\text{int}} \equiv 8 \pmod{12} \\
\text{y} \equiv 1 \pmod{6} & \text{y} \equiv 5 \pmod{6} & \text{y} \equiv 2 \pmod{3} \\
\end{array}
\]

Table 4: Groupings in triplets of trajectories generated by the function \(U(n)\)

\[
\begin{array}{ccc}
\text{x}_{\text{0(new)}} \equiv 1 \pmod{4} & \text{x}_{\text{0(new)}} \equiv 2 \pmod{2} & \text{x}_{\text{0(new)}} \equiv 3 \pmod{4} \\
\cdots & \cdots & \cdots \\
1 & \rightarrow & 1 & \rightarrow & 1 & 2 & \rightarrow & 2 & \rightarrow & 3 & 3 & \rightarrow & 3 & \rightarrow & 1 \\
5 & \rightarrow & 5 & \rightarrow & 4 & 4 & \rightarrow & 4 & \rightarrow & 6 & 7 & \rightarrow & 7 & \rightarrow & 2 \\
9 & \rightarrow & 9 & \rightarrow & 7 & 6 & \rightarrow & 6 & \rightarrow & 9 & 11 & \rightarrow & 11 & \rightarrow & 3 \\
13 & \rightarrow & 13 & \rightarrow & 10 & 8 & \rightarrow & 8 & \rightarrow & 12 & 15 & \rightarrow & 15 & \rightarrow & 4 \\
17 & \rightarrow & 17 & \rightarrow & 13 & 10 & \rightarrow & 10 & \rightarrow & 15 & 19 & \rightarrow & 19 & \rightarrow & 5 \\
21 & \rightarrow & 21 & \rightarrow & 16 & 12 & \rightarrow & 12 & \rightarrow & 18 & 23 & \rightarrow & 23 & \rightarrow & 6 \\
25 & \rightarrow & 25 & \rightarrow & 19 & 14 & \rightarrow & 14 & \rightarrow & 21 & 27 & \rightarrow & 27 & \rightarrow & 7 \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{y}_{\text{int(new)}} \equiv 1 \pmod{4} & \text{y}_{\text{int(new)}} \equiv 2 \pmod{2} & \text{y}_{\text{int(new)}} \equiv 3 \pmod{4} \\
\text{y}_{\text{new}} \equiv 1 \pmod{3} & \text{y}_{\text{new}} \equiv 3 \pmod{3} & \text{y}_{\text{new}} \equiv 1 \pmod{1} \\
\end{array}
\]

Table 5: Groupings in triplets of trajectories generated by the function \(U\)
Table 6: Tree - function U
Table 7: Tree function $U$ - odd integers
| $L = 2$ | $3 \rightarrow 1$ \hspace{1cm} modulo = 4 PG |
| $L = 3$ | $2 \rightarrow 3 \rightarrow 1$ \hspace{1cm} modulo = 8 PG |
| $L = 3$ | $9 \rightarrow 7 \rightarrow 2$ \hspace{1cm} modulo = 16 PG |
| $L = 4$ | $12 \rightarrow 18 \rightarrow 27 \rightarrow 7$ \hspace{1cm} modulo = 16 PG |
| $L = 4$ | $6 \rightarrow 9 \rightarrow 7 \rightarrow 2$ \hspace{1cm} modulo = 32 PG |
| $L = 4$ | $13 \rightarrow 10 \rightarrow 15 \rightarrow 4$ \hspace{1cm} modulo = 32 PG |
| $L = 4$ | $33 \rightarrow 25 \rightarrow 19 \rightarrow 5$ \hspace{1cm} modulo = 64 PG |
| $L = 5$ | $8 \rightarrow 12 \rightarrow 18 \rightarrow 27 \rightarrow 7$ \hspace{1cm} modulo = 32 PG |
| $L = 5$ | $4 \rightarrow 6 \rightarrow 9 \rightarrow 7 \rightarrow 2$ \hspace{1cm} modulo = 64 PG |
| $L = 5$ | $30 \rightarrow 45 \rightarrow 34 \rightarrow 51 \rightarrow 13$ \hspace{1cm} modulo = 64 PG |
| $L = 5$ | $37 \rightarrow 28 \rightarrow 42 \rightarrow 63 \rightarrow 16$ \hspace{1cm} modulo = 64 PG |
| $L = 5$ | $17 \rightarrow 13 \rightarrow 10 \rightarrow 15 \rightarrow 4$ \hspace{1cm} modulo = 128 PG |
| $L = 5$ | $22 \rightarrow 33 \rightarrow 25 \rightarrow 19 \rightarrow 5$ \hspace{1cm} modulo = 128 PG |
| $L = 5$ | $93 \rightarrow 70 \rightarrow 105 \rightarrow 79 \rightarrow 20$ \hspace{1cm} modulo = 128 PG |
| $L = 5$ | $129 \rightarrow 97 \rightarrow 73 \rightarrow 55 \rightarrow 14$ \hspace{1cm} modulo = 256 PG |
| $L = 6$ | $48 \rightarrow 72 \rightarrow 108 \rightarrow 162 \rightarrow 243 \rightarrow 61$ \hspace{1cm} modulo = 64 PP |
| $L = 6$ | $20 \rightarrow 30 \rightarrow 45 \rightarrow 34 \rightarrow 51 \rightarrow 13$ \hspace{1cm} modulo = 128 PG |
| $L = 6$ | $53 \rightarrow 40 \rightarrow 60 \rightarrow 90 \rightarrow 135 \rightarrow 34$ \hspace{1cm} modulo = 128 PG |
| $L = 6$ | $88 \rightarrow 132 \rightarrow 198 \rightarrow 297 \rightarrow 223 \rightarrow 56$ \hspace{1cm} modulo = 128 PG |
| $L = 6$ | $110 \rightarrow 165 \rightarrow 124 \rightarrow 186 \rightarrow 279 \rightarrow 70$ \hspace{1cm} modulo = 128 PG |
| $L = 6$ | $5 \rightarrow 4 \rightarrow 6 \rightarrow 9 \rightarrow 7 \rightarrow 2$ \hspace{1cm} modulo = 256 PG |
| $L = 6$ | $49 \rightarrow 37 \rightarrow 28 \rightarrow 42 \rightarrow 63 \rightarrow 16$ \hspace{1cm} modulo = 256 PG |
| $L = 6$ | $62 \rightarrow 93 \rightarrow 70 \rightarrow 105 \rightarrow 79 \rightarrow 20$ \hspace{1cm} modulo = 256 PG |
| $L = 6$ | $100 \rightarrow 150 \rightarrow 225 \rightarrow 169 \rightarrow 127 \rightarrow 32$ \hspace{1cm} modulo = 256 PG |
| $L = 6$ | $125 \rightarrow 94 \rightarrow 141 \rightarrow 106 \rightarrow 159 \rightarrow 40$ \hspace{1cm} modulo = 256 PG |
| $L = 6$ | $182 \rightarrow 273 \rightarrow 205 \rightarrow 154 \rightarrow 231 \rightarrow 58$ \hspace{1cm} modulo = 256 PG |
| $L = 6$ | $29 \rightarrow 22 \rightarrow 33 \rightarrow 25 \rightarrow 19 \rightarrow 5$ \hspace{1cm} modulo = 512 PG |
| $L = 6$ | $86 \rightarrow 129 \rightarrow 97 \rightarrow 73 \rightarrow 55 \rightarrow 14$ \hspace{1cm} modulo = 512 PG |
| $L = 6$ | $193 \rightarrow 145 \rightarrow 109 \rightarrow 82 \rightarrow 123 \rightarrow 31$ \hspace{1cm} modulo = 512 PG |
| $L = 6$ | $465 \rightarrow 349 \rightarrow 262 \rightarrow 393 \rightarrow 295 \rightarrow 74$ \hspace{1cm} modulo = 512 PG |
| $L = 6$ | $513 \rightarrow 385 \rightarrow 289 \rightarrow 217 \rightarrow 163 \rightarrow 41$ \hspace{1cm} modulo = 1024 PG |

Table 8: Sequences generated by the function $\Phi$ for $L = 2$ until $L = 6$
| L = 2 | 5 → 2 | modulo = 8 PG |
| L = 3 | 3 → 5 → 2 | modulo = 16 PG |
| L = 3 | 17 → 13 → 5 | modulo = 32 PG |
| L = 4 | 23 → 35 → 53 → 20 | modulo = 32 PG |
| L = 4 | 11 → 17 → 13 → 5 | modulo = 64 PG |
| L = 4 | 25 → 19 → 29 → 11 | modulo = 64 PG |
| L = 4 | 65 → 49 → 37 → 14 | modulo = 128 PG |
| L = 5 | 15 → 23 → 35 → 53 → 20 | modulo = 64 PG |
| L = 5 | 7 → 11 → 17 → 13 → 5 | modulo = 128 PG |
| L = 5 | 59 → 89 → 67 → 101 → 38 | modulo = 128 PG |
| L = 5 | 73 → 55 → 83 → 125 → 47 | modulo = 128 PG |
| L = 5 | 33 → 25 → 19 → 29 → 11 | modulo = 256 PG |
| L = 5 | 43 → 65 → 49 → 37 → 14 | modulo = 256 PG |
| L = 5 | 185 → 139 → 209 → 157 → 59 | modulo = 256 PG |
| L = 5 | 257 → 193 → 145 → 109 → 41 | modulo = 512 PG |
| L = 6 | 95 → 143 → 215 → 323 → 485 → 182 | modulo = 128 PG |
| L = 6 | 7 → 11 → 17 → 13 → 5 | modulo = 256 PG |
| L = 6 | 105 → 79 → 119 → 179 → 269 → 101 | modulo = 256 PG |
| L = 6 | 175 → 263 → 395 → 593 → 445 → 167 | modulo = 256 PG |
| L = 6 | 219 → 329 → 247 → 371 → 557 → 209 | modulo = 256 PG |
| L = 6 | 49 → 7 → 11 → 17 → 13 → 5 | modulo = 512 PG |
| L = 6 | 97 → 73 → 55 → 83 → 125 → 47 | modulo = 512 PG |
| L = 6 | 123 → 185 → 139 → 209 → 157 → 59 | modulo = 512 PG |
| L = 6 | 199 → 299 → 449 → 337 → 253 → 95 | modulo = 512 PG |
| L = 6 | 249 → 187 → 281 → 211 → 317 → 119 | modulo = 512 PG |
| L = 6 | 363 → 545 → 409 → 307 → 461 → 173 | modulo = 512 PG |
| L = 6 | 57 → 43 → 65 → 49 → 37 → 14 | modulo = 1024 PG |
| L = 6 | 171 → 257 → 193 → 145 → 109 → 41 | modulo = 1024 PG |
| L = 6 | 385 → 289 → 217 → 163 → 245 → 92 | modulo = 1024 PG |
| L = 6 | 929 → 697 → 523 → 785 → 589 → 221 | modulo = 1024 PG |
| L = 6 | 1025 → 769 → 577 → 433 → 325 → 122 | modulo = 2048 PG |

Table 9: Sequences generated by the function \( U(n) \) for \( L = 2 \) until \( L = 6 \)
| $L$ | $\#(DE) = 2^{L-2}$ | Modulo $BC_{n,k}$ | Dist. of integers $D_L = (1/4)(3/4)^{L-2}$ | $S_L = 1 - (3/4)^{L-1}$ | $S_L$ (%) |
|-----|---------------------|---------------------|-------------------------------------------|-----------------|---------|
| 2   | 1 mod 4             | 1/4                 | 1/4                                       | 1/4             | 25      |
|     | 2 mod 8             | 1/8                 | 1/8                                       | 3/16            | 43.75   |
|     | 1 mod 16            | 1/16                | 7/16                                      |                 |         |
|     | 2 mod 32            | 2/32                |                                          |                 |         |
|     | 1 mod 64            | 1/64                | 9/64                                      | 37/64           | 57.81   |
|     | 3 mod 128           | 3/128               |                                          |                 |         |
|     | 4 mod 256           | 1/256               | 27/256                                    | 175/256         | 68.36   |
|     |                    |                     |                                          |                 |         |
| 4   | 16 mod 64           | 1/64                |                                          |                 |         |
|     | 4 mod 128           | 4/128               |                                          |                 |         |
|     | 6 mod 256           | 6/256               |                                          |                 |         |
|     | 10 mod 512          | 4/512               |                                          |                 |         |
|     | 10 mod 1024         | 1/1024              | 81/1024                                   | 781/1024        | 76.27   |
|     | 10 mod 2048         | 5/2048              |                                          |                 |         |
|     | 10 mod 4096         | 1/4096              | 243/4096                                  | 3367/4096       | 82.20   |
| 5   | 32 mod 128          | 1/128               |                                          |                 |         |
|     | 5 mod 256           | 5/256               |                                          |                 |         |
|     | 10 mod 512          | 10/512              |                                          |                 |         |
|     | 10 mod 1024         | 10/1024             |                                          |                 |         |
|     | 10 mod 2048         | 5/2048              |                                          |                 |         |
|     | 10 mod 4096         | 1/4096              | 243/4096                                  | 3367/4096       | 82.20   |
| 6   | 64 mod 64           | ...                 |                                          |                 | ...     |
|     |                    |                     |                                          |                 |         |
| 7   | 64 mod 64           | ...                 |                                          |                 | ...     |
|     |                    |                     |                                          |                 |         |

Table 10: Distribution of first integers $x$ for $L = 2$ until $L = 7$ (function $\Delta l$)
| $L$ | $k_3 = L - 2$ | $(k_1, k_2)$ | $b = 4 \cdot 2^{k_1} \cdot 4^{k_2}$ | $a = 3^{k_3} = 3^{L - 2}$ | $x$ versus $y$ |
|-----|-------------|-------------|-----------------|-----------------|--------------|
| 2   | 0           | (0, 0)      | 4               | 1               | $x > y$      |
| 3   | 1           | (1, 0)      | 8               | 3               | $x > y$      |
|     |             | (0, 1)      | 16              |                 | $x > y$      |
| 4   | 2           | (2, 0)      | 16              | $3^2 = 9$       | $x > y$      |
|     |             | (1, 1)      | 32              |                 | $x > y$      |
|     |             | (0, 2)      | 64              |                 | $x > y$      |
| 5   | 3           | (3, 0)      | 32              | $3^3 = 27$      | $x > y$      |
|     |             | (2, 1)      | 64              |                 | $x > y$      |
|     |             | (1, 2)      | 128             |                 | $x > y$      |
|     |             | (0, 3)      | 256             |                 | $x > y$      |
| 6   | 4           | (4, 0)      | 64              | $3^4 = 81$      | $x < y$      |
|     |             | (3, 1)      | 128             |                 | $x > y$      |
|     |             | (2, 2)      | 256             |                 | $x > y$      |
|     |             | (1, 3)      | 512             |                 | $x > y$      |
|     |             | (0, 4)      | 1024            |                 | $x > y$      |
| 7   | 5           | (5, 0)      | 128             | $3^5 = 243$     | $x < y$      |
|     |             | (4, 1)      | 256             |                 | $x > y$      |
|     |             | (3, 2)      | 512             |                 | $x > y$      |
|     |             | (2, 3)      | 1024            |                 | $x > y$      |
|     |             | (1, 4)      | 2048            |                 | $x > y$      |
|     |             | (0, 5)      | 4096            |                 | $x > y$      |
| 8   | 6           | (6, 0)      | 256             | $3^6 = 727$     | $x < y$      |
|     |             | (5, 1)      | 512             |                 | $x < y$      |
|     |             | (4, 2)      | 1024            |                 | $x > y$      |
|     |             | (3, 3)      | 2048            |                 | $x > y$      |
|     |             | (2, 4)      | 4096            |                 | $x > y$      |
|     |             | (1, 5)      | 8192            |                 | $x > y$      |
|     |             | (0, 6)      | 16384           |                 | $x > y$      |
| 9   | 7           | (7, 0)      | 512             | $3^7 = 2181$    | $x < y$      |
| ... | ...         | ...         | ...             | ...             | ...          |

Table 11: Distribution of parameters $b$ and $a$ for $L = 2$ until $L = 8$ (function $U$)
| L       | Modulo Dist of integers   | \( |\frac{L_x}{y}| \) (\%) | Cumul\% | Cumul\% |
|---------|---------------------------|--------------------------|---------|---------|
| 10      | 64 | 1 7/128    | 100/128  | 82.21   |
|         | 128| 1 17/128   | 100/128  | 82.21   |
| 12      | 256| 1 17/512   | 100/512  | 96.65   |
|         | 512| 8 32/512   | 100/512  | 96.65   |
| 14      | 512| 1 17/512   | 100/512  | 96.65   |
|         | 1024| 7 17/1024 | 100/1024 | 99.99   |
| 16      | 256| 1 17/512   | 100/512  | 96.65   |
|         | 512| 8 32/512   | 100/512  | 96.65   |
| 18      | 512| 1 17/512   | 100/512  | 96.65   |
|         | 1024| 7 17/1024 | 100/1024 | 99.99   |
| 20      | 256| 1 17/512   | 100/512  | 96.65   |
|         | 512| 8 32/512   | 100/512  | 96.65   |
| 21      | 512| 1 17/512   | 100/512  | 96.65   |

Table 12: Distribution of integers for \( x < y \) with \( L = 6 \) until \( L = 20 \) (function \( \text{U} \))
Table 13: Second level sequences $x_2 \equiv 61 \pmod{81}$ (function $\Phi$)

\[
\begin{align*}
61 \rightarrow & 46 \rightarrow 69 \rightarrow 52 \rightarrow 78 \rightarrow 117 \rightarrow 88 \rightarrow 132 \rightarrow 198 \rightarrow 297 \rightarrow 223 \rightarrow 56 \quad \text{L = 12 PG} \\
142 \rightarrow & 213 \rightarrow 160 \rightarrow 240 \rightarrow 360 \rightarrow 540 \rightarrow 810 \rightarrow 1215 \rightarrow 304 \quad \text{L = 9 PP} \\
223 \rightarrow & 56 \quad \text{L = 2 PG} \\
304 \rightarrow & 456 \rightarrow 684 \rightarrow 1026 \rightarrow 1539 \rightarrow 385 \quad \text{L = 6 PP} \\
385 \rightarrow & 289 \rightarrow 217 \rightarrow 163 \rightarrow 41 \quad \text{L = 5 PG} \\
466 \rightarrow & 699 \rightarrow 175 \quad \text{L = 3 PG} \\
547 \rightarrow & 137 \quad \text{L = 2 PG} \\
628 \rightarrow & 942 \rightarrow 1413 \rightarrow 1060 \rightarrow 1590 \rightarrow 2385 \rightarrow 1789 \rightarrow 1342 \rightarrow 2013 \rightarrow 1510 \quad \text{L = 13 PG} \\
709 \rightarrow & 532 \rightarrow 798 \rightarrow 898 \rightarrow 1347 \rightarrow 337 \quad \text{L = 7 PG} \\
790 \rightarrow & 1185 \rightarrow 889 \rightarrow 667 \rightarrow 167 \quad \text{L = 5 PG} \\
871 \rightarrow & 218 \quad \text{L = 2 PG} \\
952 \rightarrow & 1428 \rightarrow 2142 \rightarrow 3213 \rightarrow 2410 \rightarrow 3615 \rightarrow 904 \quad \text{L = 7 PG} \\
1033 \rightarrow & 775 \rightarrow 194 \quad \text{L = 3 PG} \\
1114 \rightarrow & 1671 \rightarrow 418 \quad \text{L = 3 PG} \\
1195 \rightarrow & 299 \quad \text{L = 2 PG} \\
1276 \rightarrow & 1914 \rightarrow 2871 \rightarrow 718 \quad \text{L = 4 PG} \\
1357 \rightarrow & 1018 \rightarrow 1527 \rightarrow 382 \quad \text{L = 4 PG} \\
1438 \rightarrow & 2157 \rightarrow 1618 \rightarrow 2427 \rightarrow 607 \quad \text{L = 5 PG} \\
1519 \rightarrow & 380 \quad \text{L = 2 PG} \\
1600 \rightarrow & 2400 \rightarrow 3600 \rightarrow 5400 \rightarrow 8100 \rightarrow 12150 \rightarrow 1789 \rightarrow 1342 \rightarrow 2013 \rightarrow 1510 \rightarrow 904 \quad \text{L = 14 PG} \\
1681 \rightarrow & 1261 \rightarrow 946 \rightarrow 1419 \rightarrow 355 \quad \text{L = 5 PG} \\
1762 \rightarrow & 2643 \rightarrow 661 \quad \text{L = 3 PG} \\
1843 \rightarrow & 461 \quad \text{L = 2 PG} \\
l = 3 \rightarrow & 812 \quad \text{L = 5 PG} \\
2005 \rightarrow & 1504 \rightarrow 2256 \rightarrow 3384 \rightarrow 5076 \rightarrow 7614 \rightarrow 8566 \rightarrow 12849 \rightarrow 9637 \rightarrow 10842 \rightarrow 16263 \rightarrow 10438 \quad \text{L = 12 PP} \\
2086 \rightarrow & 3129 \rightarrow 2347 \rightarrow 587 \quad \text{L = 4 PG} \\
2167 \rightarrow & 542 \quad \text{L = 2 PG} \\
2248 \rightarrow & 3372 \rightarrow 5058 \rightarrow 7587 \rightarrow 1897 \quad \text{L = 5 PG} \\
2329 \rightarrow & 1747 \rightarrow 437 \quad \text{L = 3 PG} \\
2410 \rightarrow & 3615 \rightarrow 904 \quad \text{L = 3 PG} \\
2491 \rightarrow & 623 \quad \text{L = 2 PG} \\
2572 \rightarrow & 3858 \rightarrow 5787 \rightarrow 1447 \quad \text{L = 4 PG} \\
2653 \rightarrow & 1990 \rightarrow 2985 \rightarrow 2239 \rightarrow 560 \quad \text{L = 5 PG} \\
2734 \rightarrow & 4101 \rightarrow 3076 \rightarrow 4614 \rightarrow 6921 \rightarrow 5191 \rightarrow 1298 \quad \text{L = 7 PG} \\
2815 \rightarrow & 704 \quad \text{L = 2 PG} \\
2896 \rightarrow & 4344 \rightarrow 6516 \rightarrow 9774 \rightarrow 14661 \rightarrow 10996 \rightarrow 16494 \rightarrow 24741 \rightarrow 18556 \rightarrow 27834 \rightarrow 41751 \rightarrow 10438 \quad \text{L = 12 PP} \\
2977 \rightarrow & 2233 \rightarrow 1675 \rightarrow 419 \quad \text{L = 4 PG} \\
3058 \rightarrow & 4587 \rightarrow 1147 \quad \text{L = 3 PG} \\
3139 \rightarrow & 785 \quad \text{L = 2 PG} \\
3220 \rightarrow & 4830 \rightarrow 7245 \rightarrow 5076 \rightarrow 12849 \rightarrow 9637 \rightarrow 10842 \rightarrow 16263 \rightarrow 10438 \quad \text{L = 12 PP} \\
3301 \rightarrow & 2476 \rightarrow 3714 \rightarrow 5571 \rightarrow 1393 \quad \text{L = 5 PG} \\
3382 \rightarrow & 5073 \rightarrow 3805 \rightarrow 2854 \rightarrow 4281 \rightarrow 3211 \rightarrow 803 \quad \text{L = 7 PG} \\
\end{align*}
\]
level | integers beginning with 1,2,3, · · · | f generated by the programmed function $\mathcal{F}$ (equations 17, 18, 19) |
|---|---|---|
| | until $5\,000$ | until $10\,000$ | until $100\,000$ | until $1\,000\,000$ | $L_1, L_2, L_3 = 2$ to $30$ |
| 1 | 0.123049 | 0.124325 | 0.123972 | 0.123814 | 0.1236245 |
| 2 | 0.053421 | 0.051510 | 0.050701 | 0.051026 | 0.0508968 |
| 3 | 0.025010 | 0.024105 | 0.024040 | 0.024105 | 0.0240408 |
| 4 | 0.016407 | 0.015603 | 0.012790 | 0.012441 | 0.006766 |
| 5 | 0.007203 | 0.007201 | 0.007220 | 0.006766 | 0.006766 |
| 6 | 0.003201 | 0.002801 | 0.004010 | 0.003908 | 0.003908 |
| 7 | 0.003001 | 0.002100 | 0.002390 | 0.002389 | 0.002389 |
| ... | ... | ... | ... | ... | ... |

Table 14: Distribution of integers with $x < y$ (function $U$) by slices and anticipated by the programmed function $\mathcal{F}$

level | odd integers beginning with 1,3,5, · · · | f generated by the programmed function $\mathcal{F}$ |
|---|---|---|
| | until $5\,001$ | until $10\,001$ | until $100\,001$ | until $1\,000\,001$ | $L_1, L_2, L_3 = 2$ to $30$ |
| 1 | 0.200480 | 0.201640 | 0.202664 | 0.202652 | 0.2024107 |
| 2 | 0.089236 | 0.085617 | 0.084002 | 0.084262 | 0.0840688 |
| 3 | 0.047619 | 0.041208 | 0.038301 | 0.039736 | 0.0397817 |
| 4 | 0.029212 | 0.025205 | 0.020580 | 0.020810 | 0.020810 |
| 5 | 0.014806 | 0.012603 | 0.011900 | 0.011700 | 0.011700 |
| 6 | 0.006403 | 0.00409 | 0.006500 | 0.006608 | 0.006608 |
| 7 | 0.006002 | 0.004201 | 0.003780 | 0.004088 | 0.004088 |
| ... | ... | ... | ... | ... | ... |

Table 15: Distribution of odd integers with $x < y$ (function $U$) by slices and anticipated by the programmed function $\mathcal{F}$
| $k$ | $F(k)$ | $F_5(k)$ |
|-----|--------|---------|
| Terras revisited (with the table 3) | Terras revisited (as the table 3) |
| 0   | 1      | 1       |
| 1   | 0.5    | 0.5     |
| 2   | 0.5    | 0.5     |
| 3   | 0.25   | 0.375   |
| 4   | 0.25   | 0.375   |
| 5   | 0.125  | 0.375   |
| 6   | 0.125  | 0.375   |
| 7   | 0.125  | 0.2734375 | 0.3125 |
| 8   | 0.1015625 | 0.07421875 | 0.2734375 |
| 9   | 0.07421875 | 0.07421875 | 0.2734375 |
| 10  | 0.07421875 | 0.0625 | 0.259765625 | 0.2734375 |
| 20  | 2.8591 × 10^{-2} | 2.6062 × 10^{-2} | 0.2221222 | 0.2221222 |
| 30  | 1.1894 × 10^{-2} | 1.1894 × 10^{-2} | 0.2057265 | 0.2057265 |
| 40  | 6.5693 × 10^{-3} | 5.8233 × 10^{-3} | 0.19625785 | 0.1978474 |
| 50  | 3.5373 × 10^{-3} | 3.3167 × 10^{-3} | 0.191165633 | 0.191165633 |
| 60  | 1.9222 × 10^{-3} | 1.9222 × 10^{-3} | 0.18811449 | 0.18811449 |
| 70  | 1.1644 × 10^{-3} | 1.0516 × 10^{-3} | 0.18513014 | 0.185735 |
| 80  | 7.0744 × 10^{-4} | 6.6440 × 10^{-4} | 0.1831778 | 0.1831778 |
| 90  | 4.1078 × 10^{-4} | 4.1078 × 10^{-4} | 0.1819218 | 0.1819218 |
| 100 | 2.6396 × 10^{-4} | 2.3868 × 10^{-4} | 0.1806658 | 0.1806658 |

Table 16: Distribution function $F(k)$
\[
\begin{align*}
\text{Table 17: Groupings in triplets of trajectories generated by the function } U_5(n)
\end{align*}
\]

| \(x \equiv 7 \text{ (mod 8)}\) | \(x \equiv 1 \text{ (mod 4)}\) | \(x \equiv 3 \text{ (mod 32)}\) | \(x \equiv 11 \text{ (mod 16)}\) | \(x \equiv 19 \text{ (mod 32)}\) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \(\ldots\) | \(\ldots\) | \(\ldots\) | \(\ldots\) | \(\ldots\) |
| 7 | 18 | 9 | 1 | 3 | 3 | 3 | 8 | 1 | 11 | 28 | 7 | 19 | 48 | 3 |
| 15 | 38 | 19 | 5 | 13 | 13 | 35 | 88 | 11 | 27 | 68 | 17 | 51 | 128 | 8 |
| 23 | 58 | 29 | 9 | 23 | 23 | 67 | 168 | 21 | 43 | 108 | 27 | 83 | 208 | 13 |
| 31 | 78 | 39 | 13 | 33 | 33 | 99 | 248 | 31 | 59 | 148 | 37 | 115 | 288 | 18 |
| 39 | 98 | 49 | 17 | 43 | 43 | 131 | 328 | 41 | 75 | 188 | 47 | 147 | 368 | 23 |
| 47 | 118 | 59 | 21 | 58 | 53 | 163 | 408 | 51 | 91 | 228 | 57 | 179 | 448 | 28 |
| 55 | 138 | 69 | 25 | 63 | 63 | 195 | 488 | 61 | 107 | 268 | 67 | 211 | 528 | 33 |
| 63 | 158 | 79 | 29 | 73 | 73 | 227 | 568 | 71 | 123 | 308 | 77 | 243 | 608 | 38 |
| 71 | 178 | 89 | 33 | 83 | 83 | 259 | 648 | 81 | 139 | 348 | 87 | 275 | 688 | 43 |

\[
\begin{align*}
\text{\(y_{int} \equiv 18 \text{ (mod 20)}\)} & \quad \text{\(y_{int} \equiv 3 \text{ (mod 10)}\)} & \quad \text{\(y_{int} \equiv 8 \text{ (mod 80)}\)} & \quad \text{\(y_{int} \equiv 28 \text{ (mod 40)}\)} & \quad \text{\(y_{int} \equiv 48 \text{ (mod 80)}\)} \\
\text{\(y \equiv 9 \text{ (mod 10)}\)} & \quad \text{\(y \equiv 3 \text{ (mod 10)}\)} & \quad \text{\(y \equiv 1 \text{ (mod 10)}\)} & \quad \text{\(y \equiv 7 \text{ (mod 10)}\)} & \quad \text{\(y \equiv 3 \text{ (mod 5)}\)}
\end{align*}
\]

\[
\begin{align*}
\text{Table 18: Groupings in triplets of trajectories generated by the function } U_5(n)
\end{align*}
\]

| \(x \equiv 4 \text{ (mod 4)}\) | \(x \equiv 1 \text{ (mod 2)}\) | \(x \equiv 2 \text{ (mod 16)}\) | \(x \equiv 6 \text{ (mod 8)}\) | \(x \equiv 10 \text{ (mod 16)}\) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \(\ldots\) | \(\ldots\) | \(\ldots\) | \(\ldots\) | \(\ldots\) |
| 4 | 4 | 5 | 1 | 1 | 2 | 2 | 2 | 1 | 6 | 6 | 4 | 10 | 10 | 1 |
| 8 | 8 | 10 | 3 | 3 | 7 | 18 | 18 | 6 | 14 | 14 | 9 | 26 | 26 | 2 |
| 12 | 12 | 15 | 5 | 5 | 12 | 34 | 34 | 11 | 22 | 22 | 14 | 42 | 42 | 3 |
| 16 | 16 | 20 | 7 | 7 | 17 | 50 | 50 | 16 | 30 | 30 | 19 | 58 | 58 | 4 |
| 20 | 20 | 25 | 9 | 9 | 22 | 66 | 66 | 21 | 38 | 38 | 24 | 74 | 74 | 5 |
| 24 | 24 | 30 | 11 | 11 | 27 | 82 | 82 | 66 | 46 | 46 | 29 | 90 | 90 | 6 |
| 28 | 28 | 35 | 13 | 13 | 32 | 98 | 98 | 31 | 54 | 54 | 34 | 106 | 106 | 7 |
| 32 | 32 | 40 | 15 | 15 | 37 | 114 | 114 | 36 | 62 | 62 | 39 | 122 | 122 | 8 |
| 36 | 36 | 45 | 17 | 17 | 42 | 130 | 130 | 41 | 70 | 70 | 44 | 138 | 138 | 9 |

\[
\begin{align*}
\text{\(y_{int} \equiv 4 \text{ (mod 4)}\)} & \quad \text{\(y_{int} \equiv 2 \text{ (mod 2)}\)} & \quad \text{\(y_{int} \equiv 6 \text{ (mod 6)}\)} & \quad \text{\(y_{int} \equiv 10 \text{ (mod 10)}\)} \\
\text{\(y \equiv 5 \text{ (mod 5)}\)} & \quad \text{\(y \equiv 2 \text{ (mod 5)}\)} & \quad \text{\(y \equiv 1 \text{ (mod 5)}\)} & \quad \text{\(y \equiv 4 \text{ (mod 5)}\)} & \quad \text{\(y \equiv 1 \text{ (mod 1)}\)}
\end{align*}
\]