a Modified Kähler-Ricci Flow

Zhou Zhang
Department of Mathematics
University of Michigan, at Ann Arbor

Abstract: In this note, we study a Kähler-Ricci flow modified from the classic version. In the non-degenerate case, strong convergence at infinity is achieved. We also have partial results for some interesting degenerate cases.

1 Set-up and Motivation

Kähler-Ricci flow, which is nothing but Ricci flow with initial metric being Kähler, enjoys the same debut as Ricci flow in R. Hamilton’s original paper \cite{4}. H. D. Cao’s paper, \cite{1}, can be seen as the first one devoted to the study of Kähler-Ricci flow and the alternative proof of Calabi’s conjecture presented there has been bringing great interest to this object.

Though it is essentially Ricci flow, the cohomology meaning coming with Kähler condition makes it possible to transform it to an equivalent scalar flow \cite{1}, which is much simpler-looking and more flexible to study. The discussion in this note would hopefully give a flavor of the flexibility.

Let $\omega_0$ be any Kähler metric over a closed manifold $X$ (with complex dimension greater or equal to 2), and $\omega_\infty$ is any smooth real closed $(1,1)$-form. Set $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$ and consider the following flow over the level of metric potential for space-time:

$$\frac{\partial u}{\partial t} = \log \left( \frac{\omega_t + \sqrt{-1} \partial \bar{\partial} u}{\Omega} \right), \quad u(0, \cdot) = 0,$$

(1.1)

where $\Omega$ is a smooth volume form over $X$.

Let $\tilde{\omega}_t = \omega_t + \sqrt{-1} \partial \bar{\partial} u$ and the corresponding flow on the level of metric is a little bit artificially looking as follows:

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\text{Ric}(\tilde{\omega}_t) + \text{Ric}(\Omega) - e^{-t}(\omega_0 - \omega_\infty), \quad \tilde{\omega}_0 = \omega_0,$$

(1.2)

where the meaning of the form, $\text{Ric}(\Omega)$, as in \cite{9}, is a natural generalization from the Ricci form for a Kähler metric, i.e., using the volume form $\Omega$ instead of the volume form for some Kähler metric in the expression of Ricci form from classic computation in Kähler geometry.

\footnote{This statement makes use of the uniqueness and short time existence results of Ricci flow.}
Remark 1.1. The equation (1.2) doesn’t look so natural at the first sight when \( \omega_0 \neq \omega_\infty \), but it’s essentially still a Kähler-Ricci flow, and the extra term in comparison to the flow studied in [1], which is exponentially decaying, should not bring too much difference in spirit.

Our motivation to study this flow is to solve the following complex Monge-Ampère equation
\[
(\omega_\infty + \sqrt{-1} \partial \bar{\partial} u_\infty)^n = \Omega,
\]
using flow techniques. This has been done in the case of \([\omega_\infty]\) being Kähler in [1], which provides another proof of Calabi’s conjecture.

One can also solve it for some degenerate \([\omega_\infty]\) (semi-ample and big) by method of continuity using other (more direct) perturbation, which seems to be less delicate than Kähler-Ricci flow as described in [15] and [10].

The point is to allow the change of cohomology class along the flow, which is important in the consideration of \([\omega_\infty]\) being degenerate as a Kähler class. The modification of original Kähler-Ricci flow by such a term as above is inevitable from simple cohomology consideration.

Our results can be summarized in the following theorem.

Theorem 1.2. The modified Kähler-Ricci flow (1.1) (or (1.2) equivalently) exists as long as the cohomology class, \([\omega_t]\) remains Kähler.

1) When \([\omega_\infty]\) is Kähler, the flow converges exponentially smoothly to the unique solution of the corresponding Monge-Ampère equation;

2) When \([\omega_\infty]\) is semi-ample and big, we have degenerate estimates on the metrics along the flow out of the stable base locus set of \([\omega_\infty]\) uniform for all time and the volume form, \(\tilde{\omega}_t\), is bounded from above and away from 0 along the way.

3) When \([\omega_\infty]\) is “only” big, i.e., the flow exists up to a finite time \(T\), and \([\omega_T]\) is semi-ample, we have local smooth convergence of the flow out of the stable base locus set of \([\omega_T]\).

The rest part of this note will be devoted to the proof of this theorem.

Acknowledgment 1.3. I would like to thank my thesis advisor, Professor Gang Tian, for introducing this interesting field and encouragement along the way. This research was partially done during the stay at MSRI (Mathematical Sciences Research Institute) as a Postdoctor Research Fellow on academic leave. I would like to thank the institute and Department of Mathematics of University of Michigan, at Ann Arbor, for their kindness and effort to make this opportunity possible for me. Also the hospitality of the institute can not be appreciated enough.

Several people, Yanir Rubinstein and others, took the trouble to read an earlier version of this note. I would like to thank them for all their feedbacks.
2 General Facts and Basic Computation

The equation (1.1) is clearly still parabolic, and so short time existence and uniqueness is not a problem. It’s also easy to see that the smooth solution exists as long as \([\omega_t]\) remains Kähler as already being described in [15]. Simply speaking, when arguing locally in time for this range, \(\omega_t\) can be made uniform as metric which makes life very easy to follow Cao and Yau’s argument as in [1] and [13]. So the existence part of Theorem 1.2 is justified.

Convergence, or estimate uniform for time, is our main concern now. For all the expressions below, \(C\) would be a positive constant (fixed for each place). Let’s list some basic computation from (1.1) in the following.

\[
\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t}\right) = \langle \dot{\omega}_t, \frac{\partial \dot{\omega}_t}{\partial t} \rangle = \Delta_{\dot{\omega}_t} \left(\frac{\partial u}{\partial t}\right) - e^{-t} \langle \dot{\omega}_t, \omega_0 - \omega_{\infty}\rangle.
\]

\[
\frac{\partial}{\partial t} \left(\frac{\partial^2 u}{\partial t^2}\right) = \langle \dot{\omega}_t, \frac{\partial^2 \dot{\omega}_t}{\partial t^2} \rangle - \left(\frac{\partial \dot{\omega}_t}{\partial t} \cdot \frac{\partial \dot{\omega}_t}{\partial t}\right) \dot{\omega}_t \leq \Delta_{\dot{\omega}_t} \left(\frac{\partial^2 u}{\partial t^2}\right) + e^{-t} \langle \dot{\omega}_t, \omega_0 - \omega_{\infty}\rangle.
\]

Take summation of the above two and apply standard maximum principle argument to get:

\[
\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} + u\right) \leq C,
\]

and from this, it’s easy to see

\[
\frac{\partial u}{\partial t} \leq C,
\]

which gives the measure bound for \(\dot{\omega}_t = e^{\frac{\partial u}{\partial t}} \Omega\). This allows us to apply the results of Kolodziej’s (as in [7] and [8]) and our generalization (as in [14]) in respective situation of \([\omega_t]\), which provides the uniform bound for the metric potential along the flow after routine normalization in the cases under study.

**Remark 2.1.** Even in the case of \([\omega_{\infty}]\) being Kähler, the result from pluripotential theory as in [7] is used for the normalized metric potential bound, so the logic line of our argument is not quite the same as that in [1]. It would also be interesting to see whether the original argument of Cao’s can also be carried through with fewer changes.

We also need to derive some kind of low bound for \(\frac{\partial u}{\partial t}\) in search of the possible metric bound (coming from volume and Laplacian controls).

The following equation would be very useful later:

\[
\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} + u\right) = \Delta_{\dot{\omega}_t} \left(\frac{\partial u}{\partial t} + u\right) - n + \frac{\partial u}{\partial t} + (\dot{\omega}_t, \omega_{\infty}).
\]  

3 a Baby Version

Let’s start with the situation when there is no degeneration on the cohomology classes, i.e., \([\omega_{\infty}]\) is also Kähler. This is the first case in our main theorem, which could be seen as a natural generalization of Cao’s work mentioned before after allowing the change of cohomology class along the flow.
3.1 Uniform Estimates and Global Existence

Global existence of the flow only needs estimates local in time as mentioned before. Now we have to go for estimates uniform for all time. The most essential part is the $C^0$ estimates. Up to now, there is only the lower bound for $\frac{\partial u}{\partial t}$ left to be obtained.

At first, let’s assume $\omega_\infty > 0$, which will not change the problem in any essential way. We’ll remove this simplification later.

As mentioned before, by the measure bound from the previous section, we already know that $|v| \leq C$

where $v = u - \int_X u \Omega$. We have assumed $\int_X \Omega = 1$ for simplicity of notation.

We also know

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} - \int_X \frac{\partial u}{\partial t} \Omega \geq \frac{\partial u}{\partial t} - C$$

from the upper bound of $\frac{\partial u}{\partial t}$, and so the lower bound of $\frac{\partial u}{\partial t}$ would give that for $\frac{\partial v}{\partial t}$.

In fact the inverse is also true as (2.1) can be modified to be:

$$\frac{\partial}{\partial t} (\frac{\partial u}{\partial t} + v) = \Delta \omega_t (\frac{\partial u}{\partial t} + v) - n + \frac{\partial v}{\partial t} + \langle \omega_t, \omega_\infty \rangle.$$

Assuming $\frac{\partial v}{\partial t} \geq -C$, we can get a lower bound for $\frac{\partial u}{\partial t}$ by applying maximum principle as H. Tsuji did in [11] using the control of volume by the control of trace, which is nothing but the classic algebraic-geometric mean value inequality. But we can actually do better by the following more delicate maximum principle argument.

Consider the minimum value point, $p$, for $\frac{\partial u}{\partial t} + v$ for $X \times [0, T]$ with any fixed $0 < T < \infty$. As usual, we only need to study the case when that point is not at the initial time since the situation for the initial time is well under control. At $p$, we have:

$$n - \frac{\partial v}{\partial t} \geq \langle \omega_t, \omega_\infty \rangle \geq n \cdot (\frac{\omega_\infty^n}{\omega^n})^\frac{1}{n} = n \cdot (\frac{\omega_\infty^n}{e^{\frac{\omega_t}{\omega^n} \Omega}})^\frac{1}{n} > 0,$$

and so $(1 - \frac{1}{n} \frac{\partial v}{\partial t})^n \cdot e^{\frac{\partial u}{\partial t}} \geq C > 0$. Using $\frac{\partial v}{\partial t} \geq \frac{\partial u}{\partial t} - C$, one arrives at:

$$\left(C - \frac{\partial u}{\partial t} \right)^n \cdot e^{\frac{\partial u}{\partial t}} \geq C > 0$$

with $C - \frac{\partial u}{\partial t} > 0$, which gives $\frac{\partial u}{\partial t} \geq -C$ at $p$. The uniform bound for $v$ thus gives the lower bound of $\frac{\partial u}{\partial t}$ over $X$ (and also for $\frac{\partial v}{\partial t}$) from the bound at $p$.

There is another way of doing the maximum principle argument which might seems to be more direct in this case. This is also a very classic point of view when studying the flow. Basically, we examine the evolution of space-direction...
extremal value along the flow. This function, now only depending on time, would be (locally) Lipschitz simply by going through the definition, and so it’ll be legitimate to consider the first order ordinary differential inequality.

This kind of argument, in spirit, would be more delicate than what we used before. But actually for the differential inequality of interest here, the study would be as rough as before. Let’s illustrate the idea below.

Set \( A(t) = \min_{X \times \{t\}} (\partial u/\partial t + v) \). Let’s also take some \( x(t) \) where the value \( A(t) \) is achieved, but we do not assume (or need) any regularity of \( x(t) \) with respect to \( t \).

Using the sign of Laplacian, we can derive the differential inequality for the function \( A(t) \) as follows

\[
\frac{\partial A}{\partial t} \geq -n + (\omega_t, \omega_\infty)(x(t)) + (\tilde{\omega}_t, \omega_\infty)(x(t)) + Ce^{-\frac{\partial u}{\partial t} + v}(x(t))
\]

\[
= -C + A + Ce^{-A}.
\]

From this inequality, we can see that when \( A \) is sufficiently small (i.e., very negative), \( \frac{\partial A}{\partial t} \) would be big (i.e., very positive). It won’t be hard to get a lower bound for \( A \) from this mechanism. All the pieces from the previous argument are also used here, but this looks more straightforward (for people good at playing with ODE).

**Remark 3.1.** Clearly, in the degenerate version of maximum principle as what will appear later, this point of view still works as long as the point \( x(t) \) is in the regular part.

Then using classic second order estimate, we can have uniform control for the trace of \( \tilde{\omega}_t \) (i.e., Lapacian). And so, together with the volume lower bound from the bound of \( \frac{\partial u}{\partial t} \), we have them controlled uniformly as metric. Finally, high order derivatives are also uniformly controlled using classic estimates for parabolic PDE’s (including Yau’s computation and parabolic Schauder estimates). These are very standard arguments for the situation here.

Now let’s remove the assumption that \( \omega_\infty > 0 \). We always have \( \omega_\infty + \sqrt{-1} \bar{\partial} \bar{\partial} f > 0 \) for some smooth function \( f \) over \( X \) as \( [\omega_\infty] \) is Kähler. Also recall that \( \omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty) \). Let’s now set

\[
\tilde{\omega}_t = (\omega_\infty + \sqrt{-1} \bar{\partial} \bar{\partial} f) + e^{-t}(\omega_0 - (\omega_\infty + \sqrt{-1} \bar{\partial} \bar{\partial} f)) = \omega_t + (1 - e^{-t})\sqrt{-1} \bar{\partial} \bar{\partial} f,
\]

and clearly \( \tilde{\omega}_t = \tilde{\omega}_t + \sqrt{-1} \bar{\partial} \bar{\partial}(u - (1 - e^{-t})f) \). Define \( w = u - (1 - e^{-t})f \) and we have \( \tilde{\omega}_t = \tilde{\omega}_t + \sqrt{-1} \bar{\partial} \bar{\partial}w \). Clearly \( \frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} - e^{-t}f \) and taking \( t \)-derivative gives

\[
\frac{\partial}{\partial t} (\frac{\partial w}{\partial t}) = \Delta (\frac{\partial w}{\partial t}) - e^{-t} (\tilde{\omega}_t, \omega_0 - \omega_\infty - \sqrt{-1} \bar{\partial} \bar{\partial} f) + e^{-t} f.
\]
Just as before, we need the following transformation of the above equation
\[
\frac{\partial}{\partial t} \left( \frac{\partial w}{\partial t} + \bar{w} \right) = \Delta_{\bar{\omega}} \left( \frac{\partial w}{\partial t} + \bar{w} \right) - n + \langle \bar{\omega}_t, \omega_\infty + \sqrt{-1} \partial \bar{\omega} \rangle + \frac{\partial \bar{w}}{\partial t} + e^{-t} f,
\]
where \( \bar{w} \) is the normalization of \( w \) just as \( v \) for \( u \) before.

We can now apply maximum principle for the above equation at the (local in time) minimum value point of \( \frac{\partial w}{\partial t} + \bar{w} \). At that point (if not at time 0), we have
\[
n - \frac{\partial \bar{w}}{\partial t} \geq e^{-t} f + \langle \bar{\omega}_t, \omega_\infty + \sqrt{-1} \partial \bar{\omega} \rangle.
\]
Without loss of generality, we can make sure \( f > 0 \). So now one arrives at
\[
n - \frac{\partial \bar{w}}{\partial t} \geq \langle \bar{\omega}_t, \omega_\infty + \sqrt{-1} \partial \bar{\omega} \rangle \geq n \cdot \left( \frac{(\omega_\infty + \sqrt{-1} \partial \bar{\omega})^n}{\bar{\omega}_t^n} \right)^{\frac{1}{n}} = n \cdot \left( \frac{(\omega_\infty + \sqrt{-1} \partial \bar{\omega})^n}{e^{\frac{\partial u}{\partial t}} \Omega} \right)^{\frac{1}{n}} > 0,
\]
which gives \( (1 - \frac{1}{n} \frac{\partial u}{\partial t})^n \cdot e^{\frac{\partial u}{\partial t}} \geq C > 0 \). We also have \( \frac{\partial u}{\partial t} \geq \frac{\partial w}{\partial t} - C \geq \frac{\partial u}{\partial t} - C \), so we can have
\[
(C - \frac{\partial u}{\partial t})^n \cdot e^{\frac{\partial u}{\partial t}} \geq C > 0
\]
with \( C - \frac{\partial u}{\partial t} > 0 \), and still conclude that \( \frac{\partial u}{\partial t} \geq -C \) at that point, and so \( \frac{\partial w}{\partial t} \geq -C \).

It’s rather clear that \( |\bar{w}| \leq C \) from the estimates for \( v \) before since we do not need \( \omega_\infty > 0 \) there yet. Hence we see \( \frac{\partial w}{\partial t} + \bar{w} \geq -C \) globally, which gives the uniform lower bound for \( \frac{\partial w}{\partial t} \) and so for \( \frac{\partial u}{\partial t} \) (as they differ only by a bounded term \( e^{-t} f \)).

The argument for uniform higher derivatives is as standard as before.

**Remark 3.2.** The main philosophy of the above argument is that a choice of representative in a class boils down to terms like \( f \) or \( e^{-t} f \) for smooth function \( f \) over \( X \) which is clearly controlled along the flow and so should not bring any trouble. This observation is also useful when trying to apply Yau’s Laplacian estimate (as in [1]) to get second order derivative control for the current situation.

Up to now, we have got the global existence of the flow and the uniformity of the estimates allows us to use the classic Ascoli-Azela’s Theorem to get convergence for sequences of metrics along the flow. Just as in Cao’s work, we should head for stronger convergence as discussed in next subsection.

### 3.2 Convergence

H. D. Cao’s argument in [1] for convergence using Li-Yau’s Harnack Inequality should be easy to get carried through here as for the equation:
\[
\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} \right) = \Delta_{\bar{\omega}} \left( \frac{\partial u}{\partial t} \right) - e^{-t} \langle \bar{\omega}_t, \omega_0 - \omega_\infty \rangle
\]
since $⟨\tilde{\omega}_t, \omega_0 - \omega_\infty⟩$ has been uniformly controlled, and so the extra term in comparison to Cao’s situation is exponentially decreasing. Let’s illustrate some main points when adjusting his argument to the current situation in the following.

For the exponential decreasing of the oscillation of $\frac{\partial u}{\partial t}$, we’ll use Cao’s argument for the following family of auxiliary functions:

$$(\frac{\partial}{\partial t} - \Delta)\varphi_{T_0} = 0, \quad \varphi_{T_0}(T_0, \cdot) = \frac{\partial u}{\partial t}(T_0, \cdot)$$

over $[T_0, \infty) \times X$ where $T_0 \in [0, \infty)$. As we have already got the uniform estimates for $\frac{\partial u}{\partial t}$ and $\tilde{\omega}_t$, using Li-Yau’s Harnack Inequality as Cao did, we have

$\text{osc}_X \varphi_{T_0}(t) \leq C e^{-a(t-T_0)}, \quad t \in [T_0, \infty)$

where the positive constants are uniform for all $T$ as the metric control is uniform for all time.

Using the uniform estimates along the flow as mentioned before, we have

$$(\frac{\partial}{\partial t} - \Delta) (\frac{\partial u}{\partial t} + C e^{-t}) \leq 0,$$

$$(\frac{\partial}{\partial t} - \Delta) (\frac{\partial u}{\partial t} - C e^{-t}) \geq 0.$$

We also have the following equations

$$(\frac{\partial}{\partial t} - \Delta) (\varphi_{T_0 - T_0} + C e^{-T_0}) = 0,$$

$$(\frac{\partial}{\partial t} - \Delta) (\varphi_{T_0} - C e^{-T_0}) = 0.$$

Comparing them and applying maximum principle, we get the decreasing of

$$\max_X (\frac{\partial u}{\partial t} + C e^{-t} - \varphi_{T_0} - C e^{-T_0})$$

and the increasing of

$$\min_X (\frac{\partial u}{\partial t} - C e^{-t} - \varphi_{T_0} + C e^{-T_0})$$

as time increases (starting from the time $T_0$).

The values at $t = T_0$ for both quantities are 0, so we have for $t \in [T_0, \infty)$,

$$\frac{\partial u}{\partial t} \leq \varphi_{T_0} + C e^{-T_0} - C e^{-t},$$

$$\frac{\partial u}{\partial t} \geq \varphi_{T_0} - C e^{-T_0} + C e^{-t}.$$
Hence $\text{osc}_X \frac{\partial u}{\partial t} \leq \text{osc}_X \phi_{T_0} + Ce^{-T_0}$ for $t \in [T_0, \infty)$. Using the result for $\phi_{T_0}$ stated above, we have $\text{osc}_X \frac{\partial u}{\partial t} \leq Ce^{-a(t-T_0)} + Ce^{-T_0}$ for $t \geq T_0$. Taking $t = 2T_0$ and noticing this is uniform for all $T_0$, we finally arrive at

$$\text{osc}_X \frac{\partial u}{\partial t} \leq Ce^{-at}$$

for all time. Here the $a$ should differ from the previous one, but it’s still a positive constant.

This is exactly one of the essential results needed to draw the convergence for $t \to \infty$ as in Cao’s.

Set $\psi = \frac{\partial u}{\partial t} - \int_X \frac{\partial}{\partial t} \omega^n$. Clearly its difference from $\frac{\partial u}{\partial t}$ is controlled by $Ce^{-at}$, but it is more convenient for the following consideration.

We can have similar computation as in [1], for the energy,

$$E = \int_X \psi^2 \omega^n,$$

to derive a differential inequality for it. There are more terms coming out than Cao’s case, but they will all be terms controlled by $Ce^{-t}$ using the uniform estimates along the flow. Notice that though the volume is also changing along the flow, the variation is also well under control. In all, we get

$$\frac{dE}{dt} \leq -CE + Ce^{-t}$$

for large $t$. The reason to get only for large $t$ is that we need the smallness of $\psi$ from the control of oscillator of $\frac{\partial u}{\partial t}$. From this differential inequality, we can still conclude the exponential decaying of $E$.  

The final computation and argument of Cao to derive the $L^1$ convergence of the normalized metric potential can be carried through line by line in sight of the above results. Indeed, we can also justify the exponential convergence of the flow with little extra effort (just as what is in [15]).

**Remark 3.3.** In this situation, we now have a somewhat natural flow from one Ricci-flat metric to another Ricci-flat metric (in different Kähler classes of course) when $c_1(X) = 0$. Just need to choose $\Omega$ such that $\text{Ric}(\Omega) = 0$ for the flow.

### 4 Main Interest: Degenerate Case

Of course, our main interest is when $[\omega_\infty]$ is degenerate as Kähler class. In [15], we have discussed the corresponding Monge-Ampère equation using other

---

2 In fact, the exponential decaying of $E$ can be deduced from the decaying of the oscillation of $\frac{\partial u}{\partial t}$ in a more direct manner. But Cao’s method above applying the differential inequality is more delicate and can easily be adjusted for higher order Sobolev estimates.
perturbation for method of continuity. Now we want to see whether the modified flow can help us to construct a solution for the Monge-Ampère equation (as the limiting equation). At this moment, our manifold $X$ is assumed to be projective to get into algebraic geometry context for the notions of semi-ample and big.

As discussed before, we have the existence of the smooth flow as long as $[\omega_t]$ remains Kähler. There are two cases, i.e., up to infinite time and up to finite time. We discuss them separately and finish the proof of the theorem.

4.1 Infinite Time Case

Let’s assume here that $[\omega_{\infty}]$ is semi-ample and big. We still have the $L^\infty$ bound of the normalized metric potential $v$ as before using the result on degenerate Monge-Ampère equation (from [2] and [14]). Now (2.1) can be modified as:

$$\frac{\partial}{\partial t} (\frac{\partial u}{\partial t} + v - \epsilon \log |\sigma|^2) = \Delta_{\omega_t} \left( \frac{\partial u}{\partial t} + v - \epsilon \log |\sigma|^2 \right) - n + \frac{\partial v}{\partial t} + \langle \omega_t, \omega_{\infty} + \epsilon \sqrt{-1} \partial \bar{\partial} \log |\sigma|^2 \rangle$$

with $\omega_{\infty} + \epsilon \sqrt{-1} \partial \bar{\partial} \log |\sigma|^2 > 0$, where the positive $\epsilon$ can be as close to 0 as possible. The introduction of a singular term like this, as far as I know, was initiated by Tsuji in [11], which gives a natural and simple description of an algebraic geometry fact in analysis of the related PDE’s.

The following classic results from algebraic geometry are useful for us. See in [5] and [6] for related discussion. The second one is called Kodaira’s Lemma as in [12] and will be applied for the finite time case later discussed. The point of view for translating these results to the analytic statement as above is very standard as described in [3].

**Lemma 4.1.** Let $L$ be a divisor in a projective manifold $X$. If $L$ is nef and big, then there is an effective divisor $E$ and a number $a > 0$ such that $L - \epsilon E$ is Kähler for any $\epsilon \in (0, a)$.

**Lemma 4.2.** Let $L$ be a divisor in a projective manifold $X$. If $L$ is big, then there is an effective divisor $E$ such that $L - \epsilon E$ is Kähler for $\epsilon \in (a, b)$ where $0 \leq a < b < \infty$.

Similar argument as before would give a degenerate lower bound\(^3\) as

$$\frac{\partial u}{\partial t} \geq -C + \epsilon \log |\sigma|^2.$$  

Basically, we still have $\frac{\partial u}{\partial t} \geq \frac{\partial u}{\partial t} - C$. Then considering the minimum value point of the term naturally considered by the equation above, we know $\frac{\partial u}{\partial t}$ could not be too small at that point using the contradiction as for the baby version, which would essentially give the bound claimed above.

---

\(^3\)The positive constant $C$ below might depend on the other positive constant $\epsilon$. Hopefully, this won’t bring any confusion.
Now the degenerate second order estimate and high ones would still be OK by the standard procedure. More specifically, for the second order estimate, one considers the following equation

$$(\tilde{\omega}_t + \epsilon \sqrt{-1} \partial \bar{\partial} \log |\sigma|^2 + \sqrt{-1} \partial \bar{\partial} (v - \epsilon \log |\sigma|^2))^n = e^{\hat{\Omega}}.$$

Applying Yau’s computation in [13] and using degenerate maximum principle argument as in [9], we can get the degenerate Laplacian bound. Combining with the degenerate control for volume, we have achieved local (or degenerate) bound for metrics along the flow.

The treatment for higher derivatives would be standard. We provide some details at the last section.

**Remark 4.3.** There is a big difference from the situation in [12] which we want to point out. The metric potential along the flow can be bounded (though in a degenerate way) simply from the flow argument, but we can not do that here at this moment. The bound for (normalized) metric potential is coming from results proved by arguments in pluripotential theory. That’s why we need semi-ample (not just nef.) here.

Though our estimates are uniform for all time now, which gives sequence convergence for the flow, there is still this big issue about convergence along the flow which is crucial to describe the limit itself. As discussed in the baby version, the counterpart in [11] makes use of Li-Yau’s Harnack Inequality, which can be applied for the non-degenerate case as before. But the situation right now is very different. It seems to me that new method needs to be introduced for this purpose. Let’s make the conjecture about the flow convergence.

**Conjecture 4.4.** For $[\omega_\infty]$ semi-ample and big, as $t \to \infty$, this modified Kähler-Ricci flow converges weakly over $X$ and locally smoothly out of the stable base locus set of this cohomology class to the unique (bounded) solution of the limiting degenerate Monge-Ampère equation.

We can prove that for infinite time case, the volume form has uniform lower bound for all time as stated in Theorem 1.2. This might help to get the convergence of the flow and is also a nice application of a similar result for the following more canonical Kähler-Ricci flow.

Set $\hat{\omega}_t = \omega_t + \sqrt{-1} \partial \bar{\partial} \phi$. In the level of potential, consider the flow

$$\frac{\partial \phi}{\partial t} = \log \hat{\omega}_t^n \Omega - \phi, \quad \phi(0, \cdot) = 0.$$

The corresponding flow in the level of metric is the following,

$$\frac{\partial \hat{\omega}_t}{\partial t} = -\text{Ric} (\hat{\omega}_t) + \text{Ric} (\Omega) - \hat{\omega}_t + \omega_\infty, \quad \hat{\omega}_0 = \omega_0.$$

\footnote{In fact, one only needs the uniform upper bound of $\frac{\partial \phi}{\partial t}$ to carry through the Laplacian estimate by noticing the dominance of $e^{-\frac{1}{\epsilon} \hat{\Omega}}$ over $-\frac{\partial \phi}{\partial t}$ when $\frac{\partial \phi}{\partial t}$ is small.}
In the case, we have $\omega_\infty$ is big and semi-ample, so as discussed in [9] and [15], the following controls are available:

$$|\phi| \leq C, \quad \frac{\partial \phi}{\partial t} \leq C,$$

which give a lower bound for the volume form $\tilde{\omega}$ for all time and we are looking for a similar thing for $\tilde{\omega}_n$.

**Remark 4.5.** The uniform volume lower bound is a pretty interesting fact as the class $[\omega_\infty]$ is not Kähler, but somehow we have that $[\omega_\infty]^n > 0$ also makes sense in a pointwise fashion.

Let’s recall the following equations used before for the flow considered in this note. $v$ is the normalization of $u$ as before.

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = \Delta(\frac{\partial u}{\partial t}) - \langle \tilde{\omega}_t, e^{-t}(\omega_0 - \omega_\infty) \rangle,$$

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = \Delta(\frac{\partial u}{\partial t} + v) - n + \langle \tilde{\omega}_t, \omega_\infty \rangle.$$

Fix some constant $T_1 > 0$, product the first equation with $e^{-T_1}$ and taking the difference of them, we have

$$\frac{\partial}{\partial t}((1 - e^{-T_1}) \frac{\partial u}{\partial t}) = \Delta((1 - e^{-T_1}) \frac{\partial u}{\partial t} + v) - n + \langle \tilde{\omega}_t, \omega_{t+T_1} \rangle.$$

Now using the solution for the other flow, $\phi$, this equation can be transformed as follows

$$\frac{\partial}{\partial t}((1 - e^{-T_1}) \frac{\partial u}{\partial t} + v - \phi(t + T_1)) = \Delta((1 - e^{-T_1}) \frac{\partial u}{\partial t} + v - \phi(t + T_1)) - n + \frac{\partial v}{\partial t} + \frac{\partial \phi(t + T_1)}{\partial t} + \langle \tilde{\omega}_t, \omega_{t+T_1} \rangle.$$

Let $A = (1 - e^{-T_1}) \frac{\partial u}{\partial t} + v - \phi(t + T_1)$ and using the following known estimates

$$\frac{\partial v}{\partial t} \geq \frac{\partial u}{\partial t} - C, \quad \frac{\partial \phi(t + T_1)}{\partial t} \geq -C, \quad \tilde{\omega}_t^n \geq C \Omega,$$

one arrives at

$$\frac{\partial A}{\partial t} \geq \Delta A + \frac{\partial u}{\partial t} - C + C \cdot e^{-\frac{t}{10}}.$$

Use similar maximum principle argument as before, one can conclude the lower bound for $A$, and so for $\frac{\partial u}{\partial t}$, which gives the lower bound for the volume form $\tilde{\omega}_n^n$.

---

5 The lower bound of $\frac{\partial \phi}{\partial t}$ is in [15]. Basically one makes use of the essential decreasing of the volume form and the fact that for infinite time limit, the derivative of potential has to go to 0 (in the regular part).
Remark 4.6. The translation of time by $T_1$ makes the infinite time situation special. In comparison, we do not have uniform volume lower bound for finite time case for both flows (at least at this moment).

4.2 Finite Time Limit

Now we consider the case when $[\omega_\infty]$ is only big. More specifically, the flow only exists up to some finite time $T$. We also require $[\omega_T]$, which is clearly nef. and big, to be semi-ample.\footnote{This is not such a horrible assumption as it is the case, when $[\omega_\infty] = K_X$ and $[\omega_0]$ is rational, from algebraic geometry results.}

Let's first consider the situation roughly. Those degenerate estimates would still be available, though the $\epsilon$ can’t be too small now in sight of Lemma 4.2. The advantage about finite time is that the metric potential $u$ is (degenerately) bounded by itself (without normalization) using the bound for its time derivative, and it’ll also be decreasing after controllable normalization (by $-Ct$) in sight of the uniform upper bound for $\frac{\partial u}{\partial t}$. So as in [9], the (local) convergence for $t \to T$ is achieved.

This local convergence would be out of the ”stable base locus set” of $[\omega_\infty]$. Clearly, it would be more satisfying to get this with respect to $[\omega_T]$. We can do this in the same way in which we can also improve the result in [9]. Simply speaking, we can use a virtual time. Let’s get the crucial estimate for $\frac{\partial u}{\partial t}$ below.

We can easily have the following two equations.

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial t} + v) = \Delta_z(\frac{\partial u}{\partial t} + v) - n + \frac{\partial v}{\partial t} + \langle \tilde{\omega}_t, \omega_\infty \rangle,$$

$$\frac{\partial}{\partial t}(e^{-T}\frac{\partial u}{\partial t} + v) = \Delta_{\tilde{w}_t}(e^{-T}\frac{\partial u}{\partial t} + v) + e^{-T}\frac{\partial u}{\partial t} - e^{-T}\langle \tilde{\omega}_t, \omega_0 - \omega_\infty \rangle.$$

Take difference to get

$$\frac{\partial}{\partial t}((1 - e^{-T})\frac{\partial u}{\partial t} + v) = \Delta_{\tilde{w}_t}((1 - e^{-T})\frac{\partial u}{\partial t} + v) - n + \frac{\partial v}{\partial t} - e^{-T}\frac{\partial u}{\partial t} + \langle \tilde{\omega}_t, \omega_T \rangle.$$

As before, take $\sigma$ for $[\omega_T]$ such that $\omega_T + \epsilon \sqrt{-1}\partial \bar{\partial} \log|\sigma|^2 > 0$ for any positive $\epsilon$ small enough. Using it to perturb the above equation, one arrives at

$$\frac{\partial}{\partial t}((1 - e^{-T})\frac{\partial u}{\partial t} + v - \epsilon \log|\sigma|^2)$$

$$= \Delta_{\tilde{w}_t}((1 - e^{-T})\frac{\partial u}{\partial t} + v - \epsilon \log|\sigma|^2) - n + \frac{\partial v}{\partial t} - e^{-T}\frac{\partial u}{\partial t} + \langle \tilde{\omega}_t, \omega_T + \epsilon \sqrt{-1}\partial \bar{\partial} \log|\sigma|^2 \rangle.$$

Set $A = \frac{1}{1 - e^{-T}}\frac{\partial u}{\partial t} + v - \epsilon \log|\sigma|^2$. We can have, out of $\{\sigma = 0\}$,

$$\frac{\partial A}{\partial t} \geq \Delta_{\tilde{w}_t} A - C + (1 - e^{-T})\frac{\partial u}{\partial t} + C e^{-\frac{1}{1 - e^{-T}}}.$$

Then let’s do the maximum principle argument. Recall that the time $t \in [0, T)$. At the minimum value point of $A$ (assuming it is not at the initial time), which
is clearly out of \( \{ \sigma = 0 \} \), we can see \( \frac{\partial u}{\partial t} \) can not be too small (negative). Thus \( A \) can not be too small there, either. That gives

\[
(1 - e^{t-T}) \frac{\partial u}{\partial t} + v - \epsilon \log |\sigma|^2 \geq -C.
\]

The problem coming from the fact that \( 1 - e^{t-T} \) would go to 0 as \( t \to T \) can be solved by using a "virtual" time \( T_\epsilon > T \) which satisfies \( \omega_{T_\epsilon} + \epsilon \sqrt{-1} \partial \bar{\partial} \log |\sigma|^2 > 0 \) for some fixed \( \epsilon > 0 \). Then the same estimate

\[
(1 - e^{t-T_\epsilon}) \frac{\partial u}{\partial t} + v - \epsilon \log |\sigma|^2 \geq -C
\]

which gives

\[
\frac{\partial u}{\partial t} \geq -C_\epsilon + C_\epsilon \log |\sigma|^2.
\]

Notice that now the \( \sigma \) is for the class \([\omega_T]\) and so we can conclude the local convergence out of the stable base locus set of \([\omega_T]\).

One might also want to do the maximum principle argument in another flavor just as what is done for the baby version. We have to do it more carefully as follows.

The differential inequality for \( A \) is

\[
\frac{\partial A}{\partial t} \geq \Delta \sigma, A - C + (1 - e^{t-T}) \frac{\partial u}{\partial t} + C e^{-\frac{1}{n} \partial \sigma}.
\]

One wants to change the last two terms to functions on \( A \) with the right direction of control.

The last term can be treated with ease as \( e^{t-T} \frac{\partial u}{\partial t} - v + \epsilon \log |\sigma|^2 \leq C \), but it won’t be so easy for the other term as \( -\epsilon \log |\sigma|^2 \) can not be bound from above over \( X \) by any constant. In fact, the trick is to treat them together.

Set \( B = (1 - e^{t-T}) \frac{\partial u}{\partial t} + v \) and we have

\[
(1 - e^{t-T}) \frac{\partial u}{\partial t} + Ce^{-\frac{1}{n} \partial \sigma} \geq -C + B + Ce^{-\frac{B}{n}}.
\]

The function (over \( B \)), \( B + Ce^{-B} \) would be decreasing with respect to \( B \) for small enough \( B \) by derivative consideration. And so for \( B \) small enough (i.e., \( (1 - e^{t-T}) \frac{\partial u}{\partial t} \) small enough), we can change \( B \) to \( A = B - \epsilon \log |\sigma|^2 \) and that should do it.

The proof of Theorem 1.2 is finished.

5 Higher Order Estimates

We provide a short discussion on the degenerate third and higher estimates for Kähler-Ricci flow over a closed (algebraic) manifold, \( X \). It works for the
modified flow here and others (as the one in [9] appearing in Subsection 4.2) with \([\omega_\infty]\) being big.

The flow equation on the potential level is

\[
\frac{\partial u}{\partial t} = \log \left( \frac{\omega_t + \sqrt{-1}\partial\bar{\partial}u}{\Omega} \right) - u, \quad u(0, \cdot) = 0,
\]

or without the \(-u\) term on the right hand side, where \(\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)\) with \(\omega_t\) being the initial Kähler metric, \(\omega_\infty\) being a smooth representative for the (formal) infinite limiting class and \(\Omega\) being a smooth volume for over . \(\tilde{\omega}_t = \tilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}u\) is the metric solution of the flow.

The class \([\omega_\infty]\) is big and \(T \leq \infty\) is the singular time from cohomology concern with \([\omega_T]\) being nef. and big. The following estimates are available:

\[
\log|\sigma|^2 - C \leq u \leq C, \quad \log|\sigma|^2 - C \leq \frac{\partial u}{\partial t} \leq C, \quad \langle \omega_0, \tilde{\omega}_t \rangle \leq C|\sigma|^{-t},
\]

where \(E = \{\sigma = 0\}\) is a proper chosen divisor such that \([\omega_T] - \epsilon E\) is Kähler. \(\sigma\) is a holomorphic section of the line bundle, and so with a fixed hermitian metric, \(|\sigma|^2\) is a smooth function valued in \([0, C]\).

The higher estimates are discussed briefly to achieve the full local regularity. Here we would like to go for the third order estimate a little more carefully. Then the rest follows from parabolic version of Schauder estimates in a standard way. Yau’s computation in [13] is what we need.

As in Yau’s computation, the term \(S = \tilde{g}^{ij}\tilde{g}^{kl}\tilde{g}^{mn}u_{im}u_{jn}\) is considered, where the covariant derivative is with respect to uniform “background” metric.

If the flow metric control is uniform, then the parabolic version of Yau’s computation is

\[
(\Delta_{\tilde{\omega}_t} - \frac{\partial}{\partial t})S \geq -C \cdot S - C.
\]

To adjust the result, one only need to see the metric is controlled (uniformly in time) as follows

\[
|\sigma|^\beta \omega_0 \leq \tilde{\omega}_t \leq |\sigma|^{-\beta} \omega_0
\]

for large positive constant \(\beta\).

Then we know by very carefully going through Yau’s computation that

\[
|\sigma|^{2N}(\Delta_{\tilde{\omega}_t} - \frac{\partial}{\partial t})S \geq -C|\sigma|^{2N-\beta} \cdot S - C
\]

with \(N\) chosen large enough to dominate all degenerate terms.

Of course, now we want to see how \(|\sigma|^{2N}S\) is acted by the heat operator. The only additional part is from the action of \(\Delta_{\tilde{\omega}_t}\). There are two terms. One is clearly \(2\text{Re}(\nabla|\sigma|^{2N}, \nabla S)\). The other one is \(\Delta_{\tilde{\omega}_t}|\sigma|^{2N} \cdot S\).

For the first one, \(\nabla S = \nabla(|\sigma|^{2N}S|\sigma|^{-2N}) = |\sigma|^{-2N} \nabla(|\sigma|^{2N}S) - N|\sigma|^{-2S\nabla|\sigma|^2}.\)
For the second one,

\[
\Delta_{\hat{\omega}_i}|\sigma|^{2N} = \Delta_{\hat{\omega}_i}(e^{N\log|\sigma|^2}) = \langle |\sigma|^{2N} N(\log|\sigma|^2) \rangle_{\hat{\omega}_i},
\]

\[
= N^2|\sigma|^{2N} |\nabla \log|\sigma|^2|^2 + N|\sigma|^{2N} \langle \hat{\omega}_1, \sqrt{-1} \bar{T} \delta \log|\sigma|^2 \rangle
\]

\[
\geq -N|\sigma|^{2N} \langle \hat{\omega}_i, -\sqrt{-1} \bar{T} \delta \log|\sigma|^2 \rangle
\]

Out of \( E = \{ \sigma = 0 \} \), \(-\sqrt{-1} \bar{T} \delta \log|\sigma|^2 \) is nothing but the curvature form of the corresponding line bundle, still denoted by \( E \). Using the degenerate metric bound, one has

\[
\Delta_{\hat{\omega}_i}|\sigma|^{2N} \geq -N|\sigma|^{2N} \langle \hat{\omega}_i, E \rangle \geq -C|\sigma|^{2N-\beta},
\]

**Remark 5.1.** If we are in semi-ample case, with proper choice of the hermitian metric for the bundle \( \langle \cdot \rangle \) above, we can make sure that \( \Phi - \epsilon E > 0 \) (since the corresponding cohomology class is Kähler) where \( \Phi \) is the pullback of a Kähler metric from the image of the map which is constructed from the semi-ample class \( \omega_T \). In this case, we also have better zero order bounds. Moreover, using Schwarz type of estimates as in [16], we can have

\[
\Delta_{\hat{\omega}_i}|\sigma|^{2N} \geq -N|\sigma|^{2N} \langle \hat{\omega}_i, E \rangle \geq -C|\sigma|^{2N},
\]

which is better here but not going to make too much difference as one continues.

Anyway, we arrive at

\[
(\Delta_{\hat{\omega}_i} - \frac{\partial}{\partial t})(|\sigma|^{2N} S) \geq -C|\sigma|^{2N-\beta} \cdot S - C
\]

\[
+ 2 \text{Re}(\nabla|\sigma|^{2N}, |\sigma|^{-2N} \nabla(|\sigma|^{2N} S) - N|\sigma|^{-2} S|\nabla|\sigma|^2|_{\hat{\omega}_i})
\]

\[
= -C|\sigma|^{2N-\beta} S - C + 2 \text{Re}(\nabla(\log|\sigma|^{2N}), \nabla(|\sigma|^{2N} S))_{\hat{\omega}_i}
\]

\[
- N^2|\sigma|^{2N-4} S|\nabla|\sigma|^2|^2.
\]

\[
\geq -C|\sigma|^{2N-\beta} S - C + 2 \text{Re}(\nabla(\log|\sigma|^{2N}), \nabla(|\sigma|^{2N} S))_{\hat{\omega}_i},
\]

where we use \( |\nabla|\sigma|^2|^2 \leq C|\sigma|^{2-\beta} \) for the last step.

Also as in [13], we consider the \( \langle \omega_t, \hat{\omega}_i \rangle \) acted by the heat operator where \( \omega_t, \epsilon \) is the perturbation for the "background" form.

Had the metric control been uniform, one has

\[
(\Delta_{\hat{\omega}_i} - \frac{\partial}{\partial t})(\langle \omega_t, \hat{\omega}_i \rangle) \geq C \cdot S - C.
\]

For our case, similar to \( S \), we have instead

\[
|\sigma|^{2N} (\Delta_{\hat{\omega}_i} - \frac{\partial}{\partial t})(\langle \omega_t, \hat{\omega}_i \rangle) \geq C|\sigma|^{2N+\beta} S - C.
\]

\( ^7 \) The outside \(| \cdot |\) is \( \hat{\omega}_i \).
The exact same procedure as done for $S$ above gives us

$$(\Delta \omega_t - \frac{\partial}{\partial t})[|\sigma|^{2N} (\omega_{t, \epsilon}, \tilde{\omega}_t)]$$

$$\geq C|\sigma|^{2N+\beta} \cdot S - C - C|\sigma|^{2N-\beta} (\omega_{t, \epsilon}, \tilde{\omega}_t)$$

$$+ 2\text{Re}(\nabla|\sigma|^{2N}, |\sigma|^{-2N} \nabla(|\sigma|^{2N} (\omega_{t, \epsilon}, \tilde{\omega}_t)) - N|\sigma|^{-2} (\omega_{t, \epsilon}, \tilde{\omega}_t) \nabla|\sigma|^2)\omega_t$$

$$\geq C|\sigma|^{2N+\beta} S - C + 2\text{Re}(\nabla(\log|\sigma|^{2N}), \nabla(|\sigma|^{2N} (\omega_{t, \epsilon}, \tilde{\omega}_t)))\omega_t$$

$$- C|\sigma|^{2N-2-\beta} (\omega_{t, \epsilon}, \tilde{\omega}_t).$$

Properly choosing large constants $N_1 > N_2 > 0$ and $C$’s, we have

$$(\Delta \omega_t - \frac{\partial}{\partial t})[|\sigma|^{2N_1} + C|\sigma|^{2N_2} (\omega_{t, \epsilon}, \tilde{\omega}_t)]$$

$$\geq C|\sigma|^{2N_2+\beta} \cdot S - C - C|\sigma|^{2N_2-2-\beta} (\omega_{t, \epsilon}, \tilde{\omega}_t)$$

$$+ 2\text{Re}(\nabla(\log|\sigma|^{2N_1}), \nabla(|\sigma|^{2N_1} S))\omega_t + 2\text{Re}(\nabla(\log|\sigma|^{2N_2}), \nabla(C|\sigma|^{2N_2} (\omega_{t, \epsilon}, \tilde{\omega}_t)))\omega_t.$$

For $N_1$ and $N_2$, only need $2N_1 - 2 - \beta \geq 2N_2 + \beta$ at this moment. But they will be fixed later and large.

Now apply maximum principle argument. At the (local in time) maximum point, for $|\sigma|^{2N_1} S + C|\sigma|^{2N_2} (\omega_{t, \epsilon}, \tilde{\omega}_t)$, which clearly exists out of $\{\sigma = 0\}$ and assume is not at the initial time, one has $\nabla(|\sigma|^{2N_1} S) = -\nabla(C|\sigma|^{2N_2} (\omega_{t, \epsilon}, \tilde{\omega}_t))$ and

$$0 \geq C|\sigma|^{2N_2+\beta} \cdot S - C - C|\sigma|^{2N_2-2-\beta} (\omega_{t, \epsilon}, \tilde{\omega}_t)$$

$$+ 2\text{Re}(\nabla(\log|\sigma|^{2N_1}), \nabla(|\sigma|^{2N_1} S))\omega_t + 2\text{Re}(\nabla(\log|\sigma|^{2N_2}), \nabla(C|\sigma|^{2N_2} (\omega_{t, \epsilon}, \tilde{\omega}_t)))\omega_t$$

$$= C|\sigma|^{2N_2+\beta} \cdot S - C - C|\sigma|^{2N_2-2-\beta} (\omega_{t, \epsilon}, \tilde{\omega}_t)$$

$$+ 2\text{Re}(\nabla(\log|\sigma|^{2N_2}), \nabla(C|\sigma|^{2N_2} (\omega_{t, \epsilon}, \tilde{\omega}_t)))\omega_t$$

$$\geq C|\sigma|^{2N_2+\beta} \cdot S - C - C(|\nabla(\log|\sigma|^2), \nabla(|\sigma|^{2N_2} (\omega_{t, \epsilon}, \tilde{\omega}_t)))\omega_t.$$

For the last term, we have

$$|\nabla(\log|\sigma|^2), \nabla(|\sigma|^{2N_2} (\omega_{t, \epsilon}, \tilde{\omega}_t))|\omega_t|$$

$$= |(|\sigma|^{-2}\nabla|\sigma|^2, N_2|\sigma|^{2N_2-2}\nabla|\sigma|^2 (\omega_{t, \epsilon}, \tilde{\omega}_t) + |\sigma|^{2N_2} \nabla(\omega_{t, \epsilon}, \tilde{\omega}_t))|\omega_t|$$

$$\leq |(|\sigma|^{-2}\nabla|\sigma|^2, N_2|\sigma|^{2N_2-2}\nabla|\sigma|^2 (\omega_{t, \epsilon}, \tilde{\omega}_t))|\omega_t| + |(|\sigma|^{-2}\nabla|\sigma|^2, |\sigma|^{2N_2} \nabla(\omega_{t, \epsilon}, \tilde{\omega}_t))|\omega_t|$$

$$\leq C|\sigma|^{2N_2-2-2+1+1-\beta-\beta} + |\sigma|^{2N_2-2}|\nabla|\sigma|^2| \cdot |\nabla(\omega_{t, \epsilon}, \tilde{\omega}_t)|$$

$$\leq C|\sigma|^{2N_2-2-2-\beta} + C|\sigma|^{-2N_2-2+1-\frac{\beta}{2}} \cdot |\nabla(\omega_{t, \epsilon}, \tilde{\omega}_t)|$$

Now one needs to realize that

$$|\nabla(\omega_{t, \epsilon}, \tilde{\omega}_t)| = |\nabla(F + \Delta_{\omega_{t, \epsilon}} u)| \leq |\nabla F| + |\nabla \Delta_{\omega_{t, \epsilon}} (u)| \leq |\sigma|^{-\frac{\beta}{2}} + C|\sigma|^{-2\beta} S^{\frac{1}{2}}$$

with $F$ being a well controlled function.
Combining all this, we have at that maximum point,

\[ 0 \geq C|\sigma^{2N_2+\beta} \cdot S - C - C|\sigma^{2N_2-2-2\beta} - C|\sigma^{2N_2-1-\beta} - C|\sigma^{2N_2-1-2\beta} \cdot S^\frac{1}{2}. \]

For large enough \( N_2 \), we have,

\[ 0 \geq |\sigma^{2N_2+\beta} \cdot S - C(|\sigma^{2N_2+\beta} \cdot S)|^\frac{1}{2} - C, \]

and so \( |\sigma|^{2N_2+\beta} \cdot S \leq C \). For \( N_1 \) even larger, we have uniform upper bound for \( |\sigma|^{2N_1} S + C|\sigma|^{2N_2} \langle \omega_{t,\epsilon}, \tilde{\omega}_t \rangle \) at that point and so it is true globally which provides the bound

\[ S \leq C|\sigma|^{-2N_1}. \]

This gives local \( C^{2,\alpha} \) bound for the metric along the flow, then parabolic version of Schauder estimates carry though to provide all the local higher order bounds.

**References**

[1] Cao, Huaidong: Deformation of Kaehler metrics to Kaehler-Einstein metrics on compact Kaehler manifolds. Invent. Math. 81(1985), no. 2, 359–372.

[2] Philippe Eyssidieux; Vincent Guedj; Ahmed Zeriahi: Singular Kähler-Einstein metrics. ArXiv, math/0603431.

[3] Griffiths, Phillip; Harris, Joseph: Principles of algebraic geometry. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978. xii+813pp.

[4] Hamilton, Richard S.: Three-manifolds with positive Ricci curvature. J. Differential Geom. 17 (1982), no. 2, 255-306.

[5] Kawamata, Yujiro: The cone of curves of algebraic varieties. Ann. of Math. (2) 119 (1984), no.3, 603–633.

[6] Kawamata, Yujiro: A generalization of Kodaira-Ramanujam’s vanishing theorem. Math. Ann. 261 (1982), no. 1, 43–46.

[7] Kolodziej, Slawomir: The complex Monge-Ampère equation and pluripotential theory. Mem. Amer. Math. Soc. 178 (2005), no. 840, x+64 pp.

[8] Kolodziej, Slawomir: Hölder continuity of solutions to the complex Monge-Ampère equation with the right hand side in \( L^p \). Preprint.

[9] Tian, Gang; Zhang, Zhou: On the Kähler-Ricci flow on projective manifolds of general type. Chinese Annals of Mathematics - Series B, Volume 27, Number 2, 179–192.

[10] Tosatti, Valentino: Limits of Calabi-Yau metrics when the Kahler class degenerates. ArXiv: 0710.4579.
[11] Tsuji, Hajime: Existence and degeneration of Kaehler-Einstein metrics on minimal algebraic varieties of general type. Math. Ann. 281(1988), no. 1, 123–133.

[12] Tsuji, Hajime: Degenerate Monge-Ampère equation in algebraic geometry. Miniconference on Analysis and Applications (Brisbane, 1993), 209–224, Proc. Centre Math. Appl. Austral. Nat. Univ., 33, Austral. Nat. Univ., Canberra, 1994.

[13] Yau, Shing Tung: On the Ricci curvature of a compact Kaehler manifold and the complex Monge-Ampère equation. I. Comm. Pure Appl. Math. 31(1978), no. 3, 339–411.

[14] Zhang, Zhou: On Degenerate Monge-Ampère Equations over Closed Kähler Manifolds. Int. Math. Res. Not. 2006, Art. ID 63640, 18 pp.

[15] Zhang, Zhou: Degenerate Monge-Ampère Equations over Projective Manifolds. PHD Thesis at MIT, 2006.

[16] Zhang, Zhou: Scalar curvature bound for Kähler-Ricci flows over minimal manifolds of general type. ArXiv: 0801.3248.