Confounding-Robust Policy Evaluation in Infinite-Horizon Reinforcement Learning

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Abstract

Off-policy evaluation of sequential decision policies from observational data is necessary in applications of batch reinforcement learning such as education and healthcare. In such settings, however, observed actions are often confounded with transitions by unobserved variables, rendering exact evaluation of new policies impossible, i.e., unidentifiable. We develop a robust approach that estimates sharp bounds on the (unidentifiable) value of a given policy in an infinite-horizon problem given data from another policy with unobserved confounding subject to a sensitivity model. We phrase the problem precisely as computing the support function of the set of all stationary state-occupancy ratios that agree with both the data and the sensitivity model. We show how to express this set using a new partially identified estimating equation and prove convergence to the sharp bounds, as we collect more confounded data. We prove that membership in the set can be checked by solving a linear program, while the support function is given by a difficult nonconvex optimization problem. We leverage an analytical solution for the finite-state-space case to develop approximations based on nonconvex projected gradient descent. We demonstrate the resulting bounds empirically.

1 Introduction

Evaluation of sequential decision-making policies under uncertainty is a fundamental problem for learning sequential decision policies from observational data, as is necessarily the case in application areas such as education and healthcare [Jiang and Li [2016], Precup et al. [2001], Thomas and Brunskill [2016]]. However, with a few exceptions, the literature on off-policy evaluation in reinforcement learning (RL) assumes (implicitly or otherwise) the absence of unobserved confounders, auxiliary state information that affects both the policy that generated the original data as well as transitions to the next state.

Precisely in the same important domains where off-policy evaluation from a given set of observational trajectories is necessary due to the cost of or ethical constraints on experimentation, such as in education [Prasad et al. [2017], Raghu et al. [2017] or operations, it is also
generally the case that unobserved confounders are present. In the batch causal inference setting, the perils of learning from observational medical data loom large: for example, actions taken by physicians are often informed by more information than is recorded in electronic health record data. This contributes to fundamental challenges for advancing reinforcement learning in observational settings [Gottesman et al, 2019].

In this work, we initiate the study of partial identification in RL off-policy evaluation under unobserved confounding, focusing specifically on the infinite-horizon setting. Recognizing that policy value cannot actually be point-identified from confounded observational data, we propose instead to compute the sharpest bounds on policy value that can be supported by the data and any assumptions on confounding. This can then support credible conclusions about policy value from the data and can ensure safety in downstream policy learning.

Recent advancements [Gelada and Bellemare, 2019, Hallak and Mannor, 2017, Kallus and Uehara, 2019, Liu et al, 2018] improve variance reduction of unconfounded off-policy evaluation by estimating density ratios on the stationary occupancy distribution. But this assumes unconfounded data. Other advances [Kallus and Zhou, 2018] tackle partial identification of policy values from confounded data but in the logged bandit setting (single decision point) rather than the RL setting (many or infinite decision points). Our work can be framed as appropriately combining these perspectives, and our method takes the form of partially identifying the stationary density ratio via its support function, because unobserved confounding renders the ratio unidentifiable. In particular, just as considering the stationary density ratio is important for variance reduction in the unconfounded setting, it is also crucial here for preventing the “exponential conservatism” that might otherwise occur in backward-recursively applying single-decision-point robustness.

Our contributions are as follows: we establish a partially identified estimating equation that allows for the estimation of sharp bounds. We develop tractable reformulations of the resulting difficult non-convex program based on a (large) linear program or non-convex first order methods. We then demonstrate the approach on a gridworld task with unobserved confounding.

2 Problem setup

We assume data is generated from an infinite-horizon MDP with an augmented state space: $S$ is the space of the observed portion of the state and $U$ is the space of the unobserved (confounding) portion of the state. We assume the standard decision protocol for MDPs on the full-information state space $S \times U$: at each decision epoch, the system occupies state $s_t, u_t$, the decision-maker receives a reward $\Phi(s_t)$ for being in state $s_t$ and chooses an action, $a_t$, from allowable actions. Then the system transitions to the next state on $S \times U$, with the (unknown) transition probability $p(s', u' \mid s, u, a)$. The full-information MDP is represented by the tuple $M = (S \times U, \mathcal{A}, P, \Phi)$. We let $\mathcal{H}_t = \{(s_0, u_0, a_0), \ldots, (s_t, u_t, a_t)\}$ denote the (inaccessible) full-information history up to time $t$. A policy $\pi(a \mid s, u)$ is an assignment to the probability of taking action $a$ in state $(s, u)$. Any policy induces a Markov chain the history. That is, for any policy, the underlying dynamics are Markovian under full observation of states and transitions: $s_t \perp \perp \mathcal{H}_{t-2} \mid (s_{t-1}, u_{t-1}), a_{t-1}$.

In the off-policy evaluation setting, we consider the case where the observational data are
generated under an unknown behavior policy $\pi_b$, while we are interested in evaluating the (known) evaluation policy $\pi_e$, which only depends on the observed state, $\pi_e(a \mid s, u) = \pi_e(a \mid s)$. Both policies are assumed stationary (time invariant).

The observational dataset does not have full information and comprises solely of observed states and actions, that is, $(s_0, a_0), \ldots, (s_t, a_t)$.

Thus, since the action also depends on the unobserved state $u$, we have that transition to next states are confounded by $u_t$.

Notationally, we reserve $s,u$ (respectively, $s',u'$) for the random variables representing state (respectively, next state) and we refer to realized observed state values (respectively, next observed state values) using $j$ (respectively, $k$). We assume that $S$ is a discrete state space, while $U$ may be general.

We next discuss regularity conditions on the MDP structure which ensure ergodicity and that the limiting state-action occupancy frequencies exist. We assume that the Markov chain induced by $\pi_e$ and any $\pi_b$ is a positive Harris chain, so the stationary distribution exists.

**Assumption 1 (Ergodic MDP).** The MDP $M$ is ergodic: the Markov chain induced by $\pi_b$ and any $\pi_e$ is Harris recurrent and positive.

In this work, we focus on the infinite-horizon setting; we next discuss the limiting state-action-state stationary occupancy joint distribution which exists under Assumption 1. Let $p_{\pi}^t(s)$ be the distribution of state $s_t$ when executing policy $\pi$, starting from initial state $s_0$ drawn from an initial distribution over states. Then the average state-action-next-state visitation distribution is

$$p_{\pi}^\infty(s,u,a,s',u') = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \gamma^t p_{\pi}^t(s,u,a,s',u')$$

We similarly define the marginalized total-, unobserved-, and observed-state occupancy distributions as $p_{\pi}^\infty(s,u), p_{\pi}^\infty(u),$ and $p_{\pi}^\infty(s)$, given by appropriately marginalizing the above.

Under Assumption 1 the (long-run average) value of a stationary policy $\pi$ is given by

$$R_\pi = E_{s \sim p_{\pi}^\infty} [\Phi(s)].$$

We define $R_e = R_{\pi_e}$ for brevity. Notice we assumed that the reward only depends on the observed state. This does not preclude, however, dependence on action: if we are given observed-state-action reward function $\Phi'(s,a)$, we may simply define $\Phi(s) = \sum_a \pi_e(a \mid s)\Phi'(s,a)$, since $\pi_e(a \mid s)$ is assumed known. Then $R_e$ gives $\pi_e$’s value with respect to the given observed-state-action reward function.

Notationally, $E$ denotes taking expectations over the joint stationary occupancy distribution of the behavior policy, where self-evident. We denote $p_{\pi}^\infty, p_b^\infty$ for visitation distributions induced under $\pi_e, \pi_b$. Since at times it is useful to distinguish between expectation over the marginalized occupancy distribution $p_{\pi}^\infty(s,a,s')$, and total expectation over full-information transitions $p_b^\infty(s,u,a,s',u')$, we include additional subscripts on the expectation whenever this is clarifying.

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1Our model differs from typical POMDPs Kaelbling et al. 1998, since rewards are a function of observed state, as we clarify in the related work, Section 8.
If we were able to actually run the MDP using the policy $\pi_e$, which is only a function of $s$, the dynamics would be Markovian with the marginalized transition probabilities $p(k \mid j, a)$ (marginalized over the stationary state-occupancy distribution on $u$):

$$p(k \mid j, a) := \sum_{u', u''} p(k, u'' \mid j, u', a) p_e^\infty(u' \mid j)$$

Note that $p(k \mid j, a)$ is not identifiable from the observational data collected under $\pi_b$. We analogously define (partially) marginalized transition probabilities $p(k \mid j, u', a)$.

3 Off-policy evaluation under unobserved confounding

In the following, we discuss a population viewpoint, computing expectations with respect to true the stationary occupancy distribution $p_b^\infty(s, a, s')$, which is identifiable from the observational data. We discuss sampling considerations in Section 6, where show we analyze what happens when use an estimated observed-state-action occupancy distributions, $\hat{p}_e^\infty(s, a, s')$.

Given sample trajectories generated from $\pi_b$, the goal of off-policy evaluation is to estimate $R_e$, the value of a known (observed-state-dependent) evaluation policy $\pi_e(a \mid s)$. The full-information stationary density ratio is

$$w(s, u) = \frac{p_e^\infty(s, u)}{p_b^\infty(s, u)}.$$ 

If $w(s, u)$ were known, we could use it to estimate policy value under $\pi_e$ using samples from $\pi_b$ since the following equality holds by a simple density ratio argument:

$$R_e = \mathbb{E}[w(s, u)\Phi(s)].$$

From observational data, we are only able to estimate the marginalized behavior policy,

$$\pi_b(a \mid s) = \mathbb{E}[\pi_b(a \mid s, u) \mid s],$$

which is insufficient for identifying the policy value or the true marginalized transition probabilities.

3.1 Model restrictions on unobserved confounding

To make progress, we introduce restrictions on the underlying dynamics of the unobserved confounder, $u$, under which we will conduct our evaluation. In particular, we will seek to compute the range of all values of $R_e$ that match our data, encapsulated in $p_b^\infty(s, a, s')$, and the following structural assumptions.

Assumption 2.

$$\frac{p_e^\infty(s, u)}{p_b^\infty(s, u)} = \frac{p_e^\infty(s, u')}{p_b^\infty(s, u')} \quad \forall s \in S, u', u'' \in U$$

Lemma 1. Assumption 2 holds if the MDP transitions satisfy

$$p(j, u' \mid k, u, a) = p(j, \tilde{u}' \mid k, u, a), \forall u', \tilde{u}' \in U.$$
For example, Assumption 2 holds if there is just some baseline unobserved confounder for each individual, or if the unobserved confounder is exogenously drawn at each timestep. These examples are shown in Figure 1. A combination of a baseline unobserved confounder as well as exogenous confounders at each timestep is also allowed. Such MDPs would satisfy Assumption 2 for any two policies.

Assumption 2 essentially requires no time-varying confounders, i.e., confounders that are influenced by past actions. This assumption may appear strong but it is necessary: if confounders could be time-varying and the dependence on them may be arbitrary, we may need to be “exponentially conservative” in accounting for them (or even “infinitely conservative” in the infinite-horizon case). Moreover, the assumption captures important examples in healthcare where it is very often baseline confounders, such as socio-economic status or risk/toxicity/other preferences, that confound actions by affecting access to doctors and to certain medicine or operations or by affecting choices between intensive or conservative treatments.

Under Assumption 2 we simply define \( w(s) = w(s, u) \) as it does not depend on \( u \). Note that \( w(s) \) is still unidentifiable even under Assumption 2.

### 3.2 Sensitivity model

Next, we introduce a sensitivity model to control the level of assumed dependence of the behavior policy on the unobserved confounders. Following Aronow and Lee [2012], Kallus and Zhou [2018] we phrase this as bounds on the (unknown) inverse behavior policy:

\[
\beta(a \mid s, u) := \pi_b(a \mid s, u)^{-1}.
\]

In particular, we focus on having an ambiguity region given by lower and upper bounds that depend only on the observed state and action, \( l(a \mid s), m(a \mid s) \)²

\[
l(a \mid s) \leq \beta(a \mid s, u) \leq m(a \mid s) \quad \forall a, s, u.
\]

²Our approach will actually generalize to any linearly representable ambiguity sets; additional linear constraints on \( \beta \) would simply propagate additional constraints throughout the rest of the characterization.
Then let $\mathcal{B}$ consist of all functions $\beta(a \mid s, u)$ that satisfy Eq. (1).

This ambiguity set is motivated by a sensitivity model used in causal inference, which restricts how far propensities can vary pointwise from the nominal propensities \cite{Tan2012} and which has also been used in the logged bandit setting \cite{Kallus2018}. Given a sensitivity parameter $\Gamma \geq 1$ that control that amounts of allowed confounding, the marginal sensitivity model posits the following odds-ratio restriction:

$$\Gamma^{-1} \leq \frac{(1 - \pi_b(a \mid s))\pi_b(a \mid s, u)}{\pi_b(a \mid s)(1 - \pi_b(a \mid s, u))} \leq \Gamma, \quad \forall a, s, u.$$  \hspace{1cm} (2)

Eq. (2) is equivalent to saying that Eq. (1) holds with

$$l(a \mid s) = \Gamma / (\pi_b(a \mid s)) + 1 - \Gamma,$$
$$m(a \mid s) = 1 / (\Gamma \pi_b(a \mid s)) + 1 - 1/\Gamma.$$

Lastly, $\beta$ functions which are valid inverse probabilities must satisfy the next-state conditional constraints:

$$\mathbb{E}_{s,u,a,s' \sim p_b^\infty} [I[a=a'] | s' = k] = p_b^\infty(k | a') \forall k, a'$$ \hspace{1cm} (3)

We let $\tilde{\mathcal{B}}$ denote the set of all functions $\beta(a \mid s, u)$ that satisfy both Eqs. (1) and (3). Notice that, for now, if $\pi_b(a \mid s, u)$ is a probability then this is redundant. We will use this, however, after projecting this requirement away.

### 3.3 The partially identified set

Given the above restrictions, we can define the set of partially identified evaluation policy values. To do so, suppose we are given a target behavior policy $\pi_e$, the observed stationary distribution $p_b^\infty(s, a, s')$, and bounds $l(a \mid s), m(a \mid s)$ on $\beta$. We are then concerned with what $w$ could be given the non-givens. So, we define the following set for what values $w$ can take:

$$\Theta = \left\{ \frac{p_b^{(\infty)}(s, u)}{\pi_b(s, u)} : \pi_b(a \mid s, u) \text{ is a stationary policy with } \beta \in \mathcal{B} \text{ and } p_b^{(\infty)}(s, a, s') \text{ as given, } M \text{ satisfies Assumption } [1] \text{ with respect to } \pi_b \text{ and the (given) } \pi_e \right\}$$

We are then interested in determining the largest and smallest that $R_e$ can be. That is, we are interested in

$$R_e = \inf_{w \in \Theta} \mathbb{E}[w(s)\Phi(s)], \quad \overline{R}_e = \sup_{w \in \Theta} \mathbb{E}[w(s)\Phi(s)].$$  \hspace{1cm} (4)

Notice that this is equivalent to computing the support function of $\Theta$ at $-\Phi$ and $\Phi$ with respect to the $L_2$ inner product defined by $p_b^\infty(s), \langle f, g \rangle = \mathbb{E}[f(s)g(s)] = \sum_j p_b^\infty(j)f(j)g(j)$. Recall that the support function of a set $\mathcal{S}$ is given by $\psi(v) = \sup_{v \in \mathcal{S}} \langle v, s \rangle$ \cite{Rockafellar1970}.

\footnote{Note that, as defined, $\Theta$ is a set of functions of $(s, u)$ but because we enforce Assumption 2, all members are constant with respect to $u$ for each $s$; we therefore often implicitly refer to it as a set of functions of $s$ alone.}
4 Characterizing the partially identified set

In this section we proceed to simplify the above expression for $\Theta$ in more manageable terms, culminating in a linear program to check membership in $\Theta$ for any given $w$.

4.1 The partially identified estimating equation

We begin by showing that $w$ is uniquely characterized by an estimating equation characterizing its stationarity, but where some parts of the equation are not actually known.

Lemma 2. Suppose Assumptions 1 and 2 hold. Then

$$w(s) = \frac{p_k^{(\infty)}(s,u)}{p_k^{(\infty)}(s,u)} \forall s, u$$

if and only if

$$\mathbb{E}[\pi_s(a|s)w(s)\beta(a|s,u)|s'=k] = w(k) \forall k,$$

$$\mathbb{E}[w(s)] = 1.$$ (6)

The forward implication of Lemma 2 follows from Theorem 1 of Liu et al. [2018] applied to the state variable $(s,u)$ after recognizing that $w(s,u)$ only depends on $s$ under Assumption 2 and marginalizing out $u'$. The backward implications of Lemma 2 follows from the recurrence of the aggregated MDP obtained from the transform $(s,u) \mapsto s$. A complete proof appears in the appendix.

Fortunately, Eqs. (5) and (6) exactly characterize $w$. Unfortunately, Eq. (5) involves two unknowns: $\beta(a|s,u)$ and the distribution $p_k^{(\infty)}(s,u,a,s')$ with respect to which the expectation is taken. In that sense, the estimated equation is only partially identified. Nonetheless, this allows to make progress toward a tractable characterization of $\Theta$.

4.2 Marginalization

We next show that when optimizing over $\beta \in \mathcal{B}$, the sensitivity model can be reparametrized with respect to marginal weights $g_k(a|j)$

(in the following, $j, a, k$ are generic indices into $\mathcal{S}, \mathcal{A}, \mathcal{S}$, respectively):

$$g_k(a|j) := \sum_{a'} \frac{p_k^{(\infty)}(j,a',a|k)}{p_k^{(\infty)}(j,a|k)} \beta(a|j,a')$$

$$= \left(\sum_{a'} \pi_a(a|j,a')\frac{p_k^{(\infty)}(a'|j)p(k|j,a')}{p(k|j,a)}\right)^{-1}$$

Note that the values of the $g_k(a|j)$ weights are not equivalent to the confounded $\pi_a(a|s)^{-1}$: the difference is exactly the variability in the underlying full-information transition probabilities $p(k|j,a')$. We will show that $g_k(a|j) \in \mathcal{G}$ satisfies the following constraints, where Eq. (7) corresponds to Eq. (3):

$$l(a|j) \leq g_k(a|j) \leq m(a|j), \quad \forall j, a, k$$

$$p_k^{(\infty)}(k|a) = \sum_j p_k^{(\infty)}(j,a,k)g_k(a|j), \quad \forall k, a$$ (7)

In order to highlight some distinctions that arise from our approach from sensitivity analysis in the logged bandit setting, we introduce elements of the argument that verify that
using both $\tilde{B}$ and the estimating equation of Lemma 2 verify membership of a given $w$ for the partial identification set $\Theta$.

Unlike sensitivity models in causal inference, it is possible that $\Theta = \emptyset$, even if its associated sensitivity model $\tilde{B}$ is nonempty. This may occur because Eq. (3) is not feasible in the sample, and/or because $g_k(a \mid j)$ is incompatible for the marginal transitions with respect to the $\pi^\infty_b(s, a, s')$ joint distribution. From finite samples it is not possible to distinguish either case; we discuss a feasibility relaxation approach in Remark 4 to handle the former.

For any infinite-horizon MDP with transition matrix on $S \times U$, the stationary dynamics impose restrictions on the unknown full-information state-action-state visitation distribution, $\pi^\infty_b(s, u, a, s', u')$, and its observable marginalization $\pi^\infty_b(s, a, s')$. These restrictions, which are classical in studying the solution structure of infinite-horizon MDPs, yield the full-information state-action polytope (SAP) which is the set of all limiting state-action occupancy probabilities achievable under any policy, and the closely related state-action-state polytope (SASP) [Mannor and Tsitsiklis 2005, Puterman 2014].

Marginalizing the full-information constraints with respect to $\pi^\infty_b(u \mid s)$ leads to the marginalized versions $\pi^\infty_s$ and $\pi^\infty_{s'}$.

We first verify that our specification of $\tilde{B}$ exhausts the implementable implications of $\pi^\infty_s$ and $\pi^\infty_{s'}$.

Proposition 1. The implementable implications of $\pi^\infty_s$ and $\pi^\infty_{s'}$ are:

$$p^\infty_b(k \mid a) = \sum_j p^\infty_b(j, a, k) g_k(a \mid j), \forall k \in S, a \in A$$

While Proposition 1 justifies our restrictions on $\tilde{B}$, it also implies that further imposing restrictions on $\tilde{B}$ cannot ensure compatibility of $g_k(a \mid j)$ for the observed $\pi^\infty_b(s, a, s')$, where compatibility is the requirement that $\pi^\infty_b(s, a, s')$ is stationary for $g_k(a \mid j)$ (via definition of $g_k$ with respect to $\beta$). (A full statement appears in the appendix).

In the main result of this section, we verify that it is exactly the partially identified estimating equation, Lemma 2, which enforces compatibility such that combining the restrictions on $\tilde{B}$ and Lemma 2 verifies membership of $w$ in $\Theta$.

4.3 Feasibility Linear Program

Based on the above, we next show that $w \in \Theta$ can be expressed using the linear program $F(w)$ that minimizes the L1 norm of residuals of the estimating equation of Lemma 2 for a given $w$, over the sensitivity model $g \in \tilde{B}$:

$$F(w) := \min_{g \in \tilde{B}} \sum_k \left| \sum_{j, a} h_{j, a, k}(w) g_k(a \mid j) - w(k) \right|,$$

where we define the linear functions $h_{j, a, k}(w) = p^\infty_b(j, a \mid k) w(j) \pi(e)(a \mid j)$ for brevity.

Proposition 2 (Feasibility Linear Program).

$$w \in \Theta \iff F(w) \leq 0, \mathbb{E}[w(s)] = 1$$

\(^4\) In the full-information setting, the polytopes are equivalent sets under multiplication by the full-information transition probability $\pi(s', u' \mid s, u, a)$, but in the marginalized setting, such a construction is not identified, leading to Proposition 4.
Reparametrization with respect to \( g_k(a \mid j) \) follows from an optimization argument, recognizing the symmetry of optimizing a function of unknown realizations of \( u \) with respect to an unknown conditional visitation density. Reparametrization improves the scaling of the optimization program from the number of samples or trajectories to the cardinality of state and action space.

A consequence of Proposition 2 is sharpness of the partially identified interval \([R_e, R_e]\) in the sense that each point in the interval corresponds to some policy value.

**Theorem 1 (Sharpness).**

\[
\{ \mathbb{E}[w(s)\Phi(s)] : w \in \Theta \} = [R_e, R_e].
\]

5 **Optimizing over the partially identified set**

Eq. (8) suggests computing \( R_e, R_e \) by solving

\[
\inf / \sup \{ \mathbb{E}[w(s)\Phi(s)] : F(w) \leq 0, \, \mathbb{E}[w(s)] = 1 \}.
\]

However, the restriction \( F(w) = 0 \) implicitly encodes an optimization over \( g \), resulting in a hard nonconvex bilevel optimization problem. In the following, we first leverage the structure of our problem first to develop a convex but exponentially-sized reformulation. Then, using an analytical closed-form solution we develop a non-convex but reasonably-sized first-order method.

5.1 **Global optimization**

We first show that Eq. (9) can be reformulated as a finite linear program, although it will be exponentially-sized in the state space. The reparametrization uses strong LP duality and partial maximization: leveraging Proposition 2 for a fixed \( w \), dualize \( F(w) \) to obtain the dual program \( D(w, \lambda) \); then optimize finally over \( w \) on the level set \( D(w) = 0 \) in order to optimize

\[
\min / \max \{ \mathbb{E}[w(s)\Phi(s)] : w \in \Theta \}.
\]

Let \( D(w, \lambda) \) denote the dualization of \( F(w) \), parametrized by the decision variable \( w \) and further parametrized by the dual variable \( \lambda \in \{-1, 1\}^{|S|} \), e.g. a sign vector. For compactness, denote the indices to functions such as \( p_b^\infty, l, m, \pi_e \) as subscripts in the statement of \( D(w, \lambda) \):

\[
D(w, \lambda) = \max_{c \geq 0, d \geq 0, k} \sum_{j,a,k} l_{aj} c_{jak} - m_{aj} d_{jak} + \sum_{a,k} \pi_e^{h,\infty} \mu_{k|a} - \lambda^\top w
\]

s.t. \( \forall j, a, k, c_{jak} - d_{jak} + p_{ja|k}^{\infty} \pi_{aj} w_j \lambda_k + p_{j|a}^{h,\infty} \mu_{k|a} \leq 0 \)

Linearity of \( F(w) \) also holds if an instrument function is used to convert the conditional moment equality to an unconditional one, as in Eqn. 10 Liu et al. [2018], and as we use in Section 5.2 and Proposition 3.

The strategy is standard in robust optimization, e.g. Ben-Tal and Nemirovski [1999]. However, note that the variables \( g \) appear in the constraint matrix of \( F(w) \) so the usual simplification does not apply.
Algorithm 1 Nonconvex nonconvex-projected gradient descent

Input: step size $\eta_0$, step-schedule exponent $\kappa \in (0, 1)$, initial iterate $g_0$, number of iterations $N$

$w^{*}_{g_0} \in \arg \min \{||A(g_0)w||_1 : \mathbb{E}[w(s)] = 1\}$

$g_0 \in \arg \min \{||g - g_0||_1 : A(g)w^{*}_{g_0} = 0, g \in \mathcal{B}\}$

for $k = 1, \ldots, N - 1$ do

$\eta_k \leftarrow \eta_0 t^{-k}$ Update step size

$w^{*}_{g_k} \in \arg \min \{||A(g_k)w||_1 : \mathbb{E}[w(s)] = 1\}$

$\tilde{g}_k \in \arg \min \{||g - g_k||_1 : A(g)w^{*}_{g_k} = 0, g \in \mathcal{B}\}$

$g_{k+1} \leftarrow \text{Proj}_\mathcal{B}(g_k + \eta_k \cdot \nabla_{\tilde{g}_k}(\varphi^T \tilde{A}(g)^{-1}v))$

end for

Return $g_k$ with the best loss.

Given the parameter $\lambda$ and $w$, $D(w, \lambda)$ is the optimal value of a linear program. We next re-express Eq. (8) using the representation of Lemma 2 and Proposition 2 as restrictions on $D(w, \lambda)$, for all of the finitely many $\lambda$.

Theorem 2.

$$\bar{R}_c = \max_w \left\{ \mathbb{E}[w(s)\Phi(s)] : \frac{D(w, \lambda)}{\mathbb{E}[w(s)]} \geq 0, \forall \lambda \in \{-1, 1\}^{|S|} \right\}$$

The reformulation is a strong duality result. The dual variable $\lambda$ is a sign pattern; controlling all such sign patterns controls the $\ell_1$ penalty on the residuals from the estimating equation for $w$. Note that embedding $D(w, \lambda) \geq 0$ into the LP involves adding the variables $c^\lambda, d^\lambda, \mu^\lambda$ for each $\lambda$. Though exponential scaling of columns and rows in $|S|$ is intractable, Theorem 2 is crucial for proving consistency via stability of the linear program, as we show in Theorem 3.

5.2 Nonconvex nonconvex-projected gradient method

We next develop a more practical optimization approach based on non-convex first-order methods. First we restate the estimating equation Eq. (5) for a fixed $g$ as a matrix system. To evaluate expectations on the unconditional joint distribution, we introduce instrument functions $\phi_s, \phi_{s'} \in \mathbb{R}^{|S| \times 1}$, random (row) vectors which are one-hot indicators for the state random variable $s, s'$ taking on each value, $\phi_s = [\mathbb{I}[s = 0] \ldots \mathbb{I}[s = |S|]]$. Let $A(g) = \mathbb{E}[\phi_{s'}(a | s)gs'(a | s)\phi_s - \phi_{s'})^\top]$ and $b_s = p_k^{\psi}(s)$. Let $\psi$ be the set of $g \in \bar{B}$ that admit a feasible solution to the estimating equation for some $w \in \Theta$:

$$\psi := \{g \in \bar{B} : \exists w \geq 0 \text{ s.t. } A(g)w = 0, b^\top w = 1\}$$

(10)

Define $\tilde{A}(g)$ by replacing the last row of $A(g)$ by $b$ and let $v = (0, \ldots, 0, 1) \in \mathbb{R}^{|S|}$.

Proposition 3. If $g \in \psi$ then $\tilde{A}(g)$ is invertible. Moreover,

$$\Theta = \{\tilde{A}(g)^{-1}v : g \in \psi\}.$$

Proposition 3 suggests computing $\bar{R}_c, \bar{R}_c$ by solving

$$\inf / \sup \{\varphi^T \tilde{A}(g)^{-1}v : g \in \psi\},$$

(11)
where \( \varphi_s = \Phi(s)p_b^\infty(s) \). This optimization problem has both a non-convex objective and a non-convex feasible set, but it has small size. As a way to approximate \( R_{e,\ell} \), we propose a gradient descent approach to solving Eq. (11) in Algorithm 1. Since the feasible set is itself non-convex, we respectively use an approximate projection that corrects each \( g \) iterate to a feasible point but may not be a projection. This is based on taking alternating projection steps on \( w_{g_k}^* \in \arg \min \{\|A(g_k)w\|_1 : \mathbb{E}[w(s)] = 1\} \) and \( g_k \in \arg \min \{\|g - g_k\|_1 : A(g)w_{g_k}^* = 0, g \in \mathcal{B}\} \).

The above analysis considered the population setting assuming the stationary occupancy distribution \( p_b^\infty(s,a,s') \) are known. In practice, we only have an estimate based on the empirical state-action occupancy distribution observed in the finite-horizon data, \( \hat{p}_b^\infty(s,a,s') \). Define \( \hat{R}_{e,\ell}, \hat{R}_{e} \) as the corresponding values when we solve Eq. (4) with this estimate in place of \( p_b^\infty(s,a,s') \). The following establishes the consistency of the resulting estimated bounds.

**Theorem 3** (Consistency). If \( \hat{p}_b^\infty(s,a,s') \rightarrow p_b^\infty(s,a,s') \), then \( \hat{R}_{e,\ell} \rightarrow R_{e,\ell}, \hat{R}_{e} \rightarrow R_{e} \).

Since the empirical distributions satisfy \( \hat{p}_b^\infty(s,a,s') \rightarrow p_b^\infty(s,a,s') \) \citep{mannor2005approximate}, Theorem 3 and Slutsky’s theorem would then imply that \( \hat{R}_{e,\ell} \rightarrow p_{\ell}, \hat{R}_{e} \rightarrow p_{R_{e}} \).

Our consistency result relies on the LP formulation for \( R_{e} \) given by Theorem 2. The perturbation of \( \hat{p}_b^\infty \) to \( p_b^\infty \) introduces perturbations to the constraint matrix of the LP, which is beyond the realm of standard realm of sensitivity analysis for LPs \citep{bertsimas1997introduction}. Instead, our result invokes a more general stability analysis due to \citep[Theorem 1]{robinson1975perturbations}.

### 7 Empirics

**3x3 confounded windy gridworld.** We introduce unobserved confounding to a 3x3 simple windy gridworld environment, depicted in Figure 2 \citep{sutton1998reinforcement}. The agent starts at the start state (S). There is a goal state (G; green), where the agent receives \( \Phi(s) = 1 \) reward.

![Figure 2: 3x3 gridworld.](image1)

\( \Phi(s) \) of green is 1, red -0.3.

![Figure 3: Varying evaluation \( \eta \) mixture weight from uniform to \( \pi^*_b, u = 0 \).](image2)

![Figure 4: Varying evaluation \( \eta \) mixture weight from uniform to \( \pi^*_b, S \).](image3)
There are also hazard states (shaded red) where the agent receives $\Phi(s) = -0.3$. (If an action moves an agent into a wall, it simply remains in place).

We assume a binary unobserved confounder $u \in \{0, 1\}$ that represents “wind strength”. Transitions in the action direction succeed in that direction with probability $p = 0.8$, otherwise with probability 0.1 the agent goes east or west. However, when $u = 1$, the “westward wind” is strong, but if the agent takes action “east”, the agent instead stays in place (otherwise the agent transitions west). The wind is generated exogenously from all else in the environment. An optimal full-information behavior policy (agent with wind sensors) varies depending on $u_t$ by taking the left action, to avoid the penalty states. (We include details on the optimal policies, computed by value iteration with the true transition probabilities in the appendix, as well as additional empirical comparisons.)

We illustrate the bounds obtained by our approach in Figures 3 and 4. We generate a trajectory of $T = 40000$ timesteps. We let $\pi^*_{b^U}$ be the full-information optimal policy computed by value iteration with the true full-information transition matrix. The behavior policy is a mixture of $\pi^*_{b^U}$ with a uniform policy with weights 0.3, 0.7, respectively.

We illustrate bounds for evaluation policies which are mixtures to $\pi^*_{b^U}$, a suboptimal policy that is optimal for the transitions when $u = 0$ (no wind), and $\pi^*_{b^U}$, a policy that is optimal on $S$ given the true marginalized transition probabilities (which are unknown to the analyst, but known in this environment). We display the bounds as we range mixture weights on the non-uniform policy from 0.3 to 0.8. We display in a dashed line, with the same color for corresponding mixture weight $\eta$, the true value of $R_e$.

**Remark 4.** Inevitably in the infinite-horizon setting, we will have to evaluate policies based on a finite sample of trajectories. In practice, to address the former, a feasibility relaxation can be introduced on the conditional restrictions of Eq. (7)

\[ 1 - \sum_{j,k} p^\infty_b(j, a, k)g_k(a | j) \leq \epsilon_a \]

In the experiments we use $\epsilon_a = 0.02$. Further restrictions are also possible, e.g. combining bounds and sum constraints on $\sum_a \epsilon_a \leq \epsilon(n)$. Imposing feasibility relaxations can result in bounds that trade-off sharpness for feasibility.

## 8 Related work

**Off-policy evaluation in RL.** We build most directly on a line of recent work in off-policy policy evaluation which targets estimation of the stationary distribution density ratio (Gelada and Bellemare 2019, Hallak and Mannor 2017, Kallus and Uehara 2019, Liu et al. 2018), which can be highly advantageous for variance reduction, rather than stepwise importance sampling with control variates.

\[ \text{To simplify the presentation, we solve for the bounds by solving Eq. (9) directly with Gurobi version 9, and impose action-marginal constraints to favor maintaining feasibility; in the appendix we show different choices of the constraints.} \]
Sensitivity analysis in the batch setting in causal inference. Sensitivity analysis is a rich area in causal inference. A related work in approach is Kallus and Zhou [2018], which builds on Aronow and Lee [2012] and considers robust off-policy evaluation and learning in the one-decision-point setting. The identification approach in this work is very different: the partial identification region is only identified implicitly as the solution region of an estimating equation. This introduces computational intractability even for just policy evaluation. Unlike Kallus and Zhou [2018], Bennett and Kallus [2019] consider an identifiable setting where we are given a well-specified latent variable model and propose a minimax balancing solution. Finally, Kallus et al. [2018], Yadlowsky et al. [2018] study bounds for conditional average treatment effects in the one-decision-point setting. The econometrics literature has considered partial identification in a dynamic discrete choice model Chiong et al. [2016] which is reliant on structural parametrization.

Off-policy evaluation in RL with unobservables. Various recent work considers unobserved confounders in RL. Oberst and Sontag [2019] considers identification of counterfactuals of trajectories in a POMDP and SCM model. Tennenholtz et al. [2019] study off-policy evaluation in the POMDP setting, proposing a “decoupled POMDP” and leveraging the identification result of Miao et al. [2018], viewing previous and future states as negative controls. Lu et al. [2018] propose a “deconfounded RL method” that builds on the deep latent variable approach of Louizos et al. [2017], training a model with variational inference. Zhang and Bareinboim [2019] uses partial identification bounds to narrow confidence regions on the transition matrix to warm start the UCRL algorithm of Jaksch et al. [2010].

All of these are different in that they consider a setting with sufficient assumptions or data to render policy values identifiable, where in the general observational setting they are unidentifiable. Specifically, Tennenholtz et al. [2019] require an invertibility assumption that implies in a sense that we have a proxy observation for every unobserved confounder. Lu et al. [2018] assume a well-specified latent variable model, also requiring that every unobserved confounder is reflected in the proxies, and Zhang and Bareinboim [2019] consider an online setting where additional experimentation can eventually identify policy value. Our approach is complementary to these works: we focus on the time-homogeneous, infinite-horizon case, and are agnostic to distributional or support assumptions on \( u \). Our structural assumption on \( u \)'s dynamics (Assumption 2) is also new.

In contrast to POMDPs in general, which emphasize the hidden underlying state, our model is distinct in that we focus on rewards as functions of the observed state. The unobserved confounder is therefore a “nuisance” confounder which prevents us from estimating policy value, rather than the true underlying state to recover. In settings where unobserved confounders are of concern in observational data in causal inference, typically it is unclear whether or not a latent variable model such as those underlying POMDPs indeed generalizes to the time of deployment: assuming so corresponds to a structural assumption about the environment.

8In Section 10.1 we discuss a naive extension of their or other inverse-weight robust approaches to this setting and the inherent challenges of such an approach with long horizons in introducing “exponential robustness”.
Contrast to Robust MDPs. Robust MDPs, representing a model-based approach, consider policy evaluation or improvement over an ambiguity set of the transition probabilities [Iyengar 2005, Nilim and El Ghaoui 2005, Wiesemann et al. 2013]. Alternatively, some approaches build confidence regions from concentration inequalities [Petrik et al. 2016, Thomas et al. 2015] and restrict recommendations within them. Petrik and Subramanian 2014 improve performance guarantee bounds for state aggregation in MDPs; but in their setting they are able to sample additional full-information transitions unlike our fully-observational data setting. The difficulty in applying the robust MDP framework using an ambiguity set on transition matrices suggested from Lemma 4 (in the appendix) is non-rectangularity because the ambiguity set does not decompose as a product set over states, which leads to a NP-hard problem in the general case [Wiesemann et al. 2013].

9 Conclusions and future work

Our work establishes, to the best of our knowledge, the first partial identification results for policy evaluation in infinite-horizon RL under unobserved confounding. Our work focused on assumptions on the dynamics of the unobserved confounder that essentially preclude time-varying confounding, which we argued is necessary for any credible partial identification in the stationary setting. Under this restriction, we showed that the set of all policy values that agree with both the data and a marginal sensitivity model can be expressed in terms of whether the value of a linear program is non-positive. We showed how to translate this to algorithms that assess the minimal- and maximal-possible values of a policy.

Such assessments are crucial for credible conclusions about policy value it is not identified from data, and we believe such tools can be used to support safer off-policy RL in sensitive settings like medicine. Especially since data in such settings is bound to be confounded.

While checking feasibility in our partially identified set of stationary density ratios was a tractable linear program, optimizing over this set proved computationally challenging. While we provided some solutions, further algorithmic improvements may be possible. In particular, new algorithms are necessary in order to extend our results to infinite state spaces. Finally, an important next step is to translate the partial identification bounds to robust policy learning, which finds the policy in a class with minimal worst-case regret.

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10 Proofs

10.1 Relationship to finite-horizon case

We first recall an approach for robustness under unobserved confounding which might be pursued in the finite-horizon case: bounding the density ratio product of true behavior policy weights \( \prod_{t \in [H]} \pi_e(a_t | s_t) \) relative to the product of nominal inverse propensity weights, \( \prod_{t \in [H]} \pi_b(a_t | s_t, u_t) \). It is apparent by factorizing the joint distribution that the true density ratio product would identify the policy value.

\[
R_e = \mathbb{E} \left[ \sum_{h=1}^{H} r_h \mid a_1: H \sim \pi_e \right] = \mathbb{E}_b \left[ \left( \frac{p^{(0)}(s_0, u_0)}{p^{(0)}(s_0, u_0)} \prod_{t \in [H]} \frac{\pi_e(a_t | s_t) p(r_t | \mathcal{H}_t) p((s_t, u_t) | \mathcal{H}_t)}{\pi_b(a_t | s_t, u_t) p(r_t | \mathcal{H}_t) p((s_t, u_t) | \mathcal{H}_t)} \right) \sum_{h=1}^{H} r_h \right]
\]

While optimizing bounds on the range of the product of importance-sampling weights, \( \prod_{t \in [H]} \frac{\pi_e(a_t | s_t)}{\pi_b(a_t | s_t, u_t)} \) may be tractable using geometric programming Boyd et al. [2007], enforcing the moment restrictions on density ratios (as in Eq. (7)) introduces further difficulty. The core difficulty is that considering an uncertainty set which decomposes as a product set over timesteps may be too conservative to be useful in practice. While further modeling restrictions on the type of unobserved confounding, e.g. bounding the number of deviations from nominal as in Mannor et al. [2012] may be tractable to optimize, and reasonable in practice, all such modeling assumptions are necessarily imposed \textit{ex post}. Furthermore, such an approach necessarily does not extend to the infinite-horizon case.

10.2 Partial Identification

Proof of Lemma 2

Clearly, by assumption of Markovian dynamics on the full information state space, \( w(j, u) \) solves the estimating equation (state-action flow equations) on the space of \( \mathcal{S} \times \mathcal{U} \),

\[
\mathbb{E} \left[ \pi_e(a \mid s) w(s, u) \beta(a \mid s, u) \mid s' = k \right] = w(k, u) \quad \forall k \in \mathcal{S}, v \in \mathcal{U} \quad (12)
\]

\[
\mathbb{E}[w(s, u)] = 1. \quad (13)
\]

We will proceed to show that under Assumption 2, \( \tilde{w}(k) := w(k, v) \) equivalently solves the estimating equation on the observed state space \( \mathcal{S} \).

The forward implication of Lemma 2 follows from Theorem 1 of Liu et al. [2018] applied to the state variable \( (s, u) \) after recognizing that \( w(s, u) \) only depends on \( s \) under Assumption 2
and marginalizing out $u'$.

\[
\tilde{w}(k) = \frac{p_{\beta}^{\infty}(k)}{p_{b}^{\infty}(k)} = w(k, v) = \mathbb{E}_{(s,u), a, (s', u') \sim p_{\beta}^{\infty}}[w(s, u)\pi_{e}(a \mid s)\beta(a \mid s, u) \mid s' = k, u' = v] = \mathbb{E}_{(s,u), a, (s', u') \sim p_{\beta}^{\infty}}[w(s, u)\pi_{e}(a \mid s)\beta(a \mid s, u) \mid s' = k] \quad \text{by Assumption 2}
\]

\[
\tilde{w}(s) = \mathbb{E}_{(s,u), a, (s', u') \sim p_{\beta}^{\infty}}[\tilde{w}(s)\pi_{e}(a \mid s)\beta(a \mid s, u) \mid s' = k]
\]

We write out the last step to verify that $\tilde{w}(k)p_{b}^{(\infty)}(k)$ satisfies conditions on the invariant measure on $s$:

\[
\tilde{w}(k) = \frac{1}{p_{b}^{(\infty)}(k)} \sum_{j,a} \tilde{w}(j) \sum_{u} p(k \mid j, u, a)p_{b}^{(\infty)}(s, u)\pi_{b}(a \mid s, u)\frac{\pi(a \mid s)}{\pi_{b}(a \mid s, u)}, \forall k
\]

\[
= \frac{1}{p_{b}^{(\infty)}(k)} \sum_{j,a} \tilde{w}(j)\pi(a \mid j) \sum_{u} p_{b}^{(\infty)}(j, u)p(k \mid j, u, a), \forall k
\]

Therefore,

\[
\tilde{w}(k)p_{b}^{(\infty)}(k) = \sum_{j,a} \tilde{w}(j)p_{b}^{(\infty)}(j)\pi(a \mid j)p(k \mid j, a),
\]

so we conclude that $\tilde{w}(s) \propto p_{b}^{(\infty)}$, the stationary distribution on $S_{\text{obs}}$ induced by $\pi_{e}$. Finally, we argue the reverse implication; uniqueness of the solution of $\tilde{w}(s)$. Uniqueness is a consequence of the positive recurrence assumption (Assumption 1) on the full-information MDP on $S \times U$.

Note that by definition of recurrence, recurrence on the full-observation state space of the Markov process induced under $\pi$ implies recurrence of the Markov process induced under $\pi$ on its marginalized transitions $p(k \mid j, a)$. Recurrence requires that starting from any state $j, u$ in the recurrent class, the number of visits of the chain to the state is infinite. Clearly, if this is satisfied by the full-information transition matrix, this is also satisfied for the aggregated recurrent class corresponding to marginalized transitions.

Therefore, the stationary distribution exists, and is unique on $S$, under the marginalized transition matrix induced by $\pi_{e}$. The solution to the invariant measure flow equations on $S$ satisfies that $\tilde{w}(s)p_{b}^{\infty}(s) \propto p_{e}^{\infty}(s)$; and only $\tilde{w}(s)$ satisfies this requirement.

**Proof of Proposition 3** Step 1: Proving the reparametrization of $F(w)$ with respect to $g_{k}(a \mid j)$, and reformulating $g_{k}(a \mid j)$.

First, expanding the sample expectations for the estimating equation:

\[
\sum_{j} \sum_{i=1}^{N} \sum_{t=0}^{T} \sum_{a, u} \mathbb{E}[(s_{t}^{(i)} = j, u_{t}^{(i)} = u, a_{t}^{(i)} = a, s_{t+1}^{(i)} = k)] \left( \frac{w(k) - w(j)\beta^{(i)}(a \mid j, u)\pi_{e}(a \mid j)}{p(s_{t+1}^{(i)} = k)} \right) = 0, \forall k
\]
Taking limits as $T \to \infty, N \to \infty$ and multiplying by $\frac{p_b^\infty(j,a,k)}{p_b^\infty(j,a)} = 1$:

$$w(k) - \sum_j w(j) \sum_a \pi_e(a \mid j) \frac{p_b^\infty(j,a,k)}{p_b^\infty(j,a)} \sum_u \frac{p_b^\infty((j,u),a \mid k)}{p_b^\infty(j,a \mid k)} \beta(a \mid j,u) = 0, \forall k$$

Therefore, dependence on $\beta$ arises only through the marginalized weight $g_k(a \mid j)$:

$$g_k(a \mid j) = \sum_u \frac{p_b^\infty((j,u),a \mid k)}{p_b^\infty(j,a \mid k)} \beta(a \mid j,u)$$

Next, we show that optimizing over the set of marginalized weights convolved with an unknown density is equivalent to optimizing over the set of weights $\mathcal{B}$, by showing how to identify elements $\tilde{\beta} \in \tilde{\mathcal{B}}'$ with:

Although $p_b^\infty((j,u),a \mid k)$ is not identifiable from observed data, its marginalization over $u$, $p_b^\infty(j,a \mid k)$, is identifiable, so we can partially identify $\tilde{\mathcal{B}}'$ as follows:

$$\tilde{\mathcal{B}}' = \left\{ \sum_a p_b^\infty((j,u),a \mid k) \frac{p_b^\infty(j,a \mid k)}{p_b^\infty(j,a \mid k)} \beta(a \mid j,u) : \sum_u p_b^\infty((j,u),a \mid k) = p_b^\infty(j,a \mid k) \right\}$$

A simple reparametrization $q = \frac{p_b^\infty((j,u),a \mid k)}{p_b^\infty(j,a \mid k)}$ shows that optimizing over $\tilde{\mathcal{B}}'$ is equivalent to optimizing over elements of $\mathcal{B}$ averaged by unknown weights on the simplex. In the following, suppress dependence of $q_{j,u,a,k}$, $\beta(a \mid j,u)$ on $a,j$ for brevity.

$$\tilde{\mathcal{B}}' = \{ q^\top \beta : q^\top 1 = 1, 0 \leq q \leq 1, \beta \in \mathcal{B} \}$$

In particular this suggests that $q^\top \beta \in \mathcal{B}$. By convexity of $\mathcal{B}$, we can map $g_k(j,a) \in \tilde{\mathcal{B}}'$ to some $\beta \in \mathcal{B}$.

In the other direction, clearly any $\beta' \in \mathcal{B}$ is realizable by a Dirac measure, so $\beta' \in \tilde{\mathcal{B}}'$.

Lastly, we directly verify the control variate property that $\sum_j \sum_k p_b^\infty(j,a \mid k)p_b(k)g_k(j,a) = 1$:

$$\sum_j p_b^\infty(j,a,k)g_k(j,a) = \sum_{j,a,k} p_b^\infty(j,a \mid k)p_b^\infty(k) \frac{p_b^\infty(j,u,a \mid k)}{p_b^\infty(j,a \mid k)} \beta(a \mid j,u)$$

$$= \sum_{j,a} p_b^\infty(j,u) \pi_e(a \mid j,u) p(k \mid j,u) \frac{p_b^\infty(k)}{p_b^\infty(k)} \beta(a \mid j,u)$$

$$= \sum_{j,a} p_b^\infty(k,j,u \mid a)$$

so that $\sum_k \sum_j p_b^\infty(j,a,k)g_k(j,a) = 1$. 

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For interpretation of $g_k(a \mid j)$, we may further simplify and observe that

$$g_k(a \mid j) = \sum_u \frac{p_b^\infty((j,u),a \mid k)}{p_b^\infty(j,a \mid k)} \beta(a \mid j,u) = \frac{\sum_u p_b^\infty(u \mid j)p(k \mid j,u,a)\pi_b(a \mid j,u)}{\sum_u p_b^\infty(a \mid j)p(k \mid j,u,a)\pi_b(a \mid j,u)}$$

$$= \frac{p(k \mid j,a)}{\frac{1}{\sum_u \pi_b(a \mid j,u)\frac{p_b^\infty(u \mid j)p(k,u,a)}{p(k \mid j,a)}}}$$

**Step 2: Proving $F(w) = 0, \mathbb{E}[w] = 1 \implies w \in \Theta$:**

Proposition 1 shows that the specification of $\tilde{B}$ exhausts the observable implications of the sharp full information polytope of all limiting state-action-state occupancy probabilities. It remains to show that $w$ is feasible for some $g_k(a \mid j) \in \tilde{B}$ iff $g_k(a \mid j)$ satisfies Eq. (19), $p(k \mid j,a) = p_b^\infty(a,k \mid j)g_k(a \mid j), \forall j,a,k$.

Suppose $g$ is feasible for the estimating equation:

$$w(k) - \sum_j w(j) \sum_a \pi_e(a \mid j)p_b^\infty(j,a \mid k)g_k(a \mid j) = 0, \forall k \tag{14}$$

First we verify that $g_k(a \mid j)$ satisfying Eq. (19) is feasible for the estimating equation for $w$, Equation (14). By Bayes’ rule and conformability of $g_k(a \mid j)$ for $p(k \mid j,a)$,

$$p_b^\infty(j,a \mid k) = \frac{p(k \mid j,a)p_b^\infty(j)g_k(a \mid j)^{-1}}{p_b^\infty(k)}$$

so that:

$$w(k)p_b^\infty(k) - \sum_j w(j)p_b^\infty(j)\sum_a \pi_e(a \mid j)p(k \mid j,a) = 0, \forall k$$

Markovianness of the induced MDP under the true marginal transition probabilities $p(k \mid j,a)$ (follows since $p(k \mid j,a)$ corresponds to the $p^\infty(u)$-occupancy-weighted aggregation to $S$ under Assumption 2), then $w(k)p_b^\infty(k)$ is proportional to the invariant measure on $S$.

Next we show the other direction, that $w$ feasible for Equation (14) (with some $g_k(a \mid j)$) implies $g_k(a \mid j)$ satisfies Equation (19). This direction follows once we identify $w(s)p_b^\infty(s)$ feasible for Equation (14) uniquely with $p_e^\infty(s)$, which follows from Assumption 1. By Bayes’ rule and conformability of $g_k(a \mid j)$ for $p(k \mid j,a)$, $p_b^\infty(j,a \mid k) = \frac{p(k \mid j,a)p_b^\infty(j)g_k(a \mid j)^{-1}}{p_b^\infty(k)}$, feasibility implies that $w(s)p_b^\infty(s)$ satisfies compatibility under $\pi_e$ for $g_k(a \mid j)$. Uniqueness of the density ratio implies that compatibility must also hold under $\pi_b$. 

**10.3 Observable Implications, Membership Oracle, and Sharpness**

In this section, we introduce the state-action polytope and the state-action-state frequency polytope and deduce the observable implications which lead to Proposition 1, the main membership certificate result of Proposition 2 and Theorem 1.
Proof of Proposition 1: To study the observational implications (e.g., enforceable as constraints on \( \tilde{B} \)), we study the marginalized versions of both the state-action polytope and the state-action-state frequency polytope under the behavior policy as studied in Mannor and Tsitsiklis [2005], Puterman [2014]. Typically, the extremal analysis of the state-action polytope in the infinite-horizon case characterizes the structure of the optimal policy. We simply focus on its properties as a characterization of all the possible limiting state-action frequencies under any stationary policy.

While Mannor and Tsitsiklis [2005] studies more general cases, we focus on the typical case of stationary policies for simplicity. Indeed, while we state the following analysis for discrete state and action spaces (e.g., discrete \( u \)), the discussion of Altman and Shwartz [1991, Sec. 4] provides regularity results on the set of limiting state-action measures for the continuous state case with continuous \( u \). One sufficient condition for the case of continuous \( u \) is that for the transition probability density defined as

\[
P_{x,K}^\pi := \sum_{y \in K} P_{x,y}^\pi,
\]

with

\[
P_{x,y}^\pi = \text{Pr}(s_{t+1} = y \mid x),
\]

given any \( \epsilon > 0 \) there exist a finite set \( K(\epsilon) \) and an integer \( N(\epsilon) \) such that for all \( x \in X \) and \( g \in U(S) \),

\[
[(P_{x,K}^\pi)^{N(\epsilon)}]_{x,K(\epsilon)} \geq 1 - \epsilon.
\]

First, we introduce SAP, SASP as studied in Mannor and Tsitsiklis [2005].

**Definition 1 (State-action polytope).** Given an MDP, the state-action polytope SAP is defined as the set of vectors \( x \) in \( \Delta^{S \times A} \) that satisfy

\[
\sum_{j,u'} \sum_{a} p(k,u'' \mid j,u',a) p_b^\infty(j,u',a) = \sum_{a'} p_b^\infty(k,u'',a'), \quad \forall s'
\]

\( p_b^\infty(j,u',a) \in \Delta^{S \times A} \) is the limiting expected state-action frequency vector under policy \( \pi \). This constraint can be understood as a “flow conservation” constraint satisfied by any joint distribution \( p_b^\infty(s,u,a,s',u') \).

**Definition 2 (State-action-state polytope).** The state-action-state frequency polytope, SASP, is the set of vectors in \( \Delta^{S \times A \times S} \) which satisfy conformability of state transitions under transition probabilities; and flow conservation:

\[
p_b^\infty(j,u',a,k,u'') = p(k,u'' \mid j,u',a) \sum_{k',u''} p_b^\infty(j,u',a,k',u''), \quad \forall j,u',a,(k,u'')
\]

\[
\sum_{j,u'} \sum_{a} p_b^\infty(j,u',a,k,u'') = \sum_{a'} \sum_{k',u''} p_b^\infty(k,u'',a',\tilde{k},\tilde{u}''), \quad \forall(k,u'')
\]

Lemma 3.1 of Mannor and Tsitsiklis [2005] states that the two sets are equivalent: if \( p_b^\infty(s,u,a) \in \text{SAP} \), and furthermore under the transformation

\[
p_b^\infty(j,u',a,k,u'') = p_b^\infty(j,u',a) p(k,u'' \mid j,u',a),
\]

then \( p_b^\infty(j,u',a,k,u'') \in \text{SASP} \). Every element of SASP can be generated in this manner from some element of SAP.

The next result characterizes the marginalized versions of Equations (15) to (17); we present the constraints in terms of the marginalized weight that we study, \( g_k(a \mid j) \).
Lemma 3. The marginalized version of Eq. (15) is

\[ p_\infty^b(k) = \sum_a \sum_j p_\infty^b(j)p(k \mid j, a)g_k(a \mid j) - 1, \forall k \in \mathcal{S} \] (18)

The marginalized version of Eq. (16) is

\[ p(k \mid j, a) = p_\infty^b(a, k \mid j, a)g_k(a \mid j), \forall j, a, k \] (19)

and of Eq. (17) is the control variate,

\[ p_\infty^b(k \mid a) = \sum_j p_\infty^b(j, a, k)g_k(a \mid j), \forall k, a \] (20)

In order to interpret which of these constraints are observable implications and which are ultimately informative, we next leverage a structural characterization that \( g_k(a \mid j) \) can be interpreted as the function which renders the transition probabilities conformable to the joint distribution. Its proof is of independent interest in establishing the relationship to robust MDPs.

Lemma 4.

\[ \tilde{p}(k \mid j, a)\pi_b(a \mid j) = p(k \mid j, a)g_k(a \mid j) \]

Lemma 4 shows that the constraints in Equations (18) and (19) are uninformative: further restricting \( \tilde{p}(s' \mid a, s) \) within the given range of \( p(s' \mid a, s) \) is redundant. Another interpretation is that \( g_k(a \mid j) \) are precisely the weights which render the observed stationary occupancy distribution \( p_\infty^b(s, a, s') \) conformable under the unobserved true marginal transition probabilities.

Proof of Lemma 3. Marginalizing Eq. (16):

Starting from the compatibility restriction with the observed empirical state-action frequencies: \( \sum_u p_\infty^b(j, u)\pi_b(a \mid j, u)p(k, u' \mid j, u, a) = p_\infty^b(j, a, k, u') \) and marginalizing over \( u' \):

\[
\sum_u p_\infty^b(j, u)\pi_b(a \mid j, u)p(k \mid j, u, a) = \sum_u p_\infty^b(j, a, k)
\]

\[
p(k \mid j, a)p_\infty^b(j)\sum_u p_\infty^b(u \mid j)\pi_b(a \mid j, u)\frac{p(k \mid j, u, a)}{p(k \mid j, a)} = p_\infty^b(j, a, k)
\]

\[
p(k \mid j, a)g_k(a \mid j)^{-1} = p_\infty^b(a, k \mid j)
\]

where \( p(k \mid j, a) = \sum_u p(k \mid j, u, a)p_\infty^b(u \mid j) \).

This leads to the conditional compatibility constraint

\[ p(k \mid j, a) = p_\infty^b(a, k \mid j)g_k(a \mid j) \] (21)

which can also be derived as a marginalization of Eq. (16):

\[
\sum_{u, u'} p_\infty^b(j, u, a, k, u') = \sum_{u, u'} p(k, u' \mid j, u, a)\sum_{\tilde{k}, \tilde{u}'} p_\infty^b(j, u, a, \tilde{k}, \tilde{u}'), \forall j, k
\]
Marginalizing Eq. (17):
Recall that from the forward decomposition of joint distribution with respect to the transition from \( j, u \rightarrow k, u' \), but conditioning on current state and action, we have that:

\[
p_b^\infty(k, u' | a, j, u) = \frac{p_b^\infty(j, u, a, k, u')}{\pi_a(j, u)p_b^\infty(j, u)}
\]

Then marginalize the definition of \( p_b^\infty(k, u' | a) \) over \( u' \) to obtain \( p_b^\infty(k | a) \):

\[
\sum_{u'} p_b^\infty(k, u' | a) = \sum_{u'} \sum_{j,u} p_b^\infty(k, u' | a, j, u)p_b^\infty(j, u)
\]

\[
= \sum_{u'} \sum_{j,u} \frac{p_b^\infty(j, u, a, k, u')p_b^\infty(j, u)}{\pi(a | j, u)p_b^\infty(j, u)}
\]

\[
= \sum_{j} p_b^\infty(j, a, k) \sum_{u} \frac{p_b^\infty(j, u, a | k)}{p_b^\infty(j, a | k)\pi(a | j, u)}
\]

\[
= \sum_{j} p_b^\infty(j, a, k)g_k(a | j)
\]

Therefore:

\[
p_b^\infty(k | a) = \sum_{j} p_b^\infty(j, a, k)g_k(a | j), \forall k, a
\]

\[
\square
\]

Proof of Lemma 4. With full information, the transition probabilities could be estimated as

\[
p(s', u' | s, u, a) = \frac{p_b^\infty(s, u, a, s', u')}{\pi_b(a | s, u)p_b^\infty(s, u)}
\]

and similarly, the marginalized transition probabilities as \( \frac{p_b^\infty(s,u,a,s')}{\pi_b(a|s,u)p_b^\infty(s, u)} = p(s' | s, u, a) \).

But from data, we denote the biased marginalized transition probabilities \( \bar{p}(s' | s, a) \):

\[
\bar{p}(s' | s, a) := \frac{p_b^\infty(s, a, s')}{p_b^\infty(s)} \mathbb{E}[\pi_b(a | s, u) | s] = p_b^\infty(s)^{-1} \sum_{u} \frac{p_b^\infty(s, u, a, s')}{\mathbb{E}[\pi_b(a | s, u) | s]}
\]

which do not appropriately account for the transitions under the true \( \pi_b(s, u) \) policy, only its marginalization over \( s \).

A model-based perspective would partially identify the transition matrix under \( \pi_e \), deduce the bounds of \( p(s' | s, a) \) relative to \( \bar{p}(s' | s, a) \). Note that under Assumption 2

\[
p(s' | s, a) = \sum_{u} p(s' | s, u, a)p_b^\infty(u) = \sum_{u} \frac{p_b^\infty(s, u, a, s')p_b^\infty(u)}{\pi_b(a | s, u)p_b^\infty(s, u)} = p_b^\infty(s)^{-1} \sum_{u} \frac{p_b^\infty(s, u, a, s')}{\pi_b(a | s, u)}
\]

and further the distribution on unobserved confounders is independent of the policy, \( p_b^\infty(u) = p_e^\infty(u) \).

\[
\bar{p}(s' | s, a) = \frac{p_b^\infty(s, a, s')}{p_b^\infty(s)} \mathbb{E}[\pi_b(a | s, u) | s] = p_b^\infty(s)^{-1} \sum_{u} \frac{p_b^\infty(s, u, a, s')}{\mathbb{E}[\pi_b(a | s, u) | s]}
\]
Combining Equations (22) and (23) yields the statement of the lemma. We can define an ambiguity set for marginal transition probabilities on $S$ for $s, a$ state-action pairs, $P_{s' | s, a}$:

$$p(\cdot \mid s, a) \in P_{s' | s, a} := \left\{ \rho \in \Delta^{|S|} : \exists \beta \in B \text{ such that } \rho(k) = \sum_u \beta \cdot \frac{p_b^\infty(s, u, a, k)}{p_b^\infty(s)} \right\}$$

By an analogous construction as in Proposition 2, without additional restrictions on the variation of the unobserved joint visitation distribution $p_b^\infty(s, u, a, s')$,

$$P_{s' | s, a} := \left\{ \rho \in \Delta^{|S|} : \exists g \in \tilde{B} \text{ s.t. } \rho = g_{s'}(a \mid s) \cdot \tilde{p}(s' \mid s, a) \right\}$$

The ambiguity sets on transition kernels induced by bounds assumptions on the behavior policy ranges over $P \in [\Delta^{|S|}]^{|S| \times |A|}$, with $P_{s' | s, a} = p(s' \mid s, a)$.

Notably, the inverse probability restrictions on $g_k(a \mid j)$ render such an ambiguity set non-rectangular over states and actions. 

**Proof of Theorem 1.** Theorem 1 is a consequence of Proposition 1 and that computing the support function of a set (e.g. optimizing an arbitrary linear objective over $\Theta$) is equivalent to optimizing over the convex hull of $\Theta$ Rockafellar [1970]. Convexity of the interval and of $\text{conv}(\Theta)$ yields sharpness. 

### 10.4 Optimization and algorithms

**Proof of Theorem 2.** A feasibility oracle for $\Theta$, for a given $w$, is given by checking the existence of $g \in \mathcal{W}$ satisfying the moment condition. We write out the discrete expectations to take the dual appropriately.

$$F(w) := \min_{g \in \mathcal{B}, z \geq 0} \sum_k z_k$$

$$\text{s.t. } z_k \geq \sum_{j, a} p_b^{\infty}(j, a \mid k)w(j)\pi_e(a \mid j)g_k(a \mid j) - w(k), \forall k$$

$$z_k \geq -\left( \sum_{j, a} p_b^{\infty}(j, a \mid k)w(j)\pi_e(a \mid j)g_k(a \mid j) - w(k) \right), \forall k$$

$$l(a \mid j) \leq g_k(a \mid j) \leq m(a \mid j), \forall j, a, k$$

$$p_b^\infty(k \mid a) = \sum_j p_b^\infty(j, a, k)g_k(a \mid j), \forall k \in \mathcal{S}, a \in \mathcal{A}$$

Introduce $d_{j, a, k} \geq 0$ as the dual variable for $w \leq b$ and $c_{j, a, k} \geq 0$ as the dual variable for $-w \leq -a$; $\lambda$ as the free variable associated with constraints on $z$ and $\mu_{k|a}$ for equality constraints on $g_k(a \mid j)$.
The dual program is:

\[
D(w) := \max_{u \geq 0, v \geq 0, \lambda, \mu} \sum_{j,a,k} (l(a \mid j)c_{j,a,k} - m(a \mid j)d_{j,a,k}) + \sum_{k,a} \mu_{k|a} \cdot p_b^\infty(k \mid a) - \lambda^\top w
\]

\[
\text{s.t. } c_{j,a,k} - d_{j,a,k} + w(j)\pi_e(a \mid j)p_b^\infty(j, a \mid k) \cdot \lambda_k + p_b^\infty(j, a, k)\mu_a \leq 0, \forall j, a, k
\]

\[
|\lambda_k| \leq 1, \forall k
\]

The primal has \(2|\mathcal{S}| + 2|\mathcal{S}|^2|\mathcal{A}| + |\mathcal{S}||\mathcal{A}|\) constraints and \(|\mathcal{S}| + |\mathcal{S}|^2|\mathcal{A}|\) variables. The dual program has \(|\mathcal{S}| + |\mathcal{S}|^2|\mathcal{A}|\) constraints and \(2|\mathcal{S}| + 2|\mathcal{S}|^2|\mathcal{A}| + |\mathcal{S}||\mathcal{A}|\) variables, so all constraints are tight such that \(\lambda \in \{-1, 1\}\). (Degenerate solutions with \(\lambda = 0\) achieve the same solution value 0, but can be explicitly ruled out by enforcing \(\lambda\) to be binary.)

One approach is to fix a sign pattern for \(\lambda \in \{-1, 1\}\) \(|\mathcal{S}|\) and solve the LP for each value. Equivalently, let \(D(w, \lambda)\) denote the dual program for a fixed value of \(\lambda\).

\[
D(w, \lambda) := \max_{u \geq 0, v \geq 0, \mu} \sum_{j,a,k} (l(a \mid j)c_{j,a,k} - m(a \mid j)d_{j,a,k}) + \sum_{k,a} \mu_{k|a} \cdot p_b^\infty(k \mid a) - \lambda^\top w
\]

\[
\text{s.t. } c_{j,a,k} - d_{j,a,k} + w(j)\pi_e(a \mid j)p_b^\infty(j, a \mid k) \cdot \lambda_k + p_b^\infty(j, a, k)\mu_a \leq 0, \forall j, a, k
\]

By strong LP duality, the equivalence of partial minimization, and non-negativity of the value of \(F(w)\),

\[
F(w) \leq 0 \iff D(w) \geq 0 \iff \max_{|\lambda|=1} D(w, \lambda) \geq 0 \tag{24}
\]

Our central problem of interest was solving

\[
\max_w \{\mathbb{E}_{s \sim \pi_b}[w(s)\phi(s)] : F(w) \leq 0\}.
\]

While this problem is bilinear in the constraint set when we optimize over both \(\lambda\) and \(w\), using Eq. (24), we instead solve a problem with better structure, and tractable linear program subproblems.

\[
\max_{w} \{\langle w, \phi \rangle : \max_{|\lambda|=1} D[w, \lambda] \geq 0, \langle w, 1 \rangle = 1\} \tag{25}
\]

One approach is to solve the mixed integer program; which however is nonconvex (but may be solvable for small instances).

Another approach is to enumerate the linear programs, although this results in an exponentially sized linear program (exponentially many constraints; on the order of \(2^{|\mathcal{S}|}\) constraints).

\[
\max_{w} \{\langle w, \phi \rangle : D[w, \lambda] \geq 0, \forall \lambda \in \{-1, 1\}^{|\mathcal{S}|}; \langle w, 1 \rangle = 1\} \tag{26}
\]

Note that \(D[w, \lambda]\) is a linear program.

So, this problem has \(2^{|\mathcal{S}|}(|\mathcal{S}|^2|\mathcal{A}| + 1) + 1\) constraints in the extended formulation and \(2^{|\mathcal{S}|}(|\mathcal{S}|^2|\mathcal{A}| + |\mathcal{S}||\mathcal{A}|)\) variables.
Proof of Proposition 3. Let $\phi_s, \phi_t \in \mathbb{R}^{\vert \mathcal{S} \vert \times 1}$ be indicator (row) vectors for the state random variable $s, t$ taking on each value,

$$\phi_s = [I[s = 0] \ldots I[s = \mathcal{S}])]$$

such that

$$b := [p_b^\infty(s')]_{s' \in \mathcal{S}} = \mathbb{E}[\phi_s].$$

Denote the random variable for the density ratio,

$$z = \frac{\pi_e(a \mid s)}{\pi_b(a \mid s, u)}.$$

Then, observing that under assumptions of positive density of $p_b^\infty(s)$, we have that

$$\mathbb{E}_{s,a \mid s' \sim \pi_b}[w(t) - zw(j) \mid s' = t] = 0$$

is equivalent to

$$\mathbb{E}_{s,a,s' \sim \pi_b}[w(s') - zw(s)I[s' = t]] = 0$$

Then, we can encode the estimating equation that identifies $\theta$ as:

$$\{w \in \mathbb{R}^{\vert \mathcal{S} \vert}: \mathbb{E}_{s,a,t \sim \pi_b}[A]w = 0, b^T w = 1\} \quad \text{where } A = \phi_t(z \phi_s - \phi_t)^T$$

Note that $\mathbb{E}_{s,a,t \sim \pi_b}[A] = \mathbb{E}_{s,a,t \sim \pi_b}[z \phi_t \phi_s^\top] - \mathbb{E}_{s,a,t \sim \pi_b}[\phi_t \phi_t^\top]$, where the dependence on $z$ in the first term:

$$\mathbb{E}_{s,a,t \sim \pi_b}[z \phi_t \phi_s^\top] = \mathbb{E}[zI[s = j, t = k]]$$

$$= \mathbb{E}[z \mid s = j, t = k]p_b^\infty(j, k) = \sum_a p_b^\infty(j, a, k)\pi_e(a \mid j)g_k(a \mid j)$$

$A$ has rank $|\mathcal{S}| - 1$ if $g$ is feasible since satisfying the conformability constraint Equations (16) and (20) (e.g. taking expectations over the indicator vectors)

$$\sum_k (w_k - \sum_{j,a} p_b^\infty(a, j \mid k)\pi_e(a \mid j)g_k(a \mid j)w_j) = 0$$

Without loss of generality, we can construct a new matrix $\tilde{A}$ where $\tilde{A}_i = \tilde{A}_i, i < |\mathcal{S}|$, and $\tilde{A}_{|\mathcal{S}|} = b$.

Then

$$w = \tilde{A}^{-1}\begin{bmatrix} 0_{|\mathcal{S}|-1} \\ 1 \end{bmatrix}$$

Rewrite, with $J_{j,k}$ the all-zeros matrix except for a 1 in the $j, k$ component,

$$\mathbb{E}_{s,a,t \sim \pi_b}[A] = \sum_{j,k} \left( \sum_a \pi_{a,j}^e p_b^\infty(j, a, k) \cdot g_k(a \mid j) \right) J_{j,k} - \mathbb{E}_{s,a,t \sim \pi_b}[\phi_t \phi_t^\top]$$

$$\mathbb{E}_{s,a,t \sim \pi_b}[\tilde{A}] = \sum_{j \not\in \{\mathcal{S}\}, k} \left( \sum_a \pi_{a,j}^e p_b^\infty(j, a, k) \cdot g_k(a \mid j) \right) J_{j,k} + \text{diag}(b)J_{|\mathcal{S}|} - \mathbb{E}_{s,a,t \sim \pi_b}[\phi_t \phi_t^\top]$$
Now we have the following program, with $v = \begin{bmatrix} 0_{|S|-1} \\ 1 \end{bmatrix}$:

$$\min_g \{ \Phi^\top E[\tilde{A}^{-1}]v : l \leq g \leq m, \ Qg = 1 \}$$

Partial derivatives of the matrix-valued function of $g_k(a \mid j)$ follow from the matrix chain rule as

$$\frac{\partial \Phi^\top E[\tilde{A}^{-1}]v}{\partial g_k(a \mid j)} = \sum_{i,j} \frac{\partial \Phi^\top E[\tilde{A}^{-1}]v}{\partial \tilde{A}_{i,j}} \frac{\partial \tilde{A}_{i,j}}{\partial g_k(a \mid j)} = -\sum_{i,j} E[\tilde{A}]_{i,j}^{-\top} \Phi v^\top E[\tilde{A}]_{i,j}^{-\top} (\mathbb{1}[j \neq |S|] \pi_{a,j}^{\infty} p_{j,a,k}^\infty J_{j,k})$$

Proof of Theorem 3. Consistency follows from stability of the optimization problem, as reformulated in Theorem 2, in terms of the deviations of empirical probabilities from their population values. The former is a result from variational analysis/stability analysis of linear programs, which is non-standard because the perturbations occur in the constraint matrix coefficients. The latter is simply the convergence of empirical probabilities.

Stability analysis under left-hand-side perturbations [Robinson 1975]: Stability analysis establishes convergence in Hausdorff distance between $\Theta, \Theta$, the partial identification set obtained from optimizing the sample estimating equation vs. from optimizing the population estimating equation. The Hausdorff distance between two sets $A, B \subset \mathbb{R}^d$ is

$$d_H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$

where $d_H(A, B) = \inf_{b \in B} \| a - b \|; \text{ e.g. it measures the furthest distance from an arbitrary point in one of the sets to its closest neighbor in the other set.}$

The main stability analysis result we use is Theorem 1 of [Robinson 1975]. To help keep the presentation of the theorem self-contained, we state some preliminary notation. The paper considers the general case of a system of linear inequalities, where $A$ is a continuous linear operator from $X$ into $Y$ which are real Banach spaces, and $K$ is a nonempty closed convex cone in $Y$. We study

$$Ax \leq_K b, \forall x \in C,$$

with $C \subseteq X$ a convenience set to represent unperturbed constraints. We want to ascertain the stability region of the solution set $F$, which implies that for each $x_0 \in F$, for some positive number $\beta$, and for any continuous linear operator $A' : X \mapsto Y$ and any $b' \in Y$, the distance from $x_0$ to the solution set of the perturbed system $A'x \leq_K b'$ is bounded by $\beta \rho(x_0)$, with $\rho(x)$ being the residual vector,

$$\rho(x) := d(b' - A'x, K) := \inf\{ \| b' - A'x - k \| : k \in K \}$$

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For a more concise statement, introduce the augmented operator with an auxiliary dimension to homogenize the system \( Q: X \times \mathbb{R} \mapsto Y \): for finite-dimensional systems of linear equations, this is the usual homogenization.

\[
Q\left(\begin{bmatrix} x \\ \xi \end{bmatrix}\right) = \begin{cases} 
\begin{bmatrix} A & -b \\
0 
\end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + K , & \begin{bmatrix} x \\ \xi \end{bmatrix}^\top \in P, \\
0 \\
\begin{bmatrix} x \\ \xi \end{bmatrix}^\top , & \notin P 
\end{cases}
\]

\( e.g. \) \( x \) feasible iff \( 0 \in Q\left(\begin{bmatrix} x \\ 1 \end{bmatrix}\right) \).

Under this notation, \( x \in C \) satisfies \( Ax \leq K, \forall x \in C \) iff \( 0 \in Q\left(\begin{bmatrix} x \\ \xi \end{bmatrix}\right) \). We use the special properties of linear operators as special cases of convex processes (which are themselves multivalued functions between two linear spaces): If \( T \) carries \( X \) into \( Y \), with \( X,Y \) normed linear spaces, then the inverse of \( T \) is \( T^{-1} \), which is defined for \( y \in Y \) by

\[
T^{-1}y := \{ x | y \in Tx \}.
\]

\( T \) is closed if \( \text{gph}(T) := \{ (x,y) | y \in Tx \} \) is closed on product space \( X \times Y \). The norm of \( T \) is operator norm.

This approach then allows us to identify another linear operator which parametrizes the perturbation \( \Delta\left(\begin{bmatrix} x \\ \xi \end{bmatrix}\right) \) defined analogously to \( Q \) but with \( (A' - A)x - (b' - b)\xi + K \). The size of the perturbation is measured by the operator norm of this system, and a crude bound is \( \|\Delta\| \leq \|A' - A\| + \|b' - b\| \); we also have that \( \rho(x) \leq \|\Delta\| \max\{1, \|x\|\} \).

**Assumption 3.** Regularity: \( b \in \text{int}\{A(C) + K\} \) and singular otherwise.

The required regularity assumption is similar to strict consistency of [Rockafellar 1970], which states that \( 0 \in \text{int}(\text{dom})(F) \), e.g. there exists \( u,v \) such that all inequality constraints \( f < 0 \) hold strictly.

Finally, having introduced the homogenized system \( Q \) and the perturbation \( \Delta \), we state the required theorem: \( Q' = Q + \Delta \) is the perturbed augmented system.

**Theorem 5** (Linear system stability (Theorem 1, [Robinson 1975])). Suppose that the system (1) is regular. Then \( Q \) is surjective, \( \|Q^{-1}\| < +\infty \), and if \( \|Q^{-1}\| \|\Delta\| \leq 1 \), then is also regular (hence solvable), with \( \|Q'^{-1}\| \leq \|Q^{-1}\| / (1 - \|Q^{-1}\| \|\Delta\|) \). Further, if \( F' \) denotes the solution set of (2), then for any \( x \in C \) with \( \|Q'^{-1}\| \rho(x) < 1 \) we have

\[
d(x, F') \leq \frac{\|Q'^{-1}\| \rho(x)}{1 - \|Q'^{-1}\| \rho(x)} (1 + \|x\|) \quad (27)
\]

**Proving consistency via stability analysis:**
Consistency follows by applying the stability result of Theorem 5, fixing \( C = \{ \sum w = 1, w \geq 0 \} \) as the set of unperturbed residuals. Regularity, or the existence of a strictly feasible (for inequalities) solution, follows from the extended formulation as long as the bounds on \( W \) are nontrivial.
First we bound $\|Q^{-1}\|$. Without loss of generality, we consider a self-normalized estimate of the objective function such that the density ratios are optimized over $\sum_j w(j) = 1$. We observe that $\|Q^{-1}\| = \max_{\|x\| \leq 1} \{\|x\| : Qx = y, x \in C\} \leq 1$ because the unperturbed constraints include $\sum x = 1, x \geq 0$. Conceptually, we can think of $Q^{-1}y$ as “the perturbed set of $w$ identified under a perturbed estimating equation”: however, since we restrict attention to probability distributions prima facie, these too have norm 1.

Since the perturbation matrix comprises of terms $\tilde{p}^\infty_b(j,a\mid k') - p^\infty_B(j,a\mid k')$, it is sufficient to bound the operator norm of the perturbations by the sup-norm of the deviations of the empirical state-action occupancy probabilities (conditional on next state). and we have that, $\forall \lambda \in \{-1, 1\}$:

$$\|\Delta\|_1 \leq \|\tilde{p}^\infty_b(\cdot, \cdot \mid \cdot)\|_\infty$$

Note that the extreme point analysis from the proof of Theorem 2 allows us to restrict $\lambda \in \{-1, 1\}$ so that the exponentially duplicated constraints do not affect the proof once we bound by the sup-norm of the deviations.

Consistency of the empirical state-action probabilities for the population state-action probabilities yields the result, e.g. that $\hat{p}^\infty_b(j,a,k') \rightarrow_p p^\infty_B(j,a,k')$, and $\hat{p}^\infty_b(k') \rightarrow_p p^\infty_B(k')$, and by applying Slutsky’s theorem. A quantitative statement is possible using e.g. large deviation bounds for the empirical state-action probabilities and taking a union bound over the discrete state and action space. Corollary 5.6 of Altman and Zeitouni [1994] states the large deviation bound

$$\lim_{t \to \infty} \frac{1}{t} \log P_x^\gamma(f^t_s(y) \leq \Delta_1(y) - \delta^{1/2}) \leq -0.5c^2\delta.$$

11 Additional Empirics and Details

Example 1. To illustrate our model and Assumption 2, we introduce a variant of the simplest possible model in Figure 5: the confounded random walk. In the diagram, transitions out of $(s,u)$ pairs are into the larger state pairs since under Assumption 2, the unobserved confounding is exogenously generated. The confounded random walk is parametrized by the transition probabilities under action $a = 1$:

$$p(j \mid j, 1) = c_{u_1}, \quad p(k \mid j, 1) = \frac{1}{2} - c_{u_2},$$
and antisymmetrically under action $a = 2$,

$$p(j \mid j, 1) = \frac{1}{2} - c_{u_1}, \quad p(k \mid j, 1) = c_{u_2}.$$  

**Confounded gridworld**  In Section 11 we include additional examples of the bounds obtained with conditional restrictions of Equation (7) (rather than action-marginalized), and a 0.15 mixture with the behavior policy, which is the full-information optimal. (Therefore there is less uncertainty in evaluation relative to uniform).

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**Figure 6:** Varying evaluation $\eta$ mixture weight from uniform to $\pi^*, u = 0$. Behavior policy is a 0.15 mixture with optimal.  

**Figure 7:** Varying evaluation $\eta$ mixture weight from uniform to $\pi^*$. Behavior policy is a 0.15 mixture with optimal.  

A complete description of the data collection process, including sample size.

- Data: bounds computed based on a trajectory with 40000 steps, and a grid of 25 linearly spaced $\Gamma$ values from $\log(\Gamma) \in [0.1, 1.7]$ (equivalently, $\Gamma \in [1.10, 5.47]$).

- Experiments were run on 16gb Macbook Pro.  

- Packages: Python (Numpy/scipy/pandas), Gurobi Version 9