NONEXISTENCE OF GLOBAL SOLUTIONS FOR AN INHOMOGENEOUS PSEUDO-PARABOLIC EQUATION

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ABSTRACT. In the present paper, we study an inhomogeneous pseudo-parabolic equation with nonlocal nonlinearity

\[ u_t - k\Delta u_t - \Delta u = I^\gamma_0(|u|^p) + \omega(x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^N, \]

where \( p > 1, k \geq 0, \omega(x) \neq 0 \) and \( I^\gamma_0 \) is the left Riemann-Liouville fractional integral of order \( \gamma \in (0, 1) \). Based on the test function method, we have proved the blow-up result for the critical case \( \gamma = 0, p = p_c \) for \( N \geq 3 \), which answers an open question posed in [14], and in particular when \( k = 0 \) it improves the result obtained in [2]. An interesting fact is that in the case \( \gamma > 0 \), the problem does not admit global solutions for any \( p > 1 \) and \( \int_{\mathbb{R}^N} \omega(x)dx > 0 \).

1. Introduction

Recently, Zhou in [14] has investigated the inhomogeneous pseudo-parabolic equation

\[
\begin{aligned}
&u_t - k\Delta u_t - \Delta u = |u|^p + \omega(x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^N, \\
&u(0, x) = u_0(x), \quad x \in \mathbb{R}^N,
\end{aligned}
\tag{1.1}
\]

where \( p > 0, k > 0 \) and \( u_0, \omega \in C_0(\mathbb{R}^N) \).

There was studied the effect of the inhomogeneous term \( \omega(x) \) on the critical exponent \( p_c \) of the problem (1.1), and it was proven that for

\[ p_c = \begin{cases} 
\infty & \text{if } N = 1, 2, \\
\frac{N}{N - 2} & \text{if } N \geq 3,
\end{cases} \]

(a) if \( 1 < p < p_c \), \( u_0 \geq 0 \) and \( \int_{\mathbb{R}^N} \omega(x)dx > 0 \), then the solution of (1.1) blows up in finite time.
(b) if \( p > p_c \), then there exist \( u_0 \geq 0 \) and \( \omega \geq 0 \) such that the problem (1.1) admits global solutions.

Note that the critical case \( p = p_c \) was left open (see [14, Remark 4(b)]).

At first, the problem (1.1) for \( \omega(x) \equiv 0 \) has studied in [4, 11]. It is shown that there exists the critical exponent \( p_F = 1 + \frac{2}{N} \), for the pseudo-parabolic equation. This exponent coincides with the Fujita critical exponent of the semilinear heat equations, which was first introduced by Fujita in [7].

The problem (1.1) with \( k = 0 \) is considered by Bandle et. al. [2]. Namely, it was studied the cases (a), (b) and

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(c) if \( N \geq 3, p = p_c, \int_{\mathbb{R}^N} \omega(x)dx > 0, \omega(x) = O(|x|^{-\varepsilon-N}) \) as \( |x| \to \infty \) for some \( \varepsilon > 0 \), and either \( u \geq 0 \) or
\[
\int_{|x| > R} \frac{\omega^-(y)}{|x-y|^{N-2}}dy = o(1) \quad \text{as} \quad x \to \partial \mathbb{R}^N
\]
when \( R \) is enough large, then (1.1) has no global solutions.

Later on, Jleli et al. [9] generalized these results with the forcing term \( t^\sigma \omega(x) \), \( \sigma > -1 \), and showed the effects of forcing term on the critical exponents.

In this paper, we study the semilinear pseudo-parabolic equation with a forcing term depending on the space
\[
\begin{align*}
\left\{ \begin{array}{ll}
u_t - k\Delta u_t - \Delta u = I^\gamma_{0+}(|u|^p) + \omega(x), & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\
u(0, x) = u_0(x), & x \in \mathbb{R}^N, \end{array} \right.
\end{align*}
\]
where \( p > 1, k \geq 0, \omega(x) \neq 0 \) and \( I^\gamma_{0+} \) is the left Riemann-Liouville fractional integral of order \( \gamma \in [0, 1] \).

We note that the problem (1.2) for \( k = 0 \) and \( \omega(x) \equiv 0 \), was considered in [3, 5, 6, 13].

The main purpose of this paper is to prove a blow-up result for the critical case \( p = p_c \) for \( N \geq 3 \), thereby answering the open question proposed in [14]. In addition, to study the effect of nonlocal nonlinearity in time on the critical exponent.

1.1. Preliminaries.

**Definition 1.1** ([8], p. 69). The left and right Riemann-Liouville fractional integrals of order \( \gamma \in (0, 1) \) for an integrable function \( u(t), \ t \in (0, T) \) are given by
\[
I^\gamma_{0+} u(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} u(s) \, ds
\]
and
\[
I^\gamma_{T-} u(t) = \int_t^T \frac{(s-t)^{\gamma-1}}{\Gamma(\gamma)} u(s) \, ds.
\]
Since \( I^\gamma u(t) \to u(t) \) almost everywhere as \( \gamma \to 0 \) (see [8]), we can let \( I^0 u(t) = u(t) \).

**Definition 1.2** (Weak solution). We say that \( u \in L^p_{\text{loc}}([0, \infty) \times \mathbb{R}^N) \) is a global weak solution to (1.2), if
\[
\begin{align*}
\int_0^T \int_{\mathbb{R}^N} |u|^p (I^\gamma_{T-} \varphi)dxdt + \int_0^T \int_{\mathbb{R}^N} \omega \varphi dxdt &+ \int_{\mathbb{R}^N} u_0(\varphi(0, x) - k\Delta \varphi(0, x))dx \\
= - \int_0^T \int_{\mathbb{R}^N} \varphi_t dxdt + k \int_0^T \int_{\mathbb{R}^N} \varphi_t dxdt - \int_0^T \int_{\mathbb{R}^N} u \Delta \varphi dxdt,
\end{align*}
\]
holds for all \( T > 0 \) and \( \varphi \in C^{1,2}_{t,x}([0, T], \mathbb{R}^N), \varphi \geq 0, \supp \varphi \subset \subset \mathbb{R}^N \) and \( \varphi(T, \cdot) = 0 \).

**Lemma 1.3.** [12, Lemma 3.1] Let \( \omega \in L^1(\mathbb{R}^N) \) and \( \int_{\mathbb{R}^N} \omega(x)dx > 0 \). Then there exists a test function \( 0 \leq \phi \leq 1 \) such that \( \int_{\mathbb{R}^N} \omega \phi dx > 0 \).
2. Main results

In this section, we will show the blow-up of the solution to (1.2) using the test function method.

**Theorem 2.1.** Let $u_0, \omega \in L^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \omega(x) dx > 0$. Then

(i) if $\gamma > 0$, then for any $p > 1$ the problem (1.2) admits no global weak solution.

(ii) if $\gamma = 0$ and $p = p_c = \frac{N}{N-2}$, $N \geq 3$, then the problem (1.2) admits no global weak solution.

**Remark 2.2.** Note that the part (ii) of Theorem 2.1 answers to the open question posed by Zhou in [14].

**Remark 2.3.** When $k = 0$ the equation (1.2) coincides with the heat equation considered in [2], then our results remain true for the heat equation. Note that the part (i) of Theorem 2.1 in the case $k = 0$ improves the result in [1], since we do not assume that $u_0$ is positive. The part (ii) of Theorem 2.1, in case $k = 0$ improves the result in [2]. Since we do not assume some asymptotic properties of the function $\omega(x)$ as in [2], our result improves part (b) of Theorem 2.1 from [2].

**Proof of Theorem 2.1.** We present the proofs of the cases (i) and (ii) separately.

(i) **The case** $\gamma > 0$ **and** $p > 1$. The proof is done by contradiction. Assume that $u$ is a global weak solution to problem (1.2). We choose the test function in the following form

$$\varphi(t, x) = \psi(t)\xi(x),$$

with

$$\psi(t) = \left(1 - \frac{t}{T}\right)^m, \ m > \frac{p + \gamma}{p - 1}, \ t \in [0, T], \ T \in (0, \infty),$$

and

$$\xi(x) = \Phi\left(\frac{|x|^2}{R^2}\right), \ R \gg 1, \ x \in \mathbb{R}^N.$$

Let $\Phi(z) \in C_0^\infty(\mathbb{R}_+)$ be a nonincreasing function

$$\Phi(z) = \begin{cases} 1 & \text{if } 0 \leq z \leq 1, \\ \downarrow & \text{if } 1 < z < 2, \\ 0 & \text{if } z \geq 2. \end{cases}$$

Then, from (1.3) it follows that

$$\int_0^T \int_{\mathbb{R}^N} |u|^p(I_T^t \varphi) dx dt + \int_0^T \int_{\mathbb{R}^N} \omega \varphi dx dt + \int_{\mathbb{R}^N} u_0(\varphi(0, x) - k\Delta \varphi(0, x)) dx \leq \int_0^T \int_{\mathbb{R}^N} |u|\varphi_t dx dt + k \int_0^T \int_{\mathbb{R}^N} |u|\Delta \varphi_t dx dt + \int_0^T \int_{\mathbb{R}^N} |u|\Delta \varphi dx dt. \quad (2.1)$$
Using the $\varepsilon$-Young inequality in the right-side of (2.1) with $\varepsilon = \frac{p}{3}$, we obtain
\[
\int_0^T \int_{\mathbb{R}^N} |u| |\varphi_t| \, dx \, dt \leq \frac{1}{3} \int_0^T \int_{\mathbb{R}^N} |u|^p (I_{T-}^\gamma \varphi) \, dx \, dt \\
+ \frac{p - 1}{p} \left( \frac{p}{3} \right)^{-\frac{1}{p-1}} \int_0^T \int_{\mathbb{R}^N} (I_{T-}^\gamma \varphi)^{-\frac{1}{p-1}} |\varphi_t|^{\frac{p}{p-1}} \, dx \, dt.
\]

Similarly, one obtains
\[
\int_0^T \int_{\mathbb{R}^N} |u| |\Delta \varphi_t| \, dx \, dt \leq \frac{1}{3} \int_0^T \int_{\mathbb{R}^N} |u|^p (I_{T-}^\gamma \varphi) \, dx \, dt \\
+ \frac{p - 1}{p} \left( \frac{p}{3} \right)^{-\frac{1}{p-1}} \int_0^T \int_{\mathbb{R}^N} (I_{T-}^\gamma \varphi)^{-\frac{1}{p-1}} |\Delta \varphi_t|^{\frac{p}{p-1}} \, dx \, dt,
\]
and
\[
\int_0^T \int_{\mathbb{R}^N} |u| |\Delta \varphi| \, dx \, dt \leq \frac{1}{3} \int_0^T \int_{\mathbb{R}^N} |u|^p (I_{T-}^\gamma \varphi) \, dx \, dt \\
+ \frac{p - 1}{p} \left( \frac{p}{3} \right)^{-\frac{1}{p-1}} \int_0^T \int_{\mathbb{R}^N} (I_{T-}^\gamma \varphi)^{-\frac{1}{p-1}} |\Delta \varphi|^{\frac{p}{p-1}} \, dx \, dt.
\]

Therefore, we can rewrite the inequality (2.1) in the following form
\[
\int_0^T \int_{\mathbb{R}^N} \omega \varphi \, dx \, dt + \int_{\mathbb{R}^N} u_0(\varphi(0, x) - k \Delta \varphi(0, x)) \, dx \leq C(p) \left( I_1 + k I_2 + I_3 \right),
\]
where $C(p) = \left( \frac{p}{3} \right)^{-\frac{1}{p-1}}$.

Next, we estimate the integrals $I_1, I_2, I_3$. At this stage, inserting the equality
\[
(I_{T-}^\gamma \psi)(t) = \frac{\Gamma(m+1)}{\Gamma(\gamma + m + 1)} T^\gamma \left( 1 - \frac{t}{T} \right)^{m+\gamma}, \quad t \in [0, T),
\]
\[
(I_{T-}^\gamma \psi)'(t) = \frac{\Gamma(m+1)}{\Gamma(\gamma + m + 1)} T^\gamma \left( 1 - \frac{t}{T} \right)^{m+\gamma-1}, \quad t \in [0, T),
\]
to the term of the above integrals and changing the variable $y = xR$, we obtain
\[
I_1 \leq CT^{-\gamma-1} R^N, \\
I_2 \leq CT^{-\gamma-1} R^N - \frac{2p}{\gamma-1}, \\
I_3 \leq CT^{-\gamma-1} R^N - \frac{2p}{\gamma-1}.
\]
On the other hand, it follows from a simple calculation that
\[
\int_0^T \psi(t) \, dt = \int_0^T \left( 1 - \frac{t}{T} \right)^m \, dt = C(m)T.
\]
Combining (2.2)-(2.5) we arrive at
\[
\int_{\mathbb{R}^N} \omega \xi dx + C(m)T^{-1} \int_{\mathbb{R}^N} u_0(\xi - k\Delta \xi)dx \\
\leq C(p,m) \left( CT^{-\frac{\gamma}{p-1}} R^N + kCT^{-\frac{\gamma}{p-1}} R^{N-\frac{2p}{p-1}} + CT^{-\frac{\gamma}{p-1}} R^{N-\frac{2p}{p-1}} \right).
\]

Finally, fixing \( R \) and passing \( T \to +\infty \) in the last inequality and using Lemma 1.3, we deduce that \( \int_{\mathbb{R}^N} \omega \xi dx \leq 0 \), which is a contradiction.

\textbf{(ii) The critical case} \( \gamma = 0 \) and \( p = p_c = \frac{N}{N-2}, \) \( N \geq 3. \) The proof also will be done by contradiction. Suppose that \( u \) is a global weak solution to (1.2).

Now, following the idea of [10], we set the test function as
\[
\varphi(t,x) = \eta(t)\phi(x),
\]
for large enough \( R,T \)
\[
\eta(t) = \nu \left( \frac{t}{T} \right), \quad t > 0,
\]
and
\[
\phi(x) = \Psi \left( \frac{\ln \left( \frac{|x|}{\sqrt{R}} \right)}{\ln \left( \sqrt{R} \right)} \right), \quad x \in \mathbb{R}^N.
\]

Let \( \nu \in C^\infty(\mathbb{R}) \) be such that \( \nu \geq 0; \ \nu \not\equiv 0; \ \text{supp}(\nu) \subset (0,1), \) and \( \Psi : \mathbb{R} \to [0,1] \) be a smooth function satisfying
\[
\Psi(s) = \begin{cases} 
1, & \text{if } -\infty < s \leq 0, \\
0, & \text{if } s \geq 1.
\end{cases}
\]

and there exist positive constants \( \theta_1, \theta_2 \) such that
\[
|\phi''(x)| \leq \theta_1 |\phi(x)|, \quad |\phi'(x)| \leq \theta_2 |\phi(x)|.
\]

Using the fact that \( \text{supp}(\nu) \subset (0,1), \) we can easily get
\[
\int_{\mathbb{R}^N} u_0(\varphi(0,x) - k\Delta \varphi(0,x))dx = \nu(0) \int_{\mathbb{R}^N} u_0(\phi(x) - k\Delta \phi(x))dx = 0. \quad (2.10)
\]

Then, acting in the same way as in the above case, we get the following estimate
\[
\int_0^T \int_{\mathbb{R}^N} \omega \varphi dx dt \leq C(p) \left( J_1 + kJ_2 + J_3 \right), \quad (2.11)
\]

with
\[
J_1 = \int_0^T \int_{\mathbb{R}^N} \varphi^{-\frac{1}{p-1}} |\varphi|^{\frac{p}{p-1}} dx dt,
\]
\[
J_2 = \int_0^T \int_{\mathbb{R}^N} \varphi^{-\frac{1}{p-1}} |\Delta \varphi|^{\frac{p}{p-1}} dx dt,
\]
\[
J_3 = \int_0^T \int_{\mathbb{R}^N} \varphi^{-\frac{1}{p-1}} |\Delta \varphi|^{\frac{p}{p-1}} dx dt.
\]
In view of (2.6) and (2.7), let us calculate the next integral
\[
\mathcal{J}_2 = \left( \int_0^T \eta^{-\frac{1}{p-1}} |\eta_t|^{\frac{p}{p-1}} \, dt \right) \left( \int_{\mathbb{R}^N} |\Delta \phi|^{\frac{p}{p-1}} \, dx \right).
\]
(2.12)

Indeed, the function \( \phi \) is a radial, and remaining (2.9) we arrive at
\[
|\Delta \phi| = \frac{d^2 \phi}{dr^2} + \frac{N-1}{r} \frac{d\phi}{dr} = \phi'' - \frac{1}{r^2 \ln^2 \sqrt{R}} + \phi' \frac{N-2}{r^2 \ln \sqrt{R}}
\leq \theta_1 \frac{\phi}{r^2 \ln \sqrt{R}} + \theta_2 \frac{\phi}{r^2 \ln \sqrt{R}}
\leq \frac{C}{r^2 \ln R |\phi|},
\]
where \( r = |x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}} \).

Since, \( p = \frac{N}{N-2} \), by inserting the last inequality into (2.12), we can verify that
\[
\int_{\mathbb{R}^N} \phi^{-\frac{1}{p-1}} |\Delta \phi|^{\frac{p}{p-1}} \, dx \leq C\frac{N}{2} (\ln R)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \frac{|\phi|}{|x|^N} \, dx.
\]
Using (2.7) and (2.8), we get
\[
\int_{\mathbb{R}^N} \phi^{-\frac{1}{p-1}} |\Delta \phi|^{\frac{p}{p-1}} \, dx \leq C(\ln R)^{\frac{2-N}{2}}. \tag{2.13}
\]

Similarly, from (2.6) one obtains
\[
\int_0^T \eta^{-\frac{1}{p-1}} |\eta_t|^{\frac{p}{p-1}} \, dt = CT^{-\frac{N}{2}+1}. \tag{2.14}
\]
By combining (2.13)-(2.14), we can rewrite (2.12) as
\[
\mathcal{J}_2 \leq kCT^{-\frac{N}{2}+1}(\ln R)^{\frac{2-N}{2}}.
\]
Consequently, we will estimate the integrals \( \mathcal{J}_1 \) and \( \mathcal{J}_3 \), respectively, in the following form
\[
\mathcal{J}_1 \leq CT^{-\frac{N}{2}+1}R^N
\]
and
\[
\mathcal{J}_3 \leq CT^1(\ln R)^{\frac{2-N}{2}}.
\]
Finally, we deduce that
\[
\int_{\mathbb{R}^N} \omega \phi \, dx \leq C(p) \left( CT^{-\frac{N}{2}} R^N + kCT^{-\frac{N}{2}} (\ln R)^{\frac{2-N}{2}} + C(\ln R)^{\frac{2-N}{2}} \right). \tag{2.15}
\]
Now for \( T = R^j, j > 0 \), we get
\[
\int_{\mathbb{R}^N} \omega \phi \, dx \leq C \left( CR^{-\frac{N(j-2)}{2}} + kCR^{-\frac{(j-2)}{2}} (\ln R)^{\frac{2-N}{2}} + C(\ln R)^{\frac{2-N}{2}} \right).
\]
Taking \( j > 2 \) and passing to the limit as \( R \to \infty \) in the above inequality and in view of Lemma 1.3, we deduce that \( \int_{\mathbb{R}^N} \omega \phi \, dx \leq 0 \), which is a contradiction. \( \square \)
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