Binary and Multivariate Stochastic Models of Consensus Formation

Maxi San Miguel, Víctor M. Eguíluz, and Raúl Toral
IMEDEA (CSIC-UIB), Campus Universitat Illes Balears, E-07122 Palma de Mallorca, Spain
Konstantin Klemm
Department of Bioinformatics, University Leipzig, Hartelstr. 16-18, 04107 Leipzig, Germany

Introduction

A current paradigm in computer simulation studies of social sciences problems by physicists is the emergence of consensus [1, 2, 3, 4, 5, 6]. The question is to establish when the dynamics of a set of interacting agents that can choose among several options (political vote, opinion, cultural features, etc.) leads to a consensus in one of these options, or when a state with several coexisting social options prevail. The latter is called a polarized state. An important issue is to identify mechanisms producing a polarized state in spite of general convergent dynamics. When the agents are spatially distributed this problem shares many characteristics with the problem of domain growth in the kinetics of phase transitions [7]: Consensus emerges when a single spatial domain grows occupying the whole system, while polarization corresponds to a situation in which the system does not order and different spatial domains compete.

We consider here stochastic dynamic models naturally studied by computer simulations. We will first review some basic results for the voter model [8]. This is a binary option stochastic model, and probably the simplest model of collective behavior. We focus on the dynamical effect of who interacts with whom, that is on the consequences of different networks of interaction. The fact whether consensus is reached or not depends on characteristics of the network such as dimensionality. Next we consider a model proposed by Axelrod [9] for the dissemination of culture. This model can be considered as a multivariable elaboration of the voter model dynamics. Time scales of evolution scale with system size in this model in the same way as for the voter model. We also discuss for this model the role of different networks of interaction. Finally we consider the effect of exogenous stochastic perturbations that account for cultural drift.

Voter model

The voter model [8, 10, 11, 12, 13, 14, 15, 16] is defined by a set of “voters” with two opinions or spins $s_i = \pm 1$ located at the nodes of a network. The elementary dynamical step consists in randomly choosing one node (asynchronous update) and assigning to it the opinion, or spin value, of one of its nearest neighbors, also chosen at random. This mechanism of opinion formation reflects complete lack of self-confidence of the agents. It could be appropriate for describing processes of opinion formation in certain groups of teenagers in which imitation is prevalent. The dynamical rule implemented here corresponds to a node-update. An alternative
dynamics is given by a link-update rule in which the elementary dynamical step consists in randomly choosing a pair of nearest neighbor spins, i.e. a link, and randomly assigning to both nearest neighbor spins the same value if they have different values, and leaving them unchanged otherwise. These two updating rules are equivalent in a regular lattice, but they are different in a complex network in which different nodes have different number of nearest neighbors [14].

The voter model dynamics has two absorbing states, corresponding to situations in which all the spins have converged to the $s_i = 1$ or to the $s_i = -1$ consensus states. The ordering dynamics is stochastic and driven by interfacial noise. This is different of the ordering dynamics of a Glauber kinetic Ising model which is driven by minimization of surface tension. A standard order parameter to describe the ordering process [12, 13] is the average of the interface density $\rho$, defined as the density of links connecting sites with different spin value. In a disordered configuration with randomly distributed spins $\rho \simeq 1/2$, while for a completely ordered system $\rho \simeq 0$. In regular lattices of dimensionality $d \leq 2$ the system orders. This means that, in the limit of large systems, there is a coarsening process with unbounded growth of spatial domains of one of the absorbing states: consensus is reached. The asymptotic regime of approach to the ordered state is characterized in $d = 1$ by a power law $\langle \rho \rangle \sim t^{-1/2}$, while for the critical dimension $d = 2$ a logarithmic decay is found $\langle \rho \rangle \sim (\ln t)^{-1}$ [10, 12]. Here the average $\langle \cdot \rangle$ is an ensemble average.

In regular lattices with $d > 2$, as well as in small-world networks [17] and scale-free Barabási-Albert networks [18], the voter dynamics does not order the system in the thermodynamic limit of large systems [11, 13, 14]. Starting from a random initial condition and after an initial transient, the system falls in a metastable partially ordered state. In the initial transient of a given realization of the process, $\rho$ initially decreases, indicating a partial ordering of the system. After this initial transient $\rho$ fluctuates randomly around an average plateau value. In a finite system the metastable state has a finite lifetime: a finite size fluctuation takes the system from the metastable state to one of the two ordered absorbing states. In this process the fluctuation orders the system and $\rho$ changes from its metastable plateau value to $\rho = 0$ (see Fig. 1). The lifetime $\tau$ of the metastable state, for a regular lattice in $d = 3$ [11] and also for a small-world network [13], scales linearly with the system size $N$, $\tau \sim N$, while a scaling $\tau \sim N^{0.88}$ has been found [14] for the voter model in the scale-free Barabási-Albert network. The fact that a large system does not order in a small-world or scale-free network could seem counter-intuitive: one might argue that long distance links (small-world) or nodes with a large number of links (hubs in a scale-free network) should be instrumental in ordering the system. A counter argument is that what is observed corresponds to a network of large dimensionality: these complex networks have an effective infinite dimension since the average path length between two nodes grows logarithmically (or slower) with the system size.

In order to understand the different role of dimensionality and degree distribution, i.e., the probability for a node having $k$ links (degree), one can consider the voter dynamics in the
Structured scale-free (SSF) network introduced in Ref. [19]. The SSF networks are scale-free, with a degree distribution $P(k) \sim k^{-3}$ but are effectively one dimensional since the average path length scales linearly with system size $L \sim N$. Results of simulations shown in Fig. 2 indicate that the dynamics of the voter model in the SSF network or in a regular $d = 1$ network is essentially the same: the system orders with the average interface density decreasing with a power law with characteristic exponent $1/2$. This identifies dimensionality and not degree distribution as the relevant parameter to classify different classes of ordering dynamics of the voter model in complex networks.

The voter model can also be studied in other different complex networks of dimension $d > 1$ characterized by a parameter $p$ measuring the disorder of the network. This parameter is the one originally used to characterize a small-world network [17], varying continuously from $p = 0$ (regular network) to $p = 1$ (random network). One finds that network disorder decreases the lifetime of the metastable disordered states. Likewise, the lifetime of these states is decreased when the networks have nodes with a large number of links [16].

**Axelrod model**

Axelrod [9] addressed the issue of the persistence of cultural diversity asking the following question: *if people tend to become more alike in their beliefs, attitudes and behavior when they interact, why do not all differences eventually disappear?* To answer this question he proposed a model to explore mechanisms of competition between globalization (consensus) and coexistence of several cultural options (polarization). The basic premise of the model is that the more similar an actor is to a neighbor, the more likely the actor will adopt one of the neighbor’s traits. In addition to treating culture as multidimensional (not a binary option), a novelty of the model is that its dynamics takes into account the interaction between the different cultural features. The model is defined by considering $N$ agents as the nodes of a network of interaction. The state of agent $i$ is a vector of $F$ components (cultural features) $(\sigma_{i1}, \sigma_{i2}, \cdots, \sigma_{iF})$. Each $\sigma_{if}$ is one of the $q$ integer values (cultural traits) $1, \ldots, q$, initially assigned independently and with equal probability $1/q$. The time-discrete dynamics is defined as iterating the following steps:

1. Select at random a pair of sites of the network connected by a link $(i, j)$.
2. Calculate the overlap (number of shared features $\sigma_{ik} = \sigma_{jk}$) $l_{ij}$.
3. If $0 < l_{ij} < F$, the link is said to be active and sites $i$ and $j$ interact with probability $l_{ij}/F$ (similarity rule). In case of interaction, choose $g$ randomly such that $\sigma_{ig} \neq \sigma_{jg}$ and set $\sigma_{ig} = \sigma_{jg}$.

The model has $q^F$ equivalent cultural options. Consensus (global culture) is reached if a domain of one of these options occupies the whole system. For $q = 2$ Axelrod’s model can be
viewed as a set of $F$ coupled voter models. For a general value of $q$ it still shares with the voter model the basic stochastic dynamics driven by interfacial noise as shown in Fig. 3: An initial condition of a bubble of one of the $q^F$ cultures on the background of another cultural option with only one feature in common dissolves by interfacial noise. Several snapshots of the dynamical evolution from random initial conditions in a $d=2$ square lattice are shown in Fig. 4. For a given value of $F$ the evolution from initial random conditions leads to a state of global culture (consensus) or a multicultural state depending on the value of $q$. The parameter $q$ is a measure of the degree of initial disorder in the system. The fact that multicultural disordered states are reached illustrates how local convergence, enforced by the similarity rule used in the dynamics, can generate global polarization.

A systematic analysis of the dependence on $q$ can be carried out from the point of view of Statistical Physics through Monte Carlo computer simulations. Defining an order parameter as the mean value of the relative size of the largest homogeneous cultural domain $S_{\text{max}}$, one finds a nonequilibrium order-disorder transition as shown in Fig. 5 for a $d=2$ square lattice: There exists a threshold value $q_c$, such that for $q < q_c$ the system orders in a consensus monocultural uniform state ($\langle S_{\text{max}} \rangle /N \sim 1$), while for $q > q_c$ the system freezes in a polarized or multicultural state ($\langle S_{\text{max}} \rangle \ll N$). The transition becomes sharp and well defined for large systems and it is a first-order transition in $d=2$, while it becomes a continuous transition in $d=1$. In $d=1$ the Axelrod dynamics is an optimization dynamics for which a Lyapunov potential can be found. We note that $F = 2$ is a special case that we do not discuss here. We also note that $q_c$ and the transition itself is defined considering the dynamical evolution form an initial random disordered configuration and not for arbitrary initial conditions. We use here a set of uniform random initial conditions, while other authors have used a Poisson distribution for the initial random values of $q$.

**Axelrod model in complex networks**

The network of interactions among the agents accounts for the local geography in Axelrod’s model. Following our discussion of the voter model, it is natural to ask how the above results for a regular network are modified when considering a complex network of interaction. An expectation is that with random long distance interactions, the heterogeneity sustained by local interactions can no longer be maintained. For a small-world network it is found that the transition remains sharply defined as the system size increases, but it is shifted to larger values of $q$ as the disorder parameter $p$ is increased. So that, as expected, small-world connectivity favors cultural globalization. This is shown in the phase diagram of Fig. 6 in which we observe that for a given value of $q$ in which the system is in a polarized state in regular network, consensus (global culture) can be reached by increasing the disorder parameter of the network, $p$.

In a scale-free Barabási-Albert network the order-disorder transition of the Axelrod
model becomes system-size dependent and the critical value $q_c$ is shifted to larger and larger values as $N$ increases, so that a state of global culture (consensus) prevails in the limit of large systems. In addition, for a fixed large value of $N$ and fixed average connectivity $<k>$, $q_c$ is larger in a scale-free network than the limiting value of $q_c$ found for $p=1$ in a small-world network: The scale free connectivity is more efficient than a random connectivity ($p=1$) in promoting global culture. These results for the Axelrod model in small-world and scale-free networks parallel what happens for a kinetic Ising model: the small-world connectivity increases the critical temperature, while the critical temperature diverges with system size in a scale-free network.

Similarly to the discussion of the voter model, we can ask here about the specific role of the degree distribution in the fact that the transition disappears for a large systems in a scale-free Barabási-Albert network. Considering again the Structured scale-free (SSF) network introduced in Ref. [19] we find that the transition remains here well defined at a finite value of $q$ for large systems. The conclusion is that it is the spatial dimensionality of the interaction network, and not just the presence of hubs, what gives rise to the divergence of $q_c$ with $N$. On the other hand, hubs create local order in the system so that for the multicultural disordered state in a SSF network $<S_{\text{max}}>$ takes a finite value.

### Cultural drift: Exogenous perturbations in Axelrod model

Among the open questions discussed by Axelrod in his original work [9] he mentions that Perhaps the most interesting extension and at the same time, the most difficult one to analyze is cultural drift, and he suggests to model it as spontaneous changes of cultural traits. Cultural drift takes into account that there is always some influence between neighbors even when they have completely different cultures. In the language of physics simulations he is asking about the role of noise in the order-disorder transition discussed above. The stochastic dynamics giving rise to this transition can be considered as a zero temperature dynamics. The question is if this transition is robust against the presence of fluctuations, or if any finite fluctuation disorders the system, as it happens in the $d=1$ kinetic Ising model. Generally speaking noise is known to have two different effects, one is to produce disorder by accumulation of fluctuations, but another one is to help the system in finding paths in which it can escape from frozen disordered configurations, leading to ordered states. An alternative way of formulating the question is then if external perturbations acting on a frozen multicultural state can take the system to the consensus state. To address these issues we implement cultural drift in the model adding a fourth step in the iterated loop of the dynamics defined above [26]:

4. With probability $r$, perform a single feature perturbation in which randomly choosing an agent $i$ and one of its features $f$, the trait $\sigma_{if}$ is replaced by a new randomly chosen value.

Simulation results for a $d=2$ square lattice are shown in Fig. 7: We observe a transition from multicultural to consensus states controlled by an effective noise rate $r' = r(1 - 1/q)$. The
factor $(1 - 1/q)$ takes into account the probability that the single feature perturbation does not change the value of the trait. This is a noise induced transition since the control parameter is a noise property. In addition, the transition has universal scaling properties with respect to $q$: the same result is found for different values of $q$ and a consensus state is found for any value of $q$ as $r$ goes to zero. Therefore, cultural drift destroys the transition controlled by $q$ that was found in the absence of exogenous perturbations ($r = 0$). In this sense, noise is here an essential parameter that changes completely the type of transition exhibited by the system. An additional important point is the character of the states found at both sides of the noise induced transition. The disordered multicultural state found for large $r$ is no longer a frozen configuration, but rather it exhibits disordered noise-sustained dynamics. On the other hand, the consensus or ordered state found for small $r$ is metastable: Once one of the equivalent $q^F$ cultural states is reached, the systems does not stay there forever, but eventually a fluctuation takes it from this state to another one of the equivalent $q^F$ states, as shown in Fig. 8.

Why does the noise rate cause a transition? There is here a competition between two time scales, the time scale at which noise is acting $(1/r)$ and the relaxation time of perturbations $T$. For small noise rate $r$ there is time to relax and the system decays to a consensus state, while for a large noise rate, stochastic perturbations accumulate and multicultural disorder is built up. The transition is then expected to occur for $rT \sim 1$. The relaxation time $T$ of perturbations can be calculated as an exit time in a random walk [24, 26]. In a mean field approximation it is given as the time needed to reach consensus in a finite system following the voter model dynamics. For a $d = 2$ square lattice this is $T \sim N \ln N$ [11, 26]. The noise induced transition occurs then for a system size dependent value of $r$, but curves as the ones plotted in Fig. 8 for different values of $N$ collapse into a single curve when plotted versus $rN \ln N$ [26]. The general result is that in the limit of very large systems, disordered multicultural states prevail at any noise rate. Therefore cultural drift causes global polarization in large systems, but as a state with noise-sustained dynamics rather than a frozen configuration of spatially coexisting equivalent cultures.

**Summary**

We have reviewed some aspects of stochastic dynamical models of consensus formation. The simple voter model has been used to illustrate how this stochastic dynamics is very much affected by the spatial background in which it takes place: Different characteristics of the network of interactions determine if consensus grows in the system or if a polarized disordered state prevails. We have also considered these questions in a related model due to Axelrod which goes beyond the usual binary options of spin models and that also incorporates interaction among multivalued options. For this model we have also shown that exogenous stochastic perturbations are essential, since they completely change the nature of the states reached by the system in its dynamical evolution. An interesting open question for future developments is
to go beyond the static networks of interaction considered here, allowing for a co-evolution of the network and the state of the agents in the network. This general idea of co-evolution has been implemented already in other computer simulations of social dynamics [27].

We acknowledge the collaboration of K. Suchecki in the original studies of the voter model dynamics. We also acknowledge financial support from MEC (Spain) through project CONOCE2 (FIS2004-00953)

References

[1] K. Sznajd-Weron and J. Sznajd, Opinion evolution in closed community, Int.J. Mod. Phys. C, 11, 1157-1165, (2000); Physica A, Who ist left, who is light?, 351, 593-604, (2005).

[2] G. Deffuant, D. Neau, F. Amblard, G. Weisbuch, Mixing beliefs among interacting agents, Adv. Complex Syst. 3, 87-98 (2000).

[3] S. Galam, B. Chopard and M. Droz, Killer geometries in competing species dynamics, Physica A 314, 256-263 (2002).

[4] D. Stauffer, Sociophysics Simulations, Computing in Science and Engineering, 5, 71-75 (2003); How to Convince Others? Monte Carlo Simulations of the Sznajd Model, AIP Conference Proceedings, 690, 147-155 (2003).

[5] D. Stauffer, A. Sousa, and C. Schulze, Discretized Opinion Dynamics of The Deffuant Model on Scale-Free Networks, J. Artificial Societies and Social Simulation 7, issue 3, paper 7 (2004).

[6] C. Tessone, R. Toral, P. Amegual, S.H. Wio and M. San Miguel, Neighborhhod models of opinion formation, European Physical Journal B 39, 535-544 (2004).

[7] J.D. Gunton, M. San Miguel, and P.S. Sahni, in *Phase Transitions and Critical Phenomena*, Vol 8, pp. 269-466. Eds. C. Domb and J. Lebowitz (Academic Press, London 1983).

[8] T.M. Liggett, *Interacting Particle Systems* (Springer, New York 1985).

[9] R. Axelrod, The dissemination of culture: A model with local convergence and global polarization, J. Conflict Res. 41, 203-226 (1997).

[10] L. Frachebourg and P.L. Krapivsky, Exact results for kinetics of catalytic reactions, Phys. Rev. E 53, R3009-3012 (1996).
[11] P.L. Krapivsky, Kinetics of monomer-monomer surface catalytic reactions, Phys. Rev. A 45, 1067-1072 (1992).

[12] I. Dornic, H. Chaté, J. Chavé, and H. Hinrichsen, Critical Coarsening without Surface Tension: The Universality Class of the Voter Model, Phys. Rev. Lett. 87, 045701-045074 (2001).

[13] C. Castellano, D. Vilone, and A. Vespignani, Incomplete ordering of the voter model on small-world networks, Europhysics Letters 63, 153-158 (2003).

[14] K. Suchecki, V. M. Eguíluz, and M. San Miguel, Conservation laws for the voter model in complex networks, Europhysics Letters 69, 228-234 (2005).

[15] V. Sood and S. Redner, Voter Model on Heterogeneous Graphs, Phys. Rev. Lett. 94, 178701-178704 (2005).

[16] K. Suchecki, V. M. Eguíluz, and M. San Miguel, Voter model dynamics in complex networks: Role of dimensionality, disorder and degree distribution, cond-mat/0504482.

[17] D.J. Watts and S.H. Strogatz, Collective dynamics of ’small-world’ networks, Nature 393, 440-443 (1998).

[18] A. L. Barabási and R. Albert, Emergence of Scaling in Random Networks, Science 286, 509-512 (1999).

[19] K. Klemm and V.M. Eguíluz, Highly clustered scale-free networks, Phys. Rev. E 65, 036123-036127 (2002).

[20] Interactive simulations of the Axelrod model are available at http://www.imedea.uib.es/physdept/research_topics/socio/culture.html

[21] C. Castellano, M. Marsili, A Vespignani, Nonequilibrium Phase Transition in a Model for Social Influence, Phys. Rev. Lett. 85, 3536-3539 (2000).

[22] K. Klemm, V.M. Eguíluz, R. Toral, and M. San Miguel, Role of dimensionality in Axelrod’s model for the dissemination of culture, Physica A 327, 1-5 (2003).

[23] D. Vilone, A. Vespignani, and C. Castellano, Ordering phase transition in the one-dimensional Axelrod model, Eur. Phys. J. B 30, 399-406 (2002).

[24] K. Klemm, V.M. Eguíluz, R. Toral, and M. San Miguel, Globalization, polarization and cultural drift, J. Econ. Dyn. Control 29, 321-334 (2005).
[25] K. Klemm, V.M. Eguíluz, R. Toral, and M. San Miguel, Nonequilibrium transitions in complex networks: a model of social interaction, Phys. Rev. E 67, 026120(1-6) (2003).

[26] K. Klemm, V.M. Eguíluz, R. Toral, and M. San Miguel, Global culture: A noise induced transition in finite systems, Phys. Rev. E 67, 045101(1-4)(R) (2003).

[27] V.M. Eguíluz, M.G. Zimmermann, C. Cela-Conde and M. San Miguel, Role differentiation in the dynamics of social networks, American J. Sociology 110, 977-1008 (2005).
Figure 1: Interface density evolution for an individual realization in a scale-free Albert-Barabasi network with $N = 10000$ nodes and average connectivity $< k > = 8$. 
Figure 2: Mean interface density evolution in a regular $d = 1$ network and in a Structured scale-free network as indicated. The average is over 1000 realizations. $N = 10000$ and $< k > = 8$. The continuous line indicates a power law decay with exponent $-1/2$. 
Figure 3: Snapshots of the time evolution of Axelrod model at times $t = 0, 114, 272, 1331$. Different colors indicate different cultural states. System size $N = 128 \times 128$. Parameter values $F = 3, q = 15$.

Figure 4: Snapshots of the time evolution of Axelrod model from random initial conditions at times $t = 0, 1000, 3000, 6807$. At time $t = 6807$ the dynamics stops and the configuration is frozen. System size $N = 32 \times 32$. Parameter values $F = 3, q = 10$. 
Figure 5: Normalized order parameter $< S_{\text{max}} > / N$ as a function of $q$ for $d = 2$ square lattices of sizes $N = 50 \times 50$ and $N = 100 \times 100$ for $F = 10$. 
Figure 6: Phase diagram for the Axelrod model in a small-world network of size $N = 500^2$ for $F = 10$. The shaded area are $(q, p)$ parameters for which a polarized or multicultural state is reached. The other side of the continuous curve corresponds to parameters for which consensus (state of global culture) is reached [25].
Figure 7: Normalized order parameter $< S_{\text{max}} > / N$ as a function of the effective noise rate $r'$ for different values of $q$ in a $d = 2$ square lattice of size $N = 50 \times 50$ and $F = 2$ [20].
Figure 8: Snapshots of the time evolution of Axelrod model with exogenous perturbations in a $d = 2$ square lattice of size $N = 32 \times 32$ with $F = 3$, $q = 2$ and $r = 0.000017$. A random configuration is chosen at the initial time $t = 0$. Snapshots are shown at times $t = 1650, 5519, 180000, 204000$. At time $t = 1650$ the system is evolving to a metastable consensus state reached at $t = 5519$. The system remains there for a long time until a large enough fluctuation of another equivalent consensus state occurs ($t = 180000$) and takes the system to that state ($t = 204000$).