Girth and $\lambda$-choosability of graphs

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Abstract
Assume $k$ is a positive integer, $\lambda = \{k_1, k_2, \ldots, k_q\}$ is a partition of $k$ and $G$ is a graph. A $\lambda$-assignment of $G$ is a $k$-assignment $L$ of $G$ such that the colour set $\bigcup_{v \in V(G)} L(v)$ can be partitioned into $q$ subsets $C_1 \cup C_2 \cup \cdots \cup C_q$ and for each vertex $v$ of $G$, $|L(v) \cap C_i| = k_i$. We say $G$ is $\lambda$-choosable if for each $\lambda$-assignment $L$ of $G$, $G$ is $L$-colourable. In particular, if $\lambda = \{k\}$, then $\lambda$-choosable is the same as $k$-choosable, and if $\lambda = \{1, 1, \ldots, 1\}$, then $\lambda$-choosable is equivalent to $k$-colourable. For the other partitions of $k$ sandwiched between $\{k\}$ and $\{1, 1, \ldots, 1\}$ in terms of refinements, $\lambda$-choosability reveals a complex hierarchy of colourability of graphs. Assume $\lambda = \{k_1, \ldots, k_q\}$ is a partition of $k$ and $\lambda'$ is a partition of $k' \geq k$. We write $\lambda \leq \lambda'$ if there is a partition $\lambda'' = \{k''_1, \ldots, k''_q\}$ of $k'$ with $k''_i \geq k_i$ for $i = 1, 2, \ldots, q$ and $\lambda'$ is a refinement of $\lambda''$. It follows from the definition that if $\lambda \leq \lambda'$, then every $\lambda$-choosable graph is $\lambda'$-choosable. It was proved in Zhu that the converse is also true. This paper strengthens this result and proves that for any $\lambda \not\leq \lambda'$, for any integer $g$, there exists a graph of girth at least $g$ which is $\lambda$-choosable but not $\lambda'$-choosable.

Keywords
$\lambda$-choosability, girth, integer partition, partition refinement

1 | INTRODUCTION

A proper $k$-colouring of a graph $G$ is a colouring $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that adjacent vertices receive different colours. The chromatic number of $G$ is the minimum integer $k$ such that $G$ has a proper $k$-colouring. The girth of $G$ is the smallest length of a cycle in $G$. If $G$ has girth $g$, then for any vertex $v$, the subgraph of $G$ induced by vertices at distance at most $g/2 - 1$
from \( v \) is a tree. Hence large girth graphs are ‘locally’ 2-colourable. A natural question is whether locally 2-colourable graphs can have a large chromatic number. This question was answered in the affirmative by Erdős et al. \[2\]: For any positive integers \( g, k \), there exists a graph \( G \) of girth at least \( g \) and chromatic number at least \( k \). This classical result is one of the most influential results in graph theory and has been generalized in many different ways. We may view the chromatic number as a scale that measures certain complexity of graphs. Erdős’ result assures the existence of large girth graphs with given complexity with respect to this scale. By considering different measurements for graphs, one obtains various generalizations of this result \[1, 2, 7\]. For example, by considering the partial order of graph homomorphisms, it was proved in \[5\] that for any core graph \( H \) and integers \( g, t \), there exists a graph \( G \) of girth at least \( g \) such that homomorphisms from \( G \) to any graph \( H' \) of order at most \( t \) are composition of a homomorphism from \( G \) to \( H \) and a homomorphism from \( H \) to \( H' \).

This paper generalizes Erdős’ result with respect to a new measurement of colourability of graphs, which is a generalization of list colouring of graphs. An assignment of a graph \( G \) is a mapping \( L \) which assigns to each vertex \( v \) of \( G \) a set \( L(v) \) of permissible colours. A proper \( L \)-colouring of \( G \) is a proper colouring \( f \) of \( G \) such that for each vertex \( v \) of \( G \), \( f(v) \in L(v) \). We say \( G \) is \( L \)-colourable if \( G \) has a proper \( L \)-colouring. A \( k \)-assignment of \( G \) is an assignment \( L \) with \( |L(v)| = k \) for each vertex \( v \). We say \( G \) is \( k \)-choosable if \( G \) is \( L \)-colourable for any \( k \)-assignment \( L \) of \( G \). The choice number of \( G \) is the minimum integer \( k \) such that \( G \) is \( k \)-choosable.

The concept of list colouring was introduced by Erdős et al. \[3\], and independently by Vizing \[6\] in the 1970s, and provides a useful tool in many inductive proofs for upper bounds for the chromatic number of graphs, and motivates many challenging problems. There is a big gap between \( k \)-colourability and \( k \)-choosability. In particular, bipartite graphs can have arbitrary large choice number. A refinement of the concept of choosability was introduced in \[8\], which puts \( k \)-choosability and \( k \)-colourability in a same framework and considers a much more complex hierarchy of colourability of graphs.

**Definition 1.** A partition of a positive integer \( k \) is a finite multiset \( \lambda = \{k_1, k_2, ..., k_q\} \) of positive integers with \( k_1 + k_2 + \cdots + k_q = k \). Each integer \( k_i \in \lambda \) is called a part of \( \lambda \).

**Definition 2.** Assume \( \lambda = \{k_1, k_2, ..., k_q\} \) is a partition of \( k \) and \( G \) is a graph. A \( \lambda \)-assignment of \( G \) is a \( k \)-assignment \( L \) of \( G \) in which the colours in \( \cup_{x \in V(G)} L(x) \) can be partitioned into sets \( C_1, C_2, ..., C_q \) so that for each vertex \( x \) and for each \( 1 \leq i \leq q \), \( |L(x) \cap C_i| = k_i \). Each \( C_i \) is called a colour group of \( L \). We say \( G \) is \( \lambda \)-choosable if \( G \) is \( L \)-colourable for any \( \lambda \)-assignment \( L \) of \( G \).

Assume \( \lambda \) and \( \lambda' \) are two partitions of \( k \). We say \( \lambda' \) is a refinement of \( \lambda \) if \( \lambda' \) is obtained from \( \lambda \) by replacing some parts of \( \lambda \) by partitions of these parts. For example, \( \lambda' = \{2, 3, 4\} \) is a refinement of \( \lambda = \{4, 5\} \). It follows from the definition that if \( \lambda' \) is a refinement of \( \lambda \), then every \( \lambda' \)-assignment of a graph \( G \) is also a \( \lambda \)-assignment of \( G \). Hence every \( \lambda \)-choosable graph is \( \lambda' \)-choosable.

It is easy to see that if \( \lambda = \{1, 1, ..., 1\} \) consists of \( k \) copies of \( 1 \), then \( \lambda \)-choosable is the same as \( k \)-colourable. On the other hand, \( \{k\} \)-choosable is the same as \( k \)-choosable. So \( \lambda \)-choosability puts \( k \)-colourability and \( k \)-choosability of graphs under a same framework, and \( \lambda \)-choosability for those partitions \( \lambda \) of \( k \) sandwiched between \( \{k\} \) and \( \{1, 1, ..., 1\} \) (in terms of refinements) reveal a complicated hierarchy of colourability of graphs.
Definition 3. Assume $\lambda = \{k_1, ..., k_q\}$ is a partition of $k$ and $\lambda'$ is a partition of $k' \geq k$. We write $\lambda \leq \lambda'$ if there is a partition $\lambda'' = \{k''_1, ..., k''_q\}$ of $k'$ with $k''_i \geq k_i$ for $i = 1, 2, ..., q$ and $\lambda'$ is a refinement of $\lambda''$.

For example, $\lambda = \{2, 2\}$ is a partition of 4, and $\lambda' = \{1, 1, 1, 3\}$ is a partition of 6. Let $\lambda'' = \{2, 4\}$. Then $\lambda''$ is obtained from $\lambda$ by increasing one part of $\lambda$ by 2, and $\lambda'$ is a refinement of $\lambda''$. Hence $\lambda \leq \lambda'$.

If $\lambda''$ is obtained from $\lambda$ by increasing some of parts of $\lambda$, then certainly every $\lambda$-choosable graph is $\lambda''$-choosable. If $\lambda'$ is a refinement of $\lambda''$, then every $\lambda''$-choosable graph is $\lambda'$-choosable. Therefore if $\lambda \leq \lambda'$, then every $\lambda$-choosable graph is $\lambda'$-choosable. It was proved in [8] that if $\lambda \nleq \lambda'$, then there exists a graph which is $\lambda$-choosable but not $\lambda'$-choosable.

Theorem 1 (Zhu [8]). If $\lambda \leq \lambda'$, then every $\lambda$-choosable graph is $\lambda'$-choosable, and conversely, if every $\lambda$-choosable graph is $\lambda'$-choosable, then $\lambda \leq \lambda'$.

In this paper, we prove the following result, which strengthens Theorem 1, and generalizes Erdős’ result to the setting of $\lambda$-choosability of graphs.

Theorem 2. Assume $\lambda, \lambda'$ are partitions of integers and $\lambda \nleq \lambda'$. For any positive integer $g$, there exists a graph $G$ of girth at least $g$ which is $\lambda$-choosable but not $\lambda'$-choosable.

2 | PROOF OF THEOREM 2

The proof of Theorem 2 uses the probabilistic method. One new ingredient in the proof is to split vertices of a large girth graph appropriately and then add copies of some other graphs and ensure that the resulting graph still has large girth and some other required properties of a random graph.

In the calculations in our proof, the following three inequalities involving binomial coefficients will be used:

1. $\binom{a}{b} \leq \left(\frac{ea}{b}\right)^b$;
2. $\binom{a-x}{b} \binom{a}{b}^{-1} = \frac{(a-x)!}{(a-x-b)!} \frac{b!(a-b)!}{a!} = \prod_{i=0}^{b-1} \left(1 - \frac{x}{a-i}\right) \leq \left(1 - \frac{x}{a}\right)^b < e^{-bx/a}$;
3. $\binom{a-x}{b-x} \binom{a}{b}^{-1} \leq \left(\frac{b}{a}\right)^x$.

Lemma 3. For any positive integers $q, g, t$ and $0 < \varepsilon < 1/4g$, for sufficiently large $n$, there exists a $q$-partite graph $G_0$ with partite sets $V_1, V_2, ..., V_q$, which has the following properties:

1. All the parts have size $n$.
2. The girth of $G_0$ is at least $g$. 


3. For any $1 \leq i, j \leq q$ with $i \neq j$ and any subsets $A \subseteq V_i, B \subseteq V_j$ with $|A|, |B| \geq \lfloor n/t \rfloor$, there are at least $\frac{1}{2} n^{1+\varepsilon}$ edges between $A$ and $B$.

**Proof.** Let $F$ be a complete $q$-partite graph with partite set $V_1, V_2, ..., V_q$ and every part has size $n$. Let $k = \frac{q(q-1)}{2}$, then $F$ has $kn^2$ edges. Let $\mathcal{G}$ be the set of all subgraphs $G$ of $F$ with $m = \lfloor kn^{1+2\varepsilon} \rfloor$ edges. Then $|\mathcal{G}| = \left( \begin{array}{c} kn^2 \\ m \end{array} \right)$. In the following, $n$ is assumed to be sufficiently large. We consider $\mathcal{G}$ as a probability space with each member occurring with the same probability $1/|\mathcal{G}|$. □

**Claim 1.** The expected number of cycles of length less than $g$ in a graph $G \in \mathcal{G}$ is bounded by $n^\varepsilon n^{g-1}$. Thus asymptotically almost all graphs from $\mathcal{G}$ have at most $n^{2g\varepsilon}$ cycles of length $\leq g - 1$.

**Proof.** The expected number of cycles $C_l$ of length $l$ in a graph $G \in \mathcal{G}$ is at most

$$N_l = \frac{(qn)_l}{l!} \left( \frac{kn^2 - l}{m - l} \right) \left( \frac{kn^2}{m} \right)^{-1}.$$

By inequality (3),

$$\left( \frac{kn^2 - l}{m - l} \right) \left( \frac{kn^2}{m} \right)^{-1} \leq \left( \frac{m}{kn^2} \right)^l.$$

Since $m \leq kn^{1+2\varepsilon}$,

$$N_l \leq \frac{(qn)_l}{l!} \left( \frac{m}{kn^2} \right)^l < \left( \frac{qm}{kn} \right)^l \leq q^n n^{2l}.$$

Therefore

$$\sum_{l=3}^{g-1} N_l < (g - 3)q^8 n^{2(g-1)\varepsilon} < n^\varepsilon n^{2(g-1)\varepsilon} = n^{-\varepsilon} n^{2g\varepsilon}.$$

Here we assume that $n$ is large enough so that $n^\varepsilon > (g - 3)q^8 - 1$.

Let $\mathcal{G}_1$ be the set of all graphs $G \in \mathcal{G}$ with at most $n^{2g\varepsilon}$ cycles of length less than $g$. Then $|\mathcal{G}_1| \geq (1 - n^{-\varepsilon})|\mathcal{G}|$. □

For $G \in \mathcal{G}$, a **sparse pair** in $G$ is a pair $(A, B)$ of subsets of $V(G)$ such that $A \subseteq V_i$ and $B \subseteq V_j$ and $i \neq j$, $|A|, |B| = \lfloor n/t \rfloor$ and there are at most $n^{1+\varepsilon}$ edges between $A$ and $B$.

**Claim 2.** Asymptotically almost all graphs from $\mathcal{G}$ have no sparse pairs.

**Proof.** For an integer $s \leq n^{1+\varepsilon}$, denote by $M(s)$ the expected number (in a graph $G \in \mathcal{G}$) of pairs $A \subseteq V_i, B \subseteq V_j$ with $i \neq j$ such that $|A| = |B| = \lfloor n/t \rfloor$, and there are exactly $s$ edges connecting $A$ and $B$. Then
\[ M(s) = k \left( \frac{n}{m} \right)^2 \left( \frac{\left| \frac{n}{m} \right|^2}{s} \right) \left( \frac{kn^2 - \left| \frac{n}{m} \right|^2}{m - s} \right) \left( \frac{kn^2}{m} \right)^{-1}. \]

Replacing \( \left( \frac{kn^2 - \left| \frac{n}{m} \right|^2}{m - s} \right) \) by \( \left( \frac{kn^2 - \left| \frac{n}{m} \right|^2}{m} \right) \), applying inequalities (1) and (2), we have

\[ M(s) < k(\text{et})^{2n/1} \left( \frac{n}{t} \right)^{2s} e^{-(1/t^2)n^{1+2\epsilon}}. \]

Assume \( n \) is large enough so that

\[ e^{-(1/2t^2)n^{1+2\epsilon}}k(\text{et})^{2n/1} < 1. \]

Then

\[ M(s) < n^2e^{-(1/2t^2)n^{1+2\epsilon}}. \]

Hence the expected number of sparse pairs in \( G \in \mathcal{G} \) is

\[ \sum_{s<n^{1+\epsilon}} M(s) < \exp(-(1/2t^2)n^{1+2\epsilon} + 3n^{1+\epsilon} \log n) < \exp(-(1/4t^2)n^{1+2\epsilon}) < e^{-n}. \]

Let \( \mathcal{G}_2 \) be the set of graphs \( G \in \mathcal{G} \) that have no sparse pairs. Then \(|\mathcal{G}_2| > (1 - e^{-n})|\mathcal{G}|.\]

Let \( G \in \mathcal{G}_1 \cap \mathcal{G}_2 \). By deleting one edge from each cycle of \( G \) of length at most \( g - 1 \) (so at most \( n^{2\epsilon} \) edges from \( G \) are deleted), we obtain a \( q \)-partite graph \( G_0 \) of girth at least \( g \), each part \( V_i \) has \( n \) vertices, and for any \( 1 \leq i, j \leq q \) with \( i \neq j \) and any subsets \( A \subseteq V_i, B \subseteq V_j \) with \(|A|, |B| \geq |n/t|\), there is at least \( n^{1+\epsilon} - n^{2\epsilon} > \frac{1}{2}n^{1+\epsilon} \) edges between \( A \) and \( B \). This completes the proof of Lemma 3. \( \Box \)

**Lemma 4.** Let \( r \) be a positive integer and \([r] = \{1, 2, \ldots, r\}\). Let \( G_0 \) be the graph as in Lemma 3. If \( n \) is sufficiently large, then there exists a mapping \( f : E(G_0) \rightarrow [r] \times [r] \) such that the following holds:

- For any \( g : V(G_0) \rightarrow [r] \), for any \( 1 \leq i < j \leq q \), any subsets \( A \subseteq V_i, B \subseteq V_j \) with \(|A|, |B| \geq |n/t|\), there is at least one edge \( e = xy \) with \( x \in A, y \in B \) such that \( f(e) = (g(x), g(y)) \).

**Proof.** Let \( f : E(G_0) \rightarrow [r] \times [r] \) be a random mapping, where for each edge \( e = xy \), and \( g : V(G_0) \rightarrow [r] \), the probability that \( f(e) = (g(x), g(y)) \) is \( 1/r^2 \).

For two subsets \( A \subseteq V_i, B \subseteq V_j \) with \( i < j \), for \( g : V(G_0) \rightarrow [r] \), we say the pair \((A, B)\) is **bad with respect to** \( g \) if \(|A| = |B| = |n/t|\) and there is no edges \( e = xy \) with \( x \in A, y \in B \) such that \( f(e) = (g(x), g(y)) \). We say \((A, B)\) is **bad** if \((A, B)\) is bad with respect to some \( g : V(G_0) \rightarrow [r] \). To prove Lemma 4, it suffices to show that with positive probability, there is no bad pair.
By Lemma 3, for given $g : V(G_0) \to [r]$, for each subsets $A \subseteq V_i, B \subseteq V_j \ (i < j)$ with $|A|, |B| = \lfloor n/t \rfloor$, there are at least $\frac{1}{2}n^{1+\epsilon}$ edges between $A$ and $B$. For each edge $e = xy$ with $x \in A$ and $y \in B$, the probability that $f(e) \neq (g(x), g(y))$ is $1 - \frac{1}{r^2}$. Thus the probability that $(A, B)$ is bad with respect to $g$ is

$$
\left(1 - \frac{1}{r^2}\right)^{\frac{1}{2}n^{1+\epsilon}}.
$$

Let $P$ be the probability that there exists a bad pair. Then

$$
P \leq k \left(\frac{n}{\lfloor n/t \rfloor}\right)^2 r^{qn} \left(1 - \frac{1}{r^2}\right)^{\frac{1}{2}n^{1+\epsilon}} < k2^{2n}r^{qn} \left(1 - \frac{1}{r^2}\right)^{\frac{1}{2}n^{1+\epsilon}} < k(4r^q)^n \left(1 - \frac{1}{r^2}\right)^{\frac{1}{2}n^{1+\epsilon}}.
$$

Note that $k, t, q, r$ are constants (although large). So when $n$ is large enough, we have

$$
P < k(4r^q)^n \left(1 - \frac{1}{r^2}\right)^{\frac{1}{2}n^{1+\epsilon}} < 1.
$$

Hence with positive probability, there is no bad pair, and the required mapping $f$ exists. This completes the proof of Lemma 4. \hfill \Box

A graph $H$ is $d$-degenerate if each of its nonempty subgraphs has a vertex of degree at most $d$. Clearly, any $d$-degenerate graph is $(d + 1)$-choosable. The following result was proved by Kostochka and Nešetřil [4].

**Theorem 5** [4]. For any positive integers $d, g$, there is a $d$-degenerate graph with girth at least $g$ that is not $d$-colourable.

Assume $\lambda = (k_1, k_2, ..., k_q)$. For $i = 1, 2, ..., q$, let $J_i$ be a $(k_i - 1)$-degenerate graph of girth $g$ which is not $(k_i - 1)$-colourable. By adding isolated vertices, we may assume that all $J_i$ have the same number of vertices, say $|V(J_i)| = r$.

Let $f : E(G_0) \to [r] \times [r]$ be the mapping in Lemma 4, and $G'$ be the graph with vertex set $V(G_0) \times [r]$ in which $(x, s)$ is adjacent to $(y, s')$ if $e = xy \in E(G_0)$ and $x \in V_i, y \in V_j$ with $i < j$ and $f(e) = (s, s')$.

Let $G$ be the graph obtained from $G'$ as follows: For each $i = 1, 2, ..., q$, and for each vertex $v \in V_i$, add a copy of $J_i$ and identify the vertex set of this copy of $J_i$ with $\{v\} \times [r]$. Note that graph $G$ depends on $n, t, g, \lambda$. Assume $\lambda' = (k'_1, k'_2, ..., k'_p)$ and $\lambda \not\leq \lambda'$. We shall prove that

- $G$ is $\lambda$-choosable.
- $G$ has girth at least $g$.
- With appropriate choices of $n$ and $t$, $G$ is not $\lambda'$-choosable.
For each \( i \in \{1, 2, \ldots, q\} \), the subgraph \( G_i \) of \( G \) induced by \( V_i \times [r] \) consists of \( n \) vertex disjoint copies of \( J_i \). Then \( G_i \) is \((k_i - 1)\)-degenerate, and hence is \( k_i \)-choosable. As a consequence \( G \) is \( \lambda \)-choosable.

To see that \( G \) has girth at least \( g \), let \( C \) be a cycle in \( G \). If \( C \) is contained in one copy of \( J_i \) for some \( i \), then \( C \) has length at least \( g \), as \( J_i \) has girth at least \( g \). For any other cycle \( C \) in \( G \), contracting each copy of \( J_i \) to a single vertex yields a closed walk \( C' \) in \( G_0 \). Since there is at most one edge between a copy of \( J_i \) and a copy of \( J_i' \) in \( G \), each edge of \( C' \) occurs only once in \( C' \). Hence \( C' \) contains a cycle in \( G_0 \), which has length at least \( g \). As \( G_0 \) has girth at least \( g \), \( C \) has length at least \( g \) and hence \( G \) has girth at least \( g \).

Now we show that for appropriate chosen constants \( n, t, G \) is not \( \lambda' \)-choosable.

Let \( C_1', C_2', \ldots, C_p' \) be disjoint colour sets such that \( |C_j'| = 2k_j' - 1 \) for \( j = 1, 2, \ldots, p \). Let

\[
L = \left\{ \bigcup_{j=1}^{p} S_j : S_j \in \binom{C_j'}{k_j'} \right\}
\]

Here \( \binom{C_j'}{k_j'} \) is the family of all \( k_j' \)-subsets of \( C_j' \). So each element \( L = \bigcup_{j=1}^{p} S_j \) of \( L \) is a \( k' \)-set of colours, where \( k' = k_1' + k_2' + \cdots + k_p' \), and \( L \cap C_j' = k_j' \).

Let

\[
t = 2rl|L|k'.
\]

We construct a \( \lambda' \)-assignment of \( G \) as follows:

- For each vertex \( v \) of \( G_0 \), if \( v \in V_i \), then the set \( \{v\} \times [r] \) induces a copy of \( J_i \). All the vertices in each copy of \( J_i \) are assigned the same list from \( L \).
- For each \( i = 1, 2, \ldots, q \), each list from \( L \) is assigned to exactly \( \frac{n}{|L|} \) copies of \( J_i \) in \( G_i \). (We assume that \( n \) is chosen to be a multiple of \( |L| \)).

Recall that \( G_i \) is the subgraph of \( G \) induced by \( V_i \times [r] \), which consists of \( n \) copies of \( J_i \).

It follows from the definition that \( L \) is a \( \lambda' \)-assignment. We shall show that \( G \) is not \( L \)-colourable, and hence \( G \) is not \( \lambda' \)-choosable.

Assume to the contrary that there is an \( L \)-colouring \( \phi \) of \( G \). For each index \( j \in \{1, 2, \ldots, p\} \), we say a colour \( c \in \bigcup_{j=1}^{p} C_j' \) is occupied by \( G_i \) if \( c \) is used by at least \( \lceil nr/r \rceil \) vertices in \( G_i \). We say \( C_j' \) is occupied by \( G_i \) if at least \( k_j' \) colours in \( C_j' \) are occupied by \( G_i \).

For each \( i \in \{1, 2, \ldots, q\} \), let

\[
N_i = \left\{ j : C_j' \text{ is occupied by } G_i \right\}.
\]

Claim 3. For any \( i, i' \in \{1, 2, \ldots, q\} \) and \( i < i' \), we have \( N_i \cap N_{i'} = \emptyset \).
Assume \( N_i \cap N_{i'} \neq \emptyset \), say \( j \in N_i \cap N_{i'} \). By definition, there are at least \( k'_j \) colours in \( C'_j \) occupied by \( G_i \), and there are at least \( k'_j \) colours in \( C'_j \) occupied by \( G_{i'} \). As \( |C'_j| = 2k'_j - 1 \), there is a colour \( c \in C'_j \) occupied by both \( G_i \) and \( G_{i'} \), that is, there are at least \( \lfloor nr/t \rfloor \) vertices in \( G_i \) and at least \( \lfloor nr/t \rfloor \) vertices in \( G_{i'} \) coloured by \( c \).

Thus there are at least \( \lfloor n/t \rfloor \) copies of \( J_i \) containing a vertex coloured by \( c \) in \( G_i \), and at least \( \lfloor n/t \rfloor \) copies of \( J_{i'} \) containing a vertex coloured by \( c \) in \( G_{i'} \).

Let

\[
A = \{ v \in V_i : \text{some vertex in } [v] \times [r] \text{ is coloured by } c \},
\]

\[
B = \{ v \in V_i' : \text{some vertex in } [v] \times [r] \text{ is coloured by } c \}.
\]

Then \( |A|, \ |B| \geq \lfloor n/t \rfloor \). Let \( g : V(G_0) \to [r] \) be any mapping such that for all \( x \in A \cup B \), \( g(x) = a \) for some \( a \in [r] \) such that \( \phi(x, a) = c \). By Lemma 4, there exists an edge \( e = xy \) of \( G_0 \) such that \( f(e) = (g(x), g(y)) \). Hence \( G \) has an edge connecting \( (x, g(x)) \) and \( (y, g(y)) \). But both \( (x, g(x)) \) and \( (y, g(y)) \) are coloured by \( c \), a contradiction. \( \square \)

**Claim 4.** For each index \( i \in \{1, 2, ..., q\} \), \( \sum_{j \in N_i} k'_j \geq k_i \).

**Proof.** For each \( j \notin N_i \), there is a set \( D_j \) of \( k'_j \) colours in \( C'_j \), and each colour in \( D_j \) is used by less than \( \lfloor n/t \rfloor \) vertices in \( G_i \). Let \( L_0 = \bigcup_{j=1}^{n} S_j^0 \in \mathcal{L} \) be a list such that \( S_j^0 = D_j \) for each \( j \notin N_i \) and \( S_j^0 \) is an arbitrary \( k'_j \)-subset of \( C'_j \) for \( j \in N_i \).

By the definition of \( L \), there exists \( X \subseteq V_i \) such that

\[
|X| \geq \lfloor n/|\mathcal{L}| \rfloor
\]

and

\[
L(x, s) = L_0, \forall (x, s) \in X \times [r].
\]

Let

\[
Z = \{(x, s) \in X \times [r] : \phi(x, s) \in \bigcup_{j \notin N_i} D_j \}.
\]

As each colour in \( \bigcup_{j \notin N_i} D_j \) is used by less than \( \lfloor n/t \rfloor \) vertices in \( G_i \), we conclude that

\[
|Z| < \left\lfloor \frac{nr}{t} \right\rfloor \sum_{j \notin N_i} k'_j < \frac{n}{|\mathcal{L}|}.
\]

So there exists \( x \in X \) such that all vertices in \( \{x\} \times [r] \) are coloured by colours in \( \bigcup_{j \in N_i} S_j^0 \).

Since \( J_i \) is not \( (k_i - 1) \)-colourable, we conclude that

\[
\left| \bigcup_{j \in N_i} S_j^0 \right| = \sum_{j \in N_i} k'_j \geq k_i.
\]
Let $\lambda'' = \{k_1'', k_2'', ..., k_q''\}$, where $k_j'' = \sum_{j \in N} k_j'$. Then $\lambda''$ is obtained from $\lambda$ by increasing some parts of $\lambda$, and $\lambda'$ is a refinement of $\lambda''$. Hence $\lambda \leq \lambda'$, which is in contrary to our assumption. This completes the proof of Theorem 2.

Note that the structure of $G$ constructed in the proof of Theorem 2 relies more on $\lambda$. The role of $\lambda'$ is only used in choosing $t$ and $n$. Thus the same proof actually proves the following stronger result.

**Theorem 6.** Assume $\lambda$ and $\lambda_i$ ($i = 1, 2, ..., p$) are partitions of integers and $\lambda \not\leq \lambda_i$ for each $i = 1, 2, ..., p$. Then for any positive integer $g$, there exists a graph $G$ of girth $g$ which is $\lambda$-choosable, but not $\lambda_i$-choosable for $i = 1, 2, ..., p$.

On the other hand, the following question remains open.

**Question 1.** Assume $\lambda_i \not\leq \lambda$ for $i = 1, 2, ..., p$. Is it true that there exists a graph $G$ which is $\lambda_i$-choosable for $i = 1, 2, ..., p$ but not $\lambda$-choosable? If the answer to Question 1 is ‘yes’, then a natural next question is whether we can further require graph $G$ to have a large girth.

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**DATA AVAILABILITY STATEMENT**
Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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