A Contraction Approach to Model-based Reinforcement Learning

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Abstract

Model-based Reinforcement Learning has shown considerable experimental success. However, a theoretical understanding of it is still lacking. To this end, we analyze the error in cumulative reward for both stochastic and deterministic transitions using a contraction approach. We show that this approach doesn’t require strong assumptions and can recover the typical quadratic error to the horizon. We prove that branched rollouts can reduce this error and are essential for deterministic transitions to have a Bellman contraction. Our results also apply to Imitation Learning, where we prove that GAN-type learning is better than Behavioral Cloning in continuous state and action spaces.

1 Introduction

Reinforcement learning (RL) has attracted great attention recently due to its ability to learn good policies for sequential systems. Nevertheless, a common difficulty of most RL algorithms is the high sample complexity of environment queries (typically in the order of millions). This hinders the deployment of RL in practical systems. An intuitive potential solution is to learn an accurate model of the outcome given by the environment, and hence reduce the demand for environment queries. This leads to a dichotomy of RL algorithms: training without an environment model is called model-free RL and training with an environment model is called model-based RL. Model-free RL is often faulted for low exploitation of environment queries, while model-based RL suffers from model inaccuracy.

Model-based Reinforcement Learning (MBRL) is nontrivial since the sequential nature of RL allows the errors to propagate to future time-steps. This leads to the planning horizon dilemma [Wang et al. 2019]; a long horizon incurs a large cumulative error, while a short horizon results in shortsighted decisions. We hence need to better understand the fundamental limitations of model-based RL.

Most prior error analyses impose a strong assumption in their proofs; e.g., Lipschitz Value function [Luo et al. 2019; Xiao et al. 2019; Yu et al. 2020] or maximum model error [Janner et al. 2019]. In particular, the Value function is unlikely to be Lipschitz in practice because its gradient w.r.t. state can be very large. This happens when a perturbation of state is applied at the stability-instability boundary of a control system, which results in a large change in Value (performance) with a small change in state. For instance, if one perturbs a robot’s leg, it could just fall and receive lots of negative future rewards. In an effort to mitigate the cumulative reward error, Janner et al. (2019) shows experimentally that branched rollouts (short model rollouts initialized by previous real rollouts) help reduce this error and improve experimental results. However, the effectiveness of branched rollouts remains unclear since the experiments of Janner et al. (2019) are run in deterministic transitions (MuJoCo [Todorov et al. 2012]) but their error analysis only applies to stochastic transitions and contains potential errors, as discussed in section 3. Thereby, an analysis framework which applies to both stochastic and deterministic transitions is much needed.

Our main contribution is a contraction-based approach to analyze the error of MBRL that applies to both stochastic and deterministic transitions without strong assumptions. We first observe that MBRL exhibits asymmetry when learning a transition model. We then show that if the Bellman operator is a contraction w.r.t. a metric, we can analyze the error of MBRL under that metric regardless of the asymmetry. We prove that branched rollouts greatly reduce the error and are vital for deterministic transitions to have a Bellman contraction. Finally, we discuss the situation without a Bellman contraction and identify the impact of asymmetry. The resulting insight suggests the potential usefulness of the Ensemble Method [Kurutach et al. 2018] in this situation. Although this work doesn’t include any experiment, the prior work [Janner et al. 2019] has already done great experiments on branched rollouts. Since the empirical evidence is clear in the literature, we primarily focus on the theoretical understanding behind it.

In contrast to prior literature, our analysis applies to both (absolutely continuous) stochastic and deterministic transitions, mostly uses constants in expectation and doesn’t require Lipschitz assumption on Value

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functions. Thus we believe our analysis is more general. Prior work also suggests long branched rollouts may have smaller expected cumulative error than shorter ones (see section 3). Our analysis shows this can’t be true. Our contraction approach can also be applied to Imitation Learning, where we show a GAN-type learning method like GAIL \cite{Ho and Ermon 2016} is better than Behavioral Cloning. To the best of our knowledge, this is the first theoretical comparison between these methods in continuous spaces.

2 Preliminaries

Consider an infinite-horizon Markov Decision Process (MDP) represented by $(\mathcal{S}, \mathcal{A}, T, r, \gamma)$. Here $\mathcal{S}$, $\mathcal{A}$ are finite-dimensional continuous state and action spaces, $r(s, a)$ is the reward function, $\gamma \in (0, 1)$ is the discount factor, and $T(s'|s, a)$ is the transition density of $s'$ given $(s, a)$. We use $\mathcal{T}$ to denote a deterministic transition with the corresponding density $T(s'|s, a) = \delta(s' - \mathcal{T}(s, a))$.

Given an initial state distribution $\rho_0$, the goal of reinforcement learning is to learn a (stochastic) policy $\pi$ that maximizes the $\gamma$-discounted cumulative reward $R_\gamma(\rho_0, \pi, T)$, or equivalently, the expected cumulative reward, denoted as $R(\rho^{0, \pi}_T)$, under the normalized occupancy measure $\rho^{0, \pi}_T$.

$$R_\gamma(\rho_0, \pi, T) = \mathbb{E} \left[ \sum_{i=0}^{\infty} \gamma^i r(s_i, a_i) \right] \rho_0, \pi, T = \frac{1}{1 - \gamma} \mathbb{E}_{(s, a) \sim \rho^{0, \pi}_T} [r(s, a)] = R(\rho^{0, \pi}_T),$$

where $f_i(s, a | \rho_0, \pi, T)$ is the density of $(s, a)$ at step $i$ under the laws of $(\rho_0, \pi, T)$. Because the occupancy measure is uniquely defined by $(\rho_0, \pi, T, \gamma)$, we use $R(\rho^{0, \pi}_T)$ as an alternative expression of $R_\gamma(\rho_0, \pi, T)$. When $\rho_0$, $\gamma$ are fixed, we simplify the notation to $R(\pi, T)$ and $\rho^{0, \pi}_T$.

2.1 Bellman Operator

In Eq. (1), because each $f_i(s, a | \rho_0, \pi, T)$ uses the same policy, $f_i(s, a)$ and $\rho^{0, \pi}_T(s, a)$ are factorized into $f_i(s) \pi(a | s)$ and $\rho^{0, \pi}_T(s) \pi(a | s)$. This allows us to mainly focus on the state distributions. In particular, we define the normalized state occupancy measure $\rho^{0, \pi}_T(s)$ as the marginal of $\rho^{0, \pi}_T(s, a)$ and show (Fact 4 in Appendix) it satisfies a fixed-point equation characterized by a Bellman operator $B_{\pi, T}(\cdot)$.

$$\rho^{0, \pi}_T(s) = (1 - \gamma) \sum_{i=0}^{\infty} \gamma^i f_i(s) \rho_0, \pi, T = B_{\pi, T}(\rho^{0, \pi}_T(s)), \quad (2)$$

where $B_{\pi, T}(\cdot)$ is the Bellman operator generated by $(\rho_0, \pi, T)$ with discount factor $\gamma$:

$$B_{\pi, T}(\rho(s)) = (1 - \gamma) \rho_0(s) + \gamma \int T(s | s', a') \pi(a' | s') \rho(s') ds' da'.$$

As noted in Appendix, $B_{\pi, T}(\cdot)$ is a $\gamma$-contraction w.r.t. total variation distance. Thereby, $B_{\pi, T}(\cdot)$ has a unique fixed point, and according to Eq. (2), the unique fixed point is $\rho^{0, \pi}_T(s)$. This implies that the Bellman operator could be useful in analyzing the state occupancy measures. Indeed, we will introduce a lemma for Bellman operator that help upper bound the distance between state distributions (Lemma [4]), which turns out to be a key to analyze MBRL.

2.2 Model-based RL

This paper studies the following model-based RL algorithm and its variants (e.g., branched rollouts).

Line 3 deals with the storage of a dataset $\mathcal{D}$ of real transitions. Observe that each $\mathcal{D}_{i-1}$ is generated by $(\rho_0, \pi_{i-1}, T)$ and that $\mathcal{D}$ aggregates $\mathcal{D}_{i-1}$’s. The policy that generates $\mathcal{D}$, which we call the sampling policy $\pi_D$, is a mixture of previous policies. If $\mathcal{D}_{i-1}$’s have equal sizes, $\pi_D(a | s) = \sum_{j=i-1-q}^{i-1} \pi_j(a | s) \rho^{\pi}_T(s) / \sum_{j=i-1-q}^{i-1} \rho^{\pi}_T(s)$
with \( q \) being the truncation level. We need \( \mathcal{D} \) to be sufficiently large and the current policy \( \pi \) and the sampling policy \( \pi_D \) to be sufficiently close. Hence one may expect a tight truncation; e.g., \( \mathcal{D} = \mathcal{D}_{i-1} \cup \mathcal{D}_{i-2} \).

We discuss the closeness of policies in the last paragraph of this section.

Line 4 is a supervised learning task. The objective is usually the log-likelihood function for stochastic transitions or \( \ell_2 \) error for deterministic transitions. For stochastic transitions, because maximizing likelihood is equivalent to minimizing KL divergence, the total variation distance
\[
\epsilon_{TV}^{D} = \mathbb{E}_{(s,a) \sim \rho^D_T} D_{TV}(\hat{T}(s,a) \parallel T(s,a))
\]
is small by Pinsker Inequality. On the other hand, for deterministic transitions, the objective is to minimize
\[
\epsilon_{\ell_2} = \mathbb{E}_{(s,a) \sim \rho^D_T} \| \hat{T}(s,a) - T(s,a) \|_2.
\]

Line 5 is policy optimization using the learned transition \( \hat{T} \) with the goal of maximizing \( R(\pi, \hat{T}) \). However, the overall goal is to maximize \( R(\pi, T) \). Note that
\[
R(\pi, T) - R(\pi_{i-1}, T) = \underbrace{R(\pi_i, \hat{T}) - R(\pi_{i-1}, \hat{T})}_{\text{m.-b. policy improvement}} + \underbrace{R(\pi_i, T) - R(\pi_i, \hat{T}) + R(\pi_{i-1}, \hat{T}) - R(\pi_{i-1}, T)}_{\text{reward errors}}.
\]

Hence Line 5 makes an improvement (Eq. (4) > 0) if the error in cumulative reward \( |R(\pi, T) - R(\pi, \hat{T})| \) is small and the model-based policy improvement \( R(\pi_i, \hat{T}) - R(\pi_{i-1}, \hat{T}) \) is large. However, the model-based policy improvement is often theoretically intractable. This is because the policy optimization is usually conducted by deep RL algorithms (Fujimoto et al. 2018; Haarnoja et al. 2018) but the state-of-the-art provable RL algorithms are still limited in linear function approximation (Jin et al. 2020; Duan and Wang, 2020). Therefore, in this work, we assume the model-based policy improvement is sufficiently large and focus on the error in cumulative reward.

The desired closeness between \( \pi \) and \( \pi_D \) is achieved by Line 3’s truncation and Line 5’s constraint to local ball \( B_{\epsilon_D} \) of \( \pi_D \). Such closeness of policies is also used in Luo et al. (2019); Janner et al. (2019). Indeed, since \( \hat{T} \) is fitted on \( \rho^D_T \), if \( \pi \) and \( \pi_D \) are far apart, we cannot expect \( \hat{T} \) behave like \( T \) under \( \pi \).

### 3 Related Work

There have been many experimental studies of model-based RL. Evidence in Gu et al. (2016) and Nagabandi et al. (2018) suggests that for continuous control tasks, vanilla MBRL (Sutton 1991) hardly surpasses model-free RL, unless using a linear transition model or a hybrid model-based and model-free algorithm. To enhance the applicability of MBRL, the Ensemble Method is widely adopted in the literature, since it helps alleviate over-fitting in a neural network (NN) model. Instances of this approach include, an ensemble of deterministic NN transition models (Kurutach et al. 2018), an ensemble of probabilistic NN transition models (Chua et al. 2018) with model predictive control (Camacho and Bordons Alba, 2013) or ensembles of deterministic NN for means and variances of rollouts with different horizons (Buckman et al. 2018). In addition to training multiple models, Clavera et al. (2018) leverages meta-learning to train a policy that can quickly adapt to new transition models. Wang et al. (2019) provides useful benchmarks of various model-based RL methods.

On the theoretical side, for stochastic state transitions the error in the cumulative reward is quadratic in the length of model rollouts. Specifically, Janner et al. (2019, Theorem A.1) provides the bound
\[
R(\pi, T) - R(\pi, \hat{T}) \geq -\frac{2\epsilon_D^{\text{max}}}{1-\gamma}(\epsilon_m + 2\epsilon_\pi) - \frac{d\epsilon_D^{\text{max}}}{1-\gamma},
\]
where \( \epsilon_m = \max_s E_{s \sim \rho_{\pi_{D,t}}} \mathcal{D}_{TV}(T(s, a) \| \hat{T}(s, a)) \), \( \epsilon_\pi = \max_s \mathcal{D}_{TV}(\pi_{D} \| \pi(s)) \) and \( \rho_{\pi_{D,t}} \) is the density of \((s, a)\) at step \( t \) following \((\rho_0, \pi_D, T)\). For deterministic state transitions and an \( L \)-Lipschitz Value function \( V(s) \), Luo et al. (2019, Proposition 4.2) shows that

\[
|R(\pi, T) - R(\pi, \hat{T})| \leq \frac{2}{1-\gamma} L \mathbb{E}_{s \sim \rho_{\pi_{D}}} \left[ \|T(s, a) - \hat{T}(s, a)\| \right] + 2 \frac{\epsilon_\pi^2}{(1-\gamma)^2} \delta \text{diam}_S, \tag{6}
\]

where \text{diam}_S is the diameter of \( S \), and \( \delta = \mathbb{E}_{s \sim \rho_{\pi_{D}}} \sqrt{\mathcal{D}_{KL}(\pi(s) \| \pi_D(s))} \).

In practice, we enforce a degree of closeness between \( \pi_D \) and \( \pi \). Thus the dominating terms of Eq. (5) and (6) are the terms with model errors. Eq. (5) looks sharper in the sense that the model error is correlated with a linear term of the expected rollout length \((1-\gamma)^{-1}\), not a quadratic one. However, the Lipschitz constant of Value function, if exists, can be \( O((1-\gamma)^{-1}) \) because the Value function represents the cumulative reward. Therefore, it is hard to compare Eq (5) and (6) as they are in different settings. While the Lipschitzness of Value function is commonly assumed in the literature (Luo et al. 2019, Xiao et al. 2019, Yu et al. 2020), in practice, its existence is hard to verify, and, even if exists, the Lipschitz constant could be very large. To avoid strong assumptions, our work does not assume Lipschitz Value function and mostly follows Janner et al. (2019), where we will show their constants “in maxima” can be replaced by constants “in expectation”.

A major contribution of Janner et al. (2019) is the use of branched rollouts: rollouts generated by \((\rho_{\pi_D}, \pi, T)\). In particular, their Theorem 4.3 shows that branched rollouts of length \( k \) satisfy

\[
R(\pi, T) - R_{\text{branch}}(\pi) \geq -2r_m \max \left[ \frac{\epsilon_\pi^{k+1}}{(1-\gamma)^2} + \frac{\epsilon_\pi^k}{1-\gamma} + \frac{\epsilon_m}{1-\gamma} \right], \tag{7}
\]

with the same constants as Eq. (5). However, Eq. (7) implies when the model is almost perfect \( (\epsilon_m \approx 0) \), the minimal error in cumulative reward is close to zero and attained at large \( k \). This cannot be true because branched rollouts cannot undo the error made at initialization. This error propagates to the future despite using a perfect model. Also, that the minimal error is attained at large branched length \( k \) contradicts to the fact that the error accumulates over the trajectory. Even if Eq. (7) was not erroneous, it still couldn’t explain the experimental success in Janner et al. (2019) because Eq. (7) is for stochastic transitions, but their experiments were run in deterministic transitions. In contrast, our analysis shows the error of branched rollouts for both stochastic and deterministic transitions increases in the expected branched length \((1-\beta)^{-1}\), so we always favor short lengths and are free from the issues mentioned above.

### 4 Main Result

As discussed in section 2.2, we focus on the error in the cumulative reward \(|R(\pi, T) - R(\pi, \hat{T})|\) for different MBRL settings. To do so we make use of the triangle inequality

\[
|R(\pi, T) - R(\pi, \hat{T})| \leq |R(\pi, T) - R(\pi, D)| + |R(\pi, D, T) - R(\pi, \hat{T})| + |R(\pi, \hat{T}) - R(\pi, \hat{T})|. \tag{8}
\]

The RHS of Eq. (8) is determined by the discrepancies between transitions \( T, \hat{T} \) and policies \( \pi, \pi_D \). The discrepancy between policies is measured by the total variation (TV) distance.

\[
\epsilon_{\pi_D, \pi}^T = \mathbb{E}_{s \sim \rho_{\pi_D}} \mathcal{D}_{TV}(\pi_D(s) \| \pi(s)) \quad \text{and} \quad \epsilon_{\pi_D, \pi}^\hat{T} = \mathbb{E}_{s \sim \rho_{\hat{T}}} \mathcal{D}_{TV}(\pi_D(s) \| \pi(s)).
\]

The discrepancies between real and learned transitions \( T, \hat{T} \) are measured by (a) TV distance for stochastic transitions and (b) \( \ell_2 \) error for deterministic transitions.

\[
(a) \quad \epsilon_{T, \hat{T}}^{\pi_D} = \mathbb{E}_{(s, a) \sim \rho_{\pi_D}^T} \mathcal{D}_{TV}(T(s, a) \| \hat{T}(s, a)) \quad \text{and} \quad (b) \quad \epsilon_{\ell_2}^T = \mathbb{E}_{(s, a) \sim \rho_{\pi_D}^T} \|T(s, a) - \hat{T}(s, a)\|_2.
\]

By Eq. (8), the error in the cumulative reward \(|R(\pi, T) - R(\pi, \hat{T})|\) is bounded above by errors due to policy differences \( (1^{st} \text{ and } 3^{rd} \text{ term}) \) and a transition difference \( (2^{nd} \text{ term}) \). The first terms are symmetric in \((\pi, \pi_D)\) (invariant under exchange of \( \pi \) and \( \pi_D \)), and the last is symmetric in \((T, \hat{T})\). However, the terms that control them, \( \epsilon_{T, \pi_D, \pi}^{\pi_D}, \epsilon_{T, \pi_D, \pi}^\hat{T}, \epsilon_{\ell_2}^T, \epsilon_{\ell_2} \), are asymmetric; the first two in \((\pi, \pi_D)\) and the last in \((T, \hat{T})\).
To bridge the symmetric and asymmetric quantities above we will establish the following:

\[
|R(\rho_1) - R(\rho_2)| \leq C \times \{D_{TV}(\rho_1 \| \rho_2), W_1(\rho_1 \| \rho_2)\} \leq C' \times \{\epsilon_{\pi_D,\pi}^T, \epsilon_{\pi_D,\pi}^T, \epsilon_{\pi_D,\pi}^T, \epsilon_{\ell_2}\}.
\]

Inequality (*) in (9) uses either TV distance (for stochastic transitions) or 1-Wasserstein distance (Villani, 2008) (for deterministic transitions) to build a symmetric bound w.r.t. occupancy measures. Inequality (**) uses the contraction property of the Bellman operator (if it holds). While the Bellman operator is a contraction w.r.t. TV distance, this does not always hold w.r.t. \( W_1 \) distance. The situation when the Bellman operator is not a contraction is addressed in (4.4.4). Besides, although we use \( W_1 \) distance as an intermediate step to analyze deterministic transitions, we finally upper bound \( W_1 \) distance by \( \ell_2 \) error. This avoids the need to minimize \( W_1 \) error by Wasserstein GAN (Arjovsky et al., 2017), which is not usually easy to train.

In the following subsections, we first analyze the policy error, then the transition error when we have:

1. Absolutely continuous stochastic transitions,
2. Deterministic transitions with strong continuity,
3. Deterministic transitions with weak continuity.

Cases (1) and (2) have Bellman contractions yielding sharp two-sided bounds. Case (3) uses a bounding technique inspired by Syed and Schapire (2010, Lemma 2) to establish a one-sided bound. By combining the policy error with the transition errors, we obtain corresponding MBRL errors. Full proofs are in Appendix.

### 4.1 Symmetry Bridge Lemma and Policy Mismatch Error

We start by analyzing the policy error to demonstrate the bounding technique in Eq. (9). First, we introduce a key lemma, which is also a short extension of Conrad (2014, Corollary 2.4). If the Bellman operator is a contraction, the following result holds. This is the key to establishing the inequality (**) in Eq. (9).

**Lemma 1.** Let \( B \) be a Bellman operator with fixed-point \( \rho^* \). Let \( \rho \) be any state distribution. If \( B \) is a \( \eta \)-contraction w.r.t. some metric \( \| \cdot \| \), then \( \| \rho - \rho^* \| \leq \| B(\rho) \| / (1 - \eta) \).

The analysis of the policy error \( |R(\pi_D, T) - R(\pi, T)| \) uses the idea in Eq. (9). Because the Bellman operator is a contraction w.r.t. TV distance, Lemma 1 establishes inequality (**) of Eq. (9). The inequality (*) is verified by the following lemma.

**Lemma 2.** If \( 0 \leq r(s,a) \leq r_{\text{max}} \), then \( |R(\rho_1) - R(\rho_2)| \leq D_{TV}(\rho_1 \| \rho_2)r_{\text{max}}/(1 - \gamma) \).

By combining Lemmas 1 and 2, we can prove one of our main theorems.

**Theorem 1.** If \( 0 \leq r(s,a) \leq r_{\text{max}} \) and \( \epsilon^T_{\pi_D,\pi} = E_{s \sim \rho^s_T}[D_{TV}(\pi_D(\cdot|s) \| \pi(\cdot|s))], \) then

\[
|R(\pi_D, T) - R(\pi, T)| \leq \epsilon^T_{\pi_D,\pi}r_{\text{max}} \left( \frac{1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2} \right).
\]

Proof Sketch. By Lemma 2 it is enough to upper bound \( D_{TV}(\rho^D_T(s,a) \| \rho^T(s,a)) \):

\[
D_{TV}(\rho^D_T(s,a) \| \rho^T(s,a)) \leq D_{TV}(\rho^D_T(s) \| \rho^T(s)) + D_{TV}(\rho^D_T(s) \| \rho^T(s)) \leq \epsilon^T_{\pi_D,\pi} + \epsilon^T_{\pi_D,\pi} \left( \frac{1}{1 - \gamma} + \frac{\gamma}{(1 - \gamma)^2} \right)
\]

\( \square \)

Theorem 1 is of great importance because it establishes the upper bounds of policy differences (1st and 2nd term of Eq. (5)). Moreover, observe that Imitation learning (Sved and Schapire, 2010; Ho and Ermon, 2016) is matching the demonstrated policy and the generator policy. Because Theorem 1 is about the policy mismatch error, it is able to analyze Imitation learning in the following way.

Since the objectives of Behavior Cloning and GAIL are KL and JS (Jensen-Shannon) divergence respectively, we can use Pinsker’s Ineq. to translate Theorem 1 and Lemma 2 to Imitation Learning.
Corollary 1 (Error of Behavioral Cloning). Let $\pi_D$ and $\pi$ be the expert and agent policy, respectively. If $0 \leq r(s, a) \leq r^{\max}$ and $\mathbb{E}_{s \sim \rho_T^D} D_{KL}(\pi_D(\cdot|s) || \pi(\cdot|s)) \leq \epsilon_{BC}$, then

$$|R(\pi_D, T) - R(\pi, T)| \leq \sqrt{\epsilon_{BC}/2} r^{\max} \left( \frac{1}{1 - \gamma} + \frac{\gamma}{1 - \gamma} \right).$$

Corollary 2 (Error of GAIL). Let $\pi_D$ and $\pi$ be the expert and agent policy, respectively. If $0 \leq r(s, a) \leq r^{\max}$ and $D_{JS}(\rho_T^D || \rho_T^\pi) \leq \epsilon_{GAIL}$. Then

$$|R(\pi_D, T) - R(\pi, T)| \leq \sqrt{2\epsilon_{GAIL}} r^{\max}/(1 - \gamma).$$

Observe that Behavioral Cloning’s error is quadratic w.r.t. the expected horizon $(1 - \gamma)^{-1}$ while GAIL’s is linear. This suggests that GAN-style imitation learning, like GAIL, is better.

### 4.2 MBRL with Absolutely Continuous Stochastic Transitions

If the real transitions are stochastic, we can learn $\hat{T}$ by maximizing the likelihood, or equivalently by minimizing the KL divergence. To ensure the KL divergence is defined on a continuous state space, we assume the transition probability is absolutely continuous w.r.t. the state space, i.e., there is a density function and hence no discrete or singular continuous measures. Then, Theorem 2 follows from the proof of Theorem 1.

**Theorem 2.** If $0 \leq r(s, a) \leq r^{\max}$ and $\epsilon^\pi_{T, \hat{T}} = \mathbb{E}_{(s, a) \sim \rho_T^D} [D_{TV}(T(\cdot|s, a) || \hat{T}(\cdot|s, a))], then

$$|R(\pi_D, T) - R(\pi, \hat{T})| \leq \epsilon^\pi_{T, \hat{T}} r^{\max} (1 - \gamma)^{-2}.$$  

Theorems 1 and 2 combined yield the following result for MBRL with abs. cont. stochastic transitions.

**Corollary 3.** If $0 \leq r(s, a) \leq r^{\max}$, $\epsilon^\pi_{\pi_D, \pi} = \mathbb{E}_{s \sim \rho_T^D} D_{TV}(\pi_D(\cdot|s) || \pi(\cdot|s))$, and $\epsilon^\pi_{\pi_D, \pi} = \mathbb{E}_{s \sim \rho_T^D} D_{TV}(\pi(\cdot|s) || \pi(\cdot|s))$, then

$$|R(\pi, T) - R(\hat{\pi}, \hat{T})| \leq (\epsilon^\pi_{T, \hat{T}} + \epsilon^\pi_{\pi_D, \pi} + \epsilon^\pi_{\pi_D, \pi}) r^{\max} (1 - \gamma)^{-2}.$$  

Comparing Corollary 3 with previous results in Eq. 3, we indeed sharpen the bound by changing the constants from maxima to expectations.

#### 4.2.1 MBRL with Branched Rollouts

Corollary 3 indicates that the model error term $\epsilon^\pi_{T, \hat{T}} r^{\max} (1 - \gamma)^{-2}$ is quadratic w.r.t. the expected rollout length $(1 - \gamma)^{-1}$, which makes MBRL undesirable for long rollouts and leads to the planning horizon dilemma. An intuitive countermeasure is to use short rollouts that share similar distributions with the long ones. This leads to the idea of branched rollouts. Throughout the rest of the paper, $\beta > 0$ will denote the branched discount factor with $\beta < \gamma$. We define a branched rollout with discount factor $\beta$, to be a rollout following the laws of $(\rho_T^\pi, \pi, \hat{T})$. Intuitively, these are rollouts initialized on the states of previous real long rollouts, $\rho_T^\pi(s)$, and then run a few steps under policy $\pi$ and model $\hat{T}$.

The occupancy measure of branched rollouts is $\rho_T^\pi, \gamma, \beta$ where the superscripts $\rho_T^\pi, \gamma, \beta$ indicate the initial state distribution and policy, and the subscripts $\hat{T}, \beta$ indicate the transition and discount factor. Although branched rollouts are indeed short by construction, it is unclear whether their distribution is similar to that of long rollouts. This is examined in the following Lemma.

**Lemma 3.** Let $\gamma > \beta$ be discount factors of long and short rollouts. Let $\pi_D$ and $T$ be the sampling policy and the real transition, then

$$D_{TV}(\rho_T^{\pi_D, \gamma} || \rho_T^{\pi_D, \beta}) \leq (1 - \gamma)\beta/(\gamma - \beta).$$
Lemma 4.3 tells us that \( \rho^D_{\pi,T,\gamma} \) and \( \rho^D_{\pi,T,\gamma}^{\pi_D} \) are close if \( \beta \) is small. This implies once the pairs \((\pi, \pi_D)\) and \((T, \hat{T})\) are close, the distribution of branched rollouts is similar to that of the real long rollouts, so the error in cumulative reward is small. This is given in detail below.

**Corollary 4.** Let \( 0 \leq r(s, a) \leq r^{\text{max}} \),

\[
\epsilon_{T,\pi,\gamma}^D = \mathbb{E}_{s \sim \rho^D_{\pi,T,\gamma}} D_{\text{TV}}(\pi_D(\cdot|s) \parallel \pi(\cdot|s)), \quad \epsilon_{\pi,D,\pi}^T = \mathbb{E}_{s \sim \rho^D_{\pi,T,\gamma}} D_{\text{TV}}(\pi_D(\cdot|s) \parallel \pi(\cdot|s)), \quad \text{and}
\]

\[
\epsilon_{T,T}^D = \mathbb{E}_{(s,a) \sim \rho^D_{\pi,T,\gamma}} D_{\text{TV}}(T(\cdot,s,a) \parallel \hat{T}(\cdot,s,a)).
\]

Then \[
|R_\gamma(\rho_0, \pi, T) - \frac{1-\beta}{1-\gamma} R_\beta(\rho_{\pi,T,\gamma}^D, \pi, \hat{T})| \leq r^{\text{max}} \left( \frac{\epsilon_{T,T}^D}{(1-\gamma)^2} + \frac{\epsilon_{T,T}^D + \epsilon_{T,T}^D}{(1-\beta)(1-\gamma)} \right)
\]

**Proof Sketch.** Decompose the error as follows and then apply Theorems 1, 2, Lemmas 2, 3.

Notice \( \epsilon_{T,T}^D \) is controlled by supervised learning as it is evaluated on dataset \( D \). Because branched rollouts are shorter than normal rollouts, the branch cumulative reward is rescaled to \( \frac{1-\beta}{1-\gamma} R_\beta(\rho_{\pi,T,\gamma}^D, \pi, \hat{T}) \) for comparison to normal rollouts. Compared with Corollary 3, the model error term’s dependency on the rollout lengths is reduced from \( O((1-\gamma)^{-2}) \) to \( O((1-\gamma)^{-1}(1-\beta)^{-1}) \). Hence branched rollouts greatly reduce the reward error.

Corollary 4 shows the error in cumulative reward is increasing in the discount factor \( \beta \), or equivalently in the expected branched length \( (1-\beta)^{-1} \). Thus our result is free from the issue of previous work Eq. 7, as discussed in [3]. It is tempting to set \( \beta = 0 \) to minimize the error. However, if \( \beta = 0 \), each branched rollout is only composed of a single point drawn from \( \rho_{\pi,T,\gamma}^D \). This means that the branched rollouts access neither \( T \) nor \( \hat{T} \), so we will learn a policy that only optimizes the initial states and has no concern for the future. For example, the reward of MuJoCo environment [Todorov et al., 2012] is typically \( r(s,a) = \|v(s) - \|a\|^2 \). To maximize cumulative reward on branched rollouts with \( \beta = 0 \), the optimal policy \( \pi^*(a|s) \) will shortsightedly select \( a = 0 \) for any \( s \).

The branched rollout makes a trade-off between policy improvement and reward error, as discussed in [2]. The policy improvement \( R_\beta(\rho_{\pi,T,\gamma}^D, \pi, \hat{T}) - R_\beta(\rho_{\pi,T,\gamma}^D, \pi_{\pi-1}, \hat{T}) \) in branched rollouts benefits from a larger \( \beta \), while the reward error, as shown in Corollary 4 and 5, favors smaller \( \beta \). In MuJoCo, according to Janner et al. [2020, Appendix C], the branched length is chosen as 2 in early epochs and may stay small or gradually increase to 16 or 26 later. This indicates for continuous-control (MuJoCo) tasks, \( \beta \approx 0.9 \) is enough to balance policy improvement and reward error.

### 4.3 MBRL with Deterministic Transitions and Strong Lipschitz Continuity

A major difficulty in analyzing deterministic transitions is that TV distance is not suitable for comparing \( \mathcal{T} \) and \( \mathcal{T} \). Indeed, for any fixed \((s, a)\), \( D_{\text{TV}}(\delta(s'-\mathcal{T}(s,a)) \parallel \delta(s'-\mathcal{T}(s,a))) = 1 \) once \( \mathcal{T}(s,a) \neq \mathcal{T}(s,a) \). Moreover, the model error is controlled by \( \epsilon_{\mathcal{T}} = \mathbb{E}_{(s,a) \sim \rho_{\pi}} \mathbb{E}_J(s',a) \| \mathcal{T}(s,a) - \mathcal{T}(s,a) \|_2 \), but the \( \ell_2 \) error is not a distance metric for distributions. To control the distance between distributions through \( \ell_2 \) error, we should select a distance metric for distributions that can be upper bounded by \( \ell_2 \) error. The 1-Wasserstein distance is a good candidate:

\[
W_1(\rho_1(s) \parallel \rho_2(s)) = \inf_{J(s_1, s_2) \in \Pi(\rho_1, \rho_2)} \mathbb{E}_J \| s_1 - s_2 \|_2,
\]

where the infimum is over joint distributions \( J(s_1, s_2) \) with marginals \( \rho_1(s_1), \rho_2(s_2) \). To apply Eq. 9, it is crucial to use a metric on which the Bellman operator is a contraction. To ensure this holds for \( W_1 \) distance we make the following Lipschitz assumptions on the transitions and policies.
Assumption 1

(1.1) $\mathcal{T}, \tilde{\mathcal{T}}$ are $(L_{\mathcal{T},s}, L_{\mathcal{T},a}), (L_{\tilde{\mathcal{T}},s}, L_{\tilde{\mathcal{T}},a})$ Lipschitz w.r.t. states and actions.

(1.2) $\mathcal{A}$ is a convex, closed, bounded (diameter $diam(\mathcal{A})$) set in a $diam(\mathcal{A})$-dimensional space.

(1.3) $\pi(a|s) \sim P_{\mathcal{A}[N(\mu_\pi(s), \Sigma_\pi(s))]}$ and $\pi_D(a|s) \sim P_{\mathcal{A}[N(\mu_{\pi_D}(s), \Sigma_{\pi_D}(s))]}

(1.4) $\mu_\pi, \mu_{\pi_D}, \Sigma_{\pi_D}^{1/2},$ and $\Sigma_\pi^{1/2}$ are $L_{\pi, \mu}, L_{\pi_D, \mu}, L_{\pi, \Sigma}, L_{\pi_D, \Sigma}$ Lipschitz w.r.t. states.

Notice above, (1.3) $P_{\mathcal{A}}$ is the projection to $\mathcal{A}$ and (1.4) $\|\Sigma_{\pi_D}^{1/2}(s) - \Sigma_\pi^{1/2}(s')\| \leq L_{\pi, \Sigma} \|s - s\|_2$. Assumption 1 is not strong because it is easily satisfied in most continuous control tasks, as explained in §4 of Appendix. The harder assumption, which will be resolved later, is $\gamma \eta_{\pi, \tilde{\pi}} < 1$ in Lemma 4.

Lemma 4. If Assumption 1 holds, and $\eta_{\pi, \tilde{\pi}} = L_{\mathcal{T},s} + L_{\mathcal{T},a}(L_{\pi, \mu} + L_{\pi, \Sigma} \sqrt{diam(\mathcal{A})}) \gamma < 1/\beta$, then $\beta_{\pi, \tilde{\pi}}$ is a $\gamma \eta_{\pi, \tilde{\pi}}$-contraction w.r.t. 1-Wasserstein distance.

To verify there exists a nontrivial system such that $\gamma \eta_{\pi, \tilde{\pi}} < 1$ in Lemma 4 holds under Assumption 1, we take the continuous-control task as an example. The key term depends on the sample interval $\Delta$. Take $s = [x, v]^T = [\text{position}, \text{velocity}]^T, a = \text{acceleration}$. By the laws of motion,

$$s' = \begin{bmatrix} x' \\ v' \end{bmatrix} = \begin{bmatrix} x + v\Delta + \frac{1}{2}a\Delta^2 \\ v + a\Delta \end{bmatrix} = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} s + \begin{bmatrix} I \frac{1}{2}\Delta^2 \\ 0 \end{bmatrix} a = \mathcal{T}(s, a)$$

(11)

This shows $L_{\mathcal{T},s} = 1 + O(\Delta), L_{\mathcal{T},a} = O(\Delta)$ and $\eta_{\pi, \mathcal{T}} = 1 + O(\Delta)$. Therefore, we conclude that $\gamma \eta_{\pi, \tilde{\pi}} = \gamma + O(\Delta) < 1$ for small enough $\Delta$. If Lemma 4 holds for $\mathcal{T}$, we can apply the principle of Eq. (9): measure error in $W_1$ distance, apply contraction on $W_1$ to get an asymmetric bound (Lemma 4) and then upper bound $W_1$ distance by $\ell_2$ error. This gives the following Theorem for deterministic transitions.

Theorem 3. Under Lemma 4, if $r(s, a)$ is $L_r$-Lipschitz and the $\ell_2$ error is $\epsilon_{\ell_2}$, then

$$|R(\pi_D, \mathcal{T}) - R(\pi_D, \tilde{\mathcal{T}})| \leq (1 + L_{\pi_D, \mu} + L_{\pi_D, \Sigma} \sqrt{diam(\mathcal{A})}) L_r (1 + \gamma \epsilon_{\ell_2})$$

Notice the typical MuJoCo’s reward $r(s, a) = \text{velocity}(s) - ||a||^2$ is Lipschitz if the diameter $diam(\mathcal{A})$ is finite. Although Theorem 5 requires Lemma 4’s strong assumption, branched rollouts allow such restrictive assumption to be satisfied since branched rollouts use a much smaller discount factor.

Corollary 5. Let $r(s, a)$ be $L_r$-Lipschitz and bounded: $0 \leq r(s, a) \leq r_{\text{max}}$. Let $\epsilon_{\ell_2, \beta} = E_{s \sim \rho^{\pi_D}_{\gamma}, \gamma} L_{TV} (\pi_D(\cdot|s) \| \pi(\cdot|s)), \quad \epsilon_{\ell_2} = E_{s \sim \rho^{\pi_D}_{\gamma}, \gamma} L_{TV} (\pi_D(\cdot|s) \| \pi(\cdot|s)), \quad \epsilon_{\ell_2} = E_{s \sim \rho^{\pi_D}_{\gamma}, \gamma} L_{TV} (\pi_D(\cdot|s) \| \pi(\cdot|s))$, then

$$|R_\gamma(\rho_0, \pi, \mathcal{T}) - R_\gamma(\rho_0, \pi, \tilde{\mathcal{T}})| \leq r_{\text{max}} (\epsilon_{\ell_2, \beta} \gamma + \epsilon_{\ell_2} \beta) + (1 + L_{\pi_D, \mu} + L_{\pi_D, \Sigma} \sqrt{diam(\mathcal{A})}) L_r (1 + \gamma \epsilon_{\ell_2}).$$

Besides reducing model error, Corollary 5 shows another benefit of branched rollouts: it is easier to ensure $\beta \eta_{\pi_D, \tilde{\pi}} < 1$, by choosing a small $\beta$ (say 0.9). This suggests that branched rollouts are particularly useful for deterministic transitions. Such a suggestion on branched length (or equivalently, the branched discount factor $\beta$) supports the experimental success of Janner et al. (2019) and their choice of hyperparameter, as mentioned in the last paragraph of § 4.2. Also, this result is for deterministic transitions, so we resolve the issue of Janner et al. (2019), as they proved for stochastic transitions but experimented in deterministic ones.
4.4 MBRL with Deterministic Transitions and Weak Lipschitz Continuity

When Lemma \ref{lem:bellman} is invalid, there is no Bellman contraction, and we cannot use the bounding principle Eq. \ref{eq:bounding}. Here we provide another way to analyze the error, giving a weaker one-sided bound.

To begin with, we cannot expect much when $L_{\hat{T},s} \gg 1$ since the rollout diverges when being repeatedly applied to $\hat{T}$, with error growing exponentially w.r.t. rollout length. To have meaningful analysis, in this subsection we assume $L_{\hat{T},s} \leq 1 + (1 - \gamma)\iota$ with $\iota < 1$. That is, the Lipschitzness of transition w.r.t. state is only slightly higher than 1. The distance between $L_{\hat{T},s}$ and 1 is inversely proportional to the expected length $(1 - \gamma)^{-1}$; i.e., the longer the rollout length, the smoother $\hat{T}$ should be.

The following theorem reveals the impact of asymmetry when there is no Bellman contraction.

\textbf{Theorem 4.} \emph{Let $0 \leq r(s,a) \leq r_{\text{max}}$ and $\epsilon_{\ell_2} = \mathbb{E}_{(s,a) \sim \rho_{\pi_D}} \| \hat{T} - \hat{T} \|_2$. Assume that:}

(a) $\hat{T}(s,a), r(s,a), \pi_D(a|s)$ are Lipschitz in $s$ for any $a$ with constants $L_{\hat{T},s}, L_{r,s}, L_{\pi_D,s}$.

(b) $L_{\hat{T},s} \leq 1 + (1 - \gamma)\iota$ with $\iota < 1$.

(c) The action space is bounded: $\text{diam}_A < \infty$.

Then

$$R(\pi_D,\hat{T}) - R(\pi_D,\hat{T}) \leq \frac{1 + \gamma}{(1 - \gamma)^{\iota}} \sqrt{2\epsilon_{\ell_2} r_{\text{max}} L_{r} + \frac{1 + O(1)}{(1 - \gamma)^{\iota/2}} r_{\text{max}} \sqrt{2\epsilon_{\ell_2} L_{\pi_D} \text{diam}_A}}.$$ 

Theorem \ref{thm:asymmetry} is a one-sided bound resulting from the asymmetry of $\epsilon_{\ell_2} = \mathbb{E}_{(s,a) \sim \rho_{\pi_D}} \| \hat{T} - \hat{T} \|_2$: $\mathbb{E}$ is taken on $\rho_{\pi_D}$, so we can only upper bound $R(\pi_D,\hat{T})$ by $R(\pi_D,\hat{T}) + O((1 - \gamma)^{-\iota/2})$. The resulting MBRL error only ensures that a policy that works well on $\hat{T}$ also works on $\hat{T}$, but not the other way around. This one-sided nature may allow $\hat{T}$ to overfit the data. This supports the use of the Ensemble Method \cite{kurutach2018model} to mitigate model bias by training multiple independent models.

Theorem \ref{thm:asymmetry} only indicates the consequence of the asymmetry of the objective $\epsilon_{\ell_2}$, but there is a way to avoid such an issue. As discussed in Corollary \ref{cor:branching} branched rollouts provide a Bellman contraction and hence two-sided bounds.

5 Conclusion

In this paper, we analyze the error of MBRL in stochastic and deterministic transitions. We provide an analysis framework based on the contraction of Bellman operator w.r.t. distance metrics of probability distributions. We find that absolutely continuous stochastic transitions and deterministic transitions with strong Lipschitz continuity have Bellman contractions. This suggests that MBRL is better suited to these situations. We show that the difficulty of dealing with deterministic transitions that don’t yield a Bellman contraction results from the asymmetry of the objective function. Finally, we prove that branched rollouts can greatly reduce the error of MBRL and allow a Bellman contraction for deterministic transitions.

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A Appendix

A.1 Total Variations, Bellman Contraction and Symmetry Bridge

Fact 1. Let $m^1$ and $m^2$ be probability measures on $\mathbb{R}^n$ whose singular continuous parts are zero. Decompose $m^1$ and $m^2$ into their absolutely continuous and discrete parts: $m^1 = m^1_a + m^1_d$, $m^2 = m^2_a + m^2_d$. Then

$$D_{TV}(m^1||m^2) = \frac{1}{2} (\|m^1_a - m^2_a\|_1 + \|m^1_d - m^2_d\|_1) \triangleq \frac{1}{2} \|m^1 - m^2\|_1.$$ 

Proof. \cite{Hewitt and Ross 1963} Theorem 19.20 implies $D_{TV}(m^1||m^2) = D_{TV}(m^1_a||m^2_a) + D_{TV}(m^1_d||m^2_d)$, so Fact 1 is proved by combining \cite{Hewitt and Ross 1963} Theorem 19.20 and that TV distance = half of $\ell_1$ norm for absolutely continuous or discrete measures.

To avoid using a big hammer, we provide an alternative proof by revising the usual proof of “TV distance = half of $\ell_1$ norm” with the Lebesgue decomposition: $m = m_a + m_d$. Since $m^1_a$ and $m^2_a$ are absolutely continuous w.r.t. Lebesgue measure, let $d^1$, $d^2$ be the corresponding probability density functions.

Let $B = B_a \cup B_d$ where $B_a = \{x \in \text{Supp}(m^1_a) \cup \text{Supp}(m^2_a) : d^1(x) \geq d^2(x)\}$, $B_d = \{x \in \text{Supp}(m^1_d) \cup \text{Supp}(m^2_d) : m^1_d(x) \geq m^2_d(x)\}$. Since $m_a$ and $m_d$ are mutually singular, we know

$$m^1_a(B_d) = m^2_a(B_d) = 0 = m^1_d(B_a) = m^2_d(B_a) \tag{12}$$

Also, the complement operation implies

$$m^2(A^c) - m^1(A^c) = 1 - m^2(A) - 1 + m^1(A) = m^1(A) - m^2(A), \quad \text{for any measurable set } A \tag{13}$$

Hence we have an important result

$$m^1(B) - m^2(B) = m^1_a(B) - m^2_a(B) + m^1_d(B) - m^2_d(B) \tag{14}$$

(i) By definition of TV distance, we get

$$D_{TV}(m^1||m^2) \geq |m^1(B) - m^2(B)| = m^1(B) - m^2(B) \overset{\text{L1}}{=} \frac{1}{2} \|m^1 - m^2\|_1$$

(ii) For any measurable set $A$ in $\mathbb{R}^n$, we know

$$m^1(A) - m^2(A) = [m^1(A \cap B) - m^2(A \cap B)] + [m^1(A \cap B^c) - m^2(A \cap B^c)]$$
By definition of $B$, the first term is nonnegative while the second term is nonpositive; therefore

$$|m^1(A) - m^2(A)| \leq \max \left\{ m^1(A \cap B) - m^2(A \cap B), \ m^2(A \cap B^c) - m^1(A \cap B^c) \right\}$$

$$\leq \max \left\{ m^1(B) - m^2(B), \ m^2(B^c) - m^1(B^c) \right\}$$

$$\leq m^1(B) - m^2(B).$$

Taking a supremium over $A$, we arrive at

$$D_{TV}(m^1 || m^2) \leq \frac{1}{2} \| m^1 - m^2 \|_1.$$

Combining (i) and (ii), the result follows.

Due to Fact 1, in the following we will treat TV distance as the half of $\ell_1$ norm. Also, to unify the operations in discrete and continuous parts, we will consider “generalized” probability density functions where Dirac delta function is included. Thus, Fact 1 is rephrased as

$$D_{TV}(m^1 || m^2) = \frac{1}{2} \int |d^1(x) - d^2(x)| dx,$$

where $d^1, d^2$ are the generalized density functions of $m^1$ and $m^2$. This allows us to prove Fact 2.

**Fact 2.** $B_{\pi,T}$ is a $\gamma$-contraction w.r.t. total variation distance.

**Proof.** Let $p_1(s)$, $p_2(s)$ be the density functions of some state distributions.

$$D_{TV}(B_{\pi,T}(p_1) || B_{\pi,T}(p_2)) = \frac{1}{2} \int \left| B_{\pi,T}(p_1(s)) - B_{\pi,T}(p_2(s)) \right| ds$$

$$= \frac{1}{2} \int \gamma \left| \int T(s|s', a') \pi(a'|s') (p_1(s') - p_2(s')) ds' da' \right| ds$$

$$\leq \frac{\gamma}{2} \int \left| p_1(s') - p_2(s') \right| ds' da' ds$$

$$= \frac{\gamma}{2} \int \left| p_1(s') - p_2(s') \right| ds' = \gamma D_{TV}(p_1 || p_2).$$

The advantages of working on contractions are their convergence and unique fixed-point properties [Theorem 1.1, Conrad (2014)].

**Fact 3.** Let $(X,d)$ be a complete metric space and $f : X \to X$ be a map such that

$$d(f(x), f(x')) \leq cd(x, x')$$

for some $0 \leq c < 1$ and all $x, x' \in X$. Then $f$ has a unique fixed point in $X$. Moreover, for any $x_0 \in X$ the sequence of the iterates $x_0, f(x_0), f(f(x_0)), ...$ converges to the fixed point of $f$.

**Fact 4.** The normalized state occupancy measure $\rho_{\pi, T}^{\mu_0}(s)$ is a fixed point of the Bellman operator $B_{\pi,T}(\cdot)$. 

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Proof.

\[ \rho^{\rho_0, \pi}_{T, \gamma}(s) = (1 - \gamma) \sum_{i=0}^{\infty} \gamma^i f_i(s|\rho_0, \pi, T) \]

\[ = (1 - \gamma) r_0(s|\rho_0, \pi, T) + \gamma (1 - \gamma) \sum_{i=0}^{\infty} \gamma^i f_{i+1}(s|\rho_0, \pi, T) \]

\[ = (1 - \gamma) \rho_0(s) + \gamma (1 - \gamma) \int T(s'|s', a') \pi(a'|s') f_i(s'|\rho_0, \pi, T) ds'da' \]

\[ = (1 - \gamma) \rho_0(s) + \gamma \int T(s'|s', a') \pi(a'|s') (1 - \gamma) \sum_{i=0}^{\infty} \gamma^i f_i(s'|\rho_0, \pi, T) ds'da' \]

\[ = (1 - \gamma) \rho_0(s) + \gamma \int T(s'|s', a') \pi(a'|s') r^{\rho_0, \pi}_{T, \gamma}(s') ds'da' = B_{\pi, T}(\rho^{\rho_0, \pi}_{T, \gamma}(s)). \]

Together, Fact 2 and 3 imply the Bellman operator has a unique fixed point, and according to Fact 4 the unique fixed point is the state occupancy measure. The contraction and the fixed point properties are particularly useful for proving the symmetry bridge Lemma.

**Lemma 1** (symmetry bridge). Let \( B_2 \) be a Bellman operator with fixed-point \( \rho_2 \). Let \( \rho_1 \) be another state distribution. If \( B_2 \) is a \( \eta \)-contraction w.r.t. some metric \( \| \cdot \| \), then \( \| \rho_1 - \rho_2 \| \leq \| \rho_1 - B_2(\rho_1) \| / (1 - \eta) \).

**Proof.**

\[ \| \rho_1 - \rho_2 \| = \| \rho_1 - B_2^{\infty}(\rho_1) \| + \sum_{i=1}^{\infty} \| B_2^i(\rho_1) - B_2^{i+1}(\rho_1) \| \]

\[ \leq \| \rho_1 - B_2(\rho_1) \| + \sum_{i=1}^{\infty} \| \rho_1 - B_2(\rho_1) \| \eta^i = \| \rho_1 - B_2(\rho_1) \| / (1 - \eta). \]

The first line uses the fixed-point property and the triangle inequality for the distance metric \( \| \cdot \| \). The second line uses the contraction property. \( \square \)

**A.2 Error of Policies**

**Lemma 2** (Error w.r.t. TV Distance between Occupancy Measures). Let \( \rho_1(s, a) \), \( \rho_2(s, a) \) be two normalized occupancy measures of rollouts with discount factor \( \gamma \). If \( 0 \leq r(s, a) \leq r_{\text{max}} \), then \( |R(\rho_1) - R(\rho_2)| \leq D_{TV}(\rho_1\|\rho_2)r_{\text{max}}/(1 - \gamma) \). where \( D_{TV} \) is the total variation distance.

**Proof.**

\[ R(\rho_1) = \frac{1}{1 - \gamma} \int r(s, a) \rho_1(s, a) dsda \leq \frac{1}{1 - \gamma} \int r(s, a) \max \rho_1(s, a, \rho_2(s, a)) dsda \]

\[ = R(\rho_2) + \frac{1}{1 - \gamma} \int r(s, a) \left( \max \rho_1(s, a, \rho_2(s, a)) - \rho_2(s, a) \right) dsda \]

\[ \leq R(\rho_2) + \frac{r_{\text{max}}}{1 - \gamma} \int \max (\rho_1(s, a), \rho_2(s, a)) - \rho_2(s, a) dsda \]

\[ = R(\rho_2) + \frac{r_{\text{max}}}{1 - \gamma} \frac{1}{2} \| \rho_1 - \rho_2 \|_1 = R(\rho_2) + \frac{r_{\text{max}}}{1 - \gamma} D_{TV}(\rho_1\|\rho_2). \]

Because the TV distance is symmetric, we may interchange the roles of \( \rho_1 \) and \( \rho_2 \); thus we conclude that

\[ |R(\rho_1) - R(\rho_2)| \leq D_{TV}(\rho_1\|\rho_2)r_{\text{max}}/(1 - \gamma). \]

\( \square \)
**Theorem 1** (Error of Policies). If $0 \leq r(s,a) \leq r_{\text{max}}$ and the discrepancy in policies is 
\[ \epsilon^T_{\pi_D,\pi} = \mathbb{E}_{s \sim \rho_D^\pi} [D_{TV}(\pi_D(\cdot|s)||\pi(\cdot|s))], \]
then \[ R(\pi_D, T) - R(\pi, T) |\leq \epsilon^T_{\pi_D,\pi} r_{\text{max}} \left( \frac{1}{1-\gamma} + \frac{\gamma}{(1-\gamma)^2} \right). \]

**Proof.** Let $B_{\pi_D, T}, B_{\pi, T}$ be Bellman operators whose fixed points are $\rho_D^T(s), \rho_T^\pi(s)$, respectively.

According to Lemma 2, we need to upper bound $D_{TV}(\rho_D^T(s)||\rho_T^\pi(s))$. Observe that
\[
D_{TV}(\rho_D^T(s)||\rho_T^\pi(s)) = \frac{1}{2} \int |\rho_D^T(s) - \rho_T^\pi(s)| ds = \frac{1}{2} \int |\rho_D^T(s)\pi_D(a|s) - \rho_T^\pi(s)\pi(a|s)| ds 
\leq \frac{1}{2} \int \rho_D^T(s) \pi_D(a|s) - \pi(a|s) + \pi(a|s) \rho_D^T(s) - \rho_T^\pi(s) | ds
\]
\[
= \epsilon^T_{\pi_D,\pi} + D_{TV}(\rho_D^T(s)||\rho_T^\pi(s))
\]
As for the rest, by the properties of the Bellman operators, we have
\[
D_{TV}(\rho_D^T(s)||\rho_T^\pi(s)) \leq \frac{1}{1-\gamma} D_{TV}(\rho_D^T(s)||B_{\pi, T}(\rho_D^T(s)))
\leq \frac{1}{1-\gamma} D_{TV}(B_{\pi_D, T}(\rho_D^T(s)||B_{\pi, T}(\rho_D^T(s)))
\leq \frac{\gamma}{2(1-\gamma)} \int T(s'|s, a') \pi_D(a'|s') - \pi(a'|s') \rho_D^T(s') ds' da' ds
\leq \frac{\gamma}{1-\gamma} \epsilon^T_{\pi_D,\pi},
\]
where the top two lines follow from the symmetry bridge property (Lemma 1) and the fixed-point property.
Combining Eq. (15) and (16), we know $D_{TV}(\rho_D^T(s)||\rho_T^\pi(s)) \leq \epsilon^T_{\pi_D,\pi}(1 + \frac{1}{1-\gamma})$; therefore by Lemma 2
\[
|R(\pi_D, T) - R(\pi, T)| \leq \epsilon^T_{\pi_D,\pi} r_{\text{max}} \left( \frac{1}{1-\gamma} + \frac{\gamma}{(1-\gamma)^2} \right).
\]

**Corollary 1** (Error of Behavior Cloning). Let $\pi_D$ and $\pi$ be the expert policy and the agent policy. If $0 \leq r(s,a) \leq r_{\text{max}}$ and $\mathbb{E}_{s \sim \rho_D^\pi} D_{KL}(\pi_D(\cdot|s)||\pi(\cdot|s)) \leq \epsilon_{BC}$, then $R(\pi_D, T) - R(\pi, T) | \leq \sqrt{\epsilon_{BC} r_{\text{max}}(1 - \gamma)}$.

**Proof.** The result is immediate from Theorem 1 and Pinsker’s Inequality.

**Corollary 2** (Error of GAIL). Let $\pi_D$ and $\pi$ be the expert policy and the agent policy. If $0 \leq r(s,a) \leq r_{\text{max}}$ and $D_{JS}(\rho_D^T||\rho_T^\pi) \leq \epsilon_{GAIL}$. Then $|R(\pi_D, T) - R(\pi, T)| \leq \sqrt{2\epsilon_{GAIL} r_{\text{max}}(1 - \gamma)}$.

**Proof.** By definition of the JSD, for any distributions $P, Q$ and their average $M = (P + Q)/2$ we know
\[
D_{JS}(P||Q) = \frac{1}{2} [D_{KL}(P||M) + D_{KL}(Q||M)] \geq D_{TV}(P||M)^2 + D_{TV}(Q||M)^2 \geq \frac{1}{2} D_{TV}(P||Q)^2,
\]
where the first inequality follows from Pinsker’s Inequality, and the second inequality holds because that $D_{TV}(P||M) + D_{TV}(Q||M) \geq D_{TV}(P||Q)$ by triangle inequality and that $2a^2 + 2b^2 \geq c^2$ if $a + b \geq c \geq 0$.
Thus, we know $D_{TV}(\rho_D^T||\rho_T^\pi) \leq \sqrt{2\epsilon_{GAIL}}$. Applying Lemma 2 completes the proof.

**A.3 MBRL with Absolutely Continuous Stochastic Transitions**

**Theorem 2** (Error of Absolutely Continuous Stochastic Transitions). Let $\pi_D, T$ and $\hat{T}$ be the sampling policy, the real and the learned transitions. If $0 \leq r(s,a) \leq r_{\text{max}}$ and the error in one-step total variation distance is $\epsilon^{T, \hat{T}}_{T, \hat{T}} = \mathbb{E}_{s,a \sim \rho_D^T} [D_{TV}(T(\cdot|s,a)||\hat{T}(\cdot|s,a))]$, then $|R(\pi_D, T) - R(\pi_D, \hat{T})| \leq \epsilon^{T, \hat{T}}_{T, \hat{T}} r_{\text{max}}(1 - \gamma)^{-2}$. 

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Proof. If there is a upper bound for $D_T V (\rho_T^D (s, a) \| \rho_T^D (s, a))$, by Lemma 2 we are done. Also, observe that
\[
D_T V (\rho_T^D (s, a) \| \rho_T^D (s, a)) = \frac{1}{2} \sum_{i=0}^{\infty} (1-\gamma) \gamma^i f_i (s, a).
\]
so $D_T V (\rho_T^D (s, a) \| \rho_T^D (s, a))$ is of interest. Employing the properties of Bellman operator, we have
\[
D_T V (\rho_T^D (s, a) \| \rho_T^D (s, a)) \leq \frac{1}{1-\gamma} D_T V (\rho_T^D (s, a) \| B_{\pi_D, T} (\rho_T^D (s, a)))
\]
where the top two lines follows from the symmetry bridge property (Lemma 4) and the fixed-point property. Finally, from Lemma 2 we conclude that
\[
|R(\pi_D, T) - R(\pi_D, \hat{T})| \leq \epsilon_{T, \hat{T}} r_{\max} \gamma (1-\gamma)^2.
\]

Corollary 3 (Error of MBRL with Absolutely Continuous Stochastic Transition). Let $\pi_D$, $\pi$, $T$ and $\hat{T}$ be the sampling policy, the agent policy, the real transition and the learned transition. If $0 \leq r(s, a) \leq r_{\max}$ and the discrepancies are $\epsilon_{T, \hat{T}} = \mathbb{E}_{(s, a) \sim \rho_T^D} D_T V (T(\cdot|s, a)|\hat{T}(\cdot|s, a))$ and $\epsilon_{T, \hat{T}} = \mathbb{E}_{s \sim \rho_T^D} D_T V (\pi_D (\cdot|s)|\hat{T}(\cdot|s))$, then $|R(\pi, T) - R(\pi, \hat{T})| \leq (\epsilon_{T, \hat{T}} + \epsilon_{T, \hat{T}} + \epsilon_{T, \hat{T}})^{\max} \gamma (1-\gamma)^2 + (\epsilon_{T, \hat{T}} + \epsilon_{T, \hat{T}})^{\max} (1-\gamma)$.

Proof. Observe that $|R(\pi, T) - R(\pi, \hat{T})| \leq |R(\pi, T) - R(\pi_D, T)| + |R(\pi_D, T) - R(\pi_D, \hat{T})| + |R(\pi_D, \hat{T}) - R(\pi, \hat{T})|$. Combining Theorem 2 and 4 the result follows.

Lemma 3. Let $\gamma > \beta$ be discount factors of long and short rollouts. Let $\pi_D$ and $T$ be the sampling policy and the real transition, then $D_T V (\rho_T^\pi_D (s, a) \| \rho_T^\pi_D (s, a)) \leq (1-\gamma) \beta / (\gamma - \beta)$.

Proof. Since $\rho_T^\pi_D$ is generated by the triple ($\rho_0, \pi_D, T$) with discount factor $\gamma$ while $\rho_T^\pi_D$ is generated by ($\rho_T^\pi_D, \pi_D, T$) with discount factor $\beta$. By definition of the occupancy measure we have
\[
\rho_T^\pi_D (s, a) = \sum_{i=0}^{\infty} (1-\gamma) \gamma^i f_i (s, a).
\]
where $f_i (s, a)$ is the density of $(s, a)$ at time $i$ if generated by the triple ($\rho_0, \pi_D, T$). Then,
\[
D_T V (\rho_T^\pi_D (s, a) \| \rho_T^\pi_D (s, a)) \leq \frac{1}{2} \sum_{i=0}^{\infty} (1-\gamma) \gamma^i \sum_{j=0}^{i} (1-\gamma) \gamma^i-j (1-\beta) \beta^j f_i (s, a),
\]
where $f_i (s, a)$ is the density of $(s, a)$ at time $i$ generated by the triple ($\rho_0, \pi_D, T$). Then,
\[
D_T V (\rho_T^\pi_D (s, a) \| \rho_T^\pi_D (s, a)) \leq \frac{1}{2} \sum_{i=0}^{\infty} (1-\gamma) \gamma^i \sum_{j=0}^{i} (1-\gamma) \gamma^i-j (1-\beta) \beta^j f_i (s, a) = \frac{1}{2} \sum_{i=0}^{\infty} (1-\gamma) \gamma^i \sum_{j=0}^{i} (1-\gamma) \gamma^i-j (1-\beta) \beta^j f_i (s, a)
\]
where $f_i (s, a)$ is the density of $(s, a)$ at time $i$ generated by the triple ($\rho_0, \pi_D, T$). Then,
\[
D_T V (\rho_T^\pi_D (s, a) \| \rho_T^\pi_D (s, a)) \leq \frac{1}{2} \sum_{i=0}^{\infty} (1-\gamma) \gamma^i \sum_{j=0}^{i} (1-\gamma) \gamma^i-j (1-\beta) \beta^j f_i (s, a)
\]
where $f_i (s, a)$ is the density of $(s, a)$ at time $i$ generated by the triple ($\rho_0, \pi_D, T$). Then,
\[
D_T V (\rho_T^\pi_D (s, a) \| \rho_T^\pi_D (s, a)) \leq \frac{1}{2} \sum_{i=0}^{\infty} (1-\gamma) \gamma^i \sum_{j=0}^{i} (1-\gamma) \gamma^i-j (1-\beta) \beta^j f_i (s, a)
\]
where $f_i (s, a)$ is the density of $(s, a)$ at time $i$ generated by the triple ($\rho_0, \pi_D, T$). Then,
where (\*) comes from that \(-\beta(1-\gamma) + (\frac{2}{\beta})(1-\beta)\gamma\) is a strictly decreasing function in \(i\). Since \(\gamma > \beta\), its sign flips from + to - at some index; say \(M\). Finally, the sum of the absolute value are the same between \(\sum_{i=0}^{M-1}\) and \(\sum_{i=M}^{\infty}\) because the total probability is conservative, and the difference on one side is the same as that on the other.

Corollary 4 (Error of MBRL with A. C. Stochastic Transition and Branched Rollouts). Let \(\gamma > \beta\) be discount factors of long and short rollouts. Let \(\pi_T, \pi, T, \tilde{T}\) be sampling policy, agent policy, real transition and learned transition. If \(0 \leq r(s,a) \leq r_{\text{max}}\) and the discrepancies are \(\epsilon_{\pi_D,\pi} = E_{s \sim \rho_T,\gamma} D_{TV}(\pi_D(\cdot|s)||\pi(\cdot|s))\), \(\epsilon_{\pi_D,\pi} = E_{s \sim \rho_T,\gamma} D_{TV}(\pi_D(\cdot|s)||\pi(\cdot|s))\), \(\epsilon_{\pi_D} = E_{s \sim \rho_T} D_{TV}(\pi_D(T(\cdot|s,a)||\tilde{T}(\cdot|s,a))\), then

\[
|R_\gamma(p_0,\pi,T) - \frac{1-\beta}{1-\gamma}R_\beta(p_\pi^D,\pi,T)| \leq \epsilon_{\pi_D} (\frac{\pi_D^\gamma,\pi_T,\pi_D^\gamma,\pi_T}{1-\gamma})^2 + \frac{\pi_D^\gamma,\pi_T,\pi_D^\gamma,\pi_T}{1-\gamma} + \frac{\beta}{\gamma - \beta}
\]

Proof. Expand with the triangle inequality:

\[
|R_\gamma(p_0,\pi,T) - \frac{1-\beta}{1-\gamma}R_\beta(p_\pi^D,\pi,T)| \\
\leq |R_\gamma(p_0,\pi,T) - R_\gamma(p_0,\pi_D,T)| + |R_\gamma(p_0,\pi_D,T) - \frac{1-\beta}{1-\gamma}R_\beta(p_\pi^D,\pi_D,T)|
\]

By Theorem 1 the first term \(\leq \epsilon_{\pi_D,\pi} r_{\text{max}} (\frac{1}{1-\gamma} + \frac{\gamma}{1-\gamma})\).

The second term is a short extension of Lemma 2 and Lemma 3,

\[
R_\gamma(p_0,\pi_D,T) = \frac{1}{1-\gamma} \int r(s, a)p_\pi^D(s, a)dsda \leq \frac{1}{1-\gamma} \int r(s, a) \max (\rho_{\pi_D,\gamma}^D(s, a), \rho_{\pi_D,\beta}^D(s, a))dsda
\]

\[
= \frac{1-\beta}{1-\gamma} R_\beta(p_{\pi_D,\gamma}^D, \pi_D, T) + \frac{1-\gamma}{1-\gamma} \int r(s, a) \left( \max (\rho_{\pi_D,\gamma}^D(s, a), \rho_{\pi_D,\beta}^D(s, a)) - \rho_{\pi_D,\beta}^D(s, a) \right)dsda
\]

\[
\leq \frac{1-\beta}{1-\gamma} R_\beta(p_{\pi_D,\gamma}^D, \pi_D, T) + \frac{r_{\text{max}}}{1-\gamma} \int \left( \max (\rho_{\pi_D,\gamma}^D(s, a), \rho_{\pi_D,\beta}^D(s, a)) - \rho_{\pi_D,\beta}^D(s, a) \right)dsda
\]

By the symmetry of the total variation distance and Lemma 3 we obtain

\[
|R_\gamma(p_0,\pi_D,T) - \frac{1-\beta}{1-\gamma}R_\beta(p_\pi^D,\pi_D,T)| \leq \frac{r_{\text{max}}}{1-\gamma} D_{TV}(\rho_{\pi_D,\gamma}^D||\rho_{\pi_D,\beta}^D) \leq r_{\text{max}} \frac{\beta}{\gamma - \beta}.
\]

By Theorem 2 the third term \(\leq \epsilon_{\pi_D} r_{\text{max}} \frac{\beta}{(1-\beta)(1-\gamma)}\).

By Theorem 4 the fourth term \(\leq \epsilon_{\pi_D,\pi} r_{\text{max}} \left( \frac{1}{1-\gamma} + \frac{\beta}{(1-\beta)(1-\gamma)} \right)\).
The validation of Assumption 1 is below.

1.1 The real and learned transitions are Lipschitz w.r.t. states and actions. For the real transition especially in continuous control, the Lipschitzness follows from the laws of motion, as computed in Eq. (10) in the paper. For the learned transition, the Lipschitzness can be made by spectral normalization [Mivato et al., 2018] or gradient penalty [Gulrajani et al., 2017], which are some notable approaches to ensure the Lipschitzness of the discriminator in Wasserstein GAN (Arjovsky et al., 2017).

1.2 The action space is convex, closed and bounded in a finite dimensional linear space. This is a standard assumption in continuous-control and is usually satisfied (or made satisfied) in practice (Fujita and Maeda, 2018). The boundness assumption, if not naturally satisfied, is addressed in 1.3.

1.3 The policy follows truncated Gaussian, by projecting the Gaussian r.v. onto the action space. According to (Fujita and Maeda, 2018), this is a common practice in RL experiment. The Gaussian assumption is made by training some NNs for the mean and variance of the policies. As for the projection of action to a bounded convex set, it is perfectly fine in RL experiment and is largely used in most MuJoCo experiments as MuJoCo also provides the bounds for the action space. It is also a good practice since it helps stabilize the training.

1.4 The mean and covariance of the policy are Lipschitz w.r.t. state. As again noted in 1.1, the Lipschitzness can be realized by spectral normalization or gradient penalty. Since the mean and covariance of the policy are represented by some NN, this assumption can be easily made in practice.

**Lemma 4** (Conditional Contraction). Under assumption $\mathbb{I}$, if $\eta_{\pi,T} = L_{T,s} + L_{T,a}(L_{\pi,\mu} + L_{\pi,\Sigma}\sqrt{\dim A}) < 1/\gamma$, where $\gamma$ is the discount factor of $B_{\pi,T}$, then $B_{\pi,T}$ is a $\gamma_{\pi,T}$-contraction w.r.t. $1$-Wasserstein distance.

**Proof.** Recall $B_{\pi,T}(\rho(s)) = (1 - \gamma)\rho(0)(s) + \gamma \int \delta(s - T(s',a'))\pi(a'|s')\rho(s')ds'da'$. Let $\rho_1(s), \rho_2(s)$ be some distributions over states. We have

$$W_1(B_{\pi,T}(\rho_1) \| B_{\pi,T}(\rho_2)) \leq \gamma \inf_{J(s_1,a_1,s_2,a_2)\sim\Pi(\rho_1(a|s),\rho_2(a|s))} \left\| J_{s_1},a_1 \right\|_2$$

$$= \gamma \inf_{J(s_1,a_1,s_2,a_2)\sim\Pi(\rho_1(a|s),\rho_2(a|s))} \left\| J_{s_1},a_1 \right\|_2$$

$$\leq \gamma \inf_{J(s_1,a_1,s_2,a_2)\sim\Pi(\rho_1(a|s),\rho_2(a|s))} \left\| J_{s_1},a_1 \right\|_2$$

$$\leq \gamma \inf_{J(s_1,a_1,s_2,a_2)\sim\Pi(\rho_1(a|s),\rho_2(a|s))} \left\| J_{s_1},a_1 \right\|_2$$

$$\leq \gamma \inf_{J(s_1,s_2,a_2)\sim\Pi(\rho_1(a|s),\rho_2(a|s))} \left\| J_{s_1},a_1 \right\|_2$$

$$\leq \gamma \inf_{J(s_1,s_2,a_2)\sim\Pi(\rho_1(a|s),\rho_2(a|s))} \left\| J_{s_1},a_1 \right\|_2$$

$$\leq \gamma \inf_{J(s_1,s_2,a_2)\sim\Pi(\rho_1(a|s),\rho_2(a|s))} \left\| J_{s_1},a_1 \right\|_2$$

where $\inf$ takes a infimum over all joint distributions $J(s_1, s_2)$ whose marginals are $\rho_1$ and $\rho_2$.

(a) selects a joint distribution over $B_{\pi,T}(\rho_1)$ and $B_{\pi,T}(\rho_2)$ that share the same randomness of $(1 - \gamma)\rho_0$, which establishes a upper bound and allows us to cancel $(1 - \gamma)\rho_0$. (b) uses the Gaussian assumption of the policy, with $\xi_1, \xi_2$ being standard normal vectors. (c) uses the non-expansiveness property of projection onto a closed convex set. (d) selects $\xi_1 = \xi_2$ and uses the property of operator norm. (e) uses the Lipschitz assumption of $\Sigma_{\pi,1/2}(s)$ and that $\|\xi_1\| \leq \sqrt{\dim A}$ by Jensen inequality.
Lemma 5 (Error w.r.t. W1 Distance between Occupancy Measures). Let $\rho_1(s,a), \rho_2(s,a)$ be two normalized occupancy measures of rollouts with discount factor $\gamma$. If the reward is $L_r$-Lipschitz, then $|R(\rho_1) - R(\rho_2)| \leq W_1(\rho_1 \parallel \rho_2)L_r/(1 - \gamma)$.

Proof. The cumulative reward is bounded by

$$R(\rho_1) = \frac{1}{1 - \gamma} \int r(s,a) \rho_1(s,a) dsda = R(\rho_2) + \frac{1}{1 - \gamma} \int r(s,a)(\rho_1(s,a) - \rho_2(s,a)) dsda$$

$$= R(\rho_2) + \frac{L_r}{1 - \gamma} \int \frac{r(s,a)}{L_r} (\rho_1(s,a) - \rho_2(s,a)) dsda$$

$$\leq R(\rho_2) + \frac{L_r}{1 - \gamma} \sup_{\|f\|_{Lip} \leq 1} \int f(s,a)(\rho_1(s,a) - \rho_2(s,a)) dsda$$

$$= R(\rho_2) + \frac{L_r}{1 - \gamma} W_1(\rho_1 \parallel \rho_2).$$

The third line holds because $r(s,a)/L_r$ is 1-Lipschitz and the last line follows from Kantorovich-Rubinstein duality [Villani (2008)]. Since $W_1$ distance is symmetric, the same conclusion holds if interchanging $\rho_1$ and $\rho_2$; thus

$$|R(\rho_1) - R(\rho_2)| \leq W_1(\rho_1 \parallel \rho_2)L_r/(1 - \gamma).$$

Theorem 3 (Error of Deterministic Transitions with Strong Lipschitzness). Under Lemma 2, let $\mathcal{T}, \mathcal{T}, r, \pi_D$ be deterministic real transition, deterministic learned transition, reward and sampling policy. If $r(s,a)$ is $L_r$-Lipschitz and the $\ell_2$ error is $\epsilon_{\ell_2}$, then $|R(\pi_D, \mathcal{T}) - R(\pi_D, \mathcal{T})| \leq (1 + L_{\pi_D, \pi_D} + L_{\pi_D, \Sigma \sqrt{\dim A}}) L_r \frac{\gamma_{\ell_2}}{1 - \gamma_{\pi_D, \mathcal{T}}}.$

Proof. Observe that the Wasserstein distance over the joint can be upper bounded by that over the marginal.

$$W_1(\rho_{T}^{\pi_D}(s, a) \parallel \rho_{\mathcal{T}}^{\pi_D}(s, a)) = \inf_{J(s_1, a_1, s_2, a_2) \in \Pi(\rho_{T}^{\pi_D}(s, a), \rho_{\mathcal{T}}^{\pi_D}(s, a))} \mathbb{E}_J \| (s_1 - s_2, a_1 - a_2) \|_2$$

$$\leq \inf_{J(s_1, a_1, s_2, a_2) \in \Pi(\rho_{T}^{\pi_D}(s, a), \rho_{\mathcal{T}}^{\pi_D}(s, a))} \mathbb{E}_J \| s_1 - s_2 \|_2 + \| a_1 - a_2 \|_2$$

$$\leq (1 + L_{\pi_D, \pi_D} + L_{\pi_D, \Sigma \sqrt{\dim A}}) \inf_{J(s_1, s_2) \in \Pi(\rho_{T}^{\pi_D}(s), \rho_{\mathcal{T}}^{\pi_D}(s))} \mathbb{E}_J \| s_1 - s_2 \|_2$$

$$(17) = (1 + L_{\pi_D, \pi_D} + L_{\pi_D, \Sigma \sqrt{\dim A}}) W_1(\rho_{T}^{\pi_D}(s) \parallel \rho_{\mathcal{T}}^{\pi_D}(s)), $$

where (*) follows from the same analysis in Lemma 3. Also, the Wasserstein distance over the marginal is upper bounded by the $\ell_2$ error:

$$W_1(\rho_{T}^{\pi_D}(s) \parallel \rho_{\mathcal{T}}^{\pi_D}(s)) \leq \frac{1}{1 - \gamma_{\pi_D, \mathcal{T}}} W_1(\rho_{T}^{\pi_D}(s) \parallel B_{T}^{\pi_D}(\rho_{T}^{\pi_D}(s))) = \frac{1}{1 - \gamma_{\pi_D, \mathcal{T}}} W_1(B_{T}^{\pi_D}(\rho_{T}^{\pi_D}(s)) \parallel B_{T}^{\pi_D}(\rho_{\mathcal{T}}^{\pi_D}(s)))$$

$$\leq \frac{\gamma}{1 - \gamma_{\pi_D, \mathcal{T}}} \inf_{J(s_1, a_1, s_2, a_2) \sim \Pi(\rho_{T}^{\pi_D}(s) \pi_D(a|s), \rho_{\mathcal{T}}^{\pi_D}(s) \pi_D(a|s))} \mathbb{E}_J \left\| T(s_1, a_1) - \mathcal{T}(s_2, a_2) \right\|_2$$

$$\leq \frac{\gamma}{1 - \gamma_{\pi_D, \mathcal{T}}} \mathbb{E}_{(s,a) \sim \rho_{T}^{\pi_D}(s) \pi_D(a|s)} \left\| T(s,a) - \mathcal{T}(s,a) \right\|_2 = \frac{\gamma}{1 - \gamma_{\pi_D, \mathcal{T}}} \epsilon_{\ell_2}.$$
a joint distribution to cancel it. The third line builds a upper bound by choosing \((s_1, a_1) = (s_2, a_2) \sim \rho_{\pi_D}^\alpha(s)\pi_D(a|s)\). Combining Eq. (17), (18) and Lemma 5 we conclude that

\[
|R(\pi_D, T) - R(\pi_D, \hat{T})| \leq (1 + L_{\pi_D, \rho} + L_{\pi_D, \Sigma} \sqrt{\dim(A)}L_r) \frac{\gamma \epsilon_{t_2}}{(1 - \gamma)(1 - \gamma \eta_{\pi_D, \hat{T}})}.
\]

**Proof.** Modifying the proof of Corollary 4 with Theorem 3, the result follows.

**Corollary 5** (Error of MBRL with Deterministic Transition, Strong Lipschitz and Branched Rollouts).

Let \(\gamma > \beta\) be discount factors of long and short rollouts. Let \(\pi_D, \pi, T, \hat{T}\) be sampling policy, agent policy, real deterministic transition and deterministic learned transition. Under assumption \(\|\cdot\|_r\leq \max\) suppose the reward is both bounded \(0 \leq r(s, a) \leq r_{\max}\) and \(L_r\)-Lipschitz. Let \(\epsilon_{T, \gamma}^{\pi, \rho} = \mathbb{E}_{s \sim \rho_{\pi_D}^\alpha} \mathcal{D}_{TV}(\pi_D(\cdot|s)\|\pi(\cdot|s))\) and \(\epsilon_{t_2, \beta} = \mathbb{E}_{(s,a) \sim \rho_{\pi_D}^\alpha(\cdot|s)} \|\hat{T}(s,a) - T(s,a)\|_2\). Then,

\[
|\mathbb{E} \rho_0, \pi, T, \gamma \mathcal{R}(\rho_0, T, \pi, \gamma, \hat{T}) - \mathbb{E} \rho_0, \pi, T, \gamma \mathcal{R}(\rho_0, T, \pi, \gamma, \hat{T})| \leq \left(1 + L_{\pi_D, \rho} \right) \frac{\beta \epsilon_{t_2, \beta} + \beta \epsilon_{T, \gamma}^{\pi, \rho} + \beta \epsilon_{t_2, \beta} + \beta \epsilon_{T, \gamma}^{\pi, \rho} + \beta \epsilon_{t_2, \beta}}{(1 - \gamma)(1 - \gamma \eta_{\pi_D, \hat{T}})} + (1 + L_{\pi_D, \rho} + L_{\pi_D, \Sigma} \sqrt{\dim(A)}L_r) \frac{\beta \epsilon_{t_2, \beta} + \beta \epsilon_{T, \gamma}^{\pi, \rho} + \beta \epsilon_{t_2, \beta} + \beta \epsilon_{T, \gamma}^{\pi, \rho} + \beta \epsilon_{t_2, \beta}}{(1 - \gamma)(1 - \gamma \eta_{\pi_D, \hat{T}})}
\]

**Proof.** Modifying the proof of Corollary 4 with Theorem 3, the result follows.

### A.5 MBRL with Deterministic Transition and Weak Lipschitz Continuity

**Theorem 4** (One-sided Error of Deterministic Transitions). Let \(T, \hat{T}, r, \pi_D\) be deterministic real transition, deterministic learned transition, reward and sampling policy. Suppose \(0 \leq r(s, a) \leq r_{\max}\). \(\hat{T}(s,a), r(s,a)\) and \(\pi_D(a|s)\) are Lipschitz in \(s\) for any \(a\) with constants \((L_{T, \rho}, L_r, L_{\pi_D})\). Assume that \(L_{\hat{T}} \leq 1 + (1 - \gamma)\epsilon_\rho \), \(\rho\) is some upper bound and \(\epsilon_\rho < 1\) and that the action space is bounded: \(\dim(A) < \infty\). If the training loss in \(\ell_2\) error is \(\epsilon_{t_2}\), then

\[
R(\pi_D, \hat{T}) - R(\pi_D, T) \leq \frac{1 + \gamma}{(1 - \gamma)^2} \sqrt{2\epsilon_{t_2} r_{\max} L_r} + \frac{1 + O(\epsilon)}{(1 - \gamma)^{3/2}} \epsilon_{t_2} \sqrt{2\epsilon_{t_2} L_{\pi_D} \dim(A)}.
\]

**Proof.** Recall the \(\ell_2\) error is \(\mathbb{E}_{(s,a) \sim \rho_D} \|\hat{T}(s,a) - T(s,a)\|_2\). By Markov’s Inequality, for any \(\delta > 0\),

\[
\mathbb{P}_{(s,a) \sim \rho_D} \left( \|\hat{T}(s,a) - T(s,a)\|_2 \leq \delta \right) > 1 - \frac{\epsilon_{t_2}}{\delta}
\]

Eq. (19) means for a length \(H \sim \text{Geometric}(1 - \gamma)\) rollout \(\{s_i, a_i\}_{i=1}^H\) generated by \((\rho_0, \pi_D, T)\),

\[
\left\|T(s_t, a_t) - \hat{T}(s_t, a_t)\right\|_2 < \delta \quad \text{with probability greater than } 1 - \frac{\epsilon_{t_2}}{\delta}
\]

Following this idea, we say a rollout is consistent to \(\hat{T}\), if for each \(t\), \(\|s_{t+1} - \hat{T}(s_t, a_t)\|_2 < \delta\); in other words, a rollout is consistent to \(\hat{T}\) if for each time step, the state transition is similar to what \(\hat{T}\) does. Let \(P_{\hat{T}}\) be the probability measure induced by the rollout following the real transition \(T\). The cumulative reward is bounded by

\[
R(\pi_D, \hat{T}) = \int_{\text{traj}} R(\text{traj}) dP_{\hat{T}} = \int_{\text{traj consistent}} R(\text{traj}) dP_{\hat{T}} + \int_{\text{traj inconsistent}} R(\text{traj}) dP_{\hat{T}} \leq \int_{\text{traj consistent}} R(\text{traj}) dP_{\hat{T}} + \frac{\epsilon_{t_2}}{\delta} \mathbb{E}[H^2] r_{\max}.
\]
Now, we’d like to change from $P_\pi$ to $P_{\hat{\pi}}$ with the Lipschitz assumptions above. It suffices to reset the states $\{s_i\}_{i=1}^H$ so that the transition obeys $\hat{T}$. Suppose the new states are

$$s'_1 = s_1, \quad s'_i = \hat{T}(s'_{i-1}, a_{i-1}), \quad \forall \ i \geq 2. \quad (21)$$

By the Lipschitzness of $\hat{T}$, triangle inequality and $\hat{T}$-consistency, the distance between $s_i$ and $s'_i$ obeys

$$\|s_1 - s'_1\|_2 = 0$$

$$\|s_i - s'_i\|_2 \leq \|s_i - \hat{T}(s_{i-1}, a_{i-1})\|_2 + \|\hat{T}(s_{i-1}, a_{i-1}) - \hat{T}(s'_i, a_{i-1})\|_2 \leq \delta + L_T \|s_{i-1} - s'_i\|_2, \quad \forall \ i \geq 2.$$ 

That is,

$$\|s_i - s'_i\|_2 \leq \delta \sum_{j=0}^{i-2} L_T^j = \delta \frac{L_T^{i-1} - 1}{L_T - 1}, \quad \forall \ i \geq 2. \quad (22)$$

The difference of cumulative reward between $\text{traj} = \{s_i, a_i\}_{i=1}^H$ and $\text{traj}' = \{s'_i, a_i\}_{i=1}^H$ satisfies

$$R(\text{traj}) = \sum_{i=1}^{H} r(s_i, a_i) \leq r(s'_1, a_1) + \sum_{i=2}^{H} r(s'_i, a_i) + L_r \|s_i - s'_i\|_2 \quad (22) \leq R(\text{traj}') + \delta L_r \sum_{i=2}^{H} \frac{L_T^{i-1} - 1}{L_T - 1} \quad (23)$$

where (24) results from imposing $L_T = 1 + \nu(1 - \gamma) = 1 + \frac{\nu}{\gamma}$ into the exponential:

$$\sum_{i=2}^{H} \frac{L_T^{i-1} - 1}{L_T - 1} = \frac{1}{L_T - 1} \left( \frac{L_T^H - L_T}{L_T - 1} - H + 1 \right) = \frac{(1 + \frac{\nu}{\gamma})^H - \nu H}{\gamma^2} - H + 1 \leq \frac{e^{\nu H} - \nu H - 1}{(\gamma H)^2} \quad \nu(1 - \gamma) = 1 + \frac{\nu}{\gamma} \quad (24)$$

Because the transitions are deterministic, $\{s'_i\}_{i=1}^H$ are constant given $s_1, a_1, \ldots, a_H$, which means the randomness depends on $s_1, a_1, \ldots, a_H$ (with $\{s'_i\}_{i=1}^H$ being the conditions of $\pi_D$), and the density satisfies

$$P_{\hat{\pi}}(\text{traj}') = \rho_0(s'_1)\pi_D(a_1|s'_1) \prod_{i=2}^{H} \pi_D(a_i|s'_i) \geq \rho_0(s_1)\pi_D(a_1|s_1) \prod_{i=2}^{H} \left( \pi_D(a_i|s_i) - L_{\pi_D} \|s_i - s'_i\|_2 \right) \quad (22)$$

$$\geq \rho_0(s_1)\pi_D(a_1|s_1) \prod_{i=2}^{H} \left( \pi_D(a_i|s_i) + \delta L_{\pi_D} \frac{L_T^{i-1} - 1}{L_T - 1} \right) \geq P_{\hat{\pi}}(\text{traj}) \left( 1 - \sum_{i=2}^{H} \frac{\delta L_{\pi_D} L_T^{i-1} - 1}{L_T - 1} \right) \quad (25)$$
Then, conditioning on the length of rollout being \( H \), the integral term in Eq. (20) is bounded by

\[
\int_{\text{traj consistent}|H} R(\text{traj}) dP_T = \int_{s_1, a_1, \ldots, a_H \text{ consis.}} R(\text{traj}) P_T(\text{traj}) ds_1 da_1 \ldots da_H
\]  
\[
\leq \int_{s_1, a_1, \ldots, a_H \text{ consis.}} R(\text{traj}) \left( P_T(\text{traj}') + P_T(\text{traj}) \sum_{i=2}^{H} \frac{\delta L_{\pi_D}(a_i | s_i)}{\pi_D(a_i | s_i)} \frac{L^{-1}_T}{L_T - 1} \right) ds_1 da_1 \ldots da_H
\]
\[
\leq \int_{s_1, a_1, \ldots, a_H \text{ consis.}} R(\text{traj}) P_T(\text{traj}) + \int_{s_1, a_1, \ldots, a_H \text{ consis.}} R(\text{traj}) P_T(\text{traj}) \sum_{i=2}^{H} \frac{\delta L_{\pi_D}(a_i | s_i)}{\pi_D(a_i | s_i)} \frac{L^{-1}_T}{L_T - 1} ds_1 da_1 \ldots da_H
\]
\[
\leq \int_{s_1, a_1, \ldots, a_H \text{ consis.}} (R(\text{traj}') + \delta L_r(H^2/2 + (\mathbb{E}H)^2 O(i))) P_T(\text{traj}') ds_1 da_1 \ldots da_H + \int_{s_1, a_1, \ldots, a_H \text{ consis.}} H_r \max \delta L_{\pi_D} \text{diam}_A (H^3/2 + (\mathbb{E}H)^2 O(i)) ds_1 da_1 \ldots da_H
\]
\[
\leq R(\pi_D, T) + \delta L_r(H^2/2 + (\mathbb{E}H)^2 O(i)) + \delta L_{\pi_D} r_{\max} \text{diam}_A (H^3/2 + (\mathbb{E}H)^3 O(i))
\]

Combining Eq. (20) (26), by choosing

\[
\delta = \sqrt[\frac{2\epsilon_\ell}{\mathbb{E}[H^2]} H_r \mathbb{E}[H^2] + \mathbb{E}[H^2]^{3} O(i) + L_{\pi_D} r_{\max} \text{diam}_A (H^3/2 + (\mathbb{E}H)^3 O(i)),
\]

we are able to minimize:

\[
\frac{\epsilon_\ell}{\delta} \mathbb{E}[H^2] r_{\max} \delta L_r (\mathbb{E}[H^2]/2 + (\mathbb{E}H)^2 O(i)) \delta L_{\pi_D} r_{\max} \text{diam}_A (H^3/2 + (\mathbb{E}H)^3 O(i))
\]

yielding

\[
R(\pi_D, T) - R(\pi_D, \tilde{T}) \leq \mathbb{E}[H^2] \sqrt{2\epsilon_\ell r_{\max} L_r + 2\epsilon_\ell r_{\max} L_{\pi_D} \text{diam}_A (H^3/2 + (\mathbb{E}H)^3 O(i))}
\]

(a) \( \mathbb{E}[H^2] \sqrt{2\epsilon_\ell r_{\max} L_r + 2\epsilon_\ell L_{\pi_D} \text{diam}_A (H^3/2 + (\mathbb{E}H)^3 O(i))}
\]

(b) \( \mathbb{E}[H^2] \sqrt{2\epsilon_\ell r_{\max} L_r + \mathbb{E}[H^2] r_{\max} 2\epsilon_\ell L_{\pi_D} \text{diam}_A (H^3/2 + (\mathbb{E}H)^3 O(i))}
\]

(c) \( \frac{1}{(1 - \gamma)^2} \sqrt{2\epsilon_\ell r_{\max} L_r + \frac{1 + 5\gamma + 5\gamma^2 + \gamma^3 + (1 + \gamma) O(i)}{1 - \gamma^{5/2}}}
\]

(d) \( \frac{1}{(1 - \gamma)^2} \sqrt{2\epsilon_\ell r_{\max} L_r + \frac{1 + O(i)}{1 - \gamma^{5/2}}}
\]

(a) merge the two \( O(i) \) terms together. (b) uses \( \sqrt{x + y} \leq \sqrt{x} + \sqrt{y} \) for \( x, y \geq 0 \). (c) applies the identities \( \mathbb{E}[H^2] = \frac{1 + 5\gamma + 5\gamma^2 + \gamma^3}{1 - \gamma} \). (d) uses \( \sqrt{1 + x} \leq 1 + x/2 \).  

**Corollary 6** (One-sided of MBRL with Deterministic Transition and Branched Rollouts). Let \( \gamma > \beta \) be discount factors of long and short rollouts. Let \( \pi_D, \pi, \tilde{T} \) be sampling policy, agent policy, real deterministic transition and deterministic learned transition. Under the assumptions of Theorem 4 let
\[ \epsilon_{\pi_D, \pi}^\gamma = \mathbb{E}_{s \sim \mu_D} D_{TV}(\pi_D(\cdot | s) \| \pi(\cdot | s)), \quad \epsilon_{\pi_D, \pi}^\beta = \mathbb{E}_{s \sim \mu_D} D_{TV}(\pi_D(\cdot | s) \| \pi(\cdot | s)) \]

and \( \epsilon_{\ell, \beta} = \mathbb{E}_{(s, a) \sim \mu_D} \| \hat{T}(s, a) - \hat{T}(s, a) \|_2 \). Then
\[ R_{\gamma}(\rho_0, \pi, T) - \frac{1 - \beta}{1 - \gamma} R_{\beta}(\rho_D^{\gamma}, \pi, T) \leq r^{\max} \left( \frac{\epsilon_{\pi_D, \pi}^\gamma}{1 - \gamma} + \frac{\epsilon_{\pi_D, \pi}^\beta}{1 - \gamma} \right) + \frac{1 + \beta}{(1 - \beta)(1 - \gamma)} \sqrt{2\epsilon_{\ell, \beta} r^{\max} L_T} + \frac{1 + O(\iota)}{(1 - \beta)^{3/2}(1 - \gamma)} r^{\max} \sqrt{2\epsilon_{\ell, \beta} L_{\pi_D} \text{diam} A}. \]

**Proof.** Plugging in Theorem 4 with \( L_T \leq 1 + (1 - \beta)t \) to the proof of Corollary 4, the result follows. \( \square \)