Parametrization of holonomy-flux phase space in the Hamiltonian formulation of $SO(N)$ gauge field theory with $SO(D + 1)$ loop quantum gravity as an exemplification

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Abstract

The $SO(N)$ Yang-Mills gauge theory is concerned since it can be used to explore the new theory beyond the standard model of particle physics and the higher dimensional loop quantum gravity. The canonical formulation and loop quantization of $SO(N)$ Yang-Mills theory suggest a discrete $SO(N)$ holonomy-flux phase space, and the properties of the critical quantum algebras in the loop quantized $SO(N)$ Yang-Mills theory are encoded in the symplectic structure of this $SO(N)$ holonomy-flux phase space. With the $SO(D + 1)$ loop quantum gravity as an exemplification of loop quantized $SO(N)$ Yang-Mills gauge theory, we introduce a new parametrization of the $SO(D + 1)$ holonomy-flux phase space in this paper. Moreover, the symplectic structure of the $SO(D + 1)$ holonomy-flux phase space are analyzed in terms of the parametrization variables. Comparing to the Poisson algebras among the $SO(D + 1)$ holonomy-flux variables, it is shown that the Poisson algebras among the parametrization variables take a clearer formulation, i.e., the Lie algebras of $so(D + 1)$ and the Poisson algebras between angle-length pairs.

1 Introduction

Many of powerful and fundamental theories in modern physics are formulated as the Yang-Mills gauge field theories [1], e.g. Quantum electroweak theory is based on the $SU(2)_L \times U(1)_Y$ gauge theory [2–4], and Quantum chromodynamics (QCD) takes the formulation of $SU(3)$ gauge theory [5,6]. These theories provide the basement of the Standard Model (SM) of particle physics. Besides, as a non-perturbative and background-independent approach to unify general relativity (GR) and quantum mechanics, loop quantum gravity (LQG) in (1 + 3)-dimensional spacetime is developed based on $SU(2)$ gauge theory [7] [8] [9] [10], and it has made remarkable progresses in several aspects. For instance, various symmetry-reduced models are established in the framework of LQG to give the resolution of singularities [11–13], and various attempts are made in the framework of LQG to account for the BH entropy [14–17]. Nevertheless, several shortcomings of the SM of particle physics suggest new theories beyond the SM, and one of strategy to this goal is considering the gauge field theory with the higher rank gauge group, i.e. the grand unified theories with gauge group $SU(5)$ or $SO(10)$ [18,19]. Moreover, the loop quantum gravity is extended to higher dimensional spacetime to explore its potential to unify other interactions (e.g., extra dimensions and the Kakuza-Klein idea in loop quantization framework [20, 21]), and $SO(D + 1)$ gauge theory is considered in the general (1 + $D$)-dimensional LQG [22]. All of these utilization of $SO(N)$ gauge theory suggest that it is worth to study the explicit structure of the gauge group $SO(N)$ and its Lie algebra. Specifically, in the loop quantization framework of $SO(N)$ gauge theory, the information of the critical quantum algebras is encoded in the symplectic structure of the discrete $SO(N)$ holonomy-flux phase space [7,8,23,24]. Nevertheless, this symplectic structure for general $SO(N)$ is rather complicated so that it is difficult to separate out the degrees of freedom that is needed. In this article, a parametrization of the $SO(N)$ holonomy-flux phase space will be introduced, and we
will analyze the symplectic structure and the Poisson structure in terms of the parametrization variables. More explicitly, we will consider \((1 + D)\)-dimensional LQG with gauge group \(SO(D + 1)\) as an example of \(SO(N)\) gauge theory, and then introduce the parametrization of the \(SO(D + 1)\) holonomy-flux phase space. In the following part of this article, this parametrization will be referred to as twisted geometry parametrization, since it has a discrete geometry interpretation by imposing some constraints in LQG framework.

Now, let us turn to the \((1 + D)\)-dimensional LQG which is a representative example of \(SO(N)\) gauge theory. The loop quantization approach for GR in all dimensions is first developed by Bodendorfer, Thiemann and Thurn [22] [23] [25]. In detail, the all dimensional LQG is based on the connection formulation of \((1 + D)\) dimensional GR in the form of the \(SO(D + 1)\) Yang-Mills theory, with the kinematic phase space coordinatized by the canonical pairs \((A_{aIJ}, \pi^{bKL})\), consisting of the spatial \(SO(D + 1)\) connection fields \(A_{aIJ}\) and the vector fields \(\pi^{bKL}\). In this formulation, the theory is governed by the first class system of the \(SO(D + 1)\) Gaussian constraints, the \((D + 1)\)-dimensional ADM constraints and the additional simplicity constraints. Similar to the Gaussian constraints, the simplicity constraints taking the form \(S_{IJKL}^{ab} := \pi^{[aIJ}\pi^{bKL]}\) generate extra gauge symmetries in the \(SO(D + 1)\) Yang-Mills phase space. It has been shown that the connection phase space correctly reduces to the familiar ADM phase space by carrying out the symplectic reductions with respect to the simple twisted geometry coherent states [31] [32], and then it is generalized to the \((D + 1)\)-dimensional face. The Hilbert space composed by the spin-network states indicates the holonomy-flux phase space associated to each graph, with the Poisson algebras among holonomies and fluxes in the holonomy-flux phase space being isomorphic to the quantum algebras among them in the quantum Hilbert space. To look for the all-dimensional Regge ADM data encoded in the \(SO(D + 1)\) spin-network states, it is necessary to find the degrees of freedom of discrete geometries encoded in the \(SO(D + 1)\) holonomy-flux variables, by considering a gauge reduction procedure with respect to both of the \(SO(D + 1)\) Gaussian constraints and the simplicity constraints in the holonomy-flux phase space.

A series of studies in this direction is first carried out in the \(SU(2)\) formulation of \((1 + 3)\)-dimensional LQG [26] [27] [28] [29] [30], and then they are generalized to the \(SO(D + 1)\) holonomy-flux phase space in our companion paper [31, 32]. Specifically, since the simplicity constraints become anomalous at the vertices of the graphs, the reductions with respect to the Gaussian and simplicity constraints are guided by the twisted geometry parametrization of the edge simplicity constraint surface in the holonomy-flux phase space of \(SO(D + 1)\) LQG. Especially, the twisted geometry interpretation of holonomy-flux variables suggests that the Gaussian and edge simplicity constraints should be imposed strongly since they generate true gauge transformations, while the vertex simplicity constraints should be imposed weakly. The reduced space parametrized by the twisted geometric parameters give a discrete Regge geometry picture, which can be regarded as the discrete version of the ADM phase space of GR. An important application of the twisted geometry parametrization is the construction of the twisted geometry coherent state. Such kind of coherent states is firstly established in \(SU(2)\) LQG [33], and then it is generalized to the \(SO(D + 1)\) LQG with the restriction of the simple representations [34–38]. Specifically, based on the twisted geometry parameters, the simple twisted geometry coherent state in the strong solution space of quantum edge simplicity constraints is established by selecting the dominant terms (which is referred to as Perelomov type coherent state [32, 35, 39]) with simple representation of \(SO(D + 1)\) in the decomposition of the heat-kernel coherent state of \(SO(D + 1)\) [40–42]. It has been shown that the simple twisted geometry coherent states take the Gaussian superposition formulations. Especially, the simple twisted geometry coherent states provides an over-complete basis of the strong solution space of quantum edge simplicity constraints, and their wave functions have well-behaved peakedness and Ehrenfest properties in the reduced phase space with respect to the edge simplicity constraints [37, 38].

In fact, the twisted geometry parametrization of the \(SO(D + 1)\) holonomy-flux phase space discussed in Ref. [31] concerns the issues on the constraint surface of edge simplicity constraint, and the resulted twisted geometry variables only give the parametrization of the reduce phase space with respect to edge simplicity constraint. Correspondingly, the simple twisted geometry coherent states constructed based on the twisted geometry parametrization of the reduce phase space are the gauge (with respect to edge simplicity constraint) invariant coherent states [37]. In other words, the wave functions of these gauge (with respect to edge simplicity constraint) invariant coherent
states are constants along the corresponding gauge orbits, so that each of them peaks at a gauge orbit instead of a point in the phase space [43].

As we have mentioned above, the edge simplicity constraint should be imposed strongly following the twisted geometry interpretation of holonomy-flux variables. Thus, it seems that all of the studies for all dimensional \( SO(D+1) \) LQG can be completed in the strong solution space of quantum edge simplicity constraint, which is the gauge (with respect to simplicity constraint) invariant subspace of the full Hilbert space of all dimensional \( SO(D+1) \) LQG. Nevertheless, several discussions has shown that it is necessary to consider some kind of gauge fixed solution space with respect to simplicity constraint, to deal with some of the issues appeared in the all dimensional \( SO(D+1) \) LQG. Let us introduce two issues to explain this necessity. First, the regularization of the scalar constraint can be carried out by following the standard loop regularization method [22] [23] [44]. The resulted regularized scalar constraint contains the Euclidean term which is given by the antisymmetric contraction of the holonomies along some closed loops and the fluxes at the beginning and target point of these loops. Classically, this Euclidean term captures the information of both of the intrinsic and extrinsic curvature along these closed loops. However, it is shown that the Euclidean term in the quantized scalar constraint can not capture the information of those intrinsic and extrinsic curvature in the strong solution space of quantum edge simplicity constraint, since the strong imposition of quantum edge simplicity constraint leads to the gauge averaging, which vanishes some critical ingredients in the holonomies [43]. Thus, the standard loop regularization method is conflict to the strong imposition of the edge simplicity constraint. To deal with issue, one may consider the gauge fixed solution of the edge simplicity constraint to avoid the gauge averaging, so that the scalar constraint operator given by standard loop regularization method captures the information of those intrinsic and extrinsic curvature correctly. This is the first issue which points out the necessity to consider then gauge fixed solution space with respect to simplicity constraint. The second issue which points out this necessity is the the Fermion coupling problem in all dimensional LQG [45, 46]. Specifically, the strong imposition of the quantum edge simplicity constraint restricts that the holonomies in all dimensional LQG can only be represented in the simple representation space of \( SO(D+1) \), which leads that the holonomies can not transform the Fermions which take values in the spinor representation space of \( SO(D+1) \) for \( D \geq 4 \). An alternative scheme to deal with this issue is to consider the gauge fixed solution of quantum edge simplicity constraint based on the coherent states, which ensures that the holonomies could take matrixes in the spinor representation space of \( SO(D+1) \), so that they are able to describe the transformation of Fermions along edges.

Usually, in the classical theory, the gauge fixing can be realized by restricting the physical considerations on a section of the gauge orbits on the constraint surface of edge simplicity constraint. However, this is not valid in the quantum theory, since the wave functions of the quantum states which sharply converge to the constraint surface of edge simplicity constraint are always dispersed along the gauge orbits. To overcome this problem, it is reasonable to consider the coherent state whose wave function peaks at a point in the phase space, so that one could have the state whose wave function converges to both of the constraint surface of edge simplicity constraint and a section of the gauge orbits, with this convergence is controlled by the width of the wave function of the coherent state. Such kind of coherent state whose wave function peaks at a point in the \( SO(D+1) \) holonomy-flux phase space could be constructed by following a similar procedure as the construction of the simple twisted geometry coherent state in the strong solution space of quantum edge simplicity constraint [37]. More specifically, one need to consider a more generalized twisted geometry parametrization, which is able to coordinate the (almost) whole \( SO(D+1) \) holonomy-flux phase space instead of the reduced phase space. Then, based on this more generalized twisted geometry parametrization, one could decompose the heat-kernel coherent state of \( SO(D+1) \) and select some dominant terms to formulate the twisted geometry coherent state involving the non-simple representations of \( SO(D+1) \), which will be referred as to the non-simple twisted geometry coherent state in all dimensional LQG.

As the preparation step to establish the non-simple twisted geometry coherent state in all dimensional LQG, it is necessary to extend the twisted geometry parametrization to the full \( SO(D+1) \) holonomy-flux phase space. In this article, we will establish the twisted geometry parametrization of the full \( SO(D+1) \) holonomy-flux phase space, and extend this parametrization as a symplectic-morphism. Besides, we will show that the twisted geometry parametrization of edge simplicity constraint surface introduced in our previous work [31] can be regarded as a special cases of the construction in this article.
This article is organized as follows. In our brief review of the classical connection formulation of all dimensional GR in Section 2, we will also introduce the $SO(D+1)$ holonomy-flux phase space and the discretized formulation of the kinematical constraints. In Section 3 and Section 4 we will introduce the twisted geometry parametrization for the full $SO(D+1)$ phase space, and analyze the Poisson structures among the new geometric parametrization variables. Then, in Section 5 we will discuss the relation between the twisted geometry parametrizations of the edge simplicity constraint surface and the full $SO(D+1)$ holonomy-flux phase space. Finally, we will conclude with the outlook for the possible next steps of the future research.

2 Phase space of $SO(D+1)$ loop quantum gravity

2.1 Connection phase space

The classical connection formulation of GR with arbitrary spacetime dimensionality of $(1+D)$ is first developed by Bodendorfer, Thiemann and Thurn in Ref. [22]. This continuum connection phase space is coordinatized by a $so_{D+1}$ valued 1-form field $A_{abIJ}$ and a vector field $\pi^{bKL}$ on the $D$-dimensional spatial manifold $\Sigma$, with the non-trivial Poisson brackets between them being given by

$$\{A_{abIJ}(x), \pi^{bKL}(y)\} = 2\kappa \beta^{g}_{ab} \delta_{IJ}^{KL} \delta^{(D)}(x-y),$$

where $\beta$ is the Barbero-Immirzi parameter and $\kappa$ is the gravitational constant. It is known that this connection phase space correctly reduces to the familiar ADM phase space after the standard symplectic reduction procedure with respect to the first-class constraint system composed by the Gauss constraints $\mathcal{G}^{IJ} \approx 0$ and simplicity constraints $\mathcal{S}^{ab}_{IJ,KL} := \pi^{a[l} \pi^{b]}_{IJ,KL} \approx 0$. Specifically, the simplicity constraint can be solved as $\pi_{aIJ} = 2\sqrt{\pi} e_{a[I} e_{J]}$, where $e_{aI}$ is a dual D-bein field, $n^I$ satisfies $n^I n_I = 1$ determined by $e^{I}_{J}$ with $n^I e_{aI} = 0$, and $q$ is the determinant of the spatial metric $g_{ab}$ which is determined by $\pi_{aIJ}$ with $q^{ab} = e_{aI} e_{bJ}$ on the simplicity constraint surface. One can split $A_{abIJ}$ as

$$A_{abIJ} \equiv \Gamma_{abIJ}(\pi) + \beta K_{abIJ}$$

where $\Gamma_{abIJ}(\pi)$ is a functional of $\pi_{aIJ}$ and it satisfies $\Gamma_{abIJ}(\pi) = \Gamma_{abIJ}(e)$ on the simplicity constraint surface, with $\Gamma_{abIJ}(e)$ being the unique torsionless spin connection compatible with the D-bein $e_{aI}$. Then, the densitized extrinsic curvature can be given by $K^a_{b} = K_{abIJ} \pi^{bIJ}$ on the constraint surface of both Gaussian and simplicity constraint surface.

It is easy to check that the Gaussian constraint generate the standard $SO(D+1)$ gauge transformation of the connection field and its conjugate momentum. Now, let us consider the simplicity constraints from the perspectives of the corresponding gauge transformations. First, the solutions $\pi_{aIJ} = 2\sqrt{\pi} e_{a[I} e_{J]}$ to the simplicity constraint introduced above defines the constraint surface of the simplicity constraints. Then, one can verify that the infinitesimal gauge transformations induced by simplicity constraints are given by [22]

$$\delta K^P_{Q} = \{ \int_{\Sigma} d^D x f_{abIJ,KL}^{P} \pi_{bIJ}^{n} (x), K_{PQ}^{bIJ} (y) \} = 4\kappa \delta_{PQ}^{KL} a_{bIJ}(y).$$

Notice that, on the simplicity constraint surface we have $\pi^{aIJ} = 2\sqrt{\pi} e_{a[I} e_{J]}$ so that $\delta K^{IJ}_{nI} = 0$. Further, by introducing the decomposition

$$K_{abIJ} \equiv 2n_{[a} K_{b]IJ} + \bar{K}_{abIJ},$$

where $\bar{K}_{abIJ} := \bar{\eta}^{K}_{I} \bar{\eta}^{J}_{K} K_{aKL}$ with $\bar{\eta}^{I}_{J} = \delta_{J}^{I} - n^{I} n_{J}$ and $\bar{K}_{aIJ} n^{I} = 0$, we immediately find that $\bar{K}_{abIJ}$ is the pure gauge component, while the components $2n_{[a} K_{b]IJ}$ are gauge invariant with respect to the transformations given in (3). From the expressions of the ADM variables $qq^{ab} = \frac{1}{4} \pi^{aIJ} n^{IJ}$ and $\bar{K}^{ab}_{IJ} = K^{ab}_{IJ} \pi^{bIJ}$, it is easy to see that these variables are indeed gauge invariant with respect to the simplicity constraints on the constraint surface. Thus, through the symplectic gauge reduction procedure, the simplicity constraints eliminate the two parts of degrees of freedom—restricting $\pi^{aIJ} \equiv 2\sqrt{\pi} e_{a[I} e_{J]} = 0$ by the constraint equation and removing the pure-gauge components $\bar{K}_{abIJ} := \bar{\eta}^{K}_{I} \bar{\eta}^{J}_{K} K_{aKL}$. Following these results, the geometric variables constructed by the ADM variables $(q_{ab}, K^{ed})$ can be extended as functionals in the connection phase space, with their original geometric interpretation are remained on the constraints surface.
2.2 Holonomy-flux phase space

The quantization of the connection formulation of \((1 + D)\)-dimensional GR can be carried out by following the standard loop quantization procedures, which leads to a Hilbert space \(\mathcal{H}\) given by the completion of the space of cylindrical functions on the quantum configuration space \([23]\). This Hilbert space can be regarded as a union of the spaces \(\mathcal{H}_\gamma = L^2((SO(D + 1))^{E(\gamma)}, d\mu_{\text{Haar}}^{E(\gamma)})\) on all possible graphs \(\gamma\), where \(E(\gamma)\) denotes the set of edges of \(\gamma\) and \(d\mu_{\text{Haar}}^{E(\gamma)}\) denotes the product of the Haar measure on \(SO(D + 1)\). The Gaussian constraint and simplicity constraint can be promoted as constraint operators in this Hilbert space. However, it has been turned out that the quantum brackets among these constraints give an open and anomalous quantum algebra, which is distinguished with the corresponding constraint algebra of first class in connection phase space \([25]\). Hence, it is necessary to propose a proper treatment of these quantum constraints, to reduce the gauge degrees of freedom and remain the physical degrees of freedom correctly. A reasonable method to reach this goal is to construct the gauge reductions with respect to Gaussian and simplicity constraints in the holonomy-flux phase space. Through this gauge reductions, one can clarify the gauge degrees of freedom and remain the physical degrees of freedom correctly. And this approach can be regarded as a union of the spaces \(\mathcal{H}_\gamma\), where \(\gamma\) is a graph over \(D\)-dimensional faces and connection holonomies over \((D - 1)\)-dimensional edges. The Poisson algebra between \(\mathcal{H}_\gamma\) is now given by \(\Im_{SO(1, D + 1)}\), and \(\mathcal{P}\) denotes the path-ordered product. The flux \(X_{ij}^{\alpha J}\) of \(\pi_{ij}\) through the \((D - 1)\)-dimensional face dual to edge \(e\) is defined by

\[
\tau^{IJ} \equiv \frac{1}{4\beta a} A_{aIJ} \tau^{IJ}, \quad \bar{e}^a = \text{the tangent vector field of } e,
\]

where \(\tau^{IJ}\) is a basis of \(so(D + 1)\) given by \(\langle \tau^{IJ}, \chi^{KL}\rangle = 2\delta^{IJ}_{KL}\), in definition representation space of \(SO(D + 1)\), and \(\mathcal{P}\) denotes the path-ordered product. The flux \(X_{ij}^{\alpha J}\) of \(\pi_{ij}\) through the \((D - 1)\)-dimensional face dual to edge \(e\) is defined by

\[
X_{ij}^{\alpha J} := -\frac{1}{4\beta a} \text{tr} \left( \tau^{IJ} \int_{e} e^{a_1...a_{D-1}} h(\rho_e^*(\sigma))\rho_e^*(\sigma)\pi_{KL}^{\alpha} \rho_e((\rho_e^*)^{-1}) \right),
\]

where \(\alpha\) is an arbitrary but fixed constant with the dimension of length, \(e^*\) is the \((D - 1)\)-face traversed by \(e\) in the dual lattice of \(\gamma\), \(\rho_e^*(\sigma) : [0, 1] \to \Sigma\) is a path connecting the source point \(s_e\) to \(\sigma \in e^*\) such that \(\rho_e^*(\sigma) : [0, 1/2] \to e\) and \(\rho_e^*(\sigma) : [1/2, 1] \to e^*\). The Poisson algebra between the holonomy-flux variables can be induced from the Poisson bracket (1) between the connection variables, which reads

\[
\{h_e, h_{e'}\} = 0, \quad \{h_e, X_{ij}^{\alpha J}\} = \delta_{e, e'} \frac{\kappa}{a} \frac{d}{d\lambda} \left( e^{\lambda^{ij} h_e} \right) |_{\lambda=0},
\]

\[
\{X_{ij}^{\alpha J}, X_{ij'}^{\beta K}\} = \delta_{e, e'} \frac{\kappa}{2a} \left( -\delta^{ij}_{i'} X_{ij}^{\alpha K} \delta^{i'}_{jL} X_{ij'}^{\beta K} + \delta^{iJ}_{iL} X_{ij'}^{\alpha K} + \delta^{iJ}_{iK} X_{ij'}^{\alpha L} \right).
\]

Notice that \(h_e \in SO(D + 1)\), \(X_{ij}^{J} \in so(D + 1)\) and \(SO(D + 1) \times so(D + 1) \cong T^* SO(D + 1)\), the new discrete phase space called the holonomy-flux phase space of \(SO(D + 1)\) loop quantum gravity on a fixed graph, is a direct product of \(SO(D + 1)\) cotangent bundles. Finally, the complete phase
space of the theory is given by taking the union over the holonomy-flux phase spaces of all possible graphs. Similar to the $SU(2)$ case, the phase space coordinated by the holonomy-flux variables $(h_e, X_e)$ of $SO(D + 1)$ loop quantum gravity can be regarded as the discretized version of the continuum phase space.

The (discretized) Gaussian and simplicity constraints in the holonomy-flux phase space are constructed in agreement with the corresponding quantum constraints. With $X_{-e} = -h_e^{-1}X_e h_e \equiv \tilde{X}_e$, the (discretized) Gaussian constraints $G_{e}^{IJ} \approx 0$ for each vertex $v \in \gamma$ of the graph take the form [23]

$$G_{e}^{IJ} = \sum_{e | s(e) = v} X_{e}^{IJ} + \sum_{e | t(e) = v} \tilde{X}_{e}^{IJ} \approx 0,$$

where $s(e)$ and $t(e)$ denote the source and target points of the oriented edge $e$ respectively. The (discretized) simplicity constraints consist of the edge simplicity constraints $S_{e}^{IJKL} \approx 0$ and vertex simplicity constraints $S_{v,e,e',e''}^{IJKL} \approx 0$, which take the forms [23]

$$S_{e}^{IJKL} \equiv X_{e}^{[IJ} X_{e}^{KL]} \approx 0, \quad \forall e \in \gamma, \quad S_{v,e,e',e''}^{IJKL} \equiv X_{e}^{[IJ} X_{e'}^{KL]} \approx 0, \quad \forall e, e' \in \gamma, s(e) = s(e') = v. \tag{9}$$

It has been shown that, since the commutative Poisson algebra between the conjugate momentum variables $\{p^{IJKL}\}$ becomes non-commutative Poisson algebra between the flux variables $\{X_{IJKL}\}$ after the smearing, the Poisson algebra among the discrete version of simplicity constraints become non-closed and thus anomalous, which leads that the symplectic reductions in the holonomy-flux phase space becomes difficult to implement [25]. To deal with this issue, the twisted geometry parametrization of the holonomy-flux phase space is constructed, which ensures that the gauge reductions with respect to the Gaussian and simplicity constraint in the holonomy-flux phase space can be carried out with the guidance of the twisted geometric interpretation of the holonomy-flux variables [31].

The twisted geometry parametrization for the the $SU(2)$ holonomy-flux variables of $(1 + 3)$-dimensional LQG is first introduced by a series of studies following the original works by Freidel and Speziale [26] [29]. The space of the twisted geometry for $SU(2)$ LQG can undergo a symplectic reduction with respect to the discretized Gauss constraints, giving rise to a reduced phase space containing the discretized ADM data of a polyhedral Regge hypersurface. Following a similar procedure, the twisted geometry parametrization in all dimensional $SO(D + 1)$ LQG has been constructed on the edge simplicity constraint surface in the $SO(D + 1)$ holonomy-flux phase space in our companion paper [31]. It has been shown that the gauge reductions with respect to the simplicity constraints and Gaussian constraints in $SO(D + 1)$ LQG can be carried out properly in the twisted geometry parametrization space, which leads to a clear correspondence between the original holonomy-flux variables $(h_e, X_e)$ on edge simplicity constraint surface and the $D$-hypersurface discrete geometry data in Regge geometry formulation. Nevertheless, it is not enough to construct the twisted geometric parametrization on the edge simplicity constraint surface in the $SO(D + 1)$ holonomy-flux phase space. As we have mentioned in introduction, several explorations in the quantum theory of $SO(D + 1)$ LQG requires us consider the quantum states whose wave functions are dispersed beyond the edge simplicity constraint surface. Hence, it is necessary to extend the twisted geometry parametrization to interpret the phase space points which are not located in the edge simplicity constraint surface.

3 Geometric parametrization of $SO(D+1)$ holonomy-flux phase space

To ensure our statements and the notations clearer, we will first generalize the twisted geometry parametrization to $T^*SO(D+1)$ in this section. Then, it will be left to section 5 to discuss the relation between the twisted geometry parametrizations constructed in this article and previous works [31].

3.1 Beyond the edge-simplicity constraint surface

Recall the $SO(D + 1)$ holonomy-flux phase space $\times_{e \in \gamma} T^*SO(D + 1)_{e}$ associated to the given graph $\gamma$. Let us focus on the holonomy-flux phase space $T^*SO(D + 1)$ associated to a single edge
without loss of generality. To give the explicit formulation of the twisted geometric parametrization of $T^*SO(D + 1)$, let us first introduce some new notations. Consider the orthonormal basis \( \{ \delta_1, \delta_2, ..., \delta_{D+1} \} \) of \( \mathbb{R}^{D+1} \), one has the basis \( \{ \tau_{IJ} \} \) of \( so(D + 1) \) given by \( \tau_{IJ} = (\tau_{IJ})_{\text{def}} := 2\delta_I^I \delta_J^J \) in the definition representation space of \( SO(D + 1) \), where \( (\tau_{IJ})_{\text{def}} \) is the generator of the infinitely small rotation in the 2-dimensional vector space spanned by the two vectors \( \delta_I^I \) and \( \delta_J^J \). Then, let us introduce the maximum commutative sub-Lie algebra of \( so(D + 1) \) spanned by \( \{ \tau_1, \tau_2, ..., \tau_m \} \) with \( m = \left\lfloor \frac{D+1}{2} \right\rfloor \), where we define

\[
\tau_1 := \tau_{12}, \quad \tau_2 := \tau_{34}, \quad ..., \quad \tau_m := \tau_{D,D+1}
\]

for \( D + 1 \) being even, and

\[
\tau_1 := \tau_{12}, \quad \tau_2 := \tau_{34}, \quad ..., \quad \tau_m := \tau_{D-1,D}
\]

for \( D + 1 \) being odd. This maximum commutative sub-Lie algebra of \( so(D + 1) \) generates the maximum commutative subgroup \( T^m \). Then, \( SO(D + 1) \) can be regarded as a fiber bundle with the fibers \( T^m \) on the base manifold \( \mathbb{Q}_m := SO(D + 1)/T^m \), which can be also given by \( \mathbb{Q}_m = \{ V := (V_1, ..., V_m)|V_i = g_t g_i^{-1}, t \in \{1, ..., m\}, g \in SO(D + 1) \} \). One can choose a Hopf section \( n : \mathbb{Q}_m \to SO(D + 1), V \to n(V) \) and another Hopf section \( \tilde{n} : \mathbb{Q}_m \to SO(D + 1), \tilde{V} \to \tilde{n}(\tilde{V}) \) for the copy \( \mathbb{Q}_m \) of \( \mathbb{Q}_m \), which satisfy

\[
V_1 = n(V)\tau_1 n(V)^{-1}, ..., V_m = n(V)\tau_m n(V)^{-1},
\]

and

\[
\tilde{V}_1 = -\tilde{n}(\tilde{V})\tau_1 \tilde{n}(\tilde{V})^{-1}, ..., \tilde{V}_m = -\tilde{n}(\tilde{V})\tau_m \tilde{n}(\tilde{V})^{-1}
\]

with \( \mathbb{Q}_m \ni V := (V_1, ..., V_m) \) and \( \mathbb{Q}_m \ni \tilde{V} := (\tilde{V}_1, ..., \tilde{V}_m) \).

Observe that the choice for the Hopf sections is clearly non-unique, and from now on our parametrization will be given under one fixed choice of \( \{ n_e, \tilde{n}_e \} \) for each edge \( e \). Also, we will use the notations \( n_e \equiv n_e(V_e) \) and \( \tilde{n}_e \equiv \tilde{n}_e(\tilde{V}_e) \) in the following part of this article. Then, in the space \( T^*SO(D + 1) \), associated to each edge \( e \), the generalized twisted geometry parametrization can be given by the map

\[
(V_e, \tilde{V}_e, \tilde{n}_e, \xi_e) \mapsto (h_e, X_e) \in T^*SO(D + 1)_e : \quad X_e = \frac{1}{2} n_e(\eta^1_e \tau_1 + ... + \eta^m_e \tau_m) n^{-1}_e
\]

\[
h_e = n_e e^{\xi^1_e \tau_1}...e^{\xi^m_e \tau_m} \tilde{n}_e^{-1},
\]

where we defined the length parameters \( \eta^1_e =: (\eta^1_e, ..., \eta^m_e) \), \( \eta^1_e, \eta^2_e, ..., \eta^m_e \in \mathbb{R} \) with \( \eta^1_e \geq \eta^2_e \geq ... \geq |\eta^m_e| \geq 0 \) and the angle parameters \( \xi := (\xi^1_e, ..., \xi^m_e) \) with \( \xi^1_e, \xi^2_e, ..., \xi^m_e \in (0, \pi] \). Especially, \( \eta^m_e \) satisfies \( \eta^m_e \in \mathbb{R} \) for \( D + 1 \) being even, and \( \eta^m_e \geq 0 \) for \( D + 1 \) being odd. Let us introduce

\[
\eta^1_e =: \chi^1_e + ... + \frac{\chi^m_e}{2}, \quad \eta^2_e =: \chi^2_e + ... + \frac{\chi^m_e}{2},
\]

\[
..., \eta^{m-1}_e =: \chi^{m-1}_e + \frac{\chi^m_e}{2}, \quad \eta^m_e =: \frac{\chi^m_e}{2}
\]

for \( D + 1 \) being even, and

\[
\eta^1_e =: \chi^1_e + ... + \frac{-\chi^{m-1}_e}{2} + \frac{\chi^m_e}{2}, \quad \eta^2_e =: \chi^2_e + ... + \frac{-\chi^{m-1}_e}{2} + \frac{\chi^m_e}{2},
\]

\[
..., \eta^{m-1}_e =: \frac{-\chi^{m-1}_e}{2} + \frac{\chi^m_e}{2}, \quad \eta^m_e =: \frac{-\chi^m_e}{2}
\]

for \( D + 1 \) being even, one can replacing \( \eta^m_e \) by \( \chi^m_e := (\chi^1_e, ..., \chi^m_e) \) in the parametrization (14).

The twisted geometry parametrization (14) of \( T^*SO(D + 1)_e \) associated to a single edge can be directly extended to the whole graph \( \gamma \). Correspondingly, one can introduce the Levi-Civita holonomies \( [h^1_e|e \in \gamma] \) determined by the fluxes \( \{ \tilde{X}_e \in so(D + 1)|e \in \gamma \} \) and \( \{ X_e \in so(D + 1)|e \in \gamma \} \), which takes the form

\[
h^1_e \equiv n_e e^{\xi^1_e \tau_1}...e^{\xi^m_e \tau_m} \tilde{n}_e^{-1}.
\]
Note that the variables \((\zeta^1_e, ..., \zeta^n_e)\) are well-defined via the given \(h^e_\ell\) and the chosen Hopf sections, thus \((\zeta^1_e, ..., \zeta^n_e)\) are already fixed by the given \(\{X_e \in so(D+1) | e \in \gamma\}\) and \(\{\bar{X}_e \in so(D+1) | e \in \gamma\}\). Then, one can factor out \(h^e_\ell\) from \(h_e\) through the expressions

\[
h_e = \left( e^{(\xi^1_e - \zeta^1_e)n_e} ... e^{(\xi^n_e - \zeta^n_e)n_e} \right) h^e_\ell = e^{(\xi^1_e - \zeta^1_e)n_e} ... e^{(\xi^n_e - \zeta^n_e)n_e} \left( e^{(\xi^1_e - \zeta^1_e)n_e} ... e^{(\xi^n_e - \zeta^n_e)n_e} \right) \]

in the perspectives of the source point and target point of \(e\) respectively.

The above decomposition with twisted geometry parameters can be adopted to the splitting of the the Ashtekar connection as \(A_a = \Gamma_a + \beta K_a\) on a given graph. Specifically, one can consider the integral of \(A_a = \Gamma_a + \beta K_a \in so(D+1)\) along an infinitesimal edge direction \(\ell^a_e\), which leads to \(A_e \equiv A_a e_a^a, \Gamma_e \equiv \Gamma_a e_a^a\) and \(K_e \equiv K_a e_a^a\). Clearly, we can establish the following correspondence of

\[
h_e = e^{A_e} \text{ and } h^e_\ell = e^{\Gamma_e}. \tag{21}
\]

The remaining factor should account for the \(K_e\). According to the above discussion, the value of \(K_e\) may thus be expressed in the perspectives of the source point and target point of \(e\), respectively as

\[
\left( e^{(\xi^1_e - \zeta^1_e)n_e} ... e^{(\xi^n_e - \zeta^n_e)n_e} \right) = e^{\beta K_e} \tag{22}
\]

or

\[
\left( e^{(\xi^1_e - \zeta^1_e)n_e} ... e^{(\xi^n_e - \zeta^n_e)n_e} \right) = e^{\beta K_e}. \tag{23}
\]

Further, we have

\[
K_e = \frac{1}{\beta} n_e (\xi^1_e - \zeta^1_e) \tau_1 + ... + (\xi^n_e - \zeta^n_e) \tau_m \tau_1 \tag{24}
\]

or

\[
K_e = \frac{1}{\beta} \bar{n}_e (\xi^1_e - \zeta^1_e) \tau_1 + ... + (\xi^n_e - \zeta^n_e) \tau_m \tau_1 \tag{25}
\]

when it is expressed in the perspectives of the source point and target point of \(e\) respectively.

The set of the variables \(((\eta^1_1, ..., \eta^m_1), (\xi^1_e, ..., \xi^n_e), \tilde{V}_e, \tilde{V}_e)\) gives the generalization of twisted geometry parametrization for the \(SO(D+1)\) holonomy-flux phase space. Comparing with the twisted geometry parametrization for the edge-simplicity constraint surface in the \(SO(D+1)\) holonomy-flux phase space introduced in our companion paper [31], this generalized parametrization scheme covers the full \(SO(D+1)\) holonomy-flux phase space. We will now carry out an analysis of the symplectic structure of the \(SO(D+1)\) holonomy-flux phase space based on the variables \(((\eta^1_1, ..., \eta^m_1), (\xi^1_e, ..., \xi^n_e), \tilde{V}_e, \tilde{V}_e)\), before coming back to provide more support on the relation between the generalized parametrization scheme in this paper and that only for the edge simplicity constraint surface given in our companion paper [31].

4 Symplectic analysis of \(SO(D+1)\) holonomy-flux phase space

Notice that the discussions in this section only depend on each single edge of the graph. To simplify our notations, we will focus on the analysis on a single edge and omit the label \(e\) without loss of generality.

4.1 Symplectic structure of \(SO(D+1)\) holonomy-flux phase space

The symplectic structure of \(SO(D+1)\) holonomy-flux phase space has been discussed in our companion paper [31], let us give a brief review of the main notations as follows. Recall that the \(SO(D+1)\) holonomy-flux phase space associated with each edge of a given graph can be given by the group cotangent space \(T^*SO(D+1)\), as a phase space it enjoys the natural symplectic structure of the \(T^*SO(D+1)\). To give the explicit formulation of this symplectic structure, let us
introduce the function $f(h)$ on $SO(D + 1) \ni h$, and the element $p_X \in so(D + 1)^* \equiv X^{KL}Y_{KL}$ which is a linear function of $Y \in so(D + 1)$ defined by

$$p_X(Y) \equiv X^{KL}Y_{KL}.$$  
(26)

where $X = X^{KL} \in so(D + 1)$. A right-invariant vector field $\hat{X}$ associated to the Lie algebra element $X \in so(D + 1)$, acts on a function $f(h)$ via the right derivative $\nabla^R_X$ as

$$\nabla^R_X f(h) \equiv \frac{d}{dt} f(e^{-tX}h)|_{t=0};$$  
(27)

under the adjoint transformation $X \mapsto -hXh^{-1}$, we obtain the corresponding left derivative

$$\nabla^L_X f(h) \equiv \frac{d}{dt} f(he^{tX})|_{t=0} = -\nabla^R_{hXh^{-1}}f(h).$$  
(28)

One can straightforwardly show that the map from the right invariant vector fields $\hat{X}$ to the corresponding elements $X \in so(D + 1)$ is given by the algebra-valued, right-invariant 1-form $dhh^{-1}$, which reads

$$i_{\hat{X}}(dhh^{-1}) = (L_{\hat{X}}h)h^{-1} = -X,$$  
(29)

where $i$ denotes the interior product, and $L_\chi \equiv i_\chi d + di_\chi$ denotes the Lie derivative. Now, the natural symplectic potential for $T^*SO(D + 1)$ can be expressed as

$$\Theta \equiv X^{IJ}(dhh^{-1})_{IJ} \equiv \text{Tr}(Xdhh^{-1}).$$  
(30)

The symplectic 2-form then follows as

$$\Omega \equiv -d\Theta = -d\text{Tr}(Xdhh^{-1}) = \frac{1}{2}\text{Tr}(d\hat{X} \wedge h^{-1}dh - dX \wedge dhh^{-1})$$  
(31)

where we have introduced $\hat{X} \equiv -h^{-1}Xh$. From the symplectic 2-form, the Poisson brackets among the interesting phase space functions $f \equiv f(h)$ and $p_Y \equiv p_Y(X) \equiv X^{IJ}Y_{IJ}$ is given by [31]

$$\{p_Y, p_Z\} = p_{[Y, Z]}, \quad \{p_Y, f(h)\} = \nabla^R_Y f(h), \quad \{f(h), f'(h)\} = 0.$$  
(32)

One can see from the brackets (32) that the Poisson action of $\hat{p}_Y(X)$ generates left derivatives. Similarly, it is easy to check that the action of $\hat{p}_Y(X) \equiv Y^{IJ}X_{IJ}$ with $X \equiv -h^{-1}Xh$ generate the right derivative $\{\hat{p}_Y, f(h)\} = \nabla^L_Y f(h)$. Moreover, one can check the commutative relation $\{p_Y, \hat{p}_Z\} = 0$. Finally, it is easy to verify that, by setting $2\kappa/a^{D-1} = 1$, the Poisson brackets (32) given by the natural symplectic potential (30) for $T^*SO(D + 1)$ are identical with the one (7) induced by the symplectic structure (1) in the $SO(D + 1)$ connection phase space [31]. In the following part of this article, we will analyze the symplectic structure on $T^*SO(D + 1)$ based on the symplectic potential $\Theta$ without loss of generality.

### 4.2 Symplectomorphism between $SO(D + 1)$ holonomy-flux phase space and generalized twisted geometry parameter space

From now on, let us focus on the analysis on one single edge $e$ of given graph $\gamma$, and we omit the the label $e$ for all of the notations. Denote by $B := \mathbb{Q}_m \times \bar{\mathbb{Q}}_m \times (\times_{i=1}^m R_{e_i}^*) \times \mathbb{T}^m$ the collection of the generalized twisted geometric parameters $(V, \tilde{V}, \tilde{\chi}, \tilde{\xi})$. It is easy to see that the map (14) is not a one to one mapping. In fact, the map (14) is a many to one mapping on the boundary of $B$ defined by $\eta^m = 0$. One can decompose $B = B_0 \cup \hat{B}$ with

$$\hat{B} := B|_{|\eta^m| > 0}$$  
(33)

and

$$B_0 := B \setminus \hat{B}.$$  
(34)

Then, one can find that the map (14) is a one to one mapping between $\hat{B}$ and its image $\hat{B}^* \subset T^*SO(D + 1)$, while it is a many to one mapping between $B_0$ and its image $B_0^* \subset T^*SO(D + 1)$. We will first focus on the symplectic structure on $B$ in this subsection, and then go back to consider the many to one mapping between $B_0$ and its image $B_0^*$ in section 4.5.
The one to one mapping between \( \hat{B} \) and its image \( \hat{B}^* \subset T^*SO(D+1) \) is also an isomorphism
\[
\hat{B} \rightarrow \hat{B}^* \subset T^*SO(D+1). \tag{35}
\]

Based on the isomorphism (35), we may use the generalized twisted geometric parameters to express the induced symplectic structure of \( \hat{B}^* \subset T^*SO(D+1) \) inherited from the phase space \( T^*SO(D+1) \). First, the induced symplectic potential can be expressed as
\[
\Theta_{\hat{B}^*} = \mathrm{Tr}(Xdh^{-1})|_{\hat{B}^* \subset T^*SO(D+1)}
\]
\[
= \frac{1}{2} \sum_{i=1}^{m} \eta_i \mathrm{Tr}(d\tau_{i}^{-1}n^{-1}(dnn^{-1} + n(\sum d\xi_{i}^{-1} - ne\sum \xi_{i}^{-1}d\tilde{n}^{-1} - \tilde{n}e\sum \xi_{i}^{-1}d\tilde{n}^{-1})))
\]
\[
= \frac{1}{2} \sum_{i=1}^{m} \eta_i \mathrm{Tr}(V_{i}dnn^{-1}) + \sum_{i=1}^{m} \eta_i d\xi_{i} - \frac{1}{2} \sum_{i=1}^{m} \eta_i \mathrm{Tr}(\tilde{V}_{i}d\tilde{n}^{-1}).
\]

In the space \( B \), one can extend the potential \( \Theta_B = \Theta_{\hat{B}^*} \) in the limit \( |\eta_m| \rightarrow 0 \) and define
\[
\Theta_B = \frac{1}{2} \sum_{i=1}^{m} \eta_i \mathrm{Tr}(V_{i}dnn^{-1}) + \sum_{i=1}^{m} \eta_i d\xi_{i} - \frac{1}{2} \sum_{i=1}^{m} \eta_i \mathrm{Tr}(\tilde{V}_{i}d\tilde{n}^{-1}) \tag{37}
\]

as the symplectic potential on \( B \). This potential gives the symplectic form \( \Omega_B \) as
\[
\Omega_B = -d\Theta_B = \frac{1}{2} \sum_{i=1}^{m} \eta_i \mathrm{Tr}(V_{i}dnn^{-1} \wedge dnn^{-1}) - \frac{1}{2} \sum_{i=1}^{m} \eta_i \mathrm{Tr}(\tilde{V}_{i}d\tilde{n}^{-1} \wedge d\tilde{n}^{-1}) - \sum_{i=1}^{m} d\eta_i \wedge (d\xi_{i} + \frac{1}{2} \mathrm{Tr}(V_{i}dnn^{-1}) - \frac{1}{2} \mathrm{Tr}(\tilde{V}_{i}d\tilde{n}^{-1})). \tag{38}
\]

It is clear that in the \( \eta_m = 0 \) region of the above (pre-)symplectic structure is degenerate, as expected due to the degeneracy in the parametrization itself in the \( \eta_m = 0 \) region of \( T^*SO(D+1) \).

We are interested in the Poisson algebras between these twisted-geometry variables using the presymplectic form \( \Omega_B \). In order to give the explicit Poisson brackets, in the following section we will study the Hopf sections \( n(\mathcal{V}) \) and \( \tilde{n}(\mathcal{V}) \) in the perspectives of their contributions to the Hamiltonian fields on \( B \) defined by \( \Omega_B \).

### 4.3 Geometric action on the Hopf section and its decomposition

#### 4.3.1 Geometric action on the Hopf section

The Hopf map is defined as a special projection map \( \pi : SO(D+1) \rightarrow \mathbb{Q}_m \) with \( \mathbb{Q}_m := SO(D+1)/\mathbb{T}^m \), such that every element in \( \mathbb{Q}_m \) comes from an orbit generated by the maximal subgroup \( \mathbb{T}^m \) of \( SO(D+1) \) that fixed all of the elements in the set \( \{\tau_1, \tau_2, ..., \tau_m\} \). In the definition representation of \( SO(D+1) \) the Hopf map reads
\[
\pi : SO(D+1) \rightarrow \mathbb{Q}_m \quad \pi(g) = (g\tau_1 g^{-1}, g\tau_2 g^{-1}, ...).
\]

Note that \( \mathcal{V}(g) \) is invariant under \( g \rightarrow g^o_{\alpha_1, \alpha_2, ..., \alpha_m} = ge^{\alpha_1 \tau_1 + \alpha_2 \tau_2 + ... + \alpha_m \tau_m} \), thus it is a function of \( \frac{D(D+1)}{2} - \frac{D+1}{2} \) variables only. This result shows that \( SO(D+1) \) can be seen as a bundle (which is referred to as Hopf bundle) over \( \mathbb{Q}_m \) with the \( \mathbb{T}^m \) fibers. On this bundle we can introduce the Hopf sections, each as an inverse map to the above projection
\[
n : \mathbb{Q}_m \rightarrow SO(D+1) \quad \mathcal{V} \mapsto n(\mathcal{V}), \tag{40}
\]

such that \( \pi(n(\mathcal{V})) = \mathcal{V} \). This section assigns a specific \( SO(D+1) \) element \( n \) to each member of the \( \mathbb{Q}_m \), and it is easy to see that any given section \( n \) is related to all other sections via \( n^o_{\alpha_1, \alpha_2, ..., \alpha_m} \), hence the free angles \( \{\alpha_1, \alpha_2, ..., \alpha_m\} \) parametrize the set of all possible Hopf sections.
Notice that each algebra element $X \in so(D + 1)$ can be associated to a vector field $\hat{X}$ on $\mathbb{Q}_m$, which acts on a function $f(\mathcal{V})$ of $\mathbb{Q}_m$ as

$$\mathcal{L}_\hat{X} f(\mathcal{V}) := \frac{d}{dt} f(e^{-tX} \mathcal{V} e^{tX})|_{t=0}, \quad (41)$$

where $g\mathcal{V}g^{-1} := (gV_1g^{-1}, gV_2g^{-1}, \ldots, gV_mg^{-1})$ with $g \in SO(D + 1)$. Similarly, for a $so(D + 1)$ valued function $S = S(\mathcal{V})$ on $\mathbb{Q}_m$, it can be also associated to a vector field $\hat{S}$ on $\mathbb{Q}_m$, which acts on the function $f(\mathcal{V})$ of $\mathbb{Q}_m$ as

$$\mathcal{L}_\hat{S} f(\mathcal{V}) := \frac{d}{dt} f(e^{-tS\mathcal{V}e^{tS}})|_{t=0}. \quad (42)$$

Specifically, for the linear functions we have

$$\mathcal{L}_\hat{X} \mathcal{V} := (\mathcal{L}_\hat{X} V_1, \ldots, \mathcal{L}_\hat{X} V_m) = (-[X, V_1], \ldots, -[X, V_m]) =: -[X, \mathcal{V}]. \quad (43)$$

Especially, we are interested in the action of the vector fields on the Hopf section $n$. Notice that we have

$$\mathcal{L}_\hat{X} V_i(n) = (\mathcal{L}_\hat{X} n)^{-1} n^{-1} + n^{-1} (\mathcal{L}_\hat{X} n^{-1}) = ([\mathcal{L}_\hat{X} n] n^{-1}, V_i), \quad \forall i \in \{1, \ldots, m\}. \quad (44)$$

Comparing this result with (44), we deduce that

$$(\mathcal{L}_\hat{X} n)^{-1} = -X + \sum_i V_i F_X^i(\mathcal{V}), \quad (45)$$

where $F_X^i(\mathcal{V})$ are functions on $\mathbb{Q}_m$, so that $V_i F_X^i(\mathcal{V})$ commuting with the element $\mathcal{V}$ for all $i$.

**Lemma.** The solution functions $L_t^{ij} \equiv L^i : \mathbb{Q}_m \mapsto so(D + 1)$ of the equations

$$\text{Tr}(L'dnn^{-1}) = 0, \quad L_t^{ij} V_{i', ij} = \delta_{i,i'}, \quad (46)$$

appears in the Lie derivative of the Hopf map section $n(\mathcal{V})$ as,

$$L_X^i := L_t^{ij} X_{ij} = F_X^i \quad (47)$$

and it satisfies the key coherence identity

$$\mathcal{L}_{\hat{X}} L_Y^i - \mathcal{L}_{\hat{Y}} L_X^i = L_{[X,Y]}^i. \quad (48)$$

Finally, the general solution to this identity satisfying the conditions $L_t^{ij} V_{i', ij} = \delta_{i,i'}$ is given by

$$L_X^i = L_X^i + \mathcal{L}_{\hat{X}} \alpha^i \quad (49)$$

where $\alpha^i$ is a function on $\mathbb{Q}_m$.

**Proof.**

To prove Eq.(47), let us take the interior product of an arbitrary vector field $\hat{X}$ with the definition $\text{Tr}(L'dnn^{-1}) = 0$ and consider $(\mathcal{L}_\hat{X} n)^{-1} = i_{\hat{X}}(dnn^{-1})$ given by the definition of Lie derivative, we have

$$0 = i_{\hat{X}} \text{Tr}(L'dnn^{-1}) = \text{Tr}(L'(\mathcal{L}_\hat{X} n)n^{-1}) = -\text{Tr}(L'X) + \sum_{i=1}^m F_X^i \text{Tr}(L'V_i) = -L_X^i + F_X^i, \quad (50)$$

where we used $\text{Tr}(L'V_i) = L_t^{ij} V_{i', ij} = \delta_{i,i'}$ and (45). Thus, we proved $F_X^i = L_X^i$.

To prove Eq.(48), we first consider that

$$\mathcal{L}_{\hat{X}} (dnn^{-1}) = i_{\hat{X}}(dnn^{-1} \wedge dnn^{-1}) + d((\mathcal{L}_\hat{X} n)n^{-1}) \quad (51)$$

$$= [-X + \sum_i V_i L_X^i, dnn^{-1}] + d(-X + \sum_i V_i L_X^i)$$

$$= \sum_i V_i dL_X^i - [X, dnn^{-1}],$$

and

$$\mathcal{L}_{\hat{Y}} (dnn^{-1}) = i_{\hat{Y}}(dnn^{-1} \wedge dnn^{-1}) + d((\mathcal{L}_\hat{Y} n)n^{-1})$$

$$= [-Y + \sum_i V_i L_Y^i, dnn^{-1}] + d(-Y + \sum_i V_i L_Y^i)$$

$$= \sum_i V_i dL_Y^i - [Y, dnn^{-1}],$$

Then

$$\mathcal{L}_{\hat{X}} L_Y^i - \mathcal{L}_{\hat{Y}} L_X^i = L_{[X,Y]}^i.$$
where we used the definition of Lie derivative in the first equality, Eq.(45) in the second and \( dV_i = [dn^{-1}, V_i] \) in the third. Then, the above equation leads to
\[
0 = \mathcal{L}_X \text{Tr}(L'n'n^{-1}) = \text{Tr}((\mathcal{L}_X L' - [L', X])n'n^{-1}) + dL'_X
\]
by using the equalities \( \text{Tr}(L'V_i) = \delta_{i,i'} \). Further, let us take the interior product of Eq.(52) with \( \dot{Y} \) and we get
\[
\mathcal{L}_Y L'_X = \text{Tr}((\mathcal{L}_X L' - [L', X])(Y - \sum_{i'} V_{i'} L'_{i'}))
\]
(53)
\[
= \mathcal{L}_X L'_{i'} - L'_{i[X,Y]} - \sum_{i'} L'_{i'} \left( \text{Tr}((\mathcal{L}_X L')V_{i'}) - \text{Tr}(L'[X, V_{i'}]) \right)
\]
\[
= \mathcal{L}_X L'_{i'} - L'_{i[X,Y]} - \sum_{i'} L'_{i'} \mathcal{L}_X (\text{Tr}(L'V_{i'})),
\]
where the last term vanishes, thus we obtain the coherence identity (48).

To show Eq.(49), let us suppose that we have another solution \( L'' \) to the coherence identity and also the condition \( \text{Tr}(L''V_i) = L''_{i[J]V_{i,J}} = \delta_{i,i'} \). Considering the 1-form \( \phi^i = -\text{Tr}(L''dn^{-1}) \), one can see that its contraction with \( \dot{X} \) is
\[
\phi_X^n \equiv \imath_X \phi^i = -\text{Tr}(L''(\mathcal{L}_X n)n^{-1}) = L''_X - L'_X
\]
(54)
is the difference between the two solutions \( L''_X \) and \( L'_X \). Thus, \( \phi_X^n \) is also a solution to the coherence identity (48). This result together with the definition of the differential \( \imath_X \imath_Y d\phi^i = \mathcal{L}_Y \phi_X^n - \mathcal{L}_X \phi_Y^n + \phi^n_{[X,Y]} \) implies that \( d\phi^n = 0 \), which means that there exists a function \( \alpha^n \) locally at least, such that \( \phi^n = d\alpha^n \) and thus \( L''_X = L'_X + \mathcal{L}_X \alpha^n \). This proves the Eq. (49).

Finally, let us recall that the freedom in choosing the Hopf section lies in the function parameters \( \alpha^n(V) \) in the expression \( n^n(V) \equiv n^n(V)e^{\sum \alpha^n(V)\tau_i} \) for all possible choices of the sections. By applying Eq.(45) to this \( n' \), we immediately get \( L''_X = L'_X + i_X d\alpha^n \). Referring to (54), we can conclude that the function \( L' \) is exactly the function coefficient for the component of \( (dn)n^{-1} \) in the \( V_i \) direction, which is determined by a choice of the Hopf section \( n \).

### 4.3.2 Decomposition and sequence of the Hopf section

As we will see in following part of this article, the Hopf section \( n \) and the geometric action on it are closely related to the symplectic structure and the symplectic reduction on \( B \). To analyze the Hopf section \( Q_m \) more explicitly, let us consider the decomposition of the Hopf section \( n \). Recall the definition \( Q_m := SO(D + 1)/\mathbb{T}^m \), one can decompose \( Q_m \) as
\[
Q_m = D_1 \times D_2 \times \ldots \times D_m
\]
(55)
with
\[
D_1 := SO(D + 1)/(SO(2)_{\tau_1} \times SO(D - 1)_{[\tau_1]}),
\]
\[
D_2 := SO(D - 1)_{[\tau_1]}/(SO(2)_{\tau_2} \times SO(D - 3)_{[\tau_2]}),
\]
\[
\ldots
\]
\[
D_m := SO(D + 3 - 2m)_{[\tau_{m-1}]}/SO(2)_{\tau_m},
\]
(56)
(57)
(58)
(59)
where \( SO(2)_{\tau_i} \) is the group generated by \( \tau_i \) and \( SO(D + 1 - 2i)_{[\tau_i]} \) is the maximal subgroup of \( SO(D + 1) \) which preserves \( (\tau_1, ..., \tau_i) \) and has the Cartan subalgebra spanned by \( (\tau_{i+1}), ..., \tau_m \). Here one should notice that both of \( SO(2)_{\tau_i} \) and \( SO(D + 1 - 2i)_{[\tau_i]} \) preserve \( (\tau_1, ..., \tau_i) \). Then, the Hopf section \( n \) can be decomposed as
\[
n = n_1n_2...n_m.
\]
(60)

This decomposition gives a sequence of the Hopf sections, which reads
\[
n_1, n_1n_2, n_1n_2n_3, ..., n_1...n_m.
\]
(61)
For a specific one $n_1 \ldots n_i$ with $i \in \{1, \ldots, m\}$, it gives

$$n_1 \ldots n_i : \quad \mathbb{D}_1 \times \ldots \times \mathbb{D}_1 \to SO(D + 1)$$

$$(V_1, \ldots, V_i) \mapsto n_1(V_1) n_2(V_1, V_2) \ldots n_i(V_1, \ldots, V_i),$$

where

$$V_1 = n_1 n_2 \ldots n_i \tau_1 n_1^{-1} \ldots n_2^{-1} n_1^{-1} = n_1 \tau_1 n_1^{-1},$$

$$V_2 = n_1 n_2 \ldots n_i \tau_2 n_1^{-1} \ldots n_2^{-1} n_1^{-1} = n_1 n_2 \tau_2 n_1^{-1} n_1^{-1},$$

$$\ldots,$$

$$V_i = n_1 n_2 \ldots n_i \tau_i n_1^{-1} \ldots n_2^{-1} n_1^{-1}.$$

Here one should notice that the decomposition $n = n_1 \ldots n_m$ is not unique. For instance, one can carry out the transformation

$$n_i \to n_i g, n_{i+1} \to g^{-1} n_{i+1}$$

with $g \in SO(D + 1)$ being arbitrary element which preserve $(\tau_1, \ldots, \tau_i)$, and it is easy to verify that the transformation (67) preserves the Hopf section $n$ but changes $n_i$ and $n_{i+1}$ in the decomposition $n = n_1 \ldots n_m$. We can also establish the geometric actions on the Hopf sections $n_1$. Specifically, one can give

$$(\mathcal{L}_X n_1) n_1^{-1} = -X + V_1 \mathcal{L}_X^n(V_1) + \sum_{\mu} V_1^\mu \mathcal{L}_X^n(V_1)$$

based on Eqs.(43), (44) and $V_1 = n_1 \tau_1 n_1^{-1}$, where $V_1^\mu = n_1 \tilde{\tau}^\mu n_1^{-1}$ with $\{\tilde{\tau}^\mu\}$ being a basis of $so(D - 1)_{n_1}$. $\mathcal{L}_X^n(V_1) = \mathcal{L}^{IJ}_{1J}(V_1) X^{IJ}$ and $\mathcal{L}_X^n(V_1) = \mathcal{L}^{IJ}_{1J}(V_1) X^{IJ}$ are functions of $V_1 \in \mathbb{D}_1$ [31]. It has been shown that $\mathcal{L}_{1J}(V_1)$ is the solution of the equation [31]

$$\text{Tr}(\tilde{L}^1 d n_1 n_1^{-1}) = 0, \quad \text{Tr}(\tilde{L}^1 V_1) = 1, \quad \text{and Tr}(\tilde{L}^1 V_1^\mu) = 0, \forall \mu.$$ (69)

By comparing Eq.(69) and Eq.(46), it is easy to see that $\tilde{L}^1 = \tilde{L}^1$ is a solution of $L^1$ in Eq.(46). This result will be a key ingredient in discussions in the next section.

Now, by applying the results of this section to the presymplectic form $\Omega_B$, we will identify the Hamiltonian fields in $B$ and compute the Poisson brackets.

### 4.4 Computation of Hamiltonian vector fields in pre-symplectic manifold $B$

Let us recall the pre-symplectic potential $\Theta_B \equiv \frac{1}{2} \sum_{i=1}^{m} \eta_i \text{Tr}(V_i d\eta n^{-1}) + \sum_{i=1}^{m} \eta_i d\xi - \frac{1}{2} \sum_{i=1}^{m} \eta_i \text{Tr}(V_i d\eta n^{-1})$ induced from the $SO(D + 1)$ holonomy-flux phase space, which defines the pre-symplectic form $\Omega_B$ as

$$\Omega_B = -d\Theta_B = \frac{1}{2} \sum_{i=1}^{m} \eta_i \text{Tr}(V_i d\eta n^{-1} \wedge dn^{-1}) - \frac{1}{2} \sum_{i=1}^{m} \eta_i \text{Tr}(V_i d\eta n^{-1} \wedge d\eta^{-1}) - \frac{1}{2} \sum_{i=1}^{m} \eta_i \text{Tr}(V_i n d\eta n^{-1} \wedge d\eta^{-1})$$

(70)

The associated Poisson brackets can be calculated by considering the Hamiltonian vector fields on $B$. Let us denote the Hamiltonian vector field for the function $f$ as $\psi_f$, where $f \in \{\eta_i, \xi_i, p_X \equiv \frac{1}{2} \sum_i \eta_i V_X^i = \frac{1}{2} \sum_i \eta_i V^i_{1J} X^{IJ}, \tilde{p}_X \equiv \frac{1}{2} \sum_i \eta_i \tilde{V}_X^i = \frac{1}{2} \sum_i \eta_i \tilde{V}^i_{1J} X^{IJ}\}$. Then, using $i_{\psi_f} \Omega_B = -df$, the vector fields could be checked to be given by

$$\psi_{p_X} = \tilde{X} - \sum_i \mathcal{L}_X^n(V) \partial_{\xi_i}, \quad \psi_{\tilde{p}_X} = -\tilde{X} - \sum_i \tilde{L}_X^n(\tilde{V}) \partial_{\xi_i}, \quad \psi_{\eta_i} = -\partial_{\xi_i}.$$ (71)

Here $\tilde{X}$ are the vector fields generating the adjoint action on $Q_m$ labelled by $V$, associated to the algebra elements $X$. Similarly, $\tilde{X}$ are the vector fields generating the adjoint action on $Q_m$ labelled by $\tilde{V}$, associated to the algebra elements $X$. 

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Proof. The first equation of (71) can be checked by considering

$$i_X \Omega_B = -\frac{1}{2} \sum_i \text{Tr}(d(\eta_i V_i)X) + \sum_i d\eta_i L^X_i(V).$$

(72)

Notice that we have $i_{\partial_B} \Omega_B = d\eta$, the first equation of (71) follows immediately. The computation for $\psi_\bar{\psi}^X$ can be carried out similarly, with an opposite sign due to the reversal of the orientation. □

4.5 Reduction of the pre-symplectic manifold $B$

Recall that in the $\eta_m = 0$ region $\Omega_B$ is degenerate, as expected due to the degeneracy of the parametrization (14) in the $\eta_m = 0$ region. Let us now address this degeneracy to get a true symplectic manifold. We can reduce the pre-symplectic manifold $B$ with respect to the vector fields $\hat{E}$ in the kernel of $\Omega_B$, i.e. to consider the quotient manifold $\tilde{B} \equiv B/\text{Ker}(\Omega_B)$. The result would be a symplectic manifold with non-degenerate 2-form given by the quotient projection of $\Omega_B$.

In obtaining the space $\tilde{B}$, we can introduce the equivalence classes under the equivalence relation $p \sim p'$ whenever $p' = \hat{e}^p p$, with $\hat{E} \in \text{Ker}(\Omega_B)$ and $p, p' \in B$. The operation is thus determined by the vector fields in the kernel of $\Omega_B$. Since it is obvious that the vector fields $\hat{E} \in \text{Ker}(\Omega_B)$ appear in the region with $\eta_m = 0$, we look for the vector fields preserving the region while having the interior products with $\Omega_B$ proportional to $\eta$. Let us first consider the vector fields

$$\hat{E}_X \equiv \psi_{\bar{\psi}^X} - \psi_{\bar{\psi}^X},$$

(73)

where $X \in \text{so}(D + 1)$, $Y = -h^{-1}Xh$ with $h$ being a group element rotating $V^i$ to $\hat{V}^i = -h^{-1}V^i h$. Indeed, using the fact that $V^i_\Lambda = V^i_{\hat{\Lambda}}$, the interior product of the field $\hat{D}X$ with the symplectic 2-form is

$$i_{\hat{E}_X} \Omega_B = -\frac{1}{2} \sum_i d(\eta_i V^i_\Lambda - \eta_i \hat{V}^i_\Lambda) - \frac{1}{2} \sum_i \eta_i \text{Tr}(\hat{V}^i dY) = -\frac{1}{2} \sum_i \eta_i \text{Tr}([V^i, X]d\eta^{-1})).$$

(74)

Now, let us analyze the degeneracy of $i_{\hat{E}_X} \Omega_B$. Denoted by $K^i$ the subspace of $B$ defined by $\eta_i = \eta_{i+1} = \ldots = \eta_m = 0$. Consider the $\text{so}(D + 1)$ valued functions $F(V_1, \ldots, V_{(i-1)})$ on $K^i$ which satisfies

$$n_{(i-1)}^{-1}n_1^{-1}n_2^{-1}F(V_1, \ldots, V_{(i-1)})n_1n_2 \ldots n_{(i-1)} \in \text{so}(D + 3 - 2i),$$

(75)

where $n_1 \ldots n_{(i-1)}$ determined by $(V_1, \ldots, V_{(i-1)})$ is from the sequence of the Hopf sections (61), $SO(D + 1 - 2i)$ is the maximal subgroup of $SO(D + 1)$ which preserves $(\tau_1, \ldots, \tau_i)$ and has the Cartan subalgebra spanned by $(\tau_{(i+1)}, \ldots, \tau_m)$. Then, we can define the vector fields $\hat{E}^i_F$ by

$$\hat{E}^i_F := \hat{E}_X|_{X=F(V_1, \ldots, V_{(i-1)})},$$

(76)

and one can verify $i_{\hat{E}^i_F} \Omega_B = 0$ on $K^i$ by using Eq.(74). Thus, notice the relation $K^1 \subset K^2 \subset \ldots \subset K^m$, we have

$$\text{Ker}(\Omega_B) \equiv \{ \hat{E}^i_F|_{i \in \{1, \ldots, m\}} \}$$

(77)

on $K^m$.

Next, to find the equivalence class generated by the vector fields $\hat{E}^i_F$ on $K^i$, we note that the actions of the fields should rotate jointly the vectors $(V_1, \ldots, V_m)$ and $(\hat{V}_1, \ldots, \hat{V}_m)$, that is we have

$$\hat{E}^i_F (V_i) = -h^{-1}F(V_1, \ldots, V_{(i-1)}, V_i), \quad \hat{E}^i_F (\hat{V}_i) = h^{-1}F(V_1, \ldots, V_{(i-1)}, \hat{V}_i).$$

Further, the actions preserves the group element $h$, since

$$\hat{E}_X(h) = -Xh - hY = 0$$

(78)

which ensures that $\hat{E}^i_F (h) = 0$. Therefore, given $p$ and $p'$ on $K^i$, we have $p' \sim p$ if and only if the two are related by a joint rotation in $(V_1, \ldots, V_m)$ and $(\hat{V}_1, \ldots, \hat{V}_m)$ and a $h$-preserving translations in $(\xi_1, \ldots, \xi_m)$. It is easy to see that the parametrization (14) maps $p$ and $p' \sim p$ to the same image in $T^* SO(D+1)$, as expected that the equivalence class generated by the vector fields $\hat{E}^i_F$ on $K^i$ also describes the degeneracy of the parametrization (14). After the quotient with respect
to $\tilde{E}_F$ on each $K_i$, we are left with a manifold $\tilde{K}_i$ parametrized by only $(\eta_1, \ldots, \eta_{i-1})$, $(V_1, \ldots, V_m)$, $(\tilde{V}_1, \ldots, \tilde{V}_{(i-1)})$ and $(\xi_1, \ldots, \xi_m)$. Recall that $B \equiv B|_{\eta_m > 0} \cup K^m$ and $K^1 \subset K^2 \subset \ldots \subset K^m$, let us define
\[
\tilde{K}^m := K^m / \text{Ker}(\Omega_B)
\]
and then the quotient space $\hat{B} \equiv B|_{\eta_m > 0} \cup \tilde{K}^m$. Finally, we conclude that the parametrization (14) gives a one to one map between $\hat{B}$ and its image $T^*SO(D + 1)$, and it can be extended as a symplectic-morphism with $\hat{B}$ being equiped with the symplectic structure $\Omega_B$.

### 4.6 Poisson algebra among the twisted geometry parameters

Based on the Hamiltonian vector fields given by the pre-symplectic potential $\Theta_B$, the Poisson brackets between the twisted geometry parameters can be given by
\[
\{\xi_i, \eta_j\} = \delta_{i,j},
\]
\[
\{p_X, p_Y\} = p_{[X,Y]}, \quad \{\tilde{p}_X, \tilde{p}_Y\} = \tilde{p}_{[X,Y]}
\]
\[
\{V^i, \eta_j\} = \{\tilde{V}^i, \eta_j\} = 0,
\]
and
\[
\{V^i, \tilde{V}^j\} = 0.
\]
Moreover, one can show that the Poisson brackets given by $\Theta_B$ between $\xi_i$ and $p_X$, or the ones between $\xi_i$ and $\tilde{p}_X$ are non-trivial, and they are given by the function $L^i : \tilde{Q}_m \to \text{so}(D + 1)$ in the form
\[
\{\xi_i, p_X\} = L^i_X(V), \quad \{\xi_i, \tilde{p}_X\} = L^i_{\tilde{X}}(\tilde{V}),
\]
where $L^i_X \equiv \text{Tr}(L^i X)$ is the component of $L^i$ along the algebra element $X$.

Especially, the Eqs. (82) taken as the definition equations of the functions $L^i$, together with the Poisson brackets (80), already determined $L^i$ to be exactly the results of the brackets $\{\xi_i, p_X\}$ and $\{\xi_i, \tilde{p}_X\}$ given by the potential $\Theta_B$ corresponding to our choice of the Hopf sections. This result can be shown by the fact that, the function $L^i$ defined by Eqs. (82) is constrained by two conditions given by the above Poisson brackets (80), and these two conditions are exactly the definition of $L^i$ in Lemma in section 4.3.1. Let us then illustrate the details of this fact as follows. The first one of the two conditions comes from the equation
\[
p_{IJ} L^I_{ij} = p_{IJ} \{\xi_i, p^I_J\} = \frac{1}{2} \left[ \{\xi_i, p^I_J\} p_{IJ} \right] = \frac{1}{4} \{\xi_i, \sum J \eta_J^2\} = \frac{1}{2} \eta_i,
\]
with $p_{IJ} := \frac{1}{2} \sum J (\eta_J V^J)$, which gives the normalization condition $L^I_{ij} V^j_{IJ} = \delta^i_j$ in Lemma in section 4.3.1. The second one of the two conditions just comes from the Jacobi identity
\[
\{\xi_i, \{p_X, p_Y\}\} + \{p_X, \{p_Y, \xi_i\}\} + \{p_Y, \{\xi_i, p_X\}\} = 0,
\]
from which we get
\[
L^i_{[X,Y]} = \{p_X, L^i_Y\} + \{p_Y, L^i_X\} = 0,
\]
By using
\[
\{p_X, L^i_Y\} = i_{\psi_X} dL^i_Y = L_X L^i_Y,
\]
one can write the identity (85) as an identity involving Lie derivatives and we get
\[
L_X L^i_Y - L_Y L^i_X = L^i_{[X,Y]},
\]
which is just the coherence identity in Lemma in section 4.3.1. Now, it is easy to see these two conditions makes the Lemma in section 4.3.1 applicable and we can verify the result given in the beginning of this paragraph.
5 Relation with the twisted geometry parametrizations on edge simplicity constraint surface

The twisted geometry parametrization introduced in this article is constructed in the space $x\in\gamma T^* SO(D+1)$, and we also have introduced the twisted geometry parametrization of the edge simplicity constraint surface $x\in\gamma T^* SO(D+1)$ in our companion paper [31]. Thus, it is worth to discuss the relation between these two types of parametrizations.

We also focus on the twisted geometry parametrizations of the space $T^* SO(D+1)$ on a single edge without loss of generality. Then, by setting $\eta_2=\ldots=\eta_m=0$ in Eq.(14), we get

$$X = \frac{1}{2} \eta_1 n_1 \tau_1 n_1^{-1}$$

(88)

which parametrizes all of the simple fluxes satisfying $X^{[IJ}X^{KL]}=0$ in $so(D+1)$. Besides, recall the decomposition $n=n_1\ldots n_m$ of the Hopf section $n$, we get

$$X = \frac{1}{2} \eta_1 n_1 \tau_1 n_1^{-1}$$

(89)

$$h = n_1 \epsilon^{\ell_1\tau_1 n_1^{-1}}$$

with $\bar{n} = n_2\ldots n_m \epsilon^{\ell_2\tau_2} \ldots \epsilon^{\ell_m \tau_m} (\bar{n}_2\ldots \bar{n}_m)^{-1}$. Recall the edge simplicity constraint surface $T^*_{\text{es}} SO(D+1)$ defined by

$$T^*_{\text{es}} SO(D+1) = \{ (h, X) \in T^* SO(D+1) | X^{[IJ}X^{KL]}=0 \},$$

(90)

it is easy to see that $T^*_{\text{es}} SO(D+1) \subset T^* SO(D+1)$ is parametrized by $(\eta_1, \xi_1, V_1, \bar{V}_1, \bar{n})$ based on Eq.(89), where $V_1 = n_1 \tau_1 n_1^{-1}$, $\bar{V}_1 = \bar{n}_1 \tau_1 \bar{n}_1^{-1}$ with the Hopf sections $n_1$ and $\bar{n}_1$ being given by the decompositions $n = n_1\ldots n_m$ and $\bar{n} = \bar{n}_1\ldots \bar{n}_m$ respectively. Thus, by restricting the consideration on the edge simplicity constraint surface, the parametrization (14) reproduces the twisted geometry parametrization introduced in our companion paper [31].

We can further consider the symplectic reduction with respect to the edge simplicity constraint, which can be expressed as $S_{IJ,KL} \equiv p_{[IJ}K_{KL]} = 0$ with $p_{IJ} := \frac{1}{2} \sum \eta V_{IJ}^1$ in twisted geometry parameters. Notice that the Hamiltonian vector field of edge simplicity constraint is spanned by

$$\psi^S_{IJKL} = 2p_{[IJ}(\bar{X}_{KL}) - \sum L_{KL}^i \partial_{\xi_i},$$

(91)

where $\bar{X}_{KL}$ is the vector field generating the adjoint action of $X_{KL}$ on $Q_m$ labelled by $V$, with $X_{KL}$ is the $so(D+1)$ algebra element given by $X_{KL} \equiv X_{KL}^1 = \delta_{[1}^L \delta^1_{J]}$. It is easy to verify that the vector field (91) only induces the transformation of holonomy on the edge simplicity constraint surface, which reads

$$L_{\alpha^{IJKL}} \psi^S_{IJKL} h = \frac{1}{2} \eta_1 \alpha^{IJKL} V_{[IJ}^1 \tau_{KL]} h = \frac{1}{2} \eta_1 \bar{\alpha}^{IJKL} n_1 (\tau_{KL} \bar{n}) \epsilon^{\ell_1 \tau_1 n_1^{-1}},$$

(92)

where $\alpha^{IJKL}$ is an arbitrary tensor satisfying $\alpha^{IJKL} = \alpha^{[IJKL]}$ and $\bar{\alpha}^{IJKL} \equiv \alpha^{IJKL} V_{[IJ}^1 n_1^{-1} \tau_{KL]} n_1 \in so(D-1)\tau_1$. Thus, the component $\bar{n}$ is just the gauge component with respect to edge simplicity constraint. By reducing the edge simplicity constraint surface with respect to the gauge orbit generated by $\psi^S_{IJKL}$, we get the simplicity reduced phase space $B_{\text{es}}$ given by

$$B_{\text{es}} = \mathbb{R}^+ \times S^1 \times D_1 \times \bar{D}_1 \equiv \{(\eta_1, \xi_1, V_1, \bar{V}_1)\},$$

(93)

where $\eta_1 \in [0, +\infty)$, $\xi_1 \in [-\pi, \pi]$, $V_1 \in D_1$, $\bar{V}_1 \in \bar{D}_1$ with $D_1$ and $\bar{D}_1$ are defined by Eq.(56). Correspondingly, the reduced symplectic structure on $B_{\text{es}}$ gives the Poisson brackets

$$\{\bar{p}_X, \bar{p}_Y\} = \bar{p}_{[X,Y]}, \quad \{\bar{p}_X, \bar{p}_Y\} = \bar{p}_{[X,Y]}, \quad \{\xi_1, \eta_1\} = 1,$$

(94)

where $\bar{p}_X \equiv \frac{1}{2} \eta_1 V_X^1 = \frac{1}{2} \eta_1 V_{IJ} X^{IJ}$ and $\bar{p}_X \equiv \frac{1}{2} \eta_1 \bar{V}_X^1 = \frac{1}{2} \eta_1 \bar{V}_{IJ} X^{IJ}$. Specifically, the Poisson bracket between $\xi_1$ and $\{\bar{p}_X, \bar{p}_X\}$ are given by

$$\{\xi_1, \bar{p}_X\} = L_X^1(V), \quad \{\xi_1, \bar{p}_X\} = L_X^1(\bar{V}).$$

(95)
Notice these Poisson brackets is not independent of \((V_2,\ldots,V_m)\) and \((\tilde{V}_2,\ldots,\tilde{V}_m)\), since \(\xi_1\) contains the information of the choices of the Hopf section \(n\) and \(\tilde{n}\) which depend on \(V\) and \(\tilde{V}\). Recall the result of section 4.3.2, by using the decomposition \(n = n_1\ldots n_m\) and \(\tilde{n} = \tilde{n}_1\ldots\tilde{n}_m\), one can choose the Hopf sections \(n\) and \(\tilde{n}\) to ensure that

\[
L^1(V) = \bar{L}^1(V_1), \quad \text{and} \quad L^1(\tilde{V}) = \bar{L}^1(\tilde{V}_1).
\]

(96)

Then, the symplectic structure on reduce phase space \(B_{\text{red}}\) is given by the Eqs.(94), (95) and (96), which is identical with that given in our companion paper [31]. Further, the gauge reduction with respect to Gaussian constraint and the treatment of vertex simplicity constraint can be carried out following the same procedures as that in [31].

6 Conclusion and outlook

The realization of gauge fixing in quantum gauge reduction and the Fermion coupling in all dimensional LQG require us to construct the coherent state in the full Hilbert space which involving the non-simple representations of \(SO(D + 1)\). Following previous experiences, it is reasonable to consider the generalized twisted geometry coherent state and thus it is necessary to establish the twisted geometry parametrization of the full \(SO(D + 1)\) holonomy-flux phase space.

We established the generalized twisted geometry parametrization for the full \(SO(D + 1)\) holonomy-flux phase space. In particular, the twisted geometry parameters are adapted to the splitting of the Ashtekar connection to capture the degrees of freedom of the intrinsic and extrinsic part of the spatial geometry respectively. Moreover, the symplectic structure on the \(SO(D + 1)\) holonomy-flux phase space is re-expressed based on the twisted geometry parameters. Through studying the properties of the Hopf sections in \(SO(D + 1)\) Hopf fibre bundle, we obtained the Poisson algebra among the twisted geometry parameters. Especially, the relation between the twisted geometry parametrizations for the edge simplicity constraint surface and the full holonomy-flux phase space \(\times_{e\in\gamma}T^*SO(D + 1)\) are discussed. We pointed out that the twisted geometry parametrizations for \(\times_{e\in\gamma}T^*SO(D + 1)\) is equivalent to that for the edge simplicity constraint surface by carrying out the gauge reduction with respect to the edge simplicity constraint, which ensures that the treatment of the anomalous vertex simplicity constraint proposed in our companion paper [31] are still valid for the more general case considered in this article.

The twisted geometry parametrizations for \(\times_{e\in\gamma}T^*SO(D + 1)\) provides us the tool which is necessary to construct the twisted geometry coherent state in the full Hilbert space of all dimensional LQG. More explicitly, similar to the construction of twisted geometry coherent state in the solution space of edge simplicity constraint, one could decompose the heat-kernel coherent state of \(SO(D + 1)\) based on the twisted geometry parametrization for \(\times_{e\in\gamma}T^*SO(D + 1)\), and then select the terms dominated by the highest and lowest weight in each representation of \(SO(D + 1)\), to form the twisted geometry coherent state in the full Hilbert space of all dimensional LQG. This will be the subject of a follow up work [47].

It should be remarked that the twisted geometry parametrization of the \(SO(D + 1)\) holonomy-flux phase space are also valid for general \(SO(D + 1)\) (or \(SO(N)\)) Yang-Mills gauge theory, since they have the identical kinematical structure in loop quantization framework (up to the coupling constants). Though the “geometry” may be meaningless out of the framework of gravity theory, the twisted geometry parameters provide a new perspective to analyze the Poisson structure of the \(SO(D + 1)\) holonomy-flux phase space, which could help us to understand the quantum aspects of various \(SO(D + 1)\) Yang-Mills gauge theory.

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