Characterization of the ranges of wave operators for Schrödinger equations with time-dependent short-range potentials via wave packet transform

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Abstract

In this paper, we give a characterization of the ranges of the wave operators for Schrödinger equations with time-dependent “short-range” potentials by using wave packet transform, which is different from the one in Kitada–Yajima [9]. We also give an alternative proof of the existence of the wave operators for time-dependent potentials, which has been firstly proved by D. R. Yafaev [14].

1 Introduction

In this paper, we consider the following Schrödinger equation with time-dependent short-range potentials:

\[ i \frac{\partial}{\partial t} u = H(t)u, \quad H(t) = H_0 + V(t), \quad H_0 = -\frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} = -\frac{1}{2} \Delta \]

in the Hilbert space \( L^2(\mathbb{R}^n) \), where \( V(t) \) is the multiplication operator of a function \( V(t, x) \) and the domain \( D(H_0) = H^2(\mathbb{R}^n) \) is the Sobolev space of order two. We give a characterization of the ranges of the wave operators for Schrödinger equation with time-dependent potentials which are short-range in space by using wave packet transform, which is different from the characterization in Kitada–Yajima [9]. We also give an alternative proof of the existence of the wave operators, which has been firstly proved by D. R. Yafaev [14].

The wave packet transform is defined by A. Córdoba and C. Fefferman [3]. The second author, M. Kobayashi and S. Ito have introduced the physically natural representation of solutions to Schrödinger equations via wave packet transform with time dependent wave packet ([6], [7]), by which we characterize the ranges of the wave operators. If the potential \( V(t, x) \) does not depend on time, our characterization space coincides with the continuous spectral subspace.

In order to prove the existence of the wave operators, we use Cook–Kuroda’s method ([2], [10]) with the representation via the wave packet transform.

The wave operators for Schrödinger equations have been studied from 1950s. For time-independent short-range potentials, J. Cook [2] and S. T. Kuroda [10] have shown that the wave operators exist and that their ranges are the absolutely continuous spectral
subspace. E. Mourre [12] has proved the absence of singular continuous spectrum. V. Enss [14] has given an alternative proof of the existence and the completeness (coincidence of the range and the continuous spectral subspace) of the wave operators, which is called time-dependent scattering theory. For time-dependent short-range potentials, D. R. Yafaev [14] has shown the existence of the wave operators. H. Kitada–K. Yajima [9] has proved the existence of the wave operators and the modified wave operators for time-dependent short-range potentials and for time-dependent long-range potentials, respectively, and has characterized their ranges.

We assume that $V(t, x)$ satisfies the following conditions, which is called short-range.

**Assumption (A).** (i) $V(t, x)$ is a real-valued Lebesgue measurable function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.

(ii) There exist real constants $\delta > 1$ and $C > 0$ such that

$$|V(t, x)| \leq C(1 + |x|)^{-\delta}$$

for $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.

**Assumption (B).** There exists a family of unitary operators $(U(t, \tau))_{(t, \tau) \in \mathbb{R}^2}$ in $\mathcal{H}$ satisfying the following conditions.

(i) For $f \in \mathcal{H}$, $U(t, \tau)f$ is strongly continuous function with respect to $t$ and satisfies

$$U(t, \tau')U(t', \tau) = U(t, \tau), \quad U(t, t) = I \quad \text{for all } t, \tau' \in \mathbb{R},$$

where $I$ is the identity operator on $\mathcal{H}$.

(ii) For $f \in H^2(\mathbb{R}^n)$, $U(t, \tau)f$ is strongly continuously differentiable in $\mathcal{H}$ with respect to $t$ and satisfies

$$\frac{\partial}{\partial t} U(t, \tau)f = -iH(t)U(t, \tau)f \quad \text{for all } t, \tau \in \mathbb{R}.$$

**Remark 1.** If Assumption (A) is satisfied and $V(t)f$ is strongly differentiable in $\mathcal{H}$ for $f \in H^2(\mathbb{R}^n)$, Assumption (B) is satisfied (c.f. T. Kato [8]).

Let $\mathcal{S}$ be the Schwartz space of all rapidly decreasing functions on $\mathbb{R}^n$ and $\mathcal{S}'$ be the space of tempered distributions on $\mathbb{R}^n$. For positive constants $a$ and $R$, we put $\Gamma_{a,R} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid |\xi| \leq a \text{ or } |x| \geq R\}$ and $\mathcal{S}_{\text{scat}} = \{\Phi \in \mathcal{S} \mid \|\Phi\|_\mathcal{H} = 1 \text{ and } \hat{\Phi}(0) \neq 0\}$.

**Definition 1** (Wave packet transform). Let $\varphi \in \mathcal{S} \setminus \{0\}$ and $f \in \mathcal{S}'$. We define the wave packet transform $W_{\varphi}f(x, \xi)$ of $f$ with the wave packet generated by a function $\varphi$ as follows:

$$W_{\varphi}f(x, \xi) = \int_{\mathbb{R}^n} \overline{\varphi(y - x)} f(y) e^{-iy\xi} dy \quad \text{for } (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$  

Its inverse is the operator $W_{\varphi}^{-1}$ which is defined by

$$W_{\varphi}^{-1}F(x) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} \varphi(x - y) F(y, \xi) e^{iy\xi} dy d\xi$$

for $x \in \mathbb{R}^n$ and a function $F(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$.  

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Definition 2. Let $\Phi \in S_{\text{scat}}$ and we put $\Phi(t) = e^{-i t H_0} \Phi$. We define $\tilde{D}^{\pm}_\text{scat}(\tau)$ by the set of all functions in $\mathcal{H}$ such that
\[
\lim_{t \to \pm \infty} \| \chi \Gamma_{a,R} W_{\Phi(t-\tau)} [U(t, \tau) f](x + (t - \tau) \xi, \xi) \|_{L^2(\mathbb{R}_x^d \times \mathbb{R}^d_\xi)} = 0
\]
for some positive constants $a$ and $R$. For $\tau \in \mathbb{R}$, $D^{\pm}_\text{scat}(\tau)$ is defined by the closure of $\tilde{D}^{\pm}_\text{scat}(\tau)$ in the topology of $\mathcal{H}$.

The main state of this paper is the following.

Theorem 1. Suppose that (A) and (B) be satisfied. Then the wave operators
\[
W_\pm(\tau) = \lim_{t \to \pm \infty} U(\tau, t) e^{-i t (\tau - H_0)}
\]
eexist for any $\tau \in \mathbb{R}$ and their ranges $\mathcal{R}(W_\pm(\tau))$ coincide with $D^{\pm}_\text{scat}(\tau)$ for any $\Phi \in S_{\text{scat}}$. In particular, $D^{\pm}_\text{scat}(\tau)$ is independent of $\Phi$.

We use the following notations throughout the paper. $i = \sqrt{-1}$, $n \in \mathbb{N}$. For a subset $\Omega$ in $\mathbb{R}^n$ or in $\mathbb{R}^{2n}$, the inner product and the norm on $L^2(\Omega)$ are defined by $(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx$ and $\|f\|_{L^2(\Omega)} = \left(\int_\Omega |f(x)|^2dx\right)^{1/2}$ for $f, g \in L^2(\Omega)$, respectively. We write $\partial_x j = \partial / \partial x_j$, $\partial_t = \partial / \partial t$, $L_{x, \xi}^2 = L^2(\mathbb{R}_x^d \times \mathbb{R}^d_\xi)$, $(\cdot, \cdot) = (\cdot, \cdot)_{L_{x, \xi}^2}$, $\| \cdot \| = \| \cdot \|_{L_{x, \xi}^2}$.

\[
\langle t \rangle = 1 + |t|, \|f\|_{\Sigma(t)} = \sum_{a + b = n} \| x^a \partial_x^b f \|_{H^l} \quad \text{and} \quad W_\varphi u(t, x, \xi) = W_\varphi [u(t)](x, \xi).
\]
$F$ and $F^{-1}$ are the Fourier transform and the inverse Fourier transform defined by $F f(\xi) = \int_\mathbb{R}^n e^{-ix \cdot \xi} f(x)dx$ and $F^{-1} f(\xi) = (2\pi)^{-n} \int_\mathbb{R}^n e^{ix \cdot \xi} f(\xi)dx$, respectively. We often write $\{ \xi = 0 \}$ as $\{(x, \xi) \in \mathbb{R}^{2n} | \xi = 0 \}$. For sets $A$ and $B$, $A \setminus B$ denotes the set $\{ a \in A \mid a \notin B \}$. $\chi_A(x)$ the characterization function of a measurable set $A$, which is defined by $\chi_A(x) = 1$ on $A$ and $\chi_A(x) = 0$ otherwise. $F(\cdot \cdot \cdot)$ denotes the multiplication operator of a function $\chi_{\{|x| \geq \cdot \cdot \cdot \}}(x)$. For an operator $T$ on $\mathcal{H}$, $D(T)$ and $\mathcal{R}(T)$ denote the domain and the range of $T$, respectively. $\mathcal{H}_{p}(T)$ and $\mathcal{H}_{p}(T)^\perp$ denote pure point subspace of a self-adjoint operator $T$ on $\mathcal{H}$ and its orthogonal complement space, respectively.

In the preceding studies ([4], [14], [9]), the proofs of the existence of the wave operators are relied on Cook – Kuroda’s method. In our proof, we use the duality argument and the representation
\[
W_{\varphi_0} [e^{i t H_0} U(t, 0) \psi](x, \xi) = W_{\varphi_0} \psi(x, \xi) - i \int_0^t e^{i \frac{s}{2} |\xi|^2} W_{\varphi(s)} [V(s) U(s, 0) \psi](x + s \xi, \xi) ds,
\]
which is developed by the second author, M. Kobayashi and S. Ito ([6], [7]). Thus our proof can be regarded as a variation of Cook–Kuroda’s method by using the wave packet transform.

V. Enss [4] and H. Kitada–K. Yajima [9] use the phase space decomposition operators $P_{\pm,R}$ and $P_{0,R}$ as follows: $P_{\pm,R} f(x) = \int \int \int \int_{|z| \geq R} e^{i(x-y) \xi} g^\pm(z, \xi) \eta(y - z) f(y) dy dz dy dz$ and $P_{0,R} f(x) = \int \int \int \int_{|z| < R} e^{i(x-y) \xi} \eta(y - z) f(y) dy dz dy dz$. Here $g^\pm(z, \xi)$ ($g^\pm(z, \xi)$) is smooth cut-off function whose support is contained in the set that $|\xi| \leq a$ and $z \cdot \xi < 0$ ($|\xi| > b$) and $\eta$ is a smooth function such that $\int \eta dx = 1$ and supp $\tilde{\eta}$ is included in a small ball in $\mathbb{R}^n$. Since
Theorem 2 by using our characterization. The ranges of the wave operators (the latter part of Theorem 1). In section 5, we prove (the former part of Theorem 1). In section 4, we give a proof of the characterization of packet transform. In section 3, we give a proof of the existence of the wave operators.

\[ W \]

for some positive constant \( a \) and \( \Phi \) define \( Y \)

Consider the case that Remark 3.

We have

for some positive constant and some sequences (Remark 2).

On the contrary, our proof is simple. We decompose the phase space \( \mathbb{R}^n_\theta \times \mathbb{R}^n_\theta \) into only two parts \( \Gamma_{a,R} \) and \( \Gamma_{a,R}^c \), and estimate the wave packet transform of the solution in each part.

In the case that \( V \) does not depend on \( t \), the following well-known theorem holds for \( H = H_0 + V \). We give an alternative proof of the theorem by using our characterization.

**Theorem 2** (J. Cook [2], S. T. Kuroda [10], E. Mourre [12] and V. Enss [4]). Suppose that (A) be satisfied and that \( V \) do not depend on \( t \). Then the wave operators \( W_\pm = \lim_{t \to \pm \infty} e^{i(t - \tau)H} e^{-i(t - \tau)H_0} \) exist, are independent of \( \tau \) and are strongly complete:

\[
\mathcal{R}(W_\pm) = \mathcal{H}_p^+ (H).
\]

**Remark 2.** We can give "Kitada–Yajima" type characterization spaces via the wave packet transform. Let \( \Phi \in \mathcal{S}_{scat} \) and \( \mathcal{K}_{a,N} = \{ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n | |\xi| \leq a \ or \ |x| \leq N \} \). We define \( \mathcal{V}_{scat}^\pm (\tau) \) by the closure of the set of all functions in \( \mathcal{H} \) such that

\[
\lim_{N \to \pm \infty} \| \chi_{\mathcal{K}_{a,N}} W_\Phi [U(t^\pm_N, \tau)f] \| = 0
\]

for some positive constant \( a \) and some sequences \((t^\pm_N)\) tending to \( \pm \infty \) as \( N \to \infty \).

We have \( \mathcal{R}(W_\pm(\tau)) = \mathcal{V}_{scat}^\pm \), which is remarked in Section 5.

**Remark 3.** Consider the case that \( n \geq 2 \). Let \( a \) be a non-negative constant, \( \sigma \in (0, 1] \) and \( \Phi \in \mathcal{S}_{scat} \). We put

\[
\tilde{\Gamma}_{a,\sigma}^\pm = \left\{ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n | |\xi| \leq a \ or \ \mp \cos \theta(x, \xi) \geq \sigma \right\},
\]

where \( \cos \theta(x, \xi) = (x \cdot \xi)/|x||\xi| \) and double-sign corresponds. Let \( A_{scat}^\pm (\tau) \) be the closure of the set of all functions in \( \mathcal{H} \) satisfying that

\[
\lim_{t \to \pm \infty} \| \chi_{\Gamma_{a,\sigma}^\pm} W_\Phi (t)f(U(t, \tau)f)(x + (t - \tau)\xi, \xi) \| = 0
\]

for some positive constants \( a \) and \( \sigma \in (0, 1) \). Then we get

\[
A_{scat}^\pm (\tau) = \mathcal{R}(W_\pm(\tau)) = D_{scat}^\pm (\tau),
\]

since \( W_\Phi^{-1}(C^\infty_0(\mathbb{R}^{2n} \setminus \tilde{\Gamma}_{0,1}^+)) \) and \( W_\Phi^{-1}(C^\infty_0(\mathbb{R}^{2n} \setminus \tilde{\Gamma}_{0,1}^-)) \) are dense in \( \mathcal{H} \).

The plan of this paper is as follows. In section 2, we recall the properties of the wave packet transform. In section 3, we give a proof of the existence of the wave operators (the former part of Theorem 1). In section 4, we give a proof of the characterization of the ranges of the wave operators (the latter part of Theorem 1). In section 5, we prove Theorem 2 by using our characterization.
2 Wave packet transform

In this section, we briefly recall the properties of the wave packet transform and give the representation of solutions to (1) via wave packet transform, which is introduced in [6], [7].

**Proposition 1.** Let $\varphi \in S \setminus \{0\}$ and $f \in S'$. Then the wave packet transform $W_\varphi f(x, \xi)$ has the following properties:

(i) $W_\varphi f(x, \xi) \in C^\infty (\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$.

(ii) If $f$ and $\varphi$ depend on $t$ and satisfy that $f \in C^1 (\mathbb{R}; S')$ and $\varphi \in C^1 (\mathbb{R}; S)$, we have

$$W_{\varphi(t)}[\partial_t f](t, x, \xi) = -i\xi W_{\varphi(t)} f(t, x, \xi) + W_{\partial_t \varphi(t)} f(t, x, \xi)$$

in $S'$.

(iii) If $f, g \in H$ and $\psi \in S \setminus \{0\}$, we have

$$\langle W_\varphi f, W_\psi g \rangle = \langle \varphi, \psi \rangle_H (f, g)_H = \langle \psi, \varphi \rangle_H (f, g)_H.$$

(iv) The inversion formula $W_\varphi^{-1} [W_\varphi f] = f$ holds for $f \in S'$.

We can easily show Proposition 1.

Let $\varphi_0 \in S \setminus \{0\}$ and $\psi \in H$. We consider the following initial value problem of (1):

$$\begin{cases} i\partial_t u + \frac{1}{2} \Delta u - V(t, x) u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(t_0) = \psi, & x \in \mathbb{R}^n. \end{cases} (6)$$

Let $\varphi(t, x) = e^{-itH_0} \varphi_0(x)$. We have from Proposition 1

$$W_{\varphi(t)} [\Delta u](t, x, \xi) = W_{\Delta \varphi(t)} u(t, x, \xi) + 2i\xi \cdot \nabla_x W_{\varphi(t)} u(t, x, \xi) - |\xi|^2 W_{\varphi(t)} u(t, x, \xi).$$

(6) is transformed to

$$\begin{cases} \left( i\partial_t + i\xi \cdot \nabla_x - \frac{1}{2} |\xi|^2 \right) W_{\varphi(t)} u(t, x, \xi) = W_{\varphi(t)} [V(t) u(t)](x, \xi), \\ W_{\varphi(t_0)} u(t_0, x, \xi) = W_{\varphi_0} \psi(x, \xi), \end{cases}$$

since $W_{\varphi(t)} [i\partial_t u](t, x, \xi) = i\partial_t W_{\varphi(t)} u(t, x, \xi) + W_{i\partial_t \varphi(t)} u(t, x, \xi)$. We have by the method of characteristic curve that

$$W_{\varphi(t)} [U(t, t_0) \psi](x, \xi) = \frac{1}{2} e^{-\frac{1}{2} |t-t_0|^2} W_{\varphi_0} \psi(x - (t-t_0)\xi, \xi)$$

$$- i \int_{t_0}^t e^{-i\frac{1}{2} (t-s)|\xi|^2} W_{\varphi(s)} [V(s) U(s, t_0) \psi](x - (t-s)\xi, \xi) ds. \tag{7}$$

In particular, we have for the case that $V \equiv 0$

$$W_{\varphi(t)} [e^{-itH_0} \psi](x, \xi) = e^{-it\xi} W_{\varphi_0} \psi(x - t\xi, \xi). \tag{8}$$

Combining (7) and (8), we have

$$W_{\varphi(t)} [U(t, t') e^{-it'H_0} \psi](x + t'\xi, \xi) = \frac{1}{2} e^{-\frac{1}{2} |t'-t|^2} W_{\varphi_0} \psi(x + t'\xi, \xi)$$

$$+ i \int_{t}^{t'} e^{-i\frac{1}{2} (t'-s)|\xi|^2} W_{\varphi(s)} [V(s) U(s, t') e^{-it'H_0} \psi](x + s\xi, \xi) ds. \tag{9}$$
3 Existence of the wave operators

In this section, we prove the existence of the wave operators by using the wave packet transform which is defined in the previous section.

The following well-known lemma is used in the proof of Lemma 2.

**Lemma 2.** Let $f \in \mathcal{S}$. Suppose that $\text{supp} \hat{f} \subset K$ with some compact set $K$ which does not contain the origin. For any open set $K' \supset K$ and any non-negative integer $l$, there exists a positive constant $C$ such that

$$|e^{-itH_0}f(x)| \leq C(x)^{-l} \|f\|_\Sigma(l)$$

for any $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ with $x/t \notin K'$ and $t \neq 0$.

**Proof.** See [11] or [5].

Using the above lemma, we obtain the following lemma.

**Lemma 3.** Suppose that (A) be satisfied. Let $a$ and $R$ be positive constants. Then for any $L \in (0, a/6)$ and $\varphi_0 \in \mathcal{S} \setminus \{0\}$ with supp $\hat{\varphi}_0 \subset \{\xi \in \mathbb{R}^n | L/2 < |\xi| < L\}$, there exists a positive constant $C$ satisfying

$$\|W_{\varphi(s)}[V(s)\psi](x + s\xi,\xi)\|_{L^2(\mathbb{R}^{2n}\setminus \Gamma_{a,R})} \leq C(s)^{-\delta}\|\psi\|_\mathcal{H}$$

for any $s \geq 0$ and any $\psi \in \mathcal{H}$.

**Proof.** Let $\rho = a/6$ and let $l$ be an integer satisfying $l \geq \delta + (n + 1)/2$. We put $V_\rho(t,x) = \chi_0(\frac{1}{\rho|t|}x)V(t,x)$, where $\chi_0 \in C^\infty(\mathbb{R}^n)$ satisfies $\chi_0(x) = 1$ for $|x| \geq 1$ and $\chi_0(x) = 0$ for $|x| \leq 1/2$. Thus there exists a positive constant $C$ such that $|V_\rho(t,x)| \leq C(t)^{-\delta}$ for any $t \in \mathbb{R}$ and any $x \in \mathbb{R}^n$.

If $(x,\xi) \in \mathbb{R}^{2n} \setminus \Gamma_{a,R}$, $s \geq \max\{3R/a, 3\}$ and $y \in \mathbb{R}^n$ satisfying $|y - (x + s\xi)| \leq as/3$, then we have

$$|y| \geq |x + s\xi| - |y - (x + s\xi)| \geq (as - R) - \frac{as}{3} \geq \rho(s).$$

In this proof, we write $\{\cdots\}$ as $\{y \in \mathbb{R}^n | \cdots\}$. Since $V_\rho(s,y) = V(s,y)$ for $|y| \geq \rho(s)$, we have by Plancherel’s theorem and [11]

$$\left\|\int_{\{y - (x + s\xi) \leq as/3\}} e^{-isH_0\varphi_0(y - (x + s\xi))}V(s,y)\psi(y)e^{-i\xi y}dy\right\|_{L^2(\mathbb{R}^{2n}\setminus \Gamma_{a,R})}$$

$$\leq \left\|\int_{\{y - x \leq as/3\}} e^{-isH_0\varphi_0(y - x)}V_\rho(s,y)\psi(y)e^{-i\xi y}dy\right\|_{L^2(\mathbb{R}^{2n}\setminus \Gamma_{a,R})}$$

$$\leq \left\|e^{-isH_0\varphi_0(y - x)}V_\rho(s,y)\psi(y)\right\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}$$

$$\leq C(s)^{-\delta}\|\psi\|_\mathcal{H}.$$
On the other hand, Lemma 2 shows that

\[
\left\| \int_{\{|y-(x+s\xi)|>\frac{a}{2}\}} e^{-i\sigma H_0\varphi_0(y-(x+s\xi))} V(s,y)\psi(y)e^{-i\xi y} dy \right\| \\
= \left\| \left( \chi_{\{|y-x|>\frac{a}{2}\}} e^{-i\sigma H_0\varphi_0(y-x)} \right) V(s,y)\psi(y) \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}_y^2)} \\
\leq C(s)^{-1+(n+1)/2} \|\varphi_0\|_{\Sigma(l)} \left\| (y-x)^{-(n+1)/2} V(s,y)\psi(y) \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}_y^2)} \\
= C(s)^{-1+(n+1)/2} \|\varphi_0\|_{\Sigma(l)} \left\| (x)^{-(n+1)/2} \right\|_\mathcal{H} \|\psi\|_\mathcal{H} \\
\leq C(s)^{-\delta} \|\psi\|_\mathcal{H},
\]

since supp \(\hat{\varphi}_0\) \(\subset \{\xi \in \mathbb{R}^n | 0 < |\xi| < a/6\}\).

Hence (11) is obtained. \(\Box\)

We shall use the following lemma in Section 5. Let \(\Gamma_{a}^{b,\pm} = \{(x,\xi) \in \mathbb{R}^{2n} | |\xi| \geq a, |x| \geq b \text{ and } \pm x \cdot \xi \geq 0\}\).

**Lemma 4.** Suppose that (A) be satisfied. Let \(a\) and \(b\) be positive constants. Then for any \(L \in (0,a/6]\) and \(\varphi_0 \in \mathcal{S} \setminus \{0\}\) with supp \(\hat{\varphi}_0 \subset \{\xi \in \mathbb{R}^n | L/2 < |\xi| < L\}\), there exists a positive constant \(C\) independent of \(b\) satisfying

\[
\left\| W_\varphi(s) \left[ V(s)\psi \right] (x+s\xi,\xi) \right\|_{L^2(\Gamma_{a}^{b,\pm})} \leq C(s+b)^{-\delta} \|\psi\|_\mathcal{H}
\]

for any \(s \geq 0\) and any \(\psi \in \mathcal{H}\).

**Proof.** The proof is obtained by the same argument as in the proof of Lemma 3 and the estimate that

\[
|y| \geq |x+s\xi| - |y-(x+s\xi)| \\
\geq \sqrt{|x|^2 + 2x \cdot \xi + |\xi|^2} - \frac{1}{3} a(s+b) \\
\geq \frac{1}{\sqrt{2}} a(s+b) - \frac{1}{3} a(s+b) \geq \frac{1}{6} a(s+b),
\]

if \((x,\xi) \in \Gamma_{a}^{b,\pm}, s \geq (\sqrt{2}+1)/3\) and \(y \in \mathbb{R}^n\) satisfying \(|y-(x+s\xi)| \leq a(s+b)/3\). \(\Box\)

Now we give an alternative proof of the existence of the wave operators \(W_{\pm}(\tau)\) by using wave packet transform.

**Proposition 5** (D. R. Yafaev [14]). Suppose that (A) and (B) be satisfied. Then the wave operators \(W_{\pm}(\tau)\) exist for any \(\tau \in \mathbb{R}\).

**Proof.** Substituting \(V(t-\tau,x)\) for \(V(t,x)\), it suffices to show the case \(\tau = 0\). We prove the existence in the case \(t \rightarrow +\infty\) only. Let \(\Phi \in \mathcal{S}_{\text{scat}}\) and \(u_0 \in \mathcal{H}\).

First, we show the existence of \(W_{\pm}(0)u_0\) for \(W_{\Phi}u_0 \in C_0^{\infty}(\mathbb{R}^{2n} \setminus \{\xi = 0\})\). Let \(a\) and \(R\) be positive constants satisfying

\[
\text{supp } W_{\Phi}u_0 \subset \mathbb{R}^{2n} \setminus \Gamma_{a,R}
\]

(12)
and $\varphi_0 \in S \setminus \{0\}$ satisfying
\begin{equation}
\text{supp } \tilde{\varphi}_0 \subset \left\{ \xi \in \mathbb{R}^n \mid \frac{L}{2} < |\xi| < L \right\} \text{ with } 0 < L \leq \frac{a}{6} \text{ and } |(\Phi, \varphi_0)_H| > 0.
\end{equation}

By (9) and (12), we have for $t \geq 0$
\begin{equation}
(U(0, t)e^{-itH_0}u_0, \psi)_H = (u_0, e^{itH_0}U(t, 0)\psi)_H
\end{equation}
\begin{equation}
= \frac{1}{(\varphi_0, \Phi)_H} (W_\Phi u_0, W_{\varphi_0}[e^{itH_0}U(t, 0)\psi])
\end{equation}
\begin{equation}
= \frac{1}{(\varphi_0, \Phi)_H} \left( W_\Phi u_0, W_{\varphi_0} \psi - i \int_0^t \left[ V(s)U(s, 0)\psi \right] (x + s\xi, \xi) ds \right).
\end{equation}

Lemma 3 and (12) show that for $t' \geq t > 0$
\begin{equation}
\left| \left( W_\Phi u_0, -i \int_t^{t'} W_{\varphi(s)} \left[ V(s)U(s, 0)\psi \right] (x + s\xi, \xi) ds \right) \right|
\end{equation}
\begin{equation}
\leq \|W_\Phi u_0\| \int_t^{t'} \|W_{\varphi(s)} \left[ V(s)U(s, 0)\psi \right] (x + s\xi, \xi)\|_{L^2(\mathbb{R}^{2n}\setminus \Gamma, H)} ds
\end{equation}
\begin{equation}
\leq \|u_0\|_H \int_t^{t'} C(s)^{-\delta} \|U(s, 0)\psi\|_H ds
\end{equation}
\begin{equation}
\leq C(t)^{1-\delta}\|u_0\|_H \|\psi\|_H.
\end{equation}

The above inequality and (14) imply the existence of $W_+(0)u_0$ for $u_0 \in W^{-1}_\Phi(C^\infty_0(\mathbb{R}^{2n}\setminus \{\xi = 0\}))$.

For $u_0 \in H$, the existence of $W_+(0)u_0$ follows from the fact that $W^{-1}_\Phi(C^\infty_0(\mathbb{R}^{2n}\setminus \{\xi = 0\}))$ is dense in $H$. Indeed, let $\varepsilon$ be a fixed positive number. Since $C^\infty_0(\mathbb{R}^{2n}\setminus \{\xi = 0\})$ is dense in $L^2(\mathbb{R}^{2n})$, there exists $\omega \in C^\infty_0(\mathbb{R}^{2n}\setminus \{\xi = 0\})$ satisfying $\|W_\Phi u_0 - \omega\| \leq \varepsilon$. Putting $\tilde{u}_0 = W^{-1}_\Phi \omega$, we have $\|U(0, t')e^{-itH_0}u_0 - U(0, t)e^{-itH_0}u_0\|_H \leq \|U(0, t')e^{-itH_0}\tilde{u}_0 - U(0, t)e^{-itH_0}u_0\|_H + 2\varepsilon$ for any $t' \geq t > 0$. $(U(0, t)e^{-itH_0}u_0)$ is a Cauchy sequence with respect to $t$ as $t \to \infty$ in $H$, so is $(U(0, t)e^{-itH_0}u_0)$.

\section{Characterization of the Range of the Wave Operators}

In this section, we characterize the ranges of the wave operators by the wave packet transform.

\textbf{Proposition 6.} Suppose that (A) and (B) be satisfied. Then we have
\begin{equation}
\mathcal{R}(W_\pm(t)) = D^{\pm, \Phi}_{\text{scat}}(t)
\end{equation}
for any $\Phi \in S_{\text{scat}}$.

\textbf{Proof.} It suffices to prove that $\mathcal{R}(W_+(0)) = D^{+, \Phi}_{\text{scat}}(0)$, since the proposition can be proved in the same way in the other cases.
Let $\Phi \in \mathcal{S}_{scat}$ and $\varepsilon$ be a fixed positive number. Until the end of the proof, we abbreviate $W_+ = W_+(0)$, $D_{scat}^+ = D_{scat}^{\gamma} + (0)$ and $\tilde{D}_{scat}^+ = \tilde{D}_{scat}^{\gamma} + (0)$.

We first prove that $\mathcal{R}(W_+) \subset D_{scat}^+$. Let $f \in \mathcal{R}(W_+)$ and we fix $g \in W_+^{-1}(C_0^\infty(\mathbb{R}^{2n}\setminus\{\xi = 0\}))$ satisfying

\begin{equation}
\|f - W_+g\|_H \leq \varepsilon.
\end{equation}

Then there exist positive constants $a$ and $R$ such that $\chi_{\Gamma_{a,R}}(x,\xi)W_\Phi g(x,\xi) = 0$ for all $(x,\xi) \in \mathbb{R}^{2n}$. By (8) and the definition of $W_+$, we obtain

\[
\lim_{t \to \infty} \|\chi_{\Gamma_{a,R}}W_\Phi(U(t,0)W_+g)(x+it\xi,\xi)\| \\
\leq \lim_{t \to \infty} (\|\chi_{\Gamma_{a,R}}W_\Phi(t)(e^{-itH_0}g)(x+it\xi,\xi)\| + \|W_\Phi(t)[U(t,0)W_+g - e^{-itH_0}g]\|) \\
= \|\chi_{\Gamma_{a,R}}W_\Phi g\| + \lim_{t \to \infty} \|U(t,0)W_+g - e^{-itH_0}g\| \\
= 0.
\]

Hence we have $W_+ g \in \tilde{D}_{scat}^+$, which and (15) show $\mathcal{R}(W_+) \subset D_{scat}^+$. It suffices to prove that (16) exists for $u_0 \in \tilde{D}_{scat}^+$, since $D_{scat}^+$ is the closure of $\tilde{D}_{scat}^+$.

Let $u_0 \in \tilde{D}_{scat}^+$ and let $a$ and $R$ be positive constants satisfying

\begin{equation}
\lim_{t \to \infty} \|\chi_{\Gamma_{a,R}}W_\Phi(U(t,0)u_0)(x+it\xi,\xi)\| = 0.
\end{equation}

Until the end of the proof, we abbreviate $\Gamma = \Gamma_{a,R}$ and $\Gamma^c = \mathbb{R}^{2n} \setminus \Gamma$. Take $\varphi_0 \in S \setminus \{0\}$ satisfying (13). By (5), we have for $t' \geq t > 0$

\[
(e^{itH_0}U(t,0)u_0,\psi)_H = (U(t,0)u_0, e^{-itH_0}\psi)_H \\
= \frac{1}{(\varphi(t),\Phi(t))_H} \langle W_\Phi(t)[U(t,0)u_0], W_\Phi[t]e^{-itH_0}g \rangle \\
= \frac{1}{(\varphi_0,\Phi)_H} \langle \chi_\Gamma(x - t\xi,\xi)W_\Phi(t)[U(t,0)u_0], W_\Phi[t]e^{-itH_0}\psi \rangle \\
+ \frac{1}{(\varphi_0,\Phi)_H} \langle \chi_{\Gamma^c}(x - t\xi,\xi)W_\Phi(t)[U(t,0)u_0], W_\Phi[t]e^{-itH_0}\psi \rangle
\]

and

\[
(e^{it'H_0}U(t',0)u_0,\psi)_H \\
= (U(t,0)u_0, U(t,t')e^{-it'H_0}\psi)_H \\
= \frac{1}{(\varphi_0,\Phi)_H} \langle (\chi_\Gamma + \chi_{\Gamma^c})(x - t\xi,\xi)W_\Phi(t)[U(t,0)u_0], W_\Phi(t)U(t',t')e^{-it'H_0}\psi \rangle.
\]
Taking the difference between the above equalities, we have
\[
(\varphi_0, \Phi)_H \left( e^{itH_0} U(t, 0)u_0 - e^{it'H_0} U(t', 0)u_0, \psi \right)_H
\]
(18)
\[
= \left( \chi(x - t\xi, \xi) W_{\Phi(t)}[U(t, 0)u_0], W_{\varphi(t)}[e^{-itH_0}\psi - U(t, t')e^{-it'H_0}\psi] \right)
\]
+ \left( W_{\Phi(t)}[U(t, 0)u_0], \chi_{H^\infty}(x - t\xi, \xi) \left( W_{\varphi(t)}[e^{-itH_0}\psi - U(t, t')e^{-it'H_0}\psi] \right) \right).
\]

Using (17), we obtain
\[
\sup_{\|e\|_H=1} \left( \text{(the first term of the right hand side in (18))} \right)
\]
\[
\leq \sup_{\|\psi\|_H=1} \| \chi(x - t\xi, \xi) W_{\Phi(t)}[U(t, 0)u_0] \| \| W_{\varphi(t)}[e^{-itH_0}\psi - U(t, t')e^{-it'H_0}\psi] \|
\]
(19)
\[
\leq 2\|\varphi_0\|_H \| \chi_x W_{\Phi(t)}[U(t, 0)u_0](x + t\xi, \xi) \|.
\]

We have by (8) and (9)
\[
W_{\varphi(t)}[e^{-itH_0}\psi - U(t, t')e^{-it'H_0}\psi](x, \xi)
\]
\[
= -i \int_t^{t'} e^{-i\frac{\xi^2}{2}(t-s)} W_{\varphi(s)} \left[ V(s) U(s, t') e^{-it'H_0}\psi \right] (x - (t - s)\xi, \xi) ds,
\]
which shows by Lemma 3
\[
\left( \text{(the second term of the right hand side in (18))} \right)
\]
\[
= \left( W_{\Phi(t)}[U(t, 0)u_0](x + t\xi, \xi), \chi_{H^\infty}(x, \xi) \int_t^{t'} e^{-i\frac{\xi^2}{2}(t-s)} ds \right.
\]
\[
\times W_{\varphi(s)} \left[ V(s) U(s, t') e^{-it'H_0}\psi \right] (x + s\xi, \xi) ds \right)
\]
\[
\leq C \|u_0\|_H \int_t^{t'} \left\| W_{\varphi(s)} \left[ V(s) U(s, t') e^{-it'H_0}\psi \right] (x + s\xi, \xi) \right\|_{L^2(\Gamma^\infty)} ds
\]
\[
\leq C \|u_0\|_H \| \tau^{-\delta+1}\|\psi\|_H.
\]
(16) follows from (19) and (20).

Theorem 1 is obtained by Proposition 5 and 6.

5 Proof of Theorem 2 by our characterization

In this section, we give an alternative proof of Theorem 2 by using our characterization.

Proof. We shall only prove for \( \tau = 0 \) and for the case that \( t \to +\infty \). We fix \( \Phi \in S_{\text{scat}} \).
We use the same notations \( D_{\text{scat}} = D_{\text{scat}}^+(0) \) and \( \tilde{D}_{\text{scat}} = \tilde{D}_{\text{scat}}^+(0) \) as in the proof of Proposition 3

Firstly, we prove \( H_p(H)^\perp \supset D_{\text{scat}}^+ \). For \( u_0 \in \tilde{D}_{\text{scat}}^+ \), we have
\[
\lim_{\ell \to \infty} \| \chi_{\Gamma_{a,\ell}} W_{\Phi(t)}[e^{-itH} u_0](x + t\xi, \xi) \| = 0
\]
(21)
for some positive constants $a$ and $R$. On the other hand, for $\omega \in \mathcal{H}_p(H)$, we have $\omega = \sum_{n=1}^{\infty} a_n \omega_n$ where $a_n \in \mathbb{C}$ with $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ and $\omega_n$ is the normalized eigenfunction of $H$ corresponding to the eigenvalue $\lambda_n$. Then we see that $e^{-itH} \omega = \sum_{n=1}^{\infty} a_n e^{-it\lambda_n} \omega_n$ for any $t \in \mathbb{R}$. Taking $\varphi_0 \in \mathcal{S}$ satisfying (13), we get for $t \geq 0$

$$\left( e^{-itH} u_0, e^{-it\lambda_n} \omega_n \right)$$

(22)

$$= \frac{1}{\left( \varphi_0, \Phi \right)_H} \left( \chi_{\Gamma_n, R} W_{\Phi(t)}[e^{-itH} u_0](x + t\xi, \xi), e^{-it\lambda_n} W_{\Phi(t)} \omega_n(x + t\xi, \xi) \right) + \frac{1}{\left( \varphi_0, \Phi \right)_H} \left( W_{\Phi(t)}[e^{-itH} u_0](x + t\xi, \xi), \chi_{\Gamma_n, R} e^{-it\lambda_n} W_{\Phi(t)} \omega_n(x + t\xi, \xi) \right).$$

From (21), the first term of (22) is estimated by

$$\lim_{t \to +\infty} \left\{ \chi_{\Gamma_n, R} W_{\Phi(t)}[e^{-itH} u_0](x + t\xi, \xi), e^{-it\lambda_n} W_{\Phi(t)} \omega_n(x + t\xi, \xi) \right\} \leq C \|\omega_n\|_H \lim_{t \to +\infty} \|\chi_{\Gamma_n, R} W_{\Phi(t)}[e^{-itH} u_0](x + t\xi, \xi)\| = 0.$$

The second term of (22) is estimated by

$$\lim_{t \to +\infty} \left( W_{\Phi(t)}[e^{-itH} u_0](x + t\xi, \xi), \chi_{\Gamma_n, R} e^{-it\lambda_n} W_{\Phi(t)} \omega_n(x + t\xi, \xi) \right)$$

(24)

$$\leq \|u_0\|_H \lim_{t \to +\infty} \|\chi_{\Gamma_n, R} e^{-it\lambda_n} W_{\Phi(t)} \omega_n(x + t\xi, \xi)\|$$

$$\leq \|u_0\|_H \lim_{t \to +\infty} \left\{ \chi_{\Gamma_n, R} e^{-it\lambda_n} W_{\Phi(t)} \omega_n(x + t\xi, \xi) \right\}$$

$$\leq \|u_0\|_H \lim_{t \to +\infty} \|\chi_{\Gamma_n, R} e^{-it\lambda_n} W_{\Phi(t)} \omega_n(x + t\xi, \xi)\|$$

$$\leq \|u_0\|_H \lim_{t \to +\infty} \|F(|x| > at/3)\|_{L^2} \|\omega_n\|_H + \|\varphi_0\|_H \|\omega_n\|_H$$

$$= 0.$$

(21) follows from Lemma 2. (23) and (24) implies $(u_0, \omega)_H = 0$. Thus we obtain $\mathcal{H}_p(H)^{\perp} \supset D_{\text{scat}}^+$. Secondly, we prove $\mathcal{H}_p(H)^{\perp} \subset D_{\text{scat}}^+$. We put $\tilde{\mathcal{H}}_p(H)^{\perp} = \{ E_H((a', b'))f | f \in \mathcal{H}_p(H)^{\perp} \}$ and $0 < a' < b'$, then $\tilde{\mathcal{H}}_p(H)^{\perp}$ is dense in $\mathcal{H}_p(H)^{\perp}$, where $E_H(B)$ is a spectral measure of $H$ for a Borel set $B$. Thus it suffices to prove $\tilde{\mathcal{H}}_p(H)^{\perp} \subset D_{\text{scat}}^+$. Let $f \in \mathcal{H}_p(H)^{\perp}$ and be a positive constant $d$ and $\phi \in C_0^\infty([0, \infty))$ satisfying that $\phi(H)f = f$ and that $\phi \equiv 0$ on $[0, d^2/2]$. Since $w\lim_{t \to \infty} e^{-itH} f = 0$ in $\mathcal{H}$ and $(\phi(H) - \phi(H_0))$ is a compact operator on $\mathcal{H}$, we have

$$\lim_{t \to \infty} \|(\phi(H) - \phi(H_0)) e^{-itH} f\|_H = 0.$$

Let $\Xi_d = \{(x, \xi) \in \mathbb{R}^{2n} | |\xi| \leq d \}$ and $\Xi_d^c = \{(x, \xi) \in \mathbb{R}^{2n} | |\xi| > d \text{ and } |x| \leq r \}$. Then there exists a sequence $(t_N)$ tending to $\infty$ as $N \to \infty$ such that

$$\lim_{N \to \infty} \|\chi_{\Xi_d} W_{\Phi}[e^{-it_N H} f]\| = 0$$

and

$$\lim_{N \to \infty} \|\chi_{\Xi_d^c} W_{\Phi}[e^{-it_N H} f]\| = 0.$$
for any positive constant $r$. Indeed, since $\phi(\langle \xi \rangle^2/2)\chi_\Xi(y, \xi) \equiv 0$ for any $(y, \xi) \in \mathbb{R}^n$, we obtain

$$
\phi(H_0)W_\Phi^{-1}[\chi_\Xi \Psi] = \mathcal{F}^{-1} \left[ \phi \left( \frac{|\xi|^2}{2} \right) \mathcal{F}W_\Phi^{-1}[\chi_\Xi \Psi] \right]
= \mathcal{F}^{-1} \left[ \phi \left( \frac{|\xi|^2}{2} \right) \left[ \int \Phi(x-y)\chi_\Xi(y, \xi)\Psi(y, \xi)dy \right] \right] = 0.
$$

Thus we have for $t \geq 0$

$$(\chi_\Xi \Psi) e^{-itHf}(x, \xi, \Psi)
= (\phi(H)e^{-itHf}, W_\Phi^{-1}[\chi_\Xi \Psi])\mathcal{H}
= (e^{-itHf}, \phi(H_0)W_\Phi^{-1}[\chi_\Xi \Psi])\mathcal{H}
+ ((\phi(H) - \phi(H_0)) e^{-itHf}, W_\Phi^{-1}[\chi_\Xi \Psi])\mathcal{H}
= (\phi(H) - \phi(H_0)) e^{-itHf}, W_\Phi^{-1}[\chi_\Xi \Psi] \mathcal{H}.
$$

(25), (28), and the fact that $\|W_\Phi^{-1}\chi_\Xi \Psi\|_\mathcal{H} \leq \|\Psi\|$ yield (29). Take $\Phi' \in C^\infty_0(\mathbb{R}^n)$ with $|\langle \Phi', \Phi \rangle| > 0$. For any positive constant $r'$ satisfying $\chi_\Xi \Psi \in W_\Phi[e^{-itHf}(x, \xi) = \chi_\Xi \Psi \Phi'[F(|x| < r') e^{-itHf}] (x, \xi)$ for any $t \geq 0$. By the RAGE theorem (11), there exists a sequence $(t_N)$ tending to $\infty$ as $N \to \infty$ such that $\lim_{N \to \infty} \|F(|x| < r') e^{-it_NHf} \|_\mathcal{H} = 0$. Hence we get (27).

For any positive number $\varepsilon$, we take a positive constant $r$ sufficiently large so that $C\|f\|_\mathcal{H}(r)^{1-\delta} \leq \varepsilon/4$. Since $(\chi_\Xi + \chi_\Xi + \chi_{\Gamma_d} + \chi_{\Gamma_d} - 1) = 1$ for almost all $(x, \xi) \in \mathbb{R}^n$, we have for $t, t' \geq t_N$

$$
(e^{itH_0} e^{-it'f} - e^{itH_0} e^{-itHf}, \psi)\mathcal{H}
= (W_\Phi[e^{-itHf}, W_\Phi \left( (e^{-it(t-N-t')H} e^{-itH0} - e^{-it(t-N-t'H} e^{-itH0}) \psi \right)]
= ((\chi_\Xi + \chi_\Xi + \chi_{\Gamma_d} - 1) W_\Phi[e^{-itHf}, \Phi \left( (e^{-it(t-N-t')H} e^{-itH0} - e^{-it(t-N-t')H} e^{-itH0}) \psi \right)]
+ (W_\Phi[e^{-itHf}, \chi_{\Gamma_d} (I(t' - t_N; x, \xi) - I(t - t_N; x, \xi))),
$$

where $I(t; x, \xi) = -i \int_0^t e^{i(s-t)\langle |\xi|^2 \rangle W_\Phi(s) [V e^{-i(s-t)H} e^{-i(t+t_N)H} \psi] (x + s\xi, \xi) ds$.

By the equality (7) with $t = 0$ and $t_0 = -t_N$, we have

$$
(\chi_{\Gamma_d} - 1) W_\Phi[e^{-itHf}] (x, \xi)
= (\chi_{\Gamma_d} - 1) W_\Phi(U(0, -t_N)f)(x, \xi)
= (\chi_{\Gamma_d} - 1) W_\Phi(-t_N)f(x - t_N \xi, \xi)
- \int_0^{t_N} e^{it(t+s+t_N)\langle |\xi|^2 \rangle \chi_{\Gamma_d} - 1 W_\Phi(-s)[V e^{i(s+t_N)H} f](x - s\xi, \xi) ds.
$$

Lemma 4 shows that

$$
\|\chi_{\Gamma_d} - I(t - t_N; x, \xi)\| \leq C_0 \|\psi\|_\mathcal{H} \int_0^{t-t_N} (s + r)^{-\delta} ds \leq C \|\psi\|_\mathcal{H}(r)^{1-\delta}
$$
and

\[ \int_0^{t_N} \| \chi_{\Gamma_{a}^+} W_{\Phi(-s)}[Ve^{i(s+t_N)H}]f(x-s\xi,\xi)\| ds \leq C\| f \|_{H^r(\mathbb{R})}^{1-\delta}, \]

where \( C \) and \( C' \) is a positive constant and independent of \( r \) and \( N \). Taking \( \varphi_0 \in \mathcal{S} \) satisfying \((13)\) with \( a = d \), we have by Lemma 2

\[ \lim_{N \to \infty} \| \chi_{\Gamma_{a}^+} W_{\Phi(-t_N)} f(x-t_N\xi,\xi) \|
\]

By \((29)\), \((31)\) and \((32)\), we obtain

\[ \limsup_{N \to \infty} \| \chi_{\Gamma_{a}^+} W_{\Phi}[e^{-itN H} f] \| \leq \varepsilon. \]
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