Accuracy of the SWKB condition for the novel classes of exactly solvable systems

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The supersymmetric WKB (SWKB) quantization condition is supposed to be exact for all known exactly solvable quantum mechanical systems with the shape-invariant potentials. Recently, Bougie et al. [1] claimed that the SWKB formalism was not exact for the extended radial oscillator whose eigenfunctions consisted of the exceptional orthogonal polynomial. In this Letter, we examine the SWKB conditions for the two novel classes of exactly solvable systems: one has the multi-indexed Laguerre and Jacobi polynomials as the main parts of the eigenfunctions, and the other has the Krein–Adler Hermite, Laguerre and Jacobi polynomials. For all of them, the conditions are only satisfied within a finite uncertainty.

I. INTRODUCTION

The well-known WKB quantization condition is given by

$$\int_{x_l}^{x_R} \sqrt{\varepsilon_n - V(x)} \, dx = \left( n + \frac{1}{2} \right) \pi \hbar \quad (n \in \mathbb{Z}_{\geq 0}) ,$$

in which $x_l$ and $x_R$ are the “turning points”; $V(x_l) = V(x_R) = \varepsilon_n$. In the context of supersymmetric quantum mechanics (SUSY QM) [2, 3], the potential $V(x)$ is formally given by the groundstate eigenfunction of the system $\phi_0(x)$ as

$$V(x) = \hbar^2 \left[ (\partial_x \ln |\phi_0(x)|)^2 + \partial_x^2 \ln |\phi_0(x)| \right] .$$

It should be emphasized that the above potential $V(x)$ corresponds to the vanishing groundstate energy, $\varepsilon_0 = 0$.

On the other hand, a WKB-like quantization condition in SUSY QM (SWKB) proposed by [4] reads

$$\int_a^b \sqrt{\varepsilon_n - (\hbar \partial_x \ln |\phi_0(x)|)^2} \, dx = n\pi \hbar \quad (n \in \mathbb{Z}_{\geq 0}) ,$$

where $a, b$ are the roots of $(\hbar \partial_x \ln |\phi_0(x)|)^2|_{x=a,b} = \varepsilon_n$.

By construction SWKB condition is exact for the groundstate $\varepsilon_0 = 0$. For all conventional shape-invariant potentials, it has been demonstrated that the SWKB condition [3] reproduces the exact bound-state spectra by Dutt et al. [6]. It is well-known that the shape invariance is a sufficient (but not a necessary) condition for the exact solvability of the Schrödinger equation. A natural question that arises here is whether or not the shape invariance of the potential is necessary in order that the SWKB method reproduces the exact bound-state spectrum. This was already discussed by Khare et al. [6] with the Ginocchio potential and also a potential which is isospectral to the 1-d harmonic oscillator, both of which are exactly solvable but are not shape invariant. They arrived at the conjecture that the shape invariance could be a necessary condition for the SWKB condition.

Recently Bougie et al. [1] showed that the SWKB formalism is not always exact for shape-invariant potentials. Their claim is important as it shows that the shape invariance is not sufficient for the SWKB exactness. In their analysis, expansion formula with $a_i$ for a fixed parameter $a_i$ of $O(h^2)$ are used. They are not correct, which makes the derivation in [1] incorrect. Instead, in the next section, we shall numerically show that the SWKB condition is not always exact for shape-invariant potentials.

As was mentioned in [1], any qualitative argument or proof of the exactness of the SWKB condition is still absent at present, and then the conjecture of [6] has not been proved yet. We now realize that there is a possibility that the shape invariance plays no major role for the exactness of the SWKB condition. In order to come closer to the understanding of this question, it is worthwhile to examine the SWKB condition for new examples which are all exactly solvable but are not shape invariant. Ref. [6] reported the SWKB non-exactness of the above-mentioned potentials which are not shape invariant. Around the same time, it was also pointed out that the SWKB condition is neither exact nor never worse than WKB for some cases of solvable potentials by DeLaney et al. [7]. In this Letter, we study the systems with the multi-indexed Laguerre and Jacobi polynomials [8] and the Krein–Adler systems [9, 10] as the main parts of the eigenfunctions. They are obtained by deforming the harmonic oscillator (H), the radial oscillator (L) and the Pöschl–Teller potential (J) (we set $2m = 1$):

$$\mathcal{H}^{(\cdot)} = -\hbar^2 \partial_x^2 + V^{(\cdot)}(x) ,$$

$$V^{(\cdot)}(x) = \begin{cases} 
\omega^2 x^2 - \hbar \omega & = \text{H} \\
\omega^2 x^2 + \frac{\hbar^2 g(g-1)}{x^2} - \hbar \omega (2g+1) & = \text{L} \\
\frac{\hbar^2 g(g-1)}{\sin^2 x} + \frac{\hbar^2 h(h-1)}{\cos^2 x} - \hbar^2 (g+h)^2 & = \text{J}
\end{cases}$$

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with the Schrödinger equations

$$\mathcal{H}^{(s)}(\phi_n^{(s)}(x) = \mathcal{E}_n^{(s)}(\phi_n^{(s)}(x)) \quad (n \in \mathbb{Z}_{\geq 0}),$$

$$\mathcal{E}_n^{(s)} = \begin{cases} 2n\hbar \omega, & \text{if } s = H \\ 4nh \omega, & \text{if } s = L \\ 4\hbar^2 n(n + g + h), & \text{if } s = J \end{cases},$$

$$\phi_0^{(s)}(x) = \begin{cases} e^{-\frac{\hbar^2 x^2}{2}}, & \text{if } s = H \\ e^{-\frac{\hbar^2 x^2}{2}x^g}, & \text{if } s = L \\ (\sin x)^g(\cos x)^h, & \text{if } s = J \end{cases}.$$}

The shape invariance is achieved by $g \to g + 1$ and $h \to h + 1$. We note that the significant parts of the eigenfunctions $\{\phi_n(x)\}$ are described by Hermite $H_n$ for the harmonic oscillator (H), Laguerre $L_n^{(a)}$ for the radial oscillator (L) and Jacobi polynomials $P_n^{(a,b)}$ for the Pöschl–Teller potential (J) respectively. We shall see that for the Klein–Adler systems the SWKB formalism is exact only for the groundstates $n = 0$, but they could be a good approximation especially for sufficiently higher excited states $n \gg d$. We shall discuss the mathematical implications in detail.

Much of the literature employ the unit $\hbar = c = 1$ to simplify their analysis. However, the WKB formalism is discussed within the semi-classical regime $\hbar \to 0$. Thus, for the rigorous discussions, in this Letter we retain $\hbar$. However, as we will see shortly in eqs. (20), (22), (34), (35) and (36), the SWKB conditions for these systems are totally independent of $\hbar$.

II. MULTI-INDEXED SYSTEMS AND THE SWKB

In [1], the authors discussed non-exactness of the SWKB condition for the additive shape-invariant potentials. They employed the extended radial oscillator, which is equivalent to the exceptional Laguerre or the type II $X_1$-Laguerre polynomial [11][14] possessing the shape invariance. They alleged that the additive shape invariance was realized for the parameters $a_i$ such that $a_{i+1} = a_i + h$ (also in [15][16]). Their analysis was based on the expansion of the superpotential $W(a_i, h)$ in power of $h$, assuming that $W$ was independent of $h$ except through the above shift of the parameter $a_i$. For various powers of $h$, they obtained the equations that $W$ should satisfy. The lowest O($h^0$) equation has solutions of the conventional shape-invariant superpotentials. After the above formulation was done, for the computation of the SWKB the authors set $h = 1$ without any explicit reasons. The main drawback in the analysis was that they overlooked the dependence of the parameter $a_i$ on $h$; in the case of radial oscillator, $a_i \equiv h \ell$, $\ell \in \mathbb{Z}_{\geq 0}$, see eqs. (2) and (5). Therefore, we have to say that their formulation and the results are not justified.

Though their analysis seems incorrect, there still remain possibilities that the SWKB condition is not always exact for the shape-invariant potentials. We investigate the conditions for a new class of shape-invariant systems: the multi-indexed Laguerre and Jacobi systems, which contain the exceptional systems as simplest cases.

A. Multi-indexed Laguerre/Jacobi systems

The multi-indexed Laguerre and Jacobi polynomials are obtained through deformations of two of the three exactly solvable systems [1] via the virtual-state wavefunctions $\{\psi_n(x)\}$, which are defined as

$$\psi_n^{(L)}(x) := a^\frac{\hbar^2}{2} L_n^{(\frac{g}{2})}(-z),$$

$$\psi_n^{(J)}(x) := e^{-\frac{\hbar^2 x^2}{2}} L_n^{(\frac{g}{2} - h)}(z),$$

$$\phi_n^{(J)}(x) := \left(\frac{1 - y}{2}\right)^{\frac{1}{2}} \left(1 + y \frac{1}{2}\right)^{\frac{1}{2}} P_n^{(\frac{g}{2} - h)}(y),$$

$$\phi_n^{(J)}(x) := \left(\frac{1 - y}{2}\right)^{\frac{1}{2}} \left(1 + y \frac{1}{2}\right)^{\frac{1}{2}} P_n^{(\frac{g}{2} - h)}(y),$$

where

$$\xi \equiv \sqrt{\frac{\omega}{\hbar}}, \quad z \equiv \xi^2, \quad y \equiv \cos 2x,$$

and the parameters $g, h$ must satisfy

$$L: \quad g > \max \left\{ N + \frac{3}{2}, d_0 + \frac{1}{2} \right\},$$

$$J: \quad g > \max \left\{ N + 2, d_0 + \frac{1}{2} \right\},$$

$$h > \max \left\{ M + 2, d_0 + \frac{1}{2} \right\}.$$
in which \( * = L, J \) and \( W[f_1, \ldots, f_n](x) = \det (D^{-1} f_k(x))_{1 \leq j, k \leq n} \) is the Wronskian. Especially, the groundstates are written as

\[
\phi^{(M,L)}_{D,0}(x) = \frac{W_{D} \left[ \varphi^{(L)}_{d_1}, \ldots, \varphi^{(L)}_{d_M}, \varphi^{(L)}_{d_{M+1}}, \ldots, \varphi^{(L)}_{d_N}, \phi^{(L)}_{0} \right]}{W_{D} \left[ \varphi^{(L)}_{d_1}, \ldots, \varphi^{(L)}_{d_M}, \varphi^{(L)}_{d_{M+1}}, \ldots, \varphi^{(L)}_{d_N} \right]} (\xi),
\]

(17)

\[
\phi^{(M,J)}_{D,0}(x) = \frac{W_{D} \left[ \varphi^{(J)}_{d_1}, \ldots, \varphi^{(J)}_{d_M}, \varphi^{(J)}_{d_{M+1}}, \ldots, \varphi^{(J)}_{d_N}, \phi^{(J)}_{0} \right]}{W_{D} \left[ \varphi^{(J)}_{d_1}, \ldots, \varphi^{(J)}_{d_M}, \varphi^{(J)}_{d_{M+1}}, \ldots, \varphi^{(J)}_{d_N} \right]} (y) \times (1 - y^2)^{M+N}.
\]

(18)

The special cases \( |D| = \{ \ell \} \) and \( |D| = \emptyset \) and \( |D| = \emptyset \) and \( |D| = \ell \) are called the type I and the type II \( X_1 \)-Laguerre/Jacobi system, respectively. In [1], the authors discussed the problem for the type II \( X_1 \)-Laguerre system.

### B. The SWKB quantization condition for the multi-indexed systems

For the multi-indexed systems, the SWKB quantization condition [3] reads

\[
\int_{a}^{b} \sqrt{E^{(M*)}_{D,n}} - \left( \hbar \partial_x \ln \left| \phi^{(M*)}_{D,0}(x) \right| \right)^2 \, dx = n \pi \hbar
\]

\[ (n \in \mathbb{Z}_{>0}). \quad (19) \]

For the case of the radial oscillator \((L)\), with the groundstate eigenfunction [17] and the energy eigenvalue [16] reduces to

\[
I^{(M,L)} := \int_{a'}^{b'} \sqrt{4n - \left( \partial_\xi \ln \left| \phi^{(M,L)}_{D,0}(x) \right| \right)^2} \, d\xi = n \pi
\]

(20)

with \( a' \) and \( b' \) being the roots of the equation

\[
\left( \partial_\xi \ln \left| \phi^{(M,L)}_{D,0}(x) \right| \right)^2 = 4n.
\]

(21)

Note that this formula does not depend upon \( \hbar, \omega \) but depends on \( g \).

Similarly for the Pöschl–Teller potential, the SWKB condition becomes, using the groundstate eigenfunction [18] and the energy eigenvalue [16],

\[
I^{(M,J)} := \int_{a'}^{b'} \sqrt{n(n + g + h) - (1 - y^2) \left( \partial_y \ln \left| \phi^{(M,J)}_{D,0}(x) \right| \right)^2} \times \frac{dy}{\sqrt{1 - y^2}} = n \pi,
\]

(22)

which is independent of \( \hbar \).

### C. Results

We calculate the SWKB conditions numerically to see the accuracy the SWKB conditions for the multi-indexed systems. We plot the values of the integral \( I^{(M*)}/\pi \) and the relative errors for the conditions defined as

\[
\text{Err} := \frac{I^{(M*)} - n \pi}{I^{(M*)}}
\]

(23)

where \( n \) is the number of nodes. In Fig. 1 we show the result of the type II \( X_1 \)-Laguerre system [19], which corresponds to the analysis of [1]. The SWKB condition is not exact; while the errors are less than \( 10^{-2} \). For the larger parameter \( g \), the error decreases and the condition becomes closer to the exact one. The claim of [1] still holds after the explicit \( \hbar \)-dependence is properly taken into account.

Fig. 2 presents the typical examples of the analysis of the cases with the multi-indexed Laguerre and Jacobi polynomials. The plots are for the cases of \(|D| = \{1\}\) and \( |D| = \{2\} \) and \( |D| = \{1, 2\} \) and \( |D| = \{2, 3\} \), with appropriate choices of parameters. The behaviors look similar in all cases; the maximal errors occur around the smaller \( n \) with the values of orders \( \sim 10^{-3} \). For larger \( n \), it gradually reduces and in the limit \( n \to \infty \), the SWKB condition will be restored. The Laguerre system is always underestimated, while the Jacobi is overestimated.

### III. KREIN–ADLER SYSTEMS AND THE SWKB

In this section, we examine the SWKB conditions for the systems with no shape invariance, so-called the Krein–Adler systems. From an exactly solvable Hamiltonian, one can construct infinitely many variants of exactly solvable Hamiltonians and their eigenfunctions by Krein–Adler transformations. The resulting systems are, however, not shape invariant, even if the starting systems are.
FIG. 2. The accuracy of the SWKB condition for the case of (a), (b) the multi-indexed Laguerre systems with \( g = 5 \), and (c), (d) the multi-indexed Jacobi systems with \( (g, h) = (5, 6) \). We choose \( D \) as \((a),(c)\) \([D^1 = \{1\} \text{ and } D^0 = \{2\}]\), and \((b),(d)\) \([D^1 = \{1, 2\} \text{ and } D^0 = \{2, 3\}]\). The blue dots are the value of the integration \( I^{(M,L)}(20), I^{(M,J)}(22) \) and the red squares are the corresponding errors defined by eq. \ref{eq:24}, while the blue line and the red chain line mean that the SWKB condition is exact and also \( \text{Err} = 0 \). The maximal errors \( |\text{Err}| \) are (a) \( 9.7 \times 10^{-4} \), (b) \( 4.7 \times 10^{-3} \), (c) \( 1.3 \times 10^{-3} \), (d) \( 2.2 \times 10^{-3} \), respectively.

FIG. 3. The plot of the square of the superpotential \( \left( \partial_x \ln \phi_{D,n}^{(K,H)}(x) \right)^2 \) for \( d = 4 \). When \( n = 1 \), this system has more than one set of turning points.

A. Krein–Adler systems

The Krein–Adler systems are obtained by the deformations of the three exactly solvable polynomials \cite{4}. We choose the eigenfunctions with \( n \in D = \{d, d + 1\} \) and \( d \in \mathbb{Z}_{>0} \) as the seed solutions, deforming \( \{\phi_n(x)\} \) through the multiple Darboux transformation to obtain the Krein–Adler polynomials. Note that, during the transformation, the \( d \)-th and \( (d + 1) \)-th eigenstates are deleted, and the new systems are no longer shape invariant. The resulting deformed systems read:

\[
H_D^{(K,*)} := H^{(*)} - 2\hbar^2 \partial_x^2 \ln |W[\phi_d, \phi_{d+1}](x)|
\]

with

\[
H_D^{(K,*)} \phi_{D,n}^{(K,*)}(x) = \mathcal{E}_{D,n}^{(K,*)} \phi_{D,n}^{(K,*)}(x),
\]

\[
\phi_{D,n}^{(K,*)}(x) = \frac{W[\phi_d, \phi_{d+1}, \phi_n](x)}{W[\phi_d, \phi_{d+1}]}.
\]

where \( * = H, L, J \) and

\[
n := \begin{cases} n & (0 \leq n \leq d - 1) \\ n + 2 & (n \geq d) \end{cases} (n \in \mathbb{Z}_{\geq 0})
\]

with the number of nodes \( n \). Especially for the ground-state \( (n = 0) \),

\[
\phi_{D,0}^{(K,H)}(x) = e^{-\frac{\xi^2}{2}} \frac{W[H_d, H_{d+1}, 1](\xi)}{W[H_d, H_{d+1}](\xi)},
\]

\[
\phi_{D,0}^{(K,L)}(x) = e^{-\frac{\xi^2}{2}} \frac{z \frac{\partial}{\partial z} + 1}{W[L_d^{(g-\frac{1}{2})}, L_{d+1}^{(g-\frac{1}{2})}](z)},
\]

\[
\phi_{D,0}^{(K,J)}(x) = (1 - y) \frac{\partial y}{\partial y} \frac{P_d^{(g-\frac{1}{2}, h-\frac{1}{2})}, P_{d+1}^{(g-\frac{1}{2}, h-\frac{1}{2})}}{W[P_d^{(g-\frac{1}{2}, h-\frac{1}{2})}, P_{d+1}^{(g-\frac{1}{2}, h-\frac{1}{2})}]}(y),
\]

respectively.

B. The SWKB quantization conditions for the Krein–Adler systems

For the Krein–Adler systems, the SWKB quantization condition \cite{3} reads

\[
\int_a^b \sqrt{\mathcal{E}_{D,n}^{(K,*)} - \left[ \hbar \partial_x \ln \left| \phi_{D,n}^{(K,*)}(x) \right| \right]^2} \text{ d}x = n\pi \hbar (n \in \mathbb{Z}_{>0}).
\]

For the case of the harmonic oscillator \( (H) \), with the groundstate eigenfunction \( \phi_{D,0}^{(K,H)} \) and the energy eigenvalue \( \mathcal{E} \), eq. \ref{eq:3} reduces to

\[
\int_{a'}^{b'} \sqrt{2n - \left( \partial_x \ln \left| \phi_{D,0}^{(K,H)}(x) \right| \right)^2} \text{ d} \xi = n\pi
\]

with \( a' \) and \( b' \) being the roots of the equation

\[
\left( \partial_x \ln \left| \phi_{D,0}^{(K,H)}(x) \right| \right)^2 = 2n
\]

Unlike the conventional shape-invariant systems, however, this equation may possess more than two roots, i.e., there are sets of turning points: \( \{a_i', b_i'\} \) with \( a_i' < b_i' \) (see Fig. 3). The following prescription must be employed for the calculation of the integral in \ref{eq:31}. That is, the SWKB condition for the system is defined as the sum of
For the Pöschl–Teller potential, using the groundstate eigenfunction of the radial oscillator (29) and the energy eigenvalue (26), the SWKB condition is

\[ I^{(K,L)} := \sum_i \int_{a_i}^{b_i} \sqrt{\bar{n} - z} \left( \partial_x \ln \left| \phi^{(K,L)}_{D,0}(x) \right| \right)^2 \, dz = n \pi. \]  

For the Pöschl–Teller potential, using the groundstate eigenfunction (30) and the energy eigenvalue (26), we obtain the SWKB condition:

\[ I^{(K,J)} := \sum_i \int_{a_i}^{b_i} \sqrt{\bar{n} + g + h} - (1 - y^2) \left( \partial_y \ln \left| \phi^{(K,J)}_{D,0}(x) \right| \right)^2 \, dy \times \frac{dy}{\sqrt{1 - y^2}} = n \pi. \]  

For the larger value of the parameters \( g, h \), it is expected that the error decreases, and we confirm the behavior (Figs. 4(b), 6(b)). When we delete higher levels (larger \( d \)), we see different features. (See Figs. 4(b,c), 6(b,c) and 5(c).) Among these, the most distinctive behavior is seen in the

the integrations in the l.h.s. of eq. (32) for all sets of turning points \( \{a_i', b_i'\} \):

\[ I^{(K,H)} := \sum_i \int_{a_i'}^{b_i'} \sqrt{2\bar{n} - \left( \partial_x \ln \left| \phi^{(K,H)}_{D,0}(x) \right| \right)^2} \, d\xi = n \pi. \]  

We note that \( I^{(K,H)} \) does not depend upon \( h, \omega \).

For the radial oscillator (L) and the Pöschl–Teller potential (J), the formulations are done in the same manner. With the groundstate eigenfunction of the radial oscillator (29) and the energy eigenvalue (26), the SWKB condition is

\[ I^{(K,L)} := \sum_i \int_{a_i}^{b_i} \sqrt{\bar{n} - z} \left( \partial_x \ln \left| \phi^{(K,L)}_{D,0}(x) \right| \right)^2 \, dz = n \pi. \]  

\[ I^{(K,J)} \) and \( I^{(K,H)} \) are also independent of \( h \).

C. Results

We examine the SWKB conditions (34)–(36) numerically and show how accurate they are. We compute the cases of \( D = \{3, 4\}, \{15, 16\} \) and \( \{24, 25\} \) for the Hermite. We also calculate the cases of \( D = \{3, 4\} \) and \( \{15, 16\} \) with the parameters \( g = 3, 30 \) for the Laguerre, and with \( (g, h) = \{3, 4\}, \{30, 40\} \) for the Jacobi polynomials. The results are shown in Fig. 4 (Hermite), Fig. 5 (Laguerre) and Fig. 6 (Jacobi). Notable feature of these results is that the maximum of the error occurs at the vicinity of the deleted levels and as moving away from the point, the error decreases and the SWKB condition tends to be exact. Also, the errors tend to be of opposite sign between the below and the above of the deleted level. Note that the behavior at \( n \to 0 \) and \( n \to \infty \) is not symmetrical, i.e., for the smaller \( n \), the value still decreases but seems not to go to the exact condition.
Hermite polynomial. At below the deleted level, the integral value exhibits oscillating behavior around the exact one. The cases of the Laguerre, Jacobi polynomials are more moderate.

IV. CONCLUSION

In this Letter, we studied the exactness of the SWKB conditions for the new classes of exactly solvable systems: the multi-indexed Laguerre and Jacobi polynomials, and the Krein–Adler Hermite, Laguerre and the Jacobi polynomials. The problem of the exactness of the SWKB conditions were already studied for the extended radial oscillator \[ \text{[1]} \] and the results indicated that the SWKB condition was not exact. However, in their analysis, the treatment of the \( \hbar \) was incorrect and the results were problematical. We numerically computed the integrals of the SWKB conditions, and the results clearly indicated that the conditions were always not exact for the systems we study.

This Letter constitutes an initial step for a full understanding of the mathematical implications of the SWKB quantization conditions. The shape invariance of the potentials was supposed to be the key for the condition. From the results of the multi-indexed polynomials, however, we found that the shape invariance was not the sufficient condition of the exactness of the SWKB condition. In order to see the roles of the shape invariance, and the effects of SWKB in the exact solvability of the systems, we examined the SWKB conditions of the Krein–Adler polynomials, which had no explicit shape invariance. The SWKB condition is only approximately satisfied for larger \( n \). As a result, neither the exact solvability nor the shape invariance is responsible for the exactness of the SWKB condition. It is still a mystery that the SWKB conditions for these polynomials are not exact, but near to the exact value. The following problems require solutions:

- The SWKB condition is only approximately satisfied in the new types of exactly solvable polynomials studied in this Letter. It would be a good challenge to identify the mechanism causing the discrepancy.

- Several more exactly solvable systems are known: investigation of the SWKB conditions for the several known conditionally exactly solvable potentials \[ \text{[20]} \] with the parameter dependency may be valuable for the better understanding of the accuracy of the SWKB.

- If we compute the SWKB condition for some nearly exactly solvable systems, such as the quasi-exactly-solvable potentials \[ \text{[21]} \], how are the values?

These issues are currently studied, and the results will be reported in our subsequent papers.

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